M390C NOTES: GEOMETRIC LANGLANDS

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These notes were taken in UT Austin's Math 390c (Geometric Langlands) class in Fall 2016, taught by David Ben-Zvi. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to `a.debray@math.utexas.edu`.

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1. The Fourier Transform in Representation Theory: 8/25/16

"One of the traditions we have at UT is we always have to mention Tate."

The initial conception of this class was going to be more akin to a learning seminar about the geometric Langlands program, but this changed: it's now going to be an actual class, but about geometric representation theory and topological field theory. The goal is for this to turn into good lecture notes and even a book, so the class isn't the entire intended audience. As such, feedback is even more helpful than usual.

It's not entirely clear what the prerequisites for this class are; the level of background will grow as the class goes on. The actual amount of technical background needed to state things precisely is huge, and not a reasonable requirement. As such, the class will be more of a sketch and overview of the ideas and how to think about the main characters[1] in this subject. The professor's seminar (Fridays, from 2 to 4, in the same room) is probably a good place to start understanding this material more rigorously.

There will be an introduction to this class this afternoon at geometry seminar.

The Fourier transform. Do you remember Fourier series? The statement is that for L^2 functions $f : S^1 \to \mathbb{C}$,

$$f(\theta) = \sum_{n\in\mathbb{Z}} \widehat{f}(n) e^{2\pi i n\theta}.$$

This is probably the last precise formula we're going to see in this class, which may reassure you or bother you. We also will identify $S^1 \cong \mathrm{U}(1)$. The Fourier coefficients are

$$\widehat{f}(n) = \int_{S^1} f(\theta) e^{-2\pi i n\theta}\,\mathrm{d}\theta.$$

Representation theory starts with this formula.

Relatedly, for an L^2 function $f : \mathbb{R} \to \mathbb{C}$, we have a continuous combination of exponentials with coefficients $\widehat{f}(t)$:

$$f(x) = \int_{\mathbb{R}} \widehat{f}(t) e^{2\pi i x t}\,\mathrm{d}t, \tag{1.1}$$

where

$$\widehat{f}(t) = \int_{\mathbb{R}} f(x) e^{-2\pi i x t}\,\mathrm{d}x.$$

How should we think of these formulas? The exponentials $e^{2\pi i t -}$ are complex-valued functions on $\mathrm{U}(1)$ and $\mathbb{R}$, respectively. But in fact, they land in $\mathbb{C}^\times$, since they don't hit 0, and in fact they have unit norm, so

[1] Pun intended?

they are maps into $\mathrm{U}(1)$. Since $e^{a+b} = e^a e^b$, these are homomorphisms of groups. Moreover, these are the only homomorphisms: if $f(\theta_1+\theta_2) = f(\theta_1)f(\theta_2)$ for an $f : \mathrm{U}(1) \to \mathrm{U}(1)$, then $f(\theta) = e^{2\pi i x\theta}$ for some x, and similarly for functions $\mathbb{R} \to \mathrm{U}(1)$.

In other words, these functions are the *unitary characters* of the domain group: the homomorphisms to $\mathrm{U}(1) \subset \mathrm{GL}(1)$. We can recast these as representations acting through unitary matrices (also *unitary representations*), where an $x \in \mathbb{R}$ acts as multiplication by $e^{2\pi i x t}$ on the (complex) vector space $\mathbb{C}$.

From this viewpoint, we are writing general functions on $\mathrm{U}(1)$ or on $\mathbb{R}$ as linear combinations of characters. This means characters form a "basis." That is, the characters are not strictly a basis, but the space spanned by finite linear combinations of exponentials is dense in any reasonable function space L^2, C^∞, distributions, real analytic functions, L^p spaces, etc. In particular, L^2, smooth, analytic, etc. are conditions on the Fourier coefficients: $f \in L^2(S^1)$ iff $\widehat{f} \in \ell^2$ (the square-integrable sequences of numbers). f is smooth iff its Fourier coefficients are rapidly decreasing (faster than any polynomial).

This is where the analysis of Fourier series takes place: you're interested in different function spaces, and so you're interested in how the coefficients grow. But we're going to ignore it: it's deep and important for analysis, but begins a different track than representation theory. The algebraic content is that algebraic functions (Laurent series) are dense, and we're going to care more about the algebraic side than the analytic side.

Theorem 1.2 (Plancherel)**.** *If $\mathbb{R}$ denotes the x-line and $\widehat{\mathbb{R}}$ denotes the t-line, then the Fourier transform defines a unitary isomorphism $L^2(\mathbb{R}) \xrightarrow{\sim} L^2(\widehat{\mathbb{R}})$.*

This is nice, but doesn't help much for the character-theoretic viewpoint: the exponential $e^{2\pi i x t}$ is not in $L^2(\mathbb{R})$. This is where one uses Schwarz functions.

Definition 1.3. The *Schwarz space* $\mathcal{S}(\mathbb{R})$ is the space of $f \in C^\infty(\mathbb{R})$ such that f and all of its derivatives decrease more rapidly than any polynomial.

The dual space to $\mathcal{S}(\mathbb{R})$, denoted $\mathcal{S}^*$ or $\mathcal{S}'$, is called the space of *tempered distributions*. Our characters $e^{2\pi i x t}$ live in this space, and the Fourier transform extends to a linear homeomorphism $\mathcal{S}'(\mathbb{R}) \cong \mathcal{S}'(\widehat{\mathbb{R}})$.

Thus, it makes sense to define the Fourier transform of the exponential $e^{2\pi i n x}$: we obtain the delta "function" supported at n, δ_n (1 at n and 0 elsewhere), and similarly, the Fourier transform of δ_t is $e^{2\pi i x t}$. That is, the Fourier transform exchanges points and characters; in other words, $\widehat{\mathbb{R}}$ is a sort of moduli space of unitary characters of $\mathbb{R}$.

In some sense, this diagonalizes the group action: if G is either of $\mathbb{R}$ or $\mathrm{U}(1)$, then G acts on itself by translation (both left and right, since G is abelian). Thus, any space of functions on G is acted on by G: an $\alpha \in G$ sends $f \mapsto \alpha * f$ (i.e. $\alpha * f(x) = f(x+\alpha)$). If V is this function space (e.g. $L^2(G)$), then this defines an action of G on V, hence a group homomorphism $G \to \mathrm{End}(V)$. In particular, the exponential $e^{2\pi i x t}$ satisfies

$$\alpha * e^{2\pi i x t} = e^{2\pi i (x+\alpha) t)} = \big(e^{2\pi i x \alpha}\big)\big(e^{2\pi i x t}\big).$$

That is, this exponential is an eigenfunction for $\alpha * -$ for all $\alpha \in G$: characters are joint eigenfunctions, and the Fourier transform is a simultaneous diagonalization.

Succinctly, *the Fourier transform exchanges translation and multiplication*: the translation operator $\alpha * -$ is sent to the multiplication operator $\widehat{f} \mapsto \widehat{\alpha}\widehat{f}$, where $\widehat{\alpha}(t) = e^{2\pi i \alpha t}$. From the perspective of Fourier series, we have a $\mathbb{Z} \times \mathbb{Z}$ matrix with respect to the exponential basis, but only the diagonal entries $\widehat{f}(n)e^{2\pi i n\theta}$ are nonzero.

Before we make this more abstract, let's see what happens to differentiation. Since G is a Lie group, it has a Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}$, in this case $\mathbb{R} \cdot \frac{\mathrm{d}}{\mathrm{d}x}$, the infinitesimal translations at a point. The differential $\frac{\mathrm{d}}{\mathrm{d}x}$ is an infinitesimal translation, and the Fourier transform sends it to a multiplication by $(2\pi i)t$.[2]

Pontrjagin duality. We can generalize this to Pontrjagin duality, which is a kind of Fourier transform involving a locally compact abelian topological group (LCA) G, e.g. $\mathbb{R}$, $\mathbb{Z}$, S^1, $\mathbb{Z}/n$, and any finite products of these, including tori, lattices, and finite-dimensional vector spaces. More exotic examples include the p-adics. There will be more interesting examples in the algebraic world.

Definition 1.4. Let G be an LCA group; then, the *(unitary) dual* of G is $\widehat{G} = \mathrm{Hom}_{\mathsf{TopGrp}}(G, \mathrm{U}(1))$, the set of characters of G, with the topology inherited as a subset of the continuous functions $C(G) = \mathrm{Hom}_{\mathsf{Top}}(G, \mathbb{C})$.

[2] To prove this rigorously, one needs to worry about difference quotients.

We saw that if $G = \mathbb{R}$, then $\widehat{G} = \mathbb{R}$ again, and that if $G = \mathrm{U}(1)$, then $\widehat{G} = \mathbb{Z}$. Conversely, if $G = \mathbb{Z}$, then a homomorphism on G is determined by its value at 1, which can be anything in $\mathrm{U}(1)$, so $\widehat{G} = \mathrm{U}(1)$. If V is a finite-dimensional vector space, then $\widehat{V} = V^*$: any linear functional $\xi \in V^*$ defines a character $v \mapsto e^{2\pi i \langle \xi, v\rangle}$. It's a nice exercise to check that these are all the unitary characters. If $G = \Lambda$ is a lattice, then we obtain its *dual torus* T, and correspondingly a torus goes to its *dual lattice*. Lastly, we have finite abelian groups, e.g. $\mathbb{Z}/n$, which is generated by 1, so we must send 1 to an n^{th} root of unity. Thus, $(\mathbb{Z}/n)^\vee = \mu_n$, the group of n^{th} roots of unity. This is isomorphic to $\mathbb{Z}/n$ again, though in algebraic geometry, where we might not have all roots of unity, things can get more interesting, so it's useful to remember μ_n.

The claim is that the Fourier transform looks exactly the same for any LCA group; maybe we haven't defined too many exciting examples, but this is still noteworthy. We want characters on G to correspond to points on $\widehat{G}$. A point $\chi \in \widehat{G}$ defines a function on G, and correspondingly, a point $g \in G$ defines a function $\widehat{g} : \chi \mapsto \chi(g)$ on $\widehat{G}$, which looks like a nascent Fourier transform. If $g, h \in G$, then $\widehat{gh}(\chi) = \chi(gh) = \chi(g)\chi(h) = \widehat{g}\widehat{h}(\chi)$, so this transform that we're building will start from this duality of the group multiplication and the pointwise product.

One important thing to mention: $\widehat{G}$ is also a group, and in fact is locally compact abelian. The group operation is pointwise product $\chi_1 \cdot \chi_2(g) = \chi_1(g)\chi_2(g)$. This agrees with the group operations for the examples we mentioned.

Theorem 1.5 (Pontrjagin duality)**.** *The natural map $G \mapsto \widehat{\widehat{G}}$ defined by $g \mapsto \widehat{g}$ is an isomorphism of topological groups.*

Hence, this really is a duality. Nonetheless, we'll maintain the distinction between G and $\widehat{G}$: soon we'll try to generalize to nonabelian groups, and then symmetry will break.

Theorem 1.6 (Fourier transform)**.** *If G is an LCA group, then the Fourier transform map*

$$f \longmapsto \widehat{f}(\chi) = \int_G f(g) \cdot \chi(g)\, \mathrm{d}g,$$

where $\mathrm{d}g$ is the Haar measure on G,[3] *defined an isomorphism of Hilbert spaces $L^2(G) \xrightarrow{\sim} L^2(\widehat{G})$.*

Notice that, since the characters on $\mathbb{R}$ are the exponentials and the Haar measure on $\mathbb{R}$ is the usual Lesbegue measure, this generalizes (1.1).

This entire story started in Tate's thesis, which applies Pontrjagin duality to more exotic examples such as $\mathbb{Q}_p$ and $\mathbb{Q}_p^\times$ or even the group $\mathbb{A}^\times$ of *adeles*;[4] see Ramakrishnan-Valenza [1] for a modern take on this subject, including harmonic analysis on LCA groups.

We'll use this to understand all representations of G (well, nice representations). In general, not all representations of G on a space come from functions on G, but we'll be able to use Pontrjagin duality and the group algebra to do something nice.

Function theory. One important philosophy in representation theory is that the action of G on functions on G (nice functions in whichever context we're working in) is the most important, or universal, representation. We'll talk about functions and convolution from a particular perspective that will be useful several times in the class.

Let X be a finite set. Then, $F(X)$, the set of complex-valued functions on X, is unambiguous. The set of measures on X, $M(X)$, is also clear, but there's a natural bijection between them via the counting measure.

Theorem 1.7 (Finite Riesz representation theorem)**.** *There is a natural identification $F(X) = \mathrm{Hom}_{\mathbb{C}}(F(X), \mathbb{C})$.*

This comes from the inner product on $F(X)$

$$\langle f, g\rangle = \sum_{x\in X} f(x)g(x).$$

The more general Riesz representation theorem is about a Hilbert space of functions on $\mathbb{R}$, and is less trivial.

Now, suppose we have two finite sets X and Y. We can form their product, which looks like Figure 1. It's possible to identify $F(X \times Y) = F(X) \otimes F(Y)$, and via a matrix, or an "integral kernel," this space can be

[3] This is only unique up to a scalar, so we need to pick one.

[4] Not to be confused with the musician.

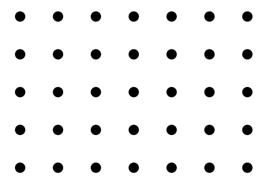

Figure 1. The product of two finite sets.

identified with $\mathrm{Hom}_{\mathbb{C}}(F(X), F(Y))$: a kernel $K(x,y) \in F(X \times Y)$ defines an operator $K * - : F(X) \to F(Y)$ defined by

$$K * f(y) = \sum x \in X K(x,y) f(x).$$

In a broader sense, let $\pi_X : X \times Y \to X$ be projection, and define π_Y similarly. Functions can pull back: $\pi_X^* f(x,y) = f(\pi_X(x,y))$, and measures can push forward by integration (or summing, since we're thinking about the counting measure) over the fibers. Thus, we can recast convolution as

$$K * f = \pi_{Y*}(K \cdot \pi_X^* f)(y) = \int_X K(x,y) f(x)\,\mathrm{d}\#.$$

Since $F(X)$ and $F(Y)$ are finite-dimensional vector spaces, K may be identified with a matrix or a linear transform, and this formula is exactly how to multiply a matrix by a vector.

A key desideratum is that, in general, all nice maps between function spaces on X and function spaces on Y come from integral kernels. For example, a map $L^2(\mathbb{R}) \to L^2(\mathrm{pt}) = \mathbb{C}$ is given by a kernel $K \in L^2(\mathbb{R} \times \mathrm{pt}) = L^2(\mathbb{R})$, realized as $f \mapsto \int K \cdot f$, by the Riesz representation theorem for L^2. Another instance of this is the Schwarz kernel theorem.

Theorem 1.8 (Schwatz kernel theorem)**.** *Let X and Y be smooth manifolds. Then,* $\mathrm{Hom}_{\mathsf{Top}}(C_c^\infty(X), \mathrm{Dist}(Y)) \cong \mathrm{Dist}(X \times Y)$.

Here, $\mathrm{Dist}(-)$ is the space of distributions, dual to compactly supported smooth functions on the manifold.

If $X = Y$ (back in the world of finite sets), then we can consider δ_Δ, the δ-function of the diagonal. In a basis, this is just the identity matrix, and convolution with K is the identity operator. More generally, if $g : X \to Y$ is a set map, then $g^* : F(Y) \to F(X)$ is represented by the kernel of the graph $\Gamma_g \subset X \times Y$: $K = \delta_{\Gamma_g}$. If this all seems a little silly, the key is that it's easier to understand over finite sets, but will work for "nice" functions in a great variety of contexts.

We can also use this to understand matrix multiplication. Given three finite sets X, Y, and Z, and kernels (functions) $K_1 : F(X) \to F(Y)$ and $K_2 : F(Y) \to F(Z)$, we can compose them. Consider the projections

$$\begin{tikzcd}
 & X \times Y \times Z \arrow[dl, "\pi_{12}"'] \arrow[dr, "\pi_{23}"] \arrow[dd, "\pi_{13}"] & \\
X \times Y & & Y \times Z \\
 & X \times Z. &
\end{tikzcd}$$

Exercise 1.9. Show that the formula for $K_2 \circ K_1$ is

$$\pi_{13*}(\pi_{12}^*(K_1) \cdot \pi_{23}^*(K_2)).$$

Relate this to matrix multiplication.

The distinction between functions and measures is irrelevant in the world of finite sets, so we can push-forward and pull back with impunity, but in a continuous setting, it's important to keep them distinct. This equates to choosing a measure (e.g. choosing a Haar measure, as we did above), and even relates to things like Poincaré duality.

2. Representation Theory as Gauge Theory: 8/25/16

Note: this talk was an overview of the class, presented at the weekly geometry seminar.

2.1. **Representation Theory.** Representation theory starts from spectral decomposition and the Fourier transform. If G is a locally compact abelian group, we attach its *unitary dual* $\widehat{G}$, the set of irreducible unitary representations of G. These are all one-dimensional, hence described by characters $\chi : G \to \mathrm{U}(1) \subset \mathbb{C}$. The key idea generalizing the Fourier transform is Pontrjagin duality, that the Fourier transform defines an isomorphism $L^2(G) \cong L^2(\widehat{G})$; there are variants for other function spaces.

Example 2.1.

- Finite Fourier series arise from $G = \mathbb{Z}/n$, for which $\widehat{G} = \mathbb{Z}/n$. The dual of $i \mapsto \zeta^i$, where ζ is a primitive root of unity, is a δ-function supported at i.
- Fourier series exchange $G = \mathrm{SO}(2)$ and $\widehat{G} = \mathbb{Z}$.
- The Fourier transform is for $G = \mathbb{R}$ and $\widehat{G} = \mathbb{R}$.

The Fourier transform takes representation theory of G, and turns it into geometry on $\widehat{G}$. For example, characters of G have turned into points of $\widehat{G}$. The group G can act in different translation-like ways: translation, differentiation (infinitesimal translation), and convolution; all of these are simultaneously diagonalized by the Fourier transform, and made into multiplication. Representations of G are turned into families of vector spaces on $\widehat{G}$, in various forms (vector bundles, sheaves, etc.), in a process called *spectral decomposition*.

This is all really nice: the Fourier transform basically solves representation theory for abelian groups. What should we do for nonabelian G?

We'd like to seek a geometry object $\widehat{G}$ parameterizing irreducible representations (unitary or other classes). This $\widehat{G}$ carries a measure, a topology, and even has algebraic geometry; this structure captures notions of families of representations.

Even though we don't know what $\widehat{G}$ is yet, we know that functions on $\widehat{G}$ should act on representations of G in a way that commutes with the G-action.

Example 2.2. If $G = \mathrm{SO}(3)$, then G acts on S^2 and hence also on $L^2(S^2)$ (the Hilbert space of a quantum free particle on a sphere). This action commutes with the spherical Laplacian Δ, and therefore we can decompose $L^2(S^2)$ into Δ-eigenspaces called *spherical harmonics*:

$$L^2(S^2) \cong \bigoplus_{n \in 2\mathbb{Z}_+} V_n.$$

This says a lot about the unitary irreducible representations of $\mathrm{SO}(3)$.

One of the huge goals of representation theory is to produce a nonabelian analogue of the Fourier transform for *arithmetic locally symmetric spaces* $X_\Gamma = \Gamma \backslash G_{\mathbb{R}}/K$. These are generalizations of the moduli space of elliptic curves: $\mathrm{SL}_2\mathbb{Z}\backslash\mathbb{H} \cong \mathrm{SL}_2\mathbb{Z}\backslash\mathrm{SL}_2\mathbb{R}/\mathrm{SO}_2$, where $\mathbb{H}$ is the upper half-plane.

For every prime p, X_Γ has a hidden p-adic symmetry group $G_{\mathbb{Q}_p}$, along with the manifest $G_{\mathbb{R}}$ symmetry. This creates a huge amount of symmetry, allowing one to define operators called Hecke operators. At almost all primes, these operators commute, so can we simultaneously diagonalize them? This is, in some sense, a goal of the Langlands program (and access the secrets of the universe, hopefully).

2.2. **Quantum field theory.** We've just seen representation theory in a nutshell; now, on to quantum field theory in a nutshell.

An n-dimensional quantum field theory $\mathcal{Z}$ attaches to every n-dimensional Riemannian manifold M a Hilbert space $\mathcal{Z}(M)$. It also has *time evolution*: an n-dimensional cobordism $N : M_1 \to M_2$ defines a linear map $\mathcal{Z}(N) : \mathcal{Z}(M_1) \to \mathcal{Z}(M_2)$. Gluing two cobordisms together corresponds to composing their linear maps.

Quantum field theory should be local, and so there's a great deal of structure that can be tracked to understand this condition.

Example 2.3 (Quantum mechanics)**.** Consider a free particle on a manifold X (e.g. $\mathbb{R}^3$), and let $n = 1$. Here, we'll let $\mathcal{Z}(M)$ be a linearization of a space of fields on M, e.g. in the *σ-model*, these fields are maps to X.

In our case, $\mathcal{Z}(\mathrm{pt})$ is the Hilbert space $L^2(X)$, and time evolution is the semigroup defined by the Hamiltonian, which is the Laplacian: $H = \Delta$. Then, the bordism $[0, T]$ is the evolution e^{iH} (so that gluing becomes composition).

Riemannian manifolds are great, but it is sometimes easier to remove the dependence on metrics. A topological field theory removes a dependence on everything but the topology of spacetime. Sometimes, these appear from another source, which is great, but other times, we have to produce these theories by forcing them. To do this, we need to kill the Hamiltonian. Supersymmetry can do this, by making H exact with respect to, e.g. the de Rham operator, and then passing to cohomology.

A *local operator* is a zero-dimensional defect, which labels measurements at a point of any spacetime. Precisely, we take tiny spheres around these points as the sphere shrinks, which defines cobordisms. For example, in quantum mechanics, local operators in quantum mechanics are the operators on the Hilbert space. These operators do not always commute, which is the statement of Heisenberg uncertainty.

We can also consider defects at higher dimensions, or singularities of higher dimensions. A *line operator* is a one-dimensional quantum mechanics living on a one-dimensional submanifold of the spacetime. These also have a huge amount of structure: they form a category.

Scaling this all the way up, a *local boundary condition* is an $(n-1)$-dimensional theory that labels boundaries in $\mathcal{Z}$. These can interface with each other in codimension 2, and there are interfaces between interfaces between interfaces… this creates the algebraic structure of an $(n-1)$-*category*.

But what does this structure buy us? There's a conjecture of Baez-Dolan, now a theorem of Lurie, that it tells us everything.

Theorem 2.4 (Cobordism hypothesis (Lurie))**.** *An n-dimensional topological field theory $\mathcal{Z}$ is uniquely determined by its higher category of boundary conditions.*

The theorem also contains an existence statement, which corresponds to a finiteness condition. You can start with your favorite $(n-1)$-category, whatever that may be, and when you try to germinate it into data on lower- and lower-dimensional manifolds, it might not be "finite enough." There's a more precise sense in the theorem statement.

In particular, a category is a 1-category, so one can think of categories as 2-dimensional TFTs, illuminating a deep geometric perspective on categories. In particular, we're interested in the category of representations of a group, so we should think about the topological field theory it describes.

2.3. **Gauge theory and moduli spaces.** Gauge theory linearizes spaces of G-bundles with connections, in a way invariant under gauge transformations. This has been tremendously influential in low-dimensional topology, producing many invariants of 3- and 4-manifolds arising from cohomology of moduli paces of G-bundles on these manifolds.

An alternate point of view is that n-dimensional gauge theories are to be understood as QFTs whose boundary conditions are $(n-1)$-dimensional QFTs with G-symmetry; that is, *gauge theories are representations of groups on field theories.*

Example 2.5 (2-dimensional Yang-Mills theory)**.** Suppose G is finite or compact. This theory more or less counts G-bundles with a connection on spacetime; the the boundary condition is quantum mechanics with symmetry group G (i.e. G acts on $L^2(X)$ and the Hamiltonian).

From the topological setting, one can just declare the boundary conditions to be the category of representation sof G, and recover a topological field theory. The local operators are given by functions on G-connections on a very small circle; linearizing this, we get the conjugacy-invariant functions on G, the *class functions* $\mathbb{C}[G/G]$.

There are various ways to compose different operators: *operator product expansion* has these two small circles get closer together and collide. Another alternative is the *little-discs* composition, where we surround two close small circles with a larger circle enclosing them; topologically, this is the same as a pair-of-pants bordism.

Because we can move these small circles around each other, local operators commute.[5] This is surprising: the quantumness of quantum mechanics, its noncommutativity, becomes commutativity in topological field theory.

So the local operators on 2-dimensional Yang-Mills theory are the class functions with convolution, which is a commutative algebra, and is in fact the center of the group algebra $\mathbb{C}[G]$. Abstractly, this is the *Bernstein*

[5]The proof, and the picture, is identical to the picture drawn to show that $\pi_n(X)$ is abelian when $n \geq 2$.

center of the category of representations of G (or, of the category of boundary conditions). That is, local operators are precisely functions on the dual $(\mathbb{C}[\widehat{G}], \cdot)$.

Just like we did with quantum mechanics, we might want to model a quantum field theory as a theory of maps to a target. The target is called the *moduli space of vacua* $\mathfrak{M}_{\mathcal{Z}}$, the universal answer to the question "if I realize my theory as a theory of maps, what does it map into?" Local operators are functions on $\mathfrak{M}_{\mathcal{Z}}$.

As in algebraic geometry, we've found a way to obtain a space $\mathfrak{M}_{\mathcal{Z}}$ from a ring (the Bernstein center of the category of boundary conditions). This sends local operators with operator product expansion to functions on $\mathfrak{M}_{\mathcal{Z}}$ with pointwise multiplication, and line defects in $\mathcal{Z}$ with operator product expansion to sheaves on $\mathfrak{M}_{\mathcal{Z}}$ with tensor product; Yang-Mills provides us an analogue of spectral decomposition.

This looks a lot like Fourier theory. However, the moduli space is discrete, which is arguably not exciting, just as there's not to say formally about the representations of compact groups. However, passing to three-dimensional theories produces a continuous moduli space, just akin to passing to representations of noncompact groups.

In this case, we replace the σ-model (a theory of maps) with a gauge model (a theory of connections). Instead of 1-forms, we have 2-forms, and in the abelian case, this is literally a Hodge star operator.

Physics teaches that in the three-dimensional case, there's a great amount of geometry on these moduli spaces:

(1) If two operators travel in linked loops, we obtain a bracket, which is a Poisson bracket: the moduli space is a Poisson variety.
(2) This has a canonical quantization, called the *Nekrasov Ω-background*.
(3) For gauge theories, or those extending to four dimensions, $\mathfrak{M}_{\mathcal{Z}}$ is what's called a *Seiberg-Witten integrable system*.

In other words, *moduli spaces of gauge theories are precisely the modern geometric setting of representation theory*: both are active research areas.

For example, the A-model is a two-dimensional TFT that measures symplectic geometry, and the B-model models complex geometry. Mirror symmetry can be thought of as a Fourier transform between these models. Since the boundary conditions of a three-dimensional TFT are two-dimensional with a group action, they can capture symmetries in symplectic and complex geometry. The analogue of a Fourier transform in this setting leads to the active program called *symplectic duality*.

2.4. **Electric-magnetic duality.** Beilinson-Drinfeld developed a geometric counterpart to the Langlands program, a kind of harmonic analysis taking place on categories of bundles on a Riemann surface. Instead of focusing on a locally symmetric space X_Γ, we focus on a moduli of bundles $\mathrm{Bun}_G(C)$; the Hilbert space of functions on X_Γ is replaced with a category of sheaves on this moduli space. Operators are replaced with functors, and prime numbers are replaced with points of the Riemann surface.

In particular, Hecke operators are functors, and should be some sort of integral operators (convolutions) on sheaves. If $\mathscr{F}$ is a sheaf and $\mathcal{P}$ is a bundle, we'd like to send $\mathscr{F}(\mathcal{P})$ to a "weighted average" of $\mathscr{F}(\mathcal{P}')$ for nearby bundles $\mathcal{P}'$. Specifically, we'd like to modify at a single point x, and keep everything else the same. This lives on the canonical non-Hausdorff space $C \amalg_{C\setminus x} C$, which has two projections down to the two copies of C. This change is called a *Hecke modification*.

Kapustin and Witten realized this can be interpreted via a 4-dimensional gauge theory: Hecke modification is the creation of a magnetic monopole in the bundle. The worldline of this monopole, called a *'t Hooft line*, is a line defect in the theory.

The fundamental question, important for harmonic analysis (or its analogue), is *why do Hecke operators commute?* There was no reason to expect this, but the operators we described do commute — for the same reason as the commutativity of local operators we described above: modifications at two different points don't interact, and modifications at the same point can be dragged off each other, swapped, and slid back onto each other: since the field theory is topological, these all describe the same operator. This is the same insight as to why higher homotopy groups commute.

Beilinson-Drinfeld axiomatized this multiplication structure, an algebraic structure encoded in collisions of points, into a *factorization algebra*. This provides a geometric theory of operators in conformal field theory (vertex algebras) and quantum field theory.

This was a very fruitful insight: Gaitsgory showed that even if there are singularities (ramified singularities), one recovers the same center as for the ramification. This is a two-dimensional solenoidal defect. This itself had applications by Bezrukavnikov and more, to modular representation theory and more.

Peter Scholze managed to port this back to number theory, producing a geometric source to the commutativity of classical Hecke operators, in the setting of $\operatorname{Spec}\mathbb{Z}$ and $\operatorname{Spec}\mathbb{Z}_p$ (which lies near the prime p in $\operatorname{Spec}\mathbb{Z}$). Very recently, this led to Faurges' conjecture, a physics-inspired conjecture shedding new light on the Langlands conjecture.

In four dimensions, the Hodge star sends 2-forms to 2-forms, so the dual of a gauge theory isn't a σ-model, but rather another gauge theory. This recalls the fact that Maxwell's equations in a vacuum (a gauge theory with gauge group $\mathrm{U}(1)$) is symmetric under Hodge star, which exchanges the roles of electricity and magnetism. A nonabelian generalization, called S-theory, relates a gauge theory with gauge group G to one for its dual group.

Kapustin and Witten interpret geometric Langlands in terms of S-duality: sheaves on the moduli of bundles are boundary conditions for $\mathcal{N} = 4$ super Yang-Mills theory for G, and the Hecke operators correspond to 't Hooft line operators. One can write down an analogue of the Fourier transform.

The physics goes up to eleven! Specifically, M-theory. But the richest known representation-theoretic structure is a six-dimensional theory, known as "theory $\mathcal{X}$."

References

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