NOTES FROM THE 2022 SUMMER SCHOOL ON GLOBAL SYMMETRIES

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Contents

Part 1. Kantaro Ohmori, Introduction to symmetries in quantum field theory

1. Extended operators and defects in functorial QFT: 6/13/22

One of the goals of this workshop is to discuss generalized symmetries in quantum field theory. For us, this means topological operators and defects. Here's a short outline of these four talks:

(1) Quantum field theory as a functor, and how to work with extended operators and defects in this formalism
(2) Topological operators and symmetries
(3) One-form symmetries in gauge theories and confinement
(4) "Non-invertible" symmetries

Let's get started. Quantum field theory means a lot of different things to a lot of different people; today, we will only focus on relativistic Euclidean QFT. When we say a d-dimensional QFT, d refers to the dimension of spacetime.

In the non-topological setting, it's not yet completely clear how to define a quantum field theory as a functor, so the following definition will be a little heuristic.

Definition 1.1. A quantum field theory is a symmetric monoidal functor $Z\colon \mathcal{B}ord_S^{\langle d,d-1\rangle} \to \mathcal{V}ect$.

Here $\mathcal{V}ect$ is the symmetric monoidal category of vector spaces with tensor product and $\mathcal{B}ord_S^{\langle d,d-1\rangle}$ is a bordism category. S refers to some kind of geometric structure we want to endow spacetime with: for example, we could ask for just a smooth structure, or a spin structure, or a Riemannian metric, or a principal G-bundle with a connection, or so on. A manifold with S-structure is called an S-manifold.

The objects of $\mathcal{B}ord_S^{\langle d,d-1\rangle}$ are closed, $(d-1)$-dimensional manifolds with an S-structure. The set of morphisms between $(d-1)$-dimensional S-manifolds M and N is the set of (diffeomorphism classes rel boundary) of S-bordisms from M to N. An *S-bordism* X from M to N is a compact, d-dimensional manifold with an identification of S-manifolds $\partial X \xrightarrow{\cong} M \amalg \overline{N}$. Here $\overline{N}$ denotes N with the opposite orientation.

To define a category, we need to compose morphisms; this is accomplished by gluing bordisms. (TODO: picture).

Remark 1.2. We haven't said precisely how to define S, so one might wonder whether it depends on d. For example, there's a difference between a framing of a manifold M (a trivialization of TM) and a stable framing (a trivialization of $TM \oplus \underline{\mathbb{R}}^k$ for some k); the former depends on d and the latter does not, and the two notions are not the same.

Freed–Hopkins [FH21] have shown that for reflection-positive topological field theories, many S-structures that appear to depend on d in fact stabilize and are independent of the dimension. ◀

If $Z: \mathcal{B}ord_S^{\langle d,d-1\rangle}$ is a quantum field theory, then for a closed $(d-1)$-dimensional S-manifold M, $Z(M)$ is a vector space. This is called the *state space* of M. If X is a bordism from M to N, then $Z(X)$ is a linear map from the state space of M to the state space of N. We often think of this map as *time evolution* of states. The fact that Z is symmetric monoidal means that $Z(M_1 \amalg M_2) \cong Z(M_1) \otimes Z(M_2)$.

Let $\tau \in (0,\infty)$. Then $M \times [0,\tau]$ is a bordism from M to M (its boundary is $M \amalg \overline{M}$). Let $U_M(\tau) \coloneqq Z(M \times [0,\tau])$, which we can think of as time evolution on M for time τ; in a Hamiltonian system we think of

$$U_M(\tau) = \exp(-\tau H_M), \tag{1.3}$$

where $H_M: Z(M) \to Z(M)$ is the Hamiltonian on M. Gluing bordisms implies $U_M(\tau_1) \circ U_M(\tau_2) = U_M(\tau_1 + \tau_2)$.

Remark 1.4. We would like to think of these operators as unitary, like in Hamiltonian quantum mechanics; making this precise from the functorial perspective is an area of active research. See for example a recent proposal of Kontevich–Segal [KS21].

To discuss unitarity we need some kind of inner product, but we did not ask for our state spaces to come with inner products. There is a one-parameter family of bilinear pairings around: the cylinder $M \times [0,\tau]$, thought of as a bordism from $M \amalg \overline{M} \to \varnothing$, induces a map $Z(M) \otimes Z(\overline{M}) \to Z(\varnothing) = \mathbb{C}$.[1] ◂

Example 1.5 (Finite gauge theory)**.** (Untwisted) finite gauge theory is a d-dimensional *topological* quantum field theory: the S-structure is topological, rather than geometric. Specifically, it is no structure at all.

Fix a finite group G and $p \in \{0,1,\dots,d-1\}$. If $p > 0$, we ask that G is abelian. Therefore we can make sense of $H^{p+1}(M;G)$ when M is a closed manifold: when G is nonabelian and $p = 0$, this is the set of isomorphism classes of principal G-bundles on M. For compact M, $H^{p+1}(M;G)$ is finite.

Let M be a closed $(d-1)$-manifold; we define the state space of finite gauge theory on M to be the vector space spanned by the finite set $H^{p+1}(M;G)$.

If W is a bordism from M to N, we define the linear map $Z(W)$ as a form of "finite path integral" — we can't make sense of the path integral in general for gauge theories, but becaue G is finite we can in this case. Fix $A \in H^{p+1}(M;G)$ and let $i_M: M \hookrightarrow W$ and $i_N: N \hookrightarrow W$ be the inclusions. Define

$$Z(W)|A\rangle \coloneqq c(W) \sum_{\substack{B \in H^{p+1}(W;G) \\ i_M^* B = A}} |i_N^* B\rangle, \tag{1.6}$$

where $c(W) \in \mathbb{R}$ is a mormalization constant that appears so that this definition is functorial when we glue bordisms. ◂

Exercise 1.7. Say $d = 2$, $p = 0$, and $G = \mathbb{Z}/n$. Calculate $Z(S^1)$, $Z(\Sigma)$, and $Z(\Sigma')$, where Σ is the pair of pants regarded as a bordism from $S^1 \amalg S^1 \to S^1$ and Σ' is Σ in the opposite direction. For bordisms, only calculate the maps up to normalization constants, since we didn't specify those constants in (1.6).

Now let's talk about extended quantum field theory. If M is a closed d-dimensional manifold, it may be regarded as a bordism $\varnothing \to \varnothing$. Applying Z, we obtain a linear map $\mathbb{C} \to \mathbb{C}$, since $Z(\varnothing) = \mathbb{C}$. This map is determined by its value on 1, which is a complex number called the *partition function* of M.

Associated to a closed $(d-1)$-manifold we have a state space. In extended QFT, we assign higher-categorical invariants to manifolds in lower dimensions; for example, to a closed $(d-2)$-dimensional manifold we assign something called a "2-vector space," which is something like a $\mathbb{C}$-linear category; and in general on a closed $(d-k)$-manifold we assign a "k-vector space," some kind of higher catgory. We will not define k-vector spaces precisely here, and indeed different researchers use different definitions.

We are especially interested in $Z(S^{q-1})$ as q varies; this is some sort of higher category. The objects of this category are the codimension-q defects or extended operators (to us, these two words mean the same thing) in the QFT Z. In non-topological theories, we need to specify the radius r of S^{q-1}; the codimension-q defects are the limit of $Z(S^{q-1}(r))$ as $r \to 0$.

There is another formalism for higher-codimension defects or operators, given by something called *decorated bordisms*. This latter approach may be easier to digest from the physics point of view. The two approaches are expected to be equivalent.

Part 2. Clay Córdova, Introduction to anomalies in quantum field theory

2. : 6/13/22

We begin by looking at degenerate ground states in quantum mechanics. The setup has a separable Hilbert space W, e.g. $L^2(\mathbb{R})$. This is the state space; the quantum states are nonzero elements of W modulo phases: we identify ψ and $\lambda\psi$ for $\lambda \in \mathbb{C}^\times$. Time evolution in this system is described using a self-adjoint positive Hermitian operator H,

[1] The fact that $Z(\varnothing) = \mathbb{C}$ is another consequence of symmetric monoidality.

called the *Hamiltonian*; $H = H^\dagger$. Some of these assumptions are because we are in the unitary setting. If $\mathcal{O}$ is an operator, it evolves under time as

$$\mathcal{O}(t) = e^{iHt}\mathcal{O}(0)e^{-iHt}. \tag{2.1}$$

Since H is self-adjoint and positive, its eigenvalues are real and nonnegative. Let E_i be the eigenvalues of H in ascending order; they represent the energy levels of the theory. We are particularly interested in the eigenvectors for the smallest eigenvalue. If this eigenspace is more than one-dimensional, we say this system has a *degenerate ground state*.

We are interested in questions related to ground state degeneracy. For example, when is there a degenerate ground state? Is this degeneracy stable under deforming H?

Example 2.2. Consider a system of n particles moving on $\mathbb{R}$ in the presence of a potential $V\colon \mathbb{R}^n \to \mathbb{R}$. Then $W = L^2(\mathbb{R}^n)$ and the Hamiltonian is

$$H \coloneqq -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + V(x_1, \dots, x_n) \tag{2.3}$$

Theorem 2.4. *Let $L^2_{loc}(R^n)$ denote the Hilbert space of functions which are square-integrable on compact subsets of $\mathbb{R}^n$. If $W = L^2_{loc}(\mathbb{R}^n)$, $V \geq 0$, and $V \to \infty$ as $|x_i| \to \infty$, then H has a nondegenerate ground state.*

The proof is an exercise, though see Reed–Simon volume IV [RS78, Chapter 8]. The assumptions cover many of the typical examples of quantum-mechanical systems, such as a double well. ⊲

The point of introducing Theorem 2.4 here is that it's not easy to produce examples of systems with ground state degeneracy.

One of the goals of this series of lectures is to develop a theory of invariants, called *anomalies*, which imply degenerate ground states in quantum mechanics and quantum field theory which are robust to perturbations.

Symmetry in quantum mechanics plays a key role. There are two kinds of symetries: *unitary transformations*, operators $U\colon W \to W$ such that $[U, W] = 0$, and *antiunitary transformations*, operators $T\colon W \to W$ such that $[H, T]$ acts on operators by conjugation by a unitary operator: $\mathcal{O} \mapsto U\mathcal{O}U^{-1}$. When T is antiunitary and $\lambda \in \mathbb{C}^\times$, $T(\lambda w) = \overline{\lambda} T(w)$.

Example 2.5. Consider a particle on a circle. Our variable x now is 2π-periodic: we identify x and $x + 2\pi$. We write down a Lagrangian

$$L = \frac{1}{2}\dot{x}^2 + \frac{\theta}{2\pi}\dot{x}, \tag{2.6}$$

where $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ is a parameter. From the Lagrangian, one does a canonical transformation to obtain the Hamiltonian; this is a standard trick, and the answer is

$$H = \frac{1}{2}\left(-i\frac{\mathrm{d}}{\mathrm{d}x} - \frac{\theta}{2\pi}\right)^2, \tag{2.7}$$

acting on the functions on the circle. The Lagrangian and Hamiltonian are quadratic, so it's easy to solve this explicitly for all energy levels at once. When we do this, we will see something interesting.

The eigenfunctions are Fourier modes: $\exp(inx)$ for $n \in \mathbb{Z}$, which has eigenvalue $E_n = (1/2)(n - \theta/2\pi)^2$. Though the energy levels are the same, the states themselves move around, so there's some sort of spectral flow. At $\theta = 0$, the ground state has energy level 0, and excited states (the eigenstates for eigenvalues larger than the smallest eigenvalue) are doubly degenerate: $\pm n$ gives you a two-dimensional eigenspace. But at $\theta = \pi$, all eigenvalues have two-dimensional eigenspaces. See Figure 1 for a picture of the spectrum as θ varies. One might think that this "level-crossing" behavior is generic, but this is not correct.

Exercise 2.8. Determine the codimension of the level-crossing loci for a multiparameter Hamiltonian (TODO: may have misunderstood this exercise?).

This system has symmetries, and let's take advantage of them. Let a be a time-independent constant; then the system is symmetric under the shift $x \mapsto x + a$. This defines unitary operators U_a which generate the Lie group U_1 under composition.

Another symmetry is reflection, $C\colon x \mapsto -x$. This acts on the Hamiltonian, and changes θ; therefore it is only actually a symmetry at $\theta = 0, \pi$. In some sense, generically we only have the U_1 symmetry, but at $\theta = 0, \pi$, the symmetry group enhances.

How do the symmetris act on the eigenstates? For translation, the answer is not so tricky:

$$U_a(w_n) = e^{ina}w_n. \tag{2.9}$$

Exercise 2.10. Convince yourself that at $\theta = 0$, $C(w_N) = w_{-n}$, and at $\theta = \pi$, $C(w_n) = w_{-n+1}$.

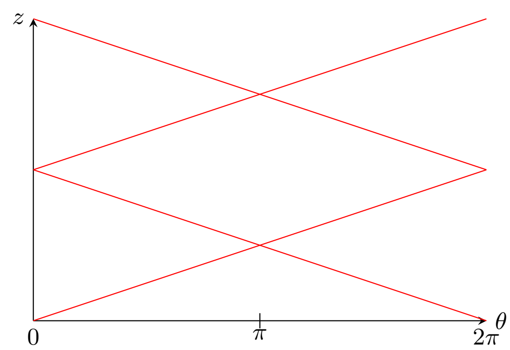

FIGURE 1. The spectrum of the particle on a circle considered in Example 2.5. The red lines are the eigenvalues and how they change under θ. Notce the crossing at $\theta = 0, \pi$, where most, resp. all eigenspaces are two-dimensional. For other values of θ, the eigenspces are all one-dimensional.

This leads to a simple-looking question: *what is the group of symmetries generated by U_a and C?* There are two different answers, depending on what exactly you mean.

(1) When acting on operators — operators are generated by x and its derivatives, and compositions thereof, so we obtain the group O_2: U_a forms SO_2, and C acts as reflections on the circle.
(2) When acting on states, however, the answer is not the same at $\theta = \pi$. There we find an additional phase:

$$CU_aC^{-1}(w_n) = CU_a(w_{-n+1}) = C\Big(e^{ia(-n+1)}w_{-n+1}\Big) = e^{ia}U_{-a}(w_n). \tag{2.11}$$

If this generated an O_2, we would've just exepcted $U_{-a}(w_n)$.

In the second setting, the phase we obtained tells us that the action of O_2 is in fact a projective representation associated to the double cover

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathrm{Pin}_2^- \longrightarrow \mathrm{O}_2 \longrightarrow 1. \tag{2.12}$$

When acting on operators, we see O_2; when acting on states, we see this double cover. ◀

Your first question might be: why is this discrepancy between states and operators even possible? It is possible because states are rays in the Hilbert space of states, not vectors. A symmetry only has to be a projective representation on the Hilbert space. Concretely, we end up with a cocycle $\mu \in Z^2(\mathrm{O}_2; \mathrm{U}_1)$ that twists the multiplication in the action on states, and μ is not cohomologous to 0.[2] Because operators are acted upon by conjugation, they are blind to this cocycle.

Another key observation: all states have the same value of μ once we fix θ: at $\theta = 0$, $\mu = 0$, and at $\theta = \pi$, we have the same nonzero cocycle.

Each eigenspace of H forms a projective representation of O_2 with given μ. At $\theta = \pi$, where μ is not cohomologous to zero, there are no one-dimensional projective representations — and therefore there is ground state degeneracy!

Projective representations are among the easiest examples of anomalies, and they are also robust under symmetry-preserving deformations: they are classified by discrete cocycles. So if we smoothly modify the Hamiltonian, we will preserve the property of ground state degeneracy.

Exercise 2.13. Once again consider the particle on the circle, and modify the Hamiltonian by adding a potential:

$$H = \frac{1}{2}\left(-i\frac{\mathrm{d}}{\mathrm{d}x} - \frac{\theta}{2\pi}\right)^2 + \frac{\lambda}{2\pi}\cos(2x). \tag{2.14}$$

(1) Show that for small λ and at $\theta = 0$, $|E_{+1} - E_{-1}| = \lambda + O(\lambda^2)$. The interpretation is that the degeneracy at $\theta = 0$ is a coincidence: we can turn on an arbitrarily small potential, deforming the Hamiltonian by an arbitrarily small amount, and undo the degeneracy.
(2) Show that at $\theta = \pi$, the ground state degeneracy persists for $\lambda \neq 0$.

The system at $\theta = \pi$ with nonzero λ is quite difficult: we don't know how to exactly solve it. But we do know the energy levels.

[2] Complex projective representations use U_1-valued cocycles; the fact that we had a $\mathbb{Z}/2$ cover, rather than a U_1 cover, comes from the fact that this cocycle is valued in the $\mathbb{Z}/2$ subgroup $\{\pm 1\} \subset \mathrm{U}_1$. TODO: so we only see the image in $Z^2(\mathrm{O}_2; \mathrm{U}_1)$; how do we tell apart pin$^+$ and pin$^-$ then?

The $\cos(2x)$ term breaks the shift symmetry from U_1 to $\mathbb{Z}/2$ (we can still shift by half of a period). This is actually all that we need to have the degeneracy.

Example 2.15. Consider a system of real fermions with a time-reversal symmetry. We'll have N fermions $\psi^i(t)$, $i = 1, \dots, N$. For simplicity assume N is even. Classically they're Grassmann variables; when we quantize we obtain a Clifford algebra: these variables' supercommutator is $\{\psi^i, \psi^j\} = 2\delta^{ij}$. We let the time-reversal symmetry T act on these operators as

$$T\psi^i(t)T^{-1} = -\psi^i(-t). \tag{2.16}$$

The Hilbert space W, a Clifford algebra, is finite-dimensional, and more precisely $\dim(W) = 2^{N/2}$. Choose the zero Hamiltonian; then all states are ground states! That's a lot of ground state degeneracy.

One typically considers a mass term, inducing a quadratic deformation of the Hamiltonian:

$$\Delta H \overset{?}{=} im\psi^1\psi^2, \tag{2.17}$$

for some $m \in \mathbb{R}$. As $T\Delta HT^{-1} = -\Delta H$, though, this mass term is incompatible with time-reversal. This crucially uses that T is an anti-unitary symmetry.

So quadratic deformations are no help. What about quartic deformations?

To answer this question, it's helpful to group the fermions into complex pairs. Let

$$a_n = \frac{1}{\sqrt{2}}\big(\psi^{2n-1} + i\psi^{2n}\big). \tag{2.18}$$

Then the creation and annihilation operators for these complex fermions satisfy $\{a_n, a_m^\dagger\} = \delta_{mn}$. Each pair generates a two-component space $W_\pm$, and

$$\begin{aligned} a(w_-) &= 0 & a^\dagger(w_+) &= 0 \\ a^\dagger(w_-) &= w_+ & a(w_+) &= w_-. \end{aligned} \tag{2.19}$$

TODO: I didn't quite follow this, sorry! In a general state, there are $N/2$ labels $w_{\pm\pm\cdots\pm}$. Consider the quartic deformation

$$\Delta H = 4q\psi^1\psi^2\psi^3\psi^4, \tag{2.20}$$

where $q > 0$. Then we can factor ΔH as

$$\Delta H = -q\Big(a_1a_1^\dagger - \frac{1}{2}\Big)\Big(a_2a_2^\dagger - \frac{1}{2}\Big). \tag{2.21}$$

There are aligned states w_{++} and w_{--} with $\Delta E = -q$, and w_{+-} and w_{-+} also are aligned. Anyways, the upshot is that there is twofold ground state degeneracy, sort of like we saw with quantum mechanics on a circle. ◀

Exercise 2.22. Show that for $N = 8$ there exists a T-invariant quartic deformation leading to a unique ground state.

Consequently, the quantity $N \bmod 8$ is protected by T. It is also an indication that there is another anomaly.

Part 3. Mike Hopkins, Lattice systems and topological field theories

3. Product states and entanglement entropy: 6/13/22

This week's talks are on work related to topology/TQFT, lattice models, and quantum information theory, including ideas growing out of a meeting at Aspen with Freed, Teleman, Freedman, Kapustin, Kitaev, Moore, and Hastings. The key idea is that a certain infrared limit of certain lattice models is described by a topological field theory. These are things we can describe mathematically through topology. But there's another way we can look at these models, from a quantum information point of view; this is a very different perspective, but manifold topology can still be useful.

Kitaev had a conjecture that for at least invertible systems, the moduli spaces of lattice models and TFTs are homotopy equivalent. In particular, given a TFT, you should be able to produce a lattice model. The map from lattice models to TFTs, which we expect to be a homotopy equivalence, should have something to do with renormalization group flow. Some of the ideas around this are related to work of Norbert, Schuch, Clement, Delcamp, Giufre, Vidal, and Jake McNamara.

A lattice system is some kind of system on Euclidean space that we think of as describing phenomena like electron hopping, etc. A TFT is more topological, described in terms of bordism theory, something maybe more familiar to topologists. The difference between these two is what makes Kitaev's conjecture more exciting.

You can think of the lattice system as some sort of material type, and connecting this to bordism and TFT, we want to build our manifold, which may have nontrivial topology, out of this material type. Imagine riveting some metal material into the shape or a torus — now you have a physical object, and you can study its electronic properties, which are a measurement of the material and is an invariant of the lattice system.

Even at this level, there are still a lot of mysteries — which lattice systems work? And at the way we currently understand it, we don't know how to apply a material to all manifolds, or say that one material on one manifold is the same as another material on another manifold. There's lots to do.

Now a little more detail. Let's consider a lattice system in $\mathbb{R}^d$. The lattice amounts to (at least) a bunch of vertices on $\mathbb{R}^d$, called *sites*. Let $S \subset \mathbb{R}^d$ be the set of sites. We choose a (complex) "local Hilbert space" $\mathcal{H}$ and put a copy of $\mathcal{H}$ on each site; the total Hilbert space is $\bigotimes_{s\in S} \mathcal{H}_s$. The Hamiltonian H is a self-adjoint operator on this total Hilbert space. We want to study the spectrum of H and its eigenspaces.

Example 3.1. Suppose the local Hilbert space is a *qubit*, a copy of $\mathbb{C}^2$. Let

$$X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.2}$$

If we work on a single site and let $H = Z$, the eigenvalues are $\{\pm 1\}$: $Z|0\rangle = |0\rangle$ and $Z|1\rangle = |1\rangle$, where $\{|0\rangle, |1\rangle\}$ is the standard basis of $\mathbb{C}^2$. ◂

Example 3.3. We coul tensor several copies of the previous example together: $\mathcal{H}_s = (\mathbb{C}^2)^{\otimes m}$. We tensor the Hamiltonians together: on two-fold tensor products define

$$H(v \otimes w) = H(v) \otimes w + v \otimes H(w), \tag{3.4}$$

and generalize to higher-fold tensor products by doing this iteratively. Now we can represent basis elements of $(\mathbb{C}^2)^{\otimes d}$ by bitstrings, e.g. $|01001\rangle$, and the spectrum is the integers $\{-d, \dots, d\}$. We could also use $H = (1+Z)/2$ and get $H|0\rangle = 0$ and $H|1\rangle = 1$, and build a spectrum on $0, \dots, d$. ◂

These examples may feel silly: we're just on a single site. Even if it's a site for sore eyes, we will want to consider multi-site lattices, and put a $\mathbb{C}^2$ (or $(\mathbb{C}^2)^{\otimes m}$) on every site and tensor them all together. In this case, we want the Hamiltonian to be a sum of local terms, meaning only using operators on nearby sites. If L denotes the length of the system, measured for example in the number of sites on a line, we want there to be a gap in the smallest two eigenvalues of H as $L \to \infty$. Defining this precisely is difficult, and we won't do it now.

Example 3.5. Let's consider a system on a line (or line segment) with sites at the integer points. The local Hilbert space will be $\mathbb{C}^4 = M_2(\mathbb{C})$. Give this the orthonormal basis e_i^j which is the zero matrix except for a 1 in position (i,j). Multiplication gives a map $M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \to M_2(\mathbb{C})$. Let the local Hamiltonian by orthogonal projection to K, the kernel of this multiplication map; then $M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) = M_2(\mathbb{C}) \oplus K$, corresponding to the eigenspaces 0 and 1 respectively. (TODO: why isn't this 1 and 0?).

Let H_i denote this operation at lattice site i, and $H := \sum_{i=1}^n H_i$. $[H_i, H_j] = 0$ for $i \neq j$, so the eigenvalues of H are easy to compute: the ground state space is the subspace of vectors annihilated by H. The ground state is the transpose of the iterated multiplication map

$$M \otimes \cdots \otimes M \longrightarrow M. \tag{3.6}$$

This has fourfold degeneracy, localized to 2 each on the ends of the chain. ◂

This is an important example, and you can generalize it to higher dimensions.

Example 3.7. Suppose we have a two-dimensional square lattice on the integer points of some rectangle. Choose vector spaces V and W, finite-dimensional, and let $M := V \otimes V^*$ and $N := W \otimes W^*$. At each site place the local Hilbert space $M \otimes N$. We want the local Hamiltonian H_p to be the kernel of orthogonal projection onto the kernel of multiplication $(M \otimes N)^4 \to (M \otimes N)^2$. TODO: there was something about composing M horizontally and N vertically? Then H is the sum of the local Hamiltonians as usual. ◂

These examples have ground states with very special properties. In the simplest examples, the ground state was something like $|0\rangle \otimes \cdots \otimes |0\rangle$, a *tensor state*. Our more sophisticated examples have more sophisticated ground states, but still with a similar feel: they are things called *matrix product states*. The idea is that instead of a vector at every site, you have a matrix of vectors at every site, and we tensor them together using matrix multiplication.

For example, if we have $\begin{pmatrix} |0\rangle & |1\rangle \\ |1\rangle & |0\rangle \end{pmatrix}$ next to $\begin{pmatrix} |0\rangle & |1\rangle \\ |1\rangle & |0\rangle \end{pmatrix}$, the total ground state for those two sites is

$$\begin{pmatrix} |0\rangle & |1\rangle \\ |1\rangle & |0\rangle \end{pmatrix} \otimes \begin{pmatrix} |0\rangle & |1\rangle \\ |1\rangle & |0\rangle \end{pmatrix} = \begin{pmatrix} |00\rangle + |11\rangle & |01\rangle + |10\rangle \\ |01\rangle + |10\rangle & |00\rangle + |11\rangle \end{pmatrix}. \tag{3.8}$$

And one can continue with longer chains of sites. This is a very special thing to be true for a ground state. And yet:

Theorem 3.9. *The ground states of the model we described in Example 3.5, as well as related models, are matrix product states.*

You can think of this as specifying tensors with one input and one output, and linking the input of one with the output of another is contracting an index, or matrix multiplication. This suggests a higher-dimensional generalization, where our tensors have more arms, corresponding to tensors with more indices, and there are thus more ways to contract them. This is relevant for the two-dimensional example Example 3.7, where we work with a product of two matrix algebras, which is the kind of object that has both horizontal and vertical outputs, which we can link up/compose in two ways. This arrangemnet of data is the same thing as a linear map from $M \otimes N$ to the local Hilbert space (TODO: I think). In fact, the ground state in this system *is* this tensor network state.

So now we have some examples to play with. There's mounting evidence that the lattice models coming from TFTs should have ground states that look like these matrix product states. Let's learn a little more about why we believe that.

One thing to keep in mind is that the area of the region we consider is not fixed: we envision it growing, to pass to some sort of limit. We should be able to map from the Hilbert space for a smaller region to the Hilbert space of a larger region, which is a process called *density matrix projection.* Given a state v in $\mathcal{H}_1 \otimes \mathcal{H}_2$, we want to measure entanglement, a quantity describing how far this state is from being a pure tensor. We can write

$$v = \sum_i \sqrt{s_i} u_i \otimes v_i, \tag{3.10}$$

where u_i and v_i are all orthonormal and $s_i \geq 0$. Associated to this quantity is the *entanglement entropy*

$$S(v) \coloneqq -\sum_i s_i \log_2(s_i). \tag{3.11}$$

This is the refinement of the rank of a projection.

Example 3.12. In $\mathbb{C}^d \otimes \mathbb{C}^d$, consider $v' \coloneqq \sum e_i \otimes e_i$ — or more precisely, the unit vector in the same direction, $v \coloneqq v'/\sqrt{d}$. Then $s_i = 1/d$ and the entanglement entropy is $S(v) \propto -\log_2(d)$. This corresponds to cutting a segment out from the line.

This was for the lattice on a line. But what if we take this on a square? More specifically, we cut the square out from the rest of the lattice. There's a d for every internal edge, and a $-\log_2(d)$ for every half-edge (the edges we cut). Now the entangleement entropy is proportional to the perimeter times $\log_2(d)$. Had we done this in some other dimension, we would've obtained the surface area times some constant, in place of the perimeter. ◀

These product states are very special in that they obey something called an *area law* like this. In dimension 1, there's a theorem due to Hastings that the entanglement entropy of a gapped system always satisfies an area law: it's a constant times the surface area. In higher dimensions, there are some results, but not a full picture. In any case, these gapped lattice systems should have an area law property. This might even be true for all Hamiltonians which are sums of local terms, not just commuting projectors.

Conversely, states which obey area laws are supposed to be close to tensor network states. This is both aspirational (we don't have a proof) and inspirational: it tells us what systems to look for, or what's out there. That is, these tensor network systems are similar to these systems that we're looking for that come from topological field theories.

Recall the example we did with $M_2(\mathbb{C})$ attached to each site on a line. If we try to take a TFT at low energy, we obtain the trivial TFT. But maybe we can fix this somehow. The map $M_2 \otimes M_2 \to M_2$ is equivariant for the action of $\mathfrak{su}_2$ on $M_2(\mathbb{C})$ by conjugation, and this action exponentiates to an action of SO_3. Now, the system we consider is *non*trivial as a system with an SO_3-symmetry; the relevant invertible TFTs are classified by characters $\mathrm{Hom}(H_2(B\mathrm{SO}_3), \mathbb{C}^\times)$, which is isomorphic to $\mathbb{Z}/2$. The TFT is computed via its partition function, an invariant of surfaces with a principal SO_3-bundle. Then the system is nontrivial, which means that you can't break the entanglement with operators that are SO_3-invariant.

There are lots of other interesting examples: the AKLT model, the Kitaev chain, and the Kitaev-Drinfeld models coming from a finite group. All of these models have ground states that can be expressed in terms of tensor network states, and the lattice models can be described, up to renormalization, can be described as these matrix product models. This gives you the special properties governing the entanglement that we discussed, as well as correlations between measurements at different places, which we'll discuss next time.

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