FURUTA'S 10/8 THEOREM

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These notes were taken in a learning seminar on Furuta's 10/8 theorem in Spring 2019. I live-TeXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Thanks to Riccardo Pedrotti for some useful comments and for the notes for §3.

Contents

1. Introduction to Seiberg-Witten theory: 1/23/19

Riccardo gave the first, introductory talk.

In 1982, Matsumoto conjectured that if M is a closed spin manifold, $b_2(M) \geq (11/8)|\sigma(M)|$. Here $b_2(M)$ is the second Betti number and $\sigma(M)$ is the signature. Equality holds for the K3 surface, so this is the best one can do.

In this seminar we'll study a theorem of Furuta which makes major progress on this conjecture.

Theorem 1.1 (10/8 theorem [Fur01])**.** *If the intersection form of M is indefinite, $b_2(M) \geq (10/8)|\sigma(M)| + 2$.*

If the intersection form is definite, work of Donaldson [Don83] says that, up to a change of orientation, the intersection form is diagonalizable, so that case is dealt with.

Furuta's proof uses both Seiberg-Witten theory and equivariant homotopy theory. It can be pushed a little bit farther, but not enough to prove the $11/8^{\text{ths}}$ conjecture, as shown recently by Hopkins-Lin-Shi-Xu [HLSX18].

Today we'll discuss some background for the proof.

Definition 1.2. Let $V \to M$ be a rank-n real oriented vector bundle. A *spin structure* on V is data $\mathfrak{s} = (P_{\mathrm{Spin}}(V), \tau)$, where $P_{\mathrm{Spin}}(V) \to M$ is a principal Spin_n-bundle and τ is an isomorphism

$$\tau\colon P_{\mathrm{Spin}}(V) \times_{\mathrm{Spin}_n} \mathbb{R}^n \xrightarrow{\cong} V.$$

A spin structure on a manifold M is a spin structure on TM.

Remark 1.3. There are other equivalent definitions of spin structures – for example, just as an orientation is a trivialization of V over the 1-skeleton of M, a spin structure is equivalent to a trivialization over the 2-skeleton. ◀

Here's a cool theorem about spin manifolds.

Theorem 1.4 (Rokhlin [Roh52]). *If M is a spin manifold, $\sigma(M) \equiv 0 \bmod 16$.*

The signature makes sense when $4 \mid \dim M$. Smoothness is crucial here; there are topological spin 4-manifolds, whatever that means, that do not satisfy this theorem. Freedman's E_8 manifold is an example.

Suppose M is a spin 4-manifold. The representation theory of Spin_4, in particular the fact that the spin representation S splits as $S^+ \oplus S^-$, leads to two quaternionic line bundles $\mathbb{S}^+, \mathbb{S}^- \to M$ with Hermitian metrics. Physics cares about these bundles, and will lead to powerful theorems in manifold topology.

These bundles have more structure: in particular, they are Clifford bundles.

Definition 1.5. Let $S \to M$ be a real vector bundle with a Euclidean metric $\langle\cdot,\cdot\rangle$. A *Clifford bundle* structure is data of, for each $x \in M$, the data of a Clifford algebra action $C\ell(T_xM)$ on S_x that varies smoothly in x, such that the Clifford action is skew-adjoint, meaning

$$\langle v \cdot s_1, s_2\rangle = -\langle s_1, v \cdot s_2\rangle.$$

We also require the existence of a connection which is compatible with the Levi-Civita connection on TM.

Given the data of a Clifford bundle, there's an operator called the *Dirac operator* D, which is the following composition:

$$C^\infty(S) \xrightarrow{\nabla^{C\ell}} C^\infty(T^*M \otimes S) \xrightarrow{\langle\cdot,\cdot\rangle} C^\infty(TM \otimes S) \xrightarrow{\text{Clifford action}} C^\infty(S). \tag{1.6}$$

This operator is denoted $\partial\!\!\!/$, a convention due to Feynman. It is a first-order, elliptic differential operator; ellipticity means that its analysis is nice.

Thus we can consider the *Seiberg-Witten equations* on a spin 4-manifold. Let $(a, \varphi) \in \Omega^1_M(i\mathbb{R}) \times \Gamma(\mathbb{S}^+)$; then the equations are

$$\partial\!\!\!/\varphi + \rho(a)(\varphi) = 0 \tag{1.7a}$$

$$\rho(\mathrm{d}^+ a) - \varphi \otimes \varphi^* + \frac{1}{2}|\varphi^2|\mathrm{id} = 0 \tag{1.7b}$$

$$\mathrm{d}^* a = 0. \tag{1.7c}$$

On a non-spin manifold, the equations are a little more complicated.

2. The monopole equations: 1/28/19

Today, Kai spoke about the monopole equations and some of their important properties, foreshadowing compactness next week. We begin with some motivation.

Recall that if M is a closed, oriented 4-manifold (in either the topological or smooth category), the intersection form $H_2(M) \times H_2(M) \to \mathbb{Z}$ is a unimodular, symmetric bilinear form.

Question 2.1. Which unimodular, symmetric bilinear forms arise as the intersection forms of smooth or topological manifolds?

For example, the intersection form of $S^2 \times S^2$ is $H := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The intersection form of $\mathbb{CP}^2$ is (1). There's an interesting bilinear form called the *E8 form*

$$\mathrm{E8} = \begin{pmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{pmatrix}. \tag{2.2}$$

Can this be realized as the intersection form of a smooth 4-manifold? Rokhlin's theorem tells us the answer is no, because such a manifold would have to be spin, and $16 \nmid \sigma(\mathrm{E8})$. However, Freedman found a topological manifold M_{E8} whose intersection form is E8!

The direct sum of two copies of E8 satisfies Rokhlin's theorem, and this form is realized by the topological 4-manifold $M_{\mathrm{E8}} \# M_{\mathrm{E8}}$. However, Donaldson showed this manifold is not smoothable: specifically, the intersection forms of smooth 4-manifolds can be diagonalized over $\mathbb{Z}$, and E8 cannot.

There's still more interesting example: consider the *K3 surface* $\{z_1^4 + z_2^4 + z_3^4 + z_4^4 = 0\} \subset \mathbb{CP}^3$; its intersection form is $-2\mathrm{E8} \oplus 3H$. So does it split as a connect sum of 3 copies of $S^2 \times S^2$ and two copies of $M_{\mathbb{E}8}$ (with the opposite orientation)? Freedman showed this is true topologically. Smoothly, of course, it can't hold, but we might still get something.

Question 2.3. Is there a smooth, oriented 4-manifold N such that, in the smooth category, $K3 \cong N \# S^2 \times s^2$?

This was a longstanding question.

Seiberg-Witten invariants allow us to answer questions such as this – though in this semester, we're more interested in the monopole map. In any case, let's define the Seiberg-Witten equations.

Let M be a smooth, oriented 4-manifold with b_2^+ odd and a Riemannian metric g, and let $\mathfrak{s}$ be a spin^c structure on M, which determines a *basic class* $K \in H^2(X)$, i.e. an integer cohomology class such that $K \equiv w_2(M) \bmod 2$. The spin^c structure $\mathfrak{s}$ defines for us spinor bundles $\mathbb{S}^+$ and $\mathbb{S}^-$. Let $\mathcal{A}_L$ denote the space of U_1-connections, $A \in \mathcal{A}_L$, and $\psi \in \Gamma(X, \mathbb{S}^+)$ (this is called a *spinor*). The Seiberg-Witten equations are

$$D_A\psi = 0 \tag{2.4a}$$

$$F_A^+ + i\delta = i\sigma(\psi). \tag{2.4b}$$

These equations have a gauge symmetry: if G denotes the group $\mathrm{Map}(X, S^1)$ with pointwise multiplication, G acts on $\mathcal{A}_L \times \Gamma(X, \mathbb{S}^+)$ on the first factor. Let B_K^+ denote the quotient minus the locus of spinors which are identically zero; then $B_K^+ \simeq \mathbb{CP}^\infty$, so we know its cohomology is isomorphic to $\mathbb{Z}[x]$, with $|x| = 2$.

Let $\mathcal{M}_K^\delta(g) \subset B_K^\times$ denote the space of solutions to the Seiberg-Witten equations. This space has dimension

$$d := \frac{1}{4}\big(K^2 - (3\sigma(M) + 2\chi(M))\big), \tag{2.5}$$

and, crucially, defines a class $[\mathcal{M}_K^\delta(g)] \in H_d(B_K^\times)$ which does not depend on g for generic choices of the metric. The *Seiberg-Witten invariants* are

$$SW_X(K) := \langle x^{d/2}, [\mathcal{M}_K^\delta(g)]\rangle \in \mathbb{Z}. \tag{2.6}$$

The fact that $b_2^+(M) = 0$ implies d is even.

This defines a map SW from the basic classes to $\mathbb{Z}$. Taubes showed two important results.

Theorem 2.7 (Vanishing theorem (Taubes))**.** *If M is diffeomorphic to a connect sum of two closed, oriented 4-manifolds $X_1 \# X_2$, $b_2^+(X_1) > 0$, and $b_2^+(X_2) > 0$, then the Seiberg-Witten equations of M vanish.*

Theorem 2.8 (Nonvanishing theorem (Taubes))**.** *If $\mathfrak{s}$ is the canonical spin^c structure associated to a complex structure on M and $b_2^+(M)$ is positive and off, then $SW(\pm c_1(M)) = \pm 1$.*

Corollary 2.9. *K3 cannot split smoothly as a connect sum.*

This leads to an interesting generalization: there are *exotic K3 surfaces*, homeomorphic but not diffeomorphic to the standard K3. They don't all admit complex structures, and many of them are not symplectic. Nonetheless, they also don't split off an $S^2 \times S^2$: this is a consequence of Furuta's 10/8 theorem, because if $K3 \cong N \# (S^2 \times S^2)$, then $b_2(N) = 20$ and $\sigma(N) = -16$, but

$$20 \not\geq \frac{10}{8}|-16| + 2. \tag{2.10}$$

Now let's discuss the monopole map. We now assume M is a spin manifold, with spin structure $\mathfrak{s}$ and spinor bundles $\mathbb{S}^\pm$. Let A denote a spin connection and consider the spaces

$$\widetilde{\mathcal{A}} := \{A + i\ker \mathrm{d}\} \times (\Gamma(\mathbb{S}^+) \oplus \Omega^1(X)) \tag{2.11}$$

$$\widetilde{C} := \{A + i\ker \mathrm{d}\} \times \big(\Gamma(\mathbb{S}^-) \oplus \Omega^0(X) \oplus H^1(X;\mathbb{R}) \oplus \Omega^+(X)\big). \tag{2.12}$$

Both of these fiber over $H^1(X;\mathbb{R})$: for $\widetilde{\mathcal{A}}$, $A + \alpha \mapsto [\alpha]$, and there is a map $\widetilde{\mu}\colon \widetilde{\mathcal{A}} \to C$ defined by

$$(A, \phi, a) \longmapsto (A, D_A\phi + ia\phi, \mathrm{d}^*a, a_{\mathrm{harm}}, \mathrm{d}^+a - \sigma(\phi)). \tag{2.13}$$

Here

- D_A is the *Dirac operator* $D_A\colon \Gamma(\mathbb{S}^+) \to \Gamma(\mathbb{S}^-)$.
- $a\phi$ denotes Clifford multiplication.

- d^* is the adjoint of d, which sends k-forms to $(k-1)$-forms, and satisfies the equation

$$\mathrm{d}^* = \star \mathrm{d} \star. \tag{2.14}$$

(This is in dimension 4; the sign convention is different in other dimensions.)
- a_{harm} is the harmonic part of a: it's a general fact that any one-form in dimension 4 splits as $a = a_{\mathrm{harm}} + \mathrm{d}^*\alpha + \mathrm{d}\beta$ for some 0-form β. A form is *harmonic* if the Laplacian $\Delta \coloneqq \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$ vanishes on it.
- $\mathrm{d}^+ a$ denotes the self-dual part of $\mathrm{d}a$.
- $\sigma(\phi)$ denotes the trace form of the endomorphism $\phi \otimes \phi^* - (1/2)\|\phi\|^2\mathrm{id}$.

Again the group G acts on $\Gamma(\mathbb{S}^\pm)$ by pointwise multiplication, using $S^1 \cong \mathrm{U}_1 \subset \mathbb{C}$. If $u \in G$, $u\colon X \to S^1$ also acts on the space of spin^c connections by $d \mapsto udu^{-1}$. Let G act trivially on forms.

Then, the map $\widetilde{\mu}$ defined in (2.13) is G-equivariant. Let G_0 denote the maps which vanish at some specified basepoint p, and let $\mathcal{A} \coloneqq \widetilde{A}/G_0$, $C \coloneqq \widetilde{C}/G_0$, and $\mu \coloneqq \widetilde{\mu}/G_0$; thus we get a map $\mu\colon A \to C$.

Now, both A and C fiber over the Picard group

$$\mathrm{Pic}^g(X) \coloneqq H^1(X;\mathbb{R})/H^1(X;\mathbb{Z}) = H^1(X;\mathbb{R})/G_0. \tag{2.15}$$

Then $S^1 = G/G_0$ acts on $\mu^{-1}(A,0,0,0,0)$, and this is the space we're interested in.

We would like to study this space, and to do so we'll need to consider Sobolev spaces. For a fixed integer $k > 2$, let A_k be the fiberwise completion of A within L_k^2 and C_{k-1} be the fiberwise completion of C within L_{k-1}^2. Then, the monopole map μ is a map $A_k \to C_{k-1}$.

Claim 2.16. This monopole map μ is S^1-equivariant, and is a compact perturbation of a linear Fredholm map.

The S^1-equivariance involves chasing through the definition but isn't bad; the rest is harder. What we can do is start by listing the terms that define a linear Fredholm map, and then check that the rest is compact. In the definition of $\widetilde{\mu}$, the terms A, $D_A\phi$, d^*a, a_{harm}, and d^+a are linear and Fredholm; thus we just have to check that $a(\phi)$ and $\sigma(\phi)$ are compact. For the first, we can use the fact that Clifford multiplication is compact, then compose with the map $C_k \to C_{k-1}$, which is also compact.

Proposition 2.17. *Let $T = \ell + c$ be a compact perturbation of a linear Fredholm map ℓ between Hilbert spaces. The restriction of T to any closed, bounded subset Ω is proper.*

This will be restated as Claim 3.5 in the next lecture, and will be proven there.

3. Compactness of the moduli space of Seiberg-Witten solutions: 2/3/19

These are Riccardo's notes on the lecture he gave, on the compactness of the moduli space of solutions to the Seiberg-Witten equations. This is a crucial step in Furuta's construction of finite-dimensional approximations, and relies on some functional analysis.

3.1. **A closer look at the Seiberg-Witten monopole map.** Let X be a oriented closed spin 4-manifold. Let $\mathfrak{s}$ be a spin structure for it. Let $\mathbb{S}^\pm$ be the positive and negative spinor bundles associated to it. Fix a spin connection A on them.

Recall the Seiberg-Witten equations can be thought as a fiber-preserving S^1-equivariant map between these two S^1-Hilbert bundles over $H^1(X;\mathbb{R})$:

$$\widetilde{\mathcal{A}} = (A + i\ker(\mathrm{d})) \times \left(\Gamma(\mathbb{S}^+) \oplus \Omega^1(X)\right) \tag{3.1a}$$

$$\widetilde{\mathcal{C}} = (A + i\ker(\mathrm{d})) \times \left(\Gamma(\mathbb{S}^-) \oplus \Omega^0(X) \oplus H^1(X;\mathbb{R}) \oplus \Omega^+(X)\right). \tag{3.1b}$$

The map $\widetilde{\mu}\colon \widetilde{\mathcal{A}} \to \widetilde{\mathcal{C}}$ is defined by

$$(A,\phi,a) \longmapsto \left(A, D_A\phi + ia\phi, \mathrm{d}^*a, a_{\mathrm{harm}}, \mathrm{d}^+a - \sigma(\phi)\right). \tag{3.2}$$

As explained in the previous seminar, $\sigma(\phi)$ denotes the trace-free endomorphism $i(\phi \otimes \phi^* - \frac{1}{2}\|\phi\|^2\mathrm{id})$ of $\mathbb{S}^+$, considered via the map ρ as a self-dual 2-form on X.

The gauge group $\mathcal{G} = \mathrm{Aut}_{\mathrm{id}}(\mathfrak{s}) \cong \mathrm{Map}(X, S^1)$ acts on spinors on the 4-manifold via multiplication with $u\colon X \to S^1$ and on Spin^c connections via addition of $ud(u^{-1})$. It acts trivially on forms.

The map $\widetilde{\mu}$ is equivariant with respect to the action of $\mathcal{G}$. Dividing by the free action of the pointed gauge group we obtain the monopole map

$$\mu = \widetilde{\mu}/\mathcal{G}_0 : \mathcal{A} \to \mathcal{C}$$

as a fiber preserving map between the bundles $\mathcal{A} = \widetilde{\mathcal{A}}/\mathcal{G}_0$ and $\mathcal{C} = \widetilde{\mathcal{C}}/\mathcal{G}_0$ over $\mathrm{Pic}^{\mathfrak{s}}(X)$. The preimage of the section $(A, 0, 0, 0, 0)$ of $\mathcal{C}$, divided by the residual S^1-action, is called the*moduli space of monopoles.*

For a fixed $k > 2$, consider the fiberwise L^2_k Sobolev completion $\mathcal{A}_k$ and the fiberwise L^2_{k-1} Sobolev completion $\mathcal{C}_{k-1}$ of $\mathcal{A}$ and $\mathcal{C}$. The monopole map extends to a continuous map $\mathcal{A}_k \to \mathcal{C}_{k-1}$ over $Pic^{\mathfrak{s}}(X)$, which will also be denoted by μ.

We will use the following properties of the monopole map.

- It is S^1-equivariant.
- Fiberwise, it is the sum $\mu = l + c$ of a linear Fredholm map l and a nonlinear compact operator c.
- Preimages of bounded sets are bounded.

Claim 3.3. The moment map is S^1-equivariant.

Proof. Equivariance is immediate. The action is the residual action of the subgroup S^1 of gauge transformations which are constant functions on X. This group acts by complex multiplication on the spaces $\Gamma(\mathbb{S}^{\pm})$ of sections of complex vector bundles and trivially on forms. ⊠

Claim 3.4. Fiberwise, the moment map is the sum $\mu = l + c$ of a linear Fredholm map ℓ and a nonlinear compact operator c.

Proof. Restricted to a fiber, the monopole map is a sum of the linear Fredholm operator ℓ, consisting of the elliptic operators D_A and $d^* + d^+$, complemented by projections to and inclusions of harmonic forms. The nonlinear part of μ is built from the bilinear terms $a\phi$ and $\sigma(\phi)$. Multiplication $\mathcal{A}_k \times \mathcal{A}_k \to \mathcal{C}_k$ is continuous for $k > 2$. Combined with the compact restriction map $\mathcal{C}_k \to \mathcal{C}_{k-1}$ (Rellich lemma, see [Per18, Lecture 19, p. 2]) we gain the claimed compactness for c: Images of bounded sets are contained in compact sets. ⊠

Now let us show the following very useful property of compact perturbations of Fredholm operators.

Claim 3.5. The restriction of a compact perturbation $l + c\colon \mathcal{U}' \to \mathcal{U}$ of a linear Fredholm map ℓ between Hilbert spaces to any bounded, closed subset is proper.

Proof. Let p denote a projection to the kernel of ℓ. Let A be a bounded closed subset of $\mathcal{U}'$. It's easy to see that we have the following commutative diagram

$$\begin{tikzcd}
A \arrow[r, "(\ell,c,p)"] \arrow[ddr, "\ell+c"'] & \mathcal{U} \times \overline{c(A)} \times \overline{p(A)} \arrow[d, "\cong"', "+"] \\
& \mathcal{U} \times \overline{c(A)} \times \overline{p(A)} \arrow[d, "\pi"] \\
& \mathcal{U}
\end{tikzcd}$$

We observe that the map $h\colon A \to \mathcal{U} \times \overline{c(A)} \times \overline{p(A)}$ given by $a \mapsto (\ell(a), c(a), p(a))$ is injective and closed. Injectivity is clear since we are projecting on the kernel.

Closedness is a little bit more involved: let $\{(\ell_n, c_n, p_n)\}_n \subset \mathrm{Im}(h)$ converge to $(\ell_\infty, c_\infty, p_\infty)$. In particular there is a sequence $\{a_n\}_n \subset A$ such that $(\ell_n, c_n, p_n) = (\ell(a_n), c(a_n), p(a_n))$. We want to prove that $(\ell_\infty, c_\infty, \rho_\infty) \in h(A)$. Since ℓ is Fredholm we have the following property: *every bounded sequence* $\{x_i\}_i$ *in the domain whose image is convergent admits a convergent subsequence* $\{x_{i_j}\}_j$. Since A is closed and bounded (and any other closed subset of it would be bounded as well hence we can directly work with A), $\{a_n\}_n$ is bounded. Since ℓ is Fredholm we can extract a convergent subsequence $\{a'_n\}_n$ converging to $a \in A$ (since A is closed). By the uniqueness of the limit, it's easy to prove

$$(\ell_\infty, c_\infty, \rho_\infty) = (\ell(a), c(a), p(a)) \tag{3.6}$$

which proves the closedness of $h(A)$. This implies that h is proper, since h is an homeomorphism onto its image.

The addition map $+\colon (u, s, e) \mapsto (u + s, s, e)$ is an homeomorphism hence proper. The projection to $\mathcal{U}$ is proper since the other two factors are compact. ⊠

3.2. **A collection of results.** We will list here some results needed for the seminar.

Let U be an open subset of $\mathbb{R}^n$. We can consider the space $C_c^\infty(U;\mathbb{R}^r)$ of compactly supported $\mathbb{R}^r$-valued functions. Fix a real number $p>1$ and an integer $k \geq 0$. The Sobolev L_k^p norm is defined by

$$\|f\|_{p,k} := \sum_{|\alpha|<k} \sup_U \|D^\alpha f\|_p. \tag{3.7}$$

The Sobolev space $L_k^p(E)$ is defined to be the completion of $\Gamma(E)$ in the L_k^p norm.

Here are the basic facts about Sobolev spaces.

Sobolev inequality: If $k \leq \ell$ then there exists a constant C such that

$$\|\cdot\|_{p,k} \leq C\|\cdot\|_{p,\ell}, \tag{3.8}$$

and hence we have a bounded inclusion of Sobolev spaces $L_k^p(E) \hookrightarrow L_\ell^p(E)$.

Rellich lemma: The inclusion $L_{k+1}^p(E) \hookrightarrow L_k^p(E)$ is a compact operator.

Morrey inequality: Suppose $\ell \geq 0$ is an integer such that $\ell < k - n/p$; then there is a constant C such that

$$\|\cdot\|_{C^\ell} \leq C\|\cdot\|_{p,k}, \tag{3.9}$$

i.e. there is a bounded inclusion

$$L_k^p(E) \hookrightarrow C^\ell(E). \tag{3.10}$$

Smoothness: One has

$$\bigcap_{k\geq k_0} L_k^p(E) = C^\infty(E). \tag{3.11}$$

Lemma 3.12. *Over a closed Riemannian 4-manifold, multiplication of smooth functions extends to a bounded map*

$$L_k^2(X) \otimes L_\ell^2(X) \to L_\ell^2(X) \tag{3.13}$$

provided that $k \geq 3$ and $k \geq \ell$. In particular, $L_k^2(X)$ is an algebra for $k \geq 3$.

There are also bounded multiplication maps for the lower regularity Sobolev spaces in 4 dimensions, but these bring in Sobolev spaces with $p > 2$.

Let now $D\colon \Gamma(E) \to \Gamma(F)$ be a differential operator of order m over a closed, oriented, Riemannian manifold (M,g). The basic point is that D extends to a bounded linear map between Hilbert spaces:

$$D\colon L_{k+m}^2(E) \to L_k^2(F). \tag{3.14}$$

Theorem 3.15 (Elliptic estimate)**.** *If D is elliptic of order m, one has estimates on the L_k^2-Sobolev norms for each $k \geq 0$:*

$$\|s\|_{2,k+m} \leq C_k(\|Ds\|_{2,k} + \|s\|_{2,k}). \tag{3.16}$$

Moreover,

$$\|s\|_{2,k+m} \leq C_k\|Ds\|_{2,k} \tag{3.17}$$

for $s \in (\ker D)^\perp$ (here $^\perp$ denotes the L^2-orthogonal complement).

There is an analogue for $L^{p,k+m}$ bounds.

As a consequence of this important theorem we have the following:

Corollary 3.18. *An elliptic operator D of order m defines a Fredholm map $L_{k+m}^2(E) \to L_k^2(F)$ for any $k \geq 0$. Its index is independent of k. Moreover, its index depends only on the symbol of D.*

Let (M,g) be an oriented Riemannian manifold. Let ∇ be an orthogonal covariant derivative in a real, Euclidean vector bundle $E \to M$. We know that ∇ has a formal adjoint ∇^*.

Proposition 3.19 (The Lichnérowicz formula)**.** *One has*

$$D^2 = \widetilde{\nabla}^*\widetilde{\nabla} + \frac{1}{4}\mathrm{scal}_g \cdot \mathrm{id}_{\mathbb{S}} + \frac{1}{2}\rho(F^\circ). \tag{3.20}$$

Lemma 3.21.

$$\frac{1}{2}\mathrm{d}^*\mathrm{d}(|s|^2) = \langle \nabla^*\nabla s, s\rangle - |\nabla s|^2. \tag{3.22}$$

Proof sketch. See [Per18, Lecture 19, Lemma 1.1]. The idea is to study the integral

$$\int_M f\langle \nabla^*\nabla s, s\rangle \,\mathrm{vol} \tag{3.23}$$

where f has compact support. ☒

It's important to remember that the one above is a pointwise equality.
Working locally one has the following result.

Lemma 3.24. *For a smooth function $f\colon M \to \mathbb{R}$ with compact support, if p is a local maximum, then $(\mathrm{d}^*\mathrm{d}f)(p) \geq 0$.*

The following lemma is an easy calculation.

Lemma 3.25. *For $\phi \in \Gamma(\mathbb{S}^+)$, one has*

$$((\phi\phi^*)_0\chi, \chi) = (\phi, \chi)^2 - \frac{1}{2}|\chi|^2|\phi|^2. \tag{3.26}$$

In particular,

$$((\phi\phi^*)_0\phi, \phi) = \frac{1}{2}|\phi|^4. \tag{3.27}$$

Proof. We have

$$\begin{aligned} ((\phi\phi^*)_0\chi, \chi) &= ((\phi\phi^*)\chi, \chi) - \frac{1}{2}(|\phi|^2\chi, \chi) \\ &= ((\phi, \chi)\phi, \chi) - \frac{1}{2}|\phi|^2|\chi|^2 \\ &= (\phi, \chi)^2 - \frac{1}{2}|\phi|^2|\chi|^2. \end{aligned}$$ ☒

Lemma 3.28. *For $\eta \in \Omega^2_X$ and $\phi \in \Gamma(\mathbb{S})$, one has $(\rho(\eta)\phi, \phi) \leq |\eta||\phi|^2$.*

Proof. It suffices to take $\eta = e \wedge f$ for orthogonal unit vectors e and f. One then has

$$\begin{aligned} (\rho(\eta)\phi, \phi) &= (\rho(e \wedge f)\phi, \phi) && (3.29)\\ &= \frac{1}{2}([\rho(e), \rho(f)]\phi, \phi) && (3.30)\\ &= -\frac{1}{2}(\rho(f)\phi, \rho(e)\phi) && (3.31)\\ &\leq |\rho(e)\phi| \cdot |\rho(f)\phi|, && (3.32) \end{aligned}$$

where in (3.31) we used the fact that ρ has image in the anti-skew-Hermitian matrices. Now since $|e| = 1$ then $|\rho(e)| = 1$ (similarly for f), and therefore we conclude. ☒

Lemma 3.33. *Let A be a Clifford connection for the spinor bundle of a spin^c structure of X. Let $a \in \Omega^1_X(i\mathbb{R})$; then*

$$D_{A+a}\phi = D_A\phi + a \cdot \phi, \tag{3.34}$$

where the last term is the Clifford multiplication between a and ϕ.

Proof. Let's work in local orthonormal coordinates of TX given by $\{e_1, \dots, e_n\}$. We have

$$\begin{aligned} D_{A+a}\phi &= \sum_i e_i \cdot (A+a)_{e_i}\phi \\ &= \sum_i e_i \cdot A_{e_i}\phi + \sum_i e_i \cdot a(e_i)\phi \\ &= D_A\phi + \sum_i e_i \cdot a(e_i)\phi \\ &= D_A\phi + a\sum_i e_i\phi \\ &= D_A\phi + a\cdot\phi. \end{aligned}$$

Notice that here we used that $a \in \Omega^1_X(i\mathbb{R})$ hence all the coefficients $a(e_i)$ are equal to each other, and without loss of generality we named then a. ⊠

3.3. **Compactness of the moduli space.** If the bundles $\mathcal{A}$ and $\mathcal{C}$ were finite-dimensional, then the boundedness property would be equivalent to properness. In this infinite-dimensional setting, the argument above can be used the same way as Heine-Borel in the finite-dimensional case to show that the boundedness condition implies properness. It turns out that the ingredients of the compactness proof for the moduli space also prove the stronger boundedness property.

Proposition 3.35. *Preimages $\mu^{-1}(B) \subset \mathcal{A}_k$ of bounded disk bundles $B \subset \mathcal{C}_{k-1}$ are contained in bounded disk bundles.*

Proof. It is sufficient to prove this fiberwise for the Sobolev completions of the restriction of the monopole map to the space $\{A\} \times (\Gamma(\mathbb{S}^+) \oplus \ker(d^*))$, which maps to $\{A\} \times \big(\Gamma(\mathbb{S}^-) \oplus \Omega^2_+(X) \oplus H^1(X;\mathbb{R})\big)$. We start by defining the following scalar product: using the elliptic operator $D = D_A + \mathrm{d}^+$ and its adjoint, define the L^2_k-norm via the scalar product on the respective function spaces through

$$(\cdot,\cdot)_i = (\cdot,\cdot)_0 + (D\cdot, D\cdot)_{i-1} \text{ for } 0 < i \leq k \tag{3.36a}$$

$$(\cdot,\cdot)_0 = \int_X \langle\cdot,\cdot\rangle. \tag{3.36b}$$

Using the elliptic estimates and continuity (i.e. boundedness) of D it's easy to see that this norm is equivalent to the classic Sobolev one. A similar definition can be extended to norms for the L^p_k-spaces. Let us take $\mu(A,\phi,a) = (A,\varphi,b,a_{\mathrm{harm}}) \in \mathcal{C}_{k-1}$ with the norm of the latter bounded by some constant R. The Lichnérowicz formula (Proposition 3.19) for a connection $A + a = A'$ reads

$$D^*_{A'}D_{A'} = A' \circ A' + \frac{1}{4}s\cdot \mathrm{id}_{\mathbb{S}} + \frac{1}{2}\rho(F^\circ_{A'}) \tag{3.37}$$

with s denoting the scalar curvature of X. As a consequence we have a pointwise estimate: using Lemma 3.21,

$$d^*d|\phi|^2 = 2\langle\nabla^*_{A'}\nabla_{A'}\phi,\phi\rangle - 2\langle\nabla_{A'}\phi,\nabla_{A'}\phi\rangle. \tag{3.38}$$

Then, removing the negative quantity on the left to obtain an inequality,

$$\leq 2\langle\nabla^*_{A'}\nabla_{A'}\phi,\phi\rangle \tag{3.39}$$

$$\leq 2\langle D^*_{A'}D_{A'}\phi - \frac{s}{4}\phi - \frac{1}{2}\rho(F^\circ_{A'})\phi,\phi\rangle. \tag{3.40}$$

Substituting in the second Seiberg-Witten equation,

$$\leq \langle 2D^*_{A'}\varphi - \frac{s}{2}\phi - (\sigma(\phi)+b)\phi,\phi\rangle \tag{3.41}$$

Now we move some terms to the left and use the equality $D_{A+a} = D_A + a$ together with the fact that the Dirac operator is self-adjoint to get

$$d^*d|\phi|^2 + \frac{s}{2}|\phi|^2 + \langle\sigma(\phi),\phi\rangle \leq \langle 2D^*_{A'}\varphi,\phi\rangle - \langle b\phi,\phi\rangle. \tag{3.42}$$

Next, use Lemma 3.25 to bound $\sigma(\phi)$ and obtain

$$d^*d|\phi|^2 + \frac{s}{2}|\phi|^2 + \frac{1}{2}|\phi|^4 \leq \langle 2D_A^*\varphi, \phi\rangle + 2\langle a\cdot\varphi, \phi\rangle - b|\phi|^2 \tag{3.43}$$

$$\leq 2\left(\|D_A^*\varphi\|_\infty + \|a\|_\infty\|\varphi\|_\infty\right)\cdot|\phi| + \|b\|_\infty\cdot|\phi|^2. \tag{3.44}$$

$$\leq c_1\left(\left(1+\|a\|_\infty\right)\|\varphi\|_{L^2_{k-1}}\cdot|\phi| + \|b\|_{L^2_{k-1}}\cdot|\phi|^2\right), \tag{3.45}$$

using the Sobolev embedding theorem (Morrey's inequality) to bound the L^∞-norm with the Sobolev norm.

Now we need to estimate $\|a\|_\infty$. First thing, for $p>4$ we get a Sobolev estimate $\|a\|_\infty \leq c_2\|a\|_{L^p_1}$ and then use the elliptic estimate:

$$\|a\|_{L^p_1} = \|a_{\mathrm{harm}} + a'\|_{L^p_1} \leq \|a_{\mathrm{harm}}\|_{L^p_1} + \|a'\|_{L^p_1} \tag{3.46}$$

$$\leq \|a_{\mathrm{harm}}\|_{L^p_0} + \|d^+a\|_{L^p_0} \tag{3.47}$$

where in (3.46) we used the Hodge decomposition of a and in (3.47) we applied the elliptic estimate to both component. Recall that $d^+(a_{\mathrm{harm}}) = 0$ and $d^+a = d^+a'$.

Combining with the equality $d^+a = b + \sigma(\phi)$ then leads to an estimate

$$\|a\|_\infty \leq c_4\left(\|a_{\mathrm{harm}}\|_{L^p_0} + \|b\|_{L^p_0} + \|\sigma(\phi)\|_{L^p_0}\right) \tag{3.48}$$

$$\leq c_5\left(\|a_{\mathrm{harm}}\|_{L^p_0} + \|b\|_{L^2_{k-1}} + \|\phi\|^2_\infty\right) \tag{3.49}$$

In the last passage we control the L^p_0-norm with the L^2_{k-1}-one, since $p>4$.

Putting these two estimates together, we get something of the form

$$d^*d|\phi|^2 + \frac{1}{2}\|s\|_\infty\|\phi\|^2_\infty + \frac{1}{2}\|\phi\|^4_\infty \leq c\left(1 + c_5\left(\|a_{\mathrm{harm}}\|_{L^p_0} + \|b\|_{L^2_{k-1}} + \|\phi\|^2_\infty\right)\right)\|\varphi\|_{L^2_{k-1}}\cdot\|\phi\|_\infty + \|b\|_{L^2_{k-1}}\cdot\|\phi\|^2_\infty \tag{3.50}$$

$$\leq K\|\phi\|^3_\infty + R\|\phi\|^2_\infty, \tag{3.51}$$

where in (3.51) we applied the bounds we had by assumption on the elements in the image.

So our inequality is now:

$$d^*d|\phi|^2 + \frac{1}{2}\|\phi\|^4_\infty \leq K\|\phi\|^3_\infty + R\|\phi\|^2_\infty - \frac{1}{2}\|s\|_\infty\|\phi\|^2_\infty \tag{3.52}$$

$$\leq K\|\phi\|^3_\infty + R\|\phi\|^2_\infty. \tag{3.53}$$

Now this inequality must hold in particular when ϕ achieves its maximum, and on that point the Laplacian is positive, hence we can forget about it and get

$$\frac{1}{2}\|\phi\|^4_\infty \leq K\|\phi\|^3_\infty + R\|\phi\|^2_\infty. \tag{3.54}$$

In particular we bound the 4^{th} power of a quantity with a polynomial in that quantity of degree 3. This implies that $\|\phi\|_\infty$ must be bounded. Therefore we can bound the L^p_0-norm of (ϕ, a) for every $p \geq 1$.

Now comes bootstrapping: for $i \leq k$, assume inductively L^2_{i-1}-bounds on (ϕ, a). To obtain L^2_i-bounds, compute:

$$\|(\phi,a)\|^2_{L^2_i} - \|(\phi,a)\|^2_{L^2_0} = \|(D_A\phi, d^+a)\|^2_{L^2_{i-1}} \tag{3.55}$$

$$= \|(\phi + ia\phi, b - \sigma(\phi)\|^2_{L^2} \tag{3.56}$$

$$= \|(\phi, b)\|^2_{L^2_{i-1}} + \|(ia\phi, \sigma(\phi))\|^2_{L^2_{i-1}}. \tag{3.57}$$

The first equality holds by our definition of the Sobolev norm. The last equality holds as $D_{A'} = D_A + a$. The summands in the last expression are bounded by the assumed L^2_{i-1}-bounds on (ϕ, a) together with the Sobolev multiplication properties. Note that the steps for $i = 2$ and 3 require special care (see [Per18, Lecture 21, p. 4]) or use Sobolev embedding together with the fact that we have control on the L^p-norms of (ϕ, a) for every p, which gives us control on the respective Sobolev norms for $p = 2$. ⊠

4. The Pin_2^--symmetry: 2/11/19

These are Arun's prepared lecture notes on the group Pin_2^-, its representations, and the Pin_2^- symmetry in the Seiberg-Witten equations associated to a spin 4-manifold.

4.1. Some avatars of Pin_2^-.

In the first part of the talk, I'll tell you some basic facts about Pin_2^-. In Seiberg-Witten theory, this group is often just called $\mathrm{Pin}(2)$, but that could be confusing: there's also Pin_2^+, which is different.

Definition 4.1. Recall that given a vector space V (over $\mathbb{R}$ or $\mathbb{C}$) and a quadratic form Q, we can form the Clifford algebra $C\ell(V,Q) := TV/(v\otimes v - Q(v)1)$. That is, we take the tensor algebra and introduce the relation $v^2 = Q(v)$. This is a $\mathbb{Z}/2$-graded algebra with the grading given by the length of a tensor mod 2; let α denote the *grading operator*, which acts on the even subspace as 1 and on the odd subspace as -1. It is common to think of V as sitting inside of $C\ell(V,Q)$ as the length-1 tensors.

The *Clifford group* $\Gamma(V,Q)$ is the group of $x \in C\ell(V,Q)^\times$ such that $\alpha(x)yx^{-1} \in V \subset C\ell(V,Q)$ for all $y \in V$.

Consider the involution $\beta\colon C\ell(V,Q) \to C\ell(V,Q)$ sending $v_1\otimes\cdots\otimes v_n \mapsto v_n \otimes\cdots\otimes v_1$. The *Clifford norm* is $N(v) := \beta(v)\cdot v$, which is a scalar on $\Gamma(V,Q)$.

The *pin group* $\mathrm{Pin}(V,Q)$ is the kernel of the Clifford norm inside $\Gamma(V,Q)$. The *spin group* $\mathrm{Spin}(V,Q)$ is the subgroup of even elements of $\mathrm{Pin}(V,Q)$. The following shorthand is standard:

- If $V=\mathbb{R}^n$ and $Q(x)=\langle x,x\rangle$, $C\ell(V,Q)$ is denoted $C\ell_n$ and $\mathrm{Pin}(V,Q)$ is denoted Pin_n^+; if $Q(x) = -\langle x,x\rangle$, they're denoted $C\ell_{-n}$ and Pin_n^-.
- The spin groups in these cases are canonically isomorphic, and denoted Spin_n.
- If $V=\mathbb{C}^n$ and $Q(x)=\langle x,x\rangle$, $\mathrm{Pin}(V,Q)$ is denoted Pin_n^c, and $\mathrm{Spin}(V,Q)$ is denoted Spin_n^c.

These are all compact, real Lie groups; there's a map $\mathrm{Spin}_n \to \mathrm{SO}_n$ which is a double cover, connected if $n\ge 2$ and universal if $n \ge 3$. Correspondingly there's a double cover $\mathrm{Pin}_n^\pm \to \mathrm{O}_n$. $\mathrm{Pin}_n^\pm$ has two components if $n>1$; $\mathrm{Pin}_1^+ \cong \mathbb{Z}/2\times\mathbb{Z}/2$ and $\mathrm{Pin}_1^- \cong \mathbb{Z}/4$.

Remark 4.2. Why would you want pin groups anyways? *A posteriori*, of course, we're going to find a Pin_2^- symmetry in the Seiberg-Witten equations of a spin 4-manifold, but there are other reasons to care. One rough answer is that there are many places in geometry and physics (index theory, fermionic QFT, ...) where one wants spin or spin^c structures, but if you want to try to study the same story on unoriented manifolds, the analogues are pin and pin^c structures. ◄

Now we focus specifically on Pin_2^-, with the hope of getting some intuition for what it is. We know it contains Spin_2 as an index-2 subgroup, and topologically is two circles.

We can get our hands on it by embedding it in Spin_3, which we do understand. Consider the map $C\ell_{-2} \hookrightarrow C\ell_{-3}^0$ (i.e. into the even part of $C\ell_{-3}$) sending $e_1\mapsto e_1e_3$ and $e_2\mapsto e_2e_3$. This also sends $1\mapsto 1$ and $e_1e_2\mapsto e_1e_2$.

There's an identification $C\ell_{-3}^0 \cong \mathbb{H}$ via $e_1e_3\mapsto i$, $e_2e_3\mapsto j$, and $e_1e_2\mapsto k$, which restricts to the (possibly familiar) isomorphism $\mathrm{Spin}_3\cong\mathrm{Sp}_1$ (which is also SU_2). This then restricts to an identification

$$\mathrm{Pin}_2^- \cong \{e^{i\theta}\}\cup\{je^{i\theta}\} \subset \mathrm{Sp}_1, \tag{4.3}$$

which is sometimes taken as a definition in this area, and which we will use heavily. The first thing it gives us is a representation of Pin_2^- on $\mathbb{H}$. We will also let $\widetilde{\mathbb{R}}$ denote the real representation of Pin_2^- which is trivial on Spin_2, and such that j acts by -1.

4.2. Appearance in the Seiberg-Witten equations.

Furuta produces the Pin_2^- symmetry in the Seiberg-Witten equations in a very elegant way, doing everything over a point, where it's close to obvious, and using the associated bundle construction to move to the tangent and spinor bundles.

Definition 4.4. Here's some notation for some representations of $\mathrm{Spin}_4\cong\mathrm{Sp}_1\times\mathrm{Sp}_1$.

- Let ${}_\pm\mathbb{H}$ denote the left action of $\mathrm{Sp}_1\times\mathrm{Sp}_1$ on the quaternions $\mathbb{H}$ by the first factor (${}_-\mathbb{H}$) or the second factor (${}_+\mathbb{H}$). These are the spinor representations.
- Let ${}_-\mathbb{H}_+$ denote the action of $\mathrm{Sp}_1\times\mathrm{Sp}_1$ on $\mathbb{H}$ by $(p,q)\cdot v = pvq^{-1}$. For Spin_4, this is the representation $\mathrm{Spin}_4 \twoheadrightarrow \mathrm{SO}_4 \hookrightarrow \mathrm{GL}_4(\mathbb{R})$.
- Let ${}_+\mathbb{H}_+$ denote the action of $\mathrm{Sp}_1\times\mathrm{Sp}_1$ by $(p,q)\cdot v = qvq^{-1}$.

Given any representation or equivariant vector bundle V, we'll let $\widetilde{V} := V \otimes \widetilde{\mathbb{R}}$.

If $(X, \mathfrak{s})$ is a 4-manifold with associated principal Spin_4-bundle $P_{\mathfrak{s}} \to X$, then we have the associated bundles

$$\mathbb{S}^{\pm} \cong P_{\mathfrak{s}} \times_{\mathrm{Spin}_4} {}_{\pm}\mathbb{H} \to X \tag{4.5a}$$

$$TX \cong P_{\mathfrak{s}} \times_{\mathrm{Spin}_4} {}_{-}\mathbb{H}_{+} \to X \tag{4.5b}$$

$$\Lambda := \mathbb{R} \oplus \Lambda^2_+ T^*X \cong P_{\mathfrak{s}} \times_{\mathrm{Spin}_4} {}_{+}\mathbb{H}_{+} \to X. \tag{4.5c}$$

Now we throw in a Pin_2^--action and extend ${}_{\pm}\mathbb{H}$ and ${}_{+}\mathbb{H}_{\pm}$ to $\mathrm{Spin}_4 \times \mathrm{Pin}_2^-$-representations:

- Using the inclusion $\mathrm{Pin}_2^- \hookrightarrow \mathrm{Sp}_1$, we define the action of $g \in \mathrm{Pin}_2^-$ on ${}_{\pm}\mathbb{H}$ to be right multiplication by g^{-1}.
- Let Pin_2^- act trivially on ${}_{\pm}\mathbb{H}_{+}$.

We need these to commute with the Spin_4-actions but that's easy, and therefore using (4.5), we have actions of Pin_2^- on the fibers of TX, $\mathbb{S}^{\pm}$, and Λ.

Proposition 4.6. *The monopole map is equivariant with respect to these* Pin_2^-*-actions.*

Proof.
(1) You can check in one line that the multiplication map ${}_{-}\mathbb{H}_{+} \times {}_{+}\mathbb{H} \to {}_{-}\mathbb{H}$ is $\mathrm{Spin}_4 \times \mathrm{Pin}_2^-$-equivariant. Passing to associated bundles, this says Clifford multiplication $C\colon \mathbb{S}^+ \to \mathbb{S}^-$ is Pin_2^--equivariant.

(2) It's just as easy to check that the map ${}_{-}\mathbb{H}_{+} \times {}_{-}\widetilde{\mathbb{H}}_{+} \to {}_{-}\widetilde{\mathbb{H}}_{+}$ sending $a, b \mapsto \overline{a}b$ is $\mathrm{Spin}_4 \times \mathrm{Pin}_2^-$-equivariant, so the map

$$\begin{aligned} \widetilde{C}\colon T^*X \times \widetilde{T}^*X &\longrightarrow \widetilde{\Lambda} \\ a, b &\longmapsto (\langle a, b\rangle, (a \wedge b)_+), \end{aligned} \tag{4.7}$$

which Furuta calls "twisted Clifford multiplication," is Pin_2^--equivariant. (Here we passed from TX to T^*X, of course using the metric to do so.)

(3) All named Pin_2^--representations have been unitary (orthogonal for $\widetilde{\mathbb{R}}$), so the actions of Pin_2^- on $\mathbb{S}^{\pm}$ are unitary (with respect to the Hermitian metric induced from the Riemannian metric on X), and on T^*X, $\widetilde{T}^*X$, Λ, and $\widetilde{\Lambda}$ are orthogonal. Therefore the covariant derivatives associated to these bundles are also Pin_2^--equivariant, hence so are the Dirac operators

$$D_1 := C \circ \nabla \colon \Gamma(\mathbb{S}^+) \longrightarrow \Gamma(\mathbb{S}^-) \tag{4.8a}$$

$$D_2 := \widetilde{C} \circ \nabla \colon \Gamma(\widetilde{T}^*X) \longrightarrow \Gamma(\widetilde{\Lambda}). \tag{4.8b}$$

(Here D_2 can be identified with $\mathrm{d}^* + \mathrm{d}^+$.) Therefore $D := D_1 \oplus D_2$ is also Pin_2^--equivariant.

(4) Now consider the map

$$\begin{aligned} {}_{+}\mathbb{H} \times {}_{-}\widetilde{\mathbb{H}}_{+} &\longrightarrow {}_{-}\mathbb{H} \times {}_{+}\widetilde{\mathbb{H}}_{+} \\ \phi, a &\longmapsto (a\phi i, \phi i \overline{\phi}). \end{aligned} \tag{4.9}$$

In a similar way, one can check this is a (nonlinear) $\mathrm{Spin}_4 \times \mathrm{Pin}_2^-$-equivariant map. It passes to a map of associated bundles $Q\colon \Gamma(\mathbb{S}^+ \oplus \widetilde{T}^*M) \to \Gamma(\mathbb{S}^- \oplus \widetilde{\Lambda})$, which is Pin_2^--equivariant.[1]

Therefore the monopole map $SW = D + Q$ is Pin_2^--equivariant. Because the Pin_2^--action is continuous, it doesn't matter what regularity we impose on sections: this fact is true both for smooth sections and their Sobolev completions. ⊠

4.3. **Some computations with the representation ring.** The proof of the $10/8^{\mathrm{ths}}$ theorem requires a few more pure representation-theoretic results, and since we have time, I'll go over them now. Let's start by listing some representations of Pin_2^-.

Example 4.10. The first representations you'd write down are the trivial representation 1 and the *sign representation* $\sigma := \widetilde{\mathbb{C}}$.

We can next define some irreducible two-dimensional representations h_d, indexed by $d \in \mathbb{Z}$, as follows: $\mathrm{Pin}_2^- = \{e^{i\theta}\} \cup \{je^{i\theta}\}$, so let the underlying complex vector space of h_d be $\mathbb{H} = \mathbb{C}^2$, with j acting in the usual

[1] In fact, since the second factor is purely imaginary, we know the image isn't just in $\mathbb{S}^- \oplus \widetilde{\Lambda}$, but in $\mathbb{S}^- \oplus \Lambda^2_+ T^*X$.

way and $e^{i\theta}$ acting by $(e^{id\theta}, e^{-id\theta})$. You can prove these are irreducible by just choosing a nonzero quaternion and pushing it around with elements of Pin_2^- until you get a basis, and this isn't hard.

As a particular example, h_1 is $\mathbb{H}$ with the Pin_2^--action restricted from the usual $\mathrm{Spin}_3 = \mathrm{Sp}_1$-action. ◀

Theorem 4.11. *The above is a complete list of isomorphism classes of irreducible representations of* Pin_2^-.

I don't know how one proves this: it's asserted by both Furuta and Bryan without proof.

Definition 4.12. The *representation ring* of a group G, denoted $RU(G)$, is the Grothendieck ring of the category of complex representations of G. That is, it is the abelian group freely generated by isomorphism classes of finite-dimensional complex representations of G modulo the relations $[V] = [V'] + [V'']$ whenever there is a short exact sequence $0 \to V' \to V \to V'' \to 0$. The ring structure is defined by $[V] \cdot [W] \coloneqq [V \otimes W]$.

Let's begin with a simple example.

Proposition 4.13. *The representation ring of* $\mathrm{Spin}_2 = \mathrm{U}_1$ *is* $\mathbb{Z}[t, t^{-1}]$, *where* $t\colon \mathrm{U}_1 \to \mathrm{U}_1$ *is the identity map.*

Proof. We can compute by taking the irreducible representations as generators and computing their relations. The irreducible representations of U_1 are indexed by $\mathbb{Z}$, with the d^{th} one χ_d sending $z \mapsto z^d$. The tensor product of one-dimensional matrices is the ordinary product in $\mathbb{C}$, so $\chi_d \otimes \chi_{d'} = \chi_{d+d'}$. Therefore $\chi_1 \mapsto t$ gives us $\mathbb{Z}[t, t^{-1}]$. ⊠

Lemma 4.14. *There's an isomorphism* $h_{d_1} \otimes h_{d_2} \cong h_{d_1+d_2} \oplus h_{d_1-d_2}$.

Proof. Inside $h_{d_1} \otimes h_{d_2} \cong \mathbb{H} \otimes_{\mathbb{C}} \mathbb{H}$, the subspace $V \coloneqq \mathrm{span}_{\mathbb{C}}\{1 \otimes 1, j \otimes j\}$ is preserved by j and $e^{i\theta}$, hence is a subrepresentation. The same applies to $W \coloneqq \mathrm{span}_{\mathbb{C}}\{1 \otimes j, j \otimes 1\}$. The vector space isomorphism $V \xrightarrow{\cong} h_{d_1+d_2}$ sending $1 \otimes 1 \mapsto e_1$ and $j \otimes j \mapsto e_2$ is Pin_2^--equivariant, which you can quickly check by hand; the same idea applies to $W \cong h_{d_1-d_2}$. ⊠

Corollary 4.15.

$$RU(\mathrm{Pin}_2^-) \cong \mathbb{Z}[\sigma, h_d \mid d \in \mathbb{Z}]/(\sigma^2, \sigma h_d = h_{-d}, h_{d_1} h_{d_2} = h_{d_1+d_2} + h_{d_1-d_2}).$$

The last thing we need to do is compute the image of the restriction map $RU(\mathrm{Pin}_2^-) \to RU(\mathrm{Spin}_2)$.

Corollary 4.16. *Under the above identifications, the map* $RU(\mathrm{Pin}_2^-) \to RU(\mathrm{Spin}_2)$ *sends* $\sigma \mapsto 1$ *and* $h_d \mapsto t^d + t^{-d}$.

5. Finite-dimensional approximations, I: 2/18/19

"These vector spaces are indexed by your favorite barnyard animals."

Today, Cameron spoke about finite-dimensional approximations to the monopole map. The idea is that taking the one-point compactification of its domain and codomain defines a Pin_2^--equivariant map between infinite-dimensional spheres. This is pretty cool, except that infinite-dimensional spheres are contractible, even equivariantly, so we need to take some finite-dimensional approximation in order to obtain homotopically interesting information.

The following theorem is the goal of the next two lectures.

Theorem 5.1. *Let* M *be a closed, spin 4-manifold such that* $b_1(M) = 0$, $b_2^+(M) > 0$, *and* $\sigma(M) < 0$. *Then there are finite-dimensional* Pin_2^-*-representations* V_λ *and* $\overline{W}_\lambda$ *and* Pin_2^-*-equivariant maps* $D_\lambda\colon V_\lambda \to \overline{W}_\lambda$ *(linear) and* $Q_\lambda\colon V_\lambda \to \overline{W}_\lambda$ *(quadratic) such that*

(1) *as* Pin_2^-*-representations,* $V_\lambda \cong \mathbb{H}^{k+m} \oplus \widetilde{\mathbb{R}}^n$ *for some* k, m, *and* n*; and*
(2) *there are* Pin_2^-*-equivariant metrics on* V_λ *and* $\overline{W}_\lambda$ *and an* $R > 0$ *such that* $(D_\lambda + Q_\lambda)(v) \neq 0$ *for all* $v \in S_R(0)$.

Recall that Pin_2^- acts on $\mathbb{H}$ through the inclusion $\mathrm{Pin}_2^- \hookrightarrow \mathrm{Spin}_3 = \mathrm{Sp}_1$ that we discussed last time, and on $\widetilde{\mathbb{R}}$ as the sign representation on $\mathbb{R}$, which is zero on $\mathrm{Spin}_2 \subset \mathrm{Pin}_2^-$, but such that the element we called j acts by -1.

Today we will prove (2), leaving the determination of the representations for next week.

There's still plenty to say about the statement of Theorem 5.1 – what's λ? How do we determine V_λ, $\overline{W}_\lambda$, D_λ, and Q_λ? What are k, m, and n? These will all be answered during the proof.

Let $S^\pm \to M$ be the spinor bundles, so we have a Clifford multiplication map $C\colon T^*M \otimes S^+ \to S^-$ and a Dirac operator $D_1\colon |Gamma(S^+) \to \Gamma(S^-)$, and a twisted Clifford multiplication map $\widetilde{C}\colon T^*M \otimes \widetilde{T}^*M \to \widetilde{\Lambda}$ which defines another Dirac operator $D_2\colon \Gamma(\widetilde{T}^*M) \to \Gamma(\widetilde{\Lambda})$. Then $D = D_1 + D_2$, as we discussed last time.

Theorem 5.2 (Weitzenböck). *D^*D is equal to $\nabla^*\nabla$ up to a zeroth-order term.*

Corollary 5.3 (Gårding's inequality). *There is some $k \geq 0$ such that*

$$\langle D^*D\psi, \psi\rangle_{L^2} + k\langle \psi, \psi\rangle_{L^2} \geq \|\psi\|^2_{L^2_1},$$

where L^2_1 denote the Sobolev norm.

Along the way we'll need another Sobolev space.

Definition 5.4. The L^2_{-1} norm of an $f \in C^\infty(E)$ is the smallest $C \in \mathbb{R}$ such that $\langle f, \psi\rangle_{L^2} \leq C\|\psi\|_{L^2_1}$ for $\psi \in L^2_1(E)$, if it exists. The completion of $C^\infty(E)$ under this norm is denoted $L^2_{-1}(E)$ (or just L^2_{-1} if E is clear from context).

Hence there is an embedding $L^2 \hookrightarrow L^2_{-1}$.

Fact. The L^2 inner product defines a continuous nondegenerate pairing $L^2_{-1} \otimes \overline{L^2_1} \to \mathbb{C}$, hence identifies L^2_{-1} as the continuous dual of L^2_1, i.e. the space of continuous linear functionals on L^2_1. ◀

Therefore we can restate Gårding's inequality as

$$\|(D^*D + k)\psi\|_{L^2_{-1}} \geq \|\psi\|_{L^2_1}. \tag{5.5}$$

Now, D^*D is a second-order differential operator, hence is a map $L^2_1 \to L^2_{-1}$. Thus (5.5) implies $D^*D + k$ is a continuous injection $L^2_1 \hookrightarrow L^2_{-1}$.

Lemma 5.6. *In fact, $D^*D + k$ is onto.*

Proof. The map $\langle\langle \cdot, \cdot\rangle\rangle\colon L^2 \times L^2 \to \mathbb{C}$ defined by

$$\langle\langle \varphi, \psi\rangle\rangle \coloneqq \langle (D^*D + k)\varphi, \psi\rangle_{L^2} \tag{5.7}$$

defines an inner product on L^2, and L^2 is complete with respect to $\langle\langle \cdot, \cdot\rangle\rangle$. Therefore, by the Riesz representation theorem, if $f \in L^2_{-1}$, then on L^2_1, $\langle f, \cdot\rangle_{L^2} = \langle\langle \varphi, \cdot\rangle\rangle$ for some $\varphi \in L^2_1$. In particular,

$$\langle (D^*D + k)\varphi, \psi\rangle = \langle f, \psi\rangle \tag{5.8}$$

for all $\psi \in L^2_1$, so $(D^*D + k)\varphi = f$. ⊠

Therefore we can invert $D^*D + k$. Consider the composition

$$T\colon L^2 \hookrightarrow L^2_{-1} L^2_1 \xhookrightarrow[\cong]{(D^*D+k)^{-1}} L^2. \tag{5.9}$$

All three maps are continuous linear maps, and the third is compact, by the Kondrachov theorem. Therefore T is compact, and since

$$\langle (D^*D + k)\varphi, \psi\rangle = \langle \varphi, (D^*D + k)\psi\rangle, \tag{5.10}$$

then T is self-adjoint. Therefore we can throw the nicest spectral theorem at T.

Theorem 5.11 (Spectral theorem for compact, self-adjoint operators). *Let $\mathcal{H}$ be a separable Hilbert space and $T\colon \mathcal{H} \to \mathcal{H}$ be a compact self-adjoint operator. Then there is an orthonormal basis*[2] *$\{e_n\}$ for $\mathcal{H}$ such that $Te_n = \mu e_n$ for a $\mu \in \mathbb{R}_{\geq 0}$, and $\mu_1 \geq \mu_2 \geq \cdots$ with $\mu_n \geq 0$ as $n \to \infty$.*

[2]This is a Hilbert basis, not an algebraic basis, in that elements of $\mathcal{H}$ can be infinite sums of basis elements.

In our case, we conclude that (TODO: I missed this calculation). Therefore if you fix a $\lambda > 0$, then there are finitely many eigenvalues $\lambda_n \le \lambda$, meaning that if V_λ denotes the span of the eigenvectors of λ_n, then V_λ is finite-dimensional.

The operator DD^* has the same spectrum as D^*D, and the eigenspace for a given eigenvalue is the same. So we can do exactly the same thing to define a finite-dimensional subspace $W_\lambda \subset W$, and $D(V_\lambda) \subset W_\lambda$; let $D_\lambda := D|_{V_\lambda}$.

TODO: what happened next (studying Q) was erased before I could write it down. Sorry about that.

Anyways, we have our operator $Q_\lambda \colon V_\lambda \to W \overset{p_\lambda}{\to} W_\lambda$. Let $p^\lambda := 1 - p_\lambda$. Then,

(1) as $\lambda \to \infty$, $\lim_{\lambda\to\infty} \|p^\lambda w\|_W \to 0$, and
(2) because every bounded sequence in a Hilbert space has a weakly convergent subsequence, then for any bounded sequence $\{v_n\} \in V$, there's a subsequence $\{v_{n_i}\}$ weakly converging to some v_∞. Because Q is compact, $\{Qv_{n_i}\}$ converges strongly to some w_∞, and $Qv_\infty = w_\infty$.

Now, since the zero set of the monopole map $D + Q$ is compact, there is some R, as promised, such that $D + Q$ does not vnaish on the sphere of radius R. To prove part (2) of Theorem 5.1, we need to bring this down to $D_\lambda + Q_\lambda$.

Lemma 5.12. *There is an $\varepsilon > 0$ such that if $\|u\|_V = R$, then $\|(D + Q)v\|_W \ge \varepsilon$.*

Proof. Assume otherwise; then there's a sequence $\{v_n\} \in S_R(0)$ such that $(D + Q)v_i \to 0$. With $\{v_{n_i}\}$, v_∞, and w_∞ as above, we know Qv_{n_i} converges strongly to Qv_∞, so $Dv_{n_i} \to -Qv_\infty$, and therefore $v_{n_i} \to v_\infty$. In particular $v_\infty \in S_R(0)$ but $(D + Q)(v_\infty) = 0$, which is a contradiction. ⊠

Lemma 5.13. *For λ sufficiently large, if $\|v\|_V = R$, then $\|p^\lambda Qv\|_W < \varepsilon$.*

Proof. Assume not; then there are sequences $\{\lambda_n\}$ and $\{v_n\}$ such that $\lambda_n \to \infty$, $v_n \in S_R(0)$, and $v_i \rightharpoonup v_\infty$, such that $\|p^{\lambda_n} Qv_n\| \ge \varepsilon$. Let $w_\infty := Qv_\infty$ as before.

Restricting to a subsequence $\{v_{n_i}\}$ as before, again $Qv_{n_i} \to \infty$, so $\lim_{n\to\infty} \|p^\lambda w_\infty\| = 0$. Therefore there's some N such that

$$\|p^{\lambda_{n_i}} Qv_\infty\| = \|p^{\lambda_{n_i}} w_\infty\| \le \frac{\varepsilon}{2} \tag{5.14}$$

whenever $i \ge N$, and therefore for $i \gg 0$,

$$\|p^{\lambda_{n_i}}(Qv_{n_i} - Qv_\infty)\| \le \frac{\varepsilon}{2}. \tag{5.15}$$

⊠

Therefore, since $Q_\lambda = p^\lambda \circ Q$, if $\|v\| = R$, TODOmissed the last part.

6. Finite-dimensional approximations, II: 2/25/19

"Remember, these are not just vector spaces but Pin_2^-*-representations, and we can pin down what they are... oh no."*

Today Leon spoke, continuing the description of the finite-dimensional approximations of the monopole map, and with the specific goal of proving part (1) of Theorem 5.1.

Today we will heavily use the Dirac operators: Furuta uses $D^1 \colon \Gamma(\mathbb{S}^+) \to \Gamma(\mathbb{S}^-)$ to denote the usual Dirac operator, and uses $D^2 \colon \Gamma(\widetilde{T}^*M) \to \Gamma(\widetilde{\Lambda})$ to denote the operator $\mathrm{d}^* \oplus \mathrm{d}^+ \colon \Omega^i(M) \to \Omega^0(M) \oplus \Omega^2_+(M)$. Then $D := D^1 \oplus D^2$; we'll discuss the quadratic part later.

Last time, we defined the finite-dimensional approximations V^i_λ and W^i_λ: given $\lambda \in \mathbb{R}$, V^i_λ is the space spanned by the eigenspaces of $(D^i)^*D^i$ corresponding to eigenvalues less than λ, and W^i_λ is defined similarly. Because D and Q are Pin_2^--equivariant, these are in fact Pin_2^--representations. We'd like to know what representations they are, and as a starting point, we'd like to know their dimensions. This is hard, but we will be able to compute the relative dimension.

Lemma 6.1. *For $i = 1, 2$, $\dim_{\mathbb{C}} V^i_\lambda - \dim_C W^i_\lambda = \mathrm{ind}_{\mathbb{C}} D^i$.*

Proof. For $\lambda' < \lambda$, let $V^i_{=\lambda'}$ denote the λ'-eigenspace of D^i, so that V^i_λ is a direct sum over all such $V^i_{=\lambda'}$; then define $W^i_{=\lambda'}$ similarly. Then

$$\begin{aligned}\operatorname{ind}_{\mathbb{C}} D^i &= \sum_{\lambda'<\lambda} \operatorname{ind}_{\mathbb{C}}(D^i|_{V^i_{=\lambda}} : V^i_{=\lambda} \to W^i_{=\lambda}) \\ &= \sum_{\lambda'<\lambda} \dim_{\mathbb{C}} V^i_{=\lambda'} - \dim_{\mathbb{C}} W^i_{=\lambda'} = \dim_{\mathbb{C}} V_\lambda - \dim_{\mathbb{C}} W_\lambda.\end{aligned} \tag{6.2}$$

⊠

Now let's recall some of the Pin_2^--representations we discussed in earlier lectures: the action on $\mathbb{H}$ via the inclusion $\mathrm{Pin}_2^- \hookrightarrow \mathrm{Spin}_3 = \mathrm{Sp}_1$, and the sign action on $\mathbb{R}$, which is denoted $\widetilde{\mathbb{R}}$; this is trivial on $\mathrm{Spin}_2 \subset \mathrm{Pin}_2^-$ but $j \in \mathrm{Pin}_2^-$ acts by -1.

Lemma 6.3. *There exist $m, m', n, n' \in \mathbb{N}$ such that $V^1_\lambda \cong \mathbb{H}^m$, $W^1_\lambda \cong \mathbb{H}^{m'}$, $V^2_\lambda \cong \widetilde{\mathbb{R}}^n$, and $W^2_\lambda \cong \widetilde{\mathbb{R}}^{n'}$.*

Proof. Because V^1_λ is a finite-dimensional subspace of $\Gamma(\mathbb{S}^+)$, there exist $p_1, \dots, p_\ell \in M$ such that the evaluation map

$$ev \colon \Gamma(\mathbb{S}^+) \longrightarrow \bigoplus_{i=1}^{\ell} (\mathbb{S}^+)_{p_i}. \tag{6.4}$$

is injective when restricted to V^1_λ. Since ev is Pin_2^--equivariant and $(\mathbb{S}_p)^+ \cong \mathbb{H}$, then we have a Pin_2^--equivariant inclusion $V^1_\lambda \hookrightarrow \mathbb{H}^M$ for some M, and since $\mathbb{H}$ is an irreducible Pin_2^--representation, this forces $V^1_\lambda \cong \mathbb{H}^m$ for some m.

The other three identities proceed in the same way. ⊠

Now we'll throw the Atiyah-Singer index theorem at D_1. This tells us that

$$\operatorname{ind}_{\mathbb{R}}(D^1) = 2\langle \widehat{A}(M), [M]\rangle, \tag{6.5}$$

where $\widehat{A}(M)$ is the *$\widehat{A}$-genus* (pronounced "*A*-hat") $\widehat{A}(M) \in H^*(M)$, a characteristic class.[3] Bordism arguments tell us that the $\widehat{A}$-genus is a linear combination of Pontrjagin classes, and in dimension 4 we get that

$$= -\left\langle \frac{p_1(M)}{12}, [M] \right\rangle, \tag{6.6}$$

which by the Hirzebruch signature theorem is

$$= -\frac{\sigma(M)}{4}. \tag{6.7}$$

Because this is an index, it's an integer, and since D^1 is quaternionic, it's divisible by 4.

Corollary 6.8 (Rokhlin's theorem)**.** *If M is a closed spin 4-manifold, $\sigma(M)$ is divisible by 16.*

Let $k := -\sigma(M)/16$. Because this is the index of D^1 over $\mathbb{H}$, we know there's an $m \in \mathbb{N}$ such that $V'_\lambda \cong \mathbb{H}^{m+k}$ and $W'_\lambda \cong \mathbb{H}^m$.

Now let's compute the index of D^2. This is

$$-\operatorname{ind} D^2 = \dim\operatorname{coker}(\mathrm{d}^* \colon \Omega^1 \to \Omega^0) + \dim\ker(\mathrm{d}^+ \colon \Omega^1 \to \Omega^+). \tag{6.9}$$

For the first factor,

$$\dim\operatorname{coker}\mathrm{d}^* = \dim\ker(\mathrm{d} \colon \Omega^0 \to \Omega^1) = b_0(M) = 1, \tag{6.10}$$

since M is connected. The other piece will be a little harder.

Lemma 6.11.

(1) *If $\alpha \in \Omega^1(M)$, then $\mathrm{d}^+\alpha = 0$ iff $\mathrm{d}\alpha = 0$.*
(2) *There's a space $\mathcal{H}^+ \subset \Omega^+_2(M)$ such that $\Omega^+_2(M) = \operatorname{Im}(\mathrm{d}^+) \oplus \mathcal{H}^+$, and*
(3) $\dim \mathcal{H}^+ = b_2^+(M)$.

We'll come back to this later, but as a corollary, we know $\operatorname{ind} D^2 = -1 - b_2^+$. Therefore we know that in Lemma 6.3, $n' = n + b_2^+ + 1$. Crucially, the indices did not depend on λ, so we're interested in k and b_2^+, moreso than m and n.

Now we turn to Q, the quadratic part of the monopole map. Recall that Q is a direct sum of two operators:

- $Q_1\colon \Gamma(\mathbb{S}^+ \oplus \widetilde{T}^*M) \to \Gamma(\mathbb{S}^-)$ sends $\phi, a \mapsto ia\phi$, and
- $Q_2\colon \Gamma(\mathbb{S}^+ \oplus \widetilde{T}^*M) \to \Gamma(\widetilde{\Lambda}) = \Omega^0(M) \oplus \Omega^2_+(M)$ sends $\phi, a \mapsto \phi a \overline{\phi}$.

TODO: I missed this part, sorry. There's W_λ and then you need one more dimension, which lands in a space called $\widetilde{W}_\lambda$.

In summary, we have the following theorem.

Theorem 6.12. *For $\lambda \gg 0$, the monopole map defines a Pin_2^--equivariant map $D_\lambda + Q_\lambda\colon V_\lambda \to \widetilde{\widetilde{W}}_\lambda$ between finite-dimensional Pin_2^--equivariant representations such that $V_\lambda \cong \mathbb{H}^{m+k} \oplus \widetilde{\mathbb{R}}^n$ and $\widetilde{W}_\lambda \cong \mathbb{H}^m \oplus \widetilde{\mathbb{R}}^{n+b_2^+}$, where m and n may depend on λ, but k doesn't. Moreover, there are Pin_2^--invariant inner products on V_λ and W_λ, and an $R > 0$ not depending on λ, such that the image of the sphere of radius R in V_λ is nonzero.*

In particular, we can restrict to $S_R(V_\lambda)$ and obtain a map between spheres with Pin_2^--actions. That is, existence of M tells us that some Pin_2^--equivariant homotopy class of maps between spheres exists; we will use some equivariant homotopy theory to show nonexistence of certain such maps, and therefore obtain the bound on the ratio of the signature and $b_2(M)$.

Proof of Lemma 6.11. For the first part, suppose $\mathrm{d}\alpha \in \Omega^-$, which means that $\star\mathrm{d}\alpha = -\mathrm{d}\alpha$. Since $\mathrm{d}^* = -\star\mathrm{d}\star$, then $\star\mathrm{d}\alpha = -\mathrm{d}^*(\star^{-1}\alpha)$, and $\star^{-1}\alpha \in \Omega^3$. Using the splitting

$$\Omega^2 = \mathrm{Im}(\mathrm{d}) \oplus \mathrm{Im}(\mathrm{d}^*) \oplus \mathcal{H}^2, \tag{6.13}$$

we get that $\mathrm{d}^*(\star^{-1}\alpha) = \mathrm{d}\alpha$.

For the second part, we use (6.13) again. Let $\pi\colon \Omega^2 \to \Omega^+$ be defined by $(1+\star)/2$. We know $\Omega^+ = \mathrm{Im}(\mathrm{d}^+) + \mathrm{Im}((\mathrm{d}^*)^+) + \mathcal{H}^+$, so it suffices to show that $\mathrm{Im}\,\mathrm{d}^+ \perp \mathcal{H}^+$ and $\mathrm{Im}\,\mathrm{d}^+ = \mathrm{Im}((\mathrm{d}^*)^+)$. For the first one, if $\mathrm{d}^+\alpha \in \mathrm{Im}(\mathrm{d}^+)$ and $\beta \in \mathcal{H}^+$, we have

$$\langle \mathrm{d}^+\alpha, \beta\rangle = \langle \mathrm{d}\alpha, \beta\rangle = \langle \alpha, \mathrm{d}^*\beta\rangle = 0. \tag{6.14}$$

For the second part, let $\alpha \in \Omega^3$; then

$$\pi(\mathrm{d}^*\alpha) = \pi(-\star\mathrm{d}\star\alpha) = -\pi\mathrm{d}(\star\alpha) = -\mathrm{d}^+(\star\alpha). \tag{6.15}$$

For the last part, recall that b_2^+ is the maximal dimension of a positive-definite subspace of $\mathcal{H}$ with respect to the pairing $\mathcal{H}^2 \otimes \mathcal{H}^2 \to \mathbb{R}$. Also recall that $\alpha \wedge \star\beta = \langle \alpha, \beta\rangle\,\mathrm{dvol}$, so if $\alpha \in \mathcal{H}^2$ is self-dual, then

$$\int \langle \alpha \wedge \alpha\rangle\,\mathrm{dvol} = \int \alpha \wedge \star\alpha = \int \langle \alpha, \alpha\rangle\,\mathrm{dvol} \geq 0. \tag{6.16}$$

with equality iff $\alpha = 0$. Therefore $\dim \mathcal{H}^+ \leq b_2^+$, and because the same argument applies to $\mathcal{H}^-$, we get $\dim \mathcal{H}^- \leq b_2^-$, so using the orthogonal splitting $\mathcal{H}^2 = \mathcal{H}^2_+ \oplus \mathcal{H}^2_-$, and therefore $\dim \mathcal{H}^+ = b_2^+$. ⊠

7. Equivariant K-theory: 3/4/19

Today, Richard spoke about equivariant K-theory, following Bryan's paper [?], which is a little simpler.

As usual, suppose X is a smooth, spin 4-manifold with $b_1 = 0$ and nonpositive signature. Bryan is interested in a generalization in which there is a spin diffeomorphism $\tau\colon X \to X$ of order 2^p or 2^{p+1}, where p is prime.

Definition 7.1. Let X be a spin n-manifold and $\pi\colon P \to X$ be the principal Spin_n-bundle of frames. A *spin diffeomorphism* is data of a diffeomorphism $\tau\colon X \to X$ and a Spin_n-equivariant map $\widetilde{\tau}\colon P \to P$ which covers τ (i.e. $\tau \circ \pi = \pi \circ \widetilde{\tau}$), and such that $\tau^*P - P = 0 \in H^1(X; \mathbb{Z}/2)$ (recalling that spin structures are a torsor over $H^1(X; \mathbb{Z}/2)$).

Example 7.2. On any spin manifold, there is a spin diffeomorphism called the *spin flip* which is the identity on the manifold but is the nontrivial automorphism of the double cover $P_{\mathrm{Spin}} \to P_{\mathrm{SO}}$. ◀

Since τ has order 2^p or 2^{p+1}, then τ^{2^p} is either the identity or the spin flip. In the former case, the symmetry group is $G_{ev} := \mathrm{Pin}_2^- \times \mathbb{Z}/2^p$, and in the latter case, it's $G_{odd} := \mathrm{Pin}_2^- \times_{\mathbb{Z}/2} \mathbb{Z}/2^{p-1}$, where $\times_{\mathbb{Z}/2}$ means modding out by the common $\mathbb{Z}/2$ subgroup.

Last time, we built a nonlinear Pin_2^--equivariant map $D_\lambda + Q_\lambda\colon V_\lambda \to \widetilde{\widetilde{W}}_\lambda$. Complexifying, we obtain a map $f\colon V_{\lambda,\mathbb{C}} \to \widetilde{\widetilde{W}}_{\lambda,\mathbb{C}}$ given by

$$f(u \otimes 1 + v \otimes i) := (D_\lambda + Q_\lambda)u \otimes 1 + (D_\lambda + Q_\lambda)v \otimes i. \tag{7.3}$$

Let $BV_{\lambda,\mathbb{C}}$ denote the unit ball in $V_{\lambda,\mathbb{C}}$ and $SV_{\lambda,\mathbb{C}}$ denote its boundary. Then, $f(SV_{\lambda,\mathbb{C}})$ does not contain zero, so we can compose with the projection

$$\widetilde{W}_{\lambda,\mathbb{C}} \longrightarrow S\widetilde{W}_{\lambda,\mathbb{C}} = (\widetilde{W}_{\lambda,\mathbb{C}} \setminus 0)/\mathbb{R}_+ \tag{7.4}$$

and obtain a Pin_2^--equivariant map $\overline{f}\colon SV_{\lambda,\mathbb{C}} \to S\widetilde{W}_{\lambda,\mathbb{C}}$, i.e. we get a map $\widetilde{f}\colon BV_{\lambda,\mathbb{C}} \to B\widetilde{W}_{\lambda,\mathbb{C}}$ which sends the boundary to the boundary. Therefore it is a Pin_2^--equivariant map of pairs

$$\widehat{f}\colon (BV_{\lambda,\mathbb{C}}, SV_{\lambda,\mathbb{C}}) \longrightarrow (B\widetilde{W}_{\lambda,\mathbb{C}}, S\widetilde{W}_{\lambda,\mathbb{C}}). \tag{7.5}$$

This is a Pin_2^--equivariant map of spheres, and we will study it using K-theory and index theorem.

Remark 7.6. The disc mod sphere construction may remind you of the Thom space construction. Given a rank-n vector bundle $E \to X$ with a Euclidean metric, let $D(E)$ denote the unit disc bundle and $S(E)$ denote the unit sphere. Then the *Thom space* of E, denoted X^E, is the quotient $D(E)/S(E)$.

If E is oriented (automatic if it's a complex vector bundle), then there is a *Thom class* $\tau_E \in H^n(X^E)$ such that the map $H^*(X) \to H^{*+n}(X^E)$ sending $a \mapsto a \smile \tau_E$ is an isomorphism, called the *Thom isomorphism.*

Definition 7.7. The *complex K-group* $K^0(X)$ of a space X (also denoted $KU^0(X)$) is the Grothendieck group of the category of vector bundles on X under direct sum, i.e. it is the group generated by isomorphism classes of vector bundles with the relation $[V \oplus W] = [V] + [W]$.

Remark 7.8. If $E \to X$ is a complex vector bundle (or more generally is even-dimensional and has a spin^c structure), there is a Thom isomorphism in K-theory $KU^0(X) \to KU^0(X^E)$. Said another way, $KU^0(E)$ is a free $KU^0(X)$-module on one generator.

Now we add equivariance to the mix.

Definition 7.9. Let G be a compact Lie group acting on a topological space X. The *G-equivariant complex K-group* $K_G^0(X)$ (or $KU_G^0(X)$) is the Grothendieck group of G-equivariant vector bundles on X under direct sum.

Theorem 7.10 (Atiyah). *If $E \to X$ is a G-equivariant complex vector bundle, there is a Thom isomorphism $K_G^0(X) \overset{\cong}{\to} K_G^0(X^E)$.*

For example, $K_G^0(\mathrm{pt}) \cong RU(G)$, the representation ring of G, and therefore the G-equivariant K-group of a sphere (the Thom space of a vector space thought of as a vector bundle over a point) is also isomorphic to $RU(G)$.

Applied to our map $f\colon V_{\lambda,\mathbb{C}} \to W_{\lambda,\mathbb{C}}$, we get a pullback map

$$f^*\colon K_G^0(B\widetilde{W}_{\lambda,\mathbb{C}}, S\widetilde{W}_{\lambda,\mathbb{C}}) \longrightarrow K_G^0(BV_{\lambda,\mathbb{C}}, SV_{\lambda,\mathbb{C}}), \tag{7.11}$$

where $G = \mathrm{Pin}_2^-$, which is a map $RU(G) \to RU(G)$. If λ_V and λ_W denote the Bott classes in $K_G^0(V)$, resp. $K_G^0(W)$,, then $f^*(\lambda_W) = \alpha_f \lambda_V$ for some (possibly virtual) representation α_f.[4] We want to understand α_f, so we will take its character, the class function $g \mapsto \mathrm{tr}(\alpha(g))$.[5]

Now let $C \subset G$ be a *topologicall cyclic subgroup* of G, i.e. either a finite cyclic subgroup or an S^1, and let c be a generator (if C is finite) or a nonzero element (if $C \cong S^1$). We will try to study the action of f^* on C-fixed points, which is a common technique in equivariant homotopy theory. Specifically, we have a commutative diagram

$$\begin{tikzcd} K_G^0(B\widetilde{W}_{\lambda,\mathbb{C}}, S\widetilde{W}_{\lambda,\mathbb{C}}) \arrow[r, "f^*"] \arrow[d, "\mathrm{res}_C"] & K_G^0(BV_{\lambda,\mathbb{C}}, SV_{\lambda,\mathbb{C}}) \arrow[d, "\mathrm{res}_C"] \\ K_C^0(B\widetilde{W}_{\lambda,\mathbb{C}}^C, S\widetilde{W}_{\lambda,\mathbb{C}}^C) \arrow[r, "(f^C)^*"] & K_C^0(BV_{\lambda,\mathbb{C}}^C, SV_{\lambda,\mathbb{C}}^C). \end{tikzcd} \tag{7.12}$$

[4] Is α_f in general an actual representation? Probably not, but we're not sure.

[5] The character formula only makes sense literally for actual representations, and α_f might be a difference of two representations, in which case we take the difference of their characters. This does not depend on how we write α_f as a difference of two actual representations.

Now C acts trivially on $V^C_{\lambda,\mathbb{C}}$ and $\widetilde{W}^C_{\lambda,\mathbb{C}}$, so

$$(f^C)^*\lambda_{\widetilde{W}^C_{\lambda,\mathbb{C}}} = d(f^C)\lambda_{V^C_{\lambda,\mathbb{C}}}, \tag{7.13}$$

where $d(f^C) = 0$ if $\dim \widetilde{W}^C_{\lambda,\mathbb{C}} \neq \dim V^C_{\lambda,\mathbb{C}}$.

Remark 7.14. α_f is called the *K-theoretic degree* of f and d is the *topological degree.* ◂

Then, using (7.12),

$$\operatorname{res}_C(\lambda_{\widetilde{W}_{\lambda,\mathbb{C}}}) = \lambda_{-1}((\widetilde{W}^C_{\lambda,\mathbb{C}})^\perp)\lambda_{\widetilde{W}^C_{\lambda,\mathbb{C}}}, \tag{7.15}$$

where

$$\lambda_{-1}\beta = \sum(-1)^i\beta_i. \tag{7.16}$$

Then we compute, following tom Dieck,

$$\begin{aligned}
(f^C)^*(\operatorname{res}_C \lambda_W) &= \operatorname{res}_C(f^*(\lambda_W))\\
(f^C)^*((\lambda_{-1}(W^C)^\perp)\lambda_{W^C}) &= \operatorname{res}_C(\alpha_f\lambda_V) = \operatorname{res}_C(\alpha_f)\lambda_V\\
(f^C)^*(\lambda_{-1}(W^C)^\perp)d(f^C)\lambda_{V^C} &= \operatorname{res}_C(\alpha_f)\lambda_{-1}((V^C)^\perp)\lambda_{V^C}.
\end{aligned}$$

If g is the generator above, then $\lambda_{-1}((V^C)^\perp)(g) \neq 0$, so

$$\alpha_f(c) = d(f^C)\lambda_{-1}((W^C)^\perp - (V^C)^\perp)(g) \tag{7.17}$$

$$\operatorname{tr}_g(\alpha_f) = d(f^g)\operatorname{tr}_g(\lambda_{-1}(W_g^\perp - V_g^\perp)). \tag{7.18}$$

Here $\lambda_{-1}(r) = 1 - r$. This second equation is very useful: we will be able to throw index theory at the degree, and representation theory at the trace, and there are good formulas for it.

8. Proof of the 10/8 theorem: 3/11/19

These are Arun's prepared lecture notes for this talk. I'll finish the proof of the 10/8 theorem and begin laying the groundwork for the rest of Bryan's results.

8.1. Recalling some facts about equivariant K-theory.

Last week, Richard told us about a character formula for equivariant K-theory. I'll write it down first, and then explain what's going on. Let G be a compact Lie group, V and W be two complex G-representations, and $f\colon (BV, SV) \to (BW, SW)$ be a G-equivariant map from the unit disc in V to the unit disc in W preserving boundaries. Then

$$\operatorname{tr}_g(\alpha_f) = d(f^g)(\lambda_{-1}((W^g)^\perp - (V^g)^\perp)). \tag{8.1}$$

Here's what the notation means.

- $\alpha_f \in RU(G)$ is the K-theoretic degree of f. The G-equivariant Thom isomorphism theorem defines $RU(G)$-module isomorphisms of $K^0_G(BV, SV)$ and $K^0_G(BW, SW)$ with $RU(G)$, so $f^*\colon K^0_G(BW, SW) \to K^0_G(BV, SV)$ is multiplication by some element of $RU(G)$, and that's α_f.
- tr_g just denotes the trace of $g \in G$ in the virtual representation α_f. That is, if you write $\alpha_f = \rho_1 - \rho_2$ in $RU(G)$, where ρ_1 and ρ_2 are bona fide G-representations, then $\operatorname{tr}_g(\alpha_f)$ is $\operatorname{tr}(\rho_1(g)) - \operatorname{tr}(\rho_2(g))$.
- f^g, V^g, and W^g denote the fixed points with respect to g. $(V^g)^\perp$ is the orthogonal complement with respect to a G-invariant inner product on V, and similarly for $(W^g)^\perp$.
- $d(f^g)$ means the ordinary degree in topology – in particular, zero if V^g and W^g have different dimensions.
- Finally, if ρ is a G-representation,

$$\lambda_{-1}(\rho) \coloneqq \sum_{i\geq 0}(-1)^i[\Lambda^i\rho]. \tag{8.2}$$

8.2. **The proof.** First, recall the representation theory of Pin_2^-:

- Two one-dimensional representations: the trivial representation 1 and the sign representation σ (which Bryan calls $\widetilde{1}$). The latter is trivial on $\mathrm{Spin}_2 \subset \mathrm{Pin}_2^-$, but j acts by -1.
- Given a $d \in \mathbb{Z}_{>0}$, let h_d be the representation of Pin_2^- on $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}$ where $e^{i\theta}$ acts by $(e^{id\theta}, e^{-id\theta})$ and j acts on $\mathbb{H}$ as normal.

These generate $RU(\mathrm{Pin}_2^-)$, and the relations are $\sigma^2 = 1$, $\sigma h_d = h_{-d}$, and $h_{d_1} h_{d_2} = h_{d_1+d_2} + h_{|d_1-d_2|}$. You can verify this by chasing around some matrices.

Therefore if $\alpha = [V_\lambda - W_\lambda]$, we can write it as

$$\alpha = \alpha_0 + \widetilde{\alpha}_0 \sigma + \sum \alpha_i h_i, \tag{8.3}$$

which is a finite sum (only finitely many α_i are nonzero).

Let $\phi \in \mathrm{Spin}_2$ generate a dense subgroup, so ϕ acts trivially on σ and nontrivially on every h_i. Recall that $V = (k+m)h_1 + n\sigma$ and $W = h_1 m + (n + b_2^+)\sigma$. Since $h_1^\phi = 0$ but ϕ acts trivially on σ, then $V^\phi = n\sigma$ and $W^\phi = (n + b_2^+)\sigma$. Because $b_2^+ > 0$, then $\dim(V^\phi) \neq \dim(W^\phi)$, and therefore the K-theory degree $d(f^\phi) = 0$. Plugging into the degree formula,

$$\mathrm{tr}_\phi(\alpha) = 0 = \alpha_0 + \widetilde{\alpha}_0 + \sum_{i>0} \alpha_i(\phi^i + \phi^{-i}), \tag{8.4}$$

recalling that $h_d|_{\mathrm{Spin}_2} \cong t^d + t^{-d}$. Therefore $\alpha_0 = \widetilde{\alpha}_0$ and $\alpha_i = 0$ for $i > 0$.

Now $j \in \mathrm{Pin}_2^-$ acts nontrivially on both h_1 and σ, so $\dim(V^j) = \dim(W^j) = 0$. This means $d(f^j) = 1$ somewhat vacuously. In particular, $(V^j)^\perp = V^j$ and $(W^j)^\perp = W^j$. Let's recall from the index-theoretic calculation that there are m and n such that

$$\begin{aligned} V &= 2(k+m)h_1 + n\sigma \\ W &= 2mh_1 + (n + b_2^+)\sigma. \end{aligned} \tag{8.5}$$

The factor of two that appears is because V and W have been complexified. Since $\sigma = \widetilde{\mathbb{R}} \otimes \mathbb{C}$, it doesn't pick up a factor of two, but $h_1 \otimes_{\mathbb{R}} \mathbb{C} = 2h$ does (it was already complex).

Now let's use the character formula applied to $j \in \mathrm{Pin}_2^-$. Since $V^j = 0$, $(V^j)^\perp = V$, and the same goes for W. Therefore

$$\mathrm{tr}_j(\alpha) = \mathrm{tr}_j(\lambda_{-1}(V - W)) = \mathrm{tr}_j(\lambda_{-1}(b_2^+\sigma - 2kh_1)). \tag{8.6}$$

Since σ is one-dimensional, $\Lambda^*\sigma = 1 + \sigma$ and $\lambda_{-1}\sigma = 1 - \sigma$. Since h_1 is two-dimensional, $\Lambda^0 h_1 = 1$, $\Lambda^1 h_1 = h_1$, and $\Lambda^{>2} h_1 = 0$; one then checks that $\det(j\colon h_1 \to h_1) = 1$ to see $\Lambda^2 h_1 = 1$ and not σ. Thus

$$= \mathrm{tr}_j\Big((1-\sigma)^{b_2^+}(2 - h_1)^{-2k}\Big). \tag{8.7}$$

Since $\mathrm{tr}_j(\sigma) = -1$ and $\mathrm{tr}_j(h_1) = 0$,

$$= 2^{b_2^+ - 2k}. \tag{8.8}$$

Comparing with (8.4),

$$\mathrm{tr}_j(\alpha) = \mathrm{tr}_j(\alpha_0(1 - \sigma)) = 2\alpha_0, \tag{8.9}$$

so the degree is $\alpha = 2^{b_2^+ - 2k - 1}$. Because this is an integer, $2k + 1 \le b_2^+$.

Now we're almost home. Recalling that $b_2 = b_2^+ + b_2^-$ and $\sigma = b_2^+ - b_2^-$, and that $k = -\sigma/16$,

$$-\frac{1}{8}\sigma + 1 \le b_2^+ \tag{8.10}$$

$$-\frac{1}{4}\sigma + 2 \le 2b_2^+ = 2b_2^+ + b_2^- - b_2^- \tag{8.11}$$

$$= b_2 + \sigma \tag{8.12}$$

$$-\frac{5}{4}\sigma + 2 \le b_2. \tag{8.13}$$

The sign on σ is kind of irrelevant: switching the orientation on X switches the sign on σ and leaves b_2 alone. So we can replace σ with $|\sigma|$ and conclude.

9. Bryan's generalization: 3/25/19

Again, these are Arun's notes, prepared in advance for this talk.

Recall that if X is a spin manifold, a *spin action* of a group G on X is an action through spin diffeomorphisms. That is, for every $g \in G$, we get a map $\varphi_g \colon X \to X$ and a map of the principal Spin_n-bundles of frames $\widehat{\varphi}_g \colon P_X \to P_X$ covering φ_g, and these must satisfy the usual axioms for an action ($\varphi_{gh} = \varphi_h \varphi_g$, etc.).

Definition 9.1. Let $(\tau, \widehat{\tau})$ be a spin diffeomorphism of X, meaning $\tau \colon X \to X$ is a diffeomorphism and $\widehat{\tau}$ is its lift to the bundle of spin frames.

- If τ and $\widehat{\tau}$ both generate a $\mathbb{Z}/2^p$, we call this a $\mathbb{Z}/2^p$*-action of even type.*
- If τ has order 2^p but $\widehat{\tau}$ has order 2^{p+1}, we call this a $\mathbb{Z}/2^p$*-action of odd type.*

The idea is: we have a $\mathbb{Z}/2^p$ of ordinary diffeomorphisms and a lift to spin diffeomorphisms. This doesn't guarantee that $\widehat{\tau}^{2^p}$ is the identity – it could be the spin flip (the nontrivial automorphism of the double cover $P_X \to B_{\mathrm{SO}}(X)$), and therefore we could get a $\mathbb{Z}/2^{p+1}$, which is what odd type is about.

Throughout let X be a closed spin 4-manifold, $k \coloneqq -\sigma(X)/16$, and $m \coloneqq b_2^+(X)$. Assume $\sigma(X) \le 0$ and $b_1(X) = 0$.

Remark 9.2. I think there is an implicit assumption on τ, in that we need quotients by groups generated by powers of τ to be manifolds. So of course it suffices for τ to act freely, but I think Bryan considers more general actions given by branched covers.

In any case, I think the assumptions we've made so far do not suffice: let ζ be a representation of $\mathbb{Z}/2^p$ in $\mathbb{C}$ sending the generator to a primitive 2^p-th root of unity and S^ζ denote its one-point compactification. Quotients of $S^\zeta \times S^\zeta$ by subgroups of $\mathbb{Z}/2^p$ are not in general manifolds. ◀

Theorem 9.3 (Bryan [**?**])**.** *Suppose τ is a spin diffeomorphism generating a $\mathbb{Z}/2^p$-action of odd type, and that τ acts freely, and let $X_i \coloneqq X/(\mathbb{Z}/2^i)$. If $m \ne 2k + b_2^+(X_1)$, $b_2^+(X_i) \ne b_2^+(X_j)$ for $i \ne j$, and $b_2^+(X_i) > 0$ for all i, then $2k + 1 + p \le m$.*

The spin structure on X_i is $P_X/(\mathbb{Z}/2^i)$. TODO: why care?

Theorem 9.4. *Let $\sigma_1, \dots, \sigma_q \colon X \to X$ be free involutions of even type, which we will think of as a $(\mathbb{Z}/2)^q$-action. Suppose $b_2^+(X/(\mathbb{Z}/2)^q) \ne 0$ and for all $g \in (\mathbb{Z}/2)^q$, $m \ne b_2^+(X/\langle g \rangle)$. Then $2k + 1 + q \le m$.*

In particular, when $q = 1$, if X admits a smooth even-type involution whose quotient has positive b_2^+ not equal to that of X, then $2k + 2 \le m$. Furuta's theorem is equivalent to $2k + 1 \le m$, so more commuting involutions strengthen the 10/8 bound.

Theorem 9.5. *If τ generates a $\mathbb{Z}/2^p$-action of either type and $\sigma(X) \ne 0$, then $b_2^-(X/\langle \tau \rangle) \ne 0$.*

Bryan then uses these somewhat esoteric-looking theorems to derive bounds for genera of embedded surfaces.

Theorem 9.6. *Let X be a compact, simply connected, oriented but not necessarily spin 4-manifold with $b_2^+(M) > 1$. Let $\Sigma \hookrightarrow M$ be a smooth embedding of a surface representing a homology class divisible by 2 and such that $[\Sigma]/2 \equiv w_2(X) \bmod 2$. Then*

$$g(\Sigma) \ge \frac{5}{4}\left(\frac{[\Sigma]^2}{4} - \sigma(M)\right) - b_2(M) + 2. \tag{9.7}$$

This is generally not as powerful as bounds coming from Seiberg-Witten theory, but it applies even when the Seiberg-Witten invariants of X vanish. So it can be used to study embedded surfaces in, for example, $\mathbb{CP}^2 \# \mathbb{CP}^2$.

9.1. The symmetries of the monopole map.

The point of Bryan's methods, of course, is that there is a larger symmetry group on all the Seiberg-Witten-theoretic data, and the monopole map is still equivariant. In the general situation, we have two Lie groups G and $\widehat{G}$ of the "downstairs" and "upstairs" actions (i.e. on X and on its bundle of spin frames, respectively) and a forgetful map $p \colon \widehat{G} \to G$. Then there are two scenarios.

(1) First suppose $\widehat{G}$ doesn't contain the spin flip.[6] Then $\widehat{G} = G$ and $p = \mathrm{id}$. In particular, G acts on all of the bundles by pullback, and D and Q are equivariant, so the group of symmetries is $\mathrm{Pin}_2^- \times G$. This in particular includes the case of an even $\mathbb{Z}/2^p$-action, so we let $G_{ev} \coloneqq \mathrm{Pin}_2^- \times \mathbb{Z}/2^p$.

[6] To say "the" spin flip does presuppose X is connected.

(2) If $\widehat{G}$ does contain the spin flip, then it's an extension

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow \widehat{G} \xrightarrow{pr} G \longrightarrow 1, \tag{9.8}$$

where the $\mathbb{Z}/2$ subgroup is generated by the spin flip. The spin flip can be identified with the action of $-1 \in \mathrm{Pin}_2^-$: they're both order-2 spin diffeomorphisms which are trivial on X. Therefore the symmetry group is the quotient $\mathrm{Pin}_2^- \times_{\mathbb{Z}/2} \widehat{G} := (\mathrm{Pin}_2^- \times \widehat{G})/(\mathbb{Z}/2)$, where $\mathbb{Z}/2$ acts diagonally. This in particular includes the case of an odd $\mathbb{Z}/2^p$-action, and we let $G_{odd} := \mathrm{Pin}_2^- \times_{\mathbb{Z}/2} \mathbb{Z}/2^{p+1}$.

9.2. **The G_{ev}- and G_{odd}-equivariant indices of the monopole map.** When we have a $\mathbb{Z}/2^p$-action of even or odd type, the same story applies to obtain finite-dimensional G_{ev}-, resp. G_{odd}-representations and an approximated monopole map between them. The goal of this section is to calculate the index of the monopole map.

Because the group $\mathbb{Z}/n$ is cyclic, all of its irreducible representations are one-dimensional. In fact, they're generated by tensor powers of a representation sending 1 to a primitive n^{th} root of unity, so $RU(\mathbb{Z}/n) \cong \mathbb{Z}[\zeta]/(\zeta^n - 1)$.

Now, what happens for G_{ev} and G_{odd}?

Proposition 9.9. *If G and H are Lie groups, the external tensor product defines an isomorphism $RU(G) \otimes RU(H) \xrightarrow{\cong} RU(G \times H)$.*

Therefore we can write a general element of $RU(G_{ev})$ as

$$\beta = \beta_0(\zeta) + \widetilde{\beta}_0(\zeta)\sigma + \sum_{i=0}^{\infty} \beta_i(\zeta) h_i, \tag{9.10}$$

where β_i and $\widetilde{\beta}_0$ are degree $\le 2^p - 1$ polynomials, only finitely many of which are nonzero.

Proposition 9.11. *Let G be a Lie group and H be a normal subgroup. Then the (isomorphism classes of) irreducible representations of G/H are in bijection with the irreducible representations of G which are trivial when restricted to H. In particular, if ξ denotes the generator of $RU(\mathbb{Z}/2^{p+1})$, the irreducible representations of G_{odd} are 1, σ, ξ^2, and $\xi^i h_j$ when $i \equiv j \bmod 2$.*

Therefore we can write a general element of $RU(G_{odd})$ as

$$\beta = \beta_0(\xi) + \widetilde{\beta}_0(\xi)\sigma + \sum_{i=0}^{\infty} \beta_i(\xi) h_i, \tag{9.12}$$

where this time, β_{2i} and $\widetilde{\beta}_0$ are even polynomials of degree at most $2^{p+1} - 2$, and β_{2i+1} are odd polynomials of degree at most $2^{p+1} - 1$.

Now we'll determine the equivariant index of the monopole map as a virtual representation; as before, it's the class of $V - W$ in $RU(G)$ ($G = G_{ev}$ or G_{odd}).

Theorem 9.13. *The equivariant index of the monopole map in the case of a $\mathbb{Z}/2^p$-action of even or of odd type is*

$$\operatorname{ind} D = s h_1 - t\sigma, \tag{9.14}$$

where for even type actions (hence G_{ev}),

$$s(\zeta) = \sum_{i=1}^{2^p} s_i \zeta^i \qquad t(\zeta) = \sum_{i=1}^{2^p} t_i \zeta^i \tag{9.15a}$$

and for odd type actions (hence G_{odd}),

$$s(\xi) = \sum_{i=1}^{2^p} s_i \xi^{2i-1} \qquad t(\zeta) = \sum_{i=1}^{2^p} t_i \zeta^{2i}, \tag{9.15b}$$

and in both cases,

$$\sum_{i=1}^{2^p} s_i = 2k \qquad \sum_{i=0 \bmod 2^j} t_i = b_2^+(X_j). \tag{9.16}$$

Proof. First we leverage what we already know: under the restriction map $r\colon RU(G) \to RU(\mathrm{Pin}_2^-)$, $r(\operatorname{ind} D) = b_2^+\sigma - 2kh_1$. This is enough to imply (9.15), once we understand this forgetful map: in the even case, $r(\zeta) = 1$, and in the odd case, $r(\xi^2) = 1$ and $r(\xi^i h_{\mathrm{odd}}) = h_{\mathrm{odd}}$ (here i is also odd). This also implies the first half of (9.16).

Now the statement on $\sum t_i$ and $b_2^+(X_j)$. In the even case, we can identify

$$\sum_{i=1}^{2^p} t_i\zeta^i = \operatorname{coker}\bigl(\mathrm{d}^* \oplus \mathrm{d}^+ \colon \Omega^1_\perp \to \Omega^0 \oplus \Omega^+\bigr), \tag{9.17}$$

as $\mathbb{Z}/2^p$-representations, where $\Omega^1_\perp$ means the orthogonal complement to the constant functions. Why is this? We basically saw this in the nonequivariant case: this corresponds to the map that Furuta calls $D_2 = \widetilde{C} \circ \nabla$. In the absence of a $\mathbb{Z}/2^p$-action, we calculated its index to be $b_2^+\sigma$, and equivariantly, we get its preimage under r. (I think the additional factor of σ goes away because we're considering $\mathrm{d}^* \oplus \mathrm{d}^+$ without the twisting by $\widetilde{\mathbb{R}}$.)

The piece invariant under $\mathbb{Z}/2^j \subset \mathbb{Z}/2^p$ is the sum over i with $i \equiv 0 \bmod 2^j$, so in particular this is dimension of the $\mathbb{Z}/2^j$-invariant part of $H^2_+(X)$. If X_j is indeed a manifold, this is (I think) equal to $H^2_+(X_j)$, so we're good.

The odd case proceeds in exactly the same way. ⊠

10. Proofs of Bryan's theorems: 4/1/19

Today, Jeffrey spoke, finishing the discussion of Bryan's paper and the proofs of his theorems.

Let's recall the notation that Bryan uses.

- We have a closed spin 4-manifold X, with $k = -\sigma(X)/16$ and $m = b_2^+(X)$.
- $G_{ev} = \mathrm{Pin}_2^- \times \mathbb{Z}/2^p$ and $G_{odd} = \mathrm{Pin}_2^- \times_{\mathbb{Z}/2} \mathbb{Z}/2^{p+1}$.
- Given a Lie group G, $RU(G)$ denotes its representation ring (Bryan calls it $R(G)$).
- In this representation ring, 1 is the trivial representation, $\tilde{1}$ denotes the sign representation of Pin_2^- (which I sometimes called σ), and h_d is the Pin_2^--representation on $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}$ where $e^{i\theta} \in \mathrm{Spin}_2 \subset \mathrm{Pin}_2^-$ acts by $(e^{id\theta}, e^{-id\theta})$. We extend these to representations of G_{ev} and G_{odd} (in the latter case tensoring with a nontrivial $\mathbb{Z}/2^{p+1}$-representation) as in the previous section.
- Let ζ be a representation of $\mathbb{Z}/2^p$ sending the generator to a primitive 2^pth root of unity, and similarly ξ be a representation of $\mathbb{Z}/2^{p+1}$ sending the generator to a primitive 2^{p+1}th root of unity.
- Let $\phi \in \mathrm{Spin}_2$ generate a dense subgroup and define $j \in \mathrm{Pin}_2^-$ via the inclusion $\mathrm{Pin}_2^- \hookrightarrow \mathbb{H}$.

Thus we have the formulas (9.10) and (9.12) writing a general element of $RU(G_{ev})$ or $RU(G_{odd})$ as a polynomial in ζ, resp. ξ.

Here's some new notation: let $\nu \in \mathbb{Z}/2^p$ and $\eta \in \mathbb{Z}/2^{p+1}$ be generators. *This is different than Bryan's notation*, but there are typos in his notation and this seems to be the best convention.

The key tool we're going to use is the trace formula (7.18). We'll let $\alpha \coloneqq \alpha_\lambda$, $V \coloneqq V_{\lambda,\mathbb{C}}$, and $W \coloneqq W_{\lambda,\mathbb{C}}$.

The first case Bryan considers is that of a spin involution.

Proof of Theorem 9.4, case $q = 1$. In this case the symmetry group is $G = G_{ev} = \mathrm{Pin}_2^- \times \mathbb{Z}/2$. In Theorem 9.13, we calculated that the G-equivariant index of the finite-dimensional approximation to the monopole map is

$$V - W = (s_1\zeta + s_2)h_1 - (t_1\zeta + t_2)\tilde{1}, \tag{10.1}$$

where $t_1 + t_2 = m$, $s_1 + s_2 = 2k$, and $t_2 = b_2^+(X/\sigma)$. By assumption, this is not equal to m, so $t_1 + t_2 \neq t_2$, and we have $t_1, t_2 \neq 0$.

Since ϕ acts nontrivially on h_1 but trivially on $\tilde{1}$, V^ϕ and W^ϕ have different dimensions, and therefore $d(f^\phi) = 0$. Hence by the trace formula, $\operatorname{tr}_\phi(\alpha) = 0$. Now $\phi\nu$ also acts nontrivially on h_1 and trivially on $\tilde{1}$, so by the same argument $\operatorname{tr}_{\phi\nu}(\alpha) = 0$.

We know α has the form

$$\alpha = \alpha_0 + \tilde{\alpha}_0\tilde{1} + \sum_i \alpha_i h_i, \tag{10.2}$$

so

$$\mathrm{tr}_\phi(\alpha) = 0 = \alpha_0(1) + \tilde{\alpha}_0(1)\tilde{1} + \sum_i \alpha_i(1)(\phi^i + \phi^{-i}), \tag{10.3}$$

so $\alpha_i(1) = 0$ for $i > 1$ and $\alpha_0(1) + \tilde{\alpha}_0(1) = 0$. In the same way, $\mathrm{tr}_{\phi\nu}(\alpha) = 0$ implies

$$\alpha_0(-1) + \tilde{\alpha}_0(-1)\tilde{1} + \sum_i \alpha_i(-1)(\phi^i + \phi^{-i}) = 0, \tag{10.4}$$

because ϕ acts trivially on ζ and ν acts by -1. Thus we also have $\alpha_i(-1) = 0$ and $\alpha_0(-1) + \tilde{\alpha}_0(-1) = 0$. Combining these,

$$\alpha = (\alpha_0^1 + \alpha_0^2)(1 - \widetilde{1}). \tag{10.5}$$

Now we look at $j\nu$, which acts trivially on $\zeta\tilde{1}$ but nontrivially on h_1 and ζh_1. Therefore $d(f^{j\nu}) = 0$ again, forcing $\alpha_0^1 = \alpha_0^2$.

Now $d(f^j) = 1$, because V^j and W^j are both zero-dimensional, so as in the proof of the 10/8 theorem we can conclude $\mathrm{tr}_j(\alpha) = 2^{m-2k}$. Thus

$$\alpha = 2^{m-2k-2}(1+\zeta)(1-\tilde{1}), \tag{10.6}$$

so $m \geq 2k+2$. The extra -2 pops up because we have $1+\zeta$; if we forget down to Pin_2^-, this is just 2. ⊠

The full proof is a generalization of this.

Proof of Theorem 9.4. Let ζ_i be the sign representation of the i^{th} summand of $(\mathbb{Z}/2)^q$. Then

$$\alpha = V - W = s(\zeta_1, \dots, \zeta_q)h_1 - t(\zeta_1, \dots, \zeta_1)\tilde{1}. \tag{10.7}$$

Let $g \in (\mathbb{Z}/2)^q$. We assumed that $M \neq b_2^+(X/g)$, so $t(\zeta_1, \dots, \zeta_q)$ has some subrepresentation in which g acts by -1. Since j acts by -1 on $\tilde{1}$, Jg has some fixed subspace in $t(\zeta_1, \dots, \zeta_q)\tilde{1}$ – but it acts nontrivially on h_1, so we can conclude $\mathrm{tr}_{jg}(\alpha) = 0$.

Now ϕg. Since $b_2^+(X/g) \neq 0$, the coefficient of 1 in $t(\zeta_1, \dots, \zeta_q)$ is nonzero, and therefore ϕg fixes some subspace of $t(\zeta_1, \dots, \zeta_q)\tilde{1}$, and again, since ϕg acts nontrivially on $s(\zeta_1, \dots, \zeta_q)h_1$, then $\mathrm{tr}_{\phi g}(\alpha) = 0$.

Again, using (10.2) and the fact that $\mathrm{tr}_{\phi g}(\alpha) = 0$ and $\mathrm{tr}_{jg}(\alpha) = 0$, we conclude $\alpha_i = 0$ for $i > 1$ and $\alpha_0 + \tilde{\alpha}_0 = 0$. Therefore α ⊠

The proof for odd type actions follows the same line of reasoning, though showing the traces vanish is a longer computation. One can generalize to other actions by groups of order 2^p, though here one must be a bit more careful about how the group acts.

11. Bauer-Furuta invariants: 4/15/19

Today, Leon spoke, giving the first of a few talks on Bauer-Furuta invariants.

Recall that the proof of the 10/8 theorem began by decomposing the monopole map as $D \oplus Q$, where D is Fredholm and linear, and Q is nonlinear, but continuous and compact; then we were able to make finite-dimensional approximations and study the Pin_2^--representation theory of the index.

Bauer-Furuta [BF04, Bau04] do something roughly similar in studying the monopole map, but with a different goal: to obtain an invariant in stable homotopy theory. That means that our finite-dimensional approximations need to be compatible in a certain way, so that we obtain an element of the (equivariant) stable stem. As before, this homotopy theory is supported by some infinite-dimensional topology and analysis, and this will dominate today's talk.

Definition 11.1. Let H and H' be Hilbert spaces and $f\colon H' \to H$ be a Fredholm map. We say f is a *compact perturbation* of a linear Fredholm map $\ell\colon H' \to H$ if there is some (possibly nonlinear) compact operator $c\colon H' \to H$ such that $f = \ell + c$.

If the preimage under f of a bounded set in H is bounded in H', then f is called *bounded*.

A continuous map $c\colon H' \to H$ not necessarily linear is *compact* if it sends bounded sets to subsets of some compact set.

Boundedness of a compact perturbation f is equivalent to the condition that f extends to one-point compactifications $f^+\colon (H')^+ \to H^+$, which are infinite-dimensional spheres.

Definition 11.2. A map $f\colon X \to Y$ is *proper* if it is closed and the preimages of points are compact.

Usually we ask for preimages of any compact sets to be compact, but this definition will be more useful today. One useful fact is that proper implies closed image.

Proposition 11.3. *If $f\colon H' \to H$ is a compact perturbation of some linear Fredholm map, then for any bounded $A' \subset H'$, $f|_{A'}$ is proper. Moreover, if f is bounded, then f and $f^+\colon (H')^+ \to H^+$ are proper.*

Proof. Let $\rho\colon H' \to \ker(\ell)$ denote the orthogonal projection onto $\ker(\ell)$, and consider the composition

$$A' \xrightarrow{f_1} H \oplus \overline{c(A')} \oplus \overline{\rho(A')} \xrightarrow{f_2} H \oplus \overline{c(A')} \oplus \overline{\rho(A)} \xrightarrow{f_3} H, \tag{11.4}$$

where $f_1(a) \coloneqq (f(a), c(a), \rho(a))$, $f_2(h, b, k) \coloneqq (h + b, b, k)$, and f_3 is projection onto the first factor. We will show that f_1 is injective and closed, f_2 is a homeomorphism, and f_3 is proper; together these will imply $f_3 \circ f_2 \circ f_1 = f$ is proper.

That f is injective isn't too bad: if $x \in \ker(f)$, then $c(x) + \ell(x) = 0$. If $c(x) \neq 0$, we see x inside $\overline{c(A')}$, and if $c(x) = \ell(x) = 0$, we see x inside $\ker(\ell)$.

Now, why is it closed? First we ignore $\overline{c(A')}$. We can consider ℓ as a map $\ker(\ell)^\perp \oplus \ker(\ell) \to \operatorname{Im}(\ell)^\perp \to \operatorname{Im}(\ell)$, and under this decomposition, ℓ has block form $\left(\begin{smallmatrix} \mathrm{id} & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Now throwing in ρ, we have a map $V_1 \oplus V_2 \to V_1 \oplus V_2 \oplus V_2$ given by $(v_1, v_2) \mapsto (v_1, 0, v_2)$, and this is closed. Now including in $\overline{c(A)}$ can follow from a general result, or we can prove it directly: TODOI missed this.

The other two pieces are pretty simple: the map $h, b, k \mapsto h - b, b, k$ is an inverse to f_2, and properness of f_2 follows directly from the definition.

Now we assume f is bounded, to prove the second statement. Let $x \in H'$ and $y \coloneqq f(x)$. Let A' be some closed and bounded set containing x; then $\overline{f(A')}$ is contained in some disc $D \subset H$. If $A'_0 \coloneqq f^{-1}(D) \cap A'$, then any $h \in \overline{f(A')}$ is also in $\overline{f(A'_0)}$. Now we can use A'_0 as our closed and bounded set, reduce to the first part of the theorem, and conclude that f is proper. (We need A'_0 so that the image is closed, which might not be true of A.)

For f^+, we know that the preimage of any point in H is compact, and $(f^+)^{-1}(\infty) = \{\infty\}$ is compact, so all we have to check is that f^+ is closed. Let $A' \subset (H')^+$ be closed(otherwise we're already done, because f is proper), and suppose $\infty \in \overline{f(A')}$; then $f(A') \setminus \{\infty\}$ is closed in H. This implies $f(A')$ is unbounded, so A' is unbounded (because $f = \ell + c$, and ℓ and c take bounded sets to bounded sets). Since A' is closed, then $\infty \in A'$ as we desired. ⊠

Remark 11.5. This kind of proof does not generalize to families of operators. That's not a problem today, but if that is something you're interested in, here's where you have to do something different. ◂

Now we want to pass to finite-dimensinal spheres, and therefore a stable cohomotopy class. Let $W \subset H$ be a finite-dimensional subspace and $S(W^\perp) \coloneqq S(H) \cap W^\perp$, where $S(H)$ denotes the unit sphere in H.

Proposition 11.6. *The projection map $W^+ \to H^+/S(W^\perp)$ is a deformation retract.*

Proof. We'll write down an inverse map. We can identify $H^+ \cong S(\mathbb{R} \oplus H)$ and $H' = W \oplus W^\perp$. The map in the proposition is explicitly, as a map $W^+ \to S(\mathbb{R} \oplus H)$,

$$h \longmapsto \frac{(|h|^2 - 1, 2h)}{|h^2 + 1|}. \tag{11.7}$$

This sends $S(W^\perp)$ to $S(0 \oplus 0 \oplus W^\perp)$, so this does descend to the quotient as claimed.

The retraction is "along latitudes": $S(\mathbb{R} \oplus W \oplus W^\perp) \setminus S(0 \oplus 0 \oplus W^\perp) \to S(\mathbb{R} \oplus W \oplus 0)$. The idea is, if we remove the "equator," we can slide latitudes off to the north and south poles, which can be seen explicitly on S^1 and S^2, and generalizes to the setting we care about. ⊠

This retraction is not very far from being a projection: $\rho_W(h) = \lambda(h)\mathrm{pr}_W(h)$, where $\lambda(h)$ is some positive number. This is explicit if we let H be one- or two-dimensional, and also holds in the general case.

The key proposition, whose proof we'll see next time, is as follows.

Proposition 11.8. *There exists a finite-dimensional $V \subset H$ such that $H = \operatorname{Im}(f) + V$ and for all finite-dimensional $W \supset V$, if $W' \coloneqq f^{-1}(W)$, then the image of $f|_{(W')^+}\colon (W')^+ \to H^+$ does not intersect $S(W^\perp)$. Moreover, if $U = V^\perp \cap W$, then $W' \cong U \oplus V'$, where $V' \coloneqq \ell^{-1}(V)$.*

Hence we can apply Proposition 11.6 and get a map of finite-dimensional spheres. And the last part implies $W^+ \cong U^+ \wedge (V')^+$, and $\rho \circ f|_{(W')^+} : (W')^+ \to W^+$ decomposes as $U^+ \wedge V^+ \to U^+ \wedge V^+$. This is the suspension compatibility that will give us a stable class.

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