CHARACTERISTIC CLASSES: EXERCISES

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These exercises are not in order. Do the ones that look the most interesting to you. Some are a lot easier than others.

(1) Show that $\mathbb{CP}^4$ cannot be embedded in $\mathbb{R}^{11}$.
(2) Show that if $E \subseteq TS^n$ is a subbundle, then either E is trivial or all of TS^n.
(3) Show that the mod 2 reduction of $p_i(V)$ is $w_i(V)^2$.
(4) A *degree-d hypersurface* in $\mathbb{CP}^{n+1}$ is a smooth (complex-)codimension-1 submanifold $X_d \subset \mathbb{CP}^n$ cut out by a degree-d homogeneous polynomial. If $S \to \mathbb{CP}^{n+1}$ denotes the tautological bundle, then the normal bundle of $X_d \hookrightarrow \mathbb{CP}^{n+1}$ is $(S^*)^{\otimes d}|_{X_d}$.
   (a) When $n = 1$, these are smooth projective curves (aka compact Riemann surfaces). What is $\chi(X_d)$? (You should get $d(3-d)$.)
   (b) Now suppose $n = 2$. Which X_d admit spin structures?
   (c) For $n = 2$, show $c_1(X_d) = 0$ iff $d = 4$. This quartic surface is known as the *K3 surface*, and generates $\Omega_4^{\mathrm{Spin}} \cong \mathbb{Z}$ (proving that is hard and not part of this exercise). What is its Euler characteristic?
   (d) Using that $\mathbb{CP}^2$ generates $\Omega_4^{\mathrm{SO}} \cong \mathbb{Z}$, show that the forgetful map $\Omega_4^{\mathrm{Spin}} \to \Omega_4^{\mathrm{SO}}$ has image $8 \cdot \Omega_4^{\mathrm{SO}}$.