SPECTRAL SEQUENCES IN (EQUIVARIANT) STABLE HOMOTOPY THEORY

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1. The homotopy fixed-point spectral sequence: 5/15/17

Today, Richard spoke about the homotopy fixed-point spectral sequence in equivariant stable homotopy theory.

We'll start with the Bousfield-Kan spectral sequence (BKSS). One good reference for this is Guillou's notes [3], and Hans Baues [1] set it up in a general model category.

We'll work in sSet, so that everything is connective. Consider a tower of fibrations

$$\begin{tikzcd} \cdots \arrow[r] & Y_s \arrow[r, "p_s"] & Y_{s-1} \arrow[r, "p_{s-1}"] & Y_{s-2} \arrow[r] & \cdots \end{tikzcd} \tag{1.1}$$

for $s \geq 0$, and let $Y \coloneqq \varprojlim Y_s$. Let F_s be the fiber of p_s.

Theorem 1.2 (Bousfield-Kan [2])**.** *In this situation, there is a spectral sequence, called the* ***Bousfield-Kan spectral sequence****, with signature*

$$E_1^{s,t} = \pi_{t-s}(F_s) \Longrightarrow \pi_{t-s}(Y).$$

If everything here is connective (which is not always the case in other model categories, as in one of our examples), this is first-quadrant. One common convention is to use the **Adams grading** $(t-s, s)$ instead of (s,t).

We can extend (1.1) into a diagram

$$\begin{tikzcd} \cdots \arrow[r] & Y_s \arrow[r, "p_s"] & Y_{s-1} \arrow[r, "p_{s-1}"] & Y_{s-2} \arrow[r] & \cdots \\ & F_s \arrow[u, "i_s"'] & F_{s-1} \arrow[u, "i_{s-1}"'] & F_{s-2}, \arrow[u, "i_{s-2}"'] & \end{tikzcd}$$

and hence into an exact couple

$$\begin{tikzcd} \cdots \arrow[r] & \pi_*(Y_{s+1}) \arrow[r] & \pi_*(Y_s) \arrow[r] \arrow[dl, "\delta"'] & \pi_*(Y_{s-1}) \arrow[r] \arrow[dl, "\delta"'] & \cdots \\ & \pi_*(F_s) \arrow[u, "i_{s*}"] & \pi_*(F_{s-1}) \arrow[u, "i_{(s-1)*}"] & \pi_*(F_{s-2}), \arrow[u, "i_{(s-2)*}"] & \end{tikzcd}$$

and the differentials are the compositions of the maps $\pi_*(F_s) \to \pi_*(Y_s) \to \pi_*(F_{s+1})$:

$$\begin{tikzcd} & \cdots \arrow[r] & \pi_*(Y_{s+1}) \arrow[r] & \pi_*(Y_s) \arrow[r] \arrow[dl, "\delta"'] & \pi_*(Y_{s-1}) \arrow[r] \arrow[dl, "\delta"'] & \cdots \\ {} & & \pi_*(F_s) \arrow[u, "i_{s*}"] \arrow[ll, red] & \pi_*(F_{s-1}) \arrow[u, "i_{(s-1)*}"] \arrow[l, red, "d_1"] & \pi_*(F_{s-2}). \arrow[u, "i_{(s-2)*}"] \arrow[l, red, "d_1"] & {} \arrow[l, red] \end{tikzcd}$$

Taking homology, we'll get a differential d_2 that jumps two steps to the left, then d_3 three steps to the left, and so on. After you check that $\operatorname{Im}(d_r) \subset \ker(d_r)$, you can define $E_r^{s,s+1} := \ker(d_r)/\operatorname{Im}(d_r)$. Let $A_s := \operatorname{Im}(\pi_0(Y_{s+r}) \to \pi_0(Y_s))$, and let $Z_r^{s,s} := (i_s)_*^{-1}(A_s)$. Then, $E_{r+1}^{s,s} = Z_r^{s,s}/d_r(E_r^{s-r,s-r+1})$.

Remark. One important caveat is that for $i \leq 2$, π_i does not produce abelian groups, but rather groups or just sets! This means that a few of the columns of this spectral sequence don't quite work, but the rest of it is normal, and the degenerate columns can still be useful. This is an example of a **fringed spectral sequence**. $\blacktriangleleft$

Bousfield and Kan cared about this spectral sequence because it allowed them to write down a useful long exact sequence, the $r^{\mathbf{th}}$ **derived homotopy sequence**: let $\pi_i Y^{(r)} := \operatorname{Im}(\pi_i(Y_{n+r}) \to \pi_i(Y_n))$; then, there's a long exact sequence

$$\cdots \longrightarrow \pi_{t-s-1} Y_{s-r-1}^{(r)} \longrightarrow E_{r+1}^{s,t} \longrightarrow \pi_{t-s} Y_s^{(r)} \xrightarrow{\delta} \pi_{t-s} Y_{s-1}^{(r)} \longrightarrow E_{r+1}^{s+r,t+r+1} \longrightarrow \pi_{t-s+1} Y_s^{(r)} \longrightarrow \cdots$$

You can do something like this in general given a spectral sequence, though you need to know how to obtain it from the exact couple.

Remark. When $r = 0$, $E_1^{s,t} = \pi_{t-s}(F_s)$, and so the first derived homotopy sequence is the long exact sequence of homotopy groups of a fibration. $\blacktriangleleft$

One nice application is to **Tot towers** ("Tot" for totalization).

Definition 1.3. Let $X^\bullet$ be a cosimplicial object in sSet. Then, its **totalization** is the complex

$$\operatorname{Tot}(X^\bullet) := \mathsf{sSet}(\Delta^\bullet, X^\bullet),$$

i.e.

$$\operatorname{Tot}_n(X^\bullet) := \mathsf{sSet}(\operatorname{sk}_n \Delta^\bullet, X^\bullet).$$

Here $(\operatorname{sk}_n \Delta^\bullet)^n := \operatorname{sk}_n \Delta^m$.

Then

$$\varprojlim \operatorname{Tot}_n(X^\bullet) = \operatorname{Tot}(X^\bullet),$$

reconciling the two definitions.

Exercise 1.4. In the Reedy model structure, $\operatorname{Tot}_n(X^\bullet) \to \operatorname{Tot}_{n-1}(X^\bullet)$ is a fibration.

Assuming this exercise, we can apply the Bousfield-Kan spectral sequence.

One place this pops up is that if $C, D \in \mathsf{C}$ and $X_\bullet \to C$ is a simplicial resolution in a simplicial category C,[1] then $\operatorname{Hom}_{\mathsf{C}}(X_\bullet, D)$ is a cosimplicial object, and this spectral sequence can be used to compute homotopically meaningful information about $\mathsf{sSet}(C, D)$.

We can use this formalism to derive the homotopy fixed point spectral sequence. Let G be a group, and X be a spectrum with a G-action. Then, the **homotopy fixed points** of X are

$$X^{hG} := F((EG)_+, X)^G,$$

i.e. the G-equivariant maps $(EG)_+ \to X$.[2] The bar construction gives us a simplicial resolution of $(EG)_+$, producing a cosimplicial object that can be plugged into the Bousfield-Kan spectral sequence. Specifically, we write $EG = B^\bullet(G, G, *)$, add a disjoint basepoint, and then take maps into X.

Theorem 1.5. *If X is a spectrum with a G-action, there's a spectral sequence, called the **homotopy fixed-point spectral sequence**, with signature*

$$E_2^{p,q} = H^p(G, \pi_q(X)) \Longrightarrow \pi_{q-p}(X^{hG}).$$

[1] Meaning that after geometrically realizing, there's an equivalence.

[2] Notationally, this is the function spectrum of maps from $\Sigma^\infty(EG)_+$ to X, or you can use the fact that spectra are cotensored over spaces.

Example 1.6. The first example is really easy. Let k be a field, and consider the Eilenberg-Mac Lane spectrum Hk. Let G act trivially on k; we want to understand $\pi_*(Hk^{hG})$. The homotopy fixed-points spectral sequence is particularly simple:

$$E_2^{p,q} = H^p(G; \pi_q(Hk)) = \begin{cases} H^p(G;k) & q = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Since this is a single row,[3] all differentials vanish, and this is also the E_∞ page. So we just have to compute $H^p(G;k)$ for $k \geq 0$.

For example, if $G = \mathbb{Z}/2$ and $k = \mathbb{F}_2$, then $H^*(\mathbb{Z}/2; \mathbb{F}_2) = H^*(\mathbb{RP}^\infty; \mathbb{F}_2) = \mathbb{F}_2[x]$, $|x| = 1$. There are no extension issues, since there's only one nonzero term in each total degree. Thus,

$$\pi_{-p}(Hk^{hG}) = H^p(G;k).$$

If you let $G = \mathbb{Z}/2$ and k be any field of odd characteristic, then $H^*(\mathbb{Z}/2;k) = k$ in degree 0, so the homotopy groups of $Hk^{h\mathbb{Z}/2}$ are all trivial except for π_0, which is k. ◀

In the context of group actions on spectra, there's another spectral sequence called the Tate spectral sequence. If X is a genuine G-spectrum, there's a norm map $X_{hG} \to X^{hG}$ whose cofiber is called the **Tate spectrum** X^{tG}.[4] This is a generalization of Tate cohomology $\widehat{H}^p$ in group cohomology. Here, $X_{hG} \coloneqq (EG_+ \wedge X)_G$ is the **homotopy orbits** of X. Then, there is a spectral sequence, called the **Tate spectral sequence**, with signature

$$E_2^{p,q} = \widehat{H}^p(G; \pi_q(X)) \Longrightarrow \pi_{q-p}(X^{tG}).$$

The similarities with the homotopy fixed point spectral sequence are no coincidence.

Example 1.7. Let C_2 act on S^1 by reflection. Then, $\pi_i(S^1)$ is trivial unless $i = 1$, in which case we get $\mathbb{Z}$. Hence,

$$E_2^{p,q} = \begin{cases} H^p(C_2;\mathbb{Z}), & q = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Under the isomorphism $\mathbb{Z}[C_2] \cong \mathbb{Z}[x]/(x^2-1)$, the $\mathbb{Z}[C_2]$-module structure on $\mathbb{Z}$ is the map $\mathbb{Z}[C_2] \to \mathbb{Z}$ sending $x \mapsto -1$, i.e. C_2 acts on $\mathbb{Z}$ through the nontrivial action. We'll let $\mathbb{Z}_\sigma$ denote $\mathbb{Z}$ with this action, and $\mathbb{Z}$ denote the integers with the trivial C_2-action. To compute the group cohomology, we need to compute a free resolution $P_\bullet \to \mathbb{Z}$ as a *trivial* $\mathbb{Z}[C_2]$-module:

$$\cdots \longrightarrow \mathbb{Z}[C_2] \xrightarrow{\cdot(x-1)} \mathbb{Z}[C_2] \xrightarrow{\cdot(x+1)} \mathbb{Z}[C_2] \xrightarrow{\cdot(x-1)} \mathbb{Z}[C_2] \xrightarrow{x \mapsto 1} \mathbb{Z} \longrightarrow 0.$$

Now we compute $\mathrm{Hom}_{\mathbb{Z}[C_2]}(P_\bullet, \mathbb{Z}_\sigma)$:

$$\cdots \longleftarrow \mathbb{Z} \xleftarrow{-2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{-2} \mathbb{Z}.$$

Taking homology, we conclude that

$$H^p(\mathbb{Z}/2, \mathbb{Z}_\sigma) = \begin{cases} \mathbb{Z}/2, & p > 0 \text{ odd} \\ 0, & \text{otherwise.} \end{cases}$$

Since the spectral sequence degenerates at page 2, this is also the homotopy groups of $(S^1)^{hC_2}$. This *a priori* doesn't make any sense, though — $(S^1)^{hC_2}$ is a space, so cannot have negative-degree homotopy groups. But since this is a fringed spectral sequence, the stuff in negative degrees doesn't apply to the calculation of homotopy groups, and $q - p = 0, 1$ mixes together in a complicated way. In this case, it tells us thagt $\pi_0((S^1)^{hC_2}) = \mathbb{Z}/2$ and higher homotopy groups vanish.[5] ◀

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[3] It's a single row in the usual grading, and a single diagonal line with slope -1 in the Adams grading.

[4] The norm map is the spectral analogue of a more concrete construction: let M be a $\mathbb{Z}[G]$-module. Then, the assignment

$$m \longmapsto \sum_{g \in G} g \cdot m$$

lands in M^G, so it factors through orbits, defining a map $M_G \to M^G$.

[5] Thanks to Tyler Lawson for this insight.

2. : 5/16/17

Our immediate goal is to apply the homotopy fixed-point spectral sequence to prove the following theorem. Let KU (resp. KO) denote the spectrum representing complex K-theory (resp. real K-theory).

Theorem 2.1. *Let C_2 act on KU by complex conjugation. Then, $KU^{hC_2} = KO$.*

Proof. By Bott periodicity,

$$\pi_*(KU) = \mathbb{Z}[\mu], |\mu| = 2 = \begin{cases} \mathbb{Z}, & \text{even degrees} \\ 0, & \text{odd degrees.} \end{cases}$$

The C_2-action on KU induces a C_2-action on $\mathbb{Z}[\mu]$, which sends $\mu \mapsto -\mu$. In particular, it's trivial in all dimensions except $q = 4k+2$, where it's multiplication by -1.

In this scenario, the homotopy fixed point spectral sequence is not fringed, and has signature

$$E_2^{p,q} = H^p(C_2, \pi_q(KU)) \Longrightarrow \pi_{q-p}(KU^{hC_2}).$$

So we need to compute some group cohomology. Let $\mathbb{Z}_\sigma$ denote $\mathbb{Z}$ as a $\mathbb{Z}[C_2]$-module with the action multiplication by -1; in Example 1.7 we showed that

$$H^p(C_2; \mathbb{Z}_\sigma) = \begin{cases} \mathbb{Z}/2, & p \text{ odd} \\ 0, & p \text{ even.} \end{cases}$$

We also need to compute $H^*(C_2; \mathbb{Z})$ (i.e. with the trivial $\mathbb{Z}/2$-action). There are a few ways to do this: for example,

$$H^p(C_2; \mathbb{Z}) = H^p(BC_2; \mathbb{Z}) = H^p(\mathbb{RP}^\infty; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & p = 0 \\ \mathbb{Z}/2, & p > 0 \text{ even} \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Alternatively, you can write down an explicit free resolution:

$$\cdots \longrightarrow \mathbb{Z}[C_2] \xrightarrow{\cdot(x+1)} \mathbb{Z}[C_2] \xrightarrow{\cdot(x-1)} \mathbb{Z}[C_2] \xrightarrow{x \mapsto 1} \mathbb{Z}.$$

Applying $\mathrm{Hom}_{\mathbb{Z}[G]}(-, \mathbb{Z})$, we get

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z},$$

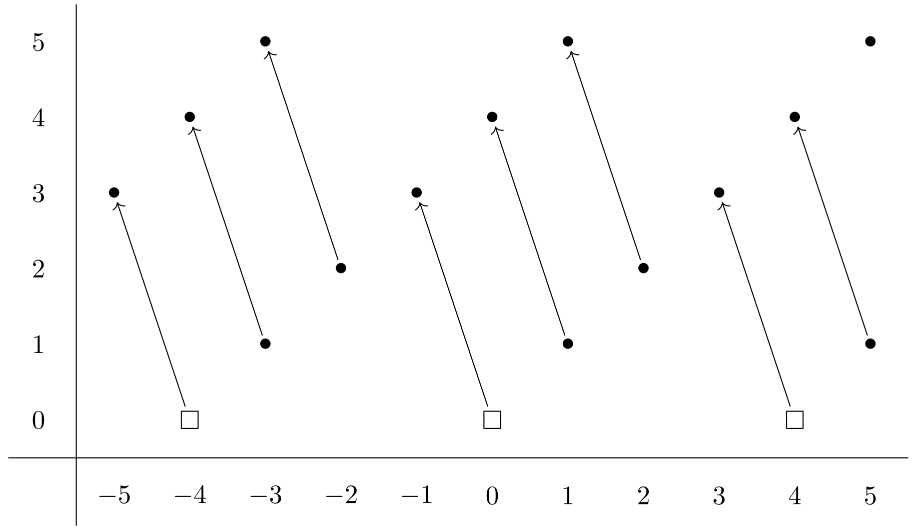
obtaining the same cohomology groups as in (2.2).

In the usual grading, the E_2 page therefore looks like this:

4	$\mathbb{Z}$		$\mathbb{Z}/2$		$\mathbb{Z}/2$	
3						
2		$\mathbb{Z}/2$		$\mathbb{Z}/2$		$\mathbb{Z}/2$
1						
0	$\mathbb{Z}$		$\mathbb{Z}/2$		$\mathbb{Z}/2$	
-1						
-2		$\mathbb{Z}/2$		$\mathbb{Z}/2$		$\mathbb{Z}/2$
	0	1	2	3	4	5

In particular, there are no differentials on the E_2 page.

Regrading by the Adams grading $(p, q) \mapsto (q - p, p)$, let $\bullet$ denote a $\mathbb{Z}/2$ and $\square$ denote a $\mathbb{Z}$; then, the spectral sequence is



We'd like to get the Bott song out of this: $\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, \mathbb{Z}, \ldots$. This means that some, but not all, of the d_3 differentials have to vanish. The multiplicative structure helps us here by translating the differentials.

We'll follow the geometric approach of Heard-Stojanoska [4]. Let ρ be the regular representation of C_2, i.e. a direct sum of the trivial and the sign representation. Let $S^{2\rho}$ denote the one-point compactification of $\rho \oplus \rho$ as a C_2-space (a space with a C_2-action). We'll write down a morphism of C_2-spectra

$$\Sigma^\infty S^{2\rho} \longrightarrow KU,$$

which, using the functoriality of the spectral sequence, induces a morphism of spectral sequences

$$\begin{array}{c} H^p(C_2; \pi_*(\Sigma^\infty S^{2\rho})) \Longrightarrow \pi_*((\Sigma^\infty S^{2\rho})^{hC_2}) \\ \Big\downarrow \\ H^p(C_2; \pi_*(KU)) \Longrightarrow \pi_*(KU^{hC_2}). \end{array} \tag{2.3}$$

This will capture enough of the structure to see a few of the differentials, and periodicity and the multiplicative structure handle the rest.

As a C_2-space, $S^{2\rho}$ is the sphere with C_2-action reflection across a plane through the origin. This is homeomorphic to $\mathbb{CP}^1$ with the conjugation action.

There's an equivariant cell structure on S^ρ: a trivial 0-cell (trivial orbit), a trivial 1-cell, and a 2-cell $D^2 \times C_2$ with C_2 switching the two components. $\mathbb{CP}^1$ includes into $\mathbb{CP}^\infty = B\mathrm{U}_1$, which maps to $B\mathrm{U}$, which maps to KU, and all of these maps are C_2-equivariant, so we have a map

$$v_1 \colon S^\rho \xrightarrow{\cong} \mathbb{CP}^1 \hookrightarrow B\mathrm{U}_1 \longrightarrow B\mathrm{U} \longrightarrow KU.$$

We can compose this with the pinch map $S^{2\rho} \to S^\rho$ (making this equivariant requires a little thinking, but is OK), so we obtain a map $v_1^2 \colon S^{2\rho} \to KU$.

The homotopy groups of $S^{2\rho}$ are hard to compute, of course, but we can compute $(S^{2\rho})^{hC_2}$ using Spanier-Whitehead duality. If X is a spectrum, its **Spanier-Whitehead dual** is $DX := F(X, \mathbb{S})$, the spectrum of maps $X \to \mathbb{S}$. This also works in the equivariant setting: there's a genuine G-equivariant sphere spectrum, also denoted $\mathbb{S}$, and if X is a G-spectrum, $DX = F(X, \mathbb{S})$, with G acting on the function spectrum through conjugation.

So by adjointness, $(S^{2\rho})^{hC_2} \cong (DS^{-2\rho})^{hC_2}$. This is homotopic to $D((S^{-2\rho})_{hC_2})$, because

$$F(X, Y)^{hC_2} \simeq F(X_{hC_2}, Y).$$

This is saying that an equivariant function is determined by its value on the orbits, though turning this into a proof requires turning it into a statement about ordinary orbits and fixed points:

$$F(X,Y)^{hC_2} \simeq F(EC_2, F(X,Y))^{C_2} \simeq F(EC_{2+} \wedge X, Y)^{C_2} \simeq F(X_{hC_2}, Y).$$

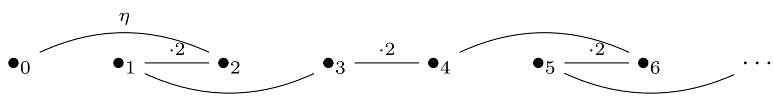
Cool. So

$$(S^{2\rho})^{hC_2} \simeq D((S^{-2\rho})_{hC_2}) \simeq D(\Sigma^{-2}(S^{-2\sigma})_{hC_2}).$$

Let $\mathbb{RP}^\infty_{-n}(\mathbb{RP}^\infty)^{-n\xi}$, i.e. the Thom spectrum of the virtual vector bundle $-n\xi$, where ξ is the tautological bundle over $\mathbb{RP}^\infty$. Thus

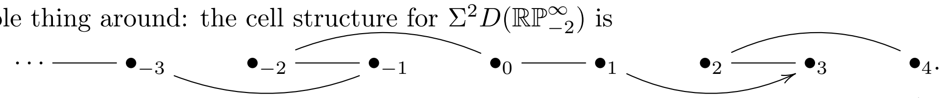
$$(S^{2\rho})^{hC_2} \simeq D(\Sigma^{-2}(\mathbb{RP}^\infty_{-2})) = \Sigma^2(D(\mathbb{RP}^\infty_{-2})).$$

Thom spectra have nice cell structures. This one has a cell in every dimension $n \geq 0$ with attaching maps as follows:



Here $\eta\colon S^3 \to S^2$ is the Hopf fibration.

Applying Σ^{-2} to this shifts the cells downward by 2 degrees. Then, applying Spanier-Whitehead duality flips the whole thing around: the cell structure for $\Sigma^2 D(\mathbb{RP}^\infty_{-2})$ is



Now we can apply this to the spectral sequence. That we have a map of spectral sequences (2.3) means that it commutes with differentials. In particular, the d_3 emerging from $(4,0)$ (in the Adams page) witnesses the attaching map η for the 4-cell. Moreover, the Adams spectral sequence, which encodes cohomology operations, acts on these spectral sequences, and this map is equivariant for this action.[6]

That the map (2.3) is nonzero comes from the fact that η lives to the E_∞ page of the Adams spectral sequence, hence defines a nonzero element of the stable stem. The rest of the argument follows from the analysis of a single differential and the multiplicative structure. For example, because adding a 4-cell to finish $\mathbb{RP}^\infty_{-n}$ kills η, then the d_3 emerging from $(4,0)$ is zero. $\boxtimes$

References

[1] Hans Joachim Baues. *Algebraic Homotopy*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1989.

[2] A.K. Bousfield and D.M. Kan. The homotopy spectral sequence of a space with coefficients in a ring. *Topology*, 11(1):79–106, 1972.

[3] Bertrand Guillou. The Bousfield-Kan spectral sequence. 2007. http://www.ms.uky.edu/~guillou/BKss.pdf.

[4] Drew Heard and Vesna Stojanoska. K-theory, reality, and duality. *Journal of K-theory*, 14(3):526–555, 2014. https://arxiv.org/abs/1401.2581.

[6]To do this, you should localize and complete at 2, so we only see 2-torsion phenomena, but this suffices.