QUANTUM TOPOLOGY AND CATEGORIFICATION SEMINAR, SPRING 2017

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CONTENTS

Part 1. Quantum topology: Chern-Simons theory and the Jones polynomial

1. THE JONES POLYNOMIAL: 1/24/17

Today, Hannah talked about the Jones polynomial, including how she sees it and why she cares about it as a topologist.

1.1. Introduction to knot theory.

Definition 1.1. A **knot** is a smooth embedding $S^1 \hookrightarrow S^3$. We can also talk about **links**, which are embeddings of finite disjoint unions of copies of S^1 into S^3.

One of the major goals of 20th-century knot theory was to classify knots up to isotopy.

Typically, a knot is presented as a **knot diagram**, a projection of $K \subset S^3$ onto a plane with "crossing information," indicating whether the knot crosses over or under itself at each crossing. Figure 1 contains an example of a knot diagram.

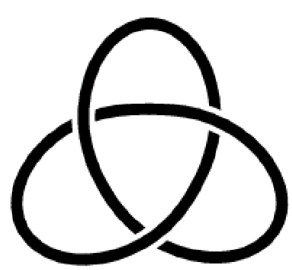

FIGURE 1. A knot diagram for the left-handed trefoil knot. Source: Wikipedia.

Given a knot in S^3, there's a theorem that a generic projection onto $\mathbb{R}^2$ is a knot diagram (i.e. all intersections are of only two pieces of the knot).

Link diagrams are defined identically to knot diagrams, but for links.

Theorem 1.2. *Any two link diagram for the same link can be related by planar isotopy and a finite sequence of* ***Reidemeister moves****.*

1.2. **Polynomials before Jones.** The first knot polynomial to be defined was the Alexander polynomial $\Delta_K(x)$, a Laurent polynomial with integer coefficients that is a knot invariant, defined in the 1920s.

Here are some properties of the Alexander polynomial:

- It's symmetric, i.e. $\Delta_K(x) = \Delta_K(x^{-1})$.
- It cannot distinguish handedness. That is, if K is a knot, its **mirror** $\overline{K}$ is the knot obtained by switching all crossings in a knot diagram, and $\Delta_K(x) = \Delta_{\overline{K}}(x)$.[1]
- The Alexander polynomial doesn't detect the unknot (which is no fun): there are explicit examples of knots 11_{34} and 11_{42} whose Alexander polynomials agree with that of the unknot.[2]

So maybe it's not so great an invariant, but it's somewhat useful.

1.3. **The Jones polynomial.** The Jones polynomial was defined much later, in the 1980s. The definition we give, in terms of skein relations, was not the original definition. There are three local models of crossings, as in Figure 2.

FIGURE 2. The three local possibilities for a crossing in a knot diagram (technically, L_0 isn't a crossing). Source: `https://en.wikipedia.org/wiki/Skein_relation`.

The idea is that, given a knot K, you could try to calculate a knot polynomial for K in terms of knot polynomials on links where one of the crossings in K has been changed from L_- to L_+ (or vice versa), or **resolved** by replacing it with an L_0. A relationship between the knot polynomials of these three links is a **skein relation**. This is a sort of inductive calculation, and the base case is the unknot. In particular, you can use the value on the unknot and the skein relations for a knot polynomial to describe the knot polynomial!

Example 1.3. The Alexander polynomial is determined by the following data.

- On the unknot, $\Delta(U) = 1$.
- The skein relation is $\Delta(L_+) - \Delta(L_-) = t\Delta(L_0)$. ◂

Definition 1.4. The **Jones polynomial** is the knot polynomial v determined by the following data.

- For the unknot, $v(U) = 1$.
- The skein relation is
$$\left(t^{1/2} - t^{-1/2}\right)v(L_0) = t^{-1}v(L_+) - tv(L_-).$$

Example 1.5. Let's calculate the Jones polynomial on a Hopf link H, two circles linked together once. The standard link diagram for it has two crossings.

- Resolving one of the crossings produces an unknot: $v(L_0) = 1$.
- Replacing the L_- with an L_- produces two unlinked circles. One more skein relation produces the unknot, so $v(L_+) = -(t^{1/2} - t^{-1/2})$.

Putting these together, one has
$$(t^{1/2} - t^{-1/2}) \cdot 1 = t^{-1}(-(t^{1/2} + t^{-1/2}) - tv(H)),$$
so $v(H) = -t^{-1/2} - t^{-5/2}$. ◂

There are many different definitions of the Jones polynomial; one of the others that we'll meet later in this seminar is via the Kauffman bracket.

Definition 1.6. The **bracket polynomial** of an unoriented link L, denoted $\langle L \rangle$, is a polynomial in a variable A defined by the skein relations

- On the unknot: $\langle O \rangle = 1$.

[1]The mirror of the left-handed trefoil is the right-handed trefoil, for example.

[2]The notation for these knots follows Rolfsen.

- There are two ways to resolve a crossing C: as two vertical lines V or two horizontal lines H. We impose the skein relation
$$\langle C\rangle = A\langle V\rangle + A^{-1}\langle H\rangle.$$
- Finally, suppose the link L is a union of one unlinked unknot and some other link L' (sometimes called the **distant union**). Then,
$$\langle L\rangle = (-A^2 - A^{-2})\langle L'\rangle.$$

Example 1.7. Once again, we'll compute the Kauffman bracket for the Hopf link. (TODO: add picture).
The result is $\langle H\rangle = -A^4 - A^{-4}$. ◀

You can show that this bracket polynomial is invariant under type II and III Reidemeister moves, but not type I. We obviously need to fix this.

Definition 1.8. Let D be an *oriented* link, and $|D|$ denote the link without an orientation. The **normalized bracket polynomial** is defined by
$$X(D) := (-A^3)^{-\omega(D)}\langle |D|\rangle).$$

Here, $\omega(D)$ is the writhe of D, an invariant defined based on a diagram. At each crossing, imagine holding your hands out in the shape of the crossing, where (shoulder → finger) is the positively oriented direction along the knot. If you hold your left hand over your right hand, the crossing is a **positive crossing**; if you hold your right hand over your left, it's a **negative crossing**.

Let ω_+ denote the number of positive crossings and ω_- denote the number of negative crossings. Then, the **writhe** of D is $\omega(D) := \omega_+ - \omega_-$. For example, the writhe of the Hopf link (with the standard orientation) is 2, and $X(H) = -A^{10} - A^2$.

Thankfully, this is invariant under all types of Reidemeister moves. The proof is somewhat annoying, however.

Theorem 1.9. *By substituting $A = t^{-1/4}$, the normalized bracket polynomial produces the Jones polynomial.*

So these two invariants are actually the same.

Here are some properties of the Jones polynomial.

- $v_{\overline{K}}(t) = v_K(t^{-1})$. Since the Jones polynomial is not symmetric, it can sometimes distinguish handedness, e.g. it can tell apart the left- and right-handed trefoils.
- It fails to distinguish all knots: once again, 11_{34} and 11_{42} have the same Jones polynomial.[3]
- It's unknown whether the Jones polynomial detects the unknot: there are no known nontrivial knots with trivial Jones polynomial.
- Computing the Jones polynomial is **P**-hard: there's no polynomial-time algorithm to compute it. (Conversely, the Alexander polynomial is one of very few knot invariants with a polynomial-time algorithm.)

If a knot does have trivial Jones polynomial, we know:

- it isn't an **alternating knot** (i.e. one where the crossings alternate between positive and negative).
- It has crossing number at least 18 (which is *big*).

One interesting application of what we'll learn in this seminar is that there are knots (9_{42} and 10_{11}) that can't be distinguished by the Jones or Alexander (or HOMFLY, or …) but *are* distinguished by SU(2)-Chern-Simons invariants.

Part 2. Categorification: Khovanov homology

References

[3] This is ultimately for the same reason as for the Alexander polynomial: there's a technical sense in which they're **mutant knots** of each other. It's notoriously hard to write down knot polynomials that detect mutations, and the Jones polynomial cannot detect them.