# M392c NOTES: ALGEBRAIC GEOMETRY

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Lecture 1.

## Some questions in algebraic geometry: 8/29/18

Office hours are Fridays from 11-1, in room 9.164 (at least for now). Today we'll talk about some questions (and some answers, too!) relating to algebraic geometry and why one might find it interesting. We're going to focus on concreteness.

Broadly speaking, algebraic geometry studies zero sets of polynomials. These could be polynomials over $\mathbb{Q}$, or $\mathbb{R}$, or $\mathbb{C}$, or finite fields, or more. The first question you might ask is, *are there solutions*? This is an *arithmetic question*: in arithmetic situations, there might not be solutions.

**Example 1.1** (Taylor-Wiles, 1994)**.** If $n \geq 3$, the polynomial $x^n + y^n = 1$ has no solutions over $\mathbb{Q}$ when $x, y \neq 0$. ◀

You might recognize this as a reformulation of Fermat's last theorem.

Another form of the same question is *can you parameterize solutions of the equation*? For example, let's try it with $x^2 + y^2 = 1$, which we know has solutions. In this case, it is possible to parameterize solutions, via the one-parameter family

$$x = \frac{\lambda^2 - 1}{\lambda^2 + 1}, \qquad y = \frac{2\lambda}{\lambda^2 + 1}. \tag{1.2}$$

These kinds of questions are called *rationality questions*. One can also ask these questions over $\mathbb{C}$ (or over other algebraically closed fields), where they can feel a bit different.

There is a general result that any quadric hypersurface with a rational point is rational. What this means is that if you assume the existence of one solution $(x_0, y_0)$ to a degree-2 polynomial in $x$ and $y$ over, say, $\mathbb{Q}$, then you can use that one solution to parameterize all other solutions. If you plot the solutions in the $xy$-plane, the parameter of another solution $(x_1, y_1)$ is the slope of the line between $(x_0, y_0)$ and $(x_1, y_1)$. Indeed, in (1.2), the parameter $\lambda$ is this slope. Because the equation is a quadric, one expects such a line to intersect in exactly two points, the first solution and another one. This is all extremely explicit, to the point that you could explain why you care to a middle schooler.

There are a few other rationality results.

**Theorem 1.3** (Segre, 1940s; Manin, 1970s; Kollár, 2000)**.** *Smooth cubics in at least three variables are rational.*

So $x^2 + y^2 = 1$ isn't rational, but $x^2 + y^2 + z^2 = 1$ is. However, this doesn't give you everything.

**Theorem 1.4** (Clemens-Griffiths, 1974)**.** *There are cubics in at least four variables which are unirational but not rational, i.e. that one cannot parameterize all solutions in a one-to-one manner.*

This was a hard theorem. How would you prove something like this?

Recent work (2012-15) by many people (Voisin, Colliot-Thèlene, Pirutka, Totaro[1]) generalizes this.

**Theorem 1.5.** *For cubics in at least five variables, one can also not parameterize solutions in a one-to-one way, even by adding additional "dummy variables."*

For four-variables cubics, this is open.

[1] If you like pictures of cats, check out Totaro's math blog: https://burttotaro.wordpress.com/.

**Schemes.** Though this result is stated completely explicitly, it was studied using some very abstract-looking machinery. In this course, we'll also work with this abstract machinery, namely the language of schemes. These are things like solutions to systems of polynomials, but not quite — they encode among other things the equivalence of such systems under changes of coordinates, which doesn't really change the underlying geometry of the solution set. Classification problems with this perspective are a big area of research, and Birkar just won a Fields medal for work in this area from 2006.

**Algebraic geometry over $\mathbb{C}$.** A third thing you could care about is specific stuff about algebraic geometry over your favorite field (typically $\mathbb{C}$, but not always). In many cases (such as $\mathbb{C}$), you have topology around, and you can ask how it interacts with the algebraic geometry we've been talking about.

For example, if $q \in \mathbb{C}^\times$ isn't a root of unity, then there's a cubic equation $y^2 = x^2 + ax + b$ whose solutions are parameterized by $\mathbb{C}^\times/q\mathbb{Z}$. This may be a bit surprising, and indicates a way in which analytic or topological information can be useful: now we can learn about the universal cover of the solution space, and other topological invariants. Then you might ask whether something like this is true in positive characteristic, which tends to be harder.

More generally, one can study the topology of algebraic varieties over $\mathbb{C}$.

**Theorem 1.6.** *The odd Betti numbers of smooth proper varieties are even.*

The proof uses the study of the Hodge Laplacian operator on a variety $X$. This needs a metric, but projective means that $X$ embeds in some $\mathbb{CP}^n$, and we can borrow its metric. There is a purely algebro-geometric proof of this, but first you need to come up with the right notion of Betti numbers (so étale cohomology, which is hard), and then invoke Deligne's proof of the Weil conjectures (also hard). Nonetheless, it's true in characteristic $p$.

More generally, the cohomology of a complex projective variety has more structure, and is much richer than that of a random manifold.[2]

**Conjecture 1.7** (Hodge conjecture, imprecise statement)**.** *The differential topology of a projective algebraic variety over $\mathbb{C}$ knows everything about its algebraic geometry.*

This is a Millennium Prize problem, meaning it comes with a \$1 million reward. You can infer that it's hard.

**Algebraic geometry over $\mathbb{Z}$.** If you work over $\mathbb{Z}$ instead of over $\mathbb{C}$, meaning your polynomial has integer coefficients, then you can reduce mod $p$ and solve it there. This is the first thing anyone does in number theory, because it often simplifies the problem to a finite question. This naturally leads one to ask, *how do the systems of equations at different primes p relate to each other?*

There's a lot to say about this, beginning with quadratic reciprocity, which is very classical yet a little weird, and continuing all the way to the Langlands program.

Supposing $X$ encodes the system of solutions to your polynomial with $\mathbb{Z}$ coefficients. Then one can define a zeta function, reminiscent of the Riemann zeta function, as follows:

$$\zeta_X(s) \coloneqq \prod_{p \text{ prime}} \exp\left(\sum \frac{1}{n}(\text{number of solutions in } \mathbb{F}_{p^n})p^{-ns}\right). \tag{1.8}$$

For $X = \operatorname{Spec}\mathbb{Z}$, corresponding to solutions to an empty set of polynomials, this recovers the usual Riemann zeta function.

For any particular $X$, one conjectures this is meromorphic (and almost entire, in some sense), and that the analogue of the Riemann hypothesis holds; for some $X$, this is known due to Deligne. There are some other related conjectures related to this known as Sato-Tate conjectures.

**Cohomology theories.** Over $\mathbb{C}$, you have topology, and therefore can invoke algebraic topology to compute cohomology of algebraic varieties. Over other fields or rings, you might not have these techniques, and there are several other approaches.

- Over an algebraically closed field, one has *étale cohomology*, whose ideas are built from covering space theory, has $\mathbb{Z}_\ell$ coefficients, where $\ell$ is a prime that's not the characteristic of the field.

[2]This doesn't require smoothness *per se*, but it's more difficult to formulate in the singular case.

- Over any field $k$, there's *de Rham cohomology*, which uses the idea that $\mathrm{d}z/z$ understands $\mathbb{C}^\times$ isn't simply connected (since $\oint \mathrm{d}z/z \neq 0$). This has coefficients in $k$.

There are others, too. One wants these to all be the same, or at least closely related; if $k = \mathbb{Q}_p$ and $\ell = p$ ($\mathbb{Q}_p$ has characteristic zero!), then these two are related by $p$-adic Hodge theory. This is related to deep and recent work by Fontaine, Scholze, and others, and relates to Scholze's Fields medal work. In 2016, Bhatt-Morrow-Scholze showed that one can sometimes interpolate between different cohomology theories. See Scholze's ICM address for more on this. The ultimate question in this corner of algebraic geometry is whether there's some universal cohomology theory interpolating between everything we have, and which is also the source of the $\zeta$-functions mentioned above.

**Degenerations.** We get additional power by studying solutions in families. For example, we can degenerate $x^2 + y^2 = 1$ to $x^2 + y^2 = 0$, which is much simpler. One asks questions such as, *what invariants are preserved under degenerations?* Therefore one might be able to use a degeneration to reduce a harder problem to an easier problem.

**Computations.** This subfield of algebraic geometry tries to make these abstract invariants concrete, by writing good algorithms to compute these invariants for explicit systems of polynomials.

**Geometric complexity theory.** This is another way to relate algebraic geometry and computer science. The goal of this field is to approach another Millennium Prize problem, P vs. NP, using algebraic geometry techniques. This roughly involves studying certain varieties and analyzing whether they're as complicated as they seem. Algebraic geometry has lots of techniques which might help, but on the other hand they haven't yet.

Probably the best way to learn algebraic geometry is to have an application or research focus in mind that you can apply the things you learn to. This method of learning tends to produce algebraic geometers.

Lecture 2.

## Defining schemes, I: 8/31/18

The goal of today's lecture is to define a scheme, first heuristically and then rigorously.

**"Definition" 2.1.** A scheme is a "space" that is a Zariski sheaf which admits an "open cover" by affine schemes.

Of course, in order to do this, we need to know what all of these words — spaces, Zariski sheaves, affine schemes, and open covers — mean in this setting.

*Remark* 2.2. There's another approach to schemes using the formalism of *locally ringed spaces*, which is followed by Hartshorne, Vakil, and many others. It's more concrete, but it makes it harder to think about what a specific scheme, such as projective space, is supposed to be. ◀

The motivation for "Definition" 2.1 is that a scheme should be something which is locally defined by algebraic equations. For example, let's look at the *Fermat equation* $X_n = \{x^n + y^n = z^n\}$. Fermat was interested in solutions in $\mathbb{Z}$, but the set of solutions makes sense in any commutative ring. This suggests our definition of space, which is not the same as a topological space.

**Definition 2.3.** A *space* is a functor $X\colon \mathcal{C}omm\mathcal{R}ing \to \mathcal{S}et$.

Concretely, this means that for every ring $A$, we get a set $X(A)$, and for every map of commutative rings $f\colon A \to B$, we get a map of sets $X(f)\colon X(A) \to X(B)$, and these morphisms should compose well (meaning that $X(f \circ g) = X(f) \circ X(g)$ and $X(\mathrm{id}) = \mathrm{id}$). For example, we could let $X_n(A)$ denote the set of solutions to the Fermat equation in the ring $A$; then, if we've solved it in $A$, we can map the solution into $B$ via $f\colon A \to B$, and we'll obtain a solution in $B$, so this defines a space $X_n$.

We should also say how spaces interact.

**Definition 2.4.** A *morphism of spaces* $f\colon X \to Y$ is data of, for all commutative rings $A$, a map $f_A\colon X(A) \to Y(A)$ such that for all ring homomorphisms $g\colon A \to B$, the diagram

$$\begin{tikzcd} X(A) \arrow[r, "f_A"] \arrow[d, "X(g)"] & Y(A) \arrow[d, "Y(g)"] \\ X(B) \arrow[r, "f_B"] & Y(B) \end{tikzcd}$$

commutes.

Schemes are special examples of spaces, in a way that feels surprisingly down-to-Earth.

Our first example of a space is the solutions to the Fermat equation in $A$, as discussed above. Here's another example.

**Example 2.5.** Let $A$ be a commutative ring. We'll define the space $\operatorname{Spec} A$ to be the functor $(\operatorname{Spec} A)(B) = \operatorname{Hom}(A, B)$; given a ring homomorphism $\varphi\colon B \to C$, we use the map $\operatorname{Hom}(A, B) \to \operatorname{Hom}(A, C)$ given by postcomposition with $\varphi$. ◀

**Definition 2.6.** An *affine scheme* is a space of the form $\operatorname{Spec} A$ for some $A$.

You don't have to be a commutative algebra expert to learn algebraic geometry, but you can see that commutative algebra is built into the definitions of algebraic geometry, so some commutative algebra knowledge is helpful.

**Example 2.7.** The space $X_n$ sending $A$ to the solutions of the Fermat equation in $A$ is an affine scheme; explicitly,

$$X_n \cong \operatorname{Spec} \mathbb{Z}[x, y, z]/(x^n + y^n - z^n).$$

This is because a ring homomorphism $\mathbb{Z}[x, y, z]/(x^n + y^n - z^n) \to A$ is exactly the data of $x, y, z \in A$ satisfying the relation $x^n + y^n - z^n = 0$. ◀

**Lemma 2.8** (Yoneda lemma)**.** *For all spaces $X$,* $\operatorname{Hom}_{\mathcal{S}paces}(\operatorname{Spec} A, X) \cong X(A)$.

*Proof sketch.* First we define a map from $\operatorname{Hom}_{\mathcal{S}paces}(\operatorname{Spec} A, X)$ to $X(A)$. Specifically, a map $f\colon \operatorname{Spec} A \to X$ is the data of for all commutative rings $B$, $\operatorname{Spec}(A)(B) \to X(B)$. Take $B = A$; then, $\operatorname{Spec}(A)(A) = \operatorname{Hom}(A)$, so take the image of the identity. It remains to check this is an equivalence. ⊠

**Corollary 2.9.** $\operatorname{Hom}_{\mathcal{S}paces}(\operatorname{Spec} A, \operatorname{Spec} B) \cong \operatorname{Hom}_{\mathcal{C}omm\mathcal{R}ing}(B, A)$.

It's interesting that the direction reverses!

*Proof.* By the Yoneda lemma, $\operatorname{Hom}_{\mathcal{S}paces}(\operatorname{Spec} A, \operatorname{Spec} B) = \operatorname{Spec}(B)(A) = \operatorname{Hom}(B, A)$. ⊠

This tells you that as long as you make sure to reverse the arrows, anything you can do with commutative rings, you can do with affine schemes, and vice versa.

**Fiber products.** This is a categorical construction which we're going to use a lot.

**Definition 2.10.** Let $X$, $Y$, and $Z$ be sets and $f\colon X \to Z$ and $g\colon Y \to Z$ be set maps. Then the *fiber product* of $X$ and $Y$ over $Z$ is

$$X \times_Z Y \coloneqq \{(x, y) \in X \times Y \mid f(x) = g(y)\}. \tag{2.11}$$

If $X$, $Y$, and $Z$ are spaces, and $f$ and $g$ are maps of spaces, then the *fiber product* of $X$ and $Y$ over $Z$ is the space defined by

$$(X \times_Z Y)(A) \coloneqq X(A) \times_{Z(A)} Y(A). \tag{2.12}$$

Technically, the notation should include $f$ and $g$, but in practice there's usually no ambiguity.

**Example 2.13.** Suppose we're given commutative rings $A$, $B$, and $C$ and maps $\operatorname{Spec} B \to \operatorname{Spec} C$ and $\operatorname{Spec} A \to \operatorname{Spec} C$ (which are equivalent data to maps $\varphi\colon C \to A$ and $\psi\colon C \to B$). Then

$$\operatorname{Spec} A \times_{\operatorname{Spec} C} \operatorname{Spec} B \cong \operatorname{Spec}(A \otimes_C B),$$

where $C$ acts on $A$, resp. $B$, through $\varphi$, resp. $\psi$. It's worth working through this one on your own, though it's not extremely hard. ◀

We'll define some properties of affine schemes with geometric names, but the definitions will rest on algebraic properties of rings. One of the real miracles of algebraic geometry is that this really works to define geometry, and even extends geometric intuition to places such as finite fields that are otherwise very hard to reason about.

**Definition 2.14.** A morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is a *closed embedding* if the induced map $A \to B$ is surjective.

Equivalently, $B = A/I$ for some ideal $I$ of $A$.

The geometric idea behind defining $\operatorname{Spec} A$ is that geometric objects have a ring of functions on them, e.g. a smooth manifold $M$ has a ring $C^\infty(M)$ of smooth $\mathbb{R}$-valued functions, and a map of manifolds $M \to N$ induces a map in the other direction by pullback: $C^\infty(N) \to C^\infty(M)$. Functional analysis results such as the Gelfand-Naimark theorem tell you what data you need to add to $C^\infty(M)$ to recover $M$ as a topological space, and we're trying to imitate this in a more abstract algebraic setting.

This context allows us to explain why Definition 2.14 deserves to be called a closed embedding: let $I = (f_1, f_2, \dots)$, so

$$\operatorname{Spec} A/I = \{f_i = 0 \text{ for all } i\} = \{f = 0 \text{ for all } f \in I\}. \tag{2.15}$$

So we think of $\operatorname{Spec} B$ as some kind of closed subspace of $\operatorname{Spec} A$, and $I$ as the ideal of functions on $\operatorname{Spec} A$ which vanish on $\operatorname{Spec} B$. This intuition can be turned into something precise.

Using fiber products, we can extend this to all spaces.

**Definition 2.16.** A map $X \to Y$ of spaces is a *closed embedding* if for all maps $\operatorname{Spec} A \to Y$, the "pullback" $\varphi$ in the fiber product diagram

$$\begin{tikzcd} \operatorname{Spec} A \times_Y X \arrow[r, "\varphi"] \arrow[d] & \operatorname{Spec} A \arrow[d] \\ X \arrow[r] & Y \end{tikzcd} \tag{2.17}$$

is a closed embedding of affine schemes. In particular, we require $\operatorname{Spec} A \times_Y X$ to be an affine scheme, which is not always satisfied.

For a quick consistency check, we should ask that Definitions 2.14 and 2.16 agree on affine schemes, and indeed, if $I \subset A$ is an ideal, and $\operatorname{Spec} B \to \operatorname{Spec} A$ is a closed embedding in the sense of Definition 2.14, then (2.17) looks like

$$\begin{tikzcd} \operatorname{Spec}(A/I \otimes_A B) \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\ \operatorname{Spec}(A/I) \arrow[r] & \operatorname{Spec} A, \end{tikzcd} \tag{2.18}$$

and since $A/I \otimes_A B \cong B/BI$, this is a closed embedding in the more general sense as well.

We'd also like to know what an open embedding is. We'd like to say that it's something whose complement is a closed embedding. Let's make this precise.

**Definition 2.19.** Let $Z \hookrightarrow X$ be a closed embedding of spaces. The *complement* $X \setminus Z$ of $Z$ in $X$ is the space with $(X \setminus Z)(A)$ the set of $x \in X(A) = \operatorname{Hom}_{\mathcal{S}paces}(\operatorname{Spec} A, X)$ such that the diagram

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & Z \arrow[d] \\ \operatorname{Spec} A \arrow[r, "x"] & X \end{tikzcd} \tag{2.20}$$

is a fiber product diagram. Here $\varnothing = \operatorname{Spec}(0)$, which sends every ring to the empty set.[3]

**Definition 2.21.** If $X = \operatorname{Spec} A$ is an affine scheme, an *open embedding* is a map of spaces $j \colon U \to X$ such that $U = X \setminus Z$ for some closed embedding $Z \hookrightarrow X$.

[3]Caution: this is only true if we work with functors on nonzero rings. However, $\varnothing = \operatorname{Spec} 0$ still counts as affine. There are other ways to correct this issue, but this is among the fastest and cheapest.

**Example 2.22.** Letting $X = \operatorname{Spec} A$, if $f \in A$ and $Z = \operatorname{Spec}(A/f)$, the map $A \twoheadrightarrow A/f$ induces a closed embedding $Z \hookrightarrow X$. Its complement is $\operatorname{Spec} A[f^{-1}]$, the *localization* of $A$ at $f$, so $\operatorname{Spec} A[f^{-1}] \to \operatorname{Spec} A$ is an open embedding. ◀

The intuition is that $f$ generates the ideal of functions that vanish precisely on the closed subset $Z$. Therefore on the complement of $Z$, they should be invertible, so we adjoin an inverse to $f$.

**Lemma 2.23.** *Let $X = \operatorname{Spec} A$ and $Z = \operatorname{Spec} A/I$. Then maps $\operatorname{Spec} B \to X \setminus Z$ correspond bijectively to maps $A \to B$ such that $B \cdot I = B$.*

*Proof.* The diagram (2.20) specializes to

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & \operatorname{Spec}(A/I) \arrow[d] \\ \operatorname{Spec} B \arrow[r] & \operatorname{Spec} A, \end{tikzcd} \tag{2.24}$$

and this fiber product is $\operatorname{Spec}(B \otimes_A A/I) = \operatorname{Spec}(B/IB) = \varnothing$, which is equivalent to $IB = B$. ⊠

**Example 2.25.** *Affine $n$-space* over $\mathbb{Z}$ is the affine scheme $\mathbb{A}^n_{\mathbb{Z}} := \operatorname{Spec} \mathbb{Z}[x_1, \dots, x_n]$, and $0 \hookrightarrow \mathbb{A}^n_{\mathbb{Z}}$ is the closed embedding corresponding to the ideal $(x_1, \dots, x_n)$. The complement $\mathbb{A}^n_{\mathbb{Z}} \setminus 0$ is not affine for $n > 1$! We'll prove that later when we have more tools. ◀

**Exercise 2.26.** Show that $\mathbb{A}^n_{\mathbb{Z}} \setminus 0$ is the space which maps a ring $A$ to the set of $n$-tuples $(x_1, \dots, x_n) \in A^n$ such that the equation $\sum x_i y_i = 1$ has a solution.

Lecture 3.

## Open covers: 9/5/18

We've been talking about functors as if they were honest geometric objects. And they *are*: the crucial reason is that we're defining open and closed subspaces of affine schemes. You can picture these as akin to open or closed sets in a topological space, and they will allow us to make sense of geometry by giving us notions of locality.

Recall that $Z \hookrightarrow X = \operatorname{Spec} A$ is a closed embedding means that this embedding is of the form $\operatorname{Spec}(A/I) \to \operatorname{Spec} A$ induced by the map $A \twoheadrightarrow A/I$, and that open embeddings are complements of closed ones. You might think of the complement as $(X \setminus Z)(B) = X(B) \setminus Z(B)$, but **this is wrong**: it's not even functorial! Instead, we want to say $(X \setminus Z)(B) = \{\operatorname{Spec} B \to X \setminus Z\}$. What this means is maps $\operatorname{Spec} B \to X$ such that the pullback $\operatorname{Spec} B \times_X Z = \varnothing$. Geometrically, this fiber product is telling you the intersection of the image of $\operatorname{Spec} B$ with $X$.

Last time, we also talked about $\mathbb{A}^1$ (also $\mathbb{A}^1_{\mathbb{Z}}$ if you want to specify the base), which is by definition $\operatorname{Spec} \mathbb{Z}[t]$. It would be nice to think of this as a line, in the sense you can draw; but it behaves more like a complex line (that is, a plane). For example, $\mathbb{A}^1$ minus a point is connected. So thinking of it as a complex line is good, but for drawing pictures you'll run out of dimensions, so the picture of a real line is also helpful.

If $B$ is a commutative ring, $\mathbb{A}^1(B) = \{\operatorname{Spec} B \to \mathbb{A}^1\}$, i.e. $\operatorname{Hom}(\mathbb{Z}[t], B) = B$, because the map is determined by where it sends $t$. This makes precise the notion that the ring of functions on $\operatorname{Spec} B$ is $B$. This is another avatar of geometry as we know it: functions on a geometric object (say, a complex manifold) are functions to a complex line, and in this setting we replace the complex line by $\mathbb{A}^1$.

Consider the embedding $0 \hookrightarrow \mathbb{A}^1_{\mathbb{Z}}$, where $0$ denotes the locus where $t = 0$, i.e. $\operatorname{Spec} \mathbb{Z}[t]/(t)$. As an affine scheme, this is isomorphic to $\operatorname{Spec} \mathbb{Z}$, because $\mathbb{Z}[t]/(t) \cong \mathbb{Z}$, but this defines a particular closed embedding $0 \hookrightarrow \mathbb{A}^1_{\mathbb{Z}}$. Last time, we discussed $\mathbb{A}^1 \setminus 0$. A map $\operatorname{Spec} B \to \mathbb{A}^1 \setminus 0$ is a function that avoids zero, which means that it's invertible.

**Exercise 3.1.** Show that $(\mathbb{A}^1 \setminus 0)(B) = B^\times$, and therefore that $\mathbb{A}^1 \setminus 0 \cong \operatorname{Spec} \mathbb{Z}[t, t^{-1}]$.

If we did this with $\mathbb{A}^2 \setminus 0$ instead of $\mathbb{A}^1 \setminus 0$, we'd obtain a nonaffine scheme.

Open coverings are another important geometric notion, and they exist in this setting too.

**Definition 3.2.** If $X = \operatorname{Spec} A$ is an affine scheme, a *(Zariski) open covering* of $X$ is a collection of open embeddings $\mathfrak{U} = \{(U, i_U : U \hookrightarrow X)\}$ such that for every nonempty $S = \operatorname{Spec} B$ and $f : S \to X$, there's some $(U, i_U) \in \mathfrak{U}$ such that $U \times_X S \neq \varnothing$.

This is the first notion of open covering in algebraic geometry; there are some others around.

The intuition behind open coverings is that points of $X$ are given by maps $\operatorname{Spec} B \to X$, and we want every point in $X$ to intersect some open embedding in the cover.

**Proposition 3.3.** *Let $X = \operatorname{Spec} A$ and $\mathfrak{U} = \{(U, i_U : U \to X)\}$ be a collection of open embeddings. The following are equivalent:*

(1) *$\mathfrak{U}$ is an open covering.*
(2) *$\mathfrak{U}$ has a finite subset $\mathfrak{V} \subset \mathfrak{U}$ which is also an open covering of $X$.*
(3) *For all fields $k$ and maps $x : \operatorname{Spec} k \to X$, there's some $(U, i_U) \in \mathfrak{U}$ such that $x$ factors through $i_U$.*
(4) *Letting $U = X \setminus Z_U$ for each $U \in \mathfrak{U}$, and writing $Z_U = \operatorname{Spec}(A/I_U)$, then*

$$\sum_{U\in\mathfrak{U}} I_U = A.$$

Point (2) is very weird coming from topology, where the open covering $\{(i-1, i+1) \mid i \in \mathbb{Z}\}$ is an open cover of $\mathbb{R}$ with no finite subcover. In other words, affine schemes feel like compact spaces from the perspective of open coverings!

The idea behind (3) is that points are affine schemes of the form $\operatorname{Spec} k$ for $k$ a field. There are different fields, and therefore different kinds of points. The reason for including (4) is that it's very useful for checking in practice. It has a similar feel to partitions of unity in manifold topology, but if you don't know what that is, that's OK.

*Proof.* We'll first show (1) $\Longrightarrow$ (4). Suppose $\mathfrak{U}$ is an embedding for which (4) does not hold. Then let

$$B := A/\sum_{U\in\mathfrak{U}} I_U. \tag{3.4}$$

By hypothesis, $B \neq 0$, and we have a closed embedding $\operatorname{Spec} B \hookrightarrow \operatorname{Spec} A$. We'll show that $\operatorname{Spec} B \times_X U = \varnothing$ for all $U \in \mathfrak{U}$.

**Lemma 3.5.** *Let $Z = \operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A = X$ be a closed embedding and $f : \operatorname{Spec} B \to X$ be a map. Then*

$$(\operatorname{Spec} B) \setminus f^{-1}(Z) = \operatorname{Spec} B \times_X (X \setminus Z).$$

This is more or less a tautology.

Returning to the claim, $\operatorname{Spec} B \times_X U$ is the complement of $Z_U \times_X \operatorname{Spec} B = \operatorname{Spec}(B/BI_U)$. But $B/BI_U = B$, so the complement of $Z_U \times_X \operatorname{Spec} B$ is the empty set.

Next, we'll show (4) $\Longrightarrow$ (3). Let $k$ be a field, and $x : \operatorname{Spec} k \to X$ be a map. We want to show this map factors through some $U$. Since $X = \operatorname{Spec} A$, $x$ corresponds to a map $\varphi : A \to k$. We claim there's a $U \in \mathfrak{U}$ with $\varphi(I_U) \neq 0$; otherwise $\varphi(\sum I_U) = 0$, and therefore $\varphi(A) = 0$. However, $\varphi(1) = 1$, so this is impossible. By Lemma 2.23, since $\varphi(I_U) \neq 0$, $k \cdot \varphi(I_U) = k$, and therefore $x : \operatorname{Spec} k \to X$ factors through $U$.

Next we'll show (3) $\Longrightarrow$ (1). Let $B$ be as in (3.4) and $f : S = \operatorname{Spec} B \to X$ be a map. We want to show that $S \times_X U \neq 0$ for some $U \in \mathfrak{U}$. Since $B \neq 0$, it has a maximal ideal $\mathfrak{m}$, and $B/\mathfrak{m}$ is a field $k$ (TODO: to be continued...) ⊠

Lecture 4.

## Defining schemes, II: 9/7/18

*"I'll let* Fun$(Y)$*, which is such a fun notation, denote..."*

Last time, we talked about open embeddings and open covers for affine schemes; today, we'll generalize this to spaces.

**Definition 4.1.** Let $X$ be a space.

(1) A map $U \to X$ of spaces is an *open embedding* if for all affine schemes $S = \operatorname{Spec} A$ and maps $f : S \to X$ of spaces, the pullback $g : U \times_X S \to S$ arising in the diagram

$$\begin{tikzcd} U \times_X S \arrow[r] \arrow[d, "g"] & U \arrow[d] \\ S \arrow[r, "f"] & X \end{tikzcd}$$

is an open embedding (since we've already define open embeddings where the target is affine).

(2) A *Zariski open covering* of $X$ is the same as in Definition 3.2, but for open embeddings of spaces, rather than affine schemes.

In this case, Proposition 3.3 need not hold: there are open coverings of some spaces $X$ (such as an infinite disjoint union of points) which have no finite subcoverings.

**Definition 4.2.** A space $X$ is a *Zariski sheaf* if for all $S = \operatorname{Spec} A$ and open coverings $\mathfrak{U}$ of $S$, the map

$$\operatorname{Hom}(S, X) \longrightarrow \{(f_U \colon U \to X \text{ for all } U) \in \mathfrak{U} \mid f_U|_{U\cap V} = f_V|_{U\cap V} \text{ for all } U, V \in \mathfrak{U}\}$$

is an isomorphism. (Here $U \cap V = U \times_X V$.)

Not everything is a Zariski sheaf, but the things that aren't are terrible, and you shouldn't worry too much about them.

Now we have all the definitions at hand to define schemes!

**Definition 4.3.** A *scheme* is a space which is a Zariski sheaf and admits an open cover $\mathfrak{U}$ such that all $U \in \mathfrak{U}$ are affine schemes.

**Exercise 4.4.** Let $X$ be the space with

$$X(A) = \{t \in A \mid t \in A^\times \text{ or } (1-t) \in A^\times\}.$$

Show that $X$ is *not* a Zariski sheaf. Also, if you know what sheafification is, show that the sheafification of $X$ is $\mathbb{A}^1$.

**Proposition 4.5.** *If $X$ is an affine scheme, then it's a scheme.*

Obviously $X$ admits an open cover by affines, given by $\mathrm{id} \colon S \to S$; the meat of the proof (or, if you prefer, tofu) is that it's a Zariski sheaf. Unlike EGA, we will start with a special case and use it to bootstrap to the general case.

Let $X = \mathbb{A}^1$.

**Definition 4.6.** A *function* on a space $Y$ is a map to $\mathbb{A}^1$. We'll let $\operatorname{Fun}(Y) \coloneqq \operatorname{Hom}(Y, \mathbb{A}^1)$.

We're explicitly trending towards geometric notation and intuition for things: one of the key processes of learning scheme theory is to start thinking geometrically rather than with commutative algebra – except when you need to prove something.

We want to show that for all affine schemes $S$ and open coverings $\mathfrak{U}$ of $S$, the map

$$\operatorname{Fun}(S) \longrightarrow \{(f_U \in \operatorname{Fun}(U)) \mid f_U|_{U\cap V} = f_V|_{U\cap V}\} \tag{4.7}$$

is an isomorphism.

First we'll prove this for a nice class of open covers.

**Lemma 4.8.** *Let $A$ be a commutative ring and $f_1, \dots, f_n \in A$. Let $D(f_i) \coloneqq \operatorname{Spec} A \setminus \operatorname{Spec}(A/(f_i))$. Then*

(1) *$D(f_i) = \operatorname{Spec} A[f_i^{-1}]$, and*

(2) *$\{D(f_i)\}$ is an open cover iff $\{f_i\}$ generates the unit ideal.*

The proof will be left as an exercise.

In the case (2) holds, the open cover $\{D(f_i)\}$ is called a *basic open cover*. It's really nice because it's an affine open cover; we'll see that there are a lot of these coverings, and enough that we will eventually be able to reduce to this case.

One can alternately characterize $D(f_i)$ as the pullback

$$\begin{tikzcd} D(f_i) \arrow[r] \arrow[d] & \operatorname{Spec} A \arrow[d] \\ \mathbb{A}^1 \setminus 0 \arrow[r] & \mathbb{A}^1. \end{tikzcd} \tag{4.9}$$

**Lemma 4.10.** *Let $f \colon M \to N$ be a map of $A$-modules. Then $f$ is injective (resp. surjective, resp. bijective) iff for all $i$, the map $M[f_i^{-1}] \to N[f_i^{-1}]$ is injective (resp. surjective, resp. bijective).*

Recall that $M[f_i^{-1}] \coloneqq M \otimes_A A[f_i^{-1}]$.

*Remark* 4.11. Let's review some facts about localization. If $M$ is a $\mathbb{Z}[t]$-module, which is equivalent data to an abelian group with an endomorphism $t\colon M \to M$, then we can form $M[t^{-1}] := M \otimes_{\mathbb{Z}[t]} \mathbb{Z}[t, t^{-1}]$. Then:

- This construction is *exact*; that is, it preserves kernels and cokernels.
- There's a natural map $M \to M[t^{-1}]$, and its kernel is the submodule of $m \in M$ with $t^n m = 0$ for some $n$.

The way you prove all of this is to write $\mathbb{Z}[t, t^{-1}]$ as the union of $t^{-n}\mathbb{Z}[t]$ for all $n$, or as the colimit of the multiplication-by-$t$ map on $\mathbb{Z}[t]$. These are all free modules, hence flat, and one can prove that filtered colimits of flat modules are flat without too much angst. ◀

Now we can get back to the lemma.

*Proof of Lemma 4.10.* For injectivity, let's compare $\ker\varphi$ with $\ker(\varphi[f_i^{-1}])$. By Remark 4.11, $(\ker\varphi)[f_i^{-1}] = \ker(\varphi[f_i^{-1}])$, so we can reduce to showing that $M = 0$ iff $M[f_i^{-1}] = 0$ for all $i$. One direction is immediate, of course; conversely, if $M[f_i^{-1}] = 0$ for all $i$, then for all $m \in M$ and all $f_i$, then $f_i^{n_i} m = 0$ for some $n_i \gg 0$. There are finitely many $f_i$, so we can let $N$ be the biggest one, and then $f_i^N m = 0$ for all $i$.

**Lemma 4.12.** $(f_1, \dots, f_n) = A$ *iff* $(f_1^N, \dots, f_n^N) = A$.

*Proof.* There's a naïve argument which isn't too bad, but the geometric reason is that $f_i$ is a function, and $f_i$ vanishes on the same locus where $f_i^N$ vanishes, and therefore $D(f_i^N) = D(f_i)$. Therefore $\{D(f_i^N)\}$ is also an open cover, which by (4.8) means $(f_1^N, \dots, f_n^N) = A$.

If this proof feels sketchy, here's a more careful one (which unfortunately masks the geometry): if $(f_1^N, \dots, f_n^N) \subsetneq A$, then it's contained in some maximal ideal $\mathfrak{m}$. Therefore for all $i$, $f_i^N = 0$ in $A/\mathfrak{m}$, and therefore $f_i = 0$ in $\mathfrak{m}$, because $A/\mathfrak{m}$ is a field; hence $f_i \in \mathfrak{m}$, which is a contradiction. ⊠

Now, using Lemma 4.12, there are some $g_1, \dots, g_n$ with $\sum_i g_i f_i^N = 1$, and therefore

$$M = \sum_i f_i f_i^N \cdot M = 0. \tag{4.13}$$

The proof of surjectivity is similar, but using cokernels instead of kernels. ⊠

This lemma is a bridge between the geometry of schemes and the linear algebra of modules. You should think of inverting $f_i$ as restricting to $D(f_i)$; we will return to this idea.

*Proof of Proposition 4.5, special case.* Now we'll prove that an affine scheme is a Zariski sheaf for basic open covers $\{D(f_i)\}$. We want to show that (4.7) is an isomorphism, and by Lemma 4.10 it suffices to show this after inverting $f_i$.

Let $g_i \in \mathrm{Fun}(D(f_i))$ be a collection of functions that agree on overlaps... TODO: I missed the last part. ⊠

Lecture 5.

## $\mathbb{A}^1$ is a Zariski sheaf: 9/10/18

Today, we're going to continue proving Proposition 4.5, that affine schemes are schemes. We're still working on the special case that $\mathbb{A}^1$ is a scheme; the key piece of the proof is showing that it's a Zariski sheaf.

**Definition 5.1.** Let $S$ be a space and $\mathfrak{U}$ be an open cover of $S$. A *refinement* of $\mathfrak{U}$ is an open covering $\mathfrak{V}$ of $S$ such that for all $U \in \mathfrak{U}$, $\mathfrak{V}_U := \{V \in \mathfrak{V} \mid V \subset U\}$ is an open covering of $U$.

The Zariski sheaf condition for maps $X \to S$ is a constraint on compatible functions on all open covers of $S$. If we only ask about a specific open cover $\mathfrak{U}$, we say "the Zariski sheaf property for $X$ with respect to $\mathfrak{U}$."

**Lemma 5.2.** *Let $X$ and $S$ be spaces, $\mathfrak{U}$ be an open cover of $S$, and $\mathfrak{V}$ be a refinement of $\mathfrak{U}$. Suppose the Zariski sheaf property holds for $X$ with respect to $\mathfrak{V}$, and for each $U \in \mathfrak{U}$ with respect to $\mathfrak{V}_U$, then it holds with respect to $\mathfrak{U}$.*

After you unwind all the definitions, this is a definition check which isn't very hard.

*Remark* 5.3. One corollary of Lemma 5.2 is that in the definition of the sheaf property, we may replace "for all affine schemes $S$" with "for all spaces $S$." All of the definitions were built from the beginning to favor affine schemes as important or special, and this is one consequence. ◀

**Definition 5.4.** A *big basic open covering* of an affine scheme $S$ is an open covering by sets of the form $D(f_i)$ as in Lemma 4.8, but over a possibly infinite indexing set.

This is only a temporary definition. The Zariski sheaf property for $X$ and every basic open covering of an affine scheme $S$ implies the Zariski sheaf for all big basic open coverings.

**Proposition 5.5.** *Let $S$ be an affine scheme and $\mathfrak{U}$ be an open cover of $S$. Then there's a big basic open covering of $S$ refining $\mathfrak{U}$.*

*Proof.* Write $S = \operatorname{Spec} A$ and for each $U \in \mathfrak{U}$, let $Z_U := S \setminus U$; the inclusion $Z_U \hookrightarrow S$ is a closed embedding, so $Z_U = \operatorname{Spec}(A/I_U)$ for some ideal $I_U \subset A$. Recall from Proposition 3.3 that since $\mathfrak{U}$ is an open covering,

$$\sum_{U \in \mathfrak{U}} I_U = A, \tag{5.6}$$

and this is an equivalent condition. Consider the big basic open cover

$$\mathfrak{V} := \{D(f) \mid f \in I_U \setminus 0 \text{ for some } U \in \mathfrak{U}\}. \tag{5.7}$$

That this is a big basic open cover is because an ideal is generated by its elements. It's also a refinement of $\mathfrak{U}$, which follows from a more general lemma.

**Lemma 5.8.** *Let $U = S \setminus Z$, where $S = \operatorname{Spec} A$ and $Z = \operatorname{Spec}(A/I)$. Then $\{D(f) \mid f \in I \setminus 0\}$ is an open cover of $U$.*

*Proof.* We want to show that for all $T = \operatorname{Spec} B$ and maps $g\colon T \to U$, the set $\mathfrak{V}_g := \{g^{-1}(D(f)) \mid f \in I \setminus 0\}$ is an open cover of $T$.[4] Then... TODO ⊠

Thus we've proven the proposition. ⊠

**Corollary 5.9.** *Let $S$ be an affine scheme with an open covering $\mathfrak{U}$. Then there's a big basic open covering $\mathfrak{V}$ refining $\mathfrak{U}$ and with the property that for all $U \in \mathfrak{U}$, $\{V \in \mathfrak{V} \mid V \subset U\}$ is a big basic open covering of $U$.*

This is the technical proposition that lets us reduce to algebra.

*Remark* 5.10. Corollary 5.9 also tells us that a big basic open covering of a space $X$ is an open covering $\mathfrak{U}$ of $X$ such that for all maps of affine schemes to $X$, the pullback of $\mathfrak{U}$ is also a big basic open covering. ◀

**Corollary 5.11.** $\mathbb{A}^1$ *is a Zariski sheaf.*

*Proof.* We showed that $\mathbb{A}^1$ is a Zariski sheaf with respect to all basic open covers of affine schemes, hence for all big basic open covers of affine schemes, hence by Remark 5.10 with respect to all spaces with big basic open covers, hence by Proposition 5.5 any affine scheme and any open cover, and therefore any space and any open cover. ⊠

**Corollary 5.12.** *Let $I$ be a set and let $\mathbb{A}^I := \operatorname{Spec} \mathbb{Z}[\{x_i \mid i \in I\}]$. Then $\mathbb{A}^I$ is a Zariski sheaf.*

*Proof.* The sheaf property is preserved under arbitrary products. ⊠

If $I$ is an $n$-element set, then $\mathbb{A}^I$ is also written $\mathbb{A}^n$.

*Proof sketch of Proposition 4.5.* We can use this to show that if $X = \operatorname{Spec} A$ is an affine scheme, then it's a Zariski sheaf. Let $I$ be a generating set for $A$ and $J \subset \mathbb{Z}[\{x_i \mid i \in I\}]$ be the ideal of relations; then, the quotient map $\mathbb{Z}[\{x_i \mid i \in I\}] \twoheadrightarrow A$ defines a closed embedding $X \subseteq \mathbb{A}^I$ cut out by $X = \{x \mid f(x) = 0 \text{ for all } f \in J\}$.

One then has to check that the sheaf property is preserved under closed embeddings, which is formal. ⊠

We'll spend the next lecture giving examples of schemes, but here are a few to start with.

- As we just showed, affine schemes are schemes.
- A *quasi-affine scheme* is an open subset of an affine scheme, such as $\mathbb{A}^2 \setminus 0$. These are indeed schemes (though not always affine): if $U$ is the complement of $\operatorname{Spec}(A/I) \subset A$, then $U$ admits a covering by $\{D(f) \mid f \in I \setminus 0\}$.

We can use this to prove $\mathbb{A}^2 \setminus 0$ isn't affine.

[4] The preimage is defined to be $g^{-1}(D(f)) := D(f) \times_U T$.

Lecture 6.

## Relative algebraic geometry: 9/12/18

One of the advantages of algebraic geometry is the ability to work relative to a given space, which generalizes choosing a base field (or ring).

**Definition 6.1.** Let $S$ be a space. A *scheme over $S$* is a space $X$ with a map $X \to S$, often just written $X/S$, such that for all affine schemes $T$ and maps $T \to S$, $X \times_S T$ is a scheme. A morphism of schemes over $S$ is a morphism of schemes which commutes with the two maps to $S$.

In the same way one can define affine schemes over $S$. If $S = \operatorname{Spec} A$, for $A$ a ring, we might write $X/A$ instead of $X/S$ and say schemes over $A$; often $A$ will be a field.

We defined spaces as functors $\mathcal{C}omm\mathcal{R}ing \to \mathcal{S}et$, and there's a similar description for schemes over $A$.

**Proposition 6.2.** *Let $A$ be a commutative ring. There's an equivalence of categories between spaces over $A$ and functors $\mathcal{C}omm\mathcal{A}lg_A \to \mathcal{S}et$ (where we ignore the zero algebra).*

*Proof sketch.* Given $X\colon \mathcal{C}omm\mathcal{A}lg_A \to \mathcal{S}et$, we can define a functor on all commutative rings by sending $B$ to the set of pairs of $(i, x)$ where $i\colon A \to B$ is an $A$-algebra structure on $B$ and $x \in X(B)$. Then the forgetful map $(i, x) \mapsto i$ defines the desired map to $\operatorname{Spec} A$.

In the other direction, let $p\colon X \to \operatorname{Spec} A$ be a scheme over $A$. We'll define a functor on commutative $A$-algebras by sending $(B, i\colon A \to B)$ to the set of maps $\varphi\colon \operatorname{Spec} B \to X$ for which the diagram

$$\begin{tikzcd} & X \arrow[d, "p"] \\ \operatorname{Spec} B \arrow[ur, "\varphi"] \arrow[r, "i^*"] & \operatorname{Spec} A \end{tikzcd} \tag{6.3}$$

commutes. ⊠

**Example 6.4.** Complex conjugation is $\mathbb{Z}$-linear (and even $\mathbb{R}$-linear) but not $\mathbb{C}$-linear, and therefore induces a map of schemes $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{C}$ which is a map of schemes over $\mathbb{R}$, but not of schemes over $\mathbb{C}$. ◂

**Proposition 6.5.** *Let $X$, $Y$, and $Z$ be schemes together with maps $X \to Z$ and $Y \to Z$. Then $X \times_Z Y$ is a scheme.*

*Proof.* If $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, and $Z = \operatorname{Spec} C$ are affine, this is certainly true: the pullback is $\operatorname{Spec} A \otimes_C B$. Now we'll show more general cases reduce to this one.

If $Y$ and $Z$ are affine but $X$ isn't, then $X$ admits an open cover $\mathfrak{U}$ by affines, and $\{U \times_Z Y \mid U \in \mathfrak{U}\}$ is an affne open cover of $X \times_Z Y$. In the same way, we may assume only that $X$ and $Z$ are affine.

Therefore if you only assume $Z$ is affine, you can pick affine open covers of $X$ and $Y$ called $\mathfrak{U}$ and $\mathfrak{V}$, respectively. Then $\{U \times_Z V \mid U \in \mathfrak{U}, V \in \mathfrak{V}\}$ is an affine open cover of $X \times_Z Y$.

Next, we assume $X$ and $Y$ are affine, but $Z$ might not be.[5] Let $\mathfrak{W}$ be an affine open cover of $Z$, and $W \in \mathfrak{W}$. By definition, the map

$$X \times_Z W \longrightarrow X \tag{6.6}$$

is an open embedding, and this implies that $X \times_Z W$ is a scheme (we called these quasi-affine): it's the complement of a closed embedding $\operatorname{Spec} A/I \to X = \operatorname{Spec} A$, and is covered by $\{D(f)\}$ where $\{f\}$ generates $I$. Anyways, then $X \times_Z Y$ is covered by

$$\mathfrak{W}' := \{(X \times_Z W) \times_W (Y \times_Z W) \mid W \in \mathfrak{W}\}. \tag{6.7}$$

Since $W$ is affine, this is a scheme by one of the earlier cases. Therefore $X \times_Z Y$ is covered by schemes, so it must be a scheme (choose an affine cover of each element of $\mathfrak{W}'$, and check this is an affine open cover of $X \times_Z Y$).

Finally, we assume none of them are affine. This is the same as the case where $X$ and $Y$ are affine, but now we can use the previous step to show that if $\mathfrak{U}$ is an open cover of $X$, $\mathfrak{V}$ is an open cover of $Y$, $U \in \mathfrak{U}$, and $V \in \mathfrak{V}$, then $U \times_Z V$ is a scheme.

[5]From here, the proof was finished up in Friday's lecture.

We've ignored the Zariski sheaf property, but it's relatively simple to show that it's preserved by fiber products. ⊠

**Corollary 6.8.** *If $S$ is a scheme, schemes over $S$ are the same thing as schemes with a map to $S$.*

*Proof.* We can check the definition on an affine open cover of $S$; Proposition 6.5 tells us that pulling back to $T$ preserves scheminess. ⊠

If $S$ is a space that's not a space, Corollary 6.8 isn't necessarily true.

**Quasicoherent sheaves and/or linear algebra.** In commutative algebra, one often studies a ring by studying its modules; these are linear-algebraic in nature, which can make them easier to reason about. The analogue for schemes is quasicoherent sheaves.

**Definition 6.9.** Let $X$ be a scheme. A *quasicoherent sheaf* (QC sheaf) $\mathcal{F}$ on $X$ is data of, for all maps $f\colon \operatorname{Spec} A \to X$, an $A$-module $\mathcal{F}_f$, and for every map $g\colon \operatorname{Spec} B \to \operatorname{Spec} A$, an isomorphism

$$\alpha_{f,g}\colon \mathcal{F}_{g\circ f} \overset{\cong}{\to} \mathcal{F}_f \otimes_A B \tag{6.10}$$

of $B$-modules, and such that a cocycle condition holds: given a triple

$$\operatorname{Spec} C \xrightarrow{h} \operatorname{Spec} B \xrightarrow{g} \operatorname{Spec} A \xrightarrow{f} X, \tag{6.11}$$

$\alpha_{f,g\circ h} = \alpha_{f\circ g,h}$ as maps $\mathcal{F}_{f\circ g\circ h} \cong (\mathcal{F}_f \otimes_A B) \otimes_B C$, using the natural isomorphism $(\mathcal{F}_f \otimes_A B) \otimes_B C \cong \mathcal{F}_f \otimes_A C$.[6] A morphism of quasicoherent sheaves $\mathcal{F} \to \mathcal{G}$ is data of maps of $A$-modules $\mathcal{F}_f \to \mathcal{G}_f$ for all $f\colon \operatorname{Spec} A \to X$, such that all induced diagrams commute. The category of QC sheaves on $X$ is denoted $\mathcal{QC}oh(X)$.

*Remark* 6.12. The word "quasicoherent" isn't really great unless you're playing Scrabble. It grew out of a generalization of coherent sheaves, which originally came from the analytic setting, where the name was more reasonable. You should think of analogues of modules when you hear QC sheaves. ◂

This is a lot of data! So we're going to find a way to express a quasicoherent sheaf with less data.

**Proposition 6.13.** *If $X = \operatorname{Spec} A$, the functor $\Gamma\colon \mathcal{QC}oh(X) \to \mathcal{M}od_A$ sending $\mathcal{F} \mapsto \mathcal{F}_{\mathrm{id}}$ is an equivalence of categories, with inverse sending an $A$-module $M$ to the sheaf $\mathcal{F}_M$ defined by $(\mathcal{F}_M)_f := M \otimes_A B$ for all $f\colon \operatorname{Spec} B \to X$.*

**Example 6.14.** For any scheme $X$, there's a quasicoherent sheaf $\mathcal{O}_X$, called the *structure sheaf* of $X$, defined to send $f\colon \operatorname{Spec} A \to X$ to $(\mathcal{O}_X)_f = A$. The maps are what you think they are. ◂

Lecture 7.

## Quasicoherent sheaves: 9/14/18

*"You know when you're looking for your phone and it was in your hand the whole time? This proof was like that."*

Here are two exercises we've been sort of implicitly using, and are good to do to get some comfort with this language.

**Exercise 7.1.**

(1) Let $U \to X$ be a map of schemes and $U$ has an open cover $\mathfrak{V}$ such that for all $V \in \mathfrak{V}$, $V \to X$ is an open embedding. Then $U \to X$ is an open embedding.

(2) If $V \to U$ and $U \to X$ are open embeddings, their composition $V \to U$ is an open embedding.

Now back to quasicoherent sheaves. On an affine scheme $X = \operatorname{Spec} A$, these are a lot like $A$-modules (in fact, exactly like $A$-modules, according to Proposition 6.13).

**Definition 7.2.** Let $f\colon X \to Y$ be a map of schemes and $\mathcal{F} \in \mathcal{QC}oh(Y)$. The *pullback* of $\mathcal{F}$, denoted $f^*\mathcal{F} \in \mathcal{QC}oh(X)$, is the quasicoherent sheaf given by the following data: for every map $g\colon \operatorname{Spec} A \to X$, $(f^*\mathcal{F})_g := \mathcal{F}_{f\circ g}$.

[6] The cocycle condition can be expressed more concisely by asking that $\mathcal{F}$ is a functor from the category of affine schemes to abelian groups.

One must check the compatibility conditions, but these aren't so bad.

If $S = \operatorname{Spec} A$ is affine, then an $A$-module $M$ defines a quasicoherent sheaf $\mathcal{M}$ by sending $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ to $\mathcal{M}_f := M \otimes_A B$. The pullback of $\mathcal{M}$ along $f$ is exactly the quasicoherent sheaf defined by the module $M \otimes_A B$.

Since we understand quasicoherent sheaves on affine schemes, let's next see how they behave on open covers. We'll start with a different-looking definition, then show it's equivalent. This second definition will be useful because it involves substantially less data.

**Definition 7.3.** Let $X$ be a scheme and $\mathfrak{U}$ be an open cover of $X$. Let $\mathcal{QC}oh(X;\mathfrak{U})$ denote the category of tuples of $\mathcal{F}_U \in \mathcal{QC}oh(U)$ for all $U \in \mathfrak{U}$ together with, for all intersecting $U, V \in \mathfrak{U}$, isomorphisms

$$\alpha_{UV}\colon \mathcal{F}_U|_{U\cap V} \xrightarrow{\cong} \mathcal{F}_V|_{U\cap V} \tag{7.4}$$

satisfying a cocycle condition on triple intersections.

This is what's sheafy about quasicoherent sheaves: they are determined from compatible local data.

There's a functor $\Phi\colon \mathcal{QC}oh(X) \to \mathcal{QC}oh(X;\mathfrak{U})$ which takes a quasicoherent sheaf and produces its pullback on all $U \in \mathfrak{U}$.

**Theorem 7.5** (Serre)**.** *The functor $\Phi$ is an equivalence of categories.*

*Proof sketch.* This will look a lot like what we did before. The first step is to reduce to the case where $X = \operatorname{Spec} A$ is affine and $\mathfrak{U}$ is a basic open cover, using a similar argument to the one from two lectures ago. The second step is similar to the proof that $\mathbb{A}^1$ is a Zariski sheaf.

Explicitly, after we've reduced to $X = \operatorname{Spec} A$ and $\mathfrak{U} = \{D(f_i) \mid (f_1, \dots, f_n) = A\}$, then a quasicoherent sheaf on $D(f_i)$ is (equivalent data to) an $A[f_i^{-1}]$-module $M_i$, together with the natural isomorphisms $\alpha_{ij}\colon M_i[f_j^{-1}] \xrightarrow{\cong} M_j[f_i^{-1}]$ as $A[(f_if_j)^{-1}]$-modules.

Given this data, we want to functorially build an $A$-module. The answer will be

$$M := \{s_i \in M_i, 1 \le i \le n \mid \text{in } M_i[f_j^{-1}] \cong M_j[f_i^{-1}],\ s_i = s_j\}. \tag{7.6}$$

Now the proof is the same as in the $\mathbb{A}^1$-setting, though there we only worried about functions, not sections. The other way is simple once one invokes the flatness of $A[f_i^{-1}]$. ⊠

We might not have defined it yet, but for a field $k$, $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$. This is slightly nicer to work with for some applications than $\mathbb{A}^2_{\mathbb{Z}}$. Let $X := \mathbb{A}^2_k \setminus 0$, our favorite non-affine scheme, with its open cover $U := \mathbb{A}^1 \times (\mathbb{A}^1 \setminus 0)$ and $V := (\mathbb{A}^1 \setminus 0) \times \mathbb{A}^1$. Then Theorem 7.5 says a quasicoherent sheaf on $\mathbb{A}^2_k \setminus 0$ is the data of

- a $k[x, x^{-1}, y]$-module $M$,
- a $k[x, y, y^{-1}]$-module $N$, and
- an isomorphism $\alpha\colon M[y^{-1}] \cong N[x^{-1}]$ of $k[x, x^{-1}, y, y^{-1}]$-modules.

Modules can be big, so it will be useful to have some finiteness hypotheses.

**Definition 7.7.** Let $X$ be a scheme and $\mathcal{F} \in \mathcal{QC}oh(X)$. Then $\mathcal{F}$ is *locally finitely generated* (l.f.g.) if for all open embeddings $j\colon \operatorname{Spec} A \to X$, $j^*\mathcal{F}$ is a finitely generated $A$-module.

**Theorem 7.8** (Nakayama's lemma)**.** *Let $\mathcal{F}$ be a locally finitely generated QC sheaf on a scheme $X$, $k$ be a field, and $x\colon \operatorname{Spec} k \to X$ be such that $x^*\mathcal{F} = 0$. Then there's an open $j\colon U \hookrightarrow X$ containing $x$ (i.e. $x$ factors through $j$) and such that $j^*\mathcal{F} = 0$.*

Geometrically, this is saying that if an l.f.g. sheaf vanishes at a point, it also vanishes in a neighborhood of that point.

*Proof.* First we'll reduce to the affine case: we know there's an affine open $V \subseteq X$ such that $x$ factors through $V$ (geometrically, the point $x$ lies in $V$), so we'll replace $X$ by $V$ (and call it $X$). Let $X = \operatorname{Spec} A$, so that $\mathcal{F}$ corresponds to a finitely generated $A$-module $M$, and $x$ corresponds to a map $\varphi\colon A \to k$. Our hypothesis means that $M \otimes_A k = 0$.

Let's induct on the number of generators of $M$. If $M$ is generated by zero elements, we're done, so assume we know it for all modules generated by $n$ elements... we'll finish this Monday. ⊠

Lecture 8.

## Quasicoherent sheaves, II: 9/17/18

We're in the middle of proving Nakayama's lemma, Theorem 7.8. We're proving it by induction on the number of generators of the $A$-module $M$, and the base case is trivial. So let's assume it's true for all modules generated by $n$ elements.[7]

*Remark* 8.1. Let's pause to ask what a finitely generated $A$-module looks like. If it has one generator, it's isomorphic to $A/I$ for some ideal $I$. If it has two generators, it's an extension of $A/I$ by $A/J$ for some ideals $I$ and $J$ of $A$. More generally, a module $M$ with $m$ generators is an extension $0 \to N \to M \to A/I \to 0$, where $N$ has $m-1$ generators.

This means some specific subcases of Nakayama's lemma, such as that for local rings, are close to trivial.[8] You could prove Theorem 7.8 by reducing to the local case, though we're using a different approach.

The fact that finitely generated modules have quotients which look like $A/I$ is the catalyst of the proof: it's untrue for modules which aren't finitely generated, such as $\mathbb{Q}$ as a $\mathbb{Z}$-module, which has no quotients of the form $\mathbb{Z}/n$. ◀

So $M$ is an extension of an $A$-module $N$ generated by $n$ elements by $A/I$:

$$0 \longrightarrow N \longrightarrow M \longrightarrow A/I \longrightarrow 0. \tag{8.2}$$

By assumption, $M \otimes_A k = 0$, which means that, since tensor product is right exact,

$$(A/I) \otimes_A k \cong k/Ik = 0. \tag{8.3}$$

Recall that we had data of a map $\varphi\colon A \to k$; since $k$ is a field, this and (8.3) imply there's some $f \in I$ wth $\varphi(f) \neq 0$. Let's localize at $f$; the map $\varphi\colon A \to k$ passes to a map $\widetilde{\varphi}\colon A[f^{-1}] \to k$, and since localization is exact, (8.2) induces a short exact sequence

$$0 \longrightarrow N[f^{-1}] \longrightarrow M[f^{-1}] \longrightarrow (A/I)[f^{-1}] \longrightarrow 0, \tag{8.4}$$

but since $A[f^{-1}] = 0$, $N[f^{-1}] \cong M[f^{-1}]$, which (crucially) is generated by $n$ elements as an $A[f^{-1}]$-module. Since $\varphi(f) \neq 0$, then $x \in D(f)$, so there's an open $U \subset D(f)$ containing $x$ such that $(\mathcal{F}|_{D(f)})|_U = 0$ by the inductive hypothesis, and that's exactly what we wanted to prove.

**Definition 8.5.** Let $M$ be an $A$-module. Then its *annihilator* $\mathrm{Ann}(M) := \{f \in A \mid f \cdot M = 0\}$, which is an ideal of $A$.

**Corollary 8.6.** *Let $X$ be a scheme and $\mathcal{F} \in \mathfrak{QCoh}(X)$ be locally finitely generated. Then the subset $U_{\mathcal{F}} := \{f\colon \operatorname{Spec} B \to X \mid f^*\mathcal{F} = 0\}$ is an open subscheme of $X$. In particular, if $X = \operatorname{Spec} A$ is affine, then $U_{\mathcal{F}}$ is the complement of the locus of $X$ on which all $f \in \mathrm{Ann}(\mathcal{F})$ vanish.*

That is, the locus where $\mathcal{F}$ vanishes is open. This fits into your intuition: if you're on $\operatorname{Spec} A$ and $\mathcal{F}$ corresponds to $A/(f)$, then $\mathcal{F}$ vanishes wherever $f$ doesn't.

*Proof.* It suffices to prove the affine statement, and this is a matter of unwinding its definition: let $X = \operatorname{Spec} A$ and $\mathcal{F}$ be an $A$-module. Given $\varphi\colon A \to B$, it suffices to prove the following are equivalent: $\mathrm{Ann}(\mathcal{F}) \cdot B = B$ and $\mathcal{F} \otimes_A B = 0$.

First, the forward implication: we know there are $f_i \in \mathrm{Ann}(\mathcal{F})$ and $g_i \in B$ such that

$$\sum_{i=1}^n \varphi(f_i) g_i = 1. \tag{8.7}$$

Therefore 1 acts by 0 on $\mathcal{F} \otimes_A B$, so that module must be the zero module.

The reverse direction is a bit harder. Suppose for a contradiction that $\mathrm{Ann}(\mathcal{F}) \cdot B \subsetneq B$, so it's contained in some maximal ideal $\mathfrak{m}$; let $k := B/\mathfrak{m}$, which is a field. Then

$$\mathcal{F} \otimes_A k = (\mathcal{F} \otimes_A B) \otimes_B k = 0. \tag{8.8}$$

[7] The theorem is true for non-affine schemes, but we've already reduced to the affine case.

[8] There are many different things called Nakayama's lemma; ours is not the most general one.

Hence, by Theorem 7.8, there's a $U \subset \operatorname{Spec} A$ containing $\operatorname{Spec} k$ such that $\mathcal{F}|_U = 0$. We can assume $U = D(f)$ for some $f \in A$, so we're assuming $\mathcal{F}[f^{-1}] = 0$. Because $\mathcal{F}$ is finitely generated, this means $f^N \mathcal{F} = 0$ for some $N \gg 0$, or $f^N \in \operatorname{Ann}(f)$. Since $\varphi(\operatorname{Ann}(\mathcal{F})) \subset \mathfrak{m}$, then $\varphi(f^N) = 0 \bmod \mathfrak{m}$, so $\varphi(f) = 0 \bmod \mathfrak{m}$, which contradicts the assumption that $\operatorname{Spec} k \in U$. ⊠

You can draw a picture of this: given a locally finitely generated sheaf $\mathcal{F}$, $\operatorname{Ann}(\mathcal{F})$ has a vanishing locus; if $\mathcal{F}$ corresponds to the module $A/I$ (here we should be on an affine scheme), then this is also the closed subset $\operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A$.

**Exercise 8.9.** Deduce every other version of Nakayama's lemma that you know (e.g. the one in Matsumara) from these versions.

**Definition 8.10.** A *vector bundle* on a scheme $X$ is a quasicoherent sheaf $\mathcal{E} \in \mathcal{QC}oh(X)$ which is locally finitely generated and *locally projective*, i.e. for some (equivalently any) affine open cover $\mathfrak{U}$ of $X$, for every $U = \operatorname{Spec} A \in \mathfrak{U}$, the pullback of $\mathcal{E}$ to $U$ is a projective $A$-module.

**Proposition 8.11.** *Let $\mathcal{E} \in \mathcal{QC}oh(X)$. The following are equivalent:*

(1) *$\mathcal{E}$ is a vector bundle.*
(2) *There is an affine open cover $\mathfrak{U}$ of $X$ such that for all $U \in \mathfrak{U}$, $\mathcal{E}|_U$ is a finitely generated free module.*

Lecture 9.

## Vector bundles: 9/19/18

*"It's not a good exercise; it's an exercise."*

Today we're going to continue talking about vector bundles, which are an absolutely crucial concept in algebraic geometry. First we'll prove Proposition 8.11, equating two definitions of vector bundles: quasicoherent sheaves which are locally projective and those which are locally free of finite rank, i.e. locally isomorphic to $\mathcal{O}_U^{\oplus r}$. This $r$ is called the *rank* of the vector bundle over $U$.

*Remark* 9.1. The rank of a vector bundle is locally constant, but doesn't have to be constant. ◀

**Lemma 9.2.** *Let $X = \operatorname{Spec} A$ and $\mathfrak{U}$ be a collection of open subsets of $X$. Then $\mathfrak{U}$ is an open cover of $X$ iff for all maximal ideals $\mathfrak{m}$ of $A$, there's a $U \in \mathfrak{U}$ such that $\operatorname{Spec}(A/\mathfrak{m}) \hookrightarrow \operatorname{Spec} A$ factors through $U$.*

The idea is that a *closed point* is an embedding $\operatorname{Spec} k \hookrightarrow X$, where $k$ is a field. So a collection of opens is an open cover if it contains every closed point, which is nice. Affineness is important here: there are general schemes with no closed points!

*Proof of Proposition 8.11.* The thing we want to prove is local, so we can immediately reduce to the case where $X = \operatorname{Spec} A$ is affine, and therefore $\mathcal{E}$ corresponds to an $A$-module, which we also denote $\mathcal{E}$. For the forward direction, we assume $\mathcal{E}$ is finitely generated and projective.

Let $\mathfrak{m}$ be a maximal idea of $A$. Then $\mathcal{E}/\mathfrak{m}$ is a finitely-generated $A/\mathfrak{m}$-module; since $A/\mathfrak{m}$ is a field, $\mathcal{E}/\mathfrak{m}$ is free, so it has a basis $\overline{s}_1, \ldots, \overline{s}_n$. We lift this to $s_1, \ldots, s_n \in \mathcal{E}$, which define a map $\tau\colon A^{\oplus n} \to \mathcal{E}$ which is surjective mod $\mathfrak{m}$. We'd like to show this is an isomorphism on some open $U$ containing $\operatorname{Spec}(A/\mathfrak{m})$.

Since $\mathcal{E}$ is a finitely generated $A$-module, so too is $\operatorname{coker}(\tau) = \mathcal{E}/\tau(A^{\oplus n})$, and since modding out by $\mathfrak{m}$ is right exact, $\operatorname{coker}(\tau)|_{\operatorname{Spec}(A/\mathfrak{m})} = 0$. Therefore by Theorem 7.8, there is some $U_0 \subset X$ containing $\operatorname{Spec}(A/\mathfrak{m})$ such that $\operatorname{coker}(\tau)|_{U_0} = 0$; without loss of generality, we can take $U_0$ to be affine, i.e. $\tau$ is surjective on $U_0$. Let's replace $X$ by $U_0$ and continue.

Because $\mathcal{E}$ is projective, the map $\tau\colon A^{\oplus n} \twoheadrightarrow \mathcal{E}$ splits; let $\sigma\colon \mathcal{E} \to A^{\oplus n}$ be a section. This means $\ker(\tau)$ is finitely generated, and therefore $\ker(\tau)|_{\operatorname{Spec}(A/\mathfrak{m})} = 0$. Now we use Nakayama's lemma again and conclude that $\tau$ is an isomorphism on some open $U_1$ containing $\operatorname{Spec}(A/\mathfrak{m})$.

The converse isn't immediately trivial: if $\mathfrak{U}$ is an affine cover of $\operatorname{Spec} A$ and $M$ is an $A$-module such that $M|_U$ is projective for all $U \in \mathfrak{U}$, why is $M$ necessarily projective? Since $A$ need not be Noetherian, we also need to show $M$ is finitely generated and presented given that its localizations are. This is not a super important point, so it's left as an exercise. Once $M$ is finitely presented, you can show that for any $f \in A$,

$$\operatorname{Hom}_A(M, N)[f^{-1}] = \operatorname{Hom}_{A[f^{-1}]}(M[f^{-1}], N[f^{-1}]). \tag{9.3}$$

This is definitely false if you don't assume finite presentation of $M$! Anyways, using this, you can recover projectivity on $\operatorname{Spec} A$ from projectivity on a basic affine open cover. ⊠

Next we'll turn to constructions with quasicoherent sheaves, and something not quite as related, affine morphisms.

**Definition 9.4.** Let $\mathcal{F}$ and $\mathcal{G}$ be quasicoherent sheaves on a scheme $X$ and $\tau\colon \mathcal{F} \to \mathcal{G}$ be a morphism. Then we can define sheaves $\ker(\tau), \mathrm{coker}(\tau) \in \mathcal{QC}oh(X)$, such that for all affine opens $U \subset X$, $\ker(\tau)|_U = \ker(\tau|_U)$ and $\mathrm{coker}(\tau)|_U = \mathrm{coker}(\tau|_U)$.

For this to make sense, we need to invoke Serre's theorem that this data actually defines a quasicoherent sheaf, along with the fact that $A \to A[f^{-1}]$ is flat, which means kernels and cokernels are preserved under pullback by an open embedding, so that gluing works.

**Definition 9.5.** With notation as before, there is a quasicoherent sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ such that on every affine open $U = \operatorname{Spec} A \hookrightarrow X$, $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})|_U = \mathcal{F}|_U \otimes_A \mathcal{G}|_U$, and on any open $V \hookrightarrow X$, $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})|_V = \mathcal{F}|_V \otimes_{\mathcal{O}_V} \mathcal{G}|_V$.

Checking that this is well-defined is easier than for the kernel and cokernel; you don't have to invoke Serre's theorem.

**Definition 9.6.** Let $X$ be a space. A map $f\colon Y \to X$ is *affine* if for all affine schemes $S$ and maps $S \to X$, the pullback $S \times_X Y$ is affine.

**Example 9.7.**

(1) Any map of affine schemes is affine, which is a rebranding of the theorem that fiber products preserve affine schemes.
(2) Closed embeddings are also affine.
(3) Not all open embeddings are affne: the standard counterexample is $\mathbb{A}^2 \setminus 0 \to \mathbb{A}^2$, because its fiber product with the identity map $\mathbb{A}^2 \to \mathbb{A}^2$ gives us back $\mathbb{A}^2 \setminus 0$, which isn't affine. ◀

Affine morphisms are nice because they have nice algebraic descriptions. Specifically, affine maps $Y \to X$, where $X$ is a scheme, correspond to commutative algebras in $\mathcal{QC}oh(X)$.

Lecture 10.

## Affine morphisms and projective space: 9/21/18

Last time, we defined affine morphisms $Y \to X$, which are those such that if you pull back by an affine scheme $\operatorname{Spec} A \to X$, $Y \times_X \operatorname{Spec} A$ is also affine. We claimed these are equivalent to commutative algebras in $\mathcal{QC}oh(X)$, akin to how affine schemes are commutative rings, but working relatively (i.e. over a scheme).

**Definition 10.1.** Let $X$ be a scheme. A *commutative algebra* in $\mathcal{QC}oh(X)$ is a quasicoherent sheaf together with an associative, commutative multiplication map $m\colon \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A} \to \mathcal{A}$ with a unit $\varepsilon\colon \mathcal{O}_X \to \mathcal{A}$.

**Definition 10.2.** Given a commutative algebra $\mathcal{A} \in \mathcal{QC}oh(X)$, we can define a scheme $\operatorname{Spec}_X(\mathcal{A})$ together with an affine map to $X$. If $B$ is a commutative ring, we let $(\operatorname{Spec}_X(\mathcal{A}))(B)$ be the set of pairs $x\colon \operatorname{Spec} B \to X$ together with maps $\rho\colon \operatorname{Spec} B \to \operatorname{Spec} x^*(\mathcal{A})$ which are sections of the canonical map arising from the $B$-algebra structure on $x^*(\mathcal{A})$.

Forgetting the section defines a map to $X$. This map is affine because if $x\colon \operatorname{Spec} B \to X$ is a map, then

$$\operatorname{Spec}_X(\mathcal{A}) \times_X \operatorname{Spec} B = \operatorname{Spec}(x^*(\mathcal{A})). \tag{10.3}$$

**Definition 10.4.** Let $\pi\colon Y \to X$ be an affine map of schemes and $\mathcal{F} \in \mathcal{QC}oh(Y)$. We will define a *pushforward* $\pi_*\mathcal{F} \in \mathcal{QC}oh(X)$ as follows: given any map $x\colon \operatorname{Spec} B \to X$, the pullback $Y \times_X \operatorname{Spec} B$ is affine, isomorphic to $y\colon \operatorname{Spec} C \to Y$ for some $C$. Then we define $x^*(\pi_*(\mathcal{F})) \coloneqq y^*(\mathcal{F})$: this is *a priori* a $C$-module, but picks up a $B$-module structure by the map $B \to C$.

**Exercise 10.5.** $\pi^*$ and $\pi_*$ are adjoint functors, i.e. for any affine map $\pi\colon X \to Y$, $\mathcal{F} \in \mathcal{QC}oh(Y)$, and $\mathcal{G} \in \mathcal{QC}oh(X)$, there's a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{QC}oh(X)}(\mathcal{G}, \pi_*(\mathcal{F})) \cong \mathrm{Hom}_{\mathcal{QC}oh(Y)}(\pi^*(\mathcal{G}), \mathcal{F}).$$

**Proposition 10.6.** *If $\pi\colon Y \to X$ is affine, then $Y = \operatorname{Spec}_X(\mathcal{A})$ for some algebra $\mathcal{A}$ in $\mathcal{QC}oh(X)$.*

*Proof sketch.* The key is that $\pi_*(\mathcal{O}_Y)$ is a commutative algebra in $\mathcal{QC}oh(X)$: by Exercise 10.5, the multiplication map is equivalent data to a map

$$\pi^*(\pi_*(\mathcal{O}_Y) \otimes_{\mathcal{O}_X} \pi_*(\mathcal{O}_Y)) = \pi^*\pi_*\mathcal{O}_Y \otimes_{\mathcal{O}_Y} \pi^*\pi_*\mathcal{O}_Y \longrightarrow \mathcal{O}_Y. \tag{10.7}$$

The unit of the adjunction is a map $\pi^*\pi_*\mathcal{O}_Y \to \mathcal{O}_Y$, so we can pass to $\mathcal{O}_Y$ and then multiply.

We then claim that as schemes over $X$ (i.e. with a map to $X$), $Y \to X$ is isomorphic to $\operatorname{Spec}_X(\pi_*(\mathcal{O}_Y)) \to X$, which one has to check. ⊠

**Exercise 10.8.** Let $X$ be a scheme. Show that closed subschemes $Z \subseteq X$ are equivalent to *ideal sheaves* $\mathcal{I} \to \mathcal{O}_X$, i.e. quasicoherent sheaves $\mathcal{I}$ with vanishing kernel.

**Projective space.** Projective space $\mathbb{P}^n$ is a scheme designed to parametrize lines in $\mathbb{A}^{n+1}$. If you try this in $\mathbb{A}^2_{\mathbb{R}}$, you notice that you get a circle; if you do this in $\mathbb{A}^2_{\mathbb{C}}$, you get a sphere (harder). But in general it looks different.

We'll describe $\mathbb{P}^n$ via its functor of points. The idea is that a map $S = \operatorname{Spec} A \to \mathbb{P}^n$ should be a line $\mathcal{L}$ together with an embedding $\mathcal{L} \to \mathbb{A}^{n+1}$, but we have to make this precise.

**Definition 10.9.** Let $S$ be a scheme. A *line bundle* on $S$ is a vector bundle of rank 1, i.e. a quasicoherent sheaf $\mathcal{L}$ on $S$ locally isomorphic to $\mathcal{O}_S$.

When $S$ is affine, these correspond to rank-1 projective $A$-modules. If $S = \operatorname{Spec} k$, this exactly covers 1-dimensional $k$-vector spaces, and this is the right generalization to commutative rings (or even to schemes).

Now we want to embed the line in $\mathbb{A}^{n+1}$, which we can think of as a map $i\colon \mathcal{L} \to \mathcal{O}_S^{\oplus(n+1)}$; "embedding" means we want $i \neq 0$.

**Proposition 10.10.** *The following are equivalent for a vector bundle $\mathcal{E} \to S$ and a map $i\colon \mathcal{L} \to \mathcal{E}$ where $\mathcal{L}$ is a line bundle.*

(1) *For all affine schemes $T$ and maps $f\colon T \to S$, $f^*(\mathcal{L}) \to f^*(\mathcal{E})$ is nonzero.*
(2) *(Will be done Monday)*
(3) *(Will be done Monday)*

**Definition 10.11.** If the conditions in Proposition 10.10 hold, $i$ is called *everywhere nonvanishing*.

**Definition 10.12.** *Projective $n$-space* $\mathbb{P}^n$ is the space whose functor of points sends an affine scheme $S$ to the set of isomorphism classes of data $(\mathcal{L}, i)$ where $\mathcal{L}$ is a line bundle on $S$ and $i\colon \mathcal{L} \to \mathcal{O}_S^{\oplus(n+1)}$ is everywhere nonvanishing.

There's something to be said about morphisms, but given a map $f\colon T \to S$, we can pull back $\mathcal{L}$ and $i$, and obtain a line bundle with an everywhere nonvanishing embedding.

We need to take isomorphism classes to ensure we get a set, not a category. This also ensures that we've modded out by rescaling (since that's an isomorphism $\mathcal{L} \to \mathcal{L}$). We'll show this is a scheme, but is usually not affine.

Lecture 11.

## Projective $n$-space and projectivizations: 9/24/18

*"These scare quotes should be less scary than those scare quotes."*

Last time, we defined projective $n$-space, $\mathbb{P}^n$, whose functor of points sends $A$ to the set of isomorphism classes of data $(\mathcal{L}, i)$, where $\mathcal{L} \to \operatorname{Spec} A$ is a line bundle and $i\colon \mathcal{L} \to \mathcal{O}_{\operatorname{Spec} A}^{\oplus(n+1)}$ is an everywhere nonvanishing map.

**Definition 11.1.** More generally, if $X$ is a scheme and $\mathcal{E}$ is a vector bundle on $X$, then we can define a space $\mathbb{P}(\mathcal{E})$, the *projectivization* of $\mathcal{E}$: if $A$ is a commutative ring, $\mathbb{P}(\mathcal{E})(A)$ is the set of isomorphism classes of data $(\mathcal{L}, x, i)$, where $\mathcal{L}$ is a line bundle on $\operatorname{Spec} A$, $x\colon \operatorname{Spec} A \to X$ is a map ("an $A$-valued point"), and $i\colon \mathcal{L} \to x^*(\mathcal{E})$ is everywhere nonvanishing.

To recover $\mathbb{P}^n$, let $X = \operatorname{Spec} \mathbb{Z}$ and $\mathcal{E} = \mathbb{Z}^{\oplus(n+1)}$.

*Remark* 11.2. Right now, schemes and line and vector bundles probably feel very abstract. That's OK; soon enough we will see many, many examples of line bundles over curves, and make them very concrete. ◀

**Example 11.3.** Consider the quasicoherent sheaf given by $k[t]/(t)$ over $\mathbb{A}^1_k := \operatorname{Spec} k[t]$. This is not a vector bundle: it's not locally free, because in a sense it's nonzero over the point 0 (a one-dimensional vector space) but vanishes everywhere else. ◂

**Proposition 11.4.** *Let $S$ be a scheme, $\mathcal{L}$ be a line bundle on $S$, $\mathcal{E}$ be a vector bundle on $S$, and $i\colon \mathcal{L} \to \mathcal{E}$. The following are equivalent:*

(1) *The induced map $\Theta(i)\colon \Theta(\mathcal{L}) \to \Theta(\mathcal{E})$ is a closed embedding.*
(2) *For all affine schemes $T$ and maps $\alpha\colon T \to S$, $\alpha^*(i)$ is nonzero.*
(3) *For all fields $k$ and maps $\alpha\colon \operatorname{Spec} k \to S$, $\alpha^*(i)$ is nonzero.*
(4) *For all affine open subschemes $U \subseteq S$ such that $\mathcal{L}|_U \cong \mathcal{O}_U$ and $\mathcal{E}|_U \cong \mathcal{O}_U^{\oplus r}$, if the induced map $\mathcal{O}_U \to \mathcal{O}_U^{\oplus r}$ sends $1 \mapsto (f_1, \dots, f_r)$, then $(f_1, \dots, f_r)$ generates the unit ideal in* $\operatorname{Fun}(U)$.
(5) *The dual map $i^\vee\colon \mathcal{E}^\vee \to \mathcal{L}^\vee$ is an epimorphism (i.e. its cokernel is zero).*[9]
(6) $\operatorname{coker}(\mathcal{L} \to \mathcal{E})$ *is a vector bundle.*

To make sense of this, we have to define $\Theta$, a way of turning vector bundles into schemes. If $P^\vee$ is unfamiliar for an $A$-module $P$, it simply means $\operatorname{Hom}_A(P, A)$.

**Exercise 11.5.** Let $P$ be a finitely generated, projective $A$-module.

(1) Show that for all $A$-modules $M$, $\operatorname{Hom}_A(P, M) \cong P^\vee \otimes_A M$.
(2) Show that $P^\vee$ is projective.
(3) Describe a natural isomorphism $P \to P^{\vee\vee}$.

**Definition 11.6.** If $\mathcal{E}$ is a vector bundle over a scheme $S$, its *dual vector bundle* $\mathcal{E}^\vee$ (sometimes also written $\mathcal{E}^*$) is the quasicoherent sheaf attaching to every affine open $i\colon U \to S$ the dual projective module of $i^*\mathcal{E}$.

By Exercise 11.5, part (1), $\mathcal{E}^\vee$ is indeed a quasicoherent sheaf.

**Example 11.7.** If $\mathcal{E} = \mathcal{O}_X^{\oplus r}$, then $\mathcal{E}^\vee \cong \mathcal{O}_X^{\oplus r}$ as well, albeit not canonically. ◂

Duality is contravariantly functorial: a map $f\colon \mathcal{F} \to \mathcal{E}$ of vector bundles induces a dual map $f^\vee\colon \mathcal{E}^\vee \to \mathcal{F}^\vee$ (do this on affines, where it's precomposition for the corresponding modules).

**Definition 11.8.** Let $\mathcal{E}$ be a vector bundle on $X$. Its *total space* is the scheme

$$\Theta(\mathcal{E}) := \operatorname{Spec}_X(\operatorname{Sym}_{\mathcal{O}_X}(\mathcal{E}^\vee)). \tag{11.9}$$

Here, $\operatorname{Sym}_{\mathcal{O}_X}(\mathcal{E})$ is the $\mathcal{O}_X$-module

$$\operatorname{Sym}_{\mathcal{O}_X}(\mathcal{E}) := \bigoplus_{n\geq 0} \operatorname{Sym}^n_{\mathcal{O}_X}(\mathcal{E}), \tag{11.10}$$

where $\operatorname{Sym}^n_{\mathcal{O}_X}(\mathcal{E}) := (\mathcal{E}^{\otimes_{\mathcal{O}_X} n})_{S_n}$, where the symmetric group $S_n$ acts by permuting the elements; more explicitly, we allow the elements in an $n$-tensor to be arbitrarily shuffled, which you can do with a quotient.

**Example 11.11.** Let $X = \operatorname{Spec} k$ and $\mathcal{E} = k^{\oplus n}$ be a $k$-vector space with basis $e_1, \dots, e_n$. Let $e^1, \dots, e^n$ denote the dual basis. Then $\operatorname{Sym}^n V$ is the vector space of degree-$n$ homogeneous polynomials in $e^1, \dots, e^r$, and $\operatorname{Sym} V = k[e^1, \dots, e^r]$. ◂

Lecture 12.

## More vector bundles: 9/26/18

Today we're in the business of proving Proposition 11.4.

**Lemma 12.1.** *Let $X$ be a scheme and $\mathcal{A}, \mathcal{B} \in \mathcal{QC}oh(X)$ be commutative algebras together with a map $f\colon \mathcal{A} \to \mathcal{B}$ of algebras. Then $\operatorname{Spec}_X(\mathcal{B}) \to \operatorname{Spec}_X(\mathcal{A})$ is a closed embedding iff $f$ is an epimorphism.*

[9] In many situations, "epimorphism" means "surjective," but these are quasicoherent sheaves, so we don't have elements to ask about preimages of.

This is a relative version of the definition of closed embeddings of affine schemes we gave awhile ago.

We also need to define the what taking the total space $\Theta$ does to morphisms. Ultimately this is because $(-)^\vee$, $\mathrm{Sym}_{\mathcal{O}_X}$, and $\mathrm{Spec}_X$ are all functors, so we know what they do on morphisms; two are contravariant and one is covariant, so we get a covariant functor.

In part (1) of Proposition 11.4, Lemma 12.1 tells us this is equivalent to $\mathrm{Sym}_{\mathcal{O}_X} \mathcal{E}^\vee \to \mathrm{Sym}_{\mathcal{O}_X} \mathcal{L}^\vee$ is an epimorphism. This is a $\mathbb{Z}$-graded sheaf, and it suffices to show this for $\mathrm{Sym}^d_{\mathcal{O}_X}$ for each $d$.

**Exercise 12.2.** Show that if $\mathcal{E}^\vee \to \mathcal{L}^\vee$, then $\mathrm{Sym}^d \mathcal{E}^\vee \to \mathrm{Sym}^d \mathcal{L}^\vee$ is too.

*Proof of Proposition 11.4, (5) $\iff$ (4).* We know condition (5) is equivalent to $\mathrm{coker}(i^\vee) = 0$. This can be checked on an open cover $\mathfrak{U}$, such as an affine open cover which trivializes $\mathcal{E}$ and $\mathcal{L}$, as in (4). In this case, for each $U$ in $\mathfrak{U}$, $i|_U \colon \mathcal{O}_U \to \mathcal{O}_U^{\oplus r}$ is determined by a row vector $(f_1, \dots, f_r)^{\mathrm{T}}$, and $i^\vee|_U \colon \mathcal{O}_U^{\oplus r} \to \mathcal{O}_U$ is the column vector $(f_1, \dots, f_r)$. Its image is the ideal generated by $(f_1, \dots, f_r)$, so the cokernel is 0 iff $(f_1, \dots, f_r) = \mathrm{Fun}(U)$. ⊠

*Proof of Proposition 11.4, (4) $\implies$ (6).* We can again assume without loss of generality that $U = \mathrm{Spec}\, B$ is affine, and now we have maps

$$B \xrightarrow{\begin{bmatrix} f_1 \\ \vdots \\ f_r \end{bmatrix}} B^{\oplus r} \xrightarrow{[g_1, \dots, g_r]} B, \tag{12.3}$$

where $\sum f_i g_i = 1$. Since the composition is the identity, the map is split, and so the cokernel is a direct summand, in particular is a direct summand of a free module, and must be free. ⊠

**Exercise 12.4.** Conversely, show that in Proposition 11.4, (6) $\implies$ (4).

Since (2) tautologically implies (3), it suffices to show (3) $\implies$ (5) and (5) $\implies$ (2); (1) is addressed by one of the exercises above, I think (TODO: I probably just missed it.)

Since $\mathrm{coker}(i)$ is locally finitely generated, Nakayama's lemma applies: if $V$ denotes the complement of the support of $\mathrm{coker}(i)$, then $V$ is open, and $S = \mathrm{Spec}\, A \to X$ factors through $V$ iff $\mathrm{coker}(i)|_S = 0$. So (5) is equivalent to $V = X$.

*Proof of Proposition 11.4, (3) $\implies$ (5).* To show $V = X$, it suffices to show any map $x \colon \mathrm{Spec}\, k \to X$ factors through $V$, since $V$ is open. One can show that $x^*$ commutes with cokernels: since $\mathrm{Spec}\, k$ is affine, it arises as a tensor product, and tensor products are right exact.[10] Therefore it suffices to show that $x^*(\mathrm{coker}(i^\vee)) = 0$. If $x^*(i^\vee)$ is nonzero, then $x^*(\mathcal{E}) \to x^*(\mathcal{L})$ is a nonzero map to a 1-dimensional vector space, hence surjective, and therefore the cokernel is zero. ⊠

*Proof of Proposition 11.4, (5) $\implies$ (2).* Let $x \colon T \to X$ be a map, where $T$ if affine. As before, $x^*$ is right exact, hence commutes with cokernels. By assumption, $\mathrm{coker}(i^\vee) = 0$, so $x^*(i^\vee) \colon x^*(\mathcal{E}^\vee) \to x^*(\mathcal{L}^\vee)$ is an epimorphism. This is the dual map to $x^*(i)$, so if $T \neq \varnothing$, this means $x^*(i) \colon x^*(\mathcal{L}) \to x^*(\mathcal{E})$ is nonzero. ⊠

These equivalent conditions are all examples of the nicest possible maps of vector bundles. It's good to have all of these different perspectives partly because they provide flexibility in the nice case, but also because it will be useful to know what happens when things go bad. When we study projective varieties, several of these conditions will come up. For example, it will be useful for showing $\mathbb{P}^n$ is a scheme!

Lecture 13.

## Line bundles on $\mathbb{P}^n$: 9/28/18

**Theorem 13.1.** *$\mathbb{P}^n$ is a scheme.*

*Proof.* First we need to check that $\mathbb{P}^n$ is a Zariski sheaf. The basic idea is that line bundles glue: if you have line bundles on each open of an open cover, together with isomorphisms on intersections satisfying a cocycle condition, you can glue them.

Next we need to cover it by affines. For $i = 0, \dots, n$, the locus $\{s_i \neq 0\} \subset \mathbb{P}^n$ is isomorphic to $\mathbb{A}^n$. We could say QED here, but let's explain what's going on.

[10] More generally, $x^*$ is a left adjoint, even not on affines; this automatically means it's right exact.

Let $U_i := \{s_i = 0\}$ be defined by saying what it means for a map $S \to \mathbb{P}^n$ factors through $U_i$. Specifically, the map $S \to \mathbb{P}^n$ is equivalent to (an isomorphism class of) data of a line bundle $\mathcal{L}$ on $S$ and a map $(s_0, \dots, s_{n+1})^{\mathrm{T}} : \mathcal{L} \to \mathcal{O}_S^{\oplus(n+1)}$ which is everywhere nonvanishing. Thus the map is described by $s_0^\vee, s_n^\vee$; we say the map factors through $U_i$ if $s_i^\vee$ is nonvanishing.[11]

First let's check that for all $i$, $U_i \subset \mathbb{P}^n$ is open. Intuitively this makes sense: we're asking for something to not vanish, which is an open condition. More precisely, we want to show that for every affine scheme $S$ with a map $S \to \mathbb{P}^n$, the pullback $U_i \times_{\mathbb{P}^n} S \to S$ is open. So let $S \to \mathbb{P}^n$ be such a map (in particular, $S$ is affine). This is equivalent data to a line bundle $\mathcal{L}$ on $S$ and $s_0, \dots, s_{n+1} : \mathcal{L} \to \mathcal{O}_S$. That is, there are sections of the map $\Theta(\mathcal{L}^\vee) \to S$. Then you can check that $U_i = S \times_{\Theta(\mathcal{L}^\vee)} \Theta(\mathcal{L}^\vee) \setminus S$, where the first map $S \to \Theta(\mathcal{L}^\vee)$ is by $s_i$ and the second map is by the zero section.

The other claim we want to check is that $U_i \cong \mathbb{A}^n$. The proof is that a map $S \to U_i$ is equivalent data to the maps $s_0, \dots, s_{i-1}, s_{i+1}, \dots, s_n : \mathcal{L} \to \mathcal{O}_S$, since we know $s_i : \mathcal{L} \to \mathcal{O}_S$ is an isomorphism. Each $s_j$, $j \neq i$, is equivalent to a function on $S$: since $s_i$ is nonvanishing, there are no further conditions on the remaining maps. Thus a map to $U_i$ is equivalent to $n$ functions, and this is natural in $S$, so $U_i \simeq \mathbb{A}^n$.

Finally, we need to show that $U_0, \dots, U_n$ is a cover of $\mathbb{P}^n$, i.e. that it pulls back to a Zariski open cover on all affines. So once again let $S \to \mathbb{P}^n$ be a map from an affine, so we have a line bundle $\mathcal{L}$ and $(n+1)$ maps $s_0, \dots, s_n : \mathcal{L} \to \mathcal{O}_S$ which collectively are nonvanishing. Without loss of generality, we can assume $\mathcal{L}$ is trivial, since it's locally trivial, and we can check that it's an open cover locally. So it suffices to show that $S \times_{\mathbb{P}^n} U_i$ is a Zariski open cover of $S$, which is more or less equivalent to $s_0, \dots, s_n$ being everywhere nonvanishing, in view of what we did last lecture. ⊠

Typically, people only use functions, rather than sections of a line bundle, to provide a first naïve definition of $\mathbb{P}^n$. In this case, the Zariski sheaf property fails: you can glue trivial line bundles to obtain something nontrivial.

**Corollary 13.2.** *Let $\mathcal{E}$ be a vector bundle on a scheme $X$. Then $P(\mathcal{E})$ is a scheme.*

*Proof.* $X$ has an affine open cover $\mathfrak{U}$ such that $\mathcal{E}|_U$ is trivial for all $U \in \mathfrak{U}$. Then $\mathbb{P}(\mathcal{E}) \times_X U \cong \mathbb{P}^n \times U$ as schemes. ⊠

$\mathbb{P}^n$ has an important line bundle called $\mathcal{O}_{\mathbb{P}^n}(1)$ (sometimes just $\mathcal{O}(1)$ if $\mathbb{P}^n$ is implicit).

**Definition 13.3.** The line bundle $\mathcal{O}_{\mathbb{P}^n}(1) \in \mathcal{QC}oh(\mathbb{P}^n)$ is the line bundle which, given a map $f : S \to \mathbb{P}^n$ which is a line bundle $\mathcal{L}$ on $S$ and the maps $s_0, \dots, s_n$, defines $f^*(\mathcal{O}_{\mathbb{P}^n}(1)) := \mathcal{L}^\vee$.

For any $n \in \mathbb{Z}$, we define $\mathcal{O}_{\mathbb{P}^n}(m) := \mathcal{O}_{\mathbb{P}^n}(1)^{\otimes m}$. If $m = 0$, we interpret the empty tensor product as $\mathcal{O}_{\mathbb{P}^n}$, the sheaf of functions; if $m < 0$, we interpret this as $((\mathcal{O}_{\mathbb{P}^n}(1))^\vee)^{\otimes(-m)}$.

**Definition 13.4.** Let $X$ be a scheme and $\mathcal{F} \in \mathcal{QC}oh(X)$. The *sections* of $X$, denoted $\Gamma(X, \mathcal{F})$, is the abelian group $\mathrm{Hom}_{\mathcal{QC}oh(X)}(\mathcal{O}_X, \mathcal{F})$.

That is, a section is compatible data, for all affines $S$ and maps $x : S \to X$, of $\sigma_x \in x^*\mathcal{F}$ regarded as a module.

The sections $\Gamma(X, \mathcal{O}_X) = \mathrm{Fun}(X)$, and if $X = \operatorname{Spec} A$, $\Gamma(X, \mathcal{F})$ is canonically the $A$-module associated to $\mathcal{F}$ (which we've just been calling $\mathcal{F}$ again).

**Definition 13.5.** If $A$ is a commutative ring, we let $\mathbb{P}^n_A := \mathbb{P}^n \times \operatorname{Spec} A$.

**Proposition 13.6.** *For $r \geq 0$, $\Gamma(\mathbb{P}^n_A, \mathcal{O}(r))$ is the $A$-module of homogeneous degree-$r$ polynomials with $A$ coefficients in $n+1$ variables.*

For $r = 1$, a map $A^{\oplus(n+1)} \to \Gamma(\mathbb{P}^n_A, \mathcal{O}(1))$ is definedc by $n+1$ maps $\mathcal{O}_{\mathbb{P}^n_A} \to \mathcal{O}_{\mathbb{P}^n_A}(1)$, i.e. for all $f : S \to \mathbb{P}^n_A$, i.e. line bundles $\mathcal{L}$ with $s_0, \dots, s_n$, compatible maps $\mathcal{O}_S \to f^*(\mathcal{O}(1))$ which identify $f^*(\mathcal{O}(1)) \cong \mathcal{L}^\vee$, and the $i^{\text{th}}$ section is identified with $s_i \in \mathcal{L}^\vee$. Following your nose with the formal stuff provides a complete proof.

Over the next few days we'll discuss finite conditions: the nicest possible schemes are smooth projective curves, and we'll discuss these soon.

[11] That is, the collection $(s_0, \dots, s_n)^{\mathrm{T}}$ is always nonvanishing, but we're asking specifically about $s_i$, which is stronger.

Lecture 14.

## Finiteness hypotheses: 10/1/18

Today, we will discuss various finiteness hypotheses one can put on a scheme.

**Definition 14.1.** A scheme $X$ is *quasicompact* if it has a finite affine open cover.

Quasicompactness refers to the usual definition of compactness with respect to the Zariski topology, but without any Hausdorff condition. On the other hand, this has *nothing* to do with compactness in the "usual" topological sense. For example, the line $\mathbb{A}^1$ is quasicompact. The analogue of compactness in the usual sense (e.g. manifold topology) is known as *properness*.

**Example 14.2.** $\mathbb{P}^n$ is quasicompact, since it has a cover by $n+1$ opens, as we saw on Friday. ◂

**Proposition 14.3.** *If $A$ is a commutative ring and $I \subset A$ is an ideal, then* $\operatorname{Spec} A \setminus \operatorname{Spec}(A/I)$ *is quasicompact if $I$ is finitely generated.*

The converse is false.

**Example 14.4.** Let $\mathbb{A}^\infty_k \coloneqq \operatorname{Spec} k[x_1, x_2, x_3, \dots]$. Then $\mathbb{A}^\infty_k \setminus 0$ is a scheme which is not quasicompact. ◂

*Remark* 14.5. More generally, a morphism $f\colon X \to Y$ is said to be quasicompact if for all affine schemes $S$ and maps $S \to Y$, $X \times_Y S$ is quasicompact. This reduces to the notion of quasicompactness for schemes when $Y = \operatorname{Spec} \mathbb{Z}$, because a map to $\operatorname{Spec} \mathbb{Z}$ is no data at all. ◂

**Definition 14.6.** A scheme $X$ is *quasiseparated* if the intersection of two affine opens in $X$ is quasicompact.

Equivalently, the diagonal map $\Delta\colon X \to X \times X$ is a quasicompact morphism. Therefore one may more generally say a quasiseparated morphism $f\colon X \to Y$ is one for which the diagonal $\Delta_f\colon X \to X \times_Y X$ is quasicompact.

**Example 14.7.** The idea is that separatedness is kind of like the Hausdorff property in topology. As such, we can import the standard counterexample, $\mathbb{A}^1$ with two origins. The idea is to take two copies of $\mathbb{A}^1$, say with coordinates $t$ and $u$, and glue them together along $\mathbb{A}^1 \setminus 0$ via the identification $t \leftrightarrow u$. (This is different from $\mathbb{P}^1$, which is separated, where we identified $t \leftrightarrow u^{-1}$.) ◂

**Exercise 14.8.** Quasicompact, quasiseparated morphisms are exactly those where the pushforward of quasicoherent sheaves is well-behaved. Specifically, given a map $f\colon X \to Y$ and a quasicoherent sheaf $\mathcal{F}$ on $X$, we define its pushforward $f_*\mathcal{F}$ on $Y$ as follows: given an affine open subscheme $j\colon U \hookrightarrow Y$, we specify

$$j^* f_* \mathcal{F} \coloneqq \Gamma(X \times_Y U, \mathcal{F}|_{X \times_Y U}).$$

Show that this is quasicoherent if $f$ is quasicompact and quasiseparated. (TODO: converse?)

Even when we restrict to quasicompact, quasiseparated things, we're still looking at extremely general objects, considerably moreso than are studied in classical algebraic geometry. So here are some more finiteness hypotheses.

**Definition 14.9.** Let $f\colon X \to Y$ be a map of schemes.

(1) Suppose $Y = \operatorname{Spec} A$. Then $f$ is *locally of finite type* (LFT) if for all affine opens $\operatorname{Spec} B \subset X$, the induced map of rings $A \to B$ makes $B$ into a finitely generated algebra, i.e. $B \cong A[x_1, \dots, x_n]/I$ for some ideal $I \subset A[x_1, \dots, x_n]$.
(2) If in addition $X$ is quasicompact, then $f$ is called *finite type*.
(3) For general $Y$, $f$ is *locally finite type* (resp. *finite type*) if for all affine opens $V \subseteq Y$, the map $X \times_Y V \to V$ is locally finite type (resp. finite type).

These properties roughly mean that you're covered by finite type algebras. It's not hard to prove that when $Y$ is affine, these definitions coincide.

**Theorem 14.10** (Hilbert's basis theorem)**.** *Suppose $A$ is a Noetherian ring amd $B$ is a finitely generated $A$-algebra. Then there is a fiber product square*

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r] \arrow[d] & \mathbb{A}^n_A \arrow[d] \\ 0 \arrow[r] & \mathbb{A}^m_A. \end{tikzcd}$$

*Here* $\mathbb{A}^k_A \coloneqq \operatorname{Spec} A[x_1, \dots, x_k]$.

This will be highly noncanonical.

*Remark* 14.11. This is telling us that $\operatorname{Spec} B$ is the zero locus of $m$ polynomials in $n$ variables with coefficients in $A$, or that more generally, a finite type Noetherian scheme is exactly one which locally admits such a description. Since this was one of our motivations for studying algebraic geometry from the beginning, this is an excellent hypothesis to have. ◀

Though you've probably already seen the proof if you know what a Noetherian ring is, it's still good to go over.

**Definition 14.12.** Let $V$ be an abelian group. A *filtration* on $V$ is a sequence

$$F_0 V \subseteq F_1 V \subseteq \cdots \subseteq V, \tag{14.13}$$

such that

$$\bigcup_{n \ge 0} F_n V = V. \tag{14.14}$$

The *associated graded* is a graded abelian group $\mathrm{gr}_\bullet(V) \coloneqq \bigoplus_{n\ge 0} \mathrm{gr}_n V$, where $\mathrm{gr}_n V \coloneqq F_n V / F_{n-1} V$, and we declare $F_{-1} V \coloneqq 0$.

Filtrations are more general than gradings, but are really nice to have, and can make some arguments a lot cleaner.

**Definition 14.15.** If $(V, F_n)$ and $(W, F'_n)$ are filtrations, a morphism of filtered abelian groups is a map $f\colon V \to W$ such that $f(F_n V) \subseteq F'_n W$.

**Exercise 14.16.** Show that if $\mathrm{gr}_\bullet(f)\colon \mathrm{gr}_\bullet(V) \to \mathrm{gr}_\bullet(W)$ is injective (resp. surjective, resp. bijective), then $f$ is injective (resp. surjective, resp. bijective).

This is a major tool in working with filtrations, especially in the (common) case where the filtered objects are complicated, but their associated gradeds are simpler.

**Definition 14.17.** A *filtered algebra* is an algebra $A$ filtered as an abelian group such that multiplication carries $F_n A \times F_m A \subseteq F_{n+m} A$.

**Example 14.18.** The algebra $A = k[x]$ is filtered by degree: we let $F_n A$ denote the polynomials of degree at most $n$. ◀

If $A$ is a filtered abelian group, $F_n A$ is also an abelian group, but if $A$ is an algebra, $F_n A$ is generally not a subalgebra, as in the above example.

If $A$ is a filtered algebra, we can make sense of the notion of filtered $A$-modules, where the filtrations given by the $A$-action and the module are compatible in the least surprising way.

**Lemma 14.19.** *If $M$ is a filtered $A$-module and $\mathrm{gr}_\bullet(M)$ is a finitely generated $\mathrm{gr}_\bullet(A)$-module, then $M$ is a finitely generated $A$-module.*

*Proof.* Let $\overline{x}_1, \dots, \overline{x}_n \in \mathrm{gr}_\bullet M$ be a generating set. We can assume they're *homogeneous*, i.e. each $\overline{x}_i$ lives in some $\mathrm{gr}_{k_i} M$. Lift $\overline{x}_i$ to some $x_i \in F_{i_k} M$; then the map $\varphi\colon A^{\oplus n} \to M$ sending the standard basis element $e_i \mapsto x_i$ is surjective after passing to the associated graded, hence by Exercise 14.16 is surjective, and therefore $\{x_1, \dots, x_n\}$ generates $M$. ☒

*Proof of Theorem 14.10.* By induction, it suffices to show that if $A$ is Noetherian, then $A[x]$ is too. Filter $A[x]$ by degree, and if $I \subseteq A$ is an ideal, let $F_n I \coloneqq I \cap F_n A[x]$.

We claim $\mathrm{gr}_\bullet I$ is finitely generated over $\mathrm{gr}_\bullet A[x]$. To see this, note that the multiplication-by-$x$ map $\mathrm{gr}_i I \to \mathrm{gr}_{i+1} I$ is an injection for all $i$, and furthermore its image in $A$ corresponds to the inclusion of an ideal. Therefore, by Noetherianness of $A$, this chain must stabilize at some $N \in \mathbb{N}$, and therefore $\mathrm{gr}_\bullet I$ is generated by $\bigoplus_{i=0}^N \mathrm{gr}_i I$. Because $A$ is Noetherian, each $\mathrm{gr}_i I$ is a finitely generated $A$-module as well, showing the claim. ⊠

Lecture 15.

## Connected and irreducible components: 10/3/18

I wasn't in class for this lecture; these notes were generously provided by Tom Gannon.

**Definition 15.1.** A scheme $X$ is *locally Noetherian* if it admits an open cover by affine open sets of the form $\operatorname{Spec} A$ for Noetherian $A$. If $X$ is also quasicompact, we say $X$ is *Noetherian*.

*Remark* 15.2. Note that if $U$ is a subset of a Noetherian scheme $X$, then $U$ is quasicompact. To see this, pick an affine open cover by Noetherian rings $\operatorname{Spec} A_i$ for $i \in \{1, \dots, m\}$; then $U^c$ is given by a finitely generated ideal. This also shows that $U$ is Noetherian. ◀

**Lemma 15.3.** *A scheme $X$ is Noetherian if and only if it is* topologically Noetherian*, that is, for all chains of closed $Z_i \subset X$, i.e. $Z_0 \supset Z_1 \supset \dots$, the $Z_i$ stabilize.*

The affine case is just rewriting the definition, and the general case just follows from compactness (exercise!). This shows one odd feature of the Zariski topology — we certainly don't have that $\mathbb{C}$ with the standard topology is Noetherian! An informal way of stating the above lemma is that taking a nontrivial closed subset is a big deal.

*Remark* 15.4. The equivalent conditions above yield the increasing chain condition for open sets but the increasing chain condition on open sets does not imply $X$ is Noetherian (for example, $X = \operatorname{Spec} k[x_i]_{i\in\mathbb{N}}/(x_i^2)$). This example also shows that there is not a bijection between closed subschemes and open subschemes, although the dual numbers also shows this. ◀

**Definition 15.5.** A scheme $X$ is *connected* if for all open covers $X = U \cup V$ such that $U \cap V = \varnothing$, either $U = X$ or $V = X$.

**Lemma 15.6.** *If $X$ is a Noetherian scheme, then $X$ can be written as the disjoint union of finitely many* connected components *of $X$, i.e. open and closed connected subschemes.*

This involves writing the definition of a disjoint union of schemes $\coprod_{i=1}^n X_i$, which we will leave as an exercise, but essentially any solution that isn't the functor $(\coprod_{i=1}^n X_i) \coloneqq \coprod_{i=1}^n X_i(A)$ will likely work.

*Proof.* If $x$ is a field valued point, then by Noetherianness there is a minimal $U_x \subset X$ which is closed and open and contains $x$. By minimality, $U_x$ is connected and $\{U_x\}_x$ is an open cover, where $x$ varies over all the field valued points, so because quasicompactness implies Zariski topology compactness (since every open cover of an affine admits a finite refinement), there are only finitely many $U_x$. ⊠

**Definition 15.7.** For a quasicompact quasiseparated (qcqs) morphism $f : X \to Y$ (for example, $f$ finite type–most importantly this is the setting where pushforward is defined on quasicoherent sheaves), we obtain by adjunction a map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ with some kernel $I$. This corresponds to a closed subscheme of $Y$, which we will denote $\overline{f(X)}$ and will call the *scheme-theoretic image*.

**Exercise 15.8.** If $X = \operatorname{Spec} A \to \operatorname{Spec} B = Y$ corresponds to a ring map $\phi : A \to B$, we can factor $\phi$ as a composite of a surjection and an injection $A \to \phi(A) \to B$.

The picture to have in mind here is that $X \to \overline{f(X)} \subset Y$, where $X \to \overline{f(X)}$ is *dominant*:

**Definition 15.9.** We say a qcqs morphism $f : X \to Y$ is *dominant* if $\overline{f(X)} = Y$.

Equivalently, $f$ is dominant if $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is a monomorphism. The idea here is that if you have a point in the closure of the image there is a function realizing this, and we have a rough equivalent between a function being dominant and the associated map on functions being injective.

**Example 15.10.** The map $\mathbb{A}^1 \setminus 0 \to \mathbb{A}^1$ is dominant.

**Definition 15.11.** A quasicompact $U \subset X$ is *dense* if $\overline{U} = X$.

As an exercise, you can look up the relationship between density and *associated primes* for a ring $A$.

**Definition 15.12.** A Noetherian scheme $X$ is *irreducible* if for every closed $Z \subset X$ with $Z \neq X$, $X \setminus Z$ is dense.

*Remark* 15.13. Other textbooks often refer to this as a scheme being *integral*. ◀

**Example 15.14.** The scheme $\operatorname{Spec} k[s,t]/(st)$ is not irreducible, which can be seen by setting $Z = \{t = 0\}$.

**Example 15.15.** The scheme $\operatorname{Spec} k[\epsilon]/(\epsilon^2)$ is not irreducible since the closed subscheme $\operatorname{Spec} k$ has empty complement (which is in particular not dense).

**Exercise 15.16.** We have that $\operatorname{Spec} A$ is irreducible if and only if $A$ is an integral domain.

**Exercise 15.17.** If $X = \operatorname{Spec} A$, then the irreducible closed subschemes of $X$ correspond to primes of $A$.

**Definition 15.18.** An *irreducible component* of $X$ is a maximal irreducible subscheme.

**Lemma 15.19.** *If $X$ is Noetherian, then there are only finitely many irreducible components and every field valued point factors through each one.*

We'll prove this next time.

Lecture 16.

## Irreducibility: 10/5/18

Today we're going to discuss a way to describe schemes as decomposed into simpler parts. One way to do this is to use connected components, but there's another notion, called reducibility, which is more general, and which we'll use more frequently. The idea is that $\operatorname{Spec} k[x,y]/(xy)$, the $x$- and $y$-axes inside $\mathbb{A}^2_k$, is a union of two $\mathbb{A}^1_k$s, so we want to call it reducible.

**Lemma 16.1.** *Let $Z \subseteq X$ be an irreducible component and $f$ be a function on $X$ with $f|_Z = 0$. Then there's a $g \in \operatorname{Fun}(X)$ with $g|_Z \neq 0$ and $f^N g = 0$ for $N \gg 0$.*

*Proof.* For $X$ affine, $Z$ is equivalent data to a prime ideal $\mathfrak{p}$ because $Z$ is irreducible, and minimal because $Z$ is a component, and $f$ must be in $\mathfrak{p}$. The localization $A_{\mathfrak{p}}$ is a local ring with the image of $f$ contained in the maximal ideal, and $A_{\mathfrak{p}}[f^{-1}] = 0$.

In general, if you're localizing at a multiplicative set and obtain zero, then some element of the set is zero. We localized with respect to $sf^N$ for $N \geq 0$ and $s \notin \mathfrak{p}$, so we conclude that there's some $g \in A \setminus \mathfrak{p}$ (so a global function on $X$ nonvanishing on $Z$), and such that $gf^N = 0$ for some $N$. ⊠

*Remark* 16.2. Localization has a geometric interpretation. Suppose $X = \operatorname{Spec} A$ and $B = A/\mathfrak{p}$, so $B$ is an integral domain. Let $k$ denote its fraction field; if $Z = \operatorname{Spec} B$, then $\operatorname{Spec} k \hookrightarrow Z$ is the *generic point* of $Z$, in the sense that in the Zariski topology, its closure contains all other points. This is perhaps a bit bizarre, but it allows for some useful constructions: the localization $\operatorname{Spec} A_{\mathfrak{p}}$ admits the geometric description of the intersection[12] of all opens $U \subseteq X$ which contain the generic point $\operatorname{Spec} k$. ◀

**Lemma 16.3.** *Let $X$ be an affine Noetherian scheme and $Z \subseteq X$ be an irreducible component. Then there's a function $g$ on $X$ with $g|_Z \neq 0$ and $g|_{X\setminus Z} = 0$.*

*Proof.* Writing $X = \operatorname{Spec} A$, $Z$ corresponds to a prime ideal $\mathfrak{p}$, necessarily finitely generated because $X$ is Noetherian: $\mathfrak{p} = (f_1, \dots, f_r)$. By Lemma 16.1, we can choose an $N > 0$ and $g_1, \dots, g_r \in \operatorname{Fun}(X)$ such that $g_i f_i^N = 0$ and $g_i|_Z \neq 0$. Letting $g = \prod g_i$, it's nonzero on $Z$, because $Z$ is Spec of a domain.

We claim $g|_{X\setminus Z} = 0$. This is because $X \setminus Z$ is covered by $D(f_i) = \{f_i \neq 0\}_{i=1,\dots,r}$, so it's enough to see on each $D(f_i)$; since $gf_i^N = 0$, then $g|_{D(f_i)} = 0$. ⊠

**Corollary 16.4.** *If $Z \subseteq X$ is an irreducible component, then $X \setminus Z$ isn't dense in $X$.*

[12]Well, actually an inverse limit, not an intersection. But in reasonable situations, these are the same thing.

*Proof.* We can easily reduce to $X$ affine. Then pick a $g$ with $g|_Z \neq 0$ and $g|_{X\setminus Z} = 0$. This means $\overline{X \setminus Z} \subseteq \{g = 0\} \subsetneq X$, since its complement $D(g)$ is an open subscheme of $X$. ⊠

**Corollary 16.5.** *If $X$ is a Noetherian scheme, it has only finitely many irreducible components.*

*Proof.* Let $\{Z_i\}_{i\in I}$ be the set of irreducible components of $X$, and let

$$Y_i := \overline{X \setminus (Z_1 \cup \cdots \cup Z_i)}. \tag{16.6}$$

We claim $Z_i \subsetneq Y_i$ but $Z_{i+1} \subseteq Y_i$ — we'll prove this in just a sec, but assuming the claim we obtain a decreasing sequence $Y_1 \supsetneq Y_2 \supsetneq \cdots$ if closed subschemes, so since $X$ is Noetherian, $|I|$ must be finite.

Now the claim. We saw that the generic point of $Z_i$ isn't in $\overline{X \setminus Z_i}$ in Corollary 16.4, and by definition $\overline{X \setminus Z_i} \supseteq Y_i$. Since $Z_{i+1} \subseteq Y_i$ and

$$Z_{i+1} \cap X \setminus (Z_1 \cup \cdots \cup Z_i) \neq \varnothing, \tag{16.7}$$

because the components $Z_i$ are distinct, then by irreducibility, $Z_{i+1}$ is contained in the closure of $X \setminus (Z_1 \cup \cdots \cup Z_i)$. ⊠

**Definition 16.8.** The *Krull dimension* $\dim X$ of a scheme $X$ is the largest integer $d$ such that there is a chain $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_d$ of irreducible subschemes of $X$. If there is no such integer, we say the dimension is infinite.

There's one main result in dimension theory.

**Theorem 16.9.** *Let $X$ and $Y$ be finite type, irreducible $k$-schemes. If $f\colon Y \to X$ is a morphism, then there's a dense open $U \subseteq X$ (equivalently, $U$ is nonempty), such that either $Y \times_X Y = \varnothing$ or for every field-valued point $x \in U$, $\dim f^{-1}(x) = \dim Y - \dim X$.*

Here $f^{-1}(x) := Y \times_X \{x\}$, as usual. A *field-valued point* is data of a field $k$ and a map $x\colon \operatorname{Spec} k \to U$. We'll prove this next time, and spend the rest of today's lecture on some corollaries.

**Corollary 16.10.** *If $X$ is irreducible and $U \subseteq X$ is dense, then* $\dim U = \dim X$.

**Corollary 16.11.** *If $X$ is irreducible and $f\colon Y \to X$ is dominant, then* $\dim Y \geq \dim X$.

Recall that dominant means it's injective on the level of functions. This means that when we tensor with the fraction field of $X$, we get something nonzero, and therefore the fiber over the generic point is nonempty.

There are nice situations in which these theorems aren't true. For example, let $A$ be a discrete valuation ring, such as $k[t]_{(t)}$ or $k[[t]]$. Then $k := A[t^{-1}]$ is the fraction field of $A$, and $\operatorname{Spec} k \hookrightarrow \operatorname{Spec} A$ is a nonempty open which is zero-dimensional, but $\operatorname{Spec} A$ is one-dimensional. So we really have to use the fact that we're finite type over a field.

Lecture 17.

## Noether normalization: 10/8/18

It will be useful to have some examples to carry around.

**Example 17.1.** The reason for irreducibility in the hypothesis of Theorem 16.9 is that dimension behaves poorly for reducible schemes: consider $Z = \{xz = 0, yz = 0\} \subset \mathbb{A}^3_k$. Geometrically, this is the union of the $xy$-plane and the $z$-axis, which clearly has two irreducible components, and $Z$ is two-dimensional. But $Z \setminus \{xy = 0\}$ is open in $Z$ and has dimension 1. We would like the dimension of open subsets to be the same as that of the entire scheme, forcing us to consider irreducibility. ◀

**Example 17.2.** For a typical, useful example, consider the map $\mathbb{A}^2 \to \mathbb{A}^2$ sending $(x, y) \mapsto (x, xy)$.[13] The fiber at a field-valued point $(x, y)$ with $x \neq 0$ is a point. The fiber at $(0, y)$ is empty for $y \neq 0$, and at the origin, the fiber is an $\mathbb{A}^1$.

So on $\mathbb{A}^2 \setminus 0$, which is an open, dense set, we get that the fibers are either empty or have the correct dimension; otherwise they could be "too big." ◀

[13] This is called the *blowup of the plane at* 0, and fits into a more general theory of blowups.

The main tool in our proof of Theorem 16.9 is the theory of finite morphisms, and in fact we'll end up reducing to Nakayama's lemma.

**Exercise 17.3.** As a warm-up for this kind of argument, use Nakayama's lemma to show that if $\mathcal{F}$ is a locally finitely generated QC sheaf on an irreducible scheme, then there exists a nonempty open $U$ such that $\mathcal{F}|_U$ is a vector bundle.

**Definition 17.4.** A morphism of schemes $f\colon Y \to X$ is *finite* if it's affine and for every affine open $U = \operatorname{Spec} A \subseteq X$, if $Y \times_X U = \operatorname{Spec} B$, then $B$ is a finitely generated $A$-module.

This is very strong: finite type was asking about finite generation as an algebra: $k[x]$ is a finitely-generated $k$-algebra but not a finitely generated $k$-module. Therefore $\mathbb{A}^1_k$ isn't finite over $\operatorname{Spec} k$!

*Remark* 17.5. If $f$ is an affine morphism, we showed that there's a sheaf of QC algebras $\mathcal{A}$ with $X = \operatorname{Spec}_Y(\mathcal{A})$. Then, $f$ is finite iff $\mathcal{A}$ is locally finitely generated as a quasicoherent sheaf ◀

**Example 17.6.**
(1) The best example to have in mind is to fix a field $k$, and let $B$ be a finite-dimensional $k$-algebra. Then $Y = \operatorname{Spec} B$ is finite over $\operatorname{Spec} k$. This implies $B$ is Artinian (which is a good exercise). So, for example, $\operatorname{Spec} \mathbb{C}$ over $\operatorname{Spec} \mathbb{R}$, or the dual numbers or other nilpotent things.
(2) A closed embedding is finite, because $A/I$ is generated as an $A$-module by $1_A$. ◀

**Proposition 17.7.** *Let $A$ be a ring and $X \coloneqq \operatorname{Spec} A$. Let $Y \subset X \times \mathbb{A}^1_A$ be a closed subscheme, corresponding to an ideal $I$ of $A[x]$, let and $f\colon Y \to X$ be the restriction of the projection map. Then $f$ is finite iff $I$ contains a monic polynomial.*

*Proof.* First assume $f$ is finite, so there are $\varphi_1, \dots, \varphi_N \in A[t]$ which generate $A[t]/I$ as an $A$-module. Choose $d$ such that $d > \deg(\varphi_i)$ for all $i$. Then there exist $a_1, \dots, a_N \in A$ such that

$$t^d = \sum_{i=1}^N a_i \varphi_i \bmod I, \tag{17.8}$$

so

$$t^d - \sum_{i=1}^N a_i \varphi_i \tag{17.9}$$

is in $I$ and is monic.

The converse is basically the same logic, but in reverse order: take your monic polynomial and lift it to get generators. ⊠

Finite morphisms are great because Nakayama's lemma applies to them. We'll see this in the proof of Theorem 16.9. We'll also need another tool, Noether normalization.

**Theorem 17.10** (Noether normalization)**.** *Let $X \subsetneq \mathbb{A}^{n+1}_k$ be a closed subscheme. Then there's a finite field extension $k \hookrightarrow k'$ and a projection map $\mathbb{A}^{n+1}_{k'} \to \mathbb{A}^n_{k'}$ such that the induced map $f\colon X_{k'} \to \mathbb{A}^n_{k'}$ is finite. If moreover $X$ is the zero locus of a single polynomial, then $f$ is dominant.*

Here $X_{k'} \coloneqq X \times_{\operatorname{Spec} k} \operatorname{Spec} k'$. By a projection, we mean a map induced by a linear surjection $(k')^{n+1} \to (k')^n$.

*Remark* 17.11. Suppose $k$ isn't a finite field. Then we don't need to pass to $k'$. (This will be evident from the proof.) ◀

**Example 17.12.** For example, consider the map $\mathbb{A}^1 \setminus 0 \hookrightarrow \mathbb{A}^2$ induced from the map $t \mapsto (t, t^{-1})$. Algebraically, we get $k[t] \hookrightarrow k[t, t^{-1}]$, which is not finite at the level of algebras (since we can take $t^{-N}$ for arbitrarily large $N$). Geometrically, you can use Nakayama's lemma to show that fibers of dominant maps over field-valued points must be nonempty, but the fiber over 0 is empty.

But you can project along any other line (except the $y$-axis), such as the diagonal, then the map is in fact finite. ◀

Lecture 18.

## Proof of Noether normalization: 10/10/18

Last time, we discussed finite morphisms and Noether normalization. The following exercise might provide some useful intuition about finite morphisms.

**Exercise 18.1.** Let $f\colon X \to Y$ be a finite, dominant map. Then for all field-valued points $x \in X$, $f^{-1}(x)$ is nonempty and zero-dimensional. Also show that $\dim X = \dim Y$.

Hint: Nakayama's lemma.

We then discussd Example 17.12, about $\{xy = 1\} \subset \mathbb{A}^2$. It doesn't project onto every line through the origin in $\mathbb{A}^1$, but everything but the $x$- and $y$-axes is good. One interesting way to think about this is that there's a $\mathbb{P}^1$ "at infinity" of $\mathbb{A}^2$, where, imprecisely speaking, we think of $\mathbb{P}^1$ as a circle of very large radiyus (though we need to identify antipodal points); a line is sent to its point of intersection with the circle. Then, we have an open subset of $\mathbb{P}^1$ (again, everything except the $x$- and $y$-axes) where the projection is finite and dominant.

More generally, suppose $X \subseteq \mathbb{A}^{n+1}$. We can embed $\mathbb{A}^{n+1} \subset \mathbb{P}^{n+1}$ as an open subscheme; let $\overline{X}$ be the closure of $X$ in $\mathbb{P}^{n+1}$. The complement of $\mathbb{A}^{n+1}$ inside $\mathbb{P}^{n+1}$ is a $\mathbb{P}^n$, and we let $\mathrm{Asym}(X) \coloneqq \overline{X} \cap \mathbb{P}^n$; we can think of this as where $X$ is going "at infinity."

If $X \neq \mathbb{A}^{n+1}$, then $\mathrm{Asym}(X) \neq \mathbb{P}^n$. We claim there exists a finite extension $k \hookrightarrow k'$ and a $k'$-point of $\mathbb{P}^n$ not in $\mathrm{Asym}(X)$, and that projecting away from this line $\ell$, the map is finite. By "projecting away from the line," we mean that there's a projection $\pi\colon \mathbb{A}^{n+1}_{k'} \to \mathbb{A}^n_{k'}$ such that $\ker(\pi) = \ell$.

*Proof of Theorem 16.9.* More explicitly, first we can reduce to the case where $X = \{f = 0\}$, for some nonzero and nonconstant $f$. Since $X \subsetneq \mathbb{A}^1$, so we can choose such an $f$ vanishing on $X$. Then $X \hookrightarrow \{f = 0\}$ is a closed embedding, hence a finite morphism. It's easy to see that finite maps are closed under compositions, so the map $\{f = 0\} \hookrightarrow \mathbb{A}^{n+1} \twoheadrightarrow \mathbb{A}^n$ is finite, and therefore the theorem for $\{f = 0\}$ implies the theorem for $X$.

Now we have $f \in k[x, y_1, \dots, y_n]$ and $X = \{f = 0\}$, which in particular is nonempty. Write

$$f = \sum_{i=0}^{d} f_i, \tag{18.2}$$

such that each $f_i$ is homogeneous of degree $i$ and $f_d$ is nonzero.

**Example 18.3.** If $f(x) = x^3 + 2x^2y$, then $f$ is homogeneous of degree 3. For $f(x) = x^3 + 1$, we'd let $f_0 = 1$, $f_1 = f_2 = 0$, and $f_3 = x^3$. ◀

**Exercise 18.4.** Show that if $X = \{f = 0\}$, $\mathrm{Asym}(X) = \{f_d = 0\} \subseteq \mathbb{P}^n$. Here we're thinking of $f_d$ as a section of $\mathcal{O}_{\mathbb{P}^n}(d)$.

**Exercise 18.5.** Show that there exists a finite extension $k \hookrightarrow k'$ and some $v \in (k')^{n+1}$ such that $f_d(v) \neq 0$. Moreover, if $k$ is infinite, we can choose $k' = k$.

This is a general fact about nonconstant polynomials. We will now write $k = k'$ for ease of notation. Moreover, up to a linear change of coordinates, we can assume $v = (1, 0, \dots, 0)$, which doesn't affect homogeneity.

If

$$f_d = ax^d + bx^{d_1}y_1 + cx^{d-1}y_2 + \cdots, \tag{18.6}$$

then $f_d(1, 0, \dots, 0) = a$, and up to scaling, we can assume $a = 1$. (We know $a \neq 0$ because $f_d(v) \neq 0$).

We can write $f = \sum_{i=0}^d g_i x^i$ for $g_i \in k[y_1, \dots, y_n]$; by construction, $g_d = 1$. Therefore, as a polynomial in $x$, $f$ is monic, and therefore by last time, $k[y_1, \dots, y_n][x]/(f)$ is finite over $k[y_1, \dots, y_n]$ (specifically, $1, x, \dots, x^{d-1}$ generate it). And since $d > 0$, $k[y_1, \dots, y_n] \hookrightarrow k[y_1, \dots, y_n][x]/(f)$, which implies dominance. Therefore we've proven the theorem. ⊠

**Corollary 18.7** (Nullstellensatz)**.** *Let $X$ be a finite-type affine scheme over $k$. Then there's a finite extension $k \hookrightarrow k'$ and a finite dominant map $X_{k'} \to \mathbb{A}^n_{k'}$ for some $n$. If $k$ is infinite, we can take $k' = k$.*

There are other theorems called the Nullstellensatz, but they're all related to each other and to this one.

*Proof.* We know $X_{k'} \subsetneq \mathbb{A}^n_{k'}$ for some $n$, and we have a finite dominant map $\pi\colon \mathbb{A}^n_{k'} \to \mathbb{A}^{n-1}_{k'}$; if $\overline{\pi(X)} = \mathbb{A}^{n+1}_{k'}$, we're done; otherwise we can repeat.

TODO: then something else happened, which I didn't quite follow. ⊠

In particular, the ring of functions on $X_{k'}$ is finite-dimensional over $k'$.

**Corollary 18.8.** $\dim_k \mathbb{A}^n_k = n$.

*Proof.* We can induct: $n = 0$ is clear, so assume it for $\mathbb{A}^n_k$, and we'll show it for $\mathbb{A}^{n+1}_k$. Let $Z \subsetneq \mathbb{A}^{n+1}$ be a closed, irreducible NTS; then $\dim Z \le n$. Since $Z \subset \{f = 0\}$ for some $f$, then it admits a finite dominant map to $\mathbb{A}^n_k$, so $\dim\{f = 0\} = n \ge \dim Z$ by induction. ⊠

Lecture 19.

## More cool facts from dimension theory: 10/12/18

Today we'll continue deducing stuff from Theorems 16.9 and 17.10. For example, at the end of the last class, we showed that $\dim \mathbb{A}^n$ is $n$, so if $f \in k[x_1, \dots, x_n]$ is nonconstant, then $\{f = 0\}$ is $(n-1)$-dimensional, and this is true for all irreducible components of $X$.

**Definition 19.1.** An irreducible scheme $X$ is *caternary* if for all closed subschemes $Z \subsetneq X$, there's a closed subscheme $Z' \subseteq X$ containing $Z$ as a closed subscheme and such that $\dim Z' = \dim X - 1$.

We basically proved the following while proving Theorem 17.10.

**Corollary 19.2.** *$\mathbb{A}^n$ is catenary.*

*Remark* 19.3. The word catenary comes from the word for "chain" in a Romance language (e.g. in Italian, it's *catena*), presumably because it gives us chains of closed subschemes. ◀

More generally:

**Corollary 19.4.** *If $X$ is an irreducible finite-type scheme over an infinite field $k$, then $X$ is catenary.*

*Proof.* We can quickly reduce to the case where $X$ is affine. Noether normalization means we may choose a finite dominant map $\pi\colon X \to \mathbb{A}^n_k$; hence $\dim X = n$. Let $Z \subsetneq X$ be maximal under closed irreducible subschemes contained in $X$. We want to show $\dim Z = n - 1$.

The restriction $\pi|_Z\colon Z \to \overline{\pi(Z)}$ is also finite dominant, so it suffices to show $\dim \overline{\pi(Z)} = n - 1$. If this is not the case, then since $\mathbb{A}^n_k$ is catenary, we can choose an $(n-1)$-dimensional irreducible $Z' \subset \mathbb{A}^n_k$ and strictly containing $\overline{\pi(Z)}$.

The theory of finite morphisms (more specifically, going-up and going-down theorems applied to $X \to \mathbb{A}^n$), $Z$ is not an irreducible component of $\pi^{-1}(Z')$, which is a contradiction. ⊠

This is an important result that's easy to take for granted — it is one of the facts about dimension that is an ansatz about any theory of dimension in geometry: all $k$-points look the same in a variety over $k$. If something like this were not true, there would have to be a different theory of dimension. It's not so surprising it reduces to studying $\mathbb{A}^n_k$, with its large symmetry group.

There are two major ways to induct on dimension for varieties over fields: project onto a lower-dimensional subscheme, and take the intersection with a hyperplane. We used the former for this proof.

*Remark* 19.5. The proof of Corollary 19.4 strongly depends on the Nullstellensatz, and in particular, is not true over more general rings: if $A$ is a DVR, then $\operatorname{Spec} A[x]$ isn't catenary. But dimension is set up to behave well over fields, so maybe this isn't so sad. ◀

**Corollary 19.6.** *If $X$ is irreducible and finite type over $k$, and $U \subseteq X$ is a nonempty open, then* $\dim U = \dim X$.

*Proof sketch.* Without loss of generality, we can assume $X = \operatorname{Spec} A$ is affine. Let $x \in U$ be a closed point, hence $\operatorname{Spec} k'$ for some extension $k \hookrightarrow k'$, and we have data of a surjective map $A \to k'$ (and, if $U = \operatorname{Spec} B$, data of a surjective map $B \to k'$). There's a closed, irreducible subscheme $Z \subsetneq X$ containing $x$ and such that $\dim Z = \dim X - 1$.

The intersection $Z \cap U$ is a nonempty, irreducible, proper closed subscheme of $U$, and is an open subscheme of $Z$. Inducting on $\dim X$,

$$\dim Z \cap U = \dim Z = \dim X - 1, \tag{19.7}$$

so $\dim U > \dim X - 1$, hence $\dim U \geq \dim X$. The other inequality is easy: take closures. ☒

**Lemma 19.8.** *If $X$ is irreducible and $Y$ is a closed subscheme of $X \times \mathbb{A}^1$, then either*

- *$Y$ is $X \times \mathbb{A}^1$, or*
- *there's a nonempty open $U \subseteq X$ such that $Y \times_X U \to U$ is finite.*

The idea is best illuminated when working over a field and $X = \operatorname{Spec} k$. There, the lemma that closed subsets of $\mathbb{A}^1$ that aren't $\mathbb{A}^1$ are just finitely many points.

*Proof.* As usual we can assume $X = \operatorname{Spec} A$ is affine, so $Y = \operatorname{Spec} A[t]/I$ for an ideal $I \subseteq A[t]$. The first option is $I = 0$, so we assume $I \neq 0$, so there's a nonzero $f \in I$, and $f = \sum_{i=0}^d a_i t^i$ for $a_i \in A$ with $a_d \neq 0$. If $U = \{a_d \neq 0\}$, then $U$ is a nonempty open subscheme of $X$. One can show that $I$ contains a monic polynomial over $U$, and we saw this is equivalent to $Y \times_X U \to U$ being finite. ☒

Lecture 20.

## The main theorem of dimension theory: 10/15/18

I wasn't in class for this lecture; these notes were generously provided by Tom Gannon.

Today, we'll prove the main theorem of dimension theory.

*Proof.* By taking an affine open subset of $X$ realizing its dimension, we can assume that $X$ is affine. Similarly, we may then take an affine open subset of $Y$ and assume that $Y$ is also an irreducible affine scheme. Then by our finite type assumptions we may write $Y$ as a closed subscheme of $X \times \mathbb{A}^n$ for some $n$.

Define $p_i \colon X \times \mathbb{A}^i \to X \times \mathbb{A}^{i-1}$ to be projection onto the first factor, $Y_n \coloneqq Y$, and for $i \in \{0, ..., n-1\}$ iteratively take $Y_i \coloneqq \overline{p_{i+1}(Y_{i+1})}$ and $\pi_i \colon Y_i \to Y_{i-1}$. This is summarized in the commutative diagram below, which made me have new found respect for Arun because I am reasonably sure he could have produced this during class:

$$\begin{tikzcd}
Y = Y_n \arrow[r, hook] \arrow[d, "\pi_n"] & X \times \mathbb{A}^n \arrow[d, "p_n"] \\
Y_{n-1} \arrow[r, hook] \arrow[d, "\pi_{n-1}"] & X \times \mathbb{A}^{n-1} \arrow[d, "p_{n-1}"] \\
\vdots \arrow[d, "\pi_1"] & \vdots \arrow[d, "p_1"] \\
X = Y_0 \arrow[r, hook] & X
\end{tikzcd}$$

Note that $\pi_i \colon Y_i \to Y_{i-1}$ embeds into the situation of the lemma. Descending inductively, we will set $U_i \subset Y_i$ with the property that $Y = Y_n \to Y_i$ has fibers of the expected dimension over $U_i$. Then taking $U = U_0$ we will be done, once we show that such $U_i$ exist. Note by the irreducibility of $Y$, we can take our "base case" $U_n = Y$.

floop

TODOFinish this proof. (This is also proven in Ravi Vakil's notes–Theorem 11.4.1) ☒

*Remark* 20.1. The idea here is to make our map $Y \to X$ as a close as possible to the projection map $X \times \mathbb{A}^1 \to X$, a situation we can study well. A key step here is that each open has the correct dimension. ◂

*Remark* 20.2. It is also true (Chevalley) that if $f \colon Y \to X$ is a dominant map between irreducible finite type $k$ -schemes and $x$ is a field valued point then $\dim(Y_x) \geq \dim(Y) - \dim(X)$. We saw this in our $(x, y) \to (x, xy)$ example. Furthermore, if $f$ is *flat* (the algebro-geometric generalization of a flat morphism of rings), all nonempty fibers are nonempty dimension. There is even a converse when $X$ and $Y$ are smooth. We will discuss what smoothness means now. ◀

The idea behind smoothness is that we can formally compute derivatives of polynomials over any field. In other words, calculus and differentials make sense formally, although some strange things occur. One main thing that comes up is that $\frac{d}{dt}t^p = 0$ in characteristic $p$.

**Definition 20.3.** For an $A$ module $M$, a *(k-linear) derivation* is a $k$-linear map $\delta\colon A \to M$ satisfying the product rule.

**Example 20.4.** If $A = k[t]$, then $\frac{\mathrm{d}}{\mathrm{d}t}\colon A \to A$ is a derivation. More generally, $\frac{\mathrm{d}}{\mathrm{d}t_i}\colon k[t_1, ..., t_n] \to k[t_1, ..., t_n]$ is a derivation. ◀

These derivations are best thought of as derivations along some vector field, at least when $M = A$. In general, it's not a terrible simplification to think of $M$ as a vector bundle and that a derivation can produce vector fields.

Lecture 21.

## Differentials and derivations: 10/17/18

*"It's not a field, but it's psychologically a field."*

Today we're going to talk about differentials and derivations, which are pretty important. For this lecture, $X = \operatorname{Spec} A$ is an affine scheme over a field $k$.

**Definition 21.1.** If $M$ is an $A$-module, a *derivation* $\delta\colon A \to M$ is an $A$-linear map satisfying the Leibniz rule

$$\delta(fg) = f\delta(g) + g\delta(f).$$

The set of derivations from $A$ to $M$ is denoted $\mathrm{Der}_A(A, M)$; it is naturally an $A$-module.

A *vector field* is a derivation $\delta\colon A \to A$.

In differential geometry, a vector field gives you a way to differentiate functions.

Derivations are corepresented by a particular $A$-module (i.e. quasicoherent sheaf on $X$) $\Omega^1_X$: that is, it's equipped with a derivation $\mathrm{d}\colon \mathcal{O}_X \to \Omega^1_X$ such that for all $A$-modules $M$, restriction along d defines an $A$-linear isomorphism

$$\mathrm{Hom}_A(\Omega^1_X, M) \xrightarrow{\cong} \mathrm{Der}_A(A, M). \tag{21.2}$$

The proof is a contruction: let $\Omega^1_X$ be generated as an $A$-module by elements $\{\mathrm{d}f \mid f \in A\}$ with relations

$$\mathrm{d}(fg) = f\,\mathrm{d}g + g\,\mathrm{d}f \qquad \text{for all } f, g \in A \tag{21.3a}$$

$$\mathrm{d}(\lambda f) = \lambda\,\mathrm{d}f \qquad \text{for all } \lambda \in k. \tag{21.3b}$$

**Lemma 21.4.** *If $f \in A$ and $n \geq 1$, then $\mathrm{d}(f^n) = nf^{n-1}\,\mathrm{d}f \in \Omega^1_X$.*

*Proof.* Induct on $n$: it's clear for $n = 1$, and assuming it for $n$, it follows for $n+1$ using the Leibniz rule on $f^{n+1} = (f^n)(f)$. ☒

As a corollary, $\mathrm{d}(1) = 0$, as $\mathrm{d}(1^n) = \mathrm{d}(1)$.

**Example 21.5.** For $X = \mathbb{A}^1_k = \operatorname{Spec} k[t]$, we have $\mathrm{d}t \in \Omega^1_{\mathbb{A}^1_k}$. We claim $\Omega^1_{\mathbb{A}^1_k}$ is freely generated by $\mathrm{d}t$, i.e. $\Omega^1_{\mathbb{A}^1_k} \cong \mathcal{O}_{\mathbb{A}^1_k} \cdot \mathrm{d}t$.

The proof is that, given a $k[t]$-module $M$ and a derivation $\delta\colon k[t] \to M$, if $f = \sum a^i_t \in k[t]$, then in $M$,

$$\delta(f) = \sum_{i\geq 1} a_i i t^{i-1}\delta(t) \in M. \tag{21.6}$$

so it's spanned by $\mathrm{d}t$. Conversely, given an element $\delta(t) \in M$, there's a unique derivation $\delta\colon k[t] \to M$ sending $t \mapsto \delta(t)$, by the universal property, so $k[t]$-linear maps $\Omega^1_X \to M$ are uniquely determined by where they send $\mathrm{d}t$. ◀

So if you boil off the abstraction, all you need is to know the derivative of a polynomial. Hopefully this is reassuring.

**Example 21.7.** Now take $X = \mathbb{A}^n_k = \operatorname{Spec} k[t_1, \dots, t_n]$. Now $\Omega^1_{\mathbb{A}^n_k}$ is a free $k[t_1, \dots, t_n]$-module of rank $n$, with a basis $\mathrm{d}t_1, \dots, \mathrm{d}t_n$, and

$$\mathrm{d}f = \sum_{i=1}^n \frac{\partial f}{\partial t_i}\,\mathrm{d}t_i. \tag{21.8}$$

The proof is essentially the same as for Example 21.5: once you know where $t_1, \dots, t_n$ go, everything else is forced by linearity and the Leibniz rule. ◀

For a general finite type affine $X = \operatorname{Spec} A$, there's a general algorithm to compute $\Omega^1_X$: first let $f_1, \dots, f_r$ generate $A$ as a $k$-algebra, and write

$$A \cong k[t_1, \dots, t_r]/(g_1, \dots, g_s) \tag{21.9}$$

for some $g_1, \dots, g_s$ encoding the relations between the $f_i$. Then $\Omega^1_X$ is generated by $\mathrm{d}f_1, \dots, \mathrm{d}f_r$ with relations $\mathrm{d}g_i|_X = 0$ for $i = 1, \dots, s$.

What's going on here? Well a map $\varphi\colon Y \to X$ of affine $k$-schemes induces a map of $A$-modules $\mathrm{d}\varphi\colon \Omega^1_X \to \varphi_*\Omega^1_Y$: take the universal differential $\mathrm{d}_Y\colon \mathcal{O}_Y \to \Omega^1_Y$ and push it forward to $X$: $\varphi_*\mathrm{d}_Y\colon \varphi_*\mathcal{O}_Y \to \varphi_*\Omega^1_Y$. Then precompose with $\varphi^*\colon \mathcal{O}_X \to \mathcal{O}_Y$ (pullback of functions); this is a differential $\mathcal{O}_X \to \varphi_*\Omega^1_Y$, hence corresponds uniquely to an $A$-module map $\Omega^1_X \to \varphi_*\Omega^1_Y$. Somewhat more explicitly, the characteristic formula is

$$\mathrm{d}\varphi(f\,\mathrm{d}g) = (f\varphi)\,\mathrm{d}(g\varphi). \tag{21.10}$$

*Remark* 21.11. Often, $\Omega^1_X$ is denoted $\Omega^1_{X/k}$: $k$-linearity is made more explicit. For example, nothing we've done so far requires $k$ to be a field, so we could work with affine schemes over a ring $B$ and study the module of differentials $\Omega^1_{X/B}$. ◀

Suppose $i\colon X \hookrightarrow Y$ is a closed embedding of affine schemes over $k$. Then one can show that the induced map $i^*\Omega^1_Y \to \Omega^1_X$ is surjective; if $I$ denotes its kernel, then $I \subseteq \mathcal{O}_Y$ is an ideal.

**Example 21.12.** $\Omega^1_X$ isn't always free; for example, suppose $k$ has characteristic zero and $X = \operatorname{Spec}(k[t]/(t^n))$. Using the above algorithm, one can show $\Omega^1_X$ is torsion. ◀

Lecture 22.

## Smoothness: 10/19/18

*"I'll stick to the party line."*

**Lemma 22.1.** *Let $k$ be a field, $A$ be a $k$-algebra, and $f \in A$. If $j\colon U \hookrightarrow X = \operatorname{Spec} A$ denotes the locus where $f \neq 0$, then restriction defines an isomorphism $j^*\Omega^1_X \xrightarrow{\cong} \Omega^1_U$.*

*Proof.* Let $M$ be an $A[f^{-1}]$-module, which is the same thing as a quasicoherent sheaf on $U$. Thinking of $M$ as an $A$-module (i.e. $j^*M$), a derivation $\delta\colon A \to M$ is the same thing as a map $j^*\Omega^1_X \to M$ by adjunction. Then $\delta$ extends uniquely to a derivation $\widetilde{\delta}$ on $A[f^{-1}]$, because we know what it has to be on $f^{-1}$ by the Leibniz rule:

$$\widetilde{\delta}(f^{-n}) = -nf^{-n-1}\widetilde{\delta}(f). \tag{22.2}$$

This is a quick inductive argument: we know $\widetilde{\delta}(1) = 0$, so

$$0 = \widetilde{\delta}(f^{-n}f^n) = f^n\widetilde{\delta}(f^{-n}) + f^{-n}\widetilde{\delta}(f^n) = f^n\widetilde{\delta}(f^{-n}) + nf^{-1}\widetilde{\delta}(f). \tag{22.3}$$

Now, given an arbitrary element $g/f^n \in A[f^{-1}]$, we define

$$\widetilde{\delta}\left(\frac{g}{f^n}\right) = g(-n)f^{-n-1}\delta(f) + f^{-n}\delta(g). \tag{22.4}$$

It's fairly straightforward to check this is well-defined, and that it gives a derivation. ⊠

As a corollary, we can define $\Omega^1_X$ as a quasicoherent sheaf on any $k$-scheme $X$: using Serre's theorem, it suffices to describe it on any open affine $j\colon U \hookrightarrow X$, where it's just $\Omega^1_U$. The above lemma guarantees this behaves correctly on intersections.

**Definition 22.5.** A $k$-scheme $X$ is *smooth* if

(1) $X$ is locally of finite type over $k$,
(2) $\Omega^1_X$ is a vector bundle, and
(3) for all irreducible components $Z \subseteq X$, the rank of $\Omega^1_X|_Z$ is equal to $\dim Z$.

*Remark* 22.6. In practice, $X$ will generally be finite type. ◂

It is a nontrivial fact that if $X$ is smooth, every irreducible component is a connected component.

*Remark* 22.7. Our definition of irreducible is slightly more restrictive than the standard definition, which allows things such as $\operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$. What we call irreducible is generally called integral. Fortunately, it doesn't make a difference in Definition 22.5, though this isn't obvious. ◂

**Example 22.8.**

(1) $\mathbb{A}^n_k$ is smooth, because $\Omega^1_{\mathbb{A}^n_k}$ is free of rank $n$.
(2) $\mathbb{P}^n_k$ is also smooth, because it has a cover by copies of $\mathbb{A}^n_k$, which is smooth.
(3) If $n \geq 2$, then $X = \operatorname{Spec} k[t]/(t^n)$ is not smooth: it's zero-dimensional, but $\Omega^1_X \neq 0$.
(4) Consider the coordinate axes in $\mathbb{A}^2_k$, $X := \operatorname{Spec} A$, where $A := k[x,y]/(xy)$. This is not smooth. It's one-dimensional, and as an $A$-module,

$$\Omega^1_X = A[\mathrm{d}x, \mathrm{d}y]/(x\,\mathrm{d}y + y\,\mathrm{d}x). \tag{22.9}$$

We have a resolution of this module defining generators and relations:

$$A \xrightarrow{1 \mapsto x\,\mathrm{d}y + y\,\mathrm{d}x} A^{\oplus 2} \xrightarrow{(\mathrm{d}x, \mathrm{d}y)} \Omega^1_X \longrightarrow 0. \tag{22.10}$$

Restricting to $(0,0) \in X$, we get

$$k \xrightarrow{0} k^{\oplus 2} \xrightarrow{\cong} \Omega^1_X|_{(0,0)} \longrightarrow 0, \tag{22.11}$$

so here it has rank 2, which is not the dimension of $X$. Therefore $X$ isn't smooth (it turns out $\Omega^1_X$ isn't a vector bundle, which is often the problem).
(5) Let $f \in k[x]$ be a *separable* polynomial, meaning it has no repeated roots over the algebraic closure $\overline{k}$ of $k$, and consider the scheme $X = \operatorname{Spec} k[x,y]/(y^2 - f(x))$.[14] From dimension theory, it's clear this is a curve (i.e. 1-dimensional); we'll show it's smooth.

This time, in the resolution of $\Omega^1_X$

$$A \xrightarrow{f} A^{\oplus 2} \xrightarrow{g} \Omega^1_X \longrightarrow 0, \tag{22.12}$$

$f(1) = \mathrm{d}(y^2 - f(x)) = 2y\,\mathrm{d}y - f'(x)\,\mathrm{d}x$, so if we have a field $K$ and $x, y \in K$ satisfying $y^2 = f(x)$, then if $2y\,\mathrm{d}y = f'(x)\,\mathrm{d}x$, then $y = 0$ and $f'(x) = 0$. Then $x$ is a root of $f$ and $f'$, but since we assumed $f$ is separable, this cannot happen. Therefore $\Omega^1_X$ is a vector bundle.

The picture is that restricting the projection $\mathbb{A}^2 \to \mathbb{A}^1$ sending $(x,y) \mapsto x$ to $X$ defines a map whose fiber at $x \in \mathbb{A}^1$ is the square roots to $f(x)$ if $f(x) \neq 0$.
(6) Our last example is an important pathology to be aware of. Suppose $k$ has characteristic $p > 0$, and suppose $\lambda \in k$ is not a $p^{\text{th}}$ power (in particular, $k$ is infinite; a typical example is $k = \mathbb{F}_p(\lambda)$, the field of rational functions in $\lambda$). Let $k' := k[t]/(t^p - \lambda)$, adjoining a $p^{\text{th}}$ root of $\lambda$; $t^p - \lambda$ is irreducible, so this is a field.

Then $\operatorname{Spec} k'$ is zero-dimensional over $k$, but $\Omega^1_{\operatorname{Spec} k'} \neq 0$, so this point is not smooth, which is weird. This is because $\Omega^1_{\operatorname{Spec} k'} = k' \cdot \mathrm{d}t/\mathrm{d}(t^p - \lambda)$, but $\mathrm{d}(t^p - \lambda) = pt^{p-1} = 0$, so $\Omega^1_{\operatorname{Spec} k'}$ is one-dimensional.

$\operatorname{Spec} k'$ is, of course, smooth over itself, i.e. as a $k'$-scheme; smoothness is relative. The related notion of regularity is intrinsic, but smoothness is always with respect to a base. Said a different way, smoothness is a property of morphisms. ◂

[14] If $\deg f \geq 5$, this is called a *hyperelliptic curve*.

Lecture 23.

## Zero-dimensional smooth varieties: 10/22/18

> *"This lemma existed 100 years ago, and will exist 100 years following. We're not a part of it."*

Today we'll classify smooth, zero-dimensional varieties over a field.

**Theorem 23.1.** *Let $X$ be a smooth, zero-dimensional finite-type scheme over a field $k$. Then*

$$X \cong \prod_{i=1}^{n} k_i, \tag{23.2}$$

*where each $k \hookrightarrow k_i$ is a separable field extension.*

*Remark* 23.3. Recall that a field extension $k \hookrightarrow k'$ is *separable* if for all $x \in k'$, the minimal polynomial of $x$ over $k$ has no repeated roots. ◄

First, one ingredient we'll need in the proof, and which will also be useful later.

**Definition 23.4.** Let $k$ be a field and $V$ be a vector space over $k$. The *split square-zero extension* associated to this data is the commutative $k$-algebra $A = k \oplus V$ with the multiplication

$$(\lambda, v) \cdot (\mu, w) \coloneqq (\lambda + \mu, \mu \cdot v + \lambda \cdot w). \tag{23.5}$$

In particular, $(0, v) \cdot (0, w) = 0$ and $(\lambda, 0) \cdot (0, v) = (0, \lambda v)$. If $V$ is one-dimensional, this recovers the dual numbers; you can think of split square-zero extensions as generalizations of the dual numbers.

*Proof of Theorem 23.1.* We can assume $X = \operatorname{Spec} A$ is affine. Since $X$ is zero-dimensional, $A$ is Artinian; from the theory of Artinian rings, $A$ is a product of Artinian local rings; since $X$ is finite type, this is a finite product. That is,

$$A = \prod_{i=1}^{n} A_i, \tag{23.6}$$

where $A_i$ is an Artinian local ring. Therefore we can reduce to the case where $A$ itself is an Artinian local ring, with maximal ideal $\mathfrak{m}$.

For the next step, we assume $A = k'$ is a finite separable extension of $k$; we'll show that $X = \operatorname{Spec} k'$ is smooth, which in this setting is equivalent to showing that $\Omega^1_{X/k} = 0$. $\Omega^1_{X/k}$ is generated by the elements $\mathrm{d}x$ for $x \in k'$; given such an $x$, let $f_x(t) \in k[t]$ denote the minimal polynomial of $x$ over $k$. Then, $f(x) = 0$, so

$$\mathrm{d}f(x) = f'(x)\,\mathrm{d}x = 0. \tag{23.7}$$

Since $f'(x) \neq 0$ by separability, then $\mathrm{d}x = 0$.

Now let $A = k \oplus V$ be a split square-zero extension. The projection map $A \to k$ is a ring map. The other projection map $A \to V$ is a derivation, which is a quick thing to check. Therefore, in particular, $A$ is a field if and only if $V = 0$.

Next, we'll show that if $A$ is an Artinian local $k$-algebra with nonzero maximal ideal $\mathfrak{m}$ such that $\mathfrak{m}^2 = 0$ and with a separable residue field $k' \coloneqq A/\mathfrak{m}$, then $\Omega^1_{\operatorname{Spec} A/k} \neq 0$. In this setting, $\mathfrak{m}$ acts trivially on itself, hence the $A$-module structure on $\mathfrak{m}$ passes to a $k'$-vector space structure.

We claim that $A$ is isomorphic to the split square-zero extension $k' \oplus \mathfrak{m}$; then this step will follow from the previous step. We'll show this by showing that the projection $\pi\colon A \to A/\mathfrak{m} = k'$ splits canonically as an algebra map. Specifically, if $x \in k'$, let $f(t) \in k[t]$ be its minimal polynomial and $\widetilde{x} \in A$ be a lift of $x$ across $\pi$. Then, there's a unique $\sigma(x) \in A$ such that $f(\sigma(x)) = 0$ and $\pi(\sigma(x)) = 0$.

If $v \in \mathfrak{m}$, then

$$f(\widetilde{x} + v) = f(\widetilde{x}) + f'(x) \cdot v, \tag{23.8}$$

which you can prove by reducing to the case where $f(t) = t^n$ and check directly using the binomial theorem and the face that $v^2 = 0$. Therefore $\pi(f(\widetilde{x})) = f(x) = 0$, so $f(\widetilde{x}) \in \mathfrak{m}$. Therefore

$$v = -\frac{1}{\pi(f'(\widetilde{x}))} \cdot f(x) \in \mathfrak{m}. \tag{23.9}$$

Because $f$ is separable, $\pi(f'(\widetilde{x})) \neq 0$.

So we've shown that if $A = k' \oplus \mathfrak{m}$ is a split square-zero extension, then $\Omega^1_{\operatorname{Spec} A/k} \neq 0$; next we'll show that if $A$ is a local $k$-algebra (not a field) with spearable residue field, then $\Omega^1_{\operatorname{Spec} A/k} \neq 0$. Now assume that $A$ is a local ring with nonzero maximal ideal $\mathfrak{m}$ with separable residue field but not necessarily assuming $\mathfrak{m}^2 = 0$. Note that the ring map $\pi\colon A \to A/\mathfrak{m}^2$ is a surjection, which implies that we have a closed embedding $i\colon \operatorname{Spec} A/\mathfrak{m}^2 \to \operatorname{Spec} A$. This closed embedding yields a surjection $i^*\Omega^1_{\operatorname{Spec} A/k} \to \Omega^1_{(\operatorname{Spec} A/\mathfrak{m}^2)/k}$, which we argued last class, and because $i^*\Omega^1_{\operatorname{Spec} A/k}$ surjects onto something nonzero, $i^*\Omega^1_{\operatorname{Spec} A/k}$ is nonzero, and thus $\Omega^1_{\operatorname{Spec} A/k}$ is nonzero as well. Next class, we will discuss the inseparable case. ⊠

Lecture 24.

## The local structure of smooth curves: 10/24/18

TODO: I may have missed stuff at the beginning. We're still proving the same theorem from last time.

**Lemma 24.1.** *Let $X \to \operatorname{Spec} k$ be a smooth scheme over a field $k$. If $k \hookrightarrow L$ is any field extension, then $X_L := X \times_{\operatorname{Spec} k} \operatorname{Spec} L$ is also smooth.*

*Proof sketch.* By Noether normalization, dimension is preserved under field extensions. If $\pi\colon X_L \to X$ is the map induced on the pullback, then

$$\Omega^1_{X_L/L} \cong \pi^*\Omega^1_{X/k} \cong \Omega^1_{X/k} \otimes_k L. \tag{24.2}$$ ⊠

Next we'll provide a useful criterion for separability.

**Lemma 24.3.** *Suppose $k \hookrightarrow k'$ is a finite, inseparable field extension, and let $\overline{k}$ be the algebraic closure of $k$. Then $B := k' \otimes_k \overline{k}$ isn't reduced.*

*Proof.* Since $B$ is an Artinian $\overline{k}$-algebra, it's reduced iff it's a finite product of fields, which would mean

$$B \cong \prod_{i=1}^{\dim_k k'} \overline{k}. \tag{24.4}$$

Assuming this, the map $k' \to B \cong \prod \overline{k}$ gives $(\dim_k k')$-many distinct embeddings of $k'$ into $k$, which means $k'$ is separable over $k$. ⊠

**Corollary 24.5.** *If $\operatorname{Spec} k'$ isn't smooth as a scheme over $k$, then $(\operatorname{Spec} k')_{\overline{k}}$ isn't smooth over $\overline{k}$.*

In this setting, $(\operatorname{Spec} k')_{\overline{k}}$ is a product of Artinian rings which are not all fields (but their residue fields are all $\overline{k}$).

Finally, now suppose $A$ is an Artinian local ring over $k$ with nonzero maximal ideal $\mathfrak{m}$. We want to show that $\Omega^1_{A/k} \neq 0$. We've showed it already if $A/\mathfrak{m}$ is separable over $k$; otherwise $\Omega^1_{A/k}$ surjects onto $\Omega^1_{(A/\mathfrak{m})/k}$, and we already know this is nonzero.

Now we'll study the local structure of smooth curves. The intuition is that, just like one-dimensional manifolds locally look like $\mathbb{R}$, a smooth curve over a field $k$ will locally look like $\mathbb{A}^1_k$.

**Definition 24.6.** A *curve* over a field $k$ is a one-dimensional, finite type scheme over $k$.

We may add more hypotheses later. For now, let's throw in the assumptions that $X = \operatorname{Spec} A$ is affine and smooth.

Here's a useful definition with somewhat weird notation, but everyone uses this notation, so it's worth getting used to.

**Definition 24.7.** Let $X = \operatorname{Spec} A$ be a smooth curve and $x$ be a closed point of $X$. Then the ideal sheaf of $x$ is denoted $\mathcal{O}_X(-x)$.

If $X = \operatorname{Spec} A$ is affine, $x = \operatorname{Spec} k'$ and the embedding determines a surjective map $A \twoheadrightarrow k'$. In this case, $\mathcal{O}_X(-x)$ is the module $\mathcal{M}_x := \ker(A \twoheadrightarrow k')$.

**Theorem 24.8.** *$\mathcal{O}_X(-x)$ is a line bundle.*

*Remark* 24.9. Assuming Theorem 24.8, we can define more line bundles $\mathcal{O}_X(nx) := \mathcal{O}_X(-x)^{\otimes -n}$, i.e. $(\mathcal{O}_X(-x)^\vee)^{\otimes n}$. ◀

In particular, we have lots of line bundles on curves!

If $X$ is a smooth curve, we know $\Omega^1_X$ is a line bundle. Locally we can trivialize it.

**Lemma 24.10.** *For every $x \in X$, there's an open $U \subset X$ containing $X$ and a map $s\colon U \to \mathbb{A}^1_k$ such that $\theta_U \cdot \mathrm{d}s \cong \Omega^1_U$.*

*Proof.* The general setup is to suppose $\mathcal{L}$ is a line bundle on $Y$, and that we have sections $s_1, \ldots, s_n \in \Gamma(Y, \mathcal{L})$ which define an epimorphism $\mathcal{O}_Y^{\oplus n} \twoheadrightarrow \mathcal{L}$. Then $\{s_i \neq 0\}_{i=1}^n$ is an open cover of $Y$.

In our setting, we know there's a neighorhood $U_0$ containing $x$ on whiche $\Omega^1_X$ is trivial, generated say by some $\omega = \sum f_i \, \mathrm{d}s_i$. As in the general case, there's an $i$ such that $\{\mathrm{d}s_i \neq 0\}$ is an open neighborhood of $x$. If $U = \{\mathrm{d}s_i \neq 0\}$, then $\mathrm{d}s_i$ generates $\Omega^1_U$. ⊠

*Remark* 24.11. If you want to apply this to higher-dimensional smooth schemes, you should replace $\Omega^1_X$ with $\Lambda^{\mathrm{top}}\Omega^1_X$. ◀

*Proof of Theorem 24.8.* The theorem is a local statement, so we can without loss of generality find an $s\colon X \to \mathbb{A}^1_k$ with $\mathrm{d}s$ generating $\Omega^1_X$.

If $k$ is algebraically closed, then $x$ is a $k$-point of $X$, so $s(x) \in \mathbb{A}^1_k = k$, which means it's a "number," in that nothing weird could happen. In this case, $t := s - s(x) \in \mathcal{O}_X(-x)$ is a uniformizer: it trivializes this line bundle in an open neighborhood of $x$.[15]

For more general $k$, $x \in X(k')$, where $k \hookrightarrow k'$ is a finite field extension. In this case we compose with the minimal polynomial... ⊠

Lecture 25.

## $\mathcal{O}_X(-x)$ is a line bundle: 10/26/18

Throughout today's lecture, $X$ is a smooth curve (this means its irreducible components all have dimension 1) over a field $k$. Let $x = \operatorname{Spec} k'$ be a closed point; we are in the middle of proving that $\mathcal{O}(-x)$, the ideal sheaf of $x$, is a line bundle. We finished the proof in the case when $k$ is algebraically closed (in which case $k' = k$).

In the general case, we constructed a $t \in \mathcal{O}(-x)$, after possibly replacing $X$ with an open $U \subset X$. The claim is that there exists a neighborhood $V \subset U$ of $x$ such that multiplication by $t$ is an isomorphism $\mathcal{O}_U|_V \xrightarrow{\cong} \mathcal{O}_U(-x)|_V$. This suffices, because $\mathcal{O}_X(-x)|_{X\setminus x} \cong \mathcal{O}_{X\setminus x}$.

Recall that we defined $t$ by first choosing an $s\colon U \to \mathbb{A}^1_k$ such that $\Omega^1_U = \mathcal{O}_X \cdot \mathrm{d}s$ (in words, $\mathrm{d}s$ trivializes $\Omega^1_U$); then $s(x) \in \mathbb{A}^1_k$ is a closed point, so here's an irreducible polynomial $g\colon \mathbb{A}^1_k \to \mathbb{A}^1_k$ with $g(s(x)) = 0$, and we defined $t := g \circ s$.

We first claim that $t$ is an epimorphism in a neighborhood of $x$. Consider the diagram

$$\begin{tikzcd} t^{-1}(0) \arrow[r] \arrow[d] & U \arrow[d, "s"] \\ s(x) \arrow[r] \arrow[d] & \mathbb{A}^1_k \arrow[d, "g"] \\ 0 \arrow[r] & \mathbb{A}^1_k. \end{tikzcd} \tag{25.1}$$

The top square is a pullback square (that's how we defined the preimage), and the bottom square is also a pullback square. Therefore the outer rectangle is also a pullback square (this is formal), which suffices because TODO.

Next, we'll show that $t^{-1}(0)$ is smooth over $\operatorname{Spec} k'$, again by abstract nonsense.

[15] This is the same thing as a uniformizer in a DVR, which (TODO) I have yet to puzzle out.

*Remark* 25.2. The definition of $\Omega^1_{S/k}$ makes sense if you replace $k$ by a more general ring, or even in much greater generality: for any map of schemes $f\colon S \to T$, we can define a sheaf of relative differentials $\Omega^1_{S/T}$. This has two important properties.

(1) Given a map $T \to \operatorname{Spec} k$, we obtain an exact sequence

$$f^*\Omega^1_{T/k} \xrightarrow{\mathrm{d}f} \Omega^1_{S/k} \longrightarrow \Omega^1_{S/T} \longrightarrow 0, \tag{25.3}$$

which encodes the fact, given a map of rings $\varphi\colon B \to A$, that a differential $\delta\colon A \to M$ is a $B$-linear iff $\delta(\operatorname{Im}(\varphi)) = 0$.

(2) Suppose we have a pullback square

$$\begin{tikzcd} S_2 \arrow[r, "g"] \arrow[d, "f"] & S_1 \arrow[d, "f_1"] \\ T_2 \arrow[r] & T_1. \end{tikzcd} \tag{25.4}$$

Then there's an isomorphism $g^*(\Omega^1_{S_1/T_1}) \cong \Omega^1_{S_2/T_2}$. One can prove this in the affine case, by looking at differentials for a pushout of rings. ◀

Applying (25.3) to our situation, we have

$$s^*\Omega^1_{\mathbb{A}^1/k} \xrightarrow{-1\mapsto \mathrm{d}s} \Omega^1_{U/k} \longrightarrow \Omega^1_{U/\mathbb{A}^1} \longrightarrow 0. \tag{25.5}$$

Since $\Omega^1_{U/\mathbb{A}^1} = 0$, then $s^*\Omega^1_{\mathbb{A}^1/k} \cong \Omega^1_{U/k}$. By base change (the second property), $\Omega^1_{t^{-1}(0)/k'} = 0$, so $t^{-1}(0)$ is smooth over $k'$.[16] Therefore $t^{-1}(0) \subseteq U$ is a smooth closed subscheme, so it's Spec of a product of fields which are separable over $k'$, or it's a disjoint union of finitely many distinct closed points. We can then let $V$ be $U$ minus those points, and we're done.

In algebra, $(t) = \mathfrak{m}_x$, so that multiplication by $t$ is a map $A \to \mathfrak{m}$; geometrically, this means $\mathcal{O}_V \to \mathcal{O}_V(-x)$ is an epimorphism.

We have just one step left: we need to show $t$ isn't a zero divisor, so that $\mathcal{O}_V \to \mathcal{O}_V(-x)$ is injective. If $V = \operatorname{Spec} A<$ then $t \in A$ and $(t) = \mathfrak{m}$.

**Lemma 25.6.** *$\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a one-dimensional $A/\mathfrak{m}$-vector space generated by $t^n$.*

*Proof.* $\mathfrak{m}^n$ is clearly generated by $t^n$, so it's at most one-dimensional. Consider the sequence

$$A/\mathfrak{m} \xrightarrow{t} \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{t} \cdots \tag{25.7}$$

The only potential failure is if $\mathfrak{m}^n = \mathfrak{m}^{n+1} = \cdots$, which is impossible by Nakayama's lemma. ⊠

The next step is that, because $A$ is Noetherian, the maps

$$A \xrightarrow{t} \mathfrak{m} \xrightarrow{t} \mathfrak{m}^2 \xrightarrow{t} \cdots \tag{25.8}$$

must stabilize: each is $A$ mod an ideal, and these ideals get bigger, hence must stabilize. Therefore for some $n \gg 0$, multiplication by $t$ is an isomorphism $\mathfrak{m}^n \to \mathfrak{m}^{n+1}$. Now consider the pair of short exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{m}^n \arrow[r] \arrow[d, "t"] & A \arrow[r] \arrow[d, "t"] & A/\mathfrak{m}^n \arrow[r] \arrow[d, "t"] & 0 \\ 0 \arrow[r] & \mathfrak{m}^{n+1} \arrow[r] & \mathfrak{m} \arrow[r] & \mathfrak{m}/\mathfrak{m}^{n+1} \arrow[r] & 0. \end{tikzcd} \tag{25.9}$$

The vertical maps are all surjections; for dimensional reasons, the rightmost vertical arrow is an isomorphism, and if $n \gg 0$, the leftmost vertical arrow is too; therefore the middle one is an isomorphism.

[16]TODO: I might have this argument slightly out of order, but this is what I think happened.

Lecture 26.

## Rational functions: 10/29/18

Last time, we showed that if $x \in X$ is a field-valued point, where $X$ is a smooth curve over a field $k$, then $\mathcal{O}_X(-x)$ is a vector bundle.

**Lemma 26.1.** *Let $x \in X$ be a closed point and $f\colon X \to \mathbb{A}^1$ be such that $f \in \bigcap_{n\geq 0} \mathcal{O}(-n\cdot x)$. Then there is an open $U \subset X$ containing $x$ such that $f|_U = 0$.*

*Proof.* Without loss of generality, we have a uniformizer $t$ for $\mathcal{O}(-x)$ in a neighborhood of $X$: $\mathcal{O}_X(-x) \cong \mathcal{O}_X \cdot t$. Therefore, since $f \in \mathcal{O}_X(-x)$, there's a unique $(f/t) \in \mathcal{O}_X$ with $t \cdot (f/t) = f$. Applying this for all $n$, we recover a unique $f/t^n$ for each $n$, hence a sequence of ideals

$$(f) \subseteq \left(\frac{f}{t}\right) \subseteq \left(\frac{f}{t^2}\right) \subseteq \dots \tag{26.2}$$

which must stabilize for some $n \gg 0$. Therefore $f/t^{n+1} = gf/t^n$, so $f(1 - tg) = 0$. Let $U \coloneqq \{1 - tg \neq 0\}$; then $x \in U$, and $U$ is the open neighborhood we wanted. ⊠

**Corollary 26.3.** *The following are equivalent:*

1. *$X$ has an open cover by irreducible schemes.*
2. *Every connected component of $f$ is irreducible.*

**Definition 26.4.** If either of the equivalent conditions in Corollary 26.3 holds, $X$ is called *locally irreducible*.

*Proof of Corollary 26.3.* Both of these are equivalent to the condition that, for all opens in $X$ and functions $f, g\colon X \to \mathbb{A}^1$ with $f \cdot g = 0$, there is an open cover $\mathfrak{U}$ of $X$ such that for all $U \in \mathfrak{U}$, $f|_U = 0$ or $g|_U = 0$.

If $x \in X$ is a closed point, it suffices to show there's a neighborhood $U$ of $x$ such that $f|_U = 0$ or $g|_U = 0$, and we can assume there's a uniformizer $t$ near $x$. By Lemma 26.1, either $f$ equals 0 in a neighborhood of $x$, or there's an $n$ such that $f \in \mathcal{O}_X(-nx)$ and $f \notin \mathcal{O}_X(-(n+1)x)$. That is, $f = t^n f_0$, and $f_0(x) \neq 0$. Our desired open is $U \coloneqq \{f_0 \neq 0\}$, which contains $x$, and $0 = fg = t^n f_0 g$; since $t^n$ isn't a zerodivisor, $f_0 g = 0$; since $f_0$ is a unit, then $g = 0$ on $U$. ⊠

We will now adopt the convention that curves are smooth and irreducible (hence also connected).

**Definition 26.5.** If $X$ is a curve over $k$, the *field of rational functions* on $X$, denoted $k(X)$, is the fraction field of $A$, where $U = \operatorname{Spec} A$ any nonempty affine open in $X$.

Think about why this is well-defined. On $\mathbb{A}^1_k$, this is $k(x)$, the field of rational functions in one variable, which is something you've seen before.

If $x \in X$ is a closed point, we can define a map $v_x\colon k(X)^\times \to \mathbb{Z}$ called *valuation at $x$*: specifically, for all $f \in k(X)^\times$, there's a unique $v_x(f) \in \mathbb{Z}$ such that $t^{-v_x(f)} \cdot f \in A^\times_{\mathfrak{m}_x}$, where $U = \operatorname{Spec} A$ is an affine open neighborhood of $x$ and $\mathfrak{m}_x$ is the maximal ideal corresponding to the closed point $x$. This $v_x(f)$ is thought of as the order of vanishing of $x$: $v_x(t) = 1$, and $v_x(t^n) = n$ for all $t \in \mathbb{Z}$ (a pole corresponds to a negative order of vanishing). We will sometimes use the convention that $v_x(0) = \infty$.

Here's why this is true. If $f \neq 0$, $f = g_1/g_2$ for $g_1, g_2 \in \operatorname{Fun}(X)$, and such that $g_1 \in \mathcal{O}_X(-nx)$, $g_1 \notin \mathcal{O}_X(-(n+1)x)$, $g_2 \in \mathcal{O}_X(-mx)$, and $g_2 \notin \mathcal{O}_X(-(m+1)x)$. Then $v_x(f) \coloneqq n - m$. This behaves well with respect to localization, hence extends well to non-affine schemes.

*Remark* 26.6. A few properties of the valuation: $v_x(fg) = v_x(f) + v_x(g)$, and $v_x(f+g) \geq \min\{v_x(f), v_x(g)\}$. ◂

**Lemma 26.7.** *If $f \in k(X)$, then $f \in \operatorname{Fun}(X) \subseteq k(X)$ iff $v_x(f) \geq 0$ for all closed points $x$ in $X$.*

The idea is that if $f$ has no poles, it's really a function.

*Proof.* Let $x \in X$ be an arbitrary closed point. It suffices to show that there's an open neighborhood $U$ of $x$ with $f \in \operatorname{Fun}(U)$. We may therefore assume $X = \operatorname{Spec} A$ is affine, so $f = g_1/g_2$ for $g_i \in \operatorname{Fun}(X)$, and $v_x(g_1) \geq v_x(g_2)$. We can assume $v_x(g_2) = 0$ by clearing out factors of the uniformizer $t$ near $x$; therefore $g_2(x) \neq 0$, so $x \in \{g_2 \neq 0\}$, and $f$ is defined as a function on this open subset. ⊠

We conclude with an important corollary.

**Corollary 26.8.** *If $X = \operatorname{Spec} A$ is affine, then $A$ is integrally closed in $k(X)$.*

(proof TODO)

Lecture 27.

## The valuative criterion for properness: 10/31/18

Today we'll say some more nice things about curves.

**Definition 27.1.** A quasicoherent sheaf $\mathcal{E}$ on a Noetherian scheme $X$ is *coherent* if it's locally finitely generated. Coherent sheaves form an abelian subcategory of $\mathcal{QC}oh(X)$ denoted $\mathcal{C}oh(X)$.

**Lemma 27.2.** *Let $X$ be a smooth curve and $\mathcal{E} \in \mathcal{C}oh(X)$. Then $\mathcal{E}$ is a vector bundle iff it's locally torsion-free, i.e. there is some cover $\mathfrak{U}$ of $X$ such that if $U \in \mathfrak{U}$, $s \in \Gamma(U, \mathcal{E})$, and $f \in \Gamma(U, \mathcal{O}_U)$ satisfy $f \cdot s = 0$, then $s = 0$ or $f = 0$.*

*Remark* 27.3. One can prove this in a more general context: using commutative algebra, a torsion-free module is flat, and Nakayama shows that a flat coherent sheaf is a vector bundle. But we'll do a shorter proof.

A general coherent sheaf might not feel like a sheaf of functions; sometimes your intuition is better spent thinking of it as a sheaf of measures, because of things such as delta "functions" in a sheaf. ◀

*Proof of Lemma 27.2.* Let $x \in X$ be a closed point. Then $x^*\mathcal{E}$ is a vector space, hence has a basis $\overline{s}_1, \dots, \overline{s}_n$. If $X$ is affine, which we can assume without loss of generality, these lift to $s_1, \dots, s_n \in \Gamma(X, \mathcal{E})$, and define a morphism

$$(s_1, \dots, s_n)^{\mathrm{T}} \colon \mathcal{O}_X \longrightarrow \mathcal{E} \tag{27.4}$$

whose cokernel is 0. Therefore by Nakayama's lemma, there's an affine open neighborhood $U$ of $x$ such that $s_1, \dots, s_n$ generate $\Gamma(U, \mathcal{E})$, and such that there's a uniformizer $t$ on $U$. We claim the restriction of (27.4) to $U$, $\mathcal{O}_U^{\oplus n} \to \mathcal{E}|_U$ is an isomorphism. Clearly it's an epimorphism, so we prove it's a monomorphism.

Suppose $f_1, \dots, f_n \in \Gamma(U, \mathcal{O}_U)$ are such that $\sum f_i s_i = 0$. Restricting to $x$,

$$\sum_{i=1}^n f_i(x)\overline{s}_i = 0, \tag{27.5}$$

but since the elements $\overline{s}_i$ are linearly independent, $f_i(x) = 0$ for all $i$, which is equialent to $f_i \in t\cdot\mathcal{O}_U = \mathcal{O}_U(-x)$. Therefore

$$\sum_{i=1}^n t\left(\frac{f_i}{t}\right) s_i = 0, \tag{27.6}$$

so by torsion-freeness, $\sum (f_i/t)s_i = 0$. Iterating, we can show that each $f_i$ vanishes to infinite order, but that can only happen when $f_i = 0$. ⊠

**Theorem 27.7** (Valuative criterion of properness)**.** *Let $U \subsetneq X$ be a nonempty open in an irreducible smooth curve $X$. Then any map $U \to \mathbb{P}^n$ extends uniquely to $X$.*

*Remark* 27.8. This is very false in higher dimensions: for example, consider the map $\mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1$ which sends a point to the line it's on. There's no way to extend this to $\mathbb{A}^2$. ◀

*Proof of Theorem 27.7.* By uniqueness, we can reduce to the case where $X = \operatorname{Spec} A$ and $U = \{f \neq 0\} \subset X$ for some $f \in A$. The map $U \to \mathbb{P}^n$ is equivalent to a line bundle $\mathcal{L}_U$ on $U$, which is equivalent to an $A[f^{-1}]$-module, which we also denote $\mathcal{L}_U$, together with $(n+1)$ sections $s_0, \dots, s_n$ everywhere nonzero, which correspond to generators over $A[f^{-1}]$.

Now let $\mathcal{L}_X \coloneqq A \cdot s_0 + \dots + A \cdot s_n \subseteq \mathcal{L}_U$, which defines a quasicoherent sheaf on $X$, and since it's a submodule of $\mathcal{L}_U$, it's torsion-free. Hence, by Lemma 27.2, it's a vector bundle, and since it's rank 1 on a dense subset, it's a line bundle. Since it comes with generators $s_0, \dots, s_n$, we get a map $X \to \mathbb{P}^n$, which clearly extends the map from $U$.

For uniqueness, suppose we have another extension, which is data of $\mathcal{L}_X$ and nonvanishing sections $\sigma_0, \dots, \sigma_n$. Then $\sigma_i \mapsto s_i$ deifnes an isomorphism $\mathcal{L}_X \cong j_*\mathcal{L}_U$. ⊠

*Remark* 27.9. The argument formally extends to yield the same conclusion with $\mathbb{P}^n$ replaced with any *projective $k$-scheme*, i.e. a $k$-scheme $Z$ with a closed embedding to $\mathbb{P}^n$ for some $n$: compose with the map to $\mathbb{P}^n$, apply the theorem, then restrict to $Z$. ◀

Next we'll discuss some variations on smoothness.

**Definition 27.10.** A curve $X$ is *regular* if $\mathcal{O}_X(-x)$ is a line bundle for every closed point $x$ of $X$.

**Definition 27.11.** An affine scheme $\operatorname{Spec} A$ is *normal* if $A$ is integrally closed in its fraction field.

The point of normality is that a finite map $X' \to X$ which is generically an isomorphism is an isomorphism if $X$ is normal. The counterexample to keep in mind is (TODOpicture).

Last class we showed that (for curves) smooth implies regular, and then that regular implies normal.

**Proposition 27.12.** *For $X$ a curve, normal implies regular, and if the ground field $k$ is perfect, then smooth, normal, and regular are equivalent.*

It's crucial that we're in dimension 1.

*Proof.* We assume $A$ is integrally closed in its fraction field $k(X)$, and want to prove regularity. Choose a maximal ideal $\mathfrak{m} \subset A$ (corresponding to some $\mathcal{O}_{\operatorname{Spec} A}(-x)$, with $x \in \operatorname{Spec} A$ a closed point), and choose some $f \in \mathfrak{m} \setminus \mathfrak{m}^2$. Then $A/f$ is a zero-dimensional ring, and without loss of generality we may assume it's local. Therefore $\mathfrak{m}^n = 0$ for some $n \gg 0$ in $A/f$, so $\mathfrak{m}^n \subset (f)$ in $A$. We'll choose $n$ minimal, and will show $n = 1$ (maybe after a further localization).

Hence, assume $n \geq 2$... TODO ⊠

Lecture 28.

## The normalization of a curve: 11/2/18

Last time, we stated that over a perfect base field $k$, a curve is smooth iff it's normal iff it's regular.

**Lemma 28.1.** *If $X$ is finite type over $k$, $k \hookrightarrow k'$ is a separable field extension, and $x \in X(k')$ is a closed point corresponding to the maximal ideal $\mathfrak{m}_x$, then the map $\mathfrak{m}_x/\mathfrak{m}_x^2 \to x^*\Omega_X^1$ is an isomorphism.*

This map arises in the following way: we have a map $\mathrm{d}\colon \mathcal{O}_X \to \Omega_X^1 \to x^*\Omega_X^1$, and this map factors through a map $\mathcal{O}_X/\mathfrak{m}_x^2 \to x^*\Omega_X^1$; then we precompose with the map $\mathfrak{m}_x/\mathfrak{m}_x^2 \to \mathcal{O}_X/\mathfrak{m}_x^2$.

*Remark* 28.2. $\mathfrak{m}/\mathfrak{m}^2$ is called the *Zariski cotangent space*, and is often one's first definition of the cotangent space. However, the definition $x^*\Omega_X^1$ has better base change properties. Of course, by the lemma, they're equivalent. ◀

*Proof of Lemma 28.1.* The first step is to assume $X = \operatorname{Spec} A$ is affine. Since $k'$ is separable, $A/\mathfrak{m}_x^2 = k' \oplus \mathfrak{m}_x \oplus \mathfrak{m}_x^2$ is a square-zero extension, giving rise to a map $\delta\colon A \to \mathfrak{m}_x/\mathfrak{m}_x^2$; it's easy to check that $\delta$ is a $k$-linear derivation. This gives rise to a map $\Omega_X^1 \to \mathfrak{m}_x/\mathfrak{m}_x^2$, hence a map $x^*\Omega_X^1 \to \mathfrak{m}_x/\mathfrak{m}_x^2$, and you can check this is the inverse map. ⊠

Why does the lemma imply our claim about normal/regular implying smooth? Well, if we have a regular curve $X$, a finite extension $k \hookrightarrow k'$, and $x \in X(k')$ closed, then $\mathfrak{m}_x/\mathfrak{m}_x^2$ is one-dimensional; since $k'$ is separable then $x^*\Omega_X^1$ is one-dimensional, so $X$ is smooth.

**Example 28.3.** Separability isn't a very restrictive hypothesis: it holds in characteristic zero, as well as for all finite fields and all algebraically closed fields. But suppose $k$ is not a perfect field and $\lambda \in k$ isn't a $p^{\text{th}}$ power. Then, consider the closed point $x = \{t^p = \lambda\} \subset \mathbb{A}_k^1$. If $k'$ is the residue field of $x$, then $k \hookrightarrow k'$ is an inseparable field extension.

The maximal ideal corresponding to $x$ is $\mathfrak{m}_x = (t^p - \lambda)$. The map $(t^p - \lambda)/(t^p - \lambda)^2 \to k' \cdot dt$ is $\mathrm{d} = 0$, so Lemma 28.1 doesn't hold in this setting. ◀

It is nonetheless useful to consider imperfect fields. "Experimental evidence" suggests that if you care about varieties, you'll probably only define them over perfect fields. But sometimes you'll want to consider families of varieties parameterized by some data, which can be thought of as a variety over some function field, and this function field need not be perfect.

Nevertheless, we now assume $k$ is perfect. We now give a construction of smooth projective curves. Start with a smooth affine curve $X \subseteq \mathbb{A}^n$. Choose one of the coordinate charts $\mathbb{A}^n \subseteq \mathbb{P}^n$; we can thus regard $X$ as a subscheme of $\mathbb{P}^n$ and take its closure $\overline{X}$. This is projective, but it might not be smooth.

Thus we will take the *normalization* of $\overline{X}$, which will yield a smooth projective variety.

*Remark* 28.4. Let $Y = \operatorname{Spec} A$, where $A$ is an integral domain of finite type over $k$. Let $k(Y)$ denote the fraction field of $A$. If $\overline{A}$ denotes the integral closure of $A$ in $k(Y)$, let $Y^{\mathrm{nm}} := \operatorname{Spec}(\overline{A})$, which is called the *normalization* of $Y$.

The normalization has a few nice properties (which we're not going to prove here): $Y^{\mathrm{nm}}$ is finite type over $k$, and hence the natural map $Y^{\mathrm{nm}} \to Y$ is finite, and an isomorphism on some open subscheme of $Y'$. Moreover, normalization localizes well on $Y$: if $U \subset Y$ is an affine open, then $U^{\mathrm{nm}} = Y^{\mathrm{nm}} \times_Y U$. Therefore normalization generalizes to irreducible $k$-schemes of finite type. ◀

So we can take the normalization $(\overline{X})^{\mathrm{nm}}$, which is finite over $\overline{X}$. It's easy to see that the composition of a finite morphism then a projective morphism is projective: more generally, using the Segre embedding, one can show that the composition of two projective morphisms is projective.

**Example 28.5.** Assume $\operatorname{char}(k) \neq 2$ and let $f(t) \in k[t]$ be a separable polynomial. Let $X = \{y^2 = f(t)\} \subseteq \mathbb{A}^2$. Then $\overline{X} \subset \mathbb{P}^2$ is not smooth if $\deg f \geq 4$. ◀

Since we care more about $(\overline{X})^{\mathrm{nm}}$ than about $\overline{X}$, we're going to make the normalization implicit.

**Corollary 28.6.** *If $X$ is a smooth affine curve over $k$, there's a smooth projective curve $\overline{X}$ and an embedding $X \hookrightarrow \overline{X}$; this data is unique up to unique isomorphism.*

We sometimes call $\overline{X}$ the *compactification* of $X$.

*Proof.* Suppose $\overline{X}_1$ and $\overline{X}_2$ are both compactifications of $X$. Because $\overline{X}_1$ is smooth, it admits a map to $\overline{X}_2$ (TODO: I think this is because we have a map $X \to \overline{X}_2$ and $X$ is a dense subscheme of $\overline{X}_1$), and in the same way we have a map $\overline{X}_2 \to \overline{X}_1$. The compositions of these maps are the identity on a dense subscheme, hence must be the identity. ⊠

*Remark* 28.7. Another way to think of $\overline{X}$ is the initial projective scheme recieving a map from $X$. This follows by the valuative criterion. ◀

One reasonable question is to what extent this generalizes.

**Definition 28.8.** A scheme $S$ is *separated* if the diagonal map $\Delta\colon S \to S \times S$ is a closed embedding.

This is the analogue of the Hausdorff condition in differential topology — and, just as in differential topology, the standard counterexample is the line with two origins.

**Example 28.9.** The line with two origins is the space $X$ whose functor of points assigns to $\operatorname{Spec} A$ the set of (isomorphism classes of) open covers $\{U, V\}$ of $\operatorname{Spec} A$ together with maps $f\colon U \to \mathbb{A}^1$ and $g\colon U \to \mathbb{A}^1$ such that $f$ and $g$ coincide when they're nonzero.

If you figure out what the closed point are, there's one for every closed point of $\mathbb{A}^1$, except there are two points corresponding to the origin. ◀

Affine schemes are separated, because the multiplication map $m\colon A \otimes A \to A$ is surjective.

**Example 28.10.** Show that $\mathbb{P}^n$ is separated. Or, more generally, any projective scheme (even any quasiprojective scheme[17]) is surjective. Hint: you'll want to go back to the definition of $\mathbb{P}^n$. ◀

Separability is a global, not local condition. As such, it can be more subtle than one expects.

**Lemma 28.11.** *Suppose $S$ is a separated scheme and $f, g\colon T \rightrightarrows S$. Then the equalizer $\{f = g\}$ is closed in $T$.*

*Proof.* $\{f = g\} = T \times_{S\times S} S$, and the inclusion from this to $T$ is the base change of the diagonal, and the pullback of a closed morphism is closed. ⊠

[17] A scheme is *quasiprojective* if it's an open subscheme of a projective scheme.

**Corollary 28.12.** *Suppose $S$ is an irreducible separated scheme with an open cover $\mathfrak{U}$, and $T$ is an affine scheme.*[18] *Suppose we have a map $S \to T$ such that for each $U \in \mathfrak{U}$, the induced map $U \to T$ is an open embedding. Then $S \to T$ is an open embedding.*

Again, the standard counterexample is the map from the affine line with two origins to $\mathbb{A}^1$.

Assuming Corollary 28.12, we can deduce a nice generalization of normalizations.

**Corollary 28.13.** *Let $X$ be a separable smooth curve over $k$. Then there's a smooth projective curve $\overline{X}$ together with an open embedding $X \hookrightarrow \overline{X}$, and this data is unique up to unique isomorphism.*

*Proof.* Cover $X$ by nonempty affines $U_i$; then we have embeddings $U_i \hookrightarrow \overline{U}_i$; if $U_i$ and $U_j$ intersect, then $\overline{U_i} = \overline{U_i \cap U_j} = \overline{U_j}$. Therefore $\overline{X} = \overline{U}_i$ for any $i$. All of the map $U_i \to \overline{X}$ coincide on the intersection, so $X \hookrightarrow \overline{X}$ is an open embedding. ⊠

Lecture 29.

## Maps between curves: 11/5/18

Let $k$ be a perfect field and $X$ be a smooth curve over $k$. Last time we saw that $X$ is separable iff it's quasiprojective (i.e. an open subscheme of a projective scheme). More specifically, we constructed a smooth projective curve $\overline{X}$ and an open embedding $X \hookrightarrow \overline{X}$.

Today, we'll discuss morphisms over curves.

**Proposition 29.1.** *Let $X$ and $Y$ be smooth, projective curves with function fields $k(X)$, resp. $k(Y)$. Then the natural map $\mathrm{Isom}_{\mathcal{S}ch/k}(X, Y) \to \mathrm{Isom}_{\mathcal{A}lg_k}(k(X), k(Y))$ is an isomorphism.*

An isomorphism $k(X) \to k(Y)$ is (equivalent data to) what's called a birational equivalence $X \dashrightarrow Y$; this says that all birational equivalences of curves come from actual isomorphisms. This fails drastically in higher dimensions, largely because the valuative criterion doesn't generalize. The associated field is known as the *minimal model program*. It also fails if you remove the hypotheses on smoothness or projectivity.

*Proof.* The inverse map is given as follows. Given some isomorphism $\iota\colon k(X) \to k(Y)$, there is a unique map $Y \to X$ which restricts to $\iota$ on the generic point. This can be proven by slightly modifying the proof of the valuative criterion for properness (which related to extending by open sets) to apply to the generic point. ⊠

**Proposition 29.2.** *Let $X$ be a separated, smooth curve. Then the map $U \mapsto \mathrm{Fun}(U)$ is an injective map from from nonempty open affine subschemes of $X$ to integrally closed, infinite-dimensional, finitely generated $k$-subalgebras $A \subseteq k(X)$ with field of fractions $k(X)$ is injective. If furthermore $X$ is projective, this map is a bijection.*

*Proof.* We'll prove the statement for projective curves. Let $A \subseteq k(X)$ and $U \coloneqq \operatorname{Spec} A$. We claim $U$ is a smooth curve over $k$: it's clearly finite type, and is irreducible, because $A$ is an integral domain. To see that $U$ is a curve, notice that its field of functions is $k(X)$ again. It's smooth because $k$ is perfect and $A$ is integrally closed.

Therefore we have a unique smooth projective curve $\overline{U}$ containing $U$ as an open subscheme. Then $k(\overline{U}) \cong k(X)$ canonically, so there's a canonical isomorphism $X \cong \overline{U}$, so we get an open embedding $U \hookrightarrow X$. ⊠

**Definition 29.3.** Let $X$ and $Y$ be finite-type schemes over $k$. A map $f\colon X \to Y$ is *constant* if its scheme-theoretic image is a closed point of $Y$.

That is, we want there to be a finite field extension $k \hookrightarrow k'$ and $f$ to factor through maps $X \to \operatorname{Spec} k'$ and $\operatorname{Spec} k' \to Y$.[19]

**Proposition 29.4.** *If $X$ and $Y$ are smooth projective curves, a map $f\colon X \to Y$ is either constant or dominant.*

*Proof.* The scheme-theoretic image of $f$ is a closed, irreducible subscheme of $Y$, hence either a closed point or all of $Y$. ⊠

[18] It might be possible to remove the affine hypothesis.
[19] In fact, the Nullstellensatz implies that $k \hookrightarrow k'$ is finite.

Furthermore, any nonconstant map of curves induces a map on function fields, which can be seen by checking affine-locally.

Now we can generalize Proposition 29.1.

**Proposition 29.5.** *Let $X$ and $Y$ be smooth projective curves. The map from the set of nonconstant functions $f\colon X \to Y$ to $\mathrm{Hom}_{\mathcal{A}lg_k}(k(Y), k(X))$ is an isomorphism.*

The proof is the same, using the valuative criterion.

**Theorem 29.6.** *Any nonconstant morphism of smooth projective irreducible curves is finite.*

In particular, this says that any nonconstant morphism of smooth projective irreducible curves is affine, which isn't obvious.

*Remark* 29.7. The projectivity assumption can't be removed–consider the map $\mathbb{A}^1 \setminus 0 \to \mathbb{A}^1$. Similarly, the separatedness is also essential–the map from $\mathbb{P}^1$ with the doubled origin to $\mathbb{P}^1$ shows this. ◀

Since constant maps are negligible, this says morphism of curves are basically field extensions.

*Proof of Theorem 29.6.* Given a nonconstant morphism $f\colon X \to Y$ of smooth projective curves, let $U = \operatorname{Spec} A$ be an affine open subset of $Y$. Then define $B$ to be the integral closure of $A$ in the fraction field $k(X)$. We claim that $\operatorname{Spec} B = f^{-1}(U)$ and that $\operatorname{Frac}(B) = k(X)$. Assuming these two claims, latter gives that $\operatorname{Spec} B$ is an affine open subset of $X$ and there's a general fact from commutative algebra which implies that $B$ is a finite $A$ module, which shows that $f$ is finite.

To see that $\operatorname{Spec} B = f^{-1}(U)$, note that for any affine open $i\colon \operatorname{Spec} C \subset X$ where $fi$ factors through $U$, we obtain a map of rings $A \to C$ to an integrally closed subring of $k(X)$. This implies there's a unique map of rings $B \to C$, which implies the claim.

To see that $\operatorname{Frac}(B) = k(X)$, first note that $k(X)/k(Y)$ is a finite extension, which is a fact about dominant morphisms between irreducible finite type $k$-schemes of the same dimension, which can be proven by generic finiteness or the "base times $\mathbb{A}^1$ or finite argument," noting that the former would increase dimension.

Now assume $f \in k(X)$. By above, we have an equation $f^n + a_1 f^{n-1} + \cdots + a_n = 0$ for $a_i \in k(Y)$. Since $\operatorname{Frac}(A) = k(Y)$, we can choose a nonzero $g \in A$ for which $ga_i \in A$ for all $i$. But this implies that $(gf)^n + ga_1(gf)^{n-1} + \cdots + g^n a_n = 0$, so $gf \in B$ (the integral closure of $A$ in $k(X)$), so $f \in \operatorname{Frac}(B)$. ⊠

Lecture 30.

## Complexes of abelian groups: 11/7/18

We have left over to prove the following lemma: TODO.

*Proof.* First, $k(X)/k(Y)$ is a finite extension. This is a general fact about dominant morphisms between irreducible finite type $k$-schemes of the same dimension; dimension theory guarantees the map is generically finite, for example.

TODO: missed what comes next.

Anyways, we get that

$$(gf)^n + ga_1(gf)^{n-1} + \cdots + g^n a_0 = 0, \tag{30.1}$$

with each $g^i a_i \in A$, so $gf$ is integral over $A$. Therefore $gf \in B$, so $f \in B[1/g] \subseteq \operatorname{Frac}(B)$. ⊠

*Remark* 30.2. It also follows that any nonconstant map $f\colon X \to Y$ of smooth projective curves is flat. This is because we can check locally, where it boils down to a question about $k$-algebras: if we've reduced to $\operatorname{Spec} A \subseteq Y$ with scheme-theoretic inverse image $\operatorname{Spec} B \subseteq X$ (guaranteed because we know $f$ is affine), then $B$ is an integral domain, so $B$ is a torsion-free $A$-module. This means it's projective, hence flat. ◀

Since $f$ is affine, $f_*\mathcal{O}_X$ is a vector bundle on $Y$.

**Corollary 30.3.** *Any smooth projective curve admits a finite flat map to $\mathbb{A}^1$.*

*Proof.* Let $U \subseteq X$ be an open affine and $f\colon U \to \mathbb{A}^1$ be a finite map (though we only really need it to be nonconstant). Therefore there exists a unique map $g\colon X \to \mathbb{P}^1$ by (I think?) Noether normalization which extends $U \to \mathbb{A}^1$, and since $f$ is nonconstant, $g$ is necessarily finite and flat. ⊠

So morphisms of smooth projective curves are really nice. One great example is the map $\mathbb{A}^1 \to \mathbb{A}^1$ induced from multiplication by $n$; of course $\mathbb{A}^1$ isn't projective, but you can turn this into a map $\mathbb{P}^1 \to \mathbb{P}^1$

Our next goal is the Riemann-Roch theorem; we will therefore need some sheaf cohomology. This is a lot of structure to take in, and we don't need all of it, so there will be a crash course in the homological algebra that we need.

**Definition 30.4.** A *complex of abelian groups* $\mathcal{F}$ (also $\mathcal{F}^\bullet$) is data of an abelian group $\mathcal{F}^i$ for each $i \in \mathbb{Z}$ together with maps $d^i \colon \mathcal{F}^i \to \mathcal{F}^{i+1}$ such that $d^{i+1}d^i = 0$ (usually written $d^2 = 0$).

For example, an abelian group $A$ defines a complex $\mathcal{F}$ "concentrated in degree zero" i.e. with $\mathcal{F}^0 = A$ and $\mathcal{F}^i = 0$ for all nonzero $i$.

We will adopt the principle that an "element" of $\mathcal{F}$ is an $x \in \mathcal{F}^0$ with $d(x) = 0$.

*Remark* 30.5. Though these look and feel like abelian groups so far, they behave very differently, in a manner reminiscent of category theory or homotopy theory. In the same way that it would be weird to ask whether two vector spaces are equal, but instead you'd want to exhibit an isomorphism between them, given elements $x$ and $y$ of $\mathcal{F}$ (i.e. in $\ker(d^0)$), a *homotopy* between them is an element $h \in \mathcal{F}^{-1}$ with $d(h) = x - y$. ◀

Now we define a few useful constructions.

**Definition 30.6.** Let $\mathcal{F}$ and $\mathcal{G}$ be chain complexes. Their *tensor product* $\mathcal{F} \otimes \mathcal{G}$ is the chain complex with

$$(\mathcal{F} \otimes \mathcal{G})^i \coloneqq \bigoplus_{j \in \mathbb{Z}} \mathcal{F}^j \otimes \mathcal{G}^{i-j}, \tag{30.7}$$

together with the differential

$$d^i(x \otimes y) = dx \otimes y + (-1)^{i-|x|} x \otimes dy. \tag{30.8}$$

Here we need to explain this definition: every element of $(\mathcal{F} \otimes \mathcal{G}^i)$ is a finite sum of homogeneous pure tensors $x \otimes y$, where $x \in \mathcal{F}^i$ and $y \in \mathcal{F}^j$; then we write $|x| = i$ and $|y| = j$ (the *degrees* of these elements). We define the differential on such elements and use linearity to extend it to all elements.

**Exercise 30.9.** Show that $d^2 = 0$, and that the sign factor is necessary: if $d'(x \otimes y) = dx \otimes y + x \otimes dy$, then $(d')^2 \neq 0$.

For example, regarding $\mathbb{Z}$ as a complex concentrated in degree zero, $\mathcal{F} \otimes \mathbb{Z} \cong \mathcal{F}$.

**Exercise 30.10.** Exhibit a canonical isomorphism $\mathcal{F} \otimes \mathcal{G} \cong \mathcal{G} \otimes \mathcal{F}$.

**Lemma 30.11.** *Let $\mathcal{F}$ and $\mathcal{G}$ be complexes. Then there is a complex $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$, unique up to unique isomorphism, such that for all complexes $\mathcal{H}$,*

$$\mathrm{Hom}_{\mathcal{C}px(\mathcal{A}b)}(\mathcal{H}, \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})) \cong \mathrm{Hom}_{\mathcal{C}px(\mathcal{A}b)}(\mathcal{F} \otimes \mathcal{H}, \mathcal{G}), \tag{30.12}$$

*and this isomorphism is functorial in $\mathcal{H}$.*

We haven't defined morphisms of complexes, but they are what you would think they are: maps $\mathcal{F}^i \to \mathcal{G}^i$ that commute with differentials.

We're not going to prove this lemma, but here's the explicit construction:

$$\underline{\mathrm{Hom}}^i(\mathcal{F}, \mathcal{G}) = \prod_{j \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{A}b}(\mathcal{F}^j, \mathcal{G}^{i+j}), \tag{30.13}$$

so maps which increase the degree by $j$ (though there's nothing about differentials yet).

**Exercise 30.14.** Given an $f \in \underline{\mathrm{Hom}}^0(\mathcal{F}, \mathcal{G})$, show that $f$ defines a map of complexes iff $df = 0$ in $\underline{\mathrm{Hom}}^1(\mathcal{F}, \mathcal{G})$.

The idea is to express $df = 0$ as $f$ commuting with the differential. You have to figure out what the differential is, but it's uniquely characterized by this property.

**Corollary 30.15.** *An element of $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{F})$ is a morphism of complexes.*

**Definition 30.16.** Therefore we define a *homotopy* between two maps $f, g \rightrightarrows \mathcal{F} \to \mathcal{G}$ is an element $h \in \underline{\mathrm{Hom}}^{-1}(\mathcal{F}, \mathcal{G})$ such that $dh = f - g$.

You can define a homotopy between homotopies by iterating this, producing an element in degree $-2$, and can interate this construction further and have a lot of fun.

We will think of two maps of complexes to be "the same" if there is a homotopy between them. But that doesn't mean the homotopy itself is unimportant.

Next we'd like to define something like kernels and cokernels, but more homotopically.

**Definition 30.17.** Let $f\colon \mathcal{F} \to \mathcal{G}$ be a map of complexes. Then there is a complex $\mathrm{hCoker}(f)$, the *homotopy cokernel* of $f$, such that for all complexes $\mathcal{H}$, $\mathrm{Hom}_{\mathcal{C}px(\mathcal{A}b)}(\mathrm{hCoker}(f), \mathcal{H})$ is equal to the set of pairs $(g, H)$ where $g\colon \mathcal{G} \to \mathcal{H}$ is a map of complexes and $H$ is a nullhomotopy of $gf$. Then $\mathrm{hCoker}(f)$ is unique up to unique isomorphism.

The idea is that we don't want to take things which are equal to zero, but instead things which are homotopic to zero, and encode this homotopy as part of the data.

Again, we will provide the construction instead of the proof: $\mathrm{hCoker}(f)^i = \mathcal{G}^i \oplus \mathcal{F}^{i+1}$, and the differential is

$$\begin{pmatrix} d_{\mathcal{G}} & f \\ 0 & -d_{\mathcal{F}} \end{pmatrix}, \tag{30.18}$$

and using this data, you can check what data of a map to $\mathcal{H}$ is, and why the extra data gives a nullhomotopy of the composition

Lecture 31.

## More homological algebra: 11/9/18

Today we're doing more homological algebra. Some references can be found at `https://web.ma.utexas.edu/users/sraskin/cft/index.html`, specifically psets 5 and 6.

Last time, we defined the homotopy cokernel $\mathrm{hCoker}(f)$ of a map $f\colon \mathcal{F} \to \mathcal{G}$ of chain complexes of abelian groups, which is universal for the data of a structure map $\mathcal{G} \to \mathrm{hCoker}(f)$ and of a nullhomotopy of the composition $\mathcal{F} \to \mathcal{G} \to \mathrm{hCoker}(f)$. Explicitly, its $i^{\text{th}}$ term is $\mathcal{G}^i \oplus \mathcal{F}^{i+1}$, and the differential sends $x \in \mathcal{G}^i$ to $(x, 0) \in \mathrm{hCoker}^i(f)$, and $y \in \mathcal{F}^i$ to $(f(x), 0)$.

Dually, we can define the *homotopy kernel* $\mathrm{hKer}(f)$ by a similar universal property: $\mathrm{hKer}(f)$ has a map $\mathrm{hKer}(f) \to \mathcal{F}$ and a nullhomotopy of the composition $\mathrm{hKer}(f) \to \mathcal{F} \to \mathcal{G}$, and is universal for this data. Explicitly, $\mathrm{hKer}(f)^i = \mathcal{G}^{i-1} \oplus \mathcal{F}^i$, with some differential similar to that for $\mathrm{hCoker}(f)$.

**Definition 31.1.** Let $n \in \mathbb{Z}$. The *shift* of a complex $\mathcal{F}$ by $n$, denoted $\mathcal{F}[n]$, is the complex whose $i^{\text{th}}$ abelian group is $\mathcal{F}^{n+i}$, and whose maps are what you would expect.

In this language, we have the following miracle:

$$\mathrm{hKer}(f)[1] \cong \mathrm{hCoker}(f). \tag{31.2}$$

**Example 31.3.** Let $f\colon A \to B$ be a map of abelian groups, regarded as a map of chain complexes in degree zero. Then its homotopy cokernel is

$$\cdots {}_{-2}0 \longrightarrow {}_{-1}A \xrightarrow{f} {}_{0}B \longrightarrow {}_{1}0 \longrightarrow \cdots \tag{31.4}$$

and the homotopy kernel is

$$\cdots {}_{-1}0 \longrightarrow {}_{0}A \xrightarrow{f} {}_{1}B \longrightarrow {}_{1}2 \longrightarrow \cdots \tag{31.5}$$

◀

One nice consequence of the miracle is that it makes commutative algebra nicer: there are lots of functors on abelian groups (or modules) which commute with kernels or cokernels, but not both. In this derived setting, there's not much of a difference between (homotopy i.e. better versions of) kernels and cokernels, so more things commute with each other up to a shift, which is nice.

Now, given a map $f\colon \mathcal{F} \to \mathcal{G}$ of complexes, let $\pi\colon \mathcal{G} \to \mathrm{hCoker}(f)$ be the induced map. We also have a nullhomotopy of the map $\pi \circ f\colon \mathcal{F} \to \mathrm{hCoker}(f)$, so by the universal property obtain a map $\mathcal{F} \to \mathrm{hKer}(\pi)$.

**Exercise 31.6.** Show that this map $\mathcal{F} \to \mathrm{hKer}(\pi)$ is a *homotopy equivalence*, i.e. there is a map $\mathrm{hKer}(\pi) \to \mathcal{F}$ such that the compositions in both directions are homotopic to the identity.

This is also a very important fact.

Recall that an element of a complex $\mathcal{F}$ is an element $x \in \mathcal{F}^0$ with $dx = 0$.

**Definition 31.7.** The *zeroth homology group* of $\mathcal{F}$, denoted $H^0(\mathcal{F})$, is the abelian group of elements of $\mathcal{F}$ modulo homotopy, i.e. $\ker(d^0)/\operatorname{Im}(d^{-1})$.

**Lemma 31.8.** *If $f\colon \mathcal{F} \to \mathcal{G}$ is a map of complexes, then the sequence*

$$H^0(\mathrm{hKer}(f)) \longrightarrow H^0(\mathcal{F}) \longrightarrow H^0(\mathcal{G}) \tag{31.9}$$

*is exact.*

That is, the image of the first map is exactly the kernel of the second. Figuring out the details of this exercise is a good way to become more familiar with cohomology.

**Corollary 31.10.** *If $f\colon \mathcal{F} \to \mathcal{G}$ is a map of complexes, the following sequence is long exact:*

(31.11)

$$\cdots \longrightarrow H^{-1}(\mathrm{hCoker}(f)) \longrightarrow H^0(\mathcal{F}) \longrightarrow H^0(\mathcal{G}) \longrightarrow H^0(\mathrm{hCoker}(f)) \longrightarrow H^1(\mathcal{F}) \longrightarrow H^1(\mathcal{G}) \longrightarrow \cdots$$

*Here $H^i(\mathcal{F}) \coloneqq H^0(\mathcal{F}[i])$ for any $i \in \mathbb{Z}$.*

*Proof sketch.* We can produce this sequence by gluing together sequences from Lemma 31.8: the first few terms come from the observation that $H^{-1}(\mathrm{hCoker}(f)) = H^0(\mathrm{hKer}(f))$; then we invoke Lemma 31.8. The next three terms come from the fact that $\mathcal{F}$ is canonically homotopic to the homotopy kernel of the map $\mathcal{G} \to \mathrm{hCoker}(f)$, then applying the lemma again. Then we apply that again and again to shifts of $\mathcal{F}$ and $\mathcal{G}$. ⊠

In addition to homotopy equivalence of complexes, there's another notion of equivalence.

**Definition 31.12.** A *quasi-isomorphism* (sometimes abbreviated qis) of chain complexes is a map $f\colon \mathcal{F} \to \mathcal{G}$ such that the induced map $f_*\colon H^i(\mathcal{F}) \to H^i(\mathcal{G})$ is an isomorphism for all $i \in \mathbb{Z}$.

*Remark* 31.13. This is equivalent to asking for $\mathrm{hCoker}(f)$ to be *acyclic*, i.e. to have trivial cohomology groups. This follows from Corollary 31.10. ◀

Importantly, a quasi-isomorphism need not have an inverse map, unlike a homotopy equivalence. For example, consider the map of complexes

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \arrow[r] & \cdots . \end{tikzcd} \tag{31.14}$$

You can check this is a quasi-isomorphism, but clearly there can be no map in the other direction. This is weird, especially because there are many contexts in which quasi-isomorphism is the right notion of equivalence. We have a weaker statement, which is that if $f\colon \mathcal{F} \to \mathcal{G}$ is a quasi-isomorphism, there must exist a suitable inverse map $\mathcal{H} \to \mathcal{F}$ and a quasi-isomorphism $\mathcal{H} \to \mathcal{G}$.

**Sheaf cohomology.** We now have all of the homological background needed to define sheaf cohomology. Specifically, given a complex $\mathcal{F}$ of quasicoherent sheaves on a quasicompact separated scheme $X$, we will define a complex of abelian groups $R\Gamma(X, \mathcal{F})$, well-defined up to canonical quasi-isomorphisms. Here the notation $R$ means this will have cohomology only in nonnegative degrees ("to the right" of zero).

This complex isn't completely well-defined — we will need to make choices, and then argue that it doesn't really depend on those choices in a precise sense. So let $\mathfrak{U}$ be a finite affine open cover of $X$, which exists because $X$ is quasicompact; because $X$ is separated, $U \cap V$ is affine for all $U, V \in \mathfrak{U}$, and this extends to higher-fold intersections.

Now we inductively define the complex $R\Gamma((X, \mathfrak{U}), \mathcal{F})$, inducting on $n \coloneqq |\mathfrak{U}|$. When $n = 1$, $X = \operatorname{Spec} A$ is affine, so we define $R\Gamma(X, \mathcal{F}) \coloneqq \Gamma(X, \mathcal{F})$, using the fact that $\mathcal{F}$ is in particular a complex of $A$-modules, hence abelian groups.

When $n = 2$ and $\mathfrak{U} = \{U, V\}$, we define

$$R\Gamma((X, \mathfrak{U}), \mathcal{F}) \coloneqq R\Gamma(U, \mathcal{F}|_U) \times^h_{R\Gamma(U \cap V, \mathcal{F}|_{U \cap V})} R\Gamma(V, \mathcal{F}|_V). \tag{31.15}$$

Here, the notation $\times^h$ denotes the *homotopy fiber product* of complexes: given maps $f\colon \mathcal{F}_1 \to \mathcal{F}_3$ and $g\colon \mathcal{F}_2 \to \mathcal{F}_3$,

$$\mathcal{F}_1 \times^h_{\mathcal{F}_3} \mathcal{F}_2 \coloneqq \mathrm{hKer}(f - g\colon \mathcal{F}_1 \oplus \mathcal{F}_2 \to \mathcal{F}_3). \tag{31.16}$$

This is universal with respect to data of a homotopy between the two maps from this homotopy fiber product to $\mathcal{F}_3$ going through $\mathcal{F}_1$ and $\mathcal{F}_2$.

Lecture 32.

## Sheaf cohomology: 11/12/18

Recall that we were defining sheaf cohomoloy $R\Gamma(\mathfrak{U}, \mathcal{F})$ inductively, where $\mathcal{F}$ is a finite affine open cover of a quasicompact separated scheme $X$ and $\mathcal{F} \in \mathfrak{QCoh}(X)$. For $n = 1$, $\mathcal{F}$ is a module over $A$ if $X = \operatorname{Spec} A$, and we let $R\Gamma = \Gamma$ in degree 0 and 0 elsewhere. The general definition is TODO.

**Lemma 32.1.** *If $X$ is affine and $\mathfrak{U}$ is a finite open cover of $X$ by affines with $U_1 \in \mathfrak{U}$ equal to $X$, then the natural map*

$$\Gamma(X, \mathcal{F}) = R\Gamma(X, \mathcal{F}) \longrightarrow R\Gamma(\mathfrak{U}, \mathcal{F}) \tag{32.2}$$

*is a homotopy equivalence.*

*Proof.* We'll induct on $n$; for $n = 1$ this is vacuous. If $n > 1$, let $\mathfrak{U}' = \mathfrak{U} \setminus U_1$; then

$$R\Gamma(\mathfrak{U}, \mathcal{F}) \coloneqq R\Gamma(\mathfrak{U}', \mathcal{F}) \times^h_{R\Gamma(\{V \cap U_1 \mid V \in \mathfrak{U}'\}, \mathcal{F})} R\Gamma(U_1, \mathcal{F}), \tag{32.3}$$

which is just

$$R\Gamma(X, \mathcal{F}) \times^h_{R\Gamma(U_1, \mathcal{F})} R\Gamma(U_1, \mathcal{F}), \tag{32.4}$$

which is homotopy equivalent to $R\Gamma(X, \mathcal{F})$. ⊠

Now let's remove the hypothesis that $X \in \mathfrak{U}$.

**Proposition 32.5.** *Let $X$ be an affine scheme and $\mathfrak{U}$ a finite cover by affines. Then $\varepsilon\colon R\Gamma(X, \mathcal{F}) \to R\Gamma(\mathfrak{U}, \mathcal{F})$ is a quasi-isomorphism.*

*Proof.* Let $X = \operatorname{Spec} A$, so that both sides are complexes of $A$-modules. For each $U \in \mathfrak{U}$, write $U = \operatorname{Spec} B_U$; we therefore have maps $A \to B_U$ for each $U$. We claim that $\varepsilon$ is a quasi-isomorphism iff it is one after tensoring with $B_U$ for all $U \in \mathfrak{U}$. Certainly, since each $B_U$ is flat over $A$, tensor product with $B_U$ preserves quasi-isomorphisms, and in fact, from Serre's theorem,

$$H^j(X, \mathcal{F}) \otimes_A B_i \cong H^j(\mathcal{F} \otimes_A B_i). \tag{32.6}$$

Moreover a complex of $A$-modules is acyclic iff it's acyclic after tensor product with $B_i$ for all $i$. So it suffices to prove the proposition on each $U \in \mathfrak{U}$, but there it reduces to Lemma 32.1. ⊠

Acyclicity is closely related to quasi-isomorphisms: $\varepsilon$ is a quasi-isomorphism iff its homotopy cokernel is acyclic.

**Corollary 32.7.** *Let $\mathfrak{U}$ and $\mathfrak{V}$ be finite open covers by affines of a quasicompact, separated scheme $X$. Then there is a canonical quasi-isomorphism $R\Gamma(\mathfrak{U}, \mathcal{F}) \to R\Gamma(\mathfrak{V}, \mathcal{F})$.*

Therefore as long as we only care about its definition up to quasi-isomorphism, we'll let $R\Gamma(X, \mathcal{F}) \coloneqq R\Gamma(\mathfrak{U}, \mathcal{F})$ for any finite affine open cover $\mathfrak{U}$ of $X$. Then we will let $H^i(X, \mathcal{F}) \coloneqq H^i(R\Gamma(X, \mathcal{F}))$.

*Proof.* Given a $U \in \mathfrak{U}$ and $V \in \mathfrak{V}$, let $W_{UV} \coloneqq U \cap V$, and let $\mathfrak{W} \coloneqq \{W_{UV}\}$. We therefore have two open covers $R\Gamma(\mathfrak{U}, \mathcal{F}) \to R\Gamma(\mathfrak{W}, \mathcal{F})$ and $R\Gamma(\mathfrak{V}, \mathcal{F}) \to R\Gamma(\mathfrak{W}, \mathcal{F})$. Both of these are quasi-isomorphisms, which follows from Proposition 32.5 and the fact that homotopy fiber products preserve quasi-isomorphisms. ⊠

It's a good exercise to work out the details of the proof that a refinement induces a quasi-isomorphism on cohomology in the case where $\mathfrak{U}$ and $\mathfrak{V}$ each have two opens.

*Remark* 32.8. There are definitions of cohomology for a scheme which isn't separated or even quasicompact, but they're less nice, ultimately requiring some sort of infinite construction. Thanks to our hypotheses, we only need to take a finite number of cones, allowing for these nice inductive proofs. ◂

**Example 32.9.** Let $\mathcal{F}$ be an abelian group regarded as a complex in degree zero. Then $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$. ◄

**Theorem 32.10.** *Let $X$ be a smooth, separated curve and $\mathcal{F} \in \mathcal{QC}oh(X)$ be a complex concentrated in degree zero. Then*

(1) $H^i(X, \mathcal{F}) = 0$ *for* $i \neq 0, 1$.
(2) *If $\mathcal{F}$ is coherent and $X$ is projective, then $H^i(X, \mathcal{F})$ is a finite-dimensional vector space over $k$.*

Projectivity is really important for the second point! For example,

$$R\Gamma(\mathbb{A}^1, \mathcal{O}_X) = k[t]. \tag{32.11}$$

*Proof of (1).* We'll first show that any smooth curve $X$ has a cover by two open affines. Choose a (the) smooth compactification $\overline{X}$ and let $\overline{f}\colon \overline{X} \to \mathbb{P}^1$ be a nonconstant map; then let $f := \overline{f}|_X$. Then $\overline{f}$ is finite, hence affine.

The embedding $j$ is also affine: there are closed points $x_1, \dots, x_n$ such that $X = \overline{X} \setminus \{x_1, \dots, x_n\}$, and therefore

$$\mathcal{O}_{\overline{X}}(x_1 + \dots + x_n) := \bigotimes_{i=1}^n \mathcal{O}_{\overline{X}}(x_i) \tag{32.12}$$

has a section, namely 1, and $X$ is precisely the nonvanishing locus of this section.

Now, $\mathbb{P}^1$ has an open cover by two affines, namely $\mathbb{A}^1 = \mathbb{P}^1 \setminus \infty$ and $\mathbb{A}^1 = \mathbb{P}^1 \setminus 0$; pull these back via $f$, which is affine, to recover an affine open cover of $X$. ⊠

The second part is more complicated, so we'll start with an example (and probably give the general proof next time).

**Proposition 32.13.** *We claim $H^0(\mathbb{P}^1; \mathcal{O}_{\mathbb{P}^1}) = k$ and $H^1(\mathbb{P}^1; \mathcal{O}_{\mathbb{P}^1}) = 0$. That is, the map $k \to R\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$ is a quasi-isomorphism.*

*Proof.* Let $\{U, V\}$ be our favorite affine open cover of $\mathbb{P}^1$. Then we can just calculate

$$\begin{aligned} R\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) &= \mathrm{hKer}(\Gamma(U, \mathcal{O}_U) \oplus \Gamma(V, \mathcal{O}_V) \longrightarrow \Gamma(U \cap V, \mathcal{O}_{U \cap V})) \\ &= \mathrm{hKer}\big((f, g) \longmapsto f - g\colon k[t] \oplus k[t^{-1}] \to k[t, t^{-1}]\big). \end{aligned} \tag{32.14}$$

So in degree 0 we get the kernel, given by the constant functions in both factors, and in degree 1 we get the cokernel. This map is surjective, though, since any Laurent polynomial is a sum of a polynomial in $t$ and a polynomial in $t^{-1}$. ⊠

Lecture 33.

## Divisors on curves: 11/14/18

TODO: about 10 to 12 minutes of divisor theory on curves. Everything today is explicitly over curves.

*Remark* 33.1. $\mathcal{O}_X(D)$ is trivialized away from the *support* of $D$, i.e. the union of the $x_i$ with nonzero multiplicity in $D$, because $1 \in \Gamma(X \setminus \mathrm{supp}(D), \mathcal{O}_X(D))$. ◄

**Proposition 33.2.** *This gives a bijection between the set of divisors on $X$ and the set of isomorpism classes of line bundles $\mathcal{L}$ on $X$ together with nonzero section $s$ of $\mathcal{L}|_{\eta_X}$, sending $D \mapsto (\mathcal{O}_X(D), 1)$.*

*Proof.* We'll construct the inverse: given a line bundle $\mathcal{L}$ on $X$ with a nonzero $s \in \Gamma(\eta_X, \mathcal{L})$, we will define a divisor $D = \sum n_i x_i$, where $n_i$ is the order of the pole of $s$ at $x_i$.

If $\mathcal{L}$ is trivial, we can choose a trivialization $s$, which is the same as a nonzero rational function $f$ on $X$. Let $x_1, \dots, x_n$ be the zeros of $f$, and define

$$D := \sum_{i=1}^n v_{x_i}(f) x_i. \tag{33.3}$$

This is independent of the choice of trivialization, because any two trivializations differ by an invertible function on $X$, which doesn't change valuations (the invertible function has no zeros and no poles).

Since divisors are compatible with restriction, we can generalize this construction to nontrivial line bundles. Now you should check this is actually an inverse to the map defined in the proposition statement. ⊠

**Corollary 33.4.** *Any line bundle is of the form $\mathcal{O}_X(D)$ for some divisor $D$.*

**Definition 33.5.** A divisor $D$ is *principal* if $D$ is the divisor of some nonzero rational function $f$, i.e. $D = \sum v_{x_i}(f)x_i$.

Equivalently, $\mathcal{O}_X(D)$ is trivial.

**Corollary 33.6.** *The set of isomorphism classes of line bundles on $X$ is in bijection with the set of divisors on $X$ modulo principal divisors, is isomorphic to the cokernel of the map $k(X)^\times \to \mathrm{Div}(X)$.*

Remember, our broader-scope goal is to show that sheaf cohomology for smooth projective curves is finite-dimensional.

**Proposition 33.7.** *Suppose $(X, \mathcal{O}_X)$ satisfies (*).*[20] *Then $(X, \mathcal{L})$ does for all line bundles $\mathcal{L}$ on $X$.*

*Proof.* By the divisor theory above, it suffices to prove this for $(X, \mathcal{L}(x))$, where $\mathcal{L}(x) := \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X(x)$. This is because if $D = 2x - y$ for distinct points $x, y \in X$, we'll show that the property for $(X, \mathcal{O}_X)$ implies it for $(X, \mathcal{O}_X(x))$, then $(X, \mathcal{O}_X(2x))$, then $(X, \mathcal{O}_X(D))$. This is because we have a short exact sequence

$$0 \longrightarrow \mathcal{L}(-x) \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}/\mathcal{L}(-x) \longrightarrow 0, \tag{33.8}$$

where $\mathcal{L}(-x) := \mathfrak{m}_x \otimes \mathcal{L}$, and $\mathcal{L}/\mathcal{L}(-x) \cong i_{x*}i_x^I\mathcal{L} = i_{x*}\mathcal{O}_x$.

A general fact about sheaf cohomology is that it preserves homotopy kernels and cokernels. In fact, it's difficult to write down functors that don't. Moreover, sheaf cohomology preserves quasi-isomorphisms, which isn't super surprising, and follows from the fact that it preserves acyclicity. Moreover, given a short exact sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{f} \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{F} \longrightarrow 0, \tag{33.9}$$

the natural map $\mathrm{hCoker}(f) \to \mathcal{G}/\mathcal{F}$ is a quasi-isomorphism (which you can check with the explicit formula). The upshot of all this is that we obtain a long exact sequence in cohomology:

$$0 \longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{G}) \longrightarrow H^0(X, \mathcal{G}/\mathcal{F}) \longrightarrow H^1(X, \mathcal{F}) \longrightarrow H^1(X, \mathcal{G}) H^1(X, \mathcal{G}/\mathcal{F}) \longrightarrow \cdots \tag{33.10}$$

Now we apply this to Equation (33.8):

$$0 \to H^0(X, \mathcal{L}(-x)) \to H^0(X, \mathcal{L}) \to H^0(X, \mathcal{L}/\mathcal{L}(-x)) \to H^1(X, \mathcal{L}(-x)) \to H^1(X, \mathcal{L}) \to H^1(X, \mathcal{L}/\mathcal{L}(-x)) \to 0 \tag{33.11}$$

Let $k' := \mathcal{O}_X/\mathfrak{m}_x$, which is a finite field extension of $k$; then $H^0(X, \mathcal{L}/\mathcal{L}(-x)) \cong k'$. Moreover, $H^1(X, \mathcal{L}/\mathcal{L}(-x)) = 0$, because of another general fact about sheaf cohomology: if $f\colon X \to Y$ is affine, $R\Gamma(X, \mathcal{F})$ is quasi-isomorphic to $R\Gamma(Y, f_*\mathcal{F})$. One can prove this directly by pulling back an affine cover. ⊠

Lecture 34.

## Finite-dimensionality of cohomology: 11/16/16

Let $X$ be a smooth projective curve over $k$. We continue with the proof of the finite-dimensionality of cohomology of $X$ valued in a coherent sheaf. So far, we've showed that it holds for $\mathbb{P}^1$ and $\mathcal{O}_{\mathbb{P}^1}$, and that $(X, \mathcal{L})$ satisfies it iff $(X, \mathcal{O}_X)$ does (here $\mathcal{L}$ is a line bundle).

**Proposition 34.1.** *If $(X, \mathcal{L})$ satisfies finite-dimensionality of cohomology for all line bundles $\mathcal{L}$, then so does $(X, \mathcal{E})$ for all vector bundles $\mathcal{E}$.*

Before we can prove this, we need a lemma.

**Lemma 34.2.** *Let $X$ be a smooth curve, $\mathcal{E} \to X$ be a vector bundle, and $U \subset X$ be a nonempty open. Suppose $\mathcal{L}_U \to \mathcal{E}|_U$ is a morphism of quasicoherent sheaves that is everywhere nonzero, where $\mathcal{L}_U$ is a line bundle on $U$. Then there's a unique extension of $\mathcal{L}_U$ to $X$ together with a nonvanishing map to $\mathcal{E}$ (where uniqueness is up to isomorphism of this data).*

[20] TODO: what's (*)?

*Proof.* Let $\mathbb{P}(\mathcal{E})$ be the projectivization of $\mathcal{E}$; the natural map $\mathbb{P}(\mathcal{E}) \to X$ is projective. The map $\mathcal{L}_U \to \mathcal{E}|_U$ defines a map $U \to \mathbb{P}(\mathcal{E})$; by the valuative criterion, this extends uniquely to $X$. ⊠

Of course, since we're using the valuative criterion, this doesn't generalize to higher-dimensional schemes!

*Proof of Proposition 34.1.* We induct on the rank $r$ of $\mathcal{E}$; for $r = 1$, $\mathcal{E}$ is a line bundle, and we've already done this case.

In general, choose a $U \subseteq X$ and an isomorphism $\mathcal{E}|_U \cong \mathcal{O}_U^{\oplus r}$. The first coordinate defines a map $\mathcal{O}_U \to \mathcal{E}|_U$, which by Lemma 34.2 extends uniquely to an everywhere nonzero embedding $\mathcal{L} \to \mathcal{E}$ on $X$. Therefore $\mathcal{E}/\mathcal{L}$ is a vector bundle, and we have a short exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}/\mathcal{L} \longrightarrow 0. \tag{34.3}$$

Since $\mathcal{L}$ is rank 1 and $\mathcal{E}/\mathcal{L}$ is rank $r-1$, both have finite-dimensional cohomology, by the inductive assumption. Therefore $\mathcal{E}$ must too, using the long exact sequence in cohomology associated to (34.3). ⊠

And here's the big fish.

**Corollary 34.4.** *If $(X, \mathcal{E})$ has finite-dimensional cohomology for every vector bundle $\mathcal{E} \to X$, then $(X, \mathcal{F})$ does for all coherent sheaves $\mathcal{F}$ on $X$.*

*Proof.* Let $U = \operatorname{Spec} A$ be an affine open in $X$; then $\mathcal{F}|_U$ is coherent, meaning it is a finitely generated $A$-module. Let $\mathcal{F}_\tau \subseteq \mathcal{F}$ be the maximal torsion module; explicitly,

$$\mathcal{F}_\tau = \{s \in \mathcal{F} \mid \text{There exists } f \in A \setminus 0 \text{ such that } fs = 0.\} \tag{34.5}$$

Then $\mathcal{F}/\mathcal{F}_\tau$ is a vector bundle. We know that on curves, vector bundles are equivalent to torsion-free coherent sheaves. If $s \in \mathcal{F}/\mathcal{F}_\tau$ is a torsion element, there's an $f \in A \setminus 0$ with $fs = 0$, so lift $s$ to some $\widetilde{s} \in \mathcal{F}$; then $fs \in \mathcal{F}_\tau$, so there's a $g \in A \setminus 0$ with $fg\widetilde{s} = 0$. Then $fg \neq 0$, so $\widetilde{s}$ is torsion. This construction globalizes, hence makes sense for non-affines.

Therefore we have a short exact sequence

$$0 \longrightarrow \mathcal{F}_\tau \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\mathcal{F}_\tau \longrightarrow 0. \tag{34.6}$$

We know $\mathcal{F}/\mathcal{F}_\tau$ is a vector bundle, hence has finite-dimensional cohomology, so it suffices to show that $\mathcal{F}_\tau$, which is a torsion coherent sheaf, has finite-dimensional cohomology. It's an exercise to show that a torsion coherent sheaf is isomorphic to a finite direct sum of skyscraper sheaves, namely those of the form $\mathcal{O}_X/\mathcal{O}_X(-nx)$ for $x \in X$ and $n \geq 0$; then you can check directly these have finite-dimensional cohomology, so we're done. ⊠

**Exercise 34.7.** As suggested in the proof, show that a torsion coherent sheaf on $X$ is isomorphic to a finite direct sum of skyscraper sheaves $\mathcal{O}_X/\mathcal{O}_X(-nx)$ for $x \in X$. (On $\mathbb{A}^1$, this is the Jordan decomposition of a finitely generated $k[t]$-module.)

The following exercise isn't directly necessary for the proof, but it's really good.

**Exercise 34.8.** Let $S$ be a Noetherian scheme. Then a coherent sheaf $\mathcal{F} \to S$ is flat iff it's a vector bundle. (Hint: Nakayama's lemma.)

Now we can prove the general theorem.

*Proof of theorem TODO.* We know the theorem for $\mathbb{P}^1$. If $X$ is a smooth projective curve, it has a finite, flat morphism $f\colon X \to \mathbb{P}^1$. Then $f_*\mathcal{O}_X$ is a coherent sheaf on $\mathbb{P}^1$ (in fact, even a vector bundle by **??**). Last time, we showed that for an affine morphism $f$, $R\Gamma(\mathbb{P}^1, f_*\mathcal{F}) \cong R\Gamma(X, \mathcal{F})$, so the theorem follows for $\mathcal{O}_X$ from the theorem for $f_*\mathcal{O}_X$. ⊠

Now, we will place an additional assumption on our curves $X$: that $k$ is algebraically closed in $k(X)$. That is, if $k' \subset k(X)$ is a finite extension of $k$, then $k' = k$. The idea is to rule out curves which really are over finite extensions of $k$. This condition is called *geometric irreducibility*, and is typically formulated differently, that if $\overline{k}$ is an algebraic closure of $k$, then $X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$ is irreducible. This is a technical assumption you don't have to think too strongly about; when $X$ is smooth and projective, this is equivalent to asking that $H^0(X, \mathcal{O}_X)$ (i.e. $\operatorname{Fun}(X)$) is isomorphic to $k$.

**Exercise 34.9.** Show that if $k \hookrightarrow k'$ is a finite extension and $\overline{k}$ is an algebraic closure of $k'$ (hence also of $k$), then $\operatorname{Spec} k' \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$ is a disjoint union of $n$ points, where $n = [k' : k]$.

**Definition 34.10.** The *genus* of a smooth projective curve $X$, denoted $g = g(X)$, is $\dim H^1(X, \mathcal{O}_X)$.

**Example 34.11.** The genus of $\mathbb{P}^1$ is zero, because we showed $H^1(\mathbb{P}^1; \mathcal{O}_{\mathbb{P}^1}) = 0$. ◀

**Corollary 34.12.** *Let $X$ be a smooth projective curve and $x \in X$ be a closed point. Then $X \setminus x$ is affine.*

*Proof.* Let $g := g(X)$ and consider $H^0(X; \mathcal{O}_X((g+1)x))$. These are functions with a pole of order at most $g$ at $x$, and has dimension at least 2. (TODO: I missed why.) This means it has at least one nonconstant function $f : X \setminus x \to \mathbb{A}^1$. By the valuative criterion, $f$ extends uniquely to a rational map $\widetilde{f} : \dashrightarrow \mathbb{P}^1$ with $f^{-1}(\mathbb{A}^1) = X \setminus x$. Since $f$ is affine, $f^{-1}(\mathbb{A}^1)$ is an affine scheme. ⊠

Lecture 35.

## : 11/19/18

Lecture 36.

## : 11/26/18

Lecture 37.

## Riemann-Roch (is in the house tonight): 11/28/18

Today we're going to discuss the Riemann-Roch theorem (which is a special case of a more general theorem, Serre duality, in the case of curves).

**Theorem 37.1** (Riemann-Roch)**.** *Let $X$ be a smooth projective curve and $\mathcal{E} \to X$ be a vector bundle. There is a quasi-isomorphism*

$$R\Gamma(X, \mathcal{E})^\vee[-1] \simeq R\Gamma(X, \mathcal{E}^\vee \otimes \Omega^1_X) \tag{37.2}$$

*functorial in $\mathcal{E}$.*

First we need to say what the dual of a complex is.

**Definition 37.3.** If $V$ is a complex of vector spaces over $k$, its *dual complex* is $V^\vee := \underline{\operatorname{Hom}}(V, k)$.

If the cohomology of $V$ is concentrated in finitely many degrees and is finite-dimensional, there is a natural quasi-isomorphism $V \xrightarrow{\simeq} (V^\vee)^\vee$ and $H^i(V^\vee) = H^i(V)^\vee$.

We don't yet have the tools to prove this, but we'll get to that. Today we'll discuss some applications.

**Corollary 37.4.** *We have $H^0(X, \mathcal{E})^\vee \cong H^1(X, \mathbb{E}^\vee \otimes \Omega^1_X)$ and $H^1(X, \mathcal{E})^\vee \cong H^0(X, \mathcal{E}^\vee \otimes \Omega^1_X)$. In particular, $H^0(X, \Omega^1_X) \cong H^1(X, \mathcal{O}_X)^\vee$.*

**Corollary 37.5.** *The genus of a smooth projective curve is $h^0(\Omega^1_X)$. Moreoger, $H^1(\Omega^1_X) \cong H^0(X, \mathcal{O}_X)^\vee = k^\vee = k$, so $h^1(\Omega^1_X) = 1$ and $\chi(\Omega^1_X) = g - 1$.*

Recall that if $\mathcal{F}$ is a coherent sheaf on a curve $X$, $\chi(\mathcal{F}) := h^0(\mathcal{F}) - h^1(\mathcal{F})$, and this is called the *Euler characteristic*.

**Corollary 37.6.** $\deg(\Omega^1_X) = 2g - 2$

This is because

$$\deg(\Omega^1_X) = \chi(\Omega^1_X) - \chi(\mathcal{O}_X) = g - 1 - (1 - g) = 2g - 2 \tag{37.7}$$

**Example 37.8.** Let's consider $X = \mathbb{P}^1$, so that $\deg(\Omega^1_X) = -2$. Though the degree of a line bundle is defined cohomologically, you can work around that using divisors: suppose $\mathcal{L} \to X$ is a line bundle and $s$ is a nonzero section on an open subscheme of $X$. Then

$$\mathcal{L} \cong \mathcal{O}_X\Big(\sum v_x(s) \cdot x\Big), \tag{37.9}$$

so

$$\deg \mathcal{L} = \sum v_x(s) \cdot \deg(x). \tag{37.10}$$

Let's put this into action for $\mathbb{P}^1$. A simple choice for $s$ is $\mathrm{d}t$ on the open $\mathbb{A}^1$, which trivializes $\Omega^1_{\mathbb{A}^1}$. Therefore $v_x(\mathrm{d}t) = 0$ for all $x \in \mathbb{A}^1$. Near the point at infinity, we have an affine open $\operatorname{Spec} k[t^{-1}]$. Since $\mathrm{d}(t^{-1}) = -t^{-2}\,\mathrm{d}t$, then away from 0 and $\infty$, $-t^2\mathrm{d}(t^{-1}) = \mathrm{d}t$. Therefore $t^{-1}$ is a uniformizer around $\infty$, so we have a pole of order 2 at $\infty$. Therefore

$$\deg(\Omega^1_{\mathbb{P}^1}) = \sum v_x(\mathrm{d}t) \cdot \deg x = -2\deg(\infty) = -2, \tag{37.11}$$

which is a nice, explicit sanity check. ◀

Recall that if $\mathcal{L}$ is a line bundle on $X$, we saw that for $n \gg 0$, $H^1(\mathcal{L}(nx)) = 0$. You might wonder how big $n$ has to be, but wonder no longer.

**Corollary 37.12.** *If $\deg \mathcal{L} > 2g-2$, then $H^1(\mathcal{L}) = 0$, and therefore we need $n\deg(x) + \deg\mathcal{L} > 2g-2$, and this bound is sharp.*

To see sharpness, let $\mathcal{L} = \Omega^1_X$, which has degree $2g-2$, and $H^1(\Omega^1_X)$ is one-dimensional.

**Exercise 37.13.** Show that this is in a sense the only such example: if $\deg\mathcal{L} = 2g-2$ and $H^1(\mathcal{L}) \neq 0$, then $\mathcal{L} \cong \Omega^1_X$.

*Proof of Corollary 37.12.* Since $H^1(\mathcal{L})^\vee = H^0(\mathcal{L}^\vee \otimes \Omega^1_X)$, then $\deg(\mathcal{L}^\vee \otimes \Omega^1_X) = \deg(\Omega^1_X) - \deg\mathcal{L} = 2g - 2 - \deg\mathcal{L} < 0$. But a section of $\mathcal{L}^\vee \otimes \Omega^1_X$ is equivalent data to a map $\Omega_X \to \mathcal{L}^\vee \otimes \Omega^1_X$; this is a map from a degree-0 line bundle to a negative-degree line bundle, so, as we showed last time, this map must be zero. ⊠

And now for something completely different. Recall that we considered the hyperelliptic curve $X_f \coloneqq \{y^2 = f(x)\} \subseteq \mathbb{A}^2$; assuming that $k$ isn't characteristic 2 and $f$ is separable implies $X_f$ is smooth, hence has a normalization $\overline{X}_f$.

If you were wondering whether curves of all genera exist, well, let $d \coloneqq \deg f$; then $g(\overline{X}_f) = \lceil d/2 \rceil - 1$. So you can get any positive number. This is difficult to compute, in part because $\overline{X}_f$ isn't the closure of $X_f$ in $\mathbb{P}^2$.

Lecture 38.

## : 11/30/18

TODO: I missed the first part.

**Lemma 38.1.** *$k[[t]]$ is a DVR, i.e. both a local ring and a PID. Moreover, $f \in k[[t]]$ is a unit iff $f(0) \neq 0$.*

**Exercise 38.2.** Let $B$ be a commutative ring and $I \subseteq B$ be an ideal with $I^n = 0$. Then $f \in B$ is a unit iff $f \bmod I$ is.

*Proof.* First, we'll prove the second part. Clearly $f$ is a unit iff it maps to a unit in each $k[t]/(t^n)$, using Exercise 38.2.

For the first part, let $I \subseteq k[[t]]$ be an ideal. If it's nonzero, there's an $f \in I \setminus 0$, so $f = t^n u$, where $u$ is a unit. Choose $f$ such that $n$ is minimal; then you can show that $I = (t^n)$. ⊠

Let $X = \operatorname{Spec} A$ and $x$ be a closed point of $X$, corresponding to a maximal ideal $\mathfrak{m}_x$ of $A$. We define $O_x \coloneqq \lim_n A/\mathfrak{m}_x^n$.

**Proposition 38.3.** *If $t$ is a uniformizer at $x$, and $k' \coloneqq A/\mathfrak{m}_x$ is the residue field at $x$, then there is a canonical isomorphism $O_x \xrightarrow{\cong} k'[[t]]$.*

*Proof.* We have compatible, surjective maps $A/\mathfrak{m}_x^n \to k'$, and we'd like the surjection to split. We can do this, in fact in a unique way, because $k \hookrightarrow k'$ is separable, $k$ is perfect, and $\mathfrak{m}_x$ is nilpotent in $A/\mathfrak{m}_x^n$. Therefore we get a map $k' \to O_x$. Since $t$ is a uniformizer at $x$, we have a map $k'[t] \to A$, and sending $t^n \mapsto 0$ is quotienting out by $\mathfrak{m}_x^n$. That is, we have maps $k'[t] \to A/\mathfrak{m}_x^n$ given by sending $t^n \mapsto 0$, and these are compatible as $n$ varies. Hence $k'[t]/(t^n) \xrightarrow{\cong} A/\mathfrak{m}_x^n$, and their limits also agree. ⊠

The map $A \to O_x$ can be thought of as a "power series expansion" around $x$. If $g \in A$ is nonzero at $x$ (i.e. $g \bmod \mathfrak{m}_x \neq 0$), then $g$ maps to a unit in $O_x$. Therefore for all neighborhoods $U \subset X$ containing $x$, we also get maps $\mathrm{Fun}(U) \to O_x$.

These maps are injective! Indeed, $g \in \mathrm{Fun}(U)$ maps to 0 iff $g \in \mathfrak{m}_x^n$ for all $n$, which implies $g = 0$. This is akin to the idea that on a connected domain in $\mathbb{C}$, a function is determined by its Taylor series near a point. We can take $\varinjlim_U \mathrm{Fun}(U)$, which is the ring of germs at $x$, and this injects into $O_x$.

Let $K_x \coloneqq \mathrm{Frac}(O_x)$. A choice of uniformizer $t$ defines an isomorphism

$$K_x \cong k'((t)) \coloneqq k'[[t]]\left[\frac{1}{t}\right] = \{\sum_{n \gg -\infty}^{\infty} a_n t^n \mid a_n \in k'\}. \tag{38.4}$$

This is called the *field of Laurent series* with coefficients in $k'$. Akin to Taylor expansion, we have a *Laurent expansion* map $k(X) \to K_x$.

*Remark* 38.5. $O_x$ is complete with respect to an $I$-adic topology, because it's an inverse limit; $K_x$ is not, because it has no ideals. This occasionally causes confusion. ◀

A variant of this construction: if $\mathcal{E}$ is a vector bundle on $X$, we can form

$$\mathcal{E}_{O_x} \coloneqq \varprojlim_n \mathcal{E}/\mathfrak{m}_x^n \cong \mathcal{E} \otimes_A O_x \tag{38.6a}$$

$$\mathcal{E}_{K_x} \coloneqq \mathcal{E}_{O_x}\left[\frac{1}{t}\right] = \mathcal{E} \otimes_A K_x. \tag{38.6b}$$

Typically we'll do this for $\mathcal{E} = \Omega_X^1$, in which case we'll think of them as $k[[t]]\,\mathrm{d}t$ and $k((t))\,\mathrm{d}t$ — but these are *not* the differentials on $O_x$ and $K_x$! There's a continuity condition.

We will use this setup to discuss a local version of the Riemann-Roch theorem.

**Theorem 38.7** (Local Riemann-Roch)**.** *There's a canonical isomorphism $(K_x)^\vee \cong \Omega_{K_x}^1$, with duality given by residues.*

We'll discuss what "residues" means in this setting. Also, importantly, $K_x^\vee$ is the continuous dual. We'll also explain what this means.

**Definition 38.8.** A *Tate vector space* over a field $k$ is a topological $k$-vector space $V$ which is the direct sum of a profinite-dimensional vector space and a discrete vector space. Here, a *profinite-dimensional* vector space is an inverse limit (as topological vector spaces) of finite-dimensional vector spaces with the discrete topology.

We want this to be the smallest class of topological vector spaces containing finite-dimensional, discrete vector spaces and their inverse limits.

Keep in mind that if $k$ usually has a topology, such as $\mathbb{C}$ or $\mathbb{Q}_p$, we're currently *not* using it.

**Example 38.9.** $C = k((t))$ is a Tate vector space. The topology is set up such that $t^n \to 0$ as $n \to \infty$. More precisely, $V = k[[t]] \oplus k((t))/k[[t]]$; the former is the inverse limit of $k[t]/(t^n)$, and the latter is discrete, with a basis $t^{-1}$, $t^{-2}$, etc. ◀

The awesome thing about Tate vector spaces is that they have duality.

**Definition 38.10.** Let $V$ be a discrete $k$-vector space. We define its *continuous dual* to be $V^\vee \coloneqq \varprojlim_{W \subseteq V \text{ f.d.}} W^\vee$ (i.e. $W$ ranges over all finite-dimensional subspaces of $V$).

The topology on $V^\vee$ is essentially the same thing as the weak-$*$ topology in functional analysis.

**Definition 38.11.** If $V \coloneqq \varprojlim V_i$, where each $V_i$ are discrete and finite-dimensional, and the maps between them are surjective, then we say $V$ is a *lattice*. In this case, we define $V^\vee \coloneqq \varinjlim V_i^\vee$, which is exactly the space of continuous maps $V \to k$.

Therefore for a general Tate vector space, we define $V^\vee$ to be the space of continuous maps $V \to k$.

Because it's true for discrete vector spaces and for latices, the natural map $V \to V^{\vee\vee}$ is a continuous isomorphism for any Tate vector space $V$. Tate vector spaces are the smallest class of vector spaces where we have this double-duality as well as discrete vector spaces.

Returning to local Riemann-Roch, let's do a plausibility check: $K = K((t))$ and $O = k[[t]]$; then $O^\vee \cong k((t))/k[[t]]$,

$$K = O \oplus k((t))/k[[t]] = \varinjlim (k[t]/(t^n))^\vee. \tag{38.12}$$

Therefore $K^\vee = k((t))/k[[t]] \oplus O = K$, which is suggestive. If you don't choose a coordinate $t$, you'll have to twist by one-forms, which is what the theorem statement says.

Lecture 39.

## Traces on Tate vector spaces: 12/3/18

Today, our goal is to construct the residue map $\mathrm{res}\colon \Omega^1_k \to k$. Let $x$ be a $k'$-valued point, $t$ be a uniformizer, and $K := k'((t))$ be the field of Laurent series at $x$.

If $k' = k$, then $t$ trivializes $\Omega^1_k$ near $x$, and

$$\mathrm{res}\left(\sum_{n=-N}^{\infty} a_n t^n\right) := a_{-1}. \tag{39.1}$$

If $k' \supsetneq k$, then we let $\mathrm{res}(\sum a_n t^n) := \mathrm{tr}_{k'/k}(a_{-1})$.

**Proposition 39.2.** *This construction is independent of the coordinate $t$.*

As a basic sanity check, let's test scale-invariance: if $t \mapsto \lambda t$, for $\lambda \in k^\times$, Laurent series change by

$$\sum a_i t^i \longmapsto \sum \frac{a_i}{\lambda^{i+1}} (\lambda t)^i \mathrm{d}(\lambda t), \tag{39.3}$$

and you can directly see that the $(-1)$-terms are the same.

If $\mathrm{char}(k) = 0$, there's a quick proof of Proposition 39.2. We first claim that $\mathrm{d}\colon K \to \Omega^1_K$ has a codimension-1 image, $\mathrm{d}t/t$ gives a basis element, and $\Omega^1_K/\operatorname{Im}(\mathrm{d}) \cong k$ with the projection map given by the residue. Then the proof is pretty easy: for $n \neq -1$,

$$t^n\,\mathrm{d}t = \mathrm{d}\left(\frac{t^{n+1}}{n+1}\right). \tag{39.4}$$

This definition of the residue map is nice for one's intuition, but not as good for proving theorems. Next we'll provide an alternate construction, due in general to Tate, which is opaque but much better for proving stuff.

First, though, a toy model. Let $A$ be a $k$-algebra acting on a finite-dimensional $k$-vector space $V$. This defines a trace map $\mathrm{tr}\colon A \to k$: the action is a map $A \to \mathrm{End}_k(V)$, and then we compose with the usual trace map $\mathrm{tr}\colon \mathrm{End}_k(V) \to k$.

We will generalize this as follows: if $A$ is now a commutative $A$-algebra acting on a Tate vector space $V$, we'll obtain a residue map $\mathrm{res}_V\colon \Omega^1_A \to k$. The natural action of $K$ on itself will define the residue map (ignoring some continuity issues that we'll have to eventually address).

**Definition 39.5.** Let $V$ be a Tate vector space. A *lattice* in $V$ is an open profinite-dimensional vector space $\Lambda \subseteq V$.

Good examples include $k[[t]] \subset k((t))$, or, more generally, $t^n k[[t]]$ for any $n \in \mathbb{Z}$.

*Remark* 39.6. If $\Lambda \subset V$ is a lattice, it's always possible to split $V$ as $V = V/\Lambda \oplus \Lambda$. Moreover, a topological vector space is Tate iff it has a lattice. Also, a lattice gives a basis for the topology of $V$. ◀

**Exercise 39.7.** Let $\Lambda, \Lambda' \subset V$ be lattices. Show that
1. $\Lambda + \Lambda'$ is a lattice,
2. $\Lambda \cap \Lambda'$ is a lattice, and
3. the quotient $(\Lambda + \Lambda')/(\Lambda \cap \Lambda')$ is finite-dimensional.

Defining the trace will be tricky, because if $V$ is an infinite-dimensional vector spaces, traces might not make any sense at all: consider the identity map $\mathrm{id}\colon \ell^2 \to \ell^2$. So we have to restrict to certain classes of operators, imposing the following restrictions.

- We will only consider continuous operators $T\colon V \to V$. This implies in particular that for all lattices $\Lambda \subseteq W$, $T^{-1}(\Lambda)$ contains a lattice in $V$. (It might not be a lattice, e.g. if $T = 0$.) The space of continuous operators $T \to W$ is denoted $\mathrm{Hom}_{\mathrm{cts}}(V, W)$.

- We say that a continuous map $T\colon V \to W$ of Tate vector spaces has *bounded image* if $\mathrm{Im}(T)$ is contained in a lattice. The space of maps $V \to W$ with bounded image is denoted $\mathrm{Hom}^\flat(V, W)$.
- Next, we let $\mathrm{Hom}^\sharp(V, W)$ denote the maps $T\colon V \to W$ with open kernel.
- A *trace-class* operator is one in both $\mathrm{Hom}^\flat(V, W)$ and $\mathrm{Hom}^\sharp(V, W)$. We denote the space of these operators $\mathrm{Hom}_{\mathrm{tr}}(V, W)$.

When $V$ and $W$ are infinite-dimensional, trace-class operators are fairly rare.

**Definition 39.8.** There exists a canonical *trace map* $\mathrm{tr}\colon \mathrm{End}_{\mathrm{tr}}(V) = \mathrm{Hom}_{\mathrm{tr}}(V, V) \to k$, given by the following construction. Since $T$ is trace-class, $\mathrm{Im}(T)$ is contained in some lattice $\Lambda$, and there is some other lattice $\Lambda' \subset \ker(V)$. Then $T$ induces a map

$$T'\colon (\Lambda + \Lambda')/(\Lambda \cap \Lambda') \longrightarrow (\Lambda + \Lambda')/(\Lambda \cap \Lambda') \tag{39.9}$$

essentially by restriction; this factors through the quotient because $\Lambda \cap \Lambda' \subset \Lambda' \subset \ker(T)$.

Using Exercise 39.7, since $\Lambda + \Lambda'$ is a lattice and $\Lambda \cap \Lambda'$ is a lattice, then (39.9) realizes $T'$ as an endomorphism of a finite-dimensional vector space, so we let $\mathrm{tr}(T) \coloneqq \mathrm{tr}(T')$.

**Exercise 39.10.** Show that this definition does not depend on our choices of $\Lambda$ and $\Lambda'$.

In view of this, it's useful to know how to produce trace-class operators.

**Exercise 39.11.** If $V$ is a Tate vector space, $\mathrm{End}^\flat(V)$ and $\mathrm{End}^\sharp(V)$ are two-sided ideals inside $\mathrm{End}_{\mathrm{cts}}(V)$, and therefore $\mathrm{End}_{\mathrm{tr}}(V)$ is also a two-sided ideal.

**Exercise 39.12.** Show that if $T\colon V \to V$ is trace-class and $S\colon V \to V$ is continuous, then $[S, T] \coloneqq ST - TS$ is trace-class, and $\mathrm{tr}([T, S]) = 0$.

**Lemma 39.13.** *Let $T\colon V \to W$ be continuous. Then there exist $T^\flat \in \mathrm{Hom}^\flat(V, W)$ and $T^\sharp \in \mathrm{Hom}^\sharp(V, W)$ such that $T = T^\flat + T^\sharp$.*

*Proof.* Let $\Lambda \subseteq W$ be a lattice, and choose a decomposition $W \cong \Lambda \oplus W/\Lambda$. Let $\pi_\Lambda\colon W \to W$ be the projection onto $\Lambda$; then let $T^\flat \coloneqq \pi_\Lambda \circ T$ and $T^\sharp \coloneqq T - T^\flat$.

Clearly $T^\flat \in \mathrm{Hom}^\flat(V, W)$; why does $T^\sharp$ have open kernel? This follows from the fact that $T^{-1}(\Lambda)$ is open, by continuity, and is contained in $\ker(T^\sharp)$. ⊠

Recall that if $A$ is an algebra, an $A$-bimodule (also called an $(A, A)$-bimodule) is a vector space with commuting left and right actions of $A$. It's often better to think of this as an $A \otimes_k A^{\mathrm{op}}$-module, and that will be a useful perspective today.

Suppose $A$ is a commutative algebra and that we have an extension of $A$-bimodules

$$0 \longrightarrow M \longrightarrow \mathcal{E} \longrightarrow A \longrightarrow 0. \tag{39.14}$$

This induces an $A$-module map

$$\delta\colon \Omega^1_A \to M \otimes_{A \otimes A} A = M/[A, M] = M/\{am - ma \mid a \in A, m \in M\}. \tag{39.15}$$

**Example 39.16.** For example, if $\mathcal{E} = A \otimes A$ and its map to $A$ is multiplication $m$, then $M = I = \ker(m)$, so we get a map

$$\Omega^1_A \longrightarrow I/[A, I] = I/I^2. \tag{39.17}$$

We claim this is an isomorphism. ◀

The construction of the map in (39.15) goes as follows: lift $1 \in A$ to some $\widetilde{1} \in \mathcal{E}$. If $f \in A$, define $\delta(f) \coloneqq f \cdot \widetilde{1} - 1\widetilde{f} \in M$, which by the universal property of $\Omega^1_A$ passes to the desired map

$$\delta\colon \Omega^1_A \to M \twoheadrightarrow M/[A, M]. \tag{39.18}$$

You can check that this doesn't depend on the choice of $\widetilde{1}$.

Definition of trace: TODO.

Lecture 40.

## Residues: 12/5/18

Warning: there may be a sign error in today's lecture.

Suppose $A$ is a commutative $k$-algebra acting on a Tate vector space $V$ by continuous maps. We would like to define a residue map $\mathrm{res}_V\colon \Omega^1_{A/k} \to k$, and first have to define the trace of a trace-class operator (since $V$ may be infinite-dimensional, and often will be in our applications). Last time, we described a canonical decomposition of a continuous operator $T\colon V \to V$ as $T = T^\flat + T^\sharp$, where $T^\flat$ has bounded image and $T^\sharp$ has open kernel.

This definition of residues follows Tate.

**Definition 40.1.** The *residue* of a differential $f\,\mathrm{d}g$ is $\mathrm{res}_V(f\,\mathrm{d}g) \coloneqq \mathrm{tr}[g, f^\flat] \in k$.

Since $\mathrm{End}^\flat(V)$ is a two-sided ideal in the algebra of continuous endomorphisms on $V$, then $[g, f^\flat] \in \mathrm{End}^\flat(V)$. Letting $f^\sharp \coloneqq f - f^\flat$, there's a lattice $\Lambda \subset V$ with $f^\sharp(\Lambda) = 0$, hence a $v \in \Lambda$ with $f(v) = f^\flat(v)$. If $v \in \Lambda \cap g^{-1}(\Lambda)$, then

$$[g, f^\flat](v) = gf^\flat(v) - f^\flat(g(v)) = [g, f](v) = 0. \tag{40.2}$$

Therefore $[g, f^\flat]$ also has open kernel, hence is trace-class, and Definition 40.1 makes sense.

Suppose that $V = A = k[[t]]$, and $A$ acts by left multiplication. Then we have another, more concrete (and coordinate-dependent) definition of residues, familiar from complex analysis, and, fortunately, Definition 40.1 agrees with this definition.

**Exercise 40.3.** Let $T \in \mathrm{End}^\flat(V)$ and $S \in \mathrm{End}^\sharp(V)$. Then $[T, S]$ is trace-class and has trace zero.

**Lemma 40.4.** *If $f \in A$, then for all $n \geq 0$, $\mathrm{res}_V(f^n\,\mathrm{d}f) = 0$.*

*Proof.* Write $f = f^\flat + f^\sharp$ as usual. Then $f^n = (f^\flat)^n + g$ where $g$ has open kernel, i.e. $(f^n)^\flat = (f^\flat)^n$. Choose a lattice $\Lambda \subset V$ and a $v \in \Lambda$ with $f(v) = f^\flat(v)$; then

$$v \in \bigcap_{i=1}^{n-1} f^{-i}(\Lambda). \tag{40.5}$$

Therefore

$$(f^\flat)^n(v) = (f^\flat)^{n-1} f^\flat(v) = (f^\flat)^{n-1} f(v) = (f^\flat)^{n-2} f^2(v) = \cdots = f^n(v). \tag{40.6}$$

Using Exercise 40.3,

$$\mathrm{res}_V(f\,\mathrm{d}g) \coloneqq \mathrm{tr}_V[g, f^\flat] = \mathrm{tr}[g^\flat, f^\flat] + \mathrm{tr}[g^\sharp, f^\flat] = \mathrm{tr}[g^\flat, f^\flat], \tag{40.7}$$

because $\mathrm{tr}[g^\sharp, f^\flat] = 0$.

Therefore

$$\mathrm{res}_V(f^n\,\mathrm{d}f) = \mathrm{tr}_V[f^\flat, (f^n)^\flat] = \mathrm{tr}[f^\flat, (f^\flat)^n] = 0, \tag{40.8}$$

because these two operators commute. ⊠

The point is: when $n$ is nonnegative, the residue behaves the way we think it should.

**Definition 40.9.** Let $\Lambda_1$ and $\Lambda_2$ be lattices in a Tate vector space $V$. The *relative dimension* of $\Lambda_1$ and $\Lambda_2$ is

$$\mathrm{rel\,dim}(\Lambda_1, \Lambda_2) \coloneqq \dim(\Lambda_1/(\Lambda_1 \cap \Lambda_2)) - \dim(\Lambda_2/(\Lambda_1 \cap \Lambda_2)).$$

This can be interpreted as an index. The quotient of a lattice by a sublattice is open and discrete in $V$, hence finite-dimensional, so this gives you an integer.

**Lemma 40.10.** *Suppose $T\colon V \to V$ is an invertible, continuous operator. Then $\mathrm{tr}_V[T^{-1}, T^\flat] = \mathrm{rel\,dim}(T^{-1}(\Lambda), \Lambda)$ for any lattice $\Lambda$.*

Two takeaways: first, that this number doesn't depend on $\Lambda$, only on $T$; and second, this will be a source of nonzero residues.

*Proof.* Choose a decomposition $V = \Lambda \oplus V/\Lambda$, and let $\pi_\Lambda\colon V \to V$ be the projection onto $\Lambda$, so that $T^\flat = \pi_\Lambda T$ by definition. Therefore

$$[T^{-1}, T^\flat] = [T^{-1}, \pi_\Lambda T] = T^{-1}\pi_\Lambda T - \pi_\Lambda T T^1 = \pi_{T^{-1}\Lambda} - \pi_\Lambda, \tag{40.11}$$

which has trace equal to $\dim(T^{-1}(\Lambda)/\Lambda \cap T^{-1}(\Lambda)) - \dim(\Lambda/\Lambda \cap T^{-1}(\Lambda))$. ⊠

We bring this to the geometric setting.

**Corollary 40.12.** *Let $X$ be a smooth curve and $x \in X$ be a closed point. Under the natural action of $A := \mathrm{Fun}(X \setminus x)$ on the Tate vector space $K_x$, for an $f \in A$,*

$$\mathrm{res}_x\left(\frac{\mathrm{d}f}{f}\right) = -\deg(x) v_x(f), \tag{40.13}$$

For example, the residue of $\mathrm{d}t/t = -1$, where $t$ is a uniformizer at $x$.

*Proof.* We'll use the lattice $O_x \subset K_x$. Lemma 40.10 applied for $T := f^{-1}$ says that $\mathrm{res}(f^{-1}\,\mathrm{d}f) = \mathrm{rel}\dim(O_x, f(O_x))$. If $v_x(f) \geq 0$, then $f(O_x) \subset O_x$, so the relative dimension is just $\dim O_x/f(O_x) = \deg(x)\cdot v_x(f)$; if $v_x(f) \leq 0$, then $f(O_x) \supset O_x$, so the relative dimension is $-\dim f(O_x)/O_x = \deg(x)\cdot v_x(f)$. ⊠

TODO: this computation is correct, so everything else is off by a minus sign. The sign in the corollary is accurate.

In this setting, the residue $\mathrm{res}_x\colon \Omega^1_{X\setminus x} \to k$ extends by continuity to $\Omega^1_{K_x}$.

Lecture 41.

## : 12/7/18

Lecture 42.

## : 12/10/18

Today, we'll prove the Riemann-Roch theorem. TODO: I missed the first part. I think $\mathbb{A}_X$ denotes the ring of adeles of $X$, but I don't know what that means yet, so hello from the other side.

...here, $\mathcal{L}_\mathbb{A} := \Gamma(X \setminus x, \mathcal{L}) \otimes_{\mathrm{Fun}(X,x)} \mathbb{A}_X$. Since $H^1(X, \Omega_X) \cong \Omega^1_{\mathbb{A}_X}/(\Omega_{O_\mathbb{A}} + \Omega^1_{k(X)})$, we get an induced residue map from $H^1(\Omega^1_X)$ to the ground field $k$: it's clear that $\mathrm{res}_\mathbb{A}(\Omega^1_{O_\mathbb{A}}) = 0$, and last time, we showed that $\mathrm{res}_\mathbb{A}(\Omega^1_{k(X)}) = 0$.

Now let $\mathcal{E}$ be a vector bundle on $X$. Then we can produce a sequence of maps

(42.1)

$$R\Gamma(X, \mathcal{E}) \otimes R\Gamma(X, \mathcal{E}^\vee \otimes \Omega^1_X)[1] \xrightarrow{\varphi} R\Gamma(X, \mathcal{E} \otimes \mathcal{E}^\vee \otimes \Omega^1)[1] \xrightarrow{\psi} R\Gamma(X, \Omega^1)[1] \xrightarrow{\chi} H^1(\Omega^1) \xrightarrow{\mathrm{res}} k,$$

where

- $\varphi$ is induced from the tensor product as an instance of a more general map $R\Gamma(X, \mathcal{E}_1)[m] \otimes R\Gamma(X, \mathcal{E}_2)[n] \to R\Gamma(X, \mathcal{E}_1 \otimes \mathcal{E}_2)[m+n]$;
- $\psi$ is induced from the evaluation map $\mathcal{E} \otimes \mathcal{E}^\vee \to \mathcal{O}_X$; and
- $\chi$ exists because $R\Gamma(X, \Omega^1_X)$ is concentrated in degrees 1 and 2, so has a natural quotient map to $H^1(X, \Omega^1_X)$.

**Theorem 42.2** (Riemann-Roch, more precise formulation)**.** *The map* (42.1) *is a perfect pairing.*

So: the cohomology groups aren't just dual; we've exhibited an explicit duality between them.

*Proof.* For notational simplicity, we will let $\mathcal{E} = \mathcal{O}_X$; ultimately because $\mathcal{E}$ is locally trivial, the general case can be reduced to this one, and the notation is much nicer. The proof will pass through the local Riemann-Roch theorem.

Since $\mathbb{A}_X$ is a Tate vector space, there's a canonical isomorphism $(\mathbb{A}_X)^\vee \cong \Omega^1_{\mathbb{A}_X}$ via the pairing $\mathbb{A}_X \otimes \Omega^1_{\mathbb{A}_X} \to k$ sending $f \otimes \omega \mapsto \mathrm{res}(f\omega)$. Recall that $\mathbb{A}_X$ has the structure

$$0 \longrightarrow O_{\mathbb{A}_X} \longrightarrow \mathbb{A}_X \longrightarrow \mathbb{A}_X/O_{\mathbb{A}_X} \longrightarrow 0, \tag{42.3}$$

where $O_{\mathbb{A}_X} \hookrightarrow \mathbb{A}_X$ is a lattice, and the quotient is discrete, hence a direct sum $\bigoplus_x K_x/O_x$ over closed points $x$. In this setting, we get a "self-duality" phenomenon: a canonical isomorphism

$$(O_{\mathbb{A}_X})^\vee \cong \Omega^1_{\mathbb{A}_X}/\Omega^1_{O_{\mathbb{A}_X}}. \tag{42.4}$$

There is another short exact sequence

$$0 \longrightarrow k(X) \longrightarrow \mathbb{A}_X \longrightarrow \mathbb{A}_X/k(X) \longrightarrow 0, \tag{42.5}$$

and this time $k(X) \hookrightarrow \mathbb{A}_X$ is discrete, and the quotient is profinite-dimensional. Analogous to (42.4), there is a map

$$\Omega^1_{k(X)} \longrightarrow (\mathbb{A}_X/k(X))^\vee. \tag{42.6}$$

**Lemma 42.7.** *The map* (42.6) *is an isomorphism.*

This will be the key lemma in the proof. First, we'll show how to use it to prove the full Riemann-Roch theorem; then we'll go back and prove it.

Let $\mathcal{L}$ be a line bundle on $X$; then we have a quasi-isomorphism

$$R\Gamma(X, \mathcal{L}) \xrightarrow{\simeq} \mathrm{hKer}\Big(\mathcal{L}_{k(X)} \oplus \mathcal{L}_{O_{\mathbb{A}_X}} \to \mathcal{L}_{\mathbb{A}_X}\Big). \tag{42.8}$$

Hence, in particular,

$$R\Gamma(X, \mathcal{O}_X) \xrightarrow{\simeq} \mathrm{hKer}(k(X) \oplus O_{\mathbb{A}_X} \to \mathbb{A}_X), \tag{42.9}$$

and, thanks to some formal isomorphisms,

$$\begin{aligned}
R\Gamma(X, \mathcal{O}_X)^\vee &\simeq \mathrm{hCoker}((\mathbb{A}_X)^\vee \to k(X)^\vee \oplus (O_{\mathbb{A}_X})^\vee) && (42.10\text{a})\\
&\simeq \mathrm{hCoker}\Big(\Omega^1_{\mathbb{A}_X} \to (\Omega^1_{\mathbb{A}_X}/\Omega^1_{k(X)}) \oplus (\Omega^1_{\mathbb{A}_X}/\Omega^1_{O_{\mathbb{A}_X}})\Big) && (42.10\text{b})\\
&\simeq \mathrm{hCoker}\Big(\Omega^1_{k(X)} \oplus \Omega^1_{O_{\mathbb{A}_X}} \to \Omega^1_{\mathbb{A}_X}\Big) && (42.10\text{c})\\
&\simeq \mathrm{hKer}\Big(\Omega^1_{k(X)} \oplus \Omega^1_{O_{\mathbb{A}_X}} \to \Omega^1_{\mathbb{A}_X}\Big)[1] && (42.10\text{d})
\end{aligned}$$

A similar analysis works for any vector bundle $\mathcal{E}$; the upshots are canonical isomorphisms

$$(\mathcal{E}_{K_x})^\vee \cong \mathcal{E}^\vee_{K_x} \otimes_{K_x} \Omega^1_{K_x} \tag{42.11}$$

and

$$(K_x)^\vee \cong \Omega^1_{K_x}. \tag{42.12}$$

We've now reduced to proving Lemma 42.7. The preliminary observation is that both sides are already $k(X)$-vector spaces, and $\Omega^1_{k(X)}$ is one-dimensional.

First, we claim the map (42.6) is nonzero. Let $\omega$ be a nonzero one-form, $x \in X$ be a closed point, and $t_x$ be a uniformizer at $x$. If $n \coloneqq v_x(\omega)$, then $t_x^{-n-1} \in K_x \hookrightarrow \mathbb{A}_{K_x}$, so $\omega$ is sent to the functional sending

$$t_x^{-n-1} \longmapsto \mathrm{res}_x(t_x^{-n-1}\omega) \neq 0. \tag{42.13}$$

Since the map is nonzero and the domain is one-dimensional, it's injective. It now suffices to show that the codomain is also one-dimensional.

Recall that

$$\mathbb{A}_X/k(X) = \varprojlim_D H^1(X, \mathcal{O}_X(-D)), \tag{42.14}$$

and therefore dually,

$$(\mathbb{A}_X/k(X))^\vee = \varinjlim_D H^1(X, \mathcal{O}_X(-D))^\vee. \tag{42.15}$$

Suppose $\lambda, \mu \in (\mathbb{A}_X/k(X))^\vee$ are linearly independent. We can assume these arise from $H^1(X; \mathcal{O}_X(-D))^\vee$ for a divisor $D$; we'll use this to show some other $H^1$ is too big.

Choose a closed point $x \in X$, and consider the pairing... TODO. ⊠