MSRI: QUANTUM SYMMETRIES INTRODUCTORY WORKSHOP

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These notes were taken at MSRI's introductory workshop on quantum symmetries in Spring 2020. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to `a.debray@math.utexas.edu`.

1. Sarah Witherspoon: Hopf algebras, I

Our perspective on Hopf algebras, their actions on rings and modules, and the structures on their categories of rings and modules, will be to think of them as generalizations of group actions and representations; groups actions are symmetries in the usual sense, and Hopf algebra actions are often related to "quantum symmetries."

We're not going to give the full definition of a Hopf algebra, because it would require drawing a lot of commutative diagrams, but we'll say enough to give the picture.

Throughout this talk we work over a field k; all tensor products are of k-vector spaces.

Definition 1.1. A *Hopf algebra* is an algebra A together with k-linear maps $\Delta\colon A \to A\otimes A$, called *comultiplication*; $\varepsilon\colon A\to k$, called the *counit*; and $S\colon A\to A$, called the *coinverse*. These maps must satisfy some properties, including that ε is an algebra homomorphism and that S is an *anti-automorphism*, i.e. that $S(xy) = S(y)S(x)$.

The definition is best understood through examples.

Example 1.2.

(1) Let G be a group. Then the group algebra $k[G]$ is a Hopf algebra, where for all $g\in G$, $\Delta(g) \coloneqq g\otimes g$, $\varepsilon(g)\coloneqq 1$, and $S(g)\coloneqq g^{-1}$. This is a key example that allows us to generalize ideas from group actions to Hopf algebra actions: whenever we define a notion for Hopf algebras, when we implement it for $k[G]$ it should recover that notion for groups.

(2) Let $\mathfrak g$ be a Lie algebra over k. Then its universal enveloping algebra $\mathcal U(\mathfrak g)$ is a Hopf algebra, where for all $x\in\mathfrak g$, $\Delta(x)\coloneqq x\otimes 1 + 1\otimes x$, $\varepsilon(x)\coloneqq 0$, and $S(x)\coloneqq -x$. Since ε is an algebra homomorphism, $\varepsilon(1_{\mathcal U(\mathfrak g)}) = 1$.

For example,

$$\mathcal U(\mathfrak{sl}_2) = k\langle e,f,h \mid ef - fe = h, he - eh = 2e, hf - fh = -2f\rangle, \tag{1.3}$$

given explicitly by the basis of $\mathfrak{sl}_2$

$$e \coloneqq \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\qquad f\coloneqq \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},\qquad h\coloneqq\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{1.4}$$ ◀

Both of these examples are classical, in that they've been known for a long time. But more recently, in the 1980s, people discovered new examples, coming from quantum groups.

Example 1.5 (Quantum $\mathfrak{sl}_2$)**.** Let $q\in k^\times\setminus\{\pm 1\}$. Then, given a simple Lie algebra $\mathfrak g$, we can define a "quantum group," $\mathcal U_q(\mathfrak g)$, which is a Hopf algebra. For example, for $\mathfrak{sl}_2$,

$$\mathcal U_q(\mathfrak{sl}_2) = k\left\langle E, F, K^{\pm 1} \mid EF - FE = \frac{K - K^{-1}}{q-q^{-1}}, KE = q^2EK, KF = q^{-2}EK\right\rangle, \tag{1.6}$$

with comultiplication

$$\Delta(E)\coloneqq E\otimes 1 + K\otimes E \tag{1.7a}$$

$$\Delta(F)\coloneqq F\otimes K^{-1} + 1\otimes F \tag{1.7b}$$

$$\Delta(K^{\pm 1})\coloneqq K^{\pm 1}\otimes K^{\pm 1} \tag{1.7c}$$

and counit $\varepsilon(E) = \varepsilon(F) = 0$ and $\varepsilon(K) = 1$. This generalizes to other simple $\mathfrak{g}$, albeit with more elaborate data. ◂

Example 1.8 (Small quantum $\mathfrak{sl}_2$)**.** Let q be an n^{th} root of unity. Then, as before, given a simple Lie algebra $\mathfrak{g}$, we can define a Hopf algebra $u_q(\mathfrak{g})$, called the *small quantum group* for $\mathfrak{g}$ and q, which is a finite-dimensional vector space over k; for $\mathfrak{sl}_2$, this is

$$u_q(\mathfrak{sl}_2) = \mathcal{U}_q(\mathfrak{sl}_2)/(E^n, F^n, K^n - 1). \tag{1.9}$$ ◂

Before we continue, we need some useful notation for comultiplication, called *Sweedler notation.* Let A be a Hopf algebra and $a \in A$; then we can symbolically write

$$\Delta(a) = \sum_{(a)} a_1 \otimes a_2. \tag{1.10}$$

Comultiplication in a Hopf algebra is *coassociative*, in that as maps $A \to A \otimes A \otimes A$,

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta. \tag{1.11}$$

Therefore when we iterate comultiplication, we can symbolically write

$$(\mathrm{id} \otimes \Delta) \circ \Delta(a) = \sum_{(a)} a_1 \otimes a_2 \otimes a_3 \tag{1.12}$$

without worrying about parentheses.

Actions on rings. Hopf algebra actions on rings generalize group actions on rings by automorphisms and actions of Lie algebras on rings by derivations. If a group G acts on a ring R, then for all $g \in G$ and $r, r' \in R$,

$$g(rr') = (gr)(gr') \tag{1.13a}$$

$$g(1_R) = 1_R. \tag{1.13b}$$

In $k[G]$, our Hopf algebra avatar of G, $\Delta(g) = g \otimes g$, and $\varepsilon(g) = 1$.

If a Lie algebra $\mathfrak{g}$ acts on a ring R by derivations, then for all $x \in \mathfrak{g}$ and $r, r' \in R$,

$$x \cdot (rr') = (x \cdot r)r' + r(x \cdot r') \tag{1.14a}$$

$$x \cdot (1_R) = 0. \tag{1.14b}$$

In $\mathcal{U}(\mathfrak{g})$, our Hopf algebra avatar of $\mathfrak{g}$, $\Delta(x) = x \otimes 1 + 1 \otimes x$, and $\varepsilon(x) = 0$. These two examples suggest how we should implement a general Hopf algebra action on a ring: comultiplication tells us how to act on the product of two elements, and the counit tells us how to act on 1.

Definition 1.15. Let A be a Hopf algebra and R be a k-algebra. An *A-module algebra structure* on R is data of an A-module structure on R such that for all $a \in A$ and $r, r' \in R$,

$$a \cdot (rr') = \sum_{(a)} (a_1 \cdot r)(a_2 \cdot r) \tag{1.16a}$$

$$a \cdot (1_R) = \varepsilon(a) 1_R. \tag{1.16b}$$

Thus a group action as in (1.13) defines an action of the Hopf algebra $k[G]$, and a Lie algebra action as in (1.14) defines an action of the Hopf algebra $\mathcal{U}(\mathfrak{g})$.

Example 1.17. The quantum analogue of the $\mathfrak{sl}_2$-action on $k[x, y]$, thought of as (functions on the) plane, there is an action of $\mathcal{U}_q(\mathfrak{sl}_2)$ on the *quantum plane*

$$R \coloneqq k\langle x, y \mid xy = qyx\rangle. \tag{1.18}$$

This is a deformation of $k[x, y]$, which is the case $q = 1$. The explicit data of the action is

$$E \cdot x = 0 \qquad F \cdot x = y \qquad K^{\pm 1} \cdot x = q^{\pm 1} x \tag{1.19}$$

$$E \cdot y = x \qquad F \cdot y = 0 \qquad K^{\pm 1} y = q^{\mp 1} y. \tag{1.20}$$

One has to check that this extends to an action satisfying Definition 1.15, but it does, and R is an A-module algebra. Here E and F act as *skew-derivations*, e.g.

$$E \cdot (rr') = (E \cdot r)r' + (K \cdot r)(E \cdot r') \tag{1.21}$$

for all $r, r' \in R$. ◀

Given a Hopf algebra action of A on R in this sense, we can construct two useful rings: the *invariant subring*

$$R^A := \{r \in R \mid a \cdot r = \varepsilon(a) \cdot r \text{ for all } a \in A\}, \tag{1.22}$$

and the *smash product ring* $R \# A$, which as a vector space is $R \otimes A$, with multiplication given by

$$(r \otimes a)(r' \otimes a') := \sum_{(a)} r(a_1 \cdot r') \otimes a_2 a'. \tag{1.23}$$

The smash product ring knows the A-module algebra structure on R. Often, rings we're interested in for other reasons are smash product rings of interesting Hopf algebra actions, and identifying this structure is useful.

Example 1.24. The *Borel subalgebra* of $\mathcal{U}_q(\mathfrak{sl}_2)$ is $k\langle E, K^{\pm 1} \mid KE = q^{-2}K\rangle$. This is isomorphic to the smash product $k[E] \# k\langle K\rangle$, where $k\langle K\rangle$ is the group algebra of the free group on the single generator K.

In fact, there's a sense in which $\mathcal{U}_q(\mathfrak{sl}_2)$ is a deformation of $k[E, F] \# k\langle K\rangle$: in this smash product ring, E and F commute, and we deform this to $\mathcal{U}_q(\mathfrak{sl}_2)$, in which they don't commute. ◀

Modules. Given a Hopf algebra A, what is the structure of its category of modules? The first thing we can do is take the tensor product of A-modules U and V using comultiplication: for $a \in A$, $u \in U$, and $v \in V$,

$$a \cdot (u \otimes v) = \sum_{(a)} a_1 \cdot u \otimes a_2 \cdot v. \tag{1.25}$$

Moreover, k has a canonical A-module structure via the counit: $a \cdot x := \varepsilon(a)x$ for $a \in A$ and $x \in k$. Finally, if U is an A-module, its vector space dual $U^* := \mathrm{Hom}_k(U, k)$ has an A-module structure via S: for all $a \in A$, $u \in U$, and $f \in U^*$, $(a \cdot f)(u) := f(S(a)u)$.

The existence of tensor products, duals, and the ground field in the world of Hopf algebra modules is a nice feature: these aren't always present for a general associative algebra. Moreover, these constructions interact well with each other.

(1) Coassociativity of Δ implies the tensor product is associative: for A-modules U, V, and W, we have a natural isomorphism $U \otimes (V \otimes W) \xrightarrow{\cong} (U \otimes V) \otimes W$.

(2) In any Hopf algebra A, we have the condition

$$\sum_{(a)} \varepsilon(a_1)a_2 = \sum_{(a)} a_1 \varepsilon(a_2) \tag{1.26}$$

for any $a_1, a_2 \in A$. This implies k, as an A-module, is the unit for the tensor product: we have natural isomorphisms $k \otimes U \cong U \cong U \otimes k$ for an A-module U.

(3) Suppose U is an A-module which is a finite-dimensional k-vector space. Then it comes with data of a *coevaluation map* $c\colon k \to U \otimes U^*$ sending

$$1 \longmapsto \sum_i u_i \otimes u_i^*, \tag{1.27}$$

where $\{u_i\}$ is a basis for U over k and $\{u_i^*\}$ is its dual basis; this map turns out to be independent of basis. We also have an *evaluation map* $e\colon U^* \otimes U \to k$ sending $f \otimes u \mapsto f(u)$. Now, not only are these A-module homomorphisms, but the composition

$$U \xrightarrow{c \otimes \mathrm{id}_U} U \otimes U^* \otimes U \xrightarrow{\mathrm{id}_U \otimes e} U \tag{1.28}$$

is the identity map.

Definition 1.29. A *tensor category*, or *monoidal category* is a category $\mathcal{C}$ together with a functor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an object $\mathbf{1} \in \mathcal{C}$ called the *unit*, and natural isomorphisms $U \otimes (V \otimes W) \cong (U \otimes V) \otimes W$ and $\mathbf{1} \otimes U \cong U \cong U \otimes \mathbf{1}$ for all objects U, V, and W in $\mathcal{C}$, subject to some coherence conditions.

Our key examples of tensor categories are the category of modules over a Hopf algebra A, as well as the subcategory of finite-dimensional modules.

If the coinverse of A is invertible, which is always the case when A is finite-dimensional over k, then $\mathcal{C} = \mathcal{M}od_A$ is a *rigid* tensor category, meaning that every object U has a *right dual* ${}^*U := \mathrm{Hom}_k(U, k)$, which means the composition (1.28) is the identity.

Remark 1.30. Notations for left and right duals differ. We're following EGNO, but Bakalov-Kirillov use a different convention; be careful! $\blacktriangleleft$

Some Hopf algebras' categories of modules have additional structure or properties: they might be semisimple, or braided, or even symmetric. This amounts to additional information on the Hopf algebra itself.

References