

M392C NOTES: BRIDGELAND STABILITY

ARUN DEBRAY
FEBRUARY 7, 2019

These notes were taken in UT Austin's M392C (Bridgeland Stability) class in Spring 2019, taught by Benjamin Schmidt. I live- \LaTeX ed them using vim, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Any mistakes in the notes are my own.

CONTENTS

1.	Introduction and quiver representations: 1/22/19	1
2.	Geometric invariant theory: 1/24/19	4
3.	Constructing moduli spaces of quiver representations: 1/29/19	6
4.	Examples of quiver varieties: 1/31/19	9
5.	Moduli spaces and slope stability: 2/5/19	11
6.	Slope stability: 2/7/19	14

Lecture 1.

Introduction and quiver representations: 1/22/19

This class will be on Bridgeland stability, though we won't get to that topic specifically for about a month. We'll follow lecture notes of Macrì-Schmidt [MS17], which are on the arXiv.

If you're pre-candidacy, make sure to do at least two exercises in this class, at least one from March or later; otherwise just make sure to show up. (If you're an undergrad who's signed up for this class, please do at least four exercises, at least two from March or later.)

Now let us enter the world of mathematics. We'll begin with two well-known theorems in algebraic geometry; we'll eventually be able to prove these using stability conditions.

Theorem 1.1 (Kodaira vanishing). *Let X be a smooth projective complex variety and L be an ample line bundle. Then for all $i > 0$, $H^i(X; L \otimes \omega_X) = 0$.*

We'll eventually give an approach in the setting where $\dim X \leq 2$. It won't be very hard once the setup is in place. In fact, there are probably plenty of other vanishing theorems one could prove using stability conditions, including some which aren't known yet.

The other theorem is over a century ago, from the Italian school of algebraic geometry.

Theorem 1.2 (Castelnuovo). *Working over an algebraically closed field, let $C \subset \mathbb{P}^3$ be a smooth curve not contained in a plane. Then $g \leq d^2/4 - d + 1$, where g is genus of C and d is its degree.*

Another goal we'll work towards:

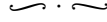
Problem 1.3. Explicitly describe some moduli spaces of vector bundles or sheaves.

Here's a concrete outline of the course.

- (1) Before we discuss any algebraic geometry, we'll study quiver theory, focusing on moduli spaces of quiver conditions. We don't need stability conditions to do this, but these spaces make great simple examples of the general story.
- (2) Next, we'll study vector bundles on curves. Bridgeland stability is a generalization of what we can say here for higher dimensions.
- (3) A crash course on derived categories and Bridgeland stability. This is pretty formal.

- (4) A crash course on intersection theory, which will be necessary for what comes later.
- (5) Surfaces.
- (6) Threefolds (if we have time).

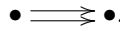
These are all mostly independent pieces, only coming together in the end, so if you get lost somewhere there's no need to panic; you'll probably be able to pick the course back up soon enough.



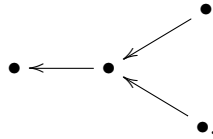
And now for the moduli of quiver representations. For this stuff, we'll follow King [Kin94], which is accessible and nice to read. Let k be an algebraically closed field.

Definition 1.4. A *quiver* is the representation theorist's word for a finite directed graph. Explicitly, a quiver Q consists of two finite sets Q_0 and Q_1 of vertices and edges, respectively, together with *tail* and *head* maps $t, h: Q_1 \rightarrow Q_0$.

Example 1.5. The *Kronecker quiver* is



The quiver of *type* D_4 is



We can also consider a quiver with a single vertex v and a single edge $e: v \rightarrow v$. ◀

Definition 1.6. A *representation* W of a quiver Q is a collection of k -vector spaces W_v for each $v \in Q_0$ and linear maps $\phi_e: W_{v_1} \rightarrow W_{v_2}$ for each edge $e: v_1 \rightarrow v_2$ in Q_1 . The vector $(\dim W_v)_{v \in Q_0}$ inside $\mathbb{C}[Q_0]$ is called the *dimension* of W .

Example 1.7. First, some trivial example. For example, here's a representation of the Kronecker quiver: $(\cdot, \cdot): k \rightrightarrows k$. A representation of the quiver with one vertex and one edge is a vector space with an endomorphism, e.g. \mathbb{C}^2 and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. ◀

Definition 1.8. Let Q be a quiver. A *morphism* of Q -representations $f: (W_v, \phi_e) \rightarrow (U_v, \psi_e)$ is a collection of linear maps $f_v: W_v \rightarrow U_v$ for each $v \in Q_0$ such that for all edges e ,

$$f_{h(e)} \circ \phi_e = \psi_e \circ f_{t(e)}.$$

If all of these linear maps are isomorphisms, f is called an *isomorphism*.

That is, data of a quiver representation includes a bunch of linear maps, and we want a morphism of quiver representations to commute with these maps.

Representation theorists want to classify quiver representations. This is really hard, so let's specialize to irreducible representations (those not a direct sum of two other ones). This is still really hard! There are classical theorems originating from the French school proving that most quivers do not admit nice classifications of their irreducible representations: some have finitely many, and some have infinitely many but nice parameterizations, and these are uncommon.

One way to make headway on these kinds of problems is to consider a moduli space of quiver representations, which may be more tractable to study.

Problem 1.9. Can you classify the (isomorphism classes of) quiver representations of the quiver with a single vertex and single edge?

Our first, naïve approach to constructing the moduli of quiver representations is to fix a dimension vector $\alpha \in \mathbb{C}[Q_0]$ and define

$$(1.10) \quad R(Q, \alpha) := \bigoplus_{e \in Q_1} \text{Hom}(W_{t(e)}, W_{h(e)}).$$

This is too big: the same isomorphism class appears at more than one point. We can mod out by a symmetry: let

$$(1.11) \quad \mathrm{GL}(\alpha) := \prod_{v \in Q_0} \mathrm{GL}(W_v)$$

act on $R(Q, \alpha)$ by a change of basis on each vector space and on ϕ_e as

$$(1.12) \quad (g\phi)_e = g_{h(e)}\phi_e g_{t(e)}^{-1}.$$

Then as a set the quotient $R(Q, \alpha)/\mathrm{GL}(\alpha)$ contains one element for each isomorphism class. But putting a geometric structure on quotients of varieties is tricky. We'll come back to this point.

Example 1.13. Let Q be the Kronecker quiver and $\alpha = (1, 1)$, so that $\mathrm{GL}(\alpha) = k^\times \times k^\times$. Pick $(t, s) \in \mathrm{GL}(\alpha)$; the action on a Q -representation $(\lambda, \mu): k \rightrightarrows k$ produces $(s\lambda t^{-1}, s\mu t^{-1}): k \rightrightarrows k$. So if $s = t$, the action is trivial. Quotienting out by the diagonal $s = t$ in $k^\times \times k^\times$, we get $k^\times: (s, t) \mapsto s/t$, and this acts on $R(Q, \alpha) = k^2$ by scalar multiplication.

This is an action we know well: the quotient is the space of lines in k^2 , also known as \mathbb{P}_k^1 – and the zero orbit. This orbit makes life more of a headache: you can't just throw it out, because then you don't get a good map to the quotient, preimages of closed things aren't always closed, etc. But the action on the zero orbit is not free. This phenomenon will appear a lot, and we'll in general have to think about what to remove. After some hard work we'll be able to take the quotient in a reasonable way and get \mathbb{P}^1 . ◀

A crash course on (linear) algebraic groups. If you want to learn more about algebraic groups, especially because we're not going to give proofs, there are several books called *Linear Algebraic Groups*: the professor recommends Humphreys' book [Hum75] with that title, and also those of Borel [Bor91] and Springer [Spr98].

Definition 1.14. An *algebraic group* is a variety G together with a group structure such that multiplication and taking inverses are morphisms of varieties.

You can guess what a morphism of algebraic groups is: a group homomorphism that's also a map of varieties.

Example 1.15. GL_n is an algebraic group. Inside the space of all $n \times n$ matrices, which is a vector space over k , GL_n is the set of matrices with nonzero determinant. This is an open condition, and the determinant can be written in terms of polynomials, so GL_n is an algebraic group.

Other examples include SL_n and elliptic curves, and we can take products, so $\mathrm{GL}(\alpha)$ is also an algebraic group. ◀

Definition 1.16. A *linear algebraic group* is an algebraic group that admits a closed embedding $G \hookrightarrow \mathrm{GL}_n$ which is also a group homomorphism.

This does not include the data of the embedding. It turns out (this is in, e.g. Humphreys) that any affine algebraic group is linear, but this is not particularly easy to show.

Exercise 1.17. Show that any algebraic group is also a smooth variety.

This does not generalize to group schemes!

We care about groups because they act. We added structure to algebraic groups, and thus care about actions which behave nicely under that structure.

Definition 1.18. A *group action* of an algebraic group G on a variety X is a morphism $\varphi: G \times X \rightarrow X$ such that for all $g, h \in G$ and $x \in X$,

- (1) $\varphi(g, \varphi(h, x)) = \varphi(gh, x)$, and
- (2) $\varphi(e, x) = x$.

Example 1.19. k^\times acts on \mathbb{A}_k^{n+1} by scalar multiplication. What's the quotient? We want \mathbb{P}_k^n , but there's also the zero orbit, and no other orbit is closed. This makes us sad; we're going to use geometric invariant theory (GIT) to address these issues and become less sad. ◀

Definition 1.20. Let G be an algebraic group.

- A *character* of G is a morphism of algebraic groups $\chi: G \rightarrow k^\times$. These form a group under pointwise multiplication, and we'll denote this group $X(G)$.
- A *one-parameter subgroup* of G , also called a *cocharacter*, is a morphism of algebraic groups $\lambda: k^\times \rightarrow G$.

Example 1.21. Since $\det(AB) = \det A \det B$, the determinant defines a character of GL_n . One example of a cocharacter is $\lambda: k^\times \rightarrow \mathrm{GL}_2$ sending $t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^2 \end{pmatrix}$. This cocharacter factors through the diagonal matrices in GL_n ; this turns out to be a general fact. ◀

Here are a few nice facts about characters and cocharacters.

Theorem 1.22.

- (1) The map $\mathbb{Z} \rightarrow X(\mathrm{GL}_n)$ sending $m \mapsto \det^m$ is an isomorphism.
- (2) If G and H are algebraic groups, the map $X(G) \times X(H) \rightarrow X(G \times H)$ sending

$$(1.23) \quad (\chi_1, \chi_2) \mapsto ((g, h) \mapsto \chi_1(g)\chi_2(h))$$
 is an isomorphism.
- (3) Up to conjugation, every cocharacter of GL_n lands in the subgroup of diagonal matrices, hence sends $t \mapsto \mathrm{diag}(t^{a_1}, \dots, t^{a_n})$ for $a_1, \dots, a_n \in \mathbb{Z}$.

We're not going to prove these: this would require a considerable detour into the theory of algebraic groups to get to, and you can read the proofs in Humphreys.

Exercise 1.24. Without using the above theorem, show that any morphism of algebraic groups $k^\times \rightarrow k^\times$ is of the form $t \mapsto t^n$ for some $n \in \mathbb{Z}$.

Lecture 2.

Geometric invariant theory: 1/24/19

Today we'll discuss some more geometric invariant theory and how to take quotients.

Nagata reinterpreted Hilbert's 14th problem as follows.

Problem 2.1. Let G be a linear algebraic group acting linearly on a finite-dimensional k -vector space V . Is the *ring of invariants*

$$\mathcal{O}(V)^G = \{f \in \mathcal{O}(V) \mid f(gx) = f(x) \text{ for all } g \in G, x \in V\}$$

finitely generated?

The elements of $\mathcal{O}(V)^G$ are called the *invariant polynomials* or *invariant functions* on V .

Nagata proved that this is not always true, though there is a positive answer with some assumptions on G . For example, GL_n and products of general linear groups satisfy this property.

Definition 2.2. A linear algebraic group G is *geometrically reductive* if for every linear action of G on a finite-dimensional vector space V (i.e. a map of algebraic groups $\varphi: G \rightarrow \mathrm{GL}(V)$) and every fixed point $v \in V$ of the G -action, there is an invariant homogeneous nonconstant polynomial f with $f(v) = 0$.

Remark 2.3. There is a different notion of a reductive group, and it is different. Sorry about that. ◀

Theorem 2.4 (Nagata [Nag63]). *If G is geometrically reductive, Problem 2.1 has a positive answer.*

If $\mathrm{char}(k) = 0$, basic facts from the theory of algebraic groups allow one to prove GL_n is geometrically reductive, and in fact in characteristic zero reductive implies geometrically reductive. This is also true in positive characteristic, but is significantly harder!

Remark 2.5. In fact, in characteristic zero, the polynomial f in the definition of geometrically reductive can be chosen such that $\deg(f) = 1$. This property is called *linearly reductive*, so in characteristic zero, reductive, geometrically reductive, and linearly reductive coincide. This is not true in positive characteristic, which is ultimately because of everyone's favorite fact about modular representation theory: representations of a group in positive characteristic need not be semisimple. ◀

Mumford conjectured the following.

Theorem 2.6 (Haboush [Hab75]). *If k is algebraically closed and G is reductive, then G is geometrically reductive.*

The difficulty was in positive characteristic.

This led to the first idea of a better quotient: take Spec of the ring of invariants; by this theorem, this gives you a variety. But sometimes this is too small: for \mathbb{C}^\times acting on \mathbb{C}^n , this tells you the closed orbits. The only closed orbit is the zero orbit, so we don't get \mathbb{P}^{n-1} , alas.

To abrogate this, we'll introduce a numerical criterion. Let G be a geometrically reductive group acting linearly on a finite-dimensional vector space V . Recall that $\mathcal{O}(V)$ is also denoted $k[V]$, the ring of polynomials on V .

Definition 2.7. Let $\chi \in X(G)$ be a character of G .

- (1) An $f \in \mathcal{O}(V)$ is *relatively invariant* of weight χ if $f(gx) = \chi(g)f(x)$ for all $x \in V$ and $g \in G$. We let $\mathcal{O}(V)^{G, \chi}$ denote the vector space of relatively invariant functions of weight χ , so that $\mathcal{O}(V)^{G, \chi^0} = \mathcal{O}(V)^G$.
- (2) Define

$$(2.8) \quad V // (G, \chi) := \text{Proj} \left(\bigoplus_{n \geq 0} \mathcal{O}(V)^{G, \chi^n} \right).$$

We let $V // G := \text{Spec}(\mathcal{O}(V)^G)$.

One can check quickly that the product of relatively invariant functions of weights χ^m and χ^n is relatively invariant of weight χ^{m+n} , so the graded abelian group in (2.8) is in fact a graded ring.

Theorem 2.9. *There's a natural map $V // (G, \chi) \rightarrow V // G$, and this map is projective.*

Example 2.10. Consider k^\times acting on k^{m+1} by scalar multiplication and $\chi: k^\times \rightarrow k^\times$. Then $k[x_0, \dots, x_m]^{k^\times, \chi^n}$ is exactly the vector space of degree- n homogeneous polynomials. Then

$$(2.11) \quad k^{m+1} // (k^\times, \text{id}) = \text{Proj}(k[x_0, \dots, x_m]) = \mathbb{P}^m,$$

where we give $k[x_0, \dots, x_m]$ its usual grading.

However, if you use other characters, you'll get something different: for $\chi = 1$ you get a single point, and for $\chi = -\text{id}$ the quotient is empty. ◀

We've been calling $V // (G, \chi)$ a "quotient," but is it really one? We'd like to say it has nice properties that a quotient should have, but in the above example, there isn't a nice map $k^{m+1} \twoheadrightarrow \mathbb{P}^m$. In general we get a nice map like that on an open subset; let's figure out what map that is.

Let Δ be the kernel of $\varphi: G \rightarrow \text{GL}(V)$.¹

Definition 2.12.

- (1) An $x \in V$ is called χ -semistable for a character χ if there is an $f \in \mathcal{O}(V)^{G, \chi^n}$ for some $n \geq 1$ such that $f(x) \neq 0$. The locus of χ -semistable points is denoted V_χ^{ss} .
- (2) If x is χ -semistable and we can choose f such that the orbit $G \cdot x \subset \{x \in V \mid f(x) \neq 0\}$ is closed, and $\dim G \cdot x = \dim G - \dim \Delta$, we call x χ -stable. The locus of χ -stable points is denoted V_χ^s .

Stability means that the orbit of x has the largest possible dimension.

Lemma 2.13. V_χ^{ss} and V_χ^s are Zariski open subsets of V .

The main theorem of geometric invariant theory in this setting² is:

Theorem 2.14 (Mumford). *There is a surjective morphism $\phi: V_\chi^{\text{ss}} \rightarrow V // (G, \chi)$ such that if $x, y \in V_\chi^{\text{ss}}$,*

- (1) *if $x, y \in V_\chi^s$, then $\phi(x) = \phi(y)$ iff $y \in G \cdot x$, and*
- (2) *in general, $\phi(x) = \phi(y)$ iff $\overline{G \cdot x} \cap \overline{G \cdot y}$ is nonempty, where closures are taken inside V_χ^{ss} .*

You can think of ϕ as the map from the original space to the quotient, but we can only see a subset of the original space. For stable points, this actually parameterizes orbits, but this isn't quite true for merely semistable points, and the problem occurs when orbits aren't closed.

¹In many references, φ is assumed to be injective.

²Mumford showed a version where G can act on any quasiprojective variety.

Definition 2.15. If $\overline{G \cdot x} \cap \overline{G \cdot y}$ is nonempty, we say x and y are S -equivalent.

Remark 2.16. S is for Seshadri, who was one of the developers of this theory. ◀

The numerical criterion we alluded to earlier is a way to find semistable points.

Definition 2.17. Let $\chi: G \rightarrow k^\times$ be a character and $\lambda: k^\times \rightarrow G$ be a cocharacter. The composition $\chi \circ \lambda: k^\times \rightarrow k^\times$ sends $t \mapsto t^n$ for some $n \in \mathbb{Z}$; we denote $\langle \chi, \lambda \rangle := n$.

Theorem 2.18 (Mumford's numerical criterion).

- (1) An $x \in V$ is χ -semistable iff $\chi(\Delta) = 1$ and for all cocharacters $\lambda: k^\times \rightarrow G$ such that $\lim_{t \rightarrow 0} \lambda(t)x$ exists, then $\langle \chi, \lambda \rangle \geq 0$.
- (2) x is χ -stable iff it's χ -semistable and if λ is as above and $\langle \chi, \lambda \rangle > 0$, then $\lambda(k^\times) \subset \Delta$.

That limit works fine in \mathbb{C} , but what about over other fields? It's obvious in formulas, and in general you can define it in terms of trying to extend to a map of varieties $k \rightarrow G$.

Proposition 2.19.

- (1) The orbit $G \cdot x$ is closed in V_χ^{ss} if for every cocharacter λ with $\langle \chi, \lambda \rangle = 0$ such that the limit $\lim_{t \rightarrow 0} \lambda(t)x$ exists, then the limit is in $G \cdot x$.
- (2) If $x, y \in V_\chi^{\text{ss}}$, then x and y are S -equivalent iff there are cocharacters λ_1, λ_2 with $\langle \chi, \lambda_1 \rangle = \langle \chi, \lambda_2 \rangle = 0$ such that $\lim_{t \rightarrow 0} \lambda_1(t)x$ and $\lim_{t \rightarrow 0} \lambda_2(t)y$ both exist and are in the same orbit.

Example 2.20. Consider $G = \text{GL}_2$ acting on the space V of 4×2 matrices: to obtain a left action by g , we multiply on the right by g^{-1} . Let $\chi: \text{GL}_2 \rightarrow k^\times$ be \det^{-1} .

What do we expect to parameterize in the quotient? A 4×2 matrix is a linear map $k^2 \rightarrow k^4$, and we're parameterizing them up to change of basis of the domain. This should morally parameterize two-dimensional subspaces of k^4 , though we never stipulated that our maps are injective. Maybe, hopefully, the open subset of semistable points are the injective maps and we'll get the Grassmannian $\text{Gr}_2(k^4)$.

We claim this is actually the case, and will use the numerical criterion to prove it. Since GL_2 acts faithfully on V , $\Delta = 1$ and the situation simplifies somewhat. We can use a group action to make the cocharacter simpler, or to make a general element of V simpler, but not both. So we'll do the former: let $\lambda = \begin{pmatrix} t^n & 0 \\ 0 & t^m \end{pmatrix}$ and

$$(2.21) \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix}, \quad \text{so that} \quad \lambda(t)A = \begin{pmatrix} a_{11}t^{-n} & a_{12}t^{-m} \\ a_{21}t^{-n} & a_{22}t^{-m} \\ a_{31}t^{-n} & a_{32}t^{-m} \\ a_{41}t^{-n} & a_{42}t^{-m} \end{pmatrix}.$$

Since $\det^{-1}(\lambda(t)) = t^{-n-m}$, $\langle \det^{-1}, \lambda \rangle = -n - m$. Therefore $\lim_{t \rightarrow 0} \lambda(t)$ exists iff either

- (1) $A = 0$, which isn't semistable, because the limit exists for every cocharacter; or
- (2) $a_{11} = \dots = a_{41} = 0$ and $a_{j2} \neq 0$ for some j , and $m \leq 0$, which is also unstable (e.g. $m = 0, n = 1$); or
- (3) $a_{2i} = 0$ for all i , and $a_{j1} \neq 0$ for some j , and $n \leq 0$, which is again unstable; or
- (4) $a_{i1} \neq 0$ for some i and $a_{j2} \neq 0$ for some j , and $m, n \leq 0$, so $\langle \chi, \lambda \rangle = -n - m \geq 0$, and these A are stable.

Now, let's look at an arbitrary cocharacter. This involves changing basis/looking at full orbits of points we found were unstable. When $A = 0$ (case (1)), this is the whole orbit, and it's unstable. For (2) and (3), A has rank 1 in the entire orbit, and therefore these are all unstable. All matrices of rank 2 are stable. ◀

Lecture 3.

Constructing moduli spaces of quiver representations: 1/29/19

Today, we're going to leverage the GIT theory we surveyed in the last lecture to define moduli spaces of quiver representations.

We begin with a quick review of Mumford's numerical criterion, since it will be an important actor today. Let G be an algebraic group acting on a k -vector space V , and let Δ denote the kernel of the associated map $\rho: G \rightarrow \text{GL}(V)$. Let $\chi: G \rightarrow k^\times$ be a character. Then, in Theorem 2.18, we saw that $x \in V$ is χ -semistable

iff $\chi(\Delta) = 1$ and for all cocharacters $\lambda: k^\times \rightarrow G$ such that $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ exists, $\langle \chi, \lambda \rangle = 0$. Moreover, x is χ -stable if in addition whenever $\langle \chi, \lambda \rangle = 0$, then $\lambda(k^\times) \subset \Delta$.

Now back to quivers. Consider a quiver Q with a set Q_0 of vertices, Q_1 of edges, and head and tail maps $h, t: Q_1 \rightrightarrows Q_0$. Let $\alpha \in \mathbb{C}[Q_0]$ be a dimension vector and vector spaces W_v of dimension $\alpha(v)$ for each $v \in Q_0$. We constructed the space

$$(3.1) \quad R(Q, \alpha) := \prod_{e \in Q_1} \text{Hom}(W_{t(e)}, W_{h(e)}),$$

but this is too big to be a moduli space of quiver representation: it contains different points that correspond to isomorphic representations. Therefore we want to take the quotient by $\text{GL}(\alpha)$ as we described, so let's apply GIT to this action and understand stability.

In this setting, the kernel Δ is the *long diagonal* $\{(t \text{id}, \dots, t \text{id}) \in \text{GL}(\alpha) \mid t \in k^\times\}$, which is isomorphic to k^\times . If $\theta \in \mathbb{Z}[Q_0]$, it defines a character $\chi: \text{GL}(\alpha) \rightarrow k^\times$ by

$$(3.2) \quad \chi_\theta(g) := \prod_{v \in Q_0} g_v^{\theta(v)}.$$

All characters of GL_n can be written in this way.

Definition 3.3. Let A be an abelian category. Its *Grothendieck group* is

$$(3.4) \quad K_0(A) := \bigoplus_{A \in A} \mathbb{Z}[A] / \sim,$$

where we quotient by an equivalence relation: for all short exact sequences $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, we say $[B] \sim [A] + [C]$.

The $\theta \in \mathbb{Z}[Q_0]$ as above can also be thought of as a function on the Grothendieck group of Rep_Q , the category of finite-dimensional representations of Q . Specifically, $\theta: K_0(\text{Rep}_Q) \rightarrow \mathbb{Z}$ is defined to send

$$(3.5) \quad M = (\{M_v\}, \{\psi_e\}) \mapsto \sum_{v \in Q_0} \theta(v) \cdot \dim(M_v).$$

We want there to be semistable points, which means we will only consider θ such that

$$(3.6) \quad \chi_\theta(\Delta) = \left\{ \prod_{v \in Q_0} t^{\theta(v)\alpha(v)} \mid t \in k^\times \right\} = \{1\},$$

i.e. such that

$$(3.7) \quad \sum_{v \in Q_0} \theta(v)\alpha(v) = 0.$$

Among other things, this means that $\theta(M) = 0$ if $\dim M = \alpha$.

Now we'll perform the GIT analysis. Let $\lambda: k^\times \rightarrow \text{GL}(\alpha)$ be a cocharacter and $M = (\{M_v\}, \{\phi_e\}) \in \text{Rep}_Q$ have dimension α . The first step will be to construct a descending \mathbb{Z} -indexed filtration

$$(3.8) \quad M \supseteq \dots \supseteq M^{(n)} \supseteq M^{(n+1)} \supseteq \dots$$

For each $v \in Q_0$, pick a decomposition

$$(3.9) \quad M_v = \bigoplus_{n \in \mathbb{Z}} M_v^{(n)},$$

such that $\lambda(t)$ acts on $M_v^{(n)}$ by multiplication by t^n . (Recall that any cocharacter of GL_n has diagonal image up to conjugation, so this makes sense.) Define

$$(3.10) \quad M_v^{(\geq n)} := \bigoplus_{m \geq n} M_v^{(m)}.$$

For each $e \in Q_1$, let $\phi_e^{(m,n)}$ denote the composition

$$(3.11) \quad M_{t(e)}^{(n)} \hookrightarrow M_{t(e)} \xrightarrow{\phi_e} M_{h(e)} \twoheadrightarrow M_{h(e)}^{(m)},$$

where the projection comes from the decomposition (3.9). Then

$$(3.12) \quad \lambda(t) \cdot \phi_e^{(m,n)} = t^m \phi_e^{(m,n)} t^{-n} = t^{m-n} \phi_e^{(m,n)},$$

which means the following are equivalent:

- $\lim_{t \rightarrow 0} \lambda(t) \phi_e$ exists,
- $\phi_e^{(m,n)} = 0$ whenever $m \leq n$, and
- ϕ_e maps $M_{t(e)}^{(\geq n)}$ into $M_{h(e)}^{(\geq n)}$.

The third condition means that $M_n := (M^{(\geq n)}, \phi_e|_{M^{(\geq n)}})$ is a subrepresentation of M , so if the limit exists it induces the desired filtration of M (3.8). In this case $W_v^{(n)} = W_v^{(\geq n)} / W_v^{(\geq n+1)}$.

Conversely, given a filtration of M as in (3.8), we can produce a cocharacter $\lambda: k^\times \rightarrow \mathrm{GL}(\alpha)$: define $\lambda(t)$ to act by t^n on $W_v^{(n)}$.

Since

$$(3.13) \quad \lim_{t \rightarrow 0} \lambda(t) \phi_e^{(m,n)} = \lim_{t \rightarrow 0} t^{m-n} \phi_e^{(m,n)} = 0$$

for $m > n$, then

$$(3.14) \quad \lim_{t \rightarrow 0} \lambda(t) \phi_e: M_{t(e)}^{(n)} \longrightarrow M_{h(e)}^{(n)} \subset M_{h(e)}^{(\geq n)}.$$

Thus the limit is the associated graded:

$$(3.15) \quad \lim_{t \rightarrow 0} \lambda(t) \cdot M = \bigoplus_{n \in \mathbb{Z}} M_n / M_{n+1}.$$

Remark 3.16. There's an n such that $M_n = 0$ and $M_n \neq M$. In other words, this filtration isn't the trivial one. This is because $\lambda(k^\times) \subset \Delta$. \blacktriangleleft

Now let's discuss (semi)stability. It will turn out to be equivalent to the following notion.

Definition 3.17. Let $M \in \mathrm{Rep}_Q$ have dimension α and be such that $\theta(M) = 0$. Then M is θ -semistable (resp. θ -stable) iff for all nonzero proper subrepresentations $N \subset M$, $\theta(N) \geq 0$ (resp. $\theta(N) > 0$).

Theorem 3.18 (King [Kin94]). A point $M \in R(Q, \alpha)$ is χ_θ -semistable (resp. χ_θ -stable) iff $M \in \mathrm{Rep}_Q$ is θ -semistable (resp. θ -stable).

Proof. First, let's show θ -(semi)stability implies GIT (semi)stability. We assumed $\sum_v \theta(v) \alpha(v) = 0$, which implies

$$(3.19) \quad \langle \chi_\theta, \lambda \rangle = \sum_{v \in Q_0} \theta(v) \sum_{n \in \mathbb{Z}} n \dim M_v^{(n)}.$$

We can change the order of summation because only finitely many $W_v^{(n)}$ are nonzero for v fixed, so

$$(3.20a) \quad \langle \chi_\theta, \lambda \rangle = \sum_{n \in \mathbb{Z}} n \sum_{v \in Q_0} \theta(v) \dim M_v^{(n)}$$

$$(3.20b) \quad = \sum_{n \in \mathbb{Z}} n \theta(M_n / M_{n+1}).$$

Since θ factors through the Grothendieck group, this is

$$(3.20c) \quad = \sum_{n \in \mathbb{Z}} n (\theta(M_n) - \theta(M_{n-1}))$$

$$(3.20d) \quad = \sum_{n \in \mathbb{Z}} \theta(M_n),$$

unwinding the telescoping series. This is nonnegative if M is θ -semistable and positive if M is θ -stable, by definition.

Conversely, suppose M is χ_θ -semistable and let N be a subrepresentation of M . Then $N \subset M$ is a filtration, hence defines a cocharacter $\lambda: k^\times \rightarrow \mathrm{GL}(\alpha)$ such that

$$(3.21) \quad 0 \stackrel{(<)}{\leq} \langle \chi_\theta, \lambda \rangle = \theta(M) + \theta(N)$$

(parentheses for stability), so $\theta(N) \geq -\theta(M) = 0$ (or $>$ for stability). \boxtimes

Now we have semistable points, and even strictly semistable points. What does S -equivalence look like in this context?

Definition 3.22. Given θ as above, let $P_\theta \subset \text{Rep}_Q$ denote the full subcategory of θ -semistable representations with $\theta(M) = 0$ (so, those objects, and all of the morphisms between them).

Lemma 3.23. P_θ is an abelian subcategory of Rep_Q . That is, let $\varphi: M \rightarrow N$ be a morphism in P_θ , i.e. $\theta(M) = \theta(N) = 0$ and M and N are θ -semistable. Then, $A := \ker(\varphi)$ and $B = \text{coker}(\varphi)$, where the kernel and cokernel are taken in Rep_Q , are in P_θ .

Proof. The kernel and cokernel fit into an exact sequence

$$(3.24) \quad 0 \longrightarrow A \longrightarrow M \longrightarrow N \longrightarrow B \longrightarrow 0.$$

Since θ is additive under short exact sequences, it in fact satisfies

$$(3.25) \quad \theta(A) + \underbrace{\theta(N)}_{=0} = \underbrace{\theta(M)}_{=0} + \theta(B),$$

so $\theta(A) = \theta(B)$. Since M is θ -semistable, $\theta(A) \geq 0$. If $K := \ker(N \rightarrow B) \hookrightarrow N$, and N is θ -semistable, then $\theta(K) \geq 0$. Therefore

$$(3.26) \quad \theta(A) = \theta(B) = \theta(N) - \theta(K) = -\theta(K) \leq 0.$$

Thus $\theta(A) = \theta(B) = 0$.

Now for semistability. Briefly, if $C \subset A$ is a subrepresentation, then it's also a subrepresentation of M , so $\theta(C) \geq 0$. The argument for B is similar. \boxtimes

Lecture 4.

Examples of quiver varieties: 1/31/19

Fix a quiver Q . Last time we explained how, given a $\theta \in \mathbb{Z}[Q_0]$, we obtain a function on objects of Rep_Q additive on short exact sequences: $\theta(M) := \sum_{v \in Q_0} \theta_v \cdot \dim(M_v)$, and we also get a character χ_θ of $\text{GL}(\alpha)$, which has weight $\theta(v)$ on the component indexed by v . In Theorem 3.18, we provided a criterion for semistability: a quiver representation M is χ_θ -semistable iff $\theta(M) = 0$ and for all $N \subsetneq M$, $\theta(N) \geq 0$. (If $\theta(N) > 0$, M is χ_θ -stable.)

We then constructed an abelian category P_θ of χ_θ -semistable objects.

Proposition 4.1. P_θ is a finite-length category, i.e. all of its objects are both Noetherian and Artinian.

Theorem 4.2 (Jordan-Hölder). *Let A be a finite-length abelian category. Then any object $M \in A$ has a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ such that M_i / M_{i-1} is simple for all i . Moreover, the associated graded*

$$(4.3) \quad \text{gr}(M) := \bigoplus_i M_i / M_{i-1}$$

is unique up to isomorphism.

This filtration satisfies a weak uniqueness condition – it's not unique, but is close to it. For example, if A and B are two simple objects, $A \oplus B$ has two filtrations $0 \rightarrow A \rightarrow A \oplus B$ and $0 \rightarrow B \rightarrow A \oplus B$. Such a filtration is called a *Jordan-Hölder filtration* of M .

Exercise 4.4. In P_θ , M is simple iff it's θ -stable.

Proposition 4.5.

- (1) A θ -semistable Q -representation M corresponds to a closed $\text{GL}(\alpha)$ orbit in $R_{\chi_\theta}^{\text{ss}}(Q, \alpha)$ iff M is semisimple (i.e. a direct sum of simple objects). Equivalently, $M \cong \text{gr}(M)$.
- (2) Two θ -semistable Q -representations M and N are S -equivalent iff $\text{gr}(M) \cong \text{gr}(N)$.

The proof was left as an exercise, but is not too difficult given how we proved things last class (and is a good way to see if you understood the proof).

We will let $M_\theta(Q, \alpha)$ denote the GIT quotient by $\text{GL}(\alpha)$ for the character χ_θ .

Example 4.6. Consider the quiver

$$(4.7) \quad Q = \bullet \rightrightarrows \bullet.$$

Let's choose $\alpha = (2, 1)$, so $R(Q, \alpha) = \text{Hom}(k^2, k)^{\oplus 4}$. Then $\text{GL}(\alpha) = \text{GL}_2 \times k^\times$. If you choose $\theta = (-1, 1)$, then $\chi_\theta = \det^{-1}$.

The long diagonal of $\text{GL}(\alpha)$ acts trivially, so let's pass to the quotient via the map $\varphi: \text{GL}(\alpha) \rightarrow \text{GL}_2$ by $(g, t) \mapsto gt^{-1}$ (here, $g \in \text{GL}_2$ and $t \in k^\times$). Now we have the same scenario as in Example 2.20 – so we leverage our work there and conclude the quiver moduli space is $\text{Gr}_2(k^4)$. \blacktriangleleft

Example 4.8. Now let's consider a slightly more interesting quiver,

$$(4.9) \quad Q = \bullet \rightrightarrows \bullet \leftrightsquigarrow \bullet.$$

Let $\alpha = (1, 1, 1)$, so $R(Q, \alpha) = \text{Hom}(k, k)^2 \times \text{Hom}(k, k)^2$. Choose the character $\theta = (a, b, c) = (a, -a - c, c)$, where $a + b + c = 0$.

Because α is small, there aren't many subrepresentations. For example, the trivial representation $k \rightrightarrows k \leftrightsquigarrow k$ has as a subrepresentation $S_2 := (0 \rightrightarrows k \leftrightsquigarrow 0)$. Since $\theta(S_2) = -a - c$, we must have $-a - c \geq 0$ or all representations are unstable.

By defining more representations, we can infer more about what constraints to put on θ to have a good moduli space. For example, given $x, y, z, w \in k$, let

$$(4.10a) \quad N_1(x, y) := k \xrightarrow[y]{x} k \leftrightsquigarrow 0$$

$$(4.10b) \quad N_2(z, w) := 0 \rightrightarrows k \xleftarrow[w]{z} k$$

$$(4.10c) \quad M(x, y, z, w) := k \xrightarrow[y]{x} k \xleftarrow[w]{z} k.$$

Thus $N_1(x, y)$ and $N_2(z, w)$ are both subrepresentations of $M(x, y, z, w)$. Because

$$(4.11a) \quad \theta(N_1(x, y)) = a + (-a - c) = -c \geq 0$$

$$(4.11b) \quad \theta(N_2(x, y)) = -a \geq 0,$$

we know both a and c must be negative.

There are a few more potential subrepresentations,

$$(4.12a) \quad S_1 := k \rightrightarrows 0 \leftrightsquigarrow 0$$

$$(4.12b) \quad S_2 := 0 \rightrightarrows 0 \leftrightsquigarrow k$$

$$(4.12c) \quad N_3 := S_1 \oplus S_2.$$

Using these, we observe that

- $S_1 \hookrightarrow M(x, y, z, w)$ iff $x = y = 0$, and in this case, we need $\theta(S_1) = a \geq 0$, so $a = 0$;
- $S_2 \hookrightarrow M(x, y, z, w)$ iff $z = w = 0$, and in this case, we need $\theta(S_2) = c \geq 0$, so $c = 0$; and
- $N_3 \hookrightarrow M(x, y, z, w)$ iff $x = y = z = w = 0$, and in this case we need $a = c = 0$.

In summary:

- (1) If $a < 0$ and $c < 0$, then $M(x, y, z, w)$ is θ -stable iff $(x, y) \neq 0$ and $(z, w) \neq 0$. Moreover, there are no *strictly semistable* (i.e. semistable but not stable) representations. At this point you might guess that the GIT quotient is $\mathbb{P}^1 \times \mathbb{P}^1$.
- (2) If $a = 0$ and $c < 0$, then $M(x, y, z, w)$ is θ -semistable iff $(z, w) \neq 0$, and there are no θ -stable representations. Now suppose $M(x, y, z, w)$ is such that $(z, w) \neq 0$; then, $M(x, y, z, w)$ and $M(x', y', z', w')$ are S -equivalent iff $(z', w') = \lambda(z, w)$ for some $\lambda \in k^\times$. In this case the GIT quotient will be a \mathbb{P}^1 , some sort of boundary where things get mushed together.
- (3) In the same way, if $c = 0$ and $a < 0$, then $M(x, y, z, w)$ is θ -semistable iff $(x, y) \neq 0$, and there are no θ -stable representations. Now suppose $M(x, y, z, w)$ is such that $(x, y) \neq 0$; then, $M(x, y, z, w)$ and $M(x', y', z', w')$ are S -equivalent iff $(x', y') = \lambda(x, y)$ for some $\lambda \in k^\times$. Again we get a \mathbb{P}^1 .
- (4) If $a = c = 0$, then all M are strictly semistable, and all of them are S -equivalent. The GIT quotient will be a point.

Even in this small case, things are complicated.

We haven't shown the statements above about S -equivalence, so let's do that. There's nothing in case (1), so let's look at (2). In this case $M(x, y, z, w) \sim M(x', y', z', w')$ iff $N(z, w) \sim N(z', w')$, and it's not hard to check that we can act precisely by scalars, so this is true iff (z', w') is a nonzero scalar multiple of (z, w) . (3) is the same. For (4), we have a three-stage Jordan-Hölder filtration:

$$(4.13) \quad 0 \subset S_2 \subset N_1(x, y) \subset M(x, y, z, w),$$

and the pieces of the associated graded are $N_1(x, y)/S_2 \cong S_1$ and $M/N_1(x, y) \cong S_3$. Therefore $\text{Gr}(M(x, y, z, w)) \cong S_1 \oplus S_2 \oplus S_3$ for any (x, y, z, w) , so they're all S -equivalent.

We know to expect $\mathbb{P}^1 \times \mathbb{P}^1$ if $a, c < 0$, or \mathbb{P}^1 if exactly one is zero, or a point if they're both zero. You can think of letting $a \rightarrow 0$ as projecting onto the first \mathbb{P}^1 , and letting $c \rightarrow 0$ as projecting onto the second \mathbb{P}^1 . But we haven't proven any of these yet! We don't even know that they're varieties *a priori*, but this is the correct answer, and it's possible to prove it. \blacktriangleleft

Remark 4.14. It turns out all quiver varieties are rational! You can get irrational varieties by imposing composition relations between arrows in the quiver; in fact, any projective variety arises in this way. \blacktriangleleft

The last thing we'll do is study some more general properties of quiver varieties.

Theorem 4.15. *If Q is acyclic quiver (i.e. it has no loops), then $M_\theta(Q, \alpha)$ is projective.*

Proof. $M_\theta(Q, \alpha) := R(Q, \alpha) // \text{GL}(\alpha)$ always has a projective morphism to $R(Q, \alpha) // \text{GL}(\alpha)$. That is, by definition, we have a projective morphism

$$(4.16) \quad \text{Proj} \left(\bigoplus_{n \geq 0} k[R(Q, \alpha)]^{\text{GL}(\alpha), \chi_\theta^n} \right) \longrightarrow \text{Spec} \left(k[R(Q, \alpha)]^{\text{GL}(\alpha)} \right).$$

It therefore suffices to prove the codomain is a point. Since $R(Q, \alpha) // \text{GL}(\alpha) = M_0(Q, \alpha)$, all representations have $\theta(M) = 0$.

Exercise 4.17. Show that if Q is acyclic and $M \in \text{Rep}_Q$ has dimension α , then $\text{gr}(M) = \bigoplus_{v \in Q_0} S_v$, where S_v is the “ δ -function”, a simple representation with k at v and 0 elsewhere.

Therefore they're all S -equivalent, and the quotient is a point. \square

Remark 4.18. The empty set is a projective variety, and may occur as the moduli space associated to an acyclic quiver. \blacktriangleleft

Lecture 5.

Moduli spaces and slope stability: 2/5/19

“19th-century math was a different world.”

So we've constructed some GIT quotient $M_Q(\alpha, \theta)$ and claimed it's the moduli space of representations of the quiver Q with dimension α , and with (semi)stability tracked by the character θ of $\text{GL}(\alpha)$. But why does this deserve to be called a moduli space anyways? We'll begin by talking about this.

Definition 5.1. Let $\mathcal{M}: \text{Sch}_k^{\text{op}} \rightarrow \text{Set}$ be a functor.

- (1) A k -scheme M is a *fine moduli space* for \mathcal{M} if there's a natural isomorphism of functors $\mathcal{M} \xrightarrow{\cong} \text{Hom}(-, M)$.³ One says that M *represents* \mathcal{M} .

³We need to specify \mathcal{M} rather than just saying “this is a moduli space,” because every scheme X is a moduli space for $\text{Hom}(-, X)$. In general, though, when it comes to moduli spaces, you “know it when you see it,” which is why people sometimes don't say what \mathcal{M} is when it's clear from context.

- (2) A k -scheme M is a *coarse moduli space* for \mathcal{M} if there's a natural transformation $\iota_M: \mathcal{M} \Rightarrow \text{Hom}(-, M)$ such that any other natural transformation $\iota_N: \mathcal{M} \Rightarrow \text{Hom}(-, N)$ for a k -scheme N factors through ι_M . That is, we ask that there's a unique map $f: M \rightarrow N$ such that the diagram

$$(5.2) \quad \begin{array}{ccc} \mathcal{M} & \xrightarrow{\iota_M} & \text{Hom}(-, M) \\ & \searrow \iota_N & \downarrow f \circ - \\ & & \text{Hom}(-, N). \end{array}$$

We'll write down a functor related to quiver representations, and $\mathcal{M}_Q(\alpha, \theta)$ will be a coarse or fine moduli space, depending on whether we have semistable points. A fine moduli space would be the nicest thing, but we won't always get it.

If B is a k -scheme and \mathcal{M} is a moduli space of doodads (whatever a doodad is), we want $\mathcal{M}(B)$ to be the set of "families of doodads parameterized by B ." What this means precisely depends on the details of the specific setting. Here's what it is for quiver representations.

Definition 5.3. Let Q be a quiver and B be a k -scheme. A *family of Q -representations on B* is data of a locally free sheaf \mathcal{W}_v for each $v \in Q_0$ and morphisms $\phi(e): \mathcal{W}_{t(a)} \rightarrow \mathcal{W}_{s(a)}$ for all $e \in Q_1$. The *rank* of a family of representations is as before: the element of $\mathbb{N}[Q_0]$ sending $v \mapsto \text{rank } \mathcal{W}_v$.

If $\alpha \in \mathbb{N}[Q_0]$ and θ is a character of $\text{GL}(\alpha)$, then a rank- α Q -representation is *θ -semistable* if all of its fibers are.

Given a family of Q -representations \mathcal{W} over B and a line bundle $L \rightarrow B$, we can tensor them together to obtain a family of Q -representation $\mathcal{W} \otimes V$ with $(\mathcal{W} \otimes V)_v := \mathcal{W}_v \otimes V$ and $\phi_{\mathcal{W} \otimes V}(e) := \phi_{\mathcal{W}} \otimes \text{id}_V$.

Therefore we have a functor $\mathcal{M}_Q(\alpha, \theta): \text{Sch}_k \rightarrow \text{Set}$ which sends a scheme B to the set of equivalence classes of families of θ -semistable Q -representations with rank α , where we say \mathcal{W} and \mathcal{W}' are equivalent if there is a line bundle $L \rightarrow B$ and an isomorphism $\mathcal{W}' \cong \mathcal{W} \otimes L$. Given a map of schemes $\phi: B \rightarrow B'$, we get a map between these sets by pulling back each \mathcal{W}_v and ϕ .

Remark 5.4. Why this equivalence relation? Well first we want to pass to isomorphism classes anyways, but the idea is that \mathcal{W} and $\mathcal{W} \otimes L$ aren't really very different with respect to the representation theory of Q , so we identify them. \triangleleft

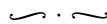
Theorem 5.5 ([Kin94]). Fix a quiver Q , dimension vector α , and character θ for $\text{GL}(\alpha)$.

- (1) $\mathcal{M}_Q(\alpha, \theta)$ is a coarse moduli space for $\mathcal{M}_Q(\alpha, \theta)$.
- (2) If α is indivisible in the ring $\mathbb{Z}[Q_0]$, then $\mathcal{M}_Q(\alpha, \theta)$ is a fine moduli space, and a smooth projective variety.

The proof isn't particularly enlightening, so we're going to skip it.

The reason we don't expect a fine moduli space in general is that semistable points can have automorphisms, which leads to a moduli stack, rather than a fine moduli space. When α is indivisible, it turns out that all representations are either θ -stable or θ -unstable, and θ -stable points have no automorphisms, making the stackiness go away.

Remark 5.6. If you restrict the functor $\mathcal{M}_Q(\alpha, \theta)$ to families of θ -stable representations, we also get a representable functor. In general, it's an open subfunctor of $\mathcal{M}_Q(\alpha, \theta)$, and often but not always is the smooth locus. \triangleleft



We're now done with quivers, and will move to other moduli spaces now. Bridgeland stability conditions, the theme of this class, will provide a general way to produce nicer moduli spaces – often you cannot make a fine moduli space of all of your objects, but you can do this for objects satisfying some kind of stability condition.

Our next class of examples will be moduli spaces of vector bundles on curves, which relates to something called slope stability. In this section, we assume the base field is \mathbb{C} ; probably these results hold in more generality, but the references were difficult to find.

Exercise 5.7. Let C be a smooth, projective curve and E be a coherent sheaf on C . Show that there is a unique exact sequence

$$(5.8) \quad 0 \longrightarrow T_E \longrightarrow E \longrightarrow F_E \longrightarrow 0,$$

where F_E is locally free and T_E is a torsion sheaf, and that this sequence noncanonically splits.

So over a curve, any coherent sheaf can be written as a direct sum of a vector bundle and a rank-zero, torsion sheaf. For this reason, on curves one often considers moduli spaces of vector bundles, where in higher dimensions one thinks about moduli spaces of sheaves.

The following theorem was probably known in the 19th century, though in its modern form it was proven by Grothendieck.

Theorem 5.9 (Grothendieck). *Let E be a vector bundle on \mathbb{P}^1 . Then there are unique integers a_1, \dots, a_n with $a_1 > \dots > a_n$ and unique nonzero vector spaces V_1, \dots, V_n such that*

$$E \cong \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i) \otimes V_i.$$

In other words, any vector bundle on \mathbb{P}^1 is a direct sum of line bundles.

Proof. First, existence. We'll induct on $r := \text{rank } E$. For $r = 1$ there's nothing to show. For $r > 1$, we'll show how to split off a line bundle, reducing to the inductive assumption. Serre duality tells us that

$$(5.10) \quad \text{Hom}(\mathcal{O}(a), E) \cong \text{Ext}^1(E, \mathcal{O}(a) \otimes \omega_{\mathbb{P}^1}) = H^1(E^\vee \otimes \mathcal{O}(a-2)) = 0 \text{ if } a \gg 0$$

by Serre vanishing. Here we're also using the fact that $\omega_{\mathbb{P}^1} = \mathcal{O}(-2)$. Let a be the largest integer such that $\text{Hom}(\mathcal{O}(a), E) \neq 0$, and fix a nonzero homomorphism $\phi: \mathcal{O}(a) \rightarrow E$.

Since $\mathcal{O}(a)$ is a line bundle and E is locally free, all map between them are either zero or injective; thus ϕ is injective, so we have a short exact sequence

$$(5.11) \quad 0 \longrightarrow \mathcal{O}(a) \xrightarrow{\phi} E \longrightarrow \text{coker}(\phi) \longrightarrow 0.$$

It suffices to show $F := \text{coker}(\phi)$ is locally free and (5.11) splits. Clearly this was not the same proof given in the 19th century!

First, that F is locally free. Using Exercise 5.7, we have a short exact sequence

$$(5.12) \quad 0 \longrightarrow T \longrightarrow F \longrightarrow G \longrightarrow 0,$$

where T is torsion and G is locally free. Assume T is nonzero. This means, because \mathbb{P}^1 is one-dimensional, T is supported in dimension zero, so there's some $x \in \mathbb{P}^1$ such that $\mathbb{C}(x)$ injects into T , where $\mathbb{C}(x)$ denotes the skyscraper sheaf with stalk \mathbb{C} at x ; in particular, since T is a subsheaf of F , $\text{Hom}(\mathbb{C}(x), F) \neq 0$.

We can pass from $\mathcal{O}(a)$ to $\mathcal{O}(a+1)$ by multiplying by an equation that cuts out x (since x has codimension 1, we only need one equation). So we have a short exact sequence

$$(5.13) \quad 0 \longrightarrow \mathcal{O}(a) \longrightarrow \mathcal{O}(a+1) \longrightarrow \mathbb{C}(x) \longrightarrow 0.$$

Applying $\text{Hom}(\mathbb{C}(x), -)$ to (5.12), we obtain a long exact sequence

$$(5.14) \quad \underbrace{\text{Hom}(\mathbb{C}(x), E)}_{=0} \longrightarrow \underbrace{\text{Hom}(\mathbb{C}(x), F)}_{\neq 0} \longrightarrow \text{Ext}^1(\mathbb{C}(x), \mathcal{O}(a)) \xrightarrow{(*)} \text{Ext}^1(\mathbb{C}(x), E).$$

We want to show that $\text{Hom}(E, \mathbb{C}(x)) \rightarrow \text{Hom}(\mathcal{O}(a), \mathbb{C}(x))$ is surjective in order to derive a contradiction; it suffices to show that $(*)$ is injective. Now weave in a piece of the long exact sequence given by applying

$\text{Hom}(-, \mathcal{O}(a))$ and $\text{Hom}(-, E)$ to (5.13):

$$\begin{array}{ccccccc}
 & & & & \text{Hom}(\mathcal{O}(a), \mathcal{O}(a)) & \xrightarrow{f} & \text{Hom}(\mathcal{O}(a), E) \\
 & & & & \downarrow g & & \downarrow h \\
 (5.15) \quad & \text{Hom}(\mathbb{C}(x), E) & \longrightarrow & \text{Hom}(\mathbb{C}(x), F) & \longrightarrow & \text{Ext}^1(\mathbb{C}(x), \mathcal{O}(a)) & \xrightarrow{(*)} \text{Ext}^1(\mathbb{C}(x), E) \\
 & \underbrace{\hspace{2cm}}_{=0} & & \underbrace{\hspace{2cm}}_{\neq 0} & & \downarrow & \\
 & & & & & \text{Ext}^1(\mathcal{O}(a+1), \mathcal{O}(a)).
 \end{array}$$

The reason that f is injective is that it sends $\text{id}_{\mathcal{O}(a)}$ to the inclusion map $\mathcal{O}(a) \hookrightarrow E$ we specified. Since $\text{Ext}^1(\mathcal{O}(a+1), \mathcal{O}(a)) = H^1(\mathbb{P}^1, \mathcal{O}(-1)) = 0$, then g is an isomorphism. Therefore $h \circ f$ is injective, so $(*)$ must be too.

Thus we've proven that F is locally free: by induction, $F = \bigoplus_j \mathcal{O}(b_j)^{\oplus r_j}$. We have left to show that $b_j \leq a$ for all j . Assume the opposite, that there's a j_0 such that $b_{j_0} > a$. Then the composition of the maps

$$(5.16) \quad \mathcal{O}_{\mathbb{P}^1}(a+1) \hookrightarrow \mathcal{O}_{\mathbb{P}^1}(b_{j_0}) \hookrightarrow F$$

is nonzero, so throwing $\text{Hom}(\mathcal{O}(a+1), -)$ at (5.11), we obtain a long exact sequence

$$(5.17) \quad \underbrace{\text{Hom}(\mathcal{O}(a+1), E)}_{\neq 0} \longrightarrow \underbrace{\text{Hom}(\mathcal{O}(a+1), F)}_{\neq 0} \longrightarrow \text{Ext}^1(\mathcal{O}(a+1), \mathcal{O}(a)) = H^1(\mathcal{O}(-1)) = 0,$$

which causes a contradiction. Uniqueness follows from another induction on r , using the fact that $\text{Hom}(\mathcal{O}(a), \mathcal{O}(b)) = 0$ if $a > b$.

Briefly, why does $0 \rightarrow \mathcal{O}(a) \rightarrow E \rightarrow \bigoplus_j \mathcal{O}(b_j)^{r_j}$ split? This is because the obstruction is $\text{Ext}^1(\mathcal{O}(b_j), \mathcal{O}(a)) = H^1(\mathcal{O}(a - b_j)) = 0$ whenever $a - b_j > 0$. \square

Lecture 6.

Slope stability: 2/7/19

The first part of the lecture involved fixing a mistake in the proof of Theorem 5.9. I included those fixes in the notes for the previous lecture; in particular, the proof that's written there should be correct.

In particular, we understand vector bundles on \mathbb{P}^1 : line bundles and what you can make out of them. But on other curves, vector bundles are more complicated, bringing in the notion of slope stability.

Let C be a smooth projective curve over \mathbb{C} (though most of this still works over any algebraically closed field).

Definition 6.1.

- (1) Let $E \rightarrow C$ be a vector bundle. The *degree* $d(E)$ of E is the degree of $\text{Det}(E) := \wedge^{\text{rank}(E)} E$ as a line bundle.
- (2) The *degree* of a torsion sheaf T , denoted $d(T)$, is the length of the scheme-theoretic support of T .
- (3) The *degree* of a coherent sheaf E is $d(E) := d(T_E) + d(F_E)$ (using Exercise 5.7).
- (4) The *rank* of a coherent sheaf E is the rank of its locally free part.

What's going on in the second definition? The idea is that any torsion sheaf on a curve is a successive extension of skyscrapers, and the length of this sequence is always the same. Both of these degrees are additive in short exact sequences.

Definition 6.2.

- (5) The *slope* of a coherent sheaf E on C is $\mu(E) = d(E)/r(E)$, if $r(E)$ is positive, and is ∞ otherwise.
- (6) A coherent sheaf E is *semistable*, resp. *stable*, if for any nonzero subsheaf $F \subsetneq E$, $\mu(F) \leq \mu(E)$ resp. $\mu(F) < \mu(E)$.

As for quiver representations, it will turn out that you can't make a moduli space out of all vector bundles, but restricting to (semi)stable ones you can.

Exercise 6.3. Show that if $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ is a short exact sequence of coherent sheaves on C , then $d(E) = d(F) + d(G)$ and $r(E) = r(F) + r(G)$.

Exercise 6.4.

- (1) Let E be a coherent sheaf on C . Show that if E is semistable, then E is torsion or locally free.
- (2) Show that E is stable, resp. semistable if for all surjective, non-isomorphic maps $E \twoheadrightarrow G$, $\mu(E) < \mu(G)$, resp. $\mu(E) \leq \mu(G)$.

Remark 6.5. The notion of degree is the first instance of Chern classes, which exist in a more general setting. \blacktriangleleft

Example 6.6. Let's consider rank-2, degree-zero bundles on \mathbb{P}^1 . These include $\mathcal{O}(-a) \oplus \mathcal{O}(a)$ for any $a \in \mathbb{Z}$, and that's it, so you get $\mathbb{Z}_{\geq 0}$. This is not a finite-type scheme, which is sad – but $\mathcal{O}(-a) \oplus \mathcal{O}(a)$ is unstable unless $a = 0$, so the moduli space of just (semi)stable bundles will end up being OK. \blacktriangleleft

Fact. This is completely trivial from the definitions we just made, but if E is a nonzero sheaf with rank zero, then $d(E) > 0$. This will create all of the issues in higher dimensions. \blacktriangleleft

But I hear you saying, you don't just care about coherent sheaves which are semistable. This is some arbitrary condition. You care about *all* coherent sheaves. Fortunately, we can approximate any coherent sheaf by semistable ones.

Theorem 6.7. Let E be a coherent sheaf on C . Then there is a unique filtration (up to isomorphism), called the Harder-Narasimhan filtration, $0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$, such that E_i/E_{i-1} is semistable for each i and $\mu(E_1/E_0) \geq \mu(E_2/E_1) \geq \cdots \geq \mu(E_n/E_{n-1})$.

We will prove this later, when we want it to hold in a general setting; the proof is the same, so there's no need to see it twice. It's a very nice proof, and you should look forward to it; it uses pretty much just the fact that $\text{Coh}(C)$ is Noetherian, and some numerical stuff.

Example 6.8. On \mathbb{P}^1 , we can see this in Theorem 5.9. We have a unique way to write

$$(6.9) \quad E \cong \mathcal{O}(a_1)^{\oplus r_1} \oplus \cdots \oplus \mathcal{O}(a_n)^{\oplus r_n}$$

such that $a_1 > \cdots > a_n$. For $1 \leq i \leq n$, let

$$(6.10) \quad E \cong \mathcal{O}(a_1)^{\oplus r_1} \oplus \cdots \oplus \mathcal{O}(a_i)^{\oplus r_i},$$

so that $E_i/E_{i-1} = \mathcal{O}(a_i)^{r_i}$, and $\mu(E_i/E_{i-1}) = r_i a_i / r_i = a_i$. \blacktriangleleft

Example 6.11. Let C be an elliptic curve and $p \in C$. Then

$$(6.12a) \quad \text{Ext}^1(\mathcal{O}_C, \mathcal{O}_C) = \text{Hom}(\mathcal{O}_C, \mathcal{O}_C) = \mathbb{C}$$

and

$$(6.12b) \quad \text{Ext}^1(\mathcal{O}_C(p), \mathcal{O}_C) = \text{Hom}(\mathcal{O}_C, \mathcal{O}_C(p)) = \mathbb{C}.$$

You can compute the second one by using the short exact sequence $0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_C(p) \rightarrow \mathcal{O}_p \rightarrow 0$ and that $H^0(\mathcal{O}_C) = H^1(\mathcal{O}_C) = \mathbb{C}$.

$\text{Ext}^1(A, B)$ parameterizes extensions of B by A ; the middle object may have automorphisms, and in both of these cases there's a \mathbb{C}^\times , so we can speak of the (isomorphism type) of the unique nontrivial extensions

$$(6.13a) \quad 0 \longrightarrow \mathcal{O}_C \longrightarrow E_1 \longrightarrow \mathcal{O}_C \longrightarrow 0$$

$$(6.13b) \quad 0 \longrightarrow \mathcal{O}_C \longrightarrow E_2 \longrightarrow \mathcal{O}_C(p) \longrightarrow 0.$$

Claim 6.14. E_1 is semistable but not stable.

Proof. We know $r(\mathcal{O}_C) = 1$, $d(\mathcal{O}_C) = 0$, and $\mu(\mathcal{O}_C) = 0$, so using Exercise 6.3, $r(E_1) = 2$, $d(E_1) = 0$, and $\mu(E_1) = 0$. Since $\mathcal{O}_C \subset \mathcal{O}_E$ is nonzero and a proper subsheaf, but $\mu(\mathcal{O}_C) = \mu(E_1)$, then E_1 is not stable.

Now let $F \subset E$ be the first quotient E_1/E_0 in the Harder-Narasimhan filtration, which is nonzero, and assume $\mu(F) > \mu(E) = 0$. Applying $\text{Hom}(F, -)$ to (6.13a), we have a left exact sequence

$$(6.15) \quad 0 \longrightarrow \text{Hom}(F, \mathcal{O}_C) \longrightarrow \text{Hom}(F, E) \longrightarrow \text{Hom}(F, \mathcal{O}_C).$$

Thus $\text{Hom}(F, \mathcal{O}_C) \neq 0$, so there's a nonzero map $\varphi: F \rightarrow \mathcal{O}_C$; let $K := \ker(\varphi)$. Since F is semistable, $\mu(F) \leq \mu(F/K)$, but $\mu(\mathcal{O}_C) = 0$ and \mathcal{O}_C is semistable, so $F/K \hookrightarrow \mathcal{O}_C$, so $\mu(F/K) < 0$ and $\mu(F/K) > \mu(F) > 0$, which is a contradiction. \square

Exercise 6.16. Show that E_2 is stable.

◀

References

- [Bor91] Armand Borel. *Linear Algebraic Groups*, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag New York, 1991. 3
- [Hab75] W. J. Haboush. Reductive groups are geometrically reductive. *Annals of Mathematics*, 102(1):67–83, 1975. <https://www.jstor.org/stable/pdf/1970974.pdf>. 4
- [Hum75] James E. Humphreys. *Linear Algebraic Groups*, volume 21 of *Graduate Texts in Mathematics*. Springer-Verlag New York, 1975. 3
- [Kin94] A.D. King. Moduli of representations of finite dimensional algebras. *The Quarterly Journal of Mathematics*, 45(4):515–530, 1994. 2, 8, 12
- [MS17] Emanuele Macrì and Benjamin Schmidt. *Lectures on Bridgeland Stability*, pages 139–211. Springer International Publishing, Cham, 2017. <https://arxiv.org/abs/1607.01262>. 1
- [Nag63] Masayoshi Nagata. Invariants of group in an affine ring. *J. Math. Kyoto Univ.*, 3(3):369–378, 1963. https://projecteuclid.org/download/pdf_1/euclid.kjm/1250524787. 4
- [Spr98] T.A. Springer. *Linear Algebraic Groups*. Modern Birkhäuser Classics. Birkhäuser Basel, 1998. 3