Pin Cobordism and Related Topics

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1. Introduction

In [1], we largely determined the structure of Ω_*^{Spin}, the Spin cobordism ring. In this paper we show how these results may be applied to study $\Omega_*^{\mathrm{Spin}}(K)$, the Spin bordism groups, in certain cases. One case of particular importance is $K = RP^\infty$ because of well-known isomorphism $\bar{\Omega}_{n+1}^{\mathrm{Spin}}(RP^\infty) \approx \Omega_n^{\mathrm{Pin}}$, where Pin is a group with two components whose component of the identity is Spin (see [2]). Our results include a determination of Ω_*^{Pin} in terms of our results on Ω_*^{Spin}. We also study the problem of determining the image of one bordism group in another, e.g. $\mathrm{Im}(\Omega_*^{\mathrm{Spin}}(K) \to \mathcal{N}_*(K))$.

Our method is to study the Adams spectral sequence for $\pi_*(K \wedge \mathbf{M}\mathrm{Spin})$. In order to do this, we must determine the $\mathscr{A}$-module structure of $H^*(K) \otimes H^*(\mathbf{M}\mathrm{Spin})$. The following is our main algebraic result which together with the results of [1], enables us to determine this structure. The $\mathscr{A}$-module structure of $H^*(K) \otimes (\mathscr{A} \otimes N)$ depends only on the $\mathscr{B}$-module structure of $H^*(K)$, where $\mathscr{B}$ is a Hopf subalgebra of $\mathscr{A}$ and N is a fixed $\mathscr{B}$-module.

2. Statements of Main Geometrical Theorems

Let K be a CW-complex. The G-bordism groups of K, $\Omega_*^G(K)$, are the homology groups of K with coefficients in the Thom spectrum $\mathbf{M}G$, that is, $\Omega_*^G(K) = H_*(K; \mathbf{M}G) = \pi_*(K^+ \wedge \mathbf{M}G)$ (see [7]). In [1], we proved that $\mathbf{M}\mathrm{Spin}$ was of the same mod two homotopy type as a wedge of three types of spectra: $\mathbf{K}(Z_2, n)$, $\mathbf{B}O\langle 8n\rangle$, and $\mathbf{B}O\langle 8n+2\rangle$. Thus, to determine Ω_*^{Pin}, we need to compute $\pi_*(RP^\infty\, \mathbf{K}(Z_2, n))$, $\pi_*(RP^\infty \wedge \mathbf{B}O\langle 8n\rangle)$, and $\pi_*(RP^\infty\, \mathbf{B}O\langle 8n+2\rangle)$. This is done explicitly in theorem 5.1; we only state the following corollary here. This corollary contradicts a proposition of C. T. C. Wall [6].

COROLLARY 2.1. The exponent of Ω_i^{Pin} is 2 if $i \equiv 0, 1, 3, 4, 5, 7, (8)$ (except in some low dimensions where it is 1), 2^{4k+3} if $i = 8k+2$, $k \geqslant 0$, and 2^{4k+4} if $i = 8k+6$, $k \geqslant 0$.

The methods we use to compute $\Omega_*^{\mathrm{Spin}}(K)$ when $K = RP^\infty$ work equally well when $K = BSO$ (see § 6).

Our other geometrical results concern the image of one bordism group in another. In § 6, we give some general considerations which have the following corollaries.

COROLLARY 2.2. $\operatorname{Im}(\Omega^{\mathrm{Pin}}\to\mathcal{N}_*)=$all cobordism classes all of whose Stiefel-Whitney numbers involving $W_2(v)$ vanish.

COROLLARY 2.3. $(\operatorname{Im}(\Omega_*(K)\to\mathcal{N}_*(K)))=$all bordism classes (M,f) all of whose Stiefel-Whitney numbers of the map involving W_1 vanish, if and only if $H_*(K;Z)$ has no 4-torsion.

COROLLARY 2.4. There is a PL-manifold M such that $[M]\notin\operatorname{Im}(\Omega_*^{PL}\to\mathcal{N}_*^{PL})$ and all mod 2 characteristic numbers of M involving W_1 vanish.

COROLLARY 2.5. $\operatorname{Im}(\Omega_*^{\mathrm{Spin}}(BSO)\to\mathcal{N}_*(BSO))=$all bordism classes (M,f) all of whose Stiefel-Whitney numbers of the map involving W_1 and W_2 vanish.

3. Modules over a Hopf Algebra

In this section we state and prove some results about the structures of tensor products of modules over a Hopf algebra which may be of independent interest. In the following sections we use these results to prove our geometric theorems.

In the following, all the objects are graded vector spaces over Z_2 of finite type and zero in negative dimensions. Let A be a connected, coassociative, cocommutative associative, locally finite Hopf algebra. We will consider the category of left modules over A tensor product over Z_2 in this category is defined using the Hopf algebra structure of A. If M is a left A-module, set $\hat{M}=M$ as a graded vector space, but with the trivial A-structure, that is $1(m)=m$, $a(m)=0$ if $a\in\bar{A}$. The following theorem is well-known, but we include it for completeness.

THEOREM 3.1. *Define* $l:\hat{M}\otimes A\to M\otimes A$ *by* $l(m\otimes 1)=m\otimes 1$ *and extend to an* A*-map. Then* l *is an isomorphism.*

Proof. We first show l is an expimorphism. Assume $m\otimes a\notin\operatorname{Im} l$, with dim a minimal with this property. Then, in $M\otimes A$,

$$a(m\otimes 1)=m\otimes a+\sum_{\dim a''<\dim a}a'(m)\otimes a'';$$

all terms but $m\otimes a\notin\operatorname{Im} l$, hence $m\otimes a\in\operatorname{Im} l$. Since $M\otimes A$ and $M\otimes A$ have the same rank as graded vector spaces, l is an isomorphism.

A acts on $M\otimes A$ on the right by $(m\otimes a)\bar{a}=m\otimes a\bar{a}$. A acts on the right of $\hat{M}$ by $\widehat{ma=x(a)m}$ where $m\in M$ and χ is the antiautomorphism of A. Define a right action of A on $M\otimes A$ via l, that is, $(m\otimes a)\bar{a}=l^{-1}(l(m\otimes a)\bar{a})$ in $M\otimes A$. By standard Hopf algebra identities one verifies.

LEMMA 3.2. This right action of A on $\hat{M}\otimes A$ is given by $(m\otimes a)\bar{a}=\sum(m)\bar{a}'\otimes a\bar{a}''$.

As a corollary of Lemma 3.2, we have the fact that the A-structure of the tensor product of two A-modules depends only on the B-structure of one of them in certain cases.

THEOREM 3.3. *Let $B\subset A$ be a Hopf subalgebra. Let M be an A-module and let N be a fixed B-module. Let $A\otimes_B N$ be a left A-module by $\bar{a}(a\otimes n)=\bar{a}a\otimes n$. Then $M\otimes(A\otimes_B N)$ depends, as an A-module, only on the B-structure of M. In particular, if $f:M_1\to M_2$ is an isomorphism of B-modules, the following composition is an isomorphism of A-modules:*

$$M_1\otimes(A\otimes_B N)\to(M_1\otimes A)\otimes_B N\xrightarrow{l_1^{-1}\otimes 1}(\hat{M}_1\otimes A)\otimes_B N\xrightarrow{f\otimes 1\otimes 1}$$
$$(\hat{M}_2\otimes A)\otimes_B N\xrightarrow{l_2\otimes 1}(M_2\otimes A)\otimes_B N\to M_2\otimes(A\otimes_B N).$$

Proof. If f is a B-map, then $f\otimes 1\otimes 1$ is an A-map by lemma 3.2. The unnamed maps are associativity isomorphisms.

In the next theorem we analize $(\hat{M}\otimes A)\otimes_B N$.

THEOREM 3.4. *If M is an A-module and N is a B-module, then $(M\otimes A)\otimes_B N\approx$ $\approx A\otimes_B(M\otimes N)$ as left A-modules, where the A action is on the first factors.*

Proof. Define the map $(\hat{M}\otimes A)\otimes_B N\to A\otimes_B(M\otimes N)$ by $m\otimes a\otimes n\to a\otimes m\otimes n$. We first show it is well defined.

$$\begin{aligned}(m\otimes a)b\otimes n&=\textstyle\sum(m)b'\otimes ab''\otimes n\to ab''\otimes\chi(b')(m)\otimes n\\&=\textstyle\sum a\otimes(b'')'\chi(b')(m)\otimes(b'')''(n)=\sum a\otimes(b')'\chi((b')'')(m)\otimes b''(n)\\&=a\otimes m\otimes b(n),\end{aligned}$$

while $m\otimes a\otimes b(n)\to a\otimes m\otimes b(n)$. Next we show it is an A-map $\bar{a}(m\otimes a\otimes n)=$ $=m\otimes aa\otimes n\to\bar{a}a\otimes m\otimes n=\bar{a}(a\otimes m\otimes n)$. The map is clearly a vector space isomorphism.

Hence, Theorem 3.3 can be restated.

COROLLARY 3.5. $M\otimes(A\otimes_B N)\approx A\otimes_B(M\otimes N)$ as A-modules, the right hand side depending only on the B-structure of M.

Finally, we have the following corollary, which is what we need for the applications.

COROLLARY 3.6. If M is an A-module and $M\supset\cdots\supset M^{[i]}\supset M^{[i-1]}\supset\cdots$ is a filtration by B-module, where $B\subset A$, then $A\otimes_B(M\otimes N)\supset\cdots A\otimes_B(M^{[i]}\otimes N)\supset\cdots$ provides on A-module filtration of $M\otimes(A\otimes_B N)$ with quotients $A\otimes_B(M^{[i]}/M^{[i-1]}\otimes N)$.

4. Tensor Products of $\mathscr{A}$-modules

We apply the results of the preceding section to some particular cases. Let $A=\mathscr{A}$, $B=\mathscr{A}_1$, where $\mathscr{A}_1$ is the subalgebra generated by Sq^1 and Sq^2.

THEOREM 4.1. *Let M be an $\mathscr{A}$-module. Assume $M\to\sum_j \mathscr{A}_1/\mathscr{A}_1(J_j)$ as an $\mathscr{A}_1$-module, where $J_j\subset\bar{\mathscr{A}}_1$. Then $M\otimes\mathscr{A}/\mathscr{A}\bar{\mathscr{A}}_1=\sum_j \mathscr{A}/\mathscr{A}(J_j)$ as an $\mathscr{A}$-module.*

Proof. This follows immediately from corollary 3.5, by taking $N=Z_2$, the trivial $\mathscr{A}_1$-module.

THEOREM 4.2. *Let M be an $\mathscr{A}$-module. Assume $M\approx\sum_j \mathscr{A}_1/\mathscr{A}_1(J_j)$ as an $\mathscr{A}_1$-module, where $J_j\subset\bar{\mathscr{A}}_1$. Assume now $\mathscr{A}_1(J_j)$, in lowest terms, is $\mathscr{A}_1(\mathrm{Sq}^2,\mathrm{Sq}^2\mathrm{Sq}^1)$, $\mathscr{A}_1(\mathrm{Sq}^3,\mathrm{Sq}^2\mathrm{Sq}^1)$, $\mathscr{A}_1(\mathrm{Sq}^2\mathrm{Sq}^1)$, $\mathscr{A}_1(\mathrm{Sq}^2\mathrm{Sq}^1,\mathrm{Sq}^5+\mathrm{Sq}^4\mathrm{Sq}^1)$, $\mathscr{A}_1(\mathrm{Sq}^3\mathrm{Sq}^1,\mathrm{Sq}^5+\mathrm{Sq}^4\mathrm{Sq}^1)$, or $\mathscr{A}_1(\mathrm{Sq}^5+\mathrm{Sq}^4\mathrm{Sq}^1)$. Then $M\otimes\mathscr{A}/\mathscr{A}(\mathrm{Sq}^3)\approx$ sum of cyclic $\mathscr{A}$-modules, as an $\mathscr{A}$-module.*

Proof. Apply corollary 3.5 with $N=\mathscr{A}_1/\mathscr{A}_1(\mathrm{Sq}^3)$. For each type of J_j one must show that $\mathscr{A}_1/\mathscr{A}_1(J_j)\otimes\mathscr{A}_1/\mathscr{A}_1(\mathrm{Sq}^3)$ is a cyclic $\mathscr{A}_1$-module. We forego giving a table of answers as we need only a few of them in the application.

A more interesting example for M is $H^*(RP^\infty)$. Let $R=\sum R^i$ be the $\mathscr{A}$-module defined by $R^i=H^{i+1}(RP^\infty)$. Let $R^{[i]}$ be the $\mathscr{A}_1$-submodule of R generated by R^j, $j\leqslant i$. The structure of R as an $\mathscr{A}_1$-module is given by the following proposition, whose proof is straightforward and left to the reader.

PROPOSITION 4.3. *As an $\mathscr{A}_1$-modules, $R^{[4i+2]}/R^{[4i-2]}\approx\mathscr{A}_1/\mathscr{A}_1(\mathrm{Sq}^1)$, $R^{[2]}/R^{[0]}\approx\mathscr{A}_1/\mathscr{A}_1(\mathrm{Sq}^1)$, and $R^{[0]}\approx\mathscr{A}_1/\mathscr{A}_1(\mathrm{Sq}^2)$. Furthermore, the extension is determined by $\mathrm{Sq}^1(r_{4i+2})=(\mathrm{Sq}^5+\mathrm{Sq}^4\mathrm{Sq}^1)(r_{4i-2})$ and $\mathrm{Sq}^1(r_2)=\mathrm{Sq}^2\mathrm{Sq}^1(r_0)$.*

THEOREM 4.4. 1) *There is an $\mathscr{A}$-module filtration, $R\otimes\mathscr{A}/\mathscr{A}\bar{\mathscr{A}}_1\supset\cdots\supset F^{(4i-2)}\supset\supset F^{(4i-2)}\supset\cdots\supset F^{(2)}\supset F^{(0)}$, with $F^{(4i+2)}/F^{(4i-2)}\approx\mathscr{A}/\mathscr{A}(\mathrm{Sq}^1)$, $F^{(2)}/F^{(0)}\approx\mathscr{A}/\mathscr{A}(\mathrm{Sq}^1)$, and $F^{(0)}\approx\mathscr{A}/\mathscr{A}(\mathrm{Sq}^2)$.* 2) *There is an $\mathscr{A}$-module filtration, $R\otimes\mathscr{A}/\mathscr{A}(\mathrm{Sq}^3)\supset\cdots\supset G^{(i)}\supset\cdots$ with $G^{(4i+1)}/G^{(4i)}\approx\mathscr{A}$, $G^{(4i+2)}/G^{(4i+1)}\approx\mathscr{A}$, $G^{(4i+3)}/G^{(4i+2)}=0$, $G^{(4i+4)}/G^{(4i+3)}\approx\mathscr{A}/\mathscr{A}(\mathrm{Sq}^1)$ and $G^{(0)}\approx\mathscr{A}/\mathscr{A}(\mathrm{Sq}^5\mathrm{Sq}^1)$.*

Proof. Apply corollary 3.6 with $N=Z_2$ or $\mathscr{A}_1/\mathscr{A}_1(\mathrm{Sq}^3)$. In the latter case, one must note that

$$\mathscr{A}_1/\mathscr{A}_1(\mathrm{Sq}^1)\otimes\mathscr{A}_1/\mathscr{A}_1(\mathrm{Sq}^3)\approx\mathscr{A}_1\oplus\mathscr{A}_1/\mathscr{A}_1(\mathrm{Sq}^1)\oplus\mathscr{A}_1$$

with generators of dimension 0, 2, and 3 respectively, while $\mathscr{A}_1/\mathscr{A}_1(\mathrm{Sq}^2)\otimes\mathscr{A}_1/\mathscr{A}_1(\mathrm{Sq}^3)\approx\mathscr{A}_1/\mathscr{A}_1(\mathrm{Sq}^5\mathrm{Sq}^1)\oplus\mathscr{A}_1$ with generators of dimension 0 and 1 respectively.

5. Pin Cobordism

By our remarks at the start of section 2, in order to compute Ω_*^{Pin}, we compute $\pi_*(RP^\infty \wedge \mathbf{K}(Z_2, n))$,

$$\pi_*(RP^\infty \wedge \mathbf{BO}\langle 8n\rangle),$$

and

$$\pi_*(RP^\infty \wedge \mathbf{BO}\langle 8n+2\rangle).$$

We note that

$$H^*(RP^\infty \wedge \mathbf{BO}\langle 8n\rangle) \approx \bar{H}^*(RP^\infty) \otimes \mathscr{A}/\mathscr{A}\bar{\mathscr{A}}_1,$$
$$H^*(RP^\infty \wedge \mathbf{BO}\langle 8n+2\rangle) \approx \bar{H}^*(RP^\infty) \otimes \mathscr{A}/\mathscr{A}(\mathrm{Sq}^3),$$

and

$$H^*(RP^\infty \wedge \mathbf{K}(Z_2, n)) \approx H^*(RP^\infty) \otimes \mathscr{A}.$$

In the previous section, we have computed the $\mathscr{A}$-module structure of these tensor products. We now compute $\mathrm{Ext}_{\mathscr{A}}$ and apply the Adams spectral sequence.

THEOREM 5.1. 1) *The contribution to* Ω_*^{Pin} *of terms* $\pi_*(RP^\infty \wedge \mathbf{K}(Z_2, n))$ *is a direct summand of* Z_2 *in each dimension* $\geqslant n$. 2) *The contribution to* Ω_*^{Pin} *of terms* $\pi_*(RP^\infty \wedge \mathbf{BO}\langle 8n\rangle)$ *is as follows:* Z_2 *in dim* $8n+i$, $i \equiv 0, 1\,(8)$; 0 *in dim* $8n+i$, $i=3, 4, 5, 7\,(8)$; ${Z_2}^{4k+3}$ *in dim* $8n+8k+2$, $k \geqslant 0$; *and* ${Z_2}^{4k+4}$ *in dim* $8n+8k+6$, $k \geqslant 0$. 3) *The contribution to* Ω_*^{Pin} *of terms* $\pi_*(RP^\infty \wedge \mathbf{BO}\langle 8n+2\rangle)$ *is as follows* Z_2 *in dim* $8n+2+i$, $i \equiv 1, 2, 5, 7\,(8)$; $Z_2 \oplus Z_2$ *in dim* $8n+2+i$, $i \equiv 6\,(8)$; O *in dim* $8n+2+i \equiv 3\,(8)$; ${Z_2}^{4k+1}$ *in dim* $8n+2+8k$, $k \geqslant 0$; *and* ${Z_2}^{4k+2}$ *in dim* $8n+2+8k+4$, $k \geqslant 0$.

Proof. Since $\hat{R} \otimes \mathscr{A} \approx R \otimes \mathscr{A}$ is a free $\mathscr{A}$-module, it contributes a Z_2 in $E_2^{0,t}$, $t \geqslant n = \dim$ of the generator of $\mathscr{A}$. This proves part 1). By a theorem of Liulevicius [4], $\mathrm{Ext}_{\mathscr{A}}(F^{(0)}, Z_2) \approx \mathrm{Ext}_{\mathscr{A}1}(\mathscr{A}_1/\mathscr{A}_1(\mathrm{Sq}^2), Z_2)$ and the latter can be computed directly. $\mathrm{Ext}_{\mathscr{A}}^{s,t}(F^{(0)}, Z_2) \approx \mathrm{Ext}^{s+1,t+2}(\mathscr{A}/\mathscr{A}\bar{\mathscr{A}}_1, Z_2)$, $s, t \geqslant 0$: and $\mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathscr{A}/\mathscr{A}\bar{\mathscr{A}}_1, Z_2) = Z_2[h_0, h_1, t, w]/\{h_0h_1, h_1^3, t^2 + h_0^2 w, h_1 t\}$, where $h_0 \in \mathrm{Ext}^{1,1}$, $h_1 \in \mathrm{Ext}^{1,2}$, $t \in \mathrm{Ext}^{3,7}$, and $w \in \mathrm{Ext}^{4,12}$. Note $\mathrm{Ext}_{\mathscr{A}}^{s,t}(\mathscr{A}/\mathscr{A}(\mathrm{Sq}^1), Z_2) = \{h_0^s\}$, $s \geqslant 0$, $t = s$. We now calculate $\mathrm{Ext}_{\mathscr{A}}(R \otimes \mathscr{A}/\mathscr{A}\bar{\mathscr{A}}_1, Z_2)$ by induction on the filtration. This is the same as taking $\sum \mathrm{Ext}_{\mathscr{A}}(F^{(4i+2)}/F^{(4i-2)}, Z_2) \oplus \mathrm{Ext}_{\mathscr{A}}(F^{(0)}\, Z_2)$ and introducing a $d_1 : \mathrm{Ext}^{s,t} \to \mathrm{Ext}^{s+1,t}$ in this direct sum. Note $H(R \otimes \mathscr{A}/\mathscr{A} \subset \mathscr{A}_1, Q_0) = H(R, Q_0) \otimes H\mathscr{A}/\mathscr{A}\bar{\mathscr{A}}_1 = 0$ as $H(R, Q_0) = 0$. Hence, be a theorem of Adams $\sum \mathrm{Ext}_{\mathscr{A}}(R \otimes \mathscr{A}/\mathscr{A}\bar{\mathscr{A}}_1, Z_2)$ has no elements of infinite height. This determines our d_1. Let x_i "generate" $F^{(4i+2)}/F^{(4i-2)}$. Then $d_1(h_0^k w^i t) = h_0^{k+4i+3} x_{2i+1}$, $i \geqslant 0$ and $d_1(h_0^k w^{i+1}) = h_0^{k+4i+4} x_{2i+2}$, $i \geqslant 0$ is the only possibility. (The elements $h_0^k w^{i+1}$ etc. denote elements in $\mathrm{Ext}_{\mathscr{A}}(F^{(0)}, Z_2)$ corresponding to the named element under the isomorphism $\mathrm{Ext}_{\mathscr{A}}^{s,t}(F^{(0)}, Z_2) \approx \mathrm{Ext}^{s+1,t+2}(\mathscr{A}/\mathscr{A}\bar{\mathscr{A}}_1, Z_2)$.) By direct computation, we note also that $h_1(w^{i+1} h_1^2) \neq 0$ in $\mathrm{Ext}_{\mathscr{A}}(R \otimes \mathscr{A}/\mathscr{A}\bar{\mathscr{A}}_1,$

Z_2). Thus, for algebraic reasons, it is trivial to check that $E_2 = E_\infty$. Reading off the homotopy gives part 2) of theorem 5.1. The calculation needed to prove part 3) is similar to that for part 2). Here we need that $\mathrm{Ext}^{s,t}(G^{(0)}, Z_2)\ \mathrm{Ext}_{\mathscr{A}}^{s-1,t-6}(\mathscr{A}/\mathscr{A}\mathscr{A}_1, Z_2)$ $s \geqslant 1$, $t \geqslant 6$, and $\mathrm{Ext}_{\mathscr{A}}^{0,0}(G^{(0)}, Z_2) = Z_2$. The calculation then proceeds as above and will be omitted.

Using the results of [1] on Ω_*^{Spin}, one can now determine Ω_*^{Pin} in any given dimension. For example

$$\Omega_{22}^{\mathrm{Pin}} = Z_{2^{12}} + Z_{2^8} + Z_{2^6} + Z_{2^4} + Z_{2^4} + Z_{2^2} + Z_{2^2} + Z_2 + Z_2 .$$

Corollary 2.1 follows immediately from theorem 5.1 as $Z^{2^{4k+3}} \subset \Omega_{8k+2}^{\mathrm{Pin}}$. We also note the following corollary of our computations.

COROLLARY 5.2. $\mathrm{Im}(\Omega_i^r \to \Omega_i^{\mathrm{Pin}}) = Z_2$ if $i \equiv 1, 2(8)$ and 0 otherwise. Furthermore, the element in $\dim 8k+2$ is divisible by 2^{4k+2}.

6. Bordism Groups

Let $\mathbf{f}: X \to \mathbf{Y}$ be a map of spectra. Assume $\mathbf{Y}$ is a wedge of $\mathbf{K}(Z_2, n)$'s. Define $G_* \subset \pi_*(\mathbf{Y})$ by $G_* = \{g \mid g \in \pi^n(\mathbf{Y})$ and $g^*(\mathrm{Ker}\,\mathbf{f}^*) = 0 \in H^*(S^n)$, where $\mathbf{f}^*: H^*(\mathbf{Y}) \to \to H^*(X)\}$. It is clear that

$$\mathrm{Im}(\pi_*(\mathbf{X}) \to \pi_*(\mathbf{Y}) \subset G_*(\pi_*(\mathbf{X}) \to \pi_*(\mathbf{Y})) \subset G_* .$$

DEFINITION 6.1. *A spectrum X has property P if given $u \in H^n(\mathbf{X})$, $u \neq O \in H^*(\mathbf{X})/\mathscr{A}H^*(\mathbf{X})$, then there exists $g \in \pi_n(\mathbf{X})$ such that $g^*(u) \neq O \in H^n(S^n)$.*

The following easy theorem is the basis for our results on bordism groups.

THEOREM 6.2. *Assume $f^*: H^*(\mathbf{Y}) \to H^*(\mathbf{X})$ is an epimorphism. Then* $\mathrm{Im}(\pi_*(\mathbf{X}) \to \to \pi_*(\mathbf{Y})) = G_*$ *if and only if X has property P.*

Proof. Let $g \in G_* \to \mathrm{Im}\,\mathbf{f}^*$. Then there exists $u \in H^*(\mathbf{Y})$ such that $g^*(u) \neq 0$ and $(\mathbf{f}\mathbf{g}')^*(u) = 0$ for all $\mathbf{g}' \in \pi_*(\mathbf{X})$ because $\mathbf{Y}$ is a wedge of Eilenberg-MacLane spectra. Since X has property P, we see that $\mathbf{f}^*(u) \in \mathscr{A}H^*(\mathbf{X})$, hence $\mathbf{f}^*(u) = \bar{a}\mathbf{f}^*v)$ because $\mathbf{f}^*$ is an epimorphism. Thus $u + \bar{a}v \in \mathrm{Ker}\,\mathbf{f}_0^*$ and $\mathbf{g}^*(u + \bar{a}v) = \mathbf{g}^*(u) = 0$, a contradiction. Conversely, let $u \in H^n(\mathbf{X})$ be such that $u \notin \mathscr{A}H^*(\mathbf{X})$. If $\mathbf{g}'^*(u) = 0$ for all $\mathbf{g} \in \pi_*(\mathbf{X})$, let $u = \mathbf{f}^*(v)$, $v \in H^*(\mathbf{Y})$, then there exists $\mathbf{g} \in \pi_*(\mathbf{Y})$ such that $\mathbf{g}^*(v) \neq 0$ and $\mathbf{g}^*(\mathrm{Ker}\,\mathbf{f}^*) = 0$. Hence $\mathbf{g} \in G_* - \mathrm{Im}\,\mathbf{f}_*$; a contradiction.

The following proposition follows easily from the structure of the Adams spectral sequence.

PROPOSITION 6.2. $\mathbf{X}$ *has property P if and only if $d_r = 0$ on $E_r^{0,t}$ in the Adams spectral sequence for $\pi_*(\mathbf{X})$ for all r and all t.*

We now prove corollary 2.2. In order to apply theorem 6.1 we must note the following fact. Let $\mathbf{f}:\mathbf{M}\mathrm{Pin}\to\mathbf{M}O$. $\mathrm{Ker}\,\mathbf{f}^*=U\cdot$ (ideal over $\mathscr{A}$ generated by w_2). Let $g\in\pi_*(\mathbf{M}O)$. If $\mathbf{g}^*(U\cdot$ ideal generated by $w_2)=0$, then $\mathbf{g}^*(\mathrm{Ker}\,\mathbf{f}^*)=0$. The proof is by induction on

$$\dim a\cdot\mathbf{g}^*\left(U\cdot a(w_2)\cdot w\right)=g^*\left(a\left(U\cdot w_2\cdot w\right)\right)+\sum_{\dim a'<\dim a} g^*\left(U\cdot a'(w_2)\cdot w^1\right)=0$$

if $\dim a\geqslant 0$. Hence $G_*=$ all cobordism classes all of whose Stiefel Whitney numbers involving $w_2(v)$ vanish. To apply theorem 6.1, we need that $\mathbf{M}\mathrm{Pin}$ has property P. This follows immediately from proposition 6.2 and the proof of theorem 5.1 where it is noted that $E_2=E_\infty$.

Corollary 2.3 is proved in a similar way. In order to apply theorem 6.1, one must show that $K\wedge\mathbf{M}SO$ has property P. It is easy to check that this is true if and only if $H_*(K;Z)$ has no 4-torsion.

Corollary 2.4 is also proved in a similar way using the results of [3] and the fact that $H_*(\mathbf{M}\,\mathrm{SPL}\colon Z)$ has 4-torsion.

A more general form of corollary 2.5 is the following result, which follows immediately from theorem 6.1.

COROLLARY 6.3. $(\mathrm{Im}(\Omega_*^{\mathrm{Spin}}(K)\to\mathscr{N}_*(K))=$ all bordism classes (M,f) all of whose Stiefel-Whitney numbers of the map involving w_1 and w_2 vanish) if and only if $K\wedge\mathbf{M}\mathrm{Spin}$ has property P.

Thus, in order to prove corollary 2.5, we must show that $E_2=E_\infty$ in the Adams spectral sequence for $\pi_*(BSO\wedge\mathbf{M}\mathrm{Spin})$. Since $H^*(BSO)\approx$ sum of cyclic $\mathscr{A}_1$-modules of type $\mathscr{A}_1$, $\mathscr{A}_1/\mathscr{A}_1(\mathrm{Sq}^3)$, and Z_2, as an $\mathscr{A}_1$-module [1], we can compute E_2 using theorem 4.1 and 4.2. We then note that $E_2=E_\infty$ for algebraic reasons.

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