DIFFERENTIAL COHOMOLOGY

ARUN DEBRAY
AUGUST 28, 2019

These notes were taken in a learning seminar on differential cohomology in Fall 2019, organized by Dan Freed. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

Contents

1. Introduction: 8/28/19

"These are the ordinary chain maps of singular homology... except they're not."

At the first meeting, Dan gave a short introduction to the ideas of differential topology.

Let X be a smooth manifold. Its first differential cohomology group is the space

$$\check{H}^1(X) := \mathrm{Map}(X, \mathbb{R}/\mathbb{Z}). \tag{1.1}$$

We should think of this as an infinite-dimensional abelian Lie group. The group structure is pointwise addition.

Treating this as an abelian Lie group, there are a few natural questions we can ask.

- First, what is its Lie algebra? The answer is the space of maps from X to $\mathbb{R}$, which is identified with $\Omega^0(X)$.
- Next, an abelian Lie group can only have two nonzero homotopy groups, π_0 and π_1. This is also true for $\check{H}^1(X)$, even though it's infinite-dimensional: $\pi_0\check{H}^1(X) \cong H^1(X)$ and $\pi_1\check{H}^1(X) \cong H^0(X)$. All higher homotopy groups of $\check{H}^1(X)$ vanish.
- The exponential map from the Lie algebra to the Lie group is $f \mapsto (f \bmod 1)$. The image is the identity component of $\check{H}^1(X)$, which is the functions that have a logarithm. The kernel of the exponential map is $\pi_1\check{H}^1(X)$.

There's a map $\omega\colon \check{H}^1(X) \to \Omega^1(X)_{\mathrm{cl}}$ which sends $\overline{f} \mapsto \mathrm{d}\overline{f}$. The "cl" means it lands in closed forms. The map isn't surjective; its image is those forms with *integral periods*, i.e. those 1-forms α such that the integral of α around any smoothly embedded circle is an integer.

Closed forms have a map dR to de Rham cohomology $H^1_{\mathrm{dR}}(X) = H^1(X;\mathbb{R})$, which is a surjective map from an infinite-dimensional vector space to a finite-dimensional vector space. $H^1(X;\mathbb{Z})$ also sits inside $H^1(X;\mathbb{R})$ as a lattice. The preimage under dR of 0 is the space of exact 1-forms, and preimages of other elements of $H^1(X;\mathbb{Z})$ form affine spaces modeled on the space of exact 1-forms. The union of all such preimages is precisely the 1-forms with integral periods.

To summarize the situation, we have maps

$$\begin{tikzcd} \check{H}^1(X) \arrow[r, "\omega"] \arrow[d, "\pi_0"] & \Omega^1(X)_{\mathrm{cl}} \arrow[d, "\mathrm{dR}"] \\ H^1(X;\mathbb{Z}) \arrow[r] & H^1(X;\mathbb{R}). \end{tikzcd} \tag{1.2}$$

This is a commutative diagram of abelian Lie groups. It is *not* a pullback diagram! This is because, for instance, ω has kernel. This tells you that $\check{H}^1(X)$ somehow contains some extra information, and this is the magic that makes differential cohomology interesting.

The fiber of ω, i.e. $\omega^{-1}(0)$, is $H^0(X;\mathbb{R}/\mathbb{Z})$, the locally constant maps from X to $\mathbb{R}/\mathbb{Z}$. This can be identified with $H^0(X;\mathbb{R})/H^0(X;\mathbb{Z})$.

We can also explicitly identify the next differential cohomology group $\check{H}^2(X)$: as a set, it is the isomorphism classes of principal $\mathbb{R}/\mathbb{Z}$-bundles on X together with a connection, or equivalently isomorphism classes of Hermitian line bundles with compatible connection; the group structure is tensor product.

To compute the Lie algebra, consider a path of connections on the trivial bundle; these are 1-forms, but because we consider them up to isomorphism, we end up with $\Omega^1(X)/\mathrm{d}\Omega^0(X)$.

The homotopy groups are similar, but shifted up once: $\pi_0\check{H}^1(X) \cong H^2(X;\mathbb{Z})$ (equivalence classes of bundles, where we forget the connection, only remembering the global information) and $\pi_1\check{H}^2(X) \cong H^1(X;\mathbb{Z})$. The higher homotopy groups vanish. Now something new can happen: $H^2(X)$ can have torsion, e.g. for $\mathbb{RP}^n$ for some n.

Again there is a map $\omega\colon \check{H}^2(X) \to \Omega^2(X)_{\mathrm{cl}}$ which sends a connection to its curvature; the image is again closed 2-forms with integral periods, again a union of affine spaces. Here, "integral periods" has a slightly different definition: using Chern-Weil theory, you can think of them as the forms which are curvatures of connections, which is an integrality condition.

The degree-2 version of (1.2) is

$$\begin{tikzcd} \check{H}^2(X) \arrow[r, "\omega"] \arrow[d, "\pi_0"] & \Omega^2(X)_{\mathrm{cl}} \arrow[d, "\mathrm{dR}"] \\ H^2(X;\mathbb{Z}) \arrow[r] & H^2(X;\mathbb{R}). \end{tikzcd} \tag{1.3}$$

Again this is a commutative diagram of abelian Lie groups but not a pullback diagram; $\ker(\omega) = H^1(X;\mathbb{R}/\mathbb{Z})$, which unlike $H^0(X;\mathbb{R}/\mathbb{Z})$, is not a torus; it need not be connected. Instead, its path components are identified with the torsion subgroup of $H^2(X;\mathbb{Z})$. Said differently, a flat connection on a circle bundles does not imply that it's trivial. There is a short exact sequence

$$0 \longrightarrow T^1(X) \coloneqq H^1(X;\mathbb{R})/H^1(X;\mathbb{Z}) \longrightarrow H^1(X;\mathbb{R}/\mathbb{Z}) \longrightarrow \mathrm{Tors}H^2(X;\mathbb{Z}) \longrightarrow 0. \tag{1.4}$$

The presence of torsion makes the description of $\check{H}^2(X)$ slightly more interesting.

Differential cohomology has more structure: multiplication and integration. Recall that integral and de Rham cohomology are rings under cup product, and differential forms are a ring under wedge product. The maps in (1.2) and (1.3) are compatible with these structures, so we might expect a map $\check{H}^1(X) \times \check{H}^1(X) \to \check{H}^2(X)$ compatible with the ring structures on differential forms and cohomology (and indeed we will get one).

This is somewhat strange, though: to define this map we want to, given $\overline{f}_1, \overline{f}_2\colon X \rightrightarrows \mathbb{R}/\mathbb{Z}$, produce a principal $\mathbb{R}/\mathbb{Z}$-bundle with connection.

If $\overline{f}_1$ has a logarithm $f_1\colon X \to \mathbb{R}$ (i.e. $f \bmod 1 = \overline{f}$), then we could take $f_1\,\mathrm{d}\overline{f}_2$ as our connection 1-form, but in general it's a little fancier. Given $\overline{f}_1, \overline{f}_2$, we can define $\overline{f}_1 \times \overline{f}_2\colon X \to \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$. There is a universal principal $\mathbb{R}/\mathbb{Z}$-bundle with connection $P \to \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ whose curvature is $\mathrm{d}x \wedge \mathrm{d}y$; then we pull that back to X via $\overline{f}_1 \times \overline{f}_2$, and that's the product $\overline{f}_1 \cdot \overline{f}_2$.

In general, we don't have as explicit geometric models for differential cohomology, and we'll have to define everything more abstractly, but for an introduction the geometric viewpoint is beneficial.

Next, let's define integration. Just as with ordinary cohomology, we'll need an orientation on X, which we now assume. Let $\check{x} = (P, \theta) \in \check{H}^2(X)$. We can integrate the curvature $\mathrm{d}\theta$ over a closed, oriented 2-manifold Σ, and because $\omega(\check{x})$ has integral periods, this is an element of $\mathbb{Z}$. Chern-Weil theory tells us this integral is topological, not geometric: it depends on P but not θ. This is an example of a *primary (topological) invariant*.

But we can also integrate $\check{x}$ over a closed, oriented 1-manifold C, which we define for now as the holonomy of θ around C. This depends on θ and lives in $\mathbb{R}/\mathbb{Z}$, and we call it a *secondary (geometric) invariant*.

In general, on a closed, oriented d-manifold, integration will be a map $\check{H}^2(X) \to \check{H}^{2-d}(\mathrm{pt})$; what we just said fits in, where we define $\check{H}^0(X) \coloneqq H^0(X)$, the space of maps to $\mathbb{Z}$.

There's plenty more to say here: what Stokes' theorem means, gluing, manifolds with boundary, etc. There's also the new stuff afforded by geometry, e.g. the Lie derivative of a form and Cartan's formulas for commutators of these operators. These enhance to the world of differential cohomology, so differential cohomology is expressing calculus of local objects which have an integrality condition.

Historically, differential cohomology was first studied by Cheeger and Simons [CS85], following the work of Chern and Simons, both in the early 1970s. Cheeger and Simons introduced something called differential characters.

Definition 1.5 (Cheeger-Simons [CS85]). Let X be a smooth manifold. A degree-k *differential character* is a homomorphism $\chi\colon Z_{k-1}(X) \to \mathbb{R}/\mathbb{Z}$ such that there exists an $\omega(\chi) \in \Omega^k(X)$ such that

$$\chi(b) = \int_C \omega(\chi) \bmod 1, \tag{1.6}$$

where $b = \partial c$ for some $c \in C_k(X)$. Here Z_{k-1} and C_k are the *smooth* chains of degrees $k-1$, resp. k: we only consider formal sums of smooth maps of the standard n-simplex into X, not just continuous ones.

Remark 1.7. Cheeger and Simons use a different degree convention in their original paper; don't get tripped up by that! ◀

It follows from the definition that ω is unique, and is closed. The degree-k differential characters define a group $\check{H}^k(X)$, and this is isomorphic to our explicit constructions of $\check{H}^1(X)$ and $\check{H}^2(X)$: the character is the map from a chain to the integral over the chain.

Deligne approached differential cohomology in a different way: fix a $k > 0$ and consider the cochain complex $\mathbb{Z}(k)$ given by

$$0 \longrightarrow \mathbb{Z} \longrightarrow \Omega^0(X) \xrightarrow{\mathrm{d}} \Omega^1(X) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^{k-1}(X) \longrightarrow 0. \tag{1.8}$$

Let's approach this with Čech cohomology, say in the case $k = 1$, where we have $0 \to \mathbb{Z} \to \Omega^0(X) \to 0$. Let $\mathfrak{U}$ be a good open cover of X; then we have a double complex

$$\begin{tikzcd} \prod\limits_{U\in\mathfrak{U}} \Omega^0(U) \arrow[r, "\delta"] & \prod\limits_{U\neq V\in\mathfrak{U}} \Omega^0(U\cap V) \arrow[r, "\delta"] & \cdots \\ \prod\limits_{U\in\mathfrak{U}} H^0(U;\mathbb{Z}) \arrow[r, "\delta"] \arrow[u] & \prod\limits_{U\neq V\in\mathfrak{U}} H^0(U\cap V) \arrow[r, "\delta"] \arrow[u] & \cdots . \end{tikzcd} \tag{1.9}$$

In most degrees, this is the usual cohomology of X, either with $\mathbb{Z}$ or $\mathbb{R}/\mathbb{Z}$ coefficients. But for degree k, this is something interesting, and you can directly check that for $k = 0, 1, 2$ we get $\check{H}^k(X)$ as we explictly described it.

Applications. One reason to like differential cohomology is to consider generalizations of the integration maps we considered on $\check{H}^1$ and $\check{H}^2$ (both the primary and secondary invariants). For example, if G is a Lie group and $\lambda \in H^k(BG;\mathbb{Z})$, then λ defines a characteristic class of principal G-bundles $P \to M$ on a closed, oriented manifold M, namely $\int_X \lambda(P) \in \mathbb{Z}$.

There is a refinement of $\lambda(P)$ to a class $\check{\lambda}(P,\theta) \in \check{H}^k(X)$ which depends both on a bundle and a connection, and in a sense all of Chern-Weil theory refines to the homomorphism $\omega\colon \check{H}^k(X) \to \Omega^k(X)_{\mathrm{cl}}$. If we integrate over a closed, oriented k-manifold, this recovers the topological invariant, but if we integrate over a closed, oriented $(k-1)$-manifold, you get the classical Chern-Simons invariant. A sufficiently robust integration theory, with a good geometric model, would yield things like integration on manifolds with boundary, or an integral over a $(k-2)$-manifold as a Hermitian line (since it should live in $\check{H}^2(\mathrm{pt})$), and this should all stitch together nicely into an invertible field theory. One might expect more examples of invertible field theories coming from other bordism invariants or cohomology theories.

Another application: let $E \to X$ be an oriented, rank-r real vector bundle, and let $c(E) \in H^r(X;\mathbb{Z})$ be the Bockstein of the Stiefel-Whitney class $w_{r-1}(E) \in H^{r-1}(X;\mathbb{Z})$. This is the Euler class, which is 0 when r is even but generally nonzero when r is odd. One can study this with differential cohomology, and this viewpoint is amenable to generalizations (e.g. to differential K- and KO-theory, where it's particularly useful).

Differential cohomology has some applications to physics. Recall Maxwell's equations for a 2-form F on a Lorentzian 4-manifold X:

$$\begin{aligned} \mathrm{d}F &= 0 \\ \mathrm{d}{\star}F &= j_E, \end{aligned} \tag{1.10}$$

where j_E is a 3-form thought of as the electric current: if we have worldlines of a bunch of particles with charges q_i, we can take the dual and obtain a (generally distributional) 3-form.[1]

In classical electromagnetism, the charges are real numbers, but Dirac pointed out that in quantum mechanics, and once we have a nonzero magnetic current, the charges must be quantized. Thus we're in need of calculus with an integrality condition, leading us to differential cohomology.

Some general theory. Let $\mathcal{M}an$ denote the category whose objects are smooth manifolds and whose morphisms are smooth maps. By a *presheaf (on $\mathcal{M}an$)* we mean a contravariant functor $F\colon \mathrm{Map}^{op} \to \mathcal{S}et$. You can think of this as something like a distribution, except instead of evaluating them on test functions we're evaluating them on test manifolds.

Remark 1.11. You may be used to presheaves on a single smooth manifold M; this is related to the general notion we defined above. Specifically, we just restrict to the subcategory of open subsets of M as objects and inclusions as morphisms. ◀

Example 1.12. Differential k-forms define a presheaf Ω^k, sending $M \mapsto \Omega^k(M)$ (which for now we only regard as a set); functoriality is by pullback. The same is true for $\check{H}^1$.

A smooth manifold X defines a presheaf F_X, with $F_X(M) \coloneqq \mathrm{Map}(M, X)$. There's a general lemma in category theory, called the Yoneda lemma, that F_X knows X: we can use that to get the set of points of X, and learn about its topology by seeing which points can be connected by a map from $\mathbb{R}$, etc. If we think only about smooth maps, it's possible to see the smooth structure on X. ◀

A presheaf naturally isomorphic to F_X for some X is called *representable*. For example, $\check{H}^1 \simeq F_{\mathbb{R}/\mathbb{Z}}$.

Example 1.13. The space $\mathrm{Map}(X, Y)$ is not a smooth manifold – it's infinite-dimensional for X, Y not discrete. But if we only need to care about finite-dimensional families, then we can use the presheaf $\mathrm{Map}(X, Y)$, whose value on M is the set $\mathrm{Map}(M \times X, Y)$. This is regarding $\mathrm{Map}(X, Y)$, and presheaves in general, as generalizations of smooth manifolds. ◀

If we want to consider things such as principal bundles and $\check{H}^2$, $\mathcal{S}et$ is not the correct target: principal bundles on X have morphisms between them, so we should really consider groupoid-valued presheaves (or more generally, presheaves valued in simplicial sets). For example, if G is a Lie group, we can let $B_\nabla(G)$ denote the groupoid of principal G-bundles with connection on M, which defines a groupoid-valued presheaf, and $\check{H}^2$ is precisely $\pi_0 B_\nabla(\mathbb{R}/\mathbb{Z})$.

But now we can ask fun questions like, what's the de Rham complex for $B_\nabla G$? This amounts to finding differential forms on each manifold compatible under pullback, or some sort of natural differential forms. This relates to early work of Thurston.

Theorem 1.14. *The de Rham complex of $B_\nabla G$ is $\mathrm{Sym}^{2\bullet}(\mathfrak{g}^*)^G$, i.e. invariant even-degree G-invariant polynomials on $\mathfrak{g}$, and the differential vanishes.*

So Chern-Weil theory sees all of the invariant differential forms, which is nice. See Freed-Hopkins [FH13] for a proof.

One possibly strange aspect of the above calculation is that the de Rham complex is levelwise finite-dimensional, which is unusual.

There are three versions of BG in the world of (pre)sheaves of groupoids:

- $B_\nabla G$, as above,
- $B_\bullet G$, which assigns the groupoid of principal bundles without any connection, and
- BG, which assigns to M the set $\mathrm{Map}(M, BG)$.[2]

[1] This is part of why differential forms where functions are replaced with distributions are called *currents*.

[2] TODO: groupoid structure?

There are maps $B_\nabla G \to B_\bullet G \to BG$.

One can also put Deligne's complex $\mathbb{Z}(k)$ into this world, e.g. using the Dold-Thom theorem to pass from a chain complex to an abelian group. For even k, there are maps

$$H^k(BG;\mathbb{Z}) \longrightarrow H^k(B_\bullet G;\mathbb{Z}(k/2)) \longrightarrow H^k(B_\nabla G;\mathbb{Z}(k)), \tag{1.15}$$

which sends $\lambda \mapsto \check{\lambda}$. For suitably chosen G, this provides a geometric construction of a certain central extension of $\mathrm{Diff}_+(S^1)$ that appears in conformal field theory. This runs into old work of Bott and Haefliger, work on characteristic classes of foliations, who certainly knew plenty of this in different language.

References

[CS85] Jeff Cheeger and James Simons. Differential characters and geometric invariants. In *Geometry and Topology*, pages 50–80, Berlin, Heidelberg, 1985. Springer Berlin Heidelberg. 3

[FH13] Daniel S. Freed and Michael J. Hopkins. Chern-Weil forms and abstract homotopy theory. 2013. http://arxiv.org/abs/1301.5959. 4