HOMOLOGY AND SENSOR NETWORKS

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"Topology! The stratosphere of human thought! In the twenty-fourth century it might possibly be of use to someone..." – In the First Circle, Aleksandr Solzhenitsyn[1]

ABSTRACT. Networks of sensors are an excellent real-world example of applied algebraic topology, as one often wants to get global information using only local sensor data. In this talk, we'll present some results of de Silva and Ghrist that use homological criteria to provide sufficient conditions for a sensor network to solve a coverage problem.

1. SENSOR NETWORKS AND THE COVERAGE PROBLEM

- What exactly is a sensor network? Not mathematically, but in the real world; provide different examples of why they're useful and what kinds of problems are difficult to solve with sensor networks (e.g. power conservation).
- These results deal specifically with sensors that have the following assumptions (list the real-world assumptions and their mathematical equivalents). In particular, we often don't know their exact locations, merely the network structure of how they can communicate.
  1. One of the strongest constraints on a sensor network is power conservation: the more often someone has to go out into the woods and change the batteries of fifty sensors, the less practical such a network will be. Generally, the biggest drain on power is radio transmission, so it's common for each sensor to have a small *broadcast radius* r_b, over which it sends its ID. Thus, if two nodes are within r_b of each other, they can detect each other's identities but nothing else (neither direction nor distance).
  2. The sensors have a *cover radius* r_c within which they can observe events. For technical reasons, we'll assume $r_c \geq r_b$, which is a reasonable (though not universal!) assumption about real-world networks. Thus, each sensor's *sensing region* is a disc of radius r_c centered at the sensor.
  3. The sensors all lie in some region $\mathcal{D} \subset \mathbb{R}^2$. We'll require $\mathcal{D}$ to be compact and connected, as is tends to be the case in the real world.
  4. We'll have to put some constraints on the boundary in order to simplify our proofs. In essence, we're going to require that we have a fence: the boundary is delineated by nodes. Specifically, these *fence nodes* are the vertices of $\partial\mathcal{D}$, which is connected and piecewise-linear; each is assumed to be within distance r_b of its neighbors on $\partial\mathcal{D}$. The identities of the fence nodes are known. This last assumption is the weakest, though it still holds in many practical systems, and we'll be able to weaken it later.

  So we have topological information: who is close to whom, but no distances or angles.
- We're going to try to solve the *covering problem*: given a domain and a network of sensors in it, is every point of the domain within sensing distance of some sensor? We'll let $\mathcal{U}$ denote the union of the sensing regions of all of our sensors, so our question can be rewritten as asking, is $\mathcal{U} \supset \mathcal{D}$?
- Other approaches include geometric and probabilistic strategies (the latter is actually pretty helpful for power conservation), but applying topology isn't ad hoc: a major motivation for trying it is that sensor network problems often require determining global properties with data about local neighborhoods, which is a common theme in topology.

2. THE RIPS COMPLEX

Definition. Let X be a collection of sensors with sensing distance r_d; then, the *Rips complex* of X, denoted $\mathcal{R}$, is the simplicial complex whose k-simplices are the $(k+1)$-tuples of nodes that can all sense each other.

[1]Ironically enough, most of the references I could find to this quote were epigraphs of papers in applied algebraic topology.

Within the Rips complex, the fence nodes generate a subcomplex $\mathcal{F}$ called the *fence subcomplex*. Specifically, one applies the Rips construction, but taking only the fence nodes instead of all of X.

Thus, the Rips complex and fence subcomplex are known from the information that the sensors have about each other. However, that isn't enough to determine, for example, the homotopy type of $\mathcal{U}$ without additional information.

3. First Result

Theorem 1 (de Silva-Ghrist (2006), [1])**.** *Let X be a set of sensor nodes in a region $\mathcal{D} \subset \mathbb{R}^2$ satisfying the assumptions listed above. Then, X solves the coverage problem for $\mathcal{D}$ if $\partial_2 : H_2(\mathcal{R}, \mathcal{F}) \to H_1(\mathcal{F})$ is nonzero.*

Proof. We can define a *simplicial realization* map $\sigma : \mathcal{R} \to \mathbb{R}^2$ first by sending nodes to their locations in $\mathbb{R}^2$. Then, each point in the simplex is a convex combination of nodes $t_1 x_1 + \cdots + t_n x_n$, so we send it to $t_1\sigma(x_1) + \cdots + t_n\sigma(x_n)$. Because we assumed that we have a well-behaved fence, $\sigma(\mathcal{F}) = \partial\mathcal{D}$, and therefore σ maps the pair $(\mathcal{R}, \mathcal{F}) \to (\mathbb{R}^2, \partial\mathcal{D})$.

Suppose that X does not solve the coverage problem, so that there exists a $p \in \mathcal{D} \setminus \mathcal{U}$. We made assumption (2) because it implies that $\sigma(\mathcal{R}) \subset \mathcal{U}$,[2] so σ maps $(\mathcal{R}, \mathcal{F})$ into $(\mathbb{R}^2 \setminus p, \partial\mathcal{D})$.

This means that, by naturality of the LES of a pair, the following diagram is commutative.

$$\begin{tikzcd} H_2(\mathcal{R}, \mathcal{F}) \arrow[r, "\partial_2"] \arrow[d, "\sigma_*"] & H_1(\mathcal{F}) \arrow[d, "\sigma_*"] \\ H_2(\mathbb{R}^2 \setminus p, \partial\mathcal{D}) \arrow[r, "\partial_2"] & H_1(\partial\mathcal{D}) \end{tikzcd} \tag{1}$$

The connecting morphism ∂_2 acts as a boundary: $\partial_2[\alpha] = [\partial\alpha]$ for an $\alpha \in H_2(\mathcal{R}, \mathcal{F})$. In particular, since ∂_2 (the top arrow) is nonzero and σ_* (the *right* arrow) is nonzero, then $\partial_2\sigma_*$ is nonzero (lower left), and so σ_* (the *left* arrow) is nonzero.

Now we use a relative Mayer-Vietoris sequence.

$$\cdots \longrightarrow H_2(A \cap B, A' \cap B') \longrightarrow H_2(A, A') \oplus H_2(B, B') \longrightarrow H_2(\mathbb{R}^2, \partial\mathcal{D}) \xrightarrow{\partial_*} H_1(A \cap B, A' \cap B') \longrightarrow \cdots$$

We'll let $A = \mathbb{R}^2 \setminus p$, B be a small disc around p, $A' = \partial\mathcal{D}$ and $B' = \varnothing$. Thus, $A \cap B \simeq S^1$ and $H_2(B, \varnothing) = 0$, so we get

$$H_2(S^1) \longrightarrow H_2(\mathbb{R}^2 \setminus p, \partial\mathcal{D}) \oplus 0 \longrightarrow H_2(\mathbb{R}^2, \partial\mathcal{D}) \xrightarrow{\partial_*} H_1(S^1).$$

But $\mathbb{R}^2$ deformation retracts onto $\mathcal{D}$ fixing $\partial\mathcal{D}$, so $H_2(\mathbb{R}^2, \partial\mathcal{D}) = H^2(\mathcal{D}, \partial\mathcal{D}) = H_2(S^2) = \mathbb{Z}$. Finally, it turns out that ∂_* sends a generator of $H_2(\mathbb{R}^2, \partial\mathcal{D})$ to a generator of $H_2(\mathbb{R}^2, \partial\mathcal{D})$,[3] so ∂_* is an isomorphism. Thus, we simplify our diagram to

$$0 \longrightarrow H_2(\mathbb{R}^2 \setminus p, \partial\mathcal{D}) \longrightarrow \mathbb{Z} \xrightarrow{\cong} \mathbb{Z},$$

so $H_2(\mathbb{R}^2 \setminus p, \partial\mathcal{D}) = 0$. Thus, in (1), $\sigma_* : H_2(\mathcal{R}, \mathcal{F}) \to H_2(\mathbb{R}^2 \setminus p, \partial\mathcal{D})$ must be the zero map, which is a contradiction. $\square$

This is nice because it's a result that can be computed programatically, since it deals with simplicial homology.

4. Another Look at the Fence Assumptions

The assumptions placed on the fence nodes in Theorem 1 are convenient and sometimes hold in reality, but not always. A more common scenario is for nodes to sense whether they are near the boundary of a region, but not have much more information (the wildlife sanctuary example is good here, because outside the boundary the area will be quite different). In order to get a homological result, though, we need a little more information about the topology. Specifically, nodes will be allowed to issue weak or strong signals; not only can a strong signal reach farther, but a node receiving a signal can detect whether it is weak or strong. This is still a fair assumption on real-world sensor networks.

[2] There's something to prove here, but I won't go into detail about it unless asked, since it's more about Euclidean geometry than anything else.

[3] A generator of the former is $\partial\mathcal{D}$, which is homologous in $(\mathbb{R}^2, \partial\mathcal{D})$ to a disc contained within B containing p, whose boundary generates $H_1(A \cap B)$.

Let's list our assumptions.

1. (1) Each node can broadcast its ID at a *weak radius* r_w or a *strong radius* r_s, such that $r_w \geq r_s \cdot \sqrt{10}$. This is a little counterintuitive: the weak radius is larger!
2. (2) The cover radius r_c must be greater than or equal to r_s.
3. (3) The key difference is that nodes can sense the boundary $\partial\mathcal{D}$ (but not its distance or direction) within a *fence distance* r_f.
4. (4) Once again, we want the nodes to lie in a compact $\mathcal{D} \subset \mathbb{R}^2$. If $\widetilde{\mathcal{D}}$ denotes the points in $\mathcal{D}$ more than $r_f + r_s/\sqrt{2}$ from the boundary, we'll assume $\widetilde{\mathcal{D}}$ is connected. (The point here is that we don't want $\mathcal{D}$ to be too "pinched.")
5. (5) Finally, if $\Sigma \subset \mathcal{D}$ consists of the points of distance at most r_f from $\partial\mathcal{D}$, there's a lower bound on its diameter. Intuitively, we know its thickness in the direction of the boundary, but want to control its diameter in other directions too.

These last two assumptions that we make on the shape of $\mathcal{D}$ and $\partial\mathcal{D}$, respectively, are reasonable assumptions to make in applications, where the boundary is probably not going to be some sort of topological monstrosity.

Once again, we can ask the same coverage problem, and once again there will be a homological result. We consider the Rips complex for the strong signal, which we'll call $\mathcal{R}_s$, and the one for the weak signal, $\mathcal{R}_w$; then, $\mathcal{R}_s$ is a subcomplex of $\mathcal{R}_w$. Similarly, we have fence subcomplexes $\mathcal{F}_s$ and $\mathcal{F}_w$ of the nodes that are able to sense the fence. (These might not be very many nodes, unlike the first scenario, though in our result we'll need some nodes to perceive the fence.)

On the other hand, relaxing the fence constraint means we can give the result in any dimension.

Theorem 2 (de Silva-Ghrist (2007), [2]). *If X is a set of sensor nodes in a region $\mathcal{D} \subset \mathbb{R}^d$ satisfying the assumptions above, then X solves the coverage problem for the restricted domain $\widetilde{\mathcal{D}}$ if $i_* : H_d(\mathcal{R}_s, \mathcal{F}_s) \to H_d(\mathcal{R}_w, \mathcal{F}_w)$ induced by inclusion is nonzero.*

Proof sketch. This proof leans much more heavily on calculations from Euclidean geometry. In the interest of time and of, well, your interest, I'm going to gloss over them, focusing on the motivation and on the topology.

We'll start in a similar way. Let $\sigma : \mathcal{R}_s \to \mathcal{D}$ be the simplicial realization map as before. If $\widetilde{\mathcal{D}}$ is contained entirely within some k-simplex, then we're done; if not, $\sigma(\mathcal{F}_s) \subset N$, where $N = \mathbb{R}^d \setminus \widetilde{\mathcal{D}}$ denotes the *extended collar* of $\mathcal{D}$. In particular, naturality of the homology long exact sequence summons for us the following commutative diagram.

$$\begin{tikzcd} H_d(\mathcal{R}_s, \mathcal{F}_s) \arrow[r, "\partial"] \arrow[d, "\sigma_*"] & H_{d-1}(\mathcal{F}_s) \arrow[d, "\sigma_*"'] \\ H_d(\mathbb{R}^d, N) \arrow[r, "\partial"] & H_{d-1}(N) \end{tikzcd}$$

Since $i_* \neq 0$, pick an $[\alpha] \in H_2(\mathcal{R}_s, \mathcal{F}_s)$ such that $i_*([\alpha]) \neq 0$. Specifically, we want $[\alpha]$ to be the class of some geometrically realizable cycle (which is okay, since these generate the singular chains, so σ_* must be nonzero on one of them).

Suppose $\sigma_*\partial[\alpha] \neq 0$ (along the upper right), so that $\sigma_*[\alpha] \neq 0$. If there's a p not covered by these sensors, then as before σ is a map $(\mathcal{R}_s, \mathcal{F}_s) \to (\mathbb{R}^d \setminus p, N)$. Alexander duality gives us an isomorphism $H_d(\mathbb{R}^d \setminus p, N) \to H^0(\mathbb{R}^d \setminus N, p) = 0$ (since $\mathbb{R}^d \setminus N$ is connected), which forces the contradiction $\sigma_*[\alpha] = 0$. Thus, in this case, the covering problem is solved.

Alternatively, if we assume that $\sigma_*\partial[\alpha] = 0$, we'll be able to derive a contradiction. As tends to happen in these arguments, it comes in the form of a big diagram chase. Specifically, we have the Rips complexes $\mathcal{R}_s$ and $\mathcal{R}_w$, but we'll also need an intermediate Rips complex $\mathcal{R}_m$ for some length m in between r_s and r_w. We'll talk about what m needs to be later; its precise value involves some messy computations that don't really provide insight into the proof. The point is, we have $\mathcal{R}_s \subset \mathcal{R}_m \subset \mathcal{R}_w$, each a subcomplex of the next, and the same thing for the fence subcomplexes, so by the naturality of the long exact sequence in homology, we have a big commutative diagram. (Note: in the talk, I intend to point and talk rather than label all the arrows; the labels are references for me further along in the proof.)

$$
\begin{tikzcd}
H_d(\mathcal{R}_s) \arrow[r] \arrow[d] & H_d(\mathcal{R}_s, \mathcal{F}_s) \arrow[r, "\partial_d^s"] \arrow[d, "i_1"] & H_{d-1}(\mathcal{F}_s) \arrow[d, "j_1"] \\
H_d(\mathcal{R}_m) \arrow[r, "\varphi_m"] \arrow[d, "h_2"] & H_d(\mathcal{R}_m, \mathcal{F}_m) \arrow[r, "\partial_d^m"] \arrow[d, "i_2"] & H_{d-1}(\mathcal{F}) \arrow[d] \\
H_d(\mathcal{R}_w) \arrow[r, "\varphi_w"] & H_d(\mathcal{R}_w, \mathcal{F}_w) \arrow[r] & H_{d-1}(\mathcal{F}_w)
\end{tikzcd}
$$

In this diagram, $i_2 \circ i_1$ is the inclusion map we hypothesized to be nonzero, so $i_2 i_1[\alpha] \neq 0$. Now, the diagram chase.

- First, let $\Sigma = \{x \in \mathbb{R}^d \mid \operatorname{dist}(x, \partial\mathcal{D}) \leq r_f\}$, the collar that marks the "fence-sensing region." We want to say that, as a geometric cycle, $\partial\alpha \in \Sigma$, but this is not quite true; if we extend on the order of r_s on each side (there is an exact value, but deriving it is cumbersome), producing a larger region S, then it's possible to show that $\partial\alpha \in S$.
- The assumptions we placed on our sensors imply that S is a thickened hypersurface; let Δ be its thickness. One can show that if
$$m = \sqrt{\Delta^2 + 2r_s^2 \frac{d-1}{d}},$$
then $j_1[\partial\alpha] = 0$ in $H_{d-1}(\mathcal{F}_m)$. The idea is that, since $\sigma_*\partial[\alpha] = 0$, then $[\partial\alpha = 0$ in $H_{d-1}(S)$, so if its image in $H_{d-1}(\mathcal{F}_m)$ is nonzero, then that image is a nontrivial cycle, and therefore encloses some $p \in S$ not covered by $\sigma(\mathcal{R}_m)$. This p must split $\sigma(\mathcal{R}_m)$ into two pieces, but we've chosen m such that each piece has radius larger than $\Delta/2$. Thus, the image of $[\partial\alpha]$ in $H_{d-1}(\mathcal{F}_m)$ is zero.
- The diagram commutes, so $\partial_d^m(i_1\alpha) = 0$, and by exactness, this pulls back to some $\zeta \in H_d(\mathcal{R}_m)$.
- The next step is to push this into the Čech complex $\check{C}_w$. de Silva and Ghrist prove that as long as $m/r_d \geq \sqrt{2d/d+1}$, there are inclusions as complexes $\mathcal{R}_m \subset \check{C}_{r_w} \subset \mathcal{R}_w$. The second inclusion is easy: if k points' sensing regions intersect each other, then each must be within sensing range of the others (which is the criterion for inclusion into the Rips complex). The first inclusion is a long proof argument using Euclidean geometry and not much topology, so I've omitted it. Anyways, the Čech theorem from before tells us that $\check{C}_w$ is homotopy equivalent to a subset of $\mathbb{R}^d$ (in this case, even an open submanifold), and therefore its d^{th} homology has to be trivial. Our inclusion map h_2 factors as $h_2 : H_d(\mathcal{R}_m) \to H_d(\check{C}_w) \to H_d(\mathcal{R}_w)$, and therefore $h_2 = 0$, and in particular $h_2(\zeta) = 0$.
- But the image of $h_2(\zeta) = 0$ under ψ_w is still 0, but by commutativity, this is exactly $i_2 i_1[\alpha]$, which was nonzero! Here's our contradiction: this case can't actually happen, so we're back in the first case, where we proved that we solved the covering problem. ☐

References

[1] de Silva, Vin and Robert Ghrist. "Coordinate-Free Coverage in Sensor Networks with Controlled Boundary via Homology." Int. J. Robot. Res. **25** 12 (2006), 1205-22. `https://www.math.upenn.edu/~ghrist/preprints/controlledboundary.pdf`.

[2] de Silva, Vin and Robert Ghrist. "Coverage in sensor networks via persistent homology." Algebr. Geom. Topol. **7** (2007), 339–358. `https://www.math.upenn.edu/~ghrist/preprints/persistent.pdf`.

[3] de Silva, Vin and Robert Ghrist. "Homological Sensor Networks." Notices of the American Mathematical Society **54** 1 (2007), 10-17. `https://www.math.upenn.edu/~ghrist/preprints/noticesdraft.pdf`.