

# FURUTA’S 10/8 THEOREM

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These notes were taken in a learning seminar on Furuta’s 10/8 theorem in Spring 2019. I live- $\text{\TeX}$ ed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to [a.debray@math.utexas.edu](mailto:a.debray@math.utexas.edu). Thanks to Riccardo Pedrotti for the notes for §3.

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## 1. INTRODUCTION TO SEIBERG-WITTEN THEORY: 1/23/19

Riccardo gave the first, introductory talk.

In 1982, Matsumoto conjectured that if  $M$  is a closed spin manifold,  $b_2(M) \geq (11/8)|\sigma(M)|$ . Here  $b_2(M)$  is the second Betti number and  $\sigma(M)$  is the signature. Equality holds for the K3 surface, so this is the best one can do.

In this seminar we’ll study a theorem of Furuta which makes major progress on this conjecture.

**Theorem 1.1** (10/8 theorem [Fur01]). *If the intersection form of  $M$  is indefinite,  $b_2(M) \geq (10/8)|\sigma(M)| + 2$ .*

If the intersection form is definite, work of Donaldson [Don83] says that, up to a change of orientation, the intersection form is diagonalizable, so that case is dealt with.

Furuta’s proof uses both Seiberg-Witten theory and equivariant homotopy theory. It can be pushed a little bit farther, but not enough to prove the  $11/8^{\text{th}}$  conjecture, as shown recently by Hopkins-Lin-Shi-Xu [HLSX18].

Today we’ll discuss some background for the proof.

**Definition 1.2.** Let  $V \rightarrow M$  be a rank- $n$  real oriented vector bundle. A *spin structure* on  $V$  is data  $\mathfrak{s} = (P_{\text{Spin}}(V), \tau)$ , where  $P_{\text{Spin}}(V) \rightarrow M$  is a principal  $\text{Spin}_n$ -bundle and  $\tau$  is an isomorphism

$$\tau: P_{\text{Spin}}(V) \times_{\text{Spin}_n} \mathbb{R}^n \xrightarrow{\cong} V.$$

A spin structure on a manifold  $M$  is a spin structure on  $TM$ .

*Remark 1.3.* There are other equivalent definitions of spin structures – for example, just as an orientation is a trivialization of  $V$  over the 1-skeleton of  $M$ , a spin structure is equivalent to a trivialization over the 2-skeleton. ◀

Here’s a cool theorem about spin manifolds.

**Theorem 1.4** (Rokhlin [Roh52]). *If  $M$  is a spin manifold,  $\sigma(M) \equiv 0 \pmod{16}$ .*

The signature makes sense when  $4 \mid \dim M$ . Smoothness is crucial here; there are topological spin 4-manifolds, whatever that means, that do not satisfy this theorem. Freedman’s  $E_8$  manifold is an example.

Suppose  $M$  is a spin 4-manifold. The representation theory of  $\text{Spin}_4$ , in particular the fact that the spin representation  $S$  splits as  $S^+ \oplus S^-$ , leads to two quaternionic line bundles  $\mathbb{S}^+, \mathbb{S}^- \rightarrow M$  with Hermitian metrics. Physics cares about these bundles, and will lead to powerful theorems in manifold topology.

These bundles have more structure: in particular, they are Clifford bundles.

**Definition 1.5.** Let  $S \rightarrow M$  be a real vector bundle with a Euclidean metric  $\langle \cdot, \cdot \rangle$ . A *Clifford bundle* structure is data of, for each  $x \in M$ , the data of a Clifford algebra action  $\text{Cl}(T_x M)$  on  $S_x$  that varies smoothly in  $x$ , such that the Clifford action is skew-adjoint, meaning

$$\langle v \cdot s_1, s_2 \rangle = -\langle s_1, v \cdot s_2 \rangle.$$

We also require the existence of a connection which is compatible with the Levi-Civita connection on  $TM$ .

Given the data of a Clifford bundle, there's an operator called the *Dirac operator*  $D$ , which is the following composition:

$$(1.6) \quad C^\infty(S) \xrightarrow{\nabla^{C\ell}} C^\infty(T^*M \otimes S) \xrightarrow{\langle \cdot, \cdot \rangle} C^\infty(TM \otimes S) \xrightarrow{\text{Clifford action}} C^\infty(S).$$

This operator is denoted  $\not{D}$ , a convention due to Feynman. It is a first-order, elliptic differential operator; ellipticity means that its analysis is nice.

Thus we can consider the *Seiberg-Witten equations* on a spin 4-manifold. Let  $(a, \varphi) \in \Omega_M^1(i\mathbb{R}) \times \Gamma(\mathbb{S}^+)$ ; then the equations are

$$(1.7a) \quad \not{D}\varphi + \rho(a)(\varphi) = 0$$

$$(1.7b) \quad \rho(d^+a) - \varphi \otimes \varphi^* + \frac{1}{2}|\varphi|^2 \text{id} = 0$$

$$(1.7c) \quad d^*a = 0.$$

On a non-spin manifold, the equations are a little more complicated.

## 2. THE MONOPOLE EQUATIONS: 1/28/19

Today, Kai spoke about the monopole equations and some of their important properties, foreshadowing compactness next week. We begin with some motivation.

Recall that if  $M$  is a closed, oriented 4-manifold (in either the topological or smooth category), the intersection form  $H_2(M) \times H_2(M) \rightarrow \mathbb{Z}$  is a unimodular, symmetric bilinear form.

**Question 2.1.** Which unimodular, symmetric bilinear forms arise as the intersection forms of smooth or topological manifolds?

For example, the intersection form of  $S^2 \times S^2$  is  $H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . The intersection form of  $\mathbb{CP}^2$  is (1). There's an interesting bilinear form called the *E8 form*

$$(2.2) \quad E8 = \begin{pmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & & & 1 & 2 \end{pmatrix}.$$

Can this be realized as the intersection form of a smooth 4-manifold? Rokhlin's theorem tells us the answer is no, because such a manifold would have to be spin, and  $16 \nmid \sigma(E8)$ . However, Freedman found a topological manifold  $M_{E8}$  whose intersection form is E8!

The direct sum of two copies of E8 satisfies Rokhlin's theorem, and this form is realized by the topological 4-manifold  $M_{E8} \# M_{E8}$ . However, Donaldson showed this manifold is not smoothable: specifically, the intersection forms of smooth 4-manifolds can be diagonalized over  $\mathbb{Z}$ , and E8 cannot.

There's still more interesting example: consider the *K3 surface*  $\{z_1^4 + z_2^4 + z_3^4 + z_4^4 = 0\} \subset \mathbb{CP}^3$ ; its intersection form is  $-2E8 \oplus 3H$ . So does it split as a connect sum of 3 copies of  $S^2 \times S^2$  and two copies of  $M_{E8}$  (with the opposite orientation)? Freedman showed this is true topologically. Smoothly, of course, it can't hold, but we might still get something.

**Question 2.3.** Is there a smooth, oriented 4-manifold  $N$  such that, in the smooth category,  $K3 \cong N \# S^2 \times S^2$ ?

This was a longstanding question.

Seiberg-Witten invariants allow us to answer questions such as this – though in this semester, we’re more interested in the monopole map. In any case, let’s define the Seiberg-Witten equations.

Let  $M$  be a smooth, oriented 4-manifold with  $b_2^+$  odd and a Riemannian metric  $g$ , and let  $\mathfrak{s}$  be a  $\text{spin}^c$  structure on  $M$ , which determines a *basic class*  $K \in H^2(X)$ , i.e. an integer cohomology class such that  $K \equiv w_2(M) \bmod 2$ . The  $\text{spin}^c$  structure  $\mathfrak{s}$  defines for us spinor bundles  $\mathbb{S}^+$  and  $\mathbb{S}^-$ . Let  $\mathcal{A}_L$  denote the space of  $U_1$ -connections,  $A \in \mathcal{A}_L$ , and  $\psi \in \Gamma(X, \mathbb{S}^+)$  (this is called a *spinor*). The Seiberg-Witten equations are

$$(2.4a) \quad D_A \psi = 0$$

$$(2.4b) \quad F_A^+ + i\delta = i\sigma(\psi).$$

These equations have a gauge symmetry: if  $G$  denotes the group  $\text{Map}(X, S^1)$  with pointwise multiplication,  $G$  acts on  $\mathcal{A}_L \times \Gamma(X, \mathbb{S}^+)$  on the first factor. Let  $B_K^+$  denote the quotient minus the locus of spinors which are identically zero; then  $B_K^+ \simeq \mathbb{CP}^\infty$ , so we know its cohomology is isomorphic to  $\mathbb{Z}[x]$ , with  $|x| = 2$ .

Let  $\mathcal{M}_K^\delta(g) \subset B_K^\times$  denote the space of solutions to the Seiberg-Witten equations. This space has dimension

$$(2.5) \quad d := \frac{1}{4}(K^2 - (3\sigma(M) + 2\chi(M))),$$

and, crucially, defines a class  $[\mathcal{M}_K^\delta(g)] \in H_d(B_K^\times)$  which does not depend on  $g$  for generic choices of the metric. The *Seiberg-Witten invariants* are

$$(2.6) \quad SW_X(K) := \langle x^{d/2}, [\mathcal{M}_K^\delta(g)] \rangle \in \mathbb{Z}.$$

The fact that  $b_2^+(M) = 0$  implies  $d$  is even.

This defines a map  $SW$  from the basic classes to  $\mathbb{Z}$ . Taubes showed two important results.

**Theorem 2.7** (Vanishing theorem (Taubes)). *If  $M$  is diffeomorphic to a connect sum of two closed, oriented 4-manifolds  $X_1 \# X_2$ ,  $b_2^+(X_1) > 0$ , and  $b_2^+(X_2) > 0$ , then the Seiberg-Witten equations of  $M$  vanish.*

**Theorem 2.8** (Nonvanishing theorem (Taubes)). *If  $\mathfrak{s}$  is the canonical  $\text{spin}^c$  structure associated to a complex structure on  $M$  and  $b_2^+(M)$  is positive and odd, then  $SW(\pm c_1(M)) = \pm 1$ .*

**Corollary 2.9.**  *$K3$  cannot split smoothly as a connect sum.*

This leads to an interesting generalization: there are *exotic K3 surfaces*, homeomorphic but not diffeomorphic to the standard K3. They don’t all admit complex structures, and many of them are not symplectic. Nonetheless, they also don’t split off an  $S^2 \times S^2$ : this is a consequence of Furuta’s 10/8 theorem, because if  $K3 \cong N \# (S^2 \times S^2)$ , then  $b_2(N) = 20$  and  $\sigma(N) = -16$ , but

$$(2.10) \quad 20 \not\geq \frac{10}{8}|-16| + 2.$$

Now let’s discuss the monopole map. We now assume  $M$  is a spin manifold, with spin structure  $\mathfrak{s}$  and spinor bundles  $\mathbb{S}^\pm$ . Let  $A$  denote a spin connection and consider the spaces

$$(2.11) \quad \tilde{\mathcal{A}} := \{A + i \ker d\} \times (\Gamma(\mathbb{S}^+) \oplus \Omega^1(X))$$

$$(2.12) \quad \tilde{\mathcal{C}} := \{A + i \ker d\} \times (\Gamma(\mathbb{S}^-) \oplus \Omega^0(X) \oplus H^1(X; \mathbb{R}) \oplus \Omega^+(X)).$$

Both of these fiber over  $H^1(X; \mathbb{R})$ : for  $\tilde{\mathcal{A}}$ ,  $A + \alpha \mapsto [\alpha]$ , and there is a map  $\tilde{\mu}: \tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{C}}$  defined by

$$(2.13) \quad (A, \phi, a) \mapsto (A, D_A \phi + ia\phi, d^*a, a_{\text{harm}}, d^+a - \sigma(\phi)).$$

Here

- $D_A$  is the *Dirac operator*  $D_A: \Gamma(\mathbb{S}^+) \rightarrow \Gamma(\mathbb{S}^-)$ .
- $a\phi$  denotes Clifford multiplication.
- $d^*$  is the adjoint of  $d$ , which sends  $k$ -forms to  $(k-1)$ -forms, and satisfies the equation

$$(2.14) \quad d^* = \star d \star.$$

(This is in dimension 4; the sign convention is different in other dimensions.)

- $a_{\text{harm}}$  is the harmonic part of  $a$ : it’s a general fact that any one-form in dimension 4 splits as  $a = a_{\text{harm}} + d^*\alpha + d\beta$  for some 0-form  $\beta$ . A form is *harmonic* if the Laplacian  $\Delta := dd^* + d^*d$  vanishes on it.

- $d^+a$  denotes the self-dual part of  $da$ .
- $\sigma(\phi)$  denotes the trace form of the endomorphism  $\phi \otimes \phi^* - (1/2)\|\phi\|^2 \text{id}$ .

Again the group  $G$  acts on  $\Gamma(\mathbb{S}^\pm)$  by pointwise multiplication, using  $S^1 \cong U_1 \subset \mathbb{C}$ . If  $u \in G$ ,  $u: X \rightarrow S^1$  also acts on the space of  $\text{spin}^c$  connections by  $d \mapsto udu^{-1}$ . Let  $G$  act trivially on forms.

Then, the map  $\tilde{\mu}$  defined in (2.13) is  $G$ -equivariant. Let  $G_0$  denote the maps which vanish at some specified basepoint  $p$ , and let  $\mathcal{A} := \tilde{A}/G_0$ ,  $\mathcal{C} := \tilde{C}/G_0$ , and  $\mu := \tilde{\mu}/G_0$ ; thus we get a map  $\mu: \mathcal{A} \rightarrow \mathcal{C}$ .

Now, both  $\mathcal{A}$  and  $\mathcal{C}$  fiber over the Picard group

$$(2.15) \quad \text{Pic}^g(X) := H^1(X; \mathbb{R})/H^1(X; \mathbb{Z}) = H^1(X; \mathbb{R})/G_0.$$

Then  $S^1 = G/G_0$  acts on  $\mu^{-1}(A, 0, 0, 0, 0)$ , and this is the space we're interested in.

We would like to study this space, and to do so we'll need to consider Sobolev spaces. For a fixed integer  $k > 2$ , let  $A_k$  be the fiberwise completion of  $A$  within  $L_k^2$  and  $C_{k-1}$  be the fiberwise completion of  $C$  within  $L_{k-1}^2$ . Then, the monopole map  $\mu$  is a map  $A_k \rightarrow C_{k-1}$ .

**Claim 2.16.** This monopole map  $\mu$  is  $S^1$ -equivariant, and is a compact perturbation of a linear Fredholm map.

The  $S^1$ -equivariance involves chasing through the definition but isn't bad; the rest is harder. What we can do is start by listing the terms that define a linear Fredholm map, and then check that the rest is compact. In the definition of  $\tilde{\mu}$ , the terms  $A$ ,  $D_A\phi$ ,  $d^*a$ ,  $a_{\text{harm}}$ , and  $d^+a$  are linear and Fredholm; thus we just have to check that  $a(\phi)$  and  $\sigma(\phi)$  are compact. For the first, we can use the fact that Clifford multiplication is compact, then compose with the map  $C_k \rightarrow C_{k-1}$ , which is also compact.

**Proposition 2.17.** Let  $T = \ell + c$  be a compact perturbation of a linear Fredholm map  $\ell$  between Hilbert spaces. The restriction of  $T$  to any closed, bounded subset  $\Omega$  is proper.

*Proof.* Let  $p$  denote projection onto  $\ker(\ell)$  and consider the commutative diagram

$$(2.18) \quad \begin{array}{ccc} \Omega \xrightarrow{(\ell, c, p)} M \times \overline{c(\Omega)} \times \overline{p(\Omega)} & \xrightarrow[\cong]{(u, s, e) \mapsto (u+a, s, e)} & M \times \overline{c(A)} \times \overline{p(A)} \xrightarrow{\text{proj}} M. \\ & \searrow \ell+c & \end{array}$$

Because the map  $(\ell, c, p)$  is injective, **TODO**.

⊠

### 3. COMPACTNESS OF THE MODULI SPACE OF SEIBERG-WITTEN SOLUTIONS: 2/3/19

These are Riccardo's notes on the lecture he gave, on the compactness of the moduli space of solutions to the Seiberg-Witten equations. This is a crucial step in Furuta's construction of finite-dimensional approximations, and relies on some functional analysis.

**3.1. A closer look at the Seiberg-Witten monopole map.** Let  $X$  be a oriented closed spin 4-manifold. Let  $\mathfrak{s}$  be a spin structure for it. Let  $\mathbb{S}^\pm$  be the positive and negative spinor bundles associated to it. Fix a spin connection  $A$  on them.

Recall the Seiberg-Witten equations can be thought as a fiber-preserving  $S^1$ -equivariant map between these two  $S^1$ -Hilbert bundles over  $H^1(X; \mathbb{R})$ :

$$(3.1a) \quad \tilde{\mathcal{A}} = (A + i \ker(d)) \times (\Gamma(\mathbb{S}^+) \oplus \Omega^1(X))$$

$$(3.1b) \quad \tilde{\mathcal{C}} = (A + i \ker(d)) \times (\Gamma(\mathbb{S}^-) \oplus \Omega^0(X) \oplus H^1(X; \mathbb{R}) \oplus \Omega^+(X)).$$

The map  $\tilde{\mu}: \tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{C}}$  is defined by

$$(3.2) \quad (A, \phi, a) \mapsto (A, D_A\phi + ia\phi, d^*a, a_{\text{harm}}, d^+a - \sigma(\phi)).$$

As explained in the previous seminar,  $\sigma(\phi)$  denotes the trace-free endomorphism  $i(\phi \otimes \phi^* - \frac{1}{2}\|\phi\|^2 \text{id})$  of  $\mathbb{S}^+$ , considered via the map  $\rho$  as a self-dual 2-form on  $X$ .

The gauge group  $\mathcal{G} = \text{Aut}_{\text{id}}(\mathfrak{s}) \cong \text{Map}(X, S^1)$  acts on spinors on the 4-manifold via multiplication with  $u: X \rightarrow S^1$  and on  $\text{spin}^c$  connections via addition of  $ud(u^{-1})$ . It acts trivially on forms.

The map  $\tilde{\mu}$  is equivariant with respect to the action of  $\mathcal{G}$ . Dividing by the free action of the pointed gauge group we obtain the monopole map

$$\mu = \tilde{\mu}/\mathcal{G}_0 : \mathcal{A} \rightarrow \mathcal{C}$$

as a fiber preserving map between the bundles  $\mathcal{A} = \tilde{\mathcal{A}}/\mathcal{G}_0$  and  $\mathcal{C} = \tilde{\mathcal{C}}/\mathcal{G}_0$  over  $\text{Pic}^s(X)$ . The preimage of the section  $(A, 0, 0, 0, 0)$  of  $\mathcal{C}$ , divided by the residual  $S^1$ -action, is called the *moduli space of monopoles*.

For a fixed  $k > 2$ , consider the fiberwise  $L_k^2$  Sobolev completion  $\mathcal{A}_k$  and the fiberwise  $L_{k-1}^2$  Sobolev completion  $\mathcal{C}_{k-1}$  of  $\mathcal{A}$  and  $\mathcal{C}$ . The monopole map extends to a continuous map  $\mathcal{A}_k \rightarrow \mathcal{C}_{k-1}$  over  $\text{Pic}^s(X)$ , which will also be denoted by  $\mu$ .

We will use the following properties of the monopole map.

- It is  $S^1$ -equivariant.
- Fiberwise, it is the sum  $\mu = l + c$  of a linear Fredholm map  $l$  and a nonlinear compact operator  $c$ .
- Preimages of bounded sets are bounded.

**Claim 3.3.** The moment map is  $S^1$ -equivariant.

*Proof.* Equivariance is immediate. The action is the residual action of the subgroup  $S^1$  of gauge transformations which are constant functions on  $X$ . This group acts by complex multiplication on the spaces  $\Gamma(\mathbb{S}^\pm)$  of sections of complex vector bundles and trivially on forms.  $\square$

**Claim 3.4.** Fiberwise, the moment map is the sum  $\mu = l + c$  of a linear Fredholm map  $l$  and a nonlinear compact operator  $c$ .

*Proof.* Restricted to a fiber, the monopole map is a sum of the linear Fredholm operator  $\ell$ , consisting of the elliptic operators  $D_A$  and  $d^* + d^+$ , complemented by projections to and inclusions of harmonic forms. The nonlinear part of  $\mu$  is built from the bilinear terms  $a\phi$  and  $\sigma(\phi)$ . Multiplication  $\mathcal{A}_k \times \mathcal{A}_k \rightarrow \mathcal{C}_k$  is continuous for  $k > 2$ . Combined with the compact restriction map  $\mathcal{C}_k \rightarrow \mathcal{C}_{k-1}$  (Rellich lemma, see page 2 Lecture 19 in [?]) we gain the claimed compactness for  $c$ : Images of bounded sets are contained in compact sets.  $\square$

Now let us show the following very useful property of compact perturbations of Fredholm operators.

**Claim 3.5.** The restriction of a compact perturbation  $l + c: \mathcal{U}' \rightarrow \mathcal{U}$  of a linear Fredholm map  $\ell$  between Hilbert spaces to any bounded, closed subset is proper.

*Proof.* Let  $p$  denote a projection to the kernel of  $\ell$ . Let  $A$  be a bounded closed subset of  $\mathcal{U}'$ . It's easy to see that we have the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{(\ell, c, p)} & \mathcal{U} \times \overline{c(A)} \times \overline{p(A)} \\ & \searrow \ell + c & \downarrow \cong + \\ & & \mathcal{U} \times \overline{c(A)} \times \overline{p(A)} \\ & & \downarrow \pi \\ & & \mathcal{U} \end{array}$$

We observe that the map  $h: A \rightarrow \mathcal{U} \times \overline{c(A)} \times \overline{p(A)}$  given by  $a \mapsto (\ell(a), c(a), p(a))$  is injective and closed. Injectivity is clear since we are projecting on the kernel.

Closedness is a little bit more involved: let  $\{(\ell_n, c_n, p_n)\}_n \subset \text{Im}(h)$  converge to  $(\ell_\infty, c_\infty, p_\infty)$ . In particular there is a sequence  $\{a_n\}_n \subset A$  such that  $(\ell_n, c_n, p_n) = (\ell(a_n), c(a_n), p(a_n))$ . We want to prove that  $(\ell_\infty, c_\infty, p_\infty) \in h(A)$ . Since  $\ell$  is Fredholm we have the following property: *every bounded sequence  $\{x_i\}_i$  in the domain whose image is convergent admits a convergent subsequence  $\{x_{i_j}\}_j$* . Since  $A$  is closed and bounded (and any other closed subset of it would be bounded as well hence we can directly work with  $A$ ),  $\{a_n\}_n$  is bounded. Since  $\ell$  is Fredholm we can extract a convergent subsequence  $\{a'_n\}_n$  converging to  $a \in A$  (since  $A$  is closed). By the uniqueness of the limit, it's easy to prove

$$(3.6) \quad (\ell_\infty, c_\infty, p_\infty) = (\ell(a), c(a), p(a))$$

which proves the closedness of  $h(A)$ . This implies that  $h$  is proper, since  $h$  is an homeomorphism onto its image.

The addition map  $+: (u, s, e) \mapsto (u + s, s, e)$  is an homeomorphism hence proper. The projection to  $\mathcal{U}$  is proper since the other two factors are compact.  $\square$

**3.2. A collection of results.** We will list here some results needed for the seminar.

Let  $U$  be an open subset of  $\mathbb{R}^n$ . We can consider the space  $C_c^\infty(U; \mathbb{R}^r)$  of compactly supported  $\mathbb{R}^r$ -valued functions. Fix a real number  $p > 1$  and an integer  $k \geq 0$ . The Sobolev  $L_k^p$  norm is defined by

$$(3.7) \quad \|f\|_{p,k} := \sum_{|\alpha| < k} \sup_U \|D^\alpha f\|_p.$$

The Sobolev space  $L_k^p(E)$  is defined to be the completion of  $\Gamma(E)$  in the  $L_k^p$  norm.

Here are the basic facts about Sobolev spaces.

**Sobolev inequality:** If  $k \leq \ell$  then there exists a constant  $C$  such that

$$(3.8) \quad \|\cdot\|_{p,k} \leq C \|\cdot\|_{p,\ell},$$

and hence we have a bounded inclusion of Sobolev spaces  $L_k^p(E) \hookrightarrow L_\ell^p(E)$ .

**Rellich lemma:** The inclusion  $L_{k+1}^p(E) \hookrightarrow L_k^p(E)$  is a compact operator.

**Morrey inequality:** Suppose  $\ell \geq 0$  is an integer such that  $\ell < k - n/p$ ; then there is a constant  $C$  such that

$$(3.9) \quad \|\cdot\|_{C^\ell} \leq C \|\cdot\|_{p,k},$$

i.e. there is a bounded inclusion

$$(3.10) \quad L_k^p(E) \hookrightarrow C^\ell(E).$$

**Smoothness:** One has

$$(3.11) \quad \bigcap_{k \geq k_0} L_k^p(E) = C^\infty(E).$$

**Lemma 3.12.** *Over a closed Riemannian 4-manifold, multiplication of smooth functions extends to a bounded map*

$$(3.13) \quad L_k^2(X) \otimes L_\ell^2(X) \rightarrow L_\ell^2(X)$$

*provided that  $k \geq 3$  and  $k \geq \ell$ . In particular,  $L_k^2(X)$  is an algebra for  $k \geq 3$ .*

There are also bounded multiplication maps for the lower regularity Sobolev spaces in 4 dimensions, but these bring in Sobolev spaces with  $p > 2$ .

Let now  $D: \Gamma(E) \rightarrow \Gamma(F)$  be a differential operator of order  $m$  over a closed, oriented, Riemannian manifold  $(M, g)$ . The basic point is that  $D$  extends to a bounded linear map between Hilbert spaces:

$$(3.14) \quad D: L_{k+m}^2(E) \rightarrow L_k^2(F).$$

**Theorem 3.15** (Elliptic estimate). *If  $D$  is elliptic of order  $m$ , one has estimates on the  $L_k^2$ -Sobolev norms for each  $k \geq 0$ :*

$$(3.16) \quad \|s\|_{2,k+m} \leq C_k (\|Ds\|_{2,k} + \|s\|_{2,k}).$$

Moreover,

$$(3.17) \quad \|s\|_{2,k+m} \leq C_k \|Ds\|_{2,k}$$

for  $s \in (\ker D)^\perp$  (here  $^\perp$  denotes the  $L^2$ -orthogonal complement).

There is an analogue for  $L^{p,k+m}$  bounds.

As a consequence of this important theorem we have the following:

**Corollary 3.18.** *An elliptic operator  $D$  of order  $m$  defines a Fredholm map  $L_{k+m}^2(E) \rightarrow L_k^2(F)$  for any  $k \geq 0$ . Its index is independent of  $k$ . Moreover, its index depends only on the symbol of  $D$ .*

Let  $(M, g)$  be an oriented Riemannian manifold. Let  $\nabla$  be an orthogonal covariant derivative in a real, Euclidean vector bundle  $E \rightarrow M$ . We know that  $\nabla$  has a formal adjoint  $\nabla^*$ .

**Proposition 3.19** (The Lichnérowicz formula). *One has*

$$(3.20) \quad D^2 = \tilde{\nabla}^* \tilde{\nabla} + \frac{1}{4} \text{scal}_g \cdot \text{id}_\mathbb{S} + \frac{1}{2} \rho(F^\circ).$$

**Lemma 3.21.**

$$(3.22) \quad \frac{1}{2}d^*d(|s|^2) = \langle \nabla^* \nabla s, s \rangle - |\nabla s|^2.$$

*Proof sketch.* See Lemma 1.1. L19 in [?]. The idea is to study the integral

$$(3.23) \quad \int_M f \langle \nabla^* \nabla s, s \rangle \text{vol}$$

where  $f$  has compact support. □

It's important to remember that the one above is a pointwise equality. Working locally one has the following result.

**Lemma 3.24.** *For a smooth function  $f: M \rightarrow \mathbb{R}$  with compact support, if  $p$  is a local maximum, then  $(d^*df)(p) \geq 0$ .*

The following lemma is an easy calculation.

**Lemma 3.25.** *For  $\phi \in \Gamma(\mathbb{S}^+)$ , one has*

$$(3.26) \quad ((\phi\phi^*)_0\chi, \chi) = (\phi, \chi)^2 - \frac{1}{2}|\chi|^2|\phi|^2.$$

*In particular,*

$$(3.27) \quad ((\phi\phi^*)_0\phi, \phi) = \frac{1}{2}|\phi|^4.$$

*Proof.* We have

$$\begin{aligned} ((\phi\phi^*)_0\chi, \chi) &= ((\phi\phi^*)\chi, \chi) - \frac{1}{2}(|\phi|^2\chi, \chi) \\ &= ((\phi, \chi)\phi, \chi) - \frac{1}{2}|\phi|^2|\chi|^2 \\ &= (\phi, \chi)^2 - \frac{1}{2}|\phi|^2|\chi|^2. \end{aligned} \quad \square$$

**Lemma 3.28.** *For  $\eta \in \Omega_X^2$  and  $\phi \in \Gamma(\mathbb{S})$ , one has  $(\rho(\eta)\phi, \phi) \leq |\eta||\phi|^2$ .*

*Proof.* It suffices to take  $\eta = e \wedge f$  for orthogonal unit vectors  $e$  and  $f$ . One then has

$$(3.29) \quad (\rho(\eta)\phi, \phi) = (\rho(e \wedge f)\phi, \phi)$$

$$(3.30) \quad = \frac{1}{2}([\rho(e), \rho(f)]\phi, \phi)$$

$$(3.31) \quad = -\frac{1}{2}(\rho(f)\phi, \rho(e)\phi)$$

$$(3.32) \quad \leq |\rho(e)\phi| \cdot |\rho(f)\phi|,$$

where in (3.31) we used the fact that  $\rho$  has image in the anti-skew-Hermitian matrices. Now since  $|e| = 1$  then  $|\rho(e)| = 1$  (similarly for  $f$ ), and therefore we conclude. □

**Lemma 3.33.** *Let  $A$  be a Clifford connection for the spinor bundle of a  $\text{spin}^c$  structure of  $X$ . Let  $a \in \Omega_X^1(i\mathbb{R})$ ; then*

$$(3.34) \quad D_{A+a}\phi = D_A\phi + a \cdot \phi,$$

where the last term is the Clifford multiplication between  $a$  and  $\phi$ .

*Proof.* Let's work in local orthonormal coordinates of  $TX$  given by  $\{e_1, \dots, e_n\}$ . We have

$$\begin{aligned}
D_{A+a}\phi &= \sum_i e_i \cdot (A+a)_{e_i} \phi \\
&= \sum_i e_i \cdot A_{e_i} \phi + \sum_i e_i \cdot a(e_i) \phi \\
&= D_A \phi + \sum_i e_i \cdot a(e_i) \phi \\
&= D_A \phi + a \sum_i e_i \phi \\
&= D_A \phi + a \cdot \phi.
\end{aligned}$$

Notice that here we used that  $a \in \Omega_X^1(i\mathbb{R})$  hence all the coefficients  $a(e_i)$  are equal to each other, and without loss of generality we named then  $a$ .  $\square$

**3.3. Compactness of the moduli space.** If the bundles  $\mathcal{A}$  and  $\mathcal{C}$  were finite-dimensional, then the boundedness property would be equivalent to properness. In this infinite-dimensional setting, the argument above can be used the same way as Heine-Borel in the finite-dimensional case to show that the boundedness condition implies properness. It turns out that the ingredients of the compactness proof for the moduli space also prove the stronger boundedness property.

**Proposition 3.35.** *Preimages  $\mu^{-1}(B) \subset \mathcal{A}_k$  of bounded disk bundles  $B \subset \mathcal{C}_{k-1}$  are contained in bounded disk bundles.*

*Proof.* It is sufficient to prove this fiberwise for the Sobolev completions of the restriction of the monopole map to the space  $\{A\} \times (\Gamma(\mathbb{S}^+) \oplus \ker(d^*))$ , which maps to  $\{A\} \times (\Gamma(\mathbb{S}^-) \oplus \Omega_+^2(X) \oplus H^1(X; \mathbb{R}))$ . We start by defining the following scalar product: using the elliptic operator  $D = D_A + d^+$  and its adjoint, define the  $L_k^2$ -norm via the scalar product on the respective function spaces through

$$(3.36a) \quad (\cdot, \cdot)_i = (\cdot, \cdot)_0 + (D\cdot, D\cdot)_{i-1} \text{ for } 0 < i \leq k$$

$$(3.36b) \quad (\cdot, \cdot)_0 = \int_X \langle \cdot, \cdot \rangle.$$

Using the elliptic estimates and continuity (i.e. boundedness) of  $D$  it's easy to see that this norm is equivalent to the classic Sobolev one. A similar definition can be extended to norms for the  $L_k^p$ -spaces. Let us take  $\mu(A, \phi, a) = (A, \varphi, b, a_{\text{harm}}) \in \mathcal{C}_{k-1}$  with the norm of the latter bounded by some constant  $R$ . The Lichn rowicz formula (Proposition 3.19) for a connection  $A + a = A'$  reads

$$(3.37) \quad D_{A'}^* D_{A'} = A' \circ A' + \frac{1}{4} s \cdot \text{id}_{\mathbb{S}} + \frac{1}{2} \rho(F_{A'}^\circ)$$

with  $s$  denoting the scalar curvature of  $X$ . As a consequence we have a pointwise estimate:

$$(3.38) \quad d^* d |\phi|^2 = 2 \langle \nabla_{A'}^* \nabla_{A'} \phi, \phi \rangle - 2 \langle \nabla_{A'} \phi, \nabla_{A'} \phi \rangle$$

$$(3.39) \quad \leq 2 \langle \nabla_{A'}^* \nabla_{A'} \phi, \phi \rangle$$

$$(3.40) \quad \leq 2 \langle D_{A'}^* D_{A'} \phi - \frac{s}{4} \phi - \frac{1}{2} \rho(F_{A'}^\circ) \phi, \phi \rangle$$

$$(3.41) \quad \leq \langle 2 D_{A'}^* \varphi - \frac{s}{2} \phi - (\sigma(\phi) + b) \phi, \phi \rangle$$

Where in 3.38 We used Lemma 3.21. In 3.39 we removed the negative quantity on the left to obtain an inequality. In 3.41 we substituted in the second S-W equation.

Now we move some terms to the left and use the equality  $D_{A+a} = D_A + a$  together with the fact that the



Dirac operator is self-adjoint to get

$$(3.42) \quad d^*d|\phi|^2 + \frac{s}{2}|\phi|^2 + \langle \sigma(\phi), \phi \rangle \leq \langle 2D_{A'}^*\varphi, \phi \rangle - \langle b\phi, \phi \rangle$$

$$(3.43) \quad d^*d|\phi|^2 + \frac{s}{2}|\phi|^2 + \frac{1}{2}|\phi|^4 \leq \langle 2D_A^*\varphi, \phi \rangle + 2\langle a \cdot \varphi, \phi \rangle - b|\phi|^2$$

$$(3.44) \quad \leq 2(\|D_A^*\varphi\|_\infty + \|a\|_\infty\|\varphi\|_\infty) \cdot |\phi| + \|b\|_\infty \cdot |\phi|^2$$

$$(3.45) \quad \leq c_1 \left( (1 + \|a\|_\infty) \|\varphi\|_{L_{k-1}^2} \cdot |\phi| + \|b\|_{L_{k-1}^2} \cdot |\phi|^2 \right)$$

Where in 3.43 we are using Lemma 3.25 to bound  $\sigma(\phi)$ . In 3.45 we used Sobolev embeddings theorem (Morrey's inequality) to bound the  $L^\infty$ -norm with the Sobolev norm.

Now we need to estimate  $\|a\|_\infty$ . First thing, for  $p > 4$  we get a Sobolev estimate  $\|a\|_\infty \leq c_2\|a\|_{L_1^p}$  and then use the elliptic estimate:

$$(3.46) \quad \|a\|_{L_1^p} = \|a_{\text{harm}} + a'\|_{L_1^p} \leq \|a_{\text{harm}}\|_{L_1^p} + \|a'\|_{L_1^p}$$

$$(3.47) \quad \leq \|a_{\text{harm}}\|_{L_0^p} + \|d^+a\|_{L_0^p}$$

where in 3.46 we used the Hodge decomposition of  $a$ , in 3.47 we applied the elliptic estimate to both component. Recall that  $d^+(a_{\text{harm}}) = 0$  and  $d^+a = d^+a'$ .

Combination with the equality  $d^+a = b + \sigma(\phi)$  then leads to an estimate

$$(3.48) \quad \|a\|_\infty \leq c_4 \left( \|a_{\text{harm}}\|_{L_0^p} + \|b\|_{L_0^p} + \|\sigma(\phi)\|_{L_0^p} \right)$$

$$(3.49) \quad \leq c_5 \left( \|a_{\text{harm}}\|_{L_0^p} + \|b\|_{L_{k-1}^2} + \|\phi\|_\infty^2 \right)$$

In the last passage we control the  $L_0^p$ -norm with the  $L_{k-1}^2$ -one, since  $p > 4$ .

Putting these two estimates together, we get something of the form

$$(3.50)$$

$$d^*d|\phi|^2 + \frac{1}{2}\|s\|_\infty\|\phi\|_\infty^2 + \frac{1}{2}\|\phi\|_\infty^4 \leq c \left( 1 + c_5 \left( \|a_{\text{harm}}\|_{L_0^p} + \|b\|_{L_{k-1}^2} + \|\phi\|_\infty^2 \right) \right) \|\varphi\|_{L_{k-1}^2} \cdot \|\phi\|_\infty + \|b\|_{L_{k-1}^2} \cdot \|\phi\|_\infty^2$$

$$(3.51) \quad \leq K\|\phi\|_\infty^3 + R\|\phi\|_\infty^2$$

Where in 3.51 we applied the bounds we had by assumption on the elements in the image.

So our inequality is now:

$$(3.52) \quad d^*d|\phi|^2 + \frac{1}{2}\|\phi\|_\infty^4 \leq K\|\phi\|_\infty^3 + R\|\phi\|_\infty^2 - \frac{1}{2}\|s\|_\infty\|\phi\|_\infty^2$$

$$(3.53) \quad \leq K\|\phi\|_\infty^3 + R\|\phi\|_\infty^2$$

Now this inequality must hold in particular when  $\phi$  achieves its maximum, and on that point the Laplacian is positive, hence we can forget about it and get:

$$(3.54) \quad \frac{1}{2}\|\phi\|_\infty^4 \leq K\|\phi\|_\infty^3 + R\|\phi\|_\infty^2$$

In particular we bound the 4-th power of a quantity with a polynomial in that quantity of degree 3. This implies that  $\|\phi\|_\infty$  must be bounded. Therefore we can bound the  $L_0^p$ -norm of  $(\phi, a)$  for every  $p \geq 1$ .

Now comes bootstrapping: for  $i \leq k$ , assume inductively  $L_{i-1}^2$ -bounds on  $(\phi, a)$ . To obtain  $L_i^2$ -bounds, compute:

$$(3.55) \quad \|(\phi, a)\|_{L_i^2}^2 - \|(\phi, a)\|_{L_0^2}^2 = \|(D_A\phi, d^+a)\|_{L_{i-1}^2}^2$$

$$(3.56) \quad = \|(\phi + ia\phi, b - \sigma(\phi))\|_{L_2}^2$$

$$(3.57) \quad = \|(\phi, b)\|_{L_{i-1}^2}^2 + \|(ia\phi, \sigma(\phi))\|_{L_{i-1}^2}^2$$

The first equality holds by our definition of the Sobolev norm. The last equality holds as  $D_{A'} = D_A + a$ . The summands in the last expression are bounded by the assumed  $L_{i-1}^2$ -bounds on  $(\phi, a)$  together with the Sobolev multiplication properties. Note that the steps for  $i = 2$  and  $3$  require special care (see [?] page 4 L21) or use Sobolev embedding together with the fact that we have control on the  $L^p$ -norms of  $(\phi, a)$  for every  $p$ , which gives us control on the respective Sobolev norms for  $p = 2$ .  $\square$

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