

M392C NOTES: A COURSE ON SEIBERG-WITTEN THEORY AND 4-MANIFOLD TOPOLOGY

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These notes were taken in UT Austin's M392C (A course on Seiberg-Witten theory and 4-manifold topology) class in Spring 2016, taught by Tim Perutz. I live-TeXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Any mistakes in the notes are my own.

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Lecture 1.

Classification problems in differential topology: 1/18/18

"This is my opinion, but it's the only reasonable opinion on this topic."

This course will be on gauge theory; specifically, it will be about Seiberg-Witten theory and its applications to the topology of 4-manifolds. The course website is <https://www.ma.utexas.edu/users/perutz/GaugeTheory.html>; consult it for the syllabus, assignments, etc.

The greatest mystery in geometric topology is: *what is the classification of smooth, compact, simply-connected four-manifolds up to diffeomorphism?* The question is wide open, and the theory behaves very differently than the theory in any other dimension.

There's a fascinating bit of partial information known, mostly via PDEs coming from gauge theory, e.g. the *instanton equation* $F_A^+ = 0$ as studied by Donaldson, Uhlenbeck, Taubes, and others. More recently, people have also studied the *Seiberg-Witten equations*

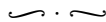
$$(1.1a) \quad D_A \psi = 0$$

$$(1.1b) \quad \rho(F_A^+) = (\psi \otimes \psi^*)_0.$$

Even without defining all of this notation, it's evident that the Seiberg-Witten equations are more complicated than the instanton equation, and indeed they were discovered later, by Seiberg and Witten in 1994. However, they're much easier to work with — after their discovery, the results of Donaldson theory were quickly reproven, and more results were found, within the decade after their discovery. This course will focus on results from Seiberg-Witten theory.

In some sense, this is a closed chapter: the stream of results on 4-manifolds has slowed to a trickle. But Seiberg-Witten theory has in the meantime found new applications to 3-manifolds, contact topology (including the remarkable proof of the Weinstein conjecture by Taubes), knots, high-dimensional topology,

Heegaard-Floer homology, and more. Throughout this constellation of applications, there are many results whose only known proofs use the Seiberg-Witten equations.



The central problem in differential topology is to classify manifolds up to diffeomorphism. To make the problem more tractable, let's restrict to smooth, compact, and boundaryless. An ideal solution would solve the following four problems for some class of manifolds (e.g. compact of a particular dimension, and maybe with some topological constraints).

- (1) Write down a set of “standard manifolds” $\{X_i\}_{i \in I}$ such that each manifold is diffeomorphic to precisely one X_i . For example, a list of diffeomorphism classes of closed oriented connected surfaces is given by the sphere and the n -holed torus for all $n \geq 0$.
- (2) Given a description of a manifold M , a way to compute invariants to decide for which $i \in I$ $M \cong X_i$. For example, if M is a closed, connected, oriented surface, we can completely classify it by its Euler characteristic.

A variant of this problem asks for an explicit algorithm to do this when M is encoded with finite information, e.g. a solution set to polynomial equations in \mathbb{R}^N with rational coefficients.

- (3) Given M and M' , compute invariants to decide whether M is diffeomorphic to M' ; once again, there's an algorithmic variant to that problem.
- (4) Understand families (fiber bundles) of manifolds diffeomorphic to M . In some sense, this means understanding the homotopy type of the topological group $\text{Diff}(M)$ of self-diffeomorphisms of M .

This is an ambitious request, but much is known in low dimensions. In dimension 1, the first three questions are trivial, and the last is nontrivial, but solved.

Example 1.2. For compact, orientable, connected surfaces, we have a complete solution: a list of diffeomorphism classes is the sphere and $(T^2)^{\#g}$ for all $g \geq 0$, and the Euler characteristic $\chi := 2 - 2g$ is a complete invariant which is algorithmically computable from any reasonable input data, solving the second and third questions. Here, “reasonable input data” could include a triangulation, a good atlas (meaning nonempty intersections are contractible), or monodromy data for holomorphic a branched covering map $\Sigma \rightarrow S^2$, where here we're thinking of surfaces as Riemann surfaces, with chosen complex structures. Here, the Riemann-Hurwitz formula can be used to compute the Euler characteristic.

For the fourth question, let $\text{Diff}^+(\Sigma)$ denote the topological group of orientation-preserving self-diffeomorphisms of Σ .

Theorem 1.3 (Earle-Eells).

- The inclusion $\text{SO}_3 \hookrightarrow \text{Diff}^+(S^2)$ is a homotopy equivalence.
- The identification $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$ defines a map $T^2 \hookrightarrow \text{Diff}^+(T^2)$ as translations; this map is a homotopy equivalence into the connected component of the identity in $\text{Diff}^+(T^2)$, and $\pi_0 \text{Diff}^+(T^2) \cong \text{SL}_2(\mathbb{Z})$.
- If $g > 1$, every connected component of $\text{Diff}^+(\Sigma_g)$ is contractible, and the mapping class group $\text{MCG}(\Sigma_g) := \pi_0 \text{Diff}^+(\Sigma_g)$ is a finitely presented infinite group which acts with finite stabilizers on a certain contractible manifold called Teichmüller space.¹

So all four questions have satisfactory answers, though understanding the mapping class groups of surfaces is still an active area of research. ◀

Example 1.4. The classification of compact, orientable 3-manifolds looks remarkably similar to the classification of surfaces (albeit much harder!), through a vision of Thurston, realized by Hamilton and Perelman. The solution is almost as complete. The proof uses geometry, and nice representatives are quotients by groups acting on hyperbolic space.

As for invariants, the fundamental group is very nearly a complete invariant.² ◀

In higher dimensions, there are a few limitations. Generally, the index set I will be uncountable. For example, there are an uncountable number of smooth 4-manifolds homeomorphic to \mathbb{R}^4 ! So there will be no nice list, and no nice moduli space either. But restricted to compact manifolds, there are countably many classes, which follows from triangulation arguments or work of Cheeger in Riemannian geometry.

¹Heuristically, but not literally, Teichmüller space is a classifying space for this group.

²The fundamental group cannot distinguish lens spaces, and that's pretty much the only exception.

The next obstacle involves the fundamental group. If M is presented as an n -handlebody (roughly, a CW complex with cells of dimension at most n), there is an induced presentation of $\pi_1(M)$, and if M is compact, this is a finite presentation (finitely many generators, and finitely many relations).

Fact. For each $n \geq 4$, all finite presentations arise from compact n -handlebodies (namely, closed n -manifolds). \blacktriangleleft

This is pretty cool, but throws a wrench in our classification goal.

Theorem 1.5 (Markov). *There is no algorithm that decides whether a given finite group presentation gives the trivial group.*

The proof shows that an algorithm which could solve this problem could be used to construct an algorithm that solves the halting problem for Turing machines.

Corollary 1.6. *There is no algorithm to decide whether a given n -handlebody, $n \geq 4$, is simply connected.*

This means that a general classification algorithm cannot possibly work for $n \geq 4$; thus, we will have to restrict what kinds of manifolds we classify.

A third issue in higher-dimensional topology is that in dimension $n \leq 3$, there are existence and uniqueness theorems of “optimal” Riemannian metrics (e.g. constraints on their isometry groups), but for $n \geq 5$, this is not true for any sense of optimal; some choices fail existence, and others fail uniqueness. This is discussed further (and more precisely) in Shmuel Weinberger’s “Computers, Rigidity, and Moduli,” which has some very interesting things to say about the utility of Riemannian geometry to classify manifolds (or lack thereof).

So four dimensions is special, but for many reasons, not just one.

Those setbacks notwithstanding, we can still say useful things.

- We will restrict to closed manifolds.
- We will focus on the simply-connected case, eliding Markov’s theorem.³

With these restrictions, we have good answers to the first three questions.

Example 1.7. There is a countable list of compact, simply-connected 5-manifolds, and invariants (cohomology, characteristic classes) which distinguish any two. \blacktriangleleft

Example 1.8. Kervaire-Milnor produced a classification of homotopy spheres in dimensions $5 \leq n \leq 18$, and a conceptual answer in higher dimensions, and further work has applied this in higher dimensions. \blacktriangleleft

There is a wider range of conceptual answers to all four questions, more or less explicit, through *surgery theory*, when $n \geq 5$ (surgery theory fails radically in dimension 4). This gives an answer to the following questions.

- Given a finite, n -dimensional CW complex X (where $n \geq 5$), when is it the homotopy type of a compact n -manifold?
- Given a simply-connected compact manifold M , what are the diffeomorphism types of manifolds homotopy equivalent to M ? (Again, we need $\dim M \geq 5$.)

Here are necessary and sufficient conditions for the existence question.

- X must be an n -dimensional *Poincaré duality space*, i.e. there is a fundamental class $[X] \in H_n(X; \mathbb{Z})$ which implements the Poincaré duality isomorphism. This basic fact about closed manifolds gets you an incredibly long way towards the answer.
- Next, X must have a tangent bundle — but it’s not clear what this means for a general Poincaré duality space. Here we mean a rank- n vector bundle $T \rightarrow X$ which is associated to the homotopy type in a certain precise sense: the unit sphere bundle of the stabilization of T , considered as a spherical fibration, has to be manifest in X in a certain way.
- If $n \equiv 0 \pmod{4}$, there’s another obstruction — a certain \mathbb{Z} -valued invariant must vanish, interpreted as asking that $T \rightarrow X$ satisfies the Hirzebruch signature theorem: the signature of the cup product form on $H^{n/2}(X)$ must be determined by the Pontrjagin classes of T .
- If $n \equiv 2 \pmod{4}$, the obstruction is a similar $\mathbb{Z}/2$ -valued invariant related to the Arf invariant of the intersection form.

³More generally, one could pick some fixed group G and ask for a classification of closed n -manifolds with $\pi_1(M) \cong G$; people do this, but we won’t worry about it.

- If n is odd, there are no further obstructions.

That's it. Uniqueness is broadly similar — once you specify a tangent bundle, there are only finitely many diffeomorphism types!

Now we turn to dimension 4, the hardest case. We want to classify smooth, closed, simply-connected 4-manifolds. The first basic invariant (even of 4-dimensional Poincaré duality spaces) is the intersection form Q_P , which we'll begin studying in detail next week. You can realize it as a unimodular matrix modulo integral equivalence. That is, it's a symmetric square matrix over \mathbb{Z} with determinant ± 1 , and integral equivalence means up to conjugation by elements of $\mathrm{GL}_b(\mathbb{Z})$.

Theorem 1.9 (Milnor). *The intersection form defines a bijection from the set of homotopy classes of 4-dimensional simply-connected Poincaré spaces to the set of unimodular matrices modulo equivalence.*

So this form captures the entire homotopy type! That's pretty cool.

Theorem 1.10 (Freedman). *The intersection form defines a bijection from the set of homeomorphism classes of 4-dimensional simply-connected topological manifolds to the set of unimodular matrices modulo equivalence.*

Thus this completely classifies (closed, simply-connected) topological four-manifolds. This theorem won Freedman a Fields medal.

The next obstruction, having a tangent bundle, is a mild constraint told to us by Rokhlin.

Theorem 1.11 (Rokhlin). *Let X be a closed 4-manifold. If Q_X has even diagonal entries, then its signature is divisible by 16.*

The signature is the number of positive eigenvalues minus the number of negative eigenvalues. Algebra tells us this is already divisible by 8, so this is just a factor-of-2 obstruction, which is not too bad.

But the rest of the story of surgery theory is just wrong in dimension 4. This is where analysis of an instanton moduli space comes in.

Theorem 1.12 (Donaldson's diagonalizability theorem). *Let X be a compact, simply-connected 4-manifold. If Q_X is positive definite, i.e. $xQ_Xx > 0$ for all nonzero $x \in \mathbb{Z}^b$, then Q_X is equivalent to the identity matrix.*

Donaldson proved this theorem as a second-year graduate student!

There's a huge number of unimodular matrices which are positive definite, but not equivalent to the identity; the first example is known as E_8 . So this is a strong constraint on their realizability by 4-manifolds.

In subsequent years, Donaldson devised invariants distinguishing infinitely many diffeomorphism types within a single homotopy class. Then, from 1994 onwards, there came new proofs of these results via Seiberg-Witten theory, which tended to be simpler,⁴ and to provide sharper, more general results. We will prove several of these in the second half of the class.

Lecture 2.

Review of the algebraic topology of manifolds: 1/23/18

Though today might be review for some students, it's important to make sure we're all on the same page, and we'll get to the good stuff soon enough. We won't do too many examples today, but will see many in the future.

Cup products. Cup products make sense in a more general sense than manifolds. Let X and Y be CW complexes; then, there is a canonical induced CW structure on $X \times Y$: the product of a pair of discs is homeomorphic to a disc, and we take the cells of $X \times Y$ to be the products of cells of X and cells of Y .

Recall that the *cellular chain complex* $C_*(X)$ is the free abelian group on the set of cells, and the *cellular cochain complex* is the dual: $C^*(X) := \mathrm{Hom}(C_*(X), \mathbb{Z})$.

Proposition 2.1 (Künneth formula). *Let X and Y be CW complexes. There is a canonical isomorphism*

$$(2.2) \quad C^*(X \times Y) \cong C^*(X) \otimes C^*(Y).$$

⁴That said, Donaldson's original proof of the diagonalizability theorem stands as one of the most beautiful things in gauge theory.

This follows because the cells of $X \times Y$ are the products of those in X and those in Y . There is an analogue of the Künneth formula for pretty much any kind of (ordinary) cohomology theory.

The *diagonal map* $\Delta: X \rightarrow X \times X$ sending $x \mapsto (x, x)$ is, annoyingly, *not* a cellular map (i.e. it does not preserve the k -skeleton). However, it is homotopic to a cellular map $\delta: X \rightarrow X \times X$.

Definition 2.3. The *cup product of cochains* is the map $\smile: C^*(X) \otimes C^*(X) \rightarrow C^*(X)$ which is the composition

$$C^*(X) \otimes C^*(X) \xrightarrow[(2.2)]{\cong} C^*(X \times X) \xrightarrow{\delta^*} C^*(X).$$

We haven't said anything about coboundaries, but the cup product plays well with them, and therefore induces a cup product on cellular cohomology, $\smile: H^*(X) \otimes H^*(X) \rightarrow H^*(X)$. This is an associative map, and it's *graded*, meaning it sends $H^i(X) \otimes H^j(X)$ into $H^{i+j}(X)$. It's unital and *graded commutative*, meaning

$$(2.4) \quad x \smile y = (-1)^{|x||y|} y \smile x.$$

This turns $H^*(X)$ into a graded commutative ring.

Remark. The cup product is *not* graded commutative on the level of cochains. However, there are coherent homotopies between $x \smile y$ and $(-1)^{|x||y|} y \smile x$. \blacktriangleleft

The fact that we had to choose $\Delta \simeq \delta$ is annoying, since it's non-explicit and non-canonical. The cup product in singular cohomology does not have this problem, as you can just work with Δ itself, but the tradeoff is that the Künneth formula is less explicit.

There are a few other incarnations of the cup product which are more geometrically transparent, and this will be useful for us when studying manifolds. These have other drawbacks, of course.

- (1) Čech cohomology is a somewhat unintuitive way to define cohomology, but has the advantage of providing a completely explicit formula for the cup product.
- (2) de Rham cohomology provides a model for the cup product which is graded-commutative on cochains, but only works with \mathbb{R} coefficients.
- (3) The intersection theory of submanifolds is a beautiful model for the cup product, but is not always available.

We'll discuss these in turn.

Čech cohomology. Let M be a manifold,⁵ and $\mathfrak{U} = \{U_i\}_{i \in I}$ be an open cover of M . For $J \subset I$, write

$$U_J := \bigcap_{i \in J} U_i.$$

Definition 2.5. We say that \mathfrak{U} is a *good cover* if it is locally finite and all U_J , $J \neq \emptyset$, are empty or contractible.

In particular, on a compact manifold, a good cover is finite.

Lemma 2.6. *Any manifold admits a good cover.*

There are two standard proofs of this — one chooses small geodesic balls around each point for a Riemannian metric on M , and the other chooses an embedding $M \hookrightarrow \mathbb{R}^N$ and then uses the intersections of small balls in \mathbb{R}^N with M .

There is also a uniqueness (really cofinality) statement.

Lemma 2.7. *Any two good covers of a manifold M admit a good common refinement.*

For $k \in \mathbb{Z}_{\geq 0}$, let $[k] := \{0, \dots, k\}$.

Definition 2.8. Let $k \in \mathbb{Z}_{\geq 0}$. A *k -simplex* of \mathfrak{U} is a way of indexing a k -fold intersection in \mathfrak{U} ; specifically, it is an injective map $\sigma: [k] \hookrightarrow I$ such that $\mathfrak{U}_{\sigma([k])}$ is nonempty. The set of k -simplices of \mathfrak{U} is denoted $S_k(\mathfrak{U})$.

There is a *boundary map* $\partial_i: S_k(\mathfrak{U}) \rightarrow S_{k-1}(\mathfrak{U})$ which deletes $\sigma(i)$.

⁵Čech cohomology works on a more general class of spaces, but we work with manifolds for simplicity.

Definition 2.9. Let A be a commutative ring. The *Čech cochain complex valued in A* is the cochain complex $\check{C}^*(M, \mathfrak{U}; A)$ defined by

$$C^k(M, \mathfrak{U}; A) := \prod_{S_k} A$$

and with differential $\delta: \check{C}^k(M, \mathfrak{U}; A) \rightarrow \check{C}^{k+1}(M, \mathfrak{U}; A)$ defined by

$$(\delta\eta)(\sigma) := \sum_{i=0}^{k+1} (-1)^{i+1} \eta(\partial_i \sigma),$$

where η is a cochain and $\sigma: [k] \hookrightarrow I$.

One can show that $\delta^2 = 0$, hence define the *Čech cohomology groups* $\check{H}^*(M, \mathfrak{U}; A) := \ker(\delta) / \text{Im}(\delta)$.

Proposition 2.10. Let \mathfrak{U} and \mathfrak{V} be good covers of a manifold M . Then, there is an isomorphism $\check{H}^*(M, \mathfrak{U}; A) \cong \check{H}^*(M, \mathfrak{V}; A)$.

Proof idea. By Lemma 2.7, \mathfrak{U} and \mathfrak{V} admit a common refinement \mathfrak{W} ; then, check that a refinement map of good covers induces an isomorphism in Čech cohomology. \square

Thus the Čech cohomology is often denoted $\check{H}^*(M; A)$.

In Čech cohomology, there is a finite, combinatorial model for the cup product: let $\alpha \in \check{C}^i$, $\beta \in \check{C}^j$, and $\sigma: [i+j] \hookrightarrow I$. Then, we let

$$(2.11) \quad (\alpha \smile \beta)(\sigma) := \alpha(\text{beginning of } \sigma) \cdot \beta(\text{end of } \sigma).$$

To be sure, this works in a more general setting (and indeed is the definition of cup product in singular cohomology), but the finiteness of Čech cochains on a compact manifold makes it a lot nicer in this setting. However, it's not at all transparent that the cup product is graded commutative on cohomology.

Theorem 2.12. There is an isomorphism of graded rings $\check{H}^*(M; A) \cong H^*(M; A)$ (where the latter means cellular cohomology).

de Rham cohomology. Recall that $\Omega^k(M)$ denotes the space of differential k -forms on a manifold M , and

$$\Omega^*(M) := \bigoplus_{k \geq 0} \Omega^k(M).$$

There is an exterior derivative $d: \Omega^*(M) \rightarrow \Omega^{*+1}(M)$ with $d^2 = 0$, so we can define the *de Rham cohomology* $H_{\text{dR}}^*(M) := \ker(d) / \text{Im}(d)$ in the usual way.

In this case, the cup product is induced by the wedge product of differential forms

$$\wedge: \Omega^i(M) \otimes \Omega^j(M) \rightarrow \Omega^{i+j}(M).$$

Proposition 2.13. The wedge product is graded commutative on differential forms, hence makes $\Omega^*(M)$ into a DGA (differential graded algebra).

This is really nice, but can only occur in characteristic zero; if you tried to do this over a field of positive characteristic, you would run into obstructions called Steenrod squares to defining a functorial graded-commutative cochain model for cohomology.

Theorem 2.14 (de Rham). Let \mathfrak{U} be a good cover of a manifold M . Then there is a natural isomorphism of graded \mathbb{R} -algebras $H_{\text{dR}}^*(M) \cong \check{H}^*(M, \mathfrak{U}; \mathbb{R})$.

There are several different ways of proving this. One is to show that they both satisfy the Eilenberg-Steenrod axioms with \mathbb{R} coefficients, and that up to natural isomorphism there is a single cohomology theory satisfying these isomorphisms. Another is to observe that Čech cohomology is a model for sheaf cohomology, and that both Čech and de Rham cohomology are derived functors of the same functor of sheaves on M applied to the constant sheaf valued in \mathbb{R} .

An alternative way to prove it, whose details can be found in Bott-Tu's book, is to form the *Čech-de Rham complex*, a double complex $\check{C}^*(M, \mathfrak{U}; \Omega^\bullet)$. Let D^* denote its *totalization*. Then there are quasi-isomorphisms $\check{C}^* \hookrightarrow D^*$ and $\Omega^*(M) \hookrightarrow D^*$ respecting products, hence inducing isomorphisms $\check{H}^* \cong H^*(D^*) \cong H_{\text{dR}}^*(M)$.

Poincaré duality and the fundamental class Poincaré duality is one of the few (relatively) easy facts about topological manifolds, and one of the only things known until the work of Kirby and Siebenmann in the 1970s. Throughout this section, X denotes a nonempty, connected topological manifold of dimension n . For a reference for this section, see May's *A Concise Course in Algebraic Topology*.

Proposition 2.15.

- (1) If $k > n$, $H_k(X) = 0$.
- (2) $H_n(X) \cong \mathbb{Z}$ if X is compact and orientable, and is 0 otherwise.

If X is compact, a choice of orientation defines a generator $[X] \in H_n(X)$, called the *fundamental class* of X . A homeomorphism $f: X \rightarrow Y$ sends $[X] \mapsto [Y]$ if f preserves orientation and $[X] \mapsto -[Y]$ if f reverses orientation. If X is a CW complex with no cells of dimension $> n$ and a single cell e_n in dimension n , then in cellular homology, $[X] = \pm[e_n]$.

There is a *trace map* or *evaluation map* $H^n(X; A) \rightarrow A$ sending $c \mapsto \text{eval}(c, [X])$ (that is, evaluate c on $[X]$); in the de Rham model on a smooth manifold, this is the integration map

$$\eta \mapsto \int_X \eta.$$

The graded abelian group $H_{-*}(X)$ is a graded module over the graded ring $H^*(X)$ via a map called the *cap product*

$$\frown: H^k(X) \otimes H_i(X) \longrightarrow H_{i-k}(X).$$

Place a CW structure on X , and recall that $\delta: X \rightarrow X \times X$ was our cellular approximation to the diagonal. Then, we can give a cellular model for the cap product:

$$C^*(X) \otimes C_*(X) \xrightarrow{\text{id} \otimes \delta_*} C^*(X) \otimes C_*(X) \otimes C_*(X) \xrightarrow{\text{eval} \otimes \text{id}} C_*(X).$$

Let X and Y be smooth n -manifolds, where X is closed and oriented. Then, $-\frown f_*[X]: H^n_{\text{dR}}(Y) \rightarrow H^n_{\text{dR}}(X)$ has the explicit model

$$\eta \mapsto \int_X f^* \eta,$$

showing how the cap product relates to the evaluation map.

Theorem 2.16 (Poincaré duality). *For X a closed, oriented manifold, the map*

$$D_X := -\frown [X]: H^*(X) \rightarrow H_{n-*}(X)$$

is an isomorphism.

For a proof, see May. In the case of smooth manifolds, there's a slick proof using Morse theory; but Poincaré duality is true for topological manifolds as well.

We will let $D^X := (D_X)^{-1}$.

Intersections of submanifolds. Intersection theory, though not its relation to the cup product, was discussed in the differential topology prelim. Let X , Y , and Z be closed, oriented manifolds of dimensions n , $n-p$, and $n-q$ respectively, and let $f: Y \rightarrow X$ and $g: Z \rightarrow X$ be smooth maps. Let $c_Y := D^X(f_*[Y]) \in H^p(X)$, and similarly let $c_Z := D^X(g_*[Z]) \in H^q(X)$. We will be able to give a nice interpretation of $c_Y \smile c_Z$.

First, let f' be a smooth map homotopic to f and transverse to g ; standard theorems in differential topology show that such a map exists. Transversality means that if $y \in Y$ and $z \in Z$ are such that $f'(y) = g(z)$, then

$$T_x X = Df'(T_y Y) + Dg(T_z Z).$$

Let $P := Y_{f'} \times_g X$, which is exactly the space of pairs (y, z) such that $f'(y) = g(z)$; transversality guarantees this is a smooth manifold of codimension $p+q$ in X . The orientations on X , Y , and Z induce one on P , and there is a canonical map $\phi: P \rightarrow X$ sending $(y, z) \mapsto f'(y) = g(z)$.

Theorem 2.17. *Let $c_P := D^X(\phi_*([P]))$. Then, $c_P = c_Y \smile c_Z$.*

If Y and Z are transverse submanifolds of X , P is exactly their intersection. We will use this result frequently.

Intersection of submanifolds gives a geometric realization of the cup product, but only for those classes represented by maps from manifolds; not all homology classes are realized in this way.

Classes of codimension at most 2 always have representatives arising from embedded submanifolds. The idea is that in general, there's a natural isomorphism

$$H^n(X) \cong [X, K(\mathbb{Z}, n)],$$

where brackets denote homotopy classes of maps and $K(\mathbb{Z}, n)$ is an *Eilenberg-Mac Lane space* for \mathbb{Z} in dimension n , i.e. a space whose only nontrivial homotopy group is $\pi_n \cong \mathbb{Z}$. These spaces always exist, and any two models for $K(\mathbb{Z}, n)$ are homotopic.

Usually Eilenberg-Mac Lane spaces are not smooth manifolds, but there are a few exceptions, including $K(\mathbb{Z}, 1) \simeq S^1$. Hence there is a bijection $[X, S^1] \rightarrow H^1(X)$. In the de Rham model, this is the map

$$[f] \mapsto f^*\omega,$$

where $\omega \in H^1(S^1) \cong \mathbb{Z}$ is the generator. Alternatively, you could think of ω as $D^{S^1}[\text{pt}]$, for any choice of $\text{pt} \in S^1$.

Thus, take $f: X \rightarrow S^1$ to be a smooth map, where X is a closed, oriented manifold. Let $H_t := f^{-1}(t) \subset X$, where $t \in S^1$ is a regular value. Then, H_t comes with a co-orientation, hence an orientation, and $[H_t] = D_X(f^*\omega)$. Thus codimension-1 submanifolds are realizable.

In this course, the case of codimension 2 will be more useful.

Proposition 2.18. *Let $\mathbb{CP}^\infty := \text{colim}_n \mathbb{CP}^n$ (the union via the inclusions $\mathbb{CP}^n \hookrightarrow \mathbb{CP}^{n+1}$). Then, \mathbb{CP}^∞ is a $K(\mathbb{Z}, 2)$.*

Hence there is a class $c \in H^2(\mathbb{CP}^\infty)$ and a natural bijection $[X, \mathbb{CP}^\infty] \rightarrow H^2(X)$ sending $[f] \mapsto f^*(c)$.

\mathbb{CP}^∞ is not a smooth manifold, but its low-dimensional skeleta are, and this leads to codimension-2 realizability. Specifically, the inclusion $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty$ defines the pullback map $H^2(\mathbb{CP}^\infty) \rightarrow H^2(\mathbb{CP}^1) \cong H_0(\mathbb{CP}^1) \cong \mathbb{Z}$. This maps the tautological class c to $[\text{pt}]$ for any $\text{pt} \in \mathbb{CP}^1$.

Let $f: \mathbb{CP}^\infty \rightarrow X$ be a map, where X is a smooth, oriented, closed manifold, which is homotopic to a smooth map $\mathbb{CP}^N \rightarrow X$ followed by the inclusion $\mathbb{CP}^N \hookrightarrow \mathbb{CP}^\infty$. Let $D \subset \mathbb{CP}^N$ be a hyperplane and $H_D := g^{-1}(D)$. Assuming $g \pitchfork D$ (which can always be done for some g in the homotopy class of f), then H_D is a codimension-2 oriented submanifold of X , and $[H_D] = D^X(g^*c)$. Thus codimension-2 homology classes are representable. We will most commonly use this in dimension 4, for which any class $a \in H^2(X)$ is represented by an embedding of a closed, oriented surface $\Sigma \hookrightarrow X$.

In general, realizability is controlled by oriented cobordism, which in higher codimension is different from cohomology, governed by maps into Thom spaces rather than Eilenberg-Mac Lane spaces. This was studied in the 1950s by Rene Thom.

Lecture 3.

Unimodular forms: 1/25/18

Today's lecture will be more algebraic in flavor, though with topology in mind; we'll be discussing the algebra that arises in the middle cohomology of even-dimensional manifolds.

Let M be a closed, oriented manifold of dimension $2n$; its middle cohomology $H^n(M)$ carries a bilinear form $\cdot: H^n(M) \otimes H^n(M) \rightarrow \mathbb{Z}$ sending

$$(3.1) \quad x, y \mapsto x \cdot y := \langle x \smile y, [M] \rangle,$$

where $\langle -, - \rangle$ denotes evaluation. If n is even (i.e. $4 \mid \dim M$), this is a symmetric form; if n is odd, it's skew-symmetric, which follows directly from the graded-commutativity of the cup product.

Poincaré duality means there are three different ways to think of this product.

- As defined, it's a pairing $H^n \otimes H^n \rightarrow \mathbb{Z}$, where the pairing is evaluation and the cup product.
- Using Poincaré duality, we could reinterpret it as a map $H^n \otimes H_n \rightarrow \mathbb{Z}$. In this case, the pairing is evaluation. This is because the Poincaré duality map is capping with the fundamental class, so

$$(x \smile y) \frown [M] = x \frown (y \frown [M]) = x \frown D_M(y).$$

- Using Poincaré duality again, it's a pairing $H_n \otimes H_n \rightarrow \mathbb{Z}$, which is the intersection product.

For an abelian group A , let $A_{\text{tors}} \subset A$ denote its torsion subgroup and $A' := A/A_{\text{tors}}$. In this case, the form (3.1) descends to a form on $H^n(M)'$, and we usually use this version of the form.

Remark. If M is 4-dimensional, the universal coefficients theorem guarantees a short exact sequence

$$0 \longrightarrow (H_1(M))_{\text{tors}} \longrightarrow H^2(M) \longrightarrow H^2(M)' \longrightarrow 0,$$

and that it splits, but non-canonically. In particular, if $H_1(M) = 0$, $H^2(M)$ is torsion-free. \blacktriangleleft

Let $\{e_i\}$ be a \mathbb{Z} -basis for $H^n(M')$ and $Q = (Q_{ij})$ be the matrix with entries $Q_{ij} := e_i \cdot e_j$. This is a symmetric matrix if n is even, and is skew-symmetric if n is odd.

The universal coefficients theorem also implies that $H^n(M') \cong \text{Hom}(H_n(M), \mathbb{Z})$, where the map sends a cohomology class y to the evaluation pairing of y and a homology class. This, plus the fact that \cdot is dual to evaluation, implies the following proposition.

Proposition 3.2. *The pairing \cdot is nondegenerate on $H^n(M)'$, i.e. the map $H^n(M)' \rightarrow \text{Hom}(H^n(M)', \mathbb{Z})$ sending $x \mapsto (y \mapsto x \cdot y)$ is an isomorphism of abelian groups.*

Corollary 3.3. $\det Q \in \{\pm 1\}$.

Now we focus on the case of manifolds which are boundaries. Suppose there is a compact oriented $(2n+1)$ -dimensional manifold N such that $M = \partial N$, and let $i: N \hookrightarrow M$ denote inclusion.

Proposition 3.4. *Let $L = \text{Im}(i^*) \subset H^n(M; \mathbb{R})$. Then,*

- (1) L is isotropic, i.e. for all $x, y \in L$, $x \cdot y = 0$.
- (2) $\dim L = (1/2) \dim H^n(M; \mathbb{R})$.

Proof. Perhaps unsurprisingly, this proof uses algebraic topology of manifolds with boundary, namely *Poincaré-Lefschetz duality*, the analogue of Poincaré duality on a compact manifold with boundary.

Part (1) follows from the fact that i^* is a ring homomorphism:

$$\begin{aligned} i^*u \cdot i^*v &= (i^*u \smile i^*v) \frown [M] \\ &= i^*(u \smile v) \frown [M] \\ &= (u \smile v) \frown i_*[M], \end{aligned}$$

but since $M = \partial N$, $[M] = \partial[M, N]$, where $\partial: H_{2n+1}(M, N; \mathbb{R}) \rightarrow H_{2n}(N; \mathbb{R})$ is the boundary map in the long exact sequence of a pair and $[M, N] \in H_{2n+1}$ is the relative fundamental class. Hence $i_*[M] = 0$, since $i_* \circ \partial = 0$.

For part (2), Poincaré-Lefschetz duality implies the following diagram is commutative with exact rows:

$$(3.5) \quad \begin{array}{ccccccc} \cdots & \longrightarrow & H^n(N; \mathbb{R}) & \xrightarrow{i^*} & H^n(M; \mathbb{R}) & \xrightarrow{\delta} & H^{n+1}(N, M; \mathbb{R}) \xrightarrow{q} H^{n+1}(N; \mathbb{R}) \longrightarrow \cdots \\ & & \cong \downarrow & & \cong \downarrow D_M & & \cong \downarrow D \\ \cdots & \longrightarrow & H_{n+1}(N, M; \mathbb{R}) & \xrightarrow{\partial} & H_n(M; \mathbb{R}) & \xrightarrow{i_*} & H_n(M; \mathbb{R}) \xrightarrow{p} H_n(N, M; \mathbb{R}) \longrightarrow \cdots \end{array}$$

Fix a complement K to L in $H^n(M; \mathbb{R})$; it suffices to show that $\dim K = \dim L$. Since $L = \text{Im}(i^*) = \ker(\delta)$, then $K \cong H^n(M; \mathbb{R}) / \ker(\delta) \cong \text{Im}(\delta)$. Since the upper row of (3.5) is exact, $\text{Im}(\delta) = \ker(q)$, and by Poincaré-Lefschetz duality this is isomorphic to $\ker(p)$. Since the lower row of (3.5) is exact, this is isomorphic to $\text{Im}(i_*) \subset H_n(N; \mathbb{R})$. However, i_* and i^* are dual (in the sense of $\text{Hom}(-, \mathbb{R})$), and linear algebra tells us that a map and its dual have the same rank.⁶ \square

Now let's focus on the case when n is even, so $4 \mid \dim(M)$.

Definition 3.6. A *unimodular lattice* (Λ, σ) is a finite-rank free abelian group Λ together with a nondegenerate symmetric bilinear form $\sigma: \Lambda \otimes \Lambda \rightarrow \mathbb{Z}$.

Therefore a closed, oriented $4m$ -manifold defines a unimodular lattice (H^{2m}, \cdot) .

Recall that for (V, σ) a symmetric bilinear form on a real vector space V , there is an orthogonal decomposition

$$(3.7) \quad V = R \oplus V^+ \oplus V^-,$$

where R is the subspace orthogonal to everything, V^+ is the subspace on which σ is positive definite, and V^- is the subspace on which σ is negative definite. Clearly (V, σ) determines $\dim V^+$ and $\dim V^-$.

⁶The matrix version of this statement is that a matrix and its transpose have the same rank.

Proposition 3.8 (Sylvester's law of inertia). *If (V, σ) is a symmetric bilinear form on a real vector space, then $\dim R$, $\dim V^+$, and $\dim V^-$ determine (V, σ) up to isomorphism.*

Definition 3.9. The *signature* of (V, σ) is $\tau := \dim V^+ - \dim V^-$. If (Λ, σ) is a unimodular lattice, then $\tau(\Lambda, \sigma) := \tau(\Lambda \otimes \mathbb{R}, \sigma \otimes \text{id}_{\mathbb{R}})$. If M is a closed, oriented, $4m$ -dimensional manifold, its signature is $\tau(M) := \tau(H^{2m}(M)', \cdot)$.

Fact. If M is a closed, oriented $4m$ -manifold, then M admits an orientation-reversing self-diffeomorphism iff $\tau(M) = 0$. \blacktriangleleft

Theorem 3.10.

- (1) *Let X_1 and X_2 be closed, oriented 4-manifolds and Y be an oriented cobordism between them, i.e. Y is a compact, oriented 5-manifold together with an orientation-preserving diffeomorphism $\partial Y^5 \cong (-X_1) \amalg X_2$.⁷ Then, $\tau(X_1) = \tau(X_2)$.*
- (2) *Conversely, if $\tau(X_1) = \tau(X_2)$, then X_1 and X_2 are cobordant.*

Therefore, in particular, the signature is a complete cobordism invariant.

Partial proof. For part (1), $H^2(-X_1 \amalg X_2) \otimes \mathbb{R}$ admits a middle-dimensional isotropic subspace by Proposition 3.4, hence has signature zero. But

$$\begin{aligned} \tau(-X_1 \amalg X_2) &= \tau((-H^2(X_1) \otimes \mathbb{R}) \oplus (H^2(X_2) \otimes \mathbb{R})) \\ &= \tau(-X_1) + \tau(X_2) \\ &= -\tau(X_1) + \tau(X_2) = 0. \end{aligned} \quad \boxtimes$$

We will not give a full proof of part (2). The idea is that cobordism classes of oriented 4-manifolds form an abelian group Ω_4^{SO} under disjoint union, and by part (1), the signature defines a homomorphism

$$\tau: \Omega_4^{\text{SO}} \longrightarrow \mathbb{Z}.$$

This map must be surjective, because $\tau(\mathbb{CP}^2) = 1$ (where the orientation is the standard one coming from its complex structure). To prove it's injective, one uses Thom's cobordism theory, which identifies Ω_4^{SO} with a homotopy group of a space called a *Thom space*, then calculates that group using the Hurewicz theorem and calculation of the homology of the Thom space in question.

Next we discuss unimodular lattices mod 2.

Definition 3.11. A *characteristic vector* c for a unimodular lattice (Λ, σ) is a $c \in \Lambda$ such that $\sigma(c, x) \equiv \sigma(x, x) \pmod{2}$.

Lemma 3.12. *The characteristic vectors form a coset of 2Λ in Λ .*

Proof. Let $\lambda = \Lambda/2\Lambda$, which is a vector space over \mathbb{F}_2 . The freshman's dream mod 2 implies that the map $\lambda \rightarrow \mathbb{Z}/2$ sending $[x] \mapsto \sigma(x, x) \pmod{2}$ is linear! Hence there is a symmetric bilinear form $\bar{\sigma}$ on λ induced by σ with determinant 1, hence is nondegenerate. Hence there exists a unique $\bar{c} \in \lambda$ such that $\bar{\sigma}(x, x) = \bar{\sigma}(\bar{c}, x) \pmod{2}$ for all $x \in \lambda$. The characteristic vectors are exactly the lifts of \bar{c} to Λ , hence are a coset of 2Λ . \boxtimes

Remark. In the case of a simply connected 4-manifold M , the element $\bar{c} \in H^2(M)/2H^2(M) \cong H^2(M; \mathbb{Z}/2)$ is exactly the second Stiefel-Whitney class $w_2(TM)$. We'll talk about characteristic classes more next lecture. This follows from the *Wu formula*.

Moreover, the characteristic vectors $c \in H^2(M; \mathbb{Z})$ are exactly the first Chern classes of spin^c structures on M , and the Seiberg-Witten invariants are functions on the set of characteristic vectors to \mathbb{Z} . We'll say more about this later.

Most of this is true even in the case of non-simply-connected manifolds, but is harder. It is not true, however, that $H^2(M)/2H^2(M) \cong H^2(M; \mathbb{Z}/2)$. \blacktriangleleft

Lemma 3.13. *Let c and c' be characteristics for (Λ, σ) . Then,*

$$\sigma(c, c) \equiv \sigma(c', c') \pmod{8}.$$

⁷Here $-X_1$ denotes X_1 with the opposite orientation.

Proof. Write $c' - c = 2x$ for some $x \in \Lambda$. Then,

$$\sigma(c', c') = \sigma(c + 2x, c + 2x) = \sigma(c, c) + 4 \underbrace{(\sigma(c, x) + \sigma(x, x))}_{(*)},$$

and $(*)$ is even. \square

Definition 3.14. Let (Λ, σ) be a unimodular form. Its type $t \in \mathbb{Z}/2$ is even if $\sigma(x, x)$ is even for all $x \in \Lambda$, and otherwise, it's odd.

Theorem 3.15 (Hasse-Minkowski theorem on unimodular forms). *An indefinite unimodular form (Λ, σ) is classified up to isomorphism by three invariants:*

- its rank $\dim_{\mathbb{R}}(\Lambda \otimes \mathbb{R})$,
- its signature $\tau \in \mathbb{Z}$, and
- its type $t \in \mathbb{Z}/2$.

For the (quite nontrivial) proof, see Serre's *A Course in Arithmetic*. The proof idea is to solve the quadratic equation $\sigma(x, x) = 0$ for $x \in (\Lambda \otimes \mathbb{Q}) \setminus 0$. This is achieved via a *local-to-global principle* which says it suffices to find solutions $x_{\infty} \in \Lambda \otimes \mathbb{R}$ and $x_p \in \Lambda \otimes \mathbb{Q}_p$ (the p -adic numbers), and this can be done using unimodularity.

This is completely different from the positive definite (equivalently, negative definite) case, for which there are finitely many isomorphism classes below a given rank r , though this number grows rapidly with r and is only known in relatively few cases.

Example 3.16. Let $I_+ := (\mathbb{Z}, 1)$ with $\sigma(x, y) = xy$ and $I_- := (\mathbb{Z}, -1)$ with $\sigma(x, y) = -xy$. Then, $I_+^{\oplus m} \oplus I_-^{\oplus n}$ has rank $m + n$, signature $m - n$, and odd type.

There is a characteristic vector $c := (1, \dots, 1)$. In particular, $c^2 = m - n - \tau$, and therefore for any characteristic vector c , $c^2 \equiv \tau \pmod{8}$. \blacktriangleleft

Example 3.17. Let

$$U := \left(\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$$

with quadratic form $(a, b)^2 := 2ab$. This is an even unimodular lattice with rank 2 and signature 0. \blacktriangleleft

The Hasse-Minkowski principle implies the following.

Corollary 3.18. *Let Λ be a unimodular lattice with signature τ and c be a characteristic vector for Λ . Then, $c \cdot c \equiv \tau \pmod{8}$. In particular, if Λ is even, then $\tau \equiv 0 \pmod{8}$.*

Proof. Either $\Lambda \oplus I_+$ or $\Lambda \oplus I_-$ is indefinite. It has odd type, and signature $\tau(\Lambda \oplus I_{\pm}) = \tau(\Lambda) \pm 1$, and if c is a characteristic vector for Λ , a characteristic vector for $\Lambda \oplus I_{\pm}$ is $c \oplus 1$ with square $c \cdot c = \pm 1$.

By Theorem 3.15, $\Lambda \oplus I_{\pm} \cong mI_+ \oplus nI_-$, so $c^2 \pm 1 \equiv \tau(\Lambda) \pm 1 \pmod{8}$. \square

Hence an even, positive-definite, unimodular lattice has rank $8k$ for some k .

Example 3.19 (E_8 lattice). The basic example is the E_8 lattice, associated to the Dynkin diagram for the exceptional simple Lie group E_8 . As a matrix, it has the form

$$(3.20) \quad \begin{pmatrix} 2 & -1 & & & & & & -1 \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ -1 & & & & & & -1 & 2 \end{pmatrix}.$$

Hasse-Minkowski implies that any even unimodular lattice Λ is isomorphic to $mU \oplus (n \pm E_8)$. The intersection forms of interesting 4-manifolds tend to have E_8 terms. \blacktriangleleft

For more on the E_8 lattice, see the professor's class notes.

Lecture 4.

The intersection form and characteristic classes: 1/30/18

Let's start by writing down the homology and cohomology of a closed, oriented 4-manifold. Poincaré duality narrows the search space considerably.

$$\begin{aligned} H_4(X) &\cong H^0(X) = \mathbb{Z} \cdot 1 \\ H_3(X) &\cong H^1(X) = \text{Hom}(\pi_1(X), \mathbb{Z}) \\ H_2(X) &\cong H^2(X) \cong \text{Hom}(H_2(X), \mathbb{Z}) \oplus H_1(X)_{\text{tors}} \\ \pi_1(X)^{\text{ab}} &= H_1(X) \cong H^3(X) \\ \mathbb{Z} \cdot [\text{pt}] &= H_0(X) \cong H^4(X). \end{aligned}$$

Additively, the homology and cohomology are determined up to isomorphism by π_1 and H_2 . Together with the intersection form Q_X on H_2 (or its torsion-free quotient), we have a lot of information, but we still don't have everything: there could be cup products from H^1 and H^2 to H^3 , and various other questions, e.g. what's the Hurewicz map $\pi_2(X) \rightarrow H_2(X)$? What about mod p coefficients?

If X is simply connected, the story is much simpler:

$$\begin{aligned} H_4(X) &\cong H^0(X) = \mathbb{Z} \cdot 1 \\ H_3(X) &\cong H^1(X) = 0 \\ H_2(X) &\cong H^2(X) \cong \text{Hom}(H_2(X), \mathbb{Z}) \\ H_1(X) &\cong H^3(X) = 0 \\ \mathbb{Z} \cdot [\text{pt}] &= H_0(X) \cong H^4(X). \end{aligned}$$

The only real information is H_2 , which determines the cohomology additively. The intersection form determines $H^*(X)$ as a graded ring and $H_*(X)$ as a graded $H^*(X)$ -module. All mod p cohomology classes are reductions of integral classes, which follows from the universal coefficient theorem. The natural map $\pi_2(X) \rightarrow H_2(X)$ is an isomorphism, by the Hurewicz theorem, which applies to any simply connected space. You can think about higher homotopy groups, and next time we'll think about the homotopy type of X , but thus far everything we can see has been determined by Q_X . (In fact, next time we'll see that it determines the homotopy type of X .)

So that knocks out the homotopy type, but we want further invariants. The next step is to investigate the tangent bundle, a distinguished rank-4 vector bundle. In particular, it has characteristic classes.

Remark. Spoiler alert: all characteristic classes we discuss today are determined by Q_X , so they don't define any new invariants.⁸ Nonetheless, they provide useful tools for computing the intersection form, and therefore will be useful to us. ◀

We now review some of the theory of characteristic classes.

Example 4.1. Let $V \rightarrow X$ be a finite-rank real vector bundle over an arbitrary topological space X . The *Stiefel-Whitney classes* are characteristic classes of V in mod 2 cohomology with the following properties.

- The i^{th} *Stiefel-Whitney class* $w_i(V) \in H^i(X; \mathbb{Z}/2)$ for $i \geq 0$. The *total Stiefel-Whitney class* is

$$w(V) := w_0(V) + w_1(V) + \cdots$$

- $w_0(V) = 1$.
- If $i > \text{rank } V$, $w_i(V) = 0$. Hence the total Stiefel-Whitney class is a finite sum.

It's a theorem that the Stiefel-Whitney classes are characterized by the following properties.

- (1) If $f: Y \rightarrow X$ is continuous, $w_i(f^*V) = f^*w_i(V)$ for all i .
- (2) If $i > \text{rank } V$, $w_i(V) = 0$.
- (3) The *Whitney sum formula*: if $U, V \rightarrow X$ are vector bundles, then $w(U \oplus V) = w(U)w(V)$.⁹

⁸In fact, Q_X determines the homotopy type of the classifying map for TX !

⁹If X is paracompact, every short exact sequence of vector bundles over X splits, so we may replace $U \oplus V$ with an extension of U by V .

- (4) If $L \rightarrow \mathbb{RP}^1$ denotes the *tautological line bundle* whose fiber over a point $\ell \in \mathbb{RP}^1$ is the line ℓ itself, $w_1(L) \neq 0$ in $H^1(\mathbb{RP}^1; \mathbb{Z}/2) \cong \mathbb{Z}/2$.

For a proof, see Hatcher's notes on vector bundles and K -theory, or Milnor-Stasheff, which provides a somewhat baffling construction in terms of Steenrod algebra on the Thom space. There are various differing constructions which make various facts about Stiefel-Whitney classes easier to prove. \blacktriangleleft

If X is path-connected, the isomorphism $H^1(X; \mathbb{Z}/2) \cong \text{Hom}(\pi_1(X); \mathbb{Z}/2)$ sends $w_1(V)$ to the orientation character of V . In particular, $w_1(V) = 0$ iff V is orientable, so if M is a manifold, M is orientable iff $w_1(TM) = 0$.

Now suppose M is a closed manifold and $V \rightarrow M$ is a rank- r vector bundle. Its top Stiefel-Whitney class $w_r(V) \in H^r(M; \mathbb{Z}/2) \cong H_{n-r}(M; \mathbb{Z}/2)$ maps to some homology class, which admits a representation by some codimension- r cycle. This cycle has an explicit description: let s be a section of V transverse to the zero section; then $s^{-1}(0)$ is codimension r and represents the homology class which is Poincaré dual to $w_r(V)$.

If V is an orientable bundle on a closed submanifold, its top Stiefel-Whitney class is the mod 2 reduction of its Euler class.

Proposition 4.2. *Let $H \in H^1(\mathbb{RP}^2; \mathbb{Z}/2) \cong \mathbb{Z}/2$ denote the generator, which is Poincaré dual to a hyperplane $\mathbb{RP}^{n-1} \subset \mathbb{RP}^n$. Then $w(T\mathbb{RP}^n) = (1 + H)^{n+1}$.*

For a proof, see Milnor-Stasheff or the professor's notes.

Now we specialize to 4-manifolds.

Theorem 4.3 (Wu). *Let X be a closed 4-manifold and $w := w_1^2(TX) + w_2(TX)$. Then w is the characteristic element of $H^2(X; \mathbb{Z}/2)$, i.e. for all $u \in H^2(X; \mathbb{Z}/2)$, $w \smile u = u \smile u$.*

There is a more general statement of Wu's theorem for manifolds in other dimensions. See Milnor-Stasheff.

For simply-connected 4-manifolds, the Stiefel-Whitney classes are determined by information we already have.

- If X is a closed, simply-connected 4-manifold, then $w_1(TX)$ vanishes, since H^1 does.
- Hence $w_2(TX) \cup u = u \smile u$ for all $u \in H^2(X; \mathbb{Z}/2)$. In this case, $H^2(X; \mathbb{Z}/2) = H^2(X)/2H^2(X)$; therefore $w_2(TX)$ is the mod 2 reduction of any characteristic vector for Q_X . Therefore w_2 provides no new information.
- $w_4(TX)$ is Poincaré dual to the zeroes of a vector field, so $w_4(TX) \smile [X]$ is the number of zeroes mod 2 of a generic vector field, i.e. the Euler characteristic mod 2. This is again not new information, since it can be read off $H^*(X)$.
- Finally, since H^3 vanishes, so must w_3 . It's a theorem of Hirzebruch and Hopf that $w_3(TX)$ vanishes for any closed, orientable 4-manifolds, and this is quite relevant for our class.

Remark. The more general version of Wu's theorem shows that on any closed, oriented 4-manifold X , $w_3(TX) = \text{Sq}^1 w_2(TX)$. Here Sq^1 is the first *Steenrod square*, a cohomology operation, which has an explicit identification as the *Bockstein map* which measures whether a mod 2 cohomology class lifts to \mathbb{Z} coefficients. In particular, $w_3(TX) = 0$ iff $w_2(TX)$ is the reduction of an integral class, and such lifts are the first Chern classes of spin^c structures. Thus the Hirzebruch-Hopf theorem is important for us because it implies that all closed, oriented 4-manifolds admit a spin^c structure. Since this theorem is trivial in the simply-connected case, though, we will not prove it. \blacktriangleleft

Example 4.4. *Chern classes* are characteristic classes $c_{2i}(E) \in H^{2i}(X; \mathbb{Z})$ for complex vector bundles $E \rightarrow X$; again the *total Chern class*

$$c(E) := c_0(E) + c_1(E) + \cdots$$

and again $c_0(E) = 1$ and $c_i(E) = 0$ for $i > \text{rank } E$. The Chern classes are uniquely characterized by similar axioms:

- (1) $c_i(E) = 0$ if $i > \text{rank } E$.
- (2) $c(E \oplus F) = c(E)c(F)$.
- (3) If $L \rightarrow \mathbb{CP}^1$ denotes the tautological line bundle $L \rightarrow \mathbb{CP}^1$, then the Poincaré dual of $c_1(L) \in H^2(\mathbb{CP}^1; \mathbb{Z})$ is $-1 \in H_0(\mathbb{CP}^1)$ (where we take as a generator any positively oriented point).

\blacktriangleleft

Proposition 4.5. If $H \in H^2(\mathbb{CP}^n)$ denotes the Poincaré dual to a hyperplane, so $H = -c_1$ of the tautological line bundle over \mathbb{CP}^n , then $c(T\mathbb{CP}^n) = (1 + H)^{n+1}$. The argument is formally identical to the real case.

Definition 4.6. Isomorphism classes of complex line bundles on X form a group under tensor product; this is called the *topological Picard group* and denoted $\text{Pic}(X)$.

If M is a closed, oriented manifold and $E \rightarrow M$ is a rank- r complex vector bundle, then the Poincaré dual of $c_r(E)$ is the zero locus of a generic section of E (i.e. transverse to the zero section).

Proposition 4.7. The first Chern class defines a homomorphism $c_1: \text{Pic}(X) \rightarrow H^2(X; \mathbb{Z})$, and this is an isomorphism.

In particular, for line bundles L_1 and L_2 , $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$. One way to use this is to use the above characterization of the Poincaré dual of the top Chern class; another is to show that BU_1 , the classifying space for complex line bundles, is a $K(\mathbb{Z}, 2)$, an Eilenberg-Mac Lane space, hence representing cohomology.

For a reference for the following theorem, see Hatcher's notes on vector bundles and K -theory.

Theorem 4.8. Let $E \rightarrow X$ be a complex vector bundle and $E_{\mathbb{R}}$ denote the underlying real vector bundle.

- $w_{2i}(E_{\mathbb{R}})$ is the mod 2 reduction of $c_i(E)$.
- $w_{2i+1}(E_{\mathbb{R}}) = 0$.

In particular, $w_2 = c_1 \bmod 2$.

In order to apply Chern classes to manifolds, we need some sort of complex structure.

Definition 4.9. Let M be an even-dimensional manifold. An *almost complex structure* on M is a $J \in \text{End}(TM)$ such that $J^2 = -\text{id}$.

This structure makes TM into a complex vector bundle, where i acts as J . Then we have access to Chern classes, albeit depending on J .

Remark. As the notation suggests, complex manifolds are almost complex. ◀

Example 4.10. One example of a complex manifold is a complex hypersurface in \mathbb{CP}^n : let F be a degree- d homogeneous polynomial in x_0, \dots, x_d and $X := \{F = 0\} \subset \mathbb{CP}^n$. Homogeneity means this makes sense in projective space; if we additionally assume that whenever $F = 0$ at least one partial derivative of F is nonzero, then X is smooth.

To study complex manifolds, we can import tools from algebraic geometry. A holomorphic hypersurface D in a complex manifold M defines an invertible sheaf (i.e. complex line bundle) $\mathcal{O}_M(D)$ whose sections over $U \subset M$ are meromorphic functions on U with only simple poles along $D \cap U$. Moreover, if $N_{D/M} := TM|_D / TD$ denotes the normal bundle to $D \hookrightarrow M$, there's an isomorphism $\mathcal{O}_M(D)|_D \cong N_{D/M}$.

On \mathbb{CP}^n , a holomorphic line bundle E is determined by its degree $d := c_1(E) \in H^2(\mathbb{CP}^n)$, because the holomorphic sheaf cohomology $H^1(\mathbb{CP}^n; \mathcal{O}_{\mathbb{CP}^n}) = 0$. If $L \rightarrow \mathbb{CP}^n$ denotes the tautological line bundle, then L^* is the positive generator of $\text{Pic } \mathbb{CP}^n$, so there's an isomorphism $E \cong (L^*)^{\otimes d}$.

It follows that $\mathcal{O}_{\mathbb{CP}^n}(X) \cong (L^*)^{\otimes d}$, so $c_1(\mathcal{O}_{\mathbb{CP}^n}(X)) = dH$, where $H = c_1(L^*)$ as before. Hence if $h = i^*H \in H^2(X)$, where $i: X \hookrightarrow \mathbb{CP}^n$ is inclusion, then

$$(4.11) \quad c_1(N_{X/\mathbb{CP}^n}) = i^*c_1(\mathcal{O}_{\mathbb{CP}^n}(X)) = dh.$$

Using the short exact sequence

$$(4.12) \quad 0 \longrightarrow TX \longrightarrow T\mathbb{CP}^n|_X \longrightarrow N_{X/\mathbb{CP}^n} \longrightarrow 0,$$

we get that

$$\begin{aligned} i^*c(T\mathbb{CP}^n) &= c(TX)c(N_{X/\mathbb{CP}^n}) \\ (1 + h)^{n+1} &= c(TX)(1 + dh), \end{aligned}$$

and therefore

$$(4.13) \quad c_j(TX) + dhc_{j-1}(TX) = \binom{n+1}{j} h^j.$$

It's possible to explicitly solve this when you have a specific X ; the base case is

$$(4.14) \quad c_1(TX) = (n+1-d)h. \quad \text{◀}$$

Example 4.15. Let $n = 3$, so X is a degree- d complex surface, hence a closed, oriented 4-manifold. In that case

$$(4.16) \quad \begin{aligned} c_1(TX) &= (4 - d)h \\ c_2(TX) &= (d^2 - 4d + 6)h^2. \end{aligned}$$

Since $[X] \in H_4(\mathbb{CP}^3) \cong H^2(\mathbb{CP}^3)$ is identified with dH under Poincaré duality, then

$$(4.17) \quad c_2(TX) \frown [X] = d(d^2 - 4d + 6).$$

Since this is again the number of zeros (with orientation) of a generic vector field, this integer is the Euler characteristic, so we have an explicit formula for the Euler characteristic of a degree- d complex surface:

$$(4.18) \quad \chi(X) = d(d^2 - 4d + 6). \quad \blacktriangleleft$$

Theorem 4.19 (Lefschetz hyperplane theorem). *Suppose $n \geq 3$ and X is a hypersurface in \mathbb{CP}^n . Then X is simply connected.*

There is a more general version of this theorem.

Therefore if X is a hypersurface in \mathbb{CP}^n , its first and third Betti numbers vanish, so

$$(4.20) \quad \chi(X) = 1 + b_2(X) + 1,$$

so

$$(4.21) \quad b_2(X) = d(d^2 - 4d + 6) - 2.$$

So we know the dimension of H^2 . It's possible to write down bases using this information, though we won't get into this.

Example 4.22. The Pontrjagin classes $p_i(V) \in H^{4i}(X)$ of a real vector bundle V are defined by

$$p_i(V) := (-1)^i c_{2i}(V \otimes \mathbb{C}). \quad \blacktriangleleft$$

Pontrjagin classes satisfy very similar axioms to Stiefel-Whitney and Chern classes; however, be aware that which ones vanish might be tricky. For example, the complexification of a rank-3 vector bundle is a rank-3 complex vector bundle, hence only has access to c_0 and c_2 for defining Pontrjagin classes.

If X is a closed oriented 4-manifold, TX has only one nonvanishing Pontrjagin class, which is $p_1(TX) \in H^4(X) \cong \mathbb{Z}$.

Lemma 4.23. *Let X be a closed, oriented 4-manifold. Then $\sigma(X) := p_1(TX) \frown [X] \in \mathbb{Z}$ is an oriented cobordism invariant.*

The basic idea is that if W is a 5-manifold bounding X , then $TW = TM \oplus \mathbb{R}$, which implies $p_1(W) = i^* p_1(M)$, where $i: M \hookrightarrow W$ is inclusion.

Lemma 4.24. *If $V \rightarrow X$ is a complex tangent bundle, $p_1(V_{\mathbb{R}}) = c_1(V)^2 - 2c_2(V)$.*

For a proof, see the notes.

Recall that this cobordism group $\Omega_4^{\text{SO}} \cong \mathbb{Z}$, and that the signature $\tau: \Omega_4^{\text{SO}} \rightarrow \mathbb{Z}$ is an isomorphism. Hence $\sigma(X)$ must be proportional to $\tau(X)$.

Theorem 4.25 (Hirzebruch signature theorem). *$\sigma = 3\tau$, i.e. on a closed, oriented 4-manifold X , $p_1(X) \frown [X] = 3\tau(X)$.*

Proof. The proof is corollary of Thom's work: we just have to check on a generator of Ω_4^{SO} , such as \mathbb{CP}^2 , which has signature 1. Then we use Lemma 4.24: $c_1(T\mathbb{CP}^2) = 3H$ and $c_2(T\mathbb{CP}^2) = \chi(\mathbb{CP}^1) = 3$, so

$$(4.26) \quad p_1(T\mathbb{CP}^2) = c_1^2(\mathbb{CP}^2) - c_2(\mathbb{CP}^2) = 9 - 2 \cdot 3 = 3. \quad \boxtimes$$

This also implies that if X_d is a degree- d hypersurface, its signature is

$$\begin{aligned} \tau(X_d) &= \frac{1}{3}(c_1^2(TX_d) - c_2(X_d)) \frown [X_d] \\ &= -\frac{1}{3}(d-2)d(d+2). \end{aligned}$$

Certainly $(d-2)d(d+2)$ is divisible by 3, so this produces an interger, even if we didn't already know that. Moreover,

$$\begin{aligned} w_2(TX_d) &= c_1(TX_d) \bmod 2 \\ &= (4-d)h \bmod 2 \\ &= dh \bmod 2. \end{aligned}$$

Hence d is even iff $w_2(TX_d) = 0$ iff X_d has even type (i.e. its intersection form has even type). For $d \geq 2$, the intersection form is indefinite, so Theorem 3.15 tells us there's a unique intersection form in this class.

Example 4.27. For $d = 4$, X_d is a *K3 surface*, and one concludes that $c_1 = 0$, $b_2 = 24$, $\tau = -16$, and the intersection form has even type. Hence by Theorem 3.15 the intersection form is $3U \oplus 2(-E_8)$. ◀

The point is that we can explicitly compute intersection forms in examples of interest, and how characteristic classes made this a bit easier.

Lecture 5.

Tangent bundles of 4-manifolds: 2/1/18

This lecture has two goals. The first is to show that if X is a closed, simply-connected manifold, then its tangent bundle is essentially determined by w_2 , τ , and χ . The second is to discuss the following theorem.

Theorem 5.1 (Rokhlin). *Let X be a closed, oriented 4-manifold. Then $16 \mid \tau(M)$.*

There are different proofs of this; today we're going to focus on its equivalence to the following fact about stable homotopy groups of the spheres.

Proposition 5.2. *Let $k \geq 5$. Then $\pi_{3+k}(S^k) \cong \mathbb{Z}/24$.*

Obstruction theory For a reference for this part of the lecture, see Hatcher's notes on vector bundles and K -theory.

Obstruction theory is about the following question: let $\pi: E \rightarrow X$ be a fiber bundle where X is a CW complex. When is there a section of X ?

At first, we will assume X is simply connected; we will be able to weaken this hypothesis later. Let F denote the typical fiber of π , i.e. the fiber over a chosen $x \in X$ (which we'll assume is in the 0-skeleton). We'll construct the section s inductively as a series of sections $s^k: X^{(k)} \rightarrow E|_{X^{(k)}}$ (where $X^{(k)}$ denotes the k -skeleton). If we're given s^k , when does it *not* extend to s^{k+1} ?

Let $\Phi: (D^{k+1}, \partial D^{k+1}) \rightarrow (X^{(k+1)}, X^{(k)})$ denote the inclusion of a $(k+1)$ -cell and $\phi := \Phi|_{\partial D^{k+1}}: S^k \rightarrow X^{(k)}$ be the attaching map. Then $\phi^*(s^k) := s^k \circ \phi$ is a section of the fiber bundle $\phi^*E \rightarrow S^k$. This bundle sits inside of $\Phi^*E \rightarrow D^{k+1}$, which is trivially trivial (i.e. canonically trivialized), because D^{k+1} is contractible. Therefore you can think of ϕ^*s^k as a map to a fiber: $S^k \rightarrow E_{\Phi(0)}$, and canonically up to homotopy, $E_{\Phi(0)} \cong F$ since X is simply connected.

Therefore we have a map $\{(k+1)\text{-cells}\} \rightarrow \pi_k F$ sending Φ to the map $\phi^*s^k: S^k \rightarrow F$. Since $\pi_k(F)$ is an abelian group (here $k > 1$), this map defines a cellular cochain

$$o^{k+1}(E, S^k) \in C^{k+1}(X; \pi_k(F)).$$

Let $\mathfrak{o}^{k+1} \in H^{k+1}(X; \pi_k(F))$ denote the cohomology class of o^{k+1} ; this depends only on the homotopy class of s^k , and if \mathfrak{o}^{k+1} vanishes, then s^k extends to a section s^{k+1} on the $(k+1)$ -skeleton.¹⁰

Definition 5.3. Suppose $\pi_i F = 0$ for $i < k$. Then the *primary obstruction* for E is $\mathfrak{o}^{k+1}(X; \pi_k(F))$.

This is an invariant of the fiber bundle, hence is easier to understand than the higher obstructions, which depend on the choices of s^k that ones makes.

Remark. If we relaxed the assumption that X is simply connected, then we'd have to use local coefficients with the action of $\pi_1(X)$ on $\pi_k(F)$. So if $\pi_1(X)$ acts trivially on $\pi_i F$ for $i \leq k$, the story continues in almost the same way. ◀

¹⁰One may have to replace s^k by a section homotopic to it.

Stiefel-Whitney classes as primary obstructions. Though they are usually presented differently, Stiefel-Whitney classes were historically discovered as obstructions to collections of sections of vector bundles.

Definition 5.4. Let $E \rightarrow X$ be a real, rank- n vector bundle with a Euclidean metric. Then $V_k(E) \rightarrow E$ denotes the *Stiefel bundle*, the fiber bundle whose fiber at x is $V_k(E_x)$, the *Stiefel manifold* of orthonormal k -frames for E_x ; standard fiber-bundle methods construct $V_k(E) \rightarrow E$ in a canonical fashion.

The typical fiber of $V_k(E)$ is $V_k(\mathbb{R}^n)$. A section of $V_k(E)$ is a k -tuple of orthonormal, hence linearly independent, sections of $E \rightarrow X$.

Stiefel and Whitney applied obstruction theory to $V_k(E)$ to understand when E admits linearly independent sections.

Proposition 5.5. $V_k(\mathbb{R}^n)$ is $(n - k - 1)$ -connected, and

$$\pi_{n-k}(V_k(\mathbb{R}^n)) \cong \begin{cases} \mathbb{Z}, & n - k \text{ even or } k = 1, \\ \mathbb{Z}/2, & \text{otherwise.} \end{cases}$$

For example, when $n = 1$, this is telling us that the first homotopy group of S^k is $\pi_k S^k = \mathbb{Z}$. We're not going to prove Proposition 5.5; the idea is to use the long exact sequence of homotopy groups associated to a fiber bundle.

Remark. It's worth thinking through what information is needed to identify a homotopy group with \mathbb{Z} ; there are two choices, so this is something like requiring an orientation. \blacktriangleleft

Hence the primary obstruction to finding a section of $V_k(E) \rightarrow X$ is an $\mathfrak{o}^{n-k+1} \in H^{n-k+1}(X; \pi_{n-k}(V_k(\mathbb{R}^n)))$. Thus we have characteristic classes for E ,

$$\mathfrak{o}_n^k(E) \in \begin{cases} H^{n-k+1}(X; \mathbb{Z}), & n - k \text{ even or } k = 1, \\ H^{n-k+1}(X; \mathbb{Z}/2), & \text{otherwise.} \end{cases}$$

The canonical nature of this construction means these are natural under pullback of vector bundles, hence are indeed characteristic classes.

In either case, there is a mod 2 characteristic class $\bar{\mathfrak{o}}_k^n \in H^{n-k+1}(X; \mathbb{Z}/2)$, and this actually makes sense irrespective of $\pi_1 X$, because $\text{Aut}(\mathbb{Z}/2) = 1$, so all $\mathbb{Z}/2$ local systems are trivial.

Theorem 5.6. $\bar{\mathfrak{o}}_k^n(E) = w_{n-k+1}(E)$.

This perspective on Stiefel-Whitney classes shows that some of them come with canonical integral lifts. For $k = 1$, $\mathfrak{o}_1^n(E) \in H^n(X; \mathbb{Z})$ is the Euler class of E , which requires a choice of orientation on E (unless you use coefficients twisted by the orientation bundle of E). You can take this as the definition of the Euler class if you wish.

Applications to 4-manifolds. For a reference, see the classical papers by Dold-Whitney and Hirzebruch-Hopf.

When $E \rightarrow X$ is a rank-4 real vector bundle, Theorem 5.6 tells us that $w_2(E)$ is the obstruction to finding 3 linearly independent sections of E over $X^{(2)}$. If ℓ denotes an orthogonal complement to these three sections given a metric on E , then $\ell \rightarrow X$ is a line subbundle of E , which is trivial iff $w_1(E) = 0$.

Corollary 5.7. Let X be a closed, oriented 4-manifold. Then $w_2(TX)$ is the complete obstruction to trivializing TX over $X \setminus \text{pt}$.

Such an X is called *almost parallelizable*.

Proof. Choose a metric g for TX and a CW structure for $Y = X \setminus \text{pt}$. Over $Y^{(2)}$, we can find orthonormal vector fields $v_1, v_2, v_3 \in \Gamma(TX|_Y)$ by Theorem 5.6, and also a unit vector $v_4 \in (v_1, v_2, v_3)^\perp$. This is because the line bundle $\ell \subset TX$ is trivial: X is orientable, so $w_1(TX) = 0$.

Let $P \rightarrow Y$ denote the principal SO_4 -bundle of oriented orthonormal frames for Y ; then, (v_1, v_2, v_3, v_4) is a section of $P|_{Y^2} \rightarrow Y^2$. The obstruction to extending this to Y^3 lies in $H^3(Y; \pi_2 \text{SO}_4)$. We'll see later that the universal cover for SO_4 is $S^3 \times S^3$, so

$$(5.8) \quad \pi_2(\text{SO}_4) = \pi_2(S^3 \times S^3) = \pi_2(S^3) \times \pi_2(S^3) = 0.$$

(It's actually true that for any Lie group G , $\pi_2(G) = 0$. But here we don't need the full, harder result.) The upshot is, the obstruction vanishes, so (v_1, \dots, v_4) extends to a section on $Y^{(3)}$. And the cohomology of a punctured 4-manifold vanishes above dimension 3, so all further obstructions vanish. \square

Theorem 5.9. *Let X be a closed, oriented 4-manifold, and suppose that $T, T' \rightarrow X$ are two rank-4 oriented vector bundles such that $w_2(T) = w_2(T') = 0$.¹¹ Then*

- (1) $T \oplus \mathbb{R} \cong T' \oplus \mathbb{R}$ iff $p_1(T) = p_1(T')$, and
- (2) $T \cong T'$ as oriented vector bundles iff $p_1(T) = p_1(T')$ and $e(T) = e(T')$.

Recall that the Euler class is the Poincaré dual to the zero set of a generic section.

To prove this, we'll need a result that will be useful again.

Lemma 5.10. *There exists a rank-4 vector bundle $E \rightarrow S^4$ with $\langle p_1(E), [S^4] \rangle = -2$ and $\langle e(E), [S^4] \rangle = 1$.*

Proof sketch. One can produce this bundle as a representative of the Bott element $\beta \in K^0(S^4)$. A more explicit construction: if $\mathbb{H}P^1$ denotes the *quaternionic projective line*, then $\mathbb{H}P^1 \cong S^4$ (for the same reason $\mathbb{R}P^1 \cong S^1$ and $\mathbb{C}P^1 \cong S^2$), and $\mathbb{H}P^1$ admits a tautological line bundle $\Lambda \rightarrow \mathbb{H}P^1$, which has the correct Pontrjagin and Euler classes. \square

Proof of Theorem 5.9. We'll use *Pontrjagin-Thom collapse* to construct a map $f: X \rightarrow S^4$ which is smooth (at least near an $x \in X$) of degree 1 with $D_x f$ an isomorphism such that $f^{-1}(f(x)) = x$. The idea is to product a map $X \rightarrow D^4$ which sends everything outside a disc neighborhood of X to ∂D^4 , then collapse by the identification $D^4/\partial D^4 \cong S^4$.

Since $w_2(T) = 0$, there's some ball $B \subset X$ containing x such that T is trivial over $X \setminus B$. Therefore $T \cong f^*U$ for some rank-4 oriented bundle $U \rightarrow S^4$. Thus $p_1(T) = f^*p_1(U)$ and $e(T) = f^*e(U)$, and of course the same is true for T' and a $U' \rightarrow S^4$. Therefore it suffices to prove the theorem on S^4 .

Bundles on the 4-sphere aren't so complicated. Let D_+ and D_- denote the two hemispheres, so $S^4 = D_+ \cup_{S^3} D_-$. $U, U' \rightarrow D_\pm$ are canonically trivialized, but the two trivializations of U over D_+ and D_- need not agree; instead, they are related by a map $S^3 \rightarrow \text{SO}_4$ called the *clutching function*, which defines an element of $\pi_3\text{SO}_4$.

Conversely, given a $[\gamma] \in \pi_3\text{SO}_4$, one can construct a vector bundle $E_\gamma \rightarrow \text{SO}_4$ by gluing the trivial bundles on D_+ and D_- by any representative γ for $[\gamma]$. In the same way, rank 5 vector bundles on S^4 correspond to elements of $\pi_3\text{SO}_5$.

Now we prove part (1). By what we've shown so far, $U \oplus \mathbb{R} \cong U' \oplus \mathbb{R}$ iff they have the same clutching function in $\pi_3\text{SO}_5 \cong \mathbb{Z}$.¹² The first Pontrjagin number defines a homomorphism

$$\begin{aligned} \pi_3\text{SO}(5) &\longrightarrow \mathbb{Z} \\ [\gamma] &\longmapsto \langle p_1(E_\gamma), [S^4] \rangle, \end{aligned}$$

and by Lemma 5.10, there exists a vector bundle on S^4 with nonzero Pontrjagin number. Hence this map is injective.

To prove part (2), we look at $\pi_3\text{SO}_4 \cong \pi_3(S^3 \times S^3) = \pi_3S^3 \times \pi_3S^3 = \mathbb{Z}^2$. Hence we can define a map

$$\begin{aligned} \pi_3(\text{SO}_4) \cong \mathbb{Z}^2 &\longrightarrow \mathbb{Z}^2 \\ [\gamma] &\longmapsto (\langle p_1(E_\gamma), [S^4] \rangle, \langle e(E_\gamma), [S^4] \rangle). \end{aligned}$$

Again, using Lemma 5.10, there's a bundle E whose image under the above map is $(-2, 1)$. For TS^4 , $p_1 = 0$ and $e = 2$, so the above map is given by the matrix

$$\begin{pmatrix} -2 & 0 \\ 1 & 2 \end{pmatrix},$$

hence is injective. \square

Remark. The theorem is true more generally assuming that $w_2(T) = w_2(T') \neq 0$: p_1 determines their stable isomorphism class and (p_1, e) determines their unstable isomorphism class. This requires a more sophisticated use of obstruction theory, and the proof is more involved. \blacktriangleleft

¹¹So we could say that T and T' are two spin vector bundles.

¹²Again, if G is any simple Lie group, $\pi_3G = \mathbb{Z}$.

Corollary 5.11. *Suppose that X and X' are closed, oriented, simply-connected 4-manifolds and $f: X' \rightarrow X$ is a homotopy equivalence. Then $f^*TX \cong TX'$ as oriented vector bundles.*

Proof. The intersection form Q_X determines $p_1 = 3\tau$ and $e = \chi$, hence the isomorphism classes of f^*TX and TX' coincide. \square

Thus TX knows no more than the (oriented) homotopy type of a closed, oriented, simply-connected manifold.

Next time we'll turn to Rokhlin's theorem.

Lecture 6.

Rokhlin's theorem and the homotopy theorem: 2/6/18

Today we'll relate Rokhlin's theorem to the homotopy-theoretic fact that $\pi_8(S^5) \cong \mathbb{Z}/24$. We will then prove a theorem of Milnor that the intersection form of a simply-connected 4-manifold determines its homotopy type (which also knows the tangent bundle, as we saw last time).

Remark. Rokhlin was Russian, and there are at least three different ways of spelling his name in English. Keep this in mind when looking for references on this theorem. \blacktriangleleft

Theorem 6.1 (Rokhlin). *Let X be a closed, oriented, smooth manifold with $w_2(TX) = 0$. Then $16 \mid \tau(X)$.*

We've seen that the signature of an even unimodular form is $0 \bmod 8$, so the key is getting another factor of 2. By the Hirzenbruch signature theorem, Rokhlin's theorem is equivalent to $48 \mid \langle p_1(TX), [X] \rangle$. This is why the $\mathbb{Z}/24$ that we mentioned arises.

Theorem 6.2. *Rokhlin's theorem is equivalent to the theorem that $\pi_8(S^5) \cong \mathbb{Z}/24$.*

We will later directly prove Rokhlin's theorem using index theory.

Remark. This is a *stable homotopy group*: for $k \geq 5$, the suspension map $\pi_{3+k}(S^k) \rightarrow \pi_{3+k+1}(S^{k+1})$ is an isomorphism, which is a nontrivial theorem in homotopy theory. For $k < 5$, different groups can arise. \blacktriangleleft

Remark. One can compute $\pi_8(S^5)$ using methods internal to homotopy theory. Rokhlin was the first to do this, albeit with a more geometric flavor. Before this, Serre invented a localization method for calculating the first p -torsion of $\pi_*(S^k)$ for p prime, which shows that $\pi_8(S^5)$ has order $2^k \cdot 3$ for some k . One can characterize the 2-primary part using, e.g., the Adams spectral sequence.

Alternatively, there's a comprehensive story due to Adams and Quillen on the image of the J -homomorphism in $\pi_*^s(S^0)$, which touches on deep and difficult ideas. This is the first nontrivial example, and in this dimension it's surjective. \blacktriangleleft

The proof of Theorem 6.2 proceeds via the Pontrjagin-Thom construction. This is an awesome thing, and you should definitely fill in the details and/or read the references.

Definition 6.3. Consider closed k -dimensional manifolds M embedded in \mathbb{R}^{k+m} . Assume we're in the *stable range*, which means $m > k$. A *normal framing* for M is a trivialization of the normal bundle $\nu_M \rightarrow M$, an isomorphism $\phi: \nu_M \rightarrow \underline{\mathbb{R}}^m$. Typically only the homotopy class of ϕ is important.

Definition 6.4. Let (M_0, ϕ_0) and (M_1, ϕ_1) be normally framed submanifolds of \mathbb{R}^{m+k} . A *framed cobordism* is data (P, Φ) , where P is a compact submanifold $P \subset \mathbb{R}^{m+k} \times [0, 1]$ with boundary $\partial P \subset \mathbb{R}^{m+k} \times \{0, 1\}$ such that

$$(6.5) \quad \partial P = (M_0 \times \{0\}) \amalg (M_1 \times \{1\}).$$

We assume a transversality condition; see the notes for details. The second piece of data is a framing $\Phi: \nu_P \rightarrow \underline{\mathbb{R}}^m$ which extends ϕ_0 and ϕ_1 .

The set of equivalence classes of framed submanifolds of \mathbb{R}^{m+k} under framed cobordism forms an abelian group under disjoint union.¹³ This group is called the *framed cobordism group*¹⁴ and denoted Ω_k^{fr} .

¹³The fact that cobordism is an equivalence relation comes from the fact that one can glue cobordisms if they share a boundary component.

¹⁴Sometimes it's instead called a *framed bordism group*.

Lemma 6.6. *As the notation suggests, the inclusion $\mathbb{R}^{m+k} \hookrightarrow \mathbb{R}^{m+k+1}$ induces an isomorphism on framed bordism groups for normally framed k -manifolds.*

This is not too hard to see; the upshot is that there's not too much homotopical information contained in a normal framing in high codimension.

Definition 6.7. There is a group homomorphism $PT: \pi_{k+m}(S^m) \rightarrow \Omega_k^{fr}$ called the *Pontrjagin-Thom homomorphism* defined as follows: given an $[f] \in \pi_{k+m}(S^m)$, represent it by a smooth map $f: S^{k+m} \rightarrow S^m$. Then, $PT(f) := f^{-1}(x)$, where x is a regular value. The framing comes from the framing of a neighborhood of x , which is diffeomorphic to \mathbb{R}^n , and this framing pulls back to $f^{-1}(x)$.

There is a lot to check here, including:

- Any map $f: S^{k+m} \rightarrow S^m$ is homotopic to a smooth map.
- The framed cobordism class of $PT(f)$ doesn't depend on the choice of smooth representative of $[f]$.
- The framed cobordism class of $PT(f)$ doesn't depend on the regular value chosen. This is because any two regular values in S^n are connected by a regular path.

Theorem 6.8 (Pontrjagin-Thom). *The Pontrjagin-Thom map is an isomorphism.*

The rough idea is to construct an inverse map, called *Pontrjagin-Thom collapse*.

Example 6.9. For $k = 0$, this is a fact you likely already know: homotopy classes of maps $S^m \rightarrow S^m$ are classified by their degree, and the degree is defined to be the number of preimages of a regular value. ◀

Another ingredient that we need for Theorem 6.2 is the J -homomorphism, a map

$$(6.10) \quad J_k^n: \pi_k(\mathrm{SO}_m) \longrightarrow \pi_{k+m}(S^m).$$

Suppose $[\theta] \in \pi_k(\mathrm{SO}_m)$, and let $\theta: (S^k, *) \rightarrow (\mathrm{SO}_m, I)$ be a map; in particular, it's a map of pointed spaces. Thus for each $x \in S^k$, $\theta(x): \mathbb{R}^m \rightarrow \mathbb{R}^m$ is an isometry, hence restricts to a map

$$(6.11) \quad \theta(x): (\overline{D}^n, \partial \overline{D}^n) \longrightarrow (\overline{D}^n, \partial \overline{D}^n),$$

where $\overline{D}^n \subset \mathbb{R}^n$ denotes the closed unit disc. Let

$$(6.12) \quad \tilde{J}(\theta): (S^k \times \overline{D}^m, S^k \times \partial \overline{D}^m) \longrightarrow (\overline{D}^m, \partial \overline{D}^m)$$

send

$$(6.13) \quad (x, y) \longmapsto \theta(x)(y).$$

This map descends to a map on the quotients, a map

$$(6.14) \quad J(\theta): (S^k \times \overline{D}^m) / (S^k \times \partial \overline{D}^m) \cong S^{k+m} \longrightarrow \overline{D}^m / \partial \overline{D}^m \cong S^m.$$

We made choices, and one can show that the homotopy class of $J(\theta)$ is independent of those choices.

Remark. In the stable range $k > m + 1$, the inclusion $\mathrm{SO}_m \hookrightarrow \mathrm{SO}_{m+1}$ induces an isomorphism on π_k , and suspension induces an isomorphism $\pi_{k+m}(S^m) \rightarrow \pi_{k+m+1}(S^{m+1})$; the J -homomorphism is compatible with these isomorphisms. ◀

The Pontrjagin-Thom theorem provides a geometric interpretation of the homotopy groups of the spheres; in view of that, we would like a geometric interpretation of the J -homomorphism. $\pi_k(\mathrm{SO}_m)$ acts simply transitively on the set of framings of $S^k \subset \mathbb{R}^{k+m}$, and there is a basepoint, namely the framing coming from the usual framing on \mathbb{R}^n . Therefore the set of framings of $S^k \subset \mathbb{R}^{k+m}$ is canonically identified with $\pi_k(\mathrm{SO}_m)$.

Proposition 6.15. *Under the identifications of $\pi_k(\mathrm{SO}_m)$ with the framings of $S^k \subset \mathbb{R}^m$ and $\pi_{m+k}(S^m) \cong \Omega_k^{fr}$, the J -homomorphism is the map sending S^k with a framing to its framed cobordism class.*

This was left as an exercise. It's not clear who originally came up with this, but it is used heavily by Kervaire and Milnor.

Now we specialize to $k = 3$ and $m = 5$, so we're interested in framings of $S^3 \subset \mathbb{R}^8$, and more generally of 3-manifolds in \mathbb{R}^8 .

Theorem 6.16. J_3 is surjective.

The proof uses spin cobordism, which we haven't discussed; a sketch is in the notes. Therefore we have a sequence

$$\pi_3 \mathrm{SO}_5 \cong \mathbb{Z} \xrightarrow{J_3} \Omega_3^{fr} \cong \pi_8(S^5) \longrightarrow 0,$$

as π_3 of any simple Lie group is \mathbb{Z} .

Proof sketch of Theorem 6.2. First, we'll assume $\pi_8(S^5) \cong \mathbb{Z}/24$, so $\ker(J_3) = 24\mathbb{Z}$. Let X be a closed, oriented 4-manifold with $w_2(TX) = 0$. By Corollary 5.7, TX is trivial after any disc D is removed from X . By the Whitney theorem, we can embed $X \hookrightarrow \mathbb{R}^9$, and over $X \setminus D$, we can choose a trivialization $\Phi: N_{X \setminus D} \rightarrow \mathbb{R}^5$.

In general, we can't extend Φ over D ; the obstruction $\mathfrak{o}(\nu, \Phi) \in \pi_3(\mathrm{SO}_5)$ is the element represented by Φ on $\partial(X \setminus D) = S^3$. In particular, $\Phi|_{S^3}$ is a framing of S^3 that lies in $\ker(J_3)$, because it's a framing of a 3-manifold that extends to a framed 4-manifold $X \setminus D$. Thus $\mathfrak{o}(\nu, \Phi) \in 24\mathbb{Z}$.

Recall from Lemma 5.10 that there's a rank-4 bundle $E \rightarrow S^4$ with $\langle p_1(E), [S^4] \rangle = -2$ and $\langle e(E), [S^4] \rangle = 1$. This implies that the clutching function of $E \oplus \mathbb{R}$ is a generator of $\pi_3(\mathrm{SO}_5)$. The upshot (there's a step or two to fill in here) is that

$$(6.17) \quad 2\mathfrak{o}(\nu, \Phi) = \pm p_1(X).$$

Since $\langle \mathfrak{o}(\nu, \Phi), [X] \rangle$ is divisible by 24, then $\langle \pi_1(X), [X] \rangle$ is divisible by 48.

Conversely, assume Rokhlin's theorem and let ϕ be a framing for S^3 , and suppose it represents a class $[a] \in \ker(J_3) \subset \mathbb{Z}$, so there is a framed 4-manifold (P, Φ) bounding (S^3, ϕ) . Let $X := P \cup_{S^3} D^4$; since $w_2(TP) = 0$, $w_2(TX) = 0$. Rokhlin's theorem implies $48 \mid \langle p_1(X), [X] \rangle$, and this is twice the obstruction a , so a is divisible by 24. Hence $\ker(J_3) \subset 24\mathbb{Z}$.

To get that it's equal, you need an example. Let X be a K3 surface (a quartic in \mathbb{CP}^3); one can show that $\langle p_1(X), [X] \rangle = -48$, which suffices. \square

Now we'll have an interlude on relative homotopy groups. Let (X, A, x) be a based pair of topological spaces, i.e. a topological space X , a subspace A , and a basepoint $x \in A$.

Definition 6.18. The n^{th} relative homotopy set of (X, A, x) is the set $\pi_n(X, A, x)$ of based maps

$$(D^n, \partial D^n, 0) \longrightarrow (X, A, x)$$

This is a pointed set, whose basepoint is the class of any map landing in A : since D^n is contractible, these are all homotopic.

For $n \geq 2$, this pointed set is a group with identity element e . The idea is that there is a collapsing map $c: D^n \rightarrow D^n \vee D^n$ (pinch the sides of the disc inward), which is a based relative map

$$(6.19) \quad (D^n, \partial D^n) \longrightarrow (D^n \vee D^n, \partial D^n \vee \partial D^n).$$

Therefore we can define $f * g := (f \vee g) \circ c$, and this defines a group structure on $\pi_n(X, A, x)$. If $n \geq 3$, this group is abelian, but it might not be for $n = 2$, for the same reason that fundamental groups need not be abelian.

We now summarize a few properties of these groups.

Proposition 6.20 (Long exact sequence of a pair). *If (X, A, x) is a based pair of spaces, there is a long exact sequence of pointed sets*

$$\cdots \longrightarrow \pi_n(A, x) \longrightarrow \pi_n(X, x) \longrightarrow \pi_n(X, A, x) \xrightarrow{\delta} \pi_{n-1}(A, x) \longrightarrow \cdots.$$

For $n \geq 3$ this is a long exact sequence of abelian groups; for $n \geq 2$ this is a long exact sequence of groups.

Definition 6.21. The Hurewicz map

$$h: \pi_n(X, A, x) \longrightarrow H_n(X, A)$$

is defined to send $[f] \mapsto f_*[D^n, \partial D^n]$.

This is a group homomorphism for $n \geq 2$.

Theorem 6.22. *The Hurewicz maps induce a commutative diagram*

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & \pi_n(A, x) & \longrightarrow & \pi_n(X, x) & \longrightarrow & \pi_n(X, A, x) \xrightarrow{\delta} \pi_{n-1}(A, x) \longrightarrow \cdots \\
 & & \downarrow h & & \downarrow h & & \downarrow h \\
 \cdots & \longrightarrow & H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, A) \xrightarrow{\delta} H_{n-1}(A) \longrightarrow \cdots
 \end{array}$$

Definition 6.23. A pair of spaces (X, A) is *n-connected* if for all $i \leq n$, every map $(D^i, \partial D^i) \rightarrow (X, A)$ is homotopic to a map $(D^i, \partial D^i) \rightarrow (A, A)$ (as these are all homotopic).

This implies that $\pi_i(X, A, x) = 0$ for $1 \leq i \leq n$ and all $x \in A$.

The key piece of this that we'll need next time is a relative version of the Hurewicz theorem.

Theorem 6.24 (Relative Hurewicz). *Let (X, A) be an $(n-1)$ -connected pair, where $n \geq 2$ and A is nonempty and simply connected. Then,*

- (1) $H_i(X, A) = 0$ for all $i < n$, and
- (2) the Hurewicz map $h: \pi_n(X, A) \rightarrow H_n(X, A)$ is an isomorphism.

That is, the first nontrivial homotopy and homology groups coincide. This is useful in similar ways to the classical Hurewicz theorem.

Next time, we'll define an isomorphism between $\pi_3((S^2)^{\vee n})$ and the $n \times n$ symmetric matrices over \mathbb{Z} , which is the key step in the homotopy theorem.

Lecture 7.

The homotopy theorem: 2/8/18

Lecture 8.

The Hodge star: 2/13/18

Lecture 9.

Hodge theory and self-duality in 4 dimensions: 2/15/18

Today we're going to talk about self-duality in dimension 4 from the perspective of Hodge theory and global differential geometry. We'll start with the general story, then specialize to dimension 4.

The Hodge theorem. Let (M, g) be a closed, oriented Riemannian manifold of dimension n . Recall from last time that this structure induces a linear map called the *Hodge star* $\star: \Lambda^k(T_x^*M) \rightarrow \Lambda^{n-k}(T_x^*M)$ for any $x \in M$. Globally this induces a map of vector bundles $\star: \Lambda^k(T^*M) \rightarrow \Lambda^{n-k}(T^*M)$.

Definition 9.1. The *co-differential* $d^*: \Omega^{k+1}(M) \rightarrow \Omega^k(M)$ is the map

$$d^* := (-1)^{k+1} \star^{-1} d \star = (-1)^{nk+1} \star d \star.$$

Since $d^2 = 0$ and $\star^2 = \pm 1$, then $(d^*)^2 = 0$.

We don't need compactness to define d^* , but if M is compact, something nice happens.

Proposition 9.2. d^* is the formal adjoint to d with respect to the L^2 inner product on $\Omega^k(M)$ defined by

$$(9.3) \quad \langle \alpha, \beta \rangle_{L^2} := \int_M g(\alpha, \beta) dx.$$

In particular, the definition of d^* was completely local, but its identification as the adjoint is global.

Proof. Applying Stokes' theorem to the identity

$$(9.4) \quad d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d\beta$$

shows that

$$(9.5) \quad \int_M d\alpha \wedge \beta = (-1)^{|\alpha|+1} \int_M \alpha \wedge d\beta,$$

so letting $\beta = \star\gamma$, we get

$$(9.6) \quad \int_M g(d\alpha, \gamma) dx = (-1)^{k+1} \int_M g(\alpha, \star^{-1}d\star\gamma) dx.$$

That is, $\langle d\alpha, \beta \rangle_{L^2} = \langle \alpha, d^*\beta \rangle_{L^2}$. □

Definition 9.7. Let (M, g) be a closed, oriented Riemannian manifold. The *Hodge Laplacian* on X is the map $\Delta: \Omega^k(M) \rightarrow \Omega^k(M)$ defined by

$$\Delta := dd^* + d^*d = (d + d^*)^2.$$

The kernel of Δ is called the space of *harmonic k -forms* and denoted \mathcal{H}_g^k .

In general, \mathcal{H}_g^k is some infinite-dimensional space; clearly, it contains $\ker(d + d^*)$.

Proposition 9.8. *If M is compact, $\mathcal{H}_g^k = \ker(d + d^*)$.*

Proof. It suffices to prove the forward inclusion. Since

$$\begin{aligned} \langle \alpha, \Delta\alpha \rangle_{L^2} &= \langle \alpha, d^*d\alpha + dd^*\alpha \rangle_{L^2} \\ &= \langle d\alpha, d\alpha \rangle_{L^2} + \langle d^*\alpha, d^*\alpha \rangle_{L^2} \\ &= \|d\alpha\|_{L^2}^2 + \|d^*\alpha\|_{L^2}^2, \end{aligned}$$

then if $\Delta\alpha = 0$, then $(d + d^*)\alpha = 0$. □

Therefore on a compact manifold, harmonic forms are also solutions to a first-order differential equation. $d + d^*$ is a kind of Dirac operator, which implies lots of nice things about harmonic forms.

There's also a variational characterization of harmonic forms. Harmonic forms are closed, hence have classes in de Rham cohomology.

Lemma 9.9. *Harmonic forms strictly minimize $\|\cdot\|_{L^2}^2$ of closed k -forms in their de Rham cohomology class. That is, if $\eta \in \mathcal{H}_g^k$ and $\tilde{\eta}$ is cohomologous to η , then $\|\eta\|_{L^2}^2 \leq \|\tilde{\eta}\|_{L^2}^2$, and equality occurs iff $\tilde{\eta} = \eta$.*

Proof. Let $\gamma \in \Omega^{k-1}(M)$. Then

$$\begin{aligned} \|\eta + d\gamma\|_{L^2}^2 - \|\eta\|_{L^2}^2 &= 2\langle \eta, d\gamma \rangle_{L^2} + \|d\gamma\|_{L^2}^2 \\ &= 2\langle d^*\eta, \gamma \rangle_{L^2} + \|d\gamma\|_{L^2}^2 \\ &= \|d\gamma\|_{L^2}^2, \end{aligned}$$

so the difference is nonzero if γ is nonzero. □

Corollary 9.10. *Every de Rham cohomology class contains at most one harmonic representative.*

Lemma 9.9 admits a converse.

Lemma 9.11. *Let $\eta \in \Omega^k(M)$ be closed. If η minimizes $\|\cdot\|_{L^2}^2$ in its de Rham cohomology class, then η is harmonic.*

Proof. Since η is a minimizer, then for any $\gamma \in \Omega^{k-1}(M)$,

$$(9.12) \quad \left. \frac{d}{dt} \right|_{t=0} \|\eta + t d\gamma\|_{L^2}^2 = 0.$$

Thus $2\langle \eta, d\gamma \rangle_{L^2} = 0$. Since γ is arbitrary, choose $\gamma = d^*\eta$. Hence

$$(9.13) \quad 0 = \langle \eta, dd^*\eta \rangle_{L^2} = \|d^*\eta\|_{L^2}^2. \quad \square$$

Remark. Even if η isn't closed, minimizing the norm still implies something interesting: that it's *co-closed* (i.e. $d^*\eta = 0$). ◀

Inside $\Omega^k(M)$, $\ker(d) \perp \text{Im}(d^*)$ under the L^2 inner product, and in fact $\ker(d) = (\text{Im}(d^*))^\perp$. This is because if $\langle \eta, d^*\omega \rangle_{L^2} = 0$ for all ω , then $\langle d\eta, \omega \rangle_{L^2} = 0$, so $\|d\eta\|_{L^2}^2 = 0$, so $d\eta = 0$.

The next theorem is powerful, and also requires more nontrivial analysis.

Theorem 9.14 (Hodge theorem). *Let (M, g) be a compact, oriented Riemannian manifold. Then there exists an L^2 -orthogonal decomposition*

$$\begin{aligned}\Omega^k(M) &= \ker(d) \oplus \operatorname{Im}(d^*) \\ \ker(d) &= \mathcal{H}_g^k \oplus \operatorname{Im}(d).\end{aligned}$$

The quotient map $\mathcal{H}_g^k \rightarrow H_{\text{dR}}^k(M)$ is an isomorphism.

Proof idea. The key part of the proof is the existence of a harmonic representative in any cohomology class. This problem breaks down into two steps.

- (1) First, one finds a *weak solution* to $\Delta\eta = 0$, meaning a solution lying in some Sobolev space L_ℓ^2 .
- (2) Second, one uses *elliptic regularity* to show that $\eta \in \bigcap_\ell L_\ell^2 = C^\infty$, and hence the solution is smooth.

Both steps use elliptic estimates to show that Δ is a bounded operator between certain Sobolev spaces. \square

So instead of working with quotient spaces of infinite-dimensional vector spaces, which can be unwieldy, you can work with finite-dimensional vector spaces, and this is sometimes nicer.

Remark. In general, the wedge product of two harmonic forms isn't harmonic, so it's difficult to see the ring structure of cohomology using Hodge theory. \blacktriangleleft

We'll have more to say about this later, but for now we'll assume the Hodge theorem and specialize to dimension 4.

(Anti)-self-dual harmonic 2-forms. Let (X, g) be a closed, oriented, Riemannian 4-manifold. Then $\star: \Lambda^2(T^*X) \rightarrow \Lambda^2(T^*X)$ squares to 1. Therefore, as in the last lecture, there is a splitting

$$(9.15) \quad \Lambda^2(T^*X) = \Lambda_{[g]}^+ \oplus \Lambda_{[g]}^-,$$

where $\Lambda_{[g]}^\pm$ is the ± 1 -eigenspace of \star on $\Lambda^2(T^*X)$. Let

$$(9.16) \quad \Omega_{[g]}^\pm := C^\infty(X, \Lambda_{[g]}^\pm);$$

these are called the *self-dual* (for 1) or *anti-self-dual* (for -1) 2-forms on X .

Definition 9.17. Let M be a manifold. A *conformal structure* on M is an equivalence class of Riemannian metrics, where $g_1 \sim g_2$ if $g_1 = fg_2$ for some function $f: M \rightarrow \mathbb{R}_{>0}$.

That is, a conformal structure is a metric up to rescaling.

Last lecture's results apply wholesale: in particular, \star only depends on the conformal class of the metric. This is special to $k = 2$: in general scaling the metric by λ scales \star by $\lambda^{k-n/2}$.

Since $d^* = -\star d \star$ in this dimension, then if $\eta \in \Omega^2(X)$, $\eta \in \ker(d + d^*)$ iff $\star\eta \in \ker(d + d^*)$. Hence if $\eta \in \mathcal{H}_g^2$, then its components

$$(9.18) \quad \eta^\pm := \frac{1}{2}(1 + \star)\eta \in \Omega_{[g]}^\pm$$

are also harmonic. Therefore if $\mathcal{H}_g^\pm := \mathcal{H}_g^2 \cap \Omega_{[g]}^\pm$, there's a splitting

$$(9.19) \quad \mathcal{H}_g^2 = \mathcal{H}_g^+ \oplus \mathcal{H}_g^-.$$

Suppose $\eta \in \mathcal{H}_g^+ \setminus 0$. Then

$$(9.20) \quad \int_X \eta \wedge \eta = \int_X \eta \wedge \star\eta = \int_X |\eta|^2 dV > 0,$$

and similarly, if $\omega \in \mathcal{H}_g^- \setminus 0$, $\int_X \omega \wedge \omega < 0$.

Therefore the conformal class of g determines maximally positive and negative definite subspaces of the quadratic form $\eta \mapsto \int_X \eta \wedge \eta$ on $H_{\text{dR}}^2(X) \cong \mathcal{H}_g^2$, namely \mathcal{H}_g^\pm . Let $b^\pm(X) := \dim \mathcal{H}_g^\pm$, which is the dimension of the maximally positive (resp. negative) subspace of the intersection pairing on de Rham cohomology.

In general, $b^+ \neq b^-$, and in fact their difference is the signature.

Remark. Let η be self-dual (resp. anti-self-dual). Then η is harmonic iff it's closed; this is because $d^* = -\star d \star$, so $d^*\eta = -\star d(\star\eta)$ and $\star\eta = \pm\eta$. \blacktriangleleft

Definition 9.21. Let (X, g) be as above and

$$d^\pm := \frac{1}{2}(1 \pm \star) \circ d.$$

The *signature complex*¹⁵ is the cochain complex

$$0 \longrightarrow \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d^+} \Omega_g^+ \longrightarrow 0.$$

We will denote the i^{th} term of this complex by \mathcal{E}^i and the differentials by D , and will denote the whole complex by (\mathcal{E}^*, D) .

Theorem 9.22. *There are isomorphisms*

$$H^i(\mathcal{E}) \cong \begin{cases} H_{\text{dR}}^i(X), & i = 0, 1 \\ \mathcal{H}_g^+(X), & i = 2. \end{cases}$$

Proof. The case $i = 0$ is true by definition. For $i = 1$, it suffices to prove that if $\alpha \in \Omega^1(X)$ is in $\ker(d^+)$, then it's closed. Since $d\alpha = d^+\alpha + d^-\alpha$, then

$$(9.23) \quad \int_X d\alpha \wedge d\alpha = \int_X |d^+\alpha|^2 dV - \int_X |d^-\alpha|^2 dV.$$

Since M is closed,

$$(9.24) \quad \int_X d\alpha \wedge d\alpha = \int_X d(\alpha \wedge d\alpha) = 0,$$

so for all α , $\|d^+\alpha\|_{L^2}^2 = \|d^-\alpha\|_{L^2}^2$. Therefore if $d^+\alpha = 0$, then $d^-\alpha = 0$ too, so $d\alpha = 0$.

For $i = 2$, we'll show that $\Omega_{[g]}^+ / \text{Im}(d^+) \cong \mathcal{H}_g^+$, or more precisely, that the composition

$$(9.25) \quad \mathcal{H}_g^+ \hookrightarrow \Omega_{[g]}^+ \longrightarrow \Omega_{[g]}^+ / \text{Im}(d^+)$$

is an isomorphism.

Let $\omega \in \Omega_{[g]}^+$; then, it has a *Hodge decomposition*

$$(9.26) \quad \omega = \omega_h + d\alpha + \star d\beta,$$

where $\omega_h \in \mathcal{H}_g^2$, $d\alpha \in \text{Im}(d)$, and $\star d\beta \in \text{Im}(d^*)$. The components are unique; hence, if ω is self-dual, $\star\omega_h = \omega_h$ and $d\beta = d\alpha$. Thus

$$(9.27) \quad \omega = \underbrace{\omega_h}_{\mathcal{H}_g^+} + \underbrace{d\alpha + \star d\alpha}_{2d^+\alpha \in \text{Im}(d^+)}. \quad \square$$

This calculation will be relevant for computing the dimensions of Seiberg-Witten moduli spaces, as will another involving Euler characteristics that we will discuss later.

Definition 9.28. Fix an $r \geq 3$ and let $\text{Conf } X$ denote the space of conformal structures on X , i.e. the space of C^r -Riemannian metrics modulo the space of positive C^r functions.

In the future, we will need to do some analysis for which C^r regularity, for which things are Banach spaces, are better than C^∞ regularity, where we merely have a Fréchet space.

Fix a $[g_0] \in \text{Conf } X$ and let $\Lambda^\pm := \Lambda_{[g_0]}^\pm$. This provides an identification of $\text{Conf } X$ with the space of C^r bundle maps $m: \Lambda^+ \rightarrow \Lambda^0 -$ such that **TODO**: given a conformal structure $[g]$, $\Lambda_{[g]}^- = \Gamma_m$, the graph of m , for some such m , and $\Lambda_{[g]}^+ = \Gamma_{m^*}$; this was discussed last time.

Hence $\text{Conf } X$ is an open subset of a Banach space, and is particular a Banach manifold. We have an identification of its tangent spaces as Banach spaces:

$$(9.29) \quad T_{[g_0]} \text{Conf } X = C^r(X, \text{Hom}(\Lambda^-, \Lambda^+)).$$

Definition 9.30. The *period map* is the map $P: \text{Conf } X \rightarrow \text{Gr}_{b-(X)} H_{\text{dR}}^2(X)$ sending $[g] \mapsto \mathcal{H}_g^- \subset \mathcal{H}_g^2 \cong H_{\text{dR}}^2(X)$.

¹⁵This is only half of the signature complex; it's all that we need, but in other sources the signature complex generally refers to something larger.

Studying the period map informs us how the space of anti-self-dual forms changes as the metric changes. The first thing we should do is compute the derivative of P .

Since $[g_0] \in \text{Conf } X$ has a neighborhood which is a neighborhood of 0 in $C^r(X, \text{Hom}(\Lambda^-, \Lambda^+))$, then a neighborhood of $\mathcal{H}_{[g_0]}^-$ is identified with a neighborhood of 0 in $\text{Hom}(\mathcal{H}_{[g_0]}^-, \mathcal{H}_{[g_0]}^+)$: a subspace is sent to a map which has it as its graph.

Thus, near $[g_0]$, we may think of P as a map $C^r(X, \text{Hom}(\Lambda^-, \Lambda^+)) \rightarrow \text{Hom}(\mathcal{H}_{g_0}^-, \mathcal{H}_{g_0}^+)$. Therefore its derivative is a map

$$(9.31) \quad D_{[g_0]}P: C^r(X, \text{Hom}(\Lambda^-, \Lambda^+)) \longrightarrow \text{Hom}(\mathcal{H}_{g_0}^-, \mathcal{H}_{g_0}^+).$$

Proposition 9.32. *Under this identification, if m is a bundle map and $\alpha^- \in \mathcal{H}_{g_0}^-$,*

$$(D_{[g_0]}P)(m)(\alpha^-) = (m(\alpha^-))_h.$$

Corollary 9.33. *The period map is a submersion.*

Thus you can move around the space of anti-self-dual forms pretty freely. This will be useful for transversality theory.