PI WORKSHOP: QFT FOR MATHEMATICIANS

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These notes were taken at the workshop on quantum field theory for mathematicians at Perimeter Institute in summer 2019. I live-TEXed these notes using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to `a.debray@math.utexas.edu`; any mistakes in the notes are my own.

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1. Theo Johnson-Freyd: Zero-dimensional QFT and Feynman Diagrams: 6/17/19

In today's TA session, Theo Johnson-Freyd spoke about Feynman diagrams and their history, and their relationship with zero-dimensional quantum field theory.

Feynman diagrams were introduced by Dirac, in a paper called "The Lagrangian in QM" in 1932, then developed further in Feynman's PhD thesis in 1938. Dirac knew there was an analogy between T^*X and $L^2(X)$, thought of as the state space for classical mechanics on X and quantum mechanics on X. Time evolution on T^*X is a symplectomorphism, meaning its graph is Lagrangian in $T^*(X\times X)$, explicitly the graph of $\mathrm{d}S$, where $S_t\in C\infty(X\times X)$. Meanwhile, in quantum mechanics on $L^2(X)$, time evolution is a unitary operator $U_t\colon L^2(X)\to L^2(X)$, and acts on operators by

$$(U_t\psi)(x_1)=\int \mathrm{d}x_0\, U_t(x_0,x_1)\psi(x_0). \tag{1.1}$$

This U_t solves the Schrödinger equation.

Exercise 1.2. $S_t(x_0-x_1)$ solves an analogous differential equation, some version of the Hamilton-Jacobi equation. What is it?

What Dirac knew was how this analogy runs: in the classical limit, as $\hbar\to 0$, $\exp(iS_t/\hbar)\approx U_t(x_0,x_1)$. This was the idea behind the path integral: we can really make sense of minimizing the action functional over the space of fields and use this to learn the physics; this approximation suggests something similar should be possible in quantum mechanics, and this is the famous path integral.

Anyways, right around then was world war 2, and many physicists worked on more applied things for a few years. But by 1948, Schwinger and Tomonaga had given a complete theory of quantum electrodynamics in terms of infinite-dimensional quantum mechanics. It was beautiful and people liked it, and there were a few infinities left, but it wasn't a big deal. Feynman then gave a lecture, not a great one, introducing Feynman diagrams, and then explained it to Dyson on a road trip from New York to Arizona. So there was plenty of time to convey the ideas. Thus in 1949 Dyson explained Feynman diagrams, proved that they gave the same theory as Schwinger and Tomonaga's description, and implemented renormalization. This is the context which this talk will live in.

The goal is: given an action S, our goal is to compute the partition function

$$Z=\int_{\mathbb{R}^n}\mathrm{d}\phi\,\exp\!\left(\frac{i}{\hbar}S(\phi)\right), \tag{1.3}$$

or more generally, if f is some function on spacetime, we want to compute a correlation function

$$\langle f\rangle = \frac{1}{Z}\int_{\mathbb{R}^n} \mathrm{d}\phi,\, f(\phi)\exp\left(\frac{i}{\hbar}S(\phi)\right). \tag{1.4}$$

There aren't many ways to solve these, so we make the *stationary phase assumption*: that in the $\hbar \to 0$ limit, the integral is supported in a formal neighborhood of the origin. This means we can pass to Taylor series: whatever a formal neighborhood of the origin is, functions on it are $\mathbb{C}[[x_1, \dots, x_n]]$. Let's also assume the minimum of S occurs at $x = 0$. Then the action has the form

$$S(x) = \frac{1}{2}a_{ij}x_ix_j + b(x), \tag{1.5}$$

where $b(x)$ contains cubic and higher terms (the *interacting part*), and $a_{ij}x_ix_j$ is the *free part.* So that the minimum is as nice as possible, we want (a_{ij}) to be positive definite, or at least invertible.

Now, how can we compute integrals? You might remember u-substitution, which is essentially cleverly choosing coordinates, or you can integrate by parts. Choose n generating functions $g_1, \dots, g_n$; then

$$0 = \frac{1}{Z}\int \frac{\partial}{\partial x_i}\left(g_i(x)\exp\left(\frac{i}{\hbar}S(x)\right)\right) \tag{1.6a}$$

$$= \left\langle \frac{\partial g_i}{\partial x_i}\right\rangle + \underbrace{\left\langle g_i(x)\frac{i}{\hbar}a_{ij}x_j\right\rangle}_{(*)} + \left\langle g_j(x)\frac{i}{\hbar}\frac{\partial}{\partial x_i}b(x)\right\rangle. \tag{1.6b}$$

If f is degree zero, then $\langle f\rangle = f$. If f is homogeneous of degree $N > 0$, then the starred term in (1.6b) is dominant, so we will choose $g_i(x)$ such that (1.6b) equals $f(x)$. One possible choice is a term in the Taylor series:

$$g_i(x)a_{ij}x_j = \frac{\hbar}{i}\frac{1}{N!}f^{(N)}_{i_1,\dots,i_N}x_{i_1}\cdots x_{i_N}, \tag{1.7a}$$

which simplifies to

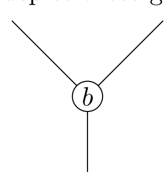
$$g_i(x) = (a^{-1})_{ij}\frac{1}{N}\frac{\partial f}{\partial x_j}. \tag{1.7b}$$

Then, explicitly,

$$\langle f\rangle = \langle (a^{-1})_{ij} i\frac{\hbar}{N}\frac{\partial^2}{\partial x_i\partial x_j}f(x)\rangle - \langle (a^{-1})_{ij}\frac{1}{N}\frac{\partial f}{\partial x_i}\frac{\partial b}{\partial x_j}\rangle. \tag{1.8}$$

Einstein once said that his greatest contribution to physics was getting rid of the summation symbol; in a similar spirit, Penrose's thesis was the first to depict these graphically. The notation

(1.9)



should be interpreted as $(1/3!)\partial_{i_3}\partial_{i_2}\partial_{i_1}b$, so in general N spokes means the N^{th} Taylor coefficient of b. We will interpret an edge between two nodes, labeled by i and j, as $(a^{-1})_{ij}$.

TODO: here's the picture of (1.8) expressed in this graphical notation.

Now I will remind you of the second trial of Hercules. He was asked to kill the Hydra, a many-headed beast such that, if you chopped off one head, two more would grow in its beast. Hercules solved this by bringing wax, and every time he chopepd off a head, he sealed it with wax so that it could not grow there. Hercules can also chop off two heads and glue them together. Plutarch, of course, didn't tell quite the same story, but an approximation to it called the Classics-al limit.

But at least in Feynman diagrams, every head that sprouts gives yet another factor of x, and we're working near 0, so x is small: successive heads are smaller and smaller, so the series converges.

Example 1.10. Suppose f is linear and b is cubic. Then the Feynman diagram expansion looks like (TODO add figure), so we obtain explicitly a sum over closed diagrams, weighted by automorphisms of the diagram, of $(i\hbar)^{\#\,\text{loops}}$ times the partial derivatives and factors of $(a^{-1})_{ij}$. ◂

From the perspective of homological algebra, it's possible to rephrase this problem, and this looks a bit like a chain complex that's close to exact, but not exact.

In quantum field theory, $\mathbb{R}^n$ is replaced with the space of fields, e.g. in a scalar field theory on M, we would take some space of functions on M. The theory of integration tells us that points in M are more or less a basis for this function space. So the points in the Feynman diagram range over M. Then a becomes the Laplacian (or $\Delta + m$, or...), and a^{-1} becomes the Green's function for the differential operator a. So a Feynman diagram has nodes located at points in M, and the edges, called *propagators*, govern how information flows between them, and how long. Then one wishes to integrate over all such choices, including how long it takes for information to propagate.

What's a little weird about this (and why Feynman's lecture wasn't so well received) is that we began with a field theory, but we ended up with pictures of interacting particles somehow!

2. Chris Elliott: Supersymmetry algebras: 6/18/19

> *"You can put 'super' in front of every noun in your sentence, and it should still be a true sentence."*

In today's TA session, Chris Elliott spoke about the Lie-algebraic structures behind supersymmetry.

First, some motivation. Let g be a pseudo-Riemannian metric on $\mathbb{R}^n$ of signature (p, q), and consider QFT on $(\mathbb{R}^n, g)$. We want to impose invariance under the symmetries of this structure.

Definition 2.1. The *Poincaré group* is $\mathrm{Iso}_{p,q} \coloneqq \mathrm{SO}_{p,q} \ltimes \mathbb{R}^n$.

The Lie algebra of the Poincaré group is $\mathfrak{iso}_{p,q} = \mathfrak{so}_{p,q} \ltimes \mathbb{R}^n$. The complexified Lie algebra, sometimes denoted $\mathfrak{iso}_n(\mathbb{C})$, is $\mathfrak{so}_n(\mathbb{C}) \ltimes \mathbb{C}^n$.

The beginning of the study of sypersymmetry algebra is a no-go theorem.

Theorem 2.2 (Coleman-Mendula)**.** *If G is a group of symmetries containing $\mathrm{Iso}_{1,n-1}$ acts on a sufficiently nice QFT, then $G = \mathrm{Iso}_{n-1,1} \times G'$ for some group G' of internal symmetries.*

So there aren't any interesting options out there. But we can exhibit interesting extensions if we consider $\mathbb{Z}/2$-graded extensions of the Poincaré Lie algebra.

Definition 2.3. An *n-dimensional super-Poincaré algebra* is a super Lie algebra (i.e. a $\mathbb{Z}/2$-graded Lie algebra with the Koszul sign rule) $\mathfrak{a}$ of the form

$$\mathfrak{a} = \mathfrak{iso}_n(\mathbb{C}) \ltimes \Pi\Sigma, \tag{2.4}$$

where $\mathfrak{so}_n(\mathbb{C})$ acts on Σ as a spinor representation and Π is parity change (so Σ and $\Pi\Sigma$ are in opposite degrees), with a bracket $\Gamma\colon \Sigma \otimes \Sigma \to \mathbb{C}^n$.

Remark 2.5*.* Analogously to Theorem 2.2, there is a classification result for super-Poincaré algebras due to Haag-Łopuszański-Sohnius. ◀

To classify super-Poincaré algebras, we need to classify the spinor representations and the pairings. The former is classical: either

- n is odd, and there's a unique irreducible spinor representation, or
- n is even, and there are two nonisomorphic spinor representations.

Remark 2.6*.* The spinor representations are precisely those representations of $\mathfrak{so}_n$ which do not arise from representations of SO_n; one can realize them as representations of Spin_n as sitting inside the Clifford algebra. Alternatively, you can obtain these representations from the Dynkin diagram: $\mathfrak{so}_n$ for n odd is type B_n, and this is the representation corresponding to the rightmost node of the Dynkin diagram; $\mathfrak{so}_n$ for n even is type D_n, and there are two rightmost nodes. ◀

Great: this tells us the possibilities for Σ.

- Odd n: $\Sigma = S \otimes W$, where W is an auxiliary finite-dimensional vector space. In this case S is the *Dirac spinor representation.*
- Even n: $\Sigma = S_+ \otimes W_+ \oplus S_- \otimes W_-$, where again $W_\pm$ are auxiliary. Here $S = S_+ \oplus S_-$, and $S_\pm$ are the *Weyl spinor representations.*

Now, how do you classify the pairings? Well, they're equivalent to symmetric maps $\Sigma^{\otimes 2} \to V = \mathbb{C}^n$, so we're looking for irreducible summands of $\mathrm{Sym}^2 \Sigma$ isomorphic to V. This can be done pretty explicitly granted a few facts about Clifford algebras.

For n odd, $S \otimes S \cong C\ell^0(V)$, the even part of the Clifford algebra; this is

$$C\ell^0(V) = \bigoplus_{k \text{ even}} \Lambda^k V = \bigoplus_{k=0}^{(n-1)/2} \Lambda^k V, \tag{2.7}$$

using $\Lambda^k V = \Lambda^{n-k} V$. In particular, there is one summand isomorphic to $\Lambda^1 V = V$.

For n even, we get the whole Clifford algebra:

$$(S_+ \oplus S_-)^{\otimes 2} \cong C\ell(V) \cong \bigoplus_{k=0}^{m} \Lambda^k V \cong 2\left(\bigoplus_{k=0}^{n/2-1} \Lambda^k V\right) \oplus \Lambda^{n/2} V, \tag{2.8}$$

so we get two copies of V. Now when we look closer, there will be interesting Bott periodicity phenomena afoot, depending on whether the spinor representations are real, complex, or quaternionic.

Lemma 2.9.

(1) *If n is odd, there's a unique irreducible summand of $S^{\otimes 2}$ isomorphic to V.*
- *For $n \equiv 1, 3 \bmod 8$, it's contained in* $\mathrm{Sym}^2 S$.
- *For $n \equiv 5, 7 \bmod 8$, it's contained in* $\Lambda^2 S$.

(2) *If $n \equiv 0, 4 \bmod 8$, there is a unique irreducible summand of $S_+ \otimes S_-$ isomorphic to V, and no such summands in $S_\pm^{\otimes 2}$.*

(3) *If $n \equiv 2, 6 \bmod 8$ (and $n > 2$), then there is a unique irreducible summand of $S_\pm^{\otimes 2}$ isomorphic to v, and no such summand in $S_+ \otimes S_-$.*
- *For $n \equiv 2 \bmod 8$ ($n > 2$), the summand is inside* $\mathrm{Sym}^2(S_\pm)$.
- *For $n \equiv 6 \bmod 8$, the summand is inside* $\Lambda^2 S_\pm$.

So this tells us that a choice of a super Poincaré algebra is a choice of

- an orthogonal vector space W (i.e. a space with a symmetric pairing) if $n \equiv 1, 3 \bmod 8$;
- a pair of orthogonal vector spaces W_+, W_-, if $n \equiv 2 \bmod 8$ (and $n > 2$);
- a single vector space W_+ with dual W_- if $n \equiv 0, 4 \bmod 8$;
- a single symplectic vector space W if $n \equiv 5, 7 \bmod 8$; or
- a pair of symplectic vector spaces W_+, W_- if $n \equiv 6 \bmod 8$.

And therefore we know all of the supersymmetry algebras.

Remark 2.10. One usually indicates a choice of super Poincaré algebra by writing $\mathcal{N} = \dim W$, or $\mathcal{N}_\pm = \dim W_\pm$. For example, the 3D $\mathcal{N} = 4$ supersymmetry algebra is

$$\mathfrak{so}_3(\mathbb{C}) \ltimes (\mathbb{C}^3 \oplus \Pi(S \otimes W)), \tag{2.11}$$

where $\dim W = 4$, and Γ appears in the Lie bracket for that direct sum.

However! When $n \equiv 5, 6, 7 \bmod 8$, i.e. in the symplectic cases, one generally writes $\mathcal{N} = (\dim W)/2$ and $\mathcal{N}_\pm = (\dim W_\pm)/2$. So $\mathcal{N} = 1$ is always the smallest amount of supersymmetry. For example, 5D $\mathcal{N} = 1$ means W is a two-dimensional symplectic vector space. ◂

This is the complex story, which is useful for applications in mathematics; the story over $\mathbb{R}$ involves real structures on these algebras, which is more complicated.

Definition 2.12. The *R-symmetry group G_R* is the group of outer automorphisms of the super Poincaré algebra $\mathfrak{a}$ which fix the even part.

In particular, these are automorphisms of W and its specified structure. Again we proceed by cases.

- For $n \equiv 1, 3 \bmod 8$, W carries an inner product, so $G_R = \mathrm{O}(W)$.
- For $n \equiv 2 \bmod 8$ ($n > 2$), we have two orthogonal vector spaces: $G_R = \mathrm{O}(W_+) \times \mathrm{O}(W_-)$.
- For $n \equiv 0, 4 \bmod 8$, we have no additional structure, so $G_R = \mathrm{GL}(W)$.
- For $n \equiv 5, 7 \bmod 8$, W has a symplectic structure, and $G_R = \mathrm{Sp}(W)$.
- For $n \equiv 6$, we have two symplectic vector spaces: $G_R = \mathrm{Sp}(W_+) \times \mathrm{Sp}(W_-)$.

This is the algebraic piece only, though; in general, not all of the R-symmetry group may actually be a symmetry of a given supersymmetric QFT. We can also take a Lie subalgebra $\mathfrak{g} \subset \mathfrak{g}_R$ and append that to the super-Poincaré algebra, taking into account that $\mathfrak{g}$ acts on Σ.

Suppose $Q \in \Sigma$ squares to zero, i.e. $\Gamma(Q, Q) = 0$. These are easy to find in some cases (e.g. W symplectic) but are not always immediate. These Q determine cohomological structures on supersymmetric QFTs, where one includes Q in the BV-BRST differential of a supersymmetric quantum field theory, which is an example of *twisting*.

Definition 2.13. If $\Gamma(Q, –)\colon \Sigma \to \mathbb{C}^n$ is surjective, we call Q *topological* (and will expect that the twisted theory is a topological field theory of some sort). These were the original twists considered by Witten.

Example 2.14 (Dimension 1)**.** $\mathfrak{so}_1(\mathbb{C})$ is trivial, so the supersymmetry algebra is just $\mathbb{C} \oplus \Pi W$, with bracket given by a bilinear pairing $\langle –, – \rangle$ on W; Q squares to zero if $\langle Q, Q\rangle = 0$.

Specifically, the 1D $\mathcal{N} = 2$ supersymmetry algebra has null vectors of $(1, \pm i)$, so these give rise to two conserved charges Q and $Q^\dagger$ that we saw yesterday. Then, turning on the differential given by Q in supersymmetric quantum mechanics tells you interesting things about de Rham cohomology or something analogous. ◀

Example 2.15 (Dimension 2)**.** First, $\mathfrak{so}_2(\mathbb{C}) \cong \mathbb{C}^\times$ and $S_\pm$ are one-dimensional with weights $\pm 1/2$. The vector representation $V = \mathbb{C}^2$ is reducible, with weight $(1, -1)$. Here Σ generally looks like $W_+ \oplus W_-$ (with weights $1/2$ and $-1/2$, respectively), and square-zero elements are pairs (w_+, w_-), where $w_\pm$ are both null. This element is topological whenever they're both nonzero (TODO: I think?), and the smallest $\mathcal{N}$ for which this happens is $\mathcal{N} = (2, 2)$. ◀

Remark 2.16. Generally, physicists consider supersymmetric theories with $\mathcal{N} \le 16$ (or $\mathcal{N} \le 32$). The ultimate reason is that people only look at particles with spin at most 2 (and spin-2 particles pretty much only in supergravity), which forces those bounds. This also means that we can only look at dimensions up to 11; outside of supergravity, where you're probably looking at $\mathcal{N} \le 16$, this restricts to dimensions up to 10. It's also common for $\mathcal{N}$ to be a power of 2. ◀

Example 2.17 (Dimension 4)**.** The exceptional isomorphism $\mathfrak{so}_4(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$; we'll denote the first copy $\mathfrak{sl}_2(\mathbb{C})_+$ and the second copy $\mathfrak{sl}_2(\mathbb{C})_-$. The spin representation $S_\pm$ is the defining representation for $\mathfrak{sl}_2(\mathbb{C})_\pm$, and $\mathfrak{sl}_2(\mathbb{C})_\mp$ acts trivially. We have $V = S_+ \otimes S_-$ and $\Sigma = S_+ \otimes W \oplus S_- \otimes W^*$. ◀

3. Natalie Paquette: 2D Yang-Mills theory: 6/19/19

In today's TA session, Natalie Paquette spoke about 2D Yang-Mills theory. This is a particularly simple field theory, which makes it a good example for studying these things; we will work in Euclidean signature and follow the standard (but far from the only) reference, Cordes-Moore-Ramgoolam. For the most part, today the gauge group will be U_N or SU_N; it will always be compact.

So, fix an oriented[1] surface Σ with a Riemannian metric, a gauge group G, a principal G-bundle P, and a connection A with curvature $F \in \Omega^2_\Sigma(\mathfrak{g})$. The action is

$$S = \frac{1}{4e^2} \int_\Sigma \mathrm{tr}(F \wedge \star F). \tag{3.1}$$

Since we're in dimension 2, we can do nice things: write $f := \star F \in \Omega^0_\Sigma(P_\mathfrak{g})$;[2] then $F = f\mu$, where μ is the area form on Σ coming from the metric. Then we can rewrite (3.1) as

$$S = \frac{1}{4e^2} \int_\Sigma \mathrm{tr}(f^2)\, \mathrm{d}\mu. \tag{3.2}$$

This has a huge symmetry group, namely $\mathrm{Diff}^+(\Sigma)$, the orientation-preserving diffeomorphisms. In some sense, this means there's not a lot of interesting theory left.

Atiyah-Bott studied this classical action, and explained in detail how to get that the equations of motion are

$$\mathrm{d}_A(\star F) = 0. \tag{3.3}$$

[1] With a little care, one can generalize to nonoriented surfaces, where we replace area forms with densities and realizing F as a section of the orientation-twisted adjoint bundle.

[2] Here $P_\mathfrak{g}$ is the associated bundle $P \times_G \mathfrak{g}$, where $\mathfrak{g}$ carries the adjoint representation.

So we're looking for covariantly constant sections of $P_{\mathfrak{g}}$.

In general, for Yang-Mills theory in dimension d, we can assign a Hilbert space to a $(d-1)$-dimensional Riemannian manifold Y, which is $L^2(\mathcal{A}/\mathcal{G})$. Here $\mathcal{A}$ is the space of connections and $\mathcal{G}$ is the group of gauge transformations, which is $\mathrm{Map}(Y, G)$. This carries a nice inner product, so we can write L^2. In dimension 2, it suffices to understand what we attach to S^1, so we just get L^2 class functions on G. This admits a Peter-Weyl decomposition:

$$L^2(G/\!\!/G) = \left(\bigoplus_{V\in\mathrm{Irr}(G)} V\otimes\overline{V}\right)^G. \tag{3.4}$$

Here, there is one summand for every isomorphism class V of irreducible unitary representations of G. The Hilbert space structure is the natural one:

$$\langle f_1, f_2\rangle = \int_G \mathrm{d}x\, \overline{f_1(x)} f_2(x), \tag{3.5}$$

where $\mathrm{d}x$ is the Haar measure, normalized such that $\mathrm{vol}(G) = 1$.

Explicitly, the Hilbert space of wavefunctions is functions on the fields, so given a field (connection) A and a class function ψ,

$$\psi[A^a(x)] = \psi\left[P\exp\left(i\int_0^L \mathrm{d}x\, A(x)\right)\right]. \tag{3.6}$$

The Hamiltonian for this theory is

$$H = \frac{e^2}{2}\int_0^L \mathrm{d}x\, \frac{\delta}{\delta A_a(x)}\frac{\delta}{\delta A_a(x)}, \tag{3.7}$$

so given a state (class function) $|R\rangle$,

$$H|R\rangle = \frac{\lambda L}{2} C_2(R)|R\rangle, \tag{3.8}$$

where $\lambda = e^2 N$ is the *'t Hooft coupling*.

It's possible to completely solve 2D Yang-Mills theory and determine the correlation functions precisely. The idea is to put the theory on a lattice, and then show that it's invariant under RG flow, so that we can calculate on the lattice rather than trying to take a continuum limit. This is a very special thing, and doesn't happen in higher dimensions.

So, triangulate Σ (polygons with more edges are OK). Let $\mathcal{V}$ denote the set of vertices; the fields are functions $\mathcal{V} \to G$. The principal bundle also gives us holonomies $U_\gamma \in G$ associated to edges γ of Σ; if $\gamma\colon x$—y, then U_γ (TODO: ?) $g_y U_\gamma g_x^{-1}$. Now, instead of the exponentiated action, the partition function is a weighted sum of the plaquettes, as first written down by Migdal. First, the local contribution to the action is:

$$\Gamma(u, a_W) = \sum_{\alpha\in\mathrm{Irr}(G)} \dim\alpha\, \chi_\alpha(u)\exp\left(a_W\frac{C_2(\alpha)}{2}\right). \tag{3.9}$$

Here a_W is the area of the plaquette W, and u is the product of the elements of G on ∂W, in an order specified by the order. As $a_W \to 0$, corresponding to a finer triangulation (which ought to be a better approximation), this looks more like $\delta(u-1)$

The total partition function is the product of all of these: letting Π denote the triangulation and a be the area of Σ_g (so Π_1 is the edges and Π_2 is the plaquettes),

$$Z_{\Sigma,\Pi}(a) = \int \prod_{\gamma\in\Pi_1}\mathrm{d}u_\gamma \prod_{i\in\Pi_2}\Gamma(U_i, a_i). \tag{3.10}$$

We'd like to show that this doesn't depend on Π! It suffices to show this in the case where you subdivide a single plaquette in two. We'll do this for a square f area a_0 with edge elements $u_1, \dots, u_4$, which we subdivide into two triangles with areas a' and a'', so of course $a_0 = a' + a''$. The contribution of the square plaquette to the partition function is

$$\Gamma - \sum_\alpha \dim\alpha\, \chi_\alpha(u_1u_2u_3u_4)\exp\left(-a_0\frac{C_2(\alpha)}{2}\right), \tag{3.11}$$

and the contribution of the two triangles is

$$\Gamma'\Gamma'' = \sum_{\alpha,\beta} \dim\alpha \dim\beta \chi_\alpha(u_1u_2v)\chi_\beta(v^{-1}u_3u_4)\exp(\dots). \tag{3.12}$$

Here v is the group element on the new edge; the orientation convention means that in one triangle, it's counted as v, and in the other, it's v^{-1}. It suffices to show that

$$\int \mathrm{d}V\, \Gamma'\Gamma'' = \Gamma, \tag{3.13}$$

which follows from the purely group-theoretic fact that

$$\int \mathrm{d}V\, \chi_\alpha(Av)\chi_\beta(v^{-1}B) = \delta_{\alpha\beta}\frac{1}{\dim\alpha}\chi_\alpha(AB), \tag{3.14}$$

which is not too hard to prove.

So you can use as coarse of a triangulation as you like, and this calculates the same answer as a much finer triangulation which in general is a better approximation of the continuum limit. In particular, you can use the coarsest possible triangulation, describing the surface as its fundamental domain with edges identified in accordance with the fundamental group. Explicitly, then we get

$$Z_\Sigma(a) = \sum_{\alpha\in\mathrm{Irr}(G)} \dim\alpha \exp\left(-\frac{aC_2(\alpha)}{2}\right)\int \mathrm{d}U_i\,\mathrm{d}V_j\,\chi_\alpha(U_1V_1U_1^{-1}V_1^{-1}\dots) = \sum_\alpha \frac{e^{-aC_2(\alpha)/2}}{(\dim\alpha)^{2g-2}}. \tag{3.15}$$

You can do something similar with surfaces with boundaries, where each boundary is labeled by some $U_i \in G$. This multiplies the α term by a product of $\chi_\alpha(U_i)$ for each boundary component i.

Remark 3.16. So 2D Yang-Mills theory depends only on the area of the surface, not the metric. This is very special, and is almost topological, so you might ask whether this has an interpretation within the functorial field theory perspective of Atiyah (for topological field theory) or Segal (for conformal field theory). Indeed, there is a formalization of 2D Yang-Mills theory functorially due to Moore-Segal. ◀

Another way to study this is in the first-order formalism, where one writes the partition function as

$$S = -\frac{1}{2}\int \mathrm{tr}(BF) + \frac{1}{2}e^2\int \mathrm{tr}(B^2)\,\mu. \tag{3.17}$$

Because only the second term depends on area, you can envision trying to integrate it out, which leads to a topological field theory called *BF theory*, which is nice, e.g. its partition function computes the symplectic volume of a moduli space related to Σ_g. It's also very nice to consider the BV version of this theory, or say more refined things related to bordisms (e.g. what's the partition function on an interval?), which has something to do with boundary conditions. These modern perspectives are fairly explicit in 2D Yang-Mills theory; depending on your choice of boundary conditions, you can obtain dg bimodules of the form $\Lambda\mathfrak{g}^k$ or $\mathrm{Sym}\,\mathfrak{g}$, etc., and see simple examples of Koszul duality.

For more on BV and BFV 2D Yang-Mills, see a recent article of Mnev-Irasu.

If you add Wilson lines to the story, you can compute the partition function in an essentially similar way to the case of surfaces with boundary, or the cutting-and-gluing considerations we used to define the partition function in general.

First, non-intersecting Wilson lines $\{\Gamma\}$ labeled by representations R_Γ: $\Sigma \setminus \amalg\Gamma$ is a disjoint union of connected surfaces Σ_c, generally with boundary. Then the answer is

$$\left\langle \prod_\Gamma W(R_\Gamma,\Gamma)\right\rangle = \int \prod_\Gamma \mathrm{d}U_\Gamma \prod_c \left(Z_{\Sigma_c}\left(a_c, U_{c_\Gamma^2=C}, U_{C_\Gamma^r=C}\right)\prod_\Gamma W(R_\Gamma,\Gamma)\right). \tag{3.18}$$

Something in this (TODO: didn't follow) is an integral in terms of Clebsch-Gordon coefficients, which are purely group-theoretical: what are the irreducible components of $V_1\otimes V_2$, where V_1 and V_2 are given irreducibles? Anyways, this can be exactly solved, as can the case of intersecting Wilson lines, in which $6j$ symbols appear somehow, which has relationships to statistical mechanics models. The relationship between intersecting Wilson lines and integrable lattice models is more complicated in dimension 4, where it's studied in recent work of Costello, Witten, and Yamazaki.

Remark 3.19. Even though no Feynman diagrams were written down in this lecture, there are still interesting perturbative considerations here, especially if you thing about the 't Hooft coupling e^2N rather than the gauge coupling, particularly in the large-N limit. This has to do with Hurwitz numbers, counting branched covers, which feels like a baby string-theory (or Gromov-Witten theory), suggesting this is a baby string theory, and this is true. ◂

4. Chris Elliott: 4D Yang-Mills theory and asymptotic freedom: 6/20/19

Today in the TA session, Chris Elliott spoke about Yang-Mills theory, first classically from the BV formalism, then its relationship to the β function and asymptotic freedom, and then quantization.

First let's look at classical Yang-Mills theory on $\mathbb{R}^4$, with fermionic matter. Let G be a compact simple Lie group and V be a representation of G. The fields of Yang-Mills theory are:

- a *gauge field* $A \in \Omega^1_{\mathbb{R}^4}(\mathfrak{g})$, and
- a *spinor* $\psi \in \Omega^0_{\mathbb{R}^4}(S \otimes V)$. Here $S = S_+ \oplus S_-$ is the Dirac spinor bundle.[3]

There's a gauge group symmetry on this theory; the infinitesimal symmetries are given by $c \in \Omega^0_{\mathbb{R}^4}(\mathfrak{g})$, which acts on the gauge field by

$$A \longmapsto A + \mathrm{d}_A(c). \tag{4.1}$$

Fix a G-invariant pairing $V \times V \to \mathbb{R}$ and a positive operator $m\colon V \to V$, which we call the *mass matrix*. The *Yang-Mills action* is

$$S(A, \psi) := \int_{\mathbb{R}^4} \frac{1}{2}\|F_A\|^2 + \mu(\psi, (\not{\mathrm{d}}_A + m)\psi). \tag{4.2}$$

Here, if $\rho\colon \Omega^1_{\mathbb{R}^4} \otimes S \to S$ is the Dirac operator (or Clifford multiplication), then $\not{\mathrm{d}}_A\psi = \rho(\mathrm{d}_A\psi)$.

There is no possible choice for the gauge fixing operator Q^{GF}, because the BRST complex is

$$(\mathcal{E}, Q) = \left(\Omega^0_{\mathbb{R}^4} \xrightarrow{\ \mathrm{d}\ } \Omega^1_{\mathbb{R}^4} \xrightarrow{\ \mathrm{d}\star\mathrm{d}\ } \Omega^3_{\mathbb{R}^4} \xrightarrow{\ \mathrm{d}\ } \Omega^4_{\mathbb{R}^4} \right). \tag{4.3}$$

To deal with this, we'll rewrite the action in the first-order formalism: fix a self-dual $\mathfrak{g}$-valued 2-form B, i.e. $B \in \Omega^2_{\mathbb{R}^4}(\mathfrak{g})$ and $\star B = B$. The first-order action is

$$S_{\mathrm{FO}}(A, B, \psi) = \int_{\mathbb{R}^4} \langle F_A, B\rangle_{L^2} - \frac{1}{2}\|B\|^2 + \mu(\psi, (\not{\mathrm{d}}_A + m)\psi). \tag{4.4}$$

Then the complex of gauge fields (and ghosts, etc.) is (TODO: did I miss something with Ω^{2+} appearing twice?)

$$\Omega^0_{\mathbb{R}^4}(\mathfrak{g}) \xrightarrow{\ \mathrm{d}\ } \Omega^1_{\mathbb{R}^4}(\mathfrak{g}) \xrightarrow{\ \mathrm{d}_+\ } \Omega^{2+}_{\mathbb{R}^4}(\mathfrak{g}) \xrightarrow{\ \mathrm{d}\ } \Omega^3_{\mathbb{R}^4}(\mathfrak{g}) \xrightarrow{\ \mathrm{d}\ } \Omega^4_{\mathbb{R}^4}(\mathfrak{g}), \tag{4.5}$$

and the spinor field complex is

$$m + \not{\mathrm{d}}\colon \Omega^0_{\mathbb{R}^4}(S \otimes V) \longrightarrow \Omega^0_{\mathbb{R}^4}(S \otimes V). \tag{4.6}$$

In particular, we can write down the interacting term: let $C \in \Omega^0_{\mathbb{R}^4}(\mathfrak{g})$, $C^\vee \in \Omega^4_{\mathbb{R}^4}(\mathfrak{g})$, $A \in \Omega^1_{\mathbb{R}^4}(\mathfrak{g})$, $A^\vee \in \Omega^3_{\mathbb{R}^4}(\mathfrak{g})$, $B, B^\vee \in \Omega^{2+}_{\mathbb{R}^4}(\mathfrak{g})$, $\psi \in \Omega^0_{\mathbb{R}^4}(S \otimes V)$, and $\psi^\vee \in \Omega^0_{\mathbb{R}^4}(S \otimes V)$. Then we get

$$I = \langle B, [A \wedge A]\rangle + \mu(\psi, \not{A}\psi) + (A^\vee, [C, A]) + ([C, \psi], \psi^\vee) + ([C, C], C^\vee). \tag{4.7}$$

This is homotopy equivalent to the second-order Yang-Mills term coupled to a trivial B, in the not-fancy sense that there is a path of theories between them.

Now we want to perform BV quantization, which involves the following steps.

1. (a) Choose a gauge fixing operator Q^{GF}, such that $[Q, Q^{\mathrm{GF}}]$ is a generalized Laplacian.
   (b) Calculate the kernel K_t mollifying the kernel for $[Q, Q^{\mathrm{GF}}]$ in $\mathcal{E} \otimes \mathcal{E}$. This splits as a sum over "particle species", which are pairs $\alpha \otimes \alpha^\vee$ paired by the symplectic form.
   (c) Calculate the propagator

$$\int_\varepsilon^L \mathrm{d}t\, (Q^{\mathrm{GF}} \otimes 1)K_t = P(\varepsilon, L). \tag{4.8}$$

Again this should split into a sum as above over $\alpha \otimes \beta$, where $|\alpha| + |\beta| + 1 = 3$.

[3] If we were formulating this on some other 4-manifold M, we would have to choose a spin structure on M.

(2) Now we want to calculate $I[L]$. Our first step is to try

$$\lim_{\varepsilon\to 0} W(P(\varepsilon, L), I), \tag{4.9a}$$

though this will be divergent. So we choose a counterterm $I^{\mathrm{CT}}(\varepsilon)$ such that the modified limit

$$\widetilde{I}[L] := \lim_{\varepsilon\to 0} W(P(\varepsilon, L), I - I^{\mathrm{CT}}(\varepsilon)) \tag{4.9b}$$

exists; there are many ways to do this.

(3) Now, we try to solve the quantum master equation. This is unobstructed for free theories, but in general there is an obstruction (it will vanish in this case). We can try to solve this by adding some term J to $\widetilde{I}[L]$; J corresponds to a potential for the failure for $\widetilde{I}[L]$ to enter the quantum master equation.

Ok, so let's look at the renormalization group flow. In lecture today, Costello explained that $R_\lambda I[L] = I[L] + \log\lambda$ plus higher-order terms (in $\hbar$).

Definition 4.10. The *β-functional* at scale L is the observable

$$\mathscr{O}_B[L] := \frac{\mathrm{d}}{\mathrm{d}\log\lambda} R_\lambda I[L]\bigg|_{\lambda=1} \in \mathcal{O}_{loc}(\mathbb{R}^4)[[\hbar]]. \tag{4.11}$$

It's a general fact that for scale-invariant theories, $\lim_{L\to 0}\mathscr{O}_\beta^{(1)}[L]$ exists and is BV closed, and therefore its cohomology class is independent of choices of $I^{\mathrm{CT}}(\varepsilon)$, etc.

Definition 4.12. In this setting, the cohomology class $\beta^{(1)} := [\lim_{L\to 0}\mathscr{O}_\beta^{(1)}[L]]$ is called the *1-loop β-function*.

Theorem 4.13 (Gross-Wilczek-Politezer '73)**.** *For Yang-Mills theory,*

$$\beta^{(1)}(g) = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}C(\mathfrak{g}) - \frac{4}{3}C(V)\right), \tag{4.14}$$

One says that asymptotic freedom holds if $\beta^{(1)}$ is negative. For example, for SU_N with f fundamental flavors and n colors, $\beta^{(1)}$ is negative if $f < 11n/2$.

Remark 4.15. Recent work of Elliott-Yoo recovers important physical results such as Theorem 4.13 from the factorization-algebraic perspective. ◀

A scale- and translation-invariant theory is strictly renormalizable at one-loop. What this means is the following lemma.

Lemma 4.16. *$\mathscr{O}_\beta^{(1)}[L]$ is cohomologous to the log part of the one-loop counterterm $I^{\mathrm{CT}}_{\log\varepsilon}(\varepsilon)$.*

We want to find Q^{GF} which means finding arrows in the opposite direction in (4.5) and (4.6). The new arrows are in blue:

$$\Omega^0_{\mathbb{R}^4}(\mathfrak{g}) \underset{\mathrm{d}^*}{\overset{\mathrm{d}}{\rightleftarrows}} \Omega^1_{\mathbb{R}^4}(\mathfrak{g}) \underset{2\mathrm{d}^*}{\overset{\mathrm{d}_+}{\rightleftarrows}} \Omega^{2+}_{\mathbb{R}^4}(\mathfrak{g}) \underset{-2\mathrm{d}^*}{\overset{\mathrm{d}}{\rightleftarrows}} \Omega^3_{\mathbb{R}^4}(\mathfrak{g}) \underset{\mathrm{d}^*}{\overset{\mathrm{d}}{\rightleftarrows}} \Omega^4_{\mathbb{R}^4}(\mathfrak{g}), \tag{4.17}$$

and for (4.6),

$$\Omega^0_{\mathbb{R}^4}(S\otimes V) \underset{\not{d}-m}{\overset{m+\not{d}}{\rightleftarrows}} \Omega^0_{\mathbb{R}^4}(S\otimes V). \tag{4.18}$$

Then $[Q, Q^{\mathrm{GF}}] = \Delta + D_{\mathrm{vert}}$, where D_{vert} gives arrows $-2\mathrm{d}^*\colon \Omega^{2+}_{\mathbb{R}^4}(\mathfrak{g}) \to \Omega^1_{\mathbb{R}^4}(\mathfrak{g})$ and $-2\mathrm{d}^*\colon \Omega^3_{\mathbb{R}^4}(\mathfrak{g}) \to \Omega^{2+}_{\mathbb{R}^4}(\mathfrak{g})$.

The heat kernel K_t splits into a sum proportioned to the scalar

$$k_t(x,y) = \frac{1}{(4\pi t)^2}\exp\left(\frac{-|x-y|^2}{4t}\right) \tag{4.19}$$

and

$$\mathcal{E} := C^\infty(\mathbb{R}^4)\otimes(Y\otimes\mathfrak{g}\oplus\mathcal{S}\otimes V), \tag{4.20}$$

where $Y\otimes\mathfrak{g}$ is the pure part and $\mathcal{S}\otimes V$ is the matter. The kernel ends up being

$$K_t = K_{AA^\vee} + K_{BB^\vee} + K_{CC^\vee} + K_{\psi\psi^\vee}. \tag{4.21}$$

This is about all that we can understand without actually computing some Feynman diagrams. The propagator applies $Q^{\mathrm{GF}} \otimes 1$ to K_t.

Lemma 4.22. *The propagator has the form*

$$P(\varepsilon, L) = \int_\varepsilon^L \mathrm{d}t \left(\frac{\partial k_t}{\partial x^i}(x,y)\big(P^i_{AB} + P^i_{A^\vee C}\big) + \frac{\partial^2 k_t}{\partial x^i \partial x^j} P^{ij}_{AA} + \frac{\partial k_t}{\partial x^i} P^i_{\psi\psi} \right). \tag{4.23}$$

The P^i_{AB} term corresponds to an edge A—B; $P^i_{A^\vee C}$ corresponds to $A^\vee$—C, P^{ij}_{AA} to A—A, and $P^i_{\psi\psi}$ to ψ—ψ.

Let Γ_k denote a one-loop Feynman diagram with k vertices.

Lemma 4.24. *The weight associated to Γ_k has no $\log(\varepsilon)$ divergence unless $k = 2$.*

When $k = 2$, there are only a few Feynman diagrams we can get (which (TODO) I wasn't able to parse or TeX down), though the one with ψ on each open edge, AA on one full edge, and $\psi\psi$ on the other is BV-exact, so we can throw it out.