THE COBORDISM HYPOTHESIS

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These notes were taken in an online learning seminar on the cobordism hypothesis in Fall 2020, organized by Araminta Amabel, Peter Haine, and Lucy Yang. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

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1. INTRODUCTION: 9/9/20

Today's talk was given by Araminta Amabel, and was an introduction/overview to the cobordism hypothesis: what is it, and why should you believe it? For today, we assume all manifolds are smooth, compact, and oriented.

1.1. **Modeling field theories.** The cobordism hypothesis is a statement about field theories. So we should begin by discussing how to model field theories mathematically. There are several ways to do this, but most of them take these key features into account:

Space: Where are we? Where does the experiment take place?
Time: How long does the experiment run for?

In relativity, these are unified into a single concept called *spacetime*. For example, if the theory takes place on a manifold X representing space, and over the time interval $[0,1]$, then spacetime is $X \times [0,1]$, though one can (and we will) consider example spacetimes which aren't products.

Fields: We won't describe the general idea of fields here, but these provide information in your theory and are associated to open subsets U inside spacetime. For example, there's a field theory called the *particle-in-a-box*. In this theory, space is X and time is $[0,1]$, and the fields on an open $U \subset X \times [0,1]$ are the maps from U into the "box," thought of as paths the particle can take.

Rules: Differential equations governing what paths are allowed. For example, in a theory called the *free massless theory*, paths must be straight lines. Often these are wrapped up into something called the *equations of motion* of the theory, such as the *Euler-Lagrange equations*.

Observables: These are the measurements you can make, such as the length of a math. In the Euler-Lagrange formalism, the observables on an open subset U, these are maps from the space of solutions to the Euler-Lagrange equations to $\mathbb{R}$.[1]

Correlation functions: These are statistical measurements that, in experimental physics, are what we actually want to compare to real-world experiments. Out of all of these, we will be most interested in something called the *partition function*.

We will work with a specific model of field theory, which is Atiyah's definition — but only of *topological* field theories. We will say what all of the above notions mean, mathematically, in Atiyah's model of TFT, but first we need some definitions.

Definition 1.1. Let $n \in \mathbb{N}$, and let $\mathcal{C}ob(n)$ denote the symmetric monoidal category given by the following data.

Objects: Closed, oriented, $(n-1)$-manifolds.

[1] This is for the classical theory; in quantum field theories this is not always true.

Morphisms: A morphism $M_1 \to M_2$ is a bordism X from M_1 to M_2, i.e. a compact, oriented n-manifold X and an equivalence class of identifications $\partial X \cong M_1 \amalg -M_2$, modulo diffeomorphisms of X that preserve the boundary. Here $-M_2$ denotes M_2 with the opposite orientation.

Composition: To compose, glue bordisms. To set this up precisely, one needs to specify collar neighborhoods of M_1 and M_2 within X, but there is a way to make this work.

Symmetric monoidal structure: The tensor product is disjoint union, and the unit is the empty set, which is a manifold of every nonnegative-integer dimension. One should specify the associator, etc., but we're not going to delve into these details right now.

Atiyah came up with this definition, building on Segal's definition of a conformal field theory.

Let $\mathcal{V}ect_k$ denote the category of vector spaces over a field k, with the symmetric monoidal structure given by tensor product.

Definition 1.2. A *topological field theory* (TFT), sometimes also *topological quantum field theory* (TQFT), is a symmetric monoidal functor $Z\colon \mathcal{C}ob(n) \to \mathcal{V}ect_k$.

So, for example, the empty set maps to k, and gluing bordisms maps to composition of linear maps.

Now let's revisit the key concepts in field theory.

Space: All objects (i.e. closed $(n-1)$-manifolds) are thought of as spaces. That is, we study this theory for all possible spaces at once!

Time: $[0,1]$.

Spacetime: All compact n-manifolds, possibly with boundary, are thought of as spacetimes. We're working with this theory for all spacetimes at the same time, which is a bit of a perspective shift from what we did before.

Observables: If the TFT is denoted Z, observables are the vector space $Z(S^{n-1})$.

We'll return to fields and equations of motion later.

The identity morphism in $\mathcal{C}ob(n)$ is the cylinder $M \times [0,1]$ (with the correct gluing data), and as Z is a functor, $Z(M \times [0,1]) = \mathrm{id}_M$. But we can do more with these bordisms: regard both M and $-M$ as incoming and $\varnothing$ as outgoing, which results in something macaroni-looking. When you hit this with Z, you get a map

$$e\colon Z(M) \otimes Z(-M) \longrightarrow k. \tag{1.3}$$

Conversely, regarding both M and $-M$ as outgoing, we get a map

$$c\colon k \longrightarrow Z(M) \otimes Z(-M). \tag{1.4}$$

Lemma 1.5 (Zorro's lemma)**.** *e is a perfect pairing.*

This is a fun exercise to do, playing with bordisms and c and e.

1.2. Classifying TFTs.

A mathematician encounters a concept, and wants to classify the possible examples. This is hard and scary in general, as far as we know right now, so let's start with a pretty simple case.

Example 1.6 ($n = 1$)**.** Objects of $\mathcal{C}ob(1)$ are finite oriented sets, i.e. finite sets with each element labeled with $+$ or $-$. Symmetric monoidality implies that if Z is a one-dimensional TFT, the value of Z on objects is determined by its values on pt_+ and π_-.

Let $V := Z(\mathrm{pt}_+)$. Then, $Z(\mathrm{pt}_-) = V^\vee$, which is ultimately because of Lemma 1.5. $Z(\mathrm{pt}_+ \amalg \pi_-) = V \otimes V^\vee = \mathrm{End}(V)$, and in general a disjoint union of n copies of pt_+ and m copies of pt_- is sent to $V^{\otimes n} \otimes (V^\vee)^{\otimes m}$.

Now what about morphisms? We know the cylinders (well, line segments) go to identity maps. The macaroni bordism $\mathrm{pt}_+ \amalg \mathrm{pt}_- \to \varnothing$ is mapped to $e\colon V \otimes V^\vee \to k$, which can be identified with the evaluation map that takes a covector ℓ and a vector v and returns $\ell(v)$. Under the identification $V \otimes V^\vee \to \mathrm{End}(V)$, this map is taken to the trace map $\mathrm{End}(V) \to k$. The opposite-direction macaroni is sent to the adjoint of this map.

All bordisms in this dimension are made of disjoint unions of these two macaronis and also circles. To determine $Z(S^1)\colon k \to k$, we factor the bordism $S^1\colon \varnothing \to \varnothing$ into two macaronis. This computes $\mathrm{tr}(\mathrm{id}_V) = \dim V$. In particular, V must be finite-dimensional; all such V determine TFTs, and V determines the TFT completely. ◀

Remark 1.7. The observables of the 1d TFT sending $\mathrm{pt}_+ \mapsto V$ are $Z(S^0) = \mathrm{End}(V)$. This is an associative algebra, and that's not a coincidence — often, the space of observables is an algebra of some sort. As homotopy theorists, we'll be interested in working with ∞-categories eventually, and the algebraic structures we'll get on observables will be quite interesting. ◀

Example 1.8 ($n = 2$). Let $Z\colon \mathscr{C}ob(2) \to \mathscr{V}ect_k$ be a TFT. Objects are closed 1-manifolds, which are all isomorphic to finite disjoint unions of S^1. Morphisms are compact, oriented, 2-manifolds with boundary. When you draw a complicated one, you can factor it as a composition and/or disjoint union of simpler bordisms, including discs with S^1 viewed as incoming or outgoing, and pairs of pants regarded as incoming or outgoing. (And cylinders, but those are identity morphisms, so not as difficult.)

The disc with S^1 incoming is often called a *cap*, and with S^1 incoming is often called a *cup*.

As S^1 has an orientation-reversing diffeomorphism, we do not need to keep track of the difference between S^1 and $-S^1$. The pair-of-pants therefore defines a multiplication-like structure on $Z(S^1)$, as a map $Z(S^1) \otimes Z(S^1) \to Z(S^1)$.

In fact, one can show this extends to a commutative algebra structure on $A \coloneqq Z(S^1)$: associativity and commutativity come from finding equivalent bordisms representing, e.g., $m(x_1, x_2)$ and $m(x_2, x_1)$. Moreover, the pair-of-pants composed with the cap is the macaroni bordism for S^1, and we already know it's a perfect pairing. So we get a *counit* map $\mathrm{tr}\colon A \to k$. Moreover, we have a unit $e\colon k \to A$, which sends $1 \mapsto 1_A$, the unit element in A.

Let's give this structure a name.

Definition 1.9. A *commutative Frobenius algebra* is a finite-dimensional commutative k-algebra A with a linear map $\mathrm{tr}\colon A \to k$ such that $a, b \mapsto \mathrm{tr}(ab)$ is a perfect pairing.

Theorem 1.10. *The map sending $Z \mapsto Z(S^1)$ is an equivalence of categories between $\mathscr{C}ob(2)$ and the category of commutative Frobenius algebras.*

This was a folklore theorem for a bit; one reference is Robbert Dijkgraaf's thesis; another is Joachim Kock's book on Frobenius algebras and TFTs.

The observables are, once again, $Z(S^1) = A$, which has an algebra structure. It's worth thinking about what Z assigns to the pants with i legs. ◀

In higher dimensions, there's way too many things to work with: $\mathscr{C}ob(3)$ has infinitely many isomorphism classes of connected objects! So in a sense it's not finitely generated. It would be nice if there were a way to simplify this, by using the fact that all closed, connected, oriented 2-manifolds are diffeomorphic to connect-sums of T^2, and to consider TFTs that "understand" this somehow. And maybe the decompositions we did of surfaces in terms of pants, cups, and caps could apply in this case. But $\mathscr{C}ob(3)$ as we defined it doesn't know how to cut in lower dimensions — it doesn't even know S^1 exists.

In general, we want to be able to cut up our manifold into simpler manifolds in a way that includes all dimensions down to 0. Why do we want this? One compelling reason is that otherwise this classification question is pretty much unapproachable, and the TFTs we get are still interesting.

The solution: higher categories! There is a higher-categorical version of $\mathscr{C}ob(n)$ which takes this desideratum into account. But: defining higher categories is hard. Defining a higher-categorical version $\mathscr{C}ob_n(n)$ of $\mathscr{C}ob(n)$, even given a nice formalism of higher categories, is still hard. We'll spend the next few lectures building these tools that we need to consider this kind of TFT. Once we do, though, we can make the following definition.

Definition 1.11. An *extended TFT* of dimension n valued in a symmetric monoidal n-category $\mathscr{C}$ is a symmetric monoidal functor between n-categories $Z\colon \mathscr{C}ob_n(n) \to \mathscr{C}$.

With this definition in hand (…eventually), we might expect that there's an equivalence of (higher) categories of extended TFTs and $\mathscr{C}$. This is wrong in two different ways: first, we need to restrict to small enough objects in $\mathscr{C}$, called "fully dualizable" ones, akin to using only finite-dimensional vector spaces in Example 1.6. Second, in dimension $n = 1$, framed is the same thing as oriented, and we miss something important: in general asking for a descent to oriented bordisms is extra data. But when we take these into account, we get:

Theorem 1.12 (Baez-Dolan cobordism hypothesis (Hopkins-Lurie, Lurie)). *There is an equivalence of n-categories $\mathscr{F}un^{\otimes}(\mathscr{C}ob_n^{fr}(n), \mathscr{C}) \xrightarrow{\simeq} C^{fd}$).*

Though Lurie provided a detailed sketch of a proof, there are more complete proofs available in special cases, e.g. in Schommer-Pries' thesis for $n \le 2$, and a nearly complete, very different approach by Ayala-Francis.

2. n-fold complete Segal spaces: 9/16/20

Today, Adrian Clough spoke.

2.1. **Complete Segal spaces.** Once we've understood complete Segal spaces, which will occupy us for the bulk of the talk, it won't be too terrible to generalize to n-fold complete Segal spaces. Complete Segal spaces are a lift of the definition of a category from set theory to homotopy theory.

Recall that the *nerve* functor $N\colon \mathscr{C}at \to s\mathscr{S}et$ sends a category $\mathscr{C}$ to the simplicial set $N_\bullet\mathscr{C}$ whose set of k-simplices is the set of strings of diagrams in $\mathscr{C}$ $A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_k} A_k$, i.e. there are k morphisms.

If you've taken the nerve of $\mathscr{C}$, you can recover the Hom-set $\mathscr{C}(c,c')$ fairly simply: it fits into a pullback diagram

$$\begin{tikzcd} \mathscr{C}(c,c') \arrow[r] \arrow[d] & C_1 \arrow[d] \\ \{(c,c')\} \arrow[r] & C_0. \end{tikzcd} \tag{2.1}$$

Here the right-hand vertical map sends a map $f : c \to c'$ to (c,c').

Generalizing, one can ask, given an arbitrary simplicial set $S_\bullet$, whether it satisfies the *Segal condition* that for all $m,n \in \mathbb{N}$,[2]

$$\begin{tikzcd} X_{m+n} \arrow[r] \arrow[d] & X_m \arrow[d] \\ X_m \arrow[r] & X_0 \end{tikzcd} \tag{2.2}$$

is a pullback diagram. Here the arrow on top sends $x_1 \to \cdots \to x_{m+n}$ to $x_m \to \cdots \to x_{m+n}$; the arrow on the left sends it to $x_0 \to \cdots \to x_m$, and the arrows to the lower right send these to x_m.

Theorem 2.3. *The essential image of the nerve functor is precisely those simplicial sets which satisfy the Segal condition.*

We will generalize this to homotopy theory, where categories will be generalized to ∞-*categories*. We can use Theorem 2.3 to characterize categories as certain simplicial sets, and we will do the same thing for ∞-categories.

Definition 2.4. A commutative square of spaces

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d, "q"] \\ Y \arrow[r, "p"] & Z \end{tikzcd} \tag{2.5}$$

is a *homotopy pullback* if the canonical map

$$W \longrightarrow X \times_Z Z^{\Delta^1} \times_Z Y = \{(x,\gamma,y) \mid \gamma\colon p(x) \to p(y)\} \tag{2.6}$$

is a weak equivalence.

What is this canonical map? Well, W maps to the usual pullback $X \times_Z Y$, and this maps to the homotopy pullback by sending $(x,y) \mapsto (x,\mathrm{id},y)$, as $p(x) = p(y)$ if $(x,y) \in X \times_Z Y$.

Now we can translate the set-theoretic Segal condition into a homotopy-theoretic definition.

Definition 2.7. A simplicial space $X : \Delta^{op} \to \mathscr{T}op$ is a *Segal space* if for all $m,n \in \mathbb{N}$, (2.2) is a homotopy pullback.

We will create Hom *spaces* for Segal spaces. This isn't all the stuff you need for enrichment, but you should think of it as: categories are tautologically enriched in sets, and Segal spaces are tautologically enriched in spaces. But we'll return to this and shape it up.

Definition 2.8. Let X be a Segal space and $x,x' \in X$. Then the *Hom space* from x to x' is the space

$$X^h(x,x') := \{(\gamma,f,\gamma') \mid \gamma\colon x \to d_1(f), \gamma'\colon d_0(f) \to x'\}, \tag{2.9}$$

Remark 2.10. Given a Segal space X, we can construct a homotopy category, whose objects are the underlying set of X and whose set of morphisms $x \to y$ is the set $\pi_0 X^h(x,y)$. $\blacktriangleleft$

[2] For us, $0 \in \mathbb{N}$.

Example 2.11. A simplicial space is *homotopically constant* if all of its face and degeneracy maps are weak equivalences. All homotopically constant simplicial spaces are Segal spaces. This in particular includes (the nerves of) categories. ◀

In this way, the homotopy theory of topological spaces embeds into Segal spaces; moreover, Segal spaces do in fact generalize (the nerves of) categories.

In a category, objects are isomorphic or they aren't. In Segal spaces, we have a new phenomenon: $x, x' \in X_0$ may be equivalent in different ways. They may be *isomorphic*, meaning there is some $(\gamma, f, \gamma') \in X^h(x, x')$ which passes to an isomorphism in the homotopy category, or they may be in the same path component in X.

Example 2.12. This is just a sketch for now, but of an idea that matters for us. There is (almost) a Segal space $\mathscr{C}ob_d$ whose space of 0-simplices is the space of closed $(d-1)$-dimensional submanifolds of $\mathbb{R}^\infty := \varinjlim \mathbb{R}^n$, and whose space of n-simplices is d-dimensional submanifolds of $\mathbb{R}^\infty \times [0, n]$ (together with a transversality requirement at $\mathbb{R}^\infty \times \{i\}$ for $i = 0, \dots, d$). These can be thought of as the time-slices in a bordism which tell you how this is a d-fold composition.

If you continue the construction here, though, you won't get degeneracy maps, because there are strictness issues coming from rescaling when you check the relations between face and degeneracy maps. There are at least four different fixes to this problem, e.g. you can realize $\mathscr{C}ob_d$ as an A_∞-category; all four fixes are at least a little awkward.

For all $M, N \in (\mathscr{C}ob_d)_0$, there is a map from the path space from M to N to $(\mathscr{C}ob_d)^h(M, N)$. This was accompanied by a cool picture. This map, thought of as a map $(\mathscr{C}ob_d)_1 \to (\mathscr{C}ob_d)_0 \times (\mathscr{C}ob_d)_0$. ◀

There are two ways to be equivalent: diffeomorphic (so in the same path in $(\mathscr{C}ob_d)_0$) or bordant via an invertible bordism. These are inequivalent — this is because in dimensions 5 (maybe 4) and above, invertible bordisms are the same thing as h-cobordisms. That is, there can be path equivalences in this Segal space which don't arise as diffeomorphisms (here we're using the h-cobordism theorem, I think).

In general, given a Segal space X and $x, x' \in X_0$, there is a space of paths $\mathrm{Path}(x, x')$ from x to x'.

Definition 2.13. A Segal space X is *complete* if for all $x, x' \in X_0$, the map $\mathrm{Path}(x, x') \to X^h(x, x')$ is a weak equivalence.

Let $\alpha_k : [1] \to [n]$ be the map sending $0 \mapsto k-1$ and $1 \mapsto k$. Given a Segal space X, let X^{inv} denote the subspace of X consisting of only the simplices $c \in X_n$ such that $\alpha_k c$ is invertible for all (TODO: which? Didn't get down in time) k.

Proposition 2.14.

(1) *If X is a Segal space, then for all $x, x' \in X_0$, $(\gamma, f, \gamma') \in X^h(x, x')$ is invertible iff $(\mathrm{const}_{d_1 f}, f, \mathrm{const}_{d_0 f})$ is invertible.*

(2) *TODO: missed.*

Rezk showed that the category of simplicial spaces has a model structure whose fibrant objects are precisely the Reedy fibrant Segal spaces, which is nice. Forcing fully faithful essentially surjective leads to a model structure in which the fibrant objects are complete. From a different perspective, this can be thought of as a univalence axiom.

Bordism categories don't quite behave as nicely; one fix is to use *flagged higher categories*, as introduced by Ayala-Francis.

(TODO: I think I missed the definition of a *category object* in a category $\mathscr{C}$).

Example 2.15.

(1) A category object in $\mathscr{S}et$ is a category.

(2) A category object in $\mathscr{T}op$ is a *topological category* – though this is not quite the same as a topologically enriched category.

(3) A category object in $\mathscr{C}at$ is called a *double category*, one model for a kind of 2-category. ◀

$\mathscr{S}et$ is a full subcategory of both $\mathscr{T}op$ and $\mathscr{C}at$, and in fact a topologically enriched category is the same as a topological category X such that $X_0 \in \mathscr{S}et$. A 2-category is the same thing as a double category with $X_0 \in \mathscr{S}et$.

2.2. n-fold complete Segal spaces.

A 2-*fold complete Segal spaces* is a simplicial object X in complete Segal spaces such that

(1) X satisfies the (homotopy) Segal condition as a simplicial object (i.e. we use homotopy pullbacks rather than strict pullbacks).[3]
(2) $X_{0\bullet}$ is a homotopy constant simplicial space.
(3) Completeness for $X_{\bullet 0}$, which is the most confusing condition of the three. The idea is to extract the underlying Segal space of X, sending $n \mapsto X_{n\bullet}^{\mathrm{inv}}$.

$$\begin{tikzcd} \vdots & \vdots & \vdots & \\ X_{20} & X_{21} & X_{22} & \cdots \\ X_{10} & X_{11} & X_{12} & \cdots \\ X_{00} \arrow[r, dash] & X_{01} \arrow[r, equal] & X_{02} & \cdots \end{tikzcd} \tag{2.16}$$

The homotopy constancy condition on $X_{0\bullet}$ tells us the lowest horizontal maps are homotopy complete. The homotopy completeness condition (in the vertical direction) tells us the leftmost vertical maps are invertible.

TODO: OK, but then what's the actual condition. . . ?

3. Fully dualizable objects: 9/23/20

Today, Jackson Van Dyke spoke about full dualizability, beginning with 1-dualizability and what it means about vector spaces; then generalizing to dualizability in a monoidal category; then categorifying to adjunctions in the 2-category of categories and what is the analogue in a general 2-category; and finally discussing full dualizability. Jackson will post his notes on his website, as well as a video from *The Mask of Zorro*, the inspiration for the colorful name "Zorro's lemma" for Lemma 1.5.

Recall that a 1-dimensional (oriented) TFT is a symmetric monoidal functor $Z\colon \mathcal{C}ob(1) \to \mathcal{V}ect_k$, where k is a field. In Example 1.6, we saw that the data of Z is determined by a vector space $V := Z(\mathrm{pt}_+)$; $\mathrm{pt}_- \mapsto V^\vee$, and the "macaroni" $[0,1]$ regarded as a bordism $\mathrm{pt}_+ \amalg \mathrm{pt}_- \to \varnothing$ is sent to the duality pairing $V \otimes V^\vee \to k$. However, Lemma 1.5, which comes from an equivalence of a Z-shaped bordism from pt_+ to pt_+ with the interval $[0,1] = \mathrm{id}\colon \mathrm{pt}_+ \to \mathrm{pt}_+$, forces V to be finite-dimensional. This motivates the first question one asks on the road to full dualizability: what is the generalization of finite-dimensionality for (nonextended) TFTs with target more general than $\mathcal{V}ect_k$?

The key thing that happened is that we have data of maps $ev\colon V \otimes V^\vee \to k$ and $coev\colon k \to V \otimes V^\vee$ coming from the interval regarded as a bordism $\mathrm{pt}_+ \amalg \mathrm{pt}_- \to \varnothing$, resp. $\varnothing \to \mathrm{pt}_+ \amalg \mathrm{pt}_-$, and applying Z. Thee is also the condition which implies Lemma 1.5, that the Z-diagram is equal to a point as morphisms in $\mathcal{C}ob(1)$. Explicitly written out, we have two maps

$$V \simeq V \otimes k \xrightarrow{\mathrm{id}\otimes coev} V \otimes V^\vee \otimes V \xrightarrow{ev\otimes \mathrm{id}} k \otimes V \simeq V \tag{3.1a}$$

$$V^\vee \simeq k \otimes V^\vee \xrightarrow{coev\otimes \mathrm{id}} V^\vee \otimes V \otimes V^\vee \xrightarrow{\mathrm{id}\otimes ev} V^\vee \otimes k \simeq V^\vee, \tag{3.1b}$$

and Lemma 1.5 is the fact that these are id_V, resp. $\mathrm{id}_{V^\vee}$. In fact, in $\mathcal{V}ect_k$, the existence of data satisfying this condition is equivalent to V being finite-dimensional.

Definition 3.2. Let V be an object in a monoidal category $(\mathcal{C}, \otimes, k)$. A *right dual* for V is data of an object $V^\vee \in \mathcal{C}$ together with maps ev and $coev$ as above satisfying (3.1). In this case, V is called the *left dual* of $V^\vee$, and both V and $V^\vee$ are called *dualizable*.

Here are some useful facts about duals.

(1) It is a fact that right and left duals, together with the data of ev and $coev$, are unique up to unique isomorphism; this justifies our use of the words "the (left or right) dual" below.

[3] The definition of a homotopy pullback in the category of complete Segal spaces is just the levelwise homotopy colimit.

(2) If $\mathscr{C}$ is symmetric monoidal, then $V^\vee$ admits a right dual and there is a canonical isomorphism from V to the right dual of $V^\vee$; therefore the notions of left and right dual coincide and we just speak of the *dual* of a dualizable object.

Definition 3.3. A monoidal category $\mathscr{C}$ *has duals* if all objects are left and right dualizable.

Now let's step up to the 2-category $\mathscr{C}at$ of categories. We're not going to define it in complete, precise generality here, but important data includes

- the objects are small categories,
- the morphisms are functors between them, and
- the 2-morphisms are natural transformations.

Adjoints will be our analogue of finite-dimensionality: something you've most likely seen before, and which will be our model for the general case.

Definition 3.4. Suppose we have functors $f : \mathscr{C} \rightleftarrows \mathscr{D} : g$ are two functors. We say f is *right adjoint* to g and g is *left adjoint* to f if there is a natural isomorphism $\mathscr{C}(gy, x) \xrightarrow{\simeq} \mathscr{D}(y, fx)$.

We can rephrase this in a way that might look suspiciously familiar: the data of an adjunction is natural transformations $u\colon \mathrm{id}_X \to g \circ f$ and $v\colon f \circ g \to \mathrm{id}_Y$ such that the 2-morphism

$$f \simeq f \circ \mathrm{id}_X \xrightarrow{\mathrm{id}\times u} f \circ g \circ f \xrightarrow{v\times \mathrm{id}} \mathrm{id}_Y \circ f \simeq f \tag{3.5}$$

is equal to the 2-morphism id_f (the identity natural transformation from f to itself), and an analogous diagram for g is equal to id_g.

Definition 3.6. Let $\mathscr{C}$ be a 2-category and f, g be 1-morphisms. We say that f is *right adjoint* to g and g is *left adjoint* to f if there exist u, v as above such that (3.5) is equal to id_f and its analogue for g is equal to id_g.

It turns out that f determines g up to unique isomorphism.

Definition 3.7. A 2-category $\mathscr{C}$ *has adjoints* if all 1-morphisms have left and right adjoints.

If $\mathscr{C}$ is a symmetric monoidal 2-category, let $X \in \mathscr{C}$. We say that X is 0-*dualizable* if it is dualizable, whence *ev*, *coev*; it is 1-*dualizable* if *ev* and *coev* have adjoints. We will generalize this to monoidal (∞, n)-categories by inductively defining k-dualizable to mean that we have $(k-1)$-dualizability, and the pair of adjoints we obtain at level k themselves have adjoints.

Speaking more precisely, if $\mathscr{C}$ is a monoidal (∞, n)-category, let $h(\mathscr{C})$ be its *homotopy category*, whose objects are the objects of $\mathscr{C}$, and whose morphisms are the isomorphism classes of 1-morphisms in $\mathscr{C}$. The monoidal structure on $\mathscr{C}$ induces one on $h(\mathscr{C})$.

Definition 3.8. An $X \in \mathscr{C}$ is 0-*dualizable* if its image in $h(\mathscr{C})$ is.

We can also *deloop* $\mathscr{C}$ into an $(\infty, n+1)$-category $B\mathscr{C}$ with a single object $\bullet$ and Hom (∞, n)-category $\mathrm{Hom}_{B\mathscr{C}}(\bullet, \bullet) := \mathscr{C}$, with composition given by tensor product.

Lemma 3.9. *$X \in \mathscr{C}$ is 0-dualizable iff X, regarded as a morphism in $B\mathscr{C}$, has both left and right adjoints.*

None of the ∞-stuff we've done so far uses anything finer than $h(\mathscr{C})$.

Stepping up, let's say $n \geq 2$. We can define a richer version of $h(\mathscr{C})$ called $h_2(\mathscr{C})$, a 2-category, whose objects are the objects in $\mathscr{C}$, whose 1-morphisms are the 1-morphisms in $\mathscr{C}$, and whose 2-morphisms are the isomorphism classes of the 2-morphisms in $\mathscr{C}$. $h_2(\mathscr{C})$ is called the *homotopy* 2-*category* of $\mathscr{C}$.

Definition 3.10. A 1-morphism f in $\mathscr{C}$ *has adjoints* if, regarded as a 1-morphism in $h_2(\mathscr{C})$, it has adjoints. (TODO: chance I missed something here.)

More generally, let s, t be k-morphisms in $\mathscr{C}$, where $n \geq k+2$. Then $\mathrm{Hom}(s, t)$ is an $(\infty, n-k)$-category and we can take its homotopy 2-category.

Definition 3.11. A $(k+1)$-morphism $\eta\colon f \to g$ *has adjoints* if η, regarded as a 1-morphism in $\mathrm{Hom}(s, t)$, has adjoints.

Definition 3.12. For $k \leq n$, an object $X \in \mathscr{C}$ is k-*dualizable* if its $(k-1)$-dualizable and the data given at level $k-1$ (either *ev* and *coev*, or adjoints) has adjoints.

Definition 3.13. An (∞, n)-category $\mathscr{C}$ *has duals* if all objects have duals and all morphisms (at all levels) have adjoints, as far up as this makes sense.

In fact, given $\mathscr{C}$, there is a subcategory $i\colon \mathscr{C}^{fd} \hookrightarrow \mathscr{C}$ satisfying the universal property

(1) $\mathscr{C}^{fd}$ has duals, and
(2) for all functors $F\colon \mathscr{D} \to \mathscr{C}$ such that $\mathscr{D}$ has duals, there is a map $f : \mathscr{D} \to \mathscr{C}^{fd}$ such that $F \simeq i \circ f$.

This $\mathscr{C}^{fd}$ is sometimes called the *subcategory of fully dualizable objects*, and it will appear again in the statement of the cobordism hypothesis.

References