

# M392C NOTES: REPRESENTATION THEORY

ARUN DEBRAY

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Lecture 1.

## Lie groups and smooth actions: 1/18/17

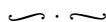
*“I’ve never even seen this many people in a graduate class... I hope it’s good.”*

Today we won’t get too far into the math, since it’s the first day, but we’ll sketch what exactly we’ll be talking about this semester.

This class is about representation theory, which is a wide subject: previous incarnations of the subject might not intersect much with what we’ll do, which is the representation theory of Lie groups, algebraic groups, and Lie algebras. There are other courses which cover Lie theory, and we’re not going to spend much time on the basics of differential geometry or topology. The basics of manifolds, topological spaces, and algebra, as covered in a first-year graduate class, will be assumed.

In fact, the class will focus on the reductive semisimple case (these words will be explained later). There will be some problem sets, maybe 2 or 3 in total. The problem sets won’t be graded, but maybe we’ll devote a class midsemester to going over solutions. If you’re a first-year graduate student, an undergraduate, or a student in another department, you should turn something in, as per usual.

Time for math.



We have to start somewhere, so let’s define Lie groups.

**Definition 1.1.** A *Lie group*  $G$  is a group object in the category of smooth manifolds. That is, it’s a smooth manifold  $G$  that is also a group, with an operation  $m : G \times G \rightarrow G$ , a  $C^\infty$  map satisfying the usual group axioms (e.g. a  $C^\infty$  inversion map, associativity).

Though in the early stages of group theory we focus on finite or at least discrete groups, such as the dihedral groups, which describe the symmetries of a polygon. These have discrete symmetries. Lie groups are the objects that describe continuous symmetries; if you’re interested in these, especially if you come from physics, these are much more fundamental.

**Example 1.2.** The group of  $n \times n$  invertible matrices (those with nonzero determinant) is called the *general linear group*  $\mathrm{GL}_n(\mathbb{R})$ . Since the determinant is multiplicative, this is a group; since  $\det(A) \neq 0$  is an open condition, as the determinant is continuous,  $\mathrm{GL}_n(\mathbb{R})$  is a manifold, and you can check that multiplication is continuous. ◀

**Example 1.3.** The *special linear group*  $\mathrm{SL}_n(\mathbb{R})$  is the group of  $n \times n$  matrices with determinant 1. This is again a group, and to check that it’s a manifold, one has to show that 1 is a regular value of  $\det : M_n(\mathbb{R}) \rightarrow \mathbb{R}$ . But this is true, so  $\mathrm{SL}_n(\mathbb{R})$  is a Lie group. ◀

**Example 1.4.** The *orthogonal group*  $\mathrm{O}(n) = \mathrm{O}(n, \mathbb{R})$  is the group of orthogonal matrices, those matrices  $A$  for which  $A^t = A^{-1}$ . Again, there’s an argument here to show this is a Lie group. ◀

You’ll notice most of these are groups of matrices, and this is a very common way for Lie groups to arise, especially in representation theory.

We can also consider matrices with complex coefficients.

**Example 1.5.** The *complex general linear group*  $\mathrm{GL}_n(\mathbb{C})$  is the group of  $n \times n$  invertible complex matrices. This has several structures.

- For the same reason as  $\mathrm{GL}_n(\mathbb{R})$ ,  $\mathrm{GL}_n(\mathbb{C})$  is a Lie group.

- $\mathrm{GL}_n(\mathbb{C})$  is also a *complex Lie group*: it's a complex manifold, and multiplication and inversion are not just smooth, but holomorphic.
- It's also a *algebraic group* over  $\mathbb{C}$ : a group object in the category of algebraic varieties. This perspective will be particularly useful for us.

We can also define the *unitary group*  $\mathrm{U}(n)$ , the group of  $n \times n$  complex matrices such that  $A^\dagger = A^{-1}$ : their inverses are their transposes. One caveat is that this is *not* a complex Lie group, as this equation isn't holomorphic. For example,  $\mathrm{U}(1) = \{z \in \mathbb{C} \text{ such that } |z| = 1\}$  is topologically  $S^1$ , and therefore is one-dimensional as a real manifold! This is also  $\mathrm{SO}(2)$  (the circle acts by rotating  $\mathbb{R}^2$ ). More generally, a *torus* is a finite product of copies of  $\mathrm{U}(1)$ . ◀

There are other examples that don't look like this, *exceptional groups* such as  $G_2$ ,  $E_6$ , and  $F_4$  which are matrix groups, yet not obviously so. We'll figure out how to get these when we discuss the classification of simple Lie algebras.

Here's an example of interest to physicists:

**Example 1.6.** Let  $q$  be a quadratic form of signature  $(1, 3)$  (corresponding to Minkowski space). Then,  $\mathrm{SO}(1, 3)$  denotes the group of matrices fixing  $q$  (origin-fixing isometries of Minkowski space), and is called the *Lorentz group*.

**Smooth actions.** If one wants to add translations, one obtains the *Poincaré group*  $\mathrm{SO}(1, 3) \ltimes \mathbb{R}^{1,3}$ . ◀

In a first course on group theory, one sees actions of a group  $G$  on a set  $X$ , usually written  $G \curvearrowright X$  and specified by a map  $G \times X \rightarrow X$ , written  $(g, x) \mapsto g \cdot x$ . Sometimes we impose additional structure; in particular, we can let  $X$  be a smooth manifold, and require  $G$  to be a Lie group and the action to be smooth, or Riemannian manifolds and isometries, etc.<sup>1</sup>

It's possible to specify this action by a continuous group homomorphism  $G \rightarrow \mathrm{Diff}(X)$  (or even smooth:  $\mathrm{Diff}(X)$  has an infinite-dimensional smooth structure, but being precise about this is technical).

**Example 1.7.**  $\mathrm{SO}(3) := \mathrm{SL}_3(\mathbb{R}) \cap \mathrm{O}(3)$  denotes the group of rotations of three-dimensional space. Rotating the unit sphere defines an action of  $\mathrm{SO}(3)$  on  $S^2$ , and this is an *action by isometries*, i.e. for all  $g \in \mathrm{SO}(3)$ , the map  $S^2 \rightarrow S^2$  defined by  $x \mapsto g \cdot x$  is an isometry. ◀

**Example 1.8.** Let  $\mathbb{H} := \{x + iy \mid y > 0\}$  denote the upper half-plane. Then,  $\mathrm{SL}_2(\mathbb{R})$  acts on  $\mathbb{H}$  by Möbius transformations. ◀

Smooth group actions arise in physics: if  $S$  is a physical system, then the symmetries of  $S$  often form a Lie group, and this group acts on the space of configurations of the system.

Where do representations come into this? Suppose a Lie group  $G$  acts on a space  $X$ . Then,  $G$  acts on the complex vector space of functions  $X \rightarrow \mathbb{C}$ , and  $G$  acts by linear maps, i.e. for each  $g \in G$ ,  $f \mapsto f(g \cdot -)$  is a linear map. This is what is meant by a representation, and for many people, choosing certain kinds of functions on  $X$  (smooth, continuous,  $L^2$ ) is a source of important representations in representation theory. Representations on  $L^2(X)$  are particularly important, as  $L^2(X)$  is a Hilbert space, and shows up as the state space in quantum mechanics, where some of this may seem familiar.

## Representations.

**Definition 1.9.** A (*linear*) *representation* of a group  $G$  is a vector space  $V$  together with an action of  $G$  on  $V$  by linear maps, i.e. a map  $G \times V \rightarrow V$  written  $(g, v) \mapsto g \cdot v$  such that for all  $g \in G$ , the map  $v \mapsto g \cdot v$  is linear.

This is equivalent to specifying a group homomorphism  $G \rightarrow \mathrm{GL}(V)$ .<sup>2</sup> Sometimes we will abuse notation and write  $V$  to mean  $V$  with this extra structure.

If  $G$  is in addition a Lie group, one might want the representation to reflect its smooth structure, i.e. requiring that the map  $G \rightarrow \mathrm{GL}(V)$  be a homomorphism of Lie groups.

The following definition, codifying the idea of a representation that's as small as can be, is key.

<sup>1</sup>What if  $X$  has singular points? It turns out the axioms of a Lie group action place strong constraints on where singularities can appear in interesting situations, though it's not completely ruled out.

<sup>2</sup>This general linear group  $\mathrm{GL}(V)$  is the group of invertible linear maps  $V \rightarrow V$ .

**Definition 1.10.** A representation  $V$  is *irreducible* if it has no nontrivial invariant subspaces. That is, if  $W \subseteq V$  is a subspace such that for all  $w \in W$  and  $g \in G$ ,  $g \cdot w \in W$ , then either  $W = 0$  or  $W = V$ .

We can now outline some of the goals of this course:

- Classify the irreducible representations of a given group.
- Classify all representations of a given group.
- Express arbitrary representations in terms of irreducibles.

These are not easy questions, especially in applications where the representations may be infinite-dimensional.

**Example 1.11** (Spherical harmonics). Here's an example of this philosophy in action.<sup>3</sup>

Let's start with the Laplacian on  $\mathbb{R}^3$ , a second-order differential operator

$$\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2,$$

which acts on  $C^\infty(\mathbb{R}^3)$ . After rewriting in spherical coordinates, the Laplacian turns out to be a sum

$$\Delta = \frac{1}{r^2} \Delta_{\text{sph}} + \Delta_{\text{rad}},$$

of spherical and radial parts independent of each other, so  $\Delta_{\text{sph}}$  acts on functions on the sphere. We're interested in the eigenfunctions for this spherical Laplacian for a few reasons, e.g. they relate to solutions to the *Schrödinger equation*

$$\dot{\psi} = \hat{H}(\psi),$$

where the Hamiltonian is

$$\hat{H} = -\Delta + V(r),$$

where  $V$  is a potential.

The action of  $\text{SO}(3)$  on the sphere by rotation defines a representation of  $\text{SO}(3)$  on  $C^\infty(S^2)$ , and we'll see that finding the eigenfunctions of the spherical Laplacian boils down to computing the irreducible components inside this representation:

$$V_0 \oplus V_2 \oplus V_4 \oplus \cdots \stackrel{\text{dense}}{\subseteq} C^\infty(S^2),$$

where the  $V_{2k}$  run through each isomorphism class of irreducible representations of  $\text{SO}(3)$ . They are also the eigenspaces for the spherical Laplacian, where the eigenvalue for  $V_{2k}$  is  $\pm k(k+1)$ , and this is not a coincidence since the spherical Laplacian is what's known as a Casimir operator for the Lie algebra  $\mathfrak{so}(3)$ . We'll see more things like this later, once we have more background. ◀

Lecture 2.

## Representation theory of compact groups: 1/20/17

First, we'll discuss some course logistics. There are course notes (namely, the ones you're reading now) and a website, [https://www.ma.utexas.edu/users/gunningham/repththeory\\_spring17.html](https://www.ma.utexas.edu/users/gunningham/repththeory_spring17.html). We won't stick to one textbook, as indicated on the website, but the textbook of Kirrilov is a good reference, and is available online. The class' office hours will be Monday from 2 to 4 pm, at least for the time being.

The course will fall into two parts.

- (1) First, we'll study finite-dimensional representations of things such as compact Lie groups (e.g.  $\text{U}(n)$  and  $\text{SU}(2)$ ) and their complexified Lie algebras, reductive Lie algebras (e.g.  $\mathfrak{gl}_n(\mathbb{C})$  and  $\mathfrak{sl}_2(\mathbb{C})$ ). There is a nice dictionary between the finite-dimensional representation theories of these objects. The algebra  $\mathfrak{sl}_2(\mathbb{C})$  is semisimple, which is stronger than reductive. Every reductive Lie algebra decomposes into a sum of a semisimple Lie algebra and an abelian Lie algebra, and abelian Lie algebras are relatively easy to understand, so we'll dedicate some time to semisimple Lie algebras.

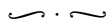
We'll also spend some time understanding the representation theory of reductive algebraic groups over  $\mathbb{C}$ , e.g.  $\text{GL}_n(\mathbb{C})$  and  $\text{SL}_2(\mathbb{C})$ . Again, there is a dictionary between the finite-dimensional representations here and those of the Lie groups and reductive Lie algebras we discussed.

All together, these form a very classical and standard subject that appears everywhere in algebra, analysis, and physics.

<sup>3</sup>No pun intended.

- (2) We'll then spend some time on the typically infinite-dimensional representations of noncompact Lie groups, such as  $\mathrm{SL}_2(\mathbb{R})$  or the Lorentz group  $\mathrm{SO}(1, 3)$ . These groups have interesting infinite-dimensional, but irreducible representations; the classification of these representations is intricate, with analytic issues, yet is still very useful, tying into among other things the Langlands program.

All these words will be defined when they appear in this course.



We'll now begin more slowly, with some basics of representations of compact groups.

**Example 2.1.** Here are some examples of compact topological groups.

- Finite groups.
- Compact Lie groups such as  $\mathrm{U}(n)$ .
- The  $p$ -adics  $\mathbb{Z}_p$  with their standard topology: two numbers are close if their difference is divisible by a large power of  $p$ .  $\mathbb{Z}_p$  is also a profinite group. ◀

**Definition 2.2.** Let  $G$  be a compact group. A (*finite-dimensional*) (*continuous*) (*complex*) *representation* of  $G$  is a finite-dimensional complex vector space  $V$  together with a continuous homomorphism  $\rho : G \rightarrow \mathrm{GL}(V)$ .

If you pick a basis,  $V \cong \mathbb{C}^n$ , so  $\mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{C})$ . These are the  $n \times n$  invertible matrices over the complex numbers, so  $\rho$  assigns a matrix to each  $g \in G$  in a continuous manner, where  $\rho(g_1 g_2) = \rho(g_1) \rho(g_2)$ , so group multiplication is sent to matrix multiplication. Sometimes it's more natural to write this through the *action map*  $G \times V \rightarrow V$  sending  $(g, v) \mapsto \rho(g) \cdot v$ .

The plethora of parentheses in Definition 2.2 comes from the fact that representations may exist over other fields, or be infinite-dimensional, or be discontinuous, but in this part of the class, when we say a representation of a compact group, we mean a finite-dimensional, complex, continuous one.

**Example 2.3.** Let  $S_3$  denote the *symmetric group on 3 letters*, the group of bijections  $\{1, 2, 3\} \rightarrow \{1, 2, 3\}$  under composition. Its elements are written in *cycle notation*:  $(1\ 2)$  is the permutation exchanging 1 and 2, and  $(1\ 2\ 3)$  sends  $1 \mapsto 2$ ,  $2 \mapsto 3$ , and  $3 \mapsto 1$ . There are six elements of  $S_3$ :  $S_3 = \{e, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$ .

For representation theory, it can be helpful to have a description in terms of generators and relations. Let  $s = (1\ 2)$  and  $t = (2\ 3)$ , so  $(1\ 3) = sts = tst$ ,  $(1\ 2\ 3) = st$ , and  $(1\ 3\ 2) = ts$ . Thus we obtain the presentation

$$(2.4) \quad S_3 = \langle s, t \mid s^2 = t^2 = e, sts = tst \rangle.$$

The relation  $sts = tst$  is an example of a *braid relation*; there exist similar presentations for all the symmetric groups, and this leads into the theory of Coxeter groups.

There's a representation you can always build for any group called the *trivial representation*, in which  $V = \mathbb{C}$  and  $\rho_{\mathrm{triv}} : G \rightarrow \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$  sends every  $g \in G$  to the identity map ( $1 \in \mathbb{C}^\times$ ).

To get another representation, let's remember that we wanted to build representations out of functions on spaces.  $S_3$  is a discrete space, so let's consider the space  $X = \{x_1, x_2, x_3\}$  (with the discrete topology). Then,  $S_3$  acts on  $X$  by permuting the indices; we want to linearize this.

Let  $V = \mathbb{C}[X] = \mathbb{C}x_1 \oplus \mathbb{C}x_2 \oplus \mathbb{C}x_3$ , a complex vector space with basis  $\{x_1, x_2, x_3\}$ . We'll have  $S_3$  act on  $V$  by permuting the basis; this is an example of a *permutation representation*. This basis defines an isomorphism  $\mathrm{GL}(V) \xrightarrow{\sim} \mathrm{GL}_3(\mathbb{C})$ , which we'll use to define a representation. Since  $s$  swaps  $x_1$  and  $x_2$ , but fixes  $x_3$ , it should map to the matrix

$$s \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly,

$$t \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(You should check that these assignments satisfy the relations in (2.4).)

Now something interesting happens. When you think of  $X$  as an  $S_3$ -space, it has no invariant subsets (save itself and the empty set). But this linearization is no longer irreducible: the element  $v = x_1 + x_2 + x_3 \in V$  is fixed by all permutations acting on  $X$ :  $\rho(g) \cdot v = v$  for all  $g \in S_3$ .

More formally, let  $W$  be the subspace of  $V$  spanned by  $v$ ; then,  $W$  is a subrepresentation of  $V$ . ◀

Let's take a break from this example and introduce some terminology.

**Definition 2.5.** Let  $G$  be a group and  $V$  be a  $G$ -representation. A *subrepresentation* or  $G$ -invariant subspace of  $V$  is a subspace  $W \subseteq V$  such that for all  $g \in G$  and  $w \in W$ ,  $g \cdot w \in W$ . If  $V$  has no nontrivial (i.e. not 0 or  $V$ ) subrepresentations,  $V$  is called *irreducible*.

This means the same  $G$ -action defines a representation on  $W$ .

If  $W \subseteq V$  is a subrepresentation, the quotient vector space  $V/W$  is a  $G$ -representation, called the *quotient representation*, and the  $G$ -action is what you think it is: in coset notation,  $g \cdot (v + W) = (gv + W)$ . This is well-defined because  $W$  is  $G$ -invariant.

We can always take quotients, but unlike for vector spaces in general, it's more intricate to try to find a complement: does the quotient split to identify  $V/W$  with a subrepresentation of  $V$ ?

Returning to Example 2.3, we found a three-dimensional representation  $V$  and a one-dimensional subrepresentation  $W$ . Let's try to find another subrepresentation  $U$  of  $V$  such that, as  $S_3$ -representations,  $V \cong W \oplus U$ . The answer turns out to be  $U = \text{span}_{\mathbb{C}}\{x_1 - x_2, x_2 - x_3\}$ .

**Claim.**  $U$  is a subrepresentation, and  $V = U \oplus W$ .

This isn't as obvious, because neither  $x_1 - x_2$  or  $x_2 - x_3$  is fixed by all elements of  $S_3$ . However, for any  $g \in S_3$ ,  $g \cdot (x_1 - x_2)$  is contained in  $U$ , and similarly for  $x_2 - x_3$ . Let's set  $U \cong \mathbb{C}^2$  with  $x_1 - x_2 \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $x_2 - x_3 \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ . Then, we can explicitly describe  $U$  in terms of the matrices for  $s$  and  $t$  in  $\text{GL}(U) \cong \text{GL}_2(\mathbb{C})$ , where the identification uses this basis.

Since  $s = \begin{pmatrix} 1 & 2 \end{pmatrix}$ , it sends  $x_1 - x_2 \mapsto x_2 - x_1 = -(x_1 - x_2)$  and  $x_2 - x_3 \mapsto x_1 - x_3 = (x_1 - x_2) + (x_2 - x_3)$ , so

$$s \mapsto \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}.$$

In the same way,

$$t \mapsto \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}.$$

The general theme of finding interesting representations inside of naturally arising representations will occur again and again in this class.

Lecture 3.

## Operations on representations: 1/23/17

*"I want to become a representation theorist!"*

*"Are you Schur?"*

Last time, we discussed representations of groups and what it means for a representation to be irreducible; today, we'll talk about some other things one can do with representations. For the time being,  $G$  can be any group; we will specialize later.

The first operation on a representation is very important.

**Definition 3.1.** A *homomorphism* of  $G$ -representations  $V \rightarrow W$  is a linear map  $\varphi : V \rightarrow W$  such that for all  $g \in G$ , the diagram

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ \downarrow g \cdot & & \downarrow g \cdot \\ V & \xrightarrow{\varphi} & W. \end{array}$$

This is also called an *intertwiner*, a  $G$ -homomorphism, or a  $G$ -equivariant map.

An *isomorphism* of representations is a homomorphism that's also a bijection.<sup>4</sup>

More explicitly, this means  $\varphi$  commutes with the  $G$ -action, in the sense that  $\varphi(g \cdot v) = g \cdot \varphi(v)$ . This is one advantage of dropping the  $\rho$ -notation: it makes this definition cleaner.

<sup>4</sup>If  $f : V \rightarrow W$  is an isomorphism of representations, then  $f^{-1} : W \rightarrow V$  is also a  $G$ -homomorphism, making this a reasonable definition. This is a useful thing to check, and doesn't take too long.

*Remark.* If  $\varphi : V \rightarrow W$  is a  $G$ -homomorphism, then  $\ker(\varphi) \subseteq V$  is a subrepresentation, and similarly for  $\text{Im}(\varphi) \subseteq W$ . ◀

The set of  $G$ -homomorphisms from  $V$  to  $W$  is a complex vector space,<sup>5</sup> denote  $\text{Hom}_G(V, W)$ .

Several constructions carry over from the world of vector spaces to the world of  $G$ -representations. Suppose  $V$  and  $W$  are  $G$ -representations.

- The direct sum  $V \oplus W$  is a  $G$ -representation with the action  $g \cdot (v, w) = (g \cdot v, g \cdot w)$ . This has dimension  $\dim V + \dim W$ .
- The tensor product  $V \otimes W$  is a  $G$ -representation: since it's generated by pure tensors, it suffices to define  $g \cdot (v \otimes w)$  to be  $(gv) \otimes (gw)$  and check that this is compatible with the relations.
- The dual space  $V^* := \text{Hom}_{\mathbb{C}}(V, \mathbb{C})$  is a  $G$ -representation: if  $\alpha \in V^*$ , we define  $(g \cdot \alpha)(v) := \alpha(g^{-1}v)$ . This might be surprising: you would expect  $\alpha(gv)$ , but this doesn't work: you want  $g \cdot (h\alpha)$  to be  $(gh) \cdot \alpha$ , but you'd get  $(hg) \cdot \alpha$ . This is why you need the  $g^{-1}$ .
- Since  $\text{Hom}_{\mathbb{C}}(V, W)$  is naturally isomorphic to  $V^* \otimes W$ ,<sup>6</sup> it inherits a  $G$ -representation structure.

**Definition 3.2.** Given a  $G$ -representation  $V$ , the *space of  $G$ -invariants* is the space

$$V^G := \{v \in V \mid g \cdot v = v\}.$$

This can be naturally identified with  $\text{Hom}_G(\mathbb{C}_{\text{triv}}, V)$ , where  $\mathbb{C}_{\text{triv}}$  is the trivial representation with action  $g \cdot z = z$  for all  $z \in \mathbb{C}$ . The identification comes by asking where 1 goes to.

These are also good reasons for using the action notation rather than writing  $\rho : G \rightarrow \text{Aut}(V)$ , which would require more complicated formulas.

There are a couple of different ways of stating Schur's lemma, but here's a good one.

**Lemma 3.3** (Schur). *Let  $V$  and  $W$  be irreducible  $G$ -representations. Then,*

$$\text{Hom}_G(V, W) = \begin{cases} 0, & \text{if } V \not\cong W \\ \mathbb{C}, & \text{if } V \cong W. \end{cases}$$

“Irreducible” is the key word here.

*Remark.* Schur's lemma requires us to work over  $\mathbb{C}$  (more generally, over any algebraically closed field). It also assumes that  $V$  and  $W$  are finite-dimensional. There are no assumptions on  $G$ ; this holds in much greater generality (e.g. over  $\mathbb{C}$ -linear categories). ◀

In general, there's a distinction between “isomorphic” and “equal” (or at least naturally isomorphic); in the latter case, there's a canonical isomorphism, namely the identity. In this case, the second piece of Lemma 3.3 can be restated as saying for any irreducible  $G$ -representation  $V$ ,

$$\text{Hom}_G(V, V) = \mathbb{C} \cdot \text{id}_V.$$

Thus, any  $G$ -homomorphism  $\varphi : V \rightarrow V$  is  $\lambda \cdot \text{id}_V$  for some  $\lambda \in \mathbb{C}$ , and in a basis is a diagonal matrix with every diagonal element equal to  $\lambda$ .

*Proof of Lemma 3.3.* Suppose  $\varphi : V \rightarrow W$  is a nonzero  $G$ -homomorphism. Thus,  $\ker(\varphi) \subset V$  isn't be all of  $V$ , so since  $V$  is irreducible it must be 0, so  $\varphi$  is injective. Similarly, since  $\text{Im}(\varphi) \subset W$  isn't 0, it must be all of  $W$ , since  $W$  is irreducible. Thus,  $\varphi$  is an isomorphism, so if  $V \not\cong W$ , the only  $G$ -homomorphism is the zero map.

Now, suppose  $\varphi : V \rightarrow V$  is a  $G$ -homomorphism. Since  $\mathbb{C}$  is algebraically closed,  $\varphi$  has an eigenvector: there's a  $\lambda \in \mathbb{C}$  and a  $v \in V$  such that  $\varphi(v) = \lambda \cdot v$ . Since  $\varphi$  and  $\lambda \cdot \text{id}_V$  are  $G$ -homomorphisms, so is  $\varphi - \lambda \text{id}_V : V \rightarrow V$ , so its kernel, the  $\lambda$ -eigenspace of  $\varphi$ , is a subrepresentation of  $V$ . Since it's nonzero, then it must be all of  $V$ , so  $V = \ker(\varphi - \lambda \text{id}_V)$ , and therefore  $\varphi = \lambda \text{id}_V$ . ◻

This is the cornerstone of representation theory, and is one of the reasons that the theory is so much nicer over  $\mathbb{C}$ .

**Corollary 3.4.** *If  $G$  is an abelian group, then any irreducible representation of  $G$  is one-dimensional.*

<sup>5</sup>Recall that we're focusing exclusively on complex representations. If we look at representations over another field  $k$ , we'll get a  $k$ -vector space.

<sup>6</sup>This depends on the fact that  $V$  and  $W$  are finite-dimensional.



*Proof.* Let  $V$  be an irreducible  $G$ -representation and  $g \in G$ . Since  $G$  is abelian,  $v \mapsto gv$  is a  $G$ -homomorphism:  $g \cdot (hv) = h(g \cdot v)$ . By Schur's lemma, the action of  $g$  is  $\lambda \cdot \text{id}_V$  for some  $\lambda \in \mathbb{C}$ , so any  $W \subseteq V$  is  $G$ -invariant. Since  $V$  is irreducible, this can only happen when  $V$  is 1-dimensional.  $\square$

**Example 3.5.** Let's talk about the irreducible representations of  $\mathbb{Z}$ . This isn't a compact group, but we'll survive. A representation of  $\mathbb{Z}$  is a homomorphism  $\mathbb{Z} \rightarrow \text{GL}(V)$  for some vector space  $V$ ; since  $\mathbb{Z}$  is a free group, this is determined by what 1 goes to, which can be chosen freely. That is, representations of  $\mathbb{Z}$  are in bijection with invertible matrices.

By Corollary 3.4, irreducible  $\mathbb{Z}$ -representations are the 1-dimensional invertible matrices, which are identified with  $\text{GL}_1(\mathbb{C}) = \mathbb{C}^\times$ , the nonzero complex numbers.  $\blacktriangleleft$

Our greater-scope goal is to understand all representations by using irreducible ones as building blocks. In the best possible case, your representation is a direct-sum of irreducibles; we'll talk about that case next time.

Lecture 4.

### Complete reducibility: 1/25/17

We've discussed what it means for a representation to be irreducible, and irreducible representations are the smallest representations; we hope to build all representations out of irreducibles. The nicest possible case is complete reducibility, which we'll discuss today.

Suppose  $G$  is a group (very generally),  $V$  is a representation, and  $W \subseteq V$  is a subrepresentation. If  $i : W \hookrightarrow V$  denotes the inclusion map, then there's a projection map onto the quotient  $V \twoheadrightarrow U := V/W$ . This is encoded in the notion of a *short exact sequence*:

$$0 \longrightarrow W \xrightarrow{i} V \xrightarrow{j} U \longrightarrow 0,$$

which means exactly that  $i$  is injective,  $j$  is surjective, and  $\text{Im}(i) = \ker(j)$ . The nicest short exact sequence is

$$0 \longrightarrow W \longrightarrow W \oplus U \longrightarrow U \longrightarrow 0,$$

where the first map is inclusion into the first factor and the second is projection onto the second factor. In this case, one says the short exact sequence *splits*. This is equivalent to specifying a projection  $V \twoheadrightarrow U$  or an inclusion  $W \hookrightarrow V$ . Since direct sums are easier to understand, this is the case we'd like to know better.

**Example 4.1.** We saw last time that a representation of  $\mathbb{Z}$  is given by the data of an invertible matrix which specifies the action of 1, akin to a discrete translational symmetry.

Consider the  $\mathbb{Z}$ -representation  $V$  on  $\mathbb{C}^2$  given by the matrix

$$A := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

This is not an irreducible representation, because  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  is an eigenvector for  $A$  with eigenvalue 1, so

$$W := \left\{ \begin{pmatrix} a \\ 0 \end{pmatrix} \mid a \in \mathbb{C} \right\}$$

is a subrepresentation of  $V$ . Since  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  has eigenvalue 1,  $W$  is the trivial representation  $\mathbb{C}_{\text{triv}}$ . Moreover, the quotient  $V/W$  is also the trivial representation, so  $V$  sits in a short exact sequence

$$0 \longrightarrow \mathbb{C}_{\text{triv}} \longrightarrow V \longrightarrow \mathbb{C}_{\text{triv}} \longrightarrow 0.$$

However,  $V$  itself is not trivial, or it would have been specified by a diagonalizable matrix. Thus,  $V$  is not a direct sum of subrepresentations (one says it's *indecomposable*), but it's not irreducible! If  $U$  is a 1-dimensional subrepresentation of  $V$ , then  $U = \mathbb{C}v$  for some nonzero  $v \in V$ . Since  $Av \in U$ ,  $Av = \lambda v$  for some  $\lambda \in \mathbb{C}$ , meaning  $v$  is an eigenvector, and in our particular case,  $v = e_1$ . Thus any direct sum of two nontrivial subrepresentations must be  $U + U = U$ , which isn't all of  $V$ .  $\blacktriangleleft$

We want to avoid these kinds of technicalities on our first trek through representation theory, and fortunately, we can.



**Definition 4.2.** A representation  $V$  of  $G$  is *completely reducible* or *semisimple* if every subrepresentation  $W \subseteq V$  has a *complement*, i.e. another subrepresentation  $U$  such that  $V \cong U \oplus W$ .

*Remark.* There are ways to make this more general, e.g. for infinite-dimensional representations, one may want closed subrepresentations. But for finite-dimensional representations, this definition suffices. ◀

A finite-dimensional semisimple representation  $V$  is a direct sum of its irreducible subrepresentations. The idea is that its subrepresentations must also be semisimple, so you can use induction.

The terminology “semisimple” arises because *simple* is a synonym for irreducible, in the context of representation theory.

So semisimple representations are nice. You might ask, *for which groups  $G$  are all representations semisimple?* To answer this question, we’ll need a few more concepts.

**Definition 4.3.** A representation  $V$  of  $G$  is called *unitary* if it admits a  $G$ -invariant inner product, i.e. a map  $B : V \times V \rightarrow \mathbb{C}$  that is:

- linear in the first factor and antilinear in the other,<sup>7</sup>
- *antisymmetric*, i.e.  $B(v, w) = \overline{B(w, v)}$ ,
- *positive definite*, i.e.  $B(v, v) \geq 0$  and  $B(v, v) = 0$  iff  $v = 0$ , and
- $G$ -invariant, meaning  $B(g \cdot v, g \cdot w) = B(v, w)$  for all  $g \in G$ .

The reason for the name is that if  $V$  is a unitary representation with form  $B$ , then as a map  $G \rightarrow \text{GL}(V) \cong \text{GL}(\mathbb{C}^n)$ , this representation factors through  $\text{U}(n)$ , the *unitary matrices*, which preserve the standard Hermitian inner product on  $\mathbb{C}^n$ . So unitary representations are representations of  $G$  into some unitary group.

**Proposition 4.4.** *Unitary representations are completely reducible.*

*Proof sketch.* How would you find a complement to a subspace? The usual way to do this is to take an orthogonal complement, and an invariant inner product is what guarantees that the orthogonal complement is a subrepresentation.

Let  $V$  be a unitary representation and  $B(-, -)$  be its invariant inner product. Let  $W \subseteq V$  be a subrepresentation, and let

$$U = W^\perp := \{v \in V \mid B(v, w) = 0 \text{ for all } w \in W\}.$$

Then,  $U$  is a subrepresentation (which you should check), and  $V = W \oplus U$ . ◻

Classifying unitary representations is a hard problem in general. Building invariant Hermitian forms isn’t too bad, but making them positive definite is for some reason much harder. In any case, for compact groups there is no trouble.

**Proposition 4.5.**

- (1) *Given an irreducible unitary representation, the invariant form  $B(-, -)$  is unique up to multiplication by a positive scalar.*
- (2) *If  $W_1, W_2 \subseteq V$ , where  $V$  is unitary and  $W_1$  and  $W_2$  are nonisomorphic irreducible subrepresentations, then  $W_1$  is orthogonal to  $W_2$ , i.e.  $B(w_1, w_2) = 0$  for all  $w_1 \in W_1$  and  $w_2 \in W_2$ .*

*Proof.* Part (1) is due to Schur’s lemma (Lemma 3.3). Consider the map  $B : V \rightarrow \overline{V}^*$  defined by  $v \mapsto B(v, -)$  (here,  $\overline{V}$  is the *conjugate space*, where the action of  $a + bi$  on  $\overline{V}$  is the action of  $a - bi$  on  $V$ ); in fact, you could use this map to define a unitary structure on a representation.  $B$  is a  $G$ -isomorphism, so by Schur’s lemma, every such isomorphism, derived from every possible choice of Hermitian form, must be a scalar multiple of this one. Since  $B$  must be positive definite, this scalar had better be positive. ◻

One particular corollary of part (1) is that if  $V$  is a unitary representation,  $V^* \cong \overline{V}$ .

So if you care about compact groups (in particular finite groups), this is all you need.

**Theorem 4.6 (Maschke).** *Any representation of a compact group admits a unitary structure.*

We’ll give a first proof for finite groups, then later one for Lie groups; we probably won’t prove the most general case.

<sup>7</sup>Some people have the opposite convention, defining the first factor to be antilinear and the second to be linear. Of course, the theory is equivalent. Such a form is called a *Hermitian form*, and if it’s antisymmetric and positive definite, it’s called a *Hermitian inner product*.

*Proof for finite groups.* Let  $G$  be a finite group and  $V$  be a  $G$ -representation. The first step to finding a  $G$ -invariant inner product is to find any Hermitian inner product, e.g. picking a basis and declaring it to be orthonormal, yielding an inner product  $B_0$  that's probably not  $G$ -invariant.

We want to average  $B_0$  over  $G$  to obtain something  $G$ -invariant, which is why we need finiteness.<sup>8</sup> That is, let

$$B(v, w) := \frac{1}{|G|} \sum_{g \in G} B_0(g \cdot v, g \cdot w).$$

Then,  $B$  is a unitary structure: it's still positive definite and Hermitian, and since multiplication by an  $h \in H$  is a bijection on  $G$ , this is  $G$ -invariant.  $\square$

Lecture 5.

### Some examples: 1/27/17

Last time, we claimed that every finite-dimensional representation of a compact group admits a unitary structure (Theorem 4.6), and therefore is completely reducible. We proved it for finite groups; later today, we'll talk about the Haar measure on a Lie group and how to use it to prove Theorem 4.6. The Haar measure also exists on topological groups more generally, but we won't construct it.

Semisimplicity is a very nice condition, and doesn't exist in general, just as most operators aren't self-adjoint, and so generally don't have discrete spectra. This is more than just a metaphor: given a compact group  $G$ , let  $\widehat{G}$  denote the set of isomorphism classes of irreducible representations of  $G$ , sometimes called its *spectrum*. (In some cases, this has topology, but when  $G$  is compact,  $\widehat{G}$  is discrete.) Using Theorem 4.6, any representation  $V$  of  $G$  is a direct sum of irreducibles in a unique way (up to permuting the factors):

$$V = \bigoplus_{W \in \widehat{G}} W^{\oplus m_W},$$

where  $m_W$  is the *multiplicity* of  $W$  in  $V$ . The summand  $W^{\oplus m_W}$  is called the  *$W$ -isotypical component* of  $V$ . The analogy with the Fourier transform can be made stronger.

**Example 5.1.** Let  $G = S_3$ . We've already seen the trivial representation  $\mathbb{C}_{\text{triv}}$  and a representation  $V = \mathbb{C}(x_1 - x_2) \oplus \mathbb{C}(x_2 - x_3)$ , where  $S_3$  acts by permuting the  $x_i$  terms; we showed that  $V$  is an irreducible representation of dimension 2.

There's a third irreducible representation  $\mathbb{C}_{\text{sign}}$ , a one-dimensional (therefore necessarily irreducible) representation  $\rho_{\text{sign}} : S_3 \rightarrow \mathbb{C}^\times$ , where  $\rho_{\text{sign}}(\sigma)$  is 1 if  $\sigma$  is even and  $-1$  if  $\sigma$  is odd.

**Exercise 5.2.** Show that  $\mathbb{C}_{\text{triv}}$ ,  $V$ , and  $\mathbb{C}_{\text{sign}}$  are all of the irreducible representations of  $S_3$  (up to isomorphism).

We'll soon see how to prove this quickly, though it's not too bad to do by hand.  $\blacktriangleleft$

**Example 5.3.** The *circle group*, variously denoted  $S^1$ ,  $U(1)$ ,  $SO(2)$ ,  $\mathbb{R}/\mathbb{Z}$ , or  $\mathbb{T}$ , is the abelian compact Lie group of real numbers under addition modulo  $\mathbb{Z}$ . Corollary 3.4 tells us all irreducible representations of  $S^1$  are 1-dimensional.

Since  $S^1$  is a quotient of the additive group  $(\mathbb{R}, +)$ , we'll approach this problem by classifying the one-dimensional representations of  $\mathbb{R}$  and seeing which ones factor through the quotient. Such a representation is a map  $(\mathbb{R}, +) \rightarrow (\mathbb{C}^\times, \times) \cong \text{GL}_1(\mathbb{C})$ ; in other words, it turns addition into multiplication.

We know a canonical way to do this: for any  $\xi \in \mathbb{C}$ , there's a representation  $\chi_\xi : t \mapsto e^{\xi t}$ . And it turns out these are all of the continuous 1-dimensional representations of  $\mathbb{R}$  up to isomorphism.

**Exercise 5.4.** Prove this: that every 1-dimensional representation  $\rho : \mathbb{R} \rightarrow \mathbb{C}^\times$  is isomorphic to some  $\chi_\xi$ . Here's a hint:

- (1) First reduce to the case where  $\rho$  is  $C^1$ . (This requires techniques we didn't ask as prerequisites.)
- (2) Now, show that  $\rho$  satisfies a differential equation  $\rho'(t) = \rho'(0)\rho(t)$ . As  $\rho$  is a homomorphism, this means  $\rho(0) = 1$ . Then, uniqueness of solutions of ODEs shows  $\rho = \chi_{\rho'(0)}$ .

<sup>8</sup>More generally, we could take a compact group, replacing the sum with an integral. Compactness is what guarantees that the integral converges.

There's something interesting going on here: representations of  $\mathbb{R}$  are determined by their derivatives at the identity. This is a shadow of a grander idea: to understand the representations of a Lie group  $G$ , look at representations of its Lie algebra  $\mathfrak{g}$ .

The noncompactness of  $\mathbb{R}$  is also reflected in its representation theory: not all of its representations are unitary.  $U(1)$  is the group of unit complex numbers under multiplication, so  $\chi_\xi$  is unitary iff  $\xi = i\eta$  for  $\eta \in \mathbb{R}$ . In this case,  $\widehat{\mathbb{R}}$  is used to denote the unitary representations, so  $\widehat{\mathbb{R}} = i\mathbb{R}$ . In particular, it's a group (since  $\mathbb{R}$  is abelian), abstractly isomorphic to  $\mathbb{R}$ , and not discrete (as  $\mathbb{R}$  is not compact).

Great; now, when does  $\chi_\xi$  descend to a representation of  $S^1$ ? This means that  $\chi_\xi(0) = \chi_\xi(1)$ , so  $\xi$  is an integer multiple of  $2\pi i$ . Thus,  $S^1$ , a compact group, has a discrete set of irreducible representations, and  $\widehat{S^1} = 2\pi i\mathbb{Z}$ . You might be able to see the Fourier series hiding in here. ◀

**Example 5.5.** Let's look at a nonabelian example,  $SU(2)$ , the set of matrices

$$SU(2) := \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mid a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}.$$

As a manifold, this is isomorphic to the unit sphere  $S^3 \subset \mathbb{C}^2$ , and indeed it's a Lie group, compact and non-abelian.

Today we'll write down some representations; later, we can use theory to prove this is all of them. The first one, as for any matrix group, comes for free: the *standard representation* or *defining representation* uses the preexisting embedding  $\rho_{\text{std}}: SU(2) \hookrightarrow GL_2(\mathbb{C})$  sending  $A \mapsto A$ . We also have the trivial representation.

We'll obtain some other representations as functions on a space  $SU(2)$  acts on. Matrix multiplication defines a smooth action of  $SU(2)$  on  $\mathbb{C}^2$ , so consider the vector space  $P$  of polynomial functions on  $\mathbb{C}^2$ . Then,  $SU(2)$  acts on  $P$  as follows: if  $f \in P$  and  $A \in SU(2)$ ,

$$A \cdot f(\mathbf{x}) := f(A^{-1}\mathbf{x}).$$

The  $A^{-1}$  term arises because you need composition to go the right way:

$$A(Bf(\mathbf{x})) = f(B^{-1}A^{-1}\mathbf{x}) = f((AB)^{-1}\mathbf{x}) = (AB)f(\mathbf{x}).$$

This is an aspect of a very general principle: if a group acts on the left on a space, then it acts on the right on its space of functions. But you can always turn right actions into left actions using inverses.

This  $P$  is an infinite-dimensional representation, but it splits into homogeneous finite-dimensional subrepresentations. Let  $P_n$  denote the homogeneous polynomials of degree  $n$ , e.g.  $P^3 = \text{span}_{\mathbb{C}}\{x^3, x^2y, xy^2, y^3\}$ .

**Proposition 5.6.** *These  $P_n$  are irreducible of dimension  $n + 1$ , and form a complete list of isomorphism types of irreducible representations of  $SU(2)$ .*

In particular,  $\widehat{SU(2)} = \mathbb{Z}_{\geq 0}$ . We get a discrete set of representations, since  $SU(2)$  is compact, but it's not a group, because  $SU(2)$  isn't abelian. Abelian groups are nice because their representations are 1-dimensional, and compact groups are nice because their representations are discrete. Abelian compact groups are even nicer, but there's not so many of those. ◀

In the last few minutes, we'll say a little about integration. Next time, we'll talk about characters and matrix coefficients.

On a topological group, finding the Haar measure is a delicate matter (but you can do it); on a Lie group, it's simpler, but requires some differential geometry. For the finite-groups case of Maschke's theorem, we averaged an inner product over  $G$  so that it was  $G$ -invariant. For compact groups, we can't sum in general, since there are infinitely many elements, but you can integrate. So what we're looking for is a measure on a Lie group. This comes naturally from a volume form, and to prove  $G$ -invariance, we'd like it to be  $G$ -invariant itself.

That is, our wish list is a left-invariant volume form on a Lie group  $G$ , an  $\omega_g \in \Lambda^{\text{top}}(T_g^*G)$ . But you can get this by choosing something at the identity and defining  $\omega_g = (\cdot g^{-1})^*\omega_1$  (that is, pull it back under left multiplication by  $g^{-1}$ ).

If  $G$  is compact, then you can use right-invariance to show the space of such forms is trivial, so the form is also unique. We'll return to this a little bit next time.

Lecture 6.

**Matrix coefficients and characters: 1/30/17**

We'll start with a review of the end of last lecture. Let  $G$  be a compact Lie group; then, there's a bi-invariant volume form  $\omega$  on  $G$ , and gives  $G$  finite volume. You can normalize so that the total volume is 1. The measure this defines is  $G$ -invariant, and is called the *Haar measure*, written  $dg$ .

*Remark.* If  $G$  is a finite group, this is  $1/|G|$  times the counting measure. You can use this to obtain the finite-groups proof of Maschke's theorem, etc. from the Lie groups one, as the integrals becomes sums.  $\blacktriangleleft$

**Example 6.1.** Let's consider the circle group  $U(1)$  again. We saw in Example 5.3 that its irreducible representations are of the form  $\chi_n : t \mapsto e^{2\pi i n t}$  for  $t \in \mathbb{R}/\mathbb{Z}$  (or  $z \mapsto z^n$  for  $z \in U(1)$ ), indexed over  $n \in \mathbb{Z}$ .

We can tie this to Fourier series. Consider  $C(U(1), \mathbb{C}) = C(U(1))$ , the space of complex-valued continuous functions on  $U(1)$ . We've already seen how a group acts on its space of functions, so  $C(U(1))$  is an infinite-dimensional representation of  $U(1)$ . We want to decompose this as a sum of irreducibles  $\chi_n$ . The  $\chi_{-n}$ -isotypical component  $\mathbb{C}\chi_n \subset C(U(1))$  (since  $\chi_n$  is itself a continuous,  $\mathbb{C}$ -valued function on  $U(1)$ ) is isomorphic to the one-dimensional representation determined by  $\chi_n$ . So we might hope to decompose  $C(U(1))$  as a sum of these  $\chi_n$  indexed by the integers.

Consider the Hermitian inner product on  $C(U(1))$  in which

$$(6.2) \quad (f_1, f_2)_{L^2} := \int_G f_1(g) \overline{f_2(g)} dg$$

(i.e. taken in the Haar measure); since  $U(1)$  is compact, this converges. The same construction may be made for any compact Lie group  $G$ . Now, we can ask how the  $\chi_{-n}$ -isotypical components fit together inside  $C(U(1))$ .

It's quick to check that, with the Haar measure normalized as above,

$$\langle \chi_n, \chi_m \rangle = \begin{cases} 1, & n = m \\ 0, & n \neq m. \end{cases}$$

If  $C_{\text{alg}}(U(1))$  denotes the space of algebraic functions on  $U(1)$ , then

$$C_{\text{alg}}(U(1)) = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}\chi_n \subset C(U(1)).$$

Then, using the Stone-Weierstrass approximation theorem (which we take as a black box),  $C_{\text{alg}}(U(1))$  is uniformly dense in  $C(U(1))$ : any continuous function  $U(1) \rightarrow \mathbb{C}$  can be approximated by sums of these  $\chi_n$ . Also, if  $L^2(U(1))$  denotes the completion of  $C(U(1))$  in this inner product,  $C_{\text{alg}}(U(1))$  is also dense in  $L^2(U(1))$ .

This recovers the Fourier-theoretic statement: the  $\chi_n$  are an orthonormal basis for  $L^2(U(1))$ , so every function has a Fourier series, an infinite sum of characters.  $\blacktriangleleft$

We want to do the same thing, but for a general compact Lie group  $G$ .

Let  $V$  be a finite-dimensional representation of  $G$ , and choose a basis of  $V$ , which provides an identification  $\text{GL}(V) \cong \text{GL}_n(\mathbb{C})$ . Thus, the representation may be considered as a map  $\rho_V : G \rightarrow \text{GL}(V) \xrightarrow{\cong} \text{GL}_n(\mathbb{C})$ . Let  $1 \leq i, j \leq n$ ; then, taking the  $ij^{\text{th}}$  component of a matrix defines a map  $\text{GL}_n(\mathbb{C}) \rightarrow \mathbb{C}$ . Composing this map with  $\rho_V$  defines the *matrix coefficients map*  $m_{V,i,j} : G \rightarrow \mathbb{C}$ .

Picking a basis is unsatisfying, so let's state this in a more invariant way. Let  $v \in V$  and  $\alpha \in V^*$ ; they will play the role of  $i$  and  $j$ . Then, we have a matrix coefficients map

$$m_{V,\alpha,v} : G \longrightarrow \mathbb{C} \\ g \longmapsto \alpha(\rho_V(g) \cdot v).$$

Another way to write this is as a map  $m_V : V^* \otimes V \rightarrow C(G)$  sending  $\alpha, v \mapsto m_{V,\alpha,v}$ . Since  $V^* \otimes V \cong \text{End}(V)$  canonically, this is determined by where it sends the identity function  $1_V$ . The resulting function, denoted  $\chi_V$ , is called the *character* of  $V$ , and is the trace of  $\rho_V(g)$ : in a basis, the formula is

$$\chi_V(g) := \sum_{i=1}^n m_{V,i,i}(g).$$

Characters are extremely useful in representation theory.

Inside  $C(G)$ , let  $C_{\text{alg}}(G)$  denote the union of  $\text{Im}(m_V)$  as  $V$  ranges over all finite-dimensional representations.

**Lemma 6.3.**  $C_{\text{alg}}(G)$  is a subring of  $C(G)$  under pointwise addition and multiplication.

*Proof sketch.* The reason is that if  $V$  and  $W$  are representations,  $v, w \in V$ , and  $\alpha, \beta \in V^*$ , then

$$\begin{aligned} m_{V, \alpha, v} + m_{W, \beta, w} &= m_{V \oplus W, \alpha + \beta, v + w} \\ m_{V, \alpha, v} m_{W, \beta, w} &= m_{V \otimes W, \alpha \otimes \beta, v \otimes w}. \end{aligned} \quad \square$$

If you like algebra, this is very nice: we'll later see that  $C_{\text{alg}}(G)$  has the structure of a commutative Hopf algebra, meaning it's the ring of functions of a complex algebraic group.

**Proposition 6.4.**

- (1) If  $V$  is an irreducible representation of  $G$ ,  $m_V$  is injective.
- (2) If  $V$  and  $W$  are non-isomorphic irreducible representations of  $G$ , then  $\text{Im}(m_V) \perp \text{Im}(m_W)$  in the  $L^2$ -inner product (6.2).
- (3) If you restruct this inner product to  $V^* \otimes V$  through  $m_V$  (where  $V$  is irreducible), it equals  $(1/\dim V)(-, -)_{V^* \otimes V}$ , where

$$(\alpha_1 \otimes v_1, \alpha_2 \otimes v_2) := (\alpha_1, \alpha_2)_{V^*} (v_1, v_2)_V,$$

for any  $G$ -invariant inner product  $(-, -)_V$  and induced dual inner product  $(-, -)_{V^*}$ .

**Exercise 6.5.** Under the canonical identification  $V^* \otimes V \cong \text{End } V$ , what does this inner product do on  $\text{End}(V)$ ?

Now, consider  $C(G)$  as an (infinite-dimensional) representation of  $G \times G$ , where the first factor acts on the left and the second acts on the right.

We'll prove the proposition next time. It turns out that  $V^* \otimes V$  is irreducible as a  $(G \times G)$ -representation, and the matrix coefficients map is  $(G \times G)$ -equivariant. Thus, we can express all of this in terms of  $(G \times G)$ -invariant bilinear forms.

Lecture 7.

## The Peter-Weyl theorem: 2/1/17

Last time, we introduced  $C_{\text{alg}}(G)$ , the subspace of continuous (complex-valued) functions on  $G$  generated by matrix coefficients of finite-dimensional representations.

**Theorem 7.1.** *There is an isomorphism of  $(G \times G)$ -representations*

$$C_{\text{alg}}(G) \cong \bigoplus_{V \in \widehat{G}}^{\perp} (V^* \otimes V),$$

where  $\bigoplus^{\perp}$  denotes an orthogonal direct sum. This isomorphism preserves an invariant Hermitian form.

*Proof sketch.* Suppose  $V \in \widehat{G}$ ; we then check  $V^* \otimes V$  is an irreducible  $(G \times G)$ -representation (which is an exercise). Last time, we saw that  $m_V : V^* \otimes V \rightarrow C(G)$  is a  $(G \times G)$ -homomorphism, and therefore must be injective (since its kernel is a subrepresentation of  $V^* \otimes V$ ). This implies orthogonality, by a lemma from last week.

It remains to check that  $(-, -)_{L^2}$ , restricted to  $V^* \otimes V$  via  $m_V$ , is  $(-, -)_{V^* \otimes V} \cdot (1/\dim V)$ . This can be computed in coordinates, choosing an orthonormal basis for  $V$ . □

This relates to a bunch of related statements called the Peter-Weyl theorem, excellently exposted by Segal and Macdonald in "Lectures on Lie groups and Lie algebras."

**Theorem 7.2** (Peter-Weyl).  $C_{\text{alg}}(G) \subseteq C(G)$  is dense in the uniform norm.

If you only care about compact matrix groups,<sup>9</sup> this density is a consequence of the Stone-Weierstrass theorem: any continuous function on a compact subset of  $\mathbb{R}^{n^2}$  can be approximated uniformly by polynomials. Then, one shows that the Peter-Weyl theorem holds for every compact Lie group by showing every compact Lie group has a faithful representation. Maybe we'll return to this point later, when we return to more analytic issues.

There's another consequence involving  $L^2$  functions.

**Corollary 7.3.**  $C_{\text{alg}}(G)$  is dense in  $L^2(G)$ , i.e.

$$L^2(G) \cong \widehat{\bigoplus (V^* \otimes V)}.$$

Here,  $\widehat{\bigoplus}$  denotes the completion of the direct sum.

You can compare this to the usual situation in Fourier analysis: you can write functions on a circle in terms of exponentials, and this is a generalization — you can write any function in terms of matrix coefficients.

If you restrict to class functions, there's a cleaner result.

**Definition 7.4.** The *class functions* on a group  $G$  are the  $L^2(G)$  functions invariant under conjugation. That is,  $G$  acts on  $L^2(G)$  by  $g \cdot f(x) = f(gxg^{-1})$ , and we consider the invariants  $L^2(G)^G$ .

**Corollary 7.5.** The class functions decompose as

$$L^2(G)^G \cong \widehat{\bigoplus_{V \in \widehat{G}} (V^* \otimes V)^G} = \widehat{\bigoplus \mathbb{C} \chi_V}.$$

Here,  $(V^* \otimes V)^G = \text{End}_G(V) = \mathbb{C} \cdot 1_V$  and  $\chi_V(g) := \text{tr}(\rho_V(g))$  is the character of  $V$ .

**Corollary 7.6.** The set  $\{\chi_V \mid V \in \widehat{G}\}$  is an orthonormal basis for the class functions  $L^2(G)^G$ . In particular,

$$(\chi_V, \chi_W) = \begin{cases} 0, & V \not\cong W \\ 1, & V \cong W. \end{cases}$$

This makes it clear that the orthogonality relations arise purely from representation theory.

This has the following surprising corollary.

**Corollary 7.7.** The isomorphism class of a representation is determined by its character.

Usually you check for things to be isomorphic by finding an isomorphism between them. But in this case, you can just compute a complete invariant, and that's pleasantly surprising.

*Proof.* Any finite-dimensional representation  $W$  can be written

$$W \cong \bigoplus_{V \in \widehat{G}} V^{\oplus m_V}.$$

Thus,

$$\chi_W = \sum_{V \in \widehat{G}} m_V \chi_V,$$

so  $m_V = \langle \chi_V, \chi_W \rangle$ . □

Maybe this is less surprising if you've already seen some character theory for finite groups.

The Peter-Weyl theorem is also useful for studying functions on homogeneous spaces. For the rest of this lecture, let  $G$  be a compact Lie group and  $H$  be a closed subgroup. Then,  $C(G/H) = C(G)^H$ , the  $H$ -invariant functions on  $G$ ,<sup>10</sup> where  $H$  has the right action on  $C(G)$ . The Peter-Weyl theorem says that

$$C(G.H) = C(G)^H \cong \left( \widehat{\bigoplus V^* \otimes V} \right)^H = \widehat{\bigoplus V^* \otimes V^H}.$$

<sup>9</sup>That is, we think of  $G \subseteq \text{U}(N)$  for some  $N$ , or equivalently,  $G$  admits a faithful, finite-dimensional representation.

<sup>10</sup>This is a different action than the conjugation action we used to write  $C(G)^G$ . Unfortunately, there's no easy way to notate these both.

**Example 7.8.** For example, consider the 2-sphere  $S^2 = \mathrm{SO}(3)/\mathrm{SO}(2)$ , or  $\mathrm{SU}(2)/\mathrm{U}(1)$  through the Hopf fibration, which is the double cover of the previous quotient.

Recall that the irreducible representations of  $\mathrm{SU}(2)$  are  $P_n$  for each  $n \geq 0$ , where  $P_n$  is the space of homogeneous polynomials of degree  $n$  in two variables (which is an  $(n+1)$ -dimensional space). Then,

$$C(S^2) = C(\mathrm{SU}(2))^{\mathrm{U}(1)} \cong \widehat{\bigoplus_{n \in \mathbb{N}} P_n^*} \otimes (P_n^{\mathrm{U}(1)}).$$

Here, we've switched from  $L^2$  functions to continuous ones, which is overlooking a little analysis, but the analysis isn't that bad in this case, so it's all right.

Anyways, what does it mean for a polynomial to be fixed by  $\mathrm{U}(1)$ ? Well,  $\mathrm{U}(1) \hookrightarrow \mathrm{SU}(2)$  through matrices such as  $\begin{pmatrix} t & 0 \\ 0 & \bar{t} \end{pmatrix}$ , and the action is

$$\begin{pmatrix} t & 0 \\ 0 & \bar{t} \end{pmatrix} \cdot x^a y^b = (t^{-1}x)^a (ty)^b = t^{b-a} x^a y^b.$$

Thus, we need  $b = a$  and  $a + b = n$ , so

$$P_n^{\mathrm{U}(1)} \cong \begin{cases} \mathbb{C}, & n \text{ even} \\ 0, & n \text{ odd.} \end{cases}$$

Therefore

$$C(S^2) = \widehat{\bigoplus_{n \in 2\mathbb{N}} P_n^*},$$

and you can go even further and figure the decomposition out explicitly.  $\blacktriangleleft$

The point of this example is that, even though we used very little information about the sphere, we put some significant constraints on functions on it, which is part of what makes representation theory cool.

*Remark.* In physics, these are indexed by  $n/2$  instead of by  $n$ , and  $n/2$  is called the *spin* of the representation. One says that only the integer-spin representation terms appear, not the half-integer ones.  $\blacktriangleleft$

Lecture 8.

### Character tables: 2/3/17

There's homework posted on the website; it's due at the end of the month.

Let  $G$  be a compact group. Then,  $C(G)$ , the space of continuous (complex-valued) functions on  $G$ , is called the *group algebra*: it has an algebra structure given by pointwise multiplication, but there's another algebra structure given by *convolution*: if  $f_1, f_2 \in C(G)$ , we define

$$(f_1 * f_2)(g) := \int_G f_1(h) f_2(h^{-1}g) \, dh,$$

where  $dh$  is the normalized Haar measure. This looks asymmetric, which is kind of strange, but what we're doing is integrating over the pairs of elements whose product equals  $g$ :

$$(f_1 * f_2)(g) = \int_{\{h_1 h_2 = g\}} f_1(h_1) f_2(h_2).$$

The identification comes by  $h \mapsto (h, h^{-1}g)$ .<sup>11</sup> This product makes  $C(G)$  into a noncommutative  $\mathbb{C}$ -algebra.<sup>12</sup>

If  $V$  is a finite-dimensional representation of  $G$ , then  $C(G)$  acts on  $V$  by the formula

$$f * v = \int_G f(g)(g \cdot v) \, dg$$

for any  $v \in V$  and  $f \in C(G)$ . You can check that this makes  $V$  into a module for  $C(G)$ . A  $\delta$ -function at some  $h \in G$  isn't continuous, unless  $G$  is finite, but the integral still makes sense distributionally, and you get  $\delta_h * v = h \cdot v$ .

You can also restrict to class functions.

<sup>11</sup>If  $G$  isn't compact, e.g.  $G = \mathbb{R}^n$ , this definition makes sense, but doesn't always converge; in this case, you can restrict to compactly supported functions, though their convolution won't be compactly supported. In this way one recovers the usual convolution operator on  $\mathbb{R}^n$ .

<sup>12</sup>There's a slight caveat here: there's no identity in this algebra, unless  $G$  is discrete.



**Exercise 8.1.** Show that class functions are central in  $(C(G), *)$ , and in fact that  $C(G)^G = Z(G)$ : the class functions are the center.

In particular, this means the class functions are a commutative algebra, and it sees the representation theory of  $G$ : the characters are all in  $C(G)^G$ , satisfy orthogonality relations, and see isomorphism classes of representations. The irreducible characters also have a nice formula under convolution.

**Proposition 8.2.** If  $V, W \in \widehat{G}$ , then

$$\chi_V * \chi_W = \begin{cases} \left(\frac{1}{\dim V}\right) \chi_V, & V = W \\ 0, & V \neq W. \end{cases}$$

*Proof.* The trick is to turn the inner product into convolution at the identity:

$$\begin{aligned} (\chi_V, \chi_W)_{L^2} &= \int_G \chi_V(g) \overline{\chi_W(g)} dg \\ &= \int_G \chi_V(g) \chi_W(g^{-1}) dg \\ &= (\chi_V * \chi_W)(1). \end{aligned}$$

This proves it at the identity; the rest is **TODO**. □

**Finite groups.** In this section, assume  $G$  is finite.

There are some particularly nice results for finite groups, since every function on a finite group is continuous. If you've been going to Tamás Hausel's talks this week (and the two more talks next week), his work on computing cohomology of moduli spaces, uses the character theory of finite groups in an essential way.

When  $G$  is finite, the group algebra  $C(G)$  is usually denoted  $\mathbb{C}[G]$ . This notation means a vector space with basis  $G$ , i.e.

$$\mathbb{C}[G] = \left\{ \sum a_g g \mid g \in G, a_g \in \mathbb{C} \right\}.$$

You can make this definition (finite weighted sums of elements of  $G$ ) for any group, but it won't be isomorphic to  $C(G)$  unless  $G$  is discrete. In any case, when  $G$  is finite,  $\mathbb{C}[G]$  is a finite-dimensional, unital algebra.

Let's think about what the Peter-Weyl theorem says in this context. Since  $\mathbb{C}[G]$  is finite-dimensional, we can ignore the completion and obtain an isomorphism

$$(8.3) \quad \mathbb{C}[G] \cong \bigoplus_{V \in \widehat{G}} (V^* \otimes V).$$

This has a number of fun corollaries. First, take the dimension of each side:

**Corollary 8.4.** Let  $G$  be a finite group. Then,  $G$  has finitely many isomorphism classes of irreducible representations, and moreover

$$|G| = \sum_{V \in \widehat{G}} (\dim V)^2.$$

For example, once you've found the trivial and sign representations of  $S_3$ , you're forced to conclude there's one more irreducible 2-dimensional representation or two more one-dimensional representations.

Looking at class functions, every function invariant on conjugacy classes is continuous now, so we have two bases for  $\mathbb{C}[G]^G$ :  $\{\chi_V \mid V \in \widehat{G}\}$  as usual, and the set of  $\delta$ -functions on conjugacy classes of  $G$ .

**Corollary 8.5.**  $|\widehat{G}|$  is equal to the number of conjugacy classes of  $G$ .

This leads to an organizational diagram called the *character table* for a finite group: across the top are the conjugacy classes  $[g]$  and down the left are the irreducible characters  $\chi$ . The entry in that row and that column is  $\chi(g)$ . This table says a lot about the representation theory of  $G$ . By Corollary 8.5, it's square.

**Example 8.6.** Let  $G = S_3$ , so conjugacy classes are cycle types (as in any symmetric group). The trivial representation has character 1 on all conjugacy classes; the sign representation has value 1 on  $e$  and  $(1\ 2\ 3)$  and  $-1$  on  $(1\ 2)$ . Then, you can compute  $\chi_V$  where  $V$  is the irreducible two-dimensional representation, and see Table 1 for the character table. Alternatively, you can use the orthogonality relations to compute the remaining entries. ◀

	$e$	$(1\ 2)$	$(1\ 2\ 3)$
$\chi_{\text{triv}}$	1	1	1
$\chi_{\text{sign}}$	1	-1	1
$\chi_V$	2	0	-1

TABLE 1. Character table of  $S_3$ .

This matrix looks like it should be unitary, but isn't quite. Nonetheless, it has some nice properties: notice that the columns are also orthogonal.

**Example 8.7.**  $S_4$  is only a little more complicated. The conjugacy classes are  $e$ ,  $(1\ 2)$ ,  $(1\ 2\ 3)$ ,  $(1\ 2)(3\ 4)$ , and  $(1\ 2\ 3\ 4)$ .

We know the trivial and sign representations, and there's the defining representation  $W$  where  $S_4$  permutes a basis of  $\mathbb{C}^4$ . This is reducible, since it fixed  $(1, 1, 1, 1)$ ; alternatively, you could use the orthogonality relation to check. So we know there's a copy of the trivial representation, so  $\chi_V := \chi_W - \chi_{\text{triv}}$ , and you can check it's irreducible.  $\blacktriangleleft$

Lecture 9.

### The character theory of $\text{SU}(2)$ : 2/6/17

Last time, we got stuck on the proof of Proposition 8.2. The key idea that was missing is that if  $m_V : V^* \otimes V \rightarrow C(G)$  is the matrix coefficients function for the representation  $V$ , then  $\text{Im}(m_V)$  is a two-sided ideal of  $C(G)$  under convolution. This follows from the formula for convolving with a matrix coefficient: given  $v \in V$ ,  $\alpha \in V^*$ , and an  $f \in C(G)$ ,

$$m_{V, \alpha, v} * f = m_{V, \alpha, v'},$$

where

$$v' = \int_G f(h)h^{-1} \cdot v \, dh.$$

This is fairly easy to prove.

Characters are examples of matrix coefficients, so if  $V, W \in \widehat{G}$ ,  $\chi_V * \chi_W \in_V (V^* \otimes V)^G \cap m_W(W^* \otimes W)^G$ . Since  $m_V(V^* \otimes V)^G \cong \text{End}_G V$ , and similarly for  $W$ , this space is trivial when  $V \not\cong W$  and is one-dimensional when  $V \cong W$ . So the proof boils down to checking  $\chi_V * \chi_V = (1/\dim V)\chi_V$ .

This means that the characters are almost orthogonal idempotents. To get actual idempotents, though, we have to normalize: let  $e_V := \dim(V)\chi_V \in C(G)^G$ , so that  $e_V * e_V = 1$  and  $e_V * e_W = 0$  if  $V \not\cong W$ . If  $U$  is any (unitary) representation of  $G$ , then  $C(G)$  acts on  $U$ , and under this action,  $e_V$  acts by projection onto the  $V$ -isotypical component of  $U$ .

For yet another way of thinking about this, we have the matrix coefficients map  $m_V : \text{End}_{\mathbb{C}}(V) \rightarrow C(G)$ , which sends  $1_V \mapsto \chi_V$ , but also the action map  $\text{act}_V : C(G) \rightarrow \text{End}(V)$ . This sends  $e_V \mapsto 1_V$ , so these maps aren't literally inverses. They are adjoint with respect to the appropriate Hermitian forms, however.

$\smile \cdot \smile$

Last time, we also worked out some examples of representations of finite groups. We managed to translate that problem into a completely different problem, of identifying certain class functions in  $C(G)^G$ . Today, we're going to try to generalize this to an infinite group, namely  $\text{SU}(2)$ .

Let's fix some notation. Concretely,

$$G = \text{SU}(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mid |a|^2 + |b|^2 = 1 \right\},$$

and inside this lies the *maximal torus*

$$T = \text{U}(1) = \left\{ \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \mid |z| = 1 \right\}.$$

Some of what we say will generalize, but  $\text{SU}(2)$  is the nicest case.

We'd like to understand the conjugacy classes of  $\text{SU}(2)$ . Let  $\tau := (\text{tr}/2) : \text{SU}(2) \rightarrow [-1, 1]$ , so  $\tau\left(\begin{smallmatrix} z & 0 \\ 0 & \bar{z} \end{smallmatrix}\right) = \text{Re}(z)$ . The fibers of  $\tau$  are exactly the conjugacy classes of  $\text{SU}(2)$ . What do they look like? Geometrically,

$SU(2)$  is a 3-sphere in  $\mathbb{R}^4$ , and we're fixing one of the real coordinates to be a particular number, so the level sets are 2-spheres, except for the poles, which are single-point conjugacy classes, and are the central elements of  $SU(2)$ .

What this means is that a class function is determined by its restriction to the maximal torus  $T$ , and for any unit complex number  $z$ ,  $(\begin{smallmatrix} z & 0 \\ 0 & \bar{z} \end{smallmatrix})$  and  $(\begin{smallmatrix} \bar{z} & 0 \\ 0 & z \end{smallmatrix})$  are in the same conjugacy class. Thus, restriction determines an isomorphism

$$(9.1) \quad C(SU(2))^{SU(2)} \xrightarrow{\cong} C(U(1))^{\mathbb{Z}/2},$$

where  $\mathbb{Z}/2$  acts on  $C(U(1))$  by conjugation. This means we can think of class functions on  $SU(2)$  in terms of their Fourier coefficients!

Given a representation  $V$  of  $SU(2)$ , we can restrict it to a  $U(1)$ -representation  $V'$ . We decompose this into *weight spaces*

$$V' = \bigoplus_{n \in \mathbb{Z}} V_n,$$

where  $z \in U(1)$  acts on  $V_n$  by  $z^n$ . Thus, identifying the character  $\chi_V \in C(SU(2))^{SU(2)}$  with its image under (9.1), we get

$$\chi_V \mapsto \sum_{n \in \mathbb{Z}} \dim(V_n) z^n.$$

Part of what made the representation theory of  $U(1)$  awesome was integration: we had a nice formula for the integral of a class function. How can we generalize this to  $SU(2)$ ? The map (9.1) doesn't preserve volume, as its fibers are spheres with different radii.

Recall that the irreducible representations of  $SU(2)$  are the spaces  $P_n$  of homogeneous polynomials in two variables of degree  $n$ . If we restrict  $P_n$  to a  $U(1)$ -representation and decompose it, we discover

$$P_n = \mathbb{C} \cdot x^n \oplus \mathbb{C} \cdot x^{n-1}y \oplus \cdots \oplus \mathbb{C} \cdot xy^{n-1} \oplus \mathbb{C} \cdot y^n.$$

The term  $x^k y^{n-k}$  has weight  $n - 2k$ , i.e.  $z$  acts as  $z^{n-2k}$  on  $\mathbb{C} \cdot x^k y^{n-k}$ . Therefore as a  $U(1)$ -representation,

$$(9.2) \quad \begin{aligned} \chi_{P_n}(z) &= z^n + z^{n-2} + \cdots + z^{-n+2} + z^{-n} \\ &= \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}}. \end{aligned}$$

(9.2) is called the *Weyl character formula* for  $SU(2)$ . There will be corresponding formulas for other groups.

We would like to use something like this to do integration: given a class function  $f \in C(SU(2))^{SU(2)}$ , let

$$f(\phi) := f \left( \begin{pmatrix} e^{i\phi} & \\ & e^{-i\phi} \end{pmatrix} \right),$$

so  $z = e^{i\phi}$ . We want to determine a function  $J(\phi)$  such that

$$\int_{SU(2)} f(g) dg = \int_0^{2\pi} f(\phi) J(\phi) d\phi.$$

The thing that powers this is that  $f$  is determined by its image under (9.1), so we should be able to determine its integral in those terms. This requires that  $f$  is a class function, and is not true in general.

Here,  $J(\phi)$  is the area of the conjugacy class whose trace is  $2 \cos \phi$ . This conjugacy class is a sphere of radius  $\sqrt{1 - \cos^2(\phi)} = |\sin \phi|$ . Thus,  $J(\phi) = C \sin^2 \phi$ , where  $C$  is such that

$$\int_0^{2\pi} C \sin^2 \phi d\phi = 1.$$

This means  $C = 1/\pi$ , so the integration formula for class functions is

$$(9.3) \quad \int_{SU(2)} f(g) dg = \frac{1}{\pi} \int_0^{2\pi} f(\phi) \sin^2 \phi d\phi.$$

**Example 9.4.** Let's try to compute this for  $n = 1$ .

$$\|\chi_{P_1}\|^2 = \frac{4}{\pi} \int_0^{2\pi} \cos^2 \phi \sin^2 \phi d\phi.$$

This is difficult but tractable, and the answer is

$$= \frac{1}{\pi} \left[ \frac{1}{2} \phi - \frac{1}{8} \sin(4\phi) \right]_0^{2\pi} = 1.$$

This is good, because we said  $P_1$  is irreducible, so the norm of its character had better be 1.  $\blacktriangleleft$

Lecture 10.

### Representation theory of Lie groups: 2/8/17

Last time, we discussed the character theory of  $SU(2)$ , using a convenient isomorphism of the algebra of class functions  $C(SU(2))^{SU(2)}$  with  $C(U(1))^{\mathbb{Z}/2}$ . If  $P_n$  denotes the irreducible  $SU(2)$ -representation of dimension  $n+1$ , and  $z = e^{i\phi}$ , the image of  $\chi_{P_n}$  in  $C(U(1))^{\mathbb{Z}/2}$  is

$$\chi_{P_n}(z) = z^n + z^{n-2} + \cdots + z^{-n+2} + z^{-n}.$$

Then we used the Weyl integration formula for  $SU(2)$ , (9.3), to show that

$$\begin{aligned} \|\chi_{P_1}\|^2 &= \frac{1}{\pi} \int_0^{2\pi} \sin^2(2\phi) d\phi \\ &= \frac{1}{\pi} \left[ \frac{\phi}{2} - \frac{1}{8} \sin(4\phi) \right]_0^{2\pi} = 1. \end{aligned}$$

Thus,  $P_1$  is an irreducible representation!

**Exercise 10.1.** Show that for every  $n$ ,  $\|\chi_{P_n}\|^2 = 1$ , so each  $P_n$  is irreducible.

So we've found some irreducible representations, and in a very curious manner, only using their characters. We then need to show there are no additional irreducible representations.

**Exercise 10.2.** Show that the functions  $\{\cos(n\phi)\}_{n \in \mathbb{Z}_{\geq 0}}$  are a basis for  $C_{\text{alg}}(U(1))^{\mathbb{Z}/2}$ .

This implies  $\{\chi_{P_n}\}$  is also a basis, and are orthonormal, so they account for all irreducible representations.

You might next wonder what the isomorphism class of  $P_n \otimes P_m$  is, as a direct sum of irreducibles. Recall that  $\chi_{V \otimes W} = \chi_V \chi_W$ , so if  $m \leq n$ ,

$$(10.3) \quad \chi_{P_n \otimes P_m} = \chi_{P_n} \chi_{P_m} = \left( \frac{z^{n+1} - z^{n-1}}{z - z^{-1}} \right) (z^m + z^{m-2} + \cdots + z^{-m}).$$

**Exercise 10.4** (Clebsch-Gordon rule). Show that (10.3) satisfies

$$\chi_{P_n} \chi_{P_m} = \sum_{k=0}^m \chi_{P_{n+m-2k}}.$$

Now that that's settled, let's look at other compact Lie groups. There's a double cover  $SU(2) \twoheadrightarrow SO(3)$ , so  $SO(3) \cong SU(2)/\pm I$ . Thus, the irreducible representations of  $SO(3)$  are given as the representations of  $SU(2)$  in which  $-I$  acts trivially. These are the  $P_n$  for even  $n$ . In other words,  $SU(2) \cong \text{Spin}(3)$ , so the irreducible representations of  $SO(3)$  have integer spin, and those of  $\text{Spin}(3)$  may have half-integer spin.

It's possible to get a little more information from this: there's a double cover  $SU(2) \times SU(2) \rightarrow SO(4)$ , so you can work out the representation theory of  $SO(4)$  in a similar way. But there's loads of other interesting groups, including  $U_n$ ,  $Sp_n$ , and many more.

**An overview of the theory.** Let  $G$  be a compact, connected Lie group.<sup>13</sup> Then,  $G$  admits a *maximal torus*  $T$ , a maximal abelian subgroup (which is necessarily isomorphic to  $U(1)^n$ ), and any  $g \in G$  is conjugate to an element of  $T$ . For example, when  $G = U(n)$ ,  $T$  is the subgroup of diagonal unitary matrices, which is isomorphic to  $U(1)^n$ ; that every  $g \in U(n)$  is conjugate to something in  $T$  means that every unitary matrix is diagonalizable. It's also true that any two maximal tori are conjugate to each other.

Thus, just like we did for  $SU(2)$  and  $U(1)$ , we can express a class function on  $G$  in terms of its restriction to  $T$ , and then use Fourier theory. Let  $\hat{T} = \text{Hom}(T, U(1))$ , which is a lattice  $\mathbb{Z}^n$ . Our goal is to write the

<sup>13</sup>Focusing on connected groups isn't too restrictive: any compact Lie group  $G$  is an extension of its identity component  $G^0$  by the finite group  $\pi_0(G)$ .

irreducible characters of  $G$  as Fourier series on  $T$ . There are still some questions, though: how do we do integration? And can we obtain a formula for the characters of the irreducible representations? The Weyl integration formula and Weyl character formula will answer these questions.

Tackling a general compact Lie group with the technology we've developed is complicated. The unitary group is within reach, but it involves knowledge of symmetric functions and representation theory of the symmetric group. So we'll start with some more general theory.

**Lie algebras.** We want to reduce the representation theory of Lie groups to pure algebra. For example, if  $G$  is a finite group, there are finitely many generators and relations, so we can express a representation as a finite set of matrices satisfying the relations. Lie groups aren't finitely (or even countably) generated, so we can't just do algebra.

There are good presentations of Lie groups that are topological, however:  $\mathrm{SO}(3)$  is generated by rotations about certain angles  $R_{x,t}$ ,  $R_{y,t}$ ,  $R_{z,t}$ , which form a frame in  $\mathbb{R}^3$ , and  $t \in \mathbb{R}/\mathbb{Z}$  is an angle. So  $\mathrm{SO}(3)$  can be described in terms of a one-parameter subgroup. We'd like to replace  $\mathrm{SO}(3)$  with its collection of one-parameter subgroups; these can be identified with tangent vectors at the identity, which is why the somewhat surprising notion of a Lie algebra is introduced.

**Definition 10.5.** A *one-parameter subgroup* of a Lie group  $G$  is a homomorphism  $\rho: \mathbb{R} \rightarrow G$ , where  $\mathbb{R}$  is a Lie group under addition.

Homomorphisms of Lie groups are always smooth. Given a one-parameter subgroup  $\rho$ , we can obtain its derivative at the identity,  $\rho'(0) \in T_1G$ . We'll let  $\mathfrak{g}$  (written `\mathfrak{g}` in `LATEX`) denote  $T_1G$ : what we're saying is that a smooth path going through the identity defines a tangent vector.

**Lemma 10.6.** The assignment  $\Phi: \rho \mapsto \rho'(0)$  is a bijection  $\mathrm{Hom}_{\mathrm{LieGrp}}(\mathbb{R}, G) \rightarrow \mathfrak{g}$ .

That is, a direction in  $\mathfrak{g}$  gives rise to a unique one-parameter subgroup in that direction.

*Proof sketch.* This is a generalization of the proof that the characters of  $\mathbb{R}$  are exponential maps: that also involves showing they're determined by their values at 0 using a differential equation.

Let  $\rho$  be a one-parameter subgroup; then, it satisfies the ODE  $\rho'(t) = \rho(t)\rho'(0)$  with the initial condition  $\rho(0) = 1_G$ . Surjectivity of  $\Phi$  follows from the existence of solutions to ODEs, and injectivity follows from uniqueness. You need a way of passing from a local solution to a global one, but this can be done.  $\square$

Now, we can define the *exponential map*  $\exp: \mathfrak{g} \rightarrow G$  sending  $A \mapsto \Phi^{-1}(A)(1)$ : given a tangent vector, move in that direction for a short time, and then return that element. This map is a local diffeomorphism (by the inverse function theorem), so the Lie algebra encodes the Lie group in some neighborhood of the identity. Then, one asks what algebraic properties of  $\mathfrak{g}$  come from the group structure on  $G$ .

— Lecture 11. —

## Lie algebras: 2/10/17

*“We’re the Baker-Campbell-Hausdorff law firm — call us if you can’t commute!”*

Let  $G$  be a Lie group. Last time, we defined its *Lie algebra*  $\mathfrak{g} := T_1G$ , and a local diffeomorphism  $\exp: \mathfrak{g} \rightarrow G$ . A local inverse to the exponential map is called a *logarithm*.

**Example 11.1.** If  $G = \mathrm{GL}_n(\mathbb{R})$ , then  $G$  is an open submanifold of  $M_n(\mathbb{R})$ , the vector space of  $n \times n$  matrices. Thus,  $T_1\mathrm{GL}_n(\mathbb{R}) = T_1M_n(\mathbb{R})$ , and the tangent space to a vector space is canonically identified with that vector space. Thus, the Lie algebra  $\mathfrak{g}$ , also written  $\mathfrak{gl}_n(\mathbb{R})$ , is  $M_n(\mathbb{R})$ . More generally, the Lie algebra of  $\mathrm{GL}(V)$  is  $\mathfrak{gl}(V) = \mathrm{End}(V)$ .

In this case, the exponential map is the matrix exponential  $\exp: M_n(\mathbb{R}) \rightarrow \mathrm{GL}_n(\mathbb{R})$  sending

$$A \mapsto e^A = \sum_{i=0}^{\infty} \frac{A^i}{i!} = 1 + A + \frac{A^2}{2} + \cdots,$$

so a one-parameter subgroup  $\rho_A$  in  $\mathrm{GL}_n(\mathbb{R})$  is given by  $\rho_A(t) = e^{tA}$  for any matrix  $A$ .  $\blacktriangleleft$

More generally, if  $G \subset \mathrm{GL}_n(\mathbb{R})$  is any Lie subgroup, the exponential map  $\mathfrak{g} \rightarrow G$  is the restriction of the matrix exponential  $M_n(\mathbb{R}) \rightarrow \mathrm{GL}_n(\mathbb{R})$  to  $\mathfrak{g} \subset M_n(\mathbb{R})$ .

**Example 11.2.**  $\mathrm{SL}_n(\mathbb{R})$  is the group of matrices  $\{A \in \mathrm{GL}_n(\mathbb{R}) \mid \det(A) = 1\}$  (since 1 is a regular value of the determinant function, this is in fact a Lie group). If you differentiate the determinant condition, you get that  $\mathrm{tr}(A) = 0$ , because  $\exp(\mathrm{tr}(A)) = \det(\exp(A))$ , and therefore the Lie algebra of  $\mathrm{SL}_n(\mathbb{R})$  is

$$\mathfrak{sl}_n(\mathbb{R}) = \{A \in \mathfrak{gl}_n(\mathbb{R}) \mid \mathrm{tr}(A) = 0\}.$$

In particular, the logarithm gives a local chart from  $\mathrm{SL}(n)$  into the vector space  $\mathfrak{sl}_n(\mathbb{R})$ , which is another way to show  $\mathrm{SL}_n(\mathbb{R})$  is a Lie group. (Kirrilov's book uses this to explain why all of the classical groups are Lie groups.)

$\mathrm{O}(n)$  is the group of  $n \times n$  matrices with  $AA^t = -I$ , so differentiating this, the Lie algebra is

$$\mathfrak{o}(n) = \{A \mid A + A^t = 0\},$$

the *skew-symmetric matrices*. This is also the Lie algebra for  $\mathrm{SO}(n)$ :  $\mathrm{SO}(n) \subset \mathrm{O}(n)$  is the connected component of the identity, and therefore  $T_1\mathrm{O}(n) = T_1\mathrm{SO}(n) = \mathfrak{o}(n)$  is again the Lie algebra of skew-symmetric matrices.

The same does not apply for  $\mathrm{SU}(n) \subset \mathrm{U}(n)$ , since  $\mathrm{U}(n) = \{A \in M_n(\mathbb{C}) \mid AA^\dagger = I\}$  is connected: its Lie algebra is

$$\mathfrak{u}(n) = \{A \mid A + A^\dagger = 0\},$$

the *skew-Hermitian matrices*. But not all of these are traceless, so  $\mathfrak{su}(n)$  is the skew-Hermitian matrices with trace zero.  $\blacktriangleleft$

The basic question is, how much does  $\mathfrak{g}$  know about  $G$ ? Clearly not everything, because  $\mathfrak{o}(n) = \mathfrak{so}(n)$ . To make this question precise, we need more structure on  $\mathfrak{g}$ .

To recover information about  $G$ , we need to know what  $\mathfrak{g}$  says about the multiplication on  $G$ . Thus, given  $A, B \in \mathfrak{g}$ , what is  $C(A, B) := \log(\exp(A)\exp(B))$ ? We can Taylor-expand it around  $A = B = 0$ . The first term is  $A + B$  (essentially by the product rule), and there must be higher-order terms unless  $G$  is abelian. Let  $b(A, B)$  be twice the second-order term, so

$$C(A, B) = A + B + \frac{1}{2}b(A, B) + O(A^2, AB, B^2).$$

This  $b: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  has some important properties.

**Proposition 11.3.**

- (1)  $b$  is bilinear.
- (2)  $b$  is skew-symmetric:  $b(A, B) = -b(B, A)$ .
- (3)  $b$  satisfies the Jacobi rule:

$$b(b(A, B), C) + b(b(B, C), A) + b(b(C, A), B) = 0.$$

This allows us to clarify what exactly we mean by a Lie algebra.

**Definition 11.4.** A *Lie algebra* is a vector space  $L$  together with an operation  $b: L \times L \rightarrow L$ , called the *Lie bracket*, satisfying the properties in Proposition 11.3. A *Lie algebra homomorphism* is a linear map between Lie algebras that commutes with the Lie bracket.

All of the Lie algebras we've defined, e.g. those in Examples 11.1 and 11.2, will be understood to come with their Lie brackets.

**Exercise 11.5.** Prove Proposition 11.3. Part of this will involve unpacking what “Taylor series” means in this context. Hint: first, notice that  $C(A, 0) = A$ ,  $C(0, B) = B$ , and  $C(-B, -A) = -C(A, B)$  by properties of the exponential map. These together give (1) and (2); the last part will also follow from properties of  $C$ .

This is pretty neat. You might wonder what additional information the higher-order terms give you, but it turns out that you don't get anything extra.

**Theorem 11.6** (Baker-Campbell-Hausdorff). *The Taylor expansion of  $C(A, B)$  can be expressed purely in terms of addition and the Lie bracket. In particular, the formula begins*

$$C(A, B) = A + B + \frac{1}{2}b(A, B) + \frac{1}{12}(b(A, b(A, B)) + b(B, b(B, A))) + \cdots$$

This is already quite surprising, and it turns out you can calculate it explicitly for most Lie groups we care about.

**Example 11.7.** If  $G = \mathrm{GL}_n(\mathbb{R})$  or  $\mathrm{GL}_n(\mathbb{C})$ , so  $\mathfrak{g} = M_n(\mathbb{R})$  (resp.  $M_n(\mathbb{C})$ ), then  $b(A, B)$  is the matrix commutator  $[A, B] = AB - BA$ . The same is true for any matrix group (i.e. Lie subgroup of  $\mathrm{GL}_n(\mathbb{R})$  or  $\mathrm{GL}_n(\mathbb{C})$ ): the commutator of two traceless matrices is traceless, so  $[\cdot, \cdot]$  also defined the Lie bracket on  $\mathfrak{sl}_n(\mathbb{R})$ ,  $\mathfrak{so}_n$ ,  $\mathfrak{su}_n$ ,  $\mathfrak{u}_n$ , and so on.  $\blacktriangleleft$

We've seen that Lie algebras can't distinguish a Lie group and the connected component of the identity. It also can't see covering maps, since the tangent space depends only on local data, so, e.g.,  $\mathfrak{spin}_n = \mathfrak{so}_n$  through the double cover  $\mathrm{Spin}(n) \rightarrow \mathrm{SO}(n)$ . However, these are the only obstructions: the Lie algebra tells you everything else.

**Theorem 11.8** (Lie's theorem). *There is an equivalence of categories between the category of connected and simply-connected Lie groups and the category of finite-dimensional Lie algebras, given by sending  $G \mapsto \mathfrak{g}$ .*

Lecture 12.

## The adjoint representations: 2/13/17

Last time, we talked about Lie's theorem, Theorem 11.8. Given a Lie group  $G$ , one obtains a Lie algebra  $\mathfrak{g}$ , its tangent space at the identity, with first-order commutation information defining the Lie bracket. Theorem 11.8 states that there's an equivalence of categories from connected, simply connected Lie groups to Lie algebras, where  $G \mapsto \mathfrak{g}$  and  $F: G_1 \rightarrow G_2$  is mapped to  $dF|_1: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ .

*Remark.*

- (1) One consequence is that for any Lie algebra  $\mathfrak{g}$ , there's a corresponding Lie group  $G$ . This is tricky in general, but easier for matrix groups. This is sort of like integration
- (2) On morphisms, if  $f: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  is a morphism of Lie algebras, how do you define  $F: G_1 \rightarrow G_2$ ? In a neighborhood of the identity, it makes sense to take  $F(\exp(x)) := \exp(f(x))$ . It turns out this suffices to define  $F$  on all of  $G$ : in particular,  $G$  is generated by any neighborhood of the identity, though this uses that  $G$  is connected and simply connected.  $\blacktriangleleft$

This has consequences on representation theory.

**Definition 12.1.** A *representation* of a Lie algebra  $\mathfrak{g}$  is a pair  $(V, \rho)$  where  $V$  is a (finite-dimensional, complex) vector space and  $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V) = \mathrm{End}_{\mathbb{C}} V$  is a Lie algebra homomorphism.

Just like for groups, we'll use  $x \cdot v$  as shorthand for  $\rho(x) \cdot v$ , representing the action map  $\mathfrak{g} \times V \rightarrow V$ . That  $\rho$  is a Lie algebra homomorphism means that for all  $x, y \in \mathfrak{g}$  and  $v \in V$ ,

$$x \cdot (y \cdot v) - y \cdot (x \cdot v) = [x, y] \cdot v.$$

Lie's theorem in particular tells us that the finite-dimensional representations of a connected, simply connected Lie group  $G$  are in natural bijection with the representations of its Lie algebra  $\mathfrak{g}$ .<sup>14</sup> And if you care about a group  $G$  that isn't simply connected, such as  $\mathrm{U}(n)$  or  $\mathrm{SO}(n)$ , you can pass to its universal cover  $\tilde{G}$ , which is, and ask about which of its representations project back down to  $G$ .

*Remark.* We've been working with Lie algebras over  $\mathbb{R}$ , but the same definition (and that of a Lie algebra representation) also works over  $\mathbb{C}$ . There are also *complex Lie groups*, which are groups in complex manifolds where multiplication and inversion are holomorphic. There is also an analogue of Lie's theorem in this setting. The category of Lie algebras over  $\mathbb{R}$  is denoted  $\mathrm{Lie}_{\mathbb{R}}$ , and the category of Lie algebras over  $\mathbb{C}$  is  $\mathrm{Lie}_{\mathbb{C}}$ .

When we talk about complex representations of real Lie groups in this setting, something interesting happens. If  $\mathfrak{h}_{\mathbb{C}}$  is a complex Lie algebra and  $\mathfrak{g}_{\mathbb{R}}$  is a real Lie algebra, then  $\mathfrak{h}_{\mathbb{C}}$  is also a Lie algebra by forgetting the complex structure, and  $\mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C}$  is a complex Lie algebra. There's an almost tautological fact that

$$(12.2) \quad \mathrm{Hom}_{\mathrm{Lie}_{\mathbb{R}}}(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{C}}) = \mathrm{Hom}_{\mathrm{Lie}_{\mathbb{C}}}(\mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C}, \mathfrak{h}_{\mathbb{C}}),$$

and this is natural in a suitable sense. This means it's possible to complexify Lie algebras, letting  $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C}$ : there's no good way to complexify a real Lie group into a complex Lie group, but on the Lie algebra level you can make this happen. When  $\mathfrak{h}_{\mathbb{C}} = \mathfrak{gl}_n(\mathbb{C})$ , the adjunction (12.2) tells us that the representations of  $\mathfrak{g}_{\mathbb{R}}$  are the same as those of  $\mathfrak{g}_{\mathbb{C}}$ , and we'll soon see these as modules over a certain algebra, called the universal enveloping algebra.  $\blacktriangleleft$

<sup>14</sup>I hear you saying, "whoa,  $\mathrm{GL}_n(\mathbb{C})$  isn't simply connected!" And you're right (its fundamental group is  $\mathbb{Z}$ ), but the statement of Lie's theorem can be suitably generalized to the case where only the source need be simply connected.



**Example 12.3.** Suppose  $G = U(n)$ , so  $\mathfrak{g} = \mathfrak{u}(n)$  is the Lie algebra of skew-Hermitian matrices (with commutator for its bracket). Its complexification is  $\mathfrak{u}(n)_{\mathbb{C}} = \mathfrak{gl}_n(\mathbb{C})$ , because every invertible complex matrix is the sum of a Hermitian matrix and  $i$  times a Hermitian matrix:

$$A = \underbrace{\left( \frac{A - A^\dagger}{2} \right)}_{\mathfrak{u}(n)} + \underbrace{\left( \frac{A + A^\dagger}{2} \right)}_{i \cdot \mathfrak{u}(n)}. \quad \blacktriangleleft$$

Notice that  $\mathfrak{gl}_n(\mathbb{R}) \otimes \mathbb{C} = \mathfrak{gl}_n(\mathbb{C})$  as well, so different real Lie algebras may map to the same complex one.

**Definition 12.4.** Let  $\mathfrak{g}_{\mathbb{C}}$  be a complex Lie algebra and  $\mathfrak{k} \subset \mathfrak{g}_{\mathbb{C}}$  be a real subalgebra such that  $\mathfrak{k}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}$ . Then,  $\mathfrak{k}$  is called a *real form* of  $\mathfrak{g}_{\mathbb{C}}$ .

To begin analyzing the theory of Lie groups, we'd like to have a few representations hanging around. Usually we defined them as acting on spaces of functions, but in general these representations are infinite-dimensional, which is unfortunate. However, there is an important finite-dimensional representation of  $G$  on  $\mathfrak{g}$ .

**Definition 12.5.** Let  $g \in G$  and  $\varphi_g: G \rightarrow G$  be the action of  $g$  by conjugation:  $h \mapsto ghg^{-1}$ , and let  $\text{Ad}_g := d\varphi_g|_1: \mathfrak{g} \rightarrow \mathfrak{g}$ . Then,  $\text{Ad}: G \rightarrow \text{GL}(\mathfrak{g})$  is a  $G$ -representation, called the *adjoint representation* of  $G$ , and its derivative  $\text{ad}: \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$  (also called the *adjoint representation*) is a Lie algebra representation of  $\mathfrak{g}$ .

The Lie algebra adjoint representation is the bracket:  $\text{ad}_x = [x, -]$ .<sup>15</sup> For example, if  $G \subset \text{GL}_n(\mathbb{R})$  is a matrix group, then  $\text{Ad}_D(A) = DAD^{-1}$ , and  $\text{ad}_B(A) = AB - BA$ . Since we care about complex representations, we'll often consider the adjoint representation of  $\mathfrak{g}_{\mathbb{C}}$ .

In Example 11.2, we found Lie algebras for a few groups. In more explicit detail,  $\mathfrak{su}(2)$  is the algebra traceless skew-Hermitian matrices, which is a 3-dimensional vector space with basis

$$i\sigma_1 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad i\sigma_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad i\sigma_3 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

The notation is because these  $\sigma_j$  are generators for Hermitian matrices. This allows for a generators-and-relations definition of  $\mathfrak{su}(2)$ :

$$\mathfrak{su}(2) = \{i\sigma_1, i\sigma_2, i\sigma_3 \mid [i\sigma_1, i\sigma_2] = -2i\sigma_3, [i\sigma_2, i\sigma_3] = -2i\sigma_1, [i\sigma_3, i\sigma_1] = -2i\sigma_2\}.$$

Skew-symmetry tells us that  $[i\sigma_i, i\sigma_i] = 0$ . In other words, we've defined  $\mathfrak{su}(2)$  to be the complex vector space spanned by these three things, and the Lie bracket is the unique skew-symmetric pairing extending the specified relation.<sup>16</sup>

The Lie algebra  $\mathfrak{so}(3)$  is the skew-orthogonal matrices, which are already traceless. Once again, this is a three-dimensional space (well, since  $\text{SU}(2)$  double covers  $\text{SO}(3)$ ,  $\mathfrak{su}(2) \cong \mathfrak{so}(3)$ ), and has a basis

$$(12.6) \quad J_x := \begin{pmatrix} 0 & & \\ & -1 & \\ & & 1 \end{pmatrix} \quad J_y := \begin{pmatrix} & 1 & \\ & 0 & \\ -1 & & \end{pmatrix} \quad J_z := \begin{pmatrix} & -1 & \\ & & \\ 1 & & 0 \end{pmatrix}.$$

These are the derivatives of rotation around the  $x$ -,  $y$ -, and  $z$ -axes, respectively, and have the same commutation relations as the generators of  $\mathfrak{su}(2)$ .

$\mathfrak{sl}_2(\mathbb{R})$  is a nonisomorphic Lie algebra, but will have the same complexification. It's also three-dimensional, generated by

$$(12.7) \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

That  $\mathfrak{so}(3) \otimes \mathbb{C} \cong \mathfrak{su}(2) \cong \mathbb{C} \cong \mathfrak{sl}_2(\mathbb{R}) \otimes \mathbb{C}$  has an interesting corollary: there's an equivalence of categories between the finite-dimensional representations of  $\text{SU}(2)$ , those of  $\mathfrak{sl}_2(\mathbb{C})$ , and those of  $\text{SL}_2(\mathbb{C})$  as a complex Lie group. Thus, we've found an algebraic way to tackle representations of Lie groups.

<sup>15</sup>This is definitely not always invertible, e.g. if  $\mathfrak{g}$  contains central elements, then they're trivial in the Lie bracket.

<sup>16</sup>If you want to be fancy, you could think of this as a quotient of a free Lie algebra by something.

Lecture 13.

**Representations of Lie algebras: 2/15/17**

There's homework on the website, and it's due at the end of February (minimum three questions). Students are encouraged to come to office hours and ask questions!

Last time, as a corollary to Theorem 11.8, we saw that there's an equivalence of categories between (complex, finite-dimensional) representations of a connected and simply-connected Lie group  $G$  and the representations of its complexified Lie algebra  $\mathfrak{g}_{\mathbb{C}}$ .

**Example 13.1.** One simple example is given by  $G = (\mathbb{R}, +) \cong (\mathbb{R}_{>0}, \cdot)$ . Then,  $\mathfrak{g} = \mathbb{R}$  with trivial Lie bracket, since  $G$  is abelian, and  $\mathfrak{g}_{\mathbb{C}} = \mathbb{C}$ .

The one-dimensional representations of  $\mathfrak{g}_{\mathbb{C}}$  are given by  $\text{Hom}_{\text{LieAlg}}(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}$ , and the one-dimensional representations of  $\mathbb{R}$  are  $\text{Hom}_{\text{LieGrp}}(\mathbb{R}, \mathbb{C}^{\times})$ , which is isomorphic to  $\mathbb{C}$  again, with  $\lambda \mapsto e^{\lambda t}$ . The one-dimensional representations are the irreducibles, so this generates the whole category. ◀

Another example, relating to what we discussed last time, was the relation between the representations of  $\text{SU}(2)$  and those of its complexified Lie algebra  $\mathfrak{su}(2)_{\mathbb{C}} \cong \mathfrak{sl}_2(\mathbb{C})$ . We wrote down the generators of  $\mathfrak{sl}_2(\mathbb{C})$  in (12.7), and those of  $\mathfrak{su}(2)$  in (12.6); they're related by  $i\sigma_1 = ie + if$ ,  $i\sigma_2 = e - f$ , and  $i\sigma_3 = ih$ . You can also take the real span of  $e$ ,  $f$ , and  $h$ , which defines the Lie algebra

$$(13.2) \quad \mathfrak{sl}_2(\mathbb{R}) = \text{span}_{\mathbb{R}}\{e, f, h \mid [e, f] = h, [h, e] = 2e, [h, f] = 2f\}.$$

Now, an  $\text{SU}(2)$ -representation  $V'$  defines an  $\mathfrak{sl}_2(\mathbb{C})$ -representation  $V$ , which in particular comes with actions of  $e$  and  $f$ . However, these do not integrate to an action of a 1-parameter subgroup of  $\text{SU}(2)$ , meaning Lie's theorem has actually introduced extra structure into the theory of  $\text{SU}(2)$ -representations that we didn't have beforehand.

**Multiplication and the universal enveloping algebra.** One thing which might be confusing is that even though elements of the Lie algebra have come to us as matrices, they're not acting by matrix multiplication. That is, there's an operator  $(x \cdot): V \rightarrow V$  for every  $x \in \mathfrak{g}$ , such that the Lie bracket  $[x, y]$  is sent to  $(x \cdot)(y \cdot) - (y \cdot)(x \cdot)$ . Multiplication of matrices isn't remembered, so even though  $h$  is of order 2 as a matrix, its eigenvalues as an operator on  $V$  could be more than  $\pm 1$ . In particular,  $x \cdot (y \cdot v) \neq (xy) \cdot v$ .

Sometimes it's nice to multiply things, though, and we can build a ring in which the above expression makes sense.

**Definition 13.3.** Let  $V$  be a vector space. Then, the *tensor algebra*  $T(V)$  is the free associative algebra on  $V$ , i.e.

$$T(V) = \bigoplus_{n \geq 0} V^{\otimes n},$$

where  $(x_1 \otimes \cdots \otimes x_n) \cdot (y_1 \otimes \cdots \otimes y_m) = x_1 \otimes x_n \otimes y_1 \otimes y_m$  extends linearly to define the multiplication.

**Definition 13.4.** Let  $\mathfrak{g}$  be a Lie algebra. Then, its *universal enveloping algebra*  $\mathcal{U}(\mathfrak{g})$  is the free associative algebra on  $\mathfrak{g}$  such that the Lie bracket is the commutator, i.e.

$$\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g}) / (x \otimes y - y \otimes x - [x, y]).$$

**Exercise 13.5.** Exhibit an equivalence of categories  $\text{Rep}_{\mathfrak{g}} \cong \text{Mod}_{\mathcal{U}(\mathfrak{g})}$ .

**The representations of  $\mathfrak{sl}_2(\mathbb{C})$ .** Let  $V$  be an  $\mathfrak{sl}_2(\mathbb{C})$ -representation, so the actions by  $e$ ,  $f$ , and  $h$  define three operators  $(e \cdot)$ ,  $(f \cdot)$ , and  $(h \cdot)$  satisfying the relations in (13.2), and this is sufficient to define the whole representation. From the perspective of Lie groups, this is incredible — there's so much information in the group structure of a Lie group, so this perspective would be impossible without passing to Lie algebras.

Let  $V[\lambda]$  denote the  $\lambda$ -eigenspace for  $h$  acting on  $V$ . Then,  $V[\lambda]$  is known as the  $\lambda$ -weight space for  $V$ , and its elements are called  $\lambda$ -weight vectors.

**Lemma 13.6.**  $e \cdot V[\lambda] \subseteq V[\lambda + 2]$  and  $f \cdot V[\lambda] \subseteq V[\lambda - 2]$ .

So  $e$  and  $f$  act by shifting the weights up and down.

*Proof.* Let  $v \in V$ , so that

$$\begin{aligned} h \cdot (e \cdot v) &= e \cdot (h \cdot v) + [h, e] \cdot v \\ &= \lambda(e \cdot v) + 2e \cdot v \\ &= (\lambda + 2)e \cdot v, \end{aligned}$$

so  $e \cdot v \in V[\lambda + 2]$ . The proof for  $f$  is identical.  $\square$

Now, assume  $V$  is irreducible and that there's a nonzero eigenvalue  $\lambda$ . Then,  $\bigoplus_{\lambda} V[\lambda] \subset V$  is a nonzero subrepresentation, and therefore must be all of  $V$ ! Thus, for irreducible representations the actions of  $e$  and  $f$  are completely determined on the weight spaces:

$$(13.7) \quad \begin{array}{ccccccc} & & \xrightarrow{e} & & \xrightarrow{e} & & \xrightarrow{e} & & \xrightarrow{e} & & \cdots \\ \cdots & \xleftarrow{f} & V[\lambda - 2] & \xleftarrow{f} & V[\lambda] & \xleftarrow{f} & V[\lambda + 2] & \xleftarrow{f} & V[\lambda + 4] & \xleftarrow{f} & \cdots \end{array}$$

Diagrams that look like (13.7) appear often in mathematics, and the representation theory of  $\mathfrak{sl}_2(\mathbb{C})$  is always hiding in the background.

Let  $\mu$  be a *highest weight*, i.e. an eigenvalue such that  $\operatorname{Re}(\lambda)$  is maximal;<sup>17</sup> a  $v \in V[\mu]$  is called a *highest weight vector*. In particular, by (13.7),  $e \cdot v = 0$ . Let

$$v^k := \frac{1}{k!} f^k \cdot v \in V[\mu - 2k].$$

You can interpret  $f^k$  either as a composition of  $(f \cdot)$   $k$  times, or as an honest element of  $\mathcal{U}(\mathfrak{sl}_2(\mathbb{C}))$ .

**Claim.**  $f \cdot v^k = v^{k+1}$ , and  $e \cdot v^{k+1} = v^k$ .

So  $e$  and  $f$  walk the ladder between the weight spaces.

**Proposition 13.8.**

- (1) *The highest weight is a nonnegative integer.*
- (2)

$$V \cong \bigoplus_{k=0}^{\mu} V[\mu - 2k],$$

and the actions of  $e$  and  $f$  move between these subspaces as in the previous claim.

Lecture 14.

## The representation theory of $\mathfrak{sl}_2(\mathbb{C})$ : 2/17/17

*“Let’s call it  $n$ ; that’s more integer-sounding.”*

Today, we’ll finish the representation theory of  $\mathfrak{sl}_2(\mathbb{C})$  that we started last time. We’ve already seen that any irreducible finite-dimensional representation of  $\mathfrak{sl}_2(\mathbb{C}) = \langle e, f, h \rangle$  has a highest weight vector, i.e. a  $v \in V[\mu]$ , where  $\mu$  is the largest eigenvalue of  $h$  (all eigenvalues are real) and  $V[\mu]$  denotes its eigenspace.

Let’s build an  $\mathcal{U}(\mathfrak{sl}_2(\mathbb{C}))$ -module  $M_\mu$ , i.e.  $\mathfrak{sl}_2(\mathbb{C})$ -representation (we may refer to these as  $\mathfrak{sl}_2(\mathbb{C})$ -modules when they’re not finite-dimensional) freely generated by a highest weight vector  $v_\mu$  of weight  $\mu$ . That is,

$$M_\mu = \mathcal{U}(\mathfrak{sl}_2(\mathbb{C})) \cdot v_\mu := \mathcal{U}(\mathfrak{sl}_2(\mathbb{C})) / (h - \mu, e).$$

That is, we take the free  $\mathcal{U}(\mathfrak{sl}_2(\mathbb{C}))$ -module, and make  $\mu$  a weight (so  $h - \mu = 0$ ), then make it the highest weight (so  $e$  acts as 0).

**Definition 14.1.**  $M_\mu$  is called the *Verma module* of highest weight  $\mu$ .

<sup>17</sup>We’ll eventually prove that all weights are real, so that  $\mu$  is unique.

We'll talk about Verma modules for other simple Lie algebras later.

Why is this good? If  $V$  is any representation of  $\mathfrak{sl}_2(\mathbb{C})$  with a highest weight vector  $v$ , then there is a homomorphism of  $\mathfrak{sl}_2(\mathbb{C})$ -modules  $M_\mu \rightarrow V$  sending  $v_\mu \mapsto v$ . In particular, if  $V$  is irreducible, then this is a quotient map, so  $V \cong M_\mu/W$  for some submodule  $W$ . This is another approach to classifying the representations of  $\mathfrak{sl}_2(\mathbb{C})$ : find all the Verma modules and then list their finite-dimensional quotients.

We can make this very explicit: let  $v_\mu^k = (1/k!)f^k v_\mu$ ; then,  $\{v_\mu^k\}$  is a basis for  $M_\mu$ : it's an infinite set, because there's no lowest weight. The action of  $f$  sends  $v_\mu^i \mapsto v_\mu^{i+1}$ , and  $e$  sends  $v_\mu^{i+1} \mapsto (\mu - i)v_\mu^i$ . Finally,  $h \cdot v_\mu^k = (\mu - 2k)v_\mu^k$ . So what we need are quotients of this.

**Lemma 14.2.** *If  $\mu$  is negative, then  $M_\mu$  is irreducible.*

The key is that  $\mu - i$  is never zero, so starting from any  $v_\mu^i$ , you can use  $e$  and  $f$  to get anywhere.

What if  $\mu = n$  is a nonnegative integer? Then,  $e \cdot v_\mu^{n+1} = 0$ . The subspace  $W = \bigoplus_{\lambda < \mu} V[\lambda]$  is a submodule, and the quotient  $L_\mu := M_\mu/W$  is a finite-dimensional  $\mathfrak{sl}_2(\mathbb{C})$ -representation. Moreover,  $W$  is again a Verma module, and is of the form  $M_{-n-2}$  (check the action of  $h$  on  $v_\mu^{n+1}$ ).

**Corollary 14.3.** *Any finite-dimensional irreducible representation of  $\mathfrak{sl}_2(\mathbb{C})$  is isomorphic to  $L_n$  for some nonnegative integer  $n$ .*

We can also compute the character:

$$\begin{aligned} \chi_{L_n}(z) &= \sum (\dim(L_n[\lambda])) e^{2\pi i \lambda t} = z^{-n} + z^{-n+2} + \cdots + z^{n-2} + z^n \\ &= \frac{z^{-n-1} - z^{n-1}}{z^{-1} - z}, \end{aligned}$$

where  $z = e^{2\pi i t}$ . Each factor of  $z^k$  corresponds to a weight space of weight  $k$ .

*Remark.* The Verma modules  $M_\mu$  are infinite-dimensional representations of  $\mathfrak{sl}_2(\mathbb{C})$ , but they do not integrate to representations of  $SU(2)$ . It's interesting that they're nonetheless part of the process of calculating the representations of  $SU(2)$ . This is an example of infinite-dimensional phenomena behaving badly. ◀

We want to generalize this to talk about representations of other Lie algebras and Lie groups. To do this, we need to discuss the classification of Lie algebras, which will occupy the next week or so. Eventually, we'll restrict to semisimple Lie algebras, but we have to get there first.

Consider an *abelian Lie algebra* (i.e. the Lie algebra of an abelian group)  $\mathfrak{t}$ : the Lie bracket is 0, so this is just the data of a finite-dimensional vector space. Therefore its universal enveloping algebra is  $\mathcal{U}(\mathfrak{t}) = \text{Sym}(\mathfrak{t})$ , the free commutative algebra on  $\mathfrak{t}$ .<sup>18</sup>

This is the nicest case; what's next? Let's consider the things that can be built from abelian Lie algebras.

**Definition 14.4.**

- A *Lie subalgebra*  $\mathfrak{h} \subseteq \mathfrak{g}$  is a vector subspace closed under Lie bracket, i.e.  $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$ .
- An *ideal* in a Lie algebra  $I \subseteq \mathfrak{g}$  is a vector subspace such that  $[I, \mathfrak{g}] \subseteq I$ .

These are the analogues of subgroups and normal subgroups, respectively; in particular, if  $I \subseteq \mathfrak{g}$  is an ideal,  $\mathfrak{g}/I$  is a Lie algebra with the bracket inherited from  $\mathfrak{g}$ .

**Definition 14.5.** A Lie algebra  $\mathfrak{b}$  is *solvable* if there is a sequence

$$\mathfrak{b} \supseteq \mathfrak{b}_1 \supseteq \mathfrak{b}_2 \supseteq \cdots \supseteq \mathfrak{b}_n \supseteq \mathfrak{b}_{n+1} = 0$$

such that each  $\mathfrak{b}_i \subseteq \mathfrak{b}_{i-1}$  is an ideal, and the quotient  $\mathfrak{b}_{i-1}/\mathfrak{b}_i$  is an abelian Lie algebra.

This is just like the definition of solvable groups.

**Example 14.6.** Let  $\mathfrak{b}$  denote the algebra of  $n \times n$  triangular matrices inside  $\mathfrak{gl}_n(\mathbb{C})$ . Then,  $\mathfrak{b}$  is solvable: let  $\mathfrak{b}_1$  be the strictly upper triangular matrices (zero on the diagonal), which is an ideal: the quotient is the Lie algebra of diagonal matrices, which is abelian. Then, one can repeat by setting the superdiagonal to 0 to define  $\mathfrak{b}_2$ , and so forth. ◀

<sup>18</sup>Alternatively, this can be identified with the algebra of polynomials on  $\mathfrak{t}^*$ .

On the other hand,  $\mathfrak{gl}_n(\mathbb{C})$  is not solvable; we'll see that in a much more general context later on.

To understand the representation theory of solvable Lie algebras, we should start with the representations of abelian Lie algebras  $\mathfrak{t}$ . We saw these are identical to  $\text{Sym}(\mathfrak{t})$ -modules, hence representations of a polynomial ring.

Schur's lemma implies that irreducible representations of an abelian Lie algebra are 1-dimensional. This is equivalent to the statement that any collection of commuting matrices has a common eigenvector: the eigenvector defines a 1-dimensional invariant subspace.

Irreducible representations of an abelian Lie algebra  $\mathfrak{t}$  are easy to classify: they are given by

$$\text{Hom}_{\text{LieAlg}}(\mathfrak{t}, \mathbb{C}) = \text{Hom}_{\text{Vect}}(\mathfrak{t}, \mathbb{C}) = \mathfrak{t}^*.$$

This is also the same as  $\text{Hom}_{\text{Alg}_{\mathbb{C}}}(\text{Sym}(\mathfrak{t}), \mathbb{C}) = \text{Spec}(\text{Sym}(\mathfrak{t}))$ . These statements, fancy as they may look, boil down to families of commuting matrices sharing a common eigenvector.

However, there's no analogue to Maschke's theorem in this context: not every representation is completely reducible. (Recall that a representation  $V$  is *completely reducible* if for every subrepresentation  $W \subset V$ , there's a complement  $U \subset V$  such that  $V = W \oplus U$ .) Completely reducible representations are direct sums of irreducibles.

If a representation cannot be written as a nontrivial direct sum of subrepresentations, it's called *indecomposable*. Irreducible representations are indecomposable, but the converse is not always true. So to understand the representation theory of a Lie algebra, it suffices to classify its indecomposable representations.

In some sense, this is the wrong question to ask: it's *wild*, meaning it's nearly impossible. This is important to understand, especially given how tractable the theory is for semisimple Lie algebras.

Lecture 15.

### Solvable and nilpotent Lie algebras: 2/20/17

Throughout today's lecture,  $\mathfrak{t}$  will denote an abelian Lie algebra,  $\mathfrak{b}$  will denote a solvable Lie algebra, and  $\mathfrak{n}$  will denote a nilpotent Lie algebra.

Last time, we talked about the representation theory of abelian Lie algebras. The irreducible representations of  $\mathfrak{t}$  are classified by  $\mathfrak{t}^* := \text{Hom}_{\mathbb{C}}(\mathfrak{t}, \mathbb{C})$ , and all irreducibles are one-dimensional: an element of  $\mathfrak{t}$  is an assignment of an eigenvalue to each operator. We wish to classify more general Lie algebras.

Given a finite-dimensional  $\mathfrak{t}$ -representation  $V$  and a  $\lambda \in \mathfrak{t}^*$ , let's consider its *generalized eigenspace*

$$V_{(\lambda)} := \{v \in V \mid \text{for all } x \in \mathfrak{t}, (x - \lambda(x))^N v = 0 \text{ for some } N \gg 0\}.$$

**Proposition 15.1.**

$$V = \bigoplus_{\lambda} V_{(\lambda)}.$$

In particular, if  $V$  is indecomposable, then  $V = V_{(\lambda)}$  for some  $\lambda \in \mathfrak{t}^*$ .

So irreducible representations are points in  $\mathfrak{t}^*$ , but indecomposables may be slightly “thicker,” though they still live over a single point. More general representations are less local.

*Remark.* One can use algebraic geometry to establish (more or less by definition) that the category of  $\mathcal{U}(\mathfrak{t})$ -modules and the category of quasicoherent sheaves on  $\mathfrak{t}^*$  are equivalent. This is useful for thinking of representations of  $\mathfrak{t}$  geometrically.

For example,  $\mathcal{U}(\mathfrak{t}) \cong \text{Sym}(\mathfrak{t}) \cong \mathbb{C}[\mathfrak{t}^*]$ , and inside these are the finite-dimensional representations, which correspond to coherent sheaves with finite support. So a representation  $V$  becomes a sheaf over the vector space  $\mathfrak{t}^*$ : the fiber over a  $\lambda \in \mathfrak{t}^*$  is the  $\lambda$ -eigenspace of  $V$ . But in algebraic geometry, it's also possible to take higher-order neighborhoods of a point, and the fiber over such a neighborhood (when  $n$  is sufficiently large) is the generalized eigenspace of  $\lambda$ . In particular, every finite-dimensional representation is a finite direct sum of indecomposables, hence is a union of skyscraper sheaves over finitely many points. This makes the classification problem more geometric: we just need to know what the representations over a single point are. ◀

So how should we classify indecomposable representations?

**Example 15.2.** Let's start with the example where  $\mathfrak{t} = \mathbb{C}$  is one-dimensional. A representation of a 1-dimensional Lie algebra is a module over  $\mathbb{C}[x]$ . This is special:  $\mathbb{C}[x]$  is a PID, and polynomial rings in more variables aren't.

We have the following tautology: the isomorphism classes of  $n$ -dimensional  $\mathfrak{t}$ -representations are in bijection with the conjugacy classes of  $n \times n$  matrices. The latter are classified by the *Jordan normal form*, whose theory implies there's a unique indecomposable representation (up to isomorphism) of  $\mathbb{C}[x]$  of dimension  $n$  generated by a single  $\lambda$ -eigenvector, where the action of  $x$  is by the matrix

$$\begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & 0 \\ & & \ddots & \vdots \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix}. \quad \blacktriangleleft$$

In this case, the representation theory of  $\mathfrak{t}$  is *tame*, in that it's easily understood. However, there's no analogue even for  $\mathfrak{t} = \mathbb{C}^k$  when  $k > 1$ , and in fact the representation theory is very complicated! So instead, the best we can do is the geometric picture described above.

One instance of this phenomenon is that the *commuting variety*, the set of  $k$ -tuples of nilpotent  $n \times n$  matrices that all commute up to conjugacy, is not discrete, and so understanding it is very complicated: much is either unknown or was resolved recently.<sup>19</sup>

**Solvable Lie algebras.** Last time, we defined a Lie algebra  $\mathfrak{b}$  to be solvable if it is built from abelian Lie algebras by successive extensions.

**Definition 15.3.** Let  $\mathfrak{g}$  be a Lie algebra. Then, its *derived ideal* is  $D(\mathfrak{g}) := [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}$ , i.e. the ideal spanned by  $[x, y]$  for  $x, y \in \mathfrak{g}$ . Then,  $\mathfrak{g}/D(\mathfrak{g})$  is called the *abelianization* of  $\mathfrak{g}$ , and the projection  $\pi: \mathfrak{g} \rightarrow \mathfrak{g}/D(\mathfrak{g})$  is initial among maps to abelian Lie algebras. That is, if  $f: \mathfrak{g} \rightarrow \mathfrak{t}$  is a morphism of Lie algebras, where  $\mathfrak{t}$  is abelian, there's a unique map  $h$  such that the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\pi} & \mathfrak{g}/D(\mathfrak{g}) \\ & \searrow f & \downarrow \exists! h \\ & & \mathfrak{t} \end{array}$$

commutes.

The *derived series* of  $\mathfrak{g}$  is

$$\mathfrak{g} \supseteq D(\mathfrak{g}) \supseteq D(D(\mathfrak{g})) \supseteq \cdots.$$

**Proposition 15.4.** A Lie algebra  $\mathfrak{g}$  is solvable iff its derived series terminates, i.e.  $D^N(\mathfrak{g}) = 0$  for some  $N \gg 0$ .

*Remark.* The use of the word “derived” is ultimately related to differentiation, as how the derived series in groups can be used to obtain infinitesimal information about a group. Also, the Jacobi identity implies that taking the bracket with something is a derivation on  $\mathcal{U}(\mathfrak{g})$ .  $\blacktriangleleft$

Solvable Lie algebras have properties generalizing abelian ones.

**Theorem 15.5 (Lie).** Let  $V$  be a finite-dimensional representation of  $\mathfrak{b}$ . Then, there's a simultaneous eigenvector  $v$  for the action of all elements of  $\mathfrak{b}$  on  $V$ .

**Corollary 15.6.** An irreducible representation of a solvable Lie algebra is one-dimensional. Moreover, the irreducible representations of  $\mathfrak{b}$  are parameterized by  $(\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}])^*$ .

*Proof.* The proof that they're one-dimensional is the same as for abelian Lie algebras. Thus, any irreducible representation is a map  $\mathfrak{b} \rightarrow \mathbb{C}$ , hence a Lie algebra homomorphism, which must factor through the abelianization.  $\square$

<sup>19</sup>If instead you ask for the matrices to be invertible, you obtain a better-understood algebraic variety called the *character variety*.

Thus, the pictorial description of the representations of abelian Lie algebras also applies to solvable Lie algebras.

Another corollary of Theorem 15.5 is that the representations of  $\mathfrak{b}$  are filtered. By a *(full) flag* for a (finite-dimensional) vector space  $V$  we mean a sequence  $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V$  where  $\dim V_i = i$ .

**Corollary 15.7.** *Let  $V$  be a  $\mathfrak{b}$ -representation. Then, there's a  $\mathfrak{b}$ -invariant flag  $0 = V_0 \subset \cdots \subset V_n = V$ , i.e. each  $V_i$  is a subrepresentation.*

Choose a basis  $\{e_1, \dots, e_n\}$  for  $V$  such that  $V_i = \text{span}\{v_j \mid j \leq i\}$ . The action of  $\mathfrak{b}$  on  $V$  in this basis is through upper triangular matrices, since  $\text{span}\{e_1, \dots, e_i\}$  is preserved for each  $i$ . That is, if  $\mathfrak{b}_n(\mathbb{C})$  denotes the Borel subalgebra of  $\mathfrak{gl}_n(\mathbb{C})$ , i.e. the algebra of upper triangular matrices, any representation  $\rho: \mathfrak{b} \rightarrow \mathfrak{gl}_n(\mathbb{C})$  factors through the inclusion  $\mathfrak{b}_n(\mathbb{C}) \hookrightarrow \mathfrak{gl}_n(\mathbb{C})$ . One says that representations of solvable Lie algebras are *upper triangulizable*.

### Nilpotent Lie algebras.

**Definition 15.8.** A Lie algebra  $\mathfrak{n}$  is *nilpotent* if there is a sequence  $\mathfrak{n} \supsetneq \mathfrak{n}_1 \supsetneq \mathfrak{n}_2 \supsetneq \cdots$  such that  $[\mathfrak{n}, \mathfrak{n}_i] \subseteq \mathfrak{n}_{i+1}$ .

You could also relax to nonstrict inclusions if you require the chain to terminate at 0 at some finite step.

**Theorem 15.9** (Engel<sup>20</sup>). *A Lie algebra  $\mathfrak{n}$  is nilpotent if the adjoint action  $\text{ad}_x$  is nilpotent for each  $x \in \mathfrak{n}$ .*

Lecture 16.

## Semisimple Lie algebras: 2/22/17

Today we'll continue to discuss the taxonomy of Lie algebras. This is a course in representation theory, rather than Lie theory, so we won't get too involved in this.

**The Jordan decomposition.** The following result may be familiar from an algebra class; it's purely linear algebra.

**Proposition 16.1.** *Let  $A: V \rightarrow V$  be a linear operator, where  $V$  is a finite-dimensional vector space. Then, there is a unique way to write  $A = A_s + A_n$ , where*

- $A_s$  is semisimple, i.e.  $V$  is a direct sum of  $A_s$ -eigenspaces;<sup>21</sup>
- $A_n$  is nilpotent, i.e.  $A_n^N = 0$  for some  $N \gg 0$ ; and
- $[A_s, A_n] = 0$ .

For example, if  $A$  is a Jordan block on  $\mathbb{C}^n$ , its semisimple part is the entries on the diagonal, and its nilpotent part is the entries on the superdiagonal. The proof for general  $A$  puts  $A$  in Jordan normal form, more or less.

**Definition 16.2.** If  $\mathfrak{g}$  is a Lie algebra, its *radical*  $\text{rad}(\mathfrak{g})$  is the largest solvable ideal in  $\mathfrak{g}$ .

This exists because the sum of two solvable ideals is solvable, so the solvable ideals form a poset which is bounded above.

**Definition 16.3.** A Lie algebra  $\mathfrak{g}$  is *semisimple* if  $\text{rad}(\mathfrak{g}) = 0$ , i.e. it has no nonzero solvable ideals.

For any Lie algebra  $\mathfrak{g}$ , there is a short exact sequence

$$(16.4) \quad 0 \longrightarrow \text{rad}(\mathfrak{g}) \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\text{rad}(\mathfrak{g}) \longrightarrow 0.$$

The third term is denoted  $\mathfrak{g}_{\text{ss}}$ , and is semisimple. The sequence (16.4) splits, albeit noncanonically, and the isomorphism  $\mathfrak{g} \cong \text{rad}(\mathfrak{g}) \times \mathfrak{g}_{\text{ss}}$  is called the *Levi decomposition*.

There's an ideal  $\text{rad}^n(\mathfrak{g}) := [\mathfrak{g}, \text{rad}(\mathfrak{g})]$ ; if  $V$  is any finite-dimensional representation of  $\mathfrak{g}$ , then  $\text{rad}^n(\mathfrak{g})$  acts by nilpotent operators. This is due to Lie's theorem (Theorem 15.5):  $\text{rad}(\mathfrak{g})$  acts by upper triangular matrices, and one can then show that  $\text{rad}^n(\mathfrak{g})$  acts by strictly upper triangular matrices.

**Definition 16.5.** A Lie algebra is called *reductive* if  $\text{rad}^n(\mathfrak{g}) = 0$ . Equivalently,  $\text{rad}(\mathfrak{g}) = Z(\mathfrak{g})$  (the center).

<sup>20</sup>Friedrich Engel the mathematician is distinct from Friedrich Engels the Marxist.

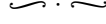
<sup>21</sup>Over  $\mathbb{C}$ , a matrix is semisimple iff it's diagonal.



There's a sense in which the difference between reductive and semisimple Lie algebras is entirely abelian Lie algebras.

**Example 16.6.**  $\mathfrak{gl}_n(\mathbb{C})$  is reductive, but not semisimple:  $\text{rad}(\mathfrak{gl}_n(\mathbb{C})) = Z(\mathfrak{gl}_n(\mathbb{C})) = \mathbb{C} \cdot 1_{nn}$ , the scalar multiples of the identity. The quotient is  $(\mathfrak{gl}_n(\mathbb{C}))_{\text{ss}} = \mathfrak{sl}_n(\mathbb{C})$ , which is semisimple, and  $\mathfrak{gl}_n(\mathbb{C}) \cong \mathfrak{sl}_n(\mathbb{C}) \oplus \mathbb{C}$ . ◀

The Levy decomposition expresses any reductive Lie algebra  $\mathfrak{g}$  as a direct sum of a semisimple Lie algebra  $\mathfrak{g}_{\text{ss}}$  and an abelian Lie algebra  $Z(\mathfrak{g})$ .



Let's apply some of to representation theory.

**Proposition 16.7.** *Let  $\mathfrak{g}$  be a complex Lie algebra. Then,  $\mathfrak{g}$  is reductive iff  $\mathfrak{g} = \text{Lie}(G)_{\mathbb{C}}$  for some compact Lie group  $G$ , and  $G$  is unique up to finite covers.*

That is, the reductive Lie algebras are exactly the ones whose representations we want to study! Semisimple Lie algebras are only a little different.

**Proposition 16.8.** *Let  $\mathfrak{g}$  be a complex Lie algebra. Then,  $\mathfrak{g}$  is semisimple iff  $\mathfrak{g} = \text{Lie}(G)_{\mathbb{C}}$  for some compact, simply connected Lie group  $G$ , and  $G$  is unique up to isomorphism.*

In both of these propositions, we mean real Lie groups  $G$ : there's a theorem that any compact complex Lie group (i.e. complex manifold with holomorphic multiplication and inversion) is a torus, in particular is abelian. If  $\mathfrak{g}$  is a semisimple Lie algebra, a compact Lie group  $G$  with  $\mathfrak{g} = \text{Lie}(G)_{\mathbb{C}}$  is called a *compact form* for  $\mathfrak{g}$ .

The upshot is that “semisimple” is apt for Lie algebras.

**Corollary 16.9.** *If  $\mathfrak{g}$  is a semisimple Lie algebra, then the category  $\text{Rep}_{\mathfrak{g}}$  of finite-dimensional complex representations of  $\mathfrak{g}$  is semisimple, i.e. every representation is a direct sum of completely reducible ones.*

Though we've arrived here through some geometry on Lie groups (“Weyl's unitary trick”), you could prove it directly.

**Example 16.10.** Let  $G = \text{SO}^+(1, 3)$  be the *Lorentz group*, i.e. the  $4 \times 4$  matrices that preserve a form of signature  $(1, 3)$  and have determinant 1; we further restrict to the identity component. This is a noncompact Lie group, and is not simply connected: as Lie groups,  $G \cong \text{PGL}_2(\mathbb{C})$ , and the universal cover of the latter is a double cover  $\text{SL}_2(\mathbb{C}) \twoheadrightarrow \text{PGL}_2(\mathbb{C})$ .<sup>22</sup> Let  $\tilde{G}$  be the connected double cover of  $G$ , which could also be written  $\text{Spin}(1, 3)$ .

Let  $\mathfrak{so}(1, 3)$  denote the Lie algebra of  $\text{SO}(1, 3)$  (as a Lie group), so as real Lie algebras,  $\mathfrak{so}(1, 3) \cong \mathfrak{sl}_2(\mathbb{C})$ , and therefore  $\mathfrak{so}(1, 3)_{\mathbb{C}} \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$ . By Lie's theorem, the category of finite-dimensional  $\tilde{G}$ -representations is equivalent to the category of finite-dimensional  $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$ -representations. This is a semisimple Lie algebra, so by Proposition 16.8 it admits a compact form  $\text{SU}(2) \times \text{SU}(2) = \text{Spin}(4)$ , a double cover of  $\text{SO}(4)$ .

The upshot is that the category of  $\tilde{G}$ -representations is semisimple (every finite-dimensional representation is completely reducible), even though  $\tilde{G}$  isn't compact. Thus, while we weren't able to prove it directly, the theory of Lie algebras is still useful here. Finally, if you want to understand representations of  $G$ , they're naturally a subcategory of  $\text{Rep}_{\tilde{G}}$  for which the kernel of the covering map acts trivially; the finite-dimensional representations of  $G$  are also completely reducible. ◀

Though the finite-dimensional representation theory of the Lorentz group looks like that of a compact group, but this is far from true for infinite-dimensional representations.

**Definition 16.11.** A Lie group  $G$  is called *reductive* (resp. *semisimple*) if its Lie algebra  $\mathfrak{g}$  is reductive (resp. semisimple).<sup>23</sup>

So finite-dimensional representations of semisimple, simply connected Lie groups are completely reducible. In fact, this also applies to “algebraically simply-connected” groups such as  $\text{SL}_2(\mathbb{R})$  (its covering spaces aren't central extensions as algebraic groups, which implies its étale fundamental group vanishes).

<sup>22</sup>Here in characteristic 0,  $\text{PGL}_2(\mathbb{C})$  and  $\text{PSL}_2(\mathbb{C})$  are the same.

<sup>23</sup>This is equivalent to asking  $\mathfrak{g}_{\mathbb{C}}$  to be reductive (resp. semisimple).

Lecture 17.

## Invariant bilinear forms on Lie algebras: 2/24/17

Today, we're going to talk about invariant bilinear forms on Lie algebras, a rich subject we haven't discussed yet.  $\mathfrak{g}$  will denote a Lie algebra over  $\mathbb{C}$ ; a lot of this will also work over  $\mathbb{R}$ .

**Definition 17.1.** Let  $B: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$  be a symmetric bilinear form.  $B$  is called *invariant* if for all  $x \in \mathfrak{g}$ ,  $B(\text{ad}_x(y), z) = -B(y, \text{ad}_x(z))$ , where  $\text{ad}$  is the action of  $\mathfrak{g}$  on itself by Lie bracket.

You can express this diagrammatically: let  $\mathfrak{o}(\mathfrak{g}, B)$  denote the Lie algebra of endomorphisms of  $\mathfrak{g}$  that preserve  $B$ , i.e.

$$\mathfrak{o}(\mathfrak{g}, B) := \{\alpha: \mathfrak{g} \rightarrow \mathfrak{g} \mid B(\alpha(-), -) = -B(-, \alpha(-))\}.$$

This is the Lie algebra of the Lie group

$$\text{O}(\mathfrak{g}, B) := \{A \in \text{GL}(\mathfrak{g}) \mid B(A(-), A(-)) = B(-, -)\}.$$

This condition looks more natural than the one for  $\mathfrak{o}(\mathfrak{g}, B)$ . For  $B$  to be invariant, we want the adjoint  $\text{ad}: \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$  to factor through  $\mathfrak{o}(\mathfrak{g}, B)$ :

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\text{ad}} & \mathfrak{gl}(\mathfrak{g}) \\ & \searrow & \uparrow \\ & & \mathfrak{o}(\mathfrak{g}, B) \end{array}$$

Equivalently,  $B$  is invariant iff the induced map  $\mathfrak{g} \rightarrow \mathfrak{g}^*$  sending  $x \mapsto B(x, -)$  is a morphism of  $\mathfrak{g}$ -representations, where  $\mathfrak{g}$  acts on itself through the adjoint action and  $\mathfrak{g}^*$  is the dual to that representation. In this context,  $\mathfrak{g}$  is called the *adjoint representation* and  $\mathfrak{g}^*$  is called the *coadjoint representation*.

**Definition 17.2.** Let  $B$  be an invariant bilinear form. Then,  $B$  is *nondegenerate* if the map  $\mathfrak{g} \rightarrow \mathfrak{g}^*$  sending  $x \mapsto B(x, -)$  is an isomorphism.

**Example 17.3.** Let  $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ . Then,  $B_{\text{std}}: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$  sending  $A, B \mapsto \text{tr}(AB)$  is a nondegenerate invariant symmetric bilinear form. ◀

This example really depends on  $\mathfrak{g}$  being  $\mathfrak{gl}(V)$ , or more generally, we can make the same construction for any Lie algebra  $\mathfrak{g}$  and a  $\mathfrak{g}$ -representation  $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ , and the representation is necessary.<sup>24</sup> Explicitly, we let  $B_V: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$  send

$$(x, y) \mapsto \text{tr}(\rho(x)\rho(y)).$$

This is always invariant and symmetric, but in general isn't nondegenerate.

In particular, we can take  $\rho$  to be the adjoint representation. In this case,  $B_{\mathfrak{g}}$  is written  $K$  and is called the *Killing form*.

**Example 17.4.** Let  $\mathfrak{g} = \mathfrak{sl}_2$  and  $V = \mathbb{C}^2$  be the standard representation. That is, we defined  $\mathfrak{sl}_2$  to be the Lie algebra of traceless,  $2 \times 2$  complex matrices, and we want it to act on  $V$  by precisely those matrices.

Recall that  $\mathfrak{sl}_2$  is generated by three matrices  $e, f$ , and  $h$  as in (12.7), so  $B_V$  can be represented by a  $3 \times 3$  matrix describing what happens to those generators. If we use the order  $(e, f, h)$ , the matrix is

$$B_V = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

This matrix is clearly diagonalizable, so  $B_V$  is nondegenerate.

Similarly, the Killing form can be described with a matrix. First, the relations (13.2) define the adjoint representation:

$$\text{ad}(e) = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{ad}(f) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{ad}(h) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then,  $K(x, y) = \text{tr}(\text{ad}(x), \text{ad}(y))$ , and you can check that  $K = 4B_V$ . ◀

<sup>24</sup>For  $\mathfrak{gl}_n(\mathbb{C})$ , we're implicitly using the defining representation.

*Remark.* That the Killing form is a multiple of  $B_V$  is no coincidence: it's possible to use Schur's lemma to show that for a simple Lie algebra  $\mathfrak{g}$ , there's a unique nondegenerate, invariant, skew-symmetric bilinear form on  $\mathfrak{g}$  up to scalar multiplication by an element of  $\mathbb{C}^\times$ . (This uses the fact that the adjoint representation is irreducible.) ◀

There's lots of things to say about this subject, and they're all really important. Here are some facts which we won't have time to prove.

**Theorem 17.5.**

- (1)  $\mathfrak{g}$  is reductive iff there exists a representation  $V$  for which  $B_V$  is nondegenerate.
- (2)  $\mathfrak{g}$  is semisimple iff the Killing form is nondegenerate.
- (3)  $\mathfrak{g}$  is solvable iff  $K(x, [y, z]) = 0$  for all  $x, y, z \in \mathfrak{g}$ .
- (4) Let  $\mathfrak{g}_{\mathbb{R}}$  be a real Lie algebra. Then,  $\mathfrak{g}_{\mathbb{R}}$  is the Lie algebra of a semisimple<sup>25</sup> compact group  $G$  iff the Killing form is negative definite.

For example, the Killing form for  $\mathfrak{gl}_n$  isn't nondegenerate, since scalar matrices lie in its kernel. This implies that  $\mathfrak{gl}_n$  is reductive but not semisimple.

The idea behind the proof of (3) is that a representation of a solvable Lie algebra can be chosen to be through upper triangular matrices; then, the bracket of two upper triangular matrices is strictly upper triangular, so causes the trace to vanish.

We can use these properties to understand what semisimple, reductive, and solvable Lie algebra actually look like. First let's organize the examples we have.

**Example 17.6** (Classical groups).

- $U(n)$ ,  $GL_n(\mathbb{R})$ , and  $GL_n(\mathbb{C})$  all have (complexified) Lie algebra  $\mathfrak{gl}_n$ . As we saw above, this Lie algebra is reductive but not semisimple.
- $SU(n)$ ,  $SL_n(\mathbb{R})$ , and  $SL_n(\mathbb{C})$  all have (complexified) Lie algebra  $\mathfrak{sl}_n$ , the Lie algebra of traceless matrices. This is reductive and semisimple, as we calculated, and in fact is an example of a *simple* Lie algebra.
- $SO(n)$ ,  $SO_{p,q}$ , and  $SO_n(\mathbb{C})$  all have (complexified) Lie algebra  $\mathfrak{so}_n$ , the Lie algebra of skew-symmetric matrices. ◀

We'll talk more about this, as well as the choice of order, next time.

Lecture 18.

## Classical Lie groups and Lie algebras: 2/27/17

We talked about how the classification of Lie algebras splits into the classification of solvable Lie algebras and semisimple Lie algebras, and we're going to discuss the classification of semisimple Lie algebras, or compact Lie groups.

Recall that we started with a compact Lie group, e.g.  $SU(2)$  or  $SO(3)$ , which is the thing we originally wanted to study. From this we obtained a reductive Lie algebra  $\mathfrak{g}$ , and it is a direct sum of an abelian part and a semisimple part  $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$ :  $\mathfrak{g} \cong Z(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$ . In particular,  $G' = [G, G]$  is a semisimple Lie group.

Thus, to understand the representation theory of compact Lie groups, it suffices to understand the representation theory of semisimple Lie algebras. In particular, given a semisimple complex Lie algebra  $\mathfrak{g}$ , one can recover a simply-connected Lie group  $G$  whose complexified Lie algebra is  $\mathfrak{g}$ , and its real Lie algebra is the unique real form of  $\mathfrak{g}$  such that the Killing form is negative definite.

**Definition 18.1.** A Lie algebra  $\mathfrak{g}$  is *simple* if it has no nontrivial ideals. By convention, the one-dimensional abelian Lie algebra is not simple.

Recall that  $\mathfrak{g}$  is semisimple if it has no solvable ideals; here we ask for no ideals at all.

**Proposition 18.2.** Any semisimple Lie algebra is a direct sum of simple Lie algebras, i.e. the bracket is defined componentwise.

<sup>25</sup>The semisimplicity hypothesis is necessary because Lie algebras cannot tell the difference between an  $n$ -dimensional torus and the additive group underlying an  $n$ -dimensional vector space.

*Proof idea.* By Corollary 16.9, if  $\mathfrak{g}$  is semisimple, its representations are completely reducible. In particular, the adjoint representation decomposes as  $\mathfrak{g} \cong \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ , and this is the decomposition of  $\mathfrak{g}$  into simple Lie algebras.  $\square$

So to understand semisimple Lie algebras, we only need to understand simple Lie algebras. Excellent. Where do simple Lie algebras come from? This is an interesting question because it has two quite distinct answers.

- From the perspective of classical groups, the Lie algebras and Lie groups we’re familiar with arise as symmetries of linear-algebraic data. For example, the unitary, orthogonal, and symplectic groups arise as the groups of endomorphisms preserving certain forms, and this is a good source of simple Lie algebras.
- The *Cartan-Killing classification* adopts the perspective that simple Lie algebras are built out of copies of  $\mathfrak{sl}_2(\mathbb{C})$  in particular combinatorial configurations.

*A priori*, these things have nothing to do with each other. Moreover, while the first perspective seems more natural, the second one is hugely important in modern physics. It will also take a little longer to explain, but it will make the whole classification make a lot more sense.

Let’s first talk about the perspective coming from classical groups.

- The unitary group  $U(n)$  is the group of  $n \times n$  complex matrices preserving the standard Hermitian form on  $\mathbb{C}^n$ . Its complexified Lie algebra is  $\mathfrak{u}_n \otimes \mathbb{C} \cong \mathfrak{gl}_n(\mathbb{C})$ , which it shares with  $GL_n(\mathbb{R})$ . This is reductive, but not semisimple, since  $Z(U(n))$  is one-dimensional. Quotienting out by this, one obtains the special unitary group  $SU(n)$ , whose complexified Lie algebra is  $\mathfrak{sl}_n$ , which is simple when  $n \geq 2$ .  $SL_2(\mathbb{R})$  has the same complexified Lie algebra.
- The special orthogonal groups are the groups preserving positive definite forms on a real vector space.  $SO(2n)$  and  $SO_{n,n}(\mathbb{R})$  have the same complexified Lie algebra  $\mathfrak{so}_{2n}$ , and  $SO(2n+1)$  and  $SO_{n,n+1}(\mathbb{R})$  have the same complexified Lie algebra  $\mathfrak{so}_{2n+1}$ ; in both cases, the former group is compact and the latter isn’t. If  $n \geq 3$ ,  $\mathfrak{so}_{2n}$  is simple, and if  $n \geq 1$ ,  $\mathfrak{so}_{2n+1}$  is simple. However,  $\mathfrak{so}_4 \cong \mathfrak{su}_2 \oplus \mathfrak{su}_2$ .
- There’s a notion of a positive definite standard Hermitian form on the free  $\mathbb{H}$ -module  $\mathbb{H}^n$ , and the *symplectic group*  $Sp(n)$  is the group of  $n \times n$  quaternionic matrices preserving this form. Confusingly, the most common convention is for its Lie algebra to be denoted  $\mathfrak{sp}_{2n}$ ; this is the algebra of *Hamiltonian matrices*, and is always simple. There’s a noncompact Lie group  $Sp_{2n}(\mathbb{R})$  with the same Lie algebra, which is the group of  $2n \times 2n$  real matrices which preserve a symplectic form (a skew-symmetric bilinear form) on  $\mathbb{R}^{2n}$ .

So we have a table of correspondences between compact groups, noncompact groups with the same Lie algebra (called “split”), and their shared complexified Lie algebra. This is all organized in Table 2, which also lists the type of each Lie algebra in the Cartan-Killing classification of simple Lie algebras.

Compact	Split	Lie algebra	Dimension	Type
$U(n)$	$GL_n(\mathbb{R})$	$\mathfrak{gl}_n(\mathbb{C})$	$n^2$	n/a
$SU(n)$	$SL_n(\mathbb{R})$	$\mathfrak{sl}_n$	$n^2 - 1$	$A_{n-1}$
$SO(2n)$	$SO_{n,n}(\mathbb{R})$	$\mathfrak{so}_{2n}$	$n(2n + 1)$	$D_n$
$SO(2n + 1)$	$SO_{n,n+1}(\mathbb{R})$	$\mathfrak{so}_{2n+1}$	$n(2n - 1)$	$B_n$
$Sp(n)$	$Sp_{2n}(\mathbb{R})$	$\mathfrak{sp}_{2n}$	$n(2n + 1)$	$C_n$

TABLE 2. A table of classical Lie groups and Lie algebras. Since  $\mathfrak{gl}_n(\mathbb{C})$  isn’t simple, it doesn’t have a Cartan-Killing type.

It seems like you should be able to turn this into a nice story where you start with  $\mathbb{R}$ ,  $\mathbb{C}$ , and  $\mathbb{H}$  and derive this table, but there’s some inevitable weirdness that creeps in. In particular, this list contains *almost* all of the simple Lie algebras — exactly five are missing! They are called  $\mathfrak{e}_6$ ,  $\mathfrak{e}_7$ ,  $\mathfrak{e}_8$ ,  $\mathfrak{f}_4$ , and  $\mathfrak{g}_2$ . Each one of these is the Lie algebra of a Lie group, so we obtain Lie groups  $E_6$ ,  $E_7$ ,  $E_8$ ,  $F_4$ , and  $G_2$ .

You can realize these geometrically from an 8-dimensional nonassociative division algebra called the *octonions*  $\mathbb{O}$ , which John Baez has written about. The octonions are constructed from the quaternions in the same way that the quaternions are constructed from  $\mathbb{C}$ , but iterating this construction further produces

more pathological objects. In any case, the exceptional Lie algebras can be obtained by looking at projective planes associated to the octonions as Riemannian symmetric spaces. Specifically,

- $\mathfrak{e}_6$  is a 78-dimensional Lie algebra arising from  $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{O}$ .
- $\mathfrak{e}_7$  is a 133-dimensional Lie algebra arising from  $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{O}$ .
- $\mathfrak{e}_8$  is a 248-dimensional Lie algebra arising from  $\mathbb{O} \otimes_{\mathbb{R}} \mathbb{O}$ .
- $\mathfrak{f}_4$  is a 52-dimensional Lie algebra arising from  $\mathbb{O}^2$ .
- $\mathfrak{g}_2$  is a 14-dimensional Lie algebra arising from  $\mathbb{O}$ , e.g.  $G_2 = \text{Aut}(\mathbb{O})$ .

These seem to come out of nowhere, but make more sense from the Cartan-Killing perspective.

*Remark.* The Lie algebras outlined in Table 2 determine their compact forms up to covering.  $\text{SU}(n)$  and  $\text{Sp}(n)$  are simply connected, but  $\pi_1(\text{SO}(m)) \cong \mathbb{Z}/2$  when  $m \geq 3$ , so there exists a unique Lie group  $\text{Spin}(m)$  that double covers  $\text{SO}(m)$  and the covering map  $\text{Spin}(m) \rightarrow \text{SO}(m)$  is a group homomorphism. ◀

*Remark.* There are some redundancies in Table 2 in low dimension.

- $\mathfrak{so}_3 \cong \mathfrak{sl}_2 \cong \mathfrak{sp}_2$ , i.e.  $A_1 = B_1 = C_1$ .
- $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \times \mathfrak{sl}_2$ , as we've seen, and therefore  $D_2 = A_1 \times A_1$ .
- $\mathfrak{so}_5 \cong \mathfrak{sp}_4$ , i.e.  $B_2 = C_2$ .
- $\mathfrak{so}_6 \cong \mathfrak{sl}_4$ , i.e.  $A_3 = D_3$ .

These last two are called *exceptional isomorphisms*. ◀

Next time, we'll begin explaining Dynkin diagrams and the Cartan-Killing perspective.

Lecture 19.

## Roots and root spaces: 3/1/17

Last time, we discussed how to obtain some simple Lie algebras from classical Lie groups  $\text{O}(n)$ ,  $\text{U}(n)$ , and  $\text{Sp}(n)$ . Today, we'll take adopt the Cartan-Killing approach of building semisimple Lie algebras from copies of  $\mathfrak{sl}_2(\mathbb{C})$ .

Throughout today's lecture,  $\mathfrak{g}$  denotes a semisimple Lie algebra over  $\mathbb{C}$ .

**Definition 19.1.** An element  $x \in \mathfrak{g}$  is called *semisimple* if  $\text{ad}(x) \in \text{End}_{\mathbb{C}}(\mathfrak{g})$  is semisimple, and is called *nilpotent* if  $\text{ad}(x)$  is nilpotent.

Later we'll see that we could choose a different representation than the adjoint.

The Jordan decomposition for matrices says that every matrix is the sum of a semisimple part and a nilpotent part that commute in a unique way. There's a similar result for semisimple Lie algebras.

**Proposition 19.2** (Jordan decomposition). *Let  $x \in \mathfrak{g}$ . Then, there exist  $x_{\text{ss}}, x_{\text{n}} \in \mathfrak{g}$  such that  $x = x_{\text{ss}} + x_{\text{n}}$ ,  $x_{\text{ss}}$  is semisimple,  $x_{\text{n}}$  is nilpotent, and  $[x_{\text{ss}}, x_{\text{n}}] = 0$ , and they are unique.*

*Proof idea.* In the adjoint representation,  $\text{ad}(x) = \text{ad}(x)_{\text{ss}} + \text{ad}(x)_{\text{n}}$ , but why do these pieces in  $\text{End}_{\mathbb{C}}(\mathfrak{g})$  come from the action of elements of  $\mathfrak{g}$ ? This is where semisimplicity of  $\mathfrak{g}$  is used in an essential way. ☐

*Remark.* The set of elements of  $\mathfrak{g}$  that are either semisimple or nilpotent are not a vector subspace, but are dense. ◀

If every element of  $\mathfrak{g}$  is nilpotent, then Theorem 15.9 would imply  $\mathfrak{g}$  is nilpotent, which is a contradiction. Thus, there exists at least one nonzero semisimple element of  $\mathfrak{g}$ . This will allow us to decompose  $\mathfrak{g}$  into eigenspaces for the semisimple elements. In fact, if we have a commuting family of semisimple elements, we can decompose  $\mathfrak{g}$  into their simultaneous eigenspaces.

**Definition 19.3.** Let  $\mathfrak{h} \subset \mathfrak{g}$  be a Lie subalgebra. Then,  $\mathfrak{h}$  is called *toral* if it's abelian and consists solely of semisimple elements of  $\mathfrak{g}$ .

Any semisimple element generates a 1-dimensional abelian Lie subalgebra, which is in particular toral. So toral subalgebras exist.

**Definition 19.4.** A *Cartan subalgebra* of a semisimple<sup>26</sup> Lie algebra is a maximal toral subalgebra.

<sup>26</sup>There's a notion of a Cartan subalgebra of any Lie algebra, but it's slightly different and not so useful for us.

Let  $C_{\mathfrak{g}}(\mathfrak{h}) := \{x \in \mathfrak{g} \mid [x, y] = 0 \text{ for all } y \in \mathfrak{h}\}$  denote the centralizer.

**Proposition 19.5.**  $\mathfrak{h} \subset \mathfrak{g}$  is a Cartan subalgebra iff it's a toral subalgebra and  $C_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$ .

Though we won't give a proof, the idea is reasonable: if we could add something to  $\mathfrak{h}$ , then whatever we added must centralize  $\mathfrak{h}$ .

**Example 19.6.** If  $\mathfrak{g} = \mathfrak{sl}_n$ , the Lie algebra of  $n \times n$  traceless matrices, then the subalgebra of diagonal traceless matrices is a Cartan subalgebra.  $\blacktriangleleft$

Let  $\mathfrak{h}$  be a Cartan subalgebra of  $\mathfrak{g}$ . Then, the restriction of the adjoint representation to  $\mathfrak{h}$  acting on  $\mathfrak{g}$  is a family of semisimple operators acting on  $\mathfrak{g}$ , which can in particular be simultaneously diagonalized. Thus, as  $\mathfrak{h}$ -representations,

$$\mathfrak{g} = \bigoplus_{\alpha} \mathfrak{g}_{\alpha}$$

for some finite set of  $\alpha \in \mathfrak{h}^*$ , and where  $h \in \mathfrak{h}$  acts on  $\mathfrak{g}_{\alpha}$  by  $\text{ad}(h) \cdot v = \alpha(h) \cdot v$  for all  $v \in \mathfrak{g}_{\alpha}$ . It's for exactly this reason that we consider Cartan subalgebras.

**Definition 19.7.** Let  $R = \{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_{\alpha} \neq 0\} \setminus \{0\}$ . The elements of  $R$  are called *roots* of  $\mathfrak{g}$ , and  $\mathfrak{g}_{\alpha}$  is called the *root space* for  $\alpha$ .

The idea behind the Cartan-Killing classification is that the data  $(\mathfrak{h}^*, K|_{\mathfrak{h}^*}, R)$ , i.e. a finite-dimensional vector space, a finite subset of it, and a bilinear form on it, is enough to reconstruct  $\mathfrak{g}$  up to isomorphism!

**Proposition 19.8.**

- (1)  $[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}] \subseteq \mathfrak{g}_{\alpha+\beta}$ .
- (2) With respect to any invariant inner product  $\langle -, - \rangle$ ,  $\mathfrak{g}_{\alpha} \perp \mathfrak{g}_{\beta}$  unless  $\alpha + \beta = 0$ , and  $\langle -, - \rangle : \mathfrak{g}_{-\alpha} \times \mathfrak{g}_{\alpha} \rightarrow \mathbb{C}$  is a perfect pairing.

**Example 19.9.** For example, let  $E_{ij}$  denote the  $n \times n$  matrix with a 1 in the  $ij^{\text{th}}$  entry and 0s everywhere else, so  $E_{ij} \in \mathfrak{sl}_n$ . Let  $\mathfrak{h}$  be the subalgebra of diagonal matrices whose entries sum to zero, which are a Cartan subalgebra for  $\mathfrak{sl}_n$ , which in particular is a subspace of  $\text{span}\{E_{ii}\}$ . In particular, if  $e_i \in \mathfrak{h}^*$  is dual to  $E_{ii}$ , then

$$\mathfrak{h}^* = \bigoplus_i \mathbb{C} \cdot e_i / \mathbb{C}(e_1 + \cdots + e_n).$$

Let  $h \in \mathfrak{h}$  be the matrix with diagonal entries  $h_1, \dots, h_n$ . Then,

$$[h, E_{ij}] = (h_i - h_j)E_{ij} = (e_i - e_j)(h)E_{ij}.$$

Thus, the roots are

$$R = \{e_i - e_j \mid i, j = 1, \dots, n, i \neq j\},$$

$$\text{anmd } \mathfrak{sl}_n(\mathbb{C})_{e_i - e_j} = \mathbb{C} \cdot E_{ij}.$$

Now we need to figure out how  $\mathfrak{sl}_2$  appears.

*Remark.* If  $\langle -, - \rangle$  is an invariant inner product on  $\mathfrak{g}$ , then its restriction to  $\mathfrak{h} \times \mathfrak{h}$  is nondegenerate, and in particular defines an isomorphism  $\mathfrak{h} \xrightarrow{\cong} \mathfrak{h}^*$ . Thus, given an  $\alpha \in R$ , let  $H_{\alpha} = \langle \alpha, - \rangle \in (\mathfrak{h}^*)^* = \mathfrak{h}$  be its dual.  $\blacktriangleleft$

$$\text{Let } h_{\alpha} = 2H_{\alpha} / \langle \alpha, \alpha \rangle \in \mathfrak{h}.$$

**Lemma 19.10.** If  $e \in \mathfrak{g}_{\alpha}$  and  $f \in \mathfrak{g}_{-\alpha}$ , then  $[e, f] = \langle e, f \rangle H_{\alpha}$ .

Thus, rescaling  $e$  and  $f$  as necessary, one can assume  $\langle e, f \rangle = 2 / \langle \alpha, \alpha \rangle$ , which in particular implies the following.

**Proposition 19.11.** The elements  $e$ ,  $f$ , and  $h_{\alpha}$  above generate a subalgebra of  $\mathfrak{g}$  isomorphic to  $\mathfrak{sl}_2(\mathbb{C})$ .

That is,  $e$ ,  $f$ , and  $h_{\alpha}$  satisfy the commutation relations we established for the generators  $e$ ,  $f$ , and  $h$  of  $\mathfrak{sl}_2(\mathbb{C})$  in (13.2).

We'll call the subalgebra generated by  $e$ ,  $f$ , and  $h_{\alpha}$   $\mathfrak{sl}_2(\mathbb{C})_{\alpha}$ . This will be really useful for us, allowing us to understand the representation theory of  $\mathfrak{g}$  in terms of that of  $\mathfrak{sl}_2$ .



Lecture 20.

**Properties of roots: 3/3/17**

*“If you take away one thing from this course, it should be the representation theory of  $\mathfrak{sl}_2$ .”*

Once again,  $\mathfrak{g}$  will denote a semisimple complex Lie algebra, and  $(-, -)$  be an invariant inner product on  $\mathfrak{g}$ .

Last time, we saw that given a Cartan subalgebra  $\mathfrak{h}$ , the roots of  $\mathfrak{g}$  are the set  $R \subset \mathfrak{h}^*$  of  $\alpha$  such that  $\mathfrak{g}_\alpha \neq 0$  (the eigenvalues in the adjoint representation). We saw that given  $\alpha$ , we can define  $e \in \mathfrak{g}_\alpha$ ,  $f \in \mathfrak{g}_{-\alpha}$ , and  $h_\alpha \in \mathfrak{h}$  that satisfy the commutation relations of  $\mathfrak{sl}_2(\mathbb{C})$ , i.e.  $[e, f] = h_\alpha$ ,  $[h_\alpha, e] = 2e$ , and  $[h_\alpha, f] = -2f$ . Thus they define an embedding  $\mathfrak{sl}_2(\mathbb{C}) \hookrightarrow \mathfrak{g}$ , whose image we called  $\mathfrak{sl}_2(\mathbb{C})_\alpha$ .

**Proposition 20.1.**

- (1) The roots span  $\mathfrak{h}^*$  as a vector space.
- (2) Each  $\mathfrak{g}_\alpha$  is one-dimensional.
- (3) If  $\alpha, \beta \in R$ ,  $n_{\alpha\beta} := 2(\alpha, \beta)/(\alpha, \alpha) \in \mathbb{Z}$ .
- (4) Let  $s_\alpha: \mathfrak{h}^* \rightarrow \mathfrak{h}^*$  be defined by

$$s_\alpha(\lambda) := \lambda - \frac{2(\alpha, \lambda)}{(\alpha, \alpha)}\alpha.$$

Then,  $s_\alpha(R) = R$ .

- (5) If  $\alpha \in R$  and  $c$  is a scalar such that  $c\alpha \in R$ , then  $c = \pm 1$ .
- (6) If  $\alpha, \beta$ , and  $\alpha + \beta$  are in  $R$ , then  $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ .

*Proof.* For (1), suppose  $h \in \{k \in \mathfrak{h} \mid \alpha(k) = 0 \text{ for all } \alpha \in R\}$ . Then,  $\text{ad}(h) = 0$ , so  $h \in Z(\mathfrak{g}) = 0$  since  $\mathfrak{g}$  is semisimple.

For (2), consider  $\mathfrak{g}$  as an  $\mathfrak{sl}_2(\mathbb{C})_\alpha$ -representation. You can check that

$$\cdots \oplus \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_\alpha \oplus \mathbb{C} \cdot h_\alpha \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha} \oplus \cdots$$

is a subrepresentation for  $\mathfrak{sl}_2(\mathbb{C})_\alpha$ , and is a decomposition into weight spaces, where  $\mathfrak{g}_{j\alpha}$  has weight  $2j$ . In particular, the zero-weight space is 1-dimensional and there are no odd weights, so  $V$  is an irreducible  $\mathfrak{sl}_2(\mathbb{C})$ -representation, and therefore all of the other weight spaces are 1-dimensional, in particular  $\mathfrak{g}_\alpha$ . Moreover, this implies  $\mathfrak{g}_\alpha = \mathbb{C} \cdot e$ , and therefore  $\text{ad}(e)(\mathfrak{g}_\alpha) = 0$ , so  $\mathfrak{g}_{2\alpha} = 0$ ,  $\mathfrak{g}_{3\alpha} = 0$ , and so forth. The same is true for  $\mathfrak{g}_{-\alpha}$ , which proves (5):  $\mathfrak{sl}_2(\mathbb{C})_\alpha = \mathfrak{g}_\alpha \cdot \mathbb{C} \cdot h_\alpha \oplus \mathfrak{g}_\alpha$ .

For (3), consider the action of  $h_\alpha \in \mathfrak{sl}_2(\mathbb{C})_\alpha$  on  $\mathfrak{g}_\beta$ ; by definition,  $h_\alpha$  acts by the weight  $\beta(h_\alpha)$ , and the only weights that can occur are integers, so  $\beta(h_\alpha) \in \mathbb{Z}$  as required. What's cool about this is that it provides a very explicit restriction on what the angles between roots can be.

For (4), consider again the action of  $\mathfrak{sl}_2(\mathbb{C})_\alpha$  on  $\mathfrak{g}$ . Then,  $h_\alpha$  acts with weight  $n_{\alpha\beta}$  on  $\mathfrak{g}_\beta$  if  $\beta \neq 0$ , and  $\text{ad}(f_\alpha)^{n_{\alpha\beta}}$  carries  $\mathfrak{g}_\beta$  to  $\mathfrak{g}_{\beta-n_{\alpha\beta}\alpha}$ , and this is an isomorphism. This “hard Lefschetz formula” occurs because this is a reflection across the origin in  $\mathfrak{g}$ , and so carries root spaces to root spaces.  $\square$

The point is that the representation theory of a semisimple Lie algebra is governed by some pretty restrictive data, because  $\mathfrak{sl}_2$  controls it. There's also some geometry hiding in the background.

Let  $\mathfrak{h}_\mathbb{R}$  denote the  $\mathbb{R}$ -span of  $h_\alpha$  for a root  $\alpha \in R$ . We now know that the adjoint action  $\text{ad}(h_\alpha)$  acts with integer eigenvalues, then  $K(h_\alpha, h_\beta) = \text{tr}(\text{ad}(h_\alpha) \text{ad}(h_\beta)) \in \mathbb{R}$  and  $K(h_\alpha, h_\alpha) = \text{tr}(\text{ad}(h_\alpha)^2) > 0$ . Thus,  $(\mathfrak{h}_\mathbb{R}, K)$  is a real inner product space, and so  $\mathfrak{h}_\mathbb{R}^*$  is spanned by  $R$  and equipped with a positive definite inner product.

**Example 20.2.** When  $\mathfrak{g} = \mathfrak{sl}_3$ ,

$$\mathfrak{h}_\mathbb{R}^* = (\mathbb{R} \cdot e_1 \oplus \mathbb{R} \cdot e_2 \oplus \mathbb{R} \cdot e_3) / \mathbb{R} \cdot (e_1 + e_2 + e_3)$$

$$\mathfrak{h}_\mathbb{R} = \{(h_1, h_2, h_3) \in \mathbb{R}^3 \mid h_1 + h_2 + h_3 = 0\}.$$

We can normalize the inner product such that  $(e_i, e_j) = \delta_{ij}$ . The roots are

$$R = \{\alpha_1 := e_1 - e_2, \alpha_2 := e_2 - e_3, \alpha_3 := e_1 - e_3, -\alpha_1, -\alpha_2, -\alpha_3\},$$

and the Killing form is  $(\alpha_1, \alpha_2) = -1$  and  $(\alpha_i, \alpha_i) = 2$ . Thus the roots have an angle of  $60^\circ$  from each other, and this data knows everything there is to know about  $\mathfrak{sl}_3$ , which we'll see next week.  $\blacktriangleleft$



Lecture 21.

## Root systems: 3/6/17

Today, we'll talk about root systems in the abstract, which will allow us to classify them for semisimple Lie algebras. Today,  $E$  will always denote a Euclidean vector space, i.e. a finite-dimensional real vector space together with an inner product  $\langle -, - \rangle$ .

We care about root systems because if  $\mathfrak{g}$  is a semisimple complex Lie algebra and  $\mathfrak{h} \subseteq \mathfrak{g}$  is a Cartan subalgebra, then  $E = \mathfrak{h}_{\mathbb{R}}^*$  with inner product  $K(-, -)$  and  $R = \{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_{\alpha} \neq 0\} \setminus \{0\}$  will be our model example for a root system.

**Definition 21.1.** A *root system* is a pair  $(E, R)$  where  $R \subseteq E$  satisfies the following axioms:

- (1)  $R$  spans  $E$  as a vector space.
- (2) For all  $\alpha, \beta \in R$ ,

$$n_{\alpha\beta} := \frac{2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \in \mathbb{Z}.$$

- (3) For all  $\alpha \in R$ , the reflection

$$s_{\alpha}(\lambda) := \lambda - \frac{2\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle} \alpha$$

maps  $R$  into itself.

If in addition whenever  $\alpha \in R$  and  $c\alpha \in R$  for some  $c \in \mathbb{R}$ , then  $c = \pm 1$ , then  $(E, R)$  is called a *reduced root system*.

In Proposition 20.1, we showed that if  $\mathfrak{g}$  is a semisimple Lie algebra, its roots  $R \subset \mathfrak{h}_{\mathbb{R}}^*$  form a reduced root system.

Even in the abstract, root systems come with quite a bit of additional data.

**Definition 21.2.** The *Weyl group*  $W_{(E, R)}$  of a root system  $(E, R)$  is the subgroup of  $O(E)$  generated by the reflections  $s_{\alpha}$  for  $\alpha \in R$ . If  $(E, R)$  is clear from context  $W_{(E, R)}$  will be denoted  $W$ .

Since each  $s_{\alpha}$  restricts to a permutation of  $R$ , you can also think of  $W$  as a subgroup of the symmetric group  $S_R$ .

*Remark.* Suppose a complex semisimple  $\mathfrak{g}$  is the Lie algebra of a compact, semisimple Lie group  $G$ . With  $(\mathfrak{h}_{\mathbb{R}}^*, R)$  as before, the Weyl group  $W_{(\mathfrak{h}_{\mathbb{R}}^*, R)} \cong N_G(H)/H$ , where  $H$  is a maximal torus of  $G$ . This equivalence is far from obvious, but sometimes is used as an alternate definition for the Weyl group. ◀

**Example 21.3.** Let  $E = \mathbb{R}^n / \mathbb{R}(1, 1, \dots, 1)$  with the inner product inherited from the usual one on  $\mathbb{R}^n$ . Then,  $R = \{e_i - e_j \mid i \neq j\} \subseteq E$  is a reduced root system, and in fact is the root system associated to  $\mathfrak{sl}_n$ , denoted  $A_n$ . ▶

Other classical examples include  $\mathfrak{so}_{2n+1}$ , which produces a root system  $B_n$ ;  $\mathfrak{sp}_{2n}$ , which gives  $C_n$ ; and  $\mathfrak{so}_{2n}$ , which gives  $D_n$ .

The number  $n_{\alpha\beta}$  prescribes the ratio of the length of projection of  $\alpha$  onto the direction of  $\beta$  to the length of  $\beta$ :  $n_{\alpha\beta}$  is twice this ratio. In fact, comparing  $n_{\alpha\beta}$  and  $n_{\beta\alpha}$  severely restricts the possible angles.

- Suppose  $n_{\alpha\beta} = 0$  and  $n_{\beta\alpha} = 0$ . Then,  $\alpha$  and  $\beta$  are orthogonal to each other.
- Suppose  $n_{\alpha\beta} = n_{\beta\alpha} = 1$ . Then,  $|\alpha| = |\beta|$ , and the angle between them is  $\pi/3$ .
- Suppose  $n_{\alpha\beta} = 2$  and  $n_{\beta\alpha} = 1$ . Then,  $|\alpha| = \sqrt{2}|\beta|$  and the angle from  $\beta$  to  $\alpha$  is  $\pi/4$ .
- Suppose  $n_{\alpha\beta} = 3$  and  $n_{\beta\alpha} = 1$ . Then,  $|\alpha| = \sqrt{3}|\beta|$  and the angle from  $\beta$  to  $\alpha$  is  $\pi/6$ .

All of these come from special right triangles.

There are more possibilities where  $n_{\alpha\beta}, n_{\beta\alpha} < 0$ :

- You can have  $n_{\alpha\beta} = n_{\beta\alpha} = -1$ , for which  $|\alpha| = |\beta|$  and the angle between them is  $2\pi/3$ .
- If  $n_{\alpha\beta} = -2$  and  $n_{\beta\alpha} = -1$ , then  $|\alpha| = \sqrt{2}|\beta|$  and the angle between them is  $3\pi/4$ .
- If  $n_{\alpha\beta} = -3$  and  $n_{\beta\alpha} = -1$ , then  $|\alpha| = \sqrt{3}|\beta|$  and the angle between them is  $5\pi/6$ .

You can check these (and  $\pi$ ) are the only possible angles between two roots!

However, it could be the case that not all possible angles are realized by actual root systems. However, a few examples will realize every case.

- $A_2$  realizes  $\pi/3$  and  $2\pi/3$ . See Figure 1 for a picture.
- $B_2 = C_2$  realizes  $\pi/4$ ,  $\pi/2$ , and  $3\pi/4$ . See Figure 2 for a picture.
- The remaining angles are realized by  $G_2$ . See Figure 4 for a picture.

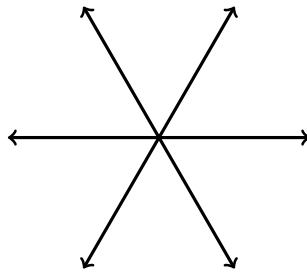


FIGURE 1. The rank-2 root system  $A_2$  realizes the angles  $\pi/3$  and  $2\pi/3$ .

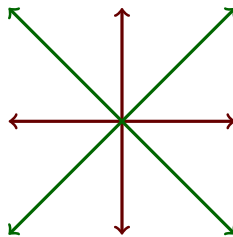


FIGURE 2. The rank-2 root systems  $B_2$  and  $C_2$  are isomorphic, and contain long (green) and short (red) roots.

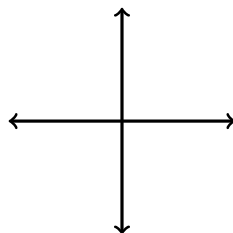


FIGURE 3. The rank-2 root system  $D_2$  splits as a product  $A_1 \times A_1$ .

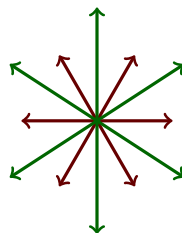


FIGURE 4. The rank-2 root system  $G_2$ , which corresponds to the exceptional Lie group  $G_2 = \text{Aut}(\mathbb{O})$ .

**Theorem 21.4.** *The root systems  $A_2$ ,  $B_2$ ,  $D_2$ , and  $G_2$  depicted in Figures 1 to 4 are a complete classification of rank-2 root systems.*

In all of these examples, it looks like one can start with two vectors and generate the rest with reflections. Let's make this precise.

**Definition 21.5.** Let  $t \in E$  be an element perpendicular to no element of  $R$ , which is a generic condition. Such a  $t$  determines a *polarization* of  $R$  as a disjoint union of  $R_+ = \{\alpha \in R \mid \langle \alpha, t \rangle > 0\}$  and  $R_- = \{\alpha \in R \mid \langle \alpha, t \rangle < 0\}$ .

Different values of  $t$  may define the same polarization, and in general there's no canonical polarization around.

**Definition 21.6.** Given a polarization  $R = R_+ \amalg R_-$ , the set  $\Pi$  of *simple roots* is the subset of  $\alpha \in R_+$  such that  $\alpha \neq \beta + \gamma$  for  $\beta, \gamma \in R_+$ .

**Proposition 21.7.** *The simple roots form a basis for  $E$ .*

Therefore in particular any  $\alpha \in R$  may be written as

$$\alpha = \sum_{\alpha_i \in \Pi} n_i \alpha_i, \quad n_i \in \mathbb{Z},$$

and if  $\alpha \in R_+$ , then  $n_i > 0$ . Let the *Cartan matrix*  $A = (a_{ij})$  be the  $r \times r$  matrix (where  $r = |\Pi| = \dim E$ ) with  $a_{ij} = n_{\alpha_j \alpha_i}$ , where  $\Pi = \{\alpha_1, \dots, \alpha_r\}$ . Next time, we'll use this to define a certain graph called the Dynkin diagram associated to  $(E, R)$ .

Lecture 22.

## Dynkin diagrams: 3/8/17

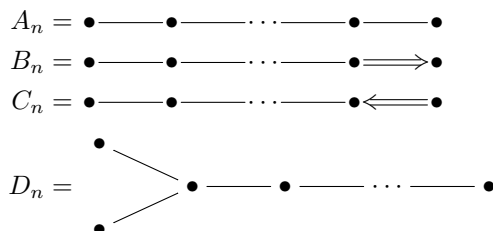
Today, we'll continue along the path that will lead us to Dynkin diagrams.

Recall that if  $E$  is a Euclidean space and  $R \subset E$  is a root system, a generic vector in  $E$  determines a splitting of  $R = R_+ \amalg R_-$ , where  $R_+$  is called the set of positive roots, and the simple roots  $\Pi \subseteq R_+$ . All of this came from the data of a compact semisimple Lie group, once upon a time.

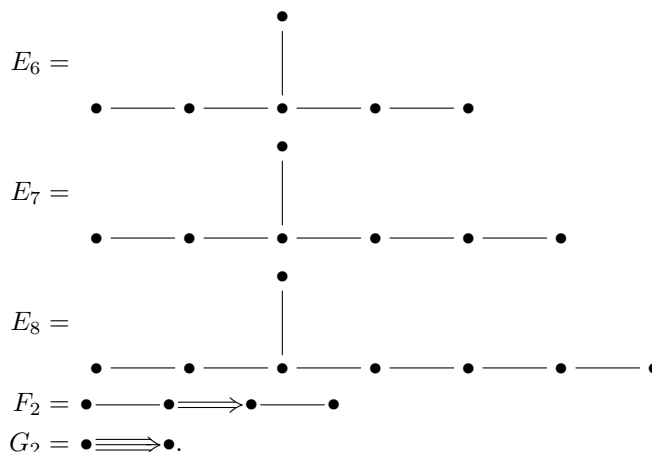
Suppose  $\Pi = \{\alpha_1, \dots, \alpha_r\}$ . Then, the root system is determined by the Cartan matrix  $A = (a_{ij})$ , where  $a_{ij} = n_{\alpha_j \alpha_i} = 2\langle \alpha_i, \alpha_j \rangle / \langle \alpha_i, \alpha_i \rangle$ . It's not necessarily symmetric, but notice that  $a_{ij} = 0$  iff  $a_{ji} = 0$ . We can encode this data in a graph called the *Dynkin diagram*, built according to the following recipe.

- (1) There will be a node  $v_i$  for each  $\alpha_i \in \Pi$ .
- (2) If  $a_{ij} = 0$  (equivalently  $a_{ji} = 0$ ), there is no edge from  $i$  to  $j$ .
- (3) If  $a_{ij} = -1$  and  $a_{ji} = -1$ , there's an edge  $\bullet - \bullet$ .
- (4) If  $a_{ij} = -1$  and  $a_{ji} = -2$ , there's an edge  $\bullet \Rightarrow \bullet$ .
- (5) If  $a_{ij} = -1$  and  $a_{ji} = -3$ , there's an edge  $\bullet \Rrightarrow \bullet$ .

It's possible to classify the irreducible root systems in terms of their Dynkin diagrams, which leads to a somewhat bizarre-looking classification: the four infinite families



and five exceptional diagrams



Though this classification looks strange, it makes the classification elementary, and could be done by an interested high-school student, though it's a bit rote.

In any case, given a root system, we'd like to build a Lie algebra. If  $\mathfrak{g}$  is a simple Lie algebra, then it splits

$$\mathfrak{g} = \bigoplus_{\alpha \in R_-} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha,$$

and the embedded copies of  $\mathfrak{sl}_2(\mathbb{C})_\alpha = \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \oplus \mathfrak{g}_\alpha$ , where  $f_\alpha \in \mathfrak{g}_{-\alpha}$ ,  $h_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_\beta]$ , and  $e_\alpha \in \mathfrak{g}_\alpha$ , with  $\langle e_\alpha, f_\alpha \rangle = 2/\langle \alpha, \alpha \rangle$ .

Let  $e_i := e_{\alpha_i}$ , and similarly for  $f_i$  and  $h_i$ . Then, these elements satisfy the *Serre relations*

$$\begin{aligned} [h_i, h_j] &= 0 & [e_i, f_j] &= \delta_{ij} h_i \\ [h_i, e_j] &= a_{ij} e_j & [h_i, f_j] &= -a_{ij} f_j \\ (\text{ad } e_i)^{1-a_{ij}} e_j &= 0 & (\text{ad } f_i)^{1-a_{ij}} f_j &= 0. \end{aligned}$$

This suffices to recover  $\mathfrak{g}$  from its root system.

**Theorem 22.1.** *Let  $R$  be a root system. Then, the free Lie algebra  $\mathfrak{g}(R)$  generated by  $e_i, f_i, h_i, i = 1, \dots, r$  satisfying the Serre relations is a finite-dimensional, semisimple Lie algebra with root system  $R$ .*

The hardest part of this theorem is showing that  $\mathfrak{g}(R)$  is finite-dimensional.

**Example 22.2.** Let's look at the rank-2 examples, which illustrate that Theorem 22.1 is nontrivial. We have  $e_1, e_2, f_1, f_2, h_1$ , and  $h_2$ , and we know

$$\begin{aligned} \text{ad}(e_2)^{1-a_{21}} e_1 &= 0 \\ \text{ad}(e_1)^{1-a_{12}} e_2 &= 0. \end{aligned}$$

But there's still quite a bit to pin down.

- For  $A_2$ , we have  $[e_2, [e_2, e_1]] = 0$  and  $[e_1, [e_1, e_2]] = 0$ .
- For  $B_2 = C_2$ ,  $\text{ad}(e_2)^3 e_1 = 0$  and  $\text{ad}(e_1)^2 e_2 = 0$ .
- For  $D_2 = A_1 \times A_1$ ,  $[e_1, e_2] = 0$ .
- For  $G_2$ ,  $\text{ad}(e_2)^4 e_1 = 0$  and  $\text{ad}(e_1)^2 e_2 = 0$ .

There are plenty of other things you could choose for  $a_{ij}$ , but only these ones work: if you asked  $\text{ad}(e_2)^5 e_1 = 0$  and  $\text{ad}(e_1)^2 e_2 = 0$ , you get something infinite-dimensional.  $\blacktriangleleft$

*Remark.* There's a notion of a *generalized Cartan matrix*, and you can define the notion of a Lie algebra associated to such matrices using the same Serre relations, which is called a *Kac-Moody algebra*. The analogue of Theorem 22.1 is that the Kac-Moody algebra is finite-dimensional iff its generalized Cartan matrix is an actual Cartan matrix.

In addition to semisimple Kac-Moody algebras, people also care about *affine Lie algebras*, whose Dynkin diagrams look like those for the ADE classification, but have one more node and additional constraints on the edges. Even after working with this for a long time, this can still seem crazy.  $\blacktriangleleft$

We've come quite far from our original approach to representation theory, but with these tools in hand we can now study the finite-dimensional representations of a semisimple Lie algebra  $\mathfrak{g}$ . This involves a classification and character formulas, which are part of the general theory, and we'll also focus on some examples, namely  $A_n$  and the orthogonal groups. The latter will lead us into spinor representations.

Anyways, let  $V$  be a finite-dimensional representation of  $\mathfrak{g}$ . Fix a Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  and a root system  $R = R_+ \amalg R_-$ , etc. Then,  $V$  decomposes into weight spaces as an  $\mathfrak{h}$ -representation:

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V[\lambda],$$

and  $h \in \mathfrak{h}$  must act semisimply on  $V$ . This is because there's a compact Lie group  $G$  whose complexified Lie algebra is  $\mathfrak{g}$ , and if  $T \subset G$  is a maximal torus,  $\mathfrak{h}$  is the complexified Lie algebra of  $T$ . This is not obvious.

The next question is: which possible weights  $\lambda \in \mathfrak{h}^*$  can occur? Each  $h_\alpha \in \mathfrak{h}$  must act with integer eigenvalues, i.e. if  $V[\lambda] \neq 0$ , then  $\lambda(h_\alpha) \in \mathbb{Z}$ . Therefore we can define the lattice of *coroots*, called the *weight lattice*

$$P := \{\lambda \in \mathfrak{h}^* \mid h_\alpha(\lambda) \in \mathbb{Z} \text{ for all } \alpha \in R\}.$$

As an element of  $\mathfrak{h}^{**} = \mathfrak{h}$ ,  $h_\alpha$  is written  $\alpha^\vee$ . So the coroots are the things which pair with roots to get integers.

There are two more lattices around: the *root lattice*  $Q \subset \mathfrak{h}^*$  and the *coroot lattice*  $Q^\vee \subset \mathfrak{h}$  defined by

$$Q := \text{span}_{\mathbb{Z}}(R), \quad Q^\vee := \text{span}_{\mathbb{Z}}(P).$$

Curiously, the dual lattice to  $Q$  isn't  $Q^\vee$ ; instead,  $P$  is dual to  $Q^\vee$ .

The root lattice has a basis given by the simple roots, i.e.  $Q = \mathbb{Z}\Pi$ . If  $\Pi = \{\alpha_1, \dots, \alpha_r\}$ , then  $Q^\vee$  has for a basis  $\{\alpha_1^\vee, \dots, \alpha_r^\vee\}$ . The dual basis to this is denoted  $\{\omega_i\}$ , i.e.  $(\alpha_i^\vee, \omega_j) = \delta_{ij}$ , and  $\{\omega_i\}$ , called the set of *fundamental weights*, is a basis for  $P$ .

**Example 22.3.** Let's look at  $A_1$ , which corresponds to  $\mathfrak{sl}_2$ . In this case, the roots are  $\alpha = 2$  and  $-\alpha = -2$ , so the root lattice is the even numbers:  $Q = 2\mathbb{Z}$ . The weight lattice is all integers. ◀

This  $\pm 2$  is why factors of 2 and  $1/2$  appear so much in these definitions.

Lecture 23.

## Representations of semisimple Lie algebras: 3/10/17

Today,  $\mathfrak{g}$  will denote a semisimple Lie algebra and  $\mathfrak{h}$  will denote a Cartan subalgebra. Let  $R$  denote the set of roots, and choose a splitting  $R = R_+ \amalg R_-$ . If  $V$  is a finite-dimensional  $\mathfrak{g}$ -representation, then it decomposes into weight spaces

$$V = \bigoplus_{\lambda \in P} V[\lambda],$$

where  $P \subseteq \mathfrak{h}^*$  is the weight lattice

$$P := \{\lambda \in \mathfrak{h}^* \mid \langle \alpha^\vee, \lambda \rangle \in \mathbb{Z} \text{ for all } \alpha \in R\}.$$

The sublattice  $Q = \mathbb{Z}R$  is contained in  $P$ .

**Example 23.1.** TODO: add picture for the root system of the two-dimensional Lie algebra  $\mathfrak{sl}_3 = A_2$ . ◀

Given a finite-dimensional representation  $V$  of  $\mathfrak{g}$ , we can define its *character* to be

$$\chi_V := \sum_{\lambda \in P} \dim(V[\lambda]) e^\lambda \in \mathbb{C}[P].$$

The term  $e^\lambda$  is used to indicate that we're thinking of this group multiplicatively. As before, we'll see that the character encodes everything about the representation.

*Remark.* If  $G$  is a simply-connected, compact Lie group whose (complexified) Lie algebra is  $\mathfrak{g}$  and  $T \subset G$  is a maximal torus whose (complexified) Lie algebra is  $\mathfrak{h}$ , then the image of the differentiation  $\text{Hom}(T, \text{U}(1)) \rightarrow \text{Hom}(\mathfrak{t}, \mathbb{C}) = \mathfrak{h}^*$  is the weight lattice  $P$ .

Given a representation  $V$  of  $G$ , we defined its character  $\chi_V \in C(G)^G \cong C(T)^W$ . If  $V'$  denotes the induced representation of  $\mathfrak{g}$ , then its character is the Fourier expansion of  $\chi_V$ , with the  $e^\lambda$  term actually an exponential. ◀

**Example 23.2.** Consider the standard representation  $V$  of  $\mathfrak{sl}_3$  on  $\mathbb{C}^3$ . We have generators  $h_1, h_2$  for  $\mathfrak{sl}_3$ , where

$$h_1 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \quad h_2 = \begin{pmatrix} & & \\ & 1 & \\ & & -1 \end{pmatrix}.$$

Thus, the eigenvalues of  $h_1$  are  $\{1, -1, 0\}$  and those of  $h_2$  are  $\{0, 1, -1\}$ . Let  $\omega_1$  and  $\omega_2$  be the *fundamental weights* (corresponding to the irreducible roots); then,

$$\chi_V = e^{\omega_1} + e^{-\omega_1 + \omega_2} + e^{-\omega_2}. \quad \blacktriangleleft$$

The character of a representation of a Lie group is invariant under the Weyl group on the maximal torus. We'd like something similar to hold. In the context of Lie algebras, we can realize the Weyl group as  $W = \langle s_\alpha \mid \alpha \in R \rangle$ .

**Proposition 23.3.**  $\chi_V \in \mathbb{C}[P]^W$ .

*Proof.* The proof once again uses  $\mathfrak{sl}_2$ -representation theory. Let  $\lambda$  be a weight of  $V$ . We'd like to show that  $\dim V[\lambda] = \dim V[s_{\alpha_i}\lambda]$ , where  $\alpha_i$  is an arbitrary simple root.

Let  $n := \langle \alpha_i^\vee, \lambda \rangle$ , which is neither nonnegative or nonpositive. First assume  $n \geq 0$ . Then,  $s_{\alpha_i}(\lambda) = \lambda - n\alpha_i$ , so we need to compare the dimensions  $V[\lambda - n\alpha_i]$  and  $V[\lambda]$ . The key is that  $e_i^n \circ f_i^n: V[\lambda] \rightarrow V[\lambda - n\alpha_i] \rightarrow V[\lambda]$  is multiplication by  $\lambda$ , so the two spaces are isomorphic.  $\square$

In fact, just as for compact Lie groups, the characters of the irreducible finite-dimensional representations of  $\mathfrak{g}$  form a basis for  $\mathbb{C}[P]^W$ !

This allows us to classify representations of  $\mathfrak{g}$ , akin to the highest-weight/Verma module classification for  $\mathfrak{sl}_2$ . All the notation we've been introducing will shine now: it would be nice for everything to live in  $\mathbb{R}$ , but since in general weights live in the root lattice, we had to introduce the terminology to address this and provide analogies.

Recall that  $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ , where  $\mathfrak{n}_+$  and  $\mathfrak{n}_-$  are the pieces corresponding to the positive and negative roots, respectively. It's not hard to check that the *Borel subalgebra*  $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}_+$  is a Lie subalgebra, and moreover a maximal solvable subalgebra.<sup>27</sup>

**Definition 23.4.** Let  $V$  be a  $\mathfrak{g}$ -representation, which could be infinite-dimensional.<sup>28</sup> A *highest weight vector* in  $\mathfrak{g}$  (with respect to  $\mathfrak{b}$ ) is a  $v \in V$  such that  $\mathfrak{n}_+ \cdot v = 0$ .

The idea is that finite-dimensional representations will have highest-weight vectors; if they're irreducible, there will be just one. In the infinite-dimensional case, one cannot expect a weight-space decomposition, nor for the weights to lie in the weight lattice.

**Proposition 23.5.** *Every finite-dimensional representation of  $\mathfrak{g}$  has a highest weight vector. In particular, if  $V$  is an irreducible, finite-dimensional representation, then it is a highest-weight representation.*

*Proof.* Let  $P(V)$  be the set of weights of  $V$ . We'll show there's a  $\lambda \in P(V)$  such that  $\lambda + \alpha \notin P(V)$  for any  $\alpha \in R_+$ . To do so, choose an  $h \in \mathfrak{h}$  such that  $\langle h, \alpha \rangle = 0$  for all  $\alpha \in R_+$ ; since there are only finitely many roots, we can choose  $\lambda$  such that  $\langle h, \lambda \rangle$  is maximal.  $\square$

The theory of Verma modules behaves like that for  $\mathfrak{sl}_2$ .

**Definition 23.6.** The *Verma module* with highest weight  $\lambda \in \mathfrak{h}^*$  is the  $\mathcal{U}(\mathfrak{g})$ -module

$$M_\lambda := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda = \mathcal{U}(\mathfrak{g}) / \left( \mathcal{U}(\mathfrak{g}) \cdot \mathfrak{n}_+ + \sum_{h \in \mathfrak{h}} \mathcal{U}(\mathfrak{g})(h - \lambda(h)) \right).$$

That is,  $M_\lambda = \mathcal{U}(\mathfrak{g}) \cdot \sigma_\lambda$ , where  $h \cdot \sigma_\lambda = \lambda(h)\sigma_\lambda$  for  $h \in \mathfrak{h}$  and  $n \cdot \sigma_\lambda = 0$  for  $n \in \mathfrak{n}_+$ .

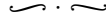
<sup>27</sup>You might have been noticing that we're making a lot of choices: a Cartan subalgebra, a set of positive roots, etc. This is okay, but there is something to check here.

<sup>28</sup>Often, one thinks of infinite-dimensional representations as  $\mathcal{U}(\mathfrak{g})$ -modules.

Lecture 24.

## Root data: 3/20/17

Today, we're going to regroup<sup>29</sup> a bit after spring break. We'd like to dummarize the whole story of representations of compact Lie groups, using the Lie algebra technology we've developed. We'll see the general Weyl character formula, which generalizes (9.2), and provide two different proofs: Weyl's original one and one using Verma modules. After that, we'll cover some examples:  $U(n)$ ,  $SU(n)$ ,  $Spin(n)$ , and so on. This will take us the next few lectures.



Let  $G$  be a compact Lie group. Just as we fixed a Cartan subalgebra of  $\mathfrak{g}$ , we'll fix a maximal torus  $T \subset G$ . As  $T$  is a compact abelian Lie group, it's a product of copies of  $U(1)$ . The complexified Lie algebra of  $T$  is a Cartan subalgebra of  $\mathfrak{g}$  (i.e. the complexified Lie algebra).

Just as semisimple Lie algebras give rise to root systems, we can describe compact Lie groups combinatorially. We do not require  $G$  to be semisimple. Recall that if  $\mathfrak{g}$  is reductive, it splits into a semisimple part  $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$  and its center  $Z(\mathfrak{g})$ . The Lie algebra decomposes over the root space:

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha.$$

We've defined lattices associated to  $\mathfrak{g}$  and  $\mathfrak{h}^*$ , and there's yet another,

$$L := L_{G,T} = \text{Hom}(T, U(1)) \rightarrow \mathfrak{h}^*,$$

where we use the standard lattice embedding  $\mathbb{Z}^n \hookrightarrow \mathbb{C}^n$ . Given a  $G$ -representation  $V$ ,  $V = \bigoplus_{\lambda \in L} V[\lambda]$ .

If  $G$  is semisimple, meaning that  $Z(G)$  is a discrete group, then there's a nice chain of inclusions of these lattices:  $Q \subset L \subset P \subset \mathfrak{h}^*$ , where  $Q$  is the root lattice and  $P$  is the weight lattice. If  $G_{ss} = G/Z(G)$ , then the weight lattice is  $L_{G_{ss}}$ , and if  $G$  is simply connected, then  $Q = L_G$ .

Recall that given an  $\alpha \in R$ , we found an embedding  $\mathfrak{sl}_2(\mathbb{C}) \hookrightarrow \mathfrak{g}$  as  $\mathfrak{g}_{-\alpha} \oplus \mathbb{C} \cdot \alpha^\vee \oplus \mathfrak{g}_\alpha$ . This integrates to a map  $SU(2) \rightarrow G$ , though this map is not always injective, e.g. we can get the covering map  $SU(2) \twoheadrightarrow SO(3)$ .

Recall that  $R^\vee := \{\alpha^\vee \mid \alpha \in R\}$ , and  $R^\vee \subset L^* := \text{Hom}(L, \mathbb{Z})$ , which is equal to  $\text{Hom}(U(1), T)$  (by restricting to the maximal torus).

So associated to the compact Lie group  $G$ , we have the lattice  $L$ , the dual lattice  $L^*$ , and the sublattice of coroots  $R^\vee$ .

**Definition 24.1.** A *root datum* is a quadruple  $(L, R, L^*, R^\vee)$ , where  $L$  and  $L^*$  are dual lattices (e.g. through the specification of a perfect pairing  $\langle -, - \rangle$  between them), and  $R \subset L$  and  $R^\vee \subset L^*$  are finite subsets, together with an explicit bijection  $(-)^\vee: R \xrightarrow{\cong} R^\vee$ , which satisfies some axioms:

- (1)  $\langle \alpha, \alpha^\vee \rangle = 2$ , and
- (2) the map  $s_\alpha: L \rightarrow L$  sending  $x \mapsto x - \langle \alpha^\vee, x \rangle \alpha$  sends  $R$  to  $R$ , and the dual map  $s_\alpha^\vee: L^* \rightarrow L^*$  sends  $R^\vee$  to  $R^\vee$ .

You could just fix  $L$  and  $R$  and get  $L^*$  and  $R^\vee$  from the duality pairing, but this is how the root datum is usually defined.

This looks different from the notion of a root system because root systems were equipped with definite bilinear forms coming from the Killing form. For  $G$  a compact Lie group, the Killing form on  $\mathfrak{g}$  need not be nondegenerate, so we don't always have an identification of  $L$  and  $L^*$ .

Also, in case you're wondering why we're doing this, it will be useful for studying the representation theory of groups that aren't simply connected, or of groups that aren't semisimple, such as  $U(n)$ .

*Remark.* We've now seen two definitions of the Weyl group associated to a Lie group  $G$ . Given a maximal torus  $T \subset G$ , we defined  $W := N_G(T)/T$ , and so  $W$  acts on  $T$ , hence on  $L$  and on  $R$ . It's nontrivial to prove, but  $W$  is generated by  $\{s_\alpha \mid \alpha \in R\} \subset S_R$  (the symmetric group on  $R$ ). We saw something like this for Lie algebras, and we also saw it in examples.

For  $G = U_n$ ,  $T = (U(1))^n$ , the subgroup of diagonal matrices. The Weyl group is  $S_n$  (the symmetric group), and  $W$  is represented explicitly by permutation matrices. If  $\{e_1, \dots, e_n\}$  is the standard basis of  $\mathbb{C}^n$ ,

<sup>29</sup>No pun intended.



then  $L = \mathbb{Z}^n$  and  $L^* = (\mathbb{Z}^n)^*$ , which contain  $R = \{e_i - e_j \mid i \neq j\}$  and  $R^\vee = \{e_i^* - e_j^* \mid i \neq j\}$ , respectively. The Weyl group acts on  $T$  by permuting entries, and  $s_{e_i - e_j} \in W$  is the reflection transposing  $i$  and  $j$ . ◀

There's a bijection between the isomorphism classes of compact Lie groups and the isomorphism classes of root data.<sup>30</sup> In particular, given a root datum, there's a way to engineer a compact Lie group out of it.

*Remark.* In fact, root data classify reductive algebraic groups over an algebraically closed field: the notion of a root datum doesn't depend on the base field being  $\mathbb{C}$ , and this in particular means that if  $k$  is an algebraically closed field, it's possible to build a reductive algebraic group over  $k$  out of a root datum over  $k$ . The category of representations of a reductive algebraic group is also given by discrete combinatorial data (up to equivalence).

If  $k = \mathbb{C}$  and  $G_{\mathbb{C}}$  is a complex reductive algebraic group, its maximal compact subgroup  $G$  is the compact Lie group that's associated to the root datum for  $G_{\mathbb{C}}$ . ◀

*Remark.* Given a root datum  $(L, R, L^*, R^\vee)$ , we can obtain another root datum by switching the roles of  $L$  and  $L^*$ , and of  $R$  and  $R^\vee$ , i.e. producing the root datum  $(L^*, R^\vee, L, R)$ ; you can check that this satisfies the axioms of a root datum. This defines an involution on the isomorphism classes of compact Lie groups, which sends a compact Lie group  $G$  to its *Langlands dual group*  $G^\vee$ . For example:

- We already saw that the unitary group is self-dual:  $U(n)^\vee = U(n)$ .
- The dual of  $SU(n)$  is  $PSU(n)$  (the simply connected and adjoint forms, respectively.).
- $SO(2n+1)$  is exchanged with  $Sp(n)$ : these are semisimple, so establish a duality on Dynkin diagrams of types  $B_n$  and  $C_n$ .
- Finally, Dynkin diagrams of type  $D_n$  are self-dual, and  $SO(2n)^\vee = SO(2n)$ .

The statements about the Lie groups are stronger than those about the Lie algebras or Dynkin diagrams, as they contain nontrivial information about groups which are not simply connected. This duality is kind of crazy: there's no obvious geometric relationship, but this duality seems to crop up in a lot of places in both math and physics: the Langlands program and Kapustin-Witten's explanation of Langlands duality as an instance of  $S$ -duality.<sup>31</sup> ◀

Lecture 25.

## Representations of $U(n)$ and $SU(n)$ : 3/22/17

*"As usual, I'm engaging in an internal struggle about notation."*

Last time, we associated to a compact Lie group  $G$  a root datum  $(L, R, L^*, R^\vee)$ . Today, we apply this to do representation theory.

**Definition 25.1.** Fix a polarization  $R = R_+ \amalg R_-$ . Then, a  $\lambda \in L$  is *dominant* if  $\langle \alpha^\vee, \lambda \rangle \in \mathbb{Z}_{\geq 0}$  for all  $\alpha \in R_+$ .

We'll let  $L_+$  denote the set of dominant weights (within the context of a specified polarization).

**Theorem 25.2** (Highest weight theorem). *For every dominant weight  $\lambda$ , there is a unique irreducible representation  $V_\lambda$  for  $G$ , and every irreducible representation of  $G$  arises in this way.*

The representation  $V_\lambda$  of  $G$  defines a highest weight representation of the complexified Lie algebra  $\mathfrak{g}$ , i.e.  $V_\lambda = \mathcal{U}(\mathfrak{g}) \cdot v_\lambda$ , where  $h \cdot v_\lambda = \lambda(h)v_\lambda$  for all  $h \in \mathfrak{h}$  and  $n \cdot v_\lambda = 0$  for all  $n \in \mathfrak{n} = \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha$ .

This is why we took a detour into Lie algebra representations: it's hard to say what this means if you only know about  $G$  and not  $\mathfrak{g}$ : there's nothing corresponding to  $\mathfrak{n}$  in the real Lie algebra for  $G$ , and in particular does not integrate to anything on  $G$ .

**Example 25.3.** Consider  $G = U(n)$ , which has for a maximal torus the diagonal matrices  $T = U(1)^n \subset T$ . The root data is  $L = L^* = \mathbb{Z}^n$  and  $R_+^\vee = R_+ = \{e_i - e_j \mid i > j\}$ . Then, the dominant weights are

$$L_+ = \{(a_1, \dots, a_n) \mid a_1 \leq a_2 \leq \dots \leq a_n\}.$$

<sup>30</sup>This can be upgraded to an equivalence of categories, though it requires some care about, e.g., the choice of maximal tori and whether maps preserve maximal tori.

<sup>31</sup>See <https://arxiv.org/abs/hep-th/0604151> for more information.

If  $G = \mathrm{SU}(n)$ , then its character lattice is a quotient of that of  $\mathrm{U}(n)$ :  $L_{\mathrm{SU}(n)} = \mathbb{Z}^n / (1, 1, \dots, 1)$  and the dominant weights are

$$L_{\mathrm{SU}(n)+} = \{(a_1, \dots, a_n) \mid a_1 \leq \dots \leq a_n\} / (1, 1, \dots, 1).$$

We can fix  $a_1 = 0$  and obtain representatives  $0 \leq a_2 \leq \dots \leq a_n$ , indexed by partitions  $\lambda = (\lambda_1, \dots, \lambda_{n-1})$ , where  $\lambda_i = a_{i+1} - a_i$ . So irreducible representations of  $\mathrm{SU}(n)$  are determined by partitions, and an irreducible representation of  $\mathrm{U}(n)$  is a partition plus an extra integer.  $\blacktriangleleft$

Given a partition, how do we construct the corresponding representation? What is its character? (Recall that the characters  $\chi_V$  live in  $C_{\mathrm{alg}}(G)^G = C_{\mathrm{alg}}(T)^W = \mathbb{C}[L]^W$ .) The irreducible characters of  $G$  form an orthonormal basis for  $C_{\mathrm{alg}}(G)^G$ , and therefore also for  $\mathbb{C}[L]^W$ , but what is the inner product on the latter space? Also, the dominant weights  $L_+$  form a fundamental domain for the action of the Weyl group on  $L$ , i.e.  $L_+ \xrightarrow{\cong} L/W$ . We'll see how to understand these explicitly next week. For example, for  $\mathrm{SU}(2)$ , the Weyl group is  $\mathbb{Z}/2$ , so acts by an involution sending a positive root to a negative root.

**Constructing  $V_\lambda$  through Verma modules.** We've successfully used Verma modules to construct the irreducible representations of  $\mathfrak{sl}_2$ , so let's apply them in other settings.

Recall that if  $\lambda \in \mathfrak{h}^*$ , we defined  $M_\lambda := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda$ , where  $\mathbb{C}_\lambda$  is the  $\mathfrak{h}$ -representation with eigenvalue  $\lambda$  and  $\mathfrak{b} = \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \rightarrow \mathfrak{h}$ .

This  $M_\lambda$  is a cyclic  $\mathcal{U}(\mathfrak{g})$ -module: there's a  $\sigma_\lambda \in M_\lambda$  and an ideal  $I \subseteq \mathcal{U}(\mathfrak{g})$  such that  $M_\lambda \cong \mathcal{U}(\mathfrak{g})/I \cong \mathcal{U}(\mathfrak{g}) \cdot \sigma_\lambda$ . In particular, as  $\mathcal{U}(\mathfrak{n}_-)$ -modules,  $\mathcal{U}(\mathfrak{n}_-) \cdot \sigma_\lambda = M_\lambda \cong \mathcal{U}(\mathfrak{n}_-)$ , which we can use to deduce the character of  $M_\lambda$ .  $\chi_{M_\lambda} \in \widehat{\mathbb{C}[L]}$  is defined by

$$(25.4) \quad \chi_{M_\lambda} = \sum_{\mu \in Q_+} K(\mu) e^{\lambda - \mu},$$

where  $K(\mu)$  is the number of ways of writing  $\mu$  as a sum of positive elements of the root lattice and

$$Q_+ = \left\{ \sum_{\alpha \in R_+} n_\alpha \alpha \mid n_\alpha \in \mathbb{Z}_{\geq 0} \right\}.$$

The monomials  $f_{\alpha_1}^{n_1} f_{\alpha_2}^{n_2} \dots f_{\alpha_r}^{n_r}$  define a basis for  $\mathcal{U}(\mathfrak{n}_-)$ , and

$$f_{\alpha_1}^{n_1} \dots f_{\alpha_r}^{n_r} \cdot v_\lambda \in M_\lambda \left[ \lambda - \sum n_i \alpha_i \right],$$

which proves (25.4).

Lecture 26.

### Example with $\mathfrak{sl}_3$ : 3/24/17

We want to construct the finite-dimensional irreducible representations of a compact Lie group  $G$  or a reductive Lie algebra  $\mathfrak{g}$ . We know that if  $V$  is such a representation, it contains a highest weight vector  $v_\lambda$ , and  $\lambda$  must be a dominant weight, i.e.  $\langle \alpha^\vee, \lambda \rangle$  is a nonnegative integer for all  $\alpha \in R_+$ . Here,  $\langle \alpha^\vee, \lambda \rangle$  is the weight of  $\sigma_\lambda$  for the action of  $\mathfrak{sl}_2(\mathbb{C})_\alpha \hookrightarrow \mathfrak{g}$ . In particular,  $V = V_\lambda$  is a quotient of the Verma module  $M_\lambda$ . We'd like to understand questions of existence and uniqueness (namely, the highest representation should exist and be unique), as well as how to compute characters. Check out Figure 5 for a picture when  $\mathfrak{g} = \mathfrak{sl}_3$ .

Let  $\lambda$  be a dominant weight for  $\mathfrak{sl}_3$ , and decompose  $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ . Let  $M_\lambda = \mathcal{U}(\mathfrak{n}_-) \cdot v_\lambda$  be a Verma module (which is a free  $\mathcal{U}(\mathfrak{n}_-)$ -module of rank 1). In  $\mathfrak{sl}_3$ ,  $\mathfrak{n}_- = \langle f_{\alpha_1}, f_{\alpha_2}, f_{\alpha_1+\alpha_2} \rangle$ , and a basis for  $\mathcal{U}(\mathfrak{n}_-)$  is  $f_{\alpha_1}^{n_1} f_{\alpha_2}^{n_2} f_{\alpha_1+\alpha_2}^{n_3}$ , where  $n_1, n_2, n_3 \geq 0$ . Pictorially, you can think of it containing all of the weights that are below and/or to the left of  $\lambda$ . You can compute the dimension of the weight space by seeing how many times a weight appears in this way.

Associated to  $\lambda$  are the integers  $\langle \alpha^\vee, \lambda \rangle$ , which are the lengths of  $\lambda$  projected onto the spaces spanned by  $\alpha$ , as  $\alpha$  ranges over the simple roots.

Anyways, what we want to do now is figure out how to find a finite-dimensional quotient of  $M_\lambda$ , which in particular will be irreducible.

For a simple root  $\alpha_i$ , let  $f_i := f_{\alpha_i}$ . Everything works just like for  $\mathfrak{sl}_2$ :  $f_i$  carries  $f_i^{n_i}(v_\lambda)$  to  $f_i^{n_i+1}(v_\lambda)$  and  $e_i$  goes back in the other direction. In particular, since this piece is a copy of  $\mathcal{U}(\mathfrak{sl}_2)$ ,  $e_i \cdot f_i^{n_i}(v_\lambda) = 0$ : the proof is the same as for  $\mathfrak{sl}_2$ .

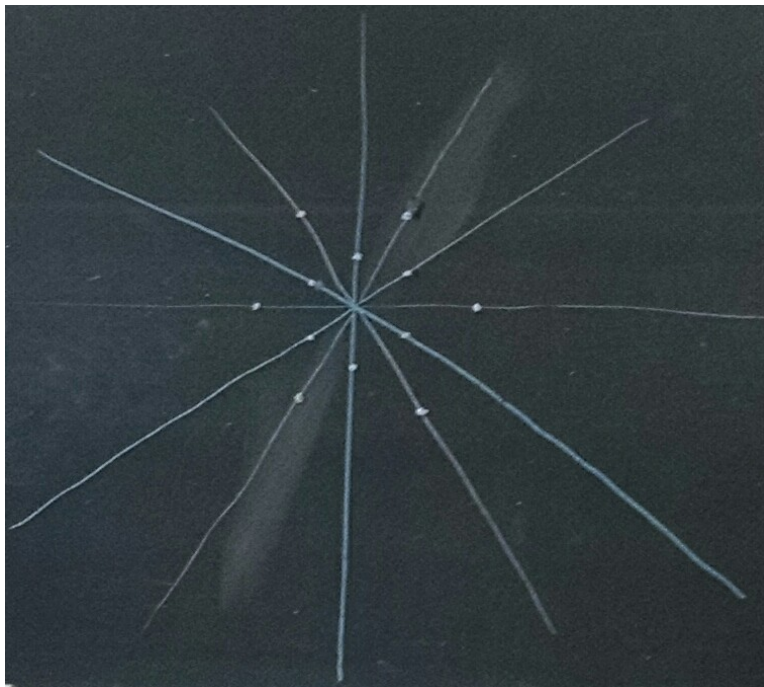


FIGURE 5. The geometry of the root system for  $\mathfrak{sl}_3$ . The white lines are spanned by the roots, and the white dots are the lattice  $\mathfrak{h}^*$ . The blue lines are chamber walls, orthogonal to the roots. The dominant weights are those between the 12:00 and 2:00 blue lines (or rather, this is the result of the choice of a polarization).

In fact,  $f_i^{n_i+1}(v_\lambda)$  is a highest weight vector, as  $e_j f_i^{n_i+1}(v_\lambda) = 0$  for all  $j = 1, \dots, r$ . So we've found more highest weight vectors. We can look at the subrepresentations they've generated, which also look like Verma modules. Thus for each  $i = 1, \dots, r$ , we have maps  $M_{s_i(\lambda) - \alpha_i} \rightarrow M_\lambda$ . Let

$$E = \text{Im} \left( \bigoplus_{i=1}^r M_{s_i(\lambda) - \alpha_i} \rightarrow M_\lambda \right).$$

That is, we take the images of all the other Verma modules. Then we throw it away.

**Proposition 26.1.**  $V_\lambda := M_\lambda / E$  is a finite-dimensional irreducible representation.

The main point is to show that  $V_\lambda$  is finite-dimensional. For each  $i$ ,  $v_\lambda$  generates a finite-dimensional  $\mathfrak{sl}_2(\mathbb{C})_{\alpha_i}$ -module.

Geometrically, in Figure 5, the Verma module is generated by the intersection of the lattice  $\mathfrak{h}^*$  and a convex set whose highest weight is  $\lambda$ , and we're removing some infinite-volume convex subsets from it.

Next we have to discuss some multiplicities; this boils down to the Weyl character formula. You get a nice equilateral, if not equilateral, hexagon and intersect it with  $\mathfrak{h}^*$ . (The picture is in Kirrilov's book.)

We'd like to compute the character, but the map  $E \rightarrow M_\lambda$  is not always injective, though the map  $M_{s_i(\lambda) - \alpha_i} \rightarrow M_\lambda$  is the beginning of a free resolution of  $V_\lambda$  as a  $\mathcal{U}(\mathfrak{g})$ -module. Specifically, the following is called the *BGG-resolution*.<sup>32</sup>

$$\cdots \longrightarrow \bigoplus_{i=1}^r M_{\alpha_i(\lambda) - \alpha_i} \longrightarrow M_\lambda \longrightarrow V_\lambda \longrightarrow 0.$$

This will be a finite resolution.

<sup>32</sup>“BGG” stands for Bernstein-Gelfand-Gelfand.

However, to get the character formula, we need to get a different, translated action of the Weyl group  $W$ : let

$$\rho := \sum_{\alpha \in R_+} \alpha$$

and the *dot action* of  $W = \langle s_\alpha \rangle$  on  $\mathfrak{h}^+$  by

$$w \cdot \lambda = w(\lambda + \rho) - \rho.$$

That is, translate by  $\rho$ , act in the usual way, and then translate back. Under this action,  $s_{\alpha_i} \cdot \lambda = s_{\alpha_i}(\lambda) - \alpha_i$ . Next time, we'll put this together into the character formula.

Lecture 27.

### The Weyl character formula: 3/27/17

Today, we're going to talk about the Weyl character formula, which we've been leading up to for a while. The setup includes a compact group  $G$  (or its complexified Lie algebra  $\mathfrak{g}$ , which is a reductive Lie algebra, and most of what we say today only depends on the semisimple part). Let  $L_+$  be the positive weights in the weight lattice  $L$  and  $V_\lambda$  be the irreducible representation of highest weight  $\lambda \in L_+$ .

**Definition 27.1.** Given an element  $w \in W := N_G(T)/T$  of the Weyl group, its *length* is

$$\ell(w) := \min\{k \mid w = s_{i_1} s_{i_2} \cdots s_{i_k}\},$$

where  $s_i$  is the reflection across  $\alpha_i^\perp$ .

**Example 27.2.** For  $G = \mathrm{U}(n)$ ,  $W = S_n$  and  $s_i$ , where  $1 \leq i \leq n-1$ ,  $s_i = (i \ i+1)$ , and the length of a permutation is how many transpositions of adjacent elements are needed to generate it. ◀

With this notation in hand, we can state the Weyl character formula.

**Theorem 27.3** (Weyl character formula). *The character  $\chi_\lambda \in \mathbb{C}[L]$  of  $V_\lambda$  is*

$$(27.4) \quad \chi_\lambda = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w \cdot \rho}}{\prod_{\alpha \in R_+} (1 - e^{-\alpha})},$$

where  $w \cdot \lambda := w(\lambda + \rho) - \rho$  is the dot action, and the exponentials are formal.

For example, the trivial representation has highest weight 0 and character  $\chi_0 = 1$ , so (27.4) reduces to the *Weyl denominator formula*

$$(27.5) \quad \underbrace{e^\rho \prod_{\alpha \in R_+} (1 - e^{-\alpha})}_{\delta} = \sum_{w \in W} (-1)^{\ell(w)} e^{w(\rho)}.$$

Here,  $\delta$  is called the *Weyl denominator*, and we've rewritten the dot action into the usual action of the Weyl group.

Using (27.5), we can rewrite the Weyl character formula: let

$$A_\lambda = \sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda)}.$$

Then,

$$\chi_\lambda = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda + \rho)}}{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\rho)}} = \frac{A_{\lambda + \rho}}{A_\rho}.$$

For example, when  $G = \mathrm{SU}_2$ , the positive weights are identified with nonnegative integers,  $\rho = 1$ , and  $\alpha = 2$ . If  $P_n$  denotes the irreducible representation with highest weight  $n$ , as explicated in Example 5.5, then

$$\chi_{P_n} = \frac{e^{n+1} - e^{-n-1}}{e^1 - e^{-1}} = e^n + e^{n-2} + \cdots + e^{-n+2} + e^{-n}.$$

In a sense, to go from formal exponentials to actual exponentials, you should multiply by a  $2\pi i$  coming from the Fourier transform.

We'd like to prove Theorem 27.3, and there are two different approaches.

*Proof sketch of Theorem 27.3.* Let  $M_\lambda$  be the Verma module with highest weight  $\lambda$ , so that its character (again as an element of  $\mathbb{C}[L]$ ) is

$$(27.6) \quad \chi_{M_\lambda} = e^\lambda \left( \prod_{\alpha \in R_+} (1 - e^{-\alpha})^{-1} \right),$$

where the inverse is formal (i.e.  $1 + e^{-\alpha} + e^{-2\alpha} + \dots$ ). Now we can use the BGG resolution

$$(27.7) \quad \cdots \longrightarrow \bigoplus_{\substack{w \in W \\ \ell(w)=2}} M_{w \cdot \lambda} \longrightarrow \bigoplus_{\substack{w \in W \\ \ell(w)=1}} M_{w \cdot \lambda} \longrightarrow M_\lambda \twoheadrightarrow V_\lambda \longrightarrow 0.$$

The character, as a trace, is a graded kind of dimension, so we can use it to figure out the character of  $V_\lambda$ . That is, since (27.7) is exact,  $\chi_{V_\lambda}$  is an alternating sum of the characters of the other terms in the resolution.

$$\chi_{V_\lambda} = \sum_{w \in W} (-1)^{\ell(w)} \chi_{M_{w \cdot \lambda}},$$

and combining this with (27.6), we're done.  $\square$

The idea of the proof is that  $\chi_{V_\lambda}$  looks nice in terms of characters of certain infinite-dimensional representations, But there are certain ways it's not satisfying, and there's an alternate proof, based on the Weyl integration formula, which has complementary strengths.

*Second proof of Theorem 27.3.* We proceed via the Weyl integration formula, using the fact that  $C(G)^G \cong C(T)^W$ . For example, class functions on  $U(n)$  are the same thing as  $S_n$ -invariant class functions on  $U(1)^n$ . When we studied the special case of  $SU(2)$ -representations, we related measures on  $G$  and on  $T$  using (9.3); it also has an analogue in this more general setting.

The engine that makes this work is that the map  $r: G/T \times T \rightarrow G$  sending  $(gT, t) \mapsto gtg^{-1}$  is generically  $|W|$ -to-1. In cases where a matrix has repeated eigenvalues, there are fewer elements in the fiber, but these matrices are the complement of a dense subset and thus have measure zero, so we can ignore them when integrating.

Concretely, for  $G = U(n)$ , this says that the set of diagonalizable matrices has full measure, so we can throw out nondiagonalizable matrices when computing integrals, then use conjugacy invariance to assume our matrices are diagonal.

Anyways, the formula is

$$\int_G f(g) dg = \frac{1}{|W|} \int_T r^* f(gT, t) J(gT, t) d(gT, t),$$

where  $J$  is the Jacobian. If you only care about class functions, this formula simplifies:  $G$  acts on  $G/T \times T$  on the first factor by left multiplication and on  $G$  by left multiplication, and  $r$  is equivariant for those actions. Then the Jacobian is the determinant of the induced map on tangent spaces

$$dr|_{(gT, t)}: T_{(gT, t)}(G/T \times T) \longrightarrow T_{gtg^{-1}}G,$$

and since we're just taking the determinant, it makes no difference if we pass to the complexified Lie algebras, giving us a map

$$dr_{\mathbb{C}}: \mathfrak{g}/\mathfrak{h} \oplus \mathfrak{h} \longrightarrow \mathfrak{h}.$$

We can identify  $\mathfrak{g} = (\mathfrak{n}_+ \oplus \mathfrak{n}_-) \oplus \mathfrak{h}$ , so we can really make sense of this map as an endomorphism and hence also make sense of its determinant.

As a block matrix,  $dr_{\mathbb{C}}$  has the form

$$dr|_C = \begin{pmatrix} \text{id} - \text{Ad}(t) & 0 \\ 0 & \text{id} \end{pmatrix}$$

and therefore its Jacobian is

$$(27.8) \quad J(gT, t) = \det(\text{id} - \text{Ad}(t)|_{\mathfrak{g}/\mathfrak{h}}) = \prod_{\alpha \in R} (1 - e^\alpha),$$

because  $e^\alpha$  is the eigenvalue of  $\text{ad } t$  acting on  $g_\alpha$ .

At this point we need to pick a polarization, which in effect is a choice of square root of the Jacobian. That is, a choice of  $R = R_+ \amalg R_-$  decomposes (27.8) into

$$\prod_{\alpha \in R} (1 - e^{-\alpha}) = \underbrace{\prod_{\alpha \in R_+} (1 - e^{-\alpha})}_{\delta} \underbrace{\prod_{\alpha \in R_-} (1 - e^{-\alpha})}_{\bar{\delta}}.$$

We'd like to define a map  $L^2(G)^G \rightarrow L^2(T)$  sending  $f \mapsto f\delta$ , and this is a bounded linear map preserving the inner product. However,  $f\delta$  isn't Weyl-invariant; instead, we get functions whose action under the Weyl group is alternating (hence the  $(-1)^\ell(w)$  terms).  $\square$

Lecture 28.

### The Weyl integration formula: 3/29/17

Lecture 29.

### Weyl modules for $U(n)$ : 3/31/17

I missed Wednesday's lecture (if you have notes, let me know). I'll also be out of town next week, so notes will appear later than usual. Sorry about that. In other news, we're also back in the 11<sup>th</sup> floor classroom.

Anyways, today we're going to talk about Weyl modules for  $U(n)$ . Recall that for  $SU(2)$ , we gave a couple of constructions for the irreducible representations  $V_n$ .

- The first construction was  $P_n$ , the space of homogeneous polynomials of degree  $n$ .
- $P_n$  is also isomorphic to  $V_n = \text{Sym}^n(\mathbb{C}^2)$ , where  $\mathbb{C}^2$  is the defining representation of  $SU(2)$ .
- Finally, we constructed them as quotients of Verma modules:  $V_n = M_n/M_{n-2}$  (where  $n$  is the highest weight).

We've generalized the last construction, but the second one sounds like it should admit a generalization, and it would be nice to have, especially for classical groups where they're easier.

If  $\lambda := (\lambda_1, \dots, \lambda_n)$ , where  $\lambda_1 \geq \dots \geq \lambda_n$  are positive integers, then there's a unique irreducible representation  $V_\lambda$  of  $U(n)$  with highest weight  $\lambda$ . How can we construct  $V_\lambda$  concretely?

Let  $\mathbb{C}^n$  be the standard representation of  $U(n)$ . Then,  $v := (1, 0, \dots, 0) \in \mathbb{C}^n$  is the highest weight vector for  $V$  with highest weight  $(1, 0, \dots, 0)$ , because  $v$  is annihilated by  $\mathfrak{n}$ , the subalgebra of strictly upper triangular matrices. Thus,  $\mathbb{C}^n = V_{(1,0,\dots,0)}$ . We'll adopt the convention to ignore trailing zeros, so  $\mathbb{C}^n = V_{(1)}$ , and its character is  $\chi_{(1)} = z_1 + \dots + z_n$ . Additionally, all weights we mention today will be highest weights.

*Remark.* Since we're considering only finite-dimensional representations, there's an equivalence of categories between the smooth representations of  $U(n)$ , the algebraic representations of  $U(n)$  (i.e. those polynomial in the entries of a matrix  $A$  and  $1/\det(A)$ ), and the algebraic representations of  $GL_n(\mathbb{C})$ . Moreover, these are also the same as the representations of the monoid of all matrices under multiplication.  $\blacktriangleleft$

We can also understand these representations through their characters. If  $\mathbb{C} = V_{(0)}$  denotes the trivial representation,  $\chi_{(0)} = 1$ , and if  $\mathbb{C}_{\det}$  denotes the *determinant representation* where a matrix acts through its determinant, then  $\mathbb{C}_{\det} = V_{(1,\dots,1)}$  and  $\chi_{(1,\dots,1)} = z_1 \cdots z_n$ . Therefore  $\mathbb{C}_{\det}^{\otimes k} = V_{(k,k,\dots,k)}$  and its character is

$$\chi_{(k,k,\dots,k)} = z_1^k z_2^k \cdots z_n^k.$$

Like for  $SU(2)$ , we'll use symmetric powers to generate other representations.

Let  $e_1, \dots, e_n$  be a basis of the standard representation. A basis for  $\text{Sym}^2(\mathbb{C}^n)$  is  $\{e_i \otimes e_j\}_{1 \leq i \leq j \leq n}$ , and similar things hold for  $\text{Sym}^d(\mathbb{C}^n)$ . It has  $e_1^d$  for a highest weight vector, and the highest weight is  $(d, 0, \dots, 0)$ , so  $\text{Sym}^d(V) = V_{(d)}$ . Its character is  $\chi_{(d)} = H_d$ , the *complete symmetric polynomial* of degree  $d$ , i.e. the sum of all degree- $d$  monomials.

For exterior powers,  $\Lambda^2(\mathbb{C}^n)$  has for a basis  $\{e_i \wedge e_j\}_{1 \leq i < j \leq n}$ , and  $\Lambda^d(V) = V_{(1,\dots,1,0,\dots,0)}$  where there are  $d$  1s. Its character is  $\chi_{(1,\dots,1)} = E_d$ , the *elementary symmetric polynomial* of degree  $d$ , i.e.

$$E_d := \sum_{i_1 < \dots < i_d} z_{i_1} z_{i_2} \cdots z_{i_d}.$$

We can also take tensor powers, but

$$V^{\otimes 2} = \text{Sym}^2(V) \oplus \Lambda^2(V).$$

In fact,  $S_2$  acts on  $V^{\otimes 2}$  by permuting the factors, and this is a decomposition into isotypic components. This doesn't generalize in the way you might expect: as  $S_3$ -representations,

$$V^{\otimes 3} = \text{Sym}^3(V) \oplus \Lambda^3(V) \oplus \tilde{V},$$

where  $\tilde{V}$  should be the term corresponding to the 2-dimensional irreducible  $S_3$ -representation.

In general,  $S_d$  acts on  $V^{\otimes d}$  by permuting the factors, and it should decompose into a sum of isotypic components indexed by the irreducible representations of  $S_d$ . Let  $M$  be an irreducible  $S_d$ -representation, and define

$$\mathbb{S}_M(V) := M \otimes_{\mathbb{C}[S_d]} V^{\otimes d},$$

where  $S_d$  acts on  $V^{\otimes d}$  by permuting the factors.  $\mathbb{S}_M$  is a functor  $\text{Rep}_{U(n)} \rightarrow \text{Vect}_{\mathbb{C}}$ , and is called the *Schur functor*. Thus,

$$V^{\otimes d} \cong \bigoplus_{M \in \hat{S}_d} \mathbb{S}_M V$$

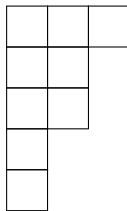
as vector spaces. But since the actions of  $U(n)$  and  $S_d$  on  $V^{\otimes d}$  commute, this is also a decomposition as  $U(n)$ -representations!

This helps us make progress on the irreducible representations of  $U(n)$ , but now we have to understand the irreducible representations of the symmetric group  $S_d$ ! Fortunately, we can now apply techniques from the representation theory of finite groups to our cause.

**Proposition 29.1.** *The irreducible representations of  $S_d$  are indexed by the partitions of  $d$ , i.e.  $\lambda = (\lambda_1, \dots, \lambda_n)$ , where  $\lambda_1 \geq \dots \geq \lambda_n$  and  $\lambda_1 + \dots + \lambda_n = d$ .*

We're not going to prove this.

Often, people describe partitions pictorially, via *Young diagrams*:



The length of the  $i^{\text{th}}$  row is  $\lambda_i$ . If you put numbers in the boxes, you get a *Young tableau* of shape  $\lambda$ . Then,  $S_d$  acts on a Young diagram by permuting the boxes.

Lecture 30.

: 4/3/17

Lecture 31.

: 4/5/17

Lecture 32.

: 4/7/17

Lecture 33.

**Representation theory of  $\text{SL}_2(\mathbb{R})$ : 4/10/17**

*"You can do this in a first course in representation theory... which I guess this is."*

I was out of town last week and missed three lectures, which I'll try to put up at some point.

Today, Sam is out of town, so David Ben-Zvi gave today's lecture, continuing the story of Borel-Weil theory, which constructs the (finite-dimensional or algebraic) irreducible representations of  $\text{SU}(2)$  geometrically. We started with the action of  $G = \text{SL}_2(\mathbb{C})$  on  $\mathbb{CP}^1 \cong G/B$ , where  $B$  is the subgroup of upper triangular matrices.



Then, if  $K \subset G$  is the copy of  $\mathrm{SU}(2)$ , this action descends to an action of  $K$  on  $K/T$ , and  $T$  is the subgroup of diagonal matrices, isomorphic to  $S^1$ .

On  $\mathbb{CP}^1$ , we have a collection of line bundles  $\mathcal{O}(n)$  for  $n \in \mathbb{Z}$ , and for  $n \geq 0$  they have nonzero global sections. Specifically, all of the representations of  $\mathrm{SU}(2)$ , and the finite-dimensional representations of  $\mathrm{SL}_2(\mathbb{C})$ , are realized as the sections of these line bundles under the induced action. The representation  $P_n$  that we cooked up arises as the representation on  $\Gamma(\mathcal{O}(n))$ .

Now we'll replace  $\mathrm{SU}(2)$  with  $\mathrm{SL}_2(\mathbb{R})$ , which is a noncompact, nonabelian group, and probably the simplest possible such example (we looked at  $\mathbb{R}$  for our first noncompact example, and  $\mathrm{SU}(2)$  for our first nonabelian example). We'd like to apply Borel-Weil theory to  $\mathrm{SL}_2(\mathbb{R})$ , but it doesn't apply *mutatis mutandis*.

The main difference is that there are different kinds of semisimple elements in  $\mathrm{SL}_2(\mathbb{R})$ , and hence different kinds of tori.

**Definition 33.1.**

- The *split torus* is the subgroup of  $g \in \mathrm{SL}_2(\mathbb{R})$  diagonalizable over  $\mathbb{C}$ , which are the matrices

$$\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, \quad a \in \mathbb{R}^\times.$$

As for compact Lie groups, these are isomorphic to the multiplicative subgroup of the field, which is reassuring.

- The *compact torus* is the subgroup of rotation matrices

$$\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$

This subgroup is isomorphic to  $S^1$ .

In particular, these two tori, both maximal abelian subgroups, are not conjugate to each other. In other noncompact Lie groups, there will be more than two kinds of tori.

Let  $\mathrm{SL}_2(\mathbb{R})^{\mathrm{rss}}$  denote the subgroup of *regular semisimple* elements, i.e. those whose eigenvalues are all distinct.  $\mathrm{SL}_2(\mathbb{R})$  acts on  $\mathrm{SL}_2(\mathbb{R})^{\mathrm{rss}}$  by conjugation, and the quotient is a disjoint union of two open subsets: the *split elements*, those conjugate to something in a split torus, and the *compact elements*, those conjugate to something in a compact torus.

There's a beautiful picture here due in this form to Harish-Chandra, with bits known earlier to Gelfand, Graev, Piatetskii, Shapiro, and Lusztig, called the *philosophy of series*. This says that if  $G$  is a semisimple<sup>33</sup> group over any field, e.g.  $\mathrm{SL}_2(\mathbb{R})$ ,  $\mathrm{SL}_2(\mathbb{C})$ ,  $\mathrm{SL}_2(\mathbb{F}_p)$ , or  $\mathrm{SL}_2(\mathbb{Q}_p)$ , its irreducible representations break up into families called *series*, and these families are labeled by conjugacy classes of maximal tori in  $G$ .<sup>34</sup> For  $\mathrm{SL}_2(\mathbb{R})$ , the two different classes of tori tell us there should be two different families of representations: the family associated to a torus  $T_i$  is “Fourier theory on the torus  $T_i$ ,” i.e. the family is in bijection with  $\widehat{T}_i$  (the characters of  $T_i$ ) up to some Weyl group symmetry. This is a generalization of the statement for compact groups, for which there's only one kind of maximal torus and therefore the highest weight theory is indexed by the characters for the maximal torus.

Heuristically, you could write this as

$$\widehat{G} \cong \coprod_{\text{tori } T_i} \widehat{T}_i / W_i.$$

And the representation theory of tori is easier to understand, even in the noncompact case. For example, over  $\mathbb{R}$ , a connected torus is of the form  $T \cong \mathbb{R}^n \times (S^1)^m$ . The representation theory of  $\mathbb{R}$  is encoded by the Fourier transform, and that of  $S^1$  by Fourier series, so  $\widehat{T}_i \cong (\mathbb{R}^n)^* \times \mathbb{Z}^m$ . In every noncompact direction, there's a continuous family of representations, and in the compact direction, there's a discrete family.

The reason this philosophy of series should be true is simple, but actually producing these representations is extremely difficult. Heuristically, you want to construct a basis for the space of class functions. Geometrically, consider these functions over elements conjugate to each kind of torus. Then, Fourier theory gives you sufficiently many class functions on each component.

<sup>33</sup>More generally,  $G$  can be reductive.

<sup>34</sup>Again, a torus means that, in the corresponding group over the algebraic closure, it's conjugate to a subgroup of diagonal matrices.

**Exercise 33.2.** Play with this in the case  $\mathrm{SL}_2(\mathbb{F}_p)$ . You can determine how many conjugacy classes there are, and therefore how many total representations there have to be, so you will see that one construction gives you most, but not all, of them.

For  $G = \mathrm{SL}_2(\mathbb{R})$ , there will be two series: the principal series and the discrete series.

The defining representation of  $\mathrm{SL}_2(\mathbb{R})$  defines an action of  $\mathrm{SL}_2(\mathbb{R})$  on  $\mathbb{R}^2 \setminus 0$ . This is a transitive action, so  $\mathbb{R}^2 \setminus 0 \cong \mathrm{SL}_2(\mathbb{R})/\mathrm{Stab}(1, 0)$ . This stabilizer is isomorphic to the subgroup  $N = \left\{ \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \right\}$ , which is isomorphic to  $\mathbb{R}$ .

There's another action on this homogeneous space by  $\mathbb{R}^\times \cong \mathbb{Z}/2 \times \mathbb{R}_+$ , from the right, where an  $a \in \mathbb{R}^\times$  acts by  $\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$ . Here, the copy of  $\mathbb{Z}/2$  is called  $M$ , and the copy of  $\mathbb{R}_+$  is called  $A$ . Thus, we can factor the homogeneous space  $\mathbb{RP}^1$  as

$$B = G/MAN = \left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix} \right\}.$$

Before quotienting, we also have  $S^1 = (\mathbb{R}^2 \setminus 0)/A = G/AN$ .

The *Iwasawa decomposition* is  $G = KAN$ , where  $K \cong \mathrm{SO}(2)$  is the group of rotation matrices from before. Thus,  $\mathbb{RP}^1 = K/N$ , and the action is transitive.

Now the line bundle: the map  $\mathbb{R}^2 \setminus 0 \rightarrow \mathbb{RP}^1$  is the tautological line bundle on  $\mathbb{RP}^1$ , minus the zero section, and is denoted  $\mathcal{O}(-1)^\times$ . We'll write the representations of  $\mathrm{SL}_2(\mathbb{R})$  as induced from the character of  $MA = \mathbb{R}^\times$  acting for  $B$ , then induced up to  $G$ .

To understand this, we need the character theory of  $MA$ . The  $A$  part is  $\mathbb{R}$ , written multiplicatively, as

$$A = \left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, a > 0 \right\}$$

send  $a \mapsto e^{s \log a} = a^s$ , which are maps  $\mathbb{R}^\times \rightarrow \mathbb{C}^\times$ , and this makes sense for any  $s \in \mathbb{C}$ . So we'll look at the functions on  $\mathbb{R}^2 \setminus 0$  that transforms under the character  $s$ , i.e. the  $f \in C^\infty(\mathbb{R}^2 \setminus 0)$  such that  $f(x \cdot a) = a^s f(x)$ . We call these functions *homogeneous of degree  $s$* . Geometrically, we've said what such an  $f$  does on each ray emerging from the origin, so  $f$  is determined by what it does to the circle: as a vector space, this space of functions is isomorphic to  $C^\infty(S^1)$ . Fourier series says that there's an isomorphism

$$C^\infty(S^1) \cong \left\{ f = \sum_{n \in \mathbb{Z}} b_n z^n \mid b_n \rightarrow 0 \text{ rapidly} \right\}.$$

The isomorphism comes from sending  $z \mapsto e^{2\pi i \theta}$ . What "rapidly" means is actually unimportant to representation theory, in a sense that Harish-Chandra figured out. As  $\mathrm{SL}_2(\mathbb{R})$  acts on  $\mathbb{R}^2 \setminus 0$ , it acts on these function spaces. This is the analogue of the Borel-Weil construction.

But there's still way too much symmetry for these to be irreducible: we haven't accounted for the action of  $M$ . Its characters are the trivial or sign representation, so the characters of  $MA$  are  $\mathbf{s} = (b, s)$ , where  $b \in \{\text{trivial}, \text{sign}\}$  and  $s \in \mathbb{C}$ . If  $x \in \mathbb{R}^\times = MA$ , then

$$x^{\mathbf{s}} = \begin{cases} |x|^s, & b = \text{trivial}, \\ x^s = \text{sign}(x)|x|^s, & b = \text{sign}. \end{cases}$$

Thus, we can define the *principal series* representations to be

$$V_{\mathbf{s}} = \{f \in C^\infty(\mathbb{R}^2 \setminus 0) \mid f(x \cdot ma) = (-1)^{b(m)} a^s f(x)\}.$$

These are indexed by the characters of  $MA$ , and in general come from  $C^\infty(G/N)$ . Alternatively, you can take the left action of  $G$  on

$$\{f \in C^\infty(G) \mid f(gman) = (-1)^{b(m)} a^s f(g)\}.$$

As a vector space, this is very simple, and has a nice description by Fourier series.

- When  $b$  is trivial, we're looking at  $C^\infty(\mathbb{RP}^1)$ , or even Fourier series

$$f = \sum_{n \in 2\mathbb{Z}} b_n z^n.$$

These are called the *spherical principal series*.

- When  $b$  is the sign representation, we get odd Fourier series, called the *nonspherical principal series*.

**Theorem 33.3.** For any Lie group  $G$ , the principal series representation  $V_{\mathbf{s}}$  is irreducible for generic  $\mathbf{s}$ .

For example, for  $\mathrm{SL}_2(\mathbb{R})$ ,  $V_{\mathbf{s}}$  is irreducible when  $s \notin \mathbb{Z}$ . This is reminiscent of Verma modules, which are irreducible when the weight is not an integer.

**Theorem 33.4** (Casselman’s subrepresentation theorem). *Every admissible irreducible representation of  $G$  appears as a subrepresentation of some principal series representation.*

We can reformulate these, at least for  $\mathrm{SL}_2(\mathbb{R})$ , in geometric terms. Namely,  $V_{\mathbf{s}}$  is the  $s$ -forms when  $b$  is the sign representation, and  $s$ -densities when  $b$  is trivial.

**Definition 33.5.**

- An  $s$ -form is a differential 1-form on  $\mathbb{RP}^1$  whose  $G$ -action is twisted by  $s$ , namely something which looks locally like  $f(x) dx^s$  and such that if  $g \in G$ ,

$$g \cdot (f(x) dx^s) = f(g^{-1}x) d(g^{-1}x)^s = f(g^{-1}x)((g^{-1})')^s dx^s.$$

- An  $s$ -density is the same thing, but for a density:

$$g \cdot (f(x)|dx|^s) = f(g^{-1}x)|d(g^{-1}x)|^s|dx|^s.$$

The idea is that  $\mathcal{O}(n)$  is the space of  $-n/2$ -forms, though one has to make sense of this notion.

Lecture 34.

### Principal series: 4/12/17

David spoke again today. Today we’ll learn about a few more concrete ways to think about principal series.

Let  $H$  be a group and  $\pi: P \rightarrow X$  be a principal  $H$ -bundle. Given a group homomorphism  $\varphi: H \rightarrow K$ , one can define an *associated  $K$ -bundle*  $(P)_K \rightarrow X = P \times_H K$ : we take the quotient of  $P \times K$  by the  $H$ -action on the right for  $P$  and on the left for  $K$ .

For us, let  $H = \mathbb{R}^\times$  and  $K = \mathbb{C}^\times$ , so an  $H$ -principal bundle is a real line bundle and a  $K$ -principal bundle is a complex line bundle. To make complex line bundles out of real ones, we need a homomorphism  $\mathbf{s}: \mathbb{R}^* \rightarrow \mathbb{C}^*$ , and these homomorphisms are the characters of  $\mathbb{R}^*$ . We decomposed  $\mathbf{s} = (b, s)$ , where  $b$  is a character for  $\mathbb{Z}/2$  and  $s$  is a character for  $\mathbb{R}^+$ . If  $b$  is trivial,  $\mathbf{s}(x) = |x|^s$ , and if  $b$  is the sign representation,  $\mathbf{s}(x) = x^s$ .<sup>35</sup> Given a real line bundle  $\mathcal{L} \rightarrow X$  and a character  $\mathbf{s}$ , we’ll call the associated complex line bundle  $\mathcal{L}^{\mathbf{s}}$ . If  $s \in \mathbb{Z}$ ,  $x^s \in \mathbb{R}$  for  $x \in \mathbb{R}$ , so this is raising  $\mathcal{L}$  to a power normally.

**Example 34.1.** Let  $M$  be an  $n$ -manifold and  $\omega_M = \Lambda^n \Omega^1 M$  be the bundle of volume forms, which is a real line bundle on  $M$ . As we mentioned last time,  $\omega_M^{\mathbf{s}}$  is  $s$ -densities if  $b$  is trivial, and  $s$ -volume forms if  $b$  is the sign representation.

An  $s$ -density has notation  $f(x) dV^s$ , which is notation that  $g \cdot (f(x) dV^s) = f(g^{-1}(x))(\det(g^{-1}))^s dV^s$ : the notation specifies how it transforms under the action. The density version is similar, but with  $|\cdot|$  signs. Ideally, both the highbrow and coordinate version make sense, but if one doesn’t feel free to think about the other.

Let  $V_{\mathbf{s}}$  be the space of  $C^\infty$   $\mathbf{s}$ -forms on  $\mathbb{RP}^1$ , i.e.  $C^\infty$  sections of  $(T^*\mathbb{RP}^1)^{\mathbf{s}}$ . You might complain that  $T^*\mathbb{RP}^1$  is trivial, and this vector space doesn’t depend on  $\mathbf{s}$  (it’s just  $C^\infty(S^1)$  as a vector space, or Fourier series on  $S^1$  with rapidly decreasing coefficients), but what’s interesting here is  $V_{\mathbf{s}}$  canonically carries an  $\mathrm{SL}_2(\mathbb{R})$ -action, and this action is nontrivial.  $\mathrm{SL}_2(\mathbb{R})$  acts on  $\mathbb{RP}^1$  by diffeomorphisms (specifically, Möbius transformations), and  $T^*\mathbb{RP}^1$  is not trivial as an equivariant line bundle. Thus, we have two one-parameter families of  $\mathrm{SL}_2(\mathbb{R})$ -representations, one for  $b$  trivial and the other for  $b$  sign.

This  $\mathrm{SL}_2(\mathbb{R})$ -action is  $\mathrm{SL}_2(\mathbb{R})$  acting on  $\mathrm{SL}_2(\mathbb{R})/B$ , where  $B$  is the Borel subgroup of upper triangular matrices. The character  $\mathbf{s}$  passes to a character of  $B$ , and therefore a one-dimensional representation of  $B$ . Then, we’re inducting it up to  $\mathrm{SL}_2(\mathbb{R})$ , where it’s realized as sections of a line bundle. This is called *smooth induction* from  $B$  to  $\mathrm{SL}_2(\mathbb{R})$ .

If  $N$  denotes the subgroup of upper triangular matrices with 1s on the diagonal, we can consider its “opposite”  $N_-$ , the lower triangular matrices with 1s on the diagonal. The Lie algebras  $\mathfrak{n}_-$  and  $\mathfrak{b}$  together span  $\mathfrak{sl}_2(\mathbb{R})$ , and on the level of groups, that means that  $N_-B$  is open (since we only have infinitesimal information), and is called an *open Bruhat cell*. This means that for a generic  $2 \times 2$  matrix, there’s a unique way to make it lower triangular with row operations that don’t permute the rows.

<sup>35</sup>To do this, you should choose the standard branch of the complex logarithm on  $\mathbb{R}^+$ .

Thinking of  $\mathbb{RP}^1 \cong G/B$ , there's a basepoint  $\mathbb{R}(1, 0)$  (the positive  $x$ -axis). Inside  $\mathbb{RP}^1$ ,  $N_- \cdot 0$  is  $B$ : upper triangular matrices are exactly those that preserve the  $x$ -axis. Then,  $N_- \cap B = 1$ , and  $N_- \cong \mathbb{R}$ . This is stereographic projection, exactly the same as for the copy of  $\mathbb{C}$  inside  $\mathbb{CP}^1$ , but it's useful to be concrete about it.

Looking back at the principal series, given an  $F \in V_{\mathbf{s}}$ , we can restrict it to an  $f \in C^\infty(\mathbb{R})$  using this copy of  $\mathbb{R} \subset \mathbb{RP}^1$ . We want to know how this transforms under the  $\mathrm{SL}_2(\mathbb{R})$ -action, given that  $F(gman) = (ma)^{\mathbf{s}} F(g)$ . Concretely, the restriction is

$$f(x) := \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}.$$

Suppose  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , so  $ad - bc = 1$  and  $g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ . Then,  $\mathrm{SL}_2(\mathbb{R})$  acts by fractional linear transformations (which might send finite points to  $\infty$ , however), and  $g^{-1}x = g^{-1}(1, x) = (ax - c)/(-bx + d)$ . The derivative is  $g' = 1/(-bx + d)^2$ , so the action is

$$(g \cdot f)(x) = F\left(g^{-1} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right),$$

and we know what  $g^{-1}$  is, so we need to do some matrix multiplication. Thus we factor the product in the opposite way, with the upper triangular matrix on the right, so we know how it transforms:

$$\begin{aligned} F\left(\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right) &= F\left(\begin{pmatrix} d - bx & -b \\ -c + ax & a \end{pmatrix}\right) \\ &= F\left(\begin{pmatrix} 1 & 0 \\ \frac{ax+c}{-bx+d} & 1 \end{pmatrix} \begin{pmatrix} d - bx & -b \\ 0 & \frac{1}{d-bx} \end{pmatrix}\right) \\ &= F\left(\begin{pmatrix} 1 & 0 \\ \frac{ax+c}{-bx+d} & 1 \end{pmatrix}\right) \cdot (-1)^{\chi(d-bx)} |d - bx|^s. \end{aligned}$$

Here we write  $\mathbf{s} = (\chi, s)$ . The absolute value is annoying, but goes away when we restrict to the spherical principal series (where  $\chi$  is trivial).<sup>36</sup> So the action is

$$f(g^{-1}x)(-1)|(g^{-1})'|^{s/2},$$

so  $f$  is a  $-s/2$ -density (or form, depending on the  $\mathbb{Z}/2$ -character).

On  $\mathbb{RP}^1$ , the tautological bundle is  $\mathcal{O}(-1)$ , and  $\mathcal{O}(2)$  is the 1-forms, so sections of  $\mathcal{O}(-1)$  are  $-1/2$ -forms, and  $V_{\mathbf{s}}$  is the sections of  $\mathcal{O}(-1)^{\mathbf{s}}$ .

One nice thing about this concrete description is that you can describe the concrete action of  $\mathfrak{sl}_2$ : we can describe the actions of  $e$ ,  $f$ , and  $h$  on functions. They'll act as vector fields on  $\mathbb{R}$  on the space of sections  $C^\infty(\mathbb{R})$  (with the  $\mathrm{SL}_2(\mathbb{R})$ -action of  $V_{\mathbf{s}}$ ).  $f$  is in  $\mathfrak{n}_-$ , so it will act by actual translations, i.e. up to a sign act by the vector field  $\frac{\partial}{\partial x}$ . In particular,  $(e^{tf} \cdot \varphi)(x) = \varphi(x - t)$ . Now we want to know the action of  $f$  in the Lie algebra, so take the derivative. This ends up being  $\frac{d}{dx}$ .<sup>37</sup> Then, in a similar way, one can show that the actions of  $e$  and  $h$  are

$$\begin{aligned} (h \cdot) &= 2x \frac{d}{dx} - s \\ (e \cdot) &= x^2 \frac{d}{dx} - sx. \end{aligned}$$

The group  $\mathrm{SL}_2(\mathbb{R})$  acts on all of  $\mathbb{RP}^1$ , and doesn't preserve  $\mathbb{R}$ . But these Lie algebra formulas are perfectly well-defined for an  $\mathbb{R}$ -action, because if  $G$  acts on a space  $X$  and  $U \subset X$  is an open subset, you can't restrict the  $G$ -action, but you can restrict the associated  $\mathfrak{g}$ -action. This is because it's always possible to restrict vector fields to open subsets.

If you run the same story for  $\mathrm{SL}_2(\mathbb{C})$  acting on  $\mathbb{CP}^1$ , you get the same Lie algebra action (for  $s = 0$ ). You can calculate the brackets for these vector fields and you'll get exactly the usual relations of  $\mathfrak{sl}_2$ . Thus, there's an  $\mathfrak{sl}_2$ -action on the space of holomorphic functions on  $\mathbb{C}$ ! For  $\mathrm{SL}_2(\mathbb{R})$ , we instead get a one-parameter family depending on  $s$ , which comes because we're looking at forms/densities, not densities.

<sup>36</sup>Warning: there's a sign error somewhere in this exposition.

<sup>37</sup>For the full calculation, see Pavel Safronov's notes from David's class a few years ago.

The *Casimir element*  $C = 2ef + 2fe + h^2 \in Z(\mathcal{U}(\mathfrak{sl}_2))$  acts by a scalar in any irreducible representation by Schur's lemma. But now we have formulas for these  $\mathfrak{sl}_2$ -actions, and you can check that all the  $\frac{d}{dx}$  terms drop out, and you get  $(s+1)^2 - 1$ .

There's no class Friday, and Sam will be back next week. We'll use the Casimir action to deduce these  $V_s$  are irreducible.

Lecture 35.

### Harish-Chandra modules: 4/17/17

Sam is back today. Today is also his birthday. Happy birthday, Sam!

We've moved into the second part of the course, on noncompact groups. This is a huge subject, so we'll probably be focusing on  $\mathrm{SL}_2(\mathbb{R})$  and  $\mathrm{SL}_2(\mathbb{C})$  as a taste.<sup>38</sup> We've seen some differences between the representation theory of compact and noncompact Lie groups:

- Representations of compact Lie groups are completely reducible (semisimple). This is not true in the non-compact case.
- The irreducible representations of compact Lie groups are finite-dimensional, but we'll construct some interesting infinite-dimensional irreducible representations of noncompact Lie groups.
- All representations of compact Lie groups may be made unitary. This is not true in the infinite-dimensional case.
- The spectrum (set of isomorphism classes of irreducibles) is discrete for a compact Lie group, but may be continuous in general, and even has an interesting measure on it, called the Plancherel measure.

Last week, we constructed representations  $V_{(s,b)}$  for  $\mathrm{SL}_2(\mathbb{R})$ , where  $s \in \mathbb{C}$  and  $b \in \{0, 1\}$  is a character of  $\mathbb{Z}/2$ . Together,  $(s, b)$  index a character for the Borel subgroup  $B$  of upper triangular matrices in  $\mathrm{SL}_2(\mathbb{R})$ . There's a decomposition  $B = MAN$ , where:

- $M$  is the group of matrices  $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ ,
- $A$  is the group of matrices  $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$  for  $a > 0$ , and
- $N$  is the group of matrices  $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$  for  $n \in \mathbb{R}$ .

This may seem like notational overkill, but this is standard notation in representation theory, and generalizes to other groups.

Though we said  $(s, b)$  index a character for  $B$ , it's more precise to say that they specify a character for  $B/N \cong A \times M$ , and this defines a character  $\mathbb{C}_{(s,b)}$  of  $B$ : namely, the action on  $\mathbb{C}$  is  $man \cdot 1 = a^s m^b$ . Then, we defined the (*non-unitary*) *principal series representation*  $V_{(s,b)} := \mathrm{Ind}_B^G(\mathbb{C}_{(s,b)})$ .

There's an explicit realization of this induced representation as a space of functions:

$$\begin{aligned} V_{(s,b)} &= \{f \in C^\infty(\mathrm{SL}_2(\mathbb{R})) \mid f(gman) = m^b a^s f(g)\} \\ &= \{f \in C^\infty(\mathrm{SL}_2(\mathbb{R})/N) \mid f(r, \theta) = r^s f(1, \theta), f(-r, \theta) = (-1)^b f(r, \theta)\} \end{aligned}$$

The use of polar coordinates in the second description is because  $\mathrm{SL}_2(\mathbb{R})/N \cong \mathbb{R}^2 \setminus 0$ . You can also think of  $V_{(s,b)}$  as the space of  $s/2$ -densities (resp.  $s/2$ -forms if  $b = 0$ ) on  $\mathbb{RP}^1$ , i.e. sections of a relevant bundles  $|\Lambda^{\mathrm{top}} T_{\mathbb{RP}^1}^*|^{s/2}$  (resp.  $(\Lambda^{\mathrm{top}} T_{\mathbb{RP}^1}^*)^{s/2}$ ). These are trivial as complex vector bundles, but have interesting  $\mathrm{SL}_2(\mathbb{R})$ -equivariant structures.

By stereographically projecting to an  $\mathbb{R} \subset \mathbb{RP}^1$ , one can begin to understand these representations in coordinates. Namely, the image of  $N_- := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  in  $G/B \cong \mathbb{RP}^1$  is the desired copy of  $\mathbb{R}$ .

Thus, we obtain a Lie algebra representation of  $\mathfrak{sl}_2$  on  $C^\infty(\mathbb{R})$ , where  $f$  acts by  $\partial_x$ ,  $h = 2x\partial_x - s$ , and  $e = x^2\partial_x - sx$ . You can use this to define the Casimir operator  $2ef + 2fe + h^2$ , which acts by a scalar:

$$C = (s+1)^2 - 1.$$

There's a problem that doesn't arise in the compact case: there aren't necessarily weight vectors for  $h$ . We'll work around this by picking a different basis for the representation using the noncompact torus

<sup>38</sup>If we think of  $\mathrm{SL}_2(\mathbb{C})$  as a complex Lie group, and ask for its holomorphic representations, we obtain the representation theory of  $\mathfrak{sl}_2(\mathbb{C})$  and hence  $\mathrm{SU}(2)$ , so we're asking for representations as a real Lie group. This also tells us about the Lorentz group, as  $\mathrm{SL}_2(\mathbb{C}) \cong \mathrm{Spin}^+(3, 1)$ .

$\mathrm{SO}(2) \subset \mathrm{SL}_2(\mathbb{R})$ , then picking a new copy of  $e$ ,  $f$ , and  $h$  inside this representation that integrates to the rotation subgroup rather than the compact torus (diagonal subgroup).

That is, there is an isomorphism  $\mathrm{SL}_2(\mathbb{R}) \cong \mathrm{SU}(1, 1)$ , the group of matrices preserving a Hermitian form of signature  $(1, 1)$  with determinant 1. The copy of  $\mathrm{SO}(2)$  inside  $\mathrm{SL}_2(\mathbb{R})$  is identified with the copy of  $\mathrm{U}(1) \subset \mathrm{SU}(1, 1)$ ; the action of  $\mathrm{SL}_2(\mathbb{R})$  on the upper half-plane  $\mathbb{H}$  by Möbius transformations is mapped to the action of  $\mathrm{SU}(1, 1)$  on the unit disc  $\mathbb{D}$ . This is ultimately because these are both conjugate subgroups of  $\mathrm{SL}_2(\mathbb{C})$ , which acts via Möbius transformations on the entire complex plane: this is no abstract coincidence.

Now we pick the basis

$$H = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad X = \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \quad Y = \frac{1}{2} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}.$$

This basis represents the natural basis for  $V_{(s,b)}$  given by Fourier coefficients: let

$$f_n := r^s e^{2\pi i n \theta},$$

which is in  $V_{(s,b)}$  where  $b \equiv n \pmod{2}$ . Then, we obtain weight spaces:

$$\cdots \begin{array}{c} \xrightarrow{X} \\ \xleftarrow{Y} \end{array} \mathbb{C} \cdot f_{n-2} \begin{array}{c} \xrightarrow{X} \\ \xleftarrow{Y} \end{array} \mathbb{C} \cdot f_n \begin{array}{c} \xrightarrow{X} \\ \xleftarrow{Y} \end{array} \mathbb{C} \cdot f_{n+2} \cdots$$

Here, the actions of  $X$  and  $Y$  send one basis vector to a scalar multiple of another:  $X \cdot f_n = (n - s)/2 \cdot f_{n+2}$  and  $Y \cdot f_n = (-n - s)/2 \cdot f_{n-2}$ . The space  $\mathbb{C} \cdot f_n$  has weight  $n$ .

We'll be able to extract a lot of information from  $V_{(s,b)}$  by looking at this weight decomposition, which is called a Harish-Chandra module associated to  $(s, b)$ .

**Definition 35.1.** Let  $G$  be a semisimple Lie group  $G$  and  $K$  be its maximal compact subgroup.<sup>39</sup> A  $(\mathfrak{g}, K)$ -module or *Harish-Chandra module* is a vector space  $V$  which carries representations of  $\mathfrak{g}$  and  $K$  that are compatible in the following senses:

- (1) the action map  $\mathfrak{g} \otimes V \rightarrow V$  is  $K$ -equivariant (where  $K$  acts on  $\mathfrak{g}$  by the adjoint  $G$ -action restricted to  $K$ ), and
- (2) the derivative of the  $K$ -action on  $V$  agrees with the restriction of the  $\mathfrak{g}$ -action to  $\mathfrak{k}$  (the Lie algebra of  $K$ ).
- (3)  $V$  is completely reducible as a  $K$ -representation.

This is the data of a  $\mathfrak{g}$ -representation that may not be integrable on all of  $G$ , but can be integrated to a group representation on the maximal compact subgroup. It's purely algebraic data that encodes a lot of the somewhat geometric information of a representation of  $G$ .

**Theorem 35.2** (Harish-Chandra). *Given a representation of a reductive Lie group  $G$  on a locally convex topological vector space  $\mathcal{V}$ . Let  $K \subset G$  denote its maximal compact subgroup and*

$$V = \mathcal{V}^{\mathrm{fin}} := \{v \in \mathcal{V} \mid \dim(\mathrm{span}(K \cdot v)) \text{ is finite}\}.$$

*Then,  $\mathcal{V}$  is irreducible (resp. unitary) iff  $V$  is irreducible (resp. unitary).*

Note that  $V$  is a  $(\mathfrak{g}, K)$ -module, not a  $G$ -representation. As an example to keep in mind, suppose  $G = S^1$  and  $\mathcal{V} = C^\infty(S^1)$ . Then,

$$V = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} \cdot e^{2\pi i n \theta},$$

the Fourier series. What's interesting is that when  $\mathcal{V} = C(S^1)$  or  $L^2(S^1)$ , we get the same Harish-Chandra module, and this is the sense in which regularity doesn't matter.

Another example is when  $G = \mathrm{SL}_2(\mathbb{R})$ , where  $\mathcal{V} = V_{(s,b)}$  and  $V$  is its corresponding  $(\mathfrak{sl}_2(\mathbb{R}), \mathrm{SO}(2))$ -module.

Next time, we'll discuss the classification of Harish-Chandra modules. There's a relationship to a certain quiver, and we'll see how this goes. We will also discuss  $\mathrm{SL}_2(\mathbb{C})$  at some point; there's extra structure coming from the complex structure, which leads to the notion of a Harish-Chandra bimodule.

<sup>39</sup>For  $G = \mathrm{SL}_2(\mathbb{R})$ ,  $K = \mathrm{SO}(2)$ . However, it is rare in general for the maximal compact subgroup to be abelian.

Lecture 36.

**Quiver representations for Harish-Chandra module categories: 4/19/17**

Last time, we saw how analytic questions about the principal series of  $\mathrm{SL}_2(\mathbb{R})$ -representations led to the notion of Harish-Chandra modules, or  $(\mathfrak{g}, K)$ -modules. Today, we're going to study these modules completely algebraically with the end goal of classifying them.

Today,  $G$  will always be  $\mathrm{SL}_2(\mathbb{R})$ , so  $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$  and  $K = \mathrm{SO}(2)$ . The center  $Z(\mathfrak{g})$  acts by a fixed scalar on any irreducible representation, and  $Z(\mathfrak{g}) = \mathbb{C}[C]$ , where  $C = 2XY + 2YX + H^2$  is the Casimir operator. It will be useful to study which modules we get when this character is fixed.

Let  $\mathrm{Mod}_{(\mathfrak{g}, K)}^\chi$  denote the category of  $(\mathfrak{g}, K)$ -modules  $M$  where  $Z(\mathfrak{g})$  acts through  $\chi$ , i.e.  $C \cdot m = \chi m$  for all  $m \in M$ . Since  $M$  is a Harish-Chandra module, it has a decomposition into weight spaces: if  $M[n]$  denotes the weight- $n$  subspace, then  $X: M[n] \rightarrow M[n+2]$  and  $Y: M[n] \rightarrow M[n-2]$ . Thus the odd and even weights don't talk to each other at all, so let  $\mathrm{Mod}_{(\mathfrak{g}, K)}^{\chi, b}$  ( $b \in \{0, 1\}$ ) denote the subcategory of  $\mathrm{Mod}_{(\mathfrak{g}, K)}^\chi$  consisting of modules whose weights are equivalent to  $b \bmod 2$ .

Recall that the Casimir operator acts on the principal series representation  $V_{s,b}$  by  $\chi = (s+1)^2 - 1$ . Let  $\lambda := s+1$ , so sometimes people say that  $C+1$  acts by  $\lambda^2$ . This shift-by-one has been very pervasive, and this again comes from taking a square root of the canonical bundle on a flag variety.

Given an  $M \in \mathrm{Mod}_{(\mathfrak{g}, K)}^\lambda$  and a  $v \in M[n]$ , then

$$\begin{aligned} (C+1)v &= \lambda^2 v = (2XY + 2YX + H^2 + 1)v \\ &= (4XY + (H-1)^2)v \\ &= 4XYv + (n-1)^2 v. \end{aligned}$$

Putting this together, we have formulas

$$\begin{aligned} XYv &= \frac{1}{2}((n-1)^2 - \lambda^2)v \\ YXv &= \frac{1}{2}((n+1)^2 - \lambda^2)v. \end{aligned}$$

Suppose  $\lambda \notin \mathbb{Z}$ . Then,  $(n \pm 1)^2 - \lambda^2 \neq 0$ , so these maps are always isomorphisms. In this case, action by  $X$  and by  $Y$  is always an isomorphism, and therefore there's exactly one isomorphism class of irreducible objects in  $\mathrm{Mod}_{(\mathfrak{g}, K)}^{\lambda, b}$ , and therefore this category is isomorphic to  $\mathrm{Vect}_{\mathbb{C}}$ . In particular, it's semisimple.

Suppose instead  $\lambda \in \mathbb{Z}$ , and let  $b \equiv \lambda \bmod 2$ . Then, again  $(n \pm 1)^2 - \lambda^2 \neq 0$ , so  $\mathrm{Mod}_{(\mathfrak{g}, K)}^{\lambda, b} \cong \mathrm{Vect}$  again.

However, if  $\lambda \in \mathbb{Z} \setminus 0$  and  $b \equiv \lambda + 1 \bmod 2$ , then  $XY$  and  $YX$  can act by 0. The picture breaks up into three pieces:

- All of the maps for weights  $n \leq -\lambda - 1$  are isomorphisms.
- All of the maps for weights between  $-\lambda + 1$  and  $\lambda - 1$  are isomorphisms.
- All of the maps for weights  $n \geq \lambda + 1$  are isomorphisms.

So the data of this category is determined by three vector spaces with some maps between them. This is encompassed in the structure of a quiver.<sup>40</sup>

**Definition 36.1.** A *quiver* is a directed graph (which may have more than one edge from  $v_i$  to  $v_j$ , and allows self-loops). Using “quiver” instead of “directed graph” indicates that you're thinking in terms of representation theory.

We can think of a quiver  $Q$  as a category  $\mathbf{Q}$  by appending an identity self-loop at every vertex and compositions of any arrows that make sense.<sup>41</sup> Then, a *quiver representation* of  $Q$  is a functor  $\mathbf{Q} \rightarrow \mathrm{Vect}$ .

<sup>40</sup>A reference for this is David Ben-Zvi's lecture notes from his class on quivers.

<sup>41</sup>Formally, there's a free-forgetful adjunction between directed graphs and small categories, where you recover the directed graph defined by the objects and morphisms a category; then,  $\mathbf{Q}$  is to be the free category on the quiver  $Q$ .



That is, a representation of a quiver is a collection of vector spaces for each vertex and linear maps for each edge. For example, consider the quiver

$$Q = \bullet_{-1} \begin{array}{c} \xrightarrow{p} \\ \xleftarrow{q} \end{array} \bullet_0 \begin{array}{c} \xrightarrow{r} \\ \xleftarrow{s} \end{array} \bullet_1.$$

A representation of  $Q$  is a diagram of vector spaces and linear maps:

$$W_{-1} \begin{array}{c} \xrightarrow{p} \\ \xleftarrow{q} \end{array} W_0 \begin{array}{c} \xrightarrow{r} \\ \xleftarrow{s} \end{array} W_1.$$

Now, looking at the maps between weight spaces for objects in  $\text{Mod}_{(\mathfrak{g}, K)}^{\lambda, b}$  in the last case we've been discussing, they look like a certain class of quiver representations.

**Proposition 36.2.** *There's an equivalence of categories from  $\text{Mod}_{(\mathfrak{g}, K)}^{\lambda, b}$  (when  $b \equiv \lambda + 1 \pmod{2}$  and  $\lambda \in \mathbb{Z} \setminus 0$ ) to the category of representations of  $Q$  such that  $pq = qp = 0$  and  $rs = sr = 0$ .*

There are three irreducible objects of this category:

$$\mathbb{C} \begin{array}{c} \xrightarrow{\quad} 0 \\ \xleftarrow{\quad} \end{array} 0, \quad 0 \begin{array}{c} \xrightarrow{\quad} \mathbb{C} \\ \xleftarrow{\quad} \end{array} 0, \quad 0 \begin{array}{c} \xrightarrow{\quad} 0 \\ \xleftarrow{\quad} \end{array} \mathbb{C}.$$

Notice that this category is not semisimple. For example, the Verma module

$$\mathbb{C} \begin{array}{c} \xrightarrow{0} \\ \xleftarrow{\sim} \end{array} \mathbb{C} \begin{array}{c} \xrightarrow{\quad} 0 \\ \xleftarrow{\quad} \end{array}$$

has the first irreducible as a subobject and the second as a quotient object, but isn't their direct sum.

The  $(\mathfrak{g}, K)$ -modules corresponding to the middle irreducible are the finite-dimensional weight spaces, and those corresponding to the first and third ones are the discrete series, or Verma modules.

Now, let's see what the principal series representation  $V_{(s, b)}$ , where  $\lambda = s + 1$ , looks like in this context. Last time, we wrote down formulas for the action of  $X$  and  $Y$ , and those tell us that we get

$$\mathbb{C} \begin{array}{c} \xrightarrow{\sim} \\ \xleftarrow{0} \end{array} \mathbb{C} \begin{array}{c} \xrightarrow{0} \\ \xleftarrow{\sim} \end{array} \mathbb{C}.$$

So in the other cases, the module category was just  $\mathbf{Vect}$  and the principal series representations were irreducible, but these are a non-semisimple category, and the  $V_{(s, b)}$  are reducible. In particular, there's a discrete series as a quotient and a finite-dimensional representation as a subrepresentation.

We're used to thinking of  $V_{(s, b)}$  as  $C^\infty$ -sections of  $\mathcal{O}(s) \rightarrow \mathbb{RP}^1$ . In this case, the finite-dimensional subrepresentation is the space of polynomial sections, by the Borel-Weil theorem.

Now suppose  $\lambda = 0$  and  $b = 0$ . In this case,  $\text{Mod}_{(\mathfrak{g}, K)}^{\lambda, b} \cong \mathbf{Vect}$ , as  $V_{(-1, 0)}$  is irreducible, and this is semisimple, but when  $\lambda = 0$  and  $b = 1$ , we instead get two pieces, and the category of modules is isomorphic to the category of representations of

$$\bullet \begin{array}{c} \xrightarrow{p} \\ \xleftarrow{q} \end{array} \bullet$$

such that  $pq = qp = 0$ . In this case,  $V_{(-1, 1)}$  corresponds to the representation

$$\mathbb{C} \begin{array}{c} \xrightarrow{0} \\ \xleftarrow{0} \end{array} \mathbb{C} \cong \left( \mathbb{C} \begin{array}{c} \xrightarrow{\quad} 0 \\ \xleftarrow{\quad} \end{array} \right) \oplus \left( 0 \begin{array}{c} \xrightarrow{\quad} \mathbb{C} \\ \xleftarrow{\quad} \end{array} \right).$$

These two pieces are called *limits of the discrete series*.

So this classifies the representations of  $\text{SL}_2(\mathbb{R})$ , more or less. If you want irreducible pieces with different central characters, semisimplicity is going to fail horribly, but we still know what the irreducible objects are.

In particular, every irreducible object of  $\text{Mod}_{(\mathfrak{g}, K)}^{\lambda, b}$  is isomorphic to a subrepresentation of  $V_{(s, b)}$  for some  $s \in \{\pm\lambda - 1\}$ . In general, this is called *Casselman's subrepresentation theorem* for other reductive groups. Another way to say this is (by adjunction/reciprocity) that every irreducible representation has coinvariants for upper triangular matrices.

So now it's a reasonable question to ask: when do these  $\text{SL}_2(\mathbb{R})$ -representations admit a unitary structure? This is known, but doing it for a general reductive group, including describing the measure on the unitary dual and proving the corresponding Plancherel theorem, is largely open. We want something like

$$L^2(G) = \int_{\widehat{G}} V,$$

whatever that means, and this is a desirable goal for doing harmonic analysis on  $G$  and its symmetric spaces. Even in the case of  $\text{SL}_2(\mathbb{R})$ , this is intricate.

Lecture 37.

### Unitary representations of $\text{SL}_2(\mathbb{R})$ : 4/24/17

Last time, we classified the irreducible  $(\mathfrak{g}, K)$ -modules for  $G = \text{SL}_2(\mathbb{R})$ , and discussed their actions on weight spaces. Today, we'll discuss which of these are unitary, i.e. act by unitary operators on a Hilbert space.

For example, any Lie group  $G$  acts unitarily on  $L^2(G)$ ; this representation often isn't irreducible. One motivation for studying unitary representations of  $G$  is harmonic analysis: the analogue of the Peter-Weyl theorem (Theorem 7.2) will be useful. However, not every irreducible representation will be unitary, unlike in the compact case.

Specifically, we'll ask about the principal series representation  $\Gamma(G/B; \mathcal{L}_{s, b})$ , where  $B$  is the Borel subgroup, in this case upper triangular matrices. We want to know when this admits a positive inner product.

**Definition 37.1.** The space of *densities* is  $\text{Dens}(X) = \Gamma(X; |\Lambda^{\text{top}} T^* X|)$ , where the absolute value means to take the absolute values of the transition functions.

Another way to describe this is as sections of the top exterior power (also called the *determinant bundle*  $\text{Det}(T^* X)$ ) tensored with the orientation bundle. You can think of these as functions on  $X$  which can be integrated and whose integrals lie in  $\mathbb{C} \cup \{\infty\}$ , but we haven't picked a measure; a measure chooses a trivialization of  $\text{Dens}(X)$ .

To make this into a Hilbert space, we have to take the square root of  $|\text{Det } X|$ , whose sections form the bundle of *half-densities*  $\text{Dens}^{1/2}(X)$ . Choose a (complex-valued) measure  $\mu$  on  $X$ ; then, there's an inner product on  $\text{Dens}^{1/2}(X)$  given by

$$\langle f_1, f_2 \rangle := \int_X f_1 \overline{f_2} d\mu,$$

and

$$L^2(X; d\mu) = \overline{\{f \in \text{Dens}^{1/2}(X) \mid \langle f, f \rangle \text{ is finite}\}}.$$

For example, let  $H \subset G$  be a Lie subgroup and  $X = G/H$ . Then,  $T^* X$  is a  $G$ -equivariant vector bundle on  $X$ , and  $T_{1H}^* X = (\mathfrak{g}/\mathfrak{h})^*$  is a representation of  $H$ . Since  $\text{Det}((\mathfrak{g}/\mathfrak{h})^*) \otimes \text{Det}(\mathfrak{h}^*) \cong \text{Det}(\mathfrak{g}^*)$ , then we can write

$$|\text{Det}((\mathfrak{g}/\mathfrak{h})^*)| = \frac{|\text{Det}(\mathfrak{g}^*)|}{|\text{Det}(\mathfrak{h}^*)|}.$$

As a  $G$ -representation,  $\Gamma(|\text{Det}(\mathfrak{g}^*)|)$  is called the *modular representation*: that it's one-dimensional says there's a unique Haar measure up to scaling, and the character of this representation  $\Delta_G$ , called the *modular character*, encodes whether it's also right-invariant.

**Proposition 37.2.** *The following are equivalent:*

- (1) *The modular character  $\Delta_G = 1$ .*
- (2)  *$G$  admits a bi-invariant volume form.*

In this case,  $G$  is called *unimodular*.

**Proposition 37.3.** *If  $G$  is semisimple, then it's unimodular.*

The idea is that the modular representation  $G \rightarrow \mathrm{GL}(\mathfrak{g}^*)$  factors through  $\mathrm{O}(\mathfrak{g}^*, K)$ , the subgroup of orthogonal matrices.

However, the Borel subgroup of  $\mathrm{SL}_2(\mathbb{R})$ , the group of upper triangular special unitary matrices, is not unimodular:

$$\Delta_B \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = |a|^{-2}.$$

We'll use this in a technique called unitary induction. Let  $W$  be an  $H$ -representation with a unitary structure, i.e. give  $W$  the structure of a Hilbert space on which  $H$  acts by unitary operators. We want to use induction to produce a unitary  $G$ -representation, but the naïve approach doesn't work.

**Definition 37.4.** The *unitary induction* of a unitary  $H$ -representation  $W$  is

$$\mathrm{UInd}_H^G(W) := \mathrm{Ind}_H^G(W \otimes \Delta_{G/H}^{1/2}).$$

This is naturally a unitary representation. This uses the fact that  $W$  carries an  $H$ -invariant inner product, and the induction of  $W \otimes \Delta_{G/H}^{1/2}$  is the space of sections of the associated bundle  $\mathcal{W} \otimes \mathrm{Dens}_{G/H}^{1/2} \rightarrow G/H$ , where  $\mathcal{W}$  is the vector bundle on  $G/H$  induced by the representation  $W$ . To do this precisely, one must restrict to  $L^2$  sections and then complete, but everything works.

Now let's apply this to  $\mathrm{SL}_2(\mathbb{R})$ .

**Theorem 37.5.** The modular character for the principal series  $B$ -representation  $\mathbb{C}_{s,b}$  is

$$\Delta_B \begin{pmatrix} a & c \\ & a^{-1} \end{pmatrix} = |a|^s \mathrm{sign}(a)^c,$$

and in particular,  $\mathbb{C}_{(s,b)}$  is unitary iff  $s$  is purely imaginary.

In particular, the unitary induction is

$$\mathrm{UInd}_B^G(\mathbb{C}_{(it,b)}) = \mathrm{Ind}_B^G(\mathbb{C}_{(-1+it,b)}) = V_{(-1+it,b)}.$$

So just as in the Weyl integration formula, there's a shift by  $-1$ , and it's there for the same reason, ultimately arising from a square root of the determinant bundle. These  $V_{(s,b)}$  are called the *unitary principal series* of  $\mathrm{SL}_2(\mathbb{R})$ .

These are not the only unitary irreducible  $\mathrm{SL}_2(\mathbb{R})$ -representations.

**Theorem 37.6.** Suppose  $V_{(s,b)}$  admits a unitary structure. Then, either

- (1)  $s = -1 + it$  and  $t \in \mathbb{R}$ , or
- (2)  $-2 < s < 0$ ,

and all of these occur.

We called (1) the unitary principal series, and (2), the exceptional cases, are the *complimentary series*. There's a natural measure on the space of unitary representations called the Plancherel measure, and the complimentary series will have measure zero.

*Proof.* We'll look at  $(\mathfrak{g}, K)$ -modules to prove that these are the only possibilities, but not address why all of them occur.

Suppose  $\lambda = s + 1$  and  $b = 0$ ,  $V_{(s,b)}$  is unitary, and  $x^\dagger$  denotes the adjoint of a Lie algebra element  $x$ . There is a copy of  $\mathfrak{sl}_2(\mathbb{R})$  inside  $\mathfrak{sl}_2(\mathbb{C})$ ; let  $\{e, f, h\}$  be its standard generators. Similarly, there's a copy of  $\mathfrak{su}(1, 1)$  inside  $\mathfrak{sl}_2(\mathbb{C})$ ; let  $\{X, Y, H\}$  be its generators. Then, as operators on  $V$ ,  $H^\dagger = H$  and  $X^\dagger = -Y$ .

Choose  $v_0 \in V[0]$ , and let  $v_{2n} = X^n v_0$ , which is in  $V[2n]$  for any  $n \in \mathbb{Z}$ . Let  $a_n = \|v_n\|$ , which is a positive number, and define  $c_n$  to be the nonzero complex number such that  $Y v_n = c_{n-2} v_{n-2}$ . Then, since  $[X, Y]v_n = H v_n$ ,  $c_{n-2} - c_n = n$ . Thus,

$$a_{n+2}^2 = \langle X v_n, v_{n+2} \rangle = -\langle v_n, Y v_{n+2} \rangle = -\bar{c}_n a_n^2,$$

so  $c_n$  is a negative real number.

Now we check the action of the Casimir operator  $C = 2XY + 2YX + H^2$ :

$$\lambda^2 v_n = (C + 1)v_n = (1 + 2c_{n-2} + 2c_n + n^2)c_n.$$

Thus, for all even  $n$ ,

$$4c_n = \lambda^2 - (n+1)^2 < 0,$$

which leads to the two possibilities in the theorem statement.  $\boxtimes$

Lecture 38.

: 4/26/17

Lecture 39.

**Harmonic analysis on the upper half-plane: 4/28/17**

A huge motivation for representation theory of (not necessarily compact) Lie groups  $G$  is to do harmonic analysis on  $G$  or homogeneous spaces  $G/K$ . More precisely this means studying  $L^2(G)$  or  $L^2(G/K)$  as representations of  $G$ , which is how we started out the whole course.

Let's fix  $G = \mathrm{SL}_2(\mathbb{R})$  and  $G/K = \mathbb{H}$  the upper half-plane.

Recall that in the compact case, we obtained spherical harmonics from the action of  $\mathrm{SO}(3)$  on  $L^2(S^2)$ , which decomposes as a completed direct sum of one copy of each irreducible  $\mathrm{SO}(3)$ -representation, and the Casimir operator  $C \in Z(\mathfrak{sl}_2)$  acts as the spherical Laplacian on  $L^2(S^2)$ .<sup>42</sup>

There's a similar picture for functions on the upper half-plane: once again (suitably normalized) the Casimir operator  $C \in Z(\mathfrak{sl}_2)$  acts as  $C = (-1/4)\Delta_{\mathrm{hyp}}$ , where

$$\Delta_{\mathrm{hyp}} = y^2(\partial_y^2 + \partial_x^2)$$

is the *hyperbolic Laplacian*, the Laplacian operator coming from the hyperbolic metric on  $\mathbb{H}$ . Then, the eigenspaces of  $C$  (or  $\Delta_{\mathrm{hyp}}$ ) are subrepresentations of  $\mathrm{SL}_2(\mathbb{R})$ , and are unitary, since they're contained in  $L^2(\mathbb{H})$ . We'd like to understand the decomposition of  $L^2(\mathbb{H})$  as irreducible representations, and what the possible eigenvalues of the Laplacian are.

**Proposition 39.1.** *As  $\mathrm{SL}_2(\mathbb{R})$ -representations,*

$$L^2(\mathbb{H}) \cong \int_{t \in \mathbb{R}_{\geq 0}} V_{(-1+it, 0)}.$$

This integral means that an  $f \in L^2(\mathbb{H})$  can be expressed as an integral over  $t$  in terms of elements of these unitary principal series, just like in the Fourier inversion formula. In particular, it is not a direct sum. The Plancherel formula could be written in this way as

$$L^2(\mathbb{R}) \cong \int_{\mathbb{R}_{\xi}} \mathbb{C}_{\xi},$$

where  $\mathbb{C}_{\xi}$  is the irreducible representation of  $\mathbb{R}$  with character  $\xi$ .

The action of the Laplacian on  $V_{(s,0)}$  is by  $(-s/2)(1+s/2) = (t^2+1)/4$ . This is a positive real eigenvalue of at least  $1/4$ , a phenomenon called the *spectral gap*. However, the action of the Laplacian on the complimentary series would precisely fill the spectral gap, but this series does not appear in  $L^2(\mathbb{H})$ .

For any given  $t$ ,  $V_{-1+it} \cong L^2(\mathbb{RP}^1)$ , and this admits maps to and from  $L^2(\mathbb{H})$ . This is called the *Poisson transform*, e.g. if you think of the hyperbolic plane as a disc, the boundary value is a function on  $\mathbb{RP}^1$ . These maps are not inverses of each other, though one is a section of the other, which is a projection.

Another thing you can ask is about the *intertwiners* of  $L^2(\mathbb{H})$ , i.e. the operators  $A: L^2(\mathbb{H}) \rightarrow L^2(\mathbb{H})$  commuting with the action of  $\mathrm{SL}_2(\mathbb{R})$ . Thinking of  $\mathbb{H} = G/K$ , the intertwiners act on the right by the algebra  $C_c(K \backslash G/K)$ , the  $K$ -biinvariant functions in the convolution algebra of  $G$ .<sup>43</sup> One surprise is that this is a commutative algebra!

There's a classification of pairs  $(G, K)$  for which  $C_c(K \backslash G/K)$  is a commutative algebra; such a pair  $(G, K)$  is called a *Gelfand pair*. There's a generalization of this to the action of a monoidal category of sheaves in the geometric Langlands program, and in this case it happens to be symmetric monoidal.

Were we to go down this route, we'd next look at discrete or arithmetic subgroups of  $G$ , e.g.  $\mathrm{SL}_2(\mathbb{Z})$ , or  $\pi_1(\Sigma_g)$ , where  $\Sigma_g$  is a Riemann surface of genus at least 2. As  $\Sigma_g$  is uniformized by the upper half-plane, we can take the quotient  $\mathrm{SL}_2(\mathbb{R})/\pi_1(\Sigma_g)$ . This leads to cool things like the theory of modular forms.

<sup>42</sup>There's some regularity issue here that we're going to not worry about today, e.g. using tempered distributions. Analogues of the Plancherel theorem mean that it does make sense in  $L^2$ , however.

<sup>43</sup>This is an example of a *Hecke algebra*, though "the" Hecke algebra (or the *spherical Hecke algebra*) is the one where  $G = \mathrm{SL}_2(\mathbb{Q}_p)$  and  $K = \mathrm{SL}_2(\mathbb{Z}_p)$ .

If you more generally think about a reductive Lie group  $G$ , this quickly becomes the Langlands program.

~ . ~

Let  $G$  be a reductive Lie group (possibly with additional hypotheses). Fix a maximal compact subgroup  $K \subset G$ , e.g.  $\mathrm{SO}(2) \subset \mathrm{SL}_2(\mathbb{R})$ . In general, it'll be nonabelian. We mentioned that, very roughly, you can go between the theory of representations of  $G$  on a Hilbert space (unitary or not) and  $(\mathfrak{g}, K)$ -modules. We discussed the passage in the forward direction, sending  $V$  to its  $K$ -finite vectors, and we may need to restrict to admissible  $G$ -representations to avoid pathologies (e.g. requiring that every irreducible representation of  $K$  appears only finitely many times in  $G$ ).

Going backwards, producing a representation from a Harish-Chandra module, is more interesting: there are multiple ways to go back, and they sometimes give you different results. But if you're interested in unitary representations, there's a strong uniqueness result. So: can we classify all admissible representations? How about all unitary ones? Or tempered ones?

These three questions have very different answers. Classifying the unitary representations in general is open. The classification of tempered representations is work of Harish-Chandra from 1960. The *Langlands classification* of all admissible representations uses the Harish-Chandra representation.

Recall that for  $\mathrm{SL}_2(\mathbb{R})$ , the subgroup of diagonal matrices  $H$  is a copy of  $\mathbb{R}^\times$ , and the maximal compact subgroup is  $T = \mathrm{SO}(2)$ . Then we had a decomposition

$$\begin{aligned}\widehat{\mathrm{SL}}_2(\mathbb{R})_{\mathrm{temp}} &= \widehat{H}/W \amalg \widehat{T} \\ &\cong (\mathbb{R} \times \mathbb{Z}/2)/\mathbb{Z}/2 \amalg \mathbb{Z}.\end{aligned}$$

In general, there will be a disjoint union across all of the Cartan subgroups, modulo some finite group symmetries.

*Remark.* A lot of this was originally studied by physicists, e.g. Wigner. Harish-Chandra was a student of Dirac. It was only later the connections to number theory were discovered. ◀

Physicists also care about the *Lorentz group*  $\mathrm{SL}_2(\mathbb{C}) \cong \mathrm{Spin}^+(1, 3)$ . There's only one conjugacy class of Cartan subgroup  $H \cong \mathbb{C}^\times$ , the diagonal matrices. As a real Lie group, this looks like  $\mathrm{U}(1) \times \mathbb{R}_{\geq 0}$ , so

$$\widehat{\mathrm{SL}}_2(\mathbb{C}) = \widehat{H}/(\mathbb{Z}/2) \cong (\mathbb{R} \times \mathbb{Z})/(\mathbb{Z}/2).$$

The Langlands classification of admissible representations expresses them in terms of *Levi subgroups*: the decomposition is the tempered representations, and irreducible quotients of principal series representations.

— Lecture 40. —

### Localization: 5/1/17

We've already seen examples of obtaining representations of Lie groups through their actions on manifolds, e.g.  $\mathrm{SL}_2(\mathbb{R})$  acting on  $\mathbb{RP}^1$ , or more generally  $G$  acting on  $G/B$ . The character  $\lambda: B \rightarrow \mathbb{C}^\times$  corresponds to an equivariant line bundle  $\mathcal{L}_\lambda$  over  $G/B$ . From this we can obtain some spaces of representations:  $G$  acts on  $\Gamma(G/B; \mathcal{L}_\lambda)$ , and differentiating,  $\mathfrak{g}$  acts on  $\Gamma(G/B; \mathcal{L}_\lambda)$  by differential operators. Therefore it extends to an action  $\mathcal{U}(\mathfrak{g}) \rightarrow \mathrm{Diff}(\mathcal{L}_\lambda, \mathcal{L}_\lambda)$ , the differential operators (e.g. derivators on the trivial bundle). But it makes sense to consider a differential operator from  $\mathcal{L}_\lambda$  to itself restricted to subsets of  $G/B$ , and so this defines a sheaf of algebras  $\mathcal{D}_{G/B}^\lambda$ , whose value on an open  $U$  is  $\mathrm{Diff}(\mathcal{L}_\lambda|_U, \mathcal{L}_\lambda|_U)$ .

Let  $\mathcal{M}$  be a  $\mathcal{D}_{G/B}^\lambda$ -module; then, taking global sections defines a  $\mathfrak{g}$ -module  $\Gamma(G/B, \mathcal{M})$ . This has a left adjoint called *localization*, which takes a  $\mathfrak{g}$ -module  $M$  to the  $\mathcal{D}_{G/B}^\lambda$ -module  $\mathrm{Loc}(M) := \mathcal{D}_{G/B}^\lambda \otimes_{\mathcal{U}(\mathfrak{g})} M$ .

This harkens back to the definitions of quasicohherent and coherent sheaves in algebraic geometry: the left adjoint to taking global sections of a (quasi)coherent sheaf on a variety defines a way to turn modules over  $\Gamma(X, \mathcal{O}_X)$  into (quasi)coherent sheaves using localization of rings. These are equivalences of categories iff  $X$  is affine. When is the corresponding representation-theoretic localization an equivalence?

**$\mathcal{D}$ -modules.** Though  $\mathcal{D}$ -modules are a notion from algebraic geometry, we'll try to keep the algebro-geometric content relatively light.

Let  $X$  be a complex manifold or smooth algebraic variety, and let  $\mathcal{O}_X$  be its sheaf of algebraic functions.

**Definition 40.1.** The *sheaf of differential operators* on  $X$  is the subsheaf of rings  $\mathcal{D}_X \subseteq \text{End}_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$  generated by

- $\mathcal{O}_X$ , acting on itself by multiplication, and
- (the sheaf of) vector fields on  $X$ , acting by derivation.

A  $\mathcal{D}$ -module on  $X$  is a  $\mathcal{D}_X$ -module.

Sometimes, to simplify the passage between algebra and geometry, we'll assume  $X$  is affine.

**Example 40.2.** When  $X = \mathbb{A}^1$ ,  $\mathcal{D}_X$  is called the *ring of differential operators in one variable* or the *Weyl algebra*. Then, functions are polynomials in  $x$ , and vector fields are generated by  $\partial_x$ . This operator satisfies the *Leibniz rule*:

$$\partial_x(x \cdot f) - x\partial_x(f) = f,$$

and therefore an explicit presentation for the Weyl algebra is

$$\mathcal{D}_{\mathbb{A}^1} = \mathbb{C}\langle x, \partial_x \rangle / [\partial_x, x] = 1.$$

This arises in many situations in mathematics and physics.

One simple example of a  $\mathcal{D}_{\mathbb{A}^1}$ -modules is  $\mathbb{C}[x]$ , where  $\partial_x$  acts trivially. But you can also encode differential equations: let  $f \in C^\infty(\mathbb{C})$ ; then, there's a  $\mathcal{D}_{\mathbb{A}^1}$ -module  $\mathcal{D}_{\mathbb{A}^1} \cdot f(x)$  which encodes the differential equation  $f$  satisfies. For example,  $\mathcal{D}_{\mathbb{A}^1} \cdot e^{\lambda x} \cong \mathcal{D}_{\mathbb{A}^1} / \mathcal{D}_{\mathbb{A}^1} \cdot (\partial_x - \lambda)$ . In this way,  $\mathcal{D}_{\mathbb{A}^1}$ -modules are stand-ins for transcendental functions which  $\mathcal{O}_X$  doesn't naturally see. ◀

**Example 40.3.** Let  $G$  be a (complex) Lie group. On  $\mathbb{A}^1$ , we took advantage of the existence of a global translation  $x$ , which doesn't exist in general on a complex manifold. But on a Lie group, we do have one, so there's a decomposition

$$\mathcal{D}_G \cong \mathcal{O}_G \# \mathcal{U}(\mathfrak{g}),$$

and  $\mathcal{U}(\mathfrak{g}) \cong (\mathcal{D}_G)^G$  and  $\#$  is a twisted tensor product, which as a vector space is the ordinary tensor product, but whose ring structure encodes the Leibniz rule. This expresses the notion that any differential operator is a function times a left-invariant differential operator. ◀

But we want to do this for the flag variety, which is not affine. Let's focus on  $\mathbb{P}^1 = G/B$  for  $G = \text{SL}_2(\mathbb{C})$ . Inside  $\mathbb{P}^1$ , there's a copy of  $\mathbb{A}^1$  with coordinate  $x$  plus a point at infinity.

The vector fields on  $\mathbb{P}^1$  are spanned by  $\partial_x$ ,  $x\partial_x$ , and  $x^2\partial_x$ , and as a sheaf  $\mathcal{Vect}(\mathbb{P}^1) \cong \mathcal{O}(2)$ . This comes from a map  $\mathfrak{sl}_2 \rightarrow \mathcal{Vect}(\mathbb{P}^1)$  sending  $f \mapsto -\partial_x$ ,  $e \mapsto -2x\partial_x$ , and  $h \mapsto -x^2\partial_x$  coming from the quotient action of  $\mathfrak{sl}_2$  on  $\text{SL}_2(\mathbb{C})/B$ .

Let  $y = 1/x$ , so that  $x$  and  $y$  are together an atlas for  $\mathbb{P}^1$ . Then,  $\partial_y(x) = \partial_y(1/y) = -y^{-2} = -x^2$ , so  $\partial_y = -x^2\partial_x$ .

The map  $\mathfrak{sl}_2 \rightarrow \mathcal{Vect}(\mathbb{P}^1)$  does not tell us everything: there's a relation that appears. We've seen that  $Z(\mathfrak{g})$  acts on  $\Gamma(G/B, \mathcal{L}_\lambda)$  by a character  $\chi_\lambda: Z(\mathfrak{g}) \rightarrow \mathbb{C}$ , and in the  $\text{SL}_2(\mathbb{C})$  example, this is  $2ef + 2fe + h^2$ . Therefore this element goes to 0, so

$$\Gamma(\mathbb{P}^1, \mathcal{D}_{\mathbb{P}^1}) = \mathcal{U}(\mathfrak{sl}_2) / (\mathcal{U}(\mathfrak{sl}_2) \cdot (2ef + 2fe + h^2)).$$

If we took twisted differential operators, i.e. those acting on the line bundle  $\mathcal{L}_\lambda$  instead of on  $\mathcal{O}_{\mathbb{P}^1}$ , then we would have quotiented out by  $\chi_\lambda$  instead, whose formula we computed.

So we've computed global sections; what about the rest of the sheaf? To what extent is a sheaf on  $\mathbb{P}^1$  determined by its global sections? This was addressed independently by Beilinson-Bernstein and Brylinski-Kashiwara on the way to proving the Kazhdan-Lusztig conjecture.

**Theorem 40.4** (Beilinson-Bernstein, Brylinski-Kashiwara). *Let  $G$  be a complex reductive group and  $\lambda \in \mathfrak{h}^*$  be a regular dominant weight.<sup>44</sup> Then, there is an adjoint equivalence of categories between  $\text{Mod}_{\mathcal{U}(\mathfrak{g})}^\lambda$  (i.e. the  $\mathcal{U}(\mathfrak{g})$ -modules where  $Z(\mathfrak{g})$  acts by  $\chi_\lambda$ ) and  $\text{Mod}_{\mathcal{D}_{G/B}^\lambda}^\lambda$  given by  $(\text{Loc}, \Gamma)$ .*

We won't give the proof, but there are two important steps.

- (1) First, it's important to verify that  $\Gamma$  is exact. That is, when  $\lambda$  is a regular dominant weight,  $\mathcal{D}_{G/B}^\lambda$ -modules can have no higher cohomology. This places severe restrictions on what these sheaves can be.

<sup>44</sup>This means that  $|w \cdot \lambda| = |w|$  under the dot action, for all  $w \in W$ . Choosing the dot action means that  $\lambda = 0$  is a regular dominant weight.

(2) Second, one must show  $\Gamma$  is faithful, and hence that  $\mathcal{D}_{G/B}^\lambda$ -modules without global sections are trivial. A variety  $X$  such that  $\Gamma$  on  $X$  is exact and faithful is called  *$\mathcal{D}$ -affine*, by analogy with quasicoherent sheaves.

**Example 40.5.** Let's look at  $\mathrm{SL}_2$  and  $\mathbb{P}^1$  again. Here,  $\rho = -1$ . When  $\lambda \in \mathbb{Z}$ , there are actual holomorphic line bundles on  $\mathbb{P}^1$  which correspond to  $\mathcal{D}_{\mathbb{P}^1}^\lambda$  — but in other cases, the line bundle doesn't quite exist, but the differential operators still make sense.

Part of the Borel-Weil-Bott theorem was about these line bundles:  $\mathcal{O}(0) = \mathcal{O}_{\mathbb{P}^1}$ ,  $\mathcal{O}(1)$ ,  $\mathcal{O}(2)$ , and so on.  $\mathcal{O}(n)$  is regular dominant for  $n \geq 0$ , singular for  $n = -1$ , and anti-dominant for  $n \leq -2$ . In particular, looking at cohomology,  $H^*(\mathbb{P}^1; \mathcal{O}(n))$  is concentrated in degree 0 if regular dominant, 1 if regular anti-dominant, and trivial for  $n = -1$ . ◀

There's a generalization of Theorem 40.4 which takes the derived version of global sections, namely cohomology. In this case one can generalize to regular anti-dominant line bundles, but it will never be possible to capture the singular bundle.