FURUTA'S 10/8 THEOREM

ARUN DEBRAY
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These notes were taken in a learning seminar on Furuta's 10/8 theorem in Spring 2019. I live-TEXed them using vim, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

Contents

1. Introduction to Seiberg-Witten theory: 1/23/19

Riccardo gave the first, introductory talk.

In 1982, Matsumoto conjectured that if M is a closed spin manifold, $b_2(M) \geq (11/8)|\sigma(M)|$. Here $b_2(M)$ is the second Betti number and $\sigma(M)$ is the signature. Equality holds for the K3 surface, so this is the best one can do.

In this seminar we'll study a theorem of Furuta which makes major progress on this conjecture.

Theorem 1.1 (10/8 theorem [Fur01])**.** *If the intersection form of M is indefinite, $b_2(M) \geq (10/8)|\sigma(M)| + 2$.*

If the intersection form is definite, work of Donaldson [Don83] says that, up to a change of orientation, the intersection form is diagonalizable, so that case is dealt with.

Furuta's proof uses both Seiberg-Witten theory and equivariant homotopy theory. It can be pushed a little bit farther, but not enough to prove the $11/8^{\text{ths}}$ conjecture, as shown recently by Hopkins-Lin-Shi-Xu [HLSX18].

Today we'll discuss some background for the proof.

Definition 1.2. Let $V \to M$ be a rank-n real oriented vector bundle. A *spin structure* on V is data $\mathfrak{s} = (P_{\mathrm{Spin}}(V), \tau)$, where $P_{\mathrm{Spin}}(V) \to M$ is a principal Spin_n-bundle and τ is an isomorphism

$$\tau\colon P_{\mathrm{Spin}}(V) \times_{\mathrm{Spin}_n} \mathbb{R}^n \xrightarrow{\cong} V.$$

A spin structure on a manifold M is a spin structure on TM.

Remark 1.3. There are other equivalent definitions of spin structures – for example, just as an orientation is a trivialization of V over the 1-skeleton of M, a spin structure is equivalent to a trivialization over the 2-skeleton. ◀

Here's a cool theorem about spin manifolds.

Theorem 1.4 (Rokhlin [Roh52])**.** *If M is a spin manifold, $\sigma(M) \equiv 0 \bmod 16$.*

The signature makes sense when $4 \mid \dim M$. Smoothness is crucial here; there are topological spin 4-manifolds, whatever that means, that do not satisfy this theorem. Freedman's E_8 manifold is an example.

Suppose M is a spin 4-manifold. The representation theory of Spin_4, in particular the fact that the spin representation S splits as $S^+ \oplus S^-$, leads to two quaternionic line bundles $\mathbb{S}^+, \mathbb{S}^- \to M$ with Hermitian metrics. Physics cares about these bundles, and will lead to powerful theorems in manifold topology.

These bundles have more structure: in particular, they are Clifford bundles.

Definition 1.5. Let $S \to M$ be a real vector bundle with a Euclidean metric $\langle \cdot, \cdot \rangle$. A *Clifford bundle* structure is data of, for each $x \in M$, the data of a Clifford algebra action $C\ell(T_x M)$ on S_x that varies smoothly in x, such that the Clifford action is skew-adjoint, meaning

$$\langle v \cdot s_1, s_2 \rangle = -\langle s_1, v \cdot s_2 \rangle.$$

We also require the existence of a connection which is compatible with the Levi-Civita connection on TM.

Given the data of a Clifford bundle, there's an operator called the *Dirac operator* D, which is the following composition:

$$C^\infty(S) \xrightarrow{\nabla^{C\ell}} C^\infty(T^*M \otimes S) \xrightarrow{\langle \cdot, \cdot \rangle} C^\infty(TM \otimes S) \xrightarrow{\text{Clifford action}} C^\infty(S). \tag{1.6}$$

This operator is denoted $\not{\partial}$, a convention due to Feynman. It is a first-order, elliptic differential operator; ellipticity means that its analysis is nice.

Thus we can consider the *Seiberg-Witten equations* on a spin 4-manifold. Let $(a, \varphi) \in \Omega^1_M(i\mathbb{R}) \times \Gamma(\mathbb{S}^+)$; then the equations are

$$\not{\partial}\varphi + \rho(a)(\varphi) = 0 \tag{1.7a}$$

$$\rho(\mathrm{d}^+ a) - \varphi \otimes \varphi^* + \frac{1}{2}|\varphi^2|\mathrm{id} = 0 \tag{1.7b}$$

$$\mathrm{d}^* a = 0. \tag{1.7c}$$

On a non-spin manifold, the equations are a little more complicated.

References

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