2019 PCMI PREPARATORY LECTURES

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These notes were taken at UT Austin as part of a learning seminar in preparation for PCMI's 2019 graduate summer school. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

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1. BPS states: 5/21/19

Today, Shehper talked about BPS states in 4D $\mathcal{N} = 2$ supersymmetric theories. This is not the only place you can have BPS states, but this is probably the one most relevant to our interests. For a reference, check out Moore's PITP lectures on BPS states.[1]

First, 4D means the dimension of the theory: we have three space coordinates and one time coordinate. There's an underlying symmetry group called the *Poincaré group* of $\mathbb{R}^{1,3}$, whose Lie algebra is

$$\mathfrak{iso}_{1,3} \cong \mathfrak{so}^+_{1,3} \ltimes \mathbb{R}^{1,3}. \tag{1.1}$$

The "+" means that we want transformations to preserve the arrow of time. That is, these transformations correspond to changes between different reference frames. In one, we have local coordinates (t, x, y, z), and from one reference frame to another, the time coordinate t is scaled by something; we want this to be a nonnegative number. The transformations coming from $\mathfrak{so}^+_{1,3}$ are called *(orthochronous special) Lorentz transformations*, but we'll call them Lorentz transformations.

Another way to describe the Poincaré group is as the group of isometries of $\mathbb{R}^{1,3}$.

Now we should clarify what "underlying symmetry" means. This is a statement about QFT, which means we have to indicate how to actually discuss or work with QFT. There are a few different formalisms, e.g. the *Hamiltonian formalism* or *canonical quantization formalism*; or the *path integral formalism*, which comes with the following data:

- A space of *field configurations* $\mathcal{F}$.
- An *action*, a function $S\colon \mathcal{F} \to \mathbb{R}$.
- A set of *local operators*.

From this data one can compute correlation functions associated to local operators $\Phi_1, \dots, \Phi_n$ at points $x_1, \dots, x_n$ in spacetime via the path integral

$$\langle \Phi_1(x_1) \cdots \Phi_n(x_n)\rangle = \int_{\mathcal{F}} \mathcal{D}\varphi\, e^{-S(\varphi)} \Phi_1 \cdots \Phi_n. \tag{1.2}$$

Of course, this is not mathematically well-defined in general, but physicists have ways of working with it which agree extremely well with experimental data.

[1] The lecture notes can be found at http://www.sns.ias.edu/pitp2/2010files/Moore_LectureNotes.rev3.pdf.

The Hamiltonian formalism in a d-dimensional quantum field theory associates to a $(d-1)$-manifold a Hilbert space $\mathcal{H}$. Elements of $\mathcal{H}$ are called *states*, because they represent states of the physical system. Inside $\mathfrak{iso}_{1,3}$, there's an element P_τ which is time translation by τ: explicitly, under the isomorphism (1.1), these are the elements in $\mathbb{R}\cdot t \subset \mathbb{R}^{1,3}$. This element acts on $\mathcal{H}$ by the Hamiltonian, and this is how the system evolves under time. An eigenvector for the Hamiltonian with eigenvalue λ is said to have *energy* λ.

Assumption 1.3. There is a unique vector $|v\rangle \in \mathcal{H}$, called the *vacuum*, with minimum energy.

There's a sense in which the vacuum generates all of the states: one can act by local operators to obtain the other states. And in this formalism, the correlation functions are given by

$$\langle \Phi_1 \cdots \Phi_n \rangle \coloneqq \langle v \mid \Phi_1 \cdots \Phi_n \mid v \rangle. \tag{1.4}$$

Explicitly, assume that $\phi(x)$ is a *Lorentz scalar*, which means it's a field transforming in the trivial representation of $\mathfrak{so}^+_{1,3}$. Here x is position, i.e. the coordinate in the spacetime manifold.

Remark 1.5. A field is not an operator, but it does determine a local operator, e.g. ϕ, as a scalar field (function), has a value at a point x. We will think of ϕ as a local operator sometimes in what follows. ◂

How do we use this to create states in $\mathcal{H}$? The first step is to Fourier transform ϕ, leading to $\widetilde{\phi}(p)$. Now this depends on the momentum p. We can act on $|v\rangle$ by $\widetilde{\phi}$ to obtain other states in $\mathcal{H}$.[2] There are things which have positive momenta and with negative momenta; these should be thought of as particle creation $\widetilde{\phi}^\dagger$, resp. particle annihilation operators $\widetilde{\phi}$ on the space of states. This is analogous to the raising and lowering operators on $\mathfrak{su}_2$-representations.

The physical interpretation is that the vacuum has no particles and no momentum. Acting by one creation operator creates a single particle with a prescribed momentum. Acting by another means two particles, and so on.

Remark 1.6. All of this is in a *free theory*, meaning the action is quadratic in the fields. In general, the story is a little more complicated. ◂

Anyways, back to "underlying symmetry." This means the following.

- The fields are all in representations of $\mathfrak{iso}_{1,3}$ (i.e. governing how it transforms under a change of coordinates).
- The Hilbert space is a unitary representation of $\mathfrak{iso}_{1,3}$. Additionally, we want every operator to be unitary, i.e. $U^\dagger U = \mathbf{1}$.

This means that Poincaré symmetries do not change the norm of states, which is important.

Example 1.7. Here are some irreducible representations of $\mathfrak{so}^+_{1,3}$.

- The *trivial* or *scalar representation* $\mathbb{C}$.
- The *vector representation*, which is the defining representation of $\mathfrak{so}^+_{1,3}$ on $\mathbb{R}^{1,3}\otimes\mathbb{C}$.
- The *tensor representations*, which are obtained from the vector representation by symmetric or exterior powers.
- The *spinor representations*, two 2-dimensional representations which are complex conjugates of each other, but are not isomorphic. In physics these are also called *Weyl spinors*; there's a different thing called a *Dirac spinor*, which transforms in the direct sum of the two spinor representations. ◂

So we've discussed what 4D QFT is. What does $\mathcal{N}=2$ mean? This is specifying "how much supersymmetry" is present in the theory. Supersymmetry means that we extend the Poincaré algebra to a $\mathbb{Z}/2$-graded Lie algebra (sometimes called a *Lie superalgebra*) $\underline{\mathfrak{g}} = \underline{\mathfrak{g}}^0 \oplus \underline{\mathfrak{g}}^1$. In our situation ($\mathcal{N}=2$), we'd like

$$\underline{\mathfrak{g}}^0 = \mathfrak{iso}_{1,3} \oplus \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_R \oplus \mathbb{C}, \tag{1.8}$$

where $\mathfrak{su}(2)_R$ denotes $\mathfrak{su}(2)$, but we write "R" to denote that this tracks something called *R-symmetry*, and likewise for $\mathfrak{u}(1)_R$ – $\mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_R$ is the *R-symmetry algebra*. Then, $\mathbb{C}$ is generated by an element Z called the *central charge* of the theory.

[2] Contextualizing this, and why we can think of this as associated to position and momentum, is really related to how quantum field theory arises via quantization from classical field theory.

Then, we want $\underline{\mathfrak{g}}^1$ to be a spinor representation of $\underline{\mathfrak{g}}^0$; specifically, for $\mathcal{N} = 2$,

$$\underline{\mathfrak{g}}^1 = (2, 1; 2)_{+1} \oplus (1, 2; 2)_{-1}. \tag{1.9}$$

The notation $(a, b; c)_d$ means the irreducible $\mathfrak{so}^+(1, 3)$-representation given by (a, b), the irreducible $\mathfrak{su}(2)_R$-representation of dimension c, and the irreducible $\mathfrak{u}(1)_R$-representation of weight d (i.e. the corresponding to the Lie group representation $\mathrm{U}_1 \to \mathrm{U}_1$ sending $z \mapsto z^d$). $\mathbb{C} \cdot Z$ and $\mathbb{R}^{1,3}$ act trivially.

Since $\underline{\mathfrak{g}}^1$ is odd, the Lie bracket restricted to $\underline{\mathfrak{g}}^1 \times \underline{\mathfrak{g}}^1$ is actually an anticommutator (or Poisson bracket), so it lives in $\mathrm{Sym}^2(\underline{\mathfrak{g}}^1)$.

Let $\{Q_\alpha^A\}$ be a basis of $(2, 1; 2)_{+1}$, where $\alpha \in \{1, 2\}$ and $A \in \{1, 2\}$; similarly, let $\{\overline{Q}_{\dot{\alpha} A}\}$ be a basis for $(1, 2; 2)_{-1}$. So we have eight basis elements in total; they're called *supercharges*.

Remark 1.10. The two-dimensional irreducible representation of $\mathfrak{su}(2)_R$ is pseudoreal. There's a notion of a complex representation being *real*, which means that it's self-conjugate – or at least, the representation and its conjugate are related through a symmetric matrix. A representation is *pseudoreal* if instead we have an antisymmetric matrix: $(M^a)^\dagger = \epsilon^{ab} M_b$ (here ϵ is the Levi-Civita tensor). ◀

The point is that complex conjugation identifies some of these basis vectors, so we have to impose the relation

$$(Q_\alpha^A)^\dagger = \overline{Q}_{\dot{\alpha} A}. \tag{1.11}$$

Once we've imposed this, we have a real 8-dimensional representation.

We can specify the commutation relations between the supercharges:

$$\{Q_\alpha^A, \overline{Q}_{\dot{\beta} B}\} = 2\sigma^m_{\alpha\dot{\beta}} P_m \delta^A{}_B \tag{1.12a}$$

$$\{Q_\alpha^A, Q_\beta^B\} = 2\epsilon_{\alpha\beta}\epsilon^{AB}\overline{Z} \tag{1.12b}$$

$$\{\overline{Q}_{\dot{\alpha} A}, \overline{Q}_{\dot{\beta} B}\} = -2\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{AB} Z. \tag{1.12c}$$

Once we understand $\mathrm{Sym}^2 \underline{\mathfrak{g}}^1$ as a representation, we can analyze this and learn, e.g. that $\sigma^m_{\alpha\dot{\beta}} P_m$ transforms in the $(2, 2)$ representation of $\mathfrak{so}^+_{1,3}$.

Definition 1.13. A *4D $\mathcal{N} = 2$ supersymmetric quantum field theory* is a QFT with an underlying symmetry algebra $\underline{\mathfrak{g}}$.

To construct BPS states, we need some representations of $\underline{\mathfrak{g}}$. We'll do this by finding an analogue of the Casimir operator inside $\mathfrak{iso}_{1,3}$ – an operator which commutes with all other operators. Explicitly, it's

$$P^2 := -P_0^2 + P_1^2 + P_2^2 + P_3^2. \tag{1.14}$$

This mimics the $\mathfrak{su}_2$ story, where the Casimir is the sum of the squares of the three Pauli matrices. In physics, P^2 is also thought of as the mass squared. For example, if the momentum is zero, this relates to the familiar equation $E^2 = M^2 c^2$ – in general momentum changes this.

Now one can choose a particular basis in which $P = (M, 0, 0, 0)$, called the *rest frame*. One place you might want this is if you want a state with particular momenta $M^\mu = (P^0, P^1, P^2, P^3)$, and can obtain it from the rest frame by a Lorentz transformation.

Anyways, once you have $(M, 0, 0, 0)$, you can act on it by $\mathfrak{so}_3$ in the last three coordinates, which produces more things of the same mass. So to create "massive" irreducible representations of $\mathfrak{iso}_{1,3}$ with a fixed mass $M > 0$, we need to look for representations of $\mathfrak{so}_3 \oplus \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_R \oplus \mathbb{C}$ as follows: we want eight generators R_α^A and T_α^A such that $\{R, R\} \neq 0$, $\{T, T\} \neq 0$, $\{R, T\} = 0$, such as

$$\{R_\alpha^A, R_\beta^B\} = 4(M - |Z|)\epsilon_{\alpha\beta}\epsilon^{AB} \tag{1.15a}$$

$$\{T_\alpha^A, T_\beta^B\} = -4(M + |Z|)\epsilon_{\alpha\beta}\epsilon^{AB} \tag{1.15b}$$

$$\{R_\alpha^A, T_\beta^B\} = 0. \tag{1.15c}$$

So we have two copies of a Clifford algebra. Explicitly, if $\zeta \in \mathfrak{u}(1) \setminus 0$,

$$R_\alpha^A := \zeta^{-1} Q_\alpha^A + \zeta \sigma^0_{\alpha\beta} \overline{Q}^{\dot{\beta} A} \tag{1.16a}$$

$$T_\alpha^A := \zeta^{-1} Q_\alpha^A - \zeta \sigma^0_{\alpha\dot{\beta}} \overline{Q}^{\dot{\beta} A}. \tag{1.16b}$$

These have reality constraints coming from those of the supercharges, e.g. $(R_1^1)^\dagger = -R_2^2$ and $(R_1^2)^\dagger = R_2^1$. This means

$$(R_1^1 = (R_1^1)^\dagger)^2 = (R_1^2 + (R_1^2)^\dagger)^2 = 4\left(M + \mathrm{Re}\left(\frac{Z}{\zeta^{-2}}\right)\right). \tag{1.17}$$

This is important for unitarity: we want $A^\dagger = A$: we want $\|A|\psi\rangle\|^2 > 0$ if $\psi \neq 0$, so we want $A^2 \geq 0$.

Suppose we choose $\zeta^{-2} = -Z/|Z|$; then the right-hand side of (1.17) simplifies to $4(M - |Z|)$. Therefore we want $M \geq |Z|$, which is called the *BPS bound*. (Other choices of ζ give you weaker constraints.) That is, in any state in a 4D $\mathcal{N} = 2$ supersymmetric theory, the mass of any state is at least $|Z|$.

There are two cases: $M = |Z|$, which is called a *BPS state*, and $M > |Z|$, which is called a *non-BPS state*. If $M = |Z|$, $\{R_\alpha^A, R_\beta^B\} = \pm 4(M - |Z|) = 0$, which acts trivially, so for BPS states, we only get one copy of the Clifford algebra (called a *short representation* rather than the usual *long representation* with two copies). In particular, the T_α^A split into creation and annihilation operators, and we get four states: $|v\rangle$, $T_\alpha^\beta|v\rangle$, $T_\gamma^\delta|v\rangle$, and $T_\alpha^\beta T_\gamma^\delta|v\rangle$. In a non-BPS state, then we'd be able to create eight states instead of four.

Great, and why do we care about BPS states? In QFT, a lot of things can happen – QFTs usually come in families, meaning there are various parameters in a quantum field theory that one can adjust. In general these parameters vary over a moduli space. If you try to move in this moduli space, short representations do not usually combine into long representations, and usually stay as they are. So BPS states are relatively rigid – or said in other words, the Hilbert space of states can change, but the spectrum of BPS states is generally invariant. Moreover, we can compute it in important situations (which is not true for the general Hilbert space), thanks to work of Gaiotto-Moore-Neitzke. In mathematics, the ways of computing BPS states have to do with things called spectral networs, which are tied to the geometry of Riemann surfaces.

The BPS representations are generally of the form $\rho\otimes s$, where ρ is the representation of $\mathfrak{so}_3 \oplus \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_R$ that we began with, and s is a short representation; similarly, the non-BPS states are in ρ tensored with a long representation. So giving representations of $\mathfrak{su}(2)_R$ and $\mathfrak{u}(1)_R$ gives you new BPS representations.

2. 3-manifold topology: 5/22/19

"You just glue noodles and pancakes to basketballs, and that's it."

Today, Charlie spoke about low-dimensional topology. Here "low-dimensional" means in dimensions 2 through 4. Today's goal is to cover surgery presentations of 3-manifolds and when two of them are equivalent. Often, interesting 3-manifold invariants are defined or calculated via surgery diagrams (such as the Witten-Reshetikhin-Turaev invariants, in particular); showing that you get an invariant is a matter of checking that it stays the same under those equivalences.

But first, dimension 2.

Definition 2.1. Let Σ be a closed, oriented surface. Let $\mathrm{Diff}^+(\Sigma)$ denote the topological group of orientation-preserving diffeomorphisms and $\mathrm{Diff}_0^+(\Sigma)$ denote the connected component of the identity, which is a normal subgroup. The *mapping class group* of Σ, denoted $\mathrm{MCG}(\Sigma)$, is $\mathrm{Diff}^+(\Sigma)/\mathrm{Diff}_0^+(\Sigma)$.

We could also have defined this using homeomorphisms instead of diffeomorphisms, and we get the same mapping class group.[3]

Theorem 2.2 (Dehn, Lickorish)**.** $\mathrm{MCG}(\Sigma)$ *is finitely generated, and is generated by Dehn twists.*

So let's talk about Dehn twists.

Definition 2.3. Let $a\colon S^1 \to \Sigma$ be an embedding and $N \cong S^1 \times I$ be a tubular neighborhood of I. The *Dehn twist* associated to a, denoted $T_a\colon \Sigma \to \Sigma$, is the (equivalence class in $\mathrm{MCG}(\Sigma)$ of) a diffeomorphism which is the identity on $\Sigma \setminus N$, and which on $N \cong S^1 \times I$ is the map

$$T_a\colon (\theta, t) \longmapsto (\theta + 2\pi t, t). \tag{2.4}$$

See Figure 1 for a picture. There's a refinement of Theorem 2.2 producing explicit generators for $\mathrm{MCG}(\Sigma)$: if Σ is connected and g denotes the genus of Σ, then we can take $3g - 1$ generators. Writing Σ as a connected sum of g tori, we can take the Dehn twists associated to two curves generating the homology of each torus, together with one other family. This is shown in the mapping class group book (*A Primer on Mapping Class Groups*).

[3]The mapping class group generalizes to other manifolds, but this fact presumably doesn't.

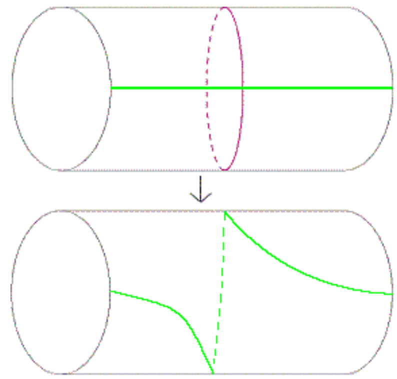

FIGURE 1. A Dehn twist. Source: https://en.wikipedia.org/wiki/Dehn_twist.

Definition 2.5. Assume Σ is connected. A curve $a\colon S^1 \hookrightarrow \Sigma$ is *separating* if $\Sigma \setminus a(S^1)$ has two components. A curve is *simple* if it doesn't intersect itself.

Question 2.6. Let Σ_g denote the closed, connected, oriented surface of genus g and $a_1, \dots, a_g, b_1, \dots, b_g \subset \Sigma_g$ be nonseparating simple closed curves such that no a_i and a_j intersect and no b_i and b_j intersect. Is there a diffeomorphism f such that $f \circ a_g = b_g$?

It seems reasonable, and in fact is true. One common trick for thinking about curves like this is – consider $\Sigma_g \setminus (a_1 \cup \dots \cup a_g)$ and $\Sigma_g \setminus (b_1 \cup \dots \cup b_g)$. These are compact surfaces, so we know their classification (each connected component is classified by its genus and number of boundary components), and you can check that these invariants match on each connected component, so there must be a diffeomorphism – and then you can extend that across the curves you removed.

Now we'll talk a little bit about handle decompositions.

Definition 2.7. A *d-dimensional k-handle* is a disc $D^d \cong D^k \times D^{n-k}$ glued to a d-manifold along the boundary $S^{k-1} \times D^{d-k} \subset \partial(D^k \times D^{d-k})$.

For example, you can build the torus out of handles: begin with a (two-dimensional) zero-handle, then attach two (two-dimensional) one-handles, then a (two-dimensional) two-handle. This is actually an instance of a general fact.

Theorem 2.8. *Any closed d-manifold can be built by attaching a series of d-dimensional handles to a d-dimensional* 0*-handle.*

Such a description is called a *handle decomposition* of the manifold.

Example 2.9. Let's write down a handle decomposition of $\mathbb{CP}^2$. Using homogeneous coordinates, we can write $\mathbb{CP}^2 = X \cup Y \cup Z$, where

$$X = \{[1 : y : z] : |y| \le 1, |z| \le 1\} \tag{2.10a}$$

$$Y = \{[x : 1 : z] : |x| \le 1, |z| \le 1\} \tag{2.10b}$$

$$Z = \{[x : y : 1] : |x| \le 1, |y| \le 1\}. \tag{2.10c}$$

These are the handles in a handle decomposition of $\mathbb{CP}^2$. Topologically, each is a $D^2 \times D^2 \cong D^4$. But which piece is which handle depends on the order you glue them in.

If we start with X, it's the zero-handle. Let's next glue in Y: $X \cap Y \cong \{[1 : y : z] : |y| = 1, |z| \le 1\}$. Thus $y \in S^1$ and $z \in D^2$, so $X \cap Y \cong S^1 \times B^2$, and therefore gluing Y to X along their intersection is attaching a 4-dimensional 2-handle. Finally let's glue in Z. Since $\partial Z = S^3$, this is attaching a 4-handle. ◂

Theorem 2.8 provides a way of classifying manifolds, at least in principle – above $d = 2$ it's intractable. But in dimension 2, it allows one to show that closed, connected surfaces are classified by whether the surface is orientable and how many handles are attached.

When $d = 3$, this is still useful, though: on a closed, connected 3-manifold, we begin with a 0-handle and some 1-handles, which maybe look like noodles that you attach to the disc. The 2-handles now look like pancakes (note: $d = 3$, so these pancakes are thick). The pancakes (i.e. 2-handles) are determined by circles on the boundary of the 1-*handlebody* (i.e. just the 0- and 1-handles glued in), and then where the 3-handle is uniquely determined. That is: given a 1-handlebody with a bunch of circles on the boundary, we know how to get a 3-manifold M out of it. Such a description is called a *Heegaard diagram* of M.

In fact, we can get more out of this. On any handlebody, you can glue the handles in reverse order, in which case k-handles become $(d-k)$-handles; this is called the *reverse handle decomposition.* So the Heegard diagram of M defines two 1-handlebodies: the one we made from the 0- and 1-handles, and the one made from the 2- and 3-handles, but for the reverse handle decomposition. These two handlebodies H_1 and H_2 have the same number of 1-handles (this number is called the *genus* of the 1-handlebody), and M is H_1 glued to H_2 across their common boundary.

Definition 2.11. A *Heegaard splitting* of a 3-manifold M is a decomposition $M = H_1 \cup H_2$, where H_1 and H_2 are 1-handlebodies and $H_1 \cap H_2$ is an embedded connected surface in M.

So we've just argued that Heegaard splittings always exist. If $\Sigma \coloneqq H_1 \cap H_2$ has genus g, you can represent M by two lists of g embedded nonseparating, nonintersecting circles in Σ, which tells us how to glue H_1 to H_2.

A reasonable next question is – can any such diagram occur? In the answer to Question 2.6, we saw that the answer is yes: there is a diffeomorphism that allows us to glue them, and we obtain a 3-manifold.

Next question: when do two Heegaard diagrams determine the same 3-manifold? An isotopy of any of the embedded discs doesn't change the diffeomorphism type of M, but the converse is false: there are nonisotopic Heegaard diagrams which define the same 3-manifold.

This motivates the notion of a *handle slide.* Looking first at $d = 2$, you could take two handles which look like ∩∩, and move one "inside" the other to obtain something that looks like ⋒. This does not change the diffeomorphism type of the surface we obtain. The same thing works when $d = 3$ – and if you trace through what happens on the Heegaard diagram, you get nonisotopic curves, but the same 3-manifold! Handle slides define a useful equivalence relation on Heegaard diagrams – but there are additional diffeomorphisms between 3-manifolds presented as Heegaard diagrams that don't come from isotopies or handle slides.

Definition 2.12. A *framed circle* in a 3-manifold is an embedded circle $\gamma\colon S^1 \hookrightarrow M$ together with a trivialization of its normal bundle.[4]

It's useful to think of a framing as a choice of normal vector field. Given a framed circle $\gamma\colon S^1 \hookrightarrow M$, let N be a tubular neighborhood of γ, which is a solid torus; let b be a longitude curve on ∂N (i.e. winding along γ) and a be a curve on ∂N winding once around γ. If you glue an $S^1 \times B^2$ to $M \setminus N$ along their boundary, where we can attach $S^1 \times D^2$ along some diffeomorphism, the resulting manifold only depends on the image of a under this diffeomorphism. Moreover, isotopic diffeomorphisms produce diffeomorphic 3-manifolds, so we really only need to ask about the image of the diffeomorphism in the mapping class group.

The mapping class group of T^2 is $\mathrm{SL}_2(\mathbb{Z})$, so $a \mapsto pa + qb$, where $p, q \in \mathbb{Z}$. One can show that the resulting 3-manifold, called *(Dehn) surgery in M along γ*, only depends on p/q. In particular, given a framed link in M, together with a rational number x_C on each component C, we get a new 3-manifold obtained by doing surgery in M along each component, with $p/q = x_C$.

Theorem 2.13. *Any closed, oriented 3-manifold is diffeomorphic to one arising as surgery on a link in S^3, and in fact we can let $p/q = \pm 1$.*

Proof sketch. Let $M = H_1 \cup H_2$ be a Heegaard splitting, where $\Sigma_g \coloneqq H_1 \cap H_2$ is a closed, connected, oriented surface of genus g. Using Theorem 2.2, we can describe the gluing along Σ_g, *a priori* an element of $\mathrm{MCG}(\Sigma_g)$ as a sequence of Dehn twists. You can think of this as a presentation of M as a sequence of bordisms: first H_1, then several copies of $\Sigma_g \times I$ glued via Dehn twists, then H_2. This kind of looks like a hamburger.

The key is to see that Dehn surgery and Dehn twists are related, which maybe isn't such a surprise given their names. Suppose in the i^{th} bordism we're gluing by a Dehn twist along the curve $a \in \Sigma_g$. A neighborhood of a looks like a thickened washer, and Dehn surgery by either 1 or -1 accomplishes the Dehn twist: a meridian curve goes once around a, in some direction, and once around the other generator. As you do successive Dehn surgeries, these curves can become linked. ⊠

[4]This definition goes through more generally for a framed submanifold in an ambient manifold.

You can think of the numbers on each component as specifying what combinations of loops you want to make contractible. The presentation in Theorem 2.13, called a *surgery diagram*, is generally not unique.

Later, we'll define a 3-manifold invariant called the *Reshetikhin-Turaev invariant*, associated to a surgery diagram. One must check that it's invariant under the equivalences, called *Kirby moves*, between surgery diagrams that generate equivalent 3-manifolds, and this is kind of a pain, but it works.

Remark 2.14. A surgery diagram whose coefficients p/q are integers also tells you how to make a compact, oriented 4-manifold W whose boundary is M: start with a D^4, so $S^3 = \partial D^4$, and attach 4-dimensional 2-handles along the components of the link in S^3; the framings needed to glue the handles are specified by the coefficients on each component of the link. ◀

This is important in the definition of the Reshetikhin-Turaev invariants: we'll have to use this 4-manifold in the definition, and the invariant in general depends on the signature of the 4-manifold we choose.

3. Khovanov homology: 5/23/19

Today, Will spoke about Khovanov homology, defining what it is and discussing how to compute it in some specific cases.

Khovanov homology is a *categorification* of the Jones polynomial – we'll see that in the end, the Jones polynomial of a link L is the graded Euler characteristic of Khovanov homology of L.

Definition 3.1. A *knot* is an embedding $K\colon S^1 \hookrightarrow S^3$ (sometimes the target is $\mathbb{R}^3$, which makes no difference). A *link* is an embedding of a closed 1-manifold into S^3, which need not be connected.

We begin with a *link diagram* D for a link L. That is, project down onto an $\mathbb{R}^2$, but where there's a self-intersection, indicate which was on top and which was below. See Figure 2 for some pictures. Our link diagrams are *oriented*, in that we've fixed a direction on each component. There can be different link diagrams for the same link.

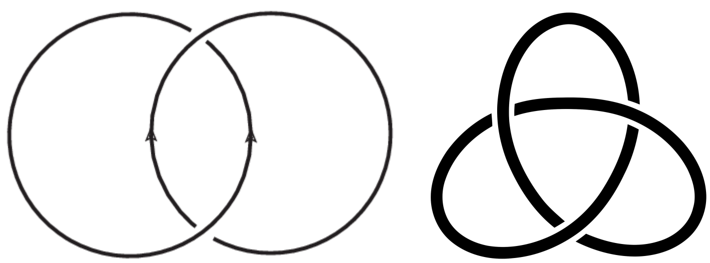

Figure 2. Link diagram for the Hopf link (left) and a trefoil (right). The left diagram is an oriented link diagram.

Definition 3.2. A self-intersection on a link diagram is called a *crossing*. We call it *positive* if, moving upwards, the left strand goes over the right strand, and otherwise call it *negative*. Given a link L, we will let $n_+ = n_+(L)$ denote the number of positive crossings and $n_- = n_-(L)$ denote the number of negative crossings.

For example, the Hopf link has $n_+ = 2$ and $n_- = 0$.

Since the goal of Khovanov homology is to refine the Jones polynomial, let's begin with the Jones polynomial. It's built from a related invariant, called the Kauffman bracket, which is defined via *smoothings*. Given a crossing on a knot diagram, one can replace it with either $=$ (called a *zero-smoothing*) or $\|$ (a *one-smoothing*). Iterating this process leads to an *unlink* (i.e. an embedding where no circle is knotted and no two circles are linked). Therefore we can recursively define knot invariants by specifying how they behave under smoothings and how they behave on unlinks.

Definition 3.3. The *Kauffman bracket* is an assignment $D \mapsto \langle D\rangle \in \mathbb{Z}[q, q^{-1}]$ of a link diagram defined recursively by the following relations:

- $\langle X \rangle = \langle = \rangle - q\langle \| \rangle$. This means that given a specific crossing X in a link L, the Kauffman bracket of L is the sum of the Kauffman bracket of the link where X is smoothed as $=$, minus q times the Kauffman bracket of the link where X is smoothed to $\|$.
- If U denotes the unknot, $\langle U^{\amalg k} \rangle = (q + q^{-1})^k$.

The *unnormalized Jones polynmial* is $\widehat{J}(D) \coloneqq (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle$, and the *Jones polynolmial* is $J(D) \coloneqq \widehat{J}(D)/(q+q^{-1})$.

Often people make the substitution $q = t^{1/2}$.

Example 3.4. Computing the Kauffman bracket in this way for the Hopf link (which I wasn't able to TEX; sorry!) shows that $\langle \text{Hopf} \rangle = q^4 + q^2 + 1 + q^{-2}$, so $\widehat{J}(\text{Hopf}) = q^6 + q^4 + q^2 + 1$. ◂

The things we've defined are clearly link diagram invariants, but a link does not have a unique diagram. How do we check this?

Theorem 3.5 (Reidemeister)**.** *Two link diagrams represent the same link iff they are related by a sequence of* Reidemeister moves*, which are the three transformations displayed in Figure 3.*

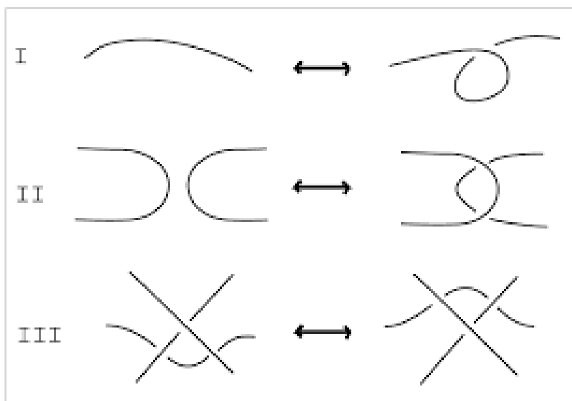

FIGURE 3. The three Reidemeister moves.

Therefore, any link diagram which is invariant under the three Reidemeister moves is in fact a link invariant.

Exercise 3.6. Use this to show that the Kauffman bracket is not a link invariant, but the Jones polynomial is.

If a link L has n crossings, then there are 2^n possible smoothings: each crossing can be resolved by a 0-smoothing or a 1-smoothing. We represent this by ordering the crossings, and then labeling each smoothing by an $\alpha \in \{0,1\}^n$. This will play into the grading of the Khovanov homology of L.

Given a smoothing α, let Γ_α denote L with all of those smoothings applied. This is necessarily an unlink; let k_α be the number of components of Γ_α. Also, let r_α denote the number of 1s in α.

We'll define a chain complex associated to L, and then the Khovanov homology will be its homology: first we assign a $\mathbb{Z}$-graded vector space to each link diagram, and then discuss the differential.

Let V denote the $\mathbb{Z}$-graded vector space spanned by two elements v_+ and v_-, where $\deg(v_\pm) = \pm 1$. The Khovanov chain complex for the unknot is V; for an unlink with k circles, we'll get $C_*((S^1)^{\amalg k}) \coloneqq V^{\otimes k}$.

Definition 3.7. Let $W = \bigoplus_i W^i$ be a graded vector space. The *quantum dimension* of W is

$$\operatorname{qdim}(W) \coloneqq \sum_{i \in \mathbb{Z}} \dim(W^m) q^m \in \mathbb{Z}[q, q^{-1}]. \tag{3.8}$$

For example, $\operatorname{qdim}(V) = q + q^{-1}$ and $\operatorname{qdim}(V^{\otimes k}) = (q+q^{-1})^k$. This looks like the Kauffman bracket, which is no coincidence. Shifting the grading corresponds to multiplying the quantum dimension by q, e.g. $\operatorname{qdim}(V[1]) = q^2 + 1$.

Now, for a general link L, the chain complex is

$$C_*(L) := \bigoplus_{\alpha \in \{0,1\}^n} C_*(\Gamma_\alpha)[r_\alpha + n_+ - 2n_-]. \tag{3.9}$$

That is, we sum together $V^{\otimes k_\alpha}$ over each α, but shifted.

The Khovanov chain complex $C_*(L)$ has an additional grading, called the *topological grading*, given by $i := r_\alpha - n_- \in \{-n_-, \dots, n_+\}$: increasing this degree corresponds to having more 1s in α.

Example 3.10. The Hopf link has two crossings, so we get four smoothings.

- $\alpha = 00$ yields two circles, separate from each other.
- $\alpha = 01$ and $\alpha = 10$ each yield one circle.
- $\alpha = 11$ yields two circles, one inside the other (though this is still the unlink).

So the Khovanov chain complex looks like this:

- in topological grading $i = 0$, we have $V^{\otimes 2}[2]$ from $\alpha = 00$,
- in topological grading $i = 1$, we have $V[3] \oplus V[3]$ from $\alpha = 01$ and $\alpha = 10$, and
- in topological grading $i = 2$, we have $V^{\otimes 2}[4]$ from $\alpha = 11$. ◁

Now we have to define the differential – the chain complex isn't a link invariant yet. First, we need some additional structure on V.

- Let $m\colon V \otimes V \to V$ be the linear map satisfying $v_+ \otimes v_+ \mapsto v_+$, $v_\pm \otimes v_\mp \mapsto v_-$, and $v_- \otimes v_- \mapsto 0$. This lowers the degree by 1.
- Let $\Delta\colon V \to V \otimes V$ be the linear map sending $v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+$ and $v_- \mapsto v_- \otimes v_-$.

The differential raises topological grading by 1. Let α, α' be smoothings which differ by a single entry, which α has as a zero and α' has as a 1. We think of this as a directed edge $e_{\alpha\alpha'}$ (sometimes denoted as α but with a $*$ in place of the changing entry) from α to α' in a directed graph of smoothings of L.

Given an edge $e_{\alpha\alpha'}$, going from Γ_α to $\Gamma_{\alpha'}$ either splits one circle into two, or combines two circles into one. Let

$$d_{\alpha\alpha'}\colon C_*(\Gamma_\alpha)[r_\alpha + n_+ + 2n_-] \longrightarrow C_*(\Gamma_{\alpha'})[r_{\alpha'} + n_+ - 2n_-] \tag{3.11}$$

be m, if $e_{\alpha\alpha'}$ combines two circles into one (so m is applied specifically to the copies of V coming from those circles); otherwise, it's Δ (again applied specifically to the copies of V coming from the circles that are changing). Then differential on the Khovanov chain complex is

$$d := \sum_{\text{tail of } e \text{ is } \alpha} \mathrm{sign}(e) d_e, \tag{3.12}$$

where $\mathrm{sign}(e)$ is equal to -1 to the number of 1s to the left of the index that's changing across e. Now, if you apply the differential twice, these signs cancel each other out, so $d^2 = 0$. Therefore we can define

Definition 3.13. The *Khovanov homology*, denoted $\mathrm{Kh}(L)$, of a link diagram is the homology of $(C_*(L), d)$.

Technically, calling this the Khovanov homology of L, rather than its diagram, is a little forward: we have to first show it's a link invariant. But this is true, and there are various ways to show this.

Remark 3.14. In fact, there are explicit examples where the Khovanov homologies of two links differ, but their Jones polynomials are the same! ◁

Example 3.15. Recall from Example 3.10 that the Khovanov chain complex for the (standard link diagram of the) Hopf link is

$$\underbrace{V^{\otimes 2}[2]}_{00} \longrightarrow \underbrace{V[3]}_{10} \oplus \underbrace{V[3]}_{01} \longrightarrow \underbrace{V^{\otimes 2}[4]}_{11}. \tag{3.16}$$

Now, the differential. Denote the elements of $V[3] \oplus V[3]$ in topological degree 1 by $x_\pm, y_\pm$, and those in $V^{\otimes 2}[4]$ in topological degree 2 by $z_\pm \otimes z_{+\|}$.

The first map must kill $v_- \otimes v_-$, so we get a generator in bidegre $(0,0)$. It also sends $v_\pm \otimes v_\mp \mapsto x_- + y_-$. But the second map sends $x_- \mapsto -z_- \otimes z_-$ and $y_- \mapsto z_- \otimes z_-$, so we only get something in the first column.

In degree $j = 4$, $v_+ \otimes v_+ \mapsto x_+ + y_+$. The second map sends $x_+ \mapsto -(z_+ \otimes z_- + z_- \otimes z_+)$ and y_+ maps to -1 times that, so again most of this goes away in homology, but there is now a generator in degree $(2,4)$.

Finally, $z_+ \otimes z_+$ in degree $(2,6)$ is a generator in homology. So the Khovanov homology of the Hopf link is $\mathbb{C}$ in grading $(0,0)$, $\mathbb{C}$ in grading $(0,2)$, $\mathbb{C}$ in grading $(2,4)$, and $\mathbb{C}$ in grading $(2,6)$. The graded Euler characteristic is $1+q^2+q^4+q^6$, so we get the (unnormalized) Jones polynomial, as we should. ◀

4. Bordism and the Pontrjagin-Thom construction: 6/4/19

To be factored in. In the meantime, see https://web.ma.utexas.edu/users/a.debray/lecture_notes/bordism.pdf.

5. Witten-Reshetikhin-Turaev invariants: 6/5/19

Today, Charlie spoke about the papers of Reshetikhin-Turaev, which sought to provide a detailed mathematical account of Witten's definition of Chern-Simons theory and its relationship to the Jones polynomial.

The input to the Reshetikhin-Turaev construction is a *ribbon category* C over a field k. This is designed to mathematically axiomatize the physics of Wilson line operators in 3D, though we're not building in the gauge group from the beginning.

- Given two Wilson lines, you can bring them close together and obtain a new one. This corresponds to a tensor product $\otimes\colon \mathsf{C}\times\mathsf{C}\to\mathsf{C}$.
- There is an empty Wilson line, which doesn't do anything under the tensor product, so we obtain a unit $I\in\mathsf{C}$. The data $(\mathsf{C},\otimes,I)$ (and a little more additional data) is a monoidal category.
- Wilson loops have a direction, and what happens when we reverse the direction? This means we want all objects $V\in\mathsf{C}$ to have *duals* $V^\vee\in\mathsf{C}$, giving us in particular a map $V^\vee\otimes V\to I$.
- In dimension 3, we can braid Wilson lines around each other, which defines a *braided monoidal structure* on $(\mathsf{C},\otimes,I)$: there is a natural transformation $c_{UV}\colon U\otimes V\to V\otimes U$ for $U,V\in\mathsf{C}$.
- The Wilson lines are framed, so it's possible to twist the framing by one full turn, defining a twist $\theta_v\colon V\to V$. When drawing pictures, it's convenient to represent the line with its framing as a ribbon, so twisting the ribbon corresponds to twisting the framing.

Working with ribbon categories directly can be a bit of a headache, so it's nice to realize them as categories of representations of something called a *ribbon Hopf algebra* A.

- The category of representations of an algebra does not always have a tensor product. We ask for a *comultiplication* map $\Delta\colon A\to A\otimes A$, which is coassociative, but not necessarily cocommutative. Then if U and V are A-representations, $A\otimes A$ acts on $U\otimes V$, and we define the A-action on $U\otimes V$ by $a\cdot(u\otimes v)\coloneqq\Delta(a)(u\otimes v)$.
- We need a *counit* map $\varepsilon\colon A\to k$ in order to define the unit object, which is A acting on k through ε.
- We need an *antipode* $\gamma\colon A\to A$ to define duals. (This antipode must satisfy some axioms expressing its compatibility with comultiplication and the counit.) If $\phi\in V^\vee\coloneqq\mathrm{Hom}_k(V,k)$, then $(a\cdot\phi)(x)\coloneqq\phi(\gamma(a)\cdot x)$ defines an A-algebra structure on $V^\vee$. This generalizes the definition of the dual representation of a finite group, where we ask g to act by g^{-1} on the dual, so that the pairing $V\otimes V^\vee\to k$ is equivariant: we get a factor of g and a factor of g^{-1}, which cancel.
- To obtain a braiding, we need an *R-matrix* $R\in A\otimes A$; then the braiding $c_{UV}(u\otimes v)\coloneqq\tau_{UV}R(u\otimes v)$, where $\tau\colon U\otimes V\to V\otimes U$ is the usual swap map. The swap map does not in general commute with the A-action, and you can think of the R-matrix as a correction term.
- The twist comes from a choice of central element $v\in A$: $\theta_v(u)\coloneqq v\cdot u$.

Example 5.1. There is a ribbon Hopf algebra $\mathcal{U}_q(\mathfrak{sl}_2)$, called the *quantized universal enveloping algebra* for $\mathfrak{sl}_2$, which is an algebra over $\mathbb{C}(q)$.[5] It's generated by elements K, K^{-1}, E, and F, satisfying the relations $KK^{-1}=K^{-1}K=1$ and

$$KEK^{-1}=q^2E \tag{5.2a}$$

$$KFK^{-1}=q^{-2}F \tag{5.2b}$$

$$[E,F]=\frac{K-K^{-1}}{q-q^{-1}}. \tag{5.2c}$$

[5] It's also possible to do this over $\mathbb{C}$, by replacing q with a generic complex number.

Here K replaces the standard $H \in \mathfrak{sl}_2$ – or, more precisely, we think of $K = e^{\hbar H}$, where $q = e^{\hbar}$. So K is sort of like a group element, but E and F are still eigenvalues, as if it were a Lie algebra element. In this sense, the quantum algebra mixes the group and algebra, which is a little weird.

To see the relationship with $\mathcal{U}(\mathfrak{sl}_2)$, take $\partial_\hbar$ at $\hbar = 0$:

$$e^{\hbar H} E e^{-\hbar H} = e^{2\hbar E} \quad \rightsquigarrow \quad [H, E] = 2E \tag{5.3a}$$

and similarly for $[H, F]$, and

$$(e^{\hbar} - e^{-\hbar})[E, F] = e^{-\hbar H} - e^{-\hbar H} \quad \rightsquigarrow \quad [E, F] = H. \tag{5.3b}$$

Formally, you can think of this as happening at $q = 1$, but to actually make sense of this one has to say it a little differently.

This defines $\mathcal{U}_q(\mathfrak{sl}_2)$ as an algebra; we would need to say more to get a ribbon Hopf algebra. ⋖

Remark 5.4. There's a different way to get at the category of representations of $\mathcal{U}_q(\mathfrak{g})$, which uses certain representations of the loop group of G. That these are equivalent is a theorem of Kazhdan and Lusztig, though at least to us it's not clear way. Maybe there are only so many ways to deform Rep_G. ⋖

We'll finish today's lecture with *graphical calculus*, a way to work concretely with objects in a ribbon category C.

Let Rib_C denote the category of ribbons labeled by objects of C. Explicitly, the objects of Rib_C are finite tuples of elements of C, and the morphisms are *ribbon graphs.* A ribbon graph from $(V_1, \dots, V_m)$ to $(V_1', \dots, V_m')$ is a finite union of oriented embeddings of $[0,1] \times [0,1]$ in $\mathbb{R}^2 \times [0,1]$. We think of $[0,1] \times [0,1]$ as a ribbon, so the first coordinate is along the length of the ribbon and the second is the width. Hence ribbons can twist, be braided, or be knots. Ribbons can form loops, or start and end at $\mathbb{R}^2 \times \{0\}$ or $\mathbb{R}^2 \times \{1\}$; if it starts at 0 or ends at 1, we want to label these endpoints by some V_i (at 0) or V_i' (at 1). If it starts at 1 or ends at 0, we label by the duals.

Given a framed knot, we get a ribbon graph from $\varnothing$ to $\varnothing$, which will evaluate to an element of $\mathbb{C}(q)$. For $\mathsf{C} = \mathsf{Rep}_{\mathcal{U}_q(\mathfrak{sl}_2)}$, Labeling with the defining $\mathfrak{sl}_2$-representation will yield the Jones polynomial. Well, more precisely, we need the corresponding $\mathcal{U}_q(\mathfrak{sl}_2)$-representation: it's generated by two elements $v_{\pm 1}$, where $E \cdot v_{-1} = v_1$, $E \cdot v_1 = 0$, $F \cdot v_1 = v_{-1}$, $F \cdot v_{-1} = 0$, and $K \cdot v_{\pm 1} = q^{\pm 1} v_{\pm 1}$. To actually implement this, we'd need to know the braiding; different people use different normalizations for it, which can be confusing.

Theorem 5.5. *There is a unique functor $F\colon \mathsf{Rib}_\mathsf{C} \to \mathsf{C}$ respecting all the structure, e.g. sending twists of ribbons to twists, duals to duals, braidings to braidings, and so on. Explicitly, $(V_1, \dots, V_k) \mapsto V_1 \otimes \cdots \otimes V_k$; the braiding of ribbons goes to the braiding C_{UV}; an unknotted, untwisted ribbon from V to $V^\vee$ goes to the pairing $V^\vee \otimes V \to k$ (and the dual version); and the twist of a ribbon to θ_v.*

Not every ribbon is obvious: if you reverse orientation on the ribbon giving the evaluation map $e\colon V^\vee \otimes V \to k$, it decomposes as

$$V \otimes V^\vee \xrightarrow{\mathrm{id} \otimes \theta} V \otimes V^\vee \xrightarrow{c_{VV^\vee}} V^\vee \otimes V \xrightarrow{e} k. \tag{5.6}$$

This is good, actually, because it allows us to define traces!

Definition 5.7. The *quantum trace* of an operator $T\colon V \to V$, denoted $\mathrm{tr}_q(T)$, is the map $k \to k$ given by the composition

$$k \xrightarrow{c} V \otimes V^\vee \xrightarrow{T \otimes \theta} V \otimes V^\vee \xrightarrow{c_{VV^\vee}} V^\vee \otimes V \xrightarrow{e} k. \tag{5.8}$$

Explicitly, evaluate F on the ribbon with an unknotted, untwisted loop labeled by V and a *coupon* labeled by T (this requires expanding our definition of ribbon graphs a bit).

In particular, the *quantum dimension* $\dim_q V \coloneqq \mathrm{tr}_q(\mathrm{id})$, which is what F assigns to an untwisted, unknotted circle labeled by V.

6. Invertible field theories: 6/6/19

To be factored in. In the meantime, see https://web.ma.utexas.edu/users/a.debray/lecture_notes/bordism.pdf.

Recall that if A is a ribbon Hopf algebra, then Rep_A is a ribbon category.

Example 7.1. If G is a finite group, $k[G]$ admits the structure of a ribbon Hopf algebra, but this is not a very interesting example: the twist is trivial, so Rep_G is symmetric monoidal. The corresponding 3-manifold invariants are thus not very interesting. ◀

Therefore one says that the first nontrivial example is $\mathcal{U}_q(\mathfrak{sl}_2)$.

In addition, given a ribbon category C, we defined colored link invariants valued in $\mathbb{C}$.[6] More generally, we have operator-valued invariants of colored ribbon graphs.

Next, to obtain something that assigns vector spaces to (decorated) surfaces, we'll need to sum over all isomorphism classes of simple objects in C. In order to do this, C must have only finitely many such isomorphism classes. This means we have to throw out some of our ribbon categories – for example, for most $q \in \mathbb{C}$, $\mathcal{U}_q(\mathfrak{sl}_2)$ has infinitely many isomorphism classes of simple representations. This is why we specialize to the case where q is a root of unity.

Definition 7.2. Let C be a ribbon category over a field k. Then $V \in \mathsf{C}$ is *simple* if $\mathrm{End}_{\mathsf{C}} V = k$. V is *dominated* by $W_1, \dots, W_n$ if all endomorphisms of V come from mapping to $W_1 \oplus \cdots \oplus W_n$, and then back to V.

Definition 7.3. A *modular category* is a ribbon category with a finie list of simple objects $\mathcal{V} := \{V_0, \dots, V_m\}$ such that $V_0 = k$ and $V_i \otimes V_j$ is dominated by $\mathcal{V}$, and such that the *S-matrix*, the k-valued matrix whose (i,j) component is F of the Hopf link labeled by V_i and V_j, is invertible.

Remark 7.4. The last condition tells us that $\mathcal{V}$ has no redundancy: if $V_i \cong V_j$ and $i \neq j$, the S-matrix contains two identical columns. ◀

Lemma 7.5 (Schur's lemma for modular categories)**.** *Let C be a modular category and $V_j, V_r \in \mathcal{V}$. If $j \neq k$,* $\mathrm{Hom}(V_j, V_k) = 0$ *o*

Proof. Let $H_{ij} \colon V_j \to V_j$ be the invariant of a single strand labeled by V_j together with a circle around it labeled by V_i. This is multiplication by some scalar $X(i,j)$. In particular, $S_{ij} = \mathrm{tr}(H_{ij}) = X(i,j) \dim V_j)$.

Now let $f \colon V_j \to V_k$. Drawing the diagrams for $f \circ H_{ij}$ and $f \circ H_{ik}$, we get isotopic diagrams, so $fX(i,j) = fX(i,k)$. Therefore $fS_{ij}/\dim(V_j) = fS_{ik}/\dim(V_k)$; the only way for this to be true for all i and j if the S-matrix is invertible is if $f = 0$. ⊠

Now, on to three-manifolds. Let M be a closed, oriented 3-manifold and $L \subset S^3$ be a framed link such that surgery on L yields M. In the first lecture, we argued that we can always choose such an L.

Let W_L be a D^4 union some 2-handles (i.e. a $D^2 \times D^2$ for every component of L) attached to a tubular neighborhood of $L \subset S^3 = \partial D^4$. To attach a handle, we need an isotopy class of an identification of the boundary of $D^2 \times D^2$ and the boundary of said tubular neighborhood, which is precisely provided by the framing of L.

It turns out that W_L is compact, and $\partial W_L = M$. This provides a geometric proof that $\Omega_3^{\mathrm{SO}} = 0$, which isn't completely trivial but is probably easier than the algebraic proof!

The intersection form of W_L is pretty simple, and can be determined geometrically, e.g. for the Hopf link is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, which has signature 2. This is telling us something interesting – we started with S^3 bounding D^4, and doing $(0,0)$-surgery on the Hopf link returns S^3 again, but now bounding a different 4-manifold with a different signature! That will be important in a second.

Here's some notation we'll need.

$$\mathcal{D} := \sum_{V_i \in \mathcal{V}} \dim(V_i)^2 \tag{7.6a}$$

$$\Delta := \sum_{V_i \in \mathcal{V}} v_{V_i}^{-1} \dim(V_i)^2 \tag{7.6b}$$

[6] Reshetikhin-Turaev worked over arbitrary commutative rings, but we're just going to work with fields today.

Given a link L, let $\Lambda(L, \mathcal{V})$ denote the set of colorings of components of L by objects in $\mathcal{V}$.

$$\{L\} := \sum_{\lambda \in \Lambda(L,\mathcal{V})} \prod_{K \in \pi_0(L)} \dim(V_{\lambda(K)}) F(L, \lambda). \tag{7.6c}$$

Here v_X is the twist on X. We'd like $\{L\}$ to be the invariant, but it's not quite, so we have to normalize it. The actual invariant we get is

$$\tau(M) := \Delta^{\sigma(W_L)} \mathcal{D}^{\sigma(L) - |\pi_0 L| - 1} \{L\}. \tag{7.7}$$

The weight on Δ appears because of what we said earlier: the signature could change, even for the same link. This is related to the anomaly. The weight on $\mathcal{D}$ occurs because quantum invariants aren't usually multiplicative under connected sum.

If Σ_g is the closed, connected, oriented, genus-g surface, then we let $\tau(\Sigma_g)$ denote the colorings of a disc with g 1-handles attached (without twists or links), or more explicitly,

$$\tau(\Sigma_g) := \bigoplus_{i_1,\dots,i_g=0}^{|\mathcal{V}|} \operatorname{Hom}\left(\bigotimes_{j=1}^{g} (V_{i_j} \otimes V_{i_j}^\vee), k \right). \tag{7.8}$$

The idea is that we get an object for every component and a morphism for the disc. It's good to think of that ribbon graph as embedded in Σ_g, where each donut hole is inside one of the 1-handles.

Example 7.9. $\tau(T^2)$ is the sum of $\operatorname{Hom}(V_i \otimes V_i^\vee, k)$ over all $V_i \in \mathcal{V}$, which has dimension $|\mathcal{V}|$. ◂

Correspondingly, let

$$\overline{\tau}(\Sigma_g) := \bigoplus_{i_1,\dots,i_g=0}^{|\mathcal{V}|} \operatorname{Hom}\left(k, \bigotimes_{j=1}^{g} (V_{i_j} \otimes V_{i_j}^\vee) \right). \tag{7.10}$$

We think of this as coming from the "upside-down" embedding of the handlebody in Σ_g; we'll denote the usual embedding (as a manifold with an embedded handlebody) H_g and this upside-down one as $\overline{H}_g$.

Now suppose M is an oriented bordism from $S_0 := \Sigma_{g_1} \amalg \cdots \amalg \Sigma_{g_\ell}$ to $S_1 := \Sigma_{h_1} \amalg \cdots \amalg \Sigma_{h_m}$. In particular, we have an identification of ∂M and $S_0 \amalg S_1$, even though we haven't given it a name. Using this identification, we can glue M to $H_{g_1} \amalg \cdots \amalg H_{g_\ell}$ and $\overline{H}_{h_1} \amalg \cdots \amalg \overline{H}_{h_m}$ for each coloring $\lambda \in \tau(\partial_- M)$ and $\lambda' \in \tau(\partial_+ M)$. (Here ∂_-, resp. ∂_+ denote the outgoing, resp. incoming boundary of M). Thus we obtain a ribbon graph, which we can evaluate on and obtain a map

$$\tau(\Sigma_{g_1}) \otimes \cdots \otimes \tau(\Sigma_{g_\ell}) \to \overline{\tau}(\Sigma_{h_1}) \otimes \cdots \otimes \overline{\tau}(\Sigma_{h_m}). \tag{7.11}$$

We can also do this if M has an embedded ribbon graph – we can still sum over colorings. You can write this as a surgery diagram with the handlebody linked to the link.

One must argue that the final answer doesn't depend on the choice of surgery diagram. It also forms a TQFT, albeit an anomalous one. You have to account for the anomaly, unless you want something which isn't functorial! The issue is that when you compose two bordisms, you might not get the value on the glued bordism. They'll only differ by a constant, and Turaev axiomatizes one way to address it and how to calculate it, but from the physics perspective it might be better to use relative field theory.

Here's a cool application. Let $\phi_S \colon T^2 \to T^2$ be the diffeomorphism switching the latitude and longitude copies of S^1. We can form a bordism $M = T^2 \times I$ where the incoming torus is attached by the identity and the outgoing T^2 is attached by ϕ_S. Then $\tau(M)$ is the S-matrix of C. Why is this? Well, if you glue two solid tori, but one with meridian and longitude switched, the 3-manifold we get is S^3, and the two fundamental cycles get interlinked as a Hopf link.

You can also use this to diagrammatically derive the Verlinde formula! Define

$$h_k^{ij} := \dim(\operatorname{Hom}(V_k, V_i \otimes V_j)). \tag{7.12}$$

Let A_{rij} denote the ribbon diagram with one circle labeled by V_r, and two separated circles linked to it with linking number one, labeled by V_i and V_j, and with V_i before V_j. Then $F(A_{rij})$ is the same as F of the Hopf link labeled by V_r and $V_i \otimes V_j$, and both of these are the trace of the ribbon graph where V_r is on a line, and V_i and V_j on two circles around it, and this is

$$X(r,j) X(r,i) \dim(V_r) = \frac{S_{rj} S_{ri}}{\dim(V_r)}. \tag{7.13}$$

But we can calculate $F(A_{rij})$ another way, using, "linearity" of F: if $V = W_1 \oplus W_2$, then F of a link with one component labeled by V is the sum of F applied to the same link, but with W_1 in place of V, plus the same but with W_2 in place of V. Therefore

$$F(A_{rij}) = \sum_k h_k^{ij} F(\text{Hopf link labeled by } V_r \text{ and } V_k) = \sum_k h_k^{ij} S_{r,k}. \tag{7.14}$$

Hence we get the Verlinde formula:

$$\sum_k h_k^{ij} S_{rk} = \frac{S_{rj} S_{ri}}{\dim(V_r)}. \tag{7.15}$$