M392C NOTES: A COURSE ON SEIBERG-WITTEN THEORY AND 4-MANIFOLD TOPOLOGY

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These notes were taken in UT Austin's M392C (A course on Seiberg-Witten theory and 4-manifold topology) class in Spring 2016, taught by Tim Perutz. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Any mistakes in the notes are my own.

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Classification problems in differential topology: 1/18/18

"This is my opinion, but it's the only reasonable opinion on this topic."

This course will be on gauge theory; specifically, it will be about Seiberg-Witten theory and its applications to the topology of 4-manifolds. The course website is https://www.ma.utexas.edu/users/perutz/GaugeTheory.html; consult it for the syllabus, assignments, etc.

The greatest mystery in geometric topology is: *what is the classification of smooth, compact, simply-connected four-manifolds up to diffeomorphism?* The question is wide open, and the thoery behaves very differently than the theory in any other dimension.

There's a fascinating bit of partial information known, mostly via PDEs coming from gauge theory, e.g. the *instanton equation* $F_A^+ = 0$ as studied by Donaldson, Uhlenbeck, Taubes, and others. More recently, people have also studied the *Seiberg-Witten equations*

$$D_A\psi = 0 \tag{1.1a}$$

$$\rho(F_A^+) = (\psi \otimes \psi^*)_0. \tag{1.1b}$$

Even without defining all of this notation, it's evident that the Seiberg-Witten equations are more complicated than the instanton equation, and indeed they were discovered later, by Seiberg and Witten in 1994. However, they're much easier to work with — after their discovery, the results of Donaldson theory were quickly reproven, and more results were found, within the decade after their discovery. This course will focus on results from Seiberg-Witten theory.

In some sense, this is a closed chapter: the stream of results on 4-manifolds has slowed to a trickle. But Seiberg-Witten theory has in the meantime found new applications to 3-manifolds, contact topology (including the remarkable proof of the Weinstein conjecture by Taubes), knots, high-dimensional topology, Heegaard-Floer homology, and more. Throughout this constellation of applications, there are many results whose only known proofs use the Seiberg-Witten equations.

The central problem in differential topology is to classify manifolds up to diffeomorphism. To make the problem more tractable, let's restrict to smooth, compact, and boundaryless. An ideal solution would solve the following four problems for some class of manifolds (e.g. compact of a particular dimension, and maybe with some topological constraints).

(1) Write down a set of "standard manifolds" $\{X_i\}_{i\in I}$ such that each manifold is diffeomorphic to precisely one X_i. For example, a list of diffeomorphism classes of closed oriented connected surfaces is given by the sphere and the n-holed torus for all $n \geq 0$.
(2) Given a description of a manifold M, a way to compute invariants to decide for which $i \in I$ $M \cong X_I$. For example, if M is a closed, connected, oriented surface, we can completely classify it by its Euler characteristic.

    A variant of this problem asks for an explicit algorithm to do this when M is encoded with finite information, e.g. a solution set to polynomial equations in $\mathbb{R}^N$ with rational coefficients.
(3) Given M and M', compute invariants to decide whether M is diffeomorphic to M'; once again, there's an algorithmic variant to that problem.
(4) Understand families (fiber bundles) of manifolds diffeomorphic to M. In some sense, this means understanding the homotopy type of the topological group $\mathrm{Diff}(M)$ of self-diffeomorphisms of M.

This is an ambitious request, but much is known in low dimensions. In dimension 1, the first three questions are trivial, and the last is nontrivial, but solved.

Example 1.2. For compact, orientable, connected surfaces, we have a complete solution: a list of diffeomorphism classes is the sphere and $(T^2)^{\# g}$ for all $g \geq 0$, and the Euler characteristic $\chi \coloneqq 2 - 2g$ is a complete invariant which is algorithmically computable from any reasonable input data, solving the second and third questions. Here, "reasonable input data" could include a triangulation, a good atlas (meaning nonempty intersections are contractible), or monodromy data for holomorphic a branched covering map $\Sigma \to S^2$, where here we're thinking of surfaces as Riemann surfaces, with chosen complex structures. Here, the Riemann-Hurwitz formula can be used to compute the Euler characteristic.

For the fourth question, let $\mathrm{Diff}^+(\Sigma)$ denote the topological group of orientation-preserving self-diffeomorphisms of Σ.

Theorem 1.3 (Earle-Eells)**.**

- *The inclusion* $\mathrm{SO}_3 \hookrightarrow \mathrm{Diff}^+(S^2)$ *is a homotopy equivalence.*
- *The identification* $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$ *defined a map* $T^2 \hookrightarrow \mathrm{Diff}^+(T^2)$ *as translations; this map is a homotopy equivalence into the connected component of the identity in* $\mathrm{Diff}^+(T^2)$*, and* $\pi_0 \mathrm{Diff}^+(T^2) \cong \mathrm{SL}_2(\mathbb{Z})$*.*
- *If* $g > 1$*, every connected component of* $\mathrm{Diff}^+(\Sigma_g)$ *is contractible, and the* mapping class group $\mathrm{MCG}(\Sigma_g) \coloneqq \pi_0 \mathrm{Diff}^+(\Sigma_g)$ *is a finitely presented infinite group which acts with finite stabilizers on a certain contractible manifold called* Teichmüller space*.*[1]

So all four questions have satisfactory answers, though understanding the mapping class groups of surfaces is still an active area of research. ◂

Example 1.4. The classification of compact, orientable 3-manifolds looks remarkably similar to the classification of surfaces (albeit much harder!), through a vision of Thurston, realized by Hamilton and Perelman. The solution is almost as complete. The proof uses geometry, and nice representatives are quotients by groups acting on hyperbolic space.

As for invariants, the fundamental group is very nearly a complete invariant.[2] ◂

In higher dimensions, there are a few limitations. Generally, the index set I will be uncountable. For example, there are an uncountable number of smooth 4-manifolds homeomorphic to $\mathbb{R}^4$! So there will be no nice list, and no nice moduli space either. But restricted to compact manifolds, there are countably many classes, which follows from triangulation arguments or work of Cheeger in Riemannian geometry.

The next obstacle involves the fundamental group. If M is presented as an *n-handlebody* (roughly, a CW complex with cells of dimension at most n), there is an induced presentation of $\pi_1(M)$, and if M is compact, this is a finite presentation (finitely many generators, and finitely many relations).

Fact. For each $n \geq 4$, all finite presentations arise from compact n-handlebodies (namely, closed n-manifolds). ◂

This is pretty cool, but throws a wrench in our classification goal.

[1]Heuristically, but not literally, Teichmüller space is a classifying space for this group.
[2]The fundamental group cannot distinguish lens spaces, and that's pretty much the only exception.

Theorem 1.5 (Markov)**.** *There is no algorithm that decides whether a given finite group presentation gives the trivial group.*

The proof shows that an algorithm which could solve this problem could be used to construct an algorithm that solves the halting problem for Turing machines.

Corollary 1.6. *There is no algorithm to decide whether a given n-handlebody, $n \geq 4$, is simply connected.*

This means that a general classification algorithm cannot possibly work for $n \geq 4$; thus, we will have to restrict what kinds of manifolds we classify.

A third issue in higher-dimensional topology is that in dimension $n \leq 3$, there are existence and uniqueness theorems of "optimal" Riemannian metrics (e.g. constraints on their isometry groups), but for $n \geq 5$, this is not true for any sense of optimal; some choices fail existence, and others fail uniqueness. This is discussed further (and more precisely) in Shmuel Weinberger's "Computers, Rigidity, and Moduli," which has some very interesting things to say about the utility of Riemannian geometry to classify manifolds (or lack thereof).

So four dimensions is special, but for many reasons, not just one.

Those setbacks notwithstanding, we can still say useful things.

- We will restrict to closed manifolds.
- We will focus on the simply-connected case, eliding Markov's theorem.[3]

With these restrictions, we have good answers to the first three questions.

Example 1.7. There is a countable list of compact, simply-connected 5-manifolds, and invariants (cohomology, characteristic classes) which distinguish any two. ◀

Example 1.8. Kervaire-Milnor produced a classification of homotopy spheres in dimensions $5 \leq n \leq 18$, and a conceptual answer in higher dimensions, and further work has applied this in higher dimensions. ◀

There is a wider range of conceptual answers to all four questions, more or less explicit, through *surgery theory*, when $n \geq 5$ (surgery theory fails radically in dimension 4). This gives an answer to the following questions.

- Given a finite, n-dimensional CW complex X (where $n \geq 5$), when is it the homotopy type of a compact n-manifold?
- Given a simply-connected compact manifold M, what are the diffeomorphism types of manifolds homotopy equivalent to M? (Again, we need $\dim M \geq 5$.)

Here are necessary and sufficient conditions for the existence question.

- X must be an n-dimensional *Poincaré duality space*, i.e. there is a fundamental class $[X] \in H_n(X;\mathbb{Z})$ which implements the Poincaré duality isomorphism. This basic fact about closed manifolds gets you an incredibly long way towards the answer.
- Next, X must have a tangent bundle — but it's not clear what this means for a general Poincaré duality space. Here we mean a rank-n vector bundle $T \to X$ which is associated to the homotopy type in a certain precise sense: the unit sphere bundle of the stablization of T, considered as a spherical fibration, has to be manifest in X in a certain way.
- If $n \equiv 0 \bmod 4$, there's another obstruction — a certain $\mathbb{Z}$-valued invariant must vanish, interpreted as asking that $T \to X$ satisfies the Hirzebruch signature theorem: the signature of the cup product form on $H^{n/2}(X)$ must be determined by the Pontrjagin classes of T.
- If $n \equiv 2 \bmod 4$, the obstruction is a similar $\mathbb{Z}/2$-valued invariant related to the Arf invariant of the intersection form.
- If n is odd, there are no further obstructions.

That's it. Uniqueness is broadly similar — once you specify a tangent bundle, there are only finitely many diffeomorphism types!

Now we turn to dimension 4, the hardest case. We want to classify smooth, closed, simply-connected 4-manifolds. The first basic invariant (even of 4-dimensional Poincaré duality spaces) is the intersection form Q_P, which we'll begin studying in detail next week. You can realize it as a unimodular matrix modulo integral equivalence. That is, it's a symmetric square matrix over $\mathbb{Z}$ with determinant ± 1, and integral equivalence means up to conjugation by elements of $\mathrm{GL}_b(\mathbb{Z})$.

[3] More generally, one could pick some fixed group G and ask for a classification of closed n-manifolds with $\pi_1(M) \cong G$; people do this, but we won't worry about it.

Theorem 1.9 (Milnor). *The intersection form defines a bijection from the set of homotopy classes of 4-dimensional simply-connected Poincaré spaces to the set of unimodular matrices modulo equivalence.*

So this form captures the entire homotopy type! That's pretty cool.

Theorem 1.10 (Freedman). *The intersection form defines a bijection from the set of homeomorphism classes of 4-dimensional simply-connected topological manifolds to the set of unimodular matrices modulo equivalence.*

Thus this completely classifies (closed, simply-connected) topological four-manifolds. This theorem won Freedman a Fields medal.

The next obstruction, having a tangent bundle, is a mild constraint told to us by Rokhlin.

Theorem 1.11 (Rokhlin). *Let X be a closed 4-manifold. If Q_X has even diagonal entries, then its signature is divisble by* 16.

The signature is the number of positive eigenvalues minus the number of negative eigenvalues. Algebra tells us this is already divisible by 8, so this is just a factor-of-2 obstruction, which is not too bad.

But the rest of the story of surgery theory is just wrong in dimension 4. This is where analysis of an instanton moduli space comes in.

Theorem 1.12 (Donaldson's diagonalizability theorem). *Let X be a compact, simply-connected 4-manifold. If Q_X is positive definite, i.e. $xQ_Xx > 0$ for all nonzero $x \in \mathbb{Z}^b$, then Q_X is equivalent to the identity matrix.*

Donaldson proved this theorem as a second-year graduate student!

There's a huge number of unimodular matrices which are positive definite, but not equivalent to the identity; the first example is known as E_8. So this is a strong constraint on their realizability by 4-manifolds.

In subsequent years, Donaldson devised invarinats distinguishing infinitely many diffeomorphism types within a single homotopy class. Then, from 1994 onwards, there came new proofs of these results via Seiberg-Witten theory, which tended to be simpler,[4] and to provide sharper, more general results. We will prove several of these in the second half of the class.

[4]That said, Donaldson's original proof of the diagonalizability theorem stands as one of the most beautiful things in gauge theory.