KÄHLER GEOMETRY

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Note: I missed the first two lectures.

3. Holomorphic line bundles: 7/12/17

Today's going to be about holomorphic vector bundles, with a focus on holomorphic line bundles.

Definition 3.1. Let X be a complex manifold. A **holomorphic vector bundle** of rank k over E is a complex manifold E and a holomorphic map $\pi\colon E \to X$ such that

- π makes $E \to X$ into a complex vector bundle of rank k, and
- E admits **holomorphic trivializations**, i.e. there's an open cover $\mathfrak{U}$ of X trivializing E such that for each $U \in \mathfrak{U}$, there's a biholomorphic map $\varphi\colon E|_U \to U \times \mathbb{C}^k$ commuting with projection to U that is complex linear on each fiber.

A rank-1 holomorphic vector bundle is called a **holomorphic line bundle**.

Equivalently, $E \to X$ is holomorphic iff admits local holomorphic sections.

Definition 3.2. A **homomorphism** of holomorphic vector bundles $f\colon E \to F$ over X is a homomorphism of complex vector bundles that is holomorphic as a map between complex manifolds.

In particular, it must commute with the projection down to X and be complex linear on each fiber. If in addition it's invertible on each fiber, f is called an **isomorphism**.

Exercise 3.3. Show that if $f\colon E \to F$ is an isomorphism of holomorphic vector bundles, it's a biholomorphism on their total spaces.

Remark. Some authors, such as Huybrechts, add an extra condition, that the dimension of the rank of a homomorphism of vector bundles is constant, thus ensuring the (fiberwise) kernel and cokernel of a morphism are again holomorphic vector bundles. Other authors, such as Griffiths-Harris, do not require this, and we'll follow that convention. ◀

In the kyperkähler geometry minicourse, we saw a different definition of holomorphic vector bundles in terms of the $\overline{\partial}_E$ operator $\overline{\partial}_E\colon C^\infty(X, E) \to C^\infty(X, T^{0,1}X \otimes E)$. This is equivalent, and one way to understand this is to use a local trivialization: given a holomorphic identification $E|_U \cong U \times \mathbb{C}^k$ (for an open $U \subset X$) and a section $\psi\colon U \to \mathbb{C}^k$, define

$$\overline{\partial}_E(\psi) := \overline{\partial}\psi = \frac{\partial\psi}{\partial\overline{z}^\alpha}\,\mathrm{d}\overline{z}^\alpha.$$

Then, check that this glues on overlaps, producing a well-defined operator on smooth sections of E.

Another way to understand holomorphic vector bundles is through transition functions.

Proposition 3.4. *There is a bijective correspondenc between the set of isomorphism classes of rank-k holomorphic vector bundles on X and the set of open covers $\mathfrak{U}$ on X and holomorphic functions $\varphi_{ab}\colon U_a \cap U_b \to \mathrm{GL}_k(\mathbb{C})$ for all $U_a, U_b \in \mathfrak{U}$ such that $\varphi_{ab}\varphi_{bc} = \varphi_{ac}$ and $\varphi_{aa} = \mathrm{id}$, modulo equivalence on a common refinement of the open cover.*

Proof sketch. Given a vector bundle E, let $\mathfrak{U}$ be an open cover for which E has holomorphic local trivializations. For $U_a, U_b \in \mathfrak{U}$ that intersect, $\varphi_{ab} \colon U_a \cap U_b \to \mathrm{GL}_n(\mathbb{C})$ is the transition function

$$\begin{tikzcd}
 & (U_a \cap U_b) \times \mathbb{C}^k \arrow[dd, "\varphi_{ab}"] \\
E|_{U_a \cap U_b} \arrow[ur, "f"] \arrow[dr, "g"'] & \\
 & (U - a \cap U_b) \times \mathbb{C}^k.
\end{tikzcd}$$

Here f is the transition function for U_a and g is the transition function for U_b.

Conversely, given the data $\mathfrak{U}$ and $\{\varphi_{ab}\}$, one can define

$$E := \coprod_{U_a \in \mathfrak{U}} U_a \times \mathbb{C}^k / (x, v) \simeq (x, \varphi_{ab} v),$$

where $x \in U_a \cap U_b$ and $v \in \mathbb{C}^k$, over all pairs $U_a, U_b \in \mathfrak{U}$. Then one must check that equivalent data defines isomophic line bundles. ⊠

For $k = 1$, this proposition identifies the set of isomorphism classes of line bundles with the first Čech cohomology $\check{H}^1(X; \mathscr{O}_X^*)$, i.e. valued in the sheaf $\mathscr{O}_X^*$ of holomorphic functions into $\mathbb{C}^\times$. This is because $\mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$.

Pretty much every natural operation you can do to vector spaces extends to holomorphic vector bundles $E, F \to X$, including

- the dual $E^* \to X$,
- the direct sum $E \oplus F \to X$,
- the tensor product $E \otimes F \to X$,
- the wedge product $\Lambda^r E \to X$,
- the pullback $f^* E \to Y$ given a holomorphic map $f \colon Y \to X$,
- and so on.

One way to prove this is to write down their transition functions: suppose $\mathfrak{U}$ is an open cover of X which holomorphically trivializes both E and F (by taking common refinements, such a cover always exists), and suppose φ_{ab} are the transition functions for $\mathfrak{U}$ for E, and ψ_{ab} are those for F. Then,

- E^* has transition functions $(\varphi_{ab}^{\mathrm{T}})^{-1}$,
- $E \oplus F$ has transition functions
$$\begin{pmatrix} \varphi_{ab} & 0 \\ 0 & \psi_{ab} \end{pmatrix},$$
- $E \otimes F$ has transition functions $\varphi_{ab} \otimes \psi_{ab}$, and
- $\Lambda^r E$ has transisiton functions $\Lambda^r \varphi_{ab}$. In particular, if $r = k = \mathrm{rank}(E)$, then $\Lambda^k \varphi_{ab} = \det(\varphi_{ab})$.
- Given a holomorphic map $f \colon Y \to X$, $f^* E$ has transition functions $\varphi_{ab} \circ f$. Hence holomorphicity of f is necessary. This uses the trivializing open cover $f^{-1}(\mathfrak{U})$.

Remark. The set of isomorphism classes of holomorphic line bundles is a group under $\otimes$, called the **Picard group** $\mathrm{Pic}(X)$. The identity is the **trivial bundle** $\underline{\mathbb{C}} := X \times \mathbb{C}$, and the inverse of a line bundle $\mathcal{L}$ is $\mathcal{L}^*$, because $\mathcal{L} \otimes \mathcal{L}^* = \mathrm{End}(\mathcal{L})$, which has a global nonvanishing section that's the identity on each fiber. Hence $\mathcal{L} \otimes \mathcal{L}^* \cong \underline{\mathbb{C}}$. ◀

Example 3.5. Let X be a complex manifold. Then, the holomorphic tangent bundle $T^{1,0}X$ and the holomorphic cotangent bundle $T^{*1,0}X$ are holomorphic vector bundles. Hence, since the wedge product of holomorphic vector bundles is holomorphic, the canonical bundle $K_X := \Lambda^{n,0} T^* X = \Lambda^n (T^{*1,0} X)$ is a holomorphic line bundle. ◀

Proof. Let (z^α) be holomorphic coordinates on (an open neighborhood of a given point in) X. This defines a local trivialization of $T^{1,0}X$, namely

$$\left(z^1, \dots, z^n, \frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n} \right).$$

If (w^β) is another set of holomorphic coordinates, the transition functions are

$$\frac{\partial}{\partial z^\alpha} = \sum_\beta \frac{\partial w^\beta}{\partial z^\alpha}\frac{\partial}{\partial w^\beta}.$$

This is the Jacobi matrix $\left(\frac{\partial w^\beta}{\partial z^\alpha}\right)$, which is holomorphic. $T^{*1,0}X$ is similar. $\boxtimes$

However, $\Lambda^{p,q}T^*X$ is *not* a holomorphic vector bundle in general! For example, the transition functions on $T^{*0,1}X$ are antiholomorphic rather than holomorphic.

Example 3.6. The **tautological bundle** on $\mathbb{CP}^n$ is

$$\mathscr{O}_{\mathbb{P}^n}(-1) := \{(\ell, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} \mid v \in \ell\},$$

i.e., a point $\ell \in \mathbb{CP}^n$ is a line in $\mathbb{C}^{n+1}$, hence we can say the fiber over ℓ is ℓ regarded as a line. This is a holomorphic line bundle. The total space looks like $\mathbb{C}^{n+1}$ with a $\mathbb{CP}^n$ "glued in" at the origin; this is the local model of a blowup.

We can describe the local trivializations explicitly. Let $U_0 = \{z_0 \neq 0\} \subset \mathbb{CP}^n$. Then, the map $U_0 \times \mathbb{C} \to \mathscr{O}_{\mathbb{P}^n}(-1)|_{U_0}$ sends

$$([z_0 : \dots : z_n], \lambda) \longmapsto \left([z_0 : \dots : z_n], \lambda \cdot \left(1, \frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}\right)\right),$$

and you can check that the transition functions for $U_0 \cap U_1 \to \mathbb{C}^\times$ (where U_1 is the locus where $z_1 \neq 0$) is the map $[z_0 : \dots : z_n] \mapsto z_1/z_0$, which is biholomorphic (and hence this actually is a holomorphic line bundle). $\blacktriangleleft$

Definition 3.7. Using the tautological bundle, we can define a bunch of other line bundles on $\mathbb{CP}^n$:

- Let $\mathscr{O}_{\mathbb{P}^n}(0) := \underline{\mathbb{C}}$, the trivial bundle.
- Let $\mathscr{O}_{\mathbb{P}^n}(1) := \mathscr{O}_{\mathbb{P}^n}(-1)$.
- If $k > 0$, let $\mathscr{O}_{\mathbb{P}^n}(k) := \mathscr{O}_{\mathbb{P}^n}(1)^{\otimes k}$ and $\mathscr{O}_{\mathbb{P}^n}(-k) = \mathscr{O}_{\mathbb{P}^n}(-1)^{\otimes k}$.

Hence $k \mapsto \mathscr{O}_{\mathbb{P}^n}(k)$ defines a group homomorphism $\Phi\colon \mathbb{Z} \to \mathrm{Pic}(\mathbb{CP}^n)$.

Theorem 3.8. *In fact,* $\Phi\colon \mathbb{Z} \to \mathrm{Pic}(\mathbb{CP}^n)$ *is an isomorphism.*

We won't prove this. It's nontrivial: for complex line bundles, you can use $H^2(\mathbb{CP}^n) \cong \mathbb{Z}$, but then you have to prove that each has a unique holomorphic structure.

Proposition 3.9. *For* $k > 0$*, the space of holomorphic sections of* $\mathscr{O}_{\mathbb{P}^n}(k)$ *is isomorphic to the space of degree-*k k*-homogeneous polynomials in* $n+1$ *variables.*

Proof sketch. Suppose we're given such a homogeneous polynomial $P(z_1, \dots, z_n)$. On the trivialization $U_0 \times \mathbb{C}$, define a section by

$$[z_0 : \dots : z_n] \longmapsto P(1, z_1/z_0, \dots, z_n/z_0) \in \mathbb{C},$$

which is holomorphic. It hence suffices to check that these local sections transform correctly according to the transition functions. On, for example, $U_0 \cap U_1$, we have that

$$P\left(1, \frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}\right) = \left(\frac{z_1}{z_0}\right)^k P\left(\frac{z_0}{z_1}, 1, \frac{z_2}{z_1}, \dots, \frac{z_n}{z_1}\right).$$

One can then show that these sections span $\Gamma(\mathbb{CP}^n, \mathscr{O}_{\mathbb{P}^n}(k))$. $\boxtimes$

Proposition 3.10. *The canonical bundle on* $\mathbb{CP}^n$ *is isomorphic to* $\mathscr{O}_{\mathbb{P}^n}(-n-1)$.

The proof idea is again to use the local trivialization U_i to define the local section

$$[1 : z_1 : \dots : z_n] \longmapsto \mathrm{d}z_1 \wedge \dots \wedge \mathrm{d}z_n$$

and compute transition functions.

Last time, we talked about holomorphic vector bundles, and in particular holomorphic line bundles. These are complex vector bundles together with the structure of a complex manifold, rather than just an almost complex manifold, on the total space.

Definition 4.1. A **Hermitian metric** on a holomorphic line bundle $\pi\colon \mathscr{L} \to X$ is a smooth function $h\colon \mathscr{L} \to \mathbb{R}_{\geq 0}$ that is **homogeneous**, in that $h(\lambda v) = |\lambda|^2 h(v)$ for all $v \in \mathscr{L}$ and $\lambda \in \mathbb{C}$.

Remark. The definition of a Hermitian metric on a general holomorphic vector bundle is usually presented differently, in terms of a smoothly varying metric on each fiber. This definition agrees for line bundles. ◄

You can think of h as the norm squared of an element of the fiber.

Homogeneity implies that $-i\partial\overline{\partial}\log h$ is the pullback by π of a smooth one-form $\omega_h \in \Omega^{1,1}(X;\mathbb{C})$, called the **curvature form**.

Definition 4.2. We say $(\mathscr{L}, h)$ is **positive** if $\omega_h(\xi, I_x\xi) > 0$ for all $\xi \in T_{\mathbb{R}}X \setminus 0$.

The idea is that ω_h defines an orientation, and we want this orientation to agree with the orientation canonically induced by the complex structure.

Lemma 4.3. *ω_h is actually a real form, i.e. $\omega_h \in \Omega^{1,1}(X;\mathbb{R})$.*

Proof. This is equivalent to ω_h being fixed by complex conjugation. And indeed,

$$\begin{aligned}\overline{\pi^*\omega_h} &= \overline{-i\partial\overline{\partial}\log h}\\ &= i\overline{\partial}\partial\log h\\ &= -i\partial\overline{\partial}\log h.\end{aligned}$$ ⊠

This makes the positivity criterion more reasonable: $\omega_h(\xi, I_x\xi) \in \mathbb{R}$.

Intuitively, positivity will mean your line bundle has lots of sections.

Example 4.4. Let's look at the tautological bundle $\mathscr{O}_{\mathbb{P}^n}(-1) \to \mathbb{CP}^n$. Recall that the points of $\mathscr{O}_{\mathbb{P}^n}(-1)$ are pairs (ℓ, v), where ℓ is a line through the origin in $\mathbb{C}^{n+1}$ and $v \in \ell$; thus, we can define a Hermitian metric on $\mathscr{O}_{\mathbb{P}^n}(-1)$ by

$$h(\ell, v) := |v|^2 = v \cdot \overline{v}.$$

Over $U_0 \subset \mathbb{CP}^n$, where $\mathscr{O}_{\mathbb{P}^n}(-1)|_{U_0}$ trivializes to $U_0 \times \mathbb{C}$, this is

$$h([z_0 : \ldots : z_n], w) = |w|^2\left(1 + \left|\frac{z_1}{z_0}\right|^2 + \cdots + \left|\frac{z_n}{z_0}\right|^2\right).$$

Hence

$$\pi^*\omega_h = -i\partial\overline{\partial}\log\left(|w|^2\left(1 + \left|\frac{z_1}{z_0}\right|^2 + \cdots + \left|\frac{z_n}{z_0}^2\right|\right)\right).$$

Homogeneity allows us to simplify this to

$$\begin{aligned}\omega_h &= -i\partial\overline{\partial}\log\left(1 + \left|\frac{z_1}{z_0}\right|^2 + \cdots + \left|\frac{z_n}{z_0}\right|^2\right)\\ &= -i\partial\overline{\partial}\log\left(|z_0|^2 + \cdots + |z_n|^2\right).\end{aligned}$$

This is a description in terms of homogeneous coordinates, and therefore makes sense on all of $\mathbb{CP}^n$, not just U_0.

When $n = 1$ (i.e. on the Riemann sphere), then on U_0,

$$\begin{aligned}\omega_h &= -i\partial\overline{\partial}\log\left(1 + |z|^2\right)\\ &= \frac{-i}{(1 + |z|^2)^2}\,\mathrm{d}z \wedge \mathrm{d}\overline{z},\end{aligned}$$

so if $z = x + iy$,

$$= \frac{-2}{(1+|z|^2)^2}\,\mathrm{d}x \wedge \mathrm{d}y,$$

so $\mathscr{O}_{\mathbb{P}^1}(-1)$ is negative! ◀

Exercise 4.5. Show that if $k > 0$, then $\mathscr{O}_{\mathbb{P}^n}(k)$ is positive. (Hint: think about how the metric on $\mathscr{O}_{\mathbb{P}^n}(-1)$ induces metric on duals and tensor products, and how the curvature form changes.)

Proposition 4.6. *Let $f\colon Y \to X$ be a holomorphic immersion between complex manifolds and $\mathscr{L} \to X$ be a holomorphic line bundle with positive metric h. Then, $f^*\mathscr{L} \to Y$, with the pullback metric, is also positive.*

Proof. The pullback metric h' on $f^*\mathscr{L}$ is defined as

$$h'(v) := h(\widetilde{f}(v)),$$

where $\widetilde{f}\colon f^*\mathscr{L} \to \mathscr{L}$ is the map on total spaces coming from the pullback. That is, there's a commutative diagram

$$\begin{tikzcd}
f^*\mathscr{L} \arrow[r, "\widetilde{f}"] \arrow[d, "\pi_{f^*}"] & \mathscr{L} \arrow[d, "\pi"] \\
Y \arrow[r, "f"] & X.
\end{tikzcd}$$

Now we compute.

$$\begin{aligned}
\pi_{f^*}^*\omega_{h'} &= -i\partial\overline{\partial}\log h' = -i\partial\overline{\partial}\log h \circ \widetilde{f} \\
&= \widetilde{f}^*(-i\partial\overline{\partial}\log h) \\
&= \widetilde{f}^*(\pi^*\omega_h) \\
&= \pi_{f^*}^* f^*\omega_h,
\end{aligned}$$

so $f^*\omega_h = \omega_h$. Hence, for all $y \in Y$ and $\xi \in T_yY \setminus 0$,

$$\begin{aligned}
f^*\omega_h(\xi, I_y\xi) &= \omega_h(f_*\xi, f_*I_y\xi) \\
&= \omega_h(f_*\xi, I_x f_*\xi)
\end{aligned}$$

because f is holomorphic. Since f is an immersion, this is positive. ☒

Remark. There's a connection[1] to Chern-Weil theory here: ω_h is a closed 1-form, and hence defines a class $[\omega_h] \in H^2(X)$. In fact, $(1/2\pi)[\omega_h] = c_1(\mathscr{L})$, the first Chern class of the line bundle. Positivity of this Chern class is a necessary condition for positivity of $\mathscr{L}$. ◀

Now we can discuss a major result, the Kodaira embedding theorem.

Definition 4.7. A complex manifold X is **projective** if it admits a holomorphic embedding into some $\mathbb{CP}^n$.

Theorem 4.8 (Kodaira embedding theorem). *Let X be a closed complex manifold. Then, X is projective iff it admits a positive line bundle.*

We'll prove the hard direction tomorrow. But the forward direction is easy: given a holomorphic embedding $f\colon X \hookrightarrow \mathbb{CP}^n$, then by Exercise 4.5, $\mathscr{O}_{\mathbb{P}^n}(1)$ is positive, and by Proposition 4.6, $f^*\mathscr{O}_{\mathbb{P}^n}(1) \to X$ is also positive.

So far we've only used that X can be immersed in $\mathbb{CP}^n$. In fact, the proof of the other direction will show that any X that immerses in $\mathbb{CP}^n$ embeds in $\mathbb{CP}^N$ for some N (which might not be n), which is independently interesting.

For the rest of today's lecture, we'll lay the groundwork for the other direction in the proof.

Definition 4.9. Let $\mathscr{L} \to X$ be a holomorphic line bundle and $\Gamma(X, \mathscr{L})$ denote the space of holomorphic sections of $\mathscr{L}$.

- A **linear system** is a complex subspace $W \subset \Gamma(X, \mathscr{L})$.
- A linear system W is **complete** if $W = \Gamma(X, \mathscr{L})$, and is a **pencil** if it has complex dimension 2.

[1] No pun intended.

Pictorially, it's helpful to think of an $s \in W$ in terms of its zero locus $s^{-1}(0)$. A pencil is spanned by two sections s_1 and s_2, so we obtain two (generically) codimension 2 submanifolds $s_1^{-1}(0)$ and $s_2^{-1}(0)$ in X, and the pencil is the span of these, which we can think about as interpolation between their zero sets. Drawing this out with two curves is nice.

Definition 4.10. Given a linear system W, its **base** or **indeterminacy locus** is

$$\mathrm{Bs}(W) := \bigcap_{s \in W} s^{-1}(0).$$

Definition 4.11. Let W be a linear system on X. The **Kodaira map** for W is $\varphi_W \colon X \setminus \mathrm{Bs}(W) \to \mathbb{P}(W^*)$ sending

$$x \longmapsto [s \longmapsto s(x)].$$

The reason we had to remove the base locus is that it maps to 0, hence doesn't make sense after projectivization.

Definition 4.12.

- A linear system W is **very ample** if its base locus is empty and $\varphi_W \colon W \to \mathbb{P}(W^*)$ is an embedding.
- A line bundle $\mathscr{L} \to X$ is **very ample** if $\Gamma(X, \mathscr{L})$ is very ample.
- $\mathscr{L}$ is **ample** if $\mathscr{L}^{\otimes k}$ is very ample for some k. (We think of this k as large, though it isn't always.)

Hence, Theorem 4.8 says that $\mathscr{L}$ is positive iff it's ample.

Very ampleness says there are lots of holomorphic sections, so much that the base locus vanishes. Ampleness says there are plenty of sections, but not quite as many. The Kodaira embedding theorem says that this is the same as positivity, so positivity means lots of sections.

Example 4.13. Let $k > 0$ and consider $\mathscr{O}_{\mathbb{P}^n}(k) \to \mathbb{CP}^n$. In Proposition 3.9, we saw that $\Gamma(\mathscr{O}_{\mathbb{P}^n}(k), \mathbb{CP}^n)$ can be identified with the space V of k-homogeneous polynomials in $n+1$ variables, which has complex dimension $\binom{n+k}{n}$. Thus we get a Kodaira map $\mathbb{CP}^n \to \mathbb{CP}^{\binom{n+k}{n}-1}$, which explicitly sends

$$z_0 : \ldots : z_n] \longmapsto [z_0^k : z_1^k : \ldots : z_0^{k-1} z_1 : z_0^{k-1} z_2 : \ldots].$$

This is an embedding, and a well-known one, called the **Veronese embedding**. For example, as an embedding $\mathbb{CP}^2 \hookrightarrow \mathbb{CP}^5$, it sends

$$[x : y : z] \longmapsto [x^2 : y^2 : z^2 : xy : xz : yz].$$

◀

Example 4.14. Consider $\mathscr{L} = (\pi_1^* \mathscr{O}_{\mathbb{P}^n}(1)) \otimes (\pi_2^* \mathscr{O}_{\mathbb{P}^m}(1))$ on $\mathbb{CP}^n \times \mathbb{CP}^m$. You can check this is very ample, hence defines an embedding of $\mathbb{CP}^n \times \mathbb{CP}^m \hookrightarrow \mathbb{CP}^{nm-1}$ sending $[z_0 : \ldots : z_n], [w_0 : \ldots : w_m]$ to the matrix $a_{ij} = z_i w_j$. This is called the **Segre embedding**, and provides a proof that a product of projective varieties is projective. ◀