M392C NOTES: SYMPLECTIC TOPOLOGY

ARUN DEBRAY
AUGUST 24, 2016

These notes were taken in UT Austin's M392C (Symplectic Topology) class in Fall 2016, taught by Robert Gompf. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

Contents

Lecture 1.

Symplectic Vector Spaces: 8/24/16

Here are a few references for this class.

- There's a book by McDuff and Salaman; in fact, there are three considerably different editions, but all are useful.
- The book by ABKLR (Aebischer, Borer, Kalin, Leuenberger, and Reimann).
- Finally, the book by the professor and Stipsicz will be useful for some parts.

As an overview, symplectic topology is the study of symplectic manifolds.

Definition 1.1. A *symplectic manifold* is a manifold together with a symplectic form.

We'll define symplectic forms in a moment, but first explain where this field arose from. One one hand, symplectic forms arise naturally from mathematical physics in the Hamiltonian formulation, and these days also appear in quantum field theory. On the other, algebraic and complex geometers found that Kähler manifolds naturally have a symplectic structure.

Intuitively, a symplectic manifold is akin to a constant-curvature Riemannian manifold, but where the symmetric bilinear form is replaced with a skew-symmetric bilinear form. (If you don't know what a Riemannian manifold is, that's okay; it will not be a prerequisite for this class.) The constant-curvature condition means that any two points have isomorphic local neighborhoods, so all questions are global; similarly, we will impose a condition on symplectic manifolds that ensures that all questions about symplectic manifolds are global.

There's also a field called symplectic geometry; it differs from symplectic topology in, among other things, also looking at local questions. But there's a reason there's no such thing as "Riemannian topology:" a Riemannian structure is very rigid, and so cutting and pasting Riemannian manifolds, especially constant-curvature ones, isn't fruitful. But symplectic manifolds have a flexibility that allows cutting and pasting to work, if you're clever. To understand this, we will have to spend a little time understanding the local structure.

Another analogy, this time with three-manifolds, is Thurston's geometrization conjecture (now a theorem, thanks to Perelman). This states that any three-manifold may be cut along sphere and tori into pieces that have natural geometry, and are almost always have constant negative curvature, hence are *hyperbolic*, so three-manifold topologists have to understand hyperbolic geometry. Symplectic topology is the analogue in the world of four-manifolds. Not all four-manifolds have symplectic structures; in fact, there exist smooth four-manifolds that are homeomorphic, but one admits a symplectic structure and the other doesn't, so they're not diffeomorphic. We've classified topological four-manifolds, but not smooth ones, so symplectic topology is a very useful tool for this. Three-manifold topologists might also care about the three-manifolds that are boundaries of four-manifolds: if the four-manifold is symplectic, its boundary has a natural structure as a *contact manifold*. The professor plans to teach a course on contact manifolds in a year.

There's a basic principle in geometry and analysis that, in order to understand nonlinear things, one first must understand linear things. Before you understand multivariable integration, you will study linear algebra and the determinant. Before understanding Riemannian geometry, you will learn about inner product spaces. In the same way, we begin with symplectic vector spaces.

Definition 1.2. A *symplectic vector space* is a finite-dimensional[1] real vector space V together with a skew-symmetric bilinear form ω that is *nondegenerate*, i.e. if $v \in V$ is such that for all $w \in V$, $\omega(v, w) = 0$, then $v = 0$.

Succinctly, nondegeneracy means every nonzero vector pairs nontrivially with something. This is a very similar condition to the ones imposed for inner product spaces as well as the indefinite forms attached to spaces in relativity theory.

Example 1.3. Our prototypical example is $\mathbb{C}^n = \mathbb{R}^{2n}$ as a real vector space. The standard inner product is the dot product $\langle -, - \rangle$; we'll define $\omega(v, w) = (iv, w)$. This is clearly still real bilinear; let's verify this is a symplectic form.

First, why is it skew-symmetric? $\omega(w, v) = \langle iw, v \rangle = \langle v, iw \rangle$. Since multiplication by i is orthogonal (it's a rotation), then it preserves the inner product, so $\langle v, iw \rangle = \langle iv, i^2 w \rangle = -\langle iv, w \rangle = -\omega v, w$, so ω is skew-symmetric.

Nondegeneracy is simple: any $v \neq 0$ has a $w \neq 0$ such that $\langle v, w \rangle \neq 0$, so $\omega v, iw = \langle iv, iw \rangle = \langle v, w \rangle \neq 0$.

If we take the standard complex basis $e_1, \dots, e_n$ for $\mathbb{C}^n$, let $f_j = ie_j$; then, $(e_1, f_1, \dots, e_n, f_n)$ is a real basis for $\mathbb{C}^n$; we will take this to be the standard basis for $\mathbb{C}^n$ as a symplectic vector space. This is because each (e_i, f_i) is a real basis for a $\mathbb{C}^1$ summand corresponding to the usual basis $(1, i)$ for $\mathbb{C}$, so this basis jives with the decomposition $\mathbb{C}^n = \mathbb{C} \oplus \mathbb{C} \oplus \cdots \oplus \mathbb{C}$.

This is a positively oriented basis, and in fact is consistent with the canonical orientation of a complex vector space, because it arises in an orientation-preserving way from the basis $(1, i)$ for $\mathbb{C}$, which defines the canonical orientation.

This basis defines a dual basis $e_1^*, f_1^*, \dots, e_n^*, f_n^*$ for the dual space $(\mathbb{R}^{2n})^*$.[2] This allows us to calculate ω in coordinates:

$$\omega = \sum_{j=1}^{n} e_j^* \wedge f_j^*. \tag{1.4}$$

Thus, $\omega(e_j, f_j) = 1 = -\omega(f_j, e_j)$, and all other pairs of basis vectors are orthogonal (evaluate to 0). This defines the same form ω because they agree on the standard basis, because $f_j = ie_j$ and $e_1, \dots, e_n$ is an orthonormal basis for the inner product.

The analogue to an orthonormal basis for a symplectic vector space is a symplectic basis, where elements come in pairs.

Definition 1.5. If $(e_1, f_1, \dots, e_n, f_n)$ is a basis for a symplectic vector space (V, ω) such that $\omega(e_j, f_j) = 1 = -\omega(f_j, e_j)$ for all j and all other pairs of basis vectors are orthogonal, then the basis is called a *symplectic basis*.

Recall that we also have a Hermitian inner product on $\mathbb{C}^n$, defined by

$$h(v, w) = \sum_{j=1}^{n} \overline{v}_j w_j.$$

This is bilinear over $\mathbb{R}$, but not over $\mathbb{C}$; it's $\mathbb{C}$-linear in the second coordinate, but conjugate linear in the first. The Hermitian analogue of the symmetry of an inner product (or the skew-symmetry of a symplectic form) is $h(w, v) = \overline{h(v, w)}$. Thus, $\operatorname{Re} h$ is symmetric, and $\operatorname{Im} h$ is skew-symmetric: $\operatorname{Re} h$ is the standard real inner product on $\mathbb{C}^n = \mathbb{R}^{2n}$, and $\operatorname{Im} h$ is the symplectic form ω.[3]

Example 1.6. As a special case of the previous example, $\mathbb{C}^1 = \mathbb{R}^2$ as a symplectic vector space has ω as the usual (positive) area form: $\omega = e \wedge f = \mathrm{d}x \wedge \mathrm{d}y$.

[1] One can talk about infinite-dimensional symplectic vector spaces, and there are useful in some contexts, but all of our symplectic vector spaces will be finite-dimensional.

[2] Recall that if V is a finite-dimensional real vector space, its *dual space* is V^*, the space of linear functions from V to $\mathbb{R}$. A basis $e_1, \dots, e_n$ of V induces a basis $e_1^*, \dots, e_n^*$ of V^*, defined by $e_j^*(e_i) = \delta_{ij}$: 1 if i and j agree, and 0 otherwise.

[3] Some authors reverse the order for h, so that it's $\mathbb{C}$-linear in the first coordinate but not the second; in this case, we'd get $\operatorname{Im} h = -\omega$. There are a lot of minus signs floating around in symplectic topology, and different authors place them in different places.

Suppose (V, ω_V) and (W, ω_W) are symplectic vector spaces; then, their direct product (or equivalently, direct sum) $V \times W$ has a symplectic structure defined by

$$\omega_{V\times W} = \pi_1^*\omega_V + \pi_2^*\omega_W,$$

where $\pi_1 : V \times W \twoheadrightarrow V$ and $\pi_2 : V \times W \twoheadrightarrow W$ are projections onto the first and second coordinates, respectively. This is a linear combination of skew-symmetric forms, hence is skew-symmetric, and if $\omega_{V\times W}(u_1, u_2) = 0$ for all $u_2 \in V \times W$, then $\pi_1 u_1 = 0$ and $\pi_2 u_1 = 0$, so $u_1 = 0$. Thus, $(V \times W, \omega_{V\times W})$ has a symplectic structure, called the *symplectic orthogonal sum* of V and W since V and W are orthogonal in it.

Not only is $\mathbb{C}^n$ the direct sum of n copies of $\mathbb{C}$, but also the standard symplectic structure on $\mathbb{C}^n$ is the symplectic orthogonal sum of n copies of the standard structure on $\mathbb{C}$: (1.4) explicitly realizes ω as a sum of pullbacks of area forms. The complex structures fit together, the orientations fit together, and the symplectic structures fit together, all nicely.

Subspaces.

Definition 1.7. Suppose V is a symplectic vector space and $W \subset V$ is a subspace. Then, its *orthogonal complement* is the subspace $W^\perp = \{v \in V \mid \omega(v, w) = 0 \text{ for all } w \in W\}$.

The definition is familiar from inner product spaces, but there are a few major differences in what happens afterwards. The first theorem is the same, though:

Theorem 1.8. *If W is a subspace of a symplectic vector space V, then* $\dim W + \dim W^\perp = \dim V$.

Proof. There's a linear map $\varphi : V \longrightarrow V^*$ assigning v to the linear transformation $\varphi(v) : V \longrightarrow \mathbb{R}$ that sends $w \mapsto \omega(v, w)$. Since ω is nondegenerate, then φ is injective. Since V and V^* have the same dimension, φ is an isomorphism. The image $\varphi(W^\perp)$ is the space of functions in V^* that vanish on W, which is isomorphic to the space of functions on V/W, i.e. $\varphi : W^\perp \longrightarrow (V/W)^*$ is injective, and in fact an isomorphism: any function on V/W lifts to a function on V vanishing on W, and then can be pulled back by φ into W. Thus, $\dim W^\perp = \dim(V/W)^* = \dim(V/W) = \operatorname{codim} W$. ⊠

The above proof also works for symmetric nondegenerate bilinear forms. What's different is that W and $W^\perp$ do not always sum to V in the symplectic case. In particular, every vector is orthogonal to itself.