INTRODUCTION TO SPECTRAL SEQUENCES

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Contents

1. Introduction to the general formalism: 5/8/17

Today, Adrian spoke about what a spectral sequence is and where they come from. The next four lectures will be interesting examples, even if today is somewhat dry.

Definition 1.1. A **(homological) spectral sequence** is the data of

- modules over a ring[1] $E^r_{p,q}$ indexed by $r \geq N$ for some positive N and $p, q \in \mathbb{Z}$, and
- maps $d_r \colon E^r_{p,q} \to E^r_{p-r,q-1+r}$, called **differentials**,

subject to the following conditions:

- $d_r^2 = 0$, and
- for all p, q, and r, $E^{r+1}_{p,q}$ is the homology of the chain complex $(E^r_{p-r\bullet,q-1+r\bullet}, d_r)$ at $E^r_{p,q}$.

The way in which the differentials affect the grading is pretty opaque, so let's see what it looks like for small r.

$$\begin{tikzcd} E^0_{p,q} \arrow[d, "d_0"] \\ E^0_{p,q-1} \end{tikzcd} \qquad \begin{tikzcd} E^1_{p-1,q} & E^1_{p,q} \arrow[l, "d_1"'] \end{tikzcd} \qquad \begin{tikzcd} E^2_{p-2,q+1} & & \\ & & E^2_{p,q}. \arrow[ull, "d_2"'] \end{tikzcd}$$

The differentials swing from downward to leftward, and comes closer and closer to pointing northwest.

This is a lot of structure, and one usually visualizes it as a book, with **pages** $E^r_{\bullet,\bullet}$, and each page is thought of as a lattice with the differentials:

$$\begin{tikzcd} & \vdots & \vdots & \vdots & \\ \cdots & E^r_{p+1,q-1} & E^r_{p+1,q} & E^r_{p+1,q+1} & \cdots \\ \cdots & E^r_{p,q-1} & E^r_{p,q} \arrow[ull] & E^r_{p,q+1} \arrow[ull] & \cdots \arrow[ull] \\ \cdots & E^r_{p-1,q-1} & E^r_{p-1,q} \arrow[ull] & E^r_{p-1,q+1}. \arrow[ull] & \cdots \arrow[ull] \\ & \vdots & \vdots & \vdots & \end{tikzcd}$$

[1] In the general setup, one has to be somewhat agnostic about what these are: in any context where one can do homological algebra, one can define spectral sequences: abelian groups, modules over a ring, objects in an abelian category...

The point of this heavy machinery is that there's a machine which takes filtered objects and functors satisfying an excision property to spectral sequences, and such pairs arise in many contexts in algebra, topology, and geometry.

Definition 1.2. Let $\mathbb{Z}$ denote the **poset category** of the integers, i.e. there's a unique arrow $m \to n$ iff $m \leq n$. Then, a **filtered object** in a category C is a functor $X \colon \mathbb{Z} \to \mathsf{C}$.

The idea is a topological space X together with inclusions $X_i \hookrightarrow X_{i+1}$, such that X is the union of all of the X_i. More generally, one can let X be the colimit over i of $X(i)$. One example is the CW filtration of a CW complex X, where $X(n)$ is the n-skeleton of X.

Definition 1.3. Let C be either Top_*, the category of pointed topological spaces, or $\mathsf{Ch}(\mathsf{Mod}_A)$, the category of chain complexes of A-modules for a ring A.

- Let $f \colon X \to Y$ be a C-morphism, so that we can take its mapping cone C_f and obtain a sequence $X \to Y \to C_f$. If we iterate this construction, $C_{Y \to C_f}$ is weakly equivalent to ΣX, and the mapping cone of this is weakly equivalent to ΣY, so we obtain a sequence
$$X \xrightarrow{f} Y \longrightarrow C_f \longrightarrow \Sigma X \xrightarrow{\Sigma f} \Sigma Y \longrightarrow \Sigma C_f \longrightarrow \dots$$
Such a sequence is called a **cofiber sequence**.[2]
- A **functor satisfying excision** is a covariant or contravariant functor $\mathsf{C} \to \mathsf{Ab}$ taking cofiber sequences to long exact sequences.[3]

To see why $C_{Y \to C_f} \simeq \Sigma X$, one can work with particularly nice maps, so that $Y \to C_f$ is an injection, and its mapping cone crushes Y to a point, producing ΣX. The cofiber C_f is the topological analogue of the quotient Y/X.

Example 1.4. Here are some examples of these functors. First, let $\mathsf{C} = \mathsf{Top}_*$:

(1) Covariant functors $\mathsf{Top}_* \to \mathsf{Ab}$ with excision include homology functors H_n.
(2) For covariant functors sending fiber sequences to long exact sequences, we have homotopy groups π_i.
(3) Contravariant functors with excision include cohomology functors H^n.

For the category of chain complexes, cofiber and fiber sequences are the same thing.

(4) Covariant functors include homology and covariant derived functors such as $\mathrm{Ext}^i(M, -)$ and $\mathrm{Tor}_i(M, -)$.
(5) Contravariant functors include cohomology and contravariant derived functors such as $\mathrm{Ext}^i(-, M)$. ◀

From here, one can draw picture of the argument for why such a functor defines a spectral sequence:

`(Diagram to be made later.)`

From this diagram, one can see how the differentials arise, and they have the grading for the E_2 page. In particular, given the filtration $\{X_p\}$ of X, we can let $E^2_{p,q} \coloneqq H_{p+q}(X_p)$.[4] Thus the E^1 page is

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
H_2(X_0) & \xleftarrow{d_1} & H_3(X_1) & \xleftarrow{d_1} & H_4(X_2) & \longleftarrow & \cdots \\
H_1(X_0) & \xleftarrow{d_1} & H_2(X_1) & \xleftarrow{d_1} & H_3(X_2) & \longleftarrow & \cdots \\
H_0(X_0) & \xleftarrow{d_1} & H_1(X_1) & \xleftarrow{d_1} & H_2(X_2). & \longleftarrow & \cdots
\end{array}$$

[2]You may prefer to call this a **cofibre sequence**.

[3]There's a version of this for functors taking fiber sequnences to long exact sequences, but we won't need to use it.

[4]Technically, we started only with one functor H, but we can define $H_{n-1}(X) \coloneqq H_n(\Sigma X)$ and extend to a family of functors, just as for homology.

The key is explaining how the differentials occur. Let h be a homology theory, $X = \{X_i\}$ be a filtration, and $C_i \coloneqq X_i/X_{i-1}$ be the cofibers. Then we have a diagram

$$\begin{tikzcd}
 & h(C_1) & h(C_2) \arrow[l] & h(C_3) \arrow[l] & \\
h(X_0) \arrow[r] & h(X_1) \arrow[r] \arrow[u] & h(X_2) \arrow[r] \arrow[u] & h(X_3) \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

Any pair $\to, \uparrow$ fits into a long exact sequence with connecting morphism $\delta\colon h(C_i) \to h(\Sigma X_{i-1})$:

$$\begin{tikzcd}
 & h(C_1) \arrow[dl, dashed, "\delta"'] & h(C_2) \arrow[l] \arrow[dl, dashed, "\delta"'] & h(C_3) \arrow[l] \arrow[dl, dashed, "\delta"'] & \\
h(X_0) \arrow[r] & h(X_1) \arrow[r] \arrow[u] & h(X_2) \arrow[r] \arrow[u] & h(X_3) \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

This is how the first differentials arise: take the connecting morphism δ, then map back $h(X_{i-1}) \to h(C_{i-1})$. Considering longer sequences of maps after taking homology gives you the higher-order differentials.

What follows was a complicated diagram chase that was hard to live-TEX.

We had the E^1 page and differentials, and after taking homology, we get the E^2 page:

$$\begin{tikzcd}
 {} & E^2_{0,2} & E^2_{1,2} & E^2_{2,2} \\
 {} & E^2_{0,1} & E^2_{1,1} \arrow[ull] & E^2_{2,1} \arrow[ull] \\
 & E^2_{0,0} & E^2_{1,0} \arrow[ull] & E^2_{2,0}. \arrow[ull]
\end{tikzcd}$$

2. The Atiyah-Hirzebruch spectral sequence: 5/9/17

Today, I'm going to talk about the Atiyah-Hirzebruch spectral sequence. Last time, we talked about how to construct a spectral sequence from a filtration of a topological space; today, we'll black-box that construction and use it to compute some stuff. Namely, we'll use the CW fibration associated to any CW complex.

Let E^* be a generalized cohomology theory and X be a CW complex. The **Atiyah-Hirzebruch spectral sequence** is a spectral sequence

$$E_2^{p,q} = H^p(X; E^q(\mathrm{pt})) \Longrightarrow E^{p+q}(X).$$

We'll explain what all this actually means.

Convergence. Sometimes you're reading a book and it feels like it goes on forever. It's nice when spectral sequences don't do that. As an example, we'll look at a **first-quadrant spectral sequence**, one where $E_2^{p,q} = 0$ when $p < 0$ or $q < 0$. In this setup, if you pick any (p, q), then after finitely many pages, the differentials are so long that they leave the first quadrant, so you get a sequence $0 \to E^r_{p,q} \to 0$, and therefore when you take homology, nothing changes. Thus it makes sense to say what the end of the spectral sequence is.

Definition 2.1. Whenever it makes sense, we'll define the E_∞**-page** of the spectral sequence to be $E_\infty^{p,q} = E^r_{p,q}$ for $r \gg 0$. One says $E_r^{p,q}$ **converges** or **abuts** to $E_\infty^{p,q}$.

Typically this is something interesting we want to calculate.

Definition 2.2. Let $A_\bullet$ be a graded abelian group together with an exhaustive filtration $\{F_p\}$.

- The **associated graded** of the filtration $\{F_i\}$ is
$$(\mathrm{gr}\, A)_{p,q} \coloneqq F_p A_{p+q}/F_{p-1}A_{p+q}.$$
- A spectral sequence $E_r^{p,q}$ **converges (weakly)** to $A_\bullet$, written
$$E_r^{p,q} \Longrightarrow A_\bullet,$$
if it has an E_∞ page and the E_∞ page is the associated graded of $A_\bullet$.

Remark. There is a notion of **conditional convergence**, due to Boardman, which essentially means "not always weakly convergent, but converges under hypotheses often met in practice." Unfortunately, defining this precisely would be a huge digression. ◀

Generalized cohomology theories. The Atiyah-Hirzebruch spectral sequence is used to compute things which behave like homology or cohomology, but are slightly different: they satisfy all of the Eilenberg-Steenrod axioms except for the dimension axiom. These generalized cohomology theories have been a huge area of focus in algebraic topology in the last half century.

Definition 2.3. A **generalized cohomology theory** (also **extraordinary cohomology theory**) is a collection of functors $h^n \colon \mathsf{Top}_* \to \mathsf{Ab}$ such that:

- Given a map $f\colon A \to X$, let X/A denote its cofiber. There is a natural transformation $\delta\colon h^n(X/A) \to h^{n+1}(A)$ such that the following sequence is long exact:

$$\cdots \longrightarrow h^n(A) \xrightarrow{h^n(f)} h^n(X) \longrightarrow h^n(X/A) \xrightarrow{\delta} h^{n+1}(A) \longrightarrow \cdots$$

- h^n takes wedge sums to direct sums: if $X = \bigvee_i X_i$, then the natural map

$$\bigoplus h^n(X_i) \longrightarrow h^n(X)$$

  is an isomorphism.

The dual notion of a **generalized homology theory** is the same, except the differentials go in the other direction. This defines a reduced homology theory, i.e. one for spaces with basepoints.

Example 2.4 (K-theory)**.** Let X be a compact Hausdorff space. Then, the set of isomorphism classes of complex vector bundles on X is a semiring, so we can take its group completion and obtain a ring $K^0(X)$.

The following theorem is foundational and beautiful.

Theorem 2.5 (Bott periodicity)**.** $K^0(\Sigma^2 X) \cong K^0(X)$.

This allows us to promote K^* into a **2-periodic** generalized cohomology theory K^*, called **complex K-theory**, by setting $K^{2n}(X) = K^0(X)$ and $K^{2n+1}(X) = K^0(\Sigma X)$.[5]

Like cohomology, K-theory is **multiplicative**, i.e. it spits out $\mathbb{Z}$-graded rings. However, $K^i(X)$ is often nonzero for negative i.

Exercise 2.6. For example, show that as graded abelian groups, $K^*(\mathrm{pt}) = \mathbb{Z}[t, t^{-1}]$, where $|t| = 2$.

K-theory admits a few variants.

- If you use real vector bundles instead of complex vector bundles, everything still works, but Bott periodicity is 8-fold periodic. Thus we obtain a periodic, multiplicative cohomology theory called **real K-theory**, denoted $KO^*(X)$. Its value on a point is encoded in the **Bott song**.
- Sometimes it will be simpler to consider a smaller variant where we only keep the negative-degree elements. This is called **connective K-theory**, and is denoted ku^* (for complex K-theory) or ko^* (for real K-theory). These are also multiplicative. ◀

Example 2.7 (Bordism)**.** Let X be a space and define $\Omega_n^{\mathrm{O}}(X)$ to be the set of equivalence classes of maps of n-manifolds $M \to X$, where $[f_0\colon M \to X] \sim [f_1\colon N \to X]$ if there's a cobordism $Y\colon M \to N$ and a map $F\colon Y \to X$ extending f_0 and f_1. This is an abelian group under disjoint union, and the collection $\{\Omega_n^{\mathrm{O}}\}$ defines a generalized homology theory called **unoriented bordism**.[6]

The following theorem was the beginning of differential topology.

Theorem 2.8 (Thom)**.** *As graded abelian groups,* $\Omega_n^{\mathrm{O}}(\mathrm{pt}) \cong \mathbb{F}_2[x_2, x_4, x_5, x_6, \dots] = \mathbb{F}_2[x_i \mid i \neq 2^j - 1]$. *Moreover,* Ω_*^{O} *is a direct sum of (suspended) ordinary cohomology theories.*

There's a lot of variations, based on whatever flavors of manifolds you consider. Using oriented manifolds produces **oriented bordism** Ω_*^{SO}, spin manifolds produce **spin bordism** Ω_*^{Spin}, and so forth. These are not direct sums of ordinary cohomology theories in general. ◀

[5]Extending from compact Hausdorff spaces to all of Top is possible, but then one loses the vector-bundle-theoretic description.

[6]The corresponding cohomology theory is called **cobordism**.

2.1. **The definition.** Recall that if X is a CW complex, it has a **CW filtration** in which X_n is the **n-skeleton**, the union of all cells of dimension $\leq n$. Then, X_n/X_{n-1} is a wedge of n-spheres indexed by the n-cells of X. Using this formalism we can define some spectral sequences.

Definition 2.9.

- Let E_* be a generalized homology theory and X be a CW complex. Then, the CW filtration on X induces a spectral sequence of homological type that strongly converges, called the **Atiyah-Hirzebruch spectral sequence**:
$$E^2_{p,q} = H_p(X; E_q(\mathrm{pt})) \Longrightarrow E_{p+q}(X).$$
- Let E^* be a generalized cohomology theory and X be a CW complex. Then, the CW filtration on X induces a spectral sequence of cohomological type that *conditionally* converges, called the **Atiyah-Hirzebruch spectral sequence**:
$$E_2^{p,q} = H^p(X; E^q(\mathrm{pt})) \Longrightarrow E^{p+q}(X).$$

Calculations.

Example 2.10. We'll use the Atiyah-Hirzebruch spectral sequence to compute $K^*(\mathbb{CP}^n)$. Recall that

$$H^p(\mathbb{CP}^k; A) = \begin{cases} A, & p \text{ even} \\ 0, & \text{odd.} \end{cases}$$

Hence

$$E_2^{p,q} = \begin{cases} \mathbb{Z}, & p, q \text{ even, } 0 \leq p \leq 2k \\ 0, & \text{otherwise.} \end{cases}$$

Thus all the differentials are zero! So $E_2^{p,q} \cong E_\infty^{p,q}$. Hence the E_∞ page has no torsion, and therefore $K^*(\mathbb{CP}^n)$ is isomorphic to its associated graded.

$$K^i(\mathbb{CP}^n) = \begin{cases} \mathbb{Z}^{n+1}, & i \text{ even} \\ 0, & \text{otherwise.} \end{cases}$$ ◀

Exercise 2.11. Let Σ be a genus-g orientable closed surface. Compute $K^*(\Sigma_g)$.

Exercise 2.12. What changes when you replace K^* with KO^*?

3. The Serre spectral sequence and computations: 5/10/17

Today, Ernie spoke on the Serre spectral sequence and some other topics.

Multiplicative structures. So far, everything we've done has been graded modules over a ring R, and often $R = \mathbb{Z}$, so we're thinking about graded abelian groups. Recall that a **graded R-module** is an R-module

$$M_\bullet = \bigoplus_{i\in\mathbb{Z}} M_i.$$

If $x \in M_i$, we say its **degree** is i, and write $|x| = i$.

Graded modules are great, as they resemble homology of spaces. Cohomology has additional structure in the form of a cup product: if $x \in H^i(X)$ and $y \in H^j(X)$, their cup product, denoted $x \smile y$ or just xy, is a class in $H^{i+j}(X)$, and $xy = (-1)^{ij}yx$. This structure is axiomatized as a graded algebra.

Definition 3.1. A **graded R-algebra** $M_\bullet$ is a graded R-module together with a **multiplication map** $\mu\colon M_\bullet \times M_\bullet \to M_\bullet$ such that

- $\mu(M_i, M_j) \subseteq M_{i+j}$ and
- if $|x| = i$ and $|y| = j$, then $\mu(x, y) = (-1)^{ij}(X)$

The structure of (a page of) a spectral sequence fits into something called a differential graded module.

Definition 3.2.

- A **bigraded R-module** is an R-module $M_{\bullet,\bullet}$ admitting a decomposition
$$M_{\bullet,\bullet} = \bigoplus_{i,j\in\mathbb{Z}} M_{i,j}.$$
The **total degree** of an $x \in M_{i,j}$, denoted $|x|$, is $i+j$. This degree turns $M_{\bullet,\bullet}$ into a singly graded R-module; this grading is called the **total grading**.
- A **differential graded R-module** is a bigraded R-module $M_{\bullet,\bullet}$ together with a map $d\colon M_{\bullet,\bullet} \to M_{\bullet,\bullet}$ such that $d^2 = 0$ and d shifts the total grading by either 1 (if $M_{\bullet,\bullet}$ is graded cohomologically) or -1 (if it's graded homologically).
- A **differential graded R-algebra** (DGA) is a differential graded R-module $M_{\bullet,\bullet}$ together with a multiplication map making $M_{\bullet,\bullet}$ a graded R-algebra with respect to the total grading and such that for all $x, y \in M_{\bullet,\bullet}$,
$$d(xy) = d(x)y + (-1)^{|x|}xd(y).$$

The multiplicative structure in cohomology is very useful: it forces a lot of information, and also can be directly useful, e.g. showing that $\mathbb{CP}^2$ and $S^2 \vee S^4$ aren't homotopic, even though they have the same homology. Similarly, a multiplicative structure on a spectral sequence will force a lot of differentials, so is an awesome thing to have in your pocket if you want to compute things with spectral sequences.

Definition 3.3. A **multiplicative spectral sequence** is a spectral sequence $E_2^{\bullet,\bullet} \Longrightarrow M_\bullet$ such that the pages $E_r^{\bullet,\bullet}$ are DGAs with respect to the grading and differential from the spectral sequence, $M_\bullet$ is a graded algebra, and the convergence reflects the multiplicative structure.

The Serre spectral sequence.

Definition 3.4. A **(Serre) fibration** $f\colon E \to X$ of topological spaces is a map such that if Δ^n denotes the n-simplex and one has commuting maps
$$\begin{tikzcd}
\Delta^n \times \{0\} \arrow[r] \arrow[d] & E \arrow[d, "f"] \\
\Delta^n \times [0,1] \arrow[r] & X,
\end{tikzcd}$$
there exists a map $G\colon \Delta^n \times [0,1] \to E$ that commutes with the maps in the diagram.

We always take X to be path-connected, in which case $f^{-1}(x) \simeq f^{-1}(x')$ for all $x, x' \in X$. This preimage is called the **fiber** of f, and is often denoted F; the triple $F \to E \to X$ is called a **fiber sequence**. We will also assume X is simply connected, which will allow us to obtain stronger results.

Example 3.5. Let M be a manifold of dimension n. Then, $TM \to M$ is a fibration, and the fiber is $\mathbb{R}^n$. ◂

Theorem 3.6 (Serre)**.** *Fix a coefficient ring R; let $f\colon E \to X$ be a fibration and F be its fiber. Then, there exists a multiplicative spectral sequence, called the* ***Serre spectral sequence***
$$E_2^{p,q} = H^p(X; H^q(F;R)) \Longrightarrow H^{p+q}(E;R).$$

Proof sketch. Let $\{X_i\}$ be the CW filtration of X, and let $E_i := f^{-1}(X_i)$, which induces an exhaustive filtration $\{E_i\}$ of E. Applying $H^q(–;R)$ defines a spectral sequence by the formalism from the first lecture. The multiplicative structure comes from the cup product on X. ⊠

Remark. Let A be a multiplicative generalized cohomology theory (e.g. K-theory). Then, we could have applied A instead of $H^q(–;R)$ and obtained a multiplicative spectral sequence
$$E_2^{p,q} = H^p(X; A^q(F)) \Longrightarrow A^{p+q}(E).$$
Letting $A = H^*(–,R)$, we recover the Serre spectral sequence, and letting $E \to X$ be the identity map $X \to X$, which is a fibration, we recover the Atiyah-Hirzebruch spectral sequence. For this reason this spectral sequence is sometimes called the **Serre-Atiyah-Hirzebruch spectral sequence**. ◂

Example 3.7. Let $PX \coloneqq \mathsf{Top}_*(I, X)$ denote the **path space**, i.e. the maps from the unit interval to X. Evaluation at 0 defines a map $\mathrm{ev}\colon PX \to X$. The path space is made of spaghetti, hence is contractible: shrink each path from time 1 to time t, and let $t \to 0$.

$\mathrm{ev}\colon PX \to X$ is a fibration, and the fiber is ΩX, the space of (based) loops in X (i.e. based maps $S^1 \to X$). Recall that a fiber sequence induces a long exact sequence in homotopy groups:

$$\cdots \longrightarrow \pi_n(F) \longrightarrow \pi_n(E) \longrightarrow \pi_n(X) \longrightarrow \pi_{n-1}(F) \longrightarrow \cdots \tag{3.8}$$

Since $\pi_n(PX) = 0$, this implies $\pi_n(X) \cong \pi_{n-1}(\Omega X)$.

Let's apply the Serre spectral sequence to this fibration in the case where $R = \mathbb{Q}$ and $X = S^3$. The Serre spectral sequence takes the form

$$E_2^{p,q} = H^p(S^3; H^q(\Omega S^3; \mathbb{Q})) \Longrightarrow H^{p+q}(PS^3; \mathbb{Q}).$$

We know the E_∞ page already: it's 0 unless $p + q = 0$, in which case it's $\mathbb{Q}$. So we're going to reverse-engineer the spectral sequence, to use the E_∞ page to compute the E_2 page.

We also know $H^*(S^3; \mathbb{Q}) = E_{\mathbb{Q}}(X)$, where $|x| = 3$, an exterior algebra in one variable. This is also isomorphic to $\mathbb{Q}[x]/x^2$, so has a $\mathbb{Q}$ in degrees 0 and 3, and is 0 elsewhere.

We know $H^0(\Omega S^3; \mathbb{Q}) = \mathbb{Q}$, so the E_2 page looks like

$$\begin{array}{ccccc}
3 & ? & 0 & 0 & ? \\
2 & ? & 0 & 0 & ? \\
1 & ? & 0 & 0 & ? \\
0 & 1 & 0 & 0 & x. \\
 & 0 & 1 & 2 & 3
\end{array}$$

We know that the $(3, 0)$ term has to vanish by the E_∞ page, so it either **supports a differential** (has a nonzero differential mapping out of it) or **receives a differential** (has a nonzero differential mapping into it). Since this is a first-quadrant spectral sequence, all differentials out of x hit 0, so it has to receive a differential. But on the E_2 page, this differential comes from the 0 in position $(1, 1)$, so it's zero, and any differentials in page 4 or above mapping into x come from the fourth quadrant, so there has to be a nonzero differential on the E_3 page mapping into x, so there's some $y \in E_2^{0,2}$, which generates a copy of $\mathbb{Q}$, such that $d_3 y = x$. There can't be more than one generator in $E_2^{0,2}$, because then either it would survive to the E_∞ page (which can't happen), or it gets killed, meaning the difference of it and y is not killed by d_3 and hence survives. Oops. Thus, $E_2^{0,2} = H^2(\Omega S^3; \mathbb{Q}) \cong \mathbb{Q}$. Hence we know $E_2^{3,2} = H^3(S^3; \mathbb{Q})$ as well, and the spectral sequence looks like

$$\begin{tikzcd}
2 & y \arrow[rrrdd, "d_3"] & 0 & 0 & \mathbb{Q} \\
1 & ? & 0 & 0 & ? \\
0 & 1 & 0 & 0 & x. \\
 & 0 & 1 & 2 & 3
\end{tikzcd}$$

We can also immediately determine $E_2^{\bullet,2}$: looking at $E_2^{0,2}$, there are no differentials that hit something nonzero, or map from something nonzero to it. So anything in it survives to the E_∞ page, and hence it must be zero. Thus $H^1(\Omega S^3; \mathbb{Q}) = 0$ and hence $E_2^{1,3} = 0$ too.

The multiplicative structure tells us that the generator of $E_2^{3,2}$ must be $y \cdot x$. Thus, the spectral sequence looks like

$$\begin{tikzcd}
2 & y \arrow[rrrdd, "d_3"] & 0 & 0 & yx \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & x. \\
 & 0 & 1 & 2 & 3
\end{tikzcd}$$

But now yx has to die, and the only way that can happen is if it's hit by d_3 of the $E_2^{0,4}$ term, which turns out to be y^2. This is because $d_3 y = x$, so

$$d_3(y^2) = d_3(y)y + (-1)^2 y d_3(y) = 2xy.$$

Thus d_3 is multiplication by 2. Hence the spectral sequence looks like

$$\begin{tikzcd}
4 & y^2 \arrow[rrrdd, "d_3"] & 0 & 0 & y^2x \\
3 & 0 & 0 & 0 & 0 \\
2 & y \arrow[rrrdd, "d_3"] & 0 & 0 & yx \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & x. \\
 & 0 & 1 & 2 & 3
\end{tikzcd}$$

But now we need y^2x to vanish, and it's hit by $y^3 \in E_2^{0,6}$ via d_3, which is multiplication by 3, and so on. Inductively we can conclude that

$$H^*(\Omega S^3; \mathbb{Q}) \cong \begin{cases} \mathbb{Q}, & n \text{ even} \\ 0, & n \text{ odd.} \end{cases}$$

Much of this argument, but not all of it, works with $\mathbb{Q}$ replaced by $\mathbb{Z}$. The difference is that multiplication by 2, 3, etc. is not an isomorphism, so we have a sequence of generators $y_1, y_2, \ldots$:

$$\begin{tikzcd}
\vdots & & & \vdots \\
y_3 \arrow[rrrdd, "d_3"] & & & y_3x \\
0 & & & 0 \\
y_2 \arrow[rrrdd, "d_3"] & & & y_2x \\
0 & & & 0 \\
y_1 \arrow[rrrdd, "d_3"] & & & y_1x \\
0 & & & 0 \\
1 & & & x. \\
0 & 1 & 2 & 3
\end{tikzcd}$$

Now we have to figure out the multiplicative structure. We know $y_1^2 = c_1 y_2$ for some $c_1 \in \mathbb{Z}$, so since d_3 is an isomorphism, let's compute: we know $d_3(y_2) = y_1 x$ by construction, and $d_3(y_1^2) = 2y_1 x$ for the same reason as over $\mathbb{Q}$, so $y_1^2 = 2y_2$.

A similar calculation in general shows that $y_1^n = n!y_n$, as

$$\begin{aligned}
d_3(y_1^n) &= d_3(y_1 y_1^{n-1}) = d_3(y_1)y_1^{n-1} + y_1(n-1)!d(y_{n-1}) \\
&= xy_1^{n-1} + y_1(n-1)!xy_{n-2} \\
&= x(n-1)!y_{n-1} + (n-1)y_{n-1}x(n-1)! \\
&= n!xy_{n-1},
\end{aligned}$$

but $d_3(n!y_n) = n!xy_{n-1}$. Hence the ring structure on $H^*(\Omega S^3)$ is a divided power algebra.

Definition 3.9. A **divided power algebra** on a single generator x in degree k, denoted $\Gamma(x)$, is the free algebra generated by $\{x_i\}_{i\geq 1}$ where $|x_i| = ki$, subject to the relations

$$x_i x_+ j = \binom{i+j}{j} x_{i+j} \qquad \text{and} \qquad x_i = \frac{x^i}{i!}.$$

Thus $H^*(\Omega S^3) \cong \Gamma(y)$ with $|y| = 2$. ◂

Exercise 3.10. The same argument works to compute $H^*(\Omega S^{2n+1})$. Work it out for $H^*(\Omega S^{2n})$, which behaves differently.

Example 3.11. Let $K(G, n)$ be an **Eilenberg-Mac Lane space**, i.e. a space with $\pi_n(K(G, n)) = G$ and all other homotopy groups vanishing. It's a theorem that these exist for all n and G (abelian when $n \geq 2$), and any two choices of a $K(G, n)$ are homotopy equivalent for given G and n. For a simple example, S^1 is a $K(\mathbb{Z}, 1)$, and for a less simple example, $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$.

Eilenberg-Mac Lane spaces with $n \geq 3$ are usually much harder to describe explicitly, but we can use the Serre spectral sequence to compute their cohomology. (3.8) tells us that $\Omega K(G, n)$ has $\pi_{n-1}(\Omega K(G, n)) = G$ and all other homotopy groups vanishing, so it's a model of $K(G, n-1)$ (here we need $n > 1$). Thus, the path space fibration is a fibration

$$K(G, n-1) \longrightarrow * \longrightarrow K(G, n).$$

You can use this to inductively compute $H^*(K(G, n))$, starting from $n = 1$, where $K(G, 1)$ often has a more explicit model.

This is useful for unerstanding **cohomology operations**, maps $H^n(-, \mathbb{Z}) \to H^p(-, \mathbb{Z})$, e.g. $x \mapsto x^2$. Since Eilenberg-Mac Lane spaces represent ordinary cohomology, these are parameterized by $[K(\mathbb{Z}, n), K(\mathbb{Z}, p)] = H^p(K(\mathbb{Z}, n))$. ◂

Example 3.12. The unitary group U_n acts on S^{2n-1} through the unit sphere embedding $S^{2n-1} \hookrightarrow \mathbb{C}^n$, and this action is transitive. The stabilizer of a point is U_{n-1}, so we obtain a fiber sequence

$$\mathrm{U}_{n-1} \longrightarrow \mathrm{U}_n \longrightarrow S^{2n-1}.$$

We'll use this to compute the cohomology of U_n. When $n = 1$, $\mathrm{U}_1 = S^1$, so $H^*(\mathrm{U}_1) = E(x_1)$, with $|x_1| = 1$.

Next let's consider $n = 2$. $H^*(S^3) = E(x_3)$, where $|x_3| = 3$, so by the Künneth formula, $H^*(\mathrm{U}_1, H^*(S^3)) = H^*(S^1) \otimes H^*(S^3) = E(x_1) \otimes E(x_3)$, and this is the E_2 page, with multiplicative structure.

$$\begin{array}{c|cccc}
1 & x & & & x_1x_3 \\
0 & 1 & & & x_3. \\
\hline
 & 0 & 1 & 2 & 3
\end{array}$$

Thus, no differentials are supported, so $E_2 = E_\infty = E(x_1, x_3)$. Thus, $H^*(\mathrm{U}_2) \cong E_{\mathbb{Z}}(x_1, x_3)$. Inductively, considering S^{2n-1} adds one more class $x_{2n-1} \in E_2^{2n-1,0}$ and no differentials can exist, so $E_2 = E_\infty = E(x_1, x_3, x_5, \ldots, x_{2n-1})$, and this is $H^*(\mathrm{U}_n)$. ◂

Example 3.13. We can apply this computation of the cohomology of U_n to obtain the cohomology of its classifying space $B\mathrm{U}_n$. This is the quotient of a contractible space $E\mathrm{U}_n$ by a free U_n-action (again, it's a theorem that this exists, and that any two choices are homotopy equivalent). Hence we get a fiber sequence $\mathrm{U}_n \to * \to B\mathrm{U}_n$.[7]

[7]This works for any Lie group G: we get a sequence $G \to EG \to BG$.

Once again, the E_∞ page vanishes, and we'll use this to determine the E_2 page. We start with column 0, which is $H^*(\mathrm{U}_n)$. But $x_1 \in E_2^{0,1}$ must die, and the only differential it can support is d_2. Thus, there's a $y_2 \in E_2^{2,0}$ with $dx_1 = y_2$. Since $|x_1|$ is odd, then the Leibniz rule means $x_1^2 = 0$, and therefore

$$d(x_1 y_2^k) = y_2 y_2^k + (-1) x_1(0) = y_2^{k+1}.$$

Thus we know part of the E_2 page:

$$\begin{tikzcd}
x_5 \\
x_1 x_3 \\
x_3 & & x_3 y_2 \\
x_1 \arrow[drr, "d_2"] & & x_1 y_2 \arrow[drr, "d_2"] & & x_1 y_2^2 \arrow[drr, "d_2"] & & \cdots \\
1 & 0 & y_2 & 0 & y_2^2. & & \cdots \\
0 & 1 & 2 & 3 & 4
\end{tikzcd}$$

Since $d_2 \colon x_1 y_2 \mapsto y_2^2$ is an isomorphism, then $d_2 \colon x_3 \mapsto 0$, and x_3 survives to the E_3 page. However, this is the last differential we can use to kill it, so $d_3 x_3$ must be some new element of $E_2^{4,0}$, which we'll call y_4. We can also compute that $d_2(x_1 x_3) = y_2 x_3$ using the Leibniz rule, so we have

$$\begin{tikzcd}
x_5 \\
x_1 x_3 \arrow[drr, "d_2"] \\
x_3 \arrow[ddrrr, "d_3"] & & x_3 y_2 \\
x_1 \arrow[drr, "d_2"] & & x_1 y_2 \arrow[drr, "d_2"] & & x_1 y_2^2 \arrow[drr, "d_2"] & & \cdots \\
1 & 0 & y_2 & 0 & y^4, y_2^2. & & \cdots \\
0 & 1 & 2 & 3 & 4
\end{tikzcd}$$

If we continue this, we inductively get generators $y_i \in H^{2i}(B\mathrm{U}_n)$, and we'll see that $d(x_i y_{i+1}^k) = y_i^{k+1}$, so there are no relations. Hence $H^*(B\mathrm{U}_n) \cong \mathbb{Z}[y_2, y_4, y_6, \ldots, y_n]$. One application of this is to characteristic classes: y_{2m} is better known as c_m, the m^{th} **Chern class** for complex vector bundles. ◀

Example 3.14. Let M be a manifold, which we'll assume to be simply connected. Let $S(M) \to M$ be the unit sphere bundle inside the tangent bundle.[8] This is a **spherical fibration**, meaning a fibration whose fiber is a sphere. Since the cohomology of a sphere is very simple, the Serre spectral sequence allows us to calculate $H^*(S(M))$.

The fibration is $S^{n-1} \to S(M) \to M$, so the E_2 page is a copy of $H^*(M)$ in row 0 and a copy in row $n-1$. One can show that if $x_{n-1} \in E_2^{0,n-1}$ is the generator, then the first and only supported differential is $d_n(x_{n-1}) = \chi(M) \cdot [M]$. You can use this to compute the E_∞ page. ◀

[8]This requires a choice of a Riemannian metric to construct it, but the resulting bundle does not depend on the choice of metric.