M392C NOTES: MATHEMATICAL GAUGE THEORY

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These notes were taken in UT Austin's M392C (Mathematical gauge theory) class in Spring 2019, taught by Dan Freed. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Any mistakes in the notes are my own. Thanks to Yixian Wu for finding and fixing a typo.

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Lecture 1.

Some useful linear algebra: 1/22/19

"Why did the typing stop?"

Today we'll discuss some basic linear algebra which, in addition to being useful on its own, is helpful for studying the self-duality equations. You should think of this as happening pointwise on the tangent space of a smooth manifold.

Let V be a real n-dimensional vector space. The exterior powers of V define more vector spaces: the scalars $\mathbb{R}$, V, $\Lambda^2 V$, and so on, up to $\Lambda^n V = \operatorname{Det} V$. We can also apply this to the dual space, defining $\mathbb{R}$, V^*, $\Lambda^2 V^*$, etc, up to $\Lambda^n V^* = \operatorname{Det} V^*$.

There is a duality pairing

$$\begin{aligned} \theta\colon \Lambda^k V^* \times \Lambda^k V &\longrightarrow \mathbb{R} \\ (v^1 \wedge \cdots \wedge v^k, v_1 \wedge \cdots \wedge v_k) &\longmapsto \det(v^i(v_j))_{i,j}, \end{aligned} \tag{1.1}$$

where $v^i \in V^*$ and $v_j \in V$.

Now fix a $\mu \in \operatorname{Det} V^* \setminus 0$, which we call a *volume form*. Then we get another duality pairing

$$\begin{aligned} \Lambda^k V \times \Lambda^{n-k} V &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto \theta(\mu, x \wedge y). \end{aligned} \tag{1.2}$$

Thus $\Lambda^k V \cong \Lambda^{n-k} V^*$.

Suppose we have additional structure: an inner product and an orientation. Let $e_1, \dots, e_n$ be an oriented, orthonormal basis of V, and $e^1, \dots, e^n$ be the dual basis. Now we can choose $\mu = e^1 \wedge \cdots \wedge e^n$.

Definition 1.3. The *Hodge star operator* is the linear operator $\star\colon \Lambda^k V^* \to \Lambda^{n-k} V^*$ characterized by

$$\alpha \wedge (\star\beta) = \langle \alpha, \beta \rangle_{\Lambda^k V} \cdot \mu. \tag{1.4}$$

The inner product on $\Lambda^k V^*$ is defined by

$$\langle v^1 \wedge \cdots \wedge v^k, w^1 \wedge \cdots \wedge w^k \rangle := \det(\langle v^i, w^j \rangle)_{i,j}. \tag{1.5}$$

The Hodge star was named after W.V.D. Hodge, a British mathematician. Notice how we've used both the metric and the orientation – it's possible to work with unoriented vector spaces (and eventually unoriented Riemannian manifolds), but one must keep track of some additional data.

Example 1.6.

- $\star(e^{i_1} \wedge \cdots \wedge e^{i_k}) = e^{j_1} \wedge \cdots \wedge e^{j_{n-k}}$ if the permutation $1, \dots, n$ to $i_1, \dots, i_k, j_1, \dots, j_{n-k}$ of $[n] := \{1, \dots, n\}$ is even. Otherwise there's a factor of -1.
- Suppose $n = 4$. Then $\star(e^1 \wedge e^2) = e^3 \wedge e^4$ and $\star(e^1 \wedge e^3) = -e^2 \wedge e^4$, and so on. ◀

Remark 1.7. The Hodge star is natural. First, you can see that we didn't make any choices when defining it, other than an orientation and a volume form, but there's also a functoriality property. Let $T\colon V \to V$ be an automorphism; this induces $(\Lambda^k T^*)^{-1}\colon \Lambda^k T^* \to \Lambda^k T^*$, and if T is an orientation-preserving isometry,

$$\star \circ (\Lambda^k T^*)^{-1} = (\Lambda^{n-k} T^*)^{-1} \circ \star. \tag{1.8}$$

Hence $\star\star\colon \Lambda^k V^* \to \Lambda^k V^*$ is some nonzero scalar multiple of the identity, and we can determine which multiple it is. Certainly we know

$$\star\star(e^1 \wedge \cdots \wedge e^k) = \star(e^{k+1} \wedge \cdots \wedge e^1) = \lambda e^1 \wedge \cdots \wedge e^k, \tag{1.9}$$

and we just have to compute the parity of these permutations: one uses k transpositions, and the other uses $n - k$. Therefore we conclude that

$$\star\star = (-1)^{k(n-k)}\colon \Lambda^k V^* \to \Lambda^k V^*. \tag{1.10}$$

◀

Now suppose $n = 2m$, so we have a middle dimension m, and $\star\star\colon \Lambda^m \to \Lambda^m$ is $(-1)^m$. This induces additional structure on $\Lambda^m V^*$.

- If m is even (so $n \equiv 0 \bmod 4$), the double Hodge star is an endomorphism squaring to 1. This defines a $\mathbb{Z}/2$-grading on $\Lambda^m V^*$, given by the ± 1-eigenspaces, which we'll denote $\Lambda^m_\pm V^*$. The $+1$-eigenspace is called *self-dual* m-forms, and the -1-eigenspace is called the *anti-self-dual* m-forms.
- If m is odd (so $n \equiv 2 \bmod 4$), the double Hodge star squares to -1, so this defines a complex structure on $\Lambda^m V^*$, where i acts by the double Hodge star.

Exercise 1.11. Especially for those interested in physics, work out this linear algebra in indefinite signature (particularly Lorentz). The signs are different, and in Lorentz signature the two bullet points above switch!

Exercise 1.12. Show that if $4 \mid n$, the direct-sum decomposition $\Lambda^m V^* = \Lambda^m_+ V^* \oplus \Lambda^m_- V^*$ is orthogonal. See if you can find the one-line proof that self-dual and anti-self-dual forms are orthogonal.

Next we introduce conformal structures. This allows the sort of geometry which knows angles, but not lengths.

Definition 1.13. A *conformal structure* on a real vector space V is a set C of inner products on V such that any $g_1, g_2 \in C$ are related by $g_1 = \lambda g_2$ for a $\lambda \in \mathbb{R}_+$.

In this setting, one can obtain g_2 from g_1 by pulling back g_1 along the dilation $T_\lambda\colon v \mapsto \lambda v$. This induces an action of $(T_\lambda^*)^{-1}$ on $\Lambda^k V^*$, which is multiplication by λ^{-k}: if μ_i is the volume form induced from g_i, so that

$$\alpha \wedge \star\beta = g_1(\alpha, \beta)\mu_1, \tag{1.14}$$

then

$$\lambda^{-2k} \alpha \wedge \star\beta = g_2(\alpha, \beta)\lambda^{-n}\mu_2. \tag{1.15}$$

Thus pulling back by dilation carries the Hodge star to $\lambda^{n-2k}\star$. Importantly, if $n = 2m$, then $\star\colon \Lambda^m V^* \to \Lambda^m V^*$ is preserved by this dilation, so it only depends on the orientation and the conformal structure.

Remark 1.16. A conformal structure is independent from an orientation. For example, on a one-dimensional vector space, a conformal structure is no information at all (all inner products are multiples of each other), but an orientation is a choice. ◀

Example 1.17. Suppose $n = 2$ and choose an orientation and a conformal structure on V. As we just saw, this is enough to define the Hodge star $\star\colon V^* \to V^*$, which defines a complex structure on V. Pick a square root i of -1 and let $\star$ act by it (there are two choices, acted on by a Galois group).

We get more structure by complexifying: $V^* \otimes \mathbb{C}$ splits as a the $\pm i$-eigenspaces of the Hodge star; we denote the i-eigenspace by $V^{(1,0)}$ (the $(1,0)$-*forms*) and the $-i$-eigenspace by $V^{(0,1)}$ (the $(0,1)$-*forms*).

Now let's globalize this: everything has been completely natural, so given an oriented, conformal 2-manifold X, it picks up a complex structure, hence is a Riemann surface, and the Hodge star is a map $\star\colon \Omega^1_X \to \Omega^1_X$. Moreover, we can do this on the complex differential forms, which split into $(1,0)$-forms and $(0,1)$-forms.

How do 1-forms most naturally appear? They're differentials of functions, so given an $f\colon X \to \mathbb{C}$, we can ask what it means for $\mathrm{d}f \in \Omega^{1,0}_X$. This is the equation

$$\star\,\mathrm{d}f = i\,\mathrm{d}f. \tag{1.18}$$

This is precisely the Cauchy-Riemann equation; its solutions are precisely the holomorphic functions on X. ◀

Remark 1.19. More generally, one can ask about functions to $\mathbb{C}^n$ or even sections of complex vector bundles; the analogue gives you notions of holomorphic sections. In this case, the equations have the notation

$$\overline{\partial} f = \left(\frac{1+i\star}{2}\right)\mathrm{d}f. \tag{1.20}$$ ◀

We'll spend some time in this class understanding a four-dimensional analogue of all of this structure.

Symmetry groups. Symmetry is a powerful perspective on geometry. If we think about V together with some structure (orientation, metric, conformal structure, some combination,...), we can ask about the symmetries of V preserving this structure. Of course, to know this, we must know V, but we can instead look at a model space $\mathbb{R}^n$ to define a *symmetry type*, and ask about its symmetry group G: then an isomorphism $\mathbb{R}^n \to V$ preserving all of the data we're interested in defines an isomorphism from G to the symmetry group of V.

Example 1.21. When $\dim V = 2$, the most general symmetry group is $\mathrm{GL}_2(\mathbb{R})$, the invertible matrices acting on $\mathbb{R}^2$. Adding more structure we get more options.

- If we restrict to orientation-preserving symmetries, we get $\mathrm{GL}_2^+(\mathbb{R})$.
- If we restrict to symmetries preserving a conformal structure, the group is called $\mathrm{CO}_2 = \mathrm{O}_2 \times \mathbb{R}^{>0}$.
- If we ask to preserve an orientation and a complex structure, we get $\mathrm{CO}_2^+ = \mathrm{SO}_2 \times \mathbb{R}^{>0}$. This is isomorphic to $\mathbb{C}^\times = \mathrm{GL}_1(\mathbb{C})$: an element of $\mathrm{SO}_2 \times \mathbb{R}^{>0}$ is rotation through some angle θ and a positive number r; this is sent to $re^{i\theta} \in \mathbb{C}^\times$.

This provides another perspective on why an orientation and a conformal structure give us a complex structure. ◀

Example 1.22. Now suppose $n = 4$, and choose a conformal structure C and an orientation on V. Then orthogonal makes sense, though orthonormal doesn't, and the Hodge star induces a $\mathbb{Z}/2$-grading on $\Lambda^2 V^* = \Lambda^2_+ V^* \oplus \Lambda^2_- V^*$, the self-dual and anti-self-dual 2-forms. The total space $\Lambda^2 V^*$ is six-dimensional, and these two subspaces are each three-dimensional.

Suppose e^1, e^2, e^3, e^4 is an orthonormal basis for some inner product in C. We can use these to define bases of $\Lambda^2_\pm V^*$, given by

$$\begin{aligned}\alpha_1^\pm &\coloneqq e^1 \wedge e^2 \pm e^3 \wedge e^4\\ \alpha_2^\pm &\coloneqq e^1 \wedge e^3 \mp e^2 \wedge e^4\\ \alpha_3^\pm &\coloneqq e^1 \wedge e^4 \pm e^2 \wedge e^3.\end{aligned} \tag{1.23}$$

Now, what symmetry groups do we have? Inside $\mathrm{GL}_4(\mathbb{R})$, preserving an orientation lands in the subgroup $\mathrm{GL}_4^+(\mathbb{R})$; preserving a conformal structure lands in $\mathrm{O}_4 \times \mathbb{R}^{>0}$; and preserving both lands in $\mathrm{SO}_4 \times \mathbb{R}^{>0}$. The first three of these act irreducibly on $\Lambda^2(\mathbb{R}^4)^*$, but the action of $\mathrm{SO}_4 \times \mathbb{R}^{>0}$ has two irreducible summands, $\Lambda^2_\pm(\mathbb{R}^4)^+$.

To understand this better, we should learn a little more about SO_4. Recall that Sp_1 is the Lie group of unit quaternions. This is isomorphic to SU_2, the group of determinant-1 unitary transformations of $\mathbb{C}^2$. This group has an irreducible 3-dimensional representation ρ in which Sp_1 acts by conjugation on the imaginary quaternions (since $\mathbb{R} \subset \mathbb{H}$ is preserved by this action).

Remark 1.24. Another way of describing ρ is: let ρ' denote the action of SU_2 on $\mathbb{C}^2$ by matrix multiplication. Then $\rho \cong \mathrm{Sym}^2 \rho'$. ◂

Proposition 1.25. *There is a double cover* $\mathrm{Sp}_1 \times \mathrm{Sp}_1 \to \mathrm{SO}_4$*. Under this cover, the* SO_4*-representation* $\Lambda^4_\pm(\mathbb{R}^4)^*$ *pulls back to a real three-dimensional representation in which one copy of* Sp_1 *acts by* ρ *and the other acts trivially.*

Proof. Let W' and W'' be two-dimensional Hermitian vector spaces with compatible quaternionic structures J', resp. J''.[1] Then, $V \coloneqq W' \otimes_{\mathbb{C}} W''$ has a real structure $J' \otimes J''$: two minuses make a plus, and compatibility of J' and J'' means the real points of V have an inner product. (These kinds of linear-algebraic spaces are things you should prove once in your life.)

By tensoring symmetries we obtain a homomorphism $\mathrm{Sp}(W') \times \mathrm{Sp}(W'') \to \mathrm{O}(V)$. This factors through $\mathrm{SO}(V) \hookrightarrow \mathrm{O}(V)$, which you can see for two reasons:

- $\mathrm{Sp}(W')$ and $\mathrm{Sp}(W'')$ are connected, so this homomorphism must factor through the identity component of $\mathrm{O}(V)$, which is $\mathrm{SO}(V)$; or
- a complex vector space has a canonical orientation, and using this we know these symmetries are orientation-preserving.

Now we want to claim this map is two-to-one. One can quickly check that $(-1, -1)$ is in the kernel; the rest is an exercise. ◂

Since Spin_n is the double cover of SO_n, this is telling us $\mathrm{Spin}_4 = \mathrm{Sp}_1 \times \mathrm{Sp}_1$. This splitting is the genesis of a lot of what we'll do in the next several lectures.

Consider the 16-dimensional space

$$V^* \otimes V^* = (W')^* \otimes (W')^* \otimes (W'')^* \otimes (W'')^*. \tag{1.26}$$

Because the map

$$\begin{aligned} \omega' \colon W' \times W' &\longrightarrow \mathbb{C} \\ \xi', \eta' &\longmapsto h'(J'\xi', \eta') \end{aligned} \tag{1.27}$$

is skew-symmetric, it lives in $\Lambda^2(W')^* \subset (W')^* \otimes (W')^*$. In particular, the embedding

$$\mathrm{Sym}^2(W')^* \oplus \mathrm{Sym}^2(W'')^* \hookrightarrow (W')^* \otimes (W')^* \otimes (W'')^* \otimes (W'')^* \tag{1.28}$$

is the map sending

$$\alpha, \beta \longmapsto \alpha \otimes w'' + \omega' \otimes \beta. \tag{1.29}$$

Remark 1.30. This story can be interpreted in terms of representations of $\mathrm{Sp}(W') \times \mathrm{Sp}(W'')$. Let $\mathbf{1}$ denote the trivial representation of Sp_1 and $\mathbf{3}$ be the three-dimensional irreducible representation we discussed above. Then (1.26) enhances to

$$V^* \otimes V^* = \mathbf{1}_{\mathrm{Sp}(W')} \otimes \mathbf{3}_{\mathrm{Sp}(W')} \otimes \mathbf{1}_{\mathrm{Sp}(W'')} \otimes \mathbf{3}_{\mathrm{Sp}(W'')}. \tag{1.31}$$

The skew-symmetric part is $\mathbf{3}_{\mathrm{Sp}(W')} \otimes \mathbf{1}_{\mathrm{Sp}(W'')} \oplus \mathbf{1}_{\mathrm{Sp}(W')} \otimes \mathbf{3}_{\mathrm{Sp}(W'')}$, and the "rest" (complement) is symmetric. ◂

The group $\mathrm{Sp}_1 \times \mathrm{Sp}_1 = \mathrm{Spin}_4$ has complex (quaternionic) two-dimensional representations $S^\pm$, the *spin representations*, and $\Lambda^2_\pm V \cong \mathrm{Sym}^2 S^\pm$.

So two-forms have self-dual and anti-self-dual parts, and curvature is a natural source of 2-forms! ◂

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[1] That is, J' is an antilinear endomorphism of W' squaring to -1, and similarly for J''. Compatible means with the Hermitian metric: h is a map $\overline{W} \times W \to \mathbb{C}$ and J is a map $W \to \overline{W}$, and if $\xi, \eta \in W'$, we want

$$h(J'\xi, \overline{J'\eta}) = \overline{h(\xi, \eta)} \qquad \text{and} \qquad h(J\xi, \eta) = -h(J\eta, \xi).$$

Lecture 2.

Fantastic 2-forms and where to find them: 1/24/19

"I've taught this before, so I know it's true."

Last time, we discussed some linear algebra which is a local model for phenomena we will study in differential geometry. For example, we saw that on an oriented even-dimensional vector space with an inner product, the Hodge star defines a self-map of the middle-dimensional part of the exterior algebra, which induces extra structure, such a splitting into self-dual and anti-self-dual pieces in dimensions divisible by 4. This therefore generalizes to a $4k$-dimensional manifold with a metric and an orientation: the space of $2k$-forms splits as an orthogonal direct sum of self-dual and anti-self-dual forms. (We also discussed other examples, such as how 1-forms on an oriented 2-manifold split into holomorphic and antiholomorphic pieces.)

We're particularly interested in the case $k = 1$, where this splitting depends only on a conformal structure, and applies to 2-forms. To study its consequences we'll discuss where one can find 2-forms in differential geometry.

Definition 2.1. A *fiber bundle* is the data of a smooth map $\pi\colon E \to X$ of smooth manifolds if for all $x \in X$ there's an open neighborhood U of x and a diffeomorphism $\varphi\colon U \times \pi^{-1}(x) \to \pi^{-1}(U)$ such that the diagram

$$\begin{tikzcd} U \times \pi^{-1}(x) \arrow[rr, "\varphi"] \arrow[dr, "\mathrm{proj}_1"'] & & \pi^{-1}(U) \arrow[dl, "\pi"] \\ & U & \end{tikzcd} \tag{2.2}$$

commutes. In this case we call X the *base space* and E the *total space*. If there is a manifold F such that in the above definition we can replace $\pi^{-1}(x)$ with F, we call π a *fiber bundle with fiber F*.[2] The map φ is called the *local trivialization*.

Example 2.3. The *trivial bundle* with fiber F is the projection map $X \times F \to X$. ◂

Remark 2.4. Fiber bundles were first defined by Steenrod in the 1940s, albeit in a different-looking way. His key insight was local triviality. There are variants depending on what kind of space you care about: for example, you can replace manifolds with spaces and smooth maps with continuous maps.

Keep in mind that a fiber bundle is data (π) and a condition. Often people say "E is a fiber bundle" when they really mean "π is a fiber bundle"; specifying E doesn't uniquely specify π. ◂

If F has more structure, such as a Lie group, torsor, vector space, algebra, Lie algebra, etc., we ask that $\varphi|_{\pi^{-1}(x)}\colon F \to \pi^{-1}(x)$ preserve this structure. For example, in a fiber bundle whose fibers are vector spaces, we want φ to be linear; in this case we call it a *vector bundle*.

Definition 2.5. If $\pi\colon E \to X$ is a vector bundle, the space of *k-forms valued in E*, denoted $\Omega^k_X(E)$, is the space of C^∞ sections of $\Lambda^k T^*X \otimes E \to X$.

For ordinary differential forms (so when E is a trivial bundle), we have the de Rham differential $\mathrm{d}\colon \Omega^k_X \to \Omega^{k+1}_X$, but we do not have this in general.

Definition 2.6. Let X be a smooth manifold.

(1) A *distribution* on X is the subbundle $E \subset TX$.
(2) A vector field ξ on X *belongs to E* if $\xi_x \in E_x \subset T_xX$ for all X.
(3) A submanifold $Y \subset X$ is an *integral submanifold* for E if for all $y \in Y$, $T_yY = E_y$ inside T_yX.

Do integral submanifolds exist? This is a local question and a global question (the latter about maximal integral submanifolds). In general, the answer is "no," as in the next example.

Example 2.7. Consider a distribution on $\mathbb{A}^3$ with coordinates (x, y, z) given by

$$E_{(x,y,z)} = \mathrm{span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}\right\}. \tag{2.8}$$

There is no integral surface for this distribution. TODO: I missed the argument, sorry. ◂

[2] Not all fiber bundles have a fiber in this sense, e.g. a fiber bundle with different fibers over different connected components.

This is the basic example that illustrates curvature. It turns out that the existence of an integral submanifold is determined completely by the (non)vanishing of a tensor.

Definition 2.9. Let $E \subset TX$ be a distribution. The *Frobenius tensor* $\phi_E\colon E \times E \to TX/E$ given by

$$\xi_1, \xi_2 \longmapsto [\xi_1, \xi_2] \bmod E.$$

Let's think about this: the Lie bracket is defined for vector fields, not vectors. So we have to extend ξ_1 and ξ_2 to vector fields (well, sections of E, since they're in E), which is a choice, and then check that what we obtain is independent of this choice. It suffices to know that this is linear over functions: that

$$[f_1\xi_1, f_2\xi_2] \overset{?}{=} f_1 f_2 [\xi_1, \xi_2]. \tag{2.10}$$

Of course, this is not what the Lie bracket does: it differentiates in both variables, so we have the extra terms $f_1(\xi \cdot f_2)\xi_2$ and $f_2(\xi \cdot f_1)\xi_1$. But both of these are sections of E, so vanish mod E, and therefore we do get a well-defined, skew-symmetric form, a section of $\Lambda^2 E^* \otimes TX/E$ – not quite a differential form.

Frobenius did many important things in mathematics, across group theory and representation theory and this theorem, which is about differential equations!

Theorem 2.11 (Frobenius theorem)**.** *An integral submanifold of E exists locally iff $\phi_E = 0$.*

This is a nonlinear ODE. As such, our proof will rely on some facts from a course on ODEs.

Lemma 2.12. *Let X be a smooth manifold, ξ be a vector field on X, and $x \in X$ be a point where ξ doesn't vanish. Then there are local coordinates $x^1, \dots, x^n$ around x such that $\xi = \partial x^1$ in this neighborhood.*

Proof. Let φ_t be the local flow generated by ξ, and choose coordinates $y^1, \dots, y^n$ near x such that $\xi_x = \frac{\partial}{\partial y^1}\Big|_x$. Define a map $U\colon \mathbb{R}^n \to X$ by

$$x^1, \dots, x^n \longmapsto \varphi_{x^1}(0, x^2, \dots, x^n). \tag{2.13}$$

The right-hand side is expressed in y-coordinates. Now we need to check this is a coordinate chart, which follows from the inverse function theorem, because the differential of φ is invertible at 0 (in fact, it's the identity). The theorem then follows because x^1 is the time direction for flow along ξ in this coordinate system. ⊠

Lemma 2.14. *With notation as above, let $\xi_1, \dots, \xi_k$ be vector fields which are linearly independent at x and suc that $[\xi_i, \xi_j] = 0$ for all $1 \le i, j \le k$. Then there exist local coordinates $x^1, \dots, x^n$ such that for $1 \le i \le k$, $\xi_i = \frac{\partial}{\partial x^i}$.*

In fact, the converse is also true, but trivially so: it's the theorem in multivariable calculus that mixed partials commute.

Proof. Let $\varphi_1, \dots, \varphi_k$ be the local flows for $\xi_1, \dots, \xi_k$. Because the pairwise Lie brackets commute, $\varphi_i\varphi_j = \varphi_j\varphi_i$. Since these vector fields are linearly independent at x, we can choose local coordinates $y^1, \dots, y^n$ around x such that $\xi_i|_x = \frac{\partial}{\partial y^i}\Big|_x$. Then, as above, define

$$x^1, \dots, x^n \longmapsto (\varphi_1)_{x_1}(\varphi_2)_{x_2} \cdots (\varphi_k)_{x_k}(0, \dots, 0, x^{k+1}, \dots, x^n). \tag{2.15}$$

You can check that $\mathrm{d}\varphi$ is invertible, so this is a change of coordinates, and then, using the fact that the flows commute, you can see that the lemma follows. ⊠

These lemmas are important theorems in their own right.

Proof of Theorem 2.11. Since the theorem statement is local, we can work in affine space $\mathbb{A}^n$. Let $\pi\colon \mathbb{A}^n \to \mathbb{A}^k$ be an affine surjection such that $\mathrm{d}\pi_0$ restricts to an isomorphism $E_0 \to \mathbb{R}^k$. Restrict to a neighborhood U of 0 in $\mathbb{A}^n$ such that $\mathrm{d}\pi_p|_{E_p}\colon E_p \to \mathbb{R}^n$ is an isomorphism for all $p \in U$, and choose $\xi_i|_p \in E_p$ such that $\mathrm{d}\pi_p(\xi_p) = \frac{\partial}{\partial y^i}$. Then, $[\xi_i, \xi_j] = 0$: we know it's in E, and

$$\mathrm{d}\pi[\xi_i, \xi_j] = [\mathrm{d}\pi(\xi_i), \mathrm{d}\pi(\xi_j)] = \left[\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right] = 0. \tag{2.16}$$

Now apply Lemma 2.14; then $\{y^{k+1} = \cdots = y^n = 0\}$ gives the desired integral submanifold. ⊠

The idea of the theorem is that it's a local normal form for an involutive distribution (one whose Frobenius tensor vanishes): locally it looks like the splitting of $\mathbb{R}^n$ into the first k coordinates and the last $(n-k)$ coordinates. And in that local model, we know what the integral manifolds are.

Consider a fiber bundle with a discrete fiber (i.e. the inverse image of every point has the discrete topology). This is also known as a *covering space*. On a "nearby fiber," whatever that means (without more data, we don't have a metric on the base space), we have some sort of parallel transport. The precise statement is that there's a neighborhood of any x on the base space such that any path in that neighborhood lifts to a path on the total space, unique if you specify a point in the fiber. More generally, you can lift families of paths, which illustrates a homotopy-theoretic generalization of a fiber bundle called a *fibration*. But globally, given an element of $\pi_1(X)$, it might lift to a nontrivial automorphism of the fiber.

We'd like to do this for more general fiber bundles $\pi\colon E \to X$, in which case we'll need more data. The kernel of $\mathrm{d}\pi$ is a distribution, and consists of the "vertical" vectors (projection down to X kills them). A complement is "horizontal".

Without any choice, we get a short exact sequence at every $e \in E$:

$$0 \longrightarrow \ker(\mathrm{d}\pi_e) \longrightarrow T_eE \xrightarrow{\mathrm{d}\pi_e} T_xX \longrightarrow 0, \tag{2.17}$$

and a splitting is exactly the choice of a complement $H_e\colon T_xX \to T_eE$. We would like to do this over the whole base, which motivates the next definition.

Definition 2.18. Let $\pi\colon E \to X$ be a fiber bundle. A *horizontal distribution* is a subbundle $H \subset TE$ transverse to $\ker(\mathrm{d}\pi)$, or equivalently a section of the (surjective) map $TE \to \pi^*TX$ of vector bundles on E.

We must address existence and uniqueness. At e the space of splittings is an affine space modeled on $\mathrm{Hom}(T_xX, \ker(\mathrm{d}\pi_e))$, because TODOsomething with a short exact sequence.

Therefore existence and uniqueness of a horizontal distribution is a question about existence and uniqueness of a section of an affine bundle over X. Using partitions of unity, we can construct many of these: existence is good, but uniqueness fails.

What about path lifting? Suppose $\gamma\colon [0,1] \to X$ is a path in X beginning at x_0 and terminating at x_1. We can pull back both E and H by γ, to obtain a rank-1 distribution γ^*H in γ^*TE, and the projection map to $T[0,1]$ is a fiberwise isomorphism. Therefore given a vector at $x_0 = \gamma(0)$ we get a unique horizontal lift along $[0,1]$ to a vector field, and therefore get a unique integral curve above γ.

Note that you cannot always lift higher-dimensional submanifolds, and again the obstruction is the Frobenius tensor, because that's the obstruction to the existence of an integral submanifold. In this context the Frobenius tensor is called *curvature* – right now it's on the total space, but in some settings we can descend it to the base.

Lecture 3.

Principal bundles, associated bundles, and the curvature 2-form: 1/29/19

"For whatever reason I'm being a little impressionistic..."

Last time, we discussed a way in which 2-forms appear in geometry: as the obstruction to integrability of a distribution $E \subset TX$. That is, a distribution contains vectors, and we can ask whether integral curves of those vectors have tangent vectors contained within E. Associated to E we defined a Frobenius tensor $\phi_E\colon \Lambda^2 E \to TX/E$ sending

$$\xi_1, \xi_2 \longmapsto [\widetilde{\xi_1}, \widetilde{\xi_2}] \bmod E, \tag{3.1}$$

where $\widetilde{\xi_i}$ is a vector field extending ξ (and we showed this doesn't depend on the choice of extension). In Theorem 2.11, we saw that ϕ_E is exactly the local obstruction to integrability; we can then move to global questions.

More generally, suppose that $\pi\colon E \to X$ is a fiber bundle. Then TE fits into a short exact sequence (2.17), and we can ask for a horizontal lift from TX to TE, which is a section H of (2.17). Then, given a vector $e \in T_xX$ and a path $\gamma\colon [0,1] \to X$ with $\gamma(0) = x$, we can pull back[3] π and H to obtain a distribution in γ^*E.

[3] In general, we can form the pullback of $[0,1] \to X \leftarrow E$ in the category of sets or spaces, but we want to put a smooth manifold structure on it. We can do it when these two maps are transverse – and since $\pi\colon E \to X$ is a submersion, this is always satisfied.

The Frobenius tensor vanishes, because $[0, 1]$ is one-dimensional, so we can extend to an integral curve and therefore parallel-transport along γ. However, if we choose different paths in a ball, there's no guarantee that parallel transport along nearby paths agree at all; the Frobenius tensor may still be nonzero on X.

Steenrod's elegant perspective on fiber bundles (see his book *The Topology of Fiber Bundles*) considered in the spirit of Felix Klein symmetry groups associated to fiber bundles. This leads to the definition of a *principal G-bundle* as a fiber bundle of right G-torsors.

Definition 3.2. Let G be a Lie group and recall that a *right G-torsor* is a smooth manifold T and a smooth right G-action on T such that the action map $T \times G \to T \times T$ sending $(t, g) \mapsto (t, t \cdot g)$ is an isomorphism.

Example 3.3. The prime example of a torsor is to let V be a real vector space; then, the manifold $\mathcal{B}(V)$ of bases of V is a $\mathrm{GL}_n(\mathbb{R})$-torsor: $\mathrm{GL}_n(\mathbb{R})$ acts by precomposition. This also works over $\mathbb{C}$ and $\mathbb{H}$. ◀

Example 3.4. Now let X be a smooth manifold. Our first example of a principal bundle spreads Example 3.3 over X: let $\mathcal{B}(X)$ be the smooth manifold[4] of pairs (x, b) where $x \in X$ and b is a basis of T_xX, i.e. an isomorphism $b\colon \mathbb{R}^n \xrightarrow{\cong} T_xX$. There's a natural forgetful map $\pi\colon \mathcal{B}(X) \to X$ sending $(x, b) \mapsto x$.

This fiber bundle is a principal $\mathrm{GL}_n(\mathbb{R})$-bundle: given $g \in \mathrm{GL}_n(\mathbb{R})$ and a basis $b\colon \mathbb{R}^n \to T_xX$, we let $b \cdot g := b \circ g\colon \mathbb{R}^n \to T_xX$, using the standard action of $\mathrm{GL}_n(\mathbb{R})$ on $\mathbb{R}^n$. This is called the *frame bundle* of X. ◀

This principal bundle controls a lot of the geometry of X, via its associated fiber bundles.

Definition 3.5. Let $\pi\colon P \to X$ be a principal G-bundle and F be a smooth (left) G-manifold. The *associated fiber bundle* with fiber F is the quotient $P \times_G F := (P \times F)/G$, which is a fiber bundle over X with fiber F. Here, G acts on $P \times F$ on the right by $(p, f) \cdot g = (p \cdot g, g^{-1} \cdot f)$.

One has to check this is a fiber bundle, and in particular that the total space is a smooth manifold. Since G acts freely on P, it acts freely on $P \times F$, but for G noncompact there's more to say.

Example 3.6. $\mathrm{GL}_n(\mathbb{R})$ acts linearly on $\mathbb{R}^n$. Because $\mathbb{R}^n$ carries the additional structure of a vector space, the associated bundle $\mathcal{B}(X) \times_{\mathrm{GL}_n(\mathbb{R})} \mathbb{R}^n$ has additional structure: it's a vector bundle. In general, additional structure on F manifests in additional structure on $P \times_G F$.

Anyways, what vector bundle do we get? (Can you guess?) An element of the fiber of $\mathcal{B}(X) \times_{\mathrm{GL}_n(\mathbb{R})} \mathbb{R}^n$ is an equivalence class of an element $v \in \mathbb{R}^n$ and a basis $p\colon \mathbb{R}^n \to T_xX$; let $\xi := p(v) \in T_xX$. Another representative of this equivalence class are represented by $g^{-1}v$ and $p \circ g$ for some $g \in \mathrm{GL}_n(\mathbb{R})$, so this pair defines the same tangent vector ξ. Therefore we recover the tangent bundle. ◀

In general, a principal bundle is telling you some internal coordinates. You know these coordinates up to some symmetry G, and the principal bundle tracks that: you have to make a choice to get coordinates, and it tells you how different choices are related.

We want to show local triviality of a principal G-bundle $\pi\colon P \to X$, which will follow from local triviality as a fiber bundle. Consider a local section $s\colon U \to \pi^{-1}(U)$, where $U \subset X$; we would like to exhibit an isomorphism of fiber bundles $U \times G \to \pi^{-1}(U)$ over U. The map is exactly

$$x, g \longmapsto s(x) \cdot g. \tag{3.7}$$

This exhibits $U \times G \cong \pi^{-1}(U)$ as principal G-bundles, so we have local trivialization. Then in every associated bundle to P, we also obtain local triviality, hence local coordinates. For example, if the bundle of frames is trivialized over U, we get local coordinates (i.e. a local trivialization of TU).

Definition 3.8. A *connection* on a principal G-bundle $\pi\colon P \to X$ is a G-invariant horizontal distribution.

Specifically, given $g \in G$, we have the right action map $R_g\colon P \to P$, and can therefore define $H_{p \cdot g} := (R_g)_*(H_p)$ for a distribution H.

In this setting, the Frobenius tensor is going to do something nice: it's a map

$$\phi_H\colon H \wedge H \to TP/H \cong \ker(\pi_*), \tag{3.9}$$

[4] You have to put a smooth manifold structure on this set! The way to do this is the only tool we have right now: work in an atlas $\mathfrak{U}$ of X which trivializes TX, do this locally, and check that the transition maps are smooth. This will also show that the map $\pi\colon \mathcal{B}(X) \to X$ is a fiber bundle.

so given two horizontal vectors, we get a vertical vector. Since H is G-invariant, the Frobenius tensor is also G-invariant, so we ought to be able to descend it to the base: there's only one piece of information on each fiber. That is, given vectors ξ_1, ξ_2 on X, we can lift them to P and compute the Frobenius tensor there, and G-invariance means it doesn't matter how we lift.

If $\mathfrak{g}$ is the Lie algebra of the Lie group G, we have an isomorphism $\underline{\mathfrak{g}} \xrightarrow{\cong} \ker(\pi_*)$ as vector bundles on P. Specifically, let $\xi \in \mathfrak{g}$, and consider the exponential map $\exp\colon \mathfrak{g} \to G$. Given $p \in P$ with $\pi(p) = x$, we get a curve in P given by $t \mapsto p \cdot \exp(t\xi)$ sending $0 \mapsto p$, and this curve is contained entirely within P_x. Therefore its tangent vector at p is in $\ker(\pi_*)$.

So the Frobenius tensor is a map $\phi_H\colon H \wedge H \to \underline{\mathfrak{g}}$. Now let's descend to the base. We'd like to claim that what we get in $\mathfrak{g}$ is invariant, but that's just not true: if $g \in G$, the action of g on $p \cdot \exp(t\xi)$ is not the same as $p \cdot g \cdot \exp(t\xi)$: the issue is that $g \exp(t\xi)$ and $\exp(t\xi) g$ may not agree. This will make it slightly more interesting to descend to the base.

First, extend ϕ_H to a map

$$\widetilde{\phi}_H\colon TP \wedge TP \longrightarrow \underline{\mathfrak{g}} \tag{3.10}$$

by projecting $p_H\colon TP \twoheadrightarrow H$, which has kernel $\ker(\pi_*)$. That is, $\widetilde{\phi}_H(\eta_1 \wedge \eta_2) \coloneqq p_H\eta_1 \wedge p_H\eta_2$. Thus $\widetilde{\phi}_H \in \Omega_P^2(\mathfrak{g})$.

Lemma 3.11. *Let $g \in G$. Then in $\Omega_P^2(\mathfrak{g})$, $R_g^*\widetilde{\phi}_H = \mathrm{Ad}_{g^{-1}} \widetilde{\phi}_H$.*

So once we choose a basis for $\mathfrak{g}$, we can think of elements of $\Omega_P^2(\mathfrak{g})$ as matrix-valued differential forms.[5] The proof of Lemma 3.11 comes from the observation above that to get from $p \cdot g \cdot \exp(t\xi)$ to

$$p \cdot \exp(t\xi) g = p \cdot g \cdot (g^{-1} \exp(t\xi) g) = p \cdot g \cdot \mathrm{Ad}_{g^{-1}}(\xi). \tag{3.12}$$

So this is exactly an example of an associated bundle to P, where the G-manifold F is $\mathfrak{g}$ with the adjoint G-bundle. So associated to P is the *adjoint bundle* $\mathfrak{g}_P \to X$ defined as $P \times_G \mathfrak{g}$. This is a vector bundle, in fact a bundle of Lie algebras because the adjoint action preserves the Lie bracket.

A section of $\mathfrak{g}_P$ is a function upstairs valued in $\mathfrak{g}$, which is exactly what $\widetilde{\phi}_H$ is.

Corollary 3.13. *$\widetilde{\phi}_H$ descends to a 2-form $-\Omega_H \in \Omega_X^2(\mathfrak{g}_P)$.*

In this case Ω_H is called the *curvature* of H. In particular, if X is a 4-manifold with a conformal structure, we can ask for this to be self-dual or anti-self-dual.

In the short exact sequence

$$0 \longrightarrow \underline{\mathfrak{g}} \longrightarrow TP \xrightarrow{\pi_*} \pi^*TX \longrightarrow 0, \tag{3.14}$$

a section $H\colon \pi^*TX \to TP$ is equivalent to a section $\Theta\colon TP \to \underline{\mathfrak{g}}$, i.e. a form $\Theta \in \Omega_P^1(\mathfrak{g})$. This is called the *connection form*, and $H = \ker(\Theta)$. It has to satisfy some properties.

- Θ must be G-invariant: $R_g^*\Theta = \mathrm{Ad}_{g^{-1}} \Theta$. This is a linear equation inside the infinite-dimensional vector space $\Omega_P^1(\mathfrak{g})$.
- The other constraint is affine: $\Theta|_{\mathrm{vertical}} = \mathrm{id}$.

So the space $\mathcal{A}_P$ of one-forms Θ satisfying these conditions is affine. This is the space of connections, and in particular tells us that there are lots of connections.

We can also interpret the Frobenius tensor in terms of Θ. Let ζ_1 and ζ_2 be horizontal vectors, and extend them to vector fields $\widetilde{\zeta}_1$ and $\widetilde{\zeta}$. Then $\zeta \cdot \Theta(\widetilde{\zeta}_{1-i}) = 0$, so

$$\begin{aligned} \mathrm{d}\Theta(\zeta_1, \zeta_2) &= \zeta_1\Theta(\widetilde{\zeta}_2) - \zeta_2\Theta(\widetilde{\zeta}_1) - \Theta([\widetilde{\zeta}_1, \widetilde{\zeta}_2]) \\ &= -\Theta([\widetilde{\zeta}_1, \widetilde{\zeta}_2]) = -\phi_H(\zeta_1, \zeta_2). \end{aligned} \tag{3.15}$$

Thus we have proved

Proposition 3.16. $\pi^*\Omega_H = -\widetilde{\phi}_H = \mathrm{d}\Theta + (1/2)[\Theta \wedge \Theta]$.

The notation $[\Theta \wedge \Theta]$ means: $\Theta \wedge \Theta \in \Omega_P^2(\mathfrak{g} \otimes \mathfrak{g})$, and this has a Lie bracket map $[\cdot]\colon \Omega_P^2(\mathfrak{g} \otimes \mathfrak{g}) \to \Omega_P^2(\mathfrak{g})$.

Corollary 3.17 (Bianchi identity)**.** $\mathrm{d}\Omega + [\Theta \wedge \Omega] = 0$.

[5] Here I suppose we need to use a Lie group G that admits a faithful finite-dimensional representation, but all compact Lie groups, and most noncompact Lie groups that you'll encounter, have this property.

Proof.

$$\begin{aligned} \mathrm{d}\Omega_H &= [\mathrm{d}\Theta \wedge \Theta] \\ &= \left[\Omega - \frac{1}{2}[\Theta\wedge\Theta]\wedge\Theta\right] \\ &= [\Omega\wedge\Theta] \end{aligned} \tag{3.18}$$

by the Jacobi identity. ⊠

This has been more theory than examples of principal bundles, but we will see plenty of examples when we delve into gauge theory.

Now given a principal G-bundle $\pi\colon P \to X$ with a connection, and any associated bundle F_P with fiber F, we get a horizontal distribution. There's a hands-on way to construct this, ot you could think of it in terms of path lifting: given an $x \in X$ and a lift $p \in P$, the connection lifts a path $\gamma\colon [0,1] \to X$ based at x to a path $\widetilde{\gamma}\colon [0,1] \to P$ based at p, so given an $f \in F$, we can define the path $\widehat{\gamma}\colon [0,1] \to F_P$ by $t \mapsto (\widetilde{\gamma}(t), f)$.

Suppose V is a G-representation, so its associated vector bundle $V_P \to X$ is a vector bundle. Then the horizontal distribution we obtain on V_P is tangent to the zero section of V_P. Let $\psi\colon X \to V_P$ be a section and $\xi \in T_xX$; we would like to differentiate ψ in the direction ξ. If ψ were valued in a fixed vector space, we could do this as usual: extend ξ to a curve $\gamma\colon (-\varepsilon, \varepsilon) \to M$, and then define

$$\nabla_\xi \psi \coloneqq \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \psi(\gamma(t)). \tag{3.19}$$

This is precisely the directional derivative. In V_P, the fibers are different vector spaces, which seems like a problem except that the connection on P defines parallel transport τ_t along γ for the fibers of V_P, and therefore we can define the directional derivative of ψ as

$$\nabla_\xi \psi \coloneqq \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \tau_{-t}\psi(\gamma(t)). \tag{3.20}$$

This is called the *covariant derivative.*

Exercise 3.21. Show that this satisfies the Leibniz rule: if f is a function on X, then

$$\nabla_\xi(f\cdot\psi) = (\xi\cdot f)\psi + f(x)\nabla_\xi\psi. \tag{3.22}$$

In other words, the existence of the horizontal distribution is somehow telling us about the Leibniz rule, though this is a somewhat mysterious fact.

Lecture 4.

Harmonic forms and (anti)-self-dual connections: 1/31/19

"The key to humor is......... timing!"

Last time, we discussed connections on principal bundles, and what they induce on associated vector bundles. We also briefly saw the covariant derivative associated to a connection. We begin with more on covariant derivatives.

Definition 4.1. Let $E \to X$ be a vector bundle. A *covariant derivative* is a linear map $\nabla\colon \Omega^0_X(E) \to \Omega^1_X(E)$ satisfying the *Leibniz rule*

$$\nabla(fs) = \mathrm{d}f\cdot s + f\nabla s, \tag{4.2}$$

where f is a smooth function on X and s is a smooth section of E.

If E is a trivial bundle with constant fiber V, the usual directional derivative is a covariant derivative, but there can be others.

We can extend ∇ to a sequence of first-order differential operators

$$0 \longrightarrow \Omega^0_X(E) \xrightarrow{\mathrm{d}_\nabla} \Omega^1_X(E) \xrightarrow{\mathrm{d}_\nabla} \Omega^2_X(E) \xrightarrow{\mathrm{d}_\nabla} \cdots \tag{4.3}$$

defined by

$$\mathrm{d}_\nabla(\omega\cdot s) \coloneqq \mathrm{d}\omega\cdot s + (-1)^k\omega\wedge\nabla s, \tag{4.4}$$

where $\omega \in \Omega^k_X$ and $s \in \Omega^0_X(E)$. Thus the first map $\mathrm{d}_\nabla\colon \Omega^0_X(E) \to \Omega^1_X(E)$ is just ∇.

Exercise 4.5. Show that $\mathrm{d}^2_\nabla(fs) = f\mathrm{d}^2_\nabla(s)$.

In other words, this says the *symbol* of d^2_∇ vanishes; this second-order operator is really a first-order operator. Therefore there exists an $F_\nabla \in \Omega^2_X(\operatorname{End} E)$, called the *curvature*, such that $\mathrm{d}^2_\nabla(s) = F_\nabla \cdot s$.

Digression 4.6. We recall what the symbol of an operator is. Let $E, F \to X$ be vector bundles and $D\colon \Omega^0_X(E) \to \Omega^0_X(F)$ be a differential operator. By definition, D is first-order if for every function f and section s,

$$D(fs) = \sigma(\mathrm{d}f)s + fDs \tag{4.7}$$

for some $\sigma\colon T^*X \otimes E \to F$, which is called the *symbol* of D. ◀

Exercise 4.8. Compute d^3_∇. (Answer: it's zero.)

Now we have two notions of curvature: the curvature associated to a covariant derivative as above, and the curvature associated to a principal bundle with connection and an associated vector bundle.

Exercise 4.9. Let G be a Lie group, $\pi\colon P \to X$ be a principal G-bundle with connection $\Theta \in \Omega^1_P(\mathfrak{g})$, and $\rho\colon G \to \operatorname{Aut}(\mathbb{E})$ be a linear representation of G. Let $E := \mathbb{E}_P = P \times_G \mathbb{E} \to X$ be the associated bundle, which carries a covariant derivative $\nabla\colon \Omega^0_X(E) \to \Omega^1_X(E)$. Compute d^2_∇ in terms of $\Omega = \mathrm{d}\Theta + (1/2)[\Theta \wedge \Theta]$.

Example 4.10. Let's think about connections on a principal $\mathbb{T}$-bundle.[6] Consider $\mathbb{C}^2$ with coordinates z^0, z^1 and metric

$$\langle (z^0, z^1), (w^0, w^1)\rangle := \overline{z^0}w^0 + \overline{z^1}w^1. \tag{4.11}$$

The circle group $\mathbb{T}$ acts on $S^3 \subset \mathbb{C}^2$ on the right by $(z^0, z^1)\cdot\lambda := (z^0\lambda, z^1\lambda)$. This is a free action, so its quotient is a smooth manifold, specifically $\mathbb{CP}^1 \cong S^2$, the manifold of complex lines through the origin in $\mathbb{C}^2$. Thus we obtain a principal $\mathbb{T}$-bundle $\pi\colon S^3 \to \mathbb{CP}^1$, called the[7] *Hopf bundle*.

Now let's put a connection on π. We want a horizontal distribution on the total space S^3. Inside $T_{(z^0,z^1)}S^3$, there's a one-dimensional subspace of vectors in the direction of the fiber $\{(z^0, z^1)\cdot\lambda\}$. The standard Riemannian metric on $\mathbb{C}^2 = \mathbb{R}^4$ allows us to choose a complementary line at each point, which is a horizontal distribution. Because $\mathbb{T}$ acts by isometries, this is an invariant distribution, hence a connection.

This is all pretty and geometric, but we need to compute the connection form $\Theta \in \Omega^1_{S^3}(i\mathbb{R})$ (the Lie algebra of $\mathbb{T}$ is a line with trivial bracket, and is more canonically $i\mathbb{R}$). Specifically,

$$\Theta = \operatorname{Im}\Big(\overline{z^0}\,\mathrm{d}z^0 + \overline{z^1}\,\mathrm{d}z^1\Big). \tag{4.12}$$

In the vertical direction, $\Theta = \mathrm{id}$, $(z^0e^{it}, z^1e^{it}) = (iz^0, iz^1)$. Looking inside the complexified tangent bundle (a four-dimensional complex vector bundle), which has basis $\{\partial_{z^0}, \partial_{\overline{z^0}}, \partial_{z^1}, \partial_{\overline{z^1}}\}$, we get

$$iz^0\frac{\partial}{\partial z^0} - i\overline{z^0}\frac{\partial}{\partial\overline{z^0}} + iz^1\frac{\partial}{\partial z^1} - i\overline{z^1}\frac{\partial}{\partial\overline{z^1}}. \tag{4.13}$$

So on vertical vectors, this is the identity. One (you) can check that on a vector normal to S^3, this vanishes – this is just linear algebra over the complex numbers, so nothing too intimidating.

Next we'd like to see

$$\Omega = \mathrm{d}\Theta = \operatorname{Im}\Big(\overline{\mathrm{d}z^0}\wedge\mathrm{d}z^0 + \overline{\mathrm{d}z^1}\wedge\mathrm{d}z^1\Big), \tag{4.14}$$

though this is already imaginary, so we can remove the 'Im' in front. You can check this descends to $\mathbb{CP}^1$. It's a 2-form on $\mathbb{C}^2$, visibly of type $(1,1)$, and we restrict it to S^3; the claim is that there's a form on $\mathbb{CP}^1$ whose pullback by π is $\Omega|_{S^3}$. This involves verifying two things: that Ω is $\mathbb{T}$-invariant, and that it's trivial in the vertical direction. This is a good practice computation.

Let Ω also denote the form on $\mathbb{CP}^1$: $\Omega \in \Omega^2_{\mathbb{CP}^1}(i\mathbb{R})$. We claim

$$\int_{\mathbb{CP}^1}\frac{1}{2\pi}i\Omega = 1. \tag{4.15}$$

[6] Here $\mathbb{T} \subset \mathbb{C}^\times$ is the group of unit-magnitude complex numbers, sometimes also denoted U_1 or S^1.

[7] Well, there's more than one Hopf bundle, and we'll see some others later, but this is the first example.

To compute this, we need some coordinates on $\mathbb{CP}^1$. We'll construct a section s of π over $\mathbb{CP}^1 \setminus \infty \cong \mathbb{C}$. Specifically, given $z \in \mathbb{C}$, which we think of as $[z : 1] \in \mathbb{CP}^1$, let

$$s(z) = \frac{(z, 1)}{\sqrt{1 + |z|^2}}. \tag{4.16}$$

The term in the denominator means that the function decays at infinity in $\mathbb{C}$, so we expect this integral to converge. (But you should still do it!) ◀

Consider a more general principal $\mathbb{T}$-bundle $\pi\colon P \to X$, where X is a smooth manifold. Is it a pullback of the Hopf bundle by a map $X \to \mathbb{CP}^1$? This need not be true, but something weaker is. Consider the generalized Hopf bundle $S^{2N+1} \to \mathbb{CP}^N$, defined in the same way as the Hopf bundle.

Theorem 4.17. *Every principal $\mathbb{T}$-bundle P over a smooth manifold X arises as a pullback of a Hopf bundle $S^{2N+1} \to \mathbb{CP}^N$ for some N.*

We can choose N independent of P, but it will depend on X. So in general you can think of pulling back from $\mathbb{CP}^\infty$.

Proof sketch. A pullback is a $\mathbb{T}$-equivariant map $\varphi\colon P \to S^{2N+1}$; the quotient by $\mathbb{T}$ defines a map $X \to \mathbb{CP}^N$ satisfying the theorem. But this is equivalent data to a section of the associated bundle $S_P^{2N+1} \to X$. This is good: there are tools in topology for constructing sections. First, using an approximation theorem, one shows that it suffices to find a continuous section. Then, one uses obstruction theory: choose a CW structure on X and a q-cell $D \to X$. We'd like to extend a section over this cell; since D is contractible, it's equivalent to ask that the map $S^{q-1} = \partial D \to S_P^{2N+1}$ is trivial (up to homotopy). This is a question about homotopy groups, and for N large enough, the relevant homotopy group vanishes. ⊠

So the next question is: can we construct universal connections Θ^{univ} on these Hopf bundles such that every connection arises as a pullback? This is finickier. Supposing it exists, and $\varphi\colon (P, \Theta) \to (S^{2N+1}, \Theta^{\text{univ}})$, then since connections form an affine space, there's an $\alpha \in \Omega_X^1(i\mathbb{R})$ such that

$$\varphi^*\Theta^{\text{univ}} - \Theta = \pi^*\alpha, \tag{4.18}$$

and hence

$$\overline{\varphi}^*\Omega^{\text{univ}} - \Omega = \mathrm{d}\alpha. \tag{4.19}$$

This therefore implies $\mathrm{d}\Omega = 0$, where $\Omega \in \Omega_X^2(i\mathbb{R})$, so it has a de Rham cohomology class $[i\Omega/2\pi] \in H^2_{\mathrm{dR}}(X)$. This is the pullback of a class $(c_1)_\mathbb{R} \in H^2_{\mathrm{dR}}(\mathbb{CP}^N)$. We can see this class explicitly; $\mathbb{CP}^N$ has a very simple CW structure with one cell in each even dimension. Therefore the cochain complex for CW cohomology with $\mathbb{Z}$ coefficients looks like $\mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to \cdots$, and we claim c_1 is the generator[8] of $H^2(\mathbb{CP}^N; \mathbb{Z})$. Then there's an argument for why these two agree, namely just calculate on $\mathbb{CP}^N$, and this is the beginning of Chern-Weil theory, relating curvature and characteristic classes.

Remark 4.20. There's a similar story for higher Chern classes, but it's sufficiently complicated enough that it's generally easier to calculate using the splitting principle to split a vector bundle as a direct sum of line bundles. ◀

Let's come back to 4-manifolds and self-duality: we let X be an oriented 4-manifold with a conformal structure $[g]$. This is enough to define the Hodge star $\star\colon \Omega_X^2 \to \Omega_X^2$, which squares to the identity. Tensoring with a vector bundle allows us to define $\star\colon \Omega_X^2(E) \to \Omega_X^2(E)$ for any vector bundle $E \to X$, which also squares to the identity; therefore we can also define self-dual and anti-self-dual forms valued in E in the same way.

Definition 4.21. Let $P \to X$ be a principal G-bundle with connection Θ and $\Omega \in \Omega_X^2(\mathfrak{g}_P)$ be the associated connection form. We say Θ is *self-dual* (resp. *anti-self-dual*) if $\star\Omega = \Omega$ (resp. $\star\Omega = -\Omega$).

As we discussed in the first lecture, this is the four-dimensional analogue of a two-dimensional question on oriented, conformal surfaces: whether a function (form, . . .) is holomorphic or antiholomorphic. The sign isn't all that intrinsic: changing the orientation on X changes it.

[8] We need to pick a sign, but this is determined by the canonical orientation of $\mathbb{CP}^N$ coming from the complex structure.

Anti-self-dual connections are of interest to physicists, since the 1970s, beginning with work of Polyakov and others looking at flat space. Uhlenbeck produced a condition guaranteeing that solutions to $\star\Omega = -\Omega$ extend over S^4, and later Atiyah, Bott, Hitchin, and Singer claimed there are more solutions, and used algebraic geometry to produce them. We will study more of this story in this class, but first some examples.

The simplest case is $G = \mathbb{T}$. Often this is called "the" abelian case, though there are certainly other abelian Lie groups, such as $\mathbb{T}^2$. Anyways, in this case Ω lives in $\Omega^2_X(i\mathbb{R})$, $\mathrm{d}\Omega = 0$, and if $\star\Omega = \pm\Omega$, then $\mathrm{d}\star\Omega = 0$ iff $\mathrm{d}^*\Omega = 0$. Together these imply that Ω is a harmonic form if X is closed.

Digression 4.22. Let M be a Riemannian manifold (though for just dimension 4, we're only going to need the conformal class of the metric.) For example, we could take $M = \mathbb{E}^n$, which denotes $\mathbb{R}^n$ with the standard Riemannian metric. Then the *Laplacian* is

$$\Delta \coloneqq -\left(\frac{\partial^2}{\partial(x^1)^2} + \cdots + \frac{\partial^2}{\partial(x^n)^2}\right). \tag{4.23}$$

Why the minus sign? This has a discrete spectrum, and we'd like it to be nonnegative rather than nonpositive.

The de Rham derivative has the form

$$\mathrm{d} = \varepsilon(\mathrm{d}x^i)\frac{\partial}{\partial x^i}, \tag{4.24}$$

where ε denotes exterior multiplication (which is its symbol). Using the metric, the formal adjoint is

$$\mathrm{d}^* = -\iota(\mathrm{d}x^i)\frac{\partial}{\partial x^i}. \tag{4.25}$$

(whose symbol is $-\iota$; here ι is interior multiplication). Then you can check that $\Delta \coloneqq \mathrm{dd}^* + \mathrm{d}^*\mathrm{d}$.

Now we can bring this to any Riemannian manifold M: we know what d is, and can define d^* by integrating by parts to construct the formal adjoint of d, or construct it locally. But, for the same reason that interior multiplication requires a metric, d^* depends on the metric. And therefore we can define the Laplacian Δ on M to be $\mathrm{dd}^* + \mathrm{d}^*\mathrm{d}$. This means the analogue of (4.23) on M in local coordinates $(x^1, \dots, x^n)$ is

$$\Delta = -\sum_{1\le i\le j\le n} -g^{ij}\frac{\partial^2}{\partial x^i \partial x^j}. \tag{4.26}$$

Here $g_{ij} \coloneqq \langle \partial_i, \partial_j\rangle$, and g^{ij} is the (components of the) inverse to the matrix $(g_{ij})_{i,j}$. If you haven't seen this before, it's good to work it out.

Now suppose M is closed and $\Delta\omega = 0$. Then

$$\begin{aligned} 0 &= \mathrm{dd}^*\omega + \mathrm{d}^*\mathrm{d}\omega \\ &= \langle \mathrm{dd}^*\omega, \omega\rangle + \langle \mathrm{d}^*\mathrm{d}\omega, \omega\rangle \\ &= \int_M (\langle \mathrm{dd}^*\omega, \omega\rangle + \langle \mathrm{d}^*\mathrm{d}\omega, \omega\rangle)\,\mathrm{dvol} \\ &= \int_M (\langle \mathrm{d}^*\omega, \mathrm{d}^*\omega\rangle + \langle \mathrm{d}\omega, \mathrm{d}\omega\rangle)\,\mathrm{dvol}. \end{aligned}$$

In fact, the converse is true.

Theorem 4.27. *On a closed Riemannian manifold,* $\mathrm{d}\omega = 0$ *and* $\mathrm{d}^*\omega = 0$ *iff* $\Delta\omega = 0$.

Such a form ω is called *harmonic*. The space of harmonic k-forms is denoted $\mathcal{H}^k_M(g) \subset \Omega^k_M$. Elliptic theory shows this is finite-dimensional,[9] and in fact more is true.

Theorem 4.28 (Hodge decomposition)**.** *There is a splitting*

$$\Omega^k_M \cong \underbrace{\mathcal{H}^k_M(g) \oplus \mathrm{Im}(\mathrm{d})}_{\text{closed}} \oplus \mathrm{Im}(\mathrm{d}^*).$$

Since harmonic forms are closed, there's a projection $\mathcal{H}^k_M(g) \to H^k_{\mathrm{dR}}(M)$, and in fact this is an isomorphism! So every cohomology class has a unique harmonic representative. ◀

[9] We'll use some elliptic theory later this semester, and will therefore go over some of the ingredients that you'd use to prove this.

And now back to 4-manifolds. If X is an oriented Riemannian 4-manifold, we have $\mathcal{H}^2_X(g) \cong H^2(X;\mathbb{R})$, and $\mathcal{H}^2_X(g)$ has two distinguished subspaces: the self-dual forms $\mathcal{H}^2_+(g)$ and the anti-self-dual forms $\mathcal{H}^2_-(g)$. These are distinct subspaces, so every harmonic 2-form ω decomposes as a sum $\omega = \omega_+ + \omega_-$, where $\omega_\pm \in \mathcal{H}^2_\pm(g)$. Explicitly,

$$\omega_\pm = \frac{\omega \pm \star\omega}{2}. \tag{4.29}$$

All of this depended on the metric, so we can ask how this changes as the metric moves, which involves some Sard-Smale theory, as we discussed in Morse theory last semester. But Chern-Weil theory tells us that if Ω comes from a connection on a principal $\mathbb{T}$-bundle, then $i\Omega/2\pi$ defines an integer-valued cohomology class. Therefore self-dual or anti-self-dual connections are the intersection of an integer lattice in H^2 with two lines $\mathcal{H}^2_\pm(g)$. Generically, this has no solutions, unless one of $\mathcal{H}^2_\pm(g)$ is zero (so all forms are self-dual, or are anti-self-dual). Perhaps that's a little disappointing.

To study this, we'll look at the intersection form, a symmetric bilinear 2-form on $H^2(X;\mathbb{Z})$ sending $c_1, c_2 \mapsto \langle c_1 \smile c_2, [X]\rangle$. Let b^2_+ (resp. b^2_-) denote the dimension of the largest subspace on which this form is positive (resp. negative). Then $b^2_+ + b^2_- = b^2(M)$, and their difference is the sigature. We'll put conditions on $b^2_\pm$ which make it possible to find (anti)-self-dual connections.

Example 4.30.

(1) On S^4, $b^2 = 0$, so $b^2_\pm = 0$. So no self-dual forms here.
(2) On $\mathbb{CP}^2$, $b^2_+ = 1$ and $b^2_- = 0$. In this case, self-dual forms exist! Hooray.
(3) But on a K3 surface, $b^2_- = 19$ and $b^2_+ = 3$, so no self-dual forms generically. ◀

This is a little annoying. Maybe we should work with a different Lie group.

The next simplest example is $\mathrm{SU}_2 = \mathrm{Sp}_1$. Associated to it is another Hopf bundle: Sp_1 acts on $S^7 \subset \mathbb{H}^2$, as (right) multiplication by unit quaternions, and the quotient is $\mathbb{HP}^1 \cong S^4$. We can use this to follow the same story as above, defining a connection geometrically and so on.

Lecture 5.

The Yang-Mills functional: 2/5/19

Let X be an oriented, conformal 4-manifold, $P \to X$ be a principal G-bundle, and $\Theta \in \Omega^1_P(\mathfrak{g})$ be a connection. We will study gauge theory in this situation; sometimes we will use a Riemannian metric in the conformal class for X.

In gauge theory, people typically use slightly different notation.

- The connection Θ is usually denoted A.
- Its curvature is denoted $F = F_A = \mathrm{d}A + (1/2)[A \wedge A] \in \Omega^2_P(\mathfrak{g})$ – but F_A is also used to denote the curvature form on X, $F_A \in \Omega^2_X(\mathfrak{g}_P)$.

Definition 5.1. We say that A is *self-dual*, resp. *anti-self-dual*, if $\star F_A = F_A$, resp. $\star F_A = -F_A$.

These are first-order nonlinear PDEs in A. Let's say something about where they come from.

Hodge theory and minimization. Let (M, g) be a closed, oriented Riemannian n-manifold.[10] Given a cohomology class $c \in H^k(M;\mathbb{R})$, what's the "best" differential form representative for c? That is, what's the "best" $\omega \in \Omega^k_M$ with $[\omega] = c$?

Well, what does "best" mean? Maybe smallest-norm: let's ask for an ω which minimizes

$$f\colon \omega \longmapsto \int_M \|\omega\|^2 \,\mathrm{dvol}_g = \int_M \omega \wedge \star\omega \tag{5.2}$$

such that $[\omega] = c$.

Remark 5.3. If M is not oriented, then we don't have a volume form, and $\omega \wedge \star\omega$ is a density. Asking to minimize this norm still makes sense. ◀

[10] One can generalize to open manifolds, but then one needs some vanishing or growth conditions at infinity, or a boundary condition. We're not going to worry about this in this motivational section.

Fix an ω_0 such that $[\omega_0] = c$. On the affine line $\omega_0 + \mathrm{d}\Omega_M^{k-1}$, consider the function

$$f(\omega_0 + \mathrm{d}\eta) = \int_M (\omega_0 + \mathrm{d}\eta) \wedge \star(\omega_0 + \mathrm{d}\eta). \tag{5.4}$$

This is a quadratic function on a real affine line. We know what those look like – parabolas. So we can find the unique minimum where the derivative of f is zero. The derivative is

$$\mathrm{d}f_{\omega_0}(\mathrm{d}\eta) = 2\int_M \mathrm{d}\eta \wedge \star\omega_0 = \pm 2\int_M \eta \wedge \mathrm{d}\star\omega_0, \tag{5.5}$$

using Stokes' theorem, since M is closed. The output equations are

$$\begin{aligned} \mathrm{d}\star\omega_0 &= 0 \\ \mathrm{d}\omega_0 &= 0. \end{aligned} \tag{5.6}$$

These are the Euler-Lagrange equations for this problem.[11] They're satisfied iff $\Delta\omega_0 = 0$, where $\Delta := \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$ is the *Laplacian*. Solutions to $\Delta\omega_0 = 0$ are called *harmonic* forms.

Lemma 5.7. *On a closed manifold M, ω is harmonic iff $\mathrm{d}\omega = 0$ and $\mathrm{d}\star\omega = 0$.*

Proof sketch. Suppose ω is harmonic. Then

$$0 = \int_M \Delta\omega \wedge \star\omega \tag{5.8}$$

$$= \int_M (\mathrm{d}\mathrm{d}^*\omega \wedge \star\omega + \mathrm{d}\mathrm{d}^*\omega \wedge \star\omega) \tag{5.9}$$

$$= \int_M \left(\|\mathrm{d}^*\omega\|^2 + \|\mathrm{d}\omega\|^2\right) \mathrm{dvol}_g. \tag{5.10}$$

Since this is the integral of a nonnegative function, that function must be 0 everywhere, so we conclude that $\mathrm{d}^*\omega = 0$ and $\mathrm{d}\omega = 0$. To get from d^* to $\mathrm{d}\star$, use the fact that the formal adjoint of d, namely d^*, is also $\pm\star\mathrm{d}\star$, where the sign depends on the degree of ω and the dimension of M.

The other direction is up to you. ⊠

Now let's apply this to connections on 4-manifolds. Let $\mathcal{A}_P$ denote the affine space of connections on the principal G-bundle $P \to X$.

Definition 5.11. The *Yang-Mills functional* $Y\colon \mathcal{A}_P \to \mathbb{R}$ is

$$Y(A) := \int_X \|F_A\|^2 \,\mathrm{dvol}_g. \tag{5.12}$$

This looks nice and all that, but we haven't yet defined everything: we need to make sense of the norm on $\Omega_P^2(\mathfrak{g})$. We'll come back to this.

Example 5.13. Suppose $G = \mathbb{T}$, so $\mathfrak{g} = i\mathbb{R}$. Then $F_A \in \Omega_X^2(i\mathbb{R})$, and we know how to take the norm of these kinds of differential forms: the Yang-Mills functional is

$$Y(A) = -\int_X F_A \wedge \star F_A. \tag{5.14}$$

We can decompose F_A into its self-dual and anti-self-dual pieces $F_A^{\pm}$: $F_A = F_A^+ + F_A^-$, and then $\star F_A = F_A^+ - F_A^-$. Thus we can rewrite the Yang-Mills functional as[12]

$$Y(A) = \int_X \underbrace{F_A^- \wedge F_A^-}_{\geq 0} - \underbrace{F_A^+ \wedge F_A^+}_{\leq 0} \tag{5.15}$$

$$\geq \int_X F_A^- \wedge F_A^- + F_A^+ \wedge F_A^+ \tag{5.16}$$

$$= \int_X F_A \wedge F_A \tag{5.17}$$

$$= 4\pi^2 \langle c_1(P)^2, [X]\rangle. \tag{5.18}$$

[11] Well, this is a little silly in this setting, since all we did is take a derivative. But in general they're more involved.

[12] If the signs look weird, keep in mind F_A is imaginary. But there may yet be sign errors.

Here $c_1(P) = (i/2\pi)[F_A] \in H^2_{\mathrm{dR}}(M)$ is the first (well only) Chern class of the principal $\mathbb{T}$-bundle P. So we obtain a lower bound on the Yang-Mills functional in terms of characteristic classes. Equality is achieved exactly when $F_A^+ \wedge F_A^+ = 0$, i.e. A is anti-self-dual.

Next, what are the critical points of Y? The differential is

$$\mathrm{d}Y_A(\dot{A}) = -2\int_X \dot{A} \wedge \mathrm{d}{\star}F_A, \tag{5.19}$$

where $A \in \mathcal{A}_P$ and $\dot{A} \in \Omega^1_X(i\mathbb{R})$ (a variation of A), so the critical points are those such that $\mathrm{d}{\star}F_A = 0$. The Bianchi identity says that, in addition, $\mathrm{d}F_A = 0$, so the critical points have harmonic curvature forms. These two PDEs are second-order (curvature is a derivative, and then we take one more derivative), but linear.

Definition 5.20. The *Yang-Mills equations* are $\mathrm{d}{\star}F_A = 0$ and $\mathrm{d}F_A = 0$.

Suppose $b_2^+(M) > 0$. Then for generic g, the minimum of Y is *not* realized: we're trying to intersect a line and a lattice. But we can get arbitrarily close, producing a sequence of connections approaching the minimum. ◀

What changes for G more general? First we need a G-invariant inner product $\langle -,-\rangle_{\mathfrak{g}}$ on $\mathfrak{g}$; this will induce an inner product $\langle -,-\rangle_{\mathfrak{g}_P}$ on $\Omega^2_X(\mathfrak{g}_P)$: we think of $\omega,\eta \mapsto \omega\wedge\eta \in \Omega^4_X(\mathfrak{g}\otimes\mathfrak{g})$, then use $\langle -,-\rangle_{\mathfrak{g}}$ to get from $\mathfrak{g}\otimes\mathfrak{g}$ to $\mathbb{R}$, and then integrate to get a number.

G acts on $\mathfrak{g}$ by the adjoint action, so we want something invariant under this action. Such a form always exists: you can take for example the Killing form. Then the Yang-Mills functional is

$$Y(A) = \int_X \langle F_A \wedge {\star}F_A\rangle_{\mathfrak{g}_P}, \tag{5.21}$$

and the above calculation generalizes to

$$\begin{aligned}
Y(A) &= \int_X \underbrace{\langle F_A^- \wedge F_A^-\rangle_{\mathfrak{g}_P}}_{\geq 0} - \underbrace{\langle F_A^+ \wedge F_A^+\rangle_{\mathfrak{g}_P}}_{\leq 0} &(5.22)\\
&\geq \int_X \langle F_A^- \wedge F_A^-\rangle_{\mathfrak{g}_P} + \langle F_A^+ \wedge F_A^+\rangle_{\mathfrak{g}_P} &(5.23)\\
&= \int_X \langle F_A \wedge F_A\rangle_{\mathfrak{g}_P} &(5.24)\\
&= 4\pi^2\langle c(P), [X]\rangle, &(5.25)
\end{aligned}$$

where $c(P)$ is some degree-4 characteristic class for principal G-bundles. In particular, this is constant, so we once again obtain a lower bound.

Exercise 5.26. Show that

$$\mathrm{d}Y_A(\dot{A}) = -2\int_X \langle \dot{A} \wedge \mathrm{d}_A \wedge {\star}F_A\rangle_{\mathfrak{g}_P},$$

where $\dot{A} \in \Omega^1_X(\mathfrak{g})$ is a variation of A.

Proof. First, let's differentiate the curvature operator $F\colon \mathcal{A}_P \to \Omega^2_X(\mathfrak{g}_P)$.

$$\begin{aligned}
\mathrm{d}F_A(\dot{A}) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F(A + t\dot{A}) &&(5.27)\\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{d}(A+t\dot{A}) + \frac{1}{2}[(A+t\dot{A})\wedge(A+t\dot{A})] &(5.28)\\
&= \mathrm{d}\dot{A} + [A\wedge\dot{A}] = \mathrm{d}_A A. &(5.29)
\end{aligned}$$

We'll use this to differentiate the Yang-Mills functional, starting with (5.21). Then

$$\begin{aligned}
\mathrm{d}Y_A(\dot{A}) &= 2\int_X \langle \mathrm{d}_A\dot{A} \wedge {\star}F_A\rangle &(5.30)\\
&= -2\int_X \langle \dot{A} \wedge \mathrm{d}_A{\star}F_A\rangle, &(5.31)
\end{aligned}$$

using the Leibniz rule

$$\mathrm{d}\langle\omega\wedge\eta\rangle = \langle\mathrm{d}_A\omega\wedge\eta\rangle \pm \langle\omega\wedge\mathrm{d}_A\eta\rangle \tag{5.32}$$

where $\omega, \eta \in \Omega^*_X(\mathfrak{g}_P)$. ⊠

Finding the critical points of the Yang-Mills functional is now a nonlinear second-order PDE. This is part of a very general story, including work of many people. One doesn't have to restrict to dimension 4; for example, Atiyah and Bott studied the Yang-Mills functional on Riemann surfaces and the topology of (certain equivalence classes of) the space of connections and its relationship with the algebraic geometry of the Riemann surface.

For the rest of today's lecture, we'll discuss some more classical geometric preliminaries, beginning with some cool facts about the conformal group. Let W be a finite-dimensional real vector space,[13] and suppose $\langle -,-\rangle\colon W\times W\to\mathbb{R}$ is a nondegenerate, symmetric bilinear form. We aren't assuming it's positive definite, e.g. it could be Lorentz. Inside W, we have the *null cone* N_W of vectors $\xi\in W$ with $\langle\xi,\xi\rangle = 0$. This is a cone because if $\xi\in N_W$, then $t\xi\in N_W$ too. If the form is definite, the null cone is just $\{0\}$.

Let Q_W denote the image of $N_W\setminus 0$ under the quotient $W\setminus 0\twoheadrightarrow \mathbb{P}W$ by $\mathbb{R}^\times$. Then Q_W is a compact real manifold, a real quadric, and carries a natural conformal structure. Suppose $\ell\in Q_W$, meaning it's a line in the null cone. Then $T_\ell Q_W\subset T_\ell\mathbb{P}W = \mathrm{Hom}(\ell, W/\ell)$, and under this identification, $T_\ell Q_W = \mathrm{Hom}(\ell,\ell^\perp/\ell)$: since ℓ is null, $\ell\subset\ell^\perp$, and in fact $\ell^\perp$ is a codimension-one subspace of W: it's cut out by one equation.

The bilinear form $\langle -,-\rangle$ descends to $\ell^\perp/\ell\times\ell^\perp/\ell\to\mathbb{R}$, and therefore defines a conformal structure on $\mathrm{Hom}(\ell,\ell^\perp/\ell)\cong\ell^\perp/\ell\otimes\ell^*$. This obviously varies smoothly in ℓ, hence defines a conformal structure on Q_W.

Suppose, for example, $W = V\oplus H$ is an orthogonal direct sum, where $\langle -,-\rangle_V$ is an inner product and H is a hyperbolic plane, so it has signature $(1,1)$ and exactly two lines. Then $\mathbb{P}H$ is a circle, and the null cone is two points inside it. The entire quadric cone splits as $Q_W = V \amalg N_V\amalg Q_V$. How does this work? Inside $\mathbb{P}W = \{[v:s:t]\}$, $V = \{[v:-\|v\|^2:1]\}$, $N_V = \{[v:1:0]\}$, and $Q_V = \{[v]:0:0\}$.

If V is n-dimensional, then N_V is $(n-1)$-dimensional and Q_V is $(n-2)$-dimensional. Therefore Q_W is a compactification of V: we've added strata of codimensions one and two to it, and picked up a conformal structure.

Example 5.33. Suppose $\langle -,-\rangle$ is an inner product. Then N_V is a point and Q_V is the empty set, so Q_W is diffeomorphic to S^n with the conformal structure induced from the usual metric.[14] In this case, the symmetry group is $\mathrm{O}(V)$, and with an orientation we can get $\mathrm{SO}(V)$. ◂

Example 5.34. More generally, let $V = \mathbb{R}^{p,q}$, where $p+q = n$, with the indefinite-signature form

$$\langle\xi,\eta\rangle = \sum_{i=1}^p \xi^i\eta^i - \sum_{i=p+1}^q \xi^i\eta^i. \tag{5.35}$$

For example, $H = \mathbb{R}^{1,1}$. In this case, $W\cong\mathbb{R}^{p+1,q+1} = \mathbb{R}^{p+1}\oplus\mathbb{R}^{q+1}$ (though the inner product on the second has a minus sign put in front). If $\xi\in\mathbb{R}^{p+1}$ and $\eta\in\mathbb{R}^{q+1}$, then $(\xi,\eta)\in N_W$ iff $\|\xi\| = \|\eta\| = 1$, so Q_W is diffeomorphic to $(S^q\times S^q)/\{\pm 1\}$, and $\mathrm{PO}(W)$ acts via conformal transformations on Q_W. Inside this we have $\mathrm{O}(V)\cong\mathrm{O}_{p,q}$. So this says that the conformal symmetries in signature (p,q) are the orthogonal symmetries in signature $(p+1,q+1)$. ◂

For example, the group of conformal symmetries of $\mathbb{R}^4$ with positive-definite inner product is $\mathrm{O}_{5,1}$, which acts on S^4, the conformal compactification of $\mathbb{R}^4$. The conformal group of $\mathbb{R}^2$, sitting inside its compactification S^2, is $\mathrm{O}_{3,1}$; with an orientation we get $\mathrm{SO}_{3,1}\cong\mathrm{PSL}_2(\mathbb{C})$, one of the low-dimensional exceptional isomorphisms of Lie groups. These act by Möbius transformations. Analogously, there's a special isomorphism $\mathrm{SO}_{5,1}\cong\mathrm{PSL}_2(\mathbb{H})$, allowing us to get at conformal symmetries in dimension 4 via the quaternions. Next time we'll apply this to the self-duality equations in dimension 4.

Lecture 6.

Spinors in low dimensions and special isomorphisms of Lie groups: 2/7/19

"I'll tell a story for three minutes and then you can go."

[13] We could work with complex vector spaces, or infinite-dimensional vector spaces.

[14] TODO: I would like to double-check this.

Today we'll continue discussing preliminaries to the ADHM construction, sometimes using them to introduce interesting nearby ideas in their own right. Today we'll discuss exceptional isomorphisms of Lie groups in low dimensions, which are applicable in other cases. For example, if you care about fermions in, e.g. supersymmetric quantum field theories in dimensions 6 or below, these ideas appear.

Fix a 4-dimensional complex vector space $\mathbb{S}$, which could be $\mathbb{C}^4$. We can take exterior powers $\Lambda^2\mathbb{S}$, $\Lambda^3\mathbb{S}$, and $\Lambda^4\mathbb{S} = \operatorname{Det}\mathbb{S}$, and similarly for $\mathbb{S}^*$. Choose a volume form $\mu \in \operatorname{Det}\mathbb{S}^* \setminus 0$; we want to consider the symmetries of $(\mathbb{S}, \mu)$. A linear map $T\colon \mathbb{S} \to \mathbb{S}$ has an associated volume $\det T$, so we're essentially asking for automorphisms with determinant 1. This is literally true for $\mathbb{S} = \mathbb{C}^4$, in which case $\operatorname{Aut}(\mathbb{S}, \mu) = \mathrm{SL}_4(\mathbb{C})$.

Now given such an automorphism T, we obtain an automorphism $\Lambda^2 T\colon \Lambda^2\mathbb{S} \to \Lambda^2\mathbb{S}$. The condition that T preserves the volume form is mapped to the bilinear pairing $B\colon \Lambda^2\mathbb{S} \times \Lambda^2\mathbb{S} \to \mathbb{C}$ sending

$$x, y \longmapsto \langle \mu, x \wedge y\rangle_{\operatorname{Det}\mathbb{S}^*, \operatorname{Det}\mathbb{S}}. \tag{6.1}$$

This is symmetric and nondegenerate, hence an inner product, so $\Lambda^2 T$ preserves this inner product! Therefore the map $\phi = \Lambda^2\colon \operatorname{Aut}(\mathbb{S}, \mu) \to \operatorname{Aut}(\Lambda^2\mathbb{S}, B)$ amounts to a homomorphism $\mathrm{SL}_4(\mathbb{C}) \to \mathrm{O}_6(\mathbb{C})$.

You can directly check that $\{\pm 1\} \subset \ker(\phi)$. Since $\mathrm{SL}_4(\mathbb{C})$ is connected, the image of ϕ is contained in $\mathrm{SO}_6(\mathbb{C})$.

Claim 6.2. $\phi\colon \mathrm{SL}_4(\mathbb{C}) \to \mathrm{SO}_6(\mathbb{C})$ is surjective.

The proof would amount to checking that it's an isomorphism on Lie algebras, so the image is open, and that the image is closed, hence all of $\mathrm{SO}_6(\mathbb{C})$. One corollary is that we've identified the spinor representation as $\mathbb{S}$.

Moreover, since $\mathrm{SL}_4(\mathbb{C})$ is simply connected, the map $\phi\colon \mathrm{SL}_4(\mathbb{C}) \to \mathrm{SO}_6(\mathbb{C})$ is the nontrivial double cover map. Therefore we have produced an isomorphism $\mathrm{SL}_4(\mathbb{C}) \cong \mathrm{Spin}_6(\mathbb{C})$.

Remark 6.3. We can define the complex spin groups in the same way as the real ones: for $n \geq 3$, $\pi_1\mathrm{SO}_n(\mathbb{C}) = \mathbb{Z}/2$, and for $n = 2$, $\pi_1\mathrm{SO}_2(\mathbb{C}) = \mathbb{Z}$, so we can ask for the connected double cover of $\mathrm{SO}_n(\mathbb{C})$, which has a canonical Lie group structure, and define it to be $\mathrm{Spin}_n(\mathbb{C})$. ◀

There can be two different realizations of this double cover, but you can prove there's a unique Lie group homomorphism between them respecting the covering map, and it's an isomorphism.

There are several other exceptional isomorphisms, and they all follow from this one.

Example 6.4. Let $J\colon \mathbb{S} \to \mathbb{S}$ be a *quaternionic structure*, i.e. an antilinear map such that $J^2 = -\mathrm{id}_{\mathbb{S}}$. Then $\Lambda^k J\colon \Lambda^k\mathbb{S} \to \Lambda^k\mathbb{S}$ is also antilinear, and squares to $(-1)^k\mathrm{id}_{\mathbb{S}}$. So on $\Lambda^2\mathbb{S}$ it defines a real structure, and on $\Lambda^3\mathbb{S}$ it's quaternionic.

We can impose another constraint, that $(\det J)^*\mu = \mu$, i.e. μ is real. That is, it's in the subspace of $\operatorname{Det}\mathbb{S}^*$ fixed by $\det J$, which is a one-dimensional real vector space. Therefore we obtain a map

$$\phi\colon \operatorname{Aut}(\mathbb{S}, J, \mu) \longrightarrow \operatorname{Aut}(\Lambda^2\mathbb{S}, B, \Lambda^2 J). \tag{6.5}$$

Now suppose $\mathbb{S} = \mathbb{C}^4$. Then the codomain is O_{p+q} for some $p + q = 6$, but B might not be positive definite. The domain is $\mathrm{SL}_2(\mathbb{H})$ – though you have to be careful with what this means. There's no determinant map $\mathrm{GL}_n(\mathbb{H}) \to \mathbb{H}^\times$, but we can take the determinant to be a complex number (via regarding quaternionic matrices as complex matrices), and ask for it to be 1.

Working out the details, the kernel will once again be $\{\pm 1\}$, and this will be a double cover, so this will identify $\mathrm{SL}_2(\mathbb{H}) \cong \mathrm{Spin}_{p,q}$, an isomorphism of 15-dimensional real Lie groups. Then $\mathbb{S}$ is the spinor representation again – but it's quaternionic, so we know we can't get Spin_6.[15]

To determine the signature, let $\{e_1, Je_1, e_2, Je_2\}$ be a basis for $\mathbb{S}$. Then we can write down a basis for $(\Lambda^@\mathbb{S})_{\mathbb{R}}$ as follows:

$$\begin{array}{ll} e_1 \wedge Je_1 & e_2 \wedge Je_2 \\ e_1 \wedge Je_2 + e_2 \wedge Je_1 & i(e_1 \wedge Je_2 - e_2 \wedge Je_1) \\ e_1 \wedge e_2 + Je_1 \wedge Je_2 & i(e_1 \wedge e_2 - Je_1 \wedge Je_2). \end{array} \tag{6.6}$$

You can check these are orthogonal. The first two form a hyperbolic pair (signature $(1,1)$), and the last four all self-pair to -1. Therefore the signature is $(1,5)$, and we conclude $\mathrm{SL}_2(\mathbb{H}) \cong \mathrm{Spin}_{1,5}$.

[15] You can also check that $Z(\mathrm{SL}_2(\mathbb{H})) \cong \mathbb{Z}/2$ but $Z(\mathrm{Spin}_6) \cong \mathbb{Z}/4$.

Taking the quotient, we also get $\mathrm{PSL}_2(\mathbb{H}) \cong \mathrm{SO}^0_{1,5}$[16] – in indefinite signature, special orthogonal groups aren't connected. Now, $\mathrm{SO}^0_{1,5}$ acts as the group of automorphisms of the conformal compactification of $\mathbb{R}^4$ with a definite metric, which gives you S^4: $\mathrm{PSL}_2(\mathbb{H})$ is the conformal group of the 4-sphere. We can also identify $\mathrm{PSL}_2(\mathbb{H}) \cong \mathrm{PGL}_2(\mathbb{H})$ via the diffeomorphism $S^4 \cong \mathbb{HP}^1$.

So in summary, we have $\mathrm{SL}_2(\mathbb{H}) \cong \mathrm{Spin}_{1,5}$ and $\mathrm{PSL}_2(\mathbb{H}) \cong \mathrm{PGL}_2(\mathbb{H}) \cong \mathrm{SO}^0_{1,5}$. ◀

Remark 6.7. If you want to imitate the above story but get Spin_6, you'll want to get the maximal compact in $\mathrm{SL}_4(\mathbb{C})$, which is SU_4. So throw out J and instead ask that your automorphisms fix a Hermitian inner product. ◀

Example 6.8. Now let's introduce a symplectic form $\omega \in \Lambda^2\mathbb{S}^*$, and let's say that $\mu = (1/2)\omega \wedge \omega$. We'll ask about the automorphisms of $\mathbb{S}$ that fix μ; if $\mathbb{S} = \mathbb{C}^4$, this group is called $\mathrm{Sp}_4(\mathbb{C})$.

Passing up to $\Lambda^2\mathbb{S}$, such an automorphism must preserve $\ker(\omega)$, which has codimension 1, and the bilinear form B from before. The automorphisms of $\ker(\omega)$ and B form the group $\mathrm{O}_5(\mathbb{C})$. As in the previous case, the map $\mathrm{Sp}_4(\mathbb{C}) \to \mathrm{O}_5(\mathbb{C})$ has image the identity component $\mathrm{SO}_5(\mathbb{C}) \subset \mathrm{O}_5(\mathbb{C})$, and is a double cover onto it. Therefore we obtain an isomorphism $\mathrm{Spin}_5(\mathbb{C}) \cong \mathrm{Sp}_4(\mathbb{C})$. ◀

Example 6.9. If you ask for automorphisms of $\mathbb{S}$ which preserve both J and ω? In this case we get a map $\mathrm{Aut}(\mathbb{S}, J, \omega) \to \mathrm{Aut}(\ker(\omega), \Lambda^2 J, B)$. The basis of $\ker(\omega)$ contains the last four vectors in (6.6), but instead of the first two we have the vector $e_1 \wedge Je_1 - e_2 \wedge Je_2$, which self-pairs to -1. Therefore we're in definite signature, so this map is identified with the map $\mathrm{Sp}_2 \to \mathrm{O}_5$. Here, Sp_2 is the symmetries of $\mathbb{H}^2$ with its standard inner product.

As usual, this only sees $\mathrm{SO}_5 \subset \mathrm{O}_5$, and is a double cover, providing for us an isomorphism $\mathrm{Spin}_5 \cong \mathrm{Sp}_2$. ◀

Let's digress from the linear algebra a little bit and talk about quaternions. A general quaternion is of the form $x = x^0 + x^1 i + x^2 j + x^3 k$.

Definition 6.10. The *conjugate* of a quaternion x as above is

$$\overline{x} \coloneqq x^0 - x^1 i - x^2 j - x^3 k.$$

The *imaginary part* of x is $\mathrm{Im}(x) \coloneqq x^1 i + x^2 j + x^3 k$.

Therefore $x\overline{x} = (x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$; this is the norm squared of x. We have $\mathrm{d}x = \mathrm{d}x^0 + \mathrm{d}x^1 i + \mathrm{d}x^2 j + \mathrm{d}x^3 k$, and therefore

$$\mathrm{d}x \wedge \mathrm{d}\overline{x} = -(\mathrm{d}x^0 \wedge \mathrm{d}x^1 + \mathrm{d}x^2 \wedge \mathrm{d}x^3)i - (\mathrm{d}x^0 \wedge \mathrm{d}x^2 - \mathrm{d}x^1 \wedge \mathrm{d}x^3)j - (\mathrm{d}x^0 \wedge \mathrm{d}x^3 + \mathrm{d}x^1 \wedge \mathrm{d}x^2)k. \tag{6.11}$$

You'll notice this is self-dual, and in fact the coeffcients are a basis for the space of self-dual 2-forms. Similarly, $\mathrm{d}\overline{x} \wedge \mathrm{d}x$ is anti-self-dual, and its coefficients are a basis for the space of anti-self-dual forms.

Example 6.12. Continuing Example 6.9, consider $S^7 \subset \mathbb{H}^2$, which is preserved by $\mathrm{Sp}_2 = \mathrm{Spin}_5$. Restricting to $\mathbb{H}^2 \setminus 0$, we have a projection down to $\mathbb{HP}^1 \cong S^4$, the quaternionic projective line.[17] This restricts to a submersion $\pi\colon S^7 \to S^4$. Two points in S^7 have the same image iff the are acted on by a unit-norm quaternion. The group of unit-norm quaternions is $\mathrm{Sp}_1 = \mathrm{SU}_3 = \mathrm{Spin}_2$. Since $S^7 = \mathrm{Sp}_2/\mathrm{Sp}_1$, $S^4 \cong \mathrm{Sp}_2/\mathrm{Sp}_1 \times \mathrm{Sp}_1$. In other words, $\pi\colon S^7 \to S^4$ is a principal Sp_1-bundle, and it is homogeneous for the left Sp_2-action. This is another example of a Hopf fibration.[18]

Now who acts on S^4? Well, SO_5 acts by isometries. The whole situation is invariant under the isometry group, so we can move the total space around by isometries of the base, and therefore SO_5 lifts to the Spin_5-action we already identified on S^7. Inside the isometries, there's a bigger group $\mathrm{SO}^0_{1,5}$ of conformal transformations. We'd like to ask whether these lift to conformal isometries on S^7. The identification $S^4 \cong \mathbb{HP}^1$ carries SO_5 to PSp_2 and $\mathrm{SO}^0_{1,5} \cong \mathrm{PSL}_2(\mathbb{H})$, and this lifts to the $\mathrm{SL}_2(\mathbb{H})$-action on $\mathbb{H}^2$.

Now we introduce connections, mimicking the basic story we already saw over $\mathbb{C}$ in §4. We want to consider a connection Θ for $S^7 \to S^4$ regarded as an Sp_1-bundle, an Sp_1-invariant horizontal distribution. Specifically, you can check that

$$\Theta \coloneqq -\,\mathrm{Im}\Big(q^0\,\overline{\mathrm{d}q^0} + q^1\,\overline{\mathrm{d}q^1}\Big) \tag{6.13}$$

[16] $\mathrm{PSL}_2(\mathbb{H})$ is defined to be $\mathrm{SL}_2(\mathbb{H})$ modulo its center $\{\pm 1\}$.

[17] Because $\mathbb{H}$ isn't commutative, we have to specify that we're taking the quotient of the right action.

[18] There are also Hopf fibrations over the reals, including the double cover $\mathbb{Z}/2 \to S^1 \to \mathbb{RP}^1$. There's also one over the octonions.

is a connection, by checking it's orthogonal to orbits everywhere. Then $\Omega_{0,1}$, restricted to the horizontal distribution, is $-\operatorname{Im}(\mathrm{d}q^0 \wedge \overline{\mathrm{d}q^0})$. But from the calculation in (6.11), we already know this is imaginary, so $\Omega_{0,1}|_{\text{horiz.}} = -\mathrm{d}q^0 \wedge \overline{\mathrm{d}q^0}$; we also know it's self-dual. ◀

So we've found one solution to the self-dual equations; we can discover others by transforming by conformal transformations. Therefore $\mathrm{SL}_2(\mathbb{H}) = \mathrm{Spin}_{1,5}$ acts on Θ to produce self-dual connections. This is an $\mathrm{SL}_2(\mathbb{H})$-orbit inside $\mathcal{A}_P$; we know it's a homogeneous manifold, so if we want to know what it is, we should compute the stabilizer of Θ. This is precisely the isometries, which are Spin_5, so the orbit is $\mathrm{Spin}_{1,5}/\mathrm{Spin}_5 = \mathrm{SO}_{1,5}/\mathrm{SO}_5$, which is a hyperbolic 5-ball M, and Θ is actually the center. As you get closer to the "edge" S^4, which we think of as the base S^4, the curvature concentrates more and more at the boundary point, and we could think of the connections at infinity as having curvature in a δ-function (this doesn't actually work, of course). The only connection which has no priviledged concentration of curvatuee is Θ, at the center.

Theorem 6.14 (Atiyah-Drinfeld-Hitchin-Manin)**.** $\mathrm{SO}_{1,5}/\mathrm{SO}_5$ *is the moduli space of self-dual* Sp_1*-instantons with Chern class* $c_2 = 1$*.*

This is the only characteristic class data; for an $\mathrm{Sp}_1 = \mathrm{SU}_2$-bundle, $c_1 = 0$, and there are no more Chern classes. On S^4, $c_2 \in H^4(S^4)$, and the orientation identifies this cohomology group with $\mathbb{Z}$, so we can make sense of $c_4 = 1$. This is an instance of fixing discrete data in a moduli problem, which is common.

This is a basic case where we can picture what's going on, and illustrates a good part of the general story. But how do you prove Theorem 6.14? How do we know that every connection is one of these? Stay tuned; we'll prove this later. Then there are other questions involving other Chern classes, other manifolds, and so on.

This kind of question was first investigated by physicists, Polyakov and others. Atiyah and Singer then checked the dimension by linearizing the equations and found that the physicists had missed some. When Simon Donaldson was a graduate student, he had the brilliant idea of taking this example, but generalizing to 4-manifolds where the intersection form is positive definite. Once again, the moduli space is five-dimensional, and you can take connections with curvature concentrated near a point, and extend by zero. Taubes show you can wiggle this a little bit and actually get a solution – and therefore you again get a copy of the manifold at infinity. Through this (and of course, plenty more) Donaldson was able to prove his first theorem – this exhibits a bordism from this manifold to something else.

Lecture 7.

Some linear algebra underlying spinors in dimension 4: 2/12/19

"Is it clear that —" "Yes, it's clear."

Recall that we've been discussing connections on S^4 with Chern number 1. Fix the usual orientation and round metric (we only need a conformal structure in dimension 4, but we have a canonical metric so we might as well). The conformal symmetries $\mathrm{SO}_{1,5} \cong \mathrm{PSL}_2(\mathbb{H})$ act on S^4, and hence also on the moduli space $\mathcal{M}$ of self-dual Sp_1-connections. This moduli space is diffeomorphic to a 5-ball.

Inside this space, there's a special point, as we discussed last time: the principal Sp_1-bundle $S^7 \to S^4$ realized as the Hopf fibration $\mathbb{H}^2 \setminus \{0\}/\mathbb{R}^{>0} \to \mathbb{HP}^1$, and $\mathbb{HP}^1 \cong S^4$. Then we can choose a Sp_1-invariant horizontal distribution on S^7, as in (6.13).

Now for some more cool facts about spinors. Previously, we discussed how to obtain several exceptional isomorphisms of Lie groups by starting with a four-dimensional complex vector space $\mathbb{S}$ and considering various wedge powers of $\mathbb{S}$ and $\mathbb{S}^*$. Now, let's suppose $\mathbb{S} = \mathbb{S}' \oplus \mathbb{S}''$, where each summand is two-dimensional. Then

$$\Lambda^2\mathbb{S}^* = \Lambda^2(\mathbb{S}')^* \oplus \Lambda^2(\mathbb{S}'')^* \oplus (\mathbb{S}')^* \otimes (\mathbb{S}'')^*. \tag{7.1}$$

The summands have dimensions 1, 1, and 4, respectively.

Suppose we fix a $\mu \in \operatorname{Det}\mathbb{S}^*$, which splits as $\epsilon' \wedge \epsilon''$, where $\epsilon' \in \Lambda^2(\mathbb{S}')^*$ and $\epsilon'' \in \Lambda^2(\mathbb{S}'')^*$. Then $\mathbb{S}' \oplus \mathbb{S}''$ has a bilinear pairing

$$B(s' \otimes s'', \widetilde{s}' \otimes \widetilde{s}'') := \epsilon'(s', \widetilde{s}')\epsilon''(s'', \widetilde{s}''). \tag{7.2}$$

Given an isometry of $\mathbb{S}'$ (i.e. preserving ϵ') and an isometry of $\mathbb{S}''$, their tensor product preserves B, so we obtain a map

$$\operatorname{Aut}(\mathbb{S}', \epsilon') \times \operatorname{Aut}(\mathbb{S}'', \epsilon'') \longrightarrow \operatorname{Aut}(\mathbb{S}' \otimes \mathbb{S}'', B), \tag{7.3a}$$

which if $\mathbb{S} = \mathbb{C}^4$ with the usual decomposition as $\mathbb{C}^2 \oplus \mathbb{C}^2$, is a map

$$\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{O}_4(\mathbb{C}), \tag{7.3b}$$

and as usual, this has image $\mathrm{SO}_4(\mathbb{C})$, and is a double cover onto it. Hence $\mathrm{Spin}_4(\mathbb{C}) \cong \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$.

We're also interested in the real spin group. Hence, fix a *quaternionic* structure $J = J' \oplus J''$, where $J' \colon \mathbb{S}' \to \mathbb{S}'$ is antilinear and squares to $-\mathrm{id}$, and similarly for J''. Then $\operatorname{Aut}(\mathbb{S}', \epsilon', J') \cong \mathrm{Sp}_1$, and similarly for $\mathbb{S}''$, and $\Lambda^2 J$ restricts to a real structure on $\mathbb{S}' \otimes \mathbb{S}''$, so we obtain a map

$$\mathrm{Sp}_1 \times \mathrm{Sp}_1 \longrightarrow \mathrm{SO}_4, \tag{7.4}$$

and this is a double cover, so $\mathrm{Spin}_4 \cong \mathrm{Sp}_1 \times \mathrm{Sp}_1$.

We can produce a real basis of $\mathbb{S}' \otimes \mathbb{S}''$ with respect to this real structure. Let e', Je' be a basis of $\mathbb{S}'$, where we ask $\epsilon'(e', Je') = 1$, and define e'' and J'' similarly. Then our real basis is

$$\begin{gathered} e' \otimes e'' + Je' \otimes Je'' \\ i(e' \otimes e'' - Je' \otimes Je'') \\ e' \otimes Je'' - Je' \otimes e'' \\ i(e' \otimes Je'' + Je' \otimes e''). \end{gathered} \tag{7.5}$$

Remark 7.6. You can also choose a real structure J on $\mathbb{S} = \mathbb{S}' \oplus \mathbb{S}''$, and stipulate that $J(\mathbb{S}') = \mathbb{S}''$, and can play a similar game as above. $\blacktriangleleft$

Now consider the exterior product $\Lambda^2\mathbb{S} \otimes \mathbb{S} \to \Lambda^3\mathbb{S}$. Restricted to $V^* := \mathbb{S}' \otimes \mathbb{S}'' \subset \Lambda^2\mathbb{S}$ we obtain a map

$$\begin{aligned} V^* \otimes (\mathbb{S}' \oplus \mathbb{S}'') &\longrightarrow \mathbb{S}' \oplus \mathbb{S}'' \\ (s' \otimes s''), (\psi', \psi'') &\longmapsto (\epsilon'(s', \psi')s'', \epsilon''(s'', \psi'')s'). \end{aligned} \tag{7.7}$$

In particular, if $v \in V^*$, $s' \in \mathbb{S}'$, and $s'' \in \mathbb{S}''$, then $v, s' \otimes 1$ lands in $\mathbb{S}''$, and correspondingly $v, 1 \otimes s''$ lands in $\mathbb{S}'$. Therefore we obtain maps $\gamma \colon V^* \otimes \mathbb{S}' \to \mathbb{S}''$ and $V^* \otimes \mathbb{S}'' \to \mathbb{S}'$. These will be the Clifford multiplication maps when we pass to associated bundles on a spin 4-manifold. The notation is suggestive: V^* will be vectors and $\mathbb{S}'$ and $\mathbb{S}''$ will be the spinors.

Proposition 7.8. *If $\theta_1, \theta_2 \in V^*$, then*

$$\gamma(\theta_1)\gamma(\theta_2) + \gamma(\theta_2)\gamma(\theta_1) = B(\theta_1, \theta_2).$$

But first we need a quick fact.

Lemma 7.9. *Let W be a two-dimensional vector space and ϵ be an area form for W. For any $w_1, w_2, w_3 \in W$,*

$$\epsilon(w_1, w_2)w_3 + \epsilon(w_2, w_3)w_1 + \epsilon(w_3, w_1)w_2 = 0.$$

Proof sketch. It suffices to check that the map

$$\begin{aligned} W \times W \times W &\longrightarrow \Lambda^2 W \otimes W \\ w_1, w_2, w_3 &\longmapsto (w_1 \wedge w_2) \otimes w_3 + (w_2 \wedge w_3) \otimes w_1 + (w_3 \wedge w_1) \otimes w_2 \end{aligned} \tag{7.10}$$

factors through $\Lambda^3 W$. $\boxtimes$

Proof of Proposition 7.8. Write $\theta_1 = s_1' \otimes s_1''$ and $\theta_2 = s_2' \otimes s_2''$. If $\psi' \in \mathbb{S}'$, then

$$\begin{aligned} \gamma(s_1' \otimes s_1'')\gamma(s_2' \otimes s_2'')\psi' &= \gamma(s_1' \otimes s_1'')\epsilon'(s_2', \psi')s_2'' \\ &= \epsilon'(s_2', \psi')\epsilon''(s_1'', s_2'')s_1'. \end{aligned} \tag{7.11a}$$

Similarly,

$$\gamma(s_2' \otimes s_2'')\gamma(s_1' \otimes s_1'') = -\epsilon'(s_1', \psi')\epsilon''(s_1'', s_2'')s_2'. \tag{7.11b}$$

Adding these together, you get

$$\gamma(s_1' \otimes s_1'')\gamma(s_2' \otimes s_2'')\psi' + \gamma(s_2' \otimes s_2'')\gamma(s_1' \otimes s_1'') = -\epsilon_1'(s_1', s_2')\epsilon''(s_1'', s_2'')\psi'. \tag{7.12}$$ $\boxtimes$

Proposition 7.13. *If $\theta_1, \theta_2 \in V^*$, then the assignment*

$$\theta_1 \wedge \theta_2 \longmapsto \gamma(\theta_1)\gamma(\theta_2) - \gamma(\theta_2)\gamma(\theta_1) \tag{7.14a}$$

defines a map

$$\Lambda^2 V^* \longrightarrow \mathrm{Aut}(\mathbb{S}') \times \mathrm{Aut}(\mathbb{S}''), \tag{7.14b}$$

such that $\Lambda^2 V_+^*$ acts by zero on $\mathbb{S}''$ and $\Lambda^2 V_-^*$ acts by zero on $\mathbb{S}'$.

Recall first that if W is a vector space, there's a canonical isomorphism

$$W \otimes W \cong \mathrm{Sym}^2 W \otimes \Lambda^2 W. \tag{7.15}$$

The general theory of decomposing a tensor product involves a tool called Young diagrams. In our setting,

$$\begin{aligned} V^* \otimes V^* &\cong \mathbb{S}' \otimes \mathbb{S}'' \otimes \mathbb{S}' \otimes \mathbb{S}'' \\ &\cong \underbrace{\mathrm{Sym}^2 \mathbb{S}' \otimes \Lambda^2 \mathbb{S}''}_{\Lambda^2_+ V^*} \oplus \underbrace{\Lambda^2 \mathbb{S}' \otimes \mathrm{Sym}^2 \mathbb{S}''}_{\Lambda^2_- V^*} \oplus \cdots \end{aligned} \tag{7.16}$$

Now choose a real subspace $L' \subset \mathbb{S}'$ such that $\mathbb{S}' = L' \oplus JL'$. Then $V*$ splits as

$$V^* = \mathbb{S}' \otimes \mathbb{S}'' = (L' \oplus JL') \otimes \mathbb{S}'' = \underbrace{L' \otimes \mathbb{S}''}_{(1,0)} \oplus \underbrace{JL' \otimes \mathbb{S}''}_{(0,1)}, \tag{7.17}$$

and J interchanges the two factors, defining a complex structure on $V_{\mathbb{R}}^*$. Similarly,

$$\begin{aligned} \Lambda^2_+ V^* &\cong \mathrm{Sym}^2 \mathbb{S}' \oplus \Lambda^2 \mathbb{S}'' \\ &\cong \underbrace{(L')^{\otimes 2} \otimes \Lambda^2 \mathbb{S}''}_{(2,0)} \oplus \underbrace{(JL')^{\otimes 2} \otimes \Lambda^2 \mathbb{S}''}_{(0,2)} \oplus \underbrace{(L' \otimes JL') \otimes \Lambda^{@} \mathbb{S}''}_{(1,1)}. \end{aligned} \tag{7.18}$$

and

$$\Lambda^2_- V^* \cong \Lambda^2 \mathbb{S}' \otimes \mathrm{Sym}^2 \mathbb{S}'' = \underbrace{(L' \otimes JL') \otimes \mathrm{Sym}^2 \mathbb{S}''}_{(1,1)}. \tag{7.19}$$

Here the annotations mean the *type* of a form in the complexificiation of a real vector space. Specifically, suppose V is a real vector space, so that $V \otimes \mathbb{C} = W \oplus \overline{W}$. Then $V^* \otimes \mathbb{C} = W^* \oplus \overline{W}^*$ too, and therefore there is a splitting

$$\Lambda^k(V^* \otimes \mathbb{C}) = \bigoplus_{p+q=k} \Lambda^p W^* \otimes \Lambda^q \overline{W}^*. \tag{7.20}$$

The forms in this summand are said to have type (p, q). Since $\dim V = 4$, then

$$\Lambda^2(V^* \otimes \mathbb{C}) = \Lambda^{2,0} \oplus \Lambda^{1,1} \oplus \Lambda^{0,2} \tag{7.21}$$

of dimensions 1, 4, and 1 respectively.

Theorem 7.22. *The intersection over all subspaces L' of the $(1,1)$-forms with respect to L' is $\Lambda^2_- V^*$.*

This follows directly from a symmetry argument: this is the only $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$-invariant subspace of $\Lambda^{1,1}$.

This is very useful, as it establishes a link between self-duality in dimension 4 and complex geometry. If you have some connection and want to know whether it's anti-self-dual, it suffices to show that it's type $(1,1)$ in every complex structure.

∽ · ∼

We're now in a position to discuss the Dirac operator, which we will do next time. In the last ten minutes of this lecture, we'll review some basics of complex geometry.

Let M be a manifold, and suppose $I \in \mathrm{End}(TM)$ squares to $-\mathrm{id}_{TM}$. This is what's called an *almost complex structure* on M, and just as above allows us to decompose

$$\Omega^k_M(\mathbb{C}) = \bigoplus_{p+q=k} \Omega^{p,q}_M. \tag{7.23}$$

In more detail, $TM \otimes \mathbb{C}$ splits as $W \oplus \overline{W}$, where W is the subspace where I acts by i (where we've chosen a square root i of -1) and $\overline{W}$ is where I acts by $-i$. Thus $\Lambda^k T^*M$ splits as a sum of $\Lambda^p W^*$ and $\Lambda^q \overline{W}^*$ over all (p, q) with $p + q = k$, and hence sections do as well, giving (7.23).

In this setting, what happens to the de Rham differential? It looks like it gets complicated, but it turns out that $\mathrm{d}_{\Omega^{1,0}}$ has no $(0,2)$-component, and more generally, $\mathrm{d}|_{\Omega^{p,q}}$ lands only in $(p+1, q)$ and $(p, q+1)$. Therefore we can let ∂ denote the part of d valued in $\Omega^{p+1,q}$ and $\overline{\partial}$ the part of d valued in $\Omega^{p,q+1}$, and we get a diagram of maps

$$\begin{tikzcd}[column sep=small]
 & & \Omega^{p,q} \arrow[dl, "\overline{\partial}"'] \arrow[dr, "\partial"] & & \\
 & \Omega^{p,q+1} \arrow[dl, "\overline{\partial}"'] \arrow[dr, "\partial"] & & \Omega^{p+1,q} \arrow[dl, "\overline{\partial}"'] \arrow[dr, "\partial"] & \\
\Omega^{p,q+2} & & \Omega^{p+1,q+1} & & \Omega^{p+2,q}
\end{tikzcd} \tag{7.24}$$

We know $\mathrm{d}^2 = 0$ iff $\partial^2 = 0$, $\overline{\partial}^2 = 0$, and $\partial\overline{\partial} + \overline{\partial}\partial = 0$, so we would like this to be true.

Claim 7.25. This condition is exactly the vanishing of the complex Frobenius tensor of the complex distribution $\overline{W} \subset TM \otimes \mathbb{C}$.

This isn't too hard to see. But the crucial equivalent condition is harder:

Theorem 7.26 (Neulander-Nirenberg)**.** *This condition holds iff we can cover M by local coordinates in which the change-of-charts map is* holomorphic.

Lecture 8.

Twistors and Dirac operators: 2/14/19

"Dimension 4 is, as always, the problem child, or the interesting child."

Last time, we discussed that if M is an almost complex manifold, meaning it comes equipped with a map $I\colon TM \to TM$ with $I^2 = -\mathrm{id}$, then the complex differential forms $\Omega^k_M(\mathbb{C})$ split into $\Omega^{p,q}_M$ indexed over p, q with $p + q = k$ based on how I acts on them. We then mentioned Theorem 7.26, which says that if d restricted to a map $\Omega^{0,1}_M \to \Omega^{2,0}_M$ is zero (which is an integrability condition), then there is an atlas for M whose change-of-charts maps are holomorphic, i.e. M is a complex manifold. In particular, in these local coordinates $z_1, \dots, z_n$, $\mathrm{d}z_1, \dots \mathrm{d}z_n$ are of type $(1, 0)$ and pointwise form a basis of $\Omega^{1,0}_M$.

Assume now that M is a complex manifold, and let $E \to M$ be a C^∞ complex vector bundle, i.e. a complex bundle in the usual sense, and not necessarily holomorphic. Suppose we have a linear operator $\overline{\partial}_E\colon \Omega^{0,0}_M(E) \to \Omega^{0,1}_M(E)$ such that

$$\overline{\partial}_E(f \cdot s) = \overline{\partial} f \cdot s + f \overline{\partial}_E s, \tag{8.1}$$

where f is a function and s is a section of E. We can then extend this to a complex

$$0 \longrightarrow \Omega^{0,0}_M(E) \xrightarrow{\overline{\partial}_E} \Omega^{0,1}_M(E) \xrightarrow{\overline{\partial}_E} \Omega^{0,2}_M(E) \longrightarrow \cdots, \tag{8.2}$$

and $\overline{\partial}^2_E\colon \Omega^{0,0}_M(E) \to \Omega^{0,2}_M(E)$ is multiplication by some tensor $\Phi_E \in \Omega^{0,2}_M(\operatorname{End} E)$.

Theorem 8.3. *If $\Phi_E = 0$, then there exists a local basis of sections $s_1, \dots, s_n$ of sections such that $\overline{\partial}_E s_j = 0$.*

This is easier than Theorem 7.26, though we'll defer the proof. In this case we can place the structure of a complex manifold on E, and E is what's called a *holomorphic vector bundle*.

The twistor approach to the anti-self-dual equations. Now we'll briefly take a peek at one approach to the anti-self-dual equations, as part of some great activity in the 1970s by researchers including the Oxford school. This involves some algebro-geometric techniques which show off the uses of the linear algebra we did last time, and is the original approach to this equation on S^4.

Let X be an oriented Riemannian 4-manifold. If $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ is a 4-dimensional complex vector space (which we'll turn into the spinor bundle soon enough) and $\dim \mathbb{S}^\pm = 2$, then $\mathrm{Sp}_1 \times \mathrm{Sp}_1$ acts on it as the spin group. If $V^+ := \mathbb{S}^+ \otimes \mathbb{S}^-$, then it's isomorphic to $(\mathbb{R}^4)^*$, and SO_4 acts on it. Choose a complex line $L \subset \mathbb{S}^+$, which as we discussed last time defines a complex structure on V^*. The space of choices is $\mathbb{P}(\mathbb{S}^+)$, which is also acted on by SO_4.

Bringing in the topology, let $\mathcal{B}_{\mathrm{SO}}(X) \to X$ be the principal SO_4-bundle of oriented orthonormal bases. The Riemannian metric defines the Levi-Civita connection Θ^{LC} on $\mathcal{B}_{\mathrm{SO}}(X)$, which is a beautiful and still somewhat mysterious fact. Because SO_4 acts on $\mathbb{P}(\mathbb{S}^+)$, we obtain an associated bundle $\mathbb{P}(\mathbb{S}^+) \to X$, and the Levi-Civita connection induces a horizontal distribution on it.

Remark 8.4. We don't have an action of SO_4 on $\mathbb{S}^+$ – that's what we'd need the spin structure for. But we do have a projective action, so the action on the projective space is well-defined. ◀

$\mathbb{P}(\mathbb{S}^+)$ is a six-dimensional real manifold.

Exercise 8.5. Choose $S^4 = \mathbb{HP}^1$ with the round metric. Then $\mathbb{P}(\mathbb{S}^+)$ is diffeomorphic to $\mathbb{CP}^3$, and the projection map $\mathbb{CP}^2 \to \mathbb{HP}^1$ is the map sending a complex line in $\mathbb{C}^2 \to \mathbb{H}^1$ to the unique quaternionic line containing it. The fiber is $\mathbb{CP}^1$.

In this case, $\mathbb{S}^+$ is a complex manifold. In general, we will only have an almost complex manifold: given a point $x, L \in \mathbb{P}(\mathbb{S}^+)$, we have the two lines T_xX and the vertical line in $T_{x,L}\mathbb{P}(\mathbb{S}^+)$ (TODO: I might have this wrong), and so we can take the usual almost complex structure where T_xX is real and the vertical line is imaginary.

Definition 8.6. If M is a Riemannian manifold of dimension at least 5, then its Riemann curvature tensor splits as a sum of three pieces: scalar curvature, Ricci curvature, and something called *Weyl curvature*, which is the piece that's invariant under conformal transformations.

In more detail, the Riemann curvature tensor has some symmetries that mean it's a section of $\mathrm{Sym}^2(\Lambda^2(T^*M))$. If V is an n-dimensional vector space with an orthogonal basis, the standard O_n-action induces an O_n-action on $\mathrm{Sym}^2(\Lambda^2 V^*)$, and this decomposes into three irreducible components, giving us the scalar curvature (for the trivial subrepresentation), the Ricci curvature, and the Weyl curvature.

The Weyl curvature requires four antisymmetric indices, so the Weyl curvature vanishes in dimensions 2 and 3: in dimension 2, the Riemannian curvature tensor is just the scalar curvature, and is called the *Gauss curvature*. In dimension 3, we also have Ricci curvature, but that's it, and so Riemannian geometry in 3 dimensions is nice, e.g. when studying Ricci flow. In dimension 4, the Weyl curvature tensor splits into two pieces: its self-dual and anti-self-dual components. This makes life interesting, as always, in dimension 4.

Theorem 8.7 (Atiyah-Hitchin-Singer)**.** *This almost complex structure is integrable iff $W_+ = 0$, where W_+ is the self-dual Weyl curvature tensor on X.*

In particular: we've passed from an integrability question on the fiber to one on the base.

Now suppose $E \to X$ is a complex vector bundle,[19] and let A be a connection on E, with curvature $F \in \Omega^2_X(\operatorname{End} E)$. Now pull back E and A to $\pi^*E \to \mathbb{P}(\mathbb{S}^+)$. Now, we're over an almost complex base, so we can decompose the curvature into its type components:

$$\pi^*F \in \Omega^2_{\mathbb{P}(\mathbb{S}^+)}(\operatorname{End} E) = \bigoplus_{p+q=2} \Omega^{p,q}(\operatorname{End} E). \tag{8.8}$$

A bit of linear algebra leads to the following lemma.

Lemma 8.9. *The connection A is anti-self-dual iff π^*F has type* $(1,1)$.

Now let $P \to M$ be a principal G-bundle with a connection $\Theta \in \Omega^1_P(\mathfrak{g})$. This satisfies two equations: if $g \in G$, then $R_g^*\Theta = \mathrm{Ad}_g^{-1}\Theta$, and if $\xi \in \mathfrak{g}$, then $\iota_{\widehat{\xi}}\Theta = \xi$.

Now let $\mathbb{E}$ be a vector space and $\rho\colon G \to \mathrm{Aut}(\mathbb{E})$ be a representation. It differentiates to a Lie algebra representation $\dot\rho\colon \mathfrak{g} \to \mathrm{End}(\mathbb{E})$. Let $E := \mathbb{E}_P$ be the associated bundle to the data of ρ, and let ψ be a section. We can think of ψ as a map $P \to \mathbb{E}$ or as an $\mathbb{E}$-valued function on P; then ψ descends to $\Omega^0_M(E)$ iff $R_g^*\psi = \rho(g^{-1}) \cdot \psi$ for all $g \in G$. More generally, if $\alpha \in \Omega^k_P(\mathbb{E})$, then it descends to $\Omega^k_M(E)$ iff

$$\begin{aligned} R_g^*\alpha &= \rho(g^{-1})\alpha \\ \iota_{\widehat{\xi}}\alpha &= 0, \end{aligned} \tag{8.10}$$

[19] As X is not necessarily a complex manifold, we can't ask for E to be holomorphic.

where as before $g \in G$ and $\xi \in \mathfrak{g}$.

The covariant derivative of ψ is

$$\nabla_\Theta \psi = \mathrm{d}\psi + \dot{\rho}(\Theta)\psi, \tag{8.11}$$

which *a priori* lives in $\Omega^1_P(\mathbb{E})$, but you can directly check that it descends, which is a useful exercise. More generally, if α is a k-form as above,

$$\mathrm{d}_\Theta \alpha = \mathrm{d}\alpha + \dot{\rho}(\Theta) \wedge \alpha, \tag{8.12}$$

and this also descends, which you can check.

Exercise 8.13 (Bianchi identity)**.** The curvature $F_\Theta = \mathrm{d}\Theta + (1/2)[\Theta \wedge \Theta]$ satisfies $\mathrm{d}F = 0 + [\mathrm{d}\Theta \wedge \Theta]$, i.e.

$$\mathrm{d}_\Theta F \coloneqq \mathrm{d}F + [\Theta \wedge F] = 0,$$

because F and $\mathrm{d}\Theta$ differ by $[\Theta \wedge \Theta]$, and the Jacobi identity shows the triple bracket you get by substituting it in is zero.

Now let M be a Riemannian n-manifold with its principal O_n-bundle bundle of frames $\mathcal{B}(M) \to M$ and the Levi-Civita connection Θ^{LC}. This therefore gives us horizontal vector fields $\partial_1, \dots, \partial_n$ on $\mathcal{B}(M)$ defined as follows: given an $x \in X$ and a basis $b\colon \mathbb{R}^n \xrightarrow{\cong} T_xM$, hence a point in $\mathcal{B}(M)$, we can take $b(e_1) \in T_xM \subset T_{(x,b)}\mathcal{B}(M)$, which is a horizontal vector, and this defines ∂_1; then $\partial_2, \dots, \partial_n$ are analogous. Moreover,

$$[\partial_k, \partial_\ell] = \underbrace{-\frac{1}{2}T^i_{k\ell}\partial_i}_{=0} - \frac{1}{2}R^i_{jk\ell}E^j_i, \tag{8.14}$$

where E^j_i is some matrix (TODO: I did not parse the definition) which is 0 everywhere except for a 1 in column i and a -1 in column j. The first term vanishes because the Levi-Civita connection is torsion-free.

We can flow along these by geodesics by solving the geodesic equation, which frames the horizontal tangent bundle on $\mathcal{B}(M)$.

We can use this to write the covariant derivative: if O_n acts on a vector space $\mathbb{S}$ and $\psi\colon \mathcal{B}(M) \to \mathbb{S}$ is an equivariant (TODO: ?) map, then

$$\nabla\psi = e^k \cdot \partial_k\psi\colon \mathcal{B}(M) \longrightarrow \mathbb{S} \otimes (\mathbb{R}^n)^*. \tag{8.15}$$

Now we can say something about the Dirac operator. Assume now that X is a 4-dimensional spin manifold with a Riemannian metric. All the linear algebra we did last time tells us that if $\widetilde{\mathcal{B}} \to X$ is the principal Spin_4-bundle of frames, we have Spin_4-equivariant maps $\psi^\pm\colon \widetilde{\mathcal{B}} \to \mathbb{S}^\pm$, where $\mathbb{S}^\pm$ are as in the previous lecture, so we obtain Clifford multiplication $\gamma\colon (\mathbb{R}^4)^* \otimes \mathbb{S}^\pm \to \mathbb{S}^\mp$, and we proved a lemma that

$$\gamma(e^i)\gamma(e^j) + \gamma(e^j)\gamma(e^i) = -\delta^{ij}. \tag{8.16}$$

Definition 8.17. Let $S^\pm$ denote the spinor bundles on X. The *Dirac operator* $D\colon \Omega^0_X(S^\pm) \to \Omega^0_X(S^\mp)$ is defined by

$$D\psi \coloneqq \gamma(e^k)\partial_k\psi.$$

It's easy to see this is a first-order differential operator: we differentiated once.

In the last ten minutes,[20] we'll compute the square of the Dirac operator.

$$\begin{aligned}
D^2\psi &= \gamma(e^k)\gamma(e^\ell)\partial_k\partial_\ell\psi && (8.18)\\
&= -\sum_{k=1}^n \partial_k^2\psi + \sum_{k<\ell}\gamma(e^k)\gamma(e^\ell)[\partial_k,\partial_\ell]\psi && (8.19)\\
&= -\nabla^*\nabla\psi + \frac{1}{4}R_{\mathrm{scal}}. && (8.20)
\end{aligned}$$

This is called the *Weitzenböck formula*.[21] Here $\nabla^*\nabla$, meaning the composition of ∇ with its formal adjoint, is called the *covariant Laplacian*, and exists for any associated bundle for the bundle of frames on a Riemannian manifold.

[20] Measured in my local frame, we have -3 minutes.

[21] The beer is spelled differently: weizenbock.

Now lets choose a Hermitian vector bundle $E \to X$ of rank N with a connection A and curvature F_A, and let $P \to X$ denote the principal U_N-bundle of unitary bases for E; then A is a connection here, in the sense that $A \in \Omega^1_P(\mathfrak{u}_N)$. Cross with the spin bundle to obtain $\widetilde{\mathcal{B}} \times_X P \to X$, which is a principal $\mathrm{Spin}_4 \times \mathrm{U}_N$-bundle, and this has a connection which heuristically is $\Theta^{\mathrm{LC}} + A$, and in particular we still have the vector fields $\partial_1, \dots, \partial_4$.

If $\mathbb{E} = \mathbb{C}^N$ is the model vector space for E, we can consider $\mathrm{Spin}_4 \times \mathrm{U}_N$-equivariant maps $\psi\colon \widetilde{\mathcal{B}} \times_X P \to \mathbb{S}^{\pm} \otimes \mathbb{E}$, and hence obtain a Dirac operator

$$D_A\colon \Omega^0_X(S^+ \otimes E) \longrightarrow \Omega^0_X(S^- \otimes E), \tag{8.21}$$

and this squares to TODO: I had to go.

Lecture 9.

A Fourier-Mukai transform for anti-self-dual connections: 2/17/19

"That's why I wrote it that way, thinking you might try to pull that one on me."

Today, we're going to discuss a kind of Fourier transform for anti-self-dual forms on a Euclidean torus, mostly following Donaldson-Kronheimer.

Definition 9.1. A *lattice* Λ in a real vector space V is a finitely generated abelian subgroup of V, necessarily free. If its rank is equal to the dimension of V, it's called *full*.

Equivalently, a full lattice is the $\mathbb{Z}$-span of a basis of V.

Now let V be a 4-dimensional oriented inner product space and $\Lambda \subset V$ be a full lattice. The model example is $V = \mathbb{R}^4$ and $\Lambda = \mathbb{Z}^4$. Then $T := V/\Lambda$ is a torus with an orientation and Riemannian metric induced from V. It is also an abelian Lie group.

Let $\Lambda^* := \mathrm{Hom}(\Lambda, \mathbb{Z})$ denote the *dual lattice*, which sits inside $V^* = \mathrm{Hom}(V, \mathbb{R})$. There is an isomorphism $\Lambda^* \xrightarrow{\cong} \mathrm{Hom}(T, \mathbb{T})$ sending $\theta\colon \Lambda \to \mathbb{Z}$ to $v \mapsto e^{2\pi i\theta(v)}$.

Definition 9.2. The *dual torus* to T is $T^* := V^*/\Lambda^*$.

There is a universal cheracter $\chi\colon T \times \Lambda^* \to \mathbb{T}$: given $\theta \in \Lambda^*$ and $x \in T$, $\chi(x, \theta) := \exp(2\pi i\theta(x))$ (essentially, "evaluate θ on x"). There are projection maps

$$\begin{tikzcd} & T \times \Lambda^* \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ T & & \Lambda^*, \end{tikzcd} \tag{9.3}$$

from which we can define a Fourier transform from (a certain class of) functions on T to (a certain class of) functions on Λ^*. Specifically, given a function f on T, we'd like to define

$$\widehat{f} := (p_2)_* \chi^{-1} p_1^* f, \tag{9.4}$$

or more explicitly,

$$\widehat{f}(e^v) = \int_T \mathrm{d}t\, e^{-2\pi i\theta(v)} \cdot f(e^v). \tag{9.5}$$

Remark 9.6. This is an instance of a more general phenomenon for locally compact abelian groups called *Pontrjagin duality*. Another example is a finite analogue: letting $\mu_n \subset \mathbb{C}$ denote the n^{th} roots of unity, we have a pairing $\mathbb{Z}/n \times \mu_n \to \mathbb{T}$ sending $(k, \lambda) \mapsto \lambda^k$. These two groups are noncanonically isomorphic,, but each is canonically isomorphic to the character group of the other. In our setting, T and Λ^* aren't isomorphic, but are still each others' character groups. ◀

We will discuss a categorified version of this, which in the algebro-geometric setting is called the Fourier-Mukai transform. We will use T^* instead of Λ^*, and push-pull along the diagram

$$\begin{tikzcd} & T \times T^* \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ T & & T^*. \end{tikzcd} \tag{9.7}$$

Instead of the universal character we will have a universal line bunde $\mathscr{L} \to T \times T^*$ with a Hermitian connection, allowing us to exchange vector bundles on T and on T^*. This $\mathscr{L}$ will be called the *Poincaré line bundle*. Heuristically, the formula will look like (9.4): if $E \to T$ is a vector bundle, we let

$$\widehat{E} = (p_2)_*(\mathscr{L} \otimes p_1^* E), \tag{9.8}$$

In order to make this precise, we have some details to figure out, namely

(1) constructing $\mathscr{L}$ and its Hermitian connection,
(2) interpreting $(p_2)_*$, and
(3) incorporating connections on E and $\widehat{E}$ into the story.

Once we do this, though, we will be able to prove some nice theorems: generic anti-self-dual connections on E pass to anti-self-dual connections on $\widehat{E}$, and there will be an inversion formula.

Remark 9.9. If we give a complex structure on V, then T and T^* acquire the structure of complex manifolds, and we can do all of this with sheaves. This is what's called the Fourier-Mukai transform. ◀

Instead of constructing the Poincaré line bundle, we'll construct a principal $\mathbb{T}$-bundle $P \to T \times T^*$ with connection; this is equivalent, via passing to the associated bundle $P \times_{\mathbb{T}} \mathbb{C} \to T \times T^*$. Begin with the trivial principal $\mathbb{T}$-bundle $V \times V^* \times \mathbb{T} \to V \times V^*$, with its trivial connection $\Theta \in \Omega^1_{V\times V^*\times\mathbb{T}}(i\mathbb{R})$; specifically,

$$\Theta_{(v,\theta,z)}(\dot{v}, \dot{v}^*, \dot{z}) = 2\pi i\theta(\dot{v}) + z^{-1}z. \tag{9.10}$$

This is a universal family of flat connections; you can check that the holonomy vanishes.

Now $\Lambda \times \Lambda^*$ acts on this trivial principal $\mathbb{T}$-bundle by

$$\lambda \cdot (v, \theta, z) \coloneqq (v + \lambda, \theta, z) \tag{9.11a}$$

$$\lambda^* \cdot (v, \theta, z) \coloneqq (v, \theta + \lambda^*, \exp(-2\pi i\lambda^*(v))z), \tag{9.11b}$$

and this covers the usual $\Lambda \times \Lambda^*$-action on the base, so it descends to a principal $\mathbb{T}$-bundle $P \to T \times T^*$ with a flat connection $\overline{\Theta}$. You can think of this as a family of flat connections on T parameterized by T^*; these bundles are trivializable,[22] but not canonically so. Similarly, you can view this as a family of flat connections on T^* parameterized by T, and these are also trivializable but not canonically trivialized. However, the total bundle is not trivial.

The connection form Ω is a translation-invariant (under the group operation) purely imaginary 2-form satisfying the formula

$$\Omega\Big((\dot{v}_1, \dot{\theta}_1), (\dot{v}_2, \dot{\theta}_2)\Big) = 2\pi i\Big(\dot{\theta}_1(\dot{v}_1) - \dot{\theta}_2(\dot{v}_2)\Big), \tag{9.12}$$

or, in other words,

$$\Omega = 2\pi i\langle \mathrm{d}\theta \wedge \mathrm{d}v\rangle. \tag{9.13}$$

Here $\mathrm{d}\theta \in \Omega^1_{T^*}(V^*)$ and $\mathrm{d}v \in \Omega^1_T(V)$ are precisely the Maurer-Cartan forms for these two tori.

Now let $E \to T$ be a Hermitian vector bundle with covariant derivative ∇_A. We can pull both of these back to $T \times T^*$ and tensor with $\mathscr{L}$, but how do we push forward? This will involve the Dirac operator! Let $\mathbb{S}^+$ and $\mathbb{S}^-$ be the spinor spaces, so that $V^* = \mathbb{S}^+ \otimes \mathbb{S}^-$. The spinor bundles over V are the trivial bundles $\underline{\mathbb{S}}^\pm \to V$, which then descend to the torus as trivial bundles. Given $\theta \in T^*$, we have Dirac operators

$$D^\pm_{A,\theta} \colon \Gamma_T(\underline{\mathbb{S}}^\pm \otimes E \otimes \mathscr{L}_\theta) \longrightarrow \Gamma_T(\underline{\mathbb{S}}^\mp \otimes E \otimes \mathscr{L}_\theta). \tag{9.14}$$

We will use this to define the pushforward $(p_2)_*$ – crucially, since T is compact, these are Fredholm operators, which play the role of sheaves in differential geometry: we have the kernel and cokernel of a Fredholm operator, but in a family these can jump. We would like to extract an honest vector bundle, so we'll ask for a hypothesis such that $\ker(D^+_{A,\theta}) = 0$. This is equivalent to saying that $D^-_{A,\theta} = (D^+_{A,\theta})^*$ is surjective.[23] Assuming this, $\ker(D^-_A) \to T^*$ (i.e. the fiber above θ is $\ker(D^-_{A,\theta})$) is a vector bundle.

[22] This is a flat connection on a torus; Chern-Weil theory implies the Chern class of P vanishes, so it's abstractly trivializable.

[23] Though this is a simple linear-algebra exercise in finite dimensions, in the infinite-dimensional Fredholm setting it's a harder theorem known as the *Fredholm alternative*.

Example 9.15. Before we discuss that hypothesis, let's look at a toy model. Consider the matrix

$$D_z \coloneqq \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} : \mathbb{C}^3 \to \mathbb{C}^2, \tag{9.16}$$

where $z \in \mathbb{C}$; hence this is a family of linear operators parametrized by $\mathbb{C}$. This is surjective for $z \neq 0$, with kernel $\mathbb{C} \cdot (0,0,1)$. At 0, the kernel is $\mathbb{C} \cdot (0,0,1) \oplus \mathbb{C} \cdot (0,1,0)$. The dimension of the kernel minus the dimension of the cokernel is always 1, but the kernel isn't a vector bundle; it does define a sheaf, however. ◂

Proposition 9.17. *If A is anti-self-dual, then $\ker(D^+_{A,\theta}) = 0$, hence we can define the pushforward.*

Proof. This is a vanishing theorem, and many vanishing theorems follow a similar proof. We will use the Weitzenböck formula

$$D_A^- D_A^+ = \nabla_A^* \nabla_A - F_A^+, \tag{9.18}$$

and we know $F_A^+ = 0$. Suppose $\psi \in \Gamma_T(\mathbb{S}^+ \otimes E \otimes \mathscr{L}_\theta)$ and $D_A^+ \psi = 0$, but $\psi \neq 0$. Then

$$0 = \int_T \langle \psi, \nabla_A^* \nabla_A \psi \rangle \, \mathrm{d}x = \int_T \|\nabla_A \psi\|^2. \tag{9.19}$$

Since $\|\nabla_A \psi\|^2$ is nonnegative, this means it's zero, so ψ is covariantly constant. This is a typical application of the Weitzenböck formula. ⊠

Spelling this out in more detail, let s_1, s_2 be a basis for $\mathbb{S}^+$. We can write

$$\psi(t) = \psi_1(t) s_1 + \psi_2(t) s_2 \tag{9.20}$$

for $t \in T$, where $\psi_i \in \Gamma_T(E \otimes \mathscr{L}_\theta)$ and $\nabla_A \psi_i = 0$. This section ψ defines a subbundle of $E \otimes \mathscr{L}_\theta$ of rank 1, and the fact that it's covariantly constant means that it splits off! Therefore there is some other bundle $\widetilde{E} \subset E \otimes \mathscr{L}_\theta$ with $E \otimes \mathscr{L}_\theta = \widetilde{E} \oplus \underline{\mathbb{C}}$; the connection form is diagonal, and is the standard connection on $\underline{\mathbb{C}}$. Now, tensoring with $\mathscr{L}_\theta^*$,

$$E = \widetilde{E} \otimes \mathscr{L}_\theta^* \oplus \mathscr{L}_\theta^* = E' \oplus \mathscr{L}_\theta^*. \tag{9.21}$$

Definition 9.22. We say E is *without flat factors* (WFF) if there is no decomposition $E = E' \oplus L$ where L is flat.

This is a generic condition.

Anyways, we have our Fourier-transformed bundle $\widehat{E} \coloneqq \ker(D_A^-)$. The next step is to define the covariant derivative on $\widehat{E}$, via a more general definition.

Definition 9.23. of vector bundles $K, L \to M$, where M is a smooth manifold, and a map $R\colon K \to L$. We will assume K and L come with covariant derivatives ∇^K, resp. ∇^L, and that either R is Fredholm, or both K and L have finite rank. Assuming that R is fiberwise surjective, then $\ker R \to M$ is a vector bundle. Choose a projection $\pi\colon K \to \ker R$ such that $\pi \circ i = \mathrm{id}_{\ker R}$, where $i\colon \ker R \hookrightarrow K$ is inclusion. Then the *compressed covariant derivative* on $\ker R$ is

$$\pi \circ \nabla^K \circ i \colon \Omega^0_M(\ker R) \longrightarrow \Omega^1_M(\ker R).$$

This general construction applies to our situation, where $K = \underline{\mathbb{S}}^- \otimes E \otimes \mathscr{L}_\theta$, $L = \underline{\mathbb{S}}^+ \otimes E \otimes \mathscr{L}_\theta$, and $R = D^-_{A,\theta}$. (TODO: how did we get π?) Therefore we pick up a connection $\widehat{A}$ on $\widehat{E} \to T^*$.

Theorem 9.24. *Assuming A is anti-self-dual and (E, ∇_A) is without flat factors, then $\widehat{A}$ is anti-self-dual.*

We want to prove that $F_{\widehat{A}} \in \Omega^{2;-}_{T^*}(\operatorname{End} \widehat{E})$, and will do so by proving that in every complex structure on T^*, $F_{\widehat{A}}$ is of type $(1,1)$, which by Theorem 7.22 suffices. The complex structures on T^* are parameterized by $\mathbb{P}(\mathbb{S}^+)$. We will need some more ingredients to do this, which we will discuss next time:

1. the Chern connection on a holomorphic, Hermitian vector bundle;
2. how to express the Dirac operator in terms of $\overline{\partial}$; and
3. how to fit these vector bundles into a family of Dolbeault complexes.

This will allow us to identify the connection with the Chern connection, which always has type $(1,1)$; we can avoid a calculation in favor of a geometric proof.