MATH 215B NOTES: ALGEBRAIC TOPOLOGY

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These notes were taken in Stanford's Math 215B class in Winter 2015, taught by Søren Galatius. I TEXed these notes up using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Thanks to Jack Petok for fixing a few mistakes.

Contents

1. Simplices, Δ-Complexes, and Homology: 1/8/15

> *"Some poor confused freshman once sat in Math 215B for twenty minutes and then asked, 'Excuse me, is this Math 51?' " – Brian Conrad*

Today's lecture was given by Dan Berwick-Evans.

We'll start with a vague notion of what this class, algebraic topology, is about: we want to study topological spaces and continuous maps between them. These form a *category*, and we'll analyze it algebraically, by defining *functors* $F : \mathsf{Spaces} \to \mathcal{C}$, where $\mathcal{C}$ will be one of the category of groups, abelian groups, rings, etc.

In more detail, a functor associates to each space X an algebraic object $F(X)$, and for each continuous map $X \to Y$, we have a homomorphism $F(f) : F(X) \to F(Y)$. There's a good reason these are the right things to study, which deals with some history and tradition, and also some results that are cleaner.

Example 1.1. The fundamental group of a space is functorial, given by $\pi_1 : \mathsf{Spaces} \to \mathsf{Grp}$. The higher homotopy groups $\pi_n : \mathsf{Spaces} \to \mathsf{AbGrp}$, for $n > 1$, are also functorial.

Our goal will be to define homology, which is a sequence of functors $H_n : \mathsf{Spaces} \to \mathsf{AbGrp}$.

A non-example of a functor is the Euler characteristic. $X \mapsto \chi(X) \in \mathbb{Z}$, but then what does one do with continuous functions?

The idea of homology is to take our space X, and consider loops in the space (unlike homotopy, they don't have to have a basepoint). For some, but not all, loops L, there is a disc $D \subset X$ such that $L = \partial D$ (e.g. a noncontractible loop in a torus doesn't have such a D). More generally, we will answer the question of when (closed) subspaces of a certain form are the boundary of some other subspace.

To make this less vague, we'll define this in terms of *simplices* and Δ-*complexes*, which are somewhat combinatorial objects that allow us to concretely define homology. The motivating example is that a torus can be thought of as a rectangle with the sides identified, but then we can split the rectangle into two triangles. Then, we can restrict our attention to loops that are along the edges of such triangles.

The advantage of this combinatorial approach is that it's somewhat mechanical; you can teach a computer to do it. But the trickery is going from triangles to higher dimensions.

Definition. An *n-simplex*, denoted $[v_0, \dots, v_n]$, is an ordered n-tuple of vectors in $\mathbb{R}^m$ such that $\{v_0 - v_1, \dots, v_0 - v_n\}$ are linearly independent. The v_i are called *vertices*, and the $(n-1)$-simplex obtained by forgetting v_j is called the j^{th} *face* of the simplex, usually denoted $[v_0, \dots, \widehat{v}_j, \dots, v_n]$. The union of the faces of an n-simplex is called its *boundary*.

This is a generalization of a triangle, even if it's not incredibly clear at first. More geometrically, one can define an n-simplex as the smallest convex subset containing $\{v_0, \dots, v_n\}$ as in the previous definition (which actually allows one to start drawing triangles, tetrahedra, etc.). But the first definition will be more useful for proving things, because the order of the vectors is important (usually).

The notion of a face is also pretty geometric: the j^{th} face is the face opposite the j^{th} vertex, and this looks familiar in 2 and 3 dimensions, though it takes a while to get used to in its generality.

Definition. The *standard simplex* is

$$\Delta^n = [e_0, \dots, e_n] = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum_i t_i = 1 \right\}.$$

For example, in $\mathbb{R}^3$, Δ^3 is the tetrahedron whose vertices are $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.

This allows one to induce coordinates on all n-simplices by sending $e_i \mapsto v_i$. These are called *barycentric coordinates*:

$$(t_0, \dots, t_n) \longmapsto \sum_i t_i v_i.$$

The beginning of the actual math is how we talk about spaces in terms of these simplices. This is where Δ-complexes come onstage.

We'll write that the interior of a space X is $\mathring{X} = X \setminus \partial X$.

Definition. A Δ-complex structure on a topological space X is a set S of maps $\sigma_\alpha : \Delta^n \to X$ such that:

(1) $\sigma_\alpha|_{\mathring{\Delta}^n} \hookrightarrow X$ (i.e. the restriction is injective) and for all $x \in X$, there's a unique α such that $x \in \operatorname{Im}(\sigma_\alpha|_{\mathring{\Delta}^n})$.
(2) The restriction of σ_α to its faces gives maps $\sigma_\beta \in S$.
(3) $A \subset X$ is open iff $\sigma_\alpha^{-1}(A)$ is open in Δ^n for all α.

The last requirement is necessary for preventing stupid things like writing the circle as an infinite union of points, which are 0-simplices. The discrete topology would be somewhat silly here. However, the set S does not have to be finite, especially if X is noncompact.

Example 1.2.

(1) We saw that the torus T^2 looks like two 2-simplices attached along their edges, with the 1- and 0-simplices induced by their boundaries.
(2) S^1 is the union of a 1-simplex (a line) and a 0-simplex (a point).
(3) The real projective plane also can be thought of as a rectangle with edges identified, so it decomposes as two 2-simplices, along with the three 1-simplices and one 0-simplex given by faces.

If X is a Δ-complex, let $\Delta_n(X)$ be the *free abelian group* on n-simplices; an element of $\Delta_n(X)$ is a formal sum $\sum_{i=1}^n n_\alpha \sigma_\alpha$ with $n_\alpha \in \mathbb{Z}$ and $\sigma_\alpha : \Delta^n \to X$. The coefficients may seem ungeometric, but they will be useful for calculations later. Elements of $\Delta_n(X)$ are called *chains*.

The key will be knowing how to take boundaries.

Definition. The *boundary* of a chain κ, denoted $\partial_n \kappa$, is determined by

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha|_{[v_0, \dots, \widehat{v}_i, \dots, v_n]},$$

and then extended linearly, so that $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ is a group homomorphism.

The signs seem a little weird, but are crucial for things to come out right; for example, the 1-simplex $[v_0]$ (corresponding to a loop) has boundary $\partial([v_0]) = v_0 - v_0 = 0$, but the boundary of a line segment from v_0 to v_1 is $v_0 + v_1$.

Way back in calculus, we learned that the boundary of a boundary is zero, sort of like Stokes' theorem. This is true in this context as well.

Lemma 1.3. $\partial_{n-1} \circ \partial_n : \Delta_n(X) \to \Delta_{n-2}(X)$ *is the zero homomorphism.*

Proof. You can get intuition about this geometrically, but let's work through the combinatorics.

$$\partial_{n-1} \circ \partial_n(\sigma) = \sum_{j<i} (-1)^i (-1)^j \sigma|_{[v_0,\ldots,\widehat{v}_j,\ldots,\widehat{v}_i,\ldots,v_n]} + \sum_{j>i} (-1)^i (-1)^{j-1} \sigma|_{[v_0,\ldots,\widehat{v}_i,\ldots,\widehat{v}_j,\ldots,v_n]}.$$

This means that the sums cancel, and the result is zero. ⊠

This looks like mumbo-jumbo, but work through it: it's the reason everything works.

Homological Algebra. So, we've introduced some abelian groups, and some maps between them. We can step back and look at the kind of algebra this produces, which is occasionally studied in its own right.

Specifically, we have abelian groups $C_n = \Delta_n(X)$ and homomorphisms $\partial_n : C_n \to C_{n-1}$ such that $\partial_{n-1} \circ \partial_n = 0$, or equivalently, $\mathrm{Im}(\partial_{n+1}) \subset \mathrm{Ker}(\partial_n)$. This is what it looks like:

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

Definition. A sequence $(C_\bullet, \partial_\bullet)$ of abelian groups and homomorphisms between them such that $\mathrm{Im}(\partial_{n+1}) \subset \mathrm{Ker}(\partial_n)$ is called a *chain complex*. Elements of $\mathrm{Ker}(\partial_n)$ are called *cycles* and elements of $\mathrm{Im}(\partial_{n+1})$ are called *boundaries*.

The words are geometrical or topological, but the concepts are pretty algebraic. Notice that the boundaries correspond to actual boundaries in the geometric sense.

Definition. The n^{th} *homology* of the chain complex $(C_\bullet, \partial_\bullet)$ is $H_n(C_\bullet, \partial_\bullet) = \mathrm{Ker}(\partial_n) / \mathrm{Im}(\partial_{n+1})$.

The equivalence class $[z]$ in H_n of a cycle is called a *homology class*, and if $[z] = [z']$, then z and z' are said to be *homologous*.

Later, it will be useful to have two more definitions.

Definition.

- $(C_\bullet, \partial_\bullet)$ is a *(long) exact sequence* if $\mathrm{Im}(\partial_{n+1}) = \mathrm{Ker}(\partial_n)$ for all n, i.e. $H_n(C_\bullet, \partial_\bullet) = 0$ for all n.
- An exact sequence of the form

$$0 \longrightarrow C_1 \longrightarrow C_2 \longrightarrow C_3 \longrightarrow 0$$

  is called *short exact*.

Topologists love short exact sequences; they imply that the map $C_1 \to C_2$ is injective, the map $C_2 \to C_3$ is surjective, and the image of the former is the kernel of the latter. This has nice properties in the category of abelian groups.

Now, we can wrap this up in its topological application.

Definition. The n^{th} *simplicial homology* of a Δ-complex is $H_n^\Delta(X) = H_n(\Delta_\bullet(X), \partial_\bullet)$.

As we promised, this is cycles mod boundaries, but in a much more algebraic, abstract framework than one may have guessed.

There are many tools for calculating homology, but we haven't developed any yet, so we have to be lucky or persistent.[1]

Example 1.4. We can realize S^1 as a single 1-simplex and a single 0-simplex, each of which has $\Delta_n(X) = \mathbb{Z}$ (one generator). Thus, the chain complex looks like

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z},$$

so $H_n^\Delta(S^1) = \mathbb{Z}$ if $n = 0, 1$ and is 0 otherwise.

H_1^Δ is related to the fundamental group (though isn't the same, as it's always abelian), and H_0^Δ will give us the number of connected components of the space.

[1] No pun intended.

Exercise 1. Show that
$$H_n^\Delta(T^2) = \begin{cases} \mathbb{Z}, & n = 0,2 \\ \mathbb{Z}\oplus\mathbb{Z}, & n = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Now, we can talk about functoriality: if $f : S^1 \to T^2$ preserves the Δ-complex structure,[2], then there will be an $H_n(f) : H_n(S^1) \to H_n(T^2)$.

Singular Homology. We can define homology in yet another way, by creating another kind of chain complex.

Definition. A *singular n-simplex* is a continuous map $\sigma : \Delta^n \to X$.

Here, "singular" signifies there are no restrictions on these maps.

Then, let C_n be the free abelian group on singular n-simplices, and define the boundary maps ∂_n in precisely the same way as above. Then, we have a chain complex $(C_\bullet, \partial_\bullet)$, and can play the same game again.

Definition. The n^{th} *singular homology* of X is $H_n(X) = H_n(C_\bullet, \partial_\bullet)$.

This is much nicer theoretically, because we don't have to worry about choosing a Δ-complex structure, and so we'll prove theorems about this one. However, calculation is a nightmare, because each C_n is generally free abelian on uncountably many generators! Thus, understanding the correspondence between H_n and H_n^Δ will be quite important.

Immediately from the definition, we can see that if X and Y are homeomorphic, then $H_n(X) = H_n(Y)$ (as they have the same singular n-simplices). We can ask some more questions (which all end up having positive answers):

(1) Is H_n homotopy invariant?
(2) Is $H_n^\Delta(X) \cong H_n(X)$?

It turns out that there are many ways to define homology, and as long as they satisfy some axioms, they will end up with the same result, which is part of their power.

2. Properties of Singular Homology: 1/13/15

> *"In these days the angel of topology and the devil of abstract algebra fight for the soul of every individual discipline of mathematics." – Hermann Weyl*

Professor Galatius is back today.

Recall that last time, we defined the *singular homology* $H_*(X)$, where X is a topological space, as well as Δ-*complexes* and the associated *simplicial homology* $H_*^\Delta(X)$, and we saw some calculations.

Today and the next lecture, we'll primarily discuss singular homology, eventually returning to simplicial homology and showing they're isomorphic. Simplicial homology is much easier (often, just possible) to calculate, and singular homology has nicer mapping properties.

Definition. A *functor* is an assignment $f \mapsto f_*$ of maps from one category to another such that $(f\circ g)_* = f_* \circ g_*$ (and this composition makes sense), and $\mathrm{id}_* = \mathrm{id}$.

If you haven't seen functors and categories before, it will be helpful to review them.

Theorem 2.1. *H_* is a functor, i.e. given a continuous map $f : X \to Y$ of topological spaces, there's an induced $f_* : H_n(X) \to H_n(Y)$ for all $n \geq 0$.*

This is the first good example of simplicial homology having nicer properties.

Proof. Recall that $H_n(X) = H_n(C_*(X), \partial) = \mathrm{Ker}(\partial_n)/\,\mathrm{Im}(\partial_{n+1})$, where $\partial_n : C_n(X) \to C_{n-1}(X)$. The proof will take two steps, first showing that $f : X \to Y$ induces a homomorphism $C_n(X) \xrightarrow{f_*} C_n(Y)$, and then showing that a map between two chain complexes induces a map on the homology groups.

Given a $\sigma : \Delta^n \to X$, let $f_*(\sigma) = \sigma \circ f : \Delta^n \to X$ (and extend linearly, giving a group homomorphism). Thus, $f_* : C_n(X) \to C_n(Y)$ is a chain map, because the following diagram commutes.

$$\begin{tikzcd} C_n(X) \arrow[r, "f_*"] \arrow[d, "\partial"] & C_n(Y) \arrow[d, "\partial"] \\ C_{n-1}(X) \arrow[r, "f_*"] & C_{n-1}(Y) \end{tikzcd}$$

[2] Later, we'll be able to use more general continuous f, without worrying about the specific Δ-complex structure.

Since all of these maps are group homomorphisms, it's sufficient to check on generators: given a $\sigma \in C_n(X)$, it's quick to check that
$$f_* \circ \partial(\sigma) = \partial \circ f_*(\sigma) = \sum (-1)^i f \circ \sigma[v_0, \dots, \widehat{v}_i, v_n].$$
Thus, since f_* commutes with the boundary maps, it induces a map of chain complexes.

Now, we'll show that if (C_*, ∂) and (C'_*, ∂') are chain complexes and $\varphi : C_* \to C'_*$ is a chain map (i.e. a group homomorphism such that $\varphi \circ \partial = \partial' \circ \varphi$ for each $\varphi_n : C_n \to C'_n$) induces $\varphi_* : H_n(C_*, \partial) \to H_n(C'_*, \partial')$. This is done by setting $\varphi_*([c]) = [\varphi_* c]$ for any $c \in C_n$; one has to check that this is well-defined, but this isn't too hard. Thus, φ_* (and f_* from before) defines a homomorphism on the homology groups. ☒

This bit may be a little bit of review from Math 210A, albeit in a different context.

Though we said that singular cohomology is impossible to calculate, there are a few silly examples.

- Suppose $X = \emptyset$. The free abelian group on no generators is the trivial group $\{0\}$ (*not* the empty set), so $H_n(\emptyset) = 0$ for all n.
- If $X = \{\bullet\}$ is a single point, then there is exactly one map $\Delta^n \to X$, and it is continuous, so $C_n(X) = \mathbb{Z}$ for all n, and ∂_n is the alternating sum as usual. However, the maps are a little more interesting: $C_1 \to C_0$ is 0 (there's no way to get this map from the previous one), but $C_2 \to C_1$ is the identity, and so on. We end up with
$$0 \longleftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{\mathrm{id}} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{\mathrm{id}} \mathbb{Z} \xleftarrow{0} \cdots$$
Thus, we can quotient out to get the homology:
$$H_n(\{\bullet\}) \cong \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n > 0. \end{cases}$$
- One can similarly explicitly calculate the singular cohomology of a disjoint union of a finite number of points.

However, in more complicated cases, this is pretty hopeless: $C_n(X)$ is often free abelian on an uncountable number of generators! It's a miracle that $H_n(X)$ is often a finitely generated abelian group, and a little bit of a mystery, connected to the mystery of how to calculate it. For example, we'll eventually show that
$$H_k(S^n) = \begin{cases} \mathbb{Z} & k = 0, n \\ 0, & \text{otherwise.} \end{cases}$$
Later, we'll see a little bit as to why these miracles hold: if X can be made into a Δ-complex with finitely many simplices, then H_* and H_*^Δ are isomorphic; later, we'll also show that H_n is homotopy-invariant (e.g. $H_n(\mathbb{R}) \cong H_n(\{\bullet\})$).

One can concoct a weaker sort of functoriality for H_n^Δ: it's only a functor with respect to the much more restrictive notion of maps of Δ-complexes, i.e. sending simplices to simplices.

Relative Homology and the Long Exact Sequence. These miracles are important to keep in mind: we don't have too much written down yet, so today will be mostly formal. The miracles and functoriality are reasons to keep in mind to care about it.

Definition. We'll use the word *pair* (X, A) to denote a topological space X and a subspace $A \subset X$.

For a pair (X, A), we have continuous inclusion $i : A \to X$, so it induces $i_* : C_n(A) \hookrightarrow C_n(X)$ (since $C_n(A)$ is the subgroup of $C_n(X)$ generated by maps $\Delta^n \to A$).

Definition. The *relative chains* $C_n(X, A)$ of this pair are defined by fitting into the short exact sequence
$$0 \longrightarrow C_n(A) \xrightarrow{i_*} C_n(X) \longrightarrow C_n(X, A) \longrightarrow 0. \tag{2.1}$$
That is, $C_n(X, A) = \operatorname{coker}(i_*)$.

Remark. There is an induced $\partial : C_n(X, A) \to C_{n-1}(X, A)$, because the following diagram commutes.
$$\begin{tikzcd} 0 \arrow[r] & C_n(A) \arrow[r, "i_*"] \arrow[d, "\partial"] & C_n(X) \arrow[r] \arrow[d, "\partial"] & C_n(X, A) \arrow[r] \arrow[d, "\partial"] & 0 \\ 0 \arrow[r] & C_{n-1}(A) \arrow[r, "i_*"'] & C_{n-1}(X) \arrow[r] & C_{n-1}(X, A) \arrow[r] & 0. \end{tikzcd}$$
Then, $(C_n(X, A), \partial)$ is a chain complex.[3]

[3]All boundary maps seem to be denoted with the same symbol. This isn't very confusing in practice, it turns out.

Definition. The homology of this complex is called the *relative homology* of (X, A): $H_n(X, A) = H_n(C_*(X, A))$.

From 210A, recall that a short exact sequence of chain complexes induces a long exact sequence in homology (or proven in Hatcher or Lang). Specifically, given a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & C'_n \arrow[r, "\varphi"] \arrow[d, "\partial'"] & C_n \arrow[r, "\psi"] \arrow[d, "\partial"] & C''_n \arrow[r] \arrow[d, "\partial''"] & 0 \\ 0 \arrow[r] & C'_{n-1} \arrow[r, "\varphi"'] & C_{n-1} \arrow[r, "\psi"'] & C''_{n-1} \arrow[r] & 0 \end{tikzcd}$$

where the rows are exact, one can define a *connecting homomorphism* $H_n(C''_*) \to H_{n-1}(C'_*)$ with a diagram chase: given a $[c] \in H_n(C''_*)$, choose a representative $c \in C''_n$, so it has a ψ-preimage $d \in C_n$, which maps to a ∂d in C_{n-1}. Since $\delta'' c = 0$ and the diagram commutes, then $\psi \partial d = \partial'' \psi d = 0$, and therefore since the bottom row is exact, then it can be pulled back to an $e \in C_{n-1}$. This is the assignment, but one must also check that it is well-defined.

The conclusion is that (well, there's a little more to show) the following sequence is long exact; the blue arrow is the connecting map.

$$\cdots \longrightarrow H_n(C'_*) \xrightarrow{\varphi_*} H_n(C_*) \xrightarrow{\psi_*} H_n(C''_*) \longrightarrow H_{n-1}(C'_*) \xrightarrow{\varphi_*} H_{n-1}(C_*) \xrightarrow{\psi_*} \cdots$$

Now, we can apply this to (2.1) to get a long exact sequence

$$H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X, A) \longrightarrow H_{n-1}(A) \longrightarrow H_{n-1}(X) \longrightarrow \cdots$$

The idea is, we can understand homology inductively: if we know it for A, we may be able to extend it to X, and understand X in terms of smaller components.

Furthermore, if $B \subset A \subset X$, then we get another short exact sequence

$$0 \longrightarrow C_n(A, B) \longrightarrow C_n(X, B) \longrightarrow C_n(X, A) \longrightarrow 0.$$

These end up being chain maps, so they induce a long exact sequence for the homology groups of this triple of spaces.

Relative homology is a generalization of absolute homology, since $H_n(X) \cong H_n(X, \emptyset)$, as the empty set generates the trivial group, and modding out by that doesn't change anything. Intuitively, homology measures holes in a surface: $\mathbb{R}^3 \setminus 0$ is the kind of hole that contributes to H_2 being nonzero, but $\mathbb{R}^3$ minus a line contributes to H_1 being nonzero. Relative homology, even more handwavily, can be a measurement of the holes in X disregarding those already in A.

Of course, defining "holes" rigorously in a topological space is difficult; one of the reasonable ways to do this is the rank of a given homology group!

Homotopy Invariance. Throughout this class, the term *map* will always refer to a continuous map.

Definition. Let $f, g : X \to Y$ be maps of topological spaces. A *homotopy* $f \simeq g$ is a map $H : I \times X \to Y$ (where $I = [0, 1]$ with the usual topology) continuous in the product topology such that $H(0, t) = f(t)$ and $H(1, t) = g(t)$.

If there exists a homotopy between f and g, one says they are *homotopic*.

Theorem 2.2. *Let $f, g : X \to Y$ be homotopic. Then, $f_*, g_* : H_n(X) \to H_n(Y)$ are equal.*

Notice that to even make this statement, we need to use the functoriality of H_n.

Definition. A map $f : X \to Y$ is a *homotopy equivalence* if there exists a $g : Y \to X$ if $f \circ g \simeq \mathrm{id}$ and $g \circ f \simeq \mathrm{id}$. Then, X and Y are said to be *homotopically equivalent*, which is also denoted $X \simeq Y$.

This is a notion somewhat like homeomorphism, but weaker.

Corollary 2.3. *If $f : X \to Y$ is a homotopy equivalence, then $f_* : H_n(X) \to H_n(Y)$ is an isomorphism.*

Proof. Since f is a homotopy equivalence, then there exists a $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}$ and $g \circ f \simeq \mathrm{id}$. Since homology is functorial, then $f_* \circ g_* = (f \circ g)_* = \mathrm{id}_* = \mathrm{id}$ as maps $H_n(X) \to H_n(X)$, and similarly $g_* \circ f_* = \mathrm{id}_* = \mathrm{id}$ as maps $H_n(Y) \to H_n(Y)$, so f_* and g_* are inverses (as group homomorphisms), so $H_n(X) \cong H_n(Y)$ for all n. ⊠

This is what we mean by the homotopy invariance of homology.

Often, this is presented as using homology to prove that two spaces *aren't* homotopy equivalent (i.e. if they have different homology groups). We'll see some examples soon.

We won't prove Theorem 2.2 today, but we can develop some ideas. The key is that $H : I \times X \to Y$ induces a *chain homotopy* from f_* to g_* on $C_*(X) \to C_*(Y)$ (it's not yet equivalence, sadly), i.e. homomorphisms $T = T_H : C_n(X) \to C_{n+1}(Y)$ such that $T \circ \partial + \partial \circ T = g_* - f_*$ (maps $C_n(X) \to C_n(Y)$). Once we show this, it's easy to finish: if $c \in C_n(X)$ is such that $\partial c = 0$, then $g_*(c) - f_*(c) = T\partial C + \partial Tc = \partial Tc$, so $[\partial Tc] = 0$, and therefore $[g_*c] = [f_*c]$.

Suppose X and Y are two spaces; then, there's a good way of combining an m-chain in X and an n-chain in Y to get an $(m+n)$-chain in $X \times Y$, called the *cross product*, denoted $C_m(X) \times C_n(Y) \xrightarrow{\times} C_{n+m}(X \times Y)$, such that:

1. $\times$ is bilinear;
2. $\partial(\sigma \times \tau) = (\partial\sigma) \times \tau + (-1)^n \sigma \times \partial\tau$; and
3. if $n = 0$ (recall that 0-chains are just formal linear combinations of points), then any $x_0 \in X$ gives $x_0 \times \sigma$ is just the Cartesian product, sending $t \mapsto (x_0, \sigma(t))$ (and $m = 0$ is similar).

This will likely be given as an exercise on the problem sets.

3. Homotopy Invariance of Singular Homology: 1/15/15

The goal of today's lecture is to prove homotopy invariance of singular homology and an excision property. These will be technical and from the definition, but after this, things become more algebraic, involving long exact sequences rather than the basic definitions.

Using these properties, we'll also be able to calculate $H_k(D^n, \partial D^n)$, the singular homology of a disc relative to its boundary; this is $\mathbb{Z}$ when $k = n$ and 0 otherwise.

Recall that $C_k(X) = \bigoplus_\Sigma \mathbb{Z}$, where Σ is the set of maps $\sigma : \Delta^n \to X$, so that a homotopy $H : I \times X \to Y$ of f and g induces a $T = T_H : C_k(X) \to C_k(Y)$, such that $T \circ \partial + \partial \circ T = g_* - f_*$. On the homework, we'll introduce a *cross product* (from p. 277), which does satisfy the universal property of the tensor product, given by $C_n(X) \otimes C_m(Y) \to C_{n+m}(Y)$, sending $\sigma \otimes \tau \mapsto \sigma \times \tau$, such that $\partial(\sigma \times \tau) = (\partial\sigma) \times \tau + (-1)^n \sigma \times (\partial\tau)$.

Continuation of proof of homotopy invariance. Showing homotopy invariance will be fairly straightforward once we write down T_H; it'll end up as the cross product with a specific element. Let $\sigma : \Delta^1 \to I$ send $(t_0, t_1) \mapsto t_1$, so that $\partial\sigma = \underline{1} - \underline{0}$. Then, given a homotopy $H : I \times X \to Y$ and a $\tau : \Delta^m \to X$, $\sigma \times \tau$ is an $(m+1)$-chain, so let $T(\tau) = H_*(\sigma \times \tau) \in C_{m+1}(Y)$.

Let's see why this satisfies the required properties:

$$\begin{aligned}\partial(T\tau) &= \partial H_*(\sigma \times \tau) = H_*(\partial(\sigma \times \tau)) \\ &= H_*((\partial\sigma) \times \tau - \sigma \times (\partial\tau)) \\ &= H_*(\underline{1} - \underline{0}) \times \tau - \underbrace{H_*(\sigma \times (\partial\tau))}_{T\partial(\tau)}.\end{aligned}$$

Thus, $(\partial T + T\partial)(\tau) = H_*(\underline{1} \times \tau) - H_*(\underline{0} \times \tau)$. But the cross product tells us how it interacts with 0-chains: $\underline{0} \times \tau : \Delta^m \to I \times X$ sends $t \mapsto (0, t)$, and therefore $H_*(\underline{0} \times \tau) : \Delta^m \to Y$ sends $t \mapsto H(0, t) = f(t)$, and $\underline{1}$ is similar. Thus, $(\partial T + T\partial)(\tau) = g_*(t) - f_*(t)$. ⊠

Note that Hatcher doesn't give the full definition of the cross product until much later, and uses the special case, which he calls the *prism operator*.

Since $\mathbb{R} \xrightarrow{\simeq} \{\bullet\}$, then we now know $H_k(\mathbb{R}) = \mathbb{Z}$ when $k = 0$ and is 0 otherwise. But to do more serious calculations we'll need the excision property, which makes precise what exactly relative homology measures.

Proposition 3.1 (Excision). *Suppose $Z \subset A \subset X$ and $\overline{Z} \subset \mathring{A}$ (i.e. the closure of Z is contained in the interior of A); then, there's an inclusion of the pair $(X \setminus Z, A \setminus Z) \subset (X, A)$; then, the induced $H_k(X \setminus Z, A \setminus Z) \to H_k(X, A)$ is an isomorphism.*

The notion of excision means cutting-out things; here, if something is "sufficiently inside" A, we can cut it out without affecting the relative homology. Surprisingly, this will be suffcient (along with homotopy invariance) to leave the definition away for a lot of calculations.

Proof of Proposition 3.1. The easiest way to show an isomorphism of homology would be to show it on the chains, though we won't be able to do that here. However, we do have a map $\varphi : C_k(X \setminus Z, A \setminus Z) \hookrightarrow C_k(X, A)$. (Recall that $C_k(X, A)$ is the free abelian group on chains $\sigma : \Delta^k \to X$ such that $\sigma(\Delta^k) \not\subseteq A$.)

Thus, these two groups are generated by almost the same sets, though on the left we also need $\sigma(\Delta^k) \cap Z = 0$. However, φ isn't surjective, e.g. it doesn't hit a path (1-chain) from a $z \in Z$ to an $x \in X \setminus A$. So we will have to pass to homology: the idea is to prove that all cycles in $C_k(X, A)$ are homologous to cycles in $C_k(X \setminus Z, A \setminus Z)$, by subdividing all simplices (*barycentric subdivision*) as in Figure 1. We want a subdivision process that replaces cycles with homologous cycles.

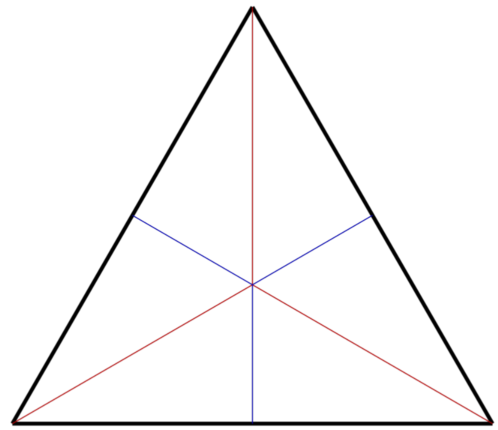

FIGURE 1. Barycentric subdivision of a 2-simplex; the process is analogous in higher dimensions. Source: http://en.wikipedia.org/wiki/Barycentric_subdivision.

This actually means we can prove something stronger than the isomorphism on homology: we'll end up proving Theorem 3.4 below.

The key is a *subdivision operator* $S : C_k(X) \to C_k(X)$ which sends a simplex to the sum of simplices on the subdivided components, and if we can show that $S \circ \partial = \partial \circ S$, then we can get that a related T satisfies $T \circ \partial + \partial \circ T = \mathbb{1} - S$ (here, $\mathbb{1} = \mathrm{id}$), so it's a chain homotopy.

First, as a special case, suppose $X \subset \mathbb{R}^n$ is a convex subset. Then, consider the *linear chains* $LC_k(X) \subset C_k(X)$ given by linear combinations of affine maps $\Delta^n \to X$. Barycentric subdivision can be defined inductively: it doesn't do anything to 0-chains, it splits a 1-chain down the middle, it splits a 2-chain as in Figure 1, and so on.

If $x \in X$, we get a map (the *cone operator*) $c_x : LC_k(X) \to LC_{k+1}(X)$ sending $[v_0, \dots, v_k] \mapsto [x, v_0, \dots, v_k]$. This is not a chain map:

$$\begin{aligned} \partial c_n[v_0, \dots, v_k] &= \partial[x, v_0, \dots, v_k] \\ &= [v_0, \dots, v_k] - \sum_{i=0}^{k} (-1)^i [x, v_0, \dots, \widehat{v}_i, \dots, v_k] \\ &= [v_0, \dots, v_k] - \sum_{i=0}^{k} c_x[v_0, \dots, \widehat{v}_i, \dots, v_k] \\ &= [v_0, \dots, v_k] - c_k(\partial[v_0, \dots, v_k]). \end{aligned}$$

Thus, $\partial \circ c_x + c_x \circ \partial = \mathbb{1}$.

Now we can define $S : LC_k(X) \to LC_k(X)$ inductively: if $k = 0$, $S\sigma = \sigma$, and in general, take the *barycenter*

$$\underline{\sigma} = \frac{1}{k+1} \sum_{i=0}^{k} v_i,$$

which is a point in X, because X is convex. Then, define $S\sigma = c_{\underline{\sigma}} S(\partial\sigma)$.

Now, we can check that S and ∂ commute. This is trivial when $k = 0$, so assume it's true up to $k - 1$, and pick a $\sigma \in LC_k(X)$. Then,

$$\begin{aligned} \partial S\sigma &= \partial(c_{\underline{\sigma}} S(\partial\sigma)) \\ &= S(\partial\sigma) - c_{\underline{\sigma}}(\partial S(\partial\sigma)) \\ &= S\partial\sigma - c_{\underline{\sigma}}(S\partial^2\sigma) = S\partial\sigma, \end{aligned}$$

using that S and ∂ commute for linear $(k-1)$-chains. Thus, they do so for k-chains as well.

Now, define $T : LC_k(X) \to LC_{k+1}(X)$ inductively: there's only one possible map when $k = -1$, and then if it's defined for $k - 1$, define it for k as follows: $T\sigma = c_{\underline{\sigma}}(\sigma - T(\partial\sigma))$. Then, (check the calculation or look it up in Hatcher), T is a chain homotopy: $T \circ \partial + \partial \circ T = \mathbb{1} - S$ as maps from $LC_k(X) \to LC_k(X)$.

Suppose $X \subset \mathbb{R}^k$ and $Y \subset \mathbb{R}^\ell$ and $\varphi : \mathbb{R}^k \to \mathbb{R}^\ell$ is affine; then, it sends simplices to simplices and preserves barycenters. In particular, the following diagram commutes.

$$\begin{tikzcd} LC_k(X) \arrow[r, "S"] \arrow[d, "\varphi_*"] & LC_k(X) \arrow[d, "\varphi_*"] \\ LC_k(Y) \arrow[r, "S"] & LC_k(Y) \end{tikzcd} \tag{3.1}$$

Categorically, this is exactly the statement that S is a *natural transformation*; a similar statement holds for T.

Now, for more general spaces X, we can extend "by naturality:" if $\sigma : \Delta^k \to X$ is in $C_k(X)$, then we have a $\sigma_* : C_k(\Delta^k) \to C_k(X)$. Let ι_k be the identity map of Δ^k; then, we can consider $\sigma \in C_k(X)$ as $\sigma_*(\iota_k)$.

But since Δ^k is a convex subset of $\mathbb{R}^k$, then we already know what $S\iota_k$ and $T\iota_k$ are, in $LC_k(\Delta^k)$ and $LC_{k+1}(\Delta^k)$, respectively. Thus, we can define $S\sigma = S(\sigma_*\iota_k) = \sigma_*(S\iota_k)$ and $T\sigma = \sigma_*(T\iota_k)$.

Notice that if X is a convex subset of $\mathbb{R}^n$, this is identical to the above definition, which boils down to (3.1), and if otherwise, it's a complete formality to verify the definitions.

$$\begin{aligned} \partial S\sigma &= \partial\sigma_*(S\iota_k) = \sigma_*(\partial S\iota_k) = \sigma_*(S\partial\iota_k) \\ S(\partial\sigma) &= S(\partial\sigma_*\iota_k) = S(\sigma_*\partial\iota_k). \end{aligned}$$

In order to make the above precise, we have to check that the more general S commutes with continuous $f : X \to Y$, i.e. it is again a natural transformation, which is equivalent to the following diagram commuting.

$$\begin{tikzcd} C_k(X) \arrow[r, "S"] \arrow[d, "f_*"] & C_k(X) \arrow[d, "f_*"] \\ C_k(y) \arrow[r, "S"] & C_k(Y) \end{tikzcd} \tag{3.2}$$

This is once again a formality: since everything is linear, we can check on generators, so if $\sigma : \Delta^k \to X$, then $f_*(S\sigma) = (f \circ \sigma)_*(S\iota_k)$, and $S(f_*\sigma) = (f \circ \sigma)_*(S\iota_k)$ by definition, so (3.2) commutes; thus, C_k is a functor and S is a natural transformation from C_k to itself.

We still have to verify that $T \circ \partial + \partial \circ T = \mathbb{1} - S$ in this general setting, but it's the same kinds of formal manipulations as for S, and will be skipped. We've done the bulk of the hard work.

Another thing which we'll have to skip due to time constraints is the following fact in Euclidean geometry: suppose $\sigma : \Delta^k \to X$, where $X \subset \mathbb{R}^n$ is convex. Then, $S\sigma$ is a linear combination of simplices whose diameter is less than $(k/(k+1)) \operatorname{diam}(\sigma)$ (which is clear if one thinks of the geometric meaning of barycentric subdivision, but still needs to be checked algebraically). This implies Corollary 3.2.

There's not much left of the proof, but we'll have to finish it next lecture.

Definition. Suppose $U = (U_i)$ is a cover of X (so that $X = \bigcup_i \mathring{U}_i$). Define $C_k^U(X) \subset C_k(X)$ (the *U-small chains*) to be the group generated by maps $\sigma : \Delta^k \to X$ such that $\sigma(\Delta^k) \subset U_i$ for some i for each σ (i can vary).

Corollary 3.2. *For any collection of subsets $U_i \subset \Delta^k$, where $\Delta_k = \bigcup \mathring{U}_i$, $S^N(\iota_k)$ is a linear combination of simplicies of diameter $(k/(k+1))^N$, which goes to 0, so if N is sufficiently large, then $S^N(\iota_k) \in C_k^U(\Delta^k)$.*

Corollary 3.3. *If U is a cover of $X = \bigcup \mathring{U}_i$, then any $c \in C_k(X)$ is a finite linear combination of U-small simplices.*

Theorem 3.4. *For any such cover $U = (U_i)$, $H_k(C_*^U(X)) \xrightarrow{\sim} H_k(C_*(X))$.*

4. Applications of Homotopy Invariance and Excision: 1/20/15

Last time, we defined *small chains* relative to a cover $U = (U_i)$ of a space X, so that $C_*^U(X) \subset C_*(X)$, and proved most of the excision theorem, Proposition 3.1.

Continuation of proof of Proposition 3.1. Recall that we constructed $S : C_k(X) \to C_k(X)$ and $T : C_k(X) \to C_{k+1}(X)$ such that $S\partial = \partial S$ (i.e. S is a chain map) and $T\partial + \partial T = \mathbb{1} - S$ (i.e. it's chain homotopic to the identity).

If $\sigma : \Delta^k \to X$ and $X = \bigcup \mathring{U}_i$, then $\Delta^k = \bigcup_i \sigma^{-1}(\mathring{U}_i)$, so $S^N\iota$ (where $\iota \in C_k(\Delta^k)$ is an inclusion map, as before) is a linear combination of simplices contained in some $\sigma^{-1}(\mathring{U}_i)$. This is so that we can subdivide simplices to prove excision; the upshot is that $S^N\iota \in C_k^U(X)$.

Now, let's attack the content of the theorem. To show surjectivity, start with a cycle $c \in C_k(X)$, so that $\partial c = 0$, and therefore $[c] = [S^N c]$ for any N, i.e. we're able to pick a small homologous cycle. In fact, by what we just said above, we're able to get it in the U-small chains, and therefore $[c] = [S^N c] \in \operatorname{Im}(H_k(C_*^U(X)))$ for large N. Thus, it's surjective.

For injectivity, suppose $c, c' \in C_k^U(X)$ are cycles and $c - c' = \partial x$ for an $x \in C_k(X)$. Then, for sufficiently large N, $S^N x \in C_k^U(X)$, and so $\partial S^N x = S^N(\partial X) = S^N c - S^N c'$, i.e. $[c] = [c']$ within $H_k(C_*^U(X))$. This isn't quite what we want, but since $c - Sc = \partial(Tc)$, and so we can do this more generally. ⊠

You can get quite a lot of mileage out of these theorems, but before that let's talk a little more about relative homology; each of the two theorems has an analogue in relative homology relying on a small formal argument.

Lemma 4.1 (Five lemma)**.** *Given a commutative diagram of abelian groups*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\varphi_1"] & B \arrow[r] \arrow[d, "\varphi_2"] & C \arrow[r] \arrow[d, "\varphi_3"] & D \arrow[r] \arrow[d, "\varphi_4"] & E \arrow[d, "\varphi_5"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

where the rows are exact, if φ_1, φ_2, φ_4, and φ_5 are isomorphisms, then φ_3 is an isomorphism as well.

The proof is a formal argument which we saw in 210A (or in Hatcher); it's important, but there's no reason to do it more than once.

This lemma is extremely useful when combining with long exact sequence: if you have a long exact sequence, you should feel some urge to approach it with the five lemma.

For example, we can prove a relative-homological version of homotopy invariance.

Proposition 4.2. *Suppose (X, A) and (Y, B) are pairs, and f is a* map of pairs, *i.e. $f : X \to Y$ is continuous and $f(A) \subset B$. If $f : X \overset{\simeq}{\to} Y$ (i.e. it's a homotopy equivalence) and $f|_A : A \overset{\simeq}{\to} B$ is also a homotopy equivalence, then $f_* : H_k(X, A) \to H_k(X, B)$ is an isomorphism.*

Proof. We'll use the absolute version and Lemma 4.1; it takes longer to write the diagram than to finish the rest of the proof. The following diagram commutes (which is quick to check).

$$\begin{tikzcd}
H_k(A) \arrow[r] \arrow[d] & H_k(X) \arrow[r] \arrow[d] & H_k(X, A) \arrow[r] \arrow[d, "f"] & H_{k-1}(A) \arrow[r] \arrow[d] & H_{k-1}(X) \arrow[d] \\
H_k(B) \arrow[r] & H_k(Y) \arrow[r] & H_k(Y, B) \arrow[r] & H_{k-1}(B) \arrow[r] & H_{k-1}(Y)
\end{tikzcd}$$

Then, by homotopy invariance for absolute homology, the blue arrows are all isomorphisms, and so f is as well. ⊠

Excision is similar, and just as easy.

Proposition 4.3 (Excision for pairs)**.** *Suppose $U_i \subseteq X$ and covers it, i.e. $X = \bigcup \mathring{U}_i$, and suppose further that $A \subset X$ and $A = \bigcup \operatorname{Int}(A \cap U_I)$. Let $U' = (A \cap U_i)$; then, $H_k(C_*^U(X, A)) \cong H_k(X, A)$.*

Proof. This time, we'll make a diagram on the chain level. The following diagram commutes,

$$\begin{tikzcd}
0 \arrow[r] & C_k^{U'}(A) \arrow[r, "\varphi"] \arrow[d, "\wr"] & C_k^U(X) \arrow[r] \arrow[d, "\wr"] & C_k^{U''}(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_k(A) \arrow[r] & C_k(X) \arrow[r] & C_k(X, A) \arrow[r] & 0
\end{tikzcd}$$

where $C_k^{U''}(X, A)$ is defined to be $\operatorname{coker}(\varphi)$. Then, both rows are exact, and all of the vertical maps are isomorphisms, so Lemma 4.1 implies the homologies are the same. ⊠

In the future, arguments such as these will be abbreviated; the five lemma is frequently used to obtain relative results from absolute ones.

Note that Proposition 4.3 is stated in a different form than Proposition 3.1 was; Hatcher restates the absolute theorem in these terms as well. It can be summarized as "small chains have the same homology." The absolute and relative versions can be summarized in a diagram: suppose $Z \subset A \subset X$, such that $\overline{Z} \subset \mathring{A}$. (Then, absolute excision says that $H_k(X \setminus Z, A \setminus Z) \xrightarrow{\sim} H_k(X, A)$.)

$$\begin{tikzcd}
C_k(X \setminus Z, A \setminus Z) \arrow[r] \arrow[d, equal] & C_k(X, A) \\
C_k(X \setminus Z, A \setminus Z) \arrow[r] & C_k^U(X, A) \arrow[u, "f"']
\end{tikzcd}$$

Here, f is guaranteed to be an isomorphism by the relative version of the theorem, and the equality on the left is by the absolute theorem. $C_k(X \setminus Z, A \setminus Z)$ is free on maps $\sigma : \Delta^k \to X$ such that $\sigma(\Delta^k) \cap Z = \emptyset$ and $\sigma(\Delta^k) \not\subset A$.

Now, we can talk about the other formulation of excision.

Definition. A *deformation retraction* is a homotopy $H : I \times U \to U$ such that $H(0, -) = \mathrm{id}$ and $H(1, -) : U \to A$, where for all $a \in A$ and $t \in I$, $H(t, a) = H(0, a)$.

Intuitively, a deformation retract is a strong form of homotopy invariance. There are some good examples in Hatcher.

Definition. A space X is *contractible* if it is homotopy equivalent to a point.

For example, the disc $\{x \in \mathbb{R}^2 : \|x\| \leq 1\}$ retracts to its center, so it is contractible.

Definition. (X, A) is a *good pair* if A is closed and there exists an open $U \subset X$ such that $A \subset U$ and U deformation retracts to A.

This seems a little bit strange, but almost anything you can think of makes a good pair. Pathological examples do exist, however.

Theorem 4.4 (Excision, version 2)**.** *If (X, A) is a good pair, the map $\pi_* : H_k(X, A) \to H_k(X/A, A/A)$ (induced by $\pi : X \to X/A$) is an isomorphism.*

This isn't an isomorphism on the chain level (unless one uses small chains).

Proof. Again, this is a consequence of Proposition 3.1 and some diagrams, such as the following one, which was induced from a commutative diagram of pairs of spaces, and is therefore commutative.

$$\begin{tikzcd}
H_k(X, A) \arrow[r] \arrow[d, "f"] & H_k(X/A, A/A) \arrow[d, "g"] \\
H_k(X, U) \arrow[r] & H_k(X/A, U/A) \\
H_k(X \setminus A, U \setminus A) \arrow[r, "\varphi"] \arrow[u, "h"] & H_k((X/A) \setminus (A/A), (U/A) \setminus (A/A)) \arrow[u, "\ell"]
\end{tikzcd}$$

Then, f and g are isomorphisms because they were induced by homotopy invariance (by Proposition 4.2), and h and ℓ are by (the first version of) excision, and φ is an isomorphism because it's induced by a homeomorphism of topological spaces. Thus, we can chase the isomorphism around the diagram. ⊠

Now, we're in business to do some calculations. Singular homology is still generated by scary-looking uncountably generated chains, but we can use these tools to get a better grasp on it.

Recall that if $B \subset A \subset X$, we get the long exact sequence of a triple, induced by the short exact sequence

$$0 \longrightarrow C_k(A, B) \longrightarrow C_k(X, B) \longrightarrow C_k(X, A) \longrightarrow 0.$$

Proposition 4.5. *Three results on homology groups:*

(1)

$$H_k(D^n, \partial D^n) \cong \begin{cases} \mathbb{Z}, & k = n \\ 0, & k \neq n. \end{cases}$$

(2)

$$H_k(S^n) = \begin{cases} \mathbb{Z}^2, & k = 0, n = 0 \\ \mathbb{Z}, & k = 0, k = n \\ 0 & \text{otherwise.} \end{cases}$$

(3)

$$H_k(S^n, \mathrm{pt}) = \begin{cases} \mathbb{Z}, & k = n \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We'll apply these ideas to $X = D^n = \{x \in \mathbb{R}^n \mid \|x\| \le 1\}$ and $A = \partial D = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$.

We computed the homology groups of a point; thus, if Y is any contractible topological space, then

$$H_k(Y) = \begin{cases} \mathbb{Z}, & k = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Since ∂D^n is a deformation retract of $D^n \setminus \{0\}$, then we have an isomorphism $H_k(D^n, \partial D^n) \xrightarrow{\sim} H_k(D^n/\partial D^n, \mathrm{pt})$. Using the triple $\mathrm{pt} \subset \partial D^n \subset D^n$, we have a long exact sequence

$$H_k(\partial D^n, \mathrm{pt}) \longrightarrow H_k(D^n, \mathrm{pt}) \longrightarrow H_k(D^n, \partial D^n) \longrightarrow H_{k-1}(\partial D^n, \mathrm{pt}) \longrightarrow H_{k-1}(D^n, \mathrm{pt}) \longrightarrow \cdots$$

where $H_n(D^n, \mathrm{pt}) = 0$, because both are contractible. Thus, we have isomorphisms $H_k(D^n, \partial D^n) \xrightarrow{\sim} H_{k-1}(\partial D^n, \mathrm{pt})$ from this long exact sequence, and by excision, $H_k(D^n, \partial D^n) \cong H_k(D^n/\partial D^n, \mathrm{pt})$. But $D^n/\partial D^n \cong S^n$.

The crucial inductive step to prove everything is that from excision, $H_k(\mathrm{pt}) \cong H_k(S^n \setminus \mathrm{pt}, \varnothing) \xrightarrow{\sim} H_k(S^n, \mathrm{pt})$. ⊠

Here's another classical application, which uses the functoriality of homology.

Theorem 4.6. *There does not exist a continuous map $r : D^n \to \partial D^n$ such that $r(x) = x$ for $x \in \partial D$.*

Proof. One way of thinking of a functor is that it sends commutative diagrams to commutative diagrams. Specifically, we have

$$\begin{tikzcd} \partial D^n \arrow[rr, "\mathrm{id}"] \arrow[dr, "\iota"'] & & \partial D^n \\ & D^n \arrow[ur, "r"'] & \end{tikzcd}$$

where $\iota : \partial D^n \to D^n$ is given by inclusion. Thus, we get a diagram

$$\begin{tikzcd} H_{n-1}(\partial D^n) = \mathbb{Z} \arrow[rr, "\mathrm{id}"] \arrow[dr, "\iota_*"'] & & H_{n-1}(\partial D^n) = \mathbb{Z} \\ & H_{n-1}(D^n) = 0, \arrow[ur, "r_*"'] & \end{tikzcd}$$

which is a contradiction: we can't commute through 0 and get the identity. ⊠

Another common first proof in algebraic topology is Brower's fixed-point theorem.

Theorem 4.7 (Brower's fixed-point)**.** *Every continuous $f : D^n \to D^n$ has a fixed point.*

Proof sketch. If not, one can construct a retract $r : D^n \to \partial D^n$ by drawing a line through x and $f(x)$, and then sending x to the nearest of the two points on ∂D that intersect that line.

One somewhat annoying question is, we know $H_n(D^n, \partial D^n) \cong \mathbb{Z}$, but can we write down an actual isomorphism? This relates to the question equating singular and simplicial homology, since it might be much easier if we knew that. We do know D^n is homeomorphic to Δ^n, though (albeit not naturally), so we can recast the question as finding a generator for $H_n(\Delta^n, \partial\Delta^n)$.

Proposition 4.8. $[\iota]$ *(the homology class of the identity) is a generator for the infinite cyclic group $H_n(\Delta^n, \partial\Delta^n)$.*

Proof. There's a reason a simplex is called Δ; it has the right shape. Thus, we can define a similarly well-named $\Lambda \subset \partial\Delta^n$, which is the union of the i^{th} faces, for $i > 0$. (Thus, when $n = 2$, it does look like a Λ.) And Λ is contractible.

We can compute the long exact sequence of $\Lambda \subset \partial\Delta^n \subset \Delta^n$, and since Λ is contractible, its homology groups go to 0, so the connecting homomorphism becomes an isomorphism: $H_k(\Delta^n, \partial\Delta^n) \xrightarrow{\sim} H_{k-1}(\partial\Delta^n, \Lambda)$, but by excision of the 0^{th} face, $H_{k-1}(\Delta^{n-1}, \partial\Delta^{n-1}) \xrightarrow{\sim} H_{k-1}(\partial\Delta^n, \Lambda)$. Specifically, this follows because both are good pairs and their quotients are homeomorphic.

Next, we can check where the identity map actually goes: it's sent to the class of the 0^{th} face, which comes from the identity map on Δ^{n-1}, and then induction deals with it. ⊠

"We're a good pair, me and you."

Recall that if (X, A) is a good pair (defined last time to be not pathological), then $H_k(X, A) \xrightarrow{\cong} H_k(X/A, A/A)$. We also stated (albeit quickly) that if both (X, A) and (Y, B) are good pairs, and $f : X \to Y$ and $f(A) \subset B$, then if f induces a homotopy equivalence $X/A \simeq Y/B$, then $H_k(X, A) \cong H_k(Y, B)$ for all k.

To prove this, we used $\Delta^{n-1} \hookrightarrow \partial\Delta^n$ as the 0^{th} face; then, we have good pairs $(\Delta^{n-1}, \partial\Delta^{n-1})$ and $(\partial\Delta^n, \Lambda)$; this was used to prove that the homology class of the identity generates $H_n(\Delta^n, \partial\Delta^n)$.

Suppose X_j are spaces indexed by a $j \in J$. Then, one can take the *disjoint union*

$$X = \coprod_{j \in J} X_j,$$

with the *disjoint union topology*, where open sets are given by the natural maps $X_j \hookrightarrow X$ (a set is open iff its intersection with each X_j is open), and each X_j is open and closed in X (no gluing). These induce homomorphisms $H_k(X_j) \to H_k(X)$, and therefore by the universal property of the direct sum, we get a map

$$\bigoplus_j H_k(X_j) \to H_k(X).$$

Each simplex $\sigma : \Delta^k \to X$ factors through some X_j (since σ must be continuous). Hence,

$$C_k(X) \xleftarrow{\cong} \bigoplus_{j \in J} C_k(X_j),$$

and it follows that

$$\bigoplus_{j \in J} H_k(X_j) \xrightarrow{\cong} H_k(X).$$

There is a similar statement for relative homology; if $A_j \subset X_j$ and $A = \bigcup A_j$, then

$$H_k(X, A) \xleftarrow{\cong} \bigoplus_{j \in J} H_k(X_j, A_j),$$

which follows, just like last time, by writing down the long exact sequence and using the five lemma.

In the same way that we defined relative singular homology, one can define *relative simplicial homology*; if $A \subset X$ is a sub-Δ-complex, then we can take the relative simplicial case $\Delta_k(X, A) = \Delta_k(X)/\Delta_k(A)$, and then define $H_k^\Delta(X, A)$ by taking the homology of the resulting chain complex. Since the empty set is a sub-Δ-complex, then this is more general, and when we prove the equivalence of singular and simplicial homology, it'll actually be easier to prove it in the relative case.

On the topic of the empty set, $\varnothing \subset X$ is a good pair, which is perhaps surprising, as is $(\varnothing, \varnothing)$. There's another common gotcha: the quotient topology X/A seems like it would identify all points in A, within X, but it actually is made by taking some extra point and identifying all points in A with it. This only matters for $A = \varnothing$, but there it could trip you up: $X/\varnothing = X \sqcup \{\text{pt}\}$, and $\varnothing/\varnothing = \{\text{pt}\}$.

Definition. If X is a Δ-complex, then its *n-skeleton* X^n is the union of the simplices in X of dimension at most n.

Theorem 5.1. *Let X be a Δ-complex. Then, $H_k^\Delta(X) \cong H_k(X)$.*

Proof. Suppose $X = \Delta^n$ and $A = \partial\Delta^n$. Then, $H_k^\Delta(\Delta^n, \partial\Delta^n) = \mathbb{Z}$ when $k = n$ and is 0 otherwise. The chain complex looks like $\cdots 0 \leftarrow 0 \leftarrow \mathbb{Z} \leftarrow 0 \leftarrow \cdots$; in this case, we computed the simplicial homology, so we know they're equal.

More generally, suppose $X = \coprod_\alpha \Delta^{n_\alpha}$, where α takes values in some indexing set, and let $A = \coprod_\alpha \partial\Delta^{n_\alpha}$. Thus, we get the following commutative diagram.

$$\begin{tikzcd}
H_k^\Delta(X, A) \arrow[r, "\sim"] & H_k(X, A) \\
\bigoplus_\alpha H_k^\Delta(\Delta^{n_\alpha}, \partial\Delta^{n_\alpha}) \arrow[u, "\wr"'] \arrow[r, "f"] & \bigoplus_\alpha H_k(\Delta^{n_\alpha}, \partial\Delta^{n_\alpha}) \arrow[u, "\wr"']
\end{tikzcd}$$

Since isomorphisms factor through direct sums, then f is an isomorphism.

More generally still, we can show that if X^n is the n-skeleton of a Δ-complex, then $H_k^\Delta(X^n, X^{n-1}) \to H_k(X^n, X^{n-1})$ is an isomorphism. It turns out that (X^n, X^{n-1}) is a good pair (the deformation retract is from the barycenter out to the faces), and therefore we have a map of good pairs

$$\left(\coprod_{n\text{-simplices of } X} \Delta^n, \coprod \partial\Delta^n \right) \longrightarrow (X^n, X^{n-1}),$$

i.e. it's continuous, restricts to a map on the smaller spaces, and even more, induces a homeomorphism on the quotients. Thus, we have the following diagram.

$$\begin{tikzcd}
H_k(\coprod \Delta^n, \coprod \partial\Delta^n) \arrow[r, "f_1"] & H_k(X^n, X^{n-1}) \\
H_k^\Delta(\coprod \Delta^n, \coprod \partial\Delta^n) \arrow[r, "f_2"] \arrow[u, "g_1"] & H_k^\Delta(X^n, X^{n-1}) \arrow[u, "g_2"]
\end{tikzcd}$$

Then, f_2 is an isomorphism because f_1 is (since we have a homeomorphism on the quotients as noted above, so just use excision), g_1 is (from the previous part), and g_2 is (which can be directly computed on the simplices).

Next, we can generalize even further: let X be any finite-dimensional Δ-complex, and $A = \emptyset$. We'll prove this by induction on $\dim(X) = n$, since $X = X^n$, and so we can assume that $H_k^\Delta(X^{n-1}) \xrightarrow{\cong} H_k(X^{n-1})$. Thus, we have two long exact sequences (we haven't talked about the long exact sequence in simplicial homology, but it's essentially the same notion) and a commutative diagram that reeks of the five lemma: the rows are exact.

$$\begin{tikzcd}
H_{k+1}^\Delta(X^n, X^{n-1}) \arrow[r] \arrow[d, "a"] & H_k^\Delta(X^{n-1}) \arrow[r] \arrow[d, "b"] & H_k^\Delta(X^n) \arrow[r] \arrow[d, "c"] & H_k^\Delta(X^n, X^{n-1}) \arrow[r] \arrow[d, "d"] & H_{k-1}^\Delta(X^{n-1}) \arrow[d, "e"] \\
H_{k+1}(X^n, X^{n-1}) \arrow[r] & H_k(X^{n-1}) \arrow[r] & H_k(X^n) \arrow[r] & H_k(X^n, X^{n-1}) \arrow[r] & H_{k-1}(X^{n-1})
\end{tikzcd}$$

Then, a and d are isomorphisms by the previous case, and b and e by induction, so by the five lemma, c is an isomorphism as well.

Now, we can address even infinite Δ-complexes X. In this case, X is the infinite union of its n-skeletons X^n over all $n \in \mathbb{N}$, so we have inclusions $X^n \hookrightarrow X$, which therefore induce maps

$$\begin{tikzcd}
\cdots \arrow[r] & H_k(X^n) \arrow[r] \arrow[dr] & H_k(X^{n+1}) \arrow[r] \arrow[d] & \cdots \\
& & H_k(X) &
\end{tikzcd} \tag{5.1}$$

Furthermore, we know that $H_k(X^n) \cong H_k(X^{n+1})$ when $n > k$, which can be checked with the long exact sequence and the relative-homological fact that $H_k(X^{n+1}, X^n) = 0$ unless $k = n + 1$.

Next, we'll need the following lemma from point-set topology.

Lemma 5.2. *If $K \subset X$ is sequentially compact, then it is contained in a finite subcomplex.*

Proof sketch. Suppose this is not true for some K; then, there is an infinite sequence $x_i \in K$ such that each x_i is in a different open simplex of X. Then, $\{x_i : i \in \mathbb{N}\}$ is closed in X, and is contained in K, but converges to something not in K, which is a contradiction to K being sequentially compact.

Recall that compactness implies sequential compactness, and in metrizable spaces the notions are equivalent.

Now, let's use it, because simplicial chains are defined as maps out of something compact.

Corollary 5.3. *Any $\sigma : \Delta^k \to X$ factors through some X^n.*

Corollary 5.4. *Any $c \in C_k(X^n)$ is in the image of some $C_k(X^n)$.*

Now, we can truly reduce to the finite case; since $H_k(X^{n-1}) \cong H_k(X^n)$ for $n > k$, then we can prove $H_k(X^n) \cong H_k(X)$: surjectivity follows from the above two corollaries by pulling chains back, and injectivity follows by chasing a $c \mapsto 0$ around. (One can make this a little fancier with direct limits.)

For simplicial homology, this is a bit easier, and is true on the chain level. Once you figure out what everything is saying, then it's not very bad. Thus, we know that

$$\begin{tikzcd} H_k^\Delta(X^{k+1}) \arrow[r, "\sim"] \arrow[d, "\wr"] & H_k(X^{k+1}) \arrow[d, "\wr"] \\ H_k^\Delta(X) \arrow[r, "f"] & H_k(X), \end{tikzcd}$$

and therefore f must be an isomorphism as well. ⊠

This gives the general absolute case of equivalence of singular and simplicial homology; for the general relative case, simply make a five lemma argument as usual.

Let's say a bit more about limits. Suppose X is a Hausorff space and $X^n \subset X^{n+1} \subset \cdots$ is a sequence of closed subspaces such that $X = \bigcup_n X^n$. Thus, we have a diagram that looks a lot like (5.1), though in this case it's more general. We can write $X = \varinjlim X^n$, and by the same argument as above, there is an isomorphism

$$\varinjlim_{n\to\infty} H_k(X^n) \xrightarrow{\sim} H_k(X) = H_k(\varinjlim_{n\to\infty} X^n).$$

One useful result of the equivalence of singular and simplicial homology is the well-definedness of simplicial homology.

Corollary 5.5. *Though* a priori *the simplicial homology should depend on the Δ-complex structure of X, it is in fact independent of it.*

Reading more into the equivalence, some finiteness results of $H_k(X)$ come out.

Theorem 5.6. *Suppose X is a compact space homotopy equivalent to a Δ-complex; then, $\bigoplus_k H_k(X)$ is a finitely generated abelian group.*

Proof. Given such an X, it's homotopy equivalent to a Δ-complex Y, and since X is compact, its image is contained in a finite Δ-subcomplex Y'. Then, by functoriality, the homotopy equivalence factors through Y', so applying $H_* = \bigoplus_k H_k$, we get that

$$\begin{tikzcd} H_*(X) \arrow[rr, "\mathrm{id}"] \arrow[dr] & & H_*(X) \\ & H_*(Y') \arrow[ur] & \\ & H_*^\Delta(Y') \arrow[u, "\wr"] & \end{tikzcd}$$

Then, $H_*^\Delta(Y')$ is finitely generated even on the chain level, and a map out of a finitely generated group has finitely generated image, so $H_*(X)$ is finitely generated. ⊠

This is nice, because this is a large class of spaces, and we have a nice classification of finitely generated abelian groups. This theorem can be used to prove that two spaces aren't homotopically equivalent.

Corollary 5.7. *Consider $X = \{0\} \cup \{1/n \mid n \in \mathbb{N}\}$, which is a compact space with the subspace topology from $\mathbb{R}$. Then, $H_0(X) \cong \bigoplus_{x\in X} \mathbb{Z}$, so X cannot be homotopy equivalent to any Δ-complex, even infinite ones.*

Remember that in topology, rather than saying two objects are isomorphic, it's generally better to say that a specific map is an isomorphism.

Interestingly, if one takes -1 instead of 0 in Corollary 5.7, the result is a discrete space, and so it is homotopy equivalent to a Δ-complex (recall that homotopies do not always preserve compactness).

Now, armed with this equivalence of the two homologies we built, we can make definitions that we wouldn't be able to otherwise, and prove useful things about them.

Definition. Let $f : S^{n-1} \to S^{n-1}$ be continuous. Then $f_* : H_{n-1}(S^{n-1}) \to H_{n-1}(S^{n-1})$ can be regarded as a map $\mathbb{Z} \to \mathbb{Z}$ by choosing the same isomorphism $H_{n-1}(S^{n-1}) \xrightarrow{\sim} \mathbb{Z}$. Then, the *degree* $\deg(f)$ is defined to be the degree of $f_* : \mathbb{Z} \to \mathbb{Z}$.

For example, $\deg(\mathrm{id}) = 1$, and the degree of a constant map is 0 (since it factors through a point, and $H_k(\mathrm{pt}) = 0$ when $k \neq 0$). Thus, we can see that the identity map isn't homotopic to a constant, which would have been somewhat difficult to prove without homology.

For proving things, it's useful to remember that doing things on the chain level is sort of a last resort, and it's nicer to use functoriality somehow, or a long exact sequence.

6. Degrees of Maps on S^n: 1/27/15

> *"After a while in algebraic topology, you get a little complacent and assume that any diagram you write down will commute, by naturality, blah, blah, blah... but this isn't always true."*

Recall that last time, we showed the equivalence of singular and simplicial homology, defined the degree $\deg(f) \in \mathbb{Z}$ of a continuous map $S^n \to S^n$, and proved Brower's fixed-point theorem.

Proposition 6.1. *If $f, g : S^n \to S^n$, then $\deg(f \circ g) = \deg(f)\deg(g)$.*

Proof. In one word, the proof would be "functoriality;" it's true that $(f \circ g)_* = f_* \circ g_*$ as maps $\mathbb{Z} \to \mathbb{Z}$, and therefore they have the same degree. ⊠

Suppose $A \in \mathrm{O}(n)$, i.e. A is an orthogonal matrix, so $A^{\mathrm{T}}A = I$. Then, $x \mapsto Ax$ is a map $S^n \to S^n$; what is its degree?

Proposition 6.2. *If A is a reflection, i.e. it's conjugate to* $B = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$, *then* $\deg(A) = -1$.

Proof. There are several way to prove this, e.g. it follows from proving that $\deg(B) = -1$ and the fact that if g is a homeomorphism, then $\deg(g^{-1} \circ f \circ g^{-1}) = \deg(f)$, because if g is a homeomorphism, then g_* is an isomorphism, and an isomorphism $\mathbb{Z} \to \mathbb{Z}$ must be multiplication by ± 1.

Now, we want to show that the degree of the map $(x_1, \dots, x_n) \mapsto (-x_1, x_2, \dots, x_n)$ is -1. Hatcher has an elegant proof involving gluing spaces together: consider the disjoint union $\Delta^n \coprod \Delta^n$, glued at the boundary (i.e. $x \in \partial\Delta_1^n \sim x \in \partial\Delta_2^n$): the result is homeomorphic to S^n, and gives S^n a Δ-complex structure with two n-simplices, an upper and a lower hemisphere.

If ι_1, ι_2 are the inclusions of Δ_1^n and Δ_2^n into the disjoint union (which was identified with S^n), then $\partial(\iota_1 - \iota_2) \in C_n(S^n)$, and therefore $\iota_1 - \iota_2 \in \Delta_n(S^n) \hookrightarrow C_n(S^n)$, and in fact $\iota_1 - \iota_2$ is a generator of $H_n(S^n) \cong \mathbb{Z}$. This is easiest to see in the simplicial chain complex $\Delta_*(S^n)$.

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\partial} \Delta_{n-1}(S^n) \longrightarrow \cdots$$

We know $\iota_1 - \iota_2$ is nontrivial; if it were, then it would be a boundary of something, and it's top-dimensional, so that doesn't work. Thus, it's in $\ker(\partial)$, which is isomorphic to $\mathbb{Z}$, since we know $H_n(S^n) \cong \mathbb{Z}$. Finally, it can't be a nontrivial multiple of anything else in $\Delta_*(S^n)$, because one can look at all of the simplices and check, and since passing to the homology doesn't involve modding out by an image (i.e. $H_n(S^n) = \ker(\partial)$), so it's still not a multiple of anything else in homology. Thus, it's a generator of $H_n(S^n)$.

Now, given $f : x \mapsto Bx$, parameterize the upper and lower hemispheres with ι_1 and ι_2; then, $f \circ \iota_1 = \iota_2$ and $f \circ \iota_2 = \iota_1$. In particular, $f_*(\iota_1 - \iota_2) = -(\iota_1 - \iota_2)$,[4] so f_* acts as -1 as a map $H_n^\Delta(S^n) \to H_n^\Delta(S^n)$, and therefore as a map $H_n(S^n) \to H_n(S^n)$ as well, so it has degree -1.

Thus, as stated above, if f is any reflection, it reduces to this case, and $\deg(f) = -1$. ⊠

Definition. If $f, g : U \to V$, then the *straight-line homotopy* from f to g is $H(x, t) = (1 - t)g(x) + tf(x)$.

This homotopy doesn't always exist (e.g. if U and V aren't convex, things can go wrong), but it's occasionally useful.

The following lemma comes from linear algebra, and we won't reprove it here.

Lemma 6.3. *Any $A \in \mathrm{O}(n)$ is a product of reflections.*

Since reflections have degree and determinant -1, then $\deg, \det : \mathrm{O}(n) \to \{\pm 1\}$; since they agree on reflections and all elements are products of reflections, then they're the same thing.

Corollary 6.4. *If $A \in \mathrm{O}(n)$, then $\deg(A) = \det(A)$ (as a map $x \mapsto Ax$).*

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[4] Note that where we say f_*, Hatcher writes $f_\sharp$, but otherwise this proof looks a lot like his.

One useful fact is that $S^{n-1} \simeq \mathbb{R}^n \setminus \{0\}$; consider the map $x \mapsto x/|x|$ from $\mathbb{R}^n \setminus 0 \to S^{n-1}$ along with inclusion in the other direction. Composing the two maps, in one direction, it's just the identity, and in the other, it's homotopy equivalent by the straight-line homotopy contracting $\mathbb{R}^n$ to S^{n-1} by lines through the origin. Thus, they are homotopy equivalent.

Thus, any $A \in \mathrm{GL}_n(\mathbb{R})$ induces a map $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ by $f(x) = Ax$. This induces a map $f_* : H_{n-1}(\mathbb{R}^n \setminus \{0\}) \to H_{n-1}(\mathbb{R}^n \setminus \{0\})$.

Proposition 6.5. *$f_* = \mathrm{sign}(\det A)$.*

Proof sketch. The proof rests on the *polar decomposition* $A = BC$, where B has positive eigenvalues, $B^{\mathrm{T}} = B$, and $\mathrm{sign}(\det B) = 1$, and $C \in \mathrm{O}(n)$. Thus, it suffices to see why multiplication by B is homotopic to the identity; this once again follows by checking the straight-line homotopy.

Corollary 6.6. *If $n > 1$, $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong \mathbb{Z}$, and by using the long exact sequence, the following diagram commutes.*

$$\begin{tikzcd}
H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \arrow[r] \arrow[d, "\cong"] & H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \arrow[d, "\cong"] \\
H_{n-1}(\mathbb{R}^n \setminus \{0\}) \arrow[r, "f_*"] & H_{n-1}(\mathbb{R}^n \setminus \{0\})
\end{tikzcd}$$

Thus, $f_* : H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \to H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ acts by $\mathrm{sign}(\det A)$.

Definition. Suppose $U \subset \mathbb{R}^n$ is open and $0 \in U$. Then, $f : U \to \mathbb{R}^n$ is C^2 if $f^{-1}(0) = 0$ and $Df(0) \in \mathrm{GL}_n(\mathbb{R})$.

That is, the derivative has to be nicely behaved. This is different from the usual definition in analysis, because we'll need $f^{-1}(0) = 0$ for topological reasons.

Claim. If $U \subset \mathbb{R}^n$ is open and $x \in U$, then $H_n(U, U \setminus \{x\}) \cong \mathbb{Z}$.

Proof. There's an inclusion of pairs $(U, U \setminus \{x\}) \hookrightarrow (\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$; then, use the relative version of the excision theorem. ⊠

Notice that it's not just abstract isomorphism; they're the same, and an isomorphism out of one gives an isomorphism out of the other.

Proposition 6.7. *If $U \subset \mathbb{R}^n$ is open and contains the origin, and if f is C^2, then $f_* : H_n(U, U \setminus \{0\}) \to H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ is given by* $\mathrm{sign}(\det Df(0)) : \mathbb{Z} \to \mathbb{Z}$.

Proof. The proof will be by Taylor expansion; one can write $f(x) = Ax + g(x)$, where $A \in \mathrm{O}(n)$ and $g(x)/|x| \to 0$ as $x \to 0$; in particular, $|g(x)| \leq |x|/2$ for $|x|$ small enough. Thus, it's possible to shrink U so that $|g(x)| \leq |x|/2$ for all $x \in U$ (which follows from excision, as long as it still contains 0).

In fact, one can assume $A = I$; otherwise, replace $f(x)$ by $\overline{f}(x) = A^{-1}f(x)$, and the same proof works. Now, since $|f(x) - x| < |x|/2$ for all x, then the line between x and $f(x)$ cannot pass through 0, and therefore one can take the straight-line homotopy from id to f. ⊠

Note that f merely being differentiable doesn't work, so be careful.

Definition. Suppose $f : S^n \to S^n$ has a point q with finite preimage $f^{-1}(q) = \{p_1, \ldots, p_k\}$, and choose an open neighborhood U_i of p_i that doesn't touch any of the other p_j.

Then, we have the following maps, though the following diagram does *not* commute!

$$\begin{tikzcd}
H_n(S^n) \arrow[rr, "f_*"] \arrow[d, "\psi_1", "\wr"'] & & H_n(S^n) \arrow[d, "\cong"] \\
H_n(S^n, S^n \setminus \{p_i\}) & & H_n(S^n, S^n \setminus \{q\}) \\
H_n(U_i, U_i \setminus \{p_i\}) \arrow[u, "\psi_2", "\wr"'] \arrow[urr, "(f|_{U_i})_*"'] & &
\end{tikzcd} \tag{6.1}$$

Here, ψ_1 and ψ_2 are isomorphisms by excision, and all are isomorphic to $\mathbb{Z}$ (if you pick one isomorphism at any point in the diagram, then all the rest are induced).

Then, the *local degree* $\deg(f|_{p_i}) \in \mathbb{Z}$ is defined to be the degree of $f|_{U_i}$ in (6.1).

Pretty much any map one can imagine from $S^n \to S^n$ has some such point q, but not all of them do.

The reason (6.1) doesn't commute is that there's no map of space $S^n \to S^n$ fixing $S^n \setminus \{p_i\} \to S^n \setminus \{q\}$ (i.e. the middle row), and therefore the associated maps of homology don't all commute. But the next diagram was induced from a commutative diagram of topological spaces, and so it *does* commute.

$$\begin{tikzcd}
H_n(S^n) \arrow[rr, "f_*"] \arrow[d, "\varphi"] & & H_n(S^n) \arrow[d] \\
H_n(S^n, S^n \setminus \{p_1, \dots, p_k\}) \arrow[rr] & & H_n(S^n, S^n \setminus \{q\}) \\
\bigoplus H_n(U_i, U_i \setminus \{p_i\}) \arrow[u, "\psi"] \arrow[urr, "\bigoplus \deg(f|_{U_i})"'] & &
\end{tikzcd} \tag{6.2}$$

ψ once again comes from excision; and φ isn't an isomorphism; in fact, it's the diagonal map $n \mapsto (n, \dots, n)$.

Corollary 6.8. *Thus, in this situation,*

$$\deg(f) = \sum_{i=1}^{k} \deg(f|_{p_i}).$$

Moreover, if f is C^1, then $\deg(f|_{p_i}) \in \{\pm 1\}$, and comes from the sign of the determinant.

This follows directly from (6.2) commuting. Thus, for most maps, one can just calculate out the degree.

Corollary 6.9. *The local degree is independent of the choice of q, as long as such a q with finite preimage exists.*

Corollary 6.10. *The degree of the* antipodal map $A : S^n \to S^n$ *sending $x \mapsto -x$ (i.e. multiplying by $-I$) is* $\deg(A) = (-1)^{n+1}$ *(since I is an $(n+1) \times (n+1)$-matrix; be careful!).*

Corollary 6.11. *$A \not\simeq \mathbb{1}$ if n is even.*

This has some amusing consequences.

Corollary 6.12 (Hairy Ball Theorem)**.** *If there exists a continuous $v : S^n \to \mathbb{R}^{n+1}$ such that $v(x) \perp x$ for all x, then n is odd.*

Proof. Given such a v, let $v'(x) = v(x)/|v(x)|$, so that $v' : S^n \to S^n$, with $v'(x) \perp x$ still. Thus, there's a homotopy $I \times S^n \to S^n$ sending $(t, x) \mapsto \cos(\pi t)x + \sin(\pi t)vx$, which rotates the identity onto the antipodal map. In particular, $\mathrm{id} \simeq A$, which means n cannot be even. ⊠

This v is often taken to be a vector field, e.g. the direction and magnitude of wind on the surface of the Earth. Since we believe this to be continuous and the surface of the Earth looks like S^2, then there must be some place where there is no wind at any given time.

Another consequence (and the reason for the name of the theorem) is that if one tries to comb the spikes on a hedgehog,[5] there will be some point where they can't be combed flat; see Figure 2.

7. The Mayer-Vietoris Sequence and Applications: 1/29/15

"When I took this class, Vietoris was still alive, and was the oldest citizen of Austria."

Recall that we've computed $H_*(D^n, \partial D^n)$ and $H_*(S^n)$, as well as $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{p\})$ and $H_*(U, U \setminus \{p\})$ for an open $U \subset \mathbb{R}^n$. We also defined the degree of a map $S^n \to S^n$ given by the degree of the induced map on homology and showed that the antipodal map has degree $(-1)^{n+1}$. Thus, there is no nonconstant, nonzero vector field on S^{2n}.

However, there is such a vector field on S^{2n-1}, since $S^{2n-1} \subset \mathbb{C}^n = \mathbb{R}^{2n}$, and therefore the map $z \mapsto iz$ (sending $(x_k, y_k) \mapsto (y_k, -x_k)$) works.

More generally, how many linearly independent vector fields can we find? On S^3, can we find two? The goal is to find $v_1, v_2 : S^3 \to \mathbb{R}^4$ with $v_i(x) \perp x$ for both $i = 1, 2$ and $\{v_1(x), v_2(x)\}$ is linearly independent for all X.

This is possible, because $S^3 \subset \mathbb{H} = \mathbb{R}^4$, with a basis $\{1, i, j, k\}$, and multiplication on $\mathbb{R} \hookrightarrow \mathbb{H}$ extends uniquely to an associative (but not commutative) multiplication on $\mathbb{H}$ with $i^2 = j^2 = k^2 = ijk = -1$. Thus, the vector fields $v_1(x) = ix$, $v_2(x) = jx$, and $v_3(x) = kx$ are all linearly independent and perpendicular to x. That is, S^3 is *parallelizable*: $T_x S^3 = x^\perp$ in $\mathbb{R}^4$, and there's a continuous change of basis. This asks for as many continuous, linearly independent vector fields as possible.

[5] "What's the English name for those pointed things on a hedgehog again?"

FIGURE 2. By Corollary 6.12, it's impossible to smoothly comb this hedgehog. It doesn't seem to mind, however. Source: https://www.pinterest.com/carmentaylorhai/hedgehogs/.

S^7 is also parallelizable, which is because it sits inside the *octonions* $\mathbb{O} = \mathbb{R}^8$, a multiplicative structure that isn't even associative, but nonetheless has enough structure for this to work. This is pretty special.

Theorem 7.1 (Kervaire-Milnor)**.** *If S^n is parallelizable, then n is one of* 0, 1, 3, *or* 7.

Then, due to Adams, there is an explicit, but complicated, formula for the maximum number of linearly independent vector fields on S^n. It requires some fancier tools than we have at the moment.

The Mayer-Vietoris Sequence. Suppose $X = U_0 \cup U_1$, where $U_0, U_1 \subset X$ are open. Then, there is a diagram of inclusions

$$\begin{tikzcd}
 & U_0 \arrow[dr, "i_0"] & \\
U_{01} = U_0 \cap U_1 \arrow[ur, hook, "j_0"] \arrow[dr, hook, "j_1"'] & & X \\
 & U_1 \arrow[ur, "i_1"'] &
\end{tikzcd}$$

Let $\varphi : \sigma \mapsto ((j_0)_*\sigma, (j_1)_*(\sigma))$; thus, there is an exact sequence of chain complexes

$$0 \longrightarrow C_*(U_{01}) \hookrightarrow C_*(U_0) \oplus C_*(U_1) \xrightarrow{(i_0)_* - (i_1)_*} C_*(X).$$

However, the rightmost arrow may not be surjective; if $\mathcal{U} = \{U_0, U_1\}$, then the image is just the $\mathcal{U}$-small chains $C_*^{\mathcal{U}}(X)$, so the following sequence is really short exact.

$$0 \longrightarrow C_*(U_{01}) \longrightarrow C_*(U_0) \oplus C_*(U_1) \longrightarrow C_*^{\mathcal{U}}(X) \longrightarrow 0$$

Thus, there is an induced long exact sequence

$$\cdots \longrightarrow H_k(U_{01}) \longrightarrow H_k(U_0) \oplus H_k(U_1) \longrightarrow H_k(X) \longrightarrow H_{k-1}(U_{01}) \longrightarrow \cdots \tag{7.1}$$

This is known as the *Mayer-Vietoris sequence*.

It's also possible to define this in terms of reduced homology, which was on the homework: one can augment the chain complex with an additional term in degree -1, i.e.

$$\cdots \xrightarrow{\partial} C_1(X) \xrightarrow{\partial} C_0(X) \longrightarrow C_0(\mathrm{pt}) \longrightarrow 0.$$

Thus, define the *reduced chains*

$$\widetilde{C}_k(X) = \begin{cases} C_k(X), & k \geq 0 \\ C_0(\mathrm{pt}) = \mathbb{Z}, & k = -1 \\ 0, & k < -1. \end{cases}$$

Then, define the *reduced homology* $\widetilde{H}_k(X)$ to be the k^{th} homology of this chain complex.

This is occasionally useful for computations; for example, if X is homotopy equivalent to a point, then $\widetilde{H}_k(X) = 0$ for all k. A lot of the same stuff works, e.g. (7.1) has a very similar form for the reduced case, because the same short exact sequence works.

Theorem 7.2 (Jordan curve theorem)**.** *For any curve $\gamma : S^1 \hookrightarrow \mathbb{R}^2$, $\mathbb{R}^2 \setminus \gamma(S^1)$ has exactly two path components.*

This boils down to the idea that every simple closed curve divides the plane into an inside and an outside. This is obvious enough until one stares at all of the strange maps which can be called continuous!

Proof. The idea is that for any space X, $H_0(X) = C_0(X)/\partial C_0(X)$.

Lemma 7.3. *Let $\mathcal{P}$ be the set of path components of X; then,*

$$H_0(X) = \bigoplus_{P \in \mathcal{P}} \mathbb{Z}.$$

This is in the textbook, and also appeared on the homework. Thus, the number of path components is the rank of $H_0(X)$; even though H_0 seems like the least weird homology group, it's still quite powerful, especially when it sits in long exact sequences with higher homology groups.

Furthermore, if $X \neq \emptyset$, then $H_0(X) = \widetilde{H}_0(X) \oplus \mathbb{Z}$, and therefore the rank of the reduced homology is one less than the number of path components.

But since path components are just a statement about homology, the theorem statement can be generalized to $\gamma : S^{n-1} \to \mathbb{R}^n$ or even, taking the one-point compactification $\mathbb{R}^n \cup \infty \approx S^n$, so one can also take $S^k \hookrightarrow S^n$. We will want to show that

$$\widetilde{H}_*(S^n \setminus \gamma(S^k)) \cong \begin{cases} \mathbb{Z}, & * = n-k-1 \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 7.4. *For any $\gamma : D^k \hookrightarrow S^n$, $\widetilde{H}_k(S^n \setminus \gamma(D^k)) = 0$.*

Proof. When $k = 0$, $S^n \setminus \{\mathrm{pt}\} \approx \mathbb{R}^n \simeq \mathrm{pt}$.

For $k = 1$, things get more complicated; there are "wild knots", injections $D^1 \to \mathbb{R}^3$ such that the complement isn't simply connected. These things are somewhat fractal. In particular, it's not always true that $S^n \setminus \gamma(D^1) \simeq \mathrm{pt}$.

Nonetheless, the homology groups are nicer, and the inductive proof has general case very similar to the case $k = 1$. Let $\gamma : D^1 \hookrightarrow S^n$, and divide $I = [0,1]$ into $[0,1/2] \cup [1/2,1]$. Let $U_0 = S^n \setminus \gamma([0,1/2])$ and $U_1 = S^n \setminus \gamma([1/2,1])$, so $U_0 \cap U_1 = S^n \setminus \gamma([0,1])$, and $U_0 \cup U_1 = S^n \setminus \gamma(1/2)$. Inductively, the reduced homology of S^n minus a point is always 0 (since it's contractible), so the reduced Mayer-Vietoris sequence collapses to a bunch of isomorphisms $\widetilde{H}_k(U_0 \cap U_1) \xrightarrow{\sim} \widetilde{H}_k(U_0) \oplus \widetilde{H}_k(U_1)$. Thus, for any nonzero $\alpha \in \widetilde{H}_k(U_0 \cap U_1)$, there's an $i \in \{0,1\}$ such that the induced map $\widetilde{H}_k(U_0 \cap U_1) \to \widetilde{H}_k(U_i)$ sends $\alpha \mapsto \alpha_1 \neq 0$.

Now, repeating the same argument with U_i, there's a nested sequence $I \supset I_1 \supset I_2 \supset \cdots$, such that $|I_\ell| = 2^{-\ell}$, and each time, $\alpha_k \mapsto \alpha_{k+1} \neq 0$.

Then, $\bigcap I_\ell = \{t\}$ for some t, so α goes to some class in $\widetilde{H}_k(S^n \setminus \gamma(t)) = 0$, so $\alpha = [c]$ for some boundary c: $c = \partial d$ for a $d \in C_{k+1}(S^1 \setminus \gamma(t))$, and by compactness, $d \in C_{k+1}(S^1 \setminus \gamma(I_\ell))$ for sufficiently large ℓ. But this means $\alpha \mapsto 0$ at some finite step, which is a contradiction. Thus, $\alpha = 0$. ⊠

By the same argument (induct as $I^m = I^{m-1} \times I$), one can see that $\widetilde{H}_k(S^n \setminus \gamma(I^m)) = 0$. In particular, $\widetilde{H}_k(S^n \setminus \gamma(D^m))$ doesn't depend on γ; this can be generalized considerably to something called Alexander duality, which we'll talk about later.

Now, we can tackle $\widetilde{H}_k(S^n \setminus \gamma(S^m))$ by decomposing $S^m = D^m_+ \cup D^m_-$ (upper and lower hemispheres), where $D^m_+ \cap D^m_- = S^{n-1}$. Let $U_0 = S^n \setminus \gamma(D^m_+)$ and $U_1 = S^n \setminus \gamma(D^m_-)$. Since $\widetilde{H}_k(U_i) = 0$, then the Mayer-Vietoris sequence reduces to isomorphisms $\widetilde{H}_k(U_0 \cap U_1) \to \widetilde{H}_{k-1}(U_0 \cap U_1)$, but the former space is $S^n \setminus S^{n-1}$ and the latter is $S^n \setminus \gamma(S^m)$. In particular, $\widetilde{H}_k(S^n \setminus \gamma(S^m)) \cong \widetilde{H}_{k+1}(S^n \setminus \gamma(S^{m-1})) \cong \cdots \cong \widetilde{H}_{k+m}(S^n \setminus \gamma(S^0))$.

But S^0 is just two points, so $S^n \setminus \gamma(S^{n-1})) \approx \mathbb{R}^n \setminus \{0\} \simeq S^{n-1}$. Thus, the homology we just looked at is $\mathbb{Z}$ when $k + m = n - 1$ and is 0 otherwise. Thus, this is also $\widetilde{H}_k(S^n \setminus \gamma(S^m))$.

Now, results start falling out: $\mathbb{R}^1 \setminus \gamma(S^0)$ has three connected components, and for $n > 1$, $\mathbb{R}^n \setminus \gamma(S^{n-1})$ and $S^n \setminus \gamma(S^{n-1})$ have the same number of path components. ⊠

Interestingly, the theorem is simple to state, but all known proofs use machinery as sophisticated as this one.

CW **complexes.** *CW* complexes are a generalization of Δ-complexes, except that one is allowed to glue n-simplices $\Delta^n \approx D^n$ along any continuous map $\partial\Delta^n \to X^{n-1}$ (where X^{n-1} is the $(n-1)$-skeleton). Thus, one can glue along more places than just boundaries.

We will need one formalism first.

Definition. Consider the following diagram of topological spaces.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

Then, consider the induced map $h : (B \coprod C)/(f(a) \sim g(a) \text{ for all } a \in A) \to D$ (where the former space has the quotient topology induced from the disjoint union topology). The diagram is said to be a *pushout diagram* if it commutes and h is a homeomorphism.

This notion, which surfaces in many other parts of mathematics, means that D is obtained by gluing B and C along f and g.

Definition. A *CW structure* on a space X is a set of continuous maps, called *cells* $e_\alpha : D^{n_\alpha} \to X$, indexed by $\alpha \in J$, such that:

- $e_\alpha(\partial D^{n_\alpha}) \subset X^{n_\alpha - 1}$, where X^k is the *k-skeleton*
$$X^k = \bigcup_{n_\alpha \le k} e_\alpha(D^{n_\alpha}).$$
- The following diagram is a pushout diagram.
$$\begin{tikzcd} \coprod\limits_{\alpha : n_\alpha = n} \partial D^{n_\alpha} \arrow[r, "\{e_\alpha|_{\partial D^{n_\alpha}}\}"] \arrow[d] & X^{n-1} \arrow[d] \\ \coprod\limits_{\alpha : n = n_\alpha} D^{n_\alpha} \arrow[r, "\{e_\alpha\}"] & X^n \end{tikzcd} \tag{7.2}$$
- $X = \bigcup_{n \in \mathbb{N}} X^n$ has the *direct limit topology*, i.e. $A \subset X$ is open (resp. closed) iff $A \cap X^n$ is open (resp. closed) in X^n for all n.

8. *CW* Complexes: 2/3/15

Last time, we defined the axioms of a *CW* complex; just like a simplicial complex, one often hears "X is a *CW* complex," even though the way in which it is realized as one is important. The extra data is a series of maps $e_\alpha : D^{n_\alpha} \to X$, the *characteristic maps* of *cells* in the complex.

Then, these maps were required to satisfy several axioms.

(1) The boundaries of the cells that generate the n-skeleton are contained within the $(n-1)$-skeleton.
(2) The diagram (7.2) is a pushout diagram.
(3) X has the direct limit topology as the union of its n-skeletons.

As a set,

$$X = \coprod_\alpha e_\alpha(D^{n_\alpha} \setminus \partial D^{n_\alpha}),$$

i.e. a disjoint union of "open cells." Thus, each such cell is homeomorphic to $\mathbb{R}^{n_\alpha}$, so $X^n \setminus X^{n-1}$ looks like a disjoint union of copies of $\mathbb{R}^n$.

One can show using point-set topology that all CW complexes X are Hausdorff (since it's true on cells), and that (X^n, X^{n-1}) is a good pair for all n. Also, just like for Δ-complexes, any compact subset $K \subset X$ meets (i.e. is nondisjoint from) only finitely many open cells (otherwise, one would have an infinite sequence in K, with each entry in the sequence in a different cell).

Inductively, X has the quotient topology from $\coprod_\alpha D^{n_\alpha} \twoheadrightarrow X$ (a set is open iff its preimage is open).

Corollary 8.1. *If $e_\alpha : D^{n_\alpha} \to X$ is one of the characteristic maps, then $e_\alpha(\partial D^{n_\alpha})$ meets only finitely many open cells.*

Definition. A *subcomplex* of a CW complex X is a union of some of the images of the characteristic maps (i.e. only using some of the α), with the condition that the resulting space has a CW structure.

The idea is that if one cell is included, the cells that contain it boundaries need to show up as well.

Corollary 8.2. *Every $e_\alpha(D^{n_\alpha}) \subset X$ is contained in a finite subcomplex.*

This follows from an inductive proof.

This is basically all that we'll use of the point-set topology of CW complexes. It's useful to remember that they can be infinite, e.g.

$$\bigvee_1^\infty [0,1] = (\mathbb{N} \times [0,1])/(\mathbb{N} \times \{0\}),$$

where $\mathbb{N}$ is given the discrete topology, $[0,1]$ has the usual topology, and the whole space has the quotient topology. This space has a CW structure with 0-cells $[\mathbb{N} \times \{0\}]$, and 1-cells $D^1 \approx [0,1] \to \mathbb{N} \times [0,1]$ sending $t \mapsto (n,t)$. This example isn't crucial, but the point is that there are often more open sets than one might expect. For example, if $x_n = (n, 1/n) \in X$, then $\lim x_n$ seems like it would be the collapsed point $[\mathbb{N} \times \{0\}]$, but in fact, it doesn't! This is because $\{(n,t) \mid t < 1/n\}$ is open in X (which follows from a definition check), and it contains the proposed limit point, but none of the x_i. In fact, this sequence doesn't converge. The point is, this is a finer topology than one might think.

Now, let's do some more algebraic topology. Since (X^n, X^{n-1}) is a good pair and (7.2) is a pushout diagram, then $X^n/X^{n-1} = \coprod D^n / \coprod \partial D^n$ (formal reasons, since it's a pushout diagram), and therefore excision implies $H_k(X^n, X^{n-1}) \xrightarrow{\sim} H_k(\coprod D^n, \coprod \partial D^n) = \bigoplus H_k(D^n, \partial D^n)$. Thus,

$$H_k(X^n, X^{n-1}) = \begin{cases} \bigoplus_{n\text{-cells}} \mathbb{Z}, & k = n \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the long exact sequence simplifies to $H_k(X^{n-1}) \xrightarrow{\sim} H_k(X^n)$, except when $k = n$, where it becomes

$$0 \longrightarrow H_n(X^{n-1}) \longrightarrow H_n(X^n) \longrightarrow H_n(X^n, X^{n-1}) \longrightarrow H_{n-1}(X^{n-1}) \longrightarrow H_{n-1}(X^n) \longrightarrow 0.$$

By induction, if $k > n$,

$$H_k(X^n) \xleftarrow{\cong} H_k(X^{n-1}) \xleftarrow{\cong} \cdots \xleftarrow{\cong} H_k(X^{-1}) = 0,$$

since $X^{-1} = \varnothing$.

Corollary 8.3. *If X is a CW complex, then $H_k(X^n) = 0$ whenever $k > n$.*

If $k < n$, then the long exact sequence shows that $H_k(X^n) \xrightarrow{\sim} H_k(X^{n+1}) \xrightarrow{\sim} H_k(X^{n+2}) \xrightarrow{\sim} \cdots$. Then, by a compactness argument, one can show that $H_k(X^n) \cong H_k(X)$ (i.e. any chain is compact, so it's contained in a finite level).

Just like simplicial homology, there's something analogous for CW complexes.

Definition. For a CW complex X, define its *cellular chains*

$$C_n^{CW} = H_n(X^n, X^{n-1}) \cong \bigoplus_{n\text{-cells}} H_n(D^n, \partial D^n) \cong \bigoplus_{n\text{-cells}} \mathbb{Z}.$$

Then, there are chain maps $\partial_n^{CW} : C_n^{CW}(X) \to C_{n-1}^{CW}(X)$ given by either of the paths across the following commutative diagram.

$$\begin{tikzcd} C_n^{CW}(X) = H_n(X^n, X^{n-1}) \arrow[rr, "a"] \arrow[dr, "b"'] & & H_{n-1}(X^{n-1}, X^{n-2}) = C_{n-1}^{CW}(X) \\ & H_{n-1}(X^{n-1}) \arrow[ur, "c"'] & \end{tikzcd}$$

Here, a is induced from the long exact sequence of the triple (X, X^{n-1}, X^{n-2}), b is from the long exact sequence of the pair (X^n, X^{n-1}), and c is from inclusion.

Theorem 8.4. *$(C_*^{CW}(X), \partial^{CW})$ is a chain complex, and there is a canonical isomorphism $H_n(C_*^{CW}(X), \partial^{CW}) \cong H_n(X)$.*

Like the singular case, the idea is that the CW complex is smaller, and therefore easier to calculate, even if it doesn't have as nice formal properties.

Proof. The proof is a diagram chase, with a relatively complicated diagram. It's somewhat easier to hear the argument than read it, but I'll do the best I can to make it work on paper.[6]

Our goal is to understand the following sequence.

$$H_{n+1}(X^{n+1}, X^n) \xrightarrow{\partial_{n+1}^{CW}} H_n(X^n, X^{n-1}) \xrightarrow{\partial_n^{CW}} H_{n-1}(X^{n-1}, X^{n-2}) \tag{8.1}$$

Each of these terms fits into a long exact sequence of pairs. Let's start with $H_{n+1}(X^{n+1}, X^n)$, which is part of the long exact sequence for the pair (X^{n+1}, X^n).

$$\cdots \longrightarrow H_{n+1}(X^{n+1}, X^n) \xrightarrow{\psi} H_n(X^n) \longrightarrow H_n(X^{n+1}) \longrightarrow H_n(X^{n+1}, X^n) \longrightarrow \cdots$$

However, we've already shown that $H_n(X^{n+1}) \cong H_n(X)$, and that $H_n(X^{n+1}, X^n) = 0$. Thus, this sequence simplifies to

$$H_{n+1}(X^{n+1}, X^n) \xrightarrow{\psi} H_n(X^n) \twoheadrightarrow H_n(X) \longrightarrow 0. \tag{8.2}$$

The middle term of (8.1), $H_n(X^n, X^{n-1})$, is part of the long exact sequence for the pair (X^n, X^{n-1}).

$$\cdots \longrightarrow H_n(X^{n-1}) \longrightarrow H_n(X^n) \xrightarrow{\varphi_1} H_n(X^n, X^{n-1}) \xrightarrow{\varphi_2} H_{n-1}(X^{n-1}) \longrightarrow \cdots$$

Here, φ_1 is induced by the inclusion $X^{n-1} \hookrightarrow X^n$. We've already shown that $H_n(X^{n-1}) = 0$, so this diagram also simplifies.

$$0 \longrightarrow H_n(X^n) \xhookrightarrow{\varphi_1} H_n(X^n, X^{n-1}) \xrightarrow{\varphi_2} H_{n-1}(X^{n-1}) \tag{8.3}$$

All right, so the last part is $H_{n-1}(X^{n-1}, X^{n-2})$, which belongs to the short exact sequence for the pair (X^{n-1}, X^{n-2}).

$$\cdots \longrightarrow H_{n-1}(X^{n-2}) \longrightarrow H_{n-1}(X^{n-1}) \xrightarrow{\chi} H_{n-1}(X^{n-1}, X^{n-2}) \longrightarrow \cdots$$

Here, χ is induced by the inclusion $X^{n-2} \hookrightarrow X^{n-1}$. We've seen that $H_{n-1}(X^{n-2}) = 0$, and therefore this sequence simplifies.

$$0 \longrightarrow H_{n-1}(X^{n-1}) \xhookrightarrow{\chi} H_{n-1}(X^{n-1}, X^{n-2}) \tag{8.4}$$

[6]This diagram was the first in some years to completely defeat my live-TeXnique, as diagonal diagrams are difficult in XY and the professor kept editing the diagram as the proof went on. Something similar happened to diagram (8.5).

Now, the diagrams (8.1), (8.2), (8.3), and (8.4) contain terms in common, so they can be stitched together into the following diagram.

$$\begin{tikzcd}[column sep=small]
& & & 0 & \\
0 \arrow[dr] & & H_n(X) \arrow[ur, two heads] & & \\
& H_n(X^n) \arrow[ur, two heads] \arrow[dr, hook, "\varphi_1"] & & & \\
H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\psi"] \arrow[rr, "\partial_{n+1}^{CW}"] & & H_n(X^n, X^{n-1}) \arrow[rr, "\partial_n^{CW}"] \arrow[dr, "\varphi_2"'] & & H_{n-1}(X^{n-1}, X^{n-2}) \\
& & & H_{n-1}(X^{n-1}) \arrow[ur, hook, "\chi"'] & \\
& & 0 \arrow[ur] & &
\end{tikzcd}$$

In this diagram, colored lines are exact sequences. Now, for the chase: since χ is injective, then φ_2 and ∂_n^{CW} have the same kernel. Since the blue line is exact, then $\ker(\varphi_2) = \operatorname{Im}(\varphi_1)$, but φ_1 is injective, so it's an isomorphism onto its image. That is, $\varphi_1 : H_n(X^n) \xrightarrow{\sim} \ker(\partial_n^{CW})$.

However, $\ker(\partial_n^{CW}) / \operatorname{Im}(\partial_{n+1}^{CW}) \cong \operatorname{coker}(\psi) \cong H_n(X)$, because the green line is exact. ⊠

It's possible to realize the boundary map $\partial_n^{CW} : C_n^{CW}(X) \to C_{n-1}^{CW}(X)$ on $H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$.

Given $e_\alpha : D^{n_\alpha} \to X^n$, the induced $e_\alpha|_{\partial D^{n_\alpha}} : \partial D^{n_\alpha} \to X^{n-1}$ is called an *attaching map*. This is helpful in realizing the boundary map explicitly. Specifically, one can draw out the following diagram.

$$\begin{tikzcd}
C_n^{CW}(X) \arrow[r, equal] \arrow[dd] & H_n(X^n, X^{n-1}) \arrow[d] & \displaystyle\bigoplus_{n\text{-cells}} H_n(D^n, \partial D^n) \arrow[l, "\cong"'] \arrow[d] & \\
& H_{n-1}(X^{n-1}) \arrow[d] & \bigoplus H_n(\partial D^n) \arrow[l, "a_*"'] & \\
C_{n-1}^{CW}(X) \arrow[r, equal] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[d, "\cong"] & \displaystyle\bigoplus_{(n-1)\text{-cells}} H_{n-1}(D^{n-1}, \partial D^{n-1}) \arrow[l, "\cong"'] & \\
& H_{n-1}(X^{n-1}/X^{n-2}, \mathrm{pt}) \arrow[r, "\cong"] & H_{n-1}\left(\coprod D^{n-1} / \coprod \partial D^{n-1}\right) \arrow[r, "\cong"] & \bigoplus H_{n-1}(D^{n-1}/\partial D^{n-1})
\end{tikzcd} \tag{8.5}$$

Here, a_* is the map induced from the attaching map on homology. The blue arrow is given by $X^{n-1}/X^{n-2} \to D^{n-1}/\partial D^{n-1}$, collapsing all but one $(n-1)$-cell, and the red arrow is basically $\bigvee S^{n-1} \twoheadrightarrow S^{n-1}$.

Corollary 8.5. *The entries of the matrix for*

$$\partial^{CW} : \bigoplus_{n\text{-cells}} H_n(D^n, \partial D^n) \longrightarrow \bigoplus_{(n-1)\text{-cells}} H_{n-1}(D^{n-1}, \partial D^{n-1})$$

are given by the degrees of the maps

$$\begin{tikzcd}
S^{n-1} = \partial D^n \arrow[r, "\varphi_1"] & X^{n-1} \arrow[d] \\
S^{n-1} = D^{n-1}/\partial D^{n-1} & X^{n-1}/X^{n-2}, \arrow[l, "\varphi_2"']
\end{tikzcd}$$

where φ_1 is induced by the attaching map and φ_2 is given by collapsing all but one $(n-1)$-cell.

Though this may look frightening, it's useful for explicitly calculating what's going on in a chain complex.

Example 8.6. The example everyone gives to justify all these calculations is complex projective space $\mathbb{C}P^n = S^{2n+1}/\,\mathrm{U}(1) \cong \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^\times$. (Here, $\mathrm{U}(1)$ means the orbit space of that action.)

It's hard to find a Δ-complex structure for $\mathbb{C}P^n$, but much easier to find a CW structure. First, we should put coordinates on $\mathbb{C}P^n$: the notation will just take $(z_0, \ldots, z^n) \in S^{2n+1} \subset \mathbb{C}^{n+1}$ and pass to the quotient, writing $[z_0 : z_1 : \cdots : z_n] \in \mathbb{C}P^n$, with at least one $z_i \neq 0$, where multiplying all coordinates by the same scalar is ignored (equivalent in the quotient). These are called *homogeneous coordinates*.

Exercise 2. Show that $\mathbb{C}P^n$ is Hausdorff.

Then, there is a map $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$ given by $[z_0 : \cdots : z_{n-1}] \mapsto [z_0 : \cdots : z_{n-1} : 0]$, and a map $e : D^{2n} \to \mathbb{C}P^n$ sending $z = (z_1, \ldots, z_{n-1}) \mapsto [z_0 : \cdots : z_{n-1} : \sqrt{(1 - |z|^2)}]$. Then, $e(\partial D^{2n}) \subset \mathbb{C}P^{n-1}$ (realized with the first map), and otherwise, it's injective: $e|_{D^{2n} \setminus \partial D^{2n}} : D^{2n} \setminus \partial D^{2n} \hookrightarrow \mathbb{C}P^n$, which has to do with finding a nice representative of a point in complex projective space.

Then, one can check that the following is a pushout diagram.

$$\begin{tikzcd}
\partial D^{2n} \arrow[r] \arrow[d] & \mathbb{C}P^{n-1} \arrow[d, hook] \\
D^{2n} \arrow[r] & \mathbb{C}P^n
\end{tikzcd}$$

This is because $\mathbb{C}P^{n-1} \cup_{\partial D^{2n}} D^{2n} \to \mathbb{C}P^n$ is a continuous bijection, but since everything involved is compact, this is a homeomorphism.

Thus, inductively, since $\mathbb{C}P^0 = \mathrm{pt}$ has an obvious CW structure, then there's a CW structure on $\mathbb{C}P^n$ with cells in dimensions $0, 2, 4, \ldots, 2n$.

To compute the homology, it's not even necessary to compute degrees, because $C_*^{CW}(\mathbb{C}P^n)$ looks like

$$0 \longleftarrow \mathbb{Z} \longleftarrow 0 \longleftarrow \mathbb{Z} \longleftarrow 0 \longleftarrow \mathbb{Z} \longleftarrow \cdots \longleftarrow 0.$$

Thus, all the boundary maps must be trivial, so

$$H_k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & k = 0, 2, \ldots, 2n \\ 0, & \text{otherwise.} \end{cases}$$

In fact, one can generalize this to an infinite CW complex, $\mathbb{C}P^\infty = \bigcup_{n=0}^\infty \mathbb{C}P^n$, with the direct limit topology. Then, the same methods prove that

$$H_k(\mathbb{C}P^\infty) \cong \begin{cases} \mathbb{Z}, & k \equiv 0 \bmod 2 \\ 0, & k \equiv 1 \bmod 2. \end{cases}$$

9. Some Loose Ends: 2/5/15

Recall that last time, we defined cellular homology $H_*^{CW}(X) \cong H_*(X)$ using the CW complex C_*^{CW}, and used this to compute $H_*(\mathbb{C}P^n)$, when $n = 0, \ldots, \infty$.

Today, we'll clean up a bunch of loose ends, things that could have been said earlier, but haven't been. For example, we'll discuss *real projective space*, $\mathbb{R}P^n = S^n/\{\pm 1\}$.

Homology with coefficients. Let X be a topological space and G be any abelian group. Then, one can define chains in X with coefficients in G, written $C_*(X; G)$ (or relative chains $C_*(X, A; G)$), where

$$C_n(X; G) = \bigoplus_{\sigma : \Delta^n \to X} G.$$

That is, the coefficients are in G, instead of $\mathbb{Z}$. A $\sigma \in C_n(X; G)$ looks like $\sum g_i \sigma_i$, where the σ_i are singular chains.

Then, since abelian groups are just $\mathbb{Z}$-modules, it makes sense to multiply by -1, so the boundary map is once again

$$\partial\sigma = \sum_{i=0}^n (-1)^i \sigma|_{[v_0, \ldots, \widehat{v}_i, \ldots, v_n]},$$

and $\partial^2 = 0$ for the same reasons as when $G = \mathbb{Z}$. Thus $C_*(X; G)$ is a chain complex, so one can define $H_n(X; G) = \ker(\partial_n)/\operatorname{Im}(\partial_{n+1})$, and $H_n(X) = H_n(X, \mathbb{Z})$.

Many of the things we've already done work just as well in this more general case, with the same proofs.

- Simplicial homology $H_*^\Delta(X; G)$ coming from simplicial chains with coefficients in G, $\Delta_*(X; G)$; the theorem proving $H_n^\Delta(X) \cong H_n(X)$ generalizes, and has exactly the same proof.

- Cellular homology $C_n^{CW}(X;G) = H_n(X^n, X^{n-1};G)$ can be defined in the same way, and $H_n^{CW}(X;G)$ is still isomorphic to $H_n(X;G)$. Again, $H^n(X^n, X^{n-1};G)$ is a direct sum of copies of G, one for each n-cell.
- A pair (X, A) induces a long exact sequence in $H_*(-, G)$.
- If $X \simeq Y$, then $H_n(X;G) \cong H_n(Y;G)$.
- Excision, in the absolute and relative cases, still exists.
- The same argument as when $G = \mathbb{Z}$ shows that
$$H_k(D^n, \partial D^n; G) = \begin{cases} G, & k = n \\ 0, & \text{otherwise.} \end{cases}$$

This is the beauty of formal arguments; they generalize really easily, even with the same proofs. For example, the proof that $H_n(X;G) \cong H_n^{CW}(X;G)$ relied on excision, a few long exact sequences of pairs, and homotopy invariance.

However, in general, $H_n(X) \not\cong H_n(X;G)$, though they're related by something called the universal coefficient theorem, which relates to the Tor functor, which we'll talk about next week.

The Formal Viewpoint. One interesting way to think about this is to axiomatize the notion of homology, so that $H_n(X)$, $H_n^\Delta(X)$, $H_n(X;G)$, and $H_n^{CW}(X)$ are all examples of some general notion. This makes the idea of "the same proofs" a little more formal.

Definition. Suppose h_n, indexed by $n \in \mathbb{Z}$, are functors h_n from pairs (X, A) to topological spaces to abelian groups $h_n(X, A)$, and suppose they are equipped with natural transformations $h_n(X, A) \to h_{n-1}(A, \varnothing)$. Then, suppose the h_n satisfy the following axioms (called the *Eilenberg-Steenrod axioms*):

- Excision: if (X, A) is a good pair, then $h_n(X, A) \xrightarrow{\sim} h_n(X/A, A/A)$ for all n. More precisely, if $\pi : X \twoheadrightarrow X/A$ is the natural map, then $h_n(\pi)$ gives the above isomorphism.
- The following is a long exact sequence.
$$\cdots \longrightarrow h_n(A, \varnothing) \longrightarrow h_n(X, \varnothing) \longrightarrow h_n(X, A) \longrightarrow h_{n-1}(X, \varnothing) \longrightarrow \cdots$$
- Homotopy invariance: if $f : X \xrightarrow{\simeq} Y$, then $h_n(f) : h_n(X, \varnothing) \xrightarrow{\cong} h_n(Y, \varnothing)$.
- $h_n(\mathrm{pt}, \varnothing) = 0$ when $n > 0$, and $h_0(\mathrm{pt}, \varnothing) = G$.

Then, h_n is said to be a *homology theory*.

The idea here is that these are the only things we knew about $H_*^{CW}(-;G)$ in the proof that it was isomorphic to $H_n(-;G)$ for all X; in particular, the same proof works for any homology theory.

Corollary 9.1 (Eilenberg & Steenrod)**.** *Therefore, if X is a finite CW complex, then there is a natural isomorphism $h_n(X) \cong H_n(X;G)$.*

Corollary 9.2.
$$h_n(D^k, \partial D^k) \cong \begin{cases} G, & k = n \\ 0, & \text{otherwise.} \end{cases}$$

This is usually taken to be a uniqueness statement; having the long exact sequence and excision are very nice for calculation, but it turns out they're powerful ways to unify the many homology theories into one more general one. Most things can be proven from very few properties.

This is one reason that going to the chain level is a last resort; this shows that anything one might want to be true can be proven from just the axioms.

Infinite CW complexes are often useful, but right now there's no way to show that different homology theories are the same on them. However, it's possible to make it work with just one more axiom.

Theorem 9.3. *Suppose that*
$$h_n\Big(\bigvee X_\alpha, *\Big) \xleftarrow{\cong} \bigoplus_\alpha h_n(X_\alpha, *).$$
Then, there's a natural isomorphism $h_n(X) \xrightarrow{\cong} H_n(X;G)$ for all CW complexes X.

One thing that can't always transfer between homology theories is the Mayer-Vietoris sequence, since this depended on details of singular homology. However, it can be defined for Δ-complexes and CW complexes; the latter works because if $X = A \cup B$ for $A, B \subset X$, then
$$\begin{array}{ccc} A \cap B & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & X \end{array}$$

is a pushout diagram, because $A/A\cap B \cong X/B$.

Example 9.4. Define *real projective space* as $\mathbb{R}P^n = S^n/\{\pm 1\}$, analogous to $\mathbb{C}P^n$, but as lines through the origin in $\mathbb{R}^n$, rather than in $\mathbb{C}^n$. It has homogeneous coordinates $[a_0 : \cdots : a_n]$ just as $\mathbb{C}P^n$ does, but this time the coordinates are real numbers not all equal to 0. Then, there's a CW structure on $\mathbb{R}P^n$ induced by the following pushout diagram.

$$\begin{tikzcd}
\partial D^n \arrow[r, "a"] \arrow[d] & \mathbb{R}P^{n-1} \arrow[d, "b"] \\
D^n \arrow[r, "c"] & \mathbb{R}P^n
\end{tikzcd}$$

Here, a is the attaching map (which is also the quotient map), $b : [a_0 : \cdots : a_{n-1}] \mapsto [a_0 : \cdots : a_{n-1} : 0]$, and $c : a = (a_0, \ldots, a_{n-1}) \mapsto [a_0 : \cdots : a_n : \sqrt{(1-|a|^2)}]$. Finally, c is the characteristic map.

Thus, there's one cell in each degree, so $C_*^{CW}(\mathbb{R}P^n)$ looks like

$$0 \longleftarrow \mathbb{Z} \longleftarrow \mathbb{Z} \longleftarrow \cdots \longleftarrow \mathbb{Z}.$$

Thus, we'll need to calculate the degree of each attaching map

$$\begin{tikzcd}
S^n = \partial D^{n+1} \arrow[r] & \mathbb{R}P^n \arrow[d] \\
D^n \approx D^n/\partial D^n \arrow[r, "\approx"] & \mathbb{R}P^n/\mathbb{R}P^{n-1}.
\end{tikzcd}$$

Now, we need to calculate degrees, which means picking a point with finite preimage. The center point maps to the center of the disc, i.e. $[0 : \cdots : 0 : 1]$ in the bottom right, and therefore also in the upper right. Thus, the two preimages in the upper left are $(0, 0, \ldots, 0, \pm 1)$, so the degree of this map φ is $\deg(\varphi \mid N) + \deg(\varphi \mid S)$.

Since we're trying to calculate the homology of the complex, we only care about the degree up to sign. φ is a local homeomorphism (since there's a local inverse, which one can write down), so the local degrees are ± 1, i.e. it's important to know whether the local degrees have the same sign or opposite signs. It's a sort of orientation question.

The antipodal map $-I : S^n \to S^n$ sends $N \leftrightarrow S$. Thus, taking a calculation at N and composing it with the antipodal map gives the same calculation at S. Thus, since degree is multiplicative with composition, and $\deg(-I) = (-1)^{n+1}$, then $\deg(\varphi) = \pm(1 + (-1)^{n+1})$, i.e. 0 if n is even, or ± 2 if n is odd. Thus, the chain complex is

$$0 \longleftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{\pm 2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{\pm 2} \mathbb{Z} \longleftarrow \cdots \xleftarrow{\pm(1+(-1)^{n+1})} \mathbb{Z} \longleftarrow 0 \longleftarrow \cdots \tag{9.1}$$

Thus,

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z}, & k = 0 \\ 0, & k > 0 \text{ is even} \\ \mathbb{Z}/2\mathbb{Z}, & k \text{ is odd}, k < n \\ \mathbb{Z}, & k = n, k \text{ is odd.} \end{cases}$$

So this answer is interesting in its own right, but also serves as a useful example for homology with coefficients: what if one calculates $C_*^{CW}(\mathbb{R}P^n; \mathbb{F}_2)$? Then, in (9.1), $\pm 2 = 0$, so the resulting homology looks much more like the case for $\mathbb{C}P^n$:

$$H_k(\mathbb{R}P^n; \mathbb{F}_2) = \begin{cases} \mathbb{F}_2, & k = 0, \ldots, n \\ 0, & \text{otherwise.} \end{cases}$$

Notice that some terms that were zero in $H_k(\mathbb{R}P^n; \mathbb{Z})$ become nonzero, and that it's not always true that $H_k(\mathbb{R}P^n; \mathbb{F}_2) = H_k(\mathbb{R}P^n) \otimes \mathbb{F}_2$, e.g. $k = n = 2m$ for some m. This will motivate the universal coefficient theorem.

Since $\mathbb{Q}$ is flat over $\mathbb{Z}$, then one can calculate that

$$H_k(\mathbb{R}P^n; \mathbb{Q}) = \begin{cases} \mathbb{Q}, & k = 0 \text{ or } k = n \text{ and } k \text{ is odd} \\ 0, & \text{otherwise.} \end{cases}$$

Here's something else that could have come earlier in the course.

Theorem 9.5 (Invariance of Domain). *Suppose $U \subset \mathbb{R}^n$ is open and $h(U) \to \mathbb{R}^n$ is injective and continuous; then, $h(U) \subset \mathbb{R}^n$ is open, and homeomorphic onto its image.*

A similar theorem exists in real analysis when U is compact, but this is much deeper and much more general; all known proofs require some fancy technology, sort of like the generalizations of the Jordan curve theorem.

Proof. The trick is to show that $h(U)$ is open; once this is true, it applies to all subsets of U, and therefore h and h^{-1} will both send open sets to open sets, so h will be a homeomorphism.

Choose some $x \in U$; without loss of generality, $x = 0$ and $D^n \subset U$ (if not, translation and scaling makes this true). Since h is injective and continuous, then $h|_{D^n}$ and $h|_{\partial D^n}$ are homeomorphic onto their images. Thus, by the Jordan curve theorem, $\mathbb{R}^n \setminus h(\partial D^n) = V \cup W$, where V and W are both open and connected, and $V \cap W \neq \emptyset$.

Thus, $h(0) \in V \coprod W$, so without loss of generality, suppose $h(0) \in V$. Then, $\mathbb{R}^n \setminus h(\partial D^n) = V \cup W = h(D^n \setminus \partial D^n) \cup (\mathbb{R}^n \setminus h(D^n))$. In particular, since $D^n \setminus \partial D^n$ is connected, then $h(D^n \setminus \partial D^n)$ is connected, so $h(D^n \setminus \partial D^n) \subset V$.

We've shown before that $\widetilde{H}_*(S^n \setminus h(D^n)) = 0$, and hence when $n \neq 1$, then $\mathbb{R}^n \setminus h(D^n)$ is connected; thus, $\mathbb{R}^n \setminus h(D^n) \subset W$.

But since the union $h(D^n \setminus \partial D^n) \cup (\mathbb{R}^n \setminus h(D^n))$ is disjoint, then $h(D^n \setminus \partial D^n) = V$, so $V \subset h(U)$; in particular, V is a path-component of an open subset in $\mathbb{R}^n$, so it's open. Thus, V is a neighborhood of $h(0)$. ⊠

This is somewhat similar to the inverse function theorem, with a C^1 map $U \to \mathbb{R}^n$; if the Jacobian is invertible, then locally the function is continuous. However, this theorem weakens the continuity requirement.

10. The Lefschetz Fixed-Point Theorem: 2/10/15

> *"It's not like a novel; in a novel, you don't want to read the last line first, but in mathematics, you want to know what you need."*

Today's lecture was given by Elizabeth Gasparim.

Theorem 10.1 (Lefschetz fixed-point theorem)**.** *If X is a finite simplicial complex, or more generally a retract of a finite simplicial complex, and if $f : X \to X$ is a continuous map with $\tau(f) \neq 0$, then f has a fixed point.*

$\tau(f)$ is the *Lefschetz number*, which we will define. Notice that CW complexes, as well as locally compact finite-dimensional spaces, are both retracts of finite simplicial complexes, so this is a quite powerful theorem.

It's also useful to know (or in our case, review) what a simplex is. The *standard simplex* in $\mathbb{R}^n$ is the convex hull of $\{0, e_1, \dots, e_n\}$, which looks like an n-dimensional version of a triangle or a tetrahedron.

Definition. A *simplicial complex* is a collection of simplices A, satisfying a few conditions:

(1) If $\sigma = \langle \sigma_0, \dots, \sigma_n \rangle$ is a simplex in A, then all of its faces belong to A.
(2) All 2-simplices in A are either disjoint, or intersect in a common face.

For example, a sphere has a simplicial structure as a tetrahedron (since they're topologically the same), with one 2-simplex, and then the 1- and 0-simplices given by its faces and vertices.

Definition. For a map $f : X \to X$, where X is a finite CW complex (or more generally, any space whose homology groups are finitely generated and vanish after finitely many dimensions), the *Lefschetz number* is

$$\tau(f) = \sum_n (-1)^n \operatorname{tr}(f_* : H_n(X) \to H_n(X)).$$

Basically, the definition is far more general than just for CW complexes; as long as there's a finite number of finite traces, then we're good.

The Euler characteristic is first defined for simplicial surfaces (or polyhedra), as the number of vertices, minus the number of edges, plus the number of faces. More generally, it's the alternating sum of the number of n-cells.

Definition. The *Euler characteristic* of a CW complex X (or more generally, any space whose homology is finitely generated and vanishes after finitely many dimensions) is

$$\chi(X) = \sum_n (-1)^n h^n(X),$$

where $h^n(X) = \dim H^n(X)$.

The Euler characteristic classifies compact, orientable Riemann surfaces, even up to homeomorphism and diffeomorphism. For a Riemann surface S, $\chi(S) = 2 - 2g$, where g is the genus of S. It has lots of useful applications.

Claim. If f is homotopy equivalent to the identity map, then $\tau(f) = \chi(X)$.

Corollary 10.2. *If X has the same homology groups as a point, then (at least modulo torsion), $\tau(f) \neq 0$, and therefore every continuous map has a fixed point!*

Of course, this is obvious for a point, but there are more interesting examples; for example, it's also true for maps $\mathbb{R}P^2 \to \mathbb{R}P^2$; you can think of this in terms of linear algebra, but it's still impressive. If $f : \mathbb{R}^3 \to \mathbb{R}^3$ is an invertible linear transformation, then it induces a map $\overline{f} : \mathbb{R}P^2 \to \mathbb{R}P^2$ (since lines through the origin are sent to lines through the origin by a linear map); then, fixed points of $\overline{f}$ are eigenvectors of f. Thus, such an eigenvector always exists; however, there are maps $\mathbb{R}^4 \to \mathbb{R}^4$ without eigenvectors, e.g. $(x_1, x_2, x_3, x_4) \mapsto (x_2, -x_1, x_4, -x_3)$, so there are maps $\mathbb{R}P^3 \to \mathbb{R}P^3$ without fixed points.

Basically, this works even if there's nonzero higher homology groups as long as they're torsion, since they don't factor into the ranks or the traces.

This is a kind of silly example, because we don't calculate τ; it's more fun to play with S^1 and a rotation, where τ will end up 0.

Proof of Theorem 10.1. Suppose $r : K \to X$ is a retraction of a finite simplicial complex K to X. Then, $f : X \to X$ has the same fixed points as $f \circ r : K \to X \hookrightarrow K$, so without loss of generality, we may assume that X is a finite simplicial complex.

Proceed by contradiction; for a finite simplicial complex X, assume f has no fixed points. Then, there is a subdivision L of X, a further subdivision K of L, and a map $g : K \to L$ homotopic to f such that $g(\sigma) \cap \sigma = \emptyset$ for each simplex $\sigma \in K$. This would imply that $\tau(f) = \tau(g) = 0$, so we want to find such a g.

Notice the trend of this proof: if you need to write down what maps do in homology, use simplicial homology, because then they become linear maps.

Choose a metric d on X (it's metrizable because it can be embedded in $\mathbb{R}^n$ for some n). Since f has no fixed points, then $d(x, f(x)) > 0$ for all $x \in X$. By compactness, there's an $\varepsilon > 0$ such that $d(x, f(x)) > \varepsilon$ for all $x \in X$.

Choose a subdivision L of X such that the *stars* of all simplices in X have diameters less than $\varepsilon/2$.[7]

Definition. The stars of $Y \subset X$, $\mathrm{St}(Y)$, is defined as the union of all closed simplices that contain Y. Then, the finer the subdivision, the smaller the stars are.

The *open stars* of Y, $\mathrm{st}(Y)$ is the same, but with open simplices instead of closed ones.

The next theorem is the next ingredient in the proof.

Theorem 10.3 (Simplicial Approximation)**.** *Given such an f, X, and L, there exists a subdivision K and a simplicial map (i.e. sending simplices to simplices, and commuting with the boundary operator) $g : K \to L$ homotopic to f.*

A generalization of this theorem exists, of course. See Hatcher.

The proof will be in Hatcher. The idea is that, by construction, g has the property that every simplex $\sigma \in K$ satisfies $f(\sigma) \subset \mathrm{st}(g(\sigma))$.

Then, $g(\sigma) \cap \sigma = \emptyset$ for all $\sigma \in K$. This is what we want, but it's not obvious at all. Thus, the diagonals of the maps in homology will be zero, and therefore the Lefschetz number is zero, which is the result. So, why can we find such a g?

Another advantage of using simplicial complexes is that one can sometimes induct on the n-skeleton. This proof uses something that we've done before, apparently, and the proof is the same, but the lecturer doesn't remember precisely which theorem. Oh, and it's with trace instead of rank. Ok.

Remark. In the statement of Theorem 10.3, subdivision is necessary, e.g. if $f : S^2 \to S^2$, it could have arbitrarily high degree, but the degree is bound by the number of simplices, so we may need to subdivide them and get more simplices to realize f as homotopic to a simplicial map.

Now, let's deal with the simplicial approximation, and therefore finish the proof. Choose a metric on K that restricts to the standard Euclidean metric on each simplex of K.

Take an open cover $\{f^{-1}(\mathrm{st}(w))$ for all vertices $w \in L\}$; one can assume that each simplex has diameter smaller than $\varepsilon/2$ as above. Then, the closed stars for each vertex have diameter less than ε, so $f(\mathrm{St}\, v) \subset \mathrm{st}(g(v))$ for all vertices $v \in K$.[8]

This is great, except that g has only been defined in small neighborhoods of vertices, and we want to define it everywhere. It's possible to define it linearly over K: each x in a simplex is a convex combination of the vertices in a unique way, so g can be extended linearly from its definitions on these vertices. Alternatively, we have f, and can take the linear homotopy between f and g at the vertices, and then use that on the rest of the simplex. ⊠

[7] A subdivision is a way of adding k-simplices for $k < n$, and breaking each n-simplex into some number of smaller n-simplices, just like barycentric subdivision that was used in the proof of excision.

[8] "My God, it's full of stars!" – Dave Bowman, *2001: A Space Odyssey*.

In summary: start with a map, approximate it with a simplicial map that it's homotopic to, and then use the simplicial map to find the contradiction we were looking for. Then, consider stars half the size of the smallest distance between x and $f(x)$, and let g separate a vertex and its star, so g has no fixed point, and therefore it's the desired map.

11. Cohomology and the Universal Coefficient Theorems: 2/12/15

"If you already saw this, you can, I don't know, update your Facebook page or something."

Last time we talked about the Lefschetz fixed-point theorem; one of many things Alexander Grothendieck was famous for was an analogous version in algebraic geometry, for algebraic varieties, unsurprisingly called the Grothendieck-Lefschetz theorem.

We've also talked about homology with coefficients in an arbitrary abelian group A, called $H_n(X, A) = H_n(C_*(X) \otimes A, \partial \otimes \mathbb{1}_A)$ (we defined it with different words, but it was equivalent, and this definition will be useful). In summary, homology with coefficients is given by taking the functor $-\otimes A$ and applying it to $C_*(X)$; then, the result was still a chain complex, so one can take its homology.

One can do this with more functors $F : \mathsf{Ab} \to \mathsf{Ab}$ (recall that a functor is an assignment of objects $A \mapsto F(A)$ and morphisms $\varphi : A \to B$ sent to $F(\varphi) : F(A) \to F(B)$, such that $F(\mathrm{id}) = \mathrm{id}$ and $F(f \circ g) = F(f) \circ F(g)$). Some functors are *contravariant*, i.e. composition takes the form $F(\varphi) : F(B) \to F(A)$ and $F(f \circ g) = F(g) \circ F(f)$ (in some sense, F reverses arrows).

The category of abelian groups has more structures; for example, one can add two morphsims $f, g : A \rightrightarrows B$ to get $f + g : A \to B$, and therefore $\mathrm{Hom}(A, B)$ has more structure than a set.

Definition. A functor $F : \mathsf{Ab} \to \mathsf{Ab}$ that preserves addition, i.e. $F(f + g) = F(f) + F(g)$, is called *additive*.

One can formalize categories that have this structure, called *additive categories*, or with a little more structure *abelian categories*, but we won't need to do this.

It turns out that if F is additive, then it sends chain complexes to chain complexes. That is, $-\otimes A$ is an additive functor, and $\mathrm{Hom}(-, B)$, sending $A \mapsto \mathrm{Hom}(A, B)$, is an additive contravariant functor.

In particular, one can apply $\mathrm{Hom}(-, A)$ to $C_*(X)$, creating the *cochains* $C^n(X; A) = \mathrm{Hom}(C_n(X), A)$. Note that since $\mathrm{Hom}(-, A)$ is contravariant, it sends a direct sum to a direct product, so since $C_n(X) = \bigoplus_{\sigma:\Delta^n \to X} \mathbb{Z}$, then $C^n(X) = \prod_\sigma A$. In some sense, $\varphi \in C^n(X; A)$ are functions of sets from $\{\sigma : \Delta^n \to X\} \to A$.

Then, the *coboundary map* $\delta : C^n(X; A) \to C^{n+1}(X; A)$ (notice the increase in index) can be obtained by just applying $\mathrm{Hom}(-, A)$ to $\partial : C_{n+1} \to C_n$; in particular, $\delta(\varphi) = \varphi \circ \partial$. However, a different convention is to define it as $(-1)^{n+1}\varphi \circ \partial$. Unfortunately, these are different, but in practice it's all right, because when we define cohomology, the definition won't depend on the sign convention.

Definition. The *cohomology groups* of X with coefficients in A are given by $H^n(X; A) = \ker(\delta : C^n(X; A) \to C^{n+1}(X; A)) / \mathrm{Im}(C^{n-1}(X; A) \to C^n(X; A))$. If $A = \mathbb{Z}$, sometimes this is merely denoted $H^n(X)$.

Once again, this is nicely functorial: if $f : X \to Y$ is continuous, then there's the induced $f_\sharp : C_*(X) \to C_*(Y)$, and then induces (via $\mathrm{Hom}(-, A)$) a map $C^n(X, A) \leftarrow C^n(Y, A)$ sometimes called $f^\sharp$, and therefore there's an induced map $f^* : H^n(Y, A) \to H^n(X, A)$.

Just as we asked the difference between $H_n(X; A)$ versus $H_n(X) \otimes A$; similarly, how does $H^n(X; A)$ differ from $\mathrm{Hom}(H_n(X), A)$? In some sense, do these functors commute with homology? There are theorems called the univeral coefficient theorems which show that $H^n(X; A)$ and $H_n(X; A)$ depend only on A and $H^n(X; \mathbb{Z})$ and $H_n(X; \mathbb{Z})$; the answer is in terms of the Ext and Tor functors. These were covered in 210A, which technically wasn't a prerequisite... we will have a quick review of them.

Tor come from the fact that $A \mapsto A \otimes B$ is right exact, but not left exact; for example, consider the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

When one tensors with $\mathbb{Z}/2$, the resulting sequence is

$$\mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{\mathrm{id}} \mathbb{Z}/2.$$

This sequence is not short exact; the right side is surjective, but the left side isn't injective. This kind of functor is called *right exact*.

Exercise 3. Show that if $F : \mathsf{Ab} \to \mathsf{Ab}$ is an additive functor that preserves short exact sequences (e.g. $-\otimes \mathbb{Q}$), then the induced homology commutes with F (for example, $H_n(X; \mathbb{Q}) = H_n(X) \otimes \mathbb{Q}$).

This will be superseded by the universal coefficient theorem in just a bit, but is interesting to think about.

Objects called *derived functors* are used to fix this; if R is a commutative[9] ring and M and N are R-modules, then $M \otimes_R N$ is also an R-module. Then, if $M' \to M \to M'' \to 0$ is exact (where they're all R-modules), then $M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N \to 0$ is still exact.

Since Eilenberg in the 1950s, there's been a standard way to work with this, called the derived functors of $-\otimes_R N$. (Since $\mathbb{Z}$-modules are abelian groups, one can think about the case where $R = \mathbb{Z}$, which will be most useful in this class.) These will be called $M \mapsto \operatorname{Tor}_n^R(M, N)$, which is a functor from R-modules to R-modules, where $\operatorname{Tor}_0^R(M, N) = M \otimes_R N$. The upshot is that a short exact sequence of R-modules $0 \to M' \to M \to M'' \to 0$ induces a long exact sequence

$$\cdots \longrightarrow \operatorname{Tor}_2(M'', N) \xrightarrow{\delta} \operatorname{Tor}_1(M', N) \longrightarrow \operatorname{Tor}_1(M, N) \longrightarrow \operatorname{Tor}_1(M'', N) \xrightarrow{\quad\delta\quad}$$
$$M' \otimes_R N \longrightarrow M \otimes_R N \longrightarrow M'' \otimes_R N \longrightarrow 0.$$

The δ are connecting homomorphisms. Note that for $R = \mathbb{Z}$, $\operatorname{Tor}_n^{\mathbb{Z}}(M, N)$ is nonzero only when $n = 0$ or $n = 1$. In this case, $\operatorname{Tor}_1^{\mathbb{Z}}(M, N)$ is simply called $\operatorname{Tor}(M, N)$.

If M and N are finitely generated, then there's a nice formula for both the tensor product and Tor.

The final upshot is the theorem we wanted in the first place.

Theorem 11.1 (Universal coefficient theorem for homology)**.** *If X is a topological space and A is an abelian group, then the following sequence is short exact.*

$$0 \longrightarrow H_n(X) \otimes A \xrightarrow{f} H_n(X; A) \longrightarrow \operatorname{Tor}(H_{n-1}(X), A) \longrightarrow 0$$

Here, $f : [c] \otimes a \mapsto [c \otimes a]$. Furthermore, this sequence splits, albeit not naturally, so $H_n(X; A) = (H_n(X) \otimes A) \oplus \operatorname{Tor}(H_{n-1}(X), A)$.

The point is, if you know $H_n(X; \mathbb{Z})$, it's possible to calculate $H_n(X; A)$ for any A, since the Tor can be calculated nicely.

Definition. If $\operatorname{Tor}_n^R(-, N) = 0$ for $n \geq 1$, then N is called *flat* (as an R-module).

For example, $\mathbb{Q}$ is flat as a $\mathbb{Z}$-module (equivalently, as an abelian group).

Corollary 11.2. *If A is flat, then $H_n(X; A) = H_n(X; \mathbb{Z})$.*

There's a similar story for $\operatorname{Hom}_R(-, A)$, when A is an R-module. If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence, then the following sequence is exact.

$$0 \longrightarrow \operatorname{Hom}_R(M'', A) \longrightarrow \operatorname{Hom}_R(M, A) \longrightarrow \operatorname{Hom}(M', A) \tag{11.1}$$

In particular, the last map may not be surjective; it's not hard to dream up examples.

Then, one does the same thing, augmenting (11.1) with a long exact sequence of functors $\operatorname{Ext}_R^i(M, A)$, but this time it's contravariant, so it looks like this.

$$\cdots \longrightarrow \operatorname{Hom}(M', A) \longrightarrow \operatorname{Ext}_R^1(M'', A) \longrightarrow \operatorname{Ext}_R^1(M, A) \longrightarrow \operatorname{Ext}_R^1(M', A) \longrightarrow \operatorname{Ext}_R^2(M'', A) \longrightarrow \cdots$$

Once again, for $R = \mathbb{Z}$, $\operatorname{Ext}_{\mathbb{Z}}^n(A, B) = 0$ for $n > 1$ and all A and B, so many topologists just call $\operatorname{Ext}_{\mathbb{Z}}^1(A, B)$ as $\operatorname{Ext}(A, B)$. Hatcher does this, and doesn't even discuss Ext more generally, but this is sometimes confusing.

Notice the notational conventions here; in topology, one tends to use upper indices wherever boundary maps and connecting homomorphisms raise the degree (e.g. Ext and cohomology), and lower indices when they lower the degree (Tor and homology).

Theorem 11.3 (Universal coefficient theorem for cohomology)**.** *If X is a topological space and A is an abelian group, then the following sequence is short exact.*

$$0 \longrightarrow \operatorname{Ext}(H_{n-1}(X), A) \longrightarrow H^n(X; A) \xrightarrow{g} \operatorname{Hom}(H_n(X), A) \longrightarrow 0$$

Here, g is given as follows: if $\delta \circ \varphi = 0$, then $[\varphi] \mapsto ([c] \to \varphi(c))$ is well-defined and surjective.

[9] This hypothesis can be relaxed in some cases, though one has to be more careful about what is said.

Note that Ext and Tor are special, because higher terms vanish; for more general functors, there ends up being a spectral sequence instead!

So this is all awesome, but now we need to be able to calculate Tor. We'll do this in the case of a general commutative ring R, because it's less confusing, less seemingly arbitrary than the special case.

Definition. An R-module P is *projective* if for any R-modules $M'' \twoheadrightarrow M$, a map $f : P \to M$ lifts to a map $\widetilde{f} : M' \to M$, so that the following diagram commutes.

$$\begin{tikzcd} & M' \arrow[d] \\ P \arrow[ur, dashed, "\widetilde{f}"] \arrow[r, "f"] & M \end{tikzcd}$$

Example 11.4. If P is a free R-module, then it's projective, since one can choose generators, and then pick lifts of all of the generators. If it's infinitely generated, you have to use the Axiom of Choice, certainly, and a lot of this wouldn't work without the assumption of the Axiom of Choice.

A standard non-example is $\mathbb{Z}/2$ as a $\mathbb{Z}$-module, since there is no lift in the following diagram.

$$\begin{tikzcd} & \mathbb{Z} \arrow[d] \\ \mathbb{Z}/2 \arrow[ur, dashed] \arrow[r, "\mathrm{id}"'] & \mathbb{Z}/2 \end{tikzcd}$$

Definition. A *projective resolution* of an R-module M is a chain complex (P_n, ∂) such that each P_n is projective and

$$H_n(P_*, \partial) = \begin{cases} M, & n = 0 \\ 0, & \text{otherwise.} \end{cases}$$

In some sense, a projective resolution looks like the following diagram, which is an isomorphism on homology.

$$\begin{tikzcd} 0 \arrow[d] & P_2 \arrow[l] \arrow[d, "\partial"] \\ 0 \arrow[d] & P_1 \arrow[l] \arrow[d, "\partial"] \\ M \arrow[d] & P_0 \arrow[l, "\varphi"'] \arrow[d] \\ 0 \arrow[r, equal] & 0 \end{tikzcd}$$

Here, all arrow are R-module homomorphisms. This induces a long exact sequence $0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots$.

Lemma 11.5. *For any R-module M, a projective resolution always exists.*

Proof. The only projective modules we know are free, so let's try that. Let P_0 be the R-module free on all of the generators of M (which is huge, but whatever); thus, $\varphi : P_0 \twoheadrightarrow M$. Next, one can choose P_1 to be free on $\ker(\varphi)$, and thus $P_1 \twoheadrightarrow \ker(\varphi) \subset P_0$, and so on. ⊠

Though these are the canonical choices, when doing calculations, you will want to use smaller resolutions (somewhat like calculating singular vs. simplicial homology).

Definition. Two maps $F : P_* \to P'_*$ and $G : P'_* \to P_*$ of chain complexes P_* and P'_* are *chain homotopic* if $F \circ G - \mathbb{1} = \partial T - T\partial$ and $G \circ F = \partial T' - T'\partial$.

Lemma 11.6. *Let M and M' be R-modules and let $\cdots \to P_1 \to P_0 \to M \to 0$ and $\cdots \to P'_1 \to P'_0 \to M' \to 0$ be projective resolutions. Then, an R-module homomorphism $f : M \to M'$ induces a chain map $F : P_* \to P'_*$, such that the following diagram commutes.*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, equal] & M' \arrow[d, equal] \\ H_0(P_*) \arrow[r, "F_*"'] & H_0(P'_*) \end{tikzcd}$$

Moreover, any two such F are chain homotopic.

Proof. This all rests upon the definition of projective; we want to find maps to fill in the dotted arrows in the following diagram, so that it commutes.

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
P_1 \arrow[r, dashed] \arrow[d] & P_1' \arrow[d] \\
P_0 \arrow[r, two heads] \arrow[d, "\varphi"] & P_0' \arrow[d, "\varphi'"] \\
M \arrow[r, "f"] \arrow[d] & M' \arrow[d] \\
0 & 0
\end{tikzcd}$$

But since P_0 is projective, the map $f \circ \varphi : P_1 \to M'$ lifts through φ' to a map $f_1 : P_0 \to P_0'$ such that the diagram commutes; then, the same logic works to create a map $P_1 \to P_0'$, which lifts to a map $P_1 \to P_1'$, since P_1 is projective, and so on.

For chain homotopy, the proof is similar; write down the condition, and then the projective assumption means that the right diagram will commute. ⊠

Corollary 11.7. *Two projective resolutions P_* and P_*' of an R-module M are chain homotopic, i.e. there's an $F : P_* \to P_*'$, and any two such chain homotopies are chain homotopic to each other.*

This is a uniqueness result; there is essentially only one projective resolution, especially once one passes to homology.

Anyways, here's where Tor comes in; one can apply $-\otimes_R N$ to a projective resolution to create a chain complex $P_* \otimes_R N$, with maps.

$$0 \longleftarrow P_0 \otimes_R N \longleftarrow P_1 \otimes_R N \longleftarrow P_2 \otimes_R N \longleftarrow \cdots$$

Then, one defines $\mathrm{Tor}_n^R(M,N) = H_n(P_* \otimes N)$. Since any two projective resolutions are chain homotopic, then they induce an isomorphism on homology, so this is well-defined.

12. The Universal Coefficient Theorems: 2/17/15

Last time, we defined the functors $\mathrm{Tor}_i^R(M,N)$, where R is a (commutative) ring and M and N are R-modules. This was defined as $H_i(P_* \otimes N)$, where $P_* \to M$ is a projective resolution.

The special thing about the integers, or any PID, is that a submodule of a free module is automatically free; this is easy enough to prove. In the finitely generated case, it follows from the classification of finitely generated modules over a PID, but is also true in the infinitely generated case. It's possible to state it more formally.

Theorem 12.1. *If F is an abelian group and has a basis, i.e. $F \approx \bigoplus \mathbb{Z}$, then if $M \le F$, then $M \approx \bigoplus \mathbb{Z}$ as well.*

The trick is for objects like $\mathbb{Q}$ or $\prod_0^\infty \mathbb{Z}$; the latter isn't free, which isn't completely obvious.

Over $\mathbb{Z}$, and over any PID, one can find for any module a free resolution of length 2: $P_1 \to P_0 \twoheadrightarrow M$. The canonical choice, which is a little wasteful, is to make P_0 free on the elements of M; then, the kernel of the surjection is free, and can be called P_1. Thus, there's a free resolution

$$0 \longrightarrow P_1 \longrightarrow P_0 \twoheadrightarrow M \longrightarrow 0.$$

Thus, $\mathrm{Tor}_0^R(M,N) \cong M \otimes_R N$, which is the cokernel of $P_1 \otimes_R N \to P_0 \otimes_R N$.

In specific example cases, one can do better than the canonical construction, e.g. if $M = \mathbb{Z}/n\mathbb{Z}$, then there's a free resolution given by

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0.$$

Thus, $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, N)$ (often just called $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, N)$) is $\ker(N \xrightarrow{\cdot n} N)$, which is the "$n$-torsion" of N. For example, $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}$.

Lemma 12.2.

(1) $\operatorname{Tor}(M, N) \cong \operatorname{Tor}(N, M)$.
(2) Tor *commutes with direct sums.*

$$\operatorname{Tor}\left(\bigoplus_\alpha M_\alpha, N\right) \cong \bigoplus_\alpha \operatorname{Tor}(M_\alpha, N).$$

(3) $\operatorname{Tor}(\mathbb{Z}, N) = 0$.

(1) ultimately comes from the natural isomorphism $M \otimes_R N \cong N \otimes_R M$, but since we made some specific choices in defining it, there's something to show.

These three properties, along with the calculation for $\operatorname{Tor}(\mathbb{Z}/n\mathbb{Z}, N)$ given above, allows one to calculate $\operatorname{Tor}(M, N)$ for any finitely generated abelian group.

Corollary 12.3. *If N is an abelian group, $N = 0$ iff $N \otimes \mathbb{Q} = 0$ and $\operatorname{Tor}(\mathbb{Z}/p\mathbb{Z}, N) = 0$ for all primes p.*

Proof. The forward direction is silly, so let's look at the reverse direction.

$\operatorname{Ker}(N \to N \otimes \mathbb{Q}) = \{x \in N \mid \text{ there exists an } n > 0 : nx = 0\}$. Thus, if $N \otimes \mathbb{Q} = 0$ but $N \neq 0$, then for any $x \in N$ with $x \neq 0$, $nx = 0$ for some n, so factor n into primes $p_1 \cdots p_m k$. In particular, for any prime p for which there's an $x' \in N \setminus 0$ such that $px' = 0$, then $x' \in \ker(\cdot n) = \operatorname{Tor}(\mathbb{Z}/p\mathbb{Z}, N)$. ⊠

Next, we'll prove the universal coefficient theorem for homology; it's similar for cohomology. Hatcher does it the other way around. First, let's restate it.

Theorem 12.4 (Universal Coefficient Theorem for homology, algebraic version)**.** *Let (C_*, ∂) be a chain complex with C_n free for all n and A be an abelian group. Then, there is a natural short exact sequence*

$$0 \longrightarrow H_n(C_*) \otimes A \longrightarrow H_n(C_* \otimes A) \longrightarrow \operatorname{Tor}(H_{n-1}(C_*), A) \longrightarrow 0,$$

and this sequence splits (albeit non-naturally).

If one applies this to singular chains on a topological space X, one gets Theorem 11.1 from last lecture.

Proof of Theorem 12.4. Let $Z_n = \ker(\partial) \subset C_n$, and $B_n = \operatorname{Im}(\partial) \subset Z_n$. Thus, B_n and Z_n are free and there's a short exact sequence

$$0 \longrightarrow Z_n \hookrightarrow C_n \xrightarrow{\partial} B_{n-1} \longrightarrow 0. \tag{12.1}$$

Then, by freeness, one can choose a splitting non-naturally: choose a basis $\{x_\alpha\}$ for B_{n-1} and $\{y_\alpha\}$ for C_n, such that $\partial y_\alpha = x_\alpha$, and use this to construct a section and split the sequence.

(12.1) can be made into a short exact sequence of chain complexes in a slightly artificial way, by letting the boundary maps on Z_n and B_n be 0, and this makes the following diagram commute.

$$\begin{tikzcd} 0 \arrow[r] & Z_n \arrow[r] \arrow[d, "0"] & C_n \arrow[r] \arrow[d, "\partial"] & B_{n-1} \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & Z_{n-1} \arrow[r] & C_{n-1} \arrow[r] & B_{n-2} \arrow[r] & 0 \end{tikzcd}$$

Thus, take the associated long exact sequence (in "homology," even though it's a weird homology for the outer two complexes) and apply $-\otimes A$:

$$\operatorname{Tor}(B_{n-1}, A) \longrightarrow Z_n \otimes A \longrightarrow C_n \otimes A \longrightarrow B_{n-1} \otimes A \longrightarrow 0,$$

but since B_{n-1} is free (and therefore flat, or equivalently it has a free resolution of itself), then this simplifies to

$$0 \longrightarrow Z_n \otimes A \longrightarrow C_n \otimes A \longrightarrow B_{n-1} \otimes A \longrightarrow 0.$$

Thus, this is a short exact sequence of chain complexes!

Since the boundary maps for Z_n are zero, then $H_n(Z_* \otimes A) = Z_n \otimes A$. The next step is to find the connecting morphism $\partial : B_{n-1} \otimes A \to Z_{n-1} \otimes A$, by chasing across this diagram.

$$\begin{tikzcd} 0 \arrow[r] & Z_n \otimes A \arrow[r] & C_n \otimes A \arrow[r, "\partial"] \arrow[d, "\partial\otimes\mathbb{1}"] & B_{n-1} \otimes A \arrow[r] & 0 \\ & Z_{n-1} \arrow[r, hook] & C_n & & \end{tikzcd}$$

It ends up that the connecting morphism is just inclusion $B_{n-1} \hookrightarrow Z_{n-1}$ tensored with the identity morphism on A.

Then, this long exact sequence becomes a bunch of short exact sequences: $0 \to B_n \hookrightarrow Z_n \to H_n(C_*) \to 0$ is a free resolution of $H_n(C_*)$, and therefore there's an induced long exact sequence

$$0 \longrightarrow \operatorname{Tor}(H_*(C_n), A) \longrightarrow B_n \otimes A \longrightarrow Z_n \otimes A \longrightarrow H_n(C_*) \otimes A \longrightarrow 0.$$

Thus, when one calculates what the kernels and cokernels actually are, $H_n(C_*) \otimes A$ is the cokernel of $B_n \otimes A \to Z_n \otimes A$, and $\operatorname{Tor}(H_n(C_*), A)$ is the kernel of $B_{n-1} \otimes A \to Z_n \otimes A$. Thus, the desired sequence exists. Then, it splits because the split exact sequence $0 \to Z_n \to C_n \to B_{n-1} \to 0$ split, and this induces a splitting everywhere else.

Specifically, a section φ induces a chain map $\varphi \otimes \mathrm{id} : C_n \otimes A \to H_*(C_*) \otimes A$ (the latter with trivial boundary maps), and $H_n(C_* \otimes A) \to H_n(C_*) \otimes A$ is given by $[c \otimes a] \mapsto [\varphi(c)] \otimes a$. Unlike the construction of the short exact sequence, this relies on the choice of splitting in the first sequence, and is thus noncanonical, but is still true as abstract groups. ⊠

Corollary 12.5. *$H_n(C_* \otimes A) \cong (H_n(C_*) \otimes A) \oplus \operatorname{Tor}(H_{n-1}(C_*), A)$, and this isomorphism is unique up to* $\operatorname{Hom}(\operatorname{Tor}(H_{n-1}, A), H_n \otimes A)$.

This is a general fact about split exact sequences: any two splittings will differ by an upper triangular matrix.

The proof for cohomology and Ext is essentially the same, but all the arrows are turned around. Dually, $\operatorname{Ext}^i_R(M, N) = H^i(\operatorname{Hom}(P_*, N))$, where $P_* \to M$ is free. Note that, unlike Tor, this doesn't work with both sides: a projective resolution $P_* \to N$ doesn't work. Instead, you need an *injective resolution*, which is a dual notion to a projective resolution. Fundamentally, the issue is that there's no such thing as a "co-free" module, and $\operatorname{Hom}(-, F)$ of a free module F doesn't always produce a free module.

Then, similarly to Tor, $\operatorname{Ext}^0_{\mathbb{Z}}(M, N) = \operatorname{Hom}(M, N)$, and $\operatorname{Ext}^1_{\mathbb{Z}}(M, N)$ tends to be called $\operatorname{Ext}(M, N)$.

Definition. A *cochain complex* is essentially the same as a chain complex except $\delta : C^n \to C^{n+1}$ increases indices, and is called the *coboundary map* or *differential*.

In the professor's opinion, this notation is unfortunate, and leads to at least part of the confusion about cohomology.

Theorem 12.6 (Universal Coefficient Theorem for cohomology, algebraic version)**.** *Let A be an abelian group and C_* be a chain complex, where C_n is free for all n. Then, the following sequence is short exact.*

$$0 \longrightarrow \operatorname{Ext}(H_{n-1}(C_*), A) \longrightarrow H^n(\operatorname{Hom}(C_*, A)) \longrightarrow \operatorname{Hom}(H_n(C_*), A) \longrightarrow 0,$$

and it splits, albeit not naturally.

The proof is basically the same.

If one combines both universal coefficient theorems and the five lemma, then one gets a nice corollary.

Corollary 12.7. *If $f : X \to Y$ induces an isomorphism $H_n(X) \to H_n(Y)$ for all n, then there are also isomorphisms $H_n(X; A) \to H_n(Y; A)$ and $H^n(Y; A) \to H^n(X; A)$ for all abelian groups A and for all n.*

This secretly uses the word "natural," in the sense that the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & H_n(X) \otimes A \arrow[r] \arrow[d, "f_* \otimes \mathrm{id}"] & H_n(X; A) \arrow[r] \arrow[d, "f_*"] & \operatorname{Tor}(H_{n-1}(X), A) \arrow[r] \arrow[d, "\operatorname{Tor}(f_*, \mathrm{id})"] & 0 \\
0 \arrow[r] & H_n(Y) \otimes A \arrow[r] & H_n(Y; A) \arrow[r] & \operatorname{Tor}(H_{n-1}(Y), A) \arrow[r] & 0
\end{tikzcd}$$

The splitting isn't natural, because adding sections to the above diagram cannot be done without making it not commute.

There is also a universal coefficient theorem for relative homology, since relative chains are still free (on a subset of generators in $C_n(X)$).

Corollary 12.8 (Universal coefficient theorem for relative homology)**.** *The following sequence is short exact, and splits, albeit not naturally.*

$$0 \longrightarrow H_n(X, Y) \otimes A \longrightarrow H_n(X, Y; A) \longrightarrow \operatorname{Tor}(H_{n-1}(X, Y), A) \longrightarrow 0.$$

Using the five lemma, one can use excision in $H_*(-; \mathbb{Z})$ to prove it for $H_*(-, A)$ and $H^*(-, A)$ for any abelian group A; this can be done as we've seen it before, but also follows quite nicely from the universal coefficient theorems.

Corollary 12.9. *If C_* is a chain complex and C_n is free for all n, then the following are equivalent.*
- $H_n(C_*) = 0$ *for all n.*
- $H_n(C_* \otimes A) = 0$ *for all n and all abelian groups A.*
- $H_n(C_* \otimes A) = 0$ *for all n and when $A = \mathbb{Q}$ and $A = \mathbb{F}_p$ for all primes p.*

Proof. Most of the equivalences are either trivial (e.g. let $A = \mathbb{Z}$ for going from the second to the first) or follow directly from the universal coefficient theorems. However, going from the third to the first is less clear.

Suppose the third is true, but $H_n(C_*) \neq 0$; then, by Corollary 12.3, either $H_n(C_*) \otimes \mathbb{Q} \neq 0$ or $\operatorname{Tor}(H_n(C_*), \mathbb{Z}/p\mathbb{Z}) \neq 0$ for some prime p, so either $H_n(C_* \otimes \mathbb{Q}) \neq 0$ or $H_{n+1}(C_* \otimes \mathbb{F}_p) \neq 0$. ⊠

Corollary 12.10. $H_n(X, Y) = 0$ *for all n iff* $H_n(X, Y; A) = 0$ *for all n and all $A = \mathbb{Q}$, $A = \mathbb{F}_p$ for p prime.*

This is quite useful: to prove the homology vanishes, one only needs to check in a bunch of fields, where things may be a lot easier to work with.

Corollary 12.11. $f : X \to Y$ *induces an isomorphism in $H_n(-, \mathbb{Z})$ for all n iff it does so in $H_n(-, A)$ for all n, and for $A = \mathbb{Q}$ and $A = \mathbb{F}_p$ for all p prime.*

Next time, the midterm is due, and in class, we'll talk about the cup product on cohomology: the real reason that cohomology is so useful is that it tends to be a ring, rather than just a group; there's a multiplication map $H^k(X; A) \otimes H^\ell(X; B) \to H^{k+\ell}(X; A \otimes B)$. Calculating the ring structure is often a stronger invariant of the space. Then, if $A = B$, one can compose with $A \otimes A \to A$ given by $a_1 \otimes a_2 \mapsto a_1 a_2$ and get a graded ring structure on the cohomology groups.

13. The Cup Product: 2/19/15

> *"The brohomology groups carry a natural flip cup product structure. The underlying space also has a get smashed product." – Bif Reiser*

Last time, we saw the details of the proof of the universal coefficient theorem for homology, and the statement of the universal coefficient theorem for cohomology (the proof is essentially the same).

Today, we'll talk about the most important difference between homology and cohomology, the cup product. Let R be a ring (in topology, this is usually $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{F}_p$ where p is prime[10]); then, the cohomology with coefficients in R, given by

$$H^*(X; R) = \bigoplus_{p=1}^{\infty} H^p(X; R),$$

has a natural ring structure. Here, R doesn't need to be commutative, though we'll usually stick to commutative examples. The ring structure is given by the *cup product*

$$H^p(X; R) \otimes H^q(X; R) \xrightarrow{\smile} H^{p+q}(X; R),$$

written $\alpha \otimes \beta \mapsto \alpha \smile \beta$, or eventually $\alpha\beta$. This creates a graded ring structure on $H^*(X; R)$.

Here are some properties of the cup product.
- It's natural: if $f : X \to Y$, then $f^*(\alpha \smile \beta) = (f^*\alpha) \smile (f^*\beta)$. This is equivalent to saying that f^* is a ring homomorphism.
- It's associative (well, since it is ring multiplication): $(\alpha \smile \beta) \smile \gamma = \alpha \smile (\beta \smile \gamma)$.
- The identity is given by $1 \in H^0(X; R)$: $1 \smile \alpha = \alpha \smile 1 = R$. Elements in cohomology are just functions, so take 1 to be the class of $(\sigma : \Delta^0 \to X) \mapsto 1$, i.e. assigning 1 to every map. It's not clear yet why this is the unit, but we'll come back and prove it.

Note that if $A \subseteq X$, then $H^*(X, A; R)$ does *not* have a ring structure; however, it's an $H^*(X; R)$-module, with a scalar multiplication also called $\smile$: $H^p(X; R) \otimes H^q(X, A; R) \to H^{p+q}(X, A; R)$. Thus, this structure is a graded module; it would be a ring but for the lack of a unit.

We're saying a lot without proving anything, but just like formal homology theory, a lot of this can be determined from the properties, which are more important than the actual construction. Nonetheless, we will provide such a construction.

The long exact sequence in cohomology increases degree, and we will prove that the connecting homomorphism $\delta : H^p(A; R) \to H^{p+1}(X, A; R)$ is $H^*(X; R)$-linear (the former is an $H^p(X; R)$-module because inclusion induces the map $H^p(X; R) \to H^p(A; R)$, and in general a ring map $A \to B$ makes B into an A-module).

[10] Hatcher calls this $\mathbb{Z}_p$, which is terrible, because those are the p-adics, which do play a role in algebraic topology!

These will be easy to prove, and in particular they will all hold on the cochain level, but the next one doesn't, and is harder to prove. This is that the cup product is *graded commutative*: if $\alpha \in H^p(X;R)$ and $\beta \in H^q(X;R)$, then $\alpha \smile \beta = (-1)^{pq}\beta \smile \alpha$.

For a concrete construction, a map $\sigma : \Delta^{p+q} \to X$ has a *front p-face* $\sigma|_{[v_0,\dots,v_p]}$ and a *back q-face* $\sigma|_{[v_p,\dots,v_{p+q}]}$; these overlap in one vertex.

Definition. Suppose $\varphi \in C^p(X;R)$ and $\psi \in C^q(X;R)$, so that they're R-valued functions on the chains. Then, the *cup product* on the cochain level is given, for $\sigma : \Delta^{p+q} \to X$, as $(\varphi \smile \psi)(\sigma) = \varphi(\sigma|_{[v_0,\dots,v_p]}) \cdot \psi(\sigma|_{[v_p,\dots,v_{p+q}]})$. Sometimes, this is called the *Alexander-Whitney formula*.

That is, use φ and ψ to get two elements of R, and then multiply them together in R.

This seems a little strange, since a lot of the information in σ is ignored, but this turns out to be a good definition. We'll be able to prove that $\delta(\varphi \smile \psi) = (\delta\varphi) \smile \psi + (-1)^p \varphi \smile (\delta\psi)$, which will imply that $\varphi \smile \psi$ is in a cocycle if φ and ψ are, and that $[\varphi] \smile [\psi] = [\varphi \smile \psi]$ is well-defined. Thus, one can prove cochain-level versions of the properties outlined above (except for commutativity, which is not true on the cochain level, though it is true up to chain homotopy), and they pass into cohomology.

There's a slight asymmetry in that we wrote the front p-face first and the back q-face last; thus, it's possible to make the other definition, which is very similar; in cohomology, in fact, it's the same, up to sign.

Proposition 13.1. *$\delta(\varphi \smile \psi) = (\delta\varphi) \smile \psi + (-1)^p \varphi \smile (\delta\psi)$.*

Proof. We have nothing but the definition, so let's apply this to a $(p+q+1)$-simplex σ. Thus,

$$\begin{aligned}(\delta(\varphi \smile \psi))(\sigma) &= (\varphi \smile \psi)(\partial\sigma)\\ &= \sum_{i=0}^{p} (-1)^i (\varphi \smile \psi)\Big(\sigma|_{[v_0,\dots,\widehat{v}_i,\dots,v_{p+q+1}]}\Big).\end{aligned}$$

The front p-face is either $\sigma|_{[v_0,\dots,v_p]}$, if $i > p$, and is $\sigma_{[v_0,\dots,\widehat{v}_i,\dots,v_p]}$ if $i \le p$; then, the q-face is similar. Thus, this expands to

$$\begin{aligned}&= \sum_{i=0}^{p+1} (-1)^i \varphi\Big(\sigma|_{[v_0,\dots,\widehat{v}_i,\dots,v_{p+1}]}\Big)\psi\Big(\sigma|_{[v_{p+1},\dots,v_{p+q+1}]}\Big) + \sum_{i=p+1}^{p+q+1} (-1)^i \varphi\Big(\sigma|_{[v_0,\dots,v_p]}\Big)\psi\Big(\sigma|_{[v_p,\dots,\widehat{v}_i,\dots,v_{p+q+1}]}\Big)\\ &= (\delta\varphi)\Big(\sigma|_{[v_0,\dots,\widehat{v}_i,\dots,v_{p+1}]}\Big)\psi\Big(\sigma|_{[v_{p+1},\dots,v_{p+q+1}]}\Big) + (-1)^p \varphi\Big(\sigma|_{[v_0,\dots,v_p]}\Big)(\delta\psi)\Big(\sigma|_{[v_p,\dots,\widehat{v}_i,\dots,v_{p+q+1}]}\Big).\end{aligned}$$ ⊠

This is just a calculation, albeit an ugly one.

Corollary 13.2. *Within cohomology, $[\varphi] \smile [\psi] = [\varphi \smile \psi]$.*

This is because $(\varphi + \delta\rho) \smile \psi = \varphi \smile \psi$, since $\delta^2 = 0$ and one can plug in the formula from Proposition 13.1.

There's no need to go into that much detail with the other properties; associativity of the cup product is just as messy: pick φ, ψ, and ρ in C^p, C^q, and C^r respectively, and then evaluate on a $\sigma \in \Delta^{p+q+r}$, which is messy but eventually ends up working.

It will be useful to talk about relative cochains. Applying $\mathrm{Hom}(-, R)$ to the short exact sequence induced by a pair yields an exact sequence, because chain complexes are free; thus, the relative cochains fit into a short exact sequence

$$0 \longrightarrow C^p(X, A; R) \xhookrightarrow{f} C^p(X;R) \longrightarrow C^p(A;R) \longrightarrow 0.$$

Thus, $C^p(X, A; R)$ consists of functions ψ on the set of simplices to R, and the relativity means that $\psi(\sigma) = 0$ whenever $\sigma(\Delta^k) \subseteq A$. Then, if $\varphi \in C^p(X, A; R)$ and $\psi \in C^q(X;R)$, then one can use f to take $\varphi \smile \psi$, but if σ is contained within A, then $(\varphi \smile \psi)(\sigma)$ is φ of something in A multiplied by something else, so it's just 0. Thus, $\varphi \smile \psi \in C^p(X, A; R)$, so this really is a $C^p(X;R)$-module, and one can pass to homology as before. Equivalently, $C^*(X, A; R)$ is an ideal of the larger ring $C^*(X;R)$, and this is merely the absorption property.

If $f : Y \to X$, then $f^*(\varphi \smile \psi) = (f^*\varphi) \smile (f^*\psi)$, which can be checked by evaluating on a $\sigma : \Delta^{p+q} \to Y$. This isn't too hard, and we won't go over the gory details.

Recall that the connecting morphism $\delta : H^p(A;R) \to H^{p+1}(X, A; R)$ is given by chasing back around the following diagram (where the implicit $C^*(-, R)$ is left out for notational simplicity).

$$\begin{tikzcd} 0 \arrow[r] & C^p(X,A) \arrow[r] & C^p(X) \arrow[r, "i^*"] \arrow[d, "\delta"] & C^p(A) \arrow[r] & 0 \\ 0 \arrow[r] & C^{p+1}(X,A) \arrow[r] & C^{p+1}(X) & & \end{tikzcd}$$

Specifically, one takes a $\varphi \in C^p(A)$, which pulls back to a $\widetilde{\varphi}$, and is pushed down to $\delta\widetilde{\varphi}$. Thus, if one takes a $\psi \in C^q(X)$, then $i^*(\widetilde{\varphi} \smile \psi) = \varphi \smile (i^*\psi) \in C^{p+q}(A)$. Then, the diagram chase boils down to

$$\delta(\widetilde{\varphi} \smile \psi) = (\delta\widetilde{\varphi}) \smile \psi + (-1)^p \widetilde{\varphi} \smile (\delta\psi) = \delta\widetilde{\varphi} \smile \psi.$$

Thus, $H^k(A;R) \to H^{k+1}(X,A;R)$ is $H^*(X;R)$-linear.

We haven't shown the unital property, but this is pretty easy to do. We also haven't done graded commutativity, which will take more than the rest of the lecture. In particular, it's not true on the cochain level.

Recall the "asymmetry" that we created by choosing to put the front face first instead of the back face. This comes from dualizing (i.e. applying $\mathrm{Hom}(-,R)$) to the map of chains $C_*(X) \to C_*(X) \otimes C_*(X)$ given by

$$\sigma \longmapsto \sum_{p+q=n} \sigma|_{[v_0,\dots,v_p]} \otimes \sigma_{[v_p,\dots,v_{p+q}]}.$$

After dualizing, this is precisely the cup product, as a map

$$\bigoplus_{p+q=n} C^p(X;R) \otimes C^q(X;R) \longrightarrow C^n(X;R).$$

Commutativity on cochains would be equivalent to the following diagram commuting.

$$\begin{tikzcd}
C_n(X) \arrow[r] \arrow[d, equal] & \bigoplus_{p+q=n} C_p(X) \otimes C_q(X) \arrow[d, "T"] \\
C_n(X) \arrow[r] & \bigoplus_{p+q=n} C_q(X) \otimes C_p(X)
\end{tikzcd}$$

Here, $T : \sigma \otimes \tau \mapsto (-1)^{pq} \tau \otimes \sigma$. However, this diagram won't commute on the cochain level, so we'll have to be craftier.

This is all a special case of the tensor product on chain complexes:[11] if (C_*, ∂) and (C'_*, ∂') are chain complexes, then one can define the n^{th}-degree term as

$$(C_* \otimes C'_*)_n = \bigoplus_{p+q=n} C_p \otimes C'_q,$$

with a boundary morphism

$$\partial_\otimes(x \otimes y) = (\partial x) \otimes y + (-1)^p x \otimes \partial' y.$$

(All of this was on the homework, so it ought to look familiar.) It's possible to prove that $\partial_\otimes^2 = 0$; then, as in the homework, one obtains a natural chain map AW (i.e. given a map $X \to Y$, the obvious diagram commutes). Finally, if $p = q = 0$, then $AW(\sigma) = \sigma \otimes \sigma$ for $\sigma : \Delta^0 \to X$.

It turns out (though it's not possible to prove this in two minutes) that there's a nice theorem about this.

Theorem 13.3. *AW is unique up to chain homotopy; if $S_0, S_1 : C_*(X) \to C_*(X) \otimes C_*(X)$ such that if $\sigma : \Delta^0 \to X$ then $S_0(\sigma) = S_1(\sigma) = \sigma \otimes \sigma$, then S_0 and S_1 are naturally chain homotopic.*

This is not the way Hatcher explains it, though a closer explanation can be found in Bredon. The proof will be given next lecture, and is relatively slick, using something called "acyclic models." This will eventually lead to a proof of graded comutativity (letting $S_0 = AW$ and S_1 be the twisted version taken by precomposing T).

14. Graded Commutativity and a Künneth Formula: 2/24/15

Today's lecture was given by Elizabeth Gasparim again.

Note that Bredon's book (e.g. chapter 6) uses a strange notational convention for chain complexes; rather than C_* and C^*, he uses Δ, so one ends up with weird symbolisms like $\Delta(\Delta)$ and the like.

When doing homotopy, it's easy to deal with the products of spaces; you just separate. But it's trickier for cohomology.

Proposition 14.1 (Künneth formula)**.** *Let F be a field and X and Y be topological spaces. Then,*

$$H_k(X \times Y; F) = \bigoplus_{i+j=k} H_i(X;F) \otimes H_j(Y;F).$$

[11] This is the tensor product in a categorically precise way, in the category of chain complexes, but this is a lot of things to show, defining Hom and several other things in this category.

The goal is eventually arrive at a proof of a more algebraic version, which has Proposition 14.1 as a corollary.

Proposition 14.2 (Künneth formula, algebraic version)**.** *Let K_* and L_* be chain complexes. Then, the following sequence is short exact.*

$$0 \longrightarrow H_n(K_*) \otimes H_n(L_*) \xrightarrow{\times} H_n(K_* \otimes L_*) \longrightarrow H_{n-1}(K_*) \otimes H_{n-1}(L_*) \longrightarrow 0$$

Definition. Let A_* and B_* be graded groups. Then, $f : A_* \to B_*$ has *degree d* if it takes $f(A_i) \subseteq B_{i+d}$ for all i.

If A_*, B_*, C_*, and D_* are graded groups, $f : A_* \to B_*$, and $g : C_* \to D_*$, then we can form the tensor product $f \otimes g : A_* \otimes C_* \to B_* \otimes D_*$ by

$$(f \otimes g)(a \otimes c) = (-1)^{\deg a \deg c} f(a) \otimes g(c).$$

In particular, we'll think about this for the chain complex

$$(C_*(X) \otimes C_*(Y))_n = \bigoplus_{i+j=n} C_i(X) \otimes C_j(Y),$$

with the boundary operartor $\partial_\otimes = \partial \otimes 1 + 1 \otimes \partial$. If a is a p-chain, then $\partial_\otimes(a \otimes b) = \partial a \otimes b + (-1)^p a \otimes \partial b$.

For example, if X and Y are points, then $(C_*(X) \otimes C_*(Y))_0 = C_0(X) \otimes C_0(Y) = \mathbb{Z}$.

Definition. Let $\epsilon : C_p(X) \to C_p(X)$ be a chain map. Then, a *chain contraction* is a map $D : C_p(X) \to C_{p+1}(X)$ which is a chain homotopy between ϵ and the identity, i.e. $\partial D + D\partial = 1 - \epsilon$.

Lemma 14.3 (Bredon, Lemma 1.1)**.** *If X and Y are contractible, then there is a chain contraction of any chain map on $C_*(X) \otimes C_*(Y)$. Consequently, $H_n(C_*(X) \otimes C_*(Y)) = 0$ for all $n > 0$, and $H_0 = \mathbb{Z}$, generated by $x_0 \otimes y_0$ for $n = 0$.*

Proof. Set $E = (D \otimes 1) \oplus (\epsilon \otimes D)$. The rest is some calculation and being careful with signs.

$$\begin{aligned} E\partial_\otimes + \partial_\otimes E &= (D \otimes 1 + \epsilon \otimes D)(\partial \otimes 1 + 1 \otimes \partial) + (\partial \otimes 1 + 1 \otimes \partial)(D \otimes 1 + \epsilon \otimes D) \\ &= D\partial \otimes 1 + D \otimes \partial - \epsilon\partial \otimes D + \epsilon \otimes D\partial + \partial D \otimes 1 + \partial\epsilon \otimes D - D \otimes \partial + \epsilon \otimes \partial D \\ &= D\partial \otimes 1 + \partial D \otimes 1 + \epsilon \otimes D\partial + \epsilon \otimes \partial D \\ &= (1 - \epsilon) \otimes 1 + \epsilon \otimes (1 - \epsilon) \\ &- 1 \otimes 1 + \epsilon \otimes \epsilon. \end{aligned}$$ ⊠

Theorem 14.4. *There exists a natural chain map $\theta : C_*(X \times Y) \to C_*(X) \otimes C_*(Y)$ whose degree-0 component is $(x, y) \mapsto x \otimes y$.*

The idea is that this is a reasonable thing to do in degree 0, and it extends well. This idea generalizes a bit; the point is to get maps in all degrees from maps in degree 0.

But first, some notation. Let $\sigma : \Delta^k \to X \times Y$ be a singular k-simplex on $X \times Y$, and let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ be the canonical projectons. Then, let $d_k : \Delta^k \to \Delta^k \times \Delta^k$ be the diagonal map. This means it's possible to write σ as a composition $\sigma = (\pi_X \circ \sigma \times \pi_Y \circ \sigma) \circ d_k$.

Then, define $\theta(\sigma) = \theta(\pi_X \circ \sigma \times \pi_Y \circ \sigma) \circ d_k$. This is reasonably intuitive: at every level, first take the diagonal map, and then use the projections to reduce to the smaller case.

Theorem 14.5 (Bredon, Thm. 1.3)**.** *Any two natural chain maps on $C_*(X \times Y)$ or $C_*(X) \times C_*(Y)$ (from one to the other, or from either one to itself) which are canonical isomorphisms in degree 0 when X and Y are points are naturally chain homotopic.*

Proof. We'll prove the case $\phi, \psi : C_p(X \times Y) \to C_p(X) \otimes C_p(Y)$; the other cases are similar.

Take D to be zero on 0-chains, and suppose inductively that D has been defined on chains of degree less than k, for some $k > 0$. Then, compute:

$$\begin{aligned} \partial_\otimes(\phi - \psi - D\partial)(d_k) &= \partial_\otimes\phi(d_k) - \partial_\otimes\psi(d_k) - \partial_\otimes D\partial d_k \\ &= \phi(\partial d_k) - \psi(\partial d_k) - ? \end{aligned}$$

We want $D\partial + \partial D = \phi - \psi$, so $? = (\phi - \psi - D\partial)(\partial d_k) = 0$.

Since $C_*(\Delta^k) \otimes C_*(\Delta^k)$ is acyclic by Lemma 14.3, there exists a $(k+1)$-chain whose boundary is $(\phi - \psi - D\partial)(d_k)$, so the above works out.

Let Dd_k be one such chain. Then, if σ is any singular simplex on $X \times Y$, then we have $\sigma = (\pi_X \circ \sigma \times \pi_Y \circ \sigma)(d_k)$, so define

$$D\sigma = ((\pi_X \circ \sigma) \otimes (\pi_Y \circ \sigma))(Dd_k),$$

and extend by linearity. Thus, $D\partial + \partial D = \phi - \psi$. ⊠

Corollary 14.6 (Eilenberg-Zilberg Theorem)**.** *The two maps $\theta : C_*(X \times Y) \to C_*(X) \otimes C_*(Y)$ and $\times : C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$ are homotopy equivalences, and are homotopy inverses of each other.*

Since this is true at the chain level, then it passes down and is also true for homology.

Corollary 14.7. $H_p(X \times Y; G) \cong H_p(C_*(X) \otimes C_*(Y); G)$.

This is a nice thing for concrete calculations: to understand the homology of the product of two spaces whose homologies are known, just take their tensor products. Bam!

Of course, there's also a dual version, though it says a little more.

Corollary 14.8. $H^p(X \times Y; G) \cong H^p(\mathrm{Hom}(C_*(X) \otimes C_*(Y); G))$.

The proof of Proposition 14.2 is involved and complicated, so we won't go into it. But it is useful to consider what happens when the coefficients aren't fields. Here, the Tor functor plays a role, since the tensor product isn't exact.

Theorem 14.9. *Let K_* and L_* be chain complexes. Then, the following sequence is short exact, and splits (albeit non-naturally).*

$$0 \longrightarrow H_n(K_*) \otimes H_n(L_*) \longrightarrow H_n(K_* \otimes L_*) \longrightarrow H_{n-1}(K_*) \otimes H_{n-1}(L_*) \longrightarrow 0$$

This is fine and algebraic, but the Tor functors pop up once you try to put in a topological space.

In general, for coefficients over a general ring R, there are torsion terms, but for fields, the result is Proposition 14.1, which is much nicer. But for general rings, coefficients in $\mathbb{Z}$ means lots of finite groups, and therefore there are lots of nontrivial Tor groups. In the *completely* general case, there are spectral sequences, but that's a story for a few years from now. . .

15. The Künneth Formula in Cohomology: 2/26/15

"This is true for all spaces, even the suspension of the Cantor set smash the irrational numbers."

Professor Galatius is back today.

Last time, we discussed that there are natural chain maps $C_*(X \times Y) \leftrightarrows C_*(X) \otimes C_*(Y)$, which, once the degree-0 part is specified, are unique up to chain homotopy; in particular, they will be inverses of each other. From the homework, two such maps included the Alexander-Whitney map AW in the forward direction and the cross product (or the *Eilenberg-Ziller product*) in the other direction. Thus, there's a chain homotopy commutativity between these two maps, where $T : x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$, where $|x|$ denotes the degree of x. This diagram does *not* commute (though it does in degree 0)! However, the two paths through the diagram are chain homotopic.

$$\begin{tikzcd} C_*(X \times Y) \arrow[r, "AW_{X\times Y}"] \arrow[d, "\text{flip}"] & C*(X) \otimes C_*(Y) \arrow[d, "T", "\wr"'] \\ C_*(Y \times X) \arrow[r, "AW_{Y\times X}"] & C_*(Y) \otimes C_*(X) \end{tikzcd} \tag{15.1}$$

However, AW is a natural transformation.

There are various things you can get out of this, though the map in the reverse direction is much nicer than the map in the forward direction.

Corollary 15.1. *The cup product is graded commutative.*

Proof. Applying the diagram (15.1) where $Y = X$, let d denote the diagonal map. Then, (15.1) boils down to the following diagram.

$$\begin{tikzcd} C_*(X) \arrow[r, "d_*"] \arrow[dr, "d_*"'] & C_*(X \times X) \arrow[r, "AW"] \arrow[d, "\text{flip}"] & C_*(X) \otimes C_*(X) \arrow[d, "T"] \\ & C_*(X \times X) \arrow[r, "AW"'] & C_*(X) \otimes C_*(X) \end{tikzcd}$$

The triangle on the left commutes, and the rest commutes up to chain homotopy. Recall that the top row sends

$$\sigma \longmapsto \sum_i \sigma|_{[v_0,\dots,v_i]} \otimes \sigma_{[v_i,\dots,v_n]}.$$

Then, apply $\mathrm{Hom}(-, R)$, where R is a ring, to obtain the following diagram, which still commutes up to chain homotopy.

$$\begin{tikzcd}
C^*(X;R) \otimes C^*(X;R) \arrow[r, "\smile"] \arrow[d, "T^*"'] & C^*(X;R) \\
C^*(X;R) \otimes C^*(X;R) \arrow[ur, "\smile"'] &
\end{tikzcd}$$

⊠

This is an example of the technique of *acyclic models*, which is a reasonably common proof technique in algebraic topology.

Fact. It is possible to prove that $\smile: C^*(X) \otimes C^*(X) \to C^*(X)$ is not chain homotopic to any strictly graded-commutative, natural map on the chain level. This leads to something calld *Steenrod operations*, which is a subject for a later class.

Strangely, the cross product $\times : C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$ is strictly commutative; it's natural, so for each pair of spaces (X, Y), there's a map $\mathrm{EZ}_{X \times Y}$, and the following diagram does commute.

$$\begin{tikzcd}
C_*(X) \otimes C_*(Y) \arrow[r, "\mathrm{EZ}_{X\times Y}"] \arrow[d, "T"'] & C_*(X \times Y) \arrow[d, "\text{flip}"] \\
C_*(Y) \otimes C_*(X) \arrow[r, "\mathrm{EZ}_{Y\times X}"] & C_*(Y \times X)
\end{tikzcd}$$

This wasn't on the homework, but it's not any more difficult. For now, it can be left as a curiosity.

Recall the Künneth formula for homology, which we didn't prove (the proof is in Hatcher, §3.B, and is about a page and a half). This states that if R is a field, then the following sequence is short exact.

$$0 \longrightarrow H_n(X;R) \otimes H_n(Y;R) \longrightarrow H_n(X \times Y;R) \longrightarrow H_{n-1}(X;R) \otimes H_{n-1}(Y;R) \longrightarrow 0$$

It turns out the only reason we need R to be a field is so that submodules of free modules are free, but this is true whenever R is a PID, so the formula holds in that case too. This holds for all spaces X and Y.

The cohomology version is a little different; it doesn't hold for all spaces.

Theorem 15.2 (Künneth theorem for cohomology)**.** *The map $H^*(X;R) \otimes H^*(Y;R) \to H^*(X \times Y;R)$ given by $\alpha \otimes \beta \mapsto (\pi_X^*\alpha) \smile (\pi_Y^*\beta)$ is an isomorphism if $H^k(X;R)$ is a finitely-generated, free R-module for all k.*

Once again, the proof is in Hatcher. The finitely generated hypothesis is necessary, because infinite-dimensional free modules don't behave as well with respect to dualizing.

Remark. This is the only theorem we've proven so far that says anything about the cup product being nontrivial!

Well, we also proved $1 \smile x = x$. But otherwise, this is the thing that rules out the cup product sending things to zero. For example, $H^1(S^1 \times S^1) = \mathbb{Z} \oplus \mathbb{Z}$, so one can think of the two copies of $\mathbb{Z}$ as pullbacks along the projections $\pi_{1,2}$ onto the two factors. Then, Theorem 15.2 implies that if $H^1(S^1) = \mathbb{Z}\alpha$, then $(\pi_1^*\alpha) \smile (\pi_2^*\alpha)$ is a generator of $H^2(S^1 \times S^1)$.

Ultimately, all Künneth theorems are theorems about abstract chain complexes.

Poincaré duality. The next theme of the class (and perhaps the last) is also a statement about the nontriviality of the cup product. This is the notion of Poincaré duality, that $\smile$ is nontrivial on the cohomology classes of manifolds.

Remark. If X is homotopy equivalent to a CW complex, then the kernel of the projection map $p : H^*(X;R) \to H^0(X;R)$ is a nilpotent ideal.

Thus, algebraic geometry may not be the best point of view on this ring, but that's not important right now. The proof isn't so hard.

The point is, it seems like a lot of time, products on the cohomology ring are zero, though not always.

Now, Poincaré duality is defined on manifolds. What is a manifold? It depends whom you ask; there are several flavors. In this class, we'll consider the most general definition; in other classes, one may require a smooth structure.

Definition. A *(topological) manifold* is a Hausdorff space M such that for all $p \in M$, there's a neighborhood U of M such that U is homeomorphic to $\mathbb{R}^n$. Then, the *dimension* of M, denoted $\dim(M)$, is said to be n.

Often, one says that M is locally homeomorphic to $\mathbb{R}^n$.

There are many examples: $\mathbb{R}^n$, S^n, $\mathbb{C}P^n$ (the n-cell is an open subset homeomorphic to $\mathbb{R}^{2n}$, and then instead of the first coordinate being nonzero, one can ask the i^{th} coordinate to be nonzero to get a covering set). A similar argument applies to $\mathbb{R}P^n$, so it's also a manifold, and there are many more examples.

One can make new examples out of old: if M and N are manifolds, and $U \subset M$ is open, then $M \times N$ and U are manifolds in their own right.

In any case, the study of manifolds has been a huge focus of algebraic topology, since the subject was founded a century ago, and particularly in the 1960s and 70s, which was a period of particularly fruitful research on high-dimensional manifolds. There was the so-called *classification problem*: given a fixed n, what is the set of manifolds M such that $\dim(M) = n$, up to homeomorphism? The goal is to describe the set in some way. Sometimes one required the manifolds to be compact or connected, but still, understanding this question, which is very difficult, has been very important.

Well, if $n = 0$, then it's not so difficult. $\mathbb{R}^0$ is a point, so a topological space is a 0-manifold iff it has the discrete topology. Well then. The only compact, connected 0-manifold is the point, up to homeomorphism.

For 1-manifolds, the answer isn't so bad, either: the only compact, connected 1-manifold is the circle (again, up to homeomorphism), which is nice. And for $n = 2$, the result is classical. A surface of genus g can be taken by taking g copies of $S^1 \times S^1$ and taking their *connected sum*, i.e. cutting out an open 2-disk on one copy and identifying it with an open 2-disk on the next, and so on. Then, this gives one family of compact, connected 2-manifolds, and another is given by doing the same thing to $\mathbb{R}P^2$; in particular, every compact, connected 2-manifold is homeomorphic to something on that list.

When $n > 2$, things get more complicated. There seems to be a global maximum in complication when $n = 4$, for when $n \geq 5$, the theory is somehow easier; there are quite strong results. In particular, if one classifies manifolds merely up to homotopy type, complete answers are known for $n \geq 5$, but not for $n = 3$ or $n = 4$.

Poincaré duality is about $H^*(M)$ when M is a compact, connected, "orientable" (which we'll define; intuitively, it means one can continuously choose a generator) manifold. Since M is locally homeomorphic to $\mathbb{R}^n$, connectedness is equivalent to path-connectedness, which is one less thing to worry about.

Note. The notation $H_n(M \mid x)$ means $H_n(M, M \setminus \{x\})$.

Suppose x is covered by a $U \approx \mathbb{R}$ within M; then, the inclusion map $U \hookrightarrow M$ induces a map in homology $H_k(M \mid x; R) \leftarrow H_k(U, U \setminus x; R)$; in particular, by the excision theorem, this is an isomorphism. Then, using excision, $H_k(U, U \setminus x; R) = R$ if $k = n$ and is 0 otherwise. Thus,

$$H_k(M \mid x; R) \cong \begin{cases} R, & k = n \\ 0, & k \neq n \end{cases}$$

However, this isomorphism is *not* canonical: it depends on the homeomorphism $U \approx \mathbb{R}^n$. This depends on orientation, so it is canonical up to sign.

Theorem 15.3. *Let M be a compact, connected, n-dimensional manifold and R be a commutative ring. Then,*

(1) $H_k(M; R) = 0$ when $k > n$.
(2) $H_n(M; R) \to H_n(M \mid x; R)$ is injective, and its image is either R or $\{r \in H_n(M \mid x) \mid 2r = 0\}$.

Definition. In part (2) of the above theorem, if the image is R, then M is called *R-orientable*.

Definition. A *fundamental class* of an R-orientable M is a class $[M] \in H_n(M; R)$ such that its image in $H_n(M \mid x; R)$ is a generator as an R-module (since it's abstractly isomorphic to R).

Such an $[M] = [c]$ (i.e. it's the class of a chain c) does something on cohomology; a cochain $[\varphi]$ can be evaluated on it, and $[\varphi] \mapsto [\varphi(c)]$ is a sort-of evaluation map $H^n(M; R) \to R$. This is a special case of something called the *cap product* $\frown$.

Combining this with the cup product $\smile$: $H^k(M; R) \otimes H^{n-k}(M) \to H^n(M; R)$, one gets a linear map $H^k(M; R) \otimes H^{n-k}(M) \to R$, i.e. a bilinear map $H^k(M; R) \times H^{n-k}(M) \to R$, and therefore a bilinear pairing q on $H^*(M) = \bigoplus_{k=0}^n H^k(M)$.

This is sort of like an inner product, though it's graded-symmetric, rather than symmetric. This leads to one version of Poincaré duality.

Theorem 15.4 (Poincaré duality)**.** *This pairing is nondegenerate. That is, if R is a field, q induces an isomorphism between $H^*(M; R) \xrightarrow{\cong} (H^*(M; R))^\vee$ (i.e. the dual vector space).*

On the other hand, if $R = \mathbb{Z}$, then q kills torsion. There isn't an isomorphism, even if one quotients out by torsion, though there is a sort of nondegeneracy condition.

Example 15.5. $M = \mathbb{C}P^n$ is $\mathbb{Z}$-orientable, and in fact, if M is simply connected, then it's R-orientable for all R.

Pick a generator $x \in H^2(\mathbb{C}P^n) \cong \mathbb{Z}$. Then, $x^{n+1} \in H^{2n+2}(\mathbb{C}P^n) = 0$, so it goes to 0 as well. In particular, there must be a unique ring map $f : \mathbb{Z}[x]/(x^{n+1}) \to H^*(\mathbb{C}P^n)$.

Theorem 15.6. *f is an isomorphism.*

Proof. This is really easy, thanks to Poincaré duality.

There's nothing to show for $n = 1$, so inductively assume it's true for $n - 1$. Restriction induces a map $H^*(\mathbb{C}P^n) \to H^*(\mathbb{C}P^{n-1})$, which we have already calculated is an (additive) isomorphism in degrees less than $2(n-2)$; however, in the highest-degree term, $H^n(\mathbb{C}P^n) = \mathbb{Z}$ and $H^n(\mathbb{C}P^{n-1}) = 0$. In particular, one can write down the following diagram; the horizontal maps are isomorphisms on degrees up to $2n - 2$, and f_{n-1} is an isomorphism by the inductive assumption, so f_n must also be an isomorphism on degrees up to $2n - 2$ (in both cases, "up to" is inclusive).

$$\begin{tikzcd} H^*(\mathbb{C}P^n) \arrow[r, "\mathrm{restr}_*"] & H^*(\mathbb{C}P^{n-1}) \\ \mathbb{Z}[x]/(x^{n+1}) \arrow[u, "f_n"] \arrow[r] & \mathbb{Z}[x]/(x^n) \arrow[u, "f_{n-1}", "\wr"'] \end{tikzcd}$$

Then, in degree $2n$, the left-hand side is $\mathbb{Z}x^n$, ad the right-hand side is $H^{2n}(\mathbb{C}P^n) = \mathbb{Z}$, so we're done once we show that $x^n \in H^n(\mathbb{C}P^n)$ is a generator. Poincaré duality on $\mathbb{Z}$ ensures that $H^2(\mathbb{C}P^n) \otimes H^{2n-2}(\mathbb{C}P^n) \to \mathbb{Z}$ is unimodular, i.e. it sends $x^{n-1} \in H^{2n-2}(\mathbb{C}P^n)$ to a generator of $H^2(\mathbb{C}P^n)^\vee = \mathrm{Hom}(H^2(\mathbb{C}P^n), \mathbb{Z})$; thus, $q(x^{n-1}, –) : H^2(\mathbb{C}P^n) \to \mathbb{Z}$ must send $x \mapsto \pm 1$ (since it needs to be both surjective and injective).

Since this pairing was really $q(x^{n-1}, x) = x^n \in H^{2n}(\mathbb{C}P^n)$ (since it came from the cup product), but this group is abstractly $\mathbb{Z}$, and $x^n \mapsto \pm 1$; in particular, x^n is a generator. ⊠

This is the sort of way Poincaré duality implies that the cup product is nontrivial on these spaces. (Here, x^n is the n-fold cup product.)

16. Poincaré Duality: 3/3/15

Last time, we stated the result of Poincaré duality, and talked about some applications, e.g. computing the ring structure of $H^*(\mathbb{C}P^n)$. We also briefly discussed the notion of orientation, which we'll expand on today.

Orientation. You may have heard this word before; let's discuss what it's trying to capture.

The basic example of a nonorientable manifold is the Möbius band, as in Figure 3; there's a sense in which it's one-sided, and there's no way to consistently pick a normal vector in a continuous direction on it.

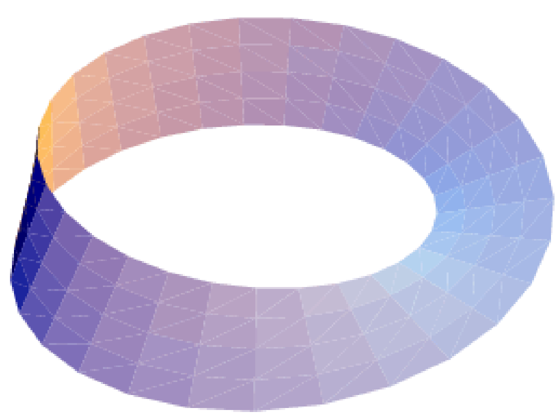

Figure 3. The Möbius band, a nonorientable surface (that is, a 2-dimensional manifold). Source: http://mathworld.wolfram.com/MoebiusStrip.html.

But this notion of orientability relies on how the manifold is embedded into $\mathbb{R}^3$, which is a little unfortunate. One could talk about a direction of rotating around a point, but this doesn't generalize to higher-dimensional manifolds.

Let M denote a Möbius band, as an abstract manifold; then, $H_2(M, M \setminus x) = H_2(M \mid x)$ is isomorphic to $\mathbb{Z}$, but not canonically; it is canonical up to sign. In other words, there is no "continuous" choice of generator. (Recall that $H_k(M \mid A; R)$ is defined to mean $H_k(M, M \setminus A; R)$.)

In general, when M is a manifold and U is a chart on M, i.e. $\mathbb{R}^n \approx U$ via a map sending an $x \in M$ to 0, let B be an open disc in U which is sent to $\mathring{D}^n$. Then, $f : M \setminus B \hookrightarrow M \setminus x$ is a homotopy equivalence, and induces back in ambient space a map $\mathbb{R}^n \setminus 0 \to \mathbb{R}^n \setminus D^n$, where $x \mapsto (x/|x|)f(|x|)$. In particular, this means $i_* : H_k(M \mid B; R) \xrightarrow{\cong} H_k(M \mid x; R)$, induced by inclusion, is a canonical isomorphism for all $x \in B$.

Definition. Let
$$M_R = \coprod_{x \in M} H_n(M \mid x; R).$$

Then, the map sending the factor of x in the disjoint union to x itself induces a map $\pi : M_R \twoheadrightarrow M$. Then, the map i_* above gives a bijection $B \times H_n(M \mid B; R) \to \pi^{-1}(B)$.

Now, for set-theoretic reasons, one can write $A \times B = \coprod_{x \in B} A$, so
$$B \times H_n(M \mid B; R) = \coprod_{x \in B} H_n(M \mid B; R) \xrightarrow{i_*} \prod_{x \in B} H_n(M \mid x; R) = \pi^{-1}(B).$$
But these things also have topologies. $B \subset M$ has a topology, under which it looks like $\mathring{D}^n \subset \mathbb{R}^n$, and $H_n(M \mid B; R)$ has the discrete topology. Then, topologize M_R such that all $\pi^{-1}(B) \subset M_R$ are open, and such that $B \times H_n(M \mid B; R) \to \pi^{-1}(B)$ is a homeomorphism.

This seems like some crazy point-set topology stuff, but apparently it's not hard to show that this is independent of the choice of chart. In particular, M_R is a topological manifold again! And $B \times \alpha$, which is homeomorphic to $\mathbb{R}^n$, is an open set.

For example, if $R = \mathbb{Z}$, then $H_2(M \mid x) \cong \mathbb{Z}$ for all x, but when M is the Möbius strip, there's no canonical identification between these for two different x. But within an open disc B, it is possible to identify them, which makes it possible to topologize that part of M_R.

Let $\widetilde{M} \subset M_{\mathbb{Z}}$ denote the generators; then, $\pi : M_{\mathbb{Z}} \to M$ restricts to a $\widetilde{\pi} : \widetilde{M} \to M$. Then, for all $x \in M$, $\widetilde{\pi}^{-1}(x)$ is a two-point set. This means it's a *double cover*.

Definition. If X is a topological space, then $\widetilde{X}$, with a map $\pi : \widetilde{X} \to X$ is a *double cover* if for every $x \in X$, there's a neighborhood $U \subset X$ containing x such that $\pi^{-1}(U) \approx U \times \{\bullet_1, \bullet_2\}$, which respects the projection map.

This is a special case of the more general notion of *covering spaces*.

Anyways, $\widetilde{M}$ is a double cover, so a set of choices for generators $\alpha(x) \in H_n(M \mid x; \mathbb{Z})$ is actually a section $\alpha : M \to \widetilde{M}$, i.e. $\widetilde{\pi} \circ \alpha = \mathrm{id}_M$. Now, we know what it means for a choice of generators to be continuous, since α can be specified to be continuous.

Definition.
- An *orientation* on a manifold M is a continuous section of $\widetilde{\pi} : \widetilde{M} \to M$.
- An *orientation along* an $A \subset M$ is a continuous section of $\widetilde{\pi}^{-1}(A) \to A$.
- An *R-orientation* is a continuous section α of $\pi_R : M_R \to M$ such that $\alpha(x) \in H_n(M \mid x; R) \approx R$ is a basis as an R-module.

The more general second and third definitions are useful on nonorientable (in the first sense) manifolds, because it may still be nice to have other ways to think about orientation.

Theorem 16.1. *Let M be a manifold and $A \subset M$ be a compact subset. Then, define $\Gamma_R(A)$ to be the set of continuous sections of $\pi_R^{-1}(A) \to A$. Then:*

(1) $H_k(M \mid A; R) = 0$ for $k > n$, and

(2) $\varphi_A : H_n(M \mid A; R) \xrightarrow{\cong} \Gamma_R(A)$, and this isomorphism is canonical.

We'll actually recycle the proof of this theorem when going back to Poincaré duality; the two are quite similar.

In (2), suppose $x \in A$ and $\alpha \in H_n(M \mid A; R)$. Then, $M \setminus A \hookrightarrow M \setminus x$ induces $H_n(M \mid A; R) \to H_n(M \mid x; R)$, sending $\alpha \mapsto \alpha(x)$. This defines a continuous section $A \mapsto \pi_R^{-1}(A) \subset M_R$.

Proof of Theorem 16.1. We'll prove this with a sort of mathematical induction known as *Mayer-Vietoris induction*. The theorem can be thought of as a function in A; the key idea will be to show that if the theorem is true for A, B, and $A \cap B$, then it will be true for $A \cup B$. Moreover, because of this induction, it's important to check both (1) and (2) at once.

In order to prove this, we'll also need the relative version of the Mayer-Vietoris sequence. We might not have stated it, but it should be no surprise given the absolute version and the five lemma. Specifically, let $U = M \setminus A$ and $V = M \setminus B$; then, there is a long exact sequence
$$\cdots \longrightarrow H_{n+1}(M, U \cup V; R) \longrightarrow H_n(M, U \cap V; R) \longrightarrow H_n(M, U; R) \oplus H_n(M, V; R) \longrightarrow H_n(M, U \cup V; R) \longrightarrow \cdots$$

The induction is actually quite nice: the inductive assumption of (1) means that $H_{n+1}(M \mid A \cup B; R) = H_{n+1}(M, U \cup V) = 0$. Using (2), which the inductive assumption shows is true for A, B, and $A \cap B$, part of the long exact sequence becomes a sequence of sections.

$$\begin{tikzcd} H_n(M, U \cap V; R) \arrow[r] \arrow[d, "\zeta"'] & H_n(M, U; R) \oplus H_n(M, V; R) \arrow[r] \arrow[d, "\varphi_A \oplus \varphi_B", "\wr"'] & H_n(M, U \cup V; R) \arrow[d, "\varphi_{A \cap B}", "\wr"'] \\ \Gamma_R(A \cup B) \arrow[r, "\psi_1"] & \Gamma_R(A) \oplus \Gamma_R(B) \arrow[r, "\psi_2"] & \Gamma_R(A \cap B) \end{tikzcd} \tag{16.1}$$

Then, φ_A is natural in that if $B \subset A$, the following diagram commutes.

$$\begin{tikzcd} H_n(M \mid A) \arrow[r, "i_*"] \arrow[d, "\varphi_A"'] & H_n(M \mid B) \arrow[d, "\varphi_B"'] \\ \Gamma_R(A) \arrow[r, "\text{restr.}"] & \Gamma_R(B) \end{tikzcd}$$

Here, i_* is the map induced by inclusion.

Now, in (16.1), the top row is already known to be exact, but the bottom row is too; specifically, let ψ_1 send a section σ to $(\rho(\sigma), -\rho(\sigma))$, where ρ is restriction, and let ψ_2 send $(\sigma, \tau) \to \rho(\sigma) - \rho(\tau)$. Then, this is exact, and ψ_1 is even injective (so the exactness extends one further back), because a pair of sections on A and B come from a section on $A \cup B$ iff they agree on the overlap $A \cap B$. This is exactly the statement that the set of sections is a *sheaf*. But then, by the exactness, ζ must be an isomorphism, so (2) follows. Then, (1) is the same, but with n replaced with k for $k > n$.

The rest of the proof is simpler. We need some sort of base case, so think about $M = \mathbb{R}^n$ and let $A \subset M$ be a compact, *convex* set.

Definition. If V is a real vector space, then $A \subset V$ is *convex* if for all $x, y \in A$, the straight line segment $[x, y]$ joining them is also contained in A.

Note that every convex set is either the empty set or contractible; it's good to remember the former option, too. In particular, it's necessary in order to state that the intersection of two convex sets is convex (which follows directly from the definition for convexity).

The theorem is clearly true for $A = \varnothing$, and if $x \in A$, then $M \setminus A \overset{\simeq}{\hookrightarrow} M \setminus x$, which follows from convexity (the homotopy inverse to inclusion is radial expansion away from x). Thus, by naturality, the following diagram commutes.

$$\begin{tikzcd} H_n(M \mid A) \arrow[r] \arrow[d, "\wr"] & \Gamma_R(A) \arrow[d, "f"] \\ H_n(M \mid x) \arrow[r, "\sim"] & \Gamma_R(\{x\}) \end{tikzcd}$$

But we know that sections over a point are just $H_n(M \mid x)$, so why is f an isomorphism? Since $M = \mathbb{R}^n$ and A is compact, then $A \subset D^n \subset \mathbb{R}^n$ for some disc D^n. We defined the topology on M_R so that $\pi_R^{-1}(D^n) \cong D^n \times H_n(M \mid D^n; R)$ (such that π_R commutes with projection to D^n, and therefore the sets of sections are the same). In particular, sections are exactly functions $A \to H_n(M \mid D^n; R)$. In particular, they're uniquely determined by a value at x, and every such value is realized.

That was the inductive beginning, with $M = \mathbb{R}^n$ and A convex. By induction, when $M = \mathbb{R}^n$, it's true for finite unions of convex sets. That is, if $A_1, \dots, A_k$ are all convex, then $A_1 \cup \dots \cup A_{k-1}$ is the union of $k-1$ convex sets, and A_k is convex, so the theorem is true for both of them, and it's also true for the intersection

$$A_k \cap \bigcup_{i=1}^{k-1} A_i = \bigcup_{i=1}^{k-1} A_k \cap A_i,$$

and this is a union of $k-1$ convex sets. This is why emphasizing the empty set was necessary. In general, in topology, one should not ignore the empty set; it's not homotopy equivalent to any other space.

Now that we've proved it for any finite union of convex sets, there's a sort of infinite induction. Specifically, if it's true for a chain $A_1 \supset A_2 \supset \cdots$, then it's true for $A = \bigcap A_i$. This follows because of direct limits, and was proven on the homework; that

$$H_k(M \mid A) = H_k\left(M, \bigcup_{i=1}^{\infty} M \setminus A_i\right) = \varinjlim H_k(M \mid A_i).$$

17. Sections and Mayer-Vietoris Induction: 3/5/15

Recall that last time, we let R be a commutative ring (with 1, as always) and M be an n-dimensional topological manifold. Then, we formed $M_R = \coprod_{x\in M} H_n(M \mid x; R)$; each index is a free R-module of rank 1, but is not naturally isomorphic to R. Then, $M_R \xrightarrow{\pi} M$, and M_R is locally homeomorphic to $B \times H_n(M \mid B; R)$, where $B \subset M$ is a ball in the chart.

In particular, we proved that if A is compact, then $H_k(M \mid A; R) = 0$ when $k > n$, and that $H_n(M \mid A; R) \xrightarrow{\cong} \Gamma_R(A)$, the set of *sections*, i.e. maps which make the following diagram commute.

$$\begin{tikzcd} & M_R \arrow[d, "\pi"] \\ A \arrow[ur, dashed, "s"] \arrow[r, hook] & M \end{tikzcd}$$

A section s can be thought of as akin to a function, but where the value $s(x)$ is in a group that depends on x.

Specifically, we managed to prove the theorem in the case where $A \subset M$ is a finite union of convex sets, but this isn't every compact set (e.g. the Cantor set cannot be written this way). Thus, we need another inductive step.

Claim. Suppose $M \supset A_1 \supset A_2 \supset A_3 \supset \cdots$ and A_1 is compact (so that all the others are too). Then, if the theorem is true for all of the A_i, then it is true for $A = \bigcap_{i=1}^\infty A_i$.

Proof. For simplicity, denote $H_k(-; R)$ as $H_k(-)$; the R is still implicitly there.

Then, $H_k(M \mid A_i) = H_k(M, M \setminus A_i)$, and $H_k(M \mid A) = H_k(M, M \setminus A)$, but $M \setminus A = \bigcup_{i=1}^\infty M \setminus A_i$.

On the homework, we proved that taking chain complexes commutes with direct limits; this was in the absolute case, but the relative case is exactly the same. In particular, the following diagram commutes.

$$\begin{tikzcd} C_k(M \setminus A) \arrow[r] & C_k(M) \arrow[d, equal] \\ \varinjlim_i C_k(M \setminus A_i) \arrow[u, "\cong"] \arrow[r] & C_k(M) \end{tikzcd}$$

That is, $\varinjlim H_k(M, M \setminus A) \xrightarrow{\cong} H_k(M, M \setminus A)$.

For the first part of the theorem, if $k > n$, then $H_k(M, M \setminus A) = \varinjlim 0 = 0$.

For the second, more difficult part of the theorem, we know the following diagram commutes.

$$\begin{tikzcd} H_n(M \mid A_1) \arrow[r, "\cong"] \arrow[d] & \Gamma_R(A_1) \arrow[d, "\text{restr.}"] \\ H_n(M \mid A_2) \arrow[r, "\cong"] \arrow[d] & \Gamma_R(A_2) \arrow[d, "\text{restr.}"] \\ \vdots & \vdots \end{tikzcd}$$

Then, the goal is whether this stack of isomorphisms passes to an isomorphism in the direct limit, comparing $H_n(M \mid A) = \varinjlim H_n(M \mid A_i)$ with $\Gamma_R(A)$. Thus, the question is whether $\varinjlim \Gamma_R(A_i) \cong \Gamma_R(A)$. Injectivity is similar to surjectivity, so we'll just show the latter.

Surjectivity would mean that any section over A extends to a section over some A_i, which looks like some kind of formality, but try drawing a picture where A is the Cantor set, and A_i is the i^{th} step in its construction — that this works is a bit unclear at first.

If Y has the discrete topology, then any continuous map $f : A \to Y$ extends to some A_i, since every point $y \in Y$ is clopen, so $f^{-1}(y)$ is open, i.e. is equal to $U_y \cap A$ for some open $U_y \subset M$. Thus, f extends to a continuous function on an open neighborhood $U = \bigcup_{y\in Y} U_y$, and therefore U must contain at least one of the A_i, since they're a descending chain of compact sets whose intersection is U: specifically, one can cover A by $f^{-1}(y)$ for all $y \in Y$, but since A is compact, one can take only finitely many of them, and then without loss of generality Y can be taken to be finite, so the extension does go through as follows.

Note that locally, sections are functions: if $B \subset M$ is a ball, then $\pi^{-1}(B) = B \times H_n(M \mid B) \cong B \times R$ (noncanonically), so sections over B are just continuous functions into R. Thus, locally, one can extend a section $A \to M_R$ for $A \hookrightarrow M$ to one on an open neighborhood of $A \subset M$; then, there are finitely many extensions, so they must agree on a small open neighborhood, an open set which contains some A_i. $\boxtimes$

Now, we can get back to the proof of the theorem by combining the inductive steps.

Continuation of Proof of Theorem 16.1. When $M = \mathbb{R}^n$, given any compact A, write it as $A = \bigcap A_i$, where $A_i \subset M$ is a compact union of finitely many convex sets.

Why is this possible? Distance on $\mathbb{R}^n$ induces a distance on the manifold, so one can write

$$A \subset \bigcup_{x \in A} B\left(x, \frac{1}{i}\right).$$

Thus, A_i is a finite union of $\overline{B(x, 1/i)}$ with $x \in A$, and their interiors cover A. Then, every point in A_i is distance at most $1/i$ from A, so the intersection of all of the A_i must be zero.

This was a takeaway from real analysis, that every compact set in $\mathbb{R}^n$ can be written as a countable intersection of finite unions of convex sets.

More generally, suppose M is a manifold and $A \subset M$. If $A \subset U \approx \mathbb{R}^n$, then it follows because the following diagram commutes.

$$\begin{tikzcd}
H_n(M \mid A) \arrow[r] & \Gamma_R(A) \arrow[d, equal] \\
H_n(U \mid A) \arrow[u, "\wr"'] \arrow[r, "\sim"] & \Gamma_R(A)
\end{tikzcd}$$

Here, the blue arrow is an isomorphism because of excision.

More generally, one may need to write $A = \bigcup_{i=1}^k A_i$, where each A_i is contained within a chart. Then, the finite induction outlined last class applies to show that it is true for A, since there are only finitely many. ⊠

Now that the theorem is done, we can discuss some consequences.

Intuitively, topological manifolds are locally very nice but globally very complicated. For example, there's a one-dimensional manifold called the *long line*,[12] which is a way of making sense of the notion of gluing uncountably many copies of the real line to each other in a way that is still somehow a manifold. This has quite terrible global properties, though each point has a neighborhood homeomorphic to $\mathbb{R}^n$.

Another example, which is in Bredon's book, is the *Plücker surface*, which is connected and path-connected, but has no dense countable set! This is a bit pathological, because one intuitively expects the countable dense set $\mathbb{Q}^n \subset \mathbb{R}^n$ to carry over into manifolds. In particular, this means the Plücker surface is not a countable union of charts.

Anyways, it's time to talk about $H_n(M \mid A) \to H_n(M \mid x)$ (where the latter is abstractly, though not canonically, isomorphic to R), where A is compact. This can be reformulated into the following diagram.

$$\begin{tikzcd}
H_n(M \mid A) \arrow[r, "f"] \arrow[d, equal] & H_n(M \mid x) \arrow[d, equal] \\
\Gamma_R(A) \arrow[r, "g"] & \Gamma_R(\{x\})
\end{tikzcd}$$

Note that if $s \in \Gamma_R(A)$, we can talk about $V_s = \{y \in M \mid s(y) = 0\}$, since near each y, s is a function; then, since these sections are continuous, then V_s is an open and closed subset of M, and being equal to 0 is also an open and closed condition. Thus, if A is connected, the only way to be 0 at x is to be 0 everywhere, and $\Gamma_R(A) \hookrightarrow \Gamma_R(X)$. (If A isn't connected, there's really not anything we could say here.) For example, if M is connected, which is the most important case, and if $A = M$, then $g : \Gamma_R(M) \hookrightarrow \Gamma_R(\{x\}) = H_n(M \mid x)$.

However, this map might not be surjective, and this has to do with whether M is orientable, as we discussed the other day. Specifically, recall that M is orientable if the covering map of $\widetilde{M} \subset M_{\mathbb{Z}}$ down to M has a section.

Proposition 17.1. *If M is orientable, so that such a section exists, then g is surjective as well, and more generally, if M is orientable along A (that is, there's a section on A), then g is still surjective.*

Proof. This is because in this case, one can choose an isomorphism $\Gamma_R(\{x\}) \cong R$, so a section α in $\Gamma_R(\{x\})$ can be realized as $\alpha \in R$. Thus, there's a map $\varphi_\alpha : \mathbb{Z} \to R$ sending $1 \mapsto \alpha$.

Thus, there's an induced map on homology, and it becomes a continuous map

$$\Phi_\alpha : \coprod H_n(M \mid x; \mathbb{Z}) \longrightarrow \coprod H_n(M \mid x; R),$$

[12]See http://en.wikipedia.org/wiki/Long_line_%28topology%29.

and the following diagram commutes.

$$\begin{tikzcd} M_{\mathbb{Z}} \arrow[rr, "\Phi_\alpha"] \arrow[dr, "\pi_{\mathbb{Z}}"'] & & M_R \arrow[dl, "\pi_R"] \\ & M & \end{tikzcd}$$

Over x, $H_n(M \mid x; \mathbb{Z}) \to H_n(M \mid x; R)$ is really $\varphi_\alpha : \mathbb{Z} \to R$.

Thus, if $\widetilde{M} \to M$ has a section over A, i.e. an $s : A \to \widetilde{M}$, then $\Phi_\alpha \circ s : A \to M_R$ is in $\Gamma_R(A)$, and has value α at x. Thus, g is surjective. ⊠

If you step back a little bit, this comes from the fact that if R is any abelian group and $a \in R$, then a comes from a homomorphism $\mathbb{Z} \to R$ (i.e. $1 \mapsto a$), which is a sort of universality of $\mathbb{Z}$: sections over $\mathbb{Z}$ become sections over any abelian group relatively easily.

Corollary 17.2. *If additionally M is connected and $A \subset M$ is compact, then the following diagram commutes and all arrows are isomorphisms.*

$$\begin{tikzcd} H_n(M \mid A) \arrow[r, "\cong"] \arrow[d, "\cong"] & \Gamma_R(A) \arrow[d, "\cong"] \\ H_n(M \mid x) \arrow[r, "\cong"] & \Gamma_R(\{x\}) \end{tikzcd} \tag{17.1}$$

In particular, $H_n(M \mid A) \cong H_n(M \mid x)$.

This just combines the two results that were just proven.

But if $A \subset M$ is compact and connected, but M is not orientable along A, (17.1) only looks like this, though it still commutes.

$$\begin{tikzcd} H_n(M \mid A) \arrow[r, "\cong"] \arrow[d, hook, "f"] & \Gamma_R(A) \arrow[d, hook] \\ H_n(M \mid x) \arrow[r, "\cong"] & \Gamma_R(\{x\}) \end{tikzcd} \tag{17.2}$$

Claim. $\mathrm{Im}(f) = \{\alpha \in H_n(M \mid x) \mid 2\alpha = 0\}$.

Proof. First, suppose $2\alpha = 0$ for some $\alpha \in H_n(M \mid x; R)$ (which is really R). By (17.2), what we really need to do is find a section, so we'll use the same trick as in the orientable case, but with $\mathbb{Z}/2$.

Choose a map $\varphi_\alpha : \mathbb{Z}/2 \to R$ sending $1 \mapsto \alpha$, which can be done because $2\alpha = 0$. Then, in the same way as the proof of Proposition 17.1, it induces a Φ_α such that the following diagram commutes.

$$\begin{tikzcd} M_{\mathbb{Z}/2} \arrow[rr, "\Phi_\alpha"] \arrow[dr, "\pi_{\mathbb{Z}/2}"'] & & M_R \arrow[dl, "\pi_R"] \\ & M & \end{tikzcd}$$

But this time, $H_n(M \mid x; \mathbb{Z}/2) \cong \mathbb{Z}/2$ canonically, because the nonzero elements have to be identified (more generally, if $R \cong \mathbb{Z}/2$ abstractly, then it is so canonically as well, since $0 \mapsto 0$ and there's only one other element). Thus, $M \times \{0,1\}$ really is $M_{\mathbb{Z}/2}$ and projects to M, so any section at $\{x\}$ extends to M by composing with Φ_α, as in Proposition 17.1, giving a section in $\mathrm{Im}(f)$.

Conversely, suppose $\alpha \in \mathrm{Im}(f)$, but $2\alpha \neq 0$. Then, $\alpha \neq -\alpha$, so look at the following diagram.

$$\begin{tikzcd} M_{\mathbb{Z}} \arrow[rr, "\Phi_\alpha"] \arrow[dr, "\pi_{\mathbb{Z}}"'] & & M_R \arrow[dl, "\pi_R"] \\ & M & \end{tikzcd}$$

Then, the composition $\widetilde{M} \hookrightarrow M_{\mathbb{Z}} \to M_R$ is injective on a fiber over x, so since A is connected, then $\widetilde{M} \to M_R$ is injective over A. Then, if $s : A \to M_R$ has $(x) = a$, then $s(A) \subset \widetilde{M}$ and this would give an orientation of A; oops. ⊠

This is more or less a complete calculation of H_n (and higher, which is zero) of a topological manifold relative to any compact set; manifolds are pretty nice.

We're getting to the point where we can actually prove Poincaré duality. If there were a single result this class was working towards, this would be it.

Recall that in the last two lectures, we have shown that if M is a manifold and A is compact, then $H_n(M \mid A) = 0$ when $k > n$, and $H_n(M \mid A; R) \xrightarrow{\cong} \Gamma_R(A)$. Moreover, if $x \in A$ for a path-connected $A \subseteq M$, then $H_n(M \mid A; R) \to H_n(M \mid x; R) \cong R$ is injective, and its image is $\{r \in R \mid 2r = 0\}$, the 2-torsion, or all of R when M is orientable along A.

If $M = A$ and it has a *triangulation*, which means a Δ-complex structure $\{e_\alpha : \Delta^{n_\alpha} \to M\}$,[13] this is all a lot easier to see: all $n_\alpha \leq \dim(M)$ (since there's no way to continuously inject $\mathbb{R}^n \hookrightarrow \mathbb{R}^m$ when $m < n$), so if $k > \dim(M)$, then $H_k(M) = H_k^\Delta(M) = 0$, as $\Delta_k(M) = 0$.

If M is orientable and compact, so one can take $M = A$, then $H_n(M; R) \cong R$. If there's a triangulation, a generator for $H_n(M; R)$ is the homology class of $\sigma = \sum \varepsilon_\alpha \sigma_\alpha$, where $\sigma_\alpha : \Delta^n \to M$ runs through all n-simplices in the triangulation (which is finite, since M is compact), and $\varepsilon_\alpha \in \{\pm 1\}$. Then, one can define whether each σ_α *preserves* or *reverses orientation*, and let $\varepsilon_\alpha = 1$ if it's orientation-preserving, and -1 if otherwise. We won't end up proving this.

Definition. If M is an n-dimensional manifold, $A \subseteq M$ is compact, and R is an abelian group, then an *R-fundamental class*, denoted $[M]_R$, is an element of $H_n(M \mid A; R)$ such that for all $x \in R$, its image (under the map induced by injection $M \setminus A \hookrightarrow M \setminus x$) is a generator for $H_n(M \mid x; R) \cong R$.

An R-fundamental class can only exist if M is orientable, or R is 2-torsion.

Another ingredient in the statement of Poincaré duality is the *evaluation map* $v : H_k(X; R) \otimes H^k(X; R) \to R$ sending $[c] \otimes [\psi] \mapsto \psi(c)$.

Let $\mathrm{TF}(A)$ denote A mod its torsion for any abelian group A, and recall the (re)statement of Poincaré duality: if M is compact and orientable, then the composition of the maps

$$\begin{tikzcd} \mathrm{TF}(H^k(M)) \otimes \mathrm{TF}(H^{n-k}(M)) \arrow[r, "\smile"] \arrow[dr] & H^n(M) \arrow[d, "v"] \\ & \mathbb{Z} \end{tikzcd}$$

is *non-degenerate*; that is there is an isomorphism $\mathrm{TF}(H^k(M)) \xrightarrow{\cong} \mathrm{Hom}(\mathrm{TF}(H^{n-k}(M)), \mathbb{Z})$.

We won't prove this, but instead reformulate it into something better and stronger. Already the definition of non-degeneracy is an improvement.

This will have some useful implications; $\mathrm{Hom}(H_{n-k}(X), \mathbb{Z})$ is necessarily torsion-free, and $\mathrm{Ext}(H_{n-k-1}(X), \mathbb{Z})$ must always be torsion, since the homology groups of X are finitely generated. Thus, in the universal coefficient theorem, the following sequence is exact.

$$0 \longrightarrow \mathrm{Ext}(H_{n-k-1}(X), \mathbb{Z}) \longrightarrow H^{n-k}(X) \longrightarrow \mathrm{Hom}(H_{n-k}(X), \mathbb{Z}) \longrightarrow 0$$

Thus, $\mathrm{TF}(H^{n-k}(X)) = \mathrm{Hom}(H_{n-k}(X), \mathbb{Z})$. This leads to another reformulation, this time encoding torsion information.

Proposition 18.1 (Poincaré duality, first reformulation)**.** *There is a natural isomorphism $H^k(M; R) \xrightarrow{\cong} H_{n-k}(M; R)$ for any abelian group R, as long as there is an R-fundamental class for M.*

In particular, either M is orientable, or R is 2-torsion.

In topology, one almost never just proves that two groups are isomorphic; it's important to know what the map is and why it's an isomorphism. In this case, there are two ingredients: an R-fundamental class $[M]_R \in H_n(M)$, and something called the cap product.

Definition. Let X be a topological space and R a ring, and let $H_k(X)$ denote $H_k(X; R)$, and similarly with cohomology. Then, the *cap product* $\frown$: $H_k(X) \otimes H^\ell(X) \to H_{k-\ell}(X)$, written $\alpha \otimes \psi \mapsto \alpha \frown \psi$, is defined as the induced map on the homology from a chain-level version $C_k(X) \otimes C^\ell(X) \to C_{k-\ell}(X)$.

Specifically, recall the Alexander-Whitney map $AW : C_*(X) \to C_*(X) \otimes C_*(X)$ and the evaluation map $v : C_k(X) \otimes C^k(X) \to R$ sending $\sigma \otimes \psi \mapsto \psi(\sigma)$. Evaluation can be thought of as a chain map, since cochains can be thought of as chains with negative degree: $C_*(X) \otimes C^{-*}(X) \to R$, where R is a chain complex as $\cdots \to 0 \to R \to 0 \to \cdots$.

[13]Sometimes, people define triangulation in a more restricted way; however, in all cases, this is part of the definition.

Then, since the tensor product of chain complexes is associative (which, technically, we should prove), then there are maps

$$C_*(X)\otimes_R C^*(X) \xrightarrow{AW\otimes \mathbb{1}} (C_*(X)\otimes_R C_*(X))\otimes_R C^*(X) = C_*(X)\otimes_R (C_*(X)\otimes_R C^*(X)) \xrightarrow{1\otimes v} C_*(X)\otimes_R R = C_*(X).$$

Since cohomology is regarded as a chain complex in negative degrees, this boils down to a map $C_k(X)\otimes C^\ell(X)\to C_{k-\ell}(X)$. This is the cap product, and it sends $\sigma\otimes\psi\mapsto \sigma\frown\psi$. Specifically,

$$\sigma\frown\psi = \underbrace{\psi\Big(\sigma|_{[v_0,\dots,v_\ell]}\Big)}_{\in R}\cdot \underbrace{\sigma|_{[v_\ell,\dots,v_k]}}_{\in C_{k-\ell}(X;R)}.$$

On the chain level, one can prove that

$$\partial(\sigma\frown\psi) = \pm(\partial\sigma)\frown\psi \pm \sigma\frown(\delta\psi),$$

and therefore this means the cap product is well-defined on homology, as a map $\frown\colon H_k(X)\otimes H^\ell(X)\to H_{k-\ell}(X)$.

This is a generalization of evaluating a cochain on a chain, e.g. if $k=\ell$ and X is path-connected, then $\sigma\frown\psi = \psi(\sigma)\cdot\sigma|_{[v_k]}$, but $\sigma|_{[v_k]}$ represents the canonical generator of $H_0(X;R)$, so $\frown\colon H_k(X;R)\otimes H^k(X;R)\to H_0(X;R)\cong R$ is evaluation.

Here are a few quick properties, which are true on the chain level and pass down to homology.

- $\sigma\frown(\phi\smile\psi) = (\sigma\frown\phi)\smile\psi$. Intuitively, this means that $\frown$ makes $C_*(X)$ into a $C^*(X)$-module, which is a good way to remember it.
- If $f: X\to Y$, then $f_*: C_*(X)\to C_*(Y)$ and $f^*: C^*(Y)\to C^*(X)$ denote the induced maps on chains and cochains, respectively. Then, $f_*(\sigma\frown f^*\psi) = (f_*\sigma)\frown\psi$ in $C_*(Y)$. In other words, $f_*: C_*(X)\to C_*(Y)$ is $C^*(Y)$-linear (that is, a map of $C^*(Y)$-modules), as $C^*(Y)$ is a module by the first property, and $C_*(X)$ is by $\frown$ and the ring map $f^*: C^*(Y)\to C^*(X)$.
- The cap product is homotopy-invariant.

There are several relative versions of the cap product. If $A\subset X$, then $C^*(X,A;R)$ is defined as the R-submodule of cochains that are 0 when evaluated on chains entirely within A. Then evaluation $v_A: C_*(X,A;R)\otimes C^*(X,A;R)\to R$ sending $\sigma\otimes\psi\mapsto\psi(\sigma)$ is still well-defined, and the Alexander-Whitney map goes through as usual, giving a map $AW_A: C_*(X,A)\to C_*(X;A)\otimes C_*(X;A)$. Thus, following the same procedure as above produces a map $C_k(X,A)\otimes C^\ell(X,A)\to C_{k-\ell}(X)$ (notice that you end up in absolute homology!), inducing $H_k(X,A)\otimes H^\ell(X,A)\xrightarrow{\frown} H_{k-\ell}(X)$.

Poincaré duality will be a kind of induction, and this will be the inductive beginning.

Theorem 18.2 (Poincaré duality, second reformulation)**.** *Let M be a compact manifold and R be an abelian group such that an R-fundamental class $[M]_R\in H_n(M;R)$ exists. Then, $[M]_R\frown - : H^k(M)\to H_{n-k}(M)$ is an isomorphism.*

We'll prove this in another kind of induction, on open sets. That means we'll have to relax the compactness hypothesis a little bit, but it also has a lot in common with the Mayer-Vietoris induction from a few lectures ago.

Example 18.3. This example will secretly be the base case for the induction.

Let $X=\Delta^n$ and $A=\partial\Delta^n$. This pair has only one interesting homology group, $H_n(\Delta^n,\partial\Delta^n)$, and one interesting cohomology group (also H^n). $[\iota]$, induced from inclusion, is a canonical generator for homology.

Just as in the absolute case, capping two things of the same degree is really just evaluation, so the map $H_n(\Delta^n,\partial\Delta^n;R)\otimes_R H^n(\Delta^n,\partial\Delta^n;R)\to H_0(\Delta^n)=R$ (since $H^n(\Delta^n,\partial\Delta^n;R)=\mathrm{Hom}(H_n(\Delta^n,\partial\Delta^n),R)$) is nondegenerate, and $[\iota]\frown - : H^n(\Delta^n,\partial\Delta^n)\to R$ is an isomorphism.

Corollary 18.4. *Using homotopy invariance, there is a similar result for $(X,A)=(\mathbb{R}^n,\mathbb{R}^n\setminus D^n)$.*

Now, there's an even better version of Theorem 18.2 that holds for noncompact manifolds.

Recall that an R-orientation of a manifold M is a continuous section $\alpha\in\Gamma_R(M)$. That is, its image in $\Gamma_R(x)\overset{\cong}{\leftarrow} H_n(M\mid x;R)\cong R$ (that is, it's free R-module of rank 1 noncanonically isomorphic to 1) is a generator for all $x\in M$. This can be used to create something akin to a fundamental class if M is R-orientable: if $A\subset M$ is compact then $\alpha_A\in\Gamma_R(A)\cong H_n(M\mid A;R)$ is sent to a generator under the map $H_n(M\mid A;R)\to H_n(M\mid x;R)$. If $A\subset B$, then this choice of α_A is compatible with the map $j_{AB}: H_n(M\mid B)\to H_n(M\mid A)$.

Given such an α_A, one obtains the map $\alpha_A \frown - : H^k(M \mid A) \to H_{n-k}(M)$ (recall that the relative cap product lands in absolute homology). Specifically, the following diagram commutes when $A \subset B \subset M$ and M is compact.

$$\begin{tikzcd} H^k(M \mid A) \arrow[r, "\alpha_A \frown -"] \arrow[d, "j_{AB}"'] & H_{n-k}(M) \\ H^k(M \mid B) \arrow[ur, "\alpha_B \frown -"'] & \end{tikzcd}$$

One might want to take the largest such compact B, but this isn't possible if M is noncompact. Thus, one can instead take the direct limit.

Definition. Let M be a manifold and $\mathcal{K} = \{K \subset M \mid K \text{ is compact}\}$, directed by inclusion. Then, the *compactly supported cohomology* of M is $H^k_C(M) = \varinjlim_{K \in \mathcal{K}} H^k(M \mid K)$.

If M is compact, then $H^k_C(M) = H^k(M)$, but the two can look very different when M is noncompact. Since compactness isn't homotopy invariant, then compactly supported cohomology isn't homotopy invariant either (see Example 18.6 for an example).

Hence, if M is R-orientable, an R-orientation $\alpha_M \in \Gamma_R(M)$ induces the map $D_M = (\alpha_A \frown -)_{A \subset M \text{ compact}}$, as $H^k_C(M) \to H_{n-k}(M)$. This allows us to relax the compactness assumption.

Theorem 18.5 (Poincaré duality, third reformulation)**.** *Let M be an n-dimensional manifold and R be a ring, such that M has an R-orientation $\alpha_M \in \Gamma_R(M)$. Then, $D_M : H^k_C(M) \to H_{n-k}(M)$ is an isomorphism.*

Example 18.6. $H^k_C(\mathbb{R}^n) = \varinjlim_{K \subset \mathbb{R}^n \text{ compact}} H^k(\mathbb{R}^n, \mathbb{R}^n \setminus K) \xleftarrow{\cong} \varinjlim_{N \in \mathbb{N}} H^n(\mathbb{R}^n, ND^n)$; this latter map is due to the cofinality of the one directed system in the other, and means that the compactly supported cohomology is R if $k = n$ and is 0 otherwise.

19. Proof of Poincaré Duality: 3/12/15

> *"And now, we should use the cap product, because we can't use formal nonsense for everything… afterwards, when we're building up larger things from smaller ones, we can use the Five Lemma uncountably many times or whatever."*

Recall that last time, we talked about the *cap product* $\frown : H_k(X) \otimes H^\ell(X) \to H_{k-\ell}(X)$, and that if R is a ring and M is an n-dimensional manifold, then let α_M be a fiberwise generator, i.e. under the map $\Gamma_R(M) \to \Gamma_R(x) = H_n(M \mid x) \approx R$ (which may not exist, and if it does, isn't necessarily unique). Then, this induced a map $D_M : H^k_C(M; R) \to H_{n-k}(M; R)$, where $H^k_C(-; R)$ is the *compactly supported cohomology*, i.e. the directed limit of $H^k(M \mid A)$ for $A \subseteq M$ compact. This works because specifying a map out of a direct limit is equivalent to specifying maps out of its constituent factors. Specifically, D_M is the induced map from $\alpha_A \frown -$ on each component factor. It depends on α, which usually isn't unique (e.g. given such an α, it can be multiplied by any unit of R, since all it has to do is be a generator).

The content of Theorem 18.5 is that D_M is an isomorphism. Then, this implies the other versions; for example, if M is compact, the compactly supported cohomology is the same as ordinary cohomology, which implies Theorem 18.2. Even if you only care about compact manifolds, which is reasonable, the inductive step in the proof, similar to the construction of the fundamental class, requires the proof to be true for open sets, and therefore the more general formulation is necessary.

Proof of Theorem 18.5. Once again, Mayer-Vietoris induction will play an important role: the goal will be to show that if the theorem holds for U, V, and $U \cap V$, then it holds for $U \cup V$. However, this time the induction will be on open subsets of M.

Remark.

(1) Any open $U \subset M$ is a manifold. This is because being a manifold is a local condition (Hausdorff and locally homeomorphic to $\mathbb{R}^n$).
(2) If $U \subset M$, then the restriction map $r_{M \to U} : \Gamma_R(M) \to \Gamma_R(U)$ sends α_M to an α_U (since restriction further to $\{x\}$ commutes with $r_{M \to U}$).

Remark. The functoriality of compactly supported cohomology is a little surprising. Ordinary cohomology is contravariant (arrow-reversing), but if $f : X \to Y$, then if $K \subset Y$ is compact, then $f^* : H^*(Y, Y \setminus K) \to H^*(X, X \setminus f^{-1}(K))$. If $f^{-1}(K)$ is compact, this maps naturally into $H^*_C(X)$ (since then $f^{-1}(K)$ is one of the members of the directed system).

Thus, if f has the property that f^{-1} of any compact set is compact, then the contravariant functoriality is preserved.

Definition. If $f : X \to Y$ is a continuous map of topological spaces, then f is said to be *proper* if for every compact $K \subseteq Y$, $f^{-1}(K)$ is compact in X.

So for a proper $f : X \to Y$, there's an induced $f^* : H_C^*(Y) \to H_C^*(X)$. In other words, there is a category of topological spaces and proper maps, and compactly supported cohomology is a functor on this category.

Furthermore, even though compactly supported cohomology isn't homotopy-invariant, there's a notion of *proper homotopy invariance* under which it is preserved.

Definition. A continuous $f : X \to Y$ is called an *open embedding* if it's a homeomorphism onto an open subset of Y.

Compactly supported cohomology is a *co*variant functor on open embeddings!

Specifically, if $K \subset X$ is compact, then $f(K) \subset Y$ is compact, so $f^* : H^k(Y, Y \setminus f(K)) \to H^k(X, X \setminus K)$ is an isomorphism, and the latter group maps into the direct limit $H_C^k(X)$. Taking the direct limit in Y gets only the compact sets that are $f(K)$ for a compact $K \subset X$, but this still maps into the overall direct limit. In other words, for any compact $C \subseteq X$, the following diagram commutes, where the vertical arrows are induced from the members of the directed set into the direct limit.

$$\begin{tikzcd}
H^k(Y, Y \setminus f(C)) \arrow[r, "\cong", "f^*"'] \arrow[d] & H^k(X, X \setminus C) \arrow[d] \\
\varinjlim_{K \subset f(X) \subset Y} H^k(Y, Y \setminus K) \arrow[r, "\cong", "\varphi"'] \arrow[d, "\psi"] & H_C^k(X) \\
H_C^k(Y) &
\end{tikzcd}$$

(All coefficients will be with respect to R, even if they're left out due to brevity or forgetfulness.)

Here, ψ is given by mapping the directed system of $K \subset f(X) \subset Y$ into the direct system of all compact $K \subset Y$, which induces the map ψ (which might not be an isomorphism) on the direct limits.

Now, the functoriality: since φ is an isomorphism, we can define $f_? = \psi \circ \varphi^{-1}$, which is a map $H_C^k(X) \to H_C^k(Y)$.

In particular, if $\iota : U \hookrightarrow M$ is inclusion, then the following diagram commutes.

$$\begin{tikzcd}
H_C^k(U; R) \arrow[r, "D_U"] \arrow[d, "\iota_?"] & H_{n-k}(U; R) \arrow[d, "\iota_*"] \\
H_C^k(M; R) \arrow[r, "D_M"] & H_{n-k}(M; R)
\end{tikzcd} \tag{19.1}$$

Step 1. The induction beginning will be when $M \approx \mathbb{R}^n$. Then, we can calculate the compactly supported cohomology (notice the difference with H^0):

$$H_C^k(\mathbb{R}^n; R) = \begin{cases} R, & k = n \\ 0, & k \neq n. \end{cases}$$

Thus, this is clearly abstractly isomorphic to $H_{n-k}(\mathbb{R}^n; R)$, but we want to show that the map given in (19.1) is an isomorphism.

The interesting example is $D_M : H_C^n(\mathbb{R}^n; R) \to H_0(\mathbb{R}^n; R) \cong R$. We know $H^n(\Delta^n, \partial\Delta^n) \to H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \to H_C^n(\mathbb{R}^n)$ is a chain of isomorphisms, and that the map $\iota \frown - : H^n(\Delta^n, \partial\Delta^n) \to H_0(\Delta^n)$ is an isomorphism, which we proved last time, and $H_0(\mathbb{R}^n) \cong H_0(\Delta^n)$, with an isomorphism we've seen before.

Thus, the following diagram commutes *up to multiplication by elements of $R^\times$*.

$$\begin{tikzcd}
H_C(\mathbb{R}^n) \arrow[r, "D_M"] & H_0(\mathbb{R}^n) \cong R \arrow[dd, "\cong"] \\
H^n(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \arrow[u, "\cong"] & \\
H^n(\Delta^n, \partial\Delta^n) \arrow[u, "\cong"] \arrow[r, "\cong"'] & H_0(\Delta^n)
\end{tikzcd}$$

Step 2. Now, we can induct: if $U, V \subset M$ are open, and the theorem is true for U, V, and $U \cap V$, then it will be true for $U \cup V$. This will use the Mayer-Vietoris sequence and the Five Lemma.

Stare at the following diagram.

$$\begin{tikzcd} H^k_C(U \cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & H^k_C(U) \oplus H^k_C(V) \arrow[r] \arrow[d, "D_U \oplus D_V"] & H^k_C(U \cup V) \arrow[d, "D_{U\cup V}"] \\ H_{n-k}(U \cap V) \arrow[r] & H_{n-k}(U) \oplus H_{n-k}(V) \arrow[r] & H_{n-k}(U \cup V) \end{tikzcd} \tag{19.2}$$

We haven't defined the Mayer-Vietoris sequence for compactly supported cohomology, but it can be built up from the relative cohomology of M relative to K for K compact; then, the direct limit of a bunch of exact sequences is still exact, so this is still valid (there's more to the proof about that, involving excision, but this is the general idea). It is also true that the maps given by these are exactly those induced by covariant functoriality, which requires a proof, but has the consequence that (19.2) commutes.

Now, to use the five lemma, we should think about a larger diagram, and the connecting homomorphism is our friend.

$$\begin{tikzcd} H^k_C(U \cap V) \arrow[r] \arrow[d, "D_{U\cap V}", "\cong"'] & H^k_C(U) \oplus H^k_C(V) \arrow[r] \arrow[d, "D_U \oplus D_V", "\cong"'] & H^k_C(U \cup V) \arrow[r] \arrow[d, "D_{U\cup V}"] & H^{k+1}_C(U \cap V) \arrow[r] \arrow[d, "D_{U\cap V}", "\cong"'] & H^{k+1}_C(U) \oplus H^{k+1}_C(V) \arrow[d, "D_U \oplus D_V", "\cong"'] \\ H_{n-k}(U \cap V) \arrow[r] & H_{n-k}(U) \oplus H_{n-k}(V) \arrow[r] & H_{n-k}(U \cup V) \arrow[r] & H_{n-k-1}(U \cap V) \arrow[r] & H_{n-k-1}(U) \oplus H_{n-k-1}(V) \end{tikzcd}$$

So the first, second, and fourth squares all commute, and we need the third one to. This is actually really messy, involving two pages of the book, going all the way back to the chain level, and using barycentric subdivision.

Anyways, it commutes, so we use the five lemma and the induction follows: the middle arrow is an isomorphism.

Step 3. Now, suppose it's true for $M = U_1 \cup \cdots \cup U_k \subset \mathbb{R}^n$, with each U_i a convex open subset of $\mathbb{R}^n$. Then, either $U_i \approx \emptyset$ or $U_i \approx \mathbb{R}^n$, so it's true for each U_i, and therefore the previous step means it's true for M.

Step 4. Finally, we use infinite induction, as in the construction of the fundamental class. Let $U_1 \subset U_2 \subset \cdots \subset M$, with each U_i open,[14] and suppose the theorem is true for all U_i. Then, we want it to be true for $U = \bigcup_i U_i$. It's a fact from point-set topology that a compact subset K of (the union of) a filtration of open sets $U_1 \subset U_2 \subset \cdots$ must be contained in some finite level: $K \subset U_N$ for some N. This ends up implying after another step or two that, since $H_*(U) = \varinjlim H_*(U_i)$, then $H^*_C(U) = \varinjlim H^*_C(U_i)$, using the covariant functoriality constructed before.

Step 5. Now, if $M \approx U \subset \mathbb{R}^n$ for any open U, then U can be written as a filtration $U_1 \subset U_2 \subset \cdots$, with $U = \bigcup_i U_i$, and each U_i a finite union of open balls. This is because U is the union of all balls in U with rational centers and rational radius (since $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$), so it's a countable union of these balls. Thus, each U_i can be taken to be the union of the first i balls.

If you believe that (which, well, it's true), then the previous two steps imply the theorem is true for U and therefore for M.

Step 6. Now, assume $M = \bigcup_{i=1}^\infty U_i$, where each $U_i \approx \mathbb{R}^n$ in M. This step will finish the proof for any manifold with a countable basis.[15]

By the previous case, it's true for U_1, U_2 and $U_1 \cap U_2$ (since this is homeomorphic $\mathbb{R}^n$ as well, or is empty), and therefore for $U_1 \cup U_2$. Now, it's true for $U_1 \cup U_2$, and U_3, and $(U_1 \cup U_2) \cap U_3$ (since it works for $U_1 \cap U_3$ and $U_2 \cap U_3$, so we use the inductive step once again), and therefore it's true for $U_1 \cup U_2 \cup U_3$, and so on.

Thus, it's true for M, and therefore for every manifold that one might encounter in practice.

Step 7. But we can generalize still further to manifolds that don't have countable charts! This relies on the version of Step 4 that holds for every direct system: then, $M = \bigcup_{i\in J} U_i$, where J the some (possibly much larger) directed system of subsets of M which can be covered by countably many charts, ordered by (what else?) inclusion. In particular, $M = \varinjlim_{i\in J} U_i$, and the theorem is true for each U_i. Specifically, let's look at the

[14] This was stated for $i \in \mathbb{Z}$, but it works for any partially ordered index set, countable or not!

[15] Some useful counterexamples include the long line, and any uncountable space with the discrete topology, which is a sort-of zero-dimensional manifold; alternatively, take the product with $\mathbb{R}$ to get a one-dimensional case.

following diagram, where the vertical arrows are the maps into the direct limits.

$$\begin{tikzcd} H_C^k(U_i) \arrow[r, "D_{U_i}", "\cong"'] \arrow[d] & H_{n-k}(U_i) \arrow[d] \\ H_C^k(M) \arrow[r] & H_{n-k}(M) \end{tikzcd}$$

This square commutes, and the direct limit of isomorphisms is an isomorphism. ⊠

Exercise 4. This proof requires M to be Hausdorff. Where is this used in the proof? Create a counterexample to Poincaré duality for the case where M isn't Hausdorff.

There are also other kinds of duality, e.g. for a compact manifold-with-boundary, e.g. cohomology can be given relative to its boundary, a notion called *Alexander duality* means that if one takes a contractible subset of $\mathbb{R}^n$ or S^n, the Poincaré duality of the complement of a compact, contractible subset can be rewritten a homology of the compact set; thus, there's an isomorphism between the cohomology of the complement and the homology of the object itself. This requires wading into point-set questions and local contractibility.

One example of Alexander duality we already saw was the Jordan curve theorem and its generalizations; since the homology only depends on the homology of the object itself, the embedding doesn't matter.