M392C NOTES: INDEX THEORY

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Lecture 1.

Overview, History, and some Linear Algebra: 1/17/18

"This formula should look fake if you haven't seen it before."

We'll start with and overview and some history of index theory. The overview will use a little bit of complex geometry, but if you don't know it that's okay; the rest of the class will not depend on it.

One of the earliest manifestations of index theory was in the theory of algebraic curves. Let M be a compact smooth connected complex curve, i.e. a Riemann surface, and let D be a divisor on M, a finite formal sum of points of M with integer coefficients. For example, if $p_1, p_2, p_3 \in M$, one divisor is $4p_1 - 2p_2 + 7p_3$.

Definition 1.1. Let f be a meromorphic function on M; then, its divisor $\mathrm{div}(f)$ is the zeros of f minus the poles of f, where both are counted with multiplicity. For $f = 0$, we let $\mathrm{div}(0) = 0$.

For example, if $M = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, then a meromorphic function on M is a rational function. If we took $f(z) = (z-1)^2/(z+2)$, then $\mathrm{div}(f) = 2 \cdot 1 - 1 \cdot (-2) - 1 \cdot \infty$: f has a double zero at 1 and a single pole at -2, and at ∞ there is a simple pole.[1]

A divisor has a *degree* which is the sum of its terms.

Theorem 1.2. *The degree of the divisor of a meromorphic function is zero.*

This is a consequence of the Cauchy integral formula.

A divisor specifies the zeros and poles of a meromorphic function, and it's a classical problem to, given a degree-zero divisor D on a Riemann surface, construct a function whose divisor is D. More generally, let $\mathscr{L}(D)$ denote the set of meromorphic f such that $\mathrm{div}(f) + D \geq 0$.[2] $\mathscr{L}(D)$ is a vector space, and if $\deg(D) < 0$, $\mathscr{L}(D) = 0$; we also have $\mathscr{L}(0) = \mathbb{C}$, given by constant functions.

Another classical question is to compute $\dim \mathscr{L}(D)$. Riemann provided an estimate:

$$\dim \mathscr{L}(D) \geq 1 - g + \deg(D), \tag{1.3}$$

[1] To see this, use the change-of-variables $z = 1/w$ and evaluate f at $w = 0$.

[2] This is missing a zero element, so one needs to adjoint 0 for everything to work.

where g is the *genus* of M, defined to be

$$g := \frac{1}{2}\operatorname{rank} H_1(X). \tag{1.4}$$

The next natural question is to identify the discrepancy, and Riemann's student Roch found the answer.

Theorem 1.5 (Riemann-Roch)**.** *here is a canonical divisor K_M such that*

$$\dim \mathscr{L}(D) - \dim \mathscr{L}(K_X - D) = 1 - g + \deg D. \tag{1.6}$$

We won't say much about K_M, though $\deg(K_M) = 2g - 2$.

Corollary 1.7. *The genus is an integer.*

A more modern interpretation of this story is that D determines a holomorphic line bundle $L \to M$, and $\mathscr{L}(D)$ is the vector space of holomorphic sections of L, i.e. $\mathscr{L}(D) \cong H^0(M; L)$. If s is any smooth section of L, s is holomorphic iff $\overline{\partial} s = 0$. That is, in local coordinates $z = x + iy$, and

$$\overline{\partial} = \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right). \tag{1.8}$$

Thus, $\overline{\partial} x = 0$ is a first-order differential equation, and computing $\dim \mathscr{L}(D)$ is asking for the dimension of the space of solutions to the equation. Thus one way you might prove Theorem 1.5 is to analyze the differential operator $\overline{\partial}$, which is a linear operator

$$\overline{\partial} \colon \Omega^{0,0}(M; L) \longrightarrow \Omega^{0,1}(M; L).$$

Then, $\mathscr{L}(D) = \ker(\overline{\partial})$ and $\mathscr{L}(K_M - D) \cong \operatorname{coker}(\overline{\partial})$.

Definition 1.9. The *index* of $\overline{\partial}$ is $\operatorname{ind}(\overline{\partial}) := \dim\ker(\overline{\partial}) - \dim\operatorname{coker}(\overline{\partial})$.

Broadly speaking, this course will be about indices of this sort, and their applications: for example, the Riemann-Roch theorem from this perspective is about computing the index of $\overline{\partial}$.

For a simpler case, let V and W be finite-dimensional vector space and $T \colon V \to W$ be a linear map. Then, $\ker(T) \subset V$ and $\operatorname{coker} T := W/T(V)$. Computing the index is a fundamental theorem in linear algebra.

Theorem 1.10.

$$\operatorname{ind}(T) := \dim(\ker T) - \dim(\operatorname{coker} T) = \dim V - \dim W.$$

In particular, it's independent of T! One way you might prove this is to observe that it's true when $T = 0$ and then try to prove that it's locally constant.

In this class, we're interested in operators between infinite-dimensional vector spaces, such as $\Omega^{p,q}(M; L)$, whose kernels and cokernels are finite-dimensional (such that the definition of an index makes sense). There will be no nice formula like Theorem 1.10, but some aspects stay the same: though the dimension of the kernel or cokernel may jump along a continuous path, their difference is constant.

Another classical subject that relates to index theory is that of the Euler number of a compact smooth n-manifold M. Betti defined *Betti numbers* $b_0, \dots, b_n$ associated to M, and Noether realized they can be identified with ranks of abelian groups (or dimensions of certain real vector spaces).[3]

Definition 1.11. The *Euler characteristic* of M is

$$\chi(M) := \sum_{i=0}^{n} (-1)^i b_i.$$

[3]These days, this would be called *categorification*: it can often be useful to identify a number as the dimension of some vector space attached to your object.

The Betti numbers are defined via simplices, and how M is built out of cells. Since M is a smooth manifold, one might want to compute them in another way, using the smooth structure of the manifold. To do this, one introduces the *de Rham complex*

$$0 \longrightarrow \Omega^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \Omega^2(M) \longrightarrow \cdots \longrightarrow \Omega^n(M) \longrightarrow 0, \tag{1.12}$$

with linear maps d such that $\mathrm{d}^2 = 0$. Unlike in the previous example, this is built out of real functions and real differential forms.

Definition 1.13. The *de Rham cohomology* of M is the sequence of real vector spaces

$$H^i_{\mathrm{dR}}(M) := \frac{\ker(\mathrm{d}\colon \Omega^i(M) \to \Omega^{i+1}(M))}{\operatorname{Im}(\mathrm{d}\colon \Omega^{i-1}(M) \to \Omega^i(M))}.$$

Theorem 1.14 (de Rham). *There is an isomorphism $H^i_{\mathrm{dR}}(M) \cong H^i(M;\mathbb{R})$, and therefore $\dim H^i_{\mathrm{dR}}(M) = b_i$.*

From this perspective, the Euler characteristic looks more like an index, where we stack together the pieces of the de Rham complex:

$$\bigoplus_{i \text{ even}} \Omega^i(M) \longrightarrow \bigoplus_{i \text{ odd}} \Omega^i(M). \tag{1.15}$$

However, the index of this is *not* the Euler characteristic! The issue is that the de Rham cohomology groups are a subquotient, not just a subspace or just a quotient. To compute the Euler characteristic as an index, we'll need some way of turning them into pure subspaces or quotients. One way to do this is to use an inner product and take orthogonal complements.

Let M be a Riemannian manifold. Then, there is a Laplace operator $\Delta\colon \Omega^i(M) \to \Omega^i(M)$, which is a linear second-order elliptic differential operator.

Remark. There are three basic kinds of differential operators studied in a typical differential equations course: elliptic, parabolic, and hyperbolic. The Laplacian is the basic example of an elliptic operator; the heat operator is the basic example of a parabolic operator; and the Schrödinger operator is the basic example of a hyperbolic operator. We will focus on elliptic operators in this course, but both the heat equation and the Schrödinger equation will appear. ◀

Example 1.16. Let $\mathbb{E}^n$ denote n-dimensional Euclidean space with coordinates $x^1, \dots, x^n$. Then, the Laplacian on $\mathbb{E}^n$ is

$$\Delta = \left(\frac{\partial}{\partial x^1}\right)^2 + \cdots + \left(\frac{\partial}{\partial x^n}\right)^2.$$

◀

For more general Riemannian manifolds, the definition of the Laplacian is more complicated, but not much more so.

Definition 1.17. If M is a Riemannian manifold, there is an L^2 inner product on $\Omega^i(M)$ defined by

$$\langle \alpha, \beta \rangle_{L^2} := \int_M \langle \alpha(M), \beta(M) \rangle \,\mathrm{dvol}_m.$$

Using these inner products, we can let $\mathrm{d}^*\colon \Omega^{i+1}(M) \to \Omega^i(M)$ be the formal adjoint to d.

Fact. d^* exists and is a first-order differential operator. ◀

Definition 1.18. The *Laplace operator* on M is $\Delta := \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$.

A form in the kernel of Δ is called *harmonic*, and the space of harmonic forms is denoted $\mathscr{H}^i(M) \subset \Delta^i(M)$.

Theorem 1.19 (Hodge theorem). *The natural map $\mathscr{H}^i(M) \to H^i_{\mathrm{dR}}(M)$ is an isomorphism. In particular,* $\dim \mathscr{H}^i(M) = b_i$.

This is how index theory enters the picture: if we can access the space of harmonic forms as kernels and cokenels of operators, we could compute the Euler characteristic as an index. And indeed, we can fix (1.15) as folows:

$$\bigoplus_{i \text{ even}} \Omega^i(M) \xrightarrow{\mathrm{d}+\mathrm{d}^*} \bigoplus_{i \text{ odd}} \Omega^i(M). \tag{1.20}$$

The index of this operator is the Euler characteristic.

A third example of index theory is the higher-dimensional Riemann-Roch theorem. Let M be a compact complex manifold; then, the $\overline{\partial}$ operator defines a *Dolbeault complex* analogous to the de Rham complex. If M is 2-(complex-)dimensional, the Euler characteristic satisfies a formula

$$\chi(M) = \frac{1}{12}\big(c_1^2(M) + c_2(M)\big)[M]. \tag{1.21}$$

Here c_1 and c_2 are examples of *characteristic classes*, which we'll start on in the next few lectures. In particular, the right-hand side is an integer. In higher dimensions, there are similar expressions with larger denominators and more characteristic classes.

These were studied by Todd and his student Egger, by Weyl, and others. But the general forms remained conjectures until 1954, when Hirzebruch proved these generalizations of the Riemann-Roch theorem, and an additional, similar result called the signature theorem. He wove together two very new pieces of mathematics: the cobordism theory of René Thom (published only earlier that year!) and the theory of sheaves.

Hirzebruch and others in this field introduced a rational combination of different characteristic numbers, called *Pontrjagin numbers*, called the $\widehat{A}$*-genus* (said "A-hat genus"). This is defined on closed oriented manifolds, and on a spin manifold is an integer.

That $\widehat{A}(M)$ is an integer is a suggestion that it's a dimension of something, and when Singer went to visit Oxford in 1963, Atiyah asked him what object has the $\widehat{A}$-genus as its dimension, and this is the problem that they solved: they constructed a differential operator called the Dirac operator on a spin manifold, and showed that its index is the $\widehat{A}$-genus.

The Dirac operator

$$D := \gamma^\mu \frac{\partial}{\partial x^\mu}$$

for some γ^μ (this notation means the index μ is implicitly summed over) is a first-order linear differential operator. We'd like this to be a square root of the Laplacian operator.

Exercise 1.22. Show that $D^2 = \Delta$ iff

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2\delta^{\mu\nu}. \tag{1.23}$$

Here, $\delta^{\mu\nu}$ means 1 if $\mu = \nu$ and 0 otherwise.

So the operator has to satisfy n^2 equations. If you try to solve this for functions on $\mathbb{E}^n$, you can show that no such γ^μ exist, but one could instead ask for vector-valued functions which satisfy (1.23), and indeed we will spend some time studying the abstract theory of matrices which satisfy this condition, rephrased as the algebraic theory of Clifford modules. In particular, we will be able to show that a spin structure is precisely what one needs to be able to construct the Dirac operator on a Riemannian manifold.

Before Atiyah and Singer told this story, Grothendieck took the Hirzebruch-Riemann-Roch theorem and generalized it still further, and Atiyah and Hirzenbruch saw how to translate his ideas from algebraic geometry to topology, and replace sheaves with vector bundles. They then defined K-theory and rapidly developed it from 1958 to 1962. When Atiyah asked Singer his question, it was in this context.

At the same time, parallel work was undertaken in the Soviet Union under Gelfand and his students. He observed that the index sometimes can be computed topologically, and asked whether this is true in general, and Atiyah-Singer's answer also incorporates this question.

Subsequently, in the 1970s, Gilkey, Patodi, and others were able to provide more rigid, simpler proofs with analytic methods, and in the 1980s Getzler made another important simplifying step to what's now called the heat equation proof of the index theorem, which we'll follow.

We'll use John Roe's book in this course. It's analytic in flavor, but also treats many other nice results, and if we go quickly enough, we'll get to see some of them, including Witten's physical treatment of Morse theory, the Lefschetz theorem, the Hodge theorem, and more.

In this class, the students will give lectures, two each week, and we hope to go through two chapters a week. You don't have to use all three hours!

On the course website (https://www.ma.utexas.edu/users/dafr/M392C/), there will be some useful information, including some old course notes, some historical background, and more to come. These will be there so that you do not forget the beauty of the material amongst all the details in the lectures.

Not everybody may know all of the prerequisites for this course, since it draws in lots of different parts of mathematics. One can ask the professor for references or talk to other students in the course.

The second half of the first day is on the first chapter of the book, reviewing some of the basics of Riemannian geometry.

Let's first start with some linear algebra and differential forms. Let V be an n-dimensional real vector space. Eventually, V will be a tangent space at a point to a manifold, and if the manifold has a Riemannian metric, V picks up an inner product.

Associated to V are several canonical vector spaces built from it: its wedge powers $\Lambda^2 V, \dots, \Lambda^n V$, and $\Lambda^0 V$, which is canonically $\mathbb{R}$. The top exterior power is also called the *determinant line*, $\operatorname{Det} V := \Lambda^n V$. Dually, there are the exterior powers of the dual space V^*: $\mathbb{R}$, V^*, $\Lambda^2 V^*, \dots, \operatorname{Det} V^*$.

An inner product on V canonically induces inner products on all of these exterior powers. One way to see this is to let $e_1, \dots, e_n$ be an orthonormal basis of V; then, there is a dual basis $e^1, \dots, e^n$ of V^*, defined by the relation

$$e^\mu(e_\nu) = \delta^\mu_\nu, \tag{1.24}$$

i.e. 1 if $\mu = \nu$ and 0 otherwise.

We specify the inner product on V^* by declaring this dual basis orthonormal, which suffices, though you have to check that if you change the orthonormal basis of V you started with, you'll end up with the same inner product nontheless.

We also obtain bases for the exterior powers of V and V^*: for $\Lambda^q V$, the basis is

$$\{e_{i_1} \wedge \cdots \wedge e_{i_q} : 1 \le i_1 < \cdots < i_q \le n\}, \tag{1.25}$$

and for $\Lambda^q V^*$, it's

$$\{e^{i_1} \wedge \cdots \wedge e^{i_q} : 1 \le i_1 < \cdots < i_q \le n\}. \tag{1.26}$$

Again we define the inner products on $\Lambda^q V$ and $\Lambda^q V^*$ by asking for these bases to be orthonormal, and again the inner product in question does not depend on the specific choice of orthonormal basis of V.

Definition 1.27. An *orientation* of V is an orientation of its determinant line. That is, $\operatorname{Det} V \setminus 0$ has two components, and an orientation is a choice of one of them.

Given n vectors $e_1, \dots, e_n \in V$, we can wedge them together to an $e_1 \wedge \cdots \wedge e_n \in \operatorname{Det} V$; $\{e_1, \dots, e_n\}$ is a basis iff $e_1 \wedge \cdots \wedge e_n \ne 0$. Thus a basis singles out one of the two rays in $\operatorname{Det} V \setminus 0$, hence defines an orientation. Since $(\operatorname{Det} V)^* = \operatorname{Det}(V^*)$ canonically, then this also defines an orientation on $(\operatorname{Det} V)^*$: the duality pairing implies there's a single $\theta \in \operatorname{Det} V^*$ which sends $e_1 \wedge \cdots \wedge e_n \mapsto 1$; we call it the *volume form* and denote it vol.

On an oriented Riemannian n-manifold, this is a differential n-form, hence can integrate it to determine the volume of the manifold. If it's not oriented, there are two at each point, which may twist globally into something called a density. Nonetheless, this can be integrated, and the volume of, e.g. $\mathbb{RP}^2$ still makes sense.

The pairing $\Lambda^q V^* \otimes \Lambda^{n-q} V^* \to \operatorname{Det} V^*$ defined by

$$\alpha, \beta \longmapsto \alpha \wedge \beta \tag{1.28}$$

is nondegenerate. An orientation of V defines a trivialization of $\operatorname{Det} V^*$ (where vol $= 1$), so this pairing is $\mathbb{R}$-valued. Therefore we obtain an isomorphism $\Lambda^q V^* \cong \Lambda^{n-q}(V)$, though it depends on the inner product and the orientation.

Example 1.29. In three dimensions, we use this frequently, to shift from the perspective of vector fields and scalars and div, ∇, and curl to differential forms. ⋖

There's also an isomorphism $\star\colon \Lambda^q V^* \to \Lambda^{n-q} V^*$ which only uses the inner product; this is called the *Hodge star*. Putting everything together, the Hodge star is defined uniquely by the stipulation that

$$\alpha_1 \wedge \star\alpha_2 = \langle \alpha_1, \alpha_2 \rangle \mathrm{vol} \tag{1.30}$$

for any $\alpha_1, \alpha_2 \in \Lambda^q V^*$.

Exercise 1.31. For example, check that $\star(e_{i_1} \wedge \cdots \wedge e_{i_q})$ is the wedge of all of the e_j not in $(i_1, \dots, i_q)$, possibly multiplied by -1.

Exercise 1.32. Show that $\star^2 = (-1)^{q(n-q)}$.

Here, "inner product" means nondegenerate inner product; much of this story still goes through for a Lorentz-signature metric, but not all of it.

Exercise 1.33. Show that on a closed, oriented Riemannian manifold M, $\mathrm{d}^* = \pm \star \mathrm{d}\star$, and determine the sign (which depends on n and q).

You can type-check that the right-hand side is a first-order differential operator which lowers the degree by 1. Solving the exercise boils down to checking that

$$\int_M \langle \mathrm{d}\alpha, \beta \rangle \,\mathrm{vol} = \pm \int_M \langle \alpha, \star\mathrm{d}\star\beta \rangle \,\mathrm{vol}.$$

You'll end up using Stokes' theorem.

Now let's think about parallelism. Let $\mathbb{A}^n$ be n-dimensional affine space (no distinguished origin), where we learn calculus. This has parallel transport: if $\xi \in \mathbb{R}^n$ is a tangent vector at some point, we can translate it everywhere to a vector field. This allows us to define differentiation: if $f : U \to \mathbb{R}$, where $U \subset \mathbb{A}^n$ is open, then we define the derivative of f at p in the direction of ξ to be

$$\xi_p f := \lim_{h \to 0} \frac{f(p + h\xi) - f(p)}{h}. \tag{1.34}$$

This uses parallelism in the expression $p + h\xi$.

More generally, if M is a smooth manifold, we don't always have a canonical parallel transport between tangent spaces for different points of the manifold, so we can't compare tangent vectors in different places and differentiate.

For example, if $\gamma\colon [a,b] \to M$ is a curve, its tangent vectors at two different points can't be compared (without extra structure), so there's no way to make the subtraction in (1.34). We'll introduce the structure that allows us to do this.

Definition 1.35. Let $V \to M$ be a vector bundle and $C^\infty(M;V)$ denote its space of smooth sections, which is a real vector space. A *covariant derivative* is a bilinear operator

$$\nabla\colon C^\infty(M;TM) \times C^\infty(M;V) \longrightarrow C^\infty(M;V),$$

denoted

$$X, s \longmapsto \nabla_X s,$$

such that

(1) $\nabla_{fX} s = f \nabla_X s$, and
(2) $\nabla_X(fs) = (X \cdot f)s + f \nabla_X s$,

where $(X \cdot f)$ is the usual directional derivative associated to a vector field.

For $V = TM$, we have the usual Lie bracket

$$[-,-]\colon C^\infty(M;TM) \times C^\infty(M;TM) \longrightarrow C^\infty(M;TM)$$

sending $X, Y \mapsto [X,Y]$; if $f, g\colon M \to \mathbb{R}$ are functions, then

$$[fX, gY] = fg[X,Y] + f(X \cdot g)Y - g(Y \cdot f)X.$$

This operator is the commutator of an infinitesimal flow of X and an infinitesimal flow of Y.

Definition 1.36. Let ∇ be a covariant derivative for the tangent bundle. Its *torsion* is

$$\tau(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y].$$

Exercise 1.37. Show that $\tau(fX, gY) = fg\tau(X,Y)$ and $\tau(X,Y) = -\tau(Y,X)$.

Let's write this out in local coordinates. There are two things we could mean – coordinates on M or on V. Since V is a vector bundle, we can use for its coordinates the coordinates of M and a (local) basis of sections $s_1, \dots, s_r$. (Global nonvanishing sections might not exist at all, e.g. $TS^2 \to S^2$). In this case, you can differentiate s_j, obtaining some linear combination of the sections depending on x in a neighborhood U:

$$\nabla_X s_j = \Gamma_j^i(x) s_i.$$

This is just parameterized linear algebra. These Γ_j^i are 1-forms on U. We can also obtain coordinates for these 1-forms: if we let

$$\nabla_{\partial/\partial x^\mu} s_j = \Gamma_{j\mu}^i s_i,$$

then $\Gamma_j^i = \Gamma_{j\mu}^i \, \mathrm{d}x^\mu$.

If $V \to M$ has an inner product (metric), a positive definite pairing $C^\infty(M;V) \times C^\infty(M;V) \to C^\infty(M)$ sending $s_1, s_2, \mapsto \langle s_1, s_2 \rangle$, we can ask how a covariant derivative interacts with it.

Definition 1.38. A covariant derivative is *compatible with the metric* if for all $X \in C^\infty(M;TM)$ and $s_1, s_2 \in C^\infty(M;V)$,

$$X \cdot \langle s_1, s_2 \rangle = \langle \nabla_X s_1, s_2 \rangle + \langle s_1, \nabla_X s_2 \rangle.$$

Definition 1.39. A section $s \in C^\infty(M;V)$ is *parallel* if $\nabla_X s = 0$ for all X.

Parallel sections exist in $\mathbb{A}^n$ but not in general; the obstruction is called the curvature.

Definition 1.40. The *curvature* of a covariant derivative ∇ is

$$K(X,Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]},$$

i.e.

$$K(X,Y)(s) := \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s.$$

If M is Riemannian, there's a beautiful theorem about how all of these structures interact.

Theorem 1.41 (Levi-Civita)**.** *Let M be a Riemannian manifold. Then, there is a unique connection on $TM \to M$ which is torsion-free and compatible with the metric.*

Exercise 1.42. Prove this theorem. The way you do this is to compute $\langle \nabla_X Y, Z \rangle$, because if you know this for all Z, you know $\nabla_X Y$. Using the torsion-free and metric compatibility conditions, you can expand it out, and after some number of steps, you'll get the answer.

This local but non-global parallelism is an important property of Riemannian manifolds.

Next we will write a local formula for this connection. Suppose we have local coordinates $x^1, \dots, x^n$ on an open set $U \subset M$; then, we obtain the symbols $\Gamma_{jk}^i \colon U \to \mathbb{R}$. If we define the inner product and the Lie bracket, we can write down formulas for them. Namely, if we let

$$g_{ij} := \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle,$$

and since

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^i} \right] = 0,$$

then we can determine equations that the Γ_{jk}^i must satisfy. These can be encoded in the Riemann curvature tensor $R(X,Y)Z$, and in coordinates, on elets

$$R_{jk\ell}^i \frac{\partial}{\partial x^i} = R\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell} \right) \frac{\partial}{\partial x^j}.$$

This tensor has a bunch of important symmetries. The curvature is a 2-form on the manifold, but valued in $\mathrm{End}(TM)$: X and Y are the two directions you're testing, and are the 2 components of the 2-form.

The symmetry $R_{jk\ell}^i = -R_{j\ell k}^i$ means that $R(X,Y)$– is a skew-symmetric endomorphism of TM.

You can also lower an index by defining

$$R_{ijk\ell} = \langle R(\partial_k, \partial_\ell) \partial_j, \partial_i \rangle, \tag{1.43}$$

and skew-symmetry means

$$R_{ijk\ell} = -R_{jik\ell}.$$

These are the two "easier" symmetries, in that they don't use much specifically about R. A more interesting one is

$$R_{ijk\ell} + R_{ik\ell j} + R_{i\ell jk} = 0,$$

and the fourth identity, which follows from the other three, is

$$R_{ijk\ell} = R_{k\ell ij}.$$

Exercise 1.44. Compute the dimension of the vector space of tensors which satisfy these identities, as a subspace of $(V^*)^{\otimes 4}$.

Lecture 2.

Principal G-bundles: 1/24/18

The first part of today's talk was given by George Torres, corresponding to the first part of Chapter 2 of Roe's book.

Definition 2.1. A *Lie group* G is a group that is also a smooth manifold, and such that multiplication $m\colon G\times G\to G$ and inversion $i\colon G\to G$ are smooth maps.

Associated to any Lie group G is its *Lie algebra* $\mathfrak g := T_eG$. There is a *Lie bracket* operation

$$[-,-]\colon \mathfrak g\times\mathfrak g \longrightarrow \mathfrak g$$

defined like last time (use multiplication on G to extend tangent vectors to G-invariant vector fields, then take their commutators).

Definition 2.2. Let G be a Lie group and M be a smooth manifold. A *principal G-bundle* is a map of smooth manifolds $\pi\colon E\to M$ together with a smooth right action of G on E whose orbits are fibers of π, such that G acts freely and transitively on each fiber.

This implies that for each $m\in M$, $\pi^{-1}(M)\cong G$ noncanonically, and $E/G\cong M$.

Connections on principal bundles are analogous to those on vector bundles; the goal is to define a horizontal subspace of the bundle, and use that and its G-translates to define parallel transport. Then, one must show that connections always exist, but this turns out to be true. This definition of connections is sometimes called an *Ehresmann connection*.

In order to define connections, we'll need a few preliminary definitions.

Definition 2.3. Let G be a Lie group. The *adjoint representation* of G is the map $\mathrm{Ad}\colon G\to\mathrm{End}(\mathfrak g)$ which sends a $g\in G$ to $d|_e\psi_g$, where $\psi_g\colon G\to G$ is conjugation by G and $d|_e$ is differentiating at the identity.

Definition 2.4. Let $\pi\colon E\to M$ be a principal G-bundle and $\rho\colon G\to\mathrm{Aut}(F)$ be a (real, finite-dimensional) representation of G. The *associated vector bundle* to E and F is the $E\times_G F := E\times F/\sim$, where $(e\cdot g, f)\sim(e, g\cdot f)$ for all $e\in E$, $f\in F$, and $g\in G$.

Exercise 2.5. Show that $E\times_G F\to M$ is indeed a vector bundle.

Definition 2.6. Let $\pi\colon E\to M$ be a principal G-bundle. A *vertical vector field* is a vector field v on E such that for all $p\in M$, $v_p = d_eA_p(u)$ for some $u\in\mathfrak g$, where $A_p\colon G\to E$ is an identification of the fiber E_p with G.[4]

An equivalent, more intuitive, definition is that a vertical vector field is contained within the *vertical subbundle* of E, i.e. the kernel of $d\pi$. A third equivalent definition is that V is vertical if for all $p\in M$, there is some $u\in\mathfrak g$ such that

$$v_p = \frac{\mathrm d}{\mathrm dt}(p\exp(tu))\bigg|_{t=0}. \tag{2.7}$$

Let $R_g\colon E\to E$ denote the right action of G, sending $e\mapsto e\cdot g$. We can pushforward by this map: let

$$(Rg)_*v := \frac{\mathrm d}{\mathrm dt}(p\exp(tu)g)\bigg|_{t=0}.$$

In particular, the pushforward of a vertical vector field is still vertical.

Horizontal differential forms are dual to vertical vector fields.

[4] TODO: I might have gotten this wrong.

Definition 2.8. Let $\alpha \in \Omega^p(E)$. Then α is *horizontal* if for all vertical vector fields $X_1, \dots, X_p$, $\alpha(X_1, \dots, X_p) = 0$. If in addition $(Rg)^*\omega = \omega$, we say α is *invariant* under the G-action.

Example 2.9. For any $\beta \in \Omega^p(M)$, $\pi^*\beta$ is an invariant horizontal form:

$$(R_g)^*\pi^*\beta = (\pi \circ R_g)^*\beta = \pi^*\beta.$$

◀

More generally, we can consider G-equivariant forms.

Definition 2.10. Let $\rho : G \to \mathrm{Aut}(F)$ be a representation and $f : E \to F$ be a smooth map. Then, f is *ρ-equivariant* if for all $e \in E$ and $g \in G$,

$$f(eg) = \rho(g^{-1})f(e).$$

Invariance is the same thing as equivariance for the trivial representation.

Lemma 2.11 (Correspondence lemma)**.** *Let $f : E \to F$ be as in the previous definition. There is a bijective correspondence between ρ-equivariant maps $f : E \to F$ and sections of $E \times_G F$.*

Proof. Let f be a ρ-equivariant map; then, we define a section s_f to send $x \in M$ to $(x, f(x)) \in E \times F$. To check that this is indeed a section, we need it to commute with the G-action on $E \times F$, and this follows because

$$\begin{aligned} s_f(x) \cdot g = (x, f(x)) \cdot g &= (xg, \rho(g^{-1})f(x)) \\ &= (xg, f(xg)) \\ &= s_f(x \cdot g). \end{aligned}$$

Conversely, let $s \colon M \to E \times_G F$ be a section, and consider the diagram

$$\begin{tikzcd} E \arrow[r, "\sigma"] \arrow[d, "\pi"] & E \times F \arrow[r, "\pi_1"] \arrow[d, "p"] & E \arrow[d, "\pi"] \\ M \arrow[r, "s"] & E \times_G F \arrow[r] & M, \end{tikzcd} \tag{2.12}$$

where p is the quotient map, π_1 is projection onto the first factor, and σ is defined such that the composition of the maps across the top is the identity. The key observations are

(1) the right-hand square is a pullback square, and
(2) $f = \pi_2\sigma$ is ρ-equivariant.

The first property is true because for any $g \in G$,

$$\begin{aligned} \pi_1(\sigma(xg)) &= xg \\ &= \pi_1(\sigma(x)) \cdot g = \pi_1(\sigma(x \cdot g)), \end{aligned}$$

and along the other corner,

$$\begin{aligned} p(\sigma(xg)) = s\pi(xg) &= s\pi(x) \\ &= p\sigma(x) \\ &= p\sigma(x \cdot g), \end{aligned}$$

so $\sigma(xg) = \sigma(x) \cdot g$. TODO: I missed the last part, that f is ρ-equivariant (and why these two properties suffice). ⊠

Exercise 2.13. Finish the proof by checking that these assignments are mutual inverses.

With these definitions in mind, we can define connections.

Definition 2.14. Let $\pi \colon E \to M$ be a principal G-bundle. A *connection* on E is a subbundle $H \subset TE$ (H for "horizontal") such that

- (splitting) there is another subbundle $V \subset TE$ such that for all $u \in E$, $T_uE = V_u \oplus H_u$, and
- (G-invariance) $d|_e R_g(H_u) = H_{u \cdot g}$.

Using the splitting lemma for vector bundles, the first condition is equivalent to the existence of a split short exact sequence

$$0 \longrightarrow V \longrightarrow TE \longrightarrow \pi^*TM \longrightarrow 0.$$

The splitting is determined by a section $\pi^*TM \to TE$ (which defines what we call "horizontal") or by a section $TE \to V$. This leads to an equivalent definition of a connection on a principal bundle, which is also useful: a connection on E is a $\mathfrak{g}$-valued 1-form ω on E, called the *connection one-form*, such that

- (G-invariance) for any $\xi \in T_uE$, $\omega(\xi \cdot g) = \mathrm{Ad}(g^{-1})\omega(\xi)$, and
- (splitting) for any $v \in \mathfrak{g}$, $\omega(X_v) = v$.

The splitting lemma guarantees you can always split short exact sequences of vector bundles. But to show that connections exist, we need to address G-invariance, which is not as immediate.

Lemma 2.15. *If G is a Lie group and $\pi\colon E \to M$ is a principal G-bundle, then there is a connection on E.*

Proof. Let $\underline{\mathfrak{g}} := E \times \mathfrak{g}$ and $V := E \times_G \mathfrak{g}$, where G acts on $\mathfrak{g}$ by the adjoint action. Then consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & \underline{\mathfrak{g}} \arrow[r] \arrow[d] & TE \arrow[r] \arrow[d] & \pi^*TM \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & V \arrow[r] & Q \arrow[r] & TM \arrow[r] & 0. \end{tikzcd} \tag{2.16}$$

Proving that this commutes takes a while, so we won't delve into the details; one reference is Atiyah, "Complex analytic connections on fibre bundles."

The point of introducing (2.16) is that if we can lift a splitting from the bottom row to the top row, it will be a G-invariant splitting, hence a connection. So choose a splitting $\sigma\colon TM \to Q$, which splits the bottom row of (2.16). Then we have a diagram

$$\begin{tikzcd} \pi^*TM \arrow[r] \arrow[d] & TM \arrow[d, "\sigma"] \\ TE \arrow[r] \arrow[d] & Q \arrow[d] \\ E \arrow[r, "\pi"] & M. \end{tikzcd} \tag{2.17}$$

The bottom rectangle is a pullback, and the total rectangle (π^*TM, TM, E, and M) is a pullback. Therefore by the universal property of pullbacks, the top rectangle also is a pullback, and this implies that σ lifts across it to something G-invariant. ⊠

There is also a more geometric proof.

Next we'll talk about exterior derivatives.

Definition 2.18. Let $\pi\colon E \to M$ be a principal G-bundle with connection H and let $p_\omega\colon \Omega^*(E) \to \Omega^*(H)$ denote projection onto the horizontal subspace. The *exterior covariant derivative* is the composition

$$\Omega^p(E) \xrightarrow{\mathrm{d}} \Omega^{p+1}(E) \xrightarrow{p_\omega} \Omega^{p+1}(H).$$

Proposition 2.19. *Let $\rho\colon G \to \mathrm{GL}(F)$ be a representation, $\alpha \in \Omega_E^p(F)$ be an F-valued, ρ-equivariant horizontal p-form on E, and ω denote the ($\mathfrak{g}$-valued) connection one-form on E. Then*

$$p_\omega(\mathrm{d}\alpha) = \mathrm{d}\alpha + \rho_*\omega \wedge \alpha.$$

Definition 2.20. Let ω be a differential 1-form. Then, the *curvature* Ω of ω is the 2-form

$$\Omega(X_1, X_2) = \mathrm{d}\omega(X_1, X_2) + [\Omega(X_1), \Omega(X_2)].$$

Exercise 2.21. With notation as above,

$$(p_\omega \mathrm{d})^2\alpha = \rho_*\Omega \wedge \alpha.$$

The point is that acting on α is the same as wedging with $\rho_*\Omega$, and this tells you something about what Ω is doing.

Remark. In some of these formulas, it's important to be careful about what the wedge products are doing. For example, we once or twice saw $\omega \wedge \alpha$, where $\omega \in \Omega^1_E(\mathfrak{g})$ and $\alpha \in \Omega^1_E(F)$. If $\mathfrak{g}$ is a matrix algebra, we can organize the components of ω into a matrix of differential forms, and F is a representation of the matrix algebra $\mathfrak{g}$, so α is a vector. In this case, the wedge product is a combination of matrix multiplication and the wedge product:

$$\begin{pmatrix} \omega^1_1 & \omega^1_2 \\ \omega^2_1 & \omega^2_2 \end{pmatrix} \wedge \begin{pmatrix} \alpha^1 \\ \alpha^2 \end{pmatrix} = \begin{pmatrix} \omega^1_1 \wedge \alpha^1 + \omega^1_2 \wedge \alpha^2 \\ \omega^2_1 \wedge \alpha^1 + \omega^2_2 \wedge \alpha^2 \end{pmatrix}. \tag{2.22}$$

The same is true for computing $\omega \wedge \omega$; in particular, this is not automatically zero.

Therefore one sometimes sees the formula for curvature written

$$\Omega := \mathrm{d}\omega + [\omega \wedge \omega]. \tag{2.23}$$

What does this mean? We have $\omega \wedge \omega \in \Omega^2_E(\mathfrak{g} \otimes \mathfrak{g})$, and the Lie bracket $[-,-]\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$. We implement this on differential forms by

$$(\omega^i \wedge \omega^j) e_i \otimes e_j \longmapsto (\omega^i \wedge \omega^j)[e_i, e_j].$$

In particular,

$$\begin{aligned} [\omega \wedge \omega](X_1, X_2) &= [\omega(X_1), \omega(X_2)] - [\omega(X_2), \omega(X_1)] \\ &= 2[\omega(X_1), \omega(X_2)], \end{aligned}$$

which is why (2.23) has an extra 1/2 in it compared to the first definition. ◂

Remark. One can even define differential forms valued in vector bundles: $\Omega^\bullet_M(E \times_G F)$ consists of sections of the exterior powers of $E \times_G F$. Alternatively, you can think of these as valued in $(E \times_G F)_p$ at a point p; the vector spaces changes as p moves, but that's okay. The quotient map $\pi\colon E \to G$ defines a pullback $\Omega^*_M(E \times_G F) \to \Omega^*_E(F)$. This provides yet another interpretation of the definition of a connection.

- Invariance is that $\alpha \in \mathrm{Im}(\pi^*)$ iff $R_g^* \alpha = \rho(g)^{-1}\alpha$ for all $g \in G$.
- Splitting comes from the fact that $\iota_\zeta \alpha = 0$ when ζ is vertical. (This denotes *contraction*: $\iota_\zeta \alpha(X) := \alpha(\zeta, X)$).

It's a good exercise to check, to get practice manipulating these vector- or bundle-valued forms. But principal bundles make some of these computations easier, by turning some bundle-valued forms into constant vector space-valued forms. ◂

Lecture 3.

Characteristic classes: 1/24/18

The second part of today's lecture was given by Riccardo Pedrotti, on characteristic classes from a geometric perspective.

The theory of characteristic classes comes from the simple question: how can we tell two vector bundles apart? For instance, how do we know that the tangent bundle to the 2-sphere is non-trivial? Characteristic classes gives a systematic approach.

Definition 3.1. A *characteristic class* c is a natural transformation which to each vector bundle V over a manifold M associates an element $c(V)$ of the cohomology group $H^*(M)$, with property that if $V_1 \simeq V_2$ then $c(V_1) = c(V_2)$.

The idea of Chern-Weil theory is the following: suppose that our bundle V is equipped with a connection. In some sense, the curvature of this connection measures the local deviation of V from flatness. Now if V is flat, and the base manifold M is simply connected, then V is trivial. This suggests that there may be a link between curvature and characteristic classes, which measure the global deviation of V from triviality. Such a link is provided by the theory of invariant polynomials.

By polynomial function we mean the following:

Definition 3.2. Let $\mathfrak{gl}_m(\mathbb{C})$ denote the Lie algebra of $m \times m$ matrices over $\mathbb{C}$. A *homogeneous polynomial function* P on $\mathfrak{gl}_m(\mathbb{C})$ is a function such that there exists a $\widetilde{P} \in \mathrm{Sym}^k(\mathbb{C}^m)^*$ such that $P(A) = \widetilde{P}(A, A, \dots, A)$. A *polynomial function* is a sum of homogeneous ones.

Definition 3.3. An *invariant* polynomial on $\mathfrak{gl}_m(\mathbb{C})$ is a polynomial function $P\colon \mathfrak{gl}_m(\mathbb{C}) \to \mathbb{C}$ such that for all $X, Y \in \mathfrak{gl}_m(\mathbb{C})$, $P(XY) = P(YX)$. An *invariant formal power series* is a formal power series over $\mathfrak{gl}_m(\mathbb{C})$ each of whose homogeneous components is an invariant polynomial.

For example, the determinant and the trace are invariant polynomials.

Lemma 3.4. *The ring of invariant polynomials on $\mathfrak{gl}_m(\mathbb{C})$ is a polynomial ring generated by the polynomials*

$$c_k(X) = (-2\pi i)^{-k} \operatorname{tr}(\Lambda^k X),$$

where $\Lambda^k X$ denotes the transformation induced by X on $\Lambda^k \mathbb{C}^m$.

Proof. Let P be any invariant polynomial. Restricting P to diagonal matrices, we see that P must be a polynomial function of the diagonal entries. Since these diagonal entries can be interchanged by conjugation, P must in fact be a *symmetric* polynomial function. Now since P is invariant under conjugation, it must be a symmetric polynomial function of the eigenvalues for all matrices with distinct eigenvalues, since by elementary linear algebra such matrices are conjugate to diagonal matrices. The set of such matrices is dense in $\mathfrak{gl}_m(\mathbb{C})$, so a continuity argument shows that P is just a symmetric polynomial function in the eigenvalues. Now it is easy to see that $\operatorname{tr}(\Lambda^k X)$ is the k^{th} elementary symmetric function in the eigenvalues of X. The main theorem on symmetric polynomials states that the ring of symmetric polynomials is itself a polynomial ring generated by the elementary symmetric functions, and this now completes the proof. ⊠

Example 3.5. To make the idea of the proof more concrete, let $m = 4$. A 4×4 diagonal matrix is of the form

$$X = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$$

for some $a, b, c, d \in \mathbb{R}$. Then, $\Lambda^2 X$ acts on $e_1 \wedge e_2$ by $Xe_1 \wedge Xe_2 = ab e_1 \wedge e_2$. Therefore $\Lambda^2 X$ is a 6×6 diagonal matrix with diagonal terms ab, ac, ad, bc, bd, and cd, and therefore its trace is

$$\operatorname{tr}(\Lambda^2 X) = ab + ac + ad + bc + bd + cd.$$

This is a quadratic polynomial, and is symmetric; it's an example of an *elementary symmetric polynomial*. There's a theorem that the ring of all symmetric polynomials are generated by these elementary symmetric polynomials. ◂

Now let V be a complex vector bundle over M with connection ∇ and curvature $K \in \Omega^2_M(\operatorname{End}(V))$. Choosing a local framing for V, we may locally identify K with a matrix of ordinary 2-forms. Hence, if P is an invariant polynomial, we can apply P to this matrix to get an even-dimensional differential form $P(K)$. *A priori*, this depends on the choice of local framing, but since P is invariant, $P(K)$ doesn't depend on the choice, and is therefore globally defined.

In terms of the principal $\mathrm{GL}_m(\mathbb{C})$-bundle E associated to V, this construction may be phrased as follows. Let Ω be the curvature form of the induced connection on E; Ω is a horizontal, equivariant 2-form on E with values in $\mathfrak{gl}_m(\mathbb{C})$, so $P(\Omega)$ is a horizontal invariant form on E. Such a form is the lift to E of a form on M, and this form is $P(K)$.

Since 2-forms are nilpotent elements in the exterior algebra $\Omega^*_M(\mathfrak{gl}_m(\mathbb{C}))$, all formal power series with 2-form-valued variables in fact converge. Thus, this construction makes good sense if P is merely an invariant formal power series.

Proposition 3.6. *For any invariant polynomial (or formal power series) P, the differential form $P(K)$ is closed, and its de Rham cohomology class is independent of the choice of connection ∇ on V.*

Proof. For the purposes of this proof call an invariant formal power series P as *respectable* if the conclusion of the proposition holds for P. Clearly the sum and product of respectable formal power series are respectable. Thus, it is enough to prove that the generators defined in Lemma 3.4 are respectable. Equivalently, since

$$\det(1 + qK) = \sum q^k \operatorname{tr}(\Lambda^k K),$$

it is enough to prove that $\det(1 + qK)$, considered as a formal power series depending on the parameter q, is respectable.

If P is a respectable formal power series with constant term a, and g is a function holomorphic in a neighborhood of a, then $g \circ P$ is also a respectable formal power series. Hence, $\det(1+qK)$ is respectable if and only if $\log\det(1+qK)$ is respectable. We will now prove directly that $\log\det(1+qK)$ is respectable.

For this purpose we will work in the associated principal $\mathrm{GL}_n(\mathbb{C})$-bundle E of frames for V, with matrix-valued connection 1-form ω and corresponding curvature 2-form Ω. Recall the formula

$$\Omega = d\omega + \omega \wedge \omega$$

where the product in the ring of matrix-valued forms is obtained by tensoring exterior product and matrix multiplication as in (2.22).

Now suppose that ω depends on a parameter t; then Ω also depends on t, and if we use a dot to denote differentiation with respect to t, then

$$\dot{\Omega} = \mathrm{d}\dot{\omega} + \omega \wedge \dot{\omega} + \dot{\omega} \wedge \omega.$$

Consider

$$\frac{\mathrm{d}}{\mathrm{d}t} \log\det(1+q\Omega) = q\,\mathrm{tr}\big(\dot{\Omega}(1+q\Omega)^{-1}\big) \tag{3.7}$$

$$= \sum_{\ell=0}^{\infty} (-1)^{\ell} q^{\ell+1}\,\mathrm{tr}\big(\Omega^{\ell}(\mathrm{d}\dot{\omega} + \omega \wedge \dot{\omega} + \dot{\omega} \wedge \omega)\big) \tag{3.8}$$

where (3.7) is justified by the formula

$$\frac{\mathrm{d}}{\mathrm{d}t} \det A(t) = (\det A(t)) \cdot \mathrm{tr}(\dot{A}(t)A(t)^{-1}),$$

and (3.8) is justified by the power series expansion

$$\frac{1}{1+z} = \sum_{i=0}^{\infty} (-1)^i z^i.$$

We also need the *second Bianchi identity*

$$\mathrm{d}\Omega = \Omega \wedge \omega - \omega \wedge \Omega,$$

which can be proven directly from the definition of the exterior derivative. Using this, plus the fact that trace is symmetric, we have that

$$\begin{aligned}\mathrm{tr}\big(\Omega^{\ell}(\omega \wedge \dot{\omega} + \dot{\omega} \wedge \omega)\big) &= \mathrm{tr}\big(\Omega^{\ell} \wedge \omega \wedge \dot{\omega} - \omega \wedge \Omega^{\ell} \wedge \dot{\omega}\big) \\ &= \mathrm{tr}\big((\mathrm{d}\Omega^{\ell}) \wedge \dot{\omega}\big).\end{aligned}$$

Therefore

$$\begin{aligned}\mathrm{tr}\big(\Omega^{\ell} \wedge (\mathrm{d}\dot{\omega} + \omega \wedge \dot{\omega} + \dot{\omega} \wedge \omega)\big) &= \mathrm{tr}\big((\mathrm{d}\Omega^{\ell}) \wedge \dot{\omega} + \Omega^{\ell} \wedge \mathrm{d}\dot{\omega}\big) \\ &= \mathrm{d}\,\mathrm{tr}\big(\Omega^{\ell} \wedge \dot{\omega}\big),\end{aligned}$$

so (3.7) simplifies to

$$\frac{\mathrm{d}}{\mathrm{d}t} \log\det(1+q\Omega) = \mathrm{d} \sum_{\ell=0}^{\infty} (-1)^{\ell} q^{\ell+1}\,\mathrm{tr}\big(\Omega^{\ell} \wedge \dot{\omega}\big), \tag{3.9}$$

and in particular is an exact form on E.

In fact, it is the exterior derivative of a horizontal and invariant form on E: $\dot{\omega}$ is horizontal and G-equivariant since it is a 1-form on M (it follows from Lemma 2.11 that the space of connections is an affine space modeled on the vector space of V-valued forms on M), Ω is horizontal and G-equivariant as well. Hence $\mathrm{tr}(\Omega^{\ell} \wedge \dot{\omega})$ is invariant since the trace is an invariant polynomial, and is horizontal since $\Omega^{\ell} \wedge \dot{\omega}$ is.

Therefore, the projection to the base manifold

$$\frac{\mathrm{d}}{\mathrm{d}t} \log\det(1+qK)$$

is also exact. Now the result follows; for since any connection can be deformed locally to flatness (i.e. $K=0$), we see that $\log\det(1+qK)$ is locally exact, hence closed, and since any two connections can be connected by a smooth path, the cohomology class of $\log\det(1+qK)$ is independent of the choice of connection, since their difference is an exact form. ⊠

It follows from the proposition that any invariant formal power series P defines a characteristic class for complex vector bundles, by the recipe "pick any connection and apply P to the curvature."

Definition 3.10. The k^{th} *Chern class* is the characteristic class corresponding to the generators c_k defined in Lemma 3.4.

Remark. We immediately see from the definition of Chern classes that if a complex vector bundle has rank m, then $c_k = 0$ for $k > m$: $\Lambda^k K$ is the linear transformation induced by K on $\Lambda^k \mathbb{C}^m$, and for $k > m$, the latter is trivial. Naturality comes from the fact that if on a local patch U_i, E has the local connection form ω_i, then on $f^{-1}(U_i)$, the curvature is $f^*\Omega_i$. ◀

Lemma 3.11. *Let V be a real vector bundle and $V_{\mathbb{C}}$ denote its complexification. Then, $c_{2k+1}(V_{\mathbb{C}}) = 0$.*

Proof. We can give V a metric and compatible connection. The curvature of such a connection is skew (i.e. $\mathfrak{o}(m)$-valued), so

$$\operatorname{tr}(\Lambda^k F) = (-1)^k \operatorname{tr}(\Lambda^k F).$$

To see this, recall that the coefficients of the characteristic equation for F are exactly $\operatorname{tr}(\Lambda^k F)$ up to a sign.[5] If λ is an eigenvalue of a skew-symmetric matrix, then $-\lambda$ is too, and on $\mathbb{C}$ this means that the characteristic polynomial is up to a constant the product of polynomials $(z^2 - \lambda^2)$, so there are no coefficients of odd index, hence proving that for k odd, $\Lambda^k F$ is traceless. ⊠

Genera. Holomorphic functions can be used to build important combinations of characteristic classes. Let $f(z)$ be any function holomorphic near $z = 0$. We can use f to construct an invariant formal power series Π_f by defining

$$\Pi_f(X) := \det\left(f\left(-\frac{1}{2\pi i}X\right)\right).$$

Again, to make sense of this, we need to sidestep convergence issues! But since we'll just be applying this to differential forms, which are nilpotent, this is okay.

The associated characteristic class is called the *Chern f-genus*. It has a few nice properties.

Lemma 3.12. *If $L \to M$ is a complex line bundle, $\Pi_f(L) = f(c_1(L))$.*

Proof. This comes from the fact that in this case the curvature is a $\mathfrak{gl}_1(\mathbb{C})$-valued 2-form, so

$$\begin{aligned}\Pi_f(L) = \Pi_f(K_L) &= \det\left(f\left(-\frac{1}{2\pi i}K_L\right)\right)\\ &= f\left(-\frac{1}{2\pi i}K_L\right)\\ &= f\left(\operatorname{tr}\left(-\frac{1}{2\pi i}K_L\right)\right) = f(c_1(L)).\end{aligned}$$ ⊠

Lemma 3.13. *For any complex vector bundles V_1 and V_2, $\Pi_f(V_1 \oplus V_2) = \Pi_f(V_1)\Pi_f(V_2)$.*

Proof sketch. Compute using a direct sum connection, which gives rise to a curvature matrix which is a block matrix. ⊠

Now observe a very useful property: if the eigenvalues of the matrix $(-1/2\pi i)X$ are $\{x_j\}$, then

$$\Pi_f(X) = \prod f(x_j) \tag{3.14}$$

is a symmetric formal power series in the x_j, and can therefore be expressed in terms of the elementary symmetric functions of the x_j. But these elementary symmetric functions are just the Chern classes. Thus in the literature the genus $\Pi_f(V)$ is often written just as in (3.14), where $x_1, \dots, x_m$ are formal variables subject to the relations

$$\begin{aligned}x_1 + x_2 + \cdots + x_m &= c_1,\\ x_1x_2 + \cdots + x_{m_1}x_{m-2} &= c_2,\end{aligned}$$

and so on.

[5] There's a sign convention here; this is true using our definition $\det(1 - qK)$. An alternative choice is to use $\det(q - K)$, in which case one must swap the indices to preserve evenness.

Example 3.15. The genus associated to $f(z) = 1 + z$ is the *total Chern class*

$$c(V) := 1 + c_1(V) + c_2(V) + c_3(V) + \cdots .$$

To see this, consider the power expansion of the determinant:

$$\det\left(1 - \frac{1}{2\pi}X\right) = \sum_k \left(-\frac{1}{2\pi}\right)^k \operatorname{tr}(\Lambda^k X) = \sum_k c_k(X).$$

From this we immediately get that $c(V_1 \oplus V_2) = c(V_1)c(V_2)$. ◀

Definition 3.16. Let $V \to M$ be a real vector bundle and g be a holomorphic function near 0, with $g(0) = 1$. Let $V_{\mathbb{C}}$ be the complexification of V. Denote by f be the branch of $z \mapsto (g(z^2))^{1/2}$ which has $f(0) = 1$; we call the genus associated to f the *Pontrjagin g-genus* of V.

Since f is an even function of z, the associated genus involves only the even Chern classes.

Lemma 3.17. *Let g be as above. Then for a real vector bundle V, the Pontrjagin g-genus is equal to*

$$\prod_j g(y_j)$$

for some formal variables y_j.

Definition 3.18. Let V be a real vector bundle. Its k^{th} *Pontrjagin class* $p_k(V)$ is the k^{th} elementary symmetric function in the formal variables y_j.

Proof of Lemma 3.17. Regard this as an identity between invariant polynomials over $\mathfrak{o}(n)$. Any matrix in $\mathfrak{o}(n)$ is similar to one in block diagonal form, where the blocks are 2×2 and are of the form

$$X = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$$

with eigenvalues $\pm i\lambda$. Since both sides of the desired identity are multiplicative for direct sums, it is enough to prove it for this block X, whose first two elementary symmetric functions are

$$c_1(X) = \left(-\frac{1}{2\pi i}\right)\operatorname{tr}(X),$$
$$c_2(X) = \left(-\frac{1}{2\pi i}\right)^2 \operatorname{tr}(\Lambda^2 X).$$

Since X is skew, then its trace vanishes, so $c_1(X) = 0$. By looking at the characteristic polynomial of X we see that $\operatorname{tr}(\Lambda^2 X) = \lambda^2$, giving

$$c_2(X) = -\frac{\lambda^2}{4\pi^2}.$$

Thus

$$y = p_1(X) = \frac{\lambda^2}{4\pi^2}.$$

On the other hand, X is similar over $\mathbb{C}$ to

$$\begin{pmatrix} -i\lambda & 0 \\ 0 & i\lambda \end{pmatrix},$$

so

$$\Pi_f(X) = f\left(-\frac{\lambda}{2\pi}\right) f\left(\frac{\lambda}{2\pi}\right) = g\left(\frac{\lambda^2}{4\pi^2}\right) = g(y)$$

as required. ⊠

Two important examples are the $\widehat{A}$-genus, which is the Pontrjagin genus associated to the holomorphic function

$$z \longmapsto \frac{\sqrt{z}/2}{\sinh(\sqrt{z}/2)}$$

and the Hirzebruch L-genus, which is the Pontrjagin genus associated with the holomorphic function

$$z \longmapsto \frac{\sqrt{z}}{\tanh(\sqrt{z})}.$$

Lecture 4.

Clifford algebras, Clifford bundles, and Dirac operators: 1/31/18

Ricky spoke today about Clifford algebras and Clifford bundles.

Let k be a field with characteristic not equal to 2. If V is a vector space over k, its *tensor algebra* is

$$T(V) := \bigoplus_{k\geq 0} V^{\otimes k}, \tag{4.1}$$

where $V^{\otimes 0} := k$.

Definition 4.2. Let V be a k-vector space with a quadratic form $Q\colon V\times V\to k$. Let $I_Q\subset T(V)$ denote the two-sided ideal generated by elements of the form $v\otimes v+Q(v)$ for $v\in V$. Then, the quotient algebra

$$C\ell(V,Q) := T(V)/I_Q$$

is called the *Clifford algebra* of V and Q.

Example 4.3. The zero function is a quadratic form, so $C\ell(V,0)$ is $T(V)/(v\otimes v=0)$, which is just the exterior algebra $\Lambda(V)$ of V. ◀

There is a natural map $i\colon V\to C\ell(V,Q)$ which is the composition

$$V = V^{\otimes 1} \hookrightarrow T(V) \xrightarrow{\pi_Q} C\ell(V,Q).$$

Lemma 4.4. *$i\colon V\to C\ell(V,Q)$ is injective.*

This is not too hard to check.

Moreover, 1 and V generate $C\ell(V,Q)$, subject to the relations $v^2=q(v,v)$. To get a smaller set of generators, we can choose a basis of V. *From now on, we assume Q is positive definite and choose an orthonormal basis $e_1,\dots,e_n$ of V.* In this case, $C\ell(V,Q)$ is generated by $1,e_1,\dots,e_n$ subject to the relations

$$\begin{gathered} e_i^2 = 1 \\ e_j\cdot e_j = -e_j\cdot e_i, \end{gathered} \tag{4.5}$$

because $e_i^2=-Q(e_i)=-1$. This implies the following fact.

Proposition 4.6. *The set*

$$\{e_{i_1}\cdots e_{i_k} \mid 1\leq i_1<\cdots<i_k\leq n, 1\leq k\leq n\}$$

is a basis for $C\ell(V,Q)$ as a vector space, and hence

$$\dim C\ell(V,Q) = 2^n = \sum_{k=0}^n \binom{n}{k}.$$

Example 4.7. Let $\langle\cdot,\cdot\rangle$ denote the standard inner product on $\mathbb{R}^n$; then, $C\ell_n := C\ell(\mathbb{R}^n,\langle\cdot,\cdot\rangle)$. In low dimensions these are familiar.

- $C\ell_1 = \langle 1\rangle\oplus\langle e\rangle$ with $e^2=1$, hence $C\ell_1\cong\mathbb{C}$.
- $C\ell_2 = \langle 1\rangle\oplus\langle e_1\rangle\oplus\langle e_2\rangle\oplus\langle e_1\cdot e_2\rangle$ with $e_1^2=e_2^2=(e_1e_2)^2=-1$, so as $\mathbb{R}$-algebras, $C\ell_2\cong\mathbb{H}$, the quaternions.

There is a sense in which real Clifford algebras are 8-fold periodict, which is an instance of *Bott periodicity*. We won't delve into this, but see Atiyah-Bott-Shapiro, "Clifford modules," for more information. ◀

Clifford algebras satisfy a universal property.

Proposition 4.8. *Let A be a k-algebra and $\varphi\colon V\to A$ be a map of vector spaces such that $\varphi(v)^2=-Q(v)\cdot 1_A$. Then, there is a unique algebra map $\widehat{\varphi}\colon C\ell(V,Q)\to A$ making the following diagram commute:*

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[dr, hook, "i"'] & A. \\ & C\ell(V,Q) \arrow[u, "\widehat{\varphi}", "\exists!"'] \end{tikzcd}$$

Proof. By the universal property of $T(V)$, there's a unique map $\psi\colon T(V)\to A$ sending

$$v_1\otimes\cdots\otimes v_k \longmapsto \varphi(v_1)\varphi(v_2)\cdots\varphi(v_k). \tag{4.9}$$

The claim follows because $\psi(I_Q)=0$, hence factors through the quotient, which is $C\ell(V,Q)$. ⊠

Let QVect_k denote the category of *quadratic spaces* over k, i.e. vector spaces together with quadratic forms; the morphisms $\varphi\colon (V_1,Q_1)\to(V_2,Q_2)$ are data of a linear map $\varphi\colon V_1\to V_2$ such that for all $v,w\in V_1$,

$$Q_1(v,w)=Q_2(\varphi(v),\varphi(w)). \tag{4.10}$$

Using Proposition 4.8, one can show that $C\ell\colon \mathsf{QVect}_k\to\mathsf{Alg}_k$ is a functor.

Another use of the universal property is to define a representation

$$\rho_{C\ell}\colon \mathrm{O}_n\to \mathrm{Aut}(C\ell_n);$$

a map $\mathbb{R}^n\to\mathbb{R}^n$ respecting the inner product defines a map $C\ell_n\to C\ell_n$, and the space of these maps is O_n.

Definition 4.11. Let (V,Q) be a quadratic space. A *Clifford module* over (V,Q) is a k-vector space S together with a k-linear map $\varphi\colon C\ell(V,Q)\to\mathrm{End}_k(S)$.

So it's just a module over the algebra $C\ell(V,Q)$.

Remark. When S is a complex vector space and V is a real vector space, then we will instead ask for the action map to be an $\mathbb{R}$-algebra homomorphism $C\ell(V,Q)\to\mathrm{End}_{\mathbb{C}}(S)$. This is equivalent to having a module over $C\ell(V,Q)\otimes_{\mathbb{R}}\mathbb{C}$. ◀

Remark. By the universal property, it suffices to specify a map $\varphi\colon V\to\mathrm{End}_k(S)$ with $\varphi(v)^2=-Q(v)\cdot\mathrm{id}$. ◀

Example 4.12.

(1) Let's consider $C\ell_1$ as a module over itself. This is the same data as a map $\varphi\colon C\ell_1\to\mathrm{End}_{\mathbb{C}}(\mathbb{C})\cong\mathbb{C}$; one choice is $1\mapsto\mathrm{id}$ and $e\mapsto i$.

(2) We can also make $\mathbb{R}^4$ into a $C\ell_2$-module by having it act on itself by left multiplication. For example, e_1 acts by the matrix

$$\begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0\end{pmatrix}.$$ ◀

Now, we apply this to geometry. Let (M,g) be a Riemannian manifold, so for each $p\in M$, $C\ell(T_pM,g_p)\cong C\ell_n$.

Definition 4.13. The *Clifford tangent bundle* is $C\ell(TM):=\mathscr{P}_{\mathrm{O}}(M)\times_{\mathrm{O}_n}C\ell_n$, where $\mathscr{P}_{\mathrm{O}}(M)$ is the principal O_n-bundle of orthonormal frames on M.

More generally, if $S\to M$ is any complex vector bundle, we can equip S with a Clifford action $c\colon C\ell(TM)\to\mathrm{End}_{\mathbb{C}}(S)$ in a similar way.

Definition 4.14. Let $S\to M$ be a complex vector bundle with a Hermitian metric $\langle\cdot,\cdot\rangle$ and a connection $\nabla\colon C^\infty(TM)\otimes C^\infty(S)\to C^\infty(S)$. This data $(S,\langle\cdot,\cdot\rangle,\nabla)$ defines a *Clifford bundle* if

- for all $X\in C^\infty(TM)$ of unit norm and $s_1,s_2\in C^\infty(S)$, $\langle X\cdot s_1,X\cdot s_2\rangle=\langle s_1,s_2\rangle$ iff $\langle X\cdot s_1,s_2\rangle+\langle s_1,X\cdot s_2\rangle=0$ and
- for all $X,Y\in C^\infty(TM)$ and $s\in C^\infty(S)$, $\nabla^L_X(Y\cdot s)=(\nabla_XY)\cdot s+Y\cdot\nabla_Xs$, where ∇^L denotes the Levi-Civita connection.[6]

Now we need to take a brief detour into something called synchronous frames.

Definition 4.15. Let (M,g) be an n-dimensional Riemannian manifold and $\varphi\colon U\to\mathbb{R}^n$ be a chart for M containing some $y\in U$. Let $e_1,\dots,e_n$ be an orthonormal basis of $T_yM\dots$

TODO: I couldn't figure out what happened here. Sorry. I'll have to fix this later.

Remark. The exponential map gives a canonical choice for a local neighborhood on a Riemannian manifold. ◀

[6] TODO: I am not completely sure I wrote this down correctly.

Dirac operators.

Definition 4.16. Let $S \to M$ be a Clifford bundle. The *Dirac operator* $D\colon C^\infty(S) \to C^\infty(S)$ is the composition

$$C^\infty(S) \xrightarrow{\nabla} C^\infty(T^*M \otimes S) \xrightarrow{g} C^\infty(TM \otimes S) \xrightarrow{C\ell} C^\infty(S).$$

In a neighborhood of a point $x \in M$, choose a local orthonormal frame $e_1, \dots, e_n$. Let $e^i := g(e_i, -)$ be the dual frame. Then, the Dirac operator in coordinates looks like

$$s \longmapsto \nabla_{(\cdot)} s = \sum e^i \otimes \nabla_{e_i} s \longmapsto \sum e_i \otimes \nabla_{e_i} s \longmapsto \sum_i e_i \cdot \nabla_{e_i} s. \tag{4.17}$$

Example 4.18. Let $\mathbb{E}^n$ denote *Euclidean space*, i.e. $\mathbb{R}^n$ with the usual flat metric. If V is a complex vector space, it canonically defines a complex vector bundle $\underline{V} \to \mathbb{E}^n$ by translation. Let $e_1, \dots, e_n$ be the standard orthonormal frame on $T\mathbb{R}^n$.

Let $\gamma := c(e_i)$, where $c\colon C\ell(TM) \to \mathrm{End}_{\mathbb{C}}(\underline{V})$ denotes the Clifford bundle action. Then, the Dirac operator $D\colon C^\infty(\underline{V}) \to C^\infty(\underline{V})$ is

$$D = \sum_i \gamma_i \cdot \partial_i, \tag{4.19}$$

where ∂_i is the usual partial derivative operator. The γ_i satisfy the *anticommutation relations*

$$\{\gamma_i, \gamma_j\} := \gamma_i \cdot \gamma_j + \gamma_j \cdot \gamma_i = -2\delta_{ij}. \tag{4.20}$$

Specifically, if $V = \mathbb{C} = C\ell_1$, then $e_1 \mapsto i$, so $\gamma = i$. Therefore $D = i\frac{\partial}{\partial x}$. ◀

The Dirac operator is self-adjoint. Well, it's *formally self-adjoint*: $C^\infty(S)$ is not a Hilbert space, so we can't talk about self-adjointness strictly speaking. One way to abrogate this problem is to take some kind of L^2 completion, but then it's an unbounded operator, so things are still a little complicated. Anyways, we'll talk about this in a bit.

Definition 4.21. Let M be a closed Riemannian manifold and V be its volume form. Then, there is an inner product on $C^\infty(S)$ defined by

$$\langle s_1, s_2 \rangle := \int_M \langle s_1(x), s_2(x) \rangle \, \mathrm{d}V.$$

Theorem 4.22. *The Dirac operator on a closed manifold is formally self-adjoint.*

Proof. That is, we want to prove that

$$\langle Ds_1, s_2 \rangle = \langle s_1, Ds_2 \rangle. \tag{4.23}$$

We will compute this locally in a synchronous frame $e_1, \dots, e_n$ for a chart in X. Then

$$\begin{aligned}
(Ds_1, s_2) - (s_1, Ds_2) &= \sum_i ((e_i \nabla_i s_1, s_2) - (s_1, e_i \nabla_i s_2)) \\
&= \sum_i \left(\nabla_i (e_i \cdot s_1) - \underbrace{(\nabla_i e_i)}_{=0}|_x \cdot s_1, s_2 \right) - (s_1, e_i \nabla_i s_2) \\
&= \sum_i \nabla_i (e_i \cdot s_1, s_2) \\
&= \sum \partial_{e_i} (e_i \cdot s_1, s_2) \\
&= \mathrm{d}^* \omega,
\end{aligned}$$

where

$$\omega_x := -\sum (e_i \cdot s_1, s_2) e^i.$$

Hence the difference is $\langle 1, \mathrm{d}^* \omega \rangle = \langle d1, \omega \rangle = 0$. ⊠

There's a local-vs.-neighborhood argument to make here, but this is the idea.

Lecture 5.

The Weitzenbock formula: 1/31/18

The next talk was by Ivan, on more Clifford bundles and Dirac operators.

Definition 5.1. Let $S \to M$ be a Clifford bundle on a Riemannian manifold (M, g), $A \in \Omega^2_M(\operatorname{End} S)$, and $\{e_i\}$ be a local synchronous orthonormal frame for M. The *Clifford contraction* of A is $\mathbb{A} \in \Omega^0_M(\operatorname{End} S)$ defined by the local formula

$$\mathbb{A} \cdot s := \sum_{i<j} c(e_i)c(e_j)A(e_i, e_j) \cdot s.$$

One should check this is independent of the choice of frame, but that is true.

Let $\mathbb{K}$ denote the Clifford contraction of the curvature for ∇ on S.

Theorem 5.2 (Weitzenbock formula)**.** *Let D denote the Dirac operator of $S \to M$. Then*

$$D^2 = \nabla^*\nabla + \mathbb{K}.$$

$\nabla^*\nabla$ is called the *covariant Laplacian*, and D^2 the *Dirac Laplacian*.

Proof. Let $p \in M$. Then

$$\begin{aligned} D^2 s|_p &= \sum_{i,j} e_i \cdot \nabla_{e_i}\Big(e_j \cdot \nabla_{e_j} s\Big)\Big|_p \\ &= \sum_{i,j} e_i \cdot e_j \cdot \nabla_{e_i}\nabla_{e_j} s|_p. \end{aligned}$$

Splitting this into the cases $i = j$ and $i \neq j$, we get

$$\begin{aligned} &= -\sum_i \nabla^2_{e_i} s|_p + \sum_{i<j} e_i e_j\Big(\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i}\Big)|_p \\ &= \nabla^*\nabla s|_p + \mathbb{K}s|_p. \end{aligned}$$

This uses the fact that we're on a synchronous frame, so $\nabla_{[e_i,e_j]} = 0$, and therefore the curvature simplifies to $\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i}$ as we used above. ⊠

This formula will be crucial for us, allowing us to supplant some of the general theory of elliptic operators in the proof of the index theorem.

Corollary 5.3 (Bochner theorem)**.** *Let $S \to M$ and $\mathbb{K}$ be as above. If $(\mathbb{K}s, s) > 0$ at some point then there are no nontrivial solutions to $D^2 s = 0$.*

Positivity makes sense because $\mathbb{K}$ is a Hermitian operator on a bundle which is fiberwise Hermitian.

Proof. Suppose that $D^2 s = 0$ and $s \neq 0$. Then, $\nabla^*\nabla s + \mathbb{K}s = 0$, so

$$0 = \underbrace{\|\nabla s\|^2}_{\geq 0} + \underbrace{\int_M \langle \mathbb{K}s, s\rangle \mathrm{d}V}_{>0}, \tag{5.4}$$

which is a contradiction. ⊠

Theorem 5.5. *Let $S \to M$ be as above, and K denote the curvature of ∇ on S. Then, $K = R^s + F^s$, where*

$$\begin{aligned} R^s(X, Y) &:= \frac{1}{4}\sum_{i,j} c(e_i)c(e_j)\langle R(X, Y)e_i, e_j\rangle \\ [F^s(X, Y), c(Z)] &= 0. \end{aligned}$$

R^s is usually called the *Riemann endomorphism*, and only depends on the Riemannian metric of the base manifold. F^s is called the *twisting curvature*.

We'll prove Theorem 5.5 in a series of lemmas. First, we find an obstruction for k being a Clifford module endomorphism.

Lemma 5.6.

$$[K(X,Y),c(Z)] = c(R(X,Y)Z).$$

Proof. Let $\{e_i\}$ denote a synchronous frame at a $p \in M$. Then it suffices to prove the lemma for $X = e_i$, $Y = e_j$, and $Z = e_k$. Since we're in a synchronous frame, the $\nabla_{[e_i,e_j]}$ component of the curvature vanishes, so

$$\begin{aligned} K(e_i,e_j)e_k \cdot s|_p &= \left(\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i}\right)(e_k \cdot s)\Big|_p \\ &= \left(R(e_i,e_j)e_k\right)\cdot s + e_k \cdot K(e_i,e_j)s\Big|_p. \end{aligned}$$

The result follows because[7]

$$\nabla_{e_i}(e_k)\nabla_{e_j} s|_p = 0. \qquad \boxtimes$$

Lemma 5.7.

$$[R^s(X,Y),c(Z)] = c(R(X,Y)Z).$$

Proof. Again let $\{e_i\}$ be an orthonormal frame, $X = e_i$, $Y = e_j$, and $Z = e_k$. Then

$$R^s(e_i,e_j)e_k \cdot s = \frac{1}{4}\sum_{\ell,m} c(e_\ell e_m e_k)\underbrace{\langle R(e_i,e_j)e_\ell, e_m\rangle}_{R_{m\ell ij}} s,$$

and similarly

$$c(e_k)R^s(e_i,e_j)s = \frac{1}{4}\sum_{\ell,m} c(e_k e_\ell e_m) R_{m\ell ij} s.$$

Hence when we put these together, we get

$$[R^s(e_i,e_j),c(e_k)] = \frac{1}{4}\sum_{\ell,m} c([e_\ell e_m, e_k]) R_{m\ell ij} s. \tag{5.8}$$

If $\ell = m$ and ℓ, m, and k are distinct, then $[e_\ell e_m, e_k] = 0$, so we only care about the cases $\ell = k \neq m$ and $k = m \neq \ell$.

Both the commutator and $R_{m\ell ij}$ are antisymmetric under the exchange of m and ℓ, so (5.8) reduces to

$$\begin{aligned} [R^s(e_i,e_j),c(e_k)] &= \frac{1}{2}\sum_\ell c([e_\ell e_k, e_k]) R_{k\ell ij} s \\ &= \sum_\ell c(e_\ell) R_{\ell kij} s \\ &= c\left(\sum_\ell R_{\ell kij} s\right). \end{aligned}$$

Since we're working in an orthonormal frame, the metric looks like the identity matrix in coordinates, so

$$\begin{aligned} &= c\left(\sum_\ell R^\ell_{kij} s\right) \\ &= c(R(e_i,e_j)e_k)s. \qquad \boxtimes \end{aligned}$$

These two lemmas suffice to prove Theorem 5.5.

Remark. Before we go on, let's review Ricci and scalar curvature, which we'll need. Let (M,g) be a Riemannian manifold. Its *Ricci curvature* is the map $\mathrm{Ric}\colon TX \times TX \to \mathbb{R}$ defined by

$$\mathrm{Ric}(X,Y) := \mathrm{tr}(Z \longmapsto R(Z,X)Y).$$

Why this trace? You could try others, but they all vanish or give you the Ricci curvature up to a sign!

Raising an index, define $\mathscr{R}\mathrm{ic}\colon TX \to TX$ by

$$\mathrm{Ric}(X,Y) = g(X, \mathscr{R}\mathrm{ic}(Y)).$$

Then, the *scalar curvature* of (M,g) is $\kappa := \mathrm{tr}(\mathscr{R}\mathrm{ic})$. In an orthonormal frame, it has the formula

$$\kappa = \sum_j \mathrm{Ric}_{jj}. \qquad \triangleleft$$

[7] TODO: maybe I missed something.

Theorem 5.9 (Improvement on Theorem 5.2)**.** *With notation as in Theorem 5.2, let $\mathbb{F}^s$ denote the Clifford contraction of F^s and κ denote the scalar curvature of (M, g). Then*

$$D^2 = \nabla^*\nabla + \mathbb{F}^s + \frac{\kappa}{4}\mathbf{1}_{\operatorname{End} S}.$$

Proof. By Theorem 5.5, $\mathbb{K} = \mathbf{R}^s + \mathbb{F}^s$, where $\mathbf{R}^s$ is the Clifford-contracted Riemann endomorphism. So all we have to show is that $\mathbf{R}^s = (\kappa/4)\mathbf{1}_{\operatorname{End} S}$. Again we compute in an orthonormal basis:

$$\begin{aligned}
\mathbf{R}^s &= \sum_{i<j} c(e_i)c(e_j)R^s(e_i, e_j)s\\
&= \frac{1}{2}\sum_{i,j} c(e_ie_j)R^s(e_i, e_j)s\\
&= \frac{1}{8}\sum_{i,j,k,\ell} c(e_ie_je_ke_\ell)\langle R(e_i, e_j)e_k, e_\ell\rangle s\\
&= \frac{1}{8}\sum_{i,j,k,\ell} c(e_ic_je_ke_\ell)R_{\ell kij}s.
\end{aligned}$$

If you decompose this into parts where various subsets of $\{i, j, k, \ell\}$ are equal to each other, the Bianchi identities allow you to simplify this sum:

$$\begin{aligned}
&= \frac{1}{4}\sum_{\ell,i,j} c(e_ic_je_ie_\ell)R_{\ell iij}s\\
&= \frac{1}{4}\sum_{\ell,j} c(e_je_\ell)\underbrace{\left(\sum_i R^i_{\ell ij}\right)}_{\mathrm{Ric}_{\ell j}}\cdot s.
\end{aligned}$$

If $\ell \neq j$, the Ricci tensor piece is antisymmetric, so does not contribute to the sum. Hence we only get the case where $k = \ell$:

$$= \frac{1}{4}\sum_j \mathrm{Ric}_{jj}\,\mathbf{1}_{\operatorname{End} S},$$

and this is indeed the scalar curvature. ⊠

Now we'll give an example of a Clifford bundle on a non-flat space.

Example 5.10. Let (M, g) be a closed Riemannian manifold and $S := \Lambda^*(T^*M)\otimes\mathbb{C}$. The Riemannian metric on M induces a Riemannian metric on Λ^*T^*M, hence a Hermitian metric on its complexification; similarly, the Levi-Civita connection induces a connection on Λ^*T^*M and therefore also on its complexification. Since the Levi-Civita connection is compatible with the metric on M, our induced connection is compatible with the Hermitian metric on S.

We define the Clifford action $c\colon TM \to \operatorname{End} S$ to satisfy

$$c(e)^2 = -g(e, e)\mathbf{1}_{\operatorname{End} S}, \tag{5.11}$$

which characterizes it uniquely. Namely, if ω is a k-form and $e \in \Gamma_M(TM)$,

$$c(e)\cdot\omega = \widetilde{e}\wedge\omega - e \lrcorner\, \omega,$$

where the first term is a $(k + 1)$-form and the second is a $(k - 1)$-form. Then

$$\begin{aligned}
c(e)^2\cdot\omega &= \underbrace{\widetilde{e}\wedge\widetilde{e}\wedge\omega}_{=0} - \widetilde{e}\wedge(e \lrcorner\, \omega) - e \lrcorner(\widetilde{e}\wedge\omega) + \underbrace{e \lrcorner(e \lrcorner\, \omega)}_{=0}\\
&= \underbrace{-e\wedge(e \lrcorner\, \omega)}_{=0} - (\underbrace{e \lrcorner\, \widetilde{e}}_{g(e,e)})\wedge\omega + \underbrace{\widetilde{e}\wedge(e \lrcorner\, \omega)}_{=0}\\
&= -g(e, e)\omega.
\end{aligned}$$

There are more things to check, including

$$g(e\cdot\omega_1, \omega_2) + g(\omega_1, e\cdot\omega_2) = 0, \tag{5.12}$$

which is left as an exercise, and the fact that

$$\nabla_X(e \cdot \omega) = (\nabla_X e) \cdot \omega + e \cdot (\nabla_X \omega). \tag{5.13}$$

One relatively quick way to prove it is to establish that

$$e \lrcorner \omega = (-1)^? \star(\widetilde{e} \wedge \star\omega). \tag{5.14}$$

This implies

$$\begin{aligned}
\nabla_X(e \lrcorner \omega) &= (-1)^? \star \nabla_X(\widetilde{e} \wedge \star\omega) \\
&= (-1)^? \star (\nabla_X(\widetilde{e}) \wedge \star\omega + \widetilde{e} \wedge \star\nabla_X \omega) \\
&= (\nabla_X e) \lrcorner \omega + e \lrcorner (\nabla_X \omega).
\end{aligned}$$

Hence

$$\begin{aligned}
\nabla_X(\widetilde{e} \wedge \omega - e \lrcorner \omega) &= (\nabla_X \widetilde{e}) \wedge \omega + \widetilde{e} \wedge \nabla_X \omega - \nabla_X(e \lrcorner \omega) \\
&= \nabla_X \widetilde{e} \wedge \omega + e \wedge \nabla_X \omega - (\nabla_X e) \lrcorner \omega - e \lrcorner \nabla_X \omega \\
&= (\nabla_X e) \cdot \omega + e \cdot (\nabla_X \omega),
\end{aligned}$$

proving (5.13). Neither side of (5.13) depends on the metric, and in fact there should be a proof that doesn't use it either.

Now we compute the Dirac operator. Let $\{e_i\}$ be a synchronous frame at p. Then

$$\begin{aligned}
D\omega|_p &= \sum_i e_i \cdot \nabla_{e_i} \omega \\
&= \sum_i e^i \wedge \nabla_{e_i} \omega \bigg|_p - e_i \lrcorner \nabla_{e_i} \omega|_p \\
&= \mathrm{d}\omega|_p - \sum_i * - 1)^? \star (e^i \wedge \nabla_{e_i} \star \omega)|_p \\
&= \mathrm{d}\omega|_p + \mathrm{d}^*\omega|_p.
\end{aligned}$$

This implies $D = \mathrm{d} + \mathrm{d}^*$, which you can again show in a more abstract way. The Laplacian is

$$\Delta := D^2 = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d},$$

called the *Hodge Laplacian*. This in particular exists on any Riemannian manifold, without any reference to Clifford bundles. ◀

There are a few more theorems proven in similar ways to the other ones above.

Theorem 5.15. *Restricted to* $\Lambda^1 T^*M$, $\Delta = \nabla^*\nabla + \mathrm{Ric}$.

Theorem 5.16 (Brchner vanishing theorem)**.** *If* (M, g) *is a Riemannian manifold,* $\mathrm{Ric} \geq 0$*, and* $\mathrm{Ric} > 0$ *at some point, then* $H^1_{\mathrm{dR}}(M) = 0$.

This one uses some Hodge theory (e.g. identifying harmonic representatives of de Rham cohomology classes).