M392C NOTES: TOPICS IN ALGEBRAIC TOPOLOGY

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These notes were taken in UT Austin's M392C (Topics in Algebraic Topology) class in Spring 2017, taught by Andrew Blumberg. I live-TEXed them using vim, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

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Lecture 1.

G-spaces: 1/17/17

This class will be an overview of equivariant stable homotopy theory. We're in the uncomfortable position where this is a big subject, a hard subject, and one that is poorly served by its textbooks. Algebraic topology is like this in general, but it's particularly acute here. Nonetheless, here are some references:

- Adams, "Prerequisites (on equivariant stable homotopy) for Carlsson's lecture." This is old, and some parts of it don't reflect how we do things now.
- The Alaska notes, edited by May, is newer, and is written by many authors. Some of it is a grab bag, and some parts (e.g. the rational equivariant bits) aren't entirely right. It's also not a textbook.
- Appendix A of Hill-Hopkins-Ravenel. This is a paper which resolved an old conjecture on manifolds using equivariant stable homotopy theory, but let this be a lesson on referee reports: the authors were asked to provide more background, and so wrote a 150-page appendix on this material. Their suffering is your gain: the introduction is well-written, albeit again not a textbook.

In the world of the professor, there are two major applications of equivariant stable homotopy theory.

- The first is trace methods in algebraic K-theory: Hochschihd homology and its topological cousins are equipped with natural S^1-actions (the same S^1-action coming from field theory). This is how people other than Quillen compute algebraic K-theory.
- The other major application is Hill-Hopkins-Ravenel's settling of the Kervaire invariant 1 conjecture.

The nice thing is, however you feel about the applications, both applications require developing new theory in equivariant stable homotopy theory. Hill-Hopkins-Ravenel in particular required a clarification of the foundations of this subject which has been enlightening.

In this class, we hope to cover the foundations of equivariant stable homotopy theory. On the one hand, this will be a modern take, insofar as we emphasize the norm and the presheaf on orbit categories (these will be explained in due time), the modern emerging consensus on how to think of these things, different than what's written in textbooks. The former is old, but has gained more attention recently; the latter is new. Moreover, there's an increasing sense that a lot of the foundations here are best done in ∞-categories. We will not take this appoach in order to avoid getting bogged down in ∞-categories; moreover, this class is supposed to be rigorous. It will sometimes be clear to some people that ∞-categories lie in the background, but we won't talk very much about them.

We'll cover some old topics such as Smith theory and the Segal conjecture, and newer ones such as trace methods and Hill-Hopkins-Ravenel, depending on student interest. We will not have time to discuss many topics, including equivariant cobordism or equivariant surgery theory.

Prerequisites. If you don't know these prerequisites, that's okay; it means you're willing to read about it on your own.

- Foundations of unstable homotopy theory at the level of May's *A Concise Course in Algebraic Topology*. For example, we'll discuss equivariant CW complexes, so it will help to know what a CW complex is.
- A little bit of category theory, e.g. found in Mac Lane.
- This class will not require much in the way of simplicial methods (simply because it's hard to reconcile simplicial methods with non-discrete Lie groups), but you will want to know the bar construction.
- A bit of abstract homotopy theory, e.g. what a model structure is.

If you don't know these, feel free to ask the professor for references. His advisor suggested that a foundation for the stable category is Lewis-May-Steinberger's account of the equivariant category and let $G = *$, but perhaps this isn't necessarily a good reference for nontrivial groups.

Unstable equivariant questions are very natural, and somewhat reasonable. But stable questions are harder; they ultimately arise from reasonable questions, but the formulation and answers are hard: even discussing the equivariant analogue of $\pi_0 S^0$ requires some representation theory — and yet of course it should. Thus there's a lot of foundations behind hard calculations. There will be problem sets; if you want to learn the material (or are an undergrad), you should do the problem sets.

Categories of topological spaces. The category of topological spaces we consider is Top, the category of compactly generated, weak Hausdorff spaces (and continuous maps); we'll also consider Top_*, the category of based, compactly generated, weak Hausdorff spaces and continuous, based maps. This is an important and old trick which eliminates some pathological behavior in quotients. It's reasonable to imagine that point-set topology shouldn't be at the heart of foundational issues, but there are various ways to motivate this, e.g. to make Top more resemble a topos or the category of simplicial sets.

Definition 1.1. Let X be a topological space.

- X is **compactly generated** if $A \subseteq X$ is **compactly closed** (i.e. $f^{-1}(A)$ is closed for every $f : Y \to X$, where Y is compact and Hausdorff), then A is closed.
- X is **weak Hausdorff** if the diagonal map $\Delta : X \to X \times X$ is closed when $X \times X$ has the compactly generated topology.

The intuition behind compact generation is that the topology is determined by compact Hausdorff spaces. The weak Hausdorff topology is strictly stronger than T_1 (points are closed), but strictly weaker than Hausdorff spaces. Any space you can think of without trying to be pathological will meet these criteria.

There is a functor k from all spaces to compactly generated spaces which adds the necessary closed sets. This has the unfortunate name of **k-ification** or **kaonification**; by putting the compactly generated topology on $X \times X$, we mean taking $k(X \times X)$. There's also a "weak Hausdorffification" functor w which makes a space weakly Hausdorff, which is some kind of quotient.[1]

When computing limits and colimits, it's often possible to compute it in the category of spaces and then apply k and w to return to Top. This is fine for limits, but for colimits, w is particularly badly behaved: you cannot compute the colimit in Top by computing it in Set and figuring out the topology; more generally, it will be some kind of quotient.

Nonetheless, there are nice theorems which make things work out anyways.

Proposition 1.2. *Let $Z = \operatorname{colim}(X_0 \to X_1 \to X_2 \to \dots)$ be a sequential colimit (sometimes called a **telescope**); if each X_i is weak Hausdorff, then so is Z.*

Proposition 1.3. *Consider a diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \\ C & \end{tikzcd}$$

where f is a closed inclusion. If A, B, and C are weakly Hausdorff, then $B \amalg_A C$ is weakly Hausdorff.

[1] The k functor is right adjoint to the forgetful map, which tells you what it does to limits.

These are the two kinds of colimits people tend to compute, so this is reassuring.

One reason we require regularity on our topological spaces is the following, which is not true for topological spaces in general.

Lemma 1.4. *Let X, Y, and Z be in* Top*; then, the natural map*

$$\mathrm{Map}(X \times Y, Z) \longmapsto \mathrm{Map}(X, \mathrm{Map}(Y, Z))$$

is a homeomorphism.

Enrichments. The categories Top and Top_* are enriched over themselves (as will categories of G-spaces, which we'll see later). This means a brief digression into enriched categories.

Definition 1.5. Let $(\mathsf{V}, \otimes, \iota)$ be a symmetric monoidal category.[2] Then, an **enrichment** of a category C over V means

- for every $x, y \in \mathsf{C}$, there is a hom-object $\underline{\mathsf{C}}(x, y)$, which is an object in V,
- for every $x \in \mathsf{C}$, there is a unit $1 \to \underline{\mathsf{C}}(x, x)$,
- composition $\underline{\mathsf{C}}(x, y) \otimes \underline{\mathsf{C}}(y, z) \to \underline{\mathsf{C}}(x, z)$ is associative and unital, and
- the underlying category is recovered as $\mathsf{C}(x, y) = \mathrm{Map}(1, \underline{\mathsf{C}}(x, y))$.

A great deal of category theory can be generalized to enriched categories, including V-enriched functors, V-enriched natural transformations, V-enriched limits and colimits, and more. The canonical reference is Kelly's notes on enriched category theory, available free and legally online. It covers just about everything we need except for the Day convolution, which can be read from Day's thesis.

Definition 1.6. Let C and D be enriched over V. Then, an **enriched functor** $F : \mathsf{C} \to \mathsf{D}$ is an assignment of objects in C to objects in D and maps $\underline{\mathsf{C}}(x, y) \to \underline{\mathsf{D}}(Fx, Fy)$ that are V-morphisms, and commute with composition.

Exercise 1.7. Work out the definition of enriched natural transformations.

This brings us to the beginning.

Let G be a group. We'll generally restrict to finite groups or compact Lie groups; this is not because these are the only interesting groups, but rather because they are the only ones we really understand. If you can come up with a good equivariant homotopy theory for discrete infinite groups, you will be famous. Throughout, keep in mind the examples C_p (the cyclic group of order p, sometimes also denoted $\mathbb{Z}/p$), C_{p^n}, the symmetric group S_n, and the circle group S^1.

There's a monad M_G on Top which sends $X \mapsto G \times X$, and analogously M_G^* on Top_* sending $X \to G_+ \wedge X$; then, one can define the category of G**-spaces** $G\mathsf{Top}$ (resp. **based** G**-spaces** $G\mathsf{Top}_*$) to be the category of algebras over M_G (resp. M_G^*). This is probably not the most explicit way to define G-spaces, but it makes it evident that $G\mathsf{Top}$ and $G\mathsf{Top}_*$ are complete and cocomplete.

More explicitly, $G\mathsf{Top}$ is the category of spaces $X \in \mathsf{Top}$ equipped with a continuous action $\mu : G \times X \to X$. That is, μ must be associative and unital. Associativity is encoded in the commutativity of the diagram

$$\begin{tikzcd} G \times G \times X \arrow[r, "1\times\mu"] \arrow[d, "m"] & G \times X \arrow[d, "\mu"] \\ G \times X \arrow[r, "\mu"] & X. \end{tikzcd}$$

The morphisms in $G\mathsf{Top}$ are the G**-equivariant** maps $f : X \to Y$, i.e. those commuting with μ:

$$\begin{tikzcd} G \times X \arrow[r] \arrow[d, "\mu_X"] & G \times Y \arrow[d, "\mu_Y"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

It's possible (but not the right idea) to let $\underline{G}$ denote the category with an object $*$ such that $\underline{G}(*, *) = G$. Then, $G\mathsf{Top}$ is also the category of functors $\underline{G} \to \mathsf{Top}$, with morphisms as natural transformations. This realizes $G\mathsf{Top}$ as a **presheaf category**; it will eventually be useful to do something like this, but not in this specific way.

[2]Briefly, this means V has a tensor product $\otimes$ and a unit ι; there are certain axioms these must satisfy.

When we write $\mathrm{Map}(X,Y)$ in $G\mathsf{Top}$ or $G\mathsf{Top}_*$, we could mean three things:

(1) The set of G-equivariant maps $X \to Y$.
(2) The space of G-equivariant maps $X \to Y$ in the subspace topology of all maps from $X \to Y$. As this suggests, $G\mathsf{Top}$ admits an enrichment over Top (resp. $G\mathsf{Top}_*$ admits an enrichment over Top_*).
(3) The G-space of all maps $X \to Y$, where G acts by conjugation: $f \mapsto g^{-1}f(g\cdot)$. This means $G\mathsf{Top}$ is enriched in itself, as is $G\mathsf{Top}_*$.

Each of these is useful in its own way: for constructions it may be important to be self-enriched, or to only look at G-equivariant maps. We will let $G\,\mathrm{Map}(X,Y)$ denote (2) or its underlying set (1), and $\mathrm{Map}(X,Y)$ or $\mathrm{Map}^G(X,Y)$ denote (3).

It turns out you can recover $G\,\mathrm{Map}$ from Map: the equivariant maps are the fixed points under conjugation of all maps. This is written $\mathrm{Map}(X,Y)^G = G\,\mathrm{Map}(X,Y)$.

Throughout this class, "subgroup" will mean "closed subgroup" unless specified otherwise.

Definition 1.8. Let X be a G-set and $H \subseteq G$ be a subgroup. Then, the **H-fixed points** of X is the space $X^H := \{x \in X \mid hx = x \text{ for all } h \in H\}$. This is naturally a WH-space, where $WH = NH/H$ (here NH is the normalizer of H in G).[3]

Definition 1.9. The **isotropy group** of an $x \in X$ is $G_x := \{h \in G \mid hx = x\}$.

These are useful in the following two ways.

(1) Often, it will be helpful to reduce questions from $G\mathsf{Top}$ to Top using $(-)^H$.
(2) It's also useful to induct over isotropy types.

Now, we'll see some examples of G-spaces.

Example 1.10. Let H be a subgroup of G; then, the **orbit space** G/H is a useful example, because it corepresents the fixed points by H. That is, $X^H = \mathrm{Map}^G(G/H,X)$. These spaces will play the role that points did when we build things such as equivariant CW complexes. ◁

Example 1.11. Let $H \subset G$ as usual and $U : G\mathsf{Top} \to H\mathsf{Top}$ be the forgetful functor. Then, U has both left and right adjoints:

- The left adjoint sends $X \mapsto G \times_H X$ (in the based case, $X \mapsto G_+ \wedge_H X$). G acts via the left action on G. This is called the **induced G-action** on X.
- The right adjoint is $F_H(G,X)$ (or $F_H(G_+,X)$ in the based case), the space of H-maps $G \to X$, with G-action $(gf)(f') = f(g'g)$. This is called the **coinduced G-action** on H. ◁

Example 1.12. Let V be a finite-dimensional real representation of G, i.e. a real inner product space on which G acts in a way compatible with the inner product. (This is specified by a group homomorphism $G \to \mathrm{O}(V)$.) The one-point compactification of V, denoted S^V, is a based G-space; the unit disc $D(V)$ and unit sphere $S(V)$ are unbased spaces, but we have a quotient sequence

$$S(V)_+ \longrightarrow D(V)_+ \longrightarrow S^V.$$

If $V = \mathbb{R}^n$ with the trivial G-action, S^V is S^n with the trivial G-action, so these generalize the usual spheres; thus, these S^V are called **representation spheres**. ◁

We will let S^n denote $S^{\mathbb{R}^n}$, namely our preferred model for the n-sphere with trivial G-action.

Beginnings of homotopy theory.

Definition 1.13. A **G-homotopy** is a map $h : X \times I \to Y$ in $G\mathsf{Top}$, where G acts trivially on I. We generally think of it, as usual, as interpolating between $h(-,0)$ and $h(-,1)$. This is the same data as a path in $G\,\mathrm{Map}(X,Y)$. A **G-homotopy equivalence** between X and Y is a map $f : X \to Y$ such that there exists a $g : Y \to X$ such that there are G-homotopies $gf \sim \mathrm{id}_X$ and $fg \sim \mathrm{id}_Y$.

The (well, a) natural question that might arise: what are G-weak equivalences and G-CW complexes? This closely relates to obstruction theory — CW complexes are test objects.

To define G-CW complexes, we need cells. One choice is $G/H \times D^{n+1}$ and $G/H \times S^n$, where the actions on D^{n+1} and S^n are trivial. This is a plausible choice (and in fact, will be the right choice), but it's not clear why — why not

[3] If $H \trianglelefteq G$, then X^H is also a G/H-space.

$G \times_H D(V)$ or $G \times_H S(V)$ for some H-representation V? Ultimately, this comes from a (quite nontrivial) theorem that these can be triangulated in terms of the cells $G/H \times D^{n+1}$ and $G/H \times S^n$.[4] This is one of several triangulation results proven in the 1970s, which are now assumed without comment, but if you like this kind of math then it's a very interesting story.

Definition 1.14. A G**-CW complex** is a sequential colimit of spaces X_n, where X_{n+1} is a pushout

$$\begin{tikzcd}
\bigvee G/H \times S^n \arrow[r] \arrow[d] & X_n \arrow[d] \\
\bigvee G/H \times D^{n+1} \arrow[r] & X_{n+1}.
\end{tikzcd}$$

That is, it's formed by attaching cells just as usual, though now we have more cells.

This immediately tells you what the homotopy groups have to be: $[G/H \times S^n, X]$, which by an adjunction game is isomorphic to $\pi_n(X^H)$. We let $\pi_n^H(X) := \pi_n(X^H)$. Thus, we can define weak equivalences.

Definition 1.15. A map $f : X \to Y$ of G-spaces is a **weak equivalence** if for all subgroups $H \subset G$, $f_* : \pi_n^H(X) \to \pi_n^H(Y)$ is an isomorphism.

These homotopy groups have a more complicated algebraic structure: they're indexed by the lattice of subgroups of G and the integers. This is fine (you can do homological algebra), but some things get more complicated, including asking what the analogue of connectedness is!

One quick question: do we need all subgroups H? What if we only want finite-index ones? The answer, in a very precise sense, is that if you're willing to use fewer subgroups, you get fewer cells $G/H \times S^n$, and that's fine, and you get a different kind of homotopy theory.

Finally, the Whitehead theorem is true for G-CW complexes. This follows for the same reason as in May's course: it follows word-for-word after proving the equivariant HELP lemma (homotopy extension lifting property), which is true by the same argument.

We'll next talk about presheaves on the orbit category, leading to Bredon cohomology.

[4]Illman's thesis is a reference, albeit not the most accessible ones.