SUMMER 2016 HOMOTOPY THEORY SEMINAR

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Contents

1. Simplicial Localizations and Homotopy Theory: 5/24/16

"It may be a little dry, but it's been raining recently, so perhaps dryness will be good to have."

Today's lecture was given by Ernie Fontes.

The point of this seminar is to study simplicial localizations. This is a somewhat dry topic; today we're going to frame it, suggesting an outline for talks and some motivation. Thus, today we'll discuss homotopy theory in broad strokes.

A good first question: *what is homotopy theory?* Relatedly, *when can we do it?* In general, homotopy theory happens whenever we have a pair of categories $(\mathbf{C}, \mathbf{W})$, where $\mathbf{W}$ is a subcategory of $\mathbf{C}$. The idea is that $\mathbf{W}$ contains morphisms that we'd like to be isomorphisms. If $\mathbf{W}$ contains all of the objects of $\mathbf{C}$, then the pair $(\mathbf{C}, \mathbf{W})$ is called a *relative category*.

Example 1.1.

(1) Often, we choose $\mathbf{C} = \mathbf{Top}$, and make $\mathbf{W}$ the category of a nice class of morphisms, e.g. π_*-isomorphisms or homotopy equivalences.
(2) Another choice is to let $\mathbf{C} = \mathbf{ch}(R)$, the category of chain complexes of R-modules, where $\mathbf{W}$ is the category of *quasi-isomorphisms* (maps which induce an isomorphism on homology).

One nice property that $\mathbf{W}$ could have is the *two-out-of-six property*: that for all triples of morphisms $f : X \to Y$, $g : Y \to Z$, and $h : Z \to Z'$ in $\mathbf{C}$, if gf and hg are in $\mathbf{W}$, then so are f, g, h, and hgf. This implies the *two-out-of-three property*, that if any two of f, g, and h are in $\mathbf{W}$, then so is the third.

Definition 1.2. If $\mathbf{W}$ satisfies the two-out-of-six property, it is called a *homotopical category*.

In either setting, we can form the *homotopy category* $\mathrm{Ho}(\mathbf{C}) = \mathbf{C}[\mathbf{W}^{-1}]$, localizing $\mathbf{C}$ at $\mathbf{W}$. This is the initial category among those categories $\mathbf{D}$ and functors $\mathbf{C} \to \mathbf{D}$ sending the arrows in $\mathbf{W}$ to isomorphisms.

Most questions in homotopy theory can be framed in terms of the homotopy category: two spaces are homotopic iff they're isomorphic in $\mathrm{Ho}(\mathbf{C})$, and the homotopy classes of maps $X \to Y$ are the hom-set $\mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}(X, Y)$ in the homotopy category.

One question which does require a little more sophistication is understanding homotopy (co)limits. Since we've inverted a lot of arrows, taking limits or colimits in a homotopy category behaves very poorly. For example, there's no pushout of the degree-2 map $S^1 \to S^1$ along with the map $S^1 \to \mathrm{pt}$, since it "should be" $\mathbb{RP}^2$ but this doesn't satisfy it. $\mathbb{RP}^2$ is the homotopy pushout, however.

Often, one obtains more structure from a homotopy category, e.g. there are some ∞-categorical notions hiding in the background here. More concretely, one often obtains a natural model category structure, where in addition to the relative category $(\mathbf{C}, \mathbf{W})$, we have classes of cofibrant and fibrant morphisms satisfying a bunch of axioms. This provides tools for computing homotopy limits and colimits, etc., but it's a lot of data; even the definition is redundant (the cofibrations and fibrations determine each other). In fact, the punchline of the three papers we're reading is that only the structure of the relative category $(\mathbf{C}, \mathbf{W})$ is necessary to recover the entire model-categorical structure! For this reason, one makes the analogy that if homotopy theory is to linear algebra, picking a model-categorical structure is akin to picking a basis.

Definition 1.3. A *simplicial set* is a simplicial object in **Set**. That is, it's a collection of sets $\{X_i\}_{i\geq 0}$ and a bunch of maps $d_{ij} : X_i \longrightarrow X_{i-1}$ for $0 \leq j \leq i$ and $s_{ij} : X_i \longrightarrow X_{i+1}$ for $0 \leq j \leq i$ satisfying some relations that look like the boundary and inclusion relations for an i-simplex inside an $(i + 1)$-simplex.

This is a vague definition, and we'll have a better one next lecture. These are akin to a "better" version of topological spaces, in that they model topological spaces very well, and can be described purely combinatorially.

Here's how the three papers of Dwyer and Kan break this information down.

(1) The first paper [DK1] constructs $\mathbf{C}[\mathbf{W}^{-1}]$, first as "just" a category, and then as a simplicially enriched category $L\mathbf{C}$, meaning that for all $X, Y \in \mathbf{C}$, $L\mathbf{C}(X, Y) \in \mathbf{sSet}$: that is, it's a simplicial set. In particular, we recover $\mathbf{C}[\mathbf{W}^{-1}]$ as the path components of this set: $\mathbf{C}[\mathbf{W}^{-1}](X, Y) = \pi_0 L\mathbf{C}(X, Y)$. There's a lot of comonadic computations here that may be confusing, but are applicable in many parts of algebra.
(2) In [DK2], Dwyer and Kan define a variant called the *hammock localization* $L^H\mathbf{C}(X, Y)_k$. The hammocks in question are commutative diagrams

$$\begin{tikzcd}
 & C_{0,1} \arrow[d, "\wr"] & C_{0,2} \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\wr"] & \cdots & C_{0,n-1} \arrow[l, "\sim"'] \arrow[d, "\wr"] \arrow[dr] & \\
X \arrow[ur] \arrow[r] \arrow[dr] & \vdots \arrow[d, "\wr"] & \vdots \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\wr"] & \cdots \arrow[r, "\sim"] & \vdots \arrow[r] \arrow[d, "\wr"] & Y \\
 & C_{k,1} \arrow[r, "\sim"] & C_{k,2} \arrow[r, "\sim"] & \cdots & C_{k,n-1} \arrow[l, "\sim"'] \arrow[ur] &
\end{tikzcd}$$

This might not seem like the best construction, but it expresses $L^H\mathbf{C}(X, Y)$ as a colimit of nerves of categories, which are easy to compute, and therefore this is surprisingly easy to work with when it comes to actually computing things. In particular, when certain weak (yet technical) properties hold, $L^H\mathbf{C}(X, Y) \simeq L\mathbf{C}(X, Y)$. The calculations in this paper are much more technical than the first, and it's worth going through more slowly.
(3) The third paper [DK3] establishes a relationship between (simplicially enriched) model categories and $L^H\mathbf{C}(X, Y)$. The takeaway is that the weak equivalences are all that you need to define a model categorical structure.

In the unlikely event we have time, there's an interesting relationship between this and algebraic K-theory: in a similar way, the algebraic K-theory of a model category actually only depends on the hammock localization, due to a paper [BM] of Blumberg and Mandell; this was a cool and surprising result.

Here's the list of planned talks; we can and should deviate from this in order to make sure we understand everything better.

(1) Simplicial sets, especially nerves and classifying spaces. This should definitely include a definition and some important constructions.
(2) Model categories; there's a lot we could talk about here, but we should talk about the definition, how to construct homotopy limits and colimits, mapping spaces, and fibrant and cofibrant replacement. This is intended to be an overview, rather than discussing complicated examples. This will be helpful to see all the structure we don't need!
(3) We then need to talk about localization in general, including the universal property for localizing rings, and discuss the discrete localization of categories. The hard version of this talk would also talk about Bousfield localization.
(4) Now, the first part of [DK1]: localization of $(\mathbf{C}, \mathbf{W})$, comonadic resolutions, and bar constructions, which detail how one constructs things. This is mostly all in the paper, and needs to be teased apart.
(5) Perhaps also it will be useful to discuss the rest of the model structure on small simple categories. Here Julie Bergner's thesis is a useful reference, as she treats this more clearly and in greater generality, though we may or may not need to refer to this.
(6) Moving to [DK2], introduce hammock localization. This is important to understand very closely; don't leave anything out of the talks.
(7) Then, we need homotopy calculus of fractions, which is useful for ensuring hammocks are small.

(8) We then need the theory of simplicial model categories; these have more structure and are more excellent than ordinary model categories. The key is understanding the axiom SM7 for a simplicial model category.
(9) Finally, we should treat the main theorem in [DK3], that $L^H\mathrm{C}(X, Y)$ models the simplicial derived mapping space in a model category.

At that point, the summer will be over, and we will be done.

2. Simplicial Sets: 5/31/16

These are Arun's prepared notes for this talk.

Simplicial sets are a combinatorial analogue of topological spaces that are often simpler to work with, yet in a sense contain the same information from the perspective of homotopy theory. At the same time, they also behave like a nonlinear analogue of chain complexes.

Two Definitions of Simplicial Sets. One definition is formal, and easier to write down; the other is more geometric, but requires more words.

Definition 2.1. The *simplex category* Δ is the category whose objects are the ordered sets $[n] = \{0, 1, \dots, n\}$ for $n \geq 0$, and whose morphisms are order-preserving functions.

Definition 2.2. A *simplicial set* is a functor $\Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$. With natural transformations as morphisms, these form the category $\mathbf{sSet}$. More generally, for any category $\mathbf{C}$, a *simplicial object* in $\mathbf{C}$ is a functor $\Delta^{\mathrm{op}} \longrightarrow \mathbf{C}$.

That is, a simplicial set X is a set X_n for each $[n]$ (called the set of *n-simplices*) with compatible actions by the morphisms in Δ. A morphism of simplicial sets $X \longrightarrow Y$ is a collection of maps $X_n \longrightarrow Y_n$ for each n that commutes with those actions.

This doesn't seem very topological or geometric; here's another definition.

Definition 2.3. A *simplicial set* X is a collection of sets X_n for each $n \geq 0$, along with functions $d_i : X_n \longrightarrow X_{n-1}$ and $s_i : X_n \longrightarrow X_{n+1}$ for $0 \leq i \leq n$, called the *face maps* and *degeneracy maps*, respectively, satisfying the relations

$$\begin{aligned}
d_i \circ d_j &= d_{j-1} \circ d_i, \quad i < j \\
s_i \circ s_j &= s_{j+1} \circ s_i, \quad i \leq j \\
d_i \circ s_j &= \begin{cases} 1, & i = j \text{ or } i = j+1 \\ s_{j-1} \circ d_i, & i < j \\ s_j \circ d_{i-1} & i > j+1. \end{cases}
\end{aligned} \tag{2.4}$$

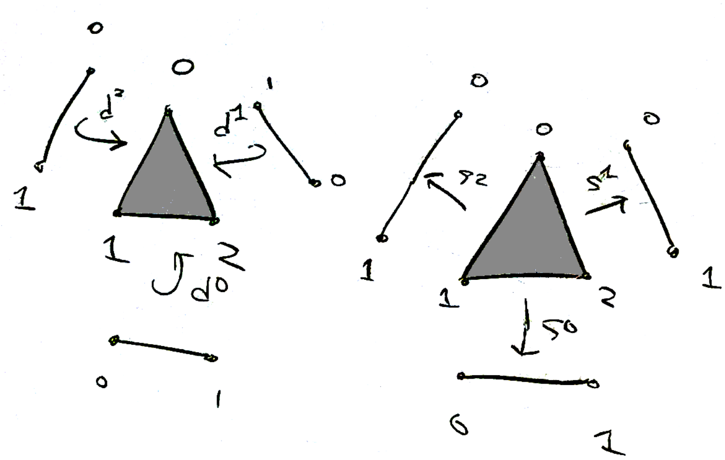

Figure 1. Examples of generating maps in Δ that induce the face and degeneracy maps of a simplicial set.

A morphism of simplicial sets $f : X \longrightarrow Y$ is a collection of maps $f_n : X_n \longrightarrow Y_n$ that commute with the face and degeneracy maps.

We can think of the object $[n] \in \mathbf{\Delta}$ as the standard n-simplex (triangle, tetrahedron, ...); in this case, the face map d_i is induced from the inclusion of the i^{th} face (which is a copy of $[n-1]$) and the degeneracy map s_i is induced from the projection onto the i^{th} face. If you play with this picture, you end up writing down the definitions in (2.4).

Example 2.5.

(1) The *standard n-simplex* is the $\Delta^n = \operatorname{Hom}_{\mathbf{\Delta}}(-,[n])$. Thus, by the Yoneda lemma, for any simplicial set X, $\operatorname{Hom}_{\mathbf{sSet}}(\Delta^n, X) = X_n$. Geometrically, think of standard n-simplex as, well, the n-dimensional simplex: the i^{th} face map is the assignment to the i^{th} face of this simplex, and the i^{th} degeneracy map realizes Δ^n as a degenerate $(n+1)$-simplex where vertices i and $i+1$ coincide. See Figure 2 for a depiction of the standard 3-simplex.

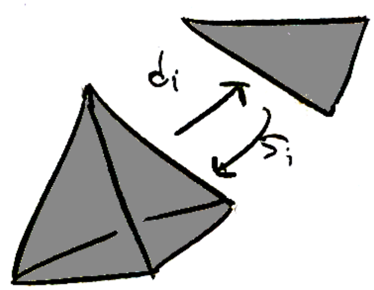

Figure 2. The standard 3-simplex, with example face and degeneracy maps.

By the Yoneda lemma, Δ^n corepresents the functor $X \mapsto X_n$. That is, there is a natural isomorphism (of sets)

$$\operatorname{Hom}_{\mathbf{sSet}}(\Delta^n, X) \cong X_n. \tag{2.6}$$

(2) Given a simplicial set X, we can form its *k-skeleton* in much the same way as for CW complexes, by preserving $X_0, \dots, X_k$ and the maps between them, but making all higher simplices degenerate.

(3) The *simplicial n-sphere*, denoted $\partial\Delta^n$, is the $(n-1)$-skeleton of Δ^n. Geometrically, this is the n-simplex minus its interior, which is a reasonable thing to call a sphere (to homotopy theorists, at least). Another equivalent formulation is to take $\Delta^n \setminus \{\mathrm{id}\}$ (regarding it as a functor), or the union (or colimit) of all of the faces of Δ^n across the morphisms gluing *their* faces (which are $(n-2)$-simplices).

(4) The *simplicial horn* Λ^n_k is the union (or colimit) of all faces of Δ^n except for the k^{th} face. The notation Λ is suggestive of the geometry. If X is a simplicial set, a *horn in X* is a map of simplicial sets $\Lambda^n_k \longrightarrow X$.

Definition 2.7. A simplicial set X is a *Kan complex* if every horn $\Lambda^n_k \longrightarrow X$ can be extended to a map $\Delta^n \longrightarrow X$, i.e. it factors through the inclusion $\Lambda^n_k \hookrightarrow \Delta^n$. If this is only true for *inner horns*, i.e. $\Lambda^n_k \hookrightarrow X$ where $0 < k < n$, then X is called a *weak Kan complex*.

One says that "every horn has a filler."

Like **Top**, the category **sSet** is complete and cocomplete: all limits and colimits exist, and in fact can be constructed levelwise. In particular, products exist.

Another nice property of this category is that we can build a simplicial set of the morphisms between two simplicial sets, rather than just a set.

Definition 2.8. Given $X, Y \in \mathbf{sSet}$, their *function complex* is the simplicial set $\mathbf{sSet}(X,Y)$ whose n-simplices are the set $\mathbf{sSet}(X,Y)_n = \operatorname{Hom}_{\mathbf{sSet}}(X \times \Delta^n, Y)$, with face and degeneracy maps induced from those on Δ^n.

For any simplicial set Y, there is an adjunction $(- \times Y, \mathbf{sSet}(Y,-))$; one says that **sSet** is *Cartesian closed.* Other Cartesian closed categories include **Set** and the category of compactly generated spaces.

Definition 2.9. A *simplicially enriched category* $\mathbf{C}$ is defined in exactly the same way as a category, but for every $X, Y \in \mathbf{C}$, there is a simplicial set (sometimes called the *function complex*) $\mathbf{C}(X,Y)$ of morphisms between them, instead of a set.

We require an associative composition law as usual; the identity is a distinguished 0-simplex in $\mathbf{C}(X,X)$ satisfying the same properties as usual.

Sometimes these are called "simplicial categories," but that term is also used to refer to simplicial objects in **Cat**; here **Cat** is the category of small categories with functors as morphisms. However, we can identify simplicially enriched categories with the simplicial categories whose objects are the same in every dimension.

If X and Y are simplicial sets, we defined their function complex $\mathbf{sSet}(X, Y)$, so **sSet** is a simplicially enriched category. The categories **Set** and **Top** can also be simplicially enriched, e.g. $\mathbf{Top}(X, Y)_n = \mathrm{Hom}_{\mathbf{Top}}(X \times |\Delta^n|, Y)$.

Geometric Realization and the Total Singular Complex. Simplicial sets are closely related to topological spaces: they're built out of n-simplices, which are manifestly topological objects. As such, there is an adjunction

$$|{-}| : \mathbf{sSet} \rightleftarrows \mathbf{Top} : S \tag{2.10}$$

relating simplicial sets and topological spaces.

The left adjoint is called *geometric realization*, and does in fact geometrically realize a simplicial set as a topological space: start with a concrete n-simplex in **Top** for every (abstract) n-simplex in X. Then, the face and degeneracy maps identify some of the faces of these n-simplices, so glue the corresponding concrete simplices together along those edges. Rigorously, "gluing" means a colimit. A simplicial set is essentially the data of n-simplices glued together in a specific way, and in particular

$$X \cong \varinjlim_{\Delta^n \to X} \Delta^n,$$

where the colimit is taken across all maps $\Delta^n \to X$ directed under arrows $\theta : \Delta^n \to \Delta^m$ that commute with the maps to X. We already know how to realize Δ^n as the standard n-simplex $|\Delta^n|$, so the geometric realization can be defined in parallel:

$$|X| = \varinjlim_{\Delta^n \to X} |\Delta^n|.$$

The geometric realization of a simplicial set is a CW complex.

The right adjoint is called the *total singular complex*, and belongs to the analogy

simplicial sets : chain complexes : : total singular complex : singular chain complex.

If Y is a topological space, we've already defined a chain complex of maps from the standard n-simplices into Y; this is a refinement. The total singular complex SY is defined by setting $SY_n = \mathrm{Hom}_{\mathbf{Top}}(\Delta^n, Y)$, the set of all continuous maps of the standard n-simplex (as a topological space) into Y. Given a map $f : \Delta^n \to Y$, we can restrict it to the i^{th} face; this is exactly what the i^{th} face map does. Applying the degeneracy map is given by collapsing Δ^{n+1} onto Δ^n at the i^{th} vertex, then composing with f, giving a map $\Delta^{n+1} \to Y$ as desired.

From this definition, the adjunction isn't too hard to see.

Proposition 2.11. *The functors* $|{-}|$ *and* S *defined above are adjoint, as in* (2.10).

Proof. We want to show for all spaces Y and simplicial sets X, there's a natural isomorphism $\mathrm{Hom}_{\mathbf{Top}}(|X|, Y) \cong \mathrm{Hom}_{\mathbf{sSet}}(X, SY)$. First, it's true when $X = \Delta^n$: $SY_n = \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, Y)$ by definition, and $SY_n = \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, SY)$ by (2.6). Since $\mathrm{Hom}_{\mathsf{C}}(A, -)$ sends colimits to limits,

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Top}}(|X|, Y) &\cong \varprojlim_{\Delta^n \to X} \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, Y) \\ &\cong \varprojlim_{\Delta^n \to X} \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, SY) \\ &\cong \mathrm{Hom}_{\mathbf{sSet}}(X, SY). \end{aligned}$$

□

One can recover the singular chain complex $C_\bullet(X)$ from the total singular complex by setting $C_n(X)$ to be the free abelian group on X_n with the boundary map

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i. \tag{2.12}$$

Fact. The total singular complex SY is a Kan complex. There's a sense in which this adjunction defines an equivalence of the homotopy theories of **sSet** and **Top**.

The Nerve of a Category. For any small category $\mathbf{C}$, we can build a simplicial set $N\mathbf{C}$, called the *nerve* of $\mathbf{C}$; this is functorial in $\mathbf{C}$, defining a functor $\mathbf{Cat} \longrightarrow \mathbf{sSet}$ (here $\mathbf{Cat}$ is the category of small categories, with functors as morphisms).

The construction is as follows:

- $N\mathbf{C}_0$ be the set of objects in $\mathbf{C}$.
- $N\mathbf{C}_1$ is the morphisms of $\mathbf{C}$.
- $N\mathbf{C}_2$ is the set of pairs of composable morphisms $X \longrightarrow Y \longrightarrow Z$.
- If $n \geq 2$, $N\mathbf{C}_n$ is the set of n-tuples of composable morphisms $X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_n$.

In other words, if $[n]$ is regarded as a poset category (so there's a unique map $i \longrightarrow j$ iff $i \leq j$), $N\mathbf{C}_n$ is the set of functors $[n] \longrightarrow \mathbf{C}$.

The degeneracy map $s_i : N\mathbf{C}_n \longrightarrow N\mathbf{C}_{n+1}$ takes a string of arrows and inserts the identity at the i^{th} position. The face map $d_i : N\mathbf{C}_n \longrightarrow N\mathbf{C}_{n-1}$ replaces the i^{th} and $(i+1)^{\text{th}}$ arrows with their composition, unless $i = 0$ or $i = n$, in which case it just drops the first or last arrow, respectively.

Fact. The nerve of a category is a weak Kan complex.

Example 2.13 (Classifying spaces)**.** If one interprets a group G as a category with a single object, its nerve will correspond to the classifying space BG.

More precisely, a discrete group G defines a category $\mathbf{G}$ with a single object $\bullet$ and $\mathrm{Hom}_{\mathbf{G}}(\bullet, \bullet) = G$, with group multiplication as composition. Its nerve $N\mathbf{G}$ is the simplicial set whose set of n-simplices is G^n: the i^{th} degeneracy map includes e at index i, and the i^{th} face map $d_i : G^n \longrightarrow G^{n-1}$ multiplies indices i and $i+1$ together (unless $i = 0$ or n, in which case that index is dropped).

Define another simplicial set X whose n-simplices are $X_n = G^{n+1}$ with the same degeneracy maps and face maps, except for d_n, which sends $(g_1, \ldots, g_n, g_{n+1}) \mapsto (g_1, \ldots, g_n g_{n+1})$ instead of dropping the last index. Then, projection onto the first n coordinates defines maps $p_n : X_n \longrightarrow N\mathbf{G}_n$ commuting with the face and degeneracy maps, so we obtain a map of simplicial sets $p : X \longrightarrow N\mathbf{G}$.

Multiplication on the last coordinate defines a right action of G on X: if $h \in G$, $(g_1, \ldots, g_{n+1}) \cdot h = (g_1, \ldots, g_n, g_{n+1}h)$. This commutes with the face and degeneracy maps of X, making it a simplicial G-set, and the fibers of p are G-torsors.

Now, we geometrically realize, suggestively defining $EG = |X|$ and $BG = |N\mathbf{G}|$. Projection $\pi = |p| : EG \longrightarrow BG$ is a fiber bundle whose fibers are G-torsors, so $\pi : EG \longrightarrow BG$ is a principal G-bundle. It's true, albeit harder to show, that EG is contractible, and therefore BG is a model for the classifying space of G.[1] Since G is discrete, BG is also a concrete model for $K(G, 1)$.

Example 2.14 (Bar construction)**.** We can generalize Example 2.13 and obtain a surprisingly useful class of simplicial objects.

Let $\mathbf{C}$ be a monoidal category, M be a monoid in $\mathbf{C}$, and $X, Y \in \mathbf{C}$ be acted on by M from the right and left, respectively.

- If $\mathbf{C} = \mathbf{Top}$, this is the notion of a continuous monoid action (from the right or the left), akin to that of a continuous group action.
- If $\mathbf{C} = \mathbf{Mod}_R$ for a commutative ring R a monoid S in $\mathbf{C}$ is an R-algebra, X is a right S-module and Y is a left S-module.

We'll build a simplicial object in $\mathbf{C}$ called the *bar construction* $B(X, M, Y)$, reminiscent of the nerve:

- The n-simplices $B_n(X, M, Y) = X \otimes M^{\otimes n} \otimes Y$ (here, $\otimes$ denotes the monoidal product; for $\mathbf{C} = \mathbf{Top}$ or $\mathbf{C} = \mathbf{Set}$, this is just Cartesian product).
- If $0 < i < n$, the i^{th} face map multiplies together the i^{th} and $(i+1)^{\text{th}}$ indices:
$$d_i : (x, m_1, \ldots, m_n, y) \mapsto (x, m_1, \ldots, m_i m_{i+1}, \ldots, m_n, y).$$
- The 0^{th} face map sends $(x, m_1, \ldots, m_n, y) \mapsto (x \cdot m_1, m_2, \ldots, m_n, y)$, and correspondingly the n^{th} face map sends $(x, m_1, \ldots, m_n, y) \mapsto (x, m_1, \ldots, m_{n-1}, m_n \cdot y)$.
- The i^{th} degeneracy map s_i inserts the identity $e \in M$ at the i^{th} index.

If we know how to geometrically realize simplicial $\mathbf{C}$-objects, then this produces a genuine object of $\mathbf{C}$.

[1] You might be wondering what happens if G isn't discrete, the case where classifying spaces are more interesting. Nearly the same story applies: we regard G as a single-object category enriched over **Top**, so $N\mathbf{G}$ is a *simplicial space* (i.e. simplicial object in **Top**). Geometric realization of simplicial spaces goes through to define the principal G-bundle $EG \longrightarrow BG$ in the same way.

- Suppose $\mathbf{C} = \mathbf{Top}$ and $M = G$ is a group. Then, $|B(*, G, *)| = BG$ and $|B(*, G, G)| = EG$ are exactly the constructions we gave in Example 2.13.
- Suppose $\mathbf{C} = \mathbf{Mod}_R$, so the monoid $M = S$ is an R-algebra. Then, $B(X, S, Y)$ is a simplicial R-module, so we can define a chain complex $K_\bullet(X, S, Y)$ of R-modules by letting the boundary map be as in (2.12). This chain complex is the usual resolution for computing $\mathrm{Tor}_S(X, Y)$!

Simplicial Homotopies. Homotopies of topological spaces are defined via unit interval $[0, 1]$; for simplicial sets, Δ^1 plays the analogous role. Everything in the next two sections comes from [GJ].

Definition 2.15. Let $f, g : X \rightrightarrows Y$ be two morphisms of simplicial sets. A *homotopy* $\eta : f \Rightarrow g$ is a morphism $\eta : X \times \Delta^1 \to Y$ such that the diagram

$$\begin{tikzcd}
X \cong X \times \Delta^0 \arrow[r, "(\mathrm{id}, d^1)"] \arrow[dr, "f"'] & X \times \Delta^1 \arrow[d, "\eta"] & X \times \Delta^0 \cong X \arrow[l, "(\mathrm{id}, d^0)"'] \arrow[dl, "g"] \\
& Y &
\end{tikzcd}$$

commutes. Here, $d^0, d^1 : \Delta^0 \rightrightarrows \Delta^1$ are the maps realizing Δ^0 as the zeroth and first vertices of Δ^1, respectively.

This is probably more or less what you were expecting. However, what's more surprising is that homotopy is not an equivalence relation! The 1-simplex Δ^1 defines a homotopy from d^0 to d^1 as maps $\Delta^0 \to \Delta^1$, but since $1 > 0$, there's no 1-simplex which can produce a homotopy from d^1 to d^0. This is awkward, but we do have the following result.

Proposition 2.16. *Homotopy is an equivalence relation on maps $X \to Y$ iff Y is a Kan complex.*

As such, we define simplicial homotopy groups for Kan complexes. Instead of picking an arbitrary basepoint, as we did in **Top**, we choose a vertex $x : \Delta^0 \to X$.

Definition 2.17. If X is a Kan complex, $x : \Delta^0 \to X$ is a vertex, and $n > 0$, we define the n^{th} *simplicial homotopy group* based at x to be the set of homotopy classes of maps $\Delta^n \to X$ that fix the boundary $\partial\Delta^n$, such that $\partial\Delta^n$ maps to x. That is, we consider maps $f : \Delta^n \to X$ fitting into a commutative diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & \Delta^0 \arrow[d, "x"] \\
\Delta^n \arrow[r, "f"] & X,
\end{tikzcd}$$

where two maps are equivalent if there is a homotopy between them fixing $\partial\Delta^n$. To define the group structure, let $f, g : \Delta^n \rightrightarrows X$ represent two elements of $\pi_n(X, x)$. Let

$$v_i = \begin{cases} x, & 0 \le i \le n-2 \\ f, & i = n-1 \\ g, & i = n+1. \end{cases}$$

Then, the assignment $i \mapsto v_i$ defines a map $\tilde{h} : \Lambda^{n+1}_n \to X$, so since X is a Kan extension, this extends to a map $h : \Delta^{n+1} \to X$. Then, we define $[a] \cdot [b] = [d_n h]$,[2] which one can show makes $\pi_n(X, x)$ into a group (the constant map to v is the identity), and an abelian group if $n \ge 2$.

We define $\pi_0(X)$, the set of *path components* of X, to be the set of homotopy classes of vertices of X.

Definition 2.18. Let X and Y be Kan complexes. A map $f : X \to Y$ is a *weak equivalence* if for all vertices x of X and $n \ge 1$, the induced map $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism, and $f_* : \pi_0(X) \to \pi_0(Y)$ is a bijection.

[2] There's a lot to check here: why is $d_n h$ constant on $\partial\Delta^n$? Why is this independent of choice of representative for $[a]$ and $[b]$?

Bisimplicial Sets. Dwyer and Kan mention in [DK1] that they "will often use, explicitly or implicitly," a result about bisimplicial sets (Proposition 2.21, below). As such, it's probably a good idea to at least explain what they're saying.

Bisimplicial sets fit into the analogy

$$\text{simplicial sets} : \text{chain complexes} :: \text{bisimplicial sets} : \text{double complexes}.$$

As double complexes are important in the genesis of spectral sequences, you might guess bisimplicial objects are too, and you'd be right.

Definition 2.19. A *bisimplicial set* is a simplicial object in **sSet**; equivalently, it is a functor $\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} = (\Delta \times \Delta)^{\mathrm{op}} \longrightarrow \mathbf{Set}$. Replacing sets with another category **C** defines the notion of a *bisimplicial object* in **C**.

Given a bisimplicial set viewed as a functor $K : (\Delta \times \Delta)^{\mathrm{op}} \longrightarrow \mathbf{Set}$, $K([m],[n])$ is written $K_{m,n}$ and called the *degree-*(m,n) *bisimplices* of K. The face and degeneracy maps are bigraded, denoted d_{ij} and s_{ij}.

Definition 2.20. If K is a bisimplicial set, its *diagonal* $\operatorname{diag} K$ is the simplicial set with n-simplices $(\operatorname{diag} K)_n = K_{n,n}$ and whose face and degeneracy maps are the diagonal maps d_{ii} and s_{ii}.

That is, if K is the functor $\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$ and $\mathrm{Diag} : \Delta^{\mathrm{op}} \longrightarrow \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$ is the diagonal functor, then $\operatorname{diag} K = K \circ \mathrm{Diag}$.

Alternatively, thinking of K as a simplicial object in **sSet**, its n-simplices are a simplicial set $K_{n,\bullet}$. These are called the *vertical simplicial sets* associated to K.

Proposition 2.21 ([GJ, Prop. IV.1.9])**.** *If $K \longrightarrow L$ is a map of bisimplicial sets such that, for every integer $n \geq 0$, the restriction $K_{n,\bullet} \longrightarrow L_{n,\bullet}$ is a homotopy equivalence, then its diagonal* $\operatorname{diag} K \longrightarrow \operatorname{diag} L$ *is also a weak homotopy equivalence.*

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