Day 1: four perspectives on characteristic classes

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Today's plan

- Brief review/introduction to vector bundles and principal G-bundles
- What are characteristic classes? And why?
- Brief introduction different perspectives on Chern classes (axiomatic definition; linear dependency of sections; Chern-Weil theory; classifying spaces)

This week's plan

- Today: four perspectives on characteristic classes
- Tomorrow: Stiefel-Whitney classes (real vector bundles, mod 2 cohomology)
- Wednesday: Steenrod squares and Wu classes (more mod 2 cohomology)
- Thursday: Chern, Pontrjagin, and Euler classes (real and complex vector bundles, $\mathbb{Z}$ cohomology)
- Friday: Chern-Weil theory (de Rham cohomology)

Vector bundles

- Idea: a continuously varying, locally trivial family of vector spaces over a base space X
- Formally, a map $\pi\colon V \to X$ such that there exists an open cover $\mathfrak{U}$ of X and homeomorphisms $\varphi_U\colon \pi^{-1}(U) \to U \times \mathbb{R}^n$ which commute with to the projections down to X
- and, for all pairs $U, V \in \mathfrak{U}$, the *transition function* $g_{UV} := \varphi_V \varphi_U^{-1}$ is $\mathrm{GL}_n(\mathbb{R})$-equivariant
- Finally, the *cocycle condition* on triple intersections $U \cap V \cap W$: $g_{WU} g_{VW} g_{UV} = \mathrm{id}$
- This defines a real vector bundle, and its *rank* is n; using $\mathbb{C}^n$ gives a complex vector bundle
  - Rank 1 vector bundles are called *line bundles*

Examples of vector bundles

- On any smooth manifold M, the tangent bundle $TM \to M$ and the cotangent bundle $T^*M \to M$
- Direct sums, tensor products, duals, etc. of vector bundles
  - but: not an abelian category
- The *pullback* of a vector bundle $V \to X$ by a continuous map $f\colon Y \to X$ is a vector bundle $f^*V \to Y$ whose fiber at $y \in Y$ is $V_{f(y)}$
- The *tautological bundle* $S \to \mathbb{RP}^n$ or $S \to \mathbb{CP}^n$, a real, resp. complex line bundle
  - A point x of $\mathbb{RP}^n$ is a line in $\mathbb{R}^{n+1}$; the fiber of S at x is that line
  - Generalizes to Grassmannians and tautological vector bundles

Vector bundles: when are we gonna have to use this?

- Much of differential geometry is stated and proven in terms of vector bundles (and things called connections on them): TM and T^*M, but also spinor bundles and the like
- In general, vector bundles interpolate between geometry and homotopy theory
  - They feel more like geometric objects (especially if you choose a connection)
  - ...but their classification depends only on the homotopy type of X
  - Upshot: allows information from differential geometry to be used in homotopy theory and vice versa!

G-torsors

- For the next few slides, G is any topological group
- A *G-torsor* is a space X with a free transitive right G-action
- Choosing a basepoint on X provides an identification $X \cong G$ — but the point is, we (usually) have no canonical choice

G-torsors: examples

- Circles are SO_2-torsors, lines are $\mathbb{R}$-torsors
- Affine n-space $\mathbb{A}^n$ is an $\mathbb{R}^n$-torsor
- The set of bases of a vector space V is a $GL(V)$-torsor
- The set of orientations on an orientable manifold M is an $H^0(M; \mathbb{Z}/2)$-torsor

Principal G-bundles

- A principal G-bundle $P \to X$ is a continuously varying family of G-torsors over X
- (so, local trivializations, continuous transition maps…)
- $\implies$ the map $P \to X$ is the quotient map for the G-action

Examples of principal bundles

- On an n-manifold M, the *frame bundle* $\mathscr{B}_{\mathrm{GL}}(M) \to M$ is the principal $\mathrm{GL}_n(\mathbb{R})$-bundle whose fiber at x is the $\mathrm{GL}_n(\mathbb{R})$-torsor of bases of T_xM
- The *orientation bundle* over a manifold M has fiber at x equal to the set of orientations of a small neighborhood of x.
  - A principal $\mathbb{Z}/2$-bundle
  - A trivialization is an orientation of M
- Unlike for vector bundles, principal G-bundles are nonlinear, so no duals, direct sums, etc.
- Like vector bundles, principal bundles pull back: given $P \to X$ and $f\colon Y \to X$, define $f^*P \to Y$ to have fiber at y equal to the fiber of P at $f(y)$

The associated bundle construction

- Input data: $P \to X$ a principal G-bundle and V a G-representation
- Output data: a vector bundle $P \times_G V \to X$, defined to be the quotient of $P \times V$ by the equivalence relation $(p \cdot g, v) \sim (p, g \cdot v)$
- Intuition: “using up” the G-actions on P and V, or maybe using the ways in which P is twisted to build a vector bundle twisted in the same ways
- Useful in geometry, where geometric aspects of a vector bundle you care about are secretly controlled by a principal G-bundle via this construction

Associated bundles

- Taking the associated bundle of a principal O_n-bundle and the standard representation of O_n on $\mathbb{R}$ defines a bijection between isomorphism classes of principal O_n-bundles over a space and rank-n real vector bundles over a space
- Likewise with U_n and complex vector bundles; can instead use $GL_n(\mathbb{R})$ and $GL_n(\mathbb{C})$ if you want
- More directly, the Gram-Schmidt algorithm defines a bijection between isomorphism classes of $GL_n(\mathbb{R})$-bundles and O_n-bundles (resp. $GL_n(\mathbb{C})$ and U_n)

Principal bundles: when are we gonna have to use this?

- Vector bundles of interest in differential geometry are all associated bundles for the bundle of frames: TM, T^*M, exterior powers; spinor bundles; and more
  - Often, one gets info on the bundle of frames, then uses the associated bundle construction to propagate that information to several vector bundles at once
- Useful for slickly defining orientations, spin structures, spin^c structures, ...
- Gauge theory is all about connections on principal bundles, both in math and in physics

Characteristic classes

- Fix some kind of vector or principal bundle (e.g. complex vector bundle; principal SU_2-bundle; etc.), a $d \in \mathbb{N}$, and a commutative ring A
- A *characteristic class* for these bundles is a procedure for associating to each bundle $E \to X$ a cohomology class $c(P) \in H^d(X; A)$ which is natural under pullback
- Naturality: given $f : Y \to X$, need $c(f^*E) = f^*c(E)$ in $H^d(Y; A)$

Ok, but why?

- More algebraic invariants of geometric or topological information
- Often detect or obstruct useful topological or geometric properties (orientability, flatness, null-bordism, ...)
- Sweet spot in “conservation of effort:” the best things in algebraic topology are both informative *and computable*

Approach 1: the axiomatic definition of Chern classes

- Chern classes are characteristic classes $c_i(V) \in H^{2i}(X;\mathbb{Z})$ for complex vector bundles $V \to X$, $i \geq 0$.
- Define them to satisfy a short list of axioms; it is a theorem of Grothendieck this uniquely characterizes them

Approach 1: the axioms

- (implicit: naturality)
- $c_0(E) = 1$
- the *Whitney sum formula* $c(E \oplus F) = c(E)c(F)$
  - Here $c(E)$ is the *total Chern class* $c_0(E) + c_1(E) + c_2(E) + \dots$
- Nontriviality: if x is the generator of $H^2(\mathbb{CP}^n; \mathbb{Z}) \cong \mathbb{Z}$, then $c(S \to \mathbb{CP}^n) = 1 - x$.
  - Here $S \to \mathbb{CP}^n$ is the tautological line bundle
  - Use the orientation of $\mathbb{CP}^n$ to pick a specific isomorphism $H^2(\mathbb{CP}^n; \mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}$

Approach 1: advantages and disadvantages

- Succinct, but no intuition for what Chern classes *are*
- Can make some computations: if $E \oplus \underline{\mathbb{C}}^k$ is trivial, $c(E) = 1$
  - Also some calculations on projective complex manifolds
  - ... but pretty inflexible for computations

Approach 2: linear dependency of generic sections

- Idea: at least on a manifold, produce submanifolds from the vector bundle data, giving classes in homology
- Then use Poincaré duality to turn these into cohomology classes
- It is a theorem that every complex vector bundle on a finite-dimensional CW complex pulls back from a vector bundle on an oriented manifold, so this suffices

Approach 2: Poincaré duality is best theorem

- An orientation of a closed n-manifold M determines a *fundamental class*, an element $[M] \in H_n(M;\mathbb{Z})$ (well, with any coefficients)
- Cap product with this class defines an isomorphism called *Poincaré duality* $H^k(M;\mathbb{Z}) \to H_{n-k}(M;\mathbb{Z})$
- For $k = n$ this corresponds to integration in de Rham theory
- With field coefficients, the universal coefficient theorem reformulates this as a duality pairing $H^k(M;k) \otimes_k H^{n-k}(M;k) \to k$
- If you only need $\mathbb{Z}/2$ coefficients, no orientation is necessary

Defining Chern classes using Poincaré duality

- Given a closed, oriented n-manifold M and a complex vector bundle $V \to M$, choose k generic sections $s_1, \ldots, s_k$ and let $N \subset M$ be the subset on which $s_1, \ldots, s_k$ are linearly dependent
  - e.g., if $k = 1$, N is the zero set of s_1
- For a generic choice of $s_1, \ldots, s_k$, N is a closed, oriented submanifold of dimension $n - 2k$. Push its fundamental class forward to define $[N] \in H_{n-2k}(M)$
  - "Generic" means suitable transversality hypotheses, etc., and is satisfied on a subset of full measure (as usual with such constructions in differential topology)
  - The homology class $[N]$ does not depend on any of the choices we made
- Now, the k^{th} *Chern class* of V is the Poincaré dual of $[N]$, which is a degree $n - (n - 2k) = 2k$ cohomology class

Approach 2: advantages and disadvantages

- Yay: makes clearer what Chern classes are measuring: an obstruction to linearly independent sections
- Non-functoriality of Poincaré duality means proving naturality is a headache
- That characteristic classes pull back from manifolds is not an obvious theorem

Approach 3: Chern-Weil theory

- Define Chern classes in de Rham theory, using concepts from differential geometry
- This is how characteristic classes tend to appear in quantum field theory

Approach 3: connections

- Let M be a smooth manifold and $V \to M$ be a real vector bundle. A *connection* on V is an $\mathbb{R}$-linear map $\nabla \colon \Gamma(TM) \otimes_{\mathbb{R}} \Gamma(V) \to \Gamma(V)$ which is $C^\infty(M)$-linear in the first argument and satisfies the *Leibniz rule*

$$\nabla_{v(f\psi)} = (v \cdot f)\psi + f\nabla_v\psi$$

  where $\nabla_v(\psi) := \nabla(v, \psi)$
- Idea: this is a way to differentiate sections of V
- Locally, $\nabla = \mathrm{d} + A$, where A is a "matrix-valued one-form," namely an element of $\Omega^1_M(\operatorname{End} E) := \Gamma(T^*M \otimes \operatorname{End} E)$
- Theorem: the space of connections on V is an infinite-dimensional affine space, and in particular nonempty

Approach 3: curvature

- The *curvature* of a connection is a matrix-valued 2-form $F_\nabla \in \Omega^2_M(\operatorname{End} E) := \Gamma(\Lambda^2 T^*M \otimes \operatorname{End} E)$ defined by the formula

$$\nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X - \nabla_{[X,Y]}$$

- Vector bundles are locally trivial, but connections are not, and the curvature measures this

Approach 3: Chern classes in de Rham cohomology

- The trace map induces a map $\mathrm{tr}\colon \Omega^k_M(\operatorname{End} E) \to \Omega^k_M$
- Given a complex vector bundle $V \to M$, choose a connection ∇ on V
- Wedge its curvature form together k times to get $(F_\nabla)^k \in \Omega^{2k}_M(\operatorname{End} E)$
- Take the trace, land in Ω^{2k}_M
- Theorem: this is a closed form, and its de Rham cohomology class does not depend on the choice of ∇!
- Define the k^{th} *Chern class* of V to be $(1/2\pi i)[\mathrm{tr}((F_\nabla)^k)] \in H^{2k}_{\mathrm{dR}}(M)$

Approach 3: advantages and disadvantages

- If you like geometry or quantum field theory, you'll probably like this approach the best
- Again, tells you something about what Chern classes actually are
- But doesn't work to get classes in integral cohomology, nor on non-manifolds, and it depends on choices
- Sometimes computable. Sometimes not.

Approach 4: the search for the universal bundle

- Idea: create a moduli space BG of principal G-bundles, carrying the “universal” or “maximally twisted” principal G-bundle $EG \to BG$
- key fact: every principal G-bundle $P \to X$ is isomorphic to $f^*EG \to X$ for some $f : X \to BG$, and moreover in a unique way (f is unique up to homotopy)
- Then, cohomology classes on BG give characteristic classes, and vice versa

Approach 4: classifying spaces

- Let G be a topological group. A *classifying space* for G, denoted BG, is any space which can be realized as the quotient of a contractible space EG by a free G-action
- Key facts about classifying spaces
  - Homotopy classes of maps $f : X \to BG$ are in natural bijection with isomorphism classes of principal G-bundles $P \to X$ via $f \mapsto (f^*EG \to X)$
  - BG always exists but is not unique! But any two choices are homotopy equivalent
  - BG is often infinite-dimensional

Approach 4: examples of classifying spaces

- S^1 is a $B\mathbb{Z}$, because $\mathbb{R}/\mathbb{Z} \cong S^1$
- $\mathbb{RP}^\infty$ is a $B\mathbb{Z}/2$, because S^∞ is contractible!
  - Can take either $S^\infty := \operatorname{colim}_n S^n$, or the unit sphere in an infinite-dimensional Banach space; this produces homotopy-equivalent but non-homeomorphic models for $B\mathbb{Z}/2$
  - Similarly, $\mathbb{CP}^\infty$ is a $B\mathrm{U}_1$
- Classifying spaces of Lie groups tend to look like infinite-dimensional Grassmannians

Approach 4: stabilization

- $H^*(B\mathrm{GL}_n(\mathbb{R}))$ gives characteristic classes of principal $\mathrm{GL}_n(\mathbb{R})$-bundles, hence real rank-n bundles via the associated bundle construction. So how do we obtain characteristic classes for all real bundles at once?
- The inclusion $\mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_{n+1}(\mathbb{R})$ induces maps $B\mathrm{GL}_n(\mathbb{R}) \to B\mathrm{GL}_{n+1}(\mathbb{R})$. Let $B\mathrm{GL}(\mathbb{R})$ denote the colimit of these maps
  - From the moduli POV, $B\mathrm{GL}_n(\mathbb{R}) \to B\mathrm{GL}_{n+1}(\mathbb{R})$ sends a vector bundle $V \to X$ to $V \oplus \underline{R} \to X$
- Thus $H^*(B\mathrm{GL}_n(\mathbb{R}))$ gives characteristic classes for all real vector bundles at once
- All this works for complex vector bundles as well

Approach 4: stabilization

- We've been using $\mathrm{GL}_n(\mathbb{R})$, but it's more common to see O_n in the literature
- There's essentially no difference (you do have to choose a metric to get a principal O_n-bundle of frames, but this is a contractible choice so don't worry about it)
- Likewise with $\mathrm{GL}_n(\mathbb{C})$ and U_n

Approach 4: defining Chern classes

- There is a theorem that $H^*(B\mathrm{GL}(\mathbb{C});\mathbb{Z}) \cong \mathbb{Z}[c_1, c_2, c_3, \dots]$ with $|c_i| = 2i$
- So we define the i^{th} Chern class of a complex vector bundle $V \to X$ to be f^*c_i, where $f\colon X \to B\mathrm{GL}(\mathbb{C})$ is in the homotopy class of maps classifying the stabilization of E

Approach 4: advantages and disadvantages

- It's nice to know that we got 'em all (for any G, or for real or complex vector bundles)
- BG is often useful for other reasons
- But, BG is mysterious, and often big
- We're also black-boxing how $H^*(BG)$ is actually computed

Foreshadowing the problem session

- Hands-on examples of vector bundles and principal bundles
- Seeing what we can compute with these different approaches
- A fifth perspective??