M392C NOTES: K-THEORY

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These notes were taken in UT Austin's Math 392C (K-theory) class in Fall 2015, taught by Dan Freed. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

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Lecture 1.

Families of Vector Spaces and Vector Bundles: 8/27/15

"Is that clear enough? I didn't hear a ding *this time."*

Let's suppose X is a topological space. Usually, when we do cohomology theory, we send in probes, n-simplicies, into the space, and then build a chain complex with a boundary map. This chain complex can be built in many ways; for general spaces we use continuous maps, but if X has the structure of a CW complex we can use a smaller complex. If we have a singular simplicial complex, a triangulation, we get other models, but they really compute the same thing.

Given a chain complex $C_\bullet$, we get a cochain complex by computing $\mathrm{Hom}(-,\mathbb{Z})$, giving us a cochain complex $C^0 \xrightarrow{\mathrm{d}} C^1 \xrightarrow{\mathrm{d}} \cdots$, giving us the cohomology groups $H^0 = H^0(X,\mathbb{Z})$.

If M is a smooth manifold, we have a cochain complex $\Omega^0_M \xrightarrow{\mathrm{d}} \Omega^1_M \xrightarrow{\mathrm{d}} \cdots$, and therefore get the de Rham cohomology $H^\bullet_{\mathrm{dR}}(M)$. de Rham's theorem states this is isomorphic to $H^\bullet(M;\mathbb{R})$, obtained by tensoring with $\mathbb{R}$.

In K-theory, we extract topological information in a very different way, using linear algebra. This in some sense gives us more powerful invariants. Consider $\mathbb{C}^n = \{(\xi^1,\dots,\xi^n) : \xi^i \in \mathbb{C}\}$. This has the canonical basis $(1,0,\dots,0)$, $(0,1,0,\dots,0)$, and so on. This is a rigid structure, in that the automorphism group of this space *with this basis* is rigid (no maps save the identity preserve the linear structure and the basis).

In general, we can consider an abstract complex vector space $(\mathbb{E},+,\cdot,0)$, and assume it's finite-dimensional. Then, $\mathrm{Aut}\,\mathbb{E}$ is an interesting group: every basis gives us an automorphism $b : \mathbb{C}^n \xrightarrow{\cong} \mathbb{E}$, and therefore gives us an isomorphism $b : \mathrm{GL}_n\,\mathbb{C} \xrightarrow{\cong} \mathrm{Aut}\,\mathbb{E}$.

We can also consider automorphisms that have some more structure; for example, $\mathbb{E}$ may have a hermitian inner product $\langle -,-\rangle : \mathbb{E}\times\mathbb{E} \to \mathbb{C}$. Then, $\mathrm{Aut}(\mathbb{E},\langle -,-\rangle) = \mathrm{U}(\mathbb{E})$, which by a basis is isomorphic to U_n, the set of $n\times n$ matrices A such that $A^*A = \mathrm{id}$ (where A^* is the conjugate transpose). U_n is a Lie group, and a subgroup of $\mathrm{GL}_n\,\mathbb{C}$.

For example, when $n=1$, $\mathrm{U}_1 \hookrightarrow \mathrm{GL}_1\,\mathbb{C}$. U_1 is the set of $\lambda\in\mathbb{C}$ such that $\overline{\lambda}\lambda = 1$, so U_1 is just the unit circle. Then, $\mathrm{GL}_1\,\mathbb{C}$ is the set of invertible complex numbers, i.e. $\mathbb{C}\setminus 0$. In fact, this means the inclusion $\mathrm{U}_1 \hookrightarrow \mathrm{GL}_1\,\mathbb{C}$ is a homotopy equivalence, and we can take the quotient to get $\mathrm{U}_1 \hookrightarrow \mathrm{GL}_1\,\mathbb{C} \twoheadrightarrow \mathbb{R}^{>0}$.

In some sense, the quotient determines the inner product structure on $\mathbb{C}$, since in this case an inner product only depends on scale. But the same behavior happens in the general case: $\mathrm{U}_n \hookrightarrow \mathrm{GL}_n \mathbb{C} \twoheadrightarrow \mathrm{GL}_n \mathbb{C}/\mathrm{U}_n$, and the quotient classifies hermitian inner products on $\mathbb{C}^n$.

Exercise. Identify the homogeneous space $\mathrm{GL}_n / \mathrm{U}_n$, and show that it's contractible. (Hint: show that it's convex.)

Now, we return to the manifold. Embedding things into the manifold is covariant: composing with $f : X \to Y$ of manifolds with something embedded into X produces something embedded into Y. K-theory will be contravariant, like cohomology: functions and differential forms on a manifold pull back contravariantly. What we'll look at is families of vector spaces parameterized by a manifold X.

Definition. A *family of vector spaces* $\pi : E \to X$ parameterized by X is a surjective, continuous map together with a continuously varying vector space structure on the fiber.

This sounds nice, but is a little vague. Any definition has data and conditions, so what are they? We have two topological spaces E and X; X is called the *base* and E is called the *total space*, as well as a continuous, surjective map $\pi : E \to X$. The condition is that the fiber $E_x = \pi^{-1}(x)$ is a vector space for each $x \in X$. Specifically, sending x to the zero element of E_x is a zero $z : X \to E$, which is a section or right inverse to π. We also have scalar multiplication $m : C \times E \to E$, which has to stay in the same fiber; thus, m commutes with π. Vector addition $+ : E \times_X E \to E$ is only defined for vectors in the same fiber, so we take the fiberwise product $E \times_X E$. Again, $+$ and π commute. Finally, what does continuously varying mean? This means that z, m, and $+$ are continuous.

Intuitively, if we let $\mathcal{V}$ be the collection of vector spaces, we might think of such a family as a function $X \to \mathcal{V}$. To each point of X, we associate a vector space, instead of, say, a number.

Example 1.1.

(1) The constant function: let $\mathbb{E}$ be a vector space. Then, $\underline{\mathbb{E}} = X \times \mathbb{E} \to X$ given by $\pi = \mathrm{pr}_1$ sends $(x, e) \mapsto x$. This is called the *constant vector bundle* or *trivial vector bundle* with fiber $\mathbb{E}$.

(2) A nonconstant bundle is the *tangent bundle* $TS^2 \to S^2$. For now, let's think of this as a family of real vector spaces; then, at each point $x \in S^2$, we have this 2-dimensional space $T_x S^2$, and different tangent spaces aren't canonically identified. Embedding $S^2 \hookrightarrow \mathbb{R}^3$ as the unit sphere, each tangent space embeds as a subspace of $\mathbb{R}^3$, and we have something called the Grassmanian. Note that $TS^2 \ncong \underline{\mathbb{R}}^2$, which we proved in algebraic topology as the hairy ball theorem.

Implicit in the second example was the definition of a map; the idea should be reasonably intuitive, but let's spell it out: if we have $\pi : E \to X$ and $\pi' : E' \to X$, a morphism is the data of a continuous $f : E \to E'$ such that the following diagram commutes.

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & X & \end{tikzcd}$$

Then, you can make all of the usual linear-algebraic constructions you like: inverses, direct sums and products, and so on.

Example 1.2. Here's an example of a rather different sort. Let $\mathbb{E}$ be a finite-dimensional complex vector space, and suppose $T : \mathbb{E} \to \mathbb{E}$ is linear. Define for any $z \in \mathbb{C}$ the map $K_z = \ker(z \cdot \mathrm{id} - T) \subset \mathbb{E}$, and let $K = \bigcup_{z \in \mathbb{C}} K_z$.

For a generic z, $z \cdot \mathrm{id} - T$ is invertible, and so $K_z = 0$. But for eigenvalues, we get something more interesting, the eigenspace. But sending $K_z \mapsto z$, we get a map $\pi : K \to \mathbb{C}$. This is interesting because the vector space is 0-dimensional except at a finite number of points, and in fact if we take

$$\varphi : \bigoplus_{z : K_z \neq 0} K_z \to \mathbb{E},$$

induced by the inclusion maps $K_z \to \mathbb{E}$, then φ is an isomorphism. This is the geometric statement of the Jorden block decomposition (or generalized eigenspace decomposition) of a vector space.

Definition. Given a family of vector spaces $\pi : E \to X$, the rank $x \mapsto \dim E_x = \pi^{-1}(X)$ is a function $\mathrm{rank} : X \to \mathbb{Z}^{\geq 0}$.

Example 1.2 seems less nice than the others, and the property that makes this explicit, developed by Norman Steenrod in the 1950s, is called local triviality.

Definition. A family of vector spaces $\pi : E \to X$ is a *vector bundle* if it has *local triviality*, i.e. for every $x \in X$, there exists an open neighborhood $U \subset X$ and isomorphism $E|_U \cong \underline{\mathbb{E}}$ for some vector space $\mathbb{E}$.

This property is sometimes also called being *locally constant*. So the fibers aren't literally equal to $\mathbb{E}$ (they're different sets), but they're isomorphic as vector spaces.

One good question is, what happens if I have two local trivializations? Suppose E_x lies above x, and we have $\varphi_x : \mathbb{E} \to E_x$ and $\varphi'_x : \mathbb{E}' \to E_x$, each defined on open neighborhoods of x in X. The function $\varphi_x^{-1} \circ \varphi'_x : \mathbb{E}' \to \mathbb{E}$ is called a *transition function*, and we can see that it must be linear, and furthermore, isomorphic.

The Clutching Construction. This leads to a way of constructing vector bundles, known as the *clutching construction*. First, consider $X = S^2$, decomposed into $B^2_+ = S^2 \setminus \{-\}$ and $B^2_- = S^2 \setminus \{+\}$ (i.e. minus the south and north poles, respectively). Each of these is diffeomorphic to the real plane, and in particular is contractible. Taking the trivial bundle $\underline{\mathbb{C}}$ over each of these, we have something like

$$\begin{tikzcd}
\underline{\mathbb{C}} \arrow[d] & & \underline{\mathbb{C}} \arrow[d] \\
B^2_+ \arrow[dr, leftrightarrow] & & B^2_- \arrow[dl, leftrightarrow] \\
& B^2_+ \cap B^2_- &
\end{tikzcd}$$

The intersection $B^2_+ \cap B^2_-$ is diffeomorphic to $\mathbb{A}^2 \setminus \{0\}$. Thus, the two structures of $\underline{\mathbb{C}}$ on this intersection are related by a map $\underline{\mathbb{C}} \to \underline{\mathbb{C}}$, which induces a map $\tau : B^2_+ \cap B^2_- \to \mathrm{Aut}(\mathbb{C}) = \mathrm{GL}_1 \mathbb{C} = \mathbb{C}^\times$. This τ has an invariant called its *winding number*, so we can construct a line bundle $L \xrightarrow{\pi} S^2$ by gluing: let L be the quotient of $(B^2_+ \times \mathbb{C}) \sqcup (B^2_- \times \mathbb{C})$ with the identification $\{x\} \times \mathbb{C} \sim \{\tau(x)\} \times \mathbb{C}$ (the former from B^2_+ and the latter from B^2_-).

More generally, if $\{U_\alpha\}_{\alpha \in A}$ is an open cover of X, then we get a map

$$\coprod_{\alpha \in A} U_\alpha \xrightarrow{p} X,$$

and so we can construct a gluing: whenever two points in the disjoint union map to the same point, we want to glue them together. The arrows linking two points to be identified have identities and compositions.

The clutching construction gives us a vector bundle over this space: given a vector bundle E_α over each U_α, we glue basepoints using those arrows, and get an associated isomorphism of vector spaces. Then, you can prove that you get a vector bundle.

Notice that maps $f : X \to Y$ of manifolds can be pulled back, and in this regard a vector bundle is a contravariant construction.

Topology and Vector Bundles. We were going to add some topology to this discussion, yes?

Theorem 1.3. *If $E \to [0,1] \times X$ is a vector bundle, then $E|_{\{0\}\times X} \cong E|_{\{1\}\times X}$.*

We'll prove this next lecture. The idea is that the isomorphism classes are homotopy-invariant, and therefore rigid or in some sense discrete. This will allow us to do topology with vector bundles.

Now, we can extract $\mathrm{Vect}^{\cong}(X)$, the set of vector bundles on X up to isomorphism. This has a 0 (the trivial bundle) and a $+$, given by direct sum of vector bundles. This gives a commutative monoid structure from X which is homotopy invariant.

Commutative monoids are a little tricky to work with; we'd rather have abelian groups. So we can complete the monoid, taking the Grothendieck group, obtaining an abelian group $K(X)$.

Using real or complex vector bundles gives $K_{\mathbb{R}}(X)$ and $K_{\mathbb{C}}(X)$, respectively (the latter is usually called $K(X)$). On S^n, one can compute that $K(S^n) = \pi_{n-1}\operatorname{GL}_N$ for some large N. These groups were computed to be periodic in both the real and complex cases, a result which is known as *Bott periodicity.*[1] This periodicity was proven in the mid-1950s.[2] This was worked into a topological theory by players such as Grothendieck and Atiyah, among others.

One of the first things we'll do in this class is provide a few different proofs of Bott periodicity.

Another interesting fact is that K-theory satisfies all of the axioms of a cohomology theory except for the values on S^n, making it a *generalized* (or *extraordinary*) *cohomology theory.* This is nice, since it means most of the computational tools of cohomology are available to help us. And since it's geometric, we can use it to attack problems in geometry, e.g. when is a manifold parallelizable?

For example, for S^n, S^0, S^1, and S^3 are parallelizable (the first two are trivial, and S^3 has a Lie group structure as the unit quaternions). It turns out there's only one more parallelizable sphere, S^7, and the rest are not; this proof by Adams in 1967 used K-theory, and is related to the question of how many division algebras there are.

Relatedly, and finer than just parallelizability, how many linearly independent vector fields are there on S^n? Even if S^n isn't parallelizable, we may have nontrivial l.i. vector fields. There are other related ideas, e.g. the Atiyah-Singer index theorem.

K-theory can proceed in different directions: we can extract modules of the ring of functions on X, and therefore using Spec, start with any ring and do algebraic K-theory. One can also intertwine K-theory and operator algebras, which is also useful in geometry. We'll focus on topological K-theory, however. There are also twistings in K-theory, which relate to representations of loop groups.

K-theory has also come into physics, both in high-energy theory and condensed matter, but we probably won't say much about it.

Nuts and bolts: this is a lecture course, so take notes. There might be notes posted on the course webpage[3], but don't count on it. There will also be plenty of readings; four are posted already.

Lecture 2.

Homotopies of Vector Bundles: 9/1/15

> *"You need a bit of Bourbaki imagination to determine the vector bundles over the empty set."*

Recall that all topological spaces in this class will be taken to be Hausdorff and paracompact.

We stated this as Theorem 1.3 last time; now, we're going to prove it.

Theorem 2.1. *Let X be a space and $E \to [0,1]\times X$ be a vector bundle. Let $j_t : X \hookrightarrow [0,1]\times X$ send $x \mapsto (t,x)$. Then, there exists a natural isomorphism $j_0^*E \xrightarrow{\cong} j_1^*E$ of vector bundles over X.*

To define the pullback more precisely, we can characterize it as fitting into the following diagram.

$$\begin{tikzcd} j^*E \arrow[r] \arrow[d] & E \arrow[d] \\ Y \arrow[r, "j"] & Z \end{tikzcd}$$

Then, j^*E is the subset of $Y \times E$ for which the diagram commutes.

We'll want to make an isomorphism of fibers and check that it is locally trivial; in the smooth case, one can use an ordinary differential equation, but in the more general continuous case, we'll do something which is in the end more elementary.

To pass between the local properties of vector bundles and a global isomorphism, we'll use partitions of unity.

Definition. Let X be a space and $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover (which can be finite, countable, or uncountable). Then, a *partition of unity* $\{\rho_\alpha\}_{\alpha\in A}$ indexed by a set A is a set of continuous functions

[1]The sequence of groups you get almost sounds musical. Maybe sing the Bott song!

[2]The professor says, "I wasn't around then, just so you know."

[3]https://www.ma.utexas.edu/users/dafr/M392C/index.html.

$X \to [0,1]$ with locally finite supports such that $\sum \rho_\alpha = 1$. This partition of unity is said to be *subordinate* to the cover $\mathcal{U}$ if there exists $i : A \to I$ such that $\operatorname{supp} \rho_\alpha \subset U_{i(\alpha)}$.

Theorem 2.2. *Let X be a Hausdorff paracompact space and $\{U_i\}_{i\in I}$ be an open cover.*

(1) *There exists a partition of unity $\{\rho_i\}_{i\in I}$ subordinate to $\{U_i\}_{i\in I}$ such that at most countably many ρ_i are not identically zero.*
(2) *There exists a partition of unity $\{\rho_\alpha\}_{\alpha\in A}$ subordinate to $\{U_i\}_{i\in I}$ such that each ρ_α is compactly supported.*
(3) *If X is a smooth manifold, we can choose ρ_α to be smooth.*

We'll only use part (1) of this theorem.

A nontrivial example is $X = \mathbb{R}$ and $U_x = (x-1, x+1)$ for $x \in \mathbb{R}$ (so an uncountable cover). In this case, we don't need every function to be nonzero; we only need a countable number.

Returning to the setup of Theorem 2.1, if X is a smooth manifold, we will set up a covariant derivative, which will allow us to define a notion of parallel. Then, parallel transport will produce the desired isomorphism. In this case, we'll call $X = M$.

Suppose first that $\mathbb{E}$ is a vector space, either real or complex. $\Omega^0_M(\mathbb{E})$ denotes the set of smooth functions $M \to \mathbb{E}$ (written as 0-forms), and we have a basic derivative operator $\mathrm{d} : \Omega^0_M(\mathbb{E}) \to \Omega^1_M(\mathbb{E})$ satisfying the Leibniz rule

$$\mathrm{d}(f\cdot e) = \mathrm{d}f \cdot e + f\,\mathrm{d}e,$$

where $f \in \Omega^0_M$ and $e \in \Omega^0_M(\mathbb{E})$ (that is, e is vector-valued and f is scalar-valued). Moreover, any other first-order differential operator (an operator $\Omega^0_M(\mathbb{E}) \to \Omega^1_M(\mathbb{E})$ that is linear and satisfies the Leibniz rule) has the form $\mathrm{d} + A$, where $A \in \Omega^1_M(\operatorname{End}\mathbb{E})$. This means that if $\mathbb{E} = \mathbb{C}^r$, then e is a column vector of $e^1, \ldots, e^r$ with $e^i \in \Omega^0(\mathbb{E})$, and $A = (A^i_j)$ is a matrix of one-forms: $A^i_j \in \Omega^1_M(\mathbb{C})$. Ultimately, this is because the difference between any two differential operators can be shown to be a tensor.

Now, let's suppose $E \to M$ is a vector bundle.

Definition. A *covariant derivative* is a linear map $\nabla : \Omega^0_M(E) \to \Omega^1_M(E)$ satisfying

$$\nabla(f\cdot e) = \mathrm{d}f\cdot e + f\cdot\nabla e$$

when $f \in \Omega^0_M$ and $e \in \Omega^0_M(\mathbb{E})$.

Here, $\Omega^0_M(E)$ is the space of sections of E. In some sense, this is a choice for functions with values in a varying vector space.

Theorem 2.3. *In this case, covariant derivatives exist, and the space of covariant derivatives is affine over $\Omega^1_M(\operatorname{End}\mathbb{E})$.*

Proof. Choose $\{U_i\}_{i\in I}$ and local trivializations $\underline{\mathbb{E}}_i \xrightarrow{\cong} E|_{U_i}$ on U_i. We have a canonical differentiation d of $\mathbb{E}_i$-valued functions on U_i to define ∇_i on the bundle $E|_{U_i} \to U_i$.

To stitch them together, choose a partition of unity $\{\rho_i\}_{i\in I}$ and define

$$\nabla e = \sum_i \rho_i \nabla(j_i^* e),$$

where $j_i : U_i \hookrightarrow M$ is inclusion. ⊠

All right, so what's parallel transport? Let $\mathcal{E} \to [0,1]$ be a vector bundle with a covariant derivative ∇. Parallel transport will be an isomorphism $\mathcal{E}_0 \xrightarrow{\sim} \mathcal{E}_1$.

Definition. A section e is *parallel* if $\nabla e = 0$.

Lemma 2.4. *The set $P \subset \Omega^0_{[0,1]}(\mathcal{E})$ of parallel sections is a subspace. Then, for any $t \in [0,1]$, the evaluation map $\mathrm{ev}_t : P \to \mathcal{E}_t$ sending $e \mapsto e(t)$ is an isomorphism.*

The first statement is just because $\nabla e = 0$ is a linear condition. The second has the interesting implication that for any $(x,t) \in \mathcal{E}$, there's a unique parallel section that extends it.

Proof. Suppose $\mathcal{E} \to [0,1]$ is trivializable, and choose a basis $e_1, \ldots, e_r$ of sections. Then, we can write

$$\nabla e_j = A^i_j e_i,$$

where we're summing over repeated indices and $A^i_j \in \Omega^1_{[0,1]}(\mathbb{C})$. Then, any section has the form $e = f^j e_j$ and the parallel transport equation is

$$\begin{aligned} 0 = \nabla e &= \nabla(()f^j e_j) \\ &= \mathrm{d}f^j e_j + f^j \nabla e_j \\ &= (\mathrm{d}f^i + A^i_j f^j) e_j. \end{aligned}$$

If we write $A^i_j = \alpha^i_j \, \mathrm{d}t$ for $\alpha^i_j \in \Omega^0_{[0,1]}(\mathbb{C})$, then the parallel transport equation is

$$\frac{\mathrm{d}f^i}{\mathrm{d}\tau} + \alpha^i_j f^j = 0. \tag{2.1}$$

This is a linear ODE on $[0,1]$, so by the fundamental theorem of ODEs, there's a unique solution to (2.1) given an initial condition.

More generally, if $\mathcal{E}$ isn't trivializable, partition it into $[0, t_1]$, $[t_1, t_2]$, and so on, so that $\mathcal{E} \to [t_i, t_{i+1}]$ is trivialiable, and compose the parallel transports on each interval. ⊠

Now, we can prove Theorem 2.1 in the smooth manifolds case.

Proof of Theorem 2.1, smooth case. Choose a covariant derivative ∇, and use parallel transport along $[0,1] \times \{x\}$ to construct an isomorphism $E_{(0,x)} \to E_{(1,x)}$. The fundamental theorem on ODEs also states that the solution smoothly depends on the initial data, so these isomorphisms vary smoothly in x. ⊠

Note that this fundamental theorem only gives local solutions, but (2.1) is linear, so a global solution exists.

In the continuous case, we can't do quite the same thing, but the same idea of parallel transport is in effect.

Proof of Theorem 2.1, continuous case. By local triviality, we can cover $[0,1] \times X$ by open sets of the form $(t_0, t) \times U$ on which $E \to [0,1] \times X$ restricts to be trivializable.

By the compactness of $[0,1]$, we can cover X by sets $\{U_i\}_{i \in I}$ such that $E|_{[0,1] \times U_i}$ is trivializable: we can get trivializations on a finite number of patches. Thus, at the finite number of boundaries, we can patch the trivialization, choosing a continuous isomorphism of vector spaces.

Choose a partition of unity $\{\rho_i\}_{i \in I}$ subordinate to $\{U_i\}_{i \in I}$ and pare down I to the countable subset of $i \in I$ such that ρ_i isn't identically zero. Let $\varphi_n = \rho_1 + \cdots + \rho_n$ for $n = 1, 2, \ldots$, and let Γ_n be the graph of φ_n, which is a subset of $[0,1] \times X$.

So now we have a countable cover, and Γ_n is only supported on $U_1 \cup \cdots \cup U_n$, and only changes from Γ_{n-1} on U_n. But since the sum of the ρ_i is 1, then the graph Γ_n must go across the whole of $[0,1] \times X$ as $n \to \infty$. But over each open set, since we've pared down I, there are only finitely many steps.[4]

Going from Γ_0 (identically 0) to Γ_1 makes a trivialization on U_1, and from Γ_1 to Γ_2 extends the trivialization further, and so on. ⊠

Corollary 2.5. *If $f : [0,1] \times X \to Y$ is continuous and $E \to Y$ is a vector bundle, then $f_0^* E \cong f_1^* E$.*

This is because $f_t(x) = f(t,x)$ is a homotopy.

Corollary 2.6. *A continuous map $f : X \to Y$ induces a pullback map $f^* : \mathrm{Vect}(Y)^{\cong} \to \mathrm{Vect}(X)^{\cong}$, and this map depends only on the homotopy type of f.*

This is a hint that we can make algebraic topology out of the sets of vector bundles of spaces. There are many homotopy-invariant sets that we attach to topological spaces, e.g. π_0, π_1, π_2, H_1, H_2, and so on; these tend to be groups and even abelian groups, and thus tend to be easier to work with.

$\mathrm{Vect}^{\cong}(X)$ is a *commutative monoid*, so there's an associative, commutative + and an identity. The identity is the isomorphism class of the bundle $\underline{\mathbb{O}}$, the zero vector space. Then, we define addition by

[4] This argument is likely confusing; it was mostly given as a picture in lecture, and can be found more clearly in Hatcher's notes on vector bundles and K-theory.

$[E] + [E'] = [E \oplus E']$. Moreover, it is a *semiring*, i.e. there's a $\times$ and a multiplicative identity 1 given by the isomorphism class of $\underline{\mathbb{C}}$. Multiplication is given by (the isomorphism class of) the tensor product.

Commutative monoids are pretty nice; a typical example is the nonnegative integers.

Example 2.7.

(1) The simplest possible space is $\emptyset$. There's a unique vector bundle over it, the zero bundle, so $\mathrm{Vect}^{\cong}(\emptyset) = 0$, the trivial monoid.
(2) Over a point, vector bundles are just finite-dimensional vector spaces, which are determined up to isomorphism by dimension, so $\mathrm{Vect}^{\cong}(\mathrm{pt}) \xrightarrow{\sim} \mathbb{Z}^{\geq 0}$.

Definition. If X is a compact space, $K(X)$ is the abelian group completion of the commutative monoid $\mathrm{Vect}^{\cong}_{\mathbb{C}}(X)$; the completion of $\mathrm{Vect}^{\cong}_{\mathbb{R}}(X)$ is denoted $KO(X)$.

This definition makes sense when X is noncompact, but doesn't give a sensible answer. We'll see other definitions in the noncompact case eventually.

We'll talk more about the abelian group completion next lecture; the idea is that for any abelian group A and homomorphism $\alpha : \mathrm{Vect}^{\cong}(X) \to A$ of commutative monoids, there should be a unique $\widetilde{\alpha}$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Vect}^{\cong}(X) \arrow[rr] \arrow[dr, "\alpha"'] & & K(X) \arrow[dl, "\widetilde{\alpha}"] \\
& A &
\end{tikzcd}$$

Another corollary of Theorem 2.1:

Corollary 2.8. *If X is contractible and $\pi : E \to X$ is a vector bundle, then π is trivializable.*

Corollary 2.9. *Let $X = U_0 \cup U_1$ for open sets U_0, U_1 and $E_i \to U_i$ be two vector bundles, and let $\alpha : [0,1] \times U_0 \cap U_1 \to \mathrm{Iso}(E_0|_{U_0 \cap U_1}, E_1|_{U_0 \cap U_1})$: that is, α is a homotopy of isomorphisms $E_0 \to E_1$ on the intersection. Then, clutching with α_t gives a vector bundle $E_t \to X$, and $E_0 \cong E_1$.*

In the last five minutes, we'll discuss a few more partition of unity arguments.

(1) Let X be a topological space, and

$$0 \longrightarrow E' \xrightarrow{\ i\ } E \xrightarrow{\ j\ } E'' \longrightarrow 0$$

be a short exact sequence of vector bundles over X. Recall that a *splitting* of this sequence is an $s : E'' \to E$ such that $j \circ s = \mathrm{id}_{E''}$. Then, splittings form a bundle of affine spaces over $\mathrm{Hom}(E'', E)$, which happens because linear maps act simply transitively on splittings (adding a linear map to a splitting is still a splitting, and any two splittings differ by a linear map).

Theorem 2.10. *Global splittings exist, i.e. the affine bundle of splittings has a global section.*

Proof. At each point, there's a section, which is a linear algebra statement, and locally on X, there's a splitting, which follows from local trivializations. Then, patch them together with a partition of unity, which works because we're in an affine space, so our partition of unity in each affine space is a weighted average (because the ρ_i are nonnegative) and therefore lies in the convex hull of the splittings. ⊠

(2) We also have Hermitian inner products. The same argument goes through, as inner products are convex (the weighted average of two inner products is convex), so one can honestly use a partition of unity in the same way as above.

Lecture 3.

Abelian Group Completions and $K(X)$: 9/3/15

"First I want to remind you about fiber bundles... (pause) *... Consider yourself reminded."*

Last time, we said that if $\mathbb{E}$ is a (real or complex) vector space, the space of its inner products is contractible. This is because we have a vector space of sesquilinear (or bilinear in the real case) maps $\mathbb{E} \times \mathbb{E} \to \mathbb{C}$ (or $\mathbb{R}$), and the inner products form a convex cone in this space.

Inner products relate to symmetry groups: the symmetry group of $\mathbb{C}^n$ is $\mathrm{GL}_n \mathbb{C}$, the set of $n \times n$ complex invertible matrices, but the symmetry group of $\mathbb{C}^n$ with an inner product $\langle -, - \rangle$ is the unitary group $\mathrm{U}_n \subset \mathrm{GL}_n \mathbb{C}$, the set of matrices A such that $A^* A = I$. In the real case, the symmetries of $\mathbb{R}^n$ are $\mathrm{GL}_n \mathbb{R}$, and the group of symmetries of $\mathbb{R}^n$ with an inner product is $\mathrm{O}_n \subset \mathrm{GL}_n \mathbb{R}$.

As a consequence, we have the following result.

Proposition 3.1. *There are deformation retractions* $\mathrm{GL}_n \mathbb{C} \to \mathrm{U}_n$ *and* $\mathrm{GL}_n \mathbb{R} \to \mathrm{O}_n$.

For example, when $n = 1$, $\mathrm{GL}_1 \mathbb{C} = \mathbb{C}^\times$, which deformation retracts onto the unit circle, which is U_1. Then, $\mathrm{GL}_1 \mathbb{R} = \mathbb{R}^\times$ and $\mathrm{O}_1 = \{\pm 1\}$, so there's a deformation retraction in the same way.

Proof. We'll give the proof in the complex case; the real case is pretty much identical.

Since the columns of an invertible matrix determine a basis of $\mathbb{C}^n$ and vice versa, identify $\mathrm{GL}_n \mathbb{C}$ with the space of bases of $\mathbb{C}^n$; then, U_n is the space of orthonormal bases of $\mathbb{C}^n$.

A general basis $e_1, \dots, e_n$ may be turned into an orthonormal basis by the Gram-Schmidt process, which is a composition of homotopies. First, we scale e_1 to have norm 1, given by the homotopy $e_1 \mapsto ((1-t) + t/|e_1|)e_1$. Then, we make $e_2 \perp e_1$, which is given by the homotopy $e_2 \mapsto e_2 - t\langle e_2, e_1 \rangle e_1$. The rest of the steps are given by scaling basis vectors and making them perpendicular to the ones we have so far, so they're also homotopies. ⊠

Group Completion. Recall that a commutative monoid is the data $(M, +, 0)$, such that $+$ is associative and commutative, and 0 is the identity for $+$.

Definition. (A, i) is a *group completion* of M if A is an abelian group, $i : M \to A$ is a homomorphism of commutative monoids, and for every abelian group B and homomorphism $f : M \to B$ of commutative monoids, there exists a unique abelian group homomorphism $\widetilde{f} : A \to B$ of abelian groups such that $\widetilde{f} \circ i = f$.

That is, we require that there exists a unique $\widetilde{f}$ such that the following diagram commutes.

$$\begin{tikzcd} M \arrow[rr, "i"] \arrow[dr, "f"'] & & A \arrow[dl, "\widetilde{f}"] \\ & B & \end{tikzcd}$$

Note that i was never specified to be injective, and in fact it often isn't.

Example 3.2.

- If $M = (\mathbb{Z}^{\geq 0}, +)$, the group completion is $A = \mathbb{Z}$.
- If $M = (\mathbb{Z}^{>0}, \times)$, we get $A = \mathbb{Q}^{>0}$.
- However, if $M = (\mathbb{Z}^{\geq 0}, \times)$, we get $A = 0$. This is because if $i : \mathbb{Z}^{\geq 0} \to A$, then there must be an $a \in A$ such that $i(0) \cdot a = 1$; thus, for any $n \geq 0$,

$$i(n) = i(n)i(1) = i(n)i(0)a = i(n \cdot 0)a = i(0)a = 1.$$

Since the group completion was defined by a universal property, we can argue for its existence and uniqueness; universal properties tend to have very strong uniqueness conditions.

We saw that the vector bundles up to isomorphism are a commutative monoid (even semiring under tensor product), and so taking the group completion can cause a loss of information, as in the last part of the above example. Though abelian groups are nicer to compute with, there are examples where information about vector bundles is lost by passing to abelian groups.

The uniqueness of the group completion is quite nice: given two group completions (A, i) and (A', i') of a commutative monoid M, there exists a unique isomorphism ϕ that commutes with the universal property.

That is, in the following diagram, $\phi \circ \widetilde{f}' = \widetilde{f}$.

$$\begin{tikzcd}
A' \arrow[dr, "\widetilde{f}'"'] & M \arrow[l, "i'"'] \arrow[r, "i"] \arrow[d, "f"] & A \arrow[ll, bend right, "\phi"'] \arrow[dl, "\widetilde{f}"] \\
& B &
\end{tikzcd}$$

To prove this, we'll apply the universal property four times. To see why ϕ is an isomorphism, putting A' in place of B and i' in place of f, we get a ϕ, and switching (A, i) with (A', i') gives us $\psi : A' \to A$. Then, in the following diagram, $i' = \phi i = (\phi\psi)i'$, which satisfies a universal property (which one?) and therefore proves ϕ and ψ are inverses.

$$\begin{tikzcd}
M \arrow[r, "i"] \arrow[dr, "i'"'] & A \arrow[d, shift left, "\phi"] \\
& A' \arrow[u, shift left, "\psi"]
\end{tikzcd}$$

For existence, define $A = M \times M/\sim$, where $(m_1, m_2) \sim (m_1 + n, m_2 + n)$ for all $m_1, m_2, n \in M$. Then, $0_A = (0_M, 0_M)$ and $-[m_1, m_2] = [m_2, m_1]$. This makes sense: it's how we get $\mathbb{Z}$ from $\mathbb{N}$, and $\mathbb{Q}$ from $\mathbb{Z}$ multiplicatively.

Often, the abelian group completion is called the *Grothendieck group* of M, called $K(M)$.

Back to K-Theory. If X is compact hausdorff, then $\mathrm{Vect}^{\cong}(X)$, the set of isomorphism classes of vector bundles over X, is a commutative monoid, with addition given by $[E'] + [E''] = [E' \oplus E'']$, and a semiring given by $[E'] \times [E''] = [E' \otimes E'']$. There's some stuff to check here.

The group completion of $\mathrm{Vect}^{\cong}_{\mathbb{C}}(X)$ is denoted $K(X)$ (sometimes $KU(X)$, with the U standing for "unitary"), and the group completion of $\mathrm{Vect}^{\cong}_{\mathbb{R}}(X)$ is denoted $KO(X)$, with the O for "orthogonal."

The map $X \mapsto K(X)$ (or $KO(X)$) is a homotopy-invariant functor; that is, if $f : X \to Y$ is continuous, then $f^* : K(Y) \to K(X)$ is a homomorphism of abelian groups. The homotopy invariance says that if $f_0 \simeq f_1$, then $f_0^* = f_1^*$. We could write $K : \mathsf{CptSpace}^{op} \to \mathsf{AbGrp}$, and mod out the homotopy.

There are plenty of other functors that look like this; for example, the n^{th} cohomology group is a contravariant functor from topological spaces (more generally than compact Hausdorff spaces) to abelian groups, and is homotopy-invariant. But this gives us a sequence of groups, indexed by $\mathbb{Z}$ (where the negative cohomology groups are zero by definition). Similarly, we'll promote the K-theory of a space to a sequence of abelian groups indexed by the integers, with $K(X)$ becoming $K^0(X)$; we'll also see that in the typical case, $K^n(X)$ is nonzero for infinitely many n.

For example, if E and E' are vector bundles, $\mathrm{Hom}(E, E') \cong E' \otimes E^*$, by the map sending $e' \otimes \theta \mapsto (e \mapsto \theta(e)e')$. There's some stuff to check; in particular, once you know it for vector spaces, it's true fiber-by-fiber. Moreover, E and E^* are isomorphic as vector bundles, because any metric $E \otimes E \to \underline{\mathbb{R}}$ induces an isomorphism $E \to E^*$; thus, in $KO(X)$, $[E] = [E']$, so $[\mathrm{Hom}(E, E')] = [E] \times [E']$.

In the complex case, the metric is a map $\overline{E} \otimes E \to \underline{\mathbb{C}}$: the conjugate bundle is defined fiber-by-fiber by the conjugate vector space $\overline{\mathbb{E}}$, identical to $\mathbb{E}$ except that scalar multiplication is composed with conjugation. Thus, there's an isomorphism $\overline{E} \xrightarrow{\sim} E^*$. This is sometimes, but not always, an isomorphism: if X is a point, then it's always an isomorphism, but the bundle $\mathbb{C}P^1 \to S^2$ isn't fixed: complex conjugation flips the winding number, and therefore produces a nonisomorphic bundle.

We said that we might lose information taking the group completion, so we want to know what kind of information we've lost. The key is the following proposition.

Proposition 3.3. *Let X be a compact Hausdorff space and $\pi : E \to X$ be a vector bundle. Then, there exists a vector bundle $\pi' : E' \to X$ such that $E \oplus E' \to X$ is trivializable.*

If $X \neq \emptyset$, then there's a map $p : X \to \mathrm{pt}$, and its pullback $p^* : K(\mathrm{pt}) \to K(X)$ is injective. That is, we have an injective map $\mathbb{Z} \hookrightarrow K(X)$, consisting of the trivial bundles (i.e. those pulled back by a point). Proposition 3.3 implies that given a $k \in K(X)$, there's a k' such that $k + k' = n$ for $n \in \mathbb{Z}$. Thus, the inverse is $-k = k' - N$.

Proof of Proposition 3.3. Since X is compact, we can cover it with a finite collection of opens $U_1, \dots, U_N$ such that $E|_{U_i}$ is trivializable for each i.

Choose a basis of sections $e_1^{(i)}, \dots, e_n^{(i)}$ on U_i, and let $\rho_1, \dots, \rho_N$ be a partition of unity subordinate to the cover $\{U_i\}$. Then, let

$$S = \left\{\rho_1 e_1^{(1)*}, \dots, \rho_1 e_n^{(1)*}, \rho_2 e_1^{(2)*}, \dots\right\} \subset C^0(X; E^*),$$

where $e_1^{(i)*}, \dots, e_r^{(i)*}$ is the dual basis of sections of $E^*|_{U_i} \to U_i$.

Then, set $V = \mathbb{C}S^*$, the set of functions $S \to \mathbb{C}$. Then, evaluation defines an injection $E \hookrightarrow \underline{V}$: evaluating at E_x determines a value on each basis element on each ρ_i that doesn't vanish there, so we get values on basis elements. Moreover, since at least one such ρ_i exists for each point, this map is injective.

Let $E' = \underline{V}/E$, so we have a short exact sequence

$$0 \longrightarrow E \longrightarrow \underline{V} \longrightarrow E' \longrightarrow 0.$$

Last time, we proved in Theorem 2.10 that all short exact sequences of vector bundles exist, so there's an isomorphism $E' \oplus E \xrightarrow{\sim} V$. ⊠

Now, we can do some stuff that will look familiar from cohomology.

Definition. The *reduced K-theory* of X is the quotient $\widetilde{K}(X) = K(X)/p^*K(\mathrm{pt})$, where $p : X \to \mathrm{pt}$.

Example 3.4. If $X = \mathrm{pt} \sqcup \mathrm{pt}$, then $K(X) \xrightarrow{\sim} \mathbb{Z} \oplus \mathbb{Z}$ sending bundles to their ranks. Then, $p^* : K(\mathrm{pt}) = \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is the diagonal map Δ, so $\widetilde{K}(X) = \mathbb{Z} \oplus \mathbb{Z}/\Delta \xrightarrow{\sim} \mathbb{Z}$.

Corollary 3.5. *Let $E, E' \to X$ be vector bundles. Then, $[E] = [E']$ in $\widetilde{K}(X)$ iff there exist $r, r' \in \mathbb{Z}^{\geq 0}$ such that $E \oplus \underline{\mathbb{C}}^r \cong E' \oplus \underline{\mathbb{C}}^{r'}$.*

In this case, we say that E and E' are *stably equivalent.* In other words, *K-theory remembers the stable equivalences of vector bundles.* This is the first inkling we have of what K-theory is about, and what the geometric meaning of group completion is.

Example 3.6. Let's look at $\widetilde{KO}(S^2)$. We have a nontrivial bundle of rank 2 over S^2, $TS^2 \to S^2$. However, $TS^2 \oplus \underline{\mathbb{R}} \to S^2$ is trivializaible!

To see this, embed $S^2 \hookrightarrow \mathbb{A}^3$; such an embedding always gives us a short exact sequence of vector bundles

$$0 \longrightarrow TS^2 \longrightarrow T\mathbb{A}^3|_{S^2} \longrightarrow \nu \longrightarrow 0.$$

The quotient ν, by definition, is the *normal bundle* of the submanifold (in this case, S^2). We know that $T\mathbb{A}^3 = \underline{\mathbb{R}}^3$ everywhere, which is almost by definition, and therefore $\nu \cong \underline{\mathbb{R}}$. This means that in $\widetilde{KO}(S^2)$, $[TS^2] = 0$.

So right now, we can calculate the K-theory of a point, and therefore of any contractible space. We want to be able to do more; a nice first step is to compute the K-theory of S^n. Just as in cohomology, this will allow us to bootstrap our calculations on CW complexes.

Definition. Recall that a *fiber bundle* is the data $\pi : E \to X$ over a topological space X such that π is surjective and local trivializations exist. E is called the *total space.*

Thus, a vector bundle is a fiber bundle where the fibers are vector spaces, *and* we require the local trivializations to respect this structure. We can do this more generally, e.g. with affine spaces and affine maps.

Example 3.7. If $V \to X$ is a vector bundle, we get some associated fiber bundles over X. For example, $\mathbb{P}V \to X$, with fiber of lines in the vector space that's the fiber of V. We can generalize to the Grassmanian $\mathrm{Gr}_k V$, which uses k-dimensional subspaces instead of lines. There are plenty more constructions.

Definition. A topological space F is *k-connected* if $Y \to F$ is null-homotopic for every CW complex Y of dimension at most k.

It actually suffices to take only the spheres for Y.

Lemma 3.8. *Let n be a positive integer and $\pi : \mathcal{E} \to X$ be a fiber bundle, where X is a CW complex with finitely many cells and of dimension at most n, and the fibers of π are $(n-1)$-connected. Then, π admits a continuous section.*

Proof. We'll do cell-by-cell induction on the skeleton $X_0 \subset X_1 \subset \cdots \subset X_n = X$. On points, π trivially has a continuous section.

Suppose we have constructed s on X_{k-1}. Then, all the k-cells are attached via maps

$$\begin{tikzcd} D^k \arrow[r, "\Phi"] & X \\ S^{k-1} \arrow[u, hook] \arrow[r, "\partial\Phi"] & X_{k-1}. \arrow[u, hook] \end{tikzcd}$$

Since $D^k \simeq \mathrm{pt}$, then $\Phi^*\mathcal{E} \to D^k$ is trivializable, so we have a map $\theta : \Phi^*\mathcal{E} \to \underline{F}$. The section on X_{k-1} pulls back and composes with θ to create a map $S^{k-1} = \partial D^k \to F$, but by hypothesis, this is null-homotopic, and therefore extends to D^k. ⊠

A different kind of induction is required when X has infinitely many cells; however, what we've proven is sufficient for the K-theory of the spheres.

Theorem 3.9. *Let $n \in \mathbb{Z}^{\geq 0}$ and $N \geq n/2$. Then, there is an isomorphism $\pi_{n-1}\,\mathrm{U}_N \to \widetilde{K}(S^n)$.*

Corollary 3.10. *The inclusion $\mathrm{U}_N \hookrightarrow \mathrm{U}_{N+1}$ induces an isomorphism $\pi_{n-1}\,\mathrm{U}_N \to \pi_{n-1}\,\mathrm{U}_{N+1}$ if $N \geq n/2$.*

Note that the theorem statement doesn't give enough information to say which map induces the isomorphism, but the proof will show that the usual inclusion does it. Specifically, thinking of U_N as a matrix group, U_N embeds in U_{n+1} on the upper left, i.e.

$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

We can take the union (direct limit) of the inclusions $\mathrm{U}_1 \subset \mathrm{U}_2 \subset \mathrm{U}_3 \subset \ldots$, and call it U_∞ (sometimes U). These sequences of homotopy groups must stabilize.

Theorem 3.11 (Bott)**.**

$$\pi_{n-1}\,\mathrm{U}_\infty \cong \widetilde{K}(S^n) = \begin{cases} \mathbb{Z}, & n \text{ even} \\ 0, & n \text{ odd}. \end{cases}$$

We have a real analogue to this theorem as well: the analogous inclusion $\mathrm{O}_1 \hookrightarrow \mathrm{O}_2 \hookrightarrow \cdots$ define a limit O_∞.

Theorem 3.12.

$$\pi_{n-1}\,\mathrm{O}_\infty \cong \widetilde{KO}(S^n) = \begin{cases} \mathbb{Z}, & n \equiv 0, 4 \bmod 8 \\ \mathbb{Z}/2\mathbb{Z}, & n \equiv 1, 2 \bmod 8 \\ 0, & n \equiv 3, 5, 6, 7 \bmod 8. \end{cases}$$

These results, known as the *Bott periodicity theorems*, are the foundations of Bott periodicity. We'll give three proofs: Bott's original proof using Morse theory, a more elementary one, and one that uses functional analysis and Fredholm operators.

Lecture 4.

Bott's Theorem: 9/8/15

"Any questions?"
"How was your weekend?"
"I was afraid of that."

We know that vector bundles always have sections (e.g. the zero section), but fiber bundles don't. For example, the following fiber bundles don't have sections.

- The orientation cover of a nonorientable manifold (e.g. the Möbius strip) is a double cover that doesn't have a section.
- The Hopf fibration $S^1 \to S^3 \to S^2$.

- Any nontrival covering map $S^1 \to S^1$.

However, sometimes sections do exist.

Theorem 4.1. *If X is a CW complex of dimension n and $\pi : \mathcal{E} \to X$ is a fiber bundle, then if the fibers of π are $(n-1)$-connected, then π admits a section.*

Definition. A *fibration* is a map $\pi : \mathcal{E} \to B$ satisfying the *homotopy lifting property*: that is, if $h : [0,1] \times S \to X$ is a homotopy and $f : \{0\} \times S \to \mathcal{E}$, then f can be lifted across the whole homotopy, i.e. there exists an $\widetilde{f} : [0,1] \times S \to \mathcal{E}$ that makes the following diagram commute.

$$\begin{tikzcd}
\{0\} \times S \arrow[r, "f"] \arrow[d, hook] & \mathcal{E} \arrow[d, "\pi"] \\
{[0,1] \times S} \arrow[r, "h"] \arrow[ur, dashed, "\widetilde{f}"] & X
\end{tikzcd}$$

Theorem 4.2. *A fiber bundle is a fibration.*

We won't prove this, but we also won't use it extremely extensively.

Theorem 4.3. *Let $N, n \in \mathbb{Z}^{\geq 0}$ and $N \geq n/2$. Then, there is an isomorphism $\varphi : \pi_{n-1} \mathrm{U}_N \to \widetilde{K}(S^n)$ defined by clutching.*

This is part of Theorem 3.11 from last time. Recall that in the reduced K-theory, two bundles are equivalent iff they are stably isomorphic: for example, over S^2, the tangent bundle is stably isomorphic to any trivial bundle, so it's equal to zero.

Proof of Theorem 4.3. We'll show that φ is a composition of three isomorphisms

$$\pi_{n-1} \mathrm{U}_N \xrightarrow{i} [S^{n-1}, \mathrm{U}_N] \xrightarrow{j} \mathrm{Vect}_N^{\cong}(S^n) \xrightarrow{k} \widetilde{K}(S^n).$$

To define i, we'll pick a basepoint $* \in S^{n-1}$; then, $\pi_{n-1} \mathrm{U}_N$ is equal to $\{f : S^{n-1} \to \mathrm{U}_N : f(*) = e\}$ up to based homotopy (U_N is naturally a pointed space, using its identity element). We want this to be isomorphic to $[S^{n-1}, \mathrm{U}_N]$, the set of maps without basepoint condition up to homotopy, so let $\phi : [S^{n-1}, \mathrm{U}_N] \to \pi_{n-1} \mathrm{U}_N$ be defined by $\phi(f) = f(*)^{-1} \cdot f$, where $f : S^{n-1} \to \mathrm{U}_N$. Then, one can check that ϕ is well-defined on homotopy classes and inverts i, so i is an isomorphism.[5]

j is defined by the clutching construction. We can write $S^n = D^n_+ \cup_{S^{n-1}} D^n_-$, and then glue $\underline{\mathbb{C}}^N \to D^n_+$ and $\underline{\mathbb{C}}^N \to D^n_-$ using $f : S^{n-1} \to \mathrm{U}_N$, because U_N is the group of isometries of $\mathbb{C}^N$. So this defines a map j, but why is it an isomorphism? We have to show that j is surjective.

Last time, we showed that the group of isomorphisms deformation retracts onto the group of isometries, so that's fine. To show that j is surjective, we could use that every vector bundle admits a Hermitian metric, or that every vector bundle over D^n is trivializable by orthogonal bases, both of which are true. That j is well-defined follows from an argument that homotopic clutching functions lead to isomorphic vector bundles. Finally, to show that j is injective, all trivializations over D^n are homotopic, since D^n is contractible and U_N is connected.

Then, k just sends a vector bundle to its stable equivalence class. For its surjectivity, we need to show that if $E \to S^n$ has rank $N \geq n/2 + 1$, then there exists an E' of rank $N - 1$ and an isomorphism of the $\underline{\mathbb{C}} \oplus E' \cong E$. In words, for large enough N, we can split off a trivial bundle from E. Equivalently, we can show that $E \to S^n$ admits a nonzero section, whose span is a line bundle $L \to X$ which is trivialized; then, we can let $E' = E/L$.

A nonzero section, normalized, is a section of the fiber bundle $S(E) \to S^{n-1}$ with fiber S^{2N-1} (the unit sphere sitting in $\mathbb{C}^N$).[6] This sphere is $(2n-2)$-connected, so by Theorem 4.1, such a section exists.

Why is k injective? We need to show that if a rank-N bundle is stably trivial in $\widetilde{K}(S^n)$, then it is actually trivial. But since it's not clear that $\mathrm{Vect}_N^{\cong}(S^n)$ is an abelian group (yet), then we'll show injectivity of sets. Let $E_0, E_1 \to S^n$ be rank-N vector bundles with an isometry $E_0 \oplus \underline{\mathbb{C}}^r \to E_1 \oplus \underline{\mathbb{C}}^r$; we'll want to produce a

[5] $[S^{n-1}, \mathrm{U}_N]$ inherits another group structure from that of U_N (i.e. pointwise multiplication of loops); one can reason about it using something called the Eckmann-Hilton argument.

[6] The *sphere bundle* $S(E)$ of a vector bundle E is the fiber bundle whose fiber over each point x is the unit sphere in the E_x.

homotopic isometry which preserves the last vector $(0,\dots,0,1)\in\mathbb{C}^r$ at each point in X. The evaluation map $\mathrm{ev}_{(0,\dots,0,1)}$ at the last basis vector is a map of fiber bundles over X; that is, the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Isom}(E_0\oplus\underline{\mathbb{C}}^r, E_1\oplus\underline{\mathbb{C}}^r) \arrow[rr, "\mathrm{ev}_{(0,\dots,0,1)}"] \arrow[dr] & & S(E_1\oplus\underline{\mathbb{C}}^r) \arrow[dl] \\
& S^n &
\end{tikzcd}$$

An isometry is a section $\varphi : S^n \to \mathrm{Isom}(E_0\oplus\underline{\mathbb{C}}^r, E_1\oplus\underline{\mathbb{C}}^r)$, so applying the evaluation map, we get a section $p\varphi : S^n \to S(E_1\oplus\underline{\mathbb{C}}^r$. We get an additional section $\xi=(0,0,\dots,0,1)$. Thus, all that's left is to construct a homotopy from $p\varphi$ to ξ, which by the homotopy lifting property defines a section of the pullback $[0,1]\times S(E_1\oplus\underline{\mathbb{C}}^r)\to[0,1]\times S^n$ over $\{0,1\}\times S^n$. ⊠

Note that, while the K-theory is a ring given by tensor product, the reduced K-theory isn't a ring in most cases.

These arguments are important to demonstrate that when N is high enough, in the stable range, we have this stability.

Corollary 4.4. *If N is in the stable range, i.e. $N\ge n/2$, then the inclusion $\mathrm{U}_N\hookrightarrow\mathrm{U}_{N+1}$ induces an isomorphism $\pi_{n-1}\mathrm{U}_N\to\pi_{n-1}\mathrm{U}_{N+1}$.*

This means that eventually $\pi_{n-1}\mathrm{U}_N$ is identical for large enough N; this group, the *stable isomorphism group* of the unitary groups, is written $\pi_{n-1}(\mathrm{U})$ (and there is a group U that makes this work, the limit of these U_N with the appropriate topology). Then, Bott's theorem, Theorem 3.11, calculates these groups: $\pi_{n-1}\mathrm{U}$ is $\mathbb{Z}$ when n is even and 0 when n is odd.

For example, a generator of $\pi_1\mathrm{U}_3$ is given by stabilizing a loop $e^{i\theta}$; that is, it's given by the map

$$e^{i\theta}\longmapsto\begin{pmatrix}e^{i\theta}&0&0\\0&1&0\\0&0&1\end{pmatrix},$$

with $\theta\in S^1$.

Outlining a Proof of Bott's Theorem. We'll move to providing different proofs of Theorem 3.11; these are explained in our readings, and so the professor won't post lecture notes for a little while.

Let's re-examine $S^2\cong\mathbb{CP}^1=\mathbb{P}(\mathbb{C}^2)$ (that is, the space of lines in $\mathbb{C}^2$). More generally, if V is a vector space, $\mathbb{P}V$ will denote its *projectivization*, the space of lines in V. Then, there is a tautological line bundle $H^*\to\mathbb{P}V$, whose fiber at a line $K\subset V$ (which is a point of $\mathbb{P}V$) is the line K.

The dual of H^* is called the *hyperplane bundle*, and denoted $H\to\mathbb{P}V$; a nonzero element of H can be identified with a hyperplane in V, and there is a canonical map $V^*\to\Gamma(\mathbb{P}V,H)$ (where $\Gamma(X,E)$ denotes the sections of $E\to X$): a linear functional on V becomes a linear functional on a line by restriction. Interestingly, if V is a complex vector bundle, then this is an isomorphism onto the holomorphic sections. In particular, the space of holomorphic sections is finite-dimensional.

In fact, if you take $\mathrm{Sym}^k V^*$, the k^{th} symmetric power of V^*, then there's a canonical map $\mathrm{Sym}^k V^*\to\Gamma(\mathbb{P}V,H^{\otimes k})$, which is again an isomorphism in the complex case.

If $V=\mathbb{C}^2$, then write $V=L\oplus\mathbb{C}$; then, L and $\mathbb{C}$ are distinguished points in our projective space. This will enable us to make a clutching-like construction in a projective space.

Let $P_\infty=\mathbb{PC}^2\setminus\{\mathbb{C}\}$ and $P_0=\mathbb{PC}^2\setminus\{L\}$; then, $P_0\cap P_\infty\cong\mathbb{PC}^2\setminus\{\mathbb{C},L\}=L^*\setminus\{0\}$. Our clutching construction will start with a vector bundle $\underline{L}\to P_0$, a vector bundle $\underline{\mathbb{C}}\to P_\infty$, and an isomorphism $\alpha:\underline{L}\to\underline{\mathbb{C}}$ over the intersection $P_0\cap P_\infty=L^*\setminus\{0\}$. Thus, we'll need to specify an isomorphism $P_0\cap P_\infty\to L^*\setminus\{0\}$ to determine how to glue $\underline{L}$ and $\underline{\mathbb{C}}$ together.

It's natural to call the identity map z^{-1}, thinking of $z\in L$, and the bundle we get is $H\to\mathbb{PC}^2$. Here again we have a punctured plane and so the winding number classifies things.

Lemma 4.5. *$H\oplus H\cong H^{\otimes 2}\oplus\underline{\mathbb{C}}$ as vector bundles over $\mathbb{CP}^1\cong S^2$.*

Proof. The two clutching maps are, respectively, $\begin{pmatrix}z^{-1}&\\&z^{-1}\end{pmatrix}$ and $\begin{pmatrix}z^{-2}&\\&1\end{pmatrix}$. Each has determinant 1, so they're both in $\mathrm{SL}_2\mathbb{C}$, which deformation-retracts onto S^2, which is simply connected. Thus, the clutching maps are homotopic. ⊠

Corollary 4.6. *If $t = [H] - 1$ in $K(S^2)$, then $t^2 = 0$.*

This is the first insight we have into the ring structure of a K-theory.

Corollary 4.7. *The map $\mathbb{Z}[t]/(t^2) \to K(S^2)$ sending $t \mapsto [H] - 1$ is an isomorphism of rings.*

Definition. Let X_1 and X_2 be topological spaces; then, there are projection maps

$$\begin{tikzcd} X_1 \times X_2 \arrow[r, "p_1"] \arrow[d, "p_2"] & X_1 \\ X_2. & \end{tikzcd}$$

Then, the *external product* is a map $K(X_1) \otimes K(X_2) \to K(X_1 \times X_2)$ defined as follows: if $u \in K(X_1)$ and $v \in K(X_2)$, then $u \otimes v \mapsto p_1^* u \cdot p_2^* v$.

Theorem 4.8. *If X is compact Hausdorff, then the external product $K(S^2) \otimes K(X) \to K(S^2 \times X)$ is an isomorphism of rings.*

We'll talk about this more next lecture; the idea is that in general distinguished basepoints of X and S^2 lift to subspaces of $S^2 \times X$.

The reason it doesn't work for S^1 is that if $X = S^1$, we get a torus $S^1 \times S^1$. Then, basepoints in S^1 give us $S^1 \vee S^1$ (the *wedge product*), and the quotient is $S^1 \wedge S^1 \simeq S^2$ (the *smash product*).

In fact, we'll bootstrap Theorem 4.8, using the smash product and reduced K-theory; then, results about smash products of spheres do a bunch of the work of periodicity for us. The proof will be elementary, in a sense, but with a lot of details about clutching functions, which is pretty explicit.

The version you'll read about in the Atiyah-Bott paper, or in Atiyah's book, is slightly more general. We want a family of S^2 parameterized by X, instead of just one, which is a fiber bundle; but we want two distinguished points, which will allow the clutching construction, and a linear structure.

Thus, more generally, if $L \to X$ is a complex line bundle, then $\mathbb{P}(L \oplus \underline{\mathbb{C}}) \to X$ is a fiber bundle with fiber S^2. We can once again form the hyperplane bundle $H \to \mathbb{P}(L \otimes \underline{\mathbb{C}})$.

Theorem 4.9. *The map $K(X)[t]/(t[L] - 1)(t - 1) \to K(\mathbb{P}(L \oplus \underline{\mathbb{C}}))$ defined by sending $t \mapsto [H]$ is an isomorphism of rings.*

Then, if $X = \mathrm{pt}$, we recover Theorem 4.8, which we'll prove next time.

Lecture 5.

: 9/10/15

Our immediate goal is to prove the following theorem.

Theorem 5.1. *Let X be compact Hausdorff. Then, the map $\mu : K(X)[t]/(1-t)^2 \to K(X) \otimes K(S^2) \to K(X \times S^2)$, defined by sending $[E] \cdot t \mapsto [E] \otimes [H]$ followed by $[E_1] \otimes [E_2] \mapsto [\pi_1^* E_1 \otimes \pi_2^* E_2]$, is an isomorphism.*

Next time, we'll introduce basepoints and use this to prove Bott periodicity, calculating the K-theory of the spheres in arbitrary dimension; we saw last time that this computes the stable homotopy groups of the unitary group.

The proof we give is due to Atiyah and Bott, and actually proves a stronger result, Theorem 4.9. Hatcher's notes provide a proof of the less general theorem.

The heuristic idea is that a bundle on S^2 is given by clutching data: two closed discs D_∞ and D_0 along with a circle $S^1 = \mathbb{T}$ (i.e. we identify it with the circle group $\mathbb{T} = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$, which is a Lie group under multiplication). Then, the final piece of clutching data is given by a group homomorphism $f : \mathbb{T} \to \mathrm{GL}_r \mathbb{C}$

Suppose f is given by a Laurent series

$$f(\lambda) = \sum_{k=-N}^{N} a_k \cdot \lambda^k,$$

with $a_k \in \operatorname{End} \mathbb{C}^r$ (i.e. they might not be invertible, but their sum is). Then, $f = \lambda^{-N} p$ for a $p \in \mathbb{C}[\lambda] \otimes \operatorname{End} \mathbb{C}^r$. Then, the K-theory class of this bundle is determined by the rank r and the winding number of $\lambda^{-N} p$, which we'll denote $\omega(\lambda^{-N}p) = -Nr + \omega(p)$. That is, it's basically determined by the winding number of a polynomial.

What is the winding number of a polynomial? For simplicity, tke $r = 1$; then, $\omega(p)$ is the number of roots of p interior to $\mathbb{T} \subset \mathbb{C}$.

In some sense, we're taking the winding number as information about S^2, but we're not getting a lot of information about X. We *categorify*: we want to find a vector space whose dimension is $\omega(p)$. Set $R = \mathbb{C}[\lambda]$, which is a commutative ring, and $M = \mathbb{C}[\lambda]$ as an R-module. (If $r > 1$, we need to tensor with $\operatorname{End}\mathbb{C}^r$ again). Then, $p : M \to M$ given by multiplication by p, has a cokernel $\operatorname{coker} p = V$, a $\deg(p)$-dimensional vector space. Thus, we can canonically decompose $V = V_+ \oplus V_-$, where V_+ is the set of roots inside the unit disc. Then, we can soup this up further when $r > 1$ and X comes back into the story.

This is essentially the way that we'll prove the theorem: the proof will construct an inverse map ν to μ. The main steps are:

- approximate an arbitrary clutching by a Laurent series, leading to a polynomial clutching
- convert a polynomial clutching to a linear clutching, and
- convert a linear clutching to a vector bundle V over X.

Proof of Theorem 5.1. The first step, approximating by Laurent series, requires some undergraduate analysis. Suppose $f : X \times \mathbb{T} \to \mathbb{C}$ is continuous. The Fourier coefficients of a function on $\mathbb{T}$ become functions parameterized by X: set

$$a_n(x) = \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} f\big(x, e^{i\theta}\big) e^{-in\theta}, \quad n \in \mathbb{Z},$$

and let $u : X \times [0,1) \times \mathbb{T} \to \mathbb{C}$ be

$$u(x, r, \lambda) = \sum_{n\in\mathbb{Z}} a_n(x) r^{|n|} \lambda^n.$$

Then, u is continuous, because

$$\|a_n\|_{C^0(X)} \le \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \|f\|_{C^0(X\times\mathbb{T})} |e^{-in}| = \|f\|_{C^0(X\times\mathbb{T})}.$$

Proposition 5.2. *$u(x, r, \lambda) \to f(x, \lambda)$ as $r \to 1$ uniformly in x and λ.*

Proof. Introduce the *Poisson kernel* $P : [0,1) \times \mathbb{T} \to \mathbb{C}$, given by

$$P(r, e^{is}) = \sum_{n\in\mathbb{Z}} r^{|n|} e^{ins} = \frac{1 - r^2}{1 - 2r\cos s + r^2}, \tag{5.1}$$

which can be proven by treating the positive and negative parts as two geometric series. Then, since it converges absolutely, we can integrate term-by-term to show that

$$\int_{\mathbb{T}} \frac{\mathrm{d}s}{2\pi} P\big(r, e^{-is}\big) = 1.$$

Additionally, if $\lambda \ne 1$, (5.1) tells us that $\lim_{r\to 1} P(r, \lambda) = 0$. Thus, $\lim_{r\to 1} = \delta_1$ in $C^0(\mathbb{T})^*$ (i.e. $\delta_1(f) = f(1)$ for $f \in C^0(\mathbb{T})$, as a distribution). Now, we can write u as a convolution on $\mathbb{T}$:

$$\begin{aligned} u\big(x, r, e^{i\theta}\big) &= \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} P\Big(r, e^{i(\theta-\phi)}\Big) f\big(x, e^{i\phi}\big) \\ &= P_\theta(r, -) *_{\mathbb{T}} f(x, -) \\ &= \langle \widetilde{P}_\theta(r, -), f(x, -)\rangle, \end{aligned}$$

where our pairing is a map $C^0(\mathbb{T})^* \times C^0(\mathbb{T}) \to \mathbb{C}$. ⊠

This will allow us to approximate a clutching function with a finite step in the Fourier series, producing a Laurent series as intended.

Corollary 5.3. *The space of Laurent functions*

$$\sum_{|k|\le N} a_k(x) \lambda^k$$

is dense in $C^0(X \times \mathbb{T})$.

Proof. If $f \in C^0(X\times\mathbb{T})$, define a_k and u as before. Given an $\varepsilon > 0$, there's an r_0 such that $\|f-u(r)\|_{C^0(\times\mathbb{T})} < \varepsilon/2$ if $r > r_0$, and an N such that

$$\sum_{|n|>N} r_0^N < \frac{\varepsilon}{2\|f\|_{C^0(X\times\mathbb{T})}}.$$

Then, one can show that the norm of the difference is less than ε. ⊠

Thus, we have our approximations of clutching bundles. Note that Hatcher's proof involves a little less "undergraduate" analysis.

Thinking about S^2 as $\mathbb{P}(\mathbb{C}_0\oplus\mathbb{C}_\infty) = \mathbb{CP}^1$, we can look at the tautological bundle. If $\lambda\in\mathbb{C}$, then the line $y = \lambda x$ in $\mathbb{C}_0\times\mathbb{C}_\infty$ projects down, e.g. $(1,\lambda)$ to 1 and λ. In particular, the tautological bundle $H^*\to\mathbb{CP}^1 = S^2$ has clutching function λ, and therefore the hyperplane bundle $H\to\mathbb{CP}^1 = S^2$ has clutching function λ^{-1}.

For a more general $\mathcal{E}\to X\times S^2$, we want to clutch $X\times D_0$ and $X\times D_\infty$ at $X\times\mathbb{T}$. Define $E\to X$ as the restriction of $\mathcal{E}\to X\times S^2$ to $X\times\{1\}$; then, E pulls back to bundles $\pi_0^*E\to X\times D_0$ and $\pi_\infty^*E\to X\times D_\infty$. Since D_0 and D_∞ are contractible, we can choose isomorphisms $\theta_0 : \pi_0^*E \xrightarrow{\cong} \mathcal{E}|_{X\times D_0}$ and $\theta_\infty : \pi_\infty^* E\to\mathcal{E}|_{X\times D_\infty}$.

Then, $f = \theta_\infty^{-1}\circ\theta_0$ is a section of the bundle $\operatorname{Aut}(\pi_{\mathbb{T}}^*E)\to X\times\mathbb{T}$. In other words, $X\times\mathbb{T}$ embeds into $X\times D_0$ and $X\times D_\infty$, and f is the clutching data from $\pi_0^*E\to\pi_\infty^*E$.

Also, we can and will choose θ_0,θ_∞ to be the identity on $X\times\{1\}$, so that f is the identity there too.

Notationally, we'll write $[\mathcal{E}] = [E, f]\in K(X\times S^2)$; we can start with an $E\to X$ and such an f, an automorphism of $E\times\mathbb{T}\to X\times\mathbb{T}$, to get a vector bundle on $X\times S^2$. For example, $[\underline{\mathbb{C}},\lambda] = [H^*]$, $[\underline{\mathbb{C}},\lambda^n] = [H^{\otimes(-n)}]$, and $[E, f\cdot\lambda^n] = [E,f]\cdot[H^{\otimes(-n)}]$ in $K(X\times S^2)$ (which one can check).

What this argument shows is the following.

Proposition 5.4. *Any vector bundle on $X\times S^2$ is isomorphic to one of the form (E,f), and any two choices of f are homotopic through normalized clutching functions.*

Here, a normalized clutching function is one homotopic through the basepoint.

Now we have our clutching function, which is continuous, and replace it with a Laurent function.

Proposition 5.5.

(1) In $K(X,S^2)$, $[E,f] = [E,\lambda^{-N}p]$ for some polynomial clutching function

$$p(x,\lambda) = \sum_{k=0}^{2n} a_k(x)\lambda^k,$$

with $a_k(x)\in\operatorname{End} E_x$.

(2) Any two such choices are homotopic via a Laurent clutching function.

Proof. The proof will show that the Laurent endomorphisms of $E\times\mathbb{T}\to X\times\mathbb{T}$. If $E = \underline{\mathbb{C}}$, the proof is the same proof with Poisson kernels at the start of the class; more generally, we'll use a partition of unity $\{\rho_i\}$ subordinate to a cover $\{U_i\}$ such that $E|_{U_i}$ is trivial. Then, $f|_{U_i}$ can be approximated by a Laurent ℓ_i, and one can check that $\sum\rho_i\ell_i$ is Laurent.

For (1), since the invertible matrices are an open set, then choose an $\varepsilon > 0$ such that $B_\varepsilon(f)$ contains only invertible functions, and choose an ℓ Laurent such that $\|f-\ell\|_{C^0(X\times\mathbb{T})} < \varepsilon$, so that ℓ is invertible and $f\simeq\ell$ by a straight-line homotopy. And we know clutching with homotopic functions doesn't change the isomorphism class of the vector bundle, hence nor the K-theory class. ⊠

Thus, we've gone from continuous to Laurent; now, we will go from Laurent to linear. Observe that $[E,f] = [E,-\lambda^N p] = [H^{\otimes N}] - [E,p]$.

Let p be a polynomial clutching function of degree at most n. Then, write

$$p(x,\lambda) = \sum_{k=0}^{n} p_k(x)\lambda^k,$$

and set

$$\mathcal{L}_p^m = \begin{pmatrix} 1 & -\lambda & & & \\ & 1 & -\lambda & & \\ & & \ddots & \ddots & \\ & & & 1 & -\lambda \\ p_n & p_{n-1} & \dots & p_1 & p_0 \end{pmatrix}.$$

This matrix of polynomials acts linearly on $E^{\oplus(n+1)} \times \mathbb{T} \to X \times \mathbb{T}$.

Proposition 5.6. $[E^{\oplus(n+1)}, \mathcal{L}_p^n] = [E,p] + [E^{\oplus n}, 1]$.

Proof. The clutching function for the right-hand side is

$$\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & p \end{pmatrix},$$

and this is exactly the matrix you get if you diagonalize $\mathcal{L}_p^n$ by elementary row and column operations. Thus, they're homotopic, and so have the same class in K-theory. ⊠

We'll then make a basic spectral construction. Suppose $T \in \operatorname{End}\mathbb{E}$ has no eigenvalues on the unit circle $\mathbb{T} \subset \mathbb{C}$. Then, take the contour integral

$$Q = \frac{1}{2\pi i} \int\limits_{|\omega|=1} (\omega - T)^{-1}\,\mathrm{d}\omega,$$

which is in $\operatorname{End}\mathbb{E}$. One can check that $Q^2 = Q$, so it's a projection, and $QT = TQ$. Thus, we can decompose $\mathbb{E} = Q\mathbb{E} \oplus (1-Q)\mathbb{E}$, which we'll denote $\mathbb{E}_+$ and $\mathbb{E}_-$, respectively. Since T commutes with Q, $T = (T_+, T_-)$, with T_+ acting on $\mathbb{E}_+$, and similarly for T_- on $\mathbb{E}_-$. This is analogous to the spectral theorem's decomposition of an operator into its generalized eigenspaces.

Proposition 5.7. *Let* $[E,q]$ *be a* K*-theory class with* $q(x,\lambda) = a(x)\lambda + b(x)$. *Then, there is a splitting* $E = E_+ \oplus E_-$ *such that* $[E,q] = [E_+,\lambda] + [E_-,1]$.

Proof. Define

$$Q = \frac{1}{2\pi i} \int\limits_{|\lambda|=1} q^{-1}\,\mathrm{d}q = \frac{1}{2\pi i} \int\limits_{|\lambda|=1} q^{-1}\frac{\partial q}{\partial \lambda}\,\mathrm{d}\lambda.$$

Choose an $\alpha \in \mathbb{R}^{>1}$ such that $q(x,\alpha)$ is an isomorphism for all x, which works because isomorphism is an open condition. Then, compose with $q(x,\alpha)^{-1}$, so we can assume $q(x,\alpha) = \mathrm{id}$. Then, $w = (1-\alpha\lambda)/(\lambda - \alpha)$ preserves $\mathbb{T}$ and D_0 as $\alpha \to \infty$. Define $q(\lambda) = (w-T)/(w+\alpha)$ with $T \in C^0(X; \operatorname{End} E)$, and $qT = Tq$. Then,

$$Q = \frac{1}{2\pi i} \int\limits_{|w|=1} (w-T)^{-1}\,\mathrm{d}w - (w+\alpha)^{-1}\,\mathrm{d}w,$$

but the last term goes away. Thus, this is the desired projection: q fails to be invertible exactly where T has an eigenvalue. Denote $q_\pm(\lambda) = a_\pm\lambda + b_\pm$, and $q_+(\lambda)$ is invertible if $\lambda \in D_\infty$ and $q_-(\lambda)$ is if $\lambda \in D_0$.

Thus, $q_+^t = a_+\lambda + tb_+$ and $q_-^t = ta_-^\lambda + b_-$ are homotopies of clutching functions, so

$$\begin{aligned} [E,q] &= [E_+, q_+] + [E_-, q_-] \\ &= [E_+, a_+\lambda] + [E_-, b_-] = [E_+,\lambda] + [E_-,1]. \end{aligned}$$ ⊠

So if we have $[E,p]$ with $\deg(p) \le n$, then

$$[E,p] = [E^{\oplus(n+1)}], \mathcal{L}_p^n] - [E^{\oplus n}, q],$$

and we just proved that a linear clutching function splits as

$$\begin{aligned} &= [V_n(E,p),\lambda] + [E^{\oplus(n+1)}, 1] - [V_n(E,p),1] - [E^{\oplus n}, 1] \\ &= [V_n(E,p)] \otimes ([H^*] - 1) + [E] \otimes 1, \end{aligned}$$

where $V_n(E, p)$ is the $+$ part of the decomposition of $E^{\oplus(n+1)}$ by $q = \mathcal{L}_p^n$. So we've gone from polynomial to linear and then split it; this will allow us to define the inverse, check it's well-defined and in fact the inverse, and so on. But this is enough of a proof sketch to follow the references and work out the details. ⊠

Even though the proof is confusing, all of the ideas are relatively elementary.