M392c NOTES: ALGEBRAIC GEOMETRY

ARUN DEBRAY
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These notes were taken in UT Austin's M392c (Algebraic geometry) class in Fall 2018, taught by Sam Raskin. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Any mistakes in the notes are my own.

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Lecture 1.

Some questions in algebraic geometry: 8/29/18

Office hours are Fridays from 11-1, in room 9.164 (at least for now). Today we'll talk about some questions (and some answers, too!) relating to algebraic geometry and why one might find it interesting. We're going to focus on concreteness.

Broadly speaking, algebraic geometry studies zero sets of polynomials. These could be polynomials over $\mathbb{Q}$, or $\mathbb{R}$, or $\mathbb{C}$, or finite fields, or more. The first question you might ask is, *are there solutions*? This is an *arithmetic question*: in arithmetic situations, there might not be solutions.

Example 1.1 (Taylor-Wiles, 1994)**.** If $n \geq 3$, the polynomial $x^n + y^n = 1$ has no solutions over $\mathbb{Q}$ when $x, y \neq 0$. ◂

You might recognize this as a reformulation of Fermat's last theorem.

Another form of the same question is *can you parameterize solutions of the equation*? For example, let's try it with $x^2 + y^2 = 1$, which we know has solutions. In this case, it is possible to parameterize solutions, via the one-parameter family

$$x = \frac{\lambda^2 - 1}{\lambda^2 + 1}, \qquad y = \frac{2\lambda}{\lambda^2 + 1}. \tag{1.2}$$

These kinds of questions are called *rationality questions*. One can also ask these questions over $\mathbb{C}$ (or over other algebraically closed fields), where they can feel a bit different.

There is a general result that any quadric hypersurface with a rational point is rational. What this means is that if you assume the existence of one solution (x_0, y_0) to a degree-2 polynomial in x and y over, say, $\mathbb{Q}$, then you can use that one solution to parameterize all other solutions. If you plot the solutions in the xy-plane, the parameter of another solution (x_1, y_1) is the slope of the line between (x_0, y_0) and (x_1, y_1). Indeed, in (1.2), the parameter λ is this slope. Because the equation is a quadric, one expects such a line to intersect in exactly two points, the first solution and another one. This is all extremely explicit, to the point that you could explain why you care to a middle schooler.

There are a few other rationality results.

Theorem 1.3 (Segre, 1940s; Manin, 1970s; Kollár, 2000)**.** *Smooth cubics in at least three variables are rational.*

So $x^2 + y^2 = 1$ isn't rational, but $x^2 + y^2 + z^2 = 1$ is. However, this doesn't give you everything.

Theorem 1.4 (Clemens-Griffiths, 1974)**.** *There are cubics in at least four variables which are unirational but not rational, i.e. that one cannot parameterize all solutions in a one-to-one manner.*

This was a hard theorem. How would you prove something like this?

Recent work (2012-15) by many people (Voisin, Colliot-Thèlene, Pirutka, Totaro[1]) generalizes this.

Theorem 1.5. *For cubics in at least five variables, one can also not parameterize solutions in a one-to-one way, even by adding additional "dummy variables."*

For four-variables cubics, this is open.

Schemes. Though this result is stated completely explicitly, it was studied using some very abstract-looking machinery. In this course, we'll also work with this abstract machinery, namely the language of schemes. These are things like solutions to systems of polynomials, but not quite — they encode among other things the equivalence of such systems under changes of coordinates, which doesn't really change the underlying geometry of the solution set. Classification problems with this perspective are a big area of research, and Birkar just won a Fields medal for work in this area from 2006.

Algebraic geometry over $\mathbb{C}$. A third thing you could care about is specific stuff about algebraic geometry over your favorite field (typically $\mathbb{C}$, but not always). In many cases (such as $\mathbb{C}$), you have topology around, and you can ask how it interacts with the algebraic geometry we've been talking about.

For example, if $q \in \mathbb{C}^\times$ isn't a root of unity, then there's a cubic equation $y^2 = x^2 + ax + b$ whose solutions are parameterized by $\mathbb{C}^\times/q\mathbb{Z}$. This may be a bit surprising, and indicates a way in which analytic or topological information can be useful: now we can learn about the universal cover of the solution space, and other topological invariants. Then you might ask whether something like this is true in positive characteristic, which tends to be harder.

More generally, one can study the topology of algebraic varieties over $\mathbb{C}$.

Theorem 1.6. *The odd Betti numbers of smooth proper varieties are even.*

The proof uses the study of the Hodge Laplacian operator on a variety X. This needs a metric, but projective means that X embeds in some $\mathbb{CP}^n$, and we can borrow its metric. There is a purely algebro-geometric proof of this, but first you need to come up with the right notion of Betti numbers (so étale cohomology, which is hard), and then invoke Deligne's proof of the Weil conjectures (also hard). Nonetheless, it's true in characteristic p.

More generally, the cohomology of a complex projective variety has more structure, and is much richer than that of a random manifold.[2]

Conjecture 1.7 (Hodge conjecture, imprecise statement)**.** *The differential topology of a projective algebraic variety over $\mathbb{C}$ knows everything about its algebraic geometry.*

This is a Millennium Prize problem, meaning it comes with a $1 million reward. You can infer that it's hard.

Algebraic geometry over $\mathbb{Z}$. If you work over $\mathbb{Z}$ instead of over $\mathbb{C}$, meaning your polynomial has integer coefficients, then you can reduce mod p and solve it there. This is the first thing anyone does in number theory, because it often simplifies the problem to a finite question. This naturally leads one to ask, *how do the systems of equations at different primes p relate to each other?*

There's a lot to say about this, beginning with quadratic reciprocity, which is very classical yet a little weird, and continuing all the way to the Langlands program.

Supposing X encodes the system of solutions to your polynomial with $\mathbb{Z}$ coefficients. Then one can define a zeta function, reminiscent of the Riemann zeta function, as follows:

$$\zeta_X(s) \coloneqq \prod_{p \text{ prime}} \exp\left(\sum \frac{1}{n}(\text{number of solutions in } \mathbb{F}_{p^n})p^{-ns}\right). \tag{1.8}$$

For $X = \operatorname{Spec}\mathbb{Z}$, corresponding to solutions to an empty set of polynomials, this recovers the usual Riemann zeta function.

For any particular X, one conjectures this is meromorphic (and almost entire, in some sense), and that the analogue of the Riemann hypothesis holds; for some X, this is known due to Deligne. There are some other related conjectures related to this known as Sato-Tate conjectures.

[1]If you like pictures of cats, check out Totaro's math blog: `https://burttotaro.wordpress.com/`.

[2]This doesn't require smoothness *per se*, but it's more difficult to formulate in the singular case.

Cohomology theories. Over $\mathbb{C}$, you have topology, and therefore can invoke algebraic topology to compute cohomology of algebraic varieties. Over other fields or rings, you might not have these techniques, and there are several other approaches.

- Over an algebraically closed field, one has *étale cohomology*, whose ideas are built from covering space theory, has $\mathbb{Z}_\ell$ coefficients, where ℓ is a prime that's not the characteristic of the field.
- Over any field k, there's *de Rham cohomology*, which uses the idea that $\mathrm{d}z/z$ understands $\mathbb{C}^\times$ isn't simply connected (since $\oint \mathrm{d}z/z \neq 0$). This has coefficients in k.

There are others, too. One wants these to all be the same, or at least closely related; if $k = \mathbb{Q}_p$ and $\ell = p$ ($\mathbb{Q}_p$ has characteristic zero!), then these two are related by p-adic Hodge theory. This is related to deep and recent work by Fontaine, Scholze, and others, and relates to Scholze's Fields medal work. In 2016, Bhatt-Morrow-Scholze showed that one can sometimes interpolate between different cohomology theories. See Scholze's ICM address for more on this. The ultimate question in this corner of algebraic geometry is whether there's some universal cohomology theory interpolating between everything we have, and which is also the source of the ζ-functions mentioned above.

Degenerations. We get additional power by studying solutions in families. For example, we can degenerate $x^2 + y^2 = 1$ to $x^2 + y^2 = 0$, which is much simpler. One asks questions such as, *what invariants are preserved under degenerations?* Therefore one might be able to use a degeneration to reduce a harder problem to an easier problem.

Computations. This subfield of algebraic geometry tries to make these abstract invariants concrete, by writing good algorithms to compute these invariants for explicit systems of polynomials.

Geometric complexity theory. This is another way to relate algebraic geometry and computer science. The goal of this field is to approach another Millennium Prize problem, P vs. NP, using algebraic geometry techniques. This roughly involves studying certain varieties and analyzing whether they're as complicated as they seem. Algebraic geometry has lots of techniques which might help, but on the other hand they haven't yet.

Probably the best way to learn algebraic geometry is to have an application or research focus in mind that you can apply the things you learn to. This method of learning tends to produce algebraic geometers.

Lecture 2.

Defining schemes: 8/31/18

The goal of today's lecture is to define a scheme, first heuristically and then rigorously.

"Definition" 2.1. A scheme is a "space" that is a Zariski sheaf which admits an "open cover" by affine schemes.

Of course, in order to do this, we need to know what all of these words — spaces, Zariski sheaves, affine schemes, and open covers — mean in this setting.

Remark 2.2. There's another approach to schemes using the formalism of *locally ringed spaces*, which is followed by Hartshorne, Vakil, and many others. It's more concrete, but it makes it harder to think about what a specific scheme, such as projective space, is supposed to be. ◀

The motivation for "Definition" 2.1 is that a scheme should be something which is locally defined by algebraic equations. For example, let's look at the *Fermat equation* $X_n = \{x^n + y^n = z^n\}$. Fermat was interested in solutions in $\mathbb{Z}$, but the set of solutions makes sense in any commutative ring. This suggests our definition of space, which is not the same as a topological space.

Definition 2.3. A *space* is a functor $X \colon \mathsf{CommRing} \to \mathsf{Set}$.

Concretely, this means that for every ring A, we get a set $X(A)$, and for every map of commutative rings $f \colon A \to B$, we get a map of sets $X(f) \colon X(A) \to X(B)$, and these morphisms should compose well (meaning that $X(f \circ g) = X(f) \circ X(g)$ and $X(\mathrm{id}) = \mathrm{id}$). For example, we could let $X_n(A)$ denote the set of solutions to the Fermat equation in the ring A; then, if we've solved it in A, we can map the solution into B via $f \colon A \to B$, and we'll obtain a solution in B, so this defines a space X_n.

We should also say how spaces interact.

Definition 2.4. A *morphism of spaces* $f\colon X \to Y$ is data of, for all commutative rings A, a map $f_A\colon X(A) \to Y(A)$ such that for all ring homomorphisms $g\colon A \to B$, the diagram

$$\begin{tikzcd} X(A) \arrow[r, "f_A"] \arrow[d, "X(g)"] & Y(A) \arrow[d, "Y(g)"] \\ X(B) \arrow[r, "f_B"] & Y(B) \end{tikzcd}$$

commutes.

Schemes are special examples of spaces, in a way that feels surprisingly down-to-Earth.

Our first example of a space is the solutions to the Fermat equation in A, as discussed above. Here's another example.

Example 2.5. Let A be a commutative ring. We'll define the space $\operatorname{Spec} A$ to be the functor $(\operatorname{Spec} A)(B) = \operatorname{Hom}(A, B)$; given a ring homomorphism $B \to C$, we use the map $\operatorname{Hom}(A, B) \to \operatorname{Hom}(B, C)$ given by postcomposition. ◀

Definition 2.6. An *affine scheme* is a space of the form $\operatorname{Spec} A$ for some A.

You don't have to be a commutative algebra expert to learn algebraic geometry, but you can see that commutative algebra is built into the definitions of algebraic geometry, so some commutative algebra knowledge is helpful.

Example 2.7. The space X_n sending A to the solutions of the Fermat equation in A is an affine scheme; explicitly,

$$X_n \cong \operatorname{Spec} \mathbb{Z}[x, y, z]/(x^n + y^n - z^n).$$

This is because a ring homomorphism $\mathbb{Z}[x, y, z]/(x^n + y^n - z^n) \to A$ is exactly the data of $x, y, z \in A$ satisfying the relation $x^n + y^n - z^n = 0$. ◀

Lemma 2.8 (Yoneda lemma)**.** *For all spaces X, $\operatorname{Hom}_{\mathsf{Spaces}}(\operatorname{Spec} A, X) \cong X(A)$.*

Proof sketch. First we define a map from $\operatorname{Hom}_{\mathsf{Spaces}}(\operatorname{Spec} A, X)$ to $X(A)$. Specifically, a map $f\colon \operatorname{Spec} A \to X$ is the data of for all commutative rings B, $\operatorname{Spec}(A)(B) \to X(B)$. Take $B = A$; then, $\operatorname{Spec}(A)(A) = \operatorname{Hom}(A)$, so take the image of the identity. It remains to check this is an equivalence. ⊠

Corollary 2.9. $\operatorname{Hom}_{\mathsf{Spaces}}(\operatorname{Spec} A, \operatorname{Spec} B) \cong \operatorname{Hom}_{\mathsf{CommRing}}(B, A)$.

It's interesting that the direction reverses!

Proof. By the Yoneda lemma, $\operatorname{Hom}_{\mathsf{Spaces}}(\operatorname{Spec} A, \operatorname{Spec} B) = \operatorname{Spec}(B)(A) = \operatorname{Hom}(B, A)$. ⊠

This tells you that as long as you make sure to reverse the arrows, anything you can do with commutative rings, you can do with affine schemes, and vice versa.

Fiber products. This is a categorical construction which we're going to use a lot.

Definition 2.10. Let X, Y, and Z be sets and $f\colon X \to Z$ and $g\colon Y \to Z$ be set maps. Then the *fiber product* of X and Y over Z is

$$X \times_Z Y := \{(x, y) \in X \times Y \mid f(x) = g(y)\}. \tag{2.11}$$

If X, Y, and Z are spaces, and f and g are maps of spaces, then the *fiber product* of X and Y over Z is the space defined by

$$(X \times_Z Y)(A) := X(A) \times_{Z(A)} Y(A). \tag{2.12}$$

Technically, the notation should include f and g, but in practice there's usually no ambiguity.

Example 2.13. Suppose we're given commutative rings A, B, and C and maps $\operatorname{Spec} B \to \operatorname{Spec} C$ and $\operatorname{Spec} A \to \operatorname{Spec} C$ (which are equivalent data to maps $\varphi\colon C \to A$ and $\psi\colon C \to B$). Then

$$\operatorname{Spec} A \times_{\operatorname{Spec} C} \operatorname{Spec} B \cong \operatorname{Spec}(A \otimes_C B),$$

where C acts on A, resp. B, through φ, resp. ψ. It's worth working through this one on your own, though it's not extremely hard. ◀

We'll define some properties of affine schemes with geometric names, but the definitions will rest on algebraic properties of rings. One of the real miracles of algebraic geometry is that this really works to define geometry, and even extends geometric intuition to places such as finite fields that are otherwise very hard to reason about.

Definition 2.14. A morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is a *closed embedding* if the induced map $A \to B$ is surjective.

Equivalently, $B = A/I$ for some ideal I of A.

The geometric idea behind defining $\operatorname{Spec} A$ is that geometric objects have a ring of functions on them, e.g. a smooth manifold M has a ring $C^\infty(M)$ of smooth $\mathbb{R}$-valued functions, and a map of manifolds $M \to N$ induces a map in the other direction by pullback: $C^\infty(N) \to C^\infty(M)$. Functional analysis results such as the Gelfand-Naimark theorem tell you what data you need to add to $C^\infty(M)$ to recover M as a topological space, and we're trying to imitate this in a more abstract algebraic setting.

This context allows us to explain why Definition 2.14 deserves to be called a closed embedding: let $I = (f_1, f_2, \dots)$, so

$$\operatorname{Spec} A/I = \{f_i = 0 \text{ for all } i\} = \{f = 0 \text{ for all } f \in I\}. \tag{2.15}$$

So we think of $\operatorname{Spec} B$ as some kind of closed subspace of $\operatorname{Spec} A$, and I as the ideal of functions on $\operatorname{Spec} A$ which vanish on $\operatorname{Spec} B$. This intuition can be turned into something precise.

Using fiber products, we can extend this to all spaces.

Definition 2.16. A map $X \to Y$ of spaces is a *closed embedding* if for all maps $\operatorname{Spec} A \to Y$, the "pullback" φ in the fiber product diagram

$$\begin{tikzcd} \operatorname{Spec} A \times_Y X \arrow[r, "\varphi"] \arrow[d] & \operatorname{Spec} A \arrow[d] \\ X \arrow[r] & Y \end{tikzcd} \tag{2.17}$$

is a closed embedding of affine schemes. In particular, we require $\operatorname{Spec} A \times_Y X$ to be an affine scheme, which is not always satisfied.

For a quick consistency check, we should ask that Definitions 2.14 and 2.16 agree on affine schemes, and indeed, if $I \subset A$ is an ideal, and $\operatorname{Spec} B \to \operatorname{Spec} A$ is a closed embedding in the sense of Definition 2.14, then (2.17) looks like

$$\begin{tikzcd} \operatorname{Spec}(A/I \otimes_A B) \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\ \operatorname{Spec}(A/I) \arrow[r] & \operatorname{Spec} A, \end{tikzcd} \tag{2.18}$$

and since $A/I \otimes_A B \cong B/BI$, this is a closed embedding in the more general sense as well.

We'd also like to know what an open embedding is. We'd like to say that it's something whose complement is a closed embedding. Let's make this precise.

Definition 2.19. Let $Z \hookrightarrow X$ be a closed embedding of spaces. The *complement* $X \setminus Z$ of Z in X is the space with $(X \setminus Z)(A)$ the set of $x \in X(A)$ such that the diagram

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & Z \arrow[d] \\ \operatorname{Spec} A \arrow[r] & X \end{tikzcd} \tag{2.20}$$

is a fiber product diagram. Here $\varnothing = \operatorname{Spec}(0)$, which sends every ring to the empty set.

Definition 2.21. If $X = \operatorname{Spec} A$ is an affine scheme, an *open embedding* is a map of spaces $j \colon U \to X$ such that $U = X \setminus Z$ for some closed embedding $Z \hookrightarrow X$.

Example 2.22. Letting $X = \operatorname{Spec} A$, if $f \in A$ and $Z = \operatorname{Spec}(A/f)$, the map $A \twoheadrightarrow A/f$ induces a closed embedding $Z \hookrightarrow X$. Its complement is $\operatorname{Spec} A[f^{-1}]$, the *localization* of A at f, so $\operatorname{Spec} A[f^{-1}] \to \operatorname{Spec} A$ is an open embedding. ◀

The intuition is that f generates the ideal of functions that vanish precisely on the closed subset Z. Therefore on the complement of Z, they should be invertible, so we adjoin an inverse to f.

Lemma 2.23. *Let $X = \operatorname{Spec} A$ and $Z = \operatorname{Spec} A/I$. Then maps $\operatorname{Spec} B \to X \setminus Z$ correspond bijectively to maps $A \to B$ such that $B \cdot I = B$.*

Proof. The diagram (2.20) specializes to

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & \operatorname{Spec}(A/I) \arrow[d] \\ \operatorname{Spec} B \arrow[r] & \operatorname{Spec} A, \end{tikzcd} \tag{2.24}$$

and this fiber product is $\operatorname{Spec}(B \otimes_A A/I) = \operatorname{Spec}(B/IB) = \varnothing$, which is equivalent to $IB = B$. ⊠

Example 2.25. *Affine n-space* over $\mathbb{Z}$ is the affine scheme $\mathbb{A}^n_{\mathbb{Z}} := \operatorname{Spec} \mathbb{Z}[x_1, \dots, x_n]$, and $0 \hookrightarrow \mathbb{A}^n_{\mathbb{Z}}$ is the closed embedding corresponding to the ideal $(x_1, \dots, x_n)$. The complement $\mathbb{A}^n_{\mathbb{Z}} \setminus 0$ is not affine! We'll prove that later when we have more tools. ◀

Exercise 2.26. Show that $\mathbb{A}^n_{\mathbb{Z}} \setminus 0$ is the space which maps a ring A to the set of n-tuples $(x_1, \dots, x_n) \in A^n$ such that the equation $\sum x_i y_i = 1$ has a solution.