M392c NOTES: p-ADIC HODGE THEORY

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These notes were taken in UT Austin's M392c (p-adic Hodge Theory) class in Fall 2019, taught by Sam Raskin. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Any mistakes in the notes are my own.

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Lecture 1.

The C^∞ de Rham comparison theorem: 8/29/19

"If you're over the age of, say, 40, the rest of the lecture's not going to be very fun for you."

Today we'll begin with some background, since there's not a lot of characteristic p work done in this department. After that, the rest of the syllabus is:

- Witt vectors, which are relatively elementary but not a lot of people probably know going into the course,
- the (derived) de Rham complex,
- perfectoid algebras,
- p-adic analytic geometry, and finally
- the main theorem: the de Rham comparison theorem.

The de Rham comparison theorem is the p-adic analogue of the de Rham theorem on differential forms. This is essentially multivariable calculus in the differential-geometric setting, but in p-adic algebraic geometry it's a harder theorem. The first versions were proved in the 1970s and 1980s, but we'll follow a more recent proof incorporating ideas of Scholze, Bhatt, and Beĭlinson, with our main reference a paper that Scholze wrote as a grad student.

Our goal will be to prove the de Rham comparison theorem in p-adic analytic geometry by analogy with the de Rham comparison theorem in the C^∞, holomorphic, and algebraic setting. That is, we'd like to generalize the following argument.

Let X be a smooth manifold.[1] Then we can attach two kinds of cohomology to X.

(1) Singular cohomology $H^*(X;\mathbb{Z})$; p-adic geometers usually call this *Betti cohomology*.
(2) de Rham cohomology $H^*_{\mathrm{dR}}(X)$.

Recall that the latter is built out of the cochain complex of differential forms:

$$0 \longrightarrow C^\infty(X) \xrightarrow{\mathrm{d}} \Omega^1(X) \xrightarrow{\mathrm{d}} \Omega^2(X) \longrightarrow \cdots \tag{1.1}$$

de Rham cohomology is the cohomology of this cochain complex. This is a chain complex of real vector spaces, and hence de Rham cohomology is also a sequence of vector spaces.

Theorem 1.2 (C^∞ de Rham comparison)**.** *There is a canonical isomorphism $H^i(X;\mathbb{Z})\otimes\mathbb{R} \cong H^i_{\mathrm{dR}}(X)$, and moreover there is a canonical quasi-isomorphism between the singular cochain complex tensored with $\mathbb{R}$ and the de Rham cochain complex.*

[1] Standing assumptions: manifolds have the homotopy type of a finite CW complex and are paracompact. Both of these mean that they're not too big.

"Canonical" is easy to make precise for cohomology groups: this is functorial in X with respect to pullback. For the cochain complexes statement, the notion of canonical is fuzzier.

Recall that the singular chain complex involves topology, with maps of simplices into X. de Rham cohomology is about calculus, so there's something nontrivial going on here.

The best-organized approach to proving Theorem 1.2 is sheaf cohomology. This uses the fact that the singular cochain complex $C_B(X;\mathbb{Z})$ is quasi-isomorphic to $\mathbf{R}\Gamma(X;\underline{\mathbb{Z}})$, where $\underline{\mathbb{Z}}$ is the (locally) constant sheaf on X valued in $\mathbb{Z}$.

There is also a complex of sheaves dR_X on X, defined as follows. For $i \geq 0$, let Ω^i_X denote the sheaf on X whose value on an open set U is $\Omega^i(U)$. As the de Rham differential commutes with pullback, it defines a map of sheaves $\mathrm{d}\colon \Omega^i_X \to \Omega^{i+1}_X$. Putting these together defines the de Rham complex:

$$\mathrm{dR}_X \coloneqq \left(0 \longrightarrow \Omega^0_X \xrightarrow{\mathrm{d}} \Omega^1_X \xrightarrow{\mathrm{d}} \Omega^2_X \longrightarrow \cdots\right) \tag{1.3}$$

Slightly abusing notation, let $\underline{\mathbb{R}}$ also denote the complex of sheaves which has $\underline{\mathbb{R}}$ in degree 0 and is 0 elsewhere. Let $\psi\colon \underline{\mathbb{R}} \to \mathrm{dR}_X$ denote the map of complexes which is the inclusion of locally constant functions as 0-forms and is zero elsewhere.

Lemma 1.4. *$\psi\colon \underline{\mathbb{R}} \to \mathrm{dR}_X$ is a quasi-isomorphism; equivalently, $H^0(\mathrm{dR}_X) = \underline{\mathbb{R}}$ and $H^i(\mathrm{dR}_X) = 0$ for $i > 0$.*

Here, the cohomology of a complex of sheaves is the cohomology of the complex, which produces sheaves.

Lemma 1.5. *$\mathbf{R}^j\Gamma(X, \Omega^i_X) = 0$ for $j > 0$, and $\mathbf{R}^0\Gamma(X, \Omega^i_X) = \Omega^i(X)$.*

Assuming these lemmas, the proof of Theorem 1.2 isn't hard.

Proof of Theorem 1.2.

$$\mathbf{R}\Gamma(X, \underline{\mathbb{Z}}) \otimes \mathbb{R} = \mathbf{R}\Gamma(X, \underline{\mathbb{R}}) \overset{(1.4)}{=} \mathbf{R}\Gamma(X, \mathrm{dR}_X) \overset{(1.5)}{=} \Gamma(X, \mathrm{dR}_X) = C_{\mathrm{dR}}(X). \tag{1.6}$$ ⊠

We will omit the proof of Lemma 1.5; the proof uses partitions of unity to split a Čech cosimplicial complex. For Lemma 1.4, the key input is the Poincaré lemma: that if $\omega \in \Omega^i(\mathbb{R}^n)$ is closed (i.e. $\mathrm{d}\omega = 0$) and $i > 0$, then ω is exact (i.e. $\omega = \mathrm{d}\eta$ for some $\eta \in \Omega^{i-1}(\mathbb{R}^n)$). Moreover, if $i = 0$ and $\mathrm{d}\omega = 0$, then ω is constant.

Proof of Lemma 1.4. Let $i > 0$ and consider the de Rham complex at degree i; let $\mathrm{d}^{i-1} \coloneqq \mathrm{d}\colon \Omega^{i-1}_X \to \Omega^i_X$ and define d^i analogously. We want to show that $\ker(\mathrm{d}^i) = \mathrm{Im}(\mathrm{d}^{i-1})$.

A section of $\ker(\mathrm{d}^i)$ on an open $U \subseteq X$ is a closed form $\omega \in \Omega^i(U)$. It suffices to show that there's an open cover $\mathfrak{V}$ of U such that for all $V \in \mathfrak{V}$, $\omega|_V \in \mathrm{Im}(\mathrm{d}^{i-1})$. Let $\mathfrak{V}$ be an atlas of U, so each $V \in \mathfrak{V}$ is diffeomorphic to V and we may apply the Poincaré lemma and conclude.

Therefore we need to prove the Poincaré lemma! To do this we really need a version of the fundamental theorem of calculus. Let t denote the first coordinate in $\mathbb{R} \times X$. If $\omega \in \Omega^i(\mathbb{R} \times X)$, the Lie derivative of ω is by definition

$$\mathcal{L}_{\partial_t}\omega \coloneqq (\mathrm{d}\iota_{\partial_t} + \iota_{\partial_t}\mathrm{d})(\omega). \tag{1.7}$$

Choose local coordinates $x_1, \dots, x_n$ on X such that we can write[2]

$$\omega = f\,\mathrm{d}t \wedge \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{i-1} \qquad \text{or} \qquad \omega = f\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_i. \tag{1.8}$$

Exercise 1.9. Show that in the first case, $\mathcal{L}_{\partial_t}\omega = (\partial_t f)\,\mathrm{d}t \wedge \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{i-1}$, and in the second case, it's $(\partial_t f)\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_i$.

We can integrate along the $\mathbb{R}$-coordinate, defining a map $\int_0^1\colon \Omega^{i+1}(\mathbb{R} \times X) \to \Omega^i(X)$; in coordinates,

$$\begin{aligned} f\,\mathrm{d}t \wedge \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_i &\longmapsto \left(\int_0^1 f\,\mathrm{d}t\right)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_i \\ f\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{i+1} &\longmapsto 0. \end{aligned} \tag{1.10}$$

The version of the fundamental theorem of calculus that we need is:

$$\int_0^1 \mathrm{d}t \wedge \mathcal{L}_{\partial t}\omega = \omega|_{\{1\}\times X} - \omega|_{\{0\}\times X} \in \Omega^i(X). \tag{1.11}$$

[2] A general such ω is a finite sum of these, but everything we need will be linear, so it doesn't matter. TODO: double-check that this footnote is necessary.

Now we can prove the Poincaré lemma. Let $X \coloneqq \mathbb{R}^n$ and $\sigma\colon \mathbb{R}\times X \to X$ be the scaling map

$$\sigma(t, x_1, \dots, x_n) \coloneqq (tx_1, \dots, tx_n). \tag{1.12}$$

Let ω be a closed i-form on X, where $i > 0$, and let

$$\eta \coloneqq -\int_0^1 \mathrm{d}t \wedge \iota_{\partial_t}\sigma^*\omega \in \Omega^{i-1}(X). \tag{1.13}$$

We'll show that $\mathrm{d}\omega = \eta$. Since we'll be applying the de Rham operator on different manifolds, let d_M denote the de Rham operator on forms on the manifold M. Then

$$\begin{aligned}
\mathrm{d}_X\eta &= -\mathrm{d}_X \int_0^1 \mathrm{d}t \wedge \iota_{\partial_t}\sigma^*\omega && (1.14)\\
&= -\int_0^1 (\mathrm{d}_{\mathbb{R}\times X} - \mathrm{d}_{\mathbb{R}})\, \mathrm{d}t \wedge \iota_{\partial_t}\sigma^*\omega && (1.15)\\
&= -\int_0^1 \mathrm{d}_{\mathbb{R}\times X}\, \mathrm{d}t \wedge \iota_{\partial_t}\sigma^*\omega && (1.16)\\
&= \int_0^1 \mathrm{d}t \wedge (\mathrm{d}_{\mathbb{R}\times X}\iota_{\partial_t}\sigma^*\omega) && (1.17)\\
&= \int_0^1 \mathrm{d}t \wedge \Big(\mathcal{L}_{\partial_t}\sigma^*\omega - \underbrace{\iota_{\partial_t}\mathrm{d}_{\mathbb{R}\times X}\sigma^*\omega}_{=0}\Big). && (1.18)
\end{aligned}$$

Now, using the fundamental theorem of calculus,

$$= \sigma^*\omega|_{\{1\}\times X} - \sigma^*\omega|_{\{0\}\times X} = \omega. \qquad \boxtimes$$

You might think of this theorem as trivial, in that some of the steps (such as the fundamental theorem of calculus) are really elementary. But if you want to prove this theorem in a different setting (say, p-adic analytic geometry), you'll have to rebuild all of this machinery, and that's less elementary.

There's a variant of the de Rham comparison theorem for complex manifolds. On a complex manifold X, let Ω_X^i denote the sheaf of holomorphic i-forms on X, and $\Omega_X^{p,q}$ denote the sheaf of smooth (p,q)-forms. For example, on a Riemann surface, $\Omega_X^{1,0}$ is 1-forms which can locally be written as $f(x,y)\,\mathrm{d}z$, and $\Omega_X^{0,1}$ is the 1-forms which can locally be written as $f(x,y)\,\mathrm{d}\overline{z}$. Let dR_X denote the holomorphic de Rham complex, defined in the same way as in the C^∞ setting, just with holomorphic differential forms.

Again $H^i(\mathrm{dR}_X) = 0$ for $i > 0$ and $H^0(\mathrm{dR}_X) \cong \underline{\mathbb{C}}$.

Exercise 1.19. Recall that the complex

$$0 \longrightarrow \Omega_X^i \longrightarrow \Omega_X^{i,0} \xrightarrow{\overline{\partial}} \Omega_X^{i,1} \xrightarrow{\overline{\partial}} \cdots \tag{1.20}$$

is acyclic. Using this, deduce the holomorphic Poincaré lemma.

Corollary 1.21. *There is a quasi-isomorphism* $\mathbf{R}\Gamma(X, \mathrm{dR}_X) \simeq C_B(X)\otimes\mathbb{C}$.

Lemma 1.22. *The analogue of Lemma 1.5, that local sections are exact, is not true here unless X is a* Stein manifold *(e.g. the space of complex points of a smooth affine variety).*

We're emphasizing the analytic perspective because that's how we're going to work in the p-adic setting.

With the de Rham comparison theorem out of the way, let's talk about the motivation for p-adic numbers. We like the rationals, but they're not complete: for some questions, we need to pass to $\mathbb{R}$ or something akin to it (including $\mathbb{C}$), and $\mathbb{Q} \hookrightarrow \mathbb{R}$. But there are other questions where we need to embed in a different field $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$, where p is a fixed prime and $\mathbb{Q}_p$ denotes the field of *p-adic numbers*.

$\mathbb{R}$ holds a priviledged position in our intuition, because we use it to measure distances and learned it earlier, but at least in this course you should not think of $\mathbb{Q} \hookrightarrow \mathbb{R}$ and $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$ as very different.

Definition 1.23. The ring of *p-adic integers* are $\mathbb{Z}_p \coloneqq \varprojlim_n \mathbb{Z}/p^n$.

You can think of a p-adic integer as an integer mod p together with information of a lift mod p^2, a lift of that mod p^3, and so on.

If you're familiar with completions with respect to an ideal, $\mathbb{Z}_p$ is the completion of $\mathbb{Z}$ with respect to (p). Since p isn't a zero divisor, $\mathbb{Z}_p$ is an integral domain.

Definition 1.24. The *p-adic numbers* $\mathbb{Q}_p$ is the field of fractions of $\mathbb{Z}_p$.

Exercise 1.25. Show that $x \in \mathbb{Z}_p$ is a unit iff its image in $\mathbb{Z}/p$ is. Hence, $\mathbb{Q}_p = \mathbb{Z}_p[1/p]$.

We want to think of the p-adic numbers as on equal footing with $\mathbb{R}$, meaning we want analysis on $\mathbb{Q}_p$.

Here's a first use of the p-adics. Let X be an algebraic variety over $\mathbb{Q}$, meaning it's locally the zero set of polynomials with rational coefficients. In general, asking whether these polynomials have a solution in rational numbers, i.e. whether $X(\mathbb{Q})$ is empty, is a tricky question. You can extend scalars and try to prove that $X(\mathbb{R}) = \varnothing$, which is sufficient (but not necessary), and might admit an analytic proof.

But now, we have many more sufficient conditions: that $X(\mathbb{Q}_p) = \varnothing$, for any p.

Example 1.26.

(1) Let $X = \{x^2 + y^2 = -1\}$. It's not hard to show that $X(\mathbb{R}) = \varnothing$, hence that $X(\mathbb{Q}) = \varnothing$.

(2) If $X = \{2x^2 + 3y^2 = 1\}$, then $X(\mathbb{R}) \neq \varnothing$, but you can show that $X(\mathbb{Q}_2) = 0$ and (harder) $X(\mathbb{Q}_3) = 0$, so there are no rational solutions. ◀

One way you might think of this is that X, as a variety over $\mathbb{Q}$, can be base-changed to $X_{\mathbb{R}}$ and $X_{\mathbb{Q}_p}$ for all p, leading to additional information, and that in some very imprecise sense, X is "amalgamated" from all of these base changes. Depending on what exactly you want from this idea, this leads to the Langlands program!

Now here's a completely different application: $\mathbb{Q}_p$ and $\mathbb{R}$ appear as coefficients for cohomology theories in algebraic geometry. There's a whole zoo of them, though some of them aren't very good.

Let X be a variety over a field k. Here are our cohomology theories.

Betti cohomology: Here, the input is an embedding $j\colon k \hookrightarrow \mathbb{C}$. Then we can define $H^*_{B,j}(X, \mathbb{Z}) := H^*_B(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Z})$, where $X^{\mathrm{an}}_{\mathbb{C}}$ denotes $X(\mathbb{C})$ with its complex topology. This is nice (yay, complex numbers!) but isn't always defined, e.g. if k has positive characteristic, or if it is characteristic zero and has larger cardinality than $\mathbb{C}$. Still, this is the first cohomology theory, and the happiest one in the settings where we can define it.

Étale cohomology: Let ℓ be a prime number. Then we can define étale cohomology groups $H^*_{\text{ét}}(X; \mathbb{Z}_\ell)$, $H^*_{\text{ét}}(X; \mathbb{Q}_\ell)$, and $H^*_{\text{ét}}(X; \mathbb{Z}/\ell^n)$. We will discuss these more later. For now, one nice feature is that if A is a finite abelian group, $H^1_{\text{ét}}(X; A)$ classifies *A-torsors*, the analogue of principal A-bundles in algebraic geometry: precisely, these are finite étale covers of X with an étale-locally simply transitive A-action. This takes as input a key idea of Serre: arbitrary covers don't port well from toplogy to geometry, but finite ones do.