MATH 215C NOTES

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These notes were taken in Stanford's Math 215C class in Spring 2015, taught by Jeremy Miller. I TEXed these notes up using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to adebray@stanford.edu. Thanks to Jack Petok for catching a few errors.

CONTENTS

1. SMOOTH MANIFOLDS: 4/7/15

"Are quals results out yet? I remember when I took this class, it was the day results came out, and so everyone was paying attention to the one smartphone, since this was seven years ago."

This class will have a take-home final, but no midterm. It's an important subject, but there aren't any really awesome theorems, which is kind of sad.

Here are some goals of this class:

- Definitions: manifolds, the tangent bundle, etc.
- Basic properties of manifolds: transversality, embedding theorems (into Euclidean space).
- Differential forms, Stokes' theorem, de Rham cohomology. Some of you may have learned this in a fancy multivariable calculus class. If 215C has a punchline, it's that de Rham cohomology is the same as regular cohomology, which is elegant but not all that helpful for doing stuff.
- Intersection theory, and the idea that intersection is dual to the cup product. We'll also talk about characteristic classes a little bit, which is supposed to be in the last quarter of a second-year graduate topology class, so we'll see what happens.
- Morse theory, which is another homology theory that ends up being the same (chain complexes built out of functions from a manifold to $\mathbb{R}$), but this is useful e.g. for using algebraic topology to provide bounds on critical points of functions.

Definition. A **manifold** is a paracompact Hausdorff space M such that for all $x \in M$, there exists an open $U \subseteq M$ such that $x \in U$ and $U \cong \mathbb{R}^n$. In this case, we say that the **dimension** of M is n.

Recall that **paracompact** means that every open cover has a subcover such that each point has a subcover containing only finitely many sets, and that **Hausdorff** means that any two points can be separated by open sets (each has an open neighborhood not containing the other).

One may also want the manifold to be **second countable**, i.e. it has a countable basis; the exceptions include things with infinitely many components. Second countability implies paracompactness, and we won't be working in the boundary between them much anyways.

Example 1.1. Let $M = \mathbb{R} \times \{0\} \coprod \mathbb{R} \times \{1\}$, where $(x,0) \sim (y,1)$ if $x = y$ and $x \neq 0$. Thus, M looks like $\mathbb{R}$, but has two copies of the origin. This (topologized with the quotient topology) is locally Euclidean, but not Hausdorff.

Example 1.2. Let ω be the first uncountable ordinal, and let $R = \omega \times [0,1)$, with the order topology. Then, R is called the **long ray**. Let L be R without its smallest point; then, both L and R are locally Euclidean and Hausdorff, but it's not paracompact, which is a confusing digression into set theory (now that you mention it, what exactly is the first uncountable ordinal?).

Later on, this will be a nice counterexample to the notion that homomorphisms of homotopy groups determine a space up to homotopy; this is only true for nicer spaces. The higher homotopy groups vanish, but it isn't contractible (which is painful to make rigorous; intuitively, it would take "too long").

In this class, though, you'll only need to know enough logic and point-set topology to know that these issues have been avoided.

Definition.

- Let M be a manifold; then, an **atlas** on M is a collection of open sets $\{U_\alpha\}$ that covers M and a collection of homeomorphisms $\varphi_\alpha : U_\alpha \xrightarrow{\cong} \mathbb{R}^n$. The pairs $(U_\alpha, \varphi_\alpha)$ are called **charts**.
- A **smooth structure** on a manifold M is an atlas $\{(U_\alpha, \varphi_\alpha)\}$ such that whenever $U_\alpha \cap U_\beta \neq \emptyset$, the map $\varphi_\beta|_{U_\alpha \cap U_\beta} \circ (\varphi_\alpha|_{U_\alpha \cap U_\beta})^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is smooth (i.e. C^∞).
- Let M be an m-dimensional manifold and N be an n-dimensional manifold. Then, a continuous $f : M \to N$ is called **smooth** (or C^∞) if for all $m \in M$, charts $(U_\alpha, \varphi_\alpha)$ containing m, and charts (V_β, φ_β) containing $f(U_\alpha)$, the map $\varphi_\beta \circ f \circ (\varphi_\alpha|_{U_\alpha})^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is smooth. In other words, we take $\varphi_\alpha(U_\alpha)$, send it back using φ_α^{-1}, then apply f and φ_β to it.

A lot of this might feel imprecise, but the basic concrete definitions eventually become second nature, so it's not super important which definition is used to start the whole thing off. For example, many authors require all atlases to be maximal (ordered by inclusion). Furthermore, even if these definitions seem painful or complicated, the idea is that smoothness is simply checked in charts: it's a local notion in $\mathbb{R}^n$, so it can be checked locally on manifolds, which locally are homeomorphic to $\mathbb{R}^n$.

There may be multiple smooth structures on a given manifold, so how do we know whether they're equivalent?

Definition. A smooth map $f : M \to N$ of manifolds is a **diffeomorphism** if there exists a smooth $g : M \to N$ with $f \circ g = \mathrm{id}$ and $g \circ f = \mathrm{id}$.

This is the notion of sameness (isomorphism) in the category of differentiable manifolds.

Tangent Bundles. The next reasonable thing to discuss is the tangent bundle, which is a specific example of fiber bundles or vector bundles.

Definition. Let E, B, and F be topological spaces. Then, a continuous map $P : E \to B$ is called a **fiber bundle with fiber** F if for all $x \in B$, there exist an open U containing x and a homeomorphism (sometimes called **change of coordinates**) $\phi_U : P^{-1}(U) \to U \times F$ such that the following diagram commutes.

$$\begin{tikzcd} P^{-1}(U) \arrow[rr, "\phi_U"] \arrow[dr, "P"'] & & U \times F \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

Here, π_1 is projection onto the first component.

The idea is that a fiber bundle locally looks like a product, but there could be some twisting, e.g. the Möbius strip locally looks like $[0,1] \times S^1$, but globally is not: ϕ_U rotates as one moves along S^1. See Figure 1 for a picture.

For a sillier example, one could just take $E = B \times F$, where the homeomorphisms can be global; this is the same sense in which $\mathbb{R}^n$ is a manifold.

Example 1.3. Another nontrivial example is where $B = \mathbb{C}P^n$ and $E \subseteq \mathbb{C}P^n \times \mathbb{C}^{n+1} = \{(\ell, \mathbf{y}) \mid \mathbf{y} \in \ell\}$; then, let $P : E \to B$ send $P(\ell, \mathbf{y}) = \ell$, so $P^{-1}(\ell) \cong \mathbb{C}$. Once again, there's some "twisting" that means the product structure only exists locally.

Definition. Let $P : E \to B$ be a fiber bundle with fiber F, and suppose that F is a real vector space. Then, P is called a **vector bundle** if the change of coordinates maps ϕ_U are linear. To be precise, there exists an open cover U_α of B with fiberwise homeomorphisms $P^{-1}(U_\alpha) \xrightarrow{\simeq} U_\alpha \times F$ such that whenever U_α and U_β intersect, $\varphi_\beta \circ \varphi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times F$ induces a map $F \to F$ for each $x \in U_\alpha \cap U_\beta$; this map is required to be linear.

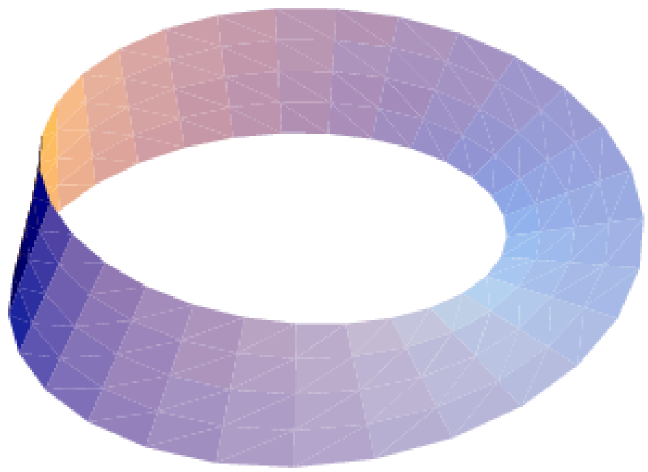

FIGURE 1. The Möbius strip, a nontrivial fiber bundle, since it locally looks like $[0,1]\times S^1$, but not globally. Source: http://mathworld.wolfram.com/MoebiusStrip.html.

For example, one can imagine a fiber bundle of $\mathbb{R}$ on S^1 (e.g. the normal lines): then, the two copies of $\mathbb{R}$ that come together to make S^1 overlap, and we have to say something on their boundary. In this case, send fibers to each other with the identity at one point, and flipped at the other; the result is the Möbius band again (if the identity was chosen in both cases, we would have had the trivial bundle again).

These definitions may seem unmotivated (perhaps this was deliberate; most of the class has seen some of this stuff already). However, the way we'll use the notion of a vector bundle is to define the tangent bundle, which is the set of tangent vectors at points in M (i.e., each fiber at x is T_xM, the tangent space to M at x). If M is embedded in Euclidean space $\mathbb{R}^N$, then the tangent bundle $TM = \{(m,\mathbf{v}) \mid \mathbf{v} \text{ is tangent to } M \text{ at } m\}$, but we want a definition that works for abstract manifolds and is more intrinsic.

Of course, since the intuition for the tangent bundle follows from the embedded case, the abstract definition isn't all that useful, but we do need it for formal arguments.

Definition. Let M be a manifold and $m\in M$. Then, the set of **tangent vectors** at m is the set of smooth functions $\gamma:\mathbb{R}\to M$ such that $\gamma(0)=m$, modulo the equivalence relation that $\gamma\sim\gamma'$ if for all smooth $f:M\to\mathbb{R}$,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\gamma(t)) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\gamma'(t)).$$

The intuition is that two functions (curves, in fact) are the same if they have the same derivative at m, but we need to add $f:M\to\mathbb{R}$ because we don't know yet how to take derivatives on manifolds. If you're familiar with germs of functions, this is a similar notion. Alternatively, this can be viewed as gluing the tangent bundles of open sets of Euclidean space together.

The goal is to have a tangent space, which means we want to turn this into a vector space somehow; tune in next time for that.

2. The Tangent Bundle: 4/9/15

"Most people like colimits better than limits, but we won't poll the audience yet."

There are several ways of defining the tangent bundle, and more interestingly putting a topolgy on it; the most low-tech way, which builds a tangent bundle on a manifold out of the trivial bundle on $\mathbb{R}^n$ by gluing, is often the best. That trivial bundle is $T\mathbb{R}^n=\mathbb{R}^n\times\mathbb{R}^n$ with projection onto $\mathbb{R}^n$ (since the tangent space at any $x\in\mathbb{R}^n$ is again isomorphic to $\mathbb{R}^n$).

But to do this, we need to pin down the notion of gluing. Suppose $\{U_\alpha\}$ is an open cover of a space B and F is a real vector space. Then, we would like the fiber to be F, but the transition maps need to respect its structure, i.e. the transition functions $t_{\alpha\beta}:U_\alpha\cap U_\beta\to\mathrm{Aut}(F)$, i.e. $\mathrm{GL}_\mathbb{R}(F)$. If F is instead a complex vector space, we would want $\mathrm{GL}_\mathbb{C}(F)$, and if it's a differentiable manifold (which is the notion of a **smooth manifold bundle**), we would like them in $\mathrm{Diff}(F)$, and so on.

Now, armed with this data, we can carry out the gluing. Define

$$E = \coprod_{\alpha} U_\alpha \times F / \Big((x \in U_\alpha, f) \sim (x \in U_\beta, t_{\alpha\beta}(x)(f)) \Big).$$

Is this a fiber bundle? We want projection, $P : E \to B$ sending $(x, f) \mapsto x$ to be well-defined.

Proposition 2.1. *Suppose that the transition functions satisfy the following conditions for all intersecting charts α, β, and δ:*

- $t_{\alpha\alpha} = \mathrm{id}$.
- $t_{\alpha\beta}(x) = t_{\beta\alpha}(x)^{-1}$.
- $t_{\alpha\beta}(x) t_{\beta\delta}(x) = t_{\alpha\delta}(x)$.

Then, $P : E \to B$ is a fiber bundle.

The intuition is that in these cases, two points in different fibers will never be identified, so projection is well-defined. The last condition is called the **cocycle condition**.

So now, we should use this for when $B = M$ is a manifold. Specifically, it's necessary to specify transition functions $U_\alpha \cap U_\beta \to \mathrm{GL}_n(\mathbb{R})$. Each U_α comes with a $\varphi_\alpha : U_\alpha \xrightarrow{\cong} \mathbb{R}^n$. Thus, there's a map $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \to U_\beta) \to \mathbb{R}^n$ is a map from an open subset of $\mathbb{R}^n$ to itself. That means we can take derivatives, and define $t_{\alpha\beta}(x) = D(\varphi_\beta \circ \varphi_\alpha^{-1})(\varphi_\alpha(x))$. Then, we can check that Proposition 2.1 holds, and sure enough, this is a tangent bundle.

On the one hand, we had to use charts which is unpleasant, but the other more intrinsic definitions aren't as easy to topologize.

Definition. Let M be a smooth manifold and $m \in M$. Then, let

$$T_m M = \{\gamma : (-\varepsilon, \varepsilon) \to M \mid \varepsilon > 0, \gamma \text{is smooth, and } \gamma(0) = m\}/\sim,$$

where $\gamma_1 \sim \gamma_2$ if for all smooth $f : M \to \mathbb{R}$,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\gamma_1(t)) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\gamma_2(t)).$$

Then, the tangent bundle is $TM = \bigcup_{m \in M} T_m M$, and the projection is $p : TM \to M$ sending $\gamma \mapsto \gamma(0)$.

In general in this class, a function between topological spaces will be assumed to be continuous, and a function of smooth manifolds will be assumed to be smooth (unless we're trying to prove this, of course, or where stated otherwise).

This second definition is a very nice definition of a set; in order to give it a topology we'll have to appeal to the first definition! In the case $M = \mathbb{R}^n$, let $L : \mathbb{R}^n \to T_m \mathbb{R}^n$ given by $L(\mathbf{v}) : \mathbb{R} \to \mathbb{R}^n$, where $L(\mathbf{v})(t) = m + t\mathbf{v}$. That is, given a vector, the result is a function whose image is that line. Then, every curve is equivalent to one of these lines, its tangent line (which is why this is called the tangent bundle). Thus, L is a bijection, and it can be promoted to a more general bijection L between our two notions of tangent bundle on $\mathbb{R}^n$, and this bijection creates the topological structure that you'd like. Then, the same notion can be defined for a general manifold M, but it'll involve some futzing around with charts.

The third definition again doesn't have an obvious natural topology, but it makes the vector-spatial structure much clearer, and it's sheafy, which algebraic geometers tend to like.

Definition. Let M be a manifold and $m \in M$. Then, define

$$\mathscr{G}(M, \mathbb{R})_m = \varinjlim_{\substack{m \in U \\ U \text{ open}}} C^\infty(U, \mathbb{R}).$$

That is, this colimit is the set of all such maps $f : U \to \mathbb{R}$ where U is an open neighborhood of M, but such that $f = g$ if there's an open neighborhood W of m such that $f|_W = g|_W$. $\mathscr{G}(M, \mathbb{R})_m$ is called the **germs of functions at** m, and $C^\infty(U, \mathbb{R})$ is the set of smooth functions from U to $\mathbb{R}$.

Note that this colimit is in the category of vector spaces, since $C^\infty(U, \mathbb{R})$ is a real vector space; moreover, it's also a ring under pointwise addition and multiplication.

Definition. Let $T \in \mathrm{Hom}_{\mathbb{R}}(\mathscr{G}(M, \mathbb{R})_m, \mathbb{R})$. Then, T is called a **derivation** if $T(fg) = f(m)T(g) + g(m)T(f)$. The set of derivations for an $m \in M$ will be denoted $T_m M$.

Proposition 2.2. *There is a natural linear homomorphism ev from the previous definition of $T_m M$ to this one.*

That is, if f is the germ of a function and $\gamma : (-\varepsilon, \varepsilon) \to M$, then $ev(\gamma)(f) \in \mathbb{R}$ is given by

$$ev(\gamma)(f) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\gamma(t)).$$

Here, f is a germ, so it's defined on a neighborhood, so its derivative exists.

When you boil down everything, the point is that these notions of the tangent bundle are equivalent; the book goes into more detail.

Definition. Suppose that $f : M \to N$ is a map of smooth manifolds and $m \in M$. Then, let $Df_m : T_m M \to T_m N$ be defined by $Df(\gamma) = f \circ \gamma$ (using the definition of equivalence classes of curves).

Proposition 2.3. *Df_m is linear, and moreover agrees with the standard ("Math 51") definition for $M, N = \mathbb{R}^n$.*

Well, now that we've defined tangent bundles and functions between them, let's use them.

Definition. $f : M \to N$ is called an **immersion** if Df_m is injective for all $m \in M$; it is called a **submersion** if Df_m is surjective for all $m \in M$.

The idea is that an immersion should have no singularities (à la $y^2 = x^3$), but it is allowed to intersect itself. Submersions are generalizations of projections.

Definition. A map $\phi : E_1 \to E_2$ is an **isomorphism of vector bundles** if it is a homeomorphism that induces linear maps on each fiber, and the following diagram commutes.

$$\begin{tikzcd} E_1 \arrow[rr, "\varphi"] \arrow[dr] & & E_2 \arrow[dl] \\ & X & \end{tikzcd}$$

Here, the arrows to X are projection.

Topological K-theory and Bott Periodicity. Many operations that we're used to from the world of vector spaces work just as well in vector bundles. For example, if $E_1 \to P$ and $E_2 \to P$ are vector bundles, then one can define $E_1 \oplus E_2 \to P$, $E_1 \otimes E_2 \to P$, and $\mathrm{Hom}(E_1, E_2) \to P$ in the reasonable way (do it fiberwise, or I guess check the universal property), $\Lambda^k E_1 \to B$, and so on.[1] Furthermore, $\oplus$ and $\otimes$ will allow us to define a ring structure on vector bundles.

We'll work out one of the cases in detail.

Definition. Let $E \to B$ be a fiber bundle and $f : X \to B$ be continuous. Then, the **pullback** of E along f is $f^*E = \{(x, e) \mid f(x) = p(e)\} \subset X \times E$. Furthermore, there's a natural map $f^*P : f^*E \to X$ given by $f^*P(x, e) = x$.

Categorically speaking, this is a fiber product.

Proposition 2.4. *$f^*P : f^*E \to X$ is a fiber bundle.*

This bundle is called the **pullback fiber bundle**.

Definition. Let E_1 and E_2 be fibers over B, and $\Delta : B \to B \times B$ be the **diagonal map**, i.e. it sends $x \mapsto (x, x)$. Then, let $E_1 \times_B E_2 = \Delta^*((E_1 \times E_2) \to (B \times B))$; if E_1 and E_2 are vector bundles, this is also denoted $E_1 \oplus E_2$.

Exercise 2.5. Show this is identical to taking the Cartesian product of the fibers over each point.

Definition.

- A **monoid** is a set with a binary operation $\times$ that is associative, but that may not have identity or inverses.
- If $\times$ is commutative, it's called a **abelian monoid**.
- If M is a monoid where $x + y = z + y$ implies $x = z$ for all $x, y, z, \in M$, then M is said to have the **cancellation property**.

Sometimes, this is called a **semigroup**, and monoids are required to have an identity.

Definition. Let M be an abelian monoid. Then, its **Grothendieck group** is the group $\mathrm{GG}(M)$ is $M \times M$ modulo the equivalence relation $(a, b) \sim (c, d)$ if $a + d + y = b + c + y$ for some $y \in M$.

[1] Note that when proving this, it may be necessary to refine or subdivide charts (make them smaller), which is a little annoying.

In some sense, this is the smallest group you can get out of M, by formally adding inverses and cancellation.

There's a natural inclusion $\iota : M \to \mathrm{GG}(M)$, which is a monoid homomorphism (i.e. the binary operation factors through it). Then, the Grothendieck group satisfies the following universal property: for any group G and map of monoids $\varphi : M \to G$, there is a unique $\phi : \mathrm{GG}(M) \to G$ such that the following diagram commutes.

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\iota"] & G \\ \mathrm{GG}(M) \arrow[ur, dashed, "\phi"'] & \end{tikzcd}$$

The intuition is that $\phi(a,b) = \varphi(a)\varphi(b)^{-1}$.

It turns out that the isomorphism classes of real vector bundles over a topological space X form a monoid, called $\mathrm{Vec}_{\mathbb{R}}(X)$, where the binary operation is the direct sum; $\mathrm{Vec}_{\mathbb{C}}(X)$ is defined analogously.

Definition. $\mathrm{KO}(X) = \mathrm{GG}(\mathrm{Vec}_{\mathbb{R}}(X))$, and $\mathrm{K}(X) = \mathrm{GG}(\mathrm{Vec}_{\mathbb{C}}(X))$, called **real** and **complex** ***K*-theory**, respectively.

The idea here is that monoids are harder to do stuff with than groups, so if we're willing to throw away some of that information, we can do more with the rest.

For example, $TS^n \oplus \mathbb{R} \cong \mathbb{R}^{n+1}$, which is an example of information that would be lost. The analogue is that there exist projective modules that aren't free.

Theorem 2.6 (Bott Periodicity)**.**
- $\mathbb{Z} \times \mathrm{KO}(X) \cong \mathrm{KO}(\Sigma^8 X)$.
- $\mathbb{Z} \times \mathrm{K}(X) \cong \mathrm{K}(\Sigma^2 X)$.

Here, Σ denotes suspension.

For example, $\mathrm{K}(\mathrm{pt}) = \mathbb{Z}$ (generated by the trivial bundle), and $\mathrm{K}(S^2)$ is generated by the trivial and tautological bundles.

3. Parallelizability: 4/14/15

"In my thesis defense, I wrote 'paralize' several times instead of 'parallelize.' "

Remark. Bott periodicity still sounds like the name of some character from the Harry Potter universe.

Definition. A **parallelization** of a smooth manifold M is a bundle isomorphism ϕ from TM to the trivial bundle $M \times \mathbb{R}^n$, i.e. a commutative diagram

$$\begin{tikzcd} TM \arrow[rr, "\phi"] \arrow[dr] & & M \times \mathbb{R}^n \arrow[dl] \\ & M & \end{tikzcd}$$

Our goal will be to show the following two theorems.

Theorem 3.1. *If M is parallelizable, then M is orientable.*

Theorem 3.2. *If M is a compact, parallelizable manifold, then $\chi(M)$, its* **Euler characteristic***, is nonzero.*

Orientability in Theorem 3.1 will be in the sense of Math 215B, i.e. for topological manifolds, though everything in this class will be smooth. We'll also discuss orientations of vector bundles.

Definition.
- If M is a manifold, then a **local orientation** at an $m \in M$ is a choice of generator of $H_n(M, M \setminus m) \cong \mathbb{Z}$.
- A **local orientation** for a k-dimensional vector bundle $V \to M$ is a choice of generator of $H_k(V_m, V_m \setminus 0)$.

Let $\widetilde{M} = \{(m, g) \mid m \in M, g$ is a local orientation at $m\}$. Similarly, let $\widetilde{M}_V = \{(m, g) \mid m \in M, g$ is a local orientation of V at $m\}$. These are covering spaces, and come with projections $P_{\widetilde{M}} : \widetilde{M} \to M$ and $P_{\widetilde{M}_V} : \widetilde{M}_V \to M$.

Specifically, if $B \subset M$ and $B \cong \mathbb{R}^n$, then $P_{\widetilde{M}}^{-1}(B)$ is the product of B and the generators of $H_n(M, M \setminus B)$, and so there's a natural bijection $H_n(M, M \setminus B) \xrightarrow{\cong} H_n(M, M \setminus b)$ for any $b \in B$. Similarly, if V is trivial over B, i.e. $P_V^{-1}(B) \cong B \times \mathbb{R}^k$, then we can do the same thing with $P_{\widetilde{M}_V}^{-1}(B)$: it's the product of B with the generators of $H_k(\mathbb{R}^k, \mathbb{R}^k \setminus 0)$.

Definition.

- M is **orientable** if $\widetilde{M} \to M$ is the trivial double cover.
- $V \to M$ is **orientable** if $\widetilde{M}_V \to M$ is the trivial double cover.

Proposition 3.3. $\widetilde{M} \cong \widetilde{M}_{TM}$.

Corollary 3.4. *M is orientable iff TM is.*

Geometrically, if we have a metric, there's a way of (topologically) identifying $T_m M$ with $B_\varepsilon(m)$ for some $\varepsilon > 0$; then, excision says that $H_n(M, M \setminus m) \cong H_n(B_\varepsilon(m), B_\varepsilon(m) \setminus m)$. But this is $H_n(TM, TM \setminus 0)$ (which does require some geometry or thinking about the exponential map). This is the intuition, but we don't have the machinery to make it rigorous; it's best to keep this one in your head.

Proof. The proof works by asking, "how do you define $\widetilde{M}$ and $\widetilde{M}_{TM}$ in terms of transition functions?" Once you write down what that actually is, they'll end up being the same.

Pick charts (U_α, ϕ_α) for M, so the $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ are homeomorphisms. We want to be able to define transition functions $t^{\widetilde{M}}_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{Z}/2$ (since $\widetilde{M}$ is a double cover of M). Why $\mathbb{Z}/2$? Because it's equal to $\operatorname{Aut}(H_n(\mathbb{R}^n, \mathbb{R}^n \setminus 0))$ (i.e. $\operatorname{Aut}(\mathbb{Z})$).

Let $t_{\alpha\beta} = (\varphi_\beta \circ \varphi_\alpha^{-1})_*$ (i.e. the induced map on homology), which is an automorphism. In higher-level wording, does $t_{\alpha\beta}$ preserve or reverse the orientation of $\mathbb{R}^n$ that is present on each chart? In order to make this work, we need a choice of orientation on $\mathbb{R}^n$, which induces orientations on $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$. Furthermore, $t_{\alpha\beta}(x)$ is an isomorphism $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \varphi_\alpha(x)) \xrightarrow{\sim} H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \varphi_\beta(x))$.

Now, we can define $t^{TM}_{\alpha\beta}$ in the same way, sending $U_\alpha \cap U_\beta \to \mathbb{Z}/2 \cong \operatorname{Aut}(T_x M, T_x M \setminus 0)$, which is an isomorphism.

Lemma 3.5. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism with $f(0) = 0$. Then, $f_* : H_n(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \to H_n(\mathbb{R}^n, \mathbb{R}^n \setminus 0)$ is the identity map iff $(Df)_* : H_n(T_0\mathbb{R}^n, T_0\mathbb{R}^n \setminus 0) \to H_n(T_0\mathbb{R}^n, T_0\mathbb{R}^n \setminus 0)$ is.*

Proof. Let

$$f_t(x) = \begin{cases} (1/t)tf(tx), & \text{if } t \in (0,1] \\ Df(x), & \text{if } t = 0. \end{cases}$$

Then, f_t is a homotopy between Df and f on $\mathbb{R}^n \setminus 0$ (which does require identifying TM with $\mathbb{R}^n$, which is fine). ⊠

In particular, this means the double cover of one is trivial iff the other is. I think. ⊠

Now, we're almost done.

Proposition 3.6. *If M is parallelizable, then $\widetilde{M}_{TM}$ is trivial.*

This is not a hard exercise, apparently.

From that, I Theorem 3.1 follows, because $\widetilde{M}$ is also trivial. Thus, we can attack Theorem 3.2.

Definition. A **vector field** is a section of $p : TM \to M$, i.e. a σ such tht $p \circ \sigma = \mathrm{id}$. If σ is smooth as a map of manifolds, the vector is said to be **smooth**.

Proposition 3.7. *If M is a parallizable, n-dimensional manifold, then there exist n vector fields $\sigma_1, \dots, \sigma_n$ such that for all $m \in M$, $\{\sigma_1(m), \dots, \sigma_n(m)\}$ are linearly independent.*

That is, parallelizability means the maximum number of linearly independent vector fields that can exist do. This is nicer since we're talking about smooth manifolds; in this class, unlike 215B, we can do analysis.

Definition. A **flow** is a map $\Phi : \mathbb{R} \times M \to M$ with $\Phi_t \circ \Phi_s = \Phi_{t+s}$ for all $s, t \in \mathbb{R}$ and $\Phi_0 = \mathrm{id}$.

Another way of saying this is that Φ is a homomorphism of topological groups from $\mathbb{R}$ into the diffeomorphism group of M, akin to a continuous group action.

Proposition 3.8. *There is a natural bijection between the set of flows and the set of vector fields.*

This can be made a homeomorphism using the compact-open topology (in the continuous case) or a Fréchet topology (in the smooth case), but that's not important right now. The idea is that the flow is given by integrating along the vector field. More precisely, given a flow Φ and an $m \in M$, there's a curve $t \mapsto \Phi_t(m)$, which lives in $T_m M$

by the definition of the tangent space (really, it's $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\Phi_t(m)$, but it's the same idea); in the other direction, given a vector field σ, one can write down a differential equation satisfying

$$\sigma(m) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\Phi_t(m),$$

with $\Phi_0 = \mathrm{id}$ as the initial condition, implies the existence of a unique solution. (This may require M to be compact.)

So the idea is that if one has a flow on a compact manifold, and to move for a very small length of time.

Proposition 3.9. *Let M be compact, and suppose $\sigma(m) \neq 0$ for all $m \in M$. Let Φ be the flow associated with σ; then, there exists an $\varepsilon > 0$ such that $\Phi_\varepsilon(m) \neq m$ for all m.*

One way to prove this is to check in charts, possibly using the implicit function theorem.

Thus, a nowhere-vanishing vector field gives us a map homotopic to id, but has no fixed points.

Recall the following theorem from Math 215B.

Theorem 3.10 (Lefschetz fixed-point)**.** *Let Y be a finite CW complex and $f : Y \to Y$ be continuous. Then, let $T^f : \bigoplus_k H_k(Y) \to \bigoplus_k H_k(Y)$ be given by $\bigoplus(-1)^k f_{*,k}$ (where $f_{*,k}$ is the map induced on H_k). Then, if $\mathrm{tr}(T^f) \neq 0$, then f has a fixed point.*

Corollary 3.11. *If there exists a nowhere-vanishing vector field σ, then the Euler characteristic is equal to zero.*

This is because $\mathrm{tr}\, T^{\mathrm{id}} = \chi(Y)$.

...right now, we haven't shown that a smooth manifold is homeomorphic to a finite *CW* complex, and the Lefschetz fixed point theorem doesn't hold on infinite *CW* complexes.

Now, the corollary implies Theorem 3.2, because if the Euler characteristic is nonzero, no nonvanishing vector fields can exist. Oops.

Both of these are examples of a notion called characteristic classes. There's a space called $B\,\mathrm{GL}_n(\mathbb{R})$, the **classifying space of vector bundles**, which can apparently be though of as a moduli space for certain stacks. It's also equal to the Grassmanian $\mathrm{Gr}(n,\infty)$. A cohomology class in the Grassmanian yields a cohomology class for every vector bundle on M; then, orientability corresponds to a class named $w_1 \in H^1(B\,\mathrm{GL}_n(\mathbb{R}))$, and there's a class called the Euler class $e \in H^n(B\,\mathrm{GL}_n(\mathbb{R}))$. The trick is, if M is parallelizable, the classifying map $M \to \mathrm{Gr}(n,\infty)$ is null-homotopic, so any cohomology class pulls back, and we know what w_1 and e are.

So how do you build this classifying map? Given a manifold M, one can embed it $M \hookrightarrow \mathbb{R}^\infty$.

4. The Whitney Embedding Theorem: 4/16/15

"I should stick to Greek letters whose names I remember."

Definition. A smooth map of manifolds $M \hookrightarrow N$ is called an **embedding** if it is an injective immersion that is a homeomorphism onto its image.

Today our goal will be to prove the following theorem.

Theorem 4.1 (Weak Whitney embedding theorem)**.** *If M is a compact n-dimensional manifold, then there exists an embedding $M \hookrightarrow \mathbb{R}^{2n+1}$.*

Remark. It's possible to remove the compactness assumption with a little more work (see the textbook), and get an embedding $M \hookrightarrow \mathbb{R}^{2n}$ with a lot more work.

The proof will require the following ingredients, which we will not prove.

Definition. Let $\{U_\alpha\}$ be an open cover of a smooth manifold M. Then, a **(smooth) partition of unity** subordinate to $\{U_\alpha\}$ is a collection of smooth functions $f_\alpha : M \to [0,1]$ such that:

- $f_\alpha^{-1}((0,1]) \subset U_\alpha$.
- For all $x \in M$, there exists an open $V_x \subseteq M$ containing x, such that $V_x \cap f_\alpha^{-1}((0,1]) \neq \emptyset$ for only finitely many α.
- $\sum_\alpha f_\alpha(x) = 1$ for all $x \in M$.

Partitions of unity are useful for turning local data or results into global ones. For example, if $f : M \to \mathbb{R}$ is a submersion, one might want to build a vector field on M that flows in the direction of $\mathbb{R}$ given by f (i.e. the flow commutes with f). One can do this locally with the implicit function theorem, and then use a partition of unity to do it globally. It still has the required property, because of the condition that the f_α sum to 1 everywhere. (This is one example; we'll use them in a different way today.)

The half-open interval in the definition arises from taking the support of f_α^{-1}, and isn't super critical to one's intuition.

Theorem 4.2. *For any smooth manifold M and open cover $\{U_\alpha\}$ of M, there exists a partition of unity subordinate to $\{U_\alpha\}$.*

This isn't too tricky to prove, and we have all of the tools, but it would distract us from nobler goals, so check out the textbook for the proof.

It's also possible to define a continuous partition of unity on a more general topological space. Unlike smooth partitions of unity on smooth manifolds, they might not always exist.

Definition. Let $f : M \to N$ be a map of smooth manifolds, where $\dim(M) = m$ and $\dim(N) = n$. Then, an $x \in M$ is a **critical point** of f if $\operatorname{rank}(Df_x) < n$, and $f(x)$ is called a **critical value**.

Theorem 4.3 (Sard)**.** *Let $f : M \to \mathbb{R}^n$ be smooth. Then, its set of critical values is measure zero in the Lesbegue measure on $\mathbb{R}^n$.*

More generally, one can replace $\mathbb{R}^n$ with any manifold N; then, however, the measure-zero criterion is replaced with the statement that the set of critical values is meager (related to the Baire category theorem).

We're also not going to prove this; once again, consult the textbook.

Proposition 4.4. *Let $K \subset M$ be closed and $f : K \to \mathbb{R}$. Assume that for all $x \in K$, there's an open neighborhood U_x (open in M) of x and a smooth $g_x : U_x \to \mathbb{R}$ such that $g_x|_{K\cap U_x} = f|_{K\cap U_x}$. Then, there exists a smooth $h : M \to \mathbb{R}$ with $h|_K = f$.*

Proof. Let $U_\alpha = \{U_x\} \cup \{M \setminus K\}$, and let f_α be a partition of unity for U_α. Let $g_\alpha = g_x$, except if $U_\alpha = M \setminus K$, where we let $g_\alpha = 0$. Finally, let

$$h = \sum_\alpha f_\alpha g_\alpha. \qquad \boxtimes$$

Theorem 4.5. *If M is a compact manifold, then there exists some (large) N such that there's an embedding $M \hookrightarrow \mathbb{R}^N$.*

Once we prove this, we'll use Sard's theorem to lower N to the desired value.

Proof. Choose two open covers $\{V_i\}_{i\in I}$ and $\{U_i\}_{i\in I}$ of M such that $\overline{V_i} \subseteq U_i$ for each i; then, since M is compact, we can choose a finite subcover $\{V_i\}_{i=1}^k$, and the corresponding $\{U_i\}_{i=1}^k$. Let $\phi_i : U_i \xrightarrow{\cong} \mathbb{R}^n$ be the chart map for U_i.

Now we have finitely many of each and $\bigcup_1^k V_i = M$ still.[2] Choose $\lambda_i : M \to \mathbb{R}$ that are 1 on $\overline{V_i}$ supported in U_i (i.e. 0 outside of U_i); these can be constructed by invoking Proposition 4.4. Then, let $\psi_i : M \to \mathbb{R}^n$ be given by $\psi_i : \lambda_i\phi_i$, and let $\theta : M \to (\mathbb{R}^n)^k \times \mathbb{R}^k$ be given by their product: $\theta = \psi_1 \times \psi_2 \times \cdots \times \psi_k \times \lambda_1 \times \cdots \times \lambda_k$.

So, why is θ an immersion? Take an $x \in M$; then, if $x \in V_i$, $\psi_i = \phi_i$ in a neighborhood of x, and ϕ_i is a diffeomorphism, so near x, θ is a product of diffeomorphisms and the zero map, so it's smooth.

θ is injective, because if $\theta(p) = \theta(q)$, then $p \in V_i$ for some i, and therefore $\lambda_i(p) = \lambda_i(q) = 1$, i.e. $q \in U_i$ (and we know $p \in U_i$ too). Thus,

$$\begin{aligned}
\phi_i(p) &= \lambda_i(p)\phi_i(p) = \psi_i(p) \\
&= \psi_i(q) = \lambda_i(q)\phi_i(q) \\
&= \phi_i(q).
\end{aligned}$$

However, we know ϕ_i is injective, so $p = q$.

Why is θ a homeomorphism onto its image? Well, M is compact and $\theta(M)$ is Hausdorff, which is a sufficient condition. Thus, θ is an embedding. $\boxtimes$

Proof of Theorem 4.1. Let $\theta : M \to \mathbb{R}^N$ be an embedding, as in Theorem 4.5, and suppose there exists a $\mathbf{w} \in \mathbb{R}^N$ such that $\mathbf{w}$ isn't tangent to $\theta(M)$ and there are no $x, y \in M$ such that $\mathbf{w} \in \operatorname{span}(\theta(x) - \theta(y))$.

Let $\pi_{\mathbf{w}^\perp}$ denote projection onto the orthogonal complement of $\mathbf{w}$; then, we will show that $\pi_{\mathbf{w}^\perp} \circ \theta$ is an embedding $M \hookrightarrow \operatorname{Im}(\pi_{\mathbf{w}^\perp}) \cong \mathbb{R}^{N-1}$. The idea is that the first condition (not tangent) makes it an immersion, and the second condition guarantees embedding. (There's more to check here, but I guess we can grind through it now without any difficult insights.)

[2] There are a couple of other ways to do this, e.g. choosing a finite cover (U_i, ϕ_i) first and then letting V_i be the support of ϕ_i.

Since M is an open submanifold of TM, then construct $\sigma : TM\backslash M \to \mathbb{R}P^{N-1}$ as follows: $D\theta : TM \to T\mathbb{R}^N$, and then the trivialization $\pi : T\mathbb{R}^N \to \mathbb{R}^N$, but if you didn't strt out in M, you won't end up at 0, so $(\pi \circ D\theta) : TM \setminus M \to \mathbb{R}^n \setminus 0$, so it's possible to projectivize, and composing $(\pi \circ D\theta)$ with this projectivization gives us the desired σ. We can also construct a $\tau : (M \times M) \setminus \Delta \to \mathbb{R}P^{N-1}$; here, $\Delta \subseteq M \times M$ is the diagonal, i.e. $\Delta = \{(x,x) \mid x \in M\}$. Thus, sending $(x, y) \mapsto \theta(x) - \theta(y)$ doesn't hit 0 if $x \neq y$, so we can send it to $\mathbb{R}P^{N-1}$; this is how τ is defined.

Observe that if $N - 1 > 2n$, then every point in the domain of σ or τ is a critical point, so their images are meager (or measure zero) in $\mathbb{R}P^{N-1}$. Thus, if $N > 2n + 1$, the **w** we sought above exists. ⊠

Theorem 4.6. *If M is an n-dimensional manifold and* $\mathrm{Emb}(M, N)$ *denotes the space of embeddings $M \hookrightarrow N$, then $\pi_i(\mathrm{Emb}(M, \mathbb{R}^N)) = 0$ for $i \le k - 1$ and $N \ge 2(n + 1 + k)$.*

Remark. If we had the better bound of an embedding into $\mathbb{R}^{2n}$, then instead we have $N \ge 2(n + k)$.

This theorem says that when N is large enough, these embeddings are connected, and in some sense clarifies that this space of embeddings is nonempty or connected. To get our hands on it, we should talk about a relative version of the Whitney embedding theorem.

Theorem 4.7 (Relative Whitney embedding theorem)**.** *Let M be an n-dimensional manifold and $L \subsetneq M$ be a submanifold. If $f : L \to \mathbb{R}^N$ with $N \ge 2n + 1$ is an embedding, then there exists an embedding $g : M \to \mathbb{R}^N$ with $g|_L = f$.*

This says that an embedding of a submanifold into $\mathbb{R}^{2N+1}$ can be lifted to an embedding of the whole manifold.

This applies to Theorem 4.6 as follows: if $f : M \hookrightarrow \mathbb{R}^N$ for $i = 1, 2$ are two embeddings, then one can embed $M \times \mathbb{R}$ into $\mathbb{R}^{N+1}$, and consider $L = M \times \{1, 2\}$ and use Theorem 4.7 to show there's an embedding that extends the f_i. This isn't entirely true (what if $f_1(M)$ intersects $f_2(M)$?), but Sard's theorem means we can control those points and fix the proof.

5. Immersions, Submersions and Tubular Neighborhoods: 4/21/15

Today, we're going to prove some honestly kind of boring technical results about immersions and submersions. But they'll be useful for all sorts of cool things like Pontryagin duality.

Theorem 5.1 (Implicit function theorem)**.** *Let $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be C^1 and $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ be such that $g(x, y) = 0$. If $i_x : \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ sends $z \mapsto (x, z)$ and $Dg \circ i_x$ is onto at y, then there exist $a, b > 0$ and an $f : B_a(x) \to B_b(y)$ such that $\{(x, f(x)) \mid x \in B_a(x)\} = \{(x, y) \in B_a(x) \times B_b(y) \mid g(x, y) = 0\}$. Moreover, if g is C^r, then so is f.*

Basically, this says that a continuous function where the derivative matrix is well-behaved can be interpreted as a level set in some small neighborhood. Alternatively, it says that if the derivative matrix is n-dimensional, then the space of solutions is what you would expect.

There's a nice way to reformulate this.

Theorem 5.2 (Inverse function theorem)**.** *Let $\theta : \mathbb{R}^n \to \mathbb{R}^m$ be C^1 and $\theta(x) = y$. If $D\theta$ is an isomorphsim at x, then there exist $a, b > 0$ and an $f : B_b(y) \to B_a(x)$ with $\theta \circ f = \mathrm{id}$. Moreover, if θ is C^r, then so is f.*

These theorems are useful because they tell us what immersions and submersions look like.

Corollary 5.3. *Let M be an m-dimensional manifold and N be an n-dimensional manifold, and let $\theta : M \to N$ be an immersion (resp. submersion) at a $p \in M$. Then, there exist open neighborhoods $U \subseteq M$ of p and $V \subseteq N$ of $f(p)$ and diffeomorphisms $\phi : U \xrightarrow{\cong} \mathbb{R}^m$ and $\psi : V \xrightarrow{\cong} \mathbb{R}^n$ that make thefollowing diagram commute.*

$$\begin{tikzcd} M \arrow[r, "\theta"] & N \\ U \arrow[u, hook] \arrow[r, "\theta|_U"] \arrow[d, "\phi", "\cong"'] & V \arrow[u, hook] \arrow[d, "\psi", "\cong"'] \\ \mathbb{R}^m \arrow[r, "\vartheta"] & \mathbb{R}^n \end{tikzcd} \tag{5.1}$$

Here, $\vartheta(x_1, \dots, x_m) = (x_1, \dots, x_m, 0, \dots, 0)$ (resp. $\vartheta(x_1, \dots, x_m) = (x_1, \dots, x_n)$, since if θ is a submersion, then $m \ge n$).

The intuition behind the proof, which we won't go into here, is that you use the implicit function theorem (resp. inverse function theorem), and then rotate.

Corollary 5.4. *If M and N are as in Corollary 5.3, $f : M \to N$ is smooth, and $y \in N$ is a regular value of f, then $f^{-1}(y)$ is an $(m-n)$-dimensional* **submanifold** *of M.*

We haven't actually defined the notion of submanifold: some authors define it as a subset of a manifold where inclusion is an immersion, and others require it to have charts so that it looks like the first m coordinates in some set of charts for the ambient manifold. In any case, Corollary 5.3 equates the two notions.

The proof of Corollary 5.4 requires both parts of Corollary 5.3 to prove, since $f : M \to N$ is a submersion at any $x \in f^{-1}(y)$, and then the submanifold is immersed in M.

Definition. Let N_1 and N_2 be submanifolds of M. Then, N_1 is **transverse** to N_2, written $N_1 \pitchfork N_2$, if for all $x \in N_1 \cap N_2$, $T_x N_1 + T_x N_2 = T_x M$.

Clearly, if N_1 doesn't intersect N_2, then they're not transverse, and if you have a point and a curve in $\mathbb{R}^n$, then they can only be transverse if they don't intersect. However, all of $\mathbb{R}^n$ intersects a point transversely.

Transversality is a way to make the hazy notion of "in general position" somewhat precise.

Theorem 5.5. *If $N_1 \pitchfork N_2$, then $N_1 \cap N_2$ is a submanifold of dimension* $\dim(N_1) + \dim(N_2) - \dim(M)$.

The proof idea is that one can find a $V \subseteq M$ and $U \subseteq N_1$ that satisfy the diagram (5.1). Then, let $\pi : N_2 \cap V \to \mathbb{R}^{\dim(M) - \dim(N_1)}$ be the projection onto the last coordinates. Thus, $\pi^{-1}(0) = N_1 \cap N_2 \cap V$, and π is a submersion, so $\pi^{-1}(0)$ is a subamanifold of the required dimension.

Basically, the only reason the professor cares about submersions is the following theorem.

Theorem 5.6. *Let $f : M \to N$ be a proper submersion with N connected. Then, $f : M \to N$ is a fiber bundle.*

One way to think of it is that the preimages are locally diffeomorphic. Another is that if a family of manifolds is smoothly dependent on a parameter t, it can only change topology at critical points, which are also where it doesn't project smoothly onto t.

Connectedness is needed because otherwise, there's a fiber bundle over every connected component, but there's no reason to assume they're the same.

Proposition 5.7. *Let $f : M \to \mathbb{R}^n$ be a proper submersion and $r > 0$. Then, there exists a diffeomorphism d such that the following diagram commutes.*

$$\begin{tikzcd} f^{-1}(B_r(\mathbf{0})) \arrow[rr, "d"] \arrow[dr, "f"'] & & f^{-1}(\mathbf{0}) \times B_r(\mathbf{0}) \arrow[dl, "\pi"] \\ & B_r(\mathbf{0}) & \end{tikzcd}$$

We'll end up using flows and partitions of unity to prove this.

Proof. We can find charts $U_\alpha \subseteq M$ and $V_\alpha \subseteq \mathbb{R}^n$ with $f(U_\alpha) = V_\alpha$ and $f|_{U_\alpha}$ fiberwise equivalent to $\pi_\alpha : V_\alpha \times \mathbb{R}^{m-n} \to V_\alpha$.

Let χ_i be the vector field on $\mathbb{R}^m$ in the i^{th} direction; then, by an abuse of notation, since $V_\alpha \subseteq \mathbb{R}^n$, we can think of χ_i as a vector field on V_α. Take χ_i^α be a vector field on U_α with $Df \chi_i^\alpha = \chi_i|_{V_\alpha}$.

Now, we would like to stitch these together: let $g_\alpha : M \to \mathbb{R}^n$ be a partition of unity subordinate to the charts $\{V_\alpha\}$, and let

$$\sigma_i = \sum_\alpha g_\alpha \chi_i^\alpha,$$

so that $Df \sigma_i = \chi_i$.

Now, choose an $r > 0$, and let $h : \mathbb{R}^m \to \mathbb{R}$ be a function that is 1 on $B_r(\mathbf{0})$ and is compactly supported. Since f is proper, then $(h \circ f)$ does too. For a $\mathbf{v} \in \mathbb{R}^m$, let

$$\sigma_{\mathbf{v}} = \sum_{i=1}^m a_i \sigma_i (h \circ f),$$

where $\mathbf{v} = (a_1, \dots, a_m)$, and let $\Phi_{\mathbf{v}} : M \to M$ be the time-1 flow along $\sigma_{\mathbf{v}}$. We want a map $\delta : f^{-1}(\mathbf{0}) \times B_r(\mathbf{0}) \to f^{-1}(B_r(\mathbf{0}))$ (and then the required d will be δ^{-1}); in fact, it will be given by $\delta(m, \mathbf{v}) = \Phi_{\mathbf{v}}(m)$. Then, we'll be able to prove this is a diffeomorphism, so we get the d we need. ⊠

Tubular Neighborhoods.

Theorem 5.8 (Tubular neighborhood)**.** *Let M be a compact, n-dimensional submanifold of $\mathbb{R}^k$ and $N_\varepsilon = \{(x, \mathbf{v}) \mid$* $\mathbf{v} \perp T_x M$ and $|\mathbf{v}| < \varepsilon\}$ *(where length and angle are measured within $\mathbb{R}^k$). Let $V_\varepsilon = \{y \in \mathbb{R}^k \mid$* there exists an $x \in M$ such that $|x - y| < \varepsilon\}$.

If $\theta : N_\varepsilon \to V_\varepsilon$ sends $(x, \mathbf{v}) \mapsto x + \mathbf{v}$, then for sufficiently small ε, θ is a diffeomorphism.

In other words, a small tubular neighborhood of M looks like $M \times \mathbb{R}^m$ for some m. We know M_ε is homotopy equivalent to M for all ε, but V_ε might not be (e.g. if the hole in a donut is filled in, in some sense).

Proof. For $x \in M$, $D\theta$ is an isomorphism at $(x, \mathbf{0})$: the dimensions line up, since then $\theta(x, \mathbf{0}) = x$. That means θ is locally a diffeomorphism (i.e. in a neighborhood for every x). But since M is compact, one can find an ε such that $D\theta$ is an isomorphism for all $(x, \mathbf{v}) \in N_\varepsilon$.

Assume that θ isn't injective for all ε; then, take $x_i, y_i \in N_{1/i}$ such that $x_i \neq y_i$ and $\theta(x_i) = \theta(y_i)$; since M is compact, we can choose a convergent subsequence, and so eventually the sequence ends up in the manifold, where it is injective.

To show that it's surjective, we know that $V_\varepsilon \supseteq \theta(N_\varepsilon)$, so suppose $y \in V_\varepsilon \setminus \theta(N_\varepsilon)$, and let $x \in M$ be the nearest point on M to it. Then, $x - y \perp T_x M$, so $\theta(x, x - y) = y$, which looks like the definition of our tubular neighborhood. ⊠

There are many ways to jazz this theorem up, e.g. replacing $\mathbb{R}^k$ with a k-dimensional manifold. This makes it trickier to define distance, but if all you care about is topology, you could talk about the normal bundle of an embedding of manifolds, which is diffeomorphic to a neighborhood. There's some tricky questions about defining the normal bundle, though it's well-defined in K-theory.

6. Cobordism: 4/23/15

"So I'm just going to quit while I'm losing."

The ideas outlined today relate to some deeper ideas about classifying manifolds up to cobordism and homotopy groups of spheres, thanks to ideas of Pontryagin and Thom.

Definition. Two smooth n-dimensional manifolds M and N are called **cobordant** if there exists a compact $(n+1)$-dimensional manifold-with-boundary W such that ∂W is diffeomorphic to $M \sqcup N$.

For example, a pair of pants is a cobordism between S^1 and $S^1 \sqcup S^1$, as in Figure 2 More generally, any number of

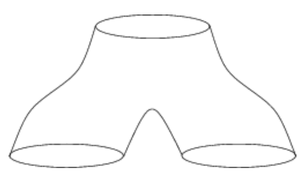

Figure 2. A cobordism between S^1 and $S^1 \sqcup S^1$, called the "pair of pants" for obvious reasons. Source: http://en.wikipedia.org/wiki/Cobordism.

circles is cobordant to any other number of circles.

Definition. Let cob^n denote the set of smooth, compact n-manifolds up to cobordism.

Fact. cob^n is a group with $[M] + [N] = [M \sqcup N]$. Moreover, $\mathrm{cob}^* = \bigoplus_n \mathrm{cob}^n$ is a graded ring, with $[M] \cdot [N] = [M \times N]$.

Theorem 6.1 (Thom)**.** $\mathrm{cob}^* \cong \mathbb{Z}/2[z_i \mid i \neq 2^j - 1]$, *where $|x_i| = i$.*

This was probably the second-best thesis in algebraic topology (after Serre's). Both Thom and Serre won Fields medals for basically their thesis work.

The reason everything is 2-torsion is that two copies of a manifold is cobordant to the empty set (by just connecting them as a sort-of cylinder, so the boundary is both copies, or equivalently both copies along with the empty set).

Corollary 6.2. *If M is a compact, smooth manifold and* $\dim(M) = 3$*, then $M = \partial W$ for some compact, smooth manifold-with-boundary W.*

Definition. Let $P(m,n) = (S^m \times \mathbb{C}P^n)/\tau$, where $\tau(x,y) = (-x, \overline{y})$.

Proposition 6.3. *In Theorem 6.1,*

$$x_i = \begin{cases} [P(i,0)] = [\mathbb{R}P^i], & \text{for } i \text{ even} \\ [P(2^r-1, s2^r)], & \text{for } i = 2^r(2s+1)-1. \end{cases}$$

The proof has two parts, one of which is reasonable for this class and the other of which is wildly inappropriate. The idea is to first find a space (well, actually a spectrum, which is weird and off-putting to some people) MO such that $\mathrm{cob}^* \cong \pi_*(MO)$; then, the second step is to calculate $\pi_*(MO)$. This is hard, because it involves calculations of homotopy groups of spheres.

Returning to Earth, let's prove some more technical lemmas needed to prove this stuff.

Theorem 6.4. *Let M be a manifold and $A, B \subseteq M$ be closed. If $f : M \to \mathbb{R}^k$ is continuous and smooth on A, then there exists a $g : M \to \mathbb{R}^k$ which is smooth on $M \setminus B$, satisfies $g|_{A\cup B} = f|_{A\cup B}$, and $g \simeq_{h_t} f\,\mathrm{rel}_{A\cup B}$, i.e there's a homotopy between g and f constant on $A \cup B$ and such that for all $x \in M$, $|h_t(x) - f(x)| < \varepsilon$.*

This seems kind of technical, but the point is that we can approximate continuous functions that are smooth on some closed set with smooth functions on most of M.

Proof sketch. Let ρ be any metric on M (in the metric space sense, not Riemannian sense; we know that all manifolds are metrizable), and let

$$\varepsilon(x) = \inf_{b\in B} \rho(x,b).$$

Since f is smooth on A, then for all $x \in A$, there exists an open neighborhood U_x of x and a smooth $g : U_x \to \mathbb{R}^k$ such that $g|_{A\cap U_x} = f|_{A\cap U_x}$. For more general $x \in M$, let V_x be an open neighborhood such that if $x \in A \setminus B$, $V_x = V_x \cap (M \setminus B)$; then, let $h_x = g_x|_{V_x}$. If $x \notin A \setminus B$, take V_x to be an open set such that $x \in V_x$ and $V_x \cap A = \emptyset$; then, take $h_x(z) = f(x)$.

The whole point of this is that these V_x should form an open cover of M; then, one can take these h_x and stitch them together with a partition of unity. ⊠

There's also a version with maps into a manifold.

Theorem 6.5. *Let M and N be manifolds with N compact, and let ρ be a metric on M. Let $A \subseteq M$ be closed and $f : M \to N$ be continuous and smooth on A. Then, for all $\varepsilon > 0$, there exists a family $h_t : M \to N$ such that*

(1) $h_0 = f$,
(2) $h_t|_A = f|_A$,
(3) h_t *is smooth for all* $t > 0$, *and*
(4) $\rho(h_t(x), f(x)) < \varepsilon$ *for all* $t \in [0,1]$.

The way to prove this is to embed $M \hookrightarrow \mathbb{R}^k$, and then take a tubular neighborhood. The compactness of N is used to guarantee that small distances in $\mathbb{R}^k$ correspond to small distances in N (using uniform continuity), so one can use distances in $\mathbb{R}^k$. Then, Theorem 6.4 gives us a smooth approximation in the tubular neighborhood, and then it projects back down into an approximation on N.

The book has a bunch of corollaries to these.

If you're wondering why we need so many smooth approximation theorems, remember that homotopy groups involve continuous maps, so to use the tools we've developed with manifolds, we need to approximate them.

Definition. Let M, X, and Y be manifolds. Then, two smooth maps $f : X \to M$ and $g : Y \to M$ are **transverse**, written $f \pitchfork g$, if for all points $x \in X$ and $y \in Y$ such that $f(x) = g(y) = z$, $Df(T_xX) + Dg(T_yY) = T_zM$. If $f : X \to M$ is an embedding and $N = f(X)$, then one also writes $g \pitchfork N$ if $g \pitchfork f$.

In particular, if the images of f and g don't intersect, then they're transverse, and if one of f or g is a submersion, then they're transverse.

Theorem 6.6. *Let $f_0 : M \to W$ be smooth and $N \subseteq W$ be a compact, smooth submanifold. Then, given a tubular neighborhood T of M, there exists a family $h_t : M \to W$ such that $h_0 = f$, $h_1 \pitchfork N$, and $h_t(x) = f(x)$ for $f(x)$ outside T.*

Definition. Let $V \to B$ be a vector bundle, and let $V^+ \to B$ be the fiberwise one-point compactification (so that V has fiber $\mathbb{R}^k$ and V^+ has fiber S^k). It's possible to think of this as saying that the transition functions for V^+ are given by the injection $\mathrm{GL}_k(\mathbb{R}) \hookrightarrow \mathrm{Diff}(S^k)$. Then, the **Thom space** is $\tau(V) = V^+/\sim$, where $x \sim y$ if x and y are both "points at infinity," i.e. added in by the one-point compactification.

How should you think about Thom spaces? Well, the Thom space of the trivial bundle $\mathbb{R} \to E \to B$ is just the suspension of B: $\tau(E) = \Sigma B$. Suspensions come up as ways of generalizing $\mathbb{R}$ in homotopy theory (e.g. in equivariant homotopy theory, where they correspond to, incredibly enough, group representations). Note that this doesn't always generalize: the trivial k-bundle $\mathbb{R}^k \to E \to B$ does not satify $\tau(E) = \Sigma^k B$ when $k = 0$; you get $B \sqcup \{\mathrm{pt}\}$, which isn't $\Sigma^0 B = B$. Some of this depends on the precise definition of the one-point compactification of zero-dimensional manifolds.

Last quarter, we learned the following isomorphism.

Proposition 6.7. $\widetilde{H}_*(\Sigma^k B) \cong \widetilde{H}_{*-k}(B)$.

Theorem 6.8 (Thom isomorphism theorem)**.** *If V is an orientable, k-dimensional vector bundle, then $H_*(B) \cong H_{*+k}(\tau(V))$*

Since all trivial bundles are orientable, this implies Proposition 6.7, and this theorem is in some sense a twisted version of that. Since Pontryagin duality only requires orientability, this is what we're looking for, though we don't need it just yet, and we'll come back to prove it later.

An embedding $M \overset{f}{\hookrightarrow} W$ gives a map of homology of M to homology of W, but Theorem 6.8 gives a map in the other direction, arising from $W \xrightarrow{g} \tau(N^f)$ (where N^f denotes the normal bundle of f), and therefore we can go between

$$H_*(W) \xrightarrow{g} H_*(\tau N^f) \xrightarrow{\cong} H_{*-\dim W + \dim M}(M).$$

This composition is written $f^!$, read "f-shriek."