M392c NOTES: TOPICS IN ALGEBRAIC GEOMETRY

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These notes were taken in UT Austin's M392c (Topics in algebraic geometry) class in Fall 2019, taught by Bernd Seibert. I live-TEXed them using vim, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Any mistakes in the notes are my own.

Contents

Lecture 1.

Historical overview of mirror symmetry, I: 8/29/19

"I saw this happening, which makes me realize how old I am."

The first two lectures will contain an overview of mirror symmetry, the broad-scope context of this class; the specific details, e.g. how fast-paced we go, will be determined by who the audience is.

There are about as many perspectives on mirror symmetry as there are researchers in mirror symmetry, but a consensus of sorts has emerged.

Recall that the *canonical bundle* of a complex manifold X is $K_X \coloneqq \operatorname{Det} T^*X$. A *Calabi-Yau manifold* is a complex manifold with a trivialization of its canonical bundle, i.e. $K_X \cong \mathcal{O}_X$. Though the definition doesn't imply it, we also often assume $b_1(X) = 0$ and that X is irreducible.

Let X be a Calabi-Yau threefold (i.e. it's a Calabi-Yau manifold of complex dimension 3).

Example 1.1. A *quintic threefold* $X \subset \mathbb{P}^4$ is the zero locus in $\mathbb{P}^4$ of a homogeneous, degree-5 polynomial f in the 5 variables $x_0, \dots, x_4$. For a generically chosen f, X is smooth. We'll prove X is Calbi-Yau.

Let $\mathcal{I}$ denote the vanishing sheaf of ideals of X, i.e. $(f) \subset \mathcal{O}_{\mathbb{P}^4}$. We therefore have a short exact sequence

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathbb{P}^4}|_X \longrightarrow \Omega_X \longrightarrow 0, \tag{1.2}$$

and since $\mathcal{I}/\mathcal{I}^2 \cong \mathcal{I} \otimes_{\mathcal{O}_{\mathbb{P}^4}} \mathcal{O}_X$, it's an invertible sheaf. Using (1.2),

$$K_{\mathbb{P}^4}|_X = \operatorname{Det} \Omega_{\mathbb{P}^4}|_X \cong \mathcal{I}/\mathcal{I}^2 \otimes K_X. \tag{1.3}$$

By standard methods, one can compute that $K_{\mathbb{P}^4} = \mathcal{O}_{\mathbb{P}^4}(-5)$, hence $K_{\mathbb{P}^4}|_X \cong \mathcal{O}_X(-5)$. Since $\mathcal{I} \hookrightarrow \mathcal{O}_{\mathbb{P}^4}$, this means $\mathcal{I} \simeq \mathcal{O}_{\mathbb{P}^4}(-5)$, and therefore $\mathcal{I}/\mathcal{I}^2 \cong \mathcal{O}_X(-5)$, and as a corollary $K_X \cong \mathcal{O}_X$. ◂

Remark 1.4. Mirror symmetry is related to string theory! If you ask physicists, even theoretical ones, they'll tell you there's plenty to do still in setting up string theory, but there are two related classes of string theories called IIA and IIB, which are supersymmetric σ-models with a target $\mathbb{R}^{1,3} \times X$, where X is some Calabi-Yau threefold. Phenomenologists are interested in the $\mathbb{R}^{1,3}$ piece, which hopes to describe our world, and X tells us some information about particle dynamics in the $\mathbb{R}^{1,3}$ factor via the Kaluza-Klein mechanism.

Now, supersymmetric σ-models are better understood in physics than string theories in general, and in fact these give you two superconformal field theories (SCFTs), one corresponding to IIA, and to IIB, with target X. Using physics arguments, you can calculate the Hodge numbers of X; since X is a Calabi-Yau threefold, you can (and we will) show that its only nonzero Hodge numbers are $h^{1,1}$ and $h^{2,1}$.

But if you do this for both the A- and B-type SCFTs, you get flipped answers: $h^{1,1}$ computed via the A-type SCFT is $h^{2,1}$ computed via the B-type SCFT. We think there's only one string theory, which is puzzling. Dixon and Lerihe-Vafa-Warner noticed that sometimes, we can find another Calabi-Yau threefold

Y such that the A-type SCFT of X is equivalent to the B-type SCFT of Y, and the A-type SCFT of Y is equivalent to the B-type SCFT for X, hence in particular $h^{1,1}(X) = h^{2,1}(Y)$ and $h^{2,1}(X) = h^{1,1}(Y)$. In fact, we'd expect the IIA string theory for X should be equivalent to the IIB string theory for Y, and likewise the IIB string theory for X should be equivalent to the IIA string theory for Y.

Greene and Plesser postulated such a duality, constructing the dual theory via an orbifolding construction. These were all in the late 1980s or early 1990s, but it was another decade before Hori-Vafa proved (at a physics level of rigor) this duality for complete intersections in toric varieties. ◂

This is good if you like physics, but what if you don't? It turns out that mirror symmetry is still useful – it helps us calculate things in pure mathematics that we didn't have access to before.

Remark 1.5. Let's address a possible source of confusion in the literature.

In 1988, Witten introduced the notion of a *topological twist* of a supersymmetric σ-model. These are topological field theories in the physical sense, not the mathematical ones: we only mean that the variation in the metric vanishes. We can obtain from this data two topologically twisted σ-models called the *A-model* $A(X)$ and the *B-model* $B(X)$, which are *a priori* unrelated to the A- and B-type SCFTs — but it turns out $A(X)$ and $B(X)$ compute certain limits, called *Yukawa couplings*, for these SCFTs. In particular, an equivalence of the A-type SCFT for X and the B-type SCFT for Y (and vice versa) implies an equivalence of $A(X)$ and $B(Y)$.

Caution: the A-model tells you about type IIB string theory, and the B-model tells you about type IIA string theory.

Some mathematicians zoom in on this, and say that mirror symmetry is just the equivalence of the $A(X)$ and $B(Y)$, and of $A(Y)$ and $B(X)$. ◂

Interestingly, the A-model only depends on the symplectic structure on X, and the B-model depends only on the complex structure.

In 1991, Candelas, de la Ossa, Greene, and Parkes studied the quintic threefold and its mirror Y_t (here t is a parameter, which we'll say more about later), and computed the Yukawa couplings F_A and F_B. Geometrically, the A-model has to do with counts of rational (i.e. genus-zero) holomorphic curves;[1] some of these were known classically. The B-model has to do with period integrals

$$F_B(t) = \int_\alpha \Omega_{Y_t}, \tag{1.6}$$

where $\alpha \in H_3(Y_t)$ and Ω_{Y_t} is a (suitably normalized) holomorphic volume form. These are generally much easier to compute. This was an astounding computation, and they made a further prediction which turned out to be true, and led to astonishing divisibility properties.

A reasonable next question is: can we do this on other Calabi-Yau threefolds? Morrison, building on ideas of Deligne, computed $F_B(Y)$ in terms of Hodge theory, giving more parameters for the Calabi-Yau moduli space. On the A-side, this led to the creation of *Gromov-Witten theory* around 1993, which makes $F_A(X)$ precise. On the symplectic side, this was the work of many people, including Y. Ruan, Tian, Fukaya-Ono, and Siebert; on the algebro-geometric side, this included work of Jun Li and Behrend-Fantechi.

Kontesvich's 1994 ICM address (and subsequent lecture notes) proposed a conjecture called *homological mirror symmetry*. In symplectic geometry, one can extract a triangulated category called the *Fukaya category* from a symplectic manifold X; if Y denotes its mirror, homological mirror symmetry postulates that this is equivalent to the bounded derived category of Y.

This was a charismatic, visionary conjecture, and people have spent a lot of time and thought on it. It's influenced many fields, to the point that people have focued less on the other contexts (e.g. the enumerative formulation). But this is a formulation, not an explanation. We don't quite have a mathematical explanation yet, though ingredients are in place to construct mirrors and make a systematic proof possible.

In 1996, Givental provided a proof of the equivalence of the counts established by Candelas, de la Ossa, Greene, and Parkes; Givental's proof was for hypersurfaces, and Lian, Liu, and Yau provided the general proof. The proof wasn't explanatory: it didn't express these equalities as being true for a reason. These proofs proceeded via localization methods: find a $\mathbb{C}^\times$-action and use methods akin to those of Atiyah-Bott and Berline-Vergne.

[1] If we don't have a complex structure, but only a symplectic structure, this seems nonsensical, but these curve counts can nonetheless be defined.

Progress on homological mirror symmetry came a little later, first established for quartic twofolds (in $\mathbb{P}^3$), i.e. for K3 surfaces. So the statement has to be modified somehow, but this can be done. This was done by Seidel in 2003, then to more general Calabi-Yau hypersurfaces by his student Nick Sheridan in 2011. This was very hard work, but was strong evidence that mirror symmetry in its various avatars is real. (One of these avatars is the geometric Langlands program.)

In the course of proving homological mirror symmetry for various cases, such as SZY-fibered symplectic manifolds on the A-side and rigid spaces on the Y-side (see Abouazid, Fukaya-Oh-Ohta-Ono), we needed a way to produce mirrors. This led to research into intrinsic construction of mirrors, and this has gone on to have applications outside of mirror symmetry: this allows for some computations to be simplified by passing to the mirror and working there. This includes work of Gross-Siebert, Gross-Hacking-Keel, and more.

This is all the genus-zero part of the story, which physicists call the *tree-level* part of the theory. People also study higher-genus (or second quantized mirror symmetry), such as Costello and Si Li, or look at the method of topolgical recursion, e.g. Eynard and Orantin.

The plan for this class is, roughly:

- Sketch the computation of Candelas, de la Ossa, Greene, and Parkes.
- Gromov-Witten theory, and its construction via virtual fundamental classes and moduli stacks.
- Potentially an introduction to toric geometry.
- Toric degenerations and mirror constructions. This has undergone several refinements, and we'll take a pretty modern perspective.
- Using this, you can compute homogeneous coordinate rings (which is a lot of information: it knows the variety, hence also the derived category). On the A-side, a result of Polischuk forces that there's only one possible Fukaya category (as an A_∞-category), which leads to a proposal for a plan to prove homological mirror symmetry in great generality. The mirror statement (using the Fukaya category and its A_∞-structure to determine the derived category of the mirror) is considered a hard open problem in symplectic geometry.
- Next, we could discuss higher-genus information. In Gromov-Witten theory, the genus is part of the input data, but we could also compute *Donaldson-Thomas invariants*, where we count ideal sheaves rather than holomorphic curves. This organizes the count differently, because curves of different genera may be part of the same count. The role of Donaldson-Thomas theory in mirror symmetry is somewhat unclear, and there's an interesting statistical-mechanics model called *crystal melting*, which ports this down to genus zero. This is work of Okounkov and others.

This can be adjusted depending on class interest.

In the last few minutes, let's begin talking about the quintic threefold, its mirror, and the work of Candelas, de la Ossa, Greene, and Parkes.

The quintic threefold comes in a big family: we're looking at degree-5 homogeneous polynomials in five variables, so to enumerate monomials, we need to know the number of ways to draw lines between five points in a line. For example, $x_0^2x_2$ corresponds to $12|345$ and $x_0x_1^2x_2$ corresponds to $1|2\|345$. The answer is

$$\binom{n+d-1}{n-1} = \binom{n+d-1}{d}, \tag{1.7}$$

which here is $\binom{9}{5} = 126$. Hence the dimension of the moduli space of quintic polynomials in $\mathbb{P}^4$ is $126-1 = 125$. However, to get the space of quintics, we need to divide out by the symmetries of the problem, which is PGL_5. This has dimension $5^2 - 1 = 24$, so the moduli space of quintic threefolds is 101-dimensional.

This is *huge* — you may think it's a long way down the road to the chemist, but that's just peanuts compared to the dimension of this moduli space. It's way too big for us to get a good grasp on.

Indeed, for a projective Calabi-Yau manifold X, the moduli space of Calabi-Yau manifolds deformation-equivalent to X is a smooth orbifold[2] of complex dimension $h^1(\Theta_X)$, where Θ_X is the holomorphic tangent bundle, and we can show that this is 101 for the quintic threefold.

[2] We'll say more about this later, but an orbifold is locally modeled on a manifold quotient by a nice group action, and you can think of it as that, as a singular topological space.