THE COBORDISM HYPOTHESIS

ARUN DEBRAY
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These notes were taken in an online learning seminar on the cobordism hypothesis in Fall 2020, organized by Araminta Amabel, Peter Haine, and Lucy Yang. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

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1. Introduction: 9/9/20

Today's talk was given by Araminta Amabel, and was an introduction/overview to the cobordism hypothesis: what is it, and why should you believe it? For today, we assume all manifolds are smooth, compact, and oriented.

1.1. **Modeling field theories.** The cobordism hypothesis is a statement about field theories. So we should begin by discussing how to model field theories mathematically. There are several ways to do this, but most of them take these key features into account:

Space: Where are we? Where does the experiment take place?
Time: How long does the experiment run for?

In relativity, these are unified into a single concept called *spacetime*. For example, if the theory takes place on a manifold X representing space, and over the time interval $[0,1]$, then spacetime is $X \times [0,1]$, though one can (and we will) consider example spacetimes which aren't products.

Fields: We won't describe the general idea of fields here, but these provide information in your theory and are associated to open subsets U inside spacetime. For example, there's a field theory called the *particle-in-a-box*. In this theory, space is X and time is $[0,1]$, and the fields on an open $U \subset X \times [0,1]$ are the maps from U into the "box," thought of as paths the particle can take.

Rules: Differential equations governing what paths are allowed. For example, in a theory called the *free massless theory*, paths must be straight lines. Often these are wrapped up into something called the *equations of motion* of the theory, such as the *Euler-Lagrange equations*.

Observables: These are the measurements you can make, such as the length of a math. In the Euler-Lagrange formalism, the observables on an open subset U, these are maps from the space of solutions to the Euler-Lagrange equations to $\mathbb{R}$.[1]

Correlation functions: These are statistical measurements that, in experimental physics, are what we actually want to compare to real-world experiments. Out of all of these, we will be most interested in something called the *partition function*.

We will work with a specific model of field theory, which is Atiyah's definition — but only of *topological* field theories. We will say what all of the above notions mean, mathematically, in Atiyah's model of TFT, but first we need some definitions.

Definition 1.1. Let $n \in \mathbb{N}$, and let $\mathscr{C}ob(n)$ denote the symmetric monoidal category given by the following data.

Objects: Closed, oriented, $(n-1)$-manifolds.

[1] This is for the classical theory; in quantum field theories this is not always true.

Morphisms: A morphism $M_1 \to M_2$ is a bordism X from M_1 to M_2, i.e. a compact, oriented n-manifold X and an equivalence class of identifications $\partial X \cong M_1 \amalg -M_2$, modulo diffeomorphisms of X that preserve the boundary. Here $-M_2$ denotes M_2 with the opposite orientation.

Composition: To compose, glue bordisms. To set this up precisely, one needs to specify collar neighborhoods of M_1 and M_2 within X, but there is a way to make this work.

Symmetric monoidal structure: The tensor product is disjoint union, and the unit is the empty set, which is a manifold of every nonnegative-integer dimension. One should specify the associator, etc., but we're not going to delve into these details right now.

Atiyah came up with this definition, building on Segal's definition of a conformal field theory.

Let $\mathscr{V}ect_k$ denote the category of vector spaces over a field k, with the symmetric monoidal structure given by tensor product.

Definition 1.2. A *topological field theory* (TFT), sometimes also *topological quantum field theory* (TQFT), is a symmetric monoidal functor $Z\colon \mathscr{C}ob(n) \to \mathscr{V}ect_k$.

So, for example, the empty set maps to k, and gluing bordisms maps to composition of linear maps.

Now let's revisit the key concepts in field theory.

Space: All objects (i.e. closed $(n-1)$-manifolds) are thought of as spaces. That is, we study this theory for all possible spaces at once!

Time: $[0,1]$.

Spacetime: All compact n-manifolds, possibly with boundary, are thought of as spacetimes. We're working with this theory for all spacetimes at the same time, which is a bit of a perspective shift from what we did before.

Observables: If the TFT is denoted Z, observables are the vector space $Z(S^{n-1})$.

We'll return to fields and equations of motion later.

The identity morphism in $\mathscr{C}ob(n)$ is the cylinder $M \times [0,1]$ (with the correct gluing data), and as Z is a functor, $Z(M \times [0,1]) = \mathrm{id}_M$. But we can do more with these bordisms: regard both M and $-M$ as incoming and $\varnothing$ as outgoing, which results in something macaroni-looking. When you hit this with Z, you get a map

$$e\colon Z(M) \otimes Z(-M) \longrightarrow k. \tag{1.3}$$

Conversely, regarding both M and $-M$ as outgoing, we get a map

$$c\colon k \longrightarrow Z(M) \otimes Z(-M). \tag{1.4}$$

Lemma 1.5 (Zorro's lemma). *e is a perfect pairing.*

This is a fun exercise to do, playing with bordisms and c and e.

1.2. **Classifying TFTs.**

A mathematician encounters a concept, and wants to classify the possible examples. This is hard and scary in general, as far as we know right now, so let's start with a pretty simple case.

Example 1.6 ($n = 1$). Objects of $\mathscr{C}ob(1)$ are finite oriented sets, i.e. finite sets with each element labeled with $+$ or $-$. Symmetric monoidality implies that if Z is a one-dimensional TFT, the value of Z on objects is determined by its values on pt_+ and π_-.

Let $V := Z(\mathrm{pt}_+)$. Then, $Z(\mathrm{pt}_-) = V^\vee$, which is ultimately because of Lemma 1.5. $Z(\mathrm{pt}_+ \amalg \pi_-) = V \otimes V^\vee = \mathrm{End}(V)$, and in general a disjoint union of n copies of pt_+ and m copies of pt_- is sent to $V^{\otimes n} \otimes (V^\vee)^{\otimes m}$.

Now what about morphisms? We know the cylinders (well, line segments) go to identity maps. The macaroni bordism $\mathrm{pt}_+ \amalg \mathrm{pt}_- \to \varnothing$ is mapped to $e\colon V \otimes V^\vee \to k$, which can be identified with the evaluation map that takes a covector ℓ and a vector v and returns $\ell(v)$. Under the identification $V \otimes V^\vee \to \mathrm{End}(V)$, this map is taken to the trace map $\mathrm{End}(V) \to k$. The opposite-direction macaroni is sent to the adjoint of this map.

All bordisms in this dimension are made of disjoint unions of these two macaronis and also circles. To determine $Z(S^1)\colon k \to k$, we factor the bordism $S^1\colon \varnothing \to \varnothing$ into two macaronis. This computes $\mathrm{tr}(\mathrm{id}_V) = \dim V$. In particular, V must be finite-dimensional; all such V determine TFTs, and V determines the TFT completely. ◁

Remark 1.7. The observables of the 1d TFT sending $\mathrm{pt}_+ \mapsto V$ are $Z(S^0) = \mathrm{End}(V)$. This is an associative algebra, and that's not a coincidence — often, the space of observables is an algebra of some sort. As homotopy theorists, we'll be interested in working with ∞-categories eventually, and the algebraic structures we'll get on observables will be quite interesting. ◁

Example 1.8 ($n = 2$)**.** Let $Z : \mathscr{C}ob(2) \to \mathscr{V}ect_k$ be a TFT. Objects are closed 1-manifolds, which are all isomorphic to finite disjoint unions of S^1. Morphisms are compact, oriented, 2-manifolds with boundary. When you draw a complicated one, you can factor it as a composition and/or disjoint union of simpler bordisms, including discs with S^1 viewed as incoming or outgoing, and pairs of pants regarded as incoming or outgoing. (And cylinders, but those are identity morphisms, so not as difficult.)

The disc with S^1 incoming is often called a *cap*, and with S^1 incoming is often called a *cup*.

As S^1 has an orientation-reversing diffeomorphism, we do not need to keep track of the difference between S^1 and $-S^1$. The pair-of-pants therefore defines a multiplication-like structure on $Z(S^1)$, as a map $Z(S^1) \otimes Z(S^1) \to Z(S^1)$.

In fact, one can show this extends to a commutative algebra structure on $A := Z(S^1)$: associativity and commutativity come from finding equivalent bordisms representing, e.g., $m(x_1, x_2)$ and $m(x_2, x_1)$. Moreover, the pair-of-pants composed with the cap is the macaroni bordism for S^1, and we already know it's a perfect pairing. So we get a *counit* map $\mathrm{tr} : A \to k$. Moreover, we have a unit $e : k \to A$, which sends $1 \mapsto 1_A$, the unit element in A.

Let's give this structure a name.

Definition 1.9. A *commutative Frobenius algebra* is a finite-dimensional commutative k-algebra A with a linear map $\mathrm{tr} : A \to k$ such that $a, b \mapsto \mathrm{tr}(ab)$ is a perfect pairing.

Theorem 1.10. *The map sending $Z \mapsto Z(S^1)$ is an equivalence of categories between $\mathscr{C}ob(2)$ and the category of commutative Frobenius algebras.*

This was a folklore theorem for a bit; one reference is Robbert Dijkgraaf's thesis; another is Joachim Kock's book on Frobenius algebras and TFTs.

The observables are, once again, $Z(S^1) = A$, which has an algebra structure. It's worth thinking about what Z assigns to the pants with i legs. ◂

In higher dimensions, there's way too many things to work with: $\mathscr{C}ob(3)$ has infinitely many isomorphism classes of connected objects! So in a sense it's not finitely generated. It would be nice if there were a way to simplify this, by using the fact that all closed, connected, oriented 2-manifolds are diffeomorphic to connect-sums of T^2, and to consider TFTs that "understand" this somehow. And maybe the decompositions we did of surfaces in terms of pants, cups, and caps could apply in this case. But $\mathscr{C}ob(3)$ as we defined it doesn't know how to cut in lower dimensions — it doesn't even know S^1 exists.

In general, we want to be able to cut up our manifold into simpler manifolds in a way that includes all dimensions down to 0. Why do we want this? One compelling reason is that otherwise this classification question is pretty much unapproachable, and the TFTs we get are still interesting.

The solution: higher categories! There is a higher-categorical version of $\mathscr{C}ob(n)$ which takes this desideratum into account. But: defining higher categories is hard. Defining a higher-categorical version $\mathscr{C}ob_n(n)$ of $\mathscr{C}ob(n)$, even given a nice formalism of higher categories, is still hard. We'll spend the next few lectures building these tools that we need to consider this kind of TFT. Once we do, though, we can make the following definition.

Definition 1.11. An *extended TFT* of dimension n valued in a symmetric monoidal n-category $\mathscr{C}$ is a symmetric monoidal functor between n-categories $Z : \mathscr{C}ob_n(n) \to \mathscr{C}$.

With this definition in hand (…eventually), we might expect that there's an equivalence of (higher) categories of extended TFTs and $\mathscr{C}$. This is wrong in two different ways: first, we need to restrict to small enough objects in $\mathscr{C}$, called "fully dualizable" ones, akin to using only finite-dimensional vector spaces in Example 1.6. Second, in dimension $n = 1$, framed is the same thing as oriented, and we miss something important: in general asking for a descent to oriented bordisms is extra data. But when we take these into account, we get:

Theorem 1.12 (Baez-Dolan cobordism hypothesis (Hopkins-Lurie, Lurie))**.** *There is an equivalence of n-categories $\mathscr{F}un^{\otimes}(\mathscr{C}ob_n^{fr}(n), \mathscr{C}) \xrightarrow{\simeq} C^{fd})$.*

Though Lurie provided a detailed sketch of a proof, there are more complete proofs available in special cases, e.g. in Schommer-Pries' thesis for $n \leq 2$, and a nearly complete, very different approach by Ayala-Francis.

References