M392C NOTES: MATHEMATICAL GAUGE THEORY

ARUN DEBRAY
JANUARY 24, 2019

These notes were taken in UT Austin's M392C (Mathematical gauge theory) class in Spring 2019, taught by Dan Freed. I live-TeXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Any mistakes in the notes are my own.

Contents

Lecture 1.

Some useful linear algebra: 1/22/19

"Why did the typing stop?"

Today we'll discuss some basic linear algebra which, in addition to being useful on its own, is helpful for studying the self-duality equations. You should think of this as happening pointwise on the tangent space of a smooth manifold.

Let V be a real n-dimensional vector space. The exterior powers of V define more vector spaces: the scalars $\mathbb{R}$, V, $\Lambda^2 V$, and so on, up to $\Lambda^n V = \operatorname{Det} V$. We can also apply this to the dual space, defining $\mathbb{R}$, V^*, $\Lambda^2 V^*$, etc, up to $\Lambda^n V^* = \operatorname{Det} V^*$.

There is a duality pairing

$$\begin{aligned} \theta\colon \Lambda^k V^* \times \Lambda^k V &\longrightarrow \mathbb{R} \\ (v^1 \wedge \cdots \wedge v^k, v_1 \wedge \cdots \wedge v_k) &\longmapsto \det(v^i(v_j))_{i,j}, \end{aligned} \tag{1.1}$$

where $v^i \in V^*$ and $v_j \in V$.

Now fix a $\mu \in \operatorname{Det} V^* \setminus 0$, which we call a *volume form*. Then we get another duality pairing

$$\begin{aligned} \Lambda^k V \times \Lambda^{n-k} V &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto \theta(\mu, x \wedge y). \end{aligned} \tag{1.2}$$

Thus $\Lambda^k V \cong \Lambda^{n-k} V^*$.

Suppose we have additional structure: an inner product and an orientation. Let $e_1, \ldots, e_n$ be an oriented, orthonormal basis of V, and $e^1, \ldots, e^n$ be the dual basis. Now we can choose $\mu = e^1 \wedge \cdots \wedge e^n$.

Definition 1.3. The *Hodge star operator* is the linear operator $\star\colon \Lambda^k V^* \to \Lambda^{n-k} V^*$ characterized by

$$\alpha \wedge (\star\beta) = \langle \alpha, \beta \rangle_{\Lambda^k V} \cdot \mu. \tag{1.4}$$

The inner product on $\Lambda^k V^*$ is defined by

$$\langle v^1 \wedge \cdots \wedge v^k, w^1 \wedge \cdots \wedge w^k \rangle := \det(\langle v^i, w^j \rangle)_{i,j}. \tag{1.5}$$

The Hodge star was named after W.V.D. Hodge, a British mathematician. Notice how we've used both the metric and the orientation – it's possible to work with unoriented vector spaces (and eventually unoriented Riemannian manifolds), but one must keep track of some additional data.

Example 1.6.

- $\star(e^{i_1} \wedge \cdots \wedge e^{i_k}) = e^{j_1} \wedge \cdots \wedge e^{j_{n-k}}$ if the permutation $1, \ldots, n$ to $i_1, \ldots, i_k, j_1, \ldots, j_{n-k}$ of $[n] := \{1, \ldots, n\}$ is even. Otherwise there's a factor of -1.
- Suppose $n = 4$. Then $\star(e^1 \wedge e^2) = e^3 \wedge e^4$ and $\star(e^1 \wedge e^3) = -e^2 \wedge e^4$, and so on. ◀

Remark 1.7. The Hodge star is natural. First, you can see that we didn't make any choices when defining it, other than an orientation and a volume form, but there's also a functoriality property. Let $T\colon V \to V$ be an automorphism; this induces $(\Lambda^k T^*)^{-1}\colon \Lambda^k T^* \to \Lambda^k T^*$, and if T is an orientation-preserving isometry,

$$\star \circ (\Lambda^k T^*)^{-1} = (\Lambda^{n-k} T^*)^{-1} \circ \star. \tag{1.8}$$

Hence $\star\star\colon \Lambda^k V^* \to \Lambda^k V^*$ is some nonzero scalar multiple of the identity, and we can determine which multiple it is. Certainly we know

$$\star\star(e^1 \wedge \cdots \wedge e^k) = \star(e^{k+1} \wedge \cdots \wedge e^1) = \lambda e^1 \wedge \cdots \wedge e^k, \tag{1.9}$$

and we just have to compute the parity of these permutations: one uses k transpositions, and the other uses $n - k$. Therefore we conclude that

$$\star\star = (-1)^{k(n-k)}\colon \Lambda^k V^* \to \Lambda^k V^*. \tag{1.10}$$

◀

Now suppose $n = 2m$, so we have a middle dimension m, and $\star\star\colon \Lambda^m \to \Lambda^m$ is $(-1)^m$. This induces additional structure on $\Lambda^m V^*$.

- If m is even (so $n \equiv 0 \bmod 4$), the double Hodge star is an endomorphism squaring to 1. This defines a $\mathbb{Z}/2$-grading on $\Lambda^m V^*$, given by the ± 1-eigenspaces, which we'll denote $\Lambda^m_\pm V^*$. The $+1$-eigenspace is called *self-dual* m-forms, and the -1-eigenspace is called the *anti-self-dual* m-forms.
- If m is odd (so $n \equiv 2 \bmod 4$), the double Hodge star squares to -1, so this defines a complex structure on $\Lambda^m V^*$, where i acts by the double Hodge star.

Exercise 1.11. Especially for those interested in physics, work out this linear algebra in indefinite signature (particularly Lorentz). The signs are different, and in Lorentz signature the two bullet points above switch!

Exercise 1.12. Show that if $4 \mid n$, the direct-sum decomposition $\Lambda^m V^* = \Lambda^m_+ V^* \oplus \Lambda^m_- V^*$ is orthogonal. See if you can find the one-line proof that self-dual and anti-self-dual forms are orthogonal.

Next we introduce conformal structures. This allows the sort of geometry which knows angles, but not lengths.

Definition 1.13. A *conformal structure* on a real vector space V is a set C of inner products on V such that any $g_1, g_2 \in C$ are related by $g_1 = \lambda g_2$ for a $\lambda \in \mathbb{R}_+$.

In this setting, one can obtain g_2 from g_1 by pulling back g_1 along the dilation $T_\lambda\colon v \mapsto \lambda v$. This induces an action of $(T_\lambda^*)^{-1}$ on $\Lambda^k V^*$, which is multiplication by λ^{-k}: if μ_i is the volume form induced from g_i, so that

$$\alpha \wedge \star\beta = g_1(\alpha, \beta)\mu_1, \tag{1.14}$$

then

$$\lambda^{-2k} \alpha \wedge \star\beta = g_2(\alpha, \beta)\lambda^{-n}\mu_2. \tag{1.15}$$

Thus pulling back by dilation carries the Hodge star to $\lambda^{n-2k}\star$. Importantly, if $n = 2m$, then $\star\colon \Lambda^m V^* \to \Lambda^m V^*$ is preserved by this dilation, so it only depends on the orientation and the conformal structure.

Remark 1.16. A conformal structure is independent from an orientation. For example, on a one-dimensional vector space, a conformal structure is no information at all (all inner products are multiples of each other), but an orientation is a choice. ◀

Example 1.17. Suppose $n = 2$ and choose an orientation and a conformal structure on V. As we just saw, this is enough to define the Hodge star $\star\colon V^* \to V^*$, which defines a complex structure on V. Pick a square root i of -1 and let $\star$ act by it (there are two choices, acted on by a Galois group).

We get more structure by complexifying: $V^* \otimes \mathbb{C}$ splits as a the $\pm i$-eigenspaces of the Hodge star; we denote the i-eigenspace by $V^{(1,0)}$ (the $(1,0)$-*forms*) and the $-i$-eigenspace by $V^{(0,1)}$ (the $(0,1)$-*forms*).

Now let's globalize this: everything has been completely natural, so given an oriented, conformal 2-manifold X, it picks up a complex structure, hence is a Riemann surface, and the Hodge star is a map $\star\colon \Omega^1_X \to \Omega^1_X$. Moreover, we can do this on the complex differential forms, which split into $(1,0)$-forms and $(0,1)$-forms.

How do 1-forms most naturally appear? They're differentials of functions, so given an $f\colon X \to \mathbb{C}$, we can ask what it means for $\mathrm{d}f \in \Omega_X^{1,0}$. This is the equation

$$\star\,\mathrm{d}f = i\,\mathrm{d}f. \tag{1.18}$$

This is precisely the Cauchy-Riemann equation; its solutions are precisely the holomorphic functions on X. ◀

Remark 1.19. More generally, one can ask about functions to $\mathbb{C}^n$ or even sections of complex vector bundles; the analogue gives you notions of holomorphic sections. In this case, the equations have the notation

$$\overline{\partial} f = \left(\frac{1+i\star}{2}\right)\mathrm{d}f. \tag{1.20}$$ ◀

We'll spend some time in this class understanding a four-dimensional analogue of all of this structure.

Symmetry groups. Symmetry is a powerful perspective on geometry. If we think about V together with some structure (orientation, metric, conformal structure, some combination,...), we can ask about the symmetries of V preserving this structure. Of course, to know this, we must know V, but we can instead look at a model space $\mathbb{R}^n$ to define a *symmetry type*, and ask about its symmetry group G: then an isomorphism $\mathbb{R}^n \to V$ preserving all of the data we're interested in defines an isomorphism from G to the symmetry group of V.

Example 1.21. When $\dim V = 2$, the most general symmetry group is $\mathrm{GL}_2(\mathbb{R})$, the invertible matrices acting on $\mathbb{R}^2$. Adding more structure we get more options.

- If we restrict to orientation-preserving symmetries, we get $\mathrm{GL}_2^+(\mathbb{R})$.
- If we restrict to symmetries preserving a conformal structure, the group is called $\mathrm{CO}_2 = \mathrm{O}_2 \times \mathbb{R}^{>0}$.
- If we ask to preserve an orientation and a complex structure, we get $\mathrm{CO}_2^+ = \mathrm{SO}_2 \times \mathbb{R}^{>0}$. This is isomorphic to $\mathbb{C}^\times = \mathrm{GL}_1(\mathbb{C})$: an element of $\mathrm{SO}_2 \times \mathbb{R}^{>0}$ is rotation through some angle θ and a positive number r; this is sent to $re^{i\theta} \in \mathbb{C}^\times$.

This provides another perspective on why an orientation and a conformal structure give us a complex structure. ◀

Example 1.22. Now suppose $n = 4$, and choose a conformal structure C and an orientation on V. Then orthogonal makes sense, though orthonormal doesn't, and the Hodge star induces a $\mathbb{Z}/2$-grading on $\Lambda^2 V^* = \Lambda_+^2 V^* \oplus \Lambda_-^2 V^*$, the self-dual and anti-self-dual 2-forms. The total space $\Lambda^2 V^*$ is six-dimensional, and these two subspaces are each three-dimensional.

Suppose e^1, e^2, e^3, e^4 is an orthonormal basis for some inner product in C. We can use these to define bases of $\Lambda_\pm^2 V^*$, given by

$$\begin{aligned}
\alpha_1^\pm &\coloneqq e^1 \wedge e^2 \pm e^3 \wedge e^4\\
\alpha_2^\pm &\coloneqq e^1 \wedge e^3 \mp e^2 \wedge e^4\\
\alpha_3^\pm &\coloneqq e^1 \wedge e^4 \pm e^2 \wedge e^3.
\end{aligned} \tag{1.23}$$

Now, what symmetry groups do we have? Inside $\mathrm{GL}_4(\mathbb{R})$, preserving an orientation lands in the subgroup $\mathrm{GL}_4^+(\mathbb{R})$; preserving a conformal structure lands in $\mathrm{O}_4 \times \mathbb{R}^{>0}$; and preserving both lands in $\mathrm{SO}_4 \times \mathbb{R}^{>0}$. The first three of these act irreducibly on $\Lambda^2(\mathbb{R}^4)^*$, but the action of $\mathrm{SO}_4 \times \mathbb{R}^{>0}$ has two irreducible summands, $\Lambda_\pm^2(\mathbb{R}^4)^+$.

To understand this better, we should learn a little more about SO_4. Recall that Sp_1 is the Lie group of unit quaternions. This is isomorphic to SU_2, the group of determinant-1 unitary transformations of $\mathbb{C}^2$. This group has an irreducible 3-dimensional representation ρ in which Sp_1 acts by conjugation on the imaginary quaternions (since $\mathbb{R} \subset \mathbb{H}$ is preserved by this action).

Remark 1.24. Another way of describing ρ is: let ρ' denote the action of SU_2 on $\mathbb{C}^2$ by matrix multiplication. Then $\rho \cong \mathrm{Sym}^2 \rho'$. ◀

Proposition 1.25. *There is a double cover* $\mathrm{Sp}_1 \times \mathrm{Sp}_1 \to \mathrm{SO}_4$. *Under this cover, the* SO_4*-representation* $\Lambda_\pm^4(\mathbb{R}^4)^*$ *pulls back to a real three-dimensional representation in which one copy of* Sp_1 *acts by* ρ *and the other acts trivially.*

Proof. Let W' and W'' be two-dimensional Hermitian vector spaces with compatible quaternionic structures J', resp. J''.[1] Then, $V := W' \otimes_{\mathbb{C}} W''$ has a real structure $J' \otimes J''$: two minuses make a plus, and compatibility of J' and J'' means the real points of V have an inner product. (These kinds of linear-algebraic spaces are things you should prove once in your life.)

By tensoring symmetries we obtain a homomorphism $\mathrm{Sp}(W') \times \mathrm{Sp}(W'') \to \mathrm{O}(V)$. This factors through $\mathrm{SO}(V) \hookrightarrow \mathrm{O}(V)$, which you can see for two reasons:

- $\mathrm{Sp}(W')$ and $\mathrm{Sp}(W'')$ are connected, so this homomorphism must factor through the identity component of $\mathrm{O}(V)$, which is $\mathrm{SO}(V)$; or
- a complex vector space has a canonical orientation, and using this we know these symmetries are orientation-preserving.

Now we want to claim this map is two-to-one. One can quickly check that $(-1, -1)$ is in the kernel; the rest is an exercise. ◂

Since Spin_n is the double cover of SO_n, this is telling us $\mathrm{Spin}_4 = \mathrm{Sp}_1 \times \mathrm{Sp}_1$. This splitting is the genesis of a lot of what we'll do in the next several lectures.

Consider the 16-dimensional space

$$V^* \otimes V^* = (W')^* \otimes (W')^* \otimes (W'')^* \otimes (W'')^*. \tag{1.26}$$

Because the map

$$\begin{aligned} \omega' \colon W' \times W' &\longrightarrow \mathbb{C} \\ \xi', \eta' &\longmapsto h'(J'\xi', \eta') \end{aligned} \tag{1.27}$$

is skew-symmetric, it lives in $\Lambda^2(W')^* \subset (W')^* \otimes (W')^*$. In particular, the embedding

$$\mathrm{Sym}^2(W')^* \oplus \mathrm{Sym}^2(W'')^* \hookrightarrow (W')^* \otimes (W')^* \otimes (W'')^* \otimes (W'')^* \tag{1.28}$$

is the map sending

$$\alpha, \beta \longmapsto \alpha \otimes w'' + \omega' \otimes \beta. \tag{1.29}$$

Remark 1.30. This story can be interpreted in terms of representations of $\mathrm{Sp}(W') \times \mathrm{Sp}(W'')$. Let $\mathbf{1}$ denote the trivial representation of Sp_1 and $\mathbf{3}$ be the three-dimensional irreducible representation we discussed above. Then (1.26) enhances to

$$V^* \otimes V^* = \mathbf{1}_{\mathrm{Sp}(W')} \otimes \mathbf{3}_{\mathrm{Sp}(W')} \otimes \mathbf{1}_{\mathrm{Sp}(W'')} \otimes \mathbf{3}_{\mathrm{Sp}(W'')}. \tag{1.31}$$

The skew-symmetric part is $\mathbf{3}_{\mathrm{Sp}(W')} \otimes \mathbf{1}_{\mathrm{Sp}(W'')} \oplus \mathbf{1}_{\mathrm{Sp}(W')} \otimes \mathbf{3}_{\mathrm{Sp}(W'')}$, and the "rest" (complement) is symmetric. ◂

The group $\mathrm{Sp}_1 \times \mathrm{Sp}_1 = \mathrm{Spin}_4$ has complex (quaternionic) two-dimensional representations $S^\pm$, the *spin representations*, and $\Lambda^2_\pm V \cong \mathrm{Sym}^2 S^\pm$.

So two-forms have self-dual and anti-self-dual parts, and curvature is a natural source of 2-forms! ◂

Lecture 2.

Fantastic 2-forms and where to find them: 1/24/19

"I've taught this before, so I know it's true."

Last time, we discussed some linear algebra which is a local model for phenomena we will study in differential geometry. For example, we saw that on an oriented even-dimensional vector space with an inner product, the Hodge star defines a self-map of the middle-dimensional part of the exterior algebra, which induces extra structure, such a splitting into self-dual and anti-self-dual pieces in dimensions divisible by 4. This therefore generalizes to a $4k$-dimensional manifold with a metric and an orientation: the space of $2k$-forms splits as an orthogonal direct sum of self-dual and anti-self-dual forms. (We also discussed other examples, such as how 1-forms on an oriented 2-manifold split into holomorphic and antiholomorphic pieces.)

[1] That is, J' is an antilinear endomorphism of W' squaring to -1, and similarly for J''. Compatible means with the Hermitian metric: h is a map $\overline{W} \times W \to \mathbb{C}$ and J is a map $W \to \overline{W}$, and if $\xi, \eta \in W'$, we want

$$h(J'\xi, \overline{J'\eta}) = \overline{h(\xi, \eta)} \qquad \text{and} \qquad h(J\xi, \eta) = -h(J\eta, \xi).$$

We're particularly interested in the case $k = 1$, where this splitting depends only on a conformal structure, and applies to 2-forms. To study its consequences we'll discuss where one can find 2-forms in differential geometry.

Definition 2.1. A *fiber bundle* is the data of a smooth map $\pi\colon E \to X$ of smooth manifolds if for all $x \in X$ there's an open neighborhood U of x and a diffeomorphism $\varphi\colon U \times \pi^{-1}(x) \to \pi^{-1}(U)$ such that the diagram

$$\begin{tikzcd} U \times \pi^{-1}(x) \arrow[rr, "\varphi"] \arrow[dr, "\mathrm{proj}_1"'] & & \pi^{-1}(U) \arrow[dl, "\pi"] \\ & U & \end{tikzcd} \tag{2.2}$$

commutes. In this case we call X the *base space* and E the *total space*. If there is a manifold F such that in the above definition we can replace $\pi^{-1}(x)$ with F, we call π a *fiber bundle with fiber* F.[2] The map φ is called the *local trivialization*.

Example 2.3. The *trivial bundle* with fiber F is the projection map $X \times F \to X$. ◀

Remark 2.4. Fiber bundles were first defined by Steenrod in the 1940s, albeit in a different-looking way. His key insight was local triviality. There are variants depending on what kind of space you care about: for example, you can replace manifolds with spaces and smooth maps with continuous maps.

Keep in mind that a fiber bundle is data (π) and a condition. Often people say "E is a fiber bundle" when they really mean "π is a fiber bundle"; specifying E doesn't uniquely specify π. ◀

If F has more structure, such as a Lie group, torsor, vector space, algebra, Lie algebra, etc., we ask that $\varphi|_{\pi^{-1}(x)}\colon F \to \pi^{-1}(x)$ preserve this structure. For example, in a fiber bundle whose fibers are vector spaces, we want φ to be linear; in this case we call it a *vector bundle*.

Definition 2.5. If $\pi\colon E \to X$ is a vector bundle, the space of *k-forms valued in E*, denoted $\Omega^k_X(E)$, is the space of C^∞ sections of $\Lambda^k T^*X \otimes E \to X$.

For ordinary differential forms (so when E is a trivial bundle), we have the de Rham differential $\mathrm{d}\colon \Omega^k_X \to \Omega^{k+1}_X$, but we do not have this in general.

Definition 2.6. Let X be a smooth manifold.

(1) A *distribution* on X is the subbundle $E \subset TX$.
(2) A vector field ξ on X *belongs to* E if $\xi_x \in E_x \subset T_xX$ for all X.
(3) A submanifold $Y \subset X$ is an *integral submanifold* for E if for all $y \in Y$, $T_yY = E_y$ inside T_yX.

Do integral submanifolds exist? This is a local question and a global question (the latter about maximal integral submanifolds). In general, the answer is "no," as in the next example.

Example 2.7. Consider a distribution on $\mathbb{A}^3$ with coordinates (x, y, z) given by

$$E_{(x,y,z)} = \mathrm{span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}\right\}. \tag{2.8}$$

There is no integral surface for this distribution. TODO: I missed the argument, sorry. ◀

This is the basic example that illustrates curvature. It turns out that the existence of an integral submanifold is determined completely by the (non)vanishing of a tensor.

Definition 2.9. Let $E \subset TX$ be a distribution. The *Frobenius tensor* $\phi_E\colon E \times E \to TX/E$ given by

$$\xi_1, \xi_2 \longmapsto [\xi_1, \xi_2] \bmod E.$$

Let's think about this: the Lie bracket is defined for vector fields, not vectors. So we have to extend ξ_1 and ξ_2 to vector fields (well, sections of E, since they're in E), which is a choice, and then check that what we obtain is independent of this choice. It suffices to know that this is linear over functions: that

$$[f_1\xi_1, f_2\xi_2] \overset{?}{=} f_1 f_2[\xi_1, \xi_2]. \tag{2.10}$$

[2]Not all fiber bundles have a fiber in this sense, e.g. a fiber bundle with different fibers over different connected components.

Of course, this is not what the Lie bracket does: it differentiates in both variables, so we have the extra terms $f_1(\xi \cdot f_2)\xi_2$ and $f_2(\xi \cdot f_1)\xi_1$. But both of these are sections of E, so vanish mod E, and therefore we do get a well-defined, skew-symmetric form, a section of $\Lambda^2 E^* \otimes TX/E$ – not quite a differential form.

Frobenius did many important things in mathematics, across group theory and representation theory and this theorem, which is about differential equations!

Theorem 2.11 (Frobenius theorem)**.** *An integral submanifold of E exists locally iff $\phi_E = 0$.*

This is a nonlinear ODE. As such, our proof will rely on some facts from a course on ODEs.

Lemma 2.12. *Let X be a smooth manifold, ξ be a vector field on X, and $x \in X$ be a point where ξ doesn't vanish. Then there are local coordinates $x^1, \dots, x^n$ around x such that $\xi = \partial x^1$ in this neighborhood.*

Proof. Let φ_t be the local flow generated by ξ, and choose coordinates $y^1, \dots, y^n$ near x such that $\xi_x = \frac{\partial}{\partial y^1}\Big|_x$. Define a map $U\colon \mathbb{R}^n \to X$ by

$$x^1, \dots, x^n \longmapsto \varphi_{x^1}(0, x^2, \dots, x^n). \tag{2.13}$$

The right-hand side is expressed in y-coordinates. Now we need to check this is a coordinate chart, which follows from the inverse function theorem, because the differential of φ is invertible at 0 (in fact, it's the identity). The theorem then follows because x^1 is the time direction for flow along ξ in this coordinate system. ⊠

Lemma 2.14. *With notation as above, let $\xi_1, \dots, \xi_k$ be vector fields which are linearly independent at x and suc that $[\xi_i, \xi_j] = 0$ for all $1 \le i, j \le k$. Then there exist local coordinates $x^1, \dots, x^n$ such that for $1 \le i \le k$, $\xi_i = \frac{\partial}{\partial x^i}$.*

In fact, the converse is also true, but trivially so: it's the theorem in multivariable calculus that mixed partials commute.

Proof. Let $\varphi_1, \dots, \varphi_k$ be the local flows for $\xi_1, \dots, \xi_k$. Because the pairwise Lie brackets commute, $\varphi_i \varphi_j = \varphi_j \varphi_i$. Since these vector fields are linearly independent at x, we can choose local coordinates $y^1, \dots, y^n$ around x such that $\xi_i|_x = \frac{\partial}{\partial y^i}\Big|_x$. Then, as above, define

$$x^1, \dots, x^n \longmapsto (\varphi_1)_{x_1}(\varphi_2)_{x_2} \cdots (\varphi_k)_{x_k}(0, \dots, 0, x^{k+1}, \dots, x^n). \tag{2.15}$$

You can check that $\mathrm{d}\varphi$ is invertible, so this is a change of coordinates, and then, using the fact that the flows commute, you can see that the lemma follows. ⊠

These lemmas are important theorems in their own right.

Proof of Theorem 2.11. Since the theorem statement is local, we can work in affine space $\mathbb{A}^n$. Let $\pi\colon \mathbb{A}^n \to \mathbb{A}^k$ be affine surjection such that $\mathrm{d}\pi_0$ restricts to an isomorphism $E_0 \to \mathbb{R}^k$. Restrict to a neighborhood U of 0 in $\mathbb{A}^n$ such that $\mathrm{d}\pi_p|_{E_p}\colon E_p \to \mathbb{R}^n$ is an isomorphism for all $p \in U$, and choose $\xi_i|_p \in E_p$ such that $\mathrm{d}\pi_p(\xi_p) = \frac{\partial}{\partial y^i}$. Then, $[\xi_i, \xi_j] = 0$: we know it's in E, and

$$\mathrm{d}\pi[\xi_i, \xi_j] = [\mathrm{d}\pi(\xi_i), \mathrm{d}\pi(\xi_j)] = \left[\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right] = 0. \tag{2.16}$$

Now apply Lemma 2.14; then $\{y^{k+1} = \cdots = y^n = 0\}$ gives the desired integral submanifold. ⊠

The idea of the theorem is that it's a local normal form for an involutive distribution (one whose Frobenius tensor vanishes): locally it looks like the splitting of $\mathbb{R}^n$ into the first k coordinates and the last $(n-k)$ coordinates. And in that local model, we know what the integral manifolds are.

Consider a fiber bundle with a discrete fiber (i.e. the inverse image of every point has the discrete topology). This is also known as a *covering space*. On a "nearby fiber," whatever that means (without more data, we don't have a metric on the base space), we have some sort of parallel transport. The precise statement is that there's a neighborhood of any x on the base space such that any path in that neighborhood lifts to a path on the total space, unique if you specify a point in the fiber. More generally, you can lift families of paths, which illustrates a homotopy-theoretic generalization of a fiber bundle called a *fibration*. But globally, given an element of $\pi_1(X)$, it might lift to a nontrivial automorphism of the fiber.

We'd like to do this for more general fiber bundles $\pi\colon E \to X$, in which case we'll need more data. The kernel of $\mathrm{d}\pi$ is a distribution, and consists of the "vertical" vectors (projection down to X kills them). A complement is "horizontal".

Without any choice, we get a short exact sequence at every $e \in E$:

$$0 \longrightarrow \ker(\mathrm{d}\pi_e) \longrightarrow T_eE \xrightarrow{\mathrm{d}\pi_e} T_xX \longrightarrow 0, \tag{2.17}$$

and a splitting is exactly the choice of a complement $H_e\colon T_xX \to T_eE$. We would like to do this over the whole base, which motivates the next definition.

Definition 2.18. Let $\pi\colon E \to X$ be a fiber bundle. A *horizontal distribution* is a subbundle $H \subset TE$ transverse to $\ker(\mathrm{d}\pi)$, or equivalently a section of the (surjective) map $TE \to \pi^*TX$ of vector bundles on E.

We must address existence and uniqueness. At e the space of splittings is an affine space modeled on $\mathrm{Hom}(T_xX, \ker(\mathrm{d}\pi_e))$, because TODOsomething with a short exact sequence.

Therefore existence and uniqueness of a horizontal distribution is a question about existence and uniqueness of a section of an affine bundle over X. Using partitions of unity, we can construct many of these: existence is good, but uniqueness fails.

What about path lifting? Suppose $\gamma\colon [0,1] \to X$ is a path in X beginning at x_0 and terminating at x_1. We can pull back both E and H by γ, to obtain a rank-1 distribution γ^*H in γ^*TE, and the projection map to $T[0,1]$ is a fiberwise isomorphism. Therefore given a vector at $x_0 = \gamma(0)$ we get a unique horizontal lift along $[0,1]$ to a vector field, and therefore get a unique integral curve above γ.

Note that you cannot always lift higher-dimensional submanifolds, and again the obstruction is the Frobenius tensor, because that's the obstruction to the existence of an integral submanifold. In this context the Frobenius tensor is called *curvature* – right now it's on the total space, but in some settings we can descend it to the base.