M392C NOTES: SPIN GEOMETRY

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These notes were taken in UT Austin's M392C (Spin Geometry) class in Fall 2016, taught by Eric Korman. I live-TeXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Thanks to Adrian Clough for fixing a few typos.

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Lecture 1.

Lie Groups: 8/25/16

There is a course website, located at https://www.ma.utexas.edu/users/ekorman/teaching/spingeometry/. There's a list of references there, none of which we'll exactly follow.

We'll assume some prerequisites for this class: definitely smooth manifolds and some basic algebraic topology. We'll use cohomology, which isn't part of our algebraic topology prelim course, but we'll review it before using it.

Introduction and motivation. Recall that a *Riemannian manifold* is a pair (M, g) where M is an n-dimensional smooth manifold and g is a *Riemannian metric* on M, i.e. a smoothly varying, positive definite inner product on each tangent space T_xM over all $x \in M$.

Definition 1.1. A *local frame* on M is a set of (locally defined) tangent vectors that give a positive basis for M, i.e. a smoothly varying set of tangent vectors that are a basis at each tangent space.

A Riemannian metric allows us to talk about *orthonormal frames*, which are those that are orthonormal with respect to the metric at all points.

Recall that the special orthogonal group is $\mathrm{SO}(n) = \{A \in M_n \mid AA^{\mathrm{T}} = I, \det A = 1\}$. This acts transitively on orthonormal, oriented bases, and therefore also acts transitively on orthonormal frames (as a frame determines an orientation). Conversely, specifying which frames are orthonormal determines the metric g.

In summary, the data of a Riemannian structure on a smooth manifold is equivalent to specifying a subset of all frames which is acted on simply transitively[1] by the group $\mathrm{SO}(n)$. This set of all frames is a *principal* $\mathrm{SO}(n)$*-bundle* over M.

By replacing $\mathrm{SO}(n)$ with another group, one obtains other kinds of geometry: using $\mathrm{GL}(n,\mathbb{C})$ instead, we get almost complex geometry, and using $\mathrm{Sp}(n)$, we get almost symplectic geometry (geometry with a specified skew-symmetric, nondegenerate form).

Remark. Let G be a Lie group and M be a manifold. Suppose we have a principal G-bundle $E \to M$ and a representation[2] $\rho : G \to V$, we naturally get a vector bundle over M.

A more surprising fact is that all[3] representations of $\mathrm{SO}(n)$ are contained in tensor products of the *defining representation* of $\mathrm{SO}(n)$ (i.e. acting on $\mathbb{R}^n$ by orientation-preserving rotations). Thus, all of the natural vector bundles are subbundles of tensor powers of the tangent bundles. That is, when we do geometry in this way, we obtain no exotic vector bundles.

If $n \geq 3$, then $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2$, so its double cover is its universal cover. Lie theory tells us this space is naturally a compact Lie group, called the *Spin group* $\mathrm{Spin}(n)$. In many ways, it's more natural to look at representations of this group. The covering map $\mathrm{Spin}(n) \twoheadrightarrow \mathrm{SO}(n)$ precomposes with any representation of $\mathrm{SO}(n)$, so any representation of $\mathrm{SO}(n)$ induces a representation of $\mathrm{Spin}(n)$. However, there are representations of the spin group that don't arise this way, so if we can refine the orthonormal frame bundle to a principal $\mathrm{Spin}(n)$-bundle, then we can create new vector bundles that don't arise as tensor powers of the tangent bundle.

Spin geometry is more or less the study of these bundles, called *bundles of spinors*; these bundles have a natural first-order differential operator called the *Dirac operator*, which relates to a powerful theorem coming out of spin geometry, the Atiyah-Singer index theorem: this is vastly more general, but has a particularly nice form for Dirac operators, and the most famous proof reduces the general case to the Dirac case. Broadly speaking, the index theorem computes the dimension of the kernel of an operator, which in various contexts is a powerful invariant. Here are a few special cases, even of just the Dirac case of the Atiyah-Singer theorem.

- The Gauss-Bonnet-Chern theorem gives an integral formula for the Euler characteristic of a manifold, which is entirely topological. In this case, the index is the Euler characteristic.
- The Hirzebruch signature theorem gives an integral formula for the signature of a manifold.
- The Grothendieck-Riemann-Roch theorem, which gives an integral formula for the Euler characteristic of a holomorphic vector bundle over a complex manifold.

Lie groups and Lie algebras.

Definition 1.2. A *Lie group* G is a smooth manifold with a group structure such that the multiplication map $G \times G \to G$ sending $g_1, g_2 \mapsto g_1 g_2$ and the inversion map $G \to G$ sending $g \mapsto g^{-1}$ are smooth.

Example 1.3.

- The *general linear group* $\mathrm{GL}(n,\mathbb{R})$ is the group of $n \times n$ invertible matrices with coefficients in $\mathbb{R}$. Similarly, $\mathrm{GL}(n,\mathbb{C})$ is the group of $n \times n$ invertible complex matrices. Most of the matrices we consider will be subgroups of these groups.
- Restricting to matrices of determinant 1 defines $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{SL}(n,\mathbb{C})$, the *special linear groups*.
- The *special unitary group* $\mathrm{SU}(n) = \{A \in \mathrm{GL}(n,\mathbb{C}) \mid A\overline{A}^{\mathrm{T}} = 1, \det A = 1\}$.
- The special orthogonal group $\mathrm{SO}(n)$, mentioned above.

Definition 1.4. A *Lie algebra* is a vector space $\mathfrak{g}$ with an anti-symmetric, bilinear pairing $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the *Jacobi identity*

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

for all $X, Y, Z \in \mathfrak{g}$.

[1] Recall that a group action on X is *transitive* if for all $x, y \in X$, there's a group element g such that $g \cdot x = y$, and is *simple* if this g is unique.

[2] A representation of a group G is a homomorphism $G \to \mathrm{GL}(V)$ for a vector space V. We'll talk more about representations later.

[3] We're only considering smooth, finite-dimensional representations.

Example 1.5. The basic and important example: if A is an algebra,[4] then A becomes a Lie algebra with the commutator bracket $[a,b] = ab - ba$. Because this algebra is associative, the Jacobi identity holds.

The Jacobi identity might seem a little vague, but here's another way to look at it: if $\mathfrak{g}$ is a Lie algebra and $X \in \mathfrak{g}$, then there's a map $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ sending $Y \mapsto [X,Y]$. The map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ sending $X \mapsto \mathrm{ad}_X$ is called the *adjoint representation* of X. The Jacobi identity says that ad intertwines the bracket of $\mathfrak{g}$ and the bracket induced from the algebra structure on $\mathrm{End}(\mathfrak{g})$ (where multiplication is composition): $\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y]$. In other words, the adjoint representation is a homomorphism of Lie algebras.

Lie groups and Lie algebras are very related: to any Lie group G, let $\mathfrak{g}$ be the set of left-invariant vector fields on G, i.e. if $L_g : G \to G$ is the map sending $h \mapsto gh$ (the *left multiplication* map), then $\mathfrak{g} = \{X \in \Gamma(TG) \mid \mathrm{d}L_g X = X \text{ for all } g \in G\}$. This is actually finite-dimensional, and has the same dimension as G.

Proposition 1.6. *If e denotes the identity of G, then the map $\mathfrak{g} \to T_eG$ sending $X \mapsto X(e)$ is an isomorphism (of vector spaces).*

The idea is that given the data at the identity, we can translate it by g to determine what its value must be everywhere. Vector fields have a Lie bracket, and the Lie bracket of two left-invariant vector fields is again left-invariant, so $\mathfrak{g}$ is naturally a Lie algebra! We will often use Proposition 1.6 to identify $\mathfrak{g}$ with the tangent space at the identity.

Example 1.7. Let's look at $\mathrm{GL}(n,\mathbb{R})$. This is an open submanifold of the vector space M_n, an n^2-dimensional vector space, as $\det A \neq 0$ is an open condition. Thus, the tangent bundle of $\mathrm{GL}(n,\mathbb{R})$ is trivial, so we can canonically identify $T_I\mathrm{GL}(n,\mathbb{R}) = M_n$. With the inherited Lie algebra structure, this space is denoted $\mathfrak{gl}(n,\mathbb{R})$.

The $n \times n$ matrices are also isomorphic to $\mathrm{End}(\mathbb{R}^n)$, since they act by linear transformations. The algebra structure defines another Lie bracket on this space.

Proposition 1.8. *Under the above identifications, these two brackets are identical, hence define the same Lie algebra structure on $\mathfrak{gl}(n,\mathbb{R})$.*

Remark. This proposition generalizes to all real matrix Lie groups (Lie subgroups of $\mathrm{GL}(n,\mathbb{R})$): the proof relies on a Lie subgroup's Lie algebra being a subalgebra of $\mathfrak{gl}(n,\mathbb{R})$.

So we can go from Lie groups to Lie algebras. What about in the other direction?

Theorem 1.9. *The correspondence sending a connected, simply-connected Lie group to its Lie algebra extends to an equivalence of categories between the category of simply connected Lie groups and finite dimensional Lie algebras over $\mathbb{R}$.*

Suppose G is any connected Lie group, not necessarily simply connected, and $\mathfrak{g}$ is its Lie algebra. If $\widetilde{G}$ denotes the universal cover of G, then $G = \widetilde{G}/\pi_1(G)$. Since $\widetilde{G}$ is simply connected, the correspondence above identifies $\mathfrak{g}$ with it, and then taking the quotient by the discrete central subgroup $\pi_1(G)$ recovers G.

The special orthogonal group. We specialize to $\mathrm{SO}(n)$, the orthogonal matrices with determinant 1. We'll usually work over $\mathbb{R}$, but sometimes $\mathbb{C}$. This is a connected Lie group.[5]

Proposition 1.10. *If $\mathfrak{so}(n)$ denotes the Lie algebra of $\mathrm{SO}(n)$, then $\mathfrak{so}(n) = \{X \in \mathfrak{gl}(n,\mathbb{R}) \mid X + X^{\mathrm{T}} = 0\}$.*

That is, $\mathfrak{so}(n)$ is the Lie algebra of skew-symmetric matrices.

Proof. If $F : M_n \to M_n$ is the function $A \mapsto A^{\mathrm{T}}A - I$, then the orthogonal group is $\mathrm{O}(n) = F^{-1}(0)$. Since $\mathrm{SO}(n)$ is the connected component of $\mathrm{O}(n)$ containing the identity, then it suffices to calculate $T_e\mathrm{O}(n)$: if 0 is a regular value of F, we can push forward by its derivative. This is in fact the case:

$$\mathrm{d}F_A(B) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F(A + tB) = A^{\mathrm{T}}B + B^{\mathrm{T}}A,$$

which is surjective for $A \in \mathrm{O}(n)$, so $\mathfrak{so}(n) = T_I\mathrm{SO}(n) = \ker(\mathrm{d}F_I) = \{B \in M_n \mid B + B^{\mathrm{T}} = 0\}$. ⊠

[4] By an *algebra* we mean a ring with a compatible vector space structure.

[5] If we only took orthogonal matrices with arbitrary determinant, we'd obtain the *orthogonal group* $\mathrm{O}(n)$, which has two connected components.

The spin group. We'll end by computing the fundamental group of SO(n); then, by general principles of Lie groups, each SO(n) has a unique, simply connected double cover, which is also a Lie group. Next time, we'll provide an *a priori* construction of this cover.

Proposition 1.11.

$$\pi_1(\mathrm{SO}(n)) = \begin{cases} \mathbb{Z}, & n = 2 \\ \mathbb{Z}/2, & n \geq 3. \end{cases}$$

Proof. If $n = 2$, $\mathrm{SO}(n) \cong S^1$ through the identification

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \longmapsto e^{i\theta},$$

and we know $\pi_1(S^1) = \mathbb{Z}$.

For $n \geq 3$, we can use a long exact sequence associated to a certain fibration, so it suffices to calculate $\pi_1(\mathrm{SO}(3))$. Specifically, we will define a Lie group structure on S^3 and a double cover map $S^3 \twoheadrightarrow \mathrm{SO}(3)$; since S^3 is simply connected, this will show $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}/2$.

We can identify S^3 with the unit sphere in the quaternions, which is naturally a group (since the product of quaternions is a polynomial, hence smooth).[6] Realize $\mathbb{R}^3$ inside the quaternions as $\mathrm{span}_{\mathbb{R}}\{i, j, k\}$ (the *imaginary quaternions*); then, we'll define $\varphi : S^3 \twoheadrightarrow \mathrm{SO}(3)$: $\varphi(q)$ for $q \in \mathbb{H}$ is the linear transformation $p \mapsto qpq^{-1} \in \mathrm{GL}(3, \mathbb{R})$, where p is an imaginary quaternion. We need to check that $\varphi(q)$ lies in SO(3), which was left as an exercise. We also need to check this is two-to-one, which is equivalent to $|\ker\varphi| = 2$, and that φ is surjective (hint: since these groups are connected, general Lie theory shows it suffices to show that the differential is an isomorphism). ⊠

Lecture 2.

Spin Groups and Clifford Algebras: 8/30/16

Last time, we gave a rushed construction of the double cover of SO(3), so let's investigate it more carefully. Recall that SO(n) is the Lie group of special orthogonal matrices, those matrices A such that $AA^{\mathrm{t}} = I$ and $\det A = 1$, i.e. those linear transformations preserving the inner product and orientation. This is a connected Lie group; we'd like to prove that for $n \geq 3$, $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2$. (For $n = 2$, $\mathrm{SO}(2) \cong S^1$, which has fundamental group $\mathbb{Z}$).

We'll prove this by explicitly constructing the double cover of SO(3), then bootstrapping it using a long exact sequence of homotopy groups to all SO(n), using the following fact.

Proposition 2.1. *Let G and H be connected Lie groups and $\varphi : G \to H$ be a Lie group homomorphism. Then, φ is a covering map iff $\mathrm{d}\varphi|_e : \mathfrak{g} \to \mathfrak{h}$ is an isomorphism.*[7]

Here $\mathfrak{g}$ is the Lie algebra of G, and $\mathfrak{h}$ is that of H. Facts like these may be found in Ziller's online notes;[8] the intuitive idea is that the condition on $\mathrm{d}\varphi|_e$ ensures an isomorphism in a neighborhood of the identity, which multiplication carries to a local isomorphism in the neighborhood of any point in G.

Now, we construct a double cover of SO(3). Recall that the *quaternions* are the noncommutative algebra $\mathbb{H} = \mathrm{span}_{\mathbb{R}}\{1, i, j, k\}$, where $i^2 = j^2 = k^2 = ijk = 1$. We can identify $\mathbb{R}^3$ with the imaginary quaternions, the span of $\{i, j, k\}$, and therefore the unit sphere S^3 goes to $\{q \in \mathbb{H} \mid |q|^2 = 1 = q\overline{q}\}$, where the conjugate exchanges i and $-i$, but also j and $-j$, and k and $-k$. This embedding means that if $v, w \in \mathbb{R}^3$, their product as quaternions is

$$vw = -\langle v, w\rangle + v \times w.$$

and in particular

$$vw + wv = -2\langle v, w\rangle. \tag{2.2}$$

If $q \in S^3$ and $v \in \mathbb{R}^3$, then $qvq^{-1} = qv\overline{q}$, i.e. $\overline{qvq^{-1}} = q\overline{v}\overline{q} = -qv\overline{q}$. That is, conjugation by something in S^3 is a linear transformation in $\mathbb{R}^3$, defining a smooth map $\varphi : S^3 \to \mathrm{GL}(3, \mathbb{R})$; we'd like to show the image lands in SO(3).

[6] This is important, because when we try to generalize to Spin_n for higher n, we'll be using Clifford algebras, which are generalizations of the quaternions.

[7] This isomorphism is as Lie algebras, but it's always a Lie algebra homomorphism, so it suffices to know that it's an isomorphism of vector spaces.

[8] https://www.math.upenn.edu/ wziller/math650/LieGroupsReps.pdf.

Let $q \in S^3$; then, we can use (2.2) to get

$$\begin{aligned}\langle \varphi(q)v, \varphi(q)w \rangle &= -\frac{1}{2}(\varphi(q)v\varphi(q)w + \varphi(q)w + \varphi(q)v)\\ &= -\frac{1}{2}\left(qvwq^{-1} + qwvq^{-1}\right)\\ &= -\frac{1}{2}\left(q(vw+wv)q^{-1}\right) = \langle v, w\rangle,\end{aligned}$$

using (2.2) again, and the fact that $\mathbb{R} = Z(\mathbb{H})$. Thus, $\mathrm{Im}(\varphi) \subset \mathrm{O}(3)$, but since S^3 is connected, its image must be connected, and its image contains the identity (since φ is a group homomorphism), so $\mathrm{Im}(\varphi)$ lies in the connected component containing the identity, which is $\mathrm{SO}(3)$.

To understand $\mathrm{d}\varphi|_1$, let's look at the Lie algebras of S^3 and $\mathrm{SO}(3)$. The embedding $S^3 \hookrightarrow \mathbb{H}$ allows us to identify $T_! S^3$ with the imaginary quaternions. If p and v are imaginary quaternions, so $\overline{p} = -p$, then

$$\begin{aligned}\mathrm{d}\varphi|_p(v) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg| |_{t=0}\varphi(e^{tp})v\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} e^{tp}ve^{-tp}\\ &= pv - vp.\end{aligned}$$

Thus, $\ker \mathrm{d}\varphi|_1 = \{p \in \mathbb{R}^3 \mid pv - vp = 0 \text{ for all imaginary quaternions } v\}$. But if something commutes with all imaginary quaternions, it commutes with all quaternions, since the imaginary quaternions and the reals (which are the center of $\mathbb{H}$) span to all of $\mathbb{H}$. Thus, the kernel is the imaginary quaternions in the center of $\mathbb{H}$, which is just $\{0\}$; hence, $\mathrm{d}\varphi|_1$ is injective, and since $T_1 S^3$ and $\mathfrak{so}(3)$ have the same dimension, it is an isomorphism. By Proposition 2.1, φ is a covering map, and $\mathrm{SO}(3) = S^3/\ker(\varphi)$.

We'll compute $|\ker\varphi|$, which will be the index of the cover. The kernel is the set of unit quaternions q such that $qvq^{-1} = v$ for all imaginary quaternions v; just as above, this must be the intersection of the real line with S^3, which is just $\{\pm 1\}$. Thus, φ is a double cover map of $\mathrm{SO}(3)$; since S^3 is simply connected, $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}/2$.

Exercise 2.3. The Lie group structure on S^3 is isomorphic to $\mathrm{SU}(2)$, the group of 2×2 special unitary matrices.

Now, what about $\pi_1(\mathrm{SO}(n))$, for $n \ge 4$? For this we use a fibration. $\mathrm{SO}(n)$ acts on $S^{n-1} \subset \mathbb{R}^n$, and the stabilizer of a point in S^n is all the rotations fixing the line containing that point, which is a copy of $\mathrm{SO}(n-1)$. This defines a fibration

$$\mathrm{SO}(n-1) \longrightarrow \mathrm{SO}(n) \longrightarrow S^{n-1}.$$

More precisely, let's fix the north pole $p = (0,0,\dots,0,1) \in S^{n-1}$; then, the map $\mathrm{SO}(n) \to S^{n-1}$ sends $A \mapsto Ap$; since A is orthogonal, Ap is a unit vector. The action of $\mathrm{SO}(n)$ is transitive, so this map is surjective. The stabilizer of p is the set of all orthogonal matrices with positive determinant such that the last column is $(0,0,\dots,0,1)$. Orthogonality forces these matrices to have block form

$$\begin{pmatrix} A & 0\\ 0 & 1\end{pmatrix},$$

where $A \in \mathrm{SO}(n-1)$; thus, the stabilizer is isomorphic to $\mathrm{SO}(n-1)$.

Now, we can use the long exact sequence in homotopy associated to a fibration:

$$\pi_2(S^{n-1}) \xrightarrow{\delta} \pi_1(\mathrm{SO}(n-1)) \longrightarrow \pi_1(\mathrm{SO}(n)) \longrightarrow \pi_1(S^{n-1}).$$

If $n \ge 4$, $\pi_2(S^{n-1})$ and $\pi + 1(S^{n-1})$ are trivial, so $\pi_1(\mathrm{SO}(n)) = \pi_1(\mathrm{SO}(n-1))$ for $n \ge 4$, so they all agree with $\mathbb{Z}?2$, so $\pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}/2$ for all $n \ge 4$.

By general Lie theory, the universal cover of a Lie group is also a Lie group.

Definition 2.4. For $n \ge 3$, the *spin group* $\mathrm{Spin}(n)$ is the unique simply-connected Lie group with Lie algebra $\mathfrak{so}(n)$. For $n = 2$, the spin group $\mathrm{Spin}(2)$ is the unique (up to isomorphism) connected double covering group of $\mathrm{SO}(2)$.

In particular, there is a double cover $\mathrm{Spin}(n) \to \mathrm{SO}(n)$, and $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$.

Right now, we do not have a concrete description of these groups; since $\mathrm{SO}(n)$ is compact, so is $\mathrm{Spin}(n)$, so we must be able to realize it as a matrix group, and we use Clifford algebras to do this.

Clifford algebras. Our goal is to replace $\mathbb{H}$ with some other algebra to realize $\mathrm{Spin}(n)$ as a subgroup of its group of units.

Recall from (2.2) that for $v, w \in \mathbb{R}^3 \hookrightarrow \mathbb{H}$, $vw + wv = -2\langle v, w\rangle$. We'll define a universal algebra for this kind of definition.

Definition 2.5. Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space. Its *Clifford algebra* is

$$\mathrm{C}\ell(V) = T(V)/(v \otimes v + \langle v, v\rangle 1).$$

Here, $T(V)$ is the tensor algebra, and we quotient by the ideal generated by the given relation.

That is, we've forced (2.2) for a vector paired with itself. That's actually sufficient to imply it for all pairs of vectors.

Remark. Though we only defined the Clifford algebra for nondegenerate inner products, the same definition can be made for all bilinear pairings. If one chooses $\langle\cdot,\cdot\rangle = 0$, one obtains the exterior algebra $\Lambda(V)$, and we'll see that Clifford algebras sometimes behave like exterior algebras.

Recall that the tensor algebra is defined by the following universal property: if A is any algebra,[9] and $f : V \to A$ is linear, then there exists a unique homomorphism of algebras $\widetilde{f} : T(V) \to A$ such that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d] & A \\ T(V). \arrow[ur, "\widetilde{f}"'] & \end{tikzcd}$$

That is, as soon as I know what happens to elements of f, I know what to do to tensors.

This implies a universal property for the Clifford algebra.

Proposition 2.6. *Let A be an algebra and $f : V \to A$ be a linear map. Then, $f(v)^2 = -\langle v, v\rangle 1_A$ iff f extends uniquely to a map $\widetilde{f} : \mathrm{C}\ell(V) \to A$ such that the following diagram commutes:*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d] & A \\ \mathrm{C}\ell(V). \arrow[ur, "\widetilde{f}"'] & \end{tikzcd}$$

The map $V \to \mathrm{C}\ell(V)$ is the composition $V \hookrightarrow T(V) \twoheadrightarrow \mathrm{C}\ell(V)$, where the last map is projection onto the quotient.

We'll end up putting lots of structure on Clifford algebras: a $\mathbb{Z}/2$-grading, a $\mathbb{Z}$-filtration, a canonical vector-space isomorphism with the exterior algebra, and so forth.

Important Example 2.7. Let $\Lambda^\bullet V$ denote the exterior algebra on V, the graded algebra whose k^{th} graded piece is wedges of k vectors: $\Lambda^k(V) = \{v_1, \dots, v_k \mid v_j \in V\}$, with the relations $v \wedge w = -w \wedge v$.

Given a $v \in V$, we can define two maps, *exterior multiplication* $\varepsilon(v) : \Lambda^\bullet(V) \to \Lambda^{\bullet-1}(V)$ defined by $\mu \mapsto v \wedge \mu$, and *interior multiplication* $i(v) : \Lambda^\bullet(V) \to \Lambda^{\bullet-1}(V)$ sending

$$v_1 \wedge \cdots \wedge v_k \mapsto \sum_{i=1}^{k} (-1)^{i-1} \langle v, v_i\rangle v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_k,$$

where $\widehat{v_i}$ means the absence of the i^{th} term.

This has a few important properties:

(1) Both of these maps are idempotents: $\varepsilon(v)^2 = i(v)^2 = 0$.

(2) If $\mu_1, \mu_2 \in \Lambda^\bullet(V)$, then

$$i(v)(\mu_1 \wedge \mu_2) = (i(v)\mu_1) \wedge \mu_2 + (-1)^{\deg \mu_1} \mu_1 \wedge i(v)\mu_2.$$

In particular,

$$\varepsilon(v) i(v) + i(v)\varepsilon(v) = \langle v, v\rangle. \tag{2.8}$$

[9] Here, an algebra is a unital ring with a compatible real vector space structure.

We can use this to define a representation of the Clifford algebra onto the exterior algebra: define a map $c : V \to \mathrm{End}(\Lambda^\bullet(V))$ by $c(v) = \varepsilon(v) - i(v)$. Then, $c(v)^2 = -(\varepsilon(v)i(v) + i(v)\varepsilon(v)) = \langle v, v\rangle$, so by the universal property, c extends to a homomorphism $c : \mathrm{C}\ell(V) \to \mathrm{End}(\Lambda^* V)$.

Given an inner product on V, there is an induced inner product on $\Lambda^\bullet V$: choose an orthonormal basis $\{e_1, \dots, e_n\}$ for V, and then declare the basis $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \mid 1 \le i_1 < \cdots < i_k \le n\}$ to be orthonormal; then, use the dot product associated to that orthonormal basis. This is coordinate-invariant, however.

Theorem 2.9. *Suppose $\{e_1, \dots, e_n\}$ is a basis for V. Then, $\{e_{i_1} e_{i_2} \cdots e_{i_k} \mid i_1 < i_2 < \cdots i_k\}$ (where the product is in the Clifford algebra) is a vector-space basis for $\mathrm{C}\ell(V)$.*

Today, we'll focus on examples, and perhaps prove this later. This tells us that v and w anticommute iff $v \perp w$, and the relations are

$$e_j e_j = \begin{cases} -e_j e_i, & i \neq j \\ -1, & i = j. \end{cases}$$

This is just like the exterior algebra, but deformed: if $i = j$, we get 1 rather than 0. Theorem 2.9 also tells us that $\dim \mathrm{C}\ell(V) = 2^{\dim V}$.

Example 2.10. $\mathrm{C}\ell(\mathbb{R}^2) \cong \mathbb{H}$ as $\mathbb{R}$-algebras: $\mathrm{C}\ell(\mathbb{R}^2)$ is generated by 1, e_1, and e_2 such that $e_1 e_2 = -e_2 e_1$ and $e_1^2 = e_2^2 = -1$. Thus, $\{1, e_1, e_2, e_1 e_2\}$ is a basis for $\mathrm{C}\ell(\mathbb{R}^2)$, and $(e_1 e_2)^2 = e_1 e_2 e_1 e_2 = -e_1^2 e_2^2 = -1$.

Thus, the isomorphism $\mathrm{C}\ell(\mathbb{R}^2) \to \mathbb{H}$ extends from $1 \mapsto 1$, $e_1 \mapsto i$, $e_2 \mapsto j$, and $e_1 e_2 \mapsto k$.

Example 2.11. Even simpler is $\mathrm{C}\ell(\mathbb{R}) \cong \mathbb{C}$, generated by 1 and e_1 such that $e_1^2 = -1$.

Example 2.12. If we consider the Clifford algebra of $\mathbb{C}$ as a complex vector space, $\mathbb{C}$ is in the center, so $\mathrm{C}\ell_{\mathbb{C}}(\mathbb{C})$ is generated by 1 and e_1 with $ie_1 = e_1 i$.

Lecture 3.

The Structure of the Clifford Algebra: 9/1/16

Last time, we started with an inner product space $(V, \langle \cdot, \cdot \rangle)$ and used it to define a Clifford algebra $\mathrm{C}\ell(V) = T(V)/(v \otimes v + \langle v, v\rangle 1)$, the free algebra generated by V such that $v^2 = -\langle v, v\rangle$.[10] For a $v \in V$, let $\widetilde{v} \in \mathrm{C}\ell(V)$ be its image under the natural map $V \to T(V) \twoheadrightarrow \mathrm{C}\ell(V)$: the first map sends a vector to a degree-1 tensor, and the second is the quotient map. It's reasonable to assume this map is injective, and in fact we'll be able to prove this, so we may identify V with its image in the Clifford algebra.

We also defined a representation of $\mathrm{C}\ell(V)$ on $\Lambda^\bullet V$, which was an algebra homomorphism $c : \mathrm{C}\ell(V) \to \mathrm{End}(\Lambda^\bullet V)$ that is defined uniquely by saying that $c(\widetilde{v}) = \varepsilon(v) - i(v)$ (exterior multiplication minus interior multiplication, also known as wedge product minus contraction). We checked that this squares to scalar multiplication by $-\langle v, v\rangle$, so it is an algebra homomorphism.

Definition 3.1. The *symbol map* is the linear map $\sigma : \mathrm{C}\ell(V) \to \Lambda^\bullet V$ defined by $u \mapsto c(u) \cdot 1$.

Theorem 2.9 defines a basis for the Clifford algebra; we can use this to prove it.

Lemma 3.2. *The map $V \to \mathrm{C}\ell(V)$ sending $v \mapsto \widetilde{v}$ is injective.*

Proof. For $v \in V$, $\sigma(v) = c(\widetilde{v})1 = \varepsilon(v) \cdot 1 - i(v) \cdot 1$. Since interior multiplication lowers degree, $i(v) = 0$, so $\sigma(v) = v$. Thus, the map $V \to \mathrm{C}\ell(V)$ is injective. ⊠

We will identify v and $\widetilde{v}$, and just think of V as a subspace of $\mathrm{C}\ell(V)$.

Proposition 3.3. *The symbol map is an isomorphism of vector spaces.*

Proof. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of V, so $e_i e_j = -e_j e_i$ unless $i = j$, in which case it's -1. So $\mathrm{C}\ell(V) = \mathrm{span}\{e_{i_1} e_{i_2} \cdots e_{i_k} \mid i_1 < i_2 < \cdots < i_k\}$. We'll show these are linearly independent, hence form a basis for $\mathrm{C}\ell(V)$, and recover Theorem 2.9 as a corollary.

[10] There are different conventions here; sometimes people work with the relation $v^2 = \langle v, v\rangle$. This is a different algebra in general over $\mathbb{R}$, but over $\mathbb{C}$ they're the same thing.

Since

$$c(e_i)e_{j_1}\wedge e_{j_2}\wedge\cdots\wedge e_{j_k} = e_i\wedge e_{j_1}\wedge\cdots\wedge e_{j_k} - i(e_i)\big(e_{j_1}\wedge\cdots\wedge e_{j_k}\big)$$
$$= e_i\wedge e_{j_1}\wedge\cdots\wedge e_{j_k}$$

if all indices are distinct, so

$$\begin{aligned} a\sigma(e_{i_1}\cdots e_{i_k}) &= c(e_{i_1})\cdots c(e_{i_k})1\\ &= c(e_1)\cdots \underbrace{c(e_{i_{k-1}})e_{i_k}}_{e_{i_{k-1}}\wedge e_{i_k}}\\ &= e_{i_1}\wedge\cdots\wedge e_{i_k}. \end{aligned}$$

As $\{i_1,\dots,i_k\}$ ranges over all k-element subsets of $\{1,\dots,n\}$, these form a basis for $\Lambda^\bullet V$. Thus, σ is surjective, and the proposed basis for $\mathrm{C}\ell(V)$ is indeed linearly independent. Thus, σ is also injective, so an isomorphism of vector spaces. ⊠

In particular, we've discovered a basis for $\mathrm{C}\ell(V)$, proving Theorem 2.9.

Remark. The symbol map is *not* an isomorphism of algebras: $\sigma(v^2)=\sigma(-\langle v,v\rangle)=-\langle v,v\rangle$, but $\sigma(v)\wedge\sigma(v)=0$. The symbol is just the highest-order data of an element of the Clifford algebra.

The proof of the following proposition is an (important) exercise.

Proposition 3.4.

$$Z(\mathrm{C}\ell(V))=\begin{cases}\mathbb{R}, & \dim V \text{ is even}\\ \mathbb{R}\oplus\mathbb{R}\gamma, & \dim V\text{ is odd,}\end{cases}$$

where $\gamma=e_1\cdots e_n$ is σ^{-1} of a volume form.

Physicists sometimes call the span of γ *pseudoscalars*, since they commute with everything (in odd degree), much like scalars.

Algebraic structures on the Clifford algebra. Recall that an algebra A is called $\mathbb{Z}$*-graded* if it has a decomposition as a vector space

$$A=\bigoplus_n \in \mathbb{Z}A_n$$

where the multiplicative structure is additive in this grading: $A_j\cdot A_k\subset A_{j+k}$. For example, $\mathbb{R}[x]$ is graded by the degree; the tensor algebra $T(V)$ is graded by degree of tensors, and $\Lambda^\bullet V$ is graded with the n^{th} piece equal to the space of n-forms.

The Clifford algebra is not graded: the square of a vector is a scalar. It admits a weaker structure, called a filtration.

Definition 3.5. An algebra A has a *filtration* (by $\mathbb{Z}$) if there is a sequence of subspaces $A^{(0)}\subset A^{(1)}\subset A^{(2)}\subset\cdots$ such that $A=\bigcup_j A^{(j)}$ and $A^{(j)}\cdot A^{(k)}\subset A^{(j+k)}$.

The key difference is that for a filtration, the different levels can intersect in more than 0.

The Clifford algebra is filtered, with $\mathrm{C}\ell(V)^{(j)}=\operatorname{span} v_1\cdots v_k\mid k\le j, v_1,\dots,v_k\in V$, the span of products of at most j vectors.

Another way we can weaken the pined-for $\mathbb{Z}$-grading is to a $\mathbb{Z}/2$-grading, which we can actually put on the Clifford algebra.

Definition 3.6. A $\mathbb{Z}/2$*-grading* of an algebra A is a decomposition $A=A^+\oplus A^-$ as vector spaces, such that $A^+A^+\subset A^+$, $A^+A^-\subset A^-$, $A^-A^+\subset A^-$, and $A^-A^-\subset A^+$. A^- is called the *odd part* or the *negative part* of A, and A^+ is called the *even part* or the *positive part*. In physics, a $\mathbb{Z}/2$-graded algebra is also called a *superalgebra*.

For the Clifford algebra, let $\mathrm{C}\ell(V)^+$ be the subspace spanned by products of odd numbers of vectors, and $\mathrm{C}\ell(V)^-$ be the subspace spanned by products of even numbers of vectors. Then, $\mathrm{C}\ell(V)=\mathrm{C}\ell(V)^+\oplus\mathrm{C}\ell(V)^-$, and this defines a $\mathbb{Z}/2$-grading.

Definition 3.7. Let $A = \bigcup_j A^{(j)}$ be a filtered algebra. Then, its *associated graded* is

$$\operatorname{gr} A = \bigoplus_j A^{(j)}/A^{(j-1)},$$

which is naturally a graded algebra with $(\operatorname{gr} A)^j = A^{(j)}/A^{(j-1)}$ and multiplication inherited from A.

Proposition 3.8. *The associated graded of the Clifford algebra* $\operatorname{gr} \mathrm{C}\ell(V) = \Lambda^\bullet V$.

This ultimately follows because the isomorphism $\mathrm{C}\ell(V)^{(j)}/\mathrm{C}\ell(V)^{(j-1)} \to \Lambda^j V$ sends $u \mapsto \sigma(u)_[j]$: the exterior algebra remembers the top part of the Clifford multiplication.[11]

Constructing spin groups. Now, we assume that the inner product on V is positive definite.

If $v \neq 0$ in V, then v^{-1} exists in $\mathrm{C}\ell(V)$ and is equal to $-v/\langle v, v\rangle$. For a $w \in V$, let $\rho_v(W)$ be conjugation: $\rho_v(W) = -vwv^{-1}$. Then, $\rho_v(v) = -v$, and if $w \perp v$, then $\rho_v(w) = -vwv^{-1} = wvv^{-1} = w$, so ρ_v preserves $\operatorname{span} v^\perp$ and sends $v \mapsto -v$. Thus, it's a reflection across $\operatorname{span} v \perp$.

Theorem 3.9. *The orthogonal group* $\mathrm{O}(n)$ *is generated by reflections, and everything in* $\mathrm{SO}(n)$ *is a product of an even number of reflections.*

Proof. Let's induct on n. When $n = 1$, $\mathrm{O}(1) = \{\pm 1\}$, for which this is vacuously true. Now, let $A \in \mathrm{O}(n)$. If A fixes an $e_1 \in \mathbb{R}^n$, then A fixes $\operatorname{span} e_1^\perp$, so by induction, $A|_{\operatorname{span} e_1^\perp} = R_1 \cdots R_k$ for some reflections $R_1, \dots, R_k \in \mathrm{O}(n-1)$. These reflections include into $\mathrm{O}(n)$ by fixing $\operatorname{span} e_1$, and are still reflections, so $A = R_1 \cdots R_k$ decomposes A as a product of reflections.

Alternatively, suppose $Ae_1 = v \neq e_1$. Let R be a reflection about $\{v - e_1\}^\perp$; then, R exchanges v and e_1. Hence, $RA \in \mathrm{O}(n)$ and fixes e_1, so by above $RA = R_1 \cdots R_k$ for some reflections, and therefore $A = RR_1 \cdots R_k$ is a product of reflections.

For $\mathrm{SO}(n)$, observe that each reflection has determinant -1, but all rotations in $\mathrm{SO}(n)$ have determinant 1, so no $A \in \mathrm{SO}(n)$ can be a product of an odd number of rotations. ⊠

We've defined reflections ρ_v in the Clifford algebra, so if we can act by orientation-preserving reflections with a $\mathbb{Z}/2$ kernel, we should have described the spin group.

This reflection ρ_v is a restriction of the *twisted adjoint action*, a representation of $\mathrm{C}\ell(V)^\times$ on $\mathrm{C}\ell(V)$: $u_1 \mapsto \rho_{u_1}$ that sends $u_2 \mapsto \alpha(u_1)u_2u_1^{-1}$ for a $u_1 \in \mathrm{C}\ell(V)^\times$ and $u_2 \in \mathrm{C}\ell(V)$. Here,

$$\alpha(u_1) = \begin{cases} u_1, & u_1 \in \mathrm{C}\ell(V)^+ \\ -u_1, & u_1 \in \mathrm{C}\ell(V)^-. \end{cases}$$

We showed that for $v \in V \setminus 0$, ρ_v preserves V and is a reflection; since $\rho_{cv} = \rho_v$ for $c \in \mathbb{R} \setminus 0$, we want to restrict to the unit circle of v such that $\langle v, v\rangle = 1$. But we will restrict further.

Definition 3.10. Let $\mathrm{Spin}(V)$ denote the subgroup of $\mathrm{C}\ell(V)^\times$ consisting of products of even numbers of unit vectors.

First question: what scalars lie in the spin group? Clearly ± 1 come from u^2 for unit vectors u, but we can do no better (after all, unit length is a strong condition on the real line).

Proposition 3.11. $\mathrm{Spin}(V) \cap \mathbb{R} \setminus 0 = \{\pm 1\}$ *inside* $\mathrm{C}\ell(V)^\times$.

Theorem 3.12. *The map* $\mathrm{Spin}(V) \to \mathrm{SO}(V)$ *sending* $u \mapsto \rho_u$ *is a nontrivial (connected) double cover when* $\dim V \geq 2$.

This implies $\mathrm{Spin}(V)$ is the unique connected double cover of $\mathrm{SO}(V)$, agreeing with the abstract construction for the spin group we constructed in the first two lectures.

Proof. We know $\rho_u \in \mathrm{SO}(V)$ because it's an even product of reflections, using Theorem 3.9, and that ρ is surjective. We also know $\ker \rho = \{u \in \mathrm{Spin}(V) \mid uv = vu \text{ for all } v \in V\}$. But since V generates $\mathrm{C}\ell(V)$ as an algebra, $\ker(\rho) = \mathrm{Spin}(V) \cap Z(\mathrm{C}\ell(V)) = \{\pm 1\}$ by Propositions 3.4 and 3.11.

Thus ρ is a double cover, so it remains to show it's nontrivial. To rule this out, it suffices to show that we can connect -1 and 1 inside $\mathrm{Spin}(V)$, because they project to the same rotation. Let $\gamma(t) = \cos(\pi t) + \sin(\pi t)e_1e_2$ (since $\dim V \geq 2$, I can take two orthogonal unit vectors). Thus, $\gamma(t) = 1$, $\gamma(1) = -1$, and $\gamma(t) = e_1(-\cos(\pi t)e_1 + \sin(\pi t)e_2)$, so it's always a product of even numbers of unit vectors, and thus a path within $\mathrm{Spin}(V)$. ⊠

[11]There is a sense in which this defines the Clifford algebra as a deformation of the exterior algebra; a fancy word for this would be *filtered quantization*. Similarly, we'll see that the symmetric algebra is the associated graded of the symmetric algebra.

This is actually the simplest proof that $\dim \mathrm{Spin}(V) = \dim \mathrm{SO}(V)$.

Next week, we'll discuss representations of the spin group.

Lecture 4.

Representations of $\mathfrak{so}(n)$ and $\mathrm{Spin}(n)$: 9/6/16

One question from last time: we constructed $\mathrm{Spin}(n)$ as a subset of the group of units of a Clifford algebra, but how does that induce a linear structure? There's two ways to do this. The first is to say that this *a priori* only constructs $\mathrm{Spin}(n)$ as a topological group; this group double covers $\mathrm{SO}(n)$, and hence must be a Lie group. Alternatively, this week, we'll explicitly realize $\mathrm{Spin}(n)$ as a closed subgroup of a matrix group, which therefore must be a Lie group.

Last time, we constructed the spin group $\mathrm{Spin}(V)$ as a subset of the units $\mathrm{C}\ell(V)^\times$, and found a double cover $\mathrm{Spin}(V(\to \mathrm{SO}(V)$. Thus, there should be an isomorphism of Lie algebras $\mathfrak{spin}(V) \xrightarrow{\sim} \mathfrak{so}(V)$. The former is a subspace of $T_1 \mathrm{C}\ell(V) \cong \mathrm{C}\ell(V)$ (since $\mathrm{C}\ell(V)$ is an affine space, as a manifold) and $\mathfrak{so}(V) \subset \mathfrak{gl}(V) = V \otimes V^*$ (and with an inner product, is also identified witrh $V \otimes V$). This identification extends to an isomorphism (of vector spaces) $\mathfrak{so}(V) \cong \Lambda^2 V$; composing with the inverse of the symbol map defines a map $\mathfrak{so}(V) \to \Lambda^2 V \to \mathrm{C}\ell(V)$.

Exercise 4.1. $\mathfrak{so}(V)$ and $\mathrm{C}\ell(V)$ both have Lie algebra structures, the former as a Lie group and the latter from the usual commutator bracket. Show that these agree, so the above map is an isomorphism of Lie algebras, and that the image of this map is $\mathfrak{spin}(V)$.

Today, we're going to discuss the representation theory (over $\mathbb{C}$) of the Lie algebra $\mathfrak{so}(V)$. Since $\mathrm{Spin}(V)$ is the simply connected Lie group with $\mathfrak{so}(V)$ as its Lie algebra, this provides a lot of information on the representation theory of $\mathrm{Spin}(V)$. In general, not all of these representations arise as representations on $\mathrm{SO}(n)$: consider the representation $\mathrm{Spin}(3) = \mathrm{SU}(2)$ on$\mathbb{C}^2$ where -1 exchanges $(1,0)$ and $(0,1)$. This doesn't descend to $\mathrm{SO}(3)$, because -1 is in the kernel of the double cover map. Such a representation is called a *spin representation*.

The name comes from physics: traditionally, physicists idenfied a Lie group with its Lie algebras, but they found that these kinds of representations didn't correspond to $\mathrm{SO}(3)$-representations. These arose in physical systems as particles with spin, in quantum mechanics:[12] a path connected -1 and 1 in $\mathrm{SU}(2)$ is a "rotation" of 260°, but isn't the identity.

Anyways, we're going to talk about the representation theory of this group; in order to do so, we should briefly discuss the representation theory of Lie groups and (semisimple) Lie algebras.

Definition 4.2. Fix $\mathbb{F}$ to be either $\mathbb{R}$ or $\mathbb{C}$.

- An $\mathbb{F}$-*representation of a Lie group* G is a Lie group homomorphism $\rho : G \to \mathrm{GL}(V, \mathbb{F})$, where V is an $\mathbb{F}$-vector space.
- An $\mathbb{F}$-*representation of a real Lie algebra* $\mathfrak{g}$ is a real Lie algebra homomorphism $\tau : \mathfrak{g} \to \mathfrak{gl}(V, k)$, where V is an $\mathbb{F}$-vector space.

We will often suppress the notation as $\rho(g)v = g \cdot v$ or $\tau(X)v = Xv$, where $g \in G$, $X \in \mathfrak{g}$, and $v \in V$, when it is unambiguous to do so. Moreover, our representations, at least for the meantime, will be finite-dimensional.

Proposition 4.3. *Let $\mathfrak{g}$ be a real Lie algebra and V be a complex vector space. Then, there is a one-to-one correspondence between representations of $\mathfrak{g}$ on V and the $\mathbb{C}$-Lie algebra homomorphisms $\mathfrak{g} \otimes \mathbb{C} \to \mathfrak{gl}(V, \mathbb{C})$.*

Here, $\mathfrak{g} \otimes \mathbb{C}$ is the complex Lie algebra whose underlying vector space is $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ with bracket extending complex linearly from the assignment

$$[X \otimes c_1, Y \otimes c_2] = [X, Y] \otimes c_1 c_2,$$

where $X, Y \in \mathfrak{g}$ and $c_1, c_2 \in \mathbb{C}$.

Proof of Proposition 4.3. Let ρ be a $\mathfrak{g}$-representation on V; then, define $\rho_{\mathbb{C}} : \mathfrak{g} \otimes \mathbb{C} \to \mathfrak{gl}(V, \mathbb{C})$ to be the unique map extending $\mathbb{C}$-linearly from $X \otimes c \mapsto c\rho(X)$.

Conversely, given a complex representation $\rho_{\mathbb{C}}$, define $\rho : \mathfrak{g} \to \mathfrak{gl}(V, \mathbb{C})$ to be $X \mapsto \rho_{\mathbb{C}}(X \otimes 1)$. ⊠

Given a Lie group representation $G \to \mathrm{GL}(V, \mathbb{F})$, one obtains a Lie algebra representation of $\mathfrak{g} = \mathrm{Lie}(G)$ by differentiation.

[12]There is one macroscopic example of spin-1/2 phenomena: see http://www.smbc-comics.com/?id=2388.

Proposition 4.4. *If G is a connected, simply-connected Lie group, then this defines a bijective correspondence between the Lie group representations of G and the Lie algebra representations of $\mathfrak{g}$.*

If G is connected, but not simply connected, let $\widetilde{G}$ denote its universal cover. Then, there's a discrete central subgroup $\Gamma \leq Z(\widetilde{G}) \leq G$ such that $G = \widetilde{G}/\Gamma$. This allows us to extend Proposition 4.4 to groups that may not be simply connected.

Proposition 4.5. *Let G be a connected Lie group, $\widetilde{G}$ be its universal cover, and Γ be such that $G = \widetilde{G}/\Gamma$. Then, differentiation defines a bijective correspondence between the representations of G and the representations of $\widetilde{G}$ on which Γ acts trivially.*

It would also be nice to understand when two representations are the same. More generally, we can ask what a homomorphism of two representations are.

Definition 4.6. Let G be a Lie group. A *homomorphism of G-representations* from $\rho_1 : G \to \mathrm{GL}(V)$ to $\rho_2 : G \to \mathrm{GL}(W)$ is a linear map $T : V \to W$ such that for all $g \in G$, $T \circ \rho_1(g) = \rho_2(g) \circ T$. If T is an isomorphism of vector spaces, this defines an *isomorphism of G-representations*.

Example 4.7.

(1) $\mathrm{SO}(n)$ can be defined as a group of $n \times n$ matrices, which act by matrix multiplication on $\mathbb{C}^n$. This is a representation, called its *defining representation*. This works for every matrix group, including $\mathrm{SL}(n)$ and $\mathrm{SU}(n)$.

(2) The determinant is a smooth map $\det : \mathrm{GL}(n, \mathbb{C}) \to \mathbb{C}^\times$ such that $\det(AB) = \det A \det B$, hence a Lie group homomorphism. Since $\mathbb{C}^\times = \mathrm{GL}(1, \mathbb{C})$, this is a one-dimensional representation of $\mathrm{GL}(n, \mathbb{C})$.

(3) Fix a $c \in \mathbb{C}$ and let $\rho_c : \mathbb{R} \to \mathfrak{gl}(1, \mathbb{C})$ send $t \mapsto ct$. We can place a Lie algebra structure on $\mathbb{R}$ where $[\cdot,\cdot] = 0$, so that ρ defines a Lie algebra representation.

The simply connected Lie group with this Lie algebra is $(\mathbb{R}, +)$, and ρ_c integrates to the Lie group representation $(\mathbb{R}, +) \to \mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^\times$ sending $s \mapsto e^{isc}$. But S^1 has the same Lie algebra as $\mathbb{R}$, and the covering map is the quotient $\mathbb{R} \twoheadrightarrow \mathbb{R}/2\pi\mathbb{Z}$. In particular, this acts trivially iff $c \in \mathbb{Z}$, which is precisely when $s \mapsto e^{isc}$ is 2π-periodic.

There are various ways to build new representations out of old ones.

Definition 4.8. Let G be a Lie group and V and W be representations of G.

- The *direct sum* of V and W is the representation on $V \otimes W$ defined by
$$g \cdot (v, w) = (g \cdot v, g \cdot w).$$
- The *tensor product* is the representation on $V \otimes W$ extending uniquely from
$$g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w).$$
- The *dual representation* to V is the representation on V^* (the dual vector space) in which g acts as its inverse transpose on $\mathrm{GL}(V^*)$.

The same definition applies *mutatis mutandis* when the Lie group G is replaced with a Lie algebra $\mathfrak{g}$, and the inverse transpose is replaced with -1 times the transpose for the dual representation.

Note that, unlike for vector spaces, it can happen that a representation isn't isomorphic to its dual, even after picking an inner product.

Definition 4.9. Let V be a representation of a group G.

- A *subrepresentation* is a subspace $W \subset V$ such that $g \cdot w \in W$ for all $w \in W$ and $g \in G$.
- V is *irreducible* if it has no nontrivial subrepresentations (here, nontrivial means "other than $\{0\}$ and V itself"). Sometimes, "irreducible representation" is abbreviated "irrep" at the chalkboard.

These definitions apply *mutatis mutandis* to representations of a Lie algebra $\mathfrak{g}$.

In nice cases, knowing the irreducible representations tells you everything.

Theorem 4.10. *For $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$, there are finitely many isomorphism classes of irreducible representations, and every representation is isomorphic to a subrepresentation of a direct sum of tensor products of these representations.*

Definition 4.11. A Lie algebra $\mathfrak{g}$ whose representations have the property from Theorem 4.10 is called *semisimple*.[13]

[13] This is equivalent to an alternate definition, where $\mathfrak{g}$ is *simple* if $\dim \mathfrak{g} > 1$ and $\mathfrak{g}$ has no nontrivial ideals, and $\mathfrak{g}$ is *semisimple* if it is a direct sum of simple Lie algebras.

In fact, we know these irreducibles explicitly: for n even, all of the irreducible representations of $\mathfrak{so}(n,\mathbb{C})$ are exterior powers of the defining representation, except for two *half-spinor representations*; for n odd, we just have one spinor representation.

Constructing the spin representations. Let V be an n-dimensional vector space over $\mathbb{R}$ with a positive definite inner product. We'll construct the spinor representations of $\mathrm{Spin}(V)$ as restrictions of the $\mathrm{C}\ell(V)$ action on a $\mathrm{C}\ell(V)$-module, which act in a way compatible with the $\mathbb{Z}/2$-grading on $\mathrm{C}\ell(V)$.

Recall that a superalgebra is a scary word for a $\mathbb{Z}/2$-graded algebra.

Definition 4.12. Let $A = A^+ \oplus A^-$ be a superalgebra. A $\mathbb{Z}/2$-*graded module* over A (or a *supermodule* for A) is an A-module with a vector-space decomposition $M = M^+ \oplus M^-$ such that $A^\pm M^\pm \subset M^+$ and $A^\pm M^\mp \subset M^0$.

Example 4.13. One quick example is that every superalgebra acts on itself by multiplication; this *regular representation* is $\mathbb{Z}/2$-graded by the product rule on a superalgebra.

Since $\mathrm{Spin}(V) \subset \mathrm{C}\ell(V)^+$, any supermodule defines two representations of $\mathrm{Spin}(V)$, one on M^+ and the other on M^-.

Since we just care about complex representations, we may as well complexify the Lie algebra, looking at $\mathrm{C}\ell(V) \otimes \mathbb{C}$.

Exercise 4.14. Show that $\mathrm{C}\ell(V) \otimes \mathbb{C} \cong \mathrm{C}\ell(V \otimes \mathbb{C})$ (the latter is the Clifford algebra on a complex vector space).

Working with this complexified Clifford algebra simplifies things a lot.

First, let's assume $n = 2m$ is even. Then, we may choose an *orthogonal complex structure* J on V, i.e. a linear map $J : V \to V$ such that $J^2 = 1$ and $\langle Jv, Jw\rangle = \langle v, w\rangle$. For example, if $\{e_1, f_1, \dots, e_m, f_m\}$ is an orthogonal basis for V, then we can define $J(e_j) = f_j$ and $J(f_j) = -e_j$. Thus, such a structure always exists; conversely, given any orthogonal complex structure J, there exists a basis on which J has this form. In other words, J allows V to be thought of as an n-dimensional complex vector space.

We'll return to this on Thursday, using it to construct the supermodule.

Lecture 5.

The Half-Spinor and Spinor Representations: 9/8/16

Let's continue where we left off from last time. We had a real inner product space V of dimension n; our goal was to construct $\mathbb{C}$-supermodules over the Clifford algebra, in order to access the spinor representations. This is most interesting when n is even; a lot of the fancy theorems we consider later in the class (Riemann-Roch, index theorems, etc.) either don't apply or are trivial when n is odd.

The even-dimensional case. Thus, we first assume $n = 2m$ is even; we can choose an orthogonal complex structure J on V, which is a linear map $J : V \to V$ that squares to 1 and is compatible with the inner product in the sense that $\langle Jv, Jw\rangle = \langle v, w\rangle$. Since $J^2 = I$, its only possible eigenvalues are $\pm i$. Let $V^{0,1}$ be the i-eigenspace for J acting on $V \otimes \mathbb{C}$, and $V^{0,1}$ be the $-i$-eigenspace; then, $\overline{V^{0,1}} = V^{1,0}$. Last time, we found a compatible orthonormal basis $\{e_1, f_1, \dots, e_m, f_m\}$, where $Je_j = f_j$ and $Jf_j = -e_j$; in this basis,

$$V^{1,0} = \mathrm{span}_{\mathbb{C}}\{e_j - if_j \mid j = 1, \dots, m\}$$
$$V^{0,1} = \mathrm{span}_{\mathbb{C}}\{e_j + if_j \mid j = 1, \dots, m\}.$$

Both $V^{0,1}$ and $V^{1,0}$ are *isotropic*, meaning the $\mathbb{C}$-linear extension of the inner product restricts to 0 on each of $V^{0,1}$ and $V^{1,0}$.

The decomposition $V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$ induces the decomposition

$$\Lambda^\bullet(V \otimes \mathbb{C}) = \Lambda^\bullet V^{1,0} \mathbin{\widehat{\otimes}} \Lambda^\bullet V^{0,1}.$$

Here, $\widehat{\otimes}$ denotes the graded tensor product, which is graded-commutative rather than commutative. $\Lambda^\bullet(V \otimes \mathbb{C})$ is a $\mathrm{C}\ell(V)$-module with the action $c(v) = \varepsilon(v) - i(v)$. We can restrict this action to $\Lambda^\bullet V^{0,1}$ and define

$$\widetilde{c}(v) = \sqrt{2}\big(\varepsilon(v^{0,1}) - i(v^{1,0})\big) \in \mathrm{End}\,\Lambda^\bullet V^{0,1}.$$

Here, we use the direct sum to uniquely write any $v \in V$ and $v = v^{1,0} + v^{0,1}$ with $v^{0,1} \in V^{0,1}$ and $v^{1,0} \in V^{1,0}$. In this case,

$$\begin{aligned}\widetilde{c}(v)^2 &= -2\big(\varepsilon(v^{0,1})i(v^{1,0}) + i(v^{1,0})\varepsilon(v^{0,1})\big)\\ &= -2\big(\varepsilon(v^{0,1})i(v^{1,0}) + \langle v^{1,0}, v^{0,1}\rangle - \varepsilon(v^{0,1})i(v^{1,0})\big)\\ &= -2\langle v^{1,0}, v^{0,1}\rangle.\end{aligned}$$

Since $V^{1,0}$ and $V^{0,1}$ are both isotropic,

$$= -\langle v^{1,0} + v^{0,1}, v^{1,0} + v^{0,1}\rangle = \langle v, v\rangle.$$

Thus, by the universal property, $\widetilde{c}$ extends to an algebra homomorphism $\widetilde{c} : \mathrm{C}\ell(V \otimes \mathbb{C}) \to \mathrm{End}_{\mathbb{C}} \Lambda^\bullet V^{0,1}$, and this is naturally compatible with the gradings: the $\mathbb{Z}$-grading on $\Lambda^\bullet V^{0,1}$ induces a $\mathbb{Z}/2$-grading into even and odd parts. Since $\widetilde{c}$ is an odd endomorphism, it gives $\Lambda^\bullet V^{0,1}$ the structure of a $\mathrm{C}\ell(V \otimes \mathbb{C})$-supermodule.

Theorem 5.1. *In fact,* $\widetilde{c} : \mathrm{C}\ell(V \otimes \mathbb{C}) \to \mathrm{End}\,\Lambda^\bullet V^{0,1}$ *is an isomorphism, explicitly realizing* $\mathrm{C}\ell(V \otimes \mathbb{C})$ *as a matrix algebra.*

Proof sketch. $\dim_{\mathbb{C}} \mathrm{C}\ell(V \otimes \mathbb{C}) = 2^n$ and $\dim_{\mathbb{C}} \mathrm{End}_{\mathbb{C}} \Lambda^\bullet V^{0,1} = \big(\dim_{\mathbb{C}} \Lambda^\bullet V^{0,1}\big)^2 = 2^{2m} = 2^n$. The dimensions match up, so it suffices to check $\widetilde{c}$ is one-to-one, which one can do inductively by checking in a basis. ⊠

By restriction, we obtain two representations of $\mathrm{Spin}(V) \subset \mathrm{C}\ell(V \otimes \mathbb{C})^\times$ on $\Lambda^{2\mathbb{Z}} V^{0,1}$ and $\Lambda^{2\mathbb{Z}+1} V^{1,0}$.

Definition 5.2. These (isomorphism classes of) representations are called the *half-spinor representations* of $\mathrm{Spin}(V)$, denoted $S^+ = \Lambda^{2\mathbb{Z}} V^{0,1}$ and $S^- = \Lambda^{2\mathbb{Z}+1} V^{0,1}$.

A priori, we don't actually know whether these are the same representation.

Proposition 5.3. *As* $\mathrm{Spin}(V)$*-representations,* $S^+ \not\cong S^-$.

Proof. The trick for this and similar statements, the trick is to look at the action of the pseudoscalar, the image of the volume element under the symbol map: $\gamma = e_1 \cdots e_m f_1 \cdots f_m \in \mathrm{Spin}(V)$. This is in the center of the Clifford algebra if n is odd, but not for n even: here $\gamma v = -v\gamma$. Thus, γ is in the center of the even part of the Clifford algebra, hence in the center of the even part of $\mathrm{End}(S^+ \oplus S^-)$. This center is isomorphic to $\mathbb{C} \oplus \mathbb{C}$, one for the center of $\mathrm{End}\, S^+$ and the other for the center of $\mathrm{End}\, S^-$. Thus, γ acts as a scalar $k_+ \in \mathbb{C}^\times$ on S^+, and a scalar $k_- \in \mathbb{C}^\times$ in S^-. We'll show these are different.

We only need to check on one element $\psi \in S^+$, so $\gamma \cdot \psi = k_+\psi$. If $v \in V$ is nonzero, then $v\psi \in S^-$, so $\gamma(v\psi) = k_- v\psi$, but $\gamma(v\psi) = (\gamma v)\psi = -v\gamma\psi = -k_+ v\psi$, so $k_- = -k_+$. Since the action of γ is nontrivial, then these constants are distinct, and therefore these representations are nonisomorphic. ⊠

Moreover, these representations don't descend to $\mathrm{SO}(n)$-representations, because -1 acts as -1 in both S^+ and S^-, but -1 generates the kernel of the double cover, but if it factored through the double cover, -1 would have to act trivially.

Theorem 5.4. *The fundamental representations of* $\mathfrak{so}(2m, \mathbb{C})$ *are:*

- S^+ *and* S^-*;*
- *the defining representation* $\mathbb{C}^n$*; and*
- *the exterior powers* $\Lambda^2\mathbb{C}^n, \dots, \Lambda^{n-2}\mathbb{C}^n$.

Recall that this means all representations of $\mathfrak{so}(2m, \mathbb{C})$ are subrepresentations of direct sums of tensor products of these representations.

Remark. If you choose a metric of a different signature, there are different groups $\mathrm{Spin}(p, q)$ and algebras $\mathfrak{so}(p, q)$. Multiplication by i can affect the signature, so complexifying eliminates the differences due to signature.

The odd-dimensional case.

Proposition 5.5. $\mathrm{C}\ell(V) \cong \mathrm{C}\ell(V \oplus \mathbb{R})^+$ *as algebras, though it does affect the grading.*

Proof. Let $\widetilde{e}$ be a unit vector in the $\mathbb{R}$-direction in $\mathrm{C}\ell(V \oplus \mathbb{R})^+$, and define $f : V \to \mathrm{C}\ell(V \oplus \mathbb{R})^+$ by $v \mapsto v\widetilde{e}$. Inside Clifford algebras, orthogonal elements anticommute, so $(v\widetilde{e})^2 = v\widetilde{e}v\widetilde{e} = -v(-1)v = v^2 = -\langle v, v\rangle$. By the universal property of Clifford algebras, this extends to an algebra homomorphism $f : \mathrm{C}\ell(V) \to \mathrm{C}\ell(V \oplus \mathbb{R})^+$.

It's reasonable to expect that f is an isomorphism, because both of these are 2^n-dimensional; let's see what happens on a basis $e_1, \dots, e_n$ of V.

- If $j < k$, $e_je_k \mapsto e_j\widetilde{e}e_k\widetilde{e} = e_je_k$.
- If $j < k < \ell$, $e_je_ke_\ell \mapsto e_je_ke_\ell\widetilde{e}$.

On $\mathrm{C}\ell(V)^+$, f is just the inclusion $\mathrm{C}\ell(V)^+ \hookrightarrow \mathrm{C}\ell(V \oplus \mathbb{R})^+$; for the odd part, we're multiplying every basis element $e_{i_1}e_{i_2}\cdots e_{i_{2k-1}}$ for $\mathrm{C}\ell(V)^-$ by $\widetilde{e}$, so it's sent to $e_{i_1}\cdots e_{i_{2k-1}}\widetilde{e}$, which hits the rest of the even part. Thus, f is an isomorphism. ⊠

So if $\dim V = 2m-1$, $\mathrm{C}\ell(V)\otimes\mathbb{C} \cong \mathrm{C}\ell(V\oplus\mathbb{R})^+\otimes\mathbb{C} \cong \mathrm{End}_\mathbb{C} S^+ \oplus \mathrm{End}_\mathbb{C} S^-$. Thus, $\mathrm{C}\ell(V)\otimes\mathbb{C}$ has two irreducible, nonisomorphic modules, S^+ and S^-. However, when we restrict to the spin group, these modules become isomorphic spin representations.

Proposition 5.6. *If* $\dim V$ *is odd, then as* $\mathrm{Spin}(V)$*-representations,* S^+ *and* S^- *are isomorphic.*

Proof. We'll define an isomorphism from S^+ to S^- that intertwines the action of $\mathrm{Spin}(V)$. The isomorphism is just multiplication by $\widetilde{e} \in \mathrm{C}\ell(V\oplus\mathbb{R})^+$, which is an isomorphism of vector spaces $S^+ \to S^-$, but this commutes with everything in $\mathrm{Spin}(V)$, as $\widetilde{e}$ commutes with all even products of basis vectors, and hence with $\mathrm{Spin}(V) \subset \mathrm{C}\ell(V\oplus\mathbb{R})^+$, i.e. it's an intertwiner, hence an isomorphism of $\mathrm{Spin}(V)$-representations. ⊠

Definition 5.7. If $\dim V$ is odd, we write S for the isomorphism classes of S^+ and S^-; this is called the *spinor representation* of $\mathrm{Spin}(V)$.

Once again, looking at the action of -1 illustrates that this doesn't pass to an $\mathrm{SO}(V)$-representation, and again this is the missing fundamental representation of $\mathfrak{so}(V\otimes\mathbb{C})$.

In summary:

- If $\dim V = 2m$ is even, we established a superalgebra isomorphism $\mathrm{C}\ell(V)\otimes\mathbb{C} \cong \mathrm{End}(S^+\oplus S^-)$, and S^+ and S^- are nonisomorphic representations of $\mathrm{Spin}(V)$ of dimension 2^{m-1}.
- if $\dim V = 2m-1$ is odd, $\mathrm{C}\ell(V)\otimes\mathbb{C} \cong \mathrm{End}(S)\oplus\mathrm{End}(S)$ as algebras, and there's a single spinor representation S of $\mathrm{Spin}(V)$ of dimension 2^{m-1}.

In either case, you can check that

$$\mathrm{C}\ell(\mathbb{C}^n) \cong \mathrm{C}\ell(\mathbb{C}^{n-2})\otimes M_2(\mathbb{C})$$

(the latter factor is the algebra of 2×2 complex matrices), but in the real case, there's a factor of 8:

$$\mathrm{C}\ell(\mathbb{R}^n) \cong \mathrm{C}\ell(\mathbb{R}^{n-8})\otimes M_{16}(\mathbb{R}).$$

This may look familiar: it's related to the stable homotopy groups of the unitary groups in the complex case and the orthogonal groups in the real case. This is a manifestation of *Bott periodicity*. Another manifestation is periodicity in K-theory: complex K-theory is 2-periodic, and KO-theory (real K-theory) is 8-periodic.

Let $R_k^\mathbb{C}$ denote the ring generated by isomorphism classes of irreducible $\mathrm{C}\ell(\mathbb{C}^k)$-supermodules, with direct sum passing to addition and tensor product passing to multiplication (this is a *representation ring* or the *K-group*), and let $R_k^\mathbb{R}$ be the same thing, but for $\mathrm{C}\ell(\mathbb{R}^k)$. Inclusion $i : \mathrm{C}\ell(\mathbb{C}^k) \hookrightarrow \mathrm{C}\ell(\mathbb{C}^{k+1})$ defines pullback maps $i^* : R_{k+1}^\mathbb{C} \to R_k^\mathbb{C}$ and $i^* : R_{k+1}^\mathbb{R} \to R_k^\mathbb{R}$. If $A_k^\mathbb{C}$ and $A_k^\mathbb{R}$ denote $\mathrm{coker}(i^*)$ in the complex cases respectively, then

$$A_k^\mathbb{C} = \pi_k \mathrm{U} = \begin{cases} 0, & k \text{ odd} \\ \mathbb{Z}, & k \text{ even,} \end{cases}$$

and $A_{k+8}^\mathbb{R} \cong A_k^\mathbb{R}$ are the homotopy groups of the orthogonal group, giving us the "Bott song" $\mathbb{Z}/2$, $\mathbb{Z}/2$, 0, $\mathbb{Z}$, 0, 0, 0, $\mathbb{Z}$.[14]

Lecture 6.

Vector Bundles and Principal Bundles: 9/13/16

Today, we'll move from representation theory into geometry, starting with a discussion of vector bundles and principal bundles. In this section, there are a lot of important exercises that are better done at home than at the board, but are important for one's understanding.

For now, when we say "manifold," we mean a smooth (C^∞) manifold without boundary.

[14] Possible tunes include "Twinkle twinkle, little star."

Definition 6.1. A *real vector bundle of rank k* over a manifold M is a manifold E together with a surjective, smooth map $\pi : E \to M$ such that every fiber $E_x = \pi^{-1}(x)$ (here $x \in M$) has the structure of a real, k-dimensional vector space, and that is locally trivial: there exists an open cover $\mathfrak{U}$ of M such that for every $U_\alpha \in \mathfrak{U}$, there is an isomorphism over M:

$$\begin{tikzcd} E|_{U_\alpha} = \pi^{-1}(U_\alpha) \arrow[rr, "\varphi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times \mathbb{R}^k \arrow[dl] \\ & U_\alpha, & \end{tikzcd}$$

meaning that for every $x \in U_\alpha$, $\varphi_\alpha|_{E_x} : E_x \to \{x\} \times \mathbb{R}^k$ is an $\mathbb{R}$-linear isomorphism of vector spaces. Replacing $\mathbb{R}$ with $\mathbb{C}$ (and real linear with complex linear) defines a *complex vector bundle* on M.

The data $\{U_\alpha, \varphi_\alpha\}$ is called a *local trivialization* of E, and can be used to give another description of a vector bundle. Given two intersecting sets $U_\alpha, U_\beta \in \mathfrak{U}$, we obtain a triangle

$$\begin{tikzcd} (U_\alpha \cap U_\beta) \times \mathbb{R}^k \arrow[rr, "\varphi_\alpha \circ \varphi_\beta^{-1}"] \arrow[dr] & & (U_\alpha \cap U_\beta) \times \mathbb{R}^k \arrow[dl] \\ & U_\alpha \times U_\beta, & \end{tikzcd}$$

so $\varphi_\alpha \circ \varphi_\beta^{-1}$ must be of the form $(x, v) \mapsto (x, g_{\alpha\beta}(x)(v))$ for a smooth map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(k)$. This $g_{\alpha\beta}$ is called a *transition function*.

Exercise 6.2. If $U_\alpha, U_\beta, U_\gamma \in \mathfrak{U}$, check that the transition functions satisfy the *cocycle condition*

$$g_{\alpha\beta}(x) g_{\beta\gamma}(x) = g_{\alpha\gamma}(x).$$

Transition functions and cocycle conditions allow recovery of the original vector bundle.

Proposition 6.3. *Let $\mathfrak{U}$ be an open cover of M and suppose we have smooth functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(k)$ for all $U_\alpha, U_\beta \in \mathfrak{U}$ satisfying the* cocycle conditions

(1) *for all $U_\alpha \in \mathfrak{U}$, $g_{\alpha\alpha} = \mathrm{id}$, and*
(2) *for all $U_\alpha, U_\beta, U_\gamma \in \mathfrak{U}$, $g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$.*

Then, the manifold

$$\coprod_{U_\alpha \in \mathfrak{U}} U_\alpha \times \mathbb{R}^k / ((x, v) \sim (x, g_{\alpha\beta}(x) v) \text{ when } x \in U_\alpha \cap U_\beta)$$

is naturally a vector bundle over M.

If we obtained the $g_{\alpha\beta}$ as transition functions from a vector bundle, what we end up with is isomorphic to the same vector bundle we started with. The cocycle condition is what guarantees that the quotient is an equivalence relation.

Operations on vector bundles.

Definition 6.4. Let $E, F \to M$ be vector bundles. Then, their *direct sum* $E \oplus F$ is the vector bundle defined by $(E \oplus F)_x = E_x \oplus F_x$ for all $x \in M$. The transition function relative to two open sets U_α, U_β in an open cover is

$$g_{\alpha\beta}^{E \oplus F} = \begin{pmatrix} g_{\alpha\beta}^E & 0 \\ 0 & g_{\alpha\beta}^F \end{pmatrix}.$$

One has to show that this exists, but it does. In the same way, the following natural operations extend to vector bundles, and there is again something to show.

Definition 6.5. The *tensor product* $E \otimes F$ is the vector bundle whose fiber over $x \in M$ is $(E \otimes F)_x = E_x \otimes F_x$. Its transition functions are Kronecker products $g_{\alpha\beta}^{E \otimes F} = g_{\alpha\beta}^E \otimes g_{\alpha\beta}^F$.

Definition 6.6. The *dual* E^* is defined to fiberwise be the dual $(E^*)_x = (E_x)^*$. Its transition function is the inverse transpose $g_{\alpha\beta}^{E^*} = ((g_{\alpha\beta}^E)^{-1})^{\mathrm{T}}$.

Definition 6.7. A *homomorphism of vector bundles* is a smooth map $T : E \to F$ commuting with the projections to M, in that the following must be a commutative diagram:

$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[dr] & & F \arrow[dl] \\ & M, & \end{tikzcd}$$

and for each $x \in M$, the map on the fiber $T|_{E_x} : E_x \to F_x$ is linear.

Suppose T is an isomorphism, so each $T|_{E_x}$ is a linear isomorphism. Let $\{U_\alpha, \varphi_\alpha^E\}$ and $\{U_\alpha, \varphi_\alpha^F\}$ be trivialization data for E and F, respectively. Then over each U_α, we can fill in the dotted line below with an isomorphism:

$$\begin{tikzcd} E|_{U_\alpha} \arrow[r, "T|_{U_\alpha}"] \arrow[d, "\sim"', "\varphi_\alpha^E"] & F|_{U_\alpha} \arrow[d, "\sim"', "\varphi_\alpha^F"] \\ U_\alpha \times \mathbb{R}^k \arrow[r, dashed] & U_\alpha \times \mathbb{R}^k. \end{tikzcd}$$

This dotted arrow must be of the form $(x, v) \mapsto (x, \lambda_\alpha(x)v)$ for some $\lambda_\alpha : U_\alpha \to \mathrm{GL}(k)$.

Exercise 6.8. Generalize this to when T is a homomorphism of vector bundles, and show that the resulting λ_α satisfy

$$g_{\alpha\beta}^F \lambda_\beta = \lambda_\alpha g_{\alpha\beta}^E.$$

From the transition-function perspective, there's a convenient generalization: we can replace $\mathrm{GL}(k)$ with an arbitrary Lie group G, and $\mathbb{R}^k$ with any space X that G acts on. That is, given a cover $\mathfrak{U} = \{U_\alpha\}$ and a collection of smooth functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ for every pair $U_\alpha, U_\beta \in \mathfrak{U}$ such that

(1) $g_{\alpha\alpha} = e$ and
(2) $g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$, then

for any G-space X we can form a space

$$E = \coprod_{U_\alpha \in \mathfrak{U}} U_\alpha \times X / ((x, y) \sim (x, g_{\alpha\beta}(x)y) \text{ when } x \in U_\alpha \cap U_\beta).$$

This will be a fiber bundle $X \to E \to M$, i.e. over M with fiber X.

The important or universal case is when X is G: G acts on itself by left multiplication. In this case, the resulting space has a *right* action of G: on each $U_\alpha \times G$, we act by right multiplication by G as $(x, g_1) \cdot g_2 = (x, g_1 g_2)$, but this is invariant under the equivalence relation, and therefore descends to a smooth right action of G on the bundle which is simply transitive on each fiber.

This data defines a principal G-bundle, but just as a vector bundle had a convenient global description, there's one for principal bundles too.

Definition 6.9. Let G be a Lie group. A *(right) principal G-bundle* over a manifold M is a fiber bundle $G \to P \to M$ with a simply-transitive, right G-action on each fiber that's locally trivial, i.e. there's a cover $\mathfrak{U}$ of P such that over each $U_\alpha \in \mathfrak{U}$, there's a diffeomorphism

$$\begin{tikzcd} P|_{U_\alpha} \arrow[rr, "\varphi_\alpha"] \arrow[dr] & & U_\alpha \times G \arrow[dl] \\ & U_\alpha, & \end{tikzcd}$$

i.e. commuting with the projections to U_α, and that intertwines the G-actions: for all $p \in P|_{U_\alpha}$ and $g \in G$, $\varphi_\alpha(pg) = \varphi_\alpha(p)g$.

In this case, G is called the *structure group* of the bundle.

As you might expect, this notion is equivalent to the notion we synthesized from transition functions.

Definition 6.10. A *homomorphism of principal G-bundles* $P_1, P_2 \to M$ is a smooth map $F : P_1 \to P_2$ commuting with the projections to M, and such that $F(pg) = F(p)g$ for all $p \in P_1$ and $g \in G$.

The following definition will be helpful when we discuss Čech cohomology.

Definition 6.11. Let $\check{Z}^1(M,G)$ denote the collection of data $\{U_\alpha, \varphi_\alpha\}$ where

- $\{U_\alpha\}$ is an open cover of M,
- $g_{\alpha\alpha} = e$, and
- $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$.

We declare two pieces of data $\{U_\alpha, g_{\alpha\beta}\}$ and $\{\widetilde{U}_\alpha, \widetilde{g}_{\alpha\beta}\}$ to be equivalent if there is a common refining cover $\mathfrak{U} = \{V_\alpha\}$ of $\{U_\alpha\}$ and $\{\widetilde{U}_\alpha\}$ and data $\lambda_\alpha : V_\alpha \to G$ such that[15]

$$\widetilde{g}_{\alpha\beta} = \lambda_\alpha g_{\alpha\beta} \lambda_\beta^{-1}. \tag{6.12}$$

The set of equivalence classes is called the *degree*-1 *non-abelian Čech cohomology* (of M, with coefficients in G), and is denoted $\check{H}^1(M;G)$.

Since we obtained these from transition data on principal bundles, perhaps the following result isn't so surprising.

Proposition 6.13. *There is a natural bijection between $\check{H}^1(M;G)$ and the set of isomorphism classes of principal G-bundles on M.*

Proposition 6.14. *Any map of principal bundles is an isomorphism.*

Suppose G acts smoothly (on the left) on a space X and P is a principal G-bundle. Then, G acts on $P \times X$ from the right by $(p,x)\cdot g = (pg, g^{-1}x)$. Define $P \times_G X = (P \times X)/G$; this is a fiber bundle over M with fiber X, and is an example of an *associated bundle construction*. In particular, if V is a G-representation, then $P \times_G V \to M$ is a vector bundle.

In particular, starting with a vector bundle E over M, we obtain transition functions $g_{\alpha\beta}$ for it, but these define a principal $\mathrm{GL}(k)$-bundle P on M. Over a point $x \in M$, P_x is the set of all bases for E_x, which is the vector space of isomorphisms from $\mathbb{R}^k$ to E_x. This bundle P is called the *frame bundle* associated to E.

Naïvely, you might expect this to be the bundle of automorphisms of E, but the right action is the subtlety: there's a natural right action of $\mathrm{GL}(k)$ on the bundle of frames, by precomposing with a linear transformation $A \in \mathrm{GL}(k)$. However, $\mathrm{GL}(k)$ acts on $\mathrm{Aut}(E)$ by conjugation, which is not a right action. However, it is true that $\mathrm{Aut}\,E = P \times_{\mathrm{GL}(k)} \mathrm{GL}(k)$, and here $\mathrm{GL}(k)$ acts on itself by conjugation.

Reduction of the structure group. Sometimes we have bundles with two different structure groups. Let $\varphi : H \to G$ be a Lie group homomorphism (often inclusion of a subgroup) and P be a principal G-bundle. When can we think of the transition functions for P being not just G-valued, but actually H-valued? For example, an *orientation* of a vector bundle is an arrangement of its transition functions to all lie in the subgroup of $\mathrm{GL}(k)$ of positive-determinant matrices.

Definition 6.15. With G, H, φ, and P as above, a *reduction of the structure group* of P to H is a principal H-bundle Q and a smooth map

$$\begin{tikzcd} Q \arrow[rr, "F"] \arrow[dr] & & P \arrow[dl] \\ & M, & \end{tikzcd}$$

such that for all $h \in H$ and $q \in Q$, $F(qh) = F(q)\varphi(h)$.

Proposition 6.16. *Reducing P to have structure group H is equivalent to finding transition functions $h_{\alpha\beta} : U_\alpha \cap U_\beta \to H$ satisfying the cocycle conditions such that $\{U_\alpha, \varphi \circ h_{\alpha\beta}\}$ is transition data for P as a principal G-bundle.*

For example, $\mathrm{GL}_+(k,\mathbb{R}) \hookrightarrow \mathrm{GL}(k,\mathbb{R})$ denotes the subgroup of matrices with positive determinant. A vector bundle E is *orientable* iff its frame bundle has a reduction of structure group to $\mathrm{GL}_+(k,\mathbb{R})$. We'll define a spin structure on a vector bundle in a similar way, by lifting from $\mathrm{SO}(n)$ to $\mathrm{Spin}(n)$.

[15] We should be careful about what we're saying: $\{U_\alpha\}$, $\{\widetilde{U}_\alpha\}$, and $\{V_\alpha\}$ aren't necessarily defined on the same index set; rather, we mean that whenever (6.12) makes sense for open sets U_α, U_β, $\widetilde{U}_\alpha$, $\widetilde{U}_\beta$, and V_α, it needs to be true.

Lecture 7.

Clifford Bundles and Spin Bundles: 9/15/16

If $\pi : E \to M$ is a vector bundle, its *space of sections* $\Gamma(M; E)$ is the vector space of sections, which are the maps $\sigma : M \to E$ such that $\pi \circ \sigma = \mathrm{id}$.

Definition 7.1. Let $E \to M$ be a real vector bundle. A *metric* on E is a smoothly-variying inner product on each fiber E_x, i.e. an element $g \in \Gamma(M; E^* \otimes E^*)$ such that for all $x \in M$,

- $g(v, v) > 0$ for all $v \in E_x \setminus 0$, and
- $g(v, w) = g(w, v)$ for all $v, w \in E_x$.

We can relate this to what we talked about last time.

Proposition 7.2. *Putting a metric on E is equivalent to reducing the structure group of E from* $\mathrm{GL}(k, \mathbb{R})$ *to* $\mathrm{O}(k)$*.*

Proof. Let's go in the reverse direction. A reduction of the structure group means we have transition functions $h_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{O}(k) \hookrightarrow \mathrm{GL}(k, \mathbb{R})$ with respect to some cover $\mathfrak{U}$ of M. In particular, we can write E as

$$E = \coprod_{U_\alpha \in \mathfrak{U}} U_\alpha \times \mathbb{R}^k / ((x, v) \sim (x, h_{\alpha\beta}(x)v))$$

like last time. On each $U_\alpha \times \mathbb{R}^k$ we define the metric

$$g_\alpha((x, v), (x, w)) = \langle v, w \rangle.$$

This metric is preserved by the transition functions: for any $U_\alpha, U_\beta \in \mathfrak{U}$, since $h_{\alpha\beta}(x) \in \mathrm{O}(k)$, then

$$\langle v, w \rangle = \langle h_{\alpha\beta}(x)v, h_{\alpha\beta}(x)w \rangle$$

and therefore this metric descends to the quotient E.

Conversely, consider the frame bundle

$$\mathrm{GL}(k, \mathbb{R}) \longrightarrow P \longrightarrow M,$$

and let $Q \subset P$ be the bundle of *orthonormal frames*, whose fiber over an $x \in M$ is the set of isometries (not just isomorphisms) from $(\mathbb{R}^k, \langle \cdot, \cdot \rangle) \to (E_x, g_x)$. Applying the Gram-Schmidt process to a local frame ensures that Q is nonempty; in fact, it's a principal $\mathrm{O}(k)$-bundle, and the inclusion $Q \hookrightarrow P$ intertwines the $\mathrm{O}(k)$-action and the $\mathrm{GL}(k, \mathbb{R})$-action. ⊠

This proof is a lot of words; the point is that in the presence of a metric, the Gram-Schmidt process converts ordinary bases (the $\mathrm{GL}(k, \mathbb{R})$-bundle) into orthonormal bases (the $\mathrm{O}(k)$-bundle).

We can cast the forward direction of the proof in the language of transition functions. In the presence of a metric, our transition functions $h_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$ might not preserve the inner product (be valued in $\mathrm{O}(k)$), but the metric allows us to find a smooth $\lambda_\alpha : U_\alpha \to \mathrm{GL}(k, \mathbb{R})$ for each $U_\alpha \in \mathfrak{U}$ such that

$$g|_{U_\alpha}(v, w) = \langle \lambda_\alpha(x)v, \lambda_\alpha(x)w \rangle,$$

so we can define new transition functions $\lambda_\alpha(x)h_{\alpha\beta}(x)\lambda_\beta^{-1}(x) \in \mathrm{O}(k)$, and this defines an isomorphic principal bundle.

So metrics allow us to reduce the structure group. It turns out we can always do this.

Proposition 7.3. *Every vector bundle* $E \to M$ *has a metric.*

Proof. Let $\mathfrak{U}$ be a trivializing open cover for E. For every $U_\alpha \in \mathfrak{U}$, we can let $E|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^k$ have the "constant metric"

$$g_\alpha((x, v), (x, w)) = \langle v, w \rangle.$$

Let $\{\psi_\alpha\}$ be a (locally finite) partition of unity subordinate to $\mathfrak{U}$; then,

$$g = \sum_\alpha \psi_\alpha g_\alpha$$

is globally defined and smooth. Then, for any $x \in M$ and $v \in E_x \setminus 0$,

$$g_x(v, v) = \sum_\alpha \psi_\alpha g_\alpha(v, v) > 0$$

since each term is nonnegative, and at least one term is positive. ⊠

Corollary 7.4. *The structure group of any real vector bundle of rank k can be reduced to* $\mathrm{O}(k)$*.*

This same argument doesn't work for complex vector bundles, since the positivity requirement is replaced with a nondegeneracy. But it is always possible to find a fiberwise Hermitian metric (meaning $h(v,v) > 0$ and $h(av, bw) = \overline{a}bh(v,w)$) and apply the same argument as above to the unitary group.

Corollary 7.5. *The structure group of any complex vector bundle of rank k can be reduced to* $\mathrm{U}(k)$*.*

Remark. Isomorphism classes of rank-k real vector bundles over a base M are in bijective correspondence with homotopy classes of maps into a *classifying space* $B\mathrm{GL}(k,\mathbb{R})$, written $[M, B\mathrm{GL}(k,\mathbb{R})]$. Similarly, isomorphism classes of rank-k complex vector bundles are in bijective correspondence with $[M, B\mathrm{GL}(k,\mathbb{C})]$. The Gram-Schmidt process defines deformation retractions $\mathrm{GL}(k,\mathbb{R}) \simeq \mathrm{O}(k)$ and $\mathrm{GL}(k,\mathbb{C}) \simeq \mathrm{U}(k)$, so real (resp. complex) rank-k vector bundles are also classified by $[M, B\mathrm{O}(k)]$ (resp. $[M, B\mathrm{U}(k)]$). This is an example of the general fact that a Lie group deformation retracts onto its maximal compact subgroup.

Now, we want to insert Clifford algebras and the spin group into this story. Clifford algebras are associated to inner products, so we need to start with a metric.

Definition 7.6. Let $E \to M$ be a real vector bundle with a metric. Its *Clifford algebra bundle* is the bundle of algebras $\mathrm{C}\ell(E) \to M$ whose fiber over an $x \in M$ is $\mathrm{C}\ell(E)_x = \mathrm{C}\ell((E, g_x))$.

In other words, if P is the principal $\mathrm{O}(k)$-bundle of orthonormal frames of E, then

$$\mathrm{C}\ell(E) = P \times_{\mathrm{O}(k)} \mathrm{C}\ell(\mathbb{R}^k),$$

where $\mathrm{O}(k)$ acts on $\mathrm{C}\ell(\mathbb{R}^k)$ by $A(v_1 \cdots v_\ell) = (Av_1) \cdots (Av_\ell)$ (here $A \in \mathrm{O}(k)$ and $v_1, \dots, v_\ell \in \mathbb{R}^k$).

Definition 7.7. A *Clifford module* for the vector bundle (E, g) over M is a vector bundle $F \to M$ with a fiberwise action $\mathrm{C}\ell(E) \otimes F \to F$. If F is $\mathbb{Z}/2$-graded and the action is compatible with the gradings on F and $\mathrm{C}\ell(E)$, meaning

- $\mathrm{C}\ell(E)^+ \cdot F^+ \subset F^+$ and $\mathrm{C}\ell(E)^- \cdot F^- \subset F^+$; and
- $\mathrm{C}\ell(E)^+ \cdot F^- \subset F^-$ and $\mathrm{C}\ell(E)^- \cdot F^+ \subset F^-$.

These fiberwise notions vary smoothly in the same way that everything else has.

Let's now assume E is oriented, which is equivalent to giving a reduction of its structure group to $\mathrm{SO}(k)$. Can we construct a $\mathrm{C}\ell(E)$-module which is fiberwise isomorphic to the spinor representation of $\mathrm{C}\ell(E_x)$? We can't use the associated bundle construction, because it's not a representation of $\mathrm{SO}(k)$, but of $\mathrm{Spin}(k)$. So we're led to the question: when can we reduce a principal $\mathrm{SO}(k)$-bundle to $\mathrm{Spin}(k)$?

Čech cohomology and Stiefel-Whitney classes. First, an easier question: when can we reduce fro $\mathrm{O}(k)$ to $\mathrm{SO}(k)$? Or, geometrically, what controls whether a real vector bundle is orientable?

Let $\{U_\alpha, g_{\alpha\beta}\}$ be transition data for a vector bundle $E \to M$, and assume that we've picked a metric to reduce the structure group to $\mathrm{O}(k)$. We want $\det g_{\alpha\beta} = 1$ for all U_α, U_β; in general, the determinant is either 1 or -1, so it's a map to $\mathbb{Z}/2$. In fact, since the determinant is a homomorphism, $\det g_{\alpha\beta}$ still satisfies the cocycle condition, so it determines a class $w_1(E) \in \check{H}^1(M; \mathbb{Z}/2)$.

This class $w_1(E)$ means there's a $\mathbb{Z}/2$-valued cocycle d_α such that $\det g_{\alpha\beta} = d_\alpha d_\beta^{-1} = d_\alpha d_\beta$ for all U_α and U_β. Let

$$\lambda_\alpha = \begin{pmatrix} \lambda_\alpha & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix},$$

so that $\det(\lambda_\alpha g_{\alpha\beta} \lambda_\beta^{-1}) = (\det g_{\alpha\beta})^2 = 1$ for all α and β, so $\{\lambda_\alpha g_{\alpha\beta} g_\beta\}$ defines an equivalent vector bundle whose transition functions lie in $\mathrm{SO}(k)$. That is, if $w_1(E) = 0$, then E is orientable! This class $w_1(E)$ is called the 1^{st} *Stiefel-Whitney class* of E.

A priori this depends on the metric.

Exercise 7.8. Show that the first Stiefel-Whitney class doesn't depend on the choice of the metric E.

Since $\mathbb{Z}/2$ is abelian, $\check{H}^1(M; \mathbb{Z}/2)$ is actually the first abelian group of a complex called the Čech cochain complex. Let's see how this works in general.

Definition 7.9. Let A be an abelian Lie group (often discrete) and $\mathfrak{U} = \{U_\alpha\}_{\alpha\in I}$ be an open cover of M indexed by some ordered set I. Then, the *Čech j-cochains (relative to $\mathfrak{U}$)* are the algebra

$$\check{C}^j(\mathfrak{U},A) = \prod_{\alpha_0<\alpha_1<\cdots<\alpha_j} C^\infty(U_{\alpha_0}\cap\cdots\cap U_{\alpha_j},A)$$

(so a product of the spaces of C^∞ functions from $U_{\alpha_0}\cap\cdots\cap U_{\alpha_j}$ to A). These fit into the *Čech cochain complex*, whose differential $\delta^j : \check{C}^j(\mathfrak{U},A)\to\check{C}^{j+1}(\mathfrak{U},A)$ defined by

$$(\delta^j\omega)_{\alpha_0\cdots\alpha_{j+1}} = \sum_{i=1}^{j+1}(-1)^i\omega_{\alpha_1\cdots\widehat{\alpha}_i\cdots\alpha_{j+1}}.$$

Here, $\widehat{\alpha}_i$ means the absence of the i^{th} term.

To honestly say this is a cochain complex, we need a lemma.

Lemma 7.10. *For any j, $\delta^{j+1}\circ\delta^j = 0$.*

This is a computation.

Definition 7.11. The j^{th} *Čech cohomology of M relative to* $\mathfrak{U}$ (and valued in A) is the quotient

$$\check{H}^j(\mathfrak{U},A) = \ker(\delta^j)/\operatorname{Im}(\delta^{j+1}).$$

The j^{th} *Čech cohomology* of M eliminates this dependence on $\mathfrak{U}$: we make the set of open covers of M directed under refinements and set

$$\check{H}^j(M,A) = \varprojlim_{\text{refinements of }\mathfrak{U}} \check{H}^j(\mathfrak{U},A).$$

Fact. If $\mathfrak{U}$ is a *good cover* of M, meaning that all intersections $U_{\alpha_0}\cap\cdots\cap U_{\alpha_\ell}$ are contractible for all ℓ, then $\check{H}^j(\mathfrak{U};A) = \check{H}^j(M;A)$.

This makes a lot of calculations easier.

For example, $\check{C}^0 = \{f = (f_\alpha)_{\alpha\in I}\}$, where $f_\alpha : U_\alpha \to A$ is smooth. The differential is

$$(\delta^0 f)_{\alpha_0\alpha_1} = f_{\alpha_1}|_{U_{\alpha_0}\cap U_{\alpha_1}} - f_{\alpha_0}|_{U_{\alpha_0}\cap U_{\alpha_1}},$$

so $\check{H}^0(M;A) = \ker(\delta^0)$ is the functions that glue together into global functions to A, or $C^\infty(M,A)$. The 1-chains are collections of functions on $U_{\alpha\beta} = U_\alpha\cap U_\beta$, and the differential is

$$(\delta^1 g)_{\alpha\beta\gamma} = g_{\alpha\beta}|_{U_\alpha\cap U_\beta\cap U_\gamma} - g_{\alpha\gamma}|_{U_\alpha\cap U_\beta\cap U_\gamma} + g_{\beta\gamma}|_{U_\alpha\cap U_\beta\cap U_\gamma}.$$

In multiplicative notation, this is exactly the cocycle condition

$$g_{\alpha\beta}g_{\beta\gamma}g_{\alpha\gamma}^{-1} = 1,$$

so we conclude that $\check{H}^1(M;A)$ is in bijection with the group of isomorphism classes of principal A-bundles of M.

We can also characterize higher $\check{H}^j$.

Theorem 7.12. *If A is a discrete group, then $\check{H}^\bullet(M;A)\cong H^\bullet(M;A)$, the singular (or cellular, etc.) cohomology of M with coefficients in A.*

For any abelian Lie group A, it's true that $\check{H}^\bullet(M;A)$ is isomorphic to the sheaf cohomology of the sheaf of smooth A-valued functions on A, $H^\bullet(C^\infty_{M;A})$.

Lecture 8.

Spin Structures and Stiefel-Whitney Classes: 9/20/16

Last time, given an abelian Lie group A, we defined Čech cohomology $\check{H}^\bullet(M;A)$ with coefficients in A. If A is discrete, this agrees with the usual cohomology (singular, cellular, etc.), and in general, it's the sheaf cohomology of the sheaf of smooth functions to A. To every real vector bundle E, we associated a Stiefel-Whitney class $w_1(E)\in H^1(M;\mathbb{Z}/2)$, and showed that it vanishes iff E is orientable.

Today, we'd like to find a class that measures the obstruction of whether a bundle's structure group may be reduced to $\mathrm{Spin}(k)$.

Let E be an oriented real vector bundle over M, so its structure group may be reduced to $\mathrm{SO}(k)$ (it admits $\mathrm{SO}(k)$-valued transition functions). If its structure may be further reduced to $\mathrm{Spin}(k)$, then such a reduction is called a *spin structure* on E. Though we used a metric to reduce to $\mathrm{SO}(k)$, spin structures are independent of the metric.

Let $\mathfrak{U}$ be an open cover of M and $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{SO}(k)$ be transition functions for E. We'd like to find functions $\widetilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Spin}(k)$ for all k satisfying the cocycle condition $\widetilde{g}_{\alpha\beta}\widetilde{g}_{\beta\gamma}\widetilde{g}_{\gamma\alpha} = 1$ on triple intersections, and that project back to $g_{\alpha\beta}$: if $\rho : \mathrm{Spin}(k) \to \mathrm{SO}(k)$ is the double cover map, then we require the following diagram to commute for any U_α and U_β:

$$\begin{tikzcd} & \mathrm{Spin}(k) \arrow[d, "\rho"] \\ U_\alpha \cap U_\beta \arrow[ur, "\widetilde{g}_{\alpha\beta}"] \arrow[r, "g_{\alpha\beta}"'] & \mathrm{SO}(k). \end{tikzcd}$$

Last time, we talked about good covers, for which all intersections of sets in the cover are contractible. One can prove that such a cover always exists; then, covering space theory shows that the functions $g_{\alpha\beta}$ always lift to some smooth functions $\widetilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Spin}(k)$, but we don't know whether this meets the cocycle condition. We do know, however, that

$$w_{\alpha\beta\gamma} = \widetilde{g}_{\alpha\beta}\widetilde{g}_{\beta\gamma}\widetilde{g}_{\gamma\alpha} \in \ker(\rho) = \{\pm 1\},$$

and if they're all $+1$, then they define a spin structure.

The class

$$w = (w_{\alpha\beta\gamma})_{U_\alpha, U_\beta, U_\gamma \in \mathfrak{U}} \in \check{C}^2(M; \mathrm{Spin}(k)),$$

i.e. it's a Čech cochain. One can check that $w = \delta\widetilde{g}$, where $\widetilde{g} = (g_{\alpha\beta})_{U_\alpha, U_\beta}$), so w is a cocycle, and $\delta w = \delta^2 g = 0$; thus, w defines a class $w_2(E) \in \check{H}^2(M; \mathbb{Z}/2)$. This is called the *second Steifel-Whitney class* of E.

Exercise 8.1. Show that $w_2(E)$ is independent of the choices of the transition functions $g_{\alpha\beta}$ and their lifts $\widetilde{g}_{\alpha\beta}$. (In fact, it's also independent of the metric.)

Proposition 8.2. *The structure group of E can be reduced to* $\mathrm{Spin}(k)$ *iff* $w_2(E) = 0$.

Proof. The forward direction is clear by construction: if there are $\widetilde{g}_{\alpha\beta}$ for all α and β satisfying the cocycle condition, then they define $w_2(E)$, but it's trivial.

Conversely, suppose $w_2(E) = 0$. Thus, there's a $t \in \check{C}^1(Z; \mathbb{Z}/2)$ such that $\delta t = 0$, so $t_{\alpha\beta}t_{\beta\gamma}t_{\gamma\alpha} = w_{\alpha\beta\gamma}$ for all $U_\alpha, U_\beta, U_\gamma \in \mathfrak{U}$, and since $t_{\alpha\beta} \in \{\pm 1\}$, then $\rho(t_{\alpha\beta}\widetilde{g}_{\alpha\beta}) = g_{\alpha\beta}$. And these satisfy the cocycle condition:

$$(t_{\beta\gamma}\widetilde{g}_{\beta\gamma})(t_{\gamma\alpha}\widetilde{g}_{\gamma\alpha})(t_{\alpha\beta}\widetilde{g}_{\alpha\beta}) = w_{\alpha\beta\gamma}\widetilde{g}_{\beta\gamma}\widetilde{g}_{\gamma\alpha}\widetilde{g}_{\alpha\beta} = w_{\alpha\beta\gamma}^2 = 1.$$ ⊠

There are higher Stiefel-Whitney classes, though they don't admit as nice of a geometric/obstruction-theoretic point of view.

Remark. From a more abstract point of view, the short exact sequence

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathrm{Spin}(k) \longrightarrow \mathrm{SO}(k) \longrightarrow 0$$

induces a long exact sequence in nonabelian Čech cohomology:

$$\cdots \longrightarrow \check{H}^1(M; \mathrm{Spin}(k)) \longrightarrow \check{H}^1(M; \mathrm{SO}(k)) \xrightarrow{\delta} \check{H}^2(M; \mathbb{Z}/2).$$

Since $\mathrm{Spin}(k)$ is in general nonabelian, the sequence stops there. But the point is that the coboundary map δ sends (the cocycle of) a vector bundle to its second Stiefel-Whitney class $w_2(E)$.

If we apply this to line bundles,[16] this tells us the following.

- The isomorphism classes of real line bundles on M are in bijection with the set of isomorphism classes of principal $\mathrm{O}(1)$-bundles: since $\mathrm{O}(1) = \mathbb{Z}/2$, this is the set $H^1(M; \mathbb{Z}/2)$. Thus, the Stiefel-Whitney class completely classifies real line bundles.

[16] A *line bundle* is a vector bundle of rank 1.

- The isomorphism classes of complex line bundles on M are in bijection with the set of isomorphism classes of principal U(1)-bundles: since $\mathrm{U}(1) \cong S^1$, this is the sheaf cohomology $H^1(M; C^\infty(M; S^1))$ (smooth functions to the circle). The exponential exact sequence
$$0 \longrightarrow \mathbb{Z} \longrightarrow C^\infty_{M;\mathbb{R}} \xrightarrow{\exp(2\pi i)} C^\infty_{M;S^1} \longrightarrow 0$$
induces an isomorphism $H^1(M; C^\infty(M; S_1)) \cong H^2(M; \mathbb{Z})$, which we'll talk about later; it implies that complex line bundles are classified by the first Chern class.

In particular, the set of isomorphism classes of (real or complex) line bundles has a group structure arising from cohomology; the group operation is tensor product of line bundles.

Example 8.3.

(1) Let's classify real line bundles on S^1. $H^1(S^1; \mathbb{Z}/2) = \mathbb{Z}/2$, so there are two line bundles.
   - The first is the trivial line bundle $S^1 \times \mathbb{R}$. The associated principal $\mathbb{Z}/2$-bundle (the bundle of unit vectors of the line bundle) is the disconnected double cover $\mathbb{Z}/2 \times S^1 = S^1 \amalg S^1 \to S^1$.
   - The second is the *Möbius line bundle* $L_M \to S^1$, whose total space is a Möbius strip. We can describe it by its transition functions: let U denote the upper half of the circle and V denote the lower half. Then, $U \cap V$ is two disconnected, short intervals I_0 and I_1. We assign an element of $\mathrm{O}(1) = \{\pm 1\}$ to each interval. If we assign the same element, we get the trivial bundle, but if we assign 1 to I_0 and -1 to I_1, we get the Möbius bundle. The assiociated frame bundle is the connected double cover of S^1, given by $S^1 \to S^1$ with the map $z \mapsto z^2$ (thinking of S^1 as the unit complex numbers).
   Let's calculate the associated Stiefel-Whitney class. We need to work on a good cover, but $U \cap V$ isn't contractible. Intead, let's take three sets U, V, and W, each covering a third of the circle. Then, $\det g_{UV} = 1$, $\det g_{VW} = 1$, and $\det g_{UW} = -1$. This collection g represents $w_1(E)$; if it were trivial, there would be $\lambda_U, \lambda_V, \lambda_W \in \mathbb{Z}/2$ whose collective coboundary is g, which implies
$$\begin{aligned}\lambda_U \lambda_V^{-1} &= 1\\ \lambda_V \lambda_W^{-1} &= 1\\ \lambda_U \lambda_W^{-1} &= -1.\end{aligned}$$
   This can't happen: the left side multiplies to 1, but the right side multiplies to -1, and therefore $w_2(E) = 1 \in H^1(S^1; \mathbb{Z}/2)$.

(2) Let's investigate the *tautological line bundle* over $\mathbb{CP}^1$. Recall that $\mathbb{CP}^1$ is the space of lines through the origin in $\mathbb{C}^2$, which can be written $(\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^\times$. Every point $\ell \in \mathbb{CP}^1$ is a complex line in $\mathbb{C}^2$, so the tautological bundle $L \to \mathbb{CP}^1$ has for its fiber over ℓ the line ℓ itself; that is,
$$L = \{(\ell, v) \mid \ell \in \mathbb{CP}^1 \text{ and } v \in L\}.$$
The standard Hermitian inner product on $\mathbb{C}^2$ induces a Hermitian metric on L, so we know its structure group may be reduced to U(1). We can use projective coordinates to write the points of $\mathbb{CP}^1$: let $[w : z]$ denote the equivalence class of the line through the origin and (w, z), where $w, z \in \mathbb{C}$. Then, let
$$\begin{aligned}U &= \{[1 : z] \mid z \in \mathbb{C}\}\\ V &= \{[w : 1] \mid w \in \mathbb{C}\}.\end{aligned}$$
Since at least one of w or z must be nonzero and we're allowed to rescale both of them by the same factor, $\{U, V\}$ is an open cover of $\mathbb{CP}^1$. On U, $L|_U$ is trivial, since it has a nonzero (in fact, unit length) section
$$[1 : z] \overset{\sigma_U}{\longmapsto} \left([1 : z], \frac{1}{1 + |z|^2}(1, z)\right).$$
Similarly, $L|_V$ is trivial, with the nonzero section
$$[w : 1] \overset{\sigma_V}{\longmapsto} \left([w : 1], \frac{1}{1 + |w|^2}(w, 1)\right).$$
We have an isomorphism $\varphi_U : L|_U \to U \times \mathbb{C}$ of vector bundles over U defined by $c\sigma_U([1 : z]) \mapsto ([1; z], c)$ for any $c \in \mathbb{C}$, and similarly an isomorphism $\varphi_V : L|_V \to V \times \mathbb{C}$. We can use these to calculate the transition

functions: on $U \cap V \simeq \mathbb{C}^\times$, $z = 1/w$, so you can check that

$$\varphi_U \circ \varphi_V^{-1}([w:1], c) = \left([w:1], c\frac{|z|}{z}\right),$$

so the transition function is $g_{UV} = w/|w| \in S^1$. The principal U(1)-bundle of frames is

$$P = \{(\ell, v) \mid v \in \ell \subset \mathbb{C}^2, |v| = 1\} \subset \mathbb{CP}^1 \times \mathbb{C}^2,$$

which is isomorphic to S^3: given $(z_1, z_2) \in S^3 \subset \mathbb{C}^2$, send it to the point $(\ell, (z_1, z_2)) \in P$, where ℓ is the line with slope z_2/z_1. Finally, the isomorphisms $P \cong S^3$ and $\mathbb{CP}^1 \cong S^2$ turn the U(1)-bundle $S^1 \hookrightarrow P \to \mathbb{CP}^1$ into a nontrivial fibration

$$S^1 \hookrightarrow S^3 \longrightarrow S^2,$$

called the *Hopf fibration*. The tautological line bundle isn't isomorphic to its dual, though it still defines the Hopf fibration.

(3) We discuss spin structures on TM, where $M = S^1 \times S^1 \times S^1$. Since M is a Lie group, TM is trivial: $TM \cong M \times \mathbb{R}^3$, and therefore the principal SO(3)-bundle of frames P_{SO} is also trivial: $P_{\mathrm{SO}} \cong M \times \mathrm{SO}(3)$. Thus, one spin structure is $P_{\mathrm{Spin}} = M \times \mathrm{Spin}(3)$ projecting onto $M \times \mathrm{SO}(3)$ by the double cover on the second factor: $P_{\mathrm{Spin}} = \mathbb{R}^3 \times \mathrm{Spin}(3)/\mathbb{Z}^3$, where $\mathbb{Z}^3$ acts only on the left factor, but we can twist this by any action of $\mathbb{Z}^3$. And if $\mathbb{Z}^3$ acts by $\{\pm 1\}$, then it doesn't affect the projection down to P_{SO}. That is, any homomorphism $\varphi : \mathbb{Z}^3 \to \mathbb{Z}/2$ defines the action

$$\mathbf{n} \cdot (\mathbf{v}, u) = (\mathbf{v} + \mathbf{n}, \varphi(\mathbf{n})u),$$

where $\mathbf{n} \in \mathbb{Z}^3$, $\mathbf{v} \in \mathbb{R}^3$, and $u \in \mathrm{Spin}(3)$. These all agree after passing to P_{SO}, and in general these are distinct.

We call two spin structures P_{Spin} and P'_{Spin} equivalent if there is an isomorphism[17] $\tau : P_{\mathrm{Spin}} \to P'_{\mathrm{Spin}}$ commuting with the projection to the $\mathrm{SO}(k)$-frame bundle.

Proposition 8.4. *Let M be a manifold; then, if TM admits a spin structure, then there is a bijection between the classes of distinct spin structures on TM and $\check{H}^1(M; \mathbb{Z}/2)$.*

This is ultimately because $\check{H}^1(M; \mathbb{Z}/2) \cong \mathrm{Hom}_{\mathrm{Grp}}(\pi_1(M), \mathbb{Z}/2)$, so we pass to what the universal cover sees. This is also a lifted version of orientations, which are classified by an obstruction $w_1 \in H^1(M; \mathbb{Z}/2)$, and which are determined by the $\mathbb{Z}/2$-valued functions on the connected components, which are in bijection with $H^0(M; \mathbb{Z}/2)$.

Lecture 9.

Connections: 9/27/16

Let $E \to M$ be a real rank-k vector bundle (with metric), where k is even, and let's assume E has spin structure, i.e. there's a principal $\mathrm{Spin}(k)$-bundle P_{Spin} such that $E \cong P_{\mathrm{Spin}} \times_{\mathrm{Spin}(k)} \mathbb{R}^k$, where $\mathrm{Spin}(k)$ acts on $\mathbb{R}^k$ through the double cover $\mathrm{Spin}(k) \twoheadrightarrow \mathrm{SO}(k)$ and the defining action of $\mathrm{SO}(k)$ on $\mathbb{R}^k$.

Let S be the spinor representation of $C\ell(\mathbb{R}^k)$, and let $S_E = P_{\mathrm{Spin}} \times_{\mathrm{Spin}(k)} S$.

Proposition 9.1. *S_E is naturally a Clifford module over E.*

Proof. We can define the Clifford bundle $C\ell(E)$ through the associated bundle construction:

$$C\ell(E) = P_{\mathrm{Spin}} \times_{\mathrm{Spin}(k)} C\ell(\mathbb{R}^k),$$

where $\mathrm{Spin}(k)$ acts on $C\ell(\mathbb{R}^k)$ by conjugation

$$u \cdot (v_1 \cdots v_\ell) = u v_1 \cdots v_\ell u^{-1},$$

so through the map $\rho : \mathrm{Spin}(k) \twoheadrightarrow \mathrm{SO}(k)$, then conjugating.

Define $C\ell(E) \otimes S_E \to S_E$ to send $[p, c] \otimes [p, \psi] \mapsto [p, c\psi]$, where $p \in P_{\mathrm{Spin}}$, $c \in C\ell(\mathbb{R}^k)$, and $\psi \in S$. This is well-defined because $[p, c] \sim [pu, u^{-1}cu]$ and $[p, \psi] \sim [pu, u^{-1}\psi]$, so sending $[p, \psi] \mapsto [p, c\psi]$ is the same as sending $[pu, u^{-1}\psi] \mapsto [pu, u^{-1}cuu^{-1}c\psi] = [pu, u^{-1}c\psi]$. ⊠

Proposition 9.2. *If $H^2(M; \mathbb{Z})$ has no 2-torsion, then S_E is independent of the choice of spin structure.*

[17] Every morphism of principal G-bundles is an isomorphism; in other words, the category of principal G-bundles is a groupoid.

Proof sketch. Note that $H^1(M;\mathbb{Z}/2)$ acts simply transitively on the set of spin structures on E, and therefore on the spinor bundles on E. $H^1(M;\mathbb{Z}/2)$ is isomorphic to the group of (isomorphism classes of) real line bundles under tensor product; the action of $H^1(M;\mathbb{Z}/2)$, and the action on the isomorphism classes of spinor bundles S_E is to tensor S_E by the complexification of a given line bundle.

Since this action uses the complexification, it's really an action by the group of complex line bundles, which is $H^2(M;\mathbb{Z})$, and using Exercise 9.3, these must be trivial.[18] ⊠

Exercise 9.3. If L is any real line bundle and $H^2(M;\mathbb{C})$ has no 2-torsion, then $L\otimes\mathbb{C}$ is trivial. (Hint: the first Chern class classifies line bundles.)

Proposition 9.4. *Any other complex Clifford module for* $\mathrm{C}\ell(E)$ *is of the form* $S_E\otimes E$*, where* W *is a complex vector bundle, and where the action of* $\mathrm{C}\ell(E)$ *is trivial on the second factor.*

Over a point, this is the statement that any $\mathrm{C}\ell(\mathbb{R}^k)$-module is $S^{\oplus\ell}$ for some ℓ. We know this because the Clifford algebra is a simple algebra, and $\mathrm{C}\ell(\mathbb{R}^k)\otimes\mathbb{C}\cong\mathrm{End}_{\mathbb{C}}S$. Here, we're definitely assuming k is even, and using the fact that the only modules over a simple algebra are matrix algebras.

Proof. Let F be a Clifford module. We claim $F\cong S_E\otimes\mathrm{Hom}_{\mathrm{C}\ell(E)}(S_E,F)$ as Clifford modules. Here, $\mathrm{Hom}_{\mathrm{C}\ell(E)}(S_E,F)$ is the vector bundle of homomorphisms that intertwine the $\mathrm{C}\ell(E)$-action. The map in question sends $\psi\otimes\varphi\mapsto\varphi(\psi)$; to show it's an isomorphism, we'll check that it's an isomorphism on everu fiber.

Over an $x\in M$, $\mathrm{C}\ell(E_x)\otimes\mathbb{C}\cong\mathrm{End}(S)$ and $F_x\cong S\otimes\mathbb{C}^\ell$ for some $\ell\in\mathbb{N}$, and therefore

$$\mathrm{Hom}_{\mathrm{C}\ell(E_x)}(S,S\otimes\mathbb{C}^\ell)\cong\mathrm{Hom}_{\mathrm{C}\ell(E)}(S,S)\otimes\mathbb{C}^\ell\cong\mathbb{C}^\ell$$

by Schur's lemma. ⊠

Example 9.5. Let M be a complex manifold with a Riemannian metric, and take $E=TM$. Since M has a complex structure, TM naturally does too; we can compose $T^*M\otimes\mathbb{C}=\Lambda^{1,0}\oplus\Lambda^{0,1}$, where $\Lambda^{1,0}$ is the i-eigenspace for the complex structure and $\Lambda^{0,1}$ is the $-i$-eigenspace.

There's a Clifford module structure on $\Lambda^{0,\bullet}=\Lambda^\bullet(\Lambda^{0,1})$, defined in the same way as the complex spin representation. Precisely, for a $v\in TM$ and $\mu\in\Lambda^{0,\bullet}$,

$$c(v)\mu=(\varepsilon(v^\flat)^{0,1}-i(v^{0,1}))\mu.$$

Here, $v\mapsto v^\flat$ is a *musical isomorphism* that "lowers" a function to a form; then, we take the part lying in $\Lambda^{0,1}$.

Fiberwise, this is the spin representation.

Exercise 9.6. Show that if TM has a spin structure, then $W=\mathrm{Hom}_{\mathrm{C}\ell}(S_{TM},\Lambda^{0,\bullet})$ satisfies $W\otimes W=\Lambda^{0,\dim_{\mathbb{C}}M}$, which is a line bundle, and therefore W is also a line bundle.

Connections. Connections will allow us to do calculus, or at least something like calculus, on Clifford modules. This leads to the theory of connections and Chern-Weil theory, a way to obtain de Rham representatives of characteristic classes using a connection.

For an arbitrary vector bundle $E\to M$, there's no natural way to identify different fibers E_x,E_y for distinct $x,y\in M$. For example, if you wanted to differentiate a section $\psi\in\Gamma(E)$ along a vector $v\in T_xM$, you'd want to define this to be

$$D_v\psi=\lim_{t\to 0}\frac{\psi(\gamma(t))-\psi(x)}{t},$$

where $\gamma:[-1,1]\to M$ is a smooth path with $\gamma(0)=x$ and $\gamma'(0)=v$ — but this doesn't make sense: $\psi(\gamma(t))$ and $\psi(x)$ are in different fibers, so we don't necessarily know how to compare them, since vector bundles may be globally nontrivial.

However, the pullback bundle $\gamma^*(E)$ is a bundle over a contractible space, and therefore is trivial. If we choose a trivialization, it would enable us to identify different fibers and compute the derivative. We think of a connection as a choice of a linear isomorphism $P^\gamma_{t_1t_2}:E_{\gamma(t_1)}\to E_{\gamma(t_2)}$ for all $\gamma:[-1,1]\to M$ and $t_1,t_2\in[-1,1]$. Such an isomorphism exists for each γ, since γ^*E is trivializable.

[18] Another way to realize this action is through the Bockstein map $H^1(M;\mathbb{Z}.2)\to H^2(M;\mathbb{Z})$ induced from the short exact sequence

$$0\longrightarrow\mathbb{Z}\xrightarrow{\cdot 2}\mathbb{Z}\longrightarrow\mathbb{Z}/2\longrightarrow 0.$$

It's easier to think about the infinitesimal version: if $\psi \in \Gamma(E)$, then define $\nabla_{\gamma'}\psi \in \Gamma(E)|_\gamma$ by

$$(\nabla_{\gamma'}\psi)(\gamma(t_0)) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} P_{tt_0}\psi(\gamma(t)).$$

One can check this depends only on the values of γ and γ' at t_0 and satisfies the Leibniz rule

$$\nabla_{\gamma'}(f\psi) = (\gamma' \cdot f)\psi + f\nabla_{\gamma'}\psi.$$

for any $f \in C^\infty(M)$.

We want to abstract this sort of operator, which will give us a rigorous notion of connection that's easy to work with.

Definition 9.7. A *connection* or *covariant derivative* on a vector bundle $E \to M$ is an operator $\nabla : \Gamma(TM) \otimes_{\mathbb{R}} \Gamma(E) \to \Gamma(E)$, written $v \otimes \psi \mapsto \nabla_v \psi$, such that[19]

(1) ∇ is $C^\infty(M)$-linear in its first argument, i.e. for all $f \in C^\infty(M)$, $v \in \Gamma(TM)$, and $\psi \in \Gamma(E)$,

$$\nabla_{fv}\psi = f\nabla_v\psi;$$

and

(2) ∇ satisfies the Leibniz rule

$$\nabla_v(f\psi) = (v \cdot f)\psi + f\nabla_v\psi.$$

Since ∇ is $C^\infty(M)$-linear in the first argument, we can dualize and think of ∇ as an $\mathbb{R}$-linear map $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes_{C^\infty(M)} E)$ such that

$$\nabla(f\psi) = \mathrm{d}f \otimes \psi + f\nabla\psi.$$

Example 9.8. If $E = \mathbb{R}$ is the trivial line bundle, the exterior derivative is an example of a connection: it turns functions into 1-forms, and satisfies a Leibniz rule.

Example 9.9. Suppose $E \to M$ is trivializable. Then, a choice of a global frame (sections $\{e_1, \dots, e_k\}$ that are a basis on every fiber) defines a connection such that $\nabla e_j = 0$ for all j. That is, a trivialization allows us to do genuine parallel transport: any $\psi \in \Gamma(E)$ is a $C^\infty(M)$-linear combination of these e_j: $\psi = \sum_j f_j e_j$, and

$$\nabla\psi = \sum_j \mathrm{d}f_j \otimes e_j.$$

We'll often refer to this connection as d.

Proposition 9.10. *Every vector bundle has a connection. In fact, the space of connections on a fixed bundle E is an affine space over $\Gamma(T^*M \otimes \operatorname{End} E)$.*

Lecture 10.

Parallel Transport: 9/29/16

Last time, we defined a connection on a real vector bundle $E \to M$ to be an $\mathbb{R}$-linear map

$$\nabla : \Gamma(E) \longrightarrow \Gamma(T^*M \otimes_{\mathbb{R}} E) \tag{10.1a}$$

obeying a Leibniz rule

$$\nabla(f\psi) = \mathrm{d}f \otimes \psi + f\nabla\psi, \tag{10.1b}$$

where $f \in C^\infty(M)$ and $\psi \in \Gamma(E)$.

Remark. The same construction works over $\mathbb{C}$, where we define a connection on a complex vector bundle to be a $\mathbb{C}$-linear map satisfying (10.1a) and (10.1b) in the same ways.

Proposition 10.2. *The space of all connections on $E \to M$ is an affine space modeled on $\Gamma(T^*M \otimes \operatorname{End} E)$.*

The key is keeping track of what is $\mathbb{R}$-linear and what is $C^\infty(M)$-linear.

[19] Though we could define the tensor product over $C^\infty(M)$, we didn't do that: we'll not be able to factor out smooth functions, just scalars.

Proof. Let ∇ be a connection on E and $A\in\Gamma(T^*M\otimes\operatorname{End}E)$. Then, for any $f\in C^\infty(M)$ and $\psi\in\Gamma(E)$,

$$\begin{aligned}(\nabla+A)(f\psi)&=\nabla(f\psi)+A(f\psi)\\&=\mathrm{d}f\otimes\psi+f\nabla\psi+fA\psi\\&=\mathrm{d}f\otimes\psi+f(\nabla+A)\psi,\end{aligned}$$

so $\nabla+A$ is a connection.

Conversely, if ∇^1 and ∇^2 are connections, then $\nabla^1-\nabla^2:\Gamma(E)\to\Gamma(T^*M\otimes E)$ is $\mathbb{R}$-linear. With f and ψ as before,

$$\begin{aligned}(\nabla^1-\nabla^2)(f\psi)&=\mathrm{d}f\otimes\psi+f\nabla^1\psi-(\mathrm{d}f\otimes\psi+f\nabla^2\psi)\\&=f(\nabla^1-\nabla^2)\psi,\end{aligned}$$

so $\nabla^1-\nabla^2$ is actually $C^\infty(M)$-linear, and therefore is an element of $\Gamma(T^*M\otimes\operatorname{End}E)$. ⊠

Exercise 10.3. Show that a connection exists on every vector bundle. (Idea: we know how to put a connection on a trivial bundle; then, patch them together on a partition of unity. You can't add connections, because of the Leibniz rule, but you can take convex combinations of them.)

Proposition 10.2 says that there is an infinite-dimensional space of connections. But there's notion of equivalence for connections called *gauge equivalence*, and there things look a little more discrete.

Supposde E is trivial on each open $U_\alpha\subset M$ for an open cover of M, and we have a choice of a framing $\underline{e}^\alpha=(e_1^\alpha,\dots,e_k^\alpha)$ (a smoothly varying basis for every fiber). These framings obey the transition functions for E: on $U_\alpha\cap U_\beta$, $\underline{e}^\alpha=g_{\alpha\beta}\underline{e}^\beta$.

If ∇ is a connection on E, then we know it's the trivial connection d, so by Proposition 10.2, there's some matrix of one-forms $A_\alpha\in\Gamma(T^*U_\alpha\otimes\operatorname{End}(E|_{U_\alpha})$ such that $\nabla=\mathrm{d}+A_\alpha$ (here, since E is trivial over U_α, $\operatorname{End}E|_{U_\alpha}=\mathfrak{gl}(k)$).

Exercise 10.4. Show that on $U_\alpha\cap U_\beta$, show that

$$A_\alpha=g_{\alpha\beta}A_\beta g_{\alpha\beta}^{-1}+(\mathrm{d}g_{\alpha\beta})g_{\alpha\beta}^{-1}.\tag{10.5}$$ [20]

Here, $\mathrm{d}g_{\alpha\beta}$ takes the exterior derivative componentwise.

These A_α are called *connection forms*, and determine the connection uniquely. The second term in (10.5) is called the *gauge term*.

Conversely, given a vector bundle E and trivializing data (the nonabelian Čech class) $(\mathfrak{U},\{g_{\alpha\beta}\})$, a collection of matrices of one-forms satisfying (10.5) uniquely determine a connection on E.

Parallel transport. Our original motivation for connections was to relate the fibers over two different points on the base space. Specifically, given a smooth path $\gamma:[0,1]\to M$, we'd like the connection to define a linear isomorphism $P_{0,t}^\gamma:E_{\gamma(0)}\to E_{\gamma(t)}$ for all $t\in[0,1]$.

Fix a $\psi_0\in E_{\gamma(0)}$; we'd like to interpolate it in a way that we obtain a section obtained along the entire curve. The connection allows us to differentiate, so we'd like this section to be constant with respect to this connection. More precisely, we have the differential equation

$$\begin{aligned}\nabla_{\gamma'}\psi&=0\\\psi(\gamma(0))&=\psi_0.\end{aligned}\tag{10.6}$$

Let U be an open neighborhood of $\gamma(0)$ and $\nabla=\mathrm{d}+A$ on U, where $A\in\Gamma(T^*U\otimes\operatorname{End}E|_U)$. Then, (10.6) becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\psi(\gamma(t))+A(\gamma')\psi(\gamma(t))&=0\\\psi(\gamma(0))&=\psi_0.\end{aligned}$$

Here, we're thinking of ψ as a smooth function $\psi:U\to\mathbb{R}^k$.

The theory of ODEs tells us that this equation has a unique solution. We define the *parallel transport* to be the value of the section ψ that we solved for at time t: $P_{0,t}^\gamma:E_{\gamma(0)}\to E_{\gamma(t)}$ is the isomorphism sending $\psi_0\mapsto\psi(\gamma(t))$.

If γ is a closed loop, then $P_{0,1}^\gamma\in\mathrm{GL}(E_{\gamma(0)})$ is called the *holonomy* of γ, often denoted $\mathrm{Hol}_\nabla(\gamma)$.

[20] The sign may be wrong for the gauge term; we weren't sure during class.

Example 10.7. Consider the real line bundles over S^1. Since $H^1(S^1;\mathbb{Z}/2) \cong \mathbb{Z}/2$, there are two isomorphism classes, the trivial bundle $L_T = S^1 \times \mathbb{R}$, and the Möbius bundle L_M, as in Example 8.3.

(1) First let's consider the trivial bundle. Since L_M is trivialized over S^1, we have the trivial connection d, and any other connection is of the form $\nabla = \mathrm{d} + A$, where $A \in \Gamma(T^*S^1 \otimes \operatorname{End} L_T) = \Gamma(T^*S^1)$, since the endomorphism bundle of a line bundle is trivial. Specifically, $A = a\,\mathrm{d}\theta$, where $a \in C^\infty(S^1)$ and $\mathrm{d}\theta$ is the usual volume form on S^1.

On S^1, a connection is determined by its holonomy around the loop. What is this holonomy? We can identify $\Gamma(L_T) = C^\infty(S^1)$; to compute the holonomy around $\gamma(t) = e^{2\pi i t}$, we need to solve

$$\begin{gathered}\nabla_{\gamma'}\psi = 0\\ \psi(\gamma(0)) = \psi_0 \in \mathbb{R},\end{gathered}$$

or equivalently, since $\mathrm{d}\theta(\gamma') = 1$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(\gamma(t)) + a(\gamma(t))\psi(\gamma(t)) = 0$$

with the same initial condition. The solution is

$$\psi(\gamma(t)) = \psi_0 \exp\left(-\int_0^t a(\psi(s))\,\mathrm{d}s\right) \tag{10.8}$$

i.e. the holonomy of ∇ around γ is multiplication by this number (10.8). It turns out this only depends on the homotopy class of the path in general, and something general called the Riemann-Hilbert correspondence associates representations of $\pi_1(M)$ to different classes of connections under holonomy. In particular, it's possible to show that on S^1, the holonomy (10.8) determines the gauge equivalence class of the connection.

(2) For the Möbius bundle, we trivialize over the northern and southern semicircles U and V, respectively; let I_1 and I_2 be the two components of $U \cap V$. Then, the transition function is

$$g_{UV} = \begin{cases} -1 & \text{on } I_1\\ 1, & \text{on } I_2.\end{cases}$$

There's no globally nonvanishing section, since L_M isn't trivial, but we can choose nonvanishing sections e_U on U and e_V on V such that $e_U = g_{UV}e_V$ on $U \cap V$.

Define a connection ∇^0 on L_M by $\nabla^0 e_U = 0$ and $\nabla^0 e_V = 0$. Because $\mathrm{d}g_{UV} = 0$, this satisfies the required compatibility condition (10.5), hence defines a connection on L_M.

To compute its holonomy, we'll again let $\gamma(t) = e^{2\pi i t}$ and fix $\psi_0 = ce_U(1) = ce_V(1)$. We solve

$$\begin{gathered}\nabla^0_{\gamma'}\psi = 0\\ \psi(\gamma(0)) = \psi_0.\end{gathered}$$

The solution to this system is

$$\psi(\gamma(t)) = \begin{cases} ce_U(\gamma(t)), & t \in [0, 1/2 + \varepsilon] \subset U\\ -ce_V(\gamma(t)), & t \in [1/2, 1] \subset V.\end{cases}$$

In particular,

$$\psi(\gamma(1)) = -ce_V(\gamma(1)) = ce_U(1) = -\psi_0.$$

Thus, the holonomy is $-1 \in \mathrm{GL}(1,\mathbb{R})$. The holonomy for any connection on the trivial bundle was positive, so this is interesting.

Any other connection $\nabla = \nabla^0 + a\,\mathrm{d}\theta$, where $a \in C^\infty(S^1)$ and $\mathrm{d}\theta$ is the volume form; then, similarly to the case for the trivial bundle, the solution to $\nabla_{\gamma'}\varphi = 0$ is

$$\varphi(\gamma(t)) = \exp\left(-\int_0^t a(\gamma(s))\,\mathrm{d}s\right)\psi(\gamma(t)),$$

so the holonomy of ∇ is

$$\mathrm{Hol}_{\nabla^0 + a\,\mathrm{d}\theta}(\gamma) = -\exp\left(-\int_0^1 a(\psi(s))\,\mathrm{d}s\right).$$

Thus, all negative numbers are possible. The Riemann-Hilbert correspondence says that we recover any representation of $\pi_1(S^1)$ into $\mathbb{R}$, which just means every number; the positive numbers correspond to the trivial bundle, and the negative numbers to the Möbius bundle.

What if we consider complex line bundles? All complex line bundles over S^1 are trivial, so any connection $\nabla = \mathrm{d} + a\,\mathrm{d}\theta$, where $a \in C^\infty(S^1, \mathbb{C})$. If $\gamma(t) = e^{2\pi i t}$, then just as for the trivial real vector bundle,

$$\mathrm{Hol}_\nabla(\gamma) = \exp\left(-\int_0^1 a(e^{2\pi i s})\,\mathrm{d}s\right),$$

so since a is complex-valued, we can get any element of $\mathrm{GL}(1,\mathbb{C})$.

Lecture 11.

Curvature: 10/4/16

Let $E \to M$ be a vector bundle, real or complex. Fix a $p \in M$; we'll ask what the connection can tell us locally around p. We choose local coordinates $x^1, \dots, x^n$ at p, so p corresponds to $x = 0$ and $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ are a basis for T_pM. Then, we have a family of parallel-transport operators

$$P_s^{\frac{\partial}{\partial x_i}} : E_{(x^1,\dots,x^n)} \longrightarrow E_{(x^1,\dots,x^i+s,\dots,x^n)}$$

which parallel-transport along $\frac{\partial}{\partial x^i}$ for time s.

On a nontrivial bundle, parallel transport may depend on the path we take: in the x^1x^2-plane, suppose we sweep out an $s \times s$ square by parallel-transporting with the operator

$$\left(P_s^{\frac{\partial}{\partial x^2}}\right)^{-1} \circ \left(P_s^{\frac{\partial}{\partial x^1}}\right)^{-1} \circ P_s^{\frac{\partial}{\partial x^2}} \circ P_s^{\frac{\partial}{\partial x^1}} \in \mathrm{GL}(E_p).$$

As $s \to 0$, this is a local measure of the dependence of parallel transport of the curve. Since

$$\frac{\partial}{\partial s} P_s^{\frac{\partial}{\partial x^i}} = \nabla_{\frac{\partial}{\partial x^i}},$$

we can replace the Lie-group-like notion with the corresponding infinitesimal, Lie-algebraic notion

$$\nabla_{\frac{\partial}{\partial x^i}} \circ \nabla_{\frac{\partial}{\partial x^j}} - \nabla_{\frac{\partial}{\partial x^j}} \circ \nabla_{\frac{\partial}{\partial x^i}}.$$

This makes sense for any vector fields, not just coordinate vector fields, so for any $X, Y \in \Gamma(TM)$, define

$$F_\nabla(X,Y) = \nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X - \nabla_{[X,Y]}.$$

This measures how much this infinitesimal parallel transport differs from that of their commutator. *A priori* this is an $\mathbb{R}$-linear (or $\mathbb{C}$-linear) map $\Gamma(E) \to \Gamma(E)$, but one can check that it's actually $C^\infty(M)$-linear in X, Y, and evaluating on functions; one says it's a *tensor*.

Proposition 11.1. $F_\nabla \in \Gamma(\Lambda^2(T^*M) \otimes \operatorname{End} E)$. *That is, it actually only depends on the fiber, not on any other information.*

Definition 11.2. This tensor F_∇ is called the *curvature* of ∇. If $F_\nabla = 0$, then ∇ is called a *flat* connection.

Theorem 11.3. *If ∇ is flat and γ is a closed curve in M starting and ending at x, then* $\mathrm{Hol}_\nabla(\gamma) \in \mathrm{GL}(E_x)$ *only depends on the homotopy class of γ.*

We know that nontrivial holonomy may exist, even if all connections are flat (e.g. on S^1).

Example 11.4.

(1) The trivial connection d on the trivial bundle $M \times \mathbb{R}^k$ is flat: $F_{\mathrm{d}}(X,Y)\psi = X \circ Y\psi - Y \circ X\psi - [X,Y]\psi = 0$, because the Lie bracket on vector fields is exactly the commutator. Thus, curvature is an obstruction to the connection being trivial.

(2) Consider TS^2. If we take the connection induced from the inclusion $S^2 \hookrightarrow \mathbb{R}^3$, one can parallel-transport from a pole to the equator, then across the equator, then back to the pole, and the result is not what we started with. Thus, we should be able to explicitly describe this not-flat connection.

To be more precise, TS^2 is a subbundle of the trivial bundle $TS^2 \oplus \nu = \underline{\mathbb{R}}^3$. Define $(\nabla_X Y)_x = \mathrm{proj}_{T_x S^2} X \cdot Y = X \cdot Y - \langle X \cdot Y, x\rangle x$, i.e. we take the directional derivative, and then project it back down to TS^2. More explictly, this is $X \cdot Y + \langle X, Y\rangle x$.

Exercise 11.5. Show that $F_\nabla(X,Y)Z = \langle Y,Z\rangle X - \langle X,Z\rangle Y \neq 0$.

For example, at the north pole, a basis for the tangent space is $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$, and

$$F_\nabla\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\frac{\partial}{\partial x} = \frac{\partial}{\partial y}.$$

You may wonder, why this connection? On a Riemannian manifold, there's a distinguished connection arising from the metric called the *Levi-Civita connection*, and this is an instance of that.

To compute the curvature locally, we can use connection forms.

Proposition 11.6. *Let A be a local connection 1-form for ∇, i.e. on an open $U \subset M$ $\nabla = \mathrm{d} + A$, where $A \in \Gamma(T^*M \otimes \operatorname{End} E|_U) = \Gamma(T^*M \otimes \mathfrak{gl}(k))$. Then, in local coordinates,*

$$(F_\nabla)|_U(X,Y) = (\mathrm{d}A)(X,Y) + [A(X), A(Y)]. \tag{11.7}$$

The proof is an exercise, and a calculation.

Corollary 11.8. *If E is a line bundle, then the curvature is canonically a 2-form $F_\nabla \in \Gamma(\Lambda^2(T^*M))$, and $F_\nabla|_U = \mathrm{d}A$.*

This is because the second term in (11.7) drops out, because $\mathfrak{gl}(1)$ is commutative. This says that F_∇ is closed, but not necessarily exact. Thus, it represents a class in $H^2_{\mathrm{dR}}(M)$, which can be shown to be independent of the choice of the connection. This class is called the *first Chern class* $c_1(E)$.

de Rham cohomology. Before we do that, let's review de Rham cohomology.

Definition 11.9. The space of p-forms is the (infinite-dimensional) vector space $\mathscr{A}^p(M) = \Gamma(\Lambda^p T^*M)$. (Notice that when $p = 0$, 0-forms are just functions on M.)

The collection of all forms, direct-summed over p, is a $\mathbb{Z}$-graded vector space.

Definition 11.10. The *exterior derivative* is the $\mathbb{R}$-linear map $\mathrm{d} : \mathscr{A}^p(M) \to \mathscr{A}^{p+1}(M)$ uniquely characterized by the following three properties.

(1) $(\mathrm{d}f)(X) = X \circ f$ when f is a function (0-form) and X is a vector field.
(2) $\mathrm{d}^2 = 0$.
(3) d obeys the Leibniz rule
$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^p \alpha \wedge \mathrm{d}\beta,$$
where $\alpha \in \mathscr{A}^p(M)$ and $\beta \in \mathscr{A}^q(M)$.

This definition is also sort of a theorem, I guess: the first and second properties define it on one-forms, then the third extends it to all forms, but it's good to have an explicit construction, in which d is dual to the Lie bracket. In local coordinates, if $\alpha \in \mathscr{A}^p(M)$, then

$$(\mathrm{d}\alpha)(X_0, \ldots, X_p) = \sum_{j=0}^p (-1)^j X_j \cdot \alpha(x_0, \ldots, \widehat{X}_j, \ldots, X_p) + \sum_{j<k} (-1)^{j+k} \alpha([X_j, X_k], \ldots, \widehat{x}_j, \ldots, \widehat{X}_k, \ldots, X_p),$$

where $X_0, \ldots, X_p \in \Gamma(TM)$. Here, a hat on an index means that we leave it out.

Definition 11.11. The differential complex $(\mathscr{A}^\bullet(M), \mathrm{d})$ is called the *de Rham complex* and its cohomology, denoted $H^\bullet_{\mathrm{dR}}(M)$, is called the *de Rham cohomology* of M.

de Rham's theorem says this cohomology group is naturally isomorphic to singular cohomology with $\mathbb{R}$ coefficients $H^\bullet(M;\mathbb{R})$, which is also isomorphic to the sheaf cohomology on M with coefficients in the constant sheaf $\underline{\mathbb{R}}$. (The de Rham cohomology gives a resolution of the constant sheaf, since the constant sheaf is the kernel of d on the zeroth-graded part.)

Chern-Weil theory. Though Stiefel-Whitney classes had to live in Čech cohomology, we'll spend more time working with Chern and Pontrjagin classes, which may be realized in de Rham cohomology. The objective of Chern-Weil theory is to quantify how nontrivial a vector bundle is: since trivial bundles always have flat connections, we can try to use the curvature or the connection to extract invariants of vector bundles.

For example, let $L \to M$ be a complex line bundle and ∇ be a connection on L. By Corollary 11.8, $F_\nabla \in \mathscr{A}^2(M)$ is closed, and therefore defines a class $[F_\nabla] \in H^2_{\mathrm{dR}}(M)$.

Proposition 11.12. *This class $[F_\nabla]$ is independent of the choice of ∇.*

Proof. Since L is a line bundle, the space of connections is an affine space over the space of one-forms on M, i.e. any other connection is $\nabla + \alpha$ for some 1-form α. Then,

$$\begin{aligned} F_{\nabla+\alpha}(X,Y) &= [\nabla_X + \alpha(X), \nabla_Y + \alpha(Y)] - \nabla_{[X,Y]} - \alpha[X,Y] \\ &= F_\nabla(X,Y) + [\nabla_X, \alpha(Y)] + [\alpha(X), \nabla_Y] - \alpha[X,Y] \\ &= F_\nabla(X,Y) + X \cdot \alpha(Y) - Y \cdot \alpha(X) - \alpha[X,Y] \\ &= F_\nabla(X,Y) + \mathrm{d}\alpha(X,Y). \end{aligned}$$

Thus, if the connection changes by α, the curvature changes by α, so when we pass to cohomology, $[F_{\nabla+\alpha}] = [F_\nabla]$. ⊠

This makes the following definition well-posed.

Definition 11.13. If $L \to M$ is a complex line bundle, its *first Chern class* is $c_1(L) = [F_\nabla] \in H^2_{\mathrm{dR}}(M) \otimes \mathbb{C}$, where ∇ is any connection on L.

Remark. The same definition applies *mutatis mutandis* to real line bundles; however, all connections on a real line bundle are flat: real line bundles are classified by $\check{H}^1(M; \mathrm{O}(1)) = \check{H}^1(M; \mathbb{Z}/2)$, so we can always find locally constant transition functions. Thus, these admit a locally trivial connection, which might not be globally trivial, but since curvature is a local quantity, this implies these bundles admit flat connections. Thus, we'd always get $c_1(E) = 0$ when E is a real line bundle.

For higher-rank real vector bundles, we'll be able to define characteristic classes in dimension $4k$, called Pontrjagin classes.

Lecture 12.

Compatibility with the Metric: 10/6/16

Last time, we saw that for a complex line bundle $L \to M$, the curvature of the connection is a 2-cocycle, and hence defines a class $[F_\nabla] \in H^2_{\mathrm{dR}}(M; \mathbb{C})$, called the first Chern class of the bundle, and that this is independent of the choice of connection. Today, we'll bring that story to vector bundles of greater dimension.

It's generally hard to describe characteristic classes purley topologically, but in this case we can define the *exponential exact sequence* of sheaves

$$0 \longrightarrow \mathbb{Z} \longrightarrow C^\infty_{\mathbb{R}} \xrightarrow{e^{2\pi i \cdot}} C^\infty_{S^1} \longrightarrow 0. \tag{12.1}$$

Exactness of sheaves means that we can make this exact on some open cover: a complex function may not globally have a logarithm, but locally it does.

Then, (12.1) induces a long exact sequence in cohomology

$$H^1(M; C^\infty_{S^1}) \xrightarrow{\sim} H^2(M; \mathbb{Z}) \xrightarrow{\cdot 2\pi i} H^2(M; \mathbb{C}) \cong H^2_{\mathrm{dR}}(M; \mathbb{C}).$$

The set of complex line bundles is identified with $H^1(M; C^\infty_{S^1})$, and the map across this diagram to $H^2_{\mathrm{dR}}(M; \mathbb{C})$ is exactly the first Chern class.[21] In general, using geometric information such as the connection allows for a clean, more elementary description of characteristic classes than the algebraic approach using classifying space, but thereby loses torsion information by passing from $H^2(M; \mathbb{Z}) \to H^2(M; \mathbb{C})$.

[21] To see a proof of this, check out Bott-Tu, "Differential Forms in Algebraic Topology," which makes it explicit where the connection forms enter the picture.

Example 12.2. Let $H \to \mathbb{CP}^1$ be the tautological line bundle, so

$$H = \{([z_1 : z_2], v) \in \mathbb{CP}^1 \times \mathbb{C}^2 \mid v \in \operatorname{span}_{\mathbb{C}}\{(z_1, z_2)\}\}.$$

Thus, $H \subset \mathbb{CP}^1 \times \mathbb{C}^2 = \underline{\mathbb{C}}^2$, the trivial $\mathbb{C}^2$-bundle on $\mathbb{CP}^1$. This has a Hermitian inner product $h : v, w \mapsto \langle v, w\rangle = \overline{v}^{\mathrm{T}} w$, which we use to define the connection as the composite

$$\nabla : \Gamma(H) \longrightarrow \Gamma(\underline{\mathbb{C}}^2) \xrightarrow{\mathrm{d}} \mathscr{A}^1(\mathbb{CP}^1; \underline{\mathbb{C}}^2) \longrightarrow \mathscr{A}^1(\mathbb{CP}^1; H).$$

The three maps are, in order,

(1) the map induced by the inclusion of each fiber $H \hookrightarrow \underline{\mathbb{C}}^2$;
(2) the exterior derivative; and
(3) projection using h.

Let $\mathfrak{U} = \{U, V\}$ be the open cover of $\mathbb{CP}^1$ where $U = \{[1 : z]\}$ and $V = \{[z : 1]\}$. Over U, we have a nonzero section $e_U([1 : z]) = ([1 : z], (1, z))$, and over V, we have the nonzero section $e_V([w : 1]) = ([w : 1], w, 1)$. Thus,

$$\begin{aligned} \nabla e_U &= \operatorname{proj}(\mathrm{d}(1, z)) = \operatorname{proj}(\mathrm{d}z \otimes (0, 1)) = \mathrm{d}z \otimes \operatorname{proj}(0, 1) \\ &= \mathrm{d}z \otimes \frac{h((1, z), (0, 1))}{h((1, z), (1, z))} \cdot e_U \\ &= \mathrm{d}z \otimes \frac{\overline{z}}{1 + |z|^2} e_U. \end{aligned}$$

That is, the connection 1-form is

$$Au = \frac{\overline{z}\, \mathrm{d}z}{1 + |z|^2}.$$

Similarly,

$$\nabla e_V = \mathrm{d}w \otimes \frac{\overline{w}}{1 + |w|^2},$$

so the connection 1-form is

$$Av = \frac{\overline{w}\, \mathrm{d}w}{1 + |w|^2}.$$

The curvature is therefore

$$F_\nabla = \mathrm{d}A_u = \frac{1}{(1 + |z|^2)^2} \mathrm{d}\overline{z} \wedge \mathrm{d}z,$$

where $z = x + iy$ and $\overline{z} = x - iy$; you can check that $\mathrm{d}A_v$ looks exactly the same. The *Fubini-Study form* on $\mathbb{CP}^1$ is the 2-form $\omega_{\mathrm{FS}} = -iF_\nabla$; it's the form that induces the complex manifold structure on $\mathbb{CP}^1$.

Finally, we can explicitly check that F_∇ is closed, but not exact:

$$\int_{\mathbb{CP}^1} F_\nabla = 2i \int_{\mathbb{R}^2} \frac{\mathrm{d}x \wedge \mathrm{d}y}{(1 + x^2 + y^2)^2} = 2\pi i,$$

using a polar transform.

As in this example, it's always true that the first Chern class is in $H^2(M; 2\pi i\mathbb{Z})$ inside $H^2(M; \mathbb{C})$.

Compatibility with the metric.

Definition 12.3. Let g be a metric (resp. Hermitian metric) on a real (resp. complex) vector bundle $E \to M$. Then, a connection ∇ on E is called *compatible with the metric* g if its Leibniz rule is compatible with g in the following sense: for all $X \in \Gamma(TM)$ and $\psi_1, \psi_2 \in \Gamma(E)$,

$$X \cdot g(\psi_1, \psi_2) = g(\nabla_X \psi_1, \psi_2) + g(\psi_1, \nabla_X \psi_2).$$

In this case, g is called *orthogonal* (resp. *Hermitian*), and one says ∇ preserves g.

For example, suppose $E = M \times \mathbb{R}^k$ is trivial and g is the standard metric $g((x, v), (x, w)) = \langle v, w\rangle$. Then, the trivial connection d is compatible with the metric.

A connection determines a connection on the entire tensor algebra; since g is a tensor, we can evaluate the connection on it. If a connection is compatible with the metric, then g is *covariantly constant*, i.e. for the induced connection on $\operatorname{Sym}^2 E^*$, $\nabla^{\operatorname{Sym}^2 E^*} g = 0$.

Proposition 12.4. *Given a metric (resp. Hermitian metric) g on a real (resp. complex) vector bundle $E \to M$, there exists a connection on M compatible with that metric.*

That said, most connections are not orthogonal, in a sense that can be made precise.

Proof. Let ∇^0 be an arbitrary connection on E, and define its *adjoint* $(\nabla^0)^*$ by

$$g(\psi_1, (\nabla^0)^*_X \psi_2) = X \cdot g(\psi_1, \psi_2) - g(\nabla^0_X \psi_1, \psi_2). \tag{12.5}$$

The nondegeneracy of g means this suffices to define $(\nabla^0)^*_X$ as a function, and it's C^∞-linear in X. What about in ψ_2?

$$\begin{aligned} g(\psi_1, (\nabla^0)^*_X(f\psi_2)) &= X \cdot g(\psi_1, f\psi_2) - g(\nabla^0_X \psi_1, f\psi_2) \\ &= (X \cdot f) g(\psi_1, \psi_2) + f X \cdot g(\psi_1, \psi_2) - f g(\nabla^0_X \psi_1, \psi_2) \\ &= (X \cdot f) g(\psi_1, \psi_2) + f g(\psi_1, (\nabla^0)^*_X \psi_2) \\ &= g(\psi_1, (Xf)\psi_2 + f(\nabla^0)^*_X \psi_2). \end{aligned}$$

This holds for all ψ_1 and ψ_2, so

$$(\nabla^0)^*_X(f\psi_2) = (X \cdot f)\psi_2 + f(\nabla^0)^*_X \psi_2$$

for all ψ_2, and therefore $(\nabla^0)^*$ is a connection.

The definition of the adjoint in (12.5) implies that ∇ is orthogonal iff $\nabla = \nabla^*$ and $(\nabla^*)^* = \nabla$. Since any convex combination of connections is a connection, we can choose

$$\nabla = \frac{1}{2}\big(\nabla^0 + (\nabla^0)^*\big),$$

which is equal to its own adjoint, and hence is orthogonal. ⊠

Exercise 12.6. Given a connection, it's *not* always true that there's a metric on M such that the connection is orthogonal. Work this out for S^1, using the Riemann-Hilbert correspondence.

Proposition 12.7. *The set of all orthogonal (resp. unitary) connections on a real (resp. complex) vector bundle is an affine space on $\mathscr{A}^1(M; \mathfrak{o}(E, g))$ (resp. $\mathscr{A}^1(M; \mathfrak{u}(E, g))$); here $\mathfrak{o}(E, g)$ is the space of skew-symmetric endomorphisms of E and $\mathfrak{u}(E, g)$ is the skew-Hermitian endomorphisms.*

Recall that given a metric on a vector bundle, we can choose transition functions that are $\mathrm{O}(k)$-valued (or $\mathrm{U}(k)$-valued in the complex case). Similarly, given (E, g) and an orthogonal connection for it, we can find an open cover $\mathfrak{U}$ of M with *orthonormal trivializations*, meaning that for each $U_\alpha \in \mathfrak{U}$, $(E|_{U_\alpha}, g|_{U_\alpha}) \cong (U_\alpha \times \mathbb{R}^k, \langle \cdot, \cdot \rangle)$ and $\nabla|_{U_\alpha} = \mathrm{d} + A_\alpha$, where $A_\alpha \in \mathscr{A}^1(U_\alpha, \mathfrak{o}(k))$ ($\mathfrak{u}(k)$ in the complex case).

Next we compute the curvature: $(\mathrm{d}A_\alpha)(X, Y)$ is in $\mathfrak{o}(E)$ because A_α is and differentiation preserves this property, and since $\mathfrak{o}(E)$ is a Lie algebra, $[A_\alpha(X), A_\alpha(Y)] \in \mathfrak{o}(E)$. Thus, $F_\nabla(X, Y) = (\mathrm{d}A_\alpha)(X, Y) + [A_\alpha(X), A_\alpha(Y)] \in \mathfrak{o}(E)$. That is:

Proposition 12.8. *If ∇ is an orthogonal (resp. unitary) connection on the real (resp. complex) bundle $E \to M$, then $F_\nabla \in \mathscr{A}^2(M; \mathfrak{o}(E))$ (resp. $F_\nabla \in \mathscr{A}^2(M; \mathfrak{u}(E))$).*

Corollary 12.9. *The first Chern class $c_1(L)$ lies in $H^2_{\mathrm{dR}}(M; i\mathbb{R}) \subset H^2_{\mathrm{dR}}(M; \mathbb{C})$.*

Proof. Since we can compute $c_1(L)$ using any connection, choose a unitary connection ∇, so that the curvature is valued in $\mathfrak{u}(1) = i\mathbb{R}$. ⊠

Remark. Another way to think of orthogonal connections is as those for which parallel transport preserves the metric.

Lecture 13.

Chern-Weil theory: 10/11/16

Today, we're going to use Chern-Weil theory to construct characteristic classes for general vector bundles, rather than just line bundles. This involves some amount of algebra with the curvature tensor F_∇ associated to a connection ∇ for a vector bundle $E \to M$. We'll define two series of classes, Chern classes and Pontrjagin classes, by taking the trace of $(F_\nabla)^k$, which defines a $2k$-form. This involves showing these classes are closed and are independent of the choice of connection, etc., as in the rank-1 case.

First we give a new perspective on curvature. We've so far thought of it geometrically, as the failure of parallel transport to be independent of path, but we can also understand it algebraically. The de Rham differential defines a complex

$$C^\infty(M) \xrightarrow{\mathrm{d}} \mathscr{A}^1(M) \xrightarrow{\mathrm{d}} \mathscr{A}^2(M) \longrightarrow \cdots,$$

and similarly, a connection defines a map $\Gamma(E) \to \mathscr{A}^1(M;E)$. We would like to extend this to a "twisted de Rham complex," and what we'll get won't quite be a chain complex, but it'll suffice for defining characteristic classes. We'll force the operator to extend the Leibniz rule using ∇.

Definition 13.1. Let $d^\nabla : \mathscr{A}^i(M;E) \to \mathscr{A}^{i+1}(M;E)$ is the unique $C^\infty(M)$-linear operator extending

$$d^\nabla(\alpha \otimes \psi) = \mathrm{d}\alpha \otimes \psi + (-1)^{\deg \alpha} \alpha \wedge \nabla\gamma,$$

where $\alpha \in \mathscr{A}^i(M)$ and $\psi \in \Gamma(E)$.[22]

First, though, we have to check this is $C^\infty(M)$-linear! It's a sum of two nonlinear terms, so we need them to cancel out. Let $f \in C^\infty(M)$; then,

$$\begin{aligned} d^\nabla(f\alpha \otimes \psi) &= (\mathrm{d}f \wedge \alpha + f\,\mathrm{d}\alpha) \otimes \psi + (-1)^{\deg\alpha} f\alpha \wedge \psi \\ d^\nabla(\alpha \otimes f\psi) &= \mathrm{d}\alpha \otimes f\psi + (-1)^{\deg\alpha}\alpha \wedge (\mathrm{d}f \otimes \psi + f\nabla\psi). \end{aligned}$$

Expanding out and applying the graded commutativity of the exterior algebra shows these are the same, just as we hoped.

More generally, $\mathscr{A}^\bullet(M;E)$ is a graded $\mathscr{A}^\bullet(M)$-module, and d^∇ is compatible with this action in the sense that

$$d^\nabla(\alpha \wedge \varphi) = \mathrm{d}\alpha \wedge \varphi + (-1)^{\deg\alpha}\alpha \wedge d^\nabla\varphi$$

for all $\alpha \in \mathscr{A}^\bullet(M)$ and $\varphi \in \mathscr{A}^\bullet(M;E)$.

This d^∇ isn't always a differential. The obstruction is curvature.

Proposition 13.2. *$F_\nabla \in \mathscr{A}^2(M;\operatorname{End}E)$ is the obstruction to d^∇ being a differential. That is, $(d^\nabla)^2 = F_\nabla$.*

Here, F_∇ acts on forms via the pairing

$$\mathscr{A}^\bullet(M;\operatorname{End}E) \otimes_{\mathscr{A}^\bullet(M)} \mathscr{A}^\bullet(M;E) \longrightarrow \mathscr{A}^\bullet(M;E)$$

defined by

$$(\alpha \otimes T, \beta \otimes \psi) = (\alpha \wedge \beta) \otimes (T\psi).$$

Here, $\alpha, \beta \in \mathscr{A}^\bullet(M)$, $T \in \operatorname{End}E$, and $\psi \in \Gamma(E)$. There's a lot of different things acting on other things, so be careful. However, because this is over the exterior algebra, you can do this all fiberwise.

Proof of Proposition 13.2. Let's first check that $(d^\nabla)^2$ is linear over differential forms. Then, it suffices to check over 0-forms. If $\alpha \in \mathscr{A}^\bullet(M)$ and $\varphi \in \mathscr{A}^\bullet(M;E)$, then

$$\begin{aligned} (d^\nabla)^2(\alpha \wedge \varphi) &= d^\nabla\big(\mathrm{d}\alpha \wedge \varphi + (-1)^{\deg\alpha}\alpha \wedge d^\nabla\varphi\big) \\ &= \mathrm{d}^2\alpha \wedge \varphi + (-1)^{1+\deg\alpha} \wedge d^\nabla\varphi + (-1)^{\deg\alpha}\mathrm{d}\alpha \wedge d^\nabla\varphi + \alpha \wedge (d^\nabla)^2\varphi \\ &= \alpha \wedge (d^\nabla)^2\varphi, \end{aligned}$$

so $(d^\nabla)^2$ is indeed $\mathscr{A}^\bullet(M)$-linear. Hence, it's sufficient to check that $(d^\nabla)^2 = F_\nabla$ when restricted to $\mathscr{A}^0(M;E) = \Gamma(E)$. Moreover, we can check this locally, i.e. on a trivializing open cover $\mathfrak{U}$ for E. For every $U_\alpha \in \mathfrak{U}$, $\mathscr{A}^\bullet(U_\alpha;E) = \mathscr{A}^\bullet(U_\alpha) \otimes \mathbb{R}^k$ as graded algebras and $d^\nabla = \mathrm{d} + A_\alpha$. Thus for a $\psi \in \Gamma(U_\alpha, E)$,

$$\begin{aligned} (d^\nabla)^2\psi &= d^\nabla(\mathrm{d}\psi + A_\alpha\psi) \\ &= \mathrm{d}(\mathrm{d}\psi + A_\alpha\psi) + A_\alpha(\mathrm{d}\psi + A_\alpha\psi) \\ &= \mathrm{d}(A_\alpha\psi) + A_\alpha(\mathrm{d}\psi) + A_\alpha^2\psi \\ &= (\mathrm{d}A_\alpha)\psi - A_\alpha \wedge \mathrm{d}\psi + A_\alpha(\mathrm{d}\psi) + A_\alpha^2\psi \\ &= (\mathrm{d}A_\alpha + A_\alpha^2)\psi = F_\nabla\psi. \end{aligned}$$

⊠

[22] Here the tensor product is taken over $C^\infty(M)$.

Proposition 13.3. *A connection ∇ on a vector bundle E naturally determines a connection $\nabla^{\operatorname{End} E}$ on $\operatorname{End} E$ by the rule*

$$\nabla_X^{\operatorname{End} E} T = [\nabla_X, T] \tag{13.4}$$

for a vector field $X \in \Gamma(TM)$ and $T \in \Gamma(\operatorname{End} E)$.

Proof. First we check that $\nabla^{\operatorname{End} E}$ is $C^\infty(M)$-linear in its first argument: since ∇ and T both are C^∞-linear, so is $\nabla^{\operatorname{End} E}$; for the Leibniz rule,

$$\begin{aligned}\nabla_X^{\operatorname{End} E}(fT)\psi &= [\nabla_X, fT]\psi\\ &= \nabla_X(fT\psi) - fT\nabla_X\psi\\ &= (Xf)(T\psi) + f\nabla_X(T\psi) - fT(\nabla_X\psi)\\ &= (Xf)(T\psi) + f(\nabla_X^{\operatorname{End} E}T)\psi.\end{aligned}$$ ⊠

Exercise 13.5. There's also a natural connection defined on the dual bundle of a bundle with connection and the tensor product of two bundles with connections. Show that the induced connection on $\operatorname{End} V$ in (13.4) is the same as the connection induced on $V \otimes V^*$ under the identification $\operatorname{End} V \cong V \otimes V^*$.

There are lots of connections on $\operatorname{End} V$, but the one defined by (13.4) is particularly nice: the identity must commute with everything, hence is covariantly constant.

In terms of connection forms, suppose (locally) $\nabla = \mathrm{d} + A$. Then, $\nabla^{\operatorname{End} E} = \mathrm{d} + [A, \cdot]$.

Remark. $\mathscr{A}^\bullet(M; \operatorname{End} E)$ is a graded algebra with product

$$(\alpha \otimes T_1) \wedge (\beta \otimes T_2) = (\alpha \wedge \beta) \otimes (T_1 T_2),$$

where $\alpha, \beta \in \mathscr{A}^\bullet(M)$ and $T_1, T_2 \in \Gamma(\operatorname{End} E)$.

Proposition 13.6. *$d^{\nabla^{\operatorname{End} E}}$ is a graded-commutative, degree-1 derivation on $\mathscr{A}^\bullet(M; \operatorname{End} E)$, i.e.*

$$d^{\nabla^{\operatorname{End} E}}(G_1 \wedge G_2) = (d^{\nabla^{\operatorname{End} E}} G_1) \wedge G_2 + (-1)^{\deg G_1} G_1 \wedge (d^{\nabla^{\operatorname{End} E}} G_2)$$

and $(d^{\nabla^{\operatorname{End} E}})^2 = [F_\nabla, \cdot]$.

We've pretty much only been using the bracket structure here, so we could replace $\operatorname{End} E$ with a Lie algebra and life would still be good.

Exercise 13.7. Show that $d^{\nabla^{\operatorname{End} E}} = [d^\nabla, \cdot]$, where we take the *supercommutator*

$$[G_1, G_2] = G_1 \wedge G_2 - (-1)^{(\deg G_1)(\deg G_2)} G_2 \wedge G_1$$

in the sense of graded commutativity.

Remark. A graded algebra A with a derivation d such that $d^2 = 0$ is called a *differential graded algebra (DGA)*; if instead d squares to a commutator $[X, \cdot]$, A is called a *curved DGA*. The curved DGAs we've discussed are the models for this class of algebras.

Now, we'll use this algebra to define characteristic classes.

Definition 13.8. The *trace* $\operatorname{tr} : \mathscr{A}^k(M; \operatorname{End} E) \to \mathscr{A}^k(M)$ is defined locally on simple tensors by

$$\operatorname{tr}(\alpha \otimes R) = (\operatorname{tr} T)\alpha,$$

and on all forms by extending linearly and working over an open cover.

Theorem 13.9 (Chern-Weil)**.** *For each $j \in \mathbb{N}$, $\operatorname{tr}((F_\nabla)^j) \in \mathscr{A}^{2j}(M)$ is closed, and its de Rham cohomology class is independent of the choice of ∇.*

Therefore $[\operatorname{tr}((F_\nabla)^k)] \in H^{2k}_{\mathrm{dR}}(M)$ is an invariant of the vector bundle E; these will help us define the Chern character, Chern classes, and Pontrjagin classes.

First we'll need a few lemmas. A defining property of the trace is that it vanishes on commutators; we have to check that it's still true in the graded sense.

Lemma 13.10. *The trace vanishes on supercommutators, i.e. for any $G_1, G_2 \in \mathscr{A}^\bullet(M; \operatorname{End} E)$, $\operatorname{tr}([G_1, G_2]) = 0$.*

Lemma 13.11. *The trace intertwines $d^{\nabla^{\operatorname{End} E}}$ and* d*, i.e. for any $G \in \mathscr{A}^\bullet(M; \operatorname{End} E)$,*

$$\mathrm{d}(\operatorname{tr} G) = \operatorname{tr}(d^{\nabla^{\operatorname{End} E}} G).$$

Proof sketch. Working locally, we can use Lemma 13.10 to write $d^{\nabla^{\operatorname{End} E}} = \mathrm{d} + [A_\alpha, \cdot]$ (here the bracket is again the supercommutator). ⊠

This works because $\nabla^{\operatorname{End} E}$ was induced from ∇ on E; it's pleasantly surprising that the result doesn't depend on the choice of connection on E.

Lemma 13.12 (Bianchi identity)**.** $d^{\nabla^{\operatorname{End} E}} F_\nabla = 0.$

Proof. This is somewhat obvious from our setup: $F_\nabla = (d^\nabla)^2$, and $d^{\nabla^{\operatorname{End} E}}$ is the supercommutator with d^∇, so

$$d^{\nabla^{\operatorname{End} E}} F_\nabla = d^\nabla \circ (d^\nabla)^2 - (d^\nabla)^2 \circ d^\nabla = 0.$$ ⊠

It's a good idea to work through this on one's own, maybe with some explicit matrices.

Proof of Theorem 13.9. By Lemma 13.11 and Proposition 13.6 (so $d^{\nabla^{\operatorname{End} E}}$ is a derivation), we have

$$\begin{aligned}
\mathrm{d}\operatorname{tr}((F_\nabla)^j) &= \operatorname{tr}(d^{\nabla^{\operatorname{End} E}}(F_\nabla)^j) \\
&= \operatorname{tr}((d^{\nabla^{\operatorname{End} E}} F_\nabla)(F_\nabla)^{j-1} + F_\nabla(d^{\nabla^{\operatorname{End} E}} F_\nabla)(F_\nabla)^{j-2}) \\
&= 0
\end{aligned}$$

by Lemma 13.12. Thus, the forms we defined are closed.

It remains to check that they're independent of the choice of connection, which is a proof similar to that of the Poincaré lemma. Let ∇^0 and ∇^1 be two connectons on E. Let $p : M \times [0,1] \to M$ be projection and pull back E to another bundle p^*E; we will think of $\Gamma(p^*E)$ as time-dependent sections of E, $\gamma_t \in \Gamma(E)$, for specific values of t. In particular, we can define a connection $\overline{\nabla}$ on p^*E by

$$\overline{\nabla} = (1-t)\nabla^0 + t\nabla^1 + \mathrm{d}t \otimes \frac{\partial}{\partial t}.$$

The last term is there so that we can differentiate in the t-direction. Explicitly,

$$(\overline{\nabla}\psi_t)_{t_0} = (1-t_0)\nabla^0\psi_{t_0} + t_0\nabla^1\psi_{t_0} + \mathrm{d}t \otimes \left.\frac{\partial}{\partial t}\right|_{t_0} \psi_t.$$

The two inclusions $i_0, i_1 : M \rightrightarrows M \times [0,1]$ send $x \mapsto (x,0)$ and $x \mapsto (x,1)$, respectively. We can pull back connections and their curvature forms, and indeed

$$\begin{aligned}
i_0^*\overline{\nabla} &= \nabla_0 & \qquad i_1^*\overline{\nabla} &= \nabla^1 \\
i_0^* F_{\overline{\nabla}} &= F_{\nabla^0} & \qquad i_1^* F_{\overline{\nabla}} &= F_{\nabla^1}.
\end{aligned}$$

The same is true for their powers $i_0^* \operatorname{tr}((F_{\overline{\nabla}})^j)$ and $i_1^* \operatorname{tr}((F_{\overline{\nabla}})^j)$. Since i_0 and i_1 are homotopic, then their pullbacks are the same map on de Rham cohomology, so

$$[\operatorname{tr}((F_{\nabla^0})^j)] = i_0^*[\operatorname{tr}((F_{\overline{\nabla}})^j)] = i_1^*[\operatorname{tr}((F_{\overline{\nabla}})^j)] = [\operatorname{tr}((F_{\nabla^1})^j)].$$ ⊠

There's a more explicit way to do this, just like for the Poincaré lemma, and this yields an explicit transgression form called the *Chern-Simons form*. The homotopy invariance of de Rham cohomology comes from integrating over the fiber and Stokes' theorem:

$$\mathrm{d}\left(\int_0^1 \operatorname{tr}(F_{\overline{\nabla}})^j\right) = \operatorname{tr}(F_{\nabla^1})^j - \operatorname{tr}(F_{\nabla^0}^j).$$

The exact form $\int_0^1 \operatorname{tr}((F_{\overline{\nabla}})^j)$ is called the Chern-Simons form $\mathrm{CS}(\nabla^0, \nabla^1) \in \mathscr{A}^{2j-1}(M)$ for these connections, and is an explicit witness for the equality of the cohomology classes we defined.

Lecture 14.

The Chern Character and the Euler Class: 10/13/16

We're going to talk about the Chern character and the Euler class. Hopefully we'll also get to Chern and Pontryagin classes, but we'll start with the ones that are the most important for us.

Last time, we used Chern-Weil theory to prove Theorem 13.9, that if $E \to M$ is a vector bundle and ∇ is a connection on E, then for all $j \geq 1$, the form $\operatorname{tr}((F_\nabla)^j) \in \mathscr{A}^{2j}(M)$ is a closed form, and its cohomology class is independent of ∇.

Recall that if $E_1, E_2 \to M$ are vector bundles with connections ∇^{E_1} and ∇^{E_2}, respectively, and $f : N \to M$ is smooth, there are induced connections on other bundles.

- On $E_1 \otimes E_2$, there's an induced connection $\nabla^{E_1\otimes E_2}$ defined by the formula
$$\nabla_X^{E_1\otimes E_2}(\psi_1 \otimes \psi_2) = \nabla_X^{E_1}\psi_1 \otimes \psi_2 + \psi_1 \otimes \nabla_X^{E_2}\psi_2,$$
and the curvature is
$$F_{\nabla^{E_1\otimes E_2}} = F_{\nabla^{E_1}} \otimes \mathrm{id} + \mathrm{id} \otimes F_{\nabla^{E_2}}.$$
- On $E_1 \oplus E_2$, the induced connection $\nabla^{E_1\oplus E_2}$ has the formula
$$\nabla_X^{E_1\oplus E_2}(\psi_1, \psi_2) = (\nabla_X^{E_1}\psi_1, \nabla_X^{E_2}\psi_2),$$
and its curvature has the block form
$$F_{\nabla^{E_1\oplus E_2}} = \begin{pmatrix} F_{\nabla^{E_1}} & 0 \\ 0 & F_{\nabla^{E_2}} \end{pmatrix}.$$
- The pullback is a little different: $\nabla^{f^*E_1}$ on f^*E_1 is defined by
$$\nabla_X^{f^*E_1} f^*\psi_1 = f^*\nabla_{f_*X}\psi_1.$$
Here, $X \in TN$, so we can push it forward using D_f. Then, the curvature is the pullback on forms and on endomorphisms: $F_{\nabla^{f^*E_1}} = f^*F_{\nabla^{E_1}}$.

Using this, we can define the Chern character.

Definition 14.1. Let $E \to \mathbb{C}$ be a complex vector bundle. Its *Chern character* is the cohomology class

$$\operatorname{ch}(E) = \left[\operatorname{tr}\left(e^{F_\nabla/2\pi i}\right)\right] = \left[\operatorname{rank}(E) + \sum_{n=1}^{\infty} \frac{\operatorname{tr}((F_\nabla)^n)}{n!(2\pi i)^n}\right] \in H_{\mathrm{dR}}^{\mathrm{even}}(M; \mathbb{C}). \tag{14.2}$$

This sum is actually finite, since M is a manifold.

Since this is independent of the choice of character, we can compute it using a unitary connection ∇, which always exists on M (Proposition 12.7). In this case, the curvature is skew-symmetric: $\overline{F}_\nabla^{\mathrm{T}} = -F_\nabla$. Thus, for any j,

$$\overline{\operatorname{tr}((iF_\nabla)^j)} = \operatorname{tr}\left(\overline{(iF_\nabla)^j}^{\mathrm{T}}\right) = \operatorname{tr}((iF_\nabla)^j),$$

so $\operatorname{tr}((iF_\nabla)^j)$ is a real differential form. Thus, $\operatorname{ch}(E) \in H_{\mathrm{dR}}^{\mathrm{even}}(M; \mathbb{Q})$.

Remark. Just as for Chern classes, it's possible to given an entirely topological derivation of the Chern character, though it's more involved. This shows that $\operatorname{ch}(E)$ is a rational class, i.e. in the image of $H^{\mathrm{even}}(M; \mathbb{Q}) \to H_{\mathrm{dR}}^{\mathrm{even}}(M; \mathbb{R})$.

Proposition 14.3.

(1) *If $\underline{\mathbb{C}}^k$ denotes the trivial line bundle, then* $\operatorname{ch}(\underline{\mathbb{C}}^k) = k \in H_{\mathrm{dR}}^0(M; \mathbb{R})$ *if M is connected.*[23]
(2) *The Chern character is additive:* $\operatorname{ch}(E_1 \oplus E_2) = \operatorname{ch}(E_1) + \operatorname{ch}(E_2)$.
(3) *The Chern character is multiplicative:* $\operatorname{ch}(E_1 \otimes E_2) = \operatorname{ch}(E_1) \wedge \operatorname{ch}(E_2)$.[24]

[23] If M isn't connected, it's a factor of k in each connected component of M.

[24] These two statements can be combined: let $K(M)$ denote the *K-theory* of M, the commutative ring generated by isomorphism classes of vector bundles of M with the relations $[E_1 \oplus E_2] = [E_1] + [E_2]$ and $[E_1 \otimes E_2] = [E_1][E_2]$. Then, the Chern character is a ring homomorphism
$$\operatorname{ch} : K(M) \longrightarrow H^{\mathrm{even}}(M; \mathbb{R}).$$
This is actually defines an isomorphism $K(M) \otimes \mathbb{Q}$ and $H^{\mathrm{even}}(M; \mathbb{Q})$.

(4) *If L is a line bundle, then*

$$\mathrm{ch}(L) = e^{c_1(L)} = 1 + c_1(L) + \frac{1}{2}c_1(L)^2 + \cdots$$

Proof. For (1), compute (14.2) using the trivial connection d, since its curvature $F_{\mathrm{d}} = 0$.

For (2), let ∇^{E_1} and ∇^{E_2} be connections on E_1 and E_2, respectively. Then,

$$\begin{aligned}
\mathrm{ch}(E_1 \oplus E_2) &= \mathrm{tr}\left(\exp\left(\frac{1}{2\pi i}\begin{pmatrix} F_{\nabla^{E_1}} & 0 \\ 0 & F_{\nabla^{E_2}} \end{pmatrix}\right)\right) \\
&= \mathrm{tr}\begin{pmatrix} e^{(1/2\pi i)F_{\nabla^{E_1}}} & 0 \\ 0 & e^{(1/2\pi i)F_{\nabla^{E_2}}} \end{pmatrix} \\
&= \mathrm{tr}\, e^{F_{\nabla^{E_1}}/2\pi i} + \mathrm{tr}\, e^{F_{\nabla^{E_2}}/2\pi i} \\
&= \mathrm{ch}(E_1) + \mathrm{ch}(E_2).
\end{aligned}$$

For (3), we can make a similar calculation, though crucially, to simplify the exponent of a product of matrices, we need them to commute. In this case, $F_{\nabla^{E_1}} \otimes \mathrm{id}$ and $\mathrm{id} \otimes F_{\nabla^{E_2}}$ do commute, so we can compute

$$\begin{aligned}
\mathrm{ch}(E_1 \otimes E_2) &= \mathrm{tr}\exp\left(\frac{1}{2\pi i}(F_{\nabla^{E_1}} \otimes \mathrm{id} + 1 \otimes F_{\nabla^{E_2}})\right) \\
&= \mathrm{tr}\left(e^{F_{\nabla^{E_1}}/2\pi i} \otimes e^{F_{\nabla^{E_1}}/2\pi i}\right) \\
&= \mathrm{tr}\left(e^{F_{\nabla^{E_1}}/2\pi i}\right) \wedge \mathrm{tr}\left(e^{F_{\nabla^{E_2}}/2\pi i}\right) \\
&= \mathrm{ch}(E_1) \wedge \mathrm{ch}(E_2).
\end{aligned}$$ ⊠

To define Pontrjagin classes, we'll need to generalize Chern-Weil theory to connections on principal G-bundles. Suppose G is a Lie group and we have a reduction of the structure group of a vector bundle E from $\mathrm{GL}(k)$ to G, i.e. a Lie group homomorphism $G \to \mathrm{GL}(k)$ and cocycles valued in G, $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ with respect to some open cover $\mathfrak{U}$ such that $\rho \circ g_{\alpha\beta}$ are transition functions for G.

Definition 14.4. The above data (reduction of structure group to G and G-valued cocycles refining the transition functions) is called a *G-structure* on E.

For example, an $\mathrm{O}(k)$-structure on E is equivalent to a metric (for a real vector bundle).

We can also ask connections to be compatible with this structure.

Definition 14.5. Let $(E, g_{\alpha\beta})$ be a vector bundle with G-structure. Then, a connection ∇ on E is called a *G-connection* if its connection forms are valued in $\mathfrak{g}$, i.e. there are $A_\alpha \in \mathscr{A}^1(M;\mathfrak{g})$ such that $\mathrm{d}\rho(A_\alpha) \in \mathscr{A}^1(M;\mathfrak{gl}(k))$ are the connection forms for ∇ with respect to this trivialization.

For example, a unitary connection is the same thing as a $\mathrm{U}(k)$-connection, which is a good thing. In general, if ∇ is a G-connection, then $F_\nabla \in \mathscr{A}^1(M;\mathfrak{g}(E))$.

So the point is that Chern-Weil theory generalizes:

Theorem 14.6 (Chern-Weil)**.** *Let $P : \mathfrak{g} \to \mathbb{R}$ (or to $\mathbb{C}$) be an invariant polynomial, i.e. $P(\mathrm{Ad}_g X) = P(X)$ for all $g \in G$ and $X \in \mathfrak{g}$. Then, for any G-connection ∇, $P(F_\nabla) \in \mathscr{A}^{\mathrm{even}}(M)$ is closed and its cohomology class is independent of the choice of ∇.*

We will care about the case $G = \mathrm{SO}(k)$. There's an invariant polynomial on $\mathfrak{o}(k)$ that isn't $\mathrm{GL}(k)$-invariant called the *Pfaffian* $\mathrm{Pf}(A)$ whose square is the determinant: for $A \in \mathfrak{o}(k)$, $\mathrm{Pf}(A) = \det(A)$. If k is odd, then $\mathrm{Pf}(A) = 0$ is forced; in low even dimensions,

$$\mathrm{Pf}\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a$$

$$\mathrm{Pf}\begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} = af - be + dc.$$

This isn't $\mathrm{GL}(k)$-invariant because

$$\mathrm{Pf}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1}\right) = \mathrm{Pf}\begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} = -a.$$

A (real) vector bundle E has an $\mathrm{SO}(k)$-construction iff E is oriented and has a metric.

Definition 14.7. Let E be an oriented, rank-k vector bundle with metric. Then, its *Euler class* is

$$e(E) = \frac{1}{(2\pi)^{k/2}}[\mathrm{Pf}(F_\nabla)] \in H^k_{\mathrm{dR}}(M;\mathbb{R}).$$

Theorem 14.6 shows this doesn't depend on the connection; it *a priori* also depends on the choice of orientation and metric, but it only depends on the orientation. This is because metrics are a convex space, so one can interpolate between them, and so the Euler class is an invariant of oriented vector bundles.

Again, there's a way to define this purely topologically, and this shows that $e(E)$ is an integral class, realized in the image of $H^k(M;\mathbb{Z}) \to H^k_{\mathrm{dR}}(M;\mathbb{R})$.

If M is an oriented manifold, then its tangent bundle is too, so we may refer to its Euler class.

Theorem 14.8 (Chern-Gauss-Bonnet)**.** *Let M be a closed, oriented manifold. Then,*

$$\int_M e(TM) = \chi(M),$$

where $\chi(M)$ is the Euler characteristic of M.

If M is a surface, this recovers the usual Gauss-Bonnet theorem relating the Gaussian curvature to the Euler characteristic. This is a special case of the index theorem for Dirac operators, which we'll begin discussing next week.

The Chern classes and Pontrjagin classes are stable classes, in that, e.g. $c_1(E \oplus \underline{\mathbb{C}}) = c_1(E)$. This is not true the Euler class, which we'll see an example of. This means the Euler class doesn't come from k-theory.

Proposition 14.9.

(1) *If $\underline{\mathbb{R}}^k$ denotes the trivial real vector bundle of rank k, then $e(\underline{\mathbb{R}}^k) = 0$.*
(2) *$e(E \oplus F) = e(E) \wedge E(F)$.*

Example 14.10. We'll calculate $e(TS^2)$; the same ideas apply to calculate $e(TS^n)$. We defined a connection ∇ on TS^2 with curvature

$$F_\nabla(X,Y)Z = \langle Y,Z\rangle X - \langle X,Z\rangle Y$$

for $X,Y,Z \in T_pS^2$. Let $\{e_1,e_2\}$ be an oriented, orthonormal basis for T_pS^2 and $\{e^1,e^2\}$ be the induced dual basis on $T_p^*S^2$. Then, $F_\nabla(e_1,e_2)e_1 = -e_2$ and $F_\nabla(e_1,e_2)e_2 = e_1$, so

$$F_\nabla = \begin{pmatrix} 0 & e^1 \wedge e^2 \\ -e^1 \wedge e^2 & 0 \end{pmatrix}.$$

Thus,

$$e(TS^2) = \frac{1}{2\pi}[\mathrm{Pf}\, F_\nabla] = \frac{1}{2\pi} e^1 \wedge e^2 = \frac{\mathrm{d}V}{2\pi},$$

where $\mathrm{d}V$ is the usual volume form on S^2 (which required a choice of orientation), and indeed

$$\int_{S^2} e(TS)^2 = \frac{1}{2\pi}\mathrm{Vol}(S^2) = 2 = \chi(S^2).$$

However, $TS^2 \oplus \underline{\mathbb{R}}$ is trivial, so $e(TS^2 \oplus \underline{\mathbb{R}}) = 0$, demonstrating that the Euler class isn't stably trivial.

In higher dimensions, $e(TS^n)$ is 0 when n is odd and nonzero when n is even (because the Euler characteristic is 0 when n is odd and 2 when n is even).

Recall that the hairy ball theorem states that TS^2 has no trivial rank-1 subbundle. We can prove something stronger using the Euler class (the "big hairy ball theorem?").

Proposition 14.11. *TS^n has no subbundles (other than 0 or itself) when n is even.*

Proof. Suppose $E \subset TS^n$ is a nontrivial subbundle; then, we can split $TS^n = E \oplus E^\perp$, so $e(TS^n) = e(E) \wedge e(E^\perp) \in H^n_{\mathrm{dR}}(S^n)$. Since $0 < \mathrm{rank}(E) < n$, then $e(E) \in H^{\mathrm{rank}\, E}_{\mathrm{dR}}(S^n) = 0$, and therefore $e(TS^n) = 0$, which is a contradiction since n is even. ⊠

Since S^n is simply connected, then $H^1(S^n; \mathbb{Z}/2) = 0$, so the first Stiefel-Whitney class $w_1(E) = 0$, and therefore all vector bundles over S^n are orientable. This is why we didn't have to fuss about orientation in the above proof.

Lecture 15.

A Crash Course in Riemannian Geometry: 10/18/16

Here's a quick roadmap for the rest of the semester.

(1) We'll start today with some Riemannian geometry.
(2) This leads naturally to Hodge theory, which allows one to choose natural representative differential forms for de Rham cohomology classes.
(3) Next, it's natural to talk about Dirac operators, and a little complex geometry; in this case, the index theorem implies the Riemann-Roch theorem.
(4) After this, we'll do some analysis of Dirac operators.
(5) The big theorem is the Atiyah-Singer index theorem, which generalizes some things we discussed at the beginning of the class (e.g. the generalized Gauss-Bonnet theorem).
(6) Then, we'll discuss some applications.
(7) Finally, we'll discuss K-theory, and provide a proof for Bott periodicity using the index theorem.
(8) After this, there's some room for topics, e.g. the index theorem for manifolds with boundary, or applications in Lie theory (e.g. the Kostant Dirac operator), or Seiberg-Witten theory, a kind of gauge theory, or maybe something else.

It's possible to teach a whole course on Riemannian geometry, but today we only have 90 minutes. It's okay, because it will build on some things we've already discussed, and spin geometry is a refinement of Riemannian geometry, albeit with a different flavor.

Definition 15.1. A *Riemannian manifold* is a pair (M, g) where M is a smooth manifold and $g \in \Gamma(S^2T^*M)$ is a metric, i.e. it's positive definite.

This g is real-valued.

There are various kinds of curvature that one can compute in Riemannian geometry, but all of them arise from the curvature of a particular connection on TM.

Proposition 15.2. *Let M be a manifold and ∇ be a connection on TM. Then, the map $T : \Gamma(TM)\times\Gamma(TM) \to \Gamma(TM)$ defined by*

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

is a tensor $T \in \Gamma(TM \otimes \Lambda^2(T^*M))$.

This tensor T is called the *torsion* of ∇; if $T = 0$, ∇ is called *torsion-free*. This only works on TM, and cannot generalize to other vector bundles.

There's also an algebraic interpretation of torsion. If ∇ is a connection on M, it naturally induces a connection on T^*M, also written ∇, defined by

$$(\nabla_X \alpha)Y = X \cdot \alpha(Y) - \alpha(\nabla_X Y),$$

where $X, Y \in \Gamma(TM)$ and $\alpha \in \mathscr{A}^1(M)$. By forcing the Leibniz rule to hold, this induces a connection on all exterior powers of T^*M:

$$\nabla_X(\alpha_1 \wedge \cdots \wedge \alpha_k) = \sum_{i=1}^k \alpha_1 \wedge \cdots \wedge \alpha_{n-1} \wedge \nabla\alpha_i \wedge \alpha_{i+1} \wedge \cdots \wedge \alpha_k.$$

Thus, we can define a sequence of maps

$$\mathscr{A}^\bullet(M) = \Gamma(\Lambda^\bullet(T^*M)) \xrightarrow{\quad\nabla\quad} \Gamma(T^*M \otimes \Lambda^\bullet(T^*M)) \xrightarrow{\alpha,\beta\mapsto\alpha\wedge\beta} \Gamma(\Lambda^{\bullet+1}(T^*M)) = \mathscr{A}^{\bullet+1}(M). \tag{15.3}$$

Proposition 15.4. *The composition* (15.3) *is the exterior derivative* d *iff the torsion of* ∇ *is* 0.

If $e_1, \dots, e_n$ is a local frame for the tangent bundle, so $e^1, \dots, e^n$ is the dual basis, then (15.3) agreeing with the exterior derivative means that for all differential forms α,

$$\mathrm{d}\alpha = \sum_{j=1}^{n} e^j \wedge \nabla_{e_j} \alpha.$$

We seek a connection for which this is true. It would also be nice if it's compatible with the metric.

Theorem 15.5. *If (M, g) is a Riemannian manifold, there exists a unique connection ∇ on TM that's torsion-free and orthogonal.*

This connection is called the *Levi-Civita connection*. The proof is a calculation which is straightforward but unenlightening.

Definition 15.6. The curvature of the Levi-Civita connection is denoted $R \in \mathscr{A}^2(M; \mathfrak{o}(TM))$, and is called the *Riemann curvature tensor*.

Since the Levi-Civita connection is determined uniquely by M and the metric, R is an invariant of the Riemannian manifold.

Definition 15.7. A *geodesic* in a Riemannian manifold (M, g) is a smooth path $\gamma : (a, b) \to M$ satisfying the *geodesic equation* $\nabla_{\gamma'} \gamma' = 0$.

That is, in the parameterization by arc-length, $\gamma(s)$ undergoes no acceleration. The geodesic equation is the Euler-Lagrange equation for the path length functional

$$\gamma \longmapsto \int_a^b g(\gamma', \gamma')\,\mathrm{d}t.$$

Thus, geodesics are critical points for this functional.

The geodesic equation is a second-order ODE. Therefore, given a point $p \in M$ and a tangent vector $v \in T_pM$, there exists a unique geodesic γ through p in the direction of v, i.e. $\gamma(0) = p$ and $\gamma'(0) = v$.

Example 15.8.

(1) *Euclidean space* $\mathbb{E}^n = (\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ is $\mathbb{R}^n$ with the usual inner product for its metric. This uses the fact that for any $p \in \mathbb{E}^n$, $T_p\mathbb{E}^n$ is canonically identified with $\mathbb{R}^n$. There are global coordinates $x^1, \dots, x^n$, and in these coordinates, the metric is
$$g = (\mathrm{d}x^1)^2 + \cdots + (\mathrm{d}x^n)^2.$$
The Levi-Civita connection is the trivial connection, given by
$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = 0.$$
This is of course covariantly constant, and is also orthogonal. Thus, the geodesic equation is just $\gamma''(t) = 0$, so the geodesics are lines: $\gamma(t) = p + tv$, for $p, v \in \mathbb{R}^n$.

(2) Consider the sphere $S^2 = \{(x^1)^2 + \cdots + (x^n)^2 = 1\} \subset \mathbb{R}^{n+1}$. We defined
$$(\nabla_X Y)_p = X \cdot Y + \langle X, Y \rangle p$$
for $X, Y \in \Gamma(TS^n)$ and $p \in S^n$. We can pull back the Euclidean metric on $\mathbb{R}^{n+1}$ to a metric g on S^n; in this metric, ∇ defined above is the Levi-Civita connection. First, we have to check that it's orthogonal: for $X, Y, Z \in \Gamma(TS^n)$ and a $p \in S^n$,
$$\begin{aligned} X \cdot \langle Y, Z \rangle - \langle \nabla_X Y, Z \rangle - \langle Y, \nabla_X Z \rangle &= X \cdot \langle Y, Z \rangle - \langle X \cdot Y + \langle X, Y \rangle p, Z \rangle - \langle Y, X \cdot Z + \langle X, Z \rangle p \rangle \\ &= X \cdot \langle Y, Z \rangle - \langle X \cdot Y, Z \rangle - \langle Y, X \cdot Z \rangle = 0. \end{aligned}$$
Next, we check that ∇ is torsion-free:
$$\begin{aligned} (\nabla_X Y)(p) - (\nabla_Y X)(p) - [X, Y](p) &= X \cdot Y + \langle X, Y \rangle p - Y \cdot X = \langle Y, X \rangle(p) - [X, Y](p) \\ &= X \cdot Y - Y \cdot X - [X, Y] = 0. \end{aligned}$$
Thus, ∇ is indeed the Levi-Civita connection. The geodesic equation is
$$\gamma''(t) + \langle \gamma'(t), \gamma'(t) \rangle \gamma(t) = 0,$$

and after applying $\langle \cdot, \gamma(t)\rangle$,

$$\langle \gamma''(t), \gamma(t)\rangle + \langle \gamma'(t), \gamma'(t)\rangle = 0,$$

i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \gamma'(t), \gamma(t)\rangle = 0.$$

Thus, $\gamma(t) \perp \gamma'(t)$, meaning the geodesics $\gamma(t)$ are great circles on the sphere. One can explicitly check that the unique solution with $\gamma(0) = p$ and $\gamma'(0) = v$ is the great circle

$$\gamma(t) = \sin(\|v\|t)\frac{v}{\|v\|} + \cos(\|v\|t)p.$$

(3) It's possible for the curvature to be nontrivial even when the tangent bundle is trivial. For example, let G be a compact Lie group and $B : S^2\mathfrak{g} \to \mathbb{R}$ be an Ad-invariant inner product (so we require it to be positive definite). Such an invariant inner product always exists, e.g. if $G = \mathrm{SO}(n)$ or $G = \mathrm{U}(n)$, we could choose the *Killing form* $B(X, Y) = -\mathrm{tr}(XY)$. Using the left-invariant trivialization of TG, we can turn G into a metric g on G, i.e. for any $X, Y \in \mathfrak{g} = T_eG$ and $h \in G$, let $\widetilde{X}_h \in TG$ be $\widetilde{X}_h = L_{h*}X$, pullback along left multiplication. Then, the metric is defined by $g(\widetilde{X}, \widetilde{Y}) = B(X, Y)$.

This trivialization gives us the trivial connection ∇^L, the connection that makes all left-invariant vector fields parallel. This preserves g but has torsion. This is the wrong connection: if we chose the right-invariant trivialization, we'd obtain the same metric, but in general a different connection, so we've broken symmetry. Explicitly, the torsion of the left-invariant connection is

$$T(\widetilde{X}, \widetilde{Y}) = \nabla^L_{\widetilde{X}}\widetilde{Y} - \nabla^L_{\widetilde{Y}}\widetilde{X} - [\widetilde{X}, \widetilde{Y}] = -\widetilde{[X, Y]} \neq 0.$$

In fact, the Levi-Civita connection is

$$\nabla_{\widetilde{X}}\widetilde{Y} = \frac{1}{2}\widetilde{[X, Y]}.$$

This is a metric connection, which we can check on any frame, hence on the left-invariant frame:

$$\begin{aligned}\widetilde{X} \cdot g(\widetilde{Y}, \widetilde{Z}) - g(\nabla_{\widetilde{X}}\widetilde{Y}, \widetilde{Z}) - g(\widetilde{Y}, \nabla_{\widetilde{X}}\widetilde{Z}) &= -\frac{1}{2}B([X, Y], Z) - \frac{1}{2}B(Y, [X, Z]) \\ &= 0\end{aligned}$$

by the Ad-invariance of B (ad_X is skew with respect to B). Thus, it only remains to check that ∇ is torsion-free:

$$\nabla_{\widetilde{X}}\widetilde{Y} - \nabla_{\widetilde{Y}}\widetilde{X} = [\widetilde{X}, \widetilde{Y}] = \frac{1}{2}\widetilde{[X, Y]} - \frac{1}{2}\widetilde{[Y, X]} - \widetilde{[X, Y]} = 0.$$

Then, one can check that

$$R(\widetilde{X}, \widetilde{Y})\widetilde{Z} = -\frac{1}{4} \sim ([[X, Y], Z]).$$

This is in general nontrivial.

Exercise 15.9. Show that if ∇^R denotes the right-invariant connection defined analogously to ∇^L, then

$$\nabla = \frac{\nabla^L + \nabla^R}{2}.$$

Then, show that on $\mathrm{SU}(2) = S^3$, this connection is the Levi-Civita connection we defined above for S^3.

Lecture 16.

Hodge Theory: 10/20/16

Recall that if (M, g) is a Riemannian manifold, so that g is a positive definite inner product on each tangent bundle, then there's a canonical connection called the Levi-Civita connection ∇ on TM, and its curvature, called the Riemann curvature tensor, is called $R \in \mathscr{A}^2(M; \mathfrak{o}(TM))$. This curvature tensor has some nice properties.

Proposition 16.1 (Properties of the Riemann curvature tensor). *Let $p \in M$ and $X, Y, Z, W \in T_pM$. Then,*

(1) *Since R is a skew-symmetric tensor,*

$$R(X, Y)Z + R(Y, X)Z = 0.$$

(2) *Since ∇ is metric,*

$$g(R(X,Y)Z,W)+g(Z,R(X,Y)W)=0.$$

(3) *(Cyclic property) Since the Levi-Civita connection is torsion-free,*

$$R(X,Y)Z+R(Y,Z)X+R(Z,X)Y=0.$$

(4)

$$g(R(X,Y)Z,W)=g(R(Z,W)X,Y).$$

The curvature tensor and the associated 4-tensor defined by contracting with the metric are both kind of hard to visualize, since they have several different endpoints with different roles. There are other tensors derived from the curvature, some of which are easier to think about.

Definition 16.2. The *Ricci curvature* of a Riemannian manifold (M,g) is the 2-tensor defined by

$$\operatorname{Ric}(Y,Z)=\operatorname{tr}(X\longmapsto R(X,Y)Z).$$

Proposition 16.3. *The Ricci tensor is a symmetric* 2*-tensor, i.e.* $\operatorname{Ric}(X,Y)=\operatorname{Ric}(Y,X)$ *and* $\operatorname{Ric}\in\Gamma(S^2(T^*M))$.

Definition 16.4. The *scalar curvature* $S^{(M)}\to\mathbb{R}$ of (M,g) is the trace of the Ricci curvature (here, we must use the metric to identify $S^2(T^*M)$ with the space of self-adjoint endomorphisms of TM).

More concretely, if $\{e_j\}\subset T_pM$ is a local orthonormal frame, then

$$S(p)=\sum_j \operatorname{Ric}(e_j,e_j).$$

This is independent of the choice of basis, but it's crucial that we restrict to orthonormal bases.

Example 16.5 (Orientable surfaces)**.** For orientable surfaces, the passage from the Riemann curvature tensor to the scalar curvature does not lose any information. You can compute this at a $p\in\Sigma$ by computing the curvature of two lines that are orthogonal to each other at p, and taking the products of the individual curvatures.

On a sphere, both lines curve away from the normal vector, so the scalar curvature is positive. On a cylinder, one is positively curved and the other is flat, so $S=0$. On a saddle, the two lines curve in opposite directions, so the scalar curvature is negative.

For a higher-dimensional manifold, the scalar curvature is a kind of average over all embedded surfaces at that point.

Hodge theory. We can use Hodge theory to cook up canonical representatives of cohomology classes, which is nice.

The metric on the tangent bundle TM induces a dual metric on T^*M, and therefore also a metric on $\Lambda^\bullet(TM)$ defined by

$$g(\alpha_1\wedge\cdots\wedge\alpha_k,\beta_1\wedge\cdots\wedge\beta_\ell)=\delta_{k\ell}\det(g(\alpha_i,\beta_j)).$$

That is, if $k\neq\ell$, it's 0. Otherwise, the matrix whose $(i,j)^{\text{th}}$ entry is $g(\alpha_i,\beta_j)$ is square, so we take its determinant.

This has the consequence that if $\{e_1,\dots,e_n\}$ is an orthonormal basis for TM, then $\{e^1,\dots,e^n\}$ is orthonormal, and $\{e^{i_1}\wedge\cdots\wedge e^{i_k}\mid 1\le i_1<\cdots<i_k\le n\}$ is an orthonormal basis for $\Lambda^\bullet T^*M$.

Suppose M is oriented. Then, there's a unique unit-length oriented element of $\mathscr{A}^n(M)$, since $\dim\Lambda^nT^*M=1$.

Definition 16.6. For (M,g) an oriented Riemannian manifold, this unique unit-length oriented element of $\mathscr{A}^n(M)$ is called the *volume form* vol_g of M.

If we have an orthonormal basis $\{e_1,\dots,e_n\}$ for TM, then the volume form is $\operatorname{vol}_g=e^1\wedge\cdots\wedge e^n$. As you might expect, this is parallel with respect to the Levi-Civita connection.

Proposition 16.7. *For any* $X\in\Gamma(TM)$*,* $\nabla_X\operatorname{vol}_g=0$*, i.e. the volume form is invariant under parallel translation.*

Proof. In an orthonormal local frame $\{e_i\}$, the connection form A is $\mathfrak{o}(n)$-valued and

$$\begin{aligned}\nabla_X \operatorname{vol}_g = \nabla_X(e^1 \wedge \cdots \wedge e^n) &= \sum_{i=1}^n e^1 \wedge \cdots \wedge \nabla e^i \wedge \cdots \wedge e^n \\ &= \sum_{i=1}^n e^1 \wedge \cdots \wedge A(x) e^i \wedge \cdots \wedge e^n \\ &= \operatorname{tr}(A(x)) e^1 \wedge \cdots \wedge e^n = 0\end{aligned}$$

because $A(x) \in \mathfrak{o}(n)$, so it's skew-symmetric. ⊠

In addition to the volume form, the metric provides us a Hodge star operator. Recall that $\Lambda^k(M)$ and $\Lambda^{n-k}(M)$ have the same dimension $\binom{n}{n-k}$, but in general there's no isomorphism between them. The Hodge star operator provides an isomorphism in the context of a metric.

Definition 16.8. The *Hodge star operator* is the isomorphism of bundles $\star : \Lambda^k(T^*M) \to \Lambda^{n-k}(T^*M)$ characterized by

$$\alpha \wedge (\star\beta) = g(\alpha, \beta) \operatorname{vol}_g$$

for $\alpha, \beta \in \Lambda^k(T^*M)$.

In an oriented orthonormal basis,

$$\star(e^1 \wedge \cdots \wedge e^k) = e^{k+1} \wedge \cdots \wedge e^n.$$

The Hodge star isn't an involution, but it's very close.

Proposition 16.9. *For any $\alpha \in \Lambda^k(T^*M)$, $\star^2 \alpha = (-1)^{k(n-k)}\alpha$.*

Hodge theory shows us how the Hodge star descends to cohomology, providing a two-line proof of Poincaré duality, for example. But nothing in life is free — proving the theorems takes some effort, and in particular some analysis.

The broad goal of Hodge theory is to use g to find preferred representatives of de Rham cohomology classes. The de Rham complex is an infinite-dimensional complex, so let's start with an easier, finite-dimensional case.

Let $V^\bullet$ denote the chain complex

$$V^0 \xrightarrow{d^0} V^1 \xrightarrow{d^1} V^2 \longrightarrow \cdots,$$

so that $d^i \circ d^{i-1} = 0$, and suppose each V^i is finite-dimensional. Then, the *cohomology* of $V^\bullet$ is defined to be $H^j(V^\bullet) = \ker(d^i)/\operatorname{Im}(d^{i-1})$. Is there a prescribed way to choose a representative $\alpha \in V^i$ for a given cohomology class $[\alpha] \in H^i(V^\bullet)$? That is, we want to canonically split the short exact sequence

$$0 \longrightarrow \operatorname{Im}(d^{i-1}) \longrightarrow \ker(d^i) \longrightarrow H^i(V^\bullet) \longrightarrow 0,$$

since that's equivalent to finding a section. Such a split always exists for vector spaces, but we need more information for it to be natural: one way to split it is to put a positive definite inner product $(\cdot,\cdot)$ on each V^j, and then take the orthogonal complement $\mathscr{H}^j$ to $\operatorname{Im}(d^{j-1})$ inside $\ker(d^j)$. That is,

$$\mathscr{H}^j = \{\alpha \in V^j \mid d^j\alpha = 0 \text{ and } (\alpha, d^{j-1}\beta) = 0 \text{ for all } \beta \in V^{j-1}\}.$$

The condition $(\alpha, d^{j-1}\beta) = 0$ is equivalent to $(d^{j-1*}\alpha, \beta) = 0$, where d^{j-1*} denotes the adjoint. That is, $\mathscr{H}^j = \ker(d^j) \cap \ker(d^{j*})$. From here out, we'll drop the indices and just refer to d and d^* for clarity. This means we're really looking inside the total space $V = \bigoplus_i V^i$ when we reference operators such as $d + d^*$.

Proposition 16.10. $\ker(d) \cap \ker(d^*) = \ker(d + d^*) = \ker(dd^* + d^*d)$.

That is, the space in question is the kernel of a first-order differential operator. We'll later see this is an example of a Dirac operator.

Proof. For the first equality, clearly $\ker(d) \cap \ker(d^*) \subset \ker(d + d^*)$. Conversely, suppose $(d + d^*)\alpha = 0$, so that $(d\alpha, d\alpha) = (-d^*\alpha, d\alpha) = -(\alpha, d^2\alpha) = 0$ because $d^2 = 0$. Thus, $d\alpha = 0$, and a similar line of argument shows $d^*\alpha = 0$, and $\alpha \in \ker(d) \cap \ker(d^*)$.

For the second equality,

$$(d + d^*)^2 = d^2 + dd^* + d^*d + (d^*)^2 = dd^* + d^*d,$$

since $d^2 = 0$. Thus, the forward direction is immediate. Conversely, suppose $(dd^* + d^*d)\alpha = 0$; then, $0 = ((d+d^*)^2\alpha, \alpha) = ((d+d^*)\alpha, (d+d^*)\alpha)$, so $(d+d^*)\alpha = 0$ as desired. ⊠

When we pass to de Rham cohomology, $d + d^*$ will be the Laplace-Beltrami operators, and $\mathscr{H}^j$ will be the space of harmonic j-forms (with respect to this operator).

For a cohomology class $[\alpha] \in H^j(V^\bullet)$, let $\alpha_H \in \mathscr{H}^j$ denote its distinguished representative in $\mathscr{H}^j$. What does this mean geometrically? $[\alpha]$ is an affine subspace of V^j, i.e. it's of the form $\alpha + d\beta$ for some $\beta \in V^{j-1}$. This subspace is modeled on $\mathrm{Im}(d^{j-1})$. With a metric, we can choose the unique element of this affine subspace with the smallest magnitude (i.e. is closest to the origin).

Proposition 16.11. *α_H is the element of the affine space $[\alpha] \subset V^j$ of the least length.*

Proof. Any element in $[\alpha]$ is of the form $\alpha_H + d\beta$ for some $\beta \in V^{j-1}$. This has norm

$$(\alpha_H + d\beta, \alpha_H + d\beta) = (\alpha_H, \alpha_H) + (d\beta, d\beta),$$

since $\alpha_H \in \mathscr{H}^j$, which is orthogonal to $\mathrm{Im}(d^{j-1})$. Thus, this value is minimized iff $d\beta = 0$, i.e. at α_H. ⊠

Such a minimum must exist because V^j is finite-dimensional. In an infinite-dimensional space, we would have to find another proof.

Let's try this out for the de Rham complex $(\mathscr{A}^\bullet(M), \mathrm{d})$, which is infinite-dimensional. The first thing we need is an inner product. Assume M is compact and oriented.

Definition 16.12. For $\alpha, \beta \in \mathscr{A}^\bullet(M)$, their inner product is

$$(\alpha, \beta) = \int_M g(\alpha, \beta)\,\mathrm{d}\,\mathrm{vol}_g = \int_M \alpha \wedge (\star\beta).$$

This defines a positive definite, nondegenerate inner product.

The next step is to consider the adjoint of d. We're not in a nice enough functional-analytic context to expect an adjoint to always exist, but d does have a formal adjoint: there exists an operator $\mathrm{d}^* : \mathscr{A}^\bullet(M) \to \mathscr{A}^{\bullet-1}(M)$ such that $(\mathrm{d}\alpha, \beta) = (\alpha, \mathrm{d}^*\beta)$. There's an explicit formula for d^* in terms of the Hodge star operator: if $\beta \in \mathscr{A}^k(M)$,

$$\mathrm{d}^*\beta = (-1)^{n(k+1)+1} \star \circ\, \mathrm{d} \circ \star\, \beta. \tag{16.13}$$

To prove this, you do the same thing as in every adjoint calculation: integrate by parts.

$$\begin{aligned}(\mathrm{d}\alpha, \beta) &= \int_M \mathrm{d}\alpha \wedge (\star\beta) \\ &= \int_M \big(\mathrm{d}(\alpha \wedge (\star\beta)) - (-1)^{k-1}\alpha \wedge \mathrm{d}(\star\beta)\big).\end{aligned}$$

By Stokes' theorem, this is

$$\begin{aligned}&= (-1)^k \int_M \alpha \wedge \mathrm{d}(\star\beta) \\ &= (-1)^k(\alpha, \star^{-1}\mathrm{d}(\star\beta)) \\ &= (-1)^{n(k+1)+1}(\alpha, \star\mathrm{d}(\star\beta))\end{aligned}$$

as desired.

Now, we mirror the definition of $\mathscr{H}^j$.

Definition 16.14. The space of *harmonic j-forms* is $\mathscr{H}^j(M) = \ker(\mathrm{d}) \cap \ker(\mathrm{d}^*) \subset \mathscr{A}^j(M)$.

Equivalently, $\mathscr{H}^j(M) = \ker(\mathrm{d} + \mathrm{d}^*)$ or $\ker(\mathrm{dd}^* + \mathrm{d}^*\mathrm{d})$. The operator

$$\Delta = \mathrm{dd}^* + \mathrm{d}^*\mathrm{d}$$

is called the *Laplace-Beltrami operator*, which is why forms in $\mathscr{H}^j$ are called Harmonic. These forms are closed, so we can inquire about their cohomology classes.

Theorem 16.15 (Hodge)**.** *The map $\mathscr{H}^j(M) \to H^j(M)$ defined by sending $\alpha_H \mapsto [\alpha_H]$ is an isomorphism.*

We won't prove this immediately, but we'll talk about some of its consequences.

Lecture 17.

Dirac Operators: 10/25/16

Recall that if (M,g) is a compact oriented Riemannian manifold, the metric defines an inner product on the real vector space $\mathscr{A}^\bullet(M)$ in which

$$(\alpha,\beta) = \int_M \alpha\wedge(\star\beta) = \int_M g(\alpha,\beta)\,\mathrm{d}\,\mathrm{vol}_g\,.$$

Hodge's theorem (Theorem 16.15) states that if $\mathscr{H}^j(M) = \{\alpha\in\mathscr{A}^j(M) \mid \mathrm{d}\alpha = \mathrm{d}^*\alpha = 0\}$, the closed and co-closed forms, then there is an isomorphism $\mathscr{H}^j(M)\cong H^j_{\mathrm{dR}}(M)$.

Using this, we can quickly prove Poincaré duality: recall that integration defines a pairing

$$P : H^j_{\mathrm{dR}}(N)\times H^{n-j}_{\mathrm{dR}}(M)\longrightarrow \mathbb{R}$$

sending

$$\alpha,\beta\longmapsto \int_M \alpha\wedge\beta. \tag{17.1}$$

Stokes' theorem guarantees this integral is independent of cohomology class.

Corollary 17.2 (Poincaré duality). (17.1) *is a perfect pairing, i.e.* $(H^j_{\mathrm{dR}}(M))^*\cong H^{n-j}_{\mathrm{dR}}(M)$.

Proof (assuming Theorem 16.15). Since P is skew-symmetric, it suffices to show that if $P([\alpha],[\beta]) = 0$ for all $[\beta]\in H^{n-k}_{\mathrm{dR}}(M)$, then $[\alpha] = 0$.

Given such an $[\alpha]\in H^j_{\mathrm{dR}}(M)$, let $\alpha_H\in\mathscr{H}^j(M)$ be its harmonic representative. Then, $\star\alpha_H\in\mathscr{A}^{n-j}(M)$ is closed (since it commutes up to sign with d by (16.13)), and

$$\begin{aligned}0 = P([\alpha],[\star\alpha]) &= \int_M \alpha + H\wedge(\star\alpha_H)\\ &= \int_M g(\alpha_H,\alpha_H)\,\mathrm{vol}_g\\ &= (\alpha_H,\alpha_H).\end{aligned}$$

Since this inner product is nondegenerate, then $\alpha_H = 0$, so $[\alpha] = 0$. ⊠

Now, let's derive some more properties of the Dirac operator $\mathrm{d}+\mathrm{d}^*$. We're going to use the *Einstein convention* for summation: in an expression where an index (i, j, etc.) appears both as an upper and a lower index, it's implicitly summed over.

Recall that the Levi-Civita connection ∇ is torsion-free, and therefore

$$\mathrm{d} = e^j\wedge\nabla_{e^j} = \varepsilon(e_j)\cdot\nabla_{e^j}$$

for every local frame $\{e_1,\dots,e_m\}$ for TM and its dual frame $\{e^1,\dots,e^n\}$ for T^*M. Here,

$$\varepsilon : \Lambda^\bullet T^*M\longrightarrow\Lambda^{\bullet+1}T^*M$$

is the bundle map

$$\varepsilon(\alpha)\beta = \alpha\wedge\beta,$$

and its adjoint is

$$i : \Lambda^\bullet(T^*M)\longrightarrow\Lambda^{\bullet-1}T^*M$$

defined to be the unique operator satisfying

(1) $i(\alpha)\beta = g(\alpha,\beta)$ if $\beta\in\Lambda^1(T^*M)$.
(2) If $\beta^1,\beta^2\in\Lambda^\bullet(T^*M)$, then

$$i(\alpha)(\beta_1\wedge\beta_2) = (i(\alpha)\beta_1)\wedge\beta_2 + (-1)^{\deg\beta_1}\beta_1\wedge i(\alpha)\beta_2.$$

We may now recharacterize d^*:

Proposition 17.3. $d^* = -i(e^j)\nabla_{e^j}$.

Proof. It suffices to show that $(\mathrm{d}\alpha,\beta)+(\alpha,i(e^j)\nabla_{e^j}\beta)=0$, so let's compute it.

$$\begin{aligned}(\mathrm{d}\alpha,\beta)+(\alpha,i(e^j)\nabla_{e^j}\beta) &= \int_M (g(e^j\wedge\nabla_{e^j}\alpha,\beta)+g(\alpha,i(e^j)\nabla_{e^j}\beta))\,\mathrm{vol}_g\\ &= \int_M (g(e^j\wedge\nabla_{e^j}\alpha,\beta)+g(e^j\wedge\alpha,\nabla_{e^j}\beta)).\end{aligned}$$

Since ∇ is a metric connection,

$$\begin{aligned}&= \int_M (g(e^j\wedge\nabla_{e^j}\alpha,\beta)+e_j\cdot g(e^j\wedge\alpha,\beta)-g(\nabla_{e_j}(e^j\wedge\alpha),\beta))\,\mathrm{vol}_g\\ &= \int_M (e_j\cdot g(e^j\wedge\alpha,\beta)-g(\nabla_{e^j}e^j\wedge\alpha\beta))\,\mathrm{vol}_g\,.\end{aligned}\tag{17.4}$$

We want to identify this integrad; it will be the divergence of a vector field.

Definition 17.5. Let X be a vector field on a Riemannian manifold (M,g), and let ∇ be its Levi-Civita connection. Then, $\nabla_X\in\Gamma(\mathrm{End}(TM))$ is a $C^\infty(M)$-linear operator, hence has a trace $\mathrm{tr}(\nabla X)\in C^\infty(M)$, called the *divergence* of X.

This agrees with the definition of the divergence in vector calculus classes; it also heavily depends on the metric.

Returning to our problem, given $\alpha\in\mathscr{A}^k(M)$ and $\beta\in\mathscr{A}^{k+1}(M)$, define a vector field $X_{\alpha\beta}$ by $\gamma(X_{\alpha\beta})=g(\gamma\wedge\alpha,\beta)$ for all $\gamma\in\mathscr{A}^1(M)$, which suffices by the nondegeneracy of the inner product. Then,

$$\begin{aligned}\mathrm{tr}(\nabla X_{\alpha\beta}) &= e^j(\nabla_{e^j}X_{\alpha\beta})\\ &= e_j\cdot(e^j(X_{\alpha\beta}))-(\nabla_{e^j}e^j)(X_{\alpha\beta})\\ &= e_j\cdot g(e^j\wedge\alpha,\beta)-g(\nabla_{e^j}e^j\wedge\alpha,\beta),\end{aligned}$$

which is the integrand in (17.4). An analogue of the divergence theorem finishes the proof.

Lemma 17.6. *If X is a vector field on M and ∇ is the Levi-Civita connection on M,*

$$\int_m \mathrm{tr}(\nabla X)\,\mathrm{vol}_g=0.$$

Proof sketch. Since ∇ is torsion-free, $\nabla_X Y-\nabla_Y X=[X,Y]$, so $\nabla_X=\mathscr{L}_X+\nabla X$ as operators. By requiring linearity and the Leibniz rule, this extends to $\mathscr{A}^\bullet(M)$; since the volume form is parallel with the Levi-Civita connection, then $0=\nabla_X\mathrm{vol}_g=\mathscr{L}_X\mathrm{vol}_g+\mathrm{tr}(\nabla X)\,\mathrm{vol}_g$. Thus, using a Cartan homotopy,

$$\begin{aligned}\int_M \mathrm{tr}(\nabla X)\,\mathrm{vol}_g &= -\int_M \mathscr{L}_X\,\mathrm{vol}_g\\ &= -\int_M (\mathrm{d}\circ i_X+i_X\circ\mathrm{d})\,\mathrm{vol}_g\\ &= -\int_M \mathrm{d}(i_X\,\mathrm{vol}_g)=0\end{aligned}$$

by Stokes' theorem. ⊠

Thus, the divergence vanishes, so (17.4) vanishes, as we wanted. ⊠

Since $\mathrm{d}^*=i(e^j)\cdot\nabla_{e^j}$, then

$$\mathrm{d}+\mathrm{d}^*=(\varepsilon(e^j)-i(e^j))\nabla_{e^j};$$

in other words, $\mathrm{d}+\mathrm{d}^*$ acts by the Clifford action of T^*M on $\Lambda^\bullet T^*M$! Specifically, for an $\alpha\in T^*M$, $c(\alpha):\Lambda^\bullet(T^*M)\to\Lambda^\bullet(T^*M)$ is defined to send

$$\beta\longmapsto\alpha\wedge\beta-i(\alpha)\beta,$$

and $c(\alpha)^2=g(\alpha,\alpha)$, thus giving $\Lambda^\bullet(T^*M)$ the structue of a $\mathrm{C}\ell(T^*M)$-module.

We can also do this invariantly: $\mathrm{d}+\mathrm{d}^*$ is the composition

$$\Gamma(\Lambda^\bullet T^*M)\xrightarrow{\nabla}\Gamma(T^*M\otimes\Lambda^\bullet T^*M)\xrightarrow{c}\Gamma(\Lambda^\bullet T^*M).$$

This suggests a generalization, replacing $\Lambda^\bullet(T^*M)$ with an arbitrary Clifford module.

Definition 17.7. Let (M, g) be a Riemannian manifold. A *Clifford module* over M is a $\mathbb{Z}/2$-graded module $E = E^+ \oplus E^-$ over $\mathrm{C}\ell(T^*M)$, i.e. a morphism of vector bundles $c : T^*M \to \operatorname{End} E$ such that $c(\alpha)^2 = -g(\alpha,\alpha)\mathrm{id}_E$ and $c(\alpha)E^\pm \subset E^\mp$, together with a metric h and an orthogonal (or unitary for a complex bundle) connection ∇^E, such that

(1) h respects the $\mathbb{Z}/2$-grading and for all $\alpha \in T^*M$ and $\psi_1, \psi_2 \in E$, $h(c(\alpha)\psi_1, \psi_2) = -h(\psi_1, c(\alpha)\psi_2)$.
(2) ∇^E must respect the $\mathbb{Z}/2$-grading and

$$\nabla^E_X(c(\alpha)\psi) = c(\nabla_X \alpha)\psi + c(\alpha)\nabla^E_X \psi.$$

That is, c should be parallel to the connection. These modules will be our main objects of study.

Exercise 17.8. Check that $E = \Lambda^\bullet(T^*M)$, $c(\alpha) = \varepsilon(\alpha) - i(\alpha)$, and ∇^E equal to the Levi-Civita connection satisfy this definition.

Definition 17.9. Let E be a Clifford module over (M, g). Then, its *Dirac operator* $D : \Gamma(E) \to \Gamma(E)$ is defined to be the composition

$$\Gamma(E) \xrightarrow{\nabla^E} \Gamma(T^*M \otimes E) \xrightarrow{c} \Gamma(E).$$

Explicitly, in coordinates, $D = c(e^j)\nabla^E_{e^j}$, and if we think about the grading,

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix},$$

where $D^\pm = D|_{\Gamma(E^\pm)}$, with an inner product given by integrating the metric over the volume form. The Hodge-theoretic example $\mathrm{d} + \mathrm{d}^*$ is the prototypical example, and is called the *de Rham-Dirac operator*. Its kernel is the space of harmonic forms, $\ker(\mathrm{d} + \mathrm{d}^*) = \mathscr{H}^\bullet(M) \cong H_{\mathrm{dR}}(M)$. Associated to this is the Euler characteristic

$$\chi(M) = \dim H^{\mathrm{even}}_{\mathrm{dR}}(M) - \dim H^{\mathrm{odd}}_{\mathrm{dR}}(M) = \dim\ker(D^+) - \dim\ker(D^-).$$

It will be helpful to think of this as an element of $K(\mathrm{pt})$.

Definition 17.10. The *index* of a Dirac operator D is

$$\mathrm{ind}(D) = \dim\ker(D^+) - \dim\ker(D^-).$$

Remark. We'll soon see that for any Dirac operator D on a compact manifold, $\ker(D)$ is always finite-dimensional, which follows from the ellipticity of these operators. More generally, the index is defined for Fredholm operators $T : V \to W$, which are those operators with finite-dimensional kernel and cokernel, and the Fredholm index $\mathrm{ind}(T) = \dim\ker(T) - \dim\operatorname{coker}(T)$ coincides with the Dirac index, because D^- and D^+ are adjoints.

If V and W are finite-dimensional, the rank-nullity theorem shows that $\mathrm{ind}(T) = \dim V - \dim W$; thus, this notion of index is only nontrivial for infinite-dimensional vector spaces.

Lecture 18.

: 10/27/16

Recall that if $(V, \langle\cdot,\cdot\rangle)$ is a real inner product space, we took its Clifford algebra $\mathrm{C}\ell(V)$, and if W is any $\mathbb{C}$-vector space, then there's a bijective equivalence between the $\mathrm{C}\ell(V)$-module structures on W and the linear maps $c : V \to \operatorname{End} W$ such that $c(v)^2 = -\langle v, v\rangle \mathrm{id}_W$.

Definition 18.1. Let W be a $\mathrm{C}\ell(V)$-module. Then, W is *unitary* if it has a Hermitian inner product h such that $c(v)$ is a skew-adjoint endomorphism of V for all $v \in V$, i.e. $h(c(v)\psi_1, \psi_2) = -h(\psi_1, c(v)\psi_2)$ for all $v \in V$ and $\psi_1, \psi_2 \in W$.

Recall that we defined $\mathrm{Spin}(V) \subset \mathrm{C}\ell(V)^\times$ to be the group generated by even products of unit vectors.

Proposition 18.2. *If W is a unitary module over $\mathrm{C}\ell(V)$, restriction defines a unitary representation* $\mathrm{Spin}(V) \to \mathrm{U}(W, h)$.

Proof. Let $u = e_1 \cdots e_{2k} \in \mathrm{Spin}(V) \subset \mathrm{C}\ell(V)$. Then,

$$\begin{aligned} h(e_1 \cdots e_{2k}\psi_1, e_1 \cdots e_{2k}\psi_2) &= (-1)^{2k} h(\psi_1, e_1^2 \cdots e_{2k}^2 \psi_2) \\ &= h(\psi_1, \psi_2), \end{aligned}$$

so the representation is indeed unitary. ⊠

Note that we can't ask for $c(v)$ to act self-adjointly,[25] because that would force

$$o \le h(c(v)\psi, c(v)\psi) = h(\psi, c(v)^2\psi) = -\langle v, v\rangle h(\psi,\psi) = 0,$$

even when ψ and v are nonzero.

Using Proposition 18.2, you can show that the regular representation of $\mathrm{C}\ell(V)$ on itself and the spin representation are both unitary.

Last time, we did this over a Riemannian manifold, but some of the terminology wrong. Here's the correct terminology.

Definition 18.3. A *Dirac bundle* over a Riemannian manifold (M, g) is a triple (E, h, ∇^E) where

- $E = E^+ \oplus E^-$ is a $\mathbb{Z}/2$-graded $\mathrm{C}\ell(T^*M)$-module, i.e. we have a bundle map $c : T^*M \to E$ such that $c(\alpha)^2 = -g(\alpha,\alpha)$ and $c(\alpha)E^\pm = E^\mp$;
- h is a Hermitian metric on E that respects the $\mathbb{Z}/2$-grading; and
- ∇^E is a connection on E that respects the $\mathbb{Z}/2$-grading,

such that

(1) $c(\alpha)$ is skew-adjoint with respect to h, and
(2) ∇^E satisfies a Leibniz rule

$$\nabla_X^E(c(\alpha)\psi) = c(\nabla_X \alpha)\psi + c(\alpha)\nabla_X^E \psi.$$

This definition encapsulates everything needed to define a Dirac operator. We'll almost always be in the case where M is even-dimensional.

Example 18.4. We talked about the example $E = \Lambda^\bullet(T^*M \otimes \mathbb{C})$, which decomposes as $\Lambda^{\mathrm{even}} T^*_{\mathbb{C}} M \oplus \Lambda^{\mathrm{odd}} T^*_{\mathbb{C}} M$. Here, h is induced from g and ∇^E is the Levi-Civita connection.

Definition 18.5. If (E, h, ∇^E) is a Dirac bundle, its associated *Dirac operator* is

$$D = c(e^j)\nabla^E_{e_j}.$$

(Here, we use the Einstein summation convention.)

The Dirac operator for the bundle in Example 18.4 is $D = \mathrm{d} + \mathrm{d}^*$, and the index of D is $\chi(M)$.

Example 18.6 (The signature operator)**.** This example will show that the $\mathbb{Z}/2$-grading imposed on a Dirac bundle is important.

Suppose M is a $4k$-dimensional manifold; then,

$$\Gamma = (-1)^k e^1 e^2 \cdots e^{4k} = (-1)^k \mathrm{vol}_g \in \mathrm{C}\ell(T^*M)$$

squares to 1. Thus, we can take $E = \Lambda^\bullet(T^*M \otimes \mathbb{C})$ as a $\mathrm{C}\ell(T^*M)$-module, but with the $\mathbb{Z}/2$-grading defined by the eigenspace decomposition of Γ. Specifically, we let

$$\Lambda^\pm T^*M = \{\alpha \in \Lambda^\bullet(T^*M) \mid \Gamma\alpha = \pm\alpha\}.$$

Then, $E = \Lambda^+ T^*M \oplus \Lambda^- T^*M$. Moreover, Γ anticommutes with T^*M inside $\mathrm{C}\ell(T^*M)$, so $c(\alpha)$ switches the odd and even parts of E, and therefore E is a super-$\mathrm{C}\ell(T^*M)$-module.

We let h and ∇^E be as in Example 18.4; these are compatible with the $\mathbb{Z}/2$-grading because Γ is a multiple of the volume form and hence parallel; thus, its eigenspaces are preserved under the connection.

This is a different bundle: the Dirac operator only depends on c and ∇, so it's still $\mathrm{d} + \mathrm{d}^*$, but since the $\mathbb{Z}/2$-grading is different, the index is different.

Recall that on a $4k$-dimensional compact, oriented manifold M we have an *intersection pairing*

$$I : H^k(M;\mathbb{R}) \times H^k(M;\mathbb{R}) \longrightarrow \mathbb{R}$$

[25] There are differing sign conventions here: some writers define Clifford algebras differently, and in that case $c(v)$ can act self-adjointly, but not skew self-adjointly.

defined by $(\alpha,\beta)\mapsto\langle\alpha\smile\beta,[M]\rangle$. On de Rham cohomology, this is

$$(\alpha,\beta)\longmapsto\int_M\alpha\wedge\beta.$$

Since $2k$ is even, I is a symmetric bilinear form, and it's nondegenerate by Poincaré duality. The key invariant of I is its *signature* $\sigma(M)=p-q$, where

- p is the maximal dimension of a subspace of $H^{2k}(M;\mathbb{R})$ on which I is positive definite, and
- q is the maximal dimension of a subspace of $H^{2k}(M;\mathbb{R})$ on which I is negative definite.

Since I is nondegenerate, $p+q=\dim H^{2k}(M;\mathbb{R})$. In particular, there is a basis $b_1,\dots,b_{p+q}$ of $H^{2k}(M;\mathbb{R})$ such that in this basis,

$$H(x^ib_i,x^ib_i)=(x^1)^2+\cdots+(x^p)^2-(x^{p+1})^2-\cdots-(x^{p+q})^2.$$

In particular,

$$\operatorname{ind}(D)=\dim\ker(D|_{E^+})-\dim\ker(D|_{E^-})=\dim\ker(D|_{E^+})-\dim\operatorname{coker}(D|_{E^+}).$$

Theorem 18.7. *The index of the Dirac operator for this Dirac bundle is $\sigma(M)$.*

Proof. We still have $\ker(D)=\ker(\mathrm{d}+\mathrm{d}^*)=\mathscr{H}^\bullet(M)$, so

$$\ker D^\pm=\{\alpha\in\mathscr{H}^\bullet(M)\mid\Gamma\alpha=\pm\alpha\}.$$

Let $\mathscr{H}^{2k,\pm}=\ker(D^\pm)\cap\mathscr{H}^{2k}(M)$ and $\mathscr{H}^{<2k}=\mathscr{H}^0(M)\oplus\cdots\oplus\mathscr{H}^{2k-1}(M)$. Then, we claim an isomorphism

$$\mathscr{H}^{<2k}\oplus\mathscr{H}^{2k,\pm}\xrightarrow{\sim}\ker(D^\pm).\tag{18.8}$$

This is because $\Gamma=\pm\star$, because

$$\Gamma(e_1\cdots e_\ell)=\pm e_1\cdots e_{4k}e_1\cdots e_\ell=\pm e_{\ell+1}\cdots e_{4k}.$$

In particular, Γ is a map $\Lambda^\ell(T^*M)\to\Lambda^{4k-\ell}(T^*M)$. The isomorphism (18.8) is the map

$$(\alpha,\beta)\longmapsto\alpha\pm\Gamma\alpha+\beta.$$

In degree $2k$, $\Gamma=\star$, so $\mathscr{H}^{2k,\pm}(M)=\{\alpha\in\mathscr{H}^{2k}(M)\mid\star\alpha=\pm\alpha\}$. Thus,

$$\begin{aligned}\operatorname{ind}(D)&=\dim\ker(D^+)-\dim\ker(D^-)\\&=\dim\mathscr{H}^{<2k}+\dim\mathscr{H}^{2k,+}-\dim\mathscr{H}^{<2k}-\dim\mathscr{H}^{2k,-}\\&=\dim\mathscr{H}^{2k,+}-\dim\mathscr{H}^{2k,-}.\end{aligned}$$

We'll be done as soon as we understand how the intersection form acts on $\mathscr{H}^{2k,\pm}$. Suppose $\alpha\in\mathscr{H}^{2k,+}$; then

$$\begin{aligned}I(\alpha,\alpha)&=\int_M\alpha\wedge\alpha=\int_M\alpha\wedge(\star\alpha)\\&=\int_M g(\alpha,\alpha)\operatorname{vol}_g,\end{aligned}$$

which is nonnegative, and equal to 0 iff $\alpha=0$. Thus, I is positive definite on $\mathscr{H}^{2k,+}$; similarly, it's negative definite on $\mathscr{H}^{2k,-}$, because we'd obtain $I(\beta,\beta)=-\int_M g(\beta,\beta)\operatorname{vol}_g$ for $\beta\in\mathscr{H}^{2k,-}$. ⊠

Definition 18.9. An oriented Riemannian manifold (M,g) is called *spin* if its tangent bundle has a spin structure. Thanks to the metric, this is equivalent to the cotangent bundle having a spin structure.

Recall that, since we have a metric and an orientation, the structure group can be reduced to $\mathrm{SO}(n)$, and a spin structure is a further reduction to $\mathrm{Spin}(n)$.

It's possible to associate a Dirac bundle called the *spin Dirac bundle* to any spin manifold: in general, a spin structure defines a Clifford module of spinors, and the remaining data of a Dirac bundle arises from the metric and Levi-Civita connection on M. Not every manifold has a spin structure, however.

Given a Dirac bundle (E,h,∇^E), it's possible to twist it by a Hermitian bundle W with Hermitian metric h_W and connection ∇^W.

Proposition 18.10. *With E and W as above, the data $(E\otimes W,h\otimes h_W,\nabla\otimes 1+1\otimes\nabla^W)$ defines a Dirac bundle.*

Here, the action is $c(\alpha)(\psi\otimes w)=(c(\alpha)\psi)\otimes w$ for $\psi\in E$, $w\in W$, and $\alpha\in T^*M$. The following theorem is the reason why we care about twisted Dirac bundles.

Theorem 18.11. *Let M be a spin manifold; then, every Dirac bundle is a twisting of the spin Dirac bundle.*

This will be an important ingredient in the index theorem.