MSRI: QUANTUM SYMMETRIES INTRODUCTORY WORKSHOP

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These notes were taken at MSRI's introductory workshop on quantum symmetries in Spring 2020. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

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Part 1. Monday, January 27

1. Sarah Witherspoon: Hopf algebras, I

Our perspective on Hopf algebras, their actions on rings and modules, and the structures on their categories of rings and modules, will be to think of them as generalizations of group actions and representations; groups actions are symmetries in the usual sense, and Hopf algebra actions are often related to "quantum symmetries."

We're not going to give the full definition of a Hopf algebra, because it would require drawing a lot of commutative diagrams, but we'll say enough to give the picture.

Throughout this talk we work over a field k; all tensor products are of k-vector spaces.

Definition 1.1. A *Hopf algebra* is an algebra A together with k-linear maps $\Delta\colon A \to A \otimes A$, called *comultiplication*; $\varepsilon\colon A \to k$, called the *counit*; and $S\colon A \to A$, called the *coinverse*. These maps must satisfy some properties, including that ε is an algebra homomorphism and that S is an *anti-automorphism*, i.e. that $S(xy) = S(y)S(x)$.

The definition is best understood through examples.

Example 1.2.

(1) Let G be a group. Then the group algebra $k[G]$ is a Hopf algebra, where for all $g \in G$, $\Delta(g) := g \otimes g$, $\varepsilon(g) := 1$, and $S(g) := g^{-1}$. This is a key example that allows us to generalize ideas from group actions to Hopf algebra actions: whenever we define a notion for Hopf algebras, when we implement it for $k[G]$ it should recover that notion for groups.

(2) Let $\mathfrak{g}$ be a Lie algebra over k. Then its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is a Hopf algebra, where for all $x \in \mathfrak{g}$, $\Delta(x) := x \otimes 1 + 1 \otimes x$, $\varepsilon(x) := 0$, and $S(x) := -x$. Since ε is an algebra homomorphism, $\varepsilon(1_{\mathcal{U}(\mathfrak{g})}) = 1$.

For example,

$$\mathcal{U}(\mathfrak{sl}_2) = k\langle e, f, h \mid ef - fe = h, he - eh = 2e, hf - fh = -2f\rangle, \tag{1.3}$$

given explicitly by the basis of $\mathfrak{sl}_2$

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.4}$$

◀

Both of these examples are classical, in that they've been known for a long time. But more recently, in the 1980s, people discovered new examples, coming from quantum groups.

Example 1.5 (Quantum $\mathfrak{sl}_2$)**.** Let $q \in k^\times \setminus \{\pm 1\}$. Then, given a simple Lie algebra $\mathfrak{g}$, we can define a "quantum group," $\mathcal{U}_q(\mathfrak{g})$, which is a Hopf algebra. For example, for $\mathfrak{sl}_2$,

$$\mathcal{U}_q(\mathfrak{sl}_2) = k\left\langle E, F, K^{\pm 1} \mid EF - FE = \frac{K - K^{-1}}{q - q^{-1}}, KE = q^2 EK, KF = q^{-2} EK \right\rangle, \tag{1.6}$$

with comultiplication

$$\Delta(E) := E \otimes 1 + K \otimes E \tag{1.7a}$$
$$\Delta(F) := F \otimes K^{-1} + 1 \otimes F \tag{1.7b}$$
$$\Delta(K^{\pm 1}) := K^{\pm 1} \otimes K^{\pm 1} \tag{1.7c}$$

and counit $\varepsilon(E) = \varepsilon(F) = 0$ and $\varepsilon(K) = 1$. This generalizes to other simple $\mathfrak{g}$, albeit with more elaborate data. ◀

Example 1.8 (Small quantum $\mathfrak{sl}_2$)**.** Let q be an n^{th} root of unity. Then, as before, given a simple Lie algebra $\mathfrak{g}$, we can define a Hopf algebra $u_q(\mathfrak{g})$, called the *small quantum group* for $\mathfrak{g}$ and q, which is a finite-dimensional vector space over k; for $\mathfrak{sl}_2$, this is

$$u_q(\mathfrak{sl}_2) = \mathcal{U}_q(\mathfrak{sl}_2)/(E^n, F^n, K^n - 1). \tag{1.9}$$

◀

Before we continue, we need some useful notation for comultiplication, called *Sweedler notation.* Let A be a Hopf algebra and $a \in A$; then we can symbolically write

$$\Delta(a) = \sum_{(a)} a_1 \otimes a_2. \tag{1.10}$$

Comultiplication in a Hopf algebra is *coassociative*, in that as maps $A \to A \otimes A \otimes A$,

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta. \tag{1.11}$$

Therefore when we iterate comultiplication, we can symbolically write

$$(\mathrm{id} \otimes \Delta) \circ \Delta(a) = \sum_{(a)} a_1 \otimes a_2 \otimes a_3 \tag{1.12}$$

without worrying about parentheses.

Actions on rings. Hopf algebra actions on rings generalize group actions on rings by automorphisms and actions of Lie algebras on rings by derivations. If a group G acts on a ring R, then for all $g \in G$ and $r, r' \in R$,

$$g(rr') = (gr)(gr') \tag{1.13a}$$

$$g(1_R) = 1_R. \tag{1.13b}$$

In $k[G]$, our Hopf algebra avatar of G, $\Delta(g) = g \otimes g$, and $\varepsilon(g) = 1$.

If a Lie algebra $\mathfrak{g}$ acts on a ring R by derivations, then for all $x \in \mathfrak{g}$ and $r, r' \in R$,

$$x \cdot (rr') = (x \cdot r)r' + r(x \cdot r') \tag{1.14a}$$

$$x \cdot (1_R) = 0. \tag{1.14b}$$

In $\mathcal{U}(\mathfrak{g})$, our Hopf algebra avatar of $\mathfrak{g}$, $\Delta(x) = x \otimes 1 + 1 \otimes x$, and $\varepsilon(x) = 0$. These two examples suggest how we should implement a general Hopf algebra action on a ring: comultiplication tells us how to act on the product of two elements, and the counit tells us how to act on 1.

Definition 1.15. Let A be a Hopf algebra and R be a k-algebra. An *A-module algebra structure* on R is data of an A-module structure on R such that for all $a \in A$ and $r, r' \in R$,

$$a \cdot (rr') = \sum_{(a)} (a_1 \cdot r)(a_2 \cdot r) \tag{1.16a}$$

$$a \cdot (1_R) = \varepsilon(a)1_R. \tag{1.16b}$$

Thus a group action as in (1.13) defines an action of the Hopf algebra $k[G]$, and a Lie algebra action as in (1.14) defines an action of the Hopf algebra $\mathcal{U}(\mathfrak{g})$.

Example 1.17. The quantum analogue of the $\mathfrak{sl}_2$-action on $k[x, y]$, thought of as (functions on the) plane, there is an action of $\mathcal{U}_q(\mathfrak{sl}_2)$ on the *quantum plane*

$$R \coloneqq k\langle x, y \mid xy = qyx\rangle. \tag{1.18}$$

This is a deformation of $k[x, y]$, which is the case $q = 1$. The explicit data of the action is

$$E \cdot x = 0 \qquad F \cdot x = y \qquad K^{\pm 1} \cdot x = q^{\pm 1}x \tag{1.19}$$

$$E \cdot y = x \qquad F \cdot y = 0 \qquad K^{\pm 1}y = q^{\mp 1}y. \tag{1.20}$$

One has to check that this extends to an action satisfying Definition 1.15, but it does, and R is an A-module algebra. Here E and F act as *skew-derivations*, e.g.

$$E \cdot (rr') = (E \cdot r)r' + (K \cdot r)(E \cdot r') \tag{1.21}$$

for all $r, r' \in R$. ◀

Given a Hopf algebra action of A on R in this sense, we can construct two useful rings: the *invariant subring*

$$R^A \coloneqq \{r \in R \mid a \cdot r = \varepsilon(a) \cdot r \text{ for all } a \in A\}, \tag{1.22}$$

and the *smash product ring* $R \# A$, which as a vector space is $R \otimes A$, with multiplication given by

$$(r \otimes a)(r' \otimes a') \coloneqq \sum_{(a)} r(a_1 \cdot r') \otimes a_2 a'. \tag{1.23}$$

The smash product ring knows the A-module algebra structure on R. Often, rings we're interested in for other reasons are smash product rings of interesting Hopf algebra actions, and identifying this structure is useful.

Example 1.24. The *Borel subalgebra* of $\mathcal{U}_q(\mathfrak{sl}_2)$ is $k\langle E, K^{\pm 1} \mid KE = q^{-2}K\rangle$. This is isomorphic to the smash product $k[E] \# k\langle K\rangle$, where $k\langle K\rangle$ is the group algebra of the free group on the single generator K.

In fact, there's a sense in which $\mathcal{U}_q(\mathfrak{sl}_2)$ is a deformation of $k[E, F] \# k\langle K\rangle$: in this smash product ring, E and F commute, and we deform this to $\mathcal{U}_q(\mathfrak{sl}_2)$, in which they don't commute. ◀

Modules. Given a Hopf algebra A, what is the structure of its category of modules? The first thing we can do is take the tensor product of A-modules U and V using comultiplication: for $a \in A$, $u \in U$, and $v \in V$,

$$a \cdot (u \otimes v) = \sum_{(a)} a_1 \cdot u \otimes a_2 \cdot v. \tag{1.25}$$

Moreover, k has a canonical A-module structure via the counit: $a \cdot x := \varepsilon(a)x$ for $a \in A$ and $x \in k$. Finally, if U is an A-module, its vector space dual $U^* := \mathrm{Hom}_k(U, k)$ has an A-module structure via S: for all $a \in A$, $u \in U$, and $f \in U^*$, $(a \cdot f)(u) := f(S(a)u)$.

The existence of tensor products, duals, and the ground field in the world of Hopf algebra modules is a nice feature: these aren't always present for a general associative algebra. Moreover, these constructions interact well with each other.

(1) Coassociativity of Δ implies the tensor product is associative: for A-modules U, V, and W, we have a natural isomorphism $U \otimes (V \otimes W) \xrightarrow{\cong} (U \otimes V) \otimes W$.
(2) In any Hopf algebra A, we have the condition

$$\sum_{(a)} \varepsilon(a_1)a_2 = \sum_{(a)} a_1\varepsilon(a_2) \tag{1.26}$$

for any $a_1, a_2 \in A$. This implies k, as an A-module, is the unit for the tensor product: we have natural isomorphisms $k \otimes U \cong U \cong U \otimes k$ for an A-module U.
(3) Suppose U is an A-module which is a finite-dimensional k-vector space. Then it comes with data of a *coevaluation map* $c\colon k \to U \otimes U^*$ sending

$$1 \longmapsto \sum_i u_i \otimes u_i^*, \tag{1.27}$$

where $\{u_i\}$ is a basis for U over k and $\{u_i^*\}$ is its dual basis; this map turns out to be independent of basis. We also have an *evaluation map* $e\colon U^* \otimes U \to k$ sending $f \otimes u \mapsto f(u)$. Now, not only are these A-module homomorphisms, but the composition

$$U \xrightarrow{c \otimes \mathrm{id}_U} U \otimes U^* \otimes U \xrightarrow{\mathrm{id}_U \otimes e} U \tag{1.28}$$

is the identity map.

Definition 1.29. A *tensor category*, or *monoidal category* is a category $\mathcal{C}$ together with a functor $\otimes\colon \mathcal{C}\times\mathcal{C} \to \mathcal{C}$, an object $\mathbf{1} \in \mathcal{C}$ called the *unit*, and natural isomorphisms $U\otimes(V\otimes W) \cong (U\otimes V)\otimes W$ and $\mathbf{1}\otimes U \cong U \cong U\otimes\mathbf{1}$ for all objects U, V, and W in $\mathcal{C}$, subject to some coherence conditions.

Our key examples of tensor categories are the category of modules over a Hopf algebra A, as well as the subcategory of finite-dimensional modules.

If the coinverse of A is invertible, which is always the case when A is finite-dimensional over k, then $\mathcal{C} = \mathcal{M}od_A$ is a *rigid* tensor category, meaning that every object U has a *right dual* ${}^*U := \mathrm{Hom}_k(U, k)$, which means the composition (1.28) is the identity.

Remark 1.30. Notations for left and right duals differ. We're following [EGNO15], but Bakalov-Kirillov [BK01] use a different convention; be careful! ◀

Some Hopf algebras' categories of modules have additional structure or properties: they might be semisimple, or braided, or even symmetric. This amounts to additional information on the Hopf algebra itself.

2. Victor Ostrik, Introduction to fusion categories, I

In the world of classical symmetries, i.e. those given by group actions, there is a particularly nice subclass: finite groups. If you know your symmetry group is finite, you can take advantage of many simplifying assumptions. Likewise, in the setting of quantum symmetries, given by, say, $\mathbb{C}$-linear tensor categories, fusion subcategories form a very nice subclass for which many simplifying assumptions hold. And indeed, if G is a finite group, its category of finite-dimensional representations is a fusion category.

Recall that a *monoidal category* is a category $\mathcal{C}$ together with a functor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a distinguished object $\mathbf{1} \in \mathcal{C}$ called the *unit*, together with natural isomorphisms implementing associativity of $\otimes$, via

$(X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z)$; and unitality of $\mathbf{1}$, via $\mathbf{1} \otimes X \xrightarrow{\cong} X \xrightarrow{\cong} X \otimes \mathbf{1}$. These must satisfy some axioms which we won't discuss in detail here; the most important one is the *pentagon axiom* on the associator.

Today, we work over an algebraically closed field k, not necessarily closed. Recall that a *k-linear category* $\mathcal{C}$ is one for which for all objects $x, y \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is a k-vector space, such that composition is bilinear. A *k-linear monoidal category* is a monoidal category that is also a k-linear category — and we also impose the consistency condition that the tensor product is a k-linear functor. we will impose a few more niceness conditions before arriving at the definition of a fusion category — in fact, as many as we can such that we still have examples!

In particular, we will only consider k-linear monoidal categories $\mathcal{C}$ such that

- all Hom-spaces are finite-dimensional over k,
- $\mathcal{C}$ is semisimple,[1]
- $\mathcal{C}$ has only finitely many isomorphism classes of simple objects,
- $\mathbf{1}$ is indecomposable, and
- $\mathcal{C}$ is *rigid*, a condition on duals of objects.

A category satisfying all of these axioms is a *fusion category.* (TODO: double-check)

There are three ways we can come to an understanding of these categories: through the definition, through realizations and examples, and through diagrammatics. We will also heavily use semisimplicity, through the principle that *k-linear functors out of $\mathcal{C}$ are determined by their values on simple objects, and all choices are allowed.*

Example 2.1. Our running example is $\mathcal{V}ec^{\omega}_{\mathbb{Z}/n}$, where n is a natural number and ω is a degree-3 cocycle for $\mathbb{Z}/n$, valued in $k^\times$.

The objects of $\mathcal{V}ec^{\omega}_{\mathbb{Z}/n}$ are the elements of $\mathbb{Z}/n$, with the tensor product $i \otimes j \coloneqq i + j$. If $\omega = 1$, then we use the obvious associator, i.e. the isomorphism

$$(i \otimes j) \otimes k \xrightarrow{\cong} i \otimes (j \otimes k) \tag{2.2}$$

which corresponds to the identity under the identifications with $i+j+k$.[2] But in general, we can do something different: choose the map (2.2) which is $\omega(i, j, k)$ times the standard one.

A priori you can use any function $\mathbb{Z}/n \times \mathbb{Z}/n \times \mathbb{Z}/n \to k^\times$, but the pentagon axiom on associativity imposes the condition that ω is a cocycle.

Exercise 2.3. If you have not seen this before, verify that the pentagon axiom forces $\partial\omega = 1$.

The simplest nontrivial example[3] is for $n = 2$ and

$$\omega(i, j, k) \coloneqq \begin{cases} 1, & \text{if } i = 0,\ j = 0, \text{ or } k = 0 \\ -1, & \text{otherwise.} \end{cases} \tag{2.4}$$

◀

$\mathbb{Z}/n$ was not special here — given any finite group G and a cocycle $\omega \in Z^3(G; k^\times)$, we obtain a fusion category $\mathcal{V}ec^{\omega}_G$ in the same way.

With ω as in (2.4), $\mathcal{V}ec^{\omega}_{\mathbb{Z}/2}$ looks like a new example, not equivalent to $\mathcal{V}ec^0_G$ for any G — but in order to understand that precisely, we need to discuss when two tensor categories are equivalent.

Definition 2.5. A *tensor equivalence* of tensor categories $\mathcal{C}$ and $\mathcal{D}$ is a monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$, i.e. a functor together with data of natural isomorphisms $F(X \otimes Y) \xrightarrow{\cong} F(X) \otimes F(Y)$ satisfying some axioms.

Choose cocycles ω and ω' for $\mathbb{Z}/n$, and let's consider tensor functors $F\colon \mathcal{V}ec^{\omega}_{\mathbb{Z}/n} \to \mathcal{V}ec^{\omega'}_{\mathbb{Z}/n}$. Furthermore, let's assume F is the identity on objects, so the data of F is the natural isomorphism $F(X \otimes Y) \cong F(X) \otimes F(Y)$. This is a choice of an element of $k^\times$ for every pair of objects, subject to some additional conditions:

Proposition 2.6. *F is a tensor functor iff $\omega = \omega' \cdot \partial\psi$.*

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[1] A k-linear category is *semisimple* if it's equivalent to the category of modules over $k \oplus \cdots \oplus k$, where there is a finite number of summands.

[2] These multiplication rules are really special, in that we were able to just write down an associator. This is generally not true; for general multiplication rules you're interested in, you'll have to work a little harder.

[3] This is nontrivial provided $\mathrm{char}(k) \neq 2$.

Corollary 2.7. *$\mathcal{V}ec^{\omega}_{\mathbb{Z}/n} \simeq \mathcal{V}ec^{\omega'}_{\mathbb{Z}/n}$ if ω and ω' are cohomologous.*

Recall that $H^3(\mathbb{Z}/n; k^\times) \cong \mathbb{Z}/n$, so we have n possibilities, some of which might coincide. If F isn't the identity on objects, it's fairly easy to see that as a function on objects, identified with a function $\mathbb{Z}/n \to \mathbb{Z}/n$, we must get a group homomorphism; if F is to be an equivalence, this homomorphism must be an isomorphism. One can run a similar argument as above and obtain a nice classification result.

Proposition 2.8. *The tensor equivalence classes of tensor categories $\mathcal{V}ec^{\omega}_{\mathbb{Z}/n}$ are in bijection with the orbits $H^3(\mathbb{Z}/n; k^\times)/\operatorname{Aut}(\mathbb{Z}/n)$, via the map sending ω to its class in cohomology.*

The action of $\operatorname{Aut}(\mathbb{Z}/n) = (\mathbb{Z}/n)^\times$ on $H^3(\mathbb{Z}/n; k^\times) \cong \mathbb{Z}/n$ is not the first action you might write down! Given $a \in (\mathbb{Z}/n)^\times$ and $s \in H^3(\mathbb{Z}/n; k^\times)$, the action is

$$a \cdot s = a^2 s. \tag{2.9}$$

This is a standard fact from group cohomology.

Now let's discuss some realizations of fusion categories. If H is a semisimple Hopf algebra, then $\mathcal{C} := \mathcal{R}ep^{fd}_H$ is a fusion category. Let $F\colon \mathcal{C} \to \mathcal{V}ec$ denote the forgetful functor to finite-dimensional vector spaces. It turns out that one can reconstruct $\mathcal{C}$ as a fusion category from F, and in fact any fusion category $\mathcal{C}$ with a tensor functor to $\mathcal{V}ec$ is equivalent to $\mathcal{R}ep^{fd}_H$ for some Hopf algebra H. The data of the tensor functor to $\mathcal{V}ec$ is crucial!

Example 2.10. For example, $\mathcal{V}ec_{\mathbb{Z}/n} \simeq \mathcal{R}ep^{fd}_{\mathbb{Z}/n}$; we saw in the previous lecture that representations of $\mathbb{Z}/n$ are equivalent to modules over the Hopf algebra $k[\mathbb{Z}/n] := k[x]/(x^n - 1)$, with comultiplication $\Delta(x) := x \otimes x$.

However, if ω is nontrivial, $\mathcal{V}ec^{\omega}_{\mathbb{Z}/n}$ admits no tensor functor to $\mathcal{V}ec$, and therefore cannot be seen using Hopf algebras. One can try to generalize the reconstruction program, using quasi-Hopf algebras, weak Hopf algebras, etc. ◀

Bimodules provide another approach to realizations: we look for a ring R and a tensor functor $F\colon \mathcal{C} \to \mathcal{B}imod_R$. Applying this to $\mathcal{V}ec^{\omega}_{\mathbb{Z}/n}$, we get (R, R)-bimodules $F(i)$ for each $i \in \mathbb{Z}/n$ and isomorphisms $F(i) \otimes_R F(j) \xrightarrow{\cong} F(i + j)$. In particular, each $F(i)$ is (tensor-)invertible.

Example 2.11. An *inner automorphism* of a ring R is conjugation by some $r \in R^\times$. Inner automorphisms form a normal subgroup of $\operatorname{Aut}(R)$, and the quotient is called the *outer automorphism group* of R and denoted $\operatorname{Out}(R)$. An *outer action* of a group G on a ring R is a group homomorphism $\varphi\colon G \to \operatorname{Out}(R)$.

Given an outer automorphism θ of R, one obtains an (R, R)-bimodule R_θ, whose left action is the R-action on R by left multiplication, and whose right action is $r \cdot x = r\theta(x)$. We need to choose an element in $\operatorname{Aut}(R)$ mapping to θ to make this definition, but different choices lead to isomorphic bimodules.

Anyways, given an outer action of $\mathbb{Z}/n$ on R, we obtain (R, R)-bimodules $R_{\varphi(i)}$ indexed by the objects $i \in \mathcal{V}ec_{\mathbb{Z}/n}$ and isomorphisms between $R_{\varphi(i)} \otimes R_{\varphi(j)} \xrightarrow{\cong} R_{\varphi(i+j)}$. This data stitches together into a tensor functor $\mathcal{V}ec_{\mathbb{Z}/n} \to \mathcal{B}imod_R$. ◀

Diagrammatics represents the objects of a fusion category $\mathcal{C}$ as points, and morphisms as lines. One can then impose relations on certain morphisms, and therefore diagrammatics provide a generators-and-relations approach to the structure of a given fusion category. Next time, we'll see how to do this for $\mathcal{V}ec^{\omega}_{\mathbb{Z}/n}$, and see more examples.

3. Eric Rowell, An introduction to modular tensor categories I

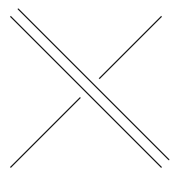
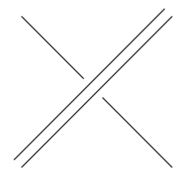
In this lecture, we'll begin with definitions and basic examples of modular tensor categories, and then use them in the next lecture. But first, let's discuss the whys of modular tensor categories.

We're often interested in knot and link invariants which are pictorial in nature, e.g. computed using a diagram. Another seemingly unrelated application is to study statistical-mechanical systems. Witten introduced TQFT into this story, extending the Jones polynomial to 3-manifold invariants using physics. Lately, there are interesting condensed-matter pheomena in topological phases. All of these are goverened by modular tensor categories in different ways, and in related ones.

(TODO: list of references, via handout)

Definition 3.1. Let $\mathcal{C}$ be a fusion category. A *braiding* on $\mathcal{C}$ (after which it's called a *braided fusion category*) is data of a natural transformation $c_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$ satisfying some relations called the *hexagon identities*.

You can think of $c_{X,Y}$ as taking strands labeled by the objects X and Y, and laying the X strand over the Y strand. The hexagon identities arise by comparing the two strands

(3.2)  and  .

Because the braiding is implemented via a natural transformation, it is functorial: we can braid morphisms as well as objects.

Example 3.3. Given a finite group G, $\mathcal{R}ep_G$ is a braided fusion category. Let V and W be representations; then the braiding $c_{V,W}(v \otimes w) \coloneqq w \otimes v$. ◀

Definition 3.4. Let $\mathcal{C}$ be a braided fusion category. The *symmetric center* or *Müger center* of $\mathcal{C}$ is the subcategory $\mathcal{C}'$ of $x \in \mathcal{C}$ such that $c_{X,Y} c_{Y,X} = \mathrm{id}_X$ for all $Y \in \mathcal{C}$.

For example, the symmetric center of $\mathcal{R}ep_G$ is once again $\mathcal{R}ep_G$.

Exercise 3.5. Why is the symmetric center of $\mathcal{C}$ a braided fusion category? In particular, why is it closed under tensor products?

Definition 3.6. If the symmetric center of $\mathcal{C}$ is itself, we call $\mathcal{C}$ *symmetric*.[4] If the symmetric center of $\mathcal{C}$ is generated by the unit object (equivalently, $\mathcal{C}' \simeq \mathcal{V}ect$), we call $\mathcal{C}$ *nondegenerate*.

Here, "generated by the unit object" means every object is isomorphic to a direct sum of copies of the unit.

Now let's put some more adjectives in front of these structures. These will make the structure nicer, as usual, but are interesting enough to have examples.

Definition 3.7. Let $\mathcal{C}$ be a braided fusion category. A *twist* on $\mathcal{C}$ is a choice of $\theta \in \mathrm{Aut}(\mathrm{id}_{\mathcal{C}})$.

Diagrammatically, we think of the twist as acting by the diagram in the first Reidemeister move, except we place right over left, not left over right. By looking at a picture of the twist on $X \otimes Y$, and untangling the picture, you can prove the *balancing equation*

$$\theta_{X \otimes Y} = c_{X,Y} \circ \theta_X \otimes \theta_Y. \tag{3.8}$$

Diagrams make it easier to picture these relations, but aren't strictly necessary. For example, the evaluation map $d_X : X^* \otimes X \to \mathbf{1}$ is represented by a diagram $\frown$ labeled by X, and coevaluation $b_X : \mathbf{1} \to X^* \otimes X$ is represented by a diagram $\smile$ labeled by X. Since braided categories aren't necessarily symmetric, one must be careful with left versus right duals.

Definition 3.9. A *ribbon structure* on a braided fusion category $\mathcal{C}$ is a twist such that $(\theta_X)^* = \theta_{X^*}$.

TODO: picture goes here. Here's where it's useful to use ribbon diagrams rather than string diagrams: really we want to keep track of the normal framings of the strings in our diagrams (thought of as embedded in $\mathbb{R}^3$), and ribbons provide a clean way to understand that.

Let $\mathrm{Irr}(\mathcal{C})$ denote the set of isomorphism classes of irreducible objects in $\mathcal{C}$. This is always a finite set; the *rank* of $\mathcal{C}$ is $\#\mathrm{Irr}(\mathcal{C})$. Choose representatives $x_1, \dots, x_r$ of the isomorphism classes of simple objects; then, by Schur's lemma, $\mathrm{Aut}(X_i) \cong \mathbb{C}^\times$. Let $\theta_i \in \mathbb{C}^\times$ denote the twist of X_i.

Now we have all the words we need to define modular tensor categories.

Definition 3.10. A *modular tensor category* is a nondegenerate ribbon fusion category.

There are other, equivalent definitions.

[4]Notice that being symmetric is a property of braided fusion categories.

Definition 3.11. A *pivotal structure* on a fusion category $\mathcal{C}$ is a natural isomorphism $j\colon X \xrightarrow{\cong} X^{**}$.

If a pivotal structure satisfies a certain niceness condition, it's called *spherical*. Then:

- A braided fusion category with a pivotal structure automatically has a twist.
- If that pivotal structure is spherical, the twist defines a ribbon structure.
- A nondegenerate braided fusion category with a spherical structure is a modular tensor category.

This still hasn't quite made contact with the usual definition.

If $\mathcal{C}$ is a ribbon fusion category, it has a canonical trace on $\mathrm{End}(X)$, valued in $\mathrm{End}(\mathbf{1}) \cong \mathbb{C}$. The *dimension* of an object $X \in \mathcal{C}$ is $\mathrm{tr}(\mathrm{id}_X)$.

Definition 3.12. The *S-matrix* of a ribbon fusion category is the matrix with entries $S_{ij} := \mathrm{tr}(c_{X_i,X_j} \circ c_{X_j,X_i})$ for $X_i, X_j \in \mathrm{Irr}(\mathcal{C})$.

Theorem 3.13 (Brugières-Müger)**.** *A ribbon tensor category $\mathcal{C}$ is modular if and only if the S-matrix is invertible.*

Now let's turn to examples.

Example 3.14. Let G be a finite abelian group and $\mathcal{V}ec_G$ be the category of G-graded vector spaces. These were discussed previously in Example 2.1, albeit in a slightly different way.

Let $c\colon G \times G \to \mathbb{C}^\times$ be a *bicharacter* of G, i.e. for all $g, h, k \in G$,

$$c(gh, k) = c(g, k)c(h, k). \tag{3.15}$$

Then we obtain a braiding on $\mathcal{V}ec_G$ by $c\colon g \otimes h \to h \otimes g$ by

$$\theta_g(v \otimes w) = c(g, h)w \otimes v. \tag{3.16}$$

For the twist, use $\theta_g := c(g, g)$. This defines a ribbon tensor category, and it is modular iff $\det((c(g,h)c(h,g))_{g,h}) \neq 0$.

Exercise 3.17. In particular, let $G := \mathbb{Z}/3$ and w be a generator. Show that $c(w, w) = \exp(2\pi i/3)$ extends to a bicharacter that defines a modular tensor structure on $\mathcal{C} := \mathcal{V}ec_G$. Show that we cannot obtain a modular structure on $\mathcal{V}ec_{\mathbb{Z}/2}$ in this way, however.

We can produce a modular structure on $\mathcal{V}ec_{\mathbb{Z}/2}$ in a different way: let z be a generator, and define $c(z, z) := i$ and $c(1, z) = c(z, 1) = c(1, 1) = 1$. This defines a modular tensor category structure on $\mathcal{V}ec_{\mathbb{Z}/2}^\omega$ whenever ω is cohomologically nontrivial; this category is of considerable interest in physics, where it's known as the *semion category*. ◀

If you tried to generalize this to G nonabelian, you would not be able to write down a braiding, because $g \otimes h \ncong h \otimes g$.

If all simple objects in $\mathcal{C}$ are invertible, $\mathcal{C}$ is called a *pointed fusion category*. It turns out these have been classified, and the underlying monoidal tensor category id $\mathcal{V}ec_G^\omega$ for some finite group G and some cocycle ω. If in addition $\mathcal{C}$ is braided, then G is abelian, and we can ask about the converse.

Theorem 3.18. *If $|G|$ is odd, $\mathcal{V}ec_G^\omega$ admits a braiding iff ω is cohomologically trivial.*

When $|G|$ is even, things are more complicated, as we saw above, but the answers are known. For $\mathbb{Z}/2$, we can get $\mathcal{R}ep_{\mathbb{Z}/2}$, and for $c(z, z) = -1$, we obtain $s\mathcal{V}ec$. Both of these are symmetric. One can generalize: Deligne [Del02] classified symmetric fusion categories, showing they're all equivalent to $\mathcal{R}ep_G$ or $\mathcal{R}ep_G(z)$, where $z \in G$ is central and order 2 (giving a super-vector space structure on G-representations). Symmetric fusion categories equivalent to $\mathcal{R}ep_G$ are called *Tannakian*; those equivalent to $\mathcal{R}ep_G(z)$ are called *super-Tannakian*.

4. Emily Peters, Subfactors and planar algebras I

Note: I (Arun) didn't fully understand this talk, and there are a lot of TODOs. Hopefully I can fix some of them soon. I'm sorry about that.

In the subject of planar algebra, one can do a lot of math by drawing pictures and reasoning carefully about them. So these talks will have plenty of pictures.

References for today's talk:

- Jones, "Planar algebras I," [Jon99] the original reference.
- The speaker's thesis.
- Heunen and Vicary, "Categories for quantum theory."

Definition 4.1. A *Temperly-Lieb diagram* of size n is an embedding of n disjoint copies of $[0,1]$ into $[0,n] \times [0,1]$, such that the boundaries of the embedded intervals lie on integer-valued points.

That is, we take an $n \times 2$ rectangle of points, and draw lines pairing them, such that no two lines cross. We identify two Temperly-Lieb diagrams which are isotopic.

Let TL_n denote the complex vector space spanned by Temberly-Lieb diagrams of size n. Addition is formal. TL_n acquires an algebra structure by *stacking*: place one diagram on top of another.

(TODO: some pictures)

The identity operator for multiplication is (TODO: diagram that looks like $||||$).

This algebra has some additional interesting structure.

- There's a trace $\mathrm{TL}_n \to \mathbb{C}$: given a Temperly-Lieb diagram, close up the embedded intervals in a process akin to a braid closure. Then TODO. (Also, TODO: a picture) I think there is a parameter δ, and if the result has n circles, we get δ^n.
- A $*$-structure, by reversing the diagram horizontally.
- This defines a Hermitian form on TL_n, by $\langle x, y\rangle \coloneqq \mathrm{tr}(y^* x)$. This is an inner product if $\delta \geq 2$.

Since the trace depends on δ, we will write $\mathrm{TL}_n(\delta)$ for the Temperly-Lieb algebra with trace given by δ.

There is an embedding $\mathrm{TL}_n \hookrightarrow \mathrm{TL}_{n+1}$, given by adding a single vertical interval on the right-hand side of a diagram. Call the colimit $\mathrm{TL}(\delta)$.

Exercise 4.2. Check that this inclusion respects multiplication, the identity, and the trace, assuming we use the same value of δ in both cases.

This is the basic example of a *planar algebra*. In general, a planar algebra is a collection of vector spaces $V_0, V_1, V_2, \dots,$ together with an action by something called the *planar operad*. Fortunately, you don't need to know what an operad is to understand the planar operad. This operad is given by (TODO: in what sense?) *planar diagrams* (which TODO: I think are also called "spaghetti-and-meatballs diagrams"). These are diagrams of embeddings of compact 1-manifolds inside many-holed annuli, together with marked points on the boundaries of the annuli. These compose in a manner reminiscent of operator product expansion. TODO: a picture is more useful than a description here.

An action of the planar operad means, for each planar diagram, a multilinear map $\bigotimes V_i \to V_0$; we also ask for these maps to be compatible with compositions.

Example 4.3. The Temperly-Lieb algebra is a planar algebra, where the planar diagrams act by insertion. ⋖

Example 4.4. The *graph planar algebra* on a simply laced graph Γ takes as its V_n the complex vector space spanned by the set of loops on Γ of length n.

There are a few different ways we can compose loops. Of course, we can concatenate loops with the same origin, as in algebraic topology; but there's another option. Assume both loops are of even length, and let p and p' be their respective halfway points; then, we can define their composition to be 0 if $p \neq p'$, and to be the first half of the first loop, then the second half of the second loop, if $p = p'$. TODO: I think that these two composition laws correspond to two planar diagrams, and these should give you the general story, so double-check this and then include those pictures.

TODO: planar diagrams enter this story somehow?

There's also a trace (TODO: picture goes here). This procedure is slightly ambiguous, so we simply sum up over all possibilities. ⋖

We've seen in a few different talks so far the idea of a monoidal category, along with many variations of their definition.

Definition 4.5. A *monoidal category* is a category $\mathcal{C}$ together with a functor $\otimes\colon \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ and a distinguished object $\mathbf{1} \in \mathcal{C}$ called the *unit*, together with data of an *associator*, a natural isomorphism $(-\otimes-)\otimes- \xrightarrow{\cong} -\otimes(-\otimes-)$ and *left and right unitors*, natural isomorphisms $\mathbf{1}\otimes- \xrightarrow{\cong} -$ and $-\otimes\mathbf{1} \xrightarrow{\cong} -$; these are subject to some coherence conditions.

The point of recalling this definition is that we'll relate it to all the pictures in not just this lecture, but also the other ones this week. This is a point that is often unclear to people — if you already know why you can do diagrammatics for various kinds of categories, it might feel not worth reviewing, but if not, it's certainly confusing.

The idea is, we can draw objects, morphisms, and equations in a monoidal category as diagrams in 2d. TODO: those diagrams.

- A morphism $f\colon A \to B$ is a box from a strand labeled by B to a strand labeled by A.
- Composition is stacking vertically.
- The tensor product is stacking horizontally, both of objects and of morphisms.
- The monoidal unit is the empty diagram.

Two diagrams which are related under planar isotopy are considered equal.

Theorem 4.6. *A well-typed equation between morphisms in a monoidal category follows from the axioms of a monoidal category iff it holds true in the graphical language described above.*

As a simple example, how do vertical and horizontal composition (namely, composition of morphisms, resp. tensor product) interact? If you do vertical, then horizontal, or horizontal, then vertical, you get the same diagrams, and therefore they must be equal: given maps $f\colon A \to B$, $g\colon B \to C$, $h\colon D \to E$, and $k\colon E \to F$,

$$(f \circ g) \otimes (h \circ k) = (f \otimes h) \circ (g \otimes k) \tag{4.7}$$

as maps $A \otimes D \to C \otimes F$.

A monoidal category is *rigid* if it has left and right duals for all objects. Evaluation and coevaluation correspond to cups and caps; thus we obtain an identity (TODO: Zorro diagram, also called snake diagram). This allows us to freely do planar isotopy. (TODO: so do we need rigidity in order for Theorem 4.6 to hold?)

Now let $\mathcal{C}$ be a rigid monoidal category and $X \in \mathcal{C}$; we will obtain a planar algebra by "zooming in" on this object X. Specifically, take $V_n := \mathrm{End}(X^{\otimes n})$. (TODO: rest of data comes from diagrammatics, I think?)

Why care? Well, the formalism of planar algebras is different enough from that of monoidal categories to lend different tools to the study of things in their intersections. For example, monoidal categories and planar algebras have different notions of smallness. For example, in a semisimple rigid monoidal category, you might measure the number of simple objects. In a planar algebra generated by X as above, smallness is more traditionally measured with the *Frobenius-Perron dimension*. This can be understood in general semisimple rigid monoidal categories $\mathcal{C}$; it is a map $K_0(\mathcal{C}) \to \mathbb{R}$ which is positive on simple eigenvalues. Specifically, suppose

$$X \otimes Y = \sum c_{XY}^Z Z, \tag{4.8}$$

where the sum is over isomorphism classs of simple objects Z of $\mathcal{C}$; this defines a matrix in the entries X and Y; its Frobenius-Perron eigenvalue is the Frobenius-Perron dimension of X.

Part 2. Tuesday, January 28

5. Victor Ostrik, Introduction to fusion categories, II

We begin by discussing diagrammatics for $\mathcal{V}ec_{\mathbb{Z}/n}^{\omega}$, following work of Agustina Czenky.

The objects of $\mathcal{V}ec_{\mathbb{Z}/n}^{\omega}$ are all tensor products of $1 \in \mathbb{Z}/n$, which in diagrammatics is denoted $\uparrow$; thus, i is denoted $\uparrow\uparrow \overset{i}{\cdots} \uparrow$. The generating morphisms are TODO(two of them), and there are some relations: TODOand TODOare the identity, and TODOis ζ times TODO, where ζ is some n^{th} root of unity determined by ω.[5]

This generators-and-relations description of $\mathcal{V}ec_{\mathbb{Z}/n}^{\omega}$ allows us to uncover a universal property.

Proposition 5.1. *Let $\mathcal{C}$ be a fusion category. Then there is a natural bijection between tensor functors $F\colon \mathcal{V}ec_{\mathbb{Z}/n}^{\omega} \to \mathcal{C}$ and isomorphism classes of data of an object $X \in \mathcal{C}$ and an isomorphism $\theta\colon X^{\otimes n} \xrightarrow{\cong} \mathbf{1}$.*

The idea is that, looking at the diagrammatics of $X := F(1)$, we have two different isomorphisms $X^{\otimes(n+1)} \xrightarrow{\cong} X$, and one must be ζ times the other.

[5]Conversely, any choice of ζ is given by some ω, though it's not always easy to work this out in practice.

The slightly more sophisticated way to say this is that functors $\mathcal{V}ec^{\omega}_{\mathbb{Z}/n} \to \mathcal{C}$ form a category, and the data (X, ϕ) as above forms a category, and the above bijection can be promoted to an equivalence of categories.

Example 5.2. Assume n is odd, so that we can use $2 \in \mathbb{Z}/n$ as a generator. Now you can compare (TODO: two diagrams), and one should be a multiple of another. It turns out the factor is ζ^4, and this gives the action of $\mathrm{Aut}(\mathbb{Z}/n)$ on $H^3(\mathbb{Z}/n; k^\times)$ from last time. ◀

Now let us use this to study tensor functors $F \colon \mathcal{V}ec^{\omega}_{\mathbb{Z}/n} \to \mathcal{B}imod_R$, where R is a k-algebra. These are classified by (R, R)-bimodules X together with an isomorphism $X^{\otimes n} \xrightarrow{\cong} R$; in particular, X is invertible. Hence we can restrict our search to $\mathrm{Pic}(R) \subset \mathcal{B}imod_R$, the subcategory of invertible bimodules. Inside $\mathrm{Pic}(R)$, we also have $\mathrm{Out}(R)$; as described last time, an outer automorphism defines an (isomorphism class of) (R, R)-bimodules.

Exercise 5.3. Assuming $Z(R) \cong k$, let $\theta \in \mathrm{Out}(R)$. Show that $\theta(g) = \zeta g$ for $g \in R^\times$ and some root of unity ζ. If this is too difficult at first, take a look at some examples; see if you can give an example for any ζ.

One big open problem in this field is to classify all fusion categories. This is of course way too hard, given that it's more difficult than the classification of finite groups, but as with the classification of finite groups, intermediate results are interesting, possible, and useful.

Theorem 5.4 (Ocneanu rigidity (Etingof-Nikshych-Ostrik [ENO05]))**.** *Fusion categories over $\mathbb{C}$ have no deformations.*

This was originally conjectured by Ocneanu. We won't say precisely what a deformation of a fusion category is, but the data of associativity in a fusion category is matrices satisfying some equations, modulo the action of some symmetry group. Ocneanu rigidity amounts to there being only finitely many orbits of solutions under this group action.

Corollary 5.5. *There are countably many tensor equivalence classes of fusion categories over $\mathbb{C}$.*

For example, $\mathcal{V}ec^{\omega}_G$ is classified by $H^3(G; \mathbb{C}^\times)$, which is a finite group.

Remark 5.6. Ocneanu rigidity is open in positive characteristic; the speaker expects it to be true, but for different reasons. The proof in characteristic zero uses a tool called *Davydov-Yetter cohomology* for fusion categories — this vanishes in characteristic zero, which implies Theorem 5.4, but is known to not vanish in characteristic p in general. ◀

Example 5.7. Another rich source of interesting examples of fusion categories are quantum groups at roots of unity. The construction is quite complicated. For example, fix an integer λ, called the *level*; then, one can build a fusion category $\mathcal{C}(\mathfrak{sl}_2, \lambda)$ as follows. There are $\lambda + 1$ simple objects $L_0, \dots, L_\lambda$, and the fusion rules are determined by

$$L_i \otimes L_1 = L_1 \otimes L_i = L_{i-1} \oplus L_{i+1}, \tag{5.8}$$

where $L_{-1} = L_{\lambda+1} = 0$; this suffices to determine the rest of the fusion rules. This looks reminiscent of the representation theory of $\mathfrak{sl}_2$, but "cut off" at λ. ◀

Any fusion category not equivalent to $\mathcal{V}ec^{\omega}_G$ or a quantum group is called *exotic*. Examples of exotic fusion categories are constructed using subfactor theory; many are related to something called *near group categories*. Such a category has as its set of simple objects $G \amalg \{X\}$, where G is a finite group. The tensor product on elements of G is just multiplication, and the remaining rules are

$$g \otimes X = X \otimes g = X \tag{5.9a}$$

$$X \otimes X = \bigoplus_{g \in G} g \oplus nX, \tag{5.9b}$$

for some $n \in \mathbb{N}$.

It's not necessarily true that one can find an associator compatible with this data, but often one can. For $n = 0$, these are *Tambara-Yamagami categories*, which are relatively well-studied. For $n > 0$, Evans-Gannon [EG14] showed that either $n = \#G - 1$ or $\#G \mid n$. Moreover, if $n \geq \#G$, then G must be abelian, which is a nice simplification! Izumi-Tucker [IT19] considered the cases where $n = \#G - 1$ and $n = \#G$ — in the latter case, there are only finitely many examples, and for $n = 2\#G$, there's a single example with

$G = \mathbb{Z}/3$. In general, it's open whether there are a finite number of examples for a fixed n as a function of $\#G$.

There are also various useful constructions which, given some fusion categories, produce more. One example is Deligne's tensor product $\mathcal{C} \boxtimes \mathcal{D}$ of fusion categories, which is again a fusion category.

Definition 5.10. An *associative algebra* in a fusion category $\mathcal{C}$ is an object $A \in \mathcal{C}$ together with $\mathcal{C}$-morphisms $m\colon A \otimes A \to A$ and a unit $i\colon \mathbf{1} \to A$, subject to axioms guaranteeing associativity of m and that i is a unit for m.

Given an associative algebra A in $\mathcal{C}$, we can define A-module and (A, A)-bimodule objects in $\mathcal{C}$, analogous to the usual case.

Exercise 5.11. Write these definitions down.

The upshot is, given an associative algebra A in $\mathcal{C}$, the category of (A, A)-bimodule objects, denoted ${}_A\mathcal{C}_A$, is a tensor category. The monoidal product is $\otimes_A$, and the unit is A. This is generally not a fusion category — and indeed, even if A is just an algebra in vector spaces, its category of bimodules generally isn't fusion! But if A satisfies an assumption called *separability*, then ${}_A\mathcal{C}_A$ is semisimple and rigid, which is pretty close to being fusion — all we need is that $\mathbf{1} \in {}_A\mathcal{C}_A$ is indecomposable. A sufficient (but not necessary) condition for this is $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A) \cong k$.

Remark 5.12. If we start with $\mathcal{C} = \mathcal{V}ec_G^\omega$ and look for bimodule categories for algebras in $\mathcal{C}$, the fusion categories we obtain are called *group-theoretical fusion categories.* ◀

Another procedure to obtain fusion categories is called *graded extensions*, building a G-graded fusion category, where G is a finite group, out of an ungraded fusion category.

Here are two extremely useful tools for classifying fusion categories.

(1) The Drinfeld center of a fusion category is a modular tensor category. Modular tensor categories have a lot of structure, and this is helpful for learning about fusion categories.
(2) Diagrammatic methods are helpful, as we saw at the beginning of today's lecture.

6. Eric Rowell, An introduction to modular tensor categories II

Last time, we discussed a few different kinds of tensor categories, in particular pointed ribbon fusion categories and pointed modular tensor categories. Both of these have been classified; the classification amounts to finding compatible twists on $\mathcal{V}ec_G$ with various braidings.

Theorem 6.1 ([EGNO15])**.**

(1) *Pointed ribbon fusion categories up to equivalence are classified by data of a finite abelian group G and a quadratic form $q\colon G \times G \to \mathbb{C}^\times$.*
(2) *Pointed modular tensor categories are classified by (G, q) as above, subject to the condition that q is nondegenerate.*

The data of (G, q) is often called a *pre-metric group*, and if q is nondegenerate, it's called a *metric group*. The quadratic form determines the 2-cocycle that specified the braiding, via

$$B(g, h) \coloneqq \frac{q(g)(q(h)}{q(gh)}. \tag{6.2}$$

This is all very nice, but we would like some more interesting examples, so we turn to quantum groups $\mathcal{C}(\mathfrak{g}, \ell)$. Here $\mathfrak{g}$ is a simple Lie algebra and $\mathcal{C}$ is the category of modules over $\mathcal{U}_q(\mathfrak{g})$, where $q \coloneqq \exp(\pi i/m \ll)$. For $m = 1$, $\mathfrak{g}$ can be ADE type; for $m = 2$, of BCF type; and for $m = 3$, $\mathfrak{g} = \mathfrak{g}_2$. Setting up the category involves some technical details, but can be done, and we obtain modular categories![6]

Example 6.3. Let's take $\mathfrak{g} = \mathfrak{so}_5$ and $\ell = 5$, so $q = e^{i\pi/10}$. The objects in $\mathcal{C}$ are described by a Weyl chamber for $\mathfrak{g}$ (TODO: it was not at all clear to me why), but $\ell = 5$ imposes that we kill all objects above a certain line. In this we have the standard representation V, the adjoint representation A, and an object at coordinates $(1/2, 1/2)$ with quantum dimension $\sqrt{5}$. The level (in the notation of the previous talk) of this category is 2, so sometimes it's also denoted $\mathrm{SO}(5)_2$. ◀

[6]Here m is important; if you leave it out, you'll always get a ribbon category, but not necessarily a modular one.

Example 6.4. Let's consider $\mathcal{C}(\mathfrak{sl}_2, 5)$. Now we look at a ray within the one-dimensional root space, and only keep the first three objects, S at 1, A at τ, and the unit. The fusion rules are $A^{\otimes 2} = \mathbf{1} \oplus A$, and $S^{\otimes 2} \cong \mathbf{1}$. Thus this category actually splits as a Deligne tensor product of the subcategory generated by S, which is called the *semion category*, and the subcategory generated by A, which is called the *Fibonacci category.* Both of these are fundamental examples. $\blacktriangleleft$

Example 6.5. $\mathcal{C}(\mathfrak{sl}_2, 4)$ is an *Ising category.* Its simple objects are $\mathbf{1}$, σ, and ψ. Here $\dim(\sigma) = \sqrt{2}$, $\dim(\psi) = 1$, $\theta_\sigma = e^{3\pi i/8}$, and $\theta_\psi = -1$. This σ particle was the first nonabelian anyon discovered, and it's reminiscent (though not the same as) to a Majorana fermion. The S-matrix is

$$S = \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}. \tag{6.6}$$ $\blacktriangleleft$

We've described examples of modular categories via their *modular data*: the S-matrix and also the T*-matrix* $T_{ij} = \delta_{ij}\theta_i$. Stay tuned for a talk later this weey by Colleen Delaney with more details.

The modular group $\mathrm{SL}_2(\mathbb{Z})$ is generated by two matrices $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. The S- and T-matrices appearing in the data of a modular category satisfy relations that imply they define a projective representation Φ of $\mathrm{SL}_2(\mathbb{Z})$.

Theorem 6.7 (Ng-Schauenburg [NS10])**.** *The image of such a representation Φ is finite. In fact, if N is the order of T, then Φ factors over* $\mathrm{SL}_2(\mathbb{Z}/n)$*.*

Classifying fusion categories is too difficult in general, but modular categories have more adjectives in front of them. Maybe we can classify them, at least for a fixed rank r that's not too large. Or even, how many of them are there?

A good first step is to consider the field $\mathbb{K}_0 \coloneqq \mathbb{Q}(s_{ij})$, which sits inside $\mathbb{Q}(\theta_i)$. Since T has finite order, $\mathbb{Q}(\theta_i)$ is a cyclotomic extension $\mathbb{Q}(\zeta_N)$ for some primitive N^{th} root of unity ζ_N. These are particularly nice Galois extensions in that:

1. Since $\mathbb{Q} \hookrightarrow \mathbb{Q}(\theta_i)$ is a cyclotomic extension, $\mathrm{Gal}(\mathbb{Q}(\theta_i)/\mathbb{Q})$ is abelian, and in particular always solvable.
2. Since we're looking at rank r, the T-matrix is $r \times r$, so we get an embedding $\mathrm{Gal}(\mathbb{K}_0/\mathbb{Q}) \hookrightarrow \mathrm{Aut}(\mathrm{Irr}(\mathcal{C})) \cong S_r$.
3. There is some k such that $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{K}_0) \cong (\mathbb{Z}/2)^k$.

Thus we have a recipe for classifying modular categories of rank r.

1. Choose an abelian subgroup A of S_r. Then, using the above facts, classify all possible S-matrices which yield the Galois group $\mathrm{Gal}(\mathbb{Q}(\mathbb{K}_0/\mathbb{Q}) \cong A \subset S_r$. For many choices of A, there are no possible S-matrices.
2. The *Verlinde formula* determines the fusion rules from the S-matrix.
3. Finally, an analogue of Ocneanu rigidity (Theorem 5.4) informs us that there are finitely many modular tensor categories with fixed fusion rules.

This has worked completely up to rank 5 so far, and is also effective in rank 6. One general question, which is still open, is *if you fix a fusion category, how do you classify its possible modular structures?* We know there can only be finitely many, but that theorem is nonconstructive. In special cases, things are known; for example, a result of Kazhdan-Wenzl [KW93] allows us to solve this for $\mathcal{C}(\mathfrak{sl}_n, \ell)$. More recent work of Nikshych [Nik19] establishes how to classify the possible braidings given fixed fusion rules. And spherical structures on a modular tensor categories are understood: they're given by invertible objects with order at most 2.

Theorem 6.8 (Rank-finiteness (Bruillard-Ng-Rowell-Wang [BNRW16]))**.** *There are finitely many modular tensor categories of a fixed rank r.*

The proof ultimately relies on results in analytic number theory, which is interesting.

Moving on, let $\mathcal{C}$ be a braided fusion category and B_n denote the braid group on n strands. Given an object $X \in \mathcal{C}$, the braiding defines a map $\psi\colon B_n \to \mathrm{Aut}(X^{\otimes n})$; if σ_i denotes the braid that switches braids i and $i+1$, then

$$\psi(\sigma_i) \coloneqq \mathrm{id}_X^{\otimes(i-1)} \otimes c_{X,X} \otimes \mathrm{id}_X^{(n-i-1)}. \tag{6.9}$$

$\mathrm{Aut}(X^{\otimes n})$ acts on

$$\mathcal{H}_n^X := \bigoplus_{Y \in \mathrm{Irr}(\mathcal{C})} \mathrm{Hom}(Y, X^{\otimes n}), \tag{6.10}$$

so we get a representation $\rho_X \colon B_n \to \mathrm{GL}(\mathcal{H}_n^X)$. In addition to being an interesting braid group representation on its own, this representation is important for implementing gates in topological quantum computation.

It's natural to ask whether the image of ρ_X is finite.

Definition 6.11. We say that $X \in \mathcal{C}$ has *property F* if the image of ρ_X is finite.

The Ising category (or rather, its nontrivial simple object) has property F, but the Fibonacci category does not.

Definition 6.12. Let X be an object in a fusion category $\mathcal{C}$ and N_X be the matrix of fusion with X on $\mathrm{Irr}(\mathcal{C})$, i.e.

$$(N_X)_{ij} = \dim \mathrm{Hom}_{\mathcal{C}}(X \otimes X_j, X_i). \tag{6.13}$$

The *Frobenius-Perron dimension* of X, denoted $\mathrm{FPdim}(X)$, is the largest eigenvalue of N_X. If X is simple and $\mathrm{FPdim}(X)^2 \in \mathbb{Z}$, X is called *weakly integral*.

Over 10 years ago, the speaker conjectured that X is weakly integral iff it has property F. This is known in special cases.

- For pointed fusion categories, this is essentially an exercise.
- For group-theoretical braided fusion categories (e.g. $\mathcal{R}ep(D^\omega G)$), this is due to Etingof-Rowell-Witherspoon [ERW08].
- For quantum groups $\mathcal{C}(\mathfrak{g}, \ell)$, this is known, thanks to work of Jones, Freedman, Larsen, Wang, Rowell, and Wenzl.
- Recently, this conjecture has been verified for weakly group-theoretical braided fusion categories by Green-Nikshych [GN19]. There is a different conjecture that weakly group-theoretical is equivalent to weakly integral.

This veracity of this conjecture is closed under taking Deligne tensor products, Drinfeld doubles, and a few other useful operations.

There are still many interesting open questions! For example, from a nondegenerate braided fusion category, one canextract an invariant called the *Witt group*, and this seems to be a rich and interesting invariant that we are still in the process of understanding.

7. Anna Beliakova, Quantum invariants of links and 3-manifolds, I

The title of this talk is inspired from Turaev's talk, but we have a different aim in mind: Turaev studies things from a very general perspective, but we're going to focus on specific examples in detail.

There is a procedure called surgery which associates to a framed link in S^3 a closed, oriented 3-manifold. A famous theorem of Lickorish-Wallace asserts that every closed, oriented 3-manifold can be realized in this way, and conversely, two framed links yield diffeomorphic 3-manifolds iff they differ by a series of known moves.

Given a ribbon Hopf algebra, one can build an invariant of framed links; for $\mathcal{U}_q(\mathfrak{sl}_2)$, for example, this is the colored Jones polynomial. Using a procedure called integration, we obtain 3-manifold invariants, in this case the Witten-Reshetikhin-Turaev invariants. And there's a way to build them directly from 3-manifolds, which uses finiteness.

There is another way to obtain framed link invariants from $\mathcal{U}_q(\mathfrak{sl}_2)$, yielding *Kashaev invariants*, which are of quantum dimension zero. These are sometimes also called *logarithmic invariants*. The corresponding 3-manifold invariants are called *Hennings CGP invariants*. (TODO: double-check this.)

More recently, these colored link invariants have been unified into a more general invariant called (TODO: double check) Hashiro's cyclotomic extension, yielding a unified Witten-Reshetikhin-Turaev invariant for 3-manifolds. We'll discuss this invariant in the second talk later this week.

Let $L \colon (S^1 \times I)^{\amalg k} \hookrightarrow S^3$ be a framed link, and let $\nu(L)$ denote its normal bundle, embedded in S^3 via the tubular neighborhood theorem. Given this data, *surgery on L* is the closed, oriented 3-manifold

$$S^3(K_f) := S^3 \setminus \nu(L) \cup_f (D^2 \times S^1)^{\amalg k}, \tag{7.1}$$

where f is the identification of $\partial(S^3 \setminus \nu(L))$ and $(D^2 \times S^1)^{\amalg k}$ given by the framing. There are two moves which change the framed link but don't change the diffeomorphism class of the 3-manifold obtained under surgery.

- The simpler move, denoted K1, exchanges a figure-8 with an empty set.
- K2 is a little more elaborate. TODO: picture.

We will define a universal $\mathfrak{sl}_2$ framed link invariant. Given $n \in \mathbb{N}$, let $\{n\} := \sigma^n - \sigma^{-n}$, where σ is a formal variable, and let $[n] = \{n\}/\{1\}$. Now, we define the quantum group

$$\mathcal{U}_q(\mathfrak{sl}_2) = \langle e, F^{(n)}, K \rangle, \tag{7.2}$$

where $F^{(n)} := F^n/[n]!$, $e = \{1\}E$, and $\sigma^H = K = \exp((h/2)H)$. Let

$$E = \sigma^{H \otimes H} 2 \sum_{n=0}^{\infty} \sigma^{\frac{n(n-1)}{2}} F^{(n)} \otimes e^n \subset \mathcal{U}_k \mathbin{\widehat{\otimes}} \mathcal{U}_k. \tag{7.3}$$

Then (TODO: not sure) R is a simple tensor: write $R = \alpha \otimes \beta$. Now we label pieces of a framed link: if, traveling upwards, left travels over right, label the left with β and the right with α; if right travels over left, label the left with $\overline{\beta}$ and the right with $\overline{\alpha}$. Label a cup (coevaluation) with k and a cap (evaluation) with k^{-1}. Call the resulting element of the universal enveloping algebra J_L.

Example 7.4. TODO: example computed from a diagram. ◁

Notice that this always lands in the center of $\mathcal{U}_k$, which is freely generated by the *Casimir element*

$$C := \{1\}FE + \sigma K + \sigma^{-1}K^{-1}. \tag{7.5}$$

Let V_n be the n-dimensional irreducible representation of $\mathcal{U}_q(\mathfrak{sl}_2)$. Then let $J_L(V_n)$ denote the action of J_L on V_n. For example, the Casimir acts on V_n by $\sigma^n + \sigma^{-n}$.

Theorem 7.6 (Habiro [Hab08]). *Let K_0 be a 0-framed knot. Then*

$$J_{K_0} = \sum_{m=0}^{\infty} a_m \sigma_m, \tag{7.7}$$

where $a_m \in \mathbb{Z}[q^{\pm 1}]$ and

$$\sigma_m = \prod_{i=1}^{m} \big(c^2 - (\sigma + \sigma^{-1})^2\big). \tag{7.8}$$

Example 7.9. For the knot 4_1, $J_{4_1} = \sum_{m=0}^{\infty} \sigma_m$. For the knot 3_1, we obtain

$$J_{3_1} = \sum_{m=0}^{\infty} (-1)^m q^{m(m-3)/2} \sigma_m. \tag{7.10}$$

◁

In general,

$$J_{K_0}(V_n) = \sum_{m=0}^{n-1} a_m \prod_{i=1}^{n} \big(q^n + q^{-n} - q^i - q^{-i}\big) = \sum a_m \prod_{i=1}^{m} \{n+i\}\{n-i\}. \tag{7.11}$$

This recovers the Witten-Reshetikhin-Turaev invariant as follows: let ξ be a p^{th} root of unity and plug in $q = \xi$. Then define

$$F_{K_a}(\xi) = \sum_{n=0}^{p-1} [n]^2 J_K(V_n)|_{q=\xi} = \sum_{n=0}^{p-1} [n]^2 q^{a(m^2-1)/4} J_{K_0}(V_n)|_{q=\xi}. \tag{7.12}$$

Then, the Witten-Reshetikhin-Turaev invariant of $S^3(K_a)$ at ξ is $F_{K_a}(\xi)/F_{\text{unknot}}(\xi)$, where the unknot has framing given by the sign of a.

Remark 7.13. There is another invariant of 3-manifolds given similar-looking data, called the *Turaev-Viro invariant*, computed by triangulating the 3-manifold and labeling tetrahedra by $6j$-symbols. Beliakova-Durhuus [BD96], Walker, and Turaev showed that the Turaev-Viro invariant of M is equal to the Reshetikhin-Turaev invariant of $M \# (-M)$, i.e. the square of the Reshetikhin-Turaev invariant of M. ◁

Theorem 7.14 (Beliakova-Chen-Lê [BCL14])**.** *For all closed, oriented 3-manifolds M and all ξ, the Witten-Reshetikhin-Turaev invariant of M at ξ is in $\mathbb{Z}[\xi]$.*

That is, we can write the Witten-Reshetikhin-Turaev invariant of M as a polynomial in ξ of degree at most $p-1$, and this is telling us that the coefficients are integers.

Now, let's write $F_{K_a}(\xi)$ using a Gauss sum:

$$F_{K_a}(\xi) = \sum_{m\geq 0} a_m \sum_{n=0}^{p-1} q^{a(m^2-1)/4}\{n+m\}\cdots[n]^2\cdots\{n-m\}. \tag{7.15}$$

This lives in $\mathbb{Z}[q^{\pm n}, q]$. Plugging in $a = \pm 1$ (TODO: I think?), we see that

$$\sum_{n=0}^{p-1} q^{a(m^2-1)/4} q^{bn} = q^{-b^2/a}\gamma_a. \tag{7.16}$$

Let $L_a(q^{bn} \coloneqq q^{-b^2/a}$ and

$$I_M = (?) \sum_{m\geq 0} a_m L_a(\{n+m\}\cdots[n]^2\cdots\{n-m\}). \tag{7.17}$$

Let $(q)_n \coloneqq (1-q)\cdots(1-q^n) \in \mathbb{Z}[q]$ and $\check{I}_n \subset \mathbb{Z}[q]$ denote the ideal spanned by $(q)_n$. Then $\check{I}_n \subset \check{I}_{n+1} \subset \check{I}_{n+2} \subset \dots$, and we can complete to

$$\widehat{\mathbb{Z}}[q] \coloneqq \varprojlim_n \mathbb{Z}[q]/(q)_n, \tag{7.18}$$

which is the ring of analytic functions on the roots of unity, and is called the *Habiro ring*. An element of $\widehat{\mathbb{Z}}[q]$ can be represented as

$$f = \sum_{k=0}^{\infty} f_k (q)_k, \tag{7.19}$$

where $f_k \in \mathbb{Z}[q]$. This defines an embedding $\widehat{\mathbb{Z}}[q] \hookrightarrow \mathbb{Z}[[1-q]]$, and $f \in \widehat{\mathbb{Z}}[q]$ is determined uniquely by its values at roots of unity. The value $\omega_\xi f$ is well-defined (TODO: figure out what this means).

Theorem 7.20 (Habiro [Hab08])**.** *If M is an integral homology sphere, there is a unique $I_M \in \widehat{\mathbb{Z}}[q]$ such that for any ξ, $\omega_\xi I_M$ is the Witten-Reshetikhin-Turaev invariant for ξ and M.*

For example, if M is the Poincaré homology sphere,

$$I_M = \frac{q}{1-q}\sum (-1)^k q^{k(k+1)/2}(q^{k+1})_{k+1}. \tag{7.21}$$

If M is a rational homology sphere with $b_1(M) > 0$ the theorem, proven by Beliakova-Bühler-Lê [BBL11], is not quite as simple. Recently, Habiro-Lê [HL16] have generalized Theorem 7.20 to the analogues of these invariants defined using an arbitrary simple Lie algebra.

Next time, we'll see how even non-semisimple invariants are determined by Witten-Reshetikhin-Turaev invariants.

8. Terry Gannon, Conformal nets I

Why care about conformal nets? Well, conformal field theory (CFT) is implicitly tied to most of the subjects in this conference, e.g. to a few talks explictly about CFTs later this week, but also relationships with modular tensor categories. Conformal nets are our current best understanding of CFT, and as such are closely related to many other topics present in this conference.

In the last three centuries, physics has given back a great deal to mathematics, first via classical mechanics leading to the study of differential equations (ordinary and partial), and then symplectic geometry; then quantum mechanics and its ramifications in functional analysis; and recently, the still ongoing mathematical understanding of quantum field theory (QFT). We are barely scratching the surface, and the mathematical understanding of quantum field theory is promising to be a much deeper gift to mathematics than classical mechanics. Witten wrote around the turn of the century that understanding QFT will be a distinguished feature of 21$^{\text{st}}$-century mathematics.

Quantum field theory is very general. We will study a very special, simple case: quantum field theories in dimension $1 + 1$ (i.e. one dimension each of space and time) which are conformally invariant. Conformal invariance is a strong condition to impose on a QFT, and we will be rewarded with nice properties and interesting examples.

The Wightman axioms lead to a focus on quantum fields, which when applied to $(1 + 1)$-dimensional CFT lead to a axiomatization of CFTs through vertex operator algebras. This is different, almost rival, to the perspective of conformal nets we will discuss today. Heisenberg argued that, since quantum fields aren't physically observable objects, we shouldn't focus on them, and instead we should axiomatize the observables, those things that one can actually (in principle) measure in a physical theory of the universe. This leads to the Haag-Kastler axioms for QFT, and when we implement this for CFT, we will see conformal nets.

In classical mechanics, the state of a system is a point in a phase space, which is a symplectic manifold. Observable data, such as the position, momentum, etc., of particles, are functions on phase space. Quantum mechanics is different. The state of a system is a ray in the phase space, which is a Hilbert space H. Observables are Hermitian operators on H, such as $(i/\hbar)\frac{\partial}{\partial x}$. Measurements amount to projecting down onto eigenspaces for different operators, and these projections tell you the different probabilities.

To discuss conformal field theories, let's first discuss conformal symmetries, which are symmetries which preserve angles infinitesimally, but might not preserve distances. For example, $z \mapsto z^{-1}$ is a conformal transformation on the Riemann sphere — one says there's a *conformal compactification* of $\mathbb{C}$, which is the Riemann sphere. The story is similar on $\mathbb{R}^{m,n}$, leading to a conformal symmetry group $\mathrm{SO}(m+1, n+1)/\{\pm 1\}$, provided that $m, n \geq 1$ and $m + n \geq 3$.

But we care about $m = n = 1$, in which things change drastically. The conformal compactification of $\mathbb{R}^{1,1}$ $S^1 \times S^1$; we add all possible light rays. The conformal transformations of $S^1 \times S^1$ are huge — this group is $\mathrm{Diff}^+(S^1) \times \mathrm{Diff}^+(S^1)$: two copies of the orientation-preserving diffeomorphisms of the circle! $\mathrm{Diff}^+(S^1)$ is a Lie group in a suitable infinite-dimensional sense, and its Lie algebra is $\mathrm{Vect}(S^1)$, the Lie algebra of vector fields on the circle.

Quantum field theory is all about representation theory; this is how it relates to tensor categories. So we will be interested in repesentations of the groups and Lie algebras we've seen so far — but since states are rays in H, rather than points, the correct notion of representation in this setting is projective representations. The best way to handle these is to pass to a central extension and obtain a *bona fide* representation, and therefore we will see central extensions of $\mathrm{Diff}^+(S^1)$ and $\mathrm{Vect}(S^1)$. This might make contact with familiar mathematics: complexify the Lie algebra and centrally extend, and what you obtain is the *Virasoro algebra*. So the representation theory of the Virasoro algebra, and those representations which lift to representations of a central extension of $\mathrm{Diff}^+(S^1)$, are important in $(1 + 1)$-dimensional CFT.

Inside $\mathrm{Diff}^+(S^1) \times \mathrm{Diff}^+(S^1)$, we could consider the somewhat special, finite-dimensional subgroup $\mathrm{SO}(2, 2)/\{\pm 1\} \cong \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$. Another fairly obvious subgroup is the subgroup of rotations, the diagonals in $\mathrm{SO}(2, 2)$.

The Virasoro algebra has a nice basis, which is the standard basis that people use when discussing it: there are elements L_n for each $n \geq 0$, and a central element k. The Lie bracket is

$$[L_m, L_n] = (m - n)L_{m+n} + \delta_{m,-n} ck, \tag{8.1}$$

where c is some constant, in fact $(m^3 - m)/12$. Since k is central, all other brackets of basis elements vanish.

There is a standard trick in conformal field theory: focus on the two factors of $\mathrm{Diff}^+(S^1)$ separately. This leads to a significant simplification — a *chiral conformal field theory* is a CFT restricted to each factor of S^1. This isn't the full story: we'd have to fit the two pieces together into one, in order to understand the full story, but there are reasonable scenarios in which this works well. It doesn't work for everything, but it will work for the examples we focus on.

Definition 8.2. A *conformal net* is data of

- a Hilbert space H, called the *state space*; and
- for every interval[7] $I \subset S^1$, a von Neumann algebra $A(I)$ of bounded linear operators on H, called the *algebra of observables* on I,

such that the algebra generated by all $A(I)$s is $B(H)$, and satisfying a crucial axiom called *locality*: if I_1 and I_2 are disjoint intervals, with $O_1 \in A(I_1)$ and $O_2 \in A(I_2)$, then $[O_1, O_2] = 0$.

[7]By an *interval* in S^1, we mean an open, connected subset.

For a conformal field theory, we need a projective representation U of $\mathrm{Diff}^+(S^1)$. This will enforce the condition of conformal invariance (well, really covariance): for every $\gamma \in \mathrm{Diff}^+(S^1)$, we get a unitary operator $U(\gamma)$, and we impose as part of the definition of a conformal net that

$$U(\gamma)A(I)U(\gamma)^* = A(\gamma(I)). \tag{8.3}$$

By differentiating, we obtain a representation of the Virasoro algebra. The *Hamiltonian* of the theory is L_0. We ask that in the Virasoro representation, L_0 is diagonalizable and has nonnegative eigenvalues. These are the possible energies in this theories, so we want these to be nonnegative. There's a final axiom, involving the vacuum.

The easiest way to get your hands on von Neumann algebras is: pick your favorite group G, which can be infinite, and a unitary representation V, maybe infinite-dimensional. Then you get lots of unitary operators; single out those which commute with the group action, the symmetries of the representation. These form a von Neumann algebra, and, up to isomorphism, all von Neumann algebras arise in this way. If you'd prefer, there's a list of axioms on a $*$-algebra giving the definition of a von Neumann algebra, but it does not get the idea across as effectively.

You might have guessed from the notation that these $A(I)$ form a net: whenever $I_1 \subset I_2$, $A(I_1) \subset A(I_2)$: if you can measure something inside a smaller space(time), you can measure it inside the bigger space(time).

Locality is asking that nothing can travel faster than the speed of light. Two regions which are separated from each other cannot influence each other infinitely fast; you can think of simultaneously performing two experiments in the different regions.

Plenty of thought went into the axioms of a conformal net, but it's clear that there's still a lot of work to do before we get to the level of mathematical comfort with this definition that we're at in, say, symplectic geometry.

Example 8.4. The silliest example involves $H = \mathbb{C}$. ◂

Example 8.5. A better example is to begin with a vertex operator algebra A. The quantum fields in this model of the CFT are the vertex operators, which are operator-valued distributions; hit them with some test function $f(\theta)$ which is supposed inside an interval I. After some difficult functional analysis, this gives operators which make up $A(I)$. This is beginning to be understood, thanks to work of Carpi, Kawahigashi, Longo, and Wiener [CKLW18]. ◂

Example 8.6. Let $L\mathrm{SU}(n)$ denote the *loop group* of $\mathrm{SU}(n)$, i.e. the infinite-dimensional Lie group of maps $S^1 \to \mathrm{SU}(n)$. If one chooses a good representation of the loop group (this is related to the conditions needed to obtain a modular tensor category of such representations). Then, in a similar way, one can build a conformal net, which was a difficult undertaking by Wasserman and others. ◂

The axioms of a conformal net are rich enough to produce some interesting phenomena. For example, if I is an interval, the interior of $S^1 \setminus I$, called I', is also an interval, and these two intervals don't overlap. *Haag duality* tells us that these two must commute, and we end up with $A(I) = (A(I'))'$. We also see that $A(I)$ is always a particular kind of irreducible von Neumann algebra (called a *factor*), type III_1. This is a very special type of von Neumann algebra, and we will see some consequences of this next time.

Conformal nets exist so that we can study their representation theory, so let's discuss what a representation is. The definition might not be surprising: a representation π of a conformal net A is data of, for each interval $I \subset S^1$, an algebra map $\pi(I)\colon A(I) \to B(K)$, where K is some Hilbert space not necessarily related to H. This is required to satisfy some axioms: notably, when $I_1 \subset I_2$, we require $\pi(I_2)|_{I_1} = \pi(I_1)$.

The tautological representation of A acting on itself by the identity map is called the *vacuum representation*. Later, when we see that representations of a conformal net form a tensor category, the vacuum representation will be the tensor unit.

Any representation of a conformal net is automatically compatible with $\mathrm{Diff}^+(S^1)$ in the following sense: given $x \in A(I)$ and $\gamma \in \mathrm{Diff}^+(S^1)$,

$$\pi(\gamma(I))(U(\gamma)xU(\gamma)^*) = U_\pi(\gamma)\pi(I)(x)U_\pi(\gamma)^*. \tag{8.7}$$

Next, we'll discuss how to build a tensor category on the category of such representations; if the CFT has a finiteness condition called *rationality*, it will be a modular tensor category. So a conformal net is a very complicated way to obtain a modular tensor category, and we will discuss this and related questions.

Part 3. **Wednesday, January 29**

9. Sarah Witherspoon, Hopf algebras, II

Today, we will spend some time discussing non-semisimple Hopf algebras and tensor categories. This makes the classification question more complicated; there can be algebras or categories of wild type, where classifying all modules or objects, even the indecomposables, is just unrealistic.

So what can you do, then? It's still possible to make coarser classifications of objects and use techniques to gain partial information. Cohomology is particularly useful.

Definition 9.1. Let A be a Hopf algebra and $n > 0$. An *n-extension* of A-modules U and V is an exact sequence of A-modules

$$0 \longrightarrow V \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_2 \longrightarrow M_1 \longrightarrow U \longrightarrow 0. \tag{9.2}$$

A morphism of n-extensions is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r] \arrow[d, equal] & M_n \arrow[r] \arrow[d] & \cdots \arrow[r] & M_n \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d] & U \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & V \arrow[r] & M'_n \arrow[r] & \cdots \arrow[r] & M'_2 \arrow[r] & M'_1 \arrow[r] & U \arrow[r] & 0, \end{tikzcd} \tag{9.3}$$

i.e. the maps on U and V are the identity. This does not define a symmetric relation on n-extensions, so define $\mathrm{Ext}^n_A(U, V)$ to be the set of n-extensions, modulo the smallest equivalence relation generated by morphisms.

There is an abelian group structure on $\mathrm{Ext}^n_A(U, V)$ induced by *Baer sum* of extensions.

Definition 9.4. The *Hopf algebra cohomology* of a Hopf algebra A over k is $H^n(A, k) := \mathrm{Ext}^n_A(k, k)$.

Hopf algebra cohomology carries a graded product structure.

Definition 9.5. Consider an m-extension and an n-extension of k by k, given respectively by

$$0 \longrightarrow k \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_1 \xrightarrow{\alpha} k \longrightarrow 0 \tag{9.6a}$$

$$0 \longrightarrow k \xrightarrow{\beta} N_n \longrightarrow \cdots \longrightarrow N_1 \longrightarrow k \longrightarrow 0\ . \tag{9.6b}$$

The *Yoneda splice* of these two extensions is the $(m + n)$-extension

$$0 \longrightarrow k \longrightarrow M_m \longrightarrow \cdots \longrightarrow M_1 \xrightarrow{\beta\circ\alpha} N_n \longrightarrow \cdots \longrightarrow N_1 \longrightarrow k \longrightarrow 0. \tag{9.7}$$

Yoneda splice defines a bilinear map $H^m(A, k) \times H^n(A, k) \to H^{m+n}(A, k)$, called the *Yoneda product* or *cup product*; this makes $H^*(A, k) := \bigoplus_n H^n(A, k)$ into a graded ring.

We haven't used the Hopf algebra structure yet, and this cohomology ring exists for a general algebra.

Theorem 9.8. *If A is a bialgebra, then $H^*(A, k)$ is graded commutative.*

That is, this theorem uses comultiplication, but not the antipode.

More generally, given a tensor category $\mathcal{C}$, one can define a graded commutative k-algebra $H^*(\mathcal{C}, \mathbf{1})$.

Conjecture 9.9 (Friedlander-Suslin, Etingof-Ostrik)**.** If A is a finite-dimensional Hopf algebra, then $H^*(A, k)$ is finitely generated, and moreover, if U and V are finite-dimensional A-modules, $\mathrm{Ext}^*_A(U, V)$ is a finitely generated module over $H^*(A, k)$.[8]

Correspondingly, if $\mathcal{C}$ is a finite tensor category,[9] $H^*(\mathcal{C}, \mathbf{1})$ is finitely generated, and for any $X, Y \in \mathcal{C}$, $\mathrm{Ext}_{\mathcal{C}}(X, Y)$ is a finitely generated $H^*(\mathcal{C}, \mathbf{1})$-module.

Somehow this conjecture needs the fact that there *is* a comultiplication, but doesn't need the specific comultiplication, which is a little surprising.

Remark 9.10. There is another cohomology theory for algebras, called *Hochschild cohomology.* However, the analogue of Conjecture 9.9 for Hochschild cohomology is false! ◀

[8]We haven't specified how to make this module structure; one way is to write $\mathrm{Ext}^*_A(U, V) \cong \mathrm{Ext}^*_A(k, U^* \otimes V)$ and form a Yoneda splice on the left.

[9]A *finite* tensor category is one satisfying a few niceness conditions, including that it has only finitely many simple objects.

Why care about Conjecture 9.9? There is a theory of "varieties for modules" which is most useful in settings where the conjecture is true. The idea is to realize modules over noncommutative objects in terms of modules over commutative objects, and then take advantage of commutativity. Recent work of Bergh-Plavnik-Witherspoon [BPW19] works out a lot of this theory for general finite tensor categories.

Conjecture 9.9 is still open, but is known in a number of cases. Here are some established results.

- For $A = k[G]$ or $\mathcal{C} = \mathcal{R}ep_G$, G a finite group, this has been known for a long time. This is only interesting in modular characteristic (i.e. $\mathrm{char}(k) = p$ divides the order of G); otherwise, $k[G]$ is semisimple and its cohomology is concentrated in degree zero. This was established in the 1960s by Golodi, Venkov, and Evans; the theory of varieties for modules in this setting followed soon after.
- In positive characteristic, if A is a *restricted enveloping algebra*, i.e. a finite-dimensional quotient of $\mathcal{U}(\mathfrak{g})$, Conjecture 9.9 was established by Friedlander-Parshall [FP86, FP87].
- In characteristic zero, Conjecture 9.9 is true for the small quantum group $u_q(\mathfrak{g})$, as shown by Ginzburg-Kumar [GK93].
- In positive characteristic, if A is a finite-dimensional cocommutative Hopf algebra, Conjecture 9.9 was shwon by Friedlander-Suslin [FS97]. This was a significant breakthrough.

Some of these papers go beyond Conjecture 9.9, establishing structural results rather than just size.

The obstruction to understanding the general case is that we don't really understand finite-dimensional Hopf algebras and finite tensor categories well enough. But there has been recent progress, including work of Gordon (2000), Mastnak-Pevtsova-Schauenburg-Witherspoon [MPSW10], Bendel-Nukana-Parshall-Pillen (2014), Nguyen-Witherspoon [NW14] for twisted group algebras, Drupieski (2016) for supergroup schemes, Vay-Stefan (2016), Friedlander-Negron (2018) on Drinfeld doubles of cocommutative algebras, Nguyen-Wang-Witherspoon [NWW17, NWW19] in positive characteristic and a few general results; Erdmann-Silberg-Wang, Negron-Plavnik [NP18] recently on some general results on finite tensor categories; and more. There's been a lot of recent progress, but finishing off the conjecture will probably require new ideas.

Ongoing work of Andrurkiewitsch-Angimo-Pevstova-Witherspoon tackles the conjecture in characteristic zero for finite-dimensional pointed Hopf algebra whose grouplike elements form an abelian group — you always get a group, but the nonabelian case is wilder and a lot harder! This relies on previous results of Nicholas and Ivan (TODO: spelling?) on the structure theory of pointed Hopf algebras.

Yetter-Drinfeld modules are an important tool in the proof.

Definition 9.11. A *Yetter-Drinfeld $k[G]$-module* is a $k[G]$-module V together with a G-grading $V = \bigoplus_{g\in G} V_g$, such that for all $g, h \in G$, $h \cdot V_g = V_{hgh^{-1}}$. The category of Yetter-Drinfeld $k[G]$-modules is denoted ${}^{k[G]}_{k[G]}\mathcal{YD}$.

In the finite-dimensional case, these are equivalent to modules over the Drinfeld double of G.

Given a Yetter-Drinfeld module V, its tensor algebra $T(V)$ is a *braided Hopf algebra*, i.e. a Hopf algebra object in ${}^{k[G]}_{k[G]}\mathcal{YD}$. There is a largest ideal $J \subset T(V)$ that is also a *coideal*, i.e.

$$\Delta(J) = J \otimes T(V) + T(V) \otimes J. \tag{9.12}$$

This ideal J is concentrated in degrees greater than 1.

Definition 9.13. The *Nichols algebra* is $T(V)/J$.

Example 9.14. If q is an n^{th} root of unity, then $u_q(\mathfrak{sl}_2)^+ := k\langle E \mid E^n = 0\rangle$ is a Nichols algebra, and we can obtain

$$u_q(\mathfrak{sl}_2)^{\geq 0} = \langle E, K \mid E^n = 0, K^n = 1, KE = q^2 EK\rangle \tag{9.15}$$

as a smash product $u_q(\mathfrak{sl}_2)^+ \# k\langle K\rangle$; in this setting, the smash product is also called the *bosonization* of $u_q(\mathfrak{sl}_2)^+$. ⋖

More generally, finite-dimensional pointed Hopf algebras whose group of grouplike elements are abelian arise not necessarily as bosonizations of Nichols algebras, but aren't far off; they're what's called *cocycle deformations*. This uses the classification of Nichols algebras, in terms of Dynkin and related diagrams.

10. Cris Negron, Finite tensor categories and Hopf algebras: a sampling

Today, we work over an algebraically closed field k.

Example 10.1 (Small quantum groups)**.** Small quantum groups are important examples of Hopf algebras. Let $k = \mathbb{C}$ and let $\mathfrak{g}$ be a simple Lie algebra. Choose Cartan data for $\mathfrak{g}$, so that we have a set Δ of positive roots, and choose $q \in \mathbb{C}^\times$ of order p. The *small quantum group* associated to this data is the algebra generated by E_α, F_α, K_α for $\alpha \in \Delta$ subject to the *q-Serre relations*

$$E_\alpha^p = F_\alpha^p = K_\alpha^p - 1 = 0 \tag{10.2a}$$

$$K_\alpha E_\beta K_\alpha^{-1} = q^{\langle\alpha,\beta\rangle} E_\alpha \tag{10.2b}$$

$$K_\alpha F_\beta K_\alpha^{-1} = q^{-\langle\alpha,\beta\rangle} F_\alpha. \tag{10.2c}$$

This is a finite-dimensional, non-semisimple Hopf algebra.

Let $u_q(\mathfrak{b})$, called the *quantum Borel*, denote the subalgebra of $u_q(\mathfrak{g})$ generated by the K_α and E_α elements; this is also finite-dimensional and non-semisimple. Let $G \subseteq u_q(\mathfrak{b})$ be the sub*group* generated by the K_α elements. ◀

One might ask: how much information is lost when we move from a Hopf algebra to its tensor category of representations?

Recall that a tensor category is an abelian, k-linear, rigid monoidal category $\mathcal{C}$ whose objects all have finite length, whose Hom spaces are finite-dimensional over k, and whose unit is simple.

Definition 10.3. Call $\mathcal{C}$ *finite* if it has finitely many simple objects and enough projectives.

This implies $\mathcal{C}$ is tensor equivalent to a category of representations of a finite-dimensional algebra. For example, the representation categories of $u_q(\mathfrak{g})$ and $u_q(\mathfrak{b})$ are finite tensor categories.

Definition 10.4. If $\mathcal{C}$ is semisimple and has finitely many simple objects, call $\mathcal{C}$ *fusion*.

There's a sequence of nested inclusions

$$\begin{aligned}\{\text{representations of finite groups over } \mathbb{C}\} &\subseteq \{\text{fusion categories}\}\\ &\subseteq \{\text{finite tensor categories}\}\\ &\subseteq \{\text{tensor categories}\}.\end{aligned} \tag{10.5}$$

For example, $\mathcal{R}ep_{\mathrm{SL}_n}$ is a tensor category that is not finite.

For Hopf algebras, taking the category of representations lands in tensor categories, and we can study how tensor equivalences of representation categories can be thought of in the language of Hopf algebras. This is sort of asking, what happens as the boundary of this map fron Hopf algebras to tensor categories?

Definition 10.6. A *Drinfeld twist* of a Hopf algebra A is a unit $J \in A \otimes A$ satisfying

$$(\varepsilon \otimes 1)H = (1 \otimes \varepsilon)J = 1 \tag{10.7}$$

and the *cocycle condition*

$$(\Delta \otimes 1)(J)(J \otimes 1) = (1 \otimes \Delta)(J)(1 \otimes J). \tag{10.8}$$

Given a Drinfeld twist J, we can build some new things.

- First, a new Hopf algebra denoted A^J, which is the same as A except the comultiplication is modified to $\Delta^J \coloneqq J\Delta(-)J$.
- We also get a new fiber functor $F_J \colon \mathcal{R}ep_A \to \mathcal{V}ect$, which is the usual forgetful functor on objects and morphisms, but whose monoidal structure is modified: when defining the map
$$F_J(V) \otimes F_J(W) \to F_J(V \otimes W), \tag{10.9}$$
take the usual map, then apply J.

Theorem 10.10. *When A is a finite-dimensional Hopf algebra,* all *fiber functors $\mathcal{R}ep_A \to \mathcal{V}ect$ arise from Drinfeld twists in this way.*

Theorem 10.11 (Ng-Schauenberg)**.** *Let A and B be finite-dimensional Hopf algebras such that $\mathcal{R}ep_A \simeq \mathcal{R}ep_B$ as tensor categories. Then there is a Drinfeld twist J of A such that, as Hopf algebras, $B \cong A^J$.*

Example 10.12 (Negron [Neg18])**.** Specializing to $A = u_q(\mathfrak{b})$, an equivalence $\mathcal{R}ep_B \simeq \mathcal{R}ep_{u_q(\mathfrak{b})}$ leads to an alternating bicharacter $J \in \mathrm{Alt}(G^\vee) \subset T_w(u_q(\mathfrak{b}))$, such that $B \cong u_q(\mathfrak{b})^J$. Since $\mathrm{Alt}(G^\vee)$ is a finite set, this is particularly nice. ◀

As we heard in Rowell's talk, the notion of being the category of representations of a Hopf algebra is not invariant under tensor equivalence, and more generally, Hopf algebra representation categories are not closed under reasonable operations on the class of tensor categories.

For example, if a group G acts on a tensor category $\mathcal{C}$, then we can *equivariantize*, building a new tensor category $\mathcal{C}^G$, the category of objects $V \in \mathcal{C}$ with compatible structural isomorphisms $g \cdot V \overset{\cong}{\to} V$ for all $g \in G$. An embedding $\mathcal{V}ect \hookrightarrow \mathcal{C}$ induces an embedding $\mathcal{R}ep_G \hookrightarrow \mathcal{C}^G$.

Theorem 10.13 (Drinfeld-Gilyaki-Nikshych-Ostrik [DGNO10])**.** *Equivariantization defines a bijection between tensor equivalence classes of tensor categories with a G-action and tensor categories with a specified embedding of $\mathcal{R}ep_G$.*

This ultimately implies that even if $\mathcal{C}$ admits a fiber functor (as representation categories of Hopf algebras must), $\mathcal{C}^G$ might not, because there are categories containing $\mathcal{R}ep_G$ but not admitting a fiber functor.

Tensor categories have connections with 2d conformal field theory, hence vertex operator algebras.

- Given a *rational conformal field theory* (equivalently, a rational vertex operator algebra), Y. Huang shows how to extract a modular fusion category.
- Given a *logathmic conformal field theory*, a series of papers by Huang-Lepowski-Zhang construct a modular tensor category, maybe with some additional assumptions. See in particular [HLZ11].

The upshot is that given an *finite* logarithmic vertex operator algebra V, its category of representations is a finite, braided tensor category which is nondegenerate and pivotal (hence ribbon). (TODO: this was a conjecture by [GR], maybe?)

Example 10.14. Given a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ and $p \in \mathbb{Z}_+$, one can construct a non-rational vertex operator algebra denoted $W_p(\mathfrak{g})$, which is cut out of a lattice model by an action of $u_q(\mathfrak{n})$ by something called short-screening operations. This was studied by (TODO: [FT]) and Lentner.

These are understood in type A_1 and, at $p = 2$, type B_n: $W_p(\mathfrak{sl}_2)$ is the *triplet model* of Kausch (1991), and $W_2(\mathrm{B}_n)$ is the *symplectic fermion model* of Kausch [Kau00]. As established by Flandoli-Lentner, these have non-semisimple, modular representation theories. ◀

TODO: by [FGST], 2005, also [AM] and Lantner?

Conjecture 10.15. There is a modular equivalence $F_\otimes$ from the category of representations of $u_q(\mathfrak{g})$ to the category of representations of $W_p(\mathfrak{g})$, where $q := \exp(i\pi/p)$.

This is mostly done for $\mathfrak{g} = \mathfrak{sl}_2$, but is completely open in general.

Remark 10.16. You should be careful with what's precisely meant by $u_q(\mathfrak{g})$. See work of Negron, TODO: also [GR, CGR, GLO]. ◀

To finish, let's talk a little bit about cohomology. Suppose $\mathcal{C}$ is a finite, but not semisimple, tensor category. Then let $\operatorname{Proj}\mathcal{C}$ denote the subcategory of projective objects in $\mathcal{C}$; this has finitely many indecomposables $P_1, \dots, P_n$, canonically labeled by the isomorphism classes of simple objects. $\operatorname{Proj}\mathcal{C}$ has a *strong* fusion rule, with

$$P_i \otimes P_j \cong \bigoplus_k P_k^{\oplus N_{ij}^k} \tag{10.17}$$

for some natural numbers N_{ij}^k. In fact, something stronger is true: $\operatorname{Proj}\mathcal{C}$ can be described by discrete/number-theoretic data. But what happens on the rest of $\mathcal{C}$?

The *stable category* of $\mathcal{C}$ is $\operatorname{Stab}\mathcal{C} := \mathcal{C}/\operatorname{Proj}\mathcal{C}$. This is not an abelian category, though it is triangulated — in particular, it has a shift functor $\Sigma\colon \operatorname{Stab}\mathcal{C} \to \operatorname{Stab}\mathcal{C}$. This data is regulated by geometry and continuous invariants, called *support theory* or *tensor-triangulated geometry*, related to the Proj variety of $\operatorname{End}^*_{\operatorname{Stab}\mathcal{C}}(\mathbf{1})$. There's a lot more that could be said about this approach to the stable category, but that is a story for another day.

Part 4. Thursday, January 30

11. Emily Peters, Subfactors and planar algebras, II

Last time we dove into planar algebras; today we'll get to subfactors too.

Definition 11.1. A *von Neumann algebra* A is a unital, $*$-closed subalgebra of $B(\mathcal{H})$, the algebra of bounded operators on some Hilbert space, such that the *double commutant* A'' of A is again A. (The double commutant is the space of operators which commute with the things which commute with A.) A *factor* is a von Neumann algebra A such that $Z(A) = A \cap A' = \mathbb{C} \cdot \mathrm{id}$. A *subfactor* is a unital inclusion of factors.

Why care about subfactors? Well, if you want to understand maps between von Neumann algebras, you can begin by trying to decompose your algebras into smaller pieces, and subfactors are the basic building blocks of maps.

Let S_∞ denote the *finitary symmetric group*, i.e. the group of permutations of a countable set which leave all but finitely many elements fixed. Alternatively, $S_\infty = \mathrm{colim}_n S_n$. In S_∞, cycle types are conjugacy classes, as with finite symmetric groups, and therefore there are infinitely many conjugacy classes.

Let $\mathcal{H} := \ell^2(S_\infty)$, which carries the left regular representation $\lambda\colon S_\infty \to B(\mathcal{H})$ by

$$(\lambda(g))\xi)(h) = \xi(g^{-1}h). \tag{11.2}$$

Define the *group von Neumann algebra* to be $L(S_\infty) := \mathbb{C}[\lambda(S_\infty)]''$. Taking the double commutant is called the *von Neumann closure*; $L(S_\infty)$ is a von Neumann algebra, and even a factor, though there is an argument to make here. This factor is called the *hyperfinite type* II_1 *factor*.

Remark 11.3. You can make this construction for any group with an infinite number of conjugacy classes (in fact, even if not, you still get a von Neumann algebra, but not a factor). For the free groups F_2 and F_3 on two, respectively three, elements, $\mathbb{C}[\lambda(F_2)] \not\cong \mathbb{C}[\lambda(F_3)]$, but it is a longstanding open question whether the group von Neumann algebras are isomorphic (i.e. after von Neumann closure).

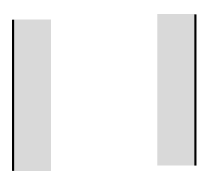
On the other hand, if G is profinite (i.e. a colimit of finite groups, as with S_∞), $L(G) \cong L(S_\infty)$. $\triangleleft$

Let G be any finite subgroup of S_∞. Then we can build a subfactor: G acts on $L(S_\infty)$, and the invariants $L(S_\infty)^G \hookrightarrow L(S_\infty)$ are a subfactor.

Given a subfactor $A \subset B$, we will build a planar algebra, at least assuming A is finite-index and irreducible. Let $X := {}_AB_B$, i.e. B as an (A, B)-bimodule, with A-action implemented through the inclusion. Correspondingly, let $\overline{X} := {}_BB_A$.

Associated to this data we have some diagrammatics. We represent X by a vertical line shaded to the right, and $\overline{X}$ by a vertical line shaded to the left.

(11.4)



X $\qquad\qquad$ $\overline{X}$

There is an evaluation (semicircle like $\frown$, shaded inside), corresponding to a map

$$X \otimes \overline{X} = {}_AB \otimes_B B_A = {}_AB_A \longrightarrow {}_AA_A. \tag{11.5}$$

In the theory of von Neumann algebras, this map is called *conditional expectation*, and has been well-studied. Correspondingly, the $\smile$-shaped semicircle, shaded inside, induces the inclusion map ${}_AA_A \hookrightarrow {}_AB_A$.

Two more diagrams: the $\smile$ shaded on the outside corresponds to a map

$${}_BB_B \longrightarrow \overline{X} \otimes X = {}_BB \otimes_A B_B, \tag{11.6}$$

which arises from the basic construction $A \subset B \subset B \otimes_A B$. the $\frown$ diagram shaded on the outside is just the multiplication map

$${}_BB \otimes_A B \longrightarrow {}_BB_B. \tag{11.7}$$

Let $V_n := \mathrm{End}(X \otimes \overline{X} \otimes \cdots)$, where there are n factors of X or $\overline{X}$. You can represent V_n by a box with n input and output wires. (TODO: I didn't get this and might have it wrong.)

From these diagrams we have a planar algebra. But the planar algebras which come from subfactors have extra structure. Because A and B are factors, the diagrams corresponding to the circle (two diagrams: shaded inside, or shaded outside) are constants. We adopt the convention to normalize such that both of them are the number $\mathrm{FPdim}(X)$.

There are two flavors of box, depending on whether you shade on the inside or outside; call them $V_{n,+}$ and $V_{n,-}$, depending on which of X or $\overline{X}$ is the first factor in the tensor product. These are all finite-dimensional, which follows from our assumption that $A \hookrightarrow B$ is finite index; and $\dim(V_{0,\pm}) = 1$. Moreover, we can take isotopies in S^2, rather than $\mathbb{R}^2$ — we can move strands through the point at infinity. (TODO: picture).

FInally, we have a positivity constraint: if $a, b \in V_{n,\pm}$, $\langle a, b\rangle \coloneqq \operatorname{tr}(b^* a)$ is positive (provided $a, b \neq 0$). As before, $*$ is reflection across a horizontal line.

Call a planar algebra satisfying these three additional constraints a *subfactor planar algebra*.

Theorem 11.8 (Jones)**.** *The planar algebras coming from subfactors are subfactor planar algebras.*

Theorem 11.9 (Popa)**.** *All subfactor planar algebras arise from subfactors.*

However, both constructions are sadly non-functorial.

The subfactor planar algebra associated to a subfactor is a nice invariant of a subfactor, but is not the only interesting one. The first invariant associated to a subfactor is Jones' *index*,

$$[B : A] \coloneqq \dim_A B. \tag{11.10}$$

This is equal to $\operatorname{FPdim}(X)^2$. Actually making sense of this dimension can be a little funny, and you may have to dip into some analysis, but it can be done.

Jones proved a theorem characterizing the possible indices which made people sit up and take notice.

Theorem 11.11 (Jones [Jon83])**.** *The index is either of the form $4\cos^2(\pi/n)$, for $n = 3, 4, 5, \dots$, or is an element of $[4, \infty]$, and all of these can arise.*

The indices of the form $4\cos^2(\pi/n)$ are "classical," less surprising, but the new, continuous ones are stranger.

The principal graph of a subfactor is another useful invariant — in fact, its ability to remember important information but not too much has led some people to think of it as the "Goldilocks invariant." It encodes the fusion rules of a tensor category. Consider a decomposition of $X \otimes \overline{X} \otimes X \otimes \cdots \otimes X$ into irreducible bimodules, then build a graph depicting $- \otimes X^{\pm}$. For example, if we had the rules

$$X \otimes \overline{X} = Z_1 \oplus \mathbf{1} \tag{11.12a}$$

$$Z_1 \otimes X = X \oplus Z_2 \oplus Z_2 \tag{11.12b}$$

$$Z_2 \otimes \overline{X} = Z_3 \otimes \overline{X} = Z_1, \tag{11.12c}$$

we would obtain the graph TODO: Dynkin diagram of type D_5, labeled $\mathbf{1}$, X, Z_1, and then Z_2 and Z_3 on the two tails.

Using this, we can build a unitary tensor category from a subfactor/subfactor planar algebra. Unitarity means that each Hom space has a positive Hermitian form, and consequently all dimensions are positive. The objects of this category are the irreducible bimodules obtained from $X \otimes \overline{X} \otimes \cdots \otimes X^{\pm}$, the morphisms are the relevant planar diagrams, and the tensor product is disjoint union of diagrams.

And this is a helpful tool for studying these tensor catgories. For small subfactor planar algebras, $\operatorname{FPdim}(X) = \sqrt{[B : A]}$ tells you about the growth rate of $\{V_{n,\pm}\}$ (assuming here that the subfactor is finite-depth or amenable). Under these assumptions, the principal graph is of ADE type, but — and a lot of work went into this — the ADE graphs that arise are D_n for n even, E_6, and E_8. One can also (TODO: which assumption is relaxed?) get extended Dynkin diagrams. These are the cases where the index is $4\cos^2(\pi/n)$ for some n, so Haagerup asked what nontrivial principal graphs can arise in the next simplest setting, where the index is above 4, but not by much. The graphs he found included TODO: H_i, AH, B_i. Bisch [Bis98] showed B_i doesn't work: its fusion rules are nonassociative. Asaeda-Haagerup [AH99] constructed subfactors realizing H_0 and AH; then Asaeda-Yasuda [AY09] showed that H_i for $i \geq 2$ cannot be realized.

This leaves one case, H_1.

Theorem 11.13 (Bigelow-Morrison-Peters-Snyder [BMPS12])**.** *There is a planar algebra realizing the graph H_1.*

This algebra is called the *extended Haagerup planar algebra*. It is a positive planar algebra generated, as a planar algebra, by $S \in V_8$ with some relations. It has Frobenius-Perron dimension about equal to 2.09218, which indicates we're just barely in the region of continuous index. The relations are described by pictures (TODO).

You can write down any random set of relations, and will probably get something zero or infinite. The work that went into Theorem 11.13 was showing that this is finite, and interesting.

12. Zhenghan Wang, Topological orders I

This talk will be a mathematics talk about topological order (in topological phases of matter), which is a subject close to physics. For a general reference for this talk, see [RW18]. We will focus on bosonic/spin intrinsic order, as opposed to fermionic order or SPT phases; if this doesn't mean anything to you, that's OK.

When we consider topological orders in dimension $2+1$, we mean two dimensions of space, and one of time, so we will usually think about surfaces, and sometimes 3-manifolds. A $(2+1)$*-dimensional topological order* is equivalent to a unitary topological modular functor, which is equivalent to a unitary modular tensor category (sometimes called an *anyon model* in physics). There are several reasons to care about these phases.

(1) At least in theory, the ability to build topological phases of matter would allow one to build a quantum computer, which has numerous applications to the real world, including making money. This is being pursued in industry, e.g. by Microsoft. However, there is still much to do, in both theory and engineering, before this can be a reality.
(2) From a theoretical physics perspective, topological phases are very interesting. These are in the subfield of condensed-matter physics, which historically understood phases of matter via Landau's group symmetry-breaking paradigm; for example, crystals are understood via discrete translation symmetries, but liquids have continuous symmetries. Phase transitions correspond to symmetry breaking. But topological phases do not follow these rules, leading to a paradigm shift in physicists' perspectives on phases, to the perspective of quantum symmetries.
(3) Finally, these are interesting objects in their own mathematical right.

Analogously to the relationship between Riemann sums and definite integrals, there are two perspectives on quantum field theory that shed mathematical insight into it: one can work in the continuum (akin to the integral) or on the lattice, which is more discrete, akin to a Riemann sum. Sometimes we use integrals to approximate Riemann sums, even though that wasn't the original way information flowed; likewise, these topological phases are QFTs on the lattice, but we can study them with continuum limits. This is a part of the general mathematical goal of understanding quantum field theory.

We will focus on two examples: the toric code and Haah's code. The toric code is very, very well-studied — almost any question you might ask about it has been answered. Haah's code is newer, and poorly understood: it's an example of a fracton model, and we think that a proper understanding of such models will lie beyond quantum field theory.

We will first study the toric code via its robust ground state degeneracy, which is a TQFT, and which is a mathematically satisfying perspective even if it's still not completely understood. In the next lecture, we'll study the elementary excitations, which lead to unitary modular tensor categories, a different perspective. But there is a mathematical theorem relating unitary TQFTs in dimension $2+1$ and unitary modular tensor categories.

You should not just accept these definitions as final — this field is still in the process of being mathematically codified. Some of these definitions are matters of convenience, so that we can actually get somewhere, even if we don't have the most correct definitions.

Definition 12.1. A *quantum theory* is a triple (L, B, H), where L is a finite-dimensional Hilbert space, B is a basis of L, and $H\colon L \to L$ is a Hermitian operator, which thanks to B we can think of as a matrix.

The basis is an unusual ingredient when one studies QFTs, but is important in the story of quantum information: it's how you represent classical information.

Many quantum theories satisfying Definition 12.1 aren't related to physics, and are therefore somewhat useless. We focus on the examples which come from physics; after all, Definition 12.1 is trying to (partially) axiomatize things physicists are interested in, right?

Definition 12.2. An *n-dimensional quantum schema* is a rule assigning to every n-dimensional manifold with a triangulation and a finite-dimensional Hilbert space, a quantum theory.

Example 12.3. The 1-dimensional Ising chain is a 1-dimensional quantum schema. Given a circle with a triangulation, the Hilbert space is the tensor product of copies of $\mathbb{C}^2$ indexed by the vertices. Assume there are N vertices, and orient the circle so we can identify them with $1, \dots, N$ in order, and call the Hilbert space L_N. Inside $\mathbb{C}^2$, let $|0\rangle$ and $|1\rangle$ be the standard basis vectors (i.e. $(1,0)$ and $(0,1)$, respectively). The Hamiltonian has the form

$$H = -\sum_{i=1}^{n-1} \sigma_i^z \sigma_{i+1}^z. \tag{12.4}$$

This is physicists' notation: let's explan what's going on. The *Pauli matrices* are the standard basis of $\mathfrak{su}_2$:

$$\sigma^x = \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^z = \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma^y = \sigma_2 := i\sigma^x\sigma^z. \tag{12.5}$$

These are Hermitian matrices which square to the identity.

The notation σ_i^z means that σ^z acts on the $\mathbb{C}^2$ at vertex i, and acts by the identity on the remaining factors, i.e. by $\mathrm{id} \otimes \mathrm{id} \otimes \cdots \otimes \sigma^z \otimes \cdots \otimes \mathrm{id}$. ◀

Given a quantum theory, the eigenvalues of the Hamiltonian $\lambda_0 \le \lambda_1 \le \dots$ are called *energies* of the theory, and the nonzero eigenvectors of λ_0 are called *ground states*. Nonzero eigenvectors for other eigenvalues are called *excited states*.

Here's the most important definition (albeit, again, not quite the real definition).

Definition 12.6. An n-dimensional Hamiltonian schema is *(sharply) gapped* if there is a constant $c > 0$ such that for all n-manifolds and triangulations, in the quantum system assigned by the schema, the eigenvalues of the Hamiltonian satisfy $\lambda_1 - \lambda_0 \ge c$.

Crucially, c does not depend on the triangulation. Sharply gapped schemas are almost topological.

Tentative definition 12.7. An n-dimensional Hamiltonian schema is *topological* if it's gapped, and if there exists a unitary topological modular functor (i.e. a unitary TQFT-like object in dimension $2+1$, which is once-extended, but isn't necessary finite enough to assign partition functions to all 3-manifolds) Z such that for any closed 2-manifold Σ, $Z(\Sigma)$ is isomorphic to the space of ground states of the Hamiltonian schema on Y for any triangulation.

We expect that a topological Hamiltonian schema represents mathematically a topological phase of matter.

Definition 12.8. A *topological phase of matter* is a path component of the space of topologically ordered Hamiltonians.

Unfortunately, we're not yet sure what the space of topologically ordered Hamiltonians is, but we want to say that two Hamiltonians are equivalent if there's a path deforming one into the other, through topological Hamiltonian schema — in particular, the path cannot close the gap: c must always be greater than some $\varepsilon > 0$. Understanding this carefully in general would require opening the can of worms called renormalization.

The toric code is the model organism in topological phases. If you want to understand just about anything about topological phases of matter, you should probably begin by thinking about it for this example.

Example 12.9 (Toric code)**.** The toric code, first studied by Kitaev [Kit03], realizes the topological order given by untwisted $\mathbb{Z}/2$-Dijkgraaf-Witten theory, corresponding to the modular tensor category $D(\mathbb{Z}/2)$, with four simple objects $\{1, e, m, \psi\}$ with $e \otimes e = m \otimes m = \psi \otimes \psi = 1$ and $e \otimes m = m \otimes e = \psi$ and twists $\theta_1 = \theta_e = \theta_m = 1$ and $\theta_\psi = -1$.[10]

The toric code schema begines with a closed surface Y and a triangulation (or more generally, a cellulation), which is often just taken to be the torus with a cellulation given by a square tiling of the plane. The Hilbert space L is a tensor product of $\mathbb{C}^2$ over all of the edges in the cellulation. Thus L is canonically identified with the group algebra for the group $(\mathbb{Z}/2)^{|E|}$, if E is the set of edges.

The Hamiltonian is

$$H = -\sum_{\text{all vertices } v} A_v - \sum_{\text{all faces } P} B_P, \tag{12.10}$$

for some operators A_v and B_P we will define.[11] A_v is the tensor product of σ^z on all of the edges adjacent to v, and the identity on the remaining edges. B_P is the tensor product of σ^x on all edges in ∂P, and the identity on the remaining edges.

[10] So the anyon ψ is a fermion, but this is still a bosonic phase, because we started with bosonic spins, or mathematically, vector spaces and not super vector spaces.

[11] Why "P"? Because in the physics literature, faces are often referred to as *plaquettes*.

Here are three important properties of the toric code.

(1) All A_v and B_P operators commute with each other. This is clear for A_v and $A_{v'}$, and for B_P and $B_{P'}$, since we just have a bunch of σ^z or σ^x operators, or for A_v and B_P when $v \notin \overline{P}$, but the interesting bit is when v and P are adjacent; then σ^x and σ^z don't commute, but they anticommute, and there is an even number of edges affected by both A_v and B_P, so two minuses make a plus and $[A_v, B_P] = 0$.

(2) The space of ground states of this model on a closed surface Σ is canonically identified with the space of $\mathbb{C}$-valued functions on $H_1(\Sigma; \mathbb{Z}/2)$. This uses the fact that the Hamiltonian is *frustration-free*, which means that the ground states are precisely those stabilized by all A_v and B_P operators. Looking at B_P gives you cycles; then looking at A_v kills boundaries.

(3) The elementary excitations for the toric code form the unitary modular tensor category $D(\mathbb{Z}/2)$. ◀

Exercise 12.11. Modify the toric code to

$$H = \sum_v \varepsilon_1 A_v + \sum_P \varepsilon_2 B_P, \tag{12.12}$$

where $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$. Which of these phases are topologically ordered, and which aren't? If you understand that, you probably understand this lecture very well.

13. Anna Beliakova, Quantum invariants of links and 3-manifolds, II

Recall that we're in the business of studying non-semisimple quantum invariants of knots and 3-manifolds. Last time, we discussed how surgery on a framed link can turn logarithmic framed link invariants into 3-manifold invariants, as studied by Hennings (1998), Beliakova-Blanchet-Geer (2017), and Costantino-Geer-Patureau-Mirand [CGPM14].

First, the algebraic data that we need. Let H be a finite-dimensional Hopf algebra, such as

$$u_\xi \coloneqq u_q(\mathfrak{sl}_2) \otimes_{\mathcal{A}} \mathbb{C}, \tag{13.1}$$

where $\mathcal{A} \coloneqq \mathbb{Z}[q^{\pm 1}]$, ξ is a p^{th} root of unity, and $\mathcal{A}$ acts on $u_q(\mathfrak{sl}_2)$ by $q \mapsto \xi$. We can also restrict to $u^{rest} \coloneqq u_\xi / \langle e^p, F^p, K^{2p} - 1\rangle$; then K^p is central.

Radford showed that there exists a unique $\mu^* \in H^\times$, called the *integral*, such that

$$(\mu \otimes \mathrm{id})\Delta(x) = \mu(x)\mathbf{1} \tag{13.2}$$

for all $x \in H$. For example, in u_ξ,

$$\mu(E^m F^n K^\ell) = \delta_{m,p-1}\delta_{n,p-1}\delta_{\ell,p+1}. \tag{13.3}$$

Theorem 13.4 (Hennings, Kauffman-Radford)**.** *Let L be a framed link and $M \coloneqq S^3(L)$. Let σ_+, resp. σ_-, be the number of positive, resp. negative eigenvalues of $\ell k(L)$. Then the Hennings invariant can be calculated as*

$$\mathrm{Hen}(M) = \frac{\mu^{\otimes |L|}(J_L)}{\mu(J_{U_+})^{\sigma_+}\mu(J_{u_-})^{\sigma_-}}. \tag{13.5}$$

The proof is surprisingly simple.

Proof. Using the Kirby move K1, write $L' = L \amalg u_+$, where u_+ indicates an unlinked unknot. Then

$$\mathrm{Hen}(M) = \frac{\mu^{\otimes |L|}(J_L)\mu(J_{u_+})}{\mu(J_{u_+}^{\sigma_+ + 1})\mu(J_{u_-})^{\sigma_-}}, \tag{13.6}$$

and we cancel out the factors of $\mu(J_{u_+})$ in the numerator and denominator. Now, perform the Kirby move K2 and use (13.2) (TODO: I didn't understand this part) and we're done. ⊠

Remark 13.7. If H is semisimple,

$$\mu = \sum \mathrm{qdim}(V_i)\, \mathrm{tr}_q^{V_i}, \tag{13.8}$$

where the sum is over the isomorphism classes of irreducible modules V_i, and the Hennings invariant for H and M and the Witten-Reshetikhin-Turaev invariant for H and M coincide. ◀

Kuperberg [Kup91] constructed a related invariant using cointegrals in a Hopf algebra.

Definition 13.9. Let H be a Hopf algebra. A *left cointegral* is an element $c \in H$ satisfying $xc = \varepsilon(x)c$ for all $x \in H$; a *right cointegral* satisfies $cx = \varepsilon(x)c$ for all $x \in H$.

A Hopf algebra in which left and right cointegrals coincide is called *unimodular*.

Kuperberg's invariant is the simplest algebraic 3-manifold invariant one can define with Hopf algebras, and this lends it its usefulness — it will probably be one of the first things we fully categorify. For example, if we chose $u_q(\mathfrak{sl}_2)$, we'd need the data of the Borel part, $\langle K, E\rangle$. Chang-Cui [CC19] showed that the Kuperberg invariant for M and H coincides with the Hennings invariant for M and $\mathcal{D}(H)$, the double of H, analogous to the relationship between Reshetikhin-Turaev invariants for a modular tensor category and Turaev-Viro invariants for its Drinfeld double.

Theorem 13.10 (Chen-Kuppum-Srinivasan [CKS09])**.** *If $b_1(M) = 0$, the Hennings invariant of M is $|H_1(M)|\, WRT(M)$; otherwise, it vanishes.*

Proof. Our proof sketch follows Habiro-Lê. But: TODO: I didn't follow what was written on the board; sorry about that. I think what happened was: the Hennings invariant of M ends up being $\sum_{i\in I} x_i \otimes y_i$, where $\{x_i\}$ and $\{y_i\}$ are both bases of H. This induces a Hopf pairing, sometimes called the *quantum Killing form*, by declaring $\langle x_i, y_j\rangle \coloneqq \delta_{ij}$. Hence if $x = \sum a_i y_i$, $\langle x, x_i\rangle = a_i$.

Let M be an integral homology sphere, so we can realize M as surgery on a knot K framed with framing ± 1. Then $I_M = \langle r^{-1}, J_K\rangle$, where r is a *ribbon element* in H. We claim that for all $x \in H$,

$$\langle r^{-1}, x\rangle = \frac{\mu(x^r)}{\mu(r)}, \tag{13.11}$$

and that $\Delta(r) = (r \otimes r)M$. This is because $(\mu \otimes \mathrm{id})\Delta(r) = \mu(r)\mathbf{1}$, so

$$\sum_i \mu(rx_i)y_i = \mu(r)\mathbf{1}, \tag{13.12}$$

and therefore

$$r^{-1} = \sum_i \frac{\mu(rx_i)}{\mu(r)} y_i \tag{13.13}$$

$$\langle r^{-1}, x_i\rangle = \frac{\mu(rx_i)}{\mu(r)}. \tag{13.14}$$

If $b_1(M) > 0$, (TODO: what happened here?) $S^2 \times S^1 = S^3(U_0)$, where U_0 denotes an unknot, and $J_{U_0} = \mathbf{1}$, so the Hennings invariant is $\mu(\mathbf{1}) = 0$. ⊠

This seems to spell doom for non-semisimple invariants, but not all of them are killed. This leads one to introduce *modified traces*, following Geer-Patureau (2008), functions $t_P\colon \operatorname{End} P \to k$, where P is an H-module, such that $t_P(fg) = t_P(gf)$ and t_P satisfies the *partial trace property* (TODO: diagram).

Remark 13.15. Now taking the invariant J_K as usual, this kind of invariant lands in

$$qHH_0(H) \coloneqq H/\langle xy0S^2(y)x \mid x, y \in H\rangle. \tag{13.16}$$ ◀

Theorem 13.17 (Beliakova-Blanchet-Gainutdinov [BBG17])**.** *Let H be a unimodular pivotal Hopf algebra. Then for $f \in \operatorname{End}_H(H)$,*

$$\mathrm{tr}_H(f) = \mu_g(f(1)), \tag{13.18}$$

where g is the pivotal element: $\mu_g(x) = \mu(gx)$. In particular, the modified trace is uniquely determined.

Then, Beliakova-Blanchet-Geer [BBG18] used this to define more invariants for a knot K_P in a 3-manifold $M = S^3(L)$: the invariant is $(\mu^{\otimes |L|} \otimes t_P)J_{L\cup K_P}$. These invariants were extended to a TQFT by De Renzi, Geer, and Patureau-Mirand [DRGPM18].

In the last few minutes, we'll discuss CGP invariants. Consider the Hopf algebra

$$u^{\text{unrolled}} \coloneqq \langle K, E, F, H\rangle/\langle E^p, F^p\rangle. \tag{13.19}$$

Given $\lambda \in |C$, we have a p-dimensional irreducible u^{unrolled}-module V_λ, and (TODO: not sure if correct) V_λ and $V_{\lambda'}$ are isomorphic iff $\lambda' - \lambda \in 2\mathbb{Z}$. So really the classification is in $\mathbb{C}/2\mathbb{Z}$. This leads to an invariant of a manifold together with a cohomology class $\lambda \in H^n(M;\mathbb{C}/2\mathbb{Z})$, given by

$$\mathrm{CGP}(M,\lambda) := \sum_{k=0}^{p-1} d^{mod}(V_{\lambda+2k}) J_K(V_{\lambda+2k}) \tag{13.20}$$

This is a surprisingly simple description of this kind of invariant, which is nice. At $p = 2$, this specializes to the Alexander polynomial and Reidemeister torsion. Blanchet, Costantino, Geer, and Patureau-Mirand [BCGPM16] extended this invariant to a TQFT in which the order of the Dehn twist is trivial. The S- and T-matrices of this TQFT are related to work of Gukov and collaborators.

14. Colleen Delaney, Modular data and beyond

This talk fits into the same line of ideas as Eric Rowell's, and to some extent also Anna Beliakova's, though here we work in the semisimple case, rather than the non-semisimple one. We always work over the ground field $k = \mathbb{C}$. We will discuss the modular data associated to a modular tensor category, how to think of them as quantum invariants, the role of the S- and T-matrices, and then invariants beyond modular data.

Let $\mathcal{C}$ be a braided fusion category; thus we have data of $\mathcal{C}$, the monoidal tensor product $\otimes$, the unit $\mathbf{1}$, and the braiding $c_{X,Y} : X \otimes Y \to Y \otimes X$. We assume the braiding is *non-degenerate*, i.e. if $c_{Y,X} \circ c_{X,Y} = \mathrm{id}_{X\otimes Y}$ for all $Y \in \mathcal{C}$, then $X \cong \mathbf{1}$.

We can introduce more niceness into $\mathcal{C}$, in two ways.

- We can consider twists $\theta \in \mathrm{Aut}(\mathrm{id}_{\mathcal{C}})$, leading to the notion of a ribbon structure.
- We can consider a pivotal structure $\phi_x : x \xrightarrow{\cong} x^{**}$. This can satisfy a niceness condition called *sphericality*.

If we have both of these, and the nondegeneracy condition, we obtain a modular tensor category. Then the S-matrix is $S_{a,b} = \mathrm{tr}(c_{b,a} \circ c_{a,b})$, as (a,b) ranges over all pairs of isomorphism classes of irreducible objects. The T-matrix is diagonal, with the diagonal entries the twists of the irreducible objects.

Modular tensor categories were first written down by Moore and Seiberg [MS89], who were investigating two-dimensional conformal field theory. To Moore and Seiberg, the S- and T-matrices were part of the data of a modular tensor category, though this is not the only approach. In particular, they extend to a representation of the entire modular group, which is the reason for their name. In this sense, modular tensor categories are a gift from quantum field theory.

Subsequently, people studied many aspects of modular tensor categories; in the 2000s there was strong focus on the invariants they define, and more recently, Bartlett, Douglas, Schommer-Pries, and Vicary [BDSPV15] showed that modular tensor categories are equivalent data to once-extended $(2+1)$-dimensional TQFTs.

There was a longstanding conjecture that the modular data is a complete invariant of the modular tensor category. This conjecture was disproven in 2017, and this and subsequent developments will be a significant part of this talk.

As a tool of convenience, we will work with *skeletal modular tensor categories*, in which there is a single object in each isomorphism class. Let $\mathcal{L}$ denote the set of irreducible objects. This dehydrates modular tensor categories to equations: the structure coefficients N_{ab}^c, i.e. the number of copies of c inside $a \otimes b$, where $a, b, c \in \mathcal{L}$; the *F-symbols* $(F_a^{abc})_{ef}$ encoding the associator, and the *R-symbols* R_c^{ab} encoding the braiding. This loses some of the elegance and coherence, but can be useful for throwing mathematical or computer tools at the problem, and this has been a successful technique.

This leads to a graphical calculus for modular tensor calculus. We follow the conventions of Barkeshli-Bonderson-Cheng-Wang [BBCW19], which applies modular tensor categories to physics. Morphisms are represented by braided trivalent ribbon graphs (TODO: diagram).

The first Reidemeister move is not allowed in this calculus, but the second and third Reidemeister moves hold. Standard results in elementary knot theory implies this defines an invariant of framed links. Our pictures don't make the framing explicit, but we use the *blackboard framing*, which is the usual one you'd have if your links were mostly on the blackboard, with no surprising framing changes at crossings.

Remark 14.1. Since modular tensor categories also define anyon models, you can think of this graphical calculus as describingp processes (such as particle creation or collision) in these physical systems. In this case the invariant we've just described calculates some sort of amplitudes. ◀

Two basic elementary invariants of modular tensor categories are the rank and the *fusion ring* (i.e. the ring generated by the isomorphism classes of simple objects, with fusion as multiplication). These are "classical" (in the mathematical sense), in that the fusion ring is the decategorification of the modular tensor category, and these invariants don't see this categorical structure.

But we also have invariants that the fusion ring doesn't see, such as the *quantum dimension* d_a of an irreducible object a, namely $\mathrm{tr}(\mathrm{id}_a)$, and the *global dimension* D, the square root of the sum of the squares of the quantum dimensions of the irreducibles. We can also evaluate any link whose strands are colored by irreducible objects.

As a simple example, consider the figure-8 colored by an irreducible a, which the graphical calculus assigns an element of $\mathrm{End}_{\mathcal{C}}(\mathbf{1})$, albeit maybe not a fast way to compute it:

$$\mathrm{tr}(c_{a,a}) = \sum_c \frac{d_c}{d_a} R_c^{aa} = \theta_a = T_{aa}. \tag{14.2}$$

So this tells you the T-matrix. In a similar but more complicated way, the graphical calculus tells you that if you label the strands of the Hopf link with a and b,

$$\frac{1}{D}\,\mathrm{tr}(c_{b,a} \circ c_{a,b}) = \frac{1}{D} \sum_c N_c^{ab} \frac{\theta_c}{\theta_a \theta_b} d_c = S_{ab}, \tag{14.3}$$

so you see the S-matrix.

Why so much fuss about modular data? Because all of the invariants we've described so far are determined by modular data, as in the following examples.

- The rank is the dimension of the S- and T-matrix.
- The fusion rules can be recovered from the formula

$$N_c^{ab} = \frac{1}{D} \sum_e \frac{S_{ea} S_{eb} S_{ec}^*}{d_e}. \tag{14.4}$$

  The mantra is "the S-matrix diagonalizes the fusion ring."
- The first column of the S-matrix is the quantum dimensions of the irreducibles, which also determine the global quantum dimension.
- The modular representation (which is a projective representation) is determined by S and T, viz. the generators s and t of $\mathrm{SL}_2(\mathbb{Z})$.
- The *central charge* c satisfies

$$e^{2\pi i c/8} = \sum_{a \in \mathcal{L}} d_a^2 \theta_a. \tag{14.5}$$

  Researchers in modular tensor categories also study higher Gauss sums, which are also determined by modular data.
- Frobenius-Schur indicators, one of the more subtle and bothersome aspects of modular tensor categories, in that they can get in the way of things you think are true but actually aren't. Anyways, they're all determined by modular data.

There are probably still yet other invariants we will apply to modular tensor categories, but which are determined by modular data.

Johnson-Freyd and Scheimbauer [JFS17] write down a 4-category of braided fusion categories. Not all of these are modular, but we can use modular data to understand this 4-category. Another thing we can do is enumerate the modular tensor categories of low rank; there are combinatorial constraints on modular data that shine a lot of light on the low-rank classification (and likewise for a related classification of "super modular tensor categories").

There are various ways to build new modular tensor categories from preexisting ones, such as the Deligne tensor product. In a Deligne tensor product, the S- and T-matrices are tensor products, so the modular data is a strong sign that your category factors as a tensor product. In other settings, it's not as clear how the modular data transforms, but this is something people are working on, and in practice partial information plus the usual combinatorial constraints are often sufficient. Some of these operations include taking the Drinfeld center, and de-equivariantization of a G-action on a modular tensor category.[12] Similarly, there's also

[12] This is not always a modular tensor category, but work of Dmitri-Nikshych shines light on when it is.

G-equivariantization, and combining these is called *gauging*; Cui-Galindo-Plavnik-Wang [CGPW16] showed this sends modular tensor categories to modular tensor categories.

So if you can find additional interesting operations, and understand how the modular data behaves under those operations, that would be excellent! This was how a newer operation called "zesting" was discovered, borne out of attempts to disambiguate modular tensor categories with similar-looking modular data.

People are also interested in studying "exotic modular tensor categories," i.e. those not built out of group theory or Lie theory in the usual ways. For example, Grossman-Izumi [GI19] construct modular data from metric groups, and Bonderson-Rowell-Wang [BRW19] discuss realizations of exotic modular data.

So it's difficult to overstate how useful modular data is. But it's not a complete invariant, and this is a recent development! We'll finish by summarizing what we know now, and how life has changed in the aftermath.

Theorem 14.6 (Mignard-Schauenberg [MS17])**.** *g For $G \coloneqq \mathbb{Z}/q \rtimes \mathbb{Z}/p$, where p and q are primes, $p \mid q-1$, and $p > 2$, the modular categories $Z(\mathcal{V}ec_G^\omega)$ are not always determined explicitly by their modular data; an explicit example is $p = 5$ and $q = 11$.*

Lots of insight and brute force went into this discovery, but the five examples for $p = 5$ and $q = 11$ (since $H^3(G; \mathrm{U}_1) \cong \mathbb{Z}/5$) are fairly explicit. These categories are rank 49, which is huge; it's open whether there are smaller examples.

One generalization of the S-matrix is the "punctured S-matrix," corresponding to adding punctures to the torus. This determines the *W-matrix*, whose (a, b) entry is what the graphical calculus assigns to the *Whitehead link* colored by the two irreducible objects a and b.

Theorem 14.7 (Bonderson-Delaney-Galindo-Rowell-Tran-Wang [BDG+18])**.** *The T- and W-matrices can distinguish the modular tensor categories from Theorem 14.6.*

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