M392c NOTES: ALGEBRAIC GEOMETRY

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Lecture 1.

Some questions in algebraic geometry: 8/29/18

Office hours are Fridays from 11-1, in room 9.164 (at least for now). Today we'll talk about some questions (and some answers, too!) relating to algebraic geometry and why one might find it interesting. We're going to focus on concreteness.

Broadly speaking, algebraic geometry studies zero sets of polynomials. These could be polynomials over $\mathbb{Q}$, or $\mathbb{R}$, or $\mathbb{C}$, or finite fields, or more. The first question you might ask is, *are there solutions*? This is an *arithmetic question*: in arithmetic situations, there might not be solutions.

Example 1.1 (Taylor-Wiles, 1994)**.** If $n \geq 3$, the polynomial $x^n + y^n = 1$ has no solutions over $\mathbb{Q}$ when $x, y \neq 0$. ◀

You might recognize this as a reformulation of Fermat's last theorem.

Another form of the same question is *can you parameterize solutions of the equation*? For example, let's try it with $x^2 + y^2 = 1$, which we know has solutions. In this case, it is possible to parameterize solutions, via the one-parameter family

$$x = \frac{\lambda^2 - 1}{\lambda^2 + 1}, \qquad y = \frac{2\lambda}{\lambda^2 + 1}. \tag{1.2}$$

These kinds of questions are called *rationality questions*. One can also ask these questions over $\mathbb{C}$ (or over other algebraically closed fields), where they can feel a bit different.

There is a general result that any quadric hypersurface with a rational point is rational. What this means is that if you assume the existence of one solution (x_0, y_0) to a degree-2 polynomial in x and y over, say, $\mathbb{Q}$, then you can use that one solution to parameterize all other solutions. If you plot the solutions in the xy-plane, the parameter of another solution (x_1, y_1) is the slope of the line between (x_0, y_0) and (x_1, y_1). Indeed, in (1.2), the parameter λ is this slope. Because the equation is a quadric, one expects such a line to intersect in exactly two points, the first solution and another one. This is all extremely explicit, to the point that you could explain why you care to a middle schooler.

There are a few other rationality results.

Theorem 1.3 (Segre, 1940s; Manin, 1970s; Kollár, 2000)**.** *Smooth cubics in at least three variables are rational.*

So $x^2 + y^2 = 1$ isn't rational, but $x^2 + y^2 + z^2 = 1$ is. However, this doesn't give you everything.

Theorem 1.4 (Clemens-Griffiths, 1974)**.** *There are cubics in at least four variables which are unirational but not rational, i.e. that one cannot parameterize all solutions in a one-to-one manner.*

This was a hard theorem. How would you prove something like this?

Recent work (2012-15) by many people (Voisin, Colliot-Thèlene, Pirutka, Totaro[1]) generalizes this.

Theorem 1.5. *For cubics in at least five variables, one can also not parameterize solutions in a one-to-one way, even by adding additional "dummy variables."*

For four-variables cubics, this is open.

Schemes. Though this result is stated completely explicitly, it was studied using some very abstract-looking machinery. In this course, we'll also work with this abstract machinery, namely the language of schemes. These are things like solutions to systems of polynomials, but not quite — they encode among other things the equivalence of such systems under changes of coordinates, which doesn't really change the underlying geometry of the solution set. Classification problems with this perspective are a big area of research, and Birkar just won a Fields medal for work in this area from 2006.

Algebraic geometry over $\mathbb{C}$. A third thing you could care about is specific stuff about algebraic geometry over your favorite field (typically $\mathbb{C}$, but not always). In many cases (such as $\mathbb{C}$), you have topology around, and you can ask how it interacts with the algebraic geometry we've been talking about.

For example, if $q \in \mathbb{C}^\times$ isn't a root of unity, then there's a cubic equation $y^2 = x^2 + ax + b$ whose solutions are parameterized by $\mathbb{C}^\times/q\mathbb{Z}$. This may be a bit surprising, and indicates a way in which analytic or topological information can be useful: now we can learn about the universal cover of the solution space, and other topological invariants. Then you might ask whether something like this is true in positive characteristic, which tends to be harder.

More generally, one can study the topology of algebraic varieties over $\mathbb{C}$.

Theorem 1.6. *The odd Betti numbers of smooth proper varieties are even.*

[1]If you like pictures of cats, check out Totaro's math blog: https://burttotaro.wordpress.com/.

The proof uses the study of the Hodge Laplacian operator on a variety X. This needs a metric, but projective means that X embeds in some $\mathbb{CP}^n$, and we can borrow its metric. There is a purely algebro-geometric proof of this, but first you need to come up with the right notion of Betti numbers (so étale cohomology, which is hard), and then invoke Deligne's proof of the Weil conjectures (also hard). Nonetheless, it's true in characteristic p.

More generally, the cohomology of a complex projective variety has more structure, and is much richer than that of a random manifold.[2]

Conjecture 1.7 (Hodge conjecture, imprecise statement)**.** *The differential topology of a projective algebraic variety over $\mathbb{C}$ knows everything about its algebraic geometry.*

This is a Millennium Prize problem, meaning it comes with a $1 million reward. You can infer that it's hard.

Algebraic geometry over $\mathbb{Z}$. If you work over $\mathbb{Z}$ instead of over $\mathbb{C}$, meaning your polynomial has integer coefficients, then you can reduce mod p and solve it there. This is the first thing anyone does in number theory, because it often simplifies the problem to a finite question. This naturally leads one to ask, *how do the systems of equations at different primes p relate to each other?*

There's a lot to say about this, beginning with quadratic reciprocity, which is very classical yet a little weird, and continuing all the way to the Langlands program.

Supposing X encodes the system of solutions to your polynomial with $\mathbb{Z}$ coefficients. Then one can define a zeta function, reminiscent of the Riemann zeta function, as follows:

$$\zeta_X(s) \coloneqq \prod_{p \text{ prime}} \exp\left(\sum \frac{1}{n}(\text{number of solutions in } \mathbb{F}_{p^n})p^{-ns}\right). \tag{1.8}$$

For $X = \operatorname{Spec} \mathbb{Z}$, corresponding to solutions to an empty set of polynomials, this recovers the usual Riemann zeta function.

For any particular X, one conjectures this is meromorphic (and almost entire, in some sense), and that the analogue of the Riemann hypothesis holds; for some X, this is known due to Deligne. There are some other related conjectures related to this known as Sato-Tate conjectures.

Cohomology theories. Over $\mathbb{C}$, you have topology, and therefore can invoke algebraic topology to compute cohomology of algebraic varieties. Over other fields or rings, you might not have these techniques, and there are several other approaches.

- Over an algebraically closed field, one has *étale cohomology*, whose ideas are built from covering space theory, has $\mathbb{Z}_\ell$ coefficients, where ℓ is a prime that's not the characteristic of the field.
- Over any field k, there's *de Rham cohomology*, which uses the idea that $\mathrm{d}z/z$ understands $\mathbb{C}^\times$ isn't simply connected (since $\oint \mathrm{d}z/z \neq 0$). This has coefficients in k.

There are others, too. One wants these to all be the same, or at least closely related; if $k = \mathbb{Q}_p$ and $\ell = p$ ($\mathbb{Q}_p$ has characteristic zero!), then these two are related by p-adic Hodge theory. This is related to deep and recent work by Fontaine, Scholze, and others, and relates to Scholze's Fields medal work. In 2016, Bhatt-Morrow-Scholze showed that one can sometimes interpolate between different cohomology theories. See Scholze's ICM address for more on this. The ultimate question in this corner of algebraic geometry is whether there's some universal cohomology theory interpolating between everything we have, and which is also the source of the ζ-functions mentioned above.

Degenerations. We get additional power by studying solutions in families. For example, we can degenerate $x^2 + y^2 = 1$ to $x^2 + y^2 = 0$, which is much simpler. One asks questions such as, *what invariants are preserved under degenerations?* Therefore one might be able to use a degeneration to reduce a harder problem to an easier problem.

Computations. This subfield of algebraic geometry tries to make these abstract invariants concrete, by writing good algorithms to compute these invariants for explicit systems of polynomials.

[2]This doesn't require smoothness *per se*, but it's more difficult to formulate in the singular case.

Geometric complexity theory. This is another way to relate algebraic geometry and computer science. The goal of this field is to approach another Millennium Prize problem, P vs. NP, using algebraic geometry techniques. This roughly involves studying certain varieties and analyzing whether they're as complicated as they seem. Algebraic geometry has lots of techniques which might help, but on the other hand they haven't yet.

Probably the best way to learn algebraic geometry is to have an application or research focus in mind that you can apply the things you learn to. This method of learning tends to produce algebraic geometers.

Lecture 2.

Defining schemes, I: 8/31/18

The goal of today's lecture is to define a scheme, first heuristically and then rigorously.

"Definition" 2.1. A scheme is a "space" that is a Zariski sheaf which admits an "open cover" by affine schemes.

Of course, in order to do this, we need to know what all of these words — spaces, Zariski sheaves, affine schemes, and open covers — mean in this setting.

Remark 2.2. There's another approach to schemes using the formalism of *locally ringed spaces*, which is followed by Hartshorne, Vakil, and many others. It's more concrete, but it makes it harder to think about what a specific scheme, such as projective space, is supposed to be. ⊲

The motivation for "Definition" 2.1 is that a scheme should be something which is locally defined by algebraic equations. For example, let's look at the *Fermat equation* $X_n = \{x^n + y^n = z^n\}$. Fermat was interested in solutions in $\mathbb{Z}$, but the set of solutions makes sense in any commutative ring. This suggests our definition of space, which is not the same as a topological space.

Definition 2.3. A *space* is a functor $X\colon \mathsf{CommRing} \to \mathsf{Set}$.

Concretely, this means that for every ring A, we get a set $X(A)$, and for every map of commutative rings $f\colon A \to B$, we get a map of sets $X(f)\colon X(A) \to X(B)$, and these morphisms should compose well (meaning that $X(f \circ g) = X(f) \circ X(g)$ and $X(\mathrm{id}) = \mathrm{id}$). For example, we could let $X_n(A)$ denote the set of solutions to the Fermat equation in the ring A; then, if we've solved it in A, we can map the solution into B via $f\colon A \to B$, and we'll obtain a solution in B, so this defines a space X_n.

We should also say how spaces interact.

Definition 2.4. A *morphism of spaces* $f\colon X \to Y$ is data of, for all commutative rings A, a map $f_A\colon X(A) \to Y(A)$ such that for all ring homomorphisms $g\colon A \to B$, the diagram

$$\begin{tikzcd} X(A) \arrow[r, "f_A"] \arrow[d, "X(g)"] & Y(A) \arrow[d, "Y(g)"] \\ X(B) \arrow[r, "f_B"] & Y(B) \end{tikzcd}$$

commutes.

Schemes are special examples of spaces, in a way that feels surprisingly down-to-Earth.

Our first example of a space is the solutions to the Fermat equation in A, as discussed above. Here's another example.

Example 2.5. Let A be a commutative ring. We'll define the space $\operatorname{Spec} A$ to be the functor $(\operatorname{Spec} A)(B) = \operatorname{Hom}(A, B)$; given a ring homomorphism $\varphi\colon B \to C$, we use the map $\operatorname{Hom}(A, B) \to \operatorname{Hom}(A, C)$ given by postcomposition with φ. ⊲

Definition 2.6. An *affine scheme* is a space of the form $\operatorname{Spec} A$ for some A.

You don't have to be a commutative algebra expert to learn algebraic geometry, but you can see that commutative algebra is built into the definitions of algebraic geometry, so some commutative algebra knowledge is helpful.

Example 2.7. The space X_n sending A to the solutions of the Fermat equation in A is an affine scheme; explicitly,

$$X_n \cong \operatorname{Spec} \mathbb{Z}[x, y, z]/(x^n + y^n - z^n).$$

This is because a ring homomorphism $\mathbb{Z}[x, y, z]/(x^n + y^n - z^n) \to A$ is exactly the data of $x, y, z \in A$ satisfying the relation $x^n + y^n - z^n = 0$. ◀

Lemma 2.8 (Yoneda lemma)**.** *For all spaces X,* $\operatorname{Hom}_{\mathsf{Spaces}}(\operatorname{Spec} A, X) \cong X(A)$.

Proof sketch. First we define a map from $\operatorname{Hom}_{\mathsf{Spaces}}(\operatorname{Spec} A, X)$ to $X(A)$. Specifically, a map $f\colon \operatorname{Spec} A \to X$ is the data of for all commutative rings B, $\operatorname{Spec}(A)(B) \to X(B)$. Take $B = A$; then, $\operatorname{Spec}(A)(A) = \operatorname{Hom}(A)$, so take the image of the identity. It remains to check this is an equivalence. ⊠

Corollary 2.9. $\operatorname{Hom}_{\mathsf{Spaces}}(\operatorname{Spec} A, \operatorname{Spec} B) \cong \operatorname{Hom}_{\mathsf{CommRing}}(B, A)$.

It's interesting that the direction reverses!

Proof. By the Yoneda lemma, $\operatorname{Hom}_{\mathsf{Spaces}}(\operatorname{Spec} A, \operatorname{Spec} B) = \operatorname{Spec}(B)(A) = \operatorname{Hom}(B, A)$. ⊠

This tells you that as long as you make sure to reverse the arrows, anything you can do with commutative rings, you can do with affine schemes, and vice versa.

Fiber products. This is a categorical construction which we're going to use a lot.

Definition 2.10. Let X, Y, and Z be sets and $f\colon X \to Z$ and $g\colon Y \to Z$ be set maps. Then the *fiber product* of X and Y over Z is

$$X \times_Z Y := \{(x, y) \in X \times Y \mid f(x) = g(y)\}. \tag{2.11}$$

If X, Y, and Z are spaces, and f and g are maps of spaces, then the *fiber product* of X and Y over Z is the space defined by

$$(X \times_Z Y)(A) := X(A) \times_{Z(A)} Y(A). \tag{2.12}$$

Technically, the notation should include f and g, but in practice there's usually no ambiguity.

Example 2.13. Suppose we're given commutative rings A, B, and C and maps $\operatorname{Spec} B \to \operatorname{Spec} C$ and $\operatorname{Spec} A \to \operatorname{Spec} C$ (which are equivalent data to maps $\varphi\colon C \to A$ and $\psi\colon C \to B$). Then

$$\operatorname{Spec} A \times_{\operatorname{Spec} C} \operatorname{Spec} B \cong \operatorname{Spec}(A \otimes_C B),$$

where C acts on A, resp. B, through φ, resp. ψ. It's worth working through this one on your own, though it's not extremely hard. ◀

We'll define some properties of affine schemes with geometric names, but the definitions will rest on algebraic properties of rings. One of the real miracles of algebraic geometry is that this really works to define geometry, and even extends geometric intuition to places such as finite fields that are otherwise very hard to reason about.

Definition 2.14. A morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is a *closed embedding* if the induced map $A \to B$ is surjective.

Equivalently, $B = A/I$ for some ideal I of A.

The geometric idea behind defining $\operatorname{Spec} A$ is that geometric objects have a ring of functions on them, e.g. a smooth manifold M has a ring $C^\infty(M)$ of smooth $\mathbb{R}$-valued functions, and a map of manifolds $M \to N$ induces a map in the other direction by pullback: $C^\infty(N) \to C^\infty(M)$. Functional analysis results such as the Gelfand-Naimark theorem tell you what data you need to add to $C^\infty(M)$ to recover M as a topological space, and we're trying to imitate this in a more abstract algebraic setting.

This context allows us to explain why Definition 2.14 deserves to be called a closed embedding: let $I = (f_1, f_2, \dots)$, so

$$\operatorname{Spec} A/I = \{f_i = 0 \text{ for all } i\} = \{f = 0 \text{ for all } f \in I\}. \tag{2.15}$$

So we think of $\operatorname{Spec} B$ as some kind of closed subspace of $\operatorname{Spec} A$, and I as the ideal of functions on $\operatorname{Spec} A$ which vanish on $\operatorname{Spec} B$. This intuition can be turned into something precise.

Using fiber products, we can extend this to all spaces.

Definition 2.16. A map $X \to Y$ of spaces is a *closed embedding* if for all maps $\operatorname{Spec} A \to Y$, the "pullback" φ in the fiber product diagram

$$\begin{tikzcd} \operatorname{Spec} A \times_Y X \arrow[r, "\varphi"] \arrow[d] & \operatorname{Spec} A \arrow[d] \\ X \arrow[r] & Y \end{tikzcd} \tag{2.17}$$

is a closed embedding of affine schemes. In particular, we require $\operatorname{Spec} A \times_Y X$ to be an affine scheme, which is not always satisfied.

For a quick consistency check, we should ask that Definitions 2.14 and 2.16 agree on affine schemes, and indeed, if $I \subset A$ is an ideal, and $\operatorname{Spec} B \to \operatorname{Spec} A$ is a closed embedding in the sense of Definition 2.14, then (2.17) looks like

$$\begin{tikzcd} \operatorname{Spec}(A/I \otimes_A B) \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\ \operatorname{Spec}(A/I) \arrow[r] & \operatorname{Spec} A, \end{tikzcd} \tag{2.18}$$

and since $A/I \otimes_A B \cong B/BI$, this is a closed embedding in the more general sense as well.

We'd also like to know what an open embedding is. We'd like to say that it's something whose complement is a closed embedding. Let's make this precise.

Definition 2.19. Let $Z \hookrightarrow X$ be a closed embedding of spaces. The *complement* $X \setminus Z$ of Z in X is the space with $(X \setminus Z)(A)$ the set of $x \in X(A) = \operatorname{Hom}_{\mathsf{Spaces}}(\operatorname{Spec} A, X)$ such that the diagram

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & Z \arrow[d] \\ \operatorname{Spec} A \arrow[r, "x"] & X \end{tikzcd} \tag{2.20}$$

is a fiber product diagram. Here $\varnothing = \operatorname{Spec}(0)$, which sends every ring to the empty set.[3]

Definition 2.21. If $X = \operatorname{Spec} A$ is an affine scheme, an *open embedding* is a map of spaces $j \colon U \to X$ such that $U = X \setminus Z$ for some closed embedding $Z \hookrightarrow X$.

Example 2.22. Letting $X = \operatorname{Spec} A$, if $f \in A$ and $Z = \operatorname{Spec}(A/f)$, the map $A \twoheadrightarrow A/f$ induces a closed embedding $Z \hookrightarrow X$. Its complement is $\operatorname{Spec} A[f^{-1}]$, the *localization* of A at f, so $\operatorname{Spec} A[f^{-1}] \to \operatorname{Spec} A$ is an open embedding. ◀

The intuition is that f generates the ideal of functions that vanish precisely on the closed subset Z. Therefore on the complement of Z, they should be invertible, so we adjoin an inverse to f.

Lemma 2.23. *Let $X = \operatorname{Spec} A$ and $Z = \operatorname{Spec} A/I$. Then maps $\operatorname{Spec} B \to X \setminus Z$ correspond bijectively to maps $A \to B$ such that $B \cdot I = B$.*

Proof. The diagram (2.20) specializes to

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & \operatorname{Spec}(A/I) \arrow[d] \\ \operatorname{Spec} B \arrow[r] & \operatorname{Spec} A, \end{tikzcd} \tag{2.24}$$

and this fiber product is $\operatorname{Spec}(B \otimes_A A/I) = \operatorname{Spec}(B/IB) = \varnothing$, which is equivalent to $IB = B$. ⊠

Example 2.25. *Affine n-space* over $\mathbb{Z}$ is the affine scheme $\mathbb{A}^n_{\mathbb{Z}} := \operatorname{Spec} \mathbb{Z}[x_1, \dots, x_n]$, and $0 \hookrightarrow \mathbb{A}^n_{\mathbb{Z}}$ is the closed embedding corresponding to the ideal $(x_1, \dots, x_n)$. The complement $\mathbb{A}^n_{\mathbb{Z}} \setminus 0$ is not affine for $n > 1$! We'll prove that later when we have more tools. ◀

[3]Caution: this is only true if we work with functors on nonzero rings. However, $\varnothing = \operatorname{Spec} 0$ still counts as affine. There are other ways to correct this issue, but this is among the fastest and cheapest.

Exercise 2.26. Show that $\mathbb{A}^n_{\mathbb{Z}} \setminus 0$ is the space which maps a ring A to the set of n-tuples $(x_1, \dots, x_n) \in A^n$ such that the equation $\sum x_i y_i = 1$ has a solution.

Lecture 3.

Open covers: 9/5/18

We've been talking about functors as if they were honest geometric objects. And they *are*: the crucial reason is that we're defining open and closed subspaces of affine schemes. You can picture these as akin to open or closed sets in a topological space, and they will allow us to make sense of geometry by giving us notions of locality.

Recall that $Z \hookrightarrow X = \operatorname{Spec} A$ is a closed embedding means that this embedding is of the form $\operatorname{Spec}(A/I) \to \operatorname{Spec} A$ induced by the map $A \twoheadrightarrow A/I$, and that open embeddings are complements of closed ones. You might think of the complement as $(X \setminus Z)(B) = X(B) \setminus Z(B)$, but **this is wrong**: it's not even functorial! Instead, we want to say $(X \setminus Z)(B) = \{\operatorname{Spec} B \to X \setminus Z\}$. What this means is maps $\operatorname{Spec} B \to X$ such that the pullback $\operatorname{Spec} B \times_X Z = \varnothing$. Geometrically, this fiber product is telling you the intersection of the image of $\operatorname{Spec} B$ with X.

Last time, we also talked about $\mathbb{A}^1$ (also $\mathbb{A}^1_{\mathbb{Z}}$ if you want to specify the base), which is by definition $\operatorname{Spec} \mathbb{Z}[t]$. It would be nice to think of this as a line, in the sense you can draw; but it behaves more like a complex line (that is, a plane). For example, $\mathbb{A}^1$ minus a point is connected. So thinking of it as a complex line is good, but for drawing pictures you'll run out of dimensions, so the picture of a real line is also helpful.

If B is a commutative ring, $\mathbb{A}^1(B) = \{\operatorname{Spec} B \to \mathbb{A}^1\}$, i.e. $\operatorname{Hom}(\mathbb{Z}[t], B) = B$, because the map is determined by where it sends t. This makes precise the notion that the ring of functions on $\operatorname{Spec} B$ is B. This is another avatar of geometry as we know it: functions on a geometric object (say, a complex manifold) are functions to a complex line, and in this setting we replace the complex line by $\mathbb{A}^1$.

Consider the embedding $0 \hookrightarrow \mathbb{A}^1_{\mathbb{Z}}$, where 0 denotes the locus where $t = 0$, i.e. $\operatorname{Spec} \mathbb{Z}[t]/(t)$. As an affine scheme, this is isomorphic to $\operatorname{Spec} \mathbb{Z}$, because $\mathbb{Z}[t]/(t) \cong \mathbb{Z}$, but this defines a particular closed embedding $0 \hookrightarrow \mathbb{A}^1_{\mathbb{Z}}$. Last time, we discussed $\mathbb{A}^1 \setminus 0$. A map $\operatorname{Spec} B \to \mathbb{A}^1 \setminus 0$ is a function that avoids zero, which means that it's invertible.

Exercise 3.1. Show that $(\mathbb{A}^1 \setminus 0)(B) = B^\times$, and therefore that $\mathbb{A}^1 \setminus 0 \cong \operatorname{Spec} \mathbb{Z}[t, t^{-1}]$.

If we did this with $\mathbb{A}^2 \setminus 0$ instead of $\mathbb{A}^1 \setminus 0$, we'd obtain a nonaffine scheme.

Open coverings are another important geometric notion, and they exist in this setting too.

Definition 3.2. If $X = \operatorname{Spec} A$ is an affine scheme, a *(Zariski) open covering* of X is a collection of open embeddings $\mathfrak{U} = \{(U, i_U : U \hookrightarrow X)\}$ such that for every nonempty $S = \operatorname{Spec} B$ and $f : S \to X$, there's some $(U, i_U) \in \mathfrak{U}$ such that $U \times_X S \neq \varnothing$.

This is the first notion of open covering in algebraic geometry; there are some others around.

The intuition behind open coverings is that points of X are given by maps $\operatorname{Spec} B \to X$, and we want every point in X to intersect some open embedding in the cover.

Proposition 3.3. *Let $X = \operatorname{Spec} A$ and $\mathfrak{U} = \{(U, i_U : U \to X)\}$ be a collection of open embeddings. The following are equivalent:*

(1) *$\mathfrak{U}$ is an open covering.*
(2) *$\mathfrak{U}$ has a finite subset $\mathfrak{V} \subset \mathfrak{U}$ which is also an open covering of X.*
(3) *For all fields k and maps $x : \operatorname{Spec} k \to X$, there's some $(U, i_U) \in \mathfrak{U}$ such that x factors through i_U.*
(4) *Letting $U = X \setminus Z_U$ for each $U \in \mathfrak{U}$, and writing $Z_U = \operatorname{Spec}(A/I_U)$, then*

$$\sum_{U \in \mathfrak{U}} I_U = A.$$

Point (2) is very weird coming from topology, where the open covering $\{(i-1, i+1) \mid i \in \mathbb{Z}\}$ is an open cover of $\mathbb{R}$ with no finite subcover. In other words, affine schemes feel like compact spaces from the perspective of open coverings!

The idea behind (3) is that points are affine schemes of the form $\operatorname{Spec} k$ for k a field. There are different fields, and therefore different kinds of points. The reason for including (4) is that it's very useful for checking in practice. It has a similar feel to partitions of unity in manifold topology, but if you don't know what that is, that's OK.

Proof. We'll first show (1) $\implies$ (4). Suppose $\mathfrak{U}$ is an embedding for which (4) does not hold. Then let

$$B \coloneqq A/\sum_{U\in\mathfrak{U}} I_U. \tag{3.4}$$

By hypothesis, $B \neq 0$, and we have a closed embedding $\operatorname{Spec} B \hookrightarrow \operatorname{Spec} A$. We'll show that $\operatorname{Spec} B \times_X U = \varnothing$ for all $U \in \mathfrak{U}$.

Lemma 3.5. *Let $Z = \operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A = X$ be a closed embedding and $f\colon \operatorname{Spec} B \to X$ be a map. Then*

$$(\operatorname{Spec} B) \setminus f^{-1}(Z) = \operatorname{Spec} B \times_X (X \setminus Z).$$

This is more or less a tautology.

Returning to the claim, $\operatorname{Spec} B \times_X U$ is the complement of $Z_U \times_X \operatorname{Spec} B = \operatorname{Spec}(B/BI_U)$. But $B/BI_U = B$, so the complement of $Z_U \times_X \operatorname{Spec} B$ is the empty set.

Next, we'll show (4) $\implies$ (3). Let k be a field, and $x\colon \operatorname{Spec} k \to X$ be a map. We want to show this map factors through some U. Since $X = \operatorname{Spec} A$, x corresponds to a map $\varphi\colon A \to k$. We claim there's a $U \in \mathfrak{U}$ with $\varphi(I_U) \neq 0$; otherwise $\varphi(\sum I_U) = 0$, and therefore $\varphi(A) = 0$. However, $\varphi(1) = 1$, so this is impossible. By Lemma 2.23, since $\varphi(I_U) \neq 0$, $k \cdot \varphi(I_U) = k$, and therefore $x\colon \operatorname{Spec} k \to X$ factors through U.

Next we'll show (3) $\implies$ (1). Let B be as in (3.4) and $f\colon S = \operatorname{Spec} B \to X$ be a map. We want to show that $S \times_X U \neq 0$ for some $U \in \mathfrak{U}$. Since $B \neq 0$, it has a maximal ideal $\mathfrak{m}$, and $B/\mathfrak{m}$ is a field k (TODO: to be continued...) $\boxtimes$

Lecture 4.

Defining schemes, II: 9/7/18

"*I'll let* $\operatorname{Fun}(Y)$*, which is such a fun notation, denote...*"

Last time, we talked about open embeddings and open covers for affine schemes; today, we'll generalize this to spaces.

Definition 4.1. Let X be a space.

(1) A map $U \to X$ of spaces is an *open embedding* if for all affine schemes $S = \operatorname{Spec} A$ and maps $f\colon S \to X$ of spaces, the pullback $g\colon U \times_X S \to S$ arising in the diagram

$$\begin{tikzcd} U \times_X S \arrow[r] \arrow[d, "g"] & U \arrow[d] \\ S \arrow[r, "f"] & X \end{tikzcd}$$

is an open embedding (since we've already define open embeddings where the target is affine).

(2) A *Zariski open covering* of X is the same as in Definition 3.2, but for open embeddings of spaces, rather than affine schemes.

In this case, Proposition 3.3 need not hold: there are open coverings of some spaces X (such as an infinite disjoint union of points) which have no finite subcoverings.

Definition 4.2. A space X is a *Zariski sheaf* if for all $S = \operatorname{Spec} A$ and open coverings $\mathfrak{U}$ of S, the map

$$\operatorname{Hom}(S, X) \longrightarrow \{(f_U\colon U \to X \text{ for all } U) \in \mathfrak{U} \mid f_U|_{U\cap V} = f_V|_{U\cap V} \text{ for all } U, V \in \mathfrak{U}\}$$

is an isomorphism. (Here $U \cap V = U \times_X V$.)

Not everything is a Zariski sheaf, but the things that aren't are terrible, and you shouldn't worry too much about them.

Now we have all the definitions at hand to define schemes!

Definition 4.3. A *scheme* is a space which is a Zariski sheaf and admits an open cover $\mathfrak{U}$ such that all $U \in \mathfrak{U}$ are affine schemes.

Exercise 4.4. Let X be the space with

$$X(A) = \{t \in A \mid t \in A^\times \text{ or } (1-t) \in A^\times\}.$$

Show that X is *not* a Zariski sheaf. Also, if you know what sheafification is, show that the sheafification of X is $\mathbb{A}^1$.

Proposition 4.5. *If X is an affine scheme, then it's a scheme.*

Obviously X admits an open cover by affines, given by $\mathrm{id}\colon S \to S$; the meat of the proof (or, if you prefer, tofu) is that it's a Zariski sheaf. Unlike EGA, we will start with a special case and use it to bootstrap to the general case.

Let $X = \mathbb{A}^1$.

Definition 4.6. A *function* on a space Y is a map to $\mathbb{A}^1$. We'll let $\mathrm{Fun}(Y) \coloneqq \mathrm{Hom}(Y, \mathbb{A}^1)$.

We're explicitly trending towards geometric notation and intuition for things: one of the key processes of learning scheme theory is to start thinking geometrically rather than with commutative algebra – except when you need to prove something.

We want to show that for all affine schemes S and open coverings $\mathfrak{U}$ of S, the map

$$\mathrm{Fun}(S) \longrightarrow \{(f_U \in \mathrm{Fun}(U)) \mid f_U|_{U\cap V} = f_V|_{U\cap V}\} \tag{4.7}$$

is an isomorphism.

First we'll prove this for a nice class of open covers.

Lemma 4.8. *Let A be a commutative ring and $f_1, \dots, f_n \in A$. Let $D(f_i) \coloneqq \operatorname{Spec} A \setminus \operatorname{Spec}(A/(f_i))$. Then*

(1) $D(f_i) = \operatorname{Spec} A[f_i^{-1}]$, *and*

(2) $\{D(f_i)\}$ *is an open cover iff* $\{f_i\}$ *generates the unit ideal.*

The proof will be left as an exercise.

In the case (2) holds, the open cover $\{D(f_i)\}$ is called a *basic open cover*. It's really nice because it's an affine open cover; we'll see that there are a lot of these coverings, and enough that we will eventually be able to reduce to this case.

One can alternately characterize $D(f_i)$ as the pullback

$$\begin{tikzcd} D(f_i) \arrow[r] \arrow[d] & \operatorname{Spec} A \arrow[d] \\ \mathbb{A}^1 \setminus 0 \arrow[r] & \mathbb{A}^1. \end{tikzcd} \tag{4.9}$$

Lemma 4.10. *Let $f\colon M \to N$ be a map of A-modules. Then f is injective (resp. surjective, resp. bijective) iff for all i, the map $M[f_i^{-1}] \to N[f_i^{-1}]$ is injective (resp. surjective, resp. bijective).*

Recall that $M[f_i^{-1}] \coloneqq M \otimes_A A[f_i^{-1}]$.

Remark 4.11. Let's review some facts about localization. If M is a $\mathbb{Z}[t]$-module, which is equivalent data to an abelian group with an endomorphism $t\colon M \to M$, then we can form $M[t^{-1}] \coloneqq M \otimes_{\mathbb{Z}[t]} \mathbb{Z}[t, t^{-1}]$. Then:

- This construction is *exact*; that is, it preserves kernels and cokernels.
- There's a natural map $M \to M[t^{-1}]$, and its kernel is the submodule of $m \in M$ with $t^n m = 0$ for some n.

The way you prove all of this is to write $\mathbb{Z}[t, t^{-1}]$ as the union of $t^{-n}\mathbb{Z}[t]$ for all n, or as the colimit of the multiplication-by-t map on $\mathbb{Z}[t]$. These are all free modules, hence flat, and one can prove that filtered colimits of flat modules are flat without too much angst. ◀

Now we can get back to the lemma.

Proof of Lemma 4.10. For injectivity, let's compare $\ker\varphi$ with $\ker(\varphi[f_i^{-1}])$. By Remark 4.11, $(\ker\varphi)[f_i^{-1}] = \ker(\varphi[f_i^{-1}])$, so we can reduce to showing that $M = 0$ iff $M[f_i^{-1}] = 0$ for all i. One direction is immediate, of course; conversely, if $M[f_i^{-1}] = 0$ for all i, then for all $m \in M$ and all f_i, then $f_i^{n_i} m = 0$ for some $n_i \gg 0$. There are finitely many f_i, so we can let N be the biggest one, and then $f_i^N m = 0$ for all i.

Lemma 4.12. $(f_1, \dots, f_n) = A$ *iff* $(f_1^N, \dots, f_n^N) = A$.

Proof. There's a naïve argument which isn't too bad, but the geometric reason is that f_i is a function, and f_i vanishes on the same locus where f_i^N vanishes, and therefore $D(f_i^N) = D(f_i)$. Therefore $\{D(f_i^N)\}$ is also an open cover, which by (4.8) means $(f_1^N, \dots, f_n^N) = A$.

If this proof feels sketchy, here's a more careful one (which unfortunately masks the geometry): if $(f_1^N, \dots, f_n^N) \subsetneq A$, then it's contained in some maximal ideal $\mathfrak{m}$. Therefore for all i, $f_i^N = 0$ in $A/\mathfrak{m}$, and therefore $f_i = 0$ in $\mathfrak{m}$, because $A/\mathfrak{m}$ is a field; hence $f_i \in \mathfrak{m}$, which is a contradiction. ⊠

Now, using Lemma 4.12, there are some $g_1, \dots, g_n$ with $\sum_i g_i f_i^N = 1$, and therefore

$$M = \sum_i f_i f_i^N \cdot M = 0. \tag{4.13}$$

The proof of surjectivity is similar, but using cokernels instead of kernels. ⊠

This lemma is a bridge between the geometry of schemes and the linear algebra of modules. You should think of inverting f_i as restricting to $D(f_i)$; we will return to this idea.

Proof of Proposition 4.5, special case. Now we'll prove that an affine scheme is a Zariski sheaf for basic open covers $\{D(f_i)\}$. We want to show that (4.7) is an isomorphism, and by Lemma 4.10 it suffices to show this after inverting f_i.

Let $g_i \in \operatorname{Fun}(D(f_i))$ be a collection of functions that agree on overlaps... TODO: I missed the last part. ⊠

Lecture 5.

$\mathbb{A}^1$ is a Zariski sheaf: 9/10/18

Today, we're going to continue proving Proposition 4.5, that affine schemes are schemes. We're still working on the special case that $\mathbb{A}^1$ is a scheme; the key piece of the proof is showing that it's a Zariski sheaf.

Definition 5.1. Let S be a space and $\mathfrak{U}$ be an open cover of S. A *refinement* of $\mathfrak{U}$ is an open covering $\mathfrak{V}$ of S such that for all $U \in \mathfrak{U}$, $\mathfrak{V}_U := \{V \in \mathfrak{V} \mid V \subset U\}$ is an open covering of U.

The Zariski sheaf condition for maps $X \to S$ is a constraint on compatible functions on all open covers of S. If we only ask about a specific open cover $\mathfrak{U}$, we say "the Zariski sheaf property for X with respect to $\mathfrak{U}$."

Lemma 5.2. *Let X and S be spaces, $\mathfrak{U}$ be an open cover of S, and $\mathfrak{V}$ be a refinement of $\mathfrak{U}$. Suppose the Zariski sheaf property holds for X with respect to $\mathfrak{V}$, and for each $U \in \mathfrak{U}$ with respect to $\mathfrak{V}_U$, then it holds with respect to $\mathfrak{U}$.*

After you unwind all the definitions, this is a definition check which isn't very hard.

Remark 5.3. One corollary of Lemma 5.2 is that in the definition of the sheaf property, we may replace "for all affine schemes S" with "for all spaces S." All of the definitions were built from the beginning to favor affine schemes as important or special, and this is one consequence. ◂

Definition 5.4. A *big basic open covering* of an affine scheme S is an open covering by sets of the form $D(f_i)$ as in Lemma 4.8, but over a possibly infinite indexing set.

This is only a temporary definition. The Zariski sheaf property for X and every basic open covering of an affine scheme S implies the Zariski sheaf for all big basic open coverings.

Proposition 5.5. *Let S be an affine scheme and $\mathfrak{U}$ be an open cover of S. Then there's a big basic open covering of S refining $\mathfrak{U}$.*

Proof. Write $S = \operatorname{Spec} A$ and for each $U \in \mathfrak{U}$, let $Z_U := S \setminus U$; the inclusion $Z_U \hookrightarrow S$ is a closed embedding, so $Z_U = \operatorname{Spec}(A/I_U)$ for some ideal $I_U \subset A$. Recall from Proposition 3.3 that since $\mathfrak{U}$ is an open covering,

$$\sum_{U \in \mathfrak{U}} I_U = A, \tag{5.6}$$

and this is an equivalent condition. Consider the big basic open cover

$$\mathfrak{V} := \{D(f) \mid f \in I_U \setminus 0 \text{ for some } U \in \mathfrak{U}\}. \tag{5.7}$$

That this is a big basic open cover is because an ideal is generated by its elements. It's also a refinement of $\mathfrak{U}$, which follows from a more general lemma.

Lemma 5.8. *Let $U = S \setminus Z$, where $S = \operatorname{Spec} A$ and $Z = \operatorname{Spec}(A/I)$. Then $\{D(f) \mid f \in I \setminus 0\}$ is an open cover of U.*

Proof. We want to show that for all $T = \operatorname{Spec} B$ and maps $g\colon T \to U$, the set $\mathfrak{V}_g := \{g^{-1}(D(f)) \mid f \in I \setminus 0\}$ is an open cover of T.[4] Then. . . TODO ⊠

Thus we've proven the proposition. ⊠

Corollary 5.9. *Let S be an affine scheme with an open covering $\mathfrak{U}$. Then there's a big basic open covering $\mathfrak{V}$ refining $\mathfrak{U}$ and with the property that for all $U \in \mathfrak{U}$, $\{V \in \mathfrak{V} \mid V \subset U\}$ is a big basic open covering of U.*

This is the technical proposition that lets us reduce to algebra.

Remark 5.10. Corollary 5.9 also tells us that a big basic open covering of a space X is an open covering $\mathfrak{U}$ of X such that for all maps of affine schemes to X, the pullback of $\mathfrak{U}$ is also a big basic open covering. ◂

Corollary 5.11. $\mathbb{A}^1$ *is a Zariski sheaf.*

Proof. We showed that $\mathbb{A}^1$ is a Zariski sheaf with respect to all basic open covers of affine schemes, hence for all big basic open covers of affine schemes, hence by Remark 5.10 with respect to all spaces with big basic open covers, hence by Proposition 5.5 any affine scheme and any open cover, and therefore any space and any open cover. ⊠

Corollary 5.12. *Let I be a set and let $\mathbb{A}^I := \operatorname{Spec} \mathbb{Z}[\{x_i \mid i \in I\}]$. Then $\mathbb{A}^I$ is a Zariski sheaf.*

Proof. The sheaf property is preserved under arbitrary products. ⊠

If I is an n-element set, then $\mathbb{A}^I$ is also written $\mathbb{A}^n$.

Proof sketch of Proposition 4.5. We can use this to show that if $X = \operatorname{Spec} A$ is an affine scheme, then it's a Zariski sheaf. Let I be a generating set for A and $J \subset \mathbb{Z}[\{x_i \mid i \in I\}]$ be the ideal of relations; then, the quotient map $\mathbb{Z}[\{x_i \mid i \in I\}] \twoheadrightarrow A$ defines a closed embedding $X \subseteq \mathbb{A}^I$ cut out by $X = \{x \mid f(x) = 0 \text{ for all } f \in J\}$.

One then has to check that the sheaf property is preserved under closed embeddings, which is formal. ⊠

We'll spend the next lecture giving examples of schemes, but here are a few to start with.

- As we just showed, affine schemes are schemes.
- A *quasi-affine scheme* is an open subset of an affine scheme, such as $\mathbb{A}^2 \setminus 0$. These are indeed schemes (though not always affine): if U is the complement of $\operatorname{Spec}(A/I) \subset A$, then U admits a covering by $\{D(f) \mid f \in I \setminus 0\}$.

We can use this to prove $\mathbb{A}^2 \setminus 0$ isn't affine.

Lecture 6.

Relative algebraic geometry: 9/12/18

One of the advantages of algebraic geometry is the ability to work relative to a given space, which generalizes choosing a base field (or ring).

Definition 6.1. Let S be a space. A *scheme over S* is a space X with a map $X \to S$, often just written X/S, such that for all affine schemes T and maps $T \to S$, $X \times_S T$ is a scheme. A morphism of schemes over S is a morphism of schemes which commutes with the two maps to S.

In the same way one can define affine schemes over S. If $S = \operatorname{Spec} A$, for A a ring, we might write X/A instead of X/S and say schemes over A; often A will be a field.

We defined spaces as functors $\mathsf{CommRing} \to \mathsf{Set}$, and there's a similar description for schemes over A.

Proposition 6.2. *Let A be a commutative ring. There's an equivalence of categories between spaces over A and functors $\mathsf{CommAlg}_A \to \mathsf{Set}$ (where we ignore the zero algebra).*

[4] The preimage is defined to be $g^{-1}(D(f)) := D(f) \times_U T$.

Proof sketch. Given $X\colon \mathsf{CommAlg}_A \to \mathsf{Set}$, we can define a functor on all commutative rings by sending B to the set of pairs of (i,x) where $i\colon A\to B$ is an A-algebra structure on B and $x\in X(B)$. Then the forgetful map $(i,x)\mapsto i$ defines the desired map to $\operatorname{Spec} A$.

In the other direction, let $p\colon X\to \operatorname{Spec} A$ be a scheme over A. We'll define a functor on commutative A-algebras by sending $(B, i\colon A\to B)$ to the set of maps $\varphi\colon \operatorname{Spec} B\to X$ for which the diagram

$$\begin{tikzcd} & X \arrow[d, "p"] \\ \operatorname{Spec} B \arrow[ur, "\varphi"] \arrow[r, "i^*"] & \operatorname{Spec} A \end{tikzcd} \tag{6.3}$$

commutes. ⊠

Example 6.4. Complex conjugation is $\mathbb{Z}$-linear (and even $\mathbb{R}$-linear) but not $\mathbb{C}$-linear, and therefore induces a map of schemes $\operatorname{Spec}\mathbb{C}\to\operatorname{Spec}\mathbb{C}$ which is a map of schemes over $\mathbb{R}$, but not of schemes over $\mathbb{C}$. ◂

Proposition 6.5. *Let X, Y, and Z be schemes together with maps $X\to Z$ and $Y\to Z$. Then $X\times_Z Y$ is a scheme.*

Proof. If $X=\operatorname{Spec} A$, $Y=\operatorname{Spec} B$, and $Z=\operatorname{Spec} C$ are affine, this is certainly true: the pullback is $\operatorname{Spec} A\otimes_C B$. Now we'll show more general cases reduce to this one.

If Y and Z are affine but X isn't, then X admits an open cover $\mathfrak{U}$ by affines, and $\{U\times_Z Y\mid U\in\mathfrak{U}\}$ is an affne open cover of $X\times_Z Y$. In the same way, we may assume only that X and Z are affine.

Therefore if you only assume Z is affine, you can pick affine open covers of X and Y called $\mathfrak{U}$ and $\mathfrak{V}$, respectively. Then $\{U\times_Z V\mid U\in\mathfrak{U}, V\in\mathfrak{V}\}$ is an affine open cover of $X\times_Z Y$.

Next, we assume X and Y are affine, but Z might not be.[5] Let $\mathfrak{W}$ be an affine open cover of Z, and $W\in\mathfrak{W}$. By definition, the map

$$X\times_Z W\longrightarrow X \tag{6.6}$$

is an open embedding, and this implies that $X\times_Z W$ is a scheme (we called these quasi-affine): it's the complement of a closed embedding $\operatorname{Spec} A/I\to X=\operatorname{Spec} A$, and is covered by $\{D(f)\}$ where $\{f\}$ generates I. Anyways, then $X\times_Z Y$ is covered by

$$\mathfrak{W}' := \{(X\times_Z W)\times_W (Y\times_Z W)\mid W\in\mathfrak{W}\}. \tag{6.7}$$

Since W is affine, this is a scheme by one of the earlier cases. Therefore $X\times_Z Y$ is covered by schemes, so it must be a scheme (choose an affine cover of each element of $\mathfrak{W}'$, and check this is an affine open cover of $X\times_Z Y$).

Finally, we assume none of them are affine. This is the same as the case where X and Y are affine, but now we can use the previous step to show that if $\mathfrak{U}$ is an open cover of X, $\mathfrak{V}$ is an open cover of Y, $U\in\mathfrak{U}$, and $V\in\mathfrak{V}$, then $U\times_Z V$ is a scheme.

We've ignored the Zariski sheaf property, but it's relatively simple to show that it's preserved by fiber products. ⊠

Corollary 6.8. *If S is a scheme, schemes over S are the same thing as schemes with a map to S.*

Proof. We can check the definition on an affine open cover of S; Proposition 6.5 tells us that pulling back to T preserves scheminess. ⊠

If S is a space that's not a space, Corollary 6.8 isn't necessarily true.

Quasicoherent sheaves and/or linear algebra. In commutative algebra, one often studies a ring by studying its modules; these are linear-algebraic in nature, which can make them easier to reason about. The analogue for schemes is quasicoherent sheaves.

Definition 6.9. Let X be a scheme. A *quasicoherent sheaf* (QC sheaf) $\mathscr{F}$ on X is data of, for all maps $f\colon \operatorname{Spec} A\to X$, an A-module $\mathscr{F}_f$, and for every map $g\colon \operatorname{Spec} B\to\operatorname{Spec} A$, an isomorphism

$$\alpha_{f,g}\colon \mathscr{F}_{g\circ f}\xrightarrow{\cong}\mathscr{F}_f\otimes_A B \tag{6.10}$$

[5] From here, the proof was finished up in Friday's lecture.

of B-modules, and such that a cocycle condition holds: given a triple

$$\operatorname{Spec} C \xrightarrow{h} \operatorname{Spec} B \xrightarrow{g} \operatorname{Spec} A \xrightarrow{f} X, \tag{6.11}$$

$\alpha_{f,g\circ h} = \alpha_{f\circ g,h}$ as maps $\mathscr{F}_{f\circ g\circ h} \cong (\mathscr{F}_f \otimes_A B) \otimes_B C$, using the natural isomorphism $(\mathscr{F}_f \otimes_A B) \otimes_B C \cong \mathscr{F}_f \otimes_A C$.[6] A morphism of quasicoherent sheaves $\mathscr{F} \to \mathscr{G}$ is data of maps of A-modules $\mathscr{F}_f \to \mathscr{G}_f$ for all $f\colon \operatorname{Spec} A \to X$, such that all induced diagrams commute. The category of QC sheaves on X is denoted $\mathsf{QCoh}(X)$.

Remark 6.12. The word "quasicoherent" isn't really great unless you're playing Scrabble. It grew out of a generalization of coherent sheaves, which originally came from the analytic setting, where the name was more reasonable. You should think of analogues of modules when you hear QC sheaves. ◀

This is a lot of data! So we're going to find a way to express a quasicoherent sheaf with less data.

Proposition 6.13. *If $X = \operatorname{Spec} A$, the functor $\Gamma\colon \mathsf{QCoh}(X) \to \mathsf{Mod}_A$ sending $\mathscr{F} \mapsto \mathscr{F}_{\mathrm{id}}$ is an equivalence of categories, with inverse sending an A-module M to the sheaf $\mathscr{F}_M$ defined by $(\mathscr{F}_M)_f := M \otimes_A B$ for all $f\colon \operatorname{Spec} B \to X$.*

Example 6.14. For any scheme X, there's a quasicoherent sheaf $\mathscr{O}_X$, called the *structure sheaf* of X, defined to send $f\colon \operatorname{Spec} A \to X$ to $(\mathscr{O}_X)_f = A$. The maps are what you think they are. ◀

Lecture 7.

Quasicoherent sheaves: 9/14/18

"You know when you're looking for your phone and it was in your hand the whole time? This proof was like that."

Here are two exercises we've been sort of implicitly using, and are good to do to get some comfort with this language.

Exercise 7.1.

(1) Let $U \to X$ be a map of schemes and U has an open cover $\mathfrak{V}$ such that for all $V \in \mathfrak{V}$, $V \to X$ is an open embedding. Then $U \to X$ is an open embedding.
(2) If $V \to U$ and $U \to X$ are open embeddings, their composition $V \to U$ is an open embedding.

Now back to quasicoherent sheaves. On an affine scheme $X = \operatorname{Spec} A$, these are a lot like A-modules (in fact, exactly like A-modules, according to Proposition 6.13).

Definition 7.2. Let $f\colon X \to Y$ be a map of schemes and $\mathscr{F} \in \mathsf{QCoh}(Y)$. The *pullback* of $\mathscr{F}$, denoted $f^*\mathscr{F} \in \mathsf{QCoh}(X)$, is the quasicoherent sheaf given by the following data: for every map $g\colon \operatorname{Spec} A \to X$, $(f^*\mathscr{F})_g := \mathscr{F}_{f\circ g}$.

One must check the compatibility conditions, but these aren't so bad.

If $S = \operatorname{Spec} A$ is affine, then an A-module M defines a quasicoherent sheaf $\mathscr{M}$ by sending $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ to $\mathscr{M}_f := M \otimes_A B$. The pullback of $\mathscr{M}$ along f is exactly the quasicoherent sheaf defined by the module $M \otimes_A B$.

Since we understand quasicoherent sheaves on affine schemes, let's next see how they behave on open covers. We'll start with a different-looking definition, then show it's equivalent. This second definition will be useful because it involves substantially less data.

Definition 7.3. Let X be a scheme and $\mathfrak{U}$ be an open cover of X. Let $\mathsf{QCoh}(X;\mathfrak{U})$ denote the category of tuples of $\mathscr{F}_U \in \mathsf{QCoh}(U)$ for all $U \in \mathfrak{U}$ together with, for all intersecting $U, V \in \mathfrak{U}$, isomorphisms

$$\alpha_{UV}\colon \mathscr{F}_U|_{U\cap V} \xrightarrow{\cong} \mathscr{F}_V|_{U\cap V} \tag{7.4}$$

satisfying a cocycle condition on triple intersections.

[6] The cocycle condition can be expressed more concisely by asking that $\mathscr{F}$ is a functor from the category of affine schemes to abelian groups.

This is what's sheafy about quasicoherent sheaves: they are determined from compatible local data.

There's a functor $\Phi\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(X;\mathfrak{U})$ which takes a quasicoherent sheaf and produces its pullback on all $U \in \mathfrak{U}$.

Theorem 7.5 (Serre)**.** *The functor Φ is an equivalence of categories.*

Proof sketch. This will look a lot like what we did before. The first step is to reduce to the case where $X = \operatorname{Spec} A$ is affine and $\mathfrak{U}$ is a basic open cover, using a similar argument to the one from two lectures ago. The second step is similar to the proof that $\mathbb{A}^1$ is a Zariski sheaf.

Explicitly, after we've reduced to $X = \operatorname{Spec} A$ and $\mathfrak{U} = \{D(f_i) \mid (f_1, \dots, f_n) = A\}$, then a quasicoherent sheaf on $D(f_i)$ is (equivalent data to) an $A[f_i^{-1}]$-module M_i, together with the natural isomorphisms $\alpha_{ij}\colon M_i[f_j^{-1}] \xrightarrow{\cong} M_j[f_i^{-1}]$ as $A[(f_if_j)^{-1}]$-modules.

Given this data, we want to functorially build an A-module. The answer will be

$$M \coloneqq \{s_i \in M_i, 1 \le i \le n \mid \text{in } M_i[f_j^{-1}] \cong M_j[f_i^{-1}],\, s_i = s_j\}. \tag{7.6}$$

Now the proof is the same as in the $\mathbb{A}^1$-setting, though there we only worried about functions, not sections. The other way is simple once one invokes the flatness of $A[f_i^{-1}]$. ⊠

We might not have defined it yet, but for a field k, $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$. This is slightly nicer to work with for some applications than $\mathbb{A}^2_{\mathbb{Z}}$. Let $X \coloneqq \mathbb{A}^2_k \setminus 0$, our favorite non-affine scheme, with its open cover $U \coloneqq \mathbb{A}^1 \times (\mathbb{A}^1 \setminus 0)$ and $V \coloneqq (\mathbb{A}^1 \setminus 0) \times \mathbb{A}^1$. Then Theorem 7.5 says a quasicoherent sheaf on $\mathbb{A}^2_k \setminus 0$ is the data of

- a $k[x, x^{-1}, y]$-module M,
- a $k[x, y, y^{-1}]$-module N, and
- an isomorphism $\alpha\colon M[y^{-1}] \cong N[x^{-1}]$ of $k[x, x^{-1}, y, y^{-1}]$-modules.

Modules can be big, so it will be useful to have some finiteness hypotheses.

Definition 7.7. Let X be a scheme and $\mathscr{F} \in \mathsf{QCoh}(X)$. Then $\mathscr{F}$ is *locally finitely generated* (l.f.g.) if for all open embeddings $j\colon \operatorname{Spec} A \to X$, $j^*\mathscr{F}$ is a finitely generated A-module.

Theorem 7.8 (Nakayama's lemma)**.** *Let $\mathscr{F}$ be a locally finitely generated QC sheaf on a scheme X, k be a field, and $x\colon \operatorname{Spec} k \to X$ be such that $x^*\mathscr{F} = 0$. Then there's an open $j\colon U \hookrightarrow X$ containing x (i.e. x factors through j) and such that $j^*\mathscr{F} = 0$.*

Geometrically, this is saying that if an l.f.g. sheaf vanishes at a point, it also vanishes in a neighborhood of that point.

Proof. First we'll reduce to the affine case: we know there's an affine open $V \subseteq X$ such that x factors through V (geometrically, the point x lies in V), so we'll replace X by V (and call it X). Let $X = \operatorname{Spec} A$, so that $\mathscr{F}$ corresponds to a finitely generated A-module M, and x corresponds to a map $\varphi\colon A \to k$. Our hypothesis means that $M \otimes_A k = 0$.

Let's induct on the number of generators of M. If M is generated by zero elements, we're done, so assume we know it for all modules generated by n elements. . . we'll finish this Monday. ⊠

Lecture 8.

Quasicoherent sheaves, II: 9/17/18

We're in the middle of proving Nakayama's lemma, Theorem 7.8. We're proving it by induction on the number of generators of the A-module M, and the base case is trivial. So let's assume it's true for all modules generated by n elements.[7]

Remark 8.1. Let's pause to ask what a finitely generated A-module looks like. If it has one generator, it's isomorphic to A/I for some ideal I. If it has two generators, it's an extension of A/I by A/J for some ideals I and J of A. More generally, a module M with m generators is an extension $0 \to N \to M \to A/I \to 0$, where N has $m - 1$ generators.

[7]The theorem is true for non-affine schemes, but we've already reduced to the affine case.

This means some specific subcases of Nakayama's lemma, such as that for local rings, are close to trivial.[8] You could prove Theorem 7.8 by reducing to the local case, though we're using a different approach.

The fact that finitely generated modules have quotients which look like A/I is the catalyst of the proof: it's untrue for modules which aren't finitely generated, such as $\mathbb{Q}$ as a $\mathbb{Z}$-module, which has no quotients of the form $\mathbb{Z}/n$. ◀

So M is an extension of an A-module N generated by n elements by A/I:

$$0 \longrightarrow N \longrightarrow M \longrightarrow A/I \longrightarrow 0. \tag{8.2}$$

By assumption, $M \otimes_A k = 0$, which means that, since tensor product is right exact,

$$(A/I) \otimes_A k \cong k/Ik = 0. \tag{8.3}$$

Recall that we had data of a map $\varphi\colon A \to k$; since k is a field, this and (8.3) imply there's some $f \in I$ wth $\varphi(f) \neq 0$. Let's localize at f; the map $\varphi\colon A \to k$ passes to a map $\widetilde{\varphi}\colon A[f^{-1}] \to k$, and since localization is exact, (8.2) induces a short exact sequence

$$0 \longrightarrow N[f^{-1}] \longrightarrow M[f^{-1}] \longrightarrow (A/I)[f^{-1}] \longrightarrow 0, \tag{8.4}$$

but since $A[f^{-1}] = 0$, $N[f^{-1}] \cong M[f^{-1}]$, which (crucially) is generated by n elements as an $A[f^{-1}]$-module. Since $\varphi(f) \neq 0$, then $x \in D(f)$, so there's an open $U \subset D(f)$ containing x such that $(\mathscr{F}|_{D(f)})|_U = 0$ by the inductive hypothesis, and that's exactly what we wanted to prove.

Definition 8.5. Let M be an A-module. Then its *annihilator* $\mathrm{Ann}(M) \coloneqq \{f \in A \mid f \cdot M = 0\}$, which is an ideal of A.

Corollary 8.6. *Let X be a scheme and $\mathscr{F} \in \mathsf{QCoh}(X)$ be locally finitely generated. Then the subset $U_{\mathscr{F}} \coloneqq \{f\colon \operatorname{Spec} B \to X \mid f^*\mathscr{F} = 0\}$ is an open subscheme of X. In particular, if $X = \operatorname{Spec} A$ is affine, then $U_{\mathscr{F}}$ is the complement of the locus of X on which all $f \in \mathrm{Ann}(\mathscr{F})$ vanish.*

That is, the locus where $\mathscr{F}$ vanishes is open. This fits into your intuition: if you're on $\operatorname{Spec} A$ and $\mathscr{F}$ corresponds to $A/(f)$, then $\mathscr{F}$ vanishes wherever f doesn't.

Proof. It suffices to prove the affine statement, and this is a matter of unwinding its definition: let $X = \operatorname{Spec} A$ and $\mathscr{F}$ be an A-module. Given $\varphi\colon A \to B$, it suffices to prove the following are equivalent: $\mathrm{Ann}(\mathscr{F}) \cdot B = B$ and $\mathscr{F} \otimes_A B = 0$.

First, the forward implication: we know there are $f_i \in \mathrm{Ann}(\mathscr{F})$ and $g_i \in B$ such that

$$\sum_{i=1}^n \varphi(f_i) g_i = 1. \tag{8.7}$$

Therefore 1 acts by 0 on $\mathscr{F} \otimes_A B$, so that module must be the zero module.

The reverse direction is a bit harder. Suppose for a contradiction that $\mathrm{Ann}(\mathscr{F}) \cdot B \subsetneq B$, so it's contained in some maximal ideal $\mathfrak{m}$; let $k \coloneqq B/\mathfrak{m}$, which is a field. Then

$$\mathscr{F} \otimes_A k = (\mathscr{F} \otimes_A B) \otimes_B k = 0. \tag{8.8}$$

Hence, by Theorem 7.8, there's a $U \subset \operatorname{Spec} A$ containing $\operatorname{Spec} k$ such that $\mathscr{F}|_U = 0$. We can assume $U = D(f)$ for some $f \in A$, so we're assuming $\mathscr{F}[f^{-1}] = 0$. Because $\mathscr{F}$ is finitely generated, this means $f^N \mathscr{F} = 0$ for some $N \gg 0$, or $f^N \in \mathrm{Ann}(f)$. Since $\varphi(\mathrm{Ann}(\mathscr{F})) \subset \mathfrak{m}$, then $\varphi(f^N) = 0 \bmod \mathfrak{m}$, so $\varphi(f) = 0 \bmod \mathfrak{m}$, which contradicts the assumption that $\operatorname{Spec} k \in U$. ⊠

You can draw a picture of this: given a locally finitely generated sheaf $\mathscr{F}$, $\mathrm{Ann}(\mathscr{F})$ has a vanishing locus; if $\mathscr{F}$ corresponds to the module A/I (here we should be on an affine scheme), then this is also the closed subset $\operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A$.

Exercise 8.9. Deduce every other version of Nakayama's lemma that you know (e.g. the one in Matsumara) from these versions.

[8] There are many different things called Nakayama's lemma; ours is not the most general one.

Definition 8.10. A *vector bundle* on a scheme X is a quasicoherent sheaf $\mathscr{E} \in \mathsf{QCoh}(X)$ which is locally finitely generated and *locally projective*, i.e. for some (equivalently any) affine open cover $\mathfrak{U}$ of X, for every $U = \operatorname{Spec} A \in \mathfrak{U}$, the pullback of $\mathscr{E}$ to U is a projective A-module.

Proposition 8.11. *Let $\mathscr{E} \in \mathsf{QCoh}(X)$. The following are equivalent:*

(1) *$\mathscr{E}$ is a vector bundle.*
(2) *There is an affine open cover $\mathfrak{U}$ of X such that for all $U \in \mathfrak{U}$, $\mathscr{E}|_U$ is a finitely generated free module.*

Lecture 9.

Vector bundles: 9/19/18

"It's not a good exercise; it's an exercise."

Today we're going to continue talking about vector bundles, which are an absolutely crucial concept in algebraic geometry. First we'll prove Proposition 8.11, equating two definitions of vector bundles: quasicoherent sheaves which are locally projective and those which are locally free of finite rank, i.e. locally isomorphic to $\mathscr{O}_U^{\oplus r}$. This r is called the *rank* of the vector bundle over U.

Remark 9.1. The rank of a vector bundle is locally constant, but doesn't have to be constant. ◂

Lemma 9.2. *Let $X = \operatorname{Spec} A$ and $\mathfrak{U}$ be a collection of open subsets of X. Then $\mathfrak{U}$ is an open cover of X iff for all maximal ideals $\mathfrak{m}$ of A, there's a $U \in \mathfrak{U}$ such that $\operatorname{Spec}(A/\mathfrak{m}) \hookrightarrow \operatorname{Spec} A$ factors through U.*

The idea is that a *closed point* is an embedding $\operatorname{Spec} k \hookrightarrow X$, where k is a field. So a collection of opens is an open cover if it contains every closed point, which is nice. Affineness is important here: there are general schemes with no closed points!

Proof of Proposition 8.11. The thing we want to prove is local, so we can immediately reduce to the case where $X = \operatorname{Spec} A$ is affine, and therefore $\mathscr{E}$ corresponds to an A-module, which we also denote $\mathscr{E}$. For the forward direction, we assume $\mathscr{E}$ is finitely generated and projective.

Let $\mathfrak{m}$ be a maximal idea of A. Then $\mathscr{E}/\mathfrak{m}$ is a finitely-generated $A/\mathfrak{m}$-module; since $A/\mathfrak{m}$ is a field, $\mathscr{E}/\mathfrak{m}$ is free, so it has a basis $\overline{s}_1, \dots, \overline{s}_n$. We lift this to $s_1, \dots, s_n \in \mathscr{E}$, which define a map $\tau\colon A^{\oplus n} \to \mathscr{E}$ which is surjective mod $\mathfrak{m}$. We'd like to show this is an isomorphism on some open U containing $\operatorname{Spec}(A/\mathfrak{m})$.

Since $\mathscr{E}$ is a finitely generated A-module, so too is $\operatorname{coker}(\tau) = \mathscr{E}/\tau(A^{\oplus n})$, and since modding out by $\mathfrak{m}$ is right exact, $\operatorname{coker}(\tau)|_{\operatorname{Spec}(A/\mathfrak{m})} = 0$. Therefore by Theorem 7.8, there is some $U_0 \subset X$ containing $\operatorname{Spec}(A/\mathfrak{m})$ such that $\operatorname{coker}(\tau)|_{U_0} = 0$; without loss of generality, we can take U_0 to be affine, i.e. τ is surjective on U_0. Let's replace X by U_0 and continue.

Because $\mathscr{E}$ is projective, the map $\tau\colon A^{\oplus n} \twoheadrightarrow \mathscr{E}$ splits; let $\sigma\colon \mathscr{E} \to A^{\oplus n}$ be a section. This means $\ker(\tau)$ is finitely generated, and therefore $\ker(\tau)|_{\operatorname{Spec}(A/\mathfrak{m})} = 0$. Now we use Nakayama's lemma again and conclude that τ is an isomorphism on some open U_1 containing $\operatorname{Spec}(A/\mathfrak{m})$.

The converse isn't immediately trivial: if $\mathfrak{U}$ is an affine cover of $\operatorname{Spec} A$ and M is an A-module such that $M|_U$ is projective for all $U \in \mathfrak{U}$, why is M necessarily projective? Since A need not be Noetherian, we also need to show M is finitely generated and presented given that its localizations are. This is not a super important point, so it's left as an exercise. Once M is finitely presented, you can show that for any $f \in A$,

$$\operatorname{Hom}_A(M, N)[f^{-1}] = \operatorname{Hom}_{A[f^{-1}]}(M[f^{-1}], N[f^{-1}]). \tag{9.3}$$

This is definitely false if you don't assume finite presentation of M! Anyways, using this, you can recover projectivity on $\operatorname{Spec} A$ from projectivity on a basic affine open cover. ⊠

Next we'll turn to constructions with quasicoherent sheaves, and something not quite as related, affine morphisms.

Definition 9.4. Let $\mathscr{F}$ and $\mathscr{G}$ be quasicoherent sheaves on a scheme X and $\tau\colon \mathscr{F} \to \mathscr{G}$ be a morphism. Then we can define sheaves $\ker(\tau), \operatorname{coker}(\tau) \in \mathsf{QCoh}(X)$, such that for all affine opens $U \subset X$, $\ker(\tau)|_U = \ker(\tau|_U)$ and $\operatorname{coker}(\tau)|_U = \operatorname{coker}(\tau|_U)$.

For this to make sense, we need to invoke Serre's theorem that this data actually defines a quasicoherent sheaf, along with the fact that $A \to A[f^{-1}]$ is flat, which means kernels and cokernels are preserved under pullback by an open embedding, so that gluing works.

Definition 9.5. With notation as before, there is a quasicoherent sheaf $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ such that on every affine open $U = \operatorname{Spec} A \hookrightarrow X$, $(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})|_U = \mathscr{F}|_U \otimes_A \mathscr{G}|_U$, and on any open $V \hookrightarrow X$, $(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})|_V = \mathscr{F}|_V \otimes_{\mathscr{O}_V} \mathscr{G}|_V$.

Checking that this is well-defined is easier than for the kernel and cokernel; you don't have to invoke Serre's theorem.

Definition 9.6. Let X be a space. A map $f\colon Y \to X$ is *affine* if for all affine schemes S and maps $S \to X$, the pullback $S \times_X Y$ is affine.

Example 9.7.

(1) Any map of affine schemes is affine, which is a rebranding of the theorem that fiber products preserve affine schemes.
(2) Closed embeddings are also affine.
(3) Not all open embeddings are affine: the standard counterexample is $\mathbb{A}^2 \setminus 0 \to \mathbb{A}^2$, because its fiber product with the identity map $\mathbb{A}^2 \to \mathbb{A}^2$ gives us back $\mathbb{A}^2 \setminus 0$, which isn't affine. ◀

Affine morphisms are nice because they have nice algebraic descriptions. Specifically, affine maps $Y \to X$, where X is a scheme, correspond to commutative algebras in $\mathsf{QCoh}(X)$. TODO: I had to head out

Lecture 10.

Affine morphisms and projective space: 9/21/18

Last time, we defined affine morphisms $Y \to X$, which are those such that if you pull back by an affine scheme $\operatorname{Spec} A \to X$, $Y \times_X \operatorname{Spec} A$ is also affine. We claimed these are equivalent to commutative algebras in $\mathsf{QCoh}(X)$, akin to how affine schemes are commutative rings, but working relatively (i.e. over a scheme).

Definition 10.1. Let X be a scheme. A *commutative algebra* in $\mathsf{QCoh}(X)$ is a quasicoherent sheaf together with an associative, commutative multiplication map $m\colon \mathscr{A} \otimes_{\mathscr{O}_X} \mathscr{A} \to \mathscr{A}$ with a unit $\varepsilon\colon \mathscr{O}_X \to \mathscr{A}$.

Definition 10.2. Given a commutative algebra $\mathscr{A} \in \mathsf{QCoh}(X)$, we can define a scheme $\operatorname{Spec}_X(\mathscr{A})$ together with an affine map to X. If B is a commutative ring, we let $(\operatorname{Spec}_X(\mathscr{A}))(B)$ be the set of pairs $x\colon \operatorname{Spec} B \to X$ together with maps $\rho\colon \operatorname{Spec} B \to \operatorname{Spec} x^*(\mathscr{A})$ which are sections of the canonical map arising from the B-algebra structure on $x^*(\mathscr{A})$.

Forgetting the section defines a map to X. This map is affine because if $x\colon \operatorname{Spec} B \to X$ is a map, then

$$\operatorname{Spec}_X(\mathscr{A}) \times_X \operatorname{Spec} B = \operatorname{Spec}(x^*(\mathscr{A})). \tag{10.3}$$

Definition 10.4. Let $\pi\colon Y \to X$ be an affine map of schemes and $\mathscr{F} \in \mathsf{QCoh}(Y)$. We will define a *pushforward* $\pi_*\mathscr{F} \in \mathsf{QCoh}(X)$ as follows: given any map $x\colon \operatorname{Spec} B \to X$, the pullback $Y \times_X \operatorname{Spec} B$ is affine, isomorphic to $y\colon \operatorname{Spec} C \to Y$ for some C. Then we define $x^*(\pi_*(\mathscr{F})) := y^*(\mathscr{F})$: this is *a priori* a C-module, but picks up a B-module structure by the map $B \to C$.

Exercise 10.5. π^* and π_* are adjoint functors, i.e. for any affine map $\pi\colon X \to Y$, $\mathscr{F} \in \mathsf{QCoh}(Y)$, and $\mathscr{G} \in \mathsf{QCoh}(X)$, there's a canonical isomorphism

$$\operatorname{Hom}_{\mathsf{QCoh}(X)}(\mathscr{G}, \pi_*(\mathscr{F})) \cong \operatorname{Hom}_{\mathsf{QCoh}(Y)}(\pi^*(\mathscr{G}), \mathscr{F}).$$

Proposition 10.6. *If $\pi\colon Y \to X$ is affine, then $Y = \operatorname{Spec}_X(\mathscr{A})$ for some algebra $\mathscr{A}$ in $\mathsf{QCoh}(X)$.*

Proof sketch. The key is that $\pi_*(\mathscr{O}_Y)$ is a commutative algebra in $\mathsf{QCoh}(X)$: by Exercise 10.5, the multiplication map is equivalent data to a map

$$\pi^*(\pi_*(\mathscr{O}_Y) \otimes_{\mathscr{O}_X} \pi_*(\mathscr{O}_Y)) = \pi^*\pi_*\mathscr{O}_Y \otimes_{\mathscr{O}_Y} \pi^*\pi_*\mathscr{O}_Y \longrightarrow \mathscr{O}_Y. \tag{10.7}$$

The unit of the adjunction is a map $\pi^*\pi_*\mathscr{O}_Y \to \mathscr{O}_Y$, so we can pass to $\mathscr{O}_Y$ and then multiply.

We then claim that as schemes over X (i.e. with a map to X), $Y \to X$ is isomorphic to $\operatorname{Spec}_X(\pi_*(\mathscr{O}_Y)) \to X$, which one has to check. ⊠

Exercise 10.8. Let X be a scheme. Show that closed subschemes $Z \subseteq X$ are equivalent to *ideal sheaves* $\mathscr{I} \to \mathscr{O}_X$, i.e. quasicoherent sheaves $\mathscr{I}$ with vanishing kernel.

Projective space. Projective space $\mathbb{P}^n$ is a scheme designed to parametrize lines in $\mathbb{A}^{n+1}$. IF you try this in $\mathbb{A}^2_{\mathbb{R}}$, you notice that you get a circle; if you do this in $\mathbb{A}^2_{\mathbb{C}}$, you get a sphere (harder). But in general it looks different.

We'll describe $\mathbb{P}^n$ via its functor of points. The idea is that a map $S = \operatorname{Spec} A \to \mathbb{P}^n$ should be a line $\mathscr{L}$ together with an embedding $\mathscr{L} \to \mathbb{A}^{n+1}$, but we have to make this precise.

Definition 10.9. Let S be a scheme. A *line bundle* on S is a vector bundle of rank 1, i.e. a quasicoherent sheaf $\mathscr{L}$ on S locally isomorphic to $\mathscr{O}_S$.

When S is affine, these correspond to rank-1 projective A-modules. If $S = \operatorname{Spec} k$, this exactly covers 1-dimensional k-vector spaces, and this is the right generalization to commutative rings (or even to schemes).

Now we want to embed the line in $\mathbb{A}^{n+1}$, which we can think of as a map $i\colon \mathscr{L} \to \mathscr{O}_S^{\oplus(n+1)}$; "embedding" means we want $i \neq 0$.

Proposition 10.10. *The following are equivalent for a vector bundle $\mathscr{E} \to S$ and a map $i\colon \mathscr{L} \to \mathscr{E}$ where $\mathscr{L}$ is a line bundle.*

(1) *For all affine schemes T and maps $f\colon T \to S$, $f^*(\mathscr{L}) \to f^*(\mathscr{E})$ is nonzero.*
(2) *(Will be done Monday)*
(3) *(Will be done Monday)*

Definition 10.11. If the conditions in Proposition 10.10 hold, i is called *everywhere nonvanishing.*

Definition 10.12. *Projective n-space* $\mathbb{P}^n$ is the space whose functor of points sends an affine scheme S to the set of isomorphism classes of data $(\mathscr{L}, i)$ where $\mathscr{L}$ is a line bundle on S and $i\colon \mathscr{L} \to \mathscr{O}_S^{\oplus(n+1)}$ is everywhere nonvanishing.

There's something to be said about morphisms, but given a map $f\colon T \to S$, we can pull back $\mathscr{L}$ and i, and obtain a line bundle with an everywhere nonvanishing embedding.

We need to take isomorphism classes to ensure we get a set, not a category. This also ensures that we've modded out by rescaling (since that's an isomorphism $\mathscr{L} \to \mathscr{L}$). We'll show this is a scheme, but is usually not affine.

Lecture 11.

Projective n-space and projectivizations: 9/24/18

"These scare quotes should be less scary than those scare quotes."

Last time, we defined projective n-space, $\mathbb{P}^n$, whose functor of points sends A to the set of isomorphism classes of data $(\mathscr{L}, i)$, where $\mathscr{L} \to \operatorname{Spec} A$ is a line bundle and $i\colon \mathscr{L} \to \mathscr{O}_{\operatorname{Spec} A}^{\oplus(n+1)}$ is an everywhere nonvanishing map.

Definition 11.1. More generally, if X is a scheme and $\mathscr{E}$ is a vector bundle on X, then we can define a space $\mathbb{P}(\mathscr{E})$, the *projectivization* of $\mathscr{E}$: if A is a commutative ring, $\mathbb{P}(\mathscr{E})(A)$ is the set of isomorphism classes of data $(\mathscr{L}, x, i)$, where $\mathscr{L}$ is a line bundle on $\operatorname{Spec} A$, $x\colon \operatorname{Spec} A \to X$ is a map ("an A-valued point"), and $i\colon \mathscr{L} \to x^*(\mathscr{E})$ is everywhere nonvanishing.

To recover $\mathbb{P}^n$, let $X = \operatorname{Spec} \mathbb{Z}$ and $\mathscr{E} = \mathbb{Z}^{\oplus(n+1)}$.

Remark 11.2. Right now, schemes and line and vector bundles probably feel very abstract. That's OK; soon enough we will see many, many examples of line bundles over curves, and make them very concrete. ◀

Example 11.3. Consider the quasicoherent sheaf given by $k[t]/(t)$ over $\mathbb{A}^1_k := \operatorname{Spec} k[t]$. This is not a vector bundle: it's not locally free, because in a sense it's nonzero over the point 0 (a one-dimensional vector space) but vanishes everywhere else. ◀

Proposition 11.4. *Let S be a scheme, $\mathscr{L}$ be a line bundle on S, $\mathscr{E}$ be a vector bundle on S, and $i\colon \mathscr{L} \to \mathscr{E}$. The following are equivalent:*

(1) *The induced map $\Theta(i)\colon \Theta(\mathscr{L}) \to \Theta(\mathscr{E})$ is a closed embedding.*
(2) *For all affine schemes T and maps $\alpha\colon T \to S$, $\alpha^*(i)$ is nonzero.*
(3) *For all fields k and maps $\alpha\colon \operatorname{Spec} k \to S$, $\alpha^*(i)$ is nonzero.*

(4) *For all affine open subschemes $U \subseteq S$ such that $\mathscr{L}|_U \cong \mathscr{O}_U$ and $\mathscr{E}|_U \cong \mathscr{O}_U^{\oplus r}$, if the induced map $\mathscr{O}_U \to \mathscr{O}_U^{\oplus r}$ sends $1 \mapsto (f_1, \dots, f_r)$, then $(f_1, \dots, f_r)$ generates the unit ideal in* $\mathrm{Fun}(U)$.
(5) *The dual map $i^\vee \colon \mathscr{E}^\vee \to \mathscr{L}^\vee$ is an epimorphism (i.e. its cokernel is zero).*[9]
(6) $\mathrm{coker}(\mathscr{L} \to \mathscr{E})$ *is a vector bundle.*

To make sense of this, we have to define Θ, a way of turning vector bundles into schemes. If $P^\vee$ is unfamiliar for an A-module P, it simply means $\mathrm{Hom}_A(P, A)$.

Exercise 11.5. Let P be a finitely generated, projective A-module.
(1) Show that for all A-modules M, $\mathrm{Hom}_A(P, M) \cong P^\vee \otimes_A M$.
(2) Show that $P^\vee$ is projective.
(3) Describe a natural isomorphism $P \to P^{\vee\vee}$.

Definition 11.6. If $\mathscr{E}$ is a vector bundle over a scheme S, its *dual vector bundle* $\mathscr{E}^\vee$ (sometimes also written $\mathscr{E}^*$) is the quasicoherent sheaf attaching to every affine open $i \colon U \to S$ the dual projective module of $i^*\mathscr{E}$.

By Exercise 11.5, part (1), $\mathscr{E}^\vee$ is indeed a quasicoherent sheaf.

Example 11.7. If $\mathscr{E} = \mathscr{O}_X^{\oplus r}$, then $\mathscr{E}^\vee \cong \mathscr{O}_X^{\oplus r}$ as well, albeit not canonically. ◀

Duality is contravariantly functorial: a map $f \colon \mathscr{F} \to \mathscr{E}$ of vector bundles induces a dual map $f^\vee \colon \mathscr{E}^\vee \to \mathscr{F}^\vee$ (do this on affines, where it's precomposition for the corresponding modules).

Definition 11.8. Let $\mathscr{E}$ be a vector bundle on X. Its *total space* is the scheme

$$\Theta(\mathscr{E}) \coloneqq \mathrm{Spec}_X(\mathrm{Sym}_{\mathscr{O}_X}(\mathscr{E}^\vee)). \tag{11.9}$$

Here, $\mathrm{Sym}_{\mathscr{O}_X}(\mathscr{E})$ is the $\mathscr{O}_X$-module

$$\mathrm{Sym}_{\mathscr{O}_X}(\mathscr{E}) \coloneqq \bigoplus_{n \geq 0} \mathrm{Sym}^n_{\mathscr{O}_X}(\mathscr{E}), \tag{11.10}$$

where $\mathrm{Sym}^n_{\mathscr{O}_X}(\mathscr{E}) \coloneqq (\mathscr{E}^{\otimes_{\mathscr{O}_X} n})_{S_n}$, where the symmetric group S_n acts by permuting the elements; more explicitly, we allow the elements in an n-tensor to be arbitrarily shuffled, which you can do with a quotient.

Example 11.11. Let $X = \mathrm{Spec}\, k$ and $\mathscr{E} = k^{\oplus n}$ be a k-vector space with basis $e_1, \dots, e_n$. Let $e^1, \dots, e^n$ denote the dual basis. Then $\mathrm{Sym}^n V$ is the vector space of degree-n homogeneous polynomials in $e^1, \dots, e^r$, and $\mathrm{Sym}\, V = k[e^1, \dots, e^r]$. ◀

Lecture 12.

More vector bundles: 9/26/18

Today we're in the business of proving Proposition 11.4.

Lemma 12.1. *Let X be a scheme and $\mathscr{A}, \mathscr{B} \in \mathsf{QCoh}(X)$ be commutative algebras together with a map $f \colon \mathscr{A} \to \mathscr{B}$ of algebras. Then $\mathrm{Spec}_X(\mathscr{B}) \to \mathrm{Spec}_X(\mathscr{A})$ is a closed embedding iff f is an epimorphism.*

This is a relative version of the definition of closed embeddings of affine schemes we gave awhile ago.

We also need to define the what taking the total space Θ does to morphisms. Ultimately this is because $(-)^\vee$, $\mathrm{Sym}_{\mathscr{O}_X}$, and Spec_X are all functors, so we know what they do on morphisms; two are contravariant and one is covariant, so we get a covariant functor.

In part (1) of Proposition 11.4, Lemma 12.1 tells us this is equivalent to $\mathrm{Sym}_{\mathscr{O}_X} \mathscr{E}^\vee \to \mathrm{Sym}_{\mathscr{O}_X} \mathscr{L}^\vee$ is an epimorphism. This is a $\mathbb{Z}$-graded sheaf, and it suffices to show this for $\mathrm{Sym}^d_{\mathscr{O}_X}$ for each d.

Exercise 12.2. Show that if $\mathscr{E}^\vee \to \mathscr{L}^\vee$, then $\mathrm{Sym}^d \mathscr{E}^\vee \to \mathrm{Sym}^d \mathscr{L}^\vee$ is too.

Proof of Proposition 11.4, (5) $\iff$ (4). We know condition (5) is equivalent to $\mathrm{coker}(i^\vee) = 0$. This can be checked on an open cover $\mathfrak{U}$, such as an affine open cover which trivializes $\mathscr{E}$ and $\mathscr{L}$, as in (4). In this case, for each U in $\mathfrak{U}$, $i|_U \colon \mathscr{O}_U \to \mathscr{O}_U^{\oplus r}$ is determined by a row vector $(f_1, \dots, f_r)^{\mathrm{T}}$, and $i^\vee|_U \colon \mathscr{O}_U^{\oplus r} \to \mathscr{O}_U$ is the column vector $(f_1, \dots, f_r)$. Its image is the ideal generated by $(f_1, \dots, f_r)$, so the cokernel is 0 iff $(f_1, \dots, f_r) = \mathrm{Fun}(U)$. ⊠

[9]In many situations, "epimorphism" means "surjective," but these are quasicoherent sheaves, so we don't have elements to ask about preimages of.

Proof of Proposition 11.4, (4) $\implies$ (6). We can again assume without loss of generality that $U = \operatorname{Spec} B$ is affine, and now we have maps

$$B \xrightarrow{\begin{bmatrix} f_1 \\ \vdots \\ f_r \end{bmatrix}} B^{\oplus r} \xrightarrow{[g_1, \dots, g_r]} B, \tag{12.3}$$

where $\sum f_i g_i = 1$. Since the composition is the identity, the map is split, and so the cokernel is a direct summand, in particular is a direct summand of a free module, and must be free. ⊠

Exercise 12.4. Conversely, show that in Proposition 11.4, (6) $\implies$ (4).

Since (2) tautologically implies (3), it suffices to show (3) $\implies$ (5) and (5) $\implies$ (2); (1) is addressed by one of the exercises above, I think (TODO: I probably just missed it.)

Since $\operatorname{coker}(i)$ is locally finitely generated, Nakayama's lemma applies: if V denotes the complement of the support of $\operatorname{coker}(i)$, then V is open, and $S = \operatorname{Spec} A \to X$ factors through V iff $\operatorname{coker}(i)|_S = 0$. So (5) is equivalent to $V = X$.

Proof of Proposition 11.4, (3) $\implies$ (5). To show $V = X$, it suffices to show any map $x\colon \operatorname{Spec} k \to X$ factors through V, since V is open. One can show that x^* commutes with cokernels: since $\operatorname{Spec} k$ is affine, it arises as a tensor product, and tensor products are right exact.[10] Therefore it suffices to show that $x^*(\operatorname{coker}(i^\vee)) = 0$. If $x^*(i^\vee)$ is nonzero, then $x^*(\mathscr{E}) \to x^*(\mathscr{L})$ is a nonzero map to a 1-dimensional vector space, hence surjective, and therefore the cokernel is zero. ⊠

Proof of Proposition 11.4, (5) $\implies$ (2). Let $x\colon T \to X$ be a map, where T if affine. As before, x^* is right exact, hence commutes with cokernels. By assumption, $\operatorname{coker}(i^\vee) = 0$, so $x^*(i^\vee)\colon x^*(\mathscr{E}^\vee) \to x^*(\mathscr{L}^\vee)$ is an epimorphism. This is the dual map to $x^*(i)$, so if $T \neq \varnothing$, this means $x^*(i)\colon x^*(\mathscr{L}) \to x^*(\mathscr{E})$ is nonzero. ⊠

These equivalent conditions are all examples of the nicest possible maps of vector bundles. It's good to have all of these different perspectives partly because they provide flexibility in the nice case, but also because it will be useful to know what happens when things go bad. When we study projective varieties, several of these conditions will come up. For example, it will be useful for showing $\mathbb{P}^n$ is a scheme!

Lecture 13.

Line bundles on $\mathbb{P}^n$: 9/28/18

Theorem 13.1. *$\mathbb{P}^n$ is a scheme.*

Proof. First we need to check that $\mathbb{P}^n$ is a Zariski sheaf. The basic idea is that line bundles glue: if you have line bundles on each open of an open cover, together with isomorphisms on intersections satisfying a cocycle condition, you can glue them.

Next we need to cover it by affines. For $i = 0, \dots, n$, the locus $\{s_i \neq 0\} \subset \mathbb{P}^n$ is isomorphic to $\mathbb{A}^n$. We could say QED here, but let's explain what's going on.

Let $U_i \coloneqq \{s_i = 0\}$ be defined by saying what it means for a map $S \to \mathbb{P}^n$ factors through U_i. Specifically, the map $S \to \mathbb{P}^n$ is equivalent to (an isomorphism class of) data of a line bundle $\mathscr{L}$ on S and a map $(s_0, \dots, s_{n+1})^{\mathrm{T}}\colon \mathscr{L} \to \mathscr{O}_S^{\oplus(n+1)}$ which is everywhere nonvanishing. Thus the map is described by $s_0^\vee, s_n^\vee$; we say the map factors through U_i if $s_i^\vee$ is nonvanishing.[11]

First let's check that for all i, $U_i \subset \mathbb{P}^n$ is open. Intuitively this makes sense: we're asking for something to not vanish, which is an open condition. More precisely, we want to show that for every affine scheme S with a map $S \to \mathbb{P}^n$, the pullback $U_i \times_{\mathbb{P}^n} S \to S$ is open. So let $S \to \mathbb{P}^n$ be such a map (in particular, S is affine). This is equivalent data to a line bundle $\mathscr{L}$ on S and $s_0, \dots, s_{n+1}\colon \mathscr{L} \to \mathscr{O}_S$. That is, there are sections of the map $\Theta(\mathscr{L}^\vee) \to S$. Then you can check that $U_i = S \times_{\Theta(\mathscr{L}^\vee)} \Theta(\mathscr{L}^\vee) \setminus S$, where the first map $S \to \Theta(\mathscr{L}^\vee)$ is by s_i and the second map is by the zero section.

The other claim we want to check is that $U_i \cong \mathbb{A}^n$. The proof is that a map $S \to U_i$ is equivalent data to the maps $s_0, \dots, s_{i-1}, s_{i+1}, \dots, s_n\colon \mathscr{L} \to \mathscr{O}_S$, since we know $s_i\colon \mathscr{L} \to \mathscr{O}_S$ is an isomorphism. Each s_j, $j \neq i$,

[10] More generally, x^* is a left adjoint, even not on affines; this automatically means it's right exact.

[11] That is, the collection $(s_0, \dots, s_n)^{\mathrm{T}}$ is always nonvanishing, but we're asking specifically about s_i, which is stronger.

is equivalent to a function on S: since s_i is nonvanishing, there are no further conditions on the remaining maps. Thus a map to U_i is equivalent to n functions, and this is natural in S, so $U_i \simeq \mathbb{A}^n$.

Finally, we need to show that $U_0, \dots, U_n$ is a cover of $\mathbb{P}^n$, i.e. that it pulls back to a Zariski open cover on all affines. So once again let $S \to \mathbb{P}^n$ be a map from an affine, so we have a line bundle $\mathscr{L}$ and $(n+1)$ maps $s_0, \dots, s_n \colon \mathscr{L} \to \mathscr{O}_S$ which collectively are nonvanishing. Without loss of generality, we can assume $\mathscr{L}$ is trivial, since it's locally trivial, and we can check that it's an open cover locally. So it suffices to show that $S \times_{\mathbb{P}^n} U_i$ is a Zariski open cover of S, which is more or less equivalent to $s_0, \dots, s_n$ being everywhere nonvanishing, in view of what we did last lecture. ⊠

Typically, people only use functions, rather than sections of a line bundle, to provide a first naïve definition of $\mathbb{P}^n$. In this case, the Zariski sheaf property fails: you can glue trivial line bundles to obtain something nontrivial.

Corollary 13.2. *Let $\mathscr{E}$ be a vector bundle on a scheme X. Then $P(\mathscr{E})$ is a scheme.*

Proof. X has an affine open cover $\mathfrak{U}$ such that $\mathscr{E}|_U$ is trivial for all $U \in \mathfrak{U}$. Then $\mathbb{P}(\mathscr{E}) \times_X U \cong \mathbb{P}^n \times U$ as schemes. ⊠

$\mathbb{P}^n$ has an important line bundle called $\mathscr{O}_{\mathbb{P}^n}(1)$ (sometimes just $\mathscr{O}(1)$ if $\mathbb{P}^n$ is implicit).

Definition 13.3. The line bundle $\mathscr{O}_{\mathbb{P}^n}(1) \in \mathsf{QCoh}(\mathbb{P}^n)$ is the line bundle which, given a map $f \colon S \to \mathbb{P}^n$ which is a line bundle $\mathscr{L}$ on S and the maps $s_0, \dots, s_n$, defines $f^*(\mathscr{O}_{\mathbb{P}^n}(1)) \coloneqq \mathscr{L}^\vee$.

For any $n \in \mathbb{Z}$, we define $\mathscr{O}_{\mathbb{P}^n}(m) \coloneqq \mathscr{O}_{\mathbb{P}^n}(1)^{\otimes m}$. If $m = 0$, we interpret the empty tensor product as $\mathscr{O}_{\mathbb{P}^n}$, the sheaf of functions; if $m < 0$, we interpret this as $((\mathscr{O}_{\mathbb{P}^n}(1))^\vee)^{\otimes(-m)}$.

Definition 13.4. Let X be a scheme and $\mathscr{F} \in \mathsf{QCoh}(X)$. The *sections* of X, denoted $\Gamma(X, \mathscr{F})$, is the abelian group $\mathrm{Hom}_{\mathsf{QCoh}(X)}(\mathscr{O}_X, \mathscr{F})$.

That is, a section is compatible data, for all affines S and maps $x \colon S \to X$, of $\sigma_x \in x^*\mathscr{F}$ regarded as a module.

The sections $\Gamma(X, \mathscr{O}_X) = \mathrm{Fun}(X)$, and if $X = \operatorname{Spec} A$, $\Gamma(X, \mathscr{F})$ is canonically the A-module associated to $\mathscr{F}$ (which we've just been calling $\mathscr{F}$ again).

Definition 13.5. If A is a commutative ring, we let $\mathbb{P}^n_A \coloneqq \mathbb{P}^n \times \operatorname{Spec} A$.

Proposition 13.6. *For $r \geq 0$, $\Gamma(\mathbb{P}^n_A, \mathscr{O}(r))$ is the A-module of homogeneous degree-r polynomials with A coefficients in $n+1$ variables.*

For $r = 1$, a map $A^{\oplus(n+1)} \to \Gamma(\mathbb{P}^n_A, \mathscr{O}(1))$ is definedc by $n+1$ maps $\mathscr{O}_{\mathbb{P}^n_A} \to \mathscr{O}_{\mathbb{P}^n_A}(1)$, i.e. for all $f \colon S \to \mathbb{P}^n_A$, i.e. line bundles $\mathscr{L}$ with $s_0, \dots, s_n$, compatible maps $\mathscr{O}_S \to f^*(\mathscr{O}(1))$ which identify $f^*(\mathscr{O}(1)) \cong \mathscr{L}^\vee$, and the i^{th} section is identified with $s_i \in \mathscr{L}^\vee$. Following your nose with the formal stuff provides a complete proof.

Over the next few days we'll discuss finite conditions: the nicest possible schemes are smooth projective curves, and we'll discuss these soon.

Lecture 14.

Finiteness hypotheses: 10/1/18

I was late to class and missed a few things.

Definition 14.1. A scheme X is *quasicompact* if it has a finite affine open cover.

Idea: this is about the same as compactness in topology.

Example 14.2. $\mathbb{P}^n$ is quasicompact, since it has a cover by $n+1$ opens, as we saw on Friday. ◄

Proposition 14.3. *If A is a commutative ring and $I \subset A$ is an ideal, then* $\operatorname{Spec} A \setminus \operatorname{Spec}(A/I)$ *is quasicompact if I is finitely generated.*

The converse is false.

Example 14.4. Let $\mathbb{A}^\infty_k \coloneqq \operatorname{Spec} k[x_1, x_2, x_3, \dots]$. Then $\mathbb{A}^\infty_k \setminus 0$ is a scheme which is not quasicompact. ◄

Remark 14.5. More generally, a morphism $f\colon X \to Y$ is said to be quasicompact if for all affine schemes S and maps $S \to Y$, $X \times_Y S$ is quasicompact. This reduces to the notion of quasicompactness for schemes when $Y = \operatorname{Spec} \mathbb{Z}$, because a map to $\operatorname{Spec} \mathbb{Z}$ is no data at all. ◀

Definition 14.6. A scheme X is *quasiseparated* if the intersection of two affine opens in X is quasicompact.

Equivalently, the diagonal map $\Delta\colon X \to X \times X$ is a quasicompact morphism. Therefore one may more generally say a quasiseparated morphism $f\colon X \to Y$ is one for which the diagonal $\Delta_f\colon X \to X \times_Y X$ is quasicompact.

Example 14.7. The idea is that quasiseparated is kind of like the Hausdorff property in topology. As such, we can import the standard counterexample, $\mathbb{A}^1$ with two origins. The idea is to take two copies of $\mathbb{A}^1$, say with coordinates t and u, and glue them together along $\mathbb{A}^1 \setminus 0$ via the identification $t \leftrightarrow u$. (This is different from $\mathbb{P}^1$, which is quasiseparated, where we identified $t \leftrightarrow u^{-1}$.) ◀

This next exercise is a

Exercise 14.8. Quasicompact, quasiseparated morphisms are exactly those where the pushforward of quasicoherent sheaves is well-behaved. Specifically, given a map $f\colon X \to Y$ and a quasicoherent sheaf $\mathscr{F}$ on X, we define its pushforward $f_*\mathscr{F}$ on Y as follows: given an affine open subscheme $j\colon U \hookrightarrow Y$, we specify

$$j^* f_* \mathscr{F} := \Gamma(X \times_Y U, \mathscr{F}|_{X \times_Y U}).$$

Show that this is quasicoherent if f is quasicompact and quasiseparated. (TODO: converse?)

Even when we restrict to quasicompact, quasiseparated things, we're still looking at extremely general objects, considerably moreso than are studied in classical algebraic geometry. So here are some more finiteness hypotheses.

Definition 14.9. Let $f\colon X \to Y$ be a map of schemes.

(1) Suppose $Y = \operatorname{Spec} A$. Then f is *locally of finite type* (LFT) if for all affine opens $\operatorname{Spec} B \subset X$, the induced map of rings $A \to B$ makes B into a finitely generated algebra, i.e. $B \cong A[x_1, \dots, x_n]/I$ for some ideal $I \subset A[x_1, \dots, x_n]$.
(2) If in addition X is quasicompact, then f is called *finite type*.
(3) For general Y, f is *locally finite type* (resp. *finite type*) if for all affine opens $V \subseteq Y$, the map $X \times_Y V \to V$ is locally finite type (resp. finite type).

These properties roughly mean that you're covered by finite type algebras. It's not hard to prove that when Y is affine, these definitions coincide.

Theorem 14.10 (Hilbert's basis theorem)**.** *Suppose A is a Noetherian ring amd B is a finitely generated A-algebra. Then there is a fiber product square*

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r] \arrow[d] & \mathbb{A}^n_A \arrow[d] \\ 0 \arrow[r] & \mathbb{A}^m_A. \end{tikzcd}$$

Here $\mathbb{A}^k_A := \operatorname{Spec} A[x_1, \dots, x_k]$.

This will be highly noncanonical.

Remark 14.11. This is telling us that $\operatorname{Spec} B$ is the zero locus of m polynomials in n variables with coefficients in A, or that more generally, a finite type Noetherian scheme is exactly one which locally admits such a description. Since this was one of our motivations for studying algebraic geometry from the beginning, this is an excellent hypothesis to have. ◀

Though you've probably already seen the proof if you know what a Noetherian ring is, it's still good to go over.

Definition 14.12. Let V be an abelian group. A *filtration* on V is a sequence

$$F_0V \subseteq F_1V \subseteq \cdots \subseteq V, \tag{14.13}$$

such that

$$\bigcup_{n\geq 0} F_nV = V. \tag{14.14}$$

The *associated graded* is a graded abelian group $\mathrm{gr}_\bullet(V) \coloneqq \bigoplus_{n\geq 0}\mathrm{gr}_n V$, where $\mathrm{gr}_n V \coloneqq F_nV/F_{n-1}V$, and we declare $F_{-1}V \coloneqq 0$.

Filtrations are more general than gradings, but are really nice to have, and can make some arguments a lot cleaner.

Definition 14.15. If (V, F_n) and (W, F'_n) are filtrations, a morphism of filtered abelian groups is a map $f\colon V \to W$ such that $f(F_nV) \subseteq F'_nW$.

Exercise 14.16. Show that if $\mathrm{gr}_\bullet(f)\colon \mathrm{gr}_\bullet(V) \to \mathrm{gr}_\bullet(W)$ is injective (resp. surjective, resp. bijective), then f is injective (resp. surjective, resp. bijective).

This is a major tool in working with filtrations, especially in the (common) case where the filtered objects are complicated, but their associated gradeds are simpler.

Definition 14.17. A *filtered algebra* is an algebra A filtered as an abelian group such that multiplication carries $F_nA \times F_mA \subseteq F_{n+m}A$.

Example 14.18. The algebra $A = k[x]$ is filtered by degree: we let F_nA denote the polynomials of degree at most n. ◂

If A is a filtered abelian group, F_nA is also an abelian group, but if A is an algebra, F_nA is generally not a subalgebra, as in the above example.

If A is a filtered algebra, we can make sense of the notion of filtered A-modules, where the filtrations given by the A-action and the module are compatible in the least surprising way.

Lemma 14.19. *If M is a filtered A-module and $\mathrm{gr}_\bullet(M)$ is a finitely generated $\mathrm{gr}_\bullet(A)$-module, then M is a finitely generated A-module.*

Proof. Let $\overline{x}_1, \ldots, \overline{x}_n \in \mathrm{gr}_\bullet M$ be a generating set. We can assume they're *homogeneous*, i.e. each $\overline{x}_i$ lives in some $\mathrm{gr}_{k_i} M$. Lift $\overline{x}_i$ to some $x_i \in F_{i_k}M$; then the map $\varphi\colon A^{\oplus n} \to M$ sending the standard basis element $e_i \mapsto x_i$ is surjective after passing to the associated graded, hence by Exercise 14.16 is surjective, and therefore $\{x_1, \ldots, x_n\}$ generates M. ⊠

Proof of Theorem 14.10. By induction, it suffices to show that if A is Noetherian, then $A[x]$ is too. Filter $A[x]$ by degree, and if $I \subseteq A$ is an ideal, let $F_nI \coloneqq I \cap F_nA[x]$.

We claim $\mathrm{gr}_\bullet I$ is finitely generated over $\mathrm{gr}_\bullet A[x]$. To see this, note that the multiplication-by-x map $\mathrm{gr}_i I \to \mathrm{gr}_{i+1} I$ is an injection for all i, and furthermore its image in A corresponds to the inclusion of an ideal. Therefore, by Noetherianness of A, this chain must stabilize at some $N \in \mathbb{N}$, and therefore $\mathrm{gr}_\bullet I$ is generated by $\bigoplus_{i=0}^N \mathrm{gr}_i I$. Because A is Noetherian, each $\mathrm{gr}_i I$ is a finitely generated A-module as well, showing the claim. ⊠

Lecture 15.

Connected and irreducible components: 10/3/18

I wasn't in class for this lecture; these notes were generously provided by Tom Gannon.

Definition 15.1. A scheme X is *locally Noetherian* if it admits an open cover by affine open sets of the form $\operatorname{Spec} A$ for Noetherian A. If X is also quasicompact, we say X is *Noetherian.*

Remark 15.2. Note that if U is a subset of a Noetherian scheme X, then U is quasicompact. To see this, pick an affine open cover by Noetherian rings $\operatorname{Spec} A_i$ for $i \in \{1, \ldots, m\}$; then U^c is given by a finitely generated ideal. This also shows that U is Noetherian. ◂

Lemma 15.3. *A scheme X is Noetherian if and only if it is* topologically Noetherian, *that is, for all chains of closed $Z_i \subset X$, i.e. $Z_0 \supset Z_1 \supset \dots$, the Z_i stabilize.*

The affine case is just rewriting the definition, and the general case just follows from compactness (exercise!). This shows one odd feature of the Zariski topology — we certainly don't have that $\mathbb{C}$ with the standard topology is Noetherian! An informal way of stating the above lemma is that taking a nontrivial closed subset is a big deal.

Remark 15.4. The equivalent conditions above yield the increasing chain condition for open sets but the increasing chain condition on open sets does not imply X is Noetherian (for example, $X = \operatorname{Spec} k[x_i]_{i\in\mathbb{N}}/(x_i^2)$). This example also shows that there is not a bijection between closed subschemes and open subschemes, although the dual numbers also shows this. ◂

Definition 15.5. A scheme X is *connected* if for all open covers $X = U \cup V$ such that $U \cap V = \varnothing$, either $U = X$ or $V = X$.

Lemma 15.6. *If X is a Noetherian scheme, then X can be written as the disjoint union of finitely many* connected components *of X, i.e. open and closed connected subschemes.*

This involves writing the definition of a disjoint union of schemes $\coprod_{i=1}^n X_i$, which we will leave as an exercise, but essentially any solution that isn't the functor $(\coprod_{i=1}^n X_i) := \coprod_{i=1}^n X_i(A)$ will likely work.

Proof. If x is a field valued point, then by Noetherianness there is a minimal $U_x \subset X$ which is closed and open and contains x. By minimality, U_x is connected and $\{U_x\}_x$ is an open cover, where x varies over all the field valued points, so because quasicompactness implies Zariski topology compactness (since every open cover of an affine admits a finite refinement), there are only finitely many U_x. ⊠

Definition 15.7. For a quasicompact quasiseparated (qcqs) morphism $f : X \to Y$ (for example, f finite type–most importantly this is the setting where pushforward is defined on quasicoherent sheaves), we obtain by adjunction a map $\mathscr{O}_Y \to f_*\mathscr{O}_X$ with some kernel I. This corresponds to a closed subscheme of Y, which we will denote $\overline{f(X)}$ and will call the *scheme-theoretic image*.

Exercise 15.8. If $X = \operatorname{Spec} A \to \operatorname{Spec} B = Y$ corresponds to a ring map $\phi : A \to B$, we can factor ϕ as a composite of a surjection and an injection $A \to \phi(A) \to B$.

The picture to have in mind here is that $X \to \overline{f(X)} \subset Y$, where $X \to \overline{f(X)}$ is *dominant*:

Definition 15.9. We say a qcqs morphism $f : X \to Y$ is *dominant* if $\overline{f(X)} = Y$.

Equivalently, f is dominant if $\mathscr{O}_Y \to f_*\mathscr{O}_X$ is a monomorphism. The idea here is that if you have a point in the closure of the image there is a function realizing this, and we have a rough equivalent between a function being dominant and the associated map on functions being injective.

Example 15.10. The map $\mathbb{A}^1 \setminus 0 \to \mathbb{A}^1$ is dominant.

Definition 15.11. A quasicompact $U \subset X$ is *dense* if $\overline{U} = X$.

As an exercise, you can look up the relationship between density and *associated primes* for a ring A.

Definition 15.12. A Noetherian scheme X is *irreducible* if for every closed $Z \subset X$ with $Z \neq X$, $X \setminus Z$ is dense.

Remark 15.13. Other textbooks often refer to this as a scheme being *integral*. ◂

Example 15.14. The scheme $\operatorname{Spec} k[s,t]/(st)$ is not irreducible, which can be seen by setting $Z = \{t = 0\}$.

Example 15.15. The scheme $\operatorname{Spec} k[\epsilon]/(\epsilon^2)$ is not irreducible since the closed subscheme $\operatorname{Spec} k$ has empty complement (which is in particular not dense).

Exercise 15.16. We have that $\operatorname{Spec} A$ is irreducible if and only if A is an integral domain.

Exercise 15.17. If $X = \operatorname{Spec} A$, then the irreducible closed subschemes of X correspond to primes of A.

Definition 15.18. An *irreducible component* of X is a maximal irreducible subscheme.

Lemma 15.19. *If X is Noetherian, then there are only finitely many irreducible components and every field valued point factors through each one.*

We'll prove this next time.

Lecture 16.

Irreducibility: 10/5/18

Today we're going to discuss a way to describe schemes as decomposed into simpler parts. One way to do this is to use connected components, but there's another notion, called reducibility, which is more general, and which we'll use more frequently. The idea is that $\operatorname{Spec} k[x,y]/(xy)$, the x- and y-axes inside $\mathbb{A}^2_k$, is a union of two $\mathbb{A}^1_k$s, so we want to call it reducible.

Lemma 16.1. *Let $Z \subseteq X$ be an irreducible component and f be a function on X with $f|_Z = 0$. Then there's a $g \in \operatorname{Fun}(X)$ with $g|_Z \neq 0$ and $f^N g = 0$ for $N \gg 0$.*

Proof. For X affine, Z is equivalent data to a prime ideal $\mathfrak{p}$ because Z is irreducible, and minimal because Z is a component, and f must be in $\mathfrak{p}$. The localization $A_\mathfrak{p}$ is a local ring with the image of f contained in the maximal ideal, and $A_\mathfrak{p}[f^{-1}] = 0$.

In general, if you're localizing at a multiplicative set and obtain zero, then some element of the set is zero. We localized with respect to sf^N for $N \geq 0$ and $s \notin \mathfrak{p}$, so we conclude that there's some $g \in A \setminus \mathfrak{p}$ (so a global function on X nonvanishing on Z), and such that $gf^N = 0$ for some N. ⊠

Remark 16.2. Localization has a geometric interpretation. Suppose $X = \operatorname{Spec} A$ and $B = A/\mathfrak{p}$, so B is an integral domain. Let k denote its fraction field; if $Z = \operatorname{Spec} B$, then $\operatorname{Spec} k \hookrightarrow Z$ is the *generic point* of Z, in the sense that in the Zariski topology, its closure contains all other points. This is perhaps a bit bizarre, but it allows for some useful constructions: the localization $\operatorname{Spec} A_\mathfrak{p}$ admits the geometric description of the intersection[12] of all opens $U \subseteq X$ which contain the generic point $\operatorname{Spec} k$. ◀

Lemma 16.3. *Let X be an affine Noetherian scheme and $Z \subseteq X$ be an irreducible component. Then there's a function g on X with $g|_Z \neq 0$ and $g|_{X \setminus Z} = 0$.*

Proof. Writing $X = \operatorname{Spec} A$, Z corresponds to a prime ideal $\mathfrak{p}$, necessarily finitely generated because X is Noetherian: $\mathfrak{p} = (f_1, \dots, f_r)$. By Lemma 16.1, we can choose an $N > 0$ and $g_1, \dots, g_r \in \operatorname{Fun}(X)$ such that $g_i f_i^N = 0$ and $g_i|_Z \neq 0$. Letting $g = \prod g_i$, it's nonzero on Z, because Z is Spec of a domain.

We claim $g|_{X \setminus Z} = 0$. This is because $X \setminus Z$ is covered by $D(f_i) = \{f_i \neq 0\}_{i=1,\dots,r}$, so it's enough to see on each $D(f_i)$; since $gf_i^N = 0$, then $g|_{D(f_i)} = 0$. ⊠

Corollary 16.4. *If $Z \subseteq X$ is an irreducible component, then $X \setminus Z$ isn't dense in X.*

Proof. We can easily reduce to X affine. Then pick a g with $g|_Z \neq 0$ and $g|_{X \setminus Z} = 0$. This means $\overline{X \setminus Z} \subseteq \{g = 0\} \subsetneq X$, since its complement $D(g)$ is an open subscheme of X. ⊠

Corollary 16.5. *If X is a Noetherian scheme, it has only finitely many irreducible components.*

Proof. Let $\{Z_i\}_{i \in I}$ be the set of irreducible components of X, and let

$$Y_i \coloneqq \overline{X \setminus (Z_1 \cup \cdots \cup Z_i)}. \tag{16.6}$$

We claim $Z_i \subsetneq Y_i$ but $Z_{i+1} \subseteq Y_i$ — we'll prove this in just a sec, but assuming the claim we obtain a decreasing sequence $Y_1 \supsetneq Y_2 \supsetneq \cdots$ if closed subschemes, so since X is Noetherian, $|I|$ must be finite.

Now the claim. We saw that the generic point of Z_i isn't in $\overline{X \setminus Z_i}$ in Corollary 16.4, and by definition $\overline{X \setminus Z_i} \supseteq Y_i$. Since $Z_{i+1} \subseteq Y_i$ and

$$Z_{i+1} \cap X \setminus (Z_1 \cup \cdots \cup Z_i) \neq \varnothing, \tag{16.7}$$

because the components Z_i are distinct, then by irreducibility, Z_{i+1} is contained in the closure of $X \setminus (Z_1 \cup \cdots \cup Z_i)$. ⊠

Definition 16.8. The *Krull dimension* $\dim X$ of a scheme X is the largest integer d such that there is a chain $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_d$ of irreducible subschemes of X. If there is no such integer, we say the dimension is infinite.

There's one main result in dimension theory.

[12]Well, actually an inverse limit, not an intersection. But in reasonable situations, these are the same thing.

Theorem 16.9. *Let X and Y be finite type, irreducible k-schemes. If $f\colon Y \to X$ is a morphism, then there's a dense open $U \subseteq X$ (equivalently, U is nonempty), such that either $Y \times_X Y = \varnothing$ or for every field-valued point $x \in U$, $\dim f^{-1}(x) = \dim Y - \dim X$.*

Here $f^{-1}(x) := Y \times_X \{x\}$, as usual. A *field-valued point* is data of a field k and a map $x\colon \operatorname{Spec} k \to U$. We'll prove this next time, and spend the rest of today's lecture on some corollaries.

Corollary 16.10. *If X is irreducible and $U \subseteq X$ is dense, then $\dim U = \dim X$.*

Corollary 16.11. *If X is irreducible and $f\colon Y \to X$ is dominant, then $\dim Y \geq \dim X$.*

Recall that dominant means it's injective on the level of functions. This means that when we tensor with the fraction field of X, we get something nonzero, and therefore the fiber over the generic point is nonempty.

There are nice situations in which these theorems aren't true. For example, let A be a discrete valuation ring, such as $k[t]_{(t)}$ or $k[[t]]$. Then $k := A[t^{-1}]$ is the fraction field of A, and $\operatorname{Spec} k \hookrightarrow \operatorname{Spec} A$ is a nonempty open which is zero-dimensional, but $\operatorname{Spec} A$ is one-dimensional. So we really have to use the fact that we're finite type over a field.

Lecture 17.

Noether normalization: 10/8/18

It will be useful to have some examples to carry around.

Example 17.1. The reason for irreducibility in the hypothesis of Theorem 16.9 is that dimension behaves poorly for reducible schemes: consider $Z = \{xz = 0, yz = 0\} \subset \mathbb{A}^3_k$. Geometrically, this is the union of the xy-plane and the z-axis, which clearly has two irreducible components, and Z is two-dimensional. But $Z \setminus \{xy = 0\}$ is open in Z and has dimension 1. We would like the dimension of open subsets to be the same as that of the entire scheme, forcing us to consider irreducibility. ◀

Example 17.2. For a typical, useful example, consider the map $\mathbb{A}^2 \to \mathbb{A}^2$ sending $(x, y) \mapsto (x, xy)$.[13] The fiber at a field-valued point (x, y) with $x \neq 0$ is a point. The fiber at $(0, y)$ is empty for $y \neq 0$, and at the origin, the fiber is an $\mathbb{A}^1$.

So on $\mathbb{A}^2 \setminus 0$, which is an open, dense set, we get that the fibers are either empty or have the correct dimension; otherwise they could be "too big." ◀

The main tool in our proof of Theorem 16.9 is the theory of finite morphisms, and in fact we'll end up reducing to Nakayama's lemma.

Exercise 17.3. As a warm-up for this kind of argument, use Nakayama's lemma to show that if $\mathscr{F}$ is a locally finitely generated QC sheaf on an irreducible scheme, then there exists a nonempty open U such that $\mathscr{F}|_U$ is a vector bundle.

Definition 17.4. A morphism of schemes $f\colon Y \to X$ is *finite* if it's affine and for every affine open $U = \operatorname{Spec} A \subseteq X$, if $Y \times_X U = \operatorname{Spec} B$, then B is a finitely generated A-module.

This is very strong: finite type was asking about finite generation as an algebra: $k[x]$ is a finitely-generated k-algebra but not a finitely generated k-module. Therefore $\mathbb{A}^1_k$ isn't finite over $\operatorname{Spec} k$!

Remark 17.5. If f is an affine morphism, we showed that there's a sheaf of QC algebras $\mathscr{A}$ with $X = \operatorname{Spec}_Y(\mathscr{A})$. Then, f is finite iff $\mathscr{A}$ is locally finitely generated as a quasicoherent sheaf ◀

Example 17.6.

(1) The best example to have in mind is to fix a field k, and let B be a finite-dimensional k-algebra. Then $Y = \operatorname{Spec} B$ is finite over $\operatorname{Spec} k$. This implies B is Artinian (which is a good exercise). So, for example, $\operatorname{Spec} \mathbb{C}$ over $\operatorname{Spec} \mathbb{R}$, or the dual numbers or other nilpotent things.
(2) A closed embedding is finite, because A/I is generated as an A-module by 1_A. ◀

Proposition 17.7. *Let A be a ring and $X := \operatorname{Spec} A$. Let $Y \subset X \times \mathbb{A}^1_A$ be a closed subscheme, corresponding to an ideal I of $A[x]$, let and $f\colon Y \to X$ be the restriction of the projection map. Then f is finite iff I contains a monic polynomial.*

[13]This is called the *blowup of the plane at* 0, and fits into a more general theory of blowups.

Proof. First assume f is finite, so there are $\varphi_1, \dots, \varphi_N \in A[t]$ which generate $A[t]/I$ as an A-module. Choose d such that $d > \deg(\varphi_i)$ for all i. Then there exist $a_1, \dots, a_N \in A$ such that

$$t^d = \sum_{i=1}^{N} a_i \varphi_i \bmod I, \tag{17.8}$$

so

$$t^d - \sum_{i=1}^{N} a_i \varphi_i \tag{17.9}$$

is in I and is monic.

The converse is basically the same logic, but in reverse order: take your monic polynomial and lift it to get generators. ⊠

Finite morphisms are great because Nakayama's lemma applies to them. We'll see this in the proof of Theorem 16.9. We'll also need another tool, Noether normalization.

Theorem 17.10 (Noether normalization)**.** *Let $X \subsetneq \mathbb{A}_k^{n+1}$ be a closed subscheme. Then there's a finite field extension $k \hookrightarrow k'$ and a projection map $\mathbb{A}_{k'}^{n+1} \to \mathbb{A}_{k'}^{n}$ such that the induced map $f \colon X_{k'} \to \mathbb{A}_{k'}^{n}$ is finite. If moreover X is the zero locus of a single polynomial, then f is dominant.*

Here $X_{k'} \coloneqq X \times_{\operatorname{Spec} k} \operatorname{Spec} k'$. By a projection, we mean a map induced by a linear surjection $(k')^{n+1} \to (k')^n$.

Remark 17.11. Suppose k isn't a finite field. Then we don't need to pass to k'. (This will be evident from the proof.) ◀

Example 17.12. For example, consider the map $\mathbb{A}^1 \setminus 0 \hookrightarrow \mathbb{A}^2$ induced from the map $t \mapsto (t, t^{-1})$. Algebraically, we get $k[t] \hookrightarrow k[t, t^{-1}]$, which is not finite at the level of algebras (since we can take t^{-N} for arbitrarily large N). Geometrically, you can use Nakayama's lemma to show that fibers of dominant maps over field-valued points must be nonempty, but the fiber over 0 is empty.

But you can project along any other line (except the y-axis), such as the diagonal, then the map is in fact finite. ◀

Lecture 18.

Proof of Noether normalization: 10/10/18

Last time, we discussed finite morphisms and Noether normalization. The following exercise might provide some useful intuition about finite morphisms.

Exercise 18.1. Let $f \colon X \to Y$ be a finite, dominant map. Then for all field-valued points $x \in X$, $f^{-1}(x)$ is nonempty and zero-dimensional. Also show that $\dim X = \dim Y$.

Hint: Nakayama's lemma.

We then discussd Example 17.12, about $\{xy = 1\} \subset \mathbb{A}^2$. It doesn't project onto every line through the origin in $\mathbb{A}^1$, but everything but the x- and y-axes is good. One interesting way to think about this is that there's a $\mathbb{P}^1$ "at infinity" of $\mathbb{A}^2$, where, imprecisely speaking, we think of $\mathbb{P}^1$ as a circle of very large radiyus (though we need to identify antipodal points); a line is sent to its point of intersection with the circle. Then, we have an open subset of $\mathbb{P}^1$ (again, everything except the x- and y-axes) where the projection is finite and dominant.

More generally, suppose $X \subseteq \mathbb{A}^{n+1}$. We can embed $\mathbb{A}^{n+1} \subset \mathbb{P}^{n+1}$ as an open subscheme; let $\overline{X}$ be the closure of X in $\mathbb{P}^{n+1}$. The complement of $\mathbb{A}^{n+1}$ inside $\mathbb{P}^{n+1}$ is a $\mathbb{P}^n$, and we let $\operatorname{Asym}(X) \coloneqq \overline{X} \cap \mathbb{P}^n$; we can think of this as where X is going "at infinity."

If $X \neq \mathbb{A}^{n+1}$, then $\operatorname{Asym}(X) \neq \mathbb{P}^n$. We claim there exists a finite extension $k \hookrightarrow k'$ and a k'-point of $\mathbb{P}^n$ not in $\operatorname{Asym}(X)$, and that projecting away from this line ℓ, the map is finite. By "projecting away from the line," we mean that there's a projection $\pi \colon \mathbb{A}_{k'}^{n+1} \to \mathbb{A}_{k'}^{n}$ such that $\ker(\pi) = \ell$.

Proof of Theorem 16.9. More explicitly, first we can reduce to the case where $X = \{f = 0\}$, for some nonzero and nonconstant f. Since $X \subsetneq \mathbb{A}^1$, so we can choose such an f vanishing on X. Then $X \hookrightarrow \{f = 0\}$ is a closed embedding, hence a finite morphism. It's easy to see that finite maps are closed under compositions, so the map $\{f = 0\} \hookrightarrow \mathbb{A}^{n+1} \twoheadrightarrow \mathbb{A}^n$ is finite, and therefore the theorem for $\{f = 0\}$ implies the theorem for X.

Now we have $f \in k[x, y_1, \dots, y_n]$ and $X = \{f = 0\}$, which in particular is nonempty. Write

$$f = \sum_{i=0}^{d} f_i, \tag{18.2}$$

such that each f_i is homogeneous of degree i and f_d is nonzero.

Example 18.3. If $f(x) = x^3 + 2x^2y$, then f is homogeneous of degree 3. For $f(x) = x^3 + 1$, we'd let $f_0 = 1$, $f_1 = f_2 = 0$, and $f_3 = x^3$. ◀

Exercise 18.4. Show that if $X = \{f = 0\}$, $\mathrm{Asym}(X) = \{f_d = 0\} \subseteq \mathbb{P}^n$. Here we're thinking of f_d as a section of $\mathscr{O}_{\mathbb{P}^n}(d)$.

Exercise 18.5. Show that there exists a finite extension $k \hookrightarrow k'$ and some $v \in (k')^{n+1}$ such that $f_d(v) \neq 0$. Moreover, if k is infinite, we can choose $k' = k$.

This is a general fact about nonconstant polynomials. We will now write $k = k'$ for ease of notation. Moreover, up to a linear change of coordinates, we can assume $v = (1, 0, \dots, 0)$, which doesn't affect homogeneity.

If

$$f_d = ax^d + bx^{d_1}y_1 + cx^{d-1}y_2 + \cdots, \tag{18.6}$$

then $f_d(1, 0, \dots, 0) = a$, and up to scaling, we can assume $a = 1$. (We know $a \neq 0$ because $f_d(v) \neq 0$).

We can write $f = \sum_{i=0}^{d} g_i x^i$ for $g_i \in k[y_1, \dots, y_n]$; by construction, $g_d = 1$. Therefore, as a polynomial in x, f is monic, and therefore by last time, $k[y_1, \dots, y_n][x]/(f)$ is finite over $k[y_1, \dots, y_n]$ (specifically, $1, x, \dots, x^{d-1}$ generate it). And since $d > 0$, $k[y_1, \dots, y_n] \hookrightarrow k[y_1, \dots, y_n][x]/(f)$, which implies dominance. Therefore we've proven the theorem. ☒

Corollary 18.7 (Nullstellensatz)**.** *Let X be a finite-type affine scheme over k. Then there's a finite extension $k \hookrightarrow k'$ and a finite dominant map $X_{k'} \to \mathbb{A}^n_{k'}$ for some n. If k is infinite, we can take $k' = k$.*

There are other theorems called the Nullstellensatz, but they're all related to each other and to this one.

Proof. We know $X_{k'} \subsetneq \mathbb{A}^n_{k'}$ for some n, and we have a finite dominant map $\pi\colon \mathbb{A}^n_{k'} \to \mathbb{A}^{n-1}_{k'}$; if $\overline{\pi(X)} = \mathbb{A}^{n+1}_{k'}$, we're done; otherwise we can repeat.

TODO: then something else happened, which I didn't quite follow. ☒

In particular, the ring of functions on $X_{k'}$ is finite-dimensional over k'.

Corollary 18.8. $\dim_k \mathbb{A}^n_k = n$.

Proof. We can induct: $n = 0$ is clear, so assume it for $\mathbb{A}^n_k$, and we'll show it for $\mathbb{A}^{n+1}_k$. Let $Z \subsetneq \mathbb{A}^{n+1}$ be a closed, irreducible NTS; then $\dim Z \leq n$. Since $Z \subset \{f = 0\}$ for some f, then it admits a finite dominant map to $\mathbb{A}^n_k$, so $\dim\{f = 0\} = n \geq \dim Z$ by induction. ☒

Lecture 19.

More cool facts from dimension theory: 10/12/18

Today we'll continue deducing stuff from Theorems 16.9 and 17.10. For example, at the end of the last class, we showed that $\dim \mathbb{A}^n$ is n, so if $f \in k[x_1, \dots, x_n]$ is nonconstant, then $\{f = 0\}$ is $(n-1)$-dimensional, and this is true for all irreducible components of X.

Definition 19.1. An irreducible scheme X is *caternary* if for all closed subschemes $Z \subsetneq X$, there's a closed subscheme $Z' \subseteq X$ containing Z as a closed subscheme and such that $\dim Z' = \dim X - 1$.

We basically proved the following while proving Theorem 17.10.

Corollary 19.2. *$\mathbb{A}^n$ is catenary.*

Remark 19.3. The word catenary comes from the word for "chain" in a Romance language (e.g. in Italian, it's *catena*), presumably because it gives us chains of closed subschemes. ◀

More generally:

Corollary 19.4. *If X is an irreducible finite-type scheme over an infinite field k, then X is catenary.*

Proof. We can quickly reduce to the case where X is affine. Noether normalization means we may choose a finite dominant map $\pi\colon X \to \mathbb{A}_k^n$; hence $\dim X = n$. Let $Z \subsetneq X$ be maximal under closed irreducible subschemes contained in X. We want to show $\dim Z = n-1$.

The restriction $\pi|_Z\colon Z \to \overline{\pi(Z)}$ is also finite dominant, so it suffices to show $\dim \overline{\pi(Z)} = n-1$. If this is not the case, then since $\mathbb{A}_k^n$ is catenary, we can choose an $(n-1)$-dimensional irreducible $Z' \subset \mathbb{A}_k^n$ and strictly containing $\overline{\pi(Z)}$.

The theory of finite morphisms (more specifically, going-up and going-down theorems applied to $X \to \mathbb{A}^n$), Z is not an irreducible component of $\pi^{-1}(Z')$, which is a contradiction. ⊠

This is an important result that's easy to take for granted — it is one of the facts about dimension that is an ansatz about any theory of dimension in geometry: all k-points look the same in a variety over k. If something like this were not true, there would have to be a different theory of dimension. It's not so surprising it reduces to studying $\mathbb{A}_k^n$, with its large symmetry group.

There are two major ways to induct on dimension for varieties over fields: project onto a lower-dimensional subscheme, and take the intersection with a hyperplane. We used the former for this proof.

Remark 19.5. The proof of Corollary 19.4 strongly depends on the Nullstellensatz, and in particular, is not true over more general rings: if A is a DVR, then $\operatorname{Spec} A[x]$ isn't catenary. But dimension is set up to behave well over fields, so maybe this isn't so sad. ◀

Corollary 19.6. *If X is irreducible and finite type over k, and $U \subseteq X$ is a nonempty open, then* $\dim U = \dim X$.

Proof sketch. Without loss of generality, we can assume $X = \operatorname{Spec} A$ is affine. Let $x \in U$ be a closed point, hence $\operatorname{Spec} k'$ for some extension $k \hookrightarrow k'$, and we have data of a surjective map $A \to k'$ (and, if $U = \operatorname{Spec} B$, data of a surjective map $B \to k'$). There's a closed, irreducible subscheme $Z \subsetneq X$ containing x and such that $\dim Z = \dim X - 1$.

The intersection $Z \cap U$ is a nonempty, irreducible, proper closed subscheme of U, and is an open subscheme of Z. Inducting on $\dim X$,

$$\dim Z \cap U = \dim Z = \dim X - 1, \tag{19.7}$$

so $\dim U > \dim X - 1$, hence $\dim U \geq \dim X$. The other inequality is easy: take closures. ⊠

Lemma 19.8. *If X is irreducible and Y is a closed subscheme of $X \times \mathbb{A}^1$, then either*

- *Y is irreducible, or*
- *there's a nonempty open $U \subseteq X$ such that $Y \times_X U \to U$ is finite.*

Proof. As usual we can assume $X = \operatorname{Spec} A$ is affine, so $Y = \operatorname{Spec} A[t]/I$ for an ideal $I \subseteq A[t]$. The first option is $I = 0$, so we assume $I \neq 0$, so there's a nonzero $f \in I$, and $f = \sum_{i=0}^d a_i t^i$ for $a_i \in A$ with $a_d \neq 0$. If $U = \{a_d \neq 0\}$, then U is a nonempty open subscheme of X. One can show that I contains a monic polynomial over U, and we saw this is equivalent to $Y \times_X U \to U$ being finite. ⊠

Lecture 20.

: 10/15/18

Lecture 21.

Differentials and derivations: 10/17/18

"It's not a field, but it's psychologically a field."

Today we're going to talk about differentials and derivations, which are pretty important. For this lecture, $X = \operatorname{Spec} A$ is an affine scheme over a field k.

Definition 21.1. If M is an A-module, a *derivation* $\delta\colon A \to M$ is an A-linear map satisfying the Leibniz rule

$$\delta(fg) = f\delta(g) + g\delta(f).$$

The set of derivations from A to M is denoted $\mathrm{Der}_A(A, M)$; it is naturally an A-module.

A *vector field* is a derivation $\delta\colon A \to A$.

In differential geometry, a vector field gives you a way to differentiate functions.

Derivations are corepresented by a particular A-module (i.e. quasicoherent sheaf on X) Ω_X^1: that is, it's equipped with a derivation $\mathrm{d}\colon \mathscr{O}_X \to \Omega_X^1$ such that for all A-modules M, restriction along d defines an A-linear isomorphism

$$\mathrm{Hom}_A(\Omega_X^1, M) \xrightarrow{\cong} \mathrm{Der}_A(A, M). \tag{21.2}$$

The proof is a contruction: let Ω_X^1 be generated as an A-module by elements $\{\mathrm{d}f \mid f \in A\}$ with relations

$$\mathrm{d}(fg) = f\,\mathrm{d}g + g\,\mathrm{d}f \qquad \text{for all } f, g \in A \tag{21.3a}$$

$$\mathrm{d}(\lambda f) = \lambda\,\mathrm{d}f \qquad \text{for all } \lambda \in k. \tag{21.3b}$$

Lemma 21.4. *If $f \in A$ and $n \geq 1$, then $\mathrm{d}(f^n) = nf^{n-1}\,\mathrm{d}f \in \Omega_X^1$.*

Proof. Induct on n: it's clear for $n = 1$, and assuming it for n, it follows for $n+1$ using the Leibniz rule on $f^{n+1} = (f^n)(f)$. ⊠

As a corollary, $\mathrm{d}(1) = 0$, as $\mathrm{d}(1^n) = \mathrm{d}(1)$.

Example 21.5. For $X = \mathbb{A}_k^1 = \operatorname{Spec} k[t]$, we have $\mathrm{d}t \in \Omega^1_{\mathbb{A}_k^1}$. We claim $\Omega^1_{\mathbb{A}_k^1}$ is freely generated by $\mathrm{d}t$, i.e. $\Omega^1_{\mathbb{A}_k^1} \cong \mathscr{O}_{\mathbb{A}_k^1} \cdot \mathrm{d}t$.

The proof is that, given a $k[t]$-module M and a derivation $\delta\colon k[t] \to M$, if $f = \sum a_t^i \in k[t]$, then in M,

$$\delta(f) = \sum_{i \geq 1} a_i i t^{i-1} \delta(t) \in M. \tag{21.6}$$

so it's spanned by $\mathrm{d}t$. Conversely, given an element $\delta(t) \in M$, there's a unique derivation $\delta\colon k[t] \to M$ sending $t \mapsto \delta(t)$, by the universal property, so $k[t]$-linear maps $\Omega_X^1 \to M$ are uniquely determined by where they send $\mathrm{d}t$. ◀

So if you boil off the abstraction, all you need is to know the derivative of a polynomial. Hopefully this is reassuring.

Example 21.7. Now take $X = \mathbb{A}_k^n = \operatorname{Spec} k[t_1, \dots, t_n]$. Now $\Omega^1_{\mathbb{A}_k^n}$ is a free $k[t_1, \dots, t_n]$-module of rank n, with a basis $\mathrm{d}t_1, \dots, \mathrm{d}t_n$, and

$$\mathrm{d}f = \sum_{i=1}^n \frac{\partial f}{\partial t_i}\,\mathrm{d}t_i. \tag{21.8}$$

The proof is essentially the same as for Example 21.5: once you know where $t_1, \dots, t_n$ go, everything else is forced by linearity and the Leibniz rule. ◀

For a general finite type affine $X = \operatorname{Spec} A$, there's a general algorithm to compute Ω_X^1: first let $f_1, \dots, f_r$ generate A as a k-algebra, and write

$$A \cong k[t_1, \dots, t_r]/(g_1, \dots, g_s) \tag{21.9}$$

for some $g_1, \dots, g_s$ encoding the relations between the f_i. Then Ω_X^1 is generated by $\mathrm{d}f_1, \dots, \mathrm{d}f_r$ with relations $\mathrm{d}g_i|_X = 0$ for $i = 1, \dots, s$.

What's going on here? Well a map $\varphi\colon Y \to X$ of affine k-schemes induces a map of A-modules $\mathrm{d}\varphi\colon \Omega_X^1 \to \varphi_*\Omega_Y^1$: take the universal differential $\mathrm{d}_Y\colon \mathscr{O}_Y \to \Omega_Y^1$ and push it forward to X: $\varphi_*\mathrm{d}_Y\colon \varphi_*\mathscr{O}_Y \to \varphi_*\Omega_Y^1$. Then precompose with $\varphi^*\colon \mathscr{O}_X \to \mathscr{O}_Y$ (pullback of functions); this is a differential $\mathscr{O}_X \to \varphi_*\Omega_Y^1$, hence corresponds uniquely to an A-module map $\Omega_X^1 \to \varphi_*\Omega_Y^1$. Somewhat more explicitly, the characteristic formula is

$$\mathrm{d}\varphi(f\,\mathrm{d}g) = (f\varphi)\,\mathrm{d}(g\varphi). \tag{21.10}$$

Remark 21.11. Often, Ω^1_X is denoted $\Omega^1_{X/k}$: k-linearity is made more explicit. For example, nothing we've done so far requires k to be a field, so we could work with affine schemes over a ring B and study the module of differentials $\Omega^1_{X/B}$. ◂

Suppose $i\colon X \hookrightarrow Y$ is a closed embedding of affine schemes over k. Then one can show that the induced map $i^*\Omega^1_Y \to \Omega^1_X$ is surjective; if I denotes its kernel, then $I \subseteq \mathscr{O}_Y$ is an ideal.

Example 21.12. Ω^1_X isn't always free; for example, suppose k has characteristic zero and $X = \operatorname{Spec}(k[t]/(t^n))$. Using the above algorithm, one can show Ω^1_X is torsion. ◂

Lecture 22.

Smoothness: 10/19/18

"I'll stick to the party line."

Lemma 22.1. *Let k be a field, A be a k-algebra, and $f \in A$. If $j\colon U \hookrightarrow X = \operatorname{Spec} A$ denotes the locus where $f \neq 0$, then restriction defines an isomorphism $j^*\Omega^1_X \xrightarrow{\cong} \Omega^1_U$.*

Proof. Let M be an $A[f^{-1}]$-module, which is the same thing as a quasicoherent sheaf on U. Thinking of M as an A-module (i.e. j^*M), a derivation $\delta\colon A \to M$ is the same thing as a map $j^*\Omega^1_X \to M$ by adjunction. Then δ extends uniquely to a derivation $\widetilde{\delta}$ on $A[f^{-1}]$, because we know what it has to be on f^{-1} by the Leibniz rule:

$$\widetilde{\delta}(f^{-n}) = -nf^{-n-1}\widetilde{\delta}(f). \tag{22.2}$$

This is a quick inductive argument: we know $\widetilde{\delta}(1) = 0$, so

$$0 = \widetilde{\delta}(f^{-n}f^n) = f^n\widetilde{\delta}(f^{-n}) + f^{-n}\widetilde{\delta}(f^n) = f^n\widetilde{\delta}(f^{-n}) + nf^{-1}\widetilde{\delta}(f). \tag{22.3}$$

Now, given an arbitrary element $g/f^n \in A[f^{-1}]$, we define

$$\widetilde{\delta}\left(\frac{g}{f^n}\right) = g(-n)f^{-n-1}\delta(f) + f^{-n}\delta(g). \tag{22.4}$$

It's fairly straightforward to check this is well-defined, and that it gives a derivation. ⊠

As a corollary, we can define Ω^1_X as a quasicoherent sheaf on any k-scheme X: using Serre's theorem, it suffices to describe it on any open affine $j\colon U \hookrightarrow X$, where it's just Ω^1_U. The above lemma guarantees this behaves correctly on intersections.

Definition 22.5. A k-scheme X is *smooth* if

1. X is locally of finite type over k,
2. Ω^1_X is a vector bundle, and
3. for all irreducible components $Z \subseteq X$, the rank of $\Omega^1_X|_Z$ is equal to $\dim Z$.

Remark 22.6. In practice, X will generally be finite type. ◂

It is a nontrivial fact that if X is smooth, every irreducible component is a connected component.

Remark 22.7. Our definition of irreducible is slightly more restrictive than the standard definition, which allows things such as $\operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$. What we call irreducible is generally called integral. Fortunately, it doesn't make a difference in Definition 22.5, though this isn't obvious. ◂

Example 22.8.

1. $\mathbb{A}^n_k$ is smooth, because $\Omega^1_{\mathbb{A}^n_k}$ is free of rank n.
2. $\mathbb{P}^n_k$ is also smooth, because it has a cover by copies of $\mathbb{A}^n_k$, which is smooth.
3. If $n \geq 2$, then $X = \operatorname{Spec} k[t]/(t^n)$ is not smooth: it's zero-dimensional, but $\Omega^1_X \neq 0$.
4. Consider the coordinate axes in $\mathbb{A}^2_k$, $X \coloneqq \operatorname{Spec} A$, where $A \coloneqq k[x,y]/(xy)$. This is not smooth. It's one-dimensional, and as an A-module,

$$\Omega^1_X = A[\mathrm{d}x, \mathrm{d}y]/(x\,\mathrm{d}y + y\,\mathrm{d}x). \tag{22.9}$$

We have a resolution of this module defining generators and relations:

$$A \xrightarrow{1\mapsto x\,\mathrm{d}y + y\,\mathrm{d}x} A^{\oplus 2} \xrightarrow{(\mathrm{d}x,\mathrm{d}y)} \Omega^1_X \longrightarrow 0. \tag{22.10}$$

Restricting to $(0,0) \in X$, we get

$$k \xrightarrow{0} k^{\oplus 2} \xrightarrow{\cong} \Omega^1_X|_{(0,0)} \longrightarrow 0, \tag{22.11}$$

so here it has rank 2, which is not the dimension of X. Therefore X isn't smooth (it turns out Ω^1_X isn't a vector bundle, which is often the problem).

(5) Let $f \in k[x]$ be a *separable* polynomial, meaning it has no repeated roots over the algebraic closure $\overline{k}$ of k, and consider the scheme $X = \operatorname{Spec} k[x,y]/(y^2 - f(x))$.[14] From dimension theory, it's clear this is a curve (i.e. 1-dimensional); we'll show it's smooth.

This time, in the resolution of Ω^1_X

$$A \xrightarrow{f} A^{\oplus 2} \xrightarrow{g} \Omega^1_X \longrightarrow 0, \tag{22.12}$$

$f(1) = \mathrm{d}(y^2 - f(x)) = 2y\,\mathrm{d}y - f'(x)\,\mathrm{d}x$, so if we have a field K and $x, y \in K$ satisfying $y^2 = f(x)$, then if $2y\,\mathrm{d}y = f'(x)\,\mathrm{d}x$, then $y = 0$ and $f'(x) = 0$. Then x is a root of f and f', but since we assumed f is separable, this cannot happen. Therefore Ω^1_X is a vector bundle.

The picture is that restricting the projection $\mathbb{A}^2 \to \mathbb{A}^1$ sending $(x,y) \mapsto x$ to X defines a map whose fiber at $x \in \mathbb{A}^1$ is the square roots to $f(x)$ if $f(x) \neq 0$.

(6) Our last example is an important pathology to be aware of. Suppose k has characteristic $p > 0$, and suppose $\lambda \in k$ is not a p^{th} power (in particular, k is infinite; a typical example is $k = \mathbb{F}_p(\lambda)$, the field of rational functions in λ). Let $k' := k[t]/(t^p - \lambda)$, adjoining a p^{th} root of λ; $t^p - \lambda$ is irreducible, so this is a field.

Then $\operatorname{Spec} k'$ is zero-dimensional over k, but $\Omega^1_{\operatorname{Spec} k'} \neq 0$, so this point is not smooth, which is weird. This is because $\Omega^1_{\operatorname{Spec} k'} = k' \cdot \mathrm{d}t/\mathrm{d}(t^p - \lambda)$, but $\mathrm{d}(t^p - \lambda) = pt^{p-1} = 0$, so $\Omega^1_{\operatorname{Spec} k'}$ is one-dimensional.

$\operatorname{Spec} k'$ is, of course, smooth over itself, i.e. as a k'-scheme; smoothness is relative. The related notion of regularity is intrinsic, but smoothness is always with respect to a base. Said a different way, smoothness is a property of morphisms. ◄

Lecture 23.

Zero-dimensional smooth varieties: 10/22/18

Today we'll classify smooth, zero-dimensional varieties over a field.

Theorem 23.1. *Let X be a smooth, zero-dimensional finite-type scheme over a field k. Then*

$$X \cong \prod_{i=1}^{n} k_i, \tag{23.2}$$

where each $k \hookrightarrow k_i$ is a separable field extension.

Remark 23.3. Recall that a field extension $k \hookrightarrow k'$ is *separable* if for all $x \in k'$, the minimal polynomial of x over k has no repeated roots. ◄

First, one ingredient we'll need in the proof, and which will also be useful later.

Definition 23.4. Let k be a field and V be a vector space over k. The *split square-zero extension* associated to this data is the commutative k-algebra $A = k \oplus V$ with the multiplication

$$(\lambda, v) \cdot (\mu, w) := (\lambda + \mu, \mu \cdot v + \lambda \cdot w). \tag{23.5}$$

In particular, $(0,v) \cdot (0,w) = 0$ and $(\lambda, 0) \cdot (0, v) = (0, \lambda v)$. If V is one-dimensional, this recovers the dual numbers; you can think of split square-zero extensions as generalizations of the dual numbers.

[14] If $\deg f \geq 5$, this is called a *hyperelliptic curve.*

Proof of Theorem 23.1. We can assume $X = \operatorname{Spec} A$ is affine. Since X is zero-dimensional, A is Artinian; from the theory of Artinian rings, A is a product of Artinian local rings; since X is finite type, this is a finite product. That is,

$$A = \prod_{i=1}^{n} A_i, \tag{23.6}$$

where A_i is an Artinian local ring. Therefore we can reduce to the case where A itself is an Artinian local ring, with maximal ideal $\mathfrak{m}$.

For the next step, we assume $A = k'$ is a finite separable extension of k; we'll show that $X = \operatorname{Spec} k'$ is smooth, which in this setting is equivalent to showing that $\Omega^1_{X/k} = 0$. $\Omega^1_{X/k}$ is generated by the elements $\mathrm{d}x$ for $x \in k'$; given such an x, let $f_x(t) \in k[t]$ denote the minimal polynomial of x over k. Then, $f(x) = 0$, so

$$\mathrm{d}f(x) = f'(x)\,\mathrm{d}x = 0. \tag{23.7}$$

Since $f'(x) \neq 0$ by separability, then $\mathrm{d}x = 0$.

Now let $A = k \oplus V$ be a split square-zero extension. The projection map $A \to k$ is a ring map. The other projection map $A \to V$ is a derivation, which is a quick thing to check. Therefore, in particular, A is a field if and only if $V = 0$.

Next, we'll show that if A is an Artinian local k-algebra with nonzero maximal ideal $\mathfrak{m}$ such that $\mathfrak{m}^2 = 0$ and with a separable residue field $k' := A/\mathfrak{m}$, then $\Omega^1_{\operatorname{Spec} A/k} = 0$. In this setting, $\mathfrak{m}$ acts trivially on itself, hence the A-module structure on $\mathfrak{m}$ passes to a k'-vector space structure.

We claim that A is isomorphic to the split square-zero extension $k' \oplus \mathfrak{m}$; then this step will follow from the previous step. We'll show this by showin that the projection $\pi\colon A \to A/\mathfrak{m} = k'$ splits canonically as an algebra map. Specifically, if $x \in k'$, let $f(t) \in k[t]$ be its minimal polynomial and $\widetilde{x} \in A$ be a lift of x across π. Then, there's a unique $\sigma(x) \in A$ such that $f(\sigma(x)) = 0$ and $\pi(\sigma(x)) = 0$.

If $v \in \mathfrak{m}$, then

$$f(\widetilde{x} + v) = f(\widetilde{x}) + f'(x) \cdot v, \tag{23.8}$$

which you can prove by reducing to the case where $f(t) = t^n$ and check directly using the binomial theorem and the face that $v^2 = 0$. Therefore $\pi(f(\widetilde{x})) = f(x) = 0$, so $f(\widetilde{x}) \in \mathfrak{m}$. Therefore

$$v = -\frac{1}{\pi(f'(x))} \cdot f(x) \in \mathfrak{m}. \tag{23.9}$$

Because f is separable, $\pi(f'(\widetilde{x})) = 0$.

So we've shown that if $A = k' \oplus \mathfrak{m}$ is a split square-zero extension, then $\Omega^1_{\operatorname{Spec} A/k} \neq 0$; next we'll show that if A is a local k-algebra (not a field) with spearable residue field, then $\Omega^1_{\operatorname{Spec} A/k} \neq 0$. ⊠