

# FURUTA’S 10/8 THEOREM

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These notes were taken in a learning seminar on Furuta’s 10/8 theorem in Spring 2019. I live-T<sub>E</sub>Xed them using vim, and as such there may be typos; please send questions, comments, complaints, and corrections to [a.debray@math.utexas.edu](mailto:a.debray@math.utexas.edu). Thanks to Riccardo Pedrotti for some useful comments and for the notes for §3.

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## 1. INTRODUCTION TO SEIBERG-WITTEN THEORY: 1/23/19

Riccardo gave the first, introductory talk.

In 1982, Matsumoto conjectured that if  $M$  is a closed spin manifold,  $b_2(M) \geq (11/8)|\sigma(M)|$ . Here  $b_2(M)$  is the second Betti number and  $\sigma(M)$  is the signature. Equality holds for the K3 surface, so this is the best one can do.

In this seminar we’ll study a theorem of Furuta which makes major progress on this conjecture.

**Theorem 1.1** (10/8 theorem [Fur01]). *If the intersection form of  $M$  is indefinite,  $b_2(M) \geq (10/8)|\sigma(M)| + 2$ .*

If the intersection form is definite, work of Donaldson [Don83] says that, up to a change of orientation, the intersection form is diagonalizable, so that case is dealt with.

Furuta’s proof uses both Seiberg-Witten theory and equivariant homotopy theory. It can be pushed a little bit farther, but not enough to prove the  $11/8^{\text{th}}$  conjecture, as shown recently by Hopkins-Lin-Shi-Xu [HLSX18].

Today we’ll discuss some background for the proof.

**Definition 1.2.** Let  $V \rightarrow M$  be a rank- $n$  real oriented vector bundle. A *spin structure* on  $V$  is data  $\mathfrak{s} = (P_{\text{Spin}}(V), \tau)$ , where  $P_{\text{Spin}}(V) \rightarrow M$  is a principal  $\text{Spin}_n$ -bundle and  $\tau$  is an isomorphism

$$\tau: P_{\text{Spin}}(V) \times_{\text{Spin}_n} \mathbb{R}^n \xrightarrow{\cong} V.$$

A spin structure on a manifold  $M$  is a spin structure on  $TM$ .

*Remark 1.3.* There are other equivalent definitions of spin structures – for example, just as an orientation is a trivialization of  $V$  over the 1-skeleton of  $M$ , a spin structure is equivalent to a trivialization over the 2-skeleton. ◀

Here’s a cool theorem about spin manifolds.

**Theorem 1.4** (Rokhlin [Roh52]). *If  $M$  is a spin manifold,  $\sigma(M) \equiv 0 \pmod{16}$ .*

The signature makes sense when  $4 \mid \dim M$ . Smoothness is crucial here; there are topological spin 4-manifolds, whatever that means, that do not satisfy this theorem. Freedman’s  $E_8$  manifold is an example.

Suppose  $M$  is a spin 4-manifold. The representation theory of  $\text{Spin}_4$ , in particular the fact that the spin representation  $S$  splits as  $S^+ \oplus S^-$ , leads to two quaternionic line bundles  $\mathbb{S}^+, \mathbb{S}^- \rightarrow M$  with Hermitian metrics. Physics cares about these bundles, and will lead to powerful theorems in manifold topology.

These bundles have more structure: in particular, they are Clifford bundles.

**Definition 1.5.** Let  $S \rightarrow M$  be a real vector bundle with a Euclidean metric  $\langle \cdot, \cdot \rangle$ . A *Clifford bundle* structure is data of, for each  $x \in M$ , the data of a Clifford algebra action  $\text{Cl}(T_x M)$  on  $S_x$  that varies smoothly in  $x$ , such that the Clifford action is skew-adjoint, meaning

$$\langle v \cdot s_1, s_2 \rangle = -\langle s_1, v \cdot s_2 \rangle.$$

We also require the existence of a connection which is compatible with the Levi-Civita connection on  $TM$ .

Given the data of a Clifford bundle, there's an operator called the *Dirac operator*  $D$ , which is the following composition:

$$(1.6) \quad C^\infty(S) \xrightarrow{\nabla^{C\ell}} C^\infty(T^*M \otimes S) \xrightarrow{\langle \cdot, \cdot \rangle} C^\infty(TM \otimes S) \xrightarrow{\text{Clifford action}} C^\infty(S).$$

This operator is denoted  $\not{D}$ , a convention due to Feynman. It is a first-order, elliptic differential operator; ellipticity means that its analysis is nice.

Thus we can consider the *Seiberg-Witten equations* on a spin 4-manifold. Let  $(a, \varphi) \in \Omega_M^1(i\mathbb{R}) \times \Gamma(\mathbb{S}^+)$ ; then the equations are

$$(1.7a) \quad \not{D}\varphi + \rho(a)(\varphi) = 0$$

$$(1.7b) \quad \rho(d^+a) - \varphi \otimes \varphi^* + \frac{1}{2}|\varphi|^2 \text{id} = 0$$

$$(1.7c) \quad d^*a = 0.$$

On a non-spin manifold, the equations are a little more complicated.

## 2. THE MONOPOLE EQUATIONS: 1/28/19

Today, Kai spoke about the monopole equations and some of their important properties, foreshadowing compactness next week. We begin with some motivation.

Recall that if  $M$  is a closed, oriented 4-manifold (in either the topological or smooth category), the intersection form  $H_2(M) \times H_2(M) \rightarrow \mathbb{Z}$  is a unimodular, symmetric bilinear form.

**Question 2.1.** Which unimodular, symmetric bilinear forms arise as the intersection forms of smooth or topological manifolds?

For example, the intersection form of  $S^2 \times S^2$  is  $H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . The intersection form of  $\mathbb{CP}^2$  is (1). There's an interesting bilinear form called the *E8 form*

$$(2.2) \quad \text{E8} = \begin{pmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & & & 1 & 2 \end{pmatrix}.$$

Can this be realized as the intersection form of a smooth 4-manifold? Rokhlin's theorem tells us the answer is no, because such a manifold would have to be spin, and  $16 \nmid \sigma(\text{E8})$ . However, Freedman found a topological manifold  $M_{\text{E8}}$  whose intersection form is E8!

The direct sum of two copies of E8 satisfies Rokhlin's theorem, and this form is realized by the topological 4-manifold  $M_{\text{E8}} \# M_{\text{E8}}$ . However, Donaldson showed this manifold is not smoothable: specifically, the intersection forms of smooth 4-manifolds can be diagonalized over  $\mathbb{Z}$ , and E8 cannot.

There's still more interesting example: consider the *K3 surface*  $\{z_1^4 + z_2^4 + z_3^4 + z_4^4 = 0\} \subset \mathbb{CP}^3$ ; its intersection form is  $-2\text{E8} \oplus 3H$ . So does it split as a connect sum of 3 copies of  $S^2 \times S^2$  and two copies of  $M_{\text{E8}}$  (with the opposite orientation)? Freedman showed this is true topologically. Smoothly, of course, it can't hold, but we might still get something.

**Question 2.3.** Is there a smooth, oriented 4-manifold  $N$  such that, in the smooth category,  $K3 \cong N \# S^2 \times S^2$ ?

This was a longstanding question.

Seiberg-Witten invariants allow us to answer questions such as this – though in this semester, we’re more interested in the monopole map. In any case, let’s define the Seiberg-Witten equations.

Let  $M$  be a smooth, oriented 4-manifold with  $b_2^+$  odd and a Riemannian metric  $g$ , and let  $\mathfrak{s}$  be a  $\text{spin}^c$  structure on  $M$ , which determines a *basic class*  $K \in H^2(X)$ , i.e. an integer cohomology class such that  $K \equiv w_2(M) \pmod{2}$ . The  $\text{spin}^c$  structure  $\mathfrak{s}$  defines for us spinor bundles  $\mathbb{S}^+$  and  $\mathbb{S}^-$ . Let  $\mathcal{A}_L$  denote the space of  $U_1$ -connections,  $A \in \mathcal{A}_L$ , and  $\psi \in \Gamma(X, \mathbb{S}^+)$  (this is called a *spinor*). The Seiberg-Witten equations are

$$(2.4a) \quad D_A \psi = 0$$

$$(2.4b) \quad F_A^+ + i\delta = i\sigma(\psi).$$

These equations have a gauge symmetry: if  $G$  denotes the group  $\text{Map}(X, S^1)$  with pointwise multiplication,  $G$  acts on  $\mathcal{A}_L \times \Gamma(X, \mathbb{S}^+)$  on the first factor. Let  $B_K^+$  denote the quotient minus the locus of spinors which are identically zero; then  $B_K^+ \simeq \mathbb{CP}^\infty$ , so we know its cohomology is isomorphic to  $\mathbb{Z}[x]$ , with  $|x| = 2$ .

Let  $\mathcal{M}_K^\delta(g) \subset B_K^\times$  denote the space of solutions to the Seiberg-Witten equations. This space has dimension

$$(2.5) \quad d := \frac{1}{4}(K^2 - (3\sigma(M) + 2\chi(M))),$$

and, crucially, defines a class  $[\mathcal{M}_K^\delta(g)] \in H_d(B_K^\times)$  which does not depend on  $g$  for generic choices of the metric. The *Seiberg-Witten invariants* are

$$(2.6) \quad SW_X(K) := \langle x^{d/2}, [\mathcal{M}_K^\delta(g)] \rangle \in \mathbb{Z}.$$

The fact that  $b_2^+(M) = 0$  implies  $d$  is even.

This defines a map  $SW$  from the basic classes to  $\mathbb{Z}$ . Taubes showed two important results.

**Theorem 2.7** (Vanishing theorem (Taubes)). *If  $M$  is diffeomorphic to a connect sum of two closed, oriented 4-manifolds  $X_1 \# X_2$ ,  $b_2^+(X_1) > 0$ , and  $b_2^+(X_2) > 0$ , then the Seiberg-Witten equations of  $M$  vanish.*

**Theorem 2.8** (Nonvanishing theorem (Taubes)). *If  $\mathfrak{s}$  is the canonical  $\text{spin}^c$  structure associated to a complex structure on  $M$  and  $b_2^+(M)$  is positive and odd, then  $SW(\pm c_1(M)) = \pm 1$ .*

**Corollary 2.9.**  *$K3$  cannot split smoothly as a connect sum.*

This leads to an interesting generalization: there are *exotic  $K3$  surfaces*, homeomorphic but not diffeomorphic to the standard  $K3$ . They don’t all admit complex structures, and many of them are not symplectic. Nonetheless, they also don’t split off an  $S^2 \times S^2$ : this is a consequence of Furuta’s 10/8 theorem, because if  $K3 \cong N \# (S^2 \times S^2)$ , then  $b_2(N) = 20$  and  $\sigma(N) = -16$ , but

$$(2.10) \quad 20 \not\geq \frac{10}{8}|-16| + 2.$$

Now let’s discuss the monopole map. We now assume  $M$  is a spin manifold, with spin structure  $\mathfrak{s}$  and spinor bundles  $\mathbb{S}^\pm$ . Let  $A$  denote a spin connection and consider the spaces

$$(2.11) \quad \tilde{\mathcal{A}} := \{A + i \ker d\} \times (\Gamma(\mathbb{S}^+) \oplus \Omega^1(X))$$

$$(2.12) \quad \tilde{\mathcal{C}} := \{A + i \ker d\} \times (\Gamma(\mathbb{S}^-) \oplus \Omega^0(X) \oplus H^1(X; \mathbb{R}) \oplus \Omega^+(X)).$$

Both of these fiber over  $H^1(X; \mathbb{R})$ : for  $\tilde{\mathcal{A}}$ ,  $A + \alpha \mapsto [\alpha]$ , and there is a map  $\tilde{\mu}: \tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{C}}$  defined by

$$(2.13) \quad (A, \phi, a) \mapsto (A, D_A \phi + ia\phi, d^*a, a_{\text{harm}}, d^+a - \sigma(\phi)).$$

Here

- $D_A$  is the *Dirac operator*  $D_A: \Gamma(\mathbb{S}^+) \rightarrow \Gamma(\mathbb{S}^-)$ .
- $a\phi$  denotes Clifford multiplication.
- $d^*$  is the adjoint of  $d$ , which sends  $k$ -forms to  $(k-1)$ -forms, and satisfies the equation

$$(2.14) \quad d^* = \star d \star.$$

(This is in dimension 4; the sign convention is different in other dimensions.)

- $a_{\text{harm}}$  is the harmonic part of  $a$ : it's a general fact that any one-form in dimension 4 splits as  $a = a_{\text{harm}} + d^*\alpha + d\beta$  for some 0-form  $\beta$ . A form is *harmonic* if the Laplacian  $\Delta := dd^* + d^*d$  vanishes on it.
- $d^+a$  denotes the self-dual part of  $da$ .
- $\sigma(\phi)$  denotes the trace form of the endomorphism  $\phi \otimes \phi^* - (1/2)\|\phi\|^2 \text{id}$ .

Again the group  $G$  acts on  $\Gamma(\mathbb{S}^\pm)$  by pointwise multiplication, using  $S^1 \cong U_1 \subset \mathbb{C}$ . If  $u \in G$ ,  $u: X \rightarrow S^1$  also acts on the space of  $\text{spin}^c$  connections by  $d \mapsto udu^{-1}$ . Let  $G$  act trivially on forms.

Then, the map  $\tilde{\mu}$  defined in (2.13) is  $G$ -equivariant. Let  $G_0$  denote the maps which vanish at some specified basepoint  $p$ , and let  $\mathcal{A} := \tilde{A}/G_0$ ,  $\mathcal{C} := \tilde{C}/G_0$ , and  $\mu := \tilde{\mu}/G_0$ ; thus we get a map  $\mu: \mathcal{A} \rightarrow \mathcal{C}$ .

Now, both  $\mathcal{A}$  and  $\mathcal{C}$  fiber over the Picard group

$$(2.15) \quad \text{Pic}^g(X) := H^1(X; \mathbb{R})/H^1(X; \mathbb{Z}) = H^1(X; \mathbb{R})/G_0.$$

Then  $S^1 = G/G_0$  acts on  $\mu^{-1}(A, 0, 0, 0, 0)$ , and this is the space we're interested in.

We would like to study this space, and to do so we'll need to consider Sobolev spaces. For a fixed integer  $k > 2$ , let  $A_k$  be the fiberwise completion of  $A$  within  $L_k^2$  and  $C_{k-1}$  be the fiberwise completion of  $C$  within  $L_{k-1}^2$ . Then, the monopole map  $\mu$  is a map  $A_k \rightarrow C_{k-1}$ .

**Claim 2.16.** This monopole map  $\mu$  is  $S^1$ -equivariant, and is a compact perturbation of a linear Fredholm map.

The  $S^1$ -equivariance involves chasing through the definition but isn't bad; the rest is harder. What we can do is start by listing the terms that define a linear Fredholm map, and then check that the rest is compact. In the definition of  $\tilde{\mu}$ , the terms  $A$ ,  $D_A\phi$ ,  $d^*a$ ,  $a_{\text{harm}}$ , and  $d^+a$  are linear and Fredholm; thus we just have to check that  $a(\phi)$  and  $\sigma(\phi)$  are compact. For the first, we can use the fact that Clifford multiplication is compact, then compose with the map  $C_k \rightarrow C_{k-1}$ , which is also compact.

**Proposition 2.17.** Let  $T = \ell + c$  be a compact perturbation of a linear Fredholm map  $\ell$  between Hilbert spaces. The restriction of  $T$  to any closed, bounded subset  $\Omega$  is proper.

This will be restated as Claim 3.5 in the next lecture, and will be proven there.

### 3. COMPACTNESS OF THE MODULI SPACE OF SEIBERG-WITTEN SOLUTIONS: 2/3/19

These are Riccardo's notes on the lecture he gave, on the compactness of the moduli space of solutions to the Seiberg-Witten equations. This is a crucial step in Furuta's construction of finite-dimensional approximations, and relies on some functional analysis.

**3.1. A closer look at the Seiberg-Witten monopole map.** Let  $X$  be a oriented closed spin 4-manifold. Let  $\mathfrak{s}$  be a spin structure for it. Let  $\mathbb{S}^\pm$  be the positive and negative spinor bundles associated to it. Fix a spin connection  $A$  on them.

Recall the Seiberg-Witten equations can be thought as a fiber-preserving  $S^1$ -equivariant map between these two  $S^1$ -Hilbert bundles over  $H^1(X; \mathbb{R})$ :

$$(3.1a) \quad \tilde{\mathcal{A}} = (A + i \ker(d)) \times (\Gamma(\mathbb{S}^+) \oplus \Omega^1(X))$$

$$(3.1b) \quad \tilde{\mathcal{C}} = (A + i \ker(d)) \times (\Gamma(\mathbb{S}^-) \oplus \Omega^0(X) \oplus H^1(X; \mathbb{R}) \oplus \Omega^+(X)).$$

The map  $\tilde{\mu}: \tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{C}}$  is defined by

$$(3.2) \quad (A, \phi, a) \mapsto (A, D_A\phi + ia\phi, d^*a, a_{\text{harm}}, d^+a - \sigma(\phi)).$$

As explained in the previous seminar,  $\sigma(\phi)$  denotes the trace-free endomorphism  $i(\phi \otimes \phi^* - \frac{1}{2}\|\phi\|^2 \text{id})$  of  $\mathbb{S}^+$ , considered via the map  $\rho$  as a self-dual 2-form on  $X$ .

The gauge group  $\mathcal{G} = \text{Aut}_{\text{id}}(\mathfrak{s}) \cong \text{Map}(X, S^1)$  acts on spinors on the 4-manifold via multiplication with  $u: X \rightarrow S^1$  and on  $\text{Spin}^c$  connections via addition of  $ud(u^{-1})$ . It acts trivially on forms.

The map  $\tilde{\mu}$  is equivariant with respect to the action of  $\mathcal{G}$ . Dividing by the free action of the pointed gauge group we obtain the monopole map

$$\mu = \tilde{\mu}/\mathcal{G}_0: \mathcal{A} \rightarrow \mathcal{C}$$

as a fiber preserving map between the bundles  $\mathcal{A} = \tilde{\mathcal{A}}/\mathcal{G}_0$  and  $\mathcal{C} = \tilde{\mathcal{C}}/\mathcal{G}_0$  over  $\text{Pic}^g(X)$ . The preimage of the section  $(A, 0, 0, 0, 0)$  of  $\mathcal{C}$ , divided by the residual  $S^1$ -action, is called the *moduli space of monopoles*.

For a fixed  $k > 2$ , consider the fiberwise  $L_k^2$  Sobolev completion  $\mathcal{A}_k$  and the fiberwise  $L_{k-1}^2$  Sobolev completion  $\mathcal{C}_{k-1}$  of  $\mathcal{A}$  and  $\mathcal{C}$ . The monopole map extends to a continuous map  $\mathcal{A}_k \rightarrow \mathcal{C}_{k-1}$  over  $\text{Pic}^5(X)$ , which will also be denoted by  $\mu$ .

We will use the following properties of the monopole map.

- It is  $S^1$ -equivariant.
- Fiberwise, it is the sum  $\mu = l + c$  of a linear Fredholm map  $l$  and a nonlinear compact operator  $c$ .
- Preimages of bounded sets are bounded.

**Claim 3.3.** The moment map is  $S^1$ -equivariant.

*Proof.* Equivariance is immediate. The action is the residual action of the subgroup  $S^1$  of gauge transformations which are constant functions on  $X$ . This group acts by complex multiplication on the spaces  $\Gamma(\mathbb{S}^\pm)$  of sections of complex vector bundles and trivially on forms.  $\square$

**Claim 3.4.** Fiberwise, the moment map is the sum  $\mu = l + c$  of a linear Fredholm map  $l$  and a nonlinear compact operator  $c$ .

*Proof.* Restricted to a fiber, the monopole map is a sum of the linear Fredholm operator  $l$ , consisting of the elliptic operators  $D_A$  and  $d^* + d^+$ , complemented by projections to and inclusions of harmonic forms. The nonlinear part of  $\mu$  is built from the bilinear terms  $a\phi$  and  $\sigma(\phi)$ . Multiplication  $\mathcal{A}_k \times \mathcal{A}_k \rightarrow \mathcal{C}_k$  is continuous for  $k > 2$ . Combined with the compact restriction map  $\mathcal{C}_k \rightarrow \mathcal{C}_{k-1}$  (Rellich lemma, see [Per18, Lecture 19, p. 2]) we gain the claimed compactness for  $c$ : Images of bounded sets are contained in compact sets.  $\square$

Now let us show the following very useful property of compact perturbations of Fredholm operators.

**Claim 3.5.** The restriction of a compact perturbation  $l + c: \mathcal{U}' \rightarrow \mathcal{U}$  of a linear Fredholm map  $l$  between Hilbert spaces to any bounded, closed subset is proper.

*Proof.* Let  $p$  denote a projection to the kernel of  $l$ . Let  $A$  be a bounded closed subset of  $\mathcal{U}'$ . It's easy to see that we have the following commutative diagram

$$\begin{array}{ccc}
 A & \xrightarrow{(\ell, c, p)} & \mathcal{U} \times \overline{c(A)} \times \overline{p(A)} \\
 & \searrow \ell + c & \downarrow \cong + \\
 & & \mathcal{U} \times \overline{c(A)} \times \overline{p(A)} \\
 & & \downarrow \pi \\
 & & \mathcal{U}
 \end{array}$$

We observe that the map  $h: A \rightarrow \mathcal{U} \times \overline{c(A)} \times \overline{p(A)}$  given by  $a \mapsto (\ell(a), c(a), p(a))$  is injective and closed. Injectivity is clear since we are projecting on the kernel.

Closedness is a little bit more involved: let  $\{(\ell_n, c_n, p_n)\}_n \subset \text{Im}(h)$  converge to  $(\ell_\infty, c_\infty, p_\infty)$ . In particular there is a sequence  $\{a_n\}_n \subset A$  such that  $(\ell_n, c_n, p_n) = (\ell(a_n), c(a_n), p(a_n))$ . We want to prove that  $(\ell_\infty, c_\infty, p_\infty) \in h(A)$ . Since  $\ell$  is Fredholm we have the following property: *every bounded sequence  $\{x_i\}_i$  in the domain whose image is convergent admits a convergent subsequence  $\{x_{i_j}\}_j$* . Since  $A$  is closed and bounded (and any other closed subset of it would be bounded as well hence we can directly work with  $A$ ),  $\{a_n\}_n$  is bounded. Since  $\ell$  is Fredholm we can extract a convergent subsequence  $\{a'_n\}_n$  converging to  $a \in A$  (since  $A$  is closed). By the uniqueness of the limit, it's easy to prove

$$(3.6) \quad (\ell_\infty, c_\infty, p_\infty) = (\ell(a), c(a), p(a))$$

which proves the closedness of  $h(A)$ . This implies that  $h$  is proper, since  $h$  is an homeomorphism onto its image.

The addition map  $+: (u, s, e) \mapsto (u + s, s, e)$  is an homeomorphism hence proper. The projection to  $\mathcal{U}$  is proper since the other two factors are compact.  $\square$

**3.2. A collection of results.** We will list here some results needed for the seminar.

Let  $U$  be an open subset of  $\mathbb{R}^n$ . We can consider the space  $C_c^\infty(U; \mathbb{R}^r)$  of compactly supported  $\mathbb{R}^r$ -valued functions. Fix a real number  $p > 1$  and an integer  $k \geq 0$ . The Sobolev  $L_k^p$  norm is defined by

$$(3.7) \quad \|f\|_{p,k} := \sum_{|\alpha| < k} \sup_U \|D^\alpha f\|_p.$$

The Sobolev space  $L_k^p(E)$  is defined to be the completion of  $\Gamma(E)$  in the  $L_k^p$  norm.

Here are the basic facts about Sobolev spaces.

**Sobolev inequality:** If  $k \leq \ell$  then there exists a constant  $C$  such that

$$(3.8) \quad \|\cdot\|_{p,k} \leq C \|\cdot\|_{p,\ell},$$

and hence we have a bounded inclusion of Sobolev spaces  $L_k^p(E) \hookrightarrow L_\ell^p(E)$ .

**Rellich lemma:** The inclusion  $L_{k+1}^p(E) \hookrightarrow L_k^p(E)$  is a compact operator.

**Morrey inequality:** Suppose  $\ell \geq 0$  is an integer such that  $\ell < k - n/p$ ; then there is a constant  $C$  such that

$$(3.9) \quad \|\cdot\|_{C^\ell} \leq C \|\cdot\|_{p,k},$$

i.e. there is a bounded inclusion

$$(3.10) \quad L_k^p(E) \hookrightarrow C^\ell(E).$$

**Smoothness:** One has

$$(3.11) \quad \bigcap_{k \geq k_0} L_k^p(E) = C^\infty(E).$$

**Lemma 3.12.** *Over a closed Riemannian 4-manifold, multiplication of smooth functions extends to a bounded map*

$$(3.13) \quad L_k^2(X) \otimes L_\ell^2(X) \rightarrow L_\ell^2(X)$$

*provided that  $k \geq 3$  and  $k \geq \ell$ . In particular,  $L_k^2(X)$  is an algebra for  $k \geq 3$ .*

There are also bounded multiplication maps for the lower regularity Sobolev spaces in 4 dimensions, but these bring in Sobolev spaces with  $p > 2$ .

Let now  $D: \Gamma(E) \rightarrow \Gamma(F)$  be a differential operator of order  $m$  over a closed, oriented, Riemannian manifold  $(M, g)$ . The basic point is that  $D$  extends to a bounded linear map between Hilbert spaces:

$$(3.14) \quad D: L_{k+m}^2(E) \rightarrow L_k^2(F).$$

**Theorem 3.15** (Elliptic estimate). *If  $D$  is elliptic of order  $m$ , one has estimates on the  $L_k^2$ -Sobolev norms for each  $k \geq 0$ :*

$$(3.16) \quad \|s\|_{2,k+m} \leq C_k (\|Ds\|_{2,k} + \|s\|_{2,k}).$$

Moreover,

$$(3.17) \quad \|s\|_{2,k+m} \leq C_k \|Ds\|_{2,k}$$

for  $s \in (\ker D)^\perp$  (here  $^\perp$  denotes the  $L^2$ -orthogonal complement).

There is an analogue for  $L^{p,k+m}$  bounds.

As a consequence of this important theorem we have the following:

**Corollary 3.18.** *An elliptic operator  $D$  of order  $m$  defines a Fredholm map  $L_{k+m}^2(E) \rightarrow L_k^2(F)$  for any  $k \geq 0$ . Its index is independent of  $k$ . Moreover, its index depends only on the symbol of  $D$ .*

Let  $(M, g)$  be an oriented Riemannian manifold. Let  $\nabla$  be an orthogonal covariant derivative in a real, Euclidean vector bundle  $E \rightarrow M$ . We know that  $\nabla$  has a formal adjoint  $\nabla^*$ .

**Proposition 3.19** (The Lichnérowicz formula). *One has*

$$(3.20) \quad D^2 = \tilde{\nabla}^* \tilde{\nabla} + \frac{1}{4} \text{scal}_g \cdot \text{id}_\mathbb{S} + \frac{1}{2} \rho(F^\circ).$$

**Lemma 3.21.**

$$(3.22) \quad \frac{1}{2}d^*d(|s|^2) = \langle \nabla^* \nabla s, s \rangle - |\nabla s|^2.$$

*Proof sketch.* See [Per18, Lecture 19, Lemma 1.1]. The idea is to study the integral

$$(3.23) \quad \int_M f \langle \nabla^* \nabla s, s \rangle \text{vol}$$

where  $f$  has compact support. □

It's important to remember that the one above is a pointwise equality. Working locally one has the following result.

**Lemma 3.24.** *For a smooth function  $f: M \rightarrow \mathbb{R}$  with compact support, if  $p$  is a local maximum, then  $(d^*df)(p) \geq 0$ .*

The following lemma is an easy calculation.

**Lemma 3.25.** *For  $\phi \in \Gamma(\mathbb{S}^+)$ , one has*

$$(3.26) \quad ((\phi\phi^*)_0\chi, \chi) = (\phi, \chi)^2 - \frac{1}{2}|\chi|^2|\phi|^2.$$

*In particular,*

$$(3.27) \quad ((\phi\phi^*)_0\phi, \phi) = \frac{1}{2}|\phi|^4.$$

*Proof.* We have

$$\begin{aligned} ((\phi\phi^*)_0\chi, \chi) &= ((\phi\phi^*)\chi, \chi) - \frac{1}{2}(|\phi|^2\chi, \chi) \\ &= ((\phi, \chi)\phi, \chi) - \frac{1}{2}|\phi|^2|\chi|^2 \\ &= (\phi, \chi)^2 - \frac{1}{2}|\phi|^2|\chi|^2. \end{aligned} \quad \square$$

**Lemma 3.28.** *For  $\eta \in \Omega_X^2$  and  $\phi \in \Gamma(\mathbb{S})$ , one has  $(\rho(\eta)\phi, \phi) \leq |\eta||\phi|^2$ .*

*Proof.* It suffices to take  $\eta = e \wedge f$  for orthogonal unit vectors  $e$  and  $f$ . One then has

$$(3.29) \quad (\rho(\eta)\phi, \phi) = (\rho(e \wedge f)\phi, \phi)$$

$$(3.30) \quad = \frac{1}{2}([\rho(e), \rho(f)]\phi, \phi)$$

$$(3.31) \quad = -\frac{1}{2}(\rho(f)\phi, \rho(e)\phi)$$

$$(3.32) \quad \leq |\rho(e)\phi| \cdot |\rho(f)\phi|,$$

where in (3.31) we used the fact that  $\rho$  has image in the anti-skew-Hermitian matrices. Now since  $|e| = 1$  then  $|\rho(e)| = 1$  (similarly for  $f$ ), and therefore we conclude. □

**Lemma 3.33.** *Let  $A$  be a Clifford connection for the spinor bundle of a  $\text{spin}^c$  structure of  $X$ . Let  $a \in \Omega_X^1(i\mathbb{R})$ ; then*

$$(3.34) \quad D_{A+a}\phi = D_A\phi + a \cdot \phi,$$

where the last term is the Clifford multiplication between  $a$  and  $\phi$ .

*Proof.* Let's work in local orthonormal coordinates of  $TX$  given by  $\{e_1, \dots, e_n\}$ . We have

$$\begin{aligned}
D_{A+a}\phi &= \sum_i e_i \cdot (A+a)_{e_i} \phi \\
&= \sum_i e_i \cdot A_{e_i} \phi + \sum_i e_i \cdot a(e_i) \phi \\
&= D_A \phi + \sum_i e_i \cdot a(e_i) \phi \\
&= D_A \phi + a \sum_i e_i \phi \\
&= D_A \phi + a \cdot \phi.
\end{aligned}$$

Notice that here we used that  $a \in \Omega_X^1(i\mathbb{R})$  hence all the coefficients  $a(e_i)$  are equal to each other, and without loss of generality we named then  $a$ .  $\square$

**3.3. Compactness of the moduli space.** If the bundles  $\mathcal{A}$  and  $\mathcal{C}$  were finite-dimensional, then the boundedness property would be equivalent to properness. In this infinite-dimensional setting, the argument above can be used the same way as Heine-Borel in the finite-dimensional case to show that the boundedness condition implies properness. It turns out that the ingredients of the compactness proof for the moduli space also prove the stronger boundedness property.

**Proposition 3.35.** *Preimages  $\mu^{-1}(B) \subset \mathcal{A}_k$  of bounded disk bundles  $B \subset \mathcal{C}_{k-1}$  are contained in bounded disk bundles.*

*Proof.* It is sufficient to prove this fiberwise for the Sobolev completions of the restriction of the monopole map to the space  $\{A\} \times (\Gamma(\mathbb{S}^+) \oplus \ker(d^*))$ , which maps to  $\{A\} \times (\Gamma(\mathbb{S}^-) \oplus \Omega_+^2(X) \oplus H^1(X; \mathbb{R}))$ . We start by defining the following scalar product: using the elliptic operator  $D = D_A + d^+$  and its adjoint, define the  $L_k^2$ -norm via the scalar product on the respective function spaces through

$$(3.36a) \quad (\cdot, \cdot)_i = (\cdot, \cdot)_0 + (D\cdot, D\cdot)_{i-1} \text{ for } 0 < i \leq k$$

$$(3.36b) \quad (\cdot, \cdot)_0 = \int_X \langle \cdot, \cdot \rangle.$$

Using the elliptic estimates and continuity (i.e. boundedness) of  $D$  it's easy to see that this norm is equivalent to the classic Sobolev one. A similar definition can be extended to norms for the  $L_k^p$ -spaces. Let us take  $\mu(A, \phi, a) = (A, \varphi, b, a_{\text{harm}}) \in \mathcal{C}_{k-1}$  with the norm of the latter bounded by some constant  $R$ . The Lichnerowicz formula (Proposition 3.19) for a connection  $A + a = A'$  reads

$$(3.37) \quad D_{A'}^* D_{A'} = A' \circ A' + \frac{1}{4} s \cdot \text{id}_{\mathbb{S}} + \frac{1}{2} \rho(F_{A'}^\circ)$$

with  $s$  denoting the scalar curvature of  $X$ . As a consequence we have a pointwise estimate: using Lemma 3.21,

$$(3.38) \quad d^* d|\phi|^2 = 2\langle \nabla_{A'}^* \nabla_{A'} \phi, \phi \rangle - 2\langle \nabla_{A'} \phi, \nabla_{A'} \phi \rangle.$$

Then, removing the negative quantity on the left to obtain an inequality,

$$(3.39) \quad \leq 2\langle \nabla_{A'}^* \nabla_{A'} \phi, \phi \rangle$$

$$(3.40) \quad \leq 2\langle D_{A'}^* D_{A'} \phi - \frac{s}{4} \phi - \frac{1}{2} \rho(F_{A'}^\circ) \phi, \phi \rangle.$$

Substituting in the second Seiberg-Witten equation,

$$(3.41) \quad \leq \langle 2D_{A'}^* \varphi - \frac{s}{2} \phi - (\sigma(\phi) + b)\phi, \phi \rangle$$

Now we move some terms to the left and use the equality  $D_{A+a} = D_A + a$  together with the fact that the Dirac operator is self-adjoint to get

$$(3.42) \quad d^* d|\phi|^2 + \frac{s}{2} |\phi|^2 + \langle \sigma(\phi), \phi \rangle \leq \langle 2D_{A'}^* \varphi, \phi \rangle - \langle b\phi, \phi \rangle.$$



Next, use Lemma 3.25 to bound  $\sigma(\phi)$  and obtain

$$(3.43) \quad d^*d|\phi|^2 + \frac{s}{2}|\phi|^2 + \frac{1}{2}|\phi|^4 \leq \langle 2D_A^*\varphi, \phi \rangle + 2\langle a \cdot \varphi, \phi \rangle - b|\phi|^2$$

$$(3.44) \quad \leq 2(\|D_A^*\varphi\|_\infty + \|a\|_\infty\|\varphi\|_\infty) \cdot |\phi| + \|b\|_\infty \cdot |\phi|^2.$$

$$(3.45) \quad \leq c_1 \left( (1 + \|a\|_\infty) \|\varphi\|_{L_{k-1}^2} \cdot |\phi| + \|b\|_{L_{k-1}^2} \cdot |\phi|^2 \right),$$

using the Sobolev embedding theorem (Morrey's inequality) to bound the  $L^\infty$ -norm with the Sobolev norm.

Now we need to estimate  $\|a\|_\infty$ . First thing, for  $p > 4$  we get a Sobolev estimate  $\|a\|_\infty \leq c_2\|a\|_{L_1^p}$  and then use the elliptic estimate:

$$(3.46) \quad \|a\|_{L_1^p} = \|a_{\text{harm}} + a'\|_{L_1^p} \leq \|a_{\text{harm}}\|_{L_1^p} + \|a'\|_{L_1^p}$$

$$(3.47) \quad \leq \|a_{\text{harm}}\|_{L_0^p} + \|d^+a\|_{L_0^p}$$

where in (3.46) we used the Hodge decomposition of  $a$  and in (3.47) we applied the elliptic estimate to both component. Recall that  $d^+(a_{\text{harm}}) = 0$  and  $d^+a = d^+a'$ .

Combining with the equality  $d^+a = b + \sigma(\phi)$  then leads to an estimate

$$(3.48) \quad \|a\|_\infty \leq c_4 \left( \|a_{\text{harm}}\|_{L_0^p} + \|b\|_{L_0^p} + \|\sigma(\phi)\|_{L_0^p} \right)$$

$$(3.49) \quad \leq c_5 \left( \|a_{\text{harm}}\|_{L_0^p} + \|b\|_{L_{k-1}^2} + \|\phi\|_\infty^2 \right)$$

In the last passage we control the  $L_0^p$ -norm with the  $L_{k-1}^2$ -one, since  $p > 4$ .

Putting these two estimates together, we get something of the form

$$(3.50)$$

$$(3.51) \quad \begin{aligned} d^*d|\phi|^2 + \frac{1}{2}\|s\|_\infty\|\phi\|_\infty^2 + \frac{1}{2}\|\phi\|_\infty^4 &\leq c \left( 1 + c_5 \left( \|a_{\text{harm}}\|_{L_0^p} + \|b\|_{L_{k-1}^2} + \|\phi\|_\infty^2 \right) \right) \|\varphi\|_{L_{k-1}^2} \cdot \|\phi\|_\infty + \|b\|_{L_{k-1}^2} \cdot \|\phi\|_\infty^2 \\ &\leq K\|\phi\|_\infty^3 + R\|\phi\|_\infty^2, \end{aligned}$$

where in (3.51) we applied the bounds we had by assumption on the elements in the image.

So our inequality is now:

$$(3.52) \quad d^*d|\phi|^2 + \frac{1}{2}\|\phi\|_\infty^4 \leq K\|\phi\|_\infty^3 + R\|\phi\|_\infty^2 - \frac{1}{2}\|s\|_\infty\|\phi\|_\infty^2$$

$$(3.53) \quad \leq K\|\phi\|_\infty^3 + R\|\phi\|_\infty^2.$$

Now this inequality must hold in particular when  $\phi$  achieves its maximum, and on that point the Laplacian is positive, hence we can forget about it and get

$$(3.54) \quad \frac{1}{2}\|\phi\|_\infty^4 \leq K\|\phi\|_\infty^3 + R\|\phi\|_\infty^2.$$

In particular we bound the 4<sup>th</sup> power of a quantity with a polynomial in that quantity of degree 3. This implies that  $\|\phi\|_\infty$  must be bounded. Therefore we can bound the  $L_0^p$ -norm of  $(\phi, a)$  for every  $p \geq 1$ .

Now comes bootstrapping: for  $i \leq k$ , assume inductively  $L_{i-1}^2$ -bounds on  $(\phi, a)$ . To obtain  $L_i^2$ -bounds, compute:

$$(3.55) \quad \|(\phi, a)\|_{L_i^2}^2 - \|(\phi, a)\|_{L_0^2}^2 = \|(D_A\phi, d^+a)\|_{L_{i-1}^2}^2$$

$$(3.56) \quad = \|(\phi + ia\phi, b - \sigma(\phi))\|_{L^2}^2$$

$$(3.57) \quad = \|(\phi, b)\|_{L_{i-1}^2}^2 + \|(ia\phi, \sigma(\phi))\|_{L_{i-1}^2}^2.$$

The first equality holds by our definition of the Sobolev norm. The last equality holds as  $D_{A'} = D_A + a$ . The summands in the last expression are bounded by the assumed  $L_{i-1}^2$ -bounds on  $(\phi, a)$  together with the Sobolev multiplication properties. Note that the steps for  $i = 2$  and  $3$  require special care (see [Per18, Lecture 21, p. 4]) or use Sobolev embedding together with the fact that we have control on the  $L^p$ -norms of  $(\phi, a)$  for every  $p$ , which gives us control on the respective Sobolev norms for  $p = 2$ .  $\square$

#### 4. THE $\text{Pin}_2^-$ -SYMMETRY: 2/11/19

These are Arun's prepared lecture notes on the group  $\text{Pin}_2^-$ , its representations, and the  $\text{Pin}_2^-$  symmetry in the Seiberg-Witten equations associated to a spin 4-manifold.

**4.1. Some avatars of  $\text{Pin}_2^-$ .** In the first part of the talk, I'll tell you some basic facts about  $\text{Pin}_2^-$ . In Seiberg-Witten theory, this group is often just called  $\text{Pin}(2)$ , but that could be confusing: there's also  $\text{Pin}_2^+$ , which is different.

**Definition 4.1.** Recall that given a vector space  $V$  (over  $\mathbb{R}$  or  $\mathbb{C}$ ) and a quadratic form  $Q$ , we can form the Clifford algebra  $Cl(V, Q) := TV/(v \otimes v - Q(v)1)$ . That is, we take the tensor algebra and introduce the relation  $v^2 = Q(v)$ . This is a  $\mathbb{Z}/2$ -graded algebra with the grading given by the length of a tensor mod 2; let  $\alpha$  denote the *grading operator*, which acts on the even subspace as 1 and on the odd subspace as  $-1$ . It is common to think of  $V$  as sitting inside of  $Cl(V, Q)$  as the length-1 tensors.

The *Clifford group*  $\Gamma(V, Q)$  is the group of  $x \in Cl(V, Q)^\times$  such that  $\alpha(x)yx^{-1} \in V \subset Cl(V, Q)$  for all  $y \in V$ .

Consider the involution  $\beta: Cl(V, Q) \rightarrow Cl(V, Q)$  sending  $v_1 \otimes \cdots \otimes v_n \mapsto v_n \otimes \cdots \otimes v_1$ . The *Clifford norm* is  $N(v) := \beta(v) \cdot v$ , which is a scalar on  $\Gamma(V, Q)$ .

The *pin group*  $\text{Pin}(V, Q)$  is the kernel of the Clifford norm inside  $\Gamma(V, Q)$ . The *spin group*  $\text{Spin}(V, Q)$  is the subgroup of even elements of  $\text{Pin}(V, Q)$ . The following shorthand is standard:

- If  $V = \mathbb{R}^n$  and  $Q(x) = \langle x, x \rangle$ ,  $Cl(V, Q)$  is denoted  $Cl_n$  and  $\text{Pin}(V, Q)$  is denoted  $\text{Pin}_n^+$ ; if  $Q(x) = -\langle x, x \rangle$ , they're denoted  $Cl_{-n}$  and  $\text{Pin}_n^-$ .
- The spin groups in these cases are canonically isomorphic, and denoted  $\text{Spin}_n$ .
- If  $V = \mathbb{C}^n$  and  $Q(x) = \langle x, x \rangle$ ,  $\text{Pin}(V, Q)$  is denoted  $\text{Pin}_n^c$ , and  $\text{Spin}(V, Q)$  is denoted  $\text{Spin}_n^c$ .

These are all compact, real Lie groups; there's a map  $\text{Spin}_n \rightarrow \text{SO}_n$  which is a double cover, connected if  $n \geq 2$  and universal if  $n \geq 3$ . Correspondingly there's a double cover  $\text{Pin}_n^\pm \rightarrow \text{O}_n$ .  $\text{Pin}_n^\pm$  has two components if  $n > 1$ ;  $\text{Pin}_1^+ \cong \mathbb{Z}/2 \times \mathbb{Z}/2$  and  $\text{Pin}_1^- \cong \mathbb{Z}/4$ .

*Remark 4.2.* Why would you want pin groups anyways? *A posteriori*, of course, we're going to find a  $\text{Pin}_2^-$  symmetry in the Seiberg-Witten equations of a spin 4-manifold, but there are other reasons to care. One rough answer is that there are many places in geometry and physics (index theory, fermionic QFT, ...) where one wants spin or  $\text{spin}^c$  structures, but if you want to try to study the same story on unoriented manifolds, the analogues are pin and  $\text{pin}^c$  structures.  $\blacktriangleleft$

Now we focus specifically on  $\text{Pin}_2^-$ , with the hope of getting some intuition for what it is. We know it contains  $\text{Spin}_2$  as an index-2 subgroup, and topologically is two circles.

We can get our hands on it by embedding it in  $\text{Spin}_3$ , which we do understand. Consider the map  $Cl_{-2} \hookrightarrow Cl_{-3}^0$  (i.e. into the even part of  $Cl_{-3}$ ) sending  $e_1 \mapsto e_1 e_3$  and  $e_2 \mapsto e_2 e_3$ . This also sends  $1 \mapsto 1$  and  $e_1 e_2 \mapsto e_1 e_2$ .

There's an identification  $Cl_{-3}^0 \cong \mathbb{H}$  via  $e_1 e_3 \mapsto i$ ,  $e_2 e_3 \mapsto j$ , and  $e_1 e_2 \mapsto k$ , which restricts to the (possibly familiar) isomorphism  $\text{Spin}_3 \cong \text{Sp}_1$  (which is also  $\text{SU}(2)$ ). This then restricts to an identification

$$(4.3) \quad \text{Pin}_2^- \cong \{e^{i\theta}\} \cup \{je^{i\theta}\} \subset \text{Sp}_1,$$

which is sometimes taken as a definition in this area, and which we will use heavily. The first thing it gives us is a representation of  $\text{Pin}_2^-$  on  $\mathbb{H}$ . We will also let  $\mathbb{R}$  denote the real representation of  $\text{Pin}_2^-$  which is trivial on  $\text{Spin}_2$ , and such that  $j$  acts by  $-1$ .

**4.2. Appearance in the Seiberg-Witten equations.** Furuta produces the  $\text{Pin}_2^-$  symmetry in the Seiberg-Witten equations in a very elegant way, doing everything over a point, where it's close to obvious, and using the associated bundle construction to move to the tangent and spinor bundles.

**Definition 4.4.** Here's some notation for some representations of  $\text{Spin}_4 \cong \text{Sp}_1 \times \text{Sp}_1$ .

- Let  ${}_+\mathbb{H}$  denote the left action of  $\text{Sp}_1 \times \text{Sp}_1$  on the quaternions  $\mathbb{H}$  by the first factor ( ${}_-\mathbb{H}$ ) or the second factor ( ${}_+\mathbb{H}$ ). These are the spinor representations.
- Let  ${}_-\mathbb{H}_+$  denote the action of  $\text{Sp}_1 \times \text{Sp}_1$  on  $\mathbb{H}$  by  $(p, q) \cdot v = pvq^{-1}$ . For  $\text{Spin}_4$ , this is the representation  $\text{Spin}_4 \twoheadrightarrow \text{SO}_4 \hookrightarrow \text{GL}_4(\mathbb{R})$ .
- Let  ${}_+\mathbb{H}_+$  denote the action of  $\text{Sp}_1 \times \text{Sp}_1$  by  $(p, q) \cdot v = qvq^{-1}$ .

Given any representation or equivariant vector bundle  $V$ , we'll let  $\tilde{V} := V \otimes \tilde{\mathbb{R}}$ .

If  $(X, \mathfrak{s})$  is a 4-manifold with associated principal  $\text{Spin}_4$ -bundle  $P_{\mathfrak{s}} \rightarrow X$ , then we have the associated bundles

$$(4.5a) \quad \mathbb{S}^{\pm} \cong P_{\mathfrak{s}} \times_{\text{Spin}_4} \pm \mathbb{H} \rightarrow X$$

$$(4.5b) \quad TX \cong P_{\mathfrak{s}} \times_{\text{Spin}_4} -\mathbb{H}_+ \rightarrow X$$

$$(4.5c) \quad \Lambda := \mathbb{R} \oplus \Lambda_+^2 T^*X \cong P_{\mathfrak{s}} \times_{\text{Spin}_4} +\mathbb{H}_+ \rightarrow X.$$

Now we throw in a  $\text{Pin}_2^-$ -action and extend  $\pm \mathbb{H}$  and  $+\mathbb{H}_{\pm}$  to  $\text{Spin}_4 \times \text{Pin}_2^-$ -representations:

- Using the inclusion  $\text{Pin}_2^- \hookrightarrow \text{Sp}_1$ , we define the action of  $g \in \text{Pin}_2^-$  on  $\pm \mathbb{H}$  to be right multiplication by  $g^{-1}$ .
- Let  $\text{Pin}_2^-$  act trivially on  $\pm \mathbb{H}_+$ .

We need these to commute with the  $\text{Spin}_4$ -actions but that's easy, and therefore using (4.5), we have actions of  $\text{Pin}_2^-$  on the fibers of  $TX$ ,  $\mathbb{S}^{\pm}$ , and  $\Lambda$ .

**Proposition 4.6.** *The monopole map is equivariant with respect to these  $\text{Pin}_2^-$ -actions.*

*Proof.* (1) You can check in one line that the multiplication map  $-\mathbb{H}_+ \times +\mathbb{H} \rightarrow -\mathbb{H}$  is  $\text{Spin}_4 \times \text{Pin}_2^-$ -equivariant. Passing to associated bundles, this says Clifford multiplication  $C: \mathbb{S}^+ \rightarrow \mathbb{S}^-$  is  $\text{Pin}_2^-$ -equivariant.

- (2) It's just as easy to check that the map  $-\mathbb{H}_+ \times -\tilde{\mathbb{H}}_+ \rightarrow -\tilde{\mathbb{H}}_+$  sending  $a, b \mapsto \bar{a}b$  is  $\text{Spin}_4 \times \text{Pin}_2^-$ -equivariant, so the map

$$(4.7) \quad \begin{aligned} \tilde{C}: T^*X \times \tilde{T}^*X &\longrightarrow \tilde{\Lambda} \\ a, b &\longmapsto (\langle a, b \rangle, (a \wedge b)_+), \end{aligned}$$

which Furuta calls “twisted Clifford multiplication,” is  $\text{Pin}_2^-$ -equivariant. (Here we passed from  $TX$  to  $T^*X$ , of course using the metric to do so.)

- (3) All named  $\text{Pin}_2^-$ -representations have been unitary (orthogonal for  $\tilde{\mathbb{R}}$ ), so the actions of  $\text{Pin}_2^-$  on  $\mathbb{S}^{\pm}$  are unitary (with respect to the Hermitian metric induced from the Riemannian metric on  $X$ ), and on  $T^*X$ ,  $\tilde{T}^*X$ ,  $\Lambda$ , and  $\tilde{\Lambda}$  are orthogonal. Therefore the covariant derivatives associated to these bundles are also  $\text{Pin}_2^-$ -equivariant, hence so are the Dirac operators

$$(4.8a) \quad D_1 := C \circ \nabla: \Gamma(\mathbb{S}^+) \longrightarrow \Gamma(\mathbb{S}^-)$$

$$(4.8b) \quad D_2 := \tilde{C} \circ \nabla: \Gamma(\tilde{T}^*X) \longrightarrow \Gamma(\tilde{\Lambda}).$$

(Here  $D_2$  can be identified with  $d^* + d^+$ .) Therefore  $D := D_1 \oplus D_2$  is also  $\text{Pin}_2^-$ -equivariant.

- (4) Now consider the map

$$(4.9) \quad \begin{aligned} +\mathbb{H} \times -\tilde{\mathbb{H}}_+ &\longrightarrow -\mathbb{H} \times +\tilde{\mathbb{H}}_+ \\ \phi, a &\longmapsto (a\phi i, \phi i \bar{\phi}). \end{aligned}$$

In a similar way, one can check this is a (nonlinear)  $\text{Spin}_4 \times \text{Pin}_2^-$ -equivariant map. It passes to a map of associated bundles  $Q: \Gamma(\mathbb{S}^+ \oplus \tilde{T}^*M) \rightarrow \Gamma(\mathbb{S}^- \oplus \tilde{\Lambda})$ , which is  $\text{Pin}_2^-$ -equivariant.<sup>1</sup>

Therefore the monopole map  $SW = D + Q$  is  $\text{Pin}_2^-$ -equivariant. Because the  $\text{Pin}_2^-$ -action is continuous, it doesn't matter what regularity we impose on sections: this fact is true both for smooth sections and their Sobolev completions.  $\square$

**4.3. Some computations with the representation ring.** The proof of the  $10/8^{\text{th}}$  theorem requires a few more pure representation-theoretic results, and since we have time, I'll go over them now. Let's start by listing some representations of  $\text{Pin}_2^-$ .

**Example 4.10.** The first representations you'd write down are the trivial representation 1 and the *sign representation*  $\sigma := \mathbb{C}$ .

We can next define some irreducible two-dimensional representations  $h_d$ , indexed by  $d \in \mathbb{Z}$ , as follows:  $\text{Pin}_2^- = \{e^{i\theta}\} \cup \{je^{i\theta}\}$ , so let the underlying complex vector space of  $h_d$  be  $\mathbb{H} = \mathbb{C}^2$ , with  $j$  acting in the usual

<sup>1</sup>In fact, since the second factor is purely imaginary, we know the image isn't just in  $\mathbb{S}^- \oplus \tilde{\Lambda}$ , but in  $\mathbb{S}^- \oplus \Lambda_+^2 T^*X$ .

way and  $e^{i\theta}$  acting by  $(e^{id\theta}, e^{-id\theta})$ . You can prove these are irreducible by just choosing a nonzero quaternion and pushing it around with elements of  $\text{Pin}_2^-$  until you get a basis, and this isn't hard.

As a particular example,  $h_1$  is  $\mathbb{H}$  with the  $\text{Pin}_2^-$ -action restricted from the usual  $\text{Spin}_3 = \text{Sp}_1$ -action.  $\blacktriangleleft$

**Theorem 4.11.** *The above is a complete list of isomorphism classes of irreducible representations of  $\text{Pin}_2^-$ .*

I don't know how one proves this: it's asserted by both Furuta and Bryan without proof.

**Definition 4.12.** The *representation ring* of a group  $G$ , denoted  $RU(G)$ , is the Grothendieck ring of the category of complex representations of  $G$ . That is, it is the abelian group freely generated by isomorphism classes of finite-dimensional complex representations of  $G$  modulo the relations  $[V] = [V'] + [V'']$  whenever there is a short exact sequence  $0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0$ . The ring structure is defined by  $[V] \cdot [W] := [V \otimes W]$ .

Let's begin with a simple example.

**Proposition 4.13.** *The representation ring of  $\text{Spin}_2 = U_1$  is  $\mathbb{Z}[t, t^{-1}]$ , where  $t: U_1 \rightarrow U_1$  is the identity map.*

*Proof.* We can compute by taking the irreducible representations as generators and computing their relations. The irreducible representations of  $U_1$  are indexed by  $\mathbb{Z}$ , with the  $d^{\text{th}}$  one  $\chi_d$  sending  $z \mapsto z^d$ . The tensor product of one-dimensional matrices is the ordinary product in  $\mathbb{C}$ , so  $\chi_d \otimes \chi_{d'} = \chi_{d+d'}$ . Therefore  $\chi_1 \mapsto t$  gives us  $\mathbb{Z}[t, t^{-1}]$ .  $\square$

**Lemma 4.14.** *There's an isomorphism  $h_{d_1} \otimes h_{d_2} \cong h_{d_1+d_2} \oplus h_{d_1-d_2}$ .*

*Proof.* Inside  $h_{d_1} \otimes h_{d_2} \cong \mathbb{H} \otimes_{\mathbb{C}} \mathbb{H}$ , the subspace  $V := \text{span}_{\mathbb{C}}\{1 \otimes 1, j \otimes j\}$  is preserved by  $j$  and  $e^{i\theta}$ , hence is a subrepresentation. The same applies to  $W := \text{span}_{\mathbb{C}}\{1 \otimes j, j \otimes 1\}$ . The vector space isomorphism  $V \xrightarrow{\cong} h_{d_1+d_2}$  sending  $1 \otimes 1 \mapsto e_1$  and  $j \otimes j \mapsto e_2$  is  $\text{Pin}_2^-$ -equivariant, which you can quickly check by hand; the same idea applies to  $W \cong h_{d_1-d_2}$ .  $\square$

**Corollary 4.15.**

$$RU(\text{Pin}_2^-) \cong \mathbb{Z}[\sigma, h_d \mid d \in \mathbb{Z}] / (\sigma^2, \sigma h_d = h_{-d}, h_{d_1} h_{d_2} = h_{d_1+d_2} + h_{d_1-d_2}).$$

The last thing we need to do is compute the image of the restriction map  $RU(\text{Pin}_2^-) \rightarrow RU(\text{Spin}_2)$ .

**Corollary 4.16.** *Under the above identifications, the map  $RU(\text{Pin}_2^-) \rightarrow RU(\text{Spin}_2)$  sends  $\sigma \mapsto 1$  and  $h_d \mapsto t^d + t^{-d}$ .*

## 5. FINITE-DIMENSIONAL APPROXIMATIONS, I: 2/18/19

*"These vector spaces are indexed by your favorite barnyard animals."*

Today, Cameron spoke about finite-dimensional approximations to the monopole map. The idea is that taking the one-point compactification of its domain and codomain defines a  $\text{Pin}_2^-$ -equivariant map between infinite-dimensional spheres. This is pretty cool, except that infinite-dimensional spheres are contractible, even equivariantly, so we need to take some finite-dimensional approximation in order to obtain homotopically interesting information.

The following theorem is the goal of the next two lectures.

**Theorem 5.1.** *Let  $M$  be a closed, spin 4-manifold such that  $b_1(M) = 0$ ,  $b_2^+(M) > 0$ , and  $\sigma(M) < 0$ . Then there are finite-dimensional  $\text{Pin}_2^-$ -representations  $V_\lambda$  and  $\overline{W}_\lambda$  and  $\text{Pin}_2^-$ -equivariant maps  $D_\lambda: V_\lambda \rightarrow \overline{W}_\lambda$  (linear) and  $Q_\lambda: V_\lambda \rightarrow \overline{W}_\lambda$  (quadratic) such that*

- (1) *as  $\text{Pin}_2^-$ -representations,  $V_\lambda \cong \mathbb{H}^{k+m} \oplus \widetilde{\mathbb{R}}^n$  for some  $k, m$ , and  $n$ ; and*
- (2) *there are  $\text{Pin}_2^-$ -equivariant metrics on  $V_\lambda$  and  $\overline{W}_\lambda$  and an  $R > 0$  such that  $(D_\lambda + Q_\lambda)(v) \neq 0$  for all  $v \in S_R(0)$ .*

Recall that  $\text{Pin}_2^-$  acts on  $\mathbb{H}$  through the inclusion  $\text{Pin}_2^- \hookrightarrow \text{Spin}_3 = \text{Sp}_1$  that we discussed last time, and on  $\widetilde{\mathbb{R}}$  as the sign representation on  $\mathbb{R}$ , which is zero on  $\text{Spin}_2 \subset \text{Pin}_2^-$ , but such that the element we called  $j$  acts by  $-1$ .

Today we will prove (2), leaving the determination of the representations for next week.

There's still plenty to say about the statement of Theorem 5.1 – what's  $\lambda$ ? How do we determine  $V_\lambda$ ,  $\overline{W}_\lambda$ ,  $D_\lambda$ , and  $Q_\lambda$ ? What are  $k$ ,  $m$ , and  $n$ ? These will all be answered during the proof.

Let  $S^\pm \rightarrow M$  be the spinor bundles, so we have a Clifford multiplication map  $C: T^*M \otimes S^+ \rightarrow S^-$  and a Dirac operator  $D_1: \Gamma(S^+) \rightarrow \Gamma(S^-)$ , and a twisted Clifford multiplication map  $\tilde{C}: T^*M \otimes \tilde{T}^*M \rightarrow \tilde{\Lambda}$  which defines another Dirac operator  $D_2: \Gamma(\tilde{T}^*M) \rightarrow \Gamma(\tilde{\Lambda})$ . Then  $D = D_1 + D_2$ , as we discussed last time.

**Theorem 5.2** (Weitzenböck).  *$D^*D$  is equal to  $\nabla^*\nabla$  up to a zeroth-order term.*

**Corollary 5.3** (Gårding's inequality). *There is some  $k \geq 0$  such that*

$$\langle D^*D\psi, \psi \rangle_{L^2} + k\langle \psi, \psi \rangle_{L^2} \geq \|\psi\|_{L_1^2}^2,$$

where  $L_1^2$  denote the Sobolev norm.

Along the way we'll need another Sobolev space.

**Definition 5.4.** The  $L_{-1}^2$  norm of an  $f \in C^\infty(E)$  is the smallest  $C \in \mathbb{R}$  such that  $\langle f, \psi \rangle_{L^2} \leq C\|\psi\|_{L_1^2}$  for  $\psi \in L_1^2(E)$ , if it exists. The completion of  $C^\infty(E)$  under this norm is denoted  $L_{-1}^2(E)$  (or just  $L_{-1}^2$  if  $E$  is clear from context).

Hence there is an embedding  $L^2 \hookrightarrow L_{-1}^2$ .

*Fact.* The  $L^2$  inner product defines a continuous nondegenerate pairing  $L_{-1}^2 \otimes \overline{L_1^2} \rightarrow \mathbb{C}$ , hence identifies  $L_{-1}^2$  as the continuous dual of  $L_1^2$ , i.e. the space of continuous linear functionals on  $L_1^2$ .  $\blacktriangleleft$

Therefore we can restate Gårding's inequality as

$$(5.5) \quad \|(D^*D + k)\psi\|_{L_{-1}^2} \geq \|\psi\|_{L_1^2}.$$

Now,  $D^*D$  is a second-order differential operator, hence is a map  $L_1^2 \rightarrow L_{-1}^2$ . Thus (5.5) implies  $D^*D + k$  is a continuous injection  $L_1^2 \hookrightarrow L_{-1}^2$ .

**Lemma 5.6.** *In fact,  $D^*D + k$  is onto.*

*Proof.* The map  $\langle \langle \cdot, \cdot \rangle \rangle: L^2 \times L^2 \rightarrow \mathbb{C}$  defined by

$$(5.7) \quad \langle \langle \varphi, \psi \rangle \rangle := \langle (D^*D + k)\varphi, \psi \rangle_{L^2}$$

defines an inner product on  $L^2$ , and  $L^2$  is complete with respect to  $\langle \langle \cdot, \cdot \rangle \rangle$ . Therefore, by the Riesz representation theorem, if  $f \in L_{-1}^2$ , then on  $L_1^2$ ,  $\langle f, \cdot \rangle_{L^2} = \langle \langle \varphi, \cdot \rangle \rangle$  for some  $\varphi \in L_1^2$ . In particular,

$$(5.8) \quad \langle (D^*D + k)\varphi, \psi \rangle = \langle f, \psi \rangle$$

for all  $\psi \in L_1^2$ , so  $(D^*D + k)\varphi = f$ .  $\square$

Therefore we can invert  $D^*D + k$ . Consider the composition

$$(5.9) \quad T: L^2 \hookrightarrow L_{-1}^2 \xrightarrow{(D^*D+k)^{-1}} L_1^2 \xrightarrow{\cong} L^2.$$

All three maps are continuous linear maps, and the third is compact, by the Kondrachov theorem. Therefore  $T$  is compact, and since

$$(5.10) \quad \langle (D^*D + k)\varphi, \psi \rangle = \langle \varphi, (D^*D + k)\psi \rangle,$$

then  $T$  is self-adjoint. Therefore we can throw the nicest spectral theorem at  $T$ .

**Theorem 5.11** (Spectral theorem for compact, self-adjoint operators). *Let  $\mathcal{H}$  be a separable Hilbert space and  $T: \mathcal{H} \rightarrow \mathcal{H}$  be a compact self-adjoint operator. Then there is an orthonormal basis<sup>2</sup>  $\{e_n\}$  for  $\mathcal{H}$  such that  $Te_n = \mu e_n$  for a  $\mu \in \mathbb{R}_{\geq 0}$ , and  $\mu_1 \geq \mu_2 \geq \dots$  with  $\mu_n \geq 0$  as  $n \rightarrow \infty$ .*

<sup>2</sup>This is a Hilbert basis, not an algebraic basis, in that elements of  $\mathcal{H}$  can be infinite sums of basis elements.

In our case, we conclude that (TODO: I missed this calculation). Therefore if you fix a  $\lambda > 0$ , then there are finitely many eigenvalues  $\lambda_n \leq \lambda$ , meaning that if  $V_\lambda$  denotes the span of the eigenvectors of  $\lambda_n$ , then  $V_\lambda$  is finite-dimensional.

The operator  $DD^*$  has the same spectrum as  $D^*D$ , and the eigenspace for a given eigenvalue is the same. So we can do exactly the same thing to define a finite-dimensional subspace  $W_\lambda \subset W$ , and  $D(V_\lambda) \subset W_\lambda$ ; let  $D_\lambda := D|_{V_\lambda}$ .

TODO: what happened next (studying  $Q$ ) was erased before I could write it down. Sorry about that.

Anyways, we have our operator  $Q_\lambda: V_\lambda \rightarrow W \xrightarrow{p_\lambda} W_\lambda$ . Let  $p^\lambda := 1 - p_\lambda$ . Then,

- (1) as  $\lambda \rightarrow \infty$ ,  $\lim_{\lambda \rightarrow \infty} \|p^\lambda w\|_W \rightarrow 0$ , and
- (2) because every bounded sequence in a Hilbert space has a weakly convergent subsequence, then for any bounded sequence  $\{v_n\} \in V$ , there's a subsequence  $\{v_{n_i}\}$  weakly converging to some  $v_\infty$ . Because  $Q$  is compact,  $\{Qv_{n_i}\}$  converges strongly to some  $w_\infty$ , and  $Qv_\infty = w_\infty$ .

Now, since the zero set of the monopole map  $D + Q$  is compact, there is some  $R$ , as promised, such that  $D + Q$  does not vnaish on the sphere of radius  $R$ . To prove part (2) of Theorem 5.1, we need to bring this down to  $D_\lambda + Q_\lambda$ .

**Lemma 5.12.** *There is an  $\varepsilon > 0$  such that if  $\|u\|_V = R$ , then  $\|(D + Q)v\|_W \geq \varepsilon$ .*

*Proof.* Assume otherwise; then there's a sequence  $\{v_n\} \in S_R(0)$  such that  $(D + Q)v_i \rightarrow 0$ . With  $\{v_{n_i}\}$ ,  $v_\infty$ , and  $w_\infty$  as above, we know  $Qv_{n_i}$  converges strongly to  $Qv_\infty$ , so  $Dv_{n_i} \rightarrow -Qv_\infty$ , and therefore  $v_{n_i} \rightarrow v_\infty$ . In particular  $v_\infty \in S_R(0)$  but  $(D + Q)(v_\infty) = 0$ , which is a contradiction.  $\square$

**Lemma 5.13.** *For  $\lambda$  sufficiently large, if  $\|v\|_V = R$ , then  $\|p^\lambda Qv\|_W < \varepsilon$ .*

*Proof.* Assume not; then there are sequences  $\{\lambda_n\}$  and  $\{v_n\}$  such that  $\lambda_n \rightarrow \infty$ ,  $v_n \in S_R(0)$ , and  $v_i \rightarrow v_\infty$ , such that  $\|p^{\lambda_n} Qv_n\| \geq \varepsilon$ . Let  $w_\infty := Qv_\infty$  as before.

Restricting to a subsequence  $\{v_{n_i}\}$  as before, again  $Qv_{n_i} \rightarrow w_\infty$ , so  $\lim_{n \rightarrow \infty} \|p^\lambda w_\infty\| = 0$ . Therefore there's some  $N$  such that

$$(5.14) \quad \|p^{\lambda_{n_i}} Qv_\infty\| = \|p^{\lambda_{n_i}} w_\infty\| \leq \frac{\varepsilon}{2}$$

whenever  $i \geq N$ , and therefore for  $i \gg 0$ ,

$$(5.15) \quad \|p^{\lambda_{n_i}} (Qv_{n_i} - Qv_\infty)\| \leq \frac{\varepsilon}{2}.$$

$\square$

Therefore, since  $Q_\lambda = p^\lambda \circ Q$ , if  $\|v\| = R$ , TODO missed the last part.

## 6. FINITE-DIMENSIONAL APPROXIMATIONS, II: 2/25/19

*“Remember, these are not just vector spaces but  $\text{Pin}_2^-$ -representations, and we can pin down what they are... oh no.”*

Today Leon spoke, continuing the description of the finite-dimensional approximations of the monopole map, and with the specific goal of proving part (1) of Theorem 5.1.

Today we will heavily use the Dirac operators: Furuta uses  $D^1: \Gamma(S^+) \rightarrow \Gamma(S^-)$  to denote the usual Dirac operator, and uses  $D^2: \Gamma(\tilde{T}^*M) \rightarrow \Gamma(\tilde{\Lambda})$  to denote the operator  $d^* \oplus d^+: \Omega^i(M) \rightarrow \Omega^0(M) \oplus \Omega_+^2(M)$ . Then  $D := D^1 \oplus D^2$ ; we'll discuss the quadratic part later.

Last time, we defined the finite-dimensional approximations  $V_\lambda^i$  and  $W_\lambda^i$ : given  $\lambda \in \mathbb{R}$ ,  $V_\lambda^i$  is the space spanned by the eigenspaces of  $(D^i)^*D^i$  corresponding to eigenvalues less than  $\lambda$ , and  $W_\lambda^i$  is defined similarly. Because  $D$  and  $Q$  are  $\text{Pin}_2^-$ -equivariant, these are in fact  $\text{Pin}_2^-$ -representations. We'd like to know what representations they are, and as a starting point, we'd like to know their dimensions. This is hard, but we will be able to compute the relative dimension.

**Lemma 6.1.** *For  $i = 1, 2$ ,  $\dim_{\mathbb{C}} V_\lambda^i - \dim_{\mathbb{C}} W_\lambda^i = \text{ind}_{\mathbb{C}} D^i$ .*

*Proof.* For  $\lambda' < \lambda$ , let  $V_{=\lambda'}^i$  denote the  $\lambda'$ -eigenspace of  $D^i$ , so that  $V_{=\lambda}^i$  is a direct sum over all such  $V_{=\lambda'}^i$ ; then define  $W_{=\lambda'}^i$  similarly. Then

$$(6.2) \quad \begin{aligned} \operatorname{ind}_{\mathbb{C}} D^i &= \sum_{\lambda' < \lambda} \operatorname{ind}_{\mathbb{C}}(D^i|_{V_{=\lambda'}^i} : V_{=\lambda'}^i \rightarrow W_{=\lambda'}^i) \\ &= \sum_{\lambda' < \lambda} \dim_{\mathbb{C}} V_{=\lambda'}^i - \dim_{\mathbb{C}} W_{=\lambda'}^i = \dim_{\mathbb{C}} V_{=\lambda}^i - \dim_{\mathbb{C}} W_{=\lambda}^i. \end{aligned} \quad \square$$

Now let's recall some of the  $\operatorname{Pin}_2^-$ -representations we discussed in earlier lectures: the action on  $\mathbb{H}$  via the inclusion  $\operatorname{Pin}_2^- \hookrightarrow \operatorname{Spin}_3 = \operatorname{Sp}_1$ , and the sign action on  $\mathbb{R}$ , which is denoted  $\widetilde{\mathbb{R}}$ ; this is trivial on  $\operatorname{Spin}_2 \subset \operatorname{Pin}_2^-$  but  $j \in \operatorname{Pin}_2^-$  acts by  $-1$ .

**Lemma 6.3.** *There exist  $m, m', n, n' \in \mathbb{N}$  such that  $V_{=\lambda}^1 \cong \mathbb{H}^m$ ,  $W_{=\lambda}^1 \cong \mathbb{H}^{m'}$ ,  $V_{=\lambda}^2 \cong \widetilde{\mathbb{R}}^n$ , and  $W_{=\lambda}^2 \cong \widetilde{\mathbb{R}}^{n'}$ .*

*Proof.* Because  $V_{=\lambda}^1$  is a finite-dimensional subspace of  $\Gamma(\mathbb{S}^+)$ , there exist  $p_1, \dots, p_\ell \in M$  such that the evaluation map

$$(6.4) \quad \operatorname{ev} : \Gamma(\mathbb{S}^+) \longrightarrow \bigoplus_{i=1}^{\ell} (\mathbb{S}^+)_{p_i}.$$

is injective when restricted to  $V_{=\lambda}^1$ . Since  $\operatorname{ev}$  is  $\operatorname{Pin}_2^-$ -equivariant and  $(\mathbb{S}_p)^+ \cong \mathbb{H}$ , then we have a  $\operatorname{Pin}_2^-$ -equivariant inclusion  $V_{=\lambda}^1 \hookrightarrow \mathbb{H}^M$  for some  $M$ , and since  $\mathbb{H}$  is an irreducible  $\operatorname{Pin}_2^-$ -representation, this forces  $V_{=\lambda}^1 \cong \mathbb{H}^m$  for some  $m$ .

The other three identities proceed in the same way.  $\square$

Now we'll throw the Atiyah-Singer index theorem at  $D_1$ . This tells us that

$$(6.5) \quad \operatorname{ind}_{\mathbb{R}}(D^1) = 2\langle \widehat{A}(M), [M] \rangle,$$

where  $\widehat{A}(M)$  is the  $\widehat{A}$ -genus (pronounced “A-hat”)  $\widehat{A}(M) \in H^*(M)$ , a characteristic class.<sup>3</sup> Bordism arguments tell us that the  $\widehat{A}$ -genus is a linear combination of Pontrjagin classes, and in dimension 4 we get that

$$(6.6) \quad = -\left\langle \frac{p_1(M)}{12}, [M] \right\rangle,$$

which by the Hirzebruch signature theorem is

$$(6.7) \quad = -\frac{\sigma(M)}{4}.$$

Because this is an index, it's an integer, and since  $D^1$  is quaternionic, it's divisible by 4.

**Corollary 6.8** (Rokhlin's theorem). *If  $M$  is a closed spin 4-manifold,  $\sigma(M)$  is divisible by 16.*

Let  $k := -\sigma(M)/16$ . Because this is the index of  $D^1$  over  $\mathbb{H}$ , we know there's an  $m \in \mathbb{N}$  such that  $V_{=\lambda}^1 \cong \mathbb{H}^{m+k}$  and  $W_{=\lambda}^1 \cong \mathbb{H}^m$ .

Now let's compute the index of  $D^2$ . This is

$$(6.9) \quad -\operatorname{ind} D^2 = \dim \operatorname{coker}(d^* : \Omega^1 \rightarrow \Omega^0) + \dim \ker(d^+ : \Omega^1 \rightarrow \Omega^+).$$

For the first factor,

$$(6.10) \quad \dim \operatorname{coker} d^* = \dim \ker(d : \Omega^0 \rightarrow \Omega^1) = b_0(M) = 1,$$

since  $M$  is connected. The other piece will be a little harder.

**Lemma 6.11.**

- (1) *If  $\alpha \in \Omega^1(M)$ , then  $d^+\alpha = 0$  iff  $d\alpha = 0$ .*
- (2) *There's a space  $\mathcal{H}^+ \subset \Omega_2^+(M)$  such that  $\Omega_2^+(M) = \operatorname{Im}(d^+) \oplus \mathcal{H}^+$ , and*
- (3)  *$\dim \mathcal{H}^+ = b_2^+(M)$ .*

We'll come back to this later, but as a corollary, we know  $\operatorname{ind} D^2 = -1 - b_2^+$ . Therefore we know that in Lemma 6.3,  $n' = n + b_2^+ + 1$ . Crucially, the indices did not depend on  $\lambda$ , so we're interested in  $k$  and  $b_2^+$ , moreso than  $m$  and  $n$ .

Now we turn to  $Q$ , the quadratic part of the monopole map. Recall that  $Q$  is a direct sum of two operators:

- $Q_1: \Gamma(\mathbb{S}^+ \oplus \widetilde{T}^*M) \rightarrow \Gamma(\mathbb{S}^-)$  sends  $\phi, a \mapsto ia\phi$ , and
- $Q_2: \Gamma(\mathbb{S}^+ \oplus \widetilde{T}^*M) \rightarrow \Gamma(\widetilde{\Lambda}) = \Omega^0(M) \oplus \Omega_+^2(M)$  sends  $\phi, a \mapsto \phi a \bar{\phi}$ .

**TODO:** I missed this part, sorry. There's  $W_\lambda$  and then you need one more dimension, which lands in a space called  $\widetilde{W}_\lambda$ .

In summary, we have the following theorem.

**Theorem 6.12.** *For  $\lambda \gg 0$ , the monopole map defines a  $\text{Pin}_2^-$ -equivariant map  $D_\lambda + Q_\lambda: V_\lambda \rightarrow \widetilde{W}_\lambda$  between finite-dimensional  $\text{Pin}_2^-$ -equivariant representations such that  $V_\lambda \cong \mathbb{H}^{m+k} \oplus \widetilde{\mathbb{R}}^n$  and  $\widetilde{W}_\lambda \cong \mathbb{H}^m \oplus \widetilde{\mathbb{R}}^{n+b_2^+}$ , where  $m$  and  $n$  may depend on  $\lambda$ , but  $k$  doesn't. Moreover, there are  $\text{Pin}_2^-$ -invariant inner products on  $V_\lambda$  and  $W_\lambda$ , and an  $R > 0$  not depending on  $\lambda$ , such that the image of the sphere of radius  $R$  in  $V_\lambda$  is nonzero.*

In particular, we can restrict to  $S_R(V_\lambda)$  and obtain a map between spheres with  $\text{Pin}_2^-$ -actions. That is, existence of  $M$  tells us that some  $\text{Pin}_2^-$ -equivariant homotopy class of maps between spheres exists; we will use some equivariant homotopy theory to show nonexistence of certain such maps, and therefore obtain the bound on the ratio of the signature and  $b_2(M)$ .

*Proof of Lemma 6.11.* For the first part, suppose  $d\alpha \in \Omega^-$ , which means that  $\star d\alpha = -d\alpha$ . Since  $d^* = -\star d\star$ , then  $\star d\alpha = -d^*(\star^{-1}\alpha)$ , and  $\star^{-1}\alpha \in \Omega^3$ . Using the splitting

$$(6.13) \quad \Omega^2 = \text{Im}(d) \oplus \text{Im}(d^*) \oplus \mathcal{H}^2,$$

we get that  $d^*(\star^{-1}\alpha) = d\alpha$ .

For the second part, we use (6.13) again. Let  $\pi: \Omega^2 \rightarrow \Omega^+$  be defined by  $(1 + \star)/2$ . We know  $\Omega^+ = \text{Im}(d^+) \oplus \text{Im}((d^*)^+) \oplus \mathcal{H}^+$ , so it suffices to show that  $\text{Im } d^+ \perp \mathcal{H}^+$  and  $\text{Im } d^+ = \text{Im}((d^*)^+)$ . For the first one, if  $d^+\alpha \in \text{Im}(d^+)$  and  $\beta \in \mathcal{H}^+$ , we have

$$(6.14) \quad \langle d^+\alpha, \beta \rangle = \langle d\alpha, \beta \rangle = \langle \alpha, d^*\beta \rangle = 0.$$

For the second part, let  $\alpha \in \Omega^3$ ; then

$$(6.15) \quad \pi(d^*\alpha) = \pi(-\star d\star\alpha) = -\pi d(\star\alpha) = -d^+(\star\alpha).$$

For the last part, recall that  $b_2^+$  is the maximal dimension of a positive-definite subspace of  $\mathcal{H}$  with respect to the pairing  $\mathcal{H}^2 \otimes \mathcal{H}^2 \rightarrow \mathbb{R}$ . Also recall that  $\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \text{dvol}$ , so if  $\alpha \in \mathcal{H}^2$  is self-dual, then

$$(6.16) \quad \int \langle \alpha \wedge \alpha \rangle \text{dvol} = \int \alpha \wedge \star\alpha = \int \langle \alpha, \alpha \rangle \text{dvol} \geq 0.$$

with equality iff  $\alpha = 0$ . Therefore  $\dim \mathcal{H}^+ \leq b_2^+$ , and because the same argument applies to  $\mathcal{H}^-$ , we get  $\dim \mathcal{H}^- \leq b_2^-$ , so using the orthogonal splitting  $\mathcal{H}^2 = \mathcal{H}_+^2 \oplus \mathcal{H}_-^2$ , and therefore  $\dim \mathcal{H}^+ = b_2^+$ .  $\square$

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