GEOMETRY AND STRING THEORY SEMINAR: FALL 2019

ARUN DEBRAY
SEPTEMBER 4, 2019

These notes were taken in UT Austin's geometry and string theory seminar in Fall 2019. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

Contents

1. A Heisenberg uncertainty principle for fluxes: 9/4/19

Today Dan spoke on a Heisenberg uncertainty principle for fluxes, which will provide background for subsequent talks. The material in this lecture is based on two papers by Freed, Moore, and Segal [FMS07a, FMS07b]; a followup talk will discuss a newer paper of García-Extebarria, Heidenreich, and Regalado [GEHR19].

First, let's recall the story for electromagnetism. Let Y be a closed oriented Riemannian manifold and give $M \coloneqq \mathbb{R} \times Y$ the Lorentz metric in which $\mathbb{R}$ is timelike. The *electromagnetic field* is a two-form $F \in \Omega^2(M)$; locally this is given by six functions, three of which tell us the electric field and three of which tell us the magnetic field. We also have electric and magnetic currents $j_E, j_B \in \Omega^3(M)$, which are closed forms.

Maxwell's equations can then be written concisely as

$$\mathrm{d}F = j_B = 0 \tag{1.1a}$$

$$\mathrm{d}{\star}F = j_E. \tag{1.1b}$$

Here the Hodge star is the one on M, not on Y, so keep in mind the Lorentz signature when chasing signs.

We're interested in fluxes. Let $\Sigma \subset Y$ be a closed, oriented siurface. The *magnetic flux* through Σ is

$$\mathcal{E}^{c\ell}_{t,\Sigma}(F) = \int_{\{t\}\times\Sigma} F \tag{1.2}$$

and the *electric flux* through Σ is

$$\mathcal{B}^{c\ell}_{t,\Sigma}(F) = \int_{\{t\}\times\Sigma} {\star}F. \tag{1.3}$$

If $j_B = j_E = 0$, then these fluxes are independent of t and depend only on the homology class of Σ, by Stokes' theorem. Hence if V is the vector space of solutions to (1.1), then $\mathcal{B}^{c\ell}$ and $\mathcal{E}^{c\ell}$ define linear functions $V \to H^2_{\mathrm{dR}}(Y)$.

Along the way to quantizing this theory, we should put it in the Hamiltonian formalism. Let

$$F \coloneqq B - \mathrm{d}t \wedge E, \tag{1.4}$$

where $B(t) \in \Omega^2(Y)$ and $E(t) \in \Omega^1(Y)$. In this setting, Maxwell's equations are

$$\frac{\partial B}{\partial T} = -\mathrm{d}_Y E \tag{1.5a}$$

$$\frac{\partial {\star}_Y E}{\partial t} = \mathrm{d}_Y {\star}_Y B. \tag{1.5b}$$

We would like to express these in terms of a Poisson bracket. The Hamiltonian is

$$H \coloneqq \frac{1}{2}\int_Y \left(\|B\|^2 + \|E\|^2\right) \mathrm{vol}_Y. \tag{1.6}$$

Let $W := \Omega^2(Y)_{\mathrm{cl}} \times \Omega^2(Y)_{\mathrm{cl}}$, and define the map $\theta_t\colon V \to W$ by

$$F \longmapsto (F|_{\{t\}\times Y}, \star F|_{\{t\}\times Y}). \tag{1.7}$$

Then W carries a Poisson structure as follows:[1] let $\eta \in \Omega^1(Y)/\mathrm{d}\Omega^0(Y)$ and define

$$\ell_\eta(B, \star_Y E) := \int_Y \eta \wedge B \tag{1.8a}$$

$$\ell'_\eta(B, \star_Y E) := \int_Y \eta \wedge \star_Y E. \tag{1.8b}$$

These satisfy the Poisson anticommutation relations

$$\{\ell_{\eta_1}, \ell_{\eta_2}\} = 0 \tag{1.9a}$$

$$\{\ell'_{\eta_1}, \ell'_{\eta_2}\} = 0 \tag{1.9b}$$

$$\{\ell_{\eta_1}, \ell'_{\eta_2}\} = \int_Y \mathrm{d}\eta_1 \wedge \eta_2. \tag{1.9c}$$

The electric and magnetic fluxes define a map $\mathcal{B}^{c\ell}, \mathcal{E}^{c\ell}\colon W \to H^2_{\mathrm{dR}}(Y) \times H^2_{\mathrm{dR}}(Y)$ sending $(B, \star_Y E) \mapsto ([B], \star_Y E)$.

A key point is that $\mathcal{B}^{c\ell}$ and $\mathcal{E}^{c\ell}$ commute. TODO: this has physics implications which I missed.

Now, following Dirac, we quantize: there are some units in which both electric and magnetic charges are integers. First, we quantize $[j] \in H^3_{\mathrm{dR}}(Y)$ by requiring it to lie in the image of $H^3(Y;\mathbb{Z}) \to H^3(Y;\mathbb{R}) \cong H^3_{\mathrm{dR}}(Y)$. This is a one-dimensional vector space, but soon enough we will think about more interesting examples.

Next, refine charge to an element of $H^3(X;\mathbb{Z})$. In this setting we haven't changed very much, but in more general settings this allows them to be torsion.[2] For example, we also refine the fluxes to elements of $H^2(Y;\mathbb{Z})$, which can have torsion, e.g. for $\mathbb{RP}^3$ and lens spaces.

The refined magnetic flux fits into a diagram

$$\begin{tikzcd} & \Omega^2(Y)_{\mathrm{cl}} \arrow[d, "\mathcal{B}^{c\ell}"] \\ H^2(Y;\mathbb{Z}) \arrow[r] & H^2_{\mathrm{dR}}(Y). \end{tikzcd} \tag{1.10}$$

So you might think the right place to situate it is the fiber product of these abelian groups, but this is wrong for physics reasons: the resulting abelian group is not local, ultimately because $H^2(X;\mathbb{Z})$ isn't. Instead, one can take a homotopy fiber product in a certain setting, landing in the relevant *differential cohomology* group $\check{H}^2(Y)$, the group of isomorphism classes of principal U_1-bundles with connection:

$$\begin{tikzcd} \check{H}^2(Y) \arrow[r, "\text{curvature}"] \arrow[d, "c_1"] & \Omega^2(Y)_{\mathrm{cl}} \arrow[d, "\mathcal{B}^{c\ell}"] \\ H^2(Y;\mathbb{Z}) \arrow[r] & H^2_{\mathrm{dR}}(Y). \end{tikzcd} \tag{1.11}$$

The differential cohomology group $\check{H}^2(Y)$ can be thought of as a local refinement of the fiber product – for example, a differential cohomology class defines a circle bundle with connection, but an element of the fiber product doesn't in general. Though to be really precise, the element of $\check{H}^2(Y)$ isn't local, but rather the circle bundle with connection that defines it (which forms a groupoid). You can see this by thinking through bundles on the circle (can be nontrivial) versus bundles on the two semicircles (always trivial). So the field in physics comes from the groupoid, not the differential cohomology group.

[1]This Poisson structure doesn't come from a symplectic structure; there is a kernel. In this case there is a foliation with symplectic leaves.

[2]Now that we're no longer working over $\mathbb{R}$, there are other choices for this refinement than $H^*(-;\mathbb{Z})$, such as generalized cohomology theories. They don't appear in the Maxwell-theoretic story, but can appear in string theory.

Remark 1.12. $\check{H}^2(Y)$ is an abelian Lie group:[3] we can tensor together two principal U_1-bundles with connection into a third. It's instructive to think through its homotopy groups — though only π_0 and π_1 are nonzero. ◂

Armed with these fine refinements, let's turn back to Maxwell theory; this is a sort of semiclassical perspective.

Let A be an $\mathbb{R}/\mathbb{Z}$-connection on $M = \mathbb{R}\times Y$ with curvature $F_A \in \Omega^2(M)$. The Lagrangian is

$$L := -\frac{1}{2} F_A \wedge \star F_A \tag{1.13}$$

and Maxwell's equations can be understood as the Euler-Lagrange equation for this Lagrangian, namely

$$\mathrm{d}\star F_A = 0, \tag{1.14a}$$

and the Bianchi identity

$$\mathrm{d}F_A = 0. \tag{1.14b}$$

Now, $T(\check{H}^2(Y))$[4] plays the role of W, though it's no longer a vector space and the fluxes define maps $\mathcal{B}^{c\ell}, \mathcal{E}^{c\ell}\colon T(\check{H}^2(Y)) \to H^2_{\mathrm{dR}}(Y)$.

Lemma 1.15. *The fluxes still commute in this semiclassical setting:* $\{\mathcal{B}^{c\ell}, \mathcal{E}^{c\ell}\} = 0$.

Remark 1.16. The (isomorphism classes of) flat connections form a subgroup of $\check{H}^2(Y)$, and this subgroup is isomorphic to $H^1(Y;\mathbb{R}/\mathbb{Z})$. As Y is compact, this is a finite-dimensional Lie group. There is an isomorphism $\beta\colon \pi_0(H^1(Y;\mathbb{R}/\mathbb{Z})) \to \mathrm{Tors}H^2(Y;\mathbb{Z})$ called the *Bockstein homomorphism*.[5] More explicitly, we have a short exact sequence

$$0 \longrightarrow T^1(Y) \longrightarrow H^1(Y;\mathbb{R}/\mathbb{Z}) \xrightarrow{\beta} \mathrm{Tors}H^2(Y;\mathbb{Z}) \longrightarrow 0, \tag{1.17}$$

where $T^1(Y) := H^1(Y;\mathbb{R})/H^1(Y;\mathbb{Z})$. ◂

The above story is semiclassical in that we've quantized charges and fluxes, but haven't produced a full Hilbert space on Y. Heuristically, we would like $\mathcal{H}_Y$ to be $L^2(\check{H}^2(Y))$, but since $\check{H}^2(Y)$ is an infinite-dimensional manifold there are some nuances going into that definition.

There are two gradings on $\mathcal{H}_Y$, a *magnetic grading* indexed by $b \in H^2(Y;\mathbb{Z})$ (a decomposition involving connected components of $\check{H}^2(Y)$), and an *electric grading*, produced by an $H^1(Y;\mathbb{R}/\mathbb{Z})$-action coming from a motion in some way on $\check{H}^2(Y)$; then, we decompose into a sum of irreducible representations. Since this group is abelian, this is just $H^1(Y;\mathbb{R}/\mathbb{Z})^\vee \cong H^2(Y;\mathbb{Z})$ by Poincaré duality.

Can we make these gradings simultaneously? No, because the $H^1(Y;\mathbb{R}/\mathbb{Z})$-action reshuffles the components in accordance with the Bockstein, whenever $H^2(Y;\mathbb{Z})$ has torsion.

To try and fix this, we can grade by quotient groups of $H^2(Y;\mathbb{Z})$, such as H^2 modulo torsion, and these two gradings commute and define a bigrading.

Now in a quantum theory, these gradings should arise from the spectra of commuting operators. Given $\omega \in H^1(Y;\mathbb{R}/\mathbb{Z})$, we obtain operators $\mathcal{B}^q(\omega)$ and $\mathcal{E}^q(\omega)$, respectively multiplication by $\exp(2\pi i\langle b,\omega\rangle)$ on $\mathcal{H}^b$ (the summand associated to b in the magnetic grading) and pullback by translation by $\omega \in H^1(Y;\mathbb{R}/\mathbb{Z}) \subset \check{H}^2(Y)$ (thought of as a flat connection).

This is a version of Heisenberg's uncertainty principle: $\mathcal{B}^q$ and $\mathcal{E}^q$ don't commute.

Theorem 1.18. $[\mathcal{B}^q(\omega_1), \mathcal{E}^q(\omega_2)] = \exp(2\pi i\langle \omega_1 \smile \beta\omega_2, [Y]\rangle)\, \mathrm{d}_{\mathcal{H}_Y}$.

[3] Well, it's not a finite-dimensional manifold, but can be made into an infinite-dimensional manifold, and in that more general sense is an abelian Lie group.

[4] Here $T(\check{H}^2(Y))$ denotes the tangent bundle of the infinite-dimensional manifold $\check{H}^2(Y)$. You can think of this as initial conditions for solutions to the Maxwell equation; as these are linear wave equations, this intuition is well-beahved enough to be accurate.

[5] This is an instance of a general phenomenon: a short exact sequence of chain complexes induces a long exact sequence in cohomology. A *Bockstein homomorphism* is a connecting map in the long exact sequence induced from the short exact sequence of chain complexes corresponding to a short exact sequence of coefficient groups; this one comes from the sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$.

Here $[-,-]$ denotes the commutator in a Lie group. The cup product of a $\mathbb{Z}$-cohomology class and an $\mathbb{R}/\mathbb{Z}$-cohomology class is an $\mathbb{R}/\mathbb{Z}$-cohomology class, using the $\mathbb{Z}$-module structure on $\mathbb{R}/\mathbb{Z}$, and then we pair with the fundamental class to obtain an element of $\mathbb{R}/\mathbb{Z}$. Exponentiating, we get a number.

We will quantize, as usual, by making a Heisenberg group extension

$$0 \longrightarrow \mathbb{T} \longrightarrow \mathcal{G} \longrightarrow \mathcal{A} \coloneqq \check{H}^2(Y) \times \check{H}^2(Y) \longrightarrow 0. \tag{1.19}$$

Here $\mathbb{T}$ is central. Such a $\mathcal{G}$ is characterized up to isomorphism by the commutator map $[-,-]\colon \mathcal{A} \times \mathcal{A} \to \mathbb{T}$. The idea is that since $\mathbb{T}$ is central, the map only depends on the equivalence class of $g \in \mathcal{G}$ in $\mathcal{A}$, and what we get is central.

The commutator map is skew but not necessarily alternating, but it does define a $\mathbb{Z}/2$-grading on the Heisenberg group, then gradings on representations, etc.

In our example, the commutator map is

$$s(A_1, A_2) = \exp\left(2\pi i \int_Y A_1 \cdot A_2\right). \tag{1.20}$$

Here $A_1 \cdot A_2$ is the product in $\check{H}^\bullet(Y)$. This is a good general way to deal with abelian gauge fields, but here we can make it more explicit: any closed 3-manifold with two $\mathbb{R}/\mathbb{Z}$-connections bounds a compact 4-manifold with two $\mathbb{R}/\mathbb{Z}$-connections,[6] and then... (TODO: ?).

Next, representation theory. For finite-dimensional Heisenberg groups subject to some nondegeneracy condition, there's a unique representation extending a given representation of $\mathbb{T}$, but in infinite-dimensions, one needs a polarization on $\mathcal{G}$, and this comes from a positve energy assumption in physics. Then $\mathcal{B}^q(\omega_1)$ and $\mathcal{E}^q(\omega_2)$ are just images of elements in the Heisenberg group, and one can compute the commutator there to prove Theorem 1.18.

Remark 1.21. This story applies any time you have an abelian gauge field. The Dirac quantization we saw above first chooses some cohomology theory, which can be determined by things like anomalies or other features of the theory in question. For Maxwell theory, we chose ordinary cohomology over the integers, denoted $H\mathbb{Z}$.

But in Type II string theory on a 9-dimensional manifold Y, something different can happen: there is a *Neveu-Schwarz field* $H \in \Omega^3(Y)$ and a *Ramond field* $F \in \Omega^*(Y)$, either even or odd (TODO: I think this is the IIA/IIB distinction). When $H = 0$, Witten and Sen proposed that the right way to solve all the constraints on string theory is to choose not $H\mathbb{Z}$ but complex K-theory KU. The abelian group of fluxes is $K^0(Y)$ (in IIA) or $K^1(Y)$ (for IIB); since K-theory is 2-periodic, these are the only options. There are examples where $K^0(Y)$ has torsion subgroups, and once again the Heisenberg uncertainty principle outlined above implies that the grading only works modulo torsion.

One then builds the Heisenberg group from *differential K-theory* $\check{K}^0(Y)$, and some features of the story are different, leading to interesting physics. ◀

References

[FMS07a] Daniel S. Freed, Gregory W. Moore, and Graeme Segal. Heisenberg groups and noncommutative fluxes. *Annals of Physics*, 322(1):236–285, 2007. January Special Issue 2007. 1

[FMS07b] Daniel S. Freed, Gregory W. Moore, and Graeme Segal. The uncertainty of fluxes. *Communications in Mathematical Physics*, 271(1):247–274, Apr 2007. https://arxiv.org/abs/hep-th/0605198. 1

[GEHR19] Iñaki García-Etxebarria, Ben Heidenreich, and Diego Regalado. IIB flux non-commutativity and the global structure of field theories. 2019. http://arxiv.org/abs/1908.08027. 1

[6] This question is only interesting for bundles; once we know it there, the extension is automatic.