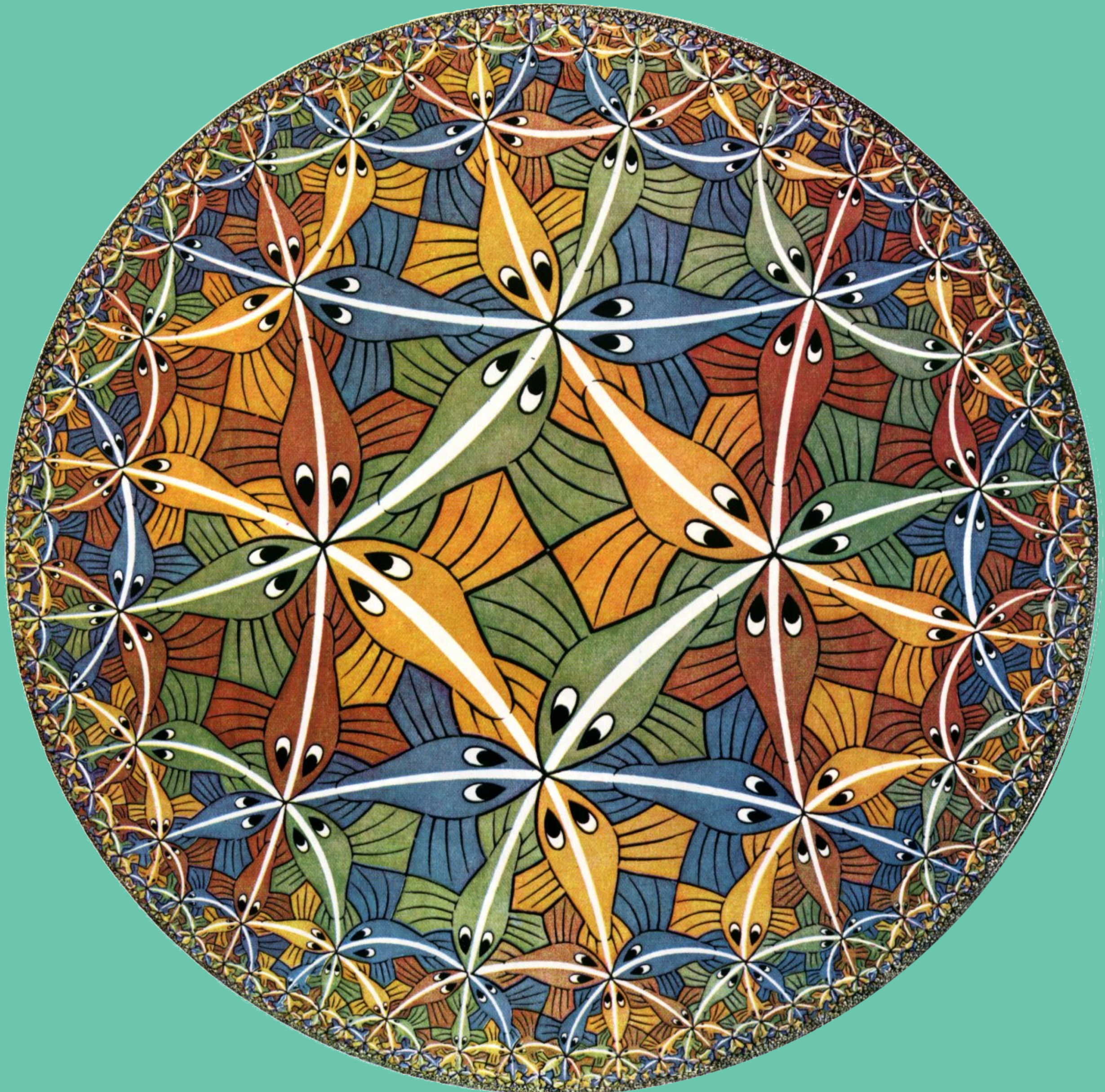
Riemann Surfaces



UT Austin, Spring 2016

M392C NOTES: RIEMANN SURFACES

ARUN DEBRAY
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These notes were taken in UT Austin's Math 392c (Riemann Surfaces) class in Spring 2016, taught by Tim Perutz. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. The image on the front cover is M.C. Escher's *Circle Limit III* (1959), sourced from http://www.wikiart.org/en/m-c-escher/circle-limit-iii.

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Lecture 1.

Review of Complex Analysis: 1/20/16

Riemann surfaces is a subject that combines the topology of structures with complex analysis: a Riemann surface is a surface endowed with a notion of holomorphic function. This turns out to be an extremely rich idea; it's closely connected to complex analysis but also to algebraic geometry. For example, the data of a compact Riemann surface along with a projective embedding specifies a proper algebraic curve over $\mathbb{C}$, in the domain of algebraic geometry.[1] In fact, the algebraic geometry course that's currently ongoing is very relevant to this one.

The theory of Riemann surfaces ties into many other domains, some of them quite applied: number theory (via modular forms), symplectic topology (pseudo-holomorphic forms), integrable systems, group theory, and so on: so a very broad range of mathematics graduate students should find it interesting.

Moreover, by comparison with algebraic geometry or the theory of complex manifolds, there's very low overhead; we will quickly be able to write down some quite nontrivial examples and prove some deep theorems: by the middle of the semester, hopefully we will prove the analytic Riemann-Roch theorem, the fundamental theorem on compact Riemann surfaces, and use it to prove a classification theorem, called the uniformization theorem.

The course textbook is S.K. Donaldson's *Riemann Surfaces*, and the course website is at http://www.ma.utexas.edu/users/perutz/RiemannSurfaces.html; it currently has notes for this week's material, a rapid review of complex function theory. We will assume a small amount of complex analysis (on the level of Cauchy's theorem; much less than the complex analysis prelim) and topology (specifically, the relationship between the fundamental group and covering spaces). Some experience with calculus on manifolds will be helpful. Some real analysis will be helpful, and midway through the semester there will be a few Hilbert spaces. Thus, though this is a topics course, the demands on your knowledge will more resemble a prelim course.

Let's warm up by (quickly) reviewing basic complex analysis; the notes on the course website will delve into more detail. For the rest of this lecture, G denotes an open set in $\mathbb{C}$.

The following definition is fundamental.

[1] This sentence is packed with jargon you're not assumed to know yet.

Definition. A function $f : G \to \mathbb{C}$ is *holomorphic* if for all $z \in G$, the complex derivative
$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$
exists. The set of holomorphic functions $G \to \mathbb{C}$ is denoted $\mathscr{O}(G)$, after the Italian *funzione olomorfa*.

Note that even though it makes sense for the limit to be infinite, this is not allowed.

First, let's establish a few basic properties.

- If $H \subset G$ is open and $f \in \mathscr{O}(G)$, then $f|_H \in \mathscr{O}(H)$.
- The sum, product, quotient, and chain rules hold for holomorphic functions, so $\mathscr{O}(G)$ is a commutative ring (with multiplication given pointwise) and in fact a commutative $\mathbb{C}$-algebra.[2]

In other words, holomorphic functions define a *sheaf* of $\mathbb{C}$-algebras on G.

By a rephrasing of the definition, then if f is holomorphic on G, then it has a *derivative* f' on G, i.e. for all $z \in G$, one can write $f(z+h) = f(z) + f'(z)h + \varepsilon_z(h)$, where $\varepsilon_z(h) \in o(h)$ (that is, $\varepsilon_z(h)/h \to 0$ as $h \to 0$). Thus, a holomorphic function is differentiable in the real sense, as a function $G \to \mathbb{R}^2$. This means that there's an $\mathbb{R}$-linear map $D_z f : \mathbb{C} \to \mathbb{C}$ such that $f(z+h) = f(z) + (D_z f)(h) + o(h)$: here, $D_z f(h) = f'(z)h$.

However, we actually know that $D_z f$ is $\mathbb{C}$-linear. This is known as the *Cauchy-Riemann condition.* Since it's *a priori* $\mathbb{R}$-linear, saying that it's $\mathbb{C}$-linear is equivalent to it commuting with multiplication by i. $D_z f$ is represented by the Jacobian matrix
$$D_z f = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}.$$
A short calculation shows that this commutes with i iff the following equations, called the *Cauchy-Riemann equations*, hold:
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1.1}$$
The content of this is exactly that $D_z f$ is complex linear.

Conversely, suppose $f : G \to \mathbb{C}$ is differentiable in the real sense. Then, if it satisfies (1.1), then $D_z f$ is complex linear. But a complex linear map $\mathbb{C} \to \mathbb{C}$ must be multiplication by a complex number $f'(z)$, so f is holomorphic, with derivative f'.

Power Series. The notation $D(c, R)$ means the open disc centered at c with radius R, i.e. all points $z \in \mathbb{C}$ such that $|z - c| < R$.

Definition. Let $A(z) = \sum_{n=0}^{\infty} a_n (z-c)^n$ be a $\mathbb{C}$-valued power series centered at a $c \in \mathbb{C}$. Then, its *radius of convergence* is $R = \sup\{|z - c| : A(z) \text{ converges}\}$, which may be 0, a positive real number, or ∞.

Theorem 1.1. *Suppose $A(z) = \sum_{n \geq 0} a_n (z-c)^n$ has radius of convergence R. Then:*

(1) $R^{-1} = \limsup |a_n|^{1/n}$*;*
(2) *$A(z)$ converges absolutely on $D(c, R)$ to a function $f(z)$;*
(3) *the convergence is uniform on smaller discs $D(c, r)$ for $r < R$;*
(4) *the series $B(z) = \sum_{n \geq 1} n a_n (z-c)^{n-1}$ has the same radius of convergence R, so converges on $D(c, R)$ to a function $g(z)$; and*
(5) *$f \in \mathscr{O}(D(c, R))$ and $f' = g$.*

These aren't extremely hard to prove: the first few rely on various series convergence tests from calculus, though the last one takes some more effort.

Paths and Cauchy's Theorem. By a *path* we mean a continuous and piecewise C^1 map $[a, b] \to \mathbb{C}$ for some real numbers $a < b$. That is, it breaks up into a finite number of chunks on which it has a continuous derivative. A *loop* is a path γ such that $\gamma(a) = \gamma(b)$.

If γ is a C^1 path in G (so its image is in G) and $f : G \to \mathbb{C}$ is continuous, we define the *integral*
$$\int_\gamma f = \int_\gamma f(z)\,\mathrm{d}z := \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t.$$

[2] A $\mathbb{C}$-*algebra* is a commutative ring A with an injective map $\mathbb{C} \hookrightarrow A$, which in this case is the constant functions.

This is a complex-valued function, because the rightmost integral has real and imaginary parts. This makes sense as a Riemann integral, because these real and imaginary parts are continuous. This is additive on the join of paths, so we can extend the definition to piecewise C^1 paths. Moreover, integrals behave the expected way under reparameterization, and so on.

Theorem 1.2 (Fundamental theorem of calculus)**.** *If $F \in \mathscr{O}(G)$ and $\gamma : [a,b] \to G$ is a path, then*

$$\int_\gamma F'(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)).$$

This is easy to deduce from the standard fundamental theorem of calculus. In particular, if γ is a loop, then the integral of a holomorphic function is 0.

Now, an extremely important theorem.

Definition. A *star-domain* is an open set $G \subset \mathbb{C}$ with a $z^* \in G$ such that for all $z \in G$, the line segment $[z^*, z]$ joining z^* and z is contained in G.

For example, any convex set is a star-domain.

Theorem 1.3 (Cauchy)**.** *If G is a star-domain, γ is a loop in G, and $f \in \mathscr{O}(G)$, then $\int_\gamma f = 0$. Indeed, $f = F'$, where*

$$F(z) = \int_{[z^*,z]} f.$$

The proof is in the notes, but the point is that you can check that this definition of F produces a holomorphic function whose derivative is f; then, you get the result. The idea is to compare $F(z+h)$ and $F(z)$ should be comparable, which depends on an explicit calculation of an integral of a holomorphic function around a triangle, which is not hard.

Cauchy didn't prove Cauchy's theorem this way; instead, he proved Green's theorem, using the Cauchy-Riemann equations. This is short and satisfying, but requires assuming that all holomorphic functions are C^1. This is true (which is great), but the standard (and easiest) way to show this is… Cauchy's theorem.

Lecture 2.

Review of Complex Analysis, II: 1/22/16

Today, we're going to continue not being too ambitious; next week we will begin to geometrify things. Last time, we stopped after Cauchy's theorem for a star domain G: for all f holomorphic on G and loops $\gamma \in G$, $\int_\gamma f = 0$, and in fact one can write down an antiderivative for f, and then apply the fundamental theorem of calculus.

Then one can bootstrap one's way up to a more powerful theorem; the next one is a version of the deformation theorem.

Corollary 2.1 (Deformation theorem)**.** *Let $G \subset \mathbb{C}$ be open and $\gamma_0, \gamma_1 : [a,b] \rightrightarrows G$ be C^1 loops that are C^1 homotopic through loops in G. Then, for all $f \in \mathscr{O}(G)$, $\int_{\gamma_0} f = \int_{\gamma_1} f$.*

Proof sketch. Fix a C^1 homotopy $\Gamma : [a,b] \times [0,1] \to G$ such that $\Gamma(a,s) = \Gamma(b,s)$ for all s, $\gamma_0(t) = \Gamma(t,0)$, and $\gamma_1(t) = \Gamma(t,1)$. Then, it is possible to divide $[a,b] \times [0,1]$ into a grid of rectangles fine enough such that the image of each rectangle is mapped under Γ to a subset of G contained in an open disc in $\mathbb{C}$, as in Figure 1. Now, by Cauchy's theorem in a disc, the integral does not depend on path within each disc, so we can apply Γ in over the rectangles from 0 to 1, showing that the two integrals are the same. ⊠

Corollary 2.2. *Cauchy's theorem holds in any simply connected open $G \subset \mathbb{C}$.*

This is considerably more general than star domains (e.g. the letter **C** is simply connected, but not a star domain). Moreover, on such a domain, any $f \in \mathscr{O}(G)$ has an antiderivative: pick some basepoint $z_0 \in G$, and let $\gamma(z_0, z)$ be a path from z_0 to z. Then,

$$F(z) = \int_{\gamma(z_0,z)} f(z)\,\mathrm{d}z$$

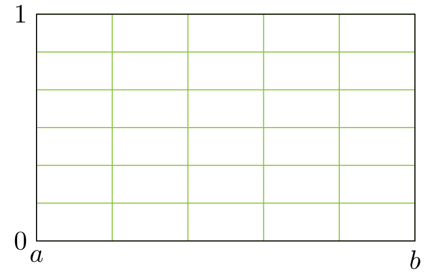

FIGURE 1. Subdividing $[a, b] \times [0, 1]$ into rectangles.

is well-defined, because any two choices of path differ by the integral of a holomorphic function on a loop, which is 0.

We can also use this to understand power series representations.

Proposition 2.3 (Cauchy's integral formula)**.** *, Let G be a domain in $\mathbb{C}$ containing the closed disc D. If $f \in \mathscr{O}(G)$, then*

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z}\, \mathrm{d}w.$$

Proof idea. Suppose D is centered at z and has radius R, and let $C(z, r)$ denote the circle centered at z and with radius r. We'll also let D^* denote the punctured disc, i.e. D minus its center point. By calculating $\int_\gamma \mathrm{d}z/z = 2\pi i$, one has that

$$\frac{1}{2\pi i} \int \partial D \frac{f(w)}{z - w}\, \mathrm{d}w - f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w) - f(z)}{w - z}\, \mathrm{d}w.$$

Using Corollary 2.1, for $r \in (0, R)$,

$$= \frac{1}{2\pi i} \int_{C(z,r)} \frac{f(w) - f(z)}{w - z}\, \mathrm{d}w,$$

and as $r \to 0$, this approaches $f'(z)$, which is bounded, and the integral over smaller and smaller circles of a bounded function tends to zero. ⊠

Theorem 2.4 (Holomorphic implies analytic)**.** *If D is a disc centered at c and $f \in \mathscr{O}(D)$, then on that disc,*

$$f(z) = \sum_{n \geq 0} a_n (z - c)^n, \qquad \text{where} \qquad a_n = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{(z - c)^{n+1}}\, \mathrm{d}z.$$

Proof sketch. For any $z \in D$, there's a $\delta > 0$ such that the closed disc $\overline{D}(z, \delta)$ of radius δ is contained in D. Hence, by Proposition 2.3,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{C(z,\delta)} \frac{f(w)}{w - z}\, \mathrm{d}w \\ &= \int_{C(c,R')} \frac{f(w)}{w - z}\, \mathrm{d}w \end{aligned}$$

for any $R' \in (0, \delta)$, by Corollary 2.1. We'd like to force a series on this. First, since

$$\frac{1}{w - z} = \frac{1}{(w - c) - (z - c)} = \frac{1}{w - c}\left(\frac{1}{1 - \frac{z-c}{w-c}}\right),$$

then

$$\begin{aligned} f(z) &= \frac{1}{3\pi i} \int_{C(c,R')} \frac{f(w)}{w - c} \frac{1}{1 - \frac{z-c}{w-c}}\, \mathrm{d}w \\ &= \frac{1}{2\pi i} \oint \frac{f(w)}{w - c} \sum_{n \geq 0} \frac{(z - c)^n}{(w - c)^n}\, \mathrm{d}w. \end{aligned}$$

Since $|(z-c)/(w-c)| < 1$ on $C(c, R')$, then this is well-defined, and since it's a geometric series, it has nice convergence properties, and so we can exchange the sum and integral to obtain

$$= \sum_{n\geq 0} \underbrace{\frac{1}{2\pi i}\left(\oint \frac{f(w)}{(w-c)^{n+1}}\,\mathrm{d}w\right)}_{a_n}(z-c)^n. \qquad \boxtimes$$

One application of this is to understand zeros of holomorphic functions. If $f \in \mathscr{O}(G)$ and $f(c) = 0$, then let $f(z) = \sum a_n(z-c)^n$ be its power series and a_m be the first nonzero coefficient. Then, in a neighborhood of c,

$$f(z) = (z-c)^m \underbrace{\sum_{n\geq m} a_n(z-c)^{n-m}}_{g(z)}.$$

This g is holomorphic and does not vanish on this neighborhood, so the takeaway is $f(z) = (z-c)^m g(z)$ near c, with g holomorphic and nonvanishing. This m is called the *multiplicity*, denoted $\mathrm{mult}(f, c)$. In particular, if $f(c) \neq 0$, then $m = 0$.

Theorem 2.5. *If G is a connected open set and $f \in \mathscr{O}(G)$ is not identically zero, then $f^{-1}(0)$ is discrete in $\mathbb{C}$.*

Proof. If $f(c) = 0$, then there's a disc D on which $f(z) = (z-c)^m g(z)$, where $m \geq 1$ and g is nonvanishing, so the only place f can vanish on D (i.e. near c) is at c itself. ⊠

Definition. A function $f \in \mathscr{O}(\mathbb{C})$, so holomorphic on the entire plane, is called *entire*.

Theorem 2.6 (Liouville)**.** *A bounded, entire function is constant.*

Proof sketch. We'll show that $f'(z) = 0$ everywhere. By Proposition 2.3, we know

$$f'(z) = \frac{1}{2\pi i}\int C(z,r)\frac{f(w)}{(w-z)^2}\,\mathrm{d}w,$$

and we can deform this loop to $C(0, R)$. Then, one bounds the integral, and the bound ends up being $O(1/R)$, so as $R \to \infty$, this necessarily goes to 0. ⊠

Lecture 3.

Meromorphic Functions and the Riemann Sphere: 1/25/16

We're still going to be doing classical function theory today, but we're going to begin to geometrify it. Recall that $G \subset \mathbb{C}$ denotes an open set.

We'll begin with the following theorem.

Theorem 3.1 (Morera)**.** *Let $f : G \to \mathbb{C}$ be a continuous function such that for all triangles $T \subset G$, $\int_{\partial T} f = 0$. Then, f is holomorphic.*

This is surprisingly easy to prove, given what we've done.

Proof. Since holomorphy is a local property, we may without loss of generality work on a disc $D(z_0, r) \subset G$. Then, define $F : D(z_0, r) \to \mathbb{C}$ by $F(z) = \int_{[z_0,z]} f$; using the hypothesis on triangles, $F' = f$. Thus, as we showed last time, this means $F \in \mathscr{O}(G)$, and so it's analytic, and therefore it has derivatives of all orders. Thus, $F' = f$ is holomorphic. ⊠

This is useful, e.g. one may have a function which is defined through an improper integral, or a pointwise limit of holomorphic functions. Then, Morera's theorem allows for an easier, indirect way to show holomorphy. Here's another application.

Definition. If $z_0 \in G$, a function $f \in \mathscr{O}(G \setminus \{z_0\})$ has a *removable singularity* at z_0 if f can be extended holomorphically to G.

Theorem 3.2. *Suppose $f \in \mathscr{O}(G \setminus \{z_0\})$ and $|f|$ is bounded near z_0. Then, f has a removable singularity at z_0.*

There are several ways to prove this quickly.

Proof. We can without loss of generality translate this to the origin, so assume $z_0 = 0$. If $g(z) = zf(z)$, then $g(z) \to 0$ as $z \to 0$, since $|f(z)|$ is bounded in a neighborhood of the origin. Thus, g extends continuously to all of G, with $g(0) = 0$.

Next, one should check that Morera's theorem applies to g; the only nontrivial example is a triangle around the origin. However, since g is holomorphic everywhere except at 0, the deformation theorem allows us to shrink the triangle as much as we want, and since $g \to 0$, the integral goes to 0 as well. If the triangle's edge or vertex touches the origin, one can use the deformation theorem to push it away again.

In particular, g is holomorphic on G and has a zero at 0, so by the discussion on multiplicities last time, $g(z) = z \cdot f(z)$, where f is holomorphic on all of G; this produces our desired extension of f. ⊠

Definition.

- If $z_0 \in G$ and $f \in \mathscr{O}(G \setminus \{z_0\})$, then f has a *pole* at z_0 if there's an $m \in \mathbb{N}$ such that $(z - z_0)^m f(z)$ is bounded near z_0 (and hence has a removable singularity there). The least such m is called the *order* of the pole.
- A *meromorphic* function on G is a pair (Δ, f) consisting of a discrete subset $\Delta \subset G$ and an $f \in \mathscr{O}(G \setminus \Delta)$ such that f has a pole at each $z \in \Delta$.

So, nothing worse than a pole happens for a meromorphic function. There are *essential singularities*, which are singularities which aren't poles, but we will not discuss them extensively; almost everything in sight will be meromorphic.

The Riemann Sphere. In some sense, the Riemann sphere is the most natural setting for meromorphic functions, and the first nontrivial example of a Riemann surface (still to be defined).

Definition. The *Riemann sphere* $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the *one-point compactification* of $\mathbb{C}$, so its topology has as its open sets (1) opens in $\mathbb{C}$, and (2) $(\mathbb{C} \setminus K) \cup \{\infty\}$, where $K \subset \mathbb{C}$ is compact.

There is a homeomorphism $\phi : \widehat{\mathbb{C}} \to S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ given by *stereographic projection*: send $\infty \mapsto (0, 0, 1)$ (the north pole), and then any other $z \in \mathbb{C}$ defines a line from z in the xy-plane to $(0, 0, 1)$ intersecting S^2 at one other point; this is $\phi(z)$. Hence, we will use $\widehat{\mathbb{C}}$ and S^2 interchangeably.

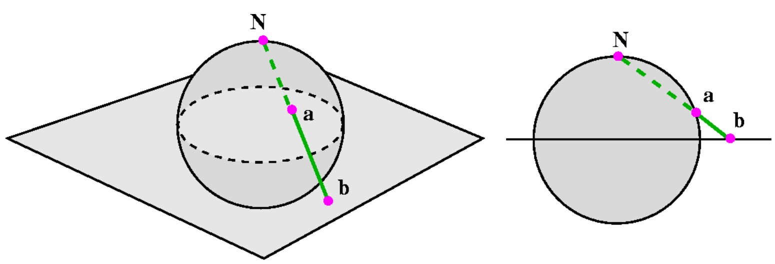

FIGURE 2. Depiction of stereographic projection, where $N = (0, 0, 1)$ is the north pole. Source: http://www.math.rutgers.edu/~greenfie/vnx/math403/diary.html.

Definition. A continuous map $f : G \to S^2$ is *holomorphic* if for all $z \in G$, either

- $f(z) \in \mathbb{C}$ (so it doesn't hit ∞) and $f : G \to \mathbb{C}$ is holomorphic, or
- if $f(z) \in \widehat{\mathbb{C}} \setminus \{0\}$, then $1/f(w) : G \to \mathbb{C}$ is holomorphic, where $1/\infty$ is understood to be 0.

If the image of f contains neither 0 not ∞, then both criteria hold, and are equivalent (since $1/z$ is holomorphic on any neighborhood not containing zero).

Proposition 3.3. *The meromorphic functions on G can be identified with the holomorphic functions $G \to S^2$.*

Proof. Suppose f is meromorphic on G, so that it has a pole of order m at z_0. Then, $f(z) = (1/(z - z_0)^m) g(z)$ for some holomorphic g with a removable singularity at z_0, and $g(z_0) \neq 0$.

By letting $1/0 = \infty$, this realizes f as a continuous map $G \to S^2$, and $1/f = (z - z_0)^m (1/g)$, which is certainly holomorphic near z_0, so f is holomorphic as a map to S^2.

The converse is quite similar, a matter of unwinding the definitions, but has been left as an exercise. ⊠

You can also define a notion of a holomorphic function coming out of S^2, not just into.

Definition. Let $G \subset S^2$ be open. A continuous $f : G \to S^2$ is *holomorphic* if one of the following is true.

- If $\infty \notin G$, then we use the same definition as above.
- If $\infty \in G$, then it's holomorphic on $G \setminus \infty$ and there's a neighborhood N of ∞ in G such that the composition
$$N^{-1} \xrightarrow{z \mapsto 1/z} N \xrightarrow{f} S^2$$
is holomorphic.

If you're used to working with manifolds, this sort of coordinate change is likely very familiar: every time we talk about ∞, we take reciprocals and talk about 0.

Example 3.4. Every rational function $p \in \mathbb{C}(z)$ is meromorphic, and extends to a holomorphic map $S^2 \to S^2$.

Now, we can talk about these geometrically: $z \mapsto z^2$ sends $e^{in\theta} \mapsto e^{2in\theta}$, so it doubles the longitude (modulo 1). In particular, it wraps the sphere twice around itself, preserving 0 and ∞, as in Figure 3.

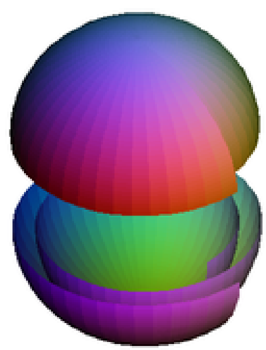

FIGURE 3. A depiction of the map $z \mapsto z^2$ on the Riemann sphere, which fixes the poles. Source: https://en.wikipedia.org/wiki/Degree_of_a_continuous_mapping.

Example 3.5. A *Möbius map* is a map
$$\mu(z) = \frac{az + b}{cz + d},$$
where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. This extends to a holomorphic map $S^2 \to S^2$ with a holomorphic inverse (the Möbius map associated to $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]^{-1}$). Thus, there's a homeomorphism $\mathrm{SL}_2(\mathbb{R})/\{\pm I\}$ to the group of Möbius transformations.

One interesting corollary is that the point at infinity is *not* special, since there's a Möbius map sending it to any other point of S^2, and indeed they act transitively on it. So we don't really have to distinguish the point at infinity from this geometric point of view.

Theorem 3.6. *If $f : S^2 \to S^2$ is holomorphic, then it's a rational function. In particular, the Möbius maps are the only invertible holomorphic maps $S^2 \to S^2$.*

The idea is to eliminate the zeros and poles by multiplying by $(z - z_0)^m$; then, one can apply Liouville's theorem to show that the result is constant.