Algebraic Geometry

UT Austin, Spring 2016

M390C NOTES: ALGEBRAIC GEOMETRY

ARUN DEBRAY
APRIL 5, 2016

These notes were taken in UT Austin's Math 390c (Algebraic Geometry) class in Spring 2016, taught by David Ben-Zvi. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Thanks to Souparna Purohit for fixing a few mistakes.

Contents

Episode I.

The Course Awakens: 1/19/16

> *"There was a mistranslation in Grothendieck's quote, 'the rising sea:' he was actually talking about raising an X-wing fighter out of a swamp using the Force."*

There are a lot of things that go under the scheme of algebraic geometry, but in this class we're going to use the slogan "algebra = geometry;" we'll try to understand algebraic objects in terms of geometry and vice versa.

There are two main bridges between algebra and geometry: to a geometric object we can associate algebra via functions, and the reverse construction might be less familiar, the notion of a spectrum. This is very similar to the notion of the spectrum of an operator.

We will follow the textbook of Ravi Vakil, *The Rising Sea*. There's also a course website.[1] The prerequisites will include some commutative algebra, but not too much category theory; some people in the class might be bored. Though we're not going to assume much about algebraic sets, basic algebraic geometry, etc., it will be helpful to have seen it.

Let's start. Suppose X is a space; then, there's generally a notion of $\mathbb{C}$-valued functions on it, and this space might be $F(X)$. For example, if X is a smooth manifold, we have $C^\infty(X)$, and if X is a complex manifold, we have the holomorphic functions $\mathrm{Hol}(X)$.[2] Another category of good examples is *algebraic sets*, $X \subset \mathbb{C}^n$ that is given by the common zero set of a bunch of polynomials: $X = \{f_1(x) = \cdots = f_k(x) = 0\}$ for some $f_1, \dots, f_k \in \mathbb{C}[x_1, \dots, x_n]$. These have a natural notion of function, *polynomial functions*, which are polynomials in $\mathbb{C}[x_1, \dots, x_n]$ restricted to X, If $I(X)$ is the functions vanishing on X, then these functions are given by $\mathbb{C}[x_1, \dots, x_n]/I$.

The point is, on all of our spaces, the functions have a natural ring structure.[3] In fact, there's more: the constant functions are a map $\mathbb{C} \to F(X)$, and since $\mathbb{C}$ is a field, this map is injective. This means $F(X)$ is a $\mathbb{C}$-algebra, i.e. it is a $\mathbb{C}$-vector space with a commutative, $\mathbb{C}$-linear multiplication.

One of the things Grothendieck emphasized is that one should never look at a space (or an anything) on its own, but consider it along with maps between spaces. For example, given a map $\pi : X \to Y$ of spaces, we always have a *pullback* homomorphism $\pi^* : F(Y) \to F(X)$: if $f : Y \to \mathbb{C}$, then its pullback is $\pi^* y(x) = y(\pi(x))$. This tells us that we have a *functor* from spaces to commutative rings.

Categories and Functors. This is all done in Vakil's book, but in case you haven't encountered any categories in the streets, let's revisit them.

Definition. A *category* C consists of a set[4] of *objects* $\mathrm{Ob}\,\mathsf{C}$; if $X \in \mathrm{Ob}\,\mathsf{C}$, we just say $X \in \mathsf{C}$. We also have for every $X, Y \in \mathsf{C}$ the set $\mathrm{Hom}_{\mathsf{C}}(X, Y)$ of *morphisms*. For every $X, Y, Z \in \mathsf{C}$, there's a *composition map* $\mathrm{Hom}_{\mathsf{C}}(X, Y) \times \mathrm{Hom}_{\mathsf{C}}(Y, Z) \to \mathrm{Hom}_{\mathsf{C}}(Y, Z)$ and a unit $1_X \in \mathrm{Hom}_{\mathsf{C}}(X, X) = \mathrm{End}_{\mathsf{C}}(X)$ satisfying a bunch of axioms that make this behave like associative function composition.

To be precise, we want categories to behave like monoids, for which the product is associative and unital. In fact, a category with one object is a monoid. Thus, we want morphisms of categories to act like morphisms of monoids: they should send composition to composition.

Definition. A *functor* $F : \mathsf{C} \to \mathsf{D}$ is a function $F : \mathrm{Ob}\,\mathsf{C} \to \mathrm{Ob}\,\mathsf{D}$ with an induced map on the morphisms:

- If the map acts as $\mathrm{Hom}_{\mathsf{C}}(X, Y) \to \mathrm{Hom}_{\mathsf{D}}(F(X), F(Y))$, F is called a *covariant* functor.
- If it sends $\mathrm{Hom}_{\mathsf{C}}(X, Y) \to \mathrm{Hom}_{\mathsf{D}}(F(Y), F(X))$, then F is *contravariant*.

When we say "functor," we always mean a covariant functor, and here's the reason. Recall that for any monoid A there's the *opposite monoid* A^{op} which has the same set, but reversed multiplication: $f \cdot_{\mathrm{op}} g = g \cdot f$. Similarly, given a category C, there's an *opposite category* C^{op} with the same objects, but $\mathrm{Hom}_{\mathsf{C}^{\mathsf{op}}}(X, Y) = \mathrm{Hom}_{\mathsf{C}}(Y, X)$. Then, a contravariant functor $\mathsf{C} \to \mathsf{D}$ is really a covariant functor $\mathsf{C}^{\mathsf{op}} \to \mathsf{D}$. Hence, in this class, we'll just refer to functors, with opposite categories where needed.

Exercise 1.1. Show that a functor $\mathsf{C}^{\mathsf{op}} \to \mathsf{D}$ induces a functor $\mathsf{C} \to \mathsf{D}^{\mathsf{op}}$.

When presented a category, you should always ask what the morphisms are; on the other hand, if someone tells you "the category of smooth manifolds," they probably mean that the morphisms are smooth functions.

Now, we see that pullback is a functor $F : \mathsf{Spaces} \to \mathsf{Ring}^{\mathsf{op}}$. One of the major goals of this class is to define a category of spaces on which this functor is an equivalence. This might not make sense, *yet*. This is the seed of "algebra = geometry."

---

[1] https://www.ma.utexas.edu/users/benzvi/teaching/alggeom/syllabus.html.

[2] The best examples here are Riemann surfaces; when the professor imagines a "typical" or example algebraic variety, he sees a Riemann surface.

[3] In this class, all rings will be commutative and have a 1. Ring homomorphisms will send 1 to 1.

[4] This is wrong. But if you already know that, you know that worrying about set-theoretic difficulties is a major distraction here, and not necessary for what we're doing, so we're not going to worry about it.

Definition. Let $F, G : \mathsf{C} \rightrightarrows \mathsf{D}$ be functors. A *natural transformation* $\eta : F \Rightarrow G$ is a collection of maps: for every $X \in \mathsf{C}$, there's a map $\eta_X : F(X) \to G(X)$ satisfying a consistency condition: for every $f : X \to Y$ in C, there's a commutative diagram

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"] & G(Y) \end{tikzcd}$$

That is, a natural transformation relates the objects and the morphisms, and reflects the structure of the category.

Definition. A natural transformation η is a *natural isomorphism* if for every $X \in \mathsf{C}$, the induced $\eta_X \in \mathrm{Hom}_{\mathsf{D}}(F(X), G(X))$ is an isomorphism.

This is equivalent to having a natural inverse to η.

So one might ask, what is the notion for which two categories are "the same?" One might naïvely suggest two functors whose composition is the identity functor, but this is bad. The set of objects isn't very useful: it doesn't capture the structure of the category. In general, asking for equality of objects is worse than asking for isomorphism of objects. Here's the right notion of sameness.

Definition. Let C and D be categories. Then, a functor $F : \mathsf{C} \to \mathsf{D}$ is an *equivalence of categories* if there's a functor $G : \mathsf{D} \to \mathsf{C}$ such that there are natural isomorphisms $FG \to \mathrm{Id}_{\mathsf{D}}$ and $GF \to \mathrm{Id}_{\mathsf{C}}$.

This is a very useful notion, and as such it will be useful to see an equivalence that is not an isomorphism.

Exercise 1.2. Let k be a field, and let $\mathsf{D} = \mathsf{fdVect}_k$, the category of finite-dimensional vector spaces and linear maps, and let C be the category whose objects are $\mathbb{Z}_{\geq 0}$, the natural numbers, with an object denoted $\langle n \rangle$, and with $\mathrm{Hom}(\langle n \rangle, \langle m \rangle) = \mathrm{Mat}_{m \times n}$. This is a category with composition given by matrix multiplication.

Let $F : \mathsf{C} \to \mathsf{D}$ send $\langle n \rangle \mapsto k^n$, and with the standard realization of matrices as linear maps. Show that F is an equivalence of categories.

This category C has only some vector spaces, but for those spaces, it has all of the morphisms.

Definition. Let $F : \mathsf{C} \to \mathsf{D}$ be a functor.

- F is *faithful* if all of the maps $\mathrm{Hom}_{\mathsf{C}}(X, Y) \hookrightarrow \mathrm{Hom}_{\mathsf{D}}(F(X), F(Y))$ are injective.
- F is *fully faithful* if all of these maps are isomorphsism.
- F is *essentially surjective* if every $X \in \mathsf{D}$ is isomorphic to $F(Z)$ for some $Z \in \mathsf{C}$.

The following theorem will also be a useful tool.

Theorem 1.3. *A functor $F : \mathsf{C} \to \mathsf{D}$ is an equivalence iff it is fully faithful and essentially surjective.*

So, to restate, we want a category of spaces that is the opposite category to the category of rings; this is what Grothendieck had in mind. In fact, let's peek a few weeks ahead and make a curious definition:

Definition. The *category of affine schemes* is $\mathsf{Rings}^{\mathsf{op}}$.

Of course, we'll make these into actual geometric objects, but categorically, this is all that we need.

Recall that if $f : M \to N$ is a set-theoretic map of manifolds, then f is smooth iff its pullback sends C^∞ functions on N to C^∞ functions on M. The first step in this direction is the following theorem, sometimes called *Gelfand duality*.

Theorem 1.4 (Gelfand-Naimark)**.** *The functor $X \mapsto C^0(X)$ (the ring of continuous functions) defines an equivalence between the category of compact Hausdorff spaces and the (opposite) category of commutative C^*-algebras.*

This is an algebro-geometric result: it identifies a category of spaces with the opposite category of a category of algebraic objects.

However, we need to think harder than Gelfand duality in terms of compact, complex manifolds or in terms of algebraic spaces: for example, for $X = \mathbb{CP}^1$, $\mathrm{Hol}(X) = \mathbb{C}$: the only holomorphic functions are constant. The issue is that there are no partitions of unity in the holomorphic or algebraic world. This means we'll need to keep track of local data too, which will lead into the next few lectures' discussions on *sheaf theory*.

Returning to the example of algebraic sets, suppose X and Y are algebraic sets. What is the set of their morphisms? We decided the ring of functions was the polynomial functions $Y \to \mathbb{C}$, so we want maps $X \to Y$ to be those whose pullbacks send polynomial functions to polynomial functions. To be precise, the *ideal of* X is $I(X) = \{f \in \mathbb{C}[x_1, \dots, x_n] \mid f|_X = 0\}$, defining a map I from algebraic subsets of $\mathbb{C}^n$) to ideals in $\mathbb{C}[x_1, \dots, x_n]$. There's also a reverse map V,[5] sending an ideal I to $V(I) = \{x \in \mathbb{C}^n \mid f(x) = 0 \text{ for all } f \in I\}$. From classical commutative algebra, it's a fact that this is finitely generated, so it's the vanishing locus of a finite number of polynomials, and therefore in fact an algebraic set.

The dictionary between algebraic sets and ideals of $\mathbb{C}[x_1, \dots, x_n]$ is one of many versions of the Nullstellensatz (more or less German for the "zero locus theorem"): if J is an ideal, $I(V(J)) = \sqrt{J}$, its radical.

Definition. Let R be a ring and $J \subset R$ be an ideal. Then, the *radical* of J is $\sqrt{J} = \{r \in R \mid r^n \in J \text{ for some } n > 0\}$. One says that J is *radical* if $J = \sqrt{J}$.

What this says is that J is radical iff R/J has no nonzero nilpotents.[6] Why are these kinds of ideals relevant? If $X \subset \mathbb{C}^n$ and f vanishes on X, then so does f^n for all n. That is, radicals encode the geometric property of vanishing, which is why $I(X)$ is a radical ideal.

This is an outline of what classical algebraic geometry studies: it starts by defining algebraic subsets, and establishing a bijection between algebraic subsets of $\mathbb{C}^n$ and radical ideals of $\mathbb{C}[x_1, \dots, x_n]$. This isn't yet an equivalence of categories. Radical ideals correspond to finitely generated $\mathbb{C}$-algebras with no (nonzero) nilpotents: an ideal I corresponds to the $\mathbb{C}$-algebra $\mathbb{C}[x_1, \dots, x_n]/I$.

This is all what the course is *not* about; we're going to replace the category of finitely generated, nilpotent-free $\mathbb{C}$-algebras with the category of *all* rings, but we want to keep some of the same intuition. This involves generalizing in a few directions at once, but we'll try to write down a dictionary; the defining principle is to identify spaces X with rings $R = F(X)$, their ring of functions.

A point $x \in X$ is a map $i_x : x \to X$, so we get a pullback $i_x^* : F(X) \to \mathbb{C}$ given by evaluation at x. Let $\mathfrak{m}_x = \ker(i_x^*)$; since $\mathbb{C}$ is a field, this is a maximal ideal.[7] If k is a field and R is a k-algebra, then R/I is also a k-algebra, so in particular if I is maximal, then $k \hookrightarrow R/I$ is a map of fields, and therefore a field extension. Thus, if k is algebraically closed (e.g. we're studying $\mathbb{C}$) and R is a finitely generated k-algebra, then maximal ideals of R are in bijection with homomorphisms $R \to k$.

Thus, given a ring R, we'll associate a set $\mathrm{MSpec}(R)$, the set of maximal ideals of R, such that R should be its ring of functions. To do this, we'll say that an $r \in R$ is a "function" on $\mathrm{MSpec}(R)$ by acting on an $\mathfrak{m}_x \subset R$ as $r \bmod \mathfrak{m}_x$. This is a "number," since it's in a field, but the notion may be different at every point in $\mathrm{MSpec}(R)$! For example, if $R = \mathbb{Z}$, then $\mathrm{MSpec}(\mathbb{Z})$ is the set of primes, and $n \in \mathbb{Z}$ is a function which at 2 is $n \bmod 2$, at 3 is $n \bmod 3$, and so on.

A perhaps nicer example is when $R = \mathbb{R}[x]$, which has maximal ideals $(x - t)$ for all $t \in \mathbb{R}$. Here, evaluation sends $f(x) \mapsto f(x) \bmod (x - t) = f(t)$. That is, this is really evaluation, and here the quotient field is $\mathbb{R}$. So these look like good old real-valued functions, but these aren't all the maximal ideals: $(x^2 + 1)$ is also a maximal ideal, and $\mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C}$. Then, we do get a kind of evaluation again, but we have to identify points and their complex conjugates.

So we'll have to find a good notion of geometry which generalizes from $\mathbb{C}$-algebras to k-algebras for any field k, to any commutative rings. We'll also have to think about nilpotents: we threw them away by thinking about zero sets, but they play a huge role in ring theory.

Episode II.

Attack of the Cones: 1/21/16

"To this end, we're going to give a crash course in category theory over the next few lectures; the door is over there."

[5] V stands for "vanishing," "variety," or maybe "vendetta."

[6] Recall that if R is a ring, an $r \in R$ is *nilpotent* if $r^n = 0$ for some n.

[7] Recall that an ideal $I \subset R$ is *maximal* iff R/I is a field. This is about the level of commutative algebra that we'll be assuming.

Remember that our general agenda is to match algebra and geometry; one way to express this idea is to take the category of rings and identify it with some category of geometric objects. However, we're going to reverse the arrows, and we'll get the category of affine schemes. These are some geometric spaces, with a contravariant functor from affine schemes to rings given by taking the ring of functions and a functor in the opposite direction called Spec.

One potential issue is that spaces may not have enough functions, e.g. $\mathbb{CP}^1$ as a complex manifold only has constant functions; as such, we'll enlarge our category to a whole category of schemes, which will also have an algebraic interpretation. Another weird aspect is that functions may take values in varying fields.

Schemes generalize geometry in three different directions: gluing spaces together to ensure we have enough functions is topology, like making manifolds; functions having varying codomains is useful for arithmetic and number theory; and allowing for rings with nilpotents feels a little like analysis.

Last time, we defined $\operatorname{MSpec}(R)$ for a ring R, the set of maximal ideals. It turns out that topology is not sufficient to understand these spaces; for example, the class of *local rings* are those with only one maximal ideal. There are many such rings, e.g. $\mathbb{C}[x]/(x^n)$, whose maximal ideal is (x). In short, MSpec doesn't see nilpotents.

To any ring R, one can attach the category Mod_R, whose objects are R-modules and morphisms are R-linear maps (those commuting with the action of R). This category is one of the more important things one studies in algebra, and we also want to express them in terms of geometric objects that are related somehow to $\operatorname{Spec} R$. This should also help us understand the algebraic properties of R-modules too.

Crash Course in Categories. There's a lot of categorical notions in algebraic geometry; it does strike one as a painful way to start a course, but hopefully we can get it out of our systems and move on to geometry knowing what we need. This corresponds to chapters 1 and 2 in the book.

We've seen several examples of categories: sets, groups, rings, etc. The next example is a useful class of categories.

Definition. A *poset* is a set S and a relation $\leq$ on S that is

- *reflexive*, so $x \leq x$ for all $x \in \mathsf{S}$;
- *transitive*, so if $x \leq y$ and $y \leq z$, then $x \leq z$; and
- *antisymmetric*, so if $x \leq y$ and $y \leq x$, then $x = y$.

S has the structure of a category: the objects are the elements of S, and $\operatorname{Hom}(x, y)$ is $\{\mathrm{pt}\}$ if $x \leq y$ and is empty otherwise.

Transitivity means that we have composition, and reflexivity gives us identity maps.

This is an unusual example compared to things like "the category of all (somethings)," but is quite useful: a functor from the poset $\bullet \to \bullet$ to another category C is a choice of $A, B \in \mathsf{C}$ and a map $A \to B$; a functor from the poset $\mathbb{N}$ is the same as an infinite sequence in C, and a commutative diagram is the same as a functor out of the category

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

into C.

Example 2.1. A particularly important example of this: if X is a topological space, then its open subsets form a poset under inclusion. Hence, they form a category, called $\mathsf{Top}(X)$. This category is important for sheaf theory, which we will say more about later. For example, if A is an abelian group and $U \subset X$ is open, then let $\mathscr{O}_A(U)$ denote the abelian group of A-valued functions on U (for example, A might be $\mathbb{C}$, so $\mathscr{O}_A(U) = C^\infty(U)$). If $V \subset U$, then restriction of functions defines a map $\operatorname{res}_U^V : \mathscr{O}_A(U) \to \mathscr{O}_A(V)$. Since restriction obeys composition, then we've defined a functor $\mathscr{O}_A : \mathsf{Top}(X)^{\mathsf{op}} \to \mathsf{Ab}$ (or perhaps to $\mathbb{C}$-algebras, or another category); this is a *presheaf of abelian groups* (or $\mathbb{C}$-algebras, etc.).

To be precise, a *presheaf* on X is a functor out of $\mathsf{Top}(X)^{\mathsf{op}}$. This is a way of organizing functions in a way that captures restriction; it will be very useful throughout this class.

Returning to category theory, one of its greatest uses is to capture structure through universal properties, rather than using explicit details of a given category. We'll give a few universal properties here.

Definition. Let C be a category.

- A *final* (or *terminal*) object in C is a $* \in \mathsf{C}$ such that for all $X \in \mathsf{C}$, there's a unique map $X \to *$.
- An *initial* object is a $* \in \mathsf{C}$ such that for all $X \in \mathsf{C}$, there's a unique map $* \to X$.

This is not the last time we'll have dual constructions produced by reversing the arrows.

Example 2.2. If C is a poset, then a terminal object is exactly a maximum element, and an initial object is a minimum element. Thus, in particular, they do not necessarily exist.

Nonetheless, if a final (or initial) object exists, it's necessarily unique.

Proposition 2.3. *Let $*$ and $*'$ be terminal objects in C; then, there's a unique isomorphism $*$ to $*'$.*

Proof. There's a unique map $* \to *$, which therefore must be the identity, and there are unique maps $* \to *'$ and $*' \to *$, so composing these, we must get the identity, so such an isomorphism exists, and it must be unique, since there's only one map $* \to *'$. ⊠

By reversing the arrows, the same thing is true for initial objects. Thus, if such an object exists, it's unique, so one often hears "the" initial or final object. These will be useful for constructing other universal properties.

Example 2.4.

(1) In the category of sets, or in the category of topological spaces, the final object is a single point: everything maps to the point. The initial object is the empty set, since there's a unique (empty) map to any set or space.
(2) In Ab or Vect_k (abelian groups and vector spaces, respectively), 0 is both initial and terminal: the unique map is the zero map. An object that is initial and final is called a *zero object*; as in the case of sets, it may not exist.
(3) In the category of rings, 0 is terminal, but not initial (since a map out of 0 must send $0 = 1$ to 0 and 1). $\mathbb{Z}$ is initial, with the unique map determined by $1 \mapsto 1$.[8]
(4) Even though we don't really understand what an affine scheme is yet, we know that $\operatorname{Spec} \mathbb{Z}$ has to be a terminal object, and $\operatorname{Spec} 0$ has to be the initial object. Since we want this to be geometric, then $\operatorname{Spec} \mathbb{Z}$ will play the role of a point. It might not look like a point, but categorically it behaves like one.
(5) The category of fields is also interesting: setting $1 = 0$ isn't allowed, so there are neither initial nor terminal objects! If we specialize to fields of a given characteristic, then we get a unique map out of $\mathbb{Q}$ or $\mathbb{F}_p$, so the category of fields of a given characteristic is initial.
(6) The poset $\mathsf{Top}(X)$ has $\varnothing$ initial and X terminal: it has top and bottom objects.

The fact that initial and terminal objects are unique means that if you characterize an object in terms of initial or terminal objects, then you know they're unique as soon as they exist.

Definition. If R is a ring, we have the category Alg_R of R-algebras (rings T with the extra structure of a map $R \to T$; morphisms must commute with this map). This is an example of something more general, called an *undercategory*: if C is a category and $X \in \mathsf{C}$, then the undercategory $X \downarrow \mathsf{C}$ is the category whose objects are data of $Y \in \mathsf{C}$ with C-morphisms $a_Y : X \to Y$ and whose morphisms are commutative diagrams

$$\begin{tikzcd} Y_1 \arrow[rr, "\varphi"] & & Y_2 \\ & X. \arrow[ul, "a_{Y_1}"] \arrow[ur, "a_{Y_2}"'] & \end{tikzcd}$$

In the same way, the *overcategory* $X \uparrow \mathsf{C}$ is the same idea, but with maps to X rather than from X (e.g. spaces over a given space X).

Thus, it's possible to concisely define $\mathsf{Alg}_R = R \downarrow \mathsf{Ring}$. We will see other examples of this.

[8] That rings and ring homomorphisms are unital is important for this to be true.

Example 2.5 (Localization)**.** Let R be a ring and $S \subset R$ be a multiplicative subset. Then, the *localization at S* is $S^{-1}R = \{r/s \mid r \in R, s \in S : r/s = r/s' \text{ when } s''(rs' - r's) = 0 \text{ for some } s'' \in S\}$. This is a construction we'll use a lot, so it will be useful to have a canonical characterization of them.

Now, let C be the category of R-algebras T with maps ($\varphi_T : R \to T$ such that (and this is a property, not structure) $\varphi_T(s)$ is invertible in T for all $s \in S$.

Exercise 2.6. Show that $S^{-1}R$ is the initial object in C.

The naïve idea that localization is "fractions in S" is true if R is an integral domain, but if we have zero divisors, the R-algebra structure map $R \to S^{-1}R$ need not be injective. But the point is that if T is an R-algebra where the elements of S become invertible, the map φ_T factors through $S^{-1}R$; this means that $S^{-1}R$ is the element of C that's "closest to R." However, you still have to concretely build it to show that it exists; however, we know already that it's determined up to unique isomorphism, so we say "the" localization.

Another very fundamental language for making constructions is that of limits and colimits. It may seem a little strange, but it's quite important.

Definition. Let I be a *small category* (so its objects form a set); in the context of limits, we will refer to it as an *index category*. Then, a functor $A : I \to \mathsf{C}$ is called a *I-shaped* (or *I-indexed*) *diagram in* C.

That is, if $m : i \to j$ is a morphism in I, then this diagram contains an arrow $A(m) : A_i \to A_j$.

Definition. Let A be an I-shaped diagram in C. Then, a *cone* on A is the data of an object $B \in \mathsf{C}$ and maps $A_i \to B$ for every $i \in I$ commuting with the morphisms in I. The cones on A form a category Cones_A,

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d] \arrow[drr, bend right=15] & A_3 \arrow[dr] & \\
A_2 \arrow[ur] \arrow[rr] & & B
\end{tikzcd}$$

FIGURE 1. A cone on a diagram A.

where the morphisms are maps $B \to B'$ commuting with all the maps in the cone.

We can also take the category of "co-cones," which are data of maps from B *into* the diagram. This is not quite the opposite category (since we want maps $B \to B'$ commuting with the maps into the diagram).[9]

Definition.

- The *colimit* $\varinjlim_I A$ is the initial object in the category of cones of A.
- The *limit* $\varprojlim_I A$ is the terminal object in the category of co-cones of A.

As before, colimits and limits may or may not exist, but if they do, they're unique up to unique isomorphism.

Colimits act like a quotient, and it's easier to map out of them. Correspondingly, limits behave like a subobject, and it's easier to map into them.

Example 2.7 (Products and Coproducts)**.** Let $I = \bullet\ \bullet$ be a two-element discrete set (no non-identity arrows). Thus, an I-shaped diagram is just a choice of two spaces A_1 and A_2, so a colimit C_1 is the data of a unique map φ_B for each $B \in \mathsf{C}$ fitting into the following diagram.

$$\begin{tikzcd}
A_1 \arrow[dr] \arrow[drr, bend left] & & \\
& C_1 \arrow[r, "\varphi_B"] & B \\
A_2 \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}$$

[9]Some people switch the definitions of cones and co-cones, but since we're not going to use these words very much, it doesn't matter all that much.

This is called the *coproduct* of A_1 and A_2, denoted $A_1 \sqcup A_2$ or $A_1 \amalg A_2$.

Similarly, the limit of A is called the *product* of A_1 and A_2, is denoted $A_1 \times A_2$, and fits into the diagram

$$\begin{tikzcd} & & A_1 \\ B \arrow[urr] \arrow[r] \arrow[drr] & A_1 \times A_2 \arrow[ur] \arrow[dr] & \\ & & A_2 \end{tikzcd}$$

In the same way, if I is a larger discrete set, we get coproducts and products of objects in C indexed by I, denoted $\coprod_I A_i$ and $\prod_I A_i$, respectively.

In the category of sets, the product is Cartesian product, and the coproduct is disjoint union. The same is true in topological spaces.

In the category of groups, the product is once again Cartesian product, but the coproduct is the free product (mapping out of it is the same as mapping out of the individual components, which is not true of the direct product). As underlying sets, this is distinct from the coproduct of sets.

In linear categories, e.g. Ab, Mod_R, or Vect_k, $V \oplus W$ is the product and coproduct, and the same is true over all finite I. However, this is *not* true when I is infinite: the coproduct is the direct sum, which takes finite sums of elements, and the product is the Cartesian product, which takes arbitrary sums of elements. It's worth working out why this is, and how it works.

Many of these categories are "sets with structure," e.g. groups, vector spaces, topological spaces, and so on. In these cases, there is a *forgetful functor* which forgets this structure: indeed, a group homomorphisms (continuous map, linear map) is a map of sets too.[10]

There's a useful principle here: *forgetful functors preserve limits*: if F is a forgetful functor, then there is a canonical isomorphism $F(\varprojlim A) \cong \varprojlim F(A)$. This is something that can be defined more rigorously and proven. But one important corollary is that if you know what the limit looks like for sets, it's the same in groups, rings, vector spaces, topological spaces, and so on. However, this is very false for coproducts, e.g. the coproduct on groups is not the same as the one on sets.

This becomes a little cooler once we see limits that aren't just products.

Example 2.8. Consider the diagram of rings

$$\cdots \longrightarrow \mathbb{Z}/p^n \longrightarrow \cdots \longrightarrow \mathbb{Z}/p^3 \longrightarrow \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p,$$

where each map is given by modding out by p. One can show that the limit exists, and it'll be the same as the limit of the underlying sets, a sequence of compatible elements; this limit is called the *p-adic integers*, denoted $\mathbb{Z}_p$. More generally, the same thing works for $\varprojlim R/I^n$ for an ideal $I \subset R$, and defines the *I-adic completion* $\widehat{R}_I$, which we'll revisit, since it has useful geometric meaning.

Episode III.

The Yoneda Chronicles: 1/26/16

"There's probably lots of notations [for this], so let me choose a bad one."

Last time, we were talking about universal propeties, which tend to correspond to terminal or initial objects. This tends to characterize an object up to unique isomorphism, so there's in a sense only one solution.

There is *not* only one object. There might be a billion! Or infinitely many. But any two are uniquely isomorphic: if we take $\mathsf{C}^{\mathsf{initial}}$, the subcategory of initial objects, it's equivalent to $*$, the category with one object and the identity morphism.[11] And we never look at categories up to isomorphism, only equivalence, so this is a better viewpoint.

We also started talking about limits and colimits last time; these are very important examples of universal properties. These are initial (resp. terminal) objects in the category of cones (resp. co-cones) of I-shaped diagrams, which are functors $I \to \mathsf{C}$. In other words, a colimit of a diagram is mapped to by every object in a diagram in a way compatible with the diagram maps, and such that any other mapped-to object factors

[10] If this seems vague, that's all right; it's possible to define and find forgetful functors more formally.

[11] The equivalence is given by any inclusion functor $* \to \mathsf{C}^{\mathsf{initial}}$, and in the other direction by projecting down onto the point.

through the limit; a limit maps to the diagram and factors through any other such map. Since these are initial or terminal objects, they are unique up to unique isomorphism, so one hears "the" (co)limit. Limits are analogous to subobjects, and colimits are more like quotients; as such, colimits tend to be more poorly behaved.

We also defined products and coproducts, which are limits and colimits, respectively, over a discrete set (made into a category by adding only the identity maps). For example, in the category of modules over a ring, the coproduct is direct sum, and the product is the Cartesian product; the difference between these is only felt at the infinite level, and the direct sum is more subtle. In the category of groups, the product of groups is the Cartesian product again (a group structure on the product as a set); on the other hand, the coproduct is *not* the coproduct of sets (disjoint union): it's the free product of groups, because maps out of G and H correspond to maps out of $G * H$. And this is different than the coproduct of abelian groups: it's direct sum (since abelian groups are $\mathbb{Z}$-modules). The patterns are: coproducts and products are quite different in general, and products are easier to understand than coproducts.

Example 3.1 (Fiber products and coproducts)**.** Let

$$I = \begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet. \end{tikzcd}$$

Limits across I are called *fiber products*, and are terminal of objects fitting into the diagram

$$\begin{tikzcd} \varprojlim A_i \arrow[r] \arrow[d] & A_1 \arrow[d, "f"] \\ A_2 \arrow[r, "g"] & A_3. \end{tikzcd} \tag{3.2}$$

The fiber product is denoted $A_1 \times_{A_3} A_2$. In Set, these exist,[12] and for (3.2), is given by $A_2 \times_{A_3} A_2 = \{a_1, a_2 \mid f(a_1) = g(a_2)\}$.

The colimit of I is A_3, since everything maps through A_3. This can be made more general; if a poset P has a maximal element m, then $\varinjlim_P A_i = A_m$, and an analogous statement holds for minimal elements and limits. In fact, a cocone on a diagram is the addition of a maximal object; a colimit is trying to be the maximum of your diagram (which might not exist, but often does), and a limit is trying to be the minimum of your diagram.

The proper way to dualize this is to take colimits across

$$I = \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet. \end{tikzcd}$$

In this case, the colimit is called the *pushout*, and fits into the following diagram.

$$\begin{tikzcd} A_1 \arrow[r, "f"] \arrow[d, "g"'] & A_2 \arrow[d] \\ A_3 \arrow[r] & \varinjlim_I A_i. \end{tikzcd}$$

This is denoted $A_2 \amalg_{A_1} A_3$; in the category of sets, this is $A_2 \amalg A_3 / \sim$, where $a_2 \sim a_3$ if there's an $a_1 \in A_1$ such that $f(a_1) = a_2$ and $g(a_1) = a_3$. Equivalence relations are a little harder to understand. And this isn't the pushout in other categories: in groups, the pushout is $G *_K H$, called the *free product with amalgamation*.

Example 3.3 (Kernels and cokernels)**.** Suppose C is a category with a zero object 0 (so 0 is both initial and terminal). For any $A, B \in \mathsf{C}$, there's a unique map $0 : A \to B$ called the *zero map*, since there's a unique map $A \to 0$ and a unique map $0 \to B$, so composing them gives us the zero map.

[12] In fact, all limits exist in the category of sets. There are some set-theoretic issues involved in the proof, but we're not going to worry about that.

Given any other $\varphi : A \to B$, we want to compare it with 0, so we're taking the (co)limit of the diagram $A \underset{\varphi}{\overset{f}{\rightrightarrows}} B$. The limit is called the *kernel*, denoted $\ker\varphi$, and the colimit is called the *cokernel*, denoted $\operatorname{coker}\varphi$.[13] Another way to think of (co)kernels is as fiber products and pushouts: they fit into the diagrams

$$\begin{tikzcd} \ker\varphi \arrow[r] \arrow[d] & A \arrow[d, "\varphi"] \\ 0 \arrow[r] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d, "\varphi"] & 0 \arrow[d] \\ B \arrow[r] & \operatorname{coker}\varphi, \end{tikzcd}$$

and this may make their non-categorical constructions more clear.

These examples are useful in algebra, but now we also know that they're unique up to unique isomorphism, which can be quite useful. It's incredible how often these come up in algebra. It's also worth remembering that (co)limits tend to play well with (co)limits, in a way that can be made precise, but provides some useful intuition about what might be true.

Example 3.4 (Completion)**.** We were also going to do algebraic geometry at some point, and one interesting algebraic construction that has a geometric analogue is *completion*: if R is a ring and $I \subset R$ is an ideal, then the completion of R at I, denoted $\widehat{R}_I$, is the limit of the diagram

$$\cdots \longrightarrow R/I^3 \longrightarrow R/I^2 \longrightarrow R/I.$$

When $R = \mathbb{Z}$ and $I = (p)$, this is the ring of *p-adic integers*, denoted $\mathbb{Z}_p$ or $\widehat{\mathbb{Z}}_{(p)}$.

In the category of sets, one can explicitly write down what the limit is, as a subset of the product:

$$\varprojlim_I A = \left\{ (a_i)_{i\in I} \in \prod_{i\in I} A_i \mid A(m)(a_i) = a_j \text{ for all } m : i \to j \in I \right\}.$$

This requires proof, but is on the homework.

Colimits in general are harder, but some colimits are easy, such as an increasing union of sets $U_1 \subset U_2 \subset U_3 \subset \cdots$. In this case, the colimit (in the category of sets) is just the union of all of these. A good example of these (albeit in a different category) is localization. $p^{-\infty}\mathbb{Z}$ is the localization $S^{-1}\mathbb{Z}$, where $S = \langle p\rangle = \{1, p, p^2, \ldots\}$. Since we know this sits inside $\mathbb{Q}$, this is an increasing union of sets

$$\mathbb{Z} \cup p^{-1}\mathbb{Z} \cup p^{-2}\mathbb{Z} \cup \cdots.$$

This means we can write it as a colimit:

$$p^{-\infty}\mathbb{Z} = \varinjlim\Big(\mathbb{Z} \longrightarrow p^{-1}\mathbb{Z} \longrightarrow p^{-2}\mathbb{Z} \longrightarrow \cdots\Big). \tag{3.5}$$

This colimit takes place in the category Ab of abelian groups, also known as the category of $\mathbb{Z}$-modules. However, as $\mathbb{Z}$-modules, $p^{-1}\mathbb{Z} \cong \mathbb{Z}$, where $1/p \mapsto 1$. In other words, (3.5) is isomorphic to the diagram

$$p^{-\infty}\mathbb{Z} = \varinjlim\left(\mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \xrightarrow{\cdot p} \cdots\right),$$

and this makes sense in more generality, in particular when we don't have something like $\mathbb{Q}$ as a reference. In particular, for any ring R and $r \in R$, we can take the limit as R-modules

$$r^{-\infty}R = \varinjlim\Big(R \xrightarrow{r} R \xrightarrow{r} R \xrightarrow{r} \cdots\Big).$$

If R is a domain, then this sits inside its field of fractions, but otherwise we don't have a reference point. And we can start the construction with an arbitrary R-module M, defining $r^{-\infty}M$ as

$$r^{-\infty}M = \varinjlim\Big(M \xrightarrow{r} M \xrightarrow{r} M \xrightarrow{r} \cdots\Big).$$

[13] This is an example of a more general construction, where one considers the diagram $f, g : A \rightrightarrows B$ for more general f and g; the limit is called the *equalizer*, and the colimit is called the *coequalizer*.

In algebraic topology, there's a notion of a spectrum, which is an infinite sequence of topological spaces. People say this is a lot of machinery with the nebulous goal of inverting the suspension functor, but this is a very similar idea: we want to invert r as many times as we can, so we have to string it out as an infinite sequence. Though this construction may look big, it has a simple purpose, which is useful to keep in mind. Localization is also given by a colimit, which we'll see in the exercise. It was already given by a universal property, but this nicer kind of universal property gives us some more information.

All of these "nice" colimits are, more precisely, examples of a notion called filtered colimits. These are the analogues to Cauchy sequences: we know the limit exists if we have this condition, and it gives us nicer comparisons of elements later on in the sequence.

Definition.

- A poset S is *filtered* if for all $x, y \in S$, there's a $z \in S$ majorizing x and y, i.e. $z \geq x$ and $z \geq y$.
- A (small nonempty) category C is *filtered* if for all $x, y \in \mathsf{C}$, there's a $z \in \mathsf{C}$ and maps $x \to z$ and $y \to z$ and any two maps $f, g : x \rightrightarrows y$ have a coequalizer $h : y \to z$ (i.e. $f \circ h = g \circ h$).[14]

A finite filtered poset necessarily has a maximum, so this only becomes interesting in the infinite case.

The upshot is: filtered colimits exist, and they tend to have nice properties. For example, localizations are filtered, and (3.5) can be seen to match the definition explicitly, and increasing unions are filtered. Moreover, forgetful functors preserve filtered colimits. However, nontrivial finite colimits (such as pushouts) will not be filtered.

In the category of sets, one can give a construction for filtered colimits: if I is a filtered category,

$$\varinjlim_I A_i = \coprod_I A_i / \sim,$$

where $a \sim b$ if they're eventually equivalent, i.e. if $a \in A_1$ and $b \in A_2$, then there's an A_3 in the diagram and maps $A_1 \to A_3$ and $A_2 \to A_3$ that map a and b to the same element.

In algebra, there are lots of statements like "localizations of direct sums are direct sums of localizations." This is true because both are colimits, and colimits play well with other colimits (though this does depend on the precise formulation of that principle). Similarly, completions of products are products of completions, because limits play well with limits. However, completions of direct sums might not do what you expect, nor localizations and products.

Yoneda's lemma.

"The Force is everywhere; it surrounds us and binds us." – Yoda

This is a slightly more mystical part of the class: we want to describe things not as they are, but as they are detected by things around them.

In fact, we get a surprising and powerful analogy from analysis: a category is much like an inner product space, where the objects of C are vectors, and the inner product is $A, B \mapsto \operatorname{Hom}(A, B)$. However, unlike inner products, this is not symmetric! This can be strange. The Yoneda lemma says, in this sense, that this pairing is nondegenerate: we can understand a "vector" completely by pairing it with other "vectors."

If C and D are categories, we can define the *functor category* $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$, whose objects are (covariant) functors $\mathsf{C} \to \mathsf{D}$, and whose morphisms are natural transformations.

To a vector space V, we define the dual space $V^* = \operatorname{Hom}(V, k)$; the inner product structure defines a map $V \to V^*$, which is an isomorphism when the inner product is nondegenerate. This nondegeneracy is somewhat weak, and in fact feels more like the sense of distributions: if $V = C_c^\infty(\mathbb{R})$, its dual space $V^* = \operatorname{Dist}(\mathbb{R})$, the linear functionals on compactly supported, smooth functions. They're not isomorphic, but there is an embedding: any compactly supported smooth function defines a distribution.[15] Distributions are nice, because they're closed under lots of operations, so you can take your PDE or whatever and solve it in the distributional sense, and then try to get a regularity result showing it was in the original space the whole time.

In category theory, we're going to do something similar. For any $X \in \mathsf{C}$, let $h_X = \operatorname{Hom}_{\mathsf{C}}(-, X)$ and $h^X : \operatorname{Hom}_{\mathsf{C}}(X, -)$. These are functors $\mathsf{C}^{\mathsf{op}} \to \mathsf{Set}$ and $\mathsf{C} \to \mathsf{Set}$, respectively, e.g. $h_X : Y \mapsto \operatorname{Hom}_{\mathsf{C}}(Y, X)$. This

[14] Another way to think of this is the following: a poset is filtered if every finite subset has a maximum, and a category is filtered if every finite subcategory has a cone (a maximal element). Then, this guarantees nice things about infinite cones.

[15] If you haven't seen distributions, this is not really necessary to understand Yoneda's lemma.

is functorial because a map $Y \to Z$ induces a map $h_X(Z) \to h_X(Y)$ by pullback, which is contravariant, and composition is covariantly functorial for h_X. These are called the *functors (co)represented by X*.

Additionally, if $f : X \to X'$, then any map $Y \to X$ induces a map $Y \to X'$ by precomposing with f. In other words, h_X is functorial in X! This defines a functor $h_{-} : \mathsf{C} \to \mathsf{Fun}(\mathsf{C}^{\mathsf{op}}, \mathsf{Set})$ sending $X \mapsto (Y \mapsto \mathrm{Hom}_{\mathsf{C}}(Y, X))$. This is weird and strange, but it's exactly like the embedding of a vector space into its dual.

We'll let $\widehat{\mathsf{C}} = \mathsf{Fun}(\mathsf{C}, \mathsf{Set})$.

Lemma 3.6 (Yoneda). *$h : \mathsf{C} \hookrightarrow \widehat{\mathsf{C}}$ is a full embedding.*

That is, for any $X, X' \in \mathsf{C}$, $\mathrm{Hom}_{\widehat{\mathsf{C}}}(h_X, h_{X'}) = \mathrm{Hom}_{\mathsf{C}}(X, X')$: we don't lose any information passing to $\widehat{\mathsf{C}}$. Or in other words, if you know all maps into X, then you know X.

For example, suppose we have a map of functors $\varphi : h_X \to h_{X'}$ in $\widehat{\mathsf{C}}$. This is a natural transformation, so for any Y, there's a map $h_X(Y) \to h_{X'}(Y)$ in a natural way. To prove the lemma, we want to construct a map $\psi : X \to X'$ which induces φ. So, how do we get such an element $\psi \in \mathrm{Hom}(X, X')$?

The only map we always have an any category is the identity, so let's look at id_X. The natural transformation φ induces a map $h_X(X) \to h_X(X')$, i.e. $\mathrm{Hom}(X, X) \to \mathrm{Hom}(X, X')$, so let $\psi = \varphi(\mathrm{id}_X)$. This assignment is a map $\mathrm{Hom}_{\widehat{\mathsf{C}}}(h_X, h_{X'}) \to \mathrm{Hom}_{\mathsf{C}}(X, X')$, and you can check this is the inverse to the map in the other direction. All this is doing is a little tautological, and as such it takes some time to sink in.

Episode IV.

The Yoneda Chronicles, II: 1/28/16

"I just like this stuff, sorry."

Last time, we were talking about the Yoneda embedding; it's kind of strange, and you have to stare at it for a bit to get it. The analogy is that if V is an inner product space, the map $v \mapsto \langle v, - \rangle$ defines an embedding $V \hookrightarrow V^*$ if the inner product is positive definite, so that $\langle v, - \rangle$ is nonzero (because $v \cdot v \neq 0$). The Yoneda embedding is sort of the same thing, but for a category C and its dual $\widehat{\mathsf{C}} = \mathsf{Fun}(\mathsf{C}^{\mathsf{op}}, \mathsf{Set})$. There's a contravariant functor $\mathsf{C} \to \widehat{\mathsf{C}}$ sending $X \mapsto h_X = \mathrm{Hom}(-, X)$, and the Yoneda lemma is that this is an embedding, or more precisely, a fully faithful functor: $\mathrm{Hom}_{\mathsf{C}}(X, Y) = \mathrm{Hom}_{\widehat{\mathsf{C}}}(h_X, h_Y)$. If you think of these as inner products, this is a "partial isometry:" there's an isometry onto the image.

The analogue of positive definiteness is that $\mathrm{id}_X \in \mathrm{Hom}_{\mathsf{C}}(X, X)$, so it must be nonempty. Then, we can transfer it around, enabling us to construct a map $X \to Y$ given a natural transformation $\varphi : h_X \to h_Y$, just by applying φ to id_X. Then, you can check that this is inverse to the map $\mathrm{Hom}_{\mathsf{C}}(X, Y) \to \mathrm{Hom}_{\widehat{\mathsf{C}}}(h_X, h_Y)$.

From the vector-spatial view, it's perhaps less surprising that you can understand the objects in a category in terms of the maps into them, but it's an extremely useful viewpoint: there are lots of operations you can perform in $\widehat{\mathsf{C}}$ (analogous to all the cool things you can do with distributions): for example, $\widehat{\mathsf{C}}$ has all limits and colimits. Then, you can try to understand how a construction in $\widehat{\mathsf{C}}$ relates to C, which is made much nicer since C sits inside of $\widehat{\mathsf{C}}$.

One example is, if X is a topological space, there's *functor of points* $h_X : \mathsf{Top}^{\mathsf{op}} \to \mathsf{Set}$ sending $Y \mapsto \mathrm{Hom}(Y, X)$. This captures a lot of the information of X: for example, the underlying set of X is captured by $\mathrm{Hom}(*, X)$; paths are given by $\mathrm{Hom}(\mathbb{R}, X)$, and so on. In this setting, the Yoneda embedding tells us something that feels a little tautological: if you know all of the maps into X, you know X. This is not minimal by any means (and in practice, you end up using a less absurd amount of data), but it's a nice perspective, courtesy of abstract nonsense. Using it, we can translate questions about a category C into questions about the category of sets.

Given a functor $\mathsf{Top}^{\mathsf{op}} \to \mathsf{Set}$, one might wonder whether it's h_X for some X. This is the question of *representability*, one of the fundamental things in Grothendieck's worldview (a space is really a collection of maps into it), and we'll develop some ways to approach this question.

For example, what does it mean for a map $f : X \to Y$ to be injective in C? There's an abstract categorical definition.

Definition. Let $f : X \to Y$ in C. Then, f is a *monomorphism* if whenever $g_1, g_2 : Z \rightrightarrows X$ and $f \circ g_1 = f \circ g_2$, then $g_1 = g_2$. A monomorphism is often denoted $f : X \hookrightarrow Y$.

The idea is: we care about X as the maps into it, so if a map out of X preserves all the information about maps into X, then it's analogous to injective.

Definition. Dually, an *epimorphism* $f : X \to Y$ in C, written $f : X \twoheadrightarrow Y$, is a map such that whenever $g_1, g_2 : Y \rightrightarrows Z$ and $g_1 \circ f = g_2 \circ g$, then $g_1 = g_2$.

The Yoneda embedding shows up as follows.

Lemma 4.1. *$f : X \to Y$ is a monomorphism iff $h_f : h_X \to h_Y$ is pointwise injective, i.e. for every $Z \in \mathsf{C}$, $h_X(Z) \hookrightarrow h_Y(Z)$. Similarly, f is an epimorphism iff $h_f : h_X \to h_Y$ is pointwise surjective.*

So we can take this strange notion of monomorphism (or epimorphism) and translate it into something nice. In functional analysis, functions have nice linear properties induced pointwise from $\mathbb{R}$, and similarly, here, morphisms can make use of the nice structure of Set.

For example, all limits and colimits exist in $\mathsf{Fun}(\mathsf{C}^{\mathsf{op}}, \mathsf{Set})$, like in the category of sets. What does this mean? (Yes, it's pretty crazily abstract.) A diagram $I \to \widehat{\mathsf{C}}$ is a diagram of functors with natural transformations between them. Then, we can define a new functor "$\varinjlim F_i$" sending an $X \in \mathsf{C}$ to $\varinjlim_I F_i(X)$ (which exists, because this limit is in Set). You should check that this is well-defined as a functor, and has the right universal property for the colimit, so we can remove the quotation marks; it's really the colimit. The point is: colimits in an abstract category might be weird or hard to define, but we know what they're like in sets, which is nice. And the same thing works for limits; the analogy is that addition or scalar multiplication of $\mathbb{R}$-valued functions on a space are done pointwise: for these functors, we're doing everything with the values of the functors. So $\mathsf{Fun}(\mathsf{C}^{\mathsf{op}}, \mathsf{Set})$ is like this nice promised land, but we need to know how to relate it to questions in C.

To understand this, let's talk about Hom. For any category C and $Z \in \mathsf{C}$, we have a functor $\mathrm{Hom}_{\mathsf{C}}(Z, –) : \mathsf{C} \to \mathsf{Set}$. This functor preserves limits: the analogy is that maps into a subspace of a given vector space are a subset,[16] so $\mathrm{Hom}(V, U) \subset \mathrm{Hom}(V, W)$. That is, "maps into a subspace is a subspace of maps." And since limits are sort of like subspaces, this can be a mneomnic for $\mathrm{Hom}(Z, –)$ preserving limits.

Things here aren't hard, just unwinding notation. The maps $\mathrm{Hom}(Z, \varprojlim A_i)$ is a cone on the diagram of the A_i: it comes with maps $Z \to A_i$ compatible with the directed maps $A_i \to A_j$ — and we said that compatible collections are exactly what limits are in the category of sets, so this is $\varprojlim \mathrm{Hom}(Z, A_i)$. That $\mathrm{Hom}(Z, –)$ preserves limits is very important, and we will use it many times.

One might wonder about $\mathrm{Hom}_{\mathsf{C}}(–, Z)$, but this is just $\mathrm{Hom}_{\mathsf{C}^{\mathsf{op}}}(Z, –)$, so we see that $\mathrm{Hom}_{\mathsf{C}}(–, Z)$ sends colimits to limits, since it's a contravariant functor. Thus, $\mathrm{Hom}_{\mathsf{C}}(\varinjlim A_i, Z) = \varprojlim \mathrm{Hom}(A_i, Z)$. The mneomnic is that maps out of a quotient $V/U \to W$ are a subspace of maps $V \to W$ (those vanishing on U).

This may feel like symbol gymnastics, but we're almost done with the Yoneda embedding for a long time. Here's the final result.

Corollary 4.2. *The Yoneda embedding* $\mathsf{C} \hookrightarrow \mathsf{Fun}(\mathsf{C}^{\mathsf{op}}, \mathsf{Set})$ *preserves limits.*

This is, again, chasing symbols: $h_{\varprojlim A_i} = \mathrm{Hom}(–, \varprojlim A_i) = \varprojlim \mathrm{Hom}(–, A_i) = \varprojlim h_{A_i}$. This is another instance of the mantra that limits are easy: you can calculate limits in any category in terms of limits of sets.

For colimits, this is completely false; this might initially seem bad, but it's actually something good. We have some word of affine schemes, which we still don't get geometrically (we will, don't worry), but categorically is $\mathsf{Ring}^{\mathsf{op}}$. Using the Yoneda embedding, we get a functor of points

$$\begin{tikzcd}
\mathsf{AffSch} \arrow[r, equal] \arrow[d, hook] & \mathsf{Ring}^{\mathsf{op}} \arrow[d, hook] \\
\mathsf{Fun}(\mathsf{AffSch}^{\mathsf{op}}, \mathsf{Set}) \arrow[r, equal] & \mathsf{Fun}(\mathsf{Ring}, \mathsf{Set}).
\end{tikzcd}$$

We will be defining schemes by gluing together affine schemes, which is a kind of colimit. Hence, it's helpful that we don't preserve colimits, so we get nontrivial schemes. In other words, *a scheme is a functor* $\mathsf{Ring} \to \mathsf{Set}$ *with certain properties*. This is useful, because not all spaces have enough functions out of them, so they're not captured by $\mathsf{Ring}^{\mathsf{op}}$, and we need to pass to the functor category.

[16] If this sounds dumb, remember that maps into a quotient are not a quotient.

Adjoint Functors. Again, the analogy to vector spaces will be instructive: if $\varphi : V \to W$ is linear, then there's an adjoint map $\varphi^\dagger : W^* \to V^*$, corresponding to matrix transpose. But if V and W are inner product spaces, then the isomorphisms $V \cong V^*$ and $W \cong W^*$ allow us to realize $\varphi^\dagger$ as a map $W \to V$, and the key property is that for any $v \in V$ and $w \in W$, $\langle \varphi(v), w\rangle_W = \langle v, \varphi^\dagger(w)\rangle_V$; this is enough to completely characterize the adjoint. These are very useful, because they're in a way the closest thing to an inverse: a map $\varphi : V \to W$ factors through an isomorphism $(\ker V)^\perp \to \mathrm{Im}(\varphi)$, and the adjoint $\varphi^\dagger : \mathrm{Im}(\varphi) \to (\ker\varphi)^\perp$ *is* the inverse to φ!

Now, we're going to do the same thing with categories, with $\langle\cdot,\cdot\rangle$ replaced with Hom again. But since this isn't symmetric, we have left- and right-flavored adjoints.

Definition. Let C and D be categories and $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ be functors. Then, (F, G) is an *adjoint pair* (order matters: F is *left adjoint* to G and G is *right adjoint* to F) if there exists a natural isomorphism $\mathrm{Hom}_{\mathsf{D}}(F(-), -) \leftrightarrows \mathrm{Hom}_{\mathsf{C}}(-, G-)$. In other words, for every $X \in \mathsf{C}$ and $Y \in \mathsf{D}$, there's an isomorphism $\mathrm{Hom}_{\mathsf{D}}(FX, Y) = \mathrm{Hom}_{\mathsf{C}}(X, GY)$ that's functorial in both X and Y.

There are other ways to rewrite this; the Wikipedia article is pretty good. For example, out of this structure there's a canonical map $\eta_X \in \mathrm{Hom}(X, GFX)$ (which doesn't have an obvious analogue in the world of vector spaces): this is the same as $\mathrm{Hom}(FX, FX)$, so let η_X be the map corresponding to id_{FX}. More precisely, there's a natural transformation $\eta : \mathrm{id}_{\mathsf{C}} \to GF$. In the same way, there's a natural transformation $\varepsilon : FG \to \mathrm{id}_{\mathsf{D}}$ given by pulling back the identity map.

Sometimes, having η and ε is more convenient than the standard definition of adjointness, so one can start with natural transformations $\eta : \mathrm{id}_{\mathsf{C}} \to FG$ and $\varepsilon : GF \to \mathrm{id}_{\mathsf{D}}$. Then, it's a theorem that if they satisfy the "mark of Zorro" axiom, that the following diagram commutes, where the first map adds GF on the right by η, and the second map collapses FG on the left by ε.

$$\begin{tikzcd} F \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & FGF \arrow[r] & F \end{tikzcd}$$

What are adjoints used for? Everything, everywhere.

Example 4.3 (Free and forgetful functors)**.** There's a pair of functors $\mathrm{Free} : \mathsf{Set} \to \mathit{Grp}$ and $\mathrm{For} : \mathit{Grp} \to \mathsf{Set}$. This is an adjunction, because a map out of a free group is determined exactly by where its generators go, so if G is a group and S is a set, then $\mathrm{Hom}_{\mathit{Grp}}(\mathrm{Free}(S), G) = \mathrm{Hom}_{\mathsf{Set}}(S, \mathrm{For}(G))$.

We can generalize this: there are lots of forgetful functors, and we can define free functors as their left adjoints; in this way one realizes the usual definition of free abelian groups, for example.

Another example: there's a forgetful functor $\mathrm{For} : \mathsf{Mod}_R \to \mathsf{Set}$, and the notion of a free R-module is a left adjoint $\mathrm{Free} : \mathsf{Set} \to \mathsf{Mod}_R$, because $\mathrm{Hom}_{\mathsf{Mod}_R}(\mathrm{Free}(S), M) = \mathrm{Hom}_{\mathsf{Set}}(S, \mathrm{For}(M))$ for any set S and R-module M. This is because a free$R$-module on a set S is R^S (the direct sum), and so the images of the generators are exactly what determine a map out of it.

But we've talked about functors to sets before: is For representable? A map $R \to M$ is determined by where it sends M: $\mathrm{For}(M) = \mathrm{Hom}_{\mathsf{Mod}_R}(R, M)$, so For is represented by R!

This is a special case of the most important adjunction.

Example 4.4. Let R be a ring and $\mathsf{C} = \mathsf{Mod}_R$. We know that $\mathrm{Hom}_R(M, N)$ isn't just a set, but is naturally an R-module (you can add and multiply maps pointwise). Since we've been using Hom to denote sets, then we'll let *inner Hom* $\underline{\mathrm{Hom}}_R(M, N)$ denote the Hom as an R-module.

Thus, we've defined a functor $\underline{\mathrm{Hom}}_R(M, -) : \mathsf{Mod}_R \to \mathsf{Mod}_R$. Does it have a left adjoint?[17] That is, we need to look at $\mathrm{Hom}(N, \underline{\mathrm{Hom}}_R(M, P))$, whose elements send $n \in N$ to an R-linear map $M \to P$. We can recast these as maps $M \times N \to P$, which must be R-linear in both M and N.

This may be looking familiar: we're looking for R-bilinear maps $M \times N \to P$ (that is, $\varphi(rm, n) = \varphi(m, rn) = r\varphi(m, n)$). And there is a universal object through which these factor through, the tensor product $M \otimes_R N$. By definition, bilinear maps $M \times N \to P$ correspond to linear maps $M \otimes_R N \to P$. You do have to construct it to show that it exists: it's the span of symbols $m \otimes n$, modded out by the equivalence relation $rm \otimes n = m \otimes rn$ (you can move scalars across the middle). There are a bunch of things to implicitly check here, some of which will be exercises for us.

[17] It turns out this does not have a right adjoint, which isn't too hard to convince yourself of.

The point is, this universal property is saying that $\mathrm{Hom}(N, \underline{\mathrm{Hom}}_R(M, P)) = \mathrm{Hom}(M \otimes_R N, P)$. Thus, $(M \otimes -, \mathrm{Hom}_R(M, -))$ is an adjoint pair! This is the real definition of the tensor product. If $R = \mathbb{Z}$, so we just have the category of abelian groups, then the tensor product will be written $M \otimes N$.

One useful thing to check is that if S is an R-algebra, the map $R \to S$ defines an S-module structure on $S \otimes_R M$. That is, we have a functor $S \otimes_R - : \mathsf{Mod}_R \to \mathsf{Mod}_S$. As one example, if $R = \mathbb{Z}$, then this specializes to S being any ring, and $A \mapsto A \otimes S$ sends A to the free module $S \otimes A$.

Tensor products are always left adjoint, and this one also has a right adjoint: given an S-module P and a map $R \to S$ (so S is an R-algebra), then forgetting to the R-module structure is functorial, and $(S \otimes_R -, \mathrm{For})$ is another adjoint pair. This is why $S \otimes A$ is regarded as free.

Example 4.5. Now, suppose S and T are both R-algebras; then, $S \otimes_R T$ is more than just an S- or T-module; in fact, it's a ring (with R-, S-, and T-algebra structures). Over $\mathbb{Z}$, this specializes to the statement that the tensor product of two rings is still a ring. As a silly example, let X be a set of m points and Y be a set of n points. Then, $S = \mathbb{C}[X]$, the functions on X, is $\mathbb{C}^m$, a commutative ring with multiplication defined pointwise. Similarly, T is functions on Y, so $T \cong \mathbb{C}^n$. Thus, $S \otimes_{\mathbb{C}} T = \mathrm{Mat}_{m,n}$: there's a complex number for every pair $(m, n) \in X \times Y$. The multiplication is the silly one, pointwise multiplication (i.e. the one you told your linear algebra students to never, ever do), because this is the ring of functions on $X \times Y$, rather than usual matrix multiplication. This might be a little more motivation for this next statement.

Exercise 4.6. $S \otimes_R T$ is the coproduct $S \amalg T$ in the category of R-algebras, and the pushout $S \amalg_R T$ in the category of rings. In other words, it fits into the following diagram.

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d] \\ T \arrow[r] & S \otimes_R T \end{tikzcd}$$

This may be confusing, because the coproduct of modules is direct sum. But the example with sets of points will be true more generally: in nice situations, $\mathrm{Fun}(X \times Y) \cong \mathrm{Fun}(X) \otimes \mathrm{Fun}(Y)$. Or in other words, $\mathrm{Hom}_{\mathsf{Ring}}(S \otimes T, U) = \mathrm{Hom}_{\mathsf{Ring}}(S, U) \times \mathrm{Hom}_{\mathsf{Ring}}(T, U)$, and there's a version with R-algebras as well.

Now, since affine schemes are the opposite category to the category of rings, then we know that $\operatorname{Spec} R \times \operatorname{Spec} S = \operatorname{Spec}(R \otimes S$: functions on $X \times Y$ are the tensor product of those on X and those on Y. Strangely, though we know what products of affine schemes are, we don't know what affine schemes are yet.

Episode V.

The Spectrum of a Ring: 2/2/16

Before we delve into the world of schemes, we have just a little more to say about adjoints.

Recall that an adjoint pair $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ is the data of a natural isomorphism $\mathrm{Hom}_{\mathsf{C}}(M, GN) \cong \mathrm{Hom}_{\mathsf{D}}(FM, N)$ for all $M \in \mathsf{C}$ and $N \in \mathsf{D}$. One important example was the adjoint $(\mathrm{Free}, \mathrm{Forget})$: free functors lie on the left, because they're very easy to map out of (just specify where the generators go). The other important example may be an instance of the first example: $(\otimes_R, \underline{\mathrm{Hom}}_R)$, because $\mathrm{Hom}_{\mathsf{Mod}_R}(M, \underline{\mathrm{Hom}}_R(N, P))$ is the R-bilinear maps $M \times N \to P$, but this is $\mathrm{Hom}_R(M \otimes_R N, P)$, by the universal property for tensor product. So tensor products can be thought of as a free construction; another example of this is that given a map $R \to S$, the functor $S \otimes_R - : \mathsf{Mod}_R \to \mathsf{Mod}_S$ is left adjoint to the forgetful functor from S-modules to R-modules.

Finally, the last piece of abstract nonsense we'll discuss is the relation between adjoints and limits. If I is an index category, then the I-shaped diagrams in a category C (the functors $I \to \mathsf{C}$) are also a category, the functor category $\mathsf{Fun}(I, \mathsf{C})$. This is also denoted C^I.

There's a natural functor $\Delta : \mathsf{C} \to \mathsf{C}^I$ sending an $M \in \mathsf{C}$ to the diagram with M at every index and the identity for every morphism, which of course commutes. Sometimes this is called the "stupid diagram," or more formally the *diagonal diagram* or *constant diagram*.

Every time you see a functor, your first question should be, *does it have an adjoint*? We can check on the left or on the right, so suppose we have a left adjoint $\Delta^\ell : \mathsf{C}^I \to \mathsf{C}$. Writing the meaning of this is less confusing than drawing pictures: if we have an $A_\bullet \in \mathsf{C}^I$ specified by

$$A_\bullet = \begin{tikzcd} A_1 \arrow[d] \arrow[dr] & \\ A_2 \arrow[r] & A_3, \end{tikzcd}$$

then the left adjoint has the property that for any $B \in \mathsf{C}$, a map $\Delta^\ell(A_\bullet) \to B$ is the data

$$\begin{tikzcd} A_1 \arrow[d] \arrow[dr] \arrow[rrr, bend left] & & & B \arrow[d, "\mathrm{id}"] \arrow[dr, "\mathrm{id}"] & \\ A_2 \arrow[r] \arrow[rrr, bend right] & A_3 \arrow[rrr, bend right] & & B_2 \arrow[r, "\mathrm{id}"] & B_3, \end{tikzcd}$$

but if we collapse the diagonal B-diagram, this is exactly a cone! Thus, $\Delta^\ell = \varinjlim$ (which, recall, may not always exist), and similarly, a right adjoint Δ^r to Δ is $\varprojlim$. In fact, there's also a way to realize adjoints as certain kinds of limits; then, the following proposition is just a consequence of the principle that "(co)limits commute with (co)limits."

Proposition 5.1. *Right adjoints (resp. left adjoints) commutes with limits (resp. colimits). That is, if (F, G) is an adjunction, then $F(\varinjlim A_i) = \varinjlim F(A_i)$ and $G(\varprojlim(B_i) = \varprojlim G(B_i)$.*

Proof. There's nothing particularly tricky here. If the adjoints are on the categories C and D, then for any $A \in \mathsf{C}$, consider $\mathrm{Hom}_{\mathsf{C}}(A, \varprojlim(G(B_i)))$. If we can show this is the same as $\mathrm{Hom}_{\mathsf{C}}(A, G(\varprojlim(B_i)))$, then the Yoneda embedding says that $\varprojlim G(B_i) = G(\varprojlim B_i)$: we can show two things are the same by showing the maps into them are the same.

First, we said last time that Hom commutes with limits, so $\mathrm{Hom}_{\mathsf{C}}(A, \varprojlim G(B_i)) = \varprojlim \mathrm{Hom}_{\mathsf{C}}(A, G(B_i)) = \varprojlim \mathrm{Hom}_{\mathsf{D}}(F(A), B_i)$ by the adjunction. But since Hom commutes with limits, then this can be rewritten as $\mathrm{Hom}_{\mathsf{D}}(F(A), \varprojlim B_i) = \mathrm{Hom}_{\mathsf{C}}(A, G(\varprojlim B_i))$, again by the adjunction.

Then, the proof for left adjoints is the same, but in the opposite category. ⊠

To formalize this, you'd want to say why it's functorial in A, but this isn't the core content of the proof.

Proposition 5.1 is useful everywhere. For example, we said that "forgetful functors preserve limits;" since forgetful functors are right adjoint to free functors, then they must preserve limits. In particular, products, fiber products, and kernels are all preserved by forgetful functors.

Another application: since localization is a colimit and $S \otimes_R -$ is a right adjoint, then it should commute with localization. In particular, there's a natural isomorphism $S^{-1}M \otimes_R N \cong S^{-1}(M \otimes_R N)$. In particular, $S^{-1}R \otimes_R M \cong S^{-1}(R \otimes_R M) = S^{-1}M$, since there's a natural isomorphism $R \otimes_R M = M$. That is, localization of modules, as a functor, is $S^{-1}R \otimes_R -$. Another way to see this is that there's a forgetful functor $\mathsf{Mod}_{S^{-1}R} \to \mathsf{Mod}_R$, and the left adjoint functor is $S^{-1}R \otimes_R -$, the localization functor.

So localization is a tensor product, and therefore a localization. Thus, it commutes with arbitrary colimits: for example, since direct sums are colimits, then $S^{-1}(\bigoplus M_i) \cong \bigoplus S^{-1}M_i$ canonically, and tensor products commute with arbitrary direct sums. Moreover, pushouts, cokernels, and coequalizers all pass through tensor products.

On the other hand, completion cannot be written as a tensor product; it's a limit. Thus, it does not necessarily commute with direct sums, etc.

Introduction to Schemes. We can't really define a scheme yet (we're missing a key ingredient), but we can still talk a lot about them. Remember that our plan was to associate an affine scheme $\mathrm{Spec}\, R$ to a ring R, in a way that is a contravariant equivalence of categories. This is quite a strong desideratum, and so we want a strong construction. We'll find this has the following three ingredients: a set of points, a topology, and a structure sheaf of functions. We'll discuss the set today, and then move to the other two later.

Each ingredient is very necessary: for example, if k is a field, then $\mathrm{Spec}\, k$ will be a point. There's only one topology here, but there are many nonisomorphic fields, so the structure sheaf will have to do something interesting. Why is this $\mathrm{Spec}\, k$? A point is the terminal object in Set, and fields have no interesting ideals: every map $k \to R$ for a ring R is necessarily injective, hence a monomorphism. Hence, all maps $\mathrm{Spec}\, R \to \mathrm{Spec}\, k$ should be epimorphisms in Set, hence surjective. This is not a proof, just an *ansatz*.

More generally, we'd like points in $\operatorname{Spec} R$ should correspond to maps $\mathrm{pt} \to \operatorname{Spec} R$, which will correspond to a ring homomorphism $R \to K$, for a field k. How do we organize these homomorphisms?

Definition. If R is a ring, define the set $\operatorname{Spec} R$ to be the set of prime ideals[18] $\mathfrak{p} \subset R$.

One's first naïve idea of what you'd want is the set of maximal ideals, which is a subset (after all, a maximal ideal is a prime ideal), but if $\mathfrak{m} \subset R$ is maximal, that's the same as a surjection $R \twoheadrightarrow k$. But if $\mathfrak{p}$ is a prime ideal, then the surjection $R \twoheadrightarrow R/\mathfrak{p}$ is onto an integral domain. So prime ideals are surjections onto integral domains.

Wait, why are we talking about integral domains? An integral domain means exactly having a field of fractions: if I is an integral domain, let $S = I \setminus 0$, which is multiplicative, so we get a field $S^{-1}I$, and an injective map $I \hookrightarrow S^{-1}I$. And a subring of a field must be an integral domain (since fields have no zero divisors). Hence, integral domains are exactly the rings which are subrings of fields. Thus, prime ideals give maps to fields, even if they may not be injective: if $\mathfrak{p}$ is a prime ideal, then $R \twoheadrightarrow R/\mathfrak{p} \hookrightarrow \operatorname{Frac}(R/\mathfrak{p})$, and the composite map may not be surjective, but its image generates the field $\operatorname{Frac}(R/\mathfrak{p})$.

In other words, a prime ideal is the same as a homomorphism $R \to k$ which generates k as a field. The field associated to a prime ideal is called its *residue field*. This is one reason why prime ideals are still somehow reasonable. One can also define an equivalence relation on maps from R to fields (there are many of these, thanks to e.g. field extensions), and prime ideals represent equivalence classes. So one might think that "prime ideals of R are the ways in which R talks to fields."

Now, suppose $r \in R$ and $\mathfrak{p} \subset R$ is prime (we'll think of it as a point $x \in \operatorname{Spec} R$). Then, there's an evaluation map $r(x) = r \bmod \mathfrak{p} \in R/\mathfrak{p}$, or even inside $\operatorname{Frac}(R/\mathfrak{p})$. So we can think of R as the set of "regular functions" on $\operatorname{Spec} R$. The codomain field of the function $r(x)$ depends on the point x, which is quite strange, but we'll eventually pin down precisely what such a function means; meanwhile, this issue is one of the main weirdnesses of schemes you'll have to work with at first.

Hence, if k is a field, then $\operatorname{Spec} k = \{(0)\} = \mathrm{pt}$, and any $r \in k$ gives a k-valued function on the point (0), which is $r(\mathrm{pt}) = r$. Moreover, if R is the zero ring, then $\operatorname{Spec} R = \varnothing$; this makes sense, because 0 is terminal in the category of rings, and $\varnothing$ is initial in the category of sets.

Example 5.2. Let's have a more interesting example, $\mathbb{A}^1_{\mathbb{C}}$, the *affine line over* $\mathbb{C}$, defined to be $\operatorname{Spec} \mathbb{C}[x]$.[19]

The maximal ideals in $\mathbb{C}[x]$ are exactly the irreducible (nonconstant) polynomials $\langle (x-t) \rangle \subset \mathbb{C}[x]$, and a $t \in \mathbb{C}$ defines a function on them which is precisely evaluation at t. However, there's one more prime ideal, the zero ideal.[20]

Lemma 5.3. *0 and $\langle (x-t) \rangle$ for $t \in \mathbb{C}$ are all of the prime ideals of $\mathbb{C}[x]$.*

Proof. Suppose $\mathfrak{p} \subset \mathbb{C}[x]$ is prime and nonzero. Then, let $f \in \mathfrak{p}$ be a polynomial of minimal degree in $\mathfrak{p}$. Then, f must be nonconstant (if it were constant, it would be invertible, so $\mathfrak{p} = \mathbb{C}[x]$, which isn't the case). However, the degree of f must be 1: if $\deg f > 1$, then since $\mathbb{C}$ is algebraically closed, then f has a root, so if $\deg f > 1$, then $f = gh$, with $\deg g, \deg h > 0$. Thus, $g \in \mathfrak{p}$ or $h \in \mathfrak{p}$, because $\mathfrak{p}$ is prime, but both of them have degrees less than that of f, which is a contradiction, so $\deg f = 1$.

Thus, $\mathfrak{p} \supset \langle f \rangle$: *a priori*, it could be bigger. We'll use the property that $\mathbb{C}[x]$ is a Euclidean domain, so we can do polynomial long division, so if $g \in \mathfrak{p}$, then $g = f \cdot m + r$, for some $m, r \in \mathbb{C}[x]$ with $\deg r < \deg f$. But since $f, g \in \mathfrak{p}$, then $r = g - fm \in \mathfrak{p}$ as well, but $\deg r < \deg f <$ so $\deg r = 0$, and therefore $f \mid g$, i.e. $g \in \langle f \rangle$. ⊠

So now we know what $\mathbb{A}^1_{\mathbb{C}}$ is as a set. One can draw a picture of it: for every $t \in \mathbb{C}$, there's a point $\langle (x-t) \rangle \in \mathbb{A}^1_{\mathbb{C}}$, so we have a bunch of point; then, we have the point corresponding to (0), which is "bigger." In general, if R is an integral domain, the point corresponding to (0) in $\operatorname{Spec} R$ will be called the *generic point*. Then, the residue field associated to each point $t \in \mathbb{C}$ is $\mathbb{C}$ again, and for the zero ideal we get $\operatorname{Frac}(\mathbb{C}[x]/0) = \mathbb{C}(x)$, the rational functions in $\mathbb{C}$.

[18] For the purposes of this definition, R is not an ideal of itself; we're only looking at proper ideals.

[19] More generally, if k is any field, the *affine line over* k is $\mathbb{A}^1_k = \operatorname{Spec} k[x]$.

[20] The condition that 0 is a prime ideal is equivalent to a ring being an integral domain, so in these cases we do have a distinguished prime ideal.

Since the proof of Lemma 5.3 only depended on $\mathbb{C}$ being an algebraically closed field, the above example works just as well for $\operatorname{Spec} k$, when k is any algebraically closed field: for every $t \in k$ we have a point with residue field k, and then the generic point (0) with residue field $k(x)$, the rational functions on $\operatorname{Spec} k[x] = \mathbb{A}^1_k$.

Example 5.4 ($\operatorname{Spec}\mathbb{Z}$)**.** Since $\mathbb{Z}$ is initial in the category of rings, then $\operatorname{Spec}\mathbb{Z}$ will be final in the category of affine schemes. So it will behave as a point, even though it doesn't look at all like one. Having a good geometric object corresponding to $\mathbb{Z}$ was a major motivator for Grothendieck, and was a feature of the scheme-theoretic approach over others.

The picture is a point for every prime $p \in \mathbb{Z}$, with residue field $\mathbb{F}_p$, but also the zero ideal, corresponding to the generic point, whose residue field is $\mathbb{Q}$. This point ends up being dense once we define a topology on $\operatorname{Spec}\mathbb{Z}$, so $\operatorname{Spec}\mathbb{Z}$ is connected, which is nice. The intuition is that every rational number is a function at all but finitely many points: $19/15 \in \mathbb{Q}$, so we can evaluate $(19/15)(7) = 5 \bmod 7$, and do this everywhere except 3 and 5, which are its "poles." (Its value at the generic point is $19/15$ again.)

Since we have a map $\mathbb{Z} \to \mathbb{Q}$, then we'd better have a nice map $\operatorname{Spec}\mathbb{Q} \to \operatorname{Spec}\mathbb{Z}$, corresponding to morphisms of residue fields. Since $\operatorname{Spec}\mathbb{Q}$ is a point, we can just send it to the generic point, whose residue field is $\mathbb{Q}$. This is why we need prime ideals (and generic points as a consequence); if we're trying to mimic ring theory, this is just necessary. Classical algebraic geometry tended to restrict itself to finitely generated algebras over an algebraically closed field, which means that we must miss out on some ring theory.

Example 5.5. We can also talk about $\mathbb{A}^1_{\mathbb{R}} = \operatorname{Spec}\mathbb{R}[x]$, or more generally $\mathbb{A}^1_k$ where k is not algebraically closed. This means classifying the prime ideals of $\mathbb{R}[x]$; since $\mathbb{R}$ isn't algebraically closed, it's no longer true that every prime ideal contains a linear factor. We do have $(x-t)$ for $t \in \mathbb{R}$ and (0) again, but since $\mathbb{R}[x]$ is again a Euclidean domain, then it's a PID. Thus, all of our ideals are (f) for some $f \in \mathbb{R}[x]$, and (f) is prime iff f is an irreducible polynomial.[21] This means we have to classify monic irreducible polynomials.

Over a field k, a monic irreducible polynomial is given exactly by a Galois orbit in $\overline{k}$. For $\mathbb{R}$, $\overline{\mathbb{R}} = \mathbb{C}$, and $\operatorname{Gal}(\mathbb{R}/\mathbb{C})$ is a group of order 2, generated by complex conjugation. Thus, the orbits are two points $\{z, \overline{z}\}$, the complex conjugate roots of an irreducible quadratic in $\mathbb{R}[x]$.

Thus, the picture of $\mathbb{A}^1_{\mathbb{R}}$ has a copy of $\mathbb{R}$ as usual (points with residue field $\mathbb{R}$), with a generic point (0) with residue field $\mathbb{R}(x)$, and an "upper half-plane" (which is not strictly true, since we're identifying points in $\mathbb{C}$, rather than taking a subset) of $\{z, \overline{z}\}$ with residue field $\mathbb{C}$ (since, for example, $\mathbb{R}[x](x^2+1) \cong \mathbb{C}$). The point is: for a field that's not algebraically closed, there are points in the affine line whose residue field is a nontrivial field extension.

Given any $f \in \mathbb{R}[x]$, we get a function on $\operatorname{Spec}\mathbb{R}[x]$: $f((x^2+1)) = f \bmod (x^2+1) \in \mathbb{C}$. This is a complex number, but evaluating at $(x-t)$, with $t \in \mathbb{R}$, gives you a real number. This is a little funny, but the takeaway is that we have these interesting new points, since $\mathbb{R}$ isn't algebraically closed.

A fun exercise is to draw $\mathbb{A}^1_{\mathbb{F}_p}$, because $\overline{\mathbb{F}}_p/\mathbb{F}_p$ isn't a finite field extension: each finite extension has a Galois group $\mathbb{Z}/p$, so there are $(p-1)$ Galois orbits at each stage. It's definitely a strange thing, and not what you would think of as a line: the point is that the construction of a scheme has extra points you might not expect, in order to make the connection between rings and schemes work.

Episode VI.

Functoriality of Spec: 2/4/16

Last time, we talked about $\operatorname{Spec} R$ for a ring R, the set of prime ideals. R acts like "functions" on this set: for an $r \in R$, "evaluating" it at a $\mathfrak{p} \in \operatorname{Spec} R$ returns $r \bmod \mathfrak{p}$ in $R/\mathfrak{p} \hookrightarrow \operatorname{Frac}(R/\mathfrak{p})$. $\operatorname{Spec} R$ has a lot of interesting structure, which we'll talk some more about today.

Recall that if k is a field, then $\operatorname{Spec} k$ is a point,[22] and if R is an integral domain, then (0) is prime, so there's a special point called the generic point. We also talked about a PID (actually a Euclidean domain), $k[x]$, where k is a field.

[21] There's this nice set of inclusions fields $\subset$ Euclidean domains $\subset$ PIDs $\subset$ UFDs $\subset$ integral domains.

[22] An easy way to remember this is that $\operatorname{Spec} k$ is a speck!

We also have affine n-space over a ring R, $\mathbb{A}^n_R = \operatorname{Spec} R[x_1, \dots, x_n]$. If $R = k$ is a field, then the affine line over k is $\mathbb{A}^1_k = \operatorname{Spec} k[x]$. This ring is a PID, and in particular primes correspond to irreducible polynomials, which correspond to Galois orbits of points in $\overline{k}$, along with one generic point.

The following theorem comes from commutative algebra.

Theorem 6.1.

- *If R is a PID, then it's also a UFD, i.e. every $r \in R$ can be factored as $r = uf_1 \cdots f_k$, where $u \in R^\times$ and the f_j are irreducibles, unique up to units and scaling.*
- *If R is a UFD, then $R[x]$ is also a UFD.*

For a general UFD, there may exist prime ideals which are not principal.

The takeaway is that affine n-space comes from Spec of a PID. We can use this to better understand $\mathbb{A}^2_{\mathbb{C}}$ (or $\mathbb{A}^2_k$ when k is algebraically closed): there's a generic point (0), and maximal ideals $(s, t) \in k^2$ (given by the ideal $(x_1 - s, x_2 - t) \subset \mathbb{C}[x_1, x_2]$, and is clearly maximal, because the quotient is $\mathbb{C}$). However, we also have other prime ideals: if f is any irreducible polynomial, then (f) is prime (and vice versa, since we're in a UFD). In particular, there are lots of irreducibles, and therefore lots of prime ideals.

Thus, your picture could consist of all the points in k^2, and a generic point (which is dense, so draw it everywhere, maybe?), and then lots of points which are curves: for example, because $x_1^2 + x_2^2$ is irreducible in $\mathbb{C}[x, y]$, there's a point in $\mathbb{A}^2_{\mathbb{C}}$ that is the unit circle. And all lines exist, and other algebraic curves.

Exercise 6.2. These are all of the points in $\mathbb{A}^2_{\mathbb{C}}$: curves corresponding to irreducible polynomials, the maximal ideals, and the generic point.

We can learn more about $\mathbb{A}^n_k$ from the following theorem.

Theorem 6.3 (Hilbert's Nullstellensatz)**.** *If k is a field, then the maximal ideals of $k[x_1, \dots, x_n]$ have residue field k' a finite extension of k.*

That is, if $\mathfrak{m} \subset k[x_1, \dots, x_n]$ is maximal, then $k \hookrightarrow k[x_1, \dots, x_n]/\mathfrak{m}$ is a finite field extension. This is nice, because we don't have bizarre transcendental extensions. Additionally, if k is algebraically closed, then maximal ideals are what you think of as points: their residue fields have to be just k, and in fact they correspond to evaluation functions $k[x_1, \dots, x_n] \to k$, which are in bijection with $(t_1, \dots, t_n) \in k^n$.

Theorem 6.3 is equivalent to the following statement.

Corollary 6.4. *If $k \to K$ is a field extension, then K is finitely generated as a k-algebra iff it's a finite-dimensional k-vector space.*

The idea is that finite generation corresponds to a surjection $k[x_1, \dots, x_n] \twoheadrightarrow K$, which corresponds to a maximal ideal in $k[x_1, \dots, x_n]$.

We'll later use theorems like these to put finiteness conditions on different kinds of ring morphisms.

So looking at something like $\mathbb{A}^3$, there's the generic point (0), and "two-dimensional" points (f) for irreduicble f, and "zero-dimensional" points corresponding to maximal ideals.[23] Why do we have so many strange points, rather than just k^n? The answer is functoriality.

Theorem 6.5. Spec *is a functor* $\mathsf{Ring}^{\mathsf{op}} \to \mathsf{Set}$, *i.e. given a ring homomorphism $\phi : R \to T$, there's a set map $\Phi : \operatorname{Spec} T \to \operatorname{Spec} R$.*

Proof. The points of $\operatorname{Spec} T$ are maps $T \to K$ that generate K, for a field K, so composing $R \to T \to K$ gives us a homomorphism. It might not generate K, but it does generate a subfield, so this map corresponds to a prime ideal in R. ⊠

Less abstract proof. Let $\mathfrak{p} \subset T$ be prime. Then, $\phi^{-1}(\mathfrak{p}) \subset R$ is also a prime ideal: if $rs \in \phi^{-1}(\mathfrak{p})$, then $\phi(rs) \in \mathfrak{p}$, so one of $\phi(r)$ or $\phi(s)$ is in $\mathfrak{p}$, so one of r or s is in $\phi^{-1}(\mathfrak{p})$. ⊠

The preimage of a maximal idea is not necessarily maximal, which is one of the big reasons we look at more than just maximal ideals.

So prime ideals pull back, which is nice. But since R acts as functions on $\operatorname{Spec} R$, this should really be thought of as pullback of functions.

[23] There is a sense in which this can be made rigorous, and defines dimensions of schemes.

Quotients. The functoriality has some interesting consequences for our favorite ring operations. First, suppose $I \subset R$ is an ideal, so there's a quotient map $\phi : R \twoheadrightarrow R/I$. Thus, we get a map in the opposite direction, $\Phi : \operatorname{Spec} R/I \to \operatorname{Spec} R$.

Exercise 6.6. Show that an ideal $\mathfrak{p} \subset R/I$ is prime iff. $\phi^{-1}(\mathfrak{p})$ is a prime ideal in R containing I.

That is, it's an inclusion-preserving bijection, or considering only the prime ideals containing I, this is an isomorphism of posets. In any case, quotients give rise to injections $\operatorname{Spec} R/I \hookrightarrow \operatorname{Spec} R$.

For example, (xy) is an ideal in $\mathbb{C}[x,y]$, so $\mathbb{A}^2_{\mathbb{C}} \supset \operatorname{Spec} \mathbb{C}[x,y]/(xy)$. Geometrically, this is the inclusion of the coordinate axes into $\mathbb{A}^2_{\mathbb{C}}$: $\operatorname{Spec} \mathbb{C}[x,y]/I$ will be the zero locus of I in $\mathbb{A}^2_{\mathbb{C}}$. We'll let $V(I) = \operatorname{Spec} \mathbb{C}[x,y]/I \hookrightarrow \mathbb{A}^2_{\mathbb{C}}$. This is because if $I \subset \mathfrak{p} \subset \mathbb{C}[x,y]$, then $f \equiv 0 \bmod \mathfrak{p}$ for all $f \in I$.

For example, if $I = (xy)$ again, then the zero ideal doesn't contain (x,y), so the generic point isn't in this subset. And since xy isn't irreducible, then $\mathbb{C}[x,y]/(xy)$ isn't an integral domain; it has no generic point! However, there are two components with generic points: we can quotient to $\operatorname{Spec} \mathbb{C}[x,y]/(x)$ or $\operatorname{Spec} \mathbb{C}[x,y]/(y)$; each of these is a copy of $\mathbb{A}^1$ embedded (set-theoretically for now) in $V(xy)$, which are the x- and y-axis, respectively. These do have generic points, so $V(xy)$ has two "generic-like" points, which correspond to two particularly interesting prime ideals in $\mathbb{C}[x,y]/(xy)$.

To reiterate, if $\mathfrak{p}$ is a prime ideal, then we have a point $\mathfrak{p} \in \operatorname{Spec} R$, but also the map $j : \operatorname{Spec} R/\mathfrak{p} \to \operatorname{Spec} R$. How do these relate? Inside $R/\mathfrak{p}$, there's the zero ideal, which is the generic point, but this ideal corresponds to the ideal $\mathfrak{p} \subset R$. That is, j takes the generic point of $\operatorname{Spec} R/\mathfrak{p}$ to $\mathfrak{p}$. So these funny generic-like points are just the generic points of subschemes, which may be a little nicer perspective. Generic points are still weird, but once we have a topology, they just correspond to points that aren't closed (which exist in some topological spaces, yet might be less familiar). But to make Spec functorial, we have to accept these strange generic points. However, it can be a surprisingly convenient language: there's a nice representative point for any subscheme.

We can also draw $\mathbb{A}^1_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[x]$. Whatever it is, it comes with a map to $\operatorname{Spec} \mathbb{Z}$ (since $\mathbb{Z}$ is the initial object in the category of rings), which had a point for every prime in $\mathbb{Z}$, and a generic point. This in some sense keeps tract of the characteristics of your residue fields.

For any prime of $\mathbb{Z}$, such as 7, there's a quotient $\mathbb{Z}[x] \twoheadrightarrow \mathbb{F}_7[x]$, and this commutes with the inclusion maps $\phi : \mathbb{Z} \to \mathbb{Z}[x]$ (and the same for $\mathbb{F}_7$). Hence, we get a commutative diagram of sets

$$\begin{tikzcd}
\mathbb{A}^1_{\mathbb{F}_7} \arrow[r] \arrow[d] & \mathbb{A}^1_{\mathbb{Z}} \arrow[d, "\Phi"] \\
\operatorname{Spec} \mathbb{F}_7 \arrow[r] & \operatorname{Spec} \mathbb{Z}.
\end{tikzcd}$$

So $\Phi^{-1}(7)$ is the set of prime ideals of $\mathbb{Z}[x]$ whose intersection with $\mathbb{Z}$ contains 7, i.e. (7), and these correspond to the prime ideals of $\mathbb{F}_7[x]$. That is, we have a copy of $\mathbb{A}^1_{\mathbb{F}_7}$ as fibers of the map $\mathbb{A}^1_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$, meaning $\mathbb{A}^1_{\mathbb{Z}}$ is a kind of surface, fibered over $\operatorname{Spec} \mathbb{Z}$.[24] Finally, what happens to the generic point? The fiber over (0) should correspond to maps to fields of characteristic 0, but these maps factor through $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}[x]/\mathfrak{p})$, so (after a little work) one gets that $\Phi^{-1}(0) = \mathbb{A}^1_{\mathbb{Q}}$. See Figure 2 for one depiction of this.

We haven't used anything specific about $\mathbb{A}^1$, so we have a similar fibration over $\mathbb{A}^n_{\mathbb{Z}}$. If k is a field, then a map $\operatorname{Spec} k[x_1, \ldots . x_n]/I$ will be induced by a map $\mathbb{Z} \to k[x_1, \ldots, x_n]/I$, which necessarily factors through the prime field of characteristic equal to k ($\mathbb{F}_p$ or $\mathbb{Q}$). Thus, it necessarily lives in the fiber for that prime ideal of $\operatorname{Spec} \mathbb{Z}$.

Localization. Suppose $S \subset R$ is multiplicative; then, the localization map $R \to S^{-1}R$ induces a map $\operatorname{Spec} S^{-1}R \to \operatorname{Spec} R$.

Exercise 6.7. Show that the prime ideals of $S^{-1}R$ are in bijection with the primes of R not meeting S.

Thus, localizations also give you subsets. One extreme example is $\operatorname{Spec} \mathbb{Q} \hookrightarrow \operatorname{Spec} \mathbb{Z}$ (a single point), but this is "infinitely generated" (i.e. S isn't finitely generated); we'll like finite localizations more. There are two particular examples we'll like.

[24] Eventually, when we define schemes, this will actually have geometric meaning, and will still be true for schemes.

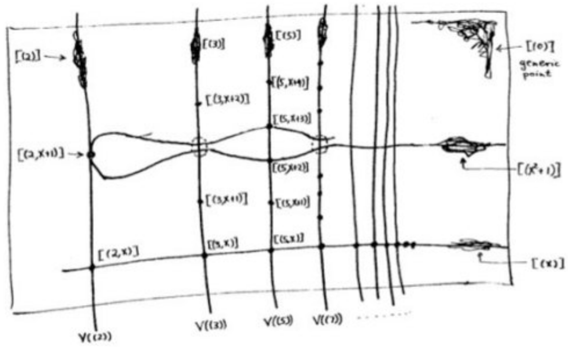

FIGURE 2. A drawing of $\mathbb{A}^1_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[x]$, from Mumford's red book on schemes.

- Given an $f \in R$, there's a multiplicative subset $S = \{1, f, f^2, \dots\}$, and the localization $S^{-1}R$ is denoted R_f or $f^{-1}R$. Then, $\operatorname{Spec} f^{-1}R = \{\mathfrak{p} \subset R : f \notin \mathfrak{p}\}$; that is, this is $\operatorname{Spec} R \setminus V(f)$ (since if $f \in \mathfrak{p}$, then $f \bmod \mathfrak{p} = 0$). This is a nice consequence of the weird definition of functions on our affine schemes: it's actually nice to know what it means for a function to be zero. You can also write $\operatorname{Spec} f^{-1}R = \operatorname{Spec} R \setminus \operatorname{Spec}(R/(f))$. This set is called $D(f)$, and when we have a topology, this will be referred to as a *distinguished open*.[25]

  For example, $\operatorname{Spec}(15)^{-1}\mathbb{Z}$ is the same as $\operatorname{Spec} \mathbb{Z}$, but with the points (3) and (5) removed. (The generic point is still in this set, since $15 \bmod (0) = 15$). Similarly, $\operatorname{Spec}((x^2+y^2)^{-1}\mathbb{C}[x,y])$ is the affine plane minus the circle.
- We also can define *localization at a prime*: if $\mathfrak{p} \subset R$ is prime, let $S = R \setminus \mathfrak{p}$, and we denote $S^{-1}R$ as $R_{\mathfrak{p}}$. This removes everything except the things that are inside $\mathfrak{p}$. That is, $\operatorname{Spec} R_{\mathfrak{p}}$ is the prime ideals contained in $\mathfrak{p}$.

  For example, look at $\operatorname{Spec} \mathbb{Z}_{(5)}$.[26] This contains (5) and (0), which can be thought of as a point with a little fuzziness around it. The rational numbers are also talking to it; it's not just $\mathbb{F}_5$. Yes, this may be a little weird.

  If we take $\operatorname{Spec} \mathbb{C}[x]_{(x-t)}$, we end up with the point $t \in \mathbb{A}^1_{\mathbb{C}}$ with a bit of fuzziness; it doesn't see any of the other points. But if we take $\operatorname{Spec} \mathbb{C}[x,y]_{(0,0)}$, there's the closed point (x,y) and the generic point (0), but for every irreducible polynomial f with $f(0,0) = 0$, its curve passes through the origin, so we get a nontrivial ideal of $\mathbb{C}[x,y]_{(0,0)}$. That is, geometrically, we also get a piece of each curve through the origin! This is the "local" of localization: ignore everything except what's happening arbitrarily close to 0.

If $\mathfrak{p}$ is prime, then $R_{\mathfrak{p}}$ is always a *local ring*, i.e. it always has a unique maximal ideal (since $\mathfrak{p}$ is the largest ideal contained in $\mathfrak{p}$).

Let's see what this looks like for a prime ideal that isn't maximal. $(x,z) \subset \mathbb{C}[x,y,z]$ is a point which is the y-axis (the generic point of $\operatorname{Spec} \mathbb{C}[y]$). Thus, $\operatorname{Spec} \mathbb{C}[x,y,z]_{(x,z)}$ has this point and the generic point of the plane. However, it will also contain the local data of surfaces intersecting this line. There's interesting ideal structure, but no other maximal ideals: there will be only one closed point.

Next time, we'll add topology to this picture; everything we did today is still true in the continuous world.

[25] More generally, one can write $\operatorname{Spec} S^{-1}R = \bigcap_{f \in S} D(f)$, where S is any multiplicative subset.

[26] The notation $\mathbb{Z}_{(5)}$ denotes localization at (5); this notation is distinct from the 5-adic numbers $\mathbb{Z}_5$, or the integers modulo 5, which are $\mathbb{Z}/5$.

Episode VII.

The Zariski Topology: 2/9/16

Last time, we associated a set $\operatorname{Spec} R$ to a ring R, which was its set of prime ideals. An element $f \in R$ defines a "function" $f(\mathfrak{p}) = f \bmod \mathfrak{p} \in \operatorname{Frac}(R/\mathfrak{p})$. This is useful because we can attach two subsets $D(f) = \{\mathfrak{p} \mid f(\mathfrak{p}) \neq 0\} \subset \operatorname{Spec} R$,[27] and its complement $V(f) = \{\mathfrak{p} \mid f(\mathfrak{p}) = 0\} \subset \operatorname{Spec} R$. Thus, $V(f)$ is the set of prime ideals containing f.

Definition. If $f \in R$, f is called *nilpotent* if $f^n = 0$ for some $n > 0$. The set of nilpotent elements forms an ideal, called the *nilradical*, denoted $\mathfrak{N}(R)$ or $\operatorname{Nil}(R)$.

Exercise 7.1. Suppose $f \in R$. Then, $f(\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \operatorname{Spec} R$ (also written $f \equiv 0$) iff f is nilpotent.

That is, functions aren't determined entirely by their values! But in integral domains, the nilradical is zero. In general, if $f \in \operatorname{Nil}(R)$, then $V(f) = \operatorname{Spec} R$ and $D(f) = \varnothing$.

We're going to define a topology on $\operatorname{Spec} R$, called the Zariski topology, based on these ideas. In topology, if a map $f : X \to \mathbb{R}$ is continuous, then $f^{-1}(0)$ is closed and $f^{-1}(\mathbb{R} \setminus 0)$ is open. We're going to do the same thing here, declaring $D(f)$ open and $V(f)$ closed. But that's not enough to describe a topology; we want to know what all the open (or closed) subsets are.

Last time, we defined an identification $V(f) = \operatorname{Spec}(R/(f))$, and therefore for any ideal $I \subset R$, $\operatorname{Spec} R/I = \bigcap_{f \in I} V(f)$. This is an intersection of closed subsets, so it should be closed. Thus, for any $S \subset R$, we can define

$$V(S) = \{x \in \operatorname{Spec} R \mid f(x) = 0 \text{ for all } f \in S\} = \bigcap_{f \in S} V(f).$$

We will declare these sets to be closed, so now we have lots of closed subsets, and they behave well under intersection.

We can try to do the same thing with opens, but it'll look a little different. For example, if S is multiplicative, then we could define "$D(S)$" to be $\operatorname{Spec} S^{-1}R = \bigcap_{f \in S} D(f)$, since we showed last time that $\{x \mid f(x) \neq 0 \text{ for all } f \in S\}$ is identified with $\operatorname{Spec} S^{-1}R$. The problem is, this is an arbitrary intersection of opens, so it might not be open. This doesn't seem like the right definition.

However, we can take advantage of de Morgan's laws: for any $S \subset R$, $\operatorname{Spec} R \setminus V(S) = \bigcup_{f \in S} D(f)$. This ought to be open, since arbitrary unions of open sets are. This breaks the symmetry between quotients and localization, but that's okay; all of these sets are unions of our basic distinguished open sets.

Definition. The *Zariski topology* on $\operatorname{Spec} R$ is the topology in which a $Z \subset \operatorname{Spec} R$ is closed if $Z = V(S)$ for some $S \subset R$.

Thus, the opens are $\operatorname{Spec} R \setminus V(S)$ for $S \subset R$. If this seems like a strange topology, the takeaway is that *a set is closed iff it's cut out by some equations.*

We can write a closed subset as the intersection of closed sets as the form $V(f)$: $Z \subset \operatorname{Spec} R$ is closed iff there's an $S \subset R$

$$Z = \bigcap_{f \in S} V(f) = \bigcap_{f : f|_Z = 0} V(f).$$

Thus, the *Zariski closure* of any $Z \subset \operatorname{Spec} R$ is just its closure in the Zariski topology:

$$\overline{Z} = \bigcap_{f \in R : f|_Z = 0} V(f).$$

Example 7.2. Suppose $R = \mathbb{C}[x]$, so $\operatorname{Spec} R = \mathbb{A}^1_{\mathbb{C}}$. Suppose $Z \subset \mathbb{A}^1_{\mathbb{C}}$ is any infinite subset; then, $\overline{\mathbb{Z}} = \mathbb{A}^1_{\mathbb{C}}$. Why is this? This is equivalent to saying "suppose I have a polynomial function vanishing on infinitely many points; then, it's equal to zero." Thus, $\overline{Z} = V(0) = \mathbb{A}^1_{\mathbb{C}}$; the only closed sets in this topology are finite. In other words, the open sets are huge!

This example works just as well for $\operatorname{Spec} \mathbb{Z}$: if $n \in \mathbb{Z}$ and $p \mid n$ for infinitely many primes, then $n = 0$. Thus, if $Z \subset \operatorname{Spec} \mathbb{Z}$ is an infinite set, then $\overline{Z} = \operatorname{Spec} \mathbb{Z}$.

[27] Here, D stands for "distinguished," though it also helps to think of it as "doesn't vanish."

The idea that the open sets are huge is true in general, and can be somewhat frustrating: this topology is quite coarse, and sometimes is hard to work with. The closed sets have formulas associated to them, and sometimes are easier to deal with.

Proposition 7.3. *The Zariski topology is indeed a topology.*

Proof. First, we need $\varnothing$ and $\operatorname{Spec} R$ to be both open and closed. They're both closed, as $\varnothing = V(1)$ and $\operatorname{Spec} R = V(0)$, and therefore both open as well.

Next, why are arbitrary unions of open sets open? This is equivalent to arbitrary intersections of closed sets being closed, but since intersections commute with each other,

$$\bigcap_{S\in\mathcal{S}} V(S) = \bigcap_{f\in\bigcup_{S\in\mathcal{S}} S} V(f).$$

Finally, we need finite intersections of opens to be open (or equivalently by induction, that the intersection of two opens is open). This is equivalent to finite unions of closed sets being closed. If $I_1, I_2 \subset R$ are sets, then $V(I_1) = V(\langle I_1\rangle)$ (we can just take the ideal generated by I_1), so we can assume I_1 and I_2 are ideals. Hence, $V(I_1) \cup V(I_2)$ is the set of $x \in \operatorname{Spec} R$ such that $i_1(x) = 0$ for all $i_1 \in I_1$ or $i_2(x) = 0$ for all $i_2 \in I_2$. This is equivalent to $x \in V(I_1 I_2)$, since these are linear combinations of products of ideals in I_1 and I_2. Thus, $V(I_1) \cup V(I_2) = V(I_1 I_2)$, which is closed, so we're happy. ⊠

Now, we can reinterpret the setwise constructions we made last week in terms of this topology. If $\mathfrak{p} \subset R$ is prime, then $\operatorname{Spec} R_\mathfrak{p} \subset \operatorname{Spec} R$, as we talked about, and the image is $\operatorname{Spec} R_\mathfrak{p} = \bigcap_{f(\mathfrak{p})\neq 0} D(f)$, i.e. this intersection is over $f \in R \setminus \mathfrak{p}$.

Zooming out a little, this topology realizes the next step in our dream: it plays nicely with the functor $\operatorname{Spec} : \mathsf{Ring}^{\mathsf{op}} \to \mathsf{Set}$, and thus provides it with more structure. More precisely:

Exercise 7.4. Spec is actually a functor $\mathsf{Ring}^{\mathsf{op}} \to \mathsf{Top}$, i.e. if $\phi : R \to T$ is a ring homomorphism, then the induced $T : \operatorname{Spec} T \to \operatorname{Spec} R$ is continuous.

Since we're defining Φ to be the same underlying map of sets, these two visions of the Spec functor commute with the forgetful functor $\mathsf{Top} \to \mathsf{Set}$.

The intuition behind why this map is continuous is that it acts as a pullback on "functions" (elements of R); the proof that shows pullbacks are continuous on manifolds provides intuition for the proof here.

Now, we have more structure, but not enough for Spec to be an equivalence (for example, $\operatorname{Spec} \mathbb{R}$ and $\operatorname{Spec} \mathbb{C}$ are both points, but $\mathbb{R} \ncong \mathbb{C}$ as rings).

Now, let's talk about one of the weirdnesses of the Zariski topology.

Claim. Let R be an integral domain. Then, the generic point $(0) \in \operatorname{Spec} R$ is dense.

Proof. $\overline{(0)}$ is the intersection of all $r \in R$ with $r \bmod 0 = 0$, i.e. of only $V(0)$.But $V(0) = \operatorname{Spec} R$, so $\overline{(0)} = \operatorname{Spec} R$. ⊠

Corollary 7.5. *Let $\mathfrak{p} \in \operatorname{Spec} R$. Then, $\overline{\mathfrak{p}} = \operatorname{Spec} R/\mathfrak{p}$ as a subset of $\operatorname{Spec} R$.*

The idea is that $R/\mathfrak{p}$ is an integral domain, so (0) is dense in $\operatorname{Spec} R/\mathfrak{p}$, and $(0) \in \operatorname{Spec} R/\mathfrak{p}$ maps to $\mathfrak{p}$ in $\operatorname{Spec} R$; since the inclusion $\operatorname{Spec} R/\mathfrak{p} \hookrightarrow \operatorname{Spec} R$ is continuous (which remains to be checked), it sends closures to closures.

Thus, prime ideals $\mathfrak{p}$ correspond to points, but can also be thought of the subschemes $R/\mathfrak{p}$.

Corollary 7.6. *A point $\mathfrak{p} \in \operatorname{Spec} R$ is closed iff it's a maximal ideal.*

The maximal ideals are the "usual" points that we're used to; closed points behave more like our intuition. All the new points, non-maximal prime ideals, are generic points of subschemes.

Another way to say this is that there's a partial order relation on points defined by inclusion. This is again unlike our usual geometric intuition.

Definition. Let $x \leftrightarrow \mathfrak{p}$ and $y \leftrightarrow \mathfrak{q}$ be points in $\operatorname{Spec} R$. Then, x is a *specialization* of y if $x \in \overline{y}$, i.e. $\mathfrak{q} \subset \mathfrak{p}$, and y is called a *generization* of x.

The minimal elements of this poset are the closed points, which correspond to the maximal ideals.

For example, in $\operatorname{Spec}\mathbb{Z}$, the generic point (0) corresponds to the image of $\operatorname{Spec}\mathbb{Q}$ induced by $\mathbb{Z} \hookrightarrow \mathbb{Q}$. It's dense, but it's not open: $\operatorname{Spec}\mathbb{Z} \setminus (0)$ is an infinie subset, and we saw above that infinite subsets aren't closed. Nonetheless, it's still Spec of a localization (the field of fractions). The takeaway is that Spec of a localization is not open in general.

Another useful fact is that the distinguished opens $D(f)$ form a basis for the Zariski topology, i.e. any open $U \subset \operatorname{Spec} R$ is a union of these open sets. This is because $U = \operatorname{Spec} R \setminus V(I)$ for some ideal $I \subset R$. Hence $\operatorname{Spec} R \setminus U = \bigcap_{f\in I} V(f)$, and hence $U = \bigcup_{f\in I} D(f)$. This is more of a tongue-twister than a proof, but it all comes down to complements of intersections becoming unions of complements.

Claim. If $f, g \in R$, then $D(f) \subset D(g)$ iff $f^n \in (g)$ for some $n > 0$.

Proof. Again, we'll unwind definitions. If $D(f) \subset D(g)$, then $f|_{V(g)} = 0$, so $f \bmod (g)$ vanishes everywhere on $\operatorname{Spec} R/(g)$. Thus, $f \in \operatorname{Nil}(R/(g))$, so $f^n \equiv 0 \bmod g$ for some n. That is, $f^n \in (g)$. The converse is analogous. ⊠

This is a nice bridge between algebra and geometry.

So we have a basis of distinguished opens and an inclusion relation between them. We also have a nice property about coverings.

Exercise 7.7. Show that if $S \subset R$, then $\operatorname{Spec} R = \bigcup_{f\in S} D(f)$ iff $(S) = R$.

That is, covering corresponds to that set generating the whole ring. But $(S) = R$ has a nice concrete meaning: there are $f_i \in S$ and $a_i \in R$ such that $\sum a_i s_i = 1$. This is necessarily a finite sum, because generating an ideal means only taking finite linear combinations. Thus, we get a curious corollary.

Corollary 7.8. $\operatorname{Spec} R$ *is quasicompact.*[28]

The proof is to replace an arbitrary cover by the finite combination that sums to 1.

There's no reasonable sense in which a scheme is compact, and certainly we don't want something like $\mathbb{A}^n_{\mathbb{C}}$ to be compact. The idea is that open sets are huge: in $\mathbb{A}^1_{\mathbb{C}}$, for example, any nonempty open is dense, and therefore any two nonempty opens have an intersection that's also dense in $\mathbb{A}^1_{\mathbb{C}}$! In other words, the Zariski topology is very far from Hausdorff.

We'd like to have notions that are compactness and Hausdorfness but for schemes, but the usual ones don't work. We'll define analogous notions (e.g. "separatedness" for the Hausdorff property), but until then, we have to be careful with keeping all the words right in the dictionary between usual geometry and algebraic geometry.

Example 7.9. If k is a field, then *infinite-dimensional affine space* over k is $X = \operatorname{Spec} k[x_1, x_2, \dots] = \mathbb{A}^\infty_k$. Then, $\mathbf{0} = V(x_1, x_2, \dots)$, so $X \setminus \mathbf{0}$ is open, but not quasicompact: the cover $X \setminus \mathbf{0} = \bigcup_{i\in\mathbb{N}} D(x_i)$ has no finite subcover. Thus, $X \setminus \mathbf{0}$ is not $\operatorname{Spec} T$ for any ring T! We'll eventually see how this is a scheme, but it isn't an affine scheme.

There are easier examples of schemes which aren't affine: the functions on $\mathbb{A}^2_{\mathbb{C}} \setminus 0$ are just $\mathbb{C}[x, y]$, which are the functions on $\mathbb{A}^2$, but saying this rigorously requires more work.

We can also talk about connectedness: when can we write $\operatorname{Spec} R = X \amalg Y$ for open and closed X and Y? The intuition is that we will be able to work in the opposite category: $R = S \times T$ iff $\operatorname{Spec} TR = \operatorname{Spec} S \amalg \operatorname{Spec} T$.

But even before that, decomposing as $X \amalg Y$ must mean there are functions i_X, i_Y on $\operatorname{Spec} R$ such that $i_X|_X = 1$, $i_Y|_X = 0$, and vice versa for Y. Thus, $i_X^2 = i_X$ and $i_X i_Y = 0$, so they're *orthogonal idempotents*; hence, connectedness corresponds to (not) finding orthogonal idempotents in R.

Episode VIII.

Connectedness, Irreducibility, and the Noetherian Condition: 2/11/16

"I just can't read my own handwriting."

[28] This just means "every open cover has a finite subcover;" in this schema, compactness is reserved for Hausdorff spaces, and we use quasicompactness to make the distinction clearer.

Today, we'll talk about properties of the Zariski topology and its relation to the structure of rings; next week, we'll cover a little general nonsense about sheaf theory, and finally get to define schemes.

Connectedness. One of the most basic questions one can ask about a topological space is whether it's connected. It turns out that for the Zariski topology, connectedness correlates very nicely with an algebraic property.

Definition. If R is a ring, then $i \in R$ is an *idempotent* if $i^2 = i$.

These are akin to projectors, and indeed, we have a complementary projector: $(1-i)^2 = 1 - 2i + i = 1 - i$, so $i^\perp = 1 - i$ is also an idempotent. Since R is commutative, then i and $i^\perp$ commute, and $i + i^\perp = 1$. Moreover, i and $i^\perp$ are *orthogonal idempotents*, in the sense that $ii^\perp = i - i = 0$.

Since these add to 1, $r = ri + ri^\perp$ for any $r \in R$, and so $R \cong iR \times i^\perp R$. Thus, the prime ideals in R are the disjoint union of the ones in iR and $i^\perp R$. Thus, $\operatorname{Spec} R = \operatorname{Spec} iR \amalg \operatorname{Spec} i^\perp R$, and this is true as topological spaces. Since $\operatorname{Spec} iR = V(i^\perp)$ and $\operatorname{Spec} i^\perp R = V(i)$, these are both clopen sets, and therefore $\operatorname{Spec} R$ isn't connected.

Conversely, we will be able to prove that if $\operatorname{Spec} R = X \amalg Y$ as topological spaces, there's an idempotent i_X which is valued 1 on X and 0 on Y, and in fact this lies in R, and so a decomposition $R = i_X R \times i_X^\perp R$. However, we don't have the techniques to prove this yet. Hence, $\operatorname{Spec} R$ is connected iff R has no idempotents.

In representation theory, one often studies associative algebras, such as the group algebra $k[G]$ associated to a finite group G over a field k. Inside A, there's a commutative ring, its center $Z(A)$. Wedderburn's theorem states that the idempotents $i \in Z(A)$, called *central idempotents*, are in bijection with the irreducible representations of G over k. Thus, commutative algebra is useful even in non-commutative algebra.

Irreducibility. The notion of irreducibility is one that doesn't come up in ordinary geometry.

Definition. A topological space X is *irreducible* if you can't write $X = Z_1 \cup Z_2$ for proper closed subsets $Z_1, Z_2 \subset X$.

In Euclidean geometry, this is absurd: consider the upper half-plane (including the x-axis) and the lower half-plane (including the x-axis) for $\mathbb{R}^2$. But the Zariski topology encodes things differently: it's a way of encoding algebraic structure on a space.

Suppose R is an integral domain. Then, there's a generic point $(0) \in \operatorname{Spec} R$ which is dense. Suppose $\operatorname{Spec} R = Z_1 \cup Z_2$ for proper closed subsets Z_1 and Z_2; then, (0) is in one of them, and so its closure is too (since they're both closed). However, that's the whole space, so one of them isn't a proper subset. Thus, if R is an integral domain, $\operatorname{Spec} R$ is irreducible.

Conversely, suppose R is any ring, and $\operatorname{Spec} R = X = Z_1 \cup Z_2$, for proper, closed subsets Z_1 and Z_2. Then, there exist functions $f_1, f_2 \in R$ such that $f_i|_{Z_i} = 0$ and $f_i \neq 0$, meaning neither f_i is nilpotent. However, $f_1 f_2 \equiv 0$, so it is nilpotent, and so $(f_1 f_2)^N = 0$. In particular, R is not an integral domain.

We must be careful, because this is not an if and only if.

Example 8.1 (Dual numbers)**.** We're going to get quite acquainted with the *dual numbers*, the ring $k[\varepsilon]/(\varepsilon^2)$, for a field k. Thus, ε is a nilpotent, so this ring isn't an integral domain. Then, $\operatorname{Spec} k[\varepsilon]/\varepsilon^2$ is a point, as is $\operatorname{Spec} k$, and so this is certainly irreducible! So integral domain implies irreducible, but not vice versa.

This ring will show up as a useful example because it's a simple example of how nilpotents work.

Another corollary is that $\operatorname{Spec} R$ as a topological space is insensitive to nilpotents in R, since nilpotents as functions are identically 0, and they don't affect the space of prime ideals. So we'll have to put a stronger structure on $\operatorname{Spec} R$ to distinguish these two rings.

Definition. A ring R is *reduced* if it has no nonzero nilpotents, i.e. $\operatorname{Nil}(R) = 0$.

Now, if we have a reduced ring R and $f, g \in R$ with $fg = 0$, $f, g \neq 0$, then $\operatorname{Spec} R = V(f) \cup V(g)$, and this is a proper decomposition (there are no nonzero nilpotents, so there are places where f and g don't vanish). The point is, *an integral domain corresponds to being reduced and irreducible*.

Recall that we have a dictionary between algebra and geometry: given an ideal $I \subset R$, there's a closed subset $V(I) \subset \operatorname{Spec} R$. Correspondingly, if $S \subset \operatorname{Spec} R$ is any subset, then the set $I(S) = \{r : r|_S \neq 0\}$ is an ideal of R. Then, $V(I(S)) = \bigcap_{f|_S=0} V(f)$, so this is just the Zariski closure of S in $\operatorname{Spec} R$.

Correspondingly, $I(R(J)) = \sqrt{J} = \{r \in R : r^N \in J, N \gg 0\}$, because if $r|_{V(J)} = 0$, then $r \bmod J$ is identically 0 on $\operatorname{Spec} R/J$, i.e. it's nilpotent in R/J (so $r^N \in J$ for some N). If $J = \sqrt{J}$, then J is called *radical*.

This correspondence thus isn't perfectly bijective, but it's a nice dictionary. In particular, we've proven the following theorem.

Theorem 8.2. *These functions I and V provide a bijection between the closed subsets of* $\operatorname{Spec} R$ *and the radical ideals of R.*

Under this bijection, prime ideals correspond to points, but also to irreducible subsets (the closure of its generic point). To be precise, $\operatorname{Spec} R/I$ is closed in $\operatorname{Spec} R$, and I is radical iff R/I is reduced. It's also irreducible iff I is prime, but we saw that both of these are equivalent to R/I being an integral domain. This can be useful: a single point can be used to understand an entire irreducible subset, which is quite precise.

For example, in $\mathbb{A}^2_{\mathbb{C}}$, irreducible subsets are in bijection with prime ideals, and irreducible polynomials give us prime ideals. However, the union of the coordinate axes is not irreducible (it's the union of the x-axis and the y-axis).

An irreducible set is automatically connected (and this translates to an algebraic statement, too), so disconnected subsets are reducible. What is this good for? Well, let $Y = \operatorname{Spec} R/I$; I want to understand this better, and so want to write this as a union of irreducible components $Y_1 \cup \cdots \cup Y_n$, so $Y_i = \operatorname{Spec} R/\mathfrak{p}_i$. Algebraically, if R is reduced, this means writing $I = \bigcap_{i=1}^n \mathfrak{p}_i$. The takeaway is that we could understand any ideal in terms of prime ideals.

This is not true in general: first of all, Y may not have a finite number of irreducible components, nor I be a finite intersection of prime ideals. For example, in infinite-dimensional affine space $\mathbb{A}^\infty_k = \operatorname{Spec} k[x_1, x_2, \dots]$, consider the set Y that's the union of all of the coordinate axes. This does not satisfy this finiteness condition, and we'd like a word for the condition that does. That word is Noetherian.

The Noetherian Condition. This should be thought of as a "finite-dimensionality." Dimension is weird enough in ordinary topology, but the weirdness of the Zariski topology allows finite-dimensionality to be easily defined.

Definition. A topological space X is *Noetherian* if it satisfies the *descending chain condition* (DCC): if $X \supset Z_1 \supset Z_2 \supset \cdots$ is an infinite sequence of closed subsets of X, then it eventually stabilizes. In other words, there's some n such that $Z_n = Z_{n+1} = Z_{n+1} = \cdots$.

This is a funny condition when one first sees it; just like irreducibility, it doesn't arise in ordinary topology. Any shrinking sequence of neighborhoods on a manifold shows that it's not Noetherian (and in the same way, there are infinite ascending chains). And these two notions are related.

Proposition 8.3. *If X is a Noetherian topological space, then any closed $Y \subset X$ can be written as a finite union of irreducibles $Y = Y_1 \cup \cdots \cup Y_n$. If we additionally specify that no Y_i contains any other Y_j, then this decomposition is unique.*

Proof. We'll prove this using a technique called *Noetherian induction*, which we'll use again.

Let S denote the set of closed subsets of X not admitting such a description; we would like to show that S is empty. If S is nonempty, we'll show there's a minimal element of S with respect to inclusion, and use it to derive a contradiction.

Suppose $Y_1 \in S$ is not minimal, then pick a $Y_2 \subset Y_1$ in S, and repeat this argument: we get a chain $Y_1 \supset Y_2 \supset Y_3 \supset \cdots$, so the Noetherian condition guarantees this stabilizes at some $Y_n = Y$, and this must be a minimal element for S.

Since Y doesn't have a decomposition into a finite number of irreducibles, then it's not irreducible, and so $Y = W \cup Z$ for proper closed subsets $W, Z \subset Y$. But since Y is minimal, then $W, Z \notin S$, so $W = W_1, \dots, W_n$ and $Z = Z_1, \dots, Z_m$ are decompositions into irreducibles. Hence, taking the union of these two, we have a finite decomposition into irreducibles for Y, which is a contradiction.

The uniqueness is pretty easy: it's very much like the uniqueness of prime factorization, which is not a coincidence. Suppose $Y = Z_1 \cup \cdots \cup Z_m = W_1 \cup \cdots \cup W_n$ are two decompositions, where no Z_i contains a Z_j, and the same for the W_i. Since $Z_1 \subset W_1 \cup \cdots \cup W_n$, then $Z_1 = (Z_1 \cap W_1) \cup \cdots \cup (Z_1 \cap W_n)$. Thus, since Z_1 is irreducible, one of these, without loss of generality $Z_1 \cap W_1$, is equal to Z_1. Thus, $Z_1 \subset W_1$, and with the Z_i and W_i switched, we have $W_1 \subset Z_2$, so $Z_1 \subset Z_2$, which we assumed was not the case unless $Z_1 = W_1$ (and then induction takes care of the rest). ⊠

The Noetherian condition arose first on rings.

Definition.

- A ring R is *Noetherian* if it satisfies the *ascending chain condition* (ACC) on ideals: if $I_1 \subset I_2 \subset I_3 \subset \cdots$ are all ideals of R, then there's some n such that $I_n = I_{n+1} = \cdots$.
- R is *Artinian* if it satisfies the descending chain condition on ideals.

One could correspondingly define Artinian topological spaces, which would have to satisfy the ascending chain condition on closed subsets. But very few spaces are Artinian (e.g. $\operatorname{Spec}\mathbb{Z}$ isn't); it suggests that the space is finite.

Exercise 8.4. Show that if R is Noetherian, then $\operatorname{Spec} R$ is a Noetherian space.

The converse is very false, since Spec doesn't detect nilpotents, so come up with a ring with an infinite ascending chain of nilpotents and you'll have a counterexample.

Lots of nice rings (or spaces) are Noetherian: all fields are, as is $\mathbb{Z}$, and quotients and localizations preserve the Noetherian condition. And there's a cleaner way to check the definition.

Exercise 8.5. Show that the following are equivalent.

(1) R is a Noetherian ring.
(2) Every ideal of R is finitely generated.
(3) For any closed $Z \subset R$ given by $Z = \operatorname{Spec} R/I$, Z is an intersection of finitely many $V(f)$ for $f \in R$.

So Noetherianness is a quite strong finite condition.

Theorem 8.6 (Hilbert basis theorem)**.** *If R is Noetherian, then so is $R[x]$.*

Combined with quotients and localizations, this is equivalent to the statement that if R is a ring, then all finitely generated R-algebras are Noetherian. This is a quite powerful statement: there are lots and lots of Noetherian rings, including pretty much any ring that one usually thinks about.

And we also get back the statement we were looking for when we defined this: if R is a Noetherian ring and $I \subset R$ is a radical ideal, then there are ideals $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$ for $\mathfrak{p}_j$ prime, or every closed subset is a finite union of irreducibles.

Proof of Theorem 8.6. Let $I \subset R[x]$ be nonzero; we'll show I is finitely generated. Pick an $f_1 \in I$ of minimal degree; then, pick an $f_2 \in I \setminus (f_1)$ of minimal degree, and $f_3 \in I \setminus (f_1, f_2)$ of minimal degree, and so forth.

Thus, we have the chain $(f_1) \subset (f_1, f_2) \subset (f_1, f_2, f_3) \subset \cdots$ in I. We will show it terminates, which means I is finitely generated, but we don't yet know $R[x]$ is Noetherian. Since we do know this for R, let a_i be the leading coefficient of f_i. Thus, we have a chain of ideals in R: $(a_1) \subset (a_1, a_2) \subset (a_1, a_2, a_3) \subset \cdots$. Since R is Noetherian, this stabilizes, so there's an n such that $a_n = \sum_{i<n} r_i a_i$ for some $r_i \in R$.

Thus, $f_n = a_n x^{\deg f_n} + \cdots$, so we can peel off the highest-degree term:

$$g = f_n - \sum_{i<n} r_i f_i x^{\deg f_n - \deg f_i}.$$

This is in I, but it's not contained in $(f_1, \ldots, f_{n-1})$, since then f_n would be (and we assumed the chain doesn't stabilize). However, it has lower degree than f_n does, and we assumed it was minimal, giving us a contradiction. ⊠

This reduction of polynomials to their coefficient rings is probably the same trick used to prove that if R is a UFD, then so is $R[x]$.

Unless you care about infinite-dimensional things, you probably won't ever have to worry about non-Noetherian spaces or rings.

As we'll see, the Zariski topology is somewhat weird, and encodes the algebra of a ring, but it doesn't pick up the geometry. It's a pictorial summary of the algebra; once we introduce some geometry, the geometry we get is much more like complex geometry.

Episode IX.

Revenge of the Sheaf: 2/16/16

This week, we're going to provide the last ingredient in the definition of an affine scheme: its sheaf of functions. To do that, we'll have to define sheaves abstractly.

Sheaves formalize the idea that functions are local: if X is a topological space, we consider functions $f : X \to \mathbb{R}$ (or to another topological space Y). This "locality" means the following things:

- If $U \subset X$ is open, we can restrict f to $f|_U$, which is a function on U. If $V \subset U$ is another open, restriction composes: $(f|_U)|_V) = f|_V$. This will generalize to the notion of a presheaf.
- If $X = U \cap V$, where U and V are open subsets, then we can glue functions that agree on the overlaps. In other words, if $\mathscr{F}(X)$ is the functions on X, then restriction gives us an injective map $\mathscr{F}(X) \to \mathscr{F}(U) \times \mathscr{F}(V)$, and the image is exactly the functions agreeing on $U \cap V$. This will generalize to a sheaf.

To define these formally, fix a category C; if it helps to be concrete, C will almost always be Set, Ab, or Ring for this class. Recall that if X is a topological space, then $\mathsf{Top}(X)$ is the category of open subsets in X, interpreted as a poset under inclusion.

Definition. The *category of presheaves* on X, $\mathsf{C}_X^{\mathrm{pre}}$ is the category whose objects are functors $\mathsf{Top}(X)^{\mathsf{op}} \to \mathsf{C}$ and whose morphisms are their natural transformations.

What does this actually mean? If $\mathscr{F}$ is a presheaf, then to any open $U \subset X$, we have its *sections* on U, $\mathscr{F}(U) \in \mathsf{C}$, and composition of morphisms means that if $W \subset V \subset U \subset X$ are open sets, then we have *restrictions maps* that commute: if res_U^V denotes restriction from U to V, then $\mathrm{res}_U^W = \mathrm{res}_V^W \circ \mathrm{res}_U^V$.

A morphism of sheaves is a natural transformation $\Phi : \mathscr{F} \to \mathscr{G}$, i.e. for all $V \subset U$ as opens of X, there's a commutative diagram of maps in C:

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\Phi(U)"] \arrow[d, "\mathrm{res}_U^V"'] & \mathscr{G}(U) \arrow[d, "\mathrm{res}_U^V"] \\
\mathscr{F}(V) \arrow[r, "\Phi(V)"] & \mathscr{G}(V).
\end{tikzcd}$$

For example, if X is a topological space, the continuous, real-valued functions $C(X;\mathbb{R})$ form a presheaf, by the assignment $U \mapsto C(U;\mathbb{R})$.

Now, we'd like to extract sheaves from this, by adding a *descent* (or *locality* or *gluing* or sheaf) axiom.

Definition. The category C_X of C-*valued sheaves* on X is the full subcategory[29] satisfying the *sheaf axiom*: let $U \subset X$ be open and $\mathfrak{U} = \{U_i\}_{i\in I}$ be an open cover of U (so U is the union of the U_i). For any $i, j \in I$, let $U_{ij} = U_i \cap U_j$; then, we require the diagram

$$\mathscr{F}(U) \xrightarrow{\prod \mathrm{res}_U^{U_i}} \prod_{i\in I} \mathscr{F}(U_i) \rightrightarrows \prod_{i,j\in I} \mathscr{F}(U_{ij}) \tag{9.1}$$

is an equalizer diagram.

Well, that was compact; let's unpack it. Suppose $U = U_1 \cap U_2$; then, $U = U_1 \amalg_{U_{12}} U_2$ in $\mathsf{Top}(X)$, which means the diagram

$$U_{12} \rightrightarrows U_1 \amalg U_2 \longrightarrow U$$

is a *co*equalizer diagram.[30] Because a presheaf is a contravariant functor, we'd like it to turn the coequalizer diagram that encodes an open cover into an equalizer diagram.

[29] "Sheaves are a full subcategory of presheaves" means that every sheaf is a presheaf, and the morphisms are the same.

[30] Something unusual is going on, because, strictly speaking, $U_1 \amalg U_2$ is not in $\mathsf{Top}(X)$. This diagram actually lives in Top. Gorthendieck reformulated a lot of this by recasting these as maps to your space, rather than subsets... but that's a story for another day.

What this actually means is that if I have objects $f_i \in \mathscr{F}(U_i)$ for each i, such that the restrictions all agree (so $\mathrm{res}^{U_{ij}}_{U_i} f_i = \mathrm{res}^{U_{ij}}_{U_j} f_j$ for all $i, j \in I$), then there exists a unique $f \in \mathscr{F}(X)$ such that $f_i = \mathrm{res}^{U_i}_{U} f$. That is, we can glue sheaves on an open cover, and can do so uniquely.

Example 9.2. Many kinds of functions on your space will form sheaves, e.g. C^∞ (smooth functions), C^ω (analytic functions), continuous functions. For any set Y, maps from X to Y form a sheaf in the category of sets.

In the future, when we say something is local, we will mean that it forms a sheaf: its values on an open cover determine it globally.

Example 9.3 (Skyscraper sheaf)**.** We can also define sheaves which don't quite look like functions. Let C be a category with a terminal object $*$ (e.g. Set), and $S \in \mathsf{C}$. Pick an $x \in X$, and define the *skyscraper sheaf* $i_{x,*}S$ by

$$i_{x,*}S(U) = \begin{cases} S, & x \in U \\ *, & x \notin U. \end{cases}$$

This is a presheaf, because this definition plays well with restriction: if $V \subset U$, then either $x \notin U$ (so we restrict $* \to *$), $x \in V$ (so we restrict $S \to S$), or $x \in U \setminus V$ (so we restrict $S \to *$), which is fine. And it's a sheaf, because if we have an open cover for an open $U \subset X$, then U contains x iff its cover does, so everything works out. The name comes from a picture where S is stacked up at x, and there's nothing anywhere else.

Example 9.4 (Constant presheaves and sheaves)**.** Let X be a space that contains two disjoint open subsets U_1 and U_2 (e.g. any nontrivial Hausdorff space). If S is a set, we can define the *constant presheaf* with value S by defining $\mathscr{F}(U) = S$ for all $U \subset X$, and the restriction maps to be the identity; this commutes with taking sections, and therefore is a presheaf.

However, this $\mathscr{F}$ is *not* a sheaf: we can pick distinct sections $s_1, s_2 \in S$, regarding $s_1 \in \mathscr{F}(U_1)$ and $s_2 \in \mathscr{F}(U_2)$. Since U_1 and U_2 are disjoint, then these have to come from a single section on $U_1 \cap U_2$ (they vacuously agree on the intersection), but the restriction maps are all the identity, so there's no way to do this.

However, we can tweak this into a sheaf. Now, for any $S \in \mathsf{Set}$, endow S with the discrete topology, and let $\underline{S}(U) = \mathrm{Hom}_{\mathsf{Top}}(U, S)$. These are continuous functions, and therefore form a sheaf $\underline{S}$, called the *constant sheaf*. Since S is totally disconnected, each map from a connected subset factors through a single point of S, and therefore the issue that the constant presheaf had doesn't arise.[31]

Example 9.5 (Sheaf of sections)**.** Let $\pi : Y \to X$ be a continuous map. Then, the *sheaf of sections* of π is defined by $\mathscr{F}(U)$ to be the sections of the map $\pi^{-1}(U) \to U$, i.e. continuous maps $s : U \to \pi^{-1}(U)$ such that $\pi \circ s = \mathrm{id}$. We can restrict sections, so this is a presheaf, but in fact sections are always a sheaf: if two sections agree on their overlap, they can be patched. That is, sections are local information. The codomain of this sheaf varies as U varies, which is unlike the previous examples.

If $Y = X \times T$, and π is projecting onto the first factor, then sections of π are just maps $X \to T$ (regarded as its graph); in other words, the sheaf of sections generalizes the sheaf of maps. In fact, we'll see later that any sheaf can be regarded in this way: sections are actually sections.

Another good example is when $\pi : Y \to X$ is a covering space with fiber Γ (e.g. $\mathbb{R} \to S^1$ with fiber $\mathbb{Z}$); then, the sections of the covering map on a sufficiently small $U \subset X$ are the same thing as maps $U \to \Gamma$, because $\pi^{-1}(U) \cong U \times \Gamma$ for small enough U. Since Γ has the discrete topology, this means that for these small U, $\mathscr{F}(U) = \underline{\Gamma}(U)$: on small enough open sets, it looks like the constant sheaf. Geometrically, this means that a section of a covering map is a choice of one of the sheets along with the inverse of the projection. However, globally, we can't map $S^1 \to \mathbb{R}$ as a section of the covering map.

Every locally constant sheaf arises from a covering space in this way, though the definition of "covering space" may need to be expanded.

[31] Again, this is a little silly with the Zariski topology, as nonempty opens all intersect. Grothendieck resolved this by defining finer topologies on schemes; we'll just not deal with constant sheaves on the Zariski topology.

Definition. Let $U \subset X$ be open. Then, there's a functor $\mathsf{C}_X \to \mathsf{C}_U$, called *restriction* (of sheaves) that sends a sheaf $\mathscr{F}$ to the sheaf $\mathscr{F}|_U$ whose value at a $V \subset U$ is $\mathscr{F}(V)$. In exactly the same way, we can define restriction of presheaves.

This makes sense: all we do is forget about the opens not contained in U. And you can check this is functorial.

This allows us to formalize the covering example just above into an extremely useful class of sheaves.

Definition. A sheaf $\mathscr{F}$ is *locally constant* if there's an open cover $\mathfrak{U}$ of X such that for every $U \in \mathfrak{U}$, $\mathscr{F}|_U \cong \underline{S}_U$ is a constant sheaf.

Another perspective is that sheaves measure twisting: we know what local data looks like, and the sheaf tells us how these are twisted and glued together to obtain the total data. One interesting way

Definition. If $\mathscr{F}$ is a (pre)sheaf, its *global sections* $\Gamma(\mathscr{F}) = \Gamma(X, \mathscr{F})$ are just $\mathscr{F}(X) = \mathsf{C}$. Taking global sections defines a functor $\Gamma : \mathsf{C}_X \to \mathsf{C}$.

To be precise, the global sections of a sheaf are sections on an open subset that agree on overlaps, for any open cover (e.g. X itself is an open cover). If $\mathscr{F} \in \mathsf{Set}_X$, so F is a sheaf of sets, we can also write $\Gamma(\mathscr{F}) = \mathrm{Hom}_{\mathsf{Set}_X}(\underline{*}, X)$. Here, $\underline{*}$ is the constant sheaf valued in a point. The idea is that a map of sheaves is the data of a map $* \to \mathscr{F}(U)$ for each open $U \subset X$, i.e. a collection of $f_U \in \mathscr{F}(U)$ which agree on overlaps, which is exactly the data we needed. This may be confusing, but is sometimes useful: we know Hom commutes with limits, so $\Gamma : \mathsf{Set}_X \to \mathsf{Set}$ preserves limits! This is a common theme: if you can write a construction as an adjoint or an instance of Hom or a limit, you already know a bunch of properties of it.

Global sections are an example of something more general: sheaves propagate from one space to another.

Definition. Let $\pi : Y \to X$ be continuous, and $\mathscr{F}$ be a sheaf (resp. presheaf) on Y. Then, we can define its *pushforward* $\pi_*\mathscr{F}$, which is a sheaf (resp. presheaf) on X, by $\pi_*\mathscr{F}(U) = \mathscr{F}(\pi^{-1}(U))$. Since π^{-1} commutes with restriction, this is a presheaf, and if $\mathscr{F}$ is a sheaf, then a cover of U pulls back under π^{-1} to a cover of its preimage, so we can glue on $\pi^{-1}(U)$ by elements of π^{-1} of its covers.

This very useful operation on sheaves defines a functor $\pi_* : \mathsf{C}_Y \to \mathsf{C}_X$, which does need to be checked: if one has a map of sheaves $\mathscr{F} \to \mathscr{G}$ and $V \subset U$, does the following diagram commute?

$$\begin{tikzcd}
\mathscr{F}(\pi^{-1}(U)) \arrow[r] \arrow[d] & \mathscr{G}(\pi^{-1}(U)) \arrow[d] \\
\mathscr{F}(\pi^{-1}(V)) \arrow[r] & \mathscr{G}(\pi^{-1}(V))
\end{tikzcd}$$

This generalizes several things we've already seen.

- Global sections are a special case of pushforward: for any topological space X, there's a unique map $\pi : X \to *$, and $\pi_* : \mathsf{C}_X \to \mathsf{C}_* = \mathsf{C}$ just takes sections that agree on all of X, i.e. $\Gamma(\mathscr{F})$.
- Skyscraper sheaves are also pushforwards: consider the map $i_x : * \hookrightarrow X$ sending $* \mapsto x$. Then, $i_{x,*} : \mathsf{C} \to \mathsf{C}_X$ is the pushforward of the constant sheaf S on $*$, as $i_{x,*}S(U) = S(i_x^{-1}(U))$, which agrees with its definition in Example 9.3.
- The sheaf of sections can also be realized in this way: if $\pi : Y \to X$ is continuous, let $\underline{*}$ denote the constant sheaf on Y; then, the sheaf of sections is just $\pi_*(\underline{*})$: for any open $V \subset Y$, $\underline{*}(V) = *$, so for any $U \subset X$, $\pi_*(\underline{*})(U) = \pi^{-1}(U)$.

We want to write any sheaf $\mathscr{F}$ as a sheaf of sections of a map $\pi : Y \to X$, and we'll do this by building Y out of the stalks of $\mathscr{F}$.

Definition. Let $\mathscr{F}$ be a (pre)sheaf and $x \in X$. Then, the *stalk* of $\mathscr{F}$ at x, $\mathscr{F}_x \in \mathsf{C}$, is the object of sections of $\mathscr{F}$ on some open subset containing X: any two neighborhoods of x intersect in a smaller neighborhood, and we would like to identify sections that agree on the intersection. If we had a minimal neighborhood of x, that would be where the stalk takes its sections, but instead we do the next best thing.

To be precise, the stalk is $\mathscr{F}_x = \varinjlim_{x \in U} \mathscr{F}(U)$. What does this mean? We have a poset of opens containing X, and if $x \in V \subset U$, then restrictions $\mathscr{F}(U) \to \mathscr{F}(V)$ define a filtered system, so we're just taking the filtered colimit, which tries to be the minimal element.

Since we worked out filtered colimits, we can write this as the quotient

$$\mathscr{F}_x = \varinjlim_{x\in U} \mathscr{F}(U) = \{f_U \in \mathscr{F}(U) \text{ where } x \in U\}/\Big(f_U \sim f_V \text{ if } \operatorname{res}^W_U f_U = \operatorname{res}^W_V f_V \text{ for some } W \subset U \cap V\Big).$$

If $\mathscr{F}$ is the shead of C^∞ (or similar) functions, then its stalks $\mathscr{F}_x$ are the *germs* of functions at x: smooth functions on neighborhoods of x, where we identify functions that agree on a neighborhood of x. Interestingly, if $\mathscr{F}$ is the sheaf of holomorphic functions on C, then by analytic continuation, $\mathscr{F}_0$ is the ring of Taylor series with nonzero radii of convergence.

Not all filtered colimits exist, but in the categories we'll care about (sets, abelian groups, rings, and such), all filtered colimits exist and are fairly well-behaved.

Episode X.

Revenge of the Sheaf, II: 2/18/16

"Ravi says some people swear by [the espace étalé]. I haven't met them."

Today, we're going to talk more about sheaves. Recall that these generalize the notion of functions on a space. If X is a topological space and C is a category, a C-valued sheaf is an association of an object of C called $F(U)$ to every open $U \subset X$, and with restriction maps $F(U) \to F(V)$ when $V \subset U$, compatible with gluing across intersecting opens. (For concreteness, you can think of everything using $\mathsf{C} = \mathsf{Set}$.)

The most common example is the sheaf of functions, $U \mapsto \operatorname{Maps}(U, \mathbb{R})$. We can also talk about "twisted functions" such as the sheaf of sections or a covering space: the local structure is $U \mapsto \operatorname{Maps}(U, T)$ for some target T, but the global structure is different. This is a common way to think about sheaves.

The most important measurement we extract from a sheaf is its stalks, which allow us to understand how the sheaf behaves on non-open subsets. For example, if $x \in X$, we'd like to understand it through the not-quite-open set $\bigcap_{x\in U} U$, and therefore we get the stalk $\mathscr{F}_x = \varinjlim_{x\in U} \mathscr{F}(U)$, where the open sets are indexed by restriction. This is a filtered colimit, and therefore can be described explicitly as equivalence classes of functions in neighborhoods, where $f \sim g$ if they're equivalent on some common neighborhood of x. Elements of a stalk are called *germs of sections*.

The notion of a stalk still makes sense for presheaves, and today we'll talk about how to determine whether a presheaf is a sheaf using its set of stalks.

Recall that $\mathbb{R}[\![x]\!]$ is the ring of power series with coefficients in $\mathbb{R}$. $C^\infty(\mathbb{R})$ is a sheaf on $\mathbb{R}$, and its stalk at the origin is C^∞_0, the germs of functions at the origin. Since every function has a Taylor series, there's a surjective ring homomorphism $\mathbb{C}^\infty_0 \twoheadrightarrow \mathbb{R}[\![x]\!]$. However, if we use the sheaf of analytic functions $C^\omega(\mathbb{R})$, the stalk C^ω_0 is the ring of Taylor series with nonzero radius of convergence, and therefore maps *in*jectively into $\mathbb{R}[\![x]\!]$.

Let S be an object of C; we'd like to understand $\operatorname{Hom}_{\mathsf{C}}(\mathscr{F}_x, S)$. We know this is $\operatorname{Hom}_{\mathsf{C}}(\varinjlim_{x\in U} \mathscr{F}(U), S) = \varprojlim \operatorname{Hom}_{\mathsf{C}}(\mathscr{F}(U), S)$, and in fact this is $\operatorname{Hom}_{\mathsf{C}_X}(\mathscr{F}, i_{x,*}S)$. This is because the sections of the skyscraper sheaf $i_{x,*}(U)$ for U containing x are just S, so these are just maps between sections of these sheaves, compatible with restriction.

The point is, *stalks are left adjoint to skyscrapers*. That is, there's an adjoint pair $-_x : \mathsf{C}_X \to \mathsf{C} : i_{x,*}$. In particular, stalks will preserve colimits (and therefore stuff like coproducts and cokernels), and skyscrapers preserve limits.

Now, given a map $\pi : Y \to X$, we'd like to understand how sheaves pass back and forth from X and Y. We already have the pushforward $\pi_* : \mathsf{C}_Y \to \mathsf{C}_X$, and it would be pretty cool if it has a left adjoint. It'll be called $\pi^{-1} : \mathsf{C}_X \to \mathsf{C}_Y$, but we can't define it in the same way: the image of an open subset may not be open, so there's no canonical open to associate with a $U \subset Y$. If π is an open embedding, then we already have $\mathscr{F} \mapsto \mathscr{F}|_Y$; we'll have to generalize this, in a way reminiscent of stalks. Given a $U \subset Y$, $\pi(U)$ may not be open, but the open subsets of X containing it is an inverse system: if $V, W \supset \pi(U)$, then $V \cap W$ does too. Since we can't literally take intersections, let's take a colimit again, and define

$$\pi^{-1}\mathscr{F}(U) = \varinjlim_{\substack{\pi(U)\subset W\subset X\\ \text{open}}} \mathscr{F}(W).$$

For example, if $Y \hookrightarrow X$ is a closed embedding, this is a notion of "germs along Y;" that is, functions that extend to some open neighborhood of Y, with the same notion of equivalence. This is very like a stalk, but along any subset.

For example, there's a unique map $\pi : Y \to \mathrm{pt}$, and any set S defines a sheaf over $*$. Then, $\pi^{-1}S(U) = S$, so $\pi^{-1}S$ is the constant presheaf. The point is, this pullback operation is only defined for presheaves. We'll have to do something else, called sheafification, to make sheaves. That said, (π^{-1}, π_*) are are still an adjoint pair on presheaves. We will be able to bump this into an adjoint pair of presheaves, and therefore conclude that π_* commutes with limits (e.g. global sections are a special case of pushforward, so as a corollary, global sections will be right exact!).

One interesting property about sheaves is that since gluing satisfies an existence and uniqueness, then $\mathscr{F}$ is determined by $\mathscr{F}(U_\alpha)$, where $\{U_\alpha\}$ is a basis for the topology on X. The sheaf property is that it's determined by open covers. As a corollary, we can think of the stalk $\mathscr{F}_x$ as the values of $\mathscr{F}$ on a "basis" of tiny open sets around x. Of course, there's no smallest such open set, but we can think of X as having this "basis" of infinitesimal open sets. This doesn't really exist, but it's motivation for the following properties of stalks. In particular, this whole idea is bunk for presheaves.

If $\mathscr{F}$ is a sheaf, then there's a map $\mathscr{F}(U) \hookrightarrow \prod_{x\in U} \mathscr{F}_x$: a section defines a germ at every point on U, and in particular this is unique: every germ is defined on some open subset, giving us a cover on which the germs agree on intersections, so it pulls back to a unique section. And if $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, functoriality gives us a map $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$, so in particular $\mathrm{Hom}_{\mathsf{C}_X}(\mathscr{F}, \mathscr{G}) \hookrightarrow \prod_{x\in X} \mathrm{Hom}_{\mathsf{C}}(\mathscr{F}_x, \mathscr{G}_x)$. We can use this to understand some properties pointwise.

Lemma 10.1. *A map $\varphi : \mathscr{F} \to \mathscr{G}$ of sheaves on X is an isomorphism iff $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ is an isomorphism for all $x \in X$.*

This will allow us to prove something we stated last lecture.

Claim. If $\mathscr{F}$ is a sheaf of sets on X, then it's isomorphic to the sheaf of sections of a map $Y_{\mathscr{F}} \to X$, called the *espace étalé* of $\mathscr{F}$.

This espace étalé tends to be more useful as a conceptual object than for doing stuff; it's completely insane unless your sheaf already looks like a cover. It will feel like a covering map, but won't be one technically.

Proof. The idea is that the fiber will be the stalks, and the fact that these are defined on a neighborhood of a point give us the topology.

The points of $Y_{\mathscr{F}}$ will be $\coprod_{x\in X} \mathscr{F}_x$, and there's a map of sets $Y_{\mathscr{F}} \to X$ sending $\mathscr{F}_x \to x$. Thus, for every open set $U \subset X$ and $f \in \mathscr{F}(U)$, we'd like the map $f : U \to \pi^{-1}(U) \subset Y_{\mathscr{F}}$ sending $x \mapsto f_x$ to be continuous. Thus, give $Y_{\mathscr{F}}$ the weakest topology making this so. Intuitively, we're parallel-transporting a germ to the points on nearby fibers that are represented by the same $f \in \mathscr{F}(U)$. Thus, it's really a covering-like topology: associated to each stalk is a copy of U, and there's no more topology. And $\mathscr{F}(U)$ is exactly the continuous maps $U \to Y_{\mathscr{F}}$ commuting with π. ⊠

In the case of a covering space, we recover the covering space again; for stuff like vector bundles, though, we end up with a similar cover, but where the fibers are discrete. This is bizarre, yes, but the point is that you can reconstruct a sheaf from its stalks. Anyways, now that we've done this, we can put it back in a box and never talk about it again.

Sheafification. Since sheaves on X are presheaves, we have a forgetful functor $\mathrm{For} : \mathsf{C}_X^{\mathrm{pre}} \to \mathsf{C}_X$. We'd like to have a 'free" functor $-_{\mathrm{sh}} : \mathsf{C}_X^{\mathrm{pre}} \to \mathsf{C}_X$ which is left adjoint to For, which will be called sheafification.

That is, this will satisfy the universal property that if $\mathscr{F}$ is a presheaf, $\mathscr{G}$ is a sheaf, and $\varphi : \mathscr{F} \to \mathscr{G}$ is a map of presheaves, there's a unique map of sheaves $\widetilde{\varphi}$ making the following diagram commute.

$$\begin{tikzcd} \mathscr{F} \arrow[r] \arrow[dr, "\varphi"'] & \mathscr{F}_{\mathrm{sh}} \arrow[d, dashed, "\exists!\ \widetilde{\varphi}"] \\ & \mathscr{G} \end{tikzcd}$$

Stated in terms of adjoints, we'd like a natural identification $\mathrm{Hom}_{\mathsf{C}_X^{\mathrm{pre}}}(\mathscr{F}, \mathrm{For}(\mathscr{G})) = \mathrm{Hom}_{\mathsf{C}_X}(\mathscr{F}_{\mathrm{sh}}, \mathscr{G})$.

From this universal property, we know that if $\mathscr{F}$ is already a sheaf, then $\mathscr{F}_{\mathrm{sh}} = \mathscr{F}$, and therefore $-_{\mathrm{sh}} \circ -_{\mathrm{sh}} = -_{\mathrm{sh}}$. In this sense, it's idempotent, so it's sort of a projection. This will be extremely useful, because we're going to do all sorts of operations on sheaves, and if they end up constructing presheaves, that's all right, and we can sheafify it right back: sheafification allows us to ignore the difference between sheaves and presheaves.

The construction won't change $\mathscr{F}$ much locally, since the issue is with gluing, which is more global. That is, we'll keep the same local data, and re-glue it to satisfy the sheaf axioms. To be precise, we'll construct a new sheaf from the stalks of $\mathscr{F}$. In fact, the espace étalé for $\mathscr{F}$ gives you such a construction: take its sheaf of sections, and you're done.

More concretely, let $\mathscr{F}_{\mathrm{sh}}(U)$ be the set of *compatible* sections $(f_x \in \mathscr{F}_x)_{x \in U} \in \prod_{x \in X} \mathscr{F}_x$; that is, for all $x \in U$, there's a $V \subset U$ containing x and an $f_V \in \mathscr{F}(V)$ such that $f_V|_y = f_y$ for all $y \in V$.

This is a way of saying that a compatible section is a collection of germs of the same function: over a small neighborhood of any point, they all come from the same section. This will allow us to glue, though one has to prove this. Another way to write this is that $\mathscr{F}_{\mathrm{sh}}(U)$ is the equalizer of $\mathscr{F}_x$ for all $x \in U$.

This is not as grotesque as it sounds. Recall that for any set S, the constant presheaf $\underline{S}_{\mathrm{pre}}$; its sheafification is the constant sheaf $\underline{S}$: compatible sections are elements of S on connected subsets of X, which is the only way to glue stalks of the constant presheaf.

Yet another way to word this: we can do everything for sheaves with stalks. Since presheaves also have stalks, this says that if you think of a presheaf through its stalks, you're really thinking of its sheafification.

Now, we have an adjunction $(-_{\mathrm{sh}}, \mathrm{For})$, so limits of sheaves are the same as the limits of the underlying presheaves. In particular, kernels are always the same. However, to calculate colimits of sheaves, you have to sheafify: sheafification preserves colimits, so you can calculate colimits in presheaves, but then you have to fix them. This is more important than it looks.

We'll describe some examples of kernels and cokernels of sheaves next time, but before that, a little more abstract nonsense. There's an analogy between the adjunction $(-_{\mathrm{sh}}, \mathrm{For})$ of presheaves and sheaves with (S^{-1}, For) between R-modules and $S^{-1}R$-modules. This is because both of these realize of the latter as a full subcategory of the former, and so the left adjoint is idempotent (if you localize twice, nothing happens). This notion is called a *categorical localization*, which can be thought of in many ways, including as an idempotent left adjoint.

To wit, let's describe localizations more categorically. Localization of modules can be thought of as a subset not just of R, but as a collection $\mathcal{S}$ of arrows $s : M \to M$ for $s \in S$ (given by multiplication by s). Then, we can *localize* the category Mod_R by making all the arrows in $\mathcal{S}$ invertible, by formally adding their inverses. The resulting category, denoted $\mathcal{S}^{-1}(\mathsf{Mod}_R)$, is equivalent to $\mathsf{Mod}_{S^{-1}R}$. This analogy exploits the same one we used for Yoneda's lemma: that a category is not unlike a noncommutative ring.

Now, we can consider the *multiplicative* (meaning closed under composition) set $\mathcal{S}$ of morphisms of presheaves that induce isomorphisms on all stalks. Then, it turns out that localizing $\mathsf{C}_X^{\mathrm{pre}}$ by $\mathcal{S}$ gives one the category of sheaves!

The same idea is used in homotopy theory, where one localizes Top at the set of maps that are *weak equivalences*, i.e. inducing isomorphisms on homotopy groups, and therefore obtains the *homotopy category*.

Episode XI.

Locally Ringed Spaces: 2/23/16

"I didn't give it a good notation because I didn't like it."

Recall that last time, we defined sheafification, which can be thought of projecting presheaves onto sheaves in a particularly nice way. This allows us to forget the difference between sheaves and presheaves, so to speak; we'll use this to understand colimits of sheaves.

Example 11.1. First, a quick digression, since we got confused last time, on the espace étalé of a skyscraper sheaf. Directly from the sheaf axioms, one can show that if $\mathscr{F}$ is a C-valued sheaf, then $\mathscr{F}(\varnothing)$ is the terminal object (a point for Set, 0 for Ab, and so on). This follows from abstract nonsense: the empty product $\prod_\varnothing S$

is necessarily the terminal object (there's more to think through here). This is what motivates the definition of the skyscraper sheaf $i_*S = i_{x,*}S$ in Example 9.3. For simplicity, assume $x \in X$ is a closed point.

Now, let's construct its espace étalé $\pi : Y_{i_*S} \to X$; for any $y \in X$, $\pi^{-1}(y)$ is the stalk of (i_*S) at y, which is S if $y = x$ or the terminal object $*$ otherwise. Thus, Y_{i_*S} is as a set a copy of X, but with S over the basepoint x instead of a single point; then, we glue each of these points of S to the rest of Y_{i_*S} as if they were all x. The result is X with multiple basepoints, so to speak, and is not at all Hausdorff. However, as topological spaces, we have a pullback diagram

$$\begin{tikzcd}
(U \setminus \{x\}) \times S \arrow[r] \arrow[d] & U \times S \arrow[d] \\
U \setminus \{x\} \arrow[r] & Y_{i_*S}.
\end{tikzcd}$$

We can also use the espace étalé to define sheafification: the sheafification $\mathscr{F}_{\mathrm{sh}}$ is just the sheaf of sections of $Y_{\mathscr{F}}$.

Kernels and Cokernels. Before discussing limits and colimits more generally, let's focus on kernels and cokernels. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of abelian groups on a space X, and for every open $U \subset X$, define $(\ker \varphi)(U) = \ker(\varphi|_U)$. It's easy to check that this is a presheaf, and a little more work to check that it's a sheaf, too. And this is actually the kernel in Ab_X, in that it satisfies the universal property: it fits into the diagram

$$\begin{tikzcd}
\ker \varphi \arrow[r] \arrow[d] & \mathscr{F} \arrow[d, "\varphi"] \\
0 \arrow[r] & \mathscr{G},
\end{tikzcd}$$

and any other sheaf $\mathscr{H}$ that fits into the same place in the above diagram has a unique map to $\ker \varphi$.

Likewise, a morphism in Ab_X is injective (meaning a monomorphism) exactly when $\varphi|_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for all open $U \subset X$.

Cokernels are a little more interesting. The sheaf assigning $U \mapsto \operatorname{coker}(\varphi|_U)$ is a presheaf, and is the cokernel in the category of presheaves, but it is *not* the cokernel in the category of sheaves; it fails to satisfy the universal property. This is where some of the interesting nuances of sheaf theory pop up.

Example 11.2. We'll let $X = \mathbb{C}$, and let $\mathscr{O}$ be the sheaf of holomorphic functions and $\mathscr{O}^*$ be the sheaf of "invertible," i.e. nonvanishing, holomorphic functions (an abelian group under multiplication). The exponential map $f(z) \mapsto e^{f(z)}$ sends holomorphic functions to nonvanishing holomorphic functions, and commutes with restriction, so it's a morphism $\exp : \mathscr{O} \to \mathscr{O}^*$ in $\mathsf{Ab}_{\mathbb{C}}$.

If a function maps to 1 in $\mathscr{O}^*$, then it must be an integer multiple of $2\pi i$, so it must be locally constant, Thus, it's constant on each connected component of the given open set. Thus, $\ker(\exp) = 2\pi i \underline{\mathbb{Z}}$: the constant sheaf, not the constant presheaf. This agrees with what we just learned about kernels.

Then, $\operatorname{Im}(\exp)(U)$ is the $f^* \in \mathscr{O}^*(U)$ such that $f = e^{2\pi i g}$ for some $g \in \mathscr{O}(U)$. That is, $\log f$ must have a well-defined branch on U. In particular, if $U = \mathbb{C}^*$ and $f = z$, then $f \notin \operatorname{Im}(\exp(U)$. This is a problem: $\mathbb{C}^*$ can be covered by simply connected open sets on which the logarithm exists, but the gluing axiom fails.

However, since $\exp : \mathscr{O} \to \mathscr{O}^*$ is surjective on simply connected open sets, then it's surjective on the level of stalks, even though it's not surjective as a map of sheaves. In other words, we want the sequence

$$\begin{tikzcd}
0 \arrow[r] & \underline{\mathbb{Z}} \arrow[r, "j"] & \mathscr{O} \arrow[r, "\exp"] & \mathscr{O}^* \arrow[r, dashed] & 0
\end{tikzcd}$$

to be a short exact sequence of sheaves, but if we naïvely define the cokernel like the kernel, it isn't. This means that *to define the sheaf cokernel, we sheafify the presheaf cokernel*. In this case, the sheafification of the presheaf cokernel $\operatorname{coker}(j)$ stitches together the stalks, but on stalks exp is surjective, so since a sheaf is completely determined by stalks, this is just $\mathscr{O}^*$ again, which jives with the idea of surjectivity. In the same way, we get that $\operatorname{coker}(\exp) = 0$, as one would expect.

In other words, a surjective map of sheaves (categorically, an epimorphism), is surjectivity on stalks, but *not* surjectivity on all open subsets. Injectivity is equivalent to injectivity on stalks and on open subsets, though.

Since sheafification preserves colimits, this can be generalized: the colimit of a diagram of sheaves is the sheafification of the presheaf colimit (which is just the colimit on every open set).

Example 11.3. This next example is in some sense the same example. Let X be a smooth manifold, $\mathscr{F}$ be the sheaf of smooth maps to S^1, C^∞ be the smooth maps to $\mathbb{R}$ (so just the smooth functions), and $\underline{\mathbb{Z}}$ is the constant sheaf (which is also smooth maps to $\mathbb{Z}$, since $\mathbb{Z}$ is discrete); each of these is a sheaf of abelian groups.

We'd like to understand that $S^1 = \mathbb{R}/\mathbb{Z}$. This comes from the sequence

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow C^\infty \longrightarrow \mathscr{F} \longrightarrow 0,$$

which is short exact. The injectivity of $\underline{\mathbb{Z}} \hookrightarrow C^\infty$ comes from the fact that every map to $\mathbb{Z}$ can be lifted to a smooth map to $\mathbb{R}$, and surjectivity comes from the fact that germs of functions can be lifted on a small neighborhood, so it's surjective on stalks. However, there are open subsets where functions can't be lifted: if $X = S^1$, then the identity map $S^1 \to S^1$ can't be lifted to a map to $\mathbb{R}$. Thus, this is surjective, even though it's not so on the level of open sets.

Example 11.4. Our next example will be the de Rham complex. Let X be a smooth manifold. Let $\underline{\mathbb{R}}$ denote the constant sheaf on $\mathbb{R}$ (locally constant functions) and Ω^1 denote the sheaf of one-forms on X. The exterior derivative gives us an exact sequence

$$0 \longrightarrow \underline{\mathbb{R}} \longrightarrow C^\infty \xrightarrow{\mathrm{d}} \Omega^1 \xrightarrow{\mathrm{d}} \Omega^2 \xrightarrow{\mathrm{d}} \dots$$

However, this is not in general short exact; if Ω^1_{cl} denotes the space of closed one-forms, then the Poincaré lemma just states that the following sequence is short exact.

$$0 \longrightarrow \underline{\mathbb{R}} \longrightarrow C^\infty \xrightarrow{\mathrm{d}} \Omega^1_{\mathrm{cl}} \longrightarrow 0$$

In other words, even considering something very simple about short exact sequences of sheaves gives us cohomology. This can be used to define sheaf cohomology, though we won't return to that anytime soon.

In fact, Example 11.2 is a special case of this, since $\mathrm{d}z/z \in \Omega^1(\mathbb{C}^*)$ is a closed form that's not exact.

Ringed Space. Anyways, we were going to talk about schemes, right? These are not just topological spaces, but ringed spaces: topological spaces with a notion of a ring of functions.

Definition. A *ringed space* is the data $(X, \mathscr{O}_X)$, where X is a topological space and $\mathscr{O}_X$ is a sheaf of rings on X.

The motivating examples are a topological space with continuous functions to $\mathbb{R}$ (since these form a ring), or a smooth manifold with the sheaf C^∞, or an analytic manifold with C^ω (analytic functions). Thus, there are definitely different notions of "function" on a manifold, but the ringed space structure means knowing what kinds of functions (geometric structure) is.

We'd also like to know how to evaluate functions on a ringed space. For an arbitrary $x \in U$ and $f \in \mathscr{O}_X(U)$, it's not clear how to define $f(x)$; we have stalks, but then what? In each of our examples (continuous functions, smooth functions, analytic functions, holomorphic functions, etc.), the stalks $\mathscr{O}_{X,x}$ aren't just rings, but local rings,[32] with the maximal ideal $\mathfrak{m}_x$ of functions which vanish at x. $\mathfrak{m}_x$ is unique, because if $f \in \mathscr{O}_{X,x} \setminus \mathfrak{m}_x$, then $f(x) \neq 0$, so it's nonzero on a neighborhood of x, and therefore invertible in that subset! Thus, $f \in \mathscr{O}_{X,x}^\times$, so $\mathfrak{m}_x$ must be unique.

The point is, evaluating at x is exactly quotienting by $\mathfrak{m}_x$, producing an element of $\mathbb{R}$. The sheaves we care about have local rings for stalks, which is what makes this evaluation work. We'll turn this into a definition of something much more useful than a ringed space.

Definition. A *locally ringed space* is a ringed space $(X, \mathscr{O}_X)$ such that for every $x \in X$, the stalk $\mathscr{O}_{X,x}$ is a local ring.

Thus, all of our basic examples are locally ringed spaces, and in general, given an $f \in \mathscr{O}_X(U)$, we can define $V(f) = \{x \in U : f(x) = 0\}$, and this will end up being a closed set.

[32] Recall that a *local ring* is a ring with a unique maximal ideal.

Schemes are particular examples of locally ringed spaces. We'll have to define how to produce a sheaf of functions, which we'll probably do next time, but we're almost there. One major takeaway is that schemes behave somewhat like these examples we already have.

We also need to define morphisms. An isomorphism is evident: $(X, \mathscr{O}_X) \cong (Y, \mathscr{O}_Y)$ is the data of a homeomorphism $f : X \to Y$ that identifies the sheaves, i.e. for all open $U \subset Y$, there's an isomorphism $f_* : \mathscr{O}_Y(U) \to \mathscr{O}_X(f^{-1}(U))$.

It's less obvious how to define morphisms in general; clearly, we need a continuous $f : X \to Y$, and we want to compare $\mathscr{O}_X$ and $\mathscr{O}_Y$. Functions pull back (because the preimage of an open set is open); in the examples we had before, we checked that the pullbacks of continuous (smooth, etc.) functions were continuous (smooth, etc.). More generally, given an open $U \subset Y$, we have the two rings $\mathscr{O}_Y(U)$ and $\mathscr{O}_X(f^{-1}(U))$, and we want the pullback of functions $f_* : \mathscr{O}_Y(U) \to \mathscr{O}_X(f^{-1}(U))$ to be a ring homomorphism. This is exactly how we defined the pushforward of a sheaf.

Definition. A *morphism of ringed spaces* is a pair $(f, f^\sharp) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ in which

- $f : X \to Y$ is continuous, and
- $f^\sharp : \mathscr{O}_Y \to f_*\mathscr{O}_X$ is a morphism in Ring_Y.

That is: for every open subset, we can pull functions back into that open subset. But we can say that more concisely with the sheaf theory we have developed.

It's worth remembering that nilpotents on affine schemes give us functions that aren't determined by their values (well, we do have to set up the structure of a locally ringed space first, but we'll get there), so a function isn't quite a bunch of values at points; it's something that we care to pull back.

This is cool, but we care about ringed spaces. What about these maximal ideals? They tell us what it means for a function to vanish. Back in the world of smooth functions, if $\varphi(y) = 0$ and $x \in f^{-1}(y)$, then $(f^*\varphi)(x) = \varphi(f(x))$ had better be 0 too. This is not preserved by morphisms of ringed spaces (since evaluation isn't defined for germs of functions on ringed spaces), so we need an additional axiom.

If $(f, f^\sharp)$ is a morphism of ringed spaces, passing to colimits induces a map $\mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$, whenever $f(x) = y$ (this is generally true for a map of sheaves, thanks to the property of colimits). Then, we want this map to send $\mathfrak{m}_y \to \mathfrak{m}_x$.

Definition. A morphism $(f, f^\sharp) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of locally ringed spaces if for every $x \in X$, $y \in Y$ such that $f(x) = y$, the induced map $f_x^\sharp : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ maps $\mathfrak{m}_y$ into $\mathfrak{m}_x$.

This is actually all the data that we'll need to define schemes: schemes are a full subcategory of locally ringed spaces; specifically, they are the ones that are locally isomorphic to $(\operatorname{Spec} R, \mathscr{O}_{\operatorname{Spec} R})$ (as soon as we define the locally ringed space structure on $\operatorname{Spec} R$), i.e. there are actual isomorphisms on an open cover.

Does this look weird? It's actually not unfamiliar: a smooth manifold is a locally ringed space that's locally isomorphic to $(\mathbb{R}^n, C^\infty)$. This encodes a lot of information; in particular, a continuous map of manifolds is smooth iff it pulls smooth functions back to smooth functions. In the same way, a topological manifold is a locally ringed space locally isomorphic to $(\mathbb{R}^n, C)$ (the sheaf of continuous functions). All the structure of an atlas is encapsulated in this notion of locally ringed spaces.

This notion is extremely general. For example, we can define a complex analytic manifold to be a locally ringed space locally isomorphic to $(U \subset \mathbb{C}^n, \mathrm{Hol})$ (since small discs in $\mathbb{C}^n$ aren't necessarily biholomorphic to all of $\mathbb{C}^n$). In all of the cases we've seen, though, $\mathscr{O}_X(U)$ is always a subset of set maps $U \to \mathbb{R}$ (or $\mathbb{C}$), and in particular functions are determined by their values. This is something that will not be true for schemes.

Next time, we will define $\operatorname{Spec} R$, as a scheme.

Episode XII.

Affine Schemes are Opposite to Rings: 2/25/16

Today, we're going to prove an important theorem, which could be called the fundamental theorem of scheme theory. In doing so, we'll have to define what an affine scheme is. This will be a first-principles motivation for why one might care about schemes; next week will be devoted to some examples.

Recall that we have a category $\mathsf{LocRing}$ of locally ringed spaces $(X, \mathscr{O}_X)$, where X is a topological space and $\mathscr{O}_X$ is a sheaf of rings on X whose stalks $\mathscr{O}_{X,x}$ are all local rings (meaning there's a unique maximal ideal $\mathfrak{m}_x$); the morphisms in this category are pairs of functions $(f, f^\sharp)$, where $f : X \to Y$ is continuous and $f^\sharp : \mathscr{O}_Y \to f_* \mathscr{O}_X$ is a map of sheaves of rings (i.e. a ring morphism on every open subset that's compatible with restriction), but such that $f^\sharp$ is local, meaning it preserves the property of a function vanishing at a point. This means that $f^\sharp(\mathfrak{m}_y) \subset \mathfrak{m}_x$.

Locally ringed spaces are a reasonably natural notion, including concepts such as manifolds, and from it we can recover the category of affine schemes. There's a functor $\Gamma : \mathsf{LocRing} \to \mathsf{Ring}^{\mathsf{op}}$ sending $(X, \mathscr{O}_X)$ to its ring of global sections $\Gamma(\mathscr{O}_X) = \mathscr{O}_X(X)$, which is pretty natural.

Theorem 12.1 ("Fundamental theorem"). *Γ has a right adjoint $\operatorname{Spec} : \mathsf{Ring}^{\mathsf{op}} \to \mathsf{LocRing}$, and there's a natural isomorphism $\Gamma(\mathscr{O}_{\operatorname{Spec} R}) \cong R$.*

That is, composing in one direction is the identity as much as it can be, somewhat like the projection examples we had for localization.

One particular consequence is that $\operatorname{Hom}_{\mathsf{LocRing}}((\operatorname{Spec} S, \mathscr{O}_{\operatorname{Spec} S}), (\operatorname{Spec} R, \mathscr{O}_{\operatorname{Spec} R})) = \operatorname{Hom}_{\mathsf{Ring}}(R, S)$, so Spec is fully faithful. This means, if we defined the category of affine schemes AffSch to be the image of Spec, then Spec defines an equivalence of categories $\mathsf{Ring}^{\mathsf{op}} \cong \mathsf{AffSch}$.

This makes some of the weirdness of the Zariski topology a little less strange: it's very coarse, so it's easier to map into it continuously (coarseness means fewer inverse images need to be open). In particular, if your idea of a geometric space is a locally ringed space, which is very reasonable, you're forced onto the strangeness of the Zariski topology.

After proving Theorem 12.1, we won't use the generality of locally ringed spaces very much, as we care more about schemes. But on schemes it's much nicer: we can identify an affine scheme with its ring of global sections, and will do so.

In any case, we need to define $\operatorname{Spec} R$ as a locally ringed space first, by defining its structure sheaf. It suffices to define it on a basis, and then checking a compatibility condition. On the topological space $\operatorname{Spec} R$, we have the basis $\{D(f)\}_{f \in R}$, and as topological spaces, $D(f) \cong \operatorname{Spec} f^{-1}R$,[33] so we define $\mathscr{O}(D(f)) = f^{-1}R$. Since distinct $f \in R$ may give the same set $D(f)$, we would like to express this invariantly.

Exercise 12.2. R_f is isomorphic to the localization of R at the set S_f of all g such that $D(f) \subset D(g)$ (equivalently, $V(g) \subset V(f)$).

Now, we would like to show this is a presheaf; suppose $D(h) \subset D(f)$; then, $\mathscr{O}(D(f)) = S_f^{-1} R = \{g(x) \neq 0, x \in D(f)\} R$ and $\mathscr{O}(D(h))$ is the inversion at all functions not vanishing on $D(h)$. Thus, we're inverting more functions, so there's a natural localization map.

To be precise, since we're going to use this more than once, suppose S and T are two multiplicative subsets of R, and $S \subset T$. Then, all the elements of S are invertible in T, so by universal property of localization, there's a natural map $S^{-1}R \to T^{-1}R$ given by inverting the elements in $T \setminus S$.

Theorem 12.3. *$\mathscr{O}_{\operatorname{Spec} R}$ is a sheaf.*

Proof. First off, we need to define the ring of functions on all subsets, not just a base. In Vakil's notes, more time is spent on sheaves on a base, but the point is that we can use a cover and then check for compatibility. Suppose $U = \bigcup_{i \in I} D(f_i)$ for some f_i. We define $\mathscr{O}(U)$ to fit into the equalizer diagram (9.1):

$$\mathscr{O}(U) \longrightarrow \prod_{i \in I} \mathscr{O}(D(f_i)) \rightrightarrows \prod_{i,j \in I} \mathscr{O}(D(f_i f_j)).$$

We need to check that this is independent of cover. This quickly reduces to showing the same assertion for $U = D(f)$ itself, by thinking about intersections, but since $D(f) = \operatorname{Spec} R_f$, we can assume $U = \operatorname{Spec} R$. The $D(f_i)$ cover R iff $1 = \sum a_i f_i$ for some $a_i \in R$; this means quasicompactness, so a finite subcover will suffice. We need to show that we have an equalizer diagram

$$R \longrightarrow \prod R_{f_i} \rightrightarrows \prod R_{f_i f_j},$$

[33] In Vakil's notes, the notation R_f is used for $f^{-1}R$.

meaning any $r \in R$ is determined by its image in R_{f_i} for all i, where the last object in the diagram is the data of the compatibility conditions.

First, we have to check the identity axiom: suppose $s \in R$ maps to 0 in each R_{f_i}; since we're not necessarily in an integral domain, this means $f_i^{m_i} s = 0$ for $m_i \gg 0$ (thinking of localization as a filtered colimit). Since $D(f_i) = D(f_i^{m_i})$, then $\{f_i^{m_i}\}$ still generates R, meaning $1 = \sum a_i' f_i^{m_i}$ (akin to a partition of unity). Thus,

$$s = s \cdot 1 = \sum a_i' f_i^{m_i} s = 0,$$

so we're good.

Gluing is harder. Suppose we have $s_i \in R_{f_i}$ that agree on overlaps: $s_i/1 \sim s_j/1$ in $R_{f_i f_j}$. We can write $s_i = a_i / f_i^{\ell_i}$, again since we may have nilpotents. Let $g_i = f_i^{\ell_i}$, so $D(f_i) = D(g_i)$, so we still have the same cover. For these to agree in $D(g_i g_j) = D(f_i f_j)$, we'd need $(g_i g_j)^{m_{ij}}(a_i - g_j - a_j g_i) = 0$.

First, let's choose a finite subcover, which we can do by quasicompactness. Thus, we can let $m = \max_{1 \le i,j \le n} m_{ij}$. Hence, $(g_i g_j)^m (a_i g_j - a_j g_i) = 0$, but this is $a_i g_i^m g_j^{m+1} - a_j g_j^m g_i^{m+1}$. That is, $a_i g_i^m / g_i^{m+1}$ is the same fraction as $a_j g_j^m / g_j^{m+1}$ in $R_{g_i g_j}$. Let $b_i = a_i g_i^m$ and $h_i = g_i^{m+1}$, and as we change notation again remember that if R were an integral domain, much less of this would be necessary. Geometrically, we're shrinking a partition of unity a few times over.

In any case, $D(h_i) = D(g_i)$, this means b_i/h_i and b_j/h_j agree on overlaps, meaning $b_i h_j - b_j h_i = 0$. Finally, we can argue something: $D(h_i)_{i=1,\dots,n}$ still covers $\operatorname{Spec} R$, so $1 = \sum r_i h_i$. Now that things are finite, we can clear denominators and glue them together: $r = r_i b_i \in R$, and we can check that $r|_{D(h_i)} = b_i/h_i$, because

$$h_j r = \sum_i h_j b_i r_i = \sum_i b_j h_i r_i = b_j \sum h_i r_i = b_j.$$

In other words, $r = b_j/h_j$ in R_{h_j}, so we can glue.

There's one last thing to worry about — what if we chose a different finite subcover? Suppose $D(f_\alpha)$ isn't in this finite subcover. Then, we just repeat the argument for the finite cover given by $\{1, \dots, n, \alpha\}$, and this leads to the construction of an $r' \in R$ where $r'_{D(f_i)} = s_i$ for $i = 1, \dots, n$ or α. But since r and r' have the same restriction on the open cover $\{D(f_i)\}_{i=1}^n$, then $r = r'$, so r restricts correctly to any of the opens we started with, which is good. ⊠

Hence, $(\operatorname{Spec} R, \mathscr{O}_{\operatorname{Spec} R})$ is a ringed space. Great! What are the stalks of $\mathscr{O}_{\operatorname{Spec} R}$?[34] An $x \in \operatorname{Spec} R$ is given by a prime ideal $\mathfrak{p} \subset R$, and therefore $\mathscr{O}_{\operatorname{Spec} R, x} = \varinjlim_{x \in U} \mathscr{O}(U)$, but we may as well just look at distinguished opens $\varinjlim_{x \in D(f)} \mathscr{O}(D(f)) = \varinjlim_{f \notin \mathfrak{p}} R_f$, and this is exactly $R_\mathfrak{p}$, which is a local ring. The stalks are local rings, so $\operatorname{Spec} R$ is a locally ringed space.

The next step is to show that $\operatorname{Spec} R$ is a functor, meaning that a map of rings gives us a map of locally ringed spaces. (We already know it's a functor into Top.) If $\phi : S \to R$ is a homomorphism of rings, we already have a continuous map $\operatorname{Spec} \phi : \operatorname{Spec} R \to \operatorname{Spec} S$, so we need to define this on sheaves as a map $\phi^\sharp : \mathscr{O}_{\operatorname{Spec} S} \to (\operatorname{Spec} \phi)_* \mathscr{O}_{\operatorname{Spec} R}$. Sections of $\mathscr{O}_{\operatorname{Spec} R}$ are called *regular functions*, and we need to understand how these pull back.

This is a lot of words, but the idea is that if $f \in S$, then we want to look at the distinguished open $D(f)$, so we want to define a ring homomorphism $S_f \to \mathscr{O}_{\operatorname{Spec} R}((\operatorname{Spec} \phi)^{-1}(D(f)))$ compatible with restrictions. But the inverse image of this distinguished open is $D(\phi(f))$, and we know $\mathscr{O}_{\operatorname{Spec} R}(D(\phi(f))) = R_{\phi(f)}$, so we need a map $S_f \to R_{\phi(f)}$, but this can be induced from ϕ using the universal property of localization.

Finally, we need to check that the map is local. If $x \in \operatorname{Spec} X$ is represented by a prime ideal $\mathfrak{p}_x$ and $\phi(x) = y$ is represented by a prime ideal $\mathfrak{p}_y \subset S$, then $\mathfrak{p}_y = \phi^{-1}(\mathfrak{p}_x)$ in S. But this is exactly what we need for the map to be local: it means we get (using the universal property again) $S_{\mathfrak{p}_y} \to R_{\mathfrak{p}_x}$. This is a little confusing, but if you spell everything out, it works.

Proof of Theorem 12.1. Now, we can tackle the adjunction. This isn't covered well in Vakil's notes, but one reference for it is §25.6 of the Stacks project.[35]

[34] In general, the first question you should ask when given a sheaf is what its stalks are.

[35] This can be found at `http://stacks.math.columbia.edu/tag/01HX`.

We need to study maps $(X, \mathscr{O}_X) \to (\operatorname{Spec} R, \mathscr{O}_{\operatorname{Spec} R})$, and we want to show these are completely determined by the maps $R \to \Gamma(\mathscr{O}_X)$.

Suppose $x \in X$, so we would like to define a corresponding prime ideal $\mathfrak{p}_x \subset \Gamma(\mathscr{O}_X)$ by $\mathfrak{p}_x = \{s \in \Gamma(\mathscr{O}_X) : s(x) = 0\}$. Then, $\mathfrak{p}_x$ is the preimage of $\mathfrak{m}_x \subset \mathscr{O}_{X,x}$ under the map sending a section to its germ $\Gamma(\mathscr{O}_X) \to \mathscr{O}_{X,x}$, so it's a pullback of a prime ideal, and therefore must be prime.

Now, if $\varphi : R \to \Gamma(\mathscr{O}_X)$ is our map, then let $\mathfrak{p}_y = \varphi^{-1}(\mathfrak{p}_x)$, which is prime and therefore defines a $y \in \operatorname{Spec} R$, meaning $f(x) = y$. That is, of all maps of sets, there's a unique map compatible with locality!

We need to show that this forces the map to be continuous, and we'll have to spill over to next time. Since we have a basis on $\operatorname{Spec} R$, it suffices to show $f^{-1}(D(r))$ is open in X. But this is $D(\varphi(r)) \subset X$, the subset of points of $\varphi(r) \in \Gamma(\mathscr{O}_X)$ doesn't vanish. Next time, we'll show that this is open for any $s \in \Gamma(\mathscr{O}_X)$, by showing $V(s)$ is closed, and then extend the map to the structure sheaf, for maybe 15 more minutes of nonsense. ⊠

Episode XIII.

Examples of Schemes: 3/1/16

Recall that we're in the middle of proving Theorem 12.1, which states that the global sections functor $\Gamma : \mathsf{LocRing} \to \mathsf{Ring}^{\mathsf{op}}$ has a right adjoint Spec.

Where are we in this process? Given a locally ringed space $(X, \mathscr{O}_X)$, suppose we have a map $\varphi : R \to \Gamma(\mathscr{O}_X)$. We need to construct a morphism of locally ringed spaces $(f, f^\sharp) : (X, \mathscr{O}_X) \to (\operatorname{Spec} R, \mathscr{O}_{\operatorname{Spec} R})$. We defined f as a map of sets: given an $x \in X$, let $\mathfrak{p}_x \subset \Gamma(\mathscr{O}_X)$ be the ideal of functions vanishing at x. Then, $\varphi^{-1}(\mathfrak{p}_x) \subset R$ is prime, so if we define that to be $f(x)$, we have a map of sets $X \to \operatorname{Spec} R$.

Next, why is this map continuous? As with affine schemes, we can define $D(s) \subset X$ to be the locus of points where $s \in \Gamma(\mathscr{O}_X)$ doesn't vanish. If $r \in R$, $D(r) \subset \operatorname{Spec} R$, and $f^{-1}(D(r)) = D(\varphi(r)) \subset X$. Since these opens are a basis for $\operatorname{Spec} R$, it suffices to show that distinguished ("doesn't-vanish") sets in X are open.

Proposition 13.1. *If $(X, \mathscr{O}_X)$ is a locally ringed space and $s \in \Gamma(\mathscr{O}_X)$, then $D(s) = \{x \in X : s(x) \neq 0\}$ is open (equivalently, $V(s) = X \setminus D(s)$ is closed).*

Proof. Suppose $x \in D(s)$, which means that the germ of s is invertible: $[s] \in \mathscr{O}_{X,x} \setminus \mathfrak{m}_x$. Thus, there's a $g \in \mathscr{O}_{X,x}$ such that $g \cdot [s] = 1$. Since s and g are both defined on some open neighborhood $U \subset X$ containing x, then $sg = 1$ on U, so s is invertible on all of U; in particular, $U \subset D(s)$. ⊠

Corollary 13.2. *s is invertible on all of $D(s)$.*

Proof. We know it's locally invertible, so we need to check that the local inverses glue together. But if U and V are open subsets of $D(s)$, f is a local inverse on U, and g is one on V, then $f|_{U\cap V} = g|_{U \cap V}$, as the inverse in a ring is unique. Thus, by the sheaf axioms, these glue into a unique section, the global inverse to s. ⊠

Returning to the adjunction, we now have a continuous map $f : X \to \operatorname{Spec} R$. Now we need to define $f^\sharp$, which means for every $D(r) \subset \operatorname{Spec} R$, we need a map $\mathscr{O}_{\operatorname{Spec} R}(D(r)) \to \mathscr{O}_X(\varphi(r)) = \mathscr{O}_X(f^{-1}(D(r)))$. This is a diagram chase: $\mathscr{O}_{\operatorname{Spec} R}(D(r)) = R_r$, so we already have a diagram

$$\begin{tikzcd} R_r \arrow[r, dashed, "?"] & \mathscr{O}_X(D(\varphi(r))) \\ R \arrow[u] \arrow[r, "\varphi"] & \Gamma(\mathscr{O}_X). \arrow[u, "\mathrm{res}"'] \end{tikzcd}$$

Now, the map $\mathrm{res} \circ \varphi$ (going along the bottom right) inverts r, because $\varphi(r)$ is invertible on $D(\varphi(r))$ by Corollary 13.2. Hence, by the universal property of localization, there's a unique map $R_r \to \mathscr{O}_X(D(\varphi(r))$ commuting with $\mathrm{res} \circ \varphi$, which is exactly what we needed, and so we're done.

We're not going to use this adjunction as a tool very much, but it's very pretty. In any case, we can generalize from affine schemes to more general schemes, which are things which locally look like affine schemes.

Definition. A *scheme* is a locally ringed space $(X, \mathscr{O}_X)$ such that there's an open cover $\mathfrak{U}$ of X such that for each $U \in \mathfrak{U}$, there's a ring R_U such that $(U, \mathscr{O}_U) \cong (\operatorname{Spec} R_U, \mathscr{O}_{\operatorname{Spec} R_U})$.

We'll spend the rest of the lecture (and much of the next lecture) giving examples.

Example 13.3. Now that we know what schemes are, let's talk about one that's not affine. Let k be a field[36] and $X = \mathbb{A}_k^2 \setminus \{0\}$ (where 0 is the point representing the maximal ideal $\mathfrak{m} = (x, y)$). This means that $X = D(x) \cup D(y)$ in $\mathbb{A}_k^2$, since the first set is all but the y-axis, and the second is all but the x-axis. In particular, X is an open subset of $\mathbb{A}_k^2$, and since it admits an open cover by affine schemes, X is a scheme.

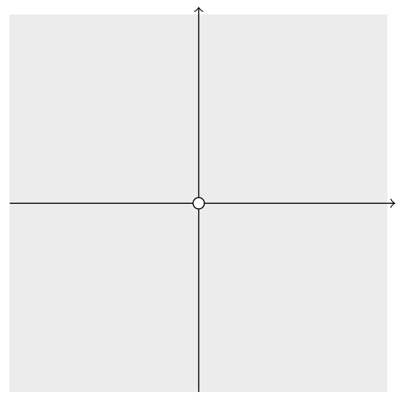

FIGURE 3. $\mathbb{A}_k^2 \setminus \{0\}$ is a scheme that's not affine.

However, X is not affine. The standard way to show this is to calculate $\Gamma(\mathscr{O}_X)$; if X were affine, $X = \operatorname{Spec}(\Gamma(\mathscr{O}_X))$, and we'll show this isn't the case. That is: affine schemes are determined by their global ring of functions.

In any case, a global function is one on $D(x)$, where x is invertible, and on $D(y)$, where y is invertible. That is, $\Gamma(\mathscr{O}_X) = k[x, y, x^{-1}] \cap k[x, y, y^{-1}]$, but this is just $k[x, y]$, and $\operatorname{Spec} k[x, y] = \mathbb{A}_k^2$.

Now, using the adjunction, we have an *affinization* map $(X, \mathscr{O}_X) \to \operatorname{Spec} \Gamma(\mathscr{O}_X)$ which is the identity on the ring of functions — it's an isomorphism iff X is affine. In this case, it's inclusion $X \hookrightarrow \mathbb{A}_k^2$, and therefore isn't even a bijection of sets, so X is not affine.

Another way to think about this: $X = \operatorname{Spec} k[x, y]_x \amalg_{\operatorname{Spec} k[x,y]_{x,y}} k[x, y]_y$, so it's a colimit of affine schemes that's not affine! On schemes, we have more gluings. However, since Spec is a right adjoint, limits in $\mathsf{Ring}^{\mathsf{op}}$, i.e. colimits in Ring, are taken to limits in $\mathsf{LocRing}$ or *Sch*. Thus, products, intersections, and fiber products of affine schemes are affine. However, unions, quotients, etc. will generally not be affine.

In the case $X = \mathbb{C}$, this is a corollary of Hartogs' theorem: a function defined everywhere except on a set of codimension 2 extends to the whole space. We'll have to define a bunch of things (including dimension) to make this precise, but it's an interesting example.

Example 13.4 (Dual numbers)**.** Consider $X = \operatorname{Spec} k[x]/(x^2) \subset \mathbb{A}_k^1$. This space is called the *dual numbers* (as is the ring that defines it). As a topological space, this is just a point, so there's something interesting in its sheaf of functions: $\mathscr{O}_X = k[x]/(x^2)$, which is a two-dimensional k-vector space.

The map $k[x] \to k[x]/(x^2)$ forgets all terms that are x^2 or higher, so $k[x]/(x^2)$ can be thought of as the $f \in k[x]$ identified as $f \sim g$ when $f(0) = g(0)$ and $f'(0) = g'(0)$ (the derivative on polynomials). Thus, the dual numbers can be conceived as a *first-order neighborhood* of $0 \in \mathbb{A}_k^1$.

More generally, for any $t \in k$, we have the maximal ideal $\mathfrak{m}_t = (x - t) \subset k[x]$, and $x \mapsto x - t$ is an automorphism $\mathbb{A}^1 \to \mathbb{A}^1$ that induces an isomorphism $\operatorname{Spec} k[x]/\mathfrak{m}_t^2 \cong \operatorname{Spec} k[x]/(x^2)$. In this case, one can picture this as a little "fuzziness" around the point t, or as the information of a tangent vector at t (which we will make precise later). That is, since we're recovering first-order behavior, nilpotent ring elements allow us to do things which resemble calculus.

[36]This construction works for any ring, but may be easier to picture for fields.

In the same way, we have inclusions

$$\operatorname{Spec} k[x]/x \hookrightarrow \operatorname{Spec} k[x]/(x^2) \hookrightarrow \operatorname{Spec} k[x]/(x^3) \hookrightarrow \cdots, \tag{13.5}$$

and so we'll define $\operatorname{Spec} k[x]/(x^{n+1})$ to be the n^{th}*-order neighborhood* of $0 \in \mathbb{A}^1_k$, which captures the first n terms in a Taylor expansion (well, of a polynomial).

More generally, for any affine scheme $\operatorname{Spec} R$ and $\mathfrak{m} \subset R$, we can define an n^{th}*-order neighborhood* of the closed point $\mathfrak{m} \in \operatorname{Spec} R$ to be $\operatorname{Spec} R/\mathfrak{m}^{n+1}$.

We can use this to given another example of something that's not an affine scheme, or even a scheme!

Example 13.6. Let's take the colimit across (13.5); we'd like to do this geometrically, so let's do it in the category of locally ringed spaces. Let $X = \varinjlim \operatorname{Spec} k[x]/(x^n)$. Thanks to our adjunction, $\Gamma(\mathscr{O}_X) = \varprojlim k[x]/(x^n)$: we're taking compatible collections of Taylor expansions of higher and higher order, meaning this is the ring $k[\![x]\!]$ of *formal power series* in k:

$$k[\![x]\!] = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in k \right\}.$$

Topologically, X is just a point. Therefore, if X is a scheme, it would have to be affine, as any open cover of a point is pretty tautological. Now, we can do something similar to Example 13.3: if X is affine, then $X \cong \operatorname{Spec} k[\![x]\!]$. $k[\![x]\!]$ is an integral domain, so (0) is a prime ideal, and therefore $\operatorname{Spec} k[\![x]\!]$ has two points, (0) (the generic point) and (x) (a closed point), so it's not isomorphic (as locally ringed spaces or even topological spaces) to X.

What's the structure of the ring of functions of $\operatorname{Spec} k[\![x]\!]$? At (0), we just get k, but at (x) we get the ring of *Laurent series*

$$k(\!(x)\!) = k[\![x]\!][x^{-1}] = \left\{ \sum_{i=-N}^{\infty} a_i x^n : N \in \mathbb{Z}, a_i \in k \right\}.$$

$\operatorname{Spec} k[\![x]\!]$ is sometimes called the *disc*, and X is sometimes called the *formal disc*. Even though it's not a scheme, we still have inclusions (well, this means something nontrivial, since as spaces they're all points) $X = \varinjlim \operatorname{Spec} k[x]/(x^n) \hookrightarrow \operatorname{Spec} k[\![x]\!] \hookrightarrow \operatorname{Spec} k[x]_{(x)}$. This last ring is $\operatorname{Spec} \mathscr{O}_{\mathbb{A}^1_k,0}$; if $k[\![x]\!]$ is the ring of Taylor series, $k[x]_{(x)}$ is the Taylor series with nonzero radii of convergence. Since the nonempty open sets are dense in $\mathbb{A}^1_k$, then a germ in $k[x]_{(x)}$ is a rational function with finitely many poles, and that's regular at 0. This does resemble calculus: functions to n^{th} order include into Taylor series include into functions which may have poles, etc.

Example 13.7. You may be wondering where things like the dual numbers arise. Let k be an algebraically closed field (of characteristic not equal to 2) and $\pi : \mathbb{A}^1_k \to \mathbb{A}^1_k$ be induced from the map $k[y] \to k[x]$ sending $y \mapsto x^2$. (For concreteness, you can do everything with $k = \mathbb{C}$). Then, every $t \in \mathbb{A}^1_k$ has two preimages (corresponding to its square roots) except 0. In particular, we want to understand the fibers of this map, not just as sets, but as schemes.

Thanks to a bunch of the abstract nonsese we've developed, it's very easy to conceptualize fibers. A fiber over t is $\pi^{-1}(t)$, which fits into the diagram of schemes

$$\begin{tikzcd} \pi^{-1}(t) \arrow[r] \arrow[d] & \mathbb{A}^1_k \arrow[d, "\pi"] \\ \{t\} \arrow[r] & \mathbb{A}^1_k. \end{tikzcd}$$

This is a pullback (or fiber product), so when we pass to Ring, we get the fiber coproduct (pushout); on a homework problem, we proved the fibered coproduct $S \amalg_T R$ is the tensor product as T-algebras: $S \otimes_T R$. In particular, if R, S, and T are rings, the following diagram is a fiber product diagram in the category of affine schemes.

$$\begin{tikzcd} \mathrm{Spec}(S \otimes_T R) \arrow[r] \arrow[d] & \mathrm{Spec}\, R \arrow[d] \\ \mathrm{Spec}\, S \arrow[r] & \mathrm{Spec}\, T \end{tikzcd}$$

Specializing to our map $\pi : \mathbb{A}^1_k \to \mathbb{A}^1_k$, the functions at $t = (y - t)$ are $k \cong k[y]/(y-t)$, so $\pi^{-1}(t) = \mathrm{Spec}(k[x] \otimes_{k[y]} k) = \mathrm{Spec}\, k[x]/(y = x^2 = t)$, i.e. $\mathrm{Spec}\, k[x]/(x^2 - t)$. What does this look like? If $t \neq 0$, this factors into $\mathrm{Spec}\, k[x]/(x - \sqrt{t}) \amalg \mathrm{Spec}\, k[x]/(x + \sqrt{t})$ — and if $t = 0$, it doesn't factor, so we have $\mathrm{Spec}\, k[x]/(x^2)$, the dual numbers! The set-theoretic fiber is a point, but the scheme-theoretic picture has more information, and many properties are nicer (e.g. the dimension of $k[x]/(x^2 - t)$ as a k-vector space is always constant).[37] Nilpotents seemed like a nuisance when we introduced the Zariski topology (functions not determined by their values?!), but they're actually very, very useful for geometric stuff such as this example.

Example 13.8. For another example of the usefulness of nilpotents, let $R = k[x, y]$ and consider the x-axis $\mathrm{Spec}\, R/(y)$ and parabola $\mathrm{Spec}\, R/(y - x^2)$. We can consider their intersection in $\mathbb{A}^2_k$; set-theoretically, it's just the origin, but scheme-theoretically, the intersection is a fiber product $\mathrm{Spec}\, R/(y - x^2) \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, R/(y)$. In Ring, this is a fiber coproduct, so a tensor product. Thus, the scheme we get is $\mathrm{Spec}(R/(y - x^2) \otimes_R R/(y)) = \mathrm{Spec}\, k[x, y]/(y, x^2) = \mathrm{Spec}\, k[x]/(x^2)$, the ring of dual numbers again. Thus, the parabola and x-axis intersect at a "double point." Since the generic intersection over an algebraically closed field of a line with a parabola will have two points, counting this as an intersection with multiplicity 2 makes for a more uniform notion of intersection.

Example 13.9. Another application is the geometry of the Chinese remainder theorem. For example, we have $\mathbb{Z}/60 \cong \mathbb{Z}/3 \times \mathbb{Z}/4 \times \mathbb{Z}/5$; what does this look like? Recall that $\mathrm{Spec}\, Z$ has a generic point and one point for each prime (this feels discrete, but it's going to be our analogue for a connected set). On this scheme, we have the function $f = 60$, e.g. $60(7) = 60 \bmod 7 = 4$. Its vanishing locus is $V(60) = \mathrm{Spec}\, \mathbb{Z}/60 \hookrightarrow \mathrm{Spec}\, \mathbb{Z}$. 60 vanishes to order 1 at 5, so $60 \in (5)$, but not in (5^2); similarly, it's in (3), but not (3^2), and in (2) and (2^2), but not (2^3). Thus, just as $\mathbb{Z}/60 \cong \mathbb{Z}/3 \times \mathbb{Z}/4 \times \mathbb{Z}/5$, $\mathrm{Spec}\, \mathbb{Z}/60 \cong \mathbb{Z}/4 \amalg \mathbb{Z}/3 \amalg \mathbb{Z}/5$.

FIGURE 4. Visualizing the Chinese remainder theorem: $\mathrm{Spec}\, \mathbb{Z}/60 = \mathrm{Spec}\, \mathbb{Z}/(2^2) \amalg \mathrm{Spec}\, \mathbb{Z}/3 \amalg \mathrm{Spec}\, \mathbb{Z}/5$.

Episode XIV.

More Examples of Schemes: 3/3/16

Last time, we talked about the ring of dual numbers $k[x]/(x^2)$; this is often denoted $k[\varepsilon]/(\varepsilon^2)$, with the sort-of analytic notion that ε is a very small number, so that ε^2 is negligible and we can ignore it. We also talked about $\mathrm{Spec}\, k[x]_{(x)}$, which sits inside $\mathbb{A}^1_k = \mathrm{Spec}\, k[x]$. This subscheme's ring of functions are the rational functions (so defined almost everywhere) that are regular at 0. In the Zariski topology, the opens are huge, so we might hope for something more local with this local ring: inside $\mathrm{Spec}\, k[x]_{(x)}$, we had $\mathrm{Spec}\, k[\![x]\!]$ and $\mathrm{Spec}\, k(\!(x)\!)$, the power series and Laurent series. Inside $\mathrm{Spec}\, k[\![x]\!]$, we also have $X = \varinjlim \mathrm{Spec}\, k[x]/(x^n)$. This X, called the *formal completion* of $\mathbb{A}^1$ at 0, is not a scheme; sometimes it's called a *formal scheme* or an ind-scheme.

Definition. An *ind-scheme* is a colimit of a diagram of schemes within $\mathsf{LocRing}$.

So ind-schemes are colimits, but they might not be schemes. Things such as formal schemes are very useful.

[37] This is an instance of something called a *flat family*.

Example 14.1. In two dimensions, things can get a bit more interesting, because there are more directions for things to happen in. In $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$ (usually referred to as $\mathbb{A}^2$ if k is clear from context), we have the maximal ideal $\mathfrak{m} = (x,y)$, corresponding to the origin. Thus, we have the *first infinitesimal neighborhood of* 0 $\operatorname{Spec} k[x.y]/\mathfrak{m}^2$. The functions on this point are functions up to equivalence if their values and first-order Taylor coefficients agree. In the same way, one can define the n^{th}*-order neighborhood* $\operatorname{Spec} k[x,y]/\mathfrak{m}^n$, where we remember more, the first n orders of the Taylor series. If we take the "union" (colimit), we get the formal completion at 0, $\widehat{\mathbb{A}}^2|_0 = \varinjlim \operatorname{Spec} k[x,y]/\mathfrak{m}^2$.

But in two dimensions, we can do something different, looking at $\operatorname{Spec} k[x,y]/(x,y^2)$ or $\operatorname{Spec} k[x,y]/(x^2,y)$. The rings are isomorphic to the ring of dual numbers, so as abstract schemes, they're the same scheme, but they're different as subschemes of the plane. In the first one, we're setting $x = 0$, but $y^2 = 0$, so on our ring of functions, we retain no information about x, but first-order information in y. It's as if we had an infinitesimal neighborhood of the origin, but only in the y-direction. In the same way, $\operatorname{Spec} k[x,y]/(x^2,y)$ defines an infinitesimal neighborhood in the x-direction, remembering one order in x and none in y.

Both of these contain the point $\operatorname{Spec} k[x,y]/\mathfrak{m}$ and are contained in $\operatorname{Spec} k[x,y]/\mathfrak{m}^2$ (these are all points topologically, but we're thinking about the rings of functions); this can be thought of as a circle of infinitesimal information around $(0,0)$.

What other things allow this to happen? We want to find an ideal $I \subset k[x,y]$ such that $\mathfrak{m} \supset I \supset \mathfrak{m}^2$. This suggests looking at $\mathfrak{m}/\mathfrak{m}^2$.

Exercise 14.2. Show that, as k-vector spaces, $\mathfrak{m}/\mathfrak{m}^2$ is a two-dimensional vector space.

Then, the image of I in $\mathfrak{m}/\mathfrak{m}^2$ is a k-subspace, so these ideals are in bijection with the lines in k^2, given by linear maps $f : \mathbb{A}^1 \to \mathbb{A}^2$; this means looking at equations such as $y = tx$ for $t \in k$. Then, the image of the dual numbers sitting in $\mathbb{A}^1$ is sent to $\operatorname{Spec} k[x,y]/I$ in $\mathbb{A}^2$.

Thinking of these as linear directions of infinitesimal information out of the origin motivates the following definition.

Definition. If k is a field, an affine scheme X is said to be *over* k if it has a map $X \to \operatorname{Spec} k$, so that if $X = \operatorname{Spec} R$, R is a k-algebra.

For example, $\mathbb{A}^n_k$ is an affine scheme over k.

Definition. Let X be an affine scheme and $x \in X$ be a closed points, so that it defines a maximal ideal $\mathfrak{m}_x \subset \mathscr{O}_X$. Then, we define the *Zariski cotangent space* at x to be $T^*_x X = \mathfrak{m}/\mathfrak{m}^2$. If X is a scheme over a field k, we can define the *Zariski tangent space* at x to be $T_x X = (\mathfrak{m}/\mathfrak{m}^2)^*$, the dual as k-vector spaces.

The reason for these definitions is the following result.

Proposition 14.3. *If X is a scheme over k and x is a closed point whose residue field is k,*[38] *there is a bijection between $T_x X$ and the maps $f : \operatorname{Spec} k[x]/(x^2) \to X$ fitting into the diagram*

$$\begin{tikzcd} \operatorname{Spec} k[x]/(x^2) \arrow[r, "f"] & X \\ \operatorname{Spec} k \arrow[u] \arrow[r] & x \arrow[u] \end{tikzcd}$$

This seems strange, but is similar to the notion of a tangent space for differential topology, where we map $\mathbb{R} \hookrightarrow M$ and identify functions that agree to first order. In algebraic geometry, it's hard to map $\mathbb{A}^1_k$ into anything, so we use the dual numbers instead.

[38] For example, if k is algebraically closed, then all closed points satisfy this.

Proof. Let $X = \operatorname{Spec} R$, where R is a k-algebra, and suppose $\varphi : R \to k[x]/(x^2)$; we can project it t $k[x].(x) \cong k$, and therefore factors through the commutative diagram

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d] & k[x]/(x^2) \arrow[d] \\ R/\mathfrak{m} \arrow[r] & k[x]/(x). \end{tikzcd}$$

Thus, if $r \in \mathfrak{m}^2$, then $\varphi(r) = 0$, and if $r \in \mathfrak{m}$, then $\varphi(r) \in (x)$. Let $\xi(r)$ be the coeffiicent of x in $\varphi(r)$ for $r \in \mathfrak{m}$, so $\xi \in (\mathfrak{m}/\mathfrak{m}^2)^*$ is a linear map $\mathfrak{m}/\mathfrak{m}^2$, and therefore defines a map $\mathfrak{m}/\mathfrak{m}^2 \to k$, which therefore defines a map $R/\mathfrak{m}^2 \to k[x]/(x^2)$ of k-vector spaces. This is because $R/\mathfrak{m} \cong k \oplus \mathfrak{m}/\mathfrak{m}^2$ and $k[x]/(x^2) \cong k \oplus k \cdot x$; we map the first summand of k to k, and then use ξ on the second component. But this is actually a ring homomorphism (everything squares to zero, so there's very little to check). $\boxtimes$

This abstraction makes things a little nicer: in differential topology, a tangent vector is a class of vectors, but here our tangent vectors are just things in $\mathfrak{m}/\mathfrak{m}^2$. Looking at things such as $k[x]/(x^3)$, etc., means doing higher-order calculus. In general, mapping nilpotents into schemes provides information about derivatives.

One logical thing to do here would be to talk about derivations, but we'll do that next lecture. Today, we'll give more examples of non-affine schemes, leading to the notion of a projective scheme. Recall that a scheme is a locally ringed space that's locally Spec of a ring (so locally affine). We can construct schemes by gluing: if X and Y are two affine schemes and we have the data of a Zariski open $U \subset X$ and a Zariski open $V \subset Y$, then an isomorphism $U \cong V$ allows us to define a new scheme $Z = X \amalg_U Y$, where we glue across $U = V$. Though you might imagine a Venn diagram or gluing manifolds, this is somewhat deceptive, since nonempty open subsets are dense. As topological spaces, gluing is what we'd expect, the usual gluing or identification. And the ring structure isn't very hard either: if $W \subset Z$ is open, either it's contained in X (so use the sheaf structure for X), contained in Y (so use the sheaf structure for Y), or contained in neither (so cover it in open sets that are in only one or the other and glue). There's something we should check here, but it's similar to what we've been doing before.

Example 14.4. Let k be a field and $X = Y = \mathbb{A}^1_k = \operatorname{Spec} k[x]$. Let $U = V = \mathbb{A}^1_k \setminus 0 = \operatorname{Spec} k[x, x^{-1}]$, since we're localizing at x.

We have two choices of isomorphism; the simplest is the identity $x \mapsto x$. Then, we're gluing $Z = \mathbb{A}^1 \amalg_{\mathbb{A}^1 \setminus 0} \mathbb{A}^1$, identifying everything except the origins. What we get is the scheme-theoretic analogue of the line with two origins, everyone's favorite non-Hausdorff space.

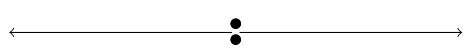

Figure 5. The line with two origins, $\mathbb{A}^1 \amalg_{\mathbb{A}^1 \setminus 0} \mathbb{A}^1$.

This is a scheme, but we can look at its sheaf of functions to show that it's not affine. $\mathscr{O}_Z(Z)$ is the set of $f \in k[x_1]$ and $g \in k[x_2]$ where we identify $f \sim g$ if they're equal on $\mathbb{A}^1 \setminus 0$. But that means that $\mathscr{O}_Z(Z) = k[x]$ (if two functions agree almost everywhere in $\mathbb{A}^1$, they have to be the same).[39] In other words, $\mathscr{O}_Z(Z) = \mathscr{O}_{\mathbb{A}^1}(\mathbb{A}^1)$, so if Z is affine, then the induced map $Z \to \mathbb{A}^1$ must be an isomorphism. However, it's not even injective as a set map, since the two origins are projected down to one. Thus, Z is not affine. This is probably one of the less useful examples of a scheme that's not affine.

This was another example of the affinization map: for any scheme, we have a natural map $X \to \operatorname{Spec}(\mathscr{O}_X(X))$, and it's an isomorphism iff X is affine.

[39] Another way to think of this is that taking global sections is a left adjoint, and therefore sends colimits to limits, so we get an intersection of rings.

Example 14.5. We can also glue $\mathbb{A}^1$ to itself in an extremely useful way: we want to identify $k[x_1, x_1^{-1}] \cong k[x_2, x_2^{-1}]$ with a more interesting map, $x_1 \mapsto x_2^{-1}$. This space will be called $\mathbb{P}^1_k$ (or $\mathbb{P}^1$ if k is known from context). In this case, we've glued more like a circle: the two origins are antipodal, as in Figure 6.

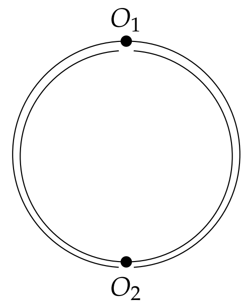

FIGURE 6. Gluing $\mathbb{A}^1_k$ to itself to form $\mathbb{P}^1_k$.

This is also not affine, but in a much stronger sense: $\Gamma(\mathscr{O}_{\mathbb{P}^1})$ is the ring of functions $f \in k[x_1]$ and $g \in k[x_2]$ that agree under $x_1 \mapsto x_2^{-1}$. But this means that f can't have any positive degrees (since they'd correspond to negative degrees in g, which aren't allowed), and vice versa, so the ring of functions is just k! Then, the affinization map is again not an isomorphism, because $\mathbb{P}^k_1$ has more than one point.

If X is a topological space, we have functions on that space, which are just continuous maps to $\mathbb{R}$. In the same way, if X is a scheme, then for any ring R, we have a natural map $\mathbb{Z} \to \mathbb{R}$ sending $1 \mapsto 1$. Hence, if $r \in R$, we can identify it with the map $\mathbb{Z}[x] \to R$ sending $x \mapsto r$. In this way, we can identify $\mathscr{O}(X) = \mathrm{Hom}_{\mathsf{Ring}}(\mathbb{Z}[x], \mathscr{O}(X)) = \mathrm{Hom}_{Sch}(X, \mathbb{A}^1_{\mathbb{Z}})$, since $\mathbb{A}^1_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[x]$.

If you don't like $\mathbb{A}^1_{\mathbb{Z}}$, we can do the same thing with a field k.

Definition. If k is a field, a *scheme over* k is a scheme X with the data of a map $X \to \operatorname{Spec} k$ (equivalently, a k-algebra structure $k \to \mathscr{O}(X)$). A morphism of schemes over k is a map $f : X \to Y$ commuting with the maps to $\operatorname{Spec} k$, and so we get a category Sch/k of schemes over k.

If X is a scheme over k, then $\mathscr{O}_X$ is a sheaf of k-algebras, and we can make the same identification of k-algebra elements as homomorphisms $k[x] \to R$; hence, $\mathscr{O}(X) = \mathrm{Hom}_{Sch/k}(X, \mathbb{A}^1_k)$. This is useful; if we're doing complex algebraic geometry, we want things to be complex linear, etc. And in general, even for $\mathbb{Z}$, we have a realization of global functions as actual functions to $\mathbb{A}^1$, as we desired.

Now, $\mathrm{Hom}_{Sch}(X, \mathbb{A}^1)$ is a ring, because $\mathscr{O}(X)$ is. Can we see this explicitly? Yes, because $\mathbb{A}^1$ is a *ring scheme*!

First, we can give $\mathbb{A}^1$ a group structure. This is more abstract than the usual definition: we have a multiplication map $\mu : G \times G \to G$, an inverse map $i : G \to G$, and an identity map $u : \bullet \to G$. Looking at $\mathbb{A}^1$ specifically, our multiplication map is a map $\mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$, so in the opposite direction, we need a map $k[z] \to k[x, y]$. The map we get is $z \mapsto x + y$. It's necessary to check associativity, i.e. the two maps $\mathbb{A}^1 \times \mathbb{A}^1 \times \mathbb{A}^1 \rightrightarrows \mathbb{A}^1$ are the same, but this is because $(x + y) + w = x + (y + w)$.

Then, the identity is $\operatorname{Spec} k \to \mathbb{A}^1$ that's the origin (induced from the ring homomorphism $k[x] \to k$ sending $x \mapsto 0$), and the inverse is induced from $x \mapsto -x$. One can check that these satisfy all the required axioms as maps of schemes, and therefore $\mathbb{A}^1_k$ is a *group object* in the category of schemes, or for short a *group scheme*. In fact, it also satisfies commutativity of multiplication, so it's abelian.

If X is a topolgical space and G is a topological group, then $\mathrm{Hom}_{\mathsf{Top}}(X, G)$ is a group; in the same way, if X is a scheme and G is a group scheme, $\mathrm{Hom}_{Sch}(X, G)$ is a group, with the structure given by

$$\begin{tikzcd}
\mathrm{Hom}(X,G) \times \mathrm{Hom}(X,G) \arrow[r, equal] \arrow[d] & \mathrm{Hom}(X, G \times G) \arrow[dl, "\mu"] \\
\mathrm{Hom}(X,G). &
\end{tikzcd}$$

This is pointwise multiplication, but in a slightly more abstract way. Thus, functions on a scheme are an abelian group, which we already knew, but it comes from this cool fact. The fact that they form a ring comes from the *ring scheme* structure on $\mathbb{A}^1$, which is induced from the multiplication on $k[x]$, and means checking some more axioms.

The reason this all this showed up now is that we also have a group scheme structure on $\mathbb{A}^1 \setminus 0$; in this context, we call it $\mathbb{G}_m$, the *multiplicative group*, which is induced by multiplication $k[x, x^{-1}] \otimes k[y, y^{-1}] \leftarrow k[z, z^{-1}]$ sending $z \mapsto xy$. This is very useful, and we'll be seeing it again.

Episode XV.

Representation Theory of the Multiplicative Group: 3/8/16

Last time, we went in several different directions, including discussing projective space $\mathbb{P}^1_k$ and the multiplicative group $\mathbb{G}_m$; today, we'll continue in those directions, possibly simultaneously.

Recall that the multiplicative group $\mathbb{G}_m$ over a field k (or over $\mathbb{Z}$) is $\operatorname{Spec} k[x, x^{-1}]$. This is a *group scheme*, meaning a *group object* in the category of schemes. More generally, a group object in a category C is an object $G \in \mathsf{C}$ and C-morphisms $m : G \times G \to G$ (multiplication), $e : \bullet \to G$ (unit), and $i : G \to G$ (inverse) that satisfy the axioms corresponding to the axioms of a group. For example, associativity is the requirement that the following diagram commutes.

$$\begin{tikzcd} G \times G \times G \arrow[r, "(m,\mathrm{id})"] \arrow[d, "(\mathrm{id},m)"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"] & G \end{tikzcd}$$

For example, a group object in the category of manifolds is a Lie group.

We can interpret this via the Yoneda lemma: as in that context, let $h_G(X) = \operatorname{Hom}_{\mathsf{C}}(X, G)$ define the functor $h_G : \mathsf{C}^{\mathsf{op}} \to \mathsf{Set}$. Then, an equivalent and alternative definition of a group object is that $h_G(X)$ is actually valued in groups, i.e. it lifts to a functor into *Grp* that commutes with the forgetful functor $\mathrm{For} : Grp \to \mathsf{Set}$.

In our case, we care about $\mathbb{G}_m$.[40] The multiplication map $\mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ is induced from the map $k[x, x^{-1}] \to k[y, y^{-1}] \otimes k[z, z^{-1}]$ that sends $x \mapsto yz$; the unit is the map $\operatorname{Spec} k \to \mathbb{G}_m$ induced from the map $x \mapsto 1$, and the inverse map $\mathbb{G}_m \to \mathbb{G}_m$ is induced from the map $k[x, x^{-1}] \to k[y, y^{-1}]$ sending $x \mapsto y^{-1}$.

However, it's more natural to think of $\mathbb{G}_m$ functorially, since it's not just a group scheme, but an affine group scheme. In other words, $h_{\mathbb{G}_m} : \mathsf{AffSch}^{\mathsf{op}} \to Grp$ is really a functor $\mathsf{Ring} \to Grp$.[41] A map in $\operatorname{Hom}_{\mathsf{AffSch}}(\operatorname{Spec} R, \mathbb{G}_m)$ corresponds directly to a map $k[x, x^{-1}] \to R$, and this is determined by where it sends x, which must map to something in $R^\times$. Hence, the group of these maps is $R^\times$, meaning as a functor, $\mathbb{G}_m(R) = R^\times$. Oftentimes, e.g. in moduli problems, this functorial perspective is more useful.

When you turn around the maps to land in Ring, an affine group scheme is the same thing as a commutative Hopf algebra.

Definition. A *Hopf algebra* over a field k[42] is a k-algebra A along with the data of k-algebra maps $\Delta : A \to A \otimes A$ (the *coproduct*), $\varepsilon : A \to k$ (the *augmentation*) and the *antipode* $S : A \to A$, such that

- Dual to associativity, the following diagram must commute.

$$A \xrightarrow{\Delta} A \otimes A \underset{\Delta \otimes \mathrm{id}_A}{\overset{\mathrm{id}_A \otimes \Delta}{\rightrightarrows}} A \otimes A \otimes A.$$

[40] If this is abstract and scary, think about the nicest case, $\mathbb{C}$; in this case, $\mathbb{G}_m = \mathbb{C}^*$.

[41] Technically, these are schemes over k, and therefore the rings are k-algebras, but if we use $\mathbb{G}_m$ over $\mathbb{Z}$, we recover this for all rings and affine schemes.

[42] Hopf algebras can also be defined over $\mathbb{Z}$ or even in the noncommutative case, which we're not doing, though essentially all the same axioms hold. One difference is that the antipode must map $A \to A^{\mathrm{op}}$, the ring with mirrored multiplication.

- Dual to identity, the following diagram must commute.

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[rr, bend right, "\mathrm{id}"'] & A \otimes A \arrow[r, "\mathrm{id}\otimes\varepsilon"] & A \end{tikzcd}$$

- Finally, there's a last diagram corresponding to the inverse axiom.

If we only have the data and conditions for Δ and ε, A would be called a *(counital) coalgebra* or a *bialgebra*.

Δ is dual to the multiplication in the group scheme, ε to the unit, and S to the inverse.

If this definition seems a little crazy, we're actually just turning the definitions of a group scheme around, so this is not something whose definition you really need to remember.

This is a lot of stuff, so why bother? Because it gives us an excuse to discuss representations of Hopf algebras, and $\mathbb{G}_m$ specifically.

Definition. A *representation* of a Hopf algebra A (or just a coalgebra) is a k-vector space V along with a *coassociative* map $a : V \to V \times A$, meaning that the diagram

$$\begin{tikzcd} V \arrow[r, "a"] & V \otimes A \arrow[r, shift left, "\mathrm{id}\otimes\Delta"] \arrow[r, shift right, "a\otimes\mathrm{id}"'] & V \otimes A \otimes A \end{tikzcd}$$

commutes, and a *counital* map ε (meaning that $\varepsilon \circ a = \mathrm{id}$).

One can also think of these as *comodules* over a group coalgebra, sort of like in ordinary representation theory.

Now, suppose V is a representation of $\mathbb{G}_m$ (regarded as a Hopf algebra with the arrows turned around). The comultiplication map $V \to V \otimes k[x, x^{-1}]$ sends $v \mapsto \sum v_n x^n$, and the counit map sends this to $v = \sum v_n$. We can define $v \in V$ to be *homogeneous* if $a(v) = vx^n$, and let V_n be the homogeneous elements of degree n. Then, $V = \bigoplus_{n\in\mathbb{Z}} V_k$, i.e. a representation of $\mathbb{G}_m$ is a *$\mathbb{Z}$-graded vector space*!

Moreover, if $V = \bigoplus V_k$ is a graded vector space, we can define $a : V \to V \otimes k[x, x^{-1}]$ by $a(v_k) = v_k x^k$. This defines a comodule structure, so representations of $\mathbb{G}_m$ are exactly graded vector spaces, and we have an even stronger result.

Theorem 15.1. *There is an equivalence of categories between representations of $\mathbb{G}_m$ ($\mathsf{Rep}_{\mathbb{G}_m}$) and $\mathbb{Z}$-graded vector spaces.*

In fact, this equivalence respects tensor products (coproducts). Recall that if $V_\bullet = \bigoplus V_k$ and $W_\bullet = \bigoplus W_k$ are $\mathbb{Z}$-graded vector spaces, then the *tensor product* $V_\bullet \otimes W_\bullet$ has k^{th}-degree component

$$(V_\bullet \otimes W_\bullet)_k = \bigoplus_{i+j=k} V_i \otimes W_j.$$

Correspondingly, if A is a Hopf algebra, the tensor product of A-comodules V and W is also an A-comodule, given by the composition of the maps

$$V \otimes W \xrightarrow{a\otimes a} (V \otimes A) \otimes (W \otimes A) = V \otimes W \otimes A \otimes A \xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes m} V \otimes W \otimes A.$$

If you define what a *tensored category* is, then the equivalence of categories in Theorem 15.1 extends to an equivalence of tensored categories. A category is a kind of weak structure (e.g. the category of graded k-vector spaces is a $\mathbb{Z}$-direct sum of Vect_k, so the tensor structure really gives it flavor). This leads into a whole subject called *Tannakian formalism*, which is a very general statement that if a group has enough faithful representations, then one can recover the group from the tensor category of its representations; this is also true back in the world of locally compact groups.

Returning to Earth (more or less), we'd like to relate this to the other thing we talked about yesterday, projective space.

First, we can grade more things than just k-vector spaces.

Definition. A *$\mathbb{Z}$-graded ring* is a graded abelian group (i.e. $\mathbb{Z}$-module) $R = \bigoplus_{i\in\mathbb{Z}} R_i$ such that multiplication $m : R \otimes R \to R$ is, as for a map of graded vector spaces, additive in the degree: on homogeneous elements, $m(R_i \otimes R_j) \subseteq R_{i+j}$.

Corollary 15.2. *If $\mathbb{G}_m$ denotes the multiplicative group over $\mathbb{Z}$, then there is an equivalence of categories between $\mathbb{Z}$-graded rings, the category of rings in $\mathsf{Rep}_{\mathbb{G}_m}$*[43]*, and the (opposite) category of affine schemes that have a $\mathbb{G}_m$-action.*

This seems kind of scary, but if you unwind what all the maps are and what structures are preserved, this is kind of a tautology. Nonetheless, it makes for an interesting and sometimes useful perspective: the algebraic notion of a $\mathbb{Z}$-graded ring is equivalent to the geometric notion of an affine scheme with a $\mathbb{G}_m$-action (i.e. a map $\operatorname{Spec} R \times \mathbb{G}_m \to \operatorname{Spec} R$ that satisfies the usual multiplication, etc.).

Projective Space. Last time, we defined $\mathbb{P}^1_k$ over a field k; we're going to make this considerably more general. This section will be much more concrete than the last one, though.

Let R be a ring; then, we'll define $\mathbb{P}^n_R$, projective n-space over R, as a scheme over $\operatorname{Spec} R$. We defined $\mathbb{P}^1$ as two copies of $\mathbb{A}^1$ glued together on their overlap, which can be sort-of thought of as a 1-simplex of $\mathbb{A}^1$s; similarly, we'll stick together an "n-simplex of $\mathbb{A}^1$s" to define $\mathbb{P}^1_R$.

Specifically, for $i = 0, \dots, n$, the i^{th} copy of $\mathbb{A}^n$ will be $U_i = \operatorname{Spec} R[x_{0/i}, x_{1/i}, \dots, x_{n/i}]/(x_{i/i} - 1)$. Then, we'll glue $D(x_{j/i})$ in U_i to $D(x_{i/j})$ in U_j, by defining $x_{k/i} = x_{k/j}/x_{i/j}$ and $x_{k/j} = x_{k/i}/x_{j/i}$ for all k.

Exercise 15.3. Show that these gluings agree on triple overlaps, so these glue together into *projective n-space* $\mathbb{P}^n_R$.

As we saw for $\mathbb{P}^1_k$, one can check that $\Gamma(\mathscr{O}_{\mathbb{P}^n_R}) = R$, the constant functions. In other words, there are no interesting global functions on projective space. We can also link this to usual projective space.

Exercise 15.4. Show that if k is an algebraically closed field, then the closed points of $\mathbb{P}^n_k$ are precisely the points $[a_0, a_1, \dots, a_n] \in k^{n+1}$ up to scaling.

We would like to say that $\mathbb{P}^n_R$ is the lines through the origin in $\mathbb{A}^{n+1}_R$, but we don't have the tools to define that yet. In any case, we see that $\mathbb{P}^n$ has no functions, but $\mathbb{A}^{n+1}$ has lots and lots of them, a whole $R[x_0, \dots, x_n]$.

We can define $0 \in \mathbb{A}^{n+1}$ as the inclusion of $\operatorname{Spec} R$ induced by the map $R[x_0, \dots, x_n] \to R$ that is evaluation at 0. Then, there is a map $\pi : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$, which we'll define patch by patch: on $D(x_i) \subset \mathbb{A}^{n+1} \setminus \{0\}$, let $\pi : D(x_i) \to U_i$ be induced by the ring map in the other direction sending $x_{k/i} \mapsto x_k/x_i$. One has to check that this is consistent on overlaps $D(x_i x_j) \to U_i \cap U_j$, but once you know this, it does glue to a map π on the whole space. And we also have a section of $\pi|_{D(x_i)}$: $\mathbb{A}^n \cong U_i$. This may be more familiar as stereographic projection: projective space is glued together from a bunch of affines, and we can actually get our hands on each affine.

Moreover, if two points live in the same ray, they get identified: $\pi(x_0, \dots, x_n) = \pi(\lambda x_0, \dots, \lambda x_n)$ for $\lambda \in R^\times$. Thus, we're close to saying that $\mathbb{P}^n$ is lines in $\mathbb{A}^{n+1}$, though not quite. But this means that $\mathbb{A}^{n+1} \setminus \{0\}$ is acted on by $\mathbb{G}_m$, and therefore $\mathbb{G}_m$ also acts on $\mathbb{A}^{n+1}$. By the abstract nonsense from earlier, this means $\mathscr{O}(\mathbb{A}^{n+1}) = R[x_0, \dots, x_n]$ is graded. The i^{th}-degree terms, $R[x_0, \dots, x_n]_i$, are the homogeneous polynomials of degree i.

Another way to think about this is that $D(x_i) = \operatorname{Spec} R[x_0, \dots, x_n][x_i^{-1}]$, which is still a $\mathbb{Z}$-graded ring (x_i^{-1} has degree -1). In other words, we have a $\mathbb{G}_m$-action on each of these affines, and the actions glue together. In other words, if $\widehat{x}_i$ denotes leaving x_i out, $D(x_i) = \operatorname{Spec} k[x_0, \dots, \widehat{x}_i, \dots, x_n][x_i, x_i^{-1}] \cong \operatorname{Spec}(k[x_0, \dots, \widehat{x}_i, \dots, x_n] \otimes k[x, x^{-1}]) = \mathbb{A}^n \times \mathbb{G}_m$, which are isomorphisms as $\mathbb{G}_m$-spaces (opposite to isomorphisms as graded rings). Thus, $\mathbb{A}^{n+1} \setminus \{0\} = \bigcup_{i=1}^n D(x_i) \cong \mathbb{A}^n \times \mathbb{G}_m$. $\mathbb{G}_m$ will act freely on each $\mathbb{A}^n$, whatever that means (something like a group G acting on $X \times G$, where the quotient is just X). In this case, the quotient is exactly U_i.

What this is trying to show is that $\mathbb{A}^{n+1}/\mathbb{G}_m = \mathbb{P}^n$. This is a little hazy: what's the quotient by a group? It turns out that a quotient of a scheme by a group action, even a free action, isn't always a scheme, but we know it when we see it: this is a good way to think of this. Next time, we'll adopt a different point of view, and discuss the Proj construction: how to make a scheme out of a $\mathbb{Z}$-graded ring.

[43] These are not exactly the *ring objects* in $\mathsf{Rep}_{\mathbb{G}_m}$; instead, we just need the object to be a ring and a $\mathbb{Z}[x, x^{-1}]$-comodule, where the counit and comultiplication maps $R \to R \otimes \mathbb{Z}[x, x^{-1}]$, etc. are ring maps, and the multiplication map $R \otimes R \to R$ to be a map of $\mathbb{Z}[x, x^{-1}]$-modules.

Episode XVI.

Projective Schemes and Proj: 3/10/16

Today, we're going to discuss projective schemes and the Proj construction.

If S is an R-algebra, then it comes with a map $R \to S$, and therefore we get a map $\operatorname{Spec} S \to \operatorname{Spec} R$, which is the notion of an (affine) scheme "over R."[44] If $x \in \operatorname{Spec} R$ is a closed point, it corresponds to some maximal ideal $\mathfrak{m}$, and the fiber of this map is $\operatorname{Spec}(S \otimes_R R/\mathfrak{m}) = \operatorname{Spec}(S/\mathfrak{m})$. Thus, one can think of a scheme over a ring as a family of schemes over the closed points, which are schemes over fields.

For example, the map $R \to R[x_1, \ldots, x_n]$ sending r to itself as a constant polynomial can be viewed in this way: over the closed point $\operatorname{Spec} k \hookrightarrow \operatorname{Spec} R$, the fiber is $\mathbb{A}^n_k$. These fibers can differ, since we may have different residue fields in R; if we used $R = \mathbb{Z}$, then we get $\mathbb{A}^n_{\mathbb{F}_p}$ for each p. This perspective also works for $\mathbb{P}^n_R$; the fiber over each closed point $\operatorname{Spec} k$ is $\mathbb{P}^n_k$.

We'll eventually define what it means for such a family to be "nice," and there are various notions of niceness that correspond to more familiar things in topology. For now, a family is just a map, and the point of mentioning this is to make geometric sense of the notion of a scheme over R when R isn't a field. In the same way, a ring may be thought of as the family of its residue fields. More generally, algebraic geometry over a ring is just a family of algebraic geometry over fields, glued together in some way.

Even though there are some theorems and questions that only make sense over a field, or even over an algebraically closed field, it is useful to do as much as possible over rings, because then we get results about familes as well.

Fix a ring R and let $\mathbb{A}^n = \mathbb{A}^n_R$ and $\mathbb{P}^n = \mathbb{P}^n_R$. Last time, we saw that projective space has very few global functions, only the "constant" functions: $\mathscr{O}(\mathbb{P}^n) = R$. However, we also constructed a map $\mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$, where the "origin" 0 is the copy of $\operatorname{Spec} R \hookrightarrow \mathbb{A}^{n+1}$ at the point $(x_0, \ldots, x_n)$, meaning taking the quotient $R[x_0, \ldots, x_n]/(x_0, \ldots, x_n)$. If we restrict the i^{th} coordinate to be nonzero, we get distinguished opens $U_i = D_{\mathbb{P}^n}(x_i) \subset \mathbb{P}^n$ and $D_{\mathbb{A}^{n+1}\setminus\{0\}}(x_i)$; in this case, the fiber of this map looks like $D_{\mathbb{P}^n}(x_i) \times \mathbb{G}_m$, because $R[x_0, \ldots, x_n][x_i^{-1}] \cong R[x_0, \ldots, \widehat{x}_i, \ldots, x_n] \otimes_R R[x_i, x_i^{-1}]$.

Since functions in $\mathbb{A}^{n+1}$ extend over the origin, then $\mathscr{O}(\mathbb{A}^{n+1} \setminus \{0\}) = R[x_0, \ldots, x_n]$, and this ring split as the direct sum

$$\mathscr{O}(\mathbb{A}^{n+1} \setminus \{0\}) = R[x_0, \ldots, x_n] = \bigoplus_{d \geq 0} R[x_0, \ldots, x_n]_d,$$

where $R[x_0, \ldots, x_n]_d$ is the homogeneous polynomials of degree d: $p(\lambda x_0, \ldots, \lambda x_n) = \lambda^d p$ for all $\lambda \in R$. That is, these polynomials are invariant under rescaling (under $\mathbb{G}_m$), which means that $V(p) \subset \mathbb{A}^{n+1} \setminus \{0\}$ is $\mathbb{G}_m$-invariant, and therefore should define a subset of $\mathbb{P}^n$, in the same way that we defined vanishing loci in affine space.

Each $U_i = \{x_i = 1\}$ is affine, specifically a copy of $\mathbb{A}^n$, so we can define the vanishing locus on each U_i and then check that they glue. Specifically, we let $V(p) \cap U_i = V(p|_{\mathbb{A}^n})$; that is, we set $x_i = 1$ and then calculate $V(p)$, thinking of p as a polynomial in $R[x_0, \ldots, \widehat{x}_i, \ldots x_n]$.

Now, we need these to glue; we might not get the same values, because things are only defined up to scaling, but the place where something vanishes is well-defined; the difference on $U_i \cap U_j$ will be x_j/x_i, and since neither x_i nor x_j is 0, then the notion of vanishing is preserved. Thus, we can actually do this, and this leads to the Proj construction.

The point is, we have very few global functions, so we can't just use functions to define vanishing subsets. However, when we discuss line bundles after break, these homogeneous polynomials will be sections of a line bundle, and sections are good enough to be functions, basically.

Constructing Proj. Whenever we discuss Proj, the term "graded ring" will mean a $\mathbb{Z}_{\geq 0}$-graded ring, with positive degree. Occasionally, we might need $\mathbb{Z}$-graded rings, but the $\mathbb{Z}$-grading will always be made explicit.

[44] More generally, a *scheme over R* is a scheme X with a map $X \to \operatorname{Spec} R$. These form a category, where the morphisms are R-linear; that is, they must commute with the maps back to $\operatorname{Spec} R$.

Hence, let $S_\bullet$ be a graded ring; thus, if S_0 denotes the terms of degree 0, then $S_\bullet$ is an S_0-algebra. We will define a scheme $\operatorname{Proj} S_\bullet \to \operatorname{Spec} S_0$; as with $\mathbb{P}^n$, this is a scheme over a ring, and therefore a family. We would like $\operatorname{Proj} R[x_0, \dots, x_n] = \mathbb{P}^n_R$, and if $S_\bullet = S_0$ only has terms in degree 0, then we'd like $\operatorname{Proj} S_\bullet = \operatorname{Spec} S_0$.

Definition. Let $S_\bullet$ be a graded ring. An ideal $I \subset S_\bullet$ is *homogeneous* if it is generated by homogeneous elements.

These are the ideals that we're going to care about for defining vanishing subsets — with one exception.

Definition. The *irrelevant ideal* is $S_+ = \bigoplus_{d>0} S_d \subset S_\bullet$. Since each S_d is homogeneous, this is a homogeneous ideal.

As graded rings, $S_\bullet/S_+ = S_0$, so the projection map $S_\bullet \to S_0$ should define a map of schemes $\operatorname{Spec} S_\bullet \setminus \{0\} \to \operatorname{Proj} S_0$, for some notion of a 0. It's this sense of forgetting the origin and allowing rescaling, as in $\mathbb{P}^n$, that is why S_+ is called irrelevant.

Definition. The *relevant primes* of $S_\bullet$ are the homogeneous prime ideals that don't contain S_+.

These will be the points of $\operatorname{Proj} S_\bullet$, which seems reasonable, but all of our familiar maximal ideals are not homogeneous; for example, if $S_\bullet = \mathbb{C}[x, y, z]$, the maximal ideals $(x - a, y - b, c - z)$ are not homogeneous. However, if $f(x, y, z) = x^2 + y^2 - z^2$, corresponding to a conic, then (f) is a relevant prime ideal.

We've defined the points of $\operatorname{Proj} S_\bullet$, so next is the open subsets.[45] Suppose $f \in S_+$ is homogeneous; then, $(S_\bullet)_f$ is a $\mathbb{Z}$-graded ring (note it has negative-degree elements, e.g. f^{-1} has degree $-\deg f$). Let $S_{\bullet,f,0} = ((S_\bullet)_f)_0$ be the degree-0 part of $(S_\bullet)_f$.

The reason we chose f to be homogeneous was so that we got a graded ring here; it can be thought of as $(\deg f)$-periodic, in some sense.

Exercise 16.1. Show that the prime ideals of $S_{\bullet,f,0}$ are in bijection with the homogeneous prime ideals of $(S_\bullet)_f$.

This is the analogue of our open cover $\{U_i\}$ of $\mathbb{P}^n$: since the homogeneous primes of $(S_\bullet)_f$ are a subset of $\operatorname{Proj} S_\bullet$, then we can think of this as $\operatorname{Spec} S_{\bullet,f,0} \hookrightarrow \operatorname{Proj} S_\bullet$, and we'd like these subsets to be a cover.

Definition. Let $T \subset S_+$ be a set of homogeneous elements. Then, we define its *vanishing set* $V(T) \subset \operatorname{Proj} S_\bullet$ to be the homogeneous primes containing T, but not S_+.

If $T = (f)$, where f is a homogeneous polynomial, then we define $D(f) = \operatorname{Proj} S_\bullet \setminus V(f)$.

As points, $D(f) \leftrightarrow \operatorname{Spec} S_{\bullet,f,0}$; we will declare this to be an isomorphism of schemes. In particular, we declare $V(T)$ to be closed (so $D(f)$ will be open).

Exercise 16.2. Show that this defines a topology on $\operatorname{Proj} S_\bullet$.

We can also see what happens when we intersect $D(f)$ and $D(g)$, where g is another homogeneous polynomial.

Exercise 16.3. Show that $D(fg) = \operatorname{Spec} S_{\bullet,fg,0} \cong D(g^{\deg f}/f^{\deg g}) \subset S_{\bullet,f,0} = D(f)$.

This gives us the information to glue these open subschemes into a scheme structure on $\operatorname{Proj} S_\bullet$.

Example 16.4. Let k be a field, and let's see what happens to $S_\bullet = k[x_0, \dots, x_n]$. If $f = x_i$, then $(S_\bullet)_f = k[x_0, \dots, x_n][x_i^{-1}] \cong k[x_0, \dots, \widehat{x}_i, \dots, x_n][x_i, x_i^{-1}]$, and $S_{\bullet,f,0} = k[x_0, \dots, \widehat{x}_i, \dots, x_n]$. Thus, saying "$S_{\bullet,f,0}$" really is akin to letting $x_i = 1$, but much more generally.

Hence, we get a map from $\operatorname{Spec} S_\bullet, f \subset \operatorname{Spec} S_\bullet \setminus \{0\}$, which is an open subset of something affine, to $D(f) = \operatorname{Spec} S_{\bullet,f,0} \subset \operatorname{Proj} S_\bullet$, and these glue together to our usual map $\mathbb{A}^{n+1} \setminus 0 \to \mathbb{P}^n = \operatorname{Proj} k[x_0, \dots, x_n]$.

Definition. A scheme X is *quasiaffine* if it's isomorphic to an open subscheme of an affine scheme.

Thus, $\mathbb{P}^n$ is not affine, but it talks to affines: we have a reasonable cover by quasiaffine schemes.

Definition.

[45] The notation here is unfortunate, but both the professor and the textbook couldn't find anything better.

- If R is a ring, a *projective scheme over* R is a scheme $X \cong \operatorname{Proj} S_\bullet$, where $S_0 = R$ and $S_\bullet$ is a finitely generated R-algebra.
- A *quasiprojective scheme* is a quasicompact open of a projective scheme.

Quasicompactness is necessary if things aren't finite-dimensional or Noetherian; this is unlikely, and most schemes one encounters fit into one of these definitions. For example, an affine scheme is quasiprojective (actually projective in a silly way, regarding a ring as a graded ring that's only interesting in degree 0), because $\mathbb{A}^n_k \subset \mathbb{P}^n_k$. Thus, quasiprojective schemes encompass both affine and projective schemes, and mean that anything you can write down with equations will probably be quasiprojective.

We defined $\mathbb{P}^n_R = \operatorname{Proj} R[x_0, \ldots, x_n]$, where each x_i has degree 1. However, if we change the degrees of x_i, we get different graded rings.

Definition. Let $d_1, \ldots, d_n \geq 1$; then, define *weighted projective space* $\mathbb{P}(d_0, \ldots, d_n) = \operatorname{Proj} R[x_0, \ldots, x_n]$, where x_i has degree d_i.

Maybe this seems silly, but the geometry can change in nontrivial ways.

Example 16.5. This example is 8.2.N in Vakil's notes.

Consider $\mathbb{P}_\mathbb{C}(1,1,2) = \operatorname{Proj} \mathbb{C}[x_1, x_2, x_3]$, where x_1 and x_2 have degree 1, and x_3 has degree 2. It turns out this is also isomorphic to $\operatorname{Proj} \mathbb{C}[u, v, w, z]/(uw - v^2)$, where all generators have degree 1.

These graded rings are not isomorphic, but their even-degree components are. And we'll show that if $S_{n\bullet}$ denotes the terms of $S_\bullet$ divisible by n, there's an isomorphism $\operatorname{Proj} S_{n\bullet} = \operatorname{Proj} S_\bullet$, which is sometimes called the *Veronese map*. Something different than for Spec is going on: we can't recover the graded ring from the projective scheme.

In fact, $\operatorname{Proj} \mathbb{C}[u, v, w, z]/(uw - v^2)$ looks like a cone in $\mathbb{A}^3 \subset \mathbb{CP}^3$. This is very different from $\mathbb{P}^3_\mathbb{C}$; in particular, it has a singularity at 0.

Next time, we'll discuss why Proj is special, and how there's a $\mathbb{G}_m$-action hidden in the background; then, we'll discuss what modules over a ring do in the context of scheme theory.

Episode XVII.

Vector Bundles and Locally Free Sheaves: 3/22/16

Last time, we proposed a riddle:

$$\text{rings} : \text{schemes} :: \text{modules} : ??$$

We've defined a nice way to pass between rings and schemes, with Spec and global sections, but one thing we like to do is study modules. What will the corresponding geometric object be in algebraic geometry?

If $(X, \mathscr{O}_X)$ is a ringed space, the sheaf gives us a ring $U \mapsto \mathscr{O}_X(U)$ for every open subset $U \subset X$. There's a notion of modules over this sheaf of rings.

Definition. If $(X, \mathscr{O}_X)$ is a ringed space, an *$\mathscr{O}_X$-module $\mathscr{M}$* is a sheaf of abelian groups on X such that for every open $U \subset X$, $\mathscr{M}(U)$ is an $\mathscr{O}_X(U)$-module and if $V \subset U$ is another open subset, the restriction map $\mathscr{M}(U) \to \mathscr{M}(V)$ is $\mathscr{O}_X(U)$-linear, meaning the following diagram commutes.

$$\begin{tikzcd}
\mathscr{O}_X(U) \times \mathscr{M}(U) \arrow[r, "\mathscr{O}_X(U)\text{-action}"] \arrow[d, "\mathrm{res} \times \mathrm{res}"] & \mathscr{M}(U) \arrow[d, "\mathrm{res}"] \\
\mathscr{O}_X(V) \times \mathscr{M}(V) \arrow[r, "\mathscr{O}_X(V)\text{-action}"] & \mathscr{M}(V)
\end{tikzcd}$$

Exercise 17.1. Show that for every $x \in X$, the stalk $\mathscr{M}_x$ is an $\mathscr{O}_{X,x}$-module.

These show up in lots of situations; the basic example is vector bundles.

Example 17.2. First, suppose M is a smooth manifold.[46] Then, a *rank-n vector bundle V* over M is the data of a smooth manifold V and a map $\pi : V \to M$ such that:

[46] This is meant to be an example that's easier to see, since it's easier to honestly draw pictures of smooth manifolds than most schemes.

- For every $x \in X$, the fiber $\pi^{-1}(x)$ is diffeomorphic to $\mathbb{R}^n$.
- For every $x \in X$, there's a neighborhood $U \subset X$ containing x such that $\pi^{-1}(U)$ is trivial, i.e. there's a diffeomorphism $h : \pi^{-1}(U) \to U \times \mathbb{R}^n$ commuting with projection to U:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "h"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^n \arrow[dl, "\mathrm{pr}_2"] \\ & U. & \end{tikzcd}$$

- For any two neighborhoods U and W on which V is trivial, the map g defined by

$$\begin{tikzcd} (U \cap W) \times \mathbb{R}^n \arrow[rr, "\mathrm{id} \times g"] \arrow[dr] & & (U \cap V) \times \mathbb{R}^n \arrow[dl] \\ & \pi^{-1}(U \cap V) & \end{tikzcd}$$

  must be a smooth map $g : U \cap V \to \mathrm{GL}_n(\mathbb{R})$.

The point is that we have a family of vector spaces over M, and they vary smoothly. The last bullet point tells us that the linear structure is locally preserved; equivalently, one can locally (but perhaps not globally) find n sections $e_1, \dots, e_n$ sending $x \mapsto e_i(x) \in \pi^{-1}(x)$, and such that on each fiber, these sections are a basis.

A vector bundle defines a sheaf $\mathscr{V}$ of $\mathbb{R}$-vector spaces on M: if $U \subset M$ is open, then $\mathscr{V}(U)$ is the space of sections of $\pi : \pi^{-1}(U) \to U$. Locally, such a section is given by $s = \sum f_i e_i$, where e_i is the local fiberwise basis and $f_i \in C^\infty(M)$, so $\mathscr{V}$ is an $\mathscr{O}_M$-module (recall that $\mathscr{O}_M$ is the sheaf of C^∞ functions).

An excellent and very useful example of a vector bundle is the *tangent bundle* TM, whose fiber is the tangent space at a point; sections of TM are vector fields.

Of course, we'd like this to motivate something in algebraic geometry. The nicest kinds of modules are free modules, so let's start with those.

Definition. Let $(X, \mathscr{O}_X)$ be a ringed space and $\mathscr{M}$ be an $\mathscr{O}_X$-module.

- $\mathscr{M}$ is a *free sheaf* if there's an isomorphism of sheaves of abelian groups $\mathscr{O}_X^{\oplus n} \cong \mathscr{M}$.
- $\mathscr{M}$ is a *locally free sheaf* of rank n if for every $x \in X$, there's an open subset $U \subset X$ such that $\mathscr{M}|_U \cong (\mathscr{O}_X|_U)^{\oplus n}$.
- A free sheaf of rank 1 is called an *invertible sheaf*, which seems like strange notation.

Vector bundles are the analogue of locally free sheaves: both are locally trivial, but might not be globally.

Example 17.3. If $(X, \mathscr{O}_X)$ is a locally ringed space and $x \in X$, let $k_x = \mathscr{O}_{X,x}/\mathfrak{m}_x$, so k_x is a field and an $\mathscr{O}_{X,x}$-module. Then, the *skyscraper* at a $p \in X$, denoted $\mathscr{O}_p$, is the $\mathscr{O}_X$-module defined by

$$\mathscr{O}_p(U) = \begin{cases} k_p, & p \in U \\ 0, & p \notin U. \end{cases}$$

This is an $\mathscr{O}_X$-module because if $p \in U$, $\mathscr{O}(U)$ acts on k_p through the map $\mathscr{O}(U) \to \mathscr{O}_{X,p}$.

Dual to this notion is a fiber above a point, which also is reminiscent of vector bundles.

Definition. If $(X, \mathscr{O}_X)$ is a locally ringed space and $\mathscr{M}$ is an $\mathscr{O}_X$-module, the *fiber* at an $x \in X$, denoted $\mathscr{M}|_x$, is $\mathscr{M}_x \otimes_{\mathscr{O}_{X,x}} k_x$, i.e. $\mathscr{M}_x/(\mathfrak{m}_x \cdot \mathscr{M}_x)$. This looks confusing, but the point is that these are the values of sections of $\mathscr{M}$ at x.

For example, if $\mathscr{M}$ is locally free of rank n, then for all $x \in X$, $\mathscr{M}|_x \cong k_x^{\oplus n}$, because $\mathscr{M}_x \cong \mathscr{O}_{X,x}^{\oplus n}$. Be careful to distinguish $\mathscr{M}_x$, which is the stalk, and $\mathscr{M}|_x$, which is the fiber!

Now, we'd like to do build a rank-n vector bundle whose corresponding sheaf is the sheaf of sections. Let X be a scheme;[47] we'd like to construct a vector bundle $V \to X$ such that locally on some open sets U, $V \cong U \times \mathbb{A}^n$. Since we're doing this locally, we may as well assume U is affine, so $U = \operatorname{Spec} R$, and suppose

[47]This construction works just as well for locally ringed spaces, but the point is that the V we construct will also be a scheme.

R is a k-algebra. (Here, k need not be a field, which might be confusing; it often is, but we could also use $\mathbb{Z}$, since every ring is a $\mathbb{Z}$-algebra.) Therefore $U \times \mathbb{A}^n_k = \operatorname{Spec}(R[x_1, \dots, x_n]) \cong \operatorname{Spec}(R \otimes_k k[x_1, \dots, x_n]) = \mathbb{A}^n_R$.

But we also need the transition functions to be k-linear, so that we respect the vector space (or $\mathbb{Z}$-module, if $k = \mathbb{Z}$) structure on $\mathbb{A}^n_k$. Suppose $\mathfrak{U}$ is a cover of X such that for each $U_i \in \mathfrak{U}$, there's an isomorphism $\varphi_i : \mathscr{M}|_{U_i} \to \mathscr{O}|^{\oplus n}_{U_i}$.[48] Thus, if $U_{ij} = U_i \cap U_j$ is nonempty, we have a transition map $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1} : \mathscr{O}(U_{ij})^{\oplus n} \to \mathscr{O}(U_{ij})^{\oplus n}$, which is an isomorphism.

If R is a ring, $\mathrm{GL}_n(R)$ will denote the set of invertible linear maps[49] $\mathbb{A}^n_R \to \mathbb{A}^n_R$, which as usual is the invertible $n \times n$ matrices with coefficients in R (since these define R-linear maps $R^n \to R^n$). These are the linear isomorphisms $\mathbb{A}^n_R \to \mathbb{A}^n_R$ because the elements $A \in \mathrm{GL}_n(R)$ are exactly the linear isomorphisms $R[x_1, \dots, x_n] \to R[x_1, \dots, x_n]$ given by

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto A \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

and so we get an induced isomorphism $\mathbb{A}^n_R \to \mathbb{A}^n_R$. Moreover, any such linear map is certainly determined by a matrix, and has an inverse, so it's an invertible matrix.

The point is that φ_{ij} must be in $\mathrm{GL}_n(\mathscr{O}(U_{ij}))$, so just as in the world of manifolds, we can glue: we have $\mathbb{A}^n_{\mathscr{O}(U_i)}$ over U_i and $\mathbb{A}_{\mathscr{O}(U_j)}$ over U_j, and on their overlap we have an isomorphism between them. This is exactly the data we needed to construct the scheme V!

We can also do this in a coordinate-free way; coordinates are a crutch, as usual. This will allow us to recast a lot of algebraic constructions in module theory as geometric things.

Definition. Let R be a ring.

- There is a forgetful functor $\mathrm{For} : \mathsf{CommAlg}_R \to \mathsf{Mod}_R$ from commutative R-algebras to R-algebras. It has a left adjoint $\mathrm{Sym} : \mathsf{Mod}_R \to \mathsf{CommAlg}_R$. If M is an R-module, $\mathrm{Sym}\, M$ is called its *symmetric algebra*; this is a commutative R-algebra that's freely generated by M, in the same sense that there's a natural identification $\mathrm{Hom}_{\mathsf{Alg}_R}(\mathrm{Sym}\, M, T) = \mathrm{Hom}_{\mathsf{Mod}_R}(M, T)$, for any R-algebra T.
- In the same way, let $T^\bullet$ denote the left adjoint to the forgetful functor $\mathsf{Alg}_R \to \mathsf{Mod}_R$; then, if M is an R-module, $T^\bullet M$ is called the *tensor algebra* on M.

As with all universal properties, we would like a construction. The tensor algebra is given by

$$T^\bullet M = \bigoplus_{i=0}^{\infty} M^{\otimes i},$$

where $M^{\otimes i} = M \otimes \cdots \otimes M$, i times, and $M^{\otimes 0} = R$. The multiplication is given by $(m_1 \otimes m_2) \cdot (m_3 \otimes m_4 \otimes m_5) = m_1 \otimes \cdots \otimes m_5$ and so forth, and extending R-linearly.

Then, we can construct the symmetric algebra from the tensor algebra: let $I \subset T^\bullet M$ be the 2-sided ideal generated by all elements of the form $x \otimes y - y \otimes x$ for $x, y \in T^\bullet M$; then, the symmetric algebra is $\mathrm{Sym}\, M = T^\bullet M / I$. That is, we take words in M, but their order no longer matters. Both $T^\bullet M$ and $\mathrm{Sym}\, M$ have natural graded structures.

We can use the symmetric algebra to define the sheaf of sections in a coordinate-free way, which is slightly more abstract, but much cleaner.

Definition. If $\mathscr{M}$ is a locally free sheaf of rank n over X, then its *dual locally free sheaf* is $\mathscr{M}^\vee = \mathscr{H}\!om_{\mathscr{O}(X)}(\mathscr{M}, \mathscr{O}_X)$.[50]

Now, if $\mathscr{M}$ is an $\mathscr{O}_X$-module, then we define its sheaf of sections to be the sheaf

$$U \longmapsto \operatorname{Spec}(\mathrm{Sym}\, \mathscr{M}^\vee(U)).$$

[48] If restrictions of sheaves are confusing to you, the we also can think of this as an isomorphism of $\mathscr{O}(U_i)$-modules $\widetilde{\varphi}_i : \mathscr{M}(U_i) \to \mathscr{O}(U_i)^{\oplus n}$.

[49] One thing which might be surprising is that we need the inverse to have coefficients in R. For example, multiplication by 2 is invertible over $\mathbb{Q}$, but not over $\mathbb{Z}$, and is not an isomorphism as a map $\mathbb{Z}^n \to \mathbb{Z}^n$.

[50] The *sheaf hom* $\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ is the sheaf defined by $U \mapsto \mathrm{Hom}_{\mathscr{O}_X(U)}(\mathscr{F}(U), \mathscr{G}(U))$.

Wow, why is this? If V is a vector space, $V^\vee$ is the linear functions on V, and therefore the polynomial functions on V can be identified with $\operatorname{Sym} V^\vee$. Thus, this construction does locally look like $\operatorname{Spec}(k[x_1, \dots, x_n]) = \mathbb{A}^n$.

This means that vector bundles are more or less the same as locally free sheaves; in particular, *we will refer to locally free sheaves and vector bundles over schemes interchangeably*, as well as *invertible sheaves and line bundles interchangeably*.

One excellent source of locally free sheaves is the Proj construction we talked about last lecture. If $S_\bullet$ is a graded ring generated by S_1, then subspaces of $\mathbb{P}^n$ correspond to quotients of $k[x_1, \dots, x_n]$ by homogeneous ideals, so one defines $(X, \mathscr{O}_X) = \operatorname{Proj} S$ to be glued together from the distinguished opens $D(x_i)$ with $\mathscr{O}_X(D(x_i)) = ((S_\bullet)_{x_i})_0$.

If $r \in \mathbb{Z}$, we'll define an $\mathscr{O}_X$-module $\mathscr{O}(r)$ by the terms in degree r: $\mathscr{O}(r)(D(x_i)) = ((S_\bullet)_{x_i})_r$; more generally, on the distinguished open $D(f)$, $\mathscr{O}(r)(D(f)) = ((S_\bullet)_f)_r$. Since S_r is an S_0-module, then $\mathscr{O}(r)$ is an $\mathscr{O}_X$-module. Moreover, $\Gamma(\mathscr{O}(r)) = S_r$ and $\Gamma(\mathscr{O}_X) = S_0$.

Since $S_\bullet$ is generated in degree 1, then each $\mathscr{O}(r)(D(x_i)) = ((S_\bullet)_{x_i})_r \cong ((S_\bullet)_{x_i})_0$, but this isomorphism is not canonical! Nonetheless, this means that each $\mathscr{O}(r)$ is a line bundle, and these are a nice class of examples. These are useful because there tend not to be many global functions on projective spaces, but at least we do have line bundles; in particular, as r increases, there are more and more homogeneous elements of degree r (and if $r < 0$, then $\mathscr{O}(r)$ is trivial, since S has no negative-degree elements). Another nice fact is that we can recover a projective scheme from all of these, because

$$S_\bullet \cong \bigoplus_{i=0}^{\infty} \Gamma(X, \mathscr{O}(r)), \tag{17.4}$$

and therefore Proj produces isomorphic projective schemes from both of them. The right-hand side isn't obviously a graded ring, but in fact has a natural graded ring structure, which is determined by the action of $\mathscr{O}(1)$ on each $\mathscr{O}(r)$.

We would also like to define the tangent bundle as a locally free sheaf, but it turns out that this only works for smooth schemes. We haven't defined smoothness yet, so we'll return to this later.

$\backsim \cdot \sim$

If $\mathscr{F}$ and $\mathscr{G}$ are locally free sheaves (vector bundles) of ranks n and m, respectively, there are a few operations we can perform on them. Specifically, the following are also locally free sheaves.

- Their direct sum $\mathscr{F} \oplus \mathscr{G}$ (as sheaves of abelian groups or locally; the notions coincide), which has rank $n + m$.
- The sheaf hom $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$, which has rank nm (since these correspond to matrices, at least locally).
- The tensor product $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$. Here one must be careful; quotients of sheaves aren't always sheaves, so we need to sheafify: if $\mathscr{H}$ denotes the sheaf $U \mapsto \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U)$, then we define $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G} = \mathscr{H}_{\mathrm{sh}}$. In other words, this is the tensor product stalkwise, and on sufficiently small open sets, but perhaps not globally. This will be a locally free sheaf of rank nm.

Claim. If $X = \operatorname{Proj} S_\bullet$, then $\mathscr{O}(1) \otimes \cdots \mathscr{O}(1)$, where we tensor r copies of $\mathscr{O}(1)$, is naturally isomorphic to $\mathscr{O}(r)$.

We can use this to refine (17.4) and see the ring structure:

$$S_\bullet = \bigoplus_{i=0}^{\infty} \Gamma(X, \mathscr{O}(1)^{\otimes r}).$$

In the same way, if X is a scheme and $\mathscr{L}$ is a line bundle over X, then there's a naturally graded ring defined by

$$S_\bullet = \bigoplus_{i=1}^{\infty} \Gamma(X, \mathscr{L}^{\otimes r}).$$

Thus, we can ask if X is projective, by asking whether $X \cong \operatorname{Proj}(S_\bullet)$; this is the analogue of asking whether X is affine by checking whether $X \cong \operatorname{Spec}(\mathscr{O}_X)$.

Next time, we'll talk about quasicoherent sheaves, and some of their uses.

Episode XVIII.

Localization and Quasicoherent Sheaves: 3/24/16

We're trying to understand modules over a ring geometrically; last lecture, we related vector bundles, modules, and locally free sheaves, which was nice, but we also have this really nice adjoint pair $(\Gamma, \operatorname{Spec})$ relating locally ringed spaces and $\mathsf{Ring}^{\mathsf{op}}$. Can we do something similar for modules?

If X is a locally ringed space, taking global sections defines a functor $\Gamma : \mathsf{Mod}_{\mathscr{O}_X} \to \mathsf{Mod}_R$ (where $R = \Gamma(\mathscr{O}_X)$, and $\mathsf{Mod}_{\mathscr{O}_X}$ is the category of $\mathscr{O}_X$-modules). Today, we'll construct a left adjoint $\Delta : \mathsf{Mod}_R \to \mathsf{Mod}_{\mathscr{O}_X}$, called localization, for reasons that we'll see. This is a geometric way of "spreading a module out over a space." And just as $\Gamma(\mathscr{O}_{\operatorname{Spec} R}) = R$, we'll see that $\Gamma(\Delta(M)) = M$ for an R-module M. The notation $\Delta(M)$ for the localization is not the only one; often one will see $\widetilde{M}$ or $\mathscr{M}$.

For most of today, we're going to focus on affine schemes, so let R be a ring and $X = \operatorname{Spec} R$. If M is an R-module, we'll define its localization $\mathscr{M}$ on distinguished opens, just as we did with $\mathscr{O}_X$. For an $f \in R$, let $\mathscr{M}(D(f)) = M_f$, or equivalently, $M \otimes_R R_f$, or $M \otimes_R \mathscr{O}_X(D(f))$; this mirrors the structure sheaf definition $\mathscr{O}_X(D(f)) = R_f$.

This construction makes it evident that $\mathscr{M}$ is an $\mathscr{O}_X$-module if it's a sheaf, but we also need to show that it's a sheaf. This is exactly the same as the argument that $\mathscr{O}_X$ is a sheaf; we need to check what happens when several distinguished opens serve as a cover for a possibly non-distinguished opens. Specifically, suppose X is covered by $\{D(f_i)\}_{i\in I}$; then, what we need to show is precisely the exactness of the following diagram:

$$0 \longrightarrow M \longrightarrow \prod_{i\in I} M_{f_i} \rightrightarrows \prod_{\substack{i,j\in I\\ i\neq j}} M_{f_i f_j}$$

Surjectivity is annoying, so just look at the argument for $\mathscr{O}_X$; for injectivity, suppose $m \in M$ maps to $0 \in \prod M_{f_i}$. Thus, $f_i^{n_i} m = 0$ for some sufficiently large n_i. But since the sets $D(f_i^{n_i})$ still cover X, then $1 = \sum r_i f_i^{n_i}$, so $m = \sum r f_i^{n_i} m = 0$.

One can think of this in a smaller package as $\widetilde{M} = M \otimes_R \mathscr{O}_X$, but this is only true on stalks and distinguished opens: the tensor product is a colimit, so we need to sheafify. Alternatively, we know that every module is a quotient of a free module, which allows us to define localization very concretely: there are sets I and J such that M fits into an exact sequence

$$R^{\oplus J} \xrightarrow{\gamma} R^{\oplus I} \longrightarrow M \longrightarrow 0.$$

Because the tensor product is right exact, we can define the localization of M to fit into the diagram

$$\mathscr{O}_X^{\oplus J} \xrightarrow{\gamma} \mathscr{O}_X^{\oplus I} \longrightarrow \mathscr{M} \longrightarrow 0.$$

However, once again we need to sheafify, since naïve quotients can behave badly.

The adjunction between Δ and Γ is a manifestation of the tensor-hom adjunction; if $\mathscr{F}$ is an $\mathscr{O}_X$-module, then $\Gamma(\mathscr{F}) = \operatorname{Hom}_{\mathscr{O}_X}(\mathscr{O}_X, \mathscr{F})$, and so plugging in $M \otimes_R \mathscr{O}_X$, we can use the tensor-hom adjunction (or really some sheafy version of it) to get M back.

Definition. Let X be a scheme.

- If $X = \operatorname{Spec} R$ is affine, then a *quasicoherent sheaf* on X is an $\mathscr{O}_X$-module $\mathscr{F}$ that's in the image of Δ up to isomorphism: $\mathscr{F} \cong \widetilde{M}$ for some R-module M.
- For a more general scheme X, a quasicoherent sheaf is an $\mathscr{O}_X$-module that is locally quasicoherent (in the sense above) on an affine cover of X.

Quasicoherent is often abbreviated QC, and the category of quasicoherent sheaves over X is denoted $\mathrm{QC}(X)$ or $\mathrm{QCoh}(X)$.

Example 18.1. Not all $\mathscr{O}_X$-modules are QC.

- Let k be a field and $\mathscr{F}$ be the skyscraper at $0 \in \mathbb{A}^1_k$ with stalk $k(x)$. We can show this isn't quasicoherent in the same way that we showed some schemes aren't affine: if $\Delta(\Gamma(\mathscr{F})) \not\cong \mathscr{F}$, then we know it can't be QC. In this case, $\Gamma(\mathscr{F}) = k(x)$, so $M = k(x)$, but then $\Delta(k(x))$ is nonzero at the generic point, so it can't be $\mathscr{F}$.

- Let $x \in X$ and $j : X \setminus x \hookrightarrow X$ be inclusion; then, we define the *extension by zero at x* to be the $\mathscr{O}_X$-module $j_!\mathscr{O}_X$ given by
$$j_!\mathscr{O}_X(U) = \begin{cases} \mathscr{O}_X(U), & x \notin U \\ 0, & x \in U. \end{cases}$$
It's not hard to check the identity and gluing axioms, and by construction, $j_!\mathscr{O}_X$ is an $\mathscr{O}_X$-module, but $\Gamma(j_!\mathscr{O}_X) = 0$, and certainly this isn't the zero sheaf, so it's not quasicoherent.

Examples of non-quasicoherent sheaves are annoying, because most natural sheaves you'd write down are quasicoherent anyways. But even though we'll probably never see those examples again, it's good to have seen them, since no every sheaf is QC.

Looking only at affine schemes, a QC sheaf is just a longwinded way to say a module over a ring, but at least there is geometry.

Example 18.2. A QC sheaf over $X = \operatorname{Spec}\mathbb{Z}$ is a $\mathbb{Z}$-module, or equivalently, an abelian group. Consider the abelian group $\mathbb{Z}/5 \oplus \mathbb{Z}/50 \cong \mathbb{Z}/5 \oplus \mathbb{Z}/5^2 \oplus \mathbb{Z}/2$. What does its localization $\mathscr{M}$ look like geometrically?

Each prime-power components is only supported at that prime, so the $\mathbb{Z}/2$-component is a skyscraper sheaf over the point (2), and $\mathbb{Z}/5$ is the skyscraper sheaf over (5). The $\mathbb{Z}/5^2$ term is annihilated by 5^2, but not 5, and so it's also only supported over (5), but there's some "fuzziness" that's associated to non-reduced behavior.

The following theorem isn't in Atiyah-Macdonald, but it is in Lang's graduate algebra book.[51]

Theorem 18.3. *Let R be a PID and M be a finitely generated module over R. Then, M is a direct sum of cyclic modules:*
$$M \cong \bigoplus_{i=1}^{n} R/(f_i),$$
where $f_i \in R$. More precisely, this splits as
$$M \cong R^{\oplus k} \oplus \bigoplus_{i=1}^{n-k} R/(f_i) :$$
the first part is free, and the second part is torsion.

The point that we care about is that a finitely generated module over a PID splits as a direct sum of its torsion and free parts.

Let k be an algebraically closed field.[52] $k[x]$ is also a PID, and the story is very similar to $\mathbb{Z}$ (and indeed any PID). Let $X = \mathbb{A}^1_k = \operatorname{Spec} k[x]$. Then, inside $\mathrm{QC}(X)$, we have the *finitely generated* QC sheaves, meaning those QC sheaves that are f.g. as modules.[53] Inside those, we also have the finitely generated torsion sheaves, which are the localizations of f.g. torsion $k[x]$-modules.

The data of an f.g. torsion $k[x]$-module M is equivalent to the data of a basis $M \cong k^n$ for a finite dimensional k-vector space and an action of $x \in \operatorname{End}(k^n)$; hence, this is equivalent to some $n \times n$ matrix A, and the $k[x]$-action is $\sum a_i x^i \mapsto \sum a_i A^i \in \operatorname{End}(M)$.

The point is, studying torsion f.g. QC sheaves over $\mathbb{A}^1_k$ is the same as doing linear algebra over k: isomorphism classes of such sheaves correspond to conjugacy classes of matrices.

In linear algebra and functional analysis there's an important theorem called the spectral theorem, which gives a geometric meaning (the spectral measure) to the eigenvalues of a linear operator; we can do something similar here.

Definition. Let R be a ring and M be an R-module. Then, the *support* of M, denoted $\operatorname{Supp} M \subset \operatorname{Spec} R$, is the set $V(r \in R : r \cdot M = 0)$.

[51] Then again, what *isn't* in Lang's graduate algebra book?

[52] Most of the story here still works if k isn't algebraically closed, because $\mathbb{A}^1_k$ still contains all of the eigenvalues of a matrix, but some of the argument is a little messier.

[53] In this case, these are exactly the *coherent sheaves*, but this is only the correct definition in the Noetherian case, and we'll give the more general definition later.

If M is a torsion R-module, then every $m \in M$ is annihilated by some element of R; if M is an f.g. torsion $k[x]$-module, then we can do this with a single polynomial (multiply together all the polynomials that annihilate the generators), so $\operatorname{Supp} M \subsetneq \mathbb{A}^1_k$.

All closed subsets of $\mathbb{A}^1_k$ are either the entire thing or finite sets of points, so this means $\operatorname{Supp} M$ must be finite.

Proposition 18.4 (Spectral theorem). *The points of* $\operatorname{Supp} M \subset \mathbb{A}^1_k$ *are the eigenvalues of its matrix A.*

For example, if $M = k[x]/(x - \lambda)$, then $A = \lambda$ (these are analogous to abelian groups of the form $\mathbb{Z}/p$). Going up to dimension 2, we could take modules such as $M = k[x]/(x - \lambda)^2$ (analogous to $\mathbb{Z}/p^2$), whose matrix is $A = \left(\begin{smallmatrix} \lambda & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

It's no coincidence that we've obtained Jordan blocks! When we take $M = k[x]/(x - \lambda)^n$, we obtain the basis 1, $x - \lambda$, $(x - \lambda)^2, \dots, (x - \lambda)^n$, and therefore $A - \lambda I$ takes each basis element to the next one, and therefore A is the Jordan block of size n for λ.

Algebraically, we know from Theorem 18.3 that these modules generate all f.g. torsion $k[x]$-modules, or equivalently that every matrix has a Jordan form. But what about the geometric perspective?

A cyclic module is a module M with a surjection $R \twoheadrightarrow M$, so these are equivalent to ideals $I \subset R$ (since $M \cong R/I$). This means M is actually a quotient ring of R, and this should correspond to a closed subscheme of $\operatorname{Spec} R$. (We haven't defined closed subschemes yet, but we will.) And we know that everything is generated by modules of this form, and we know what they look like geometrically: neighborhoods of various order over points. Thus, if R is a PID, any f.g. torsion sheaf on $\operatorname{Spec} R$ is a finite union of skyscraper sheaves with stalks $R/\mathfrak{m}^s$ for some s. Thus, over any closed point $x \in X$, if $\mathfrak{m}_x$ denotes the maximal ideal corresponding to x, then $M_x = \bigcup_{N>0} M^{\mathfrak{m}_x^N}$. Each of these factors is an $R/\mathfrak{m}_x^N$-module, and therefore M itself is a module over $\widehat{R}_x = \varprojlim R/\mathfrak{m}_x^N$. And M is a direct sum of these M_x over $x \in \operatorname{Supp} M$.

Lemma 18.5. *Let $\mathscr{M}$ and $\mathscr{N}$ be $\mathscr{O}_X$-modules such that* $\operatorname{Supp} \mathscr{M}$ *and* $\operatorname{Supp} \mathscr{N}$ *are disjoint.*[54] *Then,* $\operatorname{Hom}_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N}) = 0$.

Proof. Geometrically, this makes sense: a homomorphism $\varphi : \mathscr{M} \to \mathscr{N}$ is determined by what it does on stalks, but for every $x \in X$, either $\mathscr{M}_x = 0$ or $\mathscr{N}_x = 0$, so $\varphi_x = 0$, and hence $\varphi = 0$. ⊠

This is useful in algebra: if M and N are R-modules, they define QC sheaves on $\operatorname{Spec} R$, and if their supports are disjoint (which is a geometric notion), we know that they have no module homomorphims.

This is a facet of a very deep and powerful idea in algebra: if A is an associative algebra, one can look at its *center* $Z \subset A$, the elements that commute with A; Z is a commutative ring. If M is an A-module, then restriction gives M a Z-module structure, and therefore we can consider the support $\operatorname{Supp} M \subset \operatorname{Spec} Z$. This means the theory of representations over A has a strong dependence on the points in $\operatorname{Spec} Z$; irreducible representations are supported over a single point. This is a very important technique in representation theory.

We've defined the support of a QC sheaf $\mathscr{M} = \Delta(M)$ as a set, so there's no difference between the supports of $k[x]/(x - \lambda)$ and $k[x]/(x - \lambda)^2$. However, we can recast this by considering the subscheme $\operatorname{Spec}(R/\operatorname{Ann} M)$, which includes into $\operatorname{Spec} R$. If R is a PID, then $\operatorname{Ann}(M) = (p)$, where $p \in k[x]$ is the minimal polynomial of the matrix A determining the action of x on M.

This remembers not just the points, but the scheme-theoretic images of the possibly nonreduced factors of $\mathscr{M}$ over each point in $\mathbb{A}^1_k$. $\operatorname{Spec}(R/\operatorname{Ann} M)$ is called the *scheme-theoretic support*, and is smarter than the previous definition we had, which is sometimes called the *set-theoretic support*.

Another fun fact about this is that $\operatorname{Spec}(R/\operatorname{Ann} M) \hookrightarrow \operatorname{Spec}(R/(\chi_A))$, where χ_A is the characterisitc polynomial for A. This is a restatement of the Cayley-Hamilton theorem! (Remember, this is that $\chi_A(A) = 0$.) However, the minimal polynomial remembers more; the minimal polynomial remembers stacks of different factors over the same point, and the characteristic polynomial does not. This is a useful geometric picture of this linear algebra.

Recall also that M is a cyclic module iff $M \cong R/I$ for an ideal $I \subset R$, which is also equivalent to the characteristic polynomial being equal to the minimal polynomial. This is because in this case, there's no "stacking" behavior, so no data is lost passing to the minimal polynomial. This is equivalent to all Jordan

[54] The *support* of an $\mathscr{O}_X$-module $\mathscr{M}$ is the set of points in X such that $\mathscr{M}_x = 0$; this is a closed set.

blocks for A having distinct eigenvalues; such a matrix A is called a *regular matrix*, and these regular matrices are extremely important in representation theory.

Episode XIX.

The Hilbert Scheme of Points: 3/29/16

For the first part of today's lecture, we're going to give a few more examples of quasicoherent sheaves.

Last time, we talked about how if k is a field, $k[x]$-modules correspond to quasicoherent sheaves over $\mathbb{A}^1_k$: a finitely-generated $k[x]$-module M is a finite-dimensional k-vector space along with a matrix A which is the action of x on M. In this situation, by the fundamental theorem of modules over a PID,

$$M \cong \bigoplus_{i=1}^{s} k[x]/(x-\lambda_i)^{j_i}, \tag{19.1}$$

with each term corresponding to a Jordan block for λ_i with size j. If M is cyclic, meaning there's a surjective $k[x]$-linear homomorphism $\pi : k[x] \twoheadrightarrow M$, then we can assume that the λ_i are distinct in (19.1) over all factors. Cyclic modules correspond to ideals, where $\ker(\pi) \leftrightarrow M$; if $I \subseteq k[x]$ is an ideal, $I \leftrightarrow k[x]/I$. Geometrically, the corresponding quasicoherent sheaf is a union of skyscraper sheaves over the λ_i, but remembering terms up to order j_i (that is, nonreduced behavior). Later, when we define closed subschemes, the ideal I will correspond to the closed subscheme $\operatorname{Spec} k[x]/I \hookrightarrow \operatorname{Spec} k[x]$.

This is some very nice linear algebra, but the geometry isn't so fascinating; what if we try $\mathbb{A}^2_k$? In this case, the correspondence is between $k[x,y]$-modules M which are finite-dimensional k-vector spaces and ideals $I \subseteq k[x,y]$ of *finite codimension* (which is sort of a tautology, since codimension means dimension of the quotient $k[x,y]/I \cong M$). One can localize to think of M as a quasicoherent sheaf on $\mathbb{A}^2$; in any case, we have a surjective map $\mathscr{O}_{A^2} \twoheadrightarrow M$. The ideals of finite codimension correspond to the *finite subschemes* of $\mathbb{A}^2$, $\operatorname{Spec} k[x,y]/I \hookrightarrow \mathbb{A}^2$; the space of all these (finite codimension ideals, $k[x,y]$-modules that are finite-dimensional as vector spaces, or finite subschemes of $\mathbb{A}^2$) is known as the *Hilbert scheme of points* in $\mathbb{A}^2$. There is some justification needed to see why this is a scheme, but we're not going to work through that today; once one does this, though, this is a nice example of a moduli space. If one specializes to modules that have vector-space dimension n, this is called the *Hilbert scheme of n points* in $\mathbb{A}^2$, which is denoted $(\mathbb{A}^2)^{[n]}$.

Just as in the story for $\mathbb{A}^1$, there's a nice linear-algebraic analogue to this story. For now, assume k is algebraically closed (we might not need this, but this frees us from needing to worry about it), and let M be a $k[x,y]$-module that's a finite-dimensional k-vector space. Under the identification $M \cong k^n$ as k-vector spaces, the actions of x and y correspond to matrices $X, Y \in \mathrm{Mat}_{n\times n}(k)$, but since x and y commute in $k[x,y]$, we need X and Y to commute. This means that the map

$$\sum_{i,j=1}^{m} a_{ij} x^i y^j \longmapsto \sum_{i,j=1}^{m} a_{ij} X^i Y^j$$

is a ring homomorphism.

Now, suppose additionally that M is a cyclic module, so there's a surjective map $k[x,y] \twoheadrightarrow M$, or equivalently an isomorphism $k[x,y]/I \cong M$, where I is an ideal of $k[x,y]$. Under the chain of identifications $k[x,y]/I \cong M \cong k^n$ of k-vector spaces, let $v \in k^n$ be the image of 1. Since $k[x,y]\cdot 1_{k[x,y]/I}$ generates all of $k[x,y]/I$, then this means that $\sum a_{ij} X^i Y^j \cdot v$ generates all of k^n; in other words, $\{X^i Y^j v\}$ for some set of i and j is a basis for k^n. A v that makes that hold is a *cyclic vector*. Thus, $(\mathbb{A}^2)^{[n]}$ can be identified with the set of $X, Y \in \mathrm{Mat}_{n\times n} k$ and $v \in k^n$ such that $XY = YX$ and v is cyclic for X and Y, quotiented out by $\mathrm{GL}_n(k)$. This is a scheme, though we lack the tools to show it; if $k = \mathbb{C}$, it's also a manifold.

This stuff also arises in physics, and often the notation

$$\begin{tikzcd} \overset{n}{\bullet} \arrow[loop above] \arrow[loop below] & \overset{1}{\bullet} \arrow[l] \end{tikzcd}$$

is used;[55] this is an example of a *quiver*, and makes the Hilbert scheme of points something called a *quiver variety*.

[55] This is not technically true; we need to add arrows to express the commutativity data that X and Y commute.

Why this notation? Well, for particular X and Y this diagram specializes to

$$\begin{tikzcd} k^n \arrow[loop above, "X"] \arrow[loop below, "Y"] & k, \arrow[l, "i"'] \end{tikzcd}$$

where $i : 1 \mapsto v$. There's lots of interesting stuff to say about these, but for now we return to algebraic geometry. For concreteness, you can take $k = \mathbb{C}$, and then actually draw pictures.

The Hilbert scheme of n points should be thought of as a generalization of a configuration space: the space of n distinct closed points in $\mathbb{A}^1$ (which is an open subset of $\mathbb{A}^n$, though we don't have the tools to prove that now) is a subset of $(\mathbb{A}^1)^{[n]}$, thinking of the points $\{\lambda_1, \dots, \lambda_n\}$ as the module $k[x]/(x-\lambda_1)\cdots(x-\lambda_n)$.

Great, what about two dimensions? Let's consider the space of n distinct closed points $p_1 = (\lambda_1, \mu_1), \dots, p_n = (\lambda_n, \mu_n)$ inside $\mathbb{A}^2_k$. Through the Nullstellensatz, this corresponds to maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ of $k[x,y]$: $\mathfrak{m}_i = (x - \lambda_i, y - \mu_i)$. Then, the easiest way to come up with commuting matrices is to use diagonals: let

$$X = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \qquad \text{and} \qquad Y = \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix}.$$

Then, $v = (1, 1, \dots, 1)$ is a cyclic vector for X and Y. In this way, the configuration space of n distinct closed points of $\mathbb{A}^2$ sits inside $(\mathbb{A}^2)^{[n]}$.

At the other extreme, one could consider ideals $I \subset \mathfrak{m}_{(0,0)} = (x,y)$; then, the inclusion $\operatorname{Spec} k[x,y]/I \hookrightarrow \mathbb{A}^2$ factors through $\operatorname{Spec} k[x,y]/\mathfrak{m}_{(0,0)}^N$; that is, such ideals correspond to the different directions of "fuzziness" (meaning nonreduced behavior) at the origin; for example, if $\mathfrak{m}_{(0,0)} \supset I \supset \mathfrak{m}_{(0,0)}^2$, then I corresponds to a line in $k^2 \cong \mathfrak{m}/\mathfrak{m}^2$. Since I has codimension 2, this is part of $(\mathbb{A}^2)^{[2]}$. In the linear-algebraic world, these ideals correspond to pairs of commuting nilpotent matrices; over $k = \mathbb{C}$, this means all their eigenvalues are 0.

What happens if you allow points to collide? The two points keep track of a line (as an element of $\mathfrak{m}/\mathfrak{m}^2$, which one can regard as the cotangent space $T_0^*\mathbb{A}^2$), so in the limit, we still have a point and a direction out of it, which seems nice.

More generally, there's a nice combinatorial description of the ideals we get. We can lay out the monomials that span $k[x,y]$ as a k-vector space as follows:

$$\begin{matrix} 1 & x & x^2 & x^3 & x^4 & \cdots \\ y & xy & x^2y & x^3y & \cdots & \\ y^2 & xy^2 & x^3y & \ddots & & \\ y^3 & \vdots & & & & \\ \vdots & & & & & \end{matrix}$$

Then, an ideal of finite codimension is given by drawing a staircase from somewhere on the top right to somewhere on the bottom left; then, the ideal is the k-span of all the monomials below that divide, and the codimension is the size of everything above it. This means that the codimension-n ideals are in bijection with the partitions of n. This provides nice interpretations of what happens when points collide, as specific ideals of small codimension.

Since $k[x,y]$ is graded, one can further ask about graded ideals $I \subset k[x,y]$; in this case, $\operatorname{Spec} k[x,y]/I \hookrightarrow \mathbb{A}^2$ is invariant (as a closed subscheme) under the action of $\mathbb{G}_m \times \mathbb{G}_m$ on $\mathbb{A}^2$ (given by rescaling x and y, respectively). There's a lot more that can be said here.

Algebraically, suppose R is a PID (e.g. we've been working with $R = k[x]$). Then, finitely generated modules over R can be written as the direct sum of their free parts (isomorphic to $R^{\oplus n}$ for some n) and their torsion parts (such that every element is annihilated by some $r \in R$), and this provides a nice way to understand ideals.

However, $\mathbb{C}[x,y]$ isn't a PID, so things are less nice. A module over an integral domain is *torsion-free* if for all $v \in M$, $\operatorname{Ann}(v) = 0$ (so nothing except 0 kills anything); since $\mathbb{C}[x,y]$ is an integral domain, then its ideals are torsion-free; in particular, $\mathfrak{m}_{(0,0)} = (x,y)$ is torsion-free.

However, it's not free: it has two generators, so it would have to be free of rank 2. Let's take its fiber over the point $(5,3)$, which is $I|_{(5,3)} = I \otimes_{\mathbb{C}[x,y]} \mathbb{C}[x,y]/(x-5,y-3)$; this is a one-dimensional $\mathbb{C}$-vector space. Moreover, $\mathbb{C}[x,y]_{(x,y)} = I_{(x,y)}$. From the geometric perspective, I defines a subsheaf $\widetilde{I} \subset \mathscr{O}_{\mathbb{A}^2}$.[56] However, since $I_{(x,y)} = \mathbb{C}[x,y]/(x,y)$, then $\widetilde{I} \hookrightarrow \mathscr{O}_{\mathbb{A}^2}$ is an isomorphism away from $V(I)$; in particular, it's an isomorphism on a dense set (in fact, everywhere except the origin). Since away from the origin, $\widetilde{I}$ looks free of rank 1, but at the origin, $I|_{(0,0)} = I \otimes_{\mathbb{C}[x,y]} \mathbb{C}/I = \mathfrak{m}_{(0,0)}/\mathfrak{m}^2_{(0,0)}$. But this is a two-dimensional k-vector space.

The takeaway is that the sheaf $\widetilde{I}$ doesn't have constant dimension: its dimension looks like

$$1111111\text{\LARGE 2}1111111$$

and therefore it cannot be free.

We can define the *rank* of such a sheaf to be the the generic rank, since it's rank 1 on a dense set.

This theorem isn't relevant to the class *per se*, but it's very cool, and relevant to the stuff we were discussing earlier.

Theorem 19.2 (Atiyah-Drinfeld-Hitchin-Manin)**.** *There's a bijection between the space of $n \times n$ matrices X and Y, cyclic vectors v for them, maps $i : \mathbb{C}^k \to \mathbb{C}^n$ and $j : \mathbb{C}^n \to \mathbb{C}^k$ such that $XY - YX + ij = 0$, modulo to the action of $\mathrm{GL}_n(k)$ and the space of (generic) rank-k torsion-free sheaves on $\mathbb{A}^2$.*

These are the quiver varieties represented by the diagram

$$\begin{tikzcd} \bullet^{n} \arrow[loop above] \arrow[loop below] \arrow[r, shift right, "j"'] & \bullet^{k} \arrow[l, shift right, "i"'] \end{tikzcd}$$

This construction is very useful in physics, and is called the *ADHM construction*, after its inventors.

$\backsim \cdot \sim$

Along with the structure sheaf of a scheme X, there's another canonical sheaf, called the sheaf of differentials, which will be a quasicoherent sheaf Ω_X; we'll also get the *tangent sheaf* $T_X = \mathscr{H}om_{\mathscr{O}_X}(\Omega_X, \mathscr{O}_X)$.

Suppose B is a ring and A is a B-algebra; geometrically, this means we have a map $\operatorname{Spec} A \to \operatorname{Spec} B$; let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. If $B = \mathbb{Z}$, we just have a ring A (since $\operatorname{Spec}\mathbb{Z}$ is final in the category of schemes, the data of the map comes for free and doesn't add anything).

Definition. A *B-linear derivation of A* is a B-linear map $\partial : A \to A$ satisfying the *Leibniz rule* $\partial(fg) = f\partial g + g\partial f$.

∂ does not need to be A-linear, and generally isn't! For example, if $A = k[x,y,z]$ and $B = k$, then these look like differential operators, e.g.

$$\partial = \frac{\partial}{\partial x} + p(x,y,z)\frac{\partial}{\partial z},$$

where p is a polynomial.

We can also make the codomain more general.

Definition. Let M be an A-module; then, the space of *B-linear derivations from A to M*, denoted $\operatorname{Der}_B(A,M)$, is the space of $\partial \in \operatorname{Hom}_B(A,M)$ satisfying the Leibmiz rule.

Given such a derivation $\partial : A \to M$, we can form the map $\varphi = \mathrm{id} \oplus \partial : A \to A \oplus M$, which obeys the rule $\varphi(fg) = (fg) \oplus (f\partial g + g\partial f)$; if there was a product rule, this would also be $\varphi(fg) = (f \oplus \partial f) \cdot (g \oplus \partial g)$ for some $\cdot$. This is sort of a graded operator with only two gradings: from this definition, we know what $\cdot$ does everywhere except on two terms of degree 1; in this case, we'll define $\partial f \cdot \partial g = 0$.

[56] This is true in more generality: ideals of R correspond to subsheaves of $\mathscr{O}_{\operatorname{Spec} R}$, using the dictionary between R-modules and QC sheaves on $\operatorname{Spec} R$.

$A \oplus M$ has a ring (well, B-algebra) structure, where the multiplication map is induced by the usual multiplication map $A \otimes A \to A$ and $A \otimes M \to M$, and the zero map $M \otimes M \to 0$: these stack together into a map $(A \oplus M) \otimes (A \oplus M) \to A \oplus M$. With this structure, φ is a B-algebra homomorphism, and in fact, this identifies $\mathrm{Der}_B(A, M)$ and $\mathrm{Hom}_{\mathsf{Alg}_B}(A, A \oplus M)$.

Definition. Let $i : A \to \widetilde{A}$ be an extension of rings.

- The extension is *split* if there's a section $\psi : \widetilde{A} \to A$ (meaning $\psi \circ i = \mathrm{id}_A$).
- A split extension is *square-zero* if $(\ker \psi)^2 = 0$.

This gives us a somewhat funny result.

Proposition 19.3. *There's a one-to-one correspondence between the A-modules M and the split, square-zero extensions of A.*

The idea is that a split, square-zero extension $\widetilde{A} \cong A \oplus I$ as A-modules, and a module M is a square-zero ideal inside $A \oplus M$, which is an extension of A.

This is a surprisingly useful notion in some parts of algebra; not only does it turn derivations into ring homomorphisms, but it has good geometric properties. It illustrates a philosophy that modules are not just less fundamental than rings, but are in some sense special rings themselves.

Episode XX.

Differentials: 3/31/16

"This is a proof by intimidation."

We're going to be doing some calculus in the algebraic geometry setting over the next few weeks, starting with differentials and the language to say all the things we need to say. In order to talk about differentials, we need to spend some time talking about duals. To do this, we need to make one caveat.

Let R be a ring and M be an R-module, so we have an associated quasicoherent sheaf $\mathscr{M} = \Delta(M)$, which is an $\mathscr{O}_X$-module. We can form the dual module $M^\vee = \mathrm{Hom}_R(M, R)$, and therefore this suggests taking a dual sheaf $\mathscr{M}^\vee = \mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{O}_X)$. This is the correct notion if $\mathscr{M}$ is locally free (akin to vector bundles), but not in general.

Suppose $\mathscr{M}$ is the skyscraper sheaf at 0 on $\mathbb{A}^n$, which is the localization of the $k[x, y]$-module $M = k$, where the action of a $p \in k[x, y]$ is multiplying by $p(0)$. Thus, M is torsion, so $M^\vee = \mathrm{Hom}_{k[x,y]}(k, k[x, y]) = 0$: there are no maps from a torsion module to a free one.

There's always a map from M to its double-dual $M \to M^{\vee\vee}$ given by sending $m \in M$ to the map $(\varphi \mapsto \varphi(m))$, which is a map $\mathrm{Hom}_R(M, R) \to R$, i.e. an element of $M^{\vee\vee}$. This is a natural map, but it doesn't have to be an isomorphism: for the module that induces the skyscraper sheaf, it's zero.

Definition. An R-module M is *reflexive* if the natural map $M \to M^{\vee\vee}$ is an isomorphism.

The modules corresponding to free and locally free sheaves are reflexive. This is the sense in which the dual is actually dual; in general, it doesn't behave like you might expect.

Example 20.1. Let's consider a less silly example than the skyscraper sheaf: $\mathfrak{m}_0 = (x, y) \subset k[x, y]$ is an ideal, and therefore a $k[x, y]$-module. Then, $\mathfrak{m}_0^\vee = \mathrm{Hom}_{k[x,y]}(\mathfrak{m}_0, k[x, y])$. If $\varphi : \mathfrak{m}_0 \to k[x, y]$ is a homomorphism, then $\alpha = \varphi(x)$ and $\beta = \varphi(y)$ determine the homomorphism. However, they're not linearly independent: we need $y\alpha = x\beta$, since they're both $\varphi(x, y)$. Therefore, $\alpha = fx$ and $\beta = fy$ for some $f \in k[x, y]$, and more generally $\varphi(r) = fr$: $f = \alpha/x = \beta/y \in k[x, y]$. Thus, $\mathrm{Hom}_{k[x,y]}(\mathfrak{m}_0, k[x, y]) = k[x, y]$. And certainly $(k[x, y])^\vee = k[x, y]$, since it's free over itself, and the canonical map $\mathfrak{m}_0 \to \mathfrak{m}_0^{\vee\vee} = k[x, y]$ is the inclusion.

Thus, torsion-free does not imply reflexive! Duals are weird: they forget information.

The point of this is that to construct the ring of total sections, we took the dual, so we have to be careful. Let M be an R-module, so that $\mathrm{Sym}_R M$ is an R-algebra, given by $R \oplus M \oplus \mathrm{Sym}^2 M \oplus \mathrm{Sym}^3 M \oplus \cdots$.

Definition. If R is a ring, an R-algebra M is an *augmented R-algebra* if there's a *augmentation map* $\varepsilon : M \to R$ that's a section for the R-algebra map $\varphi : R \to M$, i.e. $\varepsilon \circ \varphi = \mathrm{id}$. $\ker(\varepsilon)$ is known as the *augmentation ideal*.

For example, quotienting out by all terms of positive degree in $\operatorname{Sym}_R M$ defines an augmentation map $\varepsilon : \operatorname{Sym}_R M \to R$, so $\operatorname{Sym}_R M$ is an augmented R-algebra, and the augmentation ideal is (generated by) the terms of positive degree.

Geometrically, taking Spec turns everything around: if $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec}\operatorname{Sym}_R M$, then the data that $\operatorname{Sym}_R M$ is an R-algebra gives us a map $Y \to X$, and $\varepsilon^* : X \to Y$ is a section for this map. $\operatorname{Sym}_R M$ is in fact a graded R-algebra, and therefore there's a $\mathbb{G}_m$-action on Y.

For example, let $R = k[x]$ and $M = k[x]/(x)$, which can be thought of as k plus an action of x. Let y be a generator of M, so that $\operatorname{Sym}_R M = k[x] \oplus k \cdot y \oplus k \cdot y^2 \oplus k \cdot y^3 \oplus \cdots \cong k[x,y]/(xy)$.

If we take Spec, this is just the union of the x- and y-axes in $\mathbb{A}^2_k$, projecting down onto $\operatorname{Spec} R = \mathbb{A}^1_k$. This isn't quite a vector bundle: the fiber over every point is a k-vector space, but at 0 it jumps (really, it's a skyscraper over 0), and this is bad.

We can recover M from the algebraic data of $\operatorname{Sym}_R M$ as the degree-1 elements, and there is also a way to do this geometrically. Explicitly, to get the terms of degree-1, take the augmentation ideal and remove all terms of degree at least 2: $M \cong I/I^2$ as R-modules.

Alternatively, one could take the ring $\operatorname{Sym}_R M/I^2$, which is the split square-zero extension $R \oplus M$, in the way that we talked about last time (so M multiplies to 0). The augmentation survives as the map $\varepsilon : \operatorname{Sym}_R M/I^2 \to \operatorname{Sym}_M I \cong R$.

This is equivalent data to what we talked about last time, but is more geometric. The maps of rings induce maps of schemes $\operatorname{Spec} R \to \operatorname{Spec}(R \oplus M) \to \operatorname{Spec}\operatorname{Sym}_R M$. Here, $\operatorname{Spec}(R \oplus M)$ is the first-order neighborhood of the "zero section" in the "total space" of M, which is $\operatorname{Spec}\operatorname{Sym}_R M$. This is because I cuts out the zero section, but we're modding out by I^2, which gives us the first-order neighborhood, as with the dual numbers. In fact, if M is free of dimension n, $\operatorname{Spec}(R \oplus M) = \operatorname{Spec} R \times (\operatorname{Spec}(k[\varepsilon]/(\varepsilon^2)))^{\oplus n}$; if M is locally free, then this is true locally.[57]

Definition. With R, M, and I as in the preceding discussion, let $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec}\operatorname{Sym}_R M$. Then, the *conormal sheaf* to $X \hookrightarrow Y$ is the sheaf associated to the R-module I/I^2.

This seems like it should be the normal sheaf (analogous to the normal bundle), but if you look carefully, this is really linear functionals, so it is more like a dual space.

For example, suppose $R = k$, so $X = \operatorname{Spec} k = \operatorname{Spec}(\operatorname{Sym}_R M/\mathfrak{m})$, where $\mathfrak{m}$ is a maximal ideal of the symmetric algebra. In this case, $I/I^2 = \mathfrak{m}/\mathfrak{m}^2$, which is the cotangent space.

We'll talk about cotangents in order to make conormals make more sense, and hence talk about derivations.

Recall that if A is a B-algebra and M is an A-module, we make $A \oplus M$ into a B-algebra as a square-zero extension, like last time. Then, the space of derivations is $\operatorname{Der}_B(A, M) = \operatorname{Hom}_B(A, A \oplus M)$. These are the B-linear functions $\partial : A \to M$ such that $\partial(fg) = f\partial g + g\partial f$. These feel like differential operators; for example, $\frac{\partial}{\partial x} \in \operatorname{Der}_k(k[x], k[x])$.

If you've been sufficiently Grothendiecized by this class, you should expect some sort of "universal" of "free" derivation given the B-algebra structure $\varphi : B \to A$. This will be an A-module $\Omega_{A/B}$ along with a map $\mathrm{d} : A \to \Omega_{A/B}$.

Definition. Define the A-module $\Omega_{A/B}$, the module of *(Kähler) differentials of A over B*, to be the A-module spanned by elements $\mathrm{d}a$ for all $a \in A$ subject to the following relations for all $a, a' \in A$:

- $\mathrm{d}a + \mathrm{d}a' = \mathrm{d}(a + a')$,
- $\mathrm{d}(aa') = a\mathrm{d}a' + a'\mathrm{d}a$, and
- $\mathrm{d}(\varphi(b)) = 0$.

Then, define $\mathrm{d} : A \to M$, the *de Rham differential*, to send $a \mapsto \mathrm{d}a$.

The first relation forces d to be A-linear, and the second is the Leibniz rule. The last rule makes this compatible with the structure of B.

[57] More generally, one can consider things such as $\operatorname{Spec}(\operatorname{Sym}_R M/I^n)$, which is something people do, though in this context it's not so useful. The point is that we're going to eventually think of I/I^2 as the conormal bundle. We'll return to things like this.

This feels nostalgically like the construction of the tensor product, and so we should expect a universal property.

Proposition 20.2. *Let M be an A-module and $\partial : A \to M$ be a derivation. Then, there is a unique derivation $\widetilde{\partial} : \Omega_{A/B} \to M$ such that the following diagram commutes.*

$$\begin{tikzcd} & \Omega_{A/B} \arrow[d, dashed, "\widetilde{\partial}\,|\,\exists!"] \\ A \arrow[ur, "\mathrm{d}"] \arrow[r, "\partial"'] & M \end{tikzcd}$$

The construction is to let $\widetilde{\partial}(\mathrm{d}a) = \partial a$ and extend A-linearly. As a consequence, $\mathrm{Hom}_A(\Omega_{A/B}, M) = \mathrm{Der}_B(A, M)$ as B-modules.

For example, $T_{A/B} = \Omega_{A/B}^\vee = \mathrm{Hom}_A(\Omega_{A/B}, A)$ can be thought of as vector fields, because it's identified with $\mathrm{Der}_B(A, A)$. Unlike in differential geometry, we're already doing everything relatively, and so these are "relative vector fields," compatible with the map $\operatorname{Spec} B \to \operatorname{Spec} A$. If you want to understand absolute vector fields, you can take $B = \mathbb{Z}$, since $\mathbb{Z}$-linearity is just additivity, which doesn't tell us anything. But the flexibility of taking something relative (which might be a point) is still very useful.

An example of $T_{A/B}$ is $A = k[x, y]$ as a $B = k[x]$-module; then, $\mathrm{Der}_{k[x]}(k[x, y], k[x, y]) \cong k[x, y]$, where $1 \mapsto \frac{\partial}{\partial y}$.

Here's a neat definition, though we haven't earned it.

Definition. A map $\operatorname{Spec} B \to \operatorname{Spec} A$ is *smooth* if the module of differentials $\Omega_{A/B}$ induced from this map is locally free.

There are many questions here: what does it mean for a module to be locally free? It's the same notion turned around, so it's free after sufficiently strong localizations. Over affine schemes, this is the same as free, but this is very far from true in general. Another nice consequence is that smoothness of any scheme is smoothness of the induced map to $\operatorname{Spec} \mathbb{Z}$. This probably isn't very enlightening; the next time we return to smoothness, we'll have the context to appreciate it more.

A vector bundle over a contractible manifold is trivial, which isn't very hard to show. Is the same true in algebraic geometry? The best example of something "contractible" is affine space, right?

Theorem 20.3 (Serre's conjecture/Quillen-Suslin theorem)**.** *If k is an algebraically closed field, all vector bundles on $\mathbb{A}_k^n$ are trivial.*

This is a scary, hard theorem: look at those big names! More seriously, the proof of this theorem was one of the first major breakthroughs demonstrating the power of algebraic K-theory. We definitely haven't earned this theorem.

Anyways, the point is, "smooth things should have tangent bundles." This is a philosophy, but we can just define it.

Definition. Let A be a k-algebra. Then, the *tangent bundle* of $X = \operatorname{Spec} A$ is $TX = \operatorname{Spec}(\operatorname{Sym}_k \Omega_{A/k})$, and the *projectivized tangent bundle* is $\mathbb{P}(TX) = \operatorname{Proj}(\operatorname{Sym}_k \Omega_{A/k})$.

TX locally looks like $X \times \mathbb{A}_k^n$, and $\mathbb{P}(TX)$ locally looks like $X \times \mathbb{P}_k^{n-1}$.

The point is that this will be a vector bundle iff X is smooth. We'll have to unwrap this later. But it advertises another good fact about algebraic geometry: from the beginning, we care about singularities, because rings have singularities. To understand smoothness in a geometric sense, we need calculus, which is why we're talking about differentials.

We defined $\Omega_{A/B}$ with a lot of generators and a lot of relations; if we have generators and relations for A as a B-algebra, we can simplify this. In particular, we can always assume $A \cong B[x_i : i \in I]/(r_j : j \in J)$. In this case, $\Omega_{A/B}$ is much simpler:

$$\Omega_{A/B} \cong \left(\bigoplus_{i \in I} \mathrm{d}x_i \right) / (\mathrm{d}r_j : j \in J).$$

The de Rham differential of a relation is given by expanding A-linearly and using the Leibniz rule, e.g. $\mathrm{d}(xy) = x\,\mathrm{d}y + y\,\mathrm{d}x$. This construction of $\Omega_{A/B}$ makes it a little more apparent that $\Omega_{A/B}$ is a "linearization" of the structure of A. It also makes some nice properties apparent.

Corollary 20.4. *If A is a finitely generated (resp. finitely presented) B-algebra, then $\Omega_{A/B}$ is a finitely generated (resp. finitely presented) A-module.*

Now, suppose $\varphi : B \twoheadrightarrow A$ is surjective, so $A \cong B/I$ for an ideal $I \subset B$; geometrically, we'd have an inclusion of schemes. Then, $\Omega_{A/B} = 0$, because every $a \in A$ is $\varphi(b)$ for some $b \in B$, so $\mathrm{d}a = \mathrm{d}(\varphi(b)) = 0$.

Localizations (open subsets) also don't have any relative differentials: if $f/g \in S^{-1}B$, then

$$\partial\left(\frac{f}{g}\right) = \frac{g\partial f - f\partial g}{g^2} = 0,$$

because $f, g \in B$, so $\partial f = \partial g = 0$. Hence, $\Omega_{S^{-1}B/B} = 0$.

Example 20.5. Suppose $A = k[x,y]/(y^2 - x^3 + x)$, which corresponds to the elliptic curve $y^2 = x^3 - x$. Then, $\Omega_{A/k} = (A\,\mathrm{d}x \oplus A\,\mathrm{d}y)/(2y\,\mathrm{d}y = (3x^2+1)\,\mathrm{d}x)$.

Is this smooth? In other words, is $\Omega_{A/k}$ locally free? If $y \neq 0$, then $\mathrm{d}x$ generates, and if $3x^2 + 1 \neq 0$, then $\mathrm{d}y$ is a generator. If $y = 0$, then $x^3 - x = 0$, so $x = 0, \pm 1$, and therefore $3x^2 + 1 \neq 0$. Thus, these cover everything, so $\Omega_{A/k}$ is a line bundle (locally free of rank 1)![58] In particular, this curve is smooth.

Example 20.6. Our favorite singular curve (well, should be singular) is $A = k[x,y]/(xy)$: the singularity is at the origin. Then, $\Omega_{A/k} = (A\,\mathrm{d}x \oplus A\,\mathrm{d}y)/(x\,\mathrm{d}y + y\,\mathrm{d}x)$. Thus, on the y-axis, $\mathrm{d}y$ generates, and on the x-axis, $\mathrm{d}x$ generates, but at the origin, we need both of them. Thus, $\Omega_{A/k}$ isn't locallty free, so this curve isn't smooth. However, $\Omega_{A/k}|_0 \cong \mathfrak{m}_0/\mathfrak{m}_0^2$, where $\mathfrak{m}_0$ is the maximal ideal corresponding to the origin.

This fact about the fiber is more general: $\Omega_{A/k}$ is a nice way to put all the contangent spaces together.

Proposition 20.7. *Let A be a k-algebra and $x \in \operatorname{Spec} A$ correspond to the maximal ideal $\mathfrak{m}_x$. If $\mathfrak{m}_x$ has residue field k, then there's an isomorphism $\delta : \mathfrak{m}_x/\mathfrak{m}_x^2 \to \Omega_{A/k}|_x = \Omega_{A/k} \otimes_A k_x$.*

Geometrically, we're tensoring with the skyscraper sheaf to obtain the fiber.

We're not going to prove this today, for a lack of time. But one can use this to construct a quasicoherent sheaf $\widetilde{\Omega}_{A/k}$, defined by $\widetilde{\Omega}_{A/k}(D(f)) = \Omega_{A_f/k}$, so localizing as usual. We'll also consider more geometric ways of understanding this.

Another useful word is the analogue of a covering space: there's no way to differentiate along the fibers.

Definition. If A and B are k-algebras, where k is characteristic zero, then if $\Omega_{A/B} = 0$, then the map $B \to A$ is called *étale*.

There is a definition of étale in positive characteristic, but this isn't the correct definition.

Episode XXI.

The Conormal and Cotangent Sequences: 4/5/16

"Even in characteristic 2 this is zero. I guess it's more zero."

Recall that last time, we were talking about differentials: if A is a B-algebra with $\varphi : B \to A$ the map inducing the B-algebra structure on A, then the module of (Kähler) differentials of A is $\Omega_{A/B}$, obtained from the free A-module generated by symbols $\mathrm{d}a$ for all $a \in A$ subject to the relations that $\mathrm{d}(\varphi(b)) = 0$, $d(aa') = a\,\mathrm{d}a' + a'\,\mathrm{d}a$, and $\mathrm{d}(a + a') = \mathrm{d}a + \mathrm{d}a'$. This also satisfies the universal property specified in Proposition 20.2: $\mathrm{Der}_B(A, M) = \mathrm{Hom}_A(\Omega_{A/B}, M)$ for all A-modules M.

It turns out that the fiber of $\Omega_{A/B}$ is the cotangent space.

Proposition 21.1. *Let k be a field and B be a k-algebra. Choose an $x \in \operatorname{Spec} B$ such that the associated maximal ideal $\mathfrak{m}_x \subset B$ has residue field $B/\mathfrak{m}_x \cong k$. Then, there is an isomorphism $\delta : \mathfrak{m}/\mathfrak{m}^2 \to \Omega_{B/k} \otimes_B k = \Omega_{B/k}|_x$.*

[58] This is actually a trivial line bundle: one can write down a nowhere-vanishing differential.

Proof. We're going to prove the dual statement, that there's a natural isomorphism of k-vector spaces $\mathrm{Hom}_k(\Omega_{B/k} \otimes_B k, k) \cong \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$. That is, we'll identify the tangent space with the dual of the fiber. Why did we restate this? The universal property means it's easier to understand maps out of the module of differentials.

We can regard k as a B-module through the quotient map $B \twoheadrightarrow B/\mathfrak{m}_x \cong k$; in this way, the data of B-linearity of a map $\Omega_{B/k} \otimes_B k \to k$ is exactly the same as k-linearity, so $\mathrm{Hom}_k(\Omega_{B/k} \otimes_B k, k) \cong \mathrm{Hom}_B(\Omega_{B/k} \otimes_B k, k)$. By the tensor-hom adjunction, this is also $\mathrm{Hom}_B(\Omega_{B/k}, \mathrm{Hom}_B(k,k)) = \mathrm{Hom}_B(\Omega_{B/k}, k) = \mathrm{Der}_k(B,k)$ by the universal property.

Let's understand this space a little more. If $\partial \in \mathrm{Der}_k(B,k)$, then $\partial|_{\mathfrak{m}_x^2} = 0$, because if $f, g \in \mathfrak{m}_x$, then $\partial(fg) = f\partial g + g\partial f$. Since $f, g \in \mathfrak{m}_x$, then they're 0 in $k = B/\mathfrak{m}_x$, and since everything in $\mathfrak{m}_x^2$ can be written as such a product, then $\partial|_{\mathfrak{m}_x^2} = 0$.

We can identify $B/\mathfrak{m}^2 = k \cdot 1 \oplus \mathfrak{m}/\mathfrak{m}^2$ as rings with trivial multiplication on the second part; after doing this, $\partial(1) = 1 \cdot \partial(1) + \partial(1) \cdot 1 = 2\partial(1)$. However, since $1 \in k = B/\mathfrak{m}$, this means $\partial(1) = 0$. Thus, ∂ is actually a map $\partial : \mathfrak{m}/\mathfrak{m}^2 \to k$, and every such map extends by 0 on k to a derivation by doing this backwards. Hence, $\mathrm{Der}_k(B,k) \cong \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$. ⊠

Here, we've defined a module of differentials for a ring; soon enough, we'll define a sheaf of differentials for a scheme. This provides a different perspective on differentials that makes this isomorphism much more explicit and geometrically intuitive.

There's a nice exact sequence we can get out of this; let's first see it geometrically. Suppose $p : M \to N$ is a submersion of manifolds. If $x \in M$ and $y = p(x)$, then what can we say about their tangent spaces? There's an induced map $\mathrm{d}p|_x : T_xM \to T_yN$; since p is a submersion, then $\mathrm{d}p|_x$ is surjective.[59] What's the kernel of $\mathrm{d}p|_x$? These are the vectors that project to 0, so they correspond to directions that are collapsed by p; in other words, the tangent vectors along the fiber $p^{-1}(y)$. Thus, there is a short exact sequence of $\mathbb{R}$-vector spaces

$$0 \longrightarrow T_x(p^{-1}(y)) \longrightarrow T_xM \xrightarrow{\mathrm{d}p|_x} T_yN \longrightarrow 0. \tag{21.2a}$$

Dualizing, we get a short exact sequence of cotangent spaces, which can be thought of as differentials. Since dualizing is contravariant, this sequence goes in the opposite direction:

$$0 \longrightarrow T_y^*N \longrightarrow T_x^*M \longrightarrow T_x^*(p^{-1}(y)) \longrightarrow 0. \tag{21.2b}$$

Definition. The *relative tangent space* of p at x is $T_{M/N}|_x = T_x(p^{-1}(y))$. These spaces fit together into a vector bundle, the *relative tangent bundle* $T^*_{M/N}$.

These allow us to restate (21.2a) and (21.2b) in terms of short exact sequences of vector bundles.

$$0 \longrightarrow T_{M/N} \longrightarrow TM \xrightarrow{p^*} p^*TN \longrightarrow 0, \tag{21.3a}$$

and the dual sequence

$$0 \longrightarrow p^*T^*N \longrightarrow T^*M \longrightarrow T^*_{M/N} \longrightarrow 0. \tag{21.3b}$$

Keep in mind that (21.3a) is only short exact because p is a submersion; for general p, it's only left exact.

We would like to restate these in terms of algebraic geometry and the module of differentials. We're aiming for a defenition of smoothness, submersions, etc. that allow us to think more geometrically, but *a priori* there's not a lot we can do. The spectrum of a ring can be very singular, and right now we don't have a lot to work with.

In algebraic geometry, everything is done over a ground ring (which can be $\mathbb{Z}$, which doesn't really change much). This is sort of true for differential geometry, though we're limited to the choices of $\mathbb{R}$ and $\mathbb{C}$; nonetheless, it definitely changes the flavor of arguments in algebraic geometry.

Let C be the ground ring and $\varphi : \operatorname{Spec} A \to \operatorname{Spec} B$ be a map of schemes over C. That is, we have a sequence of maps $C \to B \to A$. Geometrically, we'd like to carry the picture of a submersion (with fibers and all that) over, even though it might not always hold.

[59] If you haven't seen submersions before, the definition is exactly that the induced map is surjective on all tangent spaces.

In this case, we have an exact sequence (*not* a short exact sequence) similar to (21.3b), called the *relative cotangent sequence*.

$$A \otimes_B \Omega_{B/C} \xrightarrow{a \otimes \mathrm{d}b \mapsto a\,\mathrm{d}b} \Omega_{A/C} \xrightarrow{\mathrm{d}a \mapsto \mathrm{d}a} \Omega_{A/B} \longrightarrow 0. \tag{21.4}$$

This is exact because at $\Omega_{A/C}$, $a\,\mathrm{d}b \mapsto 0$, because $\mathrm{d}b$ is 0 in $\Omega_{A/B}$.

It's a fact that φ is smooth iff (21.4) is exact (or left exact). That is, *in algebraic geometry, a smooth map is analogous to a submersion*. This can be a little disorienting.

If we know more about these maps, we can conclude more about the left-exactness of (21.4), or lack thereof. Suppose $A = B/I$ and $\varphi : \operatorname{Spec} A \hookrightarrow \operatorname{Spec} B$ is the map induced by the quotient $\varphi^* : B \twoheadrightarrow B/I = A$. In this case, $\Omega_{A/B} = 0$, because φ^* is surjective. In that case, we can calculate the kernel of the first map in (21.4):

$$I/I^2 \xrightarrow{\delta} A \otimes \Omega_{B/C} \longrightarrow \Omega_{A/C} \longrightarrow \Omega_{A/B} = 0. \tag{21.5}$$

Here, $\delta = 1 \otimes \mathrm{d}i : B/I \otimes_B I \to B/I \otimes_B \Omega_{B/C}$, but $B/I \otimes_B I \cong I/I^2$. This sequence is called the *conormal sequence*.

Proposition 21.6. (21.5) *is an exact sequence of A-modules (i.e. B-modules annihilated by I).*

Proof. The only place this is in question is at $A \otimes_B \Omega_{B/C} = A[\mathrm{d}b : b \in B]/(\mathrm{d}c : c \in C)$, and $\Omega_{A/C} = A[\mathrm{d}b : b \in B]/(\mathrm{d}c : c \in C \text{ and } i \in I)$. That is, the kernel is exactly the image of $\delta = 1 \otimes i$. ⊠

This sequence is called the conormal sequence, so is there some conormal thing that it talks about? Recall that the *normal bundle* ν_f of an immersion of manifolds $f : M \to N$ is defined to fit into the short exact sequence

$$0 \longrightarrow TM \xrightarrow{\mathrm{d}f} TN \longrightarrow \nu_f \longrightarrow 0.$$

Dualizing this, we obtain a "conormal bundle"

$$0 \longrightarrow \nu_f^\vee \longrightarrow T^*N \longrightarrow T^*M \longrightarrow 0.$$

This looks suspiciously like (21.5), motivating the following definition.

Definition. The *conormal module* $N^*_{A/B}$ to the inclusion $\operatorname{Spec} B/I \hookrightarrow \operatorname{Spec} B$ is the B/I-module I/I^2.

Example 21.7. Suppose we're working over the point $\operatorname{Spec} C = \operatorname{Spec} k$ and $A = k = B/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of B. Then, $I/I^2 = \mathfrak{m}/\mathfrak{m}^2$, the cotangent space, and $\Omega_{A/k} = 0$. Thus, (21.5) becomes

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow[\delta]{\sim} \Omega_{B/k} \otimes_B k \longrightarrow 0.$$

That is, in this case, the conormal module is the contangent space.

If you like homological algebra, you probably have some questions in mind, and you might already know the answers: when you write down a left exact or right exact sequence, there really should be some long exact sequence in (co)homology, right? There should be some sort of *cotangent complex* $(\Omega_{A/B})^{\mathrm{der}}$ such that (21.5) extends to a long exact sequence

$$\cdots \longrightarrow H_1((\Omega_{A/C})^{\mathrm{der}}) \longrightarrow H_1((\Omega_{A/B})^{\mathrm{der}}) \longrightarrow A \otimes \Omega_{B/C} \longrightarrow \Omega_{B/C} \longrightarrow \Omega_{A/B} \longrightarrow 0.$$

Such a complex exists, but is beyond the scope of this course.

Pullback of differentials. We're going to have to return to earlier chapters and prove some more foundational theorems, but before we do that, let's have some properties of differentials. Suppose we have a commutative diagram

$$\begin{tikzcd} A' & A \arrow[l] \\ B' \arrow[u] & B. \arrow[l] \arrow[u] \end{tikzcd} \tag{21.8}$$

(If you like geometry, reverse all the arrows, as usual.) Then, there exists a natural A-linear map $\Omega_{A/B} \otimes A' \to \Omega_{A'/B'}$. This map is called the *pullback map*, because if $B' = B = k$, then this is the map $\Omega_{A/k} \to \Omega_{A'/k}$, which looks reasonable. The reason this exists is due to the hom-tensor adjunction, which allows us to convert this statement into a statement about derivations, and then one can compose with the map $A \to A'$.

Proposition 21.9. *If* (21.8) *specializes to the pushout square*

$$\begin{tikzcd} A \otimes_B B' & A \arrow[l] \\ B' \arrow[u] & B, \arrow[l] \arrow[u] \end{tikzcd}$$

then the pullback map is an isomorphism.

Proof idea. This is an exercise in Vakil's notes; the idea is that $\Omega_{A/B} \otimes_A A' = \Omega_{A/B} \otimes_B B' = \{b'\,\mathrm{d}a \mid b' \in B', a \in A\}/(\mathrm{d}b = 0 \text{ for all } b \in B)$; however, this is also $\Omega_{(A \otimes_B B')/B'}$. ⊠

Geometrically, this says that if you restrict differentials to a fiber, you get the differentials on the fiber. This all seems a little formal, but will become much nicer once we throw in a little sheaf theory.

There's also a nice description of this in terms of localization, which corresponds to taking an open subset on the base and a smaller open subset on its preimage: then, restricting differentials downstairs should give you all the differentials upstairs.

Proposition 21.10. *Let $\varphi : B \to A$ be a ring map, $T \subset B$ be a multiplicative subset, and $S \subset \varphi(T)$ be a multiplicative subset of A. Then, if* (21.8) *specializes to the diagram*

$$\begin{tikzcd} S^{-1}A & A \arrow[l] \\ T^{-1}B \arrow[u, "\varphi"] & B, \arrow[l] \arrow[u, "\varphi"] \end{tikzcd}$$

then the pullback map is an isomorphism.

This can be thought of as the quotient rule for derivatives, or "differentials of localization are localization of differentials." That is, if $A = B$ and $S = T = \{f, f^2, f^3, \dots\}$, then $\Omega_{A_f/k} = S^{-1}\Omega_{A/k} = A_f \otimes_A \Omega_{A/k}$, since for general modules M, $S^{-1}M = S^{-1}A \otimes_A M$.

The point of Proposition 21.10 is that Ω is a quasicoherent sheaf, because it behaves well under localization. Next time, we'll introduce the affine communication lemma to allow us to carry this over to schemes.