M392C NOTES: TOPICS IN ALGEBRAIC TOPOLOGY

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These notes were taken in UT Austin's M392C (Topics in Algebraic Topology) class in Spring 2017, taught by Andrew Blumberg. I live-TEXed them using vim, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Alternatively, these notes are hosted on Github at https://github.com/adebray/equivariant_homotopy_theory, and you can submit a pull request. Thanks to Rustam Antia-Riedel and Gill Grindstaff for catching a few errors, to Tom Gannon and Richard Wong for some clarifications, and to Yuri Sulyma for adding some remarks and references.

CONTENTS

Lecture 1.

G-spaces: 1/17/17

This class will be an overview of equivariant stable homotopy theory. We're in the uncomfortable position where this is a big subject, a hard subject, and one that is poorly served by its textbooks. Algebraic topology is like this in general, but it's particularly acute here. Nonetheless, here are some references:

- Adams, "Prerequisites (on equivariant stable homotopy) for Carlsson's lecture." [Ada84]. This is old, and some parts of it don't reflect how we do things now.
- The Alaska notes [May96], edited by May, is newer, and is written by many authors. Some of it is a grab bag, and some parts (e.g. the rational equivariant bits) aren't entirely right. It's also not a textbook.
- Appendix A of Hill-Hopkins-Ravenel [HHR16]. This is a paper which resolved an old conjecture on manifolds using equivariant stable homotopy theory, but let this be a lesson on referee reports: the authors were asked to provide more background, and so wrote a 150-page appendix on this material. Their suffering is your gain: the introduction is well-written, albeit again not a textbook.

In the world of the professor, there are two major applications of equivariant stable homotopy theory.

- The first is trace methods in algebraic K-theory: Hochschihd homology and its topological cousins are equipped with natural S^1-actions (the same S^1-action coming from field theory). This is how people other than Quillen compute algebraic K-theory.
- The other major application is Hill-Hopkins-Ravenel's settling of the Kervaire invariant 1 conjecture in [HHR16].

The nice thing is, however you feel about the applications, both applications require developing new theory in equivariant stable homotopy theory. Hill-Hopkins-Ravenel in particular required a clarification of the foundations of this subject which has been enlightening.

In this class, we hope to cover the foundations of equivariant stable homotopy theory. On the one hand, this will be a modern take, insofar as we emphasize the norm and the presheaf on orbit categories (these will be explained in due time), the modern emerging consensus on how to think of these things, different than what's written in textbooks. The former is old, but has gained more attention recently; the latter is new. Moreover, there's an increasing sense that a lot of the foundations here are best done in ∞-categories. We will not take this appoach in order to avoid getting bogged down in ∞-categories; moreover, this class is supposed to be rigorous. It will sometimes be clear to some people that ∞-categories lie in the background, but we won't talk very much about them.

We'll cover some old topics such as Smith theory and the Segal conjecture, and newer ones such as trace methods and Hill-Hopkins-Ravenel, depending on student interest. We will not have time to discuss many topics, including equivariant cobordism or equivariant surgery theory.

Prerequisites. If you don't know these prerequisites, that's okay; it means you're willing to read about it on your own.

- Foundations of unstable homotopy theory at the level of May's *A Concise Course in Algebraic Topology* [May99]. For example, we'll discuss equivariant CW complexes, so it will help to know what a CW complex is.
- A little bit of category theory, e.g. found in Mac Lane [Mac78] or Riehl [Rie16].
- This class will not require much in the way of simplicial methods (simply because it's hard to reconcile simplicial methods with non-discrete Lie groups), but you will want to know the bar construction. An excellent source for this is [Rie14, Chapter 4].
- A bit of abstract homotopy theory, e.g. what a model structure is. Good sources for model categories are [Rie14, Part III] and [Hov99].

If you don't know these, feel free to ask the professor for references. His advisor suggested that a foundation for the stable category is Lewis-May-Steinberger's account [LMS86] of the equivariant category and let $G = *$, but perhaps this isn't necessarily a good reference for nontrivial groups.

Unstable equivariant questions are very natural, and somewhat reasonable. But stable questions are harder; they ultimately arise from reasonable questions, but the formulation and answers are hard: even discussing the equivariant analogue of $\pi_0 S^0$ requires some representation theory — and yet of course it should. Thus there's a lot of foundations behind hard calculations. There will be problem sets; if you want to learn the material (or are an undergrad), you should do the problem sets.

Categories of topological spaces. The category of topological spaces we consider is Top, the category of compactly generated, weak Hausdorff spaces (and continuous maps); we'll also consider Top_*, the category of based, compactly generated, weak Hausdorff spaces and continuous, based maps. This is an important and old trick which eliminates some pathological behavior in quotients. It's reasonable to imagine that point-set topology shouldn't be at the heart of foundational issues, but there are various ways to motivate this, e.g. to make Top more resemble a topos or the category of simplicial sets.

Definition 1.1. Let X be a topological space.

- A subset $A \subseteq X$ is **compactly closed** if $f^{-1}(A)$ is closed for every $f : Y \to X$, where Y is compact and Hausdorff.
- X is **compactly generated** if every compactly closed subset of X is closed.
- X is **weak Hausdorff** if the diagonal map $\Delta : X \to X \times X$ is closed when $X \times X$ has the compactly generated topology.

The intuition behind compact generation is that the topology is determined by compact Hausdorff spaces. The weak Hausdorff topology is strictly stronger than T_1 (points are closed), but strictly weaker than Hausdorff spaces. Any space you can think of without trying to be pathological will meet these criteria.

There is a functor k from all spaces to compactly generated spaces which adds the necessary closed sets. This has the unfortunate name of **k-ification** or **kaonification**; by putting the compactly generated topology on $X \times X$, we mean taking $k(X \times X)$. There's also a "weak Hausdorffification" functor w which makes a space weakly Hausdorff, which is some kind of quotient.[1]

When computing limits and colimits, it's often possible to compute it in the category of spaces and then apply k and w to return to Top. This is fine for limits, but for colimits, w is particularly badly behaved: you cannot compute the colimit in Top by computing it in Set and figuring out the topology; more generally, it will be some kind of quotient.

Nonetheless, there are nice theorems which make things work out anyways.

Proposition 1.2. *Let $Z = \operatorname{colim}(X_0 \to X_1 \to X_2 \to \dots)$ be a sequential colimit (sometimes called a **telescope**); if each X_i is weak Hausdorff, then so is Z.*

[1] The k functor is right adjoint to the forgetful map, which tells you what it does to limits.

Proposition 1.3. *Consider a diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \\ C & \end{tikzcd}$$

where f is a closed inclusion. If A, B, and C are weakly Hausdorff, then $B \amalg_A C$ is weakly Hausdorff.

These are the two kinds of colimits people tend to compute, so this is reassuring.

One reason we require regularity on our topological spaces is the following, which is not true for topological spaces in general.

Lemma 1.4. *Let X, Y, and Z be in* Top*; then, the natural map*

$$\mathrm{Map}(X \times Y, Z) \longmapsto \mathrm{Map}(X, \mathrm{Map}(Y, Z))$$

is a homeomorphism.

Enrichments. The categories Top and Top_* are enriched over themselves (as will categories of G-spaces, which we'll see later). This means a brief digression into enriched categories.

Definition 1.5. Let $(\mathsf{V}, \otimes, \mathbf{1})$ be a symmetric monoidal category.[2] Then, an **enrichment** of a category C over V means

- for every $x, y \in \mathsf{C}$, there is a hom-object $\underline{\mathsf{C}}(x, y)$, which is an object in V,
- for every $x \in \mathsf{C}$, there is a unit $\mathbf{1} \to \underline{\mathsf{C}}(x, x)$,
- composition $\underline{\mathsf{C}}(x, y) \otimes \underline{\mathsf{C}}(y, z) \to \underline{\mathsf{C}}(x, z)$ is associative and unital, and
- the underlying category is recovered as $\mathsf{C}(x, y) = \mathrm{Map}(\mathbf{1}, \underline{\mathsf{C}}(x, y))$.

A great deal of category theory can be generalized to enriched categories, including V-enriched functors, V-enriched natural transformations, V-enriched limits and colimits, and more. The canonical reference is Kelley [Kel84], available free and legally online. It covers just about everything we need except for the Day convolution, which can be read from Day's thesis [Day70]. Another good source, with a view towards homotopy theory, is [Rie14, Chapter 3].

Definition 1.6. Let C and D be enriched over V. Then, an **enriched functor** $F : \mathsf{C} \to \mathsf{D}$ is an assignment of objects in C to objects in D and maps $\underline{\mathsf{C}}(x, y) \to \underline{\mathsf{D}}(Fx, Fy)$ that are V-morphisms, and commute with composition.

Exercise 1.7. Work out the definition of enriched natural transformations.

This brings us to the beginning.

Let G be a group. We'll generally restrict to finite groups or compact Lie groups; this is not because these are the only interesting groups, but rather because they are the only ones we really understand. If you can come up with a good equivariant homotopy theory for discrete infinite groups, you will be famous. Throughout, keep in mind the examples C_p (the cyclic group of order p, sometimes also denoted $\mathbb{Z}/p$), C_{p^n}, the symmetric group S_n, and the circle group S^1.

There's a monad M_G on Top which sends $X \mapsto G \times X$, and analogously M_G^* on Top_* sending $X \to G_+ \wedge X$; then, one can define the category of ***G*-spaces** $G\mathsf{Top}$ (resp. **based *G*-spaces** $G\mathsf{Top}_*$) to be the category of algebras over M_G (resp. M_G^*). This is probably not the most explicit way to define G-spaces, but it makes it evident that $G\mathsf{Top}$ and $G\mathsf{Top}_*$ are complete and cocomplete.

More explicitly, $G\mathsf{Top}$ is the category of spaces $X \in \mathsf{Top}$ equipped with a continuous action $\mu : G \times X \to X$. That is, μ must be associative and unital. Associativity is encoded in the commutativity of the diagram

$$\begin{tikzcd} G \times G \times X \arrow[r, "1\times\mu"] \arrow[d, "m"] & G \times X \arrow[d, "\mu"] \\ G \times X \arrow[r, "\mu"] & X. \end{tikzcd}$$

[2] Briefly, this means V has a tensor product $\otimes$ and a unit $\mathbf{1}$; there are certain axioms these must satisfy.

The morphisms in GTop are the **G-equivariant** maps $f : X \to Y$, i.e. those commuting with μ:

$$\begin{tikzcd} G \times X \arrow[r] \arrow[d, "\mu_X"] & G \times Y \arrow[d, "\mu_Y"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

It's possible (but not the right idea) to let $\underline{G}$ denote[3] the category with an object $*$ such that $\underline{G}(*,*) = G$. Then, GTop is also the category of functors $\underline{G} \to$ Top, with morphisms as natural transformations. This realizes GTop as a **presheaf category**; it will eventually be useful to do something like this, but not in this specific way.

When we write $\mathrm{Map}(X,Y)$ in GTop or $G\mathrm{Top}_*$, we could mean three things:

(1) The set of G-equivariant maps $X \to Y$.
(2) The space of G-equivariant maps $X \to Y$ in the subspace topology of all maps from $X \to Y$. As this suggests, GTop admits an enrichment over Top (resp. $G\mathrm{Top}_*$ admits an enrichment over Top_*).
(3) The G-space of all maps $X \to Y$, where G acts by conjugation: $f \mapsto g^{-1}f(g\cdot)$. This means GTop is enriched in itself, as is $G\mathrm{Top}_*$.

Each of these is useful in its own way: for constructions it may be important to be self-enriched, or to only look at G-equivariant maps. We will let $\mathrm{Map}^G(X,Y)$ or $\mathrm{Map}(X,Y)$ denote (2) or its underlying set (1), and $G\,\mathrm{Map}(X,Y)$ denote (3).

It turns out you can recover Map^G from Map: the equivariant maps are the fixed points under conjugation of all maps. This is written $\mathrm{Map}(X,Y)^G = \mathrm{Map}^G(X,Y)$.

Throughout this class, "subgroup" will mean "closed subgroup" unless specified otherwise.

Definition 1.8. Let X be a G-set and $H \subseteq G$ be a subgroup. Then, the **H-fixed points** of X is the space $X^H := \{x \in X \mid hx = x \text{ for all } h \in H\}$. This is naturally a WH-space, where $WH = NH/H$ (here NH is the normalizer of H in G).[4]

Definition 1.9. The **isotropy group** of an $x \in X$ is $G_x := \{h \in G \mid hx = x\}$.

These are useful in the following two ways.

(1) Often, it will be helpful to reduce questions from GTop to Top using $(-)^H$.
(2) It's also useful to induct over isotropy types.

Now, we'll see some examples of G-spaces.

Example 1.10. Let H be a subgroup of G; then, the **orbit space** G/H is a useful example, because it corepresents the fixed points by H. That is, $X^H \cong G\,\mathrm{Map}(G/H, X)$. These spaces will play the role that points did when we build things such as equivariant CW complexes. ◀

Example 1.11. Let $H \subset G$ as usual and $U : G\mathrm{Top} \to H\mathrm{Top}$ be the forgetful functor. Then, U has both left and right adjoints:

- The left adjoint sends X to the **balanced product** $G \times_H X := G \times X/\sim$, where $(gh, x) \sim (g, hx)$ for all $g \in G$, $h \in H$, and $x \in X$. Despite the notation, this is *not* a pullback! (In the based case, the balanced product is $G_+ \wedge_H X$.) G acts via the left action on G. This is called the **induced G-action** on X.
- The right adjoint is $F_H(G, X)$ (or $F_H(G_+, X)$ in the based case), the space of H-maps $G \to X$, with G-action $(gf)(g') = f(g'g)$. This is called the **coinduced G-action** on H.[5] ◀

Remark. Here is a categorical perspective on "change of group." Quite generally, a group homomorphism $G \xrightarrow{f} H$ induces adjunctions

$$\begin{tikzcd} G\mathrm{Top} \arrow[r, shift left=3, "f_!"] \arrow[r, phantom, shift left=1.5, "\perp"] & H\mathrm{Top} \arrow[l, "f^*" description] \arrow[l, phantom, shift left=1.5, "\perp"] \end{tikzcd} \quad G\mathrm{Top} \underset{f_*}{\overset{f_!}{\rightleftarrows}} H\mathrm{Top}.$$

[3]There isn't really a standard notation for this category, but the closest is BG. This notation emphasizes the fact that groupoids are Quillen equivalent to 1-truncated spaces.

[4]If $H \trianglelefteq G$, then X^H is also a G/H-space.

[5]This actually *is* a group action, since if $a, b, g \in G$, then $(a(bf))(g) = (bf)(ga) = f(gab) = (ab(f))(g)$.

These are given by $f_!(X) = H \times_G X$ and $f_*(X) = F_G(H, X)$ for a G-space X, where H is given the structure of a G-space by f. When $H = *$, an H-space is just a space, and $f_!(X) = X_G$ is the space of orbits while $f_*(X) = X^G$ is the space of fixed points. Observe that similar statements hold for categories of modules, given a ring homomorphism $R \xrightarrow{f} S$.

In fact, these are both cases of very general abstract nonsense. Let BG denote the category with one object $*$ with $\mathrm{Hom}(*, *) = G$; as we have said above, we can (naïvely) write $G\mathsf{Top}$ as the functor category Top^{BG}. A group homomorphism $G \xrightarrow{f} H$ induces a functor $BG \xrightarrow{F} BH$ (it is not quite true that the two are equivalent—think about why this is). Now $f^*\colon H\mathsf{Top} \to G\mathsf{Top}$ is just restriction along F:

$$\begin{tikzcd}
BG \arrow[r, dotted, "f^*(Y)"] \arrow[d, "F"'] & \mathsf{Top} \\
BH \arrow[ur, "Y"'] &
\end{tikzcd}$$

According to abstract nonsense, restriction along F has a left and right adjoint, called *left and right Kan extension along F*:

$$\begin{tikzcd}
BG \arrow[r, "X"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\eta"] & \mathsf{Top} \\
BH \arrow[ur, dotted, "f_!(X) = \mathrm{Lan}_F X"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
BG \arrow[r, "X"] \arrow[d, "F"'] \arrow[dr, phantom, "\Uparrow\epsilon"] & \mathsf{Top} \\
BH \arrow[ur, dotted, "f_*(X) = \mathrm{Ran}_F X"'] &
\end{tikzcd}$$

These diagrams do not commute, but there are natural maps $X \overset{\eta}{\Longrightarrow} f^*f_!(X)$ and $f^*f_*(X) \overset{\epsilon}{\Longrightarrow} X$. When H is the trivial group, BH is the trivial category, and it is known that left/right Kan extensions of a functor X along a functor to the trivial category pick out the colimit/limit of X. That is, still viewing a G-space X as a functor $BG \to \mathsf{Top}$, we have $X_G = \operatorname*{colim}_{BG} X$ and $X^G = \lim_{BG} X$.

For an introduction to Kan extensions, we recommend [Rie16, Chapter 6] (which is almost the same as [Rie14, Chapter 1] but with some more amusing examples). Like much of category theory, this is ultimately all trivial, but it may be highly non-trivial to understand why it is trivial. ◀

Example 1.12. Let V be a finite-dimensional real representation of G, i.e. a real inner product space on which G acts in a way compatible with the inner product. (This is specified by a group homomorphism $G \to \mathrm{O}(V)$.) The one-point compactification of V, denoted S^V, is a based G-space; the unit disc $D(V)$ and unit sphere $S(V)$ are unbased spaces, but we have a quotient sequence

$$S(V)_+ \longrightarrow D(V)_+ \longrightarrow S^V.$$

If $V = \mathbb{R}^n$ with the trivial G-action, S^V is S^n with the trivial G-action, so these generalize the usual spheres; thus, these S^V are called **representation spheres**. ◀

We will let S^n denote $S^{\mathbb{R}^n}$, namely our preferred model for the n-sphere with trivial G-action.

Beginnings of homotopy theory.

Definition 1.13. A ***G*-homotopy** is a map $h\colon X \times I \to Y$ in $G\mathsf{Top}$, where G acts trivially on I. We generally think of it, as usual, as interpolating between $h(-, 0)$ and $h(-, 1)$. This is the same data as a path in $G\mathrm{Map}(X, Y)$. A ***G*-homotopy equivalence** between X and Y is a map $f : X \to Y$ such that there exists a $g : Y \to X$ such that there are G-homotopies $gf \sim \mathrm{id}_X$ and $fg \sim \mathrm{id}_Y$.

The (well, a) natural question that might arise: what are G-weak equivalences and G-CW complexes? This closely relates to obstruction theory — CW complexes are test objects.

To define G-CW complexes, we need cells. One choice is $G/H \times D^{n+1}$ and $G/H \times S^n$, where the actions on D^{n+1} and S^n are trivial. This is a plausible choice (and in fact, will be the right choice), but it's not clear why — why not $G \times_H D(V)$ or $G \times_H S(V)$ for some H-representation V? Ultimately, this comes from a (quite nontrivial) theorem that these can be triangulated in terms of the cells $G/H \times D^{n+1}$ and $G/H \times S^n$.[6] This is one of several triangulation results proven in the 1970s, which are now assumed without comment, but if you like this kind of math then it's a very interesting story.

[6] Illman's thesis [Ill72] is a reference, albeit not the most accessible one.

Definition 1.14. A **G-CW complex** is a sequential colimit of spaces X_n, where X_{n+1} is a pushout

$$\begin{tikzcd} \bigvee G/H \times S^n \arrow[r] \arrow[d] & X_n \arrow[d] \\ \bigvee G/H \times D^{n+1} \arrow[r] & X_{n+1}, \end{tikzcd}$$

where H varies over all closed subgroups of G.

That is, it's formed by attaching cells just as usual, though now we have more cells.

This immediately tells you what the homotopy groups have to be: $[G/H \times S^n, X]$, which by an adjunction game is isomorphic to $\pi_n(X^H)$. We let $\pi_n^H(X) := \pi_n(X^H)$. Thus, we can define weak equivalences.

Definition 1.15. A map $f : X \to Y$ of G-spaces is a **weak equivalence** if for all subgroups $H \subset G$, $f_* \colon \pi_n^H(X) \to \pi_n^H(Y)$ is an isomorphism.

These homotopy groups have a more complicated algebraic structure: they're indexed by the lattice of subgroups of G and the integers. This is fine (you can do homological algebra), but some things get more complicated, including asking what the analogue of connectedness is!

One quick question: do we need all subgroups H? What if we only want finite-index ones? The answer, in a very precise sense, is that if you're willing to use fewer subgroups, you get fewer cells $G/H \times S^n$, and that's fine, and you get a different kind of homotopy theory.

Finally, the Whitehead theorem is true for G-CW complexes. This follows for the same reason as in May's course: it follows word-for-word after proving the equivariant HELP lemma (homotopy extension lifting property), which is true by the same argument.

We'll next talk about presheaves on the orbit category, leading to Bredon cohomology.

Lecture 2.

Homotopy theory of G-spaces: 1/19/17

"It's nice to write down, but oh so false."

Last time, we saw the definition of a G-CW complex, but no examples were provided. Today, we'll start with some examples.

Recall that a G-CW complex is a sequential colimit $X = \operatorname{colim}_n X_n$, where X_n is formed by attaching cells $G/H \times D^n$ along maps $G/H \times S^{n-1} \to X_{n-1}$: just like the CW complexes we know and love, but with new cells G/H indexed by the closed subgroups $H \subset G$. The idea is that you're building up a space by attaching different spaces with different isotropy groups (G/H has isotropy group H, just by construction).

Example 2.1 (Zero-dimensional complexes)**.** The zero-dimensional complexes are G/H or disjoint unions $\coprod_i G/H_i$. This is an instance of the slogan that "orbits are points." Keep in mind that if G is a compact Lie group, this might not be zero-dimensional in other, more familiar kinds of dimension. ◂

Example 2.2. Let S^1 act on $\mathbb{R}^2$ by rotation along the origin. This also induces a C_n-action, as $C_n \subseteq S^1$ as the n^{th} roots of unity. Let V denote this C_n-space.

Let $D(V)$ denote the unit disc in V, and S^V denote its one-point compactification, a representation sphere. Then, $D(V)$ looks like wedges of pie, as the origin is fixed. On S^V, the point at infinity is also fixed, so we obtain a beachball.

Now let's consider V as an S^1-space, and write down the CW structure on S^V. There are two fixed points, and each one is a 0-cell $S^1/S^1 \times *$, but there is one 1-cell $S^1 \times I$ attached to the endpoints (thought of as a meridian rotated around the sphere).

Now let's consider the beachball for C_2 on S^V, where there are two hemispheres and C_2 rotates by a half-turn. What's the G-CW structure on this?

- There are two 0-cells $C_2/C_2 \times *$, corresponding to the two fixed points, the north and south poles.
- There is a single free 1-cell $C_2 \times I$, corresponding to the boundary of the hemispheres.
- There is a single 2-cell $C_2 \times D^2$. ◂

Last time, we discussed additional cells $G \times_H S(V)$ and $G \times_H D(V)$; these can be decomposed in terms of our actual G-cells, but it's also worth mentioning that the action of G on our preferred G-cells is cellular, unlike for the other cells.

Exercise 2.3. C_2 also acts on S^2 by the antipodal map, which has no fixed points. Write a C_2-CW cell structure for this C_2-space.

Example 2.4. The torus $S^1 \times S^1$ has an S^1-action given by $z(z_1, z_2) = (zz_1, z_2)$. With this action, the torus can be viewed as an S^1-CW complex with one 0-cell $S^1/e \times *$ and one 1-cell $S^1 \times [0,1]$, with the attaching map sending 0 and 1 to $*$. Note that the largest cell we used here was a 1-cell, whereas in the nonequivariant construction of the torus, we are required to use a 2-cell. ◂

There will be additional examples of G-CW complexes on the homework, some with richer structure.

Remark. At this point in class, the professor mentioned that these notes are hosted on Github at https://github.com/adebray/equivariant_homotopy_theory. Since there aren't very many sources for learning this material, and existing ones tend to have few examples, the hope is that these notes can be turned into a good source of lecture notes for learning this material. So as you're learning this material, feel free to add examples, insert comments (e.g. "this section is confusing/unmotivated"), and let me know if you want access to the repository. ◂

Remark.

(1) There is a technical issue of a G-CW structure on a product of G-CW complexes; namely, there are technical difficulties in cleanly putting a G-CW structure on $G/H_1 \times G/H_2$ involving triangulation. We won't digress into this: it's straightforward for finite groups, but a theorem for compact Lie groups, and required revisiting the foundations. Similarly, if $H \subset G$, we'd like the forgetful functor $G\mathsf{Top} \to H\mathsf{Top}$ to send G-CW complexes to H-CW complexes. This is again possible, yet involves technicalities.

(2) A nicer fact is that computing the fixed points of a G-CW complex is straightforward. Recall that $(-)^H$ is a right adjoint, which can be seen by realizing it as the limit of the diagram

$$\begin{tikzcd} \bullet \arrow[loop above, "H"] \arrow[r] & \mathsf{Top}. \end{tikzcd}$$

Thus, we don't expect it to commute with colimits in general. However, it does commute with many important ones, as in the following proposition. ◂

Proposition 2.5. *The fixed point functor* $(-)^H$ *commutes with*

(1) *pushouts where one leg is a closed inclusion, and*
(2) *sequential colimits along closed inclusions.*

This is great, because it means we can commute $(-)^H$ through the construction of a G-CW complex! In particular, on each cell,

$$(G/K \times D^n)^H \cong (G/K)^H \times D^n,$$

so we need to understand $(G/K)^H \cong \mathrm{Map}^G(G/H, G/K)$. We will return to this important point.

Two approaches to the Whitehead theorem. We'll now discuss some homotopy theory of G-spaces and the Whitehead theorem. The first will be a hands-on proof using the HELP lemma. This is an elegant approach to unstable homotopy theory due to Peter May in which one lemma gives quick proofs of several theorems. In the equivariant case, it allows a quick reduction to the non-equivariant case; it will be useful to see a proof of this nature. Ultimately, we will take a different approach involving model categories, and this will be the second perspective.

Definition 2.6. Let $X, Y \in \mathsf{Top}$ and $f : X \to Y$ be continuous. Then, f is ***n*-connected** if $\pi_q(f) : \pi_q(X) \to \pi_q(Y)$ is an isomorphism when $q < n$ and surjective when $q = n$.

We wish to generalize this to the equivariant case.

Definition 2.7. Let $\theta : \{\text{conjugacy classes of subgroups of } G\} \to \{x \in \mathbb{Z} \mid x \geq -1\}$.

- A map $f : X \to Y$ of G-spaces is **θ-connected** if for all $H \subset G$, f^H is $\theta(H)$-connected.

- A G-CW complex is θ**-dimensional** if all cells of orbit type G/H have (nonequivariant) dimension at most $\theta(H)$.

Theorem 2.8 (Equivariant HELP lemma)**.** *Let A, X, Y, and Z be G-CW complexes such that $A \subseteq X$ is θ-dimensional and let $e : Y \to Z$ be a θ-connected G-map. Given $g : A \to Y$, $h : A \times I \to Z$, and $f : X \to Z$ such that $eg = hi_0$ and $fi = hi_1$, there exist maps $\tilde{g} : X \to Y$ and $\tilde{h} : X \times I \to Z$ that make the following diagram commute:*

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd] & & A \times I \arrow[dl, "h"'] \arrow[dd] & & A \arrow[ll, "i_1"'] \arrow[dl, "g"'] \arrow[dd] \\
& Z & & Y \arrow[ll, "e"'] & \\
X \arrow[ur, "f"] \arrow[rr, "i_0"'] & & X \times I \arrow[ul, dashed, "\tilde{h}"] & & X \arrow[ll, "i_1"] \arrow[ul, dashed, "\tilde{g}"]
\end{tikzcd}$$

This is a massive elaboration of the idea of a Hurewicz cofibration. The best way to understand this is to prove it (though it's not an easy proof).

In the non-equivariant case, one reduces to working one cell at a time, inductively extending over the cells of X not in A.[7] In this case, look at $S^{n-1} \subseteq D^n$. Now you just do it: at this point, there's no way to avoid writing down explicit homotopies.

Exercise 2.9. Think about this argument, and then read the proof in [May99].

The equivariant case is very similar: in the same way, one can reduce to inductively attaching a single cell in the case where X is a finite CW complex. This comes via a map $G/H \times S^{n-1} \to G/H \times D^n$, but the only interesting content is in the nonequivariant part, so we can reduce again to $S^{n-1} \to D^n$ with trivial G-action! This allows us to finish the proof in the same way. It also says that the homotopy theory of G-spaces is lifted from ordinary homotopy theory, in a sense that model categories will allow us to make precise.

The first consequence of Theorem 2.8 is:

Theorem 2.10. *Let $e : Y \to Z$ be a θ-connected map and $e_* : [X, Y] \to [X, Z]$ be the map induced by composition.*

- *If X has dimension less than θ, e_* is a bijection.*[8]
- *If X has dimension θ, e_* is a surjection.*

The proof is an exercise; filling in the details is a great way to get your hands on what the HELP lemma is actually doing. Hint: consider the pairs $\varnothing \to X$ and $X \times S^0 \to X \times I$, and apply the HELP lemma.

Corollary 2.11 (Equivariant Whitehead theorem)**.** *Let $e : Y \to Z$ be a weak equivalence of G-CW complexes. Then, e is a G-homotopy equivalence.*

Proof. This is also a standard argument: using Theorem 2.10, e_* is a bijection, so we can pull back $\mathrm{id}_Z \in [Z, Z]$ to an inverse $(e_*)^{-1}(\mathrm{id}_Z) \in [Z, Y]$, which is a homotopy inverse to e. ⊠

One can continue and prove the cellular approximation theorem in this way, and so forth. We won't do this, because we'll approach it from a model-categorical perspective.

One thing that's useful, not so much for this class as for enriching your life, is to learn how to approach this from the perspective of abstract homotopy theory, learning about disc complexes and so forth. You can prove theorems such as the HELP lemma and its consequences in a general setting, and then specialize them to the cases you need. This is a great way to "just do it" without needing model categories.

Anyways, we'll now define a model structure on $G\mathsf{Top}$ and $G\mathsf{Top}_*$. If you don't know what a model category is, now is a good time to review.

Proposition 2.12. *There is a model structure on $G\mathsf{Top}$ (and on $G\mathsf{Top}_*$) defined by the following data.*

Cofibrations: *The maps $f : X \to Y$ such that for all $H \subset G$, $f^H : X^H \to Y^H$ is a cofibration.*

[7] This requires reducing to the case where X is a finite CW complex, but taking a sequential colimit recovers the theorem for all CW complexes X.

[8] We say that X has dimension less than θ if for all closed subgroups $H \subset G$, all cells of orbit type G/H have (nonequivariant) dimension at most n for some $n \leq \theta(H)$.

Weak equivalences: *The maps $f : X \to Y$ such that for all $H \subset G$, $f^H : X^H \to Y^H$ is a weak equivalence.*

So we once again parametrize everything over subgroups of G and use fixed points. This is a cofibrantly generated model category; the cofibrations are specified by generators of acyclic cofibrations in a similar manner to Top. That is, in Top, one can choose generators $I = \{S^{n-1} \to D^n\}$ and $J = \{D^n \to D^n \times I\}$; in GTop, we instead take $I_G = \{G/H \times I\}$ and $J_G = \{G/H \times J\}$.

These are cells that we used to define G-CW complexes, and this is no coincidence: it's a general fact about cofibrantly generated model categories that follows from the small object argument[9] that cofibrant objects are retracts of "cell complexes" built from the things in I, and cofibrations are retracts of cellular inclusions of cell complexes. In this sense, CW complexes are inevitable.

The Whitehead theorem (Corollary 2.11) now falls out of the general theory of model categories.

Theorem 2.13 (Whitehead theorem for model categories)**.** *Let $f : X \to Y$ be a weak equivalence of cofibrant-fibrant objects in a model category. Then, f is a homotopy equivalence.*

In Top and GTop, all objects are fibrant, so this is particularly applicable.

The orbit category. We'll begin talking about the orbit category in the rest of today's lecture, and discuss the bar construction next class.

Definition 2.14. The **orbit category** $\mathscr{O}_G$ is the full subcategory of GTop on the objects G/H.

That is, its objects are the spaces G/H, where $H \subset G$ is closed, and its morphisms are $\mathrm{Map}^G(G/H, G/K) \cong (G/K)^H$. These maps are the same thing as subconjugacy relations, i.e. those of the form

$$gHg^{-1} \subseteq K, \tag{2.15}$$

since for all $h \in H, h(gK) = gK$ if and only if $K = g^{-1}hgK$ if and only if $gHg^{-1} \subseteq K$. A G-map $f : G/H \to G/K$ is completely specified by what it does to the identity coset $f(eH) = gK$, and this g implies the subconjugacy relation (2.15), since, as above, $h(gK) = gK$ for all $h \in H$.

There's another description of the orbit category.

Proposition 2.16. *Let G be a finite group. Then, the orbit category $\mathscr{O}_G$ is equivalent to the category of finite transitive G-sets and G-maps.*

The observation that ignites the proof is that if $x \in X$ has isotropy group H, then its orbit space is isomorphic to G/H.

Definition 2.17. Given a G-space X, we obtain a presheaf on the orbit category, namely a functor $X^{(-)} : \mathscr{O}_G^{\mathrm{op}} \to \mathsf{Top}$, by sending $G/H \to X^H$. This assignment itself is a functor $\psi : G\mathsf{Top} \to \mathsf{Fun}(\mathscr{O}_G^{\mathrm{op}}, \mathsf{Top})$.

Proposition 2.18. $\mathsf{Fun}(\mathscr{O}_G^{\mathrm{op}}, \mathsf{Top})$ *has a projective model structure where the weak equivalences and fibrations are taken pointwise.*

The point is the following result, a revisionist interpretation of Elmendorf's theorem.[10]

Theorem 2.19. *ψ is the right adjoint in a Quillen equivalence; the left adjoint is evaluation at G/e.*

The point is, these two model categories have the same homotopy theory.

Exercise 2.20. Check that evaluation at G/e is a left adjoint to ψ.

References

[Ada84] J.F. Adams. Prerequisites (on equivariant stable homotopy) for Carlssons's lecture. In Ib H. Madsen and Robert A. Oliver, editors, *Algebraic Topology Aarhus 1982*, chapter 25, pages 483–532. Springer Berlin Heidelberg, 1984.

[Day70] Brian Day. *Constructions in Biclosed Categories*. PhD thesis, University of New South Wales, 1970.

[HHR16] M.A. Hill, M.J. Hopkins, and D.C. Ravenel. On the nonexistence of elements of kervaire invariant one. *Annals of Mathematics*, 184(1):1–262, 2016.

[9] The small object argument is a beautiful piece of basic mathematics that everybody should know. If you don't know it, your homework is to read enough about model categories to get to that point. In general, there may be large objects and transfinite induction, but for the case we care about large cardinals won't arise.

[10] Elmendorf proved that these two categories have the same homotopy theory, but his proof was more explicit.

[Hov99] Mark Hovey. *Model Categories*. Mathematical surveys and monographs. American Mathematical Society, 1999.

[Ill72] Sören Illman. *Equivariant Algebraic Topology*. PhD thesis, Princeton University, 1972.

[Kel84] G.M. Kelley. *Basic Concepts in Enriched Category Theory*. Number 64 in Lecture Notes in Mathematics. Cambridge University Press, 1984.

[LMS86] L. Gaunce Lewis Jr., J. Peter May, and Mark Steinberger. *Equivariant Stable Homotopy Theory*. Number 1213 in Lecture Notes in Mathematics. Springer Berlin Heidelberg, 1986.

[Mac78] Saunders Mac Lane. *Categories for the Working Mathmatician*. Number 5 in Graduate Texts in Mathematics. Springer-Verlag New York, 1978.

[May96] Peter May, editor. *Equivariant Homotopy and Cohomology Theory*, volume 91 of *CBMS Regional Conference Series in Mathematics*. American Mathematical Society, 1996.

[May99] Peter May. *A Concise Course in Algebraic Topology*. Chicago Lectures in Mathematics. The University of Chicago Press, 1999.

[Rie14] Emily Riehl. *Categorical Homotopy Theory*. Cambridge University Press, 2014.

[Rie16] Emily Riehl. *Category Theory in Context*. Aurora: Dover Modern Math Originals. Dover Publications, 2016.