STACKS

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These notes were taken in Adrian Clough's minicourse in Summer 2018. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

Contents

Day 1. June 18

I missed the morning and afternoon lectures for this day, which focused on the two main themes (descent and building new spaces from old) through examples.

Day 2. June 19

1. Morning

We'll start with a review of some of yesterday's material.

Let X and R be manifolds, and $g: X \to R$ be a smooth map. Given a submersion $p\colon U \to X$, g lifts to a map $U \to R$. In this case, the pullback $U \times_X U \rightrightarrows U$ is a smooth manifold, and p is exactly the coequalizer of that diagram. This is a very simple instance of descent: we have an isomorphism between the space of smooth maps from X to R with the space of maps from U to R that coequalize $U \times_X U \rightrightarrows U$ (i.e. after composing with such a map, the two structure maps become equal).

More generally, let $\mathfrak{U}$ be an open cover of X by balls and $U := \coprod_{V \in \mathfrak{U}} V$. The story goes through with $U \times_X U \rightrightarrows U \to X$. But descent doesn't always hold: for example, if you try to lift a map $E \to X$ to U. You need additional data.

It's also useful to replace sets with groupoids. For example, if you want to consider descent data for principal G-bundles, then you can't consider the set of equivalence classes: the pullback map $\mathsf{Bun}_G(X) \to \mathsf{Bun}_G(U)$ isn't very interesting, because $\pi_0 \mathsf{Bun}_G(U)$ is trivial. Therefore in order to set up descent, we'll need to (1) consider groupoids, and (2) provide additional data on $\mathfrak{U}$ that makes interesting things possible. In this case the passage from injective to surjective to bijective is replaced with faithful to fully faithful to an equivalence of categories.

Recall that an *epimorphism* in a category C is a map $f: A \twoheadrightarrow B$ such that for any two maps $g, h\colon B \rightrightarrows C$ such that $g \circ f = h \circ f$, then $g = h$. In Set this corresponds to the usual notion of surjective.

Definition 1.1. An epimorphism $f: U \to X$ is an *effective epimorphism* if the pullback $U \times_X U$ exists and f is its coequalizer.

You can think of this as a categorified version of an equivalence relation.

It's also possible to go the other direction: given an equivalence relation $R \to U \times U$ on U and an epimorphism $U \twoheadrightarrow X$, you can ask whether $R \cong U \times_X U$.

In geometric sitations, it seems generally true that one glues by equivalence relations. This seems a little mysterious, but is one motivation for thinking about stacks in this way.

Now we turn to creating new spaces from old. In many situations you're considering a nice class of spaces (schemes, varieties) $\mathscr{C}$, but some reasonable construction starting with stuff in $\mathscr{C}$ leads you to something which does not exist in $\mathscr{C}$. For example, studying manifolds leads to asking about classifying spaces of principal G-bundles, which cannot be finite-dimensional; thus, you have to enlarge your category of spaces to get at them. Another example is fine moduli spaces in algebraic geometry, which often aren't varieties and sometimes aren't schemes.

The general approach is to (1) enlarge to a category $\widetilde{\mathscr{C}}$ of *all* objects that can reasonably be constructed from objects in $\mathscr{C}$ (i.e. by taking colimits and satiafying a descent condition), and then (2) restricting to a subcategory of nicer spaces. Depending on your application, there are different choices of nice subcategories for a given $\widetilde{\mathscr{C}}$.

Example 1.2. Let Cart be the category whose objects are open subsets of $\mathbb{R}^n$ and whose morphisms are smooth maps. Then, inside $\widetilde{\mathsf{Cart}}$, one could restrict to smooth manifolds, or more generally Fréchet manifolds, or more generally Frölicher spaces, or more generally diffeological spaces, depending on your application. ◀

To do this, we need some good candidate for $\widetilde{\mathscr{C}}$. One, which we denote $\widehat{\mathscr{C}}$, is the collection of *presheaves on* $\mathscr{C}$, i.e. contravariant functors $\mathscr{C} \to \mathsf{Set}$. We have the Yoneda embedding $\mathscr{C} \hookrightarrow \widehat{\mathscr{C}}$, and we might want to complete $\widehat{\mathscr{C}}$ under colimits to a map $\mathscr{C} \to \mathscr{D}$; $\mathscr{D}$ exists and is unique by a universal property. However, often coequalizers of the kinds we've been considering are not preserved by this construction. Here are two possible fixes for a category C and a collection τ of coequalizers we want to keep:

(1) Let $U \times_X U \rightrightarrows U \twoheadrightarrow X$ be in τ, and apply the Yoneda embedding to it, producing a diagram

$$\begin{tikzcd} \mathsf{C}(-,U) \times_{\mathsf{C}(-,X)} \mathsf{C}(-,U) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathsf{C}(-,U) \arrow[r] \arrow[dr] & \mathsf{C}(-,X). \\ & & \mathrm{Coeq}(f,g) \arrow[u, "(**)"'] \end{tikzcd} \tag{1.3}$$

The fix will be to localize the morphisms $(**)$ in $\widehat{\mathsf{C}}$, Namely, let W denote the subcategory generated by these morphisms over all diagrams in τ. The localization is denoted $\widehat{\mathsf{C}}[\mathsf{W}]$, and has an obvious universal property: given a functor $F\colon \mathsf{C} \to \mathsf{D}$, such that D is cocomplete and F preserves colimits and all coequalizers in τ, there's a unique functor $\widehat{\mathsf{C}}[\mathsf{W}] \to \mathsf{D}$. This is nice, but localization is terrible (see Remark 1.8), so it might be good to try something else.

(2) The new basic idea is that if $X \in \widetilde{\mathsf{C}}$, then we should be able to recover X from $\widetilde{\mathsf{C}}(-,X)$ restricted to C. Thus we identify X and $\widetilde{\mathsf{C}}(-,X)$; that is, we want $\widetilde{\mathsf{C}} \subseteq \widehat{\mathsf{C}}$.

Now let's think about colimits. Let $F\colon \mathsf{J} \to \mathsf{A}$ be a diagram; then $\mathsf{A}(\varinjlim F, X) \cong \varprojlim \mathsf{A}(F,X)$, or in other words, $\varinjlim X$ represents the functor $\varprojlim \mathsf{A}(F,-)$. This means that $A \rightrightarrows B \to C$ is a coequalizer iff for all $X \in \mathsf{A}$,

$$\mathsf{A}(C,X) \longrightarrow \mathsf{A}(B,X) \rightrightarrows \mathsf{A}(A,X) \tag{1.4}$$

is an *equalizer* diagram (since $\mathsf{A}(-,X)$ is contravariant).

Now assume $P \in \widetilde{\mathsf{C}} \subseteq \widehat{\mathsf{C}}$ and $R \rightrightarrows U \twoheadrightarrow X$ is in τ. Then we can apply $\widehat{\mathsf{C}}(-,P)$ to obtain a diagram

$$\begin{tikzcd} \widehat{\mathsf{C}}(X,P) \arrow[r] \arrow[d, equal] & \widehat{\mathsf{C}}(U,P) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal] & \widehat{\mathsf{C}}(R,P) \arrow[d, equal] \\ P(X) \arrow[r] & P(U) \arrow[r, shift left] \arrow[r, shift right] & P(R). \end{tikzcd} \tag{1.5}$$

Requiring this to be a coequalizer diagram is exactly a sheaf condition! For example, in the category of smooth manifolds, if $\mathfrak{U}$ is an open cover of X and $P\colon \mathsf{Man}^{\mathrm{op}} \to \mathsf{Set}$ is a presheaf, then we can apply it to the diagram

$$\coprod_{U,V \in \mathfrak{U}} U \cap V \rightrightarrows \coprod_{U \in \mathfrak{U}} U \longrightarrow X \tag{1.6}$$

to obtain (assuming $\mathsf{C} \to \widetilde{\mathsf{C}}$ preserves coproducts)

$$\begin{tikzcd} P(X) \arrow[r] \arrow[d, equal] & P\left(\coprod_{U\in\mathfrak{U}} U\right) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal] & P\left(\coprod_{U,V\in\mathfrak{U}} U\cap V\right) \arrow[d, equal] \\ P(X) \arrow[r] & \prod_{U\in\mathfrak{U}} P(U) \arrow[r, shift left] \arrow[r, shift right] & \prod_{U,V\in\mathfrak{U}} P(U\cap V), \end{tikzcd} \tag{1.7}$$

and asking for this to be a coequalizer diagram is the usual notion of a presheaf being a sheaf (e.g. you can find this diagram in Vakil).

Remark 1.8. Localization of categories is technical and messy: for example, it's easy to write down an example where after the naïve notion of localization, there are collections of morphisms between objects that are so big as not to be sets. This is bad. One of the reasons people like model categories is because they avoid this problem. ◂

2. Afternoon

"Geometry is the art of giving funny names to morphisms."

These two different ways of fixing coequalizers turn out to be the same.

Now let's suppose that $\widetilde{\mathsf{C}}$ is the category of sheaves on C with respect to the coequalizers in τ (so the second approach).

Theorem 2.1. *The inclusion $\widetilde{\mathsf{C}} \hookrightarrow \widehat{\mathsf{C}}$ admits a left adjoint $a\colon \widehat{\mathsf{C}} \to \widetilde{\mathsf{C}}$, which is precisely the localization functor from the first approach.*

This implies in particular that the localization exists and is nice (e.g. you don't have to consider arbitrarily long zigzags in it, much like in a model category). This theorem is saying something deep.

Remark 2.2. If you do the ∞-version of this, with simplicial sets and/or ∞-groupoids, this recovers localization of model categories (or ∞-categories), producing simplicial sheaves from simplicial presheaves. ◂

Theorem 2.1 holds for arbitrary τ (probably), but we'll prove it under a reasonable assumption on τ, which is also a motivation to learn about sites.

Sites.

Definition 2.3. Let C be a category.[1] Then a *coverage* on C is data of, for all $X \in \mathsf{C}$, a family $T_X = \{\{\pi_i\colon U_i \to X\}_{i\in I}\}$ of families of maps to X, such that given a map $f: X' \to X$ in C and a family $\{U_i \to X\}_{i\in I}$ in T_X, there is a family $\{U_j' \to X'\}_{j\in J}$ in $T_{X'}$ such that for all $j \in J$, there exists an $i \in I$ and a $g: U_j' \to U_i$ such that the diagram

$$\begin{tikzcd} U_j' \arrow[r, "g"] \arrow[d, "\pi_j'"] & U_i \arrow[d, "\pi_i"] \\ X' \arrow[r, "f"] & X \end{tikzcd} \tag{2.4}$$

commutes. A category together with a coverage is called a *site*.

Before we see some examples, we'll define some related notions.

Definition 2.5. Given a coverage $\mathbf{T}$ of C and a presheaf $\mathscr{F} \in \widehat{\mathsf{C}}$, let $\mathrm{Desc}_{\mathscr{F}}(\{U_i \to X\}_{i\in I})$ be the set of $(s_i) \in \prod_{i\in I} \mathscr{F}(U_i)$ such that for all $i, j \in I$ and diagrams $U_i \xleftarrow{f} V \xrightarrow{g} U_j$ such that $s_i|_V = s_j|_V$, $\mathscr{F}(f)(s_i) = s_i|_V$ and $\mathscr{F}(g)(s_j) = s_j|_V$.

This is a lot like a limit, but it's not clear how to make it exactly one nicely.

Definition 2.6. Given a category C with a coverage $\mathbf{T}$, a $\mathbf{T}$-*sheaf* $\mathscr{F}$ is a presheaf $\mathscr{F}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ such that for all $U \in \mathsf{C}$ and families $\{U_i \to U\}$ in $\mathbf{T}$, the natural map $\mathscr{F}(U) \to \mathrm{Desc}_{\mathscr{F}}(\{U_i \to U\})$ is an isomorphism.

Example 2.7.

[1] We'll assume C is small; this is not necessary for the definition, but will be true for all examples we consider.

(1) In manifolds, we could choose $\mathbf{T}$ to be coverings in the usual way, or jointly[2] surjective submersions (more in spirit of what we did yesterday), or *étale covers*, i.e. jointly surjective local homeomorphisms.
(2) Let C denote the category of open subsets of $\mathbb{R}^n$ (for all n) and smooth maps between them. Then one can once again consider covering maps, surjective submersions, or étale covers.
(3) Let Cart denote the category whose objects are $\mathbb{R}^n$ for each n and whose maps are smooth maps between them. In this case we can take surjective submersions but not open covers, because there are no coproducts.
(4) Let X be a space and C be the category of open subsets on X. Then we can take classical coverings, i.e. $\{U_i \to U\}$ is a covering iff $U \subset \bigcup U_i$. A $\mathbf{T}$-sheaf for this is a sheaf in the usual sense; this is because $\mathrm{Desc}_{\mathscr{F}}(\{U_i \subseteq U\})$ is exactly the collections of sections $\mathscr{F}(U_i)$ that agree on intersections. ◂

Looking into the last example in more detail, if you fix a manifold M, $\mathsf{Top}(M)^{\mathrm{op}} \hookrightarrow \mathsf{Man}^{\mathrm{op}}$ sending $U \mapsto U$, so we get a map from presheaves on Man to presheaves on $\mathsf{Top}(M)$, namely, restrict a presheaf $\mathscr{F}$ to open subsets of M.

Exercise 2.8. Show that a presheaf $\mathscr{F}$ on Man is a $\mathbf{T}$-sheaf (for the three choices of $\mathbf{T}$ we discussed above) iff for all manifolds M, $F|_M$ is a classical sheaf.

In fact, in Man, from any covering we can construct the classical descent situation, and we actually get an equivalence of categories $\widetilde{\mathsf{Man}} \simeq \widetilde{\mathsf{Cart}}$. This is why we want general coverages: in Cart we can't take coproducts, but we can in Man. This ultimately means that the kinds of spaces discussed earlier, namely Fréchet manifolds, Frölicher spaces, and diffeological spaces, can all be described as sheaves on Cart.

In general, suppose C admits pullbacks; then we have a similar story as for $\mathsf{Top}(X)$: descent means that given a map $f : V \to U$ and covers $\{V_i \to V\}$, $\{U_i \to U\}$, we get a diagram

$$\begin{tikzcd}
\coprod_{i,j} V_i \times_V V_j \arrow[r, "f''"] \arrow[d, shift left] \arrow[d, shift right] & \coprod_{i',j'} U_{i'} \times_U U_{j'} \arrow[d, shift left] \arrow[d, shift right] \\
\coprod_j V_j \arrow[r, "f'"] \arrow[d] & \coprod_{j'} U_{j'} \arrow[d] \\
V \arrow[r, "f"] & U.
\end{tikzcd} \tag{2.9}$$

Descent means that we should be able to reconstruct f just knowing f'' and f'. This looks a lot like descent for bundles, which seems important.

This ties to the stuff we saw this morning: in $\widetilde{\mathsf{C}}$, for $\{U_i \to U\}$ in T_U, $\coprod \mathsf{C}(-, U_i) \to \mathsf{C}(-, U)$ becomes an effective epimorphism!

Day 3. **June 20**

3. Morning

When we studied descent, we concluded that the morphisms we want to descend along are gluing morphisms, which are often effective epimorphisms: these are epimorphisms $U \to X$ which are equivalent data (via pullback $U \times_X U \rightrightarrows U$) to equivalence relations.

Then we discussed categories C of geometric objects (what a "geometric object" means depends on the application) for which we want to expand to more general objects, while preserving a collection τ of coequalizers. We suggested two ways to do this, one by localizing $\widehat{\mathsf{C}}$ in a way to get τ back, and another was to take a subcategory of $\widehat{\mathsf{C}}$, and it turns out that these two approaches are equivalent, and the localization is well behaved.

Then we discussed coverages and sheaves; in this case, τ isn't quite effective epimorphisms, but certain covering maps, and we force them to be effective epimorphisms in the presheaf category.

Today, we'll discuss more about Grothendieck topologies. The first part is technical, but it has to be: if you try to read this stuff somewhere else it will still be technical.

Let $\mathbf{T}$ be a coverage on C. Then we can ask two questions about it.

[2]For a family, this means that the map from the disjoint union of all of them is surjective.

(1) Are there any aditional coverings $\mathfrak{U}$ that we get "for free," i.e. for which any $\mathbf{T}$-sheaf satisfies the sheaf condition for $\mathfrak{U}$?
(2) Given a $\{U_i \to U\}$ in T_U, can we add more morphisms to it such that the same presheaves are $\mathbf{T}$-sheaves?

Analogy 3.1. Once again, we'll compare to the localization of rings.

Given a subset S of a ring R, we can form the localization $S^{-1}R$; if S_m denotes the multiplicative closure of S, then $S^{-1}R \cong S_m^{-1}R$; in fact, you can even take the saturated closure S_s of S, and the localization is the same.

The point is: in this setting we can add extra stuff to S and get the same localization. And we obtain a bijection between saturated subsets of R and localizations of R.

Theorem 3.2. *Let C be a category and $\mathsf{S} \subset \widehat{\mathsf{C}}$ be a subcategory. Then, S is the category of $\mathbf{T}$-sheaves for some coverage $\mathbf{T}$ iff inclusion $\mathsf{S} \hookrightarrow \widehat{\mathsf{C}}$ admits a left exact left adjoint.*

So we want a bijection between such subcategories and some kinds of coverages, which are what we'll eventually call Grothendieck topologies.

Definition 3.3. A coverage $\mathbf{T}$ is *maximal* if it is not possible to add new objects or new morphisms to it without changing the category of $\mathbf{T}$-sheaves.

Theorem 3.4. *There is a bijection between the subcategories $\mathsf{S} \subset \widehat{\mathsf{C}}$ whose inclusion admits a left exact left adjoint and maximal coverages.*

We'll prove this with a sequence of lemmas. The first is not terribly surprising.

Lemma 3.5. *Let C be a category with a coverage $\mathbf{T}$. For all $U \in \mathsf{C}$ and $P \in \widehat{\mathsf{C}}$, P satisfies the sheaf condition with respect to $\{\mathrm{id}\colon U \to U\}$.*

Lemma 3.6. *Let $U \in \mathsf{C}$ and $\{V_j \to U\}$ be a covering not necessarily in T_U. If there exists a $\{U_i \to U\} \in T_U$ such that for all j there's an i and a commutative diagram*

$$\begin{tikzcd} U_i \arrow[rr] \arrow[dr] & & V \arrow[dl] \\ & X, & \end{tikzcd} \tag{3.7}$$

then $\mathbf{T}$-sheaves satisfying the sheaf condition for $\{V_j \to U\}$ are $T \cup \{\{V_j \to U\}\}$-sheaves.

Example 3.8. Consider on Man the coverage of usual coverings by open sets, and suppose you want to add all jointly surjective submersions. Then the hypothesis of Lemma 3.6 certainly holds: locally a submersion $S \twoheadrightarrow X$ looks like a projection[3], so you can locally choose a section $\mathbb{R}^m \to S$, where $m = \dim X$; composing with the submersion produces a covering map locally, which suffices. ◀

Lemma 3.9. *If C has pullbacks, $\{U_i \to U\}$ is in T_U, and for all i, $\{U_{ij} \to U_i\}$ is in T_{U_i}, then all $\mathbf{T}$-sheaves are $\{\{U_{ij} \to U_i \to U\}\} \cup \mathbf{T}$-sheaves.*

Remark 3.10. This is stated in the Elephant without pullbacks, but Zhen Lin Low found a counterexample. ◀

The next result is a corollary of Lemma 3.6.

Corollary 3.11. *If C has pullbacks, then given $U \to V$ and $\{V_j \to V\}$ in T_V, all $\mathbf{T}$-sheaves are $\mathbf{T} \cup \{\{V_j \times_V U \to U\}\}$-sheaves.*

The idea is to let U_i be the pullback of $V_j \to V$ by $U \to V$.

Now we can define the notion of a Grothendieck pretopology; it is presented in a way that suggests a strong resemblance to the usual notion of covering maps of topological spaces.

Definition 3.12. Let C be a category with pullbacks. A coverage τ is called a *Grothendieck pretopology* if

(1) for all $U \in \mathsf{C}$, $\{\mathrm{id}\colon U \to U\} \in \tau_U$,
(2) for all $U \to V$ and $\{U_j \to V\} \in \tau_V$, the "restriction" $\{U_j \times_V U \to U\} \in \tau_U$.
(3) the conclusion of Lemma 3.9 holds.

Theorem 3.13. *Let C be a category with pullbacks and $\mathbf{T}$ be a coverage on C; then, the intersection of all Grothendieck pretopologies $\tau \supset \mathbf{T}$ is again a Grothendieck pretopology with the same sheaves as $\mathbf{T}$.*

[3]In fact, for topological manifolds this is often taken as the definition of a submersion.

4. Afternoon

Definition 4.1. Let $U \in \mathsf{C}$. A covering $\mathfrak{U}$ of U is called a *sieve* if for all $U' \to U$ in $\mathfrak{U}$ and all $U'' \to U'$, the composition $U'' \to U$ is in $\mathfrak{U}$.

So there's an obvious way to complete any covering to a sieve.

Lemma 4.2. *For any $U \in \mathsf{C}$, there's a bijection between the sieves on U and the subpresheaves of $\mathsf{C}(-, U)$.*

Proof sketch. Given a sieve S on U, you can obtain a presheaf sending U' to the set of all maps $U' \to U$ in S; the sieve property guarantees functoriality.

Conversely, a subpresheaf of $\mathsf{C}(-, U)$ identifies a certain subset of maps $U' \to U$ for each U', such that composition still lands in the corresponding subset, which is exactly the sieve condition. ⊠

We will hence identify a sieve S with the subpresheaf of $\mathsf{C}(-, U)$ it corresponds to. Then, given a map $f : U' \to U$, let f^*S denote the pullback

$$\begin{tikzcd} f^*S \arrow[r] \arrow[d] & S \arrow[d, hook] \\ \mathsf{C}(-, U) \arrow[r, "f_*"] & \mathsf{C}(-, U). \end{tikzcd} \tag{4.3}$$

Thinking about what pullbacks in Set are, f^*S is the presheaf

$$V \longmapsto \{g : V \to U' \mid g \circ f \in S(U)\}. \tag{4.4}$$

Lemma 4.5. *Let $\mathfrak{U}$ be a coverin gof U and S be the sieve it generates. Let P be a presheaf; then,*

$$\mathrm{Desc}_P(\mathfrak{U}) \cong \varprojlim_{(V \to U) \in S} P(V).$$

Proof sketch. There is a natural map $\mathrm{Desc}_P(\mathfrak{U}) \to \varprojlim P(V)$ (restrict each s_i to V) — it's injective when you pass to $\prod P(V)$, hence must have been injective. For surjectivity, there's an explicit inverse: things in the limit are indexed by larger diagrams, and we can restrict to the smaller triangles in $\mathrm{Desc}_P(\mathfrak{U})$. ⊠

Lemma 4.6. *Let $\mathbf{T}$ be a coverage and $\mathbf{T}'$ be the completion of $\mathbf{T}$ in the sense that all coverings are completed to sieves; then, $\mathbf{T}'$-sheaves are precisely $\mathbf{T}$-sheaves.*

Lemma 4.7. *Let $U \in \mathsf{C}$, $P \in \widehat{\mathsf{C}}$, and S be a sieve on U. Then, P is a sheaf with respect to S iff*

$$\widehat{\mathsf{C}}(\mathsf{C}(-, U), P) \longrightarrow \widehat{\mathsf{C}}(S, P)$$

is a bijection.

The proof is an exercise, but the key step is to show that $\widehat{\mathsf{C}}(S, P) \cong \varprojlim_{(V \to U) \in S} P(V)$.

Lemma 4.8. *With the same notation in the previous lemma, for all sieves $S' \subseteq \mathsf{C}(-, U)$ containing S, if P is a sheaf for S, then it's also a sheaf for S'.*

Definition 4.9. A coverage is *sifted* if all of its constituent coverages are sieves.

Lemma 4.10. *Let $\mathbf{T}$ be a sifted coverage on C. Then, for all $U \in \mathsf{C}$, all $f : U \to V$ in C, and all $S \in T_U$, all $\mathbf{T}$-sheaves are also sheaves with respect to f^*S.*

Now we arrive at the definition of a Grothendieck topology!

Definition 4.11. A sifted coverage τ is a *Grothendieck topology* if it satisfies the following axioms.

(1) Coverings are closed under pullback.
(2) For all $U \in \mathsf{C}$, the sieve associated to $\{\mathrm{id} : U \to U\}$ is in τ_U.
(3) For any $U \in \mathsf{C}$, $R \in \tau_U$, and $S \subset \mathsf{C}(-, U)$, and for all $f : V \to U$ in R with f^*S in τ_V, we have $S \in \tau_U$.

There's a bijective correspondence between Grothendieck topolgoies on C and subcategories $S \subset \widehat{\mathsf{C}}$ with left exact left adjoints.