M392C NOTES: K-THEORY

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SEPTEMBER 29, 2015

These notes were taken in UT Austin's Math 392c (K-theory) class in Fall 2015, taught by Dan Freed. I live-TeXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

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Lecture 1.

Families of Vector Spaces and Vector Bundles: 8/27/15

"Is that clear enough? I didn't hear a ding *this time."*

Let's suppose X is a topological space. Usually, when we do cohomology theory, we send in probes, n-simplicies, into the space, and then build a chain complex with a boundary map. This chain complex can be built in many ways; for general spaces we use continuous maps, but if X has the structure of a CW complex we can use a smaller complex. If we have a singular simplicial complex, a triangulation, we get other models, but they really compute the same thing.

Given a chain complex $C_\bullet$, we get a cochain complex by computing $\mathrm{Hom}(-, \mathbb{Z})$, giving us a cochain complex $C^0 \xrightarrow{\mathrm{d}} C^1 \xrightarrow{\mathrm{d}} \cdots$, giving us the cohomology groups $H^0 = H^0(X, \mathbb{Z})$.

If M is a smooth manifold, we have a cochain complex $\Omega^0_M \xrightarrow{\mathrm{d}} \Omega^1_M \xrightarrow{\mathrm{d}} \cdots$, and therefore get the de Rham cohomology $H^\bullet_{\mathrm{dR}}(M)$. de Rham's theorem states this is isomorphic to $H^\bullet(M; \mathbb{R})$, obtained by tensoring with $\mathbb{R}$.

In K-theory, we extract topological information in a very different way, using linear algebra. This in some sense gives us more powerful invariants. Consider $\mathbb{C}^n = \{(\xi^1, \ldots, \xi^n) : \xi^i \in \mathbb{C}\}$. This has the canonical basis $(1, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$, and so on. This is a rigid structure, in that the automorphism group of this space *with this basis* is rigid (no maps save the identity preserve the linear structure and the basis).

In general, we can consider an abstract complex vector space $(\mathbb{E}, +, \cdot, 0)$, and assume it's finite-dimensional. Then, $\mathrm{Aut}\,\mathbb{E}$ is an interesting group: every basis gives us an automorphism $b : \mathbb{C}^n \xrightarrow{\cong} \mathbb{E}$, and therefore gives us an isomorphism $b : \mathrm{GL}_n \mathbb{C} \xrightarrow{\cong} \mathrm{Aut}\,\mathbb{E}$.

We can also consider automorphisms that have some more structure; for example, $\mathbb{E}$ may have a hermitian inner product $\langle -, - \rangle : \mathbb{E} \times \mathbb{E} \to \mathbb{C}$. Then, $\mathrm{Aut}(\mathbb{E}, \langle -, - \rangle) = \mathrm{U}(\mathbb{E})$, which by a basis is isomorphic to U_n, the set of $n \times n$ matrices A such that $A^* A = \mathrm{id}$ (where A^* is the conjugate transpose). U_n is a Lie group, and a subgroup of $\mathrm{GL}_n \mathbb{C}$.

For example, when $n = 1$, $\mathrm{U}_1 \hookrightarrow \mathrm{GL}_1\,\mathbb{C}$. U_1 is the set of $\lambda \in \mathbb{C}$ such that $\overline{\lambda}\lambda = 1$, so U_1 is just the unit circle. Then, $\mathrm{GL}_1\,\mathbb{C}$ is the set of invertible complex numbers, i.e. $\mathbb{C} \setminus 0$. In fact, this means the inclusion $\mathrm{U}_1 \hookrightarrow \mathrm{GL}_1\,\mathbb{C}$ is a homotopy equivalence, and we can take the quotient to get $\mathrm{U}_1 \hookrightarrow \mathrm{GL}_1\,\mathbb{C} \twoheadrightarrow \mathbb{R}^{>0}$.

In some sense, the quotient determines the inner product structure on $\mathbb{C}$, since in this case an inner product only depends on scale. But the same behavior happens in the general case: $\mathrm{U}_n \hookrightarrow \mathrm{GL}_n\,\mathbb{C} \twoheadrightarrow \mathrm{GL}_n\,\mathbb{C}/\mathrm{U}_n$, and the quotient classifies hermitian inner products on $\mathbb{C}^n$.

Exercise. Identify the homogeneous space $\mathrm{GL}_n/\mathrm{U}_n$, and show that it's contractible. (Hint: show that it's convex.)

Now, we return to the manifold. Embedding things into the manifold is covariant: composing with $f : X \to Y$ of manifolds with something embedded into X produces something embedded into Y. K-theory will be contravariant, like cohomology: functions and differential forms on a manifold pull back contravariantly. What we'll look at is families of vector spaces parameterized by a manifold X.

Definition. A *family of vector spaces* $\pi : E \to X$ parameterized by X is a surjective, continuous map together with a continuously varying vector space structure on the fiber.

This sounds nice, but is a little vague. Any definition has data and conditions, so what are they? We have two topological spaces E and X; X is called the *base* and E is called the *total space*, as well as a continuous, surjective map $\pi : E \to X$. The condition is that the fiber $E_x = \pi^{-1}(x)$ is a vector space for each $x \in X$. Specifically, sending x to the zero element of E_x is a zero $z : X \to E$, which is a section or right inverse to π. We also have scalar multiplication $m : C \times E \to E$, which has to stay in the same fiber; thus, m commutes with π. Vector addition $+ : E \times_X E \to E$ is only defined for vectors in the same fiber, so we take the fiberwise product $E \times_X E$. Again, $+$ and π commute. Finally, what does continuously varying mean? This means that z, m, and $+$ are continuous.

Intuitively, if we let $\mathcal{V}$ be the collection of vector spaces, we might think of such a family as a function $X \to \mathcal{V}$. To each point of X, we associate a vector space, instead of, say, a number.

Example 1.1.

(1) The constant function: let $\mathbb{E}$ be a vector space. Then, $\underline{\mathbb{E}} = X \times \mathbb{E} \to X$ given by $\pi = \mathrm{pr}_1$ sends $(x, e) \mapsto x$. This is called the *constant vector bundle* or *trivial vector bundle* with fiber $\mathbb{E}$.

(2) A nonconstant bundle is the *tangent bundle* $TS^2 \to S^2$. For now, let's think of this as a family of real vector spaces; then, at each point $x \in S^2$, we have this 2-dimensional space $T_x S^2$, and different tangent spaces aren't canonically identified. Embedding $S^2 \hookrightarrow \mathbb{R}^3$ as the unit sphere, each tangent space embeds as a subspace of $\mathbb{R}^3$, and we have something called the Grassmanian. Note that $TS^2 \not\cong \mathbb{R}^2$, which we proved in algebraic topology as the hairy ball theorem.

Implicit in the second example was the definition of a map; the idea should be reasonably intuitive, but let's spell it out: if we have $\pi : E \to X$ and $\pi' : E' \to X$, a morphism is the data of a continuous $f : E \to E'$ such that the following diagram commutes.

$$
\begin{array}{ccc}
E & \xrightarrow{\ f\ } & E' \\
& \llap{\scriptstyle\pi}\searrow \quad \swarrow\rlap{\scriptstyle\pi'} & \\
& X &
\end{array}
$$

Then, you can make all of the usual linear-algebraic constructions you like: inverses, direct sums and products, and so on.

Example 1.2. Here's an example of a rather different sort. Let $\mathbb{E}$ be a finite-dimensional complex vector space, and suppose $T : \mathbb{E} \to \mathbb{E}$ is linear. Define for any $z \in \mathbb{C}$ the map $K_z = \ker(z \cdot \mathrm{id} - T) \subset \mathbb{E}$, and let $K = \bigcup_{z \in \mathbb{C}} K_z$.

For a generic z, $z \cdot \mathrm{id} - T$ is invertible, and so $K_z = 0$. But for eigenvalues, we get something more interesting, the eigenspace. But sending $K_z \mapsto z$, we get a map $\pi : K \to \mathbb{C}$. This is interesting because the vector space is 0-dimensional except at a finite number of points, and in fact if we take

$$
\varphi : \bigoplus_{z : K_z \neq 0} K_z \to \mathbb{E},
$$

induced by the inclusion maps $K_z \to E$, then φ is an isomorphism. This is the geometric statement of the Jordan block decomposition (or generalized eigenspace decomposition) of a vector space.

Definition. Given a family of vector spaces $\pi : E \to X$, the rank $x \mapsto \dim E_x = \pi^{-1}(X)$ is a function rank $: X \to \mathbb{Z}^{\geq 0}$.

Example 1.2 seems less nice than the others, and the property that makes this explicit, developed by Norman Steenrod in the 1950s, is called local triviality.

Definition. A family of vector spaces $\pi : E \to X$ is a *vector bundle* if it has *local triviality*, i.e. for every $x \in X$, there exists an open neighborhood $U \subset X$ and isomorphism $E|_U \cong \underline{\mathbb{E}}$ for some vector space $\mathbb{E}$.

This property is sometimes also called being *locally constant*. So the fibers aren't literally equal to $\mathbb{E}$ (they're different sets), but they're isomorphic as vector spaces.

One good question is, what happens if I have two local trivializations? Suppose E_x lies above x, and we have $\varphi_x : \mathbb{E} \to E_x$ and $\varphi'_x : \mathbb{E}' \to E_x$, each defined on open neighborhoods of x in X. The function $\varphi_x^{-1} \circ \varphi'_x : \mathbb{E}' \to \mathbb{E}$ is called a *transition function*, and we can see that it must be linear, and furthermore, isomorphic.

The Clutching Construction. This leads to a way of constructing vector bundles, known as the *clutching construction*. First, consider $X = S^2$, decomposed into $B_+^2 = S^2 \setminus \{-\}$ and $B_-^2 = S^2 \setminus \{+\}$ (i.e. minus the south and north poles, respectively). Each of these is diffeomorphic to the real plane, and in particular is contractible. Taking the trivial bundle $\underline{\mathbb{C}}$ over each of these, we have something like

The intersection $B_+^2 \cap B_-^2$ is diffeomorphic to $\mathbb{A}^2 \setminus \{0\}$. Thus, the two structures of $\underline{\mathbb{C}}$ on this intersection are related by a map $\underline{\mathbb{C}} \to \underline{\mathbb{C}}$, which induces a map $\tau : B_+^2 \cap B_-^2 \to \mathrm{Aut}(\mathbb{C}) = \mathrm{GL}_1\,\mathbb{C} = \mathbb{C}^\times$. This τ has an invariant called its *winding number*, so we can construct a line bundle $L \xrightarrow{\pi} S^2$ by gluing: let L be the quotient of $(B_+^2 \times \mathbb{C}) \sqcup (B_-^2 \times \mathbb{C})$ with the identification $\{x\} \times \mathbb{C} \sim \{\tau(x)\} \times \mathbb{C}$ (the former from B_+^2 and the latter from B_-^2).

More generally, if $\{U_\alpha\}_{\alpha \in A}$ is an open cover of X, then we get a map

$$\coprod_{\alpha \in A} U_\alpha \xrightarrow{p} X,$$

and so we can construct a gluing: whenever two points in the disjoint union map to the same point, we want to glue them together. The arrows linking two points to be identified have identities and compositions.

The clutching construction gives us a vector bundle over this space: given a vector bundle E_α over each U_α, we glue basepoints using those arrows, and get an associated isomorphism of vector spaces. Then, you can prove that you get a vector bundle.

Notice that maps $f : X \to Y$ of manifolds can be pulled back, and in this regard a vector bundle is a contravariant construction.

Topology and Vector Bundles. We were going to add some topology to this discussion, yes?

Theorem 1.3. *If $E \to [0,1] \times X$ is a vector bundle, then $E|_{\{0\} \times X} \cong E|_{\{1\} \times X}$.*

We'll prove this next lecture. The idea is that the isomorphism classes are homotopy-invariant, and therefore rigid or in some sense discrete. This will allow us to do topology with vector bundles.

Now, we can extract $\mathrm{Vect}^{\cong}(X)$, the set of vector bundles on X up to isomorphism. This has a 0 (the trivial bundle) and a +, given by direct sum of vector bundles. This gives a commutative monoid structure from X which is homotopy invariant.

Commutative monoids are a little tricky to work with; we'd rather have abelian groups. So we can complete the monoid, taking the Grothendieck group, obtaining an abelian group $K(X)$.

Using real or complex vector bundles gives $K_{\mathbb{R}}(X)$ and $K_{\mathbb{C}}(X)$, respectively (the latter is usually called $K(X)$). On S^n, one can compute that $K(S^n) = \pi_{n-1} \mathrm{GL}_N$ for some large N. These groups were computed to be periodic in both the real and complex cases, a result which is known as *Bott periodicity*.[1] This periodicity was proven in the mid-1950s.[2] This was worked into a topological theory by players such as Grothendieck and Atiyah, among others.

One of the first things we'll do in this class is provide a few different proofs of Bott periodicity.

Another interesting fact is that K-theory satisfies all of the axioms of a cohomology theory except for the values on S^n, making it a *generalized* (or *extraordinary*) *cohomology theory*. This is nice, since it means most of the computational tools of cohomology are available to help us. And since it's geometric, we can use it to attack problems in geometry, e.g. when is a manifold parallelizable?

For example, for S^n, S^0, S^1, and S^3 are parallelizable (the first two are trivial, and S^3 has a Lie group structure as the unit quaternions). It turns out there's only one more parallelizable sphere, S^7, and the rest are not; this proof by Adams in 1967 used K-theory, and is related to the question of how many division algebras there are.

Relatedly, and finer than just parallelizability, how many linearly independent vector fields are there on S^n? Even if S^n isn't parallelizable, we may have nontrivial l.i. vector fields. There are other related ideas, e.g. the Atiyah-Singer index theorem.

K-theory can proceed in different directions: we can extract modules of the ring of functions on X, and therefore using Spec, start with any ring and do algebraic K-theory. One can also intertwine K-theory and operator algebras, which is also useful in geometry. We'll focus on topological K-theory, however. There are also twistings in K-theory, which relate to representations of loop groups.

K-theory has also come into physics, both in high-energy theory and condensed matter, but we probably won't say much about it.

Nuts and bolts: this is a lecture course, so take notes. There might be notes posted on the course webpage[3], but don't count on it. There will also be plenty of readings; four are posted already: [2, 7, 14, 15].

Lecture 2.

Homotopies of Vector Bundles: 9/1/15

> *"You need a bit of Bourbaki imagination to determine the vector bundles over the empty set."*

Recall that all topological spaces in this class will be taken to be Hausdorff and paracompact.

We stated this as Theorem 1.3 last time; now, we're going to prove it.

Theorem 2.1. *Let X be a space and $E \to [0,1] \times X$ be a vector bundle. Let $j_t : X \hookrightarrow [0,1] \times X$ send $x \mapsto (t, x)$. Then, there exists a natural isomorphism $j_0^* E \overset{\cong}{\to} j_1^* E$ of vector bundles over X.*

To define the pullback more precisely, we can characterize it as fitting into the following diagram.

$$
\begin{array}{ccc}
j^* E & \longrightarrow & E \\
\downarrow & & \downarrow \\
Y & \overset{j}{\longrightarrow} & Z
\end{array}
$$

Then, $j^* E$ is the subset of $Y \times E$ for which the diagram commutes.

We'll want to make an isomorphism of fibers and check that it is locally trivial; in the smooth case, one can use an ordinary differential equation, but in the more general continuous case, we'll do something which is in the end more elementary.

To pass between the local properties of vector bundles and a global isomorphism, we'll use partitions of unity.

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[1] The sequence of groups you get almost sounds musical. Maybe sing the Bott song!

[2] The professor says, "I wasn't around then, just so you know."

[3] https://www.ma.utexas.edu/users/dafr/M392C/index.html.

Definition. Let X be a space and $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover (which can be finite, countable, or uncountable). Then, a *partition of unity* $\{\rho_\alpha\}_{\alpha\in A}$ indexed by a set A is a set of continuous functions $X \to [0,1]$ with locally finite supports such that $\sum \rho_\alpha = 1$. This partition of unity is said to be *subordinate* to the cover $\mathcal{U}$ if there exists $i : A \to I$ such that $\operatorname{supp} \rho_\alpha \subset U_{i(\alpha)}$.

Theorem 2.2. *Let X be a Hausdorff paracompact space and $\{U_i\}_{i\in I}$ be an open cover.*

(1) *There exists a partition of unity $\{\rho_i\}_{i\in I}$ subordinate to $\{U_i\}_{i\in I}$ such that at most countably many ρ_i are not identically zero.*

(2) *There exists a partition of unity $\{\rho_\alpha\}_{\alpha\in A}$ subordinate to $\{U_i\}_{i\in I}$ such that each ρ_α is compactly supported.*

(3) *If X is a smooth manifold, we can choose ρ_α to be smooth.*

We'll only use part (1) of this theorem.

A nontrivial example is $X = \mathbb{R}$ and $U_x = (x-1, x+1)$ for $x \in \mathbb{R}$ (so an uncountable cover). In this case, we don't need every function to be nonzero; we only need a countable number.

Returning to the setup of Theorem 2.1, if X is a smooth manifold, we will set up a covariant derivative, which will allow us to define a notion of parallel. Then, parallel transport will produce the desired isomorphism. In this case, we'll call $X = M$.

Suppose first that $\mathbb{E}$ is a vector space, either real or complex. $\Omega_M^0(\mathbb{E})$ denotes the set of smooth functions $M \to \mathbb{E}$ (written as 0-forms), and we have a basic derivative operator $\mathrm{d} : \Omega_M^0(\mathbb{E}) \to \Omega_M^1(\mathbb{E})$ satisfying the Leibniz rule

$$\mathrm{d}(f \cdot e) = \mathrm{d}f \cdot e + f\, \mathrm{d}e,$$

where $f \in \Omega_M^0$ and $e \in \Omega_M^0(\mathbb{E})$ (that is, e is vector-valued and f is scalar-valued). Moreover, any other first-order differential operator (an operator $\Omega_M^0(\mathbb{E}) \to \Omega_M^1(\mathbb{E})$ that is linear and satisfies the Leibniz rule) has the form $\mathrm{d} + A$, where $A \in \Omega_M^1(\operatorname{End}\mathbb{E})$. This means that if $\mathbb{E} = \mathbb{C}^r$, then e is a column vector of $e^1, \dots, e^r$ with $e^i \in \Omega^0(\mathbb{E})$, and $A = (A_j^i)$ is a matrix of one-forms: $A_j^i \in \Omega_M^1(\mathbb{C})$. Ultimately, this is because the difference between any two differential operators can be shown to be a tensor.

Now, let's suppose $E \to M$ is a vector bundle.

Definition. A *covariant derivative* is a linear map $\nabla : \Omega_M^0(E) \to \Omega_M^1(E)$ satisfying

$$\nabla(f \cdot e) = \mathrm{d}f \cdot e + f \cdot \nabla e$$

when $f \in \Omega_M^0$ and $e \in \Omega_M^0(\mathbb{E})$.

Here, $\Omega_M^0(E)$ is the space of sections of E. In some sense, this is a choice for functions with values in a varying vector space.

Theorem 2.3. *In this case, covariant derivatives exist, and the space of covariant derivatives is affine over $\Omega_M^1(\operatorname{End}\mathbb{E})$.*

Proof. Choose $\{U_i\}_{i\in I}$ and local trivializations $\underline{\mathbb{E}}_i \overset{\cong}{\Rightarrow} E|_{U_i}$ on U_i. We have a canonical differentiation d of $\mathbb{E}_i$-valued functions on U_i to define ∇_i on the bundle $E|_{U_i} \to U_i$.

To stitch them together, choose a partition of unity $\{\rho_i\}_{i\in I}$ and define

$$\nabla e = \sum_i \rho_i \nabla(j_i^* e),$$

where $j_i : U_i \hookrightarrow M$ is inclusion. $\boxtimes$

All right, so what's parallel transport? Let $\mathcal{E} \to [0,1]$ be a vector bundle with a covariant derivative ∇. Parallel transport will be an isomorphism $\mathcal{E}_0 \overset{\sim}{\to} \mathcal{E}_1$.

Definition. A section e is *parallel* if $\nabla e = 0$.

Lemma 2.4. *The set $P \subset \Omega_{[0,1]}^0(\mathcal{E})$ of parallel sections is a subspace. Then, for any $t \in [0,1]$, the evaluation map $\mathrm{ev}_t : P \to \mathcal{E}_t$ sending $e \mapsto e(t)$ is an isomorphism.*

The first statement is just because $\nabla e = 0$ is a linear condition. The second has the interesting implication that for any $(x, t) \in \mathcal{E}$, there's a unique parallel section that extends it.

Proof. Suppose $\mathcal{E} \to [0,1]$ is trivializable, and choose a basis $e_1, \ldots, e_r$ of sections. Then, we can write

$$\nabla e_j = A_j^i e_i,$$

where we're summing over repeated indices and $A_j^i \in \Omega^1_{[0,1]}(\mathbb{C})$. Then, any section has the form $e = f^j e_j$ and the parallel transport equation is

$$\begin{aligned} 0 = \nabla e &= \nabla(() f^j e_j) \\ &= \mathrm{d} f^j e_j + f^j \nabla e_j \\ &= (\mathrm{d} f^i + A_j^i f^j) e_j. \end{aligned}$$

If we write $A_j^i = \alpha_j^i \, \mathrm{d}t$ for $\alpha_j^i \in \Omega^0_{[0,1]}(\mathbb{C})$, then the parallel transport equation is

$$\frac{\mathrm{d} f^i}{\mathrm{d}\tau} + \alpha_j^i f^j = 0. \tag{2.1}$$

This is a linear ODE on $[0,1]$, so by the fundamental theorem of ODEs, there's a unique solution to (2.1) given an initial condition.

More generally, if $\mathcal{E}$ isn't trivializable, partition it into $[0, t_1]$, $[t_1, t_2]$, and so on, so that $\mathcal{E} \to [t_i, t_{i+1}]$ is trivialiable, and compose the parallel transports on each interval. $\boxtimes$

Now, we can prove Theorem 2.1 in the smooth manifolds case.

Proof of Theorem 2.1, smooth case. Choose a covariant derivative ∇, and use parallel transport along $[0,1] \times \{x\}$ to construct an isomorphism $E_{(0,x)} \to E_{(1,x)}$. The fundamental theorem on ODEs also states that the solution smoothly depends on the initial data, so these isomorphisms vary smoothly in x. $\boxtimes$

Note that this fundamental theorem only gives local solutions, but (2.1) is linear, so a global solution exists.

In the continuous case, we can't do quite the same thing, but the same idea of parallel transport is in effect.

Proof of Theorem 2.1, continuous case. By local triviality, we can cover $[0,1] \times X$ by open sets of the form $(t_0, t) \times U$ on which $E \to [0,1] \times X$ restricts to be trivializable.

By the compactness of $[0,1]$, we can cover X by sets $\{U_i\}_{i \in I}$ such that $E|_{[0,1] \times U_i}$ is trivializable: we can get trivializations on a finite number of patches. Thus, at the finite number of boundaries, we can patch the trivialization, choosing a continuous isomorphism of vector spaces.

Choose a partition of unity $\{\rho_i\}_{i \in I}$ subordinate to $\{U_i\}_{i \in I}$ and pare down I to the countable subset of $i \in I$ such that ρ_i isn't identically zero. Let $\varphi_n = \rho_1 + \cdots + \rho_n$ for $n = 1, 2, \ldots$, and let Γ_n be the graph of φ_n, which is a subset of $[0,1] \times X$.

So now we have a countable cover, and Γ_n is only supported on $U_1 \cup \cdots \cup U_n$, and only changes from Γ_{n-1} on U_n. But since the sum of the ρ_i is 1, then the graph Γ_n must go across the whole of $[0,1] \times X$ as $n \to \infty$. But over each open set, since we've pared down I, there are only finitely many steps.[4]

Going from Γ_0 (identically 0) to Γ_1 makes a trivialization on U_1, and from Γ_1 to Γ_2 extends the trivialization further, and so on. $\boxtimes$

Corollary 2.5. *If $f : [0,1] \times X \to Y$ is continuous and $E \to Y$ is a vector bundle, then $f_0^* E \cong f_1^* E$.*

This is because $f_t(x) = f(t, x)$ is a homotopy.

Corollary 2.6. *A continuous map $f : X \to Y$ induces a pullback map $f^* : \mathrm{Vect}(Y)^{\cong} \to \mathrm{Vect}(X)^{\cong}$, and this map depends only on the homotopy type of f.*

This is a hint that we can make algebraic topology out of the sets of vector bundles of spaces. There are many homotopy-invariant sets that we attach to topological spaces, e.g. π_0, π_1, π_2, H_1, H_2, and so on; these tend to be groups and even abelian groups, and thus tend to be easier to work with.

$\mathrm{Vect}^{\cong}(X)$ is a *commutative monoid*, so there's an associative, commutative $+$ and an identity. The identity is the isomorphism class of the bundle $\underline{\mathbb{O}}$, the zero vector space. Then, we define addition by

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[4]This argument is likely confusing; it was mostly given as a picture in lecture, and can be found more clearly in Hatcher's notes [7] on vector bundles and K-theory.

$[E] + [E'] = [E \oplus E']$. Moreover, it is a *semiring*, i.e. there's a $\times$ and a multiplicative identity 1 given by the isomorphism class of $\underline{\mathbb{C}}$. Multiplication is given by (the isomorphism class of) the tensor product.

Commutative monoids are pretty nice; a typical example is the nonnegative integers.

Example 2.7.
  (1) The simplest possible space is $\emptyset$. There's a unique vector bundle over it, the zero bundle, so $\mathrm{Vect}^{\cong}(\emptyset) = 0$, the trivial monoid.
  (2) Over a point, vector bundles are just finite-dimensional vector spaces, which are determined up to isomorphism by dimension, so $\mathrm{Vect}^{\cong}(\mathrm{pt}) \xrightarrow{\sim} \mathbb{Z}^{\geq 0}$.

Definition. If X is a compact space, $K(X)$ is the abelian group completion of the commutative monoid $\mathrm{Vect}^{\cong}_{\mathbb{C}}(X)$; the completion of $\mathrm{Vect}^{\cong}_{\mathbb{R}}(X)$ is denoted $KO(X)$.

This definition makes sense when X is noncompact, but doesn't give a sensible answer. We'll see other definitions in the noncompact case eventually.

We'll talk more about the abelian group completion next lecture; the idea is that for any abelian group A and homomorphism $\alpha : \mathrm{Vect}^{\cong}(X) \to A$ of commutative monoids, there should be a unique $\widetilde{\alpha}$ such that the following diagram commutes.

$$
\begin{array}{ccc}
\mathrm{Vect}^{\cong}(X) & \longrightarrow & K(X) \\
 & {\scriptstyle \alpha} \searrow \quad \swarrow {\scriptstyle \widetilde{\alpha}} & \\
 & A &
\end{array}
$$

Another corollary of Theorem 2.1:

Corollary 2.8. *If X is contractible and $\pi : E \to X$ is a vector bundle, then π is trivializable.*

Corollary 2.9. *Let $X = U_0 \cup U_1$ for open sets U_0, U_1 and $E_i \to U_i$ be two vector bundles, and let $\alpha : [0,1] \times U_0 \cap U_1 \to \mathrm{Iso}(E_0|_{U_0 \cap U_1}, E_1|_{U_0 \cap U_1})$: that is, α is a homotopy of isomorphisms $E_0 \to E_1$ on the intersection. Then, clutching with α_t gives a vector bundle $E_t \to X$, and $E_0 \cong E_1$.*

In the last five minutes, we'll discuss a few more partition of unity arguments.
  (1) Let X be a topological space, and

$$0 \longrightarrow E' \xrightarrow{\ i\ } E \xrightarrow{\ j\ } E'' \longrightarrow 0$$

be a short exact sequence of vector bundles over X. Recall that a *splitting* of this sequence is an $s : E'' \to E$ such that $j \circ s = \mathrm{id}_{E''}$. Then, splittings form a bundle of affine spaces over $\mathrm{Hom}(E'', E)$, which happens because linear maps act simply transitively on splittings (adding a linear map to a splitting is still a splitting, and any two splittings differ by a linear map).

  Theorem 2.10. *Global splittings exist, i.e. the affine bundle of splittings has a global section.*

  Proof. At each point, there's a section, which is a linear algebra statement, and locally on X, there's a splitting, which follows from local trivializations. Then, patch them together with a partition of unity, which works because we're in an affine space, so our partition of unity in each affine space is a weighted average (because the ρ_i are nonnegative) and therefore lies in the convex hull of the splittings. $\boxtimes$

  (2) We also have Hermitian inner products. The same argument goes through, as inner products are convex (the weighted average of two inner products is convex), so one can honestly use a partition of unity in the same way as above.

Lecture 3.

Abelian Group Completions and $K(X)$: 9/3/15

"First I want to remind you about fiber bundles... (pause) ... Consider yourself reminded."

Last time, we said that if $\mathbb{E}$ is a (real or complex) vector space, the space of its inner products is contractible. This is because we have a vector space of sesquilinear (or bilinear in the real case) maps $\mathbb{E} \times \mathbb{E} \to \mathbb{C}$ (or $\mathbb{R}$), and the inner products form a convex cone in this space.

Inner products relate to symmetry groups: the symmetry group of $\mathbb{C}^n$ is $\mathrm{GL}_n \mathbb{C}$, the set of $n \times n$ complex invertible matrices, but the symmetry group of $\mathbb{C}^n$ with an inner product $\langle -, - \rangle$ is the unitary group $\mathrm{U}_n \subset \mathrm{GL}_n \mathbb{C}$, the set of matrices A such that $A^* A = I$. In the real case, the symmetries of $\mathbb{R}^n$ are $\mathrm{GL}_n \mathbb{R}$, and the group of symmetries of $\mathbb{R}^n$ with an inner product is $\mathrm{O}_n \subset \mathrm{GL}_n \mathbb{R}$.

As a consequence, we have the following result.

Proposition 3.1. *There are deformation retractions* $\mathrm{GL}_n \mathbb{C} \to \mathrm{U}_n$ *and* $\mathrm{GL}_n \mathbb{R} \to \mathrm{O}_n$.

For example, when $n = 1$, $\mathrm{GL}_1 \mathbb{C} = \mathbb{C}^\times$, which deformation retracts onto the unit circle, which is U_1. Then, $\mathrm{GL}_1 \mathbb{R} = \mathbb{R}^\times$ and $\mathrm{O}_1 = \{\pm 1\}$, so there's a deformation retraction in the same way.

Proof. We'll give the proof in the complex case; the real case is pretty much identical.

Since the columns of an invertible matrix determine a basis of $\mathbb{C}^n$ and vice versa, identify $\mathrm{GL}_n \mathbb{C}$ with the space of bases of $\mathbb{C}^n$; then, U_n is the space of orthonormal bases of $\mathbb{C}^n$.

A general basis $e_1, \ldots, e_n$ may be turned into an orthonormal basis by the Gram-Schmidt process, which is a composition of homotopies. First, we scale e_1 to have norm 1, given by the homotopy $e_1 \mapsto ((1-t) + t/|e_1|)e_1$. Then, we make $e_2 \perp e_1$, which is given by the homotopy $e_2 \mapsto e_2 - t\langle e_2, e_1 \rangle e_1$. The rest of the steps are given by scaling basis vectors and making them perpendicular to the ones we have so far, so they're also homotopies. $\boxtimes$

Group Completion. Recall that a commutative monoid is the data $(M, +, 0)$, such that $+$ is associative and commutative, and 0 is the identity for $+$.

Definition. (A, i) is a *group completion* of M if A is an abelian group, $i : M \to A$ is a homomorphism of commutative monoids, and for every abelian group B and homomorphism $f : M \to B$ of commutative monoids, there exists a unique abelian group homomorphism $\widetilde{f} : A \to B$ of abelian groups such that $\widetilde{f} \circ i = f$.

That is, we require that there exists a unique $\widetilde{f}$ such that the following diagram commutes.

$$
\begin{array}{ccc}
M & \xrightarrow{i} & A \\
 & {\scriptstyle f} \searrow & \downarrow {\scriptstyle \widetilde{f}} \\
 & & B
\end{array}
$$

Note that i was never specified to be injective, and in fact it often isn't.

Example 3.2.

- If $M = (\mathbb{Z}^{\geq 0}, +)$, the group completion is $A = \mathbb{Z}$.
- If $M = (\mathbb{Z}^{>0}, \times)$, we get $A = \mathbb{Q}^{>0}$.
- However, if $M = (\mathbb{Z}^{\geq 0}, \times)$, we get $A = 0$. This is because if $i : \mathbb{Z}^{\geq 0} \to A$, then there must be an $a \in A$ such that $i(0) \cdot a = 1$; thus, for any $n \geq 0$,

$$i(n) = i(n)i(1) = i(n)i(0)a = i(n \cdot 0)a = i(0)a = 1.$$

Since the group completion was defined by a universal property, we can argue for its existence and uniqueness; universal properties tend to have very strong uniqueness conditions.

We saw that the vector bundles up to isomorphism are a commutative monoid (even semiring under tensor product), and so taking the group completion can cause a loss of information, as in the last part of the above example. Though abelian groups are nicer to compute with, there are examples where information about vector bundles is lost by passing to abelian groups.

The uniqueness of the group completion is quite nice: given two group completions (A, i) and (A', i') of a commutative monoid M, there exists a unique isomorphism ϕ that commutes with the universal property.

That is, in the following diagram, $\phi \circ \widetilde{f}' = \widetilde{f}$.

$$
\begin{array}{ccc}
& \xrightarrow{\quad \phi \quad} & \\
A' \xleftarrow{\;i'\;} & M \xrightarrow{\;i\;}_{\;\cong\;} & A \\
& \downarrow f & \\
\widetilde{f}' \searrow & & \swarrow \widetilde{f} \\
& B &
\end{array}
$$

To prove this, we'll apply the universal property four times. To see why ϕ is an isomorphism, putting A' in place of B and i' in place of f, we get a ϕ, and switching (A, i) with (A', i') gives us $\psi : A' \to A$. Then, in the following diagram, $i' = \phi i = (\phi \psi) i'$, which satisfies a universal property (which one?) and therefore proves ϕ and ψ are inverses.

$$
\begin{array}{ccc}
M & \xrightarrow{\;i\;} & A \\
& & \uparrow \downarrow \\
i' \searrow & & \psi \uparrow\; \downarrow \phi \\
& A' &
\end{array}
$$

For existence, define $A = M \times M / \sim$, where $(m_1, m_2) \sim (m_1 + n, m_2 + n)$ for all $m_1, m_2, n \in M$. Then, $0_A = (0_M, 0_M)$ and $-[m_1, m_2] = [m_2, m_1]$. This makes sense: it's how we get $\mathbb{Z}$ from $\mathbb{N}$, and $\mathbb{Q}$ from $\mathbb{Z}$ multiplicatively.

Often, the abelian group completion is called the *Grothendieck group* of M, called $K(M)$.

Back to K-Theory. If X is compact hausdorff, then $\mathrm{Vect}^{\cong}(X)$, the set of isomorphism classes of vector bundles over X, is a commutative monoid, with addition given by $[E'] + [E''] = [E' \oplus E'']$, and a semiring given by $[E'] \times [E''] = [E' \otimes E'']$. There's some stuff to check here.

The group completion of $\mathrm{Vect}^{\cong}_{\mathbb{C}}(X)$ is denoted $K(X)$ (sometimes $KU(X)$, with the U standing for "unitary"), and the group completion of $\mathrm{Vect}^{\cong}_{\mathbb{R}}(X)$ is denoted $KO(X)$, with the O for "orthogonal."

The map $X \mapsto K(X)$ (or $KO(X)$) is a homotopy-invariant functor; that is, if $f : X \to Y$ is continuous, then $f^* : K(Y) \to K(X)$ is a homomorphism of abelian groups. The homotopy invariance says that if $f_0 \simeq f_1$, then $f_0^* = f_1^*$. We could write $K : \mathsf{CptSpace}^{op} \to \mathsf{AbGrp}$, and mod out the homotopy.

There are plenty of other functors that look like this; for example, the nth cohomology group is a contravariant functor from topological spaces (more generally than compact Hausdorff spaces) to abelian groups, and is homotopy-invariant. But this gives us a sequence of groups, indexed by $\mathbb{Z}$ (where the negative cohomology groups are zero by definition). Similarly, we'll promote the K-theory of a space to a sequence of abelian groups indexed by the integers, with $K(X)$ becoming $K^0(X)$; we'll also see that in the typical case, $K^n(X)$ is nonzero for infinitely many n.

For example, if E and E' are vector bundles, $\mathrm{Hom}(E, E') \cong E' \otimes E^*$, by the map sending $e' \otimes \theta \mapsto (e \mapsto \theta(e)e')$. There's some stuff to check; in particular, once you know it for vector spaces, it's true fiber-by-fiber. Moreover, E and E^* are isomorphic as vector bundles, because any metric $E \otimes E \to \underline{\mathbb{R}}$ induces an isomorphism $E \to E^*$; thus, in $KO(X)$, $[E] = [E']$, so $[\mathrm{Hom}(E, E')] = [E] \times [E']$.

In the complex case, the metric is a map $\overline{E} \otimes E \to \underline{\mathbb{C}}$: the conjugate bundle is defined fiber-by-fiber by the conjugate vector space $\overline{\mathbb{E}}$, identical to $\mathbb{E}$ except that scalar multiplication is composed with conjugation. Thus, there's an isomorphism $\overline{E} \xrightarrow{\sim} E^*$. This is sometimes, but not always, an isomorphism: if X is a point, then it's always an isomorphism, but the bundle $\mathbb{C}P^1 \to S^2$ isn't fixed: complex conjugation flips the winding number, and therefore produces a nonisomorphic bundle.

We said that we might lose information taking the group completion, so we want to know what kind of information we've lost. The key is the following proposition.

Proposition 3.3. *Let X be a compact Hausdorff space and $\pi : E \to X$ be a vector bundle. Then, there exists a vector bundle $\pi' : E' \to X$ such that $E \oplus E' \to X$ is trivializable.*

If $X \neq \emptyset$, then there's a map $p : X \to \mathrm{pt}$, and its pullback $p^* : K(\mathrm{pt}) \to K(X)$ is injective. That is, we have an injective map $\mathbb{Z} \hookrightarrow K(X)$, consisting of the trivial bundles (i.e. those pulled back by a point). Proposition 3.3 implies that given a $k \in K(X)$, there's a k' such that $k + k' = n$ for $n \in \mathbb{Z}$. Thus, the inverse is $-k = k' - N$.

Proof of Proposition 3.3. Since X is compact, we can cover it with a finite collection of opens $U_1, \ldots, U_N$ such that $E|_{U_i}$ is trivializable for each i.

Choose a basis of sections $e_1^{(i)}, \ldots, e_n^{(i)}$ on U_i, and let $\rho_1, \ldots, \rho_N$ be a partition of unity subordinate to the cover $\{U_i\}$. Then, let

$$S = \left\{ \rho_1 e_1^{(1)*}, \ldots, \rho_1 e_n^{(1)*}, \rho_2 e_1^{(2)*}, \ldots \right\} \subset C^0(X; E^*),$$

where $e_1^{(i)*}, \ldots, e_r^{(i)*}$ is the dual basis of sections of $E^*|_{U_i} \to U_i$.

Then, set $V = \mathbb{C}S^*$, the set of functions $S \to \mathbb{C}$. Then, evaluation defines an injection $E \hookrightarrow \underline{V}$: evaluating at E_x determines a value on each basis element on each ρ_i that doesn't vanish there, so we get values on basis elements. Moreover, since at least one such ρ_i exists for each point, this map is injective.

Let $E' = \underline{V}/E$, so we have a short exact sequence

$$0 \longrightarrow E \longrightarrow \underline{V} \longrightarrow E' \longrightarrow 0.$$

Last time, we proved in Theorem 2.10 that all short exact sequences of vector bundles exist, so there's an isomorphism $E' \oplus E \xrightarrow{\sim} V$. $\boxtimes$

Now, we can do some stuff that will look familiar from cohomology.

Definition. The *reduced K-theory* of X is the quotient $\widetilde{K}(X) = K(X)/p^*K(\mathrm{pt})$, where $p : X \to \mathrm{pt}$.

Example 3.4. If $X = \mathrm{pt} \sqcup \mathrm{pt}$, then $K(X) \xrightarrow{\sim} \mathbb{Z} \oplus \mathbb{Z}$ sending bundles to their ranks. Then, $p^* : K(\mathrm{pt}) = \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is the diagonal map Δ, so $\widetilde{K}(X) = \mathbb{Z} \oplus \mathbb{Z}/\Delta \xrightarrow{\sim} \mathbb{Z}$.

Corollary 3.5. *Let $E, E' \to X$ be vector bundles. Then, $[E] = [E']$ in $\widetilde{K}(X)$ iff there exist $r, r' \in \mathbb{Z}^{\geq 0}$ such that $E \oplus \underline{\mathbb{C}}^r \cong E' \oplus \underline{\mathbb{C}}^{r'}$.*

In this case, we say that E and E' are *stably equivalent*. In other words, *K-theory remembers the stable equivalences of vector bundles.* This is the first inkling we have of what K-theory is about, and what the geometric meaning of group completion is.

Example 3.6. Let's look at $\widetilde{KO}(S^2)$. We have a nontrivial bundle of rank 2 over S^2, $TS^2 \to S^2$. However, $TS^2 \oplus \mathbb{R} \to S^2$ is trivializaible!

To see this, embed $S^2 \hookrightarrow \mathbb{A}^3$; such an embedding always gives us a short exact sequence of vector bundles

$$0 \longrightarrow TS^2 \longrightarrow T\mathbb{A}^3|_{S^2} \longrightarrow \nu \longrightarrow 0.$$

The quotient ν, by definition, is the *normal bundle* of the submanifold (in this case, S^2). We know that $T\mathbb{A}^3 = \mathbb{R}^3$ everywhere, which is almost by definition, and therefore $\nu \cong \underline{\mathbb{R}}$. This means that in $\widetilde{KO}(S^2)$, $[TS^2] = 0$.

So right now, we can calculate the K-theory of a point, and therefore of any contractible space. We want to be able to do more; a nice first step is to compute the K-theory of S^n. Just as in cohomology, this will allow us to bootstrap our calculations on CW complexes.

Definition. Recall that a *fiber bundle* is the data $\pi : E \to X$ over a topological space X such that π is surjective and local trivializations exist. E is called the *total space*.

Thus, a vector bundle is a fiber bundle where the fibers are vector spaces, *and* we require the local trivializations to respect this structure. We can do this more generally, e.g. with affine spaces and affine maps.

Example 3.7. If $V \to X$ is a vector bundle, we get some associated fiber bundles over X. For example, $\mathbb{P}V \to X$, with fiber of lines in the vector space that's the fiber of V. We can generalize to the Grassmanian $\mathrm{Gr}_k V$, which uses k-dimensional subspaces instead of lines. There are plenty more constructions.

Definition. A topological space F is *k-connected* if $Y \to F$ is null-homotopic for every CW complex Y of dimension at most k.

It actually suffices to take only the spheres for Y.

Lemma 3.8. *Let n be a positive integer and $\pi : \mathcal{E} \to X$ be a fiber bundle, where X is a CW complex with finitely many cells and of dimension at most n, and the fibers of π are $(n-1)$-connected. Then, π admits a continuous section.*

Proof. We'll do cell-by-cell induction on the skeleton $X_0 \subset X_1 \subset \cdots \subset X_n = X$. On points, π trivially has a continuous section.

Suppose we have constructed s on X_{k-1}. Then, all the k-cells are attached via maps

$$
\begin{array}{ccc}
D^k & \xrightarrow{\Phi} & X \\
\uparrow & & \uparrow \\
S^{k-1} & \xrightarrow{\partial\Phi} & X_{k-1}.
\end{array}
$$

Since $D^k \simeq \mathrm{pt}$, then $\Phi^* \mathcal{E} \to D^k$ is trivializable, so we have a map $\theta : \Phi^* \mathcal{E} \to \underline{F}$. The section on X_{k-1} pulls back and composes with θ to create a map $S^{k-1} = \partial D^k \to F$, but by hypothesis, this is null-homotopic, and therefore extends to D^k.   ⊠

A different kind of induction is required when X has infinitely many cells; however, what we've proven is sufficient for the K-theory of the spheres.

Theorem 3.9. *Let $n \in \mathbb{Z}^{\geq 0}$ and $N \geq n/2$. Then, there is an isomorphism $\pi_{n-1} \mathrm{U}_N \to \widetilde{K}(S^n)$.*

Corollary 3.10. *The inclusion $\mathrm{U}_N \hookrightarrow \mathrm{U}_{N+1}$ induces an isomorphism $\pi_{n-1} \mathrm{U}_N \to \pi_{n-1} \mathrm{U}_{N+1}$ if $N \geq n/2$.*

Note that the theorem statement doesn't give enough information to say which map induces the isomorphism, but the proof will show that the usual inclusion does it. Specifically, thinking of U_N as a matrix group, U_N embeds in U_{n+1} on the upper left, i.e.

$$
A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.
$$

We can take the union (direct limit) of the inclusions $\mathrm{U}_1 \subset \mathrm{U}_2 \subset \mathrm{U}_3 \subset \ldots$, and call it U_∞ (sometimes U). These sequences of homotopy groups must stabilize.

Theorem 3.11 (Bott)**.**

$$
\pi_{n-1} \mathrm{U}_\infty \cong \widetilde{K}(S^n) = \left\{ \begin{array}{ll} \mathbb{Z}, & n \text{ even} \\ 0, & n \text{ odd}. \end{array} \right.
$$

We have a real analogue to this theorem as well: the analogous inclusion $\mathrm{O}_1 \hookrightarrow \mathrm{O}_2 \hookrightarrow \cdots$ define a limit O_∞.

Theorem 3.12.

$$
\pi_{n-1} \mathrm{O}_\infty \cong \widetilde{KO}(S^n) = \left\{ \begin{array}{ll} \mathbb{Z}, & n \equiv 0, 4 \bmod 8 \\ \mathbb{Z}/2\mathbb{Z}, & n \equiv 1, 2 \bmod 8 \\ 0, & n \equiv 3, 5, 6, 7 \bmod 8. \end{array} \right.
$$

These results, known as the *Bott periodicity theorems*, are the foundations of Bott periodicity. We'll give three proofs: Bott's original proof using Morse theory, a more elementary one, and one that uses functional analysis and Fredholm operators.

Lecture 4.

Bott's Theorem: 9/8/15

> *"Any questions?"*
> *"How was your weekend?"*
> *"I was afraid of that."*

We know that vector bundles always have sections (e.g. the zero section), but fiber bundles don't. For example, the following fiber bundles don't have sections.

- The orientation cover of a nonorientable manifold (e.g. the Möbius strip) is a double cover that doesn't have a section.
- The Hopf fibration $S^1 \to S^3 \to S^2$.
- Any nontrival covering map $S^1 \to S^1$.

However, sometimes sections do exist.

Theorem 4.1. *If X is a CW complex of dimension n and $\pi : \mathcal{E} \to X$ is a fiber bundle, then if the fibers of π are $(n-1)$-connected, then π admits a section.*

Definition. A *fibration* is a map $\pi : \mathcal{E} \to B$ satisfying the *homotopy lifting property*: that is, if $h : [0,1] \times S \to X$ is a homotopy and $f : \{0\} \times S \to \mathcal{E}$, then f can be lifted across the whole homotopy, i.e. there exists an $\widetilde{f} : [0,1] \times S \to \mathcal{E}$ that makes the following diagram commute.

$$
\begin{array}{ccc}
\{0\} \times S & \xrightarrow{\ f\ } & \mathcal{E} \\
\big\downarrow & \overset{\widetilde{f}}{\nearrow} & \big\downarrow{\scriptstyle \pi} \\
[0,1] \times S & \xrightarrow{\ h\ } & X
\end{array}
$$

Theorem 4.2. *A fiber bundle is a fibration.*

We won't prove this, but we also won't use it extremely extensively.

Theorem 4.3. *Let $N, n \in \mathbb{Z}^{\geq 0}$ and $N \geq n/2$. Then, there is an isomorphism $\varphi : \pi_{n-1} \mathrm{U}_N \to \widetilde{K}(S^n)$ defined by clutching.*

This is part of Theorem 3.11 from last time. Recall that in the reduced K-theory, two bundles are equivalent iff they are stably isomorphic: for example, over S^2, the tangent bundle is stably isomorphic to any trivial bundle, so it's equal to zero.

Proof of Theorem 4.3. We'll show that φ is a composition of three isomorphisms

$$
\pi_{n-1} \mathrm{U}_N \xrightarrow{\ i\ } [S^{n-1}, \mathrm{U}_N] \xrightarrow{\ j\ } \mathrm{Vect}_N^{\cong}(S^n) \xrightarrow{\ k\ } \widetilde{K}(S^n).
$$

To define i, we'll pick a basepoint $* \in S^{n-1}$; then, $\pi_{n-1} \mathrm{U}_N$ is equal to $\{f : S^{n-1} \to \mathrm{U}_N : f(*) = e\}$ up to based homotopy (U_N is naturally a pointed space, using its identity element). We want this to be isomorphic to $[S^{n-1}, \mathrm{U}_N]$, the set of maps without basepoint condition up to homotopy, so let $\phi : [S^{n-1}, \mathrm{U}_N] \to \pi_{n-1} \mathrm{U}_N$ be defined by $\phi(f) = f(*)^{-1} \cdot f$, where $f : S^{n-1} \to \mathrm{U}_N$. Then, one can check that ϕ is well-defined on homotopy classes and inverts i, so i is an isomorphism.[5]

j is defined by the clutching construction. We can write $S^n = D^n_+ \cup_{S^{n-1}} D^n_-$, and then glue $\underline{\mathbb{C}}^N \to D^n_+$ and $\underline{\mathbb{C}}^N \to D^n_-$ using $f : S^{n-1} \to \mathrm{U}_N$, because U_N is the group of isometries of $\mathbb{C}^N$. So this defines a map j, but why is it an isomorphism? We have to show that j is surjective.

Last time, we showed that the group of isomorphisms deformation retracts onto the group of isometries, so that's fine. To show that j is surjective, we could use that every vector bundle admits a Hermitian metric, or that every vector bundle over D^n is trivializable by orthogonal bases, both of which are true. That j is well-defined follows from an argument that homotopic clutching functions lead to isomorphic vector bundles. Finally, to show that j is injective, all trivializations over D^n are homotopic, since D^n is contractible and U_N is connected.

Then, k just sends a vector bundle to its stable equivalence class. For its surjectivity, we need to show that if $E \to S^n$ has rank $N \geq n/2 + 1$, then there exists an E' of rank $N - 1$ and an isomorphism of the $\mathbb{C} \oplus E' \cong E$. In words, for large enough N, we can split off a trivial bundle from E. Equivalently, we can show that $E \to S^n$ admits a nonzero section, whose span is a line bundle $L \to X$ which is trivialized; then, we can let $E' = E/L$.

A nonzero section, normalized, is a section of the fiber bundle $S(E) \to S^{n-1}$ with fiber S^{2N-1} (the unit sphere sitting in $\mathbb{C}^N$).[6] This sphere is $(2n-2)$-connected, so by Theorem 4.1, such a section exists.

Why is k injective? We need to show that if a rank-N bundle is stably trivial in $\widetilde{K}(S^n)$, then it is actually trivial. But since it's not clear that $\mathrm{Vect}_N^{\cong}(S^n)$ is an abelian group (yet), then we'll show injectivity of sets. Let $E_0, E_1 \to S^n$ be rank-N vector bundles with an isometry $E_0 \oplus \underline{\mathbb{C}}^r \to E_1 \oplus \underline{\mathbb{C}}^r$; we'll want to produce a

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[5]$[S^{n-1}, \mathrm{U}_N]$ inherits another group structure from that of U_N (i.e. pointwise multiplication of loops); one can reason about it using something called the Eckmann-Hilton argument.

[6]The *sphere bundle* $S(E)$ of a vector bundle E is the fiber bundle whose fiber over each point x is the unit sphere in the E_x.

homotopic isometry which preserves the last vector $(0, \ldots, 0, 1) \in \mathbb{C}^r$ at each point in X. The evaluation map $\mathrm{ev}_{(0,\ldots,0,1)}$ at the last basis vector is a map of fiber bundles over X; that is, the following diagram commutes.

$$
\begin{array}{ccc}
\mathrm{Isom}(E_0 \oplus \underline{\mathbb{C}}^r, E_1 \oplus \underline{\mathbb{C}}^r) & \xrightarrow{\ \mathrm{ev}_{(0,\ldots,0,1)}\ } & S(E_1 \oplus \underline{\mathbb{C}}^r) \\
& \searrow \qquad \swarrow & \\
& S^n &
\end{array}
$$

An isometry is a section $\varphi : S^n \to \mathrm{Isom}(E_0 \oplus \underline{\mathbb{C}}^r, E_1 \oplus \underline{\mathbb{C}}^r)$, so applying the evaluation map, we get a section $p\varphi : S^n \to S(E_1 \oplus \underline{\mathbb{C}}^r)$. We get an additional section $\xi = (0, 0, \ldots, 0, 1)$. Thus, all that's left is to construct a homotopy from $p\varphi$ to ξ, which by the homotopy lifting property defines a section of the pullback $[0,1] \times S(E_1 \oplus \underline{\mathbb{C}}^r) \to [0,1] \times S^n$ over $\{0,1\} \times S^n$. $\boxtimes$

Note that, while the K-theory is a ring given by tensor product, the reduced K-theory isn't a ring in most cases.

These arguments are important to demonstrate that when N is high enough, in the stable range, we have this stability.

Corollary 4.4. *If N is in the stable range, i.e. $N \geq n/2$, then the inclusion $\mathrm{U}_N \hookrightarrow \mathrm{U}_{N+1}$ induces an isomorphism $\pi_{n-1} \, \mathrm{U}_N \to \pi_{n-1} \, \mathrm{U}_{N+1}$.*

This means that eventually $\pi_{n-1} \, \mathrm{U}_N$ is identical for large enough N; this group, the *stable isomorphism group* of the unitary groups, is written $\pi_{n-1}(\mathrm{U})$ (and there is a group U that makes this work, the limit of these U_N with the appropriate topology). Then, Bott's theorem, Theorem 3.11, calculates these groups: $\pi_{n-1} \, \mathrm{U}$ is $\mathbb{Z}$ when n is even and 0 when n is odd.

For example, a generator of $\pi_1 \, \mathrm{U}_3$ is given by stabilizing a loop $e^{i\theta}$; that is, it's given by the map

$$
e^{i\theta} \longmapsto \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
$$

with $\theta \in S^1$.

Outlining a Proof of Bott's Theorem. We'll move to providing different proofs of Theorem 3.11; these are explained in our readings, and so the professor won't post lecture notes for a little while.

Let's re-examine $S^2 \cong \mathbb{CP}^1 = \mathbb{P}(\mathbb{C}^2)$ (that is, the space of lines in $\mathbb{C}^2$). More generally, if V is a vector space, $\mathbb{P}V$ will denote its *projectivization*, the space of lines in V. Then, there is a tautological line bundle $H^* \to \mathbb{P}V$, whose fiber at a line $K \subset V$ (which is a point of $\mathbb{P}V$) is the line K.

The dual of H^* is called the *hyperplane bundle*, and denoted $H \to \mathbb{P}V$; a nonzero element of H can be identified with a hyperplane in V, and there is a canonical map $V^* \to \Gamma(\mathbb{P}V, H)$ (where $\Gamma(X, E)$ denotes the sections of $E \to X$): a linear functional on V becomes a linear functional on a line by restriction. Interestingly, if V is a complex vector bundle, then this is an isomorphism onto the holomorphic sections. In particular, the space of holomorphic sections is finite-dimensional.

In fact, if you take $\mathrm{Sym}^k V^*$, the kth symmetric power of V^*, then there's a canonical map $\mathrm{Sym}^k V^* \to \Gamma(\mathbb{P}V, H^{\otimes k})$, which is again an isomorphism in the complex case.

If $V = \mathbb{C}^2$, then write $V = L \oplus \mathbb{C}$; then, L and $\mathbb{C}$ are distinguished points in our projective space. This will enable us to make a clutching-like construction in a projective space.

Let $P_\infty = \mathbb{PC}^2 \setminus \{\mathbb{C}\}$ and $P_0 = \mathbb{PC}^2 \setminus \{L\}$; then, $P_0 \cap P_\infty \cong \mathbb{PC}^2 \setminus \{\mathbb{C}, L\} = L^* \setminus \{0\}$. Our clutching construction will start with a vector bundle $\underline{L} \to P_0$, a vector bundle $\underline{\mathbb{C}} \to P_\infty$, and an isomorphism $\alpha : \underline{L} \to \underline{\mathbb{C}}$ over the intersection $P_0 \cap P_\infty = L^* \setminus \{0\}$. Thus, we'll need to specify an isomorphism $P_0 \cap P_\infty \to L^* \setminus \{0\}$ to determine how to glue $\underline{L}$ and $\underline{\mathbb{C}}$ together.

It's natural to call the identity map z^{-1}, thinking of $z \in L$, and the bundle we get is $H \to \mathbb{PC}^2$. Here again we have a punctured plane and so the winding number classifies things.

Lemma 4.5. $H \oplus H \cong H^{\otimes 2} \oplus \mathbb{C}$ *as vector bundles over* $\mathbb{CP}^1 \cong S^2$.

Proof. The two clutching maps are, respectively, $\begin{pmatrix} z^{-1} & \\ & z^{-1} \end{pmatrix}$ and $\begin{pmatrix} z^{-2} & \\ & 1 \end{pmatrix}$. Each has determinant 1, so they're both in $\mathrm{SL}_2 \, \mathbb{C}$, which deformation-retracts onto S^2, which is simply connected. Thus, the clutching maps are homotopic. $\boxtimes$

Corollary 4.6. *If $t = [H] - 1$ in $K(S^2)$, then $t^2 = 0$.*

This is the first insight we have into the ring structure of a K-theory.

Corollary 4.7. *The map $\mathbb{Z}[t]/(t^2) \to K(S^2)$ sending $t \mapsto [H] - 1$ is an isomorphism of rings.*

Definition. Let X_1 and X_2 be topological spaces; then, there are projection maps

$$X_1 \times X_2 \xrightarrow{\;p_1\;} X_1$$
$$\downarrow{\scriptstyle p_2}$$
$$X_2.$$

Then, the *external product* is a map $K(X_1) \otimes K(X_2) \to K(X_1 \times X_2)$ defined as follows: if $u \in K(X_1)$ and $v \in K(X_2)$, then $u \otimes v \mapsto p_1^* u \cdot p_2^* v$.

Theorem 4.8. *If X is compact Hausdorff, then the external product $K(S^2) \otimes K(X) \to K(S^2 \times X)$ is an isomorphism of rings.*

We'll talk about this more next lecture; the idea is that in general distinguished basepoints of X and S^2 lift to subspaces of $S^2 \times X$.

The reason it doesn't work for S^1 is that if $X = S^1$, we get a torus $S^1 \times S^1$. Then, basepoints in S^1 give us $S^1 \vee S^1$ (the *wedge product*), and the quotient is $S^1 \wedge S^1 \simeq S^2$ (the *smash product*).

In fact, we'll bootstrap Theorem 4.8, using the smash product and reduced K-theory; then, results about smash products of spheres do a bunch of the work of periodicity for us. The proof will be elementary, in a sense, but with a lot of details about clutching functions, which is pretty explicit.

The version you'll read about in the Atiyah-Bott paper [3], or in Atiyah's book [2], is slightly more general. We want a family of S^2 parameterized by X, instead of just one, which is a fiber bundle; but we want two distinguished points, which will allow the clutching construction, and a linear structure.

Thus, more generally, if $L \to X$ is a complex line bundle, then $\mathbb{P}(L \oplus \underline{\mathbb{C}}) \to X$ is a fiber bundle with fiber S^2. We can once again form the hyperplane bundle $H \to \mathbb{P}(L \otimes \underline{\mathbb{C}})$.

Theorem 4.9 ([3])**.** *The map $K(X)[t]/(t[L] - 1)(t - 1) \to K(\mathbb{P}(L \oplus \underline{\mathbb{C}}))$ defined by sending $t \mapsto [H]$ is an isomorphism of rings.*

Then, if $X = \mathrm{pt}$, we recover Theorem 4.8, which we'll prove next time.

Lecture 5.

The K-theory of $X \times S^2$: 9/10/15

Our immediate goal is to prove the following theorem.

Theorem 5.1. *Let X be compact Hausdorff. Then, the map $\mu : K(X)[t]/(1 - t)^2 \to K(X) \otimes K(S^2) \to K(X \times S^2)$, defined by sending $[E] \cdot t \mapsto [E] \otimes [H]$ followed by $[E_1] \otimes [E_2] \mapsto [\pi_1^* E_1 \otimes \pi_2^* E_2]$, is an isomorphism.*

Next time, we'll introduce basepoints and use this to prove Bott periodicity, calculating the K-theory of the spheres in arbitrary dimension; we saw last time that this computes the stable homotopy groups of the unitary group.

The proof we give is due to Atiyah and Bott in [3], and actually proves a stronger result, Theorem 4.9. Hatcher's notes [7] provide a proof of the less general theorem.

The heuristic idea is that a bundle on S^2 is given by clutching data: two closed discs D_∞ and D_0 along with a circle $S^1 = \mathbb{T}$ (i.e. we identify it with the circle group $\mathbb{T} = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$, which is a Lie group under multiplication). Then, the final piece of clutching data is given by a group homomorphism $f : \mathbb{T} \to \mathrm{GL}_r \mathbb{C}$

Suppose f is given by a Laurent series

$$f(\lambda) = \sum_{k=-N}^{N} a_k \cdot \lambda^k,$$

with $a_k \in \mathrm{End}\,\mathbb{C}^r$ (i.e. they might not be invertible, but their sum is). Then, $f = \lambda^{-N} p$ for a $p \in \mathbb{C}[\lambda] \otimes \mathrm{End}\,\mathbb{C}^r$. Then, the K-theory class of this bundle is determined by the rank r and the winding number of $\lambda^{-N} p$,

which we'll denote $\omega(\lambda^{-N}p) = -Nr + \omega(p)$. That is, it's basically determined by the winding number of a polynomial.

What is the winding number of a polynomial? For simplicity, tke $r = 1$; then, $\omega(p)$ is the number of roots of p interior to $\mathbb{T} \subset \mathbb{C}$.

In some sense, we're taking the winding number as information about S^2, but we're not getting a lot of information about X. We *categorify*: we want to find a vector space whose dimension is $\omega(p)$. Set $R = \mathbb{C}[\lambda]$, which is a commutative ring, and $M = \mathbb{C}[\lambda]$ as an R-module. (If $r > 1$, we need to tensor with $\operatorname{End} \mathbb{C}^r$ again). Then, $p : M \to M$ given by multiplication by p, has a cokernel $\operatorname{coker} p = V$, a $\deg(p)$-dimensional vector space. Thus, we can canonically decompose $V = V_+ \oplus V_-$, where V_+ is the set of roots inside the unit disc. Then, we can soup this up further when $r > 1$ and X comes back into the story.

This is essentially the way that we'll prove the theorem: the proof will construct an inverse map ν to μ. The main steps are:

- approximate an arbitrary clutching by a Laurent series, leading to a polynomial clutching
- convert a polynomial clutching to a linear clutching, and
- convert a linear clutching to a vector bundle V over X.

Proof of Theorem 5.1. The first step, approximating by Laurent series, requires some undergraduate analysis. Suppose $f : X \times \mathbb{T} \to \mathbb{C}$ is continuous. The Fourier coefficients of a function on $\mathbb{T}$ become functions parameterized by X: set

$$a_n(x) = \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} f(x, e^{i\theta}) e^{-in\theta}, \quad n \in \mathbb{Z},$$

and let $u : X \times [0, 1) \times \mathbb{T} \to \mathbb{C}$ be

$$u(x, r, \lambda) = \sum_{n \in \mathbb{Z}} a_n(x) r^{|n|} \lambda^n.$$

Then, u is continuous, because

$$\|a_n\|_{C^0(X)} \leq \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \|f\|_{C^0(X \times \mathbb{T})} |e^{-in}| = \|f\|_{C^0(X \times \mathbb{T})}.$$

Proposition 5.2. $u(x, r, \lambda) \to f(x, \lambda)$ *as* $r \to 1$ *uniformly in x and λ.*

Proof. Introduce the *Poisson kernel* $P : [0, 1) \times \mathbb{T} \to \mathbb{C}$, given by

$$P(r, e^{is}) = \sum_{n \in \mathbb{Z}} r^{|n|} e^{ins} = \frac{1 - r^2}{1 - 2r \cos s + r^2}, \tag{5.1}$$

which can be proven by treating the positive and negative parts as two geometric series. Then, since it converges absolutely, we can integrate term-by-term to show that

$$\int_{\mathbb{T}} \frac{\mathrm{d}s}{2\pi} P(r, e^{-is}) = 1.$$

Additionally, if $\lambda \neq 1$, (5.1) tells us that $\lim_{r \to 1} P(r, \lambda) = 0$. Thus, $\lim_{r \to 1} = \delta_1$ in $C^0(\mathbb{T})^*$ (i.e. $\delta_1(f) = f(1)$ for $f \in C^0(\mathbb{T})$, as a distribution). Now, we can write u as a convolution on $\mathbb{T}$:

$$u(x, r, e^{i\theta}) = \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} P(r, e^{i(\theta - \phi)}) f(x, e^{i\phi})$$
$$= P_\theta(r, -) *_{\mathbb{T}} f(x, -)$$
$$= \langle \widetilde{P}_\theta(r, -), f(x, -) \rangle,$$

where our pairing is a map $C^0(\mathbb{T})^* \times C^0(\mathbb{T}) \to \mathbb{C}$. $\boxtimes$

This will allow us to approximate a clutching function with a finite step in the Fourier series, producing a Laurent series as intended.

Corollary 5.3. *The space of Laurent functions*

$$\sum_{|k| \leq N} a_k(x) \lambda^k$$

is dense in $C^0(X \times \mathbb{T})$.

Proof. If $f \in C^0(X \times \mathbb{T})$, define a_k and u as before. Given an $\varepsilon > 0$, there's an r_0 such that $\|f - u(r)\|_{C^0(\times \mathbb{T})} < \varepsilon/2$ if $r > r_0$, and an N such that

$$\sum_{|n| > N} r_0^N < \frac{\varepsilon}{2\|f\|_{C^0(X \times \mathbb{T})}}.$$

Then, one can show that the norm of the difference is less than ε.   ⊠

Thus, we have our approximations of clutching bundles. Note that Hatcher's proof in [7] involves a little less "undergraduate" analysis.

Thinking about S^2 as $\mathbb{P}(\mathbb{C}_0 \oplus \mathbb{C}_\infty) = \mathbb{CP}^1$, we can look at the tautological bundle. If $\lambda \in \mathbb{C}$, then the line $y = \lambda x$ in $\mathbb{C}_0 \times \mathbb{C}_\infty$ projects down, e.g. $(1, \lambda)$ to 1 and λ. In particular, the tautological bundle $H^* \to \mathbb{CP}^1 = S^2$ has clutching function λ, and therefore the hyperplane bundle $H \to \mathbb{CP}^1 = S^2$ has clutching function λ^{-1}.

For a more general $\mathcal{E} \to X \times S^2$, we want to clutch $X \times D_0$ and $X \times D_\infty$ at $X \times \mathbb{T}$. Define $E \to X$ as the restriction of $\mathcal{E} \to X \times S^2$ to $X \times \{1\}$; then, E pulls back to bundles $\pi_0^* E \to X \times D_0$ and $\pi_\infty^* E \to X \times D_\infty$. Since D_0 and D_∞ are contractible, we can choose isomorphisms $\theta_0 : \pi_0^* E \overset{\cong}{\to} \mathcal{E}|_{X \times D_0}$ and $\theta_\infty : \pi_\infty^* E \overset{\cong}{\to} \mathcal{E}|_{X \times D_\infty}$.

Then, $f = \theta_\infty^{-1} \circ \theta_0$ is a section of the bundle $\mathrm{Aut}(\pi_\mathbb{T}^* E) \to X \times \mathbb{T}$. In other words, $X \times \mathbb{T}$ embeds into $X \times D_0$ and $X \times D_\infty$, and f is the clutching data from $\pi_0^* E \to \pi_\infty^* E$.

Also, we can and will choose θ_0, θ_∞ to be the identity on $X \times \{1\}$, so that f is the identity there too.

Notationally, we'll write $[\mathcal{E}] = [E, f] \in K(X \times S^2)$; we can start with an $E \to X$ and such an f, an automorphism of $E \times \mathbb{T} \to X \times \mathbb{T}$, to get a vector bundle on $X \times S^2$. For example, $[\mathbb{C}, \lambda] = [H^*]$, $[\mathbb{C}, \lambda^n] = [H^{\otimes(-n)}]$, and $[E, f \cdot \lambda^n] = [E, f] \cdot [H^{\otimes(-n)}]$ in $K(X \times S^2)$ (which one can check).

What this argument shows is the following.

Proposition 5.4. *Any vector bundle on $X \times S^2$ is isomorphic to one of the form (E, f), and any two choices of f are homotopic through normalized clutching functions.*

Here, a normalized clutching function is one homotopic through the basepoint.

Now we have our clutching function, which is continuous, and replace it with a Laurent function.

Proposition 5.5.

(1) *In $K(X, S^2)$, $[E, f] = [E, \lambda^{-N} p]$ for some polynomial clutching function*

$$p(x, \lambda) = \sum_{k=0}^{2n} a_k(x) \lambda^k,$$

with $a_k(x) \in \mathrm{End}\, E_x$.

(2) *Any two such choices are homotopic via a Laurent clutching function.*

Proof. The proof will show that the Laurent endomorphisms of $E \times \mathbb{T} \to X \times \mathbb{T}$. If $E = \mathbb{C}$, the proof is the same proof with Poisson kernels at the start of the class; more generally, we'll use a partition of unity $\{\rho_i\}$ subordinate to a cover $\{U_i\}$ such that $E|_{U_i}$ is trivial. Then, $f|_{U_i}$ can be approximated by a Laurent ℓ_i, and one can check that $\sum \rho_i \ell_i$ is Laurent.

For (1), since the invertible matrices are an open set, then choose an $\varepsilon > 0$ such that $B_\varepsilon(f)$ contains only invertible functions, and choose an ℓ Laurent such that $\|f - \ell\|_{C^0(X \times \mathbb{T})} < \varepsilon$, so that ℓ is invertible and $f \simeq \ell$ by a straight-line homotopy. And we know clutching with homotopic functions doesn't change the isomorphism class of the vector bundle, hence nor the K-theory class.   ⊠

Thus, we've gone from continuous to Laurent; now, we will go from Laurent to linear. Observe that $[E, f] = [E, -\lambda^N p] = [H^{\otimes N}] - [E, p]$.

Let p be a polynomial clutching function of degree at most n. Then, write

$$p(x, \lambda) = \sum_{k=0}^{n} p_k(x) \lambda^k,$$

and set

$$\mathcal{L}_p^m = \begin{pmatrix} 1 & -\lambda & & & \\ & 1 & -\lambda & & \\ & & \ddots & \ddots & \\ & & & 1 & -\lambda \\ p_n & p_{n-1} & \cdots & p_1 & p_0 \end{pmatrix}.$$

This matrix of polynomials acts linearly on $E^{\oplus(n+1)} \times \mathbb{T} \to X \times \mathbb{T}$.

Proposition 5.6. $[E^{\oplus(n+1)}, \mathcal{L}_p^n] = [E, p] + [E^{\oplus n}, 1]$.

Proof. The clutching function for the right-hand side is

$$\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & p \end{pmatrix},$$

and this is exactly the matrix you get if you diagonalize $\mathcal{L}_p^n$ by elementary row and column operations. Thus, they're homotopic, and so have the same class in K-theory.   ⊠

We'll then make a basic spectral construction. Suppose $T \in \operatorname{End} \mathbb{E}$ has no eigenvalues on the unit circle $\mathbb{T} \subset \mathbb{C}$. Then, take the contour integral

$$Q = \frac{1}{2\pi i} \int\limits_{|\omega|=1} (\omega - T)^{-1} \, \mathrm{d}\omega,$$

which is in $\operatorname{End} \mathbb{E}$. One can check that $Q^2 = Q$, so it's a projection, and $QT = TQ$. Thus, we can decompose $\mathbb{E} = Q\mathbb{E} \oplus (1 - Q)\mathbb{E}$, which we'll denote $\mathbb{E}_+$ and $\mathbb{E}_-$, respectively. Since T commutes with Q, $T = (T_+, T_-)$, with T_+ acting on $\mathbb{E}_+$, and similarly for T_- on $\mathbb{E}_-$. This is analogous to the spectral theorem's decomposition of an operator into its generalized eigenspaces.

Proposition 5.7. *Let $[E, q]$ be a K-theory class with $q(x, \lambda) = a(x)\lambda + b(x)$. Then, there is a splitting $E = E_+ \oplus E_-$ such that $[E, q] = [E_+, \lambda] + [E_-, 1]$.*

Proof. Define

$$Q = \frac{1}{2\pi i} \int\limits_{|\lambda|=1} q^{-1} \, \mathrm{d}q = \frac{1}{2\pi i} \int\limits_{|\lambda|=1} q^{-1} \frac{\partial q}{\partial \lambda} \, \mathrm{d}\lambda.$$

Choose an $\alpha \in \mathbb{R}^{>1}$ such that $q(x, \alpha)$ is an isomorphism for all x, which works because isomorphism is an open condition. Then, compose with $q(x, \alpha)^{-1}$, so we can assume $q(x, \alpha) = \mathrm{id}$. Then, $w = (1 - \alpha\lambda)/(\lambda - \alpha)$ preserves $\mathbb{T}$ and D_0 as $\alpha \to \infty$. Define $q(\lambda) = (w - T)/(w + \alpha)$ with $T \in C^0(X; \operatorname{End} E)$, and $qT = Tq$. Then,

$$Q = \frac{1}{2\pi i} \int\limits_{|w|=1} (w - T)^{-1} \, \mathrm{d}w - (w + \alpha)^{-1} \, \mathrm{d}w,$$

but the last term goes away. Thus, this is the desired projection: q fails to be invertible exactly where T has an eigenvalue. Denote $q_\pm(\lambda) = a_\pm\lambda + b_\pm$, and $q_+(\lambda)$ is invertible if $\lambda \in D_\infty$ and $q_-(\lambda)$ is if $\lambda \in D_0$.

Thus, $q_+^t = a_+\lambda + tb_+$ and $q_-^t = ta_-\lambda + b_-$ are homotopies of clutching functions, so

$$\begin{aligned} [E, q] &= [E_+, q_+] + [E_-, q_-] \\ &= [E_+, a_+\lambda] + [E_-, b_-] = [E_+, \lambda] + [E_-, 1]. \end{aligned}$$ ⊠

So if we have $[E, p]$ with $\deg(p) \le n$, then

$$[E, p] = [E^{\oplus(n+1)}], \mathcal{L}_p^n] - [E^{\oplus n}, q],$$

and we just proved that a linear clutching function splits as

$$\begin{aligned} &= [V_n(E, p), \lambda] + [E^{\oplus(n+1)}, 1] - [V_n(E, p), 1] - [E^{\oplus n}, 1] \\ &= [V_n(E, p)] \otimes ([H^*] - 1) + [E] \otimes 1, \end{aligned}$$

where $V_n(E, p)$ is the + part of the decomposition of $E^{\oplus(n+1)}$ by $q = \mathcal{L}_p^n$. So we've gone from polynomial to linear and then split it; this will allow us to define the inverse, check it's well-defined and in fact the inverse, and so on. But this is enough of a proof sketch to follow the references and work out the details.   ⊠

Even though the proof is confusing, all of the ideas are relatively elementary.

Lecture 6.

The K-theory of the Spheres: 9/15/15

Recall that last time, we mostly proved Theorem 5.1, but didn't pin down our inverse. The details are mostly in [3], as well as in the expositions in [2, 7]. We'll then use it to prove Bott periodicity.[7]

Recall that the idea was to take a bundle $\mathcal{E} \to X \times S^2$ and decompose. Here's the proof at the executive summary level.

(1) Write it as (E, f) for $E \to X$ and f a clutching function, an automorphism of $(E \times \mathbb{T} \to X \times \mathbb{T})$.
(2) Homotope f to a Laurent clutching function, which is canonical: for $n \in \mathbb{Z}$, we get

$$a_n(x) = \int_{\mathbb{T}} \frac{\mathrm{d}\theta}{2\pi} f(x, e^{i\theta}) e^{-in\theta}.$$

Notice $f(x, e^{i\theta})$ is in $\operatorname{Aut} E_x$, but we're not averaging in this group, just in $\operatorname{End} E_x$, and therefore there's no guarantee that a_n is invertible. We can form

$$u_N(x, \lambda) = \sum_{|x| \leq N} a_n(x) \lambda^n,$$

with $N \in \mathbb{Z}^{>0}$. This isn't *a priori* invertible, but there's some N_0 (depending on f) such that if $N \geq N_0$, then u_N is invertible and homotopic to f through invertible clutching functions.
(3) If p is a polynomial clutching function of degree at most d on E, then we constructed a polynomial clutching function $\mathcal{L}^d p$ on $E^{\oplus(d+1)}$, and from this linear clutching function we extracted a bundle $V_d(E, P) \to X$ such that

$$[E, p] = V_d(E, p) \otimes ([H^*] - 1) + [E, 1]. \tag{6.1}$$

in $K(X \times S^2)$; note that $[E, 1] = [E] \otimes 1$.

This is all great, if we have a polynomial clutching function magically at the beginning. But from the construction we also know the following.

(i) If $p_0 \simeq p_1$ through polynomial clutching functions, then $V_d(E, p_0) \simeq V_d(E, p_1)$. This is our basic homotopy invariance.
(ii) $V_{d+1}(E, p) \cong V_d(E, p)$; this depends more explicitly on the construction we gave last time. Notice how this is consistent with (6.1).
(iii) $V_{d+1}(E, \lambda p) \cong V_d(E, p) \oplus E$.

That was all from last time; now, we'll construct an inverse, check that it's well-defined, and show that it's the inverse. The details of the proof from last time need some filling-in, but this is something we'll be able to do.

Construction of the inverse. We're going to cook up a $\nu : K(X \times S^2) \to K(X)[t]/(1-t)^2$. Given an $\mathcal{E} \to X \times S^2$, choose an f such that $\mathcal{E} \cong (E, f)$, where $E = \mathcal{E}|_{X \times \{1\}} \to X$. For N sufficiently large (greater than an N_0 depending on f), we have $f \simeq u_N = \lambda^{-N} p_N$, where p_N is a polynomial of degree at most $2N$. Then, define

$$\nu_N(E, f) = [V_{2N}(E, p_N)](t^{-1} - 1)t^N + [E]t^N.$$

First, we must check that

(1) it's independent of N given f, and then
(2) that it's independent of f,

---

so that we get a function in $\mathcal{E}$.

For (1), we'll use that $p_{N+1} \simeq \lambda p_N$ via polynomial clutching functions of degree at most $2(N+1)$ if N is sufficiently large: multiplying by λ shifts all of the coefficients, so all that changes is the top-order term λ^{2N+2} and the constant term a_{-1}. Since these are invertible when N is sufficiently large, then we can go from one to the other with a straight-line homotopy, which is polynomial. Then,

$$\nu_{N+1}(E,f)[V_{2N+2}(E, p_{N+1})](1-t)t^N + [E]t^{N+1}$$
$$= [V_{2N+2}(E, \lambda p^N)](1-t)t^N + [E]t^{N+1},$$

so using property (ii),

$$= [V_{2N+1}(E, \lambda p^N)](1-t)t^N + [E]t^{N+1}.$$

Then, using property (iii),

$$= [V_{2N}(E,p)](1-t)tt^{N-1} + [E]t^N$$
$$= [V_{2N}(E,p)](1-t)t^{N-1} + [E]t^N = \nu_N(E,f).$$

Here, we used the fact that $(1-t)t = 1-t$ in this ring, and so ν is independent of N for sufficiently large N.

To show the independence of ν from f, we'll make a truncation argument: if f_0 and f_1 are sufficiently C^0-close, then their truncations at N are also homotopy equivalent, because they'll both be invertible at the same time. Thus, this is locally constant on homotopy classes, and therefore constant on homotopy classes: $\nu(E, f_0) = \nu(E, f_1)$. In particular, ν factors through the homotopy class of f and therefore depends only on $\mathcal{E}$.

Now, we need to show that $\mu \circ \nu = \mathrm{id}_{K(X \times S^2)}$. Well, it was rigged to be the identity: look at (6.1) and the definition of ν; you get back what you started with. In the opposite direction, to check that $\nu \circ \mu = \mathrm{id}_{K(X)[t]/(1-t)^2}$, use the fact that ν is a $K(X)$-module homomorphism, and therefore some information about tensor products factors through. But then we just have to check on the generators $\nu \circ \mu(t^N)$ for $N \geq 0$. This requires one more fact, that $V_N(\mathbb{C}, 1) = 0$, which also follows from what we did last time. ⊠

Once again, this is a little more of a summary; it would be hard to give all of the details in lecture, and you can work them out using these ideas and techniques.

Computing $K(S^n)$. The rest of the lecture will deal with some elementary homotopy theory, useful not just in K-theory but also in plenty of other parts of topology. We'll use it to inductively calculate $K(S^n)$ using Theorem 5.1; note that $S^n \times S^2 \neq S^{n+2}$, but we'll be able to use smash products to do our bidding instead.

More specifically, $S^1 \times S^1$ isn't a sphere; it's a torus. But if you collapse the fundamental rectangle of the torus by two boundaries, you take $(S^1 \times S^1)/(S_1 \vee S^1)$, which gives us S^2, in a sense we'll clarify. We'll want to generalize this to inductively construct n-spheres.

Definition.

(1) A *pointed space* (X, x) is a topological space X along with some point $x \in X$.

(2) A *map of pointed spaces* $f : (X, x) \to (Y.y)$ is a continuous map $f : X \to Y$ such that $f(x) = y$. We also require homotopies $f : [0,1] \times X \to Y$ to preserve the basepoint: $f(x, t) = y$ for all $t \in [0,1]$.

Pointed spaces and their maps form a category, as well as those with additional properties, such as pointed Hausdorff spaces, pointed CW complexes, and so on.

Recall that we have defined the reduced K-theory for any space X, given by $\widetilde{K}(X) = K(X)/\mathrm{Im}(K(\mathrm{pt}) \to K(X))$, where $K(\mathrm{pt}) \to K(X)$ is induced by the unique map $\pi : X \to \mathrm{pt}$. But if (X, x) is a pointed space, then we have a splitting $\mathrm{pt} \mapsto x$ (in particular, X is nonempty). So we have two pullbacks: $x^{\cdot}K(X) \to K(\mathrm{pt})$, and $\pi^* : K(\mathrm{pt}) \to K(X)$. Thus, we can split off a summand: $K(X) \cong k(X) \oplus \pi^* K(\mathrm{pt})$, where $k(X) = \ker x^*$, and the projection $K(X) \to \widetilde{K}(X)$ restricts to an isomorphism $k(X) \xrightarrow{\cong} \widetilde{K}(X)$.

In summary, for pointed spaces, we can take the reduced K-theory to be a subspace rather than a quotient space, and specifically, the subspace that reduces to 0 at the basepoint.

This is a pretty important idea: when we're making topology out of contravariant objects, we can more generally consider the subgroup that restricts to zero at the basepoint. In K-theory specifically, vector bundles can't exactly restrict to 0, then we have that if $E^0, E^1 \to X$, then $[E^0] - [E^1] \in K(X)$ restricts to 0 at an x iff $\mathrm{rank}_x E^0 = \mathrm{rank}_x E^1$.

We can generalize this to subspaces $A \subset X$ rather than basepoints.[8] Then, we can look at things that restrict to zero on A, and use these to define a more general reduced K-theory. This is powerful: for example, if X is a CW complex, we get a filtration from its skeleton, $X_0 \subset X_1 \subset \cdots \subset X$, and this induces a filtration on K-theory.

Given two pointed spaces (X, x) and (Y, y), we can make a few useful constructions in the category of pointed spaces out of them.

Definition.
(1) The *wedge* $X \vee Y = X \coprod Y/(x \sim y)$, which is a pointed space with the identified x and y as its basepoint.
(2) The *smash*[9] is $X \wedge Y = (X \times Y)/(X \vee Y)$; once again, there is a unique image of (x, b) and (a, y), and this becomes our basepoint.
(3) The *suspension* $\Sigma X = S^1 \wedge X$. You can think of this as two cones on X collapsed by the unit interval. The unique image of the old basepoint becomes the new basepoint. Sometimes, this is called the *reduced suspension* of X.
(4) The *cone* $CX = [0, 1] \wedge X$, turned into a pointed space by taking 0 to be the basepoint of $[0, 1]$. Basically, we collapse to a point at 1; if $X = S^2$, this is the familiar cone in $\mathbb{R}^3$.

Notice that a map extending over the cone is a null homotopy; the large number of ideas that can be stated in similar terms illustrate that these can be very useful constructions.

Proposition 6.1. *Let (X, a) be a compact Hausdorff pointed space and $A \subset X$ be a subspace containing A. Then, the sequence*

$$A \overset{i}{\hookrightarrow} X \overset{q}{\twoheadrightarrow} X/A, \tag{6.2}$$

with i given by inclusion and q given by quotient, induces an exact sequence of abelian groups

$$\widetilde{K}(X/A) \overset{q^*}{\longrightarrow} \widetilde{K}(X) \overset{i^*}{\longrightarrow} \widetilde{K}(A).$$

Notice that the image of a in A, X, and X/A is our basepoint.

Proof. We'll prove it in the case of CW complexes; for a more general proof, see [7].

The composition $q \circ i$ sends A to a point, so $(q \circ i)^* = i^* \circ q^*$ has image constant vector bundles, which vanish in $\widetilde{K}(A)$. Thus, $i^* \circ q^* = 0$.

If $E \to X$ restricts to be stably trivial on A, then after adding a constant bundle, we can assume $E|_A \to A$ is trivial. So choose a trivialization; then, clutching with it produces a bundle on X/A whose image under q^* is isomorphic to E. In some sense, pt is attached to every point of A, and so we get the same fiber over every point in A, and then can clutch in that way. Certainly, we get a family of vector spaces, but we actually get a vector bundle $E \to X/A$; local triviality is only nontrivial at the basepoint (which is in a sense all of A), which follows because it's true in a deformation retract neighborhood; this exists for CW complexes.   ⊠

Now, we can employ a standard construction called the *Puppe sequence*: we'll extend (6.2) to the sequence

$$A \overset{i}{\longrightarrow} X \overset{q}{\longrightarrow} X/A \longrightarrow \Sigma A.$$

This is because $X/A \simeq X \cup_A CA$ (since replacing A with CA makes A within X null-homotopic, so we're taking the quotient by it), and $\Sigma A \simeq X \cup_A CA/X$ by definition, and we can do this by attaching a cone on X (this may be confusing; it helps to draw a picture). Thus, we can extend further to

$$A \overset{i}{\longrightarrow} X \overset{q}{\longrightarrow} X/A \longrightarrow \Sigma A \overset{\Sigma i}{\longrightarrow} \Sigma X \overset{\Sigma q}{\longrightarrow} \Sigma(X/A) \longrightarrow \Sigma^2 A \longrightarrow \cdots$$

This sequence can be made from any contravariant functor of geometric objects.

Corollary 6.2. *There exists a long exact sequence*

$$\cdots \longrightarrow \widetilde{K}(\Sigma(X/A)) \longrightarrow \widetilde{K}(\Sigma X) \longrightarrow \widetilde{K}(\Sigma A) \longrightarrow \widetilde{K}(X/A) \longrightarrow \widetilde{K}(X) \longrightarrow \widetilde{K}(A). \tag{6.3}$$

---

[8]Well, the basepoint actually sits inside A, but that won't matter so much.
[9]This can be confusing: in LATEX, $\wedge$ is called "wedge," and $\vee$ is called "vee."

This can be quite computationally useful, as in the following example.

Lemma 6.3. *Restriction induces an isomorphism $\widetilde{K}(X \vee Y) \to \widetilde{K}(X) \oplus \widetilde{K}(Y)$ for pointed spaces X and Y.*

Proof. We'll apply (6.3) to $x = y \in X \vee Y \subset X \times Y$. We have projections $\pi_1, \pi_2 : X \vee Y \rightrightarrows X, Y$, and then $\pi_1^* \oplus \pi_2^*$ is a section for the map $\widetilde{K}(X \times Y) \to \widetilde{K}(X \vee Y)$ (and similarly, there's a section $\Sigma^n s$ at every level in the long exact sequence). Thus, the long exact sequence breaks into a list of short exact sequences. TODO: what happened here? ⊠

Corollary 6.4. $\widetilde{K}(X \times Y) \cong \widetilde{K}(X) \oplus \widetilde{K}(Y) \oplus \widetilde{K}(X \wedge Y)$.

We'll be able to use this to construct a product map $\widetilde{K}(X) \otimes \widetilde{K}(Y) \to \widetilde{K}(X \wedge Y)$: if $u \in X$ and $v \in Y$, then $\pi_1^* u \cdot \pi_2^* v$ vanishes on $X \vee Y \subset X \wedge Y$, and so $\pi_1^* u \cdot \pi_2^* v \in \widetilde{K}(X \wedge Y)$ in the decomposition in Corollary 6.4.

There's a pointed version of the μ we constructed when proving Theorem 5.1.

Theorem 6.5. *The map $\beta : \widetilde{K}(X) \to \widetilde{K}(\Sigma^2 X) = \widetilde{K}(S^2 \wedge X)$ sending $u \mapsto ([H] - 1) \cdot u$ is an isomorphism.*

All of this is written up more carefully in [7]; next time, we'll turn K-theory into a cohomology theory and use it to prove a result about division algebras.

Lecture 7.

Division Algebras Over $\mathbb{R}$: 9/17/15

"I've been posting problems... they're not just for my health."

First, we'll discuss some points that were rushed through last time. if (X, x) is a pointed space, then $K(X) \cong \widetilde{K}(X) \oplus K(\{x\})$, and the latter summand is infinite cyclic. We will want to think of vector bundles that vanish at the basepoint, so we associate to a class $[E]$ the class $([E] - [\underline{E_x}]) \oplus [E_x]$ (i.e. subtract the constant bundle formed from the fiber at x).

Then, Proposition 6.2 tells us that if $A \hookrightarrow X \twoheadrightarrow X/A$ and X is compact, then we get an exact sequence $\widetilde{K}(A) \leftarrow \widetilde{K}(X) \leftarrow \widetilde{K}(X/A)$, assuming there exists a deformation retraction of a neighborhood of A in X back to A, for example when X is a CW complex and A is a subcomplex. Then, we converted this into a longer sequence called the Puppe sequence, using suspensions of A, X, and A/X.

Proposition 7.1. *If X and Y are compact Hausdorff, the sequence $X \vee Y \to X \times Y \to X \wedge Y$, which induces a split exact sequence*

$$0 \longrightarrow \widetilde{K}(X \wedge Y) \longrightarrow \widetilde{K}(X \times Y) \longrightarrow \widetilde{K}(X \vee Y) \longrightarrow 0.$$

Since $\widetilde{K}(X \vee Y) \cong \widetilde{K}(X) \oplus \widetilde{K}(Y)$, this gives us an isomorphism $\widetilde{K}(X \times Y) \cong \widetilde{K}(X \wedge Y) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y)$.

Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the canonical projections; then, we get an external product $\pi_1^* u \cdot \pi_2^* v$ for $u \in \widetilde{K}(X)$ and $v \in \widetilde{K}(Y)$. This product restricts to 0 in $X \vee Y$, and hence by the above proposition is pulled back from a unique $u * v \in \widetilde{K}(X \wedge Y)$; thus, we have a product $* : \widetilde{K}(X) \otimes \widetilde{K}(Y) \to \widetilde{K}(X \wedge Y)$.

By FOILing, the following diagram commutes.

$$
\begin{array}{ccc}
K(X) \otimes K(Y) & \xrightarrow{\cong} & (\widetilde{K}(X) \otimes \widetilde{K}(Y)) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y) \oplus \mathbb{Z} \\
\downarrow{\scriptstyle *} & & \downarrow{\scriptstyle (*,\mathrm{id},\mathrm{id},\mathrm{id})} \\
K(X \times Y) & \xrightarrow{\cong} & \widetilde{K}(X \wedge Y) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y) \oplus \mathbb{Z}
\end{array}
$$

And we proved that if X is compact Hausdorff, then $\times : K(S^2) \otimes K(X) \to K(S^2 \times X)$ is an isomorphism, and therefore $\beta : \widetilde{K}(X) \to \widetilde{K}(\Sigma^2 X)$ sending $u \mapsto ([H] - 1) * u$ is an isomorphism. Then, by induction, we get our nice result.

Corollary 7.2.

$$\widetilde{K}(S^n) = \begin{cases} 0, & n \text{ odd} \\ \mathbb{Z}, & n \text{ even.} \end{cases}$$

There are many things called Bott periodicity; this one is equivalent to Bott's original one, which used Morse theory and calculated $\pi_{n-1} U$. Things are slightly different in the real case, which we will be able to prove as well.

We'll spend the rest of this lecture and part of next lecture proving the following statements.

Proposition 7.3. *For an $n \in \mathbb{Z}^{>0}$, (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) in the following.*

    (1) $\mathbb{R}^n$ admits the structure of a division algebra.
    (2) S^{n-1} is parallelizable.
    (3) S^{n-1} is an H-space.
    (4) There exists an $f : S^{2n-1} \to S^n$ of Hopf invariant 1.

We'll define H-spaces, division algebras, and the Hopf invariant shortly. Then, we can use this to get a nice result.

Theorem 7.4. *If there exists an $f : S^{2n-1} \to S^n$ of Hopf invariant 1, then $n = 1, 2, 4,$ or 8.*

Corollary 7.5 (Milnor [11], Kervaire [9]). *$\mathbb{R}^n$ admits a divison algebra structure iff $n = 1, 2, 4,$ or 8.*

Now, what do all of these words mean?

Definition. A *unital division algebra* is a vector space A and a linear map $m : A \times A \to A$ and an $e \in A$ such that

    (1) $m(e, -) = m(-, e) = \mathrm{id}_A$.
    (2) $m(x, -)$ and $m(-, y)$ are bijective maps $A \to A$ when $x, y \neq 0$.

Notice that this is required to be neither associative nor commutative.

It's quite striking that, yet again, we're proving a theorem from pure algebra using topology! But for existence, we will have to do a little algebra.

When $n = 1$, we have $\mathbb{R}$, and when $n = 2$, we have $\mathbb{C}$, both familiar fields. When $n = 4$, we have the *quaternions* $\mathbb{H} = \mathbb{R}\{1, i, j, k\}$ with multiplication relations $i^2 = j^2 = k^2 = 1$, $ij = k$, and $ji = -k$. This multiplication is associative, but not commutative, so $\mathbb{H}$ isn't a field. Finally, when $n = 8$, we have the *octonions* or *Cayley numbers* $\mathbb{O}$, an eight-dimensional vector space over $\mathbb{R}$ with basis $\{1, e_1, e_2, \ldots, e_7\}$, with a kind of complicated multiplication table given in Figure 1. This is in some sense projective geometry over $\mathbb{F}_2$: there's a lot of interesting math to be said about this structure, and a good article to begin reading is [4].
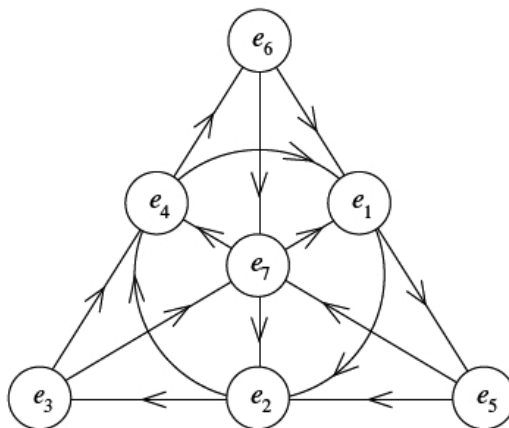

FIGURE 1. The *Fano plane*, a way to remember the rules of octonion multiplication. The rule is, $e_i^2 = -1$, and to determine $e_i \cdot e_j$, choose the third point on the line containing them, and add a minus sign if you went against the direction of the arrows. For example, $e_5 \cdot e_2 = e_3$ and $e_7 \cdot e_3 = -e_1$. Source: [4].

Definition. An *H-space* is a pointed topological space (X, e) together with an unpointed map $g : X \times X \to X$ such that $g(e, -) = g(-, e) = \mathrm{id}_X$.

This is sort of a very lax version of a topological group, with no associativity.

Finally, we'll get to the Hopf invariant later.

Partial proof of Proposition 7.3. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. Then, for any $x \in S^{n-1}$, $(x \cdot e_1)e_1, \ldots, (x \cdot e_n)e_n$ is a basis of $\mathbb{R}^n$, so use the Gram-Schmidt process to convert this into an orthonormal basis $\xi_1(x), \ldots, \xi_n(x)$ of $\mathbb{R}^n$. (For example, $\xi_1(x) = x \cdot e_1/\|x \cdot e_1\|$). Then, observe that $S^{n-1} \to S^{n-1}$ sending $x \mapsto \xi_i(x)$ is a diffeomorphism, so (1) $\implies$ (2).

For (2) $\implies$ (3), suppose $\eta_2(x), \ldots, \eta_n(x)$ is a basis of $T_x S^{n-1}$; then, use Gram-Schmidt again to get an orthonormal basis $\xi_2(x), \ldots, \xi_n(x)$ of $T_x S^{n-1}$, and therefore $x, \xi_2(x), \ldots, \xi_n(x)$ is an orthonormal basis of $\mathbb{R}$.

Then, compose with a fixed orthogonal transformation so that $(e_1, \xi_2(e_1), \ldots, \xi_n(e_1)) = (e_1, e_2, \ldots, e_n)$. Define $\alpha : S^{n-1} \to \mathrm{SO}(n)$ by $\alpha_x(e_1, \ldots, e_n) = (x, \xi_2(x), \ldots, \xi_n(x))$, and $g : S^{n-1} \times S^{n-1} \to S^{n-1}$ by $x, y \mapsto \alpha_x(y)$. Since $(e_1, y) \mapsto y$ and $(x, e_1) \mapsto x$, then this gives us an H-space structure. ⊠

The following theorem from the 1940s was originally proven with cohomology, but our K-theoretic proof of Theorem 7.4 will be a little cleaner.

Theorem 7.6 (Hopf)**.** *If $\mathbb{R}^n$ is a division algebra, then n is a power of 2.*

Proof. Multiplication $m : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ induces a map $g : S^{n-1} \times S^{n-1} \to S^{n-1}$ given by sending $x, y \mapsto (x \cdot y)/\|x \cdot y\|$. This sends antipodal points to antipodal points, so we get a quotient $\overline{g} : \mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1} \to \mathbb{RP}^{n-1}$.

We'll use $H^\bullet(\mathbb{RP}^{n-1}; \mathbb{F}_2)$; specifically, we'll use the cup product. This is a very powerful tool, but it's considerably more obscure than the K-theoretic product. Specifically, we have that $H^\bullet(\mathbb{RP}^{n-1}; \mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^n)$, with $\deg x = 1$.

For our map $\overline{g}$, let x, y, and z be the respective generators of our three copies $\mathbb{RP}^{n-1}$ (x nd y for the domain, and z for the range). Cohomology gives us a pullback $\overline{g}^*$, and in fact $\overline{g}^*(z) = x + y$: z must be sent to another 1-dimensional class, which is therefore generated by some projective lines. Looking at exactly what $\overline{g}$ is doing, we can conjugate the second to the identity, and so we get $x + y$ in cohomology.[10] Thus, by the binomial theorem,

$$0 = \overline{g}^*(z^n) = (x + y)^n$$
$$= x^n + \sum_{k=1}^{n-1} \binom{n}{k} x^{n-k} y^k + y^n$$
$$= \sum_{k=1}^{n-1} \binom{n}{k} x^{n-k} y^k,$$

which lies in $H^\bullet(\mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1}; \mathbb{F}_2)$, which by the Künneth formula, is isomorphic to $\mathbb{F}_2[x, y]/(x^n, y^n)$. In particular, the monomials x^k and y^k are all independent, and since their sum is zero mod 2, $\binom{n}{k} = 0 \pmod 2$ for eack k, and Pascal's triangle tells us that this only happens when n is a power of 2. ⊠

Let's talk about the Hopf invariant now.

Definition. Given a map $f : S^{2n-1} \to S^n$, we can take the *cone* of f, $C_f = S^n \cup_f D^{2n}$. (More generally, if $f : X \to Y$ is a continuous map, $C_f = Y \cup_f CX$; this is defined without basepoints.) For this map between spheres, C_f has the structure of a CW complex with cells e^0, e^n, and e^{2n}, and only depends on the homotopy type of f.

Using the Puppe sequence (collapsing Y gives us ΣX, which in this case is S^{2n}), we get a sequence

$$S^{2n-1} \longrightarrow S^n \longrightarrow C_f \longrightarrow S^{2n}. \tag{7.1}$$

Focusing on the latter three terms, if $n > 1$, we can deduce that $H^\bullet(C_f; \mathbb{Z}) \cong \mathbb{Z} \cdot b_m \oplus \mathbb{Z} \cdot a_{2n}$ (i.e. the generators have degrees n and $2n$, respectively). The ring structure means that $b^2 = ha$ for some $h \in \mathbb{Z}$. This h is called the *Hopf invariant*, and is determined up to sign.

By fixing orientations, we can pin down a sign for h, but we won't need to.

---

[10] In homology, the induced map sends generators to generators; this is just the dual statement.

We can give an alternative definition of the Hopf invariant using K-theory. Appling $\widetilde{K}$ to (7.1) when $n = 2m$ is even produces a split short exact sequence (because $\widetilde{K}(S^{2m+1}) = 0$)

$$0 \longrightarrow \widetilde{K}(S^{4m}) \longrightarrow \widetilde{K}(C_f) \longrightarrow \widetilde{K}(S^{2m}) \longrightarrow 0,$$

where the first map sends $([H]-1)^{2m} \mapsto \alpha$ and the second map sends β to a generator of $\widetilde{K}(S^{2m})$, $([H]-1)^m$. By exactness, this means $\beta^2 = h\alpha$ for some $h \in \mathbb{Z}$. However, h isn't well-defined; if $\beta \mapsto \beta + k\alpha$, then $\beta^2 \mapsto \beta^2 + 2k\alpha\beta = (h + 2k\ell)\alpha$, where $\alpha\beta = \ell\alpha$. We can see that h mod 2 is well-defined, though, and that's all we needed.

If n is odd, then by degree considerations, $b^2 = 0$ in $H^\bullet(C_f; \mathbb{Z})$, and so the Hopf invariant is necessarily zero.

The story behind these proofs is kind of tangled; Milnor and Kervaire. in [11] and [9], respectively, figured out the proof of Theorem 7.4 and therefore the corollary about division algebras. Milnor wrote to Bott about it in [6], and Bott was nicely surprised, so these letters were published. Then, some of the later results were published by Adams and Atiyah in [1]; one of the proofs nicely fit on a postcard. Some of these proofs depended on operations on mod 2 cohomology called Steenrod squares.

For (3) $\implies$ (4), suppose $g : S^{2m-1} \times S^{2m-1} \to S^{2m-1}$ gives S^{2m-1} an H-space structure. Then, we can view

$$S^{4m-1} = \partial(D^{4m}) = \partial(D^{2m} \times D^{2m}) = (\partial D^{2m} \times D^{2m}) \cup_{\partial D^{2m} \times \partial D^{2m}} (D^{2m} \times \partial D^{2m}).$$

Thus, we can define an $f : S^{4m-1} \to S^{2m}$ by extending from S^{2m-1} on each cone, and we'll determine its Hopf invariant next time.

Of course, this can all be found in [7].

The Splitting Principle: 9/22/15

"If this were a teaching class, I would tell you to not do what I just did."

Recall that we were in the middle of proving Proposition 7.3, which is instrumental in the K-theoretic proof that the only division algebras over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$; the key is linking in Theorem 7.4.

Soon, we'll start talking about Fredholm operators, which lead to another proof of Bott periodicity, and then move into equivariant topics, including Lie groups.

Definition. If (X, A) is a pair with $A \subset X$, the *relative K-theory* $\widetilde{K}(X, A) = \widetilde{K}(X/A)$, assuming A is nonempty.

Proof of (3) $\implies$ (4) *in Proposition 7.3.* Let $n = 2m$ and $g : S^{2n-1} \times S^{2n-1} \to S^{2m-1}$; it's easy to see that if n is odd, the Hopf invariant has to be 0, so we're assuming n is even.

This argument is straight out of [7]; read the details there (or check out the giant diagram).

We want to construct a map $f : S^{4m} \to S^{2m}$ by writing $S^{4m} = \partial(D^{4m}) = \partial(D^{2m} \times D^{2m})$, and since ∂ obeys the Leibniz rule, this is homeomorphic to $D^{2m} \times \partial D^{2m} \cup_{\partial D^{2m} \times \partial D^{2m}} \partial D^{2m} \times D^{2m}$.

We can write S^{2m} as the suspension of S^{2m-1}; thus, we can draw this as a cone with cone parameter t; to construct f, take a point on ΣS^{2m-1} with parameter t, figure out where its projection down to S^{2m-1} goes, and then send to the point above that in $S^{2m} = \Sigma S^{2m-1}$, but with the same parameter. Thus, we can use the decomposition from the previous paragraph to realize this as a map $f : S^{4m-1} \to S^{2m}$.

Let C_f denote the cone of f, which entails attaching a $4m$-cell. Thus, we get $S^{2m} \to C_f \to S^{4m}$, which as we proved gives us a short exact sequence $\widetilde{K}(S^{2m}) \leftarrow \widetilde{K}(C_f) \leftarrow \widetilde{K}(S^{4m})$; since the even-dimensional K-theory of spheres is infinite cyclic, then we've shown that $\widetilde{K}(C_f)$ is also infinite cyclic, by looking at the diagram, so if β generates it, then $\beta^2 \mapsto h\alpha$, where α generates $\widetilde{K}(S^{2m})$. It turns out (though we didn't prove

it), this is independent of the lift we chose, and in this specific case, $h = 1$, courtesy of the following diagram.

$$
\begin{array}{ccc}
\widetilde{K}(C_f) \otimes \widetilde{K}(C_f) & \xrightarrow{\quad * \quad} & \widetilde{K}(C_f) \\
\uparrow{\scriptstyle\cong} & & \uparrow \\
\widetilde{K}(C_f, D_-^{2m}) \otimes \widetilde{K}(C_f, D_+^{2m}) & \xrightarrow{\quad * \quad} & \widetilde{K}(C_f, S^{2m}) \\
\downarrow{\scriptstyle\Phi^* \otimes \Phi^*} & & \downarrow{\scriptstyle\cong}\;{\scriptstyle\Phi^*} \\
\widetilde{K}(D^{2m} \times D^{2m}, \partial D^{2m} \times D^{2m}) \otimes \widetilde{K}(D^{2m} \times D^{2m}, D^{2m} \times \partial D^{2m}) & \xrightarrow{\;*\;} & \widetilde{K}(D^{2m} \times D^{2m}, \partial(D^{2m} \times D^{2m})) \\
\downarrow{\scriptstyle\cong} & {\scriptstyle\cong} & \\
\widetilde{K}(D^{2m} \times \{e\}, \partial D^{2m} \times \{e\}) \otimes \widetilde{K}(\{e\} \times D^{2m}, \{e\} \times \partial D^{2m}) & &
\end{array}
$$

Here, the blue arrow is given by excision; in each argument, we've excised out a contractible set, so nothing changes.

What we have to prove is that β^2 is a generator, so that $\beta^2 = \alpha$. This diagram commutes, which is a fun exercise, and follows because the product of vector bundles is natural; then, this commutativity, and the isomorphisms in the diagram, allow us to show that in the uppermost map, $\beta \otimes \beta \mapsto \alpha$.   ⊠

The splitting principle. Now, we'll switch topics. The question we want to answer is: given a complex vector bundle $E \to X$, can we write E as a direct sum of line bundles? Sometimes, the answer is yet, e.g. when $X = S^2$. However, when $X = S^4$, isomorphism classes of rank-2 vector bundles over S^4 is isomorphic to $[S^3, U_2]$, but the isomorphism classes of line bundles $L \to S^4$ form $[S^3, U_1] = 0$.

Returning to rank-2 bundles, $SU_2 \hookrightarrow U_2$ creates a map $f : [S^3, SU_2] \hookrightarrow [S^3, U_2]$, and $SU_2 \cong Sp_1 \cong S^3$, as unit quaternions of length 1, and we know there are homotopically nontrivial maps $S^3 \to S^3$. That f is injective comes from the fact that $U_2 \to SU_2$ is a $2 : 1$ covering space.

We can actually produce a specific example: there's a *Hopf fibration* $S^3 \to S^7 \to S^4$[2] by choosing the vectors with unit norm $\{(q_1, q_2) : |q_1|^2 + |q_2|^2 = 1\}$. Thus, we get $S^7 \subset \mathbb{H}^2$, with fibers S^3, and projecting down to $S^4 = \mathbb{HP}^1$ produces the desired fibration.[11] This fiber bundle satisfies Steenrod local triviality; and when you pull it back by a continuous map, it can only untwist; it can't twist more. Writing as a sum of line bundles would be a kind of untwisting.

So we want to construct a map $p : \mathbb{F}(E) \to X$ (where $\mathbb{F}(E)$ is the *flag manifold*) such that

(1) the vector bundle $p^*E \to \mathbb{F}(E)$ is isomorphic to a direct sum of line bundles in the diagram

$$
\begin{array}{ccc}
p^*E & \longrightarrow & E \\
\downarrow & & \downarrow \\
\mathbb{F}(E) & \xrightarrow{\;p\;} & X,
\end{array}
$$

(2) and that the map $p^* : K(X) \to K(\mathbb{F}(E))$ is injective. This injectivity will allow us to push our isomorphism into the vector bundle.

This is known as the *splitting principle*; it's a very important argument in the theory of characteristic classes, and we're going to be doing something quite similar, though using K-theory in place of cohomology as the residence of the classes. This is a very common manuever in mathematics.

First, let's simplify the problem. We first want a map $q : \mathbb{P}(E) \to X$ such that $q^*E \supset L$ is a line bundle. This helps us because then $E \cong L \oplus E/L$, as the sequence splits; then, we have reduced the problem.

To do this, we need to make a choice of a line in each E_x. The mathematician's maneuver is to make all choices. Let $q : \mathbb{P}(E) \to X$ be defined by sending $q^{-1}(x)$ to the space of lines in E_x, which works because $\mathbb{P}(E_x) = \mathbb{P}(E)_x$.

When we do this for all x, we describe q as a fiber bundle. Then, the pullback gives the data of a line and a point in the bundle, and working with this, we get the desired line bundle L. Thus, the pullback splits as $0 \to L \to q^*E \to \mathbb{P}(E) \to 0$.

---

[11]If you replace $\mathbb{H}^2$ with $\mathbb{C}^2$, you get the more familiar Hopf fibration $S^1 \to S^3 \to S^2$.

We'd like to make it a complement, rather than just a quotient; if we have a Hermitian metric, this is easy, as we just take the orthogonal complement. We might not have this given, in which case we need to make a choice. Or, again, all choices.

Given a one-dimensional subspace L of a vector space $\mathbb{E}$, what can we say about the space of possible complements to L? If W is one complement, we can think about graphs: we can identify W with $\mathbb{E}/L$, and so given a map in $\operatorname{Hom}(\mathbb{E}/L, L)$, it's also a map $W \to L$, and this has a graph, which is a complement to L. Moreover, all such complements can be realized in this way. These complements are splittings of $0 \to L \to \mathbb{E} \to \mathbb{E}/L \to 0$, so they form an affine space, and one can work this way. Of course, it's usually simpler to choose a metric on E so that everything works.

Now we can take complements, so we can split off bundles until we run out: first, we get

$$
\begin{array}{ccc}
L_1 \oplus E_1 & \longrightarrow & E \\
\downarrow & & \downarrow \\
\mathbb{P}(E) & \longrightarrow & X
\end{array}
$$

and then repeating, we get another line bundle L_2, and so on until we run out, and so E has been written as a direct sum of line bundles.

K-theory as a cohomology theory. To get the second criterion, that p^* is injective, we need to discuss K-theory as a cohomology theory. We'll work in the category of pairs of pointed compact Hausdorff spaces (X, A) with $A \subset X$.

Definition. For $n \in \mathbb{Z}^{\geq 0}$, define $\widetilde{K}^{-n}(X, A) = \widetilde{K}(\Sigma^n(X/A))$.

If $n = 0$, $\widetilde{K}^0(X, A) = \widetilde{K}(X, A)$. We can also take $A = \emptyset$; $X/\emptyset = X_+$, defined to be $X \sqcup \mathrm{pt}$, and we can write $K^{-n}(X) = \widetilde{K}^{-n}(X_+) = \widetilde{K}(\Sigma^n(X_+))$, and $K^{-n}(X, A) = \widetilde{K}(X, A)$ if $A \neq \emptyset$.

Thus, our short exact sequence

$$\widetilde{K}^{-n}(X, A) \longrightarrow \widetilde{K}^{-n}(X) \longrightarrow \widetilde{K}^{-n}(A)$$

becomes by the Puppe sequence a long exact sequence

$$\cdots \longrightarrow \widetilde{K}^{-n}(X, A) \longrightarrow \widetilde{K}^{-n} \longrightarrow \widetilde{K}^{-n}(A) \longrightarrow \widetilde{K}^{-n+1}(X, A) \longrightarrow \widetilde{K}^{-n+1}(X) \longrightarrow \cdots \qquad (8.1)$$

Since we haven't defined $\widetilde{K}^n(X)$ for $n > 0$, this sequence terminates at $\widetilde{K}^0(A)$. However, Bott periodicity creates a map $\beta : \widetilde{K}^{-n}(X, A) \to \widetilde{K}^{-n+2}(X, A) = \widetilde{K}^{-n}(S^2 \wedge X/A)$ by $[E] \mapsto ([H] - 1) * E$. Thus (8.1) becomes a hexagon.

$$
\begin{array}{ccccc}
\widetilde{K}^{-n}(X, A) & \longrightarrow & \widetilde{K}^{-n} & \longrightarrow & \widetilde{K}^{-n}(A) \\
\uparrow \beta & & & & \downarrow \\
\widetilde{K}^{-n+2}(X, A) & & & & \\
\uparrow & & & & \downarrow \\
\widetilde{K}^{-n+1}(A) & \longleftarrow & \widetilde{K}^{-n+1}(X) & \longleftarrow & \widetilde{K}^{-n+1}(X, A)
\end{array}
$$

Now, we can define $\widetilde{K}^n(X, A) = \widetilde{K}^{n-2}(X, A)$ for any $n \in \mathbb{Z}$. For general cohomological reasons, it makes sense to think of this as graded in $\mathbb{Z}$, rather than $\mathbb{Z}/2$. Then, $K^\bullet(\mathrm{pt})$ is a $\mathbb{Z}$-graded ring, and in some sense is the "ground ring" of this cohomology theory. In fact, $K^\bullet(\mathrm{pt}) = \mathbb{Z}[u, u^{-1}]$, with $\deg(u) = 2$, as $u^{-1} = [H] \in K^{-2}(\mathrm{pt}) \cong \widetilde{K}(S^2)$.

A useful fact is that every map in the long exact sequence is compatible with the $K(X)$-module structures on $K(A)$ and $K(X, A)$.

The second part of the splitting principle (whose proof can be found in [7]), is to prove that for $q : \mathbb{P}(E) \to X$, the pullback $q^* : K(X) \to K(\mathbb{P}(E))$ is injective. We'll give part of the proof next time; it's a more sophisticated example of familiar arguments from algebraic topology. Ultimately, by the Leray-Hirsch theorem, $K(\mathbb{P}(E))$ is a free $K(X)$-module.

The Adams operations. Analogous to the Steenrod operations in cohomology, we have Adams operations in K-theory.

Theorem 8.1. *For $k \in \mathbb{Z}^{\geq 0}$ and X a compact Hausdorff space, there exists a unique a ring homomorphism $\psi^k : K^0(X) \to K^0(X)$ natural in X and satisfying $\psi^($ $[L]) = [L^{\otimes k}] = [L]^K$ for all line bundles $L \to X$. Moreover, ψ^k satisfies the following properties.*

(1) $\psi^k \psi^\ell = \psi^{k+\ell}$.
(2) If p is prime, $\psi^p(x) \equiv x^p \bmod p$ for $x \in K(X)$.
(3) ψ^k is multiplication by k^m on $\widetilde{K}(S^{2m})$.

Proof. By the splitting principle, we can reduce to direct sums of line bundles, by passing back to the flag manifold $\mathbb{F}(E)$. If $E = \bigoplus_{i=1}^r L_k$, then $\psi^k([E]) = [L_1]^k + \cdots + [L_r]^k \in K(\mathbb{F}(E))$, which certainly exists and is unique, and one can check that it descends to X.

Now we need to check all these properties. (1) is trivial: taking the sum of a bunch of kth powers followed by ℓ^{th} powers gives $(k+\ell)^{\text{th}}$ powers. For (2), set $x_i = [L_i]$, so that

$$\psi^p(x_1 + \cdots + x_r) = x_1^p + \cdots + x_r^p$$
$$\equiv (x_1 + \cdots + x_p)^p \pmod{p}.$$

For (3), when $m = 1$,

$$\psi^k([H] - 1) = [H]^k - 1$$
$$= (1 + x)^k - 1 = (1 + kx) - 1 = kx,$$

since in $K(S^2)$, the basic relation is $x = [H] - 1$, so $x^2 = ([H] - 1)^2 = 0$.   ⊠

The proof that this map descends from $\mathbb{F}(E)$ to E will be given next time; we'll also talk more about the splitting principle and characteristic classes.

But now, we can give the postcard proof of Theorem 7.4 by Adams and Atiyah in [1].

Proof of Theorem 7.4. Suppose $f : S^{4m-1} \to S^{2m}$ has Hopf invariant one, and take $C_f = S^{2m} \cup_f D^{4m}$. Then, we have $\widetilde{K}(S^{4m}) \to \widetilde{K}(C_f) \to \widetilde{K}(S^{2m})$, given respectively by maps $([H] - 1)^{2m} \mapsto \alpha$ and $\beta \mapsto ([H] - 1)^m$.

We know that $\psi^k(\alpha) = k^{2m}\alpha$ and $\psi^k(\beta) = k^m\beta + \mu_k\alpha$, with $\mu_k \in \mathbb{Z}$, so $\psi^2(\beta) = 2^m\beta + \mu_2\alpha \equiv \mu_2\alpha$ (mod 2), but this is also β^2 (mod 2), and this is $h\alpha$. Thus, μ_2 is the Hopf invariant.

Since $\psi^2\psi^3(\alpha) = \psi^3\psi^2(\alpha)$, then $2^m(2^{m-1})\mu_3 = 3^m(3^m - 1)\mu_2$; the right-hand side is odd because we wanted the Hopf invariant to be odd, and 2^m has to divide it, so $2^m \mid 3^m - 1$, which (one can check) implies m is one of 1, 2, 4, or 8.   ⊠

Lecture 9.

Flag Manifolds and Fredholm Operators: 9/24/15

> "*I see confused faces. . . speak now.*"

Next week, the professor will be gone, and Tim Perutz will deliver two lectures about Morse theory and its use in a proof of Bott periodicity. But today, we'll finish talking about flag manifolds and then introduce Fredholm operators, which we'll talk about for a few weeks.

Last time, we promoted K-theory to a cohomology theory; the following result illustrates how one might use that.

Proposition 9.1. *If $n \in \mathbb{Z}^{>0}$, then $K(\mathbb{CP}^n) = K^0(\mathbb{CP}^n)$ is a free abelian group of rank $n + 1$, and as a ring $K(\mathbb{CP}^n) \cong \mathbb{Z}[x]/(x^{n+1})$ under the identification $x \mapsto [L] - 1$, where $[L]$ is the K-theory class of the tautological bundle $L \to \mathbb{CP}^n$.*

Proof. We'll provide a proof for the group structure; then, check out [7] for the ring structure. The proof will proceed on induction on n, and also show that $K^{\text{odd}}(\mathbb{CP}^n) = 0$.

We have $\mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^n$ by attaching a single $2n$-cell (realizing it as a subcomplex), so we have a sequence $\mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^n \to S^{2n}$, and therefore the following long exact sequence.

$$\widetilde{K}^{-1}(\mathbb{CP}^{n-1}) \longrightarrow \widetilde{K}^0(S^{2n}) \longrightarrow \widetilde{K}^0(\mathbb{CP}^n) \longrightarrow \widetilde{K}^0(\mathbb{CP}^{n-1}) \longrightarrow \widetilde{K}^1(S^{2n}) \longrightarrow \cdots$$

But $\widetilde{K}^{-1}(\mathbb{CP}^{n-1}) = 0$ by hypothesis, and $\widetilde{K}^1(S^{2n}) = 0$ by our previous computations, so this is a short exact sequence. We also know that $\widetilde{K}^0(S^{2n}) = \mathbb{Z}$, and by the inductive hypothesis, $\widetilde{K}^0(\mathbb{CP}^{n-1})$ is free of rank $n-1$, so this sequence simplifies to a short exact sequence of abelian groups.

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{K}^0(\mathbb{CP}^n) \longrightarrow \mathbb{Z}^{n-1} \longrightarrow 0.$$

Thus, $\widetilde{K}^0(\mathbb{CP}^n)$ is free of rank n.

For the second half of our inductive assumption, take the following part of the long exact sequence.

$$\widetilde{K}^1(S^{2n}) \longrightarrow \widetilde{K}^1(\mathbb{CP}^n) \longrightarrow \widetilde{K}^1(\mathbb{CP}^{n-1}),$$

but we already know that $\widetilde{K}^1(S^{2n}) = 0$ and $\widetilde{K}^1(\mathbb{CP}^{n-1}) = 0$, so $\widetilde{K}^1(\mathbb{CP}^n) = 0$.   ⊠

The result for rings involves figuring out where generators go, and isn't too much more involved.

Theorem 9.2 (Leray-Hirsch). *Let $p : \mathcal{E} \to X$ be a fiber bundle with fiber F, where $\mathcal{E}$ is compact Hausdorff and X is a finite CW complex. Suppose $K^\bullet(F)$ is a free abelian group with basis $f_1, \ldots, f_N$ and we have $c_1, \ldots, c_N \in K^\bullet(\mathcal{E})$ with $c_i|_{\mathcal{E}_x} = f_i$ for all $x \in X$. Then, $K^\bullet(\mathcal{E}) \cong K^\bullet(X)[c_1, \ldots, c_n]$ as a $K^\bullet(X)$-module.*

Proof. Let $X' \subset X$ be a subcomplex, and let $[C^\bullet]^q$ denote the qth degree of the complex $C^\bullet$. Then, we have the following commutative diagram, where $\mathcal{E}' = p^{-1}(X')$.

$$\cdots \longrightarrow [K(X, X') \otimes K(F)]^q \longrightarrow [K(X) \otimes K(F)]^q \longrightarrow [K(X') \otimes K(F)]^q \longrightarrow \cdots \qquad (9.1)$$
$$\downarrow{\Psi} \qquad\qquad \downarrow{\Psi} \qquad\qquad \downarrow{\Psi}$$
$$\cdots \longrightarrow K^q(\mathcal{E}, \mathcal{E}') \longrightarrow K^q(\mathcal{E}) \longrightarrow K^q(\mathcal{E}') \longrightarrow \cdots$$

Here,

$$\Psi\left(\sum x_i \otimes f_i\right) = \sum p^*(x_i) c_i$$

for $x_i \in K^\bullet(X)$.

The rows in (9.1) are exact: the top sequence is obtained from the long exact sequence for $X' \subset X$ by tensoring with a free abelian group, and the bottom sequence is the long exact sequence for $\mathcal{E}' \subset \mathcal{E}$. Moreover, the diagram commutes, which you can check from the description of Φ, and is written out more explicitly in [7].

We'll use a typical proof technique: since there are finitely many cells floating around, we can induct on $\dim X$ plus the number of cells in each dimension in order to show that Ψ is an isomorphism.

The inductive step is $X = X' \cup_f D^n$, where $f : S^{n-1} \to X'$. We'll want to apply the five lemma to (9.1); on the right, we have Ψ acting on degree $q - 1$, so we win by the inductive assumption, and on the left, the attaching map f gives us $K(X, X') = K(D^n, S^{n-1})$, and therefore a description

$$[K(X, X') \otimes K(F)]^q \xrightarrow{\;\cong\;} [K(D^n, S^{n-1})]^q \qquad\qquad (9.2)$$
$$\downarrow{\Psi} \qquad\qquad\qquad \downarrow{\Psi}$$
$$K^q(\mathcal{E}, \mathcal{E}') \xrightarrow{\;\cong\;} K(D^n \times F, S^{n-1} \times F).$$

In other words, we've reduced to the following box:

$$\cdots \longrightarrow [K(D^n, S^{n-1}) \otimes K(F)]^q \longrightarrow [K(D^n) \otimes K(F)]^q \longrightarrow [K(S^{n-1}) \otimes K(F)]^q \longrightarrow \cdots$$
$$\downarrow{\Psi} \qquad\qquad\qquad \downarrow{\Psi} \qquad\qquad\qquad \downarrow{\Psi}$$
$$\cdots \longrightarrow K^q(D^n \times F, S^{n-1} \times F) \longrightarrow K^q(D^n \times F) \longrightarrow K^q(S^{n-1} \times F) \longrightarrow \cdots$$

Once again, the rows are exact and the diagram commutes by (9.1), but this time, D^n is contractible, so the blue arrow is an isomoprhism; then, the inductive assumptions give us the other isomorphisms we need for the five lemma, and therefore we get that the right-hand arrow in (9.2) is an isomorphism. Thus, we can apply the five lemma to (9.1), proving the theorem.   ⊠

Remark. The same proof works for $H^*(-, R)$ for coefficients in any ring R, and its use in the following discussion on splitting sequences generalizes. We can also remove the assumption that X is a CW complex, though this requires more highbrow techniques such as spectral sequences.

We'll use this to understand how complex subbundles decompose into line bundles. If $E \to X$ is a complex bundle, and we split off a line bundle L_1, so $E \cong L_1 \oplus E_1 \to \mathbb{P}(E)$. The fibers of $\mathbb{P}(E) \to X$ are $\mathbb{CP}^n$, which has free K-theory as we saw above, so we can apply the Leray-Hirsch theorem to the splitting principle.

We also talked about the Adams operations last time. Suppose we have a situation $E \to X$ and $p : \mathbb{F}(E) \to X$, where $p^*E \cong L_1 \oplus \cdots \oplus L_n$, and we have the diagram

$$
\begin{array}{ccc}
\mathbb{F}(E) & \longleftarrow & p^*E \\
{\scriptstyle p}\downarrow & & \downarrow \\
X & \longleftarrow & E.
\end{array}
$$

We want to show that for all $k \in \mathbb{Z}^{>0}$, $L_1^{\otimes k} \oplus \cdots \oplus L_n^{\oplus k} = L_1^k + \cdots + L_n^k$ has a K-theory class which descends to X.

When $n = k = 1$, this is silly, so let's consider $n = k = 2$. Here,

$$
[L_1^2 + L_2^2] = \underbrace{[(L_1 + L_2)^2]}_{[E]^2 = [E \otimes E]} - \underbrace{2[L_1 \otimes L_2]}_{2[\Lambda^2 E]}.
$$

Both of these factors descend to E, so we're good. This relies on a useful fact from linear algebra: there's a canonical isomorphism $\Lambda^2(L_1 \oplus L_2) \cong L_1 \otimes L_2$.

To see how beautiful K-theory is as opposed to singular cohomology, consider replacing L_i by its Chern class $c_1 L_i \in H^2(\mathbb{F}(E); \mathbb{Z})$. This involves a nontrivial descent argument, but the exterior powers in K-theory make the argument more smooth (heh) and more geometric.

For the general argument, recall that $\mathbb{Z}[x_1, \ldots, x_n]^{\mathrm{Sym}_n} \cong \mathbb{Z}[\sigma_1, \ldots, \sigma_n]$, where σ_i is the ith *symmetric polynomial*:

$$
\sigma_j = \sum_{i_1 < \cdots < i_j} x_{i_1} \cdots x_{i_j}.
$$

For example, when $n = 3$,

$$
\sigma_1 = x_1 + x_2 + x_3
$$
$$
\sigma_2 = x_1 x_2 + x_1 x_3 + x_2 x_3
$$
$$
\sigma_3 = x_1 x_2 x_3.
$$

Crucially, $\sigma_j(L_1, \ldots, L_n) = p^*(\Lambda^j E)$, for which the descent argument goes as in the $n = k = 2$ case. But we wanted it for $s_k = x_1^k + \cdots + x_n^k$. Thankfully, this is a classical problem, and the solution is the *Newton polynomials*: $s_1 = \sigma_1$, $s_2 = \sigma_1^2 - 2\sigma_2$, and in general,

$$
s_k - \sigma_1 s_{k-1} + \sigma_2 s_{k-2} - \cdots + (-1)^{k-1}\sigma_{k-1}s_1 + (-1)^k k\sigma_k = 0.
$$

These ideas are very similar to the theory of characteristic classes for integral cohomology, and similar descent arguments happen.

Another Approach. So far, we've represented an $x \in K(X)$ as the difference between two classes corresponding to complex vector bundles (or real vector bundles for $KO(X)$). But we'd like a more flexible way to use this in geometry, since not everything is a difference of two vector bundles. This is a very important principle for applying algebraic topology to geometry: the greater number of ways you have to realize your objects geometrically, the more powerful your theory is: for example, cohomology shows up whenever you have a CW structure on a topological space, but if you know that de Rham cohomology agrees, then you can use the same ideas in different places to simplify your proofs. Similarly, we want to make K-theory more flexible.

Let H^0 and H^1 be complex vector spaces. Then, a $T : H^0 \to H^1$ can be extended to the exact sequence

$$
0 \longrightarrow \operatorname{Ker} T \longrightarrow H^0 \overset{T}{\longrightarrow} H^1 \longrightarrow \operatorname{coker} T \longrightarrow 0.
$$

The *cokernel* is $\operatorname{coker} T = H^1 / T(H^0)$.

If H^0 and H^1 are finite-dimensional, we want to take an alternating sum, and have it equal to zero for an exact sequence. More generally, for an exact sequence

$$0 \longrightarrow E^0 \longrightarrow E^1 \longrightarrow E^2 \longrightarrow \cdots \longrightarrow E^N \longrightarrow 0,$$

we have two alternating-line results:

$$\sum_{i=0}^{N}(-1)^i \dim E^i = 0$$

$$\bigotimes_{i=0}^{N}(\det E^i)^{\otimes(-1)^i} \cong \mathbb{C}.$$

The latter is canonical.

If the E^i are vector bundles over a compact Hausdorff space X, this implies that $\sum_{i=0}^{N}(-1)^i[E^i] = 0$. For example, if $X = \mathbb{R}$ and $H^0 = H^1 = \underline{\mathbb{C}}$, we can set $x \in X$, $T_x : \mathbb{C} \to \mathbb{C}$ as multiplication by x. Then, the exact sequence degenerates except when $x = 0$, where it jumps. There, the K-theory isn't given by a difference of vector bundles... because $\mathbb{R}$ isn't compact.

This is a good motivation to generalize: we can allow H^0 and H^1 to be infinite-dimensional and approach this from the perspective we've outline above. However, we'll still require that the kernel and cokernel are finite-dimensional. A T with that stipulation is called a *Fredholm operator*, and we'll hope to build a K-theory from these operators.

There are a couple wrinkles we'll have to address, though.

- First, for infinite-dimensional vector spaces, we have topology and not just algebra: we want to talk about continuous functionals, not just linear one.
- We need to show that the Fredholms define K-theory classes when X is compact and Hausdorff.
- Then, we'll extend $K(X)$ using Fredholm operators to noncompact X.
- Finally, we'll show these make sense, by using them to prove Bott periodicity. This will bring Clifford algebras into the story, which are quite important.

We'll spend the next four lectures (not counting the two next week, where the professor is absent) on these topics. Two useful references for this section are [2, 12].

So what kind of infinite-dimensional spaces are we going to consider? Norms give us topology, and inner products give us angles (and therefore geometry). So we'll use infinite-dimensional inner product spaces; specifically *Hilbert spaces*: a vector space equipped with a bilinear (or sesquilinear in the complex case), nondegenerate pairing and that is complete.

Definition. If H^0 and H^1 are Hilbert spaces, a linear map $T : H^0 \to H^1$ is *bounded* if there exists a $C > 0$ such that for all $\xi \in H^0$, $|T\xi|_{H^1} \le C|\xi|_{H^0}$.

The following fact and the previous definition are considerably more general than just Hilbert spaces.

Fact. Let H^0 and H^1 be Hilbert spaces. Then, a linear $T : H^0 \to H^1$ is bounded iff it is continuous.

In this case, we may define the *operator norm* $\|T\|$ to be the infimum of C that work to make T bounded. This makes $\operatorname{Hom}(H^0, H^1)$, the set of continuous linear maps, into a *Banach space* (a complete normed space); in general we don't have an inner product, and we can show that if $T_1, T_2 \in \operatorname{Hom}(H^0, H^1)$, then $\|T_2 \circ T_1\| \le \|T_1\|\|T_2\|$. This makes $\operatorname{Hom}(H^0, H^1)$ into a structure called a *Banach algebra*.

We'll define $\operatorname{Hom}(H^0, H^1)^\times \subset \operatorname{Hom}(H^0, H^1)$ to be the subspace of invertible elements, i.e. homeomorphisms, but it turns out we don't need to distinguish between the two.

Theorem 9.3 (Open mapping theorem).
  (1) *If $T : H^0 \to H^1$ is bounded and bijective, then T^{-1} is bounded.*
  (2) $\operatorname{Hom}(H^0, H^1)^\times \subset \operatorname{Hom}(H^0, H^1)$ *is open (i.e. invertibility is an open condition).*

The first part is a standard theorem in functional analysis, and (2) is a fairly easy standard argument. We'll also use the following theorem.

Definition. A vector space is *separable* if there exists a countable set of vectors such that every $x \in X$ is an *infinite* linear combination of those vectors.

Theorem 9.4 (Kuiper). *If H^0 and H^1 are separable, infinite-dimensional vector spaces, then $\mathrm{Hom}(H^0, H^1)^\times$ is contractible.*

Notice that this isn't true in the finite-dimensional case.

Remark. Let H be a Hilbert space.
- If $V \subset H$ is finite-dimensional, then V is closed.
- If $V \subset H$ is closed, then since we're in a Hilbert space, we can form $V^\perp$, and therefore get a sequence $V^\perp \hookrightarrow H \twoheadrightarrow H/V$, which gives us a Hilbert space structure on $V^\perp$.

Now, we can state the main definition.

Definition. Let H^0 and H^1 be Hilbert spaces and $T : H^0 \to H^1$ be a continuous linear map. Then, T is *Fredholm* if
(1) $T(H^0) \subset H^1$ is closed,
(2) $\ker T \subset H^0$ is finite-dimensional, and
(3) $\mathrm{coker}\, T$ is finite-dimensional.

It turns out that the first requirement is superfluous.

The idea is that $\mathrm{Hom}(H^0, H^1)$ is a vector space, and therefore contractible; its topology isn't very interesting. But the space of Fredholm operators $\mathrm{Fred}(H^0, H^1)$ has a more interesting topology, and ends up being open. The space of invertible operators sits inside (since then the kernel and cokernel are trivial), and is contractible. But the space of Fredholm is not connected, and the components are indexed by the difference in dimensions of the kernel and cokernel (called the *index*) of the operators in the component. And each component is interesting, having $\pi_{2n} = \mathbb{Z}$ for all n.

We'll study this with open sets: if $W \subset H^1$ is finite-dimensional, then T is nearly surjective on operators, and we can therefore find a W such that T is transverse to it. Then, we'll reverse it, and choose a $W \subset H^1$ and consider the set of Fredholm operators that are transverse to W. This will eventually lead to constructions of K-theory classes.

Lecture 10.

Bott Periodicity and Morse-Bott Theory: 9/29/15

Today's lecture was given by Tim Perutz.

We'll talk about Bott periodicity as proved by Bott, as distinct from how it was proven by later authors.

U will denote the infinite unitary group $\mathrm{U} = \mathrm{U}(\infty) = \bigcup_n \mathrm{U}(n)$. These are infinite matrices which have block form

$$\begin{bmatrix} A & 0 \\ 0 & I_\infty \end{bmatrix},$$

where $A \in \mathrm{U}(n)$ for some n and I_∞ denotes the infinite identity matrix. Note that this is *not* the group of unitary transformations of an infinite-dimensional Hilbert space.

Bott periodicity, in a nutshell, is a homotopy equivalence $\Omega^2 \mathrm{U} \simeq \mathrm{U}$, and therefore isomorphisms $\pi_{k+2}(\mathrm{U}) \cong \pi_k(\mathrm{U})$. In particular, since U is path-connected, then $\pi_{2k}(\mathrm{U}) = \pi_0(\mathrm{U}) = 0$. The odd homotopy groups are $\pi_{2k+1}(\mathrm{U}) = \pi_1(\mathrm{U}) = \mathbb{Z}$ (because $\pi_1(\mathrm{U}(n)) = \mathbb{Z}$ for each n).

Bott talked about this in [5]. Note that this is distinct from stable homotopy theory! This is a very geometric, very down-to-Earth proof, a vindication for actual geometric methods in homotopy theory.

In the 1920s, Morse theory was developed, originally involving geodesics on Riemannian manifolds via calculus of variations. Bott used Morse theory to make detailed calculations of geodesics on Lie groups to prove Bott periodicity. He also obtained similar results for other groups: if $\mathrm{O} = \mathrm{O}(\infty)$ (infinite matrices of the same block form, but with an orthogonal matrix instead of a unitary one) and $\mathrm{Sp} = \mathrm{Sp}(\infty)$ (analogous), then $\Omega^4 \mathrm{O} = \mathrm{Sp}$ and $\Omega^4 \mathrm{Sp} = \mathrm{O}$, and therefore $\Omega^8 \mathrm{O} = \mathrm{O}$. In particular, for all k, $\pi_k(\mathrm{O}) = \pi_{k+8}(\mathrm{O})$ for all k. For example, $\pi_0(\mathrm{O}) = \mathbb{Z}/2$ and $\pi_1(\mathrm{O}) = \mathbb{Z}/2$. $\pi_2(\mathrm{O}) = 0$ (since π_2 of a Lie group is always zero, and we can use $\mathrm{O}(n)$ for our calculations). $\pi_3(\mathrm{O}) = \mathbb{Z}$ (this is true for any simple Lie group). Then, $\pi_4(\mathrm{O}) = \pi_0(\mathrm{Sp}) = 0$, $\pi_5(\mathrm{O}) = \pi_1(\mathrm{Sp}) = 0$, $\pi_6(\mathrm{O}) = \pi_2(\mathrm{Sp}) = 0$ (same deal, it's a Lie group), and $\pi_7(\mathrm{O}) = \pi_3(\mathrm{Sp}) = \mathbb{Z}$, as $\mathrm{Sp}(n)$ is simple. Then, we're back to where we started.

This periodicity was absolutely surprising, and very serendipitous.

This week, these two lectures will cover the unitary case. We'll more or less follow Milnor in [10], but we'll treat loop spaces as actual, infinite-dimensional manifolds.

Definition. A map $f : X \to Y$ of path-connected spaces is called *n-connected* if the induced maps $\pi_k X \to \pi_k Y$ are isomorphisms for $k < n$ and surjective for $k = n$.

Equivalently, for the algebraic topologists in the audience, the homotopy fiber of f is an n-connected space.

Lemma 10.1. *The inclusion* $\mathrm{U}(m) \hookrightarrow \mathrm{U}(m + n)$ *sending*

$$A \longmapsto \begin{bmatrix} A & 0 \\ 0 & I_n \end{bmatrix}$$

is $2m$-connected.

Proof sketch. First, we may without loss of generality assume $n = 1$; then, iterating that result proves it for larger n. In this case, we have a fibration sequence in which $\mathrm{U}(m + 1)$ acts on S^{2m+1} inside $\mathbb{C}^{m+1}$, so we get a sequence $\mathrm{U}(m) \to \mathrm{U}(m + 1) \to S^{2m+1}$. Then, since $\pi_k(S^{2m+1}) = 0$ for $k \leq 2m$, we can invoke the long exact sequence of homotopy groups of this fibration. ⊠

In particular, $\pi_k \mathrm{U}(m) \cong \pi_k \mathrm{U}$ when $k < 2m$ (which is called the *stable range*).

Our next step is to construct maps $j_m : \mathrm{Gr}_m(\mathbb{C}^{2m}) \to \Omega \mathrm{SU}(2m)$. let $P_m = \Omega_{I,-I} \mathrm{SU}(2m)$, i.e. the space of paths $\gamma : [0,1] \to \mathrm{SU}(2m)$ where $\gamma(0) = I$ and $\gamma(1) = -I$. That is, P_m is the space of paths from I to $-I$. Then, we'll think of the Grassmanian as follows. There is a canonical homeomorphism between $\mathrm{Gr}_m(\mathbb{C}^{2m})$ and the space of Hermitian matrices $A \in \mathrm{Mat}_{\mathbb{C}}(2m, 2m)$ whose eigenvalues are 1 and -1, each with multiplicity m.

In the reverse direction, send $A \mapsto \ker(A - I)$, and in the forward direction, we want to write an m-dimensional subspace as a matrix that acts as I on that subspace and $-I$ on its orthogonal complement, which will be Hermitian.

Now, we'll define a map $i_m : \mathrm{Gr}_m(\mathbb{C}^{2m}) \to P_m$ sending $A \mapsto (t \mapsto \exp(i\pi t A))$: $i\pi t A$ defines a one-parameter subgroup of A, and the conditions on the eigenvalues of A mean that this path starts at I and goes to $-I$. Then, we can take some reference path β in $\mathrm{SU}(2m)$ from $-I$ to I, so $a_m : P_m \to \Omega \mathrm{SU}(2m)$ sending $\gamma \mapsto \beta \circ \gamma$ is a homotopy equivalence. Finally, we'll let $j_m = a_m \circ i_m$.

Theorem 10.2. j_m *is $(2m + 1)$-connected.*

That is, the low-degree homotopy groups of the Grassmanian agree with those of the special unitary groups.

Before proving the theorem, we'll digress to talk about how this proves Bott periodicity. Theorem 10.2 provides a relationship between homotopy groups of the Grassmanian and those of unitary groups, but more classical homotopy theory provides other relationships between these groups. We can construct a map $\eta_m : \Omega \mathrm{Gr}_m(\mathbb{C}^{2m}) \to \mathrm{U}(m)$ as follows: take the tautological vector bundle $\mathbb{C}^m \to V \to \mathrm{Gr}_m(\mathbb{C}^{2m})$ (the fiber over a subspace L in the Grassmanian is just that subspace). Then, choose a Hermitian metric in V and therefore a Hermitian connection ∇.

If we're given a $\gamma \in \Omega \mathrm{Gr}_m(\mathbb{C}^{2m})$, so that $\gamma : S^1 \to \mathrm{Gr}_m(\mathbb{C}^{2m})$ is a based map, then we have a pullback vector bundle $\gamma^* V \to S^1$ and a pullback connection $\gamma^* \nabla$. Then, we'll write that $\eta_m(\gamma)$ is the holonomy of $\gamma^* \nabla$, and this is in $\mathrm{U}(m)$. Specifically, we'll try to trivialize this vector bundle over $[0, 1)$; the holonomy is the discrepancy at the basepoint (i.e. at 0 and 1), which is a unitary matrix.

Proposition 10.3. η_m *induces isomorphisms on* π_k *for* $k \leq 2m + 1$.

The proof will be omitted, but isn't too difficult: you'll write down a homotopy long exact sequence again. It's an instance of the following general fact.

Fact. Let G be a Lie group and BG be its classifying space. Then, $\Omega BG \overset{\sim}{\to} G$ (the classifying space has a canonical principal G-bundle, and the identification is obtained by pulling back to the circle and taking holonomy).

Finally, we use this to obtain Bott periodicity:

Theorem 10.4 (Bott periodicity). *There exists a map* $\mathrm{U}(m) \to \Omega^2 \mathrm{U}(2m)$ *inducing isomorphisms on* π_k *for* $k < 2m + 2$, *and hence* $\pi_k \mathrm{U} \cong \pi_{k+2} \mathrm{U}$.

Proof sketch. We have $\eta_m : \Omega \operatorname{Gr}_m(\mathbb{C}^{2m}) \to \mathrm{U}(m)$. The first thing we'll do is choose a map $\kappa_m : \mathrm{U}(m) \to \Omega \operatorname{Gr}_m(\mathbb{C}^{2m})$ which is approximately a homotopy inverse to η_m: specifically, that κ_m and η_m are inverses on π_k for $k < 2m + 2$. This is possible thanks to a version of the Whitehead theorem. Moreover, we have a map $\Omega j_m : \Omega \operatorname{Gr}_m(\mathbb{C}^{2m}) \to \Omega^2 \operatorname{SU}(2m)$, and an inclusion $\iota : \operatorname{SU}(2m) \hookrightarrow \mathrm{U}(2m)$, which is an isomorphism on π_k for all $k > 1$, thanks to the fibration

$$\operatorname{SU}(2m) \longhookrightarrow \mathrm{U}(2m) \xrightarrow{\det} \mathrm{U}(1).$$

Thus, we have the following system of maps.

$$\mathrm{U}(m) \underset{\kappa_m}{\overset{\eta_m}{\rightleftarrows}} \Omega \operatorname{Gr}_m(\mathbb{C}^{2m}) \xrightarrow{\Omega j_m} \Omega^2 \operatorname{SU}(2m) \xrightarrow{\Omega^2 \iota} \Omega^2 \mathrm{U}(2m)$$

Theorem 10.2 and Proposition 10.3 then prove that the composition of these maps induces the identity on the homotopy groups we need. $\boxtimes$

One unfortunate consequence of this proof is that we don't know how to use this to get generators of the maps. It would be an interesting exercise, but this is one of the advantages of the other proofs of Bott periodicity.

Today, we won't prove Theorem 10.2, but we'll talk about the mechanism of the proof of this theorem, which involves Morse-Bott theory. Though we want to talk about $P_m = \Omega_{I,-I} \operatorname{SU}(2m)$, which is an infinite-dimensional manifold, let's start with the finite-dimensional case.

Let M be an n-dimensional manifold and $f \in C^\infty(M)$. Let $\operatorname{crit}(f) = \{c \in M : D_c f = 0\}$, the set of critical points. For all $c \in \operatorname{crit}(f)$, we have a *Hessian* $D_c^2 f : T_c M \times T_c M \to \mathbb{R}$, which is a symmetric bilinear form defined as the second derivative in any coordinate chart centered at c (which is sort of a cheap definition, but suffices, and is indeed independent of the chart).

Definition. The *index* of a critical point $c \in \operatorname{crit}(f)$, denoted $\operatorname{ind}(f; c)$, is the index (or signature) of $D_c^2 f$, i.e. the dimension of the maximal negative-definite subspace of $T_c M$ with respect to $D_c f$ (i.e. its induced inner product).

If $\operatorname{crit}(f)$ is a submanifold of M, then f is locally constant on it, and hence the Hessian descends to the *normal spaces* $N_c = T_c M / T_c(\operatorname{crit} f)$, so we have a pairing $D_c^2 f : N_c \times N_c \to \mathbb{R}$. This doesn't change the index (we just quotiented out by a space where the form was zero).

If C is a connected component of $\operatorname{crit}(f)$, we'll write $\operatorname{ind}(f; C) = \operatorname{ind}(f, c)$ for any $c \in C$ (since it's locally constant on $\operatorname{crit}(f)$), particularly in the Morse-Bott case below.

Definition. f is said to be *Morse-Bott* if
   (1) $\operatorname{crit}(f)$ is a submanifold of M, and
   (2) for all $c \in \operatorname{crit}(f)$, the Hessian $D_c^2 f : N_c \times N_c \to \mathbb{R}$ is a non-degenerate bilinear form.

Note that for a function to be Morse, the Hessian must not be degenerate on the tangent space, and being Morse-Bott means that we can have some degeneracy, but it must vanish outside of the critical points.

Theorem 10.5. *Let M be an n-dimensional manifold, and assume the following.*
   - *We have an $f \in C^\infty(M)$ that is not only Morse-Bott, but also proper[12] and bounded below.*
   - *If $C_{\min}$ denotes the* manifold of local minima *of f, which is part of $\operatorname{crit}(f)$; we'll want to assume $C_{\min}$ is connected.*
   - *There's an ℓ such that for all conneced components C of $\operatorname{crit}(f)$ other than $C_{\min}$, $\operatorname{ind}(f; C) > \ell$.*

Then, the inclusion $C_{\min} \hookrightarrow M$ is ℓ-connected.

The idea is that all of the π_ℓ of M should come from that of $C_{\min}$. Examples won't be terribly useful right now.

We'd love to apply this to the case $M = P_m$, f is the Riemannian energy functional, and $C_{\min}$ is a path space that will be identified with the Grassmanian, but of course P_m isn't finite-dimensional. The statement is still true, of course, but just requires more work.

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[12] A smooth function f is *proper* if $f^{-1}(-\infty, c]$ is compact for all c.

First, let's set up the proof. Choose a Riemannian metric g on M, so that we have a gradient vector field $\operatorname{grad} f$. Then, $g(\operatorname{grad} f, v) = df(v)$, so if $\gamma : [0,1] \to M$, we get a nice ODE

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = -(\operatorname{grad} f) \circ \gamma. \tag{10.1}$$

The intuition is that if M is embedded in $\mathbb{R}^N$ so that f is a height function,[13] then the gradient indicates the direction of greatest increase of the function.

Then, $\operatorname{grad} f$ defines a flow $\phi_t : M \to M$, and $t \mapsto \phi_t(x)$ is a solution to (10.1), and exists at least locally, by general nonsense about differential equations. But since f is proper and bounded below, then ϕ_t exists for all $t \geq 0$! This is because the negative gradient flow points into $f^{-1}(-\infty, c]$, which is compact, so by standard long-time existince theorems on ODEs, the flow exists for all positive times.[14]

Moreover, for all starting points $x \in M$, the limit $x_\infty = \lim_{t \to \infty} \phi_t(x)$ exists, again basically due to compactness (though it does use the Morse-Bott hypothesis).

Definition. For a connected component $C \subset \operatorname{crit}(f)$, define the *stable manifold* $S_C = \{x \in M : x_\infty \in C\}$.

The stable manifold is the set of points that flows into C eventually (e.g. rolling downhill in the height function).

Lemma 10.6. *S_C is a submanifold of M, and has codimension $\operatorname{ind}(f; C)$.*

This is hard to prove.

So we want to prove that in the conditions assumed in Theorem 10.5, the manifold of minima contains all of the information about the low-dimensional homotopy groups.

Proof of Theorem 10.5. Recall that if $C \neq C_{\min}$ is a connected component of $\operatorname{crit}(f)$, then $\operatorname{ind}(f; C) > \ell$. Then, take a based map $f : S^k \to M$ where $k \leq \ell$ and the basepoint of M is taken to be in $C_{\min}$; we want to show this is homotopic to a map into $C_{\min}$.

Transversality theory tells us that h is based homotopic to a map transverse to S_C for all connected components of $\operatorname{crit}(f)$. But $\operatorname{Im}(h) \cap S_C = \emptyset$ for $C \neq C_{\min}$, as $\operatorname{Im}(h)$ has dimension at most ℓ and S_C has dimension at least ℓ, so their intersection in the general case has to be empty (i.e. we can adjust h a little bit to get an empty intersection).

Now, let $h_t : S^k \to M$ be a based map defined by $h_t = \phi_t \circ h$: we take our sphere, and flow it downwards. Notice that for all $x \in S^k$, $h(x)_\infty \in C_{\min}$, and so for $t \gg 0$, $\operatorname{Im}(h_t)$ lies in a tubular neighborhood of $C_{\min}$, which deformation retracts to $C_{\min}$. Hence, h is homotopic to some map $S^k \to C_{\min}$.

The next step is to show that $C_{\min} \hookrightarrow M$ induces injections on π_k for $k < \ell$; take an $h : S^k \to C_{\min}$ that extends to an $H : B^{k+1} \to M$, so we need to find a homotopy relative to the boundary that maps it to $C_{\min}$. As before, we may assume that h is transverse to the S_C (thanks to the relative transversality theorem), and then run the same argument; we've chosen the dimensions so that once again, it can't hit the stable manifolds except for $S_{C_{\min}}$, and so flowing once again gives us a homotopy.   ⊠

Our task for next time is to run a version of this argument in the infinite-dimensional loop space.

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[13]Though this picture is primarily for intuition, the Whitney embedding theorem means that for sufficiently large N, this is possible.

[14]There's no guarantee that it'll exist for all negative time, though.

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