M393C NOTES: PARTIAL DIFFERENTIAL EQUATIONS

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These notes were taken in UT Austin's M393C (Partial Differential Equations) class in Fall 2019, taught by Francesco Maggi. I live-TeXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Any mistakes in the notes are my own.

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Lecture 1.

Why elliptic equations?: 8/29/19

The course textbook is Evans, but we will probably diverge (hah) from it much of the time. If you're registered for the course, at the end of the course you'll have to give a presentation on a paper near the end of the course.

This course will require some familiarity with Sobolev spaces and weak solutions; if you haven't seen these before, one good reference is Evans, chapters 5 and 6.1.1, respectively.

We're going to be proving some esoteric things soon enough, so let's back up and ask: why elliptic equations?

Our first example is a simple model called the elastic membrane model. Let $\Omega \subseteq \mathbb{R}^n$ be a domain. Think of it as an elastic membrane whose boundary is fixed. The displacement of the membrane defines a function $u\colon \Omega \to \mathbb{R}$, and since the boundary is fixed, the boundary condition $u_0\colon \partial\Omega \to \mathbb{R}$ is given. The theory of *nonlinear elasticity*, understanding how this story behaves under deformations, is part of mathematical physics. For example, one can prove that if $|\nabla u|$ is small, then u will minimize the *Dirichlet energy*

$$\int_\Omega |\nabla u|^2 \tag{1.1}$$

such that $u|_{\partial\Omega} = u_0$.

The theory of Soblev spaces guarantees that there exists a unique minimizer $u \in W^{1,2}(\Omega)$, but $W^{1,2}$ functions can fail to be $L^\infty_{\ell oc}$ in dimension $n \geq 2$ – that is, they can be unbounded on every open set! The typical example is

$$u(x) = \sum_{k=1}^\infty \frac{1}{2^k} |\log|\log|x - x_k|||, \tag{1.2}$$

where $\{x_k\}_{k=1}^\infty$ is a countable dense subset of Ω (e.g. its points with rational coordinates). Sobolev spaces look simple: we have gradients and can pretend that we're acting on smooth functions, but don't forget that these pathologies sneak in.

Of course, we don't expect a minimizer of a nice physical problem to be so silly. So how can we prove strong enough regularity to avoid these pathologies? We will exploit minimality to obtain better information. That is, consider the Dirichlet energy $E(u + t\varphi)$, where φ is a test function, i.e. $\varphi \in C_c^\infty(\Omega)$. Thus $u + t\varphi$ obeys the same boundary condition, because $\varphi|_{\partial\Omega} = 0$. Thus $E(u + t\varphi)$ has a minimum at $t = 0$.

There are two consequences.

(1) The first-order consequence that, no matter what φ is,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} E(u + t\varphi) = 0. \tag{1.3}$$

(2) There's also a second-order consequence: for every φ,

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} E(u+t\varphi) \geq 0. \tag{1.4}$$

For this specific problem, the second one is trivial, and the first one is useful. For $E(u) = \int |\nabla u|^2$, we can explicitly compute the variation

$$E(u+t\varphi) = \int_\Omega |\nabla(u+t\varphi)|^2 = E(u) + 2t\int_\Omega \nabla u \cdot \nabla\varphi + t^2 E(\varphi). \tag{1.5}$$

Since $E(\varphi) \geq 0$, (1.4) contains no new information. But the first condition is an orthogonality of gradients:

$$\int_\Omega \nabla u \cdot \nabla \varphi = 0 \tag{1.6}$$

for *all* test functions φ. This seems a little bizarre, but let's integrate by parts: since $\varphi|_{\partial\Omega} = 0$,

$$\int_\Omega \nabla u \cdot \nabla\varphi = -\int_\Omega \varphi\, \mathrm{div}(\nabla u). \tag{1.7}$$

However, the right-hand side takes two (weak) derivatives of u, and *a priori* we only have one. So we can't always do this. Recall that $\mathrm{div}(\nabla u)$ is the Laplacian:

$$\mathrm{div}(\nabla u) = \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right) = \sum_{i=1}^n \frac{\partial^2 u}{\partial {x_i}^2} = \Delta u. \tag{1.8}$$

So, assuming u is $W^{2,2}$, this is telling us that u is harmonic: $\int \varphi \Delta u = 0$ for all φ, hence $\Delta u = 0$.

Example 1.9. When $n = 2$, this is telling us $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. The second derivatives tell us about concavity: at any point, if you bend some amount in the x-direction, you must bend the same amount in the opposite concavity in the y-direction. For example, such a function cannot have a local maximum in the interior of Ω! ◂

This rules out pathological examples such as (1.2): minimizers can't be just anything, but their second-order derivatives have to average to zero.

Theorem 1.10. *If $u \in W^{1,2}_{\ell oc}(\Omega)$ such that $\int_\Omega \nabla u \cdot \nabla\varphi$ for all $\varphi \in C_c^\infty(\Omega)$, then $u \in C^\infty(\Omega)$. Moreover, if u minimizes the Dirichlet energy subject to the boundary condition $u|_{\partial\Omega} = u_0$ and moreover, if u_0 has small oscillations, then $|\nabla u|$ is small, and scales linearly in the oscillation of u_0.*

The first step of the proof will be to establish something called the Caccioppoli inequality.

We know that $\int_\Omega \nabla u \cdot \nabla \varphi = 0$ for all $\varphi \in C_c^\infty(\Omega)$, and hence also for all $\varphi \in W_0^{1,2}(\Omega)$, where $W_0^{1,2}(\Omega)$ denotes the closure of $C_c^\infty(\Omega)$ in $W^{1,2}(\Omega)$ in the $W^{1,2}$-norm.

Choose a $\zeta \in C_c^\infty(\Omega)$ and $\varphi = \zeta^2 u$, which vanishes on $\partial\Omega$ and is in $W_0^{1,2}(\Omega)$. Then $\nabla\varphi = 2\zeta u \nabla\zeta + \zeta^2 \nabla u$, and therefore

$$0 = \int_\Omega \nabla u \cdot \nabla\varphi = 2\int_\Omega \zeta u \nabla\zeta \cdot \nabla u + \underbrace{\int_\Omega \zeta^2 |\nabla u^2|}_{(*)}. \tag{1.11}$$

If ζ is some sort of bump function near $x_0 \in \Omega$, then $(*)$ tells you the Dirichlet energy near x_0.[1]

Rewriting (1.11),

$$\int_\Omega \zeta^2 |\nabla u|^2 = -2\int_\Omega (u\nabla\zeta)\cdot(\zeta\nabla u). \tag{1.12}$$

u is a simpler object than ∇u, and so it will be easier to control u than ∇u. We can take the Cauchy-Schwarz inequality $ab \leq a^2/2 + b^2/2$ and throw in an epsilon expressing that we're willing to pay a lot of a and only a little b:

$$\frac{a}{\varepsilon}\cdot \varepsilon b \leq \frac{|a|^2}{2\varepsilon^2} + \frac{\varepsilon^2}{2}|b|^2. \tag{1.13}$$

[1] If ζ isn't normalized, maybe we get some multiple of this, but the distinction is not important.

Apply this to (1.12):

$$\int_\Omega \zeta^2 |\nabla u|^2 = 2\int_\Omega \underbrace{(-u\nabla\zeta)}_{a}\cdot\underbrace{(\zeta\nabla u)}_{b} \le \frac{1}{\varepsilon^2}\int_\Omega u^2|\nabla\zeta|^2 + \varepsilon^2\int_\Omega \zeta^2|\nabla u|^2. \tag{1.14}$$

Plugging in $\varepsilon = 1/2$, we have:

Corollary 1.15 (Caccioppoli inequality, preliminary version)**.** *For all $\zeta \in C_c^\infty(\Omega)$,*

$$\frac{1}{2}\int_\Omega \zeta^2|\nabla u|^2 \le 2\int_\Omega u^2|\nabla\zeta|^2. \tag{1.16}$$

This is good: we have better control over $\nabla\zeta$ than ∇u.

Pick an $x_0 \in \Omega$ and some radius $R > 0$. Let's choose as our test function ζ something which is identically 1 on $B_{R/2}(x_0)$, identically 0 on the complement of $B_R(x_0)$, and always between 0 and 1. Let's additionally ask that $\nabla\zeta$ doesn't do anything stupid: we know that if ρ is the radial direction, then

$$\frac{\mathrm{d}\zeta}{\mathrm{d}\rho} > \frac{1-0}{R-R/2} = \frac{2}{R}, \tag{1.17}$$

where equality would be attained by just decreasing linearly in ρ. This isn't smooth, but we can smooth it, and therefore we can (and do) ask that $|\nabla\zeta| \le 3/R$. Therefore

$$\int_{B_{R/2}(x_0)} |\nabla u|^2 \le \int_\Omega \zeta^2|\nabla u|^2 \le \int_\Omega u^2|\nabla\zeta|^2 \le \frac{C}{R^2}\int_{B_R(x_0)}\int u^2. \tag{1.18}$$

Here $C = 9$, but that's not so important.

Corollary 1.19 (Caccioppoli inequality)**.** *If $x_0 \in \Omega$ and R is a radius such that $B_R(x_0) \subset \Omega$, and if $u \in W^{1,2}(\Omega)$ is such that $\int_\Omega \nabla u\cdot\nabla\varphi = 0$ for all $\varphi \in C_c^\infty(\Omega)$, then there's a $C > 0$ such that*

$$\int_{B_{R/2}(x_0)} |\nabla u|^2 \le \frac{C}{R^2}\int_{B_R(x_0)} u^2. \tag{1.20}$$

This is kind of a complicated statement, but the essential fact is that the square of the gradient is controlled by the square of the function, albeit on a larger region. This is very false for general functions, and is only true because of the orthogonality condition we placed on ∇u and $\nabla\varphi$.

The next thing to think about is dimentional analysis. ∇u and u have different dimensions: the former has the dimensionality of the latter divided by length. Letting $[x]$ denote the dimension of a quantity x, this means

$$\left[\int_{B_{R/2}(x_0)} |\nabla u|^2\right] = \frac{[u]^2}{(\text{length})^2}(\text{length})^n. \tag{1.21}$$

Therefore both sides of (1.20) have dimensionality $[u]^2(\text{length})^{n-2}$, which is important: you cannot get an inequality between objects of different dimensions. If you did, you made a mistake.

We'd like to conclude that if the oscillation at the boundary of u is small, then the gradient of u is small, and we have proven something related: roughly, if u is small at the boundary, its gradient is small in the interior.

Next, if u solves $\int \nabla u\cdot\nabla\varphi = 0$, so does $u + k$ for any $k \in \mathbb{R}$, so we can choose k to minimize the right-hand side of (1.20). The best possible value of k is $⨍_{B_R(x_0)} u$ (i.e. the average of u over $B_R(x_0)$). That is,

$$\int_{B_{R/2}(x_0)} |\nabla u|^2 \le \frac{C}{R^2}\int_{B_R(x_0)} |u - (u)_{B_R(x_0)}|^2. \tag{1.22}$$

This is nice, but how are we going to get C^∞ regularity? We'll leverage the Caccioppoli inequality.

Let $\rho_\varepsilon(x) := (1/\varepsilon^n)\rho(x/\varepsilon)$ bw the mollifier of ρ, where $\rho \in C_c^\infty(B_1(x_0))$ be nonnegative and have total integral 1. Then let $u_\varepsilon := u * \rho_\varepsilon$, i.e.

$$u_\varepsilon(x) = \int u(y)\rho_\varepsilon(x-y)\,\mathrm{d}y. \tag{1.23}$$

So we really need ρ defined on the set of points of distance less than ε from Ω, but that's easy to arrange.

The equation $\int_\Omega \nabla u\cdot\nabla\varphi = 0$ is linear, so if u and v both satisfy it for all $\varphi \in C_c^\infty(\Omega)$, then so does $u+v$. This allows us to prove that $\int_\Omega \nabla u_\varepsilon\cdot\nabla\varphi = 0$ for all φ — and u_ε is smooth, so this implies $\Delta u_\varepsilon = 0$.

Since u_ε is harmonic, so is an arbitrary partial derivative w of u_ε of arbitrary order.[2] Thus $\int_\Omega \nabla w \cdot \nabla\varphi = 0$ for all $\varphi \in C_c^\infty(\Omega)$ and all w.

Now we can apply the Caccioppoli inequality to w!

$$\int_{B_{R/2}(x_0)} |\nabla w|^2 \le \frac{C}{R^2}\int_{B_R(x_0)} |w|^2. \tag{1.24}$$

Letting w range over the k^{th}-order partial derivatives of order ε, this implies

$$\int_{B_{R/2}(x_0)} |\nabla^k u_\varepsilon|^2 \le \frac{C(n,k)}{R^2}\int_{B_R} |\nabla^{k-1}u_\varepsilon|^2 \le \frac{C(n,k)}{R^4}\int_{B_{2R}(x_0)} |\nabla^{k-2}u_\varepsilon|^2 < \cdots < \frac{C(n,k)}{R^{2k}}\int_{B_{2^kR}(x_0)} u_\varepsilon^2. \tag{1.25}$$

Here, as usual, $C(n,k)$ means some constant that depends on n and k, and whose value can change between expressions. We also need $\operatorname{dist}(B_R(x_0), \partial\Omega) > \varepsilon$ so that the convolution makes sense.

Now we use a dash of the theory of Sobolev spaces: Morrey's theorem tells us that if $u \in W^{k,2}(\mathbb{R}^n)$ and $k \gg 0$ (the specific value depends on n), then $u \in L^\infty(\mathbb{R}^n)$. That is, if you have control over enough derivatives, you must be bounded. We saw that $W^{1,2}$ isn't enough, but if we pile up a few more, we're good — we can get Hölder continuity, and even better, continuous partial derivatives.

Thus, there is some k (depending on n) such that

$$C(n,k)\int_{B_R(x_0)} u_\varepsilon^2 \ge R^n |u_\varepsilon|_{L^\infty(B_{R/2}(X_0))}. \tag{1.26}$$

That is, the L^2-norm of u_ε controls the L^∞-norm.

Corollary 1.27. *Letting $\varepsilon \to 0$, $u \in W^{k,2}_{\ell oc}(\Omega)$ and*

$$\int_{B_{R/2^k}(x_0)} |\nabla^k u|^2 \le CR^{-2k}\int_{B_R(x_0)} u^2. \tag{1.28}$$

Thus, $u \in W^{k,2}(\Omega)$ for all k, so $u \in C^\infty(\Omega)$.

In the last few minutes, let's talk about oscillation. The first observation is that if u is a minimizer, so is $\min(u, \sup_{\partial\Omega} u_0)$. This controls the oscillation of the boundary data:

$$\operatorname{osc}(u_0, \partial\Omega) = \operatorname{osc}(u, \Omega). \tag{1.29}$$

Returning to the Caccioppoli inequality, this tells us

$$C(n)R^n \operatorname{osc}(u_0,\partial\Omega)^2 \ge C(n)R^n\operatorname{osc}(u,\Omega)^2 \ge \int_{B_R(x_0)} |u - (u)_{B_R(x_0)}|^2 \ge \frac{R^2}{C}\int_{B_{R/2}(x_0)} |\nabla u|^2. \tag{1.30}$$

We can also apply this to partial derivatives of u in the same way as above:

$$C(n)R^n\operatorname{osc}(u_0,\partial\Omega)^2 \ge \frac{R^{2+n}}{C(n,k)}\|\nabla u\|^2_{L^\infty(B_{R/2^k}(x_0))}. \tag{1.31}$$

Lecture 2.

General examples of elliptic equations: 9/3/19

Last time, we discussed the membrane problem (implicitly as an example of an elliptic partial differential equation), and used the Caccioppoli inequality to show that $W^{1,2}$ minimizers of the membrane problem are actually smooth and have controlled oscillation.

The membrane problem is an archetypical example of a variational problem, and variational problems are rich sources of PDEs. In general, we consider some (suitably nice) function $f : \mathbb{R}^n \to \mathbb{R}$, a domain $\Omega \subset \mathbb{R}^n$, and a boundary condition $u_0 : \partial\Omega \to \mathbb{R}$. Then we want to minimize

$$E(u) := \int_\Omega f(\nabla u) \tag{2.1}$$

[2] There's a small argument to make here.

over all $u\colon \Omega \to \mathbb{R}$ wiuth $u|_{\partial\Omega} = u_0$. Formally, we can take variations:[3] for any $\varphi \in C_c^\infty(\Omega)$,

$$E(u + t\varphi) = \int_\Omega f(\nabla u + t\nabla\varphi), \tag{2.2}$$

and if u is a minimum,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} E(u + t\varphi) = 0 \qquad \text{and} \qquad \left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} E(u + t\varphi) \ge 0. \tag{2.3}$$

More explicitly,

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\nabla u(X) + t\nabla\varphi(x)) = \nabla f(\nabla u + t\nabla\varphi)\cdot\nabla\varphi \tag{2.4a}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(\nabla u + t\nabla\varphi) = \nabla\varphi\cdot\big(\nabla^2 f(\nabla u + t\nabla\varphi)\nabla\varphi\big). \tag{2.4b}$$

Let's quickly type-check this: $\nabla^2 f(\nabla u + t\nabla\varphi)$ is an $n\times n$ matrix and $\nabla\varphi \in \mathbb{R}^n$, so we obtain a dot product of two length-n vectors, which is good.

The Euler-Lagrange equation for this problem is

$$\int_\Omega \nabla f(\nabla u)\cdot\nabla\varphi = 0 \tag{2.5}$$

for all test functions φ, and the stability condition is

$$\int_\Omega \nabla^2 f(\nabla u)\nabla\varphi\cdot\nabla\varphi \ge 0, \tag{2.6}$$

again for all $\varphi \in C_c^\infty(\Omega)$.

But when should such a minimizer exist? It's a general rule of life, albeit not a formal theorem, that we need f to be *convex*, i.e. the Hessian is positive semidefinite: $\nabla^2 f(z)w\cdot w \ge 0$ for all $w, z \in \mathbb{R}^n$. When f is convec, (2.6) is automatically true.

Example 2.7. This is sort of a fake example, but still instructive. Let A be a symmetric $n\times n$ matrix and $f(z) = Az\cdot z$ for $z \in \mathbb{R}^n$. Then f is convex iff A is nonnegative definite (i.e. its eigenvalues are nonnegative).

Then $\nabla f(z) = Az$, so the Euler-Lagrange equation is $-\operatorname{div}(A\nabla u) = 0$, which is an elliptic equation with constant coefficients.

If $v(x) := u(A^{\mathrm{T}}x)$, then $\nabla v|_x = A\nabla u|_{A^{\mathrm{T}}x}$, which implies v is harmonic. So this is kind of a simple problem, but will still inform our attempts to solve more complicated ones. Also, there's a parabolic version of this question, where the coefficients are nonconstant, and this has some interesting behavior. ◀

Another important concern, especially for PDEs coming from physics, is whether minimal solutions are unique. This is related to strict convexity, i.e. the Hessian is positive definite. In this case, there's a unique minimum amongst all $W^{1,2}$ functions satisfying the boundary condition.

Exercise 2.8. Let $f(z) := \sqrt{1+|z|^2}$. Minimizing $f(\nabla u)$ means finding a graph with minimal area given a particular value on the boundary, which is called *Plateau's problem on graphs*.

Suppose we formulate this problem on an annulus, and the boundary condition is $u_0 = 0$ on the outer boundary and equal to some positive value M on the inner boundary. For M small, there's a unique minimum, which is called a *catenoidal neck*, but large enough M cause a problem, because the slope of the catenoid in a radial direction can blow up. In this case there isn't a minimizer. This is interesting — we started with a fairly simple question, with geometric meaning, but discovered complex behavior. We have no clear existence theorem, and no nice estimates on the oscillation — these things become more delicate.

Since $\nabla f(z) = z/\sqrt{1+|z|^2}$, the Euler-Lagrange equation is

$$0 = -\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right), \tag{2.9}$$

[3]This elides an important subtlety: when you take variations, you're assuming that there *is* a minimizer. For example, the problem $\sup n : n \in \mathbb{N}$ obviously has no solution, but the Euler-Lagrange equation is $n^2 = n$, which has the solution 1, falsely suggesting that we have a solution after all.

which is the *minimal surface equation*. If we write $A(x) = 1/\sqrt{1+|\nabla u(x)|^2}\mathrm{id}$, then the Euler-Lagrange equation is $0 = -\mathrm{div}(A(x)\nabla u)$, which is a nonlinear elliptic equation with variable coefficients.

Suppose u is a Lipschitz minimizer for this problem. Then Rademacher's theorem tells us $\nabla u \in L^\infty$, in fact, bounded by the Lipschitz norm L of u, and hence

$$A(x) \geq \frac{\mathrm{id}}{\sqrt{1+L^2}}. \tag{2.10}$$

This bounded-below behavior is nice. In general, an elliptic PDE $-\mathrm{div}(A(x)\nabla u) = 0$ with $A(x) \geq \lambda\mathrm{id}$ for some λ independent of x is called a *uniformly elliptic equation* (with variable coefficients); for these equations, the Caccioppoli inequality holds, and depends on λ.

Example 2.11. Let $p \in (1, \infty)$ and consider the problem of minimizing the *p-Dirichlet energy*:

$$\inf\left(\int_\Omega |\nabla u|^p : u|_{\partial\Omega} = u_0\right). \tag{2.12}$$

This seems simpler, as we're just changing the exponent. What could possibly go wrong?

Well, the Euler-Lagrange equation is more complicated:

$$A(x) = |\nabla u(x)|^{p-2}\mathrm{id}, \tag{2.13}$$

and we set $-\mathrm{div}(A(x)\nabla u) = 0$ as usual. For $1 < p < 2$, this is no longer elliptic, as $|\nabla u|$ is too large, but for $p > 2$, this is elliptic. ◀

Example 2.14. Another fundamental example: choose some function $F : \mathbb{R} \to \mathbb{R}$ and aim to minimize

$$\int_\Omega \frac{1}{2}|\nabla u|^2 - F(u) \tag{2.15}$$

subject to some fixed boundary condition as usual. We then obtain a *semi*linear Laplace equation

$$\begin{aligned} -\Delta u &= F'(u) \\ u|_{\partial\Omega} &= u_0. \end{aligned} \tag{2.16}$$

Now the stability condition (2.6) is interesting: we're asking that for all $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega |\nabla\varphi|^2 - \underbrace{F''(u)}_{=:Q(x)}\varphi^2 \geq 0. \tag{2.17}$$

The Euler-Lagrange equation for the second variation is called the *Jacobi equation* for this problem:

$$-\Delta\varphi = Q(x)\varphi \qquad \text{on } \Omega. \tag{2.18}$$ ◀

We will discuss some very general theorems and, importantly, where they may be useful. The aim is that you won't need to open a book every time you have some elliptic equation to study, but can apply these theorems.

Example 2.19 (The Poisson equation)**.** Maybe in the setting of the above example, we know something about the regularity of $F' \circ u$ — maybe it's L^5 or something like that. That's some motivating background which (TODO somehow) leads to the *Poisson equation*

$$-\Delta u = f(x), \tag{2.20}$$

which is crucially important in physics. Specifically, consider an electrostatic system (so, there are electric fields, but everything is at rest): $f(x)$ is the charge distribution at rest, and u is the electrostatic potential. Then $\mathbf{E} := -\nabla u$ is the electric field.

What if we have a smooth solution? Then we can differentiate, letting $v := \partial_{x_i} u$, and v also satisfies a different Poisson equation:

$$-\Delta v = -\frac{\partial}{\partial x_i}(\Delta u) = \frac{\partial f}{\partial x_i} = \mathrm{div}\,\mathbf{F}, \tag{2.21}$$

where $\mathbf{F} := f(x)e_i$. ◀

The basic example of an elliptic equation will be something like

$$-\operatorname{div}(A(x)\nabla u) = f(x) + \operatorname{div}\mathbf{F}(x). \tag{2.22}$$

The weak form is

$$\int_\Omega A(x)\nabla u\cdot\nabla\varphi = \int_\Omega f(x)\varphi + \mathbf{F}(x)\cdot\nabla\varphi \tag{2.23}$$

for all $\varphi\in C_c^\infty(\Omega)$. We will generally assume $f,\mathbf{F}\in L^2(\Omega)$, which means we can automatically extend to all $\varphi\in W_0^{1,2}(\Omega)$, as we did last time. We will also assume $\lambda\mathrm{id}\le A(x)\le\Lambda\mathrm{id}$ for some $\lambda>0$ and $\Lambda<\infty$.

The Caccioppoli inequality ultimately is similar in spirit to the more specific version we proved last time in Corollary 1.19, but the differences aren't trivial.

Theorem 2.24 (Caccioppoli-Leray inequality). *Let $u\in W^{1,2}_{\ell oc}(\Omega)$ satisfy (2.23). Then*

$$\int_{B_{R/2}}|\nabla u|^2 \le C\left(n,\frac{\Lambda}{\lambda}\right)\left(\frac{1}{R^2}\int_{B_R}|u-(u)_{B_R}|^2 + R^2\int_{B_R}f^2 + \int_{B_R}|\mathbf{F}|^2\right) \tag{2.25}$$

for all balls $B_R\Subset\Omega$ of radius R, where $B_{R/2}$ denotes the ball with the same center and radius $R/2$ and $(u)_{B_R} := \fint_{B_R} u$.

Let's do some dimensional analysis: $-\Delta u$ has dimensions $[u]/L^2$, so f and $\operatorname{div}\mathbf{F}$ must as well, since $-\Delta u = f+\operatorname{div}\mathbf{F}$. Hence $[\mathbf{F}]=[u]/L$ and $[f]^2=[u^2]/L^4$. Therefore both sides of the Caccioppoli-Leray inequality have dimensioons $[u]^2/L^2$.

Consider $-\Delta u = f$ and differentiate: $-\Delta(\partial_{x_i}u) = \partial_{x_i}f$, so looking just at a Poisson equation with source $\partial_{x_i}f$, we get an estimate

$$\int\left|\nabla\left(\frac{\partial u}{\partial x_i}\right)\right|^2 \lesssim \int\left|\frac{\partial f}{\partial x_i}\right|^2, \tag{2.26}$$

but we could also look at it as a Poisson with the right-hand side as the divergence, and then we get a much better estimate:

$$\int\left|\nabla\left(\frac{\partial u}{\partial x_i}\right)\right|^2 \lesssim \int f^2. \tag{2.27}$$

This illustrates a general principle: in other settings one can try to fold terms into the divergence part or the source part, and in some settings it can be important to consider both perspectives.

Proof of Theorem 2.24. Just as last time, let $\varphi := \zeta^2 u$, where $\zeta\in C_c^\infty(\Omega)$. In particular, we will let ζ be a bump functions, so it's 1 on $B_{R/2}$, 0 outside of B_R, always between 0 and 1, and satisfies $|\nabla\zeta|\le 1/2$. Then

$$\nabla\varphi = 2\zeta u\nabla u + \zeta^2\nabla u, \tag{2.28}$$

and the bounded-below condition $A(x)\ge\lambda\mathrm{id}$ means $A(x)z\cdot z\ge\lambda|z|^2$ for all $z\in\mathbb{R}^n$. In particular, try $z=\nabla u$ and multiply by ζ^2:

$$\lambda\int_\Omega\zeta^2|\nabla u|^2 \le \int_\Omega A(x)\nabla u\cdot\nabla u\zeta^2 \tag{2.29}$$

$$= \underbrace{-2\int_\Omega \zeta u A(x)\nabla u\cdot\nabla\zeta}_{(I)} + \int_\Omega f\zeta^2 u + \int_\Omega\zeta^2\mathbf{F}\cdot\nabla u + \int_\Omega 2u\zeta\mathbf{F}\cdot\nabla\zeta. \tag{2.30}$$

This looks bad, since there are four terms, but they are terms that we can control. We have control over $|\nabla u|$ (since that's what we fed to z, and maybe we'll get a different λ but it's OK) and $|\nabla\zeta|$, so $(I)\le\Lambda|\nabla u||\nabla\zeta|$, so we get

$$(I)\lesssim\varepsilon\int_\zeta^2|\nabla u|^2 + \frac{1}{\varepsilon}\int u^2|\nabla\zeta|^2 \tag{2.31}$$

using a Cauchy-Schwarz inequality as before. In other words,

$$\int_\Omega f\zeta^2 u \lesssim R^2\int_\Omega f^2\zeta^2 + \frac{1}{R^2}\int_\Omega\zeta^2u^2. \tag{2.32}$$

Now we turn to the remaining terms of (2.30). We can bound the last term as

$$2u\zeta\mathbf{F}\cdot\nabla\zeta \lesssim \zeta^2u^2|\nabla\zeta|^2 + \zeta^2|\mathbf{F}|^2. \tag{2.33}$$

but the rest went by too quickly for me; sorry about that. (TODO) ⊠