

M392C NOTES: INDEX THEORY

ARUN DEBRAY
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These notes were taken in UT Austin's M392C (Index theory) class in Spring 2018, taught by Dan Freed. I live- \LaTeX ed them using vim, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Any mistakes in the notes are my own. Thanks to Rok Gregoric for fixing a few errors, and to Riccardo Pedrotti for providing the notes for §3.

CONTENTS

1. Overview, History, and some Linear Algebra: 1/17/18	1
2. Principal G -bundles: 1/24/18	8
3. Characteristic classes: 1/24/18	11
4. Clifford algebras, Clifford bundles, and Dirac operators: 1/31/18	16
5. The Weitzenbock formula: 1/31/18	19
6. Spin groups: 2/7/18	22
7. Spin geometry: 2/7/18	26

Lecture 1.

Overview, History, and some Linear Algebra: 1/17/18

“This formula should look fake if you haven’t seen it before.”

We’ll start with an overview and some history of index theory. The overview will use a little bit of complex geometry, but if you don’t know it that’s okay; the rest of the class will not depend on it.

One of the earliest manifestations of index theory was in the theory of algebraic curves. Let M be a compact smooth connected complex curve, i.e. a Riemann surface, and let D be a divisor on M , a finite formal sum of points of M with integer coefficients. For example, if $p_1, p_2, p_3 \in M$, one divisor is $4p_1 - 2p_2 + 7p_3$.

Definition 1.1. Let f be a meromorphic function on M ; then, its divisor $\text{div}(f)$ is the zeros of f minus the poles of f , where both are counted with multiplicity. For $f = 0$, we let $\text{div}(0) = 0$.

For example, if $M = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, then a meromorphic function on M is a rational function. If we took $f(z) = (z - 1)^2 / (z + 2)$, then $\text{div}(f) = 2 \cdot 1 - 1 \cdot (-2) - 1 \cdot \infty$: f has a double zero at 1 and a single pole at -2 , and at ∞ there is a simple pole.¹

A divisor has a *degree* which is the sum of its terms.

Theorem 1.2. *The degree of the divisor of a meromorphic function is zero.*

This is a consequence of the Cauchy integral formula.

A divisor specifies the zeros and poles of a meromorphic function, and it’s a classical problem to, given a degree-zero divisor D on a Riemann surface, construct a function whose divisor is D . More generally, let $\mathcal{L}(D)$ denote the set of meromorphic f such that $\text{div}(f) + D \geq 0$.² $\mathcal{L}(D)$ is a vector space, and if $\deg(D) < 0$, $\mathcal{L}(D) = 0$; we also have $\mathcal{L}(0) = \mathbb{C}$, given by constant functions.

Another classical question is to compute $\dim \mathcal{L}(D)$. Riemann provided an estimate:

$$(1.3) \quad \dim \mathcal{L}(D) \geq 1 - g + \deg(D),$$

¹To see this, use the change-of-variables $z = 1/w$ and evaluate f at $w = 0$.

²This is missing a zero element, so one needs to adjoin 0 for everything to work.

where g is the *genus* of M , defined to be

$$(1.4) \quad g := \frac{1}{2} \text{rank } H_1(X).$$

The next natural question is to identify the discrepancy, and Riemann's student Roch found the answer.

Theorem 1.5 (Riemann-Roch). *here is a canonical divisor K_M such that*

$$(1.6) \quad \dim \mathcal{L}(D) - \dim \mathcal{L}(K_M - D) = 1 - g + \deg D.$$

We won't say much about K_M , though $\deg(K_M) = 2g - 2$.

Corollary 1.7. *The genus is an integer.*

A more modern interpretation of this story is that D determines a holomorphic line bundle $L \rightarrow M$, and $\mathcal{L}(D)$ is the vector space of holomorphic sections of L , i.e. $\mathcal{L}(D) \cong H^0(M; L)$. If s is any smooth section of L , s is holomorphic iff $\bar{\partial}s = 0$. That is, in local coordinates $z = x + iy$, and

$$(1.8) \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right).$$

Thus, $\bar{\partial}s = 0$ is a first-order differential equation, and computing $\dim \mathcal{L}(D)$ is asking for the dimension of the space of solutions to the equation. Thus one way you might prove Theorem 1.5 is to analyze the differential operator $\bar{\partial}$, which is a linear operator

$$\bar{\partial}: \Omega^{0,0}(M; L) \longrightarrow \Omega^{0,1}(M; L).$$

Then, $\mathcal{L}(D) = \ker(\bar{\partial})$ and $\mathcal{L}(K_M - D) \cong \text{coker}(\bar{\partial})$.

Definition 1.9. The *index* of $\bar{\partial}$ is $\text{ind}(\bar{\partial}) := \dim \ker(\bar{\partial}) - \dim \text{coker}(\bar{\partial})$.

Broadly speaking, this course will be about indices of this sort, and their applications: for example, the Riemann-Roch theorem from this perspective is about computing the index of $\bar{\partial}$.

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For a simpler case, let V and W be finite-dimensional vector space and $T: V \rightarrow W$ be a linear map. Then, $\ker(T) \subset V$ and $\text{coker } T := W/T(V)$. Computing the index is a fundamental theorem in linear algebra.

Theorem 1.10.

$$\text{ind}(T) := \dim(\ker T) - \dim(\text{coker } T) = \dim V - \dim W.$$

In particular, it's independent of T ! One way you might prove this is to observe that it's true when $T = 0$ and then try to prove that it's locally constant.

In this class, we're interested in operators between infinite-dimensional vector spaces, such as $\Omega^{p,q}(M; L)$, whose kernels and cokernels are finite-dimensional (such that the definition of an index makes sense). There will be no nice formula like Theorem 1.10, but some aspects stay the same: though the dimension of the kernel or cokernel may jump along a continuous path, their difference is constant.

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Another classical subject that relates to index theory is that of the Euler number of a compact smooth n -manifold M . Betti defined *Betti numbers* b_0, \dots, b_n associated to M , and Noether realized they can be identified with ranks of abelian groups (or dimensions of certain real vector spaces).³

Definition 1.11. The *Euler characteristic* of M is

$$\chi(M) := \sum_{i=0}^n (-1)^i b_i.$$

³These days, this would be called *categorification*: it can often be useful to identify a number as the dimension of some vector space attached to your object.

The Betti numbers are defined via simplices, and how M is built out of cells. Since M is a smooth manifold, one might want to compute them in another way, using the smooth structure of the manifold. To do this, one introduces the *de Rham complex*

$$(1.12) \quad 0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \longrightarrow \cdots \longrightarrow \Omega^n(M) \longrightarrow 0,$$

with linear maps d such that $d^2 = 0$. Unlike in the previous example, this is built out of real functions and real differential forms.

Definition 1.13. The *de Rham cohomology* of M is the sequence of real vector spaces

$$H_{\text{dR}}^i(M) := \frac{\ker(d: \Omega^i(M) \rightarrow \Omega^{i+1}(M))}{\text{Im}(d: \Omega^{i-1}(M) \rightarrow \Omega^i(M))}.$$

Theorem 1.14 (de Rham). *There is an isomorphism $H_{\text{dR}}^i(M) \cong H^i(M; \mathbb{R})$, and therefore $\dim H_{\text{dR}}^i(M) = b_i$.*

From this perspective, the Euler characteristic looks more like an index, where we stack together the pieces of the de Rham complex:

$$(1.15) \quad \bigoplus_{i \text{ even}} \Omega^i(M) \longrightarrow \bigoplus_{i \text{ odd}} \Omega^i(M).$$

However, the index of this is *not* the Euler characteristic! The issue is that the de Rham cohomology groups are a subquotient, not just a subspace or just a quotient. To compute the Euler characteristic as an index, we'll need some way of turning them into pure subspaces or quotients. One way to do this is to use an inner product and take orthogonal complements.

Let M be a Riemannian manifold. Then, there is a Laplace operator $\Delta: \Omega^i(M) \rightarrow \Omega^i(M)$, which is a linear second-order elliptic differential operator.

Remark. There are three basic kinds of differential operators studied in a typical differential equations course: elliptic, parabolic, and hyperbolic. The Laplacian is the basic example of an elliptic operator; the heat operator is the basic example of a parabolic operator; and the Schrödinger operator is the basic example of a hyperbolic operator. We will focus on elliptic operators in this course, but both the heat equation and the Schrödinger equation will appear. ◀

Example 1.16. Let \mathbb{E}^n denote n -dimensional Euclidean space with coordinates x^1, \dots, x^n . Then, the Laplacian on \mathbb{E}^n is

$$\Delta = \left(\frac{\partial}{\partial x^1} \right)^2 + \cdots + \left(\frac{\partial}{\partial x^n} \right)^2.$$

For more general Riemannian manifolds, the definition of the Laplacian is more complicated, but not much more so.

Definition 1.17. If M is a Riemannian manifold, there is an L^2 inner product on $\Omega^i(M)$ defined by

$$\langle \alpha, \beta \rangle_{L^2} := \int_M \langle \alpha(M), \beta(M) \rangle \, \text{dvol}_m.$$

Using these inner products, we can let $d^*: \Omega^{i+1}(M) \rightarrow \Omega^i(M)$ be the formal adjoint to d .

Fact. d^* exists and is a first-order differential operator. ◀

Definition 1.18. The *Laplace operator* on M is $\Delta := dd^* + d^*d$.

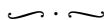
A form in the kernel of Δ is called *harmonic*, and the space of harmonic forms is denoted $\mathcal{H}^i(M) \subset \Delta^i(M)$.

Theorem 1.19 (Hodge theorem). *The natural map $\mathcal{H}^i(M) \rightarrow H_{\text{dR}}^i(M)$ is an isomorphism. In particular, $\dim \mathcal{H}^i(M) = b_i$.*

This is how index theory enters the picture: if we can access the space of harmonic forms as kernels and cokernels of operators, we could compute the Euler characteristic as an index. And indeed, we can fix (1.15) as follows:

$$(1.20) \quad \bigoplus_{i \text{ even}} \Omega^i(M) \xrightarrow{d+d^*} \bigoplus_{i \text{ odd}} \Omega^i(M).$$

The index of this operator is the Euler characteristic.



A third example of index theory is the higher-dimensional Riemann-Roch theorem. Let M be a compact complex manifold; then, the $\bar{\partial}$ operator defines a *Dolbeault complex* analogous to the de Rham complex. If M is 2-(complex-)dimensional, the Euler characteristic satisfies a formula

$$(1.21) \quad \chi(M) = \frac{1}{12}(c_1^2(M) + c_2(M))[M].$$

Here c_1 and c_2 are examples of *characteristic classes*, which we'll start on in the next few lectures. In particular, the right-hand side is an integer. In higher dimensions, there are similar expressions with larger denominators and more characteristic classes.

These were studied by Todd and his student Egger, by Weyl, and others. But the general forms remained conjectures until 1954, when Hirzebruch proved these generalizations of the Riemann-Roch theorem, and an additional, similar result called the signature theorem. He wove together two very new pieces of mathematics: the cobordism theory of René Thom (published only earlier that year!) and the theory of sheaves.

Hirzebruch and others in this field introduced a rational combination of different characteristic numbers, called *Pontrjagin numbers*, called the \hat{A} -genus (said "A-hat genus"). This is defined on closed oriented manifolds, and on a spin manifold is an integer.

That $\hat{A}(M)$ is an integer is a suggestion that it's a dimension of something, and when Singer went to visit Oxford in 1963, Atiyah asked him what object has the \hat{A} -genus as its dimension, and this is the problem that they solved: they constructed a differential operator called the Dirac operator on a spin manifold, and showed that its index is the \hat{A} -genus.

The Dirac operator

$$D := \gamma^\mu \frac{\partial}{\partial x^\mu}$$

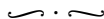
for some γ^μ (this notation means the index μ is implicitly summed over) is a first-order linear differential operator. We'd like this to be a square root of the Laplacian operator.

Exercise 1.22. Show that $D^2 = \Delta$ iff

$$(1.23) \quad \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = -2\delta^{\mu\nu}.$$

Here, $\delta^{\mu\nu}$ means 1 if $\mu = \nu$ and 0 otherwise.

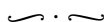
So the operator has to satisfy n^2 equations. If you try to solve this for functions on \mathbb{E}^n , you can show that no such γ^μ exist, but one could instead ask for vector-valued functions which satisfy (1.23), and indeed we will spend some time studying the abstract theory of matrices which satisfy this condition, rephrased as the algebraic theory of Clifford modules. In particular, we will be able to show that a spin structure is precisely what one needs to be able to construct the Dirac operator on a Riemannian manifold.



Before Atiyah and Singer told this story, Grothendieck took the Hirzebruch-Riemann-Roch theorem and generalized it still further, and Atiyah and Hirzebruch saw how to translate his ideas from algebraic geometry to topology, and replace sheaves with vector bundles. They then defined K -theory and rapidly developed it from 1958 to 1962. When Atiyah asked Singer his question, it was in this context.

At the same time, parallel work was undertaken in the Soviet Union under Gelfand and his students. He observed that the index sometimes can be computed topologically, and asked whether this is true in general, and Atiyah-Singer's answer also incorporates this question.

Subsequently, in the 1970s, Gilkey, Patodi, and others were able to provide more rigid, simpler proofs with analytic methods, and in the 1980s Getzler made another important simplifying step to what's now called the heat equation proof of the index theorem, which we'll follow.

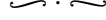


We'll use John Roe's book in this course. It's analytic in flavor, but also treats many other nice results, and if we go quickly enough, we'll get to see some of them, including Witten's physical treatment of Morse theory, the Lefschetz theorem, the Hodge theorem, and more.

In this class, the students will give lectures, two each week, and we hope to go through two chapters a week. You don't have to use all three hours!

On the course website (<https://www.ma.utexas.edu/users/dafr/M392C/>), there will be some useful information, including some old course notes, some historical background, and more to come. These will be there so that you do not forget the beauty of the material amongst all the details in the lectures.

Not everybody may know all of the prerequisites for this course, since it draws in lots of different parts of mathematics. One can ask the professor for references or talk to other students in the course.



The second half of the first day is on the first chapter of the book, reviewing some of the basics of Riemannian geometry.

Let's first start with some linear algebra and differential forms. Let V be an n -dimensional real vector space. Eventually, V will be a tangent space at a point to a manifold, and if the manifold has a Riemannian metric, V picks up an inner product.

Associated to V are several canonical vector spaces built from it: its wedge powers $\Lambda^2 V, \dots, \Lambda^n V$, and $\Lambda^0 V$, which is canonically \mathbb{R} . The top exterior power is also called the *determinant line*, $\text{Det } V := \Lambda^n V$. Dually, there are the exterior powers of the dual space V^* : $\mathbb{R}, V^*, \Lambda^2 V^*, \dots, \text{Det } V^*$.

An inner product on V canonically induces inner products on all of these exterior powers. One way to see this is to let e_1, \dots, e_n be an orthonormal basis of V ; then, there is a dual basis e^1, \dots, e^n of V^* , defined by the relation

$$(1.24) \quad e^\mu(e_\nu) = \delta_\nu^\mu,$$

i.e. 1 if $\mu = \nu$ and 0 otherwise.

We specify the inner product on V^* by declaring this dual basis orthonormal, which suffices, though you have to check that if you change the orthonormal basis of V you started with, you'll end up with the same inner product nonetheless.

We also obtain bases for the exterior powers of V and V^* : for $\Lambda^q V$, the basis is

$$(1.25) \quad \{e_{i_1} \wedge \dots \wedge e_{i_q} : 1 \leq i_1 < \dots < i_q \leq n\},$$

and for $\Lambda^q V^*$, it's

$$(1.26) \quad \{e^{i_1} \wedge \dots \wedge e^{i_q} : 1 \leq i_1 < \dots < i_q \leq n\}.$$

Again we define the inner products on $\Lambda^q V$ and $\Lambda^q V^*$ by asking for these bases to be orthonormal, and again the inner product in question does not depend on the specific choice of orthonormal basis of V .

Definition 1.27. An *orientation* of V is an orientation of its determinant line. That is, $\text{Det } V \setminus 0$ has two components, and an orientation is a choice of one of them.

Given n vectors $e_1, \dots, e_n \in V$, we can wedge them together to an $e_1 \wedge \dots \wedge e_n \in \text{Det } V$; $\{e_1, \dots, e_n\}$ is a basis iff $e_1 \wedge \dots \wedge e_n \neq 0$. Thus a basis singles out one of the two rays in $\text{Det } V \setminus 0$, hence defines an orientation. Since $(\text{Det } V)^* = \text{Det}(V^*)$ canonically, then this also defines an orientation on $(\text{Det } V)^*$: the duality pairing implies there's a single $\theta \in \text{Det } V^*$ which sends $e_1 \wedge \dots \wedge e_n \mapsto 1$; we call it the *volume form* and denote it vol .

On an oriented Riemannian n -manifold, this is a differential n -form, hence can integrate it to determine the volume of the manifold. If it's not oriented, there are two at each point, which may twist globally into something called a density. Nonetheless, this can be integrated, and the volume of, e.g. \mathbb{RP}^2 still makes sense.

The pairing $\Lambda^q V^* \otimes \Lambda^{n-q} V^* \rightarrow \text{Det } V^*$ defined by

$$(1.28) \quad \alpha, \beta \mapsto \alpha \wedge \beta$$

is nondegenerate. An orientation of V defines a trivialization of $\text{Det } V^*$ (where $\text{vol} = 1$), so this pairing is \mathbb{R} -valued. Therefore we obtain an isomorphism $\Lambda^q V^* \cong \Lambda^{n-q}(V)$, though it depends on the inner product and the orientation.

Example 1.29. In three dimensions, we use this frequently, to shift from the perspective of vector fields and scalars and div , ∇ , and curl to differential forms. ◀

There's also an isomorphism $\star: \Lambda^q V^* \rightarrow \Lambda^{n-q} V^*$ which only uses the inner product; this is called the *Hodge star*. Putting everything together, the Hodge star is defined uniquely by the stipulation that

$$(1.30) \quad \alpha_1 \wedge \star \alpha_2 = \langle \alpha_1, \alpha_2 \rangle \text{vol}$$

for any $\alpha_1, \alpha_2 \in \Lambda^q V^*$.

Exercise 1.31. For example, check that $\star(e_{i_1} \wedge \cdots \wedge e_{i_q})$ is the wedge of all of the e_j not in (i_1, \dots, i_q) , possibly multiplied by -1 .

Exercise 1.32. Show that $\star^2 = (-1)^{q(n-q)}$.

Here, “inner product” means nondegenerate inner product; much of this story still goes through for a Lorentz-signature metric, but not all of it.

Exercise 1.33. Show that on a closed, oriented Riemannian manifold M , $d^* = \pm \star d \star$, and determine the sign (which depends on n and q).

You can type-check that the right-hand side is a first-order differential operator which lowers the degree by 1. Solving the exercise boils down to checking that

$$\int_M \langle d\alpha, \beta \rangle \text{vol} = \pm \int_M \langle \alpha, \star d \star \beta \rangle \text{vol}.$$

You’ll end up using Stokes’ theorem.

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Now let’s think about parallelism. Let \mathbb{A}^n be n -dimensional affine space (no distinguished origin), where we learn calculus. This has parallel transport: if $\xi \in \mathbb{R}^n$ is a tangent vector at some point, we can translate it everywhere to a vector field. This allows us to define differentiation: if $f : U \rightarrow \mathbb{R}$, where $U \subset \mathbb{A}^n$ is open, then we define the derivative of f at p in the direction of ξ to be

$$(1.34) \quad \xi_p f := \lim_{h \rightarrow 0} \frac{f(p + h\xi) - f(p)}{h}.$$

This uses parallelism in the expression $p + h\xi$.

More generally, if M is a smooth manifold, we don’t always have a canonical parallel transport between tangent spaces for different points of the manifold, so we can’t compare tangent vectors in different places and differentiate.

For example, if $\gamma : [a, b] \rightarrow M$ is a curve, its tangent vectors at two different points can’t be compared (without extra structure), so there’s no way to make the subtraction in (1.34). We’ll introduce the structure that allows us to do this.

Definition 1.35. Let $V \rightarrow M$ be a vector bundle and $C^\infty(M; V)$ denote its space of smooth sections, which is a real vector space. A *covariant derivative* is a bilinear operator

$$\nabla : C^\infty(M; TM) \times C^\infty(M; V) \longrightarrow C^\infty(M; V),$$

denoted

$$X, s \longmapsto \nabla_X s,$$

such that

- (1) $\nabla_{fX} s = f \nabla_X s$, and
- (2) $\nabla_X (fs) = (X \cdot f)s + f \nabla_X s$,

where $(X \cdot f)$ is the usual directional derivative associated to a vector field.

For $V = TM$, we have the usual Lie bracket

$$[-, -] : C^\infty(M; TM) \times C^\infty(M; TM) \longrightarrow C^\infty(M; TM)$$

sending $X, Y \mapsto [X, Y]$; if $f, g : M \rightarrow \mathbb{R}$ are functions, then

$$[fX, gY] = f g [X, Y] + f (X \cdot g) Y - g (Y \cdot f) X.$$

This operator is the commutator of an infinitesimal flow of X and an infinitesimal flow of Y .

Definition 1.36. Let ∇ be a covariant derivative for the tangent bundle. Its *torsion* is

$$\tau(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y].$$

Exercise 1.37. Show that $\tau(fX, gY) = f g \tau(X, Y)$ and $\tau(X, Y) = -\tau(Y, X)$.

Let's write this out in local coordinates. There are two things we could mean – coordinates on M or on V . Since V is a vector bundle, we can use for its coordinates the coordinates of M and a (local) basis of sections s_1, \dots, s_r . (Global nonvanishing sections might not exist at all, e.g. $TS^2 \rightarrow S^2$). In this case, you can differentiate s_j , obtaining some linear combination of the sections depending on x in a neighborhood U :

$$\nabla_X s_j = \Gamma_j^i(x) s_i.$$

This is just parameterized linear algebra. These Γ_j^i are 1-forms on U . We can also obtain coordinates for these 1-forms: if we let

$$\nabla_{\partial/\partial x^\mu} s_j = \Gamma_{j\mu}^i s_i,$$

then $\Gamma_j^i = \Gamma_{j\mu}^i dx^\mu$.

If $V \rightarrow M$ has an inner product (metric), a positive definite pairing $C^\infty(M; V) \times C^\infty(M; V) \rightarrow C^\infty(M)$ sending $s_1, s_2 \mapsto \langle s_1, s_2 \rangle$, we can ask how a covariant derivative interacts with it.

Definition 1.38. A covariant derivative is *compatible with the metric* if for all $X \in C^\infty(M; TM)$ and $s_1, s_2 \in C^\infty(M; V)$,

$$X \cdot \langle s_1, s_2 \rangle = \langle \nabla_X s_1, s_2 \rangle + \langle s_1, \nabla_X s_2 \rangle.$$

Definition 1.39. A section $s \in C^\infty(M; V)$ is *parallel* if $\nabla_X s = 0$ for all X .

Parallel sections exist in \mathbb{A}^n but not in general; the obstruction is called the curvature.

Definition 1.40. The *curvature* of a covariant derivative ∇ is

$$K(X, Y) := [\nabla_X, \nabla_Y] - \nabla_{[X, Y]},$$

i.e.

$$K(X, Y)(s) := \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X, Y]} s.$$

If M is Riemannian, there's a beautiful theorem about how all of these structures interact.

Theorem 1.41 (Levi-Civita). *Let M be a Riemannian manifold. Then, there is a unique connection on $TM \rightarrow M$ which is torsion-free and compatible with the metric.*

Exercise 1.42. Prove this theorem. The way you do this is to compute $\langle \nabla_X Y, Z \rangle$, because if you know this for all Z , you know $\nabla_X Y$. Using the torsion-free and metric compatibility conditions, you can expand it out, and after some number of steps, you'll get the answer.

This local but non-global parallelism is an important property of Riemannian manifolds.

Next we will write a local formula for this connection. Suppose we have local coordinates x^1, \dots, x^n on an open set $U \subset M$; then, we obtain the symbols $\Gamma_{jk}^i : U \rightarrow \mathbb{R}$. If we define the inner product and the Lie bracket, we can write down formulas for them. Namely, if we let

$$g_{ij} := \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle,$$

and since

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^i} \right] = 0,$$

then we can determine equations that the Γ_{jk}^i must satisfy. These can be encoded in the Riemann curvature tensor $R(X, Y)Z$, and in coordinates, on elets

$$R_{jkl}^i \frac{\partial}{\partial x^i} = R \left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell} \right) \frac{\partial}{\partial x^j}.$$

This tensor has a bunch of important symmetries. The curvature is a 2-form on the manifold, but valued in $\text{End}(TM)$: X and Y are the two directions you're testing, and are the 2 components of the 2-form.

The symmetry $R_{jkl}^i = -R_{jlk}^i$ means that $R(X, Y) -$ is a skew-symmetric endomorphism of TM .

You can also lower an index by defining

$$(1.43) \quad R_{ijkl} = \langle R(\partial_k, \partial_\ell) \partial_j, \partial_i \rangle,$$

and skew-symmetry means

$$R_{ijkl} = -R_{jikl}.$$

These are the two “easier” symmetries, in that they don’t use much specifically about R . A more interesting one is

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0,$$

and the fourth identity, which follows from the other three, is

$$R_{ijkl} = R_{klij}.$$

Exercise 1.44. Compute the dimension of the vector space of tensors which satisfy these identities, as a subspace of $(V^*)^{\otimes 4}$.

Lecture 2.

Principal G -bundles: 1/24/18

The first part of today’s talk was given by George Torres, corresponding to the first part of Chapter 2 of Roe’s book.

Definition 2.1. A *Lie group* G is a group that is also a smooth manifold, and such that multiplication $m: G \times G \rightarrow G$ and inversion $i: G \rightarrow G$ are smooth maps.

Associated to any Lie group G is its *Lie algebra* $\mathfrak{g} := T_e G$. There is a *Lie bracket* operation

$$[-, -]: \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

defined like last time (use multiplication on G to extend tangent vectors to G -invariant vector fields, then take their commutators).

Definition 2.2. Let G be a Lie group and M be a smooth manifold. A *principal G -bundle* is a map of smooth manifolds $\pi: E \rightarrow M$ together with a smooth right action of G on E whose orbits are fibers of π , such that G acts freely and transitively on each fiber.

This implies that for each $m \in M$, $\pi^{-1}(m) \cong G$ noncanonically, and $E/G \cong M$.

Connections on principal bundles are analogous to those on vector bundles; the goal is to define a horizontal subspace of the bundle, and use that and its G -translates to define parallel transport. Then, one must show that connections always exist, but this turns out to be true. This definition of connections is sometimes called an *Ehresmann connection*.

In order to define connections, we’ll need a few preliminary definitions.

Definition 2.3. Let G be a Lie group. The *adjoint representation* of G is the map $\text{Ad}: G \rightarrow \text{End}(\mathfrak{g})$ which sends a $g \in G$ to $d|_e \psi_g$, where $\psi_g: G \rightarrow G$ is conjugation by G and $d|_e$ is differentiating at the identity.

Definition 2.4. Let $\pi: E \rightarrow M$ be a principal G -bundle and $\rho: G \rightarrow \text{Aut}(F)$ be a (real, finite-dimensional) representation of G . The *associated vector bundle* to E and F is the $E \times_G F := E \times F / \sim$, where $(e \cdot g, f) \sim (e, g \cdot f)$ for all $e \in E$, $f \in F$, and $g \in G$.

Exercise 2.5. Show that $E \times_G F \rightarrow M$ is indeed a vector bundle.

Definition 2.6. Let $\pi: E \rightarrow M$ be a principal G -bundle. A *vertical vector field* is a vector field v on E such that for all $p \in M$, $v_p = d_e A_p(u)$ for some $u \in \mathfrak{g}$, where $A_p: G \rightarrow E$ is an identification of the fiber E_p with G .⁴

An equivalent, more intuitive, definition is that a vertical vector field is contained within the *vertical subbundle* of E , i.e. the kernel of $d\pi$. A third equivalent definition is that V is vertical if for all $p \in M$, there is some $u \in \mathfrak{g}$ such that

$$(2.7) \quad v_p = \left. \frac{d}{dt} (p \exp(tu)) \right|_{t=0}.$$

Let $R_g: E \rightarrow E$ denote the right action of G , sending $e \mapsto e \cdot g$. We can pushforward by this map: let

$$(Rg)_* v := \left. \frac{d}{dt} (p \exp(tu)g) \right|_{t=0}.$$

In particular, the pushforward of a vertical vector field is still vertical.

Horizontal differential forms are dual to vertical vector fields.

⁴**TODO:** I might have gotten this wrong.

Definition 2.8. Let $\alpha \in \Omega^p(E)$. Then α is *horizontal* if for all vertical vector fields X_1, \dots, X_p , $\alpha(X_1, \dots, X_p) = 0$. If in addition $(Rg)^*\omega = \omega$, we say α is *invariant* under the G -action.

Example 2.9. For any $\beta \in \Omega^p(M)$, $\pi^*\beta$ is an invariant horizontal form:

$$(R_g)^*\pi^*\beta = (\pi \circ R_g)^*\beta = \pi^*\beta. \quad \blacktriangleleft$$

More generally, we can consider G -equivariant forms.

Definition 2.10. Let $\rho : G \rightarrow \text{Aut}(F)$ be a representation and $f : E \rightarrow F$ be a smooth map. Then, f is ρ -equivariant if for all $e \in E$ and $g \in G$,

$$f(eg) = \rho(g^{-1})f(e).$$

Invariance is the same thing as equivariance for the trivial representation.

Lemma 2.11 (Correspondence lemma). *Let $f : E \rightarrow F$ be as in the previous definition. There is a bijective correspondence between ρ -equivariant maps $f : E \rightarrow F$ and sections of $E \times_G F$.*

Proof. Let f be a ρ -equivariant map; then, we define a section s_f to send $x \in M$ to $(x, f(x)) \in E \times F$. To check that this is indeed a section, we need it to commute with the G -action on $E \times F$, and this follows because

$$\begin{aligned} s_f(x) \cdot g &= (x, f(x)) \cdot g = (xg, \rho(g^{-1})f(x)) \\ &= (xg, f(xg)) \\ &= s_f(x \cdot g). \end{aligned}$$

Conversely, let $s : M \rightarrow E \times_G F$ be a section, and consider the diagram

$$(2.12) \quad \begin{array}{ccccc} E & \xrightarrow{\sigma} & E \times F & \xrightarrow{\pi_1} & E \\ \downarrow \pi & & \downarrow p & & \downarrow \pi \\ M & \xrightarrow{s} & E \times_G F & \longrightarrow & M, \end{array}$$

where p is the quotient map, π_1 is projection onto the first factor, and σ is defined such that the composition of the maps across the top is the identity. The key observations are

- (1) the right-hand square is a pullback square, and
- (2) $f = \pi_2 \sigma$ is ρ -equivariant.

The first property is true because for any $g \in G$,

$$\begin{aligned} \pi_1(\sigma(xg)) &= xg \\ &= \pi_1(\sigma(x)) \cdot g = \pi_1(\sigma(x \cdot g)), \end{aligned}$$

and along the other corner,

$$\begin{aligned} p(\sigma(xg)) &= s\pi(xg) = s\pi(x) \\ &= p\sigma(x) \\ &= p\sigma(x \cdot g), \end{aligned}$$

so $\sigma(xg) = \sigma(x) \cdot g$. **TODO:** I missed the last part, that f is ρ -equivariant (and why these two properties suffice). \square

Exercise 2.13. Finish the proof by checking that these assignments are mutual inverses.

With these definitions in mind, we can define connections.

Definition 2.14. Let $\pi : E \rightarrow M$ be a principal G -bundle. A *connection* on E is a subbundle $H \subset TE$ (H for “horizontal”) such that

- (splitting) there is another subbundle $V \subset TE$ such that for all $u \in E$, $T_u E = V_u \oplus H_u$, and
- (G -invariance) $d|_e R_g(H_u) = H_{u \cdot g}$.

Using the splitting lemma for vector bundles, the first condition is equivalent to the existence of a split short exact sequence

$$0 \longrightarrow V \longrightarrow TE \longrightarrow \pi^*TM \longrightarrow 0.$$

The splitting is determined by a section $\pi^*TM \rightarrow TE$ (which defines what we call “horizontal”) or by a section $TE \rightarrow V$. This leads to an equivalent definition of a connection on a principal bundle, which is also useful: a connection on E is a \mathfrak{g} -valued 1-form ω on E , called the *connection one-form*, such that

- (G -invariance) for any $\xi \in T_u E$, $\omega(\xi \cdot g) = \text{Ad}(g^{-1})\omega(\xi)$, and
- (splitting) for any $v \in \mathfrak{g}$, $\omega(X_v) = v$.

The splitting lemma guarantees you can always split short exact sequences of vector bundles. But to show that connections exist, we need to address G -invariance, which is not as immediate.

Lemma 2.15. *If G is a Lie group and $\pi: E \rightarrow M$ is a principal G -bundle, then there is a connection on E .*

Proof. Let $\underline{\mathfrak{g}} := E \times \mathfrak{g}$ and $V := E \times_G \mathfrak{g}$, where G acts on \mathfrak{g} by the adjoint action. Then consider the diagram

$$(2.16) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \underline{\mathfrak{g}} & \longrightarrow & TE & \longrightarrow & \pi^*TM \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & V & \longrightarrow & Q & \longrightarrow & TM \longrightarrow 0. \end{array}$$

Proving that this commutes takes a while, so we won’t delve into the details; one reference is Atiyah, “Complex analytic connections on fibre bundles.”

The point of introducing (2.16) is that if we can lift a splitting from the bottom row to the top row, it will be a G -invariant splitting, hence a connection. So choose a splitting $\sigma: TM \rightarrow Q$, which splits the bottom row of (2.16). Then we have a diagram

$$(2.17) \quad \begin{array}{ccc} \pi^*TM & \longrightarrow & TM \\ \downarrow & & \downarrow \sigma \\ TE & \longrightarrow & Q \\ \downarrow & & \downarrow \\ E & \xrightarrow{\pi} & M. \end{array}$$

The bottom rectangle is a pullback, and the total rectangle (π^*TM , TM , E , and M) is a pullback. Therefore by the universal property of pullbacks, the top rectangle also is a pullback, and this implies that σ lifts across it to something G -invariant. \square

There is also a more geometric proof.

$\smile \cdot \smile$

Next we’ll talk about exterior derivatives.

Definition 2.18. Let $\pi: E \rightarrow M$ be a principal G -bundle with connection H and let $p_\omega: \Omega^*(E) \rightarrow \Omega^*(M)$ denote projection onto the horizontal subspace. The *exterior covariant derivative* is the composition

$$\Omega^p(E) \xrightarrow{d} \Omega^{p+1}(E) \xrightarrow{p_\omega} \Omega^{p+1}(M).$$

Proposition 2.19. Let $\rho: G \rightarrow \text{GL}(F)$ be a representation, $\alpha \in \Omega_E^p(F)$ be an F -valued, ρ -equivariant horizontal p -form on E , and ω denote the (\mathfrak{g} -valued) connection one-form on E . Then

$$p_\omega(d\alpha) = d\alpha + \rho_* \omega \wedge \alpha.$$

Definition 2.20. Let ω be a differential 1-form. Then, the *curvature* Ω of ω is the 2-form

$$\Omega(X_1, X_2) = d\omega(X_1, X_2) + [\omega(X_1), \omega(X_2)].$$

Exercise 2.21. With notation as above,

$$(p_\omega d)^2 \alpha = \rho_* \Omega \wedge \alpha.$$

The point is that acting on α is the same as wedging with $\rho_*\Omega$, and this tells you something about what Ω is doing.

Remark. In some of these formulas, it's important to be careful about what the wedge products are doing. For example, we once or twice saw $\omega \wedge \alpha$, where $\omega \in \Omega_E^1(\mathfrak{g})$ and $\alpha \in \Omega_E^1(F)$. If \mathfrak{g} is a matrix algebra, we can organize the components of ω into a matrix of differential forms, and F is a representation of the matrix algebra \mathfrak{g} , so α is a vector. In this case, the wedge product is a combination of matrix multiplication and the wedge product:

$$(2.22) \quad \begin{pmatrix} \omega_1^1 & \omega_2^1 \\ \omega_1^2 & \omega_2^2 \end{pmatrix} \wedge \begin{pmatrix} \alpha^1 \\ \alpha^2 \end{pmatrix} = \begin{pmatrix} \omega_1^1 \wedge \alpha^1 + \omega_2^1 \wedge \alpha^2 \\ \omega_1^2 \wedge \alpha^1 + \omega_2^2 \wedge \alpha^2 \end{pmatrix}.$$

The same is true for computing $\omega \wedge \omega$; in particular, this is not automatically zero.

Therefore one sometimes sees the formula for curvature written

$$(2.23) \quad \Omega := d\omega + [\omega \wedge \omega].$$

What does this mean? We have $\omega \wedge \omega \in \Omega_E^2(\mathfrak{g} \otimes \mathfrak{g})$, and the Lie bracket $[-, -]: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$. We implement this on differential forms by

$$(\omega^i \wedge \omega^j)e_i \otimes e_j \mapsto (\omega^i \wedge \omega^j)[e_i, e_j].$$

In particular,

$$\begin{aligned} [\omega \wedge \omega](X_1, X_2) &= [\omega(X_1), \omega(X_2)] - [\omega(X_2), \omega(X_1)] \\ &= 2[\omega(X_1), \omega(X_2)], \end{aligned}$$

which is why (2.23) has an extra 1/2 in it compared to the first definition. ◀

Remark. One can even define differential forms valued in vector bundles: $\Omega_M^\bullet(E \times_G F)$ consists of sections of the exterior powers of $E \times_G F$. Alternatively, you can think of these as valued in $(E \times_G F)_p$ at a point p ; the vector spaces changes as p moves, but that's okay. The quotient map $\pi: E \rightarrow G$ defines a pullback $\Omega_M^*(E \times_G F) \rightarrow \Omega_E^*(F)$. This provides yet another interpretation of the definition of a connection.

- Invariance is that $\alpha \in \text{Im}(\pi^*)$ iff $R_g^* \alpha = \rho(g)^{-1} \alpha$ for all $g \in G$.
- Splitting comes from the fact that $\iota_\zeta \alpha = 0$ when ζ is vertical. (This denotes *contraction*: $\iota_\zeta \alpha(X) := \alpha(\zeta, X)$).

It's a good exercise to check, to get practice manipulating these vector- or bundle-valued forms. But principal bundles make some of these computations easier, by turning some bundle-valued forms into constant vector space-valued forms. ◀

Lecture 3.

Characteristic classes: 1/24/18

The second part of today's lecture was given by Riccardo Pedrotti, on characteristic classes from a geometric perspective.

The theory of characteristic classes comes from the simple question: how can we tell two vector bundles apart? For instance, how do we know that the tangent bundle to the 2-sphere is non-trivial? Characteristic classes gives a systematic approach.

Definition 3.1. A *characteristic class* c is a natural transformation which to each vector bundle V over a manifold M associates an element $c(V)$ of the cohomology group $H^*(M)$, with property that if $V_1 \simeq V_2$ then $c(V_1) = c(V_2)$.

The idea of Chern-Weil theory is the following: suppose that our bundle V is equipped with a connection. In some sense, the curvature of this connection measures the local deviation of V from flatness. Now if V is flat, and the base manifold M is simply connected, then V is trivial. This suggests that there may be a link between curvature and characteristic classes, which measure the global deviation of V from triviality. Such a link is provided by the theory of invariant polynomials.

By polynomial function we mean the following:

Definition 3.2. Let $\mathfrak{gl}_m(\mathbb{C})$ denote the Lie algebra of $m \times m$ matrices over \mathbb{C} . A *homogeneous polynomial function* P on $\mathfrak{gl}_m(\mathbb{C})$ is a function such that there exists a $\tilde{P} \in \text{Sym}^k(\mathbb{C}^m)^*$ such that $P(A) = \tilde{P}(A, A, \dots, A)$. A *polynomial function* is a sum of homogeneous ones.

Definition 3.3. An *invariant polynomial* on $\mathfrak{gl}_m(\mathbb{C})$ is a polynomial function $P: \mathfrak{gl}_m(\mathbb{C}) \rightarrow \mathbb{C}$ such that for all $X, Y \in \mathfrak{gl}_m(\mathbb{C})$, $P(XY) = P(YX)$. An *invariant formal power series* is a formal power series over $\mathfrak{gl}_m(\mathbb{C})$ each of whose homogeneous components is an invariant polynomial.

For example, the determinant and the trace are invariant polynomials.

Lemma 3.4. The ring of invariant polynomials on $\mathfrak{gl}_m(\mathbb{C})$ is a polynomial ring generated by the polynomials

$$c_k(X) = (-2\pi i)^{-k} \operatorname{tr}(\Lambda^k X),$$

where $\Lambda^k X$ denotes the transformation induced by X on $\Lambda^k \mathbb{C}^m$.

Proof. Let P be any invariant polynomial. Restricting P to diagonal matrices, we see that P must be a polynomial function of the diagonal entries. Since these diagonal entries can be interchanged by conjugation, P must in fact be a *symmetric* polynomial function. Now since P is invariant under conjugation, it must be a symmetric polynomial function of the eigenvalues for all matrices with distinct eigenvalues, since by elementary linear algebra such matrices are conjugate to diagonal matrices. The set of such matrices is dense in $\mathfrak{gl}_m(\mathbb{C})$, so a continuity argument shows that P is just a symmetric polynomial function in the eigenvalues. Now it is easy to see that $\operatorname{tr}(\Lambda^k X)$ is the k^{th} elementary symmetric function in the eigenvalues of X . The main theorem on symmetric polynomials states that the ring of symmetric polynomials is itself a polynomial ring generated by the elementary symmetric functions, and this now completes the proof. \square

Example 3.5. To make the idea of the proof more concrete, let $m = 4$. A 4×4 diagonal matrix is of the form

$$X = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$$

for some $a, b, c, d \in \mathbb{R}$. Then, $\Lambda^2 X$ acts on $e_1 \wedge e_2$ by $Xe_1 \wedge Xe_2 = abe_1 \wedge e_2$. Therefore $\Lambda^2 X$ is a 6×6 diagonal matrix with diagonal terms ab, ac, ad, bc, bd , and cd , and therefore its trace is

$$\operatorname{tr}(\Lambda^2 X) = ab + ac + ad + bc + bd + cd.$$

This is a quadratic polynomial, and is symmetric; it's an example of an *elementary symmetric polynomial*. There's a theorem that the ring of all symmetric polynomials are generated by these elementary symmetric polynomials. \blacktriangleleft

Now let V be a complex vector bundle over M with connection ∇ and curvature $K \in \Omega_M^2(\operatorname{End}(V))$. Choosing a local framing for V , we may locally identify K with a matrix of ordinary 2-forms. Hence, if P is an invariant polynomial, we can apply P to this matrix to get an even-dimensional differential form $P(K)$. *A priori*, this depends on the choice of local framing, but since P is invariant, $P(K)$ doesn't depend on the choice, and is therefore globally defined.

In terms of the principal $\operatorname{GL}_m(\mathbb{C})$ -bundle E associated to V , this construction may be phrased as follows. Let Ω be the curvature form of the induced connection on E ; Ω is a horizontal, equivariant 2-form on E with values in $\mathfrak{gl}_m(\mathbb{C})$, so $P(\Omega)$ is a horizontal invariant form on E . Such a form is the lift to E of a form on M , and this form is $P(K)$.

Since 2-forms are nilpotent elements in the exterior algebra $\Omega_M^*(\mathfrak{gl}_m(\mathbb{C}))$, all formal power series with 2-form-valued variables in fact converge. Thus, this construction makes good sense if P is merely an invariant formal power series.

Proposition 3.6. For any invariant polynomial (or formal power series) P , the differential form $P(K)$ is closed, and its de Rham cohomology class is independent of the choice of connection ∇ on V .

Proof. For the purposes of this proof call an invariant formal power series P as *respectable* if the conclusion of the proposition holds for P . Clearly the sum and product of respectable formal power series are respectable. Thus, it is enough to prove that the generators defined in Lemma 3.4 are respectable. Equivalently, since

$$\det(1 + qK) = \sum q^k \operatorname{tr}(\Lambda^k K),$$

it is enough to prove that $\det(1 + qK)$, considered as a formal power series depending on the parameter q , is respectable.

If P is a respectable formal power series with constant term a , and g is a function holomorphic in a neighborhood of a , then $g \circ P$ is also a respectable formal power series. Hence, $\det(1+qK)$ is respectable if and only if $\log \det(1+qK)$ is respectable. We will now prove directly that $\log \det(1+qK)$ is respectable.

For this purpose we will work in the associated principal $\mathrm{GL}_n(\mathbb{C})$ -bundle E of frames for V , with matrix-valued connection 1-form ω and corresponding curvature 2-form Ω . Recall the formula

$$\Omega = d\omega + \omega \wedge \omega$$

where the product in the ring of matrix-valued forms is obtained by tensoring exterior product and matrix multiplication as in (2.22).

Now suppose that ω depends on a parameter t ; then Ω also depends on t , and if we use a dot to denote differentiation with respect to t , then

$$\dot{\Omega} = d\dot{\omega} + \omega \wedge \dot{\omega} + \dot{\omega} \wedge \omega.$$

Consider

$$(3.7) \quad \frac{d}{dt} \log \det(1 + q\Omega) = q \operatorname{tr}(\dot{\Omega}(1 + q\Omega)^{-1})$$

$$(3.8) \quad = \sum_{\ell=0}^{\infty} (-1)^{\ell} q^{\ell+1} \operatorname{tr}(\Omega^{\ell}(d\dot{\omega} + \omega \wedge \dot{\omega} + \dot{\omega} \wedge \omega))$$

where (3.7) is justified by the formula

$$\frac{d}{dt} \det A(t) = (\det A(t)) \cdot \operatorname{tr}(\dot{A}(t)A(t)^{-1}),$$

and (3.8) is justified by the power series expansion

$$\frac{1}{1+z} = \sum_{i=0}^{\infty} (-1)^i z^i.$$

We also need the *second Bianchi identity*

$$d\Omega = \Omega \wedge \omega - \omega \wedge \Omega,$$

which can be proven directly from the definition of the exterior derivative. Using this, plus the fact that trace is symmetric, we have that

$$\begin{aligned} \operatorname{tr}(\Omega^{\ell}(\omega \wedge \dot{\omega} + \dot{\omega} \wedge \omega)) &= \operatorname{tr}(\Omega^{\ell} \wedge \omega \wedge \dot{\omega} - \omega \wedge \Omega^{\ell} \wedge \dot{\omega}) \\ &= \operatorname{tr}((d\Omega^{\ell}) \wedge \dot{\omega}). \end{aligned}$$

Therefore

$$\begin{aligned} \operatorname{tr}(\Omega^{\ell} \wedge (d\dot{\omega} + \omega \wedge \dot{\omega} + \dot{\omega} \wedge \omega)) &= \operatorname{tr}((d\Omega^{\ell}) \wedge \dot{\omega} + \Omega^{\ell} \wedge d\dot{\omega}) \\ &= d \operatorname{tr}(\Omega^{\ell} \wedge \dot{\omega}), \end{aligned}$$

so (3.7) simplifies to

$$(3.9) \quad \frac{d}{dt} \log \det(1 + q\Omega) = d \sum_{\ell=0}^{\infty} (-1)^{\ell} q^{\ell+1} \operatorname{tr}(\Omega^{\ell} \wedge \dot{\omega}),$$

and in particular is an exact form on E .

In fact, it is the exterior derivative of a horizontal and invariant form on E : $\dot{\omega}$ is horizontal and G -equivariant since it is a 1-form on M (it follows from Lemma 2.11 that the space of connections is an affine space modeled on the vector space of V -valued forms on M), Ω is horizontal and G -equivariant as well. Hence $\operatorname{tr}(\Omega^{\ell} \wedge \dot{\omega})$ is invariant since the trace is an invariant polynomial, and is horizontal since $\Omega^{\ell} \wedge \dot{\omega}$ is.

Therefore, the projection to the base manifold

$$\frac{d}{dt} \log \det(1 + qK)$$

is also exact. Now the result follows; for since any connection can be deformed locally to flatness (i.e. $K = 0$), we see that $\log \det(1 + qK)$ is locally exact, hence closed, and since any two connections can be connected by a smooth path, the cohomology class of $\log \det(1 + qK)$ is independent of the choice of connection, since their difference is an exact form. \square

It follows from the proposition that any invariant formal power series P defines a characteristic class for complex vector bundles, by the recipe “pick any connection and apply P to the curvature.”

Definition 3.10. The k^{th} Chern class is the characteristic class corresponding to the generators c_k defined in Lemma 3.4.

Remark. We immediately see from the definition of Chern classes that if a complex vector bundle has rank m , then $c_k = 0$ for $k > m$: $\Lambda^k K$ is the linear transformation induced by K on $\Lambda^k \mathbb{C}^m$, and for $k > m$, the latter is trivial. Naturality comes from the fact that if on a local patch U_i , E has the local connection form ω_i , then on $f^{-1}(U_i)$, the curvature is $f^* \Omega_i$. \triangleleft

Lemma 3.11. Let V be a real vector bundle and $V_{\mathbb{C}}$ denote its complexification. Then, $c_{2k+1}(V_{\mathbb{C}}) = 0$.

Proof. We can give V a metric and compatible connection. The curvature of such a connection is skew (i.e. $\mathfrak{o}(m)$ -valued), so

$$\text{tr}(\Lambda^k F) = (-1)^k \text{tr}(\Lambda^k F).$$

To see this, recall that the coefficients of the characteristic equation for F are exactly $\text{tr}(\Lambda^k F)$ up to a sign.⁵ If λ is an eigenvalue of a skew-symmetric matrix, then $-\lambda$ is too, and on \mathbb{C} this means that the characteristic polynomial is up to a constant the product of polynomials $(z^2 - \lambda^2)$, so there are no coefficients of odd index, hence proving that for k odd, $\Lambda^k F$ is traceless. \boxtimes

Genera. Holomorphic functions can be used to build important combinations of characteristic classes. Let $f(z)$ be any function holomorphic near $z = 0$. We can use f to construct an invariant formal power series Π_f by defining

$$\Pi_f(X) := \det\left(f\left(-\frac{1}{2\pi i}X\right)\right).$$

Again, to make sense of this, we need to sidestep convergence issues! But since we'll just be applying this to differential forms, which are nilpotent, this is okay.

The associated characteristic class is called the *Chern f -genus*. It has a few nice properties.

Lemma 3.12. If $L \rightarrow M$ is a complex line bundle, $\Pi_f(L) = f(c_1(L))$.

Proof. This comes from the fact that in this case the curvature is a $\mathfrak{gl}_1(\mathbb{C})$ -valued 2-form, so

$$\begin{aligned} \Pi_f(L) &= \Pi_f(K_L) = \det\left(f\left(-\frac{1}{2\pi i}K_L\right)\right) \\ &= f\left(-\frac{1}{2\pi i}K_L\right) \\ &= f\left(\text{tr}\left(-\frac{1}{2\pi i}K_L\right)\right) = f(c_1(L)). \end{aligned} \quad \boxtimes$$

Lemma 3.13. For any complex vector bundles V_1 and V_2 , $\Pi_f(V_1 \oplus V_2) = \Pi_f(V_1)\Pi_f(V_2)$.

Proof sketch. Compute using a direct sum connection, which gives rise to a curvature matrix which is a block matrix. \boxtimes

Now observe a very useful property: if the eigenvalues of the matrix $(-1/2\pi i)X$ are $\{x_j\}$, then

$$(3.14) \quad \Pi_f(X) = \prod f(x_j)$$

is a symmetric formal power series in the x_j , and can therefore be expressed in terms of the elementary symmetric functions of the x_j . But these elementary symmetric functions are just the Chern classes. Thus in the literature the genus $\Pi_f(V)$ is often written just as in (3.14), where x_1, \dots, x_m are formal variables subject to the relations

$$\begin{aligned} x_1 + x_2 + \dots + x_m &= c_1, \\ x_1 x_2 + \dots + x_{m-1} x_m &= c_2, \end{aligned}$$

and so on.

⁵There's a sign convention here; this is true using our definition $\det(1 - qK)$. An alternative choice is to use $\det(q - K)$, in which case one must swap the indices to preserve evenness.

Example 3.15. The genus associated to $f(z) = 1 + z$ is the *total Chern class*

$$c(V) := 1 + c_1(V) + c_2(V) + c_3(V) + \cdots.$$

To see this, consider the power expansion of the determinant:

$$\det\left(1 - \frac{1}{2\pi}X\right) = \sum_k \left(-\frac{1}{2\pi}\right)^k \operatorname{tr}(\Lambda^k X) = \sum_k c_k(X).$$

From this we immediately get that $c(V_1 \oplus V_2) = c(V_1)c(V_2)$. ◀

Definition 3.16. Let $V \rightarrow M$ be a real vector bundle and g be a holomorphic function near 0, with $g(0) = 1$. Let $V_{\mathbb{C}}$ be the complexification of V . Denote by f be the branch of $z \mapsto (g(z^2))^{1/2}$ which has $f(0) = 1$; we call the genus associated to f the *Pontrjagin g -genus* of V .

Since f is an even function of z , the associated genus involves only the even Chern classes.

Lemma 3.17. Let g be as above. Then for a real vector bundle V , the Pontrjagin g -genus is equal to

$$\prod_j g(y_j)$$

for some formal variables y_j .

Definition 3.18. Let V be a real vector bundle. Its k^{th} Pontrjagin class $p_k(V)$ is the k^{th} elementary symmetric function in the formal variables y_j .

Proof of Lemma 3.17. Regard this as an identity between invariant polynomials over $\mathfrak{o}(n)$. Any matrix in $\mathfrak{o}(n)$ is similar to one in block diagonal form, where the blocks are 2×2 and are of the form

$$X = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$$

with eigenvalues $\pm i\lambda$. Since both sides of the desired identity are multiplicative for direct sums, it is enough to prove it for this block X , whose first two elementary symmetric functions are

$$\begin{aligned} c_1(X) &= \left(-\frac{1}{2\pi i}\right) \operatorname{tr}(X), \\ c_2(X) &= \left(-\frac{1}{2\pi i}\right)^2 \operatorname{tr}(\Lambda^2 X). \end{aligned}$$

Since X is skew, then its trace vanishes, so $c_1(X) = 0$. By looking at the characteristic polynomial of X we see that $\operatorname{tr}(\Lambda^2 X) = \lambda^2$, giving

$$c_2(X) = -\frac{\lambda^2}{4\pi^2}.$$

Thus

$$y = p_1(X) = \frac{\lambda^2}{4\pi^2}.$$

On the other hand, X is similar over \mathbb{C} to

$$\begin{pmatrix} -i\lambda & 0 \\ 0 & i\lambda \end{pmatrix},$$

so

$$\Pi_f(X) = f\left(-\frac{\lambda}{2\pi}\right)f\left(\frac{\lambda}{2\pi}\right) = g\left(\frac{\lambda^2}{4\pi^2}\right) = g(y)$$

as required. ◻

Two important examples are the \hat{A} -genus, which is the Pontrjagin genus associated to the holomorphic function

$$z \mapsto \frac{\sqrt{z}/2}{\sinh(\sqrt{z}/2)}$$

and the Hirzebruch L -genus, which is the Pontrjagin genus associated with the holomorphic function

$$z \mapsto \frac{\sqrt{z}}{\tanh(\sqrt{z})}.$$

Lecture 4.

Clifford algebras, Clifford bundles, and Dirac operators: 1/31/18

Ricky spoke today about Clifford algebras and Clifford bundles.

Let k be a field with characteristic not equal to 2. If V is a vector space over k , its *tensor algebra* is

$$(4.1) \quad T(V) := \bigoplus_{k \geq 0} V^{\otimes k},$$

where $V^{\otimes 0} := k$.

Definition 4.2. Let V be a k -vector space with a quadratic form $Q: V \times V \rightarrow k$. Let $I_Q \subset T(V)$ denote the two-sided ideal generated by elements of the form $v \otimes v + Q(v)$ for $v \in V$. Then, the quotient algebra

$$\mathcal{Cl}(V, Q) := T(V)/I_Q$$

is called the *Clifford algebra* of V and Q .

Example 4.3. The zero function is a quadratic form, so $\mathcal{Cl}(V, 0)$ is $T(V)/(v \otimes v = 0)$, which is just the exterior algebra $\Lambda(V)$ of V . ◀

There is a natural map $i: V \rightarrow \mathcal{Cl}(V, Q)$ which is the composition

$$V = V^{\otimes 1} \hookrightarrow T(V) \xrightarrow{\pi_Q} \mathcal{Cl}(V, Q).$$

Lemma 4.4. $i: V \rightarrow \mathcal{Cl}(V, Q)$ is injective.

This is not too hard to check.

Moreover, 1 and V generate $\mathcal{Cl}(V, Q)$, subject to the relations $v^2 = q(v, v)$. To get a smaller set of generators, we can choose a basis of V . From now on, we assume Q is positive definite and choose an orthonormal basis e_1, \dots, e_n of V . In this case, $\mathcal{Cl}(V, Q)$ is generated by $1, e_1, \dots, e_n$ subject to the relations

$$(4.5) \quad \begin{aligned} e_i^2 &= 1 \\ e_j \cdot e_j &= -e_j \cdot e_i, \end{aligned}$$

because $e_i^2 = -Q(e_i) = -1$. This implies the following fact.

Proposition 4.6. The set

$$\{e_{i_1} \cdots e_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n, 1 \leq k \leq n\}$$

is a basis for $\mathcal{Cl}(V, Q)$ as a vector space, and hence

$$\dim \mathcal{Cl}(V, Q) = 2^n = \sum_{k=0}^n \binom{n}{k}.$$

Example 4.7. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on \mathbb{R}^n ; then, $\mathcal{Cl}_n := \mathcal{Cl}(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$. In low dimensions these are familiar.

- $\mathcal{Cl}_1 = \langle 1 \rangle \oplus \langle e \rangle$ with $e^2 = 1$, hence $\mathcal{Cl}_1 \cong \mathbb{C}$.
- $\mathcal{Cl}_2 = \langle 1 \rangle \oplus \langle e_1 \rangle \oplus \langle e_2 \rangle \oplus \langle e_1 \cdot e_2 \rangle$ with $e_1^2 = e_2^2 = (e_1 e_2)^2 = -1$, so as \mathbb{R} -algebras, $\mathcal{Cl}_2 \cong \mathbb{H}$, the quaternions.

There is a sense in which real Clifford algebras are 8-fold periodict, which is an instance of *Bott periodicity*. We won't delve into this, but see Atiyah-Bott-Shapiro, "Clifford modules," for more information. ◀

Clifford algebras satisfy a universal property.

Proposition 4.8. Let A be a k -algebra and $\varphi: V \rightarrow A$ be a map of vector spaces such that $\varphi(v)^2 = -Q(v) \cdot 1_A$. Then, there is a unique algebra map $\widehat{\varphi}: \mathcal{Cl}(V, Q) \rightarrow A$ making the following diagram commute:

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & A \\ & \searrow i & \uparrow \widehat{\varphi} \\ & & \mathcal{Cl}(V, Q) \end{array} \quad \begin{array}{c} \\ \\ \exists! \end{array}$$

Proof. By the universal property of $T(V)$, there's a unique map $\psi: T(V) \rightarrow A$ sending

$$(4.9) \quad v_1 \otimes \cdots \otimes v_k \mapsto \varphi(v_1)\varphi(v_2)\cdots\varphi(v_k).$$

The claim follows because $\psi(I_Q) = 0$, hence factors through the quotient, which is $\mathcal{Cl}(V, Q)$. \square

Let \mathbf{QVect}_k denote the category of *quadratic spaces* over k , i.e. vector spaces together with quadratic forms; the morphisms $\varphi: (V_1, Q_1) \rightarrow (V_2, Q_2)$ are data of a linear map $\varphi: V_1 \rightarrow V_2$ such that for all $v, w \in V_1$,

$$(4.10) \quad Q_1(v, w) = Q_2(\varphi(v), \varphi(w)).$$

Using Proposition 4.8, one can show that $\mathcal{Cl}: \mathbf{QVect}_k \rightarrow \mathbf{Alg}_k$ is a functor.

Another use of the universal property is to define a representation

$$\rho_{\mathcal{Cl}}: \mathcal{O}_n \rightarrow \text{Aut}(\mathcal{Cl}_n);$$

a map $\mathbb{R}^n \rightarrow \mathbb{R}^n$ respecting the inner product defines a map $\mathcal{Cl}_n \rightarrow \mathcal{Cl}_n$, and the space of these maps is \mathcal{O}_n .

Definition 4.11. Let (V, Q) be a quadratic space. A *Clifford module* over (V, Q) is a k -vector space S together with a k -linear map $\varphi: \mathcal{Cl}(V, Q) \rightarrow \text{End}_k(S)$.

So it's just a module over the algebra $\mathcal{Cl}(V, Q)$.

Remark. When S is a complex vector space and V is a real vector space, then we will instead ask for the action map to be an \mathbb{R} -algebra homomorphism $\mathcal{Cl}(V, Q) \rightarrow \text{End}_{\mathbb{C}}(S)$. This is equivalent to having a module over $\mathcal{Cl}(V, Q) \otimes_{\mathbb{R}} \mathbb{C}$. \blacktriangleleft

Remark. By the universal property, it suffices to specify a map $\varphi: V \rightarrow \text{End}_k(S)$ with $\varphi(v)^2 = -Q(v) \cdot \text{id}$. \blacktriangleleft

Example 4.12.

- (1) Let's consider \mathcal{Cl}_1 as a module over itself. This is the same data as a map $\varphi: \mathcal{Cl}_1 \rightarrow \text{End}_{\mathbb{C}}(\mathbb{C}) \cong \mathbb{C}$; one choice is $1 \mapsto \text{id}$ and $e \mapsto i$.
- (2) We can also make \mathbb{R}^4 into a \mathcal{Cl}_2 -module by having it act on itself by left multiplication. For example, e_1 acts by the matrix

$$\begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix}.$$

Now, we apply this to geometry. Let (M, g) be a Riemannian manifold, so for each $p \in M$, $\mathcal{Cl}(T_p M, g_p) \cong \mathcal{Cl}_n$.

Definition 4.13. The *Clifford tangent bundle* is $\mathcal{Cl}(TM) := \mathcal{P}_0(M) \times_{\mathcal{O}_n} \mathcal{Cl}_n$, where $\mathcal{P}_0(M)$ is the principal \mathcal{O}_n -bundle of orthonormal frames on M .

More generally, if $S \rightarrow M$ is any complex vector bundle, we can equip S with a Clifford action $c: \mathcal{Cl}(TM) \rightarrow \text{End}_{\mathbb{C}}(S)$ in a similar way.

Definition 4.14. Let $S \rightarrow M$ be a complex vector bundle with a Hermitian metric $\langle \cdot, \cdot \rangle$ and a connection $\nabla: C^\infty(TM) \otimes C^\infty(S) \rightarrow C^\infty(S)$. This data $(S, \langle \cdot, \cdot \rangle, \nabla)$ defines a *Clifford bundle* if

- for all $X \in C^\infty(TM)$ of unit norm and $s_1, s_2 \in C^\infty(S)$, $\langle X \cdot s_1, X \cdot s_2 \rangle = \langle s_1, s_2 \rangle$ iff $\langle X \cdot s_1, s_2 \rangle + \langle s_1, X \cdot s_2 \rangle = 0$ and
- for all $X, Y \in C^\infty(TM)$ and $s \in C^\infty(S)$, $\nabla_X^L(Y \cdot s) = (\nabla_X Y) \cdot s + Y \cdot \nabla_X s$, where ∇^L denotes the Levi-Civita connection.⁶

Now we need to take a brief detour into something called synchronous frames.

Definition 4.15. Let (M, g) be an n -dimensional Riemannian manifold and $\varphi: U \rightarrow \mathbb{R}^n$ be a chart for M containing some $y \in U$. Let e_1, \dots, e_n be an orthonormal basis of $T_y M$...

TODO: I couldn't figure out what happened here. Sorry. I'll have to fix this later.

Remark. The exponential map gives a canonical choice for a local neighborhood on a Riemannian manifold. \blacktriangleleft

⁶**TODO:** I am not completely sure I wrote this down correctly.

Dirac operators.

Definition 4.16. Let $S \rightarrow M$ be a Clifford bundle. The Dirac operator $D: C^\infty(S) \rightarrow C^\infty(S)$ is the composition

$$C^\infty(S) \xrightarrow{\nabla} C^\infty(T^*M \otimes S) \xrightarrow{g} C^\infty(TM \otimes S) \xrightarrow{\text{Cl}} C^\infty(S).$$

In a neighborhood of a point $x \in M$, choose a local orthonormal frame e_1, \dots, e_n . Let $e^i := g(e_i, -)$ be the dual frame. Then, the Dirac operator in coordinates looks like

$$(4.17) \quad s \mapsto \nabla_{(\cdot)} s = \sum e^i \otimes \nabla_{e_i} s \mapsto \sum e_i \otimes \nabla_{e_i} s \mapsto \sum e_i \cdot \nabla_{e_i} s.$$

Example 4.18. Let \mathbb{E}^n denote Euclidean space, i.e. \mathbb{R}^n with the usual flat metric. If V is a complex vector space, it canonically defines a complex vector bundle $\underline{V} \rightarrow \mathbb{R}^n$ by translation. Let e_1, \dots, e_n be the standard orthonormal frame on $T\mathbb{R}^n$.

Let $\gamma := c(e_i)$, where $c: \text{Cl}(TM) \rightarrow \text{End}_{\mathbb{C}}(\underline{V})$ denotes the Clifford bundle action. Then, the Dirac operator $D: C^\infty(\underline{V}) \rightarrow C^\infty(\underline{V})$ is

$$(4.19) \quad D = \sum_i \gamma_i \cdot \partial_i,$$

where ∂_i is the usual partial derivative operator. The γ_i satisfy the anticommutation relations

$$(4.20) \quad \{\gamma_i, \gamma_j\} := \gamma_i \cdot \gamma_j + \gamma_j \cdot \gamma_i = -2\delta_{ij}.$$

Specifically, if $V = \mathbb{C} = \text{Cl}_1$, then $e_1 \mapsto i$, so $\gamma = i$. Therefore $D = i \frac{\partial}{\partial x}$. ◀

The Dirac operator is self-adjoint. Well, it's *formally self-adjoint*: $C^\infty(S)$ is not a Hilbert space, so we can't talk about self-adjointness strictly speaking. One way to abrogate this problem is to take some kind of L^2 completion, but then it's an unbounded operator, so things are still a little complicated. Anyways, we'll talk about this in a bit.

Definition 4.21. Let M be a closed Riemannian manifold and V be its volume form. Then, there is an inner product on $C^\infty(S)$ defined by

$$\langle s_1, s_2 \rangle := \int_M \langle s_1(x), s_2(x) \rangle dV.$$

Theorem 4.22. The Dirac operator on a closed manifold is formally self-adjoint.

Proof. That is, we want to prove that

$$(4.23) \quad \langle Ds_1, s_2 \rangle = \langle s_1, Ds_2 \rangle.$$

We will compute this locally in a synchronous frame e_1, \dots, e_n for a chart in X . Then

$$\begin{aligned} (Ds_1, s_2) - (s_1, Ds_2) &= \sum_i ((e_i \nabla_i s_1, s_2) - (s_1, e_i \nabla_i s_2)) \\ &= \sum_i \left(\nabla_i (e_i \cdot s_1) - \underbrace{(\nabla_i e_i)}_{=0} \cdot s_1, s_2 \right) - (s_1, e_i \nabla_i s_2) \\ &= \sum_i \nabla_i (e_i \cdot s_1, s_2) \\ &= \sum_i \partial_{e_i} (e_i \cdot s_1, s_2) \\ &= d^* \omega, \end{aligned}$$

where

$$\omega_x := - \sum (e_i \cdot s_1, s_2) e^i.$$

Hence the difference is $\langle 1, d^* \omega \rangle = \langle d1, \omega \rangle = 0$. ⊠

There's a local-vs.-neighborhood argument to make here, but this is the idea.

Lecture 5.

The Weitzenbock formula: 1/31/18

The next talk was by Ivan, on more Clifford bundles and Dirac operators.

Definition 5.1. Let $S \rightarrow M$ be a Clifford bundle on a Riemannian manifold (M, g) , $A \in \Omega_M^2(\text{End } S)$, and $\{e_i\}$ be a local synchronous orthonormal frame for M . The *Clifford contraction* of A is $\mathbb{A} \in \Omega_M^0(\text{End } S)$ defined by the local formula

$$\mathbb{A} \cdot s := \sum_{i < j} c(e_i)c(e_j)A(e_i, e_j) \cdot s.$$

One should check this is independent of the choice of frame, but that is true.

Let \mathbb{K} denote the Clifford contraction of the curvature for ∇ on S .

Theorem 5.2 (Weitzenbock formula). *Let D denote the Dirac operator of $S \rightarrow M$. Then*

$$D^2 = \nabla^* \nabla + \mathbb{K}.$$

$\nabla^* \nabla$ is called the *covariant Laplacian*, and D^2 the *Dirac Laplacian*.

Proof. Let $p \in M$. Then

$$\begin{aligned} D^2 s|_p &= \sum_{i,j} e_i \cdot \nabla_{e_i} (e_j \cdot \nabla_{e_j} s) \Big|_p \\ &= \sum_{i,j} e_i \cdot e_j \cdot \nabla_{e_i} \nabla_{e_j} s|_p. \end{aligned}$$

Splitting this into the cases $i = j$ and $i \neq j$, we get

$$\begin{aligned} &= - \sum_i \nabla_{e_i}^2 s|_p + \sum_{i < j} e_i e_j (\nabla_{e_i} \nabla_{e_j} - \nabla_{e_j} \nabla_{e_i})|_p \\ &= \nabla^* \nabla s|_p + \mathbb{K} s|_p. \end{aligned}$$

This uses the fact that we're on a synchronous frame, so $\nabla_{[e_i, e_j]} = 0$, and therefore the curvature simplifies to $\nabla_{e_i} \nabla_{e_j} - \nabla_{e_j} \nabla_{e_i}$ as we used above. \square

This formula will be crucial for us, allowing us to supplant some of the general theory of elliptic operators in the proof of the index theorem.

Corollary 5.3 (Bochner theorem). *Let $S \rightarrow M$ and \mathbb{K} be as above. If $(\mathbb{K}s, s) > 0$ at some point then there are no nontrivial solutions to $D^2 s = 0$.*

Positivity makes sense because \mathbb{K} is a Hermitian operator on a bundle which is fiberwise Hermitian.

Proof. Suppose that $D^2 s = 0$ and $s \neq 0$. Then, $\nabla^* \nabla s + \mathbb{K}s = 0$, so

$$(5.4) \quad 0 = \underbrace{\|\nabla s\|^2}_{\geq 0} + \underbrace{\int_M \langle \mathbb{K}s, s \rangle dV}_{> 0},$$

which is a contradiction. \square

Theorem 5.5. *Let $S \rightarrow M$ be as above, and K denote the curvature of ∇ on S . Then, $K = R^s + F^s$, where*

$$\begin{aligned} R^s(X, Y) &:= \frac{1}{4} \sum_{i,j} c(e_i)c(e_j) \langle R(X, Y)e_i, e_j \rangle \\ [F^s(X, Y), c(Z)] &= 0. \end{aligned}$$

R^s is usually called the *Riemann endomorphism*, and only depends on the Riemannian metric of the base manifold. F^s is called the *twisting curvature*.

We'll prove Theorem 5.5 in a series of lemmas. First, we find an obstruction for k being a Clifford module endomorphism.

Lemma 5.6.

$$[K(X, Y), c(Z)] = c(R(X, Y)Z).$$

Proof. Let $\{e_i\}$ denote a synchronous frame at a $p \in M$. Then it suffices to prove the lemma for $X = e_i$, $Y = e_j$, and $Z = e_k$. Since we're in a synchronous frame, the $\nabla_{[e_i, e_j]}$ component of the curvature vanishes, so

$$\begin{aligned} K(e_i, e_j)e_k \cdot s|_p &= \left(\nabla_{e_i} \nabla_{e_j} - \nabla_{e_j} \nabla_{e_i} \right) (e_k \cdot s) \Big|_p \\ &= (R(e_i, e_j)e_k) \cdot s + e_k \cdot K(e_i, e_j)s \Big|_p. \end{aligned}$$

The result follows because⁷

$$\nabla_{e_i}(e_k)\nabla_{e_j}s|_p = 0. \quad \square$$

Lemma 5.7.

$$[R^s(X, Y), c(Z)] = c(R(X, Y)Z).$$

Proof. Again let $\{e_i\}$ be an orthonormal frame, $X = e_i$, $Y = e_j$, and $Z = e_k$. Then

$$R^s(e_i, e_j)e_k \cdot s = \frac{1}{4} \sum_{\ell, m} c(e_\ell e_m e_k) \underbrace{\langle R(e_i, e_j)e_\ell, e_m \rangle}_{R_{m\ell ij}} s,$$

and similarly

$$c(e_k)R^s(e_i, e_j)s = \frac{1}{4} \sum_{\ell, m} c(e_k e_\ell e_m) R_{m\ell ij} s.$$

Hence when we put these together, we get

$$(5.8) \quad [R^s(e_i, e_j), c(e_k)] = \frac{1}{4} \sum_{\ell, m} c([e_\ell e_m, e_k]) R_{m\ell ij} s.$$

If $\ell = m$ and ℓ, m , and k are distinct, then $[e_\ell e_m, e_k] = 0$, so we only care about the cases $\ell = k \neq m$ and $k = m \neq \ell$.

Both the commutator and $R_{m\ell ij}$ are antisymmetric under the exchange of m and ℓ , so (5.8) reduces to

$$\begin{aligned} [R^s(e_i, e_j), c(e_k)] &= \frac{1}{2} \sum_{\ell} c([e_\ell e_k, e_k]) R_{k\ell ij} s \\ &= \sum_{\ell} c(e_\ell) R_{\ell kij} s \\ &= c \left(\sum_{\ell} R_{\ell kij} s \right). \end{aligned}$$

Since we're working in an orthonormal frame, the metric looks like the identity matrix in coordinates, so

$$\begin{aligned} &= c \left(\sum_{\ell} R_{kij}^{\ell} s \right) \\ &= c(R(e_i, e_j)e_k)s. \end{aligned} \quad \square$$

These two lemmas suffice to prove Theorem 5.5.

Remark. Before we go on, let's review Ricci and scalar curvature, which we'll need. Let (M, g) be a Riemannian manifold. Its *Ricci curvature* is the map $\text{Ric}: TX \times TX \rightarrow \mathbb{R}$ defined by

$$\text{Ric}(X, Y) := \text{tr}(Z \mapsto R(Z, X)Y).$$

Why this trace? You could try others, but they all vanish or give you the Ricci curvature up to a sign!

Raising an index, define $\mathcal{R}\text{ic}: TX \rightarrow TX$ by

$$\text{Ric}(X, Y) = g(X, \mathcal{R}\text{ic}(Y)).$$

Then, the *scalar curvature* of (M, g) is $\kappa := \text{tr}(\mathcal{R}\text{ic})$. In an orthonormal frame, it has the formula

$$\kappa = \sum_j \text{Ric}_{jj}. \quad \triangleleft$$

⁷TODO: maybe I missed something.

Theorem 5.9 (Improvement on Theorem 5.2). *With notation as in Theorem 5.2, let \mathbb{F}^s denote the Clifford contraction of F^s and κ denote the scalar curvature of (M, g) . Then*

$$D^2 = \nabla^* \nabla + \mathbb{F}^s + \frac{\kappa}{4} \mathbf{1}_{\text{End } S}.$$

Proof. By Theorem 5.5, $\mathbb{K} = \mathbf{R}^s + \mathbb{F}^s$, where \mathbf{R}^s is the Clifford-contracted Riemann endomorphism. So all we have to show is that $\mathbf{R}^s = (\kappa/4) \mathbf{1}_{\text{End } S}$. Again we compute in an orthonormal basis:

$$\begin{aligned} \mathbf{R}^s &= \sum_{i < j} c(e_i) c(e_j) R^s(e_i, e_j) s \\ &= \frac{1}{2} \sum_{i, j} c(e_i e_j) R^s(e_i, e_j) s \\ &= \frac{1}{8} \sum_{i, j, k, \ell} c(e_i e_j e_k e_\ell) \langle R(e_i, e_j) e_k, e_\ell \rangle s \\ &= \frac{1}{8} \sum_{i, j, k, \ell} c(e_i c_j e_k e_\ell) R_{\ell k i j} s. \end{aligned}$$

If you decompose this into parts where various subsets of $\{i, j, k, \ell\}$ are equal to each other, the Bianchi identities allow you to simplify this sum:

$$\begin{aligned} &= \frac{1}{4} \sum_{\ell, i, j} c(e_i c_j e_i e_\ell) R_{\ell i i j} s \\ &= \frac{1}{4} \sum_{\ell, j} c(e_j e_\ell) \left(\underbrace{\sum_i R_{\ell i j}^i}_{\text{Ric}_{\ell j}} \right) \cdot s. \end{aligned}$$

If $\ell \neq j$, the Ricci tensor piece is antisymmetric, so does not contribute to the sum. Hence we only get the case where $k = \ell$:

$$= \frac{1}{4} \sum_j \text{Ric}_{jj} \mathbf{1}_{\text{End } S},$$

and this is indeed the scalar curvature. □

Now we'll give an example of a Clifford bundle on a non-flat space.

Example 5.10. Let (M, g) be a closed Riemannian manifold and $S := \Lambda^*(T^*M) \otimes \mathbb{C}$. The Riemannian metric on M induces a Riemannian metric on $\Lambda^* T^*M$, hence a Hermitian metric on its complexification; similarly, the Levi-Civita connection induces a connection on $\Lambda^* T^*M$ and therefore also on its complexification. Since the Levi-Civita connection is compatible with the metric on M , our induced connection is compatible with the Hermitian metric on S .

We define the Clifford action $c: TM \rightarrow \text{End } S$ to satisfy

$$(5.11) \quad c(e)^2 = -g(e, e) \mathbf{1}_{\text{End } S},$$

which characterizes it uniquely. Namely, if ω is a k -form and $e \in \Gamma_M(TM)$,

$$c(e) \cdot \omega = \tilde{e} \wedge \omega - e \lrcorner \omega,$$

where the first term is a $(k+1)$ -form and the second is a $(k-1)$ -form. Then

$$\begin{aligned} c(e)^2 \cdot \omega &= \underbrace{\tilde{e} \wedge \tilde{e} \wedge \omega}_{=0} - \tilde{e} \wedge (e \lrcorner \omega) - e \lrcorner (\tilde{e} \wedge \omega) + \underbrace{e \lrcorner (e \lrcorner \omega)}_{=0} \\ &= \underbrace{-e \wedge (e \lrcorner \omega)}_{=0} - \underbrace{(e \lrcorner \tilde{e})}_{g(e, e)} \wedge \omega + \underbrace{\tilde{e} \wedge (e \lrcorner \omega)}_{=0} \\ &= -g(e, e) \omega. \end{aligned}$$

There are more things to check, including

$$(5.12) \quad g(e \cdot \omega_1, \omega_2) + g(\omega_1, e \cdot \omega_2) = 0,$$

which is left as an exercise, and the fact that

$$(5.13) \quad \nabla_X(e \cdot \omega) = (\nabla_X e) \cdot \omega + e \cdot (\nabla_X \omega).$$

One relatively quick way to prove it is to establish that

$$(5.14) \quad e \lrcorner \omega = (-1)^? \star(\tilde{e} \wedge \star \omega).$$

This implies

$$\begin{aligned} \nabla_X(e \lrcorner \omega) &= (-1)^? \star \nabla_X(\tilde{e} \wedge \star \omega) \\ &= (-1)^? \star (\nabla_X(\tilde{e}) \wedge \star \omega + \tilde{e} \wedge \star \nabla_X \omega) \\ &= (\nabla_X e) \lrcorner \omega + e \lrcorner (\nabla_X \omega). \end{aligned}$$

Hence

$$\begin{aligned} \nabla_X(\tilde{e} \wedge \omega - e \lrcorner \omega) &= (\nabla_X \tilde{e}) \wedge \omega + \tilde{e} \wedge \nabla_X \omega - \nabla_X(e \lrcorner \omega) \\ &= \nabla_X \tilde{e} \wedge \omega + e \wedge \nabla_X \omega - (\nabla_X e) \lrcorner \omega - e \lrcorner \nabla_X \omega \\ &= (\nabla_X e) \cdot \omega + e \cdot (\nabla_X \omega), \end{aligned}$$

proving (5.13). Neither side of (5.13) depends on the metric, and in fact there should be a proof that doesn't use it either.

Now we compute the Dirac operator. Let $\{e_i\}$ be a synchronous frame at p . Then

$$\begin{aligned} D\omega|_p &= \sum_i e_i \cdot \nabla_{e_i} \omega \\ &= \sum_i e^i \wedge \nabla_{e_i} \omega \Big|_p - e_i \lrcorner \nabla_{e_i} \omega \Big|_p \\ &= d\omega|_p - \sum_i (-1)^? \star(e^i \wedge \nabla_{e_i} \star \omega)|_p \\ &= d\omega|_p + d^* \omega|_p. \end{aligned}$$

This implies $D = d + d^*$, which you can again show in a more abstract way. The Laplacian is

$$\Delta := D^2 = dd^* + d^*d,$$

called the *Hodge Laplacian*. This in particular exists on any Riemannian manifold, without any reference to Clifford bundles. \blacktriangleleft

There are a few more theorems proven in similar ways to the other ones above.

Theorem 5.15. *Restricted to $\Lambda^1 T^*M$, $\Delta = \nabla^* \nabla + \text{Ric}$.*

Theorem 5.16 (Brchner vanishing theorem). *If (M, g) is a Riemannian manifold, $\text{Ric} \geq 0$, and $\text{Ric} > 0$ at some point, then $H_{\text{dR}}^1(M) = 0$.*

This one uses some Hodge theory (e.g. identifying harmonic representatives of de Rham cohomology classes).

Lecture 6.

Spin groups: 2/7/18

“The moment you might begin to think it’s wrong is when you see ‘the proof is obvious.’ ”

Today, Sebastian spoke about superalgebra and spin groups.

Superalgebra. Superalgebra, or more generally supermathematics, is the process of adding the prefix “super” to things to denote their $\mathbb{Z}/2$ -graded counterparts with the Koszul sign rule.

Definition 6.1. An algebra A (over \mathbb{R} or \mathbb{C}) is called a *superalgebra* if there is a decomposition as vector spaces $A = A_0 \oplus A_1$ such that $A_i \cdot A_j \subset A_{i+j \bmod 2}$. We call A_0 the *even elements*, A_1 the *odd elements*, and $A_0 \cup A_1$ the *homogeneous elements*.

The degree of a homogeneous element $x \in A_0 \cup A_1$ is denoted $|x| \in \mathbb{Z}/2$.

Remark. This is equivalent data to an involutive automorphism $\varepsilon : A \rightarrow A$ such that $\varepsilon(a_0 + a_1) = a_0 - a_1$; A_0 and A_1 are the ± 1 -eigenspaces of ε . \triangleleft

Definition 6.2. Let A be a superalgebra.

- The *supercommutator* of $x, y \in A$ is

$$[x, y]_s := xy - (-1)^{|x||y|}yx.$$

- The *super center* of A is

$$Z_s(A) := \{x \in A \mid [x, y]_s = 0 \text{ for all } y \in A\}.$$

Exercise 6.3. Let (V, Q) be a quadratic space. Check that $\mathcal{Cl}(V, Q)$ is a superalgebra with

$$\mathcal{Cl}(V, Q)_0 := \text{span}\{e_{i_1} \cdots e_{i_k} \mid k \text{ even}\}$$

$$\mathcal{Cl}(V, Q)_1 := \text{span}\{e_{i_1} \cdots e_{i_k} \mid k \text{ odd}\}$$

Proposition 6.4. If V is a real inner product space, $Z_s(\mathcal{Cl}(V)) \cong \mathbb{R}$ and $Z_s(\mathcal{Cl}(V) \otimes \mathbb{C}) \cong \mathbb{C}$.

Proof. Suppose $x = a + e_1 b$, where $a = a_0 + \cdots + a_{k-1}$ and $b = b_0 + \cdots + b_{k-1}$. Suppose that $x \in Z_s(\mathcal{Cl}(V))$. Then

$$\begin{aligned} 0 &= [x, e_1]_s \\ &= x e_1 - \sum_{i=1}^{k-1} (-1)^i e_i (a_i - e_1 b_i) \\ &= x e_1 - \sum_{i=0}^{k-1} (-1)^i e_1 a_i - \sum_{i=0}^{k-1} (-1)^i b_i \\ &= \underbrace{\sum_{i=0}^{k-1} (-1)^i e_1 a_i - \sum_{i=0}^{k-1} (-1)^i e_1 a_i}_{=0} + \sum_{i=0}^{k-1} (-1)^{i+1} b_i - \sum_{i=0}^{k-1} (-1)^{i+1} b_i \\ &= -2 \sum_{i=0}^{k-1} (-1)^i b_i. \end{aligned}$$

This forces $b = 0$, so x cannot have any e_1 -component. But e_1 was arbitrary. \boxtimes

The proof boils down to checking minus signs, but is not hard *per se*.

Exercise 6.5. Let V be an oriented inner product space with a positively oriented orthonormal basis $\{e_1, \dots, e_k\}$. Let

$$\omega := e_1 \cdots e_k.$$

- (1) Show that ω does not depend on the choice of basis.
- (2) Show that $\omega^2 = (-1)^{k(k+1)/2}$.
- (3) Show that for $v \in V$, $\omega v = (-1)^{k-1} v \omega$.

Hence if k is odd, $\omega \in Z(\mathcal{Cl}(V))$ (the ordinary center, not the super center). If $k = 2m$, then $\omega^{-1} = (-1)^m \omega$. Hence

$$(6.6) \quad \varepsilon(x) = \omega x \omega^{-1} = -x \omega \omega^{-1}.$$

Hence ε is an inner automorphism.

Pin and spin groups. From now on, fix \mathcal{Cl}_k to be the Clifford algebra of \mathbb{R}^k with its usual inner product, and let $\{e_1, \dots, e_k\}$ be the standard basis for \mathbb{R}^k . Any $v \in \mathbb{R}^k$ satisfies $v \cdot v = -\|v\|^2$, so $V \setminus 0 \subset \mathcal{Cl}_k^\times$.

Definition 6.7.

- The *pin*[−] group $\text{Pin}_k^- \subset \mathcal{Cl}_k^\times$ is the Lie subgroup generated by the norm-1 elements of \mathbb{R}^k .
- The *spin* group is $\text{Spin}_k := \text{Pin}_k^- \cap (\mathcal{Cl}_k)_0$.

Example 6.8. Let $k = 1$.

- $\mathcal{Cl}(1) = \mathbb{R} \oplus \mathbb{R} \cdot e_1$ where $e_1^2 = -1$, so $\mathcal{Cl}_1 \cong \mathbb{C}$.
- $\text{Pin}_1^- = \{\pm 1, \pm i\} \subset \mathbb{C}^\times$.
- $\text{Spin}_1 = \{\pm 1\} \subset \mathbb{C}^\times$.

◀

We want to study the representation theory of these groups. One way to produce representations would be to find actions of \mathcal{Cl}_k on itself that preserve \mathbb{R}^k .

Recall that $vw + wv = -2\langle v, w \rangle$, and therefore if $\|v\| = 1$, $v^{-1} = -v$. Hence

$$\begin{aligned} -vxv^{-1} &= vxv \\ &= (-xv - 2\langle x, v \rangle)v \\ &= x - 2\langle x, v \rangle v. \end{aligned}$$

Geometrically, this is the reflection of x through the hyperplane $\langle v \rangle^\perp$, as you may remember from linear algebra. This extends to a group representation

$$(6.9) \quad \begin{aligned} \rho : \text{Pin}_k^- &\longrightarrow \text{GL}_k(\mathbb{R}) \\ \rho(y)(x) &:= yx\varepsilon(y^{-1}). \end{aligned}$$

This is called the *twisted adjoint representation*. We will also let ρ denote the restriction of this representation to Spin_k . If $y = u_1 \cdots u_\ell$ for $u_1, \dots, u_\ell \in \mathbb{R}^k$, then

$$yx\varepsilon(y^{-1}) = \pm(u_1 \cdots u_\ell)x(u_1^{-1} \cdots u_\ell^{-1})$$

is also a composition of hyperplane reflections in a similar way as above. This means in particular that if $y \in \text{Spin}_k$, this is a composition of an even number of hyperplane reflections, so $\rho : \text{Spin}_k \rightarrow \text{SO}_k$, not just O_k .

Proposition 6.10. *There is a short exact sequence*

$$(6.11) \quad 1 \longrightarrow \{\pm 1\} \longrightarrow \text{Spin}_k \xrightarrow{\rho} \text{SO}_k \longrightarrow 1.$$

Proof. First, ρ is surjective, because every element of SO_k can be expressed as an even number of hyperplane reflections.

Next, let $y \in \ker(\rho)$, so that $yx\varepsilon(y^{-1}) = x$ for all x . Hence $y \in Z_s(\mathcal{Cl}(k))$, so by Proposition 6.4, $y \in \text{Spin}_k \cap \mathbb{R}$, which is $\{\pm 1\}$. \square

This implies that Spin_k is a compact simple Lie group, and a double cover of SO_k . We want to know which one it is — in particular, is it a trivial double cover? For $k > 1$, no.

Proposition 6.12. *For $k \geq 2$, Spin_k is connected. For $k \geq 3$, Spin_k is simply connected.*

Proof. The short exact sequence (6.11) is in particular a fiber sequence, hence induces a long exact sequence on homotopy groups.

$$\underbrace{\pi_1(\mathbb{Z}/2)}_1 \longrightarrow \pi_1(\text{Spin}_k) \longrightarrow \pi_1(\text{SO}_k) \longrightarrow \underbrace{\pi_0(\mathbb{Z}/2)}_{\mathbb{Z}/2} \longrightarrow \pi_0(\text{Spin}_k) \longrightarrow \underbrace{\pi_0(\text{SO}_k)}_1$$

The first assertion follows because ± 1 are connected in Spin_k , as $t \mapsto \cos t + e_1 e_2 \sin t$ connects them. Then, the second assertion follows because when $k \geq 3$, $\pi_1(\text{SO}_k) \cong \mathbb{Z}/2$, which can be checked with another long exact sequence of a fibration. \square

Since $\text{Spin}_k \twoheadrightarrow \text{SO}_k$ is a finite cover, it induces an isomorphism of the Lie algebras $\mathfrak{spin}_k \cong \mathfrak{so}_k = \{A \in M_k(\mathbb{R}) \mid A = -A^T\}$.

Lemma 6.13. \mathfrak{spin}_k can be identified with the subspace $\text{span}\{e_i e_j \mid i \neq j\} \subset \mathcal{Cl}_k$ via the map

$$A = (a_{ij}) \mapsto \frac{1}{4} \sum_{i,j} a_{ij} e_i e_j.$$

Representation theory. Let $E_k < \text{Pin}_k^-$ denote the finite subgroup $\{\pm e_1^{i_1} \cdots e_k^{i_k} \mid i_j \in \{0, 1\}\}$. Then $|E_k| = 2^{k+1}$. Let $\nu := -1 \in E_k$.

Proposition 6.14. There's a one-to-one correspondence between the irreducible representations of \mathcal{Cl}_k and the irreducible representations of E_k in which ν acts by -1 .

Remark. Warning: Roe's book has a stronger version of this theorem, which is **not** true! ◀

Since ν is involutive and central, it acts by ± 1 . If it acts by ν on some representation V , then V is a representation for $E_k / \langle \nu \rangle$. This corresponds to ignoring signs, and in particular is an abelian group of order 2^k . Therefore it has 2^k conjugacy classes, hence 2^k irreducible representations.

Therefore the conjugacy classes in E_k must be either of the form $\{g\}$, where g is central, or $\{g, g\nu\}$. Since there are 2^k conjugacy classes, you can figure out how many of each there have to be.

Lemma 6.15.

- If k is even, $Z(E_k) = \{1, \nu\}$.
- If k is odd, $Z(E_k) = \{1, \nu, \omega, \nu\omega\}$.

Recall that $\omega := e_1 \cdots e_k$. This in particular implies

- If k is even, there are $2^k + 1$ irreducible representations of E_k , hence 1 of \mathcal{Cl}_k .
- If k is odd, there are $2^k + 2$ irreducible representations of E_k , hence 2 of \mathcal{Cl}_k .

For now, assume $k = 2m$ is even, so there's a unique irreducible Δ . Since

$$(6.16) \quad |E_k| = 2^{k+1} = 2^k \cdot 1 + (\dim \Delta)^2,$$

then $\dim \Delta = 2^m$. Explicitly, it arises through a \mathcal{Cl}_k -action on an exterior algebra.

So any finite-dimensional \mathcal{Cl}_k -representation W is of the form $W \otimes_{\mathbb{C}} V$, where V is some “coefficient” vector space, or a bunch of copies of the trivial representation. It's possible to recover V from W : using Schur's lemma,⁸

$$(6.17) \quad V \cong \text{Hom}_{\mathcal{Cl}_k \otimes \mathbb{C}}(\Delta, \Delta) \cong (\Delta^* \otimes_{\mathcal{Cl}_k \otimes \mathbb{C}} \Delta) \otimes_{\mathbb{C}} V \cong \Delta^* \otimes_{\mathcal{Cl}_k \otimes \mathbb{C}} W \cong \text{Hom}_{\mathcal{Cl}_k \otimes \mathbb{C}}(\Delta, W).$$

A similar calculation constructs a natural isomorphism $\beta: \text{End}_{\mathcal{Cl}_k \otimes \mathbb{C}}(W) \xrightarrow{\cong} \text{End}_{\mathbb{C}}(V)$.

Definition 6.18. The *relative trace* $\text{tr}^{W/\Delta} F$ of an $F|_{\text{in } \text{End}_{\mathcal{Cl}(k) \otimes \mathbb{C}}(W)}$ is the trace of $\beta(F) \in \text{End}_{\mathbb{C}}(V)$.

Since the elements of Pin_k^- generate \mathcal{Cl}_k , then Δ is also an irreducible representation of Pin_k^- . One can use this to show that Spin_k is an index-2 subgroup of Pin_k^- , hence is automatically normal. Moreover, $\text{Pin}_k^- \twoheadrightarrow \text{O}_k$ is a double cover map.

Since $\text{Spin}_k \triangleleft \text{Pin}_k^-$, then Δ is also a spin representation. Since Spin_k is index 2, then either Δ is irreducible or splits as $\Delta = \Delta^+ \oplus \Delta^-$.

Proposition 6.19. As a Spin_k -representation Δ splits as $\Delta = \Delta^+ \oplus \Delta^-$.

Proof. Recall that $\omega = e_1 \cdots e_{2m}$, so $\omega^2 = (-1)^m$ and $\omega x = \varepsilon(x)\omega$. Clearly $\omega \in \text{Pin}_k^-$, and since k is even, it's also in Spin_k . Suppose $\omega\nu = \lambda\nu$. Then

$$(6.20) \quad \omega^2\nu = (-1)^m\nu = \lambda^2\nu.$$

Consider $i^m\omega \in \mathfrak{spin}_k \otimes \mathbb{C}$. Then $(i^m\omega)^2\nu = \nu = \lambda^2\nu$, so $\lambda \in \{\pm 1\}$. Let Δ_{\pm} denote the ± 1 -eigenspace of $i^m\omega$, so $\Delta = \Delta^+ \oplus \Delta^-$ as vector spaces.

Let's consider how $\mathcal{Cl}_k \otimes \mathbb{C}$ acts on this splitting. Since

$$(i^m\omega)x\nu_{\pm} = \varepsilon(x)i^m\omega\nu_{\pm} = \pm\varepsilon(x)\nu_{\pm},$$

then if x is even, $x\nu_{\pm} \in \Delta_{\pm}$, and if x is odd, $x\nu_{\pm} \in \Delta_{\mp}$. Therefore Spin_k preserves Δ_+ and Δ_- , so they're Spin_k -representations of dimension 2^{m-1} . ◻

Δ^+ (resp. Δ^-) is called the *positive* (resp. *negative*) *half-spin representation* of Spin_n . In particular, $\Delta = \Delta^+ \oplus \Delta_-$ is a $\mathbb{Z}/2$ -graded representation of \mathcal{Cl}_{2m} .

⁸This requires semisimplicity of \mathcal{Cl}_n , which is true but we haven't proven yet.

Lecture 7.

Spin geometry: 2/7/18

These are Arun's notes for his lecture on spin structures on manifolds.

Tangential structures. Let $\rho: H \rightarrow G$ be a homomorphism of Lie groups and $\pi: P \rightarrow M$ be a principal G -bundle. Recall that a *reduction of the structure group* of P to H is data $(\pi': Q \rightarrow M, \theta)$ such that

- $\pi': Q \rightarrow M$ is a principal H -bundle, and
- $\theta: Q \times_H G \rightarrow P$ is an isomorphism of principal G -bundles, where H acts on G through ρ .

An equivalence of reductions $(Q_1, \theta_1) \rightarrow (Q_2, \theta_2)$ is a map $\psi: Q_1 \rightarrow Q_2$ intertwining θ_1 and θ_2 .

Definition 7.1. Let M be a smooth n -manifold and $\rho: H \rightarrow \mathrm{GL}_n(\mathbb{R})$ be a homomorphism of Lie groups. If $\mathcal{B}(M) \rightarrow M$ denotes the principal $\mathrm{GL}_n(\mathbb{R})$ -bundle of frames on M , an H -structure on M is an equivalence class of reductions of the structure group of $\mathcal{B}(M)$ to H .

Example 7.2. Let $\rho: \mathrm{O}_n \hookrightarrow \mathrm{GL}_n(\mathbb{R})$ be inclusion. A reduction of the structure group of $\mathcal{B}(M)$ to O_n is a smoothly varying choice of which bases of $T_x M$ are orthonormal, i.e. a smoothly varying inner product on $T_x M$. Hence it's equivalent data to a Riemannian metric. The space of Riemannian metrics on M is connected, which implies that all reductions are equivalent; a manifold has a single O_n -structure. \blacktriangleleft

Example 7.3. Let $\rho: \mathrm{SO}_n \hookrightarrow \mathrm{GL}_n(\mathbb{R})$ be inclusion. In this case, a reduction of the structure group of $\mathcal{B}(M)$ to SO_n specifies which bases of $T_x M$ are oriented at every point, and therefore defines an orientation on M . Two reductions are equivalent iff they define the same orientation. Therefore an SO_n -structure on M is equivalent data to an orientation. \blacktriangleleft

In particular: an H -structure is data, and it need not always exist.

Definition 7.4. A *spin structure* on a manifold M is an H -structure for $H = \mathrm{Spin}_n$ along the map $\rho: \mathrm{Spin}_n \twoheadrightarrow \mathrm{SO}_n \hookrightarrow \mathrm{GL}_n(\mathbb{R})$. A *spin manifold* is a manifold with a specified spin structure.

Example 7.3 immediately implies that a spin structure determines an orientation.

Example 7.5 (Spin structures on the circle). The map $\mathrm{Spin}_1 \rightarrow \mathrm{GL}_1(\mathbb{R})$ factors through $\mathrm{SO}_1 = \{e\}$, so a trivialization of $\mathcal{B}(S^1)$ defines an orientation of S^1 , and a spin structure is a lift of the trivial $\{e\}$ -bundle to a principal $\mathbb{Z}/2$ -bundle. This is the same as a double cover, and there are two isomorphism classes of these.

The disconnected double cover defines a spin structure which extends over the disc, and is hence called the *bounding spin structure* S_b^1 . In physics, it's also called the *Ramond circle*.

The connected double cover defines a spin structure which does not extend over the disc, and is hence called the *nonbounding spin structure* S_{nb}^1 . In physics, this is sometimes called the *Neveu-Schwarz circle*. \blacktriangleleft

The spinor bundle. Recall from Sebastian's talk the spin representation $\rho: \mathrm{Spin}_n \rightarrow \mathrm{GL}(\Delta)$. Throughout this section, (M, g) is a Riemannian manifold with a spin structure implemented by a principal Spin_n -bundle of frames $\mathcal{B}_{\mathrm{Spin}}(M) \rightarrow M$.

Definition 7.6. The *spin bundle* or *spinor bundle* of M is $\mathcal{B}_{\mathrm{Spin}}(M) \times_{\mathrm{Spin}_n} \Delta \rightarrow M$, which is a complex vector bundle of rank n .

The spinor bundle has a lot of structure.

- The spin representation is $\mathbb{Z}/2$ -graded, and therefore the spinor bundle is as well: $S = S^0 \oplus S^1$.
- The spinor bundle has a canonical connection D on it: the Levi-Civita connection lifts from $\mathcal{B}(M)$ to $\mathcal{B}_{\mathrm{Spin}}(M)$, and therefore passes to the associated bundle.
- The spin representation factors through the unitary group $\mathrm{U}(\Delta) \hookrightarrow \mathrm{GL}(\Delta)$, and therefore there is an induced Hermitian metric on S .

Proposition 7.7. The spin connection D is compatible with h , and therefore the spinor bundle is a Clifford bundle.

Proof. If $(E, \nabla, g) \rightarrow M$ is a real vector bundle with connection and metric, compatibility with the metric means that the connection one-forms are valued in $\mathfrak{o}(E, g)$; if E is complex with a Hermitian metric h , compatibility with the metric means the connection one-forms are valued in $\mathfrak{u}(E, g)$. So this is a fun exercise in what "induced connection" actually means.

Since the Levi-Civita connection is metric-compatible, its connection one-form lives in $\Omega^1_{\mathcal{B}(M)}(\mathfrak{o}_n)$. The pullback connection on $\mathcal{B}_{\text{Spin}}(M)$ has connection one-form in $\Omega^1_{\mathcal{B}_{\text{Spin}}(M)}(\mathfrak{spin}_n)$. Since the spinor representation factors through $U(\Delta) \hookrightarrow GL(\Delta)$, the connection on the associated bundle factors through $\mathfrak{u}(\Delta) \hookrightarrow \mathfrak{gl}(\Delta)$; in particular, its connection one-form lives in $\Omega^1_{\mathfrak{S}}(\mathfrak{u}(\Delta))$, so D is compatible with the metric. \square

Proposition 7.8. *The twisting curvature of the spinor bundle is zero.*

Proof. Recall from Sebastian's talk that there's a natural identification of \mathfrak{spin}_n with the vector subspace of \mathcal{Cl}_n spanned by products $e_i e_j$ for $i \neq j$ by the assignment

$$(7.9) \quad a_{ij} \mapsto \frac{1}{4} \sum a_{ij} e_i e_j.$$

Let $\{e_k\}$ be a local orthonormal frame for TM . Therefore the curvature of the Levi-Civita connection is an \mathfrak{so}_n -valued 2-form whose (k, ℓ) entry is $\langle Re_k, e_\ell \rangle$, where R is the Riemann curvature tensor. By (7.9), the curvature 2-form of the spin connection is

$$(7.10) \quad K_{\text{Spin}} = \frac{1}{4} \sum_{k, \ell} \langle Re_k, e_\ell \rangle e_k e_\ell.$$

This acts on the spinor representation through $e_i \mapsto c(e_i)$, i.e. through

$$(7.11) \quad \frac{1}{4} \sum_{k, \ell} \langle Re_k, e_\ell \rangle c(e_k) c(e_\ell).$$

With notation as last time, this is exactly $R^s(e_j, e_i)$, so since the twisting curvature F^s satisfies $F^s = K_{\text{Spin}} - R^s$, $F^s = 0$. \square

Spin^c structures.

Definition 7.12. The Lie group Spin_n^c is defined to be $\text{Spin}_n \times_{\mathbb{Z}/2} U_1$, where $\mathbb{Z}/2 \hookrightarrow \text{Spin}_n$ as $\{\pm 1\} \subset \mathcal{Cl}_n$ and $\mathbb{Z}/2 \hookrightarrow U_1$ as $\{\pm 1\} \subset \mathbb{C}$.

This means in particular there is a short exact sequence

$$(7.13) \quad 1 \longrightarrow \mathbb{Z}/2 \longrightarrow \text{Spin}_n^c \longrightarrow \text{SO}_n \times U_1 \longrightarrow 1.$$

A *spin^c structure* on a manifold M is an H -structure for $H = \text{Spin}_n^c$ along the map to SO_n . This comes with some extra structure.

- The map $\text{Spin}_n^c \rightarrow \text{SO}_n \times U_1$ means that a Spin_n^c -structure on M determines an orientation and a principal U_1 -bundle $P \rightarrow M$. The associated complex line bundle $L := P \times_{U_1} \mathbb{C} \rightarrow M$ is called the *fundamental line bundle* associated to the spin^c structure; its Chern class is called the *Chern class* of the spin^c structure.
- Since the spin representation is a representation of the complexified Clifford algebra, it also induces a representation of Spin_n^c . In the same manner as above, a spin^c structure on M defines an associated real vector bundle $E^c \rightarrow M$, again called the *spinor bundle* S^c . Since the spin representation is unitary, the spinor bundle acquires a Hermitian metric.

However, we don't get a connection for free: instead we have to choose a connection ∇_L on L ; together with the Levi-Civita connection on $\mathcal{B}_{\text{SO}}(M)$, this induces a connection ∇^c on $\mathcal{B}_{\text{Spin}_n^c}(M) \rightarrow M$.

Proposition 7.14. *In this situation, the twisting curvature of ∇^c is $(1/2)K_L$, where K_L is the curvature of ∇_L .*

Characteristic classes. We can use characteristic classes to determine whether a manifold has a spin or spin^c structure.

First we mention a general form of the *splitting principle*, a way of making computations about characteristic classes for principal G -bundles.

Let G be a compact Lie group, T be a maximal torus for G , and $P \rightarrow X$ be a principal G -bundle over a paracompact base X . The quotient map induces a map $f : P/T \rightarrow X$, and we can pull P back along f :

$$(7.15) \quad \begin{array}{ccc} f^*P & \longrightarrow & P \\ \downarrow T & & \downarrow G \\ P/T & \xrightarrow{f} & X. \end{array}$$

In general, T splits as a product of copies of S^1 , so f^*P similarly splits. The key fact is that the map $f^*: H^*(X) \rightarrow H^*(P/T)$ is injective. Therefore any question about characteristic classes of P may be solved for f^*P , which splits, with no information lost.

Definition 7.16. If E is a real vector bundle, $\mathfrak{S}(E)$ will denote the Pontrjagin genus of E associated to

$$g(z) = \cosh\left(\frac{1}{2}\sqrt{z}\right).$$

Proposition 7.17. The Chern character of the spinor bundle on an even-dimensional spin manifold M is

$$\text{ch}(S) = 2^m \mathfrak{S}(TM).$$

Proof. We will prove a stronger result. A *spin structure* on a real vector bundle $E \rightarrow M$ is an (equivalence class of) reduction of the structure group of the principal $\text{GL}_n(\mathbb{R})$ -bundle of local frames of E across $\text{Spin}_n \rightarrow \text{GL}_n(\mathbb{R})$. Given a spin structure on E , one can define its spinor bundle S_E in the same manner as before. We will show that if E has even rank, then

$$(7.18) \quad \text{ch}(S_E) = 2^{\dim E/2} \mathfrak{S}(E).$$

If $E_1, E_2 \rightarrow X$ are even-dimensional spin vector bundles, $S_{E_1 \oplus E_2} \cong S_{E_1} \otimes S_{E_2}$, both sides of (7.18) are multiplicative under direct sum. Therefore we may apply the splitting principle for principal Spin_n -bundles to E . One choice of maximal torus in Spin_n is

- $\text{Spin}_2 \times \cdots \times \text{Spin}_2 \hookrightarrow \text{Spin}_n$ if n is even, or
- $\text{Spin}_2 \times \cdots \times \text{Spin}_2 \times \{1\} \hookrightarrow \text{Spin}_n$ if n is odd.

Thus over E/T , $\mathcal{B}_{\text{Spin}}(E)$ splits as a direct sum of principal Spin_2 -bundles, so E splits as a sum of rank-2 vector bundles with spin structure. Since (7.18) is additive, it suffices to assume $\text{rank } E = 2$.

Recall that a *classifying space* BG for a Lie group G is a space with a principal G -bundle $EG \rightarrow BG$ whose total space is contractible. This determines BG up to homotopy, and also implies that every principal G -bundle $P \rightarrow X$ is the pullback of EG along a map $X \rightarrow BG$. Since characteristic classes are natural under pullback, verifying (7.18) over $B\text{Spin}_2$ will prove it everywhere.

Given a representation V of G , we get a vector bundle $EG \times_G V \rightarrow BG$. The universal choice of $E \rightarrow \text{Spin}_2$ is the one induced from the representation $\text{Spin}_2 \rightarrow \text{SO}_2 \hookrightarrow \text{GL}_2(\mathbb{R})$, and its spinor bundle is the complex vector bundle $\Delta \rightarrow B\text{Spin}_2$ associated to the spinor representation.

Since the map $\text{Spin}_2 \rightarrow \text{SO}_2$ is isomorphic to the map $\mathbb{T} \rightarrow \mathbb{T}$ which is multiplication by 2, the map $H^*(B\text{SO}_2; \mathbb{Q}) \rightarrow H^*(B\text{Spin}_2; \mathbb{Q})$ is also multiplication by 2. Therefore if $x \in H^1(B\text{SO}_2; \mathbb{Q})$ denotes the generator, which is the Euler class of the defining representation, $H^*(B\text{SO}_2; \mathbb{Q}) \cong \mathbb{Q}[x]$ and $H^*(B\text{Spin}_2; \mathbb{Q}) \cong \mathbb{Q}[x/2]$. Since $E \rightarrow \text{Spin}_2$ comes from the defining representation of SO_2 , $c_1(E_{\mathbb{C}}) = x$.

Next we compute $\text{ch}(\Delta)$. This spinor representation is $\text{Spin}_2 \hookrightarrow \mathcal{C}\ell_2 \otimes \mathbb{C} \cong M_2(\mathbb{C}) \cong \text{End}(\mathbb{C}^{1|1})$, which acts on $\mathbb{C}^{1|1}$ and therefore induces a representation of the spin group. Since $\text{Spin}_2 \cong \text{U}_1$ abstractly, and irreducible representations of U_1 are $z \mapsto z^n$ for an $n \in \mathbb{Z}$, we will decompose Δ into characters and compare them to E .

The identification $\mathcal{C}\ell_2 \otimes \mathbb{C} \cong M_2(\mathbb{C})$ sends

$$(7.19a) \quad e_1 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$(7.19b) \quad e_2 \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

and therefore

$$(7.19c) \quad e_1 e_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This is the direct sum of Δ^+ (first entry) and Δ^- (second entry). In particular, the Δ^+ is the character $z \mapsto z$, and Δ^- is the character $z \mapsto z^{-1}$. Via the identification $H^*(B\text{Spin}_2; \mathbb{Q}) \cong \mathbb{Q}[x/2]$, $c_1(\Delta^{\pm}) = \pm x/2$, so $\text{ch}(\Delta^{\pm}) = e^{\pm x/2}$. Therefore

$$(7.20) \quad \text{ch}(\Delta) = \text{ch}(\Delta^+) - \text{ch}(\Delta^-) = e^{x/2} - e^{-x/2} = 2 \cosh\left(\frac{x}{2}\right).$$

We also have $p_1(E) = e(E)^2 = x^2$, so

$$(7.21) \quad \mathfrak{S}(E) = \cosh\left(\frac{1}{2}\sqrt{p_1(E)}\right) = \cosh\left(\frac{x}{2}\right). \quad \square$$

Definition 7.22. Let BSO_n denote the classifying space of SO_n ; there's a natural bijection between isomorphism classes of principal SO_n -bundles $P \rightarrow X$ and homotopy classes of maps $X \rightarrow BSO_n$. One can show that when $n \geq 2$, $H^2(BSO_n; \mathbb{Z}/2) \cong \mathbb{Z}/2$, with generator denoted w_2 . Given a principal SO_n -bundle $P \rightarrow X$, the pullback of w_2 by the classifying map $X \rightarrow BSO_n$ is called the *second Stiefel-Whitney class* of P , denoted $w_2(P) \in H^2(X; \mathbb{Z}/2)$.

Proposition 7.23. Let M be a closed, oriented manifold. Then, M has a spin structure iff $w_2(M) = 0$. In this case, the spin structures of M extending its given orientation are a torsor over $H^1(M; \mathbb{Z}/2)$.

Proof. We follow the proof in Roe's exercises. We'll assume $n \geq 2$; otherwise small modifications must be made, but the result is still true. Let M be a closed, oriented, manifold and $\pi: \mathcal{B}_{SO}(M) \rightarrow M$ be its principal SO_n -bundle of frames. This is a fiber bundle with fiber SO_n , hence has an associated spectral sequence

$$(7.24) \quad E_2^{p,q} = H^p(M; H^q(SO_n; \mathbb{Z}/2)) \implies H^{p+q}(\mathcal{B}_{SO}(M); \mathbb{Z}/2).$$

Associated a first-quadrant spectral sequence converging to $H^n(A)$ is its *five-term exact sequence*

$$(7.25) \quad 0 \longrightarrow E_2^{1,0} \longrightarrow H^1(A) \longrightarrow E_2^{0,1} \longrightarrow E_2^{2,0} \longrightarrow H^2(A),$$

which in this case specializes to

$$(7.26) \quad 0 \longrightarrow H^1(M; \mathbb{Z}/2) \xrightarrow{\pi^*} H^1(\mathcal{B}_{SO}(M); \mathbb{Z}/2) \xrightarrow{i^*} H^1(SO_n; \mathbb{Z}/2) \xrightarrow{\delta} H^2(M; \mathbb{Z}/2) \longrightarrow H^2(\mathcal{B}_{SO}(M); \mathbb{Z}/2).$$

We claim, and will prove in a bit, that δ sends the generator of $H^1(SO_n; \mathbb{Z}/2)$ to $w_2(M)$.

A spin structure lifting the specified orientation of M is the data of an isomorphism class of double covers $\mathcal{B}_{Spin}(M) \rightarrow \mathcal{B}_{SO}(M)$ which induce the connected double cover $Spin_n \rightarrow SO_n$ when restricted to each fiber. Isomorphism classes of covers of $\mathcal{B}_{SO}(M)$ are naturally identified with $H^1(\mathcal{B}_{SO}(M); \mathbb{Z}/2)$, and their restriction to a fiber is the map i^* in (7.26), so the set of spin structures on M is in bijection with the subset of $H^1(\mathcal{B}_{SO}(M); \mathbb{Z}/2)$ which is *not* in $\ker(i^*)$.

Suppose that $w_2(M)$ vanishes, meaning $\delta = 0$, so i^* is a surjective map onto $\mathbb{Z}/2$. Therefore $|(i^*)^{-1}(0)| = |(i^*)^{-1}(1)|$ and π^* is an injection whose image is $(i^*)^{-1}(0)$. Thus we have a noncanonical bijection $H^1(M; \mathbb{Z}/2) \rightarrow (i^*)^{-1}(1)$, the set of spin structures on M . The torsor structure arises because $(i^*)^{-1}(0) + (i^*)^{-1}(1) \subset (i^*)^{-1}(1)$.

If instead $w_2(M) \neq 0$, then δ is injective. Then $i^* = 0$, so there can be no spin structures on M .

To finish this, we had better check that δ sends the generator of $H^1(SO_n; \mathbb{Z}/2)$ to w_2 . It suffices to check this for the universal bundle $ESO_n \rightarrow BSO_n$, as all other principal SO_n -bundles are pullbacks of this one, but since ESO_n is contractible, (7.26) simplifies to

$$(7.27) \quad 0 \longrightarrow H^1(BSO_n; \mathbb{Z}/2) \longrightarrow 0 \longrightarrow H^1(SO_n; \mathbb{Z}/2) \xrightarrow{\delta} H^2(BSO_n; \mathbb{Z}/2) \longrightarrow 0.$$

so δ has to send the nonzero element to the generator of $H^2(BSO_n; \mathbb{Z}/2)$, which is w_2 , as desired. \square

Proposition 7.28. Let M be a closed manifold. Then, M has a spin^c structure iff $w_2(M)$ is the reduction of an integral class. In this case, the spin^c structures of M are a torsor over $H^2(M; \mathbb{Z})$.

Corollary 7.29. A spin structure determines a spin^c structure; the converse is not true: \mathbb{CP}^2 is a spin^c manifold which is not spin.

Partial proof. We'll prove the last part, by showing $w_2(\mathbb{CP}^2) \neq 0$. It's equal to the mod 2 reduction of $c_1(\mathbb{CP}^2)$, and $c(\mathbb{CP}^2) = (1+x)^3$, where x is the generator dual to a \mathbb{CP}^1 , so $c_1(\mathbb{CP}^2)$ is odd and therefore $w_2(\mathbb{CP}^2) \neq 0$, so \mathbb{CP}^2 is not spin, but it is spin^c . \square