M392C NOTES: SPIN GEOMETRY

ARUN DEBRAY
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These notes were taken in UT Austin's M392C (Spin Geometry) class in Fall 2016, taught by Eric Korman. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Thanks to Adrian Clough for fixing a few typos.

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Lecture 1.

Lie Groups: 8/25/16

There is a course website, located at https://www.ma.utexas.edu/users/ekorman/teaching/spingeometry/. There's a list of references there, none of which we'll exactly follow.

We'll assume some prerequisites for this class: definitely smooth manifolds and some basic algebraic topology. We'll use cohomology, which isn't part of our algebraic topology prelim course, but we'll review it before using it.

Introduction and motivation. Recall that a *Riemannian manifold* is a pair (M, g) where M is an n-dimensional smooth manifold and g is a *Riemannian metric* on M, i.e. a smoothly varying, positive definite inner product on each tangent space T_xM over all $x \in M$.

Definition 1.1. A *local frame* on M is a set of (locally defined) tangent vectors that give a positive basis for M, i.e. a smoothly varying set of tangent vectors that are a basis at each tangent space.

A Riemannian metric allows us to talk about *orthonormal frames*, which are those that are orthonormal with respect to the metric at all points.

Recall that the special orthogonal group is $\mathrm{SO}(n) = \{A \in M_n \mid AA^\mathrm{T} = I, \det A = 1\}$. This acts transitively on orthonormal, oriented bases, and therefore also acts transitively on orthonormal frames (as a frame determines an orientation). Conversely, specifying which frames are orthonormal determines the metric g.

In summary, the data of a Riemannian structure on a smooth manifold is equivalent to specifying a subset of all frames which is acted on simply transitively[1] by the group $\mathrm{SO}(n)$. This set of all frames is a *principal* $\mathrm{SO}(n)$*-bundle* over M.

By replacing $\mathrm{SO}(n)$ with another group, one obtains other kinds of geometry: using $\mathrm{GL}(n, \mathbb{C})$ instead, we get almost complex geometry, and using $\mathrm{Sp}(n)$, we get almost symplectic geometry (geometry with a specified skew-symmetric, nondegenerate form).

Remark. Let G be a Lie group and M be a manifold. Suppose we have a principal G-bundle $E \to M$ and a representation[2] $\rho : G \to V$, we naturally get a vector bundle over M.

[1] Recall that a group action on X is *transitive* if for all $x, y \in X$, there's a group element g such that $g \cdot x = y$, and is *simple* if this g is unique.
[2] A representation of a group G is a homomorphism $G \to \mathrm{GL}(V)$ for a vector space V. We'll talk more about representations later.

A more surprising fact is that all[3] representations of $\mathrm{SO}(n)$ are contained in tensor products of the *defining representation* of $\mathrm{SO}(n)$ (i.e. acting on $\mathbb{R}^n$ by orientation-preserving rotations). Thus, all of the natural vector bundles are subbundles of tensor powers of the tangent bundles. That is, when we do geometry in this way, we obtain no exotic vector bundles.

If $n \geq 3$, then $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2$, so its double cover is its universal cover. Lie theory tells us this space is naturally a compact Lie group, called the *Spin group* $\mathrm{Spin}(n)$. In many ways, it's more natural to look at representations of this group. The covering map $\mathrm{Spin}(n) \twoheadrightarrow \mathrm{SO}(n)$ precomposes with any representation of $\mathrm{SO}(n)$, so any representation of $\mathrm{SO}(n)$ induces a representation of $\mathrm{Spin}(n)$. However, there are representations of the spin group that don't arise this way, so if we can refine the orthonormal frame bundle to a principal $\mathrm{Spin}(n)$-bundle, then we can create new vector bundles that don't arise as tensor powers of the tangent bundle.

Spin geometry is more or less the study of these bundles, called *bundles of spinors*; these bundles have a natural first-order differential operator called the *Dirac operator*, which relates to a powerful theorem coming out of spin geometry, the Atiyah-Singer index theorem: this is vastly more general, but has a particularly nice form for Dirac operators, and the most famous proof reduces the general case to the Dirac case. Broadly speaking, the index theorem computes the dimension of the kernel of an operator, which in various contexts is a powerful invariant. Here are a few special cases, even of just the Dirac case of the Atiyah-Singer theorem.

- The Gauss-Bonnet-Chern theorem gives an integral formula for the Euler characteristic of a manifold, which is entirely topological. In this case, the index is the Euler characteristic.
- The Hirzebruch signature theorem gives an integral formula for the signature of a manifold.
- The Grothendieck-Riemann-Roch theorem, which gives an integral formula for the Euler characteristic of a holomorphic vector bundle over a complex manifold.

Lie groups and Lie algebras.

Definition 1.2. A *Lie group* G is a smooth manifold with a group structure such that the multiplication map $G \times G \to G$ sending $g_1, g_2 \mapsto g_1 g_2$ and the inversion map $G \to G$ sending $g \mapsto g^{-1}$ are smooth.

Example 1.3.

- The *general linear group* $\mathrm{GL}(n,\mathbb{R})$ is the group of $n \times n$ invertible matrices with coefficients in $\mathbb{R}$. Similarly, $\mathrm{GL}(n,\mathbb{C})$ is the group of $n \times n$ invertible complex matrices. Most of the matrices we consider will be subgroups of these groups.
- Restricting to matrices of determinant 1 defines $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{SL}(n,\mathbb{C})$, the *special linear groups*.
- The *special unitary group* $\mathrm{SU}(n) = \{A \in \mathrm{GL}(n,\mathbb{C}) \mid A\overline{A}^{\mathrm{T}} = 1, \det A = 1\}$.
- The special orthogonal group $\mathrm{SO}(n)$, mentioned above.

Definition 1.4. A *Lie algebra* is a vector space $\mathfrak{g}$ with an anti-symmetric, bilinear pairing $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the *Jacobi identity*

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

for all $X, Y, Z \in \mathfrak{g}$.

Example 1.5. The basic and important example: if A is an algebra,[4] then A becomes a Lie algebra with the commutator bracket $[a,b] = ab - ba$. Because this algebra is associative, the Jacobi identity holds.

The Jacobi identity might seem a little vague, but here's another way to look at it: if $\mathfrak{g}$ is a Lie algebra and $X \in \mathfrak{g}$, then there's a map $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ sending $Y \mapsto [X,Y]$. The map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ sending $X \mapsto \mathrm{ad}_X$ is called the *adjoint representation* of X. The Jacobi identity says that ad intertwines the bracket of $\mathfrak{g}$ and the bracket induced from the algebra structure on $\mathrm{End}(\mathfrak{g})$ (where multiplication is composition): $\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y]$. In other words, the adjoint representation is a homomorphism of Lie algebras.

Lie groups and Lie algebras are very related: to any Lie group G, let $\mathfrak{g}$ be the set of left-invariant vector fields on G, i.e. if $L_g : G \to G$ is the map sending $h \mapsto gh$ (the *left multiplication* map), then $\mathfrak{g} = \{X \in \Gamma(TG) \mid \mathrm{d}L_g X = X \text{ for all } g \in G\}$. This is actually finite-dimensional, and has the same dimension as G.

Proposition 1.6. *If e denotes the identity of G, then the map $\mathfrak{g} \to T_e G$ sending $X \mapsto X(e)$ is an isomorphism (of vector spaces).*

[3] We're only considering smooth, finite-dimensional representations.

[4] By an *algebra* we mean a ring with a compatible vector space structure.

The idea is that given the data at the identity, we can translate it by g to determine what its value must be everywhere. Vector fields have a Lie bracket, and the Lie bracket of two left-invariant vector fields is again left-invariant, so $\mathfrak{g}$ is naturally a Lie algebra! We will often use Proposition 1.6 to identify $\mathfrak{g}$ with the tangent space at the identity.

Example 1.7. Let's look at $\mathrm{GL}(n,\mathbb{R})$. This is an open submanifold of the vector space M_n, an n^2-dimensional vector space, as $\det A \neq 0$ is an open condition. Thus, the tangent bundle of $\mathrm{GL}(n,\mathbb{R})$ is trivial, so we can canonically identify $T_I\mathrm{GL}(n,\mathbb{R}) = M_n$. With the inherited Lie algebra structure, this space is denoted $\mathfrak{gl}(n,\mathbb{R})$.

The $n \times n$ matrices are also isomorphic to $\mathrm{End}(\mathbb{R}^n)$, since they act by linear transformations. The algebra structure defines another Lie bracket on this space.

Proposition 1.8. *Under the above identifications, these two brackets are identical, hence define the same Lie algebra structure on* $\mathfrak{gl}(n,\mathbb{R})$.

Remark. This proposition generalizes to all real matrix Lie groups (Lie subgroups of $\mathrm{GL}(n,\mathbb{R})$): the proof relies on a Lie subgroup's Lie algebra being a subalgebra of $\mathfrak{gl}(n,\mathbb{R})$.

So we can go from Lie groups to Lie algebras. What about in the other direction?

Theorem 1.9. *The correspondence sending a connected, simply-connected Lie group to its Lie algebra extends to an equivalence of categories between the category of simply connected Lie groups and finite dimensional Lie algebras over* $\mathbb{R}$.

Suppose G is any connected Lie group, not necessarily simply connected, and $\mathfrak{g}$ is its Lie algebra. If $\widetilde{G}$ denotes the universal cover of G, then $G = \widetilde{G}/\pi_1(G)$. Since $\widetilde{G}$ is simply connected, the correspondence above identifies $\mathfrak{g}$ with it, and then taking the quotient by the discrete central subgroup $\pi_1(G)$ recovers G.

The special orthogonal group. We specialize to $\mathrm{SO}(n)$, the orthogonal matrices with determinant 1. We'll usually work over $\mathbb{R}$, but sometimes $\mathbb{C}$. This is a connected Lie group.[5]

Proposition 1.10. *If* $\mathfrak{so}(n)$ *denotes the Lie algebra of* $\mathrm{SO}(n)$*, then* $\mathfrak{so}(n) = \{X \in \mathfrak{gl}(n,\mathbb{R}) \mid X + X^{\mathrm{T}} = 0\}$.

That is, $\mathfrak{so}(n)$ is the Lie algebra of skew-symmetric matrices.

Proof. If $F : M_n \to M_n$ is the function $A \mapsto A^{\mathrm{T}}A - I$, then the orthogonal group is $\mathrm{O}(n) = F^{-1}(0)$. Since $\mathrm{SO}(n)$ is the connected component of $\mathrm{O}(n)$ containing the identity, then it suffices to calculate $T_e\mathrm{O}(n)$: if 0 is a regular value of F, we can push forward by its derivative. This is in fact the case:

$$\mathrm{d}F_A(B) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F(A + tB) = A^{\mathrm{T}}B + B^{\mathrm{T}}A,$$

which is surjective for $A \in \mathrm{O}(n)$, so $\mathfrak{so}(n) = T_I\mathrm{SO}(n) = \ker(\mathrm{d}F_I) = \{B \in M_n \mid B + B^{\mathrm{T}} = 0\}$. ⊠

The spin group. We'll end by computing the fundamental group of $\mathrm{SO}(n)$; then, by general principles of Lie groups, each $\mathrm{SO}(n)$ has a unique, simply connected double cover, which is also a Lie group. Next time, we'll provide an *a priori* construction of this cover.

Proposition 1.11.

$$\pi_1(\mathrm{SO}(n)) = \begin{cases} \mathbb{Z}, & n = 2 \\ \mathbb{Z}/2, & n \geq 3. \end{cases}$$

Proof. If $n = 2$, $\mathrm{SO}(n) \cong S^1$ through the identification

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \longmapsto e^{i\theta},$$

and we know $\pi_1(S^1) = \mathbb{Z}$.

For $n \geq 3$, we can use a long exact sequence associated to a certain fibration, so it suffices to calculate $\pi_1(\mathrm{SO}(3))$. Specifically, we will define a Lie group structure on S^3 and a double cover map $S^3 \twoheadrightarrow \mathrm{SO}(3)$; since S^3 is simply connected, this will show $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}/2$.

[5] If we only took orthogonal matrices with arbitrary determinant, we'd obtain the *orthogonal group* $\mathrm{O}(n)$, which has two connected components.

We can identify S^3 with the unit sphere in the quaternions, which is naturally a group (since the product of quaternions is a polynomial, hence smooth).[6] Realize $\mathbb{R}^3$ inside the quaternions as $\mathrm{span}_{\mathbb{R}}\{i, j, k\}$ (the *imaginary quaternions*); then, we'll define $\varphi : S^3 \twoheadrightarrow \mathrm{SO}(3)$: $\varphi(q)$ for $q \in \mathbb{H}$ is the linear transformation $p \mapsto qpq^{-1} \in \mathrm{GL}(3, \mathbb{R})$, where p is an imaginary quaternion. We need to check that $\varphi(q)$ lies in SO(3), which was left as an exercise. We also need to check this is two-to-one, which is equivalent to $|\ker\varphi| = 2$, and that φ is surjective (hint: since these groups are connected, general Lie theory shows it suffices to show that the differential is an isomorphism). ⊠

Lecture 2.

Spin Groups and Clifford Algebras: 8/30/16

Last time, we gave a rushed construction of the double cover of SO(3), so let's investigate it more carefully. Recall that SO(n) is the Lie group of special orthogonal matrices, those matrices A such that $AA^{\mathrm{t}} = I$ and $\det A = 1$, i.e. those linear transformations preserving the inner product and orientation. This is a connected Lie group; we'd like to prove that for $n \geq 3$, $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2$. (For $n = 2$, $\mathrm{SO}(2) \cong S^1$, which has fundamental group $\mathbb{Z}$).

We'll prove this by explicitly constructing the double cover of SO(3), then bootstrapping it using a long exact sequence of homotopy groups to all SO(n), using the following fact.

Proposition 2.1. *Let G and H be connected Lie groups and $\varphi : G \to H$ be a Lie group homomorphism. Then, φ is a covering map iff $\mathrm{d}\varphi|_e : \mathfrak{g} \to \mathfrak{h}$ is an isomorphism.*[7]

Here $\mathfrak{g}$ is the Lie algebra of G, and $\mathfrak{h}$ is that of H. Facts like these may be found in Ziller's online notes;[8] the intuitive idea is that the condition on $\mathrm{d}\varphi|_e$ ensures an isomorphism in a neighborhood of the identity, which multiplication carries to a local isomorphism in the neighborhood of any point in G.

Now, we construct a double cover of SO(3). Recall that the *quaternions* are the noncommutative algebra $\mathbb{H} = \mathrm{span}_{\mathbb{R}}\{1, i, j, k\}$, where $i^2 = j^2 = k^2 = ijk = 1$. We can identify $\mathbb{R}^3$ with the imaginary quaternions, the span of $\{i, j, k\}$, and therefore the unit sphere S^3 goes to $\{q \in \mathbb{H} \mid |q|^2 = 1 = q\bar{q}\}$, where the conjugate exchanges i and $-i$, but also j and $-j$, and k and $-k$. This embedding means that if $v, w \in \mathbb{R}^3$, their product as quaternions is

$$vw = -\langle v, w\rangle + v \times w.$$

and in particular

$$vw + wv = -2\langle v, w\rangle. \tag{2.2}$$

If $q \in S^3$ and $v \in \mathbb{R}^3$, then $qvq^{-1} = qv\bar{q}$, i.e. $\overline{qvq^{-1}} = q\overline{vq} = -qv\bar{q}$. That is, conjugation by something in S^3 is a linear transformation in $\mathbb{R}^3$, defining a smooth map $\varphi : S^3 \to \mathrm{GL}(3, \mathbb{R})$; we'd like to show the image lands in SO(3). Let $q \in S^3$; then, we can use (2.2) to get

$$\begin{aligned}\langle \varphi(q)v, \varphi(q)w\rangle &= -\frac{1}{2}(\varphi(q)v\varphi(q)w + \varphi(q)w + \varphi(q)v)\\ &= -\frac{1}{2}\left(qvwq^{-1} + qwvq^{-1}\right)\\ &= -\frac{1}{2}\left(q(vw + wv)q^{-1}\right) = \langle v, w\rangle,\end{aligned}$$

using (2.2) again, and the fact that $\mathbb{R} = Z(\mathbb{H})$. Thus, $\mathrm{Im}(\varphi) \subset \mathrm{O}(3)$, but since S^3 is connected, its image must be connected, and its image contains the identity (since φ is a group homomorphism), so $\mathrm{Im}(\varphi)$ lies in the connected component containing the identity, which is SO(3).

[6]This is important, because when we try to generalize to Spin_n for higher n, we'll be using Clifford algebras, which are generalizations of the quaternions.

[7]This isomorphism is as Lie algebras, but it's always a Lie algebra homomorphism, so it suffices to know that it's an isomorphism of vector spaces.

[8]https://www.math.upenn.edu/ wziller/math650/LieGroupsReps.pdf.

To understand $\mathrm{d}\varphi|_1$, let's look at the Lie algebras of S^3 and SO(3). The embedding $S^3 \hookrightarrow \mathbb{H}$ allows us to identify $T_! S^3$ with the imaginary quaternions. If p and v are imaginary quaternions, so $\overline{p} = -p$, then

$$\begin{aligned} \mathrm{d}\varphi|_p(v) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg| |_{t=0}\varphi(e^{tp})v \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} e^{tp} v e^{-tp} \\ &= pv - vp. \end{aligned}$$

Thus, $\ker \mathrm{d}\varphi|_1 = \{p \in \mathbb{R}^3 \mid pv - vp = 0 \text{ for all imaginary quaternions } v\}$. But if something commutes with all imaginary quaternions, it commutes with all quaternions, since the imaginary quaternions and the reals (which are the center of $\mathbb{H}$) span to all of $\mathbb{H}$. Thus, the kernel is the imaginary quaternions in the center of $\mathbb{H}$, which is just $\{0\}$; hence, $\mathrm{d}\varphi|_1$ is injective, and since $T_1 S^3$ and $\mathfrak{so}(3)$ have the same dimension, it is an isomorphism. By Proposition 2.1, φ is a covering map, and $\mathrm{SO}(3) = S^3/\ker(\varphi)$.

We'll compute $|\ker\varphi|$, which will be the index of the cover. The kernel is the set of unit quaternions q such that $qvq^{-1} = v$ for all imaginary quaternions v; just as above, this must be the intersection of the real line with S^3, which is just $\{\pm 1\}$. Thus, φ is a double cover map of SO(3); since S^3 is simply connected, $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}/2$.

Exercise 2.3. The Lie group structure on S^3 is isomorphic to SU(2), the group of 2×2 special unitary matrices.

Now, what about $\pi_1(\mathrm{SO}(n))$, for $n \geq 4$? For this we use a fibration. $\mathrm{SO}(n)$ acts on $S^{n-1} \subset \mathbb{R}^n$, and the stabilizer of a point in S^n is all the rotations fixing the line containing that point, which is a copy of $\mathrm{SO}(n-1)$. This defines a fibration

$$\mathrm{SO}(n-1) \longrightarrow \mathrm{SO}(n) \longrightarrow S^{n-1}.$$

More precisely, let's fix the north pole $p = (0,0,\dots,0,1) \in S^{n-1}$; then, the map $\mathrm{SO}(n) \to S^{n-1}$ sends $A \mapsto Ap$; since A is orthogonal, Ap is a unit vector. The action of $\mathrm{SO}(n)$ is transitive, so this map is surjective. The stabilizer of p is the set of all orthogonal matrices with positive determinant such that the last column is $(0,0,\dots,0,1)$. Orthogonality forces these matrices to have block form

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix},$$

where $A \in \mathrm{SO}(n-1)$; thus, the stabilizer is isomorphic to $\mathrm{SO}(n-1)$.

Now, we can use the long exact sequence in homotopy associated to a fibration:

$$\pi_2(S^{n-1}) \xrightarrow{\delta} \pi_1(\mathrm{SO}(n-1)) \longrightarrow \pi_1(\mathrm{SO}(n)) \longrightarrow \pi_1(S^{n-1}).$$

If $n \geq 4$, $\pi_2(S^{n-1})$ and $\pi + 1(S^{n-1})$ are trivial, so $\pi_1(\mathrm{SO}(n)) = \pi_1(\mathrm{SO}(n-1))$ for $n \geq 4$, so they all agree with $\mathbb{Z}?2$, so $\pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}/2$ for all $n \geq 4$.

By general Lie theory, the universal cover of a Lie group is also a Lie group.

Definition 2.4. For $n \geq 3$, the *spin group* $\mathrm{Spin}(n)$ is the unique simply-connected Lie group with Lie algebra $\mathfrak{so}(n)$. For $n = 2$, the spin group Spin(2) is the unique (up to isomorphism) connected double covering group of SO(2).

In particular, there is a double cover $\mathrm{Spin}(n) \to \mathrm{SO}(n)$, and $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$.

Right now, we do not have a concrete description of these groups; since $\mathrm{SO}(n)$ is compact, so is $\mathrm{Spin}(n)$, so we must be able to realize it as a matrix group, and we use Clifford algebras to do this.

Clifford algebras. Our goal is to replace $\mathbb{H}$ with some other algebra to realize $\mathrm{Spin}(n)$ as a subgroup of its group of units.

Recall from (2.2) that for $v, w \in \mathbb{R}^3 \hookrightarrow \mathbb{H}$, $vw + wv = -2\langle v, w\rangle$. We'll define a universal algebra for this kind of definition.

Definition 2.5. Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space. Its *Clifford algebra* is

$$\mathrm{C}\ell(V) = T(V)/(v \otimes v + \langle v, v\rangle 1).$$

Here, $T(V)$ is the tensor algebra, and we quotient by the ideal generated by the given relation.

That is, we've forced (2.2) for a vector paired with itself. That's actually sufficient to imply it for all pairs of vectors.

Remark. Though we only defined the Clifford algebra for nondegenerate inner products, the same definition can be made for all bilinear pairings. If one chooses $\langle\cdot,\cdot\rangle = 0$, one obtains the exterior algebra $\Lambda(V)$, and we'll see that Clifford algebras sometimes behave like exterior algebras.

Recall that the tensor algebra is defined by the following universal property: if A is any algebra,[9] and $f : V \to A$ is linear, then there exists a unique homomorphism of algebras $\widetilde{f} : T(V) \to A$ such that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d] & A \\ T(V). \arrow[ur, "\widetilde{f}"'] & \end{tikzcd}$$

That is, as soon as I know what happens to elements of f, I know what to do to tensors.

This implies a universal property for the Clifford algebra.

Proposition 2.6. *Let A be an algebra and $f : V \to A$ be a linear map. Then, $f(v)^2 = -\langle v, v\rangle 1_A$ iff f extends uniquely to a map $\widetilde{f} : \mathrm{C}\ell(V) \to A$ such that the following diagram commutes:*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d] & A \\ \mathrm{C}\ell(V). \arrow[ur, "\widetilde{f}"'] & \end{tikzcd}$$

The map $V \to \mathrm{C}\ell(V)$ is the composition $V \hookrightarrow T(V) \twoheadrightarrow \mathrm{C}\ell(V)$, where the last map is projection onto the quotient.

We'll end up putting lots of structure on Clifford algebras: a $\mathbb{Z}/2$-grading, a $\mathbb{Z}$-filtration, a canonical vector-space isomorphism with the exterior algebra, and so forth.

Important Example 2.7. Let $\Lambda^\bullet V$ denote the exterior algebra on V, the graded algebra whose k^{th} graded piece is wedges of k vectors: $\Lambda^k(V) = \{v_1, \dots, v_k \mid v_j \in V\}$, with the relations $v \wedge w = -w \wedge v$.

Given a $v \in V$, we can define two maps, *exterior multiplication* $\varepsilon(v) : \Lambda^\bullet(V) \to \Lambda^{\bullet-1}(V)$ defined by $\mu \mapsto v \wedge \mu$, and *interior multiplication* $i(v) : \Lambda^\bullet(V) \to \Lambda^{\bullet-1}(V)$ sending

$$v_1 \wedge \cdots \wedge v_k \mapsto \sum_{i=1}^{k} (-1)^{i-1}\langle v, v_i\rangle v_1 \wedge \cdots \wedge \widehat{v}_i \wedge \cdots \wedge v_k,$$

where $\widehat{v}_i$ means the absence of the i^{th} term.

This has a few important properties:

(1) Both of these maps are idempotents: $\varepsilon(v)^2 = i(v)^2 = 0$.
(2) If $\mu_1, \mu_2 \in \Lambda^\bullet(V)$, then
$$i(v)(\mu_1 \wedge \mu_2) = (i(v)\mu_1) \wedge \mu_2 + (-1)^{\deg \mu_1} \mu_1 \wedge i(v)\mu_2.$$
In particular,

$$\varepsilon(v)i(v) + i(v)\varepsilon(v) = \langle v, v\rangle. \tag{2.8}$$

We can use this to define a representation of the Clifford algebra onto the exterior algebra: define a map $c : V \to \mathrm{End}(\Lambda^\bullet(V))$ by $c(v) = \varepsilon(v) - i(v)$. Then, $c(v)^2 = -(\varepsilon(v)i(v) + i(v)\varepsilon(v)) = \langle v, v\rangle$, so by the universal property, c extends to a homomorphism $c : \mathrm{C}\ell(V) \to \mathrm{End}(\Lambda^* V)$.

Given an inner product on V, there is an induced inner product on $\Lambda^\bullet V$: choose an orthonormal basis $\{e_1, \dots, e_n\}$ for V, and then declare the basis $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \mid 1 \le i_1 < \cdots < i_k \le n\}$ to be orthonormal; then, use the dot product associated to that orthonormal basis. This is coordinate-invariant, however.

Theorem 2.9. *Suppose $\{e_1, \dots, e_n\}$ is a basis for V. Then, $\{e_{i_1} e_{i_2} \cdots e_{i_k} \mid i_1 < i_2 < \cdots i_k\}$ (where the product is in the Clifford algebra) is a vector-space basis for $\mathrm{C}\ell(V)$.*

[9]Here, an algebra is a unital ring with a compatible real vector space structure.

Today, we'll focus on examples, and perhaps prove this later. This tells us that v and w anticommute iff $v \perp w$, and the relations are

$$e_j e_j = \begin{cases} -e_j e_i, & i \neq j \\ -1, & i = j. \end{cases}$$

This is just like the exterior algebra, but deformed: if $i = j$, we get 1 rather than 0. Theorem 2.9 also tells us that $\dim \mathrm{C}\ell(V) = 2^{\dim V}$.

Example 2.10. $\mathrm{C}\ell(\mathbb{R}^2) \cong \mathbb{H}$ as $\mathbb{R}$-algebras: $\mathrm{C}\ell(\mathbb{R}^2)$ is generated by 1, e_1, and e_2 such that $e_1 e_2 = -e_2 e_1$ and $e_1^2 = e_2^2 = -1$. Thus, $\{1, e_1, e_2, e_1 e_2\}$ is a basis for $\mathrm{C}\ell(\mathbb{R}^2)$, and $(e_1 e_2)^2 = e_1 e_2 e_1 e_2 = -e_1^2 e_2^2 = -1$.

Thus, the isomorphism $\mathrm{C}\ell(\mathbb{R}^2) \to \mathbb{H}$ extends from $1 \mapsto 1$, $e_1 \mapsto i$, $e_2 \mapsto j$, and $e_1 e_2 \mapsto k$.

Example 2.11. Even simpler is $\mathrm{C}\ell(\mathbb{R}) \cong \mathbb{C}$, generated by 1 and e_1 such that $e_1^2 = -1$.

Example 2.12. If we consider the Clifford algebra of $\mathbb{C}$ as a complex vector space, $\mathbb{C}$ is in the center, so $\mathrm{C}\ell_{\mathbb{C}}(\mathbb{C})$ is generated by 1 and e_1 with $ie_1 = e_1 i$.

Lecture 3.

The Structure of the Clifford Algebra: 9/1/16

Last time, we started with an inner product space $(V, \langle \cdot, \cdot \rangle)$ and used it to define a Clifford algebra $\mathrm{C}\ell(V) = T(V)/(v \otimes v + \langle v, v \rangle 1)$, the free algebra generated by V such that $v^2 = -\langle v, v \rangle$.[10] For a $v \in V$, let $\widetilde{v} \in \mathrm{C}\ell(V)$ be its image under the natural map $V \to T(V) \twoheadrightarrow \mathrm{C}\ell(V)$: the first map sends a vector to a degree-1 tensor, and the second is the quotient map. It's reasonable to assume this map is injective, and in fact we'll be able to prove this, so we may identify V with its image in the Clifford algebra.

We also defined a representation of $\mathrm{C}\ell(V)$ on $\Lambda^\bullet V$, which was an algebra homomorphism $c : \mathrm{C}\ell(V) \to \mathrm{End}(\Lambda^\bullet V)$ that is defined uniquely by saying that $c(\widetilde{v}) = \varepsilon(v) - i(v)$ (exterior multiplication minus interior multiplication, also known as wedge product minus contraction). We checked that this squares to scalar multiplication by $-\langle v, v \rangle$, so it is an algebra homomorphism.

Definition 3.1. The *symbol map* is the linear map $\sigma : \mathrm{C}\ell(V) \to \Lambda^\bullet V$ defined by $u \mapsto c(u) \cdot 1$.

Theorem 2.9 defines a basis for the Clifford algebra; we can use this to prove it.

Lemma 3.2. *The map $V \to \mathrm{C}\ell(V)$ sending $v \mapsto \widetilde{v}$ is injective.*

Proof. For $v \in V$, $\sigma(v) = c(\widetilde{v})1 = \varepsilon(v) \cdot 1 - i(v) \cdot 1$. Since interior multiplication lowers degree, $i(v) = 0$, so $\sigma(v) = v$. Thus, the map $V \to \mathrm{C}\ell(V)$ is injective. ⊠

We will identify v and $\widetilde{v}$, and just think of V as a subspace of $\mathrm{C}\ell(V)$.

Proposition 3.3. *The symbol map is an isomorphism of vector spaces.*

Proof. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of V, so $e_i e_j = -e_j e_i$ unless $i = j$, in which case it's -1. So $\mathrm{C}\ell(V) = \mathrm{span}\{e_{i_1} e_{i_2} \cdots e_{i_k} \mid i_1 < i_2 < \cdots < i_k\}$. We'll show these are linearly independent, hence form a basis for $\mathrm{C}\ell(V)$, and recover Theorem 2.9 as a corollary.

Since

$$\begin{aligned} c(e_i) e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_k} &= e_i \wedge e_{j_1} \wedge \cdots \wedge e_{j_k} - i(e_i)\big(e_{j_1} \wedge \cdots \wedge e_{j_k}\big) \\ &= e_i \wedge e_{j_1} \wedge \cdots \wedge e_{j_k} \end{aligned}$$

if all indices are distinct, so

$$\begin{aligned} a\sigma(e_{i_1} \cdots e_{i_k}) &= c(e_{i_1}) \cdots c(e_{i_k}) 1 \\ &= c(e_1) \cdots \underbrace{c(e_{i_{k-1}}) e_{i_k}}_{e_{i_{k-1}} \wedge e_{i_k}} \\ &= e_{i_1} \wedge \cdots \wedge e_{i_k}. \end{aligned}$$

[10] There are different conventions here; sometimes people work with the relation $v^2 = \langle v, v \rangle$. This is a different algebra in general over $\mathbb{R}$, but over $\mathbb{C}$ they're the same thing.

As $\{i_1, \ldots, i_k\}$ ranges over all k-element subsets of $\{1, \ldots, n\}$, these form a basis for $\Lambda^\bullet V$. Thus, σ is surjective, and the proposed basis for $\mathrm{C}\ell(V)$ is indeed linearly independent. Thus, σ is also injective, so an isomorphism of vector spaces. ⊠

In particular, we've discovered a basis for $\mathrm{C}\ell(V)$, proving Theorem 2.9.

Remark. The symbol map is *not* an isomorphism of algebras: $\sigma(v^2) = \sigma(-\langle v, v\rangle) = -\langle v, v\rangle$, but $\sigma(v) \wedge \sigma(v) = 0$. The symbol is just the highest-order data of an element of the Clifford algebra.

The proof of the following proposition is an (important) exercise.

Proposition 3.4.
$$Z(\mathrm{C}\ell(V)) = \begin{cases} \mathbb{R}, & \dim V \text{ is even} \\ \mathbb{R} \oplus \mathbb{R}\gamma, & \dim V \text{ is odd,} \end{cases}$$
where $\gamma = e_1 \cdots e_n$ is σ^{-1} of a volume form.

Physicists sometimes call the span of γ *pseudoscalars*, since they commute with everything (in odd degree), much like scalars.

Algebraic structures on the Clifford algebra. Recall that an algebra A is called $\mathbb{Z}$*-graded* if it has a decomposition as a vector space
$$A = \bigoplus_n \in \mathbb{Z} A_n$$
where the multiplicative structure is additive in this grading: $A_j \cdot A_k \subset A_{j+k}$. For example, $\mathbb{R}[x]$ is graded by the degree; the tensor algebra $T(V)$ is graded by degree of tensors, and $\Lambda^\bullet V$ is graded with the n^{th} piece equal to the space of n-forms.

The Clifford algebra is not graded: the square of a vector is a scalar. It admits a weaker structure, called a filtration.

Definition 3.5. An algebra A has a *filtration* (by $\mathbb{Z}$) if there is a sequence of subspaces $A^{(0)} \subset A^{(1)} \subset A^{(2)} \subset \cdots$ such that $A = \bigcup_j A^{(j)}$ and $A^{(j)} \cdot A^{(k)} \subset A^{(j+k)}$.

The key difference is that for a filtration, the different levels can intersect in more than 0.

The Clifford algebra is filtered, with $\mathrm{C}\ell(V)^{(j)} = \operatorname{span} v_1 \cdots v_k \mid k \le j, v_1, \ldots, v_k \in V$, the span of products of at most j vectors.

Another way we can weaken the pined-for $\mathbb{Z}$-grading is to a $\mathbb{Z}/2$-grading, which we can actually put on the Clifford algebra.

Definition 3.6. A $\mathbb{Z}/2$*-grading* of an algebra A is a decomposition $A = A^+ \oplus A^-$ as vector spaces, such that $A^+A^+ \subset A^+$, $A^+A^- \subset A^-$, $A^-A^+ \subset A^-$, and $A^-A^- \subset A^+$. A^- is called the *odd part* or the *negative part* of A, and A^+ is called the *even part* or the *positive part*. In physics, a $\mathbb{Z}/2$-graded algebra is also called a *superalgebra*.

For the Clifford algebra, let $\mathrm{C}\ell(V)^+$ be the subspace spanned by products of odd numbers of vectors, and $\mathrm{C}\ell(V)^-$ be the subspace spanned by products of even numbers of vectors. Then, $\mathrm{C}\ell(V) = \mathrm{C}\ell(V)^+ \oplus \mathrm{C}\ell(V)^-$, and this defines a $\mathbb{Z}/2$-grading.

Definition 3.7. Let $A = \bigcup_j A^{(j)}$ be a filtered algebra. Then, its *associated graded* is
$$\operatorname{gr} A = \bigoplus_j A^{(j)}/A^{(j-1)},$$
which is naturally a graded algebra with $(\operatorname{gr} A)^j = A^{(j)}/A^{(j-1)}$ and multiplication inherited from A.

Proposition 3.8. *The associated graded of the Clifford algebra* $\operatorname{gr} \mathrm{C}\ell(V) = \Lambda^\bullet V$.

This ultimately follows because the isomorphism $\mathrm{C}\ell(V)^{(j)}/\mathrm{C}\ell(V)^{(j-1)} \to \Lambda^j V$ sends $u \mapsto \sigma(u)_[j]$: the exterior algebra remembers the top part of the Clifford multiplication.[11]

[11]There is a sense in which this defines the Clifford algebra as a deformation of the exterior algebra; a fancy word for this would be *filtered quantization*. Similarly, we'll see that the symmetric algebra is the associated graded of the symmetric algebra.

Constructing spin groups. Now, we assume that the inner product on V is positive definite.

If $v \neq 0$ in V, then v^{-1} exists in $C\ell(V)$ and is equal to $-v/\langle v, v\rangle$. For a $w \in V$, let $\rho_v(W)$ be conjugation: $\rho_v(W) = -vwv^{-1}$. Then, $\rho_v(v) = -v$, and if $w \perp v$, then $\rho_v(w) = -vwv^{-1} = wvv^{-1} = w$, so ρ_v preserves span $v^\perp$ and sends $v \mapsto -v$. Thus, it's a reflection across span $v\perp$.

Theorem 3.9. *The orthogonal group* $\mathrm{O}(n)$ *is generated by reflections, and everything in* $\mathrm{SO}(n)$ *is a product of an even number of reflections.*

Proof. Let's induct on n. When $n = 1$, $\mathrm{O}(1) = \{\pm 1\}$, for which this is vacuously true. Now, let $A \in \mathrm{O}(n)$. If A fixes an $e_1 \in \mathbb{R}^n$, then A fixes span $e_1^\perp$, so by induction, $A|_{\operatorname{span} e_1^\perp} = R_1 \cdots R_k$ for some reflections $R_1, \dots, R_k \in \mathrm{O}(n-1)$. These reflections include into $\mathrm{O}(n)$ by fixing span e_1, and are still reflections, so $A = R_1 \cdots R_k$ decomposes A as a product of reflections.

Alternatively, suppose $Ae_1 = v \neq e_1$. Let R be a reflection about $\{v - e_1\}^\perp$; then, R exchanges v and e_1. Hence, $RA \in \mathrm{O}(n)$ and fixes e_1, so by above $RA = R_1 \cdots R_k$ for some reflections, and therefore $A = RR_1 \cdots R_k$ is a product of reflections.

For $\mathrm{SO}(n)$, observe that each reflection has determinant -1, but all rotations in $\mathrm{SO}(n)$ have determinant 1, so no $A \in \mathrm{SO}(n)$ can be a product of an odd number of rotations. ⊠

We've defined reflections ρ_v in the Clifford algebra, so if we can act by orientation-preserving reflections with a $\mathbb{Z}/2$ kernel, we should have described the spin group.

This reflection ρ_v is a restriction of the *twisted adjoint action*, a representation of $C\ell(V)^\times$ on $C\ell(V)$: $u_1 \mapsto \rho_{u_1}$ that sends $u_2 \mapsto \alpha(u_1)u_2u_1^{-1}$ for a $u_1 \in C\ell(V)^\times$ and $u_2 \in C\ell(V)$. Here,

$$\alpha(u_1) = \begin{cases} u_1, & u_1 \in C\ell(V)^+ \\ -u_1, & u_1 \in C\ell(V)^-. \end{cases}$$

We showed that for $v \in V \setminus 0$, ρ_v preserves V and is a reflection; since $\rho_{cv} = \rho_v$ for $c \in \mathbb{R} \setminus 0$, we want to restrict to the unit circle of v such that $\langle v, v\rangle = 1$. But we will restrict further.

Definition 3.10. Let $\mathrm{Spin}(V)$ denote the subgroup of $C\ell(V)^\times$ consisting of products of even numbers of unit vectors.

First question: what scalars lie in the spin group? Clearly ± 1 come from u^2 for unit vectors u, but we can do no better (after all, unit length is a strong condition on the real line).

Proposition 3.11. $\mathrm{Spin}(V) \cap \mathbb{R} \setminus 0 = \{\pm 1\}$ *inside* $C\ell(V)^\times$.

Theorem 3.12. *The map* $\mathrm{Spin}(V) \to \mathrm{SO}(V)$ *sending* $u \mapsto \rho_u$ *is a nontrivial (connected) double cover when* $\dim V \geq 2$.

This implies $\mathrm{Spin}(V)$ is the unique connected double cover of $\mathrm{SO}(V)$, agreeing with the abstract construction for the spin group we constructed in the first two lectures.

Proof. We know $\rho_u \in \mathrm{SO}(V)$ because it's an even product of reflections, using Theorem 3.9, and that ρ is surjective. We also know $\ker \rho = \{u \in \mathrm{Spin}(V) \mid uv = vu \text{ for all } v \in V\}$. But since V generates $C\ell(V)$ as an algebra, $\ker(\rho) = \mathrm{Spin}(V) \cap Z(C\ell(V)) = \{\pm 1\}$ by Propositions 3.4 and 3.11.

Thus ρ is a double cover, so it remains to show it's nontrivial. To rule this out, it suffices to show that we can connect -1 and 1 inside $\mathrm{Spin}(V)$, because they project to the same rotation. Let $\gamma(t) = \cos(\pi t) + \sin(\pi t)e_1e_2$ (since $\dim V \geq 2$, I can take two orthogonal unit vectors). Thus, $\gamma(t) = 1$, $\gamma(1) = -1$, and $\gamma(t) = e_1(-\cos(\pi t)e_1 + \sin(\pi t)e_2)$, so it's always a product of even numbers of unit vectors, and thus a path within $\mathrm{Spin}(V)$. ⊠

This is actually the simplest proof that $\dim \mathrm{Spin}(V) = \dim \mathrm{SO}(V)$.

Next week, we'll discuss representations of the spin group.

Lecture 4.

Representations of $\mathfrak{so}(n)$ and $\mathrm{Spin}(n)$: 9/6/16

One question from last time: we constructed $\mathrm{Spin}(n)$ as a subset of the group of units of a Clifford algebra, but how does that induce a linear structure? There's two ways to do this. The first is to say that this *a priori* only constructs $\mathrm{Spin}(n)$ as a topological group; this group double covers $\mathrm{SO}(n)$, and hence must be a Lie group.

Alternatively, this week, we'll explicitly realize $\mathrm{Spin}(n)$ as a closed subgroup of a matrix group, which therefore must be a Lie group.

Last time, we constructed the spin group $\mathrm{Spin}(V)$ as a subset of the units $\mathrm{C}\ell(V)^\times$, and found a double cover $\mathrm{Spin}(V(\to \mathrm{SO}(V)$. Thus, there should be an isomorphism of Lie algebras $\mathfrak{spin}(V) \xrightarrow{\sim} \mathfrak{so}(V)$. The former is a subspace of $T_1\,\mathrm{C}\ell(V) \cong \mathrm{C}\ell(V)$ (since $\mathrm{C}\ell(V)$ is an affine space, as a manifold) and $\mathfrak{so}(V) \subset \mathfrak{gl}(V) = V \otimes V^*$ (and with an inner product, is also identified witrh $V \otimes V$). This identification extends to an isomorphism (of vector spaces) $\mathfrak{so}(V) \cong \Lambda^2 V$; composing with the inverse of the symbol map defines a map $\mathfrak{so}(V) \to \Lambda^2 V \to \mathrm{C}\ell(V)$.

Exercise 4.1. $\mathfrak{so}(V)$ and $\mathrm{C}\ell(V)$ both have Lie algebra structures, the former as a Lie group and the latter from the usual commutator bracket. Show that these agree, so the above map is an isomorphism of Lie algebras, and that the image of this map is $\mathfrak{spin}(V)$.

Today, we're going to discuss the representation theory (over $\mathbb{C}$) of the Lie algebra $\mathfrak{so}(V)$. Since $\mathrm{Spin}(V)$ is the simply connected Lie group with $\mathfrak{so}(V)$ as its Lie algebra, this provides a lot of information on the representation theory of $\mathrm{Spin}(V)$. In general, not all of these representations arise as representations on $\mathrm{SO}(n)$: consider the representation $\mathrm{Spin}(3) = \mathrm{SU}(2)$ on$\mathbb{C}^2$ where -1 exchanges $(1,0)$ and $(0,1)$. This doesn't descend to $\mathrm{SO}(3)$, because -1 is in the kernel of the double cover map. Such a representation is called a *spin representation*.

The name comes from physics: traditionally, physicists idenfied a Lie group with its Lie algebras, but they found that these kinds of representations didn't correspond to $\mathrm{SO}(3)$-representations. These arose in physical systems as particles with spin, in quantum mechanics:[12] a path connected -1 and 1 in $\mathrm{SU}(2)$ is a "rotation" of $260°$, but isn't the identity.

Anyways, we're going to talk about the representation theory of this group; in order to do so, we should briefly discuss the representation theory of Lie groups and (semisimple) Lie algebras.

Definition 4.2. Fix $\mathbb{F}$ to be either $\mathbb{R}$ or $\mathbb{C}$.

- An $\mathbb{F}$-*representation of a Lie group* G is a Lie group homomorphism $\rho : G \to \mathrm{GL}(V, \mathbb{F})$, where V is an $\mathbb{F}$-vector space.
- An $\mathbb{F}$-*representation of a real Lie algebra* $\mathfrak{g}$ is a real Lie algebra homomorphism $\tau : \mathfrak{g} \to \mathfrak{gl}(V, k)$, where V is an $\mathbb{F}$-vector space.

We will often suppress the notation as $\rho(g)v = g \cdot v$ or $\tau(X)v = Xv$, where $g \in G$, $X \in \mathfrak{g}$, and $v \in V$, when it is unambiguous to do so. Moreover, our representations, at least for the meantime, will be finite-dimensional.

Proposition 4.3. *Let $\mathfrak{g}$ be a real Lie algebra and V be a complex vector space. Then, there is a one-to-one correspondence between representations of $\mathfrak{g}$ on V and the $\mathbb{C}$-Lie algebra homomorphisms $\mathfrak{g} \otimes \mathbb{C} \to \mathfrak{gl}(V, \mathbb{C})$.*

Here, $\mathfrak{g} \otimes \mathbb{C}$ is the complex Lie algebra whose underlying vector space is $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ with bracket extending complex linearly from the assignment

$$[X \otimes c_1, Y \otimes c_2] = [X, Y] \otimes c_1 c_2,$$

where $X, Y \in \mathfrak{g}$ and $c_1, c_2 \in \mathbb{C}$.

Proof of Proposition 4.3. Let ρ be a $\mathfrak{g}$-representation on V; then, define $\rho_{\mathbb{C}} : \mathfrak{g} \otimes \mathbb{C} \to \mathfrak{gl}(V, \mathbb{C})$ to be the unique map extending $\mathbb{C}$-linearly from $X \otimes c \mapsto c\rho(X)$.

Conversely, given a complex representation $\rho_{\mathbb{C}}$, define $\rho : \mathfrak{g} \to \mathfrak{gl}(V, \mathbb{C})$ to be $X \mapsto \rho_{\mathbb{C}}(X \otimes 1)$. ⊠

Given a Lie group representation $G \to \mathrm{GL}(V, \mathbb{F})$, one obtains a Lie algebra representation of $\mathfrak{g} = \mathrm{Lie}(G)$ by differentiation.

Proposition 4.4. *If G is a connected, simply-connected Lie group, then this defines a bijective correspondence between the Lie group representations of G and the Lie algebra representations of $\mathfrak{g}$.*

If G is connected, but not simply connected, let $\widetilde{G}$ denote its universal cover. Then, there's a discrete central subgroup $\Gamma \le Z(\widetilde{G}) \le G$ such that $G = \widetilde{G}/\Gamma$. This allows us to extend Proposition 4.4 to groups that may not be simply connected.

Proposition 4.5. *Let G be a connected Lie group, $\widetilde{G}$ be its universal cover, and Γ be such that $G = \widetilde{G}/\Gamma$. Then, differentiation defines a bijective correspondence between the representations of G and the representations of $\widetilde{G}$ on which Γ acts trivially.*

[12]There is one macroscopic example of spin-1/2 phenomena: see http://www.smbc-comics.com/?id=2388.

It would also be nice to understand when two representations are the same. More generally, we can ask what a homomorphism of two representations are.

Definition 4.6. Let G be a Lie group. A *homomorphism of G-representations* from $\rho_1 : G \to \mathrm{GL}(V)$ to $\rho_2 : G \to \mathrm{GL}(W)$ is a linear map $T : V \to W$ such that for all $g \in G$, $T \circ \rho_1(g) = \rho_2(g) \circ T$. If T is an isomorphism of vector spaces, this defines an *isomorphism of G-representations*.

Example 4.7.

(1) $\mathrm{SO}(n)$ can be defined as a group of $n \times n$ matrices, which act by matrix multiplication on $\mathbb{C}^n$. This is a representation, called its *defining representation*. This works for every matrix group, including $\mathrm{SL}(n)$ and $\mathrm{SU}(n)$.

(2) The determinant is a smooth map $\det : \mathrm{GL}(n, \mathbb{C}) \to \mathbb{C}^\times$ such that $\det(AB) = \det A \det B$, hence a Lie group homomorphism. Since $\mathbb{C}^\times = \mathrm{GL}(1, \mathbb{C})$, this is a one-dimensional representation of $\mathrm{GL}(n, \mathbb{C})$.

(3) Fix a $c \in \mathbb{C}$ and let $\rho_c : \mathbb{R} \to \mathfrak{gl}(1, \mathbb{C})$ send $t \mapsto ct$. We can place a Lie algebra structure on $\mathbb{R}$ where $[\cdot, \cdot] = 0$, so that ρ defines a Lie algebra representation.

 The simply connected Lie group with this Lie algebra is $(\mathbb{R}, +)$, and ρ_c integrates to the Lie group representation $(\mathbb{R}, +) \to \mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^\times$ sending $s \mapsto e^{isc}$. But S^1 has the same Lie algebra as $\mathbb{R}$, and the covering map is the quotient $\mathbb{R} \twoheadrightarrow \mathbb{R}/2\pi\mathbb{Z}$. In particular, this acts trivially iff $c \in \mathbb{Z}$, which is precisely when $s \mapsto e^{isc}$ is 2π-periodic.

There are various ways to build new representations out of old ones.

Definition 4.8. Let G be a Lie group and V and W be representations of G.

- The *direct sum* of V and W is the representation on $V \otimes W$ defined by
$$g \cdot (v, w) = (g \cdot v, g \cdot w).$$
- The *tensor product* is the representation on $V \otimes W$ extending uniquely from
$$g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w).$$
- The *dual representation* to V is the representation on V^* (the dual vector space) in which g acts as its inverse transpose on $\mathrm{GL}(V^*)$.

The same definition applies *mutatis mutandis* when the Lie group G is replaced with a Lie algebra $\mathfrak{g}$, and the inverse transpose is replaced with -1 times the transpose for the dual representation.

Note that, unlike for vector spaces, it can happen that a representation isn't isomorphic to its dual, even after picking an inner product.

Definition 4.9. Let V be a representation of a group G.

- A *subrepresentation* is a subspace $W \subset V$ such that $g \cdot w \in W$ for all $w \in W$ and $g \in G$.
- V is *irreducible* if it has no nontrivial subrepresentations (here, nontrivial means "other than $\{0\}$ and V itself"). Sometimes, "irreducible representation" is abbreviated "irrep" at the chalkboard.

These definitions apply *mutatis mutandis* to representations of a Lie algebra $\mathfrak{g}$.

In nice cases, knowing the irreducible representations tells you everything.

Theorem 4.10. *For $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$, there are finitely many isomorphism classes of irreducible representations, and every representation is isomorphic to a subrepresentation of a direct sum of tensor products of these representations.*

Definition 4.11. A Lie algebra $\mathfrak{g}$ whose representations have the property from Theorem 4.10 is called *semisimple*.[13]

In fact, we know these irreducibles explicitly: for n even, all of the irreducible representations of $\mathfrak{so}(n, \mathbb{C})$ are exterior powers of the defining representation, except for two *half-spinor representations*; for n odd, we just have one spinor representation.

[13] This is equivalent to an alternate definition, where $\mathfrak{g}$ is *simple* if $\dim \mathfrak{g} > 1$ and $\mathfrak{g}$ has no nontrivial ideals, and $\mathfrak{g}$ is *semisimple* if it is a direct sum of simple Lie algebras.

Constructing the spin representations. Let V be an n-dimensional vector space over $\mathbb{R}$ with a positive definite inner product. We'll construct the spinor representations of $\mathrm{Spin}(V)$ as restrictions of the $\mathrm{C}\ell(V)$ action on a $\mathrm{C}\ell(V)$-module, which act in a way compatible with the $\mathbb{Z}/2$-grading on $\mathrm{C}\ell(V)$.

Recall that a superalgebra is a scary word for a $\mathbb{Z}/2$-graded algebra.

Definition 4.12. Let $A = A^+ \oplus A^-$ be a superalgebra. A $\mathbb{Z}/2$-*graded module* over A (or a *supermodule* for A) is an A-module with a vector-space decomposition $M = M^+ \oplus M^-$ such that $A^\pm M^\pm \subset M^+$ and $A^\pm M^\mp \subset M^0$.

Example 4.13. One quick example is that every superalgebra acts on itself by multiplication; this *regular representation* is $\mathbb{Z}/2$-graded by the product rule on a superalgebra.

Since $\mathrm{Spin}(V) \subset \mathrm{C}\ell(V)^+$, any supermodule defines two representations of $\mathrm{Spin}(V)$, one on M^+ and the other on M^-.

Since we just care about complex representations, we may as well complexify the Lie algebra, looking at $\mathrm{C}\ell(V) \otimes \mathbb{C}$.

Exercise 4.14. Show that $\mathrm{C}\ell(V) \otimes \mathbb{C} \cong \mathrm{C}\ell(V \otimes \mathbb{C})$ (the latter is the Clifford algebra on a complex vector space).

Working with this complexified Clifford algebra simplifies things a lot.

First, let's assume $n = 2m$ is even. Then, we may choose an *orthogonal complex structure* J on V, i.e. a linear map $J : V \to V$ such that $J^2 = 1$ and $\langle Jv, Jw\rangle = \langle v, w\rangle$. For example, if $\{e_1, f_1, \dots, e_m, f_m\}$ is an orthogonal basis for V, then we can define $J(e_j) = f_j$ and $J(f_j) = -e_j$. Thus, such a structure always exists; conversely, given any orthogonal complex structure J, there exists a basis on which J has this form. In other words, J allows V to be thought of as an n-dimensional complex vector space.

We'll return to this on Thursday, using it to construct the supermodule.

Lecture 5.

The Half-Spinor and Spinor Representations: 9/8/16

Let's continue where we left off from last time. We had a real inner product space V of dimension n; our goal was to construct $\mathbb{C}$-supermodules over the Clifford algebra, in order to access the spinor representations. This is most interesting when n is even; a lot of the fancy theorems we consider later in the class (Riemann-Roch, index theorems, etc.) either don't apply or are trivial when n is odd.

The even-dimensional case. Thus, we first assume $n = 2m$ is even; we can choose an orthogonal complex structure J on V, which is a linear map $J : V \to V$ that squares to 1 and is compatible with the inner product in the sense that $\langle Jv, Jw\rangle = \langle v, w\rangle$. Since $J^2 = I$, its only possible eigenvalues are $\pm i$. Let $V^{0,1}$ be the i-eigenspace for J acting on $V \otimes \mathbb{C}$, and $V^{0,1}$ be the $-i$-eigenspace; then, $\overline{V^{0,1}} = V^{1,0}$. Last time, we found a compatible orthonormal basis $\{e_1, f_1, \dots, e_m, f_m\}$, where $Je_j = f_j$ and $Jf_j = -e_j$; in this basis,

$$\begin{aligned} V^{1,0} &= \mathrm{span}_{\mathbb{C}}\{e_j - if_j \mid j = 1, \dots, m\} \\ V^{0,1} &= \mathrm{span}_{\mathbb{C}}\{e_j + if_j \mid j = 1, \dots, m\}. \end{aligned}$$

Both $V^{0,1}$ and $V^{1,0}$ are *isotropic*, meaning the $\mathbb{C}$-linear extension of the inner product restricts to 0 on each of $V^{0,1}$ and $V^{1,0}$.

The decomposition $V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$ induces the decomposition

$$\Lambda^\bullet(V \otimes \mathbb{C}) = \Lambda^\bullet V^{1,0} \mathbin{\widehat{\otimes}} \Lambda^\bullet V^{0,1}.$$

Here, $\widehat{\otimes}$ denotes the graded tensor product, which is graded-commutative rather than commutative. $\Lambda^\bullet(V \otimes \mathbb{C})$ is a $\mathrm{C}\ell(V)$-module with the action $c(v) = \varepsilon(v) - i(v)$. We can restrict this action to $\Lambda^\bullet V^{0,1}$ and define

$$\widetilde{c}(v) = \sqrt{2}\big(\varepsilon(v^{0,1}) - i(v^{1,0})\big) \in \mathrm{End}\,\Lambda^\bullet V^{0,1}.$$

Here, we use the direct sum to uniquely write any $v \in V$ and $v = v^{1,0} + v^{0,1}$ with $v^{0,1} \in V^{0,1}$ and $v^{1,0} \in V^{1,0}$. In this case,

$$\begin{aligned} \widetilde{c}(v)^2 &= -2\big(\varepsilon(v^{0,1})i(v^{1,0}) + i(v^{1,0})\varepsilon(v^{0,1})\big) \\ &= -2\big(\varepsilon(v^{0,1})i(v^{1,0}) + \langle v^{1,0}, v^{0,1}\rangle - \varepsilon(v^{0,1})i(v^{1,0})\big) \\ &= -2\langle v^{1,0}, v^{0,1}\rangle. \end{aligned}$$

Since $V^{1,0}$ and $V^{0,1}$ are both isotropic,

$$= -\langle v^{1,0} + v^{0,1}, v^{1,0} + v^{0,1} \rangle = \langle v, v \rangle.$$

Thus, by the universal property, $\widetilde{c}$ extends to an algebra homomorphism $\widetilde{c} : \mathrm{C}\ell(V \otimes \mathbb{C}) \to \mathrm{End}_{\mathbb{C}} \Lambda^\bullet V^{0,1}$, and this is naturally compatible with the gradings: the $\mathbb{Z}$-grading on $\Lambda^\bullet V^{0,1}$ induces a $\mathbb{Z}/2$-grading into even and odd parts. Since $\widetilde{c}$ is an odd endomorphism, it gives $\Lambda^\bullet V^{0,1}$ the structure of a $\mathrm{C}\ell(V \otimes \mathbb{C})$-supermodule.

Theorem 5.1. *In fact,* $\widetilde{c} : \mathrm{C}\ell(V \otimes \mathbb{C}) \to \mathrm{End}\, \Lambda^\bullet V^{0,1}$ *is an isomorphism, explicitly realizing* $\mathrm{C}\ell(V \otimes \mathbb{C})$ *as a matrix algebra.*

Proof sketch. $\dim_{\mathbb{C}} \mathrm{C}\ell(V \otimes \mathbb{C}) = 2^n$ and $\dim_{\mathbb{C}} \mathrm{End}_{\mathbb{C}} \Lambda^\bullet V^{0,1} = \left(\dim_{\mathbb{C}} \Lambda^\bullet V^{0,1}\right)^2 = 2^{2m} = 2^n$. The dimensions match up, so it suffices to check $\widetilde{c}$ is one-to-one, which one can do inductively by checking in a basis. ⊠

By restriction, we obtain two representations of $\mathrm{Spin}(V) \subset \mathrm{C}\ell(V \otimes \mathbb{C})^\times$ on $\Lambda^{2\mathbb{Z}} V^{0,1}$ and $\Lambda^{2\mathbb{Z}+1} V^{1,0}$.

Definition 5.2. These (isomorphism classes of) representations are called the *half-spinor representations* of $\mathrm{Spin}(V)$, denoted $S^+ = \Lambda^{2\mathbb{Z}} V^{0,1}$ and $S^- = \Lambda^{2\mathbb{Z}+1} V^{0,1}$.

A priori, we don't actually know whether these are the same representation.

Proposition 5.3. *As* $\mathrm{Spin}(V)$*-representations,* $S^+ \not\cong S^-$.

Proof. The trick for this and similar statements, the trick is to look at the action of the pseudoscalar, the image of the volume element under the symbol map: $\gamma = e_1 \cdots e_m f_1 \cdots f_m \in \mathrm{Spin}(V)$. This is in the center of the Clifford algebra if n is odd, but not for n even: here $\gamma v = -v\gamma$. Thus, γ is in the center of the even part of the Clifford algebra, hence in the center of the even part of $\mathrm{End}(S^+ \oplus S^-)$. This center is isomorphic to $\mathbb{C} \oplus \mathbb{C}$, one for the center of $\mathrm{End}\, S^+$ and the other for the center of $\mathrm{End}\, S^-$. Thus, γ acts as a scalar $k_+ \in \mathbb{C}^\times$ on S^+, and a scalar $k_- \in \mathbb{C}^\times$ in S^-. We'll show these are different.

We only need to check on one element $\psi \in S^+$, so $\gamma \cdot \psi = k_+ \psi$. If $v \in V$ is nonzero, then $v\psi \in S^-$, so $\gamma(v\psi) = k_- v\psi$, but $\gamma(v\psi) = (\gamma v)\psi = -v\gamma\psi = -k_+ v\psi$, so $k_- = -k_+$. Since the action of γ is nontrivial, then these constants are distinct, and therefore these representations are nonisomorphic. ⊠

Moreover, these representations don't descend to $\mathrm{SO}(n)$-representations, because -1 acts as -1 in both S^+ and S^-, but -1 generates the kernel of the double cover, but if it factored through the double cover, -1 would have to act trivially.

Theorem 5.4. *The fundamental representations of* $\mathfrak{so}(2m, \mathbb{C})$ *are:*

- S^+ *and* S^-*;*
- *the defining representation* $\mathbb{C}^n$*; and*
- *the exterior powers* $\Lambda^2 \mathbb{C}^n, \dots, \Lambda^{n-2} \mathbb{C}^n$.

Recall that this means all representations of $\mathfrak{so}(2m, \mathbb{C})$ are subrepresentations of direct sums of tensor products of these representations.

Remark. If you choose a metric of a different signature, there are different groups $\mathrm{Spin}(p, q)$ and algebras $\mathfrak{so}(p, q)$. Multiplication by i can affect the signature, so complexifying eliminates the differences due to signature.

The odd-dimensional case.

Proposition 5.5. $\mathrm{C}\ell(V) \cong \mathrm{C}\ell(V \oplus \mathbb{R})^+$ *as algebras, though it does affect the grading.*

Proof. Let $\widetilde{e}$ be a unit vector in the $\mathbb{R}$-direction in $\mathrm{C}\ell(V \oplus \mathbb{R})^+$, and define $f : V \to \mathrm{C}\ell(V \oplus \mathbb{R})^+$ by $v \mapsto v\widetilde{e}$. Inside Clifford algebras, orthogonal elements anticommute, so $(v\widetilde{e})^2 = v\widetilde{e}v\widetilde{e} = -v(-1)v = v^2 = -\langle v, v \rangle$. By the universal property of Clifford algebras, this extends to an algebra homomorphism $f : \mathrm{C}\ell(V) \to \mathrm{C}\ell(V \oplus \mathbb{R})^+$.

It's reasonable to expect that f is an isomorphism, because both of these are 2^n-dimensional; let's see what happens on a basis $e_1, \dots, e_n$ of V.

- If $j < k$, $e_j e_k \mapsto e_j \widetilde{e} e_k \widetilde{e} = e_j e_k$.
- If $j < k < \ell$, $e_j e_k e_\ell \mapsto e_j e_k e_\ell \widetilde{e}$.

On $\mathrm{C}\ell(V)^+$, f is just the inclusion $\mathrm{C}\ell(V)^+ \hookrightarrow \mathrm{C}\ell(V \oplus \mathbb{R})^+$; for the odd part, we're multiplying every basis element $e_{i_1} e_{i_2} \cdots e_{i_{2k-1}}$ for $\mathrm{C}\ell(V)^-$ by $\widetilde{e}$, so it's sent to $e_{i_1} \cdots e_{i_{2k-1}} \widetilde{e}$, which hits the rest of the even part. Thus, f is an isomorphism. ⊠

So if $\dim V = 2m-1$, $\mathrm{C}\ell(V)\otimes\mathbb{C} \cong \mathrm{C}\ell(V\oplus\mathbb{R})^+\otimes\mathbb{C} \cong \mathrm{End}_{\mathbb{C}} S^+ \oplus \mathrm{End}_{\mathbb{C}} S^-$. Thus, $\mathrm{C}\ell(V)\otimes\mathbb{C}$ has two irreducible, nonisomorphic modules, S^+ and S^-. However, when we restrict to the spin group, these modules become isomorphic spin representations.

Proposition 5.6. *If* $\dim V$ *is odd, then as* $\mathrm{Spin}(V)$*-representations,* S^+ *and* S^- *are isomorphic.*

Proof. We'll define an isomorphism from S^+ to S^- that intertwines the action of $\mathrm{Spin}(V)$. The isomorphism is just multiplication by $\widetilde{e}\in\mathrm{C}\ell(V\oplus\mathbb{R})^+$, which is an isomorphism of vector spaces $S^+\to S^-$, but this commutes with everything in $\mathrm{Spin}(V)$, as $\widetilde{e}$ commutes with all even products of basis vectors, and hence with $\mathrm{Spin}(V)\subset\mathrm{C}\ell(V\oplus\mathbb{R})^+$, i.e. it's an intertwiner, hence an isomorphism of $\mathrm{Spin}(V)$-representations. ⊠

Definition 5.7. If $\dim V$ is odd, we write S for the isomorphism classes of S^+ and S^-; this is called the *spinor representation* of $\mathrm{Spin}(V)$.

Once again, looking at the action of -1 illustrates that this doesn't pass to an $\mathrm{SO}(V)$-representation, and again this is the missing fundamental representation of $\mathfrak{so}(V\otimes\mathbb{C})$.

In summary:

- If $\dim V = 2m$ is even, we established a superalgebra isomorphism $\mathrm{C}\ell(V)\otimes\mathbb{C}\cong\mathrm{End}(S^+\oplus S^-)$, and S^+ and S^- are nonisomorphic representations of $\mathrm{Spin}(V)$ of dimension 2^{m-1}.
- if $\dim V = 2m-1$ is odd, $\mathrm{C}\ell(V)\otimes\mathbb{C}\cong\mathrm{End}(S)\oplus\mathrm{End}(S)$ as algebras, and there's a single spinor representation S of $\mathrm{Spin}(V)$ of dimension 2^{m-1}.

In either case, you can check that

$$\mathrm{C}\ell(\mathbb{C}^n)\cong\mathrm{C}\ell(\mathbb{C}^{n-2})\otimes M_2(\mathbb{C})$$

(the latter factor is the algebra of 2×2 complex matrices), but in the real case, there's a factor of 8:

$$\mathrm{C}\ell(\mathbb{R}^n)\cong\mathrm{C}\ell(\mathbb{R}^{n-8})\otimes M_{16}(\mathbb{R}).$$

This may look familiar: it's related to the stable homotopy groups of the unitary groups in the complex case and the orthogonal groups in the real case. This is a manifestation of *Bott periodicity*. Another manifestation is periodicity in K-theory: complex K-theory is 2-periodic, and KO-theory (real K-theory) is 8-periodic.

Let $R_k^{\mathbb{C}}$ denote the ring generated by isomorphism classes of irreducible $\mathrm{C}\ell(\mathbb{C}^k)$-supermodules, with direct sum passing to addition and tensor product passing to multiplication (this is a *representation ring* or the *K-group*), and let $R_k^{\mathbb{R}}$ be the same thing, but for $\mathrm{C}\ell(\mathbb{R}^k)$. Inclusion $i:\mathrm{C}\ell(\mathbb{C}^k)\hookrightarrow\mathrm{C}\ell(\mathbb{C}^{k+1})$ defines pullback maps $i^*:R_{k+1}^{\mathbb{C}}\to R_k^{\mathbb{C}}$ and $i^*:R_{k+1}^{\mathbb{R}}\to R_k^{\mathbb{R}}$. If $A_k^{\mathbb{C}}$ and $A_k^{\mathbb{R}}$ denote $\mathrm{coker}(i^*)$ in the complex cases respectively, then

$$A_k^{\mathbb{C}}=\pi_k\mathrm{U}=\begin{cases}0, & k \text{ odd}\\ \mathbb{Z}, & k \text{ even,}\end{cases}$$

and $A_{k+8}^{\mathbb{R}}\cong A_k^{\mathbb{R}}$ are the homotopy groups of the orthogonal group, giving us the "Bott song" $\mathbb{Z}/2$, $\mathbb{Z}/2$, 0, $\mathbb{Z}$, 0, 0, 0, $\mathbb{Z}$.[14]

Lecture 6.

Vector Bundles and Principal Bundles: 9/13/16

Today, we'll move from representation theory into geometry, starting with a discussion of vector bundles and principal bundles. In this section, there are a lot of important exercises that are better done at home than at the board, but are important for one's understanding.

For now, when we say "manifold," we mean a smooth (C^∞) manifold without boundary.

Definition 6.1. A *real vector bundle of rank k* over a manifold M is a manifold E together with a surjective, smooth map $\pi:E\to M$ such that every fiber $E_x=\pi^{-1}(x)$ (here $x\in M$) has the structure of a real, k-dimensional

[14] Possible tunes include "Twinkle twinkle, little star."

vector space, and that is locally trivial: there exists an open cover $\mathfrak{U}$ of M such that for every $U_\alpha \in \mathfrak{U}$, there is an isomorphism over M:

$$\begin{tikzcd} E|_{U_\alpha} = \pi^{-1}(U_\alpha) \arrow[rr, "\varphi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times \mathbb{R}^k \arrow[dl] \\ & U_\alpha, & \end{tikzcd}$$

meaning that for every $x \in U_\alpha$, $\varphi_\alpha|_{E_x} : E_x \to \{x\} \times \mathbb{R}^k$ is an $\mathbb{R}$-linear isomorphism of vector spaces. Replacing $\mathbb{R}$ with $\mathbb{C}$ (and real linear with complex linear) defines a *complex vector bundle* on M.

The data $\{U_\alpha, \varphi_\alpha\}$ is called a *local trivialization* of E, and can be used to give another description of a vector bundle. Given two intersecting sets $U_\alpha, U_\beta \in \mathfrak{U}$, we obtain a triangle

$$\begin{tikzcd} (U_\alpha \cap U_\beta) \times \mathbb{R}^k \arrow[rr, "\varphi_\alpha \circ \varphi_\beta^{-1}"] \arrow[dr] & & (U_\alpha \cap U_\beta) \times \mathbb{R}^k \arrow[dl] \\ & U_\alpha \times U_\beta, & \end{tikzcd}$$

so $\varphi_\alpha \circ \varphi_\beta^{-1}$ must be of the form $(x, v) \mapsto (x, g_{\alpha\beta}(x)(v))$ for a smooth map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(k)$. This $g_{\alpha\beta}$ is called a *transition function*.

Exercise 6.2. If $U_\alpha, U_\beta, U_\gamma \in \mathfrak{U}$, check that the transition functions satisfy the *cocycle condition*

$$g_{\alpha\beta}(x) g_{\beta\gamma}(x) = g_{\alpha\gamma}(x).$$

Transition functions and cocycle conditions allow recovery of the original vector bundle.

Proposition 6.3. *Let $\mathfrak{U}$ be an open cover of M and suppose we have smooth functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(k)$ for all $U_\alpha, U_\beta \in \mathfrak{U}$ satisfying the* cocycle conditions

(1) *for all $U_\alpha \in \mathfrak{U}$, $g_{\alpha\alpha} = \mathrm{id}$, and*
(2) *for all $U_\alpha, U_\beta, U_\gamma \in \mathfrak{U}$, $g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$.*

Then, the manifold

$$\coprod_{U_\alpha \in \mathfrak{U}} U_\alpha \times \mathbb{R}^k / ((x, v) \sim (x, g_{\alpha\beta}(x) v) \text{ when } x \in U_\alpha \cap U_\beta)$$

is naturally a vector bundle over M.

If we obtained the $g_{\alpha\beta}$ as transition functions from a vector bundle, what we end up with is isomorphic to the same vector bundle we started with. The cocycle condition is what guarantees that the quotient is an equivalence relation.

Operations on vector bundles.

Definition 6.4. Let $E, F \to M$ be vector bundles. Then, their *direct sum* $E \oplus F$ is the vector bundle defined by $(E \oplus F)_x = E_x \oplus F_x$ for all $x \in M$. The transition function relative to two open sets U_α, U_β in an open cover is

$$g_{\alpha\beta}^{E\oplus F} = \begin{pmatrix} g_{\alpha\beta}^E & 0 \\ 0 & g_{\alpha\beta}^F \end{pmatrix}.$$

One has to show that this exists, but it does. In the same way, the following natural operations extend to vector bundles, and there is again something to show.

Definition 6.5. The *tensor product* $E \otimes F$ is the vector bundle whose fiber over $x \in M$ is $(E \otimes F)_x = E_x \otimes F_x$. Its transition functions are Kronecker products $g_{\alpha\beta}^{E\otimes F} = g_{\alpha\beta}^E \otimes g_{\alpha\beta}^F$.

Definition 6.6. The *dual* E^* is defined to fiberwise be the dual $(E^*)_x = (E_x)^*$. Its transition function is the inverse transpose $g_{\alpha\beta}^{E^*} = ((g_{\alpha\beta}^E)^{-1})^{\mathrm{T}}$.

Definition 6.7. A *homomorphism of vector bundles* is a smooth map $T : E \to F$ commuting with the projections to M, in that the following must be a commutative diagram:

$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[dr] & & F \arrow[dl] \\ & M, & \end{tikzcd}$$

and for each $x \in M$, the map on the fiber $T|_{E_x} : E_x \to F_x$ is linear.

Suppose T is an isomorphism, so each $T|_{E_x}$ is a linear isomorphism. Let $\{U_\alpha, \varphi_\alpha^E\}$ and $\{U_\alpha, \varphi_\alpha^F\}$ be trivialization data for E and F, respectively. Then over each U_α, we can fill in the dotted line below with an isomorphism:

$$\begin{tikzcd} E|_{U_\alpha} \arrow[r, "T|_{U_\alpha}"] \arrow[d, "\sim"', "\varphi_\alpha^E"] & F|_{U_\alpha} \arrow[d, "\sim"', "\varphi_\alpha^F"] \\ U_\alpha \times \mathbb{R}^k \arrow[r, dashed] & U_\alpha \times \mathbb{R}^k. \end{tikzcd}$$

This dotted arrow must be of the form $(x, v) \mapsto (x, \lambda_\alpha(x)v)$ for some $\lambda_\alpha : U_\alpha \to \mathrm{GL}(k)$.

Exercise 6.8. Generalize this to when T is a homomorphism of vector bundles, and show that the resulting λ_α satisfy

$$g_{\alpha\beta}^F \lambda_\beta = \lambda_\alpha g_{\alpha\beta}^E.$$

From the transition-function perspective, there's a convenient generalization: we can replace $\mathrm{GL}(k)$ with an arbitrary Lie group G, and $\mathbb{R}^k$ with any space X that G acts on. That is, given a cover $\mathfrak{U} = \{U_\alpha\}$ and a collection of smooth functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ for every pair $U_\alpha, U_\beta \in \mathfrak{U}$ such that

(1) $g_{\alpha\alpha} = e$ and
(2) $g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$, then

for any G-space X we can form a space

$$E = \coprod_{U_\alpha \in \mathfrak{U}} U_\alpha \times X / ((x, y) \sim (x, g_{\alpha\beta}(x)y) \text{ when } x \in U_\alpha \cap U_\beta).$$

This will be a fiber bundle $X \to E \to M$, i.e. over M with fiber X.

The important or universal case is when X is G: G acts on itself by left multiplication. In this case, the resulting space has a *right* action of G: on each $U_\alpha \times G$, we act by right multiplication by G as $(x, g_1) \cdot g_2 = (x, g_1 g_2)$, but this is invariant under the equivalence relation, and therefore descends to a smooth right action of G on the bundle which is simply transitive on each fiber.

This data defines a principal G-bundle, but just as a vector bundle had a convenient global description, there's one for principal bundles too.

Definition 6.9. Let G be a Lie group. A *(right) principal G-bundle* over a manifold M is a fiber bundle $G \to P \to M$ with a simply-transitive, right G-action on each fiber that's locally trivial, i.e. there's a cover $\mathfrak{U}$ of P such that over each $U_\alpha \in \mathfrak{U}$, there's a diffeomorphism

$$\begin{tikzcd} P|_{U_\alpha} \arrow[rr, "\varphi_\alpha"] \arrow[dr] & & U_\alpha \times G \arrow[dl] \\ & U_\alpha, & \end{tikzcd}$$

i.e. commuting with the projections to U_α, and that intertwines the G-actions: for all $p \in P|_{U_\alpha}$ and $g \in G$, $\varphi_\alpha(pg) = \varphi_\alpha(p)g$.

In this case, G is called the *structure group* of the bundle.

As you might expect, this notion is equivalent to the notion we synthesized from transition functions.

Definition 6.10. A *homomorphism of principal G-bundles* $P_1, P_2 \to M$ is a smooth map $F : P_1 \to P_2$ commuting with the projections to M, and such that $F(pg) = F(p)g$ for all $p \in P_1$ and $g \in G$.

The following definition will be helpful when we discuss Čech cohomology.

Definition 6.11. Let $\check{Z}^1(M,G)$ denote the collection of data $\{U_\alpha, \varphi_\alpha\}$ where

- $\{U_\alpha\}$ is an open cover of M,
- $g_{\alpha\alpha} = e$, and
- $g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$.

We declare two pieces of data $\{U_\alpha, g_{\alpha\beta}\}$ and $\{\widetilde{U}_\alpha, \widetilde{g}_{\alpha\beta}\}$ to be equivalent if there is a common refining cover $\mathfrak{U} = \{V_\alpha\}$ of $\{U_\alpha\}$ and $\{\widetilde{U}_\alpha\}$ and data $\lambda_\alpha : V_\alpha \to G$ such that[15]

$$\widetilde{g}_{\alpha\beta} = \lambda_\alpha g_{\alpha\beta} \lambda_\beta^{-1}. \tag{6.12}$$

The set of equivalence classes is called the *degree-*1 *non-abelian Čech cohomology* (of M, with coefficients in G), and is denoted $\check{H}^1(M;G)$.

Since we obtained these from transition data on principal bundles, perhaps the following result isn't so surprising.

Proposition 6.13. *There is a natural bijection between $\check{H}^1(M;G)$ and the set of isomorphism classes of principal G-bundles on M.*

Proposition 6.14. *Any map of principal bundles is an isomorphism.*

Suppose G acts smoothly (on the left) on a space X and P is a principal G-bundle. Then, G acts on $P \times X$ from the right by $(p,x)\cdot g = (pg, g^{-1}x)$. Define $P \times_G X = (P \times X)/G$; this is a fiber bundle over M with fiber X, and is an example of an *associated bundle construction*. In particular, if V is a G-representation, then $P \times_G V \to M$ is a vector bundle.

In particular, starting with a vector bundle E over M, we obtain transition functions $g_{\alpha\beta}$ for it, but these define a principal $\mathrm{GL}(k)$-bundle P on M. Over a point $x \in M$, P_x is the set of all bases for E_x, which is the vector space of isomorphisms from $\mathbb{R}^k$ to E_x. This bundle P is called the *frame bundle* associated to E.

Naïvely, you might expect this to be the bundle of automorphisms of E, but the right action is the subtlety: there's a natural right action of $\mathrm{GL}(k)$ on the bundle of frames, by precomposing with a linear transformation $A \in \mathrm{GL}(k)$. However, $\mathrm{GL}(k)$ acts on $\mathrm{Aut}(E)$ by conjugation, which is not a right action. However, it is true that $\mathrm{Aut}\, E = P \times_{\mathrm{GL}(k)} \mathrm{GL}(k)$, and here $\mathrm{GL}(k)$ acts on itself by conjugation.

Reduction of the structure group. Sometimes we have bundles with two different structure groups. Let $\varphi : H \to G$ be a Lie group homomorphism (often inclusion of a subgroup) and P be a principal G-bundle. When can we think of the transition functions for P being not just G-valued, but actually H-valued? For example, an *orientation* of a vector bundle is an arrangement of its transition functions to all lie in the subgroup of $\mathrm{GL}(k)$ of positive-determinant matrices.

Definition 6.15. With G, H, φ, and P as above, a *reduction of the structure group* of P to H is a principal H-bundle Q and a smooth map

$$\begin{tikzcd} Q \arrow[rr, "F"] \arrow[dr] & & P \arrow[dl] \\ & M, & \end{tikzcd}$$

such that for all $h \in H$ and $q \in Q$, $F(qh) = F(q)\varphi(h)$.

Proposition 6.16. *Reducing P to have structure group H is equivalent to finding transition functions $h_{\alpha\beta} : U_\alpha \cap U_\beta \to H$ satisfying the cocycle conditions such that $\{U_\alpha, \varphi \circ h_{\alpha\beta}\}$ is transition data for P as a principal G-bundle.*

For example, $\mathrm{GL}_+(k,\mathbb{R}) \hookrightarrow \mathrm{GL}(k,\mathbb{R})$ denotes the subgroup of matrices with positive determinant. A vector bundle E is *orientable* iff its frame bundle has a reduction of structure group to $\mathrm{GL}_+(k,\mathbb{R})$. We'll define a spin structure on a vector bundle in a similar way, by lifting from $\mathrm{SO}(n)$ to $\mathrm{Spin}(n)$.

[15]We should be careful about what we're saying: $\{U_\alpha\}$, $\{\widetilde{U}_\alpha\}$, and $\{V_\alpha\}$ aren't necessarily defined on the same index set; rather, we mean that whenever (6.12) makes sense for open sets U_α, U_β, $\widetilde{U}_\alpha$, $\widetilde{U}_\beta$, and V_α, it needs to be true.

Lecture 7.

Clifford Bundles and Spin Bundles: 9/15/16

If $\pi : E \to M$ is a vector bundle, its *space of sections* $\Gamma(M; E)$ is the vector space of sections, which are the maps $\sigma : M \to E$ such that $\pi \circ \sigma = \mathrm{id}$.

Definition 7.1. Let $E \to M$ be a real vector bundle. A *metric* on E is a smoothly-variying inner product on each fiber E_x, i.e. an element $g \in \Gamma(M; E^* \otimes E^*)$ such that for all $x \in M$,

- $g(v, v) > 0$ for all $v \in E_x \setminus 0$, and
- $g(v, w) = g(w, v)$ for all $v, w \in E_x$.

We can relate this to what we talked about last time.

Proposition 7.2. *Putting a metric on E is equivalent to reducing the structure group of E from $\mathrm{GL}(k, \mathbb{R})$ to $\mathrm{O}(k)$.*

Proof. Let's go in the reverse direction. A reduction of the structure group means we have transition functions $h_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{O}(k) \hookrightarrow \mathrm{GL}(k, \mathbb{R})$ with respect to some cover $\mathfrak{U}$ of M. In particular, we can write E as

$$E = \coprod_{U_\alpha \in \mathfrak{U}} U_\alpha \times \mathbb{R}^k / ((x, v) \sim (x, h_{\alpha\beta}(x)v))$$

like last time. On each $U_\alpha \times \mathbb{R}^k$ we define the metric

$$g_\alpha((x, v), (x, w)) = \langle v, w \rangle.$$

This metric is preserved by the transition functions: for any $U_\alpha, U_\beta \in \mathfrak{U}$, since $h_{\alpha\beta}(x) \in \mathrm{O}(k)$, then

$$\langle v, w \rangle = \langle h_{\alpha\beta}(x)v, h_{\alpha\beta}(x)w \rangle$$

and therefore this metric descends to the quotient E.

Conversely, consider the frame bundle

$$\mathrm{GL}(k, \mathbb{R}) \longrightarrow P \longrightarrow M,$$

and let $Q \subset P$ be the bundle of *orthonormal frames*, whose fiber over an $x \in M$ is the set of isometries (not just isomorphisms) from $(\mathbb{R}^k, \langle \cdot, \cdot \rangle) \to (E_x, g_x)$. Applying the Gram-Schmidt process to a local frame ensures that Q is nonempty; in fact, it's a principal $\mathrm{O}(k)$-bundle, and the inclusion $Q \hookrightarrow P$ intertwines the $\mathrm{O}(k)$-action and the $\mathrm{GL}(k, \mathbb{R})$-action. ⊠

This proof is a lot of words; the point is that in the presence of a metric, the Gram-Schmidt process converts ordinary bases (the $\mathrm{GL}(k, \mathbb{R})$-bundle) into orthonormal bases (the $\mathrm{O}(k)$-bundle).

We can cast the forward direction of the proof in the language of transition functions. In the presence of a metric, our transition functions $h_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$ might not preserve the inner product (be valued in $\mathrm{O}(k)$), but the metric allows us to find a smooth $\lambda_\alpha : U_\alpha \to \mathrm{GL}(k, \mathbb{R})$ for each $U_\alpha \in \mathfrak{U}$ such that

$$g|_{U_\alpha}(v, w) = \langle \lambda_\alpha(x)v, \lambda_\alpha(x)w \rangle,$$

so we can define new transition functions $\lambda_\alpha(x) h_{\alpha\beta}(x) \lambda_\beta^{-1}(x) \in \mathrm{O}(k)$, and this defines an isomorphic principal bundle.

So metrics allow us to reduce the structure group. It turns out we can always do this.

Proposition 7.3. *Every vector bundle $E \to M$ has a metric.*

Proof. Let $\mathfrak{U}$ be a trivializing open cover for E. For every $U_\alpha \in \mathfrak{U}$, we can let $E|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^k$ have the "constant metric"

$$g_\alpha((x, v), (x, w)) = \langle v, w \rangle.$$

Let $\{\psi_\alpha\}$ be a (locally finite) partition of unity subordinate to $\mathfrak{U}$; then,

$$g = \sum_\alpha \psi_\alpha g_\alpha$$

is globally defined and smooth. Then, for any $x \in M$ and $v \in E_x \setminus 0$,

$$g_x(v, v) = \sum_\alpha \psi_\alpha g_\alpha(v, v) > 0$$

since each term is nonnegative, and at least one term is positive. ⊠

Corollary 7.4. *The structure group of any real vector bundle of rank k can be reduced to* $\mathrm{O}(k)$.

This same argument doesn't work for complex vector bundles, since the positivity requirement is replaced with a nondegeneracy. But it is always possible to find a fiberwise Hermitian metric (meaning $h(v,v) > 0$ and $h(av, bw) = \overline{a}bh(v,w)$) and apply the same argument as above to the unitary group.

Corollary 7.5. *The structure group of any complex vector bundle of rank k can be reduced to* $\mathrm{U}(k)$.

Remark. Isomorphism classes of rank-k real vector bundles over a base M are in bijective correspondence with homotopy classes of maps into a *classifying space* $B\mathrm{GL}(k,\mathbb{R})$, written $[M, B\mathrm{GL}(k,\mathbb{R})]$. Similarly, isomorphism classes of rank-k complex vector bundles are in bijective correspondence with $[M, B\mathrm{GL}(k,\mathbb{C})]$. The Gram-Schmidt process defines deformation retractions $\mathrm{GL}(k,\mathbb{R}) \simeq \mathrm{O}(k)$ and $\mathrm{GL}(k,\mathbb{C}) \simeq \mathrm{U}(k)$, so real (resp. complex) rank-k vector bundles are also classified by $[M, B\mathrm{O}(k)]$ (resp. $[M, B\mathrm{U}(k)]$). This is an example of the general fact that a Lie group deformation retracts onto its maximal compact subgroup.

Now, we want to insert Clifford algebras and the spin group into this story. Clifford algebras are associated to inner products, so we need to start with a metric.

Definition 7.6. Let $E \to M$ be a real vector bundle with a metric. Its *Clifford algebra bundle* is the bundle of algebras $\mathrm{C}\ell(E) \to M$ whose fiber over an $x \in M$ is $\mathrm{C}\ell(E)_x = \mathrm{C}\ell((E, g_x))$.

In other words, if P is the principal $\mathrm{O}(k)$-bundle of orthonormal frames of E, then

$$\mathrm{C}\ell(E) = P \times_{\mathrm{O}(k)} \mathrm{C}\ell(\mathbb{R}^k),$$

where $\mathrm{O}(k)$ acts on $\mathrm{C}\ell(\mathbb{R}^k)$ by $A(v_1 \cdots v_\ell) = (Av_1)\cdots(Av_\ell)$ (here $A \in \mathrm{O}(k)$ and $v_1, \dots, v_\ell \in \mathbb{R}^k$).

Definition 7.7. A *Clifford module* for the vector bundle (E, g) over M is a vector bundle $F \to M$ with a fiberwise action $\mathrm{C}\ell(E) \otimes F \to F$. If F is $\mathbb{Z}/2$-graded and the action is compatible with the gradings on F and $\mathrm{C}\ell(E)$, meaning

- $\mathrm{C}\ell(E)^+ \cdot F^+ \subset F^+$ and $\mathrm{C}\ell(E)^- \cdot F^- \subset F^+$; and
- $\mathrm{C}\ell(E)^+ \cdot F^- \subset F^-$ and $\mathrm{C}\ell(E)^- \cdot F^+ \subset F^-$.

These fiberwise notions vary smoothly in the same way that everything else has.

Let's now assume E is oriented, which is equivalent to giving a reduction of its structure group to $\mathrm{SO}(k)$. Can we construct a $\mathrm{C}\ell(E)$-module which is fiberwise isomorphic to the spinor representation of $\mathrm{C}\ell(E_x)$? We can't use the associated bundle construction, because it's not a representation of $\mathrm{SO}(k)$, but of $\mathrm{Spin}(k)$. So we're led to the question: when can we reduce a principal $\mathrm{SO}(k)$-bundle to $\mathrm{Spin}(k)$?

Čech cohomology and Stiefel-Whitney classes. First, an easier question: when can we reduce fro $\mathrm{O}(k)$ to $\mathrm{SO}(k)$? Or, geometrically, what controls whether a real vector bundle is orientable?

Let $\{U_\alpha, g_{\alpha\beta}\}$ be transition data for a vector bundle $E \to M$, and assume that we've picked a metric to reduce the structure group to $\mathrm{O}(k)$. We want $\det g_{\alpha\beta} = 1$ for all U_α, U_β; in general, the determinant is either 1 or -1, so it's a map to $\mathbb{Z}/2$. In fact, since the determinant is a homomorphism, $\det g_{\alpha\beta}$ still satisfies the cocycle condition, so it determines a class $w_1(E) \in \check{H}^1(M; \mathbb{Z}/2)$.

This class $w_1(E)$ means there's a $\mathbb{Z}/2$-valued cocycle d_α such that $\det g_{\alpha\beta} = d_\alpha d_\beta^{-1} = d_\alpha d_\beta$ for all U_α and U_β. Let

$$\lambda_\alpha = \begin{pmatrix} \lambda_\alpha & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix},$$

so that $\det(\lambda_\alpha g_{\alpha\beta} \lambda_\beta^{-1}) = (\det g_{\alpha\beta})^2 = 1$ for all α and β, so $\{\lambda_\alpha g_{\alpha\beta} g_\beta\}$ defines an equivalent vector bundle whose transition functions lie in $\mathrm{SO}(k)$. That is, if $w_1(E) = 0$, then E is orientable! This class $w_1(E)$ is called the 1^{st} *Stiefel-Whitney class* of E.

A priori this depends on the metric.

Exercise 7.8. Show that the first Stiefel-Whitney class doesn't depend on the choice of the metric E.

Since $\mathbb{Z}/2$ is abelian, $\check{H}^1(M; \mathbb{Z}/2)$ is actually the first abelian group of a complex called the Čech cochain complex. Let's see how this works in general.

Definition 7.9. Let A be an abelian Lie group (often discrete) and $\mathfrak{U} = \{U_\alpha\}_{\alpha\in I}$ be an open cover of M indexed by some ordered set I. Then, the *Čech j-cochains (relative to $\mathfrak{U}$)* are the algebra

$$\check{C}^j(\mathfrak{U},A) = \prod_{\alpha_0<\alpha_1<\cdots<\alpha_j} C^\infty(U_{\alpha_0}\cap\cdots\cap U_{\alpha_j},A)$$

(so a product of the spaces of C^∞ functions from $U_{\alpha_0}\cap\cdots\cap U_{\alpha_j}$ to A). These fit into the *Čech cochain complex*, whose differential $\delta^j : \check{C}^j(\mathfrak{U},A)\to\check{C}^{j+1}(\mathfrak{U},A)$ defined by

$$(\delta^j\omega)_{\alpha_0\cdots\alpha_{j+1}} = \sum_{i=1}^{j+1}(-1)^i\omega_{\alpha_1\cdots\widehat{\alpha}_i\cdots\alpha_{j+1}}.$$

Here, $\widehat{\alpha}_i$ means the absence of the i^{th} term.

To honestly say this is a cochain complex, we need a lemma.

Lemma 7.10. *For any j, $\delta^{j+1}\circ\delta^j = 0$.*

This is a computation.

Definition 7.11. The j^{th} *Čech cohomology of M relative to* $\mathfrak{U}$ (*and valued in A) is the quotient

$$\check{H}^j(\mathfrak{U},A) = \ker(\delta^j)/\operatorname{Im}(\delta^{j+1}).$$

The j^{th} *Čech cohomology* of M eliminates this dependence on $\mathfrak{U}$: we make the set of open covers of M directed under refinements and set

$$\check{H}^j(M,A) = \varprojlim_{\text{refinements of }\mathfrak{U}} \check{H}^j(\mathfrak{U},A).$$

Fact. If $\mathfrak{U}$ is a *good cover* of M, meaning that all intersections $U_{\alpha_0}\cap\cdots\cap U_{\alpha_\ell}$ are contractible for all ℓ, then $\check{H}^j(\mathfrak{U};A) = \check{H}^j(M;A)$.

This makes a lot of calculations easier.
For example, $\check{C}^0 = \{f = (f_\alpha)_{\alpha\in I}\}$, where $f_\alpha : U_\alpha\to A$ is smooth. The differential is

$$(\delta^0 f)_{\alpha_0\alpha_1} = f_{\alpha_1}|_{U_{\alpha_0}\cap U_{\alpha_1}} - f_{\alpha_0}|_{U_{\alpha_0}\cap U_{\alpha_1}},$$

so $\check{H}^0(M;A) = \ker(\delta^0)$ is the functions that glue together into global functions to A, or $C^\infty(M,A)$. The 1-chains are collections of functions on $U_{\alpha\beta} = U_\alpha\cap U_\beta$, and the differential is

$$(\delta^1 g)_{\alpha\beta\gamma} = g_{\alpha\beta}|_{U_\alpha\cap U_\beta\cap U_\gamma} - g_{\alpha\gamma}|_{U_\alpha\cap U_\beta\cap U_\gamma} + g_{\beta\gamma}|_{U_\alpha\cap U_\beta\cap U_\gamma}.$$

In multiplicative notation, this is exactly the cocycle condition

$$g_{\alpha\beta}g_{\beta\gamma}g_{\alpha\gamma}^{-1} = 1,$$

so we conclude that $\check{H}^1(M;A)$ is in bijection with the group of isomorphism classes of principal A-bundles of M.
We can also characterize higher $\check{H}^j$.

Theorem 7.12. *If A is a discrete group, then $\check{H}^\bullet(M;A)\cong H^\bullet(M;A)$, the singular (or cellular, etc.) cohomology of M with coefficients in A.*

For any abelian Lie group A, it's true that $\check{H}^\bullet(M;A)$ is isomorphic to the sheaf cohomology of the sheaf of smooth A-valued functions on A, $H^\bullet(C^\infty_{M;A})$.