SYMMETRIC PRODUCT SPECTRA AND SPLITTINGS OF CLASSIFYING SPACES

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Introduction. The infinite symmetric product spectrum of the zero sphere—denoted $Sp^{\infty}S^0$—is equivalent to the Eilenberg-Maclane spectrum **KZ**, by the Dold-Thom theorem ([5]). Moreover, the filtration of $Sp^{\infty}S^0$ by the 2^k-fold symmetric products $Sp^{2^k}S^0$ corresponds to the length filtration on the mod 2 cohomology of **KZ** (Nakaoka [12]). These finite symmetric products have been studied by Welcher ([14]). In this paper, we show that the two-fold desuspension of the quotient spectrum $\overline{Sp}^4S^0 = Sp^4S^0/Sp^2S^0$ occurs as a stable summand of certain classifying spaces BG:

THEOREM A. *Let D_n denote the dihedral group of order 2^n, $n \geq 2$, and let q be an odd prime power with $n = \nu_2((q^2 - 1)/2)$. Then there is a 2-local stable equivalence*

$$BD_n \cong BPSL_2\mathbf{F}_q \vee \Sigma^{-2}\overline{Sp}^4S^0 \vee \Sigma^{-2}\overline{Sp}^4S^0 \vee P^{\infty} \vee P^{\infty}.$$

Here $P^{\infty} = B\mathbf{Z}/2$. The groups $PSL_2\mathbf{F}_q$ arise because D_n is a 2-Sylow subgroup of $PSL_2\mathbf{F}_q$. We note the following special cases:

(0.1) If $n = 2$, then $D_2 = (\mathbf{Z}/2)^2$. We can take $q = 3$, so that $PSL_2\mathbf{F}_3 \cong A_4$. Thus (A) gives a splitting of $P^{\infty} \times P^{\infty}$.

(0.2) If $n = 3$ then $D_3 = \mathbf{Z}/2\int\mathbf{Z}/2$ and we can take $q = 7$. Since $\mathbf{Z}/2\int\mathbf{Z}/2$ is a 2-Sylow subgroup of Σ_4, $B\Sigma_4$ is a retract of $\mathbf{Z}/2\int\mathbf{Z}/2$, and one can ask how $B\Sigma_4$ lies with respect to the splitting.

THEOREM B. *There is a 2-local stable equivalence*

$$B\Sigma_4 \cong BPSL_2\mathbf{F}_7 \vee \Sigma^{-2}\overline{Sp}^4S^0 \vee P^{\infty}.$$

Manuscript received February 5, 1982.

*Partially supported by an Achievement Rewards for College Scientists fellowship.

**Partially supported by NSF Grant MCS 78-27592.

Since $D_n \subset D_{n+1} \subset 0(2)$, we can form the direct limit $\varinjlim D_n = D_\infty$, and there is a map $BD_\infty \to BO(2)$ which is easily seen to be a stable equivalence after 2-adic completion. This suggests the following infinite version of Theorem A:

THEOREM C. *After 2-adic completion, there is a stable equivalence* $BO(2) \cong BSO(3) \vee \Sigma^{-2}\overline{Sp}^4 S^0 \vee P_1^\infty$.

The proof of Theorem C shows that $\Sigma^{-2}\overline{Sp}^4 S^0 \vee P_1^\infty \cong \Sigma^{-1}(BSO(3)/BO(2))$; $O(2)$ is contained in $SO(3)$ as the normalizer of a maximal torus. This suggests the following analogue of Theorem A:

THEOREM D. *Let Q_n be the quaternion group of order $2^n (n \geq 3)$ and let $N \subset S^3$ be the normalizer of a maximal torus. Then there is a 2-local stable equivalence*

$$BQ_n \cong BSL_2\mathbf{F}_q \vee \Sigma^{-1}(BS^3/BN) \vee \Sigma^{-1}(BS^3/BN),$$

where q is an odd prime power with $n = \nu_2(q^2 - 1)$.

In a future paper, we will prove the following partial generalization of 0.1 and 0.2.

THEOREM E. $\Sigma^{-n}\overline{Sp}^{2^n} S^0$ *is a stable summand of $B(\mathbf{Z}/2)^n$ and of $B(\int^n \mathbf{Z}/2)$. ($\int^n \mathbf{Z}/2$ is the wreath product of n copies of $\mathbf{Z}/2$).*

Except in Section 1, all cohomology groups are with $\mathbf{Z}/2$ coefficients and all spectra are implicitly completed at 2. Dimensions of the generators of the various group cohomology rings are indicated by a subscript, unless the dimension is one. In particular, x and y will always denote the generators of $H^1B(\mathbf{Z}/2)^2$.

Some special cases of the results of this paper were contained in the first author's thesis. He would like to thank his advisor, Doug Ravenel, for his generous help and guidance.

1. Preliminaries. In this section we fix a prime p. All cohomology groups are with $\mathbf{Z}/p$-coefficients, and all spectra are implicitly completed at p.

Let H be a subgroup of the finite group G. Let $i_{H,G}$ denote the inclusion $H \to G$, and let $t_{H,G}$ denote the transfer, which is a stable map $BG_+ \to BH_+$ (one or both subscripts will be omitted if the groups are clear from the context). Since $H^*BG = H^*G$ (group cohomology), we frequently write i^* in place of $(Bi)^*$. Recall (cf. [2]).

PROPOSITION 1.1. *On ordinary cohomology, $(it)^*$ is multiplication by $[G:H]$.*

COROLLARY. *If $p \nmid [G:H]$, BG is a stable summand of BH.*

Now let $N_G(H)$ denote the normalizer of H in G, and recall that the action of $N_G(H)$ on H induces an action of the Weyl group $W_G(H) = N_G(H)/H$ on the homotopy type BH. Whenever a group Γ acts on a module M, M^Γ denotes the module of Γ-invariants.

PROPOSITION 1.2. $\operatorname{Im} i^* \subseteq (H^*BH)^{W_G(H)}$.

If L is another subgroup of G, $i_L^* t_H^*$ is computed by the double coset formula [2]:

PROPOSITION 1.3. *Let $G = \cup_{i=1}^n Lx_iH$, $x_i \in G$, be a double coset decomposition of G. Let $L_i = L \cap x_i^{-1}Hx_i$, $H_i = x_iL_ix_i^{-1}$, and let ϕ_i denote the isomorphism $L_i \to H_i$ given by conjugation by x_i^{-1}. Then $i_L^* t_H^* = \Sigma_{i=1}^n t_{L_i,L}^* \phi_i^* i_{H_i,H}^*$.*

Now if L is an elementary Abelian p-group, $t_{K,L}^*$ is zero for any proper subgroup K. Hence in the double coset formula we need only consider the terms with $L_i = L$. In particular:

PROPOSITION 1.4. *If L is an elementary Abelian p-group not conjugate to a subgroup of H, $i_L^* t_H^* = 0$. If $L = H$, then $i_L^* t_H^*$ is multiplication by $\Sigma_{w \in w_G(L)} w$.*

Now suppose Γ is a subgroup of Aut H. Then Γ acts on the space BH (on the left). In the stable category maps can be added and compositions distribute over sums; hence the integral group ring $\mathbf{Z}[\Gamma]$ acts on the suspension spectrum BH. Since all our spectra have been p-adically completed, this action extends to the p-adic group ring $\mathbf{Z}_p[\Gamma]$. Consequently, as is well known, one can use idempotents in $\mathbf{Z}_p[\Gamma]$ to obtain splittings of BH; in fact, one can even use idempotents in $\mathbf{Z}/p[\Gamma]$:

PROPOSITION 1.5 (cf. F. R. Cohen [3]). *Suppose $\pi_1, \ldots, \pi_n$ are orthogonal idempotents in $\mathbf{Z}/p[\Gamma]$ with $\Sigma\pi_i = 1$. Then $BH \cong \vee_{i=1}^n X_i$ (stably), where $H^*X_i = (H^*BH)\pi_i$.*

(The proof is easy: one takes X_i = mapping telescope of $\tilde{\pi}_i$, where $\tilde{\pi}_i \in \mathbf{Z}_p[\Gamma]$ with $\tilde{\pi}_i = \pi_i \bmod p$.)

The simplest example of (1.5) is obtained by taking $H = \mathbf{Z}/p$, $\Gamma = \mathbf{Z}/(p-1)$ ([7], [10]): Fix a generator T of $\mathbf{Z}/(p-1)$. Then there are orthogonal idempotents $\pi_0, \ldots, \pi_{p-2}$ splitting $\mathbf{Z}_p[\mathbf{Z}/(p-1)]$ into the $p-1$ eigenspaces of T. Thus $B\mathbf{Z}/p$ splits into "eigenspectra" X_i,

$0 \leq i \leq p-2$. Moreover it is well known (and easy to show, using (1.2)) that X_0 (with eigenvalue 1) is $B\Sigma p$.

If π is a primitive central idempotent in $\mathbf{Z}/p[\Gamma]$, the two-sided ideal Im π is a block; i.e., $\mathbf{Z}/p[\Gamma] \cong (\operatorname{Im} \pi) \times \operatorname{Im}(1-\pi)$ (as algebras), and Im π cannot be further decomposed in this manner. If Im π happens to be a matrix ring of degree n, then π splits (non-canonically) into primitive orthogonal (non-central) idempotents $\rho_i : \pi = \Sigma_{i=1}^n \rho_i$. If X corresponds to π as in (1.5), then of course $X \cong \vee_{i=1}^n X_i$ with $H^*X_i = (H^*BH)\rho_i$. Moreover, we have the following elementary fact:

PROPOSITION 1.6. *Let ρ_1, ρ_2 be any two primitive idempotents in a matrix ring block B of $\mathbf{Z}/p[\Gamma]$. Then the corresponding stable summands X_1, X_2 of BH are equivalent.*

Proof. We have $\rho_1 = \lambda^{-1} p_2 \lambda$ for some unit $\lambda \in B$. Hence multiplication by λ on cohomology (on the right, as always) is an isomorphism Im $\rho_2 \to$ Im ρ_1. Lifting λ to a unit in $\mathbf{Z}/p[\Gamma]$ and then to any element of $\mathbf{Z}_p[\Gamma]$ yields the desired equivalence.

2. Cohomology of $PSL_2\mathbf{F}_q$. A good reference for this section is [6], Chapter 6.

The dihedral group D_n has generators s, t and relations $s^{2^{n-1}} = 1 = t^2$, $tst = s^{-1}$. For $n > 2$ the center is generated by $r = s^{2^{n-2}}$, and there are (up to conjugation) two $\mathbf{Z}/2 \oplus \mathbf{Z}/2$ subgroups: K, with basis r, t and T, with basis r, st. K and T are not conjugate, but the outer involution η of D_n defined by $\eta(s) = s^{-1}$, $\eta(t) = st$ maps T onto K. (When $n = 2$, $K = T$ and η is the automorphism with matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$). The commutator subgroup D_n' is generated by s^2, so that $D_n/D_n' \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2$ with basis s, t. Let a, b denote the dual base for $H^1(D_n)$, and let θ denote the obvious representation of D_n in $0(2)$.

PROPOSITION 2.1. *For $n > 2$, $H^*D_n = \mathbf{Z}/2[a, b, w_2]/(a^2 + ab)$, where $w_2 = w_2(\theta)$. Moreover $\{K, T\}$ is a detecting family of subgroups, and the image of H^*D_n in $H^*K \oplus H^*T$ is given by the following table:*

	a	b	w_2
K	0	y	$x^2 + xy$
T	y	y	$x^2 + xy$

Remark. By "detecting family" we mean that $\operatorname{Ker} i_K^* \cap \operatorname{Ker} i_T^* = 0$, where i_K, i_T are the inclusions. (2.1) is easily verified using the spectral sequence of the extension $D_n' \to D_n \to \mathbf{Z}/2 \oplus \mathbf{Z}/2$.

Let Q_{n+1} denote the generalized quaternion group of order 2^{n+1}, $n \geq 2$. Q_{n+1} has generators s, t and relations $s^{2^{n-1}} = t^2$, $tst^{-1} = s^{-1}$. The center has order two with generator t^2, so there is an extension $\mathbf{Z}/2 \xrightarrow{i} Q_{n+1} \xrightarrow{\pi} D_n$. $Q_{n+1}/Q_{n+1}' \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2$ with basis s, t and we let a, b denote the dual base in H^*Q_{n+1} (thus $\pi^*a = a$, etc.). There is an obvious faithful representation Ψ of Q_{n+1} in the group S^3 of unit quaternions, which is a subgroup of $0(4)$.

Proposition 2.2. *$H^*Q_{n+1} = \mathbf{Z}/2[a, b, w_4]/R$, where $w_4 = w_4(\Psi)$ and*

$$R = (a^2 + ab + b^2, a^2b + ab^2) \qquad (\textit{if } n = 2)$$

$$= (a^2 + ab, b^3) \qquad (\textit{if } n > 2).$$

Proof. We use the spectral sequence of the central extension $\mathbf{Z}/2 \xrightarrow{i} Q_{n+1} \xrightarrow{\pi} D_n$. The differentials in this spectral sequence are most easily computed by considering the commutative diagram of representations

$$\text{(2.3)} \qquad \begin{array}{ccc} Q_{n+1} & \xrightarrow{\Psi} & S^3 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \rho} \\ D_n & \xrightarrow{\theta'} & SO(3) \end{array}$$

where $\theta' = \theta \oplus \det \theta$ and ρ is the usual double covering (explicitly, ρ is obtained from the action by conjugation of S^3 on the subspace of the quaternions spanned by i, j, k). One easily checks that $w_2(\theta') = w_2 + b^2$ and $w_3(\theta') = w_2 b$ (where w_2 is replaced by $a^2 + ab$ if $n = 2$). Now let $e \in H^1(\mathbf{Z}/2)$ be the generator. From (2.3) we have

$$\pi^*(w_2(\theta')) = \pi^*(w_3(\theta')) = 0,$$

forcing the differentials $d_2 e = w_2 + b^2$, $d_3 e^2 = w_2 b$. On the other hand e^4 is a permanent cycle, since clearly $i^*(w_4(\Psi)) = e^4$. Hence the spectral sequence collapses at E_4. The proposition easily follows if one notes that $w_2 + b^2$ and $w_2 b$ are not zero divisors in the E_2 and E_3 terms, respectively.

Let q be an odd prime power. $SL_2\mathbf{F}_q$ has order $q(q^2-1)$ and contains Q_{n+1} as a 2-Sylow subgroup, where $n+1 = \nu_2(q^2-1)$. Thus

$$H^*SL_2\mathbf{F}_q \subseteq H^*Q_{n+1}.$$

THEOREM 2.4 (Quillen [13]). $H^*SL_2\mathbf{F}_q = \mathbf{Z}/2[c_3, w_4]/(c_3^2)$, *where* $c_3 = ab^2$.

Note there is a commutative diagram

(2.5)
$$\begin{tikzcd} \mathbf{Z}/2 \arrow[r] \arrow[d, equal] & SL_2\mathbf{F}_q \arrow[r, "\bar{\pi}"] & PSL_2\mathbf{F}_q \\ \mathbf{Z}/2 \arrow[r] & Q_{n+1} \arrow[r, "\pi"] \arrow[u, "i"] & D_n \arrow[u, "j"] \end{tikzcd}$$

where i, j are inclusions of 2-Sylow subgroups.

THEOREM 2.6. *There are elements* $v_2, v_3, \bar{v}_3 \in H^*PSL_2\mathbf{F}_q$ *such that* $H^*PSL_2\mathbf{F}_q \cong \mathbf{Z}/2[v_2, v_3, \bar{v}_3]/R$, where $R = (\bar{v}_3^2 + v_3\bar{v}_3)$ *if* $q = \pm 1 \bmod 8$ *and* $R = (\bar{v}_3^2 + v_3^2 + v_3\bar{v}_3 + v_2^3)$ *if* $q = \pm 3 \bmod 8$. Moreover, the image of $H^*PSL_2\mathbf{F}_q$ in $H^*K \oplus H^*T$ is given by the following table, with $z_2 = x^2 + xy + y^2$, $z_3 = x^2y + xy^2$.

$(q = \pm 1 \bmod 8)$

	v_2	v_3	$\bar{v}_3$
K	z_2	z_3	0
T	z_2	z_3	z_3

$(q = \pm 3 \bmod 8)$

	v_2	v_3	$\bar{v}_3$
K	z_2	z_3	$x^3 + x^2y + y^3$
T	--	--	-----

(Note that $q = \pm 3 \bmod 8$ is the case $n = 2$, so $K = T$.)

Proof. Consider the spectral sequence of $\mathbf{Z}/2 \to SL_2\mathbf{F}_q \xrightarrow{\bar{\pi}} PSL_2\mathbf{F}_q$. By (2.4) there are non zero elements $v_2 = d_2e$ and $v_3 = d_3e^2$. Combining (2.3) and (2.5) shows $j^*v_2 = w_2 + b^2$, $j^*v_3 = w_2b$; hence v_2, v_3 restrict to K, T as claimed by (2.1).

Furthermore, there must exist $\bar{v}_3 \in H^3PSL_2\mathbf{F}_q$ such that $\pi^*\bar{v}_3 = c_3$.

Case 1. $q = \pm 1 \bmod 8$. In dimension 3 we have $\operatorname{Ker} \pi^* = \langle b^3, bw_2, ab^2 + aw_2 \rangle$, and $\pi^*aw_2 = ab^2 = c_3$. Since $j^*v_3 = bw_2$, we can choose $\bar{v}_3$

so that $j^*\bar{v}_3 = aw_2 + \alpha(ab^2 + aw_2) + \beta b^3$ for some $\alpha, \beta \in \mathbf{Z}/2$. In fact $\alpha = \beta = 0$: For clearly $H^4PSL_2\mathbf{F}_q = \mathbf{Z}/2$ on v_2^2, and hence $Sq^1\bar{v}_3$ is either zero or v_2^2. But $j^*Sq^1\bar{v}_3 = \alpha a^2b^2 + \beta b^4 \neq j^*v_2^2$, so $Sq^1\bar{v}_3 = 0$ and $\alpha = \beta = 0$. Thus $j^*\bar{v}_3 = aw_2$ and $\bar{v}_3$ restricts to K, T as claimed. Restricting to $H^*K \oplus H^*T$ now shows that $\mathbf{Z}/2[v_2, v_3, \bar{v}_3]/R \subseteq H^*PSL_2\mathbf{F}_q$, and inspection of the spectral sequence, using (2.4), shows this inclusion must be an equality.

Case 2. $q = \pm 3 \bmod 8$. Then $n = 2$ and $D_2 = (\mathbf{Z}/2)^2$, so $H^*D_2 = \mathbf{Z}/2[a, b]$. Since (in dimension 3) Ker $\pi^* = \langle a^3, b^3, a^2b + ab^2\rangle$, $\pi^*ab^2 = c_3$ and $j^*v_3 = a^2b + ab^2$, we can choose $\bar{v}_3$ so that $j^*\bar{v}_3 = ab^2 + \alpha a^3 + \beta b^3$ for some $\alpha, \beta \in \mathbf{Z}/2$. As in the first case, $Sq^1\bar{v}_3$ is either zero or v_2^2. Since $j^*Sq^1\bar{v}_3 = a^2b^2 + \alpha a^4 + \beta b^4 \neq 0$, $Sq^1\bar{v}_3 = v_2^2$ and $\alpha = \beta = 1$. Restricting to D_2 shows $\bar{v}_3^2 + v_3^2 + v_3\bar{v}_3 + v_2^3 = 0$, and the theorem follows easily as in Case 1.

3. Symmetric product spectra. If X is a space, Sp^nX is the orbit space of the obvious Σ_n action on X^n. If X has a basepoint $*$, there are natural inclusions $Sp^nX \to Sp^{n+1}X$ obtained by inserting $*$ as the extra coordinate. These constructions carry over to spectra in a straightforward way (cf. [14]). In particular, one can define

$$Sp^nS^0 \quad \text{and} \quad Sp^\infty S^0 = \varinjlim_n Sp^nS^0.$$

By the Dold-Thom theorem [5], $Sp^\infty S^0$ is the integral Eilenberg-MacLane spectrum $K\mathbf{Z}$, so that $H^*Sp^\infty S^0 = A/ASq^1$. (Here A is the mod 2 Steenrod algebra.)

THEOREM 3.1 (Nakaoka [12]). *The inclusions $Sp^nS^0 \to Sp^\infty S^0$ are surjective on H^*, and $H^*Sp^{2^n}S^0$ has a basis given by the Sq^I with I admissible, length of $I \leq n$, and $Sq^I \notin ASq^1$. In particular, the Poincaré series of $H^*\overline{Sp}^{2^n}S^0$* is $t^{2^{n+1}-2}/\Pi_{i=1}^n (1 - t^{2^i-1})$.

The next result is well known and essentially contained in [8]. The Thom spectrum of $-\lambda$, where λ is the canonical line bundle over P^∞, is denoted P_{-1}^∞.

THEOREM 3.2. *Let $\Delta: S^0 \to Sp^2S^0$ be the map induced by the diagonals $S^n \to S^n \times S^n$. Then there is a cofibration $S^0 \xrightarrow{\Delta} Sp^2S^0 \xrightarrow{h} \Sigma P_{-1}^\infty$. Moreover, there is a commutative diagram*

$$\begin{tikzcd} Sp^2S^0 \arrow[r] \arrow[d, "h"] & \overline{Sp}^2S^0 \arrow[d, "\bar{h}"] \\ \Sigma P_{-1}^{\infty} \arrow[r, "c"] & \Sigma P_1^{\infty} \end{tikzcd}$$

where c is the usual pinch map, such that $\bar{h}$ is an equivalence. (*The main point is that the complement of the diagonal in Sp^2S^n is actually homeomorphic to the total space of the negative of the canonical line bundle over* $\mathbf{R}P^n$; *this is an amusing exercise.* The rest of (3.2) *follows easily.*)

Next, observe that the ring spectrum multiplication $Sp^{\infty}S^0 \wedge Sp^{\infty}S^0 \to Sp^{\infty}S^0$ can be described in the following filtered form: Define space maps $Sp^iS^m \wedge Sp^jS^n \to Sp^{ij}S^{m+n}$ by $(x_1, \ldots, x_i) \wedge (y_1, \ldots, y_j) \mapsto (x_1 \wedge y_1, \ldots, x_i \wedge y_j)$. These yield a map of spectra $Sp^iSp^0 \wedge Sp^jS^0 \to Sp^{ij}S^0$, and on iteration a map $\mu: \wedge^n Sp^2S^0 \to Sp^{2^n}S^0$.

PROPOSITION 3.3. *There is a commutative diagram*

$$\begin{tikzcd} \wedge^n Sp^2S^0 \arrow[r, "\mu"] \arrow[d, "\wedge^n g"] & Sp^{2^n}S^0 \arrow[d] \\ \Lambda^n \Sigma P_1^{\infty} \arrow[r, "f_n"] & \overline{Sp}^{2^n}S^0 \end{tikzcd}$$

where $g = ch$.

Proof. From (3.2) the fibre of $\wedge^n g$ is the union of the subspectra $E_k = (\wedge^k Sp^2S^0) \wedge S^0 \wedge (\wedge^{n-k-1}Sp^2S^0)$, $0 \le k < n$. One can easily check that $\mu(E_k) \subseteq Sp^{2^n-1}S^0$.

The diagram of (3.3) greatly facilitates the computation of f_n^*, since $H^*Sp^{2^n}S^0$ is cyclic over the Steenrod algebra. For $n = 2$ this is particularly easy, and we record the computation for future reference. Let f denote the composite $B(\mathbf{Z}/2)^2 \to P_1^{\infty} \wedge P_1^{\infty} \xrightarrow{f_2} \Sigma^{-2}\overline{Sp}^4S^0$. From 3.1, $H^*\overline{Sp}^4S^0$ has basis $b_{j,k} = Sq^{j+k+1}Sq^{k+1}$, $j > k > 0$.

PROPOSITION 3.4. *Let $a_{j,k} = (x + y)^k(x^jy^k + x^ky^j) \in H^*B(\mathbf{Z}/2)^2$. Then $f^*b_{j,k} = a_{j,k}$, and f^* is injective.*

Proof. We can regard $H^*(Sp^2S^0 \wedge Sp^2S^0)$ as a quotient of $H^*P_{-1}^{\infty} \wedge P_{-1}^{\infty}$, and we therefore write $x^{-1}y^{-1}$ for the generator in dimension zero. By (3.3) it is enough to compute μ^* and we have for $j > k > 0$:

$$\begin{aligned}\mu^*(b_{j,k}) &= Sq^{j+k+1}Sq^{k+1}(x^{-1}y^{-1})\\ &= Sq^{j+k+1}(x^k y^{-1} + x^{-1}y^k)\\ &= (x+y)^k x^k y^j + (x+y)^k x^j y^k\\ &= a_{j,k}.\end{aligned}$$

Since the $a_{j,k}$ are clearly independent if $j > k > 0$, f^* is injective.

4. Splitting BD_n. In this section we prove Theorem A. It will be useful to consider first the case $n = 2$. We identify $B(\mathbf{Z}/2)^2$ with $P^\infty \times P^\infty$, $(\mathbf{Z}/2)^2$ with $H_1 B(\mathbf{Z}/2)^2$, and $\operatorname{Aut}(\mathbf{Z}/2)^2$ with $GL_2 = GL_2(\mathbf{F}_2)$ in the usual way. GL_2 is isomorphic to Σ_3, and has generators $\sigma = \left(\begin{smallmatrix}0 & 1\\ 1 & 1\end{smallmatrix}\right)$ and $\tau = \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$, with relations $\sigma^3 = \tau^2 = (\sigma\tau)^2 = 1$. As usual, $x, y \in H^*B(\mathbf{Z}/2)^2$ denotes the basis dual to $(1, 0)$, $(0, 1)$. With respect to this basis, 2 by 2 matrices act on the right of H^*.

The central idempotents $\pi_1 = 1 + \sigma + \sigma^2$ and $\pi_2 = \sigma + \sigma^2$ split $\mathbf{Z}/2[GL_2]$ into blocks: $\mathbf{Z}/2[GL_2] = \operatorname{Im}\pi_1 \times \operatorname{Im}\pi_2$. Moreover one can easily check that $\operatorname{Im}\pi_1$ is an exterior algebra on one generator, while the standard representation of GL_2 on $(\mathbf{Z}/2)^2$ induces an isomorphism $\operatorname{Im}\pi_2 \cong \operatorname{End}_{\mathbf{Z}/2}(\mathbf{Z}/2)^2$. Hence π_2 splits further as in Section 1; specifically, if $\rho_1 = (1 + \tau\sigma)(1 + \tau)$ and $\rho_2 = (1 + \tau)(1 + \tau\sigma)$, then $\pi_2 = \rho_1 + \rho_2$, $\rho_1\rho_2 = 0 = \rho_2\rho_1$, and ρ_1, ρ_2 are (non-central) idempotents.

LEMMA 4.1. *$B(\mathbf{Z}/2)^2 \cong W \vee X_1 \vee X_2$ where $H^*W = \operatorname{Im}\pi_1 = (H^*B(\mathbf{Z}/2)^2)^{\mathbf{Z}/3}$, and $H^*X_i = \operatorname{Im}\rho_i$ for $i = 1, 2$. Moreover, $X_1 \cong X_2$, and if q is an odd prime power with $q = \pm 3 \bmod 8$, $W \cong BPSL_2\mathbf{F}_q$ (stably, at 2).*

Proof. Except for the last statement, this is immediate from Section 1. When $q = 3$, the Serre spectral sequence of the extension $(\mathbf{Z}/2)^2 \to A_4 \to \mathbf{Z}/3$ shows that $H^*A_4 = (H^*B(\mathbf{Z}/2)^2)^{\mathbf{Z}/3}$. It follows then from (2.6) that $\operatorname{Im}\pi_1$ is the subring generated by z_2, z_3, $\bar{z}_3$ (this can easily be proved directly). Hence if $q = \pm 3 \bmod 8$, the composite $W \to B(\mathbf{Z}/2)^2 \to BPSL_2\mathbf{F}_q$ is an equivalence.

Now let $g = (f, r_1\rho_1)\colon (P^\infty \times P^\infty \to \Sigma^{-2}\overline{Sp}^4 S^0 \vee P_1^\infty$, where f is the map of Section 3 and r_1 is projection on the first factor.

LEMMA 4.2. *g^* is an isomorphism onto* $\operatorname{Im}\rho_1$. *Thus g restricted to X_1 is an equivalence, which completes the proof of Theorem* A *for $n = 2$.*

Proof. Recall from (3.4) that f^* is injective with image $\langle a_{j,k} : j > k > 0\rangle$. On the other hand $\rho_1^* r_1^* e^j = \rho_1^* x^j = x^j + y^j = a_{j,0}$. Direct computation shows $a_{j,k}\rho_1 = a_{j,k}$ for all j, k; hence g^* is injective with $\operatorname{Im} g^* \subseteq \operatorname{Im}\rho_1^*$. But $\operatorname{Im}\rho_1^*$ has Poincaré series

$$\frac{1}{2}\left(\frac{1}{(1-t)^2} - \frac{1+t^3}{(1-t^2)(1-t^3)}\right) = \frac{t}{(1-t)(1-t^3)}$$

by (4.1) and (2.6), so $\operatorname{Im} g^* = \operatorname{Im}\rho_1^*$ follows by comparing Poincaré series.

Now suppose $n > 2$ (so $q = \pm 1 \bmod 8$), and let $F : BD_n \to BPSL_2\mathbf{F}_q \vee \Sigma^{-2}\overline{Sp}^4 S^0 \vee \Sigma^{-2}\overline{Sp}^4 S^0 \vee P_1^\infty \vee P_1^\infty$ have components $(Bj, ft_K, ft_T, r_1 t_K, r_1 t_T)$, where $j : D_n \to PSL_2\mathbf{F}_q$ denotes inclusion, $t_K, t_T : BD_n \to B(\mathbf{Z}/2)^2$ are the transfer maps (note that equivalences BK, $BT \cong B(\mathbf{Z}/2)^2$ are specified by our choice of bases for K, T in Section 2), and r_1 is the projection as before. We will show that F^* is an isomorphism. It is enough to show F^* is injective, since the Poincaré series of its target and source are equal:

$$\frac{1}{(1-t)^2} = \frac{1+t^3}{(1-t^2)(1-t^3)} + \frac{2t}{(1-t)(1-t^3)}.$$

Let $b_{j,k}$ (resp. $b'_{j,k}$) denote $Sq^{j+k+1}Sq^{k+1}$, $j > k > 0$, in the first (resp. second) copy of $\Sigma^{-2}\overline{Sp}^4$, and let $b_{j,0}$ (resp. $b'_{j,0}$) denote e^j in the first (resp. second) copy of P_1^∞.

LEMMA 4.3. *$(i_K^*, i_T^*) \cdot F^*$ is given by the following table*:

	v_2	v_3	$\bar{v}_3$	$b_{j,k}$	$b'_{j,k}$
K	z_2	z_3	0	$a_{j,k}\sigma^2$	0
T	z_2	z_3	z_3	0	$a_{j,k}\sigma^2$

Proof. The first three columns were obtained in Section 2. For the last two, observe that $W_{D_n}(K) = W_{D_n}(T) = \mathbf{Z}/2$; in both cases the generator is $s^{2^{n-3}}$ acting on $(\mathbf{Z}/2)^2$ as $\tau\sigma$. Hence by (1.4), $i_T^* t_T^* = i_K^* t_K^* = 1 + \tau\sigma$, while $i_K^* t_T^* = 0 = i_T^* t_K^*$. Since $a_{j,k}(1+\tau\sigma) = y^k(x^j(x+y)^k + x^k(x+y)^j) = a_{j,k}\sigma^2$, and $x^j(1+\tau\sigma) = x^j + (x+y)^j = a_{j,0}\sigma^2$, this completes the proof.

Inspection of the table now shows that F^* is injective. Indeed, since we have already shown that j^* and f^* are injective, the only nontrivial point to check is that Im j^* is independent of the span of the images of $b_{j,k}$ and $b'_{j,k}$. But in (4.1) and (4.2) we obtained a splitting of GL_2-modules $H^*B(\mathbf{Z}/2)^2 = \operatorname{Im} \pi_1 \oplus \operatorname{Im} \pi_2$, with $z_2, z_3 \in \operatorname{Im} \pi_1$ and $a_{j,k} \in \operatorname{Im} \pi_2$. Hence this point is also clear from the table, and the proof of Theorem A is complete.

5. Splitting $B\Sigma_4$ and $BO(2)$. Since $PSL_2\mathbf{F}_7 \cong GL_3\mathbf{F}_2$ ([4], p. 303) the obvious faithful representation $\Sigma_4 \to GL_3\mathbf{F}_2$ yields a monomorphism $p:\Sigma_4 \to PSL_2\mathbf{F}_7$ (the choice of ρ will be immaterial). Embed D_3 in Σ_4 by $s \mapsto (3421)$, $t \mapsto (13)(24)$. Recall (cf. [9], p. 61).

PROPOSITION 5.1. *$H^*\Sigma_4 = \mathbf{Z}/2[y_1, y_2, y_3]/(y_1y_3)$, where y_i is the* ith *Stiefel-Whitney class of the standard representation of Σ_4 in* $0(4)$.

Proof of Theorem B. Let $G:B\Sigma_4 \to BPSL_2\mathbf{F}_7 \vee \Sigma^{-2}\overline{Sp}^4S^0 \vee$ be the map whose components are Bp, ft_{K,Σ_4}, By_1 (considering y_1 as the non-trivial homomorphism $\Sigma_4 \to \mathbf{Z}/2$). To prove Theorem B we show G^* is an isomorphism. Again, it is enough to show G^* is injective, since the Poincaré series in question are readily seen to be equal, and again it suffices to consider the restrictions $i_K^*G^*$, $i_T^*G^*$. The image of $H^*PSL_2\mathbf{F}_7$ in $H^*K \oplus H^*T$ is given in (2.6). Since K and T are not conjugate in Σ_4, and $N_{\Sigma_4}(T) = D_3$, we find as in the proof of (4.3) that $i_K^*G^*b_{j,k} = 0$ and $i_T^*G^*b_{j,k} = a_{j,k}\sigma^2$. Finally, $i_K^*y_1 = 0$ and $i_T^*y_1 = y$. Combining these results shows G^* is injective, exactly as in the proof of Theorem A.

Proof of Theorem C. Let $g:0(2) \to SO(3)$ denote the obvious inclusion defined by $\alpha \mapsto \alpha \oplus \det\alpha$. Then the fibre of $Bg:BO(2) \to BSO(3)$ is $SO(3)/0(2) \cong \mathbf{R}P^2$. The Becker-Gottlieb transfer [1] associated to this fibration is a stable map $BSO(3) \xrightarrow{q} BO(2)$ such that i^*q^* is multiplication by $\chi(\mathbf{R}P^2) = 1$ (mod 2). (In [1] this transfer is only constructed for fibre bundles over finite complexes. However, it is easy to fit together the transfer maps corresponding to finite skeleta of $BSO(3)$, yielding q.) Thus q provides a stable splitting of Bg, and $BO(2) \cong BSO(3) \vee F$, where F is the stable fibre of Bg. To prove Theorem C we will show that the usual map $B(\mathbf{Z}/2)^2 \xrightarrow{h} BO(2)$ restricts to an equivalence $X_1 \xrightarrow{\bar{h}} F$, where $X_1 \cong \Sigma^{-2}\overline{Sp}^4S^0 \vee P_1^\infty$ is the stable summand of $B(\mathbf{Z}/2)^2$ constructed in Section 4.

Let $V = H^*B(\mathbf{Z}/2)^2 = P(x, y)$, and identify $H^*BO(2)$ with $\operatorname{Im} h^* = V^\tau =$ symmetric polynomials in x and y. Since the idempotents π_1, π_2 of Section 4 are central, we obtain a splitting $H^*BO(2) = V^\tau\pi_1 \oplus V^\tau\pi_2$. Moreover it is easy to check that $V^\tau\pi_1 = P(z_2, z_3)$ and $V^\tau\pi_2 = \operatorname{Im}\rho_1^* = H^*X_1$. Now $h^*(Bg)^*$ is an isomorphism onto $V^\tau\pi_1$ (since v_2, v_3 are the Stiefel-Whitney classes of the representation g). Hence $\bar{h}^*$ is surjective (since h^* maps onto V^τ) and comparing Poincaré series shows $\bar{h}^*$ is an isomorphism.

6. Splitting BQ_n. In this section we prove Theorem E. The standard representation $\psi_n : Q_n \to S^3$ factors through the normalizer N of the standard maximal torus $S^1 \subset S^3$. We recall that N is generated by S^1 and j, so that there is a (non-split) extension $S^1 \to N \to \mathbf{Z}/2$. The fibre of $BN \to BS^3$ is $S^3/N = \mathbf{R}P^2$; it follows that $H^*BN = \mathbf{Z}/2[y, w_4]/(y^3)$. Since $\chi(\mathbf{R}P^2) = 1$, Becker-Gottlieb transfer gives a splitting:

(6.1) $BN \cong BS^3 \vee \Sigma^{-1}(BS^3/BN)$.

The proof of Theorem E depends on the transfers $t_m : BQ_n \to BQ_m$ (n fixed, $3 \le m < n$).

LEMMA 6.2. *$t_m^* : \tilde{H}^*BQ_m \to \tilde{H}^*BQ_n$ satisfies*

(a) $t^*(w_4^k b^\epsilon) = 0\,(k \ge 0, 0 \le \epsilon \le 2)$.
(b) $t^*(w_4^k a^\epsilon) = w_4^k(a^\epsilon + b^\epsilon)\,(k \ge 0, \epsilon = 1, 2)$.
(c) $t^*(w_4^k a^2 b) = \omega_4^k a^2 b\,(k \ge 0)$.

(*see Proposition* 2.2 *for notation.*)

Proof of Theorem D. Let $G : BQ_n \to BSL_2\mathbf{F}_q \vee \Sigma^{-1}(BS^3/BN) \vee \Sigma^{-1}(BS^3/BN)$ be the map with components $(Bi, \pi B\Psi_n, \pi B(\Psi_3\alpha)t_3)$, where i is the inclusion $Q_n \to SL_2\mathbf{F}_q$, $\pi : BN \to \Sigma^{-1}(BS^3/BN)$ is the projection of (6.1), and α is the involution of Q_3 defined by $\alpha(s) = t$. We will show G^* is an isomorphism. From Theorem (2.4) we see that i^* is injective with image $\langle w_4^k a^2 b, w_4^{k+1} : k \ge 0\rangle$. By (6.1), π^* is injective with image $\langle w_4^k y^\epsilon : k \ge 0, \epsilon = 1, 2\rangle$. Thus $(B\Psi_n)^*\pi^*$ is injective with image $\langle w_4^k b^\epsilon : k \ge 0, \epsilon = 1, 2\rangle$. Finally, $\alpha^*(a) = b$; hence by (6.2) $t_3^*(B\Psi_3\alpha)^*\pi^*$ is injective with image $\langle w_4^k(a^\epsilon + b^\epsilon) : k \ge 0, \epsilon = 1, 2\rangle$. This completes the proof.

Proof of Lemma (6.2). First note that t_m^* is a map of H^*BS^3 modules; this follows from naturality of the transfer and the obvious pullback diagram

$$\begin{array}{ccc} BQ_m & \xrightarrow{1\times B\Psi_m} & BQ_m \times BS^3 \\ \downarrow & & \downarrow \\ BQ_n & \xrightarrow{1\times B\Psi_n} & BQ_n \times BS^3 \end{array}$$

Hence it is enough to verify (6.2) for the generators a, a^2, b, b^2, a^2b. Moreover we may as well assume $m = n - 1$. Let $t = t_{n-1}$ and let $i: Q_{n-1} \to Q_n$ denote inclusion. Since $t^*i^* = 0$ and $i^*b = b$, $t^*b^\epsilon = 0$ for $\epsilon = 1, 2$. From the double coset formula $i^*t^*a = b$; since $i^*a = 0$ we have $t^*a = \alpha a + b$ for some coefficient α. To determine α, consider the commutative diagram

$$\begin{array}{ccc} \mathbf{Z}/2^{n-2} & \xrightarrow{\ell} & \mathbf{Z}/2^{n-1} \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} \\ Q_{n-1} & \xrightarrow{i} & Q_n \end{array}$$

where j, ℓ are the inclusions. From naturality of the transfer (or equivalently, the double coset formula) we have $j^*t^* = u^*j^*$, where u is the transfer associated to ℓ. In dimension one we have $\operatorname{Ker} j^* = \langle b \rangle$; since u^* is an isomorphism in odd dimensions this implies $\alpha = 1$. Thus $t^*(a) = a + b$, and $t^*a^2 = Sq^1(a + b) = a^2 + b^2$. Finally, to determine $t^*(a^2b)$ we recall that $H^4(Q_n; \mathbf{Z}) = \mathbf{Z}/2^n$ and that i^* is reduction mod 2^{n-1} on $H^4(\ ; \mathbf{Z})$. Hence t^* is injective on $H^4(\ ; \mathbf{Z})$. Since $H^3(Q_n; \mathbf{Z}) = 0$ it follows that t^* is an isomorphism on H^3.

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