SPECTRAL SEQUENCES IN (EQUIVARIANT) STABLE HOMOTOPY THEORY

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Contents

1. The homotopy fixed-point spectral sequence: 5/15/17

Today, Richard spoke about the homotopy fixed-point spectral sequence in equivariant stable homotopy theory.

We'll start with the Bousfield-Kan spectral sequence (BKSS). One good reference for this is Guillou's notes [3], and Hans Baues [1] set it up in a general model category.

We'll work in sSet, so that everything is connective. Consider a tower of fibrations

$$\cdots \longrightarrow Y_s \xrightarrow{p_s} Y_{s-1} \xrightarrow{p_{s-1}} Y_{s-2} \longrightarrow \cdots \tag{1.1}$$

for $s \geq 0$, and let $Y \coloneqq \varprojlim Y_s$. Let F_s be the fiber of p_s.

Theorem 1.2 (Bousfield-Kan [2]). *In this situation, there is a spectral sequence, called the* ***Bousfield-Kan spectral sequence****, with signature*

$$E_1^{s,t} = \pi_{t-s}(F_s) \Longrightarrow \pi_{t-s}(Y).$$

If everything here is connective (which is not always the case in other model categories, as in one of our examples), this is first-quadrant. One common convention is to use the **Adams grading** $(t-s, s)$ instead of (s,t).

We can extend (1.1) into a diagram

$$\begin{tikzcd}
\cdots \arrow[r] & Y_s \arrow[r, "p_s"] & Y_{s-1} \arrow[r, "p_{s-1}"] & Y_{s-2} \arrow[r] & \cdots \\
& F_s \arrow[u, "i_s"] & F_{s-1} \arrow[u, "i_{s-1}"] & F_{s-2}, \arrow[u, "i_{s-2}"] &
\end{tikzcd}$$

and hence into an exact couple

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_*(Y_{s+1}) \arrow[r] & \pi_*(Y_s) \arrow[r] \arrow[dl, "\delta"'] & \pi_*(Y_{s-1}) \arrow[r] \arrow[dl, "\delta"'] & \cdots \\
& \pi_*(F_s) \arrow[u, "i_{s*}"] & \pi_*(F_{s-1}) \arrow[u, "i_{(s-1)*}"] & \pi_*(F_{s-2}), \arrow[u, "i_{(s-2)*}"] &
\end{tikzcd}$$

and the differentials are the compositions of the maps $\pi_*(F_s) \to \pi_*(Y_s) \to \pi_*(F_{s+1})$:

$$\begin{tikzcd}
& \cdots \arrow[r] & \pi_*(Y_{s+1}) \arrow[r] & \pi_*(Y_s) \arrow[r] \arrow[dl, "\delta"'] & \pi_*(Y_{s-1}) \arrow[r] \arrow[dl, "\delta"'] & \cdots \\
{} & & \pi_*(F_s) \arrow[u, "i_{s*}"] \arrow[ll] & \pi_*(F_{s-1}) \arrow[u, "i_{(s-1)*}"] \arrow[l, "d_1"] & \pi_*(F_{s-2}). \arrow[u, "i_{(s-2)*}"] \arrow[l, "d_1"] & {} \arrow[l]
\end{tikzcd}$$

Taking homology, we'll get a differential d_2 that jumps two steps to the left, then d_3 three steps to the left, and so on. After you check that $\operatorname{Im}(d_r) \subset \ker(d_r)$, you can define $E_r^{s,s+1} := \ker(d_r)/\operatorname{Im}(d_r)$. Let $A_s := \operatorname{Im}(\pi_0(Y_{s+r}) \to \pi_0(Y_s))$, and let $Z_r^{s,s} := (i_s)_*^{-1}(A_s)$. Then, $E_{r+1}^{s,s} = Z_r^{s,s}/d_r(E_r^{s-r,s-r+1})$.

Remark. One important caveat is that for $i \leq 2$, π_i does not produce abelian groups, but rather groups or just sets! This means that a few of the columns of this spectral sequence don't quite work, but the rest of it is normal, and the degenerate columns can still be useful. This is an example of a **fringed spectral sequence**. ⊲

Bousfield and Kan cared about this spectral sequence because it allowed them to write down a useful long exact sequence, the $r^{\mathbf{th}}$ **derived homotopy sequence**: let $\pi_i Y^{(r)} := \operatorname{Im}(\pi_i(Y_{n+r}) \to \pi_i(Y_n))$; then, there's a long exact sequence

$$\cdots \longrightarrow \pi_{t-s-1} Y_{s-r-1}^{(r)} \longrightarrow E_{r+1}^{s,t} \longrightarrow \pi_{t-s} Y_s^{(r)} \xrightarrow{\delta} \pi_{t-s} Y_{s-1}^{(r)} \longrightarrow E_{r+1}^{s+r,t+r+1} \longrightarrow \pi_{t-s+1} Y_s^{(r)} \longrightarrow \cdots$$

You can do something like this in general given a spectral sequence, though you need to know how to obtain it from the exact couple.

Remark. When $r = 0$, $E_1^{s,t} = \pi_{t-s}(F_s)$, and so the first derived homotopy sequence is the long exact sequence of homotopy groups of a fibration. ⊲

One nice application is to **Tot towers** ("Tot" for totalization).

Definition 1.3. Let $X^\bullet$ be a cosimplicial object in sSet. Then, its **totalization** is the complex

$$\operatorname{Tot}(X^\bullet) := \mathsf{sSet}(\Delta^\bullet, X^\bullet),$$

i.e.

$$\operatorname{Tot}_n(X^\bullet) := \mathsf{sSet}(\operatorname{sk}_n \Delta^\bullet, X^\bullet).$$

Here $(\operatorname{sk}_n \Delta^\bullet)^n := \operatorname{sk}_n \Delta^m$.

Then

$$\varprojlim \operatorname{Tot}_n(X^\bullet) = \operatorname{Tot}(X^\bullet),$$

reconciling the two definitions.

Exercise 1.4. In the Reedy model structure, $\operatorname{Tot}_n(X^\bullet) \to \operatorname{Tot}_{n-1}(X^\bullet)$ is a fibration.

Assuming this exercise, we can apply the Bousfield-Kan spectral sequence.

One place this pops up is that if $C, D \in \mathsf{C}$ and $X_\bullet \to C$ is a simplicial resolution in a simplicial category C,[1] then $\operatorname{Hom}_{\mathsf{C}}(X_\bullet, D)$ is a cosimplicial object, and this spectral sequence can be used to compute homotopically meaningful information about $\mathsf{sSet}(C, D)$.

We can use this formalism to derive the homotopy fixed point spectral sequence. Let G be a group, and X be a spectrum with a G-action. Then, the **homotopy fixed points** of X are

$$X^{hG} := F((EG)_+, X)^G,$$

i.e. the G-equivariant maps $(EG)_+ \to X$.[2] The bar construction gives us a simplicial resolution of $(EG)_+$, producing a cosimplicial object that can be plugged into the Bousfield-Kan spectral sequence. Specifically, we write $EG = B^\bullet(G, G, *)$, add a disjoint basepoint, and then take maps into X.

Theorem 1.5. *If X is a spectrum with a G-action, there's a spectral sequence, called the* ***homotopy fixed-point spectral sequence****, with signature*

$$E_2^{p,q} = H^p(G, \pi_q(X)) \Longrightarrow \pi_{q-p}(X^{hG}).$$

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[1] Meaning that after geometrically realizing, there's an equivalence.

[2] Notationally, this is the function spectrum of maps from $\Sigma^\infty(EG)_+$ to X, or you can use the fact that spectra are cotensored over spaces.

Example 1.6. The first example is really easy. Let k be a field, and consider the Eilenberg-Mac Lane spectrum Hk. Let G act trivially on k; we want to understand $\pi_*(Hk^{hG})$. The homotopy fixed-points spectral sequence is particularly simple:

$$E_2^{p,q} = H^p(G; \pi_q(Hk)) = \begin{cases} H^p(G;k) & q = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Since this is a single row,[3] all differentials vanish, and this is also the E_∞ page. So we just have to compute $H^p(G;k)$ for $k \geq 0$.

For example, if $G = \mathbb{Z}/2$ and $k = \mathbb{F}_2$, then $H^*(\mathbb{Z}/2;\mathbb{F}_2) = H^*(\mathbb{RP}^\infty;\mathbb{F}_2) = \mathbb{F}_2[x]$, $|x| = 1$. There are no extension issues, since there's only one nonzero term in each total degree. Thus,

$$\pi_{-p}(Hk^{hG}) = H^p(G;k).$$

If you let $G = \mathbb{Z}/2$ and k be any field of odd characteristic, then $H^*(\mathbb{Z}/2;k) = k$ in degree 0, so the homotopy groups of $Hk^{h\mathbb{Z}/2}$ are all trivial except for π_0, which is k. ◀

In the context of group actions on spectra, there's another spectral sequence called the Tate spectral sequence. If X is a genuine G-spectrum, there's a norm map $X_{hG} \to X^{hG}$ whose cofiber is called the **Tate spectrum** X^{tG}.[4] This is a generalization of Tate cohomology $\widehat{H}^p$ in group cohomology. Here, $X_{hG} \coloneqq (EG_+ \wedge X)_G$ is the **homotopy orbits** of X. Then, there is a spectral sequence, called the **Tate spectral sequence**, with signature

$$E_2^{p,q} = \widehat{H}^p(G;\pi_q(X)) \Longrightarrow \pi_{q-p}(X^{tG}).$$

The similarities with the homotopy fixed point spectral sequence are no coincidence.

Example 1.7. Let C_2 act on S^1 by reflection. Then, $\pi_i(S^1)$ is trivial unless $i = 1$, in which case we get $\mathbb{Z}$. Hence,

$$E_2^{p,q} = \begin{cases} H^p(C_2;\mathbb{Z}), & q = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Under the isomorphism $\mathbb{Z}[C_2] \cong \mathbb{Z}[x]/(x^2-1)$, the $\mathbb{Z}[C_2]$-module structure on $\mathbb{Z}$ is the map $\mathbb{Z}[C_2] \to \mathbb{Z}$ sending $x \mapsto -1$, i.e. C_2 acts on $\mathbb{Z}$ through the nontrivial action. We'll let $\mathbb{Z}_\sigma$ denote $\mathbb{Z}$ with this action, and $\mathbb{Z}$ denote the integers with the trivial C_2-action. To compute the group cohomology, we need to compute a free resolution $P_\bullet \to \mathbb{Z}$ as a *trivial* $\mathbb{Z}[C_2]$-module:

$$\cdots \longrightarrow \mathbb{Z}[C_2] \xrightarrow{\cdot(x-1)} \mathbb{Z}[C_2] \xrightarrow{\cdot(x+1)} \mathbb{Z}[C_2] \xrightarrow{\cdot(x-1)} \mathbb{Z}[C_2] \xrightarrow{x\mapsto 1} \mathbb{Z} \longrightarrow 0.$$

Now we compute $\operatorname{Hom}_{\mathbb{Z}[C_2]}(P_\bullet, \mathbb{Z}_\sigma)$:

$$\cdots \longleftarrow \mathbb{Z} \xleftarrow{-2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{-2} \mathbb{Z}.$$

Taking homology, we conclude that

$$H^p(\mathbb{Z}/2, \mathbb{Z}_\sigma) = \begin{cases} \mathbb{Z}/2, & p > 0 \text{ odd} \\ 0, & \text{otherwise.} \end{cases}$$

Since the spectral sequence degenerates at page 2, this is also the homotopy groups of $(S^1)^{hC_2}$. This doesn't make any sense, though — $(S^1)^{hC_2}$ is a space, so cannot have negative-degree homotopy groups. ◀

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[3]It's a single row in the usual grading, and a single diagonal line with slope -1 in the Adams grading.

[4]The norm map is the spectral analogue of a more concrete construction: let M be a $\mathbb{Z}[G]$-module. Then, the assignment

$$m \longmapsto \sum_{g\in G} g \cdot m$$

lands in M^G, so it factors through orbits, defining a map $M_G \to M^G$.

References

[1] Hans Joachim Baues. *Algebraic Homotopy*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1989.

[2] A.K. Bousfield and D.M. Kan. The homotopy spectral sequence of a space with coefficients in a ring. *Topology*, 11(1):79–106, 1972.

[3] Bertrand Guillou. The Bousfield-Kan spectral sequence. 2007. http://www.ms.uky.edu/~guillou/BKss.pdf.