GEOMETRY AND STRING THEORY SEMINAR: SPRING 2018

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Contents

1. Comments on Global Symmetry, Anomalies, and Duality in $(2+1)d$: 1/24/18

Today's talk was given by Val Zakharevich, on the paper [BHS17].

Definition 1.1. Let A and B be UV theories which, under renormalization group flow, flow to the same IR theory C. Then we'll say that theories A and B are dual.

Example 1.2. Let $N_f \leq N$. Then there is a conjectured duality between $\mathrm{SU}(2)_k$-Chern-Simons theory with N_f scalars, also known as *Wilson-Fischer theory*, and $\mathrm{U}(k)_{-N+N_f/2}$-Chern-Simons theory with N_f fermions. ◂

The paper [BHS17] computes the higher symmetries and anomalies of both sides of this duality and of several others; 't Hooft anomaly matching tells us that these should be the same.

This is related to our overarching goal of understanding QCD_4 with a single fermion ψ, which has a Lagrangian

$$\mathcal{L} = \mathrm{tr}(F \wedge \star F) + \overline{\psi} \not{D} \psi + m\overline{\psi}\psi, \tag{1.3}$$

where $m \in \mathbb{C}$ is a parameter whose phase diagram we're interested in. Let $\overline{m}$ denote the mass of the domain wall theory, which is a 3D QCD theory, so $\overline{m}$ is real. If m is real and negative, there's a phase transition: for $\overline{m} \ll 0$, the low energy theory is believed to be trivial, and for $\overline{m} \gg 0$ (m negative of larger magnitude), the low-energy theory is believed to be $\mathrm{SU}(N)$-Chern-Simons theory at level 1. The transition point, at $\overline{m} = 0$, should be described by $\mathrm{SU}(N)_{1/2}$ with a single fermion.

1.1. Level-rank duality.

Level-rank duality is the conjecture that $\mathrm{SU}(N)_k$-Chern-Simons theory and $\mathrm{U}(k)_{-N}$-Chern-Simons theory are isomorphic. A natural generalization is to consider $\mathrm{SU}(N)_k$ together with N_f scalar fields of mass m, where $m \in \mathbb{R}$ and $N_f < N$.

- If $m \ll 0$, the Higgs mechanism implies this should be the $\mathrm{SU}(N - N_f)_k$ theory.
- if $m \gg 0$, we should expect the $\mathrm{SU}(N)_k$ theory again.

On the dual side, let's consider $\mathrm{U}(k)_{-N-N_f/2}$ with N_f fermions of mass $m \in \mathbb{R}$.

- If $m \gg 0$, we expect to get $\mathrm{U}(k)_{-N+N_f/2}$.
- If $m \ll 0$, we expect to get $\mathrm{U}(k)_{-N}$, with Lagrangian shifted by N_f:

$$\mathcal{L} = \frac{-N + N_f}{4\pi} \mathrm{tr}\left(b\,\mathrm{d}b - \frac{2i}{3} b^3\right) + \overline{\psi}\not{D}\psi + m\psi\overline{\psi} + \text{c.c.} \tag{1.4}$$

Level-rank duality switches positive-mass scalars and negative-mass fermions, promising dualities between $\mathrm{SU}(N - N_f)_k \longleftrightarrow \mathrm{U}(k)_{-N+N_f/2}$ and $\mathrm{SU}(N)_k \longleftrightarrow \mathrm{U}(k)_{-N}$.

1.2. **Symmetries.** We now see what symmetries these theories have. First, $\mathrm{SU}(N)_k$ with N_f scalars. On a 3-manifold M, the fields are triples (P, A, Φ), where

- $P \to M$ is a principal $\mathrm{SU}(N)$-bundle with connection,
- A is a connection on P, and
- $\varphi \in \Gamma(P \times_{\mathrm{SU}(N)} (\mathbb{C}^N \otimes \mathbb{C}^{N_f}))$ is the N scalar fields.

The Lagrangian is

$$\mathcal{L}(A, \Phi) = \frac{k}{4\pi} \operatorname{tr}\left(A \wedge \mathrm{d}A + \frac{2}{3} A^3 \right) + |D_A \varphi|^2 + m|\varphi|^2 + \lambda |\varphi|^4. \tag{1.5}$$

As usual, we have an $\mathrm{SU}(N)$-gauge symmetry, and there's also a $\mathrm{U}(N_f)$-symmetry acting on $\mathbb{C}^{N_f}$, in which $e^{2\pi i/N}\mathbf{1}$ acts by a gauge symmetry. Hence the global symmetry group (for these symmetries) is $\mathrm{U}(N_f)/(\mathbb{Z}/N)$.

Ansatz 1.6. Let G be a compact Lie group and k be a level for G, and let $\mathcal{L}_{G_k}$ denote the Lagrangian for Chern-Simons theory with group G and level k. Let

$$1 \longrightarrow G \xrightarrow{\ \rho\ } H \xrightarrow{\ \sigma\ } L \longrightarrow 1 \tag{1.7}$$

be a short exact sequence of Lie groups. Then, we take as an ansatz that coupling the G_k theory to a principal L-bundle (i.e. given a principal L-bundle $P \to M$, we sum over the groupoid of all principal H-bundles which quotient to L) produces a classical gauge theory for H with Lagrangian $\mathcal{L}_{\widetilde{k}}$ such that

$$\mathcal{L}_{G_k}(P_G, A_G) = \mathcal{L}_{\widetilde{k}}((P_G, A_G) \times_G H). \tag{1.8}$$

When G is finite (so we're in the setting of Dijkgraaf-Witten theory) this is studied in [KT14].

In our setting, (1.7) specializes to $G = \mathrm{SU}(N)$, $H = (\mathrm{SU}(N) \times U(N_f))/(\mathbb{Z}/N)$, and $L = U(N_f)/(\mathbb{Z}/N)$. Chern-Simons theories for G are labeled by $H^4(BG; \mathbb{Z})$, and the map $\rho\colon G \to H$ defines a pullback

$$\rho^*\colon H^4(BH; \mathbb{Z}) \longrightarrow H^4(BG; \mathbb{Z}).$$

Given a $k \in H^4(BG; \mathbb{Z})$, we want to know whether we can implement the theory with a global L-symmetry; hence we want to know whether $k \in \mathrm{Im}(\rho^*)$; the theory is anomalous iff this is not true.

If the theory is anomalous, we'd like to compute the anomaly. Suppose that we have a $\widehat{k}_{\mathbb{R}} \in H^4(BH; \mathbb{R})$ such that $\rho^*(\widehat{k}_{\mathbb{R}}) = k_{\mathbb{R}}$ (i.e. the image of k in real cohomology). Then, we can't eliminate the anomaly, but we can couple to a bulk theory: suppose that we can extend $(P, A) \to M$ to $(P_H, A_H) \to X$, where X is a compact 4-manifold with $\partial X = M$. Then, we have an action

$$\text{“}S_{\widehat{k}_{\mathbb{R}}}(A_H)\text{”}\colon ((P_H, A_H) \to X) \longmapsto \int_X \widehat{k}_{\mathbb{R}}(F_H), \tag{1.9}$$

where F_H is the curvature of A_H.

This depends on the choice of X and $P_H \to X$ extending P, but we can hope that the dependence goes away after exponentiating the action. Let X' be another compact 4-manifold bounding M, and let $(P'_H, A') \to X'$ be another extension of (P, A). Let $\widehat{X} \coloneqq X \cup_M X'$; then, (P_H, A_H) and (P'_H, A'_H) glue to a principal G-bundle $\widetilde{P}_H \to \widetilde{X}$ with connection $\widetilde{A}$. Then we have that

$$e^{2\pi i S_{\widehat{k}_{\mathbb{R}}}(\widetilde{P}_H, \widetilde{A}_H)} = S_{\hat{\hat{k}}}(\widetilde{P}_H \times_H L) \in \mathbb{R}/2\pi i \mathbb{Z} \tag{1.10}$$

for some $\hat{\hat{k}} \in H^4(BL; \mathbb{R}/\mathbb{Z})$. This $\hat{\hat{k}}$ tells us the anomaly, so we're interested in computing it. Ultimately, this comes from a question purely in algebraic topology: we have a big commutative diagram

$$\begin{tikzcd}
H^4(BL; \mathbb{R}/\mathbb{Z}) \arrow[r] & H^4(BH; \mathbb{R}/\mathbb{Z}) \arrow[r] & H^4(BG; \mathbb{R}/\mathbb{Z}) \\
 & H^4(BH; \mathbb{R}) \arrow[r] \arrow[u] & H^4(BG; \mathbb{R}) \arrow[u] \\
 & H^4(BH; \mathbb{Z}) \arrow[r] \arrow[u] & H^4(BG; \mathbb{Z}) \arrow[u]
\end{tikzcd} \tag{1.11}$$

Then we have $\hat{\hat{k}}$ in the upper left, $\widehat{k}_{\mathbb{R}}$ in the middle, and k in the lower right. In this case the anomaly theory is purely topological. The computation for the dual theory follows a similar story, but is harder.

To actually calculate this, you can use the Leray-Serre spectral sequence; $k \in H^4(BG;\mathbb{Z})$ transgresses to something in $H^5(BL;\mathbb{Z})$, which tells you which component $\hat{\hat{k}}$ is in.

2. On Gauging Finite Subgroups: 1/31/18

Today's talk was given by Dan Freed, on Tachikawa's paper [Tac17].

Let's start with classical electromagnetism on n-dimensional Minkowski spacetime $\mathbb{M}$. Choose an $A \in \Omega^1_{\mathbb{M}}/\mathrm{d}\Omega^0_{\mathbb{M}}$, and let $F_A = \mathrm{d}A$. Maxwell's laws tell us that

$$\begin{aligned} \mathrm{d}F_A &= 0 \\ \mathrm{d}{\star}F_A &= 0, \end{aligned}$$

but more generally we could let $\mathrm{d}F_A = j_B \in \Omega^3$, a *magnetic current*, and $\mathrm{d}{\star}F_A = j_E \in \Omega^{n-1}$, an *electric current*. If both $j_E$and j_B are nonzer, the theory has an anomaly.

We next consider the quantum theory, by doing some things such as Wick rotation, charge quantization, and downshifting the degree of A.[1] The Wick-rotated quantum theory is formulated on an oriented[2] Riemannian manifold X, and A is a map $X \to \mathbb{R}\mathbb{Z}$, or its exponentiated version $\lambda\colon X \to \mathbb{T}$.

If we introduce point charges $p_1, \ldots, p_m \in X$ with charges $k_1, \ldots, k_m \in \mathbb{Z}$,[3] then the electric current, inserted in the exponentiated action, is

$$\prod_{j=1}^m \lambda(p_j)^{k_j} = \exp\left(2\pi \sum_{j=1}^m i k_j A(p_j)\right), \tag{2.1}$$

which has degree n with $\mathbb{Z}$ coefficients.

The magnetic current is defined using a circle bundle $P \to X$ with connection Θ; one can think of λ as a section of P, and this data is used to define the kinetic term. This is a degree-2 term with $\mathbb{Z}$ coefficients.

If $L := P \times_{\mathbb{T}} \mathbb{C}$ is the complex line bundle associated to P, then the *electric coupling* is

$$\prod_{i=1}^m \lambda(p_i)^{k_i} \in \bigotimes_{i=1}^m L_{p_i}^{\otimes k_i}. \tag{2.2}$$

From this perspective, the anomaly is "$\int_X j_B \cdot j_E$."

Remark. The term λ only exists if $P \to X$ is topologically trivializable.

This is akin to something that happens in topological field theory. Let Z denote the 4D oriented TQFT defined by summing the trivial theory over spin structures. Then $Z(\mathbb{CP}^2) = 0$, since $\mathbb{CP}^2$ admits no global spin structure. But if one varies the manifold in a family, interesting things may nonetheless happen. ◀

Remark. One could also replace $\mathbb{T}$ by any finite abelian group A. In this case a lot of things are still the same, though we don't choose a connection for P. In this case, the magnetic current lives in $H^1(X;A)$ rather than $H^2(X;\mathbb{Z})$ (and we could have thought of it as $H^1(X;\mathbb{T})$ for the $\mathbb{T}$-theory). However, the electric current lives in $H^n(X;A^\vee)$, where $A^\vee := \mathrm{Hom}(A,\mathbb{T})$ denotes the Pontrjagin dual of A.

We could think of the magnetic current as a map $X \to BA$; in this case the electric current is a map $X \to B^n A^\vee := K(A^\vee, n)$. In the rest of this talk, we will adopt this more abstract approach, but you should keep the rigid, geometric approach that we started with in mind for intuition or an example. ◀

From this perspective, the anomaly is a map

$$BA \times B^n A^\vee \longrightarrow B^{n+1}\mathbb{T},$$

which is induced from the pairing $A \otimes A^\vee \to \mathbb{T}$. On a closed, oriented manifold X with an electric and magnetic current we can pull this back to X and integrate it; this is the partition function for the anomaly theory.

[1] TODO: I have no idea what just happened.

[2] One could impose time-reversal symmetry and study the theory more generally on unoriented manifolds, but for our purposes this will not be necessary.

[3] TODO: I may have gotten this wrong.

Gauging. Suppose T is some kind of theory (here we probably mean a Wick-rotated field theory on Riemannian manifolds, perhaps with extra structure and background fields), and suppose it has a (global) Γ-symmetry,[4] where Γ is a finite group. This means that we can couple the theory to Γ-bundles, formulating it on manifolds with the above data and a background principal Γ-bundle.

Sometimes this symmetry gets tangled up with other symmetries, e.g. if T is a σ-model to a space X and Γ is a symmetry of X. Then, depending on how we implement the symmetry, we might end up with sections of some associated bundle.

But if this is not the case (in a σ-model sense, if $B\Gamma$ splits off from the target), then we can sum over the maps to $B\Gamma$. This process is called *gauging*.

Now suppose $A \coloneqq \Gamma$ is abelian. Then the gauged theory has a higher symmetry akin to electromagnetism, a $B^{n-2}A^\vee$ symmetry, and we can couple the theory to a background $B^{n-2}A^\vee$-field, which is exactly putting in the electric current. If you try to gauge this symmetry, you'll end up back where you started with, which is a kind of Fourier transform.

On a compact oriented manifold X, the electric coupling lives in $H^1(X;A) \times H^{n-1}(X;A^\vee)$; there's a product map

$$H^1(X;A) \times H^{n-1}(X;A^\vee) \longrightarrow H^n(X;\mathbb{T});$$

then we can evaluate on the fundamental class to obtain an element of $\mathbb{T}$, which is what one inserts into the action. This exhibits the two cohomology groups as Pontrjagin duals of each other, so the Fourier transform is an isomorphism between spaces of functions on them.

Turning to the material in the paper, let

$$1 \longrightarrow A \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

be a short exact sequence of groups, where A is abelian. In particular, A is normal in Γ; we do not assume it is central. An example (in which A is not central) is

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow S_3 \longrightarrow \mathbb{Z}/2 \longrightarrow 1.$$

We consider the situation of a theory T with a Γ-symmetry, hence an A-symmetry, and we assume we can gauge A. What happens when we do this?

The new theory should have a G-symmetry, which arises as follows: given a principal G-bundle $Q \to X$, we can sum over pairs $(P \to X, \varphi)$, where $P \to X$ is a principal Γ-bundle and

$$\begin{tikzcd} Q \arrow[rr, "\varphi", "\cong"'] \arrow[dr, "G"'] & & P/A \arrow[dl, "G"] \\ & X & \end{tikzcd}$$

is an isomorphism of principal G-bundles. In this case the magnetic current arises from the map $BG \to B^2A$ coming from extending the fiber sequence $BA \to B\Gamma \to BG$.

There's another symmetry associated to $B^{n-2}A^\vee$, which arises in a similar way as before. So, following what we did with electromagnetism, if you try to couple the theory to $BG \times B^{n-1}A^\vee$, two things might happen (and are not mutually exclusive).

(1) The two symmetries might interact nontrivially, producing an extension

$$B^{n-1}A^\vee \longrightarrow \mathcal{X} \longrightarrow BG.$$

(2) There may be an anomaly $\mathcal{X} \to B^{n+1}\mathbb{T}$.

For example, $\mathcal{X}$ might be $B\mathcal{G}$, where $\mathcal{G}$ is an extension of G by $B^{n-2}A^\vee$, so an extension of a group by a higher group.

First suppose Γ is anomaly-free, so that we can gauge it. Then, the anomaly for the theory coupled to G-symmetry is the compositions of the maps

$$BG \times B^{n-1}A^\vee \longrightarrow B^2A \times B^{n-1}A^\vee \longrightarrow B^{n+1}\mathbb{T},$$

where the first map comes from the connecting map $BG \to B^2A$ and the second is the Pontrjagin dual pairing.

[4] "Global symmetry" is redundant, because there is no other kind of symmetry.

Now suppose the Γ theory has an anomaly. There are different ways to produce anomalies, such as beginning with a map $MSO \wedge (B\Gamma)_+ \to B^{n+1}\mathbb{T}$, which produces a *gauge-gravity anomaly*, but let's begin with just $B\Gamma \to B^{n+1}\mathbb{T}$, or a *pure gauge anomaly*. This data is equivalent to a cohomology class $[\alpha] \in H^{n+1}(B\Gamma; \mathbb{T})$.

The presence or absence of the anomaly arises from a filtration on $H^{n+1}(B\Gamma; \mathbb{T})$ induced from the fiber sequence $BA \to B\Gamma \to BG$: BG has a cell structure, and we can ask whether a cohomology class over the basepoint extends over the n-skeleton. We will discuss what happens in the case when the cohomology class is in the last two pieces of the filtration, which are simpler.

To compute this, one uses the *Leray-Serre spectral sequence*

$$E_2^{p,q} := H^p(G; H^q(A; \mathbb{T})) \Longrightarrow \mathrm{gr}_F H^{p+q}(\Gamma; \mathbb{T}).$$

Here $H^q(A; \mathbb{T})$ is a nontrivial G-module from the residual G-action induced by Γ. Now let's look at the two cases.

(1) Suppose $[\alpha]$ lives in the highest filtered part; then, there's an $\overline{\alpha}\colon BG \to B^{n+1}\mathbb{T}$ lifting α across the map $B\Gamma \to BG$. In this case, $\overline{\alpha}$ is the anomaly: if you gauge the A-symmetry, the theory couples to $BG \times B^{n-1}A^\vee$, and the anomaly is a sum of the electromagnetic anomaly and $\overline{\alpha}$. This corresponds to a transgression in the spectral sequence.

(2) If $[\alpha]$ lives in the next highest filtered part, we get something in $H^n(G; \underline{A}^\vee)$ (the underline representing a nontrivial G-action). The paper [Tac17] considers only the special case, where A is central and we get a bundle

$$B\Gamma \times_{BG} \mathcal{X},$$

where $B\Gamma \to BG$ is a BA-bundle and $\mathcal{X} \to BG$ is a $B^{n-1}A^\vee$ bundle. Then we have maps

$$BG \to B^2A \times B^nA^\vee \to B^{n+2}\mathbb{T},$$

and we consider the case where this composite is null. This means it lifts to a map

$$\beta\colon B\Gamma \times_{BG} \mathcal{X} \to B^{n+1}\mathbb{T}.$$

Then, we take $\beta|_{B\Gamma}$ is the anomaly in Γ. If this was the original anomaly in Γ, then the theory couples to the extension $B^{n-1}A^\vee \to \mathcal{X} \to BG$, corresponding to some kind of extension theory, and the anomaly for this theory is $\beta|_X$. Then you could gauge the subgroup $B^{n-1}A^\vee$ and go back to the original theory.

(3) Suppose $[\alpha]$ lives in the n^{th} piece of the filtration, and assume A is central. Then the map $B\Gamma \to B\mathbb{T}^{n+1}$ doesn't descend to BG, but does descend to a map $BG \to E$, where E fits into a sequence $B^{n+1}\mathbb{T} \to E \to B^nA^\vee$. This corresponds to an anomaly in a generalized cohomology theory, albeit a relatively simple one. We could try to use this to build $\mathcal{X}$. But you also have to add the electromagnetic anomaly, and this is a little bit unclear.

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