SUMMER 2016 ALGEBRAIC GEOMETRY SEMINAR

ARUN DEBRAY
MAY 16, 2016

Contents

1. Separability, Varieties and Rational Maps: 5/16/16

This seminar has a website, located at
`https://www.ma.utexas.edu/users/toldfield/Seminars/Algebraicgeometryreading.html`.

The first half of Chapter 10 is about separated morphisms and varieties; it only took us 10 chapters! Vakil writes that he was very conflicted about leaving a proper treatment of algebraic varieties, a cornerstone of classical algebraic geometry, to so late in the notes. But from a modern perspective, our hands are tied: varieties are defined in terms of properties, which means building those properties out of other properties and out of the large amount of technology you need for modern algebraic geometry. With that technology out of the way, here we are.

One of these properties is separability. Let $\pi : X \to Y$ be a morphism of schemes; then, the **diagonal** is the induced morphism $\delta_\pi : X \to X \times_Y X$ defined by $x \mapsto (x, x)$; this maps into the fiber product because it fits into the diagram

$$\begin{tikzcd}
X \arrow[dr, "\delta_\pi"] \arrow[drr, bend left, "1_X"] \arrow[ddr, bend right, "1_X"'] & & \\
& X \times_Y X \arrow[r, "p_2"'] \arrow[d, "p_1"] & X \arrow[d, "\pi"'] \\
& X \arrow[r, "\pi"] & Y.
\end{tikzcd} \tag{1.1}$$

Here, p_1 nad p_2 are the projections onto the first and second components, respectively, and 1_X is the identity map on X.

The diagonal has a few nice properties. Suppose $V \subset Y$ is open, and $U, U' \subset \pi^{-1}(V)$ are open subsets of X. Then, $U \times_V U' = p_1^{-1}(U) \cap p_2^{-1}(U')$: we constructed fiber products such that they send open embeddings to intersections. In particular, if $U \cong \operatorname{Spec} A$, $U' \cong \operatorname{Spec} A'$, and $V \cong \operatorname{Spec} B$ are affine, $U \times_V U' \cong \operatorname{Spec}(A \otimes_B A')$. Therefore $\delta_\pi^{-1}(U \times_V U') = \delta_\pi^{-1}(p_1^{-1}(U) \cap p_2^{-1}(U')) = U \cap U'$. That is, the diagonal turns intersections into fiber products.

This argument feels like it takes place in Set, but goes through word-for-word for schemes.

Definition 1.2. A morphism $\pi : X \to Y$ of schemes is a **locally closed embedding** if it factors as $\pi = \pi_1 \circ \pi_2$, where π_2 is a closed embedding and π_1 is an open embedding.

Proposition 1.3. *For any $\pi : X \to Y$, δ_π is locally closed.*

Proof. Let $\{V_i\}$ be an affine open cover of Y, so $V_i \cong \operatorname{Spec} B_i$ for each B, and $\mathfrak{U}_i = \{U_{ij}\}$ be an affine open cover of $\pi^{-1}(V_i)$ for each i. Then, $\{U_{ij} \times_{V_i} U_{ij'} : i, j, j'\}$ covers $X \times_Y X$. More interestingly, $\{U_{ij} \times_{V_i} U_{ij} : i, j\}$ covers $\operatorname{Im}(\delta_\pi)$: this is because if $x \in U_{ij}$, then $\delta_\pi(x) \in p_1^{-1}(U_{ij})$ and in $p_2^{-1}(U_{ij})$, and $p_1^{-1}(U_{ij}) \cap p_2^{-1}(U_{ij}) = U_{ij} \times_{V_i} U_{ij}$.

Now, it suffices to show that $\delta_\pi : \delta_\pi^{-1}(U_{ij} \times_{V_i} U_{ij}) \to U_{ij} \times_{V_i} U_{ij}$ is closed, since the property of being a closed embedding is affine-local. Since each $U_{ij} \cong \operatorname{Spec} A_{ij}$ is affine, then it suffices to understand what's happening ring-theoretically: the diagonal map corresponds to the ring morphism $A_{ij} \otimes_{V_i} A_{ij} \to A_{ij}$ sending

$a \otimes a' \mapsto aa'$. This is clearly surjective, which is exactly the criterion for a morphism of schemes to be a closed embedding. $\square$

Corollary 1.4. *If X and Y are affine schemes, then δ_π is a closed embedding.*

Corollary 1.5. *If Δ denotes $\mathrm{Im}(\delta_\pi)$, then for any open $V \subset Y$ and $U \subset \pi^{-1}(V)$, $\Delta \cap (U \times_V U') \cong U \cap U'$ is a homeomorphism of topological spaces.*

This follows because a locally closed embedding is homeomorphic onto its image.

These will all be super useful once we define separability, which we'll do now.

Definition 1.6. A morphism $\pi : X \to Y$ is **separated** if $\delta_\pi : X \to X \times_Y X$ is a closed embedding.

This is weird upon first glance: why do we look at the diagonal to understand things about a morphism? The answer is that the diagonal has nice category-theoretic properties, so we can prove some useful properties by doing a few diagram chases.

More geometrically, separability corresponds to the Hausdorff property in topological spaces, and there's a criterion for this in terms of the diagonal.

Proposition 1.7. *If T is a topological space, then T is Hausdorff iff the diagonal morphism $T \to T \times T$ is a closed embedding.*

Equivalently, the image $\Delta \subset T \times T$ is a closed subspace.

Remark. Since schemes are topological spaces, you might think this proves separated schemes are Hausdorff, but this is untrue: fiber products of schemes are generally not fiber products of underlying spaces, and therefore closed embeddings of schemes are not the same as closed embeddings of their underlying spaces.

Separability is a nice property, and is good to have. But like Hausdorfness, we generally won't need to use schemes that aren't separated.

Example 1.8.

(1) By Corollary 1.4, all morphisms of affine schemes are separated.
(2) If we can cover $X \times_Y X$ by the sets $U_{ij} \times_{V_i} U_{ij}$ (with these sets as in the proof of Proposition 1.3), then π is separated.
(3) For a counterexample, let $X = \mathbb{A}^1_{(0,0)}$ be the "line with two origins" over a field k. This isn't a separated scheme: the diagonal is a "line with four origins," and these cannot be separated topologically: every open set containing one contains all of them. So take one affine piece of X, which contains exactly one origin, and therefore its image ought to contain all four, but it doesn't, so $X \to \operatorname{Spec} k$ isn't closed. This might feel a little imprecise, but one can make it fully rigorous.

We want separated morphisms to be nice: we'd like them to be preserved under base change and composition, and we'd like locally closed embeddings to be separated.

Proposition 1.9. *Locally closed embeddings are separated.*

This is the only example of a hands-on proof of a property; it's not hard, but the rest will be less abstract and easier. First, though, let's reframe it:

Proposition 1.10. *Any monomorphism of schemes is separated.*[1]

Proof. By point (2) of Example 1.8, it suffices to prove that fiber products $U_{ij} \times_{V_i} U_{ij}$ cover $X \times_Y X$ for our affine covers. So let's look at the fiber diagram (1.1) again; it tells us that $\pi \circ p_1 = \pi \circ p_2$. But since π is a monomorphism, then $p_1 = p_2$, so for any $z \in X \times_Y Z$, $p_1(z) = p_2(z)$; call this point x_z. Then, if $x_z \in U_{ij}$, $z \in p^{-1}(U_{ij})$ and $z \in p_2^{-1}(U_{ij})$, and their intersection is the fiber product. $\square$

Since locally closed embeddings are monomorphisms, Proposition 1.9 follows as a corollary.

At this point, we can define varieties, and Vakil does so, but can't do anything with them, so we'll come back to them in a little bit.

[1]More is true in general; all you need is that $p_1 = p_2$ in the diagram (1.1), which is analogous to an injectivity condition on π. Hence, it suffices that π is injective as a map of sets, but this is a weird notion for schemes, so we generally phrase it in terms of monomorphisms.

Proposition 1.11. *If A is a ring, $\mathbb{P}^n_A \to \operatorname{Spec} A$ is separated.*

The idea of the proof is to compute: we already know a cover of $\mathbb{P}^n_A$ by $n+1$ affine schemes, and can check that the induced map on rings is surjective.

The following proposition gives us an important geometric property of separability.

Proposition 1.12. *If A is a ring and $X \to \operatorname{Spec} A$ is separated, then for any affine open subsets $U, V \subset X$, $U \cap V$ is also affine.*

Proof. The diagonal is a closed embedding, so $\delta : U \times V \to U \times_A V$ is also a closed embedding. Therefore $U \times V$ is isomorphic to a closed subscheme of an affine scheme, and therefore is affine. $\square$

It's surprising how useful these arguments with the diagonal are: we got a useful and nontrivial result in one line! In general, you can prove a weirdly large amount of things by factoring them through the diagonal. In fact, le'ts use it to define another property.

Definition 1.13. A morphism $\pi : X \to Y$ is **quasiseparated** if δ_π is quasicompact.

This isn't the same as the other definition we were given, that for all affine $V \subset Y$ and $U, U' \subset \pi^{-1}(V)$, $U \cap U'$ is quasicompact. But it turns out to be equivalent.

Proposition 1.14. *$\pi : X \to Y$ is quasiseparated in the sense of Definition 1.13 iff it's quasiseparated in the sense we defined previously.*

The proof is a diagram chase involving the "magic diagram" for fiber products. This states that if $X_1, X_2 \to Y \to Z$ are maps in some category and the relevant fiber products exist, the diagram

$$\begin{tikzcd} X_1 \times_Y X_2 \arrow[r] \arrow[d] & X_1 \times_Z X_2 \arrow[d] \\ Y \arrow[r, "\delta"] & Y \times_Z Y \end{tikzcd}$$

is a fiber diagram; the proof is a diagram chase following from the associativity of products, or checking the universal property. This diagram is also very ubiquitous for proofs like these.

Proposition 1.15. *Separability and quasiseparability are preserved under base change.*

Proof. Suppose $\pi : X \to Y$ is separated and $\varphi : S \to Y$ is another map of schemes, so there's an induced morphism $\pi' : Z = X \times_Y S \to S$ fitting into the diagram

$$\begin{tikzcd} Z \arrow[r, "\pi'"] \arrow[d, "p_1"] & S \arrow[d, "\varphi"] \\ X \arrow[r, "\pi"] & Y. \end{tikzcd}$$

The magic diagram for this is the fiber diagram

$$\begin{tikzcd} Z \arrow[r, "\delta_{\pi'}"] \arrow[d] & Z \times_S Z \arrow[d] \\ X \arrow[r, "\delta_\pi"] & X \times_Y X. \end{tikzcd}$$

If π is separated, δ_π is closed, and therefore $\delta_{\pi'}$ is closed (since closed embeddings are preserved under base change), so π' is separated. The same argument works with π quasiseparated and δ_π quasicompact. $\square$

There are a few related properties that we won't prove, but whose proofs are very similar to the previous one.

Proposition 1.16. *Separability and quasiseparability are*

(1) *local on the target,*
(2) *closed under composition, and*

(3) *closed under taking products: if $\pi : X \to Y$ and $\pi' : X' \to Y'$ are separated morphisms of schemes over a scheme S, then $\pi \times \pi' : X \times_S X' \to Y \times_S Y'$ is separated; if π and π' are merely quasiseparated, so is $\pi \times \pi'$.*

Each of these is a diagram chase with the right diagram, and not a particularly hard one; the last one follows as a general categorical consequence of the others.

Now, though, we can define varieties.

Definition 1.17. Let k be a field. A **k-variety** is a k-scheme $X \to \operatorname{Spec} k$ that is reduced, separated, and of finite type. A **subvariety** of a given variety X is a reduced, locally closed subscheme.

Reducedness is a property of X, but the others are properties of the structure morphism $X \to \operatorname{Spec} k$. Notice that the affine line with doubled origin is reduced and of finite type, so separability is important for avoiding pathologies.

It's nontrivial that a subvariety $Y \subset X$ is itself a variety. X is finite type over $\operatorname{Spec} k$, so it's covered by finitely many affine opens that are schemes of finitely generated k-algebras, which are Noetherian, so X is Noetherian. Hence, $Y \hookrightarrow X$ is a finite-type morphsism into a Noetherian scheme, so Y is finite type; but we do need separability to be preserved under composition, which we just saw how to prove.

We did not require varieties to be irreducible; irreducibility doesn't behave as well as we would like, unless k is particularly nice.

Proposition 1.18. *The product of irreducible varieties over an algebraically closed field k is an irreducible k-variety.*

This follows from the nontrivial fact that if A and B are k-algebras that are integral domains, then $A \otimes_k B$ is an integral domain.

The last important thing we'll discuss today is a big meta-theorem about classes of morphisms.

Theorem 1.19 (Cancellation theorem)**.** *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "\tau"'] & & Y \arrow[dl, "\rho"] \\ & Z, & \end{tikzcd}$$

i.e. $\tau = \rho \circ \pi$, and let P be a property of morphisms preserved under base change and composition. If τ has P and δ_ρ has P, then π also has P.

The name is because we're "cancelling" ρ out of the composition.

The proof uses the notion of the graph of a morphism.

Definition 1.20. Let X and Y be schemes over a scheme S, and $\pi : X \to Y$ be a map of S-schemes. Then, the **graph** of π is the morphism $\Gamma_\pi : X \to X \times_S Y$ defined by $\Gamma_\pi; (1_X, \pi)$.

That is, this sends a point to its image on the graph. We use this because any morphism factors through its graph. Then, since δ_ρ has P, so must Γ_π, which is useful. It seems weirdly abstract and pointless, but the idea is that the nice properties of the diagonal, including locally closed embeddings, can be canceled off. In fact, if Y is separated, we can cancel off properties of closed embeddings, and if Y is quasiseparated, we can cancell off properties of quasicompact morphisms.

Rational Maps. Let's talk about rational maps, which are rational maps defined almost everywhere, and up to almost everywhere agreement. Rational maps are usually only defined on reduced varieties, since it's nearly impossible to get a hold on them otherwise; they're inherently geometric, and geometry tends to involve varieties.

Definition 1.21. A **rational map** $\pi : X \dashrightarrow Y$ is an equivalence class of morphisms $f : U \to Y$, where $U \subset X$ is a dense open subset; (f, U) and (f', U') are considered equivalent if there's a dense open set $V \subset U \cap U'$ if $f|_V = f'|_V$. One says π is **dominant** if its image is dense, or equivalently, for all nonempty opens $V \subseteq Y$, $\pi^{-1}(V) \neq \varnothing$.

Notice that dominance is well-defined, as it's independent of choice of representative.

Proposition 1.22. *Let X and Y are irreducible schemes, then $\pi : X \dashrightarrow Y$ is dominant iff the generic point of X maps to the generic point of Y.*

Proof. In the reverse direction, the generic point η_Y of Y is contained in every open subset of Y, so the preimage contains the generic point η_X of X, and in particular is nonempty.

In the other direction, suppose $\pi(\eta_X) \neq \eta_Y$; let $U = Y \setminus \overline{\pi(\eta_X)}$, which is an open subset. Thus, $\eta_X \notin \pi^{-1}(U)$, which is an open set. Since η_X is dense, it meets every nonempty open, so $\pi^{-1}(U)$ is empty, and therefore π isn't dominant. ☐

This is a pretty useful characterization of dominance. But why do we care about dominance? Because of composition.

Remark. Let $\pi : X \dashrightarrow Y$ and $\rho : Y \dashrightarrow Z$ be rational maps. If π is dominant and X is irreducible, it's possible to make sense of $\rho \circ \pi : X \dashrightarrow Z$ as a rational map, which is dominant iff ρ is.

This is nontrivial: if π isn't dominant, one might discover that the domain of ρ doesn't intersect the image of π; if they do, however, π^{-1} of the domain of definition of ρ is a nonempty open of X; since X is irreducible, it must be dense.

Definition 1.23. A rational map $\pi : X \dashrightarrow Y$ is **birational** if it's dominant and there exists a dominant $\psi : Y \dashrightarrow X$ such that as rational maps, $\pi \circ \psi \sim 1_X$ and $\psi \circ \pi \circ 1_Y$. In this case, one says π and ψ are **birational(ly equivalent)**.

Proposition 1.24. *Let X and Y be reduced schemes; then, X and Y are birational iff there exist dense open subschemes $U \subset X$ and $V \subset Y$ such that $U \cong V$.*

The idea is that we can let U and V be the domains of definition for our rational maps.

The notion of rationality is very specific to algebraic geometry; in the differentiable category, it's complete nonsense. Since any manifold can be triangulated, any two manifolds of the same dimension are birationally equivalent: remove the edges of the triangles, and you get a dense open set; clearly, any two triangles are birational. However, there exist algebraic varieties of the same dimension that aren't birationally equivalent.

Definition 1.25. A variety X over k is **rational** if it's birational to $\mathbb{A}_k^n$ for some n.

For example, $\mathbb{P}_k^n$ is rational. Rationality loses some information, but what it keeps is interesting.

Finally, let's see what dominance means in terms of ring morphisms.

Definition 1.26. Let $\varphi : \operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism of affine schemes and $\varphi^\sharp : B \to A$ be the induced map on global sections. Then, φ is dominant (i.e. as a rational map) iff $\ker(\varphi^\sharp) \subset \mathfrak{N}(A)$.

Here, $\mathfrak{N}(A)$ denotes its nilradical, the intersection of all prime ideals of A (equivalently, the ideal of nilpotent elements). That is, if A and B are reduced, dominance is equivalent to injectivity! Interestingly, this also corresponds to an inclusion of function fields, i.e. a field extension! We've reduced a geometric problem to a problem about algebra. Often, we can go in the other direction, e.g. for varieties. In this setting, birationality means isomorphism on the function fields.