HYPERKÄHLER GEOMETRY AND THE MODULI SPACE OF HIGGS BUNDLES

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Contents

1. Introduction to Kähler and hyperkähler geometry: 6/12/17

Today will be mostly preliminaries, including some complex and symplectic geometry, such as the symplectic quotient, and an introduction to Kähler and hyperkähler geometry. Over the rest of the week, we'll discuss some examples (which are usually left implicit) such as quiver varieties, introduce the moduli space of Higgs bundles, and more. A good reference for this is Andy Neitzke's lecture notes on the moduli space of Higgs bundles: https://www.ma.utexas.edu/users/neitzke/teaching/392C-higgs-bundles/higgs-bundles.pdf.

Kähler manifolds are great because they involve a whole bunch of angles towards differential geometry: Riemannian, complex, and symplectic manifolds. Riemannian manifolds are likely the most familiar, so we won't discuss them in too much detail.

1.1. Almost complex and complex geometry.

Definition 1.1. Let X be a smooth manifold. A **almost complex structure** on X is a choice of $I \in \mathrm{End}(TX)$ such that $I^2 = -1$. The pair (X, I) is called an **almost complex manifold**.

This implies that $\dim_{\mathbb{R}}(X)$ is even.

You can define a **complex manifold** as the same thing as a real manifold, but with charts valued in $\mathbb{C}^n$, and such that the change-of-charts maps are holomorphic. This implies an almost complex structure, as the tangent space is a complex vector bundle, but (as the terminology "almost" suggests) the two are not the same.

Definition 1.2. An almost complex structure (M, I) is **integrable** if there exists an open cover $\mathfrak{U}$ of M and holomorphic diffeomorphisms $\phi_U : U \to \phi_U(U) \subset \mathbb{C}^n$ for each $U \in \mathfrak{U}$.

Proposition 1.3. *An integrable almost complex structure on X is equivalent to a complex structure on X.*

If (X, I) is an almost complex manifold of dimension n, then the action of I on TX makes TX into a complex vector bundle. You can also complexify it as a real vector bundle, producing a vector bundle $T_{\mathbb{C}}X$ of rank $2n$.

Definition 1.4. Let $T^{1,0}X$ denote the eigenspace for i acting on $T_{\mathbb{C}}X$ and $T^{0,1}X$ denote the $-i$-eigenspace. Hence $T_{\mathbb{C}}X \cong T^{1,0}X \oplus T^{0,1}X$.

These are both complex vector bundles of rank n, and in fact $TX \cong T^{1,0}X$.

You can play the same game with the complexified cotangent bundle: $T^*_{\mathbb{C}}X \cong (T^*)^{1,0}X \oplus (T^*)^{0,1}X$, and $(T^*)^{1,0}X = \mathrm{Ann}(T^{0,1}X)$. More generally, one gets a **type decomposition** or **Hodge decomposition** of complexified exterior powers and complexified differential forms:

$$\Lambda^* T^*_{\mathbb{C}} X \cong \bigoplus_{p+q=n} \Lambda^{p,q} T^* X$$

$$\Omega^*_{\mathbb{C}} X \cong \bigoplus_{p,q=0}^{n} \Omega^{p,q}(X).$$

The piece of degree n is the sum of $\Lambda^{p,q}$ (resp. $\Omega^{p,q}$) for which $p+q=n$.

The intuition is that the holomorphic structure on $\mathbb{C}$ allows one to write

$$\mathrm{d}x \wedge \mathrm{d}y = -\frac{1}{2}\,\mathrm{d}z \wedge \mathrm{d}\overline{z},$$

where $\mathrm{d}z \coloneqq \mathrm{d}x + i\,\mathrm{d}y$ and $\mathrm{d}\overline{z} = \mathrm{d}x - i\,\mathrm{d}y$. The graded part $\Lambda^{p,q}T^*X$ corresponds to the pieces with $\mathrm{d}z$ in p directions and $\mathrm{d}\overline{z}$ in q directions, and similarly for $\Omega^{p,q}X$.[1]

Theorem 1.5. *The following are equivalentfor an almost complex manifold (X, I):*

(1) *I is integrable.*
(2) *There's a decomposition $\mathrm{d} = \partial + \overline{\partial}$, where $\partial\colon \Omega^{p,q} \to \Omega^{p+1,q}$ and $\overline{\partial}\colon \Omega^{p,q} \to \Omega^{p,q+1}$.*
(3) *$T^{0,1}X$ is an integrable distribution, i.e. for sections $s, t \in \Gamma(T^{0,1}X)$, $[s,t]$ is also a section.*

We'll use the first two more often than the third.

Definition 1.6. The **Dolbeault cohomology** is the homology of the complexified differential forms: $H^{p,q}_{\mathrm{Dol}}(X) \coloneqq H^q(\Omega^{p,\bullet}, \overline{\partial})$.

Definition 1.7. Let X be a complex manifold. A **holomorphic vector bundle** is a complex vector bundle $E \to X$ together with a differential $\overline{\partial}_E\colon \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$ such that

(1) $\overline{\partial}_E$ satisfies the **Leibniz rule**: if $\alpha \in \Omega^*(X)$ and $\psi \in \Omega^0(E)$,
$$\overline{\partial}_E(\alpha\psi) = (\overline{\partial}\alpha)\psi + (-1)^{|\alpha|}\alpha \wedge \overline{\partial}_E\psi.$$
(2) $\overline{\partial}_E^2 = 0$.

Exercise 1.8.

(1) If X is a complex manifold, show that $TX \to X$ is holomorphic.
(2) The **canonical line bundle** over a complex manifold is $K_X \coloneqq \Lambda^{n,0}T^*X$. Show that K_X is holomorphic.

Elements of K_N are things of the form $f(z)\,\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n$, where $f\colon X \to \mathbb{C}$ is holomorphic.

Definition 1.9. Let $E \to X$ be a holomorphic vector bundle and h be a Hermitian metric on E (i.e. a smoothly varying Hermitian metric on each fiber). The **Chern connection** is the unique connection D on E that is

- unitary with respect to h, and
- compatible with the holomorphic structue, in that $(D\psi)^{0,1} = \overline{\partial}_E\psi$ for any $\psi \in \Omega^0(E)$.

Like the Levi-Civita connection, it's a theorem that the Chern connection exists. That h is unitary means $h(Ix, y) = ih(x,y) + h(x, -Iy)$.

1.2. **Kähler geometry.** Let (X, I) be an almost complex manifold with Hermitian metric g, and let ∇ denote the Levi-Civita connection for g.

Definition 1.10. The **fundamental form** is the $\omega \in \Omega^{1,1}_{\mathbb{R}}(X)$ such that $\omega(v,w) = g(Iv, w)$.

Here, $\Omega^{p,p}_{\mathbb{R}}(X)$ is the fixed points of complex conjugation acting on $\Omega^{p,p}(X)$.

There are many different ways to define Kähler manifolds.

Definition 1.11. The triple (X, g, I) is called a **Kähler manifold** if one of the following equivalent conditions is satisfied.

(1) $\nabla I = 0$, i.e. I is **covariantly constant**.

[1]You can run this same story with tensor products, but the resulting vector bundles don't appear as often in the theory.

(2) $\nabla\omega = 0$.
(3) I is integrable and the Levi-Civita and Chern connections coincide.
(4) I is integrable and $\mathrm{d}\omega = 0$.

In this case ω is also called the **Kähler form**.

Corollary 1.12. *Let X be a Kähler manifold and $Y \subset X$ be a complex submanifold. Then, restricting g and I to Y shows that Y is also a Kähler manifold.*

Example 1.13. Everyone's first example of a Kähler manifold is $\mathbb{CP}^n$ with the **Fubini-Study metric**. ◂

You can show by a dimensional argument that any Riemann surface is automatically Kähler. Finding counterexamples is kind of tricky: combining Corollary 1.12 and Example 1.13, any smooth projective complex variety is Kähler. There are homological obstructions to being Kähler, and we'll learn more about this in the Kähler geometry minicourse later this summer.

Proposition 1.14. *Let $(X.h)$ be a Kähler manifold of complex dimension n. Then, the holonomy of the Levi-Civita connection around any point p is a subgroup of U_n.*

1.3. **Symplectic geometry.** Hyperkähler manifolds often arise as moduli spaces, which themselves often arise as symplectic quotients of things. To understand symplectic quotients we should first go over some symplectic geometry.

Definition 1.15.

- Let V be a vector space over either $\mathbb{R}$ or $\mathbb{C}$. By a **nondegenerate** 2-form we mean an $\omega \in \Lambda^2(V)$ such that $v \mapsto \omega(v, -)$ defines an isomorphism $V \to V^*$.
- A **symplectic manifold** is a pair (X, ω), where $\omega \in \Omega^2(X)$ is a closed, nondegenerate 2-form (i.e. it's nondegenerate at every $x \in X$). In this case, ω is called a **symplectic form**.
- A symplectic manifold (X, ω) is **exact** if ω is exact, i.e. there's a one-form λ such that $\mathrm{d}\lambda = \omega$.

Example 1.16. The canonical example is the cotangent bundle T^*X to a smooth manifold X, which is an exact symplectic manifold. The **Liouville form**, **Liouville differential**, or (in some contexts in physics) the **Seiberg-Witten differential** $\lambda \in \Omega^1(T^*X)$ is

$$\lambda(x, p) \cdot v = p \cdot \pi_* v,$$

where $\pi\colon T^*X \to X$ is projection. The symplectic form is $\omega = \mathrm{d}\lambda$.

In local coordinates $(p_1, q_1, \dots, p_n, q_n)$, so $p_1, \dots, p_n$ are coordinates on X (in physics, position) and $q_1, \dots, q_n$ are coordinates on the fiber (in physics, momentum), we have

$$\lambda = \sum_{i=1}^n p_i \,\mathrm{d}q_i \qquad \text{and hence} \qquad \omega = \sum_{i=1}^n \mathrm{d}p_i \wedge \mathrm{d}q_i.$$ ◂

For a general nondegenerate $\omega \in \Lambda^2(V)$, there's a **canonical basis** $\{e_1, e_2, \dots, e_n, f_1, \dots, f_n\}$ such that $\omega(e_i, e_j) = \delta_{ij}$ and $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$ for all i and j. This means that, in the canonical basis, ω is represented by the block matrix

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Moduli spaces arise as quotients. Let G be a (real) Lie group acting on a symplectic manifold (X, ω),[2] and let $\mathfrak{g}$ be the Lie algebra of G. Differentiating the action defines a Lie algebra homomorphism $\rho\colon \mathfrak{g} \to \mathcal{X}(X)$, where $\mathcal{X}(X)$ denotes the Lie algebra of vector fields on X.

We'd like to dualize this to define a map $\mu\colon X \to \mathfrak{g}^*$ that's G-equivariant (where G acts on $\mathfrak{g}^*$ through the dual of the adjoint action).

Definition 1.17. A **moment map** for the G-action on X is a map $\mu\colon X \to \mathfrak{g}^*$ such that $\iota_{\rho(Z)}\omega = \mathrm{d}(\mu \cdot Z)$ for $Z \in \mathfrak{g}$. (Here, ι is the contraction operator as in Riemannian geometry.)

This is a very important definition — it characterizes to what degree the action commutes with the Hamiltonian.

[2]Generally, G acts through **symplectomorphisms**, i.e. diffeomorphisms preserving the symplectic form, and defining the moment map will require this.

Remark.

- The moment map does not always exist. One local obstruction comes from the **Cartan formula**:
$$\mathcal{L}_v\omega = \mathrm{d}(\iota_v\omega) + \iota_v(\mathrm{d}\omega),$$
and therefore
$$\mathcal{L}_{\rho(Z)}\omega = \mathrm{d}(\iota_{\rho(Z)}\omega) + \iota_{\rho(Z)}\mathrm{d}\omega = 0,$$
so the infinitesimal action must preserve ω. This does not suffice; there are other, global obstructions.
- When a moment map exists, it need not be unique. For example, consider any $c \in [\mathfrak{g}, \mathfrak{g}]^\perp \subset \mathfrak{g}^*$;[3] then, if μ is a moment map, so is $\mu + c$. ◀

Example 1.18. Consider $X = \mathbb{C}^n$ with the standard Kähler structure
$$\omega \coloneqq \frac{i}{2}\sum_{i=1}^n \mathrm{d}z_i \wedge \mathrm{d}\overline{z}_i.$$
The easiest action you can think of is a U_1-action by rotation: $z_i \mapsto e^{i\alpha}z_i$. Thus, if we rotate z_i by $e^{i\alpha}$, we rotate $\overline{z}$ by $e^{-i\alpha}$, so this action is through local symplectomorphisms.

One moment map is
$$\mu = -\frac{1}{2}\sum \lvert \mathrm{d}z_i\rvert^2 + c$$
for any $c \in \mathbb{R}$, because the commutator subgroup of U_1 is trivial and $\mathfrak{u}_1 \cong \mathbb{R}$. ◀

Definition 1.19. Let G act on a symplectic manifold (X, ω) in such a way that a moment map μ exists. Then, the **symplectic quotient** of X by G is $X /\!/_\mu G \coloneqq \mu^{-1}(0)/G$.

The dependence on μ is a little fearsome, and the G-action on $\mu^{-1}(0)/G$ is not always free.

Proposition 1.20. *If G acts freely on $\mu^{-1}(0)$ freely, then $X /\!/_\mu G$ is a symplectic manifold. Moreover,*

1. $\dim(X /\!/_\mu G) = \dim X - 2\dim G$, *and*
2. *if ω_X denotes the symplectic form on X and $\omega_{X/\!/_\mu G}$ denotes the symplectic form on $X /\!/_\mu G$, then $\pi^*\omega_{X/\!/_\mu G} = \iota^*\omega_X$, where $\iota\colon \mu^{-1}(0) \hookrightarrow X$ is inclusion and $\pi\colon \mu^{-1}(0) \twoheadrightarrow X /\!/_\mu G$ is projection.*

Remark. For some more motivation about why this is an okay idea, let X be a compact manifold and G be a compact Lie group acting freely on X. Hence G acts on (T^*X, ω) through symplectomorphisms, and in fact a moment map always exists, and one is given by
$$\mu_Z(x, p) = -p \cdot (\rho(Z)(x)),$$
where $x \in X$ and $p \in T^*_xX$. Moreover, the action of G on $\mu^{-1}(0)$ is free, and $T^*X /\!/_\mu G \cong T^*(X/G)$ as symplectic manifolds. ◀

The point of the moment map is to generalize this nice fact.

Proposition 1.21. *Let X be a Kähler manifold and G act on X preserving g, I, and ω, and suppose that there's a moment map μ such that G acts freely on $\mu^{-1}(0)$. Then, $X /\!/_\mu G$ has a natural Kähler structure.*

Explicitly, the symplectic structure is as in Proposition 1.20, the metric is the quotient metric, and I is determined by g and ω. This fact is the reason symplectic quotients arise in the study of Kähler manifolds.

Example 1.22. Let U_1 act on $\mathbb{C}^n$ as in Example 1.18.

- If $c < 0$, then $\mu^{-1}(0) = \varnothing$. G acts freely on this, but for silly reasons.
- If $c = 0$, $\mu^{-1}(0) = \{0\}$, so the U_1-action is not free.
- If $c > 0$,
$$\mu^{-1}(0) = \left\{\sum_{i=1}^n \lvert z_i\rvert^2 = 2c\right\} \cong S^{2n-1},$$
and the quotient is $\mathbb{C}^n /\!/_\mu \mathrm{U}_1 = \mu^{-1}(0)/\mathrm{U}_1 = \mathbb{CP}^{n-1}$. The induced metric is the Fubini-Study metric.

The last case is interesting: instead of taking the quotient X/G, we took a subset of X and quotiented out by a complex analogue of G: $X \setminus \{0\}/\mathbb{C}^\times$. This applies in more general situations. ◀

[3] By this notation, we mean $[\mathfrak{g}, \mathfrak{g}]^\perp = \{f \in \mathfrak{g}^* \mid f(x) = 0 \text{ for all } x \in [\mathfrak{g}, \mathfrak{g}]\}$.

Definition 1.23. Let (X, I) be a complex manifold. A **holomorphic symplectic structure** on X is an $\Omega \in \Omega^{2,0}(X)$ such that

- $\mathrm{d}\Omega = 0$ and
- Ω is **holomorphically nondegenerate**, i.e. it induces an isomorphism $T^{1,0}X \to (T^{1,0}X)^*$.

The triple (X, I, Ω) is called a **holomorphic symplectic manifold**.

Proposition 1.24. *Let (X, I, Ω) be a holomorphic symplectic manifold. Then, $\dim_{\mathbb{R}}(X)$ is divisible by 4.*

On a holomorphic symplectic manifold, there are particularly nice holomorphic coordinates, called **Darboux coordinates** $(p_1, \dots, p_n, q_1, \dots, q_n)$, for which

$$\Omega = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i.$$

This is the analogue of a symplectic structure, but on a complex manifold.

Now we know enough to define a hyperkähler manifold.

Definition 1.25. Let (X, g) be a Riemannian manifold and I, J, and K be three complex structures on X. Then, (X, g, I, J, K) is a **hyperkähler manifold** if

- $IJ = K$ in $\mathrm{End}(TM)$, and
- (X, g, I), (X, g, J), and (X, g, K) are all Kähler manifolds.

The corresponding Kähler forms are denoted ω_I, ω_J, and ω_K. Sometimes, I_1 is used for I, I_2 for J, and I_3 for K, and ω_i for ω_{I_i}.

The relation $IJ = K$ is equivalent to imposing the relations of the quaternions onto I, J, and K, i.e. $IJ = K$, $JK = I$, and $KI = J$; and $JI = -K$, $KJ = -I$, and $IK = -J$.

If you let $\Omega_1 \coloneqq \omega_2 + i\omega_3$, then (X, I, Ω_1) is a holomorphic symplectic manifold, and hence $\dim_{\mathbb{R}} X$ must be divisible by 4.

2. Twistor space: 6/13/17

Last time, we defined a plethora of geometric structures on maps. For example, the tangent and cotangent bundles of a manifold are abstractly isomorphic, and different structures define different isomorphisms: a metric defines a symmetric form g such that $v \mapsto g(v, -)$ is an isomorphism $TX \to T^*X$. A symplectic structure defines a skew-symmetric form ω such that $v \mapsto \omega(v, -)$ is an isomorphism.

A complex structure defines a skew-symmetric involution $I\colon TX \to TX$, and a Kähler structure says this is compatible with g and ω. Heuristically, it means "$\omega = g \circ I$," but what this actually means is that

$$\omega(v, w) = g(Iv, w). \tag{2.1}$$

A hyperkähler manifold is a Kähler manifold in three ways: it comes with a Riemannian metric g and *three* almost complex structures I, J, and K such that $IJ = K$ and (X, g, I), (X, g, J), and (X, g, K) are all Kähler. Let ω_I be the symplectic form associated to (g, I), and similarly for ω_J and ω_K. Using (2.1), one can derive some relations between them, e.g.

$$\begin{aligned} \omega_K(Iv, -) &= g(KIv, -) \\ &= g(Jv, -) \\ &= \omega_J(v, -). \end{aligned}$$

Heuristically, "$\omega_K \circ I = \omega_J$," and similarly "$g = -\omega_I \circ I = -\omega_I \circ \omega_K^{-1} \circ \omega_K$." Moreover, all cyclic permutations of I, J, and K work. The quotes only mean we're not applying the symplectic forms to two arguments, but considering the map $\omega(v, -)$, which is an isomorphism.

Proposition 2.2. *Let X be a manifold with three symplectic structures ω_I, ω_J, and ω_K such that after plugging in some tangent vector,*

$$g \coloneqq -\omega_I \omega_K^{-1} \omega_J = -\omega_J \omega_I^{-1} \omega_K = -\omega_K \omega_J^{-1} \omega_I.$$

If g is positive definite, then $(g, \omega_I, \omega_J, \omega_K)$ determine a hyperkähler structure on X.

Not every symplectic manifold is hyperkähler, e.g. any Riemann surface (because every hyperkähler manifold has dimension a multiple of 4).

We said that a hyperkähler manifold has three complex structures, but in fact we get infinitely many more for free.

Proposition 2.3. *Let (X, g, I, J, K) be a hyperkähler manifold and $\mathbf{s} \in S^2 \subset \mathbb{R}^3$. Let*

$$\begin{aligned} I_{\mathbf{s}} &\coloneqq s_1 I + s_2 J + s_3 K \\ \omega_{\mathbf{s}} &\coloneqq s_1 \omega_I + s_2 \omega_J + s_3 \omega_K. \end{aligned}$$

Then, $I_{\mathbf{s}}^2 = -1$ and $(X, g, I_{\mathbf{s}})$ is Kähler.

So we have an $S^2 = \mathbb{CP}^1$ worth of complex/Kähler structures. In complex geometry, we'd use $\mathbb{CP}^1$, because we care about the complex structure on the space of complex structures.

You can think of S^2 as the unit imaginary quaternions, as $\mathbb{R}^3 = \mathrm{Im}(\mathbb{H})$. This can be used to cleverly define the (almost) complex structure on $\mathbb{CP}^1$: we want an involution $I_x \colon T_x\mathbb{CP}^1 \to T_x\mathbb{CP}^1$. At i, this is just multiplication by i: $ak + bk \mapsto i(aj + bk) = ak - bj$. That is, the almost complex structure at $x = ai + bj + ck$ is multiplying by $ai + bj + ck$, which is a rotation by 180° in $T_x\mathbb{CP}^1$. You can show this structure is integrable.

You can do the same thing with the **octonions** $\mathbb{O}$, an eight-dimensional $\mathbb{R}$-algebra whose multiplication is neither commutative nor associative, but every nonzero element is invertible. But it has just enough structure to be useful in topology and geometry. As with $\mathbb{C}$ and $\mathbb{H}$, the **imaginary octonions** $\mathrm{Im}(\mathbb{O})$ is the orthogonal complement to the subalgebra $\mathbb{R} \hookrightarrow \mathbb{O}$; $\mathrm{Im}(\mathbb{O}) \cong \mathbb{R}^7$ as vector spaces, and hence can define an almost complex structure on $S^6 \subset \mathrm{Im}(\mathbb{O})$. But this is *not* integrable! This has been known for a long time — but it's still not know whether S^6 admits a different complex structure that's integrable!

The state-of-the-art results on complex structures:

- In complex dimension 1, every almost complex structure is integrable.
- In complex dimension 2, there are examples of non-integrable complex structures, and even manifolds which admit almost complex structures but not complex structures. The examples are kind of crazy, and there are various methods to prove this, using the Hodge diamond and Pontrjagin classes.
- In complex dimensions at least 3, there are no known examples of manifolds which admit almost complex structures but no complex structures.

The twistor space is a way to encode all of the information about a hyperkähler manifold, in that you can completely recover the hyperkähler manifold from geometric information on its twistor space.

Definition 2.4. Let X be a hyperkähler manifold. Its **twistor space** is $Z \coloneqq X \times S^2$ as a manifold, with the almost complex structure $I(x, \mathbf{s}) = I_{\mathbf{s}}(x) \oplus I_{S^2}$, where we use the usual complex structure I_{S^2} on S^2 as above.

The idea is to make good use of the S^2-worth of complex structures we found by fibering all of them over S^2.

Proposition 2.5.

(1) *I is integrable, and so Z is a complex manifold.*
(2) *The projection map $\pi \colon Z \twoheadrightarrow S^2$ is holomorphic.*
(3) *Z has a **fiberwise holomorphic symplectic structure**, i.e. the fibers of π are holomorphic symplectic manifolds.*

The fiberwise holomorphic symplectic structure is dictated by an

$$\Omega \in \Lambda^2(T^{1,0}_{\mathrm{vert}}Z)^* \otimes \pi^*\mathscr{O}(2).$$

That is:

- $T_{\mathrm{vert}}Z$ is the **vertical tangent bundle** of $\pi \colon Z \to S^2$, i.e. at each $z \in Z$, the preimage of $0 \in T_{\pi(z)}S^2$. Then, apply the $(-)^{1,0}$ construction as usual.
- $\mathscr{O}(2)$ is the holomorphic line bundle of degree 2 over $\mathbb{CP}^1$, and π^* pulls it back to Z.

It's possible to understand this fiberwise holomorphic symplectic structure more explicitly. Let $\Omega_1 := \omega_2 + i\omega_3$, and similarly for Ω_2 and Ω_3. Then, above a $\zeta \in \mathbb{CP}^1$,

$$\begin{aligned}\Omega(\zeta) &= \frac{1}{2\zeta}\Omega_1 - i\omega_2 + \frac{\zeta}{2}\overline{\Omega}_1 \\ &= \frac{\omega_2 + i\omega_3}{2\zeta} - i\omega_1 + \frac{\zeta}{2}(\omega_2 - i\omega_3).\end{aligned}$$

Here, $1/\zeta$ means on the Riemann sphere: $1/0 = \infty$, $1/\infty = 0$, and everything else is as usual on $\mathbb{C}$.

Thus, we can compute at specific ζ:

$$\begin{aligned}\Omega(1) &= \omega_2 - i\omega_1 = -i\Omega_3 \\ \Omega(i) &= -\Omega_2 \\ \Omega(0) &= \Omega_1.\end{aligned}$$

Proposition 2.6. *Z has a **real structure**, i.e. an antiholomorphic involution $\rho\colon Z \to Z$, which is the map $(x, \mathbf{s}) \mapsto (x, -\mathbf{s})$. Hence it covers the antipodal map. It also satisfies $\rho^*\Omega = \overline{\Omega}$.*

Remark. There's a theorem (with a long involved proof) that a manifold Z satisfying the properties in Proposition 2.5 and Proposition 2.6 is in some sense a twistor space: one can recover a **pseudohyperkähler manifold** X (i.e. a hyperkähler manifold except that the metric might not be positive definite), which is the space of all **real holomorphic sections**, i.e. those fixed under ρ, with normal bundle $\mathscr{O}(1)^{\oplus 2n}$. Moreover, if Z is the twistor space of a hyperkähler space X', then $X' \subset X$, and in many cases $X' = X$. The point is that the twistor space all but allows you to reconstruct the ordinary hyperkähler manifold, and is the same information packaged in a different way. ◀

Now let's do some examples!

Example 2.7. $\mathbb{R}^4 \cong \mathbb{H}$ is hyperkähler: $T_p\mathbb{H} \cong \mathbb{H}$ naturally. The norm $\|q\| := q\overline{q}$ (if $q = a + bi + cj + dk$, then $\|q\| = a^2 + b^2 + c^2 + d^2$) defines a Riemannian metric on $\mathbb{H}$, and left-multiplication by i, j, and k defines the complex structures I, J, and K respectively, and they evidently preserve the norm, so everything works out and $\mathbb{H}$ is hyperkähler.

In coordinates (x_0, x_1, x_2, x_3) for the directions $(1, i, j, k)$ respectively,

$$\begin{aligned}\omega_I &= \mathrm{d}x_0 \wedge \mathrm{d}x_1 + \mathrm{d}x_2 \wedge \mathrm{d}x_3 \\ \omega_J &= \mathrm{d}x_0 \wedge \mathrm{d}x_2 + \mathrm{d}x_3 \wedge \mathrm{d}x_1 \\ \omega_K &= \mathrm{d}x_0 \wedge \mathrm{d}x_3 + \mathrm{d}x_1 \wedge \mathrm{d}x_2.\end{aligned}$$

More invariantly,

$$\omega_i = \mathrm{d}x_0 \wedge \mathrm{d}x_i + \star\mathrm{d}x_i,$$

where $\star$ is the Hodge star.

We can also write down the holomorphic symplectic forms. Let $w_1 = x_0 + ix_1$ and $z_1 = x_2 + ix_3$. Then,

$$\Omega_I = \omega_J + i\omega_K = \mathrm{d}w_1 \wedge \mathrm{d}z_1.$$

Above each $\zeta \in S^2$, I_ζ defines an isomorphsim $\mathbb{H} \cong \mathbb{C}^2$. The isomorphism depends on ζ, though: it depends on the orthogonal decomposition $\mathbb{C}^2 \cong \mathbb{C} \oplus \mathbb{C}$, which is determined by a real direction (fixed) and an imaginary direction (which is the $\mathbb{CP}^1$ of complex structures we get).

This naturally generalizes to any finite-dimensional quaternionic vector space (which is isomorphic to $\mathbb{H}^n$). ◀

Remark. The quaternions of unit norm form an $S^3 \subset \mathbb{H}$, which has a Lie group structure given by quaternion multiplication. In particular, it's SU_2 (also Spin_3 and Sp_1). Thus, ther'es an $\mathrm{SU}_2 \times \mathrm{SU}_2$-action on $\mathbb{H}$ (from the left and right, respectively) in which

$$(q_1, q_2) \cdot x = q_1 x q_2^{-1}.$$

This preserves the norm, hence defines a map $\psi\colon \mathrm{SU}_2 \times \mathrm{SU}_2 \to \mathrm{SO}_4$. It's easy to check ψ is onto, and $\ker(\psi) \cong \mathbb{Z}/2$; since $\mathrm{SU}_2 \times \mathrm{SU}_2$ is connected and $\pi_1(\mathrm{SO}_4) \cong \mathbb{Z}/2$, this implies $\mathrm{SU}_2 \times \mathrm{SU}_2 \cong \mathrm{Spin}_4$, the connected double cover of SO_4. There are other ways to show this, but using the quaternions is nice.