M392C NOTES: BRIDGELAND STABILITY

ARUN DEBRAY
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These notes were taken in UT Austin's M392C (Bridgeland Stability) class in Spring 2019, taught by Benjamin Schmidt. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Any mistakes in the notes are my own.

Contents

Lecture 1.

Introduction and quiver representations: 1/22/19

This class will be on Bridgeland stability, though we won't get to that topic specifically for about a month. We'll follow lecture notes of Macrì-Schmidt [MS17], which are on the arXiv.

If you're pre-candidacy, make sure to do at least two exercises in this class, at least one from March or later; otherwise just make sure to show up. (If you're an undergrad who's signed up for this class, please do at least four exercises, at least two from March or later.)

Now let us enter the world of mathematics. We'll begin with two well-known theorems in algebraic geometry; we'll eventually be able to prove these using stability conditions.

Theorem 1.1 (Kodaira vanishing)**.** *Let X be a smooth projective complex variety and L be an ample line bundle. Then for all $i > 0$, $H^i(X; L \otimes \omega_X) = 0$.*

We'll eventually give an approach in the setting where $\dim X \le 2$. It won't be very hard once the setup is in place. In fact, there are probably plenty of other vanishing theorems one could prove using stability conditions, including some which aren't known yet.

The other theorem is over a century ago, from the Italian school of algebraic geometry.

Theorem 1.2 (Castelnuovo)**.** *Working over an algebraically closed field, let $C \subset \mathbb{P}^3$ be a smooth curve not contained in a plane. Then $g \le d^2/4 - d + 1$, where g is genus of C and d is its degree.*

Another goal we'll work towards:

Problem 1.3. Explicitly describe some moduli spaces of vector bundles or sheaves.

Here's a concrete outline of the course.

(1) Before we discuss any algebraic geometry, we'll study quiver theory, focusing on moduli spaces of quiver conditions. We don't need stability conditions to do this, but these spaces make great simple examples of the general story.
(2) Next, we'll study vector bundles on curves. Bridgeland stability is a generalization of what we can say here for higher dimensions.
(3) A crash course on derived categories and Bridgeland stability. This is pretty formal.
(4) A crash course on intersection theory, which will be necessary for what comes later.
(5) Surfaces.

(6) Threefolds (if we have time).

These are all mostly independent pieces, only coming together in the end, so if you get lost somewhere there's no need to panic; you'll probably be able to pick the course back up soon enough.

And now for the moduli of quiver representations. For this stuff, we'll follow King [Kin94], which is accessible and nice to read. Let k be an algebraically closed field.

Definition 1.4. A *quiver* is the representation theorist's word for a finite directed graph. Explicitly, a quiver Q consists of two finite sets Q_0 and Q_1 of vertices and edges, respectively, together with *tail* and *head* maps $t, h \colon Q_1 \to Q_0$.

Example 1.5. The *Kronecker quiver* is

$$\bullet \rightrightarrows \bullet.$$

The quiver of *type* D_4 is

$$\begin{tikzcd} & & \bullet \arrow[dl] \\ \bullet & \bullet \arrow[l] & \\ & & \bullet. \arrow[ul] \end{tikzcd}$$

We can also consider a quiver with a single vertex v and a single edge $e \colon v \to v$.

Definition 1.6. A *representation* W of a quiver Q is a collection of k-vector spaces W_v for each $v \in Q_0$ and linear maps $\phi_e \colon W_{v_1} \to W_{v_2}$ for each edge $e \colon v_1 \to v_2$ in Q_1. The vector $(\dim W_v)_{v \in Q_0}$ inside $\mathbb{C}[Q_0]$ is called the *dimension* of W.

Example 1.7. First, some trivial example. For example, here's a representation of the Krokecker quiver: $(\cdot 1, \cdot 2) \colon k \rightrightarrows k$. A representation of the quiver with one vertex and one edge is a vector space with an endomorphism, e.g. $\mathbb{C}^2$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Definition 1.8. Let Q be a quiver. A *morphism* of Q-representations $f \colon (W_v, \phi_e) \to (U_v, \psi_e)$ is a collection of linear maps $f_v \colon W_v \to U_v$ for each $v \in Q_0$ such that for all edges e,

$$f_{h(e)} \circ \phi_e = \psi_e \circ f_{t(e)}.$$

If all of these linear maps are isomorphisms, f is called an *isomorphism*.

That is, data of a quiver representation includes a bunch of linear maps, and we want a morphism of quiver representations to commute with these maps.

Representations theorists want to classify quiver representations. This is really hard, so let's specialize to irreducible representations (those not a direct sum of two other ones). This is still really hard! There are classical theorems originating from the French school proving that most quivers do not admit nice classifications of their irreducible representations: some have finitely many, and some have infinitely many but nice parameterizations, and these are uncommon.

One way to make headway on these kinds of problems is to consider a moduli space of quiver representations, which may be more tractable to study.

Problem 1.9. Can you classify the (isomorphism classes of) quiver representations of the quiver with a single vertex and single edge?

Our first, naïve approach to constructing the moduli of quiver representations is to fix a dimension vector $\alpha \in \mathbb{C}[Q_0]$ and define

$$R(Q, \alpha) := \bigoplus_{e \in Q_1} \mathrm{Hom}(W_{t(e)}, W_{h(e)}). \tag{1.10}$$

This is too big: the same isomorphism class appears at more than one point. We can mod out by a symmetry: let

$$\mathrm{GL}(\alpha) := \prod_{v \in Q_0} \mathrm{GL}(W_v) \tag{1.11}$$

act on $R(Q,\alpha)$ by a change of basis on each vector space and on ϕ_e as

$$(g\phi)_e = g_{h(e)}\phi_e g_{t(e)}^{-1}. \tag{1.12}$$

Then as a set the quotient $R(Q,\alpha)/\mathrm{GL}(\alpha)$ contains one element for each isomorphism class. But putting a geometric structure on quotients of varieties is tricky. We'll come back to this point.

Example 1.13. Let Q be the Kronecker quiver and $\alpha = (1,1)$, so that $\mathrm{GL}(\alpha) = k^\times \times k^\times$. Pick $(t,s) \in \mathrm{GL}(\alpha)$; the action on a Q-representation $(\lambda,\mu)\colon k \rightrightarrows k$ produces $(s\lambda t^{-1}, s\mu t^{-1})\colon k \rightrightarrows k$. So if $s = t$, the action is trivial. Quotienting out by the diagonal $s = t$ in $k^\times \times k^\times$, we get $k^\times\colon (s,t) \mapsto s/t$, and this acts on $R(Q,\alpha) = k^2$ by scalar multiplication.

This is an action we know well: the quotient is the space of lines in k^2, also known as $\mathbb{P}^1_k$ – and the zero orbit. This orbit makes life more of a headache: you can't just throw it out, because then you don't get a good map to the quotient, preimages of closed things aren't always closed, etc. But the action on the zero orbit is not free. This phenomenon will appear a lot, and we'll in general have to think about what to remove. After some hard work we'll be able to take the quotient in a reasonable way and get $\mathbb{P}^1$. ◀

A crash course on (linear) algebraic groups. If you want to learn more about algebraic groups, especially because we're not going to give proofs, there are several books called *Linear Algebraic Groups*: the professor recommends Humphreys' book [Hum75] with that title, and also those of Borel [Bor91] and Springer [Spr98].

Definition 1.14. An *algebraic group* is a variety G together with a group structure such that multiplication and taking inverses are morphisms of varieties.

You can guess what a morphism of algebraic groups is: a group homomorphism that's also a map of varieties.

Example 1.15. GL_n is an algebraic group. Inside the space of all $n \times n$ matrices, which is a vector space over k, GL_n is the set of matrices with nonzero determinant. This is an open condition, and the determinant can be written in terms of polynomials, so GL_n is an algebraic group.

Other examples include SL_n and elliptic curves, and we can take products, so $\mathrm{GL}(\alpha)$ is also an algebraic group. ◀

Definition 1.16. A *linear algebraic group* is an algebraic group that admits a closed embedding $G \hookrightarrow \mathrm{GL}_n$ which is also a group homomorphism.

This does not include the data of the embedding. It turns out (this is in, e.g. Humphreys) that any affine algebraic group is linear, but this is not particularly easy to show.

Exercise 1.17. Show that any algebraic group is also a smooth variety.

This does not generalize to group schemes!

We care about groups because they act. We added structure to algebraic groups, and thus care about actions which behave nicely under that structure.

Definition 1.18. A *group action* of an algebraic group G on a variety X is a morphism $\varphi\colon G \times X \to X$ such that for all $g, h \in G$ and $x \in X$,

(1) $\varphi(g, \varphi(h,x)) = \varphi(gh, x)$, and
(2) $\varphi(e, x) = x$.

Example 1.19. $k^\times$ acts on $\mathbb{A}^{n+1}_k$ by scalar multiplication. What's the quotient? We want $\mathbb{P}^n_k$, but there's also the zero orbit, and no other orbit is closed. This makes us sad; we're going to use geometric invariant theory (GIT) to address these issues and become less sad. ◀

Definition 1.20. Let G be an algebraic group.

- A *character* of G is a morphism of algebraic groups $\chi\colon G \to k^\times$. These form a group under pointwise multiplication, and we'll denote this group $X(G)$.
- A *one-parameter subgroup* of G, also called a *cocharacter*, is a morphism of algebraic groups $\lambda\colon k^\times \to G$.

Example 1.21. Since $\det(AB) = \det A \det B$, the determinant defines a character of GL_n. One example of a cocharacter is $\lambda\colon k^\times \to \mathrm{GL}_2$ sending $t \mapsto \left(\begin{smallmatrix} t & 0 \\ 0 & t^2 \end{smallmatrix}\right)$. This cocharacter factors through the diagonal matrices in GL_n; this turns out to be a general fact. ◂

Here are a few nice facts about characters and cocharacters.

Theorem 1.22.
(1) *The map* $\mathbb{Z} \to X(\mathrm{GL}_n)$ *sending* $m \mapsto \det^m$ *is an isomorphism.*
(2) *If* G *and* H *are algebraic groups, the map* $X(G) \times X(H) \to X(G \times H)$ *sending*

$$(\chi_1, \chi_2) \longmapsto ((g, h) \longmapsto \chi_1(g)\chi_2(h)) \tag{1.23}$$

is an isomorphism.
(3) *Up to conjugation, every cocharacter of* GL_n *lands in the subgroup of diagonal matrices, hence sends* $t \mapsto \mathrm{diag}(t^{a_1}, \dots, t^{a_n})$ *for* $a_1, \dots, a_n \in \mathbb{Z}$.

We're not going to prove these: this would require a considerable detour into the theory of algebraic groups to get to, and you can read the proofs in Humphreys.

Exercise 1.24. Without using the above theorem, show that any morphism of algebraic groups $k^\times \to k^\times$ is of the form $t \mapsto t^n$ for some $n \in \mathbb{Z}$.

Lecture 2.

Geometric invariant theory: 1/24/19

Today we'll discuss some more geometric invariant theory and how to take quotients.

Nagata reinterpreted Hilbert's 14^{th} problem as follows.

Problem 2.1. Let G be a linear algebraic group acting linearly on a finite-dimensional k-vector space V. Is the *ring of invariants*

$$\mathscr{O}(V)^G = \{f \in \mathscr{O}(V) \mid f(gx) = f(x) \text{ for all } g \in G, x \in V\}$$

finitely generated?

The elements of $\mathscr{O}(V)^G$ are called the *invariant polynomials* or *invariant functions* on V.

Nagata proved that this is not always true, though there is a positive answer with some assumptions on G. For example, GL_n and products of general linear groups satisfy this property.

Definition 2.2. A linear algebraic group G is *geometrically reductive* if for every linear action of G on a finite-dimensional vector space V (i.e. a map of algebraic groups $\varphi\colon G \to \mathrm{GL}(V)$) and every fixed point $v \in V$ of the G-action, there is an invariant homogeneous nonconstant polynomial f with $f(v) = 0$.

Remark 2.3. There is a different notion of a reductive group, and it is different. Sorry about that. ◂

Theorem 2.4 (Nagata [Nag63])**.** *If G is geometrically reductive, Problem 2.1 has a positive answer.*

If $\mathrm{char}(k) = 0$, basic facts from the theory of algebraic groups allow one to prove GL_n is geometrically reductive, and in fact in characteristic zero reductive implies geometrically reductive. This is also true in positive characteristic, but is significantly harder!

Remark 2.5. In fact, in characteristic zero, the polynomial f in the definition of geometrically reductive can be chosen such that $\deg(f) = 1$. This property is called *linearly reductive*, so in characteristic zero, reductive, geometrically reductive, and linearly reductive coincide. This is not true in positive characteristic, which is ultimately because of everyone's favorite fact about modular representation theory: representations of a group in positive characteristic need not be semisimple. ◂

Mumford conjectured the following.

Theorem 2.6 (Haboush [Hab75])**.** *If k is algebraically closed and G is reductive, then G is geometrically reductive.*

The difficulty was in positive characteristic.

This led to the first idea of a better quotient: take Spec of the ring of invariants; by this theorem, this gives you a variety. But sometimes this is too small: for $\mathbb{C}^\times$ acting on $\mathbb{C}^n$, this tells you the closed orbits. The only closed orbit is the zero orbit, so we don't get $\mathbb{P}^{n-1}$, alas.

To abrogate this, we'll introduce a numerical criterion. Let G be a geometrically reductive group acting linearly on a finite-dimensional vector space V. Recall that $\mathscr{O}(V)$ is also denoted $k[V]$, the ring of polynomials on V.

Definition 2.7. Let $\chi \in X(G)$ be a character of G.

(1) An $f \in \mathscr{O}(V)$ is *relatively invariant* of *weight* χ if $f(gx) = \chi(g)f(x)$ for all $x \in V$ and $g \in G$. We let $\mathscr{O}(V)^{G,\chi}$ denote the vector space of relatively invariant functions of weight χ, so that $\mathscr{O}(V)^{G,\chi^0} = \mathscr{O}(V)^G$.

(2) Define

$$V /\!/ (G,\chi) := \operatorname{Proj}\left(\bigoplus_{n\geq 0} \mathscr{O}(V)^{G,\chi^n}\right). \tag{2.8}$$

We let $V /\!/ G := \operatorname{Spec}(\mathscr{O}(V)^G)$.

One can check quickly that the product of relatively invariant functions of weights χ^m and χ^n is relatively invariant of weight χ^{m+n}, so the graded abelian group in (2.8) is in fact a graded ring.

Theorem 2.9. *There's a natural map $V /\!/ (G,\chi) \to V /\!/ G$, and this map is projective.*

Example 2.10. Consider $k^\times$ acting on k^{m+1} by scalar multiplication and $\chi\colon k^\times \to k^\times$. Then $k[x_0, \dots, x_m]^{k^*,\chi^n}$ is exactly the vector space of degree-n homogeneous polynomials. Then

$$k^{m+1} /\!/ (k^\times, \mathrm{id}) = \operatorname{Proj}(k[x_0, \dots, x_m]) = \mathbb{P}^m, \tag{2.11}$$

where we give $k[x_0, \dots, x_m]$ its usual grading.

However, if you use other characters, you'll get something different: for $\chi = 1$ you get a single point, and for $\chi = -\mathrm{id}$ the quotient is empty. ◀

We've been calling $V /\!/ (G,\chi)$ a "quotient," but is it really one? We'd like to say it has nice properties that a quotient should have, but in the above example, there isn't a nice map $k^{m+1} \twoheadrightarrow \mathbb{P}^m$. In general we get a nice map like that on an open subset; let's figure out what map that is.

Let Δ be the kernel of $\varphi\colon G \to \mathrm{GL}(V)$.[1]

Definition 2.12.

(1) An $x \in V$ is called *χ-semistable* for a character χ if there is an $f \in \mathscr{O}(V)^{G,\chi^n}$ for some $n \geq 1$ such that $f(x) \neq 0$. The locus of χ-semistable points is denoted V_χ^{ss}.

(2) If x is χ-semistable and we can choose f such that the orbit $G \cdot x \subset \{x \in V \mid f(x) \neq 0\}$ is closed, and $\dim G \cdot x = \dim G - \dim \Delta$, we call x *χ-stable*. The locus of χ-stable points is denoted V_χ^s.

Stability means that the orbit of x has the largest possible dimension.

Lemma 2.13. *V_χ^{ss} and V_χ^s are Zariski open subsets of V.*

The main theorem of geometric invariant theory in this setting[2] is:

Theorem 2.14 (Mumford)**.** *There is a surjective morphism $\phi\colon V_\chi^{ss} \to V /\!/ (G,\chi)$ such that if $x, y \in V_\chi^{ss}$,*

(1) *if $x, y \in V_\chi^s$, then $\phi(x) = \phi(y)$ iff $y \in G \cdot x$, and*

(2) *in general, $\phi(x) = \phi(y)$ iff $\overline{G \cdot x} \cap \overline{G \cdot y}$ is nonempty, where closures are taken inside V_χ^{ss}.*

You can think of ϕ as the map from the original space to the quotient, but we can only see a subset of the original space. For stable points, this actually parameterizes orbits, but this isn't quite true for merely semistable points, and the problem occurs when orbits aren't closed.

[1] In many references, φ is assumed to be injective.

[2] Mumford showed a version where G can act on any quasiprojective variety.

Definition 2.15. If $\overline{G\cdot x}\cap\overline{G\cdot y}$ is nonempty, we say x and y are *S-equivalent*.

Remark 2.16. S is for Seshadri, who was one of the developers of this theory. ◀

The numerical criterion we alluded to earlier is a way to find semistable points.

Definition 2.17. Let $\chi\colon G\to k^\times$ be a character and $\lambda\colon k^\times\to G$ be a cocharacter. The composition $\chi\circ\lambda\colon k^\times\to k^\times$ sends $t\mapsto t^n$ for some $n\in\mathbb{Z}$; we denote $\langle\chi,\lambda\rangle := n$.

Theorem 2.18 (Mumford's numerical criterion)**.**

(1) *An $x\in V$ is χ-semistable iff $\chi(\Delta)=1$ and for all cocharacters $\lambda\colon k^\times\to G$ such that $\lim_{t\to0}\lambda(t)x$ exists, then $\langle\chi,\lambda\rangle\ge 0$.*
(2) *x is χ-stable iff it's χ-semistable and if λ is as above and $\langle\chi,\lambda\rangle>0$, then $\lambda(k^\times)\subset\Delta$.*

That limit works fine in $\mathbb{C}$, but what about over other fields? It's obvious in formulas, and in general you can define it in terms of trying to extend to a map of varieties $k\to G$.

Proposition 2.19.

(1) *The orbit $G\cdot x$ is closed in V^{ss}_χ if for every cocharacter λ with $\langle\chi,\lambda\rangle=0$ such that the limit $\lim_{t\to0}\lambda(t)x$ exists, then the limit is in $G\cdot x$.*
(2) *If $x,y\in V^{\mathrm{ss}}_\chi$, then x and y are S-equivalent iff there are cocharacters λ_1,λ_2 with $\langle\chi,\lambda_1\rangle=\langle\chi,\lambda_2\rangle=0$ such that $\lim_{t\to0}\lambda_1(t)x$ and $\lambda_{t\to0}\lambda_2(t)y$ both exist and are in the same orbit.*

Example 2.20. Consider $G=\mathrm{GL}_2$ acting on the space V of 4×2 matrices: to obtain a left action by g, we multiply on the right by g^{-1}. Let $\chi\colon\mathrm{GL}_2\to k^\times$ be $\det^{-1}$.

What do we expect to parameterize in the quotient? A 4×2 matrix is a linear map $k^2\to k^4$, and we're parameterizing them up to change of basis of the domain. This should morally parameterize two-dimensional subspaces of k^4, though we never stipulated that our maps are injective. Maybe, hopefully, the open subset of semistable points are the injective maps and we'll get the Grassmannian $\mathrm{Gr}_2(k^4)$.

We claim this is actually the case, and will use the numerical criterion to prove it. Since GL_2 acts faithfully on V, $\Delta=1$ and the situation simplifies somewhat. We can use a group action to make the cocharacter simpler, or to make a general element of V simpler, but not both. So we'll do the former: let $\lambda=\left(\begin{smallmatrix}t^n&0\\0&t^m\end{smallmatrix}\right)$ and

$$A=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\\a_{31}&a_{32}\\a_{41}&a_{42}\end{pmatrix},\quad\text{so that}\quad\lambda(t)A=\begin{pmatrix}a_{11}t^{-n}&a_{12}t^{-m}\\a_{21}t^{-n}&a_{22}t^{-m}\\a_{31}t^{-n}&a_{32}t^{-m}\\a_{41}t^{-n}&a_{42}t^{-m}\end{pmatrix}.\tag{2.21}$$

Since $\det^{-1}(\lambda(t))=t^{-n-m}$, $\langle\det^{-1},\lambda\rangle=-n-m$. Therefore $\lim_{t\to0}\lambda(t)$ exists iff either

(1) $A=0$, which isn't semistable, because the limit exists for every cocharacter; or
(2) $a_{11}=\dots=a_{41}=0$ and $a_{j2}\ne0$ for some j, and $m\le0$, which is also unstable (e.g. $m=0$, $n=1$); or
(3) $a_{2i}=0$ for all i, and $a_{j1}\ne0$ for some j, and $n\le0$, which is again unstable; or
(4) $a_{i1}\ne0$ for some i and $a_{j2}\ne0$ for some j, and $m,n\le0$, so $\langle\chi,\lambda\rangle=-n-m\ge0$, and these A are stable.

Now, let's look at an arbitrary cocharacter. This involves changing basis/looking at full orbits of points we found were unstable. When $A=0$ (case (1)), this is the whole orbit, and it's unstable. For (2) and (3), A has rank 1 in the entire orbit, and therefore these are all unstable. All matrices of rank 2 are stable. ◀

Lecture 3.

Constructing moduli spaces of quiver representations: 1/29/19

Today, we're going to leverage the GIT theory we surveyed in the last lecture to define moduli spaces of quiver representations.

We begin with a quick review of Mumford's numerical criterion, since it will be an important actor today. Let G be an algebraic group acting on a k-vector space V, and let Δ denote the kernel of the associated map $\rho\colon G\to\mathrm{GL}(V)$. Let $\chi\colon G\to k^\times$ be a character. Then, in Theorem 2.18, we saw that $x\in V$ is χ-semistable iff $\chi(\Delta) = 1$ and for all cocharacters $\lambda\colon k^\times \to G$ such that $\lim_{t\to 0} \lambda(t)\cdot x$ exists, $\langle \chi, \lambda\rangle = 0$. Moreover, x is χ-stable if in addition whenever $\langle\chi,\lambda\rangle = 0$, then $\lambda(k^\times) \subset \Delta$.

Now back to quivers. Consider a quiver Q with a set Q_0 of vertices, Q_1 of edges, and head and tail maps $h, t\colon Q_1 \rightrightarrows Q_0$. Let $\alpha \in \mathbb{C}[Q_0]$ be a dimension vector and vector spaces W_v of dimension $\alpha(v)$ for each $v \in Q_0$. We constructed the space

$$R(Q,\alpha) := \prod_{e\in Q_1} \mathrm{Hom}(W_{t(e)}, W_{h(e)}), \tag{3.1}$$

but this is too big to be a moduli space of quiver representation: it contains different points that correspond to isomorphic representations. Therefore we want to take the quotient by $\mathrm{GL}(\alpha)$ as we described, so let's apply GIT to this action and understand stability.

In this setting, the kernel Δ is the *long diagonal* $\{(t\mathrm{id}, \dots, t\mathrm{id}) \in \mathrm{GL}(\alpha) \mid t \in k^\times\}$, which is isomorphic to $k^\times$. If $\theta \in \mathbb{Z}[Q_0]$, it defines a character $\chi\colon \mathrm{GL}(\alpha) \to k^\times$ by

$$\chi_\theta(g) := \prod_{v\in Q_0} g_v^{\theta(v)}. \tag{3.2}$$

All characters of GL_n can be written in this way.

Definition 3.3. Let A be an abelian category. Its *Grothendieck group* is

$$K_0(\mathsf{A}) := \bigoplus_{A\in\mathsf{A}} \mathbb{Z}[A]/\sim, \tag{3.4}$$

where we quotient by an equivalence relation: for all short exact sequences $0 \to A \to B \to C \to 0$, we say $[B] \sim [A] + [C]$.

The $\theta \in \mathbb{Z}[Q_0]$ as above can also be thought of as a function on the Grothendieck group of Rep_Q, the category of finite-dimensional representations of Q. Specifically, $\theta\colon K_0(\mathsf{Rep}_Q) \to \mathbb{Z}$ is defined to send

$$M = (\{M_v\}, \{\psi_e\}) \longmapsto \sum_{v\in Q_0} \theta(v)\cdot \dim(M_v). \tag{3.5}$$

We want there to be semistable points, which means we will only consider θ such that

$$\chi_\theta(\Delta) = \left\{ \prod_{v\in Q_0} t^{\theta(v)\alpha(v)} \mid t \in k^\times \right\} = \{1\}, \tag{3.6}$$

i.e. such that

$$\sum_{v\in Q_0} \theta(v)\alpha(v) = 0. \tag{3.7}$$

Among other things, this means that $\theta(M) = 0$ if $\dim M = \alpha$.

Now we'll perform the GIT analysis. Let $\lambda\colon k^\times \to \mathrm{GL}(\alpha)$ be a cocharacter and $M = (\{M_v\}, \{\phi_e\}) \in \mathsf{Rep}_Q$ have dimension α. The first step will be to construct a descending $\mathbb{Z}$-indexed filtration

$$M \supseteq \cdots \supseteq M^{(n)} \supseteq M^{(n+1)} \supseteq \cdots. \tag{3.8}$$

For each $v \in Q_0$, pick a decomposition

$$M_v = \bigoplus_{n\in\mathbb{Z}} M_v^{(n)}, \tag{3.9}$$

such that $\lambda(t)$ acts on $M_v^{(n)}$ by multiplication by t^n. (Recall that any cocharacter of GL_n has diagonal image up to conjugation, so this makes sense.) Define

$$M_v^{(\geq n)} := \bigoplus_{m\geq n} W_v^{(m)}. \tag{3.10}$$

For each $e \in Q_1$, let $\phi_e^{(m,n)}$ denote the composition

$$M_{t(e)}^{(n)} \hookrightarrow M_{t(e)} \xrightarrow{\phi_e} M_{h(e)} \twoheadrightarrow M_{h(e)}^{(m)}, \tag{3.11}$$

where the projection comes from the decomposition (3.9). Then

$$\lambda(t) \cdot \phi_e^{(m,n)} = t^m \phi_e^{(m,n)} t^{-n} = t^{m-n} \phi_e^{(m,n)}, \tag{3.12}$$

which means the following are equivalent:

- $\lim_{t \to 0} \lambda(t) \phi_e$ exists,
- $\phi_e^{(m,n)} = 0$ whenever $m \le n$, and
- ϕ_e maps $M_{t(e)}^{(\ge n)}$ into $M_{h(e)}^{(\ge n)}$.

The third condition means that $M_n \coloneqq (M^{(\ge n)}, \phi_e|_{M^{(\ge n)}})$ is a subrepresentation of M, so if the limit exists it induces the desired filtration of M (3.8). In this case $W_v^{(n)} = W_v^{(\ge n)} / W_v^{(\ge (n+1))}$.

Conversely, given a filtration of M as in (3.8), we can produce a cocharacter $\lambda \colon k^\times \to \mathrm{GL}(\alpha)$: define $\lambda(t)$ to act by t^n on $W_v^{(n)}$.

Since

$$\lim_{t \to 0} \lambda(t) \phi_e^{(m,n)} = \lim_{t \to 0} t^{m-n} \phi_e^{(m,n)} = 0 \tag{3.13}$$

for $m > n$, then

$$\lim_{t \to 0} \lambda(t) \phi_e \colon M_{t(e)}^{(n)} \longrightarrow M_{h(e)}^{(n)} \subset M_{h(e)}^{(\ge n)}. \tag{3.14}$$

Thus the limit is the associated graded:

$$\lim_{t \to 0} \lambda(t) \cdot M = \bigoplus_{n \in \mathbb{Z}} M_n / M_{n+1}. \tag{3.15}$$

Remark 3.16. There's an n such that $M_n = 0$ and $M_n \ne M$. In other words, this filtration isn't the trivial one. This is because $\lambda(k^\times) \subset \Delta$. ◂

Now let's discuss (semi)stability. It will turn out to be equivalent to the following notion.

Definition 3.17. Let $M \in \mathsf{Rep}_Q$ have dimension α and be such that $\theta(M) = 0$. Then M is *θ-semistable* (resp. *θ-stable*) iff for all nonzero proper subrepresentations $N \subset M$, $\theta(N) \ge 0$ (resp. $\theta(N) > 0$).

Theorem 3.18 (King [Kin94])**.** *A point $M \in R(Q, \alpha)$ is χ_θ-semistable (resp. χ_θ-stable) iff $M \in \mathsf{Rep}_Q$ is θ-semistable (resp. θ-stable).*

Proof. First, let's show θ-(semi)stability implies GIT (semi)stability. We assumed $\sum_v \theta(v)\alpha(v) = 0$, which implies

$$\langle \chi_\theta, \lambda \rangle = \sum_{v \in Q_0} \theta(v) \sum_{n \in \mathbb{Z}} n \dim M_v^{(n)}. \tag{3.19}$$

We can change the order of summation because only finitely many $W_v^{(n)}$ are nonzero for v fixed, so

$$\langle \chi_\theta, \lambda \rangle = \sum_{n \in \mathbb{Z}} n \sum_{v \in Q_0} \theta(v) \dim M_v^{(n)} \tag{3.20a}$$

$$= \sum_{n \in \mathbb{Z}} n\theta(M_n / M_{n+1}). \tag{3.20b}$$

Since θ factors through the Grothendieck group, this is

$$= \sum_{n \in \mathbb{Z}} n(\theta(M_n) - \theta(M_{n-1})) \tag{3.20c}$$

$$= \sum_{n \in \mathbb{Z}} \theta(M_n), \tag{3.20d}$$

unwinding the telescoping series. This is nonnegative if M is θ-semistable and positive if M is θ-stable, by definition.

Conversely, suppose M is χ_θ-semistable and let N be a subrepresentation of M. Then $N \subset M$ is a filtration, hence defines a cocharacter $\lambda \colon k^\times \to \mathrm{GL}(\alpha)$ such that

$$0 \overset{(<)}{\le} \langle \chi_\theta, \lambda \rangle = \theta(M) + \theta(N) \tag{3.21}$$

(parentheses for stability), so $\theta(N) \geq -\theta(M) = 0$ (or $>$ for stability). ⊠

Now we have semistable points, and even strictly semistable points. What does S-equivalence look like in this context?

Definition 3.22. Given θ as above, let $\mathsf{P}_\theta \subset \mathsf{Rep}_Q$ denote the full subcategory of θ-semistable representations with $\theta(M) = 0$ (so, those objects, and all of the morphisms between them).

Lemma 3.23. *P_θ is an abelian subcategory of Rep_Q. That is, let $\varphi\colon M \to N$ be a morphism in P_θ, i.e. $\theta(M) = \theta(N) = 0$ and M and N are θ-semistable. Then, $A := \ker(\varphi)$ and $B = \mathrm{coker}(\varphi)$, where the kernel and cokernel are taken in Rep_Q, are in P_θ.*

Proof. The kernel and cokernel fit into an exact sequence

$$0 \longrightarrow A \longrightarrow M \longrightarrow N \longrightarrow B \longrightarrow 0. \tag{3.24}$$

Since θ is additive under short exact sequences, it in fact satisfies

$$\theta(A) + \underbrace{\theta(N)}_{=0} = \underbrace{\theta(M)}_{=0} + \theta(B), \tag{3.25}$$

so $\theta(A) = \theta(B)$. Since M is θ-semistable, $\theta(A) \geq 0$. If $K := \ker(N \to B) \hookrightarrow N$, and N is θ-semistable, then $\theta(K) \geq 0$. Therefore

$$\theta(A) = \theta(B) = \theta(N) - \theta(K) = -\theta(K) \leq 0. \tag{3.26}$$

Thus $\theta(A) = \theta(B) = 0$.

Now for semistability. Briefly, if $C \subset A$ is a subrepresentation, then it's also a subrepresentation of M, so $\theta(C) \geq 0$. The argument for B is similar. ⊠

Lecture 4.

Examples of quiver varieties: 1/31/19

Fix a quiver Q. Last time we explained how, given a $\theta \in \mathbb{Z}[Q_0]$, we obtain a function on objects of Rep_Q additive on short exact sequences: $\theta(M) := \sum_{v \in Q_0} \theta_v \cdot \dim(M_v)$, and we also get a character χ_θ of $\mathrm{GL}(\alpha)$, which has weight $\theta(v)$ on the component indexed by v. In Theorem 3.18, we provided a criterion for semistability: a quiver representation M is χ_θ-semistable iff $\theta(M) = 0$ and for all $N \subsetneq M$, $\theta(N) \geq 0$. (If $\theta(N) > 0$, M is χ_θ-stable.)

We then constructed an abelian category P_θ of χ_θ-semistable objects.

Proposition 4.1. *P_θ is a* finite-length category, *i.e. all of its objects are both Noetherian and Artinian.*

Theorem 4.2 (Jordan-Hölder)**.** *Let A be a finite-length abelian category. Then any object $M \in \mathsf{A}$ has a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ such that M_i/M_{i-1} is simple for all i. Moreover, the associated graded*

$$\mathrm{gr}(M) := \bigoplus_i M_i/M_{i-1} \tag{4.3}$$

is unique up to isomorphism.

This filtration satisfies a weak uniqueness condition – it's not unique, but is close to it. For example, if A and B are two simple objects, $A \oplus B$ has two filtrations $0 \to A \to A \oplus B$ and $0 \to B \to A \oplus B$. Such a filtration is called a *Jordan-Hölder filtration* of M.

Exercise 4.4. In P_θ, M is simple iff it's θ-stable.

Proposition 4.5.

(1) *A θ-semistable Q-representation M corresponds to a closed $\mathrm{GL}(\alpha)$ orbit in $R^{ss}_{\chi_\theta}(Q, \alpha)$ iff M is semisimple (i.e. a direct sum of simple objects). Equivalently, $M \cong \mathrm{gr}(M)$.*

(2) *Two θ-semistable Q-representations M and N are S-equivalent iff $\mathrm{gr}(M) \cong \mathrm{gr}(N)$.*

The proof was left as an exercise, but is not too difficult given how we proved things last class (and is a good way to see if you understood the proof).

We will let $\mathcal{M}_\theta(Q, \alpha)$ denote the GIT quotient by $\mathrm{GL}(\alpha)$ for the character χ_θ.

Example 4.6. Consider the quiver

$$Q = \bullet \mathrel{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}} \bullet. \tag{4.7}$$

Let's choose $\alpha = (2,1)$, so $R(Q,\alpha) = \mathrm{Hom}(k^2,k)^{\oplus 4}$. Then $\mathrm{GL}(\alpha) = \mathrm{GL}_2 \times k^\times$. If you choose $\theta = (-1,1)$, then $\chi_\theta = \det^{-1}$.

The long diagonal of $\mathrm{GL}(\alpha)$ acts trivially, so let's pass to the quotient via the map $\varphi\colon \mathrm{GL}(\alpha) \to \mathrm{GL}_2$ by $(g,t) \mapsto gt^{-1}$ (here, $g \in \mathrm{GL}_2$ and $t \in k^\times$). Now we have the same scenario as in Example 2.20 – so we leverage our work there and conclude the quiver moduli space is $\mathrm{Gr}_2(k^4)$. ◂

Example 4.8. Now let's consider a slightly more interesting quiver,

$$Q = \bullet \rightrightarrows \bullet \leftleftarrows \bullet. \tag{4.9}$$

Let $\alpha = (1,1,1)$, so $R(Q,\alpha) = \mathrm{Hom}(k,k)^2 \times \mathrm{Hom}(k,k)^2$. Choose the character $\theta = (a,b,c) = (a,-a-c,c)$, where $a+b+c = 0$.

Because α is small, there aren't many subrepresentations. For example, the trivial representation $k \rightrightarrows k \leftleftarrows k$ has as a subrepresentation $S_2 := (0 \rightrightarrows k \leftleftarrows 0)$. Since $\theta(S_2) = -a-c$, we must have $-a-c \geq 0$ or all representations are unstable.

By defining more representations, we can infer more about what constraints to put on θ to have a good moduli space. For example, given $x,y,z,w \in k$, let

$$N_1(x,y) := k \overset{x}{\underset{y}{\rightrightarrows}} k \leftleftarrows 0 \tag{4.10a}$$

$$N_2(z,w) := 0 \rightrightarrows k \overset{z}{\underset{w}{\leftleftarrows}} k \tag{4.10b}$$

$$M(x,y,z,w) := k \overset{x}{\underset{y}{\rightrightarrows}} k \overset{z}{\underset{w}{\leftleftarrows}} k. \tag{4.10c}$$

Thus $N_1(x,y)$ and $N_2(z,w)$ are both suprepresentations of $M(x,y,z,w)$. Because

$$\theta(N_1(x,y)) = a + (-a-c) = -c \geq 0 \tag{4.11a}$$

$$\theta(N_2(x,y)) = -a \geq 0, \tag{4.11b}$$

we know both a and c must be negative.

There are a few more potential subrepresentations,

$$S_1 := \ k \rightrightarrows 0 \leftleftarrows 0 \tag{4.12a}$$

$$S_2 := \ 0 \rightrightarrows 0 \leftleftarrows k \tag{4.12b}$$

$$N_3 := S_1 \oplus S_2. \tag{4.12c}$$

Using these, we observe that

- $S_1 \hookrightarrow M(x,y,z,w)$ iff $x = y = 0$, and in this case, we need $\theta(S_1) = a \geq 0$, so $a = 0$;
- $S_2 \hookrightarrow M(x,y,z,w)$ iff $z = w = 0$, and in this case, we need $\theta(S_3) = c \geq 0$, so $c = 0$; and
- $N_3 \hookrightarrow M(x,y,z,w)$ iff $x = y = z = w = 0$, and in this case we need $a = c = 0$.

In summary:

(1) If $a < 0$ and $c < 0$, then $M(x,y,z,w)$ is θ-stable iff $(x,y) \neq 0$ and $(z,w) \neq 0$. Moreover, there are no *strictly semistable* (i.e. semistable but not stable) representations. At this point you might guess that the GIT quotient is $\mathbb{P}^1 \times \mathbb{P}^1$.

(2) If $a = 0$ and $c < 0$, then $M(x,y,z,w)$ is θ-semistable iff $(z,w) \neq 0$, and there are no θ-stable representations. Now suppose $M(x,y,z,w)$ is such that $(z,w) \neq 0$; then, $M(x,y,z,w)$ and $M(x',y',z',w')$ are S-equivalent iff $(z',w') = \lambda(z,w)$ for some $\lambda \in k^\times$. In this case the GIT quotient will be a $\mathbb{P}^1$, some sort of boundary where things get mushed together.

(3) In the same way, if $c = 0$ and $a < 0$, then $M(x,y,z,w)$ is θ-semistable iff $(x,z) \neq 0$, and there are no θ-stable representations. Now suppose $M(x,y,z,w)$ is such that $(x,y) \neq 0$; then, $M(x,y,z,w)$ and $M(x',y',z',w')$ are S-equivalent iff $(x',y') = \lambda(x,y)$ for some $\lambda \in k^\times$. Again we get a $\mathbb{P}^1$.

(4) If $a = c = 0$, then all M are strictly semistable, and all of them are S-equivalent. The GIT quotient will be a point.

Even in this small case, things are complicated.

We haven't shown the statements above about S-equivalence, so let's do that. There's nothing in case (1), so let's look at (2). In this case $M(x,y,z,w) \sim M(x',y',z',w')$ iff $N(z,w) \sim N(z',w')$, and it's not hard to check that we can act precisely by scalars, so this is true iff (z',w') is a nonzero scalar multiple of (z,w). (3) is the same. For (4), we have a three-stage Jordan-Hölder filtration:

$$0 \subset S_2 \subset N_1(x,y) \subset M(x,y,z,w), \tag{4.13}$$

and the pieces of the associated graded are $N_1(x,y)/S_2 \cong S_1$ and $M/N_1(x,y) \cong S_3$. Therefore $\mathrm{Gr}(M(x,y,z,w)) \cong S_1 \oplus S_2 \oplus S_3$ for any (x,y,z,w), so they're all S-equivalent.

We know to expect $\mathbb{P}^1 \times \mathbb{P}^1$ if $a, c < 0$, or $\mathbb{P}^1$ if exactly one is zero, or a point if they're both zero. You can think of letting $a \to 0$ as projecting onto the first $\mathbb{P}^1$, and letting $c \to 0$ as projecting onto the second $\mathbb{P}^1$. But we haven't proven any of these yet! We don't even know that they're varieties *a priori*, but this is the correct answer, and it's possible to prove it. ◂

Remark 4.14. It turns out all quiver varieties are rational! You can get irrational varieties by imposing composition relations between arrows in the quiver; in fact, any projective variety arises in this way. ◂

The last thing we'll do is study some more general properties of quiver varieties.

Theorem 4.15. *If Q is acyclic quiver (i.e. it has no loops), then $\mathcal{M}_\theta(Q,\alpha)$ is projective.*

Proof. $\mathcal{M}_\theta(Q,\alpha) := R(Q,\alpha)/\!/\mathrm{GL}(\alpha)$ always has a projective morphsm to $R(Q,\alpha)/\!/\mathrm{GL}(\alpha)$. That is, by definition, we have a projective morphism

$$\mathrm{Proj}\left(\bigoplus_{n\ge 0} k[R(Q,\alpha)]^{\mathrm{GL}(\alpha),\chi_\theta^n}\right) \longrightarrow \mathrm{Spec}\left(k[R(Q,\alpha])^{\mathrm{GL}(\alpha)}\right). \tag{4.16}$$

It therefore suffices to prove the codomain is a point. Since $R(Q,\alpha)/\!/\mathrm{GL}(\alpha) = \mathcal{M}_0(Q,\alpha)$, all representations have $\theta(M) = 0$.

Exercise 4.17. Show that if Q is acyclic and $M \in \mathsf{Rep}_Q$ has dimension α, then $\mathrm{gr}(M) = \bigoplus_{v\in Q_0} S_v$, where S_v is the "δ-function", a simple representation with k at v and 0 elsewhere.

Therefore they're all S-equivalent, and the quotient is a point. ⊠

Remark 4.18. The empty set is a projective variety, and may occur as the moduli space associated to an acyclic quiver. ◂

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