SPRING 2017 HOMOTOPY THEORY SEMINAR

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Contents

1. Higher K-theory: 1/25/17

Today, Nicky spoke on a few approaches to higher K-theory.

Let C be a pointed ∞-category with finite colimits (as in Lurie's approach) or a category with cofibrations and weak equivalences satisfying certain axioms (as in Waldhausen's approach).

Recall that $K_0(\mathsf{C})$ was defined to be the free abelian group on isomorphism classes of objects of C modulo $[X] = [X'] + [X'']$ whenever we have a pullback

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\
* \arrow[r] & X''.
\end{tikzcd}$$

We want to generalize to a based space W such that $\pi_1(W) = K_0(\mathsf{C})$, and satisfying a universal property for C: every object X in C should determine a path p_X from $*$ to $*$ in W, and for any cofiber sequence $X' \to X \to X''$, we'd like the 2-cell bounded by the paths p_X, $p_{X'}$, and $p_{X''}$ to be contractible in W.

Remark. Given a map $f : X \to Y$, we'll let Y/X denote the cofiber of f. Waldhausen is working with maps that are already cofibrant (since he works with categories that already have special classes of maps), but the suitable cofibrant replacement also exists for ∞-categories. This notation implies that in K_0, $[Y] = [X] + [Y/X]$. ◀

Proposition 1.1. *Let $X \to Y \to Z$ be a cofiber sequence. Then, $[Z] = [X] + [Y/X] + [Z/Y]$.*

Proof. One way to prove this is to observe that $X \to Y \to Z$ means that the following two sequences are cofiber sequences:

$$X \longrightarrow Z \longrightarrow Z/X$$
$$Y/X \longrightarrow Z/X \longrightarrow Z/Y.$$

Alternatively, you could observe that that the following two sequences are cofiber sequences:

$$Y \longrightarrow Z \longrightarrow Z/Y$$
$$X \longrightarrow Y \longrightarrow Y/X.$$ □

These two proofs of this identity are two homotopies between the paths p_Z and $p_X \circ p_{Y/X} \circ p_{Z/Y}$:

(1.2)
$$\begin{tikzcd}
 & * \arrow[r, "p_{Y/X}"] \arrow[drr, Rightarrow, "p_Y"] & * \arrow[dr, "p_X"] & \\
* \arrow[ur, "p_{Z/Y}"] \arrow[urr, Rightarrow, "p_{Z/X}"'] \arrow[rrr, "p_Z"'] & & & *.
\end{tikzcd}$$

We'd like for these two homotopies to be homotopic: the two proofs of Proposition 1.1 define a map $\partial\Delta^3$ into the diagram (1.2), and we want this to extend to a map from Δ^3. In a similar way, we'd like to have a similar "coherence" statement corresponding to sequences $X_1 \to X_2 \to \cdots \to X_n$.

Waldhausen's $S_\bullet$-construction does this all formally for us. It works by gluing classifying spaces of these sequences together, which feels like a homotopy coherent nerve but isn't quite one. One way to think about is that there are choices made when making a quotient; the $S_\bullet$ construction keeps them around as simplicial data. More explicitly, given the sequence $X_0 \to X_1 \to \cdots \to X_n$, you want the 0^{th} face map to arise from a sequence $\cdots \to X_i/X_1 \to X_{i+1}/X_1 \to \cdots$, but there are choices made in picking these maps.

The formalism of the $S_\bullet$ construction will involve some homotopy theory of posets, but is nicer than last semester's stuff. Let P be a poset, and set

$$P^{(2)} := \{(i,j) \in P \times P \mid i \le j\},$$

which is also $\mathsf{Fun}(\Delta^1, P)$.

Definition 1.3. A ***P*-gapped object** in C is a functor $X : N(P^{(2)}) \to \mathsf{C}$ such that for all $i \in P$, $X(i,i) = *$ and for all $i \le j \le k$ in P, we have a pushout square

$$\begin{tikzcd}
X(i,j) \arrow[r] \arrow[d] & X(i,k) \arrow[d] \\
* = X(j,j) \arrow[r] & X(j,k).
\end{tikzcd}$$

This is a functorial notion: if $P \to Q$ is a map of posets, we get a map $N(P) \to N(Q)$, and f takes Q-gapped objects to P-gapped ones. That is, "$(-)$-gapped objects" is a functor from the simplicial indexing functor to ∞-categories. We're going to bundle this up into a simplicial set.

As usual, let $[n]$ denote the poset $\{0 < 1 < 2 < \cdots < n\}$. Let $\mathrm{Gap}_{[n]}(\mathsf{C})$ denote the ∞-category of $[n]$-gapped objects in C. Concretely, this is a diagram category for the diagram

$$\begin{tikzcd}
* = X(0,0) \arrow[r] & X(0,1) \arrow[r] \arrow[d] & X(0,2) \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & X(0,n) \arrow[d] \\
& * = X(1,1) \arrow[r] & X(1,2) \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & X(1,n) \arrow[d] \\
& & * = X(2,2) \arrow[r] & \cdots \arrow[r] \arrow[d] & X(2,n) \arrow[d] \\
& & & \vdots & \vdots \arrow[d] \\
& & & & * = X(n,n).
\end{tikzcd}$$

There's a nice animation of this available at https://www.ma.utexas.edu/users/ysulyma/.

Let $S_n(\mathsf{C})$ denote the underlying Kan complex of $\mathrm{Gap}_{[n]}(\mathsf{C})$: it's not necessarily a groupoid, but we can throw away all non-invertible arrows.[1] Thus, $S_\bullet(\mathsf{C})$ is a simplicial Kan complex,[2] and we're going to geometrically realize it. In low degrees, this recovers things we've seen before:

- $S_0(\mathsf{C}) = \mathrm{Gap}_{[0]}(\mathsf{C})$ is the full subcategory of 0-objects, which is contractible.
- $\mathrm{Gap}_{[1]}(\mathsf{C}) \cong \mathsf{C}$ (diagrams of the form $* \to X \to *$), and $S_1(\mathsf{C})$ is equivalent to the category of isomorphisms in C.

[1] Alternatively, if you're working with categories of weak equivalences, rather than ∞-categories, you're throwing out everything but the weak equivalences.

[2] By a **simplicial Kan complex**, we mean a bisimplicial set such that each $S_n(\mathsf{C})$ is a Kan complex.

- $\mathrm{Gap}_{[2]}(\mathsf{C})$ is equivalent to the ∞-category of cofiber sequences in C.

Now, we can define K-theory.

Definition 1.4. The **algebraic K-theory** of C is

$$K(\mathsf{C}) := \Omega|S_\bullet(\mathsf{C})|.$$

Because $S_\bullet(\mathsf{C})$ is a simplicial Kan complex, we must specify the geometric realization; you can either geometrically realize the diagonal or geometrically realize one axis with topology on the sets of simplices.

The K-**groups** of C are $K_n(\mathsf{C}) := \pi_n K(\mathsf{C}) = \pi_{n+1}|S_\bullet(\mathsf{C})|$.

These agree with the K-groups we defined in low dimensions, but this is a theorem.

Remark.

- Eventually, we will see how to promote this from a space to a spectrum.
- If $S_\bullet(\mathsf{C})$ is contractible, then $|S_\bullet(\mathsf{C})|$ is conncted.
- Let $F : \mathsf{C} \to \mathsf{D}$ be a functor between suitably nice[3] ∞-categories; then, we obtain a map $K(\mathsf{C}) \to K(\mathsf{D})$.
- The projections $\mathsf{C} \leftarrow \mathsf{C} \times \mathsf{D} \to \mathsf{D}$ are nice, so

$$K(\mathsf{C} \times \mathsf{D}) \cong K(\mathsf{C}) \times K(\mathsf{D}).$$

- The coproduct functor $\amalg : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ is sufficiently nice, so there's a multiplication map $m : K(\mathsf{C}) \times K(\mathsf{C}) \to K(\mathsf{C})$, which is coherently associative and commutative. In fact, $K(\mathsf{C})$ has an E_∞-structure. ◀

We'd like to compare the new K_0 and the old K_0. $|S_\bullet(\mathsf{C})|$ is a direct limit across the **skeleton functors** sk_i sending X to its i-skeleton:

$$|S_\bullet(\mathsf{C})| = \mathrm{colim}\big(\mathrm{sk}_0|S_\bullet(\mathsf{C})| \longrightarrow \mathrm{sk}_1|S_\bullet(\mathsf{C})| \longrightarrow \mathrm{sk}_2|S_\bullet(\mathsf{C})| \longrightarrow \cdots\big).$$

- We know the 0-skeleton: $\mathrm{sk}_0|S_\bullet(\mathsf{C})| = S_0(\mathsf{C})$ is contractible.
- For the 1-skeleton, $\mathrm{sk}_1|S_\bullet(\mathsf{C})| = \Sigma S_1\mathsf{C} = \Sigma\,\mathrm{iso}\,\mathsf{C}$. Thus, we have a map $\Sigma\,\mathrm{iso}\,\mathsf{C} \to |S_\bullet(\mathsf{C})|$ whose adjoint begins a sequence

$$\mathrm{iso}\,\mathsf{C} \longrightarrow \Omega|S_\bullet(\mathsf{C})| \longrightarrow \Omega^2|S_\bullet S_\bullet(\mathsf{C})| \longrightarrow \cdots$$

These are the maps that will define the K-theory spectrum.

Thus, we know $K_0(\mathsf{C}) = \pi_0(K(\mathsf{C})) = \pi_1(S_\bullet(\mathsf{C}))$, which is generated by isomorphims classes of objects in C. The relations are generated by things in $\mathrm{sk}_2|S_\bullet(\mathsf{C})|$: we've glued in 2-cells in $S_2(\mathsf{C})$ to introduce relations. That is, the relations are defined by $\pi_0(S_2(\mathsf{C}))$, which is the set of cofiber sequences. Thus, $K_0(\mathsf{C})$ is the abelian group generated by objects and modulo cofiber sequences, as desired.

Algebraic K-theory as a spectrum. Since $\mathrm{sk}_1|S_\bullet(\mathsf{C})|$ is obtained from $\mathrm{sk}_0|S_\bullet(\mathsf{C})|$ by attaching $S_1\mathsf{C} \times \Delta^1$ and $\mathfrak{so}_0|S_\bullet(\mathsf{C})|$ is contractible, then $\mathrm{sk}_1|S_\bullet(\mathsf{C})| \simeq \Sigma S_1(\mathsf{C}) \simeq \Sigma\,\mathrm{iso}\,\mathsf{C}$.

The 1-skeleton includes into the whole space, so by adjunction, we have an inclusion $\mathrm{iso}\,\mathsf{C} \hookrightarrow \Omega|S_\bullet(\mathsf{C})|$. Thus we can begin to define a spectrum, in fact an Ω-spectrum.

Definition 1.5. The **algebraic K-theory spectrum** $\widetilde{K}(\mathsf{C})$ is the spectrum assigning

$$n \longmapsto |\mathrm{iso}\,\underbrace{S_\bullet S_\bullet \cdots S_\bullet}_{n}\mathsf{C}|,$$

with the maps induced from the adjunction above.

Remark. There's a technicality here with basepoints. Waldhausen solved this by requiring exact functors to be based, but typically for ∞-categories, one requires a functor to send a zero object to a zero object. This is an issue for setting up functoriality, which is worth being aware of. One way to solve this is to quotient out by these choices. In practice, however, exact functors tend to strictly preserve the basepoint. ◀

[3] Probably pointed and with finite colimits.

Some things to notice here: the n^{th} term is $(n-1)$-connected (since the 0-skeleton of the $S_\bullet$-construction is contractible). This is an ingredient in the additivity theorem, an important result that will be presented next time. This will imply that the K-theory spectrum is an Ω-spectrum, allowing a more concise definition of K-theory space:

$$K(\mathsf{C}) = \varinjlim_n \Omega^n |S_\bullet^{(n)}(\mathsf{C})| = \Omega^\infty |S_\bullet^{(\infty)}(\mathsf{C})|.$$

This is helpful because it shows that $K(\mathsf{C})$ is an infinite loop space, and this is how we get the E_∞ structure. The point is, the Ω-spectrum structure gives you the infinite loop space structure on the nose; you don't have to take a colimit.

Remark. The $S_\bullet$-construction looks a little bit like a suspension, and there's a way in which this can be made precise. Another way of looking at this is that if you don't shift up and deloop, you have an Ω-spectrum after the 0^{th} level. This relates to the fact that the $S_\bullet$-construction is not a Kan complex, but after one subdivision, it becomes one. The class of simplicial sets with this property is formally interesting. ◂