NOTES FROM THE 2022 SUMMER SCHOOL ON GLOBAL SYMMETRIES

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Contents

Part 1. Kantaro Ohmori, Introduction to symmetries in quantum field theory

1. Extended operators and defects in functorial QFT: 6/13/22

One of the goals of this workshop is to discuss generalized symmetries in quantum field theory. For us, this means topological operators and defects. Here's a short outline of these four talks:

(1) Quantum field theory as a functor, and how to work with extended operators and defects in this formalism
(2) Topological operators and symmetries
(3) One-form symmetries in gauge theories and confinement
(4) "Non-invertible" symmetries

Let's get started. Quantum field theory means a lot of different things to a lot of different people; today, we will only focus on relativistic Euclidean QFT. When we say a d-dimensional QFT, d refers to the dimension of spacetime.

In the non-topological setting, it's not yet completely clear how to define a quantum field theory as a functor, so the following definition will be a little heuristic.

Definition 1.1. A quantum field theory is a symmetric monoidal functor $Z\colon \mathcal{B}ord_S^{\langle d,d-1\rangle} \to \mathcal{V}ect$.

Here $\mathcal{V}ect$ is the symmetric monoidal category of vector spaces with tensor product and $\mathcal{B}ord_S^{\langle d,d-1\rangle}$ is a bordism category. S refers to some kind of geometric structure we want to endow spacetime with: for example, we could ask for just a smooth structure, or a spin structure, or a Riemannian metric, or a principal G-bundle with a connection, or so on. A manifold with S-structure is called an S-manifold.

The objects of $\mathcal{B}ord_S^{\langle d,d-1\rangle}$ are closed, $(d-1)$-dimensional manifolds with an S-structure. The set of morphisms between $(d-1)$-dimensional S-manifolds M and N is the set of (diffeomorphism classes rel boundary) of S-bordisms from M to N. An *S-bordism* X from M to N is a compact, d-dimensional manifold with an identification of S-manifolds $\partial X \xrightarrow{\cong} M \amalg \overline{N}$. Here $\overline{N}$ denotes N with the opposite orientation.

To define a category, we need to compose morphisms; this is accomplished by gluing bordisms. (TODO: picture).

Remark 1.2. We haven't said precisely how to define S, so one might wonder whether it depends on d. For example, there's a difference between a framing of a manifold M (a trivialization of TM) and a stable framing (a trivialization of $TM \oplus \underline{\mathbb{R}}^k$ for some k); the former depends on d and the latter does not, and the two notions are not the same.

Freed–Hopkins [FH21] have shown that for reflection-positive topological field theories, many S-structures that appear to depend on d in fact stabilize and are independent of the dimension. ◀

If $Z\colon \mathcal{B}ord_S^{\langle d,d-1\rangle}$ is a quantum field theory, then for a closed $(d-1)$-dimensional S-manifold M, $Z(M)$ is a vector space. This is called the *state space* of M. If X is a bordism from M to N, then $Z(X)$ is a linear map from the state space of M to the state space of N. We often think of this map as *time evolution* of states. The fact that Z is symmetric monoidal means that $Z(M_1 \amalg M_2) \cong Z(M_1) \otimes Z(M_2)$.

Let $\tau \in (0,\infty)$. Then $M \times [0,\tau]$ is a bordism from M to M (its boundary is $M \amalg \overline{M}$). Let $U_M(\tau) \coloneqq Z(M \times [0,\tau])$, which we can think of as time evolution on M for time τ; in a Hamiltonian system we think of

$$U_M(\tau) = \exp(-\tau H_M), \tag{1.3}$$

where $H_M\colon Z(M) \to Z(M)$ is the Hamiltonian on M. Gluing bordisms implies $U_M(\tau_1) \circ U_M(\tau_2) = U_M(\tau_1 + \tau_2)$.

Remark 1.4. We would like to think of these operators as unitary, like in Hamiltonian quantum mechanics; making this precise from the functorial perspective is an area of active research. See for example a recent proposal of Kontevich–Segal [KS21].

To discuss unitarity we need some kind of inner product, but we did not ask for our state spaces to come with inner products. There is a one-parameter family of bilinear pairings around: the cylinder $M \times [0,\tau]$, thought of as a bordism from $M \amalg \overline{M} \to \varnothing$, induces a map $Z(M) \otimes Z(\overline{M}) \to Z(\varnothing) = \mathbb{C}$.[1] ◀

Example 1.5 (Finite gauge theory)**.** (Untwisted) finite gauge theory is a d-dimensional *topological* quantum field theory: the S-structure is topological, rather than geometric. Specifically, it is no structure at all.

Fix a finite group G and $p \in \{0, 1, \dots, d-1\}$. If $p > 0$, we ask that G is abelian. Therefore we can make sense of $H^{p+1}(M;G)$ when M is a closed manifold: when G is nonabelian and $p = 0$, this is the set of isomorphism classes of principal G-bundles on M. For compact M, $H^{p+1}(M;G)$ is finite.

Let M be a closed $(d-1)$-manifold; we define the state space of finite gauge theory on M to be the vector space spanned by the finite set $H^{p+1}(M;G)$.

If W is a bordism from M to N, we define the linear map $Z(W)$ as a form of "finite path integral" — we can't make sense of the path integral in general for gauge theories, but becaue G is finite we can in this case. Fix $A \in H^{p+1}(M;G)$ and let $i_M\colon M \hookrightarrow W$ and $i_N\colon N \hookrightarrow W$ be the inclusions. Define

$$Z(W)|A\rangle \coloneqq c(W) \sum_{\substack{B \in H^{p+1}(W;G)\\ i_M^* B = A}} |i_N^* B\rangle, \tag{1.6}$$

where $c(W) \in \mathbb{R}$ is a mormalization constant that appears so that this definition is functorial when we glue bordisms. ◀

Exercise 1.7. Say $d = 2$, $p = 0$, and $G = \mathbb{Z}/n$. Calculate $Z(S^1)$, $Z(\Sigma)$, and $Z(\Sigma')$, where Σ is the pair of pants regarded as a bordism from $S^1 \amalg S^1 \to S^1$ and Σ' is Σ in the opposite direction. For bordisms, only calculate the maps up to normalization constants, since we didn't specify those constants in (1.6).

Now let's talk about extended quantum field theory. If M is a closed d-dimensional manifold, it may be regarded as a bordism $\varnothing \to \varnothing$. Applying Z, we obtain a linear map $\mathbb{C} \to \mathbb{C}$, since $Z(\varnothing) = \mathbb{C}$. This map is determined by its value on 1, which is a complex number called the *partition function* of M.

Associated to a closed $(d-1)$-manifold we have a state space. In extended QFT, we assign higher-categorical invariants to manifolds in lower dimensions; for example, to a closed $(d-2)$-dimensional manifold we assign something called a "2-vector space," which is something like a $\mathbb{C}$-linear category; and in general on a closed $(d-k)$-manifold we assign a "k-vector space," some kind of higher catgory. We will not define k-vector spaces precisely here, and indeed different researchers use different definitions.

We are especially interested in $Z(S^{q-1})$ as q varies; this is some sort of higher category. The objects of this category are the codimension-q defects or extended operators (to us, these two words mean the same thing) in the QFT Z. In non-topological theories, we need to specify the radius r of S^{q-1}; the codimension-q defects are the limit of $Z(S^{q-1}(r))$ as $r \to 0$.

There is another formalism for higher-codimension defects or operators, given by something called *decorated bordisms*. This latter approach may be easier to digest from the physics point of view. The two approaches are expected to be equivalent.

[1] The fact that $Z(\varnothing) = \mathbb{C}$ is another consequence of symmetric monoidality.

2. : 6/14/22

As Dan mentioned in his lecture, we should think of defects in functorial TFT as labeled by elements of $\mathrm{Hom}(\mathbf{1}, Z(S^{q-1}))$, where Z is our TFT, q is the codimension of the defect, and $\mathbf{1}$ is the tensor unit.

TODO: pictures of stratified manifolds corresponding to certain kinds of defects. The key thing to keep in mind, I think, is that if we put defects in on a codimension-zero manifold, we obtain a complex number, and if we put them in in a codimension-one manifold, the manifold with all its defects has a state space. We can aslo put defects into a bordism between two manifolds with defects in them (TODO: picture), providing a linear map between what's assigned to those manifolds with defects.

Let's look at finite (higher)gauge theory for simplicity/conceteness. There is a fluctuating field $A \in H^{p+1}(M; G)$, and as we discussed last time, G is abelian if $p > 0$. These fields are identified with the maps from M to B^pG.

In this theory, there are two kinds of defects.

- *Wilson-type defects* are $(p+1)$-dimensional defects, or codimension-$(d-p-1)$ defects, labeled by a character $\phi \in G^\vee := \mathrm{Hom}(G, \mathrm{U}_1)$. You evaluate ϕ by (TODO: I didn't follow). I think you multiply by $\int_D A$, and then sum like usual in the finite gauge theory.
- *Disorder-type defects* only exist when G is abelian. Let D be a $(p+2)$-dimensional submanifold of spacetime; then the link of D is S^{p+1}. We modify the fields A to enforce the condition that

$$\int_{S^{p+1}} A = g \tag{2.1}$$

  for some fixed $g \in G$. To say this a little more carefully, we allow the fields to be singular at D (so the fields are defined only on $M \setminus D$), and enforce the boundary condition (2.1). Then, like usual in finite gauge theory, you sum over these fields with this boundary condition.

Exercise 2.2. Let Z be finite gauge theory for G and p as above. Consider the "generalized Hopf link" in S^d where we link S^{p+1} and S^{d-p-2}. Label S^{p+1} by a Wilson-type defect and S^{d-p-2} by a disorder-type defect. Compute the ratio of Z evaluated on S^d with these defects over $Z(S^d)$.

TODO: missed a lot here. I'm really sorry. This is on the relationship between invertible topological operators of codimension p and $(p-1)$-form symmetries: specifically, these two things are equivalent. Noether's theorem made an appearance.

Given a conventional zero-form symmetry, corresponding to codimension-1 invertible topological operators, we need to figure out how to characterize data of a manifold with a codimension-1 defect network. This data is some codimension-1 submanifolds labeled with elements of G, and there's a cocycle condition [TODO: what is the condition? something about bringing manifolds together]. The upshot is that our defect data defines a cocycle $B \in Z^1(M; G)$.

If we instead used a positive-dimensional Lie group G, then there exist G-connections which are not flat and the above story becomes more complicated. The corresponding symmetry has a current j and the defect operators act by

$$S \longmapsto S + i \int B \wedge j + \cdots \tag{2.3}$$

3. : 6/15/22

4. Confinement and one-form symmetries

Let's say we have a system with a one-form symmetry and a Wilson line W, and suppose the one-form symmetry defects are of the form U_g. For concreteness assume we're in $\mathbb{R}^3$, the Wilson line is along the x-axis, and the symmetry defects U_g and $U_{g^{-1}}$ are on two lines close to the y-axis. We say that [TODO: what exactly?] is *confined* if $\langle \widetilde{W} \rangle \neq 0$, where $\widetilde{W}$ is the operator formed by the product of the Wilson operator and some local term $\exp\left(c \int_\gamma \mathrm{d}\ell\right)$. In this case $U_g|0\rangle \not\propto |0\rangle$. The Wilson line satisfies a perimeter law (its expectation depends on the perimeter of the region). This is a form of one-form spontaneous symmetry breaking.

For example, consider Maxwell theory with potential $V(L) \propto 1/L$. This has an electric one-form symmetry arising from the center of U_1 (which is U_1 again). It's deconfined [TODO: confined? wasn't clear to me; I'm sorry] We expect spontaneous symmetry breaking and therefore a gapless mode, which is the photon as a Nambu–Goldstone boson. Before the generalized global symmetries paper of Gaiotto–Kapustin–Seiberg–Willett [GKSW15], this was known, but not recognized as a form of symmetry breaking.

In confined theories, W_γ is proportional to a quantity with a term including the area of a surface Σ with $\partial\Sigma = \gamma$. There is no local contribution to cancel it. This implies the limit of W applied to larger and larger rectangles vanishes! So the one-form symmetry is preserved, rather than broken. We expect for Yang–Mills in 4d that at low-temperature the theory is confined, though this is a Millennium Prize problem… at higher temperatures we expect a sharp phase transition and then deconfinement. So the one-form symmetry exists at low temperatures, but is spontaneously broken at temperatures larger than the critical temperature.

If we consider gauge theory with matter ψ, the electric one-form symmetry is explicitly broken from $Z(G)$ to those $g \in Z(G)$ which preserve ψ. The specific subgroup depends on the representation ψ transforms in.

Exercise 4.1. Consider a zero-form $\mathbb{Z}/n$ symmetry. Show that if this symmetry is spontaneously broken and gapped, then what we expect, instead of the Goldstone theorem, is n vacua. What is the analogue of the statement for a one-form $\mathbb{Z}/n$ symmetry? Hint: consider U_1 gauge theory with a charge-n scalar field. What is the electric one-form? Can you find a phase where [TODO: unclear to me, sorry]. What is the IR theory?

Now let's change subjects a bit. We're going to talk about a noninvertible chiral symmetry in 4d massless QED. This is a U_1 gauge theory with a charge-1 massless electron $\Psi = (\psi_R, \psi_L)$. There is a classical chiral symmetry

$$\begin{aligned} \psi_R &\longmapsto e^{i\alpha}\psi_R \\ \psi_L &\longmapsto \psi_L, \end{aligned} \tag{4.2}$$

for $\alpha \in \mathrm{U}_1$. This symmetry has an ABJ anonaly; let j be the Noether current and F be the dynamical curvature. Then

$$\mathrm{d}j = \frac{1}{8\pi^2}F^2. \tag{4.3}$$

If you prefer uppercase J, they're related by $j = \star J$.

The ABJ anomaly can be seen from a one-loop Feynman diagram, and this is standard in textbooks. The chiral U_1 symmetry is broken in the quantum theory. Locally $F^2 = \mathrm{d}CS$, where $CS = A\,\mathrm{d}A$ is an abelian Chern–Simons term, so let's locally modify the current as follows:

$$j \longmapsto j - \frac{1}{\pi^2}CS, \tag{4.4}$$

so that $\mathrm{d}j' = 0$.

Exercise 4.5. In general, a U_1-bundle on a general 4-manifold M can have nonzero $\int_M F^2 \neq 0$. Find an example.

Such an example is called a *nonzero instanton configuration*. In this case,

$$\begin{aligned} \langle \mathrm{d}j(x)\mathcal{O}(y)\mathcal{O}(z)\rangle &= \sum_{\mathcal{I}} \langle \mathrm{d}j(x)\mathcal{O}(y)\mathcal{O}(z)\rangle_{\mathcal{I}} \\ &= \underbrace{\langle\ldots\rangle_{\mathcal{I}=0}}_{=0} + \underbrace{\langle\ldots\rangle_{\mathcal{I}\neq 0}}_{\neq 0}. \end{aligned} \tag{4.6}$$

On $\mathbb{R}^4$ or S^4, there are no nonzero instanton configurations, so the selection rule on $\mathbb{R}^4$ is satisfied. But if we told the analogous story with a nonabelian gauge group, there can be nonzero instantons even on $\mathbb{R}^4$.

Now consider the non-topological operator

$$C \coloneqq \exp\left(i\alpha \int_\Sigma j\right), \tag{4.7}$$

where $\alpha \in \mathrm{U}_1$ and Σ is a surface in spacetime. We would like to be able to say

$$\text{“}D\alpha = \exp\left(i\alpha\left(j - \int_\Sigma CS[A]\right)\right)\text{”}, \tag{4.8}$$

but the Chern–Simons term is not globally gauge-invariant. However, if $\alpha \in 2\pi\mathbb{Z}$, then we can correctly quantize the Chern–Simons term:

$$\widetilde{D}_\alpha \coloneqq \exp\left(i\alpha \int_\Sigma CS[A]\right) \tag{4.9}$$

[TODO: $\alpha/??$ maybe?] defines an invertible defect. This appears in the integer quantum Hall effect. If $\alpha = 2\pi p/q$, with $p, q \in \mathbb{Z}$, we obtain the fractional quantum Hall effect. In this case the defect $D_\alpha^{[\Sigma]}$ can be described in terms of C_α and an abelian TFT.

In abelian gaue theories, the ABJ-anomalous chiral symmetry is indeed a symmetry. In non-abelian gauge theories, this doesn't work as well.

Part 2. Clay Córdova, Introduction to anomalies in quantum field theory

5. : 6/13/22

We begin by looking at degenerate ground states in quantum mechanics. The setup has a separable Hilbert space W, e.g. $L^2(\mathbb{R})$. This is the state space; the quantum states are nonzero elements of W modulo phases: we identify ψ and $\lambda\psi$ for $\lambda \in \mathbb{C}^\times$. Time evolution in this system is described using a self-adjoint positive Hermitian operator H, called the *Hamiltonian*; $H = H^\dagger$. Some of these assumptions are because we are in the unitary setting. If $\mathcal{O}$ is an operator, it evolves under time as

$$\mathcal{O}(t) = e^{iHt}\mathcal{O}(0)e^{-iHt}. \tag{5.1}$$

Since H is self-adjoint and positive, its eigenvalues are real and nonnegative. Let E_i be the eigenvalues of H in ascending order; they represent the energy levels of the theory. We are particularly interested in the eigenvectors for the smallest eigenvalue. If this eigenspace is more than one-dimensional, we say this system has a *degenerate ground state*.

We are interested in questions related to ground state degeneracy. For example, when is there a degenerate ground state? Is this degeneracy stable under deforming H?

Example 5.2. Consider a system of n particles moving on $\mathbb{R}$ in the presence of a potential $V\colon \mathbb{R}^n \to \mathbb{R}$. Then $W = L^2(\mathbb{R}^n)$ and the Hamiltonian is

$$H \coloneqq -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + V(x_1, \dots, x_n) \tag{5.3}$$

Theorem 5.4. *Let $L^2_{loc}(R^n)$ denote the Hilbert space of functions which are square-integrable on compact subsets of $\mathbb{R}^n$. If $W = L^2_{loc}(\mathbb{R}^n)$, $V \ge 0$, and $V \to \infty$ as $|x_i| \to \infty$, then H has a nondegenerate ground state.*

The proof is an exercise, though see Reed–Simon volume IV [RS78, Chapter 8]. The assumptions cover many of the typical examples of quantum-mechanical systems, such as a double well. ◂

The point of introducing Theorem 5.4 here is that it's not easy to produce examples of systems with ground state degeneracy.

One of the goals of this series of lectures is to develop a theory of invariants, called *anomalies*, which imply degenerate ground states in quantum mechanics and quantum field theory which are robust to perturbations.

Symmetry in quantum mechanics plays a key role. There are two kinds of symetries: *unitary transformations*, operators $U\colon W \to W$ such that $[U, W] = 0$, and *antiunitary transformations*, operators $T\colon W \to W$ such that $[H, T]$ acts on operators by conjugation by a unitary operator: $\mathcal{O} \mapsto U\mathcal{O}U^{-1}$. When T is antiunitary and $\lambda \in \mathbb{C}^\times$, $T(\lambda w) = \overline{\lambda}T(w)$.

Example 5.5. Consider a particle on a circle. Our variable x now is 2π-periodic: we identify x and $x + 2\pi$. We write down a Lagrangian

$$L = \frac{1}{2}\dot{x}^2 + \frac{\theta}{2\pi}\dot{x}, \tag{5.6}$$

where $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ is a parameter. From the Lagrangian, one does a canonical transformation to obtain the Hamiltonian; this is a standard trick, and the answer is

$$H = \frac{1}{2}\left(-i\frac{\mathrm{d}}{\mathrm{d}x} - \frac{\theta}{2\pi}\right)^2, \tag{5.7}$$

acting on the functions on the circle. The Lagrangian and Hamiltonian are quadratic, so it's easy to solve this explicitly for all energy levels at once. When we do this, we will see something interesting.

The eigenfunctions are Fourier modes: $\exp(inx)$ for $n \in \mathbb{Z}$, which has eigenvalue $E_n = (1/2)(n - \theta/2\pi)^2$. Though the energy levels are the same, the states themselves move around, so there's some sort of spectral flow. At $\theta = 0$, the ground state has energy level 0, and excited states (the eigenstates for eigenvalues larger than the smallest eigenvalue) are doubly degenerate: $\pm n$ gives you a two-dimensional eigenspace. But at $\theta = \pi$, all eigenvalues have two-dimensional eigenspaces. See Figure 1 for a picture of the spectrum as θ varies. One might think that this "level-crossing" behavior is generic, but this is not correct.

Exercise 5.8. Determine the codimension of the level-crossing loci for a multiparameter Hamiltonian (TODO: may have misunderstood this exercise?).

This system has symmetries, and let's take advantage of them. Let a be a time-independent constant; then the system is symmetric under the shift $x \mapsto x + a$. This defines unitary operators U_a which generate the Lie group U_1 under composition.

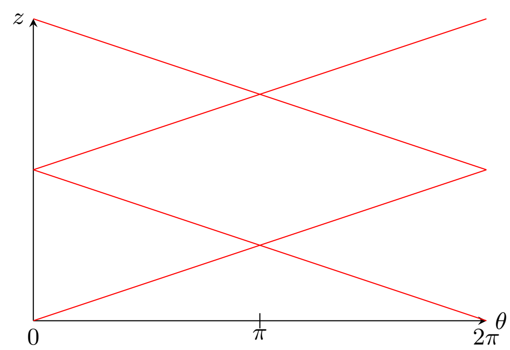

Figure 1. The spectrum of the particle on a circle considered in Example 5.5. The red lines are the eigenvalues and how they change under θ. Notce the crossing at $\theta = 0, \pi$, where most, resp. all eigenspaces are two-dimensional. For other values of θ, the eigenspces are all one-dimensional.

Another symmetry is reflection, $C\colon x \mapsto -x$. This acts on the Hamiltonian, and changes θ; therefore it is only actually a symmetry at $\theta = 0, \pi$. In some sense, generically we only have the U_1 symmetry, but at $\theta = 0, \pi$, the symmetry group enhances.

How do the symmetris act on the eigenstates? For translation, the answer is not so tricky:

$$U_a(w_n) = e^{ina} w_n. \tag{5.9}$$

Exercise 5.10. Convince yourself that at $\theta = 0$, $C(w_N) = w_{-n}$, and at $\theta = \pi$, $C(w_n) = w_{-n+1}$.

This leads to a simple-looking question: *what is the group of symmetries generated by U_a and C?* There are two different answers, depending on what exactly you mean.

(1) When acting on operators — operators are generated by x and its derivatives, and compositions thereof, so we obtain the group O_2: U_a forms SO_2, and C acts as reflections on the circle.
(2) When acting on states, however, the answer is not the same at $\theta = \pi$. There we find an additional phase:

$$CU_aC^{-1}(w_n) = CU_a(w_{-n+1}) = C\Big(e^{ia(-n+1)} w_{-n+1}\Big) = e^{ia} U_{-a}(w_n). \tag{5.11}$$

If this generated an O_2, we would've just exepcted $U_{-a}(w_n)$.

In the second setting, the phase we obtained tells us that the action of O_2 is in fact a projective representation associated to the double cover

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathrm{Pin}_2^- \longrightarrow \mathrm{O}_2 \longrightarrow 1. \tag{5.12}$$

When acting on operators, we see O_2; when acting on states, we see this double cover. ◀

Your first question might be: why is this discrepancy between states and operators even possible? It is possible because states are rays in the Hilbert space of states, not vectors. A symmetry only has to be a projective representation on the Hilbert space. Concretely, we end up with a cocycle $\mu \in Z^2(\mathrm{O}_2; \mathrm{U}_1)$ that twists the multiplication in the action on states, and μ is not cohomologous to 0.[2] Because operators are acted upon by conjugation, they are blind to this cocycle.

Another key observation: all states have the same value of μ once we fix θ: at $\theta = 0$, $\mu = 0$, and at $\theta = \pi$, we have the same nonzero cocycle.

Each eigenspace of H forms a projective representation of O_2 with given μ. At $\theta = \pi$, where μ is not cohomologous to zero, there are no one-dimensional projective representations — and therefore there is ground state degeneracy!

Projective representations are among the easiest examples of anomalies, and they are also robust under symmetry-preserving deformations: they are classified by discrete cocycles. So if we smoothly modify the Hamiltonian, we will preserve the property of ground state degeneracy.

Exercise 5.13. Once again consider the particle on the circle, and modify the Hamiltonian by adding a potential:

$$H = \frac{1}{2}\left(-i\frac{\mathrm{d}}{\mathrm{d}x} - \frac{\theta}{2\pi}\right)^2 + \frac{\lambda}{2\pi}\cos(2x). \tag{5.14}$$

[2] Complex projective representations use U_1-valued cocycles; the fact that we had a $\mathbb{Z}/2$ cover, rather than a U_1 cover, comes from the fact that this cocycle is valued in the $\mathbb{Z}/2$ subgroup $\{\pm 1\} \subset \mathrm{U}_1$. TODO: so we only see the image in $Z^2(\mathrm{O}_2; \mathrm{U}_1)$; how do we tell apart pin$^+$ and pin$^-$ then?

(1) Show that for small λ and at $\theta = 0$, $|E_{+1} - E_{-1}| = \lambda + O(\lambda^2)$. The interpretation is that the degeneracy at $\theta = 0$ is a coincidence: we can turn on an arbitrarily small potential, deforming the Hamiltonian by an arbitrarily small amount, and undo the degeneracy.
(2) Show that at $\theta = \pi$, the ground state degeneracy persists for $\lambda \neq 0$.

The system at $\theta = \pi$ with nonzero λ is quite difficult: we don't know how to exactly solve it. But we do know the energy levels.

The $\cos(2x)$ term breaks the shift symmetry from U_1 to $\mathbb{Z}/2$ (we can still shift by half of a period). This is actually all that we need to have the degeneracy.

Example 5.15. Consider a system of real fermions with a time-reversal symmetry. We'll have N fermions $\psi^i(t)$, $i = 1, \dots, N$. For simplicity assume N is even. Classically they're Grassmann variables; when we quantize we obtain a Clifford algebra: these variables' supercommutator is $\{\psi^i, \psi^j\} = 2\delta^{ij}$. We let the time-reversal symmetry T act on these operators as

$$T\psi^i(t)T^{-1} = -\psi^i(-t). \tag{5.16}$$

The Hilbert space W, a Clifford algebra, is finite-dimensional, and more precisely $\dim(W) = 2^{N/2}$. Choose the zero Hamiltonian; then all states are ground states! That's a lot of ground state degeneracy.

One typically considers a mass term, inducing a quadratic deformation of the Hamiltonian:

$$\Delta H \overset{?}{=} im\psi^1\psi^2, \tag{5.17}$$

for some $m \in \mathbb{R}$. As $T\Delta H T^{-1} = -\Delta H$, though, this mass term is incompatible with time-reversal. This crucially uses that T is an anti-unitary symmetry.

So quadratic deformations are no help. What about quartic deformations?

To answer this question, it's helpful to group the fermions into complex pairs. Let

$$a_n = \frac{1}{\sqrt{2}}\big(\psi^{2n-1} + i\psi^{2n}\big). \tag{5.18}$$

Then the creation and annihilation operators for these complex fermions satisfy $\{a_n, a_m^\dagger\} = \delta_{mn}$. Each pair generates a two-component space $W_\pm$, and

$$\begin{aligned} a(w_-) &= 0 & a^\dagger(w_+) &= 0\\ a^\dagger(w_-) &= w_+ & a(w_+) &= w_-. \end{aligned} \tag{5.19}$$

TODO: I didn't quite follow this, sorry! In a general state, there are $N/2$ labels $w_{\pm\pm\cdots\pm}$. Consider the quartic deformation

$$\Delta H = 4q\psi^1\psi^2\psi^3\psi^4, \tag{5.20}$$

where $q > 0$. Then we can factor ΔH as

$$\Delta H = -q\Big(a_1a_1^\dagger - \frac{1}{2}\Big)\Big(a_2a_2^\dagger - \frac{1}{2}\Big). \tag{5.21}$$

There are aligned states w_{++} and w_{--} with $\Delta E = -q$, and w_{+-} and w_{-+} also are aligned. Anyways, the upshot is that there is twofold ground state degeneracy, sort of like we saw with quantum mechanics on a circle. ◀

Exercise 5.22. Show that for $N = 8$ there exists a T-invariant quartic deformation leading to a unique ground state.

Consequently, the quantity $N \bmod 8$ is protected by T. It is also an indication that there is another anomaly.

6. Anomalies and inflow: 6/14/22

Yesterday we approached anomalies in quantum field theory from a very bottom-up perspective. Today, our perspective is from very high up; and in the remaining two lectures we'll try to make these two approaches meet in the middle.

Our setting is d-dimensional quantum field theory. As a useful example, it may help to let $d = 1$, where you get quantum mechanics. The symmetry structure amounts to defining classical background fields; we let A refer to the collection of these fields. Then, given such fields on a spacetime manifold, we obtain a partition function $Z[A]$.

There are two kinds of these symmetries.

(1) "Internal symmetries," for a finite group, a compact Lie group, a higher group, etc. In this case A is data of a connection on a bundle for the group. One can also introduce operators: when G is finite (more generally π-finite), a flat G-connection is equivalent to studying networks of symmetry defects; the equivalence passes through Poincaré duality. For G infinite, e.g. a compact Lie group, there is both local and global data. The local data is the curvature, which manifests in physics as the current correlators; and the global data appears as the topology of the bundle.

Right now, we are not summing over connections, or in other words we are not gauging the symmetry. The connection is a background field.

(2) "Spacetime symmetries." For example, a Lorentz symmetry would correspond to the data of a metric; a fermion number symmetry would correspond to a spin structure; and a time-reversal symmetry would correspond to allowing spacetime to be unoriented, or even unorientable.

We will use A to collectively denote all of this background data.

We will want to understand how background fields transform under gauge transformations: if λ is a gauge parameter, we should make explicit the transformation $A \to A^{\lambda}$. For connections, this is standard: if we have a U_1 global symmetry, so A is a connection on a U_1-bundle, this transformation is

$$A \longmapsto A + \mathrm{d}\lambda. \tag{6.1}$$

We want to know to what extent the partition function is invariant under these transformations.

Example 6.2. If we have a finite G-symmetry, we may have an explicit cocycle representing the connection, and this transformation will amount to shifting the cocycle by a coboundary. ◂

A key point in defining $Z[A]$ is that one often chooses explicit representatives, e.g. explicit connections in the case of a background G-symmetry. You use that connection to define $Z[A]$ — but then, you have to ask, is $Z[A]$ gauge-invariant? Is $Z[A] = Z[A^{\lambda}]$? If not, how badly does it fail? This is the question we're going to explore today.

The first thing to notice is that gauge-invariance of the partition function is closely related to topology-invariance of the symmetry defects. We've heard many times that symmetries are topological operators; right now we're focusing on invertible/grouplike symmetries, the simplest case.

For example, if the symmetry group is U_1, choose $\exp(i\varphi) \in \mathrm{U}_1$. This defines a codimension-1 defect, which we would like to insert at time τ_0. The associated background field for this slice is just

$$A = \varphi \cdot \delta(\tau - \tau_0)\,\mathrm{d}\tau. \tag{6.3}$$

If ℓ is the dual Wilson line to this defect, then

$$\exp\left(i \oint_{\ell} A\right) = \exp(i\varphi). \tag{6.4}$$

The defect at the specific time τ_0 requires an explicit choice of connection A, as is visible in (6.4). If we want to move the defect to another time τ_1, we need a gauge transformation. Let Θ denote the Heaviside step function, equal to 1 on positive numbers, 0 at 0, and -1 on negative numbers. Consider the step function gauge transformation

$$\lambda = \varphi(\Theta(\tau_0 - \tau_1) - \Theta(\tau - \tau_0)). \tag{6.5}$$

This is compactly supported.

If we find that $Z[A] \neq Z[A^{\lambda}]$, that is equivalent to the breakdown of the topological invariance of symmetry defects. This sounds scary, but before we panic, we should ask: how badly does it break down? What kinds of failures are possible?

A key idea in physics is the difference between *separated points* and *coincident points*; as long as the defects are separated (that is, their supports are not coincident), topological invariance holds exactly. That is, *failure of topological invariance can only happen when defects intersect!*

For example, say we have two defects labeled by $g, h \in G$. The problem can only occur when we bring them together — when we do that, do we get gh? Or is there some sort of projective representation?

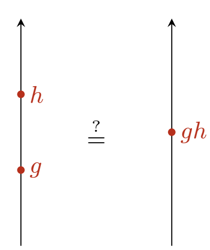

FIGURE 2. Invariance of the partition function (or the lack thereof) in quantum mechanics, which can only happen when defects are brought together. TODO: similar two-dimensional example.

That is, we allow the partition function to not be gauge-invariant, but we enforce that it is gauge-invariant at separated points. This leads to an important ansatz which is a gateway to a more modern point of view on anomalies.

Ansatz 6.6. The value of the partition function on a spacetime manifold X under a gauge transformation λ transforms as

$$Z[A^\lambda] = Z[A]\exp\left(-\int_X \alpha(\lambda, A)\right), \tag{6.7}$$

where α is a *local* functional, meaning that it obeys some sort of cutting and gluing axiom.

It may be helpful to think this through in the U_1 case: coupling to A means the action contains a term of the form

$$S = \cdots + \int_X A \wedge \star J, \tag{6.8}$$

where J is the one-form conserved current. The variation of this term is

$$\delta S = \int_X \mathrm{d}\lambda \wedge \star J \sim \int_X \lambda \wedge \mathrm{d}(\star J). \tag{6.9}$$

The quantity $\mathrm{d}(\star J) = \mathrm{d}^\mu J_\mu$ vanishes at separated points: when $x \neq y_i$,

$$\langle \mathrm{d}(\star J)\, \mathcal{O}(x)\, \mathcal{O}(y_1) \cdots \mathcal{O}(y_n)\rangle. \tag{6.10}$$

We allow *contact terms*, which are nonzero at coincident points. Since α is local, the ambiguity/failure only occurs at coincident points.

Another consequence of Ansatz 6.6 is that $|Z[A^\lambda]| = |Z[A]|$. This has a probabilistic interpretation: the modulus of a state has a probabilistic meaning, much like rays in quantum mechanics. But the exact value of $Z[A]$ is subject to an ambiguity. In general, a transformation

$$Z[A] \longmapsto Z[A]\exp\left(2\pi i \int_X \beta(A)\right), \tag{6.11}$$

where β is some sort of local term, is called a *scheme change*. In physics, trying to pin down the partition function exactly, rather than just up to a phase, is called *fixing a scheme*.

So we have a cohomology problem: finding $\{\alpha\}/\{\alpha \sim \alpha + \delta\beta\}$. This is the conventional definition of the possible anomalies. *Anomaly inflow* is a more modern point of view. Depending on whom you ask, this is a hypothesis, a definition, or a theorem. The goal of anomaly inflow is to capture the equivalence class of the anomaly α as the data of a $(d+1)$-dimensional *invertible field theory*. This has diffeent avatars in different parts of the world: the condensed-matter people may call it an SPT (symmetry-protected topological theory); you may also hear it called a classical theory. The Lagrangian/partition function of this invertible theory on a $(d+1)$-dimensional manifold Y is

$$\exp\left(2\pi i \int_Y \omega(A)\right), \tag{6.12}$$

such that if $\partial Y = X$, then

$$\exp\left(2\pi i \int_Y \omega(A^\lambda) - 2\pi i \int_Y \omega(A)\right) = \exp\left(2\pi i \int_X \alpha(\lambda, A)\right). \tag{6.13}$$

It may be helpful to draw a geometric picture to understand what this formula means. Specifically, let's let Y be the mapping torus of the transformation λ; that is, take $X \times [0,1]$ and glue 0 to 1, where we twist at time 1 by λ. That is, Y is a fiber bumdle over a circle with fiber X, and the monodromy around the circle is precisely λ. In this case,

$$\exp\left(2\pi i \int_X \alpha(\lambda, A)\right) = \exp\left(2\pi i \int_Y \omega\left(\widetilde{A}\right)\right), \tag{6.14}$$

where $\widetilde{A}$ is the data of the background fields glued by λ on the mapping torus. The right-hand side of (6.14) is the partition function of the $(d+1)$-dimensional invertible theory on the mapping torus Y.

This allows us to define a fully gauge-invariant partition function

$$\widetilde{Z}[A] := Z[A]\exp\left(2\pi i \int_Y \omega(A)\right), \tag{6.15}$$

where $\partial Y = X$, assuming we can do this. Then by construction, $\widetilde{Z}[A^\lambda] = \widetilde{Z}[A]$.

This leads to the paradigm that having an anomaly means living at the boundary of an invertible field theory in one dimension higher: anoamlies can be defined by invertible field theories. Part of the puzzle is to try to figure this out given the boundary theory, or vice versa.

TODO: I missed a few minutes of the talk; sorry about that.

Let's revisit the case of the particle on S^1, which has Lagrangian

$$L = \frac{1}{2}\dot{x}^2 + \frac{\theta}{2\pi}\dot{x}. \tag{7.1}$$

When $\theta = 0, \pi$, the symmetry enhances to O_2. Couple to a connection for $\mathrm{U}_1 \subset \mathrm{O}_2$, which we write as $A = A_0\,\mathrm{d}t$. The modified Lagrangian is

$$L = \frac{1}{2}(\dot{x} + A_0)^2 + \frac{\theta}{2\pi}(\dot{x} + A_0) + kA_0. \tag{7.2}$$

This is invariant under $x \mapsto x - \lambda$, $A_0 \to A_0 + \frac{\mathrm{d}\lambda}{\mathrm{d}t}$.

Is the exponentiated action invariant under C (which, recall, was the antiunitary involution we used to enhance from U_1 to O_2)? There are two cases to consider.

When $\theta = 0$, the Lagrangian is invariant if $k = 0$.

Exercise 7.3. Show that at $\theta = \pi$, C transforms the Euclidean action S by

$$e^{-S} \longmapsto e^{-S} \exp\left(i(2k+1)\oint A_0\,\mathrm{d}t\right). \tag{7.4}$$

As a consequence of the exercise, C-invariance forces $2k + 1 = 0$, which is a problem because $k \in \mathbb{Z}$.

The failure of C-invariance in the presence of the connection A is the anomaly. The variation in the action,

$$\delta S = (2k+1)i \int A_0\,\mathrm{d}t, \tag{7.5}$$

is a classical phase. The perspective of anomaly inflow says to extend to a compact, oriented 2-manifold Σ with $\partial\Sigma = S^1$, and to extend A into Σ.[3] Then let

$$\widetilde{Z}[A] \coloneqq Z[A] \cdot \exp\left(\frac{i}{2}\int_\Sigma F\right), \tag{7.6}$$

where $F \coloneqq \mathrm{d}A$. You can verify that $\widetilde{Z}[A]$ is C-invariant.

More generally, suppose G is a finite symmetry that acts linearly on operators, but projectively on states, and let $\mu \in H^2(G; \mathbb{R}/\mathbb{Z})$ be the cocycle classifying the projective representation. View $[A] \in H^1(X; BG)$ as a map $X \to BG$, so we can pull back μ to $A^*(\mu) \in H^2(X; \mathbb{R}/\mathbb{Z})$. Integrating this class defines an invertible theory with partition function

$$\exp\left(2\pi i \int_\Sigma A^*(\mu)\right). \tag{7.7}$$

Next let's look at a system of time-reversal-invariant fermions in 1d. We consider N real fermions $\psi^1, \dots, \psi^N$ satisfying $\{\psi^i, \psi^j\} = 2\delta^{ij}$. It turns out that imposing time-reversal invariance forces there to be a degenerate ground state unless $N \equiv 0 \bmod 8$.

In fact, if $T^2 = 1$, we can put the theory on manifolds with a pin$^-$ structure. This is surprising: in the group Pin_k^-, the lift of a reflection from O_n doesn't square to 1, so you might expect pin$^+$. What happens instead is that Wick rotation inserts a subtle i factor, and the upshot is that after Wick-rotating, we have a reflection R squaring to $(-1)^F$.

If you haven't seen pin structures before, here's a quick crash course. $w_i \in H^i(X; \mathbb{Z}/2)$ denotes the i^{th} *Stiefel-Whitney class* of the tangent bundle of X.

- An orientation is an SO-structure. We enforce $w_1 = 0$, and there is no condition on w_2.
- An O-structure is no data: every manifold has a unique O-structure. There is no constraint on characteristic classes.
- A spin structure enforces $w_1 = 0$ and $w_2 = 0$.

Pin structures are unoriented analogues of spin structures: we trivialize w_2 or something related without asking for anything about w_1.

- For a pin$^+$ structure, we need $w_2 = 0$, and there is no condition on w_1.
- For a pin$^-$ structure, we need $w_2 = w_1^2$, and there is no other condition on w_1.

If Σ is a 2-manifold, the Wu formula proves that $w_2 = w_1^2$, so Σ is pin$^-$.

Exercise 7.8. Which closed 2-manifolds are pin$^+$? Hint: the second Stiefel-Whitney class is the mod 2 reduction of the Euler class e.

[3] A priori there is an obstruction to extending a principal bundle into the bulk, but in this case this extension is always possible.

Anomaly inflow tells us that the anomaly of this system of time-reversal invariant fermions is a 2d reflection-positive invertible theory on pin$^-$ manifolds. These theories are classified by their partition functions, which are bordism invariants. The relevant bordism group is $\Omega_2^{\mathrm{Pin}^-} \cong \mathbb{Z}/8$, generated by $\mathbb{RP}^2$ with either of its two pin$^-$ structures. The bordism invariants are $\mathrm{Hom}(\Omega_2^{\mathrm{Pin}^-}, \mathrm{U}_1)$, again $\mathbb{Z}/8$; the generator is the *Arf-Brown-Kervaire invariant*. The key result is that for N real fermions with T-symmetry squaring to 1, the anomaly theory is N times the Arf-Brown-Kervaire generator.

Here is a more physical picture, due to Fidkowski–Kitaev [FK10, FK11]. Consider N 2d real Majorana fermions χ^i with Lagrangian

$$L = i\chi \not{\partial} X + im\chi_L\chi_R, \tag{7.9}$$

where χ_L and χ_R have two chiralities.

Exercise 7.10. Check that this system is time-reversal invariant, even with the mass.

Now consider this system on a compact 2-manifold Σ with nonempty boundary. Then

$$\chi_L^i|_{\partial\Sigma} = \chi_R^i|_{\partial\Sigma} = \psi^i \tag{7.11}$$

gives us a system of N real fermions on the boundary. So we can see the bulk-boundary system that anomaly inflow told us about from a more directly physical perspective.

Take m large, which is equivalent to a low-energy limit. Then χ is massive, so the limit of the partition function is invertible, and is a certain eta invariant. This is a direct construction of the invertible theory which is the anomaly of the 1d free fermion theory.

Remark 7.12. Some of the elements in $\mathrm{Hom}(\Omega_2^{\mathrm{Pin}^-}, \mathrm{U}_1)$ are nonzero, but vanish on all mapping tori. For a long time, physicists weren't collectively sure whether an anomaly that vanishes on mapping tori could still be nonzero. But the evidence suggests that they are: that an anomaly is simply an invertible theory in one dimension higher, and being zero means being zero on all manifolds, not just mapping tori. ◂

Now let's discuss some general consequences of anomalies.

One is that anomalies protect "non-triviality" of families of QFTs related by continuous deformations preserving the symmetry type. This includes several things we often do to QFTs:

- adjusting a coupling constant,
- adding more fields (CMT = spectators), or
- RG flow triggered by a symmetry-generating operator.

In a sense, RG flow is built out of operations of the first two kinds. In any case, these operations preserve the anomaly. If λ denotes some parameter in a family,

$$\widetilde{Z}_\lambda[A] = Z_\lambda[A] \exp\left(2\pi i \int_Y \omega_\lambda(A)\right), \tag{7.13}$$

where ω_λ is a family of invertible field theories, the deformation class $[\omega_\lambda]$, which by definition is any invertible field theory continuously connected to ω_λ, is an invariant of this family of theories. And in many situations, ω is rigid, so we get an even stronger invarint. For example, in the pin$^-$ example we saw, the elements of $\mathbb{Z}/8$ do not have deformations: this is a discrete set.

This is the concept that is sometimes called "anomaly matching" in physics.

An invertible theory does not need to be at the boundary of a system in one dimension higher, so if you have a nonzero deformation class of your anomaly, you know that no matter the value of your parameter, your theory is never invertible.

Let's unpack this for RG flow a little bit. Let λ be a distance scale, so $\lambda = \infty$ in the IR. The possible IR realizations of relativistic QFTs via a mass gap:

(1) The theory is gapless: the massless fields define an interacting conformal field theory.
(2) The theory is gapped: all partices have mass. This can happen either in
    (a) a TQFT, or
    (b) a topological invertible theory.

The presence of an anomaly rules out the second possibility for a gapped theory.

Part 3. Mike Hopkins, Lattice systems and topological field theories

8. Product states and entanglement entropy: 6/13/22

This week's talks are on work related to topology/TQFT, lattice models, and quantum information theory, including ideas growing out of a meeting at Aspen with Freed, Teleman, Freedman, Kapustin, Kitaev, Moore, and Hastings. The key idea is that a certain infrared limit of certain lattice models is described by a topological field theory. These are things we can describe mathematically through topology. But there's another way we can look at these models, from a quantum information point of view; this is a very different perspective, but manifold topology can still be useful.

Kitaev had a conjecture that for at least invertible systems, the moduli spaces of lattice models and TFTs are homotopy equivalent. In particular, given a TFT, you should be able to produce a lattice model. The map from lattice models to TFTs, which we expect to be a homotopy equivalence, should have something to do with renormalization group flow. Some of the ideas around this are related to work of Norbert, Schuch, Clement, Delcamp, Giufre, Vidal, and Jake McNamara.

A lattice system is some kind of system on Euclidean space that we think of as describing phenomena like electron hopping, etc. A TFT is more topological, described in terms of bordism theory, something maybe more familiar to topologists. The difference between these two is what makes Kitaev's conjecture more exciting.

You can think of the lattice system as some sort of material type, and connecting this to bordism and TFT, we want to build our manifold, which may have nontrivial topology, out of this material type. Imagine riveting some metal material into the shape or a torus — now you have a physical object, and you can study its electronic properties, which are a measurement of the material and is an invariant of the lattice system.

Even at this level, there are still a lot of mysteries — which lattice systems work? And at the way we currently understand it, we don't know how to apply a material to all manifolds, or say that one material on one manifold is the same as another material on another manifold. There's lots to do.

Now a little more detail. Let's consider a lattice system in $\mathbb{R}^d$. The lattice amounts to (at least) a bunch of vertices on $\mathbb{R}^d$, called *sites*. Let $S \subset \mathbb{R}^d$ be the set of sites. We choose a (complex) "local Hilbert space" $\mathcal{H}$ and put a copy of $\mathcal{H}$ on each site; the total Hilbert space is $\bigotimes_{s\in S} \mathcal{H}_s$. The Hamiltonian H is a self-adjoint operator on this total Hilbert space. We want to study the spectrum of H and its eigenspaces.

Example 8.1. Suppose the local Hilbert space is a *qubit*, a copy of $\mathbb{C}^2$. Let

$$X \coloneqq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad Z \coloneqq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8.2}$$

If we work on a single site and let $H = Z$, the eigenvalues are $\{\pm 1\}$: $Z|0\rangle = |0\rangle$ and $Z|1\rangle = |1\rangle$, where $\{|0\rangle, |1\rangle\}$ is the standard basis of $\mathbb{C}^2$. ◀

Example 8.3. We coul tensor several copies of the previous example together: $\mathcal{H}_s = (\mathbb{C}^2)^{\otimes m}$. We tensor the Hamiltonians together: on two-fold tensor products define

$$H(v \otimes w) = H(v) \otimes w + v \otimes H(w), \tag{8.4}$$

and generalize to higher-fold tensor products by doing this iteratively. Now we can represent basis elements of $(\mathbb{C}^2)^{\otimes d}$ by bitstrings, e.g. $|01001\rangle$, and the spectrum is the integers $\{-d, \dots, d\}$. We could also use $H = (1+Z)/2$ and get $H|0\rangle = 0$ and $H|1\rangle = 1$, and build a spectrum on $0, \dots, d$. ◀

These examples may feel silly: we're just on a single site. Even if it's a site for sore eyes, we will want to consider multi-site lattices, and put a $\mathbb{C}^2$ (or $(\mathbb{C}^2)^{\otimes m}$) on every site and tensor them all together. In this case, we want the Hamiltonian to be a sum of local terms, meaning only using operators on nearby sites. If L denotes the length of the system, measured for example in the number of sites on a line, we want there to be a gap in the smallest two eigenvalues of H as $L \to \infty$. Defining this precisely is difficult, and we won't do it now.

Example 8.5. Let's consider a system on a line (or line segment) with sites at the integer points. The local Hilbert space will be $\mathbb{C}^4 = M_2(\mathbb{C})$. Give this the orthonormal basis e_i^j which is the zero matrix except for a 1 in position (i, j). Multiplication gives a map $M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \to M_2(\mathbb{C})$. Let the local Hamiltonian by orthogonal projection to K, the kernel of this multiplication map; then $M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) = M_2(\mathbb{C}) \oplus K$, corresponding to the eigenspaces 0 and 1 respectively. (TODO: why isn't this 1 and 0?).

Let H_i denote this operation at lattice site i, and $H \coloneqq \sum_{i=1}^n H_i$. $[H_i, H_j] = 0$ for $i \neq j$, so the eigenvalues of H are easy to compute: the ground state space is the subspace of vectors annihilated by H. The ground state is the transpose of the iterated multiplication map

$$M \otimes \cdots \otimes M \longrightarrow M. \tag{8.6}$$

This has fourfold degeneracy, localized to 2 each on the ends of the chain. ◀

This is an important example, and you can generalize it to higher dimensions.

Example 8.7. Suppose we have a two-dimensional square lattice on the integer points of some rectangle. Choose vector spaces V and W, finite-dimensional, and let $M \coloneqq V \otimes V^*$ and $N \coloneqq W \otimes W^*$. At each site place the local Hilbert space $M \otimes N$. We want the local Hamiltonian H_p to be the kernel of orthogonal projection onto the kernel of multiplication $(M \otimes N)^4 \to (M \otimes N)^2$. TODO: there was something about composing M horizontally and N vertically? Then H is the sum of the local Hamiltonians as usual. ◀

These examples have ground states with very special properties. In the simplest examples, the ground state was something like $|0\rangle \otimes \cdots \otimes |0\rangle$, a *tensor state.* Our more sophisticated examples have more sophisticated ground states, but still with a similar feel: they are things called *matrix product states.* The idea is that instead of a vector at every site, you have a matrix of vectors at every site, and we tensor them together using matrix multiplication.

For example, if we have $\begin{pmatrix} |0\rangle & |1\rangle \\ |1\rangle & |0\rangle \end{pmatrix}$ next to $\begin{pmatrix} |0\rangle & |1\rangle \\ |1\rangle & |0\rangle \end{pmatrix}$, the total ground state for those two sites is

$$\begin{pmatrix} |0\rangle & |1\rangle \\ |1\rangle & |0\rangle \end{pmatrix} \otimes \begin{pmatrix} |0\rangle & |1\rangle \\ |1\rangle & |0\rangle \end{pmatrix} = \begin{pmatrix} |00\rangle + |11\rangle & |01\rangle + |10\rangle \\ |01\rangle + |10\rangle & |00\rangle + |11\rangle \end{pmatrix}. \tag{8.8}$$

And one can continue with longer chains of sites. This is a very special thing to be true for a ground state. And yet:

Theorem 8.9. *The ground states of the model we described in Example 8.5, as well as related models, are matrix product states.*

You can think of this as specifying tensors with one input and one output, and linking the input of one with the output of another is contracting an index, or matrix multiplication. This suggests a higher-dimensional generalization, where our tensors have more arms, corresponding to tensors with more indices, and there are thus more ways to contract them. This is relevant for the two-dimensional example Example 8.7, where we work with a product of two matrix algebras, which is the kind of object that has both horizontal and vertical outputs, which we can link up/compose in two ways. This arrangemnet of data is the same thing as a linear map from $M \otimes N$ to the local Hilbert space (TODO: I think). In fact, the ground state in this system *is* this tensor network state.

So now we have some examples to play with. There's mounting evidence that the lattice models coming from TFTs should have ground states that look like these matrix product states. Let's learn a little more about why we believe that.

One thing to keep in mind is that the area of the region we consider is not fixed: we envision it growing, to pass to some sort of limit. We should be able to map from the Hilbert space for a smaller region to the Hilbert space of a larger region, which is a process called *density matrix projection.* Given a state v in $\mathcal{H}_1 \otimes \mathcal{H}_2$, we want to measure entanglement, a quantity describing how far this state is from being a pure tensor. We can write

$$v = \sum_i \sqrt{s_i} u_i \otimes v_i, \tag{8.10}$$

where u_i and v_i are all orthonormal and $s_i \geq 0$. Associated to this quantity is the *entanglement entropy*

$$S(v) \coloneqq -\sum_i s_i \log_2(s_i). \tag{8.11}$$

This is the refinement of the rank of a projection.

Example 8.12. In $\mathbb{C}^d \otimes \mathbb{C}^d$, consider $v' \coloneqq \sum e_i \otimes e_i$ — or more precisely, the unit vector in the same direction, $v \coloneqq v'/\sqrt{d}$. Then $s_i = 1/d$ and the entanglement entropy is $S(v) \propto -\log_2(d)$. This corresponds to cutting a segment out from the line.

This was for the lattice on a line. But what if we take this on a square? More specifically, we cut the square out from the rest of the lattice. There's a d for every internal edge, and a $-\log_2(d)$ for every half-edge (the edges we cut). Now the entangleement entropy is proportional to the perimeter times $\log_2(d)$. Had we done this in some other dimension, we would've obtained the surface area times some constant, in place of the perimeter. ◀

These product states are very special in that they obey something called an *area law* like this. In dimension 1, there's a theorem due to Hastings that the entanglement entropy of a gapped system always satisfies an area law: it's a constant times the surface area. In higher dimensions, there are some results, but not a full picture. In any case, these gapped lattice systems should have an area law property. This might even be true for all Hamiltonians which are sums of local terms, not just commuting projectors.

Conversely, states which obey area laws are supposed to be close to tensor network states. This is both aspirational (we don't have a proof) and inspirational: it tells us what systems to look for, or what's out there. That is, these tensor network systems are similar to these systems that we're looking for that come from topological field theories.

Recall the example we did with $M_2(\mathbb{C})$ attached to each site on a line. If we try to take a TFT at low energy, we obtain the trivial TFT. But maybe we can fix this somehow. The map $M_2 \otimes M_2 \to M_2$ is equivariant for the action of $\mathfrak{su}_2$ on $M_2(\mathbb{C})$ by conjugation, and this action exponentiates to an action of SO_3. Now, the system we consider is *non*trivial as a system with an SO_3-symmetry; the relevant invertible TFTs are classified by characters $\mathrm{Hom}(H_2(B\mathrm{SO}_3), \mathbb{C}^\times)$, which is isomorphic to $\mathbb{Z}/2$. The TFT is computed via its partition function, an invariant of surfaces with a principal SO_3-bundle. Then the system is nontrivial, which means that you can't break the entanglement with operators that are SO_3-invariant.

There are lots of other interesting examples: the AKLT model, the Kitaev chain, and the Kitaev-Drinfeld models coming from a finite group. All of these models have ground states that can be expressed in terms of tensor network states, and the lattice models can be described, up to renormalization, can be described as these matrix product models. This gives you the special properties governing the entanglement that we discussed, as well as correlations between measurements at different places, which we'll discuss next time.

9. Renormalization in lattice systems: 6/15/22

Today we continue with aspects of lattice models. This may seem lowbrow, but in the remaining two lectures, we will make contact with ideas from the other lectures.

We'll start with an example: the *Affleck–Kennedy–Lieb–Tasaki model*, or AKLT model, introduced in [AKLT88]. This is a one-dimensional example; Affleck–Kennedy–Lieb–Tasaki defined it on any graph, but let's consider it on an interval of some length with sites on the integers. The local Hilbert space at each site s is a $\mathbb{C}^2$, and the total Hilbert space is the tensor product of all the local Hilbert spaces like usual. The Hamiltonian is a sum of local Hamiltonians, each of which is a projection operator. This is an example of something called a *valence bond solid*.

This system has some symmetry around that we can use. SU_2 acts on $\mathbb{C}^2$ via the defining representation, which is said to be spin-1/2. The representation $\mathrm{Sym}^n(\mathbb{C}^2)$ is spin-$n/2$. Let V_n denote the spin-n representation; thus $V_{1/2} = \mathbb{C}^2$ and $V_{1/2} \otimes V_{1/2} \cong V_1 \oplus V_0$. If $\{|\uparrow\rangle, |\downarrow\rangle\}$ is a basis of $V_{1/2}$, and we represent basis vectors in the tensor product by strings of arrows, the $V_0 \subset V_{1/2} \otimes V_{1/2}$ is spanned by $|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$.

Using this, we specify the local Hamiltonian for the AKLT model. Place a copy of V_1 at each site. On each edge, we have nearby local Hilbert spaces $V_1 \otimes V_1 = V_2 \otimes V_1 \otimes V_0$; the local Hamiltonian at the edge is projection onto V_2. Then the total Hamiltonian is a sum of local terms as usual.

Now that we have a lattice model, we can ask the same things we asked last time, including understanding the ground state and entanglement. But there's plenty more we can do, including a few things considered by AKLT. For example, this system's correlation functions have interesting properties. In order to discuss this, let's review what correlation functions do in quantum mechanics.

Let's say A is an *observable*, i.e. a self-adjoint operator on the Hilbert space $\mathcal{H}$ and $|\psi\rangle \in \mathcal{H}$. Then the *expected value* of A is the quantity

$$E(A) = \omega(A) \coloneqq \frac{\langle \psi \mid A \mid \psi \rangle}{\langle \psi \mid \psi \rangle}. \tag{9.1}$$

If A and B are two observables, their *correlator* is

$$\mathrm{corr}(A, B) \coloneqq \omega(AB) - \omega(A)\omega(B). \tag{9.2}$$

Exercise 9.3. AKLT showed that if A and B are separated by r links in the chain, then in the ground state of the AKLT system, the correlator between A and B decays exponentially as $r \to \infty$. Prove it.

In general, Lieb–Robinson [LR72] proved a bound on the speed information can travel in a gapped lattice system, and this should imply that correlators decay exponentially in the distance between observables in general, though that may not be a theorem yet. Some of these observations will help us on the way to determining when to expect a lattice model comes from a topological field theory.

Our next question: is the ground state a matrix poduct state? If not, how close is it to being one?

To address this, we need to dig in a little deeper. Let $\psi_{11} \coloneqq |\uparrow\uparrow\rangle$, $\psi_{22} \coloneqq |\downarrow\downarrow\rangle$, and

$$\psi_{21} = \psi_{12} \coloneqq \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}. \tag{9.4}$$

Then the ground state of the AKLT model is in fact a matrix product state. At sites this alternates between matrices A and B, where

$$A \coloneqq \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} \qquad \text{and} \qquad B \coloneqq \begin{pmatrix} \psi_{22} & -\psi_{12} \\ -\psi_{21} & \psi_{11} \end{pmatrix}. \tag{9.5}$$

We want to group the sites in neighboring pairs, where one site has matrix A and the other has B. Zoom out and imagine these two sites are actually one larger site. The local Hilbert space is $V_1 \otimes V_1$, and the ground state is still a matrix product state. However, something interesting happens: previously, we had a 2×2 matrix and a three-dimensional local Hilbert space; the entries of the matrix aren't linearly independent, but they span V_1. When we tensor together, we now have four elements in our matrix for a nine-dimensional space, namely $V_1 \otimes V_1$, so we don't span, though we are linearly independent. Therefore we can throw out the irrelevant parts of $V_1 \otimes V_1$, replacing it with the subspace spanned by the matrices. This process, combining sites and throwing away the irrelevant parts of the Hilbert space, is a form of renormalization.

Now we can ask, how do correlation functions behave under this operation? It's a useful exercise to show that if $r > 1$, the correlation after we renormalize between A at site s and B at site $s + r$ is zero. You can even coax this out of a diagram chase, via a perspective we will discuss next time. This is different than the exponential decay we saw before we renormalized. What's the deal? The answer is that the entries of the new, post-renormalization matrix are not orthonormal.

The takeaway is expected to be that in some sort of limit of an iterated renormalization process of the AKLT model, we would obtain the 2d matrix model from the previous lecture! This matrix model is fixed under renormalization as we performed it above.

The picture is that renormalization defines a flow, called *renormalization group flow*, on the space of lattice models, and one envisions good lattice models flowing to fixed points. This is expected to happen most of the time. And these "special" lattice models ought to correspond to certian "special" TFTs as well. The refined equivalence, conjecturally still, is that renormalization limits of lattice models should correspond to TFTs. TFTs, after all, are scale-invariant, so can't contain anything sensitive to renormalization.

SO if we start with a TFT and build a lattice model, it had better be a renormalization fixed point — and because of exponential decay of correlation functions, we need to make our correlation functions vanish for separated operators. Tensor network models are fixed under renormalization and have these nice properties.

Example 9.6. Let's look at a class of $1+1$d models, which as usual exist on an interval with sites at the integers. Label each edge $[i, i+1]$ with an algebra $A_{i,i+1}$, over $\mathbb{C}$ — maybe $\mathbb{Z}/2$-graded, maybe not; we want them to be finite-dimensional C^*-algebras. Choose $(A_{k-1,k}, A_{k,k+1})$-bimodules M_k, which are the local Hilbert spaces; the local Hamiltonians $H_{s,s+1}$ are the projections onto the kernel of the maps

$$M_s \otimes M \longrightarrow M_s \otimes_{A_{s,s+1}} M_{s+1}. \tag{9.7}$$

If you're familiar with Hochschild homology, this constriction might look familiar when you place it on a circle.

This class of examples can be described using matrix product states.

Now let's renormalize. The new Hilbert space at combined sites 0 and 1 is $M_0 \otimes M_1$, which decomposes into the ground state tensor the kernel. This decomposes the entire system into the ground states and some other stuff, the latter of which is some other system which has higher energy! So we can throw it away if we're only interested in low-energy questions.

One specific example we could use is $M := M_2(\mathbb{C})$; alternate our algebras between $\mathbb{C}$ and M and alternate our bimodules between $V := \mathbb{C}^2$ and V^*. When we renormalize, we obtain $V \otimes V^* = M$, and we increase entanglement. This is not ideal: renormalization is supposed to decrease or preserve entanglement. One of the takeaways is that we have to be careful how we renormalize.

Since we can also renormalize tensoring over M, by switching from combining i and $i+1$ to $i-1$ and i, we could obtain the trivial system when we renormalize. Since the low-energy TFT is expected to be preserved under renormalization group flow, the original system also has trivial low-energy TFT.

However, let's add the SO_3-symmetry acting on M. Then we can't renormalize to the trivial theory in a way compatible with this symmetry, and in fact this system is nontrivial! We discussed last time briefly the classification, and that can be used to describe which class this system is in. ◂

10. : 6/16/22

In the last two lectures, we worked with lattice models. The goal is to connect with topological field theories; we begin finding the connections today.

Recall that the mathematical approach to TFT begins with the bordism category $\mathcal{B}ord_n$, a symmetric monoidal (∞, n)-category with duals. We won't decorate the bordisms yet with anything, save for an n-framing of the tangent bundle, i.e. an isomorphism $TM \oplus \underline{\mathbb{R}}^{n-k} \cong \underline{\mathbb{R}}^n$. In a sense this is the universal choice, though it appears less common in physics applications.

The symmetric monoidal structure on $\mathcal{B}ord_n$ is from disjoint union $\amalg$ of manifolds and bordisms. Be careful with notation: in the bordism category, this is *not* the coproduct! So we may instead refer to it as $\otimes$.

A more general and flexible version of this is to consider a bordism category $\mathcal{B}ord_n^X$, where we fix a space X and a map $\xi\colon X \to B\mathrm{O}_n$. This bordism category is of manifolds and bordisms equipped with a map to X and a trivialization of $TM \oplus \underline{\mathbb{R}}^k \oplus \xi^*(\mathrm{taut})$, where $\mathrm{taut} \to B\mathrm{O}_n$ denotes the tautological bundle.

Example 10.1. If $X = BG$ and ξ is constant, then $\mathcal{B}ord_n^X$ consists of n-framed manifolds together with a principal G-bundle. ◂

When ξ is constant, there is a sense in which $\mathcal{B}ord_n^X$ is a colimit of $\mathcal{B}ord_n$ over a diagram shaped like X, in the category of symmetric monoidal (∞, n)-categories. This is a theorem due to Lurie [Lur09]. Lurie also has a formula for when ξ is not constant, which is more complicated.

The big theorem in that paper, a proof of a conjecture of Baez–Dolan [BD95], is that $\mathcal{B}ord_n$, the n-framed version, is the free (∞, n)-category with duals on a single object.

Definition 10.2. A *fully extended, n-dimensional topological quantum field theory* is a symmetric monoidal functor $Z\colon \mathcal{B}ord_n^X \to \mathcal{C}$, where $\mathcal{C}$ is some symmetric monoidal (∞, n)-category.

Thus structural results about $\mathcal{B}ord_n^X$ classify topological field theories as well.

Remark 10.3. Having duals is a property of a symmetric monoidal (higher)category, not structure. And the image of any dualizable object under a symmetric monoidal functor is dualizable. So we may without loss of generality assume $\mathcal{C}$ has duals. ◄

In particular, if $X = BG$ and ξ is constant, maps from the colimit over BG of $\mathcal{B}ord_n$ to $\mathcal{C}$ are the same thing as maps from $\mathcal{B}ord_n$ to $\mathrm{Fun}(BG, \mathcal{C})$. This is the data of a $\mathcal{C}$-object with a group action.

You can think of a topological field theory as a lot of data, very hard and messy, or you can think of it as a point in $\mathcal{C}$, maybe a little more data, and push the complexity into category-land. It's often a good idea to do this so that you can think without getting dragged down by details.

Now we should fill in a detail: what's $\mathcal{C}$? Well really, there should be one $\mathcal{C}_n$ for each n. This was the subject of a conjecture of Hopkins–Singer [HS05]; it is still open, but might not be for long. Moreover, we want these to be related in some way. Inside $\mathcal{B}ord_n$ is a category of less-extended TFTs starting in one dimension higher; this should have target $\mathcal{C}_{n-1}$. This suggests that our family of targets should satisfy $\Omega\mathcal{C}_n \cong \mathcal{C}_{n-1}$, where $\Omega(\text{–}) := \mathrm{Hom}_{\text{–}}(1, 1)$. One says that $\mathcal{C}_{n+1}$ is a *delooping* of $\mathcal{C}_n$.

Another thing we would like to have in $\mathcal{C}_n$ is a finite path integral, where given a finite group G (or more generally, a π-finite space) and a TFT of manifolds with a principal G-bundle, we would like to be able to sum over G-bundles and obtain a new TFT which does not depend on the gauge bundle. This is a sort of pushforward

$$\begin{tikzcd} \mathcal{B}ord_n^{BG} \arrow[r] \arrow[d, "\sum"] & \mathcal{C}_n. \\ \mathcal{B}ord_n \arrow[ur] & \end{tikzcd} \tag{10.4}$$

If we tease apart what this entails, we learn that $\mathcal{C}_n$ must have all limits and colimits over π-finite spaces! This leads to a heirarchy of conditions we must have on $\mathcal{C}_n$. For example, one first learns that $\mathcal{C}_n$ has an initial object 0 and a terminal object $*$.

Exercise 10.5. Show that, because we have all finite products and coproducts in $\mathcal{C}_n$, and because objects in $\mathcal{C}_n$ have duals, the canonical map $0 \to *$ is an isomorphism.

Proceeding in a similar way, but with trickier proofs, one can show that in $\mathcal{C}_n$, the canonical map

$$\coprod_{i\in I} X_i \xrightarrow{\simeq} \prod_{i\in I} X_i \tag{10.6}$$

is an equivalence, provided the indexing set I is finite. This suffices to show that $\mathcal{C}_n$ is already additive. This is the beginning of a heirarchy of properties Lurie defined called *semiadditivity*. This notion first appears in [HL13], and has been the subject of further research progress recently thanks to Harpaz [Har20] and Carmeli–Schlank–Yanovski [CSY21, CSY22].

We define m-semiadditivity ireratively. FOr $m = 1$, $\mathcal{C}_n$ is 1-semiadditive if the map $0 \to *$ is an isomorphism — that is, if $\mathcal{C}_n$ has a zero object. 2-semiadditivity means that the natural map (10.6) is an isomorphism. [TODO: do I have an off-by-one? am I missing limits?]

2-semiadditivty is about colimits and limits over points agreeing. The next level is about colimits and limits over 1-truncated spaces agreeing. Then over 2-truncated spaces... so in general,m-semiadditivity is asking for the natural map from a colimit to a limit, indexed over a space which is m-finite [TODO: off-by-one, probably], is an equivalence. If you ask for semiadditivty in all levels, you get an additive category.

We would like the finite path integral to exist, and therefore we want $\mathcal{C}_n$ to be n-semiadditive.

Example 10.7. $\mathcal{C}_0$ is the set of complex numbers. Whatever $\mathcal{C}_1$ is, we asked that $\mathrm{End}_{\mathcal{C}_1}(1) \cong \mathbb{C}$. For any $V \in \mathcal{C}_1$, $\mathrm{Hom}_{\mathcal{C}_1}(1, V)$ is a module over $\mathrm{End}(1) = \mathbb{C}$, so all Hom-spaces are complex vector spaces. This doesn't tell us what $\mathcal{C}_1$ is yet, though. ◄

We still don't have enough information to guide us to $\mathcal{C}_n$, so let's get some more insight from physics. The invertible theories are those that land in $\mathcal{C}_n^\times$, the subcategory of objects X such that there's an object Y with $X \otimes Y = 1$, and morphisms and higher morphisms which are invertible under composition.

Example 10.8. $\mathcal{V}ect_{\mathbb{C}}^\times \cong B\mathbb{C}^\times$: there is a single invertible vector space up to isomorphism, which is the one-dimensional one; and the invertible maps from $\mathbb{C}$ to itself are the nonzero ones, giving us $\pi_0 = 1$ and $\pi_1 = \mathbb{C}^\times$. ◄

The fact that $\mathcal{C}_{n-1} \simeq \Omega\mathcal{C}_n$, where Ω means taking endomorphisms of the unit, implies that as spaces, $\mathcal{C}_{n-1}^\times \simeq \Omega\mathcal{C}_n^\times$, where Ω means based loop space! And this is enough data to fit all these categories together into a spectrum, which we call I. This is a spectrum in the sense of algebraic topology. We know $\pi_0(I) = \pi_0\mathcal{C}_0^\times = \mathbb{C}^\times$. This leads to an interesting spectrum first studied by Brown–Comenetz:

Theorem 10.9 (Brown–Comenetz)**.** *There is a spectrum $I_{\mathbb{C}^\times}$ characterized up to equivalence by the universal property $[E, I_{\mathbb{C}^\times}] = \mathrm{Hom}(\pi_0(E), \mathbb{C}^\times)$.*

This suggests a conjecture that there is a map $I \to I_{\mathbb{C}^\times}$ which is an isomorphism on at least π_0. This conjecture appears in [FH21].

Progress on this conjecture has been happening! There is work of Johnson-Freyd and Johnson-Freyd–Reutter, Burklund–Schlank–Yuan, and Carmeli–Schlank–Yanovski on this conjecture and trying to understand $\mathcal{C}_n$ given what we know about the spectrum I.

One concrete consequence of this conjecture is that the partition function of an invertible topological field theory determines the theory. This is in a sense a statement about nature!

We know that $\mathcal{C}_1$ can't just be vector spaces or some trivial delooping of it. This is because $\pi_{-1} I_{\mathbb{C}^\times} = [\pi_1(\mathbb{S}), \mathbb{C}^\times] = \mathbb{Z}/2$. And this $\mathbb{Z}/2$ is also $\pi_0 \Omega^\infty \Sigma I_{\mathbb{C}^\times}$, which should be the group of invertible objects in the symmetric monoidal 1-category $\mathcal{C}_1$. So we need something with two isomorphism classes of invertible objects.

So we have $\pi_0 \mathcal{C}_1^\times = \mathbb{Z}/2$ and $\pi_1 \mathcal{C}_1^\times = \mathbb{C}^\times$. There should be a Postnikov invariant $k\colon B\mathbb{Z}/2 \to K(\mathbb{C}^\times, 3)$ describing a fiber bundle of $K(\mathbb{C}^\times, 2)$ over $B\mathbb{Z}/2$ whose total space is $\mathcal{C}_1^\times$. But more than that, it's the homotopical version of an extension of abelian groups: an extension of infinite loop spaces.

The upshot is that we get a functor from $B\mathbb{Z}/2$ to the 2-category of $\mathbb{C}$-linear categories, and we should take the colimit of this functor. What we obtain is an $\mathbb{C}$-linear category, and because the functor is an infinite loop map, the colimit gets a symmetric monoidal structure. What we get is the category of super vector spaces – and sure enough, the invertible super vector spaces have the correct π_0, π_1, and Postnikov invariant! But it's interesting that, even if we didn't have this category a priori, we would be able to build this category out of the $B\mathbb{Z}/2$-action.

This leads to a notion of $\mathcal{C}_n$ being separably, or algebraically, closed. There is a rough procedure beginning with $\mathcal{C}_n$, then forming the $(n+1)$-category of dualizable $\mathcal{C}_n$-module categories; this is automatically m-semiadditive if $\mathcal{C}_n$ is. Then take the algebraic closure, and that should yield $\mathcal{C}_{n+1}$. Then one would have to check that the invertible subcategory yields $I_{\mathbb{C}^\times}$.

We've already seen $\mathcal{C}_0 = \mathbb{C}$, and $\mathcal{C}_1 = s\mathcal{V}ect$. This from a physics point of view came from the desire to work with fermions, for which there is the Pauli exclusion principle. $\mathcal{C}_2$ is the Morita category of superalgebras.

11. : 6/17/22

Recall that we've been discussing lattice systems and putting them on manifolds, like wrapping some sort of material on a manifold, and a relationship with topological field theories, functors $\mathcal{B}ord_n^X \to \mathcal{C}$. If M is an $(n-1)$-manifold and $F\colon \mathcal{B}ord_n \to \mathcal{C}$ is a TFT, we can consider $\mathcal{B}ord_n^M = \mathcal{B}ord_n \otimes M$. M canonically has this structure by the identity map.

Loop this $n-1$ times: $\Omega^{n-1}\mathcal{B}ord_n \otimes M \to \Omega^{n-1}\mathcal{C}_n \otimes M \to \mathcal{C}_1 \otimes M$. This last category is modeled by lattice systems on M; $\mathcal{C}_1$ is the category of super vector spaces, so something we understand.

If you're interested in lattice systems on $\mathbb{R}^n$ specifically, you get

$$\underline{\mathcal{C}}_1 \otimes \mathbb{R}^n = \varprojlim \underline{\mathcal{C}}_1 \otimes \mathbb{R}^n/(\mathbb{R}^n \cup \mathrm{pt}) = \underline{\mathcal{C}}^1(S^n, \mathrm{pt}) = \underline{\mathcal{C}}_{n+1}. \tag{11.1}$$

In a sense, we've seen that thanks to the Baez–Dolan cobordism hypothesis we discussed in yesterday's lecture, the objects of $\mathcal{C}_{n+1}$ are the same thing as the symmetric monoidal functors $\mathcal{B}ord_n \to \mathcal{C}_{n+1}$. What about the morphisms?

The answer is also nice, thanks again to Lurie's perspective on TFTs. Consider the inclusion of $\mathcal{B}ord_n$ into $\mathcal{B}ord_{n+1}$ as morphisms from the empty set to the empty set. The cone of this morphism (the pushout of this and the map $\mathcal{B}ord_n \to *$)admits the interpretation of the bordism category whose objects are $(n+1)$-manifolds with boundaries and whose morphisms are suitable bordisms with corners between them. Maps $\mathbf{1} \to F$ are left boundary theories for F, and maps $F \to \mathbf{1}$ are right boundary theories. This tells us some morphisms; the rest are of the form $F \to G$, which is equivalent using the symmetric monoidal structure to maps $\mathbf{1} \to G \otimes F^\vee$; these are domain walls between F and G.

Recall that $\mathcal{C}_0 = \mathbb{C}$ and $\mathcal{C}_1 = s\mathcal{V}ect$. There is a map $\mathcal{C}_1 \to \mathcal{M}od_{\mathcal{C}_0}$ described by taking $x \mapsto \mathrm{Hom}(\mathbf{1}, x)$. This is a concrete fact about $\mathbb{Z}/2$-graded vector spaces, but we also expect it to generalize to higher category number: $\mathrm{Hom}(\mathbf{1}, –)$ should determine a map $\mathcal{C}_{n+1} \to \mathcal{M}od_{\mathcal{C}_n}$. In physics speak, we would describe this as sending a theory F to its category of boundary conditions.

We will call the image of this functor — the objects of $\mathcal{C}_n$ which are $\mathrm{Hom}(\mathbf{1}, x)$ for an $x \in \mathcal{C}_{n+1}$ — the *colocalization* of $\mathcal{C}_{n+1}$ at $\mathbf{1}$. A $\mathcal{C}_n$-module in this image is called *local*. Don't get too attached to this notation: it's not some standard notation in math, and we just need a word here for a little bit.

If F is local, then by an abstract nonsense argument,

$$\mathrm{Hom}_{\mathcal{C}_{n+1}}(F, G) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{M}od_{\mathcal{C}_n}}(\mathrm{Hom}(\mathbf{1}, F), \mathrm{Hom}(\mathbf{1}, G)). \tag{11.2}$$

Now choose $D \in \mathrm{Hom}(\mathbf{1}, F)$. D^* and D are an adjoint pair, and $DD^*\colon \mathbf{1} \to \mathbf{1} \in \mathcal{C}_n$ is an algebra: the unit map of the adjunction has signature $DD^*DD^* \to DD^*$.

Definition 11.3. D is a *Dirichlet boundary condition* if $\mathrm{Hom}_{\mathcal{C}_{n+1}}(F, G)$ is isomorphic to a DD^*-module in $\mathrm{Hom}_{\mathcal{C}_{n+1}}(\mathbf{1}, G)$.

This is equivalent to the notion of a Dirichlet boundary condition in Freed's lectures, but this categorical perspective will be a little easier to use in our setting.

Example 11.4. For $\mathcal{C}_1 = s\mathcal{V}ect$, the adjoint pair between $\mathbf{1}$ and $s\mathcal{V}ect$ is V^* and V. Tensoring them together, we obtain $A := \mathrm{End}(V)$, and the Dirichlet boundary conditions are A-modules. ◂

We have to be careful about overspecifying things in category theory – we may not have something like A [TODO: wasn't sure exactly what wasn't specified, might not've been A] but only after some more information, or only its category of modules. This is a common occurrence in category theory, and you want to be careful and honest — if you said your dog ate your homework, fine; but if you said you were walking down the street and your dog was distracted by your friend rollerblading by and then ate your homework? Too much information, it starts to sound suspicious!

In category theory, the technical term for being overly specific in this way is *evil.* Really. Imagine if I said "the set with one element." Well, there's lots of sets with a single element, such as {you} and {Satan} — am I saying that these sets are equal? Hopefully not! They are isomorphic, which sees the structure but not the precise objects themselves, and is not an evil thing to say.

Anyways, one cool thing we can do with our Dirichlet boundary condition D, and its adjoint D^*, is to glue our material on the lattice to the trivial phase along boundaries. You can imagine "zooming out" of this striped system with the stripes labeled by $\mathbf{1}$ and our material, and thinking of this as some other phase, some other material.

For example, if we do this in $\mathcal{C}_3$, $\mathbf{1} = \mathcal{V}ect$, and we can consider $\mathcal{V}ect$-modules such as the product of n copies of $\mathcal{V}ect$. Morphisms here are "higher matrices:" matrices whose entries are vector spaces. Composition is matrix multiplication using tensor product. These multiplications correspond to placing different materials next to each other, gluing by the Dirichlet boundary condition [TODO: I think]. In the first lecture, we saw a 2d tensor network where $W \otimes W^* \otimes V \otimes V^*$ is the local Hilbert space, which suggests that this module and Dirichlet perspective is a good way to think about tensor networks and their relationships to TFTs. This example is Morita trivial, as we discussed in a previous lecture, but this is still a useful proof of concept.

TODO: there was a picture of a bordism with corners, which kind of looked like a solid Eiffel Tower from below, and I wasn't sure how it fit into the story. Sorry about that!

Example 11.5 (Toric code)**.** We should be able to recover the toric code from the 2-category $\mathcal{V}ect[G]$ (or maybe more specifically, the monoidal category $\mathcal{V}ect[G]$. One can think of monoidal categories as 2-categories with a single object). Let's say $G = \mathbb{Z}/2$ for concreteness… TODO: I did not follow what happened next. I'm really sorry about that.

Note: this was inspired by a construction of Freed–Teleman [FT18] in the statistical mechanics case, related to their perpective on Turaev–Viro theories. ◂

These TFTs with Dirichlet boundary data all the way down are not everything. This is much like how we work with nice spaces in topology: we don't want to really grapple with Sierpiński spaces when the point of the question is about manifolds. So not all TFTs have all these boundary conditions, but it's a nice class of TFTs to work with and it's a great place to start.

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