THE GROTHENDIECK GROUP

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ABSTRACT. In the red corner, topological K-theory! The study of stable equivalences of vector bundles over a topological space, and the engine behind Bott periodicity, a result whose reverberations are felt throughout algebraic topology. In the blue corner, modular representation theory! The story of representations in positive characteristic, where the CDE triangle powers applications in group theory and number theory. What do these subjects have to say about each other? Are they just two examples of the same construction? And... why is representation theory like Game of Thrones? Tune in this Friday at 5 p.m. CDT to find out, only on Sophex.

1. DEFINITION AND FIRST EXAMPLES

There are many times in mathematics where some class of objects has a nice additive structure, except that one can't always invert elements. One approach is to say, "oh well," and develop the theory of monoids; another, which we'll discuss today, adds in formal inverses, in effect pretending that they exist, and uses them to recover something useful.

Definition. Recall that a *commutative monoid* $(M, +)$ is a set M with an associative, commutative binary operator $+ : M \times M \to M$ that has an identity: $e \in M$ such that $e + x = x + e = x$ for all $x \in M$.

That is, it is an abelian group, except (possibly) for inverses. The prototypical example is $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, the nonnegative integers under addition, and of course any abelian group satisfies the definition too.

We want a way to turn a monoid into an abelian group, and we'd like to make it as well-behaved as possible. It would be nice if we could make the following work.

(1) If we start with an abelian group A, we should end up with something isomorphic to A.
(2) If we start with $\mathbb{N}$, we should get $\mathbb{Z}$ (after all, this is what happened in set theory).
(3) We'd like this to be well-behaved under mapping: monoid homomorphisms should correspond to group homomorphisms.

It turns out we can do this.[1]

Definition. If M is a commutative monoid, the *Grothendieck group* of M, denoted $K(M)$, is the abelian group satisfying the following universal property: there is a monoid homomorphism $i : M \to K(M)$ such that if A is an abelian group and $f : M \to A$ is a monoid homomorphism, there is a unique group homomorphism $g : K(M) \to A$ making the following diagram commute.

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[dr, "f"'] & K(M) \arrow[d, "g"] \\ & A \end{tikzcd}$$

This is a fine definition, but it may also be nice to have an explicit construction: consider the equivalence relation on $M \times M$ where $(m_1, n_1) \sim (m_2, n_2)$ if $m_1 + n_2 + \ell = m_2 + n_1 + \ell$ for some $\ell \in M$.[2] Addition is well-defined on equivalence classes, and has an abelian group structure: (e, e) is the identity, and the inverse of (m, n) is (n, m). You can write (m, n) as $m - n$, which illustrates what we're actually doing here.

Proposition 1. *This construction satisfies the universal property.*

[1]Perhaps this isn't a great surprise, since this construction *is* the central focus of my talk...

[2]We need to add ℓ because M might not be *cancellative*: $m + n = m + \ell$ doesn't imply $n = \ell$. Thus, just as with localization of rings, we need to add an extra term to actually get an equivalence relation).

Proof. Let $i : M \to K(M)$ send $m \mapsto (m,0))$, and let $f : M \to A$ be a monoid homomorphism to an abelian group A. Then, define $g : K(M) \to A$ by $g((m,n)) = f(m) - f(n)$, which is well-defined on equivalence classes because f is a monoid homomorphism, and is a group homomorphism. Moreover, g is unique, because we need $g((m,0)) = f(m)$, but this automatically determines g on all of $K(M)$, since $g((m,n)) = g((m,0)) - f((n,0))$. $\square$

A combination of the universal property and this explicit construction should tell us that we've met all the niceness requirements we wanted to satisfy.

Here are some more cool facts about the Grothendieck group that I won't prove (but you can find in [1]):

- This construction is covariantly functorial, defining for us a functor $K : \mathsf{CMon} \to \mathsf{Ab}$. Moreover, it's left adjoint to the *forgetful functor* $\mathsf{Ab} \to \mathsf{CMon}$ that sends an abelian group to its underlying commutative monoid.
- If M is a *semiring* to begin with, meaning it looks just like a ring but without additive inverses, we get a ring structure on $K(M)$.

Example 2.

(1) You might have already noticed that if you start with $\mathbb{N}$ under addition, you end up with $\mathbb{Z}$. In this example, $\mathbb{N}$ was already a semiring.

(2) If you start with $\mathbb{N}$ under multiplication, you get the zero group! Oops. If we instead take the positive integers under multiplication, the Grothendieck group is $\mathbb{Q}$.

2. K-THEORY

"K-theory is the study of spaces with hair." – Max Reistenberg

The first application of the Grothendieck group is to construct K-theory, a generalized cohomology theory. K-theory is built out of vector bundles on topological spaces.

Definition. A real (resp. complex) *vector bundle* over a topological space X is, intuitively, a continuous family of vector spaces parameterized by X. Precisely, it is a space E and a surjective map $p : E \twoheadrightarrow X$ such that for each $x \in X$, $p^{-1}(X)$ is a finite-dimensional real (resp. complex) vector space, and there are *local trivializations*: for every $x \in X$, there's an open neighborhood U of x such that there exists a homeomorphism $\psi : p^{-1}(U) \to U \times \mathbb{R}^k$ such that the following diagram commutes.[3]

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\psi"] \arrow[dr, "p"'] & & U \times \mathbb{R}^k \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

A map of vector bundles $f : (E,p) \to (F,p')$ is a continuous map that commutes with projection: $p' \circ f = p$.

For example, we have the *trivial bundle* $\underline{\mathbb{R}}^n = X \times \mathbb{R}^n$ or $\underline{\mathbb{C}}^n = X \times \mathbb{C}^n$, but also the tangent bundle TX, the cotangent bundle T^*X, etc.

Your favorite functorial constructions from the land of finite-dimensional vector spaces tend to carry over to vector bundles, by applying them pointwise and then checking that the result still is locally trivial. For example, given two vector bundles E and F over a space X, their direct sum $E \oplus F$, tensor product $E \otimes F$, Hom $\mathrm{Hom}(E,F)$, and dual E^* are all vector bundles over X.[4] Given a continuous map $g : X \to Y$ and a vector bundle $p : E \to Y$, one can also define the *pullback* bundle g^*E over X, whose fiber over a point $x \in X$ is $p^{-1}(g(x)) \subset E$.

In particular, just like for vector spaces, direct sum is associative and commutative, and $X \times 0 \to X$ is the identity. Thus, isomorphism classes of vector bundles on a space X form a commutative monoid $\mathrm{Vect}(X)$, and in fact a semiring with tensor product. It sounds like now is a good time to take the Grothendieck group.

Definition. The *K-theory* of a compact, Hausdorff space X, denoted $K(X)$, is the Grothendieck group of $\mathrm{Vect}_{\mathbb{C}}(X)$. The *real K-theory* of X, denoted $KO(X)$, is the Grothendieck group of $\mathrm{Vect}_{\mathbb{R}}(X)$.

If $E \to X$ is a vector bundle, then $[E]$ denotes the class of E in $K(X)$ or $KO(X)$.

Here are a few quick facts about K-theory.

[3] In the complex case, of course, we'd use $\mathbb{C}^k$ instead of $\mathbb{R}^k$.

[4] You can also prove some of these by checking that they satisfy the appropriate universal properties.

- Since pullback of vector bundles is a contravariant functor, then K-theory is a contravariant functor $K : \mathsf{CptHaus} \to \mathsf{Ab}$.
- K-theory is homotopy-invariant.[5]

This high-level abstract stuff actually has quite a useful geometric interpretation.

Definition. For any topological space X, there's a unique map $p : X \to \bullet$ sending everything to a point. Pulling back along it, we define the *reduced K-theory* $\widetilde{K}(X) = K(X)/p^*K(\bullet)$. We can define $\widetilde{KO}(X)$ analogously.

A vector bundle over a point is just a vector space, and we have one of those per dimension, so $K(\bullet) \cong \mathbb{Z}$. The pullback of the trivial vector bundle of rank r over a point is the trivial bundle of rank r over X, so in reduced K-theory, we've identified all trivial bundles as 0. This means that if E and E' are vector bundles over X and $[E] = [E']$ in $K(X)$, then $E \oplus \underline{\mathbb{C}}^r \cong E' \oplus \underline{\mathbb{C}}^{r'}$: if you add enough copies of the trivial bundle, they're the same. This is called *stable equivalence*, and is the geometric meaning of reduced K-theory: it's the group of stable abelian classes of vector bundles. This also might provide some intuition for what the Grothendieck group construction does.

Example 3. As an example, consider the tangent bundle on S^2, TS^2. The normal bundle ν fits into a short exact sequence

$$0 \longrightarrow TS^2 \longrightarrow T\mathbb{R}^3|_{S^2} \longrightarrow \nu \longrightarrow 0,$$

and $T\mathbb{R}^3 = \mathbb{R}^3$, so we see that $TS^2 \oplus \underline{\mathbb{R}} \cong \underline{\mathbb{R}}^3$. Thus, in $\widetilde{KO}(S^2)$, $[TS^2] = 0$.

By taking the Grothendieck group, we've lost some information (now we only know stable equivalences, not just isomorphism classes), but computing with abelian groups is much easier than computing with monoids.

So far, we have a contravariant functor $K : \mathsf{CptHaus} \to \mathsf{Ab}$ that's invariant under homotopy and has a reduced version. This sounds like cohomology, which is not a coincidence. But before I talk about that, I should mention the most fascinating aspect of K-theory: Bott periodicity.

Theorem 4 (Bott [2, §2.2])**.**

$$\widetilde{K}(S^n) = \begin{cases} \mathbb{Z}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

$$\widetilde{KO}(S^n) = \begin{cases} \mathbb{Z}, & n \equiv 0,4 \bmod 8 \\ \mathbb{Z}/2, & n \equiv 1,2 \bmod 8 \\ 0, & n \equiv 3,5,6,7 \bmod 8. \end{cases}$$

This result, while interesting on its own, implies a lot of other things. One proof deals with the stable homotopy theory of Lie groups, and is connected via the Adams conjecture[6] to the stable homotopy groups of spheres. Other proofs link it to Morse theory, loop spaces of infinite Lie groups, Clifford algebras, and even spaces of Fredholm operators on a Hilbert space.

One corollary is that there is an isomorphism $\widetilde{K}(X) \to \widetilde{K}(\Sigma^2 X)$, where Σ denotes the suspension of a space. In general, if we have a pair of spaces (X, A) with $A \subset X$, then suspension allows us to define the following long exact sequence.

$$\cdots \longrightarrow \widetilde{K}(\Sigma^n A) \longrightarrow \widetilde{K}(\Sigma^n X) \longrightarrow \widetilde{K}(\Sigma^n (X/A)) \xrightarrow{\delta} \widetilde{K}(\Sigma^{n+1} A) \longrightarrow \cdots$$

In this case, we can use that to make something that looks like cohomology. Let $\widetilde{K}^{2n}(X) = \widetilde{K}(X)$ and $\widetilde{K}^{2n+1}(X) = \widetilde{K}(\Sigma(X))$. This recasts our long exact sequence as follows.

$$\cdots \longrightarrow \widetilde{K}^n(A) \longrightarrow \widetilde{K}^n(X) \longrightarrow \widetilde{K}^n(X, A) \xrightarrow{\delta} \widetilde{K}^{n+1}(A) \longrightarrow \cdots$$

This is the key construction in showing that reduced K-theory is almost a cohomology theory: it satisfies all but one of the axioms of a cohomology theory, which means it behaves just like cohomology, but with different values. This makes it something called a *generalized cohomology theory*, and K-theory was one of the first of these to be discovered.[7]

Finally, it's worth pointing out that there's an electronic hip-hop band named "K Theory." They're reasonable.

[5] If you pull back a vector bundle by two homotopic maps, the two pullbacks are isomorphic, ultimately because finite-dimensional vector spaces are invariant under homotopy. This is the reason for the homotopy-invariance of K-theory.

[6] Now a theorem, thanks to Quillen.

[7] Here things still work similarly for $\widetilde{KO}(X)$; however, since its Bott periodicity has period 8, then we have to do everything mod 8 instead of mod 2. But there's still a long exact sequence of a pair, and it's still a generalized cohomology theory.

3. Modular Representation Theory

"Representation theory is like Game of Thrones. There are lots and lots of characters; most of them are complex, and many of them are unfaithful. But everything gets a lot tensor: there are a lot of duals, and some of the characters end up decomposing!"

The Grothendieck group also appears in representation theory, though in a slightly different form. Recall that a representation V of a finite group G over a field k can be regarded as a $k[G]$-module and vice versa, because if I know what every $g \in G$ does to V, I know what linear combinations of elements of G do to V, and vice versa. Thus, irreducible representations correspond to simple $k[G]$-modules.

Most of the time, people choose $k = \mathbb{C}$. Algebraic closure and characteristic zero make a lot of things much nicer.

Theorem 5 (Maschke [3, §1.4])**.** *If V is a finite-dimensional complex representation of a finite group G, then it splits as a sum of irreducible representations.*

In module theory, this means every finitely generated $\mathbb{C}[G]$-module is a direct sum of simple modules. These kinds of modules are called *semisimple*.

Once again, finite-dimensional, complex G-representations have a direct sum and tensor product (that is: we have the direct sum and tensor product of $\mathbb{C}[G]$-modules), so we can pass to their Grothendieck group, which in this context is called $R_{\mathbb{C}}(G)$, the *representation ring* of G. Now, Maschke's theorem tells us that this is a free abelian group whose basis is the classes of irreducible representations. The ring structure is often more complicated, however.

But there's another way to look at this: a complex representation $\rho : G \to \mathrm{Aut}(V)$ is uniquely determined by its character, the function $\chi_\rho : g \mapsto \mathrm{Tr}(\rho(g))$. The group generated by these functions under addition also has a ring structure given by pointwise multiplication of functions, and is called the *character ring* of G. Since $\chi_{\rho_1 \oplus \rho_1} = \chi_{\rho_1} + \chi_{\rho_2}$, $\chi_{\rho_1 \otimes \rho_2} = \chi_{\rho_1} \cdot \chi_{\rho_2}$, and the characters of irreducible representations generate the character ring, then we actually obtain an isomorphism between the representation ring and the character ring.[8]

Example 6. S_3 acts on $\mathbb{C}^3$ by permuting the basis vectors. In this case, the character χ can be calculated as $\chi(e) = \mathrm{Tr}(I_3) = 3$; χ of a transposition is 1, since there's a single 1 on the diagonal; and χ of $(a\ b\ c)$ is 0, since there are no nonzero entries on the diagonal.

However, under this representation, the space spanned by $(1,1,1)$ is invariant, so we get two subrepresentations, that one (which is a trivial representation) and its complement, which is a two-dimensional irreducible representation.

This is all really nice, so let's set it aside and study what happens in characteristic p.[9] If $p \mid |G|$, then Maschke's theorem can fail.[10] This means that irreducible representations are no longer a basis for the representation in positive characteristic, which suggests that maybe we should try a different definition.

Consider a short exact sequence of representations

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0. \tag{1}$$

Maschke's theorem means that if we're in characteristic 0, then $M \cong M' \oplus M''$, so the Grothendieck group is also the group generated by the isomorphism classes of G-representations with relations $[M] = [M'] + [M'']$ induced by *short exact sequences*, rather than direct sums, and it's this viewpoint that will be more useful to us.

Definition. Over any field k, define the *representation ring* of G, denoted $R_k(G)$, to be the group generated by isomorphism classes of G-representations over k (i.e. finitely generated $k[G]$-modules), with relations $[M] = [M'] + [M'']$ for all short exact sequences of the form (1).

We can also define this for other classes of representations. For example, $P_k(G)$ denotes the same construction, but using the class of projective finitely generated $k[G]$-modules, rather than all finitely generated $k[G]$-modules.

[8] All of this works just as well for compact topological groups.

[9] A standard reference for representation theory in positive characteristic is [3, Part 3].

[10] Let V be the two-dimensional representation of $\mathbb{Z}/p = \langle x \rangle$ where x^n acts as $\left(\begin{smallmatrix} 1 & 0 \\ n & 1 \end{smallmatrix}\right)$, so that

$$\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ nx + y \end{pmatrix}.$$

Thus, the only $\mathbb{Z}/p$-stable subspace is the space of $\left(\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right)$. Thus, V isn't irreducible, since it has a $\mathbb{Z}/p$-stable subspace, but we can't write it as the direct sum of that subspace and anything else.

This agrees with $R_{\mathbb{C}}(G)$ as we've defined it, and always has the property that the irreducible representations are a basis for it. However, over positive characteristic, we've done something new: in the Grothendieck group, we've identified a representation V with the direct sum of all of its irreducible components. This is its *semisimplification*: we're pretending that V is semisimple, and the nearest semisimple representation is the one that has the same irreducible components.

The semisimplification has some nice properties: one can define an analogue to ordinary group characters called *Brauer characters*,[11] and the semisimplification of a representation uniquely determines its Brauer character. It also uniquely determines the characteristic polynomials of the elements of the group. So the representation ring characterizes a lot of the things we would want to know about a representation.

The most useful thing about this construction is that it allows us to stitch together the stories of representation theory in characteristic 0 and characteristic p. This is done via the CDE triangle, which is a commutative diagram and a bunch of theorems about it.

$$\begin{tikzcd} P_{\overline{\mathbb{F}}_p}(G) \arrow[rr, "c"] \arrow[dr, "e"'] & & R_{\overline{\mathbb{F}}_p}(G) \\ & R_{\overline{\mathbb{Q}}_p}(G) \arrow[ur, "d"'] & \end{tikzcd}$$

So first of all, yes, there are p-adic numbers, but the only reason we use them is because there's a really nice map $\mathbb{Q}_p \twoheadrightarrow \mathbb{F}_p$, reducing mod p, and nothing like that exists for $\mathbb{C}$. So over $\overline{\mathbb{Q}}_p$, the representation theory is essentially the same as over $\mathbb{C}$, but we can reduce mod p to obtain a representation over $\overline{\mathbb{F}}_p$. This is exactly what the map d does. Then, c sends a projective $\overline{\mathbb{F}}_p[G]$-module to its class in the representation ring, and e is a "lifting" map: it turns out that one can lift a projective representation from positive characteristic to characteristic 0. Here are some of the results about the diagram [3, §16.1]:

- d is surjective, and c and e are injective.
- There's an algorithm for obtaining the irreducible Brauer characters of a group: start with the irreducible representations in characteristic zero; then, reduce them. Some may not be irreducible anymore, so decompose them into irreducibles. Since d is surjective, this gives us everything.
- Since c is injective, then if two projective $\overline{\mathbb{F}}_p[G]$-modules have the same semisimplifications, then they're isomorphic.
- Since d is surjective, then you can lift characteristic p representations to characteristic 0 representations, as long as you're willing to allow "virtual representations" (so possibly negative coefficients). This is actually an important point: it's hard to control what's going on in positive characteristic, but this tells us that characteristic zero "sees everything," and that, though we get new irreducibles in positive characteristic, they're not that weird.

These results are used in the classification of finite simple groups, in number theory, and, in a connection I was pretty pleasantly surprised about, Quillen's proof of the Adams conjecture mentioned earlier.

4. Final Notes

These examples are more related than they might seem: in algebraic geometry, there's an analogy between vector bundles and projective modules. In fact, the original motivation for Grothendieck's construction was a generalization of the Riemann-Roch theorem applied to coherent sheaves on a space, which act like both vector bundles and modules. So in the end, maybe the interconnectedness of these examples isn't a huge surprise.

References

[1] "Grothendieck Group" (Version 16). nLab. http://ncatlab.org/nlab/revision/Grothendieck+group%2F16.
[2] Hatcher, Allen. Vector Bundles and K-theory. 2003. http://www.math.cornell.edu/~hatcher/VBKT/VB.pdf.
[3] J.P. Serre, *Linear Representations of Finite Groups* (trans. L.L. Scott), Grad. Texts in Math. **42**, Springer-Verlag, New York, 1977.

[11] Over a field of characteristic p, Brauer characters, or *modular characters*, are class functions on the set of *p-regular* conjugacy classes of G; that is, the conjugacy classes whose orders don't divide p. The idea is that if ρ is a representation, the eigenvalues of ρ must be m^{th} roots of unity, where m is the least common multiple of the orders of the p-regular elements. Then, there's an isomorphic copy of the m^{th} roots of unity in $\overline{\mathbb{Q}}_p$, so the Brauer character is defined by using those roots instead of the ones from $\overline{\mathbb{F}}_p$, producing a class function valued in $\overline{\mathbb{Q}}_p$, which is characteristic zero.