

M392C NOTES: SPIN GEOMETRY

ARUN DEBRAY
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Lecture 1.

Lie Groups: 8/25/16

There is a course website, located at <https://www.ma.utexas.edu/users/ekorman/teaching/spingeometry/>. There's a list of references there, none of which we'll exactly follow.

We'll assume some prerequisites for this class: definitely smooth manifolds and some basic algebraic topology. We'll use cohomology, which isn't part of our algebraic topology prelim course, but we'll review it before using it.

Introduction and motivation. Recall that a *Riemannian manifold* is a pair (M, g) where M is an n -dimensional smooth manifold and g is a *Riemannian metric* on M , i.e. a smoothly varying, positive definite inner product on each tangent space $T_x M$ over all $x \in M$.

Definition 1.1. A *local frame* on M is a set of (locally defined) tangent vectors that give a positive basis for M , i.e. a smoothly varying set of tangent vectors that are a basis at each tangent space.

A Riemannian metric allows us to talk about *orthonormal frames*, which are those that are orthonormal with respect to the metric at all points.

Recall that the special orthogonal group is $SO(n) = \{A \in M_n \mid AA^T = I, \det A = 1\}$. This acts transitively on orthonormal, oriented bases, and therefore also acts transitively on orthonormal frames (as a frame determines an orientation). Conversely, specifying which frames are orthonormal determines the metric g .

In summary, the data of a Riemannian structure on a smooth manifold is equivalent to specifying a subset of all frames which is acted on simply transitively¹ by the group $SO(n)$. This set of all frames is a *principal $SO(n)$ -bundle* over M .

By replacing $SO(n)$ with another group, one obtains other kinds of geometry: using $GL(n, \mathbb{C})$ instead, we get almost complex geometry, and using $Sp(n)$, we get almost symplectic geometry (geometry with a specified skew-symmetric, nondegenerate form).

Remark. Let G be a Lie group and M be a manifold. Suppose we have a principal G -bundle $E \rightarrow M$ and a representation² $\rho : G \rightarrow V$, we naturally get a vector bundle over M .

A more surprising fact is that all³ representations of $SO(n)$ are contained in tensor products of the *defining representation* of $SO(n)$ (i.e. acting on \mathbb{R}^n by orientation-preserving rotations). Thus, all of the natural vector bundles are subbundles of tensor powers of the tangent bundles. That is, when we do geometry in this way, we obtain no exotic vector bundles.

¹Recall that a group action on X is *transitive* if for all $x, y \in X$, there's a group element g such that $g \cdot x = y$, and is *simple* if this g is unique.

²A representation of a group G is a homomorphism $G \rightarrow GL(V)$ for a vector space V . We'll talk more about representations later.

³We're only considering smooth, finite-dimensional representations.

If $n \geq 3$, then $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2$, so its double cover is its universal cover. Lie theory tells us this space is naturally a compact Lie group, called the *Spin group* $\mathrm{Spin}(n)$. In many ways, it's more natural to look at representations of this group. The covering map $\mathrm{Spin}(n) \rightarrow \mathrm{SO}(n)$ precomposes with any representation of $\mathrm{SO}(n)$, so any representation of $\mathrm{SO}(n)$ induces a representation of $\mathrm{Spin}(n)$. However, there are representations of the spin group that don't arise this way, so if we can refine the orthonormal frame bundle to a principal $\mathrm{Spin}(n)$ -bundle, then we can create new vector bundles that don't arise as tensor powers of the tangent bundle.

Spin geometry is more or less the study of these bundles, called *bundles of spinors*; these bundles have a natural first-order differential operator called the *Dirac operator*, which relates to a powerful theorem coming out of spin geometry, the Atiyah-Singer index theorem: this is vastly more general, but has a particularly nice form for Dirac operators, and the most famous proof reduces the general case to the Dirac case. Broadly speaking, the index theorem computes the dimension of the kernel of an operator, which in various contexts is a powerful invariant. Here are a few special cases, even of just the Dirac case of the Atiyah-Singer theorem.

- The Gauss-Bonnet-Chern theorem gives an integral formula for the Euler characteristic of a manifold, which is entirely topological. In this case, the index is the Euler characteristic.
- The Hirzebruch signature theorem gives an integral formula for the signature of a manifold.
- The Grothendieck-Riemann-Roch theorem, which gives an integral formula for the Euler characteristic of a holomorphic vector bundle over a complex manifold.

Lie groups and Lie algebras.

Definition 1.2. A Lie group G is a smooth manifold with a group structure such that the multiplication map $G \times G \rightarrow G$ sending $g_1, g_2 \mapsto g_1 g_2$ and the inversion map $G \rightarrow G$ sending $g \mapsto g^{-1}$ are smooth.

Example 1.3.

- The general linear group $\mathrm{GL}(n, \mathbb{R})$ is the group of $n \times n$ invertible matrices with coefficients in \mathbb{R} . Similarly, $\mathrm{GL}(n, \mathbb{C})$ is the group of $n \times n$ invertible complex matrices. Most of the matrices we consider will be subgroups of these groups.
- Restricting to matrices of determinant 1 defines $\mathrm{SL}(n, \mathbb{R})$ and $\mathrm{SL}(n, \mathbb{C})$, the *special linear groups*.
- The *special unitary group* $\mathrm{SU}(n) = \{A \in \mathrm{GL}(n, \mathbb{C}) \mid A \overline{A}^T = 1, \det A = 1\}$.
- The special orthogonal group $\mathrm{SO}(n)$, mentioned above.

Definition 1.4. A Lie algebra is a vector space \mathfrak{g} with an anti-symmetric, bilinear pairing $[-, -] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

for all $X, Y, Z \in \mathfrak{g}$.

Example 1.5. The basic and important example: if A is an algebra,⁴ then A becomes a Lie algebra with the commutator bracket $[a, b] = ab - ba$. Because this algebra is associative, the Jacobi identity holds.

The Jacobi identity might seem a little vague, but here's another way to look at it: if \mathfrak{g} is a Lie algebra and $X \in \mathfrak{g}$, then there's a map $\mathrm{ad}_X : \mathfrak{g} \rightarrow \mathfrak{g}$ sending $Y \mapsto [X, Y]$. The map $\mathrm{ad} : \mathfrak{g} \rightarrow \mathrm{End}(\mathfrak{g})$ sending $X \mapsto \mathrm{ad}_X$ is called the *adjoint representation* of X . The Jacobi identity says that ad intertwines the bracket of \mathfrak{g} and the bracket induced from the algebra structure on $\mathrm{End}(\mathfrak{g})$ (where multiplication is composition): $\mathrm{ad}_{[X, Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y]$. In other words, the adjoint representation is a homomorphism of Lie algebras.

Lie groups and Lie algebras are very related: to any Lie group G , let \mathfrak{g} be the set of left-invariant vector fields on G , i.e. if $L_g : G \rightarrow G$ is the map sending $h \mapsto gh$ (the *left multiplication map*), then $\mathfrak{g} = \{X \in \Gamma(TG) \mid dL_g X = X \text{ for all } g \in G\}$. This is actually finite-dimensional, and has the same dimension as G .

Proposition 1.6. If e denotes the identity of G , then the map $\mathfrak{g} \rightarrow T_e G$ sending $X \mapsto X(e)$ is an isomorphism (of vector spaces).

The idea is that given the data at the identity, we can translate it by g to determine what its value must be everywhere. Vector fields have a Lie bracket, and the Lie bracket of two left-invariant vector fields is again left-invariant, so \mathfrak{g} is naturally a Lie algebra! We will often use Proposition 1.6 to identify \mathfrak{g} with the tangent space at the identity.

⁴By an algebra we mean a ring with a compatible vector space structure.

Example 1.7. Let's look at $GL(n, \mathbb{R})$. This is an open submanifold of the vector space M_n , an n^2 -dimensional vector space, as $\det A \neq 0$ is an open condition. Thus, the tangent bundle of $GL(n, \mathbb{R})$ is trivial, so we can canonically identify $T_1 GL(n, \mathbb{R}) = M_n$. With the inherited Lie algebra structure, this space is denoted $\mathfrak{gl}(n, \mathbb{R})$.

The $n \times n$ matrices are also isomorphic to $\text{End}(\mathbb{R}^n)$, since they act by linear transformations. The algebra structure defines another Lie bracket on this space.

Proposition 1.8. *Under the above identifications, these two brackets are identical, hence define the same Lie algebra structure on $\mathfrak{gl}(n, \mathbb{R})$.*

Remark. This proposition generalizes to all real matrix Lie groups (Lie subgroups of $GL(n, \mathbb{R})$): the proof relies on a Lie subgroup's Lie algebra being a subalgebra of $\mathfrak{gl}(n, \mathbb{R})$.

So we can go from Lie groups to Lie algebras. What about in the other direction?

Theorem 1.9. *The correspondence sending a connected, simply-connected Lie group to its Lie algebra extends to an equivalence of categories between the category of simply connected Lie groups and finite dimensional Lie algebras over \mathbb{R} .*

Suppose G is any connected Lie group, not necessarily simply connected, and \mathfrak{g} is its Lie algebra. If \tilde{G} denotes the universal cover of G , then $G = \tilde{G}/\pi_1(G)$. Since \tilde{G} is simply connected, the correspondence above identifies \mathfrak{g} with it, and then taking the quotient by the discrete central subgroup $\pi_1(G)$ recovers G .

The special orthogonal group. We specialize to $SO(n)$, the orthogonal matrices with determinant 1. We'll usually work over \mathbb{R} , but sometimes \mathbb{C} . This is a connected Lie group.⁵

Proposition 1.10. *If $\mathfrak{so}(n)$ denotes the Lie algebra of $SO(n)$, then $\mathfrak{so}(n) = \{X \in \mathfrak{gl}(n, \mathbb{R}) \mid X + X^T = 0\}$.*

That is, $\mathfrak{so}(n)$ is the Lie algebra of skew-symmetric matrices.

Proof. If $F : M_n \rightarrow M_n$ is the function $A \mapsto A^T A - I$, then the orthogonal group is $O(n) = F^{-1}(0)$. Since $SO(n)$ is the connected component of $O(n)$ containing the identity, then it suffices to calculate $T_e O(n)$: if 0 is a regular value of F , we can push forward by its derivative. This is in fact the case:

$$dF_A(B) = \left. \frac{d}{dt} \right|_{t=0} F(A + tB) = A^T B + B^T A,$$

which is surjective for $A \in O(n)$, so $\mathfrak{so}(n) = T_1 SO(n) = \ker(dF_1) = \{B \in M_n \mid B + B^T = 0\}$. \square

The spin group. We'll end by computing the fundamental group of $SO(n)$; then, by general principles of Lie groups, each $SO(n)$ has a unique, simply connected double cover, which is also a Lie group. Next time, we'll provide an *a priori* construction of this cover.

Proposition 1.11.

$$\pi_1(SO(n)) = \begin{cases} \mathbb{Z}, & n = 2 \\ \mathbb{Z}/2, & n \geq 3. \end{cases}$$

Proof. If $n = 2$, $SO(n) \cong S^1$ through the identification

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \mapsto e^{i\theta},$$

and we know $\pi_1(S^1) = \mathbb{Z}$.

For $n \geq 3$, we can use a long exact sequence associated to a certain fibration, so it suffices to calculate $\pi_1(SO(3))$. Specifically, we will define a Lie group structure on S^3 and a double cover map $S^3 \twoheadrightarrow SO(3)$; since S^3 is simply connected, this will show $\pi_1(SO(3)) = \mathbb{Z}/2$.

We can identify S^3 with the unit sphere in the quaternions, which is naturally a group (since the product of quaternions is a polynomial, hence smooth).⁶ Realize \mathbb{R}^3 inside the quaternions as $\text{span}_{\mathbb{R}}\{i, j, k\}$ (the *imaginary quaternions*); then, we'll define $\varphi : S^3 \twoheadrightarrow SO(3)$: $\varphi(q)$ for $q \in \mathbb{H}$ is the linear transformation $p \mapsto qpq^{-1} \in GL(3, \mathbb{R})$,

⁵If we only took orthogonal matrices with arbitrary determinant, we'd obtain the *orthogonal group* $O(n)$, which has two connected components.

⁶This is important, because when we try to generalize to Spin_n for higher n , we'll be using Clifford algebras, which are generalizations of the quaternions.

where p is an imaginary quaternion. We need to check that $\varphi(q)$ lies in $SO(3)$, which was left as an exercise. We also need to check this is two-to-one, which is equivalent to $|\ker \varphi| = 2$, and that φ is surjective (hint: since these groups are connected, general Lie theory shows it suffices to show that the differential is an isomorphism). \square