ALGEBRAIC TOPOLOGY

ARUN DEBRAY
AUGUST 13, 2015

Abstract. This talk was given by Hang Lu Su at Dropbox, and typeset by me (Arun Debray). Please drop me an email at a.debray@gmail.com if you find any typos or errors.

"So I fell into a ditch while trying to picture this, so the point is, it takes a while."

First, we'll define a topological space.

Definition. A *topological space* is a set X with a collection τ of subsets such that

- $X, \emptyset \in \tau$,
- if $x_i \in \tau$ for some $i \in I$, then $\bigcup_{i\in I} x_i \in \tau$, and
- $\bigcap_{i=1}^N x_i \in \tau$ if $x_1, \ldots, x_N$ are.

The elements of τ are called *open sets*. Usually, people think of them as neighborhoods, but not always: for example, one could take $X = \{1, 2, 3\}$ and $\tau = \{\emptyset, \{1, 2\}, X\}$, which doesn't seem too neighborhood-like.

The open sets of a topological space are what is called a *topological property*, in that a homeomorphism between them preserves open sets. For example, mathematicians notoriously can't tell coffee cups and donuts apart, because as topological spaces, they're identical.

Definition.

- A map $(X, \tau_X) \to (Y, \tau_Y)$ is *continuous* if the preimage of an open set is open; that is, if $S \in \tau_Y$, then $f^{-1}(Y) \in \tau_X$.
- A *homeomorphism* between two topological spaces $X \to Y$ is a continuous, bijective mapping such that the image of an open set is open.

This means that a homeomorphism exactly preserves the structure of the open sets of X and Y, so topologically they're "the same."

For example, the injection $(0, 1) \hookrightarrow \mathbb{R}$, given by the map sending $x \mapsto x$.

Hausdorfness is another important topological property.

Definition. A topological space X is *Hausdorff* if for any $x, y \in X$, they can be separated by disjoint open sets; that is, there exist open sets U_x and U_y such that $x \in U_x$, $y \in U_y$, and $U_x \cap U_y = \emptyset$.

Another common term is a *neighborhood* of an $x \in X$, which is just an open set containing X.

There's another important property topological property called *compactness*, which we won't discuss in detail, but is an important finiteness condition.[1]

A *closed manifold* is a topological space that is compact and locally looks like $\mathbb{R}^n$ for some n (called the *dimension* of the manifold), i.e. every point has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^n$.

The real line isn't compact, and therefore isn't a closed manifold, but it is a manifold. Note that "closed" doesn't mean the complement of open!

Up to homeomorphism, there is only one closed one-dimensional manifold, the circle S^1. For two-dimensional manifolds, we need the notion of *orientability*, which can be a nuanced thing to define. Basically, a manifold is orientable if you can continuously choose a notion of "clockwise." The sphere is orientable, but the Möbius strip isn't.

We classify orientable surfaces by their genus: the sphere S^2 has genus 0, and the torus has genus 1. Then, genus 2 and up are tori but with more holes (e.g. for 2 it looks like 2 tori smushed together, and so on).

Interestingly, if you identify antipodal points of the orientable surface of genus n, you get the nonorientable surface of genus n: for S^2, we get a surface called the projective plane; for the torus, we get the Klein bottle; and so on. (These are already quite hard to visualize!)[2]

[1] Formally, any open cover has a finite subcover, but in this case, it means you can think of it as having finite area, volume, etc.

[2] Interesting question: how does one define an antipodal point on a surface of genus 2 or higher?

So why does the torus give us the Klein bottle? Look carefully at what happens when antipodal points are identified: it's like gluing the ends of a half-torus together. However, they have opposite orientations, so instead of a torus, we have to twist one end around, giving us the familiar Klein bottle.

We can realize these with *CW complexes*, which will make computing fundamental groups much easier. Intuitively, a CW complex is a bunch of n-balls, glued at their boundaries to $(n-1)$-balls. For example, we can inductively construct S^n as a sequence of CW complexes.

S^0 is two 0-balls; then, S^1 can be made by taking two lines, or 1-balls, and gluing them together at their boundaries. S^2 can be made by gluing together two disks (2-balls) along their boundary. Then, we can imagine S^3 similarly, even if we can't visualize it directly.

Since CW complexes are basically inductively defined, they lend themselves very nicely to proofs by induction and the like.

We can also construct the torus as a CW complex, in a slightly more complicated way. Start with a single 0-cell and glue two 1-cells to it, to create a figure-8. Then, attach a 2-cell to these; that is, gluing opposite sides together with the same orientation. If you had done this with a hexagon, you always get a torus, but if you used an octagon, you can also get the real projective plane.

It turns out all closed manifolds can be given a CW complex structure.

The Fundamental Group. Another important topological property is the *fundamental group*, which is interesting because out of all of the properties we've discussed, it's the first actually algebraic property.

Intuitively, the fundamental group $\pi_1(X, x_0)$ of a connected space X at point x_0 is the set of paths in X from x_0 to itself, up to path homotopy (i.e. a continuous transformation of one path to another).

For example, on $\mathbb{R}^2 \setminus 0$, a path around the origin isn't path homotopic to a trivial path, because it's not possible to go around the puncture (the missing origin). But more interestingly, looping it around twice isn't homotopic to either going around 0 or 1 times, and so on. When you make the logic more rigorous, this shows there's a loop for every number of times you can go around the hole, with sign corresponding to direction. Thus, $\pi_1(\mathbb{R}^2 \setminus 0, x_0) = \mathbb{Z}$.

A *path* is defined as a continuous map $[0,1] \to X$, so we can concatenate paths, and in fact this is exactly the group multiplication operation. This is associative and preserves homotopies, but it isn't commutative. Then, the inverse of a path is just its reverse.

The fundamental group isn't always abelian; for example, the fundamental group of the plane minus two points is the free group on two generators: going around each hole can't be untangled.

However, the torus has fundamental group $\langle a, b \mid aba^{-1}b^{-1} \rangle$, which is just $\mathbb{Z} \times \mathbb{Z}$. Intuitively, this is because we can go around the donut or around the hole, and the two loops are "independent."

We can understand this a lot easier with CW complexes: the 1-cells are the generators, and the 2-cells are the relations! That is, start going around the 2-cell in a direction, and if the 1-cell a has the same orientation as you're traveling in, add a; otherwise, add a^{-1}. For example, the fundamental group of the genus-2 orientable surface is $\langle a, b, c, d \mid abcda^{-1}b^{-1}c^{-1}d^{-1} \rangle$.

This has the other interesting consequence that every group G has a presentation, so you can make a CW complex whose fundamental group is G.

We can then talk about the Brower fixed-point theorem. This will use a tool known as a deformation retraction, which is also super helpful for calculating fundamental groups.

Definition.

- A *retraction* is a continuous map r from a topological space X to a subspace $A \subseteq X$ such that $r|_A = \mathrm{id}_X$. For example, a disc retracts onto a disc half its size by shrinking the rest of it onto the boundary.
- A *deformation retraction*is "even more continuous" in that it's a map $X \times [0,1 \to A$, where $A \subseteq X$. The copy of $[0,1]$ is what topologists call the *time coordinate*. For example, a filled-in letter **A** deformation retracts onto its skeleton A.

Retracting a circle onto a point is an example of a retraction that isn't a deformation retraction, since it can't be made continuous on $[0,1]$.

The reason we like these is because if X deformation retracts onto A, then A and X have the same fundamental group! This is because if such a retraction exists, then each loop is homotopic to something in the retracted subspace, and this identification is unique up to path homotopy.[3]

For example, suppose I want to compute the fundamental group of a torus, it retracts onto a figure-8. But the fundamental group of the figure-8 is not too hard to find; it's the free group on two generators.

The converse isn't true.

[3]You may be wondering how hard it is to make all of this rigorous. The answer is, not particularly easy, but it can be a bit of a chore.

Exercise. The Möbius strip has the same fundamental group as its boundary; however, prove that it doesn't deformation retract onto its boundary.

The fundamental group is something called a functor, which comes from the world of category theory. Specifically, it's a functor from the category of pointed topological spaces to the category of groups.

Definition. A *functor* F from a category C to a category D associates to each object $X \in \mathsf{C}$ to an object $F(X) \in \mathsf{D}$, and associates each morphism $f : X \to Y$ in C to a morphism $F(f) : F(X) \to F(Y)$ in D, such that:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ (i.e. it respects the identity), and
- $F(g \circ f) = F(g) \circ F(f)$, so it preserves compositions. Here, $f : X \to Y$ and $g : Y \to Z$ are morphisms in C.

The morphisms in the category of topological spaces are continuous maps, but a continuous map $f : X \to Y$ also sends paths to paths and preserves path homotopies, so it induces a map on fundamental groups, and one can check this is a group homomorphism. Moreover, the identity is the path that stays at a given point, so it is sent to the identity again, and you can check that composition is preserved. (This can get tricky, because you have path concatenation and function composition in the same expressions, so notation can be a hazard!)

We'll need a lemma, but it's a cool lemma, and it'll use functoriality.

Lemma 1. *There is no retraction from the disk D^2 to its boundary S^1.*

Proof. Assume there is a retraction $r : D^2 \to S^1$; then, using the inclusion $g : S^1 \hookrightarrow D^2$, they compose to the identity $r \circ g = \mathrm{id}$ on S^1.

Note that the fundamental group of the circle or the disk at any point is the same, so we can just say $\pi_1(D^2)$ and $\pi_1(S^1)$. So we have a diagram:

$$\begin{tikzcd} S^1 \arrow[r, hook, "g"] \arrow[d, mapsto] & D^2 \arrow[r, "r"] \arrow[d, mapsto] & S^1 \arrow[d, mapsto] \\ \pi_1(S^1) \arrow[r] & \pi_1(D^2) \arrow[r] & \pi_1(S^1) \end{tikzcd}$$

We know the composition of the maps on the top is the identity on S^1, so the composition of the maps on the bottom must be the identity $\mathbb{Z} \to \mathbb{Z}$ as well. But $\pi_1(D^2)$ is trivial, and a map $\mathbb{Z} \to 0 \to \mathbb{Z}$ must be the zero map, so it cannot be the identity. ⊠

Now, we can attack Brower's fixed-point theorem.

Theorem 2 (Brower's fixed-point theorem)**.** *For any continuous map $f : D^2 \to D^2$, there exists an $x \in D^2$ such that $f(x) = x$, i.e. x is a* fixed point.

Proof. Suppose for the sake of contradiction that there exists an $f : D^2 \to D^2$ that is continuous and such that $f(x) \neq x$ for all $x \in D^2$.

Thus, it's possible to draw a line between x and $f(x)$. Let $s(x)$ be the point on the intersection of this line and the boundary of the disc that is closer to x. Then, s is a retraction. Oops. ⊠