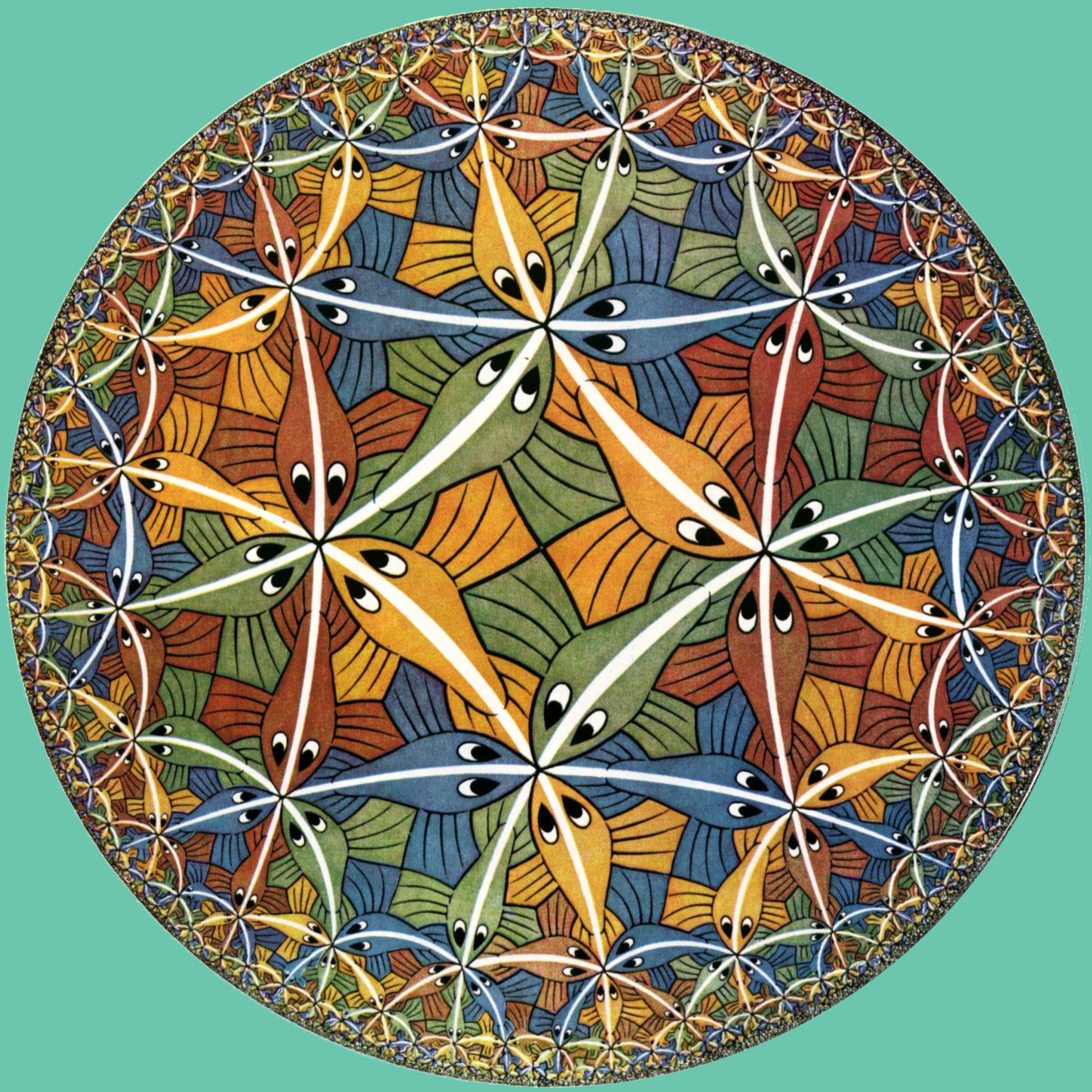
Riemann Surfaces



UT Austin, Spring 2016

M392C NOTES: RIEMANN SURFACES

ARUN DEBRAY
FEBRUARY 15, 2016

These notes were taken in UT Austin's Math 392c (Riemann Surfaces) class in Spring 2016, taught by Tim Perutz. I live-TEXed them using vim, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. The image on the front cover is M.C. Escher's *Circle Limit III* (1959), sourced from http://www.wikiart.org/en/m-c-escher/circle-limit-iii. Thanks to Adrian Clough for finding a few typos.

Contents

Lecture 1.

Review of Complex Analysis: 1/20/16

Riemann surfaces is a subject that combines the topology of structures with complex analysis: a Riemann surface is a surface endowed with a notion of holomorphic function. This turns out to be an extremely rich idea; it's closely connected to complex analysis but also to algebraic geometry. For example, the data of a compact Riemann surface along with a projective embedding specifies a proper algebraic curve over $\mathbb{C}$, in the domain of algebraic geometry.[1] In fact, the algebraic geometry course that's currently ongoing is very relevant to this one.

The theory of Riemann surfaces ties into many other domains, some of them quite applied: number theory (via modular forms), symplectic topology (pseudo-holomorphic forms), integrable systems, group theory, and so on: so a very broad range of mathematics graduate students should find it interesting.

Moreover, by comparison with algebraic geometry or the theory of complex manifolds, there's very low overhead; we will quickly be able to write down some quite nontrivial examples and prove some deep theorems: by the middle of the semester, hopefully we will prove the analytic Riemann-Roch theorem, the fundamental theorem on compact Riemann surfaces, and use it to prove a classification theorem, called the uniformization theorem.

The course textbook is S.K. Donaldson's *Riemann Surfaces*, and the course website is at http://www.ma.utexas.edu/users/perutz/RiemannSurfaces.html; it currently has notes for this week's material, a rapid review of complex function theory. We will assume a small amount of complex analysis (on the level of Cauchy's theorem; much less than the complex analysis prelim) and topology (specifically, the relationship between the fundamental group and covering spaces). Some experience with calculus on manifolds will be

[1] This sentence is packed with jargon you're not assumed to know yet.

helpful. Some real analysis will be helpful, and midway through the semester there will be a few Hilbert spaces. Thus, though this is a topics course, the demands on your knowledge will more resemble a prelim course.

Let's warm up by (quickly) reviewing basic complex analysis; the notes on the course website will delve into more detail. For the rest of this lecture, G denotes an open set in $\mathbb{C}$.

The following definition is fundamental.

Definition. A function $f : G \to \mathbb{C}$ is *holomorphic* if for all $z \in G$, the complex derivative

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists. The set of holomorphic functions $G \to \mathbb{C}$ is denoted $\mathscr{O}(G)$, after the Italian *funzione olomorfa*.

Note that even though it makes sense for the limit to be infinite, this is not allowed.

First, let's establish a few basic properties.

- If $H \subset G$ is open and $f \in \mathscr{O}(G)$, then $f|_H \in \mathscr{O}(H)$.
- The sum, product, quotient, and chain rules hold for holomorphic functions, so $\mathscr{O}(G)$ is a commutative ring (with multiplication given pointwise) and in fact a commutative $\mathbb{C}$-algebra.[2]

In other words, holomorphic functions define a *sheaf* of $\mathbb{C}$-algebras on G.

By a rephrasing of the definition, then if f is holomorphic on G, then it has a *derivative* f' on G, i.e. for all $z \in G$, one can write $f(z+h) = f(z) + f'(z)h + \varepsilon_z(h)$, where $\varepsilon_z(h) \in o(h)$ (that is, $\varepsilon_z(h)/h \to 0$ as $h \to 0$). Thus, a holomorphic function is differentiable in the real sense, as a function $G \to \mathbb{R}^2$. This means that there's an $\mathbb{R}$-linear map $D_z f : \mathbb{C} \to \mathbb{C}$ such that $f(z+h) = f(z) + (D_z f)(h) + o(h)$: here, $D_z f(h) = f'(z)h$.

However, we actually know that $D_z f$ is $\mathbb{C}$-linear. This is known as the *Cauchy-Riemann condition.* Since it's *a priori* $\mathbb{R}$-linear, saying that it's $\mathbb{C}$-linear is equivalent to it commuting with multiplication by i. $D_z f$ is represented by the Jacobian matrix

$$D_z f = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}.$$

A short calculation shows that this commutes with i iff the following equations, called the *Cauchy-Riemann equations*, hold:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1.1}$$

The content of this is exactly that $D_z f$ is complex linear.

Conversely, suppose $f : G \to \mathbb{C}$ is differentiable in the real sense. Then, if it satisfies (1.1), then $D_z f$ is complex linear. But a complex linear map $\mathbb{C} \to \mathbb{C}$ must be multiplication by a complex number $f'(z)$, so f is holomorphic, with derivative f'.

Power Series. The notation $D(c, R)$ means the open disc centered at c with radius R, i.e. all points $z \in \mathbb{C}$ such that $|z - c| < R$.

Definition. Let $A(z) = \sum_{n=0}^\infty a_n (z-c)^n$ be a $\mathbb{C}$-valued power series centered at a $c \in \mathbb{C}$. Then, its *radius of convergence* is $R = \sup\{|z - c| : A(z) \text{ converges}\}$, which may be 0, a positive real number, or ∞.

Theorem 1.1. *Suppose $A(z) = \sum_{n \ge 0} a_n (z-c)^n$ has radius of convergence R. Then:*

1. $R^{-1} = \limsup |a_n|^{1/n}$;
2. *$A(z)$ converges absolutely on $D(c, R)$ to a function $f(z)$;*
3. *the convergence is uniform on smaller discs $D(c, r)$ for $r < R$;*
4. *the series $B(z) = \sum_{n \ge 1} n a_n (z-c)^{n-1}$ has the same radius of convergence R, so converges on $D(c, R)$ to a function $g(z)$; and*
5. *$f \in \mathscr{O}(D(c, R))$ and $f' = g$.*

These aren't extremely hard to prove: the first few rely on various series convergence tests from calculus, though the last one takes some more effort.

[2] A $\mathbb{C}$-*algebra* is a commutative ring A with an injective map $\mathbb{C} \hookrightarrow A$, which in this case is the constant functions.

Paths and Cauchy's Theorem. By a *path* we mean a continuous and piecewise C^1 map $[a, b] \to \mathbb{C}$ for some real numbers $a < b$. That is, it breaks up into a finite number of chunks on which it has a continuous derivative. A *loop* is a path γ such that $\gamma(a) = \gamma(b)$.

If γ is a C^1 path in G (so its image is in G) and $f : G \to \mathbb{C}$ is continuous, we define the *integral*

$$\int_\gamma f = \int_\gamma f(z)\,\mathrm{d}z := \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t.$$

This is a complex-valued function, because the rightmost integral has real and imaginary parts. This makes sense as a Riemann integral, because these real and imaginary parts are continuous. This is additive on the join of paths, so we can extend the definition to piecewise C^1 paths. Moreover, integrals behave the expected way under reparameterization, and so on.

Theorem 1.2 (Fundamental theorem of calculus)**.** *If $F \in \mathscr{O}(G)$ and $\gamma : [a, b] \to G$ is a path, then*

$$\int_\gamma F'(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)).$$

This is easy to deduce from the standard fundamental theorem of calculus. In particular, if γ is a loop, then the integral of a holomorphic function is 0.

Now, an extremely important theorem.

Definition. A *star-domain* is an open set $G \subset \mathbb{C}$ with a $z^* \in G$ such that for all $z \in G$, the line segment $[z^*, z]$ joining z^* and z is contained in G.

For example, any convex set is a star-domain.

Theorem 1.3 (Cauchy)**.** *If G is a star-domain, γ is a loop in G, and $f \in \mathscr{O}(G)$, then $\int_\gamma f = 0$. Indeed, $f = F'$, where*

$$F(z) = \int_{[z^*,z]} f.$$

The proof is in the notes, but the point is that you can check that this definition of F produces a holomorphic function whose derivative is f; then, you get the result. The idea is to compare $F(z + h)$ and $F(z)$ should be comparable, which depends on an explicit calculation of an integral of a holomorphic function around a triangle, which is not hard.

Cauchy didn't prove Cauchy's theorem this way; instead, he proved Green's theorem, using the Cauchy-Riemann equations. This is short and satisfying, but requires assuming that all holomorphic functions are C^1. This is true (which is great), but the standard (and easiest) way to show this is... Cauchy's theorem.

Lecture 2.

Review of Complex Analysis, II: 1/22/16

Today, we're going to continue not being too ambitious; next week we will begin to geometrify things. Last time, we stopped after Cauchy's theorem for a star domain G: for all f holomorphic on G and loops $\gamma \in G$, $\int_\gamma f = 0$, and in fact one can write down an antiderivative for f, and then apply the fundamental theorem of calculus.

Then one can bootstrap one's way up to a more powerful theorem; the next one is a version of the deformation theorem.

Corollary 2.1 (Deformation theorem)**.** *Let $G \subset \mathbb{C}$ be open and $\gamma_0, \gamma_1 : [a, b] \rightrightarrows G$ be C^1 loops that are C^1 homotopic through loops in G. Then, for all $f \in \mathscr{O}(G)$, $\int_{\gamma_0} f = \int_{\gamma_1} f$.*

Proof sketch. Fix a C^1 homotopy $\Gamma : [a, b] \times [0, 1] \to G$ such that $\Gamma(a, s) = \Gamma(b, s)$ for all s, $\gamma_0(t) = \Gamma(t, 0)$, and $\gamma_1(t) = \Gamma(t, 1)$. Then, it is possible to divide $[a, b] \times [0, 1]$ into a grid of rectangles fine enough such that the image of each rectangle is mapped under Γ to a subset of G contained in an open disc in $\mathbb{C}$, as in Figure 1. Now, by Cauchy's theorem in a disc, the integral does not depend on path within each disc, so we can apply Γ in over the rectangles from 0 to 1, showing that the two integrals are the same. ⊠

Corollary 2.2. *Cauchy's theorem holds in any simply connected open $G \subset \mathbb{C}$.*

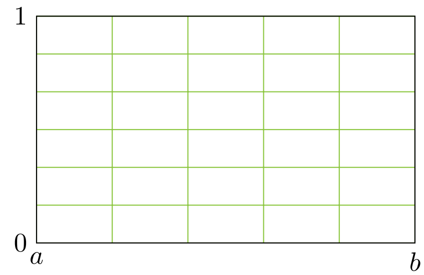


FIGURE 1. Subdividing $[a, b] \times [0, 1]$ into rectangles.

This is considerably more general than star domains (e.g. the letter **C** is simply connected, but not a star domain). Moreover, on such a domain, any $f \in \mathscr{O}(G)$ has an antiderivative: pick some basepoint $z_0 \in G$, and let $\gamma(z_0, z)$ be a path from z_0 to z. Then,

$$F(z) = \int_{\gamma(z_0,z)} f(z)\,\mathrm{d}z$$

is well-defined, because any two choices of path differ by the integral of a holomorphic function on a loop, which is 0.

We can also use this to understand power series representations.

Proposition 2.3 (Cauchy's integral formula)**.** *, Let G be a domain in $\mathbb{C}$ containing the closed disc D. If $f \in \mathscr{O}(G)$, then*

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w - z}\,\mathrm{d}w.$$

Proof idea. Suppose D is centered at z and has radius R, and let $C(z, r)$ denote the circle centered at z and with radius r. We'll also let D^* denote the punctured disc, i.e. D minus its center point. By calculating $\int_\gamma \mathrm{d}z/z = 2\pi i$, one has that

$$\frac{1}{2\pi i}\int \partial D \frac{f(w)}{z - w}\,\mathrm{d}w - f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w) - f(z)}{w - z}\,\mathrm{d}w.$$

Using Corollary 2.1, for $r \in (0, R)$,

$$= \frac{1}{2\pi i}\int_{C(z,r)} \frac{f(w) - f(z)}{w - z}\,\mathrm{d}w,$$

and as $r \to 0$, this approaches $f'(z)$, which is bounded, and the integral over smaller and smaller circles of a bounded function tends to zero. ⊠

Theorem 2.4 (Holomorphic implies analytic)**.** *If D is a disc centered at c and $f \in \mathscr{O}(D)$, then on that disc,*

$$f(z) = \sum_{n\geq 0} a_n(z - c)^n, \qquad \textit{where} \qquad a_n = \frac{1}{2\pi i}\int_{\partial D} \frac{f(z)}{(z - c)^{n+1}}\,\mathrm{d}z.$$

Proof sketch. For any $z \in D$, there's a $\delta > 0$ such that the closed disc $\overline{D}(z, \delta)$ of radius δ is contained in D. Hence, by Proposition 2.3,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{C(z,\delta)} \frac{f(w)}{w - z}\,\mathrm{d}w \\ &= \int_{C(c,R')} \frac{f(w)}{w - z}\,\mathrm{d}w \end{aligned}$$

for any $R' \in (0, \delta)$, by Corollary 2.1. We'd like to force a series on this. First, since

$$\frac{1}{w - z} = \frac{1}{(w - c) - (z - c)} = \frac{1}{w - c}\left(\frac{1}{1 - \frac{z-c}{w-c}}\right),$$

then

$$f(z) = \frac{1}{3\pi i}\int_{C(c,R')} \frac{f(w)}{w-c}\frac{1}{1-\frac{z-c}{w-c}}\,\mathrm{d}w$$
$$= \frac{1}{2\pi i}\oint \frac{f(w)}{w-c}\sum_{n\geq 0}\frac{(z-c)^n}{(w-c)^n}\,\mathrm{d}w.$$

Since $|(z-c)/(w-c)| < 1$ on $C(c,R')$, then this is well-defined, and since it's a geometric series, it has nice convergence properties, and so we can exchange the sum and integral to obtain

$$= \sum_{n\geq 0}\underbrace{\frac{1}{2\pi i}\left(\oint \frac{f(w)}{(w-c)^{n+1}}\,\mathrm{d}w\right)}_{a_n}(z-c)^n. \qquad \boxtimes$$

One application of this is to understand zeros of holomorphic functions. If $f \in \mathscr{O}(G)$ and $f(c) = 0$, then let $f(z) = \sum a_n(z-c)^n$ be its power series and a_m be the first nonzero coefficient. Then, in a neighborhood of c,

$$f(z) = (z-c)^m \underbrace{\sum_{n\geq m} a_n(z-c)^{n-m}}_{g(z)}.$$

This g is holomorphic and does not vanish on this neighborhood, so the takeaway is $f(z) = (z-c)^m g(z)$ near c, with g holomorphic and nonvanishing. This m is called the *multiplicity*, denoted $\mathrm{mult}(f, c)$. In particular, if $f(c) \neq 0$, then $m = 0$.

Theorem 2.5. *If G is a connected open set and $f \in \mathscr{O}(G)$ is not identically zero, then $f^{-1}(0)$ is discrete in $\mathbb{C}$.*

Proof. If $f(c) = 0$, then there's a disc D on which $f(z) = (z-c)^m g(z)$, where $m \geq 1$ and g is nonvanishing, so the only place f can vanish on D (i.e. near c) is at c itself. ⊠

Definition. A function $f \in \mathscr{O}(\mathbb{C})$, so holomorphic on the entire plane, is called *entire*.

Theorem 2.6 (Liouville)**.** *A bounded, entire function is constant.*

Proof sketch. We'll show that $f'(z) = 0$ everywhere. By Proposition 2.3, we know

$$f'(z) = \frac{1}{2\pi i}\int C(z,r)\frac{f(w)}{(w-z)^2}\,\mathrm{d}w,$$

and we can deform this loop to $C(0,R)$. Then, one bounds the integral, and the bound ends up being $O(1/R)$, so as $R \to \infty$, this necessarily goes to 0. ⊠

Lecture 3.

Meromorphic Functions and the Riemann Sphere: 1/25/16

We're still going to be doing classical function theory today, but we're going to begin to geometrify it. Recall that $G \subset \mathbb{C}$ denotes an open set.

We'll begin with the following theorem.

Theorem 3.1 (Morera)**.** *Let $f : G \to \mathbb{C}$ be a continuous function such that for all triangles $T \subset G$, $\int_{\partial T} f = 0$. Then, f is holomorphic.*

This is surprisingly easy to prove, given what we've done.

Proof. Since holomorphy is a local property, we may without loss of generality work on a disc $D(z_0, r) \subset G$. Then, define $F : D(z_0, r) \to \mathbb{C}$ by $F(z) = \int_{[z_0,z]} f$; using the hypothesis on triangles, $F' = f$. Thus, as we showed last time, this means $F \in \mathscr{O}(G)$, and so it's analytic, and therefore it has derivatives of all orders. Thus, $F' = f$ is holomorphic. ⊠

This is useful, e.g. one may have a function which is defined through an improper integral, or a pointwise limit of holomorphic functions. Then, Morera's theorem allows for an easier, indirect way to show holomorphy. Here's another application.

Definition. If $z_0 \in G$, a function $f \in \mathscr{O}(G \setminus \{z_0\})$ has a *removable singularity* at z_0 if f can be extended holomorphically to G.

Theorem 3.2. *Suppose $f \in \mathscr{O}(G \setminus \{z_0\})$ and $|f|$ is bounded near z_0. Then, f has a removable singularity at z_0.*

There are several ways to prove this quickly.

Proof. We can without loss of generality translate this to the origin, so assume $z_0 = 0$. If $g(z) = zf(z)$, then $g(z) \to 0$ as $z \to 0$, since $|f(z)|$ is bounded in a neighborhood of the origin. Thus, g extends continuously to all of G, with $g(0) = 0$.

Next, one should check that Morera's theorem applies to g; the only nontrivial example is a triangle around the origin. However, since g is holomorphic everywhere except at 0, the deformation theorem allows us to shrink the triangle as much as we want, and since $g \to 0$, the integral goes to 0 as well. If the triangle's edge or vertex touches the origin, one can use the deformation theorem to push it away again.

In particular, g is holomorphic on G and has a zero at 0, so by the discussion on multiplicities last time, $g(z) = z \cdot f(z)$, where f is holomorphic on all of G; this produces our desired extension of f. ⊠

Definition.

- If $z_0 \in G$ and $f \in \mathscr{O}(G \setminus \{z_0\})$, then f has a *pole* at z_0 if there's an $m \in \mathbb{N}$ such that $(z - z_0)^m f(z)$ is bounded near z_0 (and hence has a removable singularity there). The least such m is called the *order* of the pole.
- A *meromorphic* function on G is a pair (Δ, f) consisting of a discrete subset $\Delta \subset G$ and an $f \in \mathscr{O}(G \setminus \Delta)$ such that f has a pole at each $z \in \Delta$.

So, nothing worse than a pole happens for a meromorphic function. There are *essential singularities*, which are singularities which aren't poles, but we will not discuss them extensively; almost everything in sight will be meromorphic.

The Riemann Sphere. In some sense, the Riemann sphere is the most natural setting for meromorphic functions, and the first nontrivial example of a Riemann surface (still to be defined).

Definition. The *Riemann sphere* $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the *one-point compactification* of $\mathbb{C}$, so its topology has as its open sets (1) opens in $\mathbb{C}$, and (2) $(\mathbb{C} \setminus K) \cup \{\infty\}$, where $K \subset \mathbb{C}$ is compact.

There is a homeomorphism $\phi : \widehat{\mathbb{C}} \to S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ given by *stereographic projection*: send $\infty \mapsto (0, 0, 1)$ (the north pole), and then any other $z \in \mathbb{C}$ defines a line from z in the xy-plane to $(0, 0, 1)$ intersecting S^2 at one other point; this is $\phi(z)$. Hence, we will use $\widehat{\mathbb{C}}$ and S^2 interchangeably.

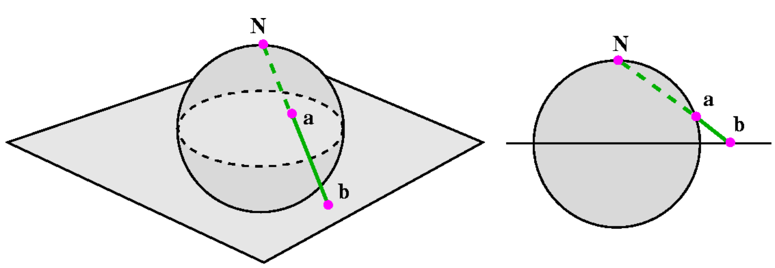


Figure 2. Depiction of stereographic projection, where $N = (0, 0, 1)$ is the north pole. Source: http://www.math.rutgers.edu/~greenfie/vnx/math403/diary.html.

Definition. A continuous map $f : G \to S^2$ is *holomorphic* if for all $z \in G$, either

- $f(z) \in \mathbb{C}$ (so it doesn't hit ∞) and $f : G \to \mathbb{C}$ is holomorphic, or
- if $f(z) \in \widehat{\mathbb{C}} \setminus \{0\}$, then $1/f(w) : G \to \mathbb{C}$ is holomorphic, where $1/\infty$ is understood to be 0.

If the image of f contains neither 0 not ∞, then both criteria hold, and are equivalent (since $1/z$ is holomorphic on any neighborhood not containing zero).

Proposition 3.3. *The meromorphic functions on G can be identified with the holomorphic functions $G \to S^2$.*

Proof. Suppose f is meromorphic on G, so that it has a pole of order m at z_0. Then, $f(z) = (1/(z-z_0)^m)g(z)$ for some holomorphic g with a removable singularity at z_0, and $g(z_0) \neq 0$.

By letting $1/0 = \infty$, this realizes f as a continuous map $G \to S^2$, and $1/f = (z-z_0)^m(1/g)$, which is certainly holomorphic near z_0, so f is holomorphic as a map to S^2.

The converse is quite similar, a matter of unwinding the definitions, but has been left as an exercise. ⊠

You can also define a notion of a holomorphic function coming out of S^2, not just into.

Definition. Let $G \subset S^2$ be open. A continuous $f : G \to S^2$ is *holomorphic* if one of the following is true.

- If $\infty \notin G$, then we use the same definition as above.
- If $\infty \in G$, then it's holomorphic on $G \setminus \infty$ and there's a neighborhood N of ∞ in G such that the composition
$$N^{-1} \xrightarrow{z \mapsto 1/z} N \xrightarrow{f} S^2$$
is holomorphic.

If you're used to working with manifolds, this sort of coordinate change is likely very familiar: every time we talk about ∞, we take reciprocals and talk about 0.

Example 3.4. Every rational function $p \in \mathbb{C}(z)$ is meromorphic, and extends to a holomorphic map $S^2 \to S^2$.

Now, we can talk about these geometrically: $z \mapsto z^2$ sends $e^{in\theta} \mapsto e^{2in\theta}$, so it doubles the longitude (modulo 1). In particular, it wraps the sphere twice around itself, preserving 0 and ∞, as in Figure 3.

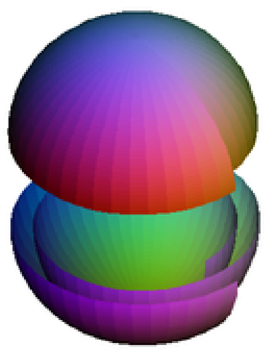


FIGURE 3. A depiction of the map $z \mapsto z^2$ on the Riemann sphere, which fixes the poles. Source: `https://en.wikipedia.org/wiki/Degree_of_a_continuous_mapping`.

Example 3.5. A *Möbius map* is a map
$$\mu(z) = \frac{az+b}{cz+d},$$
where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. This extends to a holomorphic map $S^2 \to S^2$ with a holomorphic inverse (the Möbius map associated to $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]^{-1}$). Thus, there's a homeomorphism $\mathrm{SL}_2(\mathbb{R})/\{\pm I\}$ to the group of Möbius transformations.

One interesting corollary is that the point at infinity is *not* special, since there's a Möbius map sending it to any other point of S^2, and indeed they act transitively on it. So we don't really have to distinguish the point at infinity from this geometric point of view.

Theorem 3.6. *If $f : S^2 \to S^2$ is holomorphic, then it's a rational function. In particular, the Möbius maps are the only invertible holomorphic maps $S^2 \to S^2$.*

The idea is to eliminate the zeros and poles by multiplying by $(z-z_0)^m$; then, one can apply Liouville's theorem to show that the result is constant.

Lecture 4.

Analytic Continuation: 1/27/16

This corresponds to §1.1 in the textbook, and is one of the classical motivations for Riemann surfaces.

The problem is: if $G \subset \mathbb{C}$ is open and $f \in \mathscr{O}(G)$, then we would like to extend f holomorphically, or maybe meromorphically, to a larger domain $H \supset G$. Such extensions are called *analytic* (resp. *meromorphic*) *continuations* of f.[3]

Remark. If H is connected, then there exists at most one meromorphic continuation of f to H, because the difference of two continuations vanishes on the open set G, and hence vanishes everywhere.

Example 4.1. Let $f(z) = \sum_{n\geq 0} z^n$, which converges on the open unit disc, but diverges when $|z| \geq 1$. At first sight, this suggests we'll never get any farther than the disc, but this turns out to merely be an artifact of this presentation of f: we could instead write it as $f(z) = 1/(1-z)$, which meromorphically extends f to the whole of $\mathbb{C}$ (with a single pole at $z = 1$). Thus, this power series representation is not *per se* intrinsic.

One can take this further and define analytic continuations of general functions defined by power series.

Example 4.2. This example is more sophisticated, and will take longer; it reflects a common theme in this subject, that the examples are nontrivial and are worth taking seriously. Define the Γ-*function*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,\mathrm{d}t$$

on the open set $\operatorname{Re} z > 0$. This integral is doubly improper, since there's a singularity at 0 and it's unbounded on the right, so we really should rewrite it as

$$\Gamma(z) = \lim_{\varepsilon\to 0^+} \int_\varepsilon^1 t^{z-1}e^{-t}\,\mathrm{d}t + \lim_{T\to\infty}\int_1^T t^{z-1}e^{-t}\,\mathrm{d}t.$$

Let $H_a = \{z \mid \operatorname{Re} z > a\}$. We're going to show that Γ extends to the entire plane, but first we need to show that it's holomorphic on the right half-plane.

Proposition 4.3. $\Gamma \in \mathscr{O}(H_0)$.

Proof sketch. Since we need to realize $\Gamma(z)$ as a limit, let

$$g_n(z) = \int_{1/n}^n t^{z-1}e^{-t}\,\mathrm{d}t.$$

This is an integral of a holomorphic function, so $g_n \in \mathscr{O}(\mathbb{C})$ and

$$g_n'(z) = \int_{1/n}^n \frac{\partial}{\partial z}\left(t^{z-1}e^{-t}\right)\mathrm{d}t = (z-1)g_n(z-1).$$

If $a > 0$, then g converges uniformly on the strip $a < \operatorname{Re} z < b$ — the goal is to show that g_n is uniformly Cauchy on this strip (the details of which are left to the reader) by comparing to the integral of $e^{-t/2}$ for $t \gg 0$, the point being that $e^{x-1}e^{-t} \leq e^{-t/2}$ for t sufficiently large. For $t < 1$, one should compare it to the integral of t^{x-1}. Then, we need to use the following theorem.

Theorem 4.4. *If $f_n \in \mathscr{O}(G)$ and $f_n(z) \to f(z)$ locally uniformly, then $f \in \mathscr{O}(G)$.*

The proof uses Morera's theorem (Theorem 3.1) and can be found in the review notes (or Rudin, etc.). In any case, this means $\Gamma = \lim_{n\to\infty} g_n$ is holomorphic on the right half-plane. ⊠

Now, we can talk about extending Γ.

Theorem 4.5. Γ *has a meromorphic continuation to $\mathbb{C}$, whose only poles are simple poles*[4] *at 0, -1, -2, and so on.*

[3] Though "holomorphic continuation" would make more sense, tradition gives us the term "analytic continuation."

[4] A pole is *simple* if it's degree 1.

Proof. Since the gamma function is given by an integral, let Γ_0 be that integral from 0 to 1, and Γ_∞ be the integral from 1 to ∞. Then, the argument above shows that $\Gamma_\infty \in \mathscr{O}(\mathbb{C})$, so the only extension that we actually need to make is of

$$\Gamma_0(z) = \int_0^1 t^{z-1} e^{-t}\,\mathrm{d}t.$$

The cunning idea is that we're going to look at the n^{th}-order Taylor polynomial for e^{-t}, which provides an integral we can actually do, and then treat everything else separately. Specifically, let

$$e_n(t) = \sum_{j=0}^{n-1} \frac{(-t)^j}{j!},$$

so that

$$\begin{aligned}\Gamma_0(z) &= \underbrace{\int_0^1 t^{z-1}\big(e^{-t} - e_n(t)\big)\,\mathrm{d}t}_{\Gamma_n(z)} + \int_0^1 t^{z-1} e_n(t)\,\mathrm{d}t.\\ &= \Gamma_n(z) + \sum_{j=0}^{n-1} \frac{(-1)^j}{j!(z+j)}.\end{aligned}$$

The $(z+j)$ in the denominator on the right gives us simple poles at $0, -1, -2, \ldots, -n+1$. But $e^{-t} - e_n(t)$ has a zero of order n at $t = 0$, so

$$\int_0^1 t^{z-1}\big(e^{-t} - e_n(t)\big)\,\mathrm{d}t$$

exists on H_{-n}, so $\Gamma_n \in \mathscr{O}(H_{-n})$. Thus, we can extend Γ meromorphically to all of $\mathbb{C}$, because any $z \in \mathbb{C}$ is in some H_{-n}, so we can work this with Γ_n. ⊠

It goes without saying that Γ is one of the most prominent functions in analytic number theory.

These two successful examples of meromorphic continuation are in some sense atypical; in general, there is a problem of multi-valuedness or monodromy.

Example 4.6. For an algebraic example of this problem, consider

$$f(z) = \sum_{n\geq 0} \binom{1/2}{n} z^n,$$

where

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}.$$

By a generalized binomial theorem (or checking that it satisfies the right differential equation), one can show that f converges on $D(0,1)$ to a branch of $\sqrt{1+z}$. We can extend holomorphically to the *cut plane* $\mathbb{C} \setminus (-\infty, -1]$ by writing $f(z) = \exp((1/2)\log(1+z))$, where we can choose a branch of $\log(1+z)$ in this cut plane, such as $\log(re^{i\theta}) = \log r + i\theta$, with $\theta \in (-\pi, \pi)$.

There's nothing particularly special about this branch cut. Plenty of other branch cuts (paths from -1 to $-\infty$ whose complements are simply connected) work just as fine — but we cannot extend further, because as we go around a loop around -1, $f(z)$ flips $-f(z)$ (the other branch of $\sqrt{1+z}$), since the logarithm changes by $2\pi i$. This is a little unsatisfactory, since we can't go further.

A similar story holds for just about any algebraic function, since one has to take a branch cut to resolve the ambiguity of multiple roots.

The Riemann surfaces way to approach this is instead of making arbitrary branch cuts, it's more canonical instead to study the equation $w^2 - (1-z) = 0$, which implicitly defines w as a square root of $1+z$. Then, we consider the set

$$X = \{(z,w) \in \mathbb{C}^2 \mid P(z,w) = 0\},$$

where $P(z,w) = w^2 - (1+z)$. Soon, we will see that this X is a Riemann surface. We can play exactly the same game with any $P(z,w) : \mathbb{C}^2 \to \mathbb{C}$ that is holomorphic in each variable separately, includin any polynomial in z and w. This defines for us its zero set $X = \{P(z,w) = 0\}$.

Then, we have an implicit function theorem, which is a major classical motivation for the theory of Riemann surfaces, just as the implicit function theorem on $\mathbb{R}^n$ is a major classical motivation for defining abstract manifolds.

Theorem 4.7 (Implicit function theorem)**.** *If $(z_0, w_0) \in X$ and $\frac{\partial P}{\partial w}(z_0, w_0) \neq 0$, then there's a disc $D_1 \subset \mathbb{C}$ centered at $z_0 \in \mathbb{C}$ and a disc $D_2 \subset \mathbb{C}$ centered at w_0, and a holomorphic $\phi : D_1 \to D_2$ such that $\phi(z_0) = w_0$ and $X \cap (D_1 \times D_2)$ is the graph of ϕ, i.e. $\{(z, \phi(z)) \mid z \in D_1\}$.*

An analogue of this function holds for C^1 real functions (or C^∞ ones), and this version can be extracted from that, but it has a simpler, direct proof.

Proof. This proof hinges on a theorem called the *argument principle*, that if $f \in \mathscr{O}(G)$ and $\overline{D}$ is a closed disc in G with $f(z) \neq 0$ on $\partial\overline{D}$, then

$$\frac{1}{2\pi i} \int_{\partial\overline{D}} \frac{f'(w)}{f(w)}\, \mathrm{d}w = \sum_{\substack{z \in D \\ f(z)=0}} \operatorname{mult}(f; z). \tag{4.1}$$

That is, integrating the logarithmic derivative counts the zeros inside D, with multiplicity. There's also the related formula

$$\frac{1}{2\pi i} \int_{\partial\overline{D}} \frac{w f'(w)}{f(w)}\, \mathrm{d}w = \sum_{\substack{z \in D \\ f(z)=0}} z \operatorname{mult}(f; z). \tag{4.2}$$

These are nice exercises in residue calculus.

Returning to the implicit function theorem, let $f_z = P(z, \cdot)$, so $f_{z_0}(w_0) = 0$, but $f'_{z_0}(w_0) \neq 0$. Thus, $\operatorname{mult}(f_{z_0}; w_0) = 1$, and therefore by isolation of zeros, there's a disc D_2 centered at w_0 such that w_0 is the only zero of f_{z_0} in $\overline{D}_2$. Hence, by (4.1),

$$\frac{1}{2\pi i} \int_{\partial D_2} \frac{f'_{z_0}}{f_{z_0}} = 1.$$

Since $f_{z_0} \neq 0$ on the boundary, then there's a $\delta > 0$ such that $|f_{z_0}| > 2\delta > 0$ on ∂D_2. Thus, there's a disc D_1 centered at z_0 such that for all $z \in D_1$, $|f_z| > \delta$ on ∂D_2 because P is continuous. Hence,

$$\frac{1}{2\pi i} \int_{\partial D_2} \frac{f'_z}{f_z} = 1,$$

or, by (4.1), there's a unique solution $w = \phi(z)$ to $P(z, w) = 0$ with $z \in D_1$ and $w \in D_2$. Thus, we need only to show that ϕ is holomorphic. By (4.2),

$$\phi(z) = \frac{1}{2\pi i} \int_{\partial D_2} \frac{w f'_z(w)}{f_z(w)}\, \mathrm{d}w = \frac{1}{2\pi i} \int_{\partial D_2} \frac{w}{P(z,w)} \frac{\partial P}{\partial w}(z, w)\, \mathrm{d}w.$$

Hence, ϕ is holomorphic in z (since its derivative is given by differentiating under the integral sign). ☒

Thus, even working just with zero sets of algebraic functions, Riemann surfaces show up very nicely.

Lecture 5.

Analytic Continuation Along Paths: 1/29/16

Today, we're going to talk about analytic continuation along paths and the interesting things that result. There's also a more classical Weierstrass way to look at this.

Definition. If ϕ is a holomorphic function defined on a neighborhood U of a $z_0 \in \mathbb{C}$ and $\gamma : [0, 1] \to \mathbb{C}$ is a path with $\gamma(0) = z_0$, then an *analytic continuation* of ϕ along γ consists of a pair (U_t, ϕ_t) for all $t \in [0, 1]$, where U_t is a neighborhood of $\gamma(t)$ and $\phi_t \in \mathscr{O}(U_t)$ such that:

- $\phi_0 = \phi$ on $U_0 \cap U$, and
- the different ϕ_t should agree, in the sense that for all $s \in [0, 1]$, there's a $\delta > 0$ such that if $|t - s| < \delta$, then ϕ_s and ϕ_t agree on $U_s \cap U_t$.

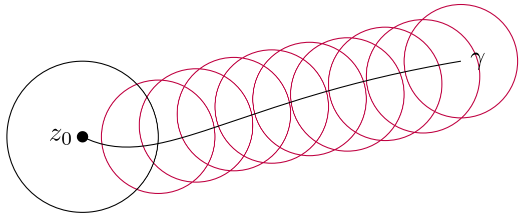


FIGURE 4. Analytic continuation along a path; on sufficiently close circles, the extensions must agree.

Note, however, that if γ intersects itself, then there's no requirement for the extensions to agree on those overlaps (if δ is sufficiently small, for example). Weierstrass said this is how one should think of complex analytic functions, and this confused a lot of people, but did lead to Weyl's work that we'll discuss in a few lectures.

Example 5.1. The logarithm is a very good example. Start with a branch of log defined on some open set U_0, so $\log(re^{i\theta}) = \log r + i\theta$, or $\log z = \log|z| + i \arg z$, for some continuous, real-valued $\arg : U_0 \to \mathbb{R}/2\pi\mathbb{Z}$.

Then, for any $\gamma : [0,1] \to \mathbb{C}^*$ with $\gamma(0) = z_0 \in U_0$, we can uniquly lift $\arg\circ\gamma : [0,1] \to \mathbb{R}/2\pi\mathbb{Z}$ to $\mathbb{R}$ consistent with $\arg(z_0)$; this lift will be called $\arg_\gamma$.[5] Then, define $\log_{\gamma_t}(z) = \log|z| + \arg_{\gamma(t)}(z)$, which defines a continuation of the logarithm around γ.

Example 5.2. For a more algebraic example, let

$$\phi(z) = \sum_{j\ge 0} \binom{1/2}{j} z^j$$

on the unit disc $D(0,1)$, so $\phi(z)^2 = z+1$. Then, one can continue along any γ with image in $\mathbb{C}\setminus\{-1\}$ by setting $\phi_t(z) = \exp((1/2)\log_{\gamma_t}(1+z))$. However, if $\gamma(t) = -1 + e^{2\pi i t}$, then γ winds around -1, and when it returns to a point, the extension of ϕ has a different value!

Example 5.3. Analytic continuation along paths works particularly well with differential equations: let p and q be meromorphic functions. Then, we want to find a $u(z)$ such that $u'' + p(z)u' + q(z)u = 0$, which we'll call $\boxed{p,q}$.[6] If you think differential equations are boring, questions like these are still motivated by study of $\mathcal{D}$-modules and the like in algebraic geometry.

Let's work near a point z_0 where p and q are holomorphic, so z_0 is a *regular point*, and without loss of generality make $z_0 = 0$. We're going to look for *series solutions*: set $p(z) = \sum_{n\ge0} p_n z^n$ and $q(z) = \sum q_n z^n$ on $D(0,R)$ for some R, and we want to find $u(z) = \sum_{n\ge0} u_n z^n$. Equating the coefficients of z_n in $\boxed{p,q}$, one obtains the recurrence relation

$$(n+1)(n+2)u_{n+1} + \sum_{i=0}^{n}(n+i-1)p_i u_{n+1-i} + \sum_{j=0}^{n} q_j u_{n-j} = 0.$$

By induction, one shows that all of the u_j are determined by a choice of $(u_0,u_1) \in \mathbb{C}^2$.

Proposition 5.4. *$\sum u_n z^n$ converges in the same disc $D(0,R)$.*

The detailed proof is a homework assignment, and depends on the following lemma, due to an idea of Cauchy.

Lemma 5.5 (Majorization)**.** *Say $|p_n| \le P_n$ and $|q_n| \le Q_n$. Then, let $P(z) = \sum P_n z^n$ and $Q(z) = \sum Q_n z^n$. If $u = \sum u_n z^n$ is a solution to $\boxed{p,q}$ and $U_n = \sum U_n z^n$ is a solution to $\boxed{P,Q}$, and if $U_0 = |u_0|$ and $U_1 = |u_1|$, then $|u_n| \le |U_n|$.*

The proof involves some straightforward estimates after the recurrence formula.

[5] One can think of this in terms of the theory of covering spaces, which is one reason this function lifts.

[6] "If you're typing notes, feel free to call it something else, like $L_{p.q}$."

Proof sketch of Proposition 5.4. Let's work on $\overline{D(0,r)}$ where $r < R$. Then, we have estimates like $|p_n| \le M/r^n$ and $|q_n| \le M/r^n$, where $M = \sup_{z \in \overline{D(0,r)}}\{|p(z)|, |q(z)|\}$, which follows from Cauchy's estimates (which themselves are corollaries of the Cauchy integral formula, Proposition 2.3).

Now, using the majorization lemma, we can compare $\boxed{p, q}$ to

$$\boxed{\sum_{n\ge 0}|p_n|z^n, \sum_{n \ge 0}|q_n|z^n} \qquad \text{and} \qquad \boxed{\sum \frac{M}{r^n}z^n, \sum \frac{M}{r^n}z^n}.$$

It makes sense to compare this to $\boxed{M/(1-z/r), M/(1-z/r)^2}$, i.e. the equation

$$u'' + \frac{Mu'}{1-z/r} + \frac{Mu}{(1-z/r)^2} = 0.$$

This last equation has an explicit solution $\mu/(1-z/r)$ for some μ, and its Taylor series converges on $D(0,r)$; now, using the majorization lemma, the coefficients of our original series are smaller, and therefore it converges. $\boxtimes$

Thus, we have a 2-dimensional $\mathbb{C}$-vector space V of solutions near z_0. The tie-in to the rest of lecture is the following proposition/exercise.

Exercise. Show that if $p, q \in \mathscr{O}(G)$ and $\gamma : [0,1] \to G$, then any solution to $\boxed{p, q}$ has a solution along γ through solutions to $\boxed{p, q}$.

Monodromy. If γ is now a loop in G, so $\gamma(0) = \gamma(1) = z_0$, then analytic continuation around γ defines a linear map $M_\gamma : V \to V$ called the *monodromy map*: you go around and end up not where you started, and it's easy to see that this dependence is linear.

Exercise. M_γ depends only on the homotopy class of γ (relative to basepoints).

Thus, this is only interesting if G isn't simply connected, so in general we get interesting examples of monodromy by going around poles of p and q. In particular, there's the oxymoronic-sounding notion of regular singular points. The prototype is the following, simpler equation:

$$u'' + \frac{A}{z}u' + \frac{B}{z^2}u = 0, \tag{5.1}$$

where $A, B \in \mathbb{C}$ are just constants. We seek solutions of the form $u(z) = z^\alpha$, where $\alpha \in \mathbb{C}$; this is defined initially near 1, and then analytically continued along paths in $\mathbb{C}^*$. If you write down the left-hand side, you end up getting

$$u'' + \frac{A}{z}u' + \frac{B}{z^2}u = \underbrace{(\alpha(\alpha-1) + A\alpha + B)}_{I(\alpha)} z^{\alpha-2}.$$

In other words, to get a solution, we need $I(\alpha) = 0$; this is called the *indicial equation*. Since it's a quadratic, then there's one or two roots: if the roots α_1 and α_2 are distinct, then $(z^{\alpha_1}, z^{\alpha_2})$ is a basis for V (the solutions near 1), and if γ is the unit circle, then the monodromy matrix in this basis is

$$M_\gamma = \begin{bmatrix} e^{2\pi i\alpha_1} & 0 \\ 0 & e^{2\pi i \alpha_2}\end{bmatrix}. \tag{5.2}$$

If α is a related root, the basis we get is $(z^\alpha, z^\alpha \log z)$, and the monodromy matrix is a nontrivial Jordan block:

$$M_\gamma = e^{2\pi i\alpha}\begin{bmatrix} 1 & 2\pi i \\ 0 & 1\end{bmatrix}.$$

One takeaway is that even an equation as simple as (5.1) has monodromy.

This generalizes quite naturally.

Definition. A $z_0 \in \mathbb{C}$ is a *regular singular point* of $u'' + pu' + qu = 0$ if p has a pole of order at most 1 and q has a pole of order at most 2 at z_0.

One seeks solutions via the *Frobenius method*: since p has a simple pole and q has a double pole, then there are $\widetilde{p}, \widetilde{q}$ holomorphic in a neighborhood of 0 such that $p(z) = A/z + \widetilde{p}(z)$ and $q(z) = B/z^2 + C/z + \widetilde{q}(z)$. Thus, the indicial equation is $\alpha(\alpha-1) + A\alpha + B = 0$.

Proposition 5.6. *If there are indicial roots α_1 and α_2 such that $\alpha_1 - \alpha_2 \notin \mathbb{Z}$, then there are solutions $u_1, u_2 \in V$ such that $u_1 = z^{\alpha_1} w_1$ and $u_2 = z^{\alpha_2} w_2$, and the monodromy matrix is as in (5.2).*

Lecture 6.

Riemann Surfaces and Holomorphic Maps: 2/3/16

Today, we'll begin with section 3.1 of the book, defining Riemann surfaces properly. This may be very routine to you or far from it; in any case, the notion of a manifold is central to mathematics, and now's as good a time as any to see it.

Definition. A *Riemann surface* (abbreviated R.S.) is the data of

- a Hausdorff topological space X, along with
- an *atlas*; that is, a collection $\{(U_\alpha, \phi_\alpha) \mid \alpha \in A\}$ where the $U_\alpha \subset X$ are open, $\bigcup_{\alpha \in A} U_\alpha = X$, and $\phi_\alpha : U_\alpha \to \mathbb{C}$ is a homeomorphism onto its image;[7] we require that for all $\alpha, \beta \in A$, the transition map $\tau_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \mathbb{C}$ is holomorphic.

We deem two atlases on X to define the same surface if their union is also an atlas satisfying the above conditions.

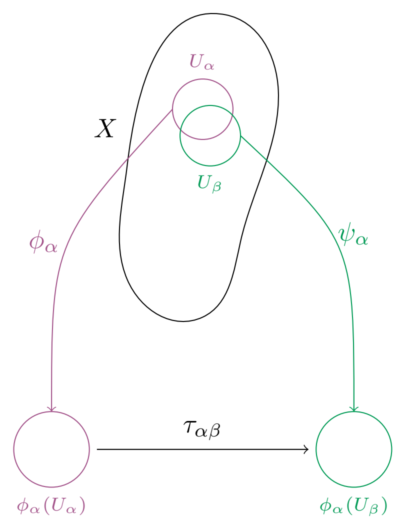


FIGURE 5. A transition map between charts for a Riemann surface X.

The rest of this week will be devoted to examples of Riemann surfaces, and unwinding this definition.

Remark.

- If $p \in U_\alpha$, we can think of ϕ_α as defining a holomorphic coordinate z neat p on X. The definition forces us to work only with notions that are independent of the particular coordinate we chose; for example, it does make sense to ask for a holomorphic function's order of vanishing at p.
- There are many variants of this definition of a Riemann surface given by replacing holomorphicity with something else. If one instead requires the maps to be smooth, the resulting definition is for a *smooth surface*; if we require smoothness with positive Jacobian, it's a *smooth oriented surface*; and many more.
- A $\mathbb{C}$-linear map $\mathbb{C} \to \mathbb{C}$ has non-negative Jacobian, because the map $\mathbb{C} \to \mathbb{C}$ sending $z \mapsto az$ acts on $\mathbb{R}^2$ by $\left[\begin{smallmatrix} \operatorname{Re} a & -\operatorname{Im} a \\ \operatorname{Im} a & \operatorname{Re} a \end{smallmatrix}\right]$, which has det $|a|^2$. In particular, a Riemann surface is also a smooth oriented surface. Thus, by the classification of compact, smooth, oriented surfaces, any connected, compact Riemann surface is equivalent to a standard genus-g surface.

[7] Each pair (U_α, ϕ_α) is called a *chart*.

- In the conventional definition of smooth surfaces, one generally assumes that a smooth surface has a countable atlas; equivalently, one may take the space to be paracompact or second-countable. There are tricky counterexamples if you don't include this (e.g. they do not admit partitions of unity, and hence Riemannian metrics). However, this isn't necessary in the world of Riemann surfaces.

Theorem 6.1 (Radó)**.** *Any connected Riemann surface has a countable holomorphic atlas.*

Thus, unlike in differential geometry, where we care only about nicer surfaces, here we get that our surfaces are nice already.[8]

Now, we want to know not just what these are, but also how to map between them.

Definition. Let $(X, \{(U_\alpha, \phi_\alpha)\})$ and $(Y, \{(V_\beta, \psi_\beta)\})$ be Riemann surfaces. Then, a *holomorphic map* $f : X \to Y$ is a continuous map such that for all charts ϕ_α and ψ_β, $\psi_\beta \circ f \circ \phi_\beta^{-1}$ is holomorphic. An invertible holomorphic map is called *biholomorphic*.

Courtesy of the implicit function theorem for holomorphic functions, Theorem 4.7, the inverse of a biholomorphic function is also holomorphic.

Example 6.2.

(1) Any open set $\Omega \subset \mathbb{C}$ is a Riemann surface with just one chart $\phi : \Omega \to \mathbb{C}$ given by inclusion (the only translation functions are the identity, which is holomorphic).

(2) The Riemann sphere $S^2 = \widehat{\mathbb{C}}$ is a Riemann surface with an atlas of two charts: the copy of $\mathbb{C}$ inside $\widehat{\mathbb{C}}$ is sent to $\mathbb{C}$ by the identity, and $\mathbb{C}^* \cup \{\infty\}$ is sent to $\mathbb{C}$ by $z \mapsto 1/z$; the transition map is $z \mapsto 1/z$ on $\mathbb{C}^*$, which is holomorphic. The Möbius maps $\mu : S^2 \to S^2$ given by $\mu(z) = (az + b)/(cz + d)$, where $ad - bc = 1$, are biholomorphic.

(3) $\mathbb{D}$ will denote the *unit disc* $D(0, 1)$, and $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$, the *upper half-plane*. The Möbius function $\mu : \mathbb{H} \to \mathbb{D}$ sending $\mu(z) = (z - i)/(z + i)$ is biholomorphic (since it's the restriction of a Möbius map on S^2), and it sends $0 \mapsto -1$, $\infty \mapsto 1$, and $1 \mapsto i$, so $\mu(\mathbb{R} \cup \infty)$ is the unit circle., Then, $\mu(i) = 0$, so $\mu(\mathbb{H}) = \mathbb{D}$ (it has to be either the inside or the outside of $\mathbb{D}$, since μ is continuous).

(4) Let $P(z, w)$ be holomorphic in both $z \in \mathbb{C}$ and $w \in \mathbb{C}$. Then, $X = \{(z, w) \in \mathbb{C}^2 \mid P(z, w) = 0\}$; if we know that there's no $(z, w) \in X$ where both partial derivatives of P vanish, then X is naturally a Riemann surface.

    We proved that if $\frac{\partial P}{\partial w}(w_0, w_0) \neq 0$, then there are discs $D_1 \subset \mathbb{C}$ and $D_2 \subset \mathbb{C}$ centered at z_0 and w_0, respectively, and a holomorphic $\phi : D_1 \to D_2$ with $\phi(z_0) = w_0$ and $U = X \cap (D_1 \times D_2) = \mathrm{graph}(\phi)$. We can use this to get a chart $\psi : U \to D_1$ given by projection onto the first factor. Alternatively, if $\frac{\partial P}{\partial w}$ vanishes at (z_0, w_0), then $\frac{\partial P}{\partial z}$ doesn't, so we can do the same thing, but with $\phi : D_2 \to D_1$. Then, our map is projection onto the second coordinate.

    Now, let's look at the change-of-charts maps. If both charts have the same type, the transition map is just the identity on some open set in $\mathbb{C}$, so let's look at what happens on a chart map between the two types, where X is locally the graph of an $f : D_1 \to D_2$. Then, ϕ^{-1} sends $z \mapsto (z, f(z))$ and ψ sends $(z, w) \mapsto w$, so the transition map is $z \mapsto f(z)$, which by construction was holomorphic; hence, X is a Riemann surface.

Lecture 7.

More Sources of Riemann Surfaces: 2/5/16

Today we'll talk about three more sources of Riemann surfaces, with more to come on Monday.

Covering Spaces. The first, easiest example is covering spaces. Recall that a continuous map $\pi : Y \to X$ is a *covering map* if X is the union of open sets U such that $\pi^{-1}(U) \to U$ is equivalent to $U \times D \to U$, where D has the discrete topology.

If X is a Riemann surface, then Y acquires a Riemann surface structure that makes π holomorphic. The idea is that the charts $X \to \mathbb{C}$ lift to several disjoint copies in Y. Each of these maps homeomorphically onto the chart, and then composing with the chart map gives a chart structure on Y. There's something to be

[8] Later in the class, we'll prove the uniformization theorem, which says that every connected Riemann surface is equivalent to a quotient of $\mathbb{C}$, the sphere, or the hyperbolic plane by a group action. This implies Radó's theorem, but is now how Radó originally proved it.

fleshed out here, but it's straightforward; in fact, the requirement that π is holomorphic pretty much forces one's hand.

Suppose X is path-connected, with a basepoint x_0. Then, we can construct a *universal cover* $\pi : \widetilde{X} \to X$, with $\widetilde{X}$ simply connected. We'll see this again, so it's useful to remember the construction: $\widetilde{X} = \{(x, \gamma) \mid x \in X, \gamma : [0,1] \to X, \gamma(0) = x_0, \gamma(x) = x\}$ modulo the equivalence relation $(x, \gamma) \sim (x', \gamma')$ if $x = x'$ and γ and γ' are homotopic. The topology on $\widetilde{X}$ is chosen to make it a covering map.

The fundamental group of X, $\pi_1(X, x_0)$, acts on X by *deck transformations*, maps $g : \widetilde{X} \to \widetilde{X}$ that commute with the projection to X; if X and Y are Riemann surfaces, the deck transformations are biholomorphic. Moreover, any connected and path-connected covering space of X takes the form $\widetilde{X}/G$, where $G \le \pi_1(X, x_0)$.

In summary, there's nothing new caused by making X and Y Riemann surfaces; the whole theory maps nicely into the category of Riemann surfaces and holomorphic maps.

The Riemann Surface of a Holomorphic Function. The idea is that we'll construct a "maximal analytic continuation" of a prescribed holomorphic function. The domain will be a Riemann surface, and not always an open set in the plane. It's the realization of Weierstrass' idea of considering all possible branches of a holomorphic function.

The input data will be an open $U \subset \mathbb{C}$, a $z_0 \in U$, and an $f \in \mathscr{O}(U)$. Then, an *abstract analytic continuation* (AAC) of f is $\mathscr{X} = (X, x_0, \pi, F)$, where X is a Riemann surface, $x_0 \in X$, $\pi : X \to \mathbb{C}$ sends $x_0 \mapsto z_0$ and is a *local homeomorphism* (meaning $\pi'(z)$ never vanishes). We require that if σ is a local right inverse to π near z_0 (so $\pi \circ \sigma = \mathrm{id}$), then we require that $F \circ \sigma = f$ in a neighborhood of z_0.[9]

There's a natural notion of a morphism between two abstract analytic continuations $\mathscr{X}$ and $\mathscr{X}'$ of (U, f) (respectively given by (X, x_0, π, F) and (X', x_0', π', F')): a holomorphic map $\phi : X \to X'$ respecting all the structure, i.e. it intertwines π and π', as well as F and F'. In particular, the AACs are a category C_f.

Definition. A *terminal object* in a category C is an $X \in \mathsf{C}$ such that any $X' \in \mathsf{C}$ maps to X in a unique way.

Proposition 7.1. *C_f has a terminal object $\mathscr{X}_f = (X_f, x_0, \pi, F)$.*

You should think of this as a sort of maximal object. More concretely, a path $\gamma : [0,1] \to \mathbb{C}$ with $\gamma(0) = x_0$ lifts to a $\widetilde{\gamma} : [0,1] \to X$ (meaning $\pi \circ \widetilde{\gamma} = \gamma$) iff f has an analytic continuation along the path γ.

$\mathscr{X}_f$ is called the *Riemann surface of the function* f. It can be given a more concrete construction: the set of paths $\gamma : [0,1] \to \mathbb{C}$ with $\gamma(0) = z_0$ and f admits an analytic continuation f_γ along γ, modulo an equivalence relation, where $\gamma \sim \gamma'$ if $\gamma(1) = \gamma'(1)$ and $f_\gamma(\gamma(1)) = f_{\gamma'}(\gamma'(1))$. That is, since there's at most one analytic continuation along any path (up to some equivalence about the size of the neighborhoods, which is irrelevant), we identify the "same" analytic continuations: in particular, homotopic paths are identified. Thus, X_f is a quotient of the universal cover of an open set in $\mathbb{C}$, so it is itself a cover: we have a covering map $\pi : X_f \to \mathbb{C}$ sending $[\gamma] \to \gamma(1)$. The basepoint $x_0 \in X_f$ is the class of the constant path at z_0, and the map $F : X_f \to \mathbb{C}$ sends $\gamma \mapsto f_\gamma(\gamma(1))$.[10]

This seems a little abstract, but working through it is probably helpful. As an example, though, suppose f is a branch of $\sqrt{z}$ on some open $U \subset \mathbb{C}$. We know that $X = \{w^2 - z = 0\} \subset \mathbb{C}^2$ is a Riemann surface (some partial derivative-checking should be done here); then, X_f will be the subset of X where $z \neq 0$. Then, $X_f \twoheadrightarrow \mathbb{C}^*$ by $(z, w) \mapsto z$, which is a double cover.

Historically, this is one of the important examples for constructing Riemann surfaces, though "terminal object in a category" isn't the language one would have heard/

Plane Projective Algebraic Curves. This is also an extremely important class of examples.

Recall that *complex projective space*, $\mathbb{C}P^n$, is the set of one-dimensional (complex) vector subspaces in $\mathbb{C}^{n+1}$. These are given by points in $\mathbb{C}^{n+1}$ modulo the action of $\mathbb{C}^*$ acting by scaling (which doesn't change the line through a point). That is, $\mathbb{C}P^n = (\mathbb{C}^{n+1} \setminus 0)/\mathbb{C}^*$, and carries the quotient topology, making it a topological space, and it's compact (since it can also be realized as the quotient topology on $S^{2n+1}/\mathrm{U}(1)$, by scaling each vector in $\mathbb{C}^{n+1} \setminus 0$ to a unit vector).

Points in $\mathbb{C}P^n$ are usually written in *homogeneous coordinates* $[z_0 : z_1 : \cdots : z_n]$, which represents the equivalence class (modulo scaling) of $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \setminus 0$.

---

[9] Up to making the neighborhood smaller, σ is unique anyways; thus, it's unique as the germ of a function.

[10] In fact, another way to define X_f is as a quotient of the universal cover, subject to some conditions, but then it's less apparent that it's unique.

We can write $\mathbb{C}P^n = U_0 \cup U_1 \cup \cdots \cup U_n$, where U_j is the set of classes of points where $z_j \neq 0$, so after rescaling, $[z_0 : \cdots : z_{j-1} : 1 : z_{j+1} : \cdots : z_n]$. Thus, looking at all the other coordinates, it's identified with $\mathbb{C}^n$, and this places a (complex) manifold structure on $\mathbb{C}P^n$.

One can pass back and forth between polynomials $p \in \mathbb{C}[z_1, z_2]$ and homogeneous polynomials $P(z_0, z_1, z_2)$ in three variables: if

$$p(z_1, z_2) = \sum_{i,j} a_{ij} z_1^i z_2^j,$$

and d is the largest degree $(i + j))$ in p, then it corresponds to

$$P(Z_0, Z_1, Z_2) = \sum_{i,j} a_{ij} Z_0^{d-i-j} Z_1^i Z_2^j,$$

which is homogeneous of degree d. For example $z_1^2 + z_2^3 + 1$ is homogenized to $Z_0 Z_1^2 - Z_2^3 + Z_0^3$. Thus, we can think of $X = \{(z_1, z_2) \mid p(z_1, z_2) = 0\} \subset \mathbb{C}^2$ as a subset in $U_0 \subset \mathbb{C}^2$; then, the closure of X in $\mathbb{C}P^2$ is the compact space $\overline{X} = \{(Z_0, Z_1, Z_2) \mid P(Z_0, Z_1, Z_2) = 0\}$. Next time, we'll prove the following proposition.

Proposition 7.2. *Suppose that for all $q \in \overline{X}$, $\frac{\partial P}{\partial Z_j}$ is nonvanishing for some j. Then, the Riemann structure on $X \subset U_0$ extends to a Riemann surface structure on $\overline{X} \subset \mathbb{C}P^2$.*

Lecture 8.

Projective Surfaces and Quotients: 2/8/16

Today, we'll discuss two fundamental and important examples of Riemann surfaces: plane projective curves and quotients.

Plane Projective Curves. We discussed this a little bit last time, but we have a correspondence between polynomials $p(z_1, z_2) \in \mathbb{C}[z_1, z_2]$ and homogeneous polynomials $P(Z_0, Z_1, Z_2) \in \mathbb{C}[Z_0, Z_1, Z_2]$: $z_1^3 - z_2 z_1 + 1$ is sent to $Z_1^3 - Z_0 Z_1 Z_2 + Z_0^3$. Then, if $X = V(p) = \{(z_1, z_2) \mid p(z_1, z_2) = 0\} \subset \mathbb{C}^2$, then since $\mathbb{C}^2 \hookrightarrow \mathbb{C}P^2$ by $(z, w) \mapsto [1 : z_1 : z_2]$, then $X \subset \overline{X}$, which is $\{[Z_0 : Z_1 : Z_2] \mid P(Z_0, Z_1, Z_2) = 0\}$ (where p and P are identified by the above correspondence). Since $\mathbb{C}P^2$ is compact and $\overline{X}$ is closed, then $\overline{X}$ is compact.

We'd like to make $\overline{X}$ into a Riemann surface with $X \hookrightarrow \overline{X}$ holomorphic (in other words, extending the Riemann surface structure on X), which is the content of Proposition 7.2.

Proof of Proposition 7.2. Take $q = [Z_0 : Z_1 : Z_2] \in \overline{X}$. One of the Z_j is nonzero, so by the cyclic symmetry of this problem, we can assume $Z_0 \neq 0$, and scale to set $Z_0 = 1$. *Euler's identity on homogeneous polynomials* tells us that if $P(Z_1, \dots, Z_n)$ is a homogeneous polynomial (or, more generally, a homogeneous function), then

$$\sum Z_j \frac{\partial P}{\partial Z_j} = \deg P \cdot P(q).$$

The proof is but two lines, coming down to the chain rule, but is left as an exercise.

When we apply this to our choice of q, the takeaway is that

$$-\frac{\partial P}{\partial Z_0} = Z_1 \frac{\partial P}{\partial Z_1} + Z_2 \frac{\partial P}{\partial Z_2}.$$

In particular, one of $\frac{\partial P}{\partial Z_1}$ or $\frac{\partial P}{\partial Z_2}$ does not vanish. Since $p(z_1, z_2) = P(1, z_1, z_2)$, then this defines a Riemann surface $X_0 \subset U_0 = \mathbb{C}^2$, as we showed last time. In the same way, we can define Riemann surfaces $X_1 \subset \overline{X}$, where $Z_1 \neq 0$ and $Z_2 \subset \overline{X}$, where $Z_2 \neq 0$, and we know that $\overline{X} = X_0 \cup X_1 \cup X_2$.

Finally, one needs to check that the transition functions between these three components are holomorphic, which has been left as an exercise. ⊠

In a sense, $\overline{X}$ is just X along with the "points at infinity," which are the points in $\overline{X}$ where $Z_0 = 0$. If you start out with $X = V(p)$, where

$$p(z_1, z_2) = \sum_{i,j} a_{ij} z_1^i z_2^j, \qquad \text{so that} \qquad P(Z_0, Z_1, Z_2) = \sum_{i,j=0}^{d} a_{ij} Z_0^{d-i-j} Z_1^i Z_2^j,$$

and we can explicitly see what these points at infinity are:

$$P(0, Z_1, Z_2) = \sum_{i+j=d} a_{ij} Z_0^i Z_1^j.$$

In some sense, we keep only the terms of maximal degree. For instance, if $p(z_1, z_2) = z_1^2 - z_2(z_2+1)(z_2-1)$, which is a classic example of an elliptic curve, then $P(0, Z_1, Z_2) = -Z_2^3$, so the only point at infinity is $[0:1:0]$ (since $Z_2^3 = 0$). These points correspond to asymptotic behavior of your original polynomial (since the highest-degree terms dominate), which makes the name of "points at infinity" make sense.

This is a very geometric construction, depending on how you embed your Riemann surface into $\mathbb{C}$; as such, it doesn't have a whole lot of categorical significance.

Quotients. In enough detail, one could really spend an entire semester on quotients of the upper half-plane; many interesting Riemann surfaces can be realized as quotients of other Riemann surfaces by groups of biholomorphic maps.[11] The general construction is kind of hairy, but the idea can be conveyed well through a few examples. First, though, recall the following facts from complex analysis.

(1) $\mathrm{Aut}(S^2) = \mathrm{PSL}_2(\mathbb{C})$, which is also the group of Möbius maps. This is ultimately because any holomorphic map $S^2 \to S^2$ is a rational function.
(2) $\mathrm{Aut}\,\mathbb{C}$ is the set of maps in $\mathrm{PSL}_2(\mathbb{C})$ that send $\infty \mapsto \infty$, and these are therefore the maps $z \mapsto az + b$ with $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$.[12]
(3) Then, $\mathrm{Aut}(\mathbb{H}) = \{\mu \in \mathrm{PSL}_2(\mathbb{C}) \mid \mu(\mathbb{H}) = \mathbb{H}\}$, which is also $\mathrm{PSL}_2(\mathbb{R})$. Thus, understanding Riemann surfaces often really boils down to understanding free subgroups of this group. This also subsumes $\mathrm{Aut}(\mathbb{D})$, which is the same, because $\mathbb{H} \cong \mathbb{D}$ under a suitable Möbius map, so $\mathrm{Aut}(\mathbb{D})$ and $\mathrm{Aut}(\mathbb{H})$ are conjugate in $\mathrm{PSL}_2(\mathbb{C})$. In fact, $\mathrm{Aut}(\mathbb{D}) = \mathrm{PSU}(1,1)$; here, $\mathrm{SU}(1,1)$ is the group of unitary matrices preserving a signature-$(1,1)$ quadratic form. That is, if $\langle \underline{z}, \underline{w} \rangle = z_1\overline{w}_1 - z_2\overline{w}_2$, for $z, w \in \mathbb{C}^2$, then $\mathrm{SU}(1,1) = \{A \in \mathrm{SL}_2(\mathbb{C}) \mid \langle A\underline{z}, A\underline{w} \rangle = \langle \underline{z}, \underline{w} \rangle\}$, or more explicitly,
$$\mathrm{SU}(1,1) = \left\{ \begin{bmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{bmatrix} \mid |\alpha|^2 - |\beta|^2 = 1 \right\}.$$
Then, $\mathrm{PSU}(1,1)$ is the image of this in $\mathrm{PGL}_2(\mathbb{C})$, i.e. $\mathrm{SU}(1,1)/\{\pm I\}$.[13]

Now, what quotients do we get? If $\mu \neq \mathrm{id}$ is in $\mathrm{Aut}(S^2)$, then it fixes exactly two points on S^2, so no nontrivial subgroup of $\mathrm{Aut}(S^2)$ acts freely.

So instead, let's look at $\mathrm{Aut}(\mathbb{C})$. For example, $\mathbb{Z} \hookrightarrow \mathrm{Aut}(\mathbb{C})$ by $n \cdot z = z + n\lambda$, for a fixed $\lambda \in \mathbb{C}^*$. That is, $\mathbb{Z}$ acts by translation, scaled by λ. If $\Gamma < \mathrm{Aut}(\mathbb{C})$ is this subgroup, then $X = \mathbb{C}/\Gamma$, meaning the orbit space, is a Riemann surface. This looks like a cylinder, as small subsets of $\mathbb{C}$ project homeomorphically onto X, so we can create a chart structure by passing such images up to $\mathbb{C}$ and taking charts for them.

For example, if $r = |\lambda|/3$ and $z \in \mathbb{C}$, let $D_z = D(z, r)$; then, the quotient π maps D_z homeomorphically onto its image in X (since any two points whose image in the quotient is the same are at least $|\lambda|$ apart from each other). Then, the charts are $\pi(D_z) \to D_z$, since $\pi(D_z) \subset \mathbb{C}$, and the change-of-charts maps are just translations by $h\lambda$, which are smooth.

More generally, suppose $\Lambda \subset \mathbb{C}$ is a *lattice*: $\Lambda = \mathbb{Z} \oplus \mathbb{Z} \cdot \lambda$, where $\mathrm{Im}\,\lambda > 0$, as in Figure 6. This again acts on $\mathbb{C}$ by translation, and the same construction gives the quotient $X = \mathbb{C}/\Lambda$. Once again, taking $r = 1/3 \cdot \min(1, \mathrm{Im}\,\lambda))$ and $D_z = D(z, r)$ as the images of charts gives X a Riemann surface structure. Topologically, X looks like a torus.

Lecture 9.

Fuchsian Groups: 2/10/16

"There's an elephant in the room, and it is hyperbolic geometry."

[11] This is kind of a lame statement, since they *all* arise as quotients of S^2, $\mathbb{C}$, or $\mathbb{H}$, courtesy of the uniformization theorem.

[12] You can prove this using the Casorati-Weierstrass theorem on essential singularities, or think of this as holomorphic rational functions.

[13] Usually, the story runs in reverse: using the Schwarz lemma, one discovers that all of the automorphisms of the disc are Möbius transformations, and then uses this to obtain $\mathrm{Aut}(\mathbb{H})$, $\mathrm{Aut}(S^2)$, and $\mathrm{Aut}(\mathbb{C})$.

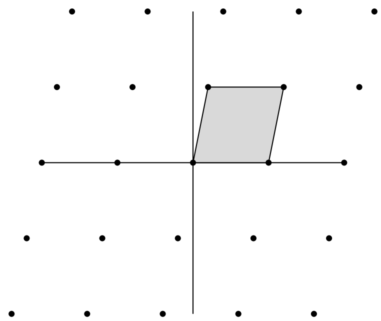


FIGURE 6. A lattice $\Lambda \subset \mathbb{C}$ and a fundamental domain for the quotient, which is a Riemann surface.

Today, we'll consider a specific example of quotient Riemann surfaces, quotients by the actions of Fuchsian groups. These are subgroups of $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{Aut}(\mathbb{H})$, or, equivalently, $\mathrm{PSU}(1,1) = \mathrm{Aut}(\mathbb{D})$, as we established an isomorphism between these two groups, and in fact a conjugacy inside $\mathrm{PSL}_2(\mathbb{C})$.[14]

These groups have a natural topology to them. First, $\mathrm{SL}_2(\mathbb{C}) \subset \mathbb{C}^4$ (since it's a group of 2×2 matrices), so it has the subspace topology. Thus, $\mathrm{PSL}_2(\mathbb{C})$ has the quotient topology, and as subspaces, $\mathrm{Aut}(\mathbb{D})$ and $\mathrm{Aut}(\mathbb{H})$ gain the subspace topology.

Definition. A *Fuchsian group*[15] Γ is a discrete subgroup of $\mathrm{Aut}(\mathbb{H})$.

The conjugacy of $\mathrm{Aut}(\mathbb{H})$ with $\mathrm{Aut}(\mathbb{D})$ means that this is equivalent to defining the conjugate subgroup $\Gamma' \leq \mathrm{Aut}(\mathbb{D})$.

We'd like to study quotients $\mathbb{H}/\Gamma$, or $\mathbb{D}/\Gamma'$; these turn out to all be nice Riemann surfaces. In general, we should use the hyperbolic structure on $\mathbb{H}$ or on $\mathbb{D}$ when talking about these quotients (remarkably, the biholomorphic functions exactly correspond with the isometries with respect to these hyperbolic structures). However, since we haven't done any differential geometry in this class, we'll adopt this perhaps more pedestrian, but more understandable approach.

To understand a $\gamma \in \mathrm{Aut}(\mathbb{H})$, we can think of it as a map $S^2 \to S^2$, and think about its fixed points. We'd like none of them to be in $\mathbb{H}$, so that the action is free and its quotient is a Riemann surface. γ is a fractional linear transformation $\gamma(z) = (az+b)/(cz+d)$, where $ad - bc = 1$.

- First, it's easy to check that $\gamma(\infty) = \infty$ iff $c = 0$.
- If $z \in \mathbb{C}$ is fixed, then $z = (az+b)/(cz+d)$, so $cz^2 + (d-a)z - b = 0$. If $c \neq 0$ and $\gamma \neq \mathrm{id}$, then after some case-checking, one sees that there's at most one fixed point in $\mathbb{C}$, which is actually in $\mathbb{R}$.
- The more interesting case is where $c \neq 0$, so the discriminant is $\Delta = (d-a)^2 + 4bc = (\operatorname{tr} A)^2 - 4$, where $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. Thus, there are three cases:
  1. γ is an *elliptic element* if $\Delta < 0$, i.e. $\mathrm{tr}^2(A) < 4$. In this case, $c \neq 0$, and there are two fixed points, one in $\mathbb{H}$ and its conjugate in $\overline{\mathbb{H}}$ (that is, the lower half-plane). By conjugating into $\gamma' \,\mathrm{Aut}(\mathbb{D})$, there's a unique fixed point in $\mathbb{D}$, and after a conjugation this is 0. But this means γ' is a rotation of the disc (e.g. by the Schwarz lemma); there's not quite such a simple description of $\gamma \in \mathrm{Aut}(\mathbb{H})$, but the point is that elliptic elements are conjugates of rotations. In particular, they may have finite or infinite order.
  2. γ is a *parabolic element* if $\Delta = 0$, i.e. $\mathrm{tr}^2(A) = 4$. If $c \neq 0$, then γ has one fixed point which is in $\mathbb{R}$. If $c = 0$, then $A = \left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right]$, so ∞ is fixed. Thus, in either case, there's one fixed point, and it's in $\mathbb{R} \cup \{\infty\}$, and γ is conjugate in $\mathrm{PSL}_2(\mathbb{R})$ to a μ fixing infinity and sending 0 to ± 1 (i.e. $\mu(z) = z \pm 1$). In particular, parabolic elements have infinite order.
  3. γ is a *hyperbolic element* if $\Delta > 0$, so $\mathrm{tr}^2(A) > 4$. In this case, either $c \neq 0$, so there are two distinct fixed points in $\mathbb{R}$, or $c = 0$ and $a \neq d$, so there's one fixed point in $\mathbb{R}$, and ∞ is also fixed. Thus, in either case, there are two distinct fixed points in $\mathbb{R} \cup \{\infty\}$; such a γ is conjugate in $\mathrm{PSL}_2(\mathbb{R})$ to a μ fixing both 0 and ∞. Thus, $\mu(z) = \lambda z$, where $\lambda > 0$ and $\lambda \neq 1$. Thus, γ has infinite order.

[14] $\mathrm{SL}_2(\mathbb{C})$ is a four(-complex)-dimensional, complex Lie group, and $\mathrm{Aut}(\mathbb{D})$ and $\mathrm{Aut}(\mathbb{H})$ are 3-dimensional, noncompact, real Lie groups.

[15] These were named by Poincaré, not Fuchs, though Fuchs did study them.

This is somewhat elementary, but a complete description, and we can use it to talk about Fuchsian groups.

Example 9.1. Let p be a prime number, and let $\widetilde{\Gamma}_p = \{A \in \mathrm{SL}_2(\mathbb{Z}) \mid A \equiv \pm i \bmod p\} \subset \mathrm{SL}_2(\mathbb{R})$. Then, let $\Gamma_p = \widetilde{\Gamma}_p / \pm I$, which is a subgroup of $\mathrm{PSL}_2(\mathbb{R})$. An element $\gamma \in \widetilde{\Gamma}_p$ has the form

$$\gamma = \pm \begin{bmatrix} ap+1 & * \\ * & bp+1 \end{bmatrix},$$

where $a, b \in \mathbb{Z}$ and we don't know what the off-diagonal entries are. Then, $\operatorname{tr}\gamma = \pm((a+b)p+2)$, which is generally not in $(-2, 2)$. In fact, if $p \geq 5$, then $|\operatorname{tr}\gamma| \geq 2$.[16] Thus, all elements of Γ_p are elliptic or hyperbolic, and therefore have no fixed points in $\mathbb{H}$. Hence, Γ_p acts freely on $\mathbb{H}$.

Proposition 9.2. *Let $\Gamma \subset \mathrm{Aut}(\mathbb{H})$ be a Fuchsian group.*

(1) *For all $z \in \mathbb{H}$, there's a neighborhood $N \subset \mathbb{H}$ of z such that if $q_1, q_2 \in N$ and $\gamma \in \Gamma$ satisfy $\gamma(q_1) = q_2$, then $\gamma(z) = z$ (i.e. $\gamma \in \mathrm{stab}_\Gamma(z)$).*
(2) *For all $q_1, q_2 \in \mathbb{H}$ such that $q_2 \notin \Gamma \cdot q_1$, there exist neighborhoods N_1 of q_1 and N_2 of q_2 such that $N_2 \cap \Gamma \cdot N_1 = 0$.*

Note. Part (2) says that the quotient is Hausdorff; if further Γ acts freely, then the condition from last lecture holds, and implies that the quotient is a Riemann surface. So we're always at least Hausdorff, and often a Riemann surface.

Proof of Proposition 9.2, part (1). A really satisfying proof of this proposition would employ hyperbolic geometry, but we can give a hands-on proof of its first part.

We can work in $\mathbb{D}$ and without loss of generality assume $z = 0$ (since we can always conjugate by an element moving $z \mapsto 0$). Now, let $D = D(0, \varepsilon)$ (the disc of radius ε) and suppose $q, \gamma(q) \in D$ for some $\gamma \in \Gamma$.

$\gamma(z) = \alpha z + \beta/(\overline{\beta} z + \overline{\alpha})$, where $|\alpha|^2 - |\beta|^2 = 1$, so since $|q| < \varepsilon$ and $|\gamma(q)| < \varepsilon$, then

$$|\alpha q + \beta| \leq \varepsilon|\overline{\beta} q + \overline{\alpha}| \leq \varepsilon(|\beta|\varepsilon + |\alpha|).$$

We can use this to bound $|\beta|$, again by the triangle inequality: $|\beta| \leq \varepsilon(|\beta|\varepsilon + |\alpha|) + |\alpha|\varepsilon$, or $|\beta| \leq 2\varepsilon|\alpha|/(1-\varepsilon^2)$. But since $2\varepsilon/(1-\varepsilon^2) \to 0$ as $\varepsilon \to 0$, meaning we can make $|\beta|$ arbiitrarily small relative to $|\alpha|$.

The hypothesis we haven't used yet is that Γ is discrete, so suppose there is a sequence $\gamma_n \in \Gamma$ such that $|\beta_n|/|\alpha_n| \to 0$ as $n \to \infty$. Since $|\alpha_n|^2 - |\beta_n|^2 = 1$, then $|\alpha_n| \to 1$ and $|\beta_n| \to 0$. But since Γ is discrete, then eventually $|\alpha_n| = 1$ and $\beta_n = 0$. In other words, there's a $k \ll 1$ such that $|\beta| \leq\ k|\alpha|$: so $\beta = 0$ (e.g. take ε such that $2\varepsilon/(1-\varepsilon^) < k$), and thus $\gamma(z) = (\alpha/\overline{\alpha})z$, so it's a rotation about 0, and therefore fixes 0. ⊠

The proof of the second part can be found in the textbook.

The next surprising thing is that for *any* Fuchsian group, acting freely or not, $\mathbb{H}/\Gamma$ has the structure of a Riemann surface making the projection holomorphic. By part (2) of Proposition 9.2, we know it's Hausdorff, and we know how to make charts where the action is free, so we have to address the case where $\mathrm{stab}_\Gamma(z) \neq 1$.

The model case will be where $\Gamma_n = \mathbb{Z}/n$, which acts on $D(0; r)$ by $k \cdot z = e^{2\pi k/n} z$. Hence, $D(0; r)/\Gamma_n \cong D(0; r^n)$, by sending $[z] \mapsto z^n$ (this defines a well-defined homomorphism). This will be compatible with charts near the fixed point 0, so this quotient is a Riemann surface.

In general, if Γ is any Fuchsian group, then $\mathrm{stab}_\Gamma(0) = \Gamma_n$: it has to be a finite group of rotations (since it's a discrete subgroup of $\mathrm{U}(1)$), and for a general $z \in \mathbb{D}$, one can move it to 0 by conjugating to get a chart for it.

Lecture 10.

Properties of Holomorphic Maps: 2/15/16

First: there's no lecture Wednesday, and there may be lecture Friday. Also, read §5.1 of the textbook; it reviews calculus on manifolds: tangent vectors, cotangent vectors, and two-forms. We'll bootstrap it into calculus on Riemann surfaces later in this class.

Today, though, we're going to talk about holomorphic maps. We've seen a lot of ways in which Riemann surfaces arise (and in fact constructed all of them, thanks to the uniformization theorem). Today, the main focus will be on the structure of proper holomorphic maps.

Some properties from complex analysis generalize straightforwardly.

[16] In fact, there are no elliptic elements if $p = 2$ or $p = 3$, and this requires a small but different argument.

Lemma 10.1. *Let $f : X \to Y$ be a holomorphic map between Riemann surfaces, and $x \in X$. Then, the following are equivalent.*

(1) *f maps a neighborhood U of x homeomorphically to its image $V = f(U)$, and the inverse $f^{-1} : V \to U$ is holomorphic.*
(2) *In local coordinates near x and $f(x)$, $f'(x) \neq 0$.*[17]

(1) $\implies$ (2) by the chain rule: $f^{-1} \circ f = \mathrm{id}$, and then use the chain rule to show that $f'(z)$ is also invertible, so nonzero. (2) $\implies$ (1) relates to the inverse function theorem, and is proven using the argument principle in the same way as Theorem 4.7.

We have another lemma about the local behavior of holomorphic maps.

Lemma 10.2. *Again let $f : X \to Y$ be a holomorphic map and $x \in X$. If ψ is a holomorphic chart near $f(x)$, then there's a holomorphic chart ϕ near x such that $\widetilde{f} = \psi \circ f \circ \phi^{-1}$ takes the form $\widetilde{f}(z) = z^k$, for some integer $k \geq 0$ independent of the chart.*

So holomorphic maps look very simple, at least locally. This is a much stronger constraint than for smooth maps on smooth surfaces; Lemma 10.1 has an analogue for real manifolds, but Lemma 10.2 doesn't.

Proof of Lemma 10.2. If $f'(x) \neq 0$, this reduces to Lemma 10.1: the homeomorphism defines charts for which $\widetilde{f}(z) = z$.

Otherwise, fix coordinates ψ near $f(x)$, and fix an initial choice of coordinates around x; by translation, we assume $x = 0$. In these charts,

$$\widetilde{f}(z) = \sum_{n \geq k} a_n z^n = a_k z^k \underbrace{\sum_{m \geq 0} \left(\frac{a_{m+k}}{a_k} \right) z^m}_{g(z)},$$

where $k > 1$ and $a_k \neq 0$, so $\widetilde{f}(0) = \widetilde{f}'(0) = 0$. Thus, $g(z)$ is holomorphic and $g(0) = 1$. Thus, we can define $h(z)$ to be a k^{th} root of $g(z)$, which is continuous and satisfies $h(0) = 1$, so if $a_k^{1/k}$ is any k^{th} root of a_k, then let $\phi(z) = a_k^{1/k} z h(z)$, so $\phi(z)^k = f(z)$, $\phi(0) = 0$, and $\phi'(0) = a_k^{1/k} \neq 0$, so by Lemma 10.2, ϕ is the desired coordinate chart.

We need to show this is independent of k, but this follows because $k = \min\{\ell \geq 1 \mid f^{(\ell)}(x) \neq 0\}$; this is invariant, so we're good. ⊠

This lemma is really part of complex analysis, but generalizes quite readily to Riemann surfaces.

Definition. If $x \in X$ is such that this $k \neq 1$, then x is called a *critical point* of f. The set of critical points is called $\mathrm{crit}(f)$.

These are exactly the same as the places where f' vanishes, and as the critical points of f regarded as a map between smooth manifolds.

Definition. A *critical value* of Y is a point in the image of $\mathrm{crit}(f)$. A *regular value* of f is a $y \in Y$ that's not a critical value. The set of regular values is denoted Y_0.

In differential topology, there's also the useful notion of the degree of a map; we'll find it useful and actually be able to reprove it in the holomorphic setting.

Definition. A continuous map $f : X \to Y$ of topological spaces is *proper* if whenever $K \subset Y$ is compact, $f^{-1}(K)$ is compact.

Fact. Let S and T be smooth, oriented surfaces, $f : S \to T$ be a proper smooth map, and $y \in Y$ be a regular value of f. For any $x \in f^{-1}(y)$, let

$$\varepsilon_x = \begin{cases} 1, & \text{if } f \text{ preserves orientation near } x, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Then, we define $\deg_y(f) = \sum_{x \in f^{-1}(y)} \varepsilon_x$. The cool fact is that this is independent of y, and is denoted $\deg(f)$.

[17] There is a way to state this in a way that's independent of local coordinates, using the tangent bundle, and we'll get there in a few weeks.

Returning to the world of Riemann surfaces, if $f : X \to Y$ is a proper, holomorphic map of Riemann surfaces. Since $\mathrm{crit}(f)$ is the zero set of the holomorphic f', then it's discrete (assuming f is nonzero). Thus, $\Delta = f(\mathrm{crit}\, f)$, the critical values, is also discrete.[18] Thus, if $y \in Y \setminus \Delta$ is a critical value, then for all $x \in f^{-1}(y)$, $f'(x) \neq 0$, so f is a local homeomorphism near x preserving orientation, so $\deg f = \sum_{x\in f^{-1}(y)} = |f^{-1}(y)|$: it just counts points in the preimage!

We can also understand this as follows: there's a fact from topology that any proper local homeomorphism is a covering map, so if $Y_0 = Y \setminus \Delta$, $X_0 = f^{-1}(Y_0)$, and $f_0 = f|_{X_0}$, then f_0 is a proper local homeomorphism, so a covering map with $\deg(f)$ sheets. Now, if $y \in Y$ is arbitrary (perhaps not regular), then $\deg(f) = \sum_{x\in f^{-1}(y)} k_x$, where k_x is the value for x coming from Lemma 10.2. At this point, the proof of this is very simple: since f locally looks like $z \mapsto z^{k_x}$ near x, then its degree there is just k_x, which is the sum of the preimages, or k_x^{th} roots, of a $y \neq 0$. Again, notice how much cleaner this is than for smooth functions.

As a consequence, we have the following theorem.

Theorem 10.3. *Let X be a compact, connected Riemann surface and $f : X \to S^2$ be a holomorphic function with just one pole*[19] *which is simple, then f is biholomorphic.*

Proof. The hypotheses imply that $\deg f = \deg f_\infty = 1$, so for any $y \in Y$, a positively weighted counts in $f^{-1}(y)$, so $f^{-1}(y)$ is always a single point. Thus, f is bijective, so the result follows from Lemma 10.1. ⊠

Remark. One can state this for the preimage of any $y \in S^2$, and even replace S^2 with any other compact Riemann surface, but S^2 is the place in which this is the most useful, because it corresponds to meromorphic functions on Riemann surfaces.

With f as before, $f_0 : X_0 \to Y_0 = Y \setminus \Delta$ is a covering map with $d = \deg f$ sheets. Then, lifting of paths is a group homomorphism called *monodromy*, $\mathrm{Mon} : \pi_1(Y_0, y_0) \to \mathrm{Aut}\, F$, where $F = f^{-1}(y_0)$; if X is connected, then this action is transitive on F. Hence, the data from a proper holomorphic map $f : X \to Y$ consists of a discrete set of critical values and a transitive homomorphism on the permutations of the fiber, which is a pretty nice topological thing to extract. Next time, we'll prove Riemann's existence theorem, which provides a converse to this.

[18] A theorem from general topology shows that the image of a discrete set under a proper map must also be discrete. This is thus considerably stronger than Sard's theorem.

[19] Recall that a *pole* of a function into S^2 is a point in the preimage of ∞.