M392C NOTES: BRIDGELAND STABILITY

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These notes were taken in UT Austin's M392C (Bridgeland Stability) class in Spring 2019, taught by Benjamin Schmidt. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to `a.debray@math.utexas.edu`. Any mistakes in the notes are my own.

CONTENTS

Lecture 1.

Introduction and quiver representations: 1/22/19

This class will be on Bridgeland stability, though we won't get to that topic specifically for about a month. We'll follow lecture notes of Macrì-Schmidt [MS17], which are on the arXiv.

If you're pre-candidacy, make sure to do at least two exercises in this class, at least one from March or later; otherwise just make sure to show up. (If you're an undergrad who's signed up for this class, please do at least four exercises, at least two from March or later.)

Now let us enter the world of mathematics. We'll begin with two well-known theorems in algebraic geometry; we'll eventually be able to prove these using stability conditions.

Theorem 1.1 (Kodaira vanishing)**.** *Let X be a smooth projective complex variety and L be an ample line bundle. Then for all $i > 0$, $H^i(X; L \otimes \omega_X) = 0$.*

We'll eventually give an approach in the setting where $\dim X \leq 2$. It won't be very hard once the setup is in place. In fact, there are probably plenty of other vanishing theorems one could prove using stability conditions, including some which aren't known yet.

The other theorem is over a century ago, from the Italian school of algebraic geometry.

Theorem 1.2 (Castelnuovo)**.** *Working over an algebraically closed field, let $C \subset \mathbb{P}^3$ be a smooth curve not contained in a plane. Then $g \leq d^2/4 - d + 1$, where g is genus of C and d is its degree.*

Another goal we'll work towards:

Problem 1.3. Explicitly describe some moduli spaces of vector bundles or sheaves.

Here's a concrete outline of the course.

(1) Before we discuss any algebraic geometry, we'll study quiver theory, focusing on moduli spaces of quiver conditions. We don't need stability conditions to do this, but these spaces make great simple examples of the general story.
(2) Next, we'll study vector bundles on curves. Bridgeland stability is a generalization of what we can say here for higher dimensions.
(3) A crash course on derived categories and Bridgeland stability. This is pretty formal.
(4) A crash course on intersection theory, which will be necessary for what comes later.
(5) Surfaces.
(6) Threefolds (if we have time).

These are all mostly independent pieces, only coming together in the end, so if you get lost somewhere there's no need to panic; you'll probably be able to pick the course back up soon enough.

And now for the moduli of quiver representations. For this stuff, we'll follow King [Kin94], which is accessible and nice to read. Let k be an algebraically closed field.

Definition 1.4. A *quiver* is the representation theorist's word for a finite directed graph. Explicitly, a quiver Q consists of two finite sets Q_0 and Q_1 of vertices and edges, respectively, together with *tail* and *head* maps $t, h\colon Q_1 \to Q_0$.

Example 1.5. The *Kronecker quiver* is

$$\bullet \rightrightarrows \bullet.$$

The quiver of *type* D_4 is

$$\begin{tikzcd} & & \bullet \arrow[dl] \\ \bullet & \bullet \arrow[l] & \\ & & \bullet. \arrow[ul] \end{tikzcd}$$

We can also consider a quiver with a single vertex v and a single edge $e\colon v \to v$. $\blacktriangleleft$

Definition 1.6. A *representation* W of a quiver Q is a collection of k-vector spaces W_v for each $v \in Q_0$ and linear maps $\phi_e\colon W_{v_1} \to W_{v_2}$ for each edge $e\colon v_1 \to v_2$ in Q_1. The vector $(\dim W_v)_{v\in Q_0}$ inside $\mathbb{C}[Q_0]$ is called the *dimension* of W.

Example 1.7. First, some trivial example. For example, here's a representation of the Krokecker quiver: $(\cdot 1, \cdot 2)\colon k \rightrightarrows k$. A representation of the quiver with one vertex and one edge is a vector space with an endomorphism, e.g. $\mathbb{C}^2$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. $\blacktriangleleft$

Definition 1.8. Let Q be a quiver. A *morphism* of Q-representations $f\colon (W_v, \phi_e) \to (U_v, \psi_e)$ is a collection of linear maps $f_v\colon W_v \to U_v$ for each $v \in Q_0$ such that for all edges e,

$$f_{h(e)} \circ \phi_e = \psi_e \circ f_{t(e)}.$$

If all of these linear maps are isomorphisms, f is called an *isomorphism*.

That is, data of a quiver representation includes a bunch of linear maps, and we want a morphism of quiver representations to commute with these maps.

Representations theorists want to classify quiver representations. This is really hard, so let's specialize to irreducible representations (those not a direct sum of two other ones). This is still really hard! There are classical theorems originating from the French school proving that most quivers do not admit nice classifications of their irreducible representations: some have finitely many, and some have infinitely many but nice parameterizations, and these are uncommon.

One way to make headway on these kinds of problems is to consider a moduli space of quiver representations, which may be more tractable to study.

Problem 1.9. Can you classify the (isomorphism classes of) quiver representations of the quiver with a single vertex and single edge?

Our first, naïve approach to constructing the moduli of quiver representations is to fix a dimension vector $\alpha \in \mathbb{C}[Q_0]$ and define

$$R(Q,\alpha) := \bigoplus_{e\in Q_1} \mathrm{Hom}(W_{t(e)}, W_{h(e)}). \tag{1.10}$$

This is too big: the same isomorphism class appears at more than one point. We can mod out by a symmetry: let

$$\mathrm{GL}(\alpha) := \prod_{v\in Q_0} \mathrm{GL}(W_v) \tag{1.11}$$

act on $R(Q,\alpha)$ by a change of basis on each vector space and on ϕ_e as

$$(g\phi)_e = g_{h(e)}\phi_e g_{t(e)}^{-1}. \tag{1.12}$$

Then as a set the quotient $R(Q,\alpha)/\mathrm{GL}(\alpha)$ contains one element for each isomorphism class. But putting a geometric structure on quotients of varieties is tricky. We'll come back to this point.

Example 1.13. Let Q be the Kronecker quiver and $\alpha = (1,1)$, so that $\mathrm{GL}(\alpha) = k^\times \times k^\times$. Pick $(t,s) \in \mathrm{GL}(\alpha)$; the action on a Q-representation $(\lambda,\mu)\colon k \rightrightarrows k$ produces $(s\lambda t^{-1}, s\mu t^{-1})\colon k \rightrightarrows k$. So if $s = t$, the action is trivial. Quotienting out by the diagonal $s = t$ in $k^\times \times k^\times$, we get $k^\times$: $(s,t) \mapsto s/t$, and this acts on $R(Q,\alpha) = k^2$ by scalar multiplication.

This is an action we know well: the quotient is the space of lines in k^2, also known as $\mathbb{P}^1_k$ – and the zero orbit. This orbit makes life more of a headache: you can't just throw it out, because then you don't get a good map to the quotient, preimages of closed things aren't always closed, etc. But the action on the zero orbit is not free. This phenomenon will appear a lot, and we'll in general have to think about what to remove. After some hard work we'll be able to take the quotient in a reasonable way and get $\mathbb{P}^1$. $\blacktriangleleft$

A crash course on (linear) algebraic groups. If you want to learn more about algebraic groups, especially because we're not going to give proofs, there are several books called *Linear Algebraic Groups*: the professor recommends Humphreys' book [Hum75] with that title, and also those of Borel [Bor91] and Springer [Spr98].

Definition 1.14. An *algebraic group* is a variety G together with a group structure such that multiplication and taking inverses are morphisms of varieties.

You can guess what a morphism of algebraic groups is: a group homomorphism that's also a map of varieties.

Example 1.15. GL_n is an algebraic group. Inside the space of all $n \times n$ matrices, which is a vector space over k, GL_n is the set of matrices with nonzero determinant. This is an open condition, and the determinant can be written in terms of polynomials, so GL_n is an algebraic group.

Other examples include SL_n and elliptic curves, and we can take products, so $\mathrm{GL}(\alpha)$ is also an algebraic group. $\blacktriangleleft$

Definition 1.16. A *linear algebraic group* is an algebraic group that admits a closed embedding $G \hookrightarrow \mathrm{GL}_n$ which is also a group homomorphism.

This does not include the data of the embedding. It turns out (this is in, e.g. Humphreys) that any affine algebraic group is linear, but this is not particularly easy to show.

Exercise 1.17. Show that any algebraic group is also a smooth variety.

This does not generalize to group schemes!

We care about groups because they act. We added structure to algebraic groups, and thus care about actions which behave nicely under that structure.

Definition 1.18. A *group action* of an algebraic group G on a variety X is a morphism $\varphi\colon G \times X \to X$ such that for all $g, h \in G$ and $x \in X$,

(1) $\varphi(g, \varphi(h,x)) = \varphi(gh, x)$, and
(2) $\varphi(e, x) = x$.

Example 1.19. $k^\times$ acts on $\mathbb{A}_k^{n+1}$ by scalar multiplication. What's the quotient? We want $\mathbb{P}_k^n$, but there's also the zero orbit, and no other orbit is closed. This makes us sad; we're going to use geometric invariant theory (GIT) to address these issues and become less sad. ◀

Definition 1.20. Let G be an algebraic group.

- A *character* of G is a morphism of algebraic groups $\chi\colon G \to k^\times$. These form a group under pointwise multiplication, and we'll denote this group $X(G)$.
- A *one-parameter subgroup* of G, also called a *cocharacter*, is a morphism of algebraic groups $\lambda\colon k^\times \to G$.

Example 1.21. Since $\det(AB) = \det A \det B$, the determinant defines a character of GL_n. One example of a cocharacter is $\lambda\colon k^\times \to \mathrm{GL}_2$ sending $t \mapsto \left(\begin{smallmatrix} t & 0 \\ 0 & t^2 \end{smallmatrix}\right)$. This cocharacter factors through the diagonal matrices in GL_n; this turns out to be a general fact. ◀

Here are a few nice facts about characters and cocharacters.

Theorem 1.22.

(1) *The map $\mathbb{Z} \to X(\mathrm{GL}_n)$ sending $m \mapsto \det^m$ is an isomorphism.*
(2) *If G and H are algebraic groups, the map $X(G) \times X(H) \to X(G \times H)$ sending*

$$(\chi_1, \chi_2) \longmapsto ((g,h) \longmapsto \chi_1(g)\chi_2(h)) \tag{1.23}$$

is an isomorphism.
(3) *Up to conjugation, every cocharacter of GL_n lands in the subgroup of diagonal matrices, hence sends $t \mapsto \mathrm{diag}(t^{a_1}, \dots, t^{a_n})$ for $a_1, \dots, a_n \in \mathbb{Z}$.*

We're not going to prove these: this would require a considerable detour into the theory of algebraic groups to get to, and you can read the proofs in Humphreys.

Exercise 1.24. Without using the above theorem, show that any morphism of algebraic groups $k^\times \to k^\times$ is of the form $t \mapsto t^n$ for some $n \in \mathbb{Z}$.

Lecture 2.

Geometric invariant theory: 1/24/19

Today we'll discuss some more geometric invariant theory and how to take quotients.

Nagata reinterpreted Hilbert's 14th problem as follows.

Problem 2.1. Let G be a linear algebraic group acting linearly on a finite-dimensional k-vector space V. Is the *ring of invariants*

$$\mathscr{O}(V)^G = \{f \in \mathscr{O}(V) \mid f(gx) = f(x) \text{ for all } g \in G, x \in V\}$$

finitely generated?

The elements of $\mathscr{O}(V)^G$ are called the *invariant polynomials* or *invariant functions* on V.

Nagata proved that this is not always true, though there is a positive answer with some assumptions on G. For example, GL_n and products of general linear groups satisfy this property.

Definition 2.2. A linear algebraic group G is *geometrically reductive* if for every linear action of G on a finite-dimensional vector space V (i.e. a map of algebraic groups $\varphi\colon G \to \mathrm{GL}(V)$) and every fixed point $v \in V$ of the G-action, there is an invariant homogeneous nonconstant polynomial f with $f(v) = 0$.

Remark 2.3. There is a different notion of a reductive group, and it is different. Sorry about that. ◀

Theorem 2.4 (Nagata [Nag63])**.** *If G is geometrically reductive, Problem 2.1 has a positive answer.*

If $\mathrm{char}(k) = 0$, basic facts from the theory of algebraic groups allow one to prove GL_n is geometrically reductive, and in fact in characteristic zero reductive implies geometrically reductive. This is also true in positive characteristic, but is significantly harder!

Remark 2.5. In fact, in characteristic zero, the polynomial f in the definition of geometrically reductive can be chosen such that $\deg(f) = 1$. This property is called *linearly reductive*, so in characteristic zero, reductive, geometrically reductive, and linearly reductive coincide. This is not true in positive characteristic, which is ultimately because of everyone's favorite fact about modular representation theory: representations of a group in positive characteristic need not be semisimple. ◀

Mumford conjectured the following.

Theorem 2.6 (Haboush [Hab75])**.** *If k is algebraically closed and G is reductive, then G is geometrically reductive.*

The difficulty was in positive characteristic.

This led to the first idea of a better quotient: take Spec of the ring of invariants; by this theorem, this gives you a variety. But sometimes this is too small: for $\mathbb{C}^\times$ acting on $\mathbb{C}^n$, this tells you the closed orbits. The only closed orbit is the zero orbit, so we don't get $\mathbb{P}^{n-1}$, alas.

To abrogate this, we'll introduce a numerical criterion. Let G be a geometrically reductive group acting linearly on a finite-dimensional vector space V. Recall that $\mathscr{O}(V)$ is also denoted $k[V]$, the ring of polynomials on V.

Definition 2.7. Let $\chi \in X(G)$ be a character of G.

(1) An $f \in \mathscr{O}(V)$ is *relatively invariant* of *weight* χ if $f(gx) = \chi(g)f(x)$ for all $x \in V$ and $g \in G$. We let $\mathscr{O}(V)^{G,\chi}$ denote the vector space of relatively invariant functions of weight χ, so that $\mathscr{O}(V)^{G,\chi^0} = \mathscr{O}(V)^G$.

(2) Define

$$V /\!/ (G,\chi) := \operatorname{Proj}\left(\bigoplus_{n\ge 0} \mathscr{O}(V)^{G,\chi^n}\right). \tag{2.8}$$

We let $V /\!/ G := \operatorname{Spec}(\mathscr{O}(V)^G)$.

One can check quickly that the product of relatively invariant functions of weights χ^m and χ^n is relatively invariant of weight χ^{m+n}, so the graded abelian group in (2.8) is in fact a graded ring.

Theorem 2.9. *There's a natural map $V /\!/ (G,\chi) \to V /\!/ G$, and this map is projective.*

Example 2.10. Consider $k^\times$ acting on k^{m+1} by scalar multiplication and $\chi\colon k^\times \to k^\times$. Then $k[x_0,\dots,x_m]^{k^*,\chi^n}$ is exactly the vector space of degree-n homogeneous polynomials. Then

$$k^{m+1} /\!/ (k^\times, \mathrm{id}) = \operatorname{Proj}(k[x_0,\dots,x_m]) = \mathbb{P}^m, \tag{2.11}$$

where we give $k[x_0,\dots,x_m]$ its usual grading.

However, if you use other characters, you'll get something different: for $\chi = 1$ you get a single point, and for $\chi = -\mathrm{id}$ the quotient is empty. ◀

We've been calling $V /\!/ (G,\chi)$ a "quotient," but is it really one? We'd like to say it has nice properties that a quotient should have, but in the above example, there isn't a nice map $k^{m+1} \twoheadrightarrow \mathbb{P}^m$. In general we get a nice map like that on an open subset; let's figure out what map that is.

Let Δ be the kernel of $\varphi\colon G \to \mathrm{GL}(V)$.[1]

Definition 2.12.

(1) An $x \in V$ is called *χ-semistable* for a character χ if there is an $f \in \mathscr{O}(V)^{G,\chi^n}$ for some $n \ge 1$ such that $f(x) \ne 0$. The locus of χ-semistable points is denoted V_χ^{ss}.

(2) If x is χ-semistable and we can choose f such that the orbit $G\cdot x \subset \{x \in V \mid f(x) \ne 0\}$ is closed, and $\dim G\cdot x = \dim G - \dim \Delta$, we call x *χ-stable*. The locus of χ-stable points is denoted V_χ^s.

Stability means that the orbit of x has the largest possible dimension.

Lemma 2.13. *V_χ^{ss} and V_χ^s are Zariski open subsets of V.*

The main theorem of geometric invariant theory in this setting[2] is:

[1] In many references, φ is assumed to be injective.

[2] Mumford showed a version where G can act on any quasiprojective variety.

Theorem 2.14 (Mumford)**.** *There is a surjective morphism $\phi\colon V_\chi^{ss} \to V /\!/ (G,\chi)$ such that if $x,y \in V_\chi^{ss}$,*

(1) if $x,y \in V_\chi^s$, then $\phi(x) = \phi(y)$ iff $y \in G\cdot x$, and
(2) in general, $\phi(x) = \phi(y)$ iff $\overline{G\cdot x} \cap \overline{G\cdot y}$ is nonempty, where closures are taken inside V_χ^{ss}.

You can think of ϕ as the map from the original space to the quotient, but we can only see a subset of the original space. For stable points, this actually parameterizes orbits, but this isn't quite true for merely semistable points, and the problem occurs when orbits aren't closed.

Definition 2.15. If $\overline{G\cdot x} \cap \overline{G\cdot y}$ is nonempty, we say x and y are *S-equivalent*.

Remark 2.16. S is for Seshadri, who was one of the developers of this theory. ◀

The numerical criterion we alluded to earlier is a way to find semistable points.

Definition 2.17. Let $\chi\colon G \to k^\times$ be a character and $\lambda\colon k^\times \to G$ be a cocharacter. The composition $\chi\circ\lambda\colon k^\times \to k^\times$ sends $t \mapsto t^n$ for some $n \in \mathbb{Z}$; we denote $\langle \chi, \lambda\rangle := n$.

Theorem 2.18 (Mumford's numerical criterion)**.**

(1) An $x \in V$ is χ-semistable iff $\chi(\Delta) = 1$ and for all cocharacters $\lambda\colon k^\times \to G$ such that $\lim_{t\to 0}\lambda(t)x$ exists, then $\langle\chi,\lambda\rangle \ge 0$.
(2) x is χ-stable iff it's χ-semistable and if λ is as above and $\langle\chi,\lambda\rangle > 0$, then $\lambda(k^\times) \subset \Delta$.

That limit works fine in $\mathbb{C}$, but what about over other fields? It's obvious in formulas, and in general you can define it in terms of trying to extend to a map of varieties $k \to G$.

Proposition 2.19.

(1) The orbit $G\cdot x$ is closed in V_χ^{ss} if for every cocharacter λ with $\langle\chi,\lambda\rangle = 0$ such that the limit $\lim_{t\to 0}\lambda(t)x$ exists, then the limit is in $G\cdot x$.
(2) If $x,y \in V_\chi^{ss}$, then x and y are S-equivalent iff there are cocharacters λ_1, λ_2 with $\langle\chi,\lambda_1\rangle = \langle\chi,\lambda_2\rangle = 0$ such that $\lim_{t\to 0}\lambda_1(t)x$ and $\lambda_{t\to 0}\lambda_2(t)y$ both exist and are in the same orbit.

Example 2.20. Consider $G = \mathrm{GL}_2$ acting on the space V of 4×2 matrices: to obtain a left action by g, we multiply on the right by g^{-1}. Let $\chi\colon \mathrm{GL}_2 \to k^\times$ be $\det^{-1}$.

What do we expect to parameterize in the quotient? A 4×2 matrix is a linear map $k^2 \to k^4$, and we're parameterizing them up to change of basis of the domain. This should morally parameterize two-dimensional subspaces of k^4, though we never stipulated that our maps are injective. Maybe, hopefully, the open subset of semistable points are the injective maps and we'll get the Grassmannian $\mathrm{Gr}_2(k^4)$.

We claim this is actually the case, and will use the numerical criterion to prove it. Since GL_2 acts faithfully on V, $\Delta = 1$ and the situation simplifies somewhat. We can use a group action to make the cocharacter simpler, or to make a general element of V simpler, but not both. So we'll do the former: let $\lambda = \left(\begin{smallmatrix} t^n & 0 \\ 0 & t^m\end{smallmatrix}\right)$ and

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42}\end{pmatrix}, \quad \text{so that} \quad \lambda(t)A = \begin{pmatrix} a_{11}t^{-n} & a_{12}t^{-m} \\ a_{21}t^{-n} & a_{22}t^{-m} \\ a_{31}t^{-n} & a_{32}t^{-m} \\ a_{41}t^{-n} & a_{42}t^{-m}\end{pmatrix}. \tag{2.21}$$

Since $\det^{-1}(\lambda(t)) = t^{-n-m}$, $\langle \det^{-1}, \lambda\rangle = -n-m$. Therefore $\lim_{t\to 0}\lambda(t)$ exists iff either

(1) $A = 0$, which isn't semistable, because the limit exists for every cocharacter; or
(2) $a_{11} = \cdots = a_{41} = 0$ and $a_{j2} \ne 0$ for some j, and $m \le 0$, which is also unstable (e.g. $m = 0$, $n = 1$); or
(3) $a_{2i} = 0$ for all i, and $a_{j1} \ne 0$ for some j, and $n \le 0$, which is again unstable; or
(4) $a_{i1} \ne 0$ for some i and $a_{j2} \ne 0$ for some j, and $m,n \le 0$, so $\langle\chi,\lambda\rangle = -n-m \ge 0$, and these A are stable.

Now, let's look at an arbitrary cocharacter. This involves changing basis/looking at full orbits of points we found were unstable. When $A = 0$ (case (1)), this is the whole orbit, and it's unstable. For (2) and (3), A has rank 1 in the entire orbit, and therefore these are all unstable. All matrices of rank 2 are stable. ◀

Lecture 3.

Constructing moduli spaces of quiver representations: 1/29/19

Today, we're going to leverage the GIT theory we surveyed in the last lecture to define moduli spaces of quiver representations.

We begin with a quick review of Mumford's numerical criterion, since it will be an important actor today. Let G be an algebraic group acting on a k-vector space V, and let Δ denote the kernel of the associated map $\rho\colon G \to \mathrm{GL}(V)$. Let $\chi\colon G \to k^\times$ be a character. Then, in Theorem 2.18, we saw that $x \in V$ is χ-semistable iff $\chi(\Delta) = 1$ and for all cocharacters $\lambda\colon k^\times \to G$ such that $\lim_{t\to 0} \lambda(t)\cdot x$ exists, $\langle \chi, \lambda\rangle = 0$. Moreover, x is χ-stable if in addition whenever $\langle \chi, \lambda\rangle = 0$, then $\lambda(k^\times) \subset \Delta$.

Now back to quivers. Consider a quiver Q with a set Q_0 of vertices, Q_1 of edges, and head and tail maps $h, t\colon Q_1 \rightrightarrows Q_0$. Let $\alpha \in \mathbb{C}[Q_0]$ be a dimension vector and vector spaces W_v of dimension $\alpha(v)$ for each $v \in Q_0$. We constructed the space

$$R(Q,\alpha) := \prod_{e\in Q_1} \mathrm{Hom}(W_{t(e)}, W_{h(e)}), \tag{3.1}$$

but this is too big to be a moduli space of quiver representation: it contains different points that correspond to isomorphic representations. Therefore we want to take the quotient by $\mathrm{GL}(\alpha)$ as we described, so let's apply GIT to this action and understand stability.

In this setting, the kernel Δ is the *long diagonal* $\{(t\mathrm{id}, \dots, t\mathrm{id}) \in \mathrm{GL}(\alpha) \mid t \in k^\times\}$, which is isomorphic to $k^\times$. If $\theta \in \mathbb{Z}[Q_0]$, it defines a character $\chi\colon \mathrm{GL}(\alpha) \to k^\times$ by

$$\chi_\theta(g) := \prod_{v\in Q_0} g_v^{\theta(v)}. \tag{3.2}$$

All characters of GL_n can be written in this way.

Definition 3.3. Let A be an abelian category. Its *Grothendieck group* is

$$K_0(\mathsf{A}) := \bigoplus_{A\in\mathsf{A}} \mathbb{Z}[A]/\sim, \tag{3.4}$$

where we quotient by an equivalence relation: for all short exact sequences $0 \to A \to B \to C \to 0$, we say $[B] \sim [A] + [C]$.

The $\theta \in \mathbb{Z}[Q_0]$ as above can also be thought of as a function on the Grothendieck group of Rep_Q, the category of finite-dimensional representations of Q. Specifically, $\theta\colon K_0(\mathsf{Rep}_Q) \to \mathbb{Z}$ is defined to send

$$M = (\{M_v\}, \{\psi_e\}) \longmapsto \sum_{v\in Q_0} \theta(v)\cdot \dim(M_v). \tag{3.5}$$

We want there to be semistable points, which means we will only consider θ such that

$$\chi_\theta(\Delta) = \left\{ \prod_{v\in Q_0} t^{\theta(v)\alpha(v)} \mid t \in k^\times \right\} = \{1\}, \tag{3.6}$$

i.e. such that

$$\sum_{v\in Q_0} \theta(v)\alpha(v) = 0. \tag{3.7}$$

Among other things, this means that $\theta(M) = 0$ if $\dim M = \alpha$.

Now we'll perform the GIT analysis. Let $\lambda\colon k^\times \to \mathrm{GL}(\alpha)$ be a cocharacter and $M = (\{M_v\}, \{\phi_e\}) \in \mathsf{Rep}_Q$ have dimension α. The first step will be to construct a descending $\mathbb{Z}$-indexed filtration

$$M \supseteq \cdots \supseteq M^{(n)} \supseteq M^{(n+1)} \supseteq \cdots. \tag{3.8}$$

For each $v \in Q_0$, pick a decomposition

$$M_v = \bigoplus_{n\in\mathbb{Z}} M_v^{(n)}, \tag{3.9}$$

such that $\lambda(t)$ acts on $M_v^{(n)}$ by multiplication by t^n. (Recall that any cocharacter of GL_n has diagonal image up to conjugation, so this makes sense.) Define

$$M_v^{(\geq n)} := \bigoplus_{m\geq n} W_v^{(m)}. \tag{3.10}$$

For each $e \in Q_1$, let $\phi_e^{(m,n)}$ denote the composition

$$M_{t(e)}^{(n)} \hookrightarrow M_{t(e)} \xrightarrow{\phi_e} M_{h(e)} \twoheadrightarrow M_{h(e)}^{(m)}, \tag{3.11}$$

where the projection comes from the decomposition (3.9). Then

$$\lambda(t)\cdot\phi_e^{(m,n)} = t^m\phi_e^{(m,n)}t^{-n} = t^{m-n}\phi_e^{(m,n)}, \tag{3.12}$$

which means the following are equivalent:

- $\lim_{t\to 0}\lambda(t)\phi_e$ exists,
- $\phi_e^{(m,n)} = 0$ whenever $m \leq n$, and
- ϕ_e maps $M_{t(e)}^{(\geq n)}$ into $M_{h(e)}^{(\geq n)}$.

The third condition means that $M_n := (M^{(\geq n)}, \phi_e|_{M^{(\geq n)}})$ is a subrepresentation of M, so if the limit exists it induces the desired filtration of M (3.8). In this case $W_v^{(n)} = W_v^{(\geq n)}/W_v^{(\geq(n+1))}$.

Conversely, given a filtration of M as in (3.8), we can produce a cocharacter $\lambda\colon k^\times \to \mathrm{GL}(\alpha)$: define $\lambda(t)$ to act by t^n on $W_v^{(n)}$.

Since

$$\lim_{t\to 0}\lambda(t)\phi_e^{(m,n)} = \lim_{t\to 0} t^{m-n}\phi_e^{(m,n)} = 0 \tag{3.13}$$

for $m > n$, then

$$\lim_{t\to 0}\lambda(t)\phi_e\colon M_{t(e)}^{(n)} \longrightarrow M_{h(e)}^{(n)} \subset M_{h(e)}^{(\geq n)}. \tag{3.14}$$

Thus the limit is the associated graded:

$$\lim_{t\to 0}\lambda(t)\cdot M = \bigoplus_{n\in\mathbb{Z}} M_n/M_{n+1}. \tag{3.15}$$

Remark 3.16. There's an n such that $M_n = 0$ and $M_n \neq M$. In other words, this filtration isn't the trivial one. This is because $\lambda(k^\times) \subset \Delta$. ◄

Now let's discuss (semi)stability. It will turn out to be equivalent to the following notion.

Definition 3.17. Let $M \in \mathsf{Rep}_Q$ have dimension α and be such that $\theta(M) = 0$. Then M is *θ-semistable* (resp. *θ-stable*) iff for all nonzero proper subrepresentations $N \subset M$, $\theta(N) \geq 0$ (resp. $\theta(N) > 0$).

Theorem 3.18 (King [Kin94]). *A point $M \in R(Q,\alpha)$ is χ_θ-semistable (resp. χ_θ-stable) iff $M \in \mathsf{Rep}_Q$ is θ-semistable (resp. θ-stable).*

Proof. First, let's show θ-(semi)stability implies GIT (semi)stability. We assumed $\sum_v \theta(v)\alpha(v) = 0$, which implies

$$\langle\chi_\theta,\lambda\rangle = \sum_{v\in Q_0}\theta(v)\sum_{n\in\mathbb{Z}} n\dim M_v^{(n)}. \tag{3.19}$$

We can change the order of summation because only finitely many $W_v^{(n)}$ are nonzero for v fixed, so

$$\langle\chi_\theta,\lambda\rangle = \sum_{n\in\mathbb{Z}} n\sum_{v\in Q_0}\theta(v)\dim M_v^{(n)} \tag{3.20a}$$

$$= \sum_{n\in\mathbb{Z}} n\theta(M_n/M_{n+1}). \tag{3.20b}$$

Since θ factors through the Grothendieck group, this is

$$= \sum_{n\in\mathbb{Z}} n(\theta(M_n) - \theta(M_{n-1})) \tag{3.20c}$$

$$= \sum_{n\in\mathbb{Z}} \theta(M_n), \tag{3.20d}$$

unwinding the telescoping series. This is nonnegative if M is θ-semistable and positive if M is θ-stable, by definition.

Conversely, suppose M is χ_θ-semistable and let N be a subrepresentation of M. Then $N \subset M$ is a filtration, hence defines a cocharacter $\lambda\colon k^\times \to \mathrm{GL}(\alpha)$ such that

$$0 \overset{(<)}{\leq} \langle \chi_\theta, \lambda\rangle = \theta(M) + \theta(N) \tag{3.21}$$

(parentheses for stability), so $\theta(N) \geq -\theta(M) = 0$ (or $>$ for stability). ⊠

Now we have semistable points, and even strictly semistable points. What does S-equivalence look like in this context?

Definition 3.22. Given θ as above, let $\mathsf{P}_\theta \subset \mathsf{Rep}_Q$ denote the full subcategory of θ-semistable representations with $\theta(M) = 0$ (so, those objects, and all of the morphisms between them).

Lemma 3.23. P_θ *is an abelian subcategory of* Rep_Q*. That is, let* $\varphi\colon M \to N$ *be a morphism in* P_θ*, i.e.* $\theta(M) = \theta(N) = 0$ *and* M *and* N *are* θ*-semistable. Then,* $A := \ker(\varphi)$ *and* $B = \operatorname{coker}(\varphi)$*, where the kernel and cokernel are taken in* Rep_Q*, are in* P_θ*.*

Proof. The kernel and cokernel fit into an exact sequence

$$0 \longrightarrow A \longrightarrow M \longrightarrow N \longrightarrow B \longrightarrow 0. \tag{3.24}$$

Since θ is additive under short exact sequences, it in fact satisfies

$$\theta(A) + \underbrace{\theta(N)}_{=0} = \underbrace{\theta(M)}_{=0} + \theta(B), \tag{3.25}$$

so $\theta(A) = \theta(B)$. Since M is θ-semistable, $\theta(A) \geq 0$. If $K := \ker(N \to B) \hookrightarrow N$, and N is θ-semistable, then $\theta(K) \geq 0$. Therefore

$$\theta(A) = \theta(B) = \theta(N) - \theta(K) = -\theta(K) \leq 0. \tag{3.26}$$

Thus $\theta(A) = \theta(B) = 0$.

Now for semistability. Briefly, if $C \subset A$ is a subrepresentation, then it's also a subrepresentation of M, so $\theta(C) \geq 0$. The argument for B is similar. ⊠

Lecture 4.

Examples of quiver varieties: 1/31/19

Fix a quiver Q. Last time we explained how, given a $\theta \in \mathbb{Z}[Q_0]$, we obtain a function on objects of Rep_Q additive on short exact sequences: $\theta(M) := \sum_{v\in Q_0} \theta_v \cdot \dim(M_v)$, and we also get a character χ_θ of $\mathrm{GL}(\alpha)$, which has weight $\theta(v)$ on the component indexed by v. In Theorem 3.18, we provided a criterion for semistability: a quiver representation M is χ_θ-semistable iff $\theta(M) = 0$ and for all $N \subsetneq M$, $\theta(N) \geq 0$. (If $\theta(N) > 0$, M is χ_θ-stable.)

We then constructed an abelian category P_θ of χ_θ-semistable objects.

Proposition 4.1. P_θ *is a* finite-length category*, i.e. all of its objects are both Noetherian and Artinian.*

Theorem 4.2 (Jordan-Hölder)**.** *Let* A *be a finite-length abelian category. Then any object* $M \in \mathsf{A}$ *has a filtration* $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ *such that* M_i/M_{i-1} *is simple for all* i*. Moreover, the associated graded*

$$\mathrm{gr}(M) := \bigoplus_i M_i/M_{i-1} \tag{4.3}$$

is unique up to isomorphism.

This filtration satisfies a weak uniqueness condition – it's not unique, but is close to it. For example, if A and B are two simple objects, $A \oplus B$ has two filtrations $0 \to A \to A \oplus B$ and $0 \to B \to A \oplus B$. Such a filtration is called a *Jordan-Hölder filtration* of M.

Exercise 4.4. In P_θ, M is simple iff it's θ-stable.

Proposition 4.5.
(1) A θ-semistable Q-representation M corresponds to a closed $\mathrm{GL}(\alpha)$ orbit in $R^{ss}_{\chi_\theta}(Q,\alpha)$ iff M is semisimple (i.e. a direct sum of simple objects). Equivalently, $M \cong \mathrm{gr}(M)$.
(2) Two θ-semistable Q-representations M and N are S-equivalent iff $\mathrm{gr}(M) \cong \mathrm{gr}(N)$.

The proof was left as an exercise, but is not too difficult given how we proved things last class (and is a good way to see if you understood the proof).

We will let $M_\theta(Q,\alpha)$ denote the GIT quotient by $\mathrm{GL}(\alpha)$ for the character χ_θ.

Example 4.6. Consider the quiver

$$Q = \bullet \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow}\; \bullet. \tag{4.7}$$

Let's choose $\alpha = (2,1)$, so $R(Q,\alpha) = \mathrm{Hom}(k^2,k)^{\oplus 4}$. Then $\mathrm{GL}(\alpha) = \mathrm{GL}_2 \times k^\times$. If you choose $\theta = (-1,1)$, then $\chi_\theta = \det^{-1}$.

The long diagonal of $\mathrm{GL}(\alpha)$ acts trivially, so let's pass to the quotient via the map $\varphi\colon \mathrm{GL}(\alpha) \to \mathrm{GL}_2$ by $(g,t) \mapsto gt^{-1}$ (here, $g \in \mathrm{GL}_2$ and $t \in k^\times$). Now we have the same scenario as in Example 2.20 – so we leverage our work there and conclude the quiver moduli space is $\mathrm{Gr}_2(k^4)$. ◀

Example 4.8. Now let's consider a slightly more interesting quiver,

$$Q = \bullet \rightrightarrows \bullet \leftleftarrows \bullet. \tag{4.9}$$

Let $\alpha = (1,1,1)$, so $R(Q,\alpha) = \mathrm{Hom}(k,k)^2 \times \mathrm{Hom}(k,k)^2$. Choose the character $\theta = (a,b,c) = (a,-a-c,c)$, where $a+b+c = 0$.

Because α is small, there aren't many subrepresentations. For example, the trivial representation $k \rightrightarrows k \leftleftarrows k$ has as a subrepresentation $S_2 := (0 \rightrightarrows k \leftleftarrows 0)$. Since $\theta(S_2) = -a-c$, we must have $-a-c \ge 0$ or all representations are unstable.

By defining more representations, we can infer more about what constraints to put on θ to have a good moduli space. For example, given $x,y,z,w \in k$, let

$$N_1(x,y) := k \overset{x}{\underset{y}{\rightrightarrows}} k \leftleftarrows 0 \tag{4.10a}$$

$$N_2(z,w) := 0 \rightrightarrows k \overset{z}{\underset{w}{\leftleftarrows}} k \tag{4.10b}$$

$$M(x,y,z,w) := k \overset{x}{\underset{y}{\rightrightarrows}} k \overset{z}{\underset{w}{\leftleftarrows}} k. \tag{4.10c}$$

Thus $N_1(x,y)$ and $N_2(z,w)$ are both suprepresentations of $M(x,y,z,w)$. Because

$$\theta(N_1(x,y)) = a + (-a-c) = -c \ge 0 \tag{4.11a}$$

$$\theta(N_2(x,y)) = -a \ge 0, \tag{4.11b}$$

we know both a and c must be negative.

There are a few more potential subrepresentations,

$$S_1 := \; k \rightrightarrows 0 \leftleftarrows 0 \tag{4.12a}$$

$$S_2 := \; 0 \rightrightarrows 0 \leftleftarrows k \tag{4.12b}$$

$$N_3 := S_1 \oplus S_2. \tag{4.12c}$$

Using these, we observe that

- $S_1 \hookrightarrow M(x,y,z,w)$ iff $x = y = 0$, and in this case, we need $\theta(S_1) = a \ge 0$, so $a = 0$;
- $S_2 \hookrightarrow M(x,y,z,w)$ iff $z = w = 0$, and in this case, we need $\theta(S_3) = c \ge 0$, so $c = 0$; and
- $N_3 \hookrightarrow M(x,y,z,w)$ iff $x = y = z = w = 0$, and in this case we need $a = c = 0$.

In summary:

(1) If $a < 0$ and $c < 0$, then $M(x, y, z, w)$ is θ-stable iff $(x, y) \neq 0$ and $(z, w) \neq 0$. Moreover, there are no *strictly semistable* (i.e. semistable but not stable) representations. At this point you might guess that the GIT quotient is $\mathbb{P}^1 \times \mathbb{P}^1$.

(2) If $a = 0$ and $c < 0$, then $M(x, y, z, w)$ is θ-semistable iff $(z, w) \neq 0$, and there are no θ-stable representations. Now suppose $M(x, y, z, w)$ is such that $(z, w) \neq 0$; then, $M(x, y, z, w)$ and $M(x', y', z', w')$ are S-equivalent iff $(z', w') = \lambda(z, w)$ for some $\lambda \in k^\times$. In this case the GIT quotient will be a $\mathbb{P}^1$, some sort of boundary where things get mushed together.

(3) In the same way, if $c = 0$ and $a < 0$, then $M(x, y, z, w)$ is θ-semistable iff $(x, z) \neq 0$, and there are no θ-stable representations. Now suppose $M(x, y, z, w)$ is such that $(x, y) \neq 0$; then, $M(x, y, z, w)$ and $M(x', y', z', w')$ are S-equivalent iff $(x', y') = \lambda(x, y)$ for some $\lambda \in k^\times$. Again we get a $\mathbb{P}^1$.

(4) If $a = c = 0$, then all M are strictly semistable, and all of them are S-equivalent. The GIT quotient will be a point.

Even in this small case, things are complicated.

We haven't shown the statements above about S-equivalence, so let's do that. There's nothing in case (1), so let's look at (2). In this case $M(x, y, z, w) \sim M(x', y', z', w')$ iff $N(z, w) \sim N(z', w')$, and it's not hard to check that we can act precisely by scalars, so this is true iff (z', w') is a nonzero scalar multiple of (z, w). (3) is the same. For (4), we have a three-stage Jordan-Hölder filtration:

$$0 \subset S_2 \subset N_1(x, y) \subset M(x, y, z, w), \tag{4.13}$$

and the pieces of the associated graded are $N_1(x, y)/S_2 \cong S_1$ and $M/N_1(x, y) \cong S_3$. Therefore $\mathrm{Gr}(M(x, y, z, w)) \cong S_1 \oplus S_2 \oplus S_3$ for any (x, y, z, w), so they're all S-equivalent.

We know to expect $\mathbb{P}^1 \times \mathbb{P}^1$ if $a, c < 0$, or $\mathbb{P}^1$ if exactly one is zero, or a point if they're both zero. You can think of letting $a \to 0$ as projecting onto the first $\mathbb{P}^1$, and letting $c \to 0$ as projecting onto the second $\mathbb{P}^1$. But we haven't proven any of these yet! We don't even know that they're varieties *a priori*, but this is the correct answer, and it's possible to prove it. ◀

Remark 4.14. It turns out all quiver varieties are rational! You can get irrational varieties by imposing composition relations between arrows in the quiver; in fact, any projective variety arises in this way. ◀

The last thing we'll do is study some more general properties of quiver varieties.

Theorem 4.15. *If Q is acyclic quiver (i.e. it has no loops), then $M_\theta(Q, \alpha)$ is projective.*

Proof. $M_\theta(Q, \alpha) := R(Q, \alpha) /\!/ \mathrm{GL}(\alpha)$ always has a projective morphsm to $R(Q, \alpha) /\!/ \mathrm{GL}(\alpha)$. That is, by definition, we have a projective morphism

$$\mathrm{Proj}\left(\bigoplus_{n \geq 0} k[R(Q, \alpha)]^{\mathrm{GL}(\alpha), \chi_\theta^n}\right) \longrightarrow \mathrm{Spec}\Big(k[R(Q, \alpha)]^{\mathrm{GL}(\alpha)}\Big). \tag{4.16}$$

It therefore suffices to prove the codomain is a point. Since $R(Q, \alpha) /\!/ \mathrm{GL}(\alpha) = M_0(Q, \alpha)$, all representations have $\theta(M) = 0$.

Exercise 4.17. Show that if Q is acyclic and $M \in \mathsf{Rep}_Q$ has dimension α, then $\mathrm{gr}(M) = \bigoplus_{v \in Q_0} S_v$, where S_v is the "δ-function", a simple representation with k at v and 0 elsewhere.

Therefore they're all S-equivalent, and the quotient is a point. ⊠

Remark 4.18. The empty set is a projective variety, and may occur as the moduli space associated to an acyclic quiver. ◀

Lecture 5.

Moduli spaces and slope stability: 2/5/19

"19^{th}*-century math was a different world.*"

So we've constructed some GIT quotient $M_Q(\alpha, \theta)$ and claimed it's the moduli space of representations of the quiver Q with dimension α, and with (semi)stability tracked by the character θ of $\mathrm{GL}(\alpha)$. But why does this deserve to be called a moduli space anyways? We'll begin by talking about this.

Definition 5.1. Let $\mathcal{M}\colon \mathsf{Sch}_k^{\mathsf{op}} \to \mathsf{Set}$ be a functor.

(1) A k-scheme M is a *fine moduli space* for $\mathcal{M}$ if there's a natural isomorphism of functors $\mathcal{M} \overset{\cong}{\Rightarrow} \mathrm{Hom}(\text{–}, M)$.[3] One says that M *represents* $\mathcal{M}$.
(2) A k-scheme M is a *coarse moduli space* for $\mathcal{M}$ if there's a natural transformation $\iota_M\colon \mathcal{M} \Rightarrow \mathrm{Hom}(\text{–}, M)$ such that any other natural transformation $\iota_N\colon \mathcal{M} \Rightarrow \mathrm{Hom}(\text{–}, N)$ for a k-scheme N factors through ι_M. That is, we ask that there's a unique map $f\colon M \to N$ such that the diagram

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\iota_M"] \arrow[dr, "\iota_N"'] & \mathrm{Hom}(\text{–}, M) \arrow[d, "f\circ\text{–}"] \\ & \mathrm{Hom}(\text{–}, N). \end{tikzcd} \tag{5.2}$$

We'll write down a functor related to quiver representations, and $\mathcal{M}_Q(\alpha, \theta)$ will be a coarse or fine moduli space, depending on whether we have semistable points. A fine moduli space would be the nicest thing, but we won't always get it.

If B is a k-scheme and $\mathcal{M}$ is a moduli space of doodads (whatever a doodad is), we want $\mathcal{M}(B)$ to be the set of "families of doodads parameterized by B." What this means precisely depends on the details of the specific setting. Here's what it is for quiver representations.

Definition 5.3. Let Q be a quiver and B be a k-scheme. A *family of Q-representations on B* is data of a locally free sheaf $\mathscr{W}_v$ for each $v \in Q_0$ and morphisms $\phi(e)\colon \mathscr{W}_{t(a)} \to \mathscr{W}_{s(a)}$ for all $e \in Q_1$. The *rank* of a family of representations is as before: the element of $\mathbb{N}[Q_0]$ sending $v \mapsto \operatorname{rank} \mathscr{W}_v$.

If $\alpha \in \mathbb{N}[Q_0]$ and θ is a character of $\mathrm{GL}(\alpha)$, then a rank-α Q-representation is *θ-semistable* if all of its fibers are.

Given a family of Q-representations $\mathscr{W}$ over B and a line bundle $L \to B$, we can tensor them together to obtain a family of Q-representation $\mathscr{W} \otimes V$ with $(\mathscr{W} \otimes V)_v := \mathscr{W}_v \otimes V$ and $\phi_{\mathscr{W}\otimes V}(e) := \phi_{\mathscr{W}} \otimes \mathrm{id}_V$.

Therefore we have a functor $\mathcal{M}_Q(\alpha, \theta)\colon \mathsf{Sch}_k \to \mathsf{Set}$ which sends a scheme B to the set of equivalence classes of families of θ-semistable Q-representations with rank α, where we say $\mathscr{W}$ and $\mathscr{W}'$ are equivalent if there is a line bundle $L \to B$ and an isomorphism $\mathscr{W}' \cong \mathscr{W} \otimes L$. Given a map of schemes $\phi\colon B \to B'$, we get a map between these sets by pulling back each $\mathscr{W}_v$ and ϕ.

Remark 5.4. Why this equivalence relation? Well first we want to pass to isomorphism classes anyways, but the idea is that $\mathscr{W}$ and $\mathscr{W} \otimes L$ aren't really very different with respect to the representation theory of Q, so we identify them. ◂

Theorem 5.5 ([Kin94]). *Fix a quiver Q, dimension vector α, and character θ for* $\mathrm{GL}(\alpha)$.

(1) $M_Q(\alpha, \theta)$ is a coarse moduli space for $\mathcal{M}_Q(\alpha, \theta)$.
(2) If α is indivisible in the ring $\mathbb{Z}[Q_0]$, then $M_Q(\alpha, \theta)$ is a fine moduli space, and a smooth projective variety.

The proof isn't particularly enlightening, so we're going to skip it.

The reason we don't expect a fine moduli space in general is that semistable points can have automorphisms, which leads to a moduli stack, rather than a fine moduli space. When α is indivisible, it turns out that all representations are either θ-stable or θ-unstable, and θ-stable points have no automorphisms, making the stackiness go away.

Remark 5.6. If you restrict the functor $\mathcal{M}_Q(\alpha, \theta)$ to families of θ-stable representations, we also get a representable functor. In general, it's an open subfunctor of $\mathcal{M}_Q(\alpha, \theta)$, and often but not always is the smooth locus. ◂

[3] We need to specify $\mathcal{M}$ rather than just saying "this is a moduli space," because every scheme X is a moduli space for $\mathrm{Hom}(\text{–}, X)$. In general, though, when it comes to moduli spaces, you "know it when you see it," which is why people sometimes don't say what $\mathcal{M}$ is when it's clear from context.

We're now done with quivers, and will move to other moduli spaces now. Bridgeland stability conditions, the theme of this class, will provide a general way to produce nicer moduli spaces – often you cannot make a fine moduli space of all of your objects, but you can do this for objects satisfying some kind of stability condition.

Our next class of examples will be moduli spaces of vector bundles on curves, which relates to something called slope stability. In this section, we assume the base field is $\mathbb{C}$; probably these results hold in more generality, but the references were difficult to find.

Exercise 5.7. Let C be a smooth, projective curve and E be a coherent sheaf on C. Show that there is a unique exact sequence

$$0 \longrightarrow T_E \longrightarrow E \longrightarrow F_E \longrightarrow 0, \tag{5.8}$$

where F_E is locally free and T_E is a torsion sheaf, and that this sequence noncanonically splits.

So over a curve, any coherent sheaf can be written as a direct sum of a vector bundle and a rank-zero, torsion sheaf. For this reason, on curves one often considers moduli spaces of vector bundles, where in higher dimensions one thinks about moduli spaces of sheaves.

The following theorem was robably known in the 19^{th} century, though in its modern form it was proven by Grothendieck.

Theorem 5.9 (Grothendieck)**.** *Let E be a vector bundle on $\mathbb{P}^1$. Then there are unique integers $a_1, \dots, a_n$ with $a_1 > \cdots > a_n$ and unique nonzero vector spaces $V_1, \dots, V_n$ such that*

$$E \cong \bigoplus_{i=1}^{n} \mathscr{O}_{\mathbb{P}^1}(a_i) \otimes V_1.$$

In other words, any vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles.

Proof. First, existence. We'll induct on $r := \operatorname{rank} E$. For $r = 1$ there's nothing to show. For $r > 1$, we'll show how to split off a line bundle, reducing to the inductive assumption. Serre duality tells us that

$$\operatorname{Hom}(\mathscr{O}(a), E) \cong \operatorname{Ext}^1(E, \mathscr{O}(a) \otimes \omega_{\mathbb{P}^1}) = H^1(E^\vee \otimes \mathscr{O}(a-2)) = 0 \text{ if } a \gg 0 \tag{5.10}$$

by Serre vanishing. Here we're also using the fact that $\omega_{\mathbb{P}^1} = \mathscr{O}(-2)$. Let a be the largest integer such that $\operatorname{Hom}(\mathscr{O}(a), E) \neq 0$, and fix a nonzero homomorphism $\phi\colon \mathscr{O}(a) \to E$.

Since $\mathscr{O}(a)$ is a line bundle and E is locally free, all map between them are either zero or injective; thus ϕ is injective, so we have a short exact sequence

$$0 \longrightarrow \mathscr{O}(a) \xrightarrow{\ \phi\ } E \longrightarrow \operatorname{coker}(\phi) \longrightarrow 0. \tag{5.11}$$

It suffices to show $F := \operatorname{coker}(\phi)$ is locally free and (5.11) splits. Clearly this was not the same proof given in the 19^{th} century!

First, that F is locally free. Using Exercise 5.7, we have a short exact sequence

$$0 \longrightarrow T \longrightarrow F \longrightarrow G \longrightarrow 0, \tag{5.12}$$

where T is torsion and G is locally free. Assume T is nonzero. This means, because $\mathbb{P}^1$ is one-dimensional, T is supported in dimension zero, so there's some $x \in \mathbb{P}^1$ such that $\mathbb{C}(x)$ injects into T, where $\mathbb{C}(x)$ denotes the skyscraper sheaf with stalk $\mathbb{C}$ at x; in particular, since T is a subsheaf of F, $\operatorname{Hom}(\mathbb{C}(x), F) \neq 0$.

We can pass from $\mathscr{O}(a)$ to $\mathscr{O}(a+1)$ by multiplying by an equation that cuts out x (since x has codimension 1, we only need one equation). So we have a short exact sequence

$$0 \longrightarrow \mathscr{O}(a) \longrightarrow \mathscr{O}(a+1) \longrightarrow \mathbb{C}(x) \longrightarrow 0. \tag{5.13}$$

Applying $\operatorname{Hom}(\mathbb{C}(x), –)$ to (5.12), we obtain a long exact sequence

$$\underbrace{\operatorname{Hom}(\mathbb{C}(x), E)}_{=0} \longrightarrow \underbrace{\operatorname{Hom}(\mathbb{C}(x), F)}_{\neq 0} \longrightarrow \operatorname{Ext}^1(\mathbb{C}(x), \mathscr{O}(a)) \xrightarrow{(*)} \operatorname{Ext}^1(\mathbb{C}(x), E). \tag{5.14}$$

We want to show that $\mathrm{Hom}(E, \mathbb{C}(x)) \to \mathrm{Hom}(\mathscr{O}(a), \mathbb{C}(x))$ is surjective in order to derive a contradiction; it suffices to show that $(*)$ is injective. Now weave in a piece of the long exact sequence given by applying $\mathrm{Hom}(–, \mathscr{O}(a))$ and $\mathrm{Hom}(–, E)$ to (5.13):

(5.15)
$$\begin{tikzcd} & & \mathrm{Hom}(\mathscr{O}(a), \mathscr{O}(a)) \arrow[r, hook, "f"] \arrow[d, "g"] & \mathrm{Hom}(\mathscr{O}(a), E) \arrow[d, hook, "h"] \\ \underbrace{\mathrm{Hom}(\mathbb{C}(x), E)}_{=0} \arrow[r] & \underbrace{\mathrm{Hom}(\mathbb{C}(x), F)}_{\neq 0} \arrow[r] & \mathrm{Ext}^1(\mathbb{C}(x), \mathscr{O}(a)) \arrow[r, "(*)"] \arrow[d] & \mathrm{Ext}^1(\mathbb{C}(x), E) \\ & & \mathrm{Ext}^1(\mathscr{O}(a+1), \mathscr{O}(a)). & \end{tikzcd}$$

The reason that f is injective is that it sends $\mathrm{id}_{\mathscr{O}(a)}$ to the inclusion map $\mathscr{O}(a) \hookrightarrow E$ we specified. Since $\mathrm{Ext}^1(\mathscr{O}(a+1), \mathscr{O}(a)) = H^1(\mathbb{P}^1, \mathscr{O}(-1)) = 0$, then g is an isomorphism. Therefore $h \circ f$ is injective, so $(*)$ must be too.

Thus we've proven that F is locally free: by induction, $F = \bigoplus_j \mathscr{O}(b_j)^{\oplus r_j}$. We have left to show that $b_j \le a$ for all j. Assume the opposite, that there's a j_0 such hat $b_{j_0} > a$. Then the composition of the maps

(5.16)
$$\mathscr{O}_{\mathbb{P}^1}(a+1) \hookrightarrow \mathscr{O}_{\mathbb{P}^1}(b_j) \hookrightarrow F$$

is nonzero, so throwing $\mathrm{Hom}(\mathscr{O}(a+1), –)$ at (5.11), we obtain a long exact sequence

(5.17)
$$\underbrace{\mathrm{Hom}(\mathscr{O}(a+1), E)}_{\neq 0} \longrightarrow \underbrace{\mathrm{Hom}(\mathscr{O}(a+1), F)}_{\neq 0} \longrightarrow \mathrm{Ext}^1(\mathscr{O}(a+1), \mathscr{O}(a)) = H^1(\mathscr{O}(-1)) = 0,$$

which causes a contradiction. Uniqueness follows from another induction on r, using the fact that $\mathrm{Hom}(\mathscr{O}(a), \mathscr{O}(b)) = 0$ if $a > b$.

Briefly, why does $0 \to \mathscr{O}(a) \to E \to \bigoplus_j \mathscr{O}(b_j)^{r_j}$ split? This is because the obstruction is $\mathrm{Ext}^1(\mathscr{O}(b_j), \mathscr{O}(a)) = H^1(\mathscr{O}(a - b_j) = 0$ whenever $a - b_j > 0$. ⊠

Lecture 6.

Slope stability: 2/7/19

The first part of the lecture involved fixing a mistake in the proof of Theorem 5.9. I included those fixes in the notes for the previous lecture; in particular, the proof that's written there should be correct.

In particular, we understand vector bundles on $\mathbb{P}^1$: line bundles and what you can make out of them. But on other curves, vector bundles are more complicated, bringing in the notion of slope stability.

Let C be a smooth projective curve over $\mathbb{C}$ (though most of this still works over any algebraically closed field).

Definition 6.1.

(1) Let $E \to C$ be a vector bundle. The *degree* $d(E)$ of E is the degree of $\mathrm{Det}(E) := \Lambda^{\mathrm{rank}(E)}E$ as a line bundle.
(2) The *degree* of a torsion sheaf T, denoted $d(T)$, is the length of the scheme-theoretic support of T.
(3) The *degree* of a coherent sheaf E is $d(E) := d(T_E) + d(F_E)$ (using Exercise 5.7).
(4) The *rank* of a coherent sheaf E is the rank of its locally free part.

What's going on in the second definition? The idea is that any torsion sheaf on a curve is a successive extension of skyscrapers, and the length of this sequence is always the same. Both of these degrees are additive in short exact sequences.

Definition 6.2.

(5) The *slope* of a coherent sheaf E on C is $\mu(E) = d(E)/r(E)$, if $r(E)$ is positive, and is ∞ otherwise.

(6) A coherent sheaf E is *semistable*, resp. *stable*, if for any nonzero subsheaf $F \subsetneq E$, $\mu(F) \le \mu(E)$ resp. $\mu(F) < \mu(E)$.

As for quiver representations, it will turn out that you can't make a moduli space out of all vector bundles, but restricting to (semi)stable ones you can.

Exercise 6.3. Show that if $0 \to F \to E \to G \to 0$ is a short exact sequence of coherent sheaves on C, then $d(E) = d(F) + d(G)$ and $r(E) = r(F) + r(G)$.

Exercise 6.4.

(1) Let E be a coherent sheaf on C. Show that if E is semistable, then E is torsion or locally free.
(2) Show that E is stable, resp. semistable if for all surjective, non-isomorphic maps $E \twoheadrightarrow G$, $\mu(E) < \mu(G)$, resp. $\mu(E) \le \mu(G)$.

Remark 6.5. The notion of degree is the first instance of Chern classes, which exist in a more general setting. ◂

Example 6.6. Let's consider rank-2, degree-zero bundles on $\mathbb{P}^1$. These include $\mathscr{O}(-a) \oplus \mathscr{O}(a)$ for any $a \in \mathbb{Z}$, and that's it, so you get $\mathbb{Z}_{\ge 0}$. This is not a finite-type scheme, which is sad – but $\mathscr{O}(-a) \oplus \mathscr{O}(a)$ is unstable unless $a = 0$, so the moduli space of just (semi)stable bundles will end up being OK. ◂

Fact. This is completely trivial from the definitions we just made, but if E is a nonzero sheaf with rank zero, then $d(E) > 0$. This will create all of the issues in higher dimensions. ◂

But I hear you saying, you don't just care about coherent sheaves which are semistable. This is some arbitrary condition. You care about *all* coherent sheaves. Fortunately, we can approximate any coherent sheaf by semistable ones.

Theorem 6.7. *Let E be a coherent sheaf on C. Then there is a unique filtration (up to isomorphism), called the* Harder-Narasimhan filtration, $0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$, *such that E_i/E_{i-1} is semistable for each i and $\mu(E_1/E_0) \ge \mu(E_2/E_1) \ge \cdots \ge \mu(E_n/E_{n-1})$.*

We will prove this later in Proposition 8.18, when we want it to hold in a general setting; the proof is the same, so there's no need to see it twice. It's a very nice proof, and you should look forward to it; it uses pretty much just the fact that $\mathsf{Coh}(C)$ is Noetherian, and some numerical stuff.

Example 6.8. On $\mathbb{P}^1$, we can see this in Theorem 5.9. We have a unique way to write

$$E \cong \mathscr{O}(a_1)^{\oplus r_1} \oplus \cdots \oplus \mathscr{O}(a_n)^{\oplus r_n} \tag{6.9}$$

such that $a_1 > \cdots > a_n$. For $1 \le i \le n$, let

$$E \cong \mathscr{O}(a_1)^{\oplus r_1} \oplus \cdots \oplus \mathscr{O}(a_i)^{\oplus r_i}, \tag{6.10}$$

so that $E_i/E_{i-1} = \mathscr{O}(a_i)^{r_i}$, and $\mu(E_i/E_{i-1}) = r_i a_i / r_i = a_i$. ◂

Example 6.11. Let C be an elliptic curve and $p \in C$. Then

$$\mathrm{Ext}^1(\mathscr{O}_C, \mathscr{O}_C) = \mathrm{Hom}(\mathscr{O}_C, \mathscr{O}_C) = \mathbb{C} \tag{6.12a}$$

and

$$\mathrm{Ext}^1(\mathscr{O}_C(p), \mathscr{O}_C) = \mathrm{Hom}(\mathscr{O}_C, \mathscr{O}_C(p)) = \mathbb{C}. \tag{6.12b}$$

You can compute the second one by using the short exact sequence $0 \to \mathscr{O}_C \to \mathscr{O}_C(p) \to \mathscr{O}_p \to 0$ and that $H^0(\mathscr{O}_C) = H^1(\mathscr{O}_C) = \mathbb{C}$.

$\mathrm{Ext}^1(A, B)$ parameterizes extensions of B by A; the middle object may have automorphisms, and in both of these cases there's a $\mathbb{C}^\times$, so we can speak of the (isomorphism type) of the unique nontrivial extensions

$$0 \longrightarrow \mathscr{O}_C \longrightarrow E_1 \longrightarrow \mathscr{O}_C \longrightarrow 0 \tag{6.13a}$$

$$0 \longrightarrow \mathscr{O}_C \longrightarrow E_2 \longrightarrow \mathscr{O}_C(p) \longrightarrow 0. \tag{6.13b}$$

Claim 6.14. E_1 is semistable but not stable.

Proof. We know $r(\mathscr{O}_C) = 1$, $d(\mathscr{O}_C) = 0$, and $\mu(\mathscr{O}_C) = 0$, so using Exercise 6.3, $r(E_1) = 2$, $d(E_1) = 0$, and $\mu(E_1) = 0$. Since $\mathscr{O}_C \subset \mathscr{O}_E$ is nonzero and a proper subsheaf, but $\mu(\mathscr{O}_C) = \mu(E_1)$, then E_1 is not stable.

Now let $F \subset E$ be the first quotient E_1/E_0 in the Harder-Narasimhan filtration, which is nonzero, and assume $\mu(F) > \mu(E) = 0$. Applying $\mathrm{Hom}(F, -)$ to (6.13a), we have a left exact sequence

$$0 \longrightarrow \mathrm{Hom}(F, \mathscr{O}_C) \longrightarrow \mathrm{Hom}(F, E) \longrightarrow \mathrm{Hom}(F, \mathscr{O}_C). \tag{6.15}$$

Thus $\mathrm{Hom}(F, \mathscr{O}_C) \neq 0$, so there's a nonzero map $\varphi\colon F \to \mathscr{O}_C$; let $K := \ker(\varphi)$. Since F is semistable, $\mu(F) \leq \mu(F/K)$, but $\mu(\mathscr{O}_C) = 0$ and $\mathscr{O}_C$ is semistable, so $F/K \hookrightarrow \mathscr{O}_C$, so $\mu(F/K) < 0$ and $\mu(F/K) > \mu(F) > 0$, which is a contradiction. ⊠

Exercise 6.16. Show that E_2 is stable.

◀

Lecture 7.

The moduli space of vector bundles on curves: 2/12/19

Last time, we defined the slope $\mu(E)$ of a coherent sheaf E on a curve C, which is the degree divided by the rank. For a vector bundle, the degree is the class of $\operatorname{Det} E$ in $\mathrm{Pic}(C) \cong \mathbb{Z}$, and for a torsion sheaf it's the length of the scheme-theoretic support. The idea is that if you plot a line through $(0,0)$ and $(r(E), d(E))$, the slope of E is literaly the slope of the line, which is something I wish someone had told me earlier.

Anyways, we then defined (semi)stability for E: E is (semi)stable if for all nonzero proper subsheaves $F \subsetneq E$, $\mu(F) < \mu(E)$ ($\mu(F) \leq \mu(E)$). Then we talked about the Harder-Narasimhan filtration of a coherent sheaf, which is unique up to isomorphism, and such that the quotients are semistable. In fact, we can construct a different filtration.

Theorem 7.1. *Fix a slope $\mu \in \mathbb{Q}$ and let $\mathsf{P}(\mu)$ denote the full subcategory of $\mathsf{Coh}(C)$ of semistable sheaves E with $\mu(E) = \mu$ is a finite-length abelian category, and its simple objects are the stable sheaves.*

We will defer this proof to later, where we'll prove it in a more general setting.

Corollary 7.2. *If E is a semistable sheaf, then there is a Jordan-Hölder filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

such that the quotients E_i/E_{i-1} are all stable, and $\mu(E_i/E_{i-1}) = \mu(E)$. Moreover, the associated graded of this filtration is unique up to isomorphism.

The uniqueness statement is equivalent to the quotients E_i/E_{i-1} being unique up to permutation.

Definition 7.3. Let E and E' be two semistable coherent sheaves on C with the same slope. We say E and E' are *S-equivalent* if $\mathrm{gr}(E) \cong \mathrm{gr}(E')$.

Now let's talk about moduli spaces of vector bundles on curves. For $\mathbb{P}^1$, the fact that vector bundles split as direct sums of line bundles means that everything is pretty easy: if you fix the degree and the rank, the moduli space is empty or a point. This doesn't mean that the whole category is semisimple, though.

We again let C be a smooth curve over $\mathbb{C}$; probably most of this works over a more general algebraically closed field, but the references are harder to find. The cases $g(C) = 1$ and $g(C) \geq 2$ behave very different. We're going to focus on the $g(C) \geq 2$ case; if you'd like to understand the genus 1 case, consult Atiyah [Ati57] or Polishchuk [Pol03].

Thus, we now assume $g(C) \geq 2$. Fix a positive integer r, a $d \in \mathbb{Z}$, and a degree-d line bundle L; we'll try to construct the moduli space of degree-d, rank-r coherent sheaves E with top exterior power isomorphic to L.

Definition 7.4. Given a $\mathbb{C}$-scheme B, define $\mathcal{M}_C(r, d)(B)$ to be the set of vector bundles E on $C \times B$ such that for all $b \in B(\mathbb{C})$, $E|_{C \times \{b\}}$ is semistable of rank r and degree d, modulo the equivalence relation where $E \sim E'$ if $E \cong E' \otimes \pi^* M$ for some $M \in \mathrm{Pic}(B)$, where $\pi\colon E \times B \to B$ is projection onto the second factor.

Pullbacks preserve this equivalence relation, defining a functor $\mathcal{M}_C(r, d)\colon \mathsf{Sch}_{\mathbb{C}}^{\mathsf{op}} \to \mathsf{Set}$.

Definition 7.5. Given a $\mathbb{C}$-scheme B, define $\mathcal{M}_C(r, L)(B)$ to be the set of vector bundles E on $C \times B$ such that for all $b \in B(\mathbb{C})$, $E|_{C\times\{b\}}$ is semistable of rank r and $\operatorname{Det}(E|_{C\times\{b\}}) \cong L$, modulo the equivalence relation where $E \sim E'$ if $E \cong E' \otimes \pi^* M$ for some $M \in \operatorname{Pic}(B)$, where $\pi\colon E \times B \to B$ is projection onto the second factor.

Pullbacks preserve this equivalence relation, defining a functor $\mathcal{M}_C(r, L)\colon \mathsf{Sch}_{\mathbb{C}}^{\mathsf{op}} \to \mathsf{Set}$.

These moduli functors were studied by many people in around the 1970s.

Theorem 7.6 (Drézet, Mumford, Narasimhan, Ramanan, Seshadri, . . .)**.**

(1) *There is a coarse moduli space $M_C(r, d)$ for $\mathcal{M}_C(r, d)$ parameterizing equivalence classes of semistable vector bundles on C, which has the following properties.*
  (a) *It's nonempty.*[4]
  (b) *It's an integral, normal, factorial, and projective variety over $\mathbb{C}$, with dimension $r^2(g-1)+1$.*
  (c) *There is an open (sometimes empty) subset $M_C^s(r, d) \subset M_C(r, s)$ which parameterizes stable vector bundles.*
  (d) *Except for the case when $g = 2$, $r = 2$, and d is even, $M_C(r, d) \setminus M_C^s(r, d)$ is precisely the set of singular points. In particular, if there are no semistable-but-not-stable bundles, the moduli space is smooth.*
  (e) *There's an isomorphism $\operatorname{Pic}(M_C(r, d)) \cong \operatorname{Pic}(\operatorname{Pic}^d(C)) \times \mathbb{Z}$.*
  (f) *If $\gcd(r, d) = 1$, then $M_C(r, d) = M_C^s(r, d)$, and this is a fine moduli space.*

(2) *There are similar statements for $M_C(r, L)$, except:*
  (a) *Its dimension is $(r^2-1)(g-1)$.*
  (b) *$\operatorname{Pic}(M_C(r, L)) = \mathbb{Z}\cdot\theta$, where θ is ample.*
  (c) *If $u := \gcd(r, d)$, then $\omega_{M_C(r,L)} = -2u\theta$.*

Here $\operatorname{Pic}^d(C)$ is the *Jacobian*, a moduli space of degree-d line bundles. Also, keep in mind that if (r, d) share a common factor p, you can always find a semistable object that's not stable: using nonemptiness of the moduli space $M_C(r/p, d/p)$, choose an element E; then $E^{\oplus p}$ is semistable but not stable for r and d.

The proof would take us a long time to go through in complete detail, though in broad strokes it's similar to the one for quiver representations. Two good references for the proof are Le Portier [Por97] and Newstead [New11].

Abelian categories. We'll discuss general stability conditions on abelian categories. In order to do this, we'll first discuss some generalities on abelian categories. Maybe you've seen some of this before, but probably just for categories of modules and such. Instead, though, we'll deal with more general examples, where injective is not necessarily the same as monomorphic. The Stacks project [Sta19, Tags 09SE and 00ZX] is a good reference for this material.

Definition 7.7. A category A is *additive* if

- for all objects $A, B, C \in \mathsf{A}$, $\operatorname{Hom}_{\mathsf{A}}(A, B)$ is an abelian group, and the composition map $\operatorname{Hom}(A, B) \times \operatorname{Hom}(B, C) \to \operatorname{Hom}(A, C)$ sending $f, g \mapsto g \circ f$ is bilinear;
- there is an object $0 \in \mathsf{A}$ such that $\operatorname{Hom}_{\mathsf{A}}(0, 0) = \{0\}$; and
- for all $A, B \in \mathsf{A}$, the coproduct $A \oplus B$ and product $A \times B$ exist and are isomorphic.

Exercise 7.8. Show that 0 is the *zero object*, meaning it's both initial and terminal: for any $A \in \mathsf{A}$, there are unique maps $0 \to A$ and $A \to 0$.

Lots of categories are additive – vector spaces over a field, vector bundles over a base, sheaves over a base, etc. The category of schemes is not additive: binary products and coproducts need not coincide.

From now on, A is an additive category.

Definition 7.9. Let $f\colon A \to B$ be a morphism in A.

(1) A *kernel* of f is an object $K \in \mathsf{A}$ together with a morphism $i\colon K \hookrightarrow A$ such that $f \circ i = 0$, and such that for any map $i'\colon K' \to A$ such that $f \circ i' = 0$ factors uniquely through i.
(2) A *cokernel* of f is an object $C \in \mathsf{A}$ together with a morphism $\pi\colon B \to C$ such that $\pi \circ f = 0$ and such that any map $\pi'\colon B \to C'$ such that $\pi' \circ f = 0$ factors uniquely through π.

[4] There are choices of rank and degree in genus 0 and 1 where this is not true! It's also a significant headache in higher dimensions.

Kernels and cokernels need not exist, but if they do they're unique up to unique isomorphism, as is standard for universal properties. Therefore one usually speaks of *the* kernel and *the* cokernel, which are denoted $\ker(f)$ and $\operatorname{coker}(f)$, respectively.

Definition 7.10. Let f be as in Definition 7.9.

(3) If $\ker(f)$ exists, then the *coimage* of f, denoted $\operatorname{coim}(f)$, is $\operatorname{coker}(i)$, where $i\colon \ker(f) \to A$ is the map specified in the definition of a kernel.
(4) If $\operatorname{coker}(f)$ exists, the *image* of f, denoted $\operatorname{Im}(f)$, is $\ker(\pi)$, where $\pi\colon B \to \operatorname{coker}(f)$ is the map specified in the definition of a cokernel.

Exercise 7.11. Suppose both the kernel and cokernel of f exist. Then show that there exists a unique morphism $h\colon \operatorname{coim}(f) \to \operatorname{Im}(f)$ such that the following diagram commutes.

$$\begin{tikzcd} \ker(f) \arrow[r, "i"] & A \arrow[r, "f"] \arrow[d] & B \arrow[r] & \operatorname{coker}(f) \\ & \operatorname{coim}(f) \arrow[r, dashed, "h", "\exists!"'] & \operatorname{Im}(f) \arrow[u] & \end{tikzcd} \tag{7.12}$$

Definition 7.13. An additive category A is *abelian* if the kernel and cokernel of every morphism exists, and the natural map $\operatorname{coim}(f) \to \operatorname{Im}(f)$ is always an isomorphism.

The usual theorems of homological algebra (five lemma, snake lemma, etc.) hold in abelian categories.

Example 7.14.

(1) If R is any ring (not necessarily commutative), then Mod_R is abelian. These are the standard examples of abelian categories; conversely, every abelian category is a subcategory of a category of modules, but the embedding may be terrible.
(2) The categories of coherent or quasicoherent sheaves over any scheme.
(3) The category of representations of a quiver. ◂

The categories of vector bundles over a positive-dimensional manifold as well as the derived category of an abelian category are examples of additive but not abelian categories.

Lecture 8.

Stability in an abelian category: 2/14/19

Let A be an abelian category.

Definition 8.1. Let $Z\colon K_0(\mathsf{A}) \to \mathbb{C}$ be an additive homomorphism. Suppose that for all nonzero $E \in \mathsf{A}$,

(1) $\operatorname{Im}(Z(E)) \geq 0$, and
(2) $\operatorname{Im}(Z(E)) = 0$ implies $\operatorname{Re}(Z(E)) < 0$.

Then Z is called a *stability function*, $R(E) := \operatorname{Im}(Z(E))$ is called the *generalized rank* of E, and $D(E) := -\operatorname{Re}(Z(E))$ is called the *generalized degree* of E. Then $M(E) = R(E)/D(E)$ is called the *generalized slope* of E. Sometimes $Z(E)$ is called the *central charge* of E.

For example, for coherent sheaves E on a smooth projective curve, $ir(E) - d(E)$ is a stability function, because degree and rank are additive in short exact sequences, and the zero bundle is the only coherent sheaf with both rank and degree zero.

Definition 8.2. An object $E \in \mathsf{A}$ is *Z-stable* (resp. *Z-stable*) if for all nonzero $F \subsetneq E$, $M(F) < M(E)$ (resp. $M(F) \leq M(E)$).

Example 8.3. Another example of this is when A is the category of representations of a quiver Q. Given a $\theta \in \mathbb{Z}[Q_0]$, we get a stability function $Z_\theta(E) := \theta(E) + i \dim E$, where we define $\dim E := \sum_{v\in Q_0} \dim(E_v)$. ◂

This is not quite the same notion as we looked at before for quiver representations, as we haven't fixed a dimension α. But if $\theta(E) = 0$, then E is θ-semistable iff E is Z_θ-semistable, since $M(F) = -\theta(F)/\dim F \leq M(E) = 0$ iff $\theta(F) > 0$, so in this case our two notions of stability coincide. But the more general notion of (semi)stability is useful in Harder-Narasimhan filtrations, in which not all objects have the same slope.

We've defined (semi)stability in terms of subobjects; there's also an equivalent formulation in terms of quotients.

Exercise 8.4. Show that $E \in \mathsf{A}$ is Z-semistable (resp. Z-stable) iff for all quotients $E \twoheadrightarrow G$ with nonzero kernel, $M(E) \le M(G)$ (resp. $M(E) < M(G)$).

Lemma 8.5. *Let $A, B \in \mathsf{A}$ be Z-semistable objects with $M(A) > M(B)$. Then $\mathrm{Hom}_{\mathsf{A}}(A, B) = 0$.*

Proof. Let $f\colon A \to B$ be an A-morphism. Then it factors as $A \twoheadrightarrow \mathrm{Im}(f) \hookrightarrow B$, but if $\mathrm{Im}(f) \ne 0$, then by semistability of A and B,

$$M(A) \le M(\mathrm{Im}(f)) \le M(B) < M(A). \tag{8.6}$$ ⊠

So far Harder-Narasimhan filtrations have been very useful, so we're going to ask for them in the general setting. Under some mild hypotheses, we'll show that they exist in general.

Definition 8.7. The pair (A, Z) as above is called a *stability condition* if any nonzero object has a *Harder-Narasimhan filtration* much like before: a filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$ such that E_i/E_{i-1} is Z-semistable and $\mu(E_{i+1}/E_i) > \mu(E_i/E_{i-1})$ for all i.

Exercise 8.8. Show that Harder-Narasimhan filtrations are unique up to unique isomorphism if they exist.

To get this condition, we need a few assumptions on A.

Definition 8.9. An abelian category A is *Noetherian* if for any ascending chain $A_0 \subset A_1 \subset \cdots \subset A$, we have $A_i = A_{i+1}$ for sufficiently large i.[5] The analogous condition with ascending chains replaced by descending chains is called *Artinian*.

For example, a category of representations of a quiver is finite-length, hence both Noetherian and Artinian. The category of coherent sheaves on a curve is Noetherian, but the category of quasicoherent sheaves on a variety is not.

Lemma 8.10. *Let $Z\colon K_0(\mathsf{A}) \to \mathbb{C}$ be a stability function. Assume A is Noetherian and $R\colon K_0(\mathsf{A}) \to \mathbb{R}$ has discrete image. Then for any $E \in \mathsf{A}$, the generalized degrees of subobjects of E are bounded above.*

Proof. We induct on the generalized rank. If $R(E) = 0$, then $R(F) = 0$ for all $F \subset E$, and

$$0 < D(F) = D(E) - D(E/F) < D(E), \tag{8.11}$$

so we can let $D_E := D(E)$ be the upper bound.

More generally, assume $R(E) > 0$, and suppose $F_N \subset E$ be a sequence of objects with $\lim_{n\to\infty} D(F_n) = \infty$. If for some n, $R(F_n) = R(E)$, then $R(E/F_n) = 0$ and $D(E/F_n) \ge 0$, so

$$D(F_n) = D(E) - D(E/F_n) \le D(E), \tag{8.12}$$

so we may without loss of generality assume $R(F_n) < R(E)$ for all n. We will use these assumptions to construct an increasing sequence $(n_k)_{k\in\mathbb{N}}$ such that

$$D(F_{n_1} + \cdots + F_{n_k}) > k + D(E) \tag{8.13a}$$

$$R(F_{n_1} + \cdots + F_{n_k}) < R(E). \tag{8.13b}$$

Here "+" refers to the internal sum: the smallest subobject of E containing the given subobjects. By assumption, we can choose some n_1 such that $D(F_{n_1}) \ge 1 + D(E)$. Now inductively assume we have constructed $n_1, \dots, n_{k-1}$. For $n > n_{k-1}$ there's a short exact sequence

(8.14)
$$0 \longrightarrow (F_{n_1} + \cdots + F_{n_{k-1}}) \cap F_n \longrightarrow (F_{n_1} + \cdots + F_{n_{k-1}}) \oplus F_n \longrightarrow F_{n_1} + \cdots + F_{n_{k-1}} + F_n \longrightarrow 0,$$

and therefore

$$D(F_{n_1} + \cdots + F_{n_{k-1}} + F_n) = D(F_{n_1} + \cdots + F_{n_{k-1}}) + D(F_n) - D((F_{n_1} + \cdots + F_{n_k}) \cap F_n). \tag{8.15}$$

By our first inductive assumption, $D((F_{n_1} + \cdots + F_{n_{k-1}}) \cap F_n)$ is bounded above, so

$$\lim_{n\to\infty} D(F_{n_1} + \cdots + F_{n_{k-1}} + F_n) = \infty. \tag{8.16}$$

[5] Here the notation $\subset$ means the kernel is zero.

Therefore we can choose some $n_k > n_{k-1}$ satisfying (8.13a). To check (8.13b), suppose that $R(F_{n_1} + \cdots + F_{n_k}) = R(E)$ (it's not possible for it to be greater); then,

$$k + D(E) < D(F_{n_1} + \cdots + F_{n_k}) < D(E), \tag{8.17}$$

which of course is a contradiction. ⊠

Now that that's out of the way, we can get at Harder-Narasimhan filtrations.

Proposition 8.18. *As above, let $Z\colon K_0(\mathsf{A}) \to \mathbb{C}$ be a stability function such that A is Noetherian and the image of the generalized rank function is discrete. Assume in addition that the image of D is discrete.*[6] *Then Harder-Narasimhan filtrations exist, i.e. (A, Z) is a stability condition.*

Proof. Let E be a nonzero object of A and let $\mathcal{H}(E)$ denote the convex hull of $\{Z(F) \mid F \subset E\}$ in $\mathbb{C}$. By Lemma 8.10, $\mathcal{H}(E)$ is bounded from the left.

Let $\mathcal{H}_\ell$ denote the half-plane to the left of the line frpom 0 to $Z(E)$. If E is semistable, then $0 \subset E$ is a Harder-Narasimhan filtration and we're done. Otherwise, $P(E) := \mathcal{H}(E) \cap H_\ell$ is a finite-vertex, bounded polygon; let $v_0 = 0, v_1, \dots, v_n = Z(E)$ be the vertices of this polygon in increasing order. Choose $F_i \subset E$ wth $Z(F_i) = v_i$ for $i = 0, \dots, n-1$. We claim

(1) $F_{i-1} \subset F_i$ for $i = 1, \dots, n$ (where $F_n = E$),
(2) $G_i := F_i/F_{i-1}$ is semistable, and
(3) $\mu(G_i) > \mu(G_{i+1})$ for each i.

For (1), by definition $Z(F_{i-1} \cap F_i)$ and $Z_(F_{i-1} + F_i)$ are in $\mathcal{H}(E)$ by definition. Moreover, we know that

$$R(F_{i-1} \cap F_i) \le R(F_{i-1}) < R(F_i) \le R(F_{i-1} + F_i), \tag{8.19}$$

so using that generalized degree and rank are additive in the short exact sequence

$$0 \longrightarrow F_{i-1} \cap F_i \longrightarrow F_{i-1} \oplus F_i \longrightarrow F_{i-1} + F_i \longrightarrow 0, \tag{8.20}$$

$Z(F_{i-1} \cap F_i) + Z(F_{i-1} + F_i) = v_{i-1} + v_i$. Therefore

$$Z(F_{i-1} \cap F_i) - Z(F_{i-1} + F_i) = (v_{i-1} - Z(F_{i-1} + F_i)) + (v_i - Z(F_{i-1} + F_i)). \tag{8.21}$$

This can't work unless $Z(F_{i-1} + F_i) = v_i$ and $Z(F_{i-1} \cap F_i) = v_i - 1$.

We'll finish parts (2) and (3) next time.

Lecture 9.

Triangulated categories: 2/19/19

We're in the middle of proving that Harder-Narasimhan filtrations exist for stability conditions $Z\colon K_0(\mathsf{A}) \to \mathbb{C}$ in a general abelian category A exist, as long as

(1) A is abelian,
(2) the image of $R\colon K_0(\mathsf{A}) \to \mathbb{R}$ is discrete, and
(3) the image of $D\colon K_0(\mathsf{A}) \to \mathbb{R}$ discrete.

Though condition (3) is necessary in our proof, it's actually superfluous. However, especially once we study vector bundles on surfaces, we'll see plenty of examples of stability conditions for which the generalized rank and degree maps is not discrete, and we will have to work around this.

Continuation of the proof of Proposition 8.18. Recall that we defined $\mathcal{H}(E)$ to be the convex hull of $\{Z(F) \mid F \subset E\}$, where $E \in \mathsf{A}$. This has infinite area in general, so we cut it off with a line $\mathcal{H}_\ell$ to obtain a genuine polygon $P(E)$ with vertices $v_1, \dots, v_n$, under assumptions (2) and (3) above. This also allows us to choose $F_i \subset E$ with $Z(F_i) = v_i$.

It suffices to prove that $F_{i-1} \subset F_i$ for each i (which we did last time), that the quotients $G_i := F_i/F_{i-1}$ are semistable (which we'll do now), and that $M(G_i) > M(G_{i+1})$ for each i (which we'll do now). Fortunately, the hardest part is already behind us.

[6] This hypothesis, though not strictly necessary, makes the proof much simpler.

Let $\overline{A} \subset G_i$ be a nonzero subobject, and $A \subset F_i$ be its preimage under the quotient $F_i \twoheadrightarrow G_i$. Then $Z(A) \in \mathcal{H}(E)$ and $R(F_{i-1}) \le R(A) \le R(F_i)$. Then $Z(\overline{A}) = Z(A) - Z(F_{i-1})$, which has smaller or equal slope than $Z(G_i) = Z(F_i) - Z(F_{i-1})$.

The last thing we need to prove, that the slopes decrease, is because we're going clockwise around a convex polygon with a vertex at 0, from a vertex above and to the left of 0; therefore you can see that the slopes decrease. ⊠

The fact that the polygon is bounded from the left is the most difficult part; after that it's basically convex geometry.

Exercise 9.1. If you're interested, give a proof without assumption (3).

In particular, we've proven the existence of Harder-Narasimhan filtrations in the settings we promised to: for coherent sheaves on a smooth projective curve (Theorem 6.7) and for quiver representations.

Definition 9.2. A *very weak stability function*[7] is a homomorphism $Z\colon K_0(\mathsf{A})$ such that for all objects E of A, $\mathrm{Im}(Z(E)) \ge 0$ and if $\mathrm{Im}(Z(E)) = 0$, then $\mathrm{Re}(Z(E)) \le 0$.

So it's exactly the same as a stability condition, except that we can have generalized degree zero. We still consider the slope of something with generalized rank and degree zero to be infinity.

Remark 9.3. Proposition 8.18 still holds in this setting, but there's no longer a unique choice for $F_i \subset E$ with $Z(F_i) = v_i$; instead, we must choose the largest one, which is fine, because A is Noetherian. ◂

We will encounter very weak stability conditions when studying slopes of vector bundles on surfaces. In general, stability conditions will form a complex manifold, once formulated in the setting of derived categories, and the geometry of this complex manifold will inform us about stability conditions.

∽ · ∼

To continue with stability conditions, we will need to work with the derived category of an abelian category. The notion of a derived category is due to Verdier [Ver96] from his thesis; we will (concisely) follow Huybrechts [Huy06].

Let A be an abelian category. We would like to construct out of A another category $D(\mathsf{A})$ with two properties:

(1) the objects of $D(\mathsf{A})$ are complexes of objects in A, and
(2) if $F^\bullet \to A$ is a "resolution" (in whatever sense) of an object $A \in \mathsf{A}$, then in $D(\mathsf{A})$, we want $F^\bullet \simeq A$ (here A is regarded as a complex concentrated in degree zero).

As an illustration of the philosophy behind the second point, recall that to compute derived functors of f on A, you resolve A by a complex $F^\bullet$ of nice objects, and apply f to this complex. We'd like to say this is the same as applying the derived functors to A.

One's first guess would be the category of complexes in A, with morphisms commuting with the complex maps, but this doesn't satisfy the second condition. In fact, what we get won't even be abelian, though it will be additive, and with the extra structure of a triangulated category.

Definition 9.4. Let D be a (small) additive category. The structure of a *triangulated category* on D is given by an additive equivalence $T\colon \mathsf{D} \to \mathsf{D}$,[8] and a set of *distinguished triangles*, which are sequences of objects $A \to B \to C \to T(A)$ in D satisfying the axioms **TR1** through **TR4** below.

First, though, we need some notation: if $A, B \in \mathsf{D}$ and $f\colon A \to B$ is a morphism in D, we let $A[n] := T^n A$ and $f[n]\colon A[n] \to B[n]$ denote $T^n f$. A *morphism* of distinguished triangles from $A \to B \to C \to A[1]$ to $A' \to B' \to C' \to A'[1]$ is morphisms $f\colon A \to A'$, $g\colon B \to B'$, and $h\colon C \to C'$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & A[1] \arrow[d, "f[1]"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1]. \end{tikzcd} \tag{9.5}$$

[7] There is a preexisting notion of a weak stability function, and this is different.
[8] In other settings, T is sometimes denoted $[1]$ or Σ.

If f, g, and h are isomorphisms, we call this an *isomorphism* of distinguished triangles.

TR1. (a) For every object $A \in \mathsf{D}$, the triangle

$$A \xrightarrow{\mathrm{id}} A \longrightarrow 0 \longrightarrow A[1] \tag{9.6}$$

is distinguished.

(b) The set of distinguished triangles is closed under isomorphisms.

(c) Any map $f\colon A \to B$ can be completed to a distinguished triangle $A \xrightarrow{f} B \to C \to A[1]$. Sometimes C is called the *cone* of f, and denoted $C(f)$.

TR2. The triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1] \tag{9.7a}$$

is distinguished if and only if the triangle

$$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{f[1]} B[1] \tag{9.7b}$$

is distinguished.

TR3. Suppose $A \to B \to C \to A[1]$ and $A' \to B' \to C' \to A'[1]$ are distinguished triangles and $f\colon A \to A'$ and $g\colon B \to B'$ are maps such that the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] & A[1] \arrow[d, "f[1]"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd} \tag{9.8}$$

commutes. Then there exists an $h\colon C \to C'$ such that when h is inserted in the above diagram, it still commutes.

TR4. The *octahedral axiom*: given $f\colon A \to B$ and $g\colon B \to C$ in D, there is a distinguished triangle

$$C(f) \xrightarrow{\alpha} C(g\circ f) \xrightarrow{\beta} C(g) \xrightarrow{\gamma} C(f)[1] \tag{9.9}$$

such that the following diagram is commutative:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r] \arrow[d, "g"] & C(f) \arrow[r] \arrow[d, "\alpha"] & A[1] \arrow[d, equal] \\ A \arrow[r, "g\circ f"] \arrow[d] & C \arrow[r] \arrow[d] & C(g\circ f) \arrow[r] \arrow[d, "\beta"] & A[1] \arrow[d] \\ 0 \arrow[r] \arrow[d] & C(g) \arrow[r, equal] \arrow[d] & C(g) \arrow[r] \arrow[d, "\gamma"] & 0 \arrow[d] \\ A[1] \arrow[r, "f[1]"] & B[1] \arrow[r] & C(f)[1] \arrow[r] & A[2]. \end{tikzcd} \tag{9.10}$$

Distinguished triangles are the analogue of exact sequences in a triangulated category. You can think of requirement (**TR1**c) as saying that we can take cokernels, even though we're not in an abelian category. Condition **TR3** posits existence but not uniqueness, which is very unusual in category theory; thus some category theorists suggest that perhaps we should be using something different, but this definition is in fact flexible enough to serve our purposes. The octahedral axiom is somewhat tricky to write down, but will be essential when we discuss abelian categories inside our triangulated category, via t-structures and hearts.[9]

This is a long definition, but the intuition you should keep in mind is that we're trying to have an abelian category but we don't quite have one. So, for example, to gain intuition for the octahedral axiom you can make sense of it in an abelian category A: let $f\colon A \hookrightarrow B$ and $g\colon B \hookrightarrow C$ be two morphisms in A; then

[9]However, it's currently open whether there are any categories which satisfy **TR1** through **TR3** but not **TR4**!

$C(f) = B/A$, $C(g) = C/B$, and $C(g \circ f) = C/A$. Then the analogue of **TR4** is the existence of a short exact sequence

$$0 \longrightarrow B/A \longrightarrow C/A \longrightarrow C/B \longrightarrow 0, \tag{9.11}$$

which certainly exists. But here it's canonical, whereas in a triangulated category one has to choose the maps α, β, and γ.

We will not prove all of the basic facts about homological algebra, but instead reference Huybrechts' book mentioned above. But there are a few important facts to remember.

Exercise 9.12. If $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow A[1]$ is a distinguished triangle, then $g \circ f = 0$.

Proposition 9.13. *If $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow A[1]$ is a distinguished triangle and $A_0 \in \mathsf{D}$, then there are exact sequences of abelian groups*

$$\mathrm{Hom}(A_0, A) \xrightarrow{f\circ -} \mathrm{Hom}(A_0, B) \xrightarrow{g\circ -} \mathrm{Hom}(A_0, C) \tag{9.14a}$$

$$\mathrm{Hom}(C, A_0) \xrightarrow[-\circ g]{} \mathrm{Hom}(B, A_0) \xrightarrow[-\circ f]{} \mathrm{Hom}(A, A_0). \tag{9.14b}$$

Proof. We will prove (9.14a); the proof of (9.14b) is similar, but with the arrows in the other direction.

Given $a\colon A_0 \to A$, its image in $\mathrm{Hom}(A_0, C)$ is $g \circ f \circ a = 0$ by Exercise 9.12, and therefore $\mathrm{Im}(f \circ -) \subset \ker(g \circ -)$. Now suppose $b\colon A_0 \to B \in \ker(g \circ -)$. Using **TR3**, we can choose some a such that the diagram

$$\begin{tikzcd} A_0[-1] \arrow[r] \arrow[d, "b[-1]"] & 0 \arrow[r] \arrow[d] & A_0 \arrow[r, equal] \arrow[d, "a"] & A_0 \arrow[d, "b"] \\ B[-1] \arrow[r] & C[-1] \arrow[r] & A \arrow[r] & B \end{tikzcd} \tag{9.15}$$

commutes, and then we have that $b = g \circ a$, so it's in the image as we desired. ⊠

Exercise 9.16. If $A \to B \to C \to A[1]$ is distinguished, show $A \to B$ is an isomorphism iff $C \cong 0$.

Lecture 10.

Derived categories: 2/21/19

Before we get on to derived categories, there's a few more facts about triangulated categories to mention. But once we get them out of the way, we'll only focus on derived categories, never general triangulated ones.

Exercise 10.1 (Five lemma for triangulated categories)**.** In a triangulated category D, consider a morphism of distinguished triangles

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & A[1] \arrow[d, "f[1]"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1]. \end{tikzcd} \tag{10.2}$$

Show that if two of f, g, and h are isomorphisms, then so is the third.

Definition 10.3. An additive functor[10] $F\colon \mathsf{D} \to \mathsf{D}'$ is *exact* if it maps distinguished triangles to distinguished triangles and $F \circ [1]_{\mathsf{D}} = [1]_{\mathsf{D}'} \circ F$.

These are the structure-preserving functors for triangulated categories.

[10] Recall that in an additive category, the hom-sets are actually abelian groups; for an additive functor $F\colon \mathsf{C} \to \mathsf{D}$, the map $\mathrm{Hom}_{\mathsf{C}}(A, B) \to \mathrm{Hom}_{\mathsf{D}}(F(A), F(B))$ must be a group homomorphism.

Remark 10.4. In all of the examples we care about, $\mathrm{Hom}_{\mathsf{D}}(A, B)$ is more than an abelian group, but is naturally a k-vector space (meaning composition is k-linear). Then in all definitions we can replace abelian groups with k-vector spaces, obtaining the notions of *k-linear abelian categories* and *k-linear triangulated categories*. This extra structure will be important later when we construct moduli spaces and want them to be k-schemes. ◂

Now, on to the crucial example, and why we care about all this triangulated abstraction anyways.

Definition 10.5. Let A be an abelian category.

- A complex $A^\bullet$ of objects in A is called *bounded* if $A^i = 0$ for $i \ll 0$ and $i \gg 0$.
- We let $\mathsf{Kom}^b(\mathsf{A})$ denote the full subcategory of bounded complexes (so: objects are bounded complexes, and we allow all morphisms between them). This is an abelian category.
- A morphism $A^\bullet \to B^\bullet$ in $\mathsf{Kom}^b(\mathsf{A})$ is a *quasi-isomorphism* if the induced map $\mathcal{H}^i(A^\bullet) \to \mathcal{H}^i(B^\bullet)$[11] is an isomorphism for all i.
- The *bounded derived category* of D, denoted $D^b(\mathsf{A})$, is the localization of $\mathsf{Kom}^b(\mathsf{A})$ at the subcategory of quasi-isomorphisms.[12]

Theorem 10.6 (Verdier [Ver96])**.** *There is a category $D^b(\mathsf{A})$ and a functor $Q\colon \mathsf{Kom}^b(\mathsf{A}) \to D^b(\mathsf{A})$ such that*

(1) if $f\colon A^\bullet \to B^\bullet$ is a quasi-isomorphism in $\mathsf{Kom}^b(\mathsf{A})$, then Qf is an isomorphism, and
(2) any functor $F\colon \mathsf{A} \to \mathsf{C}$ such that F of any quasi-isomorphism is invertible factors uniquely (up to natural isomorphisms) through Q.

This doesn't just mean adding inverses to quasi-isomorphisms; in general one has to add more arrows. But in the actual construction, we don't add any objects.

Corollary 10.7. *If $f\colon A^\bullet \to B^\bullet$ is an isomorphism in $D^b(\mathsf{A})$, then $f_*\colon \mathcal{H}^i(A^\bullet) \to \mathcal{H}^i(B^\bullet)$ is the identity map. Therefore $\mathcal{H}^i\colon D^b(\mathsf{A}) \to \mathsf{A}$ is a functor.*

Corollary 10.8. *The full subcategory of $A^\bullet \in D^b(\mathsf{A})$ with the property that $\mathcal{H}^i(A^\bullet) = 0$ for $i \neq 0$ is equivalent to* A*.*

We think of this as embedding A into $D^b(\mathsf{A})$.

So what are the morphisms in $D^b(\mathsf{A})$? A general morphism $A^\bullet \to B^\bullet$ factors as two maps in $\mathsf{Kom}^b(\mathsf{A})$, $A^\bullet \leftarrow C^\bullet \to B^\bullet$, where the first map is a quasi-isomorphism. Defining composition is a little involved, though.

We would like to make $D^b(\mathsf{A})$ into a triangulated category. First we must define the shift functor, and then the distinguished triangles. The shift functor is easy: let $[1]\colon D^b(\mathsf{A}) \to D^b(\mathsf{A})$ be defined by $E[1]^i := E^{i+1}$.

Recall that an abelian category *has enough injectives* if every object embeds into an injective object.

Lemma 10.9. *Assume either that A has enough injectives*[13] *or $\mathsf{A} = \mathsf{Coh}(X)$. Then for $A, B \in \mathsf{A}$, there's a natural isomorphism $\mathrm{Hom}_{D^b(\mathsf{A})}(A, B[n]) = \mathrm{Ext}^1_{\mathsf{A}}(A, B)$.*

This might make distinguished triangles $A \to B \to C \to A[1]$ make more sense. The map $C \to A[1]$ is a little weird, maybe, and the point is that $\mathrm{Hom}(C, A[1]) = \mathrm{Ext}^1(C, A)$, which may be easier to get a handle on.

Definition 10.10. Let $f\colon A^\bullet \to B^\bullet$ be a morphism in $\mathsf{Kom}^b(\mathsf{A})$. We define its *cone* $C(f) \in \mathsf{Kom}^b(\mathsf{A})$ by $C(f)^i = A^{i+1} \oplus B^i$ with the differential

$$\begin{pmatrix} -d_A^{i+1} & 0 \\ f & d_B^i \end{pmatrix}. \tag{10.11}$$

[11] We'll use $\mathcal{H}^*$ to denote the cohomology of a complex and H^* to denote sheaf cohomology.

[12] This means that we want to consider, in a precise sense, the smallest category containing $\mathsf{Kom}^b(\mathsf{A})$ but such that all quasi-isomorphisms have inverses. Doing this naïvely leads to set-theoretic difficulties, but there is a way to make it work correctly.

[13] If this isn't true, the way to work around this is to embed in some other abelian category which does have enough injectives. For example, $\mathsf{Coh}(X)$ has neither enough projectives nor injectives in general, so we work inside $\mathsf{QCoh}(X)$.

The morphisms $B^i \hookrightarrow A^{i+1} \oplus B^i \twoheadrightarrow A^i$ induce maps of complexes

$$A^\bullet \xrightarrow{f} B^\bullet \longrightarrow C(f)^\bullet \longrightarrow A^\bullet[1]. \tag{10.12}$$

Definition 10.13. A sequence of morphisms $F \to G \to H \to F[1]$ in $D^b(\mathsf{A})$ is a *distinguished triangle* if it is isomorphic in $D^b(\mathsf{A})$ to (the functor Q applied to) a sequence of morphisms of the form (10.12) for some $f \in \mathrm{Hom}_{\mathsf{Kom}^b(\mathsf{A})}(A, B)$.

In particular, a short exact sequence of complexes defines a distinguished triangle. In practice, this is where the triangles we care about come from.

Theorem 10.14. *The distinguished triangles from Definition 10.13 and the shift functor above provide the structure of a triangulated category on $D^b(\mathsf{A})$.*

The proof involves a *lot* of gory details and diagram chases. You can find a complete proof in Gelfand and Manin's homological algebra book [GM03].

Remark 10.15. There are several variants on this construction; for example, you can start with complexes which are unbounded below but bounded above (so $A^i = 0$ for $i \gg 0$), defining a triangulated category $D^-(\mathsf{A})$; or complexes which are bounded below but not above, defining a triangulated category $D^+(\mathsf{A})$; or complexes which need not be bounded in either direction, defining a triangulated category $D(\mathsf{A})$. All of these are somewhat relevant, though the latter less so; complexes unbounded in both directions are pretty annoying to work with.

The categories we will consider all have finite homological dimension, which means that we essentially don't have to worry about the differences between these. ◂

Now we will discuss derived functors. The basic idea is essentially the same as in your homological algebra class, but instead of a sequence of derived functors on A, we will obtain one derived functor on $D^\pm(\mathsf{A})$. Certain properties of derived functors that are complicated to state from the former perspective, such as Grothendieck-Verdier duality, are much nicer and more concice in the language of derived categories.

Lemma 10.16 (Existence of injective resolutions)**.** *Let* A *be an abelian category with enough injectives. Then any* $\mathsf{A}^\bullet \in D^+(\mathsf{A})$ *is isomorphiuc to an* $I^\bullet \in D^+(\mathsf{A})$*, where* I^i *is injective for all* i*.*

We actually need something slightly stronger, which is that there is a construction to replace $A^\bullet$ with $I^\bullet$ that defines a functor $fr\colon D^+(\mathsf{A}) \to D^+(\mathsf{A})$.

Definition 10.17. Let A and B be abelian categories and $F\colon \mathsf{A} \to \mathsf{B}$ be left exact. Then the *right derived functor* of F, denoted $\mathbf{R}F\colon D^+(\mathsf{A}) \to D^+(\mathsf{B})$, is $F \circ fr$.

That is, replace $A^\bullet$ with an injective resolution and apply F to it.

Remark 10.18. Dually, if A has enough projectives and $F\colon \mathsf{A} \to \mathsf{B}$ is right exact, there is a *left exact functor* $\mathbf{L}F\colon D^-(\mathsf{A}) \to D^-(\mathsf{B})$ which takes a functorial projective resolution, then applies F. ◂

Problem: the category of coherent sheaves on a smooth projective variety X in general has neither enough projectives nor enough injectives! This happens even if X is a curve. The latter problem can be satisfied by working in $\mathsf{QCoh}(X)$, and then hoping that whatever you get out is coherent. But $\mathsf{QCoh}(X)$ still doesn't have enough projectives, which is a problem. The issue is really projectivity – for affines, this is the same as thinking about modules over a ring, which is fine. For example, projectives in $\mathsf{QCoh}(X)$ are free; locally free does not imply projective.

But we still want functors on derived categories of coherent sheaves, and even more, we want them to be on $D^b(\mathsf{Coh}(X))$, rather than on $D^\pm(\mathsf{Coh}(X))$. This presents an additional challenge.

Let X be a smooth projective variety over k,[14] and let $D^b(X) := D^b(\mathsf{Coh}(X))$.

Proposition 10.19. $\mathsf{QCoh}(X)$ *has enough injectives.*

[14] If you care about more general things, particularly for the case when X isn't smooth, consult Huybrechts [Huy06].

Lecture 11.

Derived categories of coherent sheaves: 2/26/19

Let X be a smooth projective variety over an algebraically closed field k. Last time, we defined the bounded derived category of X, $D^b(X)$, to be $D^b(\mathsf{Coh}(X))$. This behaves poorly if we remove niceness hypotheses on X, and even in this setting we have neither enough projectives nor injectives, but $\mathsf{QCoh}(X)$ has enough injectives, at least. So if we want to define right derived functors, we can work in the derived category of $\mathsf{QCoh}(X)$ – but we might land in $D^+(\mathsf{QCoh}(X))$. So typically we'd have to run the general story and then check that what we end up with is small enough to land in $D^b(X)$.

Today, we're going to study several examples of derived functors, which are all triangulated functors (i.e. the analogue of exactness but in a triangulated category) as first studied by Verdier, and use them to cleanly state a duality theorem which is inaccessible at the level of cohomology.

Proposition 11.1. *The inclusion functor $D^b(\mathsf{QCoh}(X)) \to D^+(\mathsf{QCoh}(X))$ induces a triangulated equivalence onto the full subcategory of complexes whose cohomology is bounded.*

"Cohomology is bounded" means that $\mathcal{H}^i(A) = 0$ for $i \gg 0$ (since we're in bounded-below complexes, we already know this for $i \ll 0$).

Proposition 11.2. *The functor $D^b(X) \to D^b(\mathsf{QCoh}(X))$ induces an equivalence onto the full subcategory of complexes whose cohomology sheaves are coherent.*

With these two propositions, we'll be able to figure out when we're back in $D^b(X)$.

Now, we'll give several examples of important derived functors and when they satisfy some finiteness conditions (e.g. landing in$D^b(X)$ rather than $D^+(\mathsf{QCoh}(X))$). There is a lot of hard work in basic scheme theory that goes into these proofs.

Theorem 11.3 (Serre duality). *Suppose $n := \dim X$. Then there's a natural isomorphism*

$$\mathrm{Hom}(A^\bullet, B^\bullet[i]) \cong \mathrm{Hom}(B^\bullet, A^\bullet \otimes \omega_X[n-i])^\vee$$

for all $A^\bullet, B^\bullet \in D^b(X)$.

We also need a way to pass from cohomology sheaves to cohomology groups. Specifically, the map $X \to \operatorname{Spec} k$ induces a pushforward/global sections map $\Gamma\colon \mathsf{QCoh}(X) \to \mathsf{Vect}_k$, which is left exact but not exact in general. Hence it has a right derived functor

$$\mathbf{R}\Gamma\colon D^+(\mathsf{QCoh}(X)) \longrightarrow D^+(\mathsf{Vect}_k), \tag{11.4}$$

and the *sheaf cohomology* of $\mathscr{F} \in D^+(\mathsf{QCoh}(X))$ is defined to the sequence of vector spaces $H^i(\mathscr{F}) := \mathcal{H}^i(\mathbf{R}\Gamma(\mathscr{F}))$.

Theorem 11.5 (Grothendieck). *For all $\mathscr{F} \in \mathsf{QCoh}(X)$, if $i > \dim X$, $H^i(X, \mathscr{F}) = 0$.*

Theorem 11.6 (Serre). *If $\mathscr{F} \in \mathsf{Coh}(X)$, then $H^i(X, \mathscr{F})$ is finite-dimensional for all i.*

This crucially uses the fact that X is projective. Even $H^0(\mathscr{F}) = \Gamma(\mathscr{F})$ may be infinite-dimensional on an affine scheme.

So our first example is $\mathbf{R}\Gamma$ restricted to $D^b(X)$, which is a functor

$$H^*\colon D^b(X) \longrightarrow D^b(\mathsf{Vect}_k^{\mathrm{fd}}), \tag{11.7}$$

where the target is the bounded derived category of finite-dimensional vector spaces.

Our next example is the direct image. Let Y be another smooth projective k-scheme and $f\colon X \to Y$ be a morphism. Then we have a *direct image* functor $f_*\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(Y)$: if $\mathscr{F} \in \mathsf{QCoh}(X)$ and $U \subset Y$, then $f_*\mathscr{F}(U) := \mathscr{F}(f^{-1}(U))$. This is left exact but not exact, so we have a derived functor

$$\mathbf{R}f_*\colon D^+(\mathsf{QCoh}(X)) \longrightarrow D^+(\mathsf{QCoh}(Y)). \tag{11.8}$$

Theorem 11.9. *If f is proper, $\mathbf{R}f_*$ restricts to a functor $D^b(X) \to D^b(Y)$.*

The third example is sheaf Hom. Recall that if $\mathscr{F}$ and $\mathscr{G}$ are two quasicoherent sheaves on X, we can define $\mathscr{H}om(\mathscr{F},\mathscr{G})$, whose value on an open U is $\mathrm{Hom}(\mathscr{F}(U),\mathscr{G}(U))$. Therefore we have a functor $\mathscr{H}om(\mathscr{F},–)\colon \mathsf{QCoh}(X)\to\mathsf{QCoh}(X)$, which is left exact. Therefore we obtain a right derived functor

$$\mathbf{R}\,\mathscr{H}om(\mathscr{F},–)\colon D^+(\mathsf{QCoh}(X))\longrightarrow D^+(\mathsf{QCoh}(X)). \tag{11.10a}$$

If $\mathscr{F}$ and $\mathscr{G}$ are coherent, so is $\mathscr{H}om(\mathscr{F},\mathscr{G})$, so when $\mathscr{F}$ is coherent, this is actually a functor

$$\mathbf{R}\,\mathscr{H}om(\mathscr{F},–)\colon D^+(\mathsf{Coh}(X))\longrightarrow D^+(\mathsf{Coh}(X)). \tag{11.10b}$$

Moreover, using (crucially) that X is smooth, this is actually a map

$$\mathbf{R}\,\mathscr{H}om(\mathscr{F},–)\colon D^+(X)\longrightarrow D^+(X). \tag{11.10c}$$

This is the derived-categorical version of Ext.

Definition 11.11. The *derived dual* of an $\mathscr{F}\in\mathsf{Coh}(X)$ is $\mathscr{F}^\vee := \mathbf{R}\,\mathscr{H}om(\mathscr{F},\mathscr{O}_X)$.

The next example is the derived tensor product.

Proposition 11.12 (Existence of locally free resolutions)**.** *Any complex $A^\bullet\in D^b(X)$ is isomorphic to some complex $F^\bullet\in D^b(X)$ such that F^i is locally free for each i.*

This proposition crucially uses smoothnes of X.

Locally free sheaves are flat, so we can use a locally free resolution to define the left derived functor of $A^\bullet\otimes –\colon \mathsf{Coh}(X)\to\mathsf{Coh}(X)$ as a functor

$$A^\bullet\otimes^{\mathbf{L}} –\colon D^b(X)\longrightarrow D^b(X). \tag{11.13}$$

This therefore generalizes to taking the tensor product with any $A^\bullet\in D^b(X)$. This is the derived-categorical version of Tor.

The final example we need for now is pullback: given a map $f\colon X\to Y$, where Y is as above, we get a map $f^*\colon \mathsf{Coh}(Y)\to\mathsf{Coh}(X)$. This is in general right exact, but it's exact on locally free sheaves, so we can take a locally free resolution and define a left derived functor

$$\mathbf{L}f^*\colon D^b(Y)\longrightarrow D^b(X). \tag{11.14}$$

These functors satisfy a whole bunch of compatibility conditions. Normally these involve a bunch of different hypotheses on the sheaves in question, but we've skirted around that by putting fairly strict hypotheses on X.

Proposition 11.15. *Let $f\colon X\to Y$ be a map of smooth projective varieties (as above), A, B, and $C\in D^b(X)$, and E and $F\in D^b(Y)$.*

(1) $–\otimes^{\mathbf{L}} –$ is associative and commutative up to natural isomorphism.

(2) (Projection formula) There's a natural isomorphism

$$\mathbf{R}f_*(A)\otimes^{\mathbf{L}} E\simeq \mathbf{R}f_*(A\otimes^{\mathbf{L}}\mathbf{L}f^*(E)). \tag{11.16a}$$

(3) There's a natural isomorphism

$$\mathbf{L}f^*(E\otimes^{\mathbf{L}} F)\simeq \mathbf{L}f^*(E)\otimes^{\mathbf{L}}\mathbf{L}f^*(F). \tag{11.16b}$$

(4) $\mathbf{L}f^*$ and $\mathbf{R}f_*$ are adjoint functors, i.e. there's a natural isomorphism

$$\mathrm{Hom}(\mathbf{L}f^*(E),A)\simeq \mathrm{Hom}(E,\mathbf{R}f_*(A)). \tag{11.16c}$$

(5) There's a natural isomorphism

$$\mathbf{R}\,\mathscr{H}om(A,B)\otimes^{\mathbf{L}} C\simeq \mathbf{R}\,\mathscr{H}om(A,B\otimes^{\mathbf{L}} C). \tag{11.16d}$$

(6) $–\otimes^{\mathbf{L}} B$ and $\mathbf{R}\,\mathscr{H}om(B,–)$ are adjoint functors, and moreover there's a natural isomorphism

$$\mathbf{R}\,\mathscr{H}om(A\otimes^{\mathbf{L}} B,C)\simeq \mathbf{R}\,\mathscr{H}om(A,\mathbf{R}\,\mathscr{H}om(B,C)). \tag{11.16e}$$

(7) There's a natural isomorphism

$$\mathbf{R}\,\mathscr{H}om(A,B\otimes^{\mathscr{L}} C)\simeq \mathbf{R}\,\mathscr{H}om(\mathbf{R}\,\mathscr{H}om(B,A),C). \tag{11.16f}$$

(8) There's a natural isomorphism $A^\vee\otimes^{\mathbf{L}} B\simeq \mathbf{R}\,\mathscr{H}om(A,B)$.

These will be helpful for computations, even though we haven't given their proofs. If you'd like to prove them, start with the version for modules over a ring, where things are easier, and try to say the same proof for coherent sheaves.

In this setting we have a very general duality theorem.

Theorem 11.17 (Grothendieck-Verdier duality)**.** *Let $f\colon X \to Y$ be a morphism of smooth projective schemes of relative dimension* $\dim f := \dim X - \dim Y$. *Let $\omega_f := \omega_f \otimes f^*\omega_Y^\vee$, $A \in D^b(X)$, and $B \in D^b(Y)$. Then there is a natural isomorphism*

$$\mathbf{R}f_*\mathbf{R}\mathscr{H}\!om(A, \mathbf{L}f^*(B) \otimes \omega_f[\dim(f)]) \simeq \mathbf{R}\mathscr{H}\!om(\mathbf{R}f_*A, B). \tag{11.18}$$

Despite its abstract-looking nature, this is occasionally useful for computations.

Now we'll be able to begin discussing Bridgeland stability, though we won't be able to give a definition until the next lecture. This is a notion of stability conditions on a triangulated category, rather than an abelian category. Bridgeland's original article [Bri07] is very readable, and a good reference.

But before we get to that, we need to discuss a little more triangulated category theory, specifically the notion of a heart of a t-structure. We're not going to discuss the notion of a t-structure in complete generality; if you'd like to read it, consult Beĭlinson-Bernstein-Deligne-Gabber [BBDG83], though they write in French.

The idea of (the heart of) a t-structure on a triangulated category is motivated by the following question.

Question 11.19. Suppose A and B are abelian categories and $F\colon D^b(\mathsf{A}) \to D^b(\mathsf{B})$ is a triangulated equivalence. This does *not* necessarily restrict to an equivalence $\mathsf{A} \to \mathsf{B}$ on complexes concentrated in degree zero, so what is the structure of B in $B^b(\mathsf{A})$?

We'll skirt the generality of bounded t-structures by defining the heart directly.

Definition 11.20. Let C be a triangulated category. The *heart of a bounded t-structure* in C is a full additive subcategory $\mathsf{C}^\heartsuit$ such that

(1) if $A, B \in \mathsf{C}^\heartsuit$ and $i, j \in \mathbb{Z}$ with $i > j$, then

$$\mathrm{Ext}^{j-i}(A, B) := \mathrm{Hom}_{\mathsf{C}}(A[i], B[j]) = 0. \tag{11.21}$$

(2) For all $E \in \mathsf{C}$, there are integers $k_1 > \cdots > k_m$, objects $E_0, \dots, E_m \in \mathsf{C}$, and a collection of distinguished triangles $E_{i-1} \to E_i \to A_i[k_i] \to E_{i-1}[1]$, where $A_i \in \mathsf{C}^\heartsuit$, $E_0 = 0$, and $E_m = E$.

The second condition is a kind of Harder-Narasimhan filtration, albeit in the derived sense. It turns out that these two axioms are enough to guarantee that $\mathsf{C}^\heartsuit$ is an abelian category!

Lecture 12.

The heart of a bounded t-structure: 2/28/19

Last time, we defined the heart $\mathsf{C}^\heartsuit$ of a bounded t-structure on a a triangulated category C – even though we didn't define bounded t-structures.

Example 12.1. Today, we'll begin with the standard example, an abelian category A sitting inside its bounded derived category as complexes concentrated in degree zero. When you learn about t-structures, this will be the heart of the *standard t-structure* on $D^b(\mathsf{A})$.

We need to show two things. First, that $\mathrm{Hom}_{D^b(\mathsf{A})}(A[i], B[j]) \neq 0$ for any $i > j$ and $A, B \in \mathsf{A}$. This is true, because we have two complexes concentrated at i and j, respectively, but $i \neq j$ so there are no maps between them.

The second thing we need to check is the existence of a Harder-Narasimhan-esque filtration of objects in $D^b(\mathsf{A})$ by objects in A. Let $E^\bullet \in D^b(\mathsf{A})$, so by definition $\mathcal{H}^i(E^\bullet)$ vanishes for $i \gg 0$ and $i \ll 0$. If $E^\bullet$ has vanishing cohomology, there's nothing to prove. Let r be the length of the largest interval $[i, j] \subset \mathbb{R}$ such that $\mathcal{H}^i(E^\bullet) \neq 0$ and $\mathcal{H}^j(E^\bullet) \neq 0$; we will induct on r. When $r = 0$, there's a single integer k such that $\mathcal{H}^{-k}(E^\bullet) \neq 0$. Define a map of complexes $A^\bullet \to E^\bullet$ by

$$\begin{tikzcd}
\cdots \arrow[r] & E^{-k-1} \arrow[r, "d^{-k-1}"] & E^{-k} \arrow[r, "d^{-k}"] & E^{-k+1} \arrow[r] & \cdots \\
\cdots \arrow[r] & E^{-k-1} \arrow[r] \arrow[u, equal] & \ker(d^{-k}) \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots ,
\end{tikzcd} \tag{12.2}$$

which commutes (there's not much to check), and moreover, is a quasi-isomorphism! We know the cohomology is zero above degree $-k$; at $-k$, we still get $\ker(d^{-k}) / \operatorname{Im}(d^{-k-1})$; and in lower degrees the complexes are the same. So we've cut off the right side of $E^\bullet$; now let's cut off the left by defining another quasi-isomorphism $A^\bullet \overset{\simeq}{\to} \mathcal{H}^{-k}(E^\bullet)[k]$:

$$\begin{tikzcd}
\cdots \arrow[r] & E^{-k-1} \arrow[r] \arrow[d] & \ker(d^{-k}) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \mathcal{H}^{-k}(E^\bullet) \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd} \tag{12.3}$$

In all degrees except for $-k$, this induces the zero map on cohomology, but cohomology is zero, so this is an isomorphism. In degree $-k$, the map is modding out by the image, which also is an isomorphism on cohomology. Thus, $E^\bullet \in \mathsf{A}[k]$ and we're set.

Now, for $r > 0$, let k be the smallest integer with $\mathcal{H}^{-k}(E^\bullet) \neq 0$. We will again "cut off" part of $E^\bullet$, though not quite via quasi-isomorphisms. Consider the two morphisms of complexes

$$\begin{tikzcd}
F^\bullet = \Big(\cdots \arrow[r] & E^{-k-2} \arrow[r] \arrow[d, equal] & \ker(d^{-k-1}) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \Big) \\
E^\bullet = \Big(\cdots \arrow[r] & E^{-k-2} \arrow[r] \arrow[d] & E^{-k-1} \arrow[r] \arrow[d] & E^{-k} \arrow[r] \arrow[d] & E^{-k+1} \arrow[r] \arrow[d] & \cdots \Big) \\
\mathcal{H}^{-k}(E^\bullet) \simeq \Big(\cdots \arrow[r] & 0 \arrow[r] & E^{-k-1} / \ker(d^{-k-1}) \arrow[r] & E^{-k} \arrow[r] & E^{-k-1} \arrow[r] & \cdots \Big)
\end{tikzcd} \tag{12.4}$$

The map $F^\bullet \to E^\bullet$ is *not* a quasi-isomorphism, but this *is* a short exact sequence of complexes, which means it induces a distinguished triangle $F^\bullet \to E^\bullet \to \mathcal{H}^{-k}(E^\bullet)[k]$, and $\mathcal{H}^{-k}(E^\bullet)[k]$ lives in $\mathsf{A}[k]$ as we want. Therefore it suffices to apply this to F, but F has one fewer nonzero cohomology group than E so we apply the inductive assumption.

In fact, we've proven something stronger than the statement about the filtration in Definition 11.20, namely that we can choose the pieces of the filtration to be cohomology groups of $E^\bullet$. Slicing off pieces of the heart! Sounds painful. ◀

Remark 12.5. If you take A inside $D^{\pm}(\mathsf{A})$, this argument doesn't work, which is what the "bounded" in "bounded t-structure" is all about. ◀

Exercise 12.6. Show that Harder-Narasimhan filtrations for Example 12.1 are unique, in the usual sense. Though this is not asked for by the axioms of Definition 11.20, it is implied by them.

Definition 12.7. Let $\mathsf{C}^\heartsuit$ be the heart of a bounded t-structure for a triangulated category C. Then the *cohomology with respect to* $\mathsf{C}^\heartsuit$ of an object $E^\bullet \in \mathsf{C}$ can be defined as follows: let $E_0 \to \cdots \to E_n$ be the Harder-Narasimhan filtration of $E^\bullet$, with the distinguished triangles $E_{i-1} \to E_i \to A_i[k_i] \to E_{i-1}[1]$. Then, define

$$\mathcal{H}^k_{\mathsf{C}^\heartsuit}(E^\bullet) = \begin{cases} A_i, & \text{if } k = k_i \text{ for some } i \\ 0, & \text{otherwise.} \end{cases}$$

Remark 12.8. **Warning:** we originally motivated t-structures as describing equivalences $D^b(\mathsf{A}) \to D^b(\mathsf{B})$ that don't restrict to equivalences $\mathsf{A} \to \mathsf{B}$. You might think, "oh, in that setting, the notion of cohomology with respect to the heart is the usual cohomology in the other category" but that is not true! ◂

Theorem 12.9. *The heart $\mathsf{C}^\heartsuit$ of a bounded t-structure is an abelian category. If $A \to B \to C \to A[1]$ is a distinguished triangle in C such that $A, B, C \in \mathsf{C}^\heartsuit$, then $0 \to A \to B \to C \to 0$ is a short exact sequence in $\mathsf{C}^\heartsuit$, and conversely.*

Proof sketch. Recall that if $f\colon A \to B$ is a morphism in $\mathsf{C}^\heartsuit$, and $C(f)$ denotes the cone on f, then the triangle $A \to B \to C(f) \to A[1]$ is distinguished. Then $C(f)$ is not always in $\mathsf{C}^\heartsuit$, but we claim it has a two-term Harder-Narasimhan filtration, which makes it accessible.

(1) First, show that $\mathcal{H}^i_{\mathsf{C}^\heartsuit}(C(f)) = 0$ unless $i = -1$ or $i = 0$. This uses the long exact sequence in cohomology associated to a distinguished triangle.
(2) Next, show that $\mathcal{H}^{-1}_{\mathsf{C}^\heartsuit}(C(f)) \cong \ker(f)$ and that $\mathcal{H}^0_{\mathsf{C}^\heartsuit}(C(f)) = \operatorname{coker}(f)$.

So if $C(f)$ *is* in $\mathsf{C}^\heartsuit$, we conclude that $\mathcal{H}^{-1}_{\mathsf{C}^\heartsuit}(C(f)) = 0$, so f is injective, and $\operatorname{coker}(f) = \mathcal{H}^0_{\mathsf{C}^\heartsuit}(C(f)) = C(f)$, so we have exactness at B. Running a similar argument again for the distinguished triangle $B \to C(f) \to A[1] \to B[1]$ shows that the map $B \to C(f)$ is surjective. ⊠

Let's now look at an example that's different from Example 12.1.

Example 12.10. Let A denote the smallest full additive subcategory of $D^b(\mathbb{P}^1)$ containing both $\mathscr{O}_{\mathbb{P}^1}(1)$ and $\mathscr{O}_{\mathbb{P}^1}[2]$ and which is *closed under extensions*, i.e. if $A \to B \to C \to A[1]$ is a distinguished triangle with $A, C \in \mathsf{A}$, then $B \in \mathsf{A}$ too.

We claim that there are no nontrivial extensions, i.e.

$$\mathsf{A} = \{\mathscr{O}_{\mathbb{P}^1}^{\oplus a}[2] \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus b} \mid a, b \in \mathbb{Z}_{\geq 0}\}. \tag{12.11}$$

This is a statement about sheaves not admitting extensions, so we need to compute some Ext groups:

$$\operatorname{Ext}^1(\mathscr{O}_{\mathbb{P}^1}(1), \mathscr{O}_{\mathbb{P}^1}[2]) = H^3(\mathbb{P}^1, \mathscr{O}(-1)) = 0 \tag{12.12}$$

because cohomology on a curve vanishes above degree 1, and

$$\operatorname{Ext}^1(\mathscr{O}_{\mathbb{P}^1}[2], \mathscr{O}_{\mathbb{P}^1}(1)) = H^{-1}(\mathbb{P}^1, \mathscr{O}(1)) = 0. \tag{12.13}$$

Exercise 12.14. Show that A is the heart of a bounded t-structure. Hint: use Theorem 5.9: that any $E \in \mathsf{Coh}(\mathbb{P}^1)$ splits as $T \oplus \mathscr{O}(a_1) \oplus \cdots \oplus \mathscr{O}(a_n)$, where T is torsion and $a_1, \ldots, a_n \in \mathbb{Z}$. You will also want to use the fact that the *Euler sequence*

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^1}(-2) \xrightarrow{\binom{x}{y}} \mathscr{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \longrightarrow 0 \longrightarrow 0, \tag{12.15}$$

where x and y are the two homogeneous coordinates on $\mathbb{P}^1$, is short exact (and then by tensoring around, you can shift this to other $\mathscr{O}(n)$).

Next, we claim that $\mathsf{A} \simeq \mathsf{Vect}_k \oplus \mathsf{Vect}_k$. This is clear on objects by (12.11), and for morphisms we need to check there are no "cross morphisms" $\mathscr{O}_{\mathbb{P}^1}[2] \rightleftarrows \mathscr{O}_{\mathbb{P}^1}$. This is again a fact about cohomology:

$$\operatorname{Hom}(\mathscr{O}_{\mathbb{P}^1}[2], \mathscr{O}_{\mathbb{P}^1}(1)) = H^{-2}(\mathbb{P}^1, \mathscr{O}(1)) = 0 \tag{12.16a}$$

$$\operatorname{Hom}(\mathscr{O}_{\mathbb{P}^1}(1), \mathscr{O}_{\mathbb{P}^1}[2]) = H^2(\mathbb{P}^1, \mathscr{O}(-2)) = 0, \tag{12.16b}$$

and the claim follows.

And most interestingly, we'll show there's no equivalence $D^b(\mathsf{A}) \to D^b(\mathbb{P}^1)$ that restricts to the identity on A. This time, we use Ext: compare

$$\operatorname{Ext}_{D^b(\mathbb{P}^1)}(\mathscr{O}_{\mathbb{P}^1}[2], \mathscr{O}_{\mathbb{P}^1}(1)) = H^0(\mathscr{O}_{\mathbb{P}^1}(1)) \cong k^2, \tag{12.17}$$

but since A is semisimple (it's a direct sum of two copies of Vect_k), all of its extensions vanish. In particular, this is not equivaleny to the standard t-structure on $D^b(\mathbb{P}^1)$. ◂

Definition 12.18. Let C be a triangulated category. Its *Grothendieck group* $K_0(\mathsf{C})$ is defined to be the quotient of the free abelian group on the objects of C by the triangles: if $A \to B \to C \to A[1]$ is a distinguished triangle in C, impose the re;lation $[A] + [C] = [B]$ in $K_0(\mathsf{C})$.

Exercise 12.19. Show that if $\mathsf{C}^\heartsuit$ is the heart of a bounded t-structure for a triangulated category C, then the inclusion map $\mathsf{C}^\heartsuit \hookrightarrow \mathsf{C}$ induces an isomorphism $K_0(\mathsf{C}^\heartsuit) \xrightarrow{\cong} K_0(\mathsf{C})$.

In particular, no matter what heart you choose, its Grothendieck group is the same. When we define stability conditions, this means the map $K_0(\mathsf{C}) \to \mathbb{C}$ always has the same domain.

The next definition looks a lot like the definition of a heart of a bounded t-structure, and is indeed very similar, but is different: instead of indexing by $\mathbb{Z}$, we index by $\mathbb{R}$.

Definition 12.20. Let C be a triangulated category. A *slicing* $\mathcal{P}$ of C is a collection of full additive subcategories $\mathcal{P}(\phi) \subset \mathsf{C}$ for all $\phi \in \mathbb{R}$, such that

(1) $\mathcal{P}(\phi)[1] = \mathcal{P}(\phi + 1)$;
(2) if $\phi_1 > \phi_2$ and $A \in \mathcal{P}(\phi_1)$ and $B \in \mathcal{P}(\phi_2)$, then $\mathrm{Hom}_{\mathsf{C}}(A, B) = 0$; and
(3) for all $E \in \mathsf{C}$ there are real numbers $\phi_1 > \phi_2 > \cdots > \phi_n$ and objects $E_i \in \mathsf{C}$ and $A_i \in \mathcal{P}(\phi_i)$ for $i = 1, \dots, n$, together with distinguished triangles $E_i \to E_{i+1} \to A_i \to E_i[1]$, such that $E_0 = 0$ and $E_n = E$.

Definition 12.21. The filtration $0 = E_0 \to E_1 \to \cdots \to E_n$ is again called a *Harder-Narasimhan filtration*. We will write $\phi^+(E) := \phi_1$ and $\phi^-(E) := \phi_n$.

As always, Harder-Narasimhan filtrations are unique.

Proposition 12.22. *For any $a \in \mathbb{R}$, the subcategory $\mathcal{P}((a, a+1])$, defined to be the full subcategory of objects $E \in \mathsf{C}$ such that $\phi^+(E) \leq a + 1$ and $\phi^-(E) > a$, is the heart of a bounded t-structure on C.*

We're almost at the definition of a Bridgeland stability condition, and will get there early next lecture. But first, two more exercises.

Exercise 12.23. Let $\mathsf{C}^{\heartsuit_1}, \mathsf{C}^{\heartsuit_2}$ are two hearts of bounded t-structures on a triangulated category C, and suppose $\mathsf{C}^{\heartsuit_1} \subset \mathsf{C}^{\heartsuit_2}$ (that is, as a full subcategory, though in this case fullness is automatic from the definition of the heart). Then $\mathsf{C}^{\heartsuit_1} = \mathsf{C}^{\heartsuit_2}$.

Exercise 12.24. Show similarly that if $\mathcal{P}_1$ and $\mathcal{P}_2$ are two slicings of a triangulated category C such that $\mathcal{P}_1(\phi) \subset \mathcal{P}_2(\phi)$ for all $\phi \in \mathbb{R}$, then $\mathcal{P}_1 = \mathcal{P}_2$.

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