GEOMETRY AND STRING THEORY SEMINAR: SPRING 2018

ARUN DEBRAY
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Contents

1. Comments on Global Symmetry, Anomalies, and Duality in $(2+1)d$: 1/24/18

Today's talk was given by Val Zakharevich, on the paper [BHS17].

Definition 1.1. Let A and B be UV theories which, under renormalization group flow, flow to the same IR theory C. Then we'll say that theories A and B are dual.

Example 1.2. Let $N_f \le N$. Then there is a conjectured duality between $\mathrm{SU}(2)_k$-Chern-Simons theory with N_f scalars, also known as *Wilson-Fischer theory*, and $\mathrm{U}(k)_{-N+N_f/2}$-Chern-Simons theory with N_f fermions. ◀

The paper [BHS17] computes the higher symmetries and anomalies of both sides of this duality and of several others; 't Hooft anomaly matching tells us that these should be the same.

This is related to our overarching goal of understanding QCD_4 with a single fermion ψ, which has a Lagrangian

$$\mathcal{L} = \mathrm{tr}(F \wedge \star F) + \overline{\psi} \not{D} \psi + m \overline{\psi} \psi, \tag{1.3}$$

where $m \in \mathbb{C}$ is a parameter whose phase diagram we're interested in. Let $\overline{m}$ denote the mass of the domain wall theory, which is a 3D QCD theory, so $\overline{m}$ is real. If m is real and negative, there's a phase transition: for $\overline{m} \ll 0$, the low energy theory is believed to be trivial, and for $\overline{m} \gg 0$ (m negative of larger magnitude), the low-energy theory is believed to be $\mathrm{SU}(N)$-Chern-Simons theory at level 1. The transition point, at $\overline{m} = 0$, should be described by $\mathrm{SU}(N)_{1/2}$ with a single fermion.

1.1. Level-rank duality.

Level-rank duality is the conjecture that $\mathrm{SU}(N)_k$-Chern-Simons theory and $\mathrm{U}(k)_{-N}$-Chern-Simons theory are isomorphic. A natural generalization is to consider $\mathrm{SU}(N)_k$ together with N_f scalar fields of mass m, where $m \in \mathbb{R}$ and $N_f < N$.

- If $m \ll 0$, the Higgs mechanism implies this should be the $\mathrm{SU}(N - N_f)_k$ theory.
- if $m \gg 0$, we should expect the $\mathrm{SU}(N)_k$ theory again.

On the dual side, let's consider $\mathrm{U}(k)_{-N-N_f/2}$ with N_f fermions of mass $m \in \mathbb{R}$.

- If $m \gg 0$, we expect to get $\mathrm{U}(k)_{-N+N_f/2}$.
- If $m \ll 0$, we expect to get $\mathrm{U}(k)_{-N}$, with Lagrangian shifted by N_f:

$$\mathcal{L} = \frac{-N+N_f}{4\pi}\,\mathrm{tr}\left(b\,\mathrm{d}b - \frac{2i}{3}b^3\right) + \overline{\psi}\not{D}\psi + m\psi\overline{\psi} + \mathrm{c.c.} \tag{1.4}$$

Level-rank duality switches positive-mass scalars and negative-mass fermions, promising dualities between $\mathrm{SU}(N-N_f)_k \longleftrightarrow \mathrm{U}(k)_{-N+N_f/2}$ and $\mathrm{SU}(N)_k \longleftrightarrow \mathrm{U}(k)_{-N}$.

1.2. **Symmetries.** We now see what symmetries these theories have. First, $\mathrm{SU}(N)_k$ with N_f scalars. On a 3-manifold M, the fields are triples (P, A, Φ), where

- $P \to M$ is a principal $\mathrm{SU}(N)$-bundle with connection,
- A is a connection on P, and
- $\varphi \in \Gamma(P \times_{\mathrm{SU}(N)} (\mathbb{C}^N \otimes \mathbb{C}^{N_f}))$ is the N scalar fields.

The Lagrangian is

$$\mathcal{L}(A, \Phi) = \frac{k}{4\pi} \operatorname{tr}\left(A \wedge \mathrm{d}A + \frac{2}{3} A^3 \right) + |D_A \varphi|^2 + m|\varphi|^2 + \lambda |\varphi|^4. \tag{1.5}$$

As usual, we have an $\mathrm{SU}(N)$-gauge symmetry, and there's also a $\mathrm{U}(N_f)$-symmetry acting on $\mathbb{C}^{N_f}$, in which $e^{2\pi i/N}\mathbf{1}$ acts by a gauge symmetry. Hence the global symmetry group (for these symmetries) is $\mathrm{U}(N_f)/(\mathbb{Z}/N)$.

Ansatz 1.6. Let G be a compact Lie group and k be a level for G, and let $\mathcal{L}_{G_k}$ denote the Lagrangian for Chern-Simons theory with group G and level k. Let

$$1 \longrightarrow G \xrightarrow{\ \rho\ } H \xrightarrow{\ \sigma\ } L \longrightarrow 1 \tag{1.7}$$

be a short exact sequence of Lie groups. Then, we take as an ansatz that coupling the G_k theory to a principal L-bundle (i.e. given a principal L-bundle $P \to M$, we sum over the groupoid of all principal H-bundles which quotient to L) produces a classical gauge theory for H with Lagrangian $\mathcal{L}_{\widetilde{k}}$ such that

$$\mathcal{L}_{G_k}(P_G, A_G) = \mathcal{L}_{\widetilde{k}}((P_G, A_G) \times_G H). \tag{1.8}$$

When G is finite (so we're in the setting of Dijkgraaf-Witten theory) this is studied in [KT14].

In our setting, (1.7) specializes to $G = \mathrm{SU}(N)$, $H = (\mathrm{SU}(N) \times U(N_f))/(\mathbb{Z}/N)$, and $L = U(N_f)/(\mathbb{Z}/N)$. Chern-Simons theories for G are labeled by $H^4(BG; \mathbb{Z})$, and the map $\rho\colon G \to H$ defines a pullback

$$\rho^*\colon H^4(BH; \mathbb{Z}) \longrightarrow H^4(BG; \mathbb{Z}).$$

Given a $k \in H^4(BG; \mathbb{Z})$, we want to know whether we can implement the theory with a global L-symmetry; hence we want to know whether $k \in \operatorname{Im}(\rho^*)$; the theory is anomalous iff this is not true.

If the theory is anomalous, we'd like to compute the anomaly. Suppose that we have a $\widehat{k}_{\mathbb{R}} \in H^4(BH; \mathbb{R})$ such that $\rho^*(\widehat{k}_{\mathbb{R}}) = k_{\mathbb{R}}$ (i.e. the image of k in real cohomology). Then, we can't eliminate the anomaly, but we can couple to a bulk theory: suppose that we can extend $(P, A) \to M$ to $(P_H, A_H) \to X$, where X is a compact 4-manifold with $\partial X = M$. Then, we have an action

$$\text{“}S_{\widehat{k}_{\mathbb{R}}}(A_H)\text{”}\colon ((P_H, A_H) \to X) \longmapsto \int_X \widehat{k}_{\mathbb{R}}(F_H), \tag{1.9}$$

where F_H is the curvature of A_H.

This depends on the choice of X and $P_H \to X$ extending P, but we can hope that the dependence goes away after exponentiating the action. Let X' be another compact 4-manifold bounding M, and let $(P'_H, A') \to X'$ be another extension of (P, A). Let $\widehat{X} \coloneqq X \cup_M X'$; then, (P_H, A_H) and (P'_H, A'_H) glue to a principal G-bundle $\widetilde{P}_H \to \widetilde{X}$ with connection $\widetilde{A}$. Then we have that

$$e^{2\pi i S_{\widehat{k}_{\mathbb{R}}}(\widetilde{P}_H, \widetilde{A}_H)} = S_{\hat{\hat{k}}}(\widetilde{P}_H \times_H L) \in \mathbb{R}/2\pi i\mathbb{Z} \tag{1.10}$$

for some $\hat{\hat{k}} \in H^4(BL; \mathbb{R}/\mathbb{Z})$. This $\hat{\hat{k}}$ tells us the anomaly, so we're interested in computing it. Ultimately, this comes from a question purely in algebraic topology: we have a big commutative diagram

$$\begin{tikzcd}
H^4(BL;\mathbb{R}/\mathbb{Z}) \arrow[r] & H^4(BH;\mathbb{R}/\mathbb{Z}) \arrow[r] & H^4(BG;\mathbb{R}/\mathbb{Z}) \\
& H^4(BH;\mathbb{R}) \arrow[r] \arrow[u] & H^4(BG;\mathbb{R}) \arrow[u] \\
& H^4(BH;\mathbb{Z}) \arrow[r] \arrow[u] & H^4(BG;\mathbb{Z}) \arrow[u]
\end{tikzcd} \tag{1.11}$$

Then we have $\hat{\hat{k}}$ in the upper left, $\widehat{k}_{\mathbb{R}}$ in the middle, and k in the lower right. In this case the anomaly theory is purely topological. The computation for the dual theory follows a similar story, but is harder.

To actually calculate this, you can use the Leray-Serre spectral sequence; $k \in H^4(BG; \mathbb{Z})$ transgresses to something in $H^5(BL; \mathbb{Z})$, which tells you which component $\hat{\hat{k}}$ is in.

References

[BHS17] Francesco Benini, Po-Shen Hsin, and Nathan Seiberg. Comments on global symmetries, anomalies, and duality in $(2+1)d$. *Journal of High Energy Physics*, 2017(4):135, Apr 2017. https://arxiv.org/pdf/1702.07035.pdf. 1

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