M392C NOTES: REPRESENTATION THEORY

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Lecture 1.

Lie groups and smooth actions: 1/18/17

"I've never even seen this many people in a graduate class... I hope it's good."

Today we won't get too far into the math, since it's the first day, but we'll sketch what exactly we'll be talking about this semester.

This class is about representation theory, which is a wide subject: previous incarnations of the subject might not intersect much with what we'll do, which is the representation theory of Lie groups, algebraic groups, and Lie algebras. There are other courses which cover Lie theory, and we're not going to spend much time on the basics of differential geometry or topology. The basics of manifolds, topological spaces, and algebra, as covered in a first-year graduate class, will be assumed.

In fact, the class will focus on the reductive semisimple case (these words will be explained later). There will be some problem sets, maybe 2 or 3 in total. The problem sets won't be graded, but maybe we'll devote a class midsemester to going over solutions. If you're a first-year graduate student, an undergraduate, or a student in another department, you should turn something in, as per usual.

Time for math.

We have to start somewhere, so let's define Lie groups.

Definition 1.1. A *Lie group* G is a group object in the category of smooth manifolds. That is, it's a smooth manifold G that is also a group, with an operation $m : G \times G \to G$, a C^∞ map satisfying the usual group axioms (e.g. a C^∞ inversion map, associativity).

Though in the early stages of group theory we focus on finite or at least discrete groups, such as the dihedral groups, which describe the symmetries of a polygon. These have discrete symmetries. Lie groups are the objects that describe continuous symmetries; if you're interested in these, especially if you come from physics, these are much more fundamental.

Example 1.2. The group of $n \times n$ invertible matrices (those with nonzero determinant) is called the *general linear group* $\mathrm{GL}_n(\mathbb{R})$. Since the determinant is multiplicative, this is a group; since $\det(A) \neq 0$ is an open condition, as the determinant is continuous, $\mathrm{GL}_n(\mathbb{R})$ is a manifold, and you can check that multiplication is continuous. ◀

Example 1.3. The *special linear group* $\mathrm{SL}_n(\mathbb{R})$ is the group of $n \times n$ matrices with determinant 1. This is again a group, and to check that it's a manifold, one has to show that 1 is a regular value of $\det : M_n(\mathbb{R}) \to \mathbb{R}$. But this is true, so $\mathrm{SL}_n(\mathbb{R})$ is a Lie group. ◀

Example 1.4. The *orthogonal group* $\mathrm{O}(n) = \mathrm{O}(n, \mathbb{R})$ is the group of orthogonal matrices, those matrices A for which $A^t = A^{-1}$. Again, there's an argument here to show this is a Lie group. ◀

You'll notice most of these are groups of matrices, and this is a very common way for Lie groups to arise, especially in representation theory.

We can also consider matrices with complex coefficients.

Example 1.5. The *complex general linear group* $\mathrm{GL}_n(\mathbb{C})$ is the group of $n \times n$ invertible complex matrices. This has several structures.

- For the same reason as $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$ is a Lie group.
- $\mathrm{GL}_n(\mathbb{C})$ is also a *complex Lie group*: it's a complex manifold, and multiplication and inversion are not just smooth, but holomorphic.
- It's also a *algebraic group* over $\mathbb{C}$: a group object in the category of algebraic varieties. This perspective will be particularly useful for us.

We can also define the *unitary group* $\mathrm{U}(n)$, the group of $n \times n$ complex matrices such that $A^\dagger = A^{-1}$: their inverses are their transposes. One caveat is that this is *not* a complex Lie group, as this equation isn't holomorphic. For example, $\mathrm{U}(1) = \{z \in \mathbb{C} \text{ such that } |z| = 1\}$ is topologically S^1, and therefore is one-dimensional as a real manifold! This is also $\mathrm{SO}(2)$ (the circle acts by rotating $\mathbb{R}^2$). More generally, a *torus* is a finite product of copies of $\mathrm{U}(1)$. ◀

There are other examples that don't look like this, *exceptional groups* such as G_2, E_6, and F_4 which are matrix groups, yet not obviously so. We'll figure out how to get these when we discuss the classification of simple Lie algebras.

Here's an example of interest to physicists:

Example 1.6. Let q be a quadratic form of signature $(1, 3)$ (corresponding to Minkowski space). Then, $\mathrm{SO}(1, 3)$ denotes the group of matrices fixing q (origin-fixing isometries of Minkowski space), and is called the *Lorentz group*.

Smooth actions. If one wants to add translations, one obtains the *Poincaré group* $\mathrm{SO}(1, 3) \ltimes \mathbb{R}^{1,3}$. ◀

In a first course on group theory, one sees actions of a group G on a set X, usually written $G \curvearrowright X$ and specified by a map $G \times X \to X$, written $(g, x) \mapsto g \cdot x$. Sometimes we impose additional structure; in particular, we can let X be a smooth manifold, and require G to be a Lie group and the action to be smooth, or Riemannian manifolds and isometries, etc.[1]

It's possible to specify this action by a continuous group homomorphism $G \to \mathrm{Diff}(X)$ (or even smooth: $\mathrm{Diff}(X)$ has an infinite-dimensional smooth structure, but being precise about this is technical).

Example 1.7. $\mathrm{SO}(3) := \mathrm{SL}_3(\mathbb{R}) \cap \mathrm{O}(3)$ denotes the group of rotations of three-dimensional space. Rotating the unit sphere defines an action of $\mathrm{SO}(3)$ on S^2, and this is an *action by isometries*, i.e. for all $g \in \mathrm{SO}(3)$, the map $S^2 \to S^2$ defined by $x \mapsto g \cdot x$ is an isometry. ◀

Example 1.8. Let $\mathbb{H} := \{x + iy \mid y > 0\}$ denote the upper half-plane. Then, $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ by Möbius transformations. ◀

Smooth group actions arise in physics: if S is a physical system, then the symmetries of S often form a Lie group, and this group acts on the space of configurations of the system.

Where do representations come into this? Suppose a Lie group G acts on a space X. Then, G acts on the complex vector space of functions $X \to \mathbb{C}$, and G acts by linear maps, i.e. for each $g \in G$, $f \mapsto f(g \cdot -)$ is a linear map. This is what is meant by a representation, and for many people, choosing certain kinds of functions on X (smooth, continuous, L^2) is a source of important representations in representation theory. Representations on $L^2(X)$ are particularly important, as $L^2(X)$ is a Hilbert space, and shows up as the state space in quantum mechanics, where some of this may seem familiar.

Representations.

Definition 1.9. A *(linear) representation* of a group G is a vector space V together with an action of G on V by linear maps, i.e. a map $G \times V \to V$ written $(g, v) \mapsto g \cdot v$ such that for all $g \in G$, the map $v \mapsto g \cdot v$ is linear.

This is equivalent to specifying a group homomorphism $G \to \mathrm{GL}(V)$.[2] Sometimes we will abuse notation and write V to mean V with this extra structure.

If G is in addition a Lie group, one might want the representation to reflect its smooth structure, i.e. requiring that the map $G \to \mathrm{GL}(V)$ be a homomorphism of Lie groups.

The following definition, codifying the idea of a representation that's as small as can be, is key.

Definition 1.10. A representation V is *irreducible* if it has no nontrivial invariant subspaces. That is, if $W \subseteq V$ is a subspace such that for all $w \in W$ and $g \in G$, $g \cdot w \in W$, then either $W = 0$ or $W = V$.

We can now outline some of the goals of this course:

- Classify the irreducible representations of a given group.
- Classify all representations of a given group.
- Express arbitrary representations in terms of irreducibles.

These are not easy questions, especially in applications where the representations may be infinite-dimensional.

Example 1.11 (Spherical harmonics)**.** Here's an example of this philosophy in action.[3]

Let's start with the Laplacian on $\mathbb{R}^3$, a second-order differential operator

$$\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2,$$

which acts on $C^\infty(\mathbb{R}^3)$. After rewriting in spherical coordinates, the Laplacian turns out to be a sum

$$\Delta = \frac{1}{r^2}\Delta_{\mathrm{sph}} + \Delta_{\mathrm{rad}},$$

of spherical and radial parts independent of each other, so Δ_{sph} acts on functions on the sphere. We're interested in the eigenfunctions for this spherical Laplacian for a few reasons, e.g. they relate to solutions to the *Schrödinger equation*

$$\dot{\psi} = \widehat{H}(\psi),$$

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[1] What if X has singular points? It turns out the axioms of a Lie group action place strong constraints on where singularities can appear in interesting situations, though it's not completely ruled out.

[2] This general linear group $\mathrm{GL}(V)$ is the group of invertible linear maps $V \to V$.

[3] No pun intended.

where the Hamiltonian is

$$\widehat{H} = -\Delta + V(r),$$

where V is a potential.

The action of SO(3) on the sphere by rotation defines a representation of SO(3) on $C^\infty(S^2)$, and we'll see that finding the eigenfunctions of the spherical Laplacian boils down to computing the irreducible components inside this representation:

$$V_0 \oplus V_2 \oplus V_4 \oplus \cdots \overset{\text{dense}}{\subseteq} C^\infty(S^2),$$

where the V_{2k} run through each isomorphism class of irreducible representations of SO(3). They are also the eigenspaces for the spherical Laplacian, where the eigenvalue for V_{2k} is $\pm k(k+1)$, and this is not a coincidence since the spherical Laplacian is what's known as a Casimir operator for the Lie algebra $\mathfrak{so}(3)$. We'll see more things like this later, once we have more background. ◀

Lecture 2.

Representations theory of compact groups: 1/20/17

First, we'll discuss some course logistics. There are course notes (namely, the ones you're reading now) and a website, https://www.ma.utexas.edu/users/gunningham/reptheory_spring17.html. We won't stick to one textbook, as indicated on the website, but the textbook of Kirrilov is a good reference, and is available online. The class' office hours will be Monday from 2 to 4 pm, at least for the time being.

The course will fall into two parts.

(1) First, we'll study finite-dimensional representations of things such as compact Lie groups (e.g. $\mathrm{U}(n)$ and $\mathrm{SU}(2)$) and their complexified Lie algebras, reductive Lie algebras (e.g. $\mathfrak{gl}_n(\mathbb{C})$ and $\mathfrak{sl}_2(\mathbb{C})$). There is a nice dictionary between the finite-dimensional representation theories of these objects. The algebra $\mathfrak{sl}_2(\mathbb{C})$ is semisimple, which is stronger than reductive. Every reductive Lie algebra decomposes into a sum of a semisimple Lie algebra and an abelian Lie algebra, and abelian Lie algebras are relatively easy to understand, so we'll dedicate some time to semisimple Lie algebras.

We'll also spend some time understanding the representation theory of reductive algebraic groups over $\mathbb{C}$, e.g. $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{SL}_2(\mathbb{C})$. Again, there is a dictionary between the finite-dimensional representations here and those of the Lie groups and reductive Lie algebras we discussed.

All together, these form a very classical and standard subject that appears everywhere in algebra, analysis, and physics.

(2) We'll then spend some time on the typically infinite-dimensional representations of noncompact Lie groups, such as $\mathrm{SL}_2(\mathbb{R})$ or the Lorentz group $\mathrm{SO}(1,3)$. These groups have interesting infinite-dimensional, but irreducible representations; the classification of these representations is intricate, with analytic issues, yet is still very useful, tying into among other things the Langlands program.

All these words will be defined when they appear in this course.

We'll now begin more slowly, with some basics of representations of compact groups.

Example 2.1. Here are some examples of compact topological groups.

- Finite groups.
- Compact Lie groups such as $\mathrm{U}(n)$.
- The p-adics $\mathbb{Z}_p$ with their standard topology: two numbers are close if their difference is divisible by a large power of p. $\mathbb{Z}_p$ is also a profinite group. ◀

Definition 2.2. Let G be a compact group. A *(finite-dimensional) (continuous) (complex) representation* of G is a finite-dimensional complex vector space V together with a continuous homomorphism $\rho : G \to \mathrm{GL}(V)$.

If you pick a basis, $V \cong \mathbb{C}^n$, so $\mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{C})$. These are the $n \times n$ invertible matrices over the complex numbers, so ρ assigns a matrix to each $g \in G$ in a continuous manner, where $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$, so group multiplication is sent to matrix multiplication. Sometimes it's more natural to write this through the *action map* $G \times V \to V$ sending $(g, v) \mapsto \rho(g) \cdot v$.

The plethora of parentheses in Definition 2.2 comes from the fact that representations may exist over other fields, or be infinite-dimensional, or be discontinuous, but in this part of the class, when we say a representation of a compact group, we mean a finite-dimensional, complex, continuous one.

Example 2.3. Let S_3 denote the *symmetric group on* 3 *letters*, the group of bijections $\{1,2,3\} \to \{1,2,3\}$ under composition. Its elements are written in *cycle notation*: $(1\ 2)$ is the permutation exchanging 1 and 2, and $(1\ 2\ 3)$ sends $1 \mapsto 2$, $2 \mapsto 3$, and $3 \mapsto 1$. There are six elements of S_3: $S_3 = \{e, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$.

For representation theory, it can be helpful to have a description in terms of generators and relations. Let $s = (1\ 2)$ and $t = (2\ 3)$, so $(1\ 3) = sts = tst$, $(1\ 2\ 3) = st$, and $(1\ 3\ 2) = ts$. Thus we obtain the presentation

$$S_3 = \langle s, t \mid s^2 = t^2 = e, sts = tst\rangle. \tag{2.4}$$

The relation $sts = tst$ is an example of a *braid relation*; there exist similar presentations for all the symmetric groups, and this leads into the theory of Coxeter groups.

There's a representation you can always build for any group called the *trivial representation*, in which $V = \mathbb{C}$ and $\rho_{\mathrm{triv}} : G \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$ sends every $g \in G$ to the identity map ($1 \in \mathbb{C}^\times$).

To get another representation, let's remember that we wanted to build representations out of functions on spaces. S_3 is a discrete space, so let's consider the space $X = \{x_1, x_2, x_3\}$ (with the discrete topology). Then, S_3 acts on X by permuting the indices; we want to linearize this.

Let $V = \mathbb{C}[X] = \mathbb{C}x_1 \oplus \mathbb{C}x_2 \oplus \mathbb{C}x_3$, a complex vector space with basis $\{x_1, x_2, x_3\}$. We'll have S_3 act on V by permuting the basis; this is an example of a *permutation representation.* This basis defines an isomorphism $\mathrm{GL}(V) \xrightarrow{\sim} \mathrm{GL}_3(\mathbb{C})$, which we'll use to define a representation. Since s swaps x_1 and x_2, but fixes x_3, it should map to the matrix

$$s \longmapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly,

$$t \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(You should check that these assignments satisfy the relations in (2.4).)

Now something interesting happens. When you think of X as an S_3-space, it has no invariant subsets (save itself and the empty set). But this linearization is no longer irreducible: the element $v = x_1 + x_2 + x_3 \in V$ is fixed by all permutations acting on X: $\rho(g) \cdot v = v$ for all $g \in S_3$.

More formally, let W be the subspace of V spanned by v; then, W is a subrepresentation of V. ◀

Let's take a break from this example and introduce some terminology.

Definition 2.5. Let G be a group and V be a G-representation. A *subrepresentation* or G*-invariant subspace* of V is a subspace $W \subseteq V$ such that for all $g \in G$ and $w \in W$, $g \cdot w \in W$. If V has no nontrivial (i.e. not 0 or V) subrepresentations, V is called *irreducible.*

This means the same G-action defines a representation on W.

If $W \subseteq V$ is a subrepresentation, the quotient vector space V/W is a G-representation, called the *quotient representation*, and the G-action is what you think it is: in coset notation, $g \cdot (v + W) = (gv + W)$. This is well-defined because W is G-invariant.

We can always take quotients, but unlike for vector spaces in general, it's more intricate to try to find a complement: does the quotient split to identify V/W with a subrepresentation of V?

Returning to Example 2.3, we found a three-dimensional representation V and a one-dimensional subrepresentation W. Let's try to find another subrepresentation U of V such that, as S_3-representations, $V \cong W \oplus U$. The answer turns out to be $U = \mathrm{span}_{\mathbb{C}}\{x_1 - x_2, x_2 - x_3\}$.

Claim. U is a subrepresentation, and $V = U \oplus W$.

This isn't as obvious, because neither $x_1 - x_2$ or $x_2 - x_3$ is fixed by all elements of S_3. However, for any $g \in S_3$, $g \cdot (x_1 - x_2)$ is contained in U, and similarly for $x_2 - x_3$. Let's set $U \cong \mathbb{C}^2$ with $x_1 - x_2 \mapsto \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $x_2 - x_3 \mapsto \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$. Then, we can explicitly describe U in terms of the matrices for s and t in $\mathrm{GL}(U) \cong \mathrm{GL}_2(\mathbb{C})$, where the identification uses this basis.

Since $s = (1\ 2)$, it sends $x_1 - x_2 \mapsto x_2 - x_1 = -(x_1 - x_2)$ and $x_2 - x_3 \mapsto x_1 - x_3 = (x_1 - x_2) + (x_2 - x_3)$, so
$$s \longmapsto \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}.$$
In the same way,
$$t \longmapsto \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}.$$
The general theme of finding interesting representations inside of naturally arising representations will occur again and again in this class.

Lecture 3.

Operations on representations: 1/23/17

"I want to become a representation theorist!"
"Are you Schur?"

Last time, we discussed representations of groups and what it means for a representation to be irreducible; today, we'll talk about some other things one can do with representations. For the time being, G can be any group; we will specialize later.

The first operation on a representation is very important.

Definition 3.1. A *homomorphism* of G-representations $V \to W$ is a linear map $\varphi : V \to W$ such that for all $g \in G$, the diagram
$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "g\cdot"] & W \arrow[d, "g\cdot"] \\ V \arrow[r, "\varphi"] & W. \end{tikzcd}$$
This is also called an *intertwiner*, a *G-homomorphism*, or a *G-equivariant* map.

An *isomorphism* of representations is a homomorphism that's also a bijection.[4]

More explicitly, this means φ commutes with the G-action, in the sense that $\varphi(g \cdot v) = g \cdot \varphi(v)$. This is one advantage of dropping the ρ-notation: it makes this definition cleaner.

Remark. If $\varphi : V \to W$ is a G-homomorphism, then $\ker(\varphi) \subseteq V$ is a subrepresentation, and similarly for $\mathrm{Im}(\varphi) \subseteq W$. ◂

The set of G-homomorphisms from V to W is a complex vector space,[5] denote $\mathrm{Hom}_G(V, W)$.

Several constructions carry over from the world of vector spaces to the world of G-representations. Suppose V and W are G-representations.

- The direct sum $V \oplus W$ is a G-representation with the action $g \cdot (v, w) = (g \cdot v, g \cdot w)$. This has dimension $\dim V + \dim W$.
- The tensor product $V \otimes W$ is a G-representation: since it's generated by pure tensors, it suffices to define $g \cdot (v \otimes w)$ to be $(gv) \otimes (gw)$ and check that this is compatible with the relations.
- The dual space $V^* := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is a G-representation: if $\alpha \in V^*$, we define $(g \cdot \alpha)(v) := \alpha(g^{-1}v)$. This might be surprising: you would expect $\alpha(gv)$, but this doesn't work: you want $g \cdot (h\alpha)$ to be $(gh) \cdot \alpha$, but you'd get $(hg) \cdot \alpha$. This is why you need the g^{-1}.
- Since $\mathrm{Hom}_{\mathbb{C}}(V, W)$ is naturally isomorphic to $V^* \otimes W$,[6] it inherits a G-representation structure.

Definition 3.2. Given a G-representation V, the *space of G-invariants* is the space
$$V^G := \{v \in V \mid g \cdot v = v\}.$$

[4] If $f : V \to W$ is an isomorphism of representations, then $f^{-1} : W \to V$ is also a G-homomorphism, making this a reasonable definition. This is a useful thing to check, and doesn't take too long.

[5] Recall that we're focusing exclusively on complex representations. If we look at representations over another field k, we'll get a k-vector space.

[6] This depends on the fact that V and W are finite-dimensional.

This can be naturally identified with $\mathrm{Hom}_G(\mathbb{C}_{\mathrm{triv}}, V)$, where $\mathbb{C}_{\mathrm{triv}}$ is the trivial representation with action $g \cdot z = z$ for all $z \in \mathbb{C}$. The identification comes by asking where 1 goes to.

These are also good reasons for using the action notation rather than writing $\rho : G \to \mathrm{Aut}(V)$, which would require more complicated formulas.

There are a couple of different ways of stating Schur's lemma, but here's a good one.

Lemma 3.3 (Schur)**.** *Let V and W be irreducible G-representations. Then,*

$$\mathrm{Hom}_G(V, W) = \begin{cases} 0, & \text{if } V \not\cong W \\ \mathbb{C}, & \text{if } V \cong W. \end{cases}$$

"Irreducible" is the key word here.

Remark. Schur's lemma requires us to work over $\mathbb{C}$ (more generally, over any algebraically closed field). It also assumes that V and W are finite-dimensional. There are no assumptions on G; this holds in much greater generality (e.g. over $\mathbb{C}$-linear categories). ◀

In general, there's a distinction between "isomorphic" and "equal" (or at least naturally isomorphic); in the latter case, there's a canonical isomorphism, namely the identity. In this case, the second piece of Lemma 3.3 can be restated as saying for any irreducible G-representation V,

$$\mathrm{Hom}_G(V, V) = \mathbb{C} \cdot \mathrm{id}_V.$$

Thus, any G-homomorphism $\varphi : V \to V$ is $\lambda \cdot \mathrm{id}_V$ for some $\lambda \in \mathbb{C}$, and in a basis is a diagonal matrix with every diagonal element equal to λ.

Proof of Lemma 3.3. Suppose $\varphi : V \to W$ is a nonzero G-homomorphism. Thus, $\ker(\varphi) \subset V$ isn't be all of V, so since V is irreducible it must be 0, so φ is injective. Similarly, since $\mathrm{Im}(\varphi) \subset W$ isn't 0, it must be all of W, since W is irreducible, Thus, φ is an isomorphism, so if $V \not\cong W$, the only G-homomorphism is the zero map.

Now, suppose $\varphi : V \to V$ is a G-homomorphism. Since $\mathbb{C}$ is algebraically closed, φ has an eigenvector: there's a $\lambda \in \mathbb{C}$ and a $v \in V$ such that $\varphi(v) = \lambda \cdot v$. Since φ and $\lambda \cdot \mathrm{id}_V$ are G-homomorphisms, so is $\varphi - \lambda\mathrm{id}_V : V \to V$, so its kernel, the λ-eigenspace of φ, is a subrepresentation of V. Since it's nonzero, then it must be all of V, so $V = \ker(\varphi - \lambda\mathrm{id}_V)$, and therefore $\varphi = \lambda\mathrm{id}_V$. ⊠

This is the cornerstone of representation theory, and is one of the reasons that the theory is so much nicer over $\mathbb{C}$.

Corollary 3.4. *If G is an abelian group, then any irreducible representation of G is one-dimensional.*

Proof. Let V be an irreducible G-representation and $g \in G$. Since G is abelian, $v \mapsto gv$ is a G-homomorphism: $g \cdot (hv) = h(g \cdot v)$. By Schur's lemma, the action of g is $\lambda \cdot \mathrm{id}_V$ for some $\lambda \in \mathbb{C}$, so any $W \subseteq V$ is G-invariant. Since V is irreducible, this can only happen when V is 1-dimensional. ⊠

Example 3.5. Let's talk about the irreducible representations of $\mathbb{Z}$. This isn't a compact group, but we'll survive. A representation of $\mathbb{Z}$ is a homomorphism $\mathbb{Z} \to \mathrm{GL}(V)$ for some vector space V; since $\mathbb{Z}$ is a free group, this is determined by what 1 goes to, which can be chosen freely. That is, representations of $\mathbb{Z}$ are in bijection with invertible matrices.

By Corollary 3.4, irreducible $\mathbb{Z}$-representations are the 1-dimensional invertible matrices, which are identified with $\mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$, the nonzero complex numbers. ◀

Our greater-scope goal is to understand all representations by using irreducible ones as building blocks. In the best possible case, your representation is a direct-sum of irreducibles; we'll talk about that case next time.

Lecture 4.

Complete reducibility: 1/25/17

We've discussed what it means for a representation to be irreducible, and irreducible representations are the smallest representations; we hope to build all representations out of irreduicbles. The nicest possible case is complete reducibility, which we'll discuss today.

Suppose G is a group (very generally), V is a representation, and $W \subseteq V$ is a subrepresentation. If $i : W \hookrightarrow V$ denotes the inclusion map, then there's a projection map onto the quotient $V \twoheadrightarrow U := V/W$. This is encoded in the notion of a *short exact sequence*:

$$0 \longrightarrow W \xrightarrow{i} V \xrightarrow{j} U \longrightarrow 0,$$

which means exactly that i is injective, j is surjective, and $\mathrm{Im}(i) = \ker(j)$. The nicest short exact sequence is

$$0 \longrightarrow W \longrightarrow W \oplus U \longrightarrow U \longrightarrow 0,$$

where the first map is inclusion into the first factor and the second is projection onto the second factor. In this case, one says the short exact sequence *splits*. This is equivalent to specifying a projection $V \twoheadrightarrow U$ or an inclusion $W \hookrightarrow V$. Since direct sums are easier to understand, this is the case we'd like to know better.

Example 4.1. We saw last time that a representation of $\mathbb{Z}$ is given by the data of an invertible matrix which specifies the action of 1, akin to a discrete translational symmetry.

Consider the $\mathbb{Z}$-representation V on $\mathbb{C}^2$ given by the matrix

$$A := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

This is not an irreducible representation, because $\binom{1}{0}$ is an eigenvector for A with eigenvalue 1, so

$$W := \left\{ \begin{pmatrix} a \\ 0 \end{pmatrix} \mid a \in \mathbb{C} \right\}$$

is a subrepresentation of V. Since $\binom{1}{0}$ has eigenvalue 1, W is the trivial representation $\mathbb{C}_{\mathrm{triv}}$. Moreover, the quotient V/W is also the trivial representation, so V sits in a short exact sequence

$$0 \longrightarrow \mathbb{C}_{\mathrm{triv}} \longrightarrow V \longrightarrow \mathbb{C}_{\mathrm{triv}} \longrightarrow 0.$$

However, V itself is not trivial, or it would have been specified by a diagonalizable matrix. Thus, V is not a direct sum of subrepresentations (one says it's *indecomposable*), but it's not irreducible! If U is a 1-dimensional subrepresentation of V, then $U = \mathbb{C}v$ for some nonzero $v \in V$. Since $Av \in U$, $Av = \lambda v$ for some $\lambda \in \mathbb{C}$, meaning v is an eigenvector, and in our particular case, $v = e_1$. Thus any direct sum of two nontrivial subrepresentations must be $U + U = U$, which isn't all of V. ◂

We want to avoid these kinds of technicalities on our first trek through representation theory, and fortunately, we can.

Definition 4.2. A representation V of G is *completely reducible* or *semisimple* if every subrepresentation $W \subseteq V$ has a *complement*, i.e. another subrepresentation U such that $V \cong U \oplus W$.

Remark. There are ways to make this more general, e.g. for infinite-dimensional representations, one may want closed subrepresentations. But for finite-dimensional representations, this definition suffices. ◂

A finite-dimensional semisimple representation V is a direct sum of its irreducible subrepresentations. The idea is that its subrepresentations must also be semisimple, so you can use induction.

The terminology "semisimple" arises because *simple* is a synonym for irreducible, in the context of representation theory.

So semisimple representations are nice. You might ask, *for which groups G are all representations semisimple*? To answer this question, we'll need a few more concepts.

Definition 4.3. A representation V of G is called *unitary* if it admits a G-invariant inner product, i.e. a map $B : V \times V \to \mathbb{C}$ that is:

- linear in the first factor and antilinear in the other,[7]
- *antisymmetric*, i.e. $B(v, w)) = \overline{B(w, v)}$,
- *positive definite*, i.e. $B(v, v) \geq 0$ and $B(v, v) = 0$ iff $v = 0$, and
- *G-invariant*, meaning $B(g \cdot v, g \cdot w) = B(v, w)$ for all $g \in G$.

[7]Some people have the opposite convention, defining the first factor to be antilinear and the second to be linear. Of course, the theory is equivalent. Such a form is called a *Hermitian form*, and if it's antisymmetric and positive definite, it's called a *Hermitian inner product*.

The reason for the name is that if V is a unitary representation with form B, then as a map $G \to \mathrm{GL}(V) \cong \mathrm{GL}(\mathbb{C}^n)$, this representation factors through $\mathrm{U}(n)$, the *unitary matrices*, which preserve the standard Hermitian inner product on $\mathbb{C}^n$. So unitary representations are representations of G into some unitary group.

Proposition 4.4. *Unitary representations are completely reducible.*

Proof sketch. How would you find a complement to a subspace? The usual way to do this is to take an orthogonal complement, and an invariant inner product is what guarantees that the orthogonal complement is a subrepresentation.

Let V be a unitary representation and $B(-,-)$ be its invariant inner product. Let $W \subseteq V$ be a subrepresentation, and let

$$U = W^\perp := \{v \in V \mid B(v,w) = 0 \text{ for all } w \in W\}.$$

Then, U is a subrepresentation (which you should check), and $V = W \oplus U$. ⊠

Classifying unitary representations is a hard problem in general. Building invariant Hermitian forms isn't too bad, but making them positive definite is for some reason much harder. In any case, for compact groups there is no trouble.

Proposition 4.5.

(1) *Given an irreducible unitary representation, the invariant form $B(-,-)$ is unique up to multiplication by a positive scalar.*
(2) *If $W_1, W_2 \subseteq V$, where V is unitary and W_1 and W_2 are nonisomorphic irreducible subrepresentations, then W_1 is orthogonal to W_2, i.e. $B(w_1, w_2) = 0$ for all $w_1 \in W_1$ and $w_2 \in W_2$.*

Proof. Part (1) is due to Schur's lemma (Lemma 3.3). Consider the map $B : V \to \overline{V}^*$ defined by $v \mapsto B(v,-)$ (here, $\overline{V}$ is the *conjugate space*, where the action of $a+bi$ on $\overline{V}$ is the action of $a-bi$ on V); in fact, you could use this map to define a unitary structure on a representation. B is a G-isomorphism, so by Schur's lemma, every such isomorphism, derived from every possible choice of Hermitian form, must be a scalar multiple of this one. Since B must be positive definite, this scalar had better be positive. ⊠

One particular corollary of part (1) is that if V is a unitary representation, $V^* \cong \overline{V}$.

So if you care about compact groups (in particular finite groups), this is all you need.

Theorem 4.6 (Maschke)**.** *Any representation of a compact group admits a unitary structure.*

We'll give a first proof for finite groups, then later one for Lie groups; we probably won't prove the most general case.

Proof for finite groups. Let G be a finite group and V be a G-representation. The first step to finding a G-invariant inner product is to find any Hermitian inner prodict, e.g. picking a basis and declaring it to be orthonormal, yielding an inner product B_0 that's probably not G-invariant.

We want to average B_0 over G to obtain something G-invariant, which is why we need finiteness.[8] That is, let

$$B(v,w) := \frac{1}{|G|} \sum_{g \in G} B_0(g \cdot v, g \cdot w).$$

Then, B is a unitary structure: it's still positive definite and Hermitian, and since multiplication by an $h \in H$ is a bijection on G, this is G-invariant. ⊠

Lecture 5.

Some examples: 1/27/17

Last time, we claimed that every finite-dimensional representation of a compact group admits a unitary structure (Theorem 4.6), and therefore is completely reducible. We proved it for finite groups; later today, we'll talk about the Haar measure on a Lie group and how to use it to prove Theorem 4.6. The Haar measure also exists on topological groups more generally, but we won't construct it.

[8]More generally, we could take a compact group, replacing the sum with an integral. Compactness is what guarantees that the integral converges.

Semisimplicity is a very nice condition, and doesn't exist in general, just as most operators aren't self-adjoint, and so generally don't have discrete spectra. This is more than just a metaphor: given a compact group G, let $\widehat{G}$ denote the set of isomorphism classes of irreducible representations of G, sometimes called its *spectrum*. (In some cases, this has topology, but when G is compact, $\widehat{G}$ is discrete.) Using Theorem 4.6, any representation V of G is a direct sum of irreducibles in a unique way (up to permuting the factors):

$$V = \bigoplus_{W \in \widehat{G}} W^{\oplus m_W},$$

where m_W is the *multiplicity* of W in V. The summand $W^{\oplus m_W}$ is called the W*-isotypical component* of V. The analogy with the Fourier transform can be made stronger.

Example 5.1. Let $G = S_3$. We've already seen the trivial representation $\mathbb{C}_{\mathrm{triv}}$ and a representation $V = \mathbb{C}(x_1 - x_2) \oplus \mathbb{C}(x_2 - x_3)$, where S_3 acts by permuting the x_i terms; we showed that V is an irreducible representation of dimension 2.

There's a third irreducible representation $\mathbb{C}_{\mathrm{sign}}$, a one-dimensional (therefore necessarily irreducible) representation $\rho_{\mathrm{sign}} : S_3 \to \mathbb{C}^\times$, where $\rho_{\mathrm{sign}}(\sigma)$ is 1 if σ is even and -1 if σ is odd.

Exercise 5.2. Show that $\mathbb{C}_{\mathrm{triv}}$, V, and $\mathbb{C}_{\mathrm{sign}}$ are all of the irreducible representations of S_3 (up to isomorphism).

We'll soon see how to prove this quickly, though it's not too bad to do by hand. ◀

Example 5.3. The *circle group*, variously denoted S^1, $\mathrm{U}(1)$, $\mathrm{SO}(2)$, $\mathbb{R}/\mathbb{Z}$, or $\mathbb{T}$, is the abelian compact Lie group of real numbers under addition modulo $\mathbb{Z}$. Corollary 3.4 tells us all irreducible representations of S^1 are 1-dimensional.

Since S^1 is a quotient of the additive group $(\mathbb{R}, +)$, we'll approach this problem by classifying the one-dimensional representations of $\mathbb{R}$ and seeing which ones factor through the quotient. Such a representation is a map $(\mathbb{R}, +) \to (\mathbb{C}^\times, \times) \cong \mathrm{GL}_1(\mathbb{C})$; in other words, it turns addition into multiplication.

We know a canonical way to do this: for any $\xi \in \mathbb{C}$, there's a representation $\chi_\xi : t \mapsto e^{\xi t}$. And it turns out these are all of the continuous 1-dimensional representations of $\mathbb{R}$ up to isomorphism.

Exercise 5.4. Prove this: that every 1-dimensional representation $\rho : \mathbb{R} \to \mathbb{C}^\times$ is isomorphic to some χ_ξ. Here's a hint:

(1) First reduce to the case where ρ is C^1. (This requires techniques we didn't ask as prerequisites.)
(2) Now, show that ρ satisfies a differential equation $\rho'(t) = \rho'(0)\rho(t)$. As ρ is a homomorphism, this means $\rho(0) = 1$. Then, uniqueness of solutions of ODEs shows $\rho = \chi_{\rho'(0)}$.

There's something interesting going on here: representations of $\mathbb{R}$ are determined by their derivatives at the identitty. This is a shadow of a grander idea: to understand the representations of a Lie group G, look at representations of its Lie algebra $\mathfrak{g}$.

The noncompactness of $\mathbb{R}$ is also reflected in its representation theory: not all of its representations are unitary. $\mathrm{U}(1)$ is the group of unit complex numbers under multiplication, so χ_ξ is unitary iff $\xi = i\eta$ for $\eta \in \mathbb{R}$. In this case, $\widehat{\mathbb{R}}$ is used to denote the unitary representations, so $\widehat{\mathbb{R}} = i\mathbb{R}$. In particular, it's a group (since $\mathbb{R}$ is abelian), abstractly isomorphic to $\mathbb{R}$, and not discrete (as $\mathbb{R}$ is not compact).

Great; now, when does χ_ξ descend to a representation of S^1? This means that $\chi_\xi(0) = \chi_\xi(1)$, so ξ is an integer multiple of $2\pi i$. Thus, S^1, a compact group, has a discrete set of irreducible representations, and $\widehat{S}^1 = 2\pi i\mathbb{Z}$. You might be able to see the Fourier series hiding in here. ◀

Example 5.5. Let's look at a nonabelian example, $\mathrm{SU}(2)$, the set of matrices

$$\mathrm{SU}(2) := \left\{ \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \mid a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}.$$

As a manifold, this is isomorphic to the unit sphere $S^3 \subset \mathbb{C}^2$, and indeed it's a Lie group, compact and non-abelian.

Today we'll write down some representations; later, we can use theory to prove this is all of them. The first one, as for any matrix group, comes for free: the *standard representation* or *defining representation* uses the preexisting embedding $\rho_{\mathrm{std}} : \mathrm{SU}(2) \hookrightarrow \mathrm{GL}_2(\mathbb{C})$ sending $A \mapsto A$. We also have the trivial representation.

We'll obtain some other representations as functions on a space SU(2) acts on. Matrix multiplication defines a smooth action of SU(2) on $\mathbb{C}^2$, so consider the vector space P of polynomial functions on $\mathbb{C}^2$. Then, SU(2) acts on P as follows: if $f \in P$ and $A \in \mathrm{SU}(2)$,

$$A \cdot f(\mathbf{x}) := f(A^{-1}\mathbf{x}).$$

The A^{-1} term arises because you need composition to go the right way:

$$A(Bf(\mathbf{x})) = f(B^{-1}A^{-1}\mathbf{x}) = f((AB)^{-1}\mathbf{x}) = (AB)f(\mathbf{x}).$$

This is an aspect of a very general principle: if a group acts on the left on a space, then it acts on the right on its space of functions. But you can always turn right actions into left actions using inverses.

This P is an infinite-dimensional representation, but it splits into homogeneous finite-dimensional subrepresentations. Let P_n denote the homogeneous polynomials of degree n, e.g. $P^3 = \mathrm{span}_{\mathbb{C}}\{x^3, x^2y, xy^2, y^3\}$.

Proposition 5.6. *These P_n are irreducible of dimension $n+1$, and form a complete list of isomorphism types of irreducible representations of* SU(2).

In particular, $\widehat{\mathrm{SU}(2)} = \mathbb{Z}_{\geq 0}$. We get a discrete set of representations, since SU(2) is compact, but it's not a group, because SU(2) isn't abelian. Abelian groups are nice because their representations are 1-dimensional, and compact groups are nice because their representations are discrete. Abelian compact groups are even nicer, but there's not so many of those. ◂

In the last few minutes, we'll say a little about integration. Next time, we'll talk about characters and matrix coefficients.

On a topological group, finding the Haar measure is a delicate matter (but you can do it); on a Lie group, it's simpler, but requires some differential geometry. For the finite-groups case of Maschke's theorem, we averaged an inner product over G so that it was G-invariant. For compact groups, we can't sum in general, since there are infinitely many elements, but you can integrate. So what we're looking for is a measure on a Lie group. This comes naturally from a volume form, and to prove G-invariance, we'd like it to be G-invariant itself.

That is, our wish list is a left-invariant volume form on a Lie group G, an $\omega_g \in \Lambda^{\mathrm{top}}(T_g^*G)$. But you can get this by choosing something at the identity and defining $\omega_g = (\cdot g^{-1})^*\omega_1$ (that is, pull it back under left multiplication by g^{-1}).

If G is compact, then you can use right-invariance to show the space of such forms is trivial, so the form is also unique. We'll return to this a little bit next time.

Lecture 6.

Matrix coefficients and characters: 1/30/17

We'll start with a review of the end of last lecture. Let G be a compact Lie group; then, there's a bi-invariant volume form ω on G, and gives G finite volume. You can normalize so that the total volume is 1. The measure this defines is G-invariant, and is called the *Haar measure*, written $\mathrm{d}g$.

Remark. If G is a finite group, this is $1/|G|$ times the counting measure. You can use this to obtain the finite-groups proof of Maschke's theorem, etc. from the Lie groups one, as the integrals becomes sums. ◂

Example 6.1. Let's consider the circle group U(1) again. We saw in Example 5.3 that its irreducible representations are of the form $\chi_n : t \mapsto e^{2\pi i n t}$ for $t \in \mathbb{R}/\mathbb{Z}$ (or $z \mapsto z^n$ for $z \in \mathrm{U}(1)$), indexed over $n \in \mathbb{Z}$.

We can tie this to Fourier series. Consider $C(\mathrm{U}(1), \mathbb{C}) = C(\mathrm{U}(1))$, the space of complex-valued continuous functions on U(1). We've already seen how a group acts on its space of functions, so $C(\mathrm{U}(1))$ is an infinite-dimensional representation of U(1). We want to decompose this as a sum of irreducibles χ_n. The χ_{-n}-isotypical component $\mathbb{C}\chi_n \subset C(\mathrm{U}(1))$ (since χ_n is itself a continuous, $\mathbb{C}$-valued function on U(1)) is isomorphic to the one-dimensional representation determined by χ_n. So we might hope to decompose $C(\mathrm{U}(1))$ as a sum of these χ_n indexed by the integers.

Consider the Hermitian inner product on $C(\mathrm{U}(1))$ in which

$$(f_1, f_2)_{L^2} := \int_G f_1(g)\overline{f_2(g)}\,\mathrm{d}g \tag{6.2}$$

(i.e. taken in the Haar measure); since U(1) is compact, this converges. The same construction may be made for any compact Lie group G. Now, we can ask how the χ_{-n}-isotypic components fit together inside $C(\mathrm{U}(1))$.

It's quick to check that, with the Haar measure normalized as above,

$$\langle \chi_n, \chi_m \rangle = \begin{cases} 1, & n = m \\ 0, & n \neq m. \end{cases}$$

If $C_{\mathrm{alg}}(\mathrm{U}(1))$ denotes the space of algebraic functions on U(1), then

$$C_{\mathrm{alg}}(\mathrm{U}(1)) = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}\chi_n \subset C(\mathrm{U}(1)).$$

Then, using the Stone-Weierstrass approximation theorem (which we take as a black box), $C_{\mathrm{alg}}(\mathrm{U}(1))$ is uniformly dense in $C(\mathrm{U}(1))$: any continuous function $\mathrm{U}(1) \to \mathbb{C}$ can be approximated by sums of these χ_n. Also, if $L^2(\mathrm{U}(1))$ denotes the completion of $C(\mathrm{U}(1))$ in this inner product, $C_{\mathrm{alg}}(\mathrm{U}(1))$ is also dense in $L^2(\mathrm{U}(1))$.

This recovers the Fourier-theoretic statement: the χ_n are an orthonormal basis for $L^2(\mathrm{U}(1))$, so every function has a Fourier series, an infinite sum of characters. ◂

We want to do the same thing, but for a general compact Lie group G.

Let V be a finite-dimensional representation of G, and choose a basis of V, which provides an identification $\mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{C})$. Thus, the representation may be considered as a map $\rho_V : G \to \mathrm{GL}(V) \xrightarrow{\cong} \mathrm{GL}_n(\mathbb{C})$. Let $1 \leq i, j \leq n$; then, taking the ij^{th} component of a matrix defines a map $\mathrm{GL}_n(\mathbb{C}) \to \mathbb{C}$. Composing this map with ρ_V defines the *matrix coefficients map* $m_{V,i,j} : G \to \mathbb{C}$.

Picking a basis is unsatisfying, so let's state this in a more invariant way. Let $v \in V$ and $\alpha \in V^*$; they will play the role of i and j. Then, we have a matrix coefficients map

$$\begin{aligned} m_{V,\alpha,v} \colon G &\longrightarrow \mathbb{C} \\ g &\longmapsto \alpha(\rho_V(g)\cdot v). \end{aligned}$$

Another way to write this is as a map $m_V \colon V^* \otimes V \to C(G)$ sending $\alpha, v \mapsto m_{V,\alpha,v}$. Since $V^* \otimes V \cong \mathrm{End}(V)$ canonically, this is determined by where it sends the identity function 1_V. The resulting function, denoted χ_V, is called the *character* of V, and is the trace of $\rho_V(g)$: in a basis, the formula is

$$\chi_V(g) \coloneqq \sum_{i=1}^n m_{V,i,i}(g).$$

Characters are extremely useful in representation theory.

Inside $C(G)$, let $C_{\mathrm{alg}}(G)$ denote the union of $\mathrm{Im}(m_V)$ as V ranges over all finite-dimensional representations.

Lemma 6.3. *$C_{\mathrm{alg}}(G)$ is a subring of $C(G)$ under pointwise addition and multiplication.*

Proof sketch. The reason is that if V and W are representations, $v, w \in V$, and $\alpha, \beta \in V^*$, then

$$\begin{aligned} m_{V,\alpha,v} + m_{W,\beta,w} &= m_{V\oplus W,\alpha+\beta,v+w} \\ m_{V,\alpha,v} m_{W,\beta,w} &= m_{V\otimes W,\alpha\otimes\beta,v\otimes w}. \end{aligned}$$ ⊠

If you like algebra, this is very nice: we'll later see that $C_{\mathrm{alg}}(G)$ has the structure of a commutative Hopf algebra, meaning it's the ring of functions of a complex algebraic group.

Proposition 6.4.

(1) *If V is an irreducible representation of G, m_V is injective.*
(2) *If V and W are non-isomorphic irreducible representations of G, then $\mathrm{Im}(m_V) \perp \mathrm{Im}(m_W)$ in the L^2-inner product (6.2).*
(3) *If you restruct this inner product to $V^* \otimes V$ through m_V (where V is irreducible), it equals $(1/\dim V)(\text{–},\text{–})_{V^*\otimes V}$, where*

$$(\alpha_1 \otimes v_1, \alpha_2 \otimes v_2) \coloneqq (\alpha_1, \alpha_2)_{V^*}(v_1, v_2)_V,$$

for any G-invariant inner product $(\text{–},\text{–})_V$ and induced dual inner product $(\text{–},\text{–})_{V^*}$.

Exercise 6.5. Under the canonical identification $V^* \otimes V \cong \operatorname{End} V$, what does this inner product do on $\operatorname{End}(V)$?

Now, consider $C(G)$ as an (infinite-dimensional) representation of $G \times G$, where the first factor acts on the left and the second acts on the right.

We'll prove the proposition next time. It turns out that $V^* \otimes V$ is irreducible as a $(G \times G)$-representation, and the matrix coefficients map is $(G \times G)$-equivariant. Thus, we can express all of this in terms of $(G \times G)$-invariant bilinear forms.

Lecture 7.

The Peter-Weyl theorem: 2/1/17

Last time, we introduced $C_{\mathrm{alg}}(G)$, the subspace of continuous (complex-valued) functions on G generated by matrix coefficients of finite-dimensional representations.

Theorem 7.1. *There is an isomorphism of $(G \times G)$-representations*

$$C_{\mathrm{alg}}(G) \cong \overset{\perp}{\bigoplus_{V \in \widehat{G}}} (V^* \otimes V),$$

where $\overset{\perp}{\oplus}$ denotes an orthogonal direct sum. This isomorphism preserves an invariant Hermitian form.

Proof sketch. Suppose $V \in \widehat{G}$; we then check $V^* \otimes V$ is an irreducible $(G \times G)$-representation (which is an exercise). Last time, we saw that $m_V : V^* \otimes V \to C(G)$ is a $(G \times G)$-homomorphism, and therefore must be injective (since its kernel is a subrepresentation of $V^* \otimes V$). This implies orthogonality, by a lemma from last week.

It remains to check that $(-,-)_{L^2}$, restricted to $V^* \otimes V$ via m_V, is $(-,-)_{V^* \otimes V} \cdot (1/\dim V)$. This can be computed in coordinates, choosing an orthonormal basis for V. ⊠

This relates to a bunch of related statements called the Peter-Weyl theorem, excellently exposited by Segal and Macdonald in "Lectures on Lie groups and Lie algebras."

Theorem 7.2 (Peter-Weyl)**.** *$C_{\mathrm{alg}}(G) \subseteq C(G)$ is dense in the uniform norm.*

If you only care about compact matrix groups,[9] this density is a consequence of the Stone-Weierstrass theorem: any continuous function on a compact subset of $\mathbb{R}^{n^2}$ can be approximated uniformly by polynomials. Then, one shows that the Peter-Weyl theorem holds for every compact Lie group by showing every compact Lie group has a faithful representation. Maybe we'll return to this point later, when we return to more analytic issues.

There's another consequence involving L^2 functions.

Corollary 7.3. *$C_{\mathrm{alg}}(G)$ is dense in $L^2(G)$, i.e.*

$$L^2(G) \cong \widehat{\bigoplus}(V^* \otimes V).$$

Here, $\widehat{\oplus}$ denotes the completion of the direct sum.

You can compare this to the usual situation in Fourier analysis: you can write functions on a circle in terms of exponentials, and this is a generalization — you can write any function in terms of matrix coefficients.

If you restrict to class functions, there's a cleaner result.

Definition 7.4. The *class functions* on a group G are the $L^2(G)$ functions invariant under conjugation. That is, G acts on $L^2(G)$ by $g \cdot f(x) = f(gxg^{-1})$, and we consider the invariants $L^2(G)^G$.

Corollary 7.5. *The class functions decompose as*

$$L^2(G)^G \cong \widehat{\bigoplus}_{V \in \widehat{G}} (V^* \otimes V)^G = \widehat{\bigoplus} \mathbb{C}\chi_V.$$

Here, $(V^* \otimes V)^G = \operatorname{End}_G(V) = \mathbb{C} \cdot 1_V$ and $\chi_V(g) := \operatorname{tr}(\rho_V(g))$ is the character of V.

[9] That is, we think of $G \subseteq \mathrm{U}(N)$ for some N, or equivalently, G admits a faithful, finite-dimensional representation.

Corollary 7.6. *The set $\{\chi_V \mid V \in \widehat{G}\}$ is an orthonormal basis for the class functions $L^2(G)^G$. In particular,*

$$(\chi_V, \chi_W) = \begin{cases} 0, & V \not\cong W \\ 1, & V \cong W. \end{cases}$$

This makes it clear that the orthogonality relations arise purely from representation theory.

This has the following surprising corollary.

Corollary 7.7. *The isomorphism class of a representation is determined by its character.*

Usually you check for things to be isomorphic by finding an isomorphism between them. But in this case, you can just compute a complete invariant, and that's pleasantly surprising.

Proof. Any finite-dimensional representation W can be written

$$W \cong \bigoplus_{V \in \widehat{G}} V^{\oplus m_V}.$$

Thus,

$$\chi_W = \sum_{V \in \widehat{G}} m_V \chi_V,$$

so $m_V = \langle \chi_V, \chi_W \rangle$. ⊠

Maybe this is less surprising if you've already seen some character theory for finite groups.

The Peter-Weyl theorem is also useful for studying functions on homogeneous spaces. For the rest of this lecture, let G be a compact Lie group and H be a closed subgroup. Then, $C(G/H) = C(G)^H$, the H-invariant functions on G,[10] where H has the right action on $C(G)$. The Peter-Weyl theorem says that

$$C(G.H) = C(G)^H \cong \left(\widehat{\bigoplus} V^* \otimes V \right)^H = \widehat{\bigoplus} V^* \otimes V^H.$$

Example 7.8. For example, consider the 2-sphere $S^2 = \mathrm{SO}(3)/\mathrm{SO}(2)$, or $\mathrm{SU}(2)/\mathrm{U}(1)$ through the Hopf fibration, which is the double cover of the previous quotient.

Recall that the irreducible representations of SU(2) are P_n for each $n \geq 0$, where P_n is the space of homogeneous polynomials of degree n in two variabls (which is an $(n+1)$-dimensional space). Then,

$$C(S^2) = C(\mathrm{SU}(2))^{\mathrm{U}(1)} \cong \widehat{\bigoplus_{n \in \mathbb{N}}} P_n^* \otimes (P_n^{\mathrm{U}(1)}).$$

Here, we've switched from L^2 functions to continuous ones, which is overlooking a little analysis, but the analysis isn't that bad in this case, so it's all right.

Anyways, what does it mean for a polynomial to be fixed by U(1)? Well, $\mathrm{U}(1) \hookrightarrow \mathrm{SU}(2)$ through matrices such as $\left(\begin{smallmatrix} t & 0 \\ 0 & \bar{t} \end{smallmatrix}\right)$, and the action is

$$\begin{pmatrix} t & 0 \\ 0 & \bar{t} \end{pmatrix} \cdot x^a y^b = (t^{-1}x)^a (ty)^b = t^{b-a} x^a y^b.$$

Thus, we need $b = a$ and $a + b = n$, so

$$P_n^{\mathrm{U}(1)} \cong \begin{cases} \mathbb{C}, & n \text{ even} \\ 0, & n \text{ odd}. \end{cases}$$

Therefore

$$C(S^2) = \widetilde{\bigoplus}_{n \in 2\mathbb{N}} P_n^*,$$

and you can go even further and figure the decomposition out explicitly. ◀

The point of this example is that, even though we used very little information about the sphere, we put some significant constraints on functions on it, which is part of what makes representation theory cool.

Remark. In physics, these are indexed by $n/2$ instead of by n, and $n/2$ is called the *spin* of the representation. One says that only the integer-spin representation terms appear, not the half-integer ones. ◀

[10] This is a different action than the conjugation action we used to write $C(G)^G$. Unfortunately, there's no easy way to notate these both.

Lecture 8.

Character tables: 2/3/17

There's homework posted on the website; it's due at the end of the month.

Let G be a compact group. Then, $C(G)$, the space of continuous (complex-valued) functions on G, is called the *group algebra*: it has an algebra structure given by pointwise multiplication, but there's another algebra structure given by *convolution*: if $f_1, f_2 \in C(G)$, we define

$$(f_1 * f_2)(g) := \int_G f_1(h) f_2(h^{-1}g)\,\mathrm{d}h,$$

where $\mathrm{d}h$ is the normalized Haar measure. This looks asymmetric, which is kind of strange, but what we're doing is integrating over the pairs of elements whose product equals g:

$$(f_1 * f_2)(g) = \int_{\{h_1 h_2 = g\}} f_1(h_1) f_2(h_2).$$

The identification comes by $h \mapsto (h, h^{-1}g)$.[11] This product makes $C(G)$ into a noncommutative $\mathbb{C}$-algebra.[12]

If V is a finite-dimensional representation of G, then $C(G)$ acts on V by the formula

$$f * v = \int_G f(g)(g \cdot v)\,\mathrm{d}g$$

for any $v \in V$ and $f \in C(G)$. You can check that this makes V into a module for $C(G)$. A δ-function at some $h \in G$ isn't continuous, unless G is finite, but the integral still makes sense distributionally, and you get $\delta_h * v = h \cdot v$.

You can also restrict to class functions.

Exercise 8.1. Show that class functions are central in $(C(G), *)$, and in fact that $C(G)^G = Z(G)$: the class functions are the center.

In particular, this means the class functions are a commutative algebra, and it sees the representation theory of G: the characters are all in $C(G)^G$, satisfy orthogonality relations, and see isomorphism classes of representations. The irreducible characters also have a nice formula under convolution.

Proposition 8.2. *If $V, W \in \widehat{G}$, then*

$$\chi_V * \chi_W = \begin{cases} \left(\dfrac{1}{\dim V}\right)\chi_V, & V = W \\ 0, & V \neq W. \end{cases}$$

Proof. The trick is to turn the inner product into convolution at the identity:

$$\begin{aligned}(\chi_V, \chi_W)_{L^2} &= \int_G \chi_V(g)\overline{\chi_W(g)}\,\mathrm{d}g \\ &= \int_G \chi_V(g)\chi_W(g^{-1})\,\mathrm{d}g \\ &= (\chi_V * \chi_W)(1).\end{aligned}$$

This proves it at the identity; the rest is TODO. ⊠

Finite groups. In this section, assume G is finite.

There are some particularly nice results for finite groups, since every function on a finite group is continuous. If you've been going to Tamás Hausel's talks this week (and the two more talks next week), his work on computing cohomology of moduli spaces, uses the character theory of finite groups in an essential way.

When G is finite, the group algebra $C(G)$ is usually denoted $\mathbb{C}[G]$. This notation means a vector space with basis G, i.e.

$$\mathbb{C}[G] = \left\{ \sum a_g g \mid g \in G, a_g \in \mathbb{C} \right\}.$$

[11] If G isn't compact, e.g. $G = \mathbb{R}^n$, this definition makes sense, but doesn't always converge; in this case, you can restrict to compactly supported functions, though their convolution won't be compactly supported. In this way one recovers the usual convolution operator on $\mathbb{R}^n$.

[12] There's a slight caveat here: there's no identity in this algebra, unless G is discrete.

You can make this definition (finite weighted sums of elements of G) for any group, but it won't be isomorphic to $C(G)$ unless G is discrete. In any case, when G is finite, $\mathbb{C}[G]$ is a finite-dimensional, unital algebra.

Let's think about what the Peter-Weyl theorem says in this context. Since $\mathbb{C}[G]$ is finite-dimensional, we can ignore the completion and obtain an isomorphism

$$\mathbb{C}[G] \cong \bigoplus_{V\in\widehat{G}} (V^* \otimes V). \tag{8.3}$$

This has a number of fun corollaries. First, take the dimension of each side:

Corollary 8.4. *Let G be a finite group. Then, G has finitely many isomorphism classes of irreducible representations, and moreover*

$$|G| = \sum_{V\in\widehat{G}} (\dim V)^2.$$

For example, once you've found the trivial and sign representations of S_3, you're forced to conclude there's one more irreducible 2-dimensional representation or two more one-dimensional representations.

Looking at class functions, every function invariant on conjugacy classes is continuous now, so we have two bases for $\mathbb{C}[G]^G$: $\{\chi_V \mid V \in \widehat{G}\}$ as usual, and the set of δ-functions on conjugacy classes of G.

Corollary 8.5. $|\widehat{G}|$ *is equal to the number of conjugacy classes of G.*

This leads to an organizational diagram called the *character table* for a finite group: across the top are the conjugacy classes $[g]$ and down the left are the irreducible characters χ. The entry in that row and that column is $\chi(g)$. This table says a lot about the representation theory of G. By Corollary 8.5, it's square.

Example 8.6. Let $G = S_3$, so conjugacy classes are cycle types (as in any symmetric group). The trivial

	e	$(1\ 2)$	$(1\ 2\ 3)$
χ_{triv}	1	1	1
χ_{sign}	1	-1	1
χ_V	2	0	-1

TABLE 1. Character table of S_3.

representation has character 1 on all conjugacy classes; the sign representation has value 1 on e and $(1\ 2\ 3)$ and -1 on $(1\ 2)$. Then, you can compute χ_V where V is the irreducible two-dimensional representation, and see Table 1 for the character table. Alternatively, you can use the orthogonality relations to compute the remaining entries. ◂

This matrix looks like it should be unitary, but isn't quite. Nonetheless, it has some nice properties: notice that the columns are also orthogonal.

Example 8.7. S_4 is only a little more complicated. The conjugacy classes are e, $(1\ 2)$, $(1\ 2\ 3)$, $(1\ 2)(3\ 4)$, and $(1\ 2\ 3\ 4)$.

We know the trivial and sign representations, and there's the defining representation W where S_4 permutes a basis of $\mathbb{C}^4$. This is reducible, since it fixed $(1,1,1,1)$; alternatively, you could use the orthogonality relation to check. So we know there's a copy of the trivial representation, so $\chi_V \coloneqq \chi_W - \chi_{\mathrm{triv}}$, and you can check it's irreducible. ◂

Lecture 9.

The character theory of $\mathrm{SU}(2)$: 2/6/17

Last time, we got stuck on the proof of Proposition 8.2. The key idea that was missing is that if $m_V : V^*\otimes V \to C(G)$ is the matrix coefficients function for the representation V, then $\mathrm{Im}(m_V)$ is a two-sided ideal of $C(G)$ under convolution. This follows from the formula for convolving with a matrix coefficient: given $v \in V$, $\alpha \in V^*$, and an $f \in C(G)$,

$$m_{V,\alpha,v} * f = m_{V,\alpha,v'},$$

where

$$v' = \int_G f(h)h^{-1} \cdot v \, \mathrm{d}h.$$

This is fairly easy to prove.

Characters are examples of matrix coefficients, so if $V, W \in \widehat{G}$, $\chi_V * \chi_W \in_V (V^* \otimes V)^G \cap m_W(W^* \otimes W)^G$. Since $m_V(V^* \otimes V)^G \cong \mathrm{End}_G V$, and similarly for W, this space is trivial when $V \ncong W$ and is one-dimensional when $V \cong W$. So the proof boils down to checking $\chi_V * \chi_V = (1/\dim V)\chi_V$.

This means that the characters are almost orthogonal idempotents. To get actual idempotents, though, we have to normalize: let $e_V := \dim(V)\chi_V \in C(G)^G$, so that $e_V * e_V = 1$ and $e_V * e_W = 0$ if $V \ncong W$. If U is any (unitary) representation of G, then $C(G)$ acts on U, and under this action, e_V acts by projection onto the V-isotypical component of U.

For yet another way of thinking about this, we have the matrix coefficients map $m_V : \mathrm{End}_{\mathbb{C}}(V) \to C(G)$, which sends $1_V \mapsto \chi_V$, but also the action map $\mathrm{act}_V : C(G) \to \mathrm{End}(V)$. This sends $e_V \mapsto 1_V$, so these maps aren't literally inverses. They are adjoint with respect to the appropriate Hermitian forms, however.

Last time, we also worked out some examples of representations of finite groups. We managed to translate that problem into a completely different problem, of identifying certain class functions in $C(G)^G$. Today, we're going to try to generalize this to an infinite group, namely SU(2).

Let's fix some notation. Concretely,

$$G = \mathrm{SU}(2) = \left\{ \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \mid |a|^2 + |b|^2 = 1 \right\},$$

and inside this lies the *maximal torus*

$$T = \mathrm{U}(1) = \left\{ \begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix} \mid |z| = 1 \right\}.$$

Some of what we say will generalize, but SU(2) is the nicest case.

We'd like to understand the conjugacy classes of SU(2). Let $\tau := (\mathrm{tr}/2) : \mathrm{SU}(2) \to [-1, 1]$, so $\tau\left(\begin{smallmatrix} z & 0 \\ 0 & \overline{z} \end{smallmatrix}\right) = \mathrm{Re}(z)$. The fibers of τ are exactly the conjugacy classes of SU(2). What do they look like? Geometrically, SU(2) is a 3-sphere in $\mathbb{R}^4$, and we're fixing one of the real coordinates to be a particular number, so the level sets are 2-spheres, except for the poles, which are single-point conjugacy classes, and are the central elements of SU(2).

What this means is that a class function is determined by its restriction to the maximal torus T, and for any unit complex number z, $\left(\begin{smallmatrix} z & 0 \\ 0 & \overline{z} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \overline{z} & 0 \\ 0 & z \end{smallmatrix}\right)$ are in the same conjugacy class. Thus, restriction determines an isomorphism

$$C(\mathrm{SU}(2))^{\mathrm{SU}(2)} \xrightarrow{\cong} C(\mathrm{U}(1))^{\mathbb{Z}/2}, \tag{9.1}$$

where $\mathbb{Z}/2$ acts on $C(\mathrm{U}(1))$ by conjugation. This means we can think of class functions on SU(2) in terms of their Fourier coefficients!

Given a representation V of SU(2), we can restrict it to a U(1)-representation V'. We decompose this into *weight spaces*

$$V' = \bigoplus_{n \in \mathbb{Z}} V_n,$$

where $z \in \mathrm{U}(1)$ acts on V_n by z^n. Thus, identifying the character $\chi_V \in C(\mathrm{SU}(2))^{\mathrm{SU}(2)}$ with its image under (9.1), we get

$$\chi_V \longmapsto \sum_{n \in \mathbb{Z}} \dim(V_n) z^n.$$

Part of what made the representation theory of U(1) awesome was integration: we had a nice formula for the integral of a class function. How can we generalize this to SU(2)? The map (9.1) doesn't preserve volume, as its fibers are spheres with different radii.

Recall that the irreducible representations of SU(2) are the spaces P_n of homogeneous polynomials in two variables of degree n. If we restrict P_n to a U(1)-representation and decompose it, we discover

$$P_n = \mathbb{C} \cdot x^n \oplus \mathbb{C} \cdot x^{n-1} y \oplus \cdots \oplus \mathbb{C} \cdot x y^{n-1} \oplus \mathbb{C} \cdot y^n.$$

The term $x^k y^{n-k}$ has weight $n-2k$, i.e. z acts as z^{n-2k} on $\mathbb{C}\cdot x^k y^{n-k}$. Therefore as a U(1)-representation,

$$\begin{aligned}\chi_{P_n}(z) &= z^n + z^{n-2} + \cdots + z^{-n+2} + z^{-n}\\ &= \frac{z^{n+1}-z^{-n-1}}{z-z^{-1}}.\end{aligned}\tag{9.2}$$

(9.2) is called the *Weyl character formula* for SU(2). There will be corresponding formulas for other groups.

We would like to use something like this to do integration: given a class function $f\in C(\mathrm{SU}(2))^{\mathrm{SU}(2)}$, let

$$f(\phi) := f\begin{pmatrix} e^{i\phi} & \\ & e^{-i\phi}\end{pmatrix},$$

so $z=e^{i\phi}$. We want to determine a function $J(\phi)$ such that

$$\int_{\mathrm{SU}(2)} f(g)\,\mathrm{d}g = \int_0^{2\pi} f(\phi)J(\phi)\,\mathrm{d}\phi.$$

The thing that powers this is that f is determined by its image under (9.1), so we should be able to determine its integral in those terms. This requires that f is a class function, and is not true in general.

Here, $J(\phi)$ is the area of the conjugacy class whose trace is $2\cos\phi$. This conjugacy class is a sphere of radius $\sqrt{1-\cos^2(\phi)} = |\sin\phi|$. Thus, $J(\phi) = C\sin^2\phi$, where C is such that

$$\int_0^{2\pi} C\sin^2\phi\,\mathrm{d}\phi = 1.$$

This means $C=1/\pi$, so the integration formula for class functions is

$$\int_{\mathrm{SU}(2)} f(g)\,\mathrm{d}g = \frac{1}{\pi}\int_0^{2\pi} f(\phi)\sin^2\phi\,\mathrm{d}\phi.\tag{9.3}$$

Example 9.4. Let's try to compute this for $n=1$.

$$\|\chi_{P_1}\|^2 = \frac{4}{\pi}\int_0^{2\pi}\cos^2\phi\sin^2\phi\,\mathrm{d}\phi.$$

This is difficult but tractable, and the answer is

$$= \frac{1}{\pi}\left[\frac{1}{2}\phi - \frac{1}{8}\sin(4\phi)\right]_0^{2\pi} = 1.$$

This is good, because we said P_1 is irreducible, so the norm of its character had better be 1. ◀

Lecture 10.

Representation theory of Lie groups: 2/8/17

Last time, we discussed the character theory of SU(2), using a convenient isomorphism of the algebra of class functions $C(\mathrm{SU}(2))^{\mathrm{SU}(2)}$ with $C(\mathrm{U}(1))^{\mathbb{Z}/2}$. If P_n denotes the irreducible SU(2)-representation of dimension $n+1$, and $z=e^{i\phi}$, the image of χ_{P_n} in $C(\mathrm{U}(1))^{\mathbb{Z}/2}$ is

$$\chi_{P_n}(z) = z^n + z^{n-2} + \cdots + z^{-n+2} + z^{-n}.$$

Then we used the Weyl integration formula for SU(2), (9.3), to show that

$$\begin{aligned}\|\chi_{P_1}\|^2 &= \frac{1}{\pi}\int_0^{2\pi}\sin^2(2\phi)\,\mathrm{d}\phi\\ &= \frac{1}{\pi}\left[\frac{\phi}{2} - \frac{1}{8}\sin(4\phi)\right]_0^{2\pi} = 1.\end{aligned}$$

Thus, P_1 is an irreducible representation!

Exercise 10.1. Show that for every n, $\|\chi_{P_n}\|^2 = 1$, so each P_n is irreducible.

So we've found some irreducible representations, and in a very curious manner, only using their characters. We then need to show there are no additional irreducible representations.

Exercise 10.2. Show that the functions $\{\cos(n\phi)\}_{n\in\mathbb{Z}_{\geq 0}}$ are a basis for $C_{\mathrm{alg}}(\mathrm{U}(1))^{\mathbb{Z}/2}$.

This implies $\{\chi_{P_n}\}$ is also a basis, and are orthonormal, so they account for all irreducible representations.

You might next wonder what the isomorphism class of $P_n \otimes P_m$ is, as a direct sum of irreducibles. Recall that $\chi_{V\otimes W} = \chi_V \chi_W$, so if $m \leq n$,

$$\chi_{P_n\otimes P_m} = \chi_{P_n}\chi_{P_m} = \left(\frac{z^{n+1} - z^{n-1}}{z - z^{-1}}\right)\left(z^m + z^{m-2} + \cdots + z^{-m}\right). \tag{10.3}$$

Exercise 10.4 (Clebsch-Gordon rule)**.** Show that (10.3) satisfies

$$\chi_{P_n}\chi_{P_m} = \sum_{k=0}^{m} \chi_{P_{n+m-2k}}.$$

Now that that's settled, let's look at other compact Lie groups. There's a double cover $\mathrm{SU}(2) \twoheadrightarrow \mathrm{SO}(3)$, so $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\pm I$. Thus, the irreducible representations of SO(3) are given as the representations of SU(2) in which $-I$ acts trivially. These are the P_n for even n. In other words, $\mathrm{SU}(2) \cong \mathrm{Spin}(3)$, so the irreducible representations of SO(3) have integer spin, and those of Spin(3) may have half-integer spin.

It's possible to get a little more information from this: there's a double cover $\mathrm{SU}(2)\times\mathrm{SU}(2) \to \mathrm{SO}(4)$, so you can work out the representation theory of SO(4) in a similar way. But there's loads of other interesting groups, including U_n, Sp_n, and many more.

An overview of the theory. Let G be a compact, connected Lie group.[13] Then, G admits a *maximal torus* T, a maximal abelian subgroup (which is necessarily isomorphic to $\mathrm{U}(1)^n$), and any $g \in G$ is conjugate to an element of T. For example, when $G = \mathrm{U}(n)$, T is the subgroup of diagonal unitary matrices, which is isomorphic to $\mathrm{U}(1)^n$; that every $g \in \mathrm{U}(n)$ is conjugate to something in T means that every unitary matrix is diagonalizable. It's also true that any two maximal tori are conjugate to each other.

Thus, just like we did for SU(2) and U(1), we can express a class function on G in terms of its restriction to T, and then use Fourier theory. Let $\widehat{T} = \mathrm{Hom}(T, \mathrm{U}(1))$, which is a lattice $\mathbb{Z}^n$. Our goal is to write the irreducible characters of G as Fourier series on T. There are still some questions, though: how do we do integration? And can we obtain a formula for the characters of the irreducible representations? The Weyl integration formula and Weyl character formula will answer these questions.

Tackling a general compact Lie group with the technology we've developed is complicated. The unitary group is within reach, but it involves knowledge of symmetric functions and representation theory of the symmetric group. So we'll start with some more general theory.

Lie algebras. We want to reduce the representation theory of Lie groups to pure algebra. For example, if G is a finite group, there are finitely many generators and relations, so we can express a representation as a finite set of matrices satisfying the relations. Lie groups aren't finitely (or even countably) generated, so we can't just do algebra.

There are good presentations of Lie groups that are topological, however: SO(3) is generated by rotations about certain angles $R_{x,t}$, $R_{y,t}$, $R_{z,t}$, which form a frame in $\mathbb{R}^3$, and $t \in \mathbb{R}/\mathbb{Z}$ is an angle. So SO(3) can be described in terms of a one-parameter subgroup. We'd like to replace SO(3) with its collection of one-parameter subgroups; these can be identified with tangent vectors at the identity, which is why the somewhat surprising notion of a Lie algebra is introduced.

Definition 10.5. A *one-parameter subgroup* of a Lie group G is a homomorphism $\rho\colon \mathbb{R} \to G$, where $\mathbb{R}$ is a Lie group under addition.

Homomorphisms of Lie groups are always smooth. Given a one-parameter subgroup ρ, we can obtain its derivative at the identity, $\rho'(0) \in T_1G$. We'll let $\mathfrak{g}$ (written `\mathfrak g` in LaTeX) denote T_1G: what we're saying is that a smooth path going through the identity defines a tangent vector.

Lemma 10.6. *The assignment* $\Phi\colon \rho \mapsto \rho'(0)$ *is a bijection* $\mathrm{Hom}_{\mathsf{LieGrp}}(\mathbb{R}, G) \to \mathfrak{g}$.

That is, a direction in $\mathfrak{g}$ gives rise to a unique one-parameter subgroup in that direction.

[13]Focusing on connected groups isn't too restrictive: any compact Lie group G is an extension of its identity component G^0 by the finite group $\pi_0(G)$.

Proof sketch. This is a generalization of the proof that the characters of $\mathbb{R}$ are exponential maps: that also involves showing they're determined by their values at 0 using a differential equation.

Let ρ be a one-parameter subgroup; then, it satisfies the ODE $\rho'(t) = \rho(t)\rho'(0)$ with the initial condition $\rho(0) = 1_G$. Surjectivity of Φ follows from the existence of solutions to ODEs, and injectivity follows from uniqueness. You need a way of passing from a local solution to a global one, but this can be done. ⊠

Now, we can define the *exponential map* $\exp\colon \mathfrak{g} \to G$ sending $A \mapsto \Phi^{-1}(A)(1)$: given a tangent vector, move in that direction for a short time, and then return that element. This map is a local diffeomorphism (by the inverse function theorem), so the Lie algebra encodes the Lie group in some neighborhood of the identity. Then, one asks what algebraic properties of $\mathfrak{g}$ come from the group structure on G.

Lecture 11.

Lie algebras: 2/10/17

"We're the Baker-Campbell-Hausdorff law firm — call us if you can't commute!"

Let G be a Lie group. Last time, we defined its *Lie algebra* $\mathfrak{g} := T_1G$, and a local diffeomorphism $\exp\colon \mathfrak{g} \to G$. A local inverse to the exponential map is called a *logarithm.*

Example 11.1. If $G = \mathrm{GL}_n(\mathbb{R})$, then G is an open submanifold of $M_n(\mathbb{R})$, the vector space of $n \times n$ matrices. Thus, $T_1\mathrm{GL}_n(\mathbb{R}) = T_1M_n(\mathbb{R})$, and the tangent space to a vector space is canonically identified with that vector space. Thus, the Lie algebra $\mathfrak{g}$, also written $\mathfrak{gl}_n(\mathbb{R})$, is $M_n(\mathbb{R})$. More generally, the Lie algebra of $\mathrm{GL}(V)$ is $\mathfrak{gl}(V) = \mathrm{End}(V)$.

In this case, the exponential map is the matrix exponential $\exp\colon M_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ sending

$$A \longmapsto e^A = \sum_{i=0}^{\infty} \frac{A^i}{i!} = 1 + A + \frac{A^2}{2} + \cdots,$$

so a one-parameter subgroup ρ_A in $\mathrm{GL}_n(\mathbb{R})$ is given by $\rho_A(t) = e^{tA}$ for any matrix A. ◀

More generally, if $G \subset \mathrm{GL}_n(\mathbb{R})$ is any Lie subgroup, the exponential map $\mathfrak{g} \to G$ is the restriction of the matrix exponential $M_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ to $\mathfrak{g} \subset M_n(\mathbb{R})$.

Example 11.2. $\mathrm{SL}_n(\mathbb{R})$ is the group of matrices $\{A \in \mathrm{GL}_n(\mathbb{R}) \mid \det(A) = 1\}$ (since 1 is a regular value of the determinant function, this is in fact a Lie group). If you differentiate the determinant condition, you get that $\mathrm{tr}(A) = 0$, because $\exp(\mathrm{tr}(A)) = \det(\exp(A))$, and therefore the Lie algebra of $\mathrm{SL}_n(\mathbb{R})$ is

$$\mathfrak{sl}_n(\mathbb{R}) = \{A \in \mathfrak{gl}_n(\mathbb{R}) \mid \mathrm{tr}(A) = 0\}.$$

In particular, the logarithm gives a local chart from $\mathrm{SL}(n)$ into the vector space $\mathfrak{sl}_n(\mathbb{R})$, which is another way to show $\mathrm{SL}_n(\mathbb{R})$ is a Lie group. (Kirrilov's book uses this to explain why all of the classical groups are Lie groups.)

$\mathrm{O}(n)$ is the group of $n \times n$ matrices with $AA^t = I$, so differentiating this, the Lie algebra is

$$\mathfrak{o}(n) = \{A \mid A + A^t = 0\},$$

the *skew-symmetric matrices.* This is also the Lie algebra for $\mathrm{SO}(n)$: $\mathrm{SO}(n) \subset \mathrm{O}(n)$ is the connected component of the identity, and therefore $T_1\mathrm{O}(n) = T_1\mathrm{SO}(n) = \mathfrak{o}(n)$ is again the Lie algebra of skew-symmetric matrices.

The same does not apply for $\mathrm{SU}(n) \subset \mathrm{U}(n)$, since $\mathrm{U}(n) = \{A \in M_n(\mathbb{C}) \mid AA^\dagger = I\}$ is connected: its Lie algebra is

$$\mathfrak{u}(n) = \{A \mid A + A^\dagger\} = 0,$$

the *skew-Hermitian matrices.* But not all of these are traceless, so $\mathfrak{su}(n)$ is the skew-Hermitian matrices with trace zero. ◀

The basic question is, how much does $\mathfrak{g}$ know about G? Clearly not everything, because $\mathfrak{o}(n) = \mathfrak{so}(n)$. To make this question precise, we need more structure on $\mathfrak{g}$.

To recover information about G, we need to know what $\mathfrak{g}$ says about the multiplication on G. Thus, given $A, B \in \mathfrak{g}$, what is $C(A, B) := \log(\exp(A)\exp(B))$? We can Taylor-expand it around $A = B = 0$. The first term is $A+B$ (essentially by the product rule), and there must be higher-order terms unless G is abelian. Let $b(A,B)$ be twice the second-order term, so

$$C(A,B) = A + B = \frac{1}{2}b(A,B) + O(A^2, AB, B^2).$$

This $b\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ has some important properties.

Proposition 11.3.
(1) *b is bilinear.*
(2) *b is* skew-symmetric*: $b(A,B) = -b(B,A)$.*
(3) *b satisfies the* Jacobi rule*:*

$$b(b(A,B),C) + b(b(B,C),A) + b(b(C,A),B) = 0.$$

This allows us to clarify what exactly we mean by a Lie algebra.

Definition 11.4. A *Lie algebra* is a vector space L together with an operation $b\colon L\times L\to L$, called the *Lie bracket*, satisfying the properties in Proposition 11.3. A *Lie algebra homomorphism* is a linear map between Lie algebras that commutes with the Lie bracket.

All of the Lie algebras we've defined, e.g. those in Examples 11.1 and 11.2, will be understood to come with their Lie brackets.

Exercise 11.5. Prove Proposition 11.3. Part of this will involve unpacking what "Taylor series" means in this context. Hint: first, notice that $C(A,0)=A$, $C(0,B)=B$, and $C(-B,-A) = -C(A,B)$ by properties of the exponential map. These together give (1) and (2); the last part will also follow from properties of C.

This is pretty neat. You might wonder what additional information the higher-order terms give you, but it turns out that you don't get anything extra.

Theorem 11.6 (Baker-Campbell-Hausdorff)**.** *The Taylor expansion of $C(A,B)$ can be expressed purely in terms of addition and the Lie bracket. In particular, the formula begins*

$$C(A,B) = A + B + \frac{1}{2}b(A,B) + \frac{1}{12}(b(A,b(A,B)) + b(B,b(B,A))) + \cdots$$

This is already quite surprising, and it turns out you can calulate it explicitly for most Lie groups we care about.

Example 11.7. If $G = \mathrm{GL}_n(\mathbb{R})$ or $\mathrm{GL}_n(\mathbb{C})$, so $\mathfrak{g} = M_n(\mathbb{R})$ (resp. $M_n(\mathbb{C})$), then $b(A,B)$ is the matrix commutator $[A,B] = AB - BA$. The same is true for any matrix group (i.e. Lie subgroup of $\mathrm{GL}_n(\mathbb{R})$ or $\mathrm{GL}_n(\mathbb{C})$): the commutator of two traceless matrices is traceless, so $[,]$ also defined the Lie bracket on $\mathfrak{sl}_n(\mathbb{R})$, $\mathfrak{so}_n$, $\mathfrak{su}_n$, $\mathfrak{u}_n$, and so on. ◀

We've seen that Lie algebras can't distinguish a Lie group and the connected component of the identity. It also can't see covering maps, since the tangent space depends only on local data, so, e.g., $\mathfrak{spin}_n = \mathfrak{so}_n$ through the double cover $\mathrm{Spin}(n) \twoheadrightarrow \mathrm{SO}(n)$. However, these are the only obstructions: the Lie algebra tells you everything else.

Theorem 11.8 (Lie's theorem)**.** *There is an equivalence of categories between the category of connected and simply-connected Lie groups and the category of finite-dimensional Lie algebras, given by sending $G\mapsto\mathfrak{g}$.*

Lecture 12.

The adjoint representations: 2/13/17

Last time, we talked about Lie's theorem, Theorem 11.8. Given a Lie group G, one obtains a Lie algebra $\mathfrak{g}$, its tangent space at the identity, with first-order commutation information defining the Lie bracket. Theorem 11.8 states that there's an equivalence of categories from connected, simply connected Lie groups to Lie algebras, where $G\mapsto\mathfrak{g}$ and $F\colon G_1\to G_2$ is mapped to $\mathrm{d}F|_1\colon \mathfrak{g}_1\to\mathfrak{g}_2$.

Remark.
(1) One consequence is that for any Lie algebra $\mathfrak{g}$, there's a corresponding Lie group G. This is tricky in general, but easier for matrix groups. This is sort of like integration

(2) On morphisms, if $f\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is a morphisms of Lie algebras, how do you define $F\colon G_1 \to G_2$? In a neighborhood of the identity, it makes sense to take $F(\exp(x)) := \exp(f(x))$. It turns out this suffices to define F on all of G: in particular, G is generated by any neighborhood of the identity, though this uses that G is connected and simply connected. ◀

This has consequences on representation theory.

Definition 12.1. A *representation* of a Lie algebra $\mathfrak{g}$ is a pair (V, ρ) where V is a (finite-dimensional, complex) vector space and $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V) = \mathrm{End}_{\mathbb{C}} V$ is a Lie algebra homomorphism.

Just like for groups, we'll use $x \cdot v$ as shorthand for $\rho(x) \cdot v$, representing the action map $\mathfrak{g} \times V \to V$. That ρ is a Lie algebra homomorphism means that for all $x, y \in \mathfrak{g}$ and $v \in V$,

$$x \cdot (y \cdot v) - y \cdot (x \cdot v) = [x, y] \cdot v.$$

Lie's theorem in particular tells us that the finite-dimensional representations of a connected, simply connected Lie group G are in natural bijection with the representations of its Lie algebra $\mathfrak{g}$.[14] And if you care about a group G that isn't simply connected, such as $\mathrm{U}(n)$ or $\mathrm{SO}(n)$, you can pass to its universal cover $\widetilde{G}$, which is, and ask about which of its representations project back down to G.

Remark. We've been working with Lie algebras over $\mathbb{R}$, but the same definition (and that of a Lie algebra representation) also works over $\mathbb{C}$. There are also *complex Lie groups*, which are groups in complex manifolds where multiplication and inversion are holomorphic. There is also an analogue of Lie's theorem in this setting. The category of Lie algebras over $\mathbb{R}$ is denoted $\mathsf{Lie}_{\mathbb{R}}$, and the category of Lie algebras over $\mathbb{C}$ is $\mathsf{Lie}_{\mathbb{C}}$.

When we talk about complex representations of real Lie groups in this setting, something interesting happens. If $\mathfrak{h}_{\mathbb{C}}$ is a complex Lie algebra and $\mathfrak{g}_{\mathbb{R}}$ is a real Lie algebra, then $\mathfrak{h}_{\mathbb{C}}$ is also a Lie algebra by forgetting the complex structure, and $\mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C}$ is a complex Lie algebra. There's an almost tautological fact that

$$\mathrm{Hom}_{\mathsf{Lie}_{\mathbb{R}}}(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{C}}) = \mathrm{Hom}_{\mathsf{Lie}_{\mathbb{C}}}(\mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C}, \mathfrak{h}_{\mathbb{C}}), \tag{12.2}$$

and this is natural in a suitable sense. This means it's possible to complexify Lie algebras, letting $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C}$: there's no good way to complexify a real Lie group into a complex Lie group, but on the Lie algebra level you can make this happen. When $\mathfrak{h}_{\mathbb{C}} = \mathfrak{gl}_n(\mathbb{C})$, the adjunction (12.2) tells us that the representationss of $\mathfrak{g}_{\mathbb{R}}$ are the same as those of $\mathfrak{g}_{\mathbb{C}}$, and we'll soon see these as modules over a certain algebra, called the universal enveloping algebra. ◀

Example 12.3. Suppose $G = \mathrm{U}(n)$, so $\mathfrak{g} = \mathfrak{u}(n)$ is the Lie algebra of skew-Hermitian matrices (with commutator for its bracket). Its complexification is $\mathfrak{u}(n)_{\mathbb{C}} = \mathfrak{gl}_n(\mathbb{C})$, because every invertible complex matrix is the sum of a Hermitian matrix and i times a Hermitian matrix:

$$A = \underbrace{\left(\frac{A - A^\dagger}{2}\right)}_{\mathfrak{u}(n)} + \underbrace{\left(\frac{A + A^\dagger}{2}\right)}_{i \cdot \mathfrak{u}(n)}.$$ ◀

Notice that $\mathfrak{gl}_n(\mathbb{R}) \otimes \mathbb{C} = \mathfrak{gl}_n(\mathbb{C})$ as well, so different real Lie algebras may map to the same complex one.

Definition 12.4. Let $\mathfrak{g}_{\mathbb{C}}$ be a complex Lie algebra and $\mathfrak{k} \subset \mathfrak{g}_{\mathbb{C}}$ be a real subalgebra such that $\mathfrak{k}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}$. Then, $\mathfrak{k}$ is called a *real form* of $\mathfrak{g}_{\mathbb{C}}$.

To begin analyzing the theory of Lie groups, we'd like to have a few representations hanging around. Usually we defined them as acting on spaces of functions, but in general these representations are infinite-dimensional, which is unfortunate. However, there is an important finite-dimensional representation of G on $\mathfrak{g}$.

Definition 12.5. Let $g \in G$ and $\varphi_g\colon G \to G$ be the action of g by conjugation: $h \mapsto ghg^{-1}$, and let $\mathrm{Ad}_g := \mathrm{d}\varphi_g|_1\colon \mathfrak{g} \to \mathfrak{g}$. Then, $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ is a G-representation, called the *adjoint representation* of G, and its derivative $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ (also called the *adjoint representation*) is a Lie algebra representation of $\mathfrak{g}$.

The Lie algebra adjoint representation is the bracket: $\mathrm{ad}_x = [x, -]$.[15] For example, if $G \subset \mathrm{GL}_n(\mathbb{R})$ is a matrix group, then $\mathrm{Ad}_D(A) = DAD^{-1}$, and $\mathrm{ad}_B(A) = AB - BA$. Since we care about complex representations, we'll often consider the adjoint representation of $\mathfrak{g}_{\mathbb{C}}$.

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[14] I hear you saying, "whoa, $\mathrm{GL}_n(\mathbb{C})$ isn't simply connected!" And you're right (its fundamental group is $\mathbb{Z}$), but the statement of Lie's theorem can be suitably generalized to the case where only the source need be simply connected.

[15] This is definitely not always invertible, e.g. if $\mathfrak{g}$ contains central elements, then they're trivial in the Lie bracket.

In Example 11.2, we found Lie algebras for a few groups. In more explicit detail, $\mathfrak{su}(2)$ is the algebra traceless skew-Hermitian matrices, which is a 3-dimensional vector space with basis

$$i\sigma_1 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad i\sigma_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad i\sigma_3 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

The notation is because these σ_j are generators for Hermitian matrices. This allows for a generators-and-relations definition of $\mathfrak{su}(2)$:

$$\mathfrak{su}(2) = \{i\sigma_1, i\sigma_2, i\sigma_3 \mid [i\sigma_1, i\sigma_2] = -2i\sigma_3, [i\sigma_2, i\sigma_3] = -2i\sigma_1, [i\sigma_3, i\sigma_1] = -2i\sigma_2\}.$$

Skew-symmetry tells us that $[i\sigma_i, i\sigma_i] = 0$. In other words, we've defined $\mathfrak{su}(2)$ to be the complex vector space spanned by these three things, and the Lie bracket is the unique skew-symmetric pairing extending the specified relation.[16]

The Lie algebra $\mathfrak{so}(3)$ is the skew-orthogonal matrices, which are already traceless. Once again, this is a three-dimensional space (well, since SU(2) double covers SO(3), $\mathfrak{su}(2) \cong \mathfrak{so}(3)$), and has a basis

$$J_x := \begin{pmatrix} 0 & & \\ & & -1 \\ & 1 & \end{pmatrix} \qquad J_y := \begin{pmatrix} & & 1 \\ & 0 & \\ -1 & & \end{pmatrix} \qquad J_z := \begin{pmatrix} & -1 & \\ 1 & & \\ & & 0 \end{pmatrix}. \tag{12.6}$$

These are the derivatives of rotation around the x-, y-, and z-axes, respectively, and have the same commutation relations as the generators of $\mathfrak{su}(2)$.

$\mathfrak{sl}_2(\mathbb{R})$ is a nonisomorphic Lie algebra, but will have the same complexification. It's also three-dimensional, generated by

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{12.7}$$

That $\mathfrak{so}(3) \otimes \mathbb{C} \cong \mathfrak{su}(2) \cong \mathbb{C} \cong \mathfrak{sl}_2(\mathbb{R}) \otimes \mathbb{C}$ has an interesting corollary: there's an equivalence of categories between the finite-dimensional representations of SU(2), those of $\mathfrak{sl}_2(\mathbb{C})$, and those of $\mathrm{SL}_2(\mathbb{C})$ as a complex Lie group. Thus, we've found an algebraic way to tackle representations of Lie groups.

Lecture 13.

Representations of Lie algebras: 2/15/17

There's homework on the website, and it's due at the end of February (minimum three questions). Students are encouraged to come to office hours and ask questions!

Last time, as a corollary to Theorem 11.8, we saw that there's an equivalence of categories between (complex, finite-dimensional) representations of a connected and simply-connected Lie group G and the representations of its complexified Lie algebra $\mathfrak{g}_\mathbb{C}$.

Example 13.1. One simple example is given by $G = (\mathbb{R}, +) \cong (\mathbb{R}_{>0}, \cdot)$. Then, $\mathfrak{g} = \mathbb{R}$ with trivial Lie bracket, since G is abelian, and $\mathfrak{g}_\mathbb{C} = \mathbb{C}$.

The one-dimensional representations of $\mathfrak{g}_\mathbb{C}$ are given by $\mathrm{Hom}_{\mathsf{LieAlg}}(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}$, and the one-dimensional representations of $\mathbb{R}$ are $\mathrm{Hom}_{\mathsf{LieGrp}}(\mathbb{R}, \mathbb{C}^\times)$, which is isomorphic to $\mathbb{C}$ again, with $\lambda \mapsto e^{\lambda t}$. The one-dimensional representations are the irreducibles, so this generates the whole category. ◀

Another example, relating to what we discussed last time, was the relation between the representations of SU(2) and those of its complexified Lie algebra $\mathfrak{su}(2)_\mathbb{C} \cong \mathfrak{sl}_2(\mathbb{C})$. We wrote down the generators of $\mathfrak{sl}_2(\mathbb{C})$ in (12.7), and those of $\mathfrak{su}(2)$ in (12.6); they're related by $i\sigma_1 = ie + if$, $i\sigma_2 = e - f$, and $i\sigma_3 = ih$. You can also take the real span of e, f, and h, which defines the Lie algebra

$$\mathfrak{sl}_2(\mathbb{R}) = \mathrm{span}_\mathbb{R}\{e, f, h \mid [e, f] = h, [h, e] = 2e, [h, f] = 2f\}. \tag{13.2}$$

Now, an SU(2)-representation V' defines an $\mathfrak{sl}_2(\mathbb{C})$-representation V, which in particular comes with actions of e and f. However, these do not integrate to an action of a 1-parameter subgroup of SU(2), meaning Lie's theorem has actually introduced extra structure into the theory of SU(2)-representations that we didn't have beforehand.

[16] If you want to be fancy, you could think of this as a quotient of a free Lie algebra by something.

Multiplication and the universal enveloping algebra. One thing which might be confusing is that even though elements of the Lie algebra have come to us as matrices, they're not acting by matrix multiplication. That is, there's an operator $(x\cdot)\colon V \to V$ for every $x \in \mathfrak{g}$, such that the Lie bracket $[x, y]$ is sent to $(x\cdot)(y\cdot) - (y\cdot)(x\cdot)$. Multiplication of matrices isn't remembered, so even though h is of order 2 as a matrix, its eigenvalues as an operator on V could be more than ± 1. In particular, $x \cdot (y \cdot v) \neq (xy) \cdot v$.

Sometimes it's nice to multiply things, though, and we can build a ring in which the above expression makes sense.

Definition 13.3. Let V be a vector space. Then, the *tensor algebra* $T(V)$ is the free associative algebra on V, i.e.

$$T(V) = \bigoplus_{n\geq 0} V^{\otimes n},$$

where $(x_1 \otimes \cdots \otimes x_n) \cdot (y_1 \otimes \cdots \otimes y_m) = x_1 \otimes x_n \otimes y_1 \otimes y_m$ extends linearly to define the multiplication.

Definition 13.4. Let $\mathfrak{g}$ be a Lie algebra. Then, its *universal enveloping algebra* $\mathcal{U}(\mathfrak{g})$ is the free associative algebra on $\mathfrak{g}$ such that the Lie bracket is the commutator, i.e.

$$\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g})/(x \otimes y - y \otimes x - [x, y]).$$

Exercise 13.5. Exhibit an equivalence of categories $\mathsf{Rep}_{\mathfrak{g}} \cong \mathsf{Mod}_{\mathcal{U}(\mathfrak{g})}$.

The representations of $\mathfrak{sl}_2(\mathbb{C})$. Let V be an $\mathfrak{sl}_2(\mathbb{C})$-representation, so the actions by e, f, and h define three operators $(e\cdot)$, $(f\cdot)$, and $(h\cdot)$ satisfying the relations in (13.2), and this is sufficient to define the whole representation. From the perspective of Lie groups, this is incredible — there's so much information in the group structure of a Lie group, so this perspective would be impossible without passing to Lie algebras.

Let $V[\lambda]$ denote the λ-eigenspace for h acting on V. Then, $V[\lambda]$ is known as the λ-*weight space* for V, and its elements are called λ-*weight vectors*.

Lemma 13.6. *$e \cdot V[\lambda] \subseteq V[\lambda + 2]$ and $f \cdot V[\lambda] \subseteq V[\lambda - 2]$.*

So e and f act by shifting the weights up and down.

Proof. Let $v \in V$, so that

$$\begin{aligned} h \cdot (e \cdot v) &= e \cdot (h \cdot v) + [h, e] \cdot v \\ &= \lambda(e \cdot v) + 2e \cdot v \\ &= (\lambda + 2)e \cdot v, \end{aligned}$$

so $e \cdot v \in V[\lambda + 2]$. The proof for f is identical. ⊠

Now, assume V is irreducible and that there's a nonzero eigenvalue λ. Then, $\bigoplus_\lambda V[\lambda] \subset V$ is a nonzero subrepresentation, and therefore must be all of V! Thus, for irreducible representations the actions of e and f are completely determined on the weight spaces:

(13.7)
$$\begin{tikzcd}
\cdots \arrow[r, bend left] & V[\lambda-2] \arrow[l, bend left] \arrow[r, bend left, "e"] & V[\lambda] \arrow[l, bend left, "f"] \arrow[r, bend left, "e"] & V[\lambda+2] \arrow[l, bend left, "f"] \arrow[r, bend left, "e"] & V[\lambda+4] \arrow[l, bend left, "f"] \arrow[r, bend left] & \cdots \arrow[l, bend left]
\end{tikzcd}$$

Diagrams that look like (13.7) appear often in mathematics, and the representation theory of $\mathfrak{sl}_2(\mathbb{C})$ is always hiding in the background.

Let μ be a *highest weight*, i.e. an eigenvalue such that $\mathrm{Re}(\lambda)$ is maximal;[17] a $v \in V[\mu]$ is called a *highest weight vector*. In particular, by (13.7), $e \cdot v = 0$. Let

$$v^k := \frac{1}{k!} f^k \cdot v \in V[\mu - 2k].$$

You can interpret f^k either as a composition of $(f\cdot)$ k times, or as an honest element of $\mathcal{U}(\mathfrak{sl}_2(\mathbb{C}))$.

Claim. $f \cdot v^k = v^{k+1}$, and $e \cdot v^{k+1} = v^k$.

[17] We'll eventually prove that all weights are real, so that μ is unique.

So e and f walk the ladder between the weight spaces.

Proposition 13.8.

(1) *The highest weight is a nonnegative integer.*
(2)

$$V \cong \bigoplus_{k=0}^{\mu} V[\mu - 2k],$$

and the actions of e and f move between these subspaces as in the previous claim.

Lecture 14.

The representation theory of $\mathfrak{sl}_2(\mathbb{C})$: 2/17/17

"Let's call it n; that's more integer-sounding."

Today, we'll finish the representation theory of $\mathfrak{sl}_2(\mathbb{C})$ that we started last time. We've already seen that any irreducible finite-dimensional representation of $\mathfrak{sl}_2(\mathbb{C}) = \langle e, f, h\rangle$ has a highest weight vector, i.e. a $v \in V[\mu]$, where μ is the largest eigenvector of h (all eigenvectors are real) and $V[\mu]$ denotes its eigenspace.

Let's build an $\mathcal{U}(\mathfrak{sl}_2(\mathbb{C}))$-module M_μ, i.e. $\mathfrak{sl}_2(\mathbb{C})$-representation (we may refer to these as $\mathfrak{sl}_2(\mathbb{C})$-modules when they're not finite-dimensional) freely generated by a highest weight vector v_μ of weight μ. That is,

$$M_\mu = \mathcal{U}(\mathfrak{sl}_2(\mathbb{C})) \cdot v_\mu \coloneqq \mathcal{U}(\mathfrak{sl}_2(\mathbb{C}))/(h - \mu, e).$$

That is, we take the free $\mathcal{U}(\mathfrak{sl}_2(\mathbb{C}))$-module, and make μ a weight (so $h - \mu = 0$), then make it the highest weight (so e acts as 0).

Definition 14.1. M_μ is called the *Verma module* of highest weight μ.

We'll talk about Verma modules for other simple Lie algebras later.

Why is this good? If V is any representation of $\mathfrak{sl}_2(\mathbb{C})$ with a highest weight vector v, then there is a homomorphism of $\mathfrak{sl}_2(\mathbb{C})$-modules $M_\mu \to V$ sending $v_\mu \mapsto v$. In particular, if V is irreducible, then this is a quotient map, so $V \cong M_\mu/W$ for some submodule W. This is another approach to classifying the representations of $\mathfrak{sl}_2(\mathbb{C})$: find all the Verma modules and then list their finite-dimensional quotients.

We can make this very explicit: let $v_\mu^k = (1/k!)f^k v_\mu$; then, $\{v_\mu^k\}$ is a basis for M_μ: it's an infinite set, because there's no lowest weight. The action of f sends $v_\mu^i \mapsto v_\mu^{i+1}$, and e sends $v_\mu^{i+1} \mapsto (\mu - i)v_\mu^i$. Finally, $h \cdot v_\mu^k = (\mu - 2k)v_\mu^k$. So what we need are quotients of this.

Lemma 14.2. *If μ is negative, then M_μ is irreducible.*

The key is that $\mu - i$ is never zero, so starting from any v_μ^i, you can use e and f to get anywhere.

What if $\mu = n$ is a nonnegative integer? Then, $e \cdot v_\mu^{n+1} = 0$. The subspace $W = \bigoplus_{\lambda<\mu} V[\lambda]$ is a submodule, and the quotient $L_\mu \coloneqq M_\mu/W$ is a finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-representation. Moreover, W is again a Verma module, and is of the form M_{-n-2} (check the action of h on v_μ^{n+1}).

Corollary 14.3. *ANy finite-dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ is isomorphic to L_n for some nonnegative integer n.*

We can also compute the character:

$$\begin{aligned}\chi_{L_n}(z) &= \sum (\dim(L_n[\lambda]))e^{2\pi i\lambda t} = z^{-n} + z^{-n+2} + \cdots + z^{n-2} + z^n \\ &= \frac{z^{-n-1} - z^{n-1}}{z^{-1} - z},\end{aligned}$$

where $z - e^{2\pi it}$. Each factor of z^k corresponds to a weight space of weight k.

Remark. The Verma modules M_μ are infinite-dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$, but they do not integrate to representations of SU(2). It's interesting that they're nonetheless part of the process of calculating the representations of SU(2). This is an example of infinite-dimensional phenomena behaving badly. ◂

We want to generalize this to talk about representations of other Lie algebras and Lie groups. To do this, we need to discuss the classification of Lie algebras, which will occupy the next week or so. Eventually, we'll restrict to semisimple Lie algebras, but we have to get there first.

Consider an *abelian Lie algebra* (i.e. the Lie algebra of an abelian group) $\mathfrak{t}$: the Lie bracket is 0, so this is just the data of a finite-dimensional vector space. Therefore its universal enveloping algebra is $\mathcal{U}(\mathfrak{t}) = \mathrm{Sym}(\mathfrak{t})$, the free commutative algebra on $\mathfrak{t}$.[18]

This is the nicest case; what's next? Let's consider the things that can be built from abelian Lie algebras.

Definition 14.4.

- A *Lie subalgebra* $\mathfrak{h} \subseteq \mathfrak{g}$ is a vector subspace closed under Lie bracket, i.e. $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$.
- An *ideal* in a Lie algebra $I \subseteq \mathfrak{g}$ is a vector subspace such that $[I, \mathfrak{g}] \subseteq I$.

These are the analogues of subgroups and normal subgroups, respectively; in particular, if $I \subseteq \mathfrak{g}$ is an ideal, $\mathfrak{g}/I$ is a Lie algebra with the bracket inherited from $\mathfrak{g}$.

Definition 14.5. A Lie algebra $\mathfrak{b}$ is *solvable* if there is a sequence

$$\mathfrak{b} \supseteq \mathfrak{b}_1 \supseteq \mathfrak{b}_2 \supseteq \cdots \supseteq \mathfrak{b}_n \supseteq \mathfrak{b}_{n+1} = 0$$

such that each $\mathfrak{b}_i \subset \mathfrak{b}_{i-1}$ is an ideal, and the quotient $\mathfrak{b}_{i-1}/\mathfrak{b}_i$ is an abelian Lie algebra.

This is just like the definition of solvable groups.

Example 14.6. Let $\mathfrak{b}$ denote the algebra of $n \times n$ triangular matrices inside $\mathfrak{gl}_n(\mathbb{C})$. Then, $\mathfrak{b}$ is solvable: let $\mathfrak{b}_1$ be the strictly upper triangular matrices (zero on the diagonal), which is an ideal: the quotient is the Lie algebra of diagonal matrices, which is abelian. Then, one can repeat by setting the superdiagonal to 0 to define $\mathfrak{b}_2$, and so forth. ◀

On the other hand, $\mathfrak{gl}_n(\mathbb{C})$ is not solvable; we'll see that in a much more general context later on.

To understand the representation theory of solvable Lie algebras, we should start with the representations of abelian Lie algebras $\mathfrak{t}$. We saw these are identical to $\mathrm{Sym}(\mathfrak{t})$-modules, hence representations of a polynomial ring.

Schur's lemma implies that irreducible representations of an abelian Lie algebra are 1-dimensional. This is equivaent to the statement that any collection of commuting matrices has a common eigenvector: the eigenvector defines a 1-dimensional invariant subspace.

Irreducible representations of an abelian Lie algebra $\mathfrak{t}$ are easy to classify: they are given by

$$\mathrm{Hom}_{\mathsf{LieAlg}}(\mathfrak{t}, \mathbb{C}) = \mathrm{Hom}_{\mathsf{Vect}}(\mathfrak{t}, \mathbb{C}) = \mathfrak{t}^*.$$

This is also the same as $\mathrm{Hom}_{\mathsf{Alg}_\mathbb{C}}(\mathrm{Sym}(\mathfrak{t}), \mathbb{C}) = \mathrm{Spec}(\mathrm{Sym}(\mathfrak{t}))$. These statements, fancy as they may look, boil down to families of commuting matrices sharing a common eigenvector.

However, there's no analogue to Maschke's theorem in this context: not every representation is completely reducible. (Recall that a representation V is *completely reducible* if for every subrepresentation $W \subset V$, there's a complement $U \subset V$ such that $V = W \oplus V$.) Completely reducible representations are direct sums of irreducibles.

If a representation cannot be written as a nontrivial direct sum of subrepresentations, it's called *indecomposable*. Irreducible representations are indecomposable, but the converse is not always true. So to understand the representation theory of a Lie algebra, it suffices to classify its indecomposable representations.

In some sense, this is the wrong question to ask: it's *wild*, meaning it's nearly impossible. This is important to understand, especially given how tractable the theory is for semisimple Lie algebras.

Lecture 15.

Solvable and nilpotent Lie algebras: 2/20/17

Throughout today's lecture, $\mathfrak{t}$ will denote an abelian Lie algebra, $\mathfrak{b}$ will denote a solvable Lie algebra, and $\mathfrak{n}$ will denote a nilpotent Lie algebra.

Last time, we talked about the representation theory of abelian Lie algebras. The irreducible representations of $\mathfrak{t}$ are classified by $\mathfrak{t}^* := \mathrm{Hom}_\mathbb{C}(\mathfrak{t}, \mathbb{C})$, and all irreducibles are one-dimensional: an element of $\mathfrak{t}$ is an assignment of an eigenvalue to each operator. We wish to classify more general Lie algebras.

[18] Alternatively, this can be identified with the algebra of polynomials on $\mathfrak{t}^*$.

Given a finite-dimensional $\mathfrak{t}$-representation V and a $\lambda \in \mathfrak{t}^*$, let's consider its *generalized eigenspace*

$$V_{(\lambda)} \coloneqq \{v \in V \mid \text{for all } x \in \mathfrak{t},\ (x - \lambda(x))^N \nu = 0 \text{ for some } N \gg 0\}.$$

Proposition 15.1.

$$V = \bigoplus_{\lambda} V_{(\lambda)}.$$

In particular, if V is indecomposable, then $V = V_{(\lambda)}$ for some $\lambda \in \mathfrak{t}^*$.

So irreducible representations are points in $\mathfrak{t}^*$, but indecomposables may be slightly "thicker," though they still live over a single point. More general representations are less local.

Remark. One can use algebraic geometry to establish (more or less by definition) that the category of $\mathcal{U}(\mathfrak{t})$-modules and the category of quasicoherent sheaves on $\mathfrak{t}^*$ are equivalent. This is useful for thinking of representations of $\mathfrak{t}$ geometrically.

For example, $\mathcal{U}(\mathfrak{t}) \cong \operatorname{Sym}(\mathfrak{t}) \cong \mathbb{C}[\mathfrak{t}^*]$, and inside these are the finite-dimensional representations, which correspond to coherent sheaves with finite support. So a representation V becomes a sheaf over the vector space $\mathfrak{t}^*$: the fiber over a $\lambda \in \mathfrak{t}^*$ is the λ-eigenspace of V. But in algebraic geometry, it's also possible to take higher-order neighborhoods of a point, and the fiber over such a neighborhood (when n is sufficiently large) is the generalized eigenspace of λ. In particular, every finite-dimensional representation is a finite direct sum of indecomposables, hence is a union of skyscraper sheaves over finitely many points. This makes the classification problem more geometric: we just need to know what the representations over a single point are. ⊲

So how should we classify indecomposable representations?

Example 15.2. Let's start with the example where $\mathfrak{t} = \mathbb{C}$ is one-dimensional. A representation of a 1-dimensional Lie algebra is a module over $\mathbb{C}[x]$. This is special: $\mathbb{C}[x]$ is a PID, and polynomial rings in more variables aren't.

We have the following tautology: the isomorphism classes of n-dimensional $\mathfrak{t}$-representations are in bijection with the conjugacy classes of $n \times n$ matrices. The latter are classified by the *Jordan normal form*, whose theory implies there's a unique indecomposable representation (up to isomorphism) of $\mathbb{C}[x]$ of dimension n generated by a single λ-eigenvector, where the action of x is by the matrix

$$\begin{pmatrix} \lambda & 1 & & & 0 \\ & \lambda & 1 & & 0 \\ & & \ddots & \ddots & \vdots \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix}.$$ ⊲

In this case, the representation theory of $\mathfrak{t}$ is *tame*, in that it's easily understood. However, there's no analogue even for $\mathfrak{t} = \mathbb{C}^k$ when $k > 1$, and in fact the representation theory is very complicated! So instead, the best we can do is the geometric picture described above.

One instance of this phenomenon is that the *commuting variety*, the set of k-tuples of nilpotent $n \times n$ matrices that all commute up to conjugacy, is not discrete, and so understanding it is very complicated: much is either unknown or was resolved recently.[19]

Solvable Lie algebras. Last time, we defined a Lie algebra $\mathfrak{b}$ to be solvable if it is built from abelian Lie algebras by successive extensions.

Definition 15.3. Let $\mathfrak{g}$ be a Lie algebra. Then, its *derived ideal* is $D(\mathfrak{g}) \coloneqq [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}$, i.e. the ideal spanned by $[x, y]$ for $x, y \in \mathfrak{g}$. Then, $\mathfrak{g}/D(\mathfrak{g})$ is called the *abelianization* of $\mathfrak{g}$, and the projection $\pi\colon \mathfrak{g} \to \mathfrak{g}/D(\mathfrak{g})$ is initial among maps to abelian Lie algebras. That is, if $f\colon \mathfrak{g} \to \mathfrak{t}$ is a morphism of Lie algebras, where $\mathfrak{t}$ is

[19] If instead you ask for the matrices to be invertible, you obtain a better-understood algebraic variety called the *character variety*.

abelian, there's a unique map h such that the diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, two heads, "\pi"] \arrow[dr, "f"'] & \mathfrak{g}/D(\mathfrak{g}) \arrow[d, dashed, "\exists!\, h"] \\ & \mathfrak{k} \end{tikzcd}$$

commutes.

The *derived series* of $\mathfrak{g}$ is

$$\mathfrak{g} \supseteq D(\mathfrak{g}) \supseteq D(D(\mathfrak{g})) \supseteq \cdots .$$

Proposition 15.4. *A Lie algebra $\mathfrak{g}$ is solvable iff its derived series terminates, i.e. $D^N(\mathfrak{g}) = 0$ for some $N \gg 0$.*

Remark. The use of the word "derived" is ultimately related to differentiation, as how the derived series in groups can be used to obtain infinitesimal information about a group. Also, the Jacobi identity implies that taking the bracket with something is a derivation on $\mathcal{U}(\mathfrak{g})$. ◀

Solvable Lie algebras have properties generalizing abelian ones.

Theorem 15.5 (Lie)**.** *Let V be a finite-dimensional representation of $\mathfrak{b}$. Then, there's a simultaneous eigenvector v for the action of all elements of $\mathfrak{b}$ on V.*

Corollary 15.6. *An irreducible representation of a solvable Lie algebra is one-dimensional. Moreover, the irreducible representations of $\mathfrak{b}$ are parameterized by $(\mathfrak{b}/[\mathfrak{b},\mathfrak{b}])^*$.*

Proof. The proof that they're one-dimensional is the same as for abelian Lie algebras. Thus, any irreducible representation is a map $\mathfrak{b} \to \mathbb{C}$, hence a Lie algebra homomorphism, which must factor through the abelianization. ⊠

Thus, the pictorial description of the representations of abelian Lie algebras also applies to solvable Lie algebras.

Another corollary of Theorem 15.5 is that the representations of $\mathfrak{b}$ are filtered. By a *(full) flag* for a (finite-dimensional) vector space V we mean a sequence $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V$ where $\dim V_i = i$.

Corollary 15.7. *Let V be a $\mathfrak{b}$-representation. Then, there's a $\mathfrak{b}$-invariant flag $0 = V_0 \subset \cdots \subset V_n = V$, i.e. each V_i is a subrepresentation.*

Choose a basis $\{e_1, \dots, e_n\}$ for V such that $V_i = \operatorname{span}\{v_j \mid j \le i\}$. The action of $\mathfrak{b}$ on V in this basis is through upper triangular matrices, since $\operatorname{span}\{e_1, \dots, e_i\}$ is preserved for each i. That is, if $\mathfrak{b}_n(\mathbb{C})$ denotes the *Borel subalgebra* of $\mathfrak{gl}_n(\mathbb{C})$, i.e. the algebra of upper triangular matrices, any representation $\rho\colon \mathfrak{b} \to \mathfrak{gl}_n(\mathbb{C})$ factors through the inclusion $\mathfrak{b}_n(\mathbb{C}) \hookrightarrow \mathfrak{gl}_n(\mathbb{C})$. One says that representations of solvable Lie algebras are *upper triangulizable.*

Nilpotent Lie algebras.

Definition 15.8. A Lie algebra $\mathfrak{n}$ is *nilpotent* if there is a sequence $\mathfrak{n} \supsetneq \mathfrak{n}_1 \supsetneq \mathfrak{n}_i \supsetneq \cdots$ such that $[\mathfrak{n}, \mathfrak{n}_i] \subseteq \mathfrak{n}_{i+1}$.

You could also relax to nonstrict inclusions if you require the chain to terminate at 0 at some finite step.

Theorem 15.9 (Engel[20])**.** *A Lie algebra $\mathfrak{n}$ is nilpotent if the adjoint action ad_x is nilpotent for each $x \in \mathfrak{n}$.*

Lecture 16.

Semisimple Lie algebras: 2/22/17

Today we'll continue to discuss the taxonomy of Lie algebras. This is a course in representation theory, rather than Lie theory, so we won't get too involved in this.

[20] Friedrich Engel the mathematician is distinct from Friedrich Engels the Marxist.

The Jordan decomposition. The following result may be familiar from an algebra class; it's purely linear algebra.

Proposition 16.1. *Let $A\colon V \to V$ be a linear operator, where V is a finite-dimensional vector space. Then, there is a unique way to write $A = A_s + A_n$, where*

- *A_s is* semisimple*, i.e. V is a direct sum of A_s-eigenspaces;*[21]
- *A_n is* nilpotent*, i.e. $A_n^N = 0$ for some $N \gg 0$; and*
- *$[A_s, A_n] = 0$.*

For example, if A is a Jordan block on $\mathbb{C}^n$, its semisimple part is the entries on the diagonal, and its nilpotent part is the entries on the superdiagonal. The proof for general A puts A in Jordan normal form, more or less.

Definition 16.2. If $\mathfrak{g}$ is a Lie algebra, its *radical* $\mathrm{rad}(\mathfrak{g})$ is the largest solvable ideal in $\mathfrak{g}$.

This exists because the sum of two solvable ideals is solvable, so the solvable ideals form a poset which is bounded above.

Definition 16.3. A Lie algebra $\mathfrak{g}$ is *semisimple* if $\mathrm{rad}(\mathfrak{g}) = 0$, i.e. it has no nonzero solvable ideals.

For any Lie algebra $\mathfrak{g}$, there is a short exact sequence

$$0 \longrightarrow \mathrm{rad}(\mathfrak{g}) \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathrm{rad}(\mathfrak{g}) \longrightarrow 0. \tag{16.4}$$

The third term is denoted $\mathfrak{g}_{\mathrm{ss}}$, and is semisimple. The sequence (16.4) splits, albeit noncanonically, and the isomorphism $\mathfrak{g} \cong \mathrm{rad}(\mathfrak{g}) \times \mathfrak{g}_{\mathrm{ss}}$ is called the *Levi decomposition.*

There's an ideal $\mathrm{rad}^n(\mathfrak{g}) \coloneqq [\mathfrak{g}, \mathrm{rad}(\mathfrak{g})]$; if V is any finite-dimensional representation of $\mathfrak{g}$, then $\mathrm{rad}^n(\mathfrak{g})$ acts by nilpotent operators. This is due to Lie's theorem (Theorem 15.5): $\mathrm{rad}(\mathfrak{g})$ acts by upper triangular matrices, and one can then show that $\mathrm{rad}^n(\mathfrak{g})$ acts by strictly upper triangular matrices.

Definition 16.5. A Lie algebra is called *reductive* if $\mathrm{rad}^n(\mathfrak{g}) = 0$. Equivalently, $\mathrm{rad}(\mathfrak{g}) = Z(\mathfrak{g})$ (the center).

There's a sense in which the difference between reductive and semisimple Lie algebras is entirely abelian Lie algebras.

Example 16.6. $\mathfrak{gl}_n(\mathbb{C})$ is reductive, but not semisimple: $\mathrm{rad}(\mathfrak{gl}_n(\mathbb{C})) = Z(\mathfrak{gl}_n(\mathbb{C})) = \mathbb{C}\cdot 1_{nn}$, the scalar multiples of the identity. The quotient is $(\mathfrak{gl}_n(\mathbb{C}))_{\mathrm{ss}} = \mathfrak{sl}_n(\mathbb{C})$, which is semisimple, and $\mathfrak{gl}_n(\mathbb{C}) \cong \mathfrak{sl}_n(\mathbb{C}) \oplus \mathbb{C}$. ◀

The Levy decomposition expresses any reductive Lie algebra $\mathfrak{g}$ as a direct sum of a semisimple Lie algebra $\mathfrak{g}_{\mathrm{ss}}$ and an abelian Lie algebra $Z(\mathfrak{g})$.

Let's apply some of to representation theory.

Proposition 16.7. *Let $\mathfrak{g}$ be a complex Lie algebra. Then, $\mathfrak{g}$ is reductive iff $\mathfrak{g} = \mathrm{Lie}(G)_{\mathbb{C}}$ for some compact Lie group G, and G is unique up to finite covers.*

That is, the reductive Lie algebras are exactly the ones whose representations we want to study! Semisimple Lie algebras are only a little different.

Proposition 16.8. *Let $\mathfrak{g}$ be a complex Lie algebra. Then, $\mathfrak{g}$ is semisimple iff $\mathfrak{g} = \mathrm{Lie}(G)_{\mathbb{C}}$ for some compact, simply connected Lie group G, and G is unique up to isomorphism.*

In both of these propositions, we mean real Lie groups G: there's a theorem that any compact complex Lie group (i.e. complex manifold with holomorphic multiplication and inversion) is a torus, in particular is abelian. If $\mathfrak{g}$ is a semisimple Lie algebra, a compact Lie group G with $\mathfrak{g} = \mathrm{Lie}(G)_{\mathbb{C}}$ is called a *compact form* for $\mathfrak{g}$.

The upshot is that "semisimple" is apt for Lie algebras.

Corollary 16.9. *If $\mathfrak{g}$ is a semisimple Lie algebra, then the category $\mathsf{Rep}_{\mathfrak{g}}$ of finite-dimensional complex representations of $\mathfrak{g}$ is semisimple, i.e. every representation is a direct sum of completely reducible ones.*

[21] Over $\mathbb{C}$, a matrix is semisimple iff it's diagonal.

Though we've arrived here through some geometry on Lie groups ("Weyl's unitary trick"), you could prove it directly.

Example 16.10. Let $G = \mathrm{SO}^+(1,3)$ be the *Lorentz group*, i.e. the 4×4 matrices that preserve a form of signature $(1,3)$ and have determinant 1; we further restrict to the identity component. This is a noncompact Lie group, and is not simply connected: as Lie groups, $G \cong \mathrm{PGL}_2(\mathbb{C})$, and the universal cover of the latter is a double cover $\mathrm{SL}_2(\mathbb{C}) \twoheadrightarrow \mathrm{PGL}_2(\mathbb{C})$.[22] Let $\widetilde{G}$ be the connected double cover of G, which could also be written $\mathrm{Spin}(1,3)$.

Let $\mathfrak{so}(1,3)$ denote the Lie algebra of $\mathrm{SO}(1,3)$ (as a Lie group), so as real Lie algebras, $\mathfrak{so}(1,3) \cong \mathfrak{sl}_2(\mathbb{C})$, and therefore $\mathfrak{so}(1,3)_{\mathbb{C}} \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$. By Lie's theorem, the category of finite-dimensional $\widetilde{G}$-representations is equivalent to the category of finite-dimensional $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$-representations. This is a semisimple Lie algebra, so by Proposition 16.8 it admits a compact form $\mathrm{SU}(2)\times \mathrm{SU}(2) = \mathrm{Spin}(4)$, a double cover of $\mathrm{SO}(4)$.

The upshot is that the category of $\widetilde{G}$-representations is semisimple (every finite-dimensional representation is completely reducible), even though $\widetilde{G}$ isn't compact. Thus, while we weren't able to prove it directly, the theory of Lie algebras is still useful here. Finally, if you want to understand representations of G, they're naturally a subcategory of $\mathsf{Rep}_{\widetilde{G}}$ for which the kernel of the covering map acts trivially; the finite-dimensional representations of G are also completely reducible. ◀

Though the finite-dimensional representation theory of the Lorentz group looks like that of a compact group, but this is far from true for infinite-dimensional representations.

Definition 16.11. A Lie group G is called *reductive* (resp. *semisimple*) if its Lie algebra $\mathfrak{g}$ is reductive (resp. semisimple).[23]

So finite-dimensional representations of semisimple, simply connected Lie groups are completely reducible. In fact, this also applies to "algebraically simply-connected" groups such as $\mathrm{SL}_2(\mathbb{R})$ (its covering spaces aren't central extensions as algebraic groups, which implies its étale fundamental group vanishes).

Lecture 17.

Invariant bilinear forms on Lie algebras: 2/24/17

Today, we're going to talk about invariant bilinear forms on Lie algebras, a rich subject we haven't discussed yet. $\mathfrak{g}$ will denote a Lie algebra over $\mathbb{C}$; a lot of this will also work over $\mathbb{R}$.

Definition 17.1. Let $B\colon \mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ be a symmetric bilinear form. B is called *invariant* if for all $x\in\mathfrak{g}$, $B(\mathrm{ad}_x(y), z) = -B(y, \mathrm{ad}_x(Z))$, where ad is the action of $\mathfrak{g}$ on itself by Lie bracket.

You can express this diagrammatically: let $\mathfrak{o}(\mathfrak{g}, B)$ denote the Lie algebra of endomorphisms of $\mathfrak{g}$ that preserve B, i.e.

$$\mathfrak{o}(\mathfrak{g}, B) := \{\alpha\colon \mathfrak{g}\to\mathfrak{g} \mid B(\alpha(\text{–}), \text{–}) = -B(\text{–}, \alpha(\text{–}))\}.$$

This is the Lie algebra of the Lie group

$$\mathrm{O}(\mathfrak{g}, B) := \{A \in \mathrm{GL}(\mathfrak{g}) \mid B(A(\text{–}), A(\text{–})) = B(\text{–},\text{–})\}.$$

This condition looks more natural than the one for $\mathfrak{o}(\mathfrak{g}, B)$. For B to be invariant, we want the adjoint $\mathrm{ad}\colon \mathfrak{g}\to\mathfrak{gl}(\mathfrak{g})$ to factor through $\mathfrak{o}(\mathfrak{g}, B)$:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{ad}"] \arrow[dr, dashed] & \mathfrak{gl}(\mathfrak{g}) \\ & \mathfrak{o}(\mathfrak{g}, B) \arrow[u, hook] \end{tikzcd}$$

Equivalently, B is invariant iff the induced map $\mathfrak{g}\to\mathfrak{g}^*$ sending $x\mapsto B(x, \text{–})$ is a morphism of $\mathfrak{g}$-representations, where $\mathfrak{g}$ acts on itself through the adjoint action and $\mathfrak{g}^*$ is the dual to that representation. In this context, $\mathfrak{g}$ is called the *adjoint representation* and $\mathfrak{g}^*$ is called the *coadjoint representation.*

[22]Here in characteristic 0, $\mathrm{PGL}_2(\mathbb{C})$ and $\mathrm{PSL}_2(\mathbb{C})$ are the same.

[23]This is equivalent to asking $\mathfrak{g}_{\mathbb{C}}$ to be reductive (resp. semisimple).

Definition 17.2. Let B be an invariant bilinear form. Then, B is *nondegenerate* if the map $\mathfrak{g} \to \mathfrak{g}^*$ sending $x \mapsto B(x, -)$ is an isomorphism.

Example 17.3. Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. Then, $B_{\mathrm{std}} \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ sending $A, B \mapsto \mathrm{tr}(AB)$ is a nondegenerate invariant symmetric bilinear form. ◀

This example really depends on $\mathfrak{g}$ being $\mathfrak{gl}(V)$, or more generally, we can make the same construction for any Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-representation $\rho \colon \mathfrak{g} \to \mathfrak{gl}(V)$, and the representation is necessary.[24] Explicitly, we let $B_V \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ send

$$(x, y) \longmapsto \mathrm{tr}(\rho(x)\rho(y)).$$

This is always invariant and symmetric, but in general isn't nondegenerate.

In particular, we can take ρ to be the adjoint representation. In this case, $B_{\mathfrak{g}}$ is written K and is called the *Killing form*.

Example 17.4. Let $\mathfrak{g} = \mathfrak{sl}_2$ and $V = \mathbb{C}^2$ be the standard representation. That is, we defined $\mathfrak{sl}_2$ to be the Lie algebra of traceless, 2×2 complex matrices, and we want it to act on V by precisely those matrices.

Recall that $\mathfrak{sl}_2$ is generated by three matrices e, f, and h as in (12.7), so B_V can be represented by a 3×3 matrix describing what happens to those generators. If we use the order (e, f, h), the matrix is

$$B_V = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

This matrix is clearly diagonalizable, so B_V is nondegenerate.

Similarly, the Killing form can be described with a matrix. First, the relations (13.2) define the adjoint representation:

$$\mathrm{ad}(e) = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \mathrm{ad}(f) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix} \qquad \mathrm{ad}(h) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, $K(x, y) = \mathrm{tr}(\mathrm{ad}(x), \mathrm{ad}(y))$, and you can check that $K = 4B_V$. ◀

Remark. That the Killing form is a multiple of B_V is no coincidence: it's possible to use Schur's lemma to show that for a simple Lie algebra $\mathfrak{g}$, there's a unique nondegenerate, invariant, skew-symmetric bilinear form on $\mathfrak{g}$ up to scalar multiplication by an element of $\mathbb{C}^\times$. (This uses the fact that the adjoint representation is irreducible.) ◀

There's lots of things to say about this subject, and they're all really important. Here are some facts which we won't have time to prove.

Theorem 17.5.

(1) $\mathfrak{g}$ *is reductive iff there exists a representation V for which B_V is nondegenerate.*
(2) $\mathfrak{g}$ *is semisimple iff the Killing form is nondegenerate.*
(3) $\mathfrak{g}$ *is solvable iff $K(x, [y, z]) = 0$ for all $x, y, z \in \mathfrak{g}$.*
(4) *Let $\mathfrak{g}_{\mathbb{R}}$ be a real Lie algebra. Then, $\mathfrak{g}_{\mathbb{R}}$ is the Lie algebra of a semisimple*[25] *compact group G iff the Killing form is negative definite.*

For example, the Killing form for $\mathfrak{gl}_n$ isn'n nondegenerate, since scalar matrices lie in its kernel. This implies that $\mathfrak{gl}_n$ is reductive but not semisimple.

The idea behind the proof of (3) is that a representation of a solvable Lie algebra can be chosen to be through upper triangular matrices; then, the bracket of two upper triangular matrices is strictly upper triangular, so causes the trace to vanish.

We can use these properties to understand what semisimple, reductive, and solvable Lie algebra actually look like. First let's organize the examples we have.

Example 17.6 (Classical groups)**.**

[24] For $\mathfrak{gl}_n(\mathbb{C})$, we're implicitly using the defining representation.

[25] The semisimplicity hypothesis is necessary because Lie algebras cannot tell the difference between an n-dimensional torus and the additive group underlying an n-dimensional vector space.

- $\mathrm{U}(n)$, $\mathrm{GL}_n(\mathbb{R})$, and $\mathrm{GL}_n(\mathbb{C})$ all have (complexified) Lie algebra $\mathfrak{gl}_n$. As we saw above, this Lie algebra is reductive but not semisimple.
- $\mathrm{SU}(n)$, $\mathrm{SL}_n(\mathbb{R})$, and $\mathrm{SL}_n(\mathbb{C})$ all have (complexified) Lie algebra $\mathfrak{sl}_n$, the Lie algebra of traceless matrices. This is reductive and semisimple, as we calculated, and in fact is an example of a *simple* Lie algebra.
- $\mathrm{SO}(n)$, $\mathrm{SO}_{p,q}$, and $\mathrm{SO}_n(\mathbb{C})$ all have (complexified) Lie algebra $\mathfrak{so}_n$, the Lie algebra of skew-symmetric matrices. ◂

We'll talk more about this, as well as the choice of order, next time.

Lecture 18.

Classical Lie groups and Lie algebras: 2/27/17

We talked about how the classification of Lie algebras splits into the classification of solvable Lie algebras and semisimple Lie algebras, and we're going to discuss the classification of semisimple Lie algebras, or compact Lie groups.

Recall that we started with a compact Lie group, e.g. $\mathrm{SU}(2)$ or $\mathrm{SO}(3)$, which is the thing we originally wanted to study. From this we obtained a reductive Lie algebra $\mathfrak{g}$, and it is a direct sum of an abelian part and a semisimple part $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$: $\mathfrak{g} \cong Z(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$. In particular, $G' = [G, G]$ is a semisimple Lie group.

Thus, to understand the representation theory of compact Lie groups, it suffices to understand the representation theory of semisimple Lie algebras. In particular, given a semisimple complex Lie algebra $\mathfrak{g}$, one can recover a simply-connected Lie group G whose complexified Lie algebra is $\mathfrak{g}$, and its real Lie algebra is the unique real form of $\mathfrak{g}$ such that the Killing form is negative definite.

Definition 18.1. A Lie algebra $\mathfrak{g}$ is *simple* if it has no nontrivial ideals. By convention, the one-dimensional abelian Lie algebra is not simple.

Recall that $\mathfrak{g}$ is semisimple if it has no solvable ideals; here we ask for no ideals at all.

Proposition 18.2. *Any semisimple Lie algebra is a direct sum of simple Lie algebras, i.e. the bracket is defined componentwise.*

Proof idea. By Corollary 16.9, if $\mathfrak{g}$ is semisimple, its representations are completely reducible. In particular, the adjoint representation decomposes as $\mathfrak{g} \cong \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$, and this is the decomposition of $\mathfrak{g}$ into simple Lie algebras. ⊠

So to understand semisimple Lie algebras, we only need to understand simple Lie algebras. Excellent. Where do simple Lie algebras come from? This is an interesting question because it has two quite distinct answers.

- From the perspective of classical groups, the Lie algebras and Lie groups we're familiar with arise as symmetries of linear-algebraic data. For example, the unitary, orthogonal, and symplectic groups arise as the groups of endomorphisms preserving certain forms, and this is a good source of simple Lie algebras.
- The *Cartan-Killing classification* adopts the perspective that simple Lie algebras are built out of copies of $\mathfrak{sl}_2(\mathbb{C})$ in particular combinatorial configurations.

A priori, these things have nothing to do with each other. Moreover, while the first perspective seems more natural, the second one is hugely important in modern physics. It will also take a little longer to explain, but it will make the whole classification make a lot more sense.

Let's first talk about the perspective coming from classical groups.

- The unitary group $\mathrm{U}(n)$ is the group of $n \times n$ complex matrices preserving the standard Hermitian form on $\mathbb{C}^n$. Its complexified Lie algebra is $\mathfrak{u}_n \otimes \mathbb{C} \cong \mathfrak{gl}_n(\mathbb{C})$, which it shares with $\mathrm{GL}_n(\mathbb{R})$. This is reductive, but not semisimple, since $Z(\mathrm{U}(n))$ is one-dimensional. Quotienting out by this, one obtains the special unitary group $\mathrm{SU}(n)$, whose complexified Lie algebra is $\mathfrak{sl}_n$, which is simple when $n \geq 2$. $\mathrm{SL}_2(\mathbb{R})$ has the same complexified Lie algebra.
- The special orthogonal groups are the groups preserving positive definite forms on a real vector space. $\mathrm{SO}(2n)$ and $\mathrm{SO}_{n,n}(\mathbb{R})$ have the same complexified Lie algebra $\mathfrak{so}_{2n}$, and $\mathrm{SO}(2n+1)$ and $\mathrm{SO}_{n,n+1}(\mathbb{R})$

have the same complexified Lie algebra $\mathfrak{so}_{2n+1}$; in both cases, the former group is compact and the latter isn't. If $n \geq 3$, $\mathfrak{so}_{2n}$ is simple, and if $n \geq 1$, $\mathfrak{so}_{2n+1}$ is simple. However, $\mathfrak{so}_4 \cong \mathfrak{su}_2 \oplus \mathfrak{su}_2$.

- There's a notion of a positive definite standard Hermitian form on the free $\mathbb{H}$-module $\mathbb{H}^n$, and the *symplectic group* $\mathrm{Sp}(n)$ is the group of $n \times n$ quaternionic matrices preserving this form. Confusingly, the most common convention is for its Lie algebra to be denoted $\mathfrak{sp}_{2n}$; this is the algebra of *Hamiltonian matrices*, and is always simple. There's a noncompact Lie group $\mathrm{Sp}_{2n}(\mathbb{R})$ with the same Lie algebra, which is the group of $2n \times 2n$ real matrices which preserve a symplectic form (a skew-symmetric bilinear form) on $\mathbb{R}^{2n}$.

So we have a table of correspondences between compact groups, noncompact groups with the same Lie algebra (called "split"), and their shared complexified Lie algebra. This is all organized in Table 2, which also lists the type of each Lie algebra in the Cartan-Killing classification of simple Lie algebras.

Compact	Split	Lie algebra	Dimension	Type
$\mathrm{U}(n)$	$\mathrm{GL}_n(\mathbb{R})$	$\mathfrak{gl}_n(\mathbb{C})$	n^2	n/a
$\mathrm{SU}(n)$	$\mathrm{SL}_n(\mathbb{R})$	$\mathfrak{sl}_n$	n^2-1	A_{n-1}
$\mathrm{SO}(2n)$	$\mathrm{SO}_{n,n}(\mathbb{R})$	$\mathfrak{so}_{2n}$	$n(2n+1)$	D_n
$\mathrm{SO}(2n+1)$	$\mathrm{SO}_{n,n+1}(\mathbb{R})$	$\mathfrak{so}_{2n+1}$	$n(2n-1)$	B_n
$\mathrm{Sp}(n)$	$\mathrm{Sp}_{2n}(\mathbb{R})$	$\mathfrak{sp}_{2n}$	$n(2n+1)$	C_n

Table 2. A table of classical Lie groups and Lie algebras. Since $\mathfrak{gl}_n(\mathbb{C})$ isn't simple, it doesn't have a Cartan-Killing type.

It seems like you should be able to turn this into a nice story where you start with $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ and derive this table, but there's some inevitable weirdness that creeps in. In particular, this list contains *almost* all of the simple Lie algebras — exactly five are missing! They are called $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$, $\mathfrak{f}_4$, and $\mathfrak{g}_2$. Each one of these is the Lie algebra of a Lie group, so we obtain Lie groups E_6, E_7, E_8, F_4, and G_2.

You can realize these geometrically from an 8-dimensional nonassociative division algebra called the *octonions* $\mathbb{O}$, which John Baez has written about. The octonions are constructed from the quaterions in the same way that the quaterions are constructed from $\mathbb{C}$, but iterating this construction further produces more pathological objects. In any case, the exceptional Lie algebras can be obtained by looking at projective planes associated to the octonions as Riemannian symmetric spaces. Specifically,

- $\mathfrak{e}_6$ is a 78-dimensional Lie algebra arising from $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{O}$.
- $\mathfrak{e}_7$ is a 133-dimensional Lie algebra arising from $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{O}$.
- $\mathfrak{e}_8$ is a 248-dimensional Lie algebra arising from $\mathbb{O} \otimes_{\mathbb{R}} \mathbb{O}$.
- $\mathfrak{f}_4$ is a 52-dimensional Lie algebra arising from $\mathbb{O}^2$.
- $\mathfrak{g}_2$ is a 14-dimensional Lie algebra arising from $\mathbb{O}$, e.g. $\mathrm{G}_2 = \mathrm{Aut}(\mathbb{O})$.

These seem to come out of nowhere, but make more sense from the Cartan-Killing perspective.

Remark. The Lie algebras outlined in Table 2 determine their compact forms up to covering. $\mathrm{SU}(n)$ and $\mathrm{Sp}(n)$ are simply connected, but $\pi_1(\mathrm{SO}(m)) \cong \mathbb{Z}/2$ when $m \geq 3$, so there exists a unique Lie group $\mathrm{Spin}(m)$ that double covers $\mathrm{SO}(m)$ and the covering map $\mathrm{Spin}(m) \twoheadrightarrow \mathrm{SO}(m)$ is a group homomorphism. ◀

Remark. There are some redundancies in Table 2 in low dimension.

- $\mathfrak{so}_3 \cong \mathfrak{sl}_2 \cong \mathfrak{sp}_2$, i.e. $A_1 = B_1 = C_1$.
- $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \times \mathfrak{sl}_2$, as we've seen, and therefore $D_2 = A_1 \times A_1$.
- $\mathfrak{so}_5 \cong \mathfrak{sp}_4$, i.e. $B_2 = C_2$.
- $\mathfrak{so}_6 \cong \mathfrak{sl}_4$, i.e. $A_3 = D_3$.

These last two are called *exceptional isomorphisms.* ◀

Next time, we'll begin explaining Dynkin diagrams and the Cartan-Killing perspective.

Lecture 19.

Roots and root spaces: 3/1/17

Last time, we discussed ho to obtain some simple Lie algebras from classical Lie groups $\mathrm{O}(n)$, $\mathrm{U}(n)$, and $\mathrm{Sp}(n)$. Today, we'll take adopt the Cartan-Killing approach of building semisimple Lie algebras from copies of $\mathfrak{sl}_2(\mathbb{C})$.

Throughout today's lecture, $\mathfrak{g}$ denotes a semisimple Lie algebra over $\mathbb{C}$.

Definition 19.1. An element $x \in \mathfrak{g}$ is called *semisimple* if $\mathrm{ad}(x) \in \mathrm{End}_{\mathbb{C}}(\mathfrak{g})$ is semisimple, and is called *nilpotent* if $\mathrm{ad}(x)$ is nilpotent.

Later we'll see that we could choose a different representation than the adjoint.

The Jordan decomposition for matrices says that every matrix is the sum of a semisimple part and a nilpotent part that commute in a unique way. There's a similar result for semisimple Lie algebras.

Proposition 19.2 (Jordan decomposition). *Let $x \in \mathfrak{g}$. Then, there exist $x_{\mathrm{ss}}, x_{\mathrm{n}} \in \mathfrak{g}$ such that $x = x_{\mathrm{ss}} + x_{\mathrm{n}}$, x_{ss} is semisimple, x_{n} is nilpotent, and $[x_{\mathrm{ss}}, x_{\mathrm{n}}] = 0$, and they are unique.*

Proof idea. In the adjoint representation, $\mathrm{ad}(x) = \mathrm{ad}(x)_{\mathrm{ss}} + \mathrm{ad}(x)_{\mathrm{n}}$, but why do these pieces in $\mathrm{End}_{\mathbb{C}}(\mathfrak{g})$ come from the action of elements of $\mathfrak{g}$? This is where semisimplicity of $\mathfrak{g}$ is used in an essential way. ⊠

Remark. The set of elements of $\mathfrak{g}$ that are either semisimple or nilpotent are not a vector subspace, but are dense. ◀

If every element of $\mathfrak{g}$ is nilpotent, then Theorem 15.9 would imply $\mathfrak{g}$ is nilpotent, which is a contradiction. Thus, there exists at least one nonzero semisimple element of $\mathfrak{g}$. This will allow us to decompose $\mathfrak{g}$ into eigenspaces for the semisimple elements. In fact, if we have a commuting family of semisimple elements, we can decompose $\mathfrak{g}$ into their simultaneous eigenspaces.

Definition 19.3. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra. Then, $\mathfrak{h}$ is called *toral* if it's abelian and consists solely of semisimple elements of $\mathfrak{g}$.

Any semisimple element generates a 1-dimensional abelian Lie subalgebra, which is in particular toral. So toral subalgebras exist.

Definition 19.4. A *Cartan subalgebra* of a semisimple[26] Lie algebra is a maximal toral subalgebra.

Let $C_{\mathfrak{g}}(\mathfrak{h}) \coloneqq \{x \in \mathfrak{g} \mid [x, y] = 0 \text{ for all } y \in \mathfrak{h}\}$ denote the centralizer.

Proposition 19.5. *$\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra iff it's a toral subalgebra and $C_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.*

Though we won't give a proof, the idea is reasonable: if we could add something to $\mathfrak{h}$, then whatever we added must centralize $\mathfrak{h}$.

Example 19.6. If $\mathfrak{g} = \mathfrak{sl}_n$, the Lie algebra of $n \times n$ traceless matrices, then the subalgebra of diagonal traceless matrices is a Cartan subalgebra. ◀

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then, the restriction of the adjoint representation to $\mathfrak{h}$ acting on $\mathfrak{g}$ is a family of semisimple operators acting on $\mathfrak{g}$, which can in particular be simultaneously diagonalized. Thus, as $\mathfrak{h}$-representations,

$$\mathfrak{g} = \bigoplus_{\alpha} \mathfrak{g}_\alpha$$

for some finite set of $\alpha \in \mathfrak{h}^*$, and where $h \in \mathfrak{h}$ acts on $\mathfrak{g}_\alpha$ by $\mathrm{ad}(h) \cdot v = \alpha(h) \cdot v$ for all $v \in \mathfrak{g}_\alpha$. It's for exactly this reason that we consider Cartan subalgebras.

Definition 19.7. Let $R = \{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0\} \setminus \{0\}$. The elements of R are called *roots* of $\mathfrak{g}$, and $\mathfrak{g}_\alpha$ is called the *root space* for α.

The idea behind the Cartan-Killing classification is that the data $(\mathfrak{h}^*, K|_{\mathfrak{h}^*}, R)$, i.e. a finite-dimensional vector space, a finite subset of it, and a bilinear form on it, is enough to reconstruct $\mathfrak{g}$ up to isomorphism!

[26] There's a notion of a Cartan subalgebra of any Lie algebra, but it's slightly different and not so useful for us.

Proposition 19.8.

(1) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.
(2) *With respect to any invariant inner product* $\langle -, - \rangle$, $\mathfrak{g}_\alpha \perp \mathfrak{g}_\beta$ *unless* $\alpha+\beta = 0$, *and* $\langle -, - \rangle \colon \mathfrak{g}_{-\alpha} \times \mathfrak{g}_\alpha \to \mathbb{C}$ *is a perfect pairing.*

Example 19.9. For example, let E_{ij} denote the $n \times n$ matrix with a 1 in the ij^{th} entry and 0s everywhere else, so $E_{ij} \in \mathfrak{sl}_n$. Let $\mathfrak{h}$ be the subalgebra of diagonal matrices whose entries sum to zero, which are a Cartan subalgebra for $\mathfrak{sl}_n$, which in particular is a subspace of $\operatorname{span}\{E_{ii}\}$. In particular, if $e_i \in \mathfrak{h}^*$ is dual to E_{ii}, then

$$\mathfrak{h}^* = \bigoplus_i \mathbb{C} \cdot e_i / \mathbb{C}(e_1 + \cdots + e_n).$$

Let $h \in \mathfrak{h}$ be the matrix with diagonal entries $h_1, \ldots, h_n$. Then,

$$[h, E_{ij}] = (h_i - h_j)E_{ij} = (e_i - e_j)(h)E_{ij}.$$

Thus, the roots are

$$R = \{e_i - e_j \mid i, j = 1, \ldots, n, i \neq j\},$$

anmd $\mathfrak{sl}_n(\mathbb{C})_{e_i - e_j} = \mathbb{C} \cdot E_{ij}$. ◀

Now we need to figure out how $\mathfrak{sl}_2$ appears.

Remark. If $\langle -, - \rangle$ is an invariant inner product on $\mathfrak{g}$, then its restriction to $\mathfrak{h} \times \mathfrak{h}$ is nondegenerate, and in particular defines an isomorphism $\mathfrak{h} \xrightarrow{\cong} \mathfrak{h}^*$. Thus, given an $\alpha \in R$, let $H_\alpha = \langle \alpha, - \rangle \in (\mathfrak{h}^*)^* = \mathfrak{h}$ be its dual. ◀

Let $h_\alpha = 2H_\alpha / \langle \alpha, \alpha \rangle \in \mathfrak{h}$.

Lemma 19.10. *If* $e \in \mathfrak{g}_\alpha$ *and* $f \in \mathfrak{g}_{-\alpha}$, *then* $[e, f] = \langle e, f \rangle H_\alpha$.

Thus, rescaling e and f as necessary, one can assume $\langle e, f \rangle = 2/\langle \alpha, \alpha \rangle$, which in particular implies the following.

Proposition 19.11. *The elements* e, f, *and* h_α *above generate a subalgebra of* $\mathfrak{g}$ *isomorphic to* $\mathfrak{sl}_2(\mathbb{C})$.

That is, e, f, and h_α satisfy the commutation relations we established for the generators e, f, and h of $\mathfrak{sl}_2(\mathbb{C})$ in (13.2).

We'll call the subalgebra generated by e, f, and h_α $\mathfrak{sl}_2(\mathbb{C})_\alpha$. This will be really useful for us, allowing us to understand the representation theory of $\mathfrak{g}$ in terms of that of $\mathfrak{sl}_2$.

Lecture 20.

Properties of roots: 3/3/17

"If you take away one thing from this course, it should be the representation theory of $\mathfrak{sl}_2$*."*

Once again, $\mathfrak{g}$ will denote a semisimple complex Lie algebra, and $(-, -)$ be an invariant inner product on $\mathfrak{g}$.

Last time, we saw that given a Cartan subalgebra $\mathfrak{h}$, the roots of $\mathfrak{g}$ are the set $R \subset \mathfrak{h}^*$ of α such that $\mathfrak{g}_\alpha \neq 0$ (the eigenvalues in the adjoint representation). We saw that given α, we can define $e \in \mathfrak{g}_\alpha$, $f \in \mathfrak{g}_{-\alpha}$, and $h_\alpha \in \mathfrak{h}$ that satisfy the commutation relations of $\mathfrak{sl}_2(\mathbb{C})$, i.e. $[e, f] = h_\alpha$, $[h_\alpha, e] = 2e$, and $[h_\alpha, f] = -2f$. Thus they define an embedding $\mathfrak{sl}_2(\mathbb{C}) \hookrightarrow \mathfrak{g}$, whose image we called $\mathfrak{sl}_2(\mathbb{C})_\alpha$.

Proposition 20.1.

(1) *The roots span* $\mathfrak{h}^*$ *as a vector space.*
(2) *Each* $\mathfrak{g}_\alpha$ *is one-dimensional.*
(3) *If* $\alpha, \beta \in R$, $n_{\alpha\beta} \coloneqq 2(\alpha, \beta)/(\alpha, \alpha) \in \mathbb{Z}$.
(4) *Let* $s_\alpha \colon \mathfrak{h}^* \to \mathfrak{h}^*$ *be defined by*

$$s_\alpha(\lambda) \coloneqq \lambda - \frac{2(\alpha, \lambda)}{(\alpha, \alpha)}\alpha.$$

Then, $s_\alpha(R) = R$.
(5) *If* $\alpha \in R$ *and* c *is a scalar such that* $c\alpha \in \mathbb{R}$, *then* $c = \pm 1$.
(6) *If* α, β, *and* $\alpha + \beta$ *are in* R, *then* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

Proof. For (1), suppose $h \in \{k \in \mathfrak{h} \mid \alpha(k) = 0 \text{ for all } \alpha \in R\}$. Then, $\mathrm{ad}(h) = 0$, so $h \in Z(\mathfrak{g}) = 0$ since $\mathfrak{g}$ is semisimple.

For (2), consider $\mathfrak{g}$ as an $\mathfrak{sl}_2(\mathbb{C})_\alpha$-representation. You can check that

$$\cdots \oplus \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_\alpha \oplus \mathbb{C} \cdot h_\alpha \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha} \oplus \cdots$$

is a subrepresentation for $\mathfrak{sl}_2(\mathbb{C})_\alpha$, and is a decomposition into weight spaces, where $\mathfrak{g}_{j\alpha}$ has weight $2j$. In particular, the zero-weight space is 1-dimensional and there are no odd weights, so V is an irreducible $\mathfrak{sl}_2(\mathbb{C})$-representation, and therefore all of the other weight spaces are 1-dimensional, in particular $\mathfrak{g}_\alpha$. Moreover, this implies $\mathfrak{g}_\alpha = \mathbb{C} \cdot e$, and therefore $\mathrm{ad}(e)(\mathfrak{g}_\alpha) = 0$, so $\mathfrak{g}_{2\alpha} = 0$, $\mathfrak{g}_{3\alpha} = 0$, and so forth. The same is true for $\mathfrak{g}_{-\alpha}$, which proves (5): $\mathfrak{sl}_2(\mathbb{C})_\alpha = \mathfrak{g}_\alpha \cdot \mathbb{C} \cdot h_\alpha \oplus \mathfrak{g}_\alpha$.

For (3), consider the action of $h_\alpha \in \mathfrak{sl}_2(\mathbb{C})_\alpha$ on $\mathfrak{g}_\beta$; by definition, h_α acts by the weight $\beta(h_\alpha)$, and the only weights that can occur are integers, so $\beta(h_\alpha) \in \mathbb{Z}$ as required. What's cool about this is that it provides a very explicit restriction on what the angles between roots can be.

For (4), consider again the action of $\mathfrak{sl}_2(\mathbb{C})_\alpha$ on $\mathfrak{g}$. Then, h_α acts with weight $n_{\alpha\beta}$ on $\mathfrak{g}_\beta$ if $\beta \neq 0$, and $\mathrm{ad}(f_\alpha)^{n_{\alpha\beta}}$ carries $\mathfrak{g}_\beta$ to $\mathfrak{g}_{\beta - n_{\alpha\beta}\alpha}$, and this is an isomorphism. This "hard Lefschetz formula" occurs because this is a reflection across the origin in $\mathfrak{g}$, and so carries root spaces to root spaces. ⊠

The point is that the representation theory of a semisimple Lie algebra is governed by some pretty restrictive data, because $\mathfrak{sl}_2$ controls it. There's also some geometry hiding in the background.

Let $\mathfrak{h}_\mathbb{R}$ denote the $\mathbb{R}$-span of h_α for a root $\alpha \in R$. We now know that the adjoint action $\mathrm{ad}(h_\alpha)$ acts with integer eigenvalues, then $K(h_\alpha, h_\beta) = \mathrm{tr}(\mathrm{ad}(h_\alpha)\,\mathrm{ad}(h_\beta)) \in \mathbb{R}$ and $K(h_\alpha h_\alpha) = \mathrm{tr}(\mathrm{ad}(h_\alpha)^2) > 0$. Thus, $(\mathfrak{h}_\mathbb{R}, K)$ is a real inner product space, and so $\mathfrak{h}_\mathbb{R}^*$ is spanned by R and equipped with a positive definite inner product.

Example 20.2. When $\mathfrak{g} = \mathfrak{sl}_3$,

$$\mathfrak{h}_\mathbb{R}^* = (\mathbb{R} \cdot e_1 \oplus \mathbb{R} \cdot e_2 \oplus \mathbb{R} \cdot e_3)/\mathbb{R} \cdot (e_1 + e_2 + e_3)$$
$$\mathfrak{h}_\mathbb{R} = \{(h_1, h_2, h_3) \in \mathbb{R}^3 \mid h_1 + h_2 + h_3 = 0\}.$$

We can normalize the inner product such that $(e_i, e_j) = \delta_{ij}$. The roots are

$$R = \{\alpha_1 \coloneqq e_1 - e_2, \alpha_2 \coloneqq e_2 - e_3, \alpha_3 \coloneqq e_1 - e_3, -\alpha_1, -\alpha_2, -\alpha_3\},$$

and the Killing form is $(\alpha_1, \alpha_2) = -1$ and $(\alpha_i, \alpha_i) = 2$. Thus the roots have an angle of 60° from each other, and this data knows everything there is to know about $\mathfrak{sl}_3$, which we'll see next week. ◀

Lecture 21.

Root systems: 3/6/17

Today, we'll talk about root systems in the abstract, which will allow us to classify them for semisimple Lie algebras. Today, E will always denote a Euclidean vector space, i.e. a finite-dimensional real vector space together with an inner product $\langle -, - \rangle$.

We care about root systems because if $\mathfrak{g}$ is a semisimple complex Lie algebra and $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra, then $E = \mathfrak{h}_\mathbb{R}^*$ with inner product $K(-,-)$ and $R = \{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0\} \setminus \{0\}$ will be our model example for a root system.

Definition 21.1. A *root system* is a pair (E, R) where $R \subseteq E$ satisfies the following axioms:

(1) R spans E as a vector space.
(2) For all $\alpha, \beta \in R$,
$$n_{\alpha\beta} \coloneqq \frac{2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \in \mathbb{Z}.$$
(3) For all $\alpha \in R$, the reflection
$$s_\alpha(\lambda) \coloneqq \lambda - \frac{2\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle}\alpha$$
maps R into itself.

If in addition whenever $\alpha \in R$ and $c\alpha \in R$ for some $c \in \mathbb{R}$, then $c = \pm 1$, then (E, R) is called a *reduced root system.*

In Proposition 20.1, we showed that if $\mathfrak{g}$ is a semisimple Lie algebra, its roots $R \subset \mathfrak{h}_{\mathbb{R}}^*$ form a reduced root system.

Even in the abstract, root systems come with quite a bit of additional data.

Definition 21.2. The *Weyl group* $W_{(E,R)}$ of a root system (E, R) is the subgroup of $\mathrm{O}(E)$ generated by the reflections s_α for $\alpha \in R$. If (E, R) is clear from context $W_{(E,R)}$ will be denoted W.

Since each s_α restricts to a permutation of R, you can also think of W as a subgroup of the symmetric group S_R.

Remark. Suppose a complex semisimple $\mathfrak{g}$ is the Lie algebra of a compact, semisimple Lie group G. With $(\mathfrak{h}_{\mathbb{R}}^*, R)$ as before, the Weyl group $W_{(\mathfrak{h}_{\mathbb{R}}^*,R)} \cong N_G(H)/H$, where H is a maximal torus of G. This equivalence is far from obvious, but sometimes is used as an alternate definition for the Weyl group. ◀

Example 21.3. Let $E = \mathbb{R}^n/\mathbb{R}(1, 1, \dots, 1)$ with the inner product inherited from the usual one on $\mathbb{R}^n$. Then, $R = \{e_i - e_j \mid i \neq j\} \subseteq E$ is a reduced root system, and in fact is the root system associated to $\mathfrak{sl}_n$, denoted A_n. ◀

Other classical examples include $\mathfrak{so}_{2n+1}$, which produces a root system B_n; $\mathfrak{sp}_{2n}$, which gives C_n; and $\mathfrak{so}_{2n}$, which gives D_n.

The number $n_{\alpha\beta}$ prescribes the ratio of the length of projection of α onto the direction of β to the length of β: $n_{\alpha\beta}$ is twice this ratio. In fact, comparing $n_{\alpha\beta}$ and $n_{\beta\alpha}$ severely restricts the possible angles.

- Suppose $n_{\alpha\beta} = 0$ and $n_{\beta\alpha} = 0$. Then, α and β are orthogonal to each other.
- Suppose $n_{\alpha\beta} = n_{\beta\alpha} = 1$. Then, $|\alpha| = |\beta|$, and the angle between them is $\pi/3$.
- Suppose $n_{\alpha\beta} = 2$ and $n_{\beta\alpha} = 1$. Then, $|\alpha| = \sqrt{2}|\beta|$ and the angle from β to α is $\pi/4$.
- Suppose $n_{\alpha\beta} = 3$ and $n_{\beta\alpha} = 1$. Then, $|\alpha| = \sqrt{3}|\beta|$ and the angle from β to α is $\pi/6$.

All of these come from special right triangles.

There are more possibilities where $n_{\alpha\beta}, n_{\beta\alpha} < 0$:

- You can have $n_{\alpha\beta} = n_{\beta\alpha} = -1$, for which $|\alpha| = |\beta|$ and the angle between them is $2\pi/3$.
- If $n_{\alpha\beta} = -2$ and $n_{\beta\alpha} = -1$, then $|\alpha| = \sqrt{2}|\beta|$ and the angle between them is $3\pi/4$.
- if $n_{\alpha\beta} = -3$ and $n_{\beta\alpha} = -1$, then $|\alpha| = \sqrt{3}|\beta|$ and the angle between them is $5\pi/6$.

You can check these (and π) are the only possible angles between two roots!

However, it could be the case that not all possible angles are realized by actual root systems. However, a few examples will realize every case.

- A_2 realizes $\pi/3$ and $2\pi/3$. See Figure 1 for a picture.
- $B_2 = C_2$ realizes $\pi/4$, $\pi/2$, and $3\pi/4$. See Figure 2 for a picture.
- The remaining angles are realized by G_2. See Figure 4 for a picture.

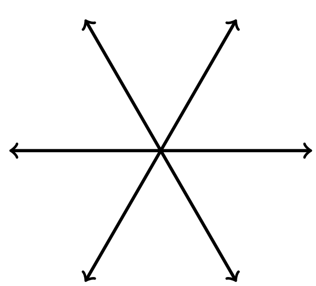


Figure 1. The rank-2 root system A_2 realizes the angles $\pi/3$ and $2\pi/3$.

Theorem 21.4. *The root systems A_2, B_2, D_2, and G_2 depicted in Figures 1 to 4 are a complete classification of rank-2 root systems.*

In all of these examples, it looks like one can start with two vectors and generate the rest with reflections. Let's make this precise.

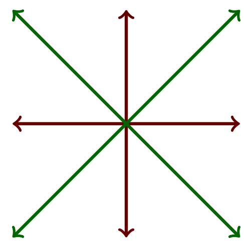


FIGURE 2. The rank-2 root systems B_2 and C_2 are isomorphic, and contain long (green) and short (red) roots.

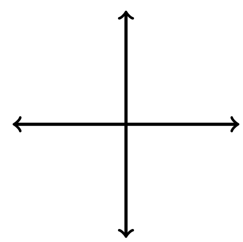


FIGURE 3. The rank-2 root system D_2 splits as a product $A_1 \times A_1$.

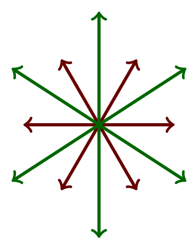


FIGURE 4. The rank-2 root system G_2, which corresponds to the exceptional Lie group $\mathrm{G}_2 = \mathrm{Aut}(\mathbb{O})$.

Definition 21.5. Let $t \in E$ be an element perpendicular to no element of R, which is a generic condition. Such a t determines a *polarization* of R as a disjoint union of $R_+ = \{\alpha \in R \mid \langle \alpha, t\rangle > 0\}$ and $R_- = \{\alpha \in R \mid \langle \alpha, t\rangle < 0\}$.

Different values of t may define the same polarization, and in general there's no canonical polarization around.

Definition 21.6. Given a polarization $R = R_+ \amalg R_-$, the set Π of *simple roots* is the subset of $\alpha \in R_+$ such that $\alpha \neq \beta + \gamma$ for $\beta, \gamma \in R_+$.

Proposition 21.7. *The simple roots form a basis for E.*

Therefore in particular any $\alpha \in R$ may be written as

$$\alpha = \sum_{\alpha_i \in \Pi} n_i \alpha_i, \qquad n_i \in \mathbb{Z},$$

and if $\alpha \in R_+$, then $n_i > 0$. Let the *Cartan matrix* $A = (a_{ij})$ be the $r \times r$ matrix (where $r = |\Pi| = \dim E$) with $a_{ij} = n_{\alpha_j \alpha_i}$, where $\Pi = \{\alpha_1, \dots, \alpha_r\}$. Next time, we'll use this to define a certain graph called the Dynkin diagram associated to (E, R).

Lecture 22.

Dynkin diagrams: 3/8/17

Today, we'll continue along the path that will lead us to Dynkin diagrams.

Recall that if E is a Euclidean space and $R \subset E$ is a root system, a generic vector in E determines a splitting of $R = R_+ \amalg R_-$, where R_+ is called the set of positive roots, and the simple roots $\Pi \subseteq R_+$. All of this came from the data of a compact semisimple Lie group, once upon a time.

Suppose $\Pi = \{\alpha_1, \dots, \alpha_r\}$. Then, the root system is determined by the Cartan matrix $A = (a_{ij})$, where $a_{ij} = n_{\alpha_j \alpha_i} = 2\langle \alpha_i, \alpha_j \rangle / \langle \alpha_i, \alpha_i \rangle$. It's not necessarily symmetric, but notice that $a_{ij} = 0$ iff $a_{ji} = 0$. We can encode this data in a graph called the *Dynkin diagram*, built according the the following recipe.

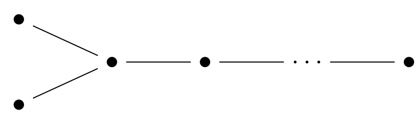
(1) There will be a node v_i for each $\alpha_i \in \Pi$.
(2) If $a_{ij} = 0$ (equivalently $a_{ji} = 0$), there is no edge from i to j.
(3) If $a_{ij} = -1$ and $a_{ji} = -1$, there's an edge $\bullet - \bullet$.
(4) If $a_{ij} = -1$ and $a_{ji} = -2$, there's an edge $\bullet \Rightarrow \bullet$.
(5) If $a_{ij} = -1$ and $a_{ji} = -3$, there's an edge $\bullet \Rrightarrow \bullet$.

It's possible to classify the irreducible root systems in terms of their Dynkin diagrams, which leads to a somewhat bizarre-looking classification: the four infinite families

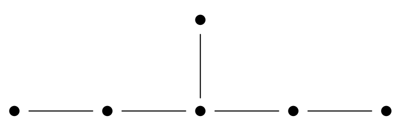
$$A_n = \bullet \text{---} \bullet \text{---} \cdots \text{---} \bullet \text{---} \bullet$$

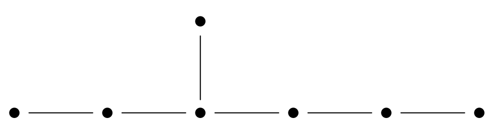
$$B_n = \bullet \text{---} \bullet \text{---} \cdots \text{---} \bullet \Longrightarrow \bullet$$

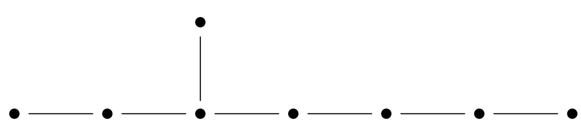
$$C_n = \bullet \text{---} \bullet \text{---} \cdots \text{---} \bullet \Longleftarrow \bullet$$

$D_n =$ 

and five exceptional diagrams

$E_6 =$ 

$E_7 =$ 

$E_8 =$ 

$$F_2 = \bullet \text{---} \bullet \Longrightarrow \bullet \text{---} \bullet$$

$$G_2 = \bullet \Rrightarrow \bullet.$$

Though this classification looks strange, it makes the classification elementary, and could be done by an interested hihg-school student, though it's a bit rote.

In any case, given a root system, we'd like to build a Lie algebra. If $\mathfrak{g}$ is a simple Lie algebra, then it splits

$$\mathfrak{g} = \bigoplus_{\alpha \in R_-} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha,$$

and the embedded copies of $\mathfrak{sl}_2(\mathbb{C})_\alpha = \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \oplus \mathfrak{g}_\alpha$, where $f_\alpha \in \mathfrak{g}_{-\alpha}$, $h_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_\beta]$, and $e_\alpha \in \mathfrak{g}_\alpha$, with $\langle e_\alpha, f_\alpha \rangle = 2/\langle \alpha, \alpha \rangle$.

Let $e_i := e_{\alpha_i}$, and similarly for f_i and h_i. Then, these elements satisfy the *Serre relations*

$$\begin{aligned} [h_i, h_j] &= 0 & \qquad [e_i, f_j] &= \delta_{ij} h_i \\ [h_i, e_j] &= a_{ij} e_j & [h_i, f_j] &= -a_{ij} f_j \\ (\operatorname{ad} e_i)^{1-a_{ij}} e_j &= 0 & (\operatorname{ad} f_i)^{1-a_{ij}} f_j &= 0. \end{aligned}$$

This suffices to recover $\mathfrak{g}$ from its root system.

Theorem 22.1. *Let R be a root system. Then, the free Lie algebra $\mathfrak{g}(R)$ generated by $e_i, f_i, h_i, i = 1, \dots, r$ satisfying the Serre relations is a finite-dimensional, semisimple Lie algebra with root system R.*

The hardest part of this theorem is showing that $\mathfrak{g}(R)$ is finite-dimensional.

Example 22.2. Let's look at the rank-2 examples, which illustrate that Theorem 22.1 is nontrivial. We have e_1, e_2, f_1, f_2, h_1, and h_2, and we know

$$\operatorname{ad}(e_2)^{1-a_{21}}e_1 = 0$$
$$\operatorname{ad}(e+1)^{1-a_{12}}e_2 = 0.$$

But there's still quite a bit to pin down.

- For A_2, we have $[e_2,[e_2,e_1]] = 0$ and $[e_1,[e_1,e_2]] = 0$.
- For $B_2 = C_2$, $\operatorname{ad}(e_2)^3e_1 = 0$ and $\operatorname{ad}(e_1)^2e_2 = 0$.
- For $D_2 = A_1 \times A_1$, $[e_1, e_2] = 0$.
- For G_2, $\operatorname{ad}(e_2)^4e_1 = 0$ and $\operatorname{ad}(e_1)^2e_2 = 0$.

There are plenty of other things you could choose for a_{ij}, but only these ones work: if you asked $\operatorname{ad}(e_2)^5e_1 = 0$ and $\operatorname{ad}(e_1)^2e_2 = 0$, you get something infinite-dimensional. ◀

Remark. There's a notion of a *generalized Cartan matrix*, and you can define the notion of a Lie algebra associated to such matrices using the same Serre relations, which is called a *Kac-Moody algebra*. The analogue of Theorem 22.1 is that the Kac-Moody algebra is finite-dimensional iff its generalized Cartan matrix is an actual Cartan matrix.

In addition to semisimple Kac-Moody algebras, people also care about *affine Lie algebras*, whose Dynkin diagrams look like those for the ADE classification, but have one more node and additional constraints on the edges. Even after working with this for a long time, this can still seem crazy. ◀

We've come quite far from our original approach to representation theory, but with these tools in hand we can now study the finite-dimensional representations of a semisimple Lie algebra $\mathfrak{g}$. This involves a classification and character formulas, which are part of the general theory, and we'll also focus on some examples, namely A_n and the orthogonal groups. The latter will lead us into spinor representations.

Anyways, let V be a finite-dimensional representation of $\mathfrak{g}$. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a root system $R = R_+ \amalg R_-$, etc. Then, V decomposes into weight spaces as an $\mathfrak{h}$-representation:

$$V = \bigoplus_{\lambda\in\mathfrak{h}^*} V[\lambda],$$

and $h \in \mathfrak{h}$ must act semisimply on V. This is because there's a compact Lie group G whose complexified Lie algebra is $\mathfrak{g}$, and if $T \subset G$ is a maximal torus, $\mathfrak{h}$ is the complexified Lie algebra of T. This is not obvious.

The next question is: which possible weights $\lambda \in \mathfrak{h}^*$ can occur? Each $h_\alpha \in \mathfrak{h}$ must act with integer eigenvalues, i.e. if $V[\lambda] \neq 0$, then $\lambda(h_\alpha) \in \mathbb{Z}$. Therefore we can define the lattice of *coroots*, called the *weight lattice*

$$P := \{\lambda \in \mathfrak{h}^* \mid h_\alpha(\lambda) \in \mathbb{Z} \text{ for all } \alpha \in R\}.$$

As an element of $\mathfrak{h}^{**} = \mathfrak{h}$, h_α is written $\alpha^\vee$. So the coroots are the things which pair with roots to get integers.

There are two more lattices around: the *root lattice* $Q \subset \mathfrak{h}^*$ and the *coroot lattice* $Q^\vee \subset \mathfrak{h}$ defined by

$$Q := \operatorname{span}_{\mathbb{Z}}(R), \qquad Q^\vee := \operatorname{span}_{\mathbb{Z}}(P).$$

Curiously, the dual lattice to Q isn't $Q^\vee$; instead, P is dual to $Q^\vee$.

The root lattice has a basis given by the simple roots, i.e. $Q = \mathbb{Z}\Pi$. If $\Pi = \{\alpha_1, \dots, \alpha_r\}$, then $Q^\vee$ has for a basis $\{\alpha_1^\vee, \dots, \alpha_n^\vee\}$. The dual basis to this is denoted $\{\omega_i\}$, i.e. $(\alpha_i^\vee, \omega_j) = \delta_{ij}$, and $\{\omega_i\}$, called the set of *fundamental weights*, is a basis for P.

Example 22.3. Let's look at A_1, which corresponds to $\mathfrak{sl}_2$. In this case, the roots are $\alpha = 2$ and $-\alpha = -2$, so the root lattice is the even numbers: $Q = 2\mathbb{Z}$. The weight lattice is all integers. ◀

This ± 2 is why factors of 2 and $1/2$ appear so much in these definitions.