# HYPERKÄHLER GEOMETRY AND THE MODULI SPACE OF HIGGS BUNDLES

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Contents



## 1. Introduction to Kähler and hyperkähler geometry: 6/12/17

Today will be mostly preliminaries, including some complex and symplectic geometry, such as the symplectic quotient, and an introduction to Kähler and hyperkähler geometry. Over the rest of the week, we'll discuss some examples (which are usually left implicit) such as quiver varieties, introduce the moduli space of Higgs bundles, and more. A good reference for this is Andy Neitzke's lecture notes on the moduli space of Higgs bundles: https://www.ma.utexas.edu/users/neitzke/teaching/392C-higgs-bundles/higgs-bundles.pdf.

Kähler manifolds are great because they involve a whole bunch of angles towards differential geometry: Riemannian, complex, and symplectic manifolds. Riemannian manifolds are likely the most familiar, so we won't discuss them in too much detail.

### 1.1. Almost complex and complex geometry.

**Definition 1.1.** Let $X$ be a smooth manifold. A **almost complex structure** on $X$ is a choice of $I \in \mathrm{End}(TX)$ such that $I^2 = -1$. The pair $(X, I)$ is called an **almost complex manifold**.

This implies that $\dim_{\mathbb{R}}(X)$ is even.

You can define a **complex manifold** as the same thing as a real manifold, but with charts valued in $\mathbb{C}^n$, and such that the change-of-charts maps are holomorphic. This implies an almost complex structure, as the tangent space is a complex vector bundle, but (as the terminology "almost" suggests) the two are not the same.

**Definition 1.2.** An almost complex structure $(M, I)$ is **integrable** if there exists an open cover $\mathfrak{U}$ of $M$ and holomorphic diffeomorphisms $\phi_U \colon U \to \phi_U(U) \subset \mathbb{C}^n$ for each $U \in \mathfrak{U}$.

**Proposition 1.3.** *An integrable almost complex structure on $X$ is equivalent to a complex structure on $X$.*

If $(X, I)$ is an almost complex manifold of dimension $n$, then the action of $I$ on $TX$ makes $TX$ into a complex vector bundle. You can also complexify it as a real vector bundle, producing a vector bundle $T_{\mathbb{C}}X$ of rank $2n$.

**Definition 1.4.** Let $T^{1,0}X$ denote the eigenspace for $i$ acting on $T_{\mathbb{C}}X$ and $T^{0,1}X$ denote the $-i$-eigenspace. Hence $T_{\mathbb{C}}X \cong T^{1,0}X \oplus T^{0,1}X$.

These are both complex vector bundles of rank $n$, and in fact $TX \cong T^{1,0}X$.

You can play the same game with the complexified cotangent bundle: $T^*_{\mathbb{C}}X \cong (T^*)^{1,0}X \oplus (T^*)^{0,1}X$, and $(T^*)^{1,0}X = \mathrm{Ann}(T^{0,1}X)$. More generally, one gets a **type decomposition** or **Hodge decomposition** of complexified exterior powers and complexified differential forms:

$$\Lambda^* T^*_{\mathbb{C}} X \cong \bigoplus_{p+q=n} \Lambda^{p,q} T^* X$$

$$\Omega^*_{\mathbb{C}} X \cong \bigoplus_{p,q=0}^{n} \Omega^{p,q}(X).$$

The piece of degree $n$ is the sum of $\Lambda^{p,q}$ (resp. $\Omega^{p,q}$) for which $p+q=n$.

The intuition is that the holomorphic structure on $\mathbb{C}$ allows one to write

$$\mathrm{d}x \wedge \mathrm{d}y = -\frac{1}{2}\,\mathrm{d}z \wedge \mathrm{d}\overline{z},$$

where $\mathrm{d}z \coloneqq \mathrm{d}x + i\,\mathrm{d}y$ and $\mathrm{d}\overline{z} = \mathrm{d}x - i\,\mathrm{d}y$. The graded part $\Lambda^{p,q}T^*X$ corresponds to the pieces with $\mathrm{d}z$ in $p$ directions and $\mathrm{d}\overline{z}$ in $q$ directions, and similarly for $\Omega^{p,q}X$.[1]

**Theorem 1.5.** *The following are equivalentfor an almost complex manifold $(X, I)$:*

(1) *$I$ is integrable.*
(2) *There's a decomposition $\mathrm{d} = \partial + \overline{\partial}$, where $\partial\colon \Omega^{p,q} \to \Omega^{p+1,q}$ and $\overline{\partial}\colon \Omega^{p,q} \to \Omega^{p,q+1}$.*
(3) *$T^{0,1}X$ is an integrable distribution, i.e. for sections $s, t \in \Gamma(T^{0,1}X)$, $[s,t]$ is also a section.*

We'll use the first two more often than the third.

**Definition 1.6.** The **Dolbeault cohomology** is the homology of the complexified differential forms: $H^{p,q}_{\mathrm{Dol}}(X) \coloneqq H^q(\Omega^{p,\bullet}, \overline{\partial})$.

**Definition 1.7.** Let $X$ be a complex manifold. A **holomorphic vector bundle** is a complex vector bundle $E \to X$ together with a differential $\overline{\partial}_E\colon \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$ such that

(1) $\overline{\partial}_E$ satisfies the **Leibniz rule**: if $\alpha \in \Omega^*(X)$ and $\psi \in \Omega^0(E)$,
$$\overline{\partial}_E(\alpha\psi) = (\overline{\partial}\alpha)\psi + (-1)^{|\alpha|}\alpha \wedge \overline{\partial}_E\psi.$$
(2) $\overline{\partial}_E^2 = 0$.

**Exercise 1.8.**

(1) If $X$ is a complex manifold, show that $TX \to X$ is holomorphic.
(2) The **canonical line bundle** over a complex manifold is $K_X \coloneqq \Lambda^{n,0}T^*X$. Show that $K_X$ is holomorphic.

Elements of $K_N$ are things of the form $f(z)\,\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n$, where $f\colon X \to \mathbb{C}$ is holomorphic.

**Definition 1.9.** Let $E \to X$ be a holomorphic vector bundle and $h$ be a Hermitian metric on $E$ (i.e. a smoothly varying Hermitian metric on each fiber). The **Chern connection** is the unique connection $D$ on $E$ that is

- unitary with respect to $h$, and
- compatible with the holomorphic structue, in that $(D\psi)^{0,1} = \overline{\partial}_E\psi$ for any $\psi \in \Omega^0(E)$.

Like the Levi-Civita connection, it's a theorem that the Chern connection exists. That $h$ is unitary means $h(Ix, y) = ih(x, y) + h(x, -Iy)$.

1.2. **Kähler geometry.** Let $(X, I)$ be an almost complex manifold with Hermitian metric $g$, and let $\nabla$ denote the Levi-Civita connection for $g$.

**Definition 1.10.** The **fundamental form** is the $\omega \in \Omega^{1,1}_{\mathbb{R}}(X)$ such that $\omega(v, w) = g(Iv, w)$.

Here, $\Omega^{p,p}_{\mathbb{R}}(X)$ is the fixed points of complex conjugation acting on $\Omega^{p,p}(X)$.

There are many different ways to define Kähler manifolds.

**Definition 1.11.** The triple $(X, g, I)$ is called a **Kähler manifold** if one of the following equivalent conditions is satisfied.

(1) $\nabla I = 0$, i.e. $I$ is **covariantly constant**.

[1] You can run this same story with tensor products, but the resulting vector bundles don't appear as often in the theory.

(2) $\nabla\omega = 0$.
(3) $I$ is integrable and the Levi-Civita and Chern connections coincide.
(4) $I$ is integrable and $\mathrm{d}\omega = 0$.

In this case $\omega$ is also called the **Kähler form**.

**Corollary 1.12.** *Let $X$ be a Kähler manifold and $Y \subset X$ be a complex submanifold. Then, restricting $g$ and $I$ to $Y$ shows that $Y$ is also a Kähler manifold.*

**Example 1.13.** Everyone's first example of a Kähler manifold is $\mathbb{CP}^n$ with the **Fubini-Study metric**. ◂

You can show by a dimensional argument that any Riemann surface is automatically Kähler. Finding counterexamples is kind of tricky: combining Corollary 1.12 and Example 1.13, any smooth projective complex variety is Kähler. There are homological obstructions to being Kähler, and we'll learn more about this in the Kähler geometry minicourse later this summer.

**Proposition 1.14.** *Let $(X.h)$ be a Kähler manifold of complex dimension $n$. Then, the holonomy of the Levi-Civita connection around any point $p$ is a subgroup of $\mathrm{U}_n$.*

1.3. **Symplectic geometry.** Hyperkähler manifolds often arise as moduli spaces, which themselves often arise as symplectic quotients of things. To understand symplectic quotients we should first go over some symplectic geometry.

**Definition 1.15.**

- Let $V$ be a vector space over either $\mathbb{R}$ or $\mathbb{C}$. By a **nondegenerate** 2-form we mean an $\omega \in \Lambda^2(V)$ such that $v \mapsto \omega(v, -)$ defines an isomorphism $V \to V^*$.
- A **symplectic manifold** is a pair $(X, \omega)$, where $\omega \in \Omega^2(X)$ is a closed, nondegenerate 2-form (i.e. it's nondegenerate at every $x \in X$). In this case, $\omega$ is called a **symplectic form**.
- A symplectic manifold $(X, \omega)$ is **exact** if $\omega$ is exact, i.e. there's a one-form $\lambda$ such that $\mathrm{d}\lambda = \omega$.

**Example 1.16.** The canonical example is the cotangent bundle $T^*X$ to a smooth manifold $X$, which is an exact symplectic manifold. The **Liouville form**, **Liouville differential**, or (in some contexts in physics) the **Seiberg-Witten differential** $\lambda \in \Omega^1(T^*X)$ is

$$\lambda(x,p) \cdot v = p \cdot \pi_* v,$$

where $\pi\colon T^*X \to X$ is projection. The symplectic form is $\omega = \mathrm{d}\lambda$.

In local coordinates $(p_1, q_1, \dots, p_n, q_n)$, so $p_1, \dots, p_n$ are coordinates on $X$ (in physics, position) and $q_1, \dots, q_n$ are coordinates on the fiber (in physics, momentum), we have

$$\lambda = \sum_{i=1}^n p_i \,\mathrm{d}q_i \qquad \text{and hence} \qquad \omega = \sum_{i=1}^n \mathrm{d}p_i \wedge \mathrm{d}q_i. \qquad ◂$$

For a general nondegenerate $\omega \in \Lambda^2(V)$, there's a **canonical basis** $\{e_1, e_2, \dots, e_n, f_1, \dots, f_n\}$ such that $\omega(e_i, e_j) = \delta_{ij}$ and $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$ for all $i$ and $j$. This means that, in the canonical basis, $\omega$ is represented by the block matrix

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Moduli spaces arise as quotients. Let $G$ be a (real) Lie group acting on a symplectic manifold $(X, \omega)$,[2] and let $\mathfrak{g}$ be the Lie algebra of $G$. Differentiating the action defines a Lie algebra homomorphism $\rho\colon \mathfrak{g} \to \mathcal{X}(X)$, where $\mathcal{X}(X)$ denotes the Lie algebra of vector fields on $X$.

We'd like to dualize this to define a map $\mu\colon X \to \mathfrak{g}^*$ that's $G$-equivariant (where $G$ acts on $\mathfrak{g}^*$ through the dual of the adjoint action).

**Definition 1.17.** A **moment map** for the $G$-action on $X$ is a map $\mu\colon X \to \mathfrak{g}^*$ such that $\iota_{\rho(Z)}\omega = \mathrm{d}(\mu \cdot Z)$ for $Z \in \mathfrak{g}$. (Here, $\iota$ is the contraction operator as in Riemannian geometry.)

This is a very important definition — it characterizes to what degree the action commutes with the Hamiltonian.

[2] Generally, $G$ acts through **symplectomorphisms**, i.e. diffeomorphisms preserving the symplectic form, and defining the moment map will require this.

*Remark.*

- The moment map does not always exist. One local obstruction comes from the **Cartan formula**:
$$\mathcal{L}_v\omega = \mathrm{d}(\iota_v\omega) + \iota_v(\mathrm{d}\omega),$$
and therefore
$$\mathcal{L}_{\rho(Z)}\omega = \mathrm{d}(\iota_{\rho(Z)}\omega) + \iota_{\rho(Z)}\mathrm{d}\omega = 0,$$
so the infinitesimal action must preserve $\omega$. This does not suffice; there are other, global obstructions.
- When a moment map exists, it need not be unique. For example, consider any $c \in [\mathfrak{g}, \mathfrak{g}]^\perp \subset \mathfrak{g}^*$;[3] then, if $\mu$ is a moment map, so is $\mu + c$. ◂

**Example 1.18.** Consider $X = \mathbb{C}^n$ with the standard Kähler structure
$$\omega := \frac{i}{2}\sum_{i=1}^n \mathrm{d}z_i \wedge \mathrm{d}\overline{z}_i.$$
The easiest action you can think of is a $\mathrm{U}_1$-action by rotation: $z_i \mapsto e^{i\alpha}z_i$. Thus, if we rotate $z_i$ by $e^{i\alpha}$, we rotate $\overline{z}$ by $e^{-i\alpha}$, so this action is through local symplectomorphisms.

One moment map is
$$\mu = -\frac{1}{2}\sum \lvert \mathrm{d}z_i\rvert^2 + c$$
for any $c \in \mathbb{R}$, because the commutator subgroup of $\mathrm{U}_1$ is trivial and $\mathfrak{u}_1 \cong \mathbb{R}$. ◂

**Definition 1.19.** Let $G$ act on a symplectic manifold $(X, \omega)$ in such a way that a moment map $\mu$ exists. Then, the **symplectic quotient** of $X$ by $G$ is $X /\!/_\mu G := \mu^{-1}(0)/G$.

The dependence on $\mu$ is a little fearsome, and the $G$-action on $\mu^{-1}(0)/G$ is not always free.

**Proposition 1.20.** *If $G$ acts freely on $\mu^{-1}(0)$ freely, then $X /\!/_\mu G$ is a symplectic manifold. Moreover,*

(1) $\dim(X /\!/_\mu G) = \dim X - 2\dim G$, *and*
(2) *if $\omega_X$ denotes the symplectic form on $X$ and $\omega_{X/\!/_\mu G}$ denotes the symplectic form on $X /\!/_\mu G$, then $\pi^*\omega_{X/\!/_\mu G} = \iota^*\omega_X$, where $\iota\colon \mu^{-1}(0) \hookrightarrow X$ is inclusion and $\pi\colon \mu^{-1}(0) \twoheadrightarrow X /\!/_\mu G$ is projection.*

*Remark.* For some more motivation about why this is an okay idea, let $X$ be a compact manifold and $G$ be a compact Lie group acting freely on $X$. Hence $G$ acts on $(T^*X, \omega)$ through symplectomorphisms, and in fact a moment map always exists, and one is given by
$$\mu_Z(x, p) = -p \cdot (\rho(Z)(x)),$$
where $x \in X$ and $p \in T_x^*X$. Moreover, the action of $G$ on $\mu^{-1}(0)$ is free, and $T^*X /\!/_\mu G \cong T^*(X/G)$ as symplectic manifolds. ◂

The point of the moment map is to generalize this nice fact.

**Proposition 1.21.** *Let $X$ be a Kähler manifold and $G$ act on $X$ preserving $g$, $I$, and $\omega$, and suppose that there's a moment map $\mu$ such that $G$ acts freely on $\mu^{-1}(0)$. Then, $X /\!/_\mu G$ has a natural Kähler structure.*

Explicitly, the symplectic structure is as in Proposition 1.20, the metric is the quotient metric, and $I$ is determined by $g$ and $\omega$. This fact is the reason symplectic quotients arise in the study of Kähler manifolds.

**Example 1.22.** Let $\mathrm{U}_1$ act on $\mathbb{C}^n$ as in Example 1.18.

- If $c < 0$, then $\mu^{-1}(0) = \varnothing$. $G$ acts freely on this, but for silly reasons.
- If $c = 0$, $\mu^{-1}(0) = \{0\}$, so the $\mathrm{U}_1$-action is not free.
- If $c > 0$,
$$\mu^{-1}(0) = \left\{\sum_{i=1}^n \lvert z_i\rvert^2 = 2c\right\} \cong S^{2n-1},$$
and the quotient is $\mathbb{C}^n /\!/_\mu \mathrm{U}_1 = \mu^{-1}(0)/\mathrm{U}_1 = \mathbb{CP}^{n-1}$. The induced metric is the Fubini-Study metric.

The last case is interesting: instead of taking the quotient $X/G$, we took a subset of $X$ and quotiented out by a complex analogue of $G$: $X \setminus \{0\}/\mathbb{C}^\times$. This applies in more general situations. ◂

[3] By this notation, we mean $[\mathfrak{g}, \mathfrak{g}]^\perp = \{f \in \mathfrak{g}^* \mid f(x) = 0 \text{ for all } x \in [\mathfrak{g}, \mathfrak{g}]\}$.

**Definition 1.23.** Let $(X, I)$ be a complex manifold. A **holomorphic symplectic structure** on $X$ is an $\Omega \in \Omega^{2,0}(X)$ such that

- $\mathrm{d}\Omega = 0$ and
- $\Omega$ is **holomorphically nondegenerate**, i.e. it induces an isomorphism $T^{1,0}X \to (T^{1,0}X)^*$.

The triple $(X, I, \Omega)$ is called a **holomorphic symplectic manifold**.

**Proposition 1.24.** *Let $(X, I, \Omega)$ be a holomorphic symplectic manifold. Then, $\dim_{\mathbb{R}}(X)$ is divisible by 4.*

On a holomorphic symplectic manifold, there are particularly nice holomorphic coordinates, called **Darboux coordinates** $(p_1, \dots, p_n, q_1, \dots, q_n)$, for which

$$\Omega = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i.$$

This is the analogue of a symplectic structure, but on a complex manifold.

Now we know enough to define a hyperkähler manifold.

**Definition 1.25.** Let $(X, g)$ be a Riemannian manifold and $I$, $J$, and $K$ be three complex structures on $X$. Then, $(X, g, I, J, K)$ is a **hyperkähler manifold** if

- $IJ = K$ in $\mathrm{End}(TM)$, and
- $(X, g, I)$, $(X, g, J)$, and $(X, g, K)$ are all Kähler manifolds.

The corresponding Kähler forms are denoted $\omega_I$, $\omega_J$, and $\omega_K$. Sometimes, $I_1$ is used for $I$, $I_2$ for $J$, and $I_3$ for $K$, and $\omega_i$ for $\omega_{I_i}$.

The relation $IJ = K$ is equivalent to imposing the relations of the quaternions onto $I$, $J$, and $K$, i.e. $IJ = K$, $JK = I$, and $KI = J$; and $JI = -K$, $KJ = -I$, and $IK = -J$.

If you let $\Omega_1 \coloneqq \omega_2 + i\omega_3$, then $(X, I, \Omega_1)$ is a holomorphic symplectic manifold, and hence $\dim_{\mathbb{R}} X$ must be divisible by 4.

## 2. Twistor space: 6/13/17

Last time, we defined a plethora of geometric structures on maps. For example, the tangent and cotangent bundles of a manifold are abstractly isomorphic, and different structures define different isomorphisms: a metric defines a symmetric form $g$ such that $v \mapsto g(v, -)$ is an isomorphism $TX \to T^*X$. A symplectic structure defines a skew-symmetric form $\omega$ such that $v \mapsto \omega(v, -)$ is an isomorphism.

A complex structure defines a skew-symmetric involution $I\colon TX \to TX$, and a Kähler structure says this is compatible with $g$ and $\omega$. Heuristically, it means "$\omega = g \circ I$," but what this actually means is that

$$\omega(v, w) = g(Iv, w). \tag{2.1}$$

A hyperkähler manifold is a Kähler manifold in three ways: it comes with a Riemannian metric $g$ and *three* almost complex structures $I$, $J$, and $K$ such that $IJ = K$ and $(X, g, I)$, $(X, g, J)$, and $(X, g, K)$ are all Kähler. Let $\omega_I$ be the symplectic form associated to $(g, I)$, and similarly for $\omega_J$ and $\omega_K$. Using (2.1), one can derive some relations between them, e.g.

$$\begin{aligned}\omega_K(Iv, -) &= g(KIv, -)\\ &= g(Jv, -)\\ &= \omega_J(v, -).\end{aligned}$$

Heuristically, "$\omega_K \circ I = \omega_J$," and similarly "$g = -\omega_I \circ I = -\omega_I \circ \omega_K^{-1} \circ \omega_K$." Moreover, all cyclic permutations of $I$, $J$, and $K$ work. The quotes only mean we're not applying the symplectic forms to two arguments, but considering the map $\omega(v, -)$, which is an isomorphism.

**Proposition 2.2.** *Let $X$ be a manifold with three symplectic structures $\omega_I$, $\omega_J$, and $\omega_K$ such that after plugging in some tangent vector,*

$$g \coloneqq -\omega_I \omega_K^{-1} \omega_J = -\omega_J \omega_I^{-1} \omega_K = -\omega_K \omega_J^{-1} \omega_I.$$

*If $g$ is positive definite, then $(g, \omega_I, \omega_J, \omega_K)$ determine a hyperkähler structure on $X$.*

Not every symplectic manifold is hyperkähler, e.g. any Riemann surface (because every hyperkähler manifold has dimension a multiple of 4).

We said that a hyperkähler manifold has three complex structures, but in fact we get infinitely many more for free.

**Proposition 2.3.** *Let $(X, g, I, J, K)$ be a hyperkähler manifold and $\mathbf{s} \in S^2 \subset \mathbb{R}^3$. Let*

$$I_{\mathbf{s}} := s_1 I + s_2 J + s_3 K$$
$$\omega_{\mathbf{s}} := s_1 \omega_I + s_2 \omega_J + s_3 \omega_K.$$

*Then, $I_{\mathbf{s}}^2 = -1$ and $(X, g, I_{\mathbf{s}})$ is Kähler.*

So we have an $S^2 = \mathbb{CP}^1$ worth of complex/Kähler structures. In complex geometry, we'd use $\mathbb{CP}^1$, because we care about the complex structure on the space of complex structures.

You can think of $S^2$ as the unit imaginary quaternions, as $\mathbb{R}^3 = \mathrm{Im}(\mathbb{H})$. This can be used to cleverly define the (almost) complex structure on $\mathbb{CP}^1$: we want an involution $I_x \colon T_x\mathbb{CP}^1 \to T_x\mathbb{CP}^1$. At $i$, this is just multiplication by $i$: $ak + bk \mapsto i(aj + bk) = ak - bj$. That is, the almost complex structure at $x = ai + bj + ck$ is multiplying by $ai + bj + ck$, which is a rotation by $180^\circ$ in $T_x\mathbb{CP}^1$. You can show this structure is integrable.

You can do the same thing with the **octonions** $\mathbb{O}$, an eight-dimensional $\mathbb{R}$-algebra whose multiplication is neither commutative nor associative, but every nonzero element is invertible. But it has just enough structure to be useful in topology and geometry. As with $\mathbb{C}$ and $\mathbb{H}$, the **imaginary octonions** $\mathrm{Im}(\mathbb{O})$ is the orthogonal complement to the subalgebra $\mathbb{R} \hookrightarrow \mathbb{O}$; $\mathrm{Im}(\mathbb{O}) \cong \mathbb{R}^7$ as vector spaces, and hence can define an almost complex structure on $S^6 \subset \mathrm{Im}(\mathbb{O})$. But this is *not* integrable! This has been known for a long time — but it's still not know whether $S^6$ admits a different complex structure that's integrable!

The state-of-the-art results on complex structures:

- In complex dimension 1, every almost complex structure is integrable.
- In complex dimension 2, there are examples of non-integrable complex structures, and even manifolds which admit almost complex structures but not complex structures. The examples are kind of crazy, and there are various methods to prove this, using the Hodge diamond and Pontrjagin classes.
- In complex dimensions at least 3, there are no known examples of manifolds which admit almost complex structures but no complex structures.

The twistor space is a way to encode all of the information about a hyperkähler manifold, in that you can completely recover the hyperkähler manifold from geometric information on its twistor space.

**Definition 2.4.** Let $X$ be a hyperkähler manifold. Its **twistor space** is $Z := X \times S^2$ as a manifold, with the almost complex structure $I(x, \mathbf{s}) = I_{\mathbf{s}}(x) \oplus I_{S^2}$, where we use the usual complex structure $I_{S^2}$ on $S^2$ as above.

The idea is to make good use of the $S^2$-worth of complex structures we found by fibering all of them over $S^2$.

**Proposition 2.5.**

(1) *$I$ is integrable, and so $Z$ is a complex manifold.*
(2) *The projection map $\pi \colon Z \twoheadrightarrow S^2$ is holomorphic.*
(3) *$Z$ has a **fiberwise holomorphic symplectic structure**, i.e. the fibers of $\pi$ are holomorphic symplectic manifolds.*

The fiberwise holomorphic symplectic structure is dictated by an

$$\Omega \in \Lambda^2(T^{1,0}_{\mathrm{vert}} Z)^* \otimes \pi^*\mathscr{O}(2).$$

That is:

- $T_{\mathrm{vert}}Z$ is the **vertical tangent bundle** of $\pi \colon Z \to S^2$, i.e. at each $z \in Z$, the preimage of $0 \in T_{\pi(z)}S^2$. Then, apply the $(-)^{1,0}$ construction as usual.
- $\mathscr{O}(2)$ is the holomorphic line bundle of degree 2 over $\mathbb{CP}^1$, and $\pi^*$ pulls it back to $Z$.

It's possible to understand this fiberwise holomorphic symplectic structure more explicitly. Let $\Omega_1 \coloneqq \omega_2 + i\omega_3$, and similarly for $\Omega_2$ and $\Omega_3$. Then, above a $\zeta \in \mathbb{CP}^1$,

$$\begin{aligned}\Omega(\zeta) &= \frac{1}{2\zeta}\Omega_1 - i\omega_2 + \frac{\zeta}{2}\overline{\Omega}_1\\ &= \frac{\omega_2 + i\omega_3}{2\zeta} - i\omega_1 + \frac{\zeta}{2}(\omega_2 - i\omega_3).\end{aligned}$$

Here, $1/\zeta$ means on the Riemann sphere: $1/0 = \infty$, $1/\infty = 0$, and everything else is as usual on $\mathbb{C}$.

Thus, we can compute at specific $\zeta$:

$$\begin{aligned}\Omega(1) &= \omega_2 - i\omega_1 = -i\Omega_3\\ \Omega(i) &= -\Omega_2\\ \Omega(0) &= \Omega_1.\end{aligned}$$

**Proposition 2.6.** *$Z$ has a **real structure**, i.e. an antiholomorphic involution $\rho\colon Z \to Z$, which is the map $(x, \mathbf{s}) \mapsto (x, -\mathbf{s})$. Hence it covers the antipodal map. It also satisfies $\rho^*\Omega = \overline{\Omega}$.*

*Remark.* There's a theorem (with a long involved proof) that a manifold $Z$ satisfying the properties in Proposition 2.5 and Proposition 2.6 is in some sense a twistor space: one can recover a **pseudohyperkähler manifold** $X$ (i.e. a hyperkähler manifold except that the metric might not be positive definite), which is the space of all **real holomorphic sections**, i.e. those fixed under $\rho$, with normal bundle $\mathscr{O}(1)^{\oplus 2n}$. Moreover, if $Z$ is the twistor space of a hyperkähler space $X'$, then $X' \subset X$, and in many cases $X' = X$. The point is that the twistor space all but allows you to reconstruct the ordinary hyperkähler manifold, and is the same information packaged in a different way. ◀

Now let's do some examples!

**Example 2.7.** $\mathbb{R}^4 \cong \mathbb{H}$ is hyperkähler: $T_p\mathbb{H} \cong \mathbb{H}$ naturally. The norm $\|q\| \coloneqq q\overline{q}$ (if $q = a + bi + cj + dk$, then $\|q\| = a^2 + b^2 + c^2 + d^2$) defines a Riemannian metric on $\mathbb{H}$, and left-multiplication by $i$, $j$, and $k$ defines the complex structures $I$, $J$, and $K$ respectively, and they evidently preserve the norm, so everything works out and $\mathbb{H}$ is hyperkähler.

In coordinates $(x_0, x_1, x_2, x_3)$ for the directions $(1, i, j, k)$ respectively,

$$\begin{aligned}\omega_I &= \mathrm{d}x_0 \wedge \mathrm{d}x_1 + \mathrm{d}x_2 \wedge \mathrm{d}x_3\\ \omega_J &= \mathrm{d}x_0 \wedge \mathrm{d}x_2 + \mathrm{d}x_3 \wedge \mathrm{d}x_1\\ \omega_K &= \mathrm{d}x_0 \wedge \mathrm{d}x_3 + \mathrm{d}x_1 \wedge \mathrm{d}x_2.\end{aligned}$$

More invariantly,

$$\omega_i = \mathrm{d}x_0 \wedge \mathrm{d}x_i + \star\mathrm{d}x_i,$$

where $\star$ is the Hodge star.

We can also write down the holomorphic symplectic forms. Let $w_1 = x_0 + ix_1$ and $z_1 = x_2 + ix_3$. Then,

$$\Omega_I = \omega_J + i\omega_K = \mathrm{d}w_1 \wedge \mathrm{d}z_1.$$

Above each $\zeta \in S^2$, $I_\zeta$ defines an isomorphsim $\mathbb{H} \cong \mathbb{C}^2$. The isomorphism depends on $\zeta$, though: it depends on the orthogonal decomposition $\mathbb{C}^2 \cong \mathbb{C} \oplus \mathbb{C}$, which is determined by a real direction (fixed) and an imaginary direction (which is the $\mathbb{CP}^1$ of complex structures we get).

This naturally generalizes to any finite-dimensional quaternionic vector space (which is isomorphic to $\mathbb{H}^n$). ◀

*Remark.* The quaternions of unit norm form an $S^3 \subset \mathbb{H}$, which has a Lie group structure given by quaternion multiplication. In particular, it's $\mathrm{SU}_2$ (also $\mathrm{Spin}_3$ and $\mathrm{Sp}_1$). Thus, ther'es an $\mathrm{SU}_2 \times \mathrm{SU}_2$-action on $\mathbb{H}$ (from the left and right, respectively) in which

$$(q_1, q_2) \cdot x = q_1 x q_2^{-1}.$$

This preserves the norm, hence defines a map $\psi\colon \mathrm{SU}_2 \times \mathrm{SU}_2 \to \mathrm{SO}_4$. It's easy to check $\psi$ is onto, and $\ker(\psi) \cong \mathbb{Z}/2$; since $\mathrm{SU}_2 \times \mathrm{SU}_2$ is connected and $\pi_1(\mathrm{SO}_4) \cong \mathbb{Z}/2$, this implies $\mathrm{SU}_2 \times \mathrm{SU}_2 \cong \mathrm{Spin}_4$, the connected double cover of $\mathrm{SO}_4$. There are other ways to show this, but using the quaternions is nice.

Let $\Gamma$ be a finite subgroup of $\mathrm{SU}_2$, and let it act on $\mathbb{H}$ through the right action of $\mathrm{SU}_2$. Suppose this action is free (except at the origin). The quotient space $\mathbb{H}/\Gamma$ is called a **hyperkähler orbifold** (the action is not free at the origin, and hence this is not a manifold). If you remove the origin, the quotient $(\mathbb{H} \setminus 0)/\Gamma$ is a hyperkähler manifold. A blowup at the origin is also a way to address this issue; in this case, it's called a **resolution of a hyperkähler singularity**. ◂

Given a $T \in \mathrm{SU}_2$, one can pull back the complex structure $I_{\mathbf{s}}$ along the action of $T$, through the left $\mathrm{SU}_2$-action. Let $T$ act on $\mathrm{SU}_2$ by $T \cdot \mathbf{s} = T^{-1}\mathbf{s}T$. Then, $T^*I_{\mathbf{s}} = I_{T\cdot\mathbf{s}}$.

We can also compactify one of the directions in Example 2.7.

**Example 2.8.** Consider $\mathbb{R}^3 \times S^1 \cong \mathbb{R}^4/2\pi\mathbb{Z}$. Let's start with a complex structure given by

$$\Omega_1 = -i\frac{\mathrm{d}\chi_1}{\chi_1} \wedge \mathrm{d}z_1,$$

where $\chi_1 = \exp(i(x_0 + ix_1))$ and $z_1 = x_2 + ix_3$. Using this, as complex manifolds, $(\mathbb{R}^3 \times S^1, I_1) \cong \mathbb{C} \times \mathbb{C}^\times$: a complex plane cross a cylinder. If you apply this to more general $\zeta$, it changes: there are two distinguished points, complex conjugates of each other, where you get different complex structures on the same smooth manifolds.

You can run the rest of the story and get a hyperkähler structure on $\mathbb{R}^3 \times S^1$. ◂

**Example 2.9** (Gibbons-Hawking)**.** Let $U \subset \mathbb{R}^3$ be an open set and $E \to \mathbb{R}^3$ be a principal $\mathrm{U}_1$-bundle.[4] Then, there's a procedure to endow a hyperkähler structure on $E$ given a harmonic function $V\colon U \to \mathbb{R}_{>0}$, called the **potential function**, satisfying some techical conditions (e.g. topological obstructions). The space $E$ is called a **Gibbons-Hawking space**, and is a hyperkähler manifold with a $\mathrm{U}_1$-action.

You'd want to pick $U = \mathbb{R}^3$, but there's no positive harmonic function on all of $\mathbb{R}^3$, so intead we'll choose $\mathbb{R}^3 \setminus 0$ and $V(x) = 1/4\pi|x|$. If you run the argument, you get $E = \mathbb{R}^4 \setminus 0$ with the usual metric on $\mathbb{R}^4$. This extends over 0 as a hyperkähler manifold $\overline{E}$, but not as a principal $\mathrm{U}_1$-bundle: the point at 0 is not free. We say that the fiber over 0 **degenerates** or 0 is a **bad point**; the quotient $\overline{E}/\mathrm{U}_1 \cong \mathbb{R}^3$ though. ◂

## 3. More examples of hyperkähler manifolds: 6/14/17

Recall that a Kähler manifold is a manifold $X$ with a Riemannian metric $g$ and a complex structure $I$ such that one of several equivalent conditions is true:

- the complex structure is covariantly constant (i.e. $\nabla I = 0$),
- the symplectic form induced by $I$ and $g$ is covariantly constant, or
- the holonomy induced by $g$ is a subgroup of $\mathrm{U}_n$.

Calabi-Yau manifolds are particular examplex of Kähler manifolds.

**Definition 3.1.** A **Calabi-Yau manifold** is a compact Riemannian manifold whose holonomy is a subgroup of $\mathrm{SU}_n$.

These are meant to be generalizations of $K3$ surfaces.

Recall that a hyperkähler manifold is the data of $(X, g, I, J, K)$, the quaternionic analogue of a Kähler manifold: $(x, g)$ is Riemannian, and each of $I$, $J$, and $K$ is a complex structure such that $(X, g, I)$, $(X, g, J)$, and $(X, g, K)$ are all Kähler, and $I$, $J$, and $K$ satisfy the quaternionic relations. Some things true about $\mathbb{C}$ for Kähler manifolds are true for $\mathbb{H}$ and hyperkähler manifolds: for example, the tangent space to any hyperkähler manifold is naturally a quaternionic vector space.

Because $\mathrm{Sp}_k \subset \mathrm{SU}_{2k} \subset \mathrm{U}_{2k}$ for any $k$, then a compact hyperkähler manifold is automatically Calabi-Yau, and a Calabi-Yau manifold is automatically Kähler. Thanks to a low-dimensional coincidence, $\mathrm{Sp}_1 \cong \mathrm{SU}_2$, so in has real dimension 4, a manifold is compact hyperkähler iff it's Calabi-Yau.

**Example 3.2.** In Example 2.7, we discussed $\mathbb{H}$ as a hyperkähler manifold. More generally, if $\Gamma \subset \mathrm{SU}_2$ is a finite subgroup, $(\mathbb{H} \setminus 0)/\Gamma$ is hyperkähler. There's an ADE-type classification of finite subgroups of $\mathrm{SU}_2$ using Dynkin diagrams, and the $A$-type subgroups are

$$\Gamma_k = \left\{ \begin{pmatrix} e^{2\pi in/k} & 0 \\ 0 & e^{-2\pi in/k} \end{pmatrix} \right\} \cong \mathbb{Z}/k. \tag{3.3}$$

[4]More generally, we can assume $\mathrm{U}_1$ acts on $E$, which admits a projection to $\mathbb{R}^3$ trivializing the action; the action need not be free.

This would be a quotient by a rotation. ◀

The first dimension in which hyperkähler manifolds can arise is dimension 4, and there's a complete classification. A **K3 surface** is a compact, simply connected smooth complex surface (i.e. compact dimension 2) with a trivial canonical bundle.

**Theorem 3.4** (Kodaira)**.** *Every Calabi-Yau manifold of real dimension 4 is either a $K3$ surface or a compact torus $(S^1)^4$. Equivalently, every compact hyperkähler manifold of real dimension 4 is either a $K3$ surface or a compact torus.*

There are plenty of other examples:

- If $X$ is a $K3$ surface, its Hilbert scheme of points $\mathrm{Hilb}(X)$ is also hyperkähler.
- **ALE spaces** (asymptotically locally Euclidean) spaces are noncompact, complete hyperkähler spaces, whose metric, symplectic, and complex structures all are asymptotic to $(\mathbb{H} \setminus 0)/\Gamma$. These arise in physics as the moduli space for gravitational instantons, and physics motivations revived the study of hyperkähler geometry.
- Certain gauge-theoretic PDEs have moduli spaces of solutions that are hyperkähler, including the moduli space of instantons, monopole moduli spaces, the solutions to Hitchin's self-duality equations on a Riemann surface, the solutions to Nahm's equations, and so on. There's a general physics principle that in any four-dimensional $\mathcal{N} = 2$ supersymmetric field theory, the moduli of vacua is hyperkähler, and several of the above examples arise in this way.
- **Nakajima quiver varieties** are varieties parameterizing representations of quivers. Here a **quiver** $Q$ is a directed graph regarded as a category, and **quiver representation** is a functor $Q \to \mathsf{Vect}$. Quiver varieties often arise as quotients like other moduli spaces.

Now we'll say some more about the Gibbons-Hawking construction of Example 2.9: given an open subset $U \subset \mathbb{R}^3$, a principal $\mathrm{U}_1$-bundle $X \to U$, and a harmonic function $V \colon U \to \mathbb{R}_{>0}$, plus some technical conditions, there's a ay to place a hyperkähler structure on $X$.

For example, we can take $U = \mathbb{R}^3 \setminus 0$ and $V = 1/4\pi\|x\|$: on $\mathbb{R}^3$, $\Delta V = -\delta(x)$, so when we remove the origin, $V$ is harmonic. If we restrict to the unit sphere $S^2 \subset U$, the principal $\mathrm{U}_1$-bundle structure we get is the Hopf fibration $S^1 \to S^3 \to S^2$.

If you try to extend over the origin, you'll get $\overline{X} = H \supset X$, but the projection to $\mathbb{R}^3$ has a degenerate fiber over 0.

There are a couple of variations of this:

(1) If you instead take $V = k/4\pi\|x\|$, you instead get $X = (\mathbb{H} \setminus 0)/\Gamma_k$, where $\Gamma_k$ is as in (3.3).

(2) Let $x_1, \dots, x_k \subset \mathbb{R}^3$ and take $U = \mathbb{R}^3 \setminus \{x_1, \dots, x_k\}$ with potential function

$$V(x) = \sum_{i=1}^{k} \frac{1}{4\pi\|x - x_i\|}.$$

Then, you get a hyperkähler manifold $X \to U$, which you can extend to a hyperkähler manifold $\overline{X}$ over all of $\mathbb{R}^3$, but the fibers over each $x_i$ are degenerate: the fibers are $\mathrm{U}_1$ everywhere except the $x_i$, where they're points.

When $n = 2$, this last example is called an **Eguchi-Henson space**; it was also studied by Calabi. The fiber over the line from $x_1$ to $x_2$ is an $S^2$. In the complex structure $I(\zeta)$ for any $\zeta = |x_1 x_2|$, this $S^2$ is a complex submanifold, and in particular, by Corollary 1.12, is Kähler. There's a biholomorphism between $\overline{X}$ and $T * \mathbb{CP}^1$.

In any other complex structure $I(\zeta')$, however, $\overline{X}$ has no compact complex submanifolds, and therefore is not $T^*\mathbb{CP}^1$.

For $n > 2$, you can play the same game: suppose no three of the $x_i$ are colinear. Then, the line between $x_i$ and $x_j$ has the same property: its fiber is an $S^2$, and the corresponding complex structure on $\overline{X}$ will be biholomophric to $T^*\mathbb{CP}^1$ for $\zeta = |x_i x_j|$, and will be very different otherwise. This is called the **multi-Eguchi-Henson space**.

The degenerate case is where some points are colinear. The worst possibility is where all $k$ points lie on a line, in which case the fiber over this line is a wedge sum of spheres, which have intersection numbr 1 if they're adjacent and 0 otherwise. The line segment looks like a Dynkin diagram of type $A_{k-1}$, and this is a useful perspective. This also has to do with a minimal resolution of a **da Val singularity** of type $A_{k-1}$. So this degenerate case is not as great as multi-Eguchi-Henson space, but from far away, it looks very similar, and asymptotically, it just looks like $(\mathbb{H} \setminus 0)/\Gamma_k$. Thus, it's an example of an ALE space.

**Example 3.5.** The **Taub-NUT space** is another example coming from physics via the Gibbons-Hawking construction. Take

$$V(x) = 1 + \frac{1}{4\pi\|x\|}$$

on $U = \mathbb{R}^3 \setminus 0$. Thus, the potential doesn't vanish at $\infty$, and one can show that, asymptotically, this converges to $\mathbb{R}^3 \times S^1$ rather than $\mathbb{H}$. However, in any complex structure, the total space is biholomorphic to $\mathbb{C}^2$. The metric is asymptotically

$$g \approx V\,|\mathrm{d}x|^2 + \frac{1}{V}\,|\mathrm{d}\Theta|^2.$$

◀

**Example 3.6.** This stuff is very different for different structures, so let's take another look at a simple example, $\mathbb{R}^2 \times T^2$.

The complex structure we'll put on $\mathbb{R}^2$ is just the usual identification with $\mathbb{C}$, but there's a bunch of complex structures on $T^2$, which form the fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$. Explicitly, one takes the quotient of $\mathbb{C}$ by a lattice. Let $\mathcal{U}$ denote the upper half-plane and $\tau \in \mathcal{U}$. Let $\Lambda_\tau := (2\pi/\operatorname{Im}\tau)\mathbb{Z} \oplus \mathbb{Z}\tau$, which is a lattice in $\mathbb{C}$. Then, $T^2_\tau := \mathbb{C}/\Lambda_\tau$ is a complex structure on the torus with volume $(2\pi)^2$.

Hence $\mathbb{R}^2 \times T^2$, with this complex structure specified by $\tau$, is a quotient of $\mathbb{H}$ by a free action, and inherits that hyperkähler structure. Explicitly:

- The complex structure specified by $I$ is biholomorphic to $\mathbb{C} \times T^2_\tau$.
- The complex structure specified by $-I$ is biholomorphic to $\mathbb{C} \times T^2_{-\overline{\tau}}$. This is because the complex structure on $\mathbb{H}$ is just $\mathbb{C}^2$, and $i$ sends $(1, i, j, k)$ to $(1, -i, j, -k)$, so we're taking the quotient after flipping $\tau$ across the $y$-axis, producing $-\overline{\tau}$.

What about the other $\zeta \in \mathbb{C}^\times$? Define

$$\begin{aligned}\chi_A &= \exp\left(\frac{z_1}{2\zeta} - ix_1 + \frac{\overline{z}_1\zeta}{2}\right)\\ \chi_B &= \exp\left(\frac{iz_1}{2\zeta} + ix_0 - \frac{i\overline{z}_1\zeta}{2}\right),\end{aligned}$$

where $z_1 = x_2 + ix_3$, and $x_0$ and $x_1$ are the real and imaginary coordinates on $\mathbb{C}$. Then, $(\chi_A, \chi_B)$ gives you holomorphic coordinates on $\mathbb{R}^2 \times T^2$. This is again a twistor-like construction. At $\zeta = 1$, for example, $\chi_A = \exp(x_2 - ix_1)$ and $\chi_B = \exp(-x_3 + ix_0)$.

The rest of the data:

$$\begin{aligned}\omega_i &= \mathrm{d}x_0 \wedge x_i + \star(\mathrm{d}x_i)\\ I_j &= x_0 + ix_j, x_{j+1 \bmod 3} + jx_{j+2 \bmod 3}.\end{aligned} \tag{3.7}$$

That is,

$$\begin{aligned}I_1 = I &= x_0 + ix_1, x_2 + ix_3\\ I_2 = J &= x_0 + ix_2, x_3 + ix_1\\ I_3 = K &= x_0 + ix_3, x_1 + ix_2.\end{aligned} \tag{3.8}$$

The Hodge star in (3.7) is taken in $\mathbb{R}^3$.

At $\zeta = -i$, we get

$$\chi_A = \exp(-x_3 - ix_1) = \exp(-(x_3 + ix_1))$$
$$\chi_B * = \exp(-x_2 + ix_0) = \exp(i(x_0 + ix_2)),$$

whicn by comparing with (3.8), is explicitly a set of holomorphic coordinates for the complex structure $I_2$.

The holomorphic symplectic form is

$$\Omega(\zeta) = \frac{1}{2\zeta}\Omega_1 - i\omega_1 + \frac{\zeta}{2}\overline{\Omega}_1 = \frac{\omega_2 + i\omega_3}{2\zeta} - i\omega_1 + \frac{\zeta}{2}(\omega_2 - i\omega_3),$$

and therefore

$$\frac{\mathrm{d}\chi_A \wedge \mathrm{d}\chi_B}{\chi_A \chi_B} = -i\Omega(\zeta).$$

Therefore $\mathbb{R}^2 \times T^2$ with the complex structure $I_\zeta$ is biholomorphic to $\mathbb{C}^\times \times \mathbb{C}^\times$. ◀

The point of these examples, which quickly get more notationally complicated, is that the $\mathbb{CP}^1$ worth of complex structures means that any identification of your hyperkähler manifold with some other complex manifold depends finely on your choice of complex structure.

## 4. The moduli space of Higgs bundles: 6/15/17

Today, we're going to discuss a particularly important example, the moduli space of Higgs bundles, and some preliminaries about complex vector bundles on a Riemann surface. Throughout today's and tomorrow's lectures, fix a closed Riemann surface $C$ and a Kähler form $\omega_C$ such that $\int_C \omega_C = 1$.

Given a complex vector bundle $E \to C$, we can extract a few invariants.

- The **rank** is the dimension of the fibers of $E$ (if $C$ is disconnected, we assume all fibers have the same dimension).
- The **degree** $\deg(E) \in \mathbb{Z}$ can be defined in several equivalent ways:
  - In algebraic topology, one can define
  $$\deg E := \langle c_1(E), [C] \rangle,$$
  the pairing of the first Chern class $c_1(E) \in H^2(C)$ and the fundamental class $[C] \in H_0(M)$. This is not terribly easy to compute, but makes evident that the degree is an integer.
  - The **Chern-Weil formula**: fix a connection $D$ on $E$ and let $F_D \in \Omega^2(\operatorname{End} E)$ be its curvature. Then,
  $$\deg E := \frac{i}{2\pi} \int_C \operatorname{tr}(F_D).$$
  This is more computable.
  - If $E$ is a holomorphic vector bundle, let $s$ be a meromorphic section; then, $\deg(E)$ is the number of zeros of $s$ minus the number of poles of $s$, where both zeros and poles are counted with multiplicity.

**Example 4.1.** When $C = \mathbb{CP}^1$, there is a line bundle $\mathscr{O}(n) \to \mathbb{CP}^1$ with degree $n$ for all $n \in \mathbb{Z}$. It's defined via transition functions: using stereographic projection, $\mathbb{CP}^1$ is covered by two copies of $\mathbb{C}$ (the complements of 0 and $\infty$, respectively), and on their intersection, the transition function for $\mathscr{O}(n)$ is $z \mapsto z^{-n}$.

Alternatively, one can define $\mathscr{O}(0)$ to be the trivial bundle, $\mathscr{O}(1)$ to be the tangent bundle, and for $n > 0$, $\mathscr{O}(n) := \mathscr{O}(1)^{\otimes n}$. For $n < 0$, let $\mathscr{O}(n) := \mathscr{O}(-n)^*$. In particular, $\mathscr{O}(-1)$ is the tautological bundle. ◀

**Proposition 4.2.** *On $C$, all complex vector bundles of rank $k$ and degree $d$ are isomorphic.*

The hypothesis that we're over a Riemann surface is very important!

One reinterpretation of this result is that the moduli space of rank $k$, degree-$d$ vector bundles is a point. This is not very interesting. So to get a cooler moduli space, we'll endow our vector bundles with extra structure. Here are some ways to do this.

(1) What if we consider the moduli space $\mathcal{N}_{k,d}$ of *holomorphic* vector bundles of rank $k$ and degree $d$? This is a complex variety, and its complex structure depends on that of $C$. It's not smooth, however.

(2) We could also consider the moduli space $\mathcal{N}'_{k,d}$ of complex, degree-$d$, rank $k$ vector bundles with unitary connections $D$ plus a **harmonicity condition**: the curvature $F_D$ must satisfy
$$F_D = -2\pi i \frac{d}{k} \omega_C.$$
For example, in degree 0, we're asking for flat connections. This is equivalent to considering the moduli space of representations $\pi_1(C) \to \mathrm{U}_k$, the representation arises as the holonomy, and the other direction involves passing to the universal cover. There are analogous constructions in higher degree.

As a moduli space of representations, it's evident that it doesn't depend on the complex structure. The moduli space is symplectic, but not obviously complex. If $k$ and $d$ are coprime (or $d = 0$), however, it turns out $\mathcal{N}_{k,d} \cong \mathcal{N}'_{k,d}$, and the complex and symplectic structures are compatible, meaning $\mathcal{N}_{k,d}$ is Kähler! This is not obvious: there are subtle analytic details.

For various reasons, it's more interesting to consider moduli spaces of vector bundles satisfying a stability condition.

**Definition 4.3.** Let $E \to C$ be a holomorphic vector bundle.

- Its **slope** is $\mu(E) \coloneqq \deg(E)/\operatorname{rank}(E) \in \mathbb{Q}$.
- $E$ is **stable** if for any holomorphic subbundle $E' \subset E$, $\mu(E') < \mu(E)$.
- $E$ is **semistable** is for any holomorphic subbundle $E' \subset E$, $\mu(E') \le \mu(E)$.
- $E$ is **polystable** if it's a sum of stable bundles that all have the same slope.

In particular, stable bundles are polystable, and polystable bundles are semistable.

*Remark.* If $k$ and $d$ are coprime, the notions of stability, semistability, and polystability coincide. ◂

Now we'll proceed to the moduli space of Higgs bundles. In the following, intuition may be easier for $d = 0$.

**Definition 4.4.** A **Higgs bundle** is a triple $(E, \overline{\partial}_E, \varphi)$ where

- $E$ is a complex vector bundle of rank $k \ge 2$ and degree $d$,
- $\overline{\partial}_E$ is a holomorphic structure on $E$, and
- $\varphi \in H^0(C, \operatorname{End} E \otimes K_C)$ is a **Higgs field**; here, $K_C$ is the canonical bundle.

In local coordinates, $\varphi$ is specified by a $k \times k$ matrix $A$ whose entries are holomorphic functions; then, $\varphi = A\,\mathrm{d}z$. One can also pull $\mathrm{d}z$ back in, and obtain a matrix of holomorphic one-forms.

**Definition 4.5.** A **Higgs subbundle** of $(E, \overline{\partial}_E, \varphi)$ is a holomorphic subbundle $E' \subset E$ such that $\varphi(E') \subset E' \otimes K_C$.

The goal will be to understand the moduli space of stable (or polystable, or semistable) Higgs bundles $\mathcal{M}_{k,d}$, modulo equivalence. Since a Higgs bundle is additional data on a holomorphic bundle, we expect this to project down to $\mathcal{N}_{k,d}$ via a forgetful map. The fibers of this map should be $H^0(\operatorname{End}(E) \otimes K_C)$, but is this map a fiber bundle? The space of Higgs fields on a given complex bundle $(E, \overline{\partial}_D)$ is the same as the space of harmonic forms $\Phi_Z \in \mathcal{H}^{1,0}_D(\operatorname{End} E)$, which has something to do with Hodge theory. This tells you that when you vary the complex structure, the fiber varies in an interesting way.

Moreover, $T_D\mathcal{N}_{k,d}$ will be the space of antiholomorphic forms, i.e. elements of $\mathcal{H}^{0,1}_D(\operatorname{End} E)$. There's a pairing between $\mathcal{H}^{0,1}_D$ and $\mathcal{H}^{1,0}_D$:

$$\dot{A}_{\overline{z}}, \Phi_Z \longmapsto \int_C \operatorname{tr}(\dot{A}_{\overline{z}} \wedge \Phi_z).$$

Hence they're isomorphic, and therefore the moduli space of Higgs bundles is isomorphic to $T^*\mathcal{N}_{k,d}$.

In particular, it has a lot of structure: it's the cotangent space to a complex manifold, so it's already a holomorphic symplectic manifold, and therefore has dimension divisible by 4. Moreover, if $k$ and $d$ are coprime, then $\mathcal{N}_{k,d}$ is Kähler, so the moduli space of Higgs bundles is hyperkähler.

*Remark.* Using the moment map, one can show that the symplectic quotient of a symplectic manifold is still symplectic. There's an analogue for hyperkähler manifolds: given a hyperkähler manifold $X$ and a compact Lie group $G$, one can define a **hyperkähler monent map**, which is a triple of moment maps $\mu$, and form the **hyperkähler quotient** $X /\!/\!/\!/_\mu G$, which has dimension $\dim X - 4\dim G$. For example, Gibbons-Hawking spaces (Example 2.9) are acted on by $\mathrm{U}_1$, and are 4-dimensional, so their hyperkähler quotients are points. But you can also construct the moduli space of Higgs bundles as a hyperkähler quotient, and this approach is interesting. ◂

There's another way to realize $\mathcal{M}_{k,d}$ as the total space of a (more or less) fibration.

**Definition 4.6.** The **Hitchin base** is

$$B = B_k(C) \coloneqq \bigoplus_{i=0}^{k} H^0(C, K_c^{\otimes i}).$$

The **Hitchin fibration** is the map

$$\rho\colon \mathcal{M}_{k,d} \longrightarrow B_k(C)$$

which sends

$$(\overline{\partial}_E, \varphi) \longmapsto \sum_{i=1}^{k} \operatorname{tr}(\varphi^i).$$

The fiber is generically a complex, compact torus, but it can degenerate.

This map is closely related to the characteristic polynomial: if $\lambda$ is the Liouville 1-form for $T^*C$ and

$$\det(\lambda - \varphi) = \lambda^k + \sum_{i=1}^{k} \phi_i \lambda^{k-i},$$

then

$$\rho([D, \varphi]) = (\phi_1, \phi_2, \dots, \phi_k).$$

One way to keep track of this data is to use spectral curves.

**Definition 4.7.** The **spectral curve** for $\boldsymbol{\phi}$ is

$$\Sigma_{\boldsymbol{\phi}} := \left\{ \lambda^k - \sum_{i=1}^{k} \phi_i \lambda^{k-i} = 0 \right\} \subset T^*C.$$

The name is because, well, it's a curve which tracks the spectrum of $\boldsymbol{\phi}$. There's a $k$-fold ramified cover $\Sigma_{\boldsymbol{\phi}} \to C$.

**Proposition 4.8.** *The Hitchin fibration $\rho\colon \mathcal{M}_{k,d} \to B$ is holomorphic, where $\mathcal{M}$ has the complex structure $I$.*

Next: where do the fibers of $\rho$ degenerate? The total space is smooth, so we do actually have to look at $\rho$. Consider the **discriminant**

$$\Delta := \prod_{i>j} (\lambda_i - \lambda_j)^2.$$

The indices $i$ and $j$ are only locally defined, since the covering map for $\Sigma_{\boldsymbol{\phi}}$ have monodromy in general.

**Example 4.9.** Let $k = 2$. Then, the characteristic equation is $\lambda^2 + \phi_1 \lambda + \phi_2 = 0$. If this factors as $(\lambda - \lambda_1)(\lambda - \lambda_2)$, then

$$\begin{aligned} \lambda_1 &= -(\lambda_1 + \lambda_2) \\ \lambda_2 &= \lambda_1 \lambda_2. \end{aligned}$$

Hence

$$\begin{aligned} (\lambda_1 - \lambda_2)^2 &= (\lambda_1 + \lambda_2)^2 - 4\lambda_1\lambda_2 \\ &= \phi_1^2 - 4\phi_2, \end{aligned}$$

and this is the discriminant.

In higher dimensions, the equations are more complicated but the ideas are the same. ◀

**Definition 4.10.** The **smooth locus** $B' \subset B$ is the subset of $\boldsymbol{\phi}$ for which the discriminant $\Delta_{\boldsymbol{\phi}} \in H^0(C, K_C^{\otimes k(k-1)})$ has only simple zeros.

This is called the smooth locus because for $\boldsymbol{\phi} \in B'$, the spectral curve $\Sigma_{\boldsymbol{\phi}}$ is smooth.

**Proposition 4.11.** *Let $\boldsymbol{\phi} \in B'$. Then, $\Sigma_{\boldsymbol{\phi}}$ has $2k(k-1)(g-1)$ branch points, where $g$ is the genus of $C$.*

This is proven with the Gauss-Bonnet theorem.

**Proposition 4.12.** *Let $\boldsymbol{\phi} \in B'$. Then, $\rho^{-1}(\boldsymbol{\phi}) = \mathcal{N}_{1,d'}(\Sigma_{\boldsymbol{\phi}})$ (i.e. the moduli space for the curve $\Sigma_{\boldsymbol{\phi}}$), where $d' := d - k(k-1)(g-1)$ and $g$ is the genus of $C$.*

**Proposition 4.13.** *The Hitchin fibration is proper and surjective.*

The geometric structure of the moduli space of Higgs bundles is complicated, but we can also describe it topologically.

**Definition 4.14.** The **nilpotent cone** is $\rho^{-1}(0) \subset \mathcal{M}_{k,d}$.

This has half the dimension of $\mathcal{M}_{k,d}$.

**Proposition 4.15.** *There is a deformation retraction of $\mathcal{M}_{k,d}$ onto its nilpotent cone.*

The point is: all the information that algebraic topology can see (homotopy groups, homology groups, and so on) is equivalent on the nilpotent cone and the entire space. The geometry may be different, e.g. there's a deformation retraction of $\mathbb{C}^\times$ onto the unit circle.

On the smooth locus, the fiber of the Hitchin fibration is the Jacobian of the spectral curve.

**Proposition 4.16.** *$T^*\mathcal{N}^s_{k,d}$ is an open, dense subset of $\mathcal{M}^s_{k,d}$ (here, the $s$ means "stable").*

So the moduli space of Higgs bundles is a partial compactification of the cotangent space.

**Proposition 4.17.** *If $k$ and $d$ are coprime, the metric on $\mathcal{M}_{k,d}$ is complete.*

This is not entirely natural: the cotangent bundle to a Kähler manifold is hyperkähler, but is not in general complete. Completeness means all geodesics exist for all time, starting from any point and going in any direction. For example, the open disc in $\mathbb{R}^2$ is not complete, but $\mathbb{R}^2$ is.

**Proposition 4.18.** *$\mathcal{M}_{k,d}$ is a **complex integrable system**, i.e. the fiber over any $\boldsymbol{\phi} \in B'$ is a Lagrangian submanifold with respect to the holomorphic symplectic structure. In particular, the dimension of the base equals the dimension of the fiber.*