PI WORKSHOP: QFT FOR MATHEMATICIANS

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These notes were taken at the workshop on quantum field theory for mathematicians at Perimeter Institute in summer 2019. I live-TEXed these notes using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to `a.debray@math.utexas.edu`; any mistakes in the notes are my own.

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1. Theo Johnson-Freyd: Zero-dimensional QFT and Feynman Diagrams: 6/17/19

In today's TA session, Theo Johnson-Freyd spoke about Feynman diagrams and their history, and their relationship with zero-dimensional quantum field theory.

Feynman diagrams were introduced by Dirac, in a paper called "The Lagrangian in QM" in 1932, then developed further in Feynman's PhD thesis in 1938. Dirac knew there was an analogy between T^*X and $L^2(X)$, thought of as the state space for classical mechanics on X and quantum mechanics on X. Time evolution on T^*X is a symplectomorphism, meaning its graph is Lagrangian in $T^*(X \times X)$, explicitly the graph of $\mathrm{d}S$, where $S_t \in C\infty(X \times X)$. Meanwhile, in quantum mechanics on $L^2(X)$, time evolution is a unitary operator $U_t \colon L^2(X) \to L^2(X)$, and acts on operators by

$$(U_t\psi)(x_1) = \int \mathrm{d}x_0\, U_t(x_0, x_1)\psi(x_0). \tag{1.1}$$

This U_t solves the Schrödinger equation.

Exercise 1.2. $S_t(x_0 - x_1)$ solves an analogous differential equation, some version of the Hamilton-Jacobi equation. What is it?

What Dirac knew was how this analogy runs: in the classical limit, as $\hbar \to 0$, $\exp(iS_t/\hbar) \approx U_t(x_0, x_1)$. This was the idea behind the path integral: we can really make sense of minimizing the action functional over the space of fields and use this to learn the physics; this approximation suggests something similar should be possible in quantum mechanics, and this is the famous path integral.

Anyways, right around then was world war 2, and many physicists worked on more applied things for a few years. But by 1948, Schwinger and Tomonaga had given a complete theory of quantum electrodynamics in terms of infinite-dimensional quantum mechanics. It was beautiful and people liked it, and there were a few infinities left, but it wasn't a big deal. Feynman then gave a lecture, not a great one, introducing Feynman diagrams, and then explained it to Dyson on a road trip from New York to Arizona. So there was plenty of time to convey the ideas. Thus in 1949 Dyson explained Feynman diagrams, proved that they gave the same theory as Schwinger and Tomonaga's description, and implemented renormalization. This is the context which this talk will live in.

The goal is: given an action S, our goal is to compute the partition function

$$Z = \int_{\mathbb{R}^n} \mathrm{d}\phi \, \exp\left(\frac{i}{\hbar} S(\phi)\right), \tag{1.3}$$

or more generally, if f is some function on spacetime, we want to compute a correlation function

$$\langle f \rangle = \frac{1}{Z} \int_{\mathbb{R}^n} \mathrm{d}\phi, f(\phi) \exp\left(\frac{i}{\hbar} S(\phi)\right). \tag{1.4}$$

There aren't many ways to solve these, so we make the *stationary phase assumption*: that in the $\hbar \to 0$ limit, the integral is supported in a formal neighborhood of the origin. This means we can pass to Taylor series: whatever a formal neighborhood of the origin is, functions on it are $\mathbb{C}[[x_1, \ldots, x_n]]$. Let's also assume the minimum of S occurs at $x = 0$. Then the action has the form

$$S(x) = \frac{1}{2} a_{ij} x_i x_j + b(x), \tag{1.5}$$

where $b(x)$ contains cubic and higher terms (the *interacting part*), and $a_{ij} x_i x_j$ is the *free part*. So that the minimum is as nice as possible, we want (a_{ij}) to be positive definite, or at least invertible.

Now, how can we compute integrals? You might remember u-substitution, which is essentially cleverly choosing coordinates, or you can integrate by parts. Choose n generating functions $g_1, \ldots, g_n$; then

$$0 = \frac{1}{Z} \int \frac{\partial}{\partial x_i} \left(g_i(x) \exp\left(\frac{i}{\hbar} S(x)\right)\right) \tag{1.6a}$$

$$= \left\langle \frac{\partial g_i}{\partial x_i} \right\rangle + \underbrace{\left\langle g_i(x) \frac{i}{\hbar} a_{ij} x_j \right\rangle}_{(*)} + \left\langle g_j(x) \frac{i}{\hbar} \frac{\partial}{\partial x_i} b(x) \right\rangle. \tag{1.6b}$$

If f is degree zero, then $\langle f \rangle = f$. If f is homogeneous of degree $N > 0$, then the starred term in (1.6b) is dominant, so we will choose $g_i(x)$ such that (1.6b) equals $f(x)$. One possible choice is a term in the Taylor series:

$$g_i(x) a_{ij} x_j = \frac{\hbar}{i} \frac{1}{N!} f^{(N)}_{i_1, \ldots, i_N} x_{i_1} \cdots x_{i_N}, \tag{1.7a}$$

which simplifies to

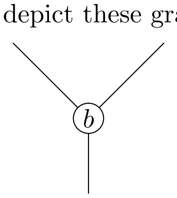
$$g_i(x) = (a^{-1})_{ij} \frac{1}{N} \frac{\partial f}{\partial x_j}. \tag{1.7b}$$

Then, explicitly,

$$\langle f \rangle = \langle (a^{-1})_{ij} i \frac{\hbar}{N} \frac{\partial^2}{\partial x_i \partial x_j} f(x) \rangle - \langle (a^{-1})_{ij} \frac{1}{N} \frac{\partial f}{\partial x_i} \frac{\partial b}{\partial x_j} \rangle. \tag{1.8}$$

Einstein once said that his greatest contribution to physics was getting rid of the summation symbol; in a similar spirit, Penrose's thesis was the first to depict these graphically. The notation

(1.9)



should be interpreted as $(1/3!)\partial_{i_3}\partial_{i_2}\partial_{i_1} b$, so in general N spokes means the N^{th} Taylor coefficient of b. We will interpret an edge between two nodes, labeled by i and j, as $(a^{-1})_{ij}$.

TODO: here's the picture of (1.8) expressed in this graphical notation.

Now I will remind you of the second trial of Hercules. He was asked to kill the Hydra, a many-headed beast such that, if you chopped off one head, two more would grow in its beast. Hercules solved this by bringing wax, and every time he chopepd off a head, he sealed it with wax so that it could not grow there. Hercules can also chop off two heads and glue them together. Plutarch, of course, didn't tell quite the same story, but an approximation to it called the Classics-al limit.

But at least in Feynman diagrams, every head that sprouts gives yet another factor of x, and we're working near 0, so x is small: successive heads are smaller and smaller, so the series converges.

Example 1.10. Suppose f is linear and b is cubic. Then the Feynman diagram expansion looks like (TODO add figure), so we obtain explicitly a sum over closed diagrams, weighted by automorphisms of the diagram, of $(i\hbar)^{\#\,\text{loops}}$ times the partial derivatives and factors of $(a^{-1})_{ij}$. ◀

From the perspective of homological algebra, it's possible to rephrase this problem, and this looks a bit like a chain complex that's close to exact, but not exact.

In quantum field theory, $\mathbb{R}^n$ is replaced with the space of fields, e.g. in a scalar field theory on M, we would take some space of functions on M. The theory of integration tells us that points in M are more or less a basis for this function space. So the points in the Feynman diagram range over M. Then a becomes the Laplacian (or $\Delta + m$, or...), and a^{-1} becomes the Green's function for the differential operator a. So a Feynman diagram has nodes located at points in M, and the edges, called *propagators*, govern how information flows between them, and how long. Then one wishes to integrate over all such choices, including how long it takes for information to propagate.

What's a little weird about this (and why Feynman's lecture wasn't so well received) is that we began with a field theory, but we ended up with pictures of interacting particles somehow!

2. Chris Elliott: Supersymmetry algebras: 6/18/19

> *"You can put 'super' in front of every noun in your sentence, and it should still be a true sentence."*

In today's TA session, Chris Elliott spoke about the Lie-algebraic structures behind supersymmetry.

First, some motivation. Let g be a pseudo-Riemannian metric on $\mathbb{R}^n$ of signature (p, q), and consider QFT on $(\mathbb{R}^n, g)$. We want to impose invariance under the symmetries of this structure.

Definition 2.1. The *Poincaré group* is $\mathrm{Iso}_{p,q} \coloneqq \mathrm{SO}_{p,q} \ltimes \mathbb{R}^n$.

The Lie algebra of the Poincaré group is $\mathfrak{iso}_{p,q} = \mathfrak{so}_{p,q} \ltimes \mathbb{R}^n$. The complexified Lie algebra, sometimes denoted $\mathfrak{iso}_n(\mathbb{C})$, is $\mathfrak{so}_n(\mathbb{C}) \ltimes \mathbb{C}^n$.

The beginning of the study of sypersymmetry algebra is a no-go theorem.

Theorem 2.2 (Coleman-Mendula)**.** *If G is a group of symmetries containing $\mathrm{Iso}_{1,n-1}$ acts on a sufficiently nice QFT, then $G = \mathrm{Iso}_{n-1,1} \times G'$ for some group G' of internal symmetries.*

So there aren't any interesting options out there. But we can exhibit interesting extensions if we consider $\mathbb{Z}/2$-graded extensions of the Poincaré Lie algebra.

Definition 2.3. An *n-dimensional super-Poincaré algebra* is a super Lie algebra (i.e. a $\mathbb{Z}/2$-graded Lie algebra with the Koszul sign rule) $\mathfrak{a}$ of the form

$$\mathfrak{a} = \mathfrak{iso}_n(\mathbb{C}) \ltimes \Pi\Sigma, \tag{2.4}$$

where $\mathfrak{so}_n(\mathbb{C})$ acts on Σ as a spinor representation and Π is parity change (so Σ and $\Pi\Sigma$ are in opposite degrees), with a bracket $\Gamma\colon \Sigma \otimes \Sigma \to \mathbb{C}^n$.

Remark 2.5. Analogously to Theorem 2.2, there is a classification result for super-Poincaré algebras due to Haag-Łopuszański-Sohnius. ◀

To classify super-Poincaré algebras, we need to classify the spinor representations and the pairings. The former is classical: either

- n is odd, and there's a unique irreducible spinor representation, or
- n is even, and there are two nonisomorphic spinor representations.

Remark 2.6. The spinor representations are precisely those representations of $\mathfrak{so}_n$ which do not arise from representations of SO_n; one can realize them as representations of Spin_n as sitting inside the Clifford algebra. Alternatively, you can obtain these representations from the Dynkin diagram: $\mathfrak{so}_n$ for n odd is type B_n, and this is the representation corresponding to the rightmost node of the Dynkin diagram; $\mathfrak{so}_n$ for n even is type D_n, and there are two rightmost nodes. ◀

Great: this tells us the possibilities for Σ.

- Odd n: $\Sigma = S \otimes W$, where W is an auxiliary finite-dimensional vector space. In this case S is the *Dirac spinor representation.*

- Even n: $\Sigma = S_+ \otimes W_+ \oplus S_- \otimes W_-$, where again $W_\pm$ are auxiliary. Here $S = S_+ \oplus S_-$, and $S_\pm$ are the *Weyl spinor representations*.

Now, how do you classify the pairings? Well, they're equivalent to symmetric maps $\Sigma^{\otimes 2} \to V = \mathbb{C}^n$, so we're looking for irreducible summands of $\mathrm{Sym}^2 \Sigma$ isomorphic to V. This can be done pretty explicitly granted a few facts about Clifford algebras.

For n odd, $S \otimes S \cong C\ell^0(V)$, the even part of the Clifford algebra; this is

$$C\ell^0(V) = \bigoplus_{k \text{ even}} \Lambda^k V = \bigoplus_{k=0}^{(n-1)/2} \Lambda^k V, \tag{2.7}$$

using $\Lambda^k V = \Lambda^{n-k} V$. In particular, there is one summand isomorphic to $\Lambda^1 V = V$.

For n even, we get the whole Clifford algebra:

$$(S_+ \oplus S_-)^{\otimes 2} \cong C\ell(V) \cong \bigoplus_{k=0}^{m} \Lambda^k V \cong 2\left(\bigoplus_{k=0}^{n/2-1} \Lambda^k V\right) \oplus \Lambda^{n/2} V, \tag{2.8}$$

so we get two copies of V. Now when we look closer, there will be interesting Bott periodicity phenomena afoot, depending on whether the spinor representations are real, complex, or quaternionic.

Lemma 2.9.

(1) *If n is odd, there's a unique irreducible summand of $S^{\otimes 2}$ isomorphic to V.*
   - *For $n \equiv 1, 3 \bmod 8$, it's contained in $\mathrm{Sym}^2 S$.*
   - *For $n \equiv 5, 7 \bmod 8$, it's contained in $\Lambda^2 S$.*

(2) *If $n \equiv 0, 4 \bmod 8$, there is a unique irreducible summand of $S_+ \otimes S_-$ isomorphic to V, and no such summands in $S_\pm^{\otimes 2}$.*

(3) *If $n \equiv 2, 6 \bmod 8$ (and $n > 2$), then there is a unique irreducible summand of $S_\pm^{\otimes 2}$ isomorphic to v, and no such summand in $S_+ \otimes S_-$.*
   - *For $n \equiv 2 \bmod 8$ ($n > 2$), the summand is inside $\mathrm{Sym}^2(S_\pm)$.*
   - *For $n \equiv 6 \bmod 8$, the summand is inside $\Lambda^2 S_\pm$.*

So this tells us that a choice of a super Poincaré algebra is a choice of

- an orthogonal vector space W (i.e. a space with a symmetric pairing) if $n \equiv 1, 3 \bmod 8$;
- a pair of orthogonal vector spaces W_+, W_-, if $n \equiv 2 \bmod 8$ (and $n > 2$);
- a single vector space W_+ with dual W_- if $n \equiv 0, 4 \bmod 8$;
- a single symplectic vector space W if $n \equiv 5, 7 \bmod 8$; or
- a pair of symplectic vector spaces W_+, W_- if $n \equiv 6 \bmod 8$.

And therefore we know all of the supersymmetry algebras.

Remark 2.10. One usually indicates a choice of super Poincaré algebra by writing $\mathcal{N} = \dim W$, or $\mathcal{N}_\pm = \dim W_\pm$. For example, the 3D $\mathcal{N} = 4$ supersymmetry algebra is

$$\mathfrak{so}_3(\mathbb{C}) \ltimes (\mathbb{C}^3 \oplus \Pi(S \otimes W)), \tag{2.11}$$

where $\dim W = 4$, and Γ appears in the Lie bracket for that direct sum.

However! When $n \equiv 5, 6, 7 \bmod 8$, i.e. in the symplectic cases, one generally writes $\mathcal{N} = (\dim W)/2$ and $\mathcal{N}_\pm = (\dim W_\pm)/2$. So $\mathcal{N} = 1$ is always the smallest amount of supersymmetry. For example, 5D $\mathcal{N} = 1$ means W is a two-dimensional symplectic vector space. ◀

This is the complex story, which is useful for applications in mathematics; the story over $\mathbb{R}$ involves real structures on these algebras, which is more complicated.

Definition 2.12. The *R-symmetry group* G_R is the group of outer automorphisms of the super Poincaré algebra $\mathfrak{a}$ which fix the even part.

In particular, these are automorphisms of W and its specified structure. Again we proceed by cases.

- For $n \equiv 1, 3 \bmod 8$, W carries an inner product, so $G_R = \mathrm{O}(W)$.
- For $n \equiv 2 \bmod 8$ ($n > 2$), we have two orthogonal vector spaces: $G_R = \mathrm{O}(W_+) \times \mathrm{O}(W_-)$.
- For $n \equiv 0, 4 \bmod 8$, we have no additional structure, so $G_R = \mathrm{GL}(W)$.
- For $n \equiv 5, 7 \bmod 8$, W has a symplectic structure, and $G_R = \mathrm{Sp}(W)$.

- For $n \equiv 6$, we have two symplectic vector spaces: $G_R = \mathrm{Sp}(W_+) \times \mathrm{Sp}(W_-)$.

This is the algebraic piece only, though; in general, not all of the R-symmetry group may actually be a symmetry of a given supersymmetric QFT. We can also take a Lie subalgebra $\mathfrak{g} \subset \mathfrak{g}_R$ and append that to the super-Poincaré algebra, taking into account that $\mathfrak{g}$ acts on Σ.

Suppose $Q \in \Sigma$ squares to zero, i.e. $\Gamma(Q, Q) = 0$. These are easy to find in some cases (e.g. W symplectic) but are not always immediate. These Q determine cohomological structures on supersymmetric QFTs, where one includes Q in the BV-BRST differential of a supersymmetric quantum field theory, which is an example of *twisting*.

Definition 2.13. If $\Gamma(Q, –)\colon \Sigma \to \mathbb{C}^n$ is surjective, we call Q *topological* (and will expect that the twisted theory is a topological field theory of some sort). These were the original twists considered by Witten.

Example 2.14 (Dimension 1)**.** $\mathfrak{so}_1(\mathbb{C})$ is trivial, so the supersymmetry algebra is just $\mathbb{C} \oplus \Pi W$, with bracket given by a bilinear pairing $\langle –, – \rangle$ on W; Q squares to zero if $\langle Q, Q \rangle = 0$.

Specifically, the 1D $\mathcal{N} = 2$ supersymmetry algebra has null vectors of $(1, \pm i)$, so these give rise to two conserved charges Q and $Q^\dagger$ that we saw yesterday. Then, turning on the differential given by Q in supersymmetric quantum mechanics tells you interesting things about de Rham cohomology or something analogous. ◀

Example 2.15 (Dimension 2)**.** First, $\mathfrak{so}_2(\mathbb{C}) \cong \mathbb{C}^\times$ and $S_\pm$ are one-dimensional with weights $\pm 1/2$. The vector representation $V = \mathbb{C}^2$ is reducible, with weight $(1, -1)$. Here Σ generally looks like $W_+ \oplus W_-$ (with weights $1/2$ and $-1/2$, respectively), and square-zero elements are pairs (w_+, w_-), where $w_\pm$ are both null. This element is topological whenever they're both nonzero (TODO: I think?), and the smallest $\mathcal{N}$ for which this happens is $\mathcal{N} = (2, 2)$. ◀

Remark 2.16. Generally, physicists consider supersymmetric theories with $\mathcal{N} \le 16$ (or $\mathcal{N} \le 32$). The ultimate reason is that people only look at particles with spin at most 2 (and spin-2 particles pretty much only in supergravity), which forces those bounds. This also means that we can only look at dimensions up to 11; outside of supergravity, where you're probably looking at $\mathcal{N} \le 16$, this restricts to dimensions up to 10. It's also common for $\mathcal{N}$ to be a power of 2. ◀

Example 2.17 (Dimension 4)**.** The exceptional isomorphism $\mathfrak{so}_4(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$; we'll denote the first copy $\mathfrak{sl}_2(\mathbb{C})_+$ and the second copy $\mathfrak{sl}_2(\mathbb{C})_-$. The spin representation $S_\pm$ is the defining representation for $\mathfrak{sl}_2(\mathbb{C})_\pm$, and $\mathfrak{sl}_2(\mathbb{C})_\mp$ acts trivially. We have $V = S_+ \otimes S_-$ and $\Sigma = S_+ \otimes W \oplus S_- \otimes W^*$. ◀

3. Natalie Paquette: 2D Yang-Mills theory: 6/19/19

In today's TA session, Natalie Paquette spoke about 2D Yang-Mills theory. This is a particularly simple field theory, which makes it a good example for studying these things; we will work in Euclidean signature and follow the standard (but far from the only) reference, Cordes-Moore-Ramgoolam. For the most part, today the gauge group will be U_N or SU_N; it will always be compact.

So, fix an oriented[1] surface Σ with a Riemannian metric, a gauge group G, a principal G-bundle P, and a connection A with curvature $F \in \Omega^2_\Sigma(\mathfrak{g})$. The action is

$$S = \frac{1}{4e^2} \int_\Sigma \mathrm{tr}(F \wedge \star F). \tag{3.1}$$

Since we're in dimension 2, we can do nice things: write $f \coloneqq \star F \in \Omega^0_\Sigma(P_\mathfrak{g})$;[2] then $F = f\mu$, where μ is the area form on Σ coming from the metric. Then we can rewrite (3.1) as

$$S = \frac{1}{4e^2} \int_\Sigma \mathrm{tr}(f^2)\, \mathrm{d}\mu. \tag{3.2}$$

This has a huge symmetry group, namely $\mathrm{Diff}^+(\Sigma)$, the orientation-preserving diffeomorphisms. In some sense, this means there's not a lot of interesting theory left.

[1] With a little care, one can generalize to nonoriented surfaces, where we replace area forms with densities and realizing F as a section of the orientation-twisted adjoint bundle.

[2] Here $P_\mathfrak{g}$ is the associated bundle $P \times_G \mathfrak{g}$, where $\mathfrak{g}$ carries the adjoint representation.

Atiyah-Bott studied this classical action, and explained in detail how to get that the equations of motion are

$$\mathrm{d}_A(\star F) = 0. \tag{3.3}$$

So we're looking for covariantly constant sections of $P_{\mathfrak{g}}$.

In general, for Yang-Mills theory in dimension d, we can assign a Hilbert space to a $(d-1)$-dimensional Riemannian manifold Y, which is $L^2(\mathcal{A}/\mathcal{G})$. Here $\mathcal{A}$ is the space of connections and $\mathcal{G}$ is the group of gauge transformations, which is $\mathrm{Map}(Y, G)$. This carries a nice inner product, so we can write L^2. In dimension 2, it suffices to understand what we attach to S^1, so we just get L^2 class functions on G. This admits a Peter-Weyl decomposition:

$$L^2(G/\!\!/G) = \left(\bigoplus_{V\in\mathrm{Irr}(G)} V \otimes \overline{V}\right)^G. \tag{3.4}$$

Here, there is one summand for every isomorphism class V of irreducible unitary representations of G. The Hilbert space structure is the natural one:

$$\langle f_1, f_2\rangle = \int_G \mathrm{d}x\, \overline{f_1(x)} f_2(x), \tag{3.5}$$

where $\mathrm{d}x$ is the Haar measure, normalized such that $\mathrm{vol}(G) = 1$.

Explicitly, the Hilbert space of wavefunctions is functions on the fields, so given a field (connection) A and a class function ψ,

$$\psi[A^a(x)] = \psi\left[P\exp\left(i\int_0^L \mathrm{d}x\, A(x)\right)\right]. \tag{3.6}$$

The Hamiltonian for this theory is

$$H = \frac{e^2}{2}\int_0^L \mathrm{d}x\, \frac{\delta}{\delta A_a(x)}\frac{\delta}{\delta A_a(x)}, \tag{3.7}$$

so given a state (class function) $|R\rangle$,

$$H|R\rangle = \frac{\lambda L}{2} C_2(R)|R\rangle, \tag{3.8}$$

where $\lambda = e^2 N$ is the *'t Hooft coupling*.

It's possible to completely solve 2D Yang-Mills theory and determine the correlation functions precisely. The idea is to put the theory on a lattice, and then show that it's invariant under RG flow, so that we can calculate on the lattice rather than trying to take a continuum limit. This is a very special thing, and doesn't happen in higher dimensions.

So, triangulate Σ (polygons with more edges are OK). Let $\mathcal{V}$ denote the set of vertices; the fields are functions $\mathcal{V} \to G$. The principal bundle also gives us holonomies $U_\gamma \in G$ associated to edges γ of Σ; if $\gamma\colon x$—y, then U_γ (TODO: ?) $g_y U_\gamma g_x^{-1}$. Now, instead of the exponentiated action, the partition function is a weighted sum of the plaquettes, as first written down by Migdal. First, the local contribution to the action is:

$$\Gamma(u, a_W) = \sum_{\alpha\in\mathrm{Irr}(G)} \dim\alpha\, \chi_\alpha(u) \exp\left(a_W \frac{C_2(\alpha)}{2}\right). \tag{3.9}$$

Here a_W is the area of the plaquette W, and u is the product of the elements of G on ∂W, in an order specified by the order. As $a_W \to 0$, corresponding to a finer triangulation (which ought to be a better approximation), this looks more like $\delta(u-1)$

The total partition function is the product of all of these: letting Π denote the triangulation and a be the area of Σ_g (so Π_1 is the edges and Π_2 is the plaquettes),

$$Z_{\Sigma,\Pi}(a) = \int \prod_{\gamma\in\Pi_1} \mathrm{d}u_\gamma \prod_{i\in\Pi_2} \Gamma(U_i, a_i). \tag{3.10}$$

We'd like to show that this doesn't depend on Π! It suffices to show this in the case where you subdivide a single plaquette in two. We'll do this for a square f area a_0 with edge elements $u_1, \dots, u_4$, which we subdivide into two triangles with areas a' and a'', so of course $a_0 = a' + a''$. The contribution of the square plaquette to the partition function is

$$\Gamma - \sum_\alpha \dim\alpha \chi_\alpha(u_1u_2u_3u_4)\exp\left(-a_0\frac{C_2(\alpha)}{2}\right), \tag{3.11}$$

and the contribution of the two triangles is

$$\Gamma'\Gamma'' = \sum_{\alpha,\beta} \dim\alpha\dim\beta\chi_\alpha(u_1u_2v)\chi_\beta(v^{-1}u_3u_4)\exp(\dots). \tag{3.12}$$

Here v is the group element on the new edge; the orientation convention means that in one triangle, it's counted as v, and in the other, it's v^{-1}. It suffices to show that

$$\int \mathrm{d}V\,\Gamma'\Gamma'' = \Gamma, \tag{3.13}$$

which follows from the purely group-theoretic fact that

$$\int \mathrm{d}V\,\chi_\alpha(Av)\chi_\beta(v^{-1}B) = \delta_{\alpha\beta}\frac{1}{\dim\alpha}\chi_\alpha(AB), \tag{3.14}$$

which is not too hard to prove.

So you can use as coarse of a triangulation as you like, and this calculates the same answer as a much finer triangulation which in general is a better approximation of the continuum limit. In particular, you can use the coarsest possible triangulation, describing the surface as its fundamental domain with edges identified in accordance with the fundamental group. Explicitly, then we get

$$Z_\Sigma(a) = \sum_{\alpha\in\mathrm{Irr}(G)} \dim\alpha\exp\left(-\frac{aC_2(\alpha)}{2}\right)\int \mathrm{d}U_i\,\mathrm{d}V_j\,\chi_\alpha(U_1V_1U_1^{-1}V_1^{-1}\dots) = \sum_\alpha \frac{e^{-aC_2(\alpha)/2}}{(\dim\alpha)^{2g-2}}. \tag{3.15}$$

You can do something similar with surfaces with boundaries, where each boundary is labeled by some $U_i \in G$. This multiplies the α term by a product of $\chi_\alpha(U_i)$ for each boundary component i.

Remark 3.16. So 2D Yang-Mills theory depends only on the area of the surface, not the metric. This is very special, and is almost topological, so you might ask whether this has an interpretation within the functorial field theory perspective of Atiyah (for topological field theory) or Segal (for conformal field theory). Indeed, there is a formalization of 2D Yang-Mills theory functorially due to Moore-Segal. ◀

Another way to study this is in the first-order formalism, where one writes the partition function as

$$S = -\frac{1}{2}\int \mathrm{tr}(BF) + \frac{1}{2}e^2\int \mathrm{tr}(B^2)\,\mu. \tag{3.17}$$

Because only the second term depends on area, you can envision trying to integrate it out, which leads to a topological field theory called *BF theory*, which is nice, e.g. its partition function computes the symplectic volume of a moduli space related to Σ_g. It's also very nice to consider the BV version of this theory, or say more refined things related to bordisms (e.g. what's the partition function on an interval?), which has something to do with boundary conditions. These modern perspectives are fairly explicit in 2D Yang-Mills theory; depending on your choice of boundary conditions, you can obtain dg bimodules of the form $\Lambda\mathfrak{g}^k$ or $\mathrm{Sym}\,\mathfrak{g}$, etc., and see simple examples of Koszul duality.

For more on BV and BFV 2D Yang-Mills, see a recent article of Mnev-Irasu.

If you add Wilson lines to the story, you can compute the partition function in an essentially similar way to the case of surfaces with boundary, or the cutting-and-gluing considerations we used to define the partition function in general.

First, non-intersecting Wilson lines $\{\Gamma\}$ labeled by representations R_Γ: $\Sigma \setminus \amalg\Gamma$ is a disjoint union of connected surfaces Σ_c, generally with boundary. Then the answer is

$$\left\langle \prod_\Gamma W(R_\Gamma,\Gamma)\right\rangle = \int \prod_\Gamma \mathrm{d}U_\Gamma \prod_c \left(Z_{\Sigma_c}\left(a_c, U_{c_\Gamma^2=C}, U_{C_\Gamma^r=C}\right)\prod_\Gamma W(R_\Gamma,\Gamma)\right). \tag{3.18}$$

Something in this (TODO: didn't follow) is an integral in terms of Clebsch-Gordon coefficients, which are purely group-theoretical: what are the irreducible components of $V_1\otimes V_2$, where V_1 and V_2 are given irreducibles? Anyways, this can be exactly solved, as can the case of intersecting Wilson lines, in which $6j$

symbols appear somehow, which has relationships to statistical mechanics models. The relationship between intersecting Wilson lines and integrable lattice models is more complicated in dimension 4, where it's studied in recent work of Costello, Witten, and Yamazaki.

Remark 3.19. Even though no Feynman diagrams were written down in this lecture, there are still interesting perturbative considerations here, especially if you thing about the 't Hooft coupling e^2N rather than the gauge coupling, particularly in the large-N limit. This has to do with Hurwitz numbers, counting branched covers, which feels like a baby string-theory (or Gromov-Witten theory), suggesting this is a baby string theory, and this is true. ⋖

4. Chris Elliott: 4D Yang-Mills theory and asymptotic freedom: 6/20/19

Today in the TA session, Chris Elliott spoke about Yang-Mills theory, first classically from the BV formalism, then its relationship to the β function and asymptotic freedom, and then quantization.

First let's look at classical Yang-Mills theory on $\mathbb{R}^4$, with fermionic matter. Let G be a compact simple Lie group and V be a representation of G. The fields of Yang-Mills theory are:

- a *gauge field* $A \in \Omega^1_{\mathbb{R}^4}(\mathfrak{g})$, and
- a *spinor* $\psi \in \Omega^0_{\mathbb{R}^4}(S \otimes V)$. Here $S = S_+ \oplus S_-$ is the Dirac spinor bundle.[3]

There's a gauge group symmetry on this theory; the infinitesimal symmetries are given by $c \in \Omega^0_{\mathbb{R}^4}(\mathfrak{g})$, which acts on the gauge field by

$$A \longmapsto A + \mathrm{d}_A(c). \tag{4.1}$$

Fix a G-invariant pairing $V \times V \to \mathbb{R}$ and a positive operator $m\colon V \to V$, which we call the *mass matrix*. The *Yang-Mills action* is

$$S(A, \psi) := \int_{\mathbb{R}^4} \frac{1}{2}\|F_A\|^2 + \mu(\psi, (\not{\mathrm{d}}_A + m)\psi). \tag{4.2}$$

Here, if $\rho\colon \Omega^1_{\mathbb{R}^4} \otimes S \to S$ is the Dirac operator (or Clifford multiplication), then $\not{\mathrm{d}}_A\psi = \rho(\mathrm{d}_A\psi)$.

There is no possible choice for the gauge fixing operator Q^{GF}, because the BRST complex is

$$(\mathcal{E}, Q) = \left(\Omega^0_{\mathbb{R}^4} \xrightarrow{\ \mathrm{d}\ } \Omega^1_{\mathbb{R}^4} \xrightarrow{\ \mathrm{d}\star\mathrm{d}\ } \Omega^3_{\mathbb{R}^4} \xrightarrow{\ \mathrm{d}\ } \Omega^4_{\mathbb{R}^4} \right). \tag{4.3}$$

To deal with this, we'll rewrite the action in the first-order formalism: fix a self-dual $\mathfrak{g}$-valued 2-form B, i.e. $B \in \Omega^2_{\mathbb{R}^4}(\mathfrak{g})$ and $\star B = B$. The first-order action is

$$S_{\mathrm{FO}}(A, B, \psi) = \int_{\mathbb{R}^4} \langle F_A, B\rangle_{L^2} - \frac{1}{2}\|B\|^2 + \mu(\psi, (\not{\mathrm{d}}_A + m)\psi). \tag{4.4}$$

Then the complex of gauge fields (and ghosts, etc.) is (TODO: did I miss something with Ω^{2+} appearing twice?)

$$\Omega^0_{\mathbb{R}^4}(\mathfrak{g}) \xrightarrow{\ \mathrm{d}\ } \Omega^1_{\mathbb{R}^4}(\mathfrak{g}) \xrightarrow{\ \mathrm{d}_+\ } \Omega^{2+}_{\mathbb{R}^4}(\mathfrak{g}) \xrightarrow{\ \mathrm{d}\ } \Omega^3_{\mathbb{R}^4}(\mathfrak{g}) \xrightarrow{\ \mathrm{d}\ } \Omega^4_{\mathbb{R}^4}(\mathfrak{g}), \tag{4.5}$$

and the spinor field complex is

$$m + \not{\mathrm{d}}\colon \Omega^0_{\mathbb{R}^4}(S \otimes V) \longrightarrow \Omega^0_{\mathbb{R}^4}(S \otimes V). \tag{4.6}$$

In particular, we can write down the interacting term: let $C \in \Omega^0_{\mathbb{R}^4}(\mathfrak{g})$, $C^\vee \in \Omega^4_{\mathbb{R}^4}(\mathfrak{g})$, $A \in \Omega^1_{\mathbb{R}^4}(\mathfrak{g})$, $A^\vee \in \Omega^3_{\mathbb{R}^4}(\mathfrak{g})$, $B, B^\vee \in \Omega^{2+}_{\mathbb{R}^4}(\mathfrak{g})$, $\psi \in \Omega^0_{\mathbb{R}^4}(S \otimes V)$, and $\psi^\vee \in \Omega^0_{\mathbb{R}^4}(S \otimes V)$. Then we get

$$I = \langle B, [A \wedge A]\rangle + \mu(\psi, \not{A}\psi) + (A^\vee, [C, A]) + ([C, \psi], \psi^\vee) + ([C, C], C^\vee). \tag{4.7}$$

This is homotopy equivalent to the second-order Yang-Mills term coupled to a trivial B, in the not-fancy sense that there is a path of theories between them.

Now we want to perform BV quantization, which involves the following steps.

(1) (a) Choose a gauge fixing operator Q^{GF}, such that $[Q, Q^{\mathrm{GF}}]$ is a generalized Laplacian.
    (b) Calculate the kernel K_t mollifying the kernel for $[Q, Q^{\mathrm{GF}}]$ in $\mathcal{E} \otimes \mathcal{E}$. This splits as a sum over "particle species", which are pairs $\alpha \otimes \alpha^\vee$ paired by the symplectic form.

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[3] If we were formulating this on some other 4-manifold M, we would have to choose a spin structure on M.

(c) Calculate the propagator

$$\int_\varepsilon^L \mathrm{d}t\,(Q^{\mathrm{GF}}\otimes 1)K_t = P(\varepsilon, L). \tag{4.8}$$

Again this should split into a sum as above over $\alpha\otimes\beta$, where $|\alpha|+|\beta|+1=3$.

(2) Now we want to calculate $I[L]$. Our first step is to try

$$\lim_{\varepsilon\to 0} W(P(\varepsilon,L), I), \tag{4.9a}$$

though this will be divergent. So we choose a counterterm $I^{\mathrm{CT}}(\varepsilon)$ such that the modified limit

$$\widetilde{I}[L] \coloneqq \lim_{\varepsilon\to 0} W(P(\varepsilon,L), I - I^{\mathrm{CT}}(\varepsilon)) \tag{4.9b}$$

exists; there are many ways to do this.

(3) Now, we try to solve the quantum master equation. This is unobstructed for free theories, but in general there is an obstruction (it will vanish in this case). We can try to solve this by adding some term J to $\widetilde{I}[L]$; J corresponds to a potential for the failure for $\widetilde{I}[L]$ to enter the quantum master equation.

Ok, so let's look at the renormalization group flow. In lecture today, Costello explained that $R_\lambda I[L] = I[L] + \log\lambda$ plus higher-order terms (in $\hbar$).

Definition 4.10. The *β-functional* at scale L is the observable

$$\mathscr{O}_B[L] \coloneqq \frac{\mathrm{d}}{\mathrm{d}\log\lambda} R_\lambda I[L]\bigg|_{\lambda=1} \in \mathcal{O}_{loc}(\mathbb{R}^4)[[\hbar]]. \tag{4.11}$$

It's a general fact that for scale-invariant theories, $\lim_{L\to 0}\mathscr{O}_\beta^{(1)}[L]$ exists and is BV closed, and therefore its cohomology class is independent of choices of $I^{\mathrm{CT}}(\varepsilon)$, etc.

Definition 4.12. In this setting, the cohomology class $\beta^{(1)} \coloneqq [\lim_{L\to 0}\mathscr{O}_\beta^{(1)}[L]]$ is called the *1-loop β-function.*

Theorem 4.13 (Gross-Wilczek-Politezer '73)**.** *For Yang-Mills theory,*

$$\beta^{(1)}(g) = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}C(\mathfrak{g}) - \frac{4}{3}C(V)\right), \tag{4.14}$$

One says that asymptotic freedom holds if $\beta^{(1)}$ is negative. For example, for SU_N with f fundamental flavors and n colors, $\beta^{(1)}$ is negative if $f < 11n/2$.

Remark 4.15. Recent work of Elliott-Yoo recovers important physical results such as Theorem 4.13 from the factorization-algebraic perspective. ⊲

A scale- and translation-invariant theory is strictly renormalizable at one-loop. What this means is the following lemma.

Lemma 4.16. *$\mathscr{O}_\beta^{(1)}[L]$ is cohomologous to the log part of the one-loop counterterm $I^{\mathrm{CT}}_{\log\varepsilon}(\varepsilon)$.*

We want to find Q^{GF} which means finding arrows in the opposite direction in (4.5) and (4.6). The new arrows are in blue:

$$\Omega^0_{\mathbb{R}^4}(\mathfrak{g}) \underset{\mathrm{d}^*}{\overset{\mathrm{d}}{\rightleftarrows}} \Omega^1_{\mathbb{R}^4}(\mathfrak{g}) \underset{2\mathrm{d}^*}{\overset{\mathrm{d}_+}{\rightleftarrows}} \Omega^{2+}_{\mathbb{R}^4}(\mathfrak{g}) \underset{-2\mathrm{d}^*}{\overset{\mathrm{d}}{\rightleftarrows}} \Omega^3_{\mathbb{R}^4}(\mathfrak{g}) \underset{\mathrm{d}^*}{\overset{\mathrm{d}}{\rightleftarrows}} \Omega^4_{\mathbb{R}^4}(\mathfrak{g}), \tag{4.17}$$

and for (4.6),

$$\Omega^0_{\mathbb{R}^4}(S\otimes V) \underset{\not{d}-m}{\overset{m+\not{d}}{\rightleftarrows}} \Omega^0_{\mathbb{R}^4}(S\otimes V). \tag{4.18}$$

Then $[Q, Q^{\mathrm{GF}}] = \Delta + D_{\mathrm{vert}}$, where D_{vert} gives arrows $-2\mathrm{d}^*\colon \Omega^{2+}_{\mathbb{R}^4}(\mathfrak{g}) \to \Omega^1_{\mathbb{R}^4}(\mathfrak{g})$ and $-2\mathrm{d}^*\colon \Omega^3_{\mathbb{R}^4}(\mathfrak{g}) \to \Omega^{2+}_{\mathbb{R}^4}(\mathfrak{g})$.

The heat kernel K_t splits into a sum proportioned to the scalar

$$k_t(x,y) = \frac{1}{(4\pi t)^2}\exp\left(\frac{-|x-y|^2}{4t}\right) \tag{4.19}$$

and

$$\mathcal{E} := C^\infty(\mathbb{R}^4) \otimes (Y \otimes \mathfrak{g} \oplus \mathcal{S} \otimes V), \tag{4.20}$$

where $Y \otimes \mathfrak{g}$ is the pure part and $\mathcal{S} \otimes V$ is the matter. The kernel ends up being

$$K_t = K_{AA^\vee} + K_{BB^\vee} + K_{CC^\vee} + K_{\psi\psi^\vee}. \tag{4.21}$$

This is about all that we can understand without actually computing some Feynman diagrams. The propagator applies $Q^{\mathrm{GF}} \otimes 1$ to K_t.

Lemma 4.22. *The propagator has the form*

$$P(\varepsilon, L) = \int_\varepsilon^L \mathrm{d}t \left(\frac{\partial k_t}{\partial x^i}(x,y)\big(P^i_{AB} + P^i_{A^\vee C}\big) + \frac{\partial^2 k_t}{\partial x^i \partial x^j} P^{ij}_{AA} + \frac{\partial k_t}{\partial x^i} P^i_{\psi\psi} \right). \tag{4.23}$$

The P^i_{AB} term corresponds to an edge A—B; $P^i_{A^\vee C}$ corresponds to $A^\vee$—C, P^{ij}_{AA} to A—A, and $P^i_{\psi\psi}$ to ψ—ψ.

Let Γ_k denote a one-loop Feynman diagram with k vertices.

Lemma 4.24. *The weight associated to Γ_k has no $\log(\varepsilon)$ divergence unless $k = 2$.*

When $k = 2$, there are only a few Feynman diagrams we can get (which (TODO) I wasn't able to parse or TeX down), though the one with ψ on each open edge, AA on one full edge, and $\psi\psi$ on the other is BV-exact, so we can throw it out.

5. Du Pei: Verlinde algebras and 2D TQFTs: 6/21/19

"And now we have the degree-k line bundle over $\mathbb{CP}^3$, which is OK..."

Today, Du Pei spoke about the Verlinde formula and 2D TQFTs, which is a continuation of Natalie's talk from Wednesday.

Two-dimensional Yang-Mills theory is almost topological: it depends on $e^2 a(\Sigma)$, where a is the area. In the limit $e^2 \to 0$, where one obtains a BF theory, this seems to be topological, as it has no area-dependence. However, this is not a mathematicians' TQFT (in the style of Atiyah-Segal): the space of states attached to a circle is infinite-dimensional, which means the partition function $Z(T^2) = \infty$; similarly, $Z(S^2)$ diverges. However, this not-quite-TQFT is still useful; it computes the volume of the moduli space $\mathcal{M}_\flat(\Sigma, G)$ of flat G-connections on Σ.

Today, given a compact, simple, simply-connected Lie group G, we'd like to define a family of interesting TQFTs in the sense of Atiyah-Segal on oriented surfaces, and such that these TQFTs are at least as interesting as 2D Yang-Mills theory. The theories will be parameterized by G and a *level* $k \in \mathbb{Z}_{\geq 0}$.

The data of a 2D TQFT is equivalent to the data of a finite-dimensional, commutative Frobenius algebra. For (G, k) given, this algebra will be the *Verlinde algebra* for G_k. This algebra arises from the land of 2D conformal field theory (or, more or less equivalently, vertex operator algebras, or affine Lie algebras), where it computes the fusion rules for the Wess-Zumino-Witten (WZW) model associated to the same data G and k. The Verlinde algebra also appears in 3D quantum field theory, where it's the algebra of line defects in Chern-Simons theory (again for the same input data G_k). In particular, this talk will teach you how to do computations with line operators in Chern-Simons theory!

Remark 5.1. These two appearances of the Verlinde algebra in quantum field theory are related through a relationship between Chern-Simons theory and the WZW model for G_k. We'll likely hear more about this next week. ◀

There's a third appearance of the Verlinde algebra, as allowing one to compute nonabelian Θ-functions on $\mathcal{M}_\flat(\Sigma, G)$ – and still others, e.g. a relationship with the quantum cohomology and quantum K-theory of Grassmannians.

Given G and k, let's now build the Frobenius algebra $(V, \star, (\cdot,\cdot), \mathbf{1})$.

- V denotes a finite-dimensional complex vector space.
- $\star\colon V \times V \to V$, called the *fusion rule*, is symmetric and bilinear.
- $(\cdot,\cdot)\colon V \times V \to \mathbb{C}$ is an inner product.
- $\mathbf{1}\colon \mathbb{C} \to V$ is the identity for the multiplication.

These must be compatible in the following ways:

- $(V, \star, \mathbf{1})$ is an associative, commutative, unital algebra.
- $(\cdot, \cdot)$ is a nondegenerate pairing and $(a \star b, c) = (a, b \star c)$.

Example 5.2. For $\mathrm{SU}(2)_k$, V is $(k+1)$-dimensional, and there's a nice basis $e_0, \dots, e_k$, indexed by the integrable representations of $L\mathrm{SU}(2)_k$ associated with the highest weight $i \in \Lambda_{\mathrm{weight}}(\mathrm{SU}(2)) = \mathbb{Z}$. Here $L\mathrm{SU}_2$ is the *loop group* of SU(2), i.e. the space of maps $S^1 \to \mathrm{SU}(2)$ with group structure given by pointwise multiplication.

The data of the multiplication and the inner product will be encoded as follows: write

$$e_i \star e_j = f_{ij}^{\ell} e_\ell \tag{5.3a}$$

$$(e_i, e_j) = \eta_{ij} \tag{5.3b}$$

$$f_{ij\ell} = f_{ij}^{m} \eta_{m\ell}. \tag{5.3c}$$

Let

$$\Delta(i, j, \ell) = \max\{i - j - \ell, j - i - \ell, \ell - i - j, i + j + \ell - 2k\}. \tag{5.4}$$

This has an interpretation as something to do with a triangle fitting into SU(2) – unfortunately, I did not understand exactly how.

Anyways, then

$$f_{ij\ell} = \begin{cases} 1, & \text{if } i + j + \ell \text{ is even and } \Delta(i, j, \ell) \le 0 \\ 0, & \text{otherwise.} \end{cases} \tag{5.5}$$

And $\eta_{ij} = \delta_{ij}$, which means we can identify lower and upper indices. This has defined almost all of the data of the Verlinde algebra – all we have to do is specify the unit, which is e_0.

Then one must check associativity, commutativity, and the Frobenius condition; associativity is relatively easy, and commutativity is harder. ◀

The data of a Frobenius algebra can be made very explicit from the bordism-theoretic viewpoint: V is what the TQFT Z assigns to S^1; $\star$ is what's assigned to a pair of pants, $(\cdot, \cdot)$ is what's assigned to a cylinder as a bordism $S^1 \times S^1 \to \varnothing$; and $\mathbf{1}$ is what's assigned to the disc as a bordism $\varnothing \to S^1$. Every surface can be cut into a series of these bordisms and their duals (switch the incoming and outgoing components).

Therefore, for example, if you want to know what the $\mathrm{SU}(2)_k$ TQFT assigns to a genus-2 surface, cut it into pieces to conclude

$$Z_{\mathrm{SU}(2)_k}(\Sigma_2) = \sum_{i,j,k} f_{ijk} = \frac{1}{6}k^3 + k^2 + \frac{11}{6}k + 1. \tag{5.6}$$

This is the $(k+1)^{\mathrm{st}}$ *tetrahedral number*. That is, if you stack balls in the shape of a tetrahedron with $k+1$ balls along an edge, (5.6) is the total number of balls in the tetrahedron. The 1/6 in front can be identified with other interesting numbers, e.g. you can relate it to the volume of $\mathbb{CP}^3$ in its usual metric, or as $(1/\pi^2)(\zeta(2))$.

The first identification, as the volume of $\mathbb{CP}^3$, uses the fact that $\mathcal{M}_\flat(\Sigma_2, \mathrm{SU}(2)) = \mathbb{CP}^3$. If $\omega \in H^2(\mathbb{CP}^3; \mathbb{Z})$ is the Atiyah-Bott form associated with this, then

$$\int_{\mathbb{CP}^3} e^\omega = \int_{\mathbb{CP}^3} \frac{\omega^3}{6} = \frac{1}{6}. \tag{5.7}$$

Now, how does this relate to 2D Yang-Mills theory? The partition function is

$$Z_{\mathrm{YM}}(G, \Sigma_2) = \frac{1}{\pi^2} \sum_{\text{irreps } R} \frac{1}{(\dim R)^2} = \frac{1}{\pi^2}\zeta(2), \tag{5.8}$$

which does agree. (Here and elsewhere, we're taking the area-zero limit of Yang-Mills, which might also be denoted Z_{BF}.)

One thing that's interesting is taking $k \to \infty$ and asking how the partition function on Σ_g behaves. To obtain an actual number, we need to normalize by k^{-3g-3}; then we get

$$\lim_{k\to\infty} \frac{Z_{G_k}(\Sigma_g)}{k^{3g-3}} = \mathrm{vol}(\mathcal{M}_\flat(\Sigma_g, G) \tag{5.9a}$$

When $G = \mathrm{SU}(2)$,

$$= \frac{2}{(2\pi^2)^{g-1}} Z_{\mathrm{YM}}(\Sigma_g, \mathrm{SU}_2). \tag{5.9b}$$

Remark 5.10. For any G, k, and Σ, $Z_{G_k}(\Sigma_g)$ is an integer, even though it looks at first like some rational number (as in (5.6)). This is a suggestion that it counts something, and indeed, it does: $\mathcal{M}_\flat(\Sigma_g, G)$ can be identified with the moduli space of holomorphic $G_\mathbb{C}$-bundles over Σ, where $G_\mathbb{C}$ is the complexification of G. Let $\mathcal{L}$ denote the determinant line bundle; then

$$Z_{G_k} = h^0(\mathcal{M}_\flat; \mathcal{L}^{\otimes k}). \tag{5.11}$$

(Here, $h^0 := \dim H^0$.) For example, when $G = \mathrm{U}(1)$, $G_\mathbb{C} = \mathbb{C}^\times$; then $\mathcal{M}_\flat(\Sigma, \mathrm{U}(1))$ is the *Jacobi variety* $\mathrm{Jac}(\Sigma)$; holomorphic sections of $\mathcal{L}^k$ are called *Riemann theta functions of order k*.

Hence, more generally, $H^0(\mathcal{M}_\flat(\Sigma, G), \mathcal{L}^{\otimes k})$ is called the space of *nonabelian theta functions* for G of order k. ◀

For example, let's look at the $\mathrm{SU}(2)_2$ theory. The identification $\mathcal{M}_\flat(\Sigma_2, \mathrm{SU}(2)) \cong \mathbb{CP}^3$ sends the determinant bundle to $\mathscr{O}(1)$, so the partition function is $h^0(\mathbb{CP}^3; \mathscr{O}(k))$. This is the dimension of the space of homogeneous polynomials in four variables of degree k, which does agree with (5.6). The index theorem tells us

$$Z_{\mathrm{SU}(2)_k}(\Sigma_2) = h^0(\mathbb{CP}^3; \mathscr{O}(k)) = \chi(\mathbb{CP}^3; \mathscr{O}(k)) = \int_{\mathbb{CP}^3} \mathrm{td}(\mathbb{CP}^3) e^{k\omega}, \tag{5.12}$$

which makes it less surprising that one obtains the symplectic volume. Here td denotes the Todd class.

Now, what happens for more general G? Let $T \subset G$ be a maximal torus and $\mathfrak{t} \subset \mathfrak{g}$ be the associated Cartan. Let h be the *dual Coxeter number* and $\langle -, - \rangle$ denote the Killing form, and consider the map

$$(h + k)\langle -, - \rangle \colon \mathfrak{t} \to \mathfrak{t}^*, \tag{5.13}$$

which exponentiates to a map $\chi\colon T \to T^*$. Let $F := \chi^{-1}(1)$.

The underlying vector space of the Verlinde algebra of G_k is $V := V[F^{\mathrm{reg}}/W]$; remove the singular elements and quotient by the Weyl group W. The algebra structure is multiplication of functions! We'll choose a basis

$$\Theta_\lambda := \sum_{w \in W} \frac{e^{w(\lambda)}}{\prod_{\alpha > 0}(1 - \exp(-w(\alpha)))} \in V, \tag{5.14}$$

where λ ranges over the dominant weights such that if θ is the highest root, $\langle \lambda, \theta \rangle \le k$. The inner product is $(\Theta_{\lambda_1}, \Theta_{\Lambda_2}) = \delta_{\lambda_1 \lambda_2^*}$, and the unit is the constant function.

If you want, you can start with this definition and reproduce the fusion rule stated before! But this presentation makes it clearer that one can diagonalize the fusion rule, i.e. writing down another basis $\{W_\lambda\}$ such that $W_\lambda \star W_{\lambda'} \propto \delta_{\lambda\lambda'} W_\lambda$. If we can find this, computing the partition function will be really easy, because we've essentially written the algebra as a direct sum.

It's clear from (5.14) that such a basis exists. It's possible to identify F^{reg}/W with the set of integrable representations of the loop group at level k, and λ is sent (back to F^{reg}/W) to

$$\exp((\lambda + \rho)^\vee/(k + h)). \tag{5.15}$$

So let f_μ be the δ-function on $\mu \in F^{\mathrm{reg}}/W$. If we basis change to $\Theta_\lambda(\mu)$, we can define $S_{\lambda\mu} := c\Theta_\lambda(\mu)$, with c a constant for which $SS^\dagger = \mathbf{1}$, then these $S_{\lambda\mu}$ define a matrix called the *S-matrix*. This theory is the decategrification of a 3D theory, and this S-matrix is the S-matrix of that theory in the usual sense.

Theorem 5.16 (Verlinde formula)**.**

$$Z_{G_k}(\Sigma_g) = \sum_\lambda (S_{0\lambda})^{2-2g}.$$

For example, for $G = \mathrm{SU}(2)$,

$$S_{\lambda\mu} = \left(\frac{2}{k+2}\right)^{1/2} \sin\left(\frac{\pi(\lambda+1)(\mu+1)}{k+2}\right), \tag{5.17}$$

and you can evaluate this and recover (5.6).

Today, Dylan Butson spoke about the AKSZ formalism and boundary theories, with an eye towards building examples and making calculations (there might not be a whole lot today, but there'll be another lecture on this material). a In this lecture, everything will be classical, until maybe right at the end.

Definition 6.1. Let X be a derived manifold (or other geometric object). An *n-shifted symplectic structure* on X is a nondegenerate, closed 2-form $\omega_X : T_X \otimes T_X \to k[n]$.

Here k is the base field.

Example 6.2. Let $S \in \mathscr{O}(Y)$, thought of as an action functional. Its derived critical locus $d\mathrm{Crit}(S) := \mathscr{O}(T^*[-1]Y)$ with differential $\iota_{\mathrm{d}S}$, is (-1)-shifted symplectic. We've seen this example in the free scalar field theory, where $Y = C^\infty(M)$ and $T^*[-1]Y = (\mathrm{d}\star\mathrm{d}\colon C^\infty(M) \to \Omega^n(M))$. ◀

Example 6.3. Let G be a Lie group and choose a nondegenerate ad-invariant 2-form $\mathfrak{g}\otimes\mathfrak{g} \to k$. Then BG is 2-shifted symplectic, thanks to the identification $T_0BG = \mathfrak{g}[1]^{\otimes 2}]$, and the form maps to $k[2]$. ◀

Again, these definitions and examples are a little telegraphic.

Definition 6.4. A *d-orientation* on X is a nondegenerate map $\int_M \colon \Gamma(X, \mathscr{O}_X) \to k[-d]$.

Example 6.5. If M is a smooth manifold, its *de Rham space* M_{dR} is a locally dg-ringed space whose underlying topological space is M and whose sheaf of dg algebras is (Ω_M^*, d). If M is closed, oriented, and d-dimensional, M_{dR} is d-oriented. ◀

Example 6.6. If X is a Calabi-Yau variety of dimension d, then Serre duality defines a pairing $\Gamma(X, \mathscr{O}_X) \to k[-d]$, and hence X is d-oriented. ◀

The key idea of the AKSZ construction is that if X is an n-shifted symplectic space and M is d-oriented, then the space $\mathrm{Maps}(M, X)$ is $(n-d)$-shifted symplectic. The reason for this is an identification

$$T_f\mathrm{Maps}(M,X)^{\otimes 2} = \Gamma(M, f^*T_X)^{\otimes 2} \xrightarrow{f^*\omega_X} \Gamma(M, \mathscr{O}_M)[n] \xrightarrow{\int_M} k[n-d]. \tag{6.7}$$

Example 6.8.

(1) If V is a symplectic vector space, $T_0\mathrm{Maps}(\mathbb{R}_{\mathrm{dR}}, V) \simeq \Omega_{\mathbb{R}}^\bullet \otimes V$.
(2) One can construct classical Chern-Simons theory from this perspective, where we fix G and the 3-manifold M. Because BG is 2-shifted and M is 3-dimensional, $T_0\mathrm{Maps}(M_{\mathrm{dR}}, BG) = \Omega_M^\bullet \otimes \mathfrak{g}[1]$ is (-1)-shifted symplectic, as expected for the classical phase space.
(3) One can run the above example replacing BG by a holomorphic symplectic variety X, which defines a theory called the *Rozansky-Witten twist* of the 3D $\mathcal{N}=4$ theory. But this theory isn't perfectly well-behaved; it's only $\mathbb{Z}/2$-graded, rather than $\mathbb{Z}$-graded.
(4) Let X be a complex variety. Then the AKSZ construction can recover the B-model for X on a surface Σ: $\mathrm{Maps}(\Sigma_{\mathrm{dR}}, T^*[1]X)$ is (-1)-shifted symplectic.
(5) One can consider variants of Chern-Simons theory where some of the directions on spacetime are made holomorphic. So one obtains a four-dimensional topological-holomorphic Chern-Simons theory with space $\mathrm{Maps}((\mathbb{C}\times\mathbb{R}^2)_{\mathrm{dR}}, BG)$, a five-dimensional topological-holomorphic Chern-Simons theory with space $\mathrm{Maps}((\mathbb{C}^2\times\mathbb{R})_{\mathrm{dR}}, BG)$, and a six-dimensional holomorphic Chern-Simons theory with space $\mathrm{Maps}(X_{\mathrm{dR}}, BG)$, where X is a complex manifold of dimension 3. These theories have been studied by Costello.
(6) The *Kapustin-Witten B-twist* on a 4-manifold M for a group G is built by the AKSZ construction to yield the space $\mathrm{Maps}(M_{\mathrm{dR}}, T^*[3]BG)$.
(7) There's an interesting twist of the 5D $\mathcal{N}=2$ theory, studied by Elliott and Raghavendran, which is topological in three directions and holomorphic in one (complex) direction. Given a 3-manifold M, the space of classical solutions is $\mathrm{Maps}(M_{\mathrm{dR}}\times\mathbb{C}, T^*[3]BG)$. ◀

Moving on to classical field theory on manifolds with boundary. Let M be a compact, oriented manifold of dimension d and X be a $(d-1)$-shifted symplectic space. Then $\mathrm{Maps}(M_{\mathrm{dR}}, X)$ is no longer symplectic, but $\mathrm{Maps}((\partial M)_{dR}, X)$ is 0-shifted symplectic, and moreover the restriction map $\mathrm{Maps}(M_{\mathrm{dR}}, X) \to \mathrm{Maps}((\partial M)_{\mathrm{dR}}, X)$ is Lagrangian!

It's a general fact from shifted symplectic geometry that if $L_1, L_2 \to X$ are Lagrangians, where X is k-shifted symplectic, then $L_1 \times_X L_2$ is $(k-1)$-shifted. So choose some Lagrangian $L \to \mathrm{Maps}((\partial M)_{\mathrm{dR}}, X)$; then

$$\mathrm{Maps}(M_{\mathrm{dR}}, X) \times_{\mathrm{Maps}((\partial M)_{\mathrm{dR}}, X)} L \tag{6.9}$$

is (-1)-shifted symplectic, which is what we need for the BV formalism.

Thus we can define a boundary theory. Let $\mathscr{E}(U)$ be the space of fields of the resulting theory (TODO: is that correct?). If $U \subset M$ is open, we let $\mathrm{Obs}^{c\ell}(U) = \mathscr{O}(\mathscr{E}(U) \times_{\mathscr{E}(U \cap \partial M)} L)$ if $U \cap \partial M$ is nonempty, and if it's empty, we just use $\mathscr{E}(U)$. This motivates the following definition.

Definition 6.10. A *local boundary condition* for $\mathscr{E}$ is a Lagrangian subbundle of $\mathscr{E}|_{\partial M}$ which is compatible with the equations of motion in a specific sense.

There's an important idea in physics that, when you have a boundary condition for a theory, sometimes the boundary values (fields which survive on ∂M) themselves have dynamics, so you have a boundary theory. That is, when the equations of motion are well-posed with respect to the Lagrangian (meaning there's a unique solution to the problem of extending to the boundary once the boundary condition is chosen), we get an isomorphism $L(U) \cong \mathscr{E}^L(V)$, where $\mathscr{E}^L$ is the extension to ∂M, $U \subset \partial M$, and $V \subset M$ satisfies $V \cap \partial M = U$. Therefore $\mathscr{O}(L)$ defines a $\mathbb{P}_0$-factorization algebra on N, which we identify as the boundary field theory.

Let's make explicit this $\mathbb{P}_0$-structure. Let X be a symplectic manifold. More or less by definition, if $\mathscr{E} := \mathrm{Maps}(\mathbb{R}^{\geq 0}_{\mathrm{dR}}, X)$, then $\mathscr{E}|_{\partial X}$ is zero-shifted symplectic. If $L \hookrightarrow X$ is a Lagrangian, then it carries a Poisson structure – though this is only really interesting in this derived setting. We will consider the formal completion of X around L, denoted $X^\wedge_L$. This will look a lot like the normal bundle ν_L of L, but with a deformation; there's a theorem called deformation to the normal cone which makes this precise. In this setting, the trick is that we've added some homological vector field Q to the differential.

Since L is Lagrangian, we can use ω to identify this with $T^*L, Q = \{S, -\} = \mathscr{O}(T^*L)[1]$, which we can think of as polyvector fields on L. Here $S = Q = \Pi + \Pi^{(3)} + \dots$, where $\Pi^{(k)}$ are higher terms that often vanish in practice. The classical master equation guarantees that these define a homotopy $\mathbb{P}_0$-structure, and if $\Pi^{(\geq 3)}$ vanish, then this is an actual $\mathbb{P}_0$ structure.

Example 6.11 (B-model to BG)**.** The space of fields is $\mathscr{E} := T_0\mathrm{Maps}(\Sigma_{\mathrm{dR}}, T^*[1]BG)$. Concretely, this lives in degrees $-1, \dots, 2$: in degree -1, we have $\Omega^0_\Sigma(\mathfrak{g})$, in degree 0, we have $\Omega^1_\Sigma(\mathfrak{g}) \oplus \Omega^0_\Sigma(\mathfrak{g}^*)$, in degree 1, we have $\Omega^2_\Sigma(\mathfrak{g}) \oplus \Omega^1_\Sigma(\mathfrak{g}^*)$, and in degree 2, we have $\Omega^2_\Sigma(\mathfrak{g}^*)$. The differentials are induced from the de Rham differential.

The action for a $\mathfrak{g}$-valued differential form A and an $\mathfrak{g}^*$-valued differential form B is

$$S(A, B) := \int_\Sigma B\left(\mathrm{d}A + \frac{1}{2}[A \wedge A]\right). \tag{6.12}$$

So because of the curvature term, this is also called a *BF theory.* In both this way and in the complex of fields, this looks a bit like a deformed Chern-Simons theory.

There are two boundary conditions, given by Lagrangians L_A and L_B which kill off the B-fields and A-fields, respectively.

On the boundary, $\mathscr{E}|_{\partial\Sigma}$ kills off $\Omega^2_{\partial\Sigma}(\mathfrak{g})$ and $\Omega^2_{\partial\Sigma}(\mathfrak{g}^*)$, and everything else is the same. If we work with the Lagrangian L_B, we can describe $\mathscr{E}|_{\partial\Sigma}$ as T^*L_B with an additional $\int B[A.A]$ term in the action; this term lives in $L_B^* \otimes L_B$, hence can be interpreted as a bivector with a linear coefficient. Moreover, the algebra of operators $\mathscr{O}_\hbar(L_B) \cong \mathscr{O}_\hbar(\mathfrak{g}^*) = \mathcal{U}_\hbar(\mathfrak{g})$, the enveloping operator. This is $\mathrm{End}_{\mathcal{U}(\mathfrak{g})}(\mathcal{U}(\mathfrak{g}), \mathcal{U}(\mathfrak{g}))$, which makes sense: this should be the algebra of endomorphism of some sort of the boundary conditions.

This L_B is the Dirichlet boundary condition, and L_A is the Neumann boundary condition; analogously we get $\mathscr{E}|_{\partial\Sigma} = T^*L_A$, and $\mathscr{O}(L_A) \cong C^*(\mathfrak{g})$, which is consistent with seeing this as the B-model to (formal) BG, as this is also $\mathbf{R}\,\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(\mathbb{C}, \mathbb{C})$: it's again an algebra of endomorphisms. ◀

You can get many more examples in this way, e.g. starting with boundary conditions to Chern-Simons theory, one can obtain the Kac-Moody algebra and the affine W-algebra for the same input data, and starting from a twist of the 5D $\mathcal{N} = 2$ theory, one can recover 4D Chern-Simons theory, and these are pretty explicit and not too difficult computations.

7. Justin Hilburn: 3D and 4D examples: 6/25/19

Today, Justin Hilburn discussed higher-dimensional examples of supersymmetry. Simply knowing how the supersymmetry algebras behave as one passes between different dimensions is a good way to learn what questions to ask in mathematics.

First, a review of what Si Li discussed. Let's consider the 2D $\mathcal{N} = (2,2)$ supersymmetry algebra, but complexified: we get

$$\underbrace{\mathbb{C}^\times}_{\mathrm{Spin}_2(\mathbb{C})} \times \underbrace{\mathbb{C}^\times_A \times \mathbb{C}^\times_B}_{R\text{-symmetry}} \ltimes V_{2d} \cdot \Pi\big(\mathbb{C}^2 \otimes S_+^{2d} \oplus \mathbb{C}^2 \otimes S_-\big). \tag{7.1}$$

We need an "inner product" (actually a polarization) on these two factors of $\mathbb{C}^2$: we use the one $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

As usual, the even subalgebra acts on the odd piece: $\mathrm{Spin}_2(\mathbb{C})$ acts on V_{2d} with weights $(1,-1)$, and on $S_\pm^{2d}$ with weight $\pm 1/2$. The R-symmetry acts as follows: $\mathbb{C}^\times_A$ acts antidiagonally, and $\mathbb{C}^\times_B$ acts diagonally.

Let's notate the supercharges by $Q_\alpha^{\beta\gamma}$, where α is the weight under the $\mathrm{Spin}_2(\mathbb{C})$-representation and β,γ are the weights under the R-symmetry. These are valued in ± 1, but their product must be -1, so there are four supercharges, not eight.

Si described two topological twists this morning:

$$Q_A = Q_-^{++} + Q_+^{-+} \tag{7.2a}$$

$$Q_B = Q_-^{++} + Q_+^{+-}. \tag{7.2b}$$

The *holomorphic twist* is given by $Q_H \coloneqq Q_-^{++}$.

Exercise 7.3. Show that the image of $[Q_A, -]$ is all of V_{2d}, and the same is true for $[Q_B, -]$. Show that $[Q_H, -]$ is one-dimensional.

The reason for the name "holomorphic twist" is that the things you consider all end up satisfying $\partial_{\bar z} = 0$, so they're holomorphic.

After twisting, one gets that the solutions to the equations of motion are as follows, for the σ-model with Kähler target X.

- For the holomorphic twist, we get $\mathrm{Maps}(\Sigma_{\mathrm{Dol}}, T^*[1]X)$, where $\Sigma_{\mathrm{Dol}} \coloneqq T[1]\Sigma$.
- For the B-twist, we get $\mathrm{Maps}(\Sigma_{\mathrm{dR}}, T^*[1]X)$. Here Σ_{dR} has the same underlying supermanifold $T[1]\Sigma$, but with the de Rham differential.
- The A-twist is a little harder, and in the end we get $\mathrm{Maps}(\Sigma, T^*_\pi T[-1]X)$. Here $T^*_\pi T[-1]X$ means the *π-twisted cotangent bundle*, where π is a canonical Poisson vector. This means the differential is bracketing with $\{\pi, -\}$. If this is too confusing, don't worry about it, as you're not the only one; the A-model is difficult to understand from this perspective.

Let's talk about 3D $\mathcal{N} = 4$ supersymmetry. The algebra, once again complexified, is

$$\mathrm{Spin}_3(\mathbb{C}) \times G_R \ltimes \big(V_{3d} \oplus \Pi\big(W^{3d} \otimes S_{3d}\big)\big). \tag{7.4}$$

Here W^{3d} is a four-dimensional vector space which carries a symmetric form. As usual, the even subalgebra acts on the odd piece. We can identify $\mathrm{Spin}_3(\mathbb{C}) \cong \mathrm{SL}_2(\mathbb{C})$; then the spinor representation S_{3d} is identified with the defining $\mathrm{SL}_2(\mathbb{C})$-representation, which is two-dimensional, and $V_{3d} \cong \mathrm{Sym}^2 S_{3d}$.

The R-symmetry group is $G_R = \mathrm{SL}_2(\mathbb{C})_A \times \mathrm{SL}_2(\mathbb{C})_B$. Thus the last thing we need to completely describe the supersymmetry algebra is W^{3d}, in a way that is invariant under the R-symmetry. One option is to consider $\mathbb{C}_A^2$ with its usual symplectic form ω, and tensor it with $\mathbb{C}_B^2$ and its symplectic form; $\omega \otimes \omega$ is a symmetric bilinear form.[4] The supercharges $Q_\alpha^{\beta\gamma} = e^\alpha \otimes e^\beta \otimes e_\gamma$, where these tensor factors are weight vectors for these representations: e_α for $\mathbb{C}_A^2$, e_β for $\mathbb{C}_B^2$, and e_γ for S_{3d}. These three indices live in $\{\pm 1\}$. Thus, $e_\gamma e_{\gamma'}$ is a weight vector for $\mathrm{Sym}^2(S_{3d})$, so in this incarnation call it $v_{\gamma\gamma'}$.

This allows us to compute, for example, $Q_+^{+-} \cdot Q_-^{-+}$: we'll have a v_{+-} term from γ_+ and γ_-, and then have to multiply it by $\omega(e^+, e^-)$ (on A) and $\omega(e^-, e^+)$ (on B). In general,

$$[Q_\gamma^{\alpha\dot\alpha}, Q_\mu^{\beta\dot\beta}] = \epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} \cdot e_\gamma e_\mu. \tag{7.5}$$

[4] Under the identification $\mathrm{Spin}_4(\mathbb{C}) \cong \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$, W^{3d} is *not* the spinor representation, but the four-dimensional vector representation.

We claim that the 2D $\mathcal{N} = (2,2)$ supersymmetry algebra embeds into the 3D $\mathcal{N} = 4$ supersymmetry algebra. This is more or less forced: this embedding has to play nicely with the symmetries, and that doesn't give us many choices.

First, we need the embedding $\mathrm{Spin}_2(\mathbb{C}) \hookrightarrow \mathrm{Spin}_3(\mathbb{C})$, which we might as well take to be the standard maximal torus $\mathbb{C}^\times \hookrightarrow \mathrm{SL}_2(\mathbb{C})$. The R-symmetries will embed in the same way: $\mathbb{C}_A^\times \hookrightarrow \mathrm{SL}_2(\mathbb{C})_A$ embeds as the standard maximal torus, and same for $\mathbb{C}_B^\times \hookrightarrow \mathrm{SL}_2(\mathbb{C})_B$. We next need to embed $V_{2d} \hookrightarrow V_{3d}$ as follows: if $V_{2d} = \mathrm{span}\{e_+e_+, e_-e_-\}$ and $V_{3d} = \mathrm{span}\{e_+e_+, e_+e_- = e_-e_+, e_-e_-\}$, then the embedding is the one defined by the notation. Finally, we need our maps

$$\mathbb{C}^2 \otimes S_\pm^{2d} \hookrightarrow (\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes S^{3d} = S_+^{2d} \oplus S_-^{2d}. \tag{7.6}$$

The embedding is chosen for us by the R-symmetry. We have $Q_+^{-+} = e^+ \otimes e_+$, $Q_+^{+-} = e^- \otimes e_+$, $Q_-^{++} = e^+ \otimes e_-$, and $Q_-^{--} = e^- \otimes e_-$. Therefore the embedding is the one that's consistent with the notation – this was weird notation in 2D, but was designed to make this work out. The reason to expect something this nice is that these elements are completely determined by their transformation properties.

Exercise 7.7. Check the commutation relations. In order to do this, yu'll have to use the somewhat weird inner products we defined on $\mathbb{C}^2$ and on $\mathbb{C}_A^2 \otimes \mathbb{C}_B^2$.

Corollary 7.8. *All of these supercharges embed into the 3D $\mathcal{N} = 4$ supersymmetry algebra.*

This is because, well, we built a Lie superalgebra homomorphism from the 2D $\mathcal{N} = (2,2)$ algebra into the 3D $\mathcal{N} = 4$ one.

Exercise 7.9. Compute the image of $[Q, –]$ in V_{3d} in these examples.

Thus you can ask how many translation symmetries get killed when you do the analogous twists. The answer will be: 3 for Q_A and Q_B, and 2 for Q_H. Therefore the A- and B-twists give you fully topological theories, and the other twist is a *holomorphic-topological twist*: in one direction, it's translation-invariant, but in the other two (or one complex direction) it's holomorphic (you can think of this as topological in the $\bar{z}$-direction).

Exercise 7.10. Show that, along a similar line of reasoning, the 3D $\mathcal{N} = 4$ algebra embeds into the 4D $\mathcal{N} = 4$ algebra. Once again, you'll use an exceptional isomorphism, specifically $\mathrm{Spin}_6(\mathbb{C}) \cong \mathrm{SL}_4(\mathbb{C})$, and how the embedding $\mathrm{Spin}_3(\mathbb{C}) \hookrightarrow \mathrm{Spin}_6(\mathbb{C})$ looks in that model.

There are other embeddings of superalgebras, but these are some of the most interesting ones to mathematicians.

In conclusion, what does this buy you? In Davide Gaiotto's lectures, we learned about boundary conditions for 2D $\mathcal{N} = (2,2)$ theories, and showed that they're related to topological quantum mechanics (either the Riemannian or Hermitian versions), which can be explained by an embedding of the supersymmetry algebras. Similarly, what we've learned in this lecture is that if you want to study boundary conditions for a 3D $\mathcal{N} = 4$ theory, one nice class is those which preserve a 2D $\mathcal{N} = (2,2)$ supersymmetry at the boundary. The computation explains (in part) how these two things are related.

Mathematically, this predicts relationships between ordiunary mirror symmetry, quantum cohomology, etc.

8. Dylan Butson: The AKSZ formalism and boundary theories, II: 6/26/19

Today, Dylan Butson spoke, continuing his talk from Monday.

So far, one of the main things we've learned from Kevin Costello and Philsang Yoo's lectures is that quantum field theories on M are the same thing as factorization BD_0-algebras on M, and classical field theories on M are the same thing as $\mathbb{P}_0$-algebras over M. A classical field theory with space $\mathscr{E}$ of fields and action S defines the $\mathbb{P}_0$-algebra

$$U \longmapsto \mathscr{O}(\mathscr{E}(U)), \{S, –\}, \tag{8.1}$$

and the corresponding quantum field theory is the factorization BD_0-algebra

$$U \longmapsto \mathscr{O}(\mathscr{E}(U))[[\hbar]], \{S, –\} + \hbar\Delta. \tag{8.2}$$

But we can also directly think of these as factorization E_0-algebras (well, an E_0-algebra isn't much data, so you can think of this as just a factorization algebra valued in cochain complexes). This passage from $\mathbb{P}_0$ to E_0, which might seem a little funny, is sort of analogous to the passage from $\mathbb{P}_1$-algebras to E_1-algebras, which are just associative algebras.[5]

Now, topological field theories over M are the same thing as locally constant factorization algebras. When $M = \mathbb{R}^n$, this is the same thing as E_n-algebras, where the product structure comes from the inclusion of two small discs into a larger disc. You can think of this as the topological operator product expansion map, which doesn't depend on how big anything is because this is a topological field theory.

Now let's throw boundaries into the story. Consider a locally constant factorization algebra $\mathscr{F}$ on $\mathbb{R}_{\leq 0}$. We have two kinds of open intervals: if U doesn't intersect the boundary point, then the same story as on all of $\mathbb{R}$ means that $\mathscr{F}(U) = \mathscr{F}((-\infty, -\varepsilon))$, which is some E_1-algebra A (so, an associative algebra). But if U does contain the boundary point, whatever space H we assign to it is an A-module, coming from the factorization map for including U and some disjoint open (which is assigned A, since it doesn't contain zero) into some larger open set, giving us a map $\rho\colon A \otimes H \to H$. Since $\mathscr{F}$ is locally constant, the choices we made in defining ρ do not matter.

In higher dimensions, things are a little more elaborate, e.g. in $(-\infty, 0] \times \mathbb{R}$, a factorization algebra amounts to an E_2-algebra A in the bulk, together with a factorization E_1-algebra on the boundary, together with a compatible module structure.

In general, we will assume our theories are topological in the direction normal to the boundary. So working on $M \coloneqq (-\infty, 0] \times M$, part of the data of a factorization algebra on M is a factorization algebra on N. Being topological normal to the boundary means that what we get on N is a factorization E_1-algebra, and therefore an inclusion of factorization E_1-algebras on N inside all factorization algebras on M. We saw this on $\mathbb{R}_{\leq 0}$, where we had an E_1-algebra A on the bulk and an E_0-algebra (vector space) H on the boundary, which was an A-module.

So at the classical level, we expect that factorization $\mathbb{P}_1$-algebras on N include into factorization $\mathbb{P}_0$-algebras in M. There is a sense in which the inclusion of $\mathbb{P}_1$-algebras on N into $\mathbb{P}_0$-algebras on N is Lagrangian.

Explicitly, to build the algebra on the bulk from the algebra on the boundary, we can take $\mathscr{E}_M \coloneqq \Omega^\bullet_{\mathbb{R}_{\leq 0}} \boxtimes \mathscr{E}_N$. For example, in Chern-Simons theory, where M is a 3-manifold and N is a surface, $\mathscr{E}_N = \Omega^\bullet_N(\mathfrak{g})$; functions on this are $\mathbb{P}_1$, and the algebra on M is

$$\Omega^\bullet_M(\mathfrak{g}) = \Omega^\bullet_{\mathbb{R}_{\leq 0}} \boxtimes \Omega^\bullet_N(\mathfrak{g}). \tag{8.3}$$

Another way to say this is that

$$\mathrm{Maps}(M, BG) = \mathrm{Maps}(\mathbb{R}_{\leq 0}, \mathrm{Maps}(\Sigma, BG)). \tag{8.4}$$

All of this motivates the following definition.

Definition 8.5. A *(regular embedded) boundary condition* for $\mathscr{E}$ is a Lagrangian subbundle $L \hookrightarrow \mathscr{E}^\partial$ over ∂M.

Here, $\mathscr{E}$ is a classical field theory, hence includes data of a (-1)-shifted symplectic structure. Therefore $\mathscr{E}^\partial \coloneqq \mathscr{E}|_{\partial M}$ is 0-shifted symplectic, and we're asking for (TODO: derived?) Lagrangians in this sense.

The upshot is that for $U \subset N$, the sheaf $U \mapsto \mathscr{O}(L(U))$ is a factorization $\mathbb{P}_0$-algebra on N. We have an isomorphism $\mathrm{PV}_{-1}(L)[1] = \mathscr{O}(T^*L)[1]$, at least of graded vector spaces (TODO: iron this out), which means it's almost true that $\mathscr{E}^\partial$ is quasi-isomorphic to T^*L – and it's exactly true if we use a different differential $Q \coloneqq \{S, –\}$. We call this the *deformed cotangent bundle* to L, and denote it $T^*_\Pi L$; here Π is the image of S under the isomorphism $\mathscr{O}(T^*L)[1] \xrightarrow{\cong} \mathrm{PV}_{-1}(L)[1]$. There are some conditions on S, which amount to asking for Π to be Poisson.

So to summarize, we took $(\mathscr{E}, L \to \mathscr{E}^\partial)$ and produced Π, as we wanted. But it's also worth noticing that if you begin with (L, Π), you can reconstruct $\mathscr{E}^\partial$ by $T^*_\Pi L$, and then can reconstruct $\mathscr{E} = \Omega^\bullet_{\mathbb{R}} \boxtimes \mathscr{E}^\partial$. That is, given a theory, we can obtain from it a theory in one dimension higher, together with a canonical boundary condition of our original theory. This new theory is called the *universal bulk theory*.

Remark 8.6. In general, $T^*_\Pi L$ is an example of the *Poisson center* of (L, Π), which is a $\mathbb{P}_0$ analogue of Hochschild cochains. ◀

[5] TODO: what is a $\mathbb{P}_1$-algebra?

The next thing we'll talk about is interval compactifications. These have already come up here and there during the talks. The question is: what happens if you have two boundary conditions M_0 and M_1 at 0 and 1, and a bulk theory A on the unit interval? (Maybe we work on the interval times some closed manifold M). We assumed the theory is topological normal to the boundary components, so we can shrink the interval and think of this as more or less a theory just on M. The algebraic idea is that A is a factorization E_1-algebra, M_0 is a right A-module, and M_1 is a left A-module; shrinking the interval corresponds to considering the theory given by the factorization algebra $M_0 \otimes_A M_1$.

Now, classically these boundary conditions were given by Lagrangians $L_0, L_1 \to \mathscr{E}^\partial$, and we want to compute their (derived) intersection $L \times_{\mathscr{E}^\partial} L = L \times_{\mathscr{E}^\partial} \mathscr{E} \times_{\mathscr{E}^\partial} L$. Since $\mathscr{E}^\partial$ is 0-shifted symplectic, this pullback is (-1)-shifted symplectic, meaning (TODO: I think) we can quantize, and this corresponds to $M_1 \otimes_A M_2$ as we saw before.

Example 8.7 (B-model)**.** In the B-model with target X, $\mathscr{E} = T_0\mathrm{Maps}([0,1]_{\mathrm{dR}} \times \mathbb{R}_{\mathrm{dR}}, T^*[1]X)$, so $\mathscr{E}^\partial = T_0\mathrm{Maps}(\mathbb{R}_{\mathrm{dR}}, T^*[1]X)$. Choose the Lagrangian $L \coloneqq T_0\mathrm{Maps}(\mathbb{R}_{\mathrm{dR}}, mX)$; then

$$L \times_{\mathscr{E}^\partial} L = T_0\mathrm{Maps}(\mathbb{R}_{\mathrm{dR}}, T^*X). \tag{8.8}$$

A more general Lagrangian is the graph of a closed 1-form: W: in this notation, $\widetilde{L} \coloneqq T_0\mathrm{Maps}(\mathbb{R}_{\mathrm{dR}}, \Gamma(\mathrm{d}W))$. Then we can compute the intersection

$$L \times_{\mathscr{E}^\partial} \widetilde{L} = T_0\mathrm{Maps}(\mathbb{R}_{\mathrm{dR}}, \mathrm{Crit}(W)). \tag{8.9}$$

This looks like supersymmetric quantum mechanics with superpotential W, albeit twisted slightly.

In a 2D theory, the boundary conditions should form a category, and this is true for the B-model: (8.8) is computing $\mathrm{Hom}(L, L)$ in this category, and (8.9) is $\mathrm{Hom}(L, \widetilde{L})$. $\blacktriangleleft$

Suppose that, instead of just having a boundary, we have a defect, so the theory extends past where the lower-dimensional theory is. Then in general we get some sort of bimodule. This is reminiscent of something that happens in string theory, where there are things called branes which are sort of analogous to defects, and several of them are called a stack of branes, e.g. maybe a GL_n gauge theory for n branes. This leads to some sort of gauge theory, and boundary conditions for the gauge theory where branes end, or factorization bimodules where two stacks of branes meet. This is more than an analogy: we can use string theory to better understand constructions in quantum field theory, and zooming out corresponds to an iterated tensor product in factorization algebras.

For example, if you have a stack of 3 branes, then a stack of 2, then a stack of 4, we get a quiver

$$\begin{tikzcd} 3 \arrow[r, no head] & 2 \arrow[r, no head] & 1 \end{tikzcd} \tag{8.10}$$

and then can build some sort of field theory using an AKSZ construction (somehow related to the Nakajima quiver variety), and this theory is the one obtained by tensoring the relevant bimodules together. This leads to interesting predictions of mathematical constructions.

9. Justin Hilburn: Defects in higher-dimensional supersymmetric field theories: 6/25/19

We've learned that the 2D $\mathcal{N} = (2,2)$ supersymmetry algebra embeds into the 3D $\mathcal{N} = 4$ supersymmetry algebra, which in turn embeds into the 4D $\mathcal{N} = 4$ supersymmetry algebra. The 2D $\mathcal{N} = (2,2)$ superalgebra has three square-zero supercharges, which means they can be used to define topological twists; under these embeddings, their images still square to zero, and so we can still use them to twist. To determine whether the twisted theories are topological, we should ask how many translations remain in cohomology.

Example 9.1 (Warmup: B-model)**.** Let X be a Calabi-Yau manifold of dimension n, so that we have an isomorphism $\mathscr{O}_X \cong \omega_X[-d]$. The 2D $\mathcal{N} = (2,2)$ sigma model has as space of solutions to the equations of motion the space

$$\mathrm{Maps}(\Sigma_{\mathrm{dR}}, T^*[1]X). \tag{9.2}$$

Here $\Sigma_{\mathrm{dR}} \coloneqq (T[1]\Sigma, \mathrm{d}_{\mathrm{dR}})$.

We would like to produce from this a 2D TQFT, in the mathematical sense of a functor from the bordism category, from this data. The general ansatz is that this TQFT should attach to M the geometric quantization of that space of solutions to the equations of motion on $M \times \mathbb{R}^d$; we'd also like to do extended TQFT, valued in the 2-category of (maybe derived) categories. Here we need $\dim M = 2 - d$.

Thus far, you might only know how to do geometric quantization in codimension 1, to produce a vector space. But we would also like to extract categories from it. Specifically, $Z(\mathrm{pt})$ should be a category, and the as usual, $Z(S^1)$ should be a vector space, and Z of a surface should be a number.

The AKSZ formalism tells us that if M is a compact d-manifold, then

$$\mathrm{Maps}(M_{\mathrm{dR}} \times \mathbb{R}^{2-d}_{\mathrm{dR}}, T^*[1]X) \tag{9.3}$$

carries a symplectic form of degree $1-d$. The $\mathbb{R}^{2-d}_{\mathrm{dR}}$ isn't terribly important; it's only there because we need some sort of thickening of M.

We can also consider (oriented) manifolds M with incoming and outgoing boundaries $\partial_0 M$ and $\partial_1 M$, respectively. This will produce a Lagrangian correspondence $\mathrm{EOM}(\partial_0 M) \leftarrow \mathrm{EOM}(M) \rightarrow \mathrm{EOM}(\partial_1 M)$. ◀

Remark 9.4. We didn't explain why we needed X to be Calabi-Yau; this is what allows us to produce an oriented TQFT, rather than a framed one. This geometric story is less clear in the framed setting. ◀

Doing this geometric quantization in general is a bit of a pain, involving a bunch of constructions involving gerbes and stuff like that, but there's a nice ansatz that will make this all much easier, and which will work in most situations of interest. Specifically, we expect that geometric quantization of $T^*[n]X$ should product something like $n\mathsf{Coh}(X)$, which is an n-category.

- For $n=-1$, we expect to obtain a number.
- For $n=0$, what we'll get is the sheaf $\mathscr{O}_X$.
- For $n=1$, we'll get the category $\mathsf{Coh}(X)$ (or $\mathsf{QCoh}(X)$; we're not worried about the difference today).
- For $n=2$, what we get are quasicoherent sheaves of catgories, i.e. module categories for $(\mathsf{QCoh}(X), \otimes)$.

This makes life easier, at least at this schematic level (which will make the details easier later, too). Returning to the B-model, $Z(\mathrm{pt})$ is the geometric quantization of $T^*[1]X$, which should give us $\mathrm{QCoh}(X)$.

Then, let's ask about the space of states on S^1. This is the geometric quantization of

$$\mathrm{Maps}(S^1_{\mathrm{dR}} \times \mathbb{R}_{\mathrm{dR}}, T^*[1]X), \tag{9.5}$$

which is 0-shifted symplectic. You can rewrite this as

$$\cong T^*\mathrm{Maps}(S^1_{\mathrm{dR}} \times \mathbb{R}_{\mathrm{dR}}, X). \tag{9.6}$$

Ignoring the noncompact direction that doesn't really matter, this is $T^*\mathcal{L}X$, and $\mathcal{L}X = T^*[-1]X$. So when we geometrically quantize, we get $\Gamma(\mathcal{L}X, \mathscr{O}_{\mathcal{L}X})$, which is dual to the polyvector fields on X. This is the Hochschild homology of the category $Z(\mathrm{pt})$ (which the Calabi-Yau structure allows us to identify with Hochschild cohomology, at least up to a shift).

Remark 9.7. In general, when X is not Calabi-Yau, one must consider 2-framings of the circle, i.e. a trivialization of $TS^1 \oplus \underline{\mathbb{R}}$. These are given by homotopy classes of maps $\pi_1 S^1 \to \mathrm{SO}_2$, so we have $\mathbb{Z}$ of them. If you want to quantize the n^{th} framed circle, you do almost the same thing, but instead of $\Gamma(\mathcal{L}X, \mathscr{O}_{\mathcal{L}X})$, you obtain $\Gamma(\mathcal{L}X, \pi^! \omega_X^{\otimes n})$, where $\pi\colon \mathcal{L}X \to X$ is the evaluation map. You have to approach this differently than we have been, however. ◀

Now let's talk about line operators in dimension 2. A physicist would suggest that, because of the state-operator correspondence, we should consider the link of the line in the surface, which is an S^0, and then try to rewrite the theory under study as maps into a space of maps from the link. In our example, we can think of spacetime (locally at least) as $\mathbb{R}_{>0} \times S^0 \times \mathbb{R}$.[6] Supposing S^0 has a small radius R, we can rewrite the theory as

$$\mathrm{Maps}((\mathbb{R}_{\geq 0} \times \mathbb{R})_{\mathrm{dR}}, \mathrm{Maps}(S^0_{\mathrm{dR}}, T^*[1]X)). \tag{9.8}$$

This follows from some sort of mapping stack adjunction.

Now, S^0_{dR} is pretty simple, so $\mathrm{Maps}(S^0_{\mathrm{dR}}, T^*[1]X) \simeq T^*[1](X \times X)$. And when we geometrically quantize, we get $\mathrm{QCoh}(X \times X)$, which is indeed $Z(S^0)$.

Now, what would a physicist mean when they say the trivial line? This would mean an inclusion $S^0 \hookrightarrow I$ inducing a map $\mathrm{EOM}(I) \to \mathrm{EOM}(S^0) = T^*[1](X \times X)$. The domain is also $T^*[1]X$, since I is contractible, and what we end up getting is just the diagonal map

$$T^*[1]X \xrightarrow{\Delta} T^*[1](X \times X). \tag{9.9}$$

[6] As Theo pointed out, the green direction is the vertical direction.

You can also identify $T^*[1]X$ with the conormal $N^*_X[1](X\times X)$ using a dash of symplectic geometry.

It's time for another ansatz: given a map $f\colon Y\to X$, the geometric quantization of $N^*_Y[n]X$ should be $f_*n\mathsf{Coh}(Y)$, so e.g. $f_*\mathscr{O}_Y$ for $n=1$, $f_*\mathsf{QCoh}(Y)$ for $n=2$, etc. Therefore applying geometric quantization to the interval should produce $\Delta_*\mathscr{O}_X$, thought of as an integral kernel from the geometric quantization of the point to itself.

Remark 9.10. This is another consistency check: the category of lines is a monoidal category under convolution (induced from a one-dimensional version of the pair-of-pants bordism), and whatever we attach to an interval should be the monoidal unit, and this is indeed what happens. ◀

In Davide Gaiotto's talk today, he discussed compactifying boundary conditions on a circle. Let's make sense of this. As we saw in Dylan Butson's talk, the AKSZ formalism tells us that

$$\mathrm{Maps}(M, N^*_Y[1]X) = N^*_{\mathrm{Maps}(M,Y)}[1]\mathrm{Maps}(M,X) \longrightarrow \mathrm{Maps}(M, T^*[1]X) \tag{9.11}$$

is Lagrangian, and its geometric quantizatin is $f_*\mathscr{O}_{Y\times Y}$.

Example 9.12 (3D $\mathcal{N}=4$ σ-models and the B-twist)**.** Now let's study this in a less trivial example, namely 3D $\mathcal{N}=4$ theories, with the same ansatz to guide us.

The data needed to define a 3D $\mathcal{N}=4$ σ-model is T^*X, which we want to be holomorphic symplectic (if you prefer complex geometry) or hyperKähler (if you like real geometry). Let M be a 3-manifold; the equations of motion of the B-twist on M are[7]

$$\mathrm{EOM}_B(M) = \mathrm{Maps}(M_{\mathrm{dR}}, T^*[2]X), \tag{9.13}$$

though there are some annoying issues with respect to a $\mathbb{Z}$-grading versus a $\mathbb{Z}/2$-grading. Anyways, this should yield a TQFT $Z\colon \mathsf{Bord}_3\to\mathsf{2Cat}$, and we can use this ansatz to obtain a good approximation of what this theory will spit out.

First, what are the local operators? A local operator integrates on the link around a point, which is an S^2. So we should figure out $Z(S^2)$ (well, OK, $S^2\times\mathbb{R}$, but we're not going to worry about that right now). Anyways, $Z(S^2)$ is the geometric quantization of

$$\mathrm{Maps}(S^2_{\mathrm{dR}}, T^*[2]X), \tag{9.14}$$

and it's a fact that, since X is a scheme, this is $T^*T^*[2]X$. Therefore $Z(S^2)=\mathscr{O}_{T^*[2]X}$.

Next, line operators! The link of a line operator is S^1, so we'll do a similar trick as before to conclude that $Z(S^1)$ is the geometric quantization of

$$\mathrm{Maps}(S^1_{\mathrm{dR}}, T^*[2]X) = T^*[1]\mathrm{Maps}(S^1_{\mathrm{dR}}, X) = T^*[1]T^*[2]X. \tag{9.15}$$

Occasionally, as above, there are multiple ways to rewrite this as a cotangent bundle, leading to different but equivalent descriptions of $Z(S^1)$ (here this is about changing the polarization). The first rewriting is quite general, but the latter requires X to be a scheme. In this case, the two answers are $\mathsf{QCoh}(T^*[2]X)$ and $\mathsf{QCoh}(T[-1]X)$, and Ben-Zvi and Nadler showed these two are Koszul dual, hence equivalent.[8] This Koszul duality is a categorified analogue of the Fourier transform exchanging position and momentum.

In these two different ways of writing this category of lines, there should be two ways of writing the trivial line, and they'd better be equivalent. This might tell us some constraints on the ansatz. The first approach, beginning with $S^1\hookrightarrow D^2$, tells us that the trivial line maps to

$$T^*[2]X \to \mathrm{Maps}(S^1_{\mathrm{dR}}, T^*[2]X) = T^*[1]T^*[2]X, \tag{9.16}$$

and in the other polarization, we get

$$N_{\mathrm{Maps}(D^2_{\mathrm{dR}},X)}[1]\mathrm{Maps}(S^1_{\mathrm{dR}}, X) \to T^*[1]\mathrm{Maps}(S^1_{\mathrm{dR}}, X). \tag{9.17}$$

If you pay careful attention to the functional analysis, you'll see that things end up slightly different – the natural polarizations in these two sides disagree, and you'll be off by some factor of the canonical bundle at the end. But you can track this carefully, and recover the correct answer. ◀

[7]There are two versions of this story: this one and one which looks less like the B-model.

[8]There is some functional analysis that's been swept under the rug, but it does work out.

Example 9.18 (3D A-model)**.** Briefly, in the A-twist of a 3D $\mathcal{N} = 4$ σ-model with target X, the theory looks like

$$T^*[-1]\mathrm{Maps}(C \times M^1_{\mathrm{dR}}, X_{\mathrm{dR}}), \tag{9.19}$$

where C is an algebraic curve: the theory is partially holomorphic. Hence, to find the category of line operators, we can't just write down S^2 and go wild: we have to use a different model, as two copies of the disc glued over the punctured disc, which in algebraic geometry is

$$\text{“}S^2\text{”} = \operatorname{Spec}\mathbb{C}[[t]] \cup_{\operatorname{Spec}\mathbb{C}((t))} \operatorname{Spec}\mathbb{C}[[t]]. \tag{9.20}$$

This is what led to the BFN construction, recovering the local operators and the equations of motion. And then, as we've been doing, you can proceed naïvely as we have been. ◀