MATH 215C NOTES

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These notes were taken in Stanford's Math 215C class in Spring 2015, taught by Jeremy Miller. I TeXed these notes up using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to adebray@stanford.edu. Thanks to Jack Petok for catching a few errors.

Contents

1. Smooth Manifolds: 4/7/15

> *"Are quals results out yet? I remember when I took this class, it was the day results came out, and so everyone was paying attention to the one smartphone, since this was seven years ago."*

This class will have a take-home final, but no midterm. It's an important subject, but there aren't any really awesome theorems, which is kind of sad.

Here are some goals of this class:

- Definitions: manifolds, the tangent bundle, etc.
- Basic properties of manifolds: transversality, embedding theorems (into Euclidean space).
- Differential forms, Stokes' theorem, de Rham cohomology. Some of you may have learned this in a fancy multivariable calculus class. If 215C has a punchline, it's that de Rham cohomology is the same as regular cohomology, which is elegant but not all that helpful for doing stuff.
- Intersection theory, and the idea that intersection is dual to the cup product. We'll also talk about characteristic classes a little bit, which is supposed to be in the last quarter of a second-year graduate topology class, so we'll see what happens.
- Morse theory, which is another homology theory that ends up being the same (chain complexes built out of functions from a manifold to $\mathbb{R}$), but this is useful e.g. for using algebraic topology to provide bounds on critical points of functions.

Definition. A **manifold** is a paracompact Hausdorff space M such that for all $x \in M$, there exists an open $U \subseteq M$ such that $x \in U$ and $U \cong \mathbb{R}^n$. In this case, we say that the **dimension** of M is n.

Recall that **paracompact** means that every open cover has a subcover such that each point has a subcover containing only finitely many sets, and that **Hausdorff** means that any two points can be separated by open sets (each has an open neighborhood not containing the other).

One may also want the manifold to be **second countable**, i.e. it has a countable basis; the exceptions include things with infinitely many components. Second countability implies paracompactness, and we won't be working in the boundary between them much anyways.

Example 1.1. Let $M = \mathbb{R}\times\{0\}\coprod\mathbb{R}\times\{1\}$, where $(x,0)\sim(y,1)$ if $x = y$ and $x \neq 0$. Thus, M looks like $\mathbb{R}$, but has two copies of the origin. This (topologized with the quotient topology) is locally Euclidean, but not Hausdorff.

Example 1.2. Let ω be the first uncountable ordinal, and let $R = \omega\times[0,1)$, with the order topology. Then, R is called the **long ray**. Let L be R without its smallest point; then, both L and R are locally Euclidean and Hausdorff, but it's not paracompact, which is a confusing digression into set theory (now that you mention it, what exactly is the first uncountable ordinal?).

Later on, this will be a nice counterexample to the notion that homomorphisms of homotopy groups determine a space up to homotopy; this is only true for nicer spaces. The higher homotopy groups vanish, but it isn't contractible (which is painful to make rigorous; intuitively, it would take "too long").

In this class, though, you'll only need to know enough logic and point-set topology to know that these issues have been avoided.

Definition.

- Let M be a manifold; then, an **atlas** on M is a collection of open sets $\{U_\alpha\}$ that covers M and a collection of homeomorphisms $\varphi_\alpha : U_\alpha \xrightarrow{\cong} \mathbb{R}^n$. The pairs $(U_\alpha, \varphi_\alpha)$ are called **charts**.
- A **smooth structure** on a manifold M is an atlas $\{(U_\alpha,\varphi_\alpha)\}$ such that whenever $U_\alpha\cap U_\beta \neq \emptyset$, the map $\varphi_\beta|_{U_\alpha\cap U_\beta}\circ(\varphi_\alpha|_{U_\alpha\cap U_\beta})^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is smooth (i.e. C^∞).
- Let M be an m-dimensional manifold and N be an n-dimensional manifold. Then, a continuous $f : M \to N$ is called **smooth** (or C^∞) if for all $m \in M$, charts $(U_\alpha,\varphi_\alpha)$ containing m, and charts (V_β,φ_β) containing $f(U_\alpha)$, the map $\varphi_\beta\circ f\circ(\varphi_\alpha|_{U_\alpha})^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is smooth. In other words, we take $\varphi_\alpha(U_\alpha)$, send it back using φ_α^{-1}, then apply f and φ_β to it.

A lot of this might feel imprecise, but the basic concrete definitions eventually become second nature, so it's not super important which definition is used to start the whole thing off. For example, many authors require all atlases to be maximal (ordered by inclusion). Furthermore, even if these definitions seem painful or complicated, the idea is that smoothness is simply checked in charts: it's a local notion in $\mathbb{R}^n$, so it can be checked locally on manifolds, which locally are homeomorphic to $\mathbb{R}^n$.

There may be multiple smooth structures on a given manifold, so how do we know whether they're equivalent?

Definition. A smooth map $f : M \to N$ of manifolds is a **diffeomorphism** if there exists a smooth $g : M \to N$ with $f\circ g = \mathrm{id}$ and $g\circ f = \mathrm{id}$.

This is the notion of sameness (isomorphism) in the category of differentiable manifolds.

Tangent Bundles. The next reasonable thing to discuss is the tangent bundle, which is a specific example of fiber bundles or vector bundles.

Definition. Let E, B, and F be topological spaces. Then, a continuous map $P : E \to B$ is called a **fiber bundle with fiber** F if for all $x \in B$, there exist an open U containing x and a homeomorphism (sometimes called **change of coordinates**) $\phi_U : P^{-1}(U) \to U\times F$ such that the following diagram commutes.

$$\begin{tikzcd} P^{-1}(U) \arrow[rr, "\phi_U"] \arrow[dr, "P"'] & & U\times F \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

Here, π_1 is projection onto the first component.

The idea is that a fiber bundle locally looks like a product, but there could be some twisting, e.g. the Möbius strip locally looks like $[0,1]\times S^1$, but globally is not: ϕ_U rotates as one moves along S^1. See Figure 1 for a picture.

For a sillier example, one could just take $E = B\times F$, where the homeomorphisms can be global; this is the same sense in which $\mathbb{R}^n$ is a manifold.

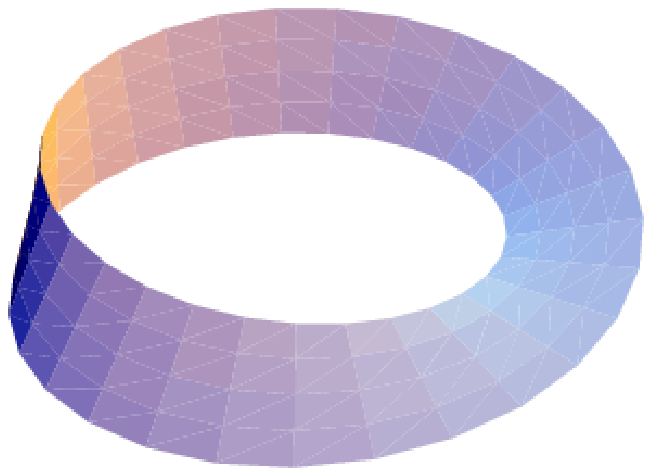

FIGURE 1. The Möbius strip, a nontrivial fiber bundle, since it locally looks like $[0,1]\times S^1$, but not globally. Source: http://mathworld.wolfram.com/MoebiusStrip.html.

Example 1.3. Another nontrivial example is where $B=\mathbb{C}P^n$ and $E\subseteq \mathbb{C}P^n\times\mathbb{C}^{n+1}=\{(\ell,\mathbf{y})\mid \mathbf{y}\in\ell\}$; then, let $P:E\to B$ send $P(\ell,\mathbf{y})=\ell$, so $P^{-1}(\ell)\cong\mathbb{C}$. Once again, there's some "twisting" that means the product structure only exists locally.

Definition. Let $P:E\to B$ be a fiber bundle with fiber F, and suppose that F is a real vector space. Then, P is called a **vector bundle** if the change of coordinates maps ϕ_U are linear. To be precise, there exists an open cover U_α of B with fiberwise homeomorphisms $P^{-1}(U_\alpha)\xrightarrow{\simeq}U_\alpha\times F$ such that whenever U_α and U_β intersect, $\varphi_\beta\circ\varphi_\alpha^{-1}:(U_\alpha\cap U_\beta)\times F$ induces a map $F\to F$ for each $x\in U_\alpha\cap U_\beta$; this map is required to be linear.

For example, one can imagine a fiber bundle of $\mathbb{R}$ on S^1 (e.g. the normal lines): then, the two copies of $\mathbb{R}$ that come together to make S^1 overlap, and we have to say something on their boundary. In this case, send fibers to each other with the identity at one point, and flipped at the other; the result is the Möbius band again (if the identity was chosen in both cases, we would have had the trivial bundle again).

These definitions may seem unmotivated (perhaps this was deliberate; most of the class has seen some of this stuff already). However, the way we'll use the notion of a vector bundle is to define the tangent bundle, which is the set of tangent vectors at points in M (i.e., each fiber at x is T_xM, the tangent space to M at x). If M is embedded in Euclidean space $\mathbb{R}^N$, then the tangent bundle $TM=\{(m,\mathbf{v})\mid \mathbf{v}\text{ is tangent to }M\text{ at }m\}$, but we want a definition that works for abstract manifolds and is more intrinsic.

Of course, since the intuition for the tangent bundle follows from the embedded case, the abstract definition isn't all that useful, but we do need it for formal arguments.

Definition. Let M be a manifold and $m\in M$. Then, the set of **tangent vectors** at m is the set of smooth functions $\gamma:\mathbb{R}\to M$ such that $\gamma(0)=m$, modulo the equivalence relation that $\gamma\sim\gamma'$ if for all smooth $f:M\to\mathbb{R}$,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\gamma(t))=\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\gamma'(t)).$$

The intuition is that two functions (curves, in fact) are the same if they have the same derivative at m, but we need to add $f:M\to\mathbb{R}$ because we don't know yet how to take derivatives on manifolds. If you're familiar with germs of functions, this is a similar notion. Alternatively, this can be viewed as gluing the tangent bundles of open sets of Euclidean space together.

The goal is to have a tangent space, which means we want to turn this into a vector space somehow; tune in next time for that.

2. THE TANGENT BUNDLE: 4/9/15

"Most people like colimits better than limits, but we won't poll the audience yet."

There are several ways of defining the tangent bundle, and more interestingly putting a topolgy on it; the most low-tech way, which builds a tangent bundle on a manifold out of the trivial bundle on $\mathbb{R}^n$ by gluing, is often the best. That trivial bundle is $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ with projection onto $\mathbb{R}^n$ (since the tangent space at any $x \in \mathbb{R}^n$ is again isomorphic to $\mathbb{R}^n$).

But to do this, we need to pin down the notion of gluing. Suppose $\{U_\alpha\}$ is an open cover of a space B and F is a real vector space. Then, we would like the fiber to be F, but the transition maps need to respect its structure, i.e. the transition functions $t_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Aut}(F)$, i.e. $\mathrm{GL}_{\mathbb{R}}(F)$. If F is instead a complex vector space, we would want $\mathrm{GL}_{\mathbb{C}}(F)$, and if it's a differentiable manifold (which is the notion of a **smooth manifold bundle**), we would like them in $\mathrm{Diff}(F)$, and so on.

Now, armed with this data, we can carry out the gluing. Define

$$E = \coprod_\alpha U_\alpha \times F / \Big((x \in U_\alpha, f) \sim (x \in U_\beta, t_{\alpha\beta}(x)(f)) \Big).$$

Is this a fiber bundle? We want projection, $P : E \to B$ sending $(x, f) \mapsto x$ to be well-defined.

Proposition 2.1. *Suppose that the transition functions satisfy the following conditions for all intersecting charts α, β, and δ:*

- $t_{\alpha\alpha} = \mathrm{id}$.
- $t_{\alpha\beta}(x) = t_{\beta\alpha}(x)^{-1}$.
- $t_{\alpha\beta}(x) t_{\beta\delta}(x) = t_{\alpha\delta}(x)$.

Then, $P : E \to B$ is a fiber bundle.

The intuition is that in these cases, two points in different fibers will never be identified, so projection is well-defined. The last condition is called the **cocycle condition**.

So now, we should use this for when $B = M$ is a manifold. Specifically, it's necessary to specify transition functions $U_\alpha \cap U_\beta \to \mathrm{GL}_n(\mathbb{R})$. Each U_α comes with a $\varphi_\alpha : U_\alpha \xrightarrow{\cong} \mathbb{R}^n$. Thus, there's a map $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \to U_\beta) \to \mathbb{R}^n$ is a map from an open subset of $\mathbb{R}^n$ to itself. That means we can take derivatives, and define $t_{\alpha\beta}(x) = D(\varphi_\beta \circ \varphi_\alpha^{-1})(\varphi_\alpha(x))$. Then, we can check that Proposition 2.1 holds, and sure enough, this is a tangent bundle.

On the one hand, we had to use charts which is unpleasant, but the other more intrinsic definitions aren't as easy to topologize.

Definition. Let M be a smooth manifold and $m \in M$. Then, let

$$T_m M = \{\gamma : (-\varepsilon, \varepsilon) \to M \mid \varepsilon > 0, \gamma \text{is smooth, and } \gamma(0) = m\} / \sim,$$

where $\gamma_1 \sim \gamma_2$ if for all smooth $f : M \to \mathbb{R}$,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\gamma_1(t)) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\gamma_2(t)).$$

Then, the tangent bundle is $TM = \bigcup_{m \in M} T_m M$, and the projection is $p : TM \to M$ sending $\gamma \mapsto \gamma(0)$.

In general in this class, a function between topological spaces will be assumed to be continuous, and a function of smooth manifolds will be assumed to be smooth (unless we're trying to prove this, of course, or where stated otherwise).

This second definition is a very nice definition of a set; in order to give it a topology we'll have to appeal to the first definition! In the case $M = \mathbb{R}^n$, let $L : \mathbb{R}^n \to T_m \mathbb{R}^n$ given by $L(\mathbf{v}) : \mathbb{R} \to \mathbb{R}^n$, where $L(\mathbf{v})(t) = m + t\mathbf{v}$. That is, given a vector, the result is a function whose image is that line. Then, every curve is equivalent to one of these lines, its tangent line (which is why this is called the tangent bundle). Thus, L is a bijection, and it can be promoted to a more general bijection L between our two notions of tangent bundle on $\mathbb{R}^n$, and this bijection creates the topological structure that you'd like. Then, the same notion can be defined for a general manifold M, but it'll involve some futzing around with charts.

The third definition again doesn't have an obvious natural topology, but it makes the vector-spatial structure much clearer, and it's sheafy, which algebraic geometers tend to like.

Definition. Let M be a manifold and $m \in M$. Then, define

$$\mathscr{G}(M, \mathbb{R})_m = \varinjlim_{\substack{m \in U \\ U \text{ open}}} C^\infty(U, \mathbb{R}).$$

That is, this colimit is the set of all such maps $f : U \to \mathbb{R}$ where U is an open neighborhood of M, but such that $f = g$ if there's an open neighborhood W of m such that $f|_W = g|_W$. $\mathscr{G}(M,\mathbb{R})_m$ is called the **germs of functions at** m, and $C^\infty(U,\mathbb{R})$ is the set of smooth functions from U to $\mathbb{R}$.

Note that this colimit is in the category of vector spaces, since $C^\infty(U,\mathbb{R})$ is a real vector space; moreover, it's also a ring under pointwise addition and multiplication.

Definition. Let $T \in \mathrm{Hom}_\mathbb{R}(\mathscr{G}(M,\mathbb{R})_m, \mathbb{R})$. Then, T is called a **derivation** if $T(fg) = f(m)T(g) + g(m)T(f)$. The set of derivations for an $m \in M$ will be denoted $T_m M$.

Proposition 2.2. *There is a natural linear homomorphism ev from the previous definition of $T_m M$ to this one.*

That is, if f is the germ of a function and $\gamma : (-\varepsilon,\varepsilon) \to M$, then $ev(\gamma)(f) \in \mathbb{R}$ is given by

$$ev(\gamma)(f) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\gamma(t)).$$

Here, f is a germ, so it's defined on a neighborhood, so its derivative exists.

When you boil down everything, the point is that these notions of the tangent bundle are equivalent; the book goes into more detail.

Definition. Suppose that $f : M \to N$ is a map of smooth manifolds and $m \in M$. Then, let $Df_m : T_m M \to T_m N$ be defined by $Df(\gamma) = f \circ \gamma$ (using the definition of equivalence classes of curves).

Proposition 2.3. *Df_m is linear, and moreover agrees with the standard ("Math 51") definition for $M, N = \mathbb{R}^n$.*

Well, now that we've defined tangent bundles and functions between them, let's use them.

Definition. $f : M \to N$ is called an **immersion** if Df_m is injective for all $m \in M$; it is called a **submersion** if Df_m is surjective for all $m \in M$.

The idea is that an immersion should have no singularities (à la $y^2 = x^3$), but it is allowed to intersect itself. Submersions are generalizations of projections.

Definition. A map $\phi : E_1 \to E_2$ is an **isomorphism of vector bundles** if it is a homeomorphism that induces linear maps on each fiber, and the following diagram commutes.

$$\begin{tikzcd} E_1 \arrow[rr, "\varphi"] \arrow[dr] & & E_2 \arrow[dl] \\ & X & \end{tikzcd}$$

Here, the arrows to X are projection.

Topological K-theory and Bott Periodicity. Many operations that we're used to from the world of vector spaces work just as well in vector bundles. For example, if $E_1 \to P$ and $E_2 \to P$ are vector bundles, then one can define $E_1 \oplus E_2 \to P$, $E_1 \otimes E_2 \to P$, and $\mathrm{Hom}(E_1, E_2) \to P$ in the reasonable way (do it fiberwise, or I guess check the universal property), $\Lambda^k E_1 \to B$, and so on.[1] Furthermore, $\oplus$ and $\otimes$ will allow us to define a ring structure on vector bundles.

We'll work out one of the cases in detail.

Definition. Let $E \to B$ be a fiber bundle and $f : X \to B$ be continuous. Then, the **pullback** of E along f is $f^*E = \{(x,e) \mid f(x) = p(e)\} \subset X \times E$. Furthermore, there's a natural map $f^*P : f^*E \to X$ given by $f^*P(x,e) = x$.

Categorically speaking, this is a fiber product.

Proposition 2.4. *$f^*P : f^*E \to X$ is a fiber bundle.*

This bundle is called the **pullback fiber bundle**.

Definition. Let E_1 and E_2 be fibers over B, and $\Delta : B \to B \times B$ be the **diagonal map**, i.e. it sends $x \mapsto (x,x)$. Then, let $E_1 \times_B E_2 = \Delta^*((E_1 \times E_2) \to (B \times B))$; if E_1 and E_2 are vector bundles, this is also denoted $E_1 \oplus E_2$.

Exercise 2.5. Show this is identical to taking the Cartesian product of the fibers over each point.

[1] Note that when proving this, it may be necessary to refine or subdivide charts (make them smaller), which is a little annoying.

Definition.

- A **monoid** is a set with a binary operation $\times$ that is associative, but that may not have identity or inverses.
- If $\times$ is commutative, it's called a **abelian monoid**.
- If M is a monoid where $x+y=z+y$ implies $x=z$ for all $x,y,z,\in M$, then M is said to have the **cancellation property**.

Sometimes, this is called a **semigroup**, and monoids are required to have an identity.

Definition. Let M be an abelian monoid. Then, its **Grothendieck group** is the group $\mathrm{GG}(M)$ is $M\times M$ modulo the equivalence relation $(a,b)\sim(c,d)$ if $a+d+y=b+c+y$ for some $y\in M$.

In some sense, this is the smallest group you can get out of M, by formally adding inverses and cancellation.

There's a natural inclusion $\iota : M\to \mathrm{GG}(M)$, which is a monoid homomorphism (i.e. the binary operation factors through it). Then, the Grothendieck group satisfies the following universal property: for any group G and map of monoids $\varphi : M\to G$, there is a unique $\phi : \mathrm{GG}(M)\to G$ such that the following diagram commutes.

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\iota"'] & G \\ \mathrm{GG}(M) \arrow[ur, dashed, "\phi"'] & \end{tikzcd}$$

The intuition is that $\phi(a,b)=\varphi(a)\varphi(b)^{-1}$.

It turns out that the isomorphism classes of real vector bundles over a topological space X form a monoid, called $\mathrm{Vec}_{\mathbb{R}}(X)$, where the binary operation is the direct sum; $\mathrm{Vec}_{\mathbb{C}}(X)$ is defined analogously.

Definition. $\mathrm{KO}(X)=\mathrm{GG}(\mathrm{Vec}_{\mathbb{R}}(X))$, and $\mathrm{K}(X)=\mathrm{GG}(\mathrm{Vec}_{\mathbb{C}}(X))$, called **real** and **complex K-theory**, respectively.

The idea here is that monoids are harder to do stuff with than groups, so if we're willing to throw away some of that information, we can do more with the rest.

For example, $TS^n\oplus\mathbb{R}\cong\mathbb{R}^{n+1}$, which is an example of information that would be lost. The analogue is that there exist projective modules that aren't free.

Theorem 2.6 (Bott Periodicity)**.**

- $\mathbb{Z}\times\mathrm{KO}(X)\cong\mathrm{KO}(\Sigma^8 X)$.
- $\mathbb{Z}\times\mathrm{K}(X)\cong\mathrm{K}(\Sigma^2 X)$.

Here, Σ denotes suspension.

For example, $\mathrm{K}(\mathrm{pt})=\mathbb{Z}$ (generated by the trivial bundle), and $\mathrm{K}(S^2)$ is generated by the trivial and tautological bundles.

3. Parallelizability: 4/14/15

"In my thesis defense, I wrote 'paralize' several times instead of 'parallelize.' "

Remark. Bott periodicity still sounds like the name of some character from the Harry Potter universe.

Definition. A **parallelization** of a smooth manifold M is a bundle isomorphism ϕ from TM to the trivial bundle $M\times\mathbb{R}^n$, i.e. a commutative diagram

$$\begin{tikzcd} TM \arrow[rr, "\phi"] \arrow[dr] & & M\times\mathbb{R}^n \arrow[dl] \\ & M & \end{tikzcd}$$

Our goal will be to show the following two theorems.

Theorem 3.1. *If M is parallelizable, then M is orientable.*

Theorem 3.2. *If M is a compact, parallelizable manifold, then $\chi(M)$, its* **Euler characteristic**, *is nonzero.*

Orientability in Theorem 3.1 will be in the sense of Math 215B, i.e. for topological manifolds, though everything in this class will be smooth. We'll also discuss orientations of vector bundles.

Definition.

- If M is a manifold, then a **local orientation** at an $m \in M$ is a choice of generator of $H_n(M, M \setminus m) \cong \mathbb{Z}$.
- A **local orientation** for a k-dimensional vector bundle $V \to M$ is a choice of generator of $H_k(V_m, V_m \setminus 0)$.

Let $\widetilde{M} = \{(m, g) \mid m \in M, g \text{ is a local orientation at } m\}$. Similarly, let $\widetilde{M}_V = \{(m, g) \mid m \in M, g \text{ is a local orientation of } V \text{ at } m\}$. These are covering spaces, and come with projections $P_{\widetilde{M}} : \widetilde{M} \to M$ and $P_{\widetilde{M}_V} : \widetilde{M}_V \to M$.

Specifically, if $B \subset M$ and $B \cong \mathbb{R}^n$, then $P_{\widetilde{M}}^{-1}(B)$ is the product of B and the generators of $H_n(M, M \setminus B)$, and so there's a natural bijection $H_n(M, M \setminus B) \xrightarrow{\cong} H_n(M, M \setminus b)$ for any $b \in B$. Similarly, if V is trivial over B, i.e. $P_V^{-1}(B) \cong B \times \mathbb{R}^k$, then we can do the same thing with $P_{\widetilde{M}_V}^{-1}(B)$: it's the product of B with the generators of $H_k(\mathbb{R}^k, \mathbb{R}^k \setminus 0)$.

Definition.

- M is **orientable** if $\widetilde{M} \to M$ is the trivial double cover.
- $V \to M$ is **orientable** if $\widetilde{M}_V \to M$ is the trivial double cover.

Proposition 3.3. $\widetilde{M} \cong \widetilde{M}_{TM}$.

Corollary 3.4. *M is orientable iff TM is.*

Geometrically, if we have a metric, there's a way of (topologically) identifying $T_m M$ with $B_\varepsilon(m)$ for some $\varepsilon > 0$; then, excision says that $H_n(M, M \setminus m) \cong H_n(B_\varepsilon(m), B_\varepsilon(m) \setminus m)$. But this is $H_n(TM, TM \setminus 0)$ (which does require some geometry or thinking about the exponential map). This is the intuition, but we don't have the machinery to make it rigorous; it's best to keep this one in your head.

Proof. The proof works by asking, "how do you define $\widetilde{M}$ and $\widetilde{M}_{TM}$ in terms of transition functions?" Once you write down what that actually is, they'll end up being the same.

Pick charts (U_α, ϕ_α) for M, so the $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ are homeomorphisms. We want to be able to define transition functions $t_{\alpha\beta}^{\widetilde{M}} : U_\alpha \cap U_\beta \to \mathbb{Z}/2$ (since $\widetilde{M}$ is a double cover of M). Why $\mathbb{Z}/2$? Because it's equal to $\mathrm{Aut}(H_n(\mathbb{R}^n, \mathbb{R}^n \setminus 0))$ (i.e. $\mathrm{Aut}(\mathbb{Z})$).

Let $t_{\alpha\beta} = (\varphi_\beta \circ \varphi_\alpha^{-1})_*$ (i.e. the induced map on homology), which is an automorphism. In higher-level wording, does $t_{\alpha\beta}$ preserve or reverse the orientation of $\mathbb{R}^n$ that is present on each chart? In order to make this work, we need a choice of orientation on $\mathbb{R}^n$, which induces orientations on $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$. Furthermore, $t_{\alpha\beta}(x)$ is an isomorphism $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \varphi_\alpha(x)) \xrightarrow{\sim} H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \varphi_\beta(x))$.

Now, we can define $t_{\alpha\beta}^{TM}$ in the same way, sending $U_\alpha \cap U_\beta \to \mathbb{Z}/2 \cong \mathrm{Aut}(T_x M, T_x M \setminus 0)$, which is an isomorphism.

Lemma 3.5. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism with $f(0) = 0$. Then, $f_* : H_n(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \to H_n(\mathbb{R}^n, \mathbb{R}^n \setminus 0)$ is the identity map iff $(Df)_* : H_n(T_0\mathbb{R}^n, T_0\mathbb{R}^n \setminus 0) \to H_n(T_0\mathbb{R}^n, T_0\mathbb{R}^n \setminus 0)$ is.*

Proof. Let

$$f_t(x) = \begin{cases} (1/t) t f(tx), & \text{if } t \in (0, 1] \\ Df(x), & \text{if } t = 0. \end{cases}$$

Then, f_t is a homotopy between Df and f on $\mathbb{R}^n \setminus 0$ (which does require identifying TM with $\mathbb{R}^n$, which is fine). ⊠

In particular, this means the double cover of one is trivial iff the other is. I think. ⊠

Now, we're almost done.

Proposition 3.6. *If M is parallelizable, then $\widetilde{M}_{TM}$ is trivial.*

This is not a hard exercise, apparently.

From that, I Theorem 3.1 follows, because $\widetilde{M}$ is also trivial. Thus, we can attack Theorem 3.2.

Definition. A **vector field** is a section of $p : TM \to M$, i.e. a σ such tht $p \circ \sigma = \mathrm{id}$. If σ is smooth as a map of manifolds, the vector is said to be **smooth**.

Proposition 3.7. *If M is a parallizable, n-dimensional manifold, then there exist n vector fields $\sigma_1, \dots, \sigma_n$ such that for all $m \in M$, $\{\sigma_1(m), \dots, \sigma_n(m)\}$ are linearly independent.*

That is, parallelizability means the maximum number of linearly independent vector fields that can exist do. This is nicer since we're talking about smooth manifolds; in this class, unlike 215B, we can do analysis.

Definition. A **flow** is a map $\Phi : \mathbb{R} \times M \to M$ with $\Phi_t \circ \Phi_s = \Phi_{t+s}$ for all $s, t \in \mathbb{R}$ and $\Phi_0 = \mathrm{id}$.

Another way of saying this is that Φ is a homomorphism of topological groups from $\mathbb{R}$ into the diffeomorphism group of M, akin to a continuous group action.

Proposition 3.8. *There is a natural bijection between the set of flows and the set of vector fields.*

This can be made a homeomorphism using the compact-open topology (in the continuous case) or a Fréchet topology (in the smooth case), but that's not important right now. The idea is that the flow is given by integrating along the vector field. More precisely, given a flow Φ and an $m \in M$, there's a curve $t \mapsto \Phi_t(m)$, which lives in $T_m M$ by the definition of the tangent space (really, it's $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \Phi_t(m)$, but it's the same idea); in the other direction, given a vector field σ, one can write down a differential equation satisfying

$$\sigma(m) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \Phi_t(m),$$

with $\Phi_0 = \mathrm{id}$ as the initial condition, implies the existence of a unique solution. (This may require M to be compact.)

So the idea is that if one has a flow on a compact manifold, and to move for a very small length of time.

Proposition 3.9. *Let M be compact, and suppose $\sigma(m) \neq 0$ for all $m \in M$. Let Φ be the flow associated with σ; then, there exists an $\varepsilon > 0$ such that $\Phi_\varepsilon(m) \neq m$ for all m.*

One way to prove this is to check in charts, possibly using the implicit function theorem.

Thus, a nowhere-vanishing vector field gives us a map homotopic to id, but has no fixed points.

Recall the following theorem from Math 215B.

Theorem 3.10 (Lefschetz fixed-point)**.** *Let Y be a finite CW complex and $f : Y \to Y$ be continuous. Then, let $T^f : \bigoplus_k H_k(Y) \to \bigoplus_k H_k(Y)$ be given by $\bigoplus (-1)^k f_{*,k}$ (where $f_{*,k}$ is the map induced on H_k). Then, if $\mathrm{tr}(T^f) \neq 0$, then f has a fixed point.*

Corollary 3.11. *If there exists a nowhere-vanishing vector field σ, then the Euler characteristic is equal to zero.*

This is because $\mathrm{tr}\, T^{\mathrm{id}} = \chi(Y)$.

... right now, we haven't shown that a smooth manifold is homeomorphic to a finite *CW* complex, and the Lefschetz fixed point theorem doesn't hold on infinite *CW* complexes.

Now, the corollary implies Theorem 3.2, because if the Euler characteristic is nonzero, no nonvanishing vector fields can exist. Oops.

Both of these are examples of a notion called characteristic classes. There's a space called $B\,\mathrm{GL}_n(\mathbb{R})$, the **classifying space of vector bundles**, which can apparently be though of as a moduli space for certain stacks. It's also equal to the Grassmanian $\mathrm{Gr}(n,\infty)$. A cohomology class in the Grassmanian yields a cohomology class for every vector bundle on M; then, orientability corresponds to a class named $w_1 \in H^1(B\,\mathrm{GL}_n(\mathbb{R}))$, and there's a class called the Euler class $e \in H^n(B\,\mathrm{GL}_n(\mathbb{R}))$. The trick is, if M is parallelizable, the classifying map $M \to \mathrm{Gr}(n,\infty)$ is null-homotopic, so any cohomology class pulls back, and we know what w_1 and e are.

So how do you build this classifying map? Given a manifold M, one can embed it $M \hookrightarrow \mathbb{R}^\infty$.

4. The Whitney Embedding Theorem: 4/16/15

"I should stick to Greek letters whose names I remember."

Definition. A smooth map of manifolds $M \hookrightarrow N$ is called an **embedding** if it is an injective immersion that is a homeomorphism onto its image.

Today our goal will be to prove the following theorem.

Theorem 4.1 (Weak Whitney embedding theorem)**.** *If M is a compact n-dimensional manifold, then there exists an embedding $M \hookrightarrow \mathbb{R}^{2n+1}$.*

Remark. It's possible to remove the compactness assumption with a little more work (see the textbook), and get an embedding $M \hookrightarrow \mathbb{R}^{2n}$ with a lot more work.

The proof will require the following ingredients, which we will not prove.

Definition. Let $\{U_\alpha\}$ be an open cover of a smooth manifold M. Then, a **(smooth) partition of unity** subordinate to $\{U_\alpha\}$ is a collection of smooth functions $f_\alpha : M \to [0,1]$ such that:

- $f_\alpha^{-1}((0,1]) \subset U_\alpha$.

- For all $x \in M$, there exists an open $V_x \subseteq M$ containing x, such that $V_x \cap f_\alpha^{-1}((0,1]) \neq \emptyset$ for only finitely many α.
- $\sum_\alpha f_\alpha(x) = 1$ for all $x \in M$.

Partitions of unity are useful for turning local data or results into global ones. For example, if $f : M \to \mathbb{R}$ is a submersion, one might want to build a vector field on M that flows in the direction of $\mathbb{R}$ given by f (i.e. the flow commutes with f). One can do this locally with the implicit function theorem, and then use a partition of unity to do it globally. It still has the required property, because of the condition that the f_α sum to 1 everywhere. (This is one example; we'll use them in a different way today.)

The half-open interval in the definition arises from taking the support of f_α^{-1}, and isn't super critical to one's intuition.

Theorem 4.2. *For any smooth manifold M and open cover $\{U_\alpha\}$ of M, there exists a partition of unity subordinate to $\{U_\alpha\}$.*

This isn't too tricky to prove, and we have all of the tools, but it would distract us from nobler goals, so check out the textbook for the proof.

It's also possible to define a continuous partition of unity on a more general topological space. Unlike smooth partitions of unity on smooth manifolds, they might not always exist.

Definition. Let $f : M \to N$ be a map of smooth manifolds, where $\dim(M) = m$ and $\dim(N) = n$. Then, an $x \in M$ is a **critical point** of f if $\operatorname{rank}(Df_x) < n$, and $f(x)$ is called a **critical value**.

Theorem 4.3 (Sard)**.** *Let $f : M \to \mathbb{R}^n$ be smooth. Then, its set of critical values is measure zero in the Lesbegue measure on $\mathbb{R}^n$.*

More generally, one can replace $\mathbb{R}^n$ with any manifold N; then, however, the measure-zero criterion is replaced with the statement that the set of critical values is meager (related to the Baire category theorem).

We're also not going to prove this; once again, consult the textbook.

Proposition 4.4. *Let $K \subset M$ be closed and $f : K \to \mathbb{R}$. Assume that for all $x \in K$, there's an open neighborhood U_x (open in M) of x and a smooth $g_x : U_x \to \mathbb{R}$ such that $g_x|_{K\cap U_x} = f|_{K\cap U_x}$. Then, there exists a smooth $h : M \to \mathbb{R}$ with $h|_K = f$.*

Proof. Let $U_\alpha = \{U_x\} \cup \{M \setminus K\}$, and let f_α be a partition of unity for U_α. Let $g_\alpha = g_x$, except if $U_\alpha = M \setminus K$, where we let $g_\alpha = 0$. Finally, let

$$h = \sum_\alpha f_\alpha g_\alpha. \qquad \boxtimes$$

Theorem 4.5. *If M is a compact manifold, then there exists some (large) N such that there's an embedding $M \hookrightarrow \mathbb{R}^N$.*

Once we prove this, we'll use Sard's theorem to lower N to the desired value.

Proof. Choose two open covers $\{V_i\}_{i\in I}$ and $\{U_i\}_{i\in I}$ of M such that $\overline{V_i} \subseteq U_i$ for each i; then, since M is compact, we can choose a finite subcover $\{V_i\}_{i=1}^k$, and the corresponding $\{U_i\}_{i=1}^k$. Let $\phi_i : U_i \xrightarrow{\cong} \mathbb{R}^n$ be the chart map for U_i.

Now we have finitely many of each and $\bigcup_1^k V_i = M$ still.[2] Choose $\lambda_i : M \to \mathbb{R}$ that are 1 on $\overline{V_i}$ supported in U_i (i.e. 0 outside of U_i); these can be constructed by invoking Proposition 4.4. Then, let $\psi_i : M \to \mathbb{R}^n$ be given by $\psi_i : \lambda_i \phi_i$, and let $\theta : M \to (\mathbb{R}^n)^k \times \mathbb{R}^k$ be given by their product: $\theta = \psi_1 \times \psi_2 \times \cdots \times \psi_k \times \lambda_1 \times \cdots \times \lambda_k$.

So, why is θ an immersion? Take an $x \in M$; then, if $x \in V_i$, $\psi_i = \phi_i$ in a neighborhood of x, and ϕ_i is a diffeomorphism, so near x, θ is a product of diffeomorphisms and the zero map, so it's smooth.

θ is injective, because if $\theta(p) = \theta(q)$, then $p \in V_i$ for some i, and therefore $\lambda_i(p) = \lambda_i(q) = 1$, i.e. $q \in U_i$ (and we know $p \in U_i$ too). Thus,

$$\begin{aligned}\phi_i(p) &= \lambda_i(p)\phi_i(p) = \psi_i(p)\\ &= \psi_i(q) = \lambda_i(q)\phi_i(q)\\ &= \phi_i(q).\end{aligned}$$

However, we know ϕ_i is injective, so $p = q$.

Why is θ a homeomorphism onto its image? Well, M is compact and $\theta(M)$ is Hausdorff, which is a sufficient condition. Thus, θ is an embedding. $\boxtimes$

[2] There are a couple of other ways to do this, e.g. choosing a finite cover (U_i, ϕ_i) first and then letting V_i be the support of ϕ_i.

Proof of Theorem 4.1. Let $\theta : M \to \mathbb{R}^N$ be an embedding, as in Theorem 4.5, and suppose there exists a $\mathbf{w} \in \mathbb{R}^N$ such that $\mathbf{w}$ isn't tangent to $\theta(M)$ and there are no $x, y \in M$ such that $\mathbf{w} \in \operatorname{span}(\theta(x) - \theta(y))$.

Let $\pi_{\mathbf{w}^\perp}$ denote projection onto the orthogonal complement of $\mathbf{w}$; then, we will show that $\pi_{\mathbf{w}^\perp} \circ \theta$ is an embedding $M \hookrightarrow \operatorname{Im}(\pi_{\mathbf{w}^\perp}) \cong \mathbb{R}^{N-1}$. The idea is that the first condition (not tangent) makes it an immersion, and the second condition guarantees embedding. (There's more to check here, but I guess we can grind through it now without any difficult insights.)

Since M is an open submanifold of TM, then construct $\sigma : TM \backslash M \to \mathbb{R}P^{N-1}$ as follows: $D\theta : TM \to T\mathbb{R}^N$, and then the trivialization $\pi : T\mathbb{R}^N \to \mathbb{R}^N$, but if you didn't strt out in M, you won't end up at 0, so $(\pi \circ D\theta) : TM \setminus M \to \mathbb{R}^n \setminus 0$, so it's possible to projectivize, and composing $(\pi \circ D\theta)$ with this projectivization gives us the desired σ. We can also construct a $\tau : (M \times M) \setminus \Delta \to \mathbb{R}P^{N-1}$; here, $\Delta \subseteq M \times M$ is the diagonal, i.e. $\Delta = \{(x, x) \mid x \in M\}$. Thus, sending $(x, y) \mapsto \theta(x) - \theta(y)$ doesn't hit 0 if $x \neq y$, so we can send it to $\mathbb{R}P^{N-1}$; this is how τ is defined.

Observe that if $N - 1 > 2n$, then every point in the domain of σ or τ is a critical point, so their images are meager (or measure zero) in $\mathbb{R}P^{N-1}$. Thus, if $N > 2n + 1$, the $\mathbf{w}$ we sought above exists. ⊠

Theorem 4.6. *If M is an n-dimensional manifold and* $\operatorname{Emb}(M, N)$ *denotes the space of embeddings $M \hookrightarrow N$, then* $\pi_i(\operatorname{Emb}(M, \mathbb{R}^N)) = 0$ *for $i \le k - 1$ and $N \ge 2(n + 1 + k)$.*

Remark. If we had the better bound of an embedding into $\mathbb{R}^{2n}$, then instead we have $N \ge 2(n + k)$.

This theorem says that when N is large enough, these embeddings are connected, and in some sense clarifies that this space of embeddings is nonempty or connected. To get our hands on it, we should talk about a relative version of the Whitney embedding theorem.

Theorem 4.7 (Relative Whitney embedding theorem)**.** *Let M be an n-dimensional manifold and $L \subseteq M$ be a submanifold. If $f : L \to \mathbb{R}^N$ with $N \ge 2n + 1$ is an embedding, then there exists an embedding $g : M \to \mathbb{R}^N$ with $g|_L = f$.*

This says that an embedding of a submanifold into $\mathbb{R}^{2N+1}$ can be lifted to an embedding of the whole manifold.

This applies to Theorem 4.6 as follows: if $f : M \hookrightarrow \mathbb{R}^N$ for $i = 1, 2$ are two embeddings, then one can embed $M \times \mathbb{R}$ into $\mathbb{R}^{N+1}$, and consider $L = M \times \{1, 2\}$ and use Theorem 4.7 to show there's an embedding that extends the f_i. This isn't entirely true (what if $f_1(M)$ intersects $f_2(M)$?), but Sard's theorem means we can control those points and fix the proof.

5. Immersions, Submersions and Tubular Neighborhoods: 4/21/15

Today, we're going to prove some honestly kind of boring technical results about immersions and submersions. But they'll be useful for all sorts of cool things like Pontryagin duality.

Theorem 5.1 (Implicit function theorem)**.** *Let $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be C^1 and $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ be such that $g(x, y) = 0$. If $i_x : \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ sends $z \mapsto (x, z)$ and $Dg \circ i_x$ is onto at y, then there exist $a, b > 0$ and an $f : B_a(x) \to B_b(y)$ such that $\{(x, f(x)) \mid x \in B_a(x)\} = \{(x, y) \in B_a(x) \times B_b(y) \mid g(x, y) = 0\}$. Moreover, if g is C^r, then so is f.*

Basically, this says that a continuous function where the derivative matrix is well-behaved can be interpreted as a level set in some small neighborhood. Alternatively, it says that if the derivative matrix is n-dimensional, then the space of solutions is what you would expect.

There's a nice way to reformulate this.

Theorem 5.2 (Inverse function theorem)**.** *Let $\theta : \mathbb{R}^n \to \mathbb{R}^m$ be C^1 and $\theta(x) = y$. If $D\theta$ is an isomorphsim at x, then there exist $a, b > 0$ and an $f : B_b(y) \to B_a(x)$ with $\theta \circ f = \mathrm{id}$. Moreover, if θ is C^r, then so is f.*

These theorems are useful because they tell us what immersions and submersions look like.

Corollary 5.3. *Let M be an m-dimensional manifold and N be an n-dimensional manifold, and let $\theta : M \to N$ be an immersion (resp. submersion) at a $p \in M$. Then, there exist open neighborhoods $U \subseteq M$ of p and $V \subseteq N$ of $f(p)$*

and diffeomorphisms $\phi : U \xrightarrow{\cong} \mathbb{R}^m$ *and* $\psi : V \xrightarrow{\cong} \mathbb{R}^n$ *that make thefollowing diagram commute.*

$$
\begin{tikzcd}
M \arrow[r, "\theta"] & N \\
U \arrow[u, hook] \arrow[r, "\theta|_U"] \arrow[d, "\phi", "\cong"'] & V \arrow[u, hook] \arrow[d, "\psi", "\cong"'] \\
\mathbb{R}^m \arrow[r, "\vartheta"] & \mathbb{R}^n
\end{tikzcd}
\tag{5.1}
$$

Here, $\vartheta(x_1,\dots,x_m) = (x_1,\dots,x_m,0,\dots,0)$ *(resp.* $\vartheta(x_1,\dots,x_m) = (x_1,\dots,x_n)$*, since if* θ *is a submersion, then* $m \geq n$*).*

The intuition behind the proof, which we won't go into here, is that you use the implicit function theorem (resp. inverse function theorem), and then rotate.

Corollary 5.4. *If* M *and* N *are as in Corollary 5.3,* $f : M \to N$ *is smooth, and* $y \in N$ *is a regular value of* f*, then* $f^{-1}(y)$ *is an* $(m-n)$*-dimensional* **submanifold** *of* M.

We haven't actually defined the notion of submanifold: some authors define it as a subset of a manifold where inclusion is an immersion, and others require it to have charts so that it looks like the first m coordinates in some set of charts for the ambient manifold. In any case, Corollary 5.3 equates the two notions.

The proof of Corollary 5.4 requires both parts of Corollary 5.3 to prove, since $f : M \to N$ is a submersion at any $x \in f^{-1}(y)$, and then the submanifold is immersed in M.

Definition. Let N_1 and N_2 be submanifolds of M. Then, N_1 is **transverse** to N_2, written $N_1 \pitchfork N_2$, if for all $x \in N_1 \cap N_2$, $T_x N_1 + T_x N_2 = T_x M$.

Clearly, if N_1 doesn't intersect N_2, then they're not transverse, and if you have a point and a curve in $\mathbb{R}^n$, then they can only be transverse if they don't intersect. However, all of $\mathbb{R}^n$ intersects a point transversely.

Transversality is a way to make the hazy notion of "in general position" somewhat precise.

Theorem 5.5. *If* $N_1 \pitchfork N_2$*, then* $N_1 \cap N_2$ *is a submanifold of dimension* $\dim(N_1) + \dim(N_2) - \dim(M)$.

The proof idea is that one can find a $V \subseteq M$ and $U \subseteq N_1$ that satisfy the diagram (5.1). Then, let $\pi : N_2 \cap V \to \mathbb{R}^{\dim(M)-\dim(N_1)}$ be the projection onto the last coordinates. Thus, $\pi^{-1}(0) = N_1 \cap N_2 \cap V$, and π is a submersion, so $\pi^{-1}(0)$ is a subamanifold of the required dimension.

Basically, the only reason the professor cares about submersions is the following theorem.

Theorem 5.6. *Let* $f : M \to N$ *be a proper submersion with* N *connected. Then,* $f : M \to N$ *is a fiber bundle.*

One way to think of it is that the preimages are locally diffeomorphic. Another is that if a family of manifolds is smoothly dependent on a parameter t, it can only change topology at critical points, which are also where it doesn't project smoothly onto t.

Connectedness is needed because otherwise, there's a fiber bundle over every connected component, but there's no reason to assume they're the same.

Proposition 5.7. *Let* $f : M \to \mathbb{R}^n$ *be a proper submersion and* $r > 0$*. Then, there exists a diffeomorphism* d *such that the following diagram commutes.*

$$
\begin{tikzcd}
f^{-1}(B_r(\mathbf{0})) \arrow[rr, "d"] \arrow[dr, "f"'] & & f^{-1}(\mathbf{0}) \times B_r(\mathbf{0}) \arrow[dl, "\pi"] \\
& B_r(\mathbf{0}) &
\end{tikzcd}
$$

We'll end up using flows and partitions of unity to prove this.

Proof. We can find charts $U_\alpha \subseteq M$ and $V_\alpha \subseteq \mathbb{R}^n$ with $f(U_\alpha) = V_\alpha$ and $f|_{U_\alpha}$ fiberwise equivalent to $\pi_\alpha : V_\alpha \times \mathbb{R}^{m-n} \to V_\alpha$.

Let χ_i be the vector field on $\mathbb{R}^m$ in the i^{th} direction; then, by an abuse of notation, since $V_\alpha \subseteq \mathbb{R}^n$, we can think of χ_i as a vector field on V_α. Take χ_i^α be a vector field on U_α with $Df\chi_i^\alpha = \chi_i|_{V_\alpha}$.

Now, we would like to stitch these together: let $g_\alpha : M \to \mathbb{R}^n$ be a partition of unity subordinate to the charts $\{V_\alpha\}$, and let

$$\sigma_i = \sum_\alpha g_\alpha \chi_i^\alpha,$$

so that $Df\sigma_i = \chi_i$.

Now, choose an $r > 0$, and let $h : \mathbb{R}^m \to \mathbb{R}$ be a function that is 1 on $B_r(\mathbf{0})$ and is compactly supported. Since f is proper, then $(h \circ f)$ does too. For a $\mathbf{v} \in \mathbb{R}^m$, let

$$\sigma_{\mathbf{v}} = \sum_{i=1}^m a_i \sigma_i (h \circ f),$$

where $\mathbf{v} = (a_1, \dots, a_m)$, and let $\Phi_{\mathbf{v}} : M \to M$ be the time-1 flow along $\sigma_{\mathbf{v}}$. We want a map $\delta : f^{-1}(\mathbf{0}) \times B_r(\mathbf{0}) \to f^{-1}(B_r(\mathbf{0}))$ (and then the required d will be δ^{-1}); in fact, it will be given by $\delta(m, \mathbf{v}) = \Phi_{\mathbf{v}}(m)$. Then, we'll be able to prove this is a diffeomorphism, so we get the d we need. ⊠

Tubular Neighborhoods.

Theorem 5.8 (Tubular neighborhood)**.** *Let M be a compact, n-dimensional submanifold of $\mathbb{R}^k$ and $N_\varepsilon = \{(x, \mathbf{v}) \mid \mathbf{v} \perp T_x M$ and $|\mathbf{v}| < \varepsilon\}$ (where length and angle are measured within $\mathbb{R}^k$). Let $V_\varepsilon = \{y \in \mathbb{R}^k \mid$ there exists an $x \in M$ such that $|x - y| < \varepsilon\}$.*

If $\theta : N_\varepsilon \to V_\varepsilon$ sends $(x, \mathbf{v}) \mapsto x + \mathbf{v}$, then for sufficiently small ε, θ is a diffeomorphism.

In other words, a small tubular neighborhood of M looks like $M \times \mathbb{R}^m$ for some m. We know M_ε is homotopy equivalent to M for all ε, but V_ε might not be (e.g. if the hole in a donut is filled in, in some sense).

Proof. For $x \in M$, $D\theta$ is an isomorphism at $(x, \mathbf{0})$: the dimensions line up, since then $\theta(x, \mathbf{0}) = x$. That means θ is locally a diffeomorphism (i.e. in a neighborhood for every x). But since M is compact, one can find an ε such that $D\theta$ is an isomorphism for all $(x, \mathbf{v}) \in N_\varepsilon$.

Assume that θ isn't injective for all ε; then, take $x_i, y_i \in N_{1/i}$ such that $x_i \neq y_i$ and $\theta(x_i) = \theta(y_i)$; since M is compact, we can choose a convergent subsequence, and so eventually the sequence ends up in the manifold, where it is injective.

To show that it's surjective, we know that $V_\varepsilon \supseteq \theta(N_\varepsilon)$, so suppose $y \in V_\varepsilon \setminus \theta(N_\varepsilon)$, and let $x \in M$ be the nearest point on M to it. Then, $x - y \perp T_x M$, so $\theta(x, x - y) = y$, which looks like the definition of our tubular neighborhood. ⊠

There are many ways to jazz this theorem up, e.g. replacing $\mathbb{R}^k$ with a k-dimensional manifold. This makes it trickier to define distance, but if all you care about is topology, you could talk about the normal bundle of an embedding of manifolds, which is diffeomorphic to a neighborhood. There's some tricky questions about defining the normal bundle, though it's well-defined in K-theory.

6. Cobordism: 4/23/15

"So I'm just going to quit while I'm losing."

The ideas outlined today relate to some deeper ideas about classifying manifolds up to cobordism and homotopy groups of spheres, thanks to ideas of Pontryagin and Thom.

Definition. Two smooth n-dimensional manifolds M and N are called **cobordant** if there exists a compact $(n+1)$-dimensional manifold-with-boundary W such that ∂W is diffeomorphic to $M \sqcup N$.

For example, a pair of pants is a cobordism between S^1 and $S^1 \sqcup S^1$, as in Figure 2 More generally, any number of

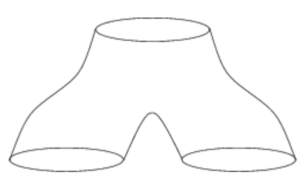

Figure 2. A cobordism between S^1 and $S^1 \sqcup S^1$, called the "pair of pants" for obvious reasons. Source: http://en.wikipedia.org/wiki/Cobordism.

circles is cobordant to any other number of circles.

Definition. Let cob^n denote the set of smooth, compact n-manifolds up to cobordism.

Fact. cob^n is a group with $[M]+[N]=[M\sqcup N]$. Moreover, $\mathrm{cob}^*=\bigoplus_n \mathrm{cob}^n$ is a graded ring, with $[M]\cdot[N]=[M\times N]$.

Theorem 6.1 (Thom)**.** $\mathrm{cob}^*\cong \mathbb{Z}/2[z_i \mid i\neq 2^j-1]$*, where* $|x_i|=i$.

This was probably the second-best thesis in algebraic topology (after Serre's). Both Thom and Serre won Fields medals for basically their thesis work.

The reason everything is 2-torsion is that two copies of a manifold is cobordant to the empty set (by just connecting them as a sort-of cylinder, so the boundary is both copies, or equivalently both copies along with the empty set).

Corollary 6.2. *If M is a compact, smooth manifold and* $\dim(M)=3$*, then $M=\partial W$ for some compact, smooth manifold-with-boundary W.*

Definition. Let $P(m,n)=(S^m\times\mathbb{C}P^n)/\tau$, where $\tau(x,y)=(-x,\overline{y})$.

Proposition 6.3. *In Theorem 6.1,*

$$x_i=\begin{cases}[P(i,0)]=[\mathbb{R}P^i], & \text{for } i \text{ even}\\ [P(2^r-1,s2^r)], & \text{for } i=2^r(2s+1)-1.\end{cases}$$

The proof has two parts, one of which is reasonable for this class and the other of which is wildly inappropriate. The idea is to first find a space (well, actually a spectrum, which is weird and off-putting to some people) MO such that $\mathrm{cob}^*\cong\pi_*(MO)$; then, the second step is to calculate $\pi_*(MO)$. This is hard, because it involves calculations of homotopy groups of spheres.

Returning to Earth, let's prove some more technical lemmas needed to prove this stuff.

Theorem 6.4. *Let M be a manifold and $A,B\subseteq M$ be closed. If $f:M\to\mathbb{R}^k$ is continuous and smooth on A, then there exists a $g:M\to\mathbb{R}^k$ which is smooth on $M\setminus B$, satisfies $g|_{A\cup B}=f|_{A\cup B}$, and $g\simeq_{h_t} f\ \mathrm{rel}_{A\cup B}$, i.e there's a homotopy between g and f constant on $A\cup B$ and such that for all $x\in M$, $|h_t(x)-f(x)|<\varepsilon$.*

This seems kind of technical, but the point is that we can approximate continuous functions that are smooth on some closed set with smooth functions on most of M.

Proof sketch. Let ρ be any metric on M (in the metric space sense, not Riemannian sense; we know that all manifolds are metrizable), and let

$$\varepsilon(x)=\inf_{b\in B}\rho(x,b).$$

Since f is smooth on A, then for all $x\in A$, there exists an open neighborhood U_x of x and a smooth $g_:U_x\to\mathbb{R}^k$ such that $g|_{A\cap U_x}=f|_{A\cap U_x}$. For more general $x\in M$, let V_x be an open neighborhood such that if $x\in A\setminus B$, $V_x=V_x\cap(M\setminus B)$; then, let $h_x=g_x|_{V_x}$. If $x\notin A\setminus B$, take V_x to be an open set such that $x\in V_x$ and $V_x\cap A=\emptyset$; then, take $h_x(z)=f(x)$.

The whole point of this is that these V_x should form an open cover of M; then, one can take these h_x and stitch them together with a partition of unity. ⊠

There's also a version with maps into a manifold.

Theorem 6.5. *Let M and N be manifolds with N compact, and let ρ be a metric on M. Let $A\subseteq M$ be closed and $f:M\to N$ be continuous and smooth on A. Then, for all $\varepsilon>0$, there exists a family $h_t:M\to N$ such that*

(1) $h_0=f$,
(2) $h_t|_A=f|_A$,
(3) h_t *is smooth for all* $t>0$*, and*
(4) $\rho(h_t(x),f(x))<\varepsilon$ *for all* $t\in[0,1]$.

The way to prove this is to embed $M\hookrightarrow\mathbb{R}^k$, and then take a tubular neighborhood. The compactness of N is used to guarantee that small distances in $\mathbb{R}^k$ correspond to small distances in N (using uniform continuity), so one can use distances in $\mathbb{R}^k$. Then, Theorem 6.4 gives us a smooth approximation in the tubular neighborhood, and then it projects back down into an approximation on N.

The book has a bunch of corollaries to these.

If you're wondering why we need so many smooth approximation theorems, remember that homotopy groups involve continuous maps, so to use the tools we've developed with manifolds, we need to approximate them.

Definition. Let M, X, and Y be manifolds. Then, two smooth maps $f : X \to M$ and $g : Y \to M$ are **transverse**, written $f \pitchfork g$, if for all points $x \in X$ and $y \in Y$ such that $f(x) = g(y) = z$, $Df(T_x X) + Dg(T_y Y) = T_z M$. If $f : X \to M$ is an embedding and $N = f(X)$, then one also writes $g \pitchfork N$ if $g \pitchfork f$.

In particular, if the images of f and g don't intersect, then they're transverse, and if one of f or g is a submersion, then they're transverse.

Theorem 6.6. *Let $f_0 : M \to W$ be smooth and $N \subseteq W$ be a compact, smooth submanifold. Then, given a tubular neighborhood T of M, there exists a family $h_t : M \to W$ such that $h_0 = f$, $h_1 \pitchfork N$, and $h_t(x) = f(x)$ for $f(x)$ outside T.*

Definition. Let $V \to B$ be a vector bundle, and let $V^+ \to B$ be the fiberwise one-point compactification (so that V has fiber $\mathbb{R}^k$ and V^+ has fiber S^k). It's possible to think of this as saying that the transition functions for V^+ are given by the injection $\mathrm{GL}_k(\mathbb{R}) \hookrightarrow \mathrm{Diff}(S^k)$. Then, the **Thom space** is $\tau(V) = V^+/\sim$, where $x \sim y$ if x and y are both "points at infinity," i.e. added in by the one-point compactification.

How should you think about Thom spaces? Well, the Thom space of the trivial bundle $\mathbb{R} \to E \to B$ is just the suspension of B: $\tau(E) = \Sigma B$. Suspensions come up as ways of generalizing $\mathbb{R}$ in homotopy theory (e.g. in equivariant homotopy theory, where they correspond to, incredibly enough, group representations). Note that this doesn't always generalize: the trivial k-bundle $\mathbb{R}^k \to E \to B$ does not satify $\tau(E) = \Sigma^k B$ when $k = 0$; you get $B \sqcup \{\mathrm{pt}\}$, which isn't $\Sigma^0 B = B$. Some of this depends on the precise definition of the one-point compactification of zero-dimensional manifolds.

Last quarter, we learned the following isomorphism.

Proposition 6.7. $\widetilde{H}_*(\Sigma^k B) \cong \widetilde{H}_{*-k}(B)$.

Theorem 6.8 (Thom isomorphism theorem)**.** *If V is an orientable, k-dimensional vector bundle, then $H_*(B) \cong H_{*+k}(\tau(V))$*

Since all trivial bundles are orientable, this implies Proposition 6.7, and this theorem is in some sense a twisted version of that. Since Pontryagin duality only requires orientability, this is what we're looking for, though we don't need it just yet, and we'll come back to prove it later.

An embedding $M \overset{f}{\hookrightarrow} W$ gives a map of homology of M to homology of W, but Theorem 6.8 gives a map in the other direction, arising from $W \xrightarrow{g} \tau(N^f)$ (where N^f denotes the normal bundle of f), and therefore we can go between

$$H_*(W) \xrightarrow{g} H_*(\tau N^f) \xrightarrow{\cong} H_{*-\dim W + \dim M}(M).$$

This composition is written $f^!$, read "f-shriek."

7. Homotopy Theory: 4/28/15

Definition. Let (X, x_0) be a based (i.e. pointed) topological space.

- Then, $\Omega X = \mathrm{Map}^*(S^1, X)$; that is, the continuous maps of based topological spaces. ΩX, read "loop X," is called the **loop space**.
- The **reduced suspension** ΣX is $(X \times I)/(\{0,1\} \times X \cup I \times \{x_0\})$, i.e. taking the double cone and collapsing the basepoint.
- The **unreduced suspension** is given by $(X \times I)/(\{0,1\} \times X)$.

For all reasonable spaces, the reduced and unreduced suspensions are homotopic. However, the following adjunction is useful.

Proposition 7.1. $\Phi : \mathrm{Map}^*(\Sigma X, Y) \to \mathrm{Map}^*(X, \Omega Y)$ *given by* $\Phi(f)(x)(t) = f(t, x)$ *is a homeomorphism.*

This may require some annoying point-set topological stuff, but it's certainly a homotopy equivalence, and categorically we're set either way.

Corollary 7.2. $\Omega\Omega Y = \mathrm{Map}^*(S^2, Y)$ *(and so on with $\Omega^n Y$ and S^n).*

By categorical nonsense, $\mathrm{id} \in \mathrm{Map}^*(\Sigma X, \Sigma X)$ maps to some unit in $\mathrm{Map}^*(X, \Omega\Sigma X)$, so there is a natural map $u : X \to \Omega\Sigma X$, and $\mathrm{id} \in \mathrm{Map}^*(\Omega X, \Omega X)$ is sent to a counit in $\mathrm{Map}^*(\Sigma\Omega X, X)$, which is a canonical map $\Sigma\Omega X \to X$ (evaluate a point on an interval as that length along a loop).

Definition. A topological space X is n**-connected** if $\pi_i(X) = 0$ for $i \le n$.

Definition. The **Hurewicz map** maps from homotopy to homology: $h : \pi_i(X) \to H_i(X)$ is defined by $h(f) = f_*[S^n]$, where $f : S^n \to X$.

Thus, $H_0(X)$ is the free abelian group on $\pi_0(X)$ (the connected components of X). If X is connected, then $H_1(X) = \pi_1(X)^{\mathrm{ab}}$.

Theorem 7.3 (Hurewicz). *If X is $(n-1)$-connected, then $h : \pi_n(X) \to H_n(X)$ is an isomorphism.*

The idea is that h can be very near an isomorphism: even in lower dimensions, it does the least it can to get from one to the other (e.g. taking the free group or abelianization).

Theorem 7.4 (Freudenthal suspension theorem). *Let X be n-connected. Then, the natural map $u : X \to \Omega\Sigma X$ induces an isomorphism on π_i for $i \le 2n$, and is surjective for $i = 2n+1$.*

We'll return to this theorem when we discuss stable homotopy theory; in fact, it is the reason stable homotopy groups exist. The next theorem is also relevant.

Theorem 7.5 (Bott-Samuelson). *$H_*(\Omega\Sigma X) \cong T(\widetilde{H}_*(X))$, where T denotes the tensor algebra.*

We know T is the adjoint functor to the forgetful functor from graded rings to graded vector spaces, so we have a ring, but why is $H_*(\Omega\Sigma X)$ a ring? It turns out to come from the concatenation map $\Omega Y \times \Omega Y \to \Omega Y$, so when you take homology, this turns into a map $H_*(\Omega Y) \otimes H_*(\Omega Y) \to H_*(\Omega Y)$.

This theorem can be used to reason out a proof for Theorem 7.4, if you think about it a lot: if V_* is a graded vector space, the natural map $V_* \to T(V_*)$ doesn't hit everything, and the first thing it doesn't hit is something in $2n$ (the square of something), so invoking the theorem, the first place it isn't an isomorphism wouldbe $2n+1$.

Later, we'll return to this with Morse theory, and prove that $H_*(\Omega S^{n+1}) = \mathbb{Z}[x]$, where $|x| = n$.

Stable Homotopy Theory. The idea here is that one might want to "de-suspend" spaces, which is where spectra come in.

Definition. A **spectrum** $\mathscr{X}$ is a sequence of based spaces X_n and maps $f_n : \Sigma X_n \to X_{n+1}$.

Maps of spectra are somewhat difficult, and this model isn't very good for them, and so on, so we won't really talk about them. Sometimes this definition is termed "naïve spectra" or "pre-spectra," because the real one is yet more complicated!

Example 7.6. If X is a based space, let $\Sigma^\infty X$ be the spectrum with $(\Sigma^\infty X)_n = \Sigma^n X$, with $f_n : \Sigma\Sigma^n X \to \Sigma^{n+1} X$ is the identity. Thus, Σ^∞ is a functor from based spaces to spectra.

However, not all spectra come from spaces.

Example 7.7. Let G be an abelian group and $K(G,n)$ be the **Eilenberg-MacLane space** on G (i.e. the topological space uniquely characterized by $\pi_i(K(G,n))$ being 0 when $i \ne n$ and G when $i = n$). Then, there's a spectrum H_G with $(H_G)_n = K(G,n)$. Since loops lower homotopy by 1, $\Omega K(G,n+1) \cong K(G,n)$, so adjointness guarantees us a map $\Sigma K(G,n) \to K(G,n+1)$, making H_G into a spectrum.

Thus, one can think of spectra as the minimal category that contains both abelian groups and based spaces. That's kind of weird, but interesting!

Definition. There's a functor Ω^∞ (read "loops-infinity") from spectra to spaces given by $\Omega^\infty \mathscr{X} = \varinjlim \Omega^n X_n$.

Since Ω and Σ are adjoints, then Ω^∞ and Σ^∞ are adjoints. We also have that $\Omega^\infty H_G = G$, endowed with the discrete topology.

Definition. Let $\mathscr{X}$ be a spectrum; then, $\Sigma^{-1}\mathscr{X}$ is the spectrum with $(\Sigma^{-1}X)_n = X_{n-1}$, and $(\Sigma^{-1}X)_0 = \mathrm{pt}$.

We haven't talked about homotopy equivalence of spectra, but changing any finite number of terms makes no difference (which is a digression we do not have time for). It's quite difficult to actually define the right category so that everything works out cleanly.

Thus, $\Sigma^{-1}\Sigma^\infty\Sigma X = \Sigma^\infty X$. Getting the direction of the shift right can be confusing. However, this means that in the category of spectra, we've inverted suspension! And in particular, there are negative homotopy and homology groups, at least as soon as we define homotopy and homology of spectra.

Definition. Let $\mathscr{X}$ be a spectrum. Then, $H_i(X) = \varinjlim H_{n+i}(X_n)$.

Proposition 7.8. $H_i(\Sigma^\infty X) \cong \widetilde{H}_i(X)$.

This is because of a fact we proved in Math 215B: if $\Sigma X_n \to X_{n+1}$, then $\widetilde{H}_j(X_n) \to \widetilde{H}_{j+1}(X_{n+1})$.

Definition. If $\mathscr{X}$ is a spectrum, then $\pi_i(\mathscr{X}) = \varinjlim \pi_{n+i}(X_n)$.

The homotopy groups are directed with the following maps: $\pi_{n+i}(X_n) \to \pi_{n+i}(\Omega X_{n+1}) = \pi_{n+i+1}(X_{n+1})$.

Definition. If X is a topological space, its **stable homotopy groups** are $\pi_i^{\mathrm{st}}(X) = \pi_i(\Sigma^\infty X)$.

The following calculation tells us that homology is reasonably defined.

Proposition 7.9. $\pi_i(\Omega^\infty \mathscr{X}) = \pi_i(\mathscr{X})$.

The following is kind of unimportant to our discussion, but it's cool.

Theorem 7.10. *Let S_∞ denote the symmetric group on a countably infinite set; then, its group homology $H_i(S_\infty) = H_i(\Omega^\infty \Sigma^\infty S^0)$.*

Example 7.11. Returning to Earth (somewhat), we can discuss Thom spectra. There are two notions of these spectra, but they're closely related. If $E \to B$ is a vector bundle, then let $(B^E)_n = \tau(E \oplus \mathbb{R}^n)$; then, $\tau(E \oplus R) = \Sigma\tau(E)$, so the **Thom spectrum** of E is $\Sigma^\infty \tau(E)$.

What makes these interesting is that we can define them for virtual vector bundles (i.e. formal sums of vector bundles over a space): if $E \oplus F = \mathbb{R}^N$, then $B^{-E} = \Sigma^{-N}\Sigma^\infty \tau(F)$.

Definition.

- The **Grassmanian** of k-planes in $\mathbb{R}^n$, $\mathrm{Gr}_k(\mathbb{R}^n)$, is the set of k-dimensional planes in $\mathbb{R}^n$, with the subspace topology from $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^k, \mathbb{R}^n)/\mathrm{GL}_k(\mathbb{R}^n)$.
- We can also define a canonical vector bundle $\gamma_k^n \to \mathrm{Gr}_k(\mathbb{R}^n)$ with $\gamma_k^n \subset \mathbb{R}^n \times \mathrm{Gr}_k(\mathbb{R}^n)$: specifically, $\gamma_k^n = \{(\mathbf{v}, T) \mid \mathbf{v} \in \mathrm{Im}(T)\}$. Then, the projection map $\pi : \gamma_k^n \to \mathrm{Gr}_k(\mathbb{R}^n)$ is given by $\pi(\mathbf{v}, T) = T$.

Definition. The space $B\mathrm{O}(k)$ (sometimes called $B\mathrm{GL}_k(\mathbb{R})$) is defined to be $B\mathrm{O}(k) = \mathrm{Gr}_k(\mathbb{R}^\infty) = \varinjlim \mathrm{Gr}_k(\mathbb{R}^n)$, where each $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ is fixed, and there's a resulting vector bundle γ_k.

γ_k should be thought of as a "universal vector bundle," because other bundles come from its pullbacks: if $f : X \to B\mathrm{O}(k)$, then we can send $f \mapsto f^*\gamma_k$.

Theorem 7.12. *Let X be a finite CW complex. Then, the set of k-dimensional vector bundles on X up to isomorphism is equal to* $\pi_0(\mathrm{Map}(X, B\mathrm{O}(k)))$.

The theorem is likely true for more general X, but we won't need that.

Now, if E is a vector bundle, pick an embedding $e : E \to \mathbb{R}^\infty$ with e affine on each fiber (the embedding guarantees that e is injective on each fiber). This will give us data for a map $X \to B\mathrm{O}(k)$, though using some version of the Whitney embedding theorem for vector bundles, apparently. This map sends a point $x \in X$ to $e(\pi^{-1}(X))$.

This is a lot easier to visualize for a manifold and its tangent bundle (since we know we can injectively and affinely embed that into $\mathbb{R}^N$ for some N, and therefore also into $\mathbb{R}^\infty$).

$B\mathrm{O}(k)$ is called the **classifying space of vector bundles** because the data of a vector bundle (up to isomorphism) is the same as a map from X into $B\mathrm{O}(k)$.

8. The Pontryagin-Thom Theorem: 4/30/15

"This should be obvious."

Recall that we defined $B\mathrm{O}(n) = \mathrm{Gr}_n(\mathbb{R}^\infty)$, with the canonical line bundle $\gamma_n \to B\mathrm{O}(n)$.

Definition. Let $\mathrm{MO}(n) = \tau(\gamma_n)$.

Recall that if one has a map $f : X \to Y$, a bundle $E \to Y$ can be pulled back to a bundle $f^*E \to X$. This induces pullbacks on the Thom spaces, where the point at infinity is sent to the point at infinity, so we have $\tau(f^*E) \to \tau(E)$.

We also have an injection $i : B\mathrm{O}(i) \hookrightarrow B\mathrm{O}(i+1)$: if $V \subseteq \mathbb{R}^\infty$, then $V \hookrightarrow \mathbb{R} \oplus V \subseteq \mathbb{R} \oplus \mathbb{R}^\infty \cong \mathbb{R}^\infty$. This plays nicely with the canonical bundles: $i^*\gamma_{n+1} = \mathbb{R} \oplus \gamma_n$.

Definition. Let MO be the spectrum with $(\mathrm{MO})_n = \mathrm{MO}(n)$.

This is notation, I guess, but the point is that the pullback of γ_{n+1} makes the axioms for a spectrum hold: specificially, one can check that $\tau(\mathbb{R}\oplus V)=\Sigma\tau(V)$. In particular, $\Sigma\mathrm{MO}(n)=\Sigma\tau(\gamma_n)=\tau(\mathbb{R}\oplus\gamma_n)=\tau(i^*\gamma_{n+1})$, and this maps into $\tau(\gamma_{n+1})$, which is just $\mathrm{MO}(n+1)$.

Intuitively, we would want an infinite-dimensional vector bundle over the direct limit of the $B\mathrm{O}(n)$, and then to take the Thom space of that. But we would want them to be shifted downwards somehow, and so spectra make the whole thing somewhat cleaner.

Definition. The **Pontryagin-Thom map** $\theta:\mathrm{cob}^n\to\pi_n(\mathrm{MO})$ is defined as follows. Given a compact n-dimensional manifold, let $e:M\hookrightarrow\mathbb{R}^{n+k}$ be an embedding. Thus, there is an induced map $M\to\mathrm{Gr}_k(\mathbb{R}^{n+k})$: since the Grassmanian classifies vector bundles, we send each point of M to the class of its normal bundle, which is a k-plane in $\mathbb{R}^{n+k}$. Then, let N be a tubular neighborhood of M, and let ν be its normal bundle. There is a map $S^{n+k}=\mathbb{R}^{n+k}\cup\{\infty\}\to\overline{N}/\partial\overline{N}=\tau(\nu)=\mathrm{MO}(k)$ which is the identity on N, and sending things not in N to the extra point at infinity. We also have the map $e':M\to B\mathrm{O}(k)$ (given from the map into the Grassmanian), and $e'^*\gamma_k=\nu$.

With all of this structure, this map sends the fundamental class $[M]$ of M into $\pi_{n+k}(\mathrm{MO}(k))$, and therefore into $\pi_n(\mathrm{MO})$. This is our map θ.

There are many things to check about this definition.

- θ doesn't depend on your choice of e.
- If M and M' are cobordant, then $\theta(M)=\theta(M')$.
- θ is an isomorphism.

Claim. This map does not depend on one's choice of e.

Proof sketch. It should be clear (his words, not mine) that if two embeddings are isotopic, the induced θ is the same, because isotopic embeddings induce the same map on homotopy. In low dimensions, not all embeddings are isotopic (e.g. $S^1\hookrightarrow\mathbb{R}^3$ as a circle or a trefoil knot), but in high dimension it's not a problem. So in a sufficiently high enough dimension, so that the space of embeddings is connected, it's independent of the choice of embedding, even if it might still depend on dimension.

To deal with the dimension, you can check that $M\hookrightarrow\mathbb{R}^{n+k}\to\mathbb{R}^{n+k+1}$ induces the map $\pi_{n+k}(\mathrm{MO}(k))\to\pi_{n+k+1}(\mathrm{MO}(k+1))$ that we get from the spectrum. ⊠

Along the way, you might have been wondering why we care that M is compact: this makes the maps into maps of based topological spaces, which is pretty helpful.

You may also want to know the group operation on $\pi_n(X)$: if $f,g:(I^n,0)\to(X,x_0)$; then, you attach f and g along a boundary ($f(1)$ and $g(0)$, so to speak), and then embed it into your space. This leads to a nice pictoral proof that higher homotopy groups are abelian (they can be massaged into switching around, since we only care up to homotopy); thus, all homotopy groups of spectra ar abelian.

To show θ is a homomorphism is to show that $\theta(M\sqcup M')=\theta(M)+\theta(M')$. This uses the tubular neighborhoods: they're sent "next to" each other (in the attaching sense needed for the group law on π_n), so addition is sent to addition. Strictly speaking, though, it's not a homomorphism until we know it's invariant on cobordism classes.

Claim. θ is invariant on cobordism classes of compact manifolds.

Proof. Let W be an $(n+1)$-dimensional manifold-with-boundary whose boundary is $M\sqcup M'$. The Whitney embedding theorem also has an analogue for manifolds with boundary, so we can embed $W\hookrightarrow\mathbb{R}^{n+k}\times[0,\infty)$, such that $\partial W\hookrightarrow\mathbb{R}^{n+k}\times\{0\}$. If we take the one-point compactification of the last coordinate, then we have embedded the whole thing into D^{n+k+1}. Thus, if we apply the whole Pontryagin-Thom construction to $B\mathrm{O}(k)$, then the S^{n+k} inside D^{n+k+1} is also sent to $B\mathrm{O}(k)$, and this map ends up being $\theta(\partial W)$. However, we know the map from the n-sphere is null-homotopic, since it extends to the disc, so $\theta(M\sqcup M')=0$. ⊠

To prove that it's an isomorphism, we'll describe an inverse, getting a manifold from an element of the homotopy group. Consider the following diagram.

$$\begin{tikzcd}
S^{n+k} \arrow[r, "f"] \arrow[dr, "f_1"] \arrow[dr, bend right, "f_2"'] & \mathrm{MO}(k) \\
 & \tau(\gamma_k^n\to\mathrm{Gr}_k(\mathbb{R}^n)) \arrow[u]
\end{tikzcd}$$

We also have $i : \mathrm{Gr}_k(\mathbb{R}^n) \to \tau(\gamma_k^n)$. We can make f_1 transverse to i, first by a smooth approximation, then by a transverse approximation. This requires a little thinking, because a Thom space may not be smooth at the point at infinity, but it ends up working. Furthermore, we place the following restrictions on f_2.

- f_2 is homotopic to f_1.
- f_2 is smooth everywhere except possibly at the point at infinity.
- $f_2 \pitchfork i$.

Claim. If $M = f_2^{-1}(\mathrm{Gr}_k(\mathbb{R}^n)) \subseteq S^{n+k}$, then $\theta(M) = f$.

This is scary, but the idea is that if you wind through the Pontryagin-Thom construction, the map ends up being f_2 compased with the inclusion $\tau(\gamma_k^n) \hookrightarrow \mathrm{MO}(k)$, so it's all good. Presumably. So this means that it's surjective. We also have injectivity, which is equivalent to saying that the backwards map is well-defined, so that equivalent maps create cobordant manifolds.

It's nice that unoriented cobordism is 2-torsion, because we can treat $+$ and $-$ signs a little more cavalierly.[3] In the oriented case, this isn't true, and things are somewhat trickier.

To prove injectivity, we'll show that if $\theta(M) = 0$, then we have a homotopy H_t where $H_0 = \theta(M)$ and H_1 is the constant map at the basepoint (i.e. $\theta(\emptyset)$). Then, we'll use this homotopy to prove that M is the boundary.

Let $H' : I \times S^{n+k} \to \mathrm{Gr}_k(\mathbb{R}^n)$ such that the following are true.

- $H \simeq H'$.
- $H' = \mathrm{Hom}(\{0,1\} \times S^{n+k})$.
- H' is smooth away from ∞.
- $H' \pitchfork \mathrm{Gr}_k(\mathbb{R}^n)$.

That is, we can smoothly approximate, and then transversely approximate, the homotopy so that it has the right properties.

Cobordism can be defined on any class of manifolds with tangential structure; for example, if one uses oriented manifolds, much (but not all) of the discussion continues, but with the group $B\mathrm{SO}(k) = \mathrm{Gr}_k^{\mathrm{or}}(\mathbb{R}^\infty)$ (the oriented k-planes in $\mathbb{R}^\infty$). One can also use stably framed manifolds: the normal bundle of an abstract manifold isn't well-defined, but it is stably defined. Calculating this presupposes that stable homotopy groups of the spectrum we get are relatively easy to calculate, but, well, this isn't always true. Anyways, we get an equivalence relationship $\mathrm{cob}_{\mathrm{fr}}^n = \{M, f\}/\sim$ (where f is our frame). In this case, we end up with the stable homotopy groups of spheres: $\mathrm{cob}_{\mathrm{fr}}^* = \pi_*(\Sigma^\infty S^0) = \pi_*^{\mathrm{st}}(S^0)$. This doesn't help is calculate the cobordism, but the idea is that this might give us insights into the stable homotopy groups of spheres, and allows one to calculate that $\pi_0^{\mathrm{st}}(S^0) = \mathbb{Z}$ and $\pi_1^{\mathrm{st}}(S^0) = \mathbb{Z}/2$.

There are varying degrees of how much geometric intuition you can get from this idea; it's not all just really weird homotopy theory. Specifically, we're classifying frames on S^1, which is quite reasonable to think about.

9. Differential Forms: 5/5/15

"I assigned [that homework problem] by accident."

Multilinear Algebra. A lot of the introductory stuff (tensor and wedge products, and so on) is likely to be review from undergrad. But then again, so might be differential forms.

Even though we're going to cover this in the case of real vector spaces, a lot of it holds true more generally for modules over a ring.

Definition. Let V and W be real vector spaces. Then, $V \otimes_{\mathbb{R}} W$, the **tensor product** of V and W, is the free module on the set $V \times W$ modulo the relations

$$\begin{gathered}(rv)\otimes w - r(v\otimes w),\\ (rv)\otimes w - v\otimes(rw),\\ v\otimes(w+w') - v\otimes w - v\otimes w', \text{ and}\\ (v+v')\otimes w - v\otimes w - v'\otimes w,\end{gathered}$$

where (v,w) is written as $v \otimes w$. Here, $v, v' \in V$, $w, w' \in W$, and $r \in \mathbb{R}$. Sometimes, the tensor product is just written $V \otimes W$.

[3] I mean, we've been treating everything pretty cavalierly in this class.

The point is to turn bilinear maps out of $V \times W$ into linear maps out of $V \otimes W$. A fancier way to describe this is to consider a category $\mathscr{C}$ whose objects are real vector spaces U along with bilinear maps $V \times W \to U$, and whose morphisms are functions $T : U_1 \to U_2$ making the following diagram commute.

$$\begin{tikzcd} & V \times W \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ U_1 \arrow[rr, "T"] & & U_2 \end{tikzcd}$$

Then, the tensor product is defined to be the initial object in $\mathscr{C}$. This approach has the advantage that initial objects are always unique up to unique isomorphism, though we would still have to use the construction to show that it exists.

A little more concretely, the tensor product also satisfies the following universal mapping property, which is just obtained by unwinding the categorical definition.

Proposition 9.1. *If $f : V \times W \to U$ is bilinear, then there exists a unique linear map $h : V \otimes W \to U$ such that the following diagram commutes.*

$$\begin{tikzcd} V \times W \arrow[r, "f"] \arrow[d, "e"] & U \\ V \otimes W \arrow[ur, "h"'] & \end{tikzcd}$$

Here, $e : (v, w) \mapsto v \otimes w$.

Proposition 9.2. *There is a natural isomorphism $(V \otimes W) \otimes U \cong V \otimes (U \otimes W)$.*

Proposition 9.3. *If $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of V and $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ is a basis of W, then $\{\mathbf{v}_i \otimes \mathbf{w}_j \mid 1 \le i \le n, 1 \le j \le m\}$ is a basis for $V \otimes W$.*

Corollary 9.4. $\dim(V \otimes W) = \dim(V)\dim(W)$.

Proposition 9.5. *There is an isomorphism*

$$\Phi : \mathrm{Hom}_{\mathbb{R}}(V \otimes_{\mathbb{R}} W, U) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{R}}(V, \mathrm{Hom}_{\mathbb{R}}(W, U))$$

given as follows: if $f : V \otimes W \to U$, then we get $\widehat{f} : V \times W \to U$ given by $\widehat{f}(v, w) = f(v \otimes w)$ according to Proposition 9.1; then, let $\Phi(f) : v \mapsto (w \mapsto \widehat{f}(v, w))$.

Definition. If V is a real vector space, its **dual space** is $V^* = \mathrm{Hom}(V, \mathbb{R})$, the space of linear real-valued functions on V.

Proposition 9.6. *If V and W are finite-dimensional vector spaces, there is a natural isomorphism $V \otimes W^* \cong \mathrm{Hom}(W, V)$.*

This might hold in more generality; the professor doesn't remember right now.

Definition. If V is a real vector space, the k-fold **wedge product** is the space

$$\Lambda^k V = \left(\bigotimes_{i=1}^{k} V \right) / M,$$

where M is generated by all relations of the form

$$v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_n + v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_i \otimes \cdots \otimes v_n.$$

One also writes $A^k(V) = (\Lambda^k V)^*$.

This means you can transpose two entries in the tensor, but the sign is switched.

The wedge product also satisfies a universal property.

Proposition 9.7. *Let $f : V^n \to U$ be an alternating n-linear map. Then, there exists a unique h making the following diagram commute.*

$$\begin{tikzcd} V^n \arrow[r, "f"] \arrow[d] & U \\ \Lambda^n V \arrow[ur, "h"'] & \end{tikzcd}$$

These constructions are also all functorial: here's an example for tensor product, but the same thing works for wedge products.

Definition. Let $f_1 : V_1 \to W_1$ and $f_2 : V_2 \to W_2$ be maps of real vector spaces. Then, $f_1 \otimes f_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$ is the unique map making the following diagram commute.

$$\begin{tikzcd} V_1 \times V_2 \arrow[r, "f_1 \times f_2"] \arrow[d] & W_1 \times W_2 \arrow[d] \\ V_1 \otimes V_2 \arrow[r, dashed, "f_1 \otimes f_2"] & W_1 \otimes W_2 \end{tikzcd}$$

Here, the vertical arrows send $(x_1, x_2) \mapsto x_1 \otimes x_2$.

Though this definition looks like it has something to prove, the idea is that composing along the upper right of the square creates a bilinear map $V_1 \times V_2 \to W_1 \otimes W_2$, so the mapping property makes the definition work.

Finally, we can define tensor algebras more generally.

Definition. By associativity of the tensor product (Proposition 9.2), there is a natural map $V^{\otimes k} \otimes V^{\otimes \ell} \to V^{\otimes (k+\ell)}$. Using this multiplication, the **tensor algebra** is the graded ring

$$T(V) = \bigoplus_{n=0}^{\infty} \left(\bigotimes_{j=1}^{n} V \right).$$

Similarly, we have a graded ring structure on

$$A(V) = \bigoplus_{n=0}^{\infty} A^n(V).$$

If $\omega \in A^k(V)$ and $\eta \in A^\ell(V)$, then we can define $\omega \wedge \eta \in A^{k+\ell}(V)$ by

$$\omega \wedge \eta(\mathbf{v}_1, \dots, \mathbf{v}_{k+\ell}) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \omega(\mathbf{v}_{\sigma(1)}, \dots, \mathbf{v}_{\sigma(k)}) \eta(\mathbf{v}_{\sigma(k+1)}, \dots, \mathbf{v}_{\sigma(k+\ell)}).$$

These can also be defined in terms of universal properties, taking free vector spaces and restricting them in some reasonably natural way.

In order for $A(V)$ to be a graded ring, we need to check one more thing.

Proposition 9.8. *$(\omega, \eta) \mapsto \omega \wedge \eta$ is graded commutative, i.e. $\omega \wedge \eta = (-1)^{\ell k} \eta \wedge \omega$. Thus, $A(V)$ is a graded ring.*

This should remind you of the graded commutativity of the cup product and how it turns cohomology into a graded ring.

The next useful construction for us will be the determinant.

Proposition 9.9.

$$\dim(\Lambda^k V) = \binom{\dim V}{k},$$

and if $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of V, then

$$\{\mathbf{v}_{m_1} \wedge \mathbf{v}_{m_2} \wedge \cdots \wedge \mathbf{v}_{m_k} \mid m_1 < m_2 < \cdots < m_k\}$$

is a basis of $\Lambda^k V$.

In particular, if V is n-dimensional, then $\Lambda^n V$ is one-dimensional (isomorphic, but not canonically, to $\mathbb{R}$), and $\Lambda^m V$ is zero-dimensional if $m > n$.

Definition. Let V be an n-dimensional vector space and $f : V \to V$. Then, the induced $f_* = f^{\vee n} : \Lambda^n V \to \Lambda^n V$ is an element of the one-dimensional vector space $\mathrm{Hom}_{\mathbb{R}}(\Lambda^n V, \Lambda^n V)$, so $f_* = \lambda \cdot \mathrm{id}_*$. Then, the **determinant** of f is defined to be $\det(f) = \lambda$.

Vector Bundles. We want to generalize these notions from vector spaces to vector bundles, so let $E_i \to M$ for $i = 1,2$ be vector bundles with transition functions $t^i_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_{n_i}(\mathbb{R})$. Without loss of generality, assume the charts U_α locally trivialize both E_1 and E_2, which we can always do (trivialize one, then use those to trivialize the other).

For $x \in U_\alpha \cap U_\beta$, $t^1_{\alpha\beta}(x) \otimes t^2_{\alpha\beta}(x) \in \mathrm{GL}_{n_1 \times n_2}(\mathbb{R})$: each component is invertible, so their tensor product is.[4]

Definition. The **tensor product** of E_1 and E_2, denoted $E_1 \otimes E_2$, is the vector bundle whose fibers are given by the fiberwise tensor products of E_1 and E_2 and whose transition functions are $t^1_{\alpha\beta} \otimes t^2_{\alpha\beta}$.

The same operation works for wedge products, Hom, and dual spaces, thanks to functoriality. This can be made general, as long as applying it to the transition functions preserves invertibility. In particular, we also have constructions of:

- $E_1 \oplus E_2$ (the **direct sum**), given by the direct sum on fibers,
- $\Lambda^k E_1$, also defined fiber-by-fiber,
- $\mathrm{Hom}(E_1, E_2)$, defined fiberwise, and
- the **dual bundle** $E_1^* = \mathrm{Hom}(E_1, \mathbb{R})$ (here, $\mathbb{R}$ is the trivial bundle).

Differential Forms. Given a vector bundle $\pi : E \to B$, then the space of sections is written $\Gamma(E) = \{f : B \to E \mid \pi \circ f = \mathrm{id}_B\}$. This just means that for every p, $f(p)$ lies in the bundle above p.

Definition. The space of **differential p-forms** is $\Omega^p(M) = \Gamma(A^p(TM))$, i.e. the smooth sections $f : M \to A^p(TM)$ such that $f(p)$ lies in the fiber above p. Then, define

$$\Omega(M) = \bigoplus_{p=0}^{\infty} \Omega^p(M).$$

This is a graded ring, under pointwise multiplication.

For example, 0-forms are just functions and $A^1(V) = V^*$ (alternating doesn't mean anything if there's only one coordinate). Given an $f : M \to \mathbb{R}$, taking $Df_x : T_x M \to T_{f(x)}\mathbb{R} = \mathbb{R}$. This is a fiber for a 1-form, so given a 0-form f, Df is a 1-form (it's not hard to check that it's smooth). In this context, we'll denote it $\mathrm{d}f \in \Omega^1(M)$.[5]

We will be able to generalize this to higher forms, to obtain a map $d : \Omega^p(M) \to \Omega^{p+1}(M)$. In the book, there's a definition using coordinates, which is messy, and a coordinate-free definition using vector fields and Lie brackets, which is at least coordinate-free but somehow messier. Unfortunately, there doesn't seem to be an easy way around this, which is weird; most constructions in geometry are either elegant but impossible to compute with or easier to get a handle on but depend on charts, and most objects seem to have both constructions. Anyways, here's the coordinate-dependent definition.

Definition. Let $f_i : \mathbb{R}^n \to \mathbb{R}$ be the i^{th} coordinate; then, $\mathrm{d}x_i \in \Omega^1(\mathbb{R}^n)$ is just notation for $\mathrm{d}f_i$.

This can be thought of as the covector in the i^{th} direction. In particular, if $\omega \in \Omega^p(\mathbb{R}^n)$, then

$$\omega = \sum_{1 \le i_1 < \cdots < i_p \le n} f_{i_1 \dots i_p} \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{i_p}, \tag{9.1}$$

where $f_{i_1 \dots i_p} : V \to \mathbb{R}$, which follows from the construction of the dual basis and the basis for $\Lambda^p V$.

Definition. If ω is given as in (9.1), then let $\mathrm{d}\omega \in \Omega^{p+1}(\mathbb{R}^n)$ be defined by

$$\mathrm{d}\omega = \sum_{1 \le i_1 < \cdots < i_p \le n} \mathrm{d}f_{i_1 \dots i_p} \wedge \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{i_p}.$$

This map d is called the **exterior derivative**.

---

[4]Strictly speaking, the tensor product of the transition functions is in $\mathrm{GL}(R^{n_1} \otimes \mathbb{R}^{n_2})$, which is a little more natural, but the dimensions end up the same anyways.

[5]Implicitly stated somewhere in here is that sections of a trivial bundle are just maps into the fiber. This ultimately underscores why functions are 0-forms.

This already looks very coordinate-dependent, and for manifolds it gets even hairier. But in both cases, they are independent of choice of coordinates.

On manifolds, we want to define the exterior derivative locally, which means pulling back from $\mathbb{R}^n$ and stitching the results together using a partition of unity.

Definition. Let M be a smooth manifold and $\omega \in \Omega^(M)$. Let (ϕ, U) be a chart on M and suppose $m \in U$. If $\mathbf{v}_1, \dots, \mathbf{v}_{p+1} \in T_p M$, then let

$$\mathrm{d}\omega(\mathbf{v}_1, \dots, \mathbf{v}_{p+1}) = (\mathrm{d}(\phi^* \omega))(D\phi^{-1}(\mathbf{v}_1), \dots, D\phi^{-1}(\mathbf{v}_{p+1})).$$

This is a reasonable definition because D is locally a bijection, so we get a $\mathrm{d}\omega \in \Omega^{p+1}(M)$.

There are a couple questions of well-definedness here, which will be on the homework.

10. de Rham Cohomology, Integration of Differential Forms, and Stokes' Theorem: 5/7/15

Recall that last time, we defined the space $\Omega^p(M)$ of differential forms to be the space of sections of the dual space $A^p(M) = (\Lambda^p TM)^*$, and defined an operator $\mathrm{d} : \Omega^p(M) \to \Omega^{p+1}(M)$.

This operator will turn $\Omega^\bullet(M)$ into a chain complex.

Proposition 10.1. $\mathrm{d}^2 = 0$.

Proof. Since d is determined locally, then it suffices to prove it for $\mathbb{R}^n$, and since it's linear, it suffices to prove it on forms that look like

$$\omega = f\, \mathrm{d}x_{m_1} \wedge \cdots \wedge \mathrm{d}x_{m_p}.$$

Then,

$$\begin{aligned} \mathrm{d}\omega &= \mathrm{d}f \wedge \mathrm{d}x_{m_1} \wedge \cdots \wedge \mathrm{d}x_{m_p} \\ &= \left(\sum_i \frac{\partial f}{\partial x_i} \mathrm{d}x_i \right) \wedge \mathrm{d}x_{m_1} \wedge \cdots \wedge x_{m_p}, \end{aligned}$$

so

$$\mathrm{d}(\mathrm{d}\omega) = \sum_i \sum_j \frac{\partial^2 f}{\partial x_i \partial x_j} (\mathrm{d}x_j \wedge \mathrm{d}x_i) \wedge (\mathrm{d}x_{m_1} \wedge \cdots \wedge x_{m_p}). \tag{10.1}$$

Looking at (10.1) more closely, notice that if you switch i and j as indices in the sum, you get the same result, but if you switch $\mathrm{d}x_i \wedge \mathrm{d}x_j$ to $\mathrm{d}x_j \wedge \mathrm{d}x_i$, you have to add a minus sign. Thus, $\mathrm{d}(\mathrm{d}\omega) = -\mathrm{d}(\mathrm{d}\omega)$, so it's equal to zero. ⊠

The correct thing to do with a chain complex is to take its cohomology.

Definition.

- A p-form ω is called **closed** if $\mathrm{d}\omega = 0$ (i.e. it lies in ker(d)).
- A p-form ω is called **exact** if $\omega = \mathrm{d}\eta$ for a $(p-1)$-form η (i.e. it lies in Im(d)).
- The **de Rham cohomology** of a manifold M is the closed forms modulo the exact ones, i.e.

$$H^p_{\mathrm{dR}}(M) = \frac{\ker(\mathrm{d} : \Omega^p(M) \to \Omega^{p+1}(M))}{\operatorname{Im}(\mathrm{d} : \Omega^{p-1}(M) \to \Omega^p(M))}.$$

This corresponds to other kinds of cohomology, e.g. the obstruction to a closed form being non-exact has a lot to do with nontrivial loops in the surface. In fact, we'll prove the following theorem next lecture.

Theorem 10.2 (de Rham)**.** $H^p_{\mathrm{dR}}(M) \cong H^p(M; \mathbb{R})$.

Differential forms can naturally be integrated: a p-form defines a p-dimensional volume on p-dimensional submanifolds of M. Thus, we want some sort of integration map to be the isomorphism in Theorem 10.2, and what we'll end up using is

$$(\omega, f : \Delta^p \to M) \longmapsto \int_{\Delta^p} f^* \omega,$$

which defines a map $\Omega^p(M) \otimes C^p(M) \to \mathbb{R}$, which allow adjointness to prove things for us. So it behooves us to learn how to integrate p-forms.

One interesting corollary is that de Rham cohomology, even though it looks like it depends on the smooth structure of the manifold, is actually homotopy invariant! If you were looking for something stronger, well, maybe it's sad.

For integration, we'll always want M to be an oriented, n-dimensional manifold. There are many definitions of orientability; the book defines it as having charts with transition functions with positive determinant, and that ends up being equivalent to the one we gave earlier in the class.

Definition. Let $\Omega_c^n(M) = \Gamma_c(A^n(M) \to M)$, i.e. the **compactly supported n-forms**, which are equal to the zero section outside of a compact subset of M.

Our goal will be to define a map $\int : \Omega_c^n(M) \to \mathbb{R}$. We'll do it in steps.

Let $U \subseteq \mathbb{R}^n$ be open and $\omega \in \Omega_c^n(U)$. Then, there's a unique $f : \mathbb{R}^n \to \mathbb{R}$ such that $f(\mathbf{x}) = 0$ when $\mathbf{x} \notin U$ and $\omega = f\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$. Then, define

$$\int_U \omega = \int_{\mathbb{R}^n} f\,\mathrm{d}x_1\,\mathrm{d}x_2 \cdots \mathrm{d}x_n.$$

Notice that this depends on the choice of orientation of $\mathbb{R}^n$, so if you switch the order of x_1 and x_2, the sign changes.

Proposition 10.3. *If $V \subseteq \mathbb{R}^n$ and $\varphi : V \to U$ is an orientation-preserving diffeomorphism, then*

$$\int_V \varphi^* \omega = \int_U \omega.$$

This will end up being true for general manifolds as well.

Proof.

$$\begin{aligned}
\varphi^*(f\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n) &= (f \circ \varphi)(\varphi^*(\mathrm{d}x_1) \wedge \cdots \wedge \varphi^*(\mathrm{d}x_n)) \\
&= (f \circ \varphi)(\mathrm{d}x_1 D\varphi \wedge \cdots \wedge \mathrm{d}x_n D\varphi) \\
&= (f \circ \varphi)\det(D\varphi^{\mathrm{T}})(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n),
\end{aligned}$$

which just follows from the definition of the determinant as coming from the top exterior power. But in multivariable calculus, we learned that

$$\int_V f \circ \varphi |\det D\varphi|\,\mathrm{d}x_1 \cdots \mathrm{d}x_n = \int_U f\,\mathrm{d}x_1 \cdots \mathrm{d}x_n, \tag{10.2}$$

and since φ is orientation-preserving, then $\det(D\varphi^{\mathrm{T}}) = |\det D\varphi|$, so the left side of (10.2) is equal to $\int_V \varphi^* \omega$ and the right side is equal to $\int_U \omega$. ⊠

Notice that this proof heavily leans on the fact that $T\mathbb{R}^n$ has a trivialization, and the multivariable calculus fact of (10.2). On the other hand, in physics classes people prove this by drawing pictures and waving their hands, so I guess it could be worse (well, is less rigorous and more intuitive worse? It depends on who you are).

Definition. Let M be an n-dimensional manifold and $\phi : \mathbb{R}^n \to M$ be an orientation-preserving chart map. If $\omega \in \Omega^p(M)$ such that $\operatorname{supp}(\omega) \subseteq \operatorname{Im}(\phi)$, then define

$$\int_M \omega = \int_{\mathbb{R}^n} \phi^* \omega.$$

Proposition 10.4. *The above definition is well-defined, independent of the choice of chart ϕ.*

Proof. Let ϕ_1 and ϕ_2 be two orientation-preserving chart maps whose images both contain $\operatorname{supp}\omega$. Let $V = \phi_1(\mathbb{R}^n) \cap \phi_2(\mathbb{R}^n)$, so $\phi_1^{-1} \circ \phi_2 : \phi_1^{-1}(V) \to \phi_2^{-1}(V)$. Then,

$$\begin{aligned}
\int_{\mathbb{R}^n} \phi_1^* \omega &= \int_{\phi^{-1}(V)} \phi_1^* \omega \\
&= \int_{\phi_2^{-1}(V)} (\phi_1^{-1} \circ \phi_2)^* (\phi_1^* \omega) \\
&= \int_{\phi_2^{-1}(V)} \phi_2^* \omega = \int_{\mathbb{R}^n} \phi_2^* \omega.
\end{aligned}$$

⊠

Notice that, even though we've been doing stuff that's in theory smooth, we can do everything a little more generally, e.g. integrating continuous functions or forms, rather than just smooth ones.

Now, let's finally define the integral in full generality, as opposed to for forms supported by just one chart.

Definition. Let $\omega \in \Omega_c^n(M)$ for an oriented n-dimensional manifold M. Let $\{U_i\}$ be a locally finite covering of M be charts, and $\{f_i\}$ be a partition of unity subordinate to $\{U_i\}$. Then, we can define

$$\int_M \omega = \sum_i \int_M f_i \omega.$$

Proposition 10.5. *The above definition is independent of the choice of* $\{U_i\}$ *and* $\{f_i\}$.

Proof. Let $\{V_i\}$ be another such locally finite covering by charts and $\{g_i\}$ be a partition of unity subordinate to $\{V_i\}$. Then,

$$\begin{aligned}\sum_i \int_M f_i \omega &= \sum_i \int_M \left(\sum_j g_j\right) f_i \omega \\ &= \sum_{i,j} \int_M g_j f_i \omega \\ &= \sum_j \int_M \left(\sum_i f_i\right) g_j \omega \\ &= \sum_j \int_M g_j \omega.\end{aligned}$$ ⊠

By unwinding the definitions, we can apply Proposition 10.3 more generally.

Corollary 10.6. *If* M *and* N *are* n*-dimensional manifolds and* $\omega : M \to N$ *is an orientation-preserving diffeomorphism and* $\omega \in \Omega_c^n(N)$*, then*

$$\int_M \varphi^* \omega = \int_N \omega.$$

The remainder of the class will cover Stokes' theorem. This involves manifolds-with-boundary, so we have to define integration of forms on manifolds-with-boundary. It turns out this works in almost exactly the same way; we just get subsets of $\mathbb{R}^n$ that might not be open, but that's fine.

Proposition 10.7. *Let* M *be an oriented manifold-with-boundary; then, there is a canonical choice of orientation for* ∂M.

Proof. Let $M' = M \cup_{\partial M} (\partial M \times [0,1))$; that is, we glue along the boundary. The proof of the tubular neighborhood theorem implies that $M' \cong M \setminus \partial M$. M' can be thought of as having a collar around the boundary of M.

An orientation of M canonically determines an orientation of M', because if $(x,r) \in M'$, there are canonical isomorphisms

$$H_n(M', M' \setminus \{(x,r)\}) \cong H_n(\partial M \times (0,1), \partial M \times (0,1) \setminus (x,r)) \cong H_{n-1}(\partial M, \partial M \setminus x).$$ ⊠

More geometrically, this is akin to orienting ∂M along the outward-pointing unit normal.

Now, we can state the theorem itself.

Theorem 10.8 (Stokes)**.** *Let* $\omega \in \Omega_c^{n-1}(M)$*. Then,*

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega.$$

Proof. Of course, we'll use a partition of unity for this. It will allow us to distill the proof into two cases: charts that don't intersect the boundary (where $\int \mathrm{d}\omega = 0$), and those where it does intersect the boundary, where it will just take value on the boundary.

Lemma 10.9. *Let* $\omega \in \Omega_c^{n-1}(\mathbb{R}^n)$*. Then,* $\int_{\mathbb{R}^n} \mathrm{d}\omega = 0$.

Proof. Write ω as

$$\omega = \sum_i \underbrace{f_i \,\mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n}_{\omega_i},$$

where $\widehat{\mathrm{d}x_i}$ indicates that index is missing. Then, we throw calculus at it:

$$\int_{\mathbb{R}^n} \mathrm{d}\omega_i = \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_i} \,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$
$$= \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \frac{\partial f}{\partial x_i} \,\mathrm{d}x_i \right) \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n.$$

However,

$$\int_{\mathbb{R}} \frac{\partial f}{\partial x_i} \,\mathrm{d}x_i = \lim_{r\to\infty} \int_{-r}^{r} \frac{\partial f}{\partial x_i} \,\mathrm{d}x_i$$
$$= \lim_{r\to\infty} f(x_1, \dots, r, \dots, x_n) - f(x_1, \dots, -r, x_n)$$
$$= 0,$$

since ω is compactly supported. ⊠

Something very similar happens for the charts on the boundary.

Lemma 10.10. *Let $H = \{x_1 \ge 0\} \subseteq \mathbb{R}^n$. Then,*

$$\int_H \mathrm{d}(f \,\mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n) = \int_{\partial H} f \,\mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n.$$

This has nearly the same proof; just treat $i = 1$ separately, to recover the boundary, and as a corollary, for any compactly supported ω,

$$\int_H \mathrm{d}\omega = \int_{\partial H} \omega.$$

Then, stitch these together using a partition of unity. ⊠

11. de Rham's Theorem: 5/12/15

"I won't say anything negative about group theory."

Recall that we were in the middle of proving Stokes' theorem, that given an inclusion $i : \partial M \to M$ which determines the orientation of ∂M from an orientation of M and an $\omega \in \Omega_c^{n-1}(M)$, we have that

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega.$$

We were dealing with the case where $M = H = \{(x_1, \dots, x_n) \mid x_1 \ge 0\} \subseteq \mathbb{R}^n$. If $\omega \in \Omega_c^{n-1}(H)$, then

$$\omega = \sum f_i \,\mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n,$$

so

$$i^*\omega = (f_1|_{\partial H}) \,\mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n.$$

Then, integrate.

$$\int_{\partial H} i^*\omega = \int_{\partial H} -f_1|_{\partial H} \,\mathrm{d}x_2 \,\mathrm{d}x_3 \cdots \mathrm{d}x_n.$$

The sign comes from the induced orientation of ∂H.

Next, we calculate

$$\mathrm{d}\omega = \sum_i \left(\sum_j \frac{\partial f_i}{\partial x_j} \,\mathrm{d}x_j \right) \wedge \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n,$$

which is only nonzero when no index is repeated, i.e. $i = j$; then, rearranging the forms requires a sign factor:

$$= \sum_i (-1)^{i+1} \frac{\partial f_i}{\partial x_i} \,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n.$$

For $i \ge 2$, we know that

$$\int_H \frac{\partial f_i}{\partial x_i} \,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n = 0,$$

and for $i=1$, we integrate by parts.

$$\begin{aligned}
\int_H \frac{\partial f_1}{\partial x_1}\,\mathrm{d}x_1\wedge\cdots\wedge \mathrm{d}x_n &= \int_H \frac{\partial f_1}{\partial x_1}\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_n\\
&= \int_{\partial H}\left(\int_0^\infty \frac{\partial f_1}{\partial x_1}\,\mathrm{d}x_1\right)\mathrm{d}x_2\cdots\mathrm{d}x_n\\
&= \int_{\partial H}\left(\lim_{r\to\infty}\frac{\partial f_1}{\partial x_1}(r,x_2,\dots,x_n)-\frac{\partial f_1}{\partial x_1}(0,x_2,\dots,x_n)\right)\mathrm{d}x_2\cdots\mathrm{d}x_n\\
&= \int_{\partial H} -\left.\frac{\partial f_1}{\partial x_1}\right|_{\partial H}\mathrm{d}x_2\cdots\mathrm{d}x_n.
\end{aligned}$$

Then, the general case follows using a partition of unity.

de Rham's theorem. In order to show that de Rham cohomology is isomorphic to singular cohomology, we'll start by defining cohomology on smooth simplices $H^*_{C^\infty}(M;\mathbb{R})$. Then, inclusion of smooth simplices into all simplices creates a wrong-way map $H^*(M;\mathbb{R})\to H^*_{C^\infty}(M;\mathbb{R})$, and integration will give us a map $H^*_{\mathrm{dR}}(M)\to H^*_{C^\infty}(M)$. Next, we'll show that all three satisfy a Mayer-Vietoris sequence and that they agree on $\mathbb{R}^n$, which gives us enough material to make an inductive gluing argument.

Proposition 11.1. *Let U and V be open in M, and let $i_U : U\hookrightarrow U\cup V$, $i_V : U\hookrightarrow U\cup V$, $j_U : U\cap V\hookrightarrow U$, and $j_V : U\cap V\hookrightarrow V$ be inclusions. Then, the following sequence is short exact.*

$$0\longrightarrow \Omega^p(U\cup V)\xrightarrow{i_U^*\oplus i_V^*}\Omega^p(U)\oplus\Omega^p(V)\xrightarrow{j_U^*-j_V^*}\Omega(U\cap V)\longrightarrow 0$$

A short exact sequence on the chain level induces a long exact sequence in homology, which we already know from homological algebra, so this proposition is all we need for the Mayer-Vietoris sequence to exist.

Proof. Clearly, $i_U^*\oplus i_V^*$ is injective, and the sequence is exact at $\Omega^p(U)\oplus\Omega^p(V)$, because something is in $\mathrm{Im}(i_U^*\oplus i_V^*)$ iff its two coordinates agree on $U\cap V$, which is equivalent to $j_U^*-j_V^*$ sending it to zero. Thus, all that's left is surjectivity.

Let $\omega\in\Omega^p(U\cap V)$, and let $f : M\to[0,1]$ be a smooth function which is 0 on a neighborhood of $U\setminus V$ and 1 on a neighborhood of $V\setminus U$; in particular, if $\eta = f\omega+(1-f)\omega$, then $f\omega$ extends to a form $\alpha\in\Omega^p(U)$ and $-(1-f)\omega$ extends to a form $\beta\in\Omega^p(V)$, so $(j_U^*-j_V^*)(\alpha,\beta)=\eta$.[6] ⊠

Corollary 11.2. *There is a long exact sequence*

$$\cdots\longrightarrow H^p_{\mathrm{dR}}(U\cup V)\longrightarrow H^p_{\mathrm{dR}}(U)\oplus H^p_{\mathrm{dR}}(V)\longrightarrow H_{\mathrm{dR}}(U\cap V)\longrightarrow H^{p-1}_{\mathrm{dR}}(U\cup V)\longrightarrow\cdots$$

We'll call this sequence the **Mayer-Vietoris sequence**, by analogy with singular cohomology.

Lemma 11.3 (Poincaré)**.**

$$H^*_{\mathrm{dR}}(\mathbb{R}^n)=\begin{cases}\mathbb{R}, & *=0\\ 0, & *>0.\end{cases}$$

We'll show this by first showing de Rham cohomology is invariant under smooth homotopy, and then calculating the de Rham cohomology of a point.

Proposition 11.4.

$$H^*_{\mathrm{dR}}(\mathrm{pt})=\begin{cases}\mathbb{R}, & *=0\\ 0, & *>0.\end{cases}$$

Proof. $\Omega^p(\mathrm{pt})=\Gamma(\Lambda^p(0)^*\to\mathrm{pt})$, which is only nonzero when $p=0$, where it's equal to $\mathbb{R}$. Thus, $\mathrm{d}=0$ for each p, and therefore the chain complex is equal to its own homology. ⊠

A fancy way of saying this is "a point is formal," whatever that means.

Definition.

- Two smooth functions $f,g : M\to N$ are **smoothly homotopic** if there exists a smooth $H : M\times\mathbb{R}\to N$ with $H(-,0)=f$ and $H(-,1)=g$.

[6] It's important to show that such a f exists; in particular, we might have gotten the definition wrong in lecture.

- Let C^* and D^* be cochain complexes.[7] Then, a **chain homotopy** between $f^*, g_*: C^* \to D_*$ is a collection of maps $H_n: C_n \to D_{n-1}$ such that $f - g = dH - Hd$.

Recall the following proposition from 210A; now, it'll actually be useful!

Proposition 11.5. *Chain homotopic maps induce the same map in homology.*

And the final puzzle piece:

Proposition 11.6. *If $f, g: M \to N$ are smoothly homotopic, then $f^*, g^*: \Omega^*(N) \to \Omega^*(M)$ are chain homotopic.*

Proof of Lemma 11.3. We'll prove Poincaré's lemma assuming Proposition 11.6, and then go back and prove Proposition 11.6.

We know that $\mathrm{id}_{\mathbb{R}^n}$ is homotopic to 0 via the homotopy $H_t(\mathbf{x}) = t\mathbf{x}$. Since de Rham cohomology is functorial, then the commutative diagram

$$\begin{tikzcd} \mathbb{R}^n \arrow[rr, "\mathrm{id}"] \arrow[dr, "0"'] & & \mathbb{R}^n \\ & \mathrm{pt} \arrow[ur, "i"'] & \end{tikzcd}$$

is sent to the diagram

$$\begin{tikzcd} H^*_{\mathrm{dR}}(\mathbb{R}^n) & & H^*_{\mathrm{dR}}(\mathbb{R}^n) \arrow[ll, "\cong", "\mathrm{id}^*"'] \arrow[dl, "i^*"] \\ & H^*_{\mathrm{dR}}(\mathrm{pt}). \arrow[ul, "0^*"] & \end{tikzcd}$$

Thus, since $0^* \circ i^* = \mathrm{id}$, then i^* must be injective and 0^* must be surjective, so since these are real vector spaces, $H^*_{\mathrm{dR}}(\mathrm{pt}) \cong H^*_{\mathrm{dR}}(\mathbb{R}^n)$. ⊠

Lemma 11.7. *Let $i_t: M \to M \times \mathbb{R}$ be inclusion at t, i.e. $x \mapsto (x, t)$. Then, i_0^* is chain homotopic to i_1^*.*

Proof. Let $h: \Omega^p(M \times I) \to \Omega^{p-1}(M)$ be given by

$$h(\omega)(\mathbf{v}_1, \dots, \mathbf{v}_{p-1}) = \int_0^1 \omega\left(\frac{\mathrm{d}}{\mathrm{d}t}, \mathbf{v}_1, \dots, \mathbf{v}_{p-1}\right) \mathrm{d}t,$$

where $\mathbf{v}_1, \dots, \mathbf{v}_{p-1} \in T_m M$ and $\frac{\mathrm{d}}{\mathrm{d}t}$ is the tangent vector in the $\mathbb{R}$-direction. Then, h is a chain homotopy between $\Omega^*(M \times \mathbb{R})$ and $\Omega^*(M)$. ⊠

The proof is a calculation that you can check yourself. But we'll leverage it for the rest of the proof.

Proof of Proposition 11.6. Let $H: M \times \mathbb{R} \to N$ be a smooth homotopy from f to g, i.e. $f = H \circ i_0$ and $g = H \circ i_1$. Then, let $h^i_p: \Omega^p(M \times \mathbb{R}) \to \Omega^{p-1}(M)$ be a chain homotopy from $i_0^* \to i_1^*$, and let $h_p = h^i_p \circ H^*$. Then, since both of the ingredients are chain homotopies, then h_p is, and it's a chain homotopy between f and g. ⊠

Thus, we've now proven everything we need to for Poincaré's lemma, and these are all of the facts we need about de Rham cohomology to prove de Rham's theorem.

Definition. Let $C_p^{C^\infty}(M; G)$, the **smooth chains** on M, be the free abelian group on $C^\infty(\Delta^p \to M)$ with coefficients in G. $C_p^{C^\infty}(M; G)$ is a subcomplex of $C_p(M; G)$, so we can define **smooth homology**

$$H_p^{C^\infty}(M; G) = H_p(C^{C^\infty}(M; G))$$

and **smooth cohomology**

$$H^p_{C^\infty}(M; G) = H^p(C^{C^\infty}(M; G)).$$

[7] This means the boundary maps increase index; if they decrease index, the chain homotopy should go in the other direction.

Note that inclusion of chains gives us a map $i : H_p^{C^\infty}(M) \to H_p(M)$, and similarly for cohomology (in the same direction, moreover). We also have a map

$$\int : \Omega^p(M) \to C^p_{C^\infty}(M;\mathbb{R}) = \mathrm{Hom}_{\mathbb{Z}}(C_p^{C^\infty}(M);\mathbb{R})$$

defined by a bilinear pairing $\Omega^p(M) \times C_p^{C^\infty}(M) \to \mathbb{R}$ in which

$$(\omega \in \Omega^p(M), f : \Delta^p \to M) \mapsto \int_{\Delta^p} f^*\omega.$$

Then, using the same proof method as for Lemma 11.3, we get the following important result.

Proposition 11.8.

$$H^*_{C^\infty}(\mathbb{R}^n;\mathbb{R}) = \begin{cases} \mathbb{R}, & * = 0 \\ 0, & * > 0. \end{cases}$$

12. Curl and the Proof of de Rham's Theorem: 5/14/15

> *"I asked someone... what do [swimmers] do when there are thunderstorms here? They said there aren't thunderstorms in the Bay Area."*

Today we're going to finish the proof of de Rham's theorem, and then talk about curl.

Proposition 12.1. *If P is a statement about smooth manifolds such that:*

(1) $P(\mathbb{R}^n)$ is true;
(2) if M and N are smoothly homotopic, then $P(M)$ iff $P(N)$;
(3) if $U, V \subseteq M$ and $P(U)$, $P(V)$, and $P(U \cap V)$ are true, then $P(U \cup V)$ is true; and
(4) if $P(M_\alpha)$ is true for all $\alpha \in I$, then $P\left(\coprod_\alpha M_\alpha\right)$ is true;

then $P(M)$ is true for all smooth manifolds M.

Notice that the last two properties are different: we can take finite unions with overlap, or arbitrary disjoint unions. This isn't quite the same as the examples we had with homology in 215B, but cohomology doesn't play quite as well with colimits as homology does, so it's not quite as simple.

Corollary 12.2. *For any manifold M, $H^*_{\mathrm{dR}}(M) \xrightarrow{\int} H^*_{C^\infty}(M)$ and $H^*(M;\mathbb{R}) \xrightarrow{\cong} H^*_{C^\infty}(M)$ are both isomorphisms of graded $\mathbb{R}$-vector spaces.*

Proof. Let $P(M)$ be that this corollary is true for M; then, we'll use Proposition 12.1. The Poincaré lemma gives us (1), and Proposition 11.6 gives us (2).

For (3), we'll use the long exact sequence and the Five lemma. Specifically, we have the following commutative diagram, thanks to Corollary 11.2.

$$\begin{tikzcd}
H^k_{\mathrm{dR}}(U) \oplus H^k_{\mathrm{dR}}(V) \arrow[r] \arrow[d, "\wr"] & H^k_{\mathrm{dR}}(U \cap V) \arrow[r] \arrow[d, "\wr"] & H^{k+1}_{\mathrm{dR}}(U \cup V) \arrow[r] \arrow[d] & H^{k+1}_{\mathrm{dR}}(U) \oplus H^{k+1}_{\mathrm{dR}}(V) \arrow[r] \arrow[d, "\wr"] & H^{k+1}_{\mathrm{dR}}(U \cap V) \arrow[d, "\wr"] \\
H^k_{C^\infty}(U) \oplus H^k_{C^\infty}(V) \arrow[r] & H^k_{C^\infty}(U \cap V) \arrow[r] & H^{k+1}_{C^\infty}(U \cup V) \arrow[r] & H^{k+1}_{C^\infty}(U) \oplus H^{k+1}_{C^\infty}(V) \arrow[r] & H^{k+1}_{C^\infty}(U \cap V)
\end{tikzcd}$$

Then, the Five lemma guarantees the middle map is an isomorphism as well.

Finally, for (4), cohomology sends disjoint unions to products, since it's contravariant, so the isomorphisms go through. ⊠

This is the result of de Rham's theorem, but we need to show that our induction of open sets is valid.

Proof of Proposition 12.1. By (1) and (2), P is true for all convex open sets in $\mathbb{R}^n$, and by (3), it's true for finite unions of convex open sets in $\mathbb{R}^n$.

Let $M \subseteq \mathbb{R}^n$ be open. By taking the one-point compactification of $\mathbb{R}^n$, which is $\mathbb{R}^n \cup \{\infty\} = S^n$, we can get a smooth proper map $f : M \to [0,\infty)$, e.g. $f(x) = 1/\operatorname{dist}(x,\infty)$. Let $A_n = f^{-1}([n, n+1])$; then, it is possible to cover A_n by finitely many convex open sets U_i^n such that

$$V_n = \bigcup_i U_i^n \subseteq f^{-1}\left(\left[n - \frac{1}{3}, n + \frac{1}{3}\right]\right).$$

Thus, we know that it's true for V_n. Moreover, V_n and V_{n+2} never intersect, so we also know it's true for $\bigcup_{n \text{ odd}} V_n$ and $\bigcup_{n \text{ even}} V_n$, and it's also true for their intersection, so by (3), it's also true for their union, which is M.

The general case, where M isn't open in $\mathbb{R}^n$, isn't much different: if M is compact, then we can cover it with finitely many open sets, which is fine, and if it isn't, the same trick with a proper map $f : M \to [0,\infty)$ works. ⊠

Curl. In the textbook, curl is defined in a rather coordinate-dependent way, but we can do better than that.

Definition. An **orientation form** on an n-dimensional manifold is a nowhere-vanishing $\omega \in \Omega^n(M)$.

So why is this called an orientation form?

Proposition 12.3. *A smooth manifold is orientable iff it has an orientation form, and moreover, there is a bijection between orientations of M and orientation forms up to multiplication by a positive function.*

This equates yet another definition of orientability with ours, though there are yet more...

Proof. We know that M is orientable iff TM is, and we also know that $A^n(TM)$ is trivial iff there exists an orientation form: in the forward direction, choose the section given by 1 everywhere. In the reverse direction, given an orientation form ω, let $T_\omega : M \times \mathbb{R} \to A^n(TM)$ be defined by $T_\omega(m,r) = r\omega_m$. Thus, we get a commutative diagram that trivializes $A^n(TM)$.

$$\begin{tikzcd} \mathbb{R} \times M \arrow[r, "T_\omega"] \arrow[d] & A^n(TM) \arrow[dl] \\ M & \end{tikzcd}$$

In particular, we want that $A^n(TM)$ is orientable iff the orientation bundle TM_{or} is (i.e. the one parameterized by generators of $H_n(TM, TM \setminus 0)$). The transition functions for TM_{or} are $t^{\mathrm{or}}_{\alpha\beta} = \operatorname{sign}\det \phi_\alpha \phi_\beta^{-1}$, which defines a function $U_\alpha \cap U_\beta \to \mathbb{Z}/2 = \pi_0(\mathbb{R}^*)$, and the transition functions for $A^n(M)$ are $t^A_{\alpha\beta} = \det \phi_\alpha \phi_\beta^{-1}$, which is a function $U_\alpha \cap U_\beta \to \mathbb{R}^*$.

We'll have to finish the rest of the proof on Tuesday; it depends on a little bit of homotopy invariance. ⊠

To define curl, we also need a Riemannian metric.

Definition. A Riemannian metric is a section $\langle\cdot,\cdot\rangle$ of $TM^* \otimes TM^* \to M$ such that $\langle\cdot,\cdot\rangle_m$ is an inner product on $T_m M$ for all $m \in M$.

This allows us to define lengths of tangent vectors, curves, etc. One can show that the space of Riemannian metrics on a manifold is contractible, and in particular, one always exists (which uses convexity and a partition of unity). For the rest of this discussion, fix a Riemannian metric on our n-dimensional manifold M.

Definition. An orientation form ω is called a **volume form** if $\omega_m(b_1,\dots,b_n) = \pm 1$ for every orthonormal basis $\{b_1,\dots,b_n\}$ of $T_m M$ for every $m \in M$.

Proposition 12.4. *There exists a unique volume form equivalent to any given orientation form.*

Proof. Let $f : M \to (0,\infty)$ be given by $f(m) = |\omega_m(b_1,\dots,b_n)|$, where $\{b_1,\dots,b_n\}$ is an orthonormal basis. If $a_1,\dots,a_n$ is another orthonormal basis, then there's a linear map $T : T_m M \to T_m M$ with $T(a_i) = b_i$, and therefore T is orthogonal, so $\det(T) = \pm 1$. In particular,

$$\begin{aligned} |\omega_m(b_1,\dots,b_n)| &= |\omega_m(Ta_1,\dots,Ta_n)| \\ &= |\det(T)\omega_m(a_1,\dots,a_n)| \\ &= |\omega_m(a_1,\dots,a_n)|. \end{aligned}$$

It remains to show f is smooth, but this is true, and then our volume form is ω/f. ⊠

There's a formula for the volume form in terms of the metric, which differential geometers maybe care about.

Proposition 12.5. *Let V be a volume form; then, in local coordinates, the metric is given by $g_{ij}(\cdot,\cdot)$, and V has the formula*

$$V = \sqrt{\det(g_{ij})}\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n.$$

Definition. Fix a metric $\langle\cdot,\cdot\rangle$ and a volume form V; then, the **Hodge star** $*:\Omega^p(M)\to\Omega^{n-p}(M)$ be the unique linear map such that if $\alpha,\beta\in\Omega^p(M)$, then $\alpha\wedge(*\beta)=\langle\alpha,\beta\rangle_p V$. Here, $\langle\cdot,\cdot\rangle_p$ is the induced metric on $\Omega^p(M)$ from the Riemannian metric, using functoriality, where

$$\langle v_1\wedge\cdots\wedge v_p, w_1\wedge\cdots\wedge w_p\rangle = \begin{vmatrix} & \langle v_i, w_j\rangle & \end{vmatrix}.$$

For example, on $\mathbb{R}^3$, $*\mathrm{d}x=\mathrm{d}y\wedge\mathrm{d}z$, because $\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z$ is the standard volume form on $\mathbb{R}^3$.

Now, we can define curl in a coordinate-free way.

Definition. The **curl** of α is $\mathrm{curl}(\alpha)=*(\mathrm{d}\alpha)$.

13. Surjectivity of the Pontryagin-Thom Map: 5/19/15

Recall from a few lectures back that we talked about the Pontryagin-Thom construction.[8]

If M is an n-dimensional manifold, $M\hookrightarrow\mathbb{R}^{n+k}\subseteq S^{n+k}$, and therefore sending the manifold to the normal bundle is a map $M^n\to\mathrm{Gr}_k(\mathbb{R}^{n+k})$; then, we can embed this into $\mathrm{Gr}(\mathbb{R}^\infty)=B\mathrm{O}(k)$, to make everything suitably homotopy-invariant. There's some stuff about classifying vector bundles here, too.

We also have a map from $S^{n+k}\to\tau(N)$ (the Thom space of the normal bundle), which embeds into $\tau(\gamma_k)$, where γ_k is the canonical bundle of $\mathrm{Gr}_k(\mathbb{R}^\infty)$. But $\tau(\gamma_k)=\mathrm{MO}(k)$, and therefore we have a class in $\pi_{n+k}(\mathrm{MO}(k))$, which becomes a class in $\pi_n(\mathrm{MO})$.

So why on Earth is this surjective? Suppose $[f]\in\pi(\mathrm{MO})$; then, by the definition of a spectrum, this means $[f]\in\pi_{n+k}(\mathrm{MO}(k))$ for some n and k, i.e. we have a map $f:S^{n+k}\to\mathrm{MO}(k)$.

We would like to cook up two maps $f_1,f_2:S^{n+k}\rightrightarrows\tau(\gamma_k^s\to\mathrm{Gr}(\mathbb{R}^{k+s}))$, where $f_1\simeq f_2$, $f_2\pitchfork\mathrm{Gr}_k(\mathbb{R}^{k+s})$, and f_2 is smooth on a neighborhood of the Grassmanian. One of the important steps in the proof is showing that these maps exist. We have a map $g:M\to\mathrm{Gr}_k(\mathbb{R}^{n+k})$ that sends a point to its normal bundle, and we'll want $f_2|_M:M\to\mathrm{Gr}_k(\mathbb{R}^{k+s})$. You can obtain this by taking a tubular neighborhood N of M, and mapping N to γ_k^{k+s}, the canonical bundle of $\mathrm{Gr}_k(\mathbb{R}^{k+s})$, but this still isn't super illuminating. We want to use the fact that $T_p\gamma_k^{k+s}=T\,\mathrm{Gr}_k(\mathbb{R}^k)\oplus(\gamma_k^{k+s})_p$, which is a consequence of a more general fact about the tangent space of a vector bundle.

Once we have these maps, we'll send $[f]\mapsto f_2^{-1}(\mathrm{Gr}_k(\mathbb{R}^{k+s}))$, which is just M.

We'd like to show something akin to the following criterion, which makes intuitive sense and is true when $n=1$ and $k=1$, but may not be true more generally: that an $h:M\to\mathrm{Gr}_k(\mathbb{R}^{n+k})$ is homotopic to g if $h(m)+TM(m)=\mathbb{R}^{n+k}$.

Let's boil this down some more. The transversality of f_2 means that $Df_2(m)+T_{f_2(m)}\mathrm{Gr}_k(\mathbb{R}^{k+s})=T_{f_2(m)}\gamma_k^{k+s})$. The dimension of the Grassmanian is $k(n-k)$.

Additionally, we have the fact that $\pi_0\,\mathrm{Map}(M,B\mathrm{O}(k))$ is the set of k-dimensional vector bundles over M, up to isomorphism. Using that as a black box, we take $V_1=f_2^*\gamma_k^{k+s}\to M$, and using $i:M\to S^{n+k}$, letting $V_2=N(i)=i^*TS^{n+k}/TM\to M$, then we want to show that V_1 and V_2 are stably isomorphic, i.e. there exist $a,b\in\mathbb{N}$ such that $\mathbb{R}^b\oplus V_1\cong V_2\oplus\mathbb{R}^a$. But using the black box, finding a bundle isomorphism means that they're the same class in π_0, i.e. are homotopic.

It's possible to show that $f_2^{-1}(\gamma_k^{k+s})$ is a tubular neighborhood of M, which is a useful next step.

At this point, things got confused and we tried to start again.

We have $M\subset\mathbb{R}^{n+k}\subset S^{n+k}$, so let's look at the normal bundle N of M in this space, which is k-dimensional. M maps into the Grassmanian as the 0-section, which we can identify with $\mathrm{Gr}_k(\mathbb{R}^{k+s})$, which we will call f_2. Let $\pi:\gamma_k^{s+k}\to\mathrm{Gr}_k(\mathbb{R}^{n+k})$; then, we know that $Df_2(N_p)\subset\ker(\pi_*)$, which makes sense when you think about what normality passes to. But $\ker(\pi_*)$ can be identified with the fiber of the vertical plane at $f_2(p)$. This is the identification we were hoping for: the bundles are isomorphic, so we can use the black box from before, or even work around it, by just seeing where the maps go: $S^{n+k}\to\mathbb{R}^{n+k}\cup\infty\to\overline{N(m)}/\partial N(m)=T(v)$. Then, this is equal to $T(f_2^*\gamma_k^s)$, and this maps into a homotopy group of MO.

In this proof, we require transversality to imply that the normal bundle was mapped into $\ker(\pi_*)$.

In the last fifteen minutes of class, though, we'll state some theorems that we'll prove next time, which involve intersection theory.

Definition. Let M be a compact, orientable manifold. Then, the **intersection product** is a map $H_{n-i}(M)\otimes H_{n-j}(M)\to H_{n-i-j}(M)$ in which $A\cdot B=\mathrm{PD}(\mathrm{PD}^{-1}(A)\smile\mathrm{PD}^{-1}(B))$.

[8] This lecture was hopelessly above my level of understanding and was presented very informally. Please forgive me for the parts I left out or got wrong. – Arun

The point of this is to make the homology into a ring using Poincaré duality, which isn't a super interesting ring structure, but will be useful. Also, despite the fact that this uses the cup product, it was actually discovered first, and this definition was given later.

Theorem 13.1. *Let M be a compact, orientable manifold and N_1 and N_2 be submanifolds of M such that $N_1 \pitchfork N_2$. Then, $[N_1]\cdot[N_2] = [N_1 \cap N_2]$.*

14. The Intersection Product: 5/21/15

Our goal is to prove Theorem 13.1: that if M is a compact, oriented manifold and $L_1, L_2 \subseteq M$ are compact, oriented submanifolds such that $L_1 \pitchfork L_2$, then $[L_1]\cdot[L_2] = [L_1 \cap L_2]$, where $\cdot$ is the intersection product defined last lecture. This is the Poincaré dual to cup product, i.e. $\mathrm{PD}[L_1] \smile \mathrm{PD}[L_2] = \mathrm{PD}[L_1 \cap L_2]$.[9]

Today's lecture will be a little bit formal, so if you're looking for some actual maps or geometry, this won't be the most satisfying lecture. As a result, be mindful of typographical or notational errors.

Definition. Let $E \to X$ be a vector bundle with a Riemannian metric on the fibers. Then:

- $D(E) \to X$ denotes the **unit disc bundle**, i.e. the fiber over x is the points of length at most 1 in the fiber of E over x.
- $S(E) \to X$ denotes the **unit sphere bundle**, which is the same as above, but with the requirement that the length is exactly 1.

Proposition 14.1. *Let $E \to X$ be a vector bundle with a metric. Then, $\tau(E)$ is homeomorphic to $D(E)/D(S)$.*

However, in reasonable conditions, all vector bundles have metrics.

Theorem 14.2. *Let $E \to M$ be a smooth vector bundle over a smooth manifold. Then, there exists a Riemannian metric on E.*

On more general topological spaces, the point-set topological condition we need to construct a metric are paracompactness and Hausdorff, since we'll be constructing a partition of unity.

Proof. Let $U_i \subseteq M$ be an open cover of M such that $\pi^{-1}(U_i) \to U_i$ is trivial. On trivial bundles, fiberwise Riemannian metrics exist, so let $\langle\cdot,\cdot\rangle_i$ be a metric on $\pi^{-1}(U_i)$. Take φ_i to be a partition of unity subordinate to $\{U_i\}$, and let

$$\langle\cdot,\cdot\rangle = \sum_i \varphi_i \langle\cdot,\cdot\rangle_i.$$

It's clear that this is bilinear and symmetric, since we've just multiplied by stuff, and it's nondegenerate because we've multiplied by positive stuff, and at every point, at least one of the φ_i is positive. ⊠

Remark. You can promote this argument to show that the space of metrics is contractible, so, up to homotopy, you can just pick one. But we probably won't use this.

Theorem 14.3 (Thom isomorphism theorem)**.** *Let M be a compact, oriented manifold and $E \xrightarrow{\pi} M$ be an oriented k-dimensional vector bundle. Then, $\widetilde{H}^*(\tau(E)) \cong H^{*-k}(X)$.*

Proof. Let $n = \dim(M)$. Poincaré duality gives us an isomorphism $\mathrm{PD} : H^{*-k}(M) \xrightarrow{\sim} H_{n+k-*}(M)$. Then, let $i : X \to E$ be inclusion; i and π are homotopy equivalent, and therefore there's another isomorphism $i^* : H_{n+k-*}(M) \xrightarrow{\sim} H_{n+k-*}(E)$. But E isn't compact, so Poincaré duality gives us an isomorphism $\mathrm{PD} : H_{n+k-*}(E) \xrightarrow{\sim} H^*_c(E)$, and for nice topological spaces, compact cohomology is isomorphic to the reduced cohomology of the one-point compactification, which in this case is $\widetilde{H}^*(\tau(M))$. Composing these morphisms, we have an isomorphism $H^{*-k}(M) \to \widetilde{H}^*(\tau(E))$. ⊠

This theorem holds over more general topological spaces, but the proof involves either the Serre spectral sequence or unpleasant Čech cohomology calculations. The theorem is also useful for computing cobordism stuff, which is why Thom's name is on it.

There's actually some interesting sheaf cohomology floating around in the background when M isn't orientable, because one can take $H_{n+k-*}(M; \mathscr{O}_M)$ and $H_{n+k-*}(E; i_*\mathscr{O}_M)$.

[9] If M is nonorientable, this result, as well as most of the results involving orientation and homology in this lecture, will hold in the case of homology with $\mathbb{Z}/2$-coefficients.

Definition. Let M be a compact, oriented manifold and $\pi : E \to M$ be a k-dimensional vector bundle (with $k > 0$) over M. Let $i : M \hookrightarrow E$ be a section for π; then, the class $u = \mathrm{PD}_E^{-1}(i_*[M])$, regarded in any of $H_c^k(E)$, $H^k(\tau(E))$, or $H^k(D(E), S(E))$, is called the **Thom class** of $E \to M$.

Observation. $u \smile [D(E)] = i_*[M]$, which comes from unwinding the definitions.

Proposition 14.4. *The isomorphism in the Thom isomorphism theorem is induced by*

$$H^*(M) \xrightarrow{\pi^*} H^*(D(E)) \xrightarrow{-\smile u} H^{*+k}(D(E), S(E)).$$

Proof. The proposition is equivalent to saying that if $\beta \in H^*(M)$, then $\mathrm{PD}_{D(E)} \circ i_* \circ \mathrm{PD}_M(\beta) = u \smile \pi^*(\beta)$. Let $\alpha = \pi^*(\beta)$, so that $i^*(\beta) = \alpha$. So, since $u = \mathrm{PD}_{D(E)}^{-1} i_*(M)$, then $[D(E)] \frown u = i_*(M)$, and therefore

$$\begin{aligned} \mathrm{PD}_{D(E)}^{-1}(i_*(\mathrm{PD}_M(\beta))) &= \mathrm{PD}_{D(E)}^{-1}(i_*(i^*\alpha \frown [M])) \\ &= \mathrm{PD}_{D(E)}^{-1}(\alpha \frown (i_*[M])). \end{aligned}$$

This uses the general fact that if $f : A \to B$, with $a \in H^*(B)$ and $b \in H^*(A)$, then $f_*(f^*a \frown b) = a \frown f^*b$, which is probably in Chapter 3 of Hatcher.

$$= \mathrm{PD}_{D(E)}^{-1}(\alpha \frown (u \frown D(E))).$$

The next fact we need is that $(a \smile b) \frown c = a \frown (b \frown c)$, so that

$$\begin{aligned} &= \mathrm{PD}_{D(E)}^{-1}((\alpha \smile u) \frown [D(E)]) \\ &= \alpha \smile u \\ &= \pi^*(\beta) \smile u. \end{aligned}$$

⊠

Since we have the Thom class $u \in H^k(D(E), S(E)) = H^k(E, E \setminus M)$, then if $\mathscr{O}_E$ denotes the orientation bundle of E, then the Thom class u gives you a section of $\mathscr{O}_E \to M$, i.e. it generates the k^{th} cohomology. Thus, a manifold has a Thom class iff it is orientable; of course, we started with orientable manifolds and fundamental classes, but you can define it more generally in this way.

Definition. Let $L \subseteq M$; then, to prove the theorem, we'll need a bunch of notation. To wit:

- $i_L^M : L \hookrightarrow M$ will denote inclusion.
- N_L^M will be the normal bundle.
- $u_L \in H^*(\tau N_L^M)$ denotes the Thom class.
- Since this space is naturally isomorphic to $H^*(N_L^M, N_L^M \setminus L) = H^*(M, M \setminus L)$, let $u_L^{M, M\setminus L}$ denote its class in $H^*(M, M \setminus L)$ and u_L^M be its image under the map $H^*(M, M \setminus L) \to H^*(M)$.

Proof of Theorem 13.1. Since $L_1, L_2 \subseteq M$, we have that if $j \neq i$ for $i, j \in \{1, 2\}$, then $N_{L_1 \cap L_2}^{L_i} \cong M_{L_j}^M|_{L_1 \cap L_2}$. This is the single geometric fact we use, and can be understood by drawing a picture. This means that $u_{L_1 \cap L_2}^{L_i} = (i_{L_j}^M)^* u_{L_i}^M$.

Here's an outline of the proof.

Step 1. First, we must show that $u_{L_1 \cap L_2}^M = u_{L_1}^M \smile u_{L_2}^M$.
Step 2. Then, we'll interpret this, showing that $u_L^M = \mathrm{PD}_M^{-1}[L]$, where L is any of L_1, L_2, or $L_1 \cap L_2$.

We'll cover the second step first; it follows from naturality or functoriality of things. Specifically, if $i = \mathrm{codim}(L)$ (i.e. the dimension of N), then the following diagram commutes.

$$\begin{tikzcd} H_n(M) \otimes H^i(M, M \setminus L) \arrow[r] \arrow[d] & H_n(M, M \setminus L) \otimes H^i(M, M \setminus L) \arrow[d, ")"] & H_n(N, N \setminus L) \otimes H^i(N, N \setminus L) \arrow[l, "\sim"'] \arrow[d, ")"] \\ H_n(M) \otimes H^i(M) \arrow[r, "\frown"] & H_{n-i}(M) & H_{n-i}(N) \arrow[l] \end{tikzcd}$$

Here, the blue arrows come from excision.

To finish the proof of this section, we chase $[M] \otimes u_L^{M, M\setminus L}$ around the diagram: at $H_{n-i}(M)$ in the bottom center, it is $u_L^M \frown [M]$ along the lower left and $i_*[L]$ along the upper right. ⊠

"I don't know why I'm using so many colors."

Recall that we're in the middle of proving Theorem 13.1, that if L_1 and L_2 are compact oriented submanifolds of a compact oriented M and $L_1 \pitchfork L_2$, then $[L_1]\cdot[L_2] = [L_1 \cap L_2]$.

We also had a lot of notation floating around. For $L \hookrightarrow M$, let $i_L^M : L \hookrightarrow M$ denote the inclusion map, N_L^M be the normal bundle, $u_L \in H^*(D(N_L^M), S(N_L^M))$ be the Thom class, and $u_L^{M,M\setminus L} \in H^*(M, M\setminus L)$ denote the Thom clas in this group (which is isomorphic to $H^*(D(N_L^M), S(N_L^M))$ anyways). Let u_L^M be the image of $u_L^{M\setminus L}$ in $H^*(M)$.

This proof had exactly one piece of geometric information, which is now normal bundles behave under inclusion, i.e. that $N_{L_1\cap L_2}^{L_1} = (i_{L_1\cap L_2}^{L_2})^* N_{L_2}^M$. Everything else is functoriality and abstract nonsense. Last time, we showed that $u_L^M = \mathrm{PD}_M^{-1}[L]$ and $u_L^M \frown [M] = [L]$; thus, it suffices to show that $u_{L_1\cap L_2}^M = u_{L_1}^M \smile u_{L_2}^M$.

As a corollary to our single piece of geometric information, we know that $u_{L_1\cap L_2}^{L_1} = (i_{L_1}^M)^* u_{L_2}^M$, because we know that if $f : X \to Y$, then taking the Thom class commutes with pullback, i.e. in the diagram

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] & E \arrow[d] \\ X \arrow[r, "f"] & Y, \end{tikzcd}$$

$u_{f^*E} = f^*(u_E)$.

Then, $u_{L_2} \in H^*(\tau N_{L_2}^M) = H^*(M, L_2)$ maps to $(i_{L_1\cap L_2}^*)u_{L_2} = u_{L_1\cap L_2} \in H^*(\tau N_{L_1\cap L_2}^{L_1}) = H^*(L_1, L_1 \setminus (L_1\cap L_2))$. (This can be redrawn as a commutative diagram, which makes things a little clearer.)

By the above, it also suffices to show

$$\begin{aligned}
[M] \frown u_{L_1\cap L_2}^M &= [M] \frown (u_{L_1}^M \smile u_{L_2}^M) \\
&= (i_{L_1\cap L_2})_*[L_1 \cap L_2] \\
&= (i_{L_1}^M)^*([L_1] \frown u_{L_1\cap L_2}^{L_2}) \\
&= (i_{L_1}^M)^*([L_1] \frown (i_{L_1}^M)^* u_{L_2}^M) \\
&= ([M] \frown u_{L_1}^M) \frown u_{L_2}^M \\
&- [M] \frown (u_{L_1}^M \smile u_{L_2}^M).
\end{aligned}$$

Thus, we've finished the proof: it really is all algebraic abstraction.

One common question about vector bundles is whether they're trivial; sometimes, you can write down a trivialization, but the Euler class is an obstruction in cohomology that determines whether a vector bundle is trivial.

Definition. Let $\pi : E \to M$ be a k-dimensional oriented vector bundle and i be the zero section for π; then, the **Euler class** of E is $e(E) \in H^k(M)$ defined by $e(E) = i^* u_M^E$.

That is, we're pushing u_M^E along the following sequence.

$$\begin{tikzcd} H^*(\tau E) \arrow[r, equal] & H^k(E, E\setminus M) \arrow[r] & H^*(E) \arrow[r, "i_*"] & H^*(M) \\ u \arrow[r, mapsto] & u_M^{E,E\setminus M} \arrow[r, mapsto] & u_M^E \arrow[r, mapsto] & e(E) \end{tikzcd}$$

Proposition 15.1. *Let* $\mathrm{Vec}^k : \mathsf{Manifolds} \to \mathsf{Sets}$ *denote the functor sending a manifold M to the set of isomorphism classes of k-dimensional vector bundles; then, e gives a natural transformation between* Vec^k *and* H^k *(the latter as a functor to* Sets*).*

That is, e is a **characteristic class**. Note that since $\mathrm{Vec}^k(M) = \pi_0(\mathrm{Map}(M, BO(k)))$, then we could have just used $BO(k)$ to classify vector bundles and therefore provide a more abstract definition. This is useful when talking about moduli spaces in algebraic geometry.

Concretely, though, what Proposition 15.1 tells us is that e commutes with pullback: $e(f^*(E)) = f^*(e(E))$.

Proposition 15.2. *Let $E \xrightarrow{\pi} M$ be a vector bundle, where M is a compact, orientable manifold. If i and σ are two sections of π such that $i \pitchfork \sigma$, then* $\mathrm{PD}_M\, e(E) = [\{m \in M \mid i(m) = \sigma(m)\}]$.

That is, geometrically, the Euler class is Poincaré dual to intersections of sections.

Proof. We know that

$$[\{m \in M \mid i(m) = \sigma(m)\}] = \pi_*(i_*[M] \cdot \sigma_*[M]) = \pi_*(i_*[M] \cdot i_*[M]),$$

since i and σ must be homotopic. Moreover,

$$\begin{aligned}[M] \frown e(M) &= [M] \frown i^* u_M^E \\ &= \pi_*(i_*[M] \frown u_M^E) \\ &= \pi_*(i_*[M] \frown \mathrm{PD}_E^{-1} i_*[M]).\end{aligned}$$

Be careful; some of these will lie in compactly supported cohomology. Which ones?

It suffices to show that

$$i_*[M] \frown \mathrm{PD}_E^{-1} i_*[M] = \mathrm{PD}_E(\mathrm{PD}_E^{-1} i_*[M] \smile \mathrm{PD}_E^{-1} i_*[M]),$$

and the right-hand side simplifies as

$$\begin{aligned}\mathrm{PD}_E(\mathrm{PD}_E^{-1} i_*[M] \smile \mathrm{PD}_E^{-1} i_*[M]) &= [E] \frown (\mathrm{PD}_E^{-1} i_*[M] \smile \mathrm{PD}_E^{-1} i_*[M]) \\ &= ([DE] \frown \mathrm{PD}_E^{-1} i_*[M]) \frown \mathrm{PD}_E^{-1} i_*[M] \\ &= i_*[M] \frown \mathrm{PD}_E^{-1} i_*[M].\end{aligned}$$

⊠

Proposition 15.3. $e(TM) \frown M = \chi(M)[\mathrm{pt}]$.

This proposition also uses relatively little geometry. The idea is to consider the diagonal map $\Delta : M \to M \times M$ and a section π for it; then, $\Delta^* N_{\Delta(M)}^{M\times M} = TM$, and compute from there (this is in the textbook).

Corollary 15.4. *In $H_0(M)$, $e(TM) \frown [M]$ is equal to the number of zeros (with sign) of a vector field transverse to the zero field times* $[\mathrm{pt}]$.

Morse theory. For the next few days, we'll talk about Morse theory. The idea is to take a manifold M and consider functions $f : M \to \mathbb{R}$.

Definition. A function $f : M \to \mathbb{R}$ is called **Morse** if it has isolated critical points and whenever $Df(m) = 0$, the matrix of mixed partial derivatives at m is invertible.

We will build a chain complex out of the critical points of f, and use this to compute the homology of f.

Definition. If f is Morse and $x \in M$ is a critical point of f, then the **index** of x is the number of negative eigenvalues of the matrix of mixed partials.

Next time, we'll talk more about it, but Morse's great idea was to replace M with its space of geodesics, so we can use the homology of a manifold to gain insights about its critical points (e.g. lower bounds), and therefore solve problems seemingly unrelated to algebraic topology.