M392C NOTES: SPIN GEOMETRY

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These notes were taken in UT Austin's M392C (Spin Geometry) class in Fall 2016, taught by Eric Korman. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Thanks to Adrian Clough for fixing a few typos.

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Lecture 1.

Lie Groups: 8/25/16

There is a course website, located at https://www.ma.utexas.edu/users/ekorman/teaching/spingeometry/. There's a list of references there, none of which we'll exactly follow.

We'll assume some prerequisites for this class: definitely smooth manifolds and some basic algebraic topology. We'll use cohomology, which isn't part of our algebraic topology prelim course, but we'll review it before using it.

Introduction and motivation. Recall that a *Riemannian manifold* is a pair (M, g) where M is an n-dimensional smooth manifold and g is a *Riemannian metric* on M, i.e. a smoothly varying, positive definite inner product on each tangent space T_xM over all $x \in M$.

Definition 1.1. A *local frame* on M is a set of (locally defined) tangent vectors that give a positive basis for M, i.e. a smoothly varying set of tangent vectors that are a basis at each tangent space.

A Riemannian metric allows us to talk about *orthonormal frames*, which are those that are orthonormal with respect to the metric at all points.

Recall that the special orthogonal group is $\mathrm{SO}(n) = \{A \in M_n \mid AA^{\mathrm{T}} = I, \det A = 1\}$. This acts transitively on orthonormal, oriented bases, and therefore also acts transitively on orthonormal frames (as a frame determines an orientation). Conversely, specifying which frames are orthonormal determines the metric g.

In summary, the data of a Riemannian structure on a smooth manifold is equivalent to specifying a subset of all frames which is acted on simply transitively[1] by the group $\mathrm{SO}(n)$. This set of all frames is a *principal* $\mathrm{SO}(n)$*-bundle* over M.

By replacing $\mathrm{SO}(n)$ with another group, one obtains other kinds of geometry: using $\mathrm{GL}(n, \mathbb{C})$ instead, we get almost complex geometry, and using $\mathrm{Sp}(n)$, we get almost symplectic geometry (geometry with a specified skew-symmetric, nondegenerate form).

Remark. Let G be a Lie group and M be a manifold. Suppose we have a principal G-bundle $E \to M$ and a representation[2] $\rho : G \to V$, we naturally get a vector bundle over M.

[1] Recall that a group action on X is *transitive* if for all $x, y \in X$, there's a group element g such that $g \cdot x = y$, and is *simple* if this g is unique.

[2] A representation of a group G is a homomorphism $G \to \mathrm{GL}(V)$ for a vector space V. We'll talk more about representations later.

A more surprising fact is that all[3] representations of $\mathrm{SO}(n)$ are contained in tensor products of the *defining representation* of $\mathrm{SO}(n)$ (i.e. acting on $\mathbb{R}^n$ by orientation-preserving rotations). Thus, all of the natural vector bundles are subbundles of tensor powers of the tangent bundles. That is, when we do geometry in this way, we obtain no exotic vector bundles.

If $n \geq 3$, then $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2$, so its double cover is its universal cover. Lie theory tells us this space is naturally a compact Lie group, called the *Spin group* $\mathrm{Spin}(n)$. In many ways, it's more natural to look at representations of this group. The covering map $\mathrm{Spin}(n) \twoheadrightarrow \mathrm{SO}(n)$ precomposes with any representation of $\mathrm{SO}(n)$, so any representation of $\mathrm{SO}(n)$ induces a representation of $\mathrm{Spin}(n)$. However, there are representations of the spin group that don't arise this way, so if we can refine the orthonormal frame bundle to a principal $\mathrm{Spin}(n)$-bundle, then we can create new vector bundles that don't arise as tensor powers of the tangent bundle.

Spin geometry is more or less the study of these bundles, called *bundles of spinors*; these bundles have a natural first-order differential operator called the *Dirac operator*, which relates to a powerful theorem coming out of spin geometry, the Atiyah-Singer index theorem: this is vastly more general, but has a particularly nice form for Dirac operators, and the most famous proof reduces the general case to the Dirac case. Broadly speaking, the index theorem computes the dimension of the kernel of an operator, which in various contexts is a powerful invariant. Here are a few special cases, even of just the Dirac case of the Atiyah-Singer theorem.

- The Gauss-Bonnet-Chern theorem gives an integral formula for the Euler characteristic of a manifold, which is entirely topological. In this case, the index is the Euler characteristic.
- The Hirzebruch signature theorem gives an integral formula for the signature of a manifold.
- The Grothendieck-Riemann-Roch theorem, which gives an integral formula for the Euler characteristic of a holomorphic vector bundle over a complex manifold.

Lie groups and Lie algebras.

Definition 1.2. A *Lie group G* is a smooth manifold with a group structure such that the multiplication map $G \times G \to G$ sending $g_1, g_2 \mapsto g_1 g_2$ and the inversion map $G \to G$ sending $g \mapsto g^{-1}$ are smooth.

Example 1.3.

- The *general linear group* $\mathrm{GL}(n,\mathbb{R})$ is the group of $n \times n$ invertible matrices with coefficients in $\mathbb{R}$. Similarly, $\mathrm{GL}(n,\mathbb{C})$ is the group of $n \times n$ invertible complex matrices. Most of the matrices we consider will be subgroups of these groups.
- Restricting to matrices of determinant 1 defines $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{SL}(n,\mathbb{C})$, the *special linear groups*.
- The *special unitary group* $\mathrm{SU}(n) = \{A \in \mathrm{GL}(n,\mathbb{C}) \mid A\overline{A}^{\mathrm{T}} = 1, \det A = 1\}$.
- The special orthogonal group $\mathrm{SO}(n)$, mentioned above.

Definition 1.4. A *Lie algebra* is a vector space $\mathfrak{g}$ with an anti-symmetric, bilinear pairing $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the *Jacobi identity*

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

for all $X, Y, Z \in \mathfrak{g}$.

Example 1.5. The basic and important example: if A is an algebra,[4] then A becomes a Lie algebra with the commutator bracket $[a,b] = ab - ba$. Because this algebra is associative, the Jacobi identity holds.

The Jacobi identity might seem a little vague, but here's another way to look at it: if $\mathfrak{g}$ is a Lie algebra and $X \in \mathfrak{g}$, then there's a map $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ sending $Y \mapsto [X,Y]$. The map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ sending $X \mapsto \mathrm{ad}_X$ is called the *adjoint representation* of X. The Jacobi identity says that ad intertwines the bracket of $\mathfrak{g}$ and the bracket induced from the algebra structure on $\mathrm{End}(\mathfrak{g})$ (where multiplication is composition): $\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y]$. In other words, the adjoint representation is a homomorphism of Lie algebras.

Lie groups and Lie algebras are very related: to any Lie group G, let $\mathfrak{g}$ be the set of left-invariant vector fields on G, i.e. if $L_g : G \to G$ is the map sending $h \mapsto gh$ (the *left multiplication* map), then $\mathfrak{g} = \{X \in \Gamma(TG) \mid \mathrm{d}L_g X = X \text{ for all } g \in G\}$. This is actually finite-dimensional, and has the same dimension as G.

Proposition 1.6. *If e denotes the identity of G, then the map $\mathfrak{g} \to T_eG$ sending $X \mapsto X(e)$ is an isomorphism (of vector spaces).*

[3] We're only considering smooth, finite-dimensional representations.

[4] By an *algebra* we mean a ring with a compatible vector space structure.

The idea is that given the data at the identity, we can translate it by g to determine what its value must be everywhere. Vector fields have a Lie bracket, and the Lie bracket of two left-invariant vector fields is again left-invariant, so $\mathfrak{g}$ is naturally a Lie algebra! We will often use Proposition 1.6 to identify $\mathfrak{g}$ with the tangent space at the identity.

Example 1.7. Let's look at $\mathrm{GL}(n,\mathbb{R})$. This is an open submanifold of the vector space M_n, an n^2-dimensional vector space, as $\det A \neq 0$ is an open condition. Thus, the tangent bundle of $\mathrm{GL}(n,\mathbb{R})$ is trivial, so we can canonically identify $T_I\mathrm{GL}(n,\mathbb{R}) = M_n$. With the inherited Lie algebra structure, this space is denoted $\mathfrak{gl}(n,\mathbb{R})$.

The $n \times n$ matrices are also isomorphic to $\mathrm{End}(\mathbb{R}^n)$, since they act by linear transformations. The algebra structure defines another Lie bracket on this space.

Proposition 1.8. *Under the above identifications, these two brackets are identical, hence define the same Lie algebra structure on* $\mathfrak{gl}(n,\mathbb{R})$.

Remark. This proposition generalizes to all real matrix Lie groups (Lie subgroups of $\mathrm{GL}(n,\mathbb{R})$): the proof relies on a Lie subgroup's Lie algebra being a subalgebra of $\mathfrak{gl}(n,\mathbb{R})$.

So we can go from Lie groups to Lie algebras. What about in the other direction?

Theorem 1.9. *The correspondence sending a connected, simply-connected Lie group to its Lie algebra extends to an equivalence of categories between the category of simply connected Lie groups and finite dimensional Lie algebras over* $\mathbb{R}$.

Suppose G is any connected Lie group, not necessarily simply connected, and $\mathfrak{g}$ is its Lie algebra. If $\widetilde{G}$ denotes the universal cover of G, then $G = \widetilde{G}/\pi_1(G)$. Since $\widetilde{G}$ is simply connected, the correspondence above identifies $\mathfrak{g}$ with it, and then taking the quotient by the discrete central subgroup $\pi_1(G)$ recovers G.

The special orthogonal group. We specialize to $\mathrm{SO}(n)$, the orthogonal matrices with determinant 1. We'll usually work over $\mathbb{R}$, but sometimes $\mathbb{C}$. This is a connected Lie group.[5]

Proposition 1.10. *If $\mathfrak{so}(n)$ denotes the Lie algebra of* $\mathrm{SO}(n)$*, then* $\mathfrak{so}(n) = \{X \in \mathfrak{gl}(n,\mathbb{R}) \mid X + X^{\mathrm{T}} = 0\}$.

That is, $\mathfrak{so}(n)$ is the Lie algebra of skew-symmetric matrices.

Proof. If $F : M_n \to M_n$ is the function $A \mapsto A^{\mathrm{T}}A - I$, then the orthogonal group is $\mathrm{O}(n) = F^{-1}(0)$. Since $\mathrm{SO}(n)$ is the connected component of $\mathrm{O}(n)$ containing the identity, then it suffices to calculate $T_e\mathrm{O}(n)$: if 0 is a regular value of F, we can push forward by its derivative. This is in fact the case:

$$\mathrm{d}F_A(B) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F(A+tB) = A^{\mathrm{T}}B + B^{\mathrm{T}}A,$$

which is surjective for $A \in \mathrm{O}(n)$, so $\mathfrak{so}(n) = T_I\mathrm{SO}(n) = \ker(\mathrm{d}F_I) = \{B \in M_n \mid B + B^{\mathrm{T}} = 0\}$. ⊠

The spin group. We'll end by computing the fundamental group of $\mathrm{SO}(n)$; then, by general principles of Lie groups, each $\mathrm{SO}(n)$ has a unique, simply connected double cover, which is also a Lie group. Next time, we'll provide an *a priori* construction of this cover.

Proposition 1.11.

$$\pi_1(\mathrm{SO}(n)) = \begin{cases} \mathbb{Z}, & n = 2 \\ \mathbb{Z}/2, & n \geq 3. \end{cases}$$

Proof. If $n = 2$, $\mathrm{SO}(n) \cong S^1$ through the identification

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \longmapsto e^{i\theta},$$

and we know $\pi_1(S^1) = \mathbb{Z}$.

For $n \geq 3$, we can use a long exact sequence associated to a certain fibration, so it suffices to calculate $\pi_1(\mathrm{SO}(3))$. Specifically, we will define a Lie group structure on S^3 and a double cover map $S^3 \twoheadrightarrow \mathrm{SO}(3)$; since S^3 is simply connected, this will show $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}/2$.

[5] If we only took orthogonal matrices with arbitrary determinant, we'd obtain the *orthogonal group* $\mathrm{O}(n)$, which has two connected components.

We can identify S^3 with the unit sphere in the quaternions, which is naturally a group (since the product of quaternions is a polynomial, hence smooth).[6] Realize $\mathbb{R}^3$ inside the quaternions as $\operatorname{span}_{\mathbb{R}}\{i,j,k\}$ (the *imaginary quaternions*); then, we'll define $\varphi : S^3 \twoheadrightarrow \mathrm{SO}(3)$: $\varphi(q)$ for $q \in \mathbb{H}$ is the linear transformation $p \mapsto qpq^{-1} \in \mathrm{GL}(3,\mathbb{R})$, where p is an imaginary quaternion. We need to check that $\varphi(q)$ lies in SO(3), which was left as an exercise. We also need to check this is two-to-one, which is equivalent to $|\ker\varphi| = 2$, and that φ is surjective (hint: since these groups are connected, general Lie theory shows it suffices to show that the differential is an isomorphism). ⊠

Lecture 2.

Spin Groups and Clifford Algebras: 8/30/16

Last time, we gave a rushed construction of the double cover of SO(3), so let's investigate it more carefully. Recall that SO(n) is the Lie group of special orthogonal matrices, those matrices A such that $AA^{\mathrm{t}} = I$ and $\det A = 1$, i.e. those linear transformations preserving the inner product and orientation. This is a connected Lie group; we'd like to prove that for $n \geq 3$, $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2$. (For $n = 2$, $\mathrm{SO}(2) \cong S^1$, which has fundamental group $\mathbb{Z}$).

We'll prove this by explicitly constructing the double cover of SO(3), then bootstrapping it using a long exact sequence of homotopy groups to all SO(n), using the following fact.

Proposition 2.1. *Let G and H be connected Lie groups and $\varphi : G \to H$ be a Lie group homomorphism. Then, φ is a covering map iff $\mathrm{d}\varphi|_e : \mathfrak{g} \to \mathfrak{h}$ is an isomorphism.*[7]

Here $\mathfrak{g}$ is the Lie algebra of G, and $\mathfrak{h}$ is that of H. Facts like these may be found in Ziller's online notes;[8] the intuitive idea is that the condition on $\mathrm{d}\varphi|_e$ ensures an isomorphism in a neighborhood of the identity, which multiplication carries to a local isomorphism in the neighborhood of any point in G.

Now, we construct a double cover of SO(3). Recall that the *quaternions* are the noncommutative algebra $\mathbb{H} = \operatorname{span}_{\mathbb{R}}\{1,i,j,k\}$, where $i^2 = j^2 = k^2 = ijk = 1$. We can identify $\mathbb{R}^3$ with the imaginary quaternions, the span of $\{i,j,k\}$, and therefore the unit sphere S^3 goes to $\{q \in \mathbb{H} \mid |q|^2 = 1 = q\bar{q}\}$, where the conjugate exchanges i and $-i$, but also j and $-j$, and k and $-k$. This embedding means that if $v, w \in \mathbb{R}^3$, their product as quaternions is

$$vw = -\langle v, w\rangle + v \times w.$$

and in particular

$$vw + wv = -2\langle v, w\rangle. \tag{2.2}$$

If $q \in S^3$ and $v \in \mathbb{R}^3$, then $qvq^{-1} = qv\bar{q}$, i.e. $\overline{qvq^{-1}} = q\overline{vq} = -qv\bar{q}$. That is, conjugation by something in S^3 is a linear transformation in $\mathbb{R}^3$, defining a smooth map $\varphi : S^3 \to \mathrm{GL}(3,\mathbb{R})$; we'd like to show the image lands in SO(3). Let $q \in S^3$; then, we can use (2.2) to get

$$\begin{aligned}
\langle \varphi(q)v, \varphi(q)w\rangle &= -\frac{1}{2}(\varphi(q)v\varphi(q)w + \varphi(q)w + \varphi(q)v) \\
&= -\frac{1}{2}\big(qvwq^{-1} + qwvq^{-1}\big) \\
&= -\frac{1}{2}\big(q(vw + wv)q^{-1}\big) = \langle v, w\rangle,
\end{aligned}$$

using (2.2) again, and the fact that $\mathbb{R} = Z(\mathbb{H})$. Thus, $\mathrm{Im}(\varphi) \subset \mathrm{O}(3)$, but since S^3 is connected, its image must be connected, and its image contains the identity (since φ is a group homomorphism), so $\mathrm{Im}(\varphi)$ lies in the connected component containing the identity, which is SO(3).

[6] This is important, because when we try to generalize to Spin_n for higher n, we'll be using Clifford algebras, which are generalizations of the quaternions.

[7] This isomorphism is as Lie algebras, but it's always a Lie algebra homomorphism, so it suffices to know that it's an isomorphism of vector spaces.

[8] https://www.math.upenn.edu/ wziller/math650/LieGroupsReps.pdf.

To understand $\mathrm{d}\varphi|_1$, let's look at the Lie algebras of S^3 and SO(3). The embedding $S^3 \hookrightarrow \mathbb{H}$ allows us to identify $T_! S^3$ with the imaginary quaternions. If p and v are imaginary quaternions, so $\overline{p} = -p$, then

$$\begin{aligned}\mathrm{d}\varphi|_p(v) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg| |_{t=0}\varphi(e^{tp})v \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} e^{tp}ve^{-tp} \\ &= pv - vp.\end{aligned}$$

Thus, $\ker \mathrm{d}\varphi|_1 = \{p \in \mathbb{R}^3 \mid pv - vp = 0 \text{ for all imaginary quaternions } v\}$. But if something commutes with all imaginary quaternions, it commutes with all quaternions, since the imaginary quaternions and the reals (which are the center of $\mathbb{H}$) span to all of $\mathbb{H}$. Thus, the kernel is the imaginary quaternions in the center of $\mathbb{H}$, which is just $\{0\}$; hence, $\mathrm{d}\varphi|_1$ is injective, and since $T_1 S^3$ and $\mathfrak{so}(3)$ have the same dimension, it is an isomorphism. By Proposition 2.1, φ is a covering map, and $\mathrm{SO}(3) = S^3/\ker(\varphi)$.

We'll compute $|\ker \varphi|$, which will be the index of the cover. The kernel is the set of unit quaternions q such that $qvq^{-1} = v$ for all imaginary quaternions v; just as above, this must be the intersection of the real line with S^3, which is just $\{\pm 1\}$. Thus, φ is a double cover map of SO(3); since S^3 is simply connected, $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}/2$.

Exercise 2.3. The Lie group structure on S^3 is isomorphic to SU(2), the group of 2×2 special unitary matrices.

Now, what about $\pi_1(\mathrm{SO}(n))$, for $n \geq 4$? For this we use a fibration. $\mathrm{SO}(n)$ acts on $S^{n-1} \subset \mathbb{R}^n$, and the stabilizer of a point in S^n is all the rotations fixing the line containing that point, which is a copy of $\mathrm{SO}(n-1)$. This defines a fibration

$$\mathrm{SO}(n-1) \longrightarrow \mathrm{SO}(n) \longrightarrow S^{n-1}.$$

More precisely, let's fix the north pole $p = (0,0,\dots,0,1) \in S^{n-1}$; then, the map $\mathrm{SO}(n) \to S^{n-1}$ sends $A \mapsto Ap$; since A is orthogonal, Ap is a unit vector. The action of $\mathrm{SO}(n)$ is transitive, so this map is surjective. The stabilizer of p is the set of all orthogonal matrices with positive determinant such that the last column is $(0,0,\dots,0,1)$. Orthogonality forces these matrices to have block form

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix},$$

where $A \in \mathrm{SO}(n-1)$; thus, the stabilizer is isomorphic to $\mathrm{SO}(n-1)$.

Now, we can use the long exact sequence in homotopy associated to a fibration:

$$\pi_2(S^{n-1}) \xrightarrow{\delta} \pi_1(\mathrm{SO}(n-1)) \longrightarrow \pi_1(\mathrm{SO}(n)) \longrightarrow \pi_1(S^{n-1}).$$

If $n \geq 4$, $\pi_2(S^{n-1})$ and $\pi + 1(S^{n-1})$ are trivial, so $\pi_1(\mathrm{SO}(n)) = \pi_1(\mathrm{SO}(n-1))$ for $n \geq 4$, so they all agree with $\mathbb{Z}?2$, so $\pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}/2$ for all $n \geq 4$.

By general Lie theory, the universal cover of a Lie group is also a Lie group.

Definition 2.4. For $n \geq 3$, the *spin group* $\mathrm{Spin}(n)$ is the unique simply-connected Lie group with Lie algebra $\mathfrak{so}(n)$. For $n = 2$, the spin group Spin(2) is the unique (up to isomorphism) connected double covering group of SO(2).

In particular, there is a double cover $\mathrm{Spin}(n) \to \mathrm{SO}(n)$, and $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$.

Right now, we do not have a concrete description of these groups; since $\mathrm{SO}(n)$ is compact, so is $\mathrm{Spin}(n)$, so we must be able to realize it as a matrix group, and we use Clifford algebras to do this.

Clifford algebras. Our goal is to replace $\mathbb{H}$ with some other algebra to realize $\mathrm{Spin}(n)$ as a subgroup of its group of units.

Recall from (2.2) that for $v, w \in \mathbb{R}^3 \hookrightarrow \mathbb{H}$, $vw + wv = -2\langle v, w\rangle$. We'll define a universal algebra for this kind of definition.

Definition 2.5. Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space. Its *Clifford algebra* is

$$\mathrm{C}\ell(V) = T(V)/(v \otimes v + \langle v, v\rangle 1).$$

Here, $T(V)$ is the tensor algebra, and we quotient by the ideal generated by the given relation.

That is, we've forced (2.2) for a vector paired with itself. That's actually sufficient to imply it for all pairs of vectors.

Remark. Though we only defined the Clifford algebra for nondegenerate inner products, the same definition can be made for all bilinear pairings. If one chooses $\langle\cdot,\cdot\rangle = 0$, one obtains the exterior algebra $\Lambda(V)$, and we'll see that Clifford algebras sometimes behave like exterior algebras.

Recall that the tensor algebra is defined by the following universal property: if A is any algebra,[9] and $f : V \to A$ is linear, then there exists a unique homomorphism of algebras $\widetilde{f} : T(V) \to A$ such that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d] & A \\ T(V). \arrow[ur, "\widetilde{f}"'] & \end{tikzcd}$$

That is, as soon as I know what happens to elements of f, I know what to do to tensors.

This implies a universal property for the Clifford algebra.

Proposition 2.6. *Let A be an algebra and $f : V \to A$ be a linear map. Then, $f(v)^2 = -\langle v, v\rangle 1_A$ iff f extends uniquely to a map $\widetilde{f} : \mathrm{C}\ell(V) \to A$ such that the following diagram commutes:*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d] & A \\ \mathrm{C}\ell(V). \arrow[ur, "\widetilde{f}"'] & \end{tikzcd}$$

The map $V \to \mathrm{C}\ell(V)$ is the composition $V \hookrightarrow T(V) \twoheadrightarrow \mathrm{C}\ell(V)$, where the last map is projection onto the quotient.

We'll end up putting lots of structure on Clifford algebras: a $\mathbb{Z}/2$-grading, a $\mathbb{Z}$-filtration, a canonical vector-space isomorphism with the exterior algebra, and so forth.

Important Example 2.7. Let $\Lambda^\bullet V$ denote the exterior algebra on V, the graded algebra whose k^{th} graded piece is wedges of k vectors: $\Lambda^k(V) = \{v_1,\dots,v_k \mid v_j \in V\}$, with the relations $v\wedge w = -w\wedge v$.

Given a $v \in V$, we can define two maps, *exterior multiplication* $\varepsilon(v) : \Lambda^\bullet(V) \to \Lambda^{\bullet-1}(V)$ defined by $\mu \mapsto v\wedge\mu$, and *interior multiplication* $i(v) : \Lambda^\bullet(V) \to \Lambda^{\bullet-1}(V)$ sending

$$v_1 \wedge\cdots\wedge v_k \mapsto \sum_{i=1}^{k} (-1)^{i-1}\langle v, v_i\rangle v_1 \wedge\cdots\wedge \widehat{v}_i \wedge\cdots\wedge v_k,$$

where $\widehat{v}_i$ means the absence of the i^{th} term.

This has a few important properties:

(1) Both of these maps are idempotents: $\varepsilon(v)^2 = i(v)^2 = 0$.
(2) If $\mu_1,\mu_2 \in \Lambda^\bullet(V)$, then

$$i(v)(\mu_1\wedge\mu_2) = (i(v)\mu_1)\wedge\mu_2 + (-1)^{\deg\mu_1}\mu_1\wedge i(v)\mu_2.$$

In particular,

$$\varepsilon(v)i(v) + i(v)\varepsilon(v) = \langle v, v\rangle. \tag{2.8}$$

We can use this to define a representation of the Clifford algebra onto the exterior algebra: define a map $c : V \to \mathrm{End}(\Lambda^\bullet(V))$ by $c(v) = \varepsilon(v) - i(v)$. Then, $c(v)^2 = -(\varepsilon(v)i(v) + i(v)\varepsilon(v)) = \langle v, v\rangle$, so by the universal property, c extends to a homomorphism $c : \mathrm{C}\ell(V) \to \mathrm{End}(\Lambda^* V)$.

Given an inner product on V, there is an induced inner product on $\Lambda^\bullet V$: choose an orthonormal basis $\{e_1,\dots,e_n\}$ for V, and then declare the basis $\{e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_k} \mid 1 \le i_1 < \cdots < i_k \le n\}$ to be orthonormal; then, use the dot product associated to that orthonormal basis. This is coordinate-invariant, however.

Theorem 2.9. *Suppose $\{e_1,\dots,e_n\}$ is a basis for V. Then, $\{e_{i_1}e_{i_2}\cdots e_{i_k} \mid i_1 < i_2 < \cdots i_k\}$ (where the product is in the Clifford algebra) is a vector-space basis for $\mathrm{C}\ell(V)$.*

[9] Here, an algebra is a unital ring with a compatible real vector space structure.

Today, we'll focus on examples, and perhaps prove this later. This tells us that v and w anticommute iff $v \perp w$, and the relations are

$$e_j e_j = \begin{cases} -e_j e_i, & i \neq j \\ -1, & i = j. \end{cases}$$

This is just like the exterior algebra, but deformed: if $i = j$, we get 1 rather than 0. Theorem 2.9 also tells us that $\dim \mathrm{C}\ell(V) = 2^{\dim V}$.

Example 2.10. $\mathrm{C}\ell(\mathbb{R}^2) \cong \mathbb{H}$ as $\mathbb{R}$-algebras: $\mathrm{C}\ell(\mathbb{R}^2)$ is generated by 1, e_1, and e_2 such that $e_1 e_2 = -e_2 e_1$ and $e_1^2 = e_2^2 = -1$. Thus, $\{1, e_1, e_2, e_1 e_2\}$ is a basis for $\mathrm{C}\ell(\mathbb{R}^2)$, and $(e_1 e_2)^2 = e_1 e_2 e_1 e_2 = -e_1^2 e_2^2 = -1$.

Thus, the isomorphism $\mathrm{C}\ell(\mathbb{R}^2) \to \mathbb{H}$ extends from $1 \mapsto 1$, $e_1 \mapsto i$, $e_2 \mapsto j$, and $e_1 e_2 \mapsto k$.

Example 2.11. Even simpler is $\mathrm{C}\ell(\mathbb{R}) \cong \mathbb{C}$, generated by 1 and e_1 such that $e_1^2 = -1$.

Example 2.12. If we consider the Clifford algebra of $\mathbb{C}$ as a complex vector space, $\mathbb{C}$ is in the center, so $\mathrm{C}\ell_{\mathbb{C}}(\mathbb{C})$ is generated by 1 and e_1 with $ie_1 = e_1 i$.

Lecture 3.

The Structure of the Clifford Algebra: 9/1/16

Last time, we started with an inner product space $(V, \langle \cdot, \cdot \rangle)$ and used it to define a Clifford algebra $\mathrm{C}\ell(V) = T(V)/(v \otimes v + \langle v, v\rangle 1)$, the free algebra generated by V such that $v^2 = -\langle v, v\rangle$.[10] For a $v \in V$, let $\widetilde{v} \in \mathrm{C}\ell(V)$ be its image under the natural map $V \to T(V) \twoheadrightarrow \mathrm{C}\ell(V)$: the first map sends a vector to a degree-1 tensor, and the second is the quotient map. It's reasonable to assume this map is injective, and in fact we'll be able to prove this, so we may identify V with its image in the Clifford algebra.

We also defined a representation of $\mathrm{C}\ell(V)$ on $\Lambda^\bullet V$, which was an algebra homomorphism $c : \mathrm{C}\ell(V) \to \mathrm{End}(\Lambda^\bullet V)$ that is defined uniquely by saying that $c(\widetilde{v}) = \varepsilon(v) - i(v)$ (exterior multiplication minus interior multiplication, also known as wedge product minus contraction). We checked that this squares to scalar multiplication by $-\langle v, v\rangle$, so it is an algebra homomorphism.

Definition 3.1. The *symbol map* is the linear map $\sigma : \mathrm{C}\ell(V) \to \Lambda^\bullet V$ defined by $u \mapsto c(u) \cdot 1$.

Theorem 2.9 defines a basis for the Clifford algebra; we can use this to prove it.

Lemma 3.2. *The map $V \to \mathrm{C}\ell(V)$ sending $v \mapsto \widetilde{v}$ is injective.*

Proof. For $v \in V$, $\sigma(v) = c(\widetilde{v})1 = \varepsilon(v) \cdot 1 - i(v) \cdot 1$. Since interior multiplication lowers degree, $i(v) = 0$, so $\sigma(v) = v$. Thus, the map $V \to \mathrm{C}\ell(V)$ is injective. ⊠

We will identify v and $\widetilde{v}$, and just think of V as a subspace of $\mathrm{C}\ell(V)$.

Proposition 3.3. *The symbol map is an isomorphism of vector spaces.*

Proof. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of V, so $e_i e_j = -e_j e_i$ unless $i = j$, in which case it's -1. So $\mathrm{C}\ell(V) = \mathrm{span}\{e_{i_1} e_{i_2} \cdots e_{i_k} \mid i_1 < i_2 < \cdots < i_k\}$. We'll show these are linearly independent, hence form a basis for $\mathrm{C}\ell(V)$, and recover Theorem 2.9 as a corollary.

Since

$$\begin{aligned} c(e_i) e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_k} &= e_i \wedge e_{j_1} \wedge \cdots \wedge e_{j_k} - i(e_i)\big(e_{j_1} \wedge \cdots \wedge e_{j_k}\big) \\ &= e_i \wedge e_{j_1} \wedge \cdots \wedge e_{j_k} \end{aligned}$$

if all indices are distinct, so

$$\begin{aligned} a\sigma(e_{i_1} \cdots e_{i_k}) &= c(e_{i_1}) \cdots c(e_{i_k})1 \\ &= c(e_1) \cdots \underbrace{c(e_{i_{k-1}}) e_{i_k}}_{e_{i_{k-1}} \wedge e_{i_k}} \\ &= e_{i_1} \wedge \cdots \wedge e_{i_k}. \end{aligned}$$

[10] There are different conventions here; sometimes people work with the relation $v^2 = \langle v, v\rangle$. This is a different algebra in general over $\mathbb{R}$, but over $\mathbb{C}$ they're the same thing.

As $\{i_1,\dots,i_k\}$ ranges over all k-element subsets of $\{1,\dots,n\}$, these form a basis for $\Lambda^\bullet V$. Thus, σ is surjective, and the proposed basis for $\mathrm{C}\ell(V)$ is indeed linearly independent. Thus, σ is also injective, so an isomorphism of vector spaces. ⊠

In particular, we've discovered a basis for $\mathrm{C}\ell(V)$, proving Theorem 2.9.

Remark. The symbol map is *not* an isomorphism of algebras: $\sigma(v^2)=\sigma(-\langle v,v\rangle)=-\langle v,v\rangle$, but $\sigma(v)\wedge\sigma(v)=0$. The symbol is just the highest-order data of an element of the Clifford algebra.

The proof of the following proposition is an (important) exercise.

Proposition 3.4.

$$Z(\mathrm{C}\ell(V))=\begin{cases}\mathbb{R}, & \dim V \text{ is even}\\ \mathbb{R}\oplus\mathbb{R}\gamma, & \dim V \text{ is odd,}\end{cases}$$

where $\gamma=e_1\cdots e_n$ is σ^{-1} of a volume form.

Physicists sometimes call the span of γ *pseudoscalars*, since they commute with everything (in odd degree), much like scalars.

Algebraic structures on the Clifford algebra. Recall that an algebra A is called $\mathbb{Z}$*-graded* if it has a decomposition as a vector space

$$A=\bigoplus_n \in \mathbb{Z}A_n$$

where the multiplicative structure is additive in this grading: $A_j\cdot A_k\subset A_{j+k}$. For example, $\mathbb{R}[x]$ is graded by the degree; the tensor algebra $T(V)$ is graded by degree of tensors, and $\Lambda^\bullet V$ is graded with the n^{th} piece equal to the space of n-forms.

The Clifford algebra is not graded: the square of a vector is a scalar. It admits a weaker structure, called a filtration.

Definition 3.5. An algebra A has a *filtration* (by $\mathbb{Z}$) if there is a sequence of subspaces $A^{(0)}\subset A^{(1)}\subset A^{(2)}\subset\cdots$ such that $A=\bigcup_j A^{(j)}$ and $A^{(j)}\cdot A^{(k)}\subset A^{(j+k)}$.

The key difference is that for a filtration, the different levels can intersect in more than 0.

The Clifford algebra is filtered, with $\mathrm{C}\ell(V)^{(j)}=\operatorname{span} v_1\cdots v_k\mid k\le j, v_1,\dots,v_k\in V$, the span of products of at most j vectors.

Another way we can weaken the pined-for $\mathbb{Z}$-grading is to a $\mathbb{Z}/2$-grading, which we can actually put on the Clifford algebra.

Definition 3.6. A $\mathbb{Z}/2$*-grading* of an algebra A is a decomposition $A=A^+\oplus A^-$ as vector spaces, such that $A^+A^+\subset A^+$, $A^+A^-\subset A^-$, $A^-A^+\subset A^-$, and $A^-A^-\subset A^+$. A^- is called the *odd part* or the *negative part* of A, and A^+ is called the *even part* or the *positive part*. In physics, a $\mathbb{Z}/2$-graded algebra is also called a *superalgebra*.

For the Clifford algebra, let $\mathrm{C}\ell(V)^+$ be the subspace spanned by products of odd numbers of vectors, and $\mathrm{C}\ell(V)^-$ be the subspace spanned by products of even numbers of vectors. Then, $\mathrm{C}\ell(V)=\mathrm{C}\ell(V)^+\oplus\mathrm{C}\ell(V)^-$, and this defines a $\mathbb{Z}/2$-grading.

Definition 3.7. Let $A=\bigcup_j A^{(j)}$ be a filtered algebra. Then, its *associated graded* is

$$\operatorname{gr} A=\bigoplus_j A^{(j)}/A^{(j-1)},$$

which is naturally a graded algebra with $(\operatorname{gr}A)^j=A^{(j)}/A^{(j-1)}$ and multiplication inherited from A.

Proposition 3.8. *The associated graded of the Clifford algebra* $\operatorname{gr}\mathrm{C}\ell(V)=\Lambda^\bullet V$.

This ultimately follows because the isomorphism $\mathrm{C}\ell(V)^{(j)}/\mathrm{C}\ell(V)^{(j-1)}\to\Lambda^j V$ sends $u\mapsto\sigma(u)_[j]$: the exterior algebra remembers the top part of the Clifford multiplication.[11]

[11] There is a sense in which this defines the Clifford algebra as a deformation of the exterior algebra; a fancy word for this would be *filtered quantization*. Similarly, we'll see that the symmetric algebra is the associated graded of the symmetric algebra.

Constructing spin groups. Now, we assume that the inner product on V is positive definite.

If $v \neq 0$ in V, then v^{-1} exists in $C\ell(V)$ and is equal to $-v/\langle v, v\rangle$. For a $w \in V$, let $\rho_v(W)$ be conjugation: $\rho_v(W) = -vwv^{-1}$. Then, $\rho_v(v) = -v$, and if $w \perp v$, then $\rho_v(w) = -vwv^{-1} = wvv^{-1} = w$, so ρ_v preserves span $v^\perp$ and sends $v \mapsto -v$. Thus, it's a reflection across span v $\perp$.

Theorem 3.9. *The orthogonal group* $\mathrm{O}(n)$ *is generated by reflections, and everything in* $\mathrm{SO}(n)$ *is a product of an even number of reflections.*

Proof. Let's induct on n. When $n = 1$, $\mathrm{O}(1) = \{\pm 1\}$, for which this is vacuously true. Now, let $A \in \mathrm{O}(n)$. If A fixes an $e_1 \in \mathbb{R}^n$, then A fixes span $e_1^\perp$, so by induction, $A|_{\operatorname{span} e_1^\perp} = R_1 \cdots R_k$ for some reflections $R_1, \ldots, R_k \in \mathrm{O}(n-1)$. These reflections include into $\mathrm{O}(n)$ by fixing span e_1, and are still reflections, so $A = R_1 \cdots R_k$ decomposes A as a product of reflections.

Alternatively, suppose $Ae_1 = v \neq e_1$. Let R be a reflection about $\{v - e_1\}^\perp$; then, R exchanges v and e_1. Hence, $RA \in \mathrm{O}(n)$ and fixes e_1, so by above $RA = R_1 \cdots R_k$ for some reflections, and therefore $A = RR_1 \cdots R_k$ is a product of reflections.

For $\mathrm{SO}(n)$, observe that each reflection has determinant -1, but all rotations in $\mathrm{SO}(n)$ have determinant 1, so no $A \in \mathrm{SO}(n)$ can be a product of an odd number of rotations. ⊠

We've defined reflections ρ_v in the Clifford algebra, so if we can act by orientation-preserving reflections with a $\mathbb{Z}/2$ kernel, we should have described the spin group.

This reflection ρ_v is a restriction of the *twisted adjoint action*, a representation of $C\ell(V)^\times$ on $C\ell(V)$: $u_1 \mapsto \rho_{u_1}$ that sends $u_2 \mapsto \alpha(u_1)u_2u_1^{-1}$ for a $u_1 \in C\ell(V)^\times$ and $u_2 \in C\ell(V)$. Here,

$$\alpha(u_1) = \begin{cases} u_1, & u_1 \in C\ell(V)^+ \\ -u_1, & u_1 \in C\ell(V)^-. \end{cases}$$

We showed that for $v \in V \setminus 0$, ρ_v preserves V and is a reflection; since $\rho_{cv} = \rho_v$ for $c \in \mathbb{R} \setminus 0$, we want to restrict to the unit circle of v such that $\langle v, v\rangle = 1$. But we will restrict further.

Definition 3.10. Let $\mathrm{Spin}(V)$ denote the subgroup of $C\ell(V)^\times$ consisting of products of even numbers of unit vectors.

First question: what scalars lie in the spin group? Clearly ± 1 come from u^2 for unit vectors u, but we can do no better (after all, unit length is a strong condition on the real line).

Proposition 3.11. $\mathrm{Spin}(V) \cap \mathbb{R} \setminus 0 = \{\pm 1\}$ *inside* $C\ell(V)^\times$.

Theorem 3.12. *The map* $\mathrm{Spin}(V) \to \mathrm{SO}(V)$ *sending* $u \mapsto \rho_u$ *is a nontrivial (connected) double cover when* $\dim V \geq 2$.

This implies $\mathrm{Spin}(V)$ is the unique connected double cover of $\mathrm{SO}(V)$, agreeing with the abstract construction for the spin group we constructed in the first two lectures.

Proof. We know $\rho_u \in \mathrm{SO}(V)$ because it's an even product of reflections, using Theorem 3.9, and that ρ is surjective. We also know $\ker\rho = \{u \in \mathrm{Spin}(V) \mid uv = vu \text{ for all } v \in V\}$. But since V generates $C\ell(V)$ as an algebra, $\ker(\rho) = \mathrm{Spin}(V) \cap Z(C\ell(V)) = \{\pm 1\}$ by Propositions 3.4 and 3.11.

Thus ρ is a double cover, so it remains to show it's nontrivial. To rule this out, it suffices to show that we can connect -1 and 1 inside $\mathrm{Spin}(V)$, because they project to the same rotation. Let $\gamma(t) = \cos(\pi t) + \sin(\pi t)e_1e_2$ (since $\dim V \geq 2$, I can take two orthogonal unit vectors). Thus, $\gamma(t) = 1$, $\gamma(1) = -1$, and $\gamma(t) = e_1(-\cos(\pi t)e_1 + \sin(\pi t)e_2)$, so it's always a product of even numbers of unit vectors, and thus a path within $\mathrm{Spin}(V)$. ⊠

This is actually the simplest proof that $\dim \mathrm{Spin}(V) = \dim \mathrm{SO}(V)$.

Next week, we'll discuss representations of the spin group.

Lecture 4.

Representations of $\mathfrak{so}(n)$ and $\mathrm{Spin}(n)$: 9/6/16

One question from last time: we constructed $\mathrm{Spin}(n)$ as a subset of the group of units of a Clifford algebra, but how does that induce a linear structure? There's two ways to do this. The first is to say that this *a priori* only constructs $\mathrm{Spin}(n)$ as a topological group; this group double covers $\mathrm{SO}(n)$, and hence must be a Lie group.

Alternatively, this week, we'll explicitly realize $\mathrm{Spin}(n)$ as a closed subgroup of a matrix group, which therefore must be a Lie group.

Last time, we constructed the spin group $\mathrm{Spin}(V)$ as a subset of the units $\mathrm{C}\ell(V)^\times$, and found a double cover $\mathrm{Spin}(V(\to \mathrm{SO}(V)$. Thus, there should be an isomorphism of Lie algebras $\mathfrak{spin}(V) \xrightarrow{\sim} \mathfrak{so}(V)$. The former is a subspace of $T_1 \mathrm{C}\ell(V) \cong \mathrm{C}\ell(V)$ (since $\mathrm{C}\ell(V)$ is an affine space, as a manifold) and $\mathfrak{so}(V) \subset \mathfrak{gl}(V) = V \otimes V^*$ (and with an inner product, is also identified witrh $V \otimes V$). This identification extends to an isomorphism (of vector spaces) $\mathfrak{so}(V) \cong \Lambda^2 V$; composing with the inverse of the symbol map defines a map $\mathfrak{so}(V) \to \Lambda^2 V \to \mathrm{C}\ell(V)$.

Exercise 4.1. $\mathfrak{so}(V)$ and $\mathrm{C}\ell(V)$ both have Lie algebra structures, the former as a Lie group and the latter from the usual commutator bracket. Show that these agree, so the above map is an isomorphism of Lie algebras, and that the image of this map is $\mathfrak{spin}(V)$.

Today, we're going to discuss the representation theory (over $\mathbb{C}$) of the Lie algebra $\mathfrak{so}(V)$. Since $\mathrm{Spin}(V)$ is the simply connected Lie group with $\mathfrak{so}(V)$ as its Lie algebra, this provides a lot of information on the representation theory of $\mathrm{Spin}(V)$. In general, not all of these representations arise as representations on $\mathrm{SO}(n)$: consider the representation $\mathrm{Spin}(3) = \mathrm{SU}(2)$ on$\mathbb{C}^2$ where -1 exchanges $(1,0)$ and $(0,1)$. This doesn't descend to $\mathrm{SO}(3)$, because -1 is in the kernel of the double cover map. Such a representation is called a *spin representation*.

The name comes from physics: traditionally, physicists idenfied a Lie group with its Lie algebras, but they found that these kinds of representations didn't correspond to $\mathrm{SO}(3)$-representations. These arose in physical systems as particles with spin, in quantum mechanics:[12] a path connected -1 and 1 in $\mathrm{SU}(2)$ is a "rotation" of $260°$, but isn't the identity.

Anyways, we're going to talk about the representation theory of this group; in order to do so, we should briefly discuss the representation theory of Lie groups and (semisimple) Lie algebras.

Definition 4.2. Fix $\mathbb{F}$ to be either $\mathbb{R}$ or $\mathbb{C}$.

- An $\mathbb{F}$-*representation of a Lie group* G is a Lie group homomorphism $\rho : G \to \mathrm{GL}(V, \mathbb{F})$, where V is an $\mathbb{F}$-vector space.
- An $\mathbb{F}$-*representation of a real Lie algebra* $\mathfrak{g}$ is a real Lie algebra homomorphism $\tau : \mathfrak{g} \to \mathfrak{gl}(V, k)$, where V is an $\mathbb{F}$-vector space.

We will often suppress the notation as $\rho(g)v = g \cdot v$ or $\tau(X)v = Xv$, where $g \in G$, $X \in \mathfrak{g}$, and $v \in V$, when it is unambiguous to do so. Moreover, our representations, at least for the meantime, will be finite-dimensional.

Proposition 4.3. *Let $\mathfrak{g}$ be a real Lie algebra and V be a complex vector space. Then, there is a one-to-one correspondence between representations of $\mathfrak{g}$ on V and the $\mathbb{C}$-Lie algebra homomorphisms $\mathfrak{g} \otimes \mathbb{C} \to \mathfrak{gl}(V, \mathbb{C})$.*

Here, $\mathfrak{g} \otimes \mathbb{C}$ is the complex Lie algebra whose underlying vector space is $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ with bracket extending complex linearly from the assignment

$$[X \otimes c_1, Y \otimes c_2] = [X, Y] \otimes c_1 c_2,$$

where $X, Y \in \mathfrak{g}$ and $c_1, c_2 \in \mathbb{C}$.

Proof of Proposition 4.3. Let ρ be a $\mathfrak{g}$-representation on V; then, define $\rho_{\mathbb{C}} : \mathfrak{g} \otimes \mathbb{C} \to \mathfrak{gl}(V, \mathbb{C})$ to be the unique map extending $\mathbb{C}$-linearly from $X \otimes c \mapsto c\rho(X)$.

Conversely, given a complex representation $\rho_{\mathbb{C}}$, define $\rho : \mathfrak{g} \to \mathfrak{gl}(V, \mathbb{C})$ to be $X \mapsto \rho_{\mathbb{C}}(X \otimes 1)$. ⊠

Given a Lie group representation $G \to \mathrm{GL}(V, \mathbb{F})$, one obtains a Lie algebra representation of $\mathfrak{g} = \mathrm{Lie}(G)$ by differentiation.

Proposition 4.4. *If G is a connected, simply-connected Lie group, then this defines a bijective correspondence between the Lie group representations of G and the Lie algebra representations of $\mathfrak{g}$.*

If G is connected, but not simply connected, let $\widetilde{G}$ denote its universal cover. Then, there's a discrete central subgroup $\Gamma \le Z(\widetilde{G}) \le G$ such that $G = \widetilde{G}/\Gamma$. This allows us to extend Proposition 4.4 to groups that may not be simply connected.

Proposition 4.5. *Let G be a connected Lie group, $\widetilde{G}$ be its universal cover, and Γ be such that $G = \widetilde{G}/\Gamma$. Then, differentiation defines a bijective correspondence between the representations of G and the representations of $\widetilde{G}$ on which Γ acts trivially.*

[12] There is one macroscopic example of spin-1/2 phenomena: see http://www.smbc-comics.com/?id=2388.

It would also be nice to understand when two representations are the same. More generally, we can ask what a homomorphism of two representations are.

Definition 4.6. Let G be a Lie group. A *homomorphism of G-representations* from $\rho_1 : G \to \mathrm{GL}(V)$ to $\rho_2 : G \to \mathrm{GL}(W)$ is a linear map $T : V \to W$ such that for all $g \in G$, $T \circ \rho_1(g) = \rho_2(g) \circ T$. If T is an isomorphism of vector spaces, this defines an *isomorphism of G-representations*.

Example 4.7.

(1) $\mathrm{SO}(n)$ can be defined as a group of $n \times n$ matrices, which act by matrix multiplication on $\mathbb{C}^n$. This is a representation, called its *defining representation*. This works for every matrix group, including $\mathrm{SL}(n)$ and $\mathrm{SU}(n)$.

(2) The determinant is a smooth map $\det : \mathrm{GL}(n, \mathbb{C}) \to \mathbb{C}^\times$ such that $\det(AB) = \det A \det B$, hence a Lie group homomorphism. Since $\mathbb{C}^\times = \mathrm{GL}(1, \mathbb{C})$, this is a one-dimensional representation of $\mathrm{GL}(n, \mathbb{C})$.

(3) Fix a $c \in \mathbb{C}$ and let $\rho_c : \mathbb{R} \to \mathfrak{gl}(1, \mathbb{C})$ send $t \mapsto ct$. We can place a Lie algebra structure on $\mathbb{R}$ where $[\cdot, \cdot] = 0$, so that ρ defines a Lie algebra representation.

The simply connected Lie group with this Lie algebra is $(\mathbb{R}, +)$, and ρ_c integrates to the Lie group representation $(\mathbb{R}, +) \to \mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^\times$ sending $s \mapsto e^{isc}$. But S^1 has the same Lie algebra as $\mathbb{R}$, and the covering map is the quotient $\mathbb{R} \twoheadrightarrow \mathbb{R}/2\pi\mathbb{Z}$. In particular, this acts trivially iff $c \in \mathbb{Z}$, which is precisely when $s \mapsto e^{isc}$ is 2π-periodic.

There are various ways to build new representations out of old ones.

Definition 4.8. Let G be a Lie group and V and W be representations of G.

- The *direct sum* of V and W is the representation on $V \otimes W$ defined by
$$g \cdot (v, w) = (g \cdot v, g \cdot w).$$
- The *tensor product* is the representation on $V \otimes W$ extending uniquely from
$$g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w).$$
- The *dual representation* to V is the representation on V^* (the dual vector space) in which g acts as its inverse transpose on $\mathrm{GL}(V^*)$.

The same definition applies *mutatis mutandis* when the Lie group G is replaced with a Lie algebra $\mathfrak{g}$, and the inverse transpose is replaced with -1 times the transpose for the dual representation.

Note that, unlike for vector spaces, it can happen that a representation isn't isomorphic to its dual, even after picking an inner product.

Definition 4.9. Let V be a representation of a group G.

- A *subrepresentation* is a subspace $W \subset V$ such that $g \cdot w \in W$ for all $w \in W$ and $g \in G$.
- V is *irreducible* if it has no nontrivial subrepresentations (here, nontrivial means "other than $\{0\}$ and V itself"). Sometimes, "irreducible representation" is abbreviated "irrep" at the chalkboard.

These definitions apply *mutatis mutandis* to representations of a Lie algebra $\mathfrak{g}$.

In nice cases, knowing the irreducible representations tells you everything.

Theorem 4.10. *For $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$, there are finitely many isomorphism classes of irreducible representations, and every representation is isomorphic to a subrepresentation of a direct sum of tensor products of these representations.*

Definition 4.11. A Lie algebra $\mathfrak{g}$ whose representations have the property from Theorem 4.10 is called *semisimple*.[13]

In fact, we know these irreducibles explicitly: for n even, all of the irreducible representations of $\mathfrak{so}(n, \mathbb{C})$ are exterior powers of the defining representation, except for two *half-spinor representations*; for n odd, we just have one spinor representation.

[13] This is equivalent to an alternate definition, where $\mathfrak{g}$ is *simple* if $\dim \mathfrak{g} > 1$ and $\mathfrak{g}$ has no nontrivial ideals, and $\mathfrak{g}$ is *semisimple* if it is a direct sum of simple Lie algebras.

Constructing the spin representations. Let V be an n-dimensional vector space over $\mathbb{R}$ with a positive definite inner product. We'll construct the spinor representations of $\mathrm{Spin}(V)$ as restrictions of the $\mathrm{C}\ell(V)$ action on a $\mathrm{C}\ell(V)$-module, which act in a way compatible with the $\mathbb{Z}/2$-grading on $\mathrm{C}\ell(V)$.

Recall that a superalgebra is a scary word for a $\mathbb{Z}/2$-graded algebra.

Definition 4.12. Let $A = A^+ \oplus A^-$ be a superalgebra. A $\mathbb{Z}/2$-*graded module* over A (or a *supermodule* for A) is an A-module with a vector-space decomposition $M = M^+ \oplus M^-$ such that $A^\pm M^\pm \subset M^+$ and $A^\pm M^\mp \subset M^0$.

Example 4.13. One quick example is that every superalgebra acts on itself by multiplication; this *regular representation* is $\mathbb{Z}/2$-graded by the product rule on a superalgebra.

Since $\mathrm{Spin}(V) \subset \mathrm{C}\ell(V)^+$, any supermodule defines two representations of $\mathrm{Spin}(V)$, one on M^+ and the other on M^-.

Since we just care about complex representations, we may as well complexify the Lie algebra, looking at $\mathrm{C}\ell(V) \otimes \mathbb{C}$.

Exercise 4.14. Show that $\mathrm{C}\ell(V) \otimes \mathbb{C} \cong \mathrm{C}\ell(V \otimes \mathbb{C})$ (the latter is the Clifford algebra on a complex vector space).

Working with this complexified Clifford algebra simplifies things a lot.

First, let's assume $n = 2m$ is even. Then, we may choose an *orthogonal complex structure* J on V, i.e. a linear map $J : V \to V$ such that $J^2 = 1$ and $\langle Jv, Jw\rangle = \langle v, w\rangle$. For example, if $\{e_1, f_1, \dots, e_m, f_m\}$ is an orthogonal basis for V, then we can define $J(e_j) = f_j$ and $J(f_j) = -e_j$. Thus, such a structure always exists; conversely, given any orthogonal complex structure J, there exists a basis on which J has this form. In other words, J allows V to be thought of as an n-dimensional complex vector space.

We'll return to this on Thursday, using it to construct the supermodule.