M381C NOTES

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These notes were taken in UT Austin's Math 381c class in Fall 2015, taught by Luis Caffarelli. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

Contents

Lecture 1.

Outer Measure and the Lesbegue Measure: 8/26/15

The book for the class is *Measure and Integral*, 2^{nd} *Edition*, by Wheeden and Zygmund. This course will cover Lesbegue integration (chapters 3 to 6 of the book), function spaces (including L^p spaces; chapters 7 to 9), abstract integration and measure theory (chapters 2, 10, and 11).

In analysis, we first started with $\mathbb{R}^n$, and then started discussing functions not just as isolated entities, but as elements of a space, discussing the theory of continuous functions as a whole. For example, one might consider $C[0,1]$, the space of continuous functions on the interval $[0,1]$; then, you can talk about the distance between functions, e.g. $d(f,g) = \sup|f-g|$. For example, one can take an interval of size h around an f, so any g with $d(f,g) \le h$ is always trapped within that band. This distance is used to discuss uniform convergence: $\{f_n\}$ is said to *converge uniformly* to f if $d(f_k, f) \to 0$: that is, no matter how small you make the strip around f, for sufficiently large k, f_k is trapped in the strip.

On $\mathbb{R}^n$, the distance function is

$$d(x,y) = \Big(\sum (x_i - y_i)^2\Big)^{1/2},$$

but there are other distances, e.g.

$$d_1(x,y) = \sum |x_i - y_i| \qquad \text{and} \qquad d_\infty(x,y) = \sup_{1\le i\le n} |x_i - y_i|.$$

These are more or less the same, but for discussing distances between functions they do matter. For example, one distance between functions f and g is given by

$$d(f,g) = \int_0^1 |f-g|\,\mathrm{d}x.$$

But this means that there are sequences of continuous functions that converge to discontinuous ones, so $C[0,1]$ isn't the right space to study: it's not complete, just like the rational numbers. Analysis is about limits, so we really should use a complete space.

All right, great, so let's just use Riemann-integrable functions. This isn't sufficient either: let $\varepsilon > 0$ and let B_k be the function that traces out a triangle with base $\varepsilon/4^k$ wide and height 2^k. These are all nice, continuous and integrable functions, and the integral is always less than ε. Taking the limit should produce a function with integral zero, but choosing any partition doesn't work out, and so the Riemann integral isn't right either.

Let's pass from integration to defining the volume of a set. This is linked to integration, because the integral of the *indicator function* of a set S (that is, the function 1 on S and 0 outside). If S has a nice boundary, then the area of S is the Riemann integral of the indicator function. But if S doesn't have a nice boundary, the upper and lower Riemann sums disagree: for a trivial example, take $S = \mathbb{Q}$.

This is because we don't measure sets in a sharp enough way here; if you're allowed to take infinite partitions, we can cover each rational number with a cube of size $\varepsilon/2^k$, and this covers all of $\mathbb{Q}$ with total measure at most ε.

Given a theory of Lesbegue integration, you can define the measure of a set from the Lesbegue integral of its characteristic function. Going in the other direction, though, given a function f to integrate, add one more dimension and stretch a distance of 1 in that direction; then, the Lesbegue integral of f is the same as the Lesbegue measure of the new undergraph. So the integral and the measure are the same.

The Riemann integral greatly depends on the smoothness of the function; Lesbegue integration relies on a different idea. Given the graph of a function, we can draw horizontal bars across its undergraph, and calculate the area that way. Then, make finer and finer partitions, and take the limit. This is nice because it's monotone, an approximation below, and so it has nice convergence properties. But this means we need a very sturdy theory of measuring sets, since the undergraph could have a very complicated boundary. Specifically, we need a theory that allows us to do infinite processes.

The property we need is *countable additivity*: the sum of the measures of countably many disjoint sets is the measure of their union. Riemann integration has that property for a finite number of sets. An equivalent way to think of this is, given a family of monotonically increasing sets, the measure of the limit is the limit of the measures for the Lesbegue measure.

We can also think of this in terms of cubes covering a set S; given a countable number of cubes Q_j such that $\bigcup_{j=1}^{\infty} Q_j \supset S$, then we know that $\sum V(Q_j)$ is at least the measure of S, whatever that is. And we know $V(Q_j)$ is the product of its sides.

But this might have too much measure in it, so let's take the infimum over all covers.

The first observation we need is that we may refine the partition so that the cubes are nonoverlapping (i.e. their interiors are disjoint). Another observation is that closed and open covers are equivalent here; for any closed cover $\{Q_j\}$ we can choose a cover of open cubes $\{Q_j^*\}$, where $V(Q_j^*) \leq V(Q_j) + \varepsilon/2^j$; thus, the sum of the measures of the open cover is at most ε plus that of the closed cubes; over all measures, the infima are the same.

Definition. This is called the *exterior measure* of S, or $|S|_e$, the infimum over all such covers by a countable nunber of nonoverlapping cubes.

One can refine a partition in a process called *dyadic refinement*: given a partition, split each cube into 2^n cubes by cutting down the middle of each side, and then throwing out all of the cubes that are disjoint with S. In fact, given a standard grid of length 1 of $\mathbb{R}^n$, repeatedly doing dyadic refinement makes for a cover of S (the *dyadic cover*) that, when S is open, has measure equal to the exterior measure of S. That is:

Lemma 1.1. *If $\{Q_j\}$ is the dyadic cover of an open set S, then $|S|_e = \sum \mathrm{Vol}(Q_j)$.*

Proof. First, notice that $Q^{(m)} = \bigcup_{i=1}^{m} Q_j$ is a compact set, so suppose we have any other cover $\{R_j\}$ of S. Then, without loss of generality, by adding $\varepsilon/2^k$, we can make it an open cover. Since $Q^{(m)}$ is compact, a finite number of the R_j cover it. Thus, the volume of this cover must be larger than that of $Q^{(m)}$, since we have a nice finite number of cubes. ⊠

Thus, the exterior measure is realized by this partition; there's no extra. It's completely tight.

To recap:

(1) We defined the exterior measure $|E|_e = \inf \sum V(Q_j)$ over all covers Q_j.
(2) We can use either open or closed cubes.
(3) For an open set S, the exterior measure of S is realized by nonoverlapping cubes.

Thus, for any set E, we have $|E|_e = \inf |U|_e$, over all open sets $U \supseteq E$. Why? For any cover by cubes of E, we can enlarge them to make it open, so the union of the cubes will be an open set, which is a superset of E. But we can choose such an open cover by cubes with measure at most ε more than $|E|_e$, for any $\varepsilon > 0$. Of course, we will want open sets to be measurable sets.

We also want this to be invariant under change of coordinates; suppose $\{Q_j\}$ is one system of coordinates and $\{Q_j^*\}$ is another system of coordinates, then the exterior measures induced by them coincide, because a cube in $\{Q_j\}$ can be approximated arbitrarily well by a countable cover in $\{Q_j^*\}$.

Some of this may feel unrigorous; careful reasoning, involving some epsilons, should fix this.

So far we've just done coverings and counting; now we come to the decision making.

Definition.

- For an open set U, we define the *Lesbegue measure* of U to be $|U|_e$, its exterior measure.
- A set E is *measurable* if for all $\varepsilon > 0$, there exists an open $U \supseteq E$ such that $|U \setminus E|_e < \varepsilon$.

The idea behind measurability is that a measurable set should look a lot like an open set. Specifically, its boundary should be well-behaved: it looks like the boundary of an open set.

Nonmeasurable sets exist, as long as you're willing to invoke the Axiom of Choice. You might be used to counterexamples like the Cantor set, but that's beautiful compared to a typical nonmeasurable set. For each $x \in [0,1)$, consider the set $x + \mathbb{Q}$, which is the equivalence class of x where $x \sim y$ if $x - y \in \mathbb{Q}$. Then, construct a set S by choosing one point from each equivalence class (which requires the Axiom of Choice). Thus, for any $q \in \mathbb{Q}$, $q + S$ is disjoint from S. In particular, $\{q + S \mid q \in \mathbb{Q}\}$ is a countable disjoint family of sets whose union is $[0,1]$. Since they're all translations, they all have the same measure, and by countable additivity, the measure of their union is 1. But there's no number a such that $\sum_1^\infty a = 1$. Thus, S is nonmeasurable.

Lecture 2.

Defining the Lesbegue Measure: 8/31/15

"And since we're analysts, or, well, at least in an analysis course..."

Recall that we started off by talking about the distance between functions, with the goal of turning the space $C[0,1]$ of continuous functions on the unit interval into a Banach space. It's already a vector space, and with the uniform distance $d(f,g) = \sup|f-g|$ induced from the uniform norm, it's in fact complete. But under the L^1 norm

$$d(f,g) = \int_0^1 |f-g|\,\mathrm{d}x,$$

this space isn't complete: limits of some Cauchy sequences of continuous functions aren't continuous. This is the first difficulty that arises from working with the space of continuous functions.

More worryingly, the space of Riemann-integrable functions isn't complete in this norm either: you can produce a Cauchy sequence of continuous functions that don't converge, e.g.

$$f_k = \sum_{i=1}^k \chi_{(q_i - \varepsilon 2^{-i}, q_i + \varepsilon 2^{-i})},$$

where q_i is an enumeration of the rationals on $[0,1]$. Any interval contains a rational, so the upper Riemann sum of the limit is always 1, but the limit of the integrals will be ε.

Since integrating indicator functions comes from defining the volumes (or measures) of their indicated sets, maybe we should step back and look at measures of sets. In the above case, the union of the intervals that defined the f_k has upper Riemann measure equal to 1 and lower measure ε; the problem is that Riemann measure is essentially finite; it can only consider finite coverings, which are too imprecise to handle the f_k. Thus, Lesbegue measure generalizes this to countable coverings, which are good enough to make this work.

Measure and integral are closely tied: measure is the integral of a characteristic set, and the integral can be calculated by approximating a function with a sum of step functions.

For Lesbegue measure, let's for now talk only about bounded sets (dealing with the unbounded case is a minor modification). Let $E \subset \mathbb{R}^n$; then, for any countable covering $\{Q_j\}$, we take $\sum \mathrm{Vol}(Q_j)$ to be an upper bound for the measure of E. Thus, the infimum of all such measures is called the *exterior measure* of E, denoted $|E|_e$.

The first issue is that not every set is measurable. So we'll just restrict to the measurable sets, but we want to define a measure on as many sets as possible: open sets, closed sets, and countable unions thereof.

We have the following easy fact.

Claim. For any measurable E and F, $|E \cup F|_e \le |E|_e + |F|_e$.

Proof. If $\{Q_j\}$ is a countable cover of E and $\{Q_j'\}$ is one for F, then their union is a cover for $E \cup F$. ⊠

Last time, we also talked about the dyadic cover: given an open set U, we build a non-overlapping covering that's inside U, by dividing it into a grid and then subdividing boxes that are partially outside of U, and so on, which is a countable process. Thus, for open sets, the exterior measure can be calculated with interior subsets, so in some sense it's also an interior measure.

What Lesbegue did was to declare open sets measurable; they're the base of the σ-algebra of measurable sets. That is, if U is open, we define the measure of U to be its exterior measure: $|U| = |U|_e$. Thus, for any E, $|E|_e = \inf_{U \supset E} |U|_e$, for U open, because any covering can be arbitrarily well approximated with an open covering.

So, to recap:

- We have defined the exterior measure $|E|_e$.
- $|E \cup F|_e \le |E|_e + |F|_e$.
- $|E|_e = \inf |U|_e$ for open $U \supset E$.

Using the last point, for each $k \in \mathbb{N}$, choose a U_k such that $|U_k|_e \le |E|_e + 1/k$. Then,

$$|E|_e = \left| \bigcap U_k \right|_e,$$

and $H = \bigcap U_k$ is a G_δ set, as it's a countable intersection of open sets. Note that H itself might not be open, but the takeaway here is that G_δ sets ought to be measurable.

The problem with Riemann measure was that the boundary of E might be too complicated for Riemann approximation to work. So we see that we should look at the boundary to determine whether a set is measurable. One could approach this by declaring G_δ sets to be measurable and then defining E as above to be measurable if $|H \setminus E|_e = 0$, but it'll be better to define things differently, and recover this as a theorem.

Definition.

- A set E is *measurable* if for all $\varepsilon > 0$, there exists an open $U \supset E$ such that $|U \setminus E|_e < \varepsilon$.
- If $|E|_e = 0$, then E is therefore measurable, and is said to have *measure zero*.

Another characterization is that E is measurable if for any $\varepsilon > 0$ there exists a measurable set H such that $|E \triangle H|_e < \varepsilon$.[1]

Armed with these definitions, we can prove useful properties of the Lesbegue measure.

Definition. A σ-algebra X is a family of sets closed under countable operations: that is, if $\{U_i\}_{i=1}^\infty \in X$, then $\bigcap U_i$ and $\bigcup U_i$ must be in X, as well as $\mathcal{C}(U_i)$ (that is, its complement) for each i.

Theorem 2.1. *The family of Lesbegue-measurable sets is a σ-algebra, and the Lesbegue measure is countably additive.*

We'll prove this in a few smaller steps.

Definition. The *distance* between two sets A and B is $d(A,B) = \inf_{x \in A, y \in B} d(x,y)$.

Proposition 2.2. *Suppose A and B have positive distance (that is, $d(A,B) > 0$). Then, $|A \cup B|_e = |A|_e + |B|_e$.*

Proof. The idea is to find coverings of A and B that are disjoint. Let $\delta = d(A,B)$. Then, take a grid of length $\delta/2$ to cover A and B; then, every grid box covers A xor covers B. Let U_1 be the union of those (open) boxes that cover A, and U_2 be that for B.

For any covering $\mathcal{Q}$ of $A \cup B$, we can intersect it with U_1 and U_2 to get a covering $\mathcal{Q}'$ of $A \cup B$ that is at most as large as the original one. TODO I got really lost here. ⊠

In particular, since nonintersecting compact sets have positive distance, we can apply this to compact sets.

Proposition 2.3. *Compact sets are measurable.*

Proof. Let K be compact and $U \supset K$ be open such that $|K|_e \ge |U|_e - \varepsilon$. Then, we want to show that $|U \setminus K|$ is small. Well,

$$|U \setminus K| = \sum_1^\infty \mathrm{Vol}(Q_j)$$

[1] Here, $\triangle$ denotes symmetric difference.

for a countable set of cubes Q_j (since $U \setminus K$ is open), but if we replace the infinite sum with the partial sum $\sum_1^M \operatorname{Vol}(Q_j)$ for a sufficiently large M, we have that

$$\sum_1^M \operatorname{Vol}(Q_j) = |U \setminus K| - \varepsilon.$$

But this finite union of the closed cubes Q_j is a compact set. We'll call it L_M. Therefore, $|K \cup L_M|_e = |K|_e + |L_M|_e$, since $K \cap L_M = \emptyset$, and since $K \cup L_M \subset U$, then $|K \cup L_M|_e \le |U|_e = |K|_e + \varepsilon$, so this means that $|L_M|_e < \varepsilon$. If we let $M \to \infty$, the measure of L_M is always less than ε, i.e. the measure of $U \setminus K$, which is the union of the sets L_M as $M \to \infty$, is also less than ε, and therefore K is measurable. ⊠

In more words, the idea is that we can surround a compact set by an open set arbitrarily well, and their difference is a strip around the compact set, whose measure can be approximated by a bunch of cubes. The key idea is that, if we take only finitely many of these cubes, they're compact, so the exterior measure is additive.

Corollary 2.4.

(1) If U is open and $K \subset U$ is compact, then $|U| = |U \setminus K| + |K|$.
(2) If E is any measurable set, then for any $\varepsilon > 0$ there exists a compact $K_\varepsilon \subset E$ such that $|E| \le |K_\varepsilon| + \varepsilon$.

Proof. The first part comes directly from the proof; for the second part, given a measurable E and $\varepsilon > 0$, there exists an open $U \supset E$ such that $|U \setminus E| < \varepsilon$. Then, since U is open, we can choose a compact $K \subset E \subset U$ such that $|U \setminus K| < \varepsilon$, which therefore also controls $|U \setminus E|$. ⊠

Some abstract approaches to measure theory start by defining exterior and interior measures, with results like that one.

Now that we have compact sets, we can start talking countable additivity.

Proof of Theorem 2.1. Corollary 2.4 means that the complement of a measurable set is measurable; it says that if a set is well-approximated by an open set, its complement is at least as well approximated by an open set.

For countable additivity, let E_j be a countable collection of disjoint measurable sets and $E = \bigcup_j E_j$. We want to show that

$$|E| = \sum_j |E_j|.$$

First, if there are finitely many E_j, we know that

$$\left| \bigcup_{j=1}^N E_j \right| \le \sum_{j=1}^N |E_j|,$$

but we can also approximate with compact sets: let $K_j \subset E_j$ with $|K_j| \ge |E_j| - \varepsilon 2^{-j}$. Since the measure is additive on compact sets, then

$$\left| \bigcup_{j=1}^N K_j \right| = \sum_{j=1}^N |K_j| \ge \sum |E_j| - \varepsilon,$$

and we already know $\sum |K_j| \le \sum |E_j|$ since $K_j \subset E_j$. Then, for infinitely many sets, the same argument works, but take the limit. ⊠

Now that we know that the set of Lesbegue-measurable sets is a σ-algebra, let's try to characterize it.

Definition. The *Borel sets* are the elements of the σ-algebra generated by all open sets (i.e. the smallest σ-algebra containing all open sets, or the collection of countable unions, countable intersections, and complements generated by the open sets).

However, these aren't the Lesbegue-measurable sets! That is, there exist Lesbegue-measurable sets that aren't Borel. Let's consider the *Cantor set*, which is an uncountable set of measure zero. It's closed, and therefore Borel, but will be an important step in the construction.

The Cantor set can be constructed in the following way: start with $[0,1]$. In the first step, remove the middle third; in the next step, remove the middle third of the remaining two pieces, and so on, removing the middle third of each piece at each step.

This is a countable intersection of closed sets, and thus closed, and thus measurable. But at the k^{th} step the remaining measure is $(2/3)^k$, and this goes to 0, so the Cantor set has measure zero.

One can characterize the Cantor set as the numbers

$$\sum_{k=1}^{\infty} a_k 3^{-k}, \qquad a_k = 0 \text{ or } 2.$$

But this is uncountable, because for any $x \in [0,1]$, x has a base-2 decimal expansion of 0s and 1s; replace every 1 with a 2 to get a unique point in the Cantor set. Thus, this skinny set has the same cardinality as $\mathbb{R}$!

In particular, the set of subsets of the Cantor set has cardinality $2^{\mathbb{R}}$, but there are "only" uncountably many Borel sets, so there must exist a subset of the Cantor set that isn't Borel.