M392c NOTES: TOPICS IN ALGEBRAIC GEOMETRY

ARUN DEBRAY
SEPTEMBER 19, 2019

These notes were taken in UT Austin's M392c (Topics in algebraic geometry) class in Fall 2019, taught by Bernd Seibert. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Any mistakes in the notes are my own.

Contents

Lecture 1.

Historical overview of mirror symmetry, I: 8/29/19

"I saw this happening, which makes me realize how old I am."

The first two lectures will contain an overview of mirror symmetry, the broad-scope context of this class; the specific details, e.g. how fast-paced we go, will be determined by who the audience is.

There are about as many perspectives on mirror symmetry as there are researchers in mirror symmetry, but a consensus of sorts has emerged.

Recall that the *canonical bundle* of a complex manifold X is $K_X \coloneqq \operatorname{Det} T^*X$. A *Calabi-Yau manifold* is a complex manifold with a trivialization of its canonical bundle, i.e. $K_X \cong \mathcal{O}_X$. Though the definition doesn't imply it, we also often assume $b_1(X) = 0$ and that X is irreducible.

Let X be a Calabi-Yau threefold (i.e. it's a Calabi-Yau manifold of complex dimension 3).

Example 1.1. A *quintic threefold* $X \subset \mathbb{P}^4$ is the zero locus in $\mathbb{P}^4$ of a homogeneous, degree-5 polynomial f in the 5 variables $x_0, \dots, x_4$. For a generically chosen f, X is smooth. We'll prove X is Calabi-Yau.

Let $\mathcal{I}$ denote the vanishing sheaf of ideals of X, i.e. $(f) \subset \mathcal{O}_{\mathbb{P}^4}$. We therefore have a short exact sequence

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathbb{P}^4}|_X \longrightarrow \Omega_X \longrightarrow 0, \tag{1.2}$$

and since $\mathcal{I}/\mathcal{I}^2 \cong \mathcal{I} \otimes_{\mathcal{O}_{\mathbb{P}^4}} \mathcal{O}_X$, it's an invertible sheaf. Using (1.2),

$$K_{\mathbb{P}^4}|_X = \operatorname{Det} \Omega_{\mathbb{P}^4}|_X \cong \mathcal{I}/\mathcal{I}^2 \otimes K_X. \tag{1.3}$$

By standard methods, one can compute that $K_{\mathbb{P}^4} = \mathcal{O}_{\mathbb{P}^4}(-5)$, hence $K_{\mathbb{P}^4}|_X \cong \mathcal{O}_X(-5)$. Since $\mathcal{I} \hookrightarrow \mathcal{O}_{\mathbb{P}^4}$, this means $\mathcal{I} \simeq \mathcal{O}_{\mathbb{P}^4}(-5)$, and therefore $\mathcal{I}/\mathcal{I}^2 \cong \mathcal{O}_X(-5)$, and as a corollary $K_X \cong \mathcal{O}_X$. ◀

Remark 1.4. Mirror symmetry is related to string theory! If you ask physicists, even theoretical ones, they'll tell you there's plenty to do still in setting up string theory, but there are two related classes of string theories called IIA and IIB, which are supersymmetric σ-models with a target $\mathbb{R}^{1,3} \times X$, where X is some Calabi-Yau threefold. Phenomenologists are interested in the $\mathbb{R}^{1,3}$ piece, which hopes to describe our world, and X tells us some information about particle dynamics in the $\mathbb{R}^{1,3}$ factor via the Kaluza-Klein mechanism.

Now, supersymmetric σ-models are better understood in physics than string theories in general, and in fact these give you two superconformal field theories (SCFTs), one corresponding to IIA, and to IIB, with target X. Using physics arguments, you can calculate the Hodge numbers of X; since X is a Calabi-Yau threefold, you can (and we will) show that its only nonzero Hodge numbers are $h^{1,1}$ and $h^{2,1}$.

But if you do this for both the A- and B-type SCFTs, you get flipped answers: $h^{1,1}$ computed via the A-type SCFT is $h^{2,1}$ computed via the B-type SCFT. We think there's only one string theory, which is puzzling. Dixon and Lerihe-Vafa-Warner noticed that sometimes, we can find another Calabi-Yau threefold Y such that the A-type SCFT of X is equivalent to the B-type SCFT of Y, and the A-type SCFT of Y is equivalent to the B-type SCFT for X, hence in particular $h^{1,1}(X) = h^{2,1}(Y)$ and $h^{2,1}(X) = h^{1,1}(Y)$. In fact, we'd expect the IIA string theory for X should be equivalent to the IIB string theory for Y, and likewise the IIB string theory for X should be equivalent to the IIA string theory for Y.

Greene and Plesser postulated such a duality, constructing the dual theory via an orbifolding construction. These were all in the late 1980s or early 1990s, but it was another decade before Hori-Vafa proved (at a physics level of rigor) this duality for complete intersections in toric varieties. ◀

This is good if you like physics, but what if you don't? It turns out that mirror symmetry is still useful – it helps us calculate things in pure mathematics that we didn't have access to before.

Remark 1.5. Let's address a possible source of confusion in the literature.

In 1988, Witten introduced the notion of a *topological twist* of a supersymmetric σ-model. These are topological field theories in the physical sense, not the mathematical ones: we only mean that the variation in the metric vanishes. We can obtain from this data two topologically twisted σ-models called the *A-model* $A(X)$ and the *B-model* $B(X)$, which are *a priori* unrelated to the A- and B-type SCFTs — but it turns out $A(X)$ and $B(X)$ compute certain limits, called *Yukawa couplings*, for these SCFTs. In particular, an equivalence of the A-type SCFT for X and the B-type SCFT for Y (and vice versa) implies an equivalence of $A(X)$ and $B(Y)$.

Caution: the A-model tells you about type IIB string theory, and the B-model tells you about type IIA string theory.

Some mathematicians zoom in on this, and say that mirror symmetry is just the equivalence of the $A(X)$ and $B(Y)$, and of $A(Y)$ and $B(X)$. ◀

Interestingly, the A-model only depends on the symplectic structure on X, and the B-model depends only on the complex structure.

In 1991, Candelas, de la Ossa, Greene, and Parkes [CDGP91] studied the quintic threefold and its mirror Y_t (here t is a parameter, which we'll say more about later), and computed the Yukawa couplings F_A and F_B. Geometrically, the A-model has to do with counts of rational (i.e. genus-zero) holomorphic curves;[1] some of these were known classically. The B-model has to do with period integrals

$$F_B(t) = \int_\alpha \Omega_{Y_t}, \tag{1.6}$$

where $\alpha \in H_3(Y_t)$ and Ω_{Y_t} is a (suitably normalized) holomorphic volume form. These are generally much easier to compute. This was an astounding computation, and they made a further prediction which turned out to be true, and led to astonishing divisibility properties.

A reasonable next question is: can we do this on other Calabi-Yau threefolds? Morrison, building on ideas of Deligne, computed $F_B(Y)$ in terms of Hodge theory, giving more parameters for the Calabi-Yau moduli space. On the A-side, this led to the creation of *Gromov-Witten theory* around 1993, which makes $F_A(X)$ precise. On the symplectic side, this was the work of many people, including Y. Ruan, Tian, Fukaya-Ono, and Siebert; on the algebro-geometric side, this included work of Jun Li and Behrend-Fantechi.

Kontesvich's 1994 ICM address (and subsequent lecture notes) proposed a conjecture called *homological mirror symmetry*. In symplectic geometry, one can extract a triangulated category called the *Fukaya category* from a symplectic manifold X; if Y denotes its mirror, homological mirror symmetry postulates that this is equivalent to the bounded derived category of Y.

This was a charismatic, visionary conjecture, and people have spent a lot of time and thought on it. It's influenced many fields, to the point that people have focued less on the other contexts (e.g. the enumerative

[1] If we don't have a complex structure, but only a symplectic structure, this seems nonsensical, but these curve counts can nonetheless be defined.

formulation). But this is a formulation, not an explanation. We don't quite have a mathematical explanation yet, though ingredients are in place to construct mirrors and make a systematic proof possible.

In 1996, Givental provided a proof of the equivalence of the counts established by Candelas, de la Ossa, Greene, and Parkes; Givental's proof was for hypersurfaces, and Lian, Liu, and Yau provided the general proof. The proof wasn't explanatory: it didn't express these equalities as being true for a reason. These proofs proceeded via localization methods: find a $\mathbb{C}^\times$-action and use methods akin to those of Atiyah-Bott and Berline-Vergne.

Progress on homological mirror symmetry came a little later, first established for quartic twofolds (in $\mathbb{P}^3$), i.e. for K3 surfaces. So the statement has to be modified somehow, but this can be done. This was done by Seidel in 2003, then to more general Calabi-Yau hypersurfaces by his student Nick Sheridan in 2011. This was very hard work, but was strong evidence that mirror symmetry in its various avatars is real. (One of these avatars is the geometric Langlands program.)

In the course of proving homological mirror symmetry for various cases, such as SZY-fibered symplectic manifolds on the A-side and rigid spaces on the Y-side (see Abouazid, Fukaya-Oh-Ohta-Ono), we needed a way to produce mirrors. This led to research into intrinsic construction of mirrors, and this has gone on to have applications outside of mirror symmetry: this allows for some computations to be simplified by passing to the mirror and working there. This includes work of Gross-Siebert, Gross-Hacking-Keel, and more.

This is all the genus-zero part of the story, which physicists call the *tree-level* part of the theory. People also study higher-genus (or second quantized mirror symmetry), such as Costello and Si Li, or look at the method of topolgical recursion, e.g. Eynard and Orantin.

The plan for this class is, roughly:

- Sketch the computation of Candelas, de la Ossa, Greene, and Parkes in [CDGP91].
- Gromov-Witten theory, and its construction via virtual fundamental classes and moduli stacks.
- Potentially an introduction to toric geometry.
- Toric degenerations and mirror constructions. This has undergone several refinements, and we'll take a pretty modern perspective.
- Using this, you can compute homogeneous coordinate rings (which is a lot of information: it knows the variety, hence also the derived category). On the A-side, a result of Polischuk forces that there's only one possible Fukaya category (as an A_∞-category), which leads to a proposal for a plan to prove homological mirror symmetry in great generality. The mirror statement (using the Fukaya category and its A_∞-structure to determine the derived category of the mirror) is considered a hard open problem in symplectic geometry.
- Next, we could discuss higher-genus information. In Gromov-Witten theory, the genus is part of the input data, but we could also compute *Donaldson-Thomas invariants*, where we count ideal sheaves rather than holomorphic curves. This organizes the count differently, because curves of different genera may be part of the same count. The role of Donaldson-Thomas theory in mirror symmetry is somewhat unclear, and there's an interesting statistical-mechanics model called *crystal melting*, which ports this down to genus zero. This is work of Okounkov and others.

This can be adjusted depending on class interest.

In the last few minutes, let's begin talking about the quintic threefold, its mirror, and the work of Candelas, de la Ossa, Greene, and Parkes.

The quintic threefold comes in a big family: we're looking at degree-5 homogeneous polynomials in five variables, so to enumerate monomials, we need to know the number of ways to draw lines between five points in a line. For example, $x_0^2 x_2$ corresponds to $12|345$ and $x_0 x_1^2 x_2$ corresponds to $1|2\|345$. The answer is

$$\binom{n+d-1}{n-1} = \binom{n+d-1}{d}, \tag{1.7}$$

which here is $\binom{9}{5} = 126$. Hence the dimension of the moduli space of quintic polynomials in $\mathbb{P}^4$ is $126-1 = 125$. However, to get the space of quintics, we need to divide out by the symmetries of the problem, which is PGL_5. This has dimension $5^2 - 1 = 24$, so the moduli space of quintic threefolds is 101-dimensional.

This is *huge* — you may think it's a long way down the road to the chemist, but that's just peanuts compared to the dimension of this moduli space. It's way too big for us to get a good grasp on.

Indeed, for a projective Calabi-Yau manifold X, the moduli space of Calabi-Yau manifolds deformation-equivalent to X is a smooth orbifold[2] of complex dimension $h^1(\Theta_X)$, where Θ_X is the holomorphic tangent bundle, and we can show that this is 101 for the quintic threefold.

Lecture 2.

Hodge diamonds of Calabi-Yau threefolds: 9/3/19

Last time, we studied the quintic threefold in $\mathbb{P}^4$, which is Calabi-Yau, and whose moduli space is terribly high-dimensional, but remarkably is a smooth orbifold! (That is, the stabilizer groups are finite.) This is unusual, and related to the Calabi-Yau property — for general varieties there's a "Murphy's law" property guaranteeing all sorts of terrible singularities in the moduli space. For a general projective Calabi-Yau manifold X, the moduli of Calabi-Yau deformations of X is a smooth orbifold of dimension $h^1(\Theta_X)$; for the quintic threefold this is 101. Here Θ_X is the holomorphic tangent bundle.

We'll begin with a brief description of how to compute this number, then look at the Hodge theory of the quintic threefold and its mirror. The *Euler sequence* is the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \longrightarrow \Theta_{\mathbb{P}^n} \longrightarrow 0. \tag{2.1}$$

To describe the maps, write $\mathbb{P}^n = \operatorname{Proj}\mathbb{C}[x_0,\dots,x_n]$; then $x_i\partial_{x_i}$ is a well-defined logarithmic vector field on $\mathbb{P}^n$. Then the two maps in (2.1) are $1 \mapsto \sum e_i$ and $e_i \mapsto x_i\partial x_i$, respectively, where e_i is the i^{th} standard basis vector in $\mathcal{O}(1)^{\oplus n}$.

Remark 2.2. TODO: I (Arun) think this looks like a short exact sequence I'd recognize in differential topology relating $T\mathbb{CP}^n$ and its tautological bundle; I'd like to think this through. ◀

We also have the *conormal sequence* for any variety $X \subset \mathbb{P}^4$. Let $\mathcal{I}_X$ denote the sheaf of ideals cutting out X; then the following sequence is short exact:

$$0 \longrightarrow \mathcal{I}_X/\mathcal{I}_X^2 \xrightarrow{g\mapsto \mathrm{d}g} \Omega^1_{\mathbb{P}^4}|_X \xrightarrow{\mathrm{restr}_x} \Omega^1_X \longrightarrow 0. \tag{2.3}$$

Since $\mathcal{I}_X/\mathcal{I}_X^2$ is the conormal bundle of X, this resembles the conormal sequence in differential geometry. Dualizing, we get the *normal sequence*, which is more likely to look familiar:

$$0 \longrightarrow \Theta_X \longrightarrow \Theta_{\mathbb{P}^4}|_X \longrightarrow N_{X\subset\mathbb{P}^4} \longrightarrow 0, \tag{2.4}$$

and since X has degree 5, $N_{X\subset\mathbb{P}^4} \cong \mathcal{O}_{\mathbb{P}^4}(5)|_X$.

Finally, we have two *restriction sequences*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-5) \longrightarrow \mathcal{O}_{\mathbb{P}^4} \longrightarrow \mathcal{O}_X \longrightarrow 0 \tag{2.5a}$$

$$0 \longrightarrow \Theta_{\mathbb{P}^4}(-5) \longrightarrow \Theta_{\mathbb{P}^4} \longrightarrow \Theta_{\mathbb{P}^4}|_X \longrightarrow 0. \tag{2.5b}$$

Now take the long exact sequence in cohomology associated to (2.4):

$$H^0(\Theta_X) \longrightarrow H^0(\Theta|_{\mathbb{P}^4}|_X) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^4}(5)|_X) \longrightarrow H^1(\Theta_X) \longrightarrow H^1(\Theta_{\mathbb{P}^4}|_X) \longrightarrow \cdots \tag{2.6}$$

We will show that

(1) $H^0(\Theta_X) = 0$,
(2) $H^0(\Theta_{\mathbb{P}^4}|_X) \cong \mathbb{C}^{24}$,
(3) $H^0(\mathcal{O}_{\mathbb{P}^4}(5)|_X) \cong \mathbb{C}^{125}$, and
(4) $H^1(\Theta_{\mathbb{P}^4}|_X) = 0$,

which collectively imply that $H^1(\Theta_X) \cong \mathbb{C}^{101}$ (since $101 = 125 - 24$).

First, (4). Take the long exact sequence in cohomology associated to (2.1):

$$\underbrace{H^1(\mathcal{O}_{\mathbb{P}^4}(1))^{\oplus 5}}_{=0} \longrightarrow H^1(\Theta_{\mathbb{P}^4}) \longrightarrow \underbrace{H^2(\mathcal{O}_{\mathbb{P}^4})}_{=0}, \tag{2.7}$$

so $H^1(\Theta_{\mathbb{P}^4}) = 0$. TODO: then restrict to X.

[2] We'll say more about this later, but an orbifold is locally modeled on a manifold quotient by a nice group action, and you can think of it as that, as a singular topological space.

Now take (2.1), tensor with $\mathcal{O}(-5)$, and take the long exact sequence in cohomolgy.[3]:

$$\underbrace{H^i(\mathcal{O}_{\mathbb{P}^4}(-4))}_{=0} \longrightarrow H^i(\Theta_{\mathbb{P}^4}(-5)) \longrightarrow \underbrace{H^i(\mathcal{O}_{\mathbb{P}^4}(-5))}_{=0}, \tag{2.8}$$

and therefore $H^i(\Theta_{\mathbb{P}^4}(-5)) = 0$.

TODO: several more arguments like this, which I couldn't follow in realtime and couldn't reconstruct from the board. Sorry about that. For example, we used the first restriction sequence to use info on $H^1(\Theta_{\mathbb{P}^4})$ and $H^2(\Theta_{\mathbb{P}^4(-5)}$ to conclude $H^1(\Theta_{\mathbb{P}^4}|_X)$ vanishes...

OK, now let's discuss the Hodge diamond of the quintic threefold. On a compact Kähler manifold of complex dimension n, we have some nice facts about the Dolbeault cohomology $H^{i,j}_{\overline{\partial}} := H^j(\mathcal{A}^{i,\bullet}, \overline{\partial})$, where $\mathcal{A}^{\bullet,\bullet}$ is the sheaf of holomorphic differential forms, bigraded via ∂ and $\overline{\partial}$ as usual. Let $\Omega^i_X := (\Omega_X)^{\otimes i}$ and $K_X := \Omega^n_X$. Then,

(1) There are canonical isomorphisms $H^{i,j}_{\overline{\partial}} \cong H^j(X, \Omega^i_X) = \overline{H^{j,i}_{\overline{\partial}}}$ (i.e. the conjugate complex vector space). Hence $h^{ij} = h^{ji}$.
(2) Serre duality tells us $H^{n-j}(X, \Omega^{n-i}_X) \cong H^j(X, K_X \otimes (\Omega^{n-i}_X)^*)^* = H^j(X, \Omega^i_X)^*$, so we have a canonical isomorphism $H^{n-i,n-j}_{\overline{\partial}} \cong H^{i,j}_{\overline{\partial}}$ and $h^{i,j} = h^{n-i,n-j}$.
(3) Let $b^k := \dim_{\mathbb{C}} H^k(X;\mathbb{C}) = H^k_{\mathrm{dR}}(X) \otimes \mathbb{C}$. This group is the direct sum of $H^{i,j}_{\overline{\partial}}$ over $i+j=k$.

These facts are proven using some difficult analysis.

Now if in addition X is Calabi-Yau, $b_1 = 0$, and therefore $h^{1,0} = h^{0,1} = 0$. Moreover, $H^{n,0} \cong H^0(X, K_X) \cong H^0(X, \mathcal{O}_X) \cong \mathbb{C}$, so $h^{n,0} = h^{0,n} = 1$. We further assume X is *irreducible*: neither X nor its universal cover are a product of Calabi-Yau manifolds in a nontrivial way.[4] Beauville showed this is equivalent to $H^{k,0} = 0$, $k = 1, \ldots, n-1$.

It is traditional to arrange the Hodge numbers $h^{i,j}$ in a diamond, known as (surprise!) the *Hodge diamond*. For a 3-fold, we have

$$\begin{array}{ccccccc} & & & h^{3,3} & & & \\ & & h^{2,3} & & h^{3,2} & & \\ & h^{1,3} & & h^{2,2} & & h^{3,1} & \\ h^{0,3} & & h^{1,2} & & h^{2,1} & & h^{3,0} \\ & h^{0,2} & & h^{1,1} & & h^{2,0} & \\ & & h^{0,1} & & h^{1,0} & & \\ & & & h^{0,0}. & & & \end{array} \tag{2.9}$$

[3] Why is $\mathcal{O}(-5)$ flat?

[4] If you like Riemannian geometry and metrics of special holonomy, irreducible Calabi-Yau corresponds to having holonomy landing in SU_n.

But the Calabi-Yau condition tells us this collapses to very few parameters:

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{2,2} & & 0 & \\
1 & & h^{1,2} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1, & & &
\end{array}
\tag{2.10}
$$

and the two red values are equal, as are the two blue values. The red values are both 101 for the quintic threefold.

To get at the last piece of information in the Hodge diamond, we'll relate $h^{1,1}$ to the Picard group.

Definition 2.11. The *Néron-Severi group* $NS(X)$ is the preimage of $H^{1,1}(X) \subset H^2(X;\mathbb{C})$ under the map $H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{C})$.

In complex analytic geometry, we have the *exponential exact sequence* of sheaves of abelian groups

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{\cdot 2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \longrightarrow 1. \tag{2.12}$$

The fact that we began with 0 and ended with 1 isn't significant; it only represents that the first two sheaves of abelian groups are written additively, and the last is written multiplicatively.

Anyways, (2.12) induces a long exact sequence in cohomology.

$$H^1(X,\mathbb{Z}) \longrightarrow H^1(X,\mathcal{O}_X) \longrightarrow H^1(X,\mathcal{O}_X^\times) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathcal{O}_X). \tag{2.13}$$

We have identifications $H^1(X,\mathcal{O}_X) = H^{0,1}$, and $H^1(X,\mathcal{O}_X^\times)$ with the *Picard group* $\mathrm{Pic}(X)$, the isomorphism classes of holomorphic line bundles under tensor product.

Theorem 2.14 (Lefschetz theorem on $(1,1)$ classes)**.** *The image of $c_1\colon \mathrm{Pic}(X) \to H^2(X,\mathbb{Z})$ is exactly the Néron-Severi group.*

Thus, for a projective Calabi-Yau threefold, $h^{1,1}(X) = \operatorname{rank} NS(X)$ and $\mathrm{Pic}(X) \cong NS(X)$. This is telling you that a projective Calabi-Yau threefold has no non-projective deformations! This is not true in general, e.g. for K3 surfaces.

Remark 2.15. Serre's GAGA theorem explans why we can so cavalierly pass between the algebro-geometric and complex-analytic world: as long as we restrict to projective varieties and projective manifolds, there are appropriate equivalences of categories between the two perspectives. ◀

To actually compute $h^{1,1}$, though, we need another general theorem from Kähler geometry.

Theorem 2.16 (Lefschetz hyperplane theorem)**.** *Let $X \subset \mathbb{P}^{n+1}$ be a hypersurface, so $\dim X = n$. The map $H^k(\mathbb{P}^{n+1};\mathbb{Z}) \to H^k(X;\mathbb{Z})$ is an isomorphism for $k < n-1$ and is surjective for $k = n-1$.*

In the case of a Calabi-Yau threefold, $H^1(X;\mathbb{Z}) = H^1(\mathbb{P}^4;\mathbb{Z}) = 0$, and doing this for H^2 shows $NS(X) \cong \mathbb{Z}$, and $\mathrm{Pic}(X) = \mathbb{Z}\cdot c_1(\mathcal{O}_X(1))$. So $h^{1,1} = h^{2,2} = 1$.

For the mirror quintic, these should be swapped: we should get $h^{1,1} = h^{2,2} = 1$ and $h^{1,2} = h^{2,1} = 101$. This is a bit weird: it has a huge Picard group and a very small moduli space (it will be an orbifold $\mathbb{P}^1$).

Lecture 3.

The mirror quintic: 9/5/19

As part of mirror symmetry, we want to find a Calabi-Yau threefold Y whose Hodge diamond is the mirror of that of the quintic threefold. In particular, it should have $h^{1,1} = h^{2,2} = 1$ (small space of deformations) and $h^{1,2} = h^{2,1} = 101$ (very large Picard group).

Remark 3.1. The construction we discuss today is physically motivated by *minimal conformal field theories* and their tensor products, and by a procedure to orbifold them. ◀

Specifically, begin with the *Fermat quintic* $X \coloneqq \{x_0^5 + \cdots + x_4^5 = 0\} \subset \mathbb{P}^4$. Now $(\mathbb{Z}/5)^5$ acts on $\mathbb{P}^4$ through its action on $\mathbb{C}^5$, and the diagonal $\mathbb{Z}/5$ subgroup fixes X, so we have a $(\mathbb{Z}/5)^5/(\mathbb{Z}/5) \cong \mathbb{Z}/4$-action on X. Let $\overline{Y} \coloneqq X/(\mathbb{Z}/5)^4$.

However, we have a problem: X is smooth, by the Jacobian criterion, but $\overline{Y}$ is not: if, for example, $x_i = x_j = 0$, then the stabilizer of $\mathbf{x}$ contains a copy (TODO: possibly more?) of $\mathbb{Z}/5$. There's a curve $\widetilde{C}_{ij} = Z(x_i, x_j) \subset X$ where the local action is $\zeta \cdot (z_1, z_2, z_3) = (\zeta z_1, \overline{\zeta} z_2, z_3)$, so the singularity looks like that of $uv = w^4$ in $\mathbb{C}^3$, which is an A_4 singularity.[5]

We can do worse, however: when $x_i = x_j = x_k = 0$ for disjoint i, j, k, we get $(\mathbb{Z}/5)^2$ in the stabilizer, and this locally looks like $\mathbb{C}^3/(\mathbb{Z}/5)^2$, with the action

$$(\zeta, \xi) \cdot (z_1, z_2, z_3) = (\zeta\xi z_1, \zeta^{-1} z_2, \xi^{-1} z_3). \tag{3.2}$$

We want to resolve these singularities by blowing them up. Since we're not just blowing up points, this takes a little care. Note that $C_{01} = Z(x_0, x_1, x_2^5 + x_3^5 + x_4^5)/(\mathbb{Z}/5)^3 \simeq Z(u + v + w) \subset \mathbb{P}^2_{u,v,w}$; here $u = x_2^5$, $v = x_3^5$, and $w = x_4^5$, and this is a $\mathbb{P}^1$ inside $\overline{Y}$.

Proposition 3.3. *There exists a projective resolution $Y \to \overline{Y}$.*

One can do this by hand, or in a more general way using methods from toric geometry.

We want to count the number of independent exceptional divisors in Y. Resolving an A_4 gives four $\mathbb{P}^1$s over each C_{ij}, and similarly we'll get six over each P_{ijk}, and each $\mathbb{P}^1$ produces 10, so we have 100 linearly independent elements of $H^2(Y)$. The hyperplane class is also independent, which is how (albeit with some more work) one obtains rank 101. This is shown by hand.

Proposition 3.4. *Y is Calabi-Yau, $h^{1,1}(Y) = 101$, and $h^{2,1}(Y) = 1$.*

There's a direct proof due to S.S. Roan, and a more general approach with toric methods due to Batyrev.

Now Y fits into a one-dimensional family, and this is small enough that we might hope to write it down. In fact, this works — it's an example of a general construction called the *Dwork family*. In this case we deform with a parameter ψ and consider

$$f_\psi \coloneqq x_0^5 + \cdots + x_4^5 - 5\psi x_0 x_1 x_2 x_3 x_4 = 0. \tag{3.5}$$

This again is fixed by a diagonal $\mathbb{Z}/5$-action, giving us a $(\mathbb{Z}/5)^3$-action. Let $X_\psi = Z(f_\psi)$, which is a family over $\mathbb{P}^1$ in ψ. If we let $z \coloneqq (5\psi)^{-5}$, then $\mathbb{P}^1_\psi \to P^1_z$ is a quotient by a $\mathbb{Z}/5$-action, and Y fits into a family $\mathcal{Y} \to \mathbb{P}^1_z$ of Calabi-Yau threefolds which is smooth away from 0 and ∞. This family has some special fibers.

- At $z = 0$, $f_\psi = x_0 \cdots x_4 = 0$, so the zero locus is a union of five copies of $\mathbb{P}^3$ in $\mathbb{P}^4$, specifically the coordinate hyperplanes. This is a bad-looking degeneration! But it will be important in the computations, in that we will often consider z near zero.
- At $z = 5^{-5}$, i.e. $\psi = 1$, there's a three-dimensional A_1 singularity. To see this, let's first pass to the cover X_1, which has 125 three-dimensional A_1 singularities, which locally look like $\{x^2 + y^2 + z^2 + w^2 = 0\}$. These all live in the same $(\mathbb{Z}/5)^3$-orbit, hence all get identified in the quotient. This is called a *conifold*, and isn't a great singularity to have — it behaves like letting your complex structure go to infinity.
- The *Fermat point* $z = \infty$, which is what we started with, the Fermat quintic. This has an additional $\mathbb{Z}/5$-symmetry.

So the moduli space of mirror quintics, namely $\mathbb{P}^1_z$, is really an orbifold $\mathbb{P}^1$, with these two singularities. The singularity at $z = 0$ is called the *large complex structure (LCS)* limit point, $z = 5^{-5}$ is called the *conifold point*, and $z = \infty$ is called the *orbifold point*. All of these points have some meaning in mirror symmetry.

Physicists are interested in computing Yukawa couplings, certain numbers extracted from an effective field theory. We can compute them in two ways, either using X or using Y, and they should agree. These take the form

$$\langle h, h, h \rangle_A = \sum_{d \in \mathbb{N}} N_d d^3 q^d, \tag{3.6}$$

[5] More generally, the singularity of type A_{n-1} can be found in $\{uv = w^n\}$.

where h is the hyperplane class in $H^2(X)$ (or more precisely, its Poincaré dual). When (3.6) was first written down, people did not know what these N_d were completely mathematically, but now we know they're Gromov-Witten invariants, a count of genus-0, degree-d curves C in your Calabi-Yau threefold. The d^3 comes form fixing the points of intersection with three copies of h.

There's a subtlety in N_d: it's not a naïve integer-valued count, because there could be maps which aren't embeddings, so this *a priori* gives rational numbers. You end up with rational power series in q, expressed in terms of *primitive counts*, which aren't exactly Gromov-Witten invariants, and haven't yet been made mathematical in general. But the Gromov-Witten invariants exist, and the numbers we get out at the end agree, which was one of the first manifestations of mirror symmetry historically. Physics suggested that these are symplectic invariants (in this setting you use pseudoholomorphic curves, following Gromov, Floer, and Fukaya), and in particular should be invariant under deformations of the complex structure.

But before we knew how to define and compute Gromov-Witten invariants, the computations that people did used the B-model on the mirror quintic, which sees the complex structure but not the symplectic structure. In this setting the Yukawa coupling on the family Y_z (with $z = (5\psi)^{-5}$) is

$$\langle \partial_z, \partial_z, \partial_z\rangle_B \coloneqq \int_{Y_Z} \Omega^\nu(z) \wedge \partial_z^3 \Omega^\nu(z), \tag{3.7}$$

where Ω^ν is a (suitably normalized) holomorphic volume form: we fix $\int_{\beta_0} \Omega^\nu$ to be some constant, given $\beta_0 \in H_3(Y;\mathbb{Z})$.

Now, why is ∂_z a mirror to h? The idea is that h is equivalent data to a vector field on the moduli space of symplectic structures on X (well, really $\exp(2\pi i h)$ is that vector field). The mirror is a vector field ∂_w, a vector field on the moduli space of complex structures on Y, and it turns out

$$w = \int_{\beta_1} \Omega^\nu(z) \tag{3.8}$$

for a family of 3-cycles $\beta_1 \in H_2(Y;\mathbb{Z})$. The mirror symmetry statement is that

$$\langle h,h,h\rangle_A = \langle \partial_z, \partial_z, \partial_z\rangle_B \tag{3.9}$$

where $q = \exp(2\pi i w(z))$.

Now we want to compute these periods. We'll omit some details; there's a good account in Gross' lecture notes from the Nordfjordeid summer school.

$H_3(Y_\psi, \mathbb{Z}) \cong \mathbb{Z}^4$. Near $\psi = \infty$ (the large complex structure limit), we have a vanishing cycle. The idea of what's going on is to consider a singularity of the form $zw = t$ for t small. When $t \neq 0$, this is a one-sheeted hyperboloid, so we have a cycle diffeomorphic to S^1. When $t = 0$, there are two paraboloids, so the cycle has gone away, in a sense. We're in a higher-dimensional setting, but the basic idea is the same. We can write down an explicit choice for β_0, which will be diffeomorphic to a T^3, and next time we'll continue the computation.

Lecture 4.

Period integrals and the Picard-Fuchs equation: 9/10/19

Today we continue our discussion of the mirror quintic $\overline{Y}_\psi$, which fits into a one-dimensional family: ψ is a coordinate on an orbifold $\mathbb{P}^1$. Last time we discussed the vanishing cycle β_0, which is diffeomorphic to a T^3, and today we'll begin discussing the holomorphic 3-form Ω.

We can relatively easily write down this form by working inside $\mathbb{P}^4$, by taking the residue of a meromorphic (in fact rational) 4-form on $\mathbb{P}^4$ with simple poles along $X_\psi = Z(f_\psi)$. There are not so many choices to do this, and we might be able to guess the right answer.

$$\Omega(\psi) \coloneqq 5\psi \cdot \mathrm{res}_{X_\psi} \frac{\widetilde{\Omega}}{f_\psi} \in \Gamma(X_\psi, \Omega^3_{X_\psi}), \tag{4.1a}$$

where

$$\widetilde{\Omega} \coloneqq \sum_{i=0}^4 x_i \, \mathrm{d}x_0 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_4. \tag{4.1b}$$

(4.1a) doesn't quite make literal sense, but in homogeneous coordinates it's perfectly fine. Choose local holomorphic coordinates on X_ψ with $x_4 = 1$ and $\partial_{x_3} f \neq 0$; then

$$\Omega(\psi) = 5\psi \left.\frac{\mathrm{d}x_0 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2}{\partial_{x_3} f_\psi}\right|_{X_\psi}. \tag{4.2}$$

Now we'd like to normalize. If $\phi_0 := \int_{\beta_0} \Omega(\psi)$, then $\widetilde{\Omega} := \phi_0^{-1}\Omega(\psi)$ is normalized to have total integral 1. We can explicitly determine ϕ_0 with the (higher-dimensional) residue theorem:

$$\int_{\beta_0} \Omega(\psi) = \int_{T^4} 5\psi \frac{\mathrm{d}x_0 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3}{f_\psi} \tag{4.3}$$

$$= \int_{T^4} \frac{\mathrm{d}x_0 \wedge \cdots \wedge \mathrm{d}x_3}{x_0x_1x_2x_3}\left(\frac{1 + x_0^5 + \cdots + x_3^5}{5\psi x_0x_1x_2x_3} - 1\right)^{-1}. \tag{4.4}$$

We can expand the second term as a geometric series:

$$= -\sum_{n\geq 0}\int_{T^4} \frac{\mathrm{d}x_0 \wedge \cdots \wedge \mathrm{d}x_3}{x_0x_1x_2x_3}\frac{(1 + x_0^5 + \cdots + x_3^5)^n}{(5\psi)^n(x_0x_1x_2x_3)^n}. \tag{4.5}$$

All summands in the numerator are fifth powers, so the summands in the denominator must be as well in order to contribute:

$$= -\sum_{n\geq 0}\int_{T^4} \frac{\mathrm{d}x_0 \wedge \cdots \wedge \mathrm{d}x_3}{x_0x_1x_2x_3}\frac{(1 + x_0^5 + \cdots + x_3^5)^{5n}}{(5\psi)^{5n}(x_0x_1x_2x_3)^{5n}} \tag{4.6}$$

$$= -(2\pi i)^4 \sum_{n\geq 0} \frac{(5n)!}{(n!)^5(5\psi)^{5n}}, \tag{4.7}$$

telling us $\phi_0(z)$. This is enumerating the number of possibilities for these contributions. To completely pin it down, we'd need two more, similar integrals.

Griffiths' theory of period integrals, developed in the 1980s, allows one to compute more period integrals via the Picard-Fuchs equation. The argument goes as follows. Since cohomology is topological in nature, $H^3(Y_\psi;\mathbb{C})$ is locally constant, and is four-dimensional, so we can realize it as a trivial holomorphic vector bundle (at least over some subspace of the moduli space for the mirror quintic). The holomorphic volume form $\Omega(\psi)$ gives a section of this bundle — it trivializes $H^{3,0}$, but the Hodge structure on H^3 varies in ψ, which is part of the general story of *variation of Hodge structure*. The flat connection on H^3 is called the *Gauß-Manin connection* and denoted ∇^{GM}.

In particular, the derivatives $\partial_z^i\Omega(z)$ need not be holomorphic, since the trivialization of H^3 doesn't trivialize $H^{3,0}$. But the five elements of $\{\Omega(z), \partial\Omega(z), \partial^2\Omega(z), \partial^3\Omega(z), \partial^4\Omega(z)\}$ are sections of a four-dimensional vector bundle, hence must satisfy a relation, called the *Picard-Fuchs equation*,[6] a 4^{th}-order ODE with holomorphic coefficients.

To derive the equation, we'll produce more 3-forms from forms with higher-order poles; this part of the story is called *Griffiths' reduction of pole order*. As usual, let $X = Z(f_\psi) \subset \mathbb{P}^4$; then, associated to the pair of spaces $(\mathbb{P}^4, \mathbb{P}^4 \setminus X)$, we have the long exact sequence in cohomology

$$H^4(\mathbb{P}^4;\mathbb{C}) \longrightarrow H^4(\mathbb{P}^4 \setminus X;\mathbb{C}) \longrightarrow \underbrace{H^5(\mathbb{P}^4, \mathbb{P}^4 \setminus X;\mathbb{C})}_{(*)} \longrightarrow \underbrace{H^5(\mathbb{P}^4;\mathbb{C})}_{=0}. \tag{4.8}$$

If U is a tubular neighborhood of X, then $\dim U = 8$ and excision implies

$$(*) \cong H^5(U, U \setminus X;\mathbb{C}) \cong H^5(U, \partial U;\mathbb{C}). \tag{4.9}$$

Lefschetz duality, a version of Poincaré duality with boundary, establishes an isomorphism $H^q(M,\partial M) \cong H_{n-q}(M)$ for any compact oriented manifold M. Using this, and the fact that U retracts onto X,

$$(4.9) \cong H_3(U;\mathbb{C}) \cong H_3(X;\mathbb{C}) \cong H^3(X;\mathbb{C}), \tag{4.10}$$

where the last map is Poincaré duality.

[6]The version of the Picard-Fuchs equation that you might find on, say, Wikipedia is in the setting of elliptic curves, which is the simplest setting for variations of Hodge structures. It fits into a more general story, though today we're only going to look at 3-folds.

Returning to (4.8), we've exhibited a surjective map $H^4(\mathbb{P}^4 \setminus X; \mathbb{C}) \twoheadrightarrow H^3(X; \mathbb{C})$; moreover, the use of differential forms to represent cohomology classes (TODO: I think that's what happened) tells us $H^0(\mathbb{P}^4 \setminus X, \Omega^4_{\mathbb{P}^4 \setminus X}) \twoheadrightarrow H^4(\mathbb{P}^4 \setminus X; \mathbb{C})$, so we can represent any degree-3 cohomology class on X by differential 4-forms on $\mathbb{P}^4 \setminus X$.

Specifically, consider something of the form $g\widetilde{\Omega}/f^\ell \in H^0(\mathbb{P}^4, \Omega^4_{\mathbb{P}^4 \setminus X})$, where $\deg g = 5\ell - 5$ (and $f = f\psi$). The exact forms are those of the form

(4.11)

$$\mathrm{d}\left(\frac{\sum_{i<j}(-1)^{i+j}(x_i g_j - x_j g_i')\,\mathrm{d}x_0 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \widehat{\mathrm{d}x_j} \wedge \cdots \wedge \mathrm{d}x_4 f^\ell}{=}\left(\ell \sum g_j \partial_{x_j} f - f \sum \partial_[x_j\right) g_j\right)\frac{\widetilde{\Omega}}{f^{\ell+1}}.$$

The upshot is that the numerator is in the ideal generated by $\partial_{x_i} f$, and we can therefore reduce ℓ. Taking four derivatives seems onerous but is perfectly tractable with the help of a physicist friend or a computer, and we obtain a relation, expressing g as a linear combination of $\partial_z^i \Omega(z)$, $i = 0, \dots, 4$. The answer is actually pretty simple.

Proposition 4.12. *Any period $\phi = \int_\alpha \Omega(\psi)$ fulfills the* Picard-Fuchs equation*, the ODE*

$$\theta^4 - 5z(5\theta + 1)(5\theta + 2)(5\theta + 3)(5\theta + 4)\phi(z) = 0, \tag{PF}$$

where $\theta = z\partial_z$.

It's not too hard to verify that ϕ_0 from (4.7) fulfills the equation — you might imagine there's a simpler one, and to prove that's not true requires more work.

Remark 4.13. Generalizing this to other hypersurfaces, or in general complete intersections in toric varieties, is less of a mess than the general story. Sometimes one has to delve into the more general theory of hypergeometric functions. ◀

(PF) is an ODE with a regular single pole

$$\Theta \cdot \phi(z) = A(z) \cdot \phi(z), \tag{RS}$$

where $\phi(z) \in \mathbb{C}^5$. This fits into a beautiful theorem that's sadly absent from the modern American ODE curriculum.

Theorem 4.14. (RS) *has a fundamental system of solutions of the form $\Phi(z) = S(z) \cdot z^R$, where $S(z) \in M_s(\mathcal{O})$, $R \in M_s(\mathbb{C})$, and*

$$z^R = I + (\log z)R + (\log z)^2 R^2 + \cdots. \tag{4.15}$$

If the eigenvalues do not differ by integers, we can take $R = A(0)$.

Throw some linear algebra at (PF) and you can calculate that $A(0)$ has Jordan normal form with a single Jordan block, and $S = (\psi_0, \psi_1, \psi_2, \psi_3)$, where each ψ_i is a germ of a holomorphic function. This yields a fundamental system of solutions $\phi_0(z) = \psi_0$ — up to scaling, there's a unique single-valued (i.e. no $\log z$ terms) solution. Including logarithmic terms, we have additional solutions:

$$\phi_1(z) = \psi_0(z) \log z + \psi_1(z) \tag{4.16a}$$

$$\phi_2(z) = \psi_0(z)(\log z)^2 + \psi_1(z) \log z + \psi_2(z) \tag{4.16b}$$

$$\phi_3(z) = \psi_0(z)(\log z)^4 + \cdots + \psi_4(z). \tag{4.16c}$$

These solutions are multivalued, which means there's monodromy. This seems like a mystery, and one concludes the cycles must have monodromy. Though β_0 doesn't, everthing else has monodromy. Specifically, the monodromy of $z^{A(0)}$ refelcts the monodromy of $H^3(Y_z; \mathbb{C})$ around $z = 0$ (equivalently, $\psi = \infty$). More specifically, one can show that there's a symplectic basis $\beta_0, \beta_1, \alpha_1, \alpha_0$ of $H_3(Y_z; \mathbb{Q})$ such that the monodromy sends

$$\alpha_0 \longmapsto \alpha_1 \longmapsto \beta_1 \longmapsto \beta_0 \longmapsto 0. \tag{4.17}$$

Therefore $\phi_0 = \int_{\beta_0} \Omega(z)$, $\phi_1 = \int_{\beta_1} \Omega(z)$, $\phi_2 = \int_{\alpha_1} \Omega(z)$, and $\phi_3 = \int_{\alpha_0} \Omega(z)$. We're not far from the final computation of the Yukawa couplings!

Now let's write down the canonical coordinates. Let $q = e^{2\pi i w}$, where

$$w = \frac{\int_{\beta_1} \Omega(z)}{\int_{\beta_0} \Omega(z)} = \int_{\beta_1(z)} \widetilde{\Omega}(z). \tag{4.18}$$

Then $\phi_1(z) = \phi_0(z)\log z + \psi_1(z)$ is easy to obtain as a series solution to (PF); specifically, up to some constant,

$$\psi_1(z) = 5\sum_{n\geq 1} \frac{(5n)!}{(n!)^5}\left(\sum_{j=n+1}^{5n} \frac{1}{j}\right)^n. \tag{4.19}$$

Lecture 5.

Yukawa coupling: 9/12/19

"Typically physicists are right when it comes to numbers."

So far we've run through the computations of period integrals, and we're almost done with everything; the last step is to compute the Yukawa coupling. These computations are sometimes done by a different research community. They have to do with the (projective) special Kähler structure on the moduli space, special coordinates related to that, etc.

So we want to compute

$$\langle \partial_z, \partial_z, \partial_z\rangle_B = \int_{Y_z} \widetilde{\Omega}(z) \wedge \partial_z^3 \widetilde{\Omega}(z), \tag{5.1}$$

where $\widetilde{\Omega}(z)$ is the normalized holomorphic volume form we discussed last time.

First, because the normalization is a bit messy, let's simplify the calculation. Introduce the auxiliary quantities

$$W_k := \int_{Y_z} \Omega(z) \wedge \partial_z^k \Omega(z), \tag{5.2}$$

where $k = 0, \dots, 4$, so (5.1) is W_3. Using the Picard-Fuchs equation

$$\left(\frac{\mathrm{d}^4}{\mathrm{d}z^4} + \sum_{k=0}^{3} c_k \frac{\mathrm{d}^k}{\mathrm{d}z^k}\right)\Omega(z) = 0, \tag{5.3}$$

we have a relation between the W_k with different k:

$$W_4 + \sum_{k=0}^{3} c_k W_k = 0. \tag{5.4}$$

The next ingredient we need to make progress is *Griffiths transversality*. Let U be an open subspace of the moduli space for Y and $\mathcal{F} := H^3(Y_z;\mathbb{C}) \otimes_{\mathbb{Z}} \mathcal{O}_U$. Then $\mathcal{F}$ has the *Hodge filtration* $\mathcal{F} = \mathcal{F}^0 \supseteq \mathcal{F}^1 \supseteq \mathcal{F}^2 \supseteq \mathcal{F}^3$, where

$$\mathcal{F}^k := \bigoplus_{q\geq k} R^q \pi_* \Omega^{3-q}. \tag{5.5}$$

Recall from last time we had something called the Gauß-Manin connection ∇^{GM}; the image of $\mathcal{F}^k$ under this connection is contained in $\mathcal{F}^{k-1} \otimes \Omega_U^1$, which one can check by looking at the definition of ∇^{GM} and some Hodge theory. This is part of a general story about Kähler manifolds.

Recall that under the wedge product, $H^{p,q}$ and $H^{p',q'}$ are orthogonal unless $p' = 3-p$ and $q' = 3-q$. This immediately forces $W_0 = W_1 = W_2 = 0$. Hence $W_2''(z) = 0$ as well and so forth, and in particular $2W_3' - W_4 = 0$. Plugging this into the Picard-Fuchs equation, we conclude

$$W_3 + \frac{1}{2}c_3 W_3 = 0. \tag{5.6}$$

That this only depends on c_3 is nice, but c_3 is not necessarily easy to compute. In this form, the answer is

$$c_3(z) = \frac{6}{z} - \frac{25^5}{1-5^5 z}, \tag{5.7}$$

which is simple enough that we can solve for W_3 in closed form:

$$W_3 = \frac{C}{(2\pi i)^3 z^3 (5^2 z - 1)}, \tag{5.8}$$

where C is a constant of integration we have no control over; we can't resolve this ambiguity.

The final step is to rewrite this in terms of $q = e^{2\pi i w}$, where $w = \phi_1(z)/\phi_0(z)$ is the canonical coordinate, and then expand. The answer is

$$\langle \partial_w, \partial_w, \partial_w \rangle_B = -c_1 - 575\frac{c_1}{c_2}q - \frac{1950750}{2}\frac{c_1}{c_2^2}q^2 - \frac{10277490000}{6}\frac{c_1}{c_2^3}q^3 + \dots \tag{5.9}$$

We can rewrite this as

$$5 + \sum_{d\geq 1} d^3 n_d \frac{q^d}{1-q^d} = 5 + n_1 q + (8n_2 + n_1)q^2 + (27n_3 + n_1)q^3 + \cdots \tag{5.10}$$

Here n_d is a Gromov-Witten invariant, counting curves of degree d in the quintic. This makes the following predictions about curve counts.

- $n_1 = 2875$ is classically known.
- $n_2 = 609250$, computed by S. Katz just a few years before the initial mirror symmetry paper.
- $n_3 = 317206375$, which disagreed with a preexisting computation – but the authors of that computation took a more careful look at their computations, and it turned out that this agrees with the correct answer, as determined by Ellingswood and Strømme in 1990.

At this point, it was clear something was going on. The conjecture for all d was proven by Givental in 1996 and Lian, Lin, and Yau in 1997.

Remark 5.11. Two takeaways: first, computations are hard, of course, but another takeaway is that the prediction depends on the large complex structure limit (so the Kähler cone), though the Yukawa couplings exist in the entire moduli space, and in particular depends on the monodromy in $H^3(Y)$. These two facts seem to stand in opposition to each other. ◀

One also asks, in what generality does mirror symmetry apply? The orbifolding construction of the mirror is special to the quintic threefold; Batyrev and Borisov constructed mirrors for complete intersections in toric varieties.

We will next discuss Gromov-Witten theory, but from a somewhat broader perspective beginning with moduli spaces and stacks. This will streamline the discussions of log Gromov-Witten theory and Donaldson-Thomas theory later in the course.

Gromov-Witten theory lives within the general framework of curve counting, e.g. how many genus-zero holomorphic curves in the quintic? This is a very classical question, and also attracted the interest of the Italian school of algebraic geometry.

The rigidity of algebraic geometry sometimes works against you: for example, if you want to consider the conics through three points, there's an issue for non-generic point arrangements. Typical computations assumed the existence of enough deformations to work around this; classically, this involved "general position arguments" which were often simple — but proving that general-position objects exist at all can be quite difficult, especially in higher-dimensional enumerative questions. These are essentially transversality arguments.

From a more modern point of view, one works with topology. One considers a moduli space of objects, and each condition imposed (e.g. curve must go through a given point) is a cohomology class, and then one computes in the cohomology ring. For example, intersection theory in the Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$ governs the classical theory called *Schubert calculus.*

You can get a fair ways with this, e.g. you can compute the number of lines on a quintic using Schubert calculus, and some higher-genus counts proceed along essentially similar methods. But there is a conjecture[7] that for every curve, there's some quintic without a discrete moduli space of such curves. This is still open, and if it's true these methods will fail in general.

[7]This conjecture is often attributed to Klemens, except that Klemens claims to not have conjectured it!

Generally, moduli spaces of curves on a quintic don't have the expected dimension. For example, on the Dwork family $\{f_\psi = 0\}$, the moduli space of lines contains 375 isolated lines (each isomorphic to $\mathbb{P}^1$), e.g.

$$(u, v, -\zeta^k u, -\zeta^\ell v, 0), \tag{5.12}$$

where $[u : v] \in \mathbb{P}^1$, ζ is a primitive fifth root of unity, and $0 \le k, \ell \le 4$. But there are also two irreducible families. How do we count in the absence of general deformations?

Moreover, in higher degree, there are multiple covers, e.g. the degree-d cover $\mathbb{P}^1 \to \mathbb{P}^1$. How do we count these?

The answer is to use virtual counts, which produces invariants which are constant in families of targets.

Moduli spaces. The basic idea of a moduli space is to consider the set of isomorphism classes of some object under interest, and give it a topology, and, ideally, the structure of an algebraic variety, ideally in some systematic way. We want the set of closed points of this space $\mathcal{M}$ to be the set of isomorphism classes of the geometric objects in question (e.g. varieties).

To upgrade this into a variety or scheme, you need to know what the structure sheaf is, or equivalently what all the holomorphic functions are. We can test this by pulling back along maps $T \to \mathcal{M}$. This tells us an object for each point of T, and we want this map to be holomorphic if these objects vary algebraically in T.

Example 5.13. The *Hilbert scheme*[8] is the moduli space of closed subschemes $Z \subset \mathbb{P}^N$, where N is fixed. Often one also fixes the Hilbert polynomial of Z. This forms a moduli space $\mathrm{Hilb}(\mathbb{P}^N)$, with a canonical subscheme $\mathcal{Z} \subset \mathrm{Hilb}(\mathbb{P}^N) \times \mathbb{P}^N$, and together these satisfy a universal property. Given any flat proper map $Z \to T$, where $Z \subset T \times \mathbb{P}^N$ is closed, there is a unique map $\varphi\colon T \to \mathrm{Hilb}(\mathbb{P}^N)$ such that the pullback of $\mathcal{Z} \subset \mathrm{Hilb}(\mathbb{P}^N) \times \mathbb{P}^N$ along φ is identified with the inclusion $Z \subset T \times \mathbb{P}^N$. ◀

This is just about the nicest moduli problem out there; things are usually more difficult (or, depending on your perspective, more interesting).

To set this in more general terms, let $\mathcal{S}ch$ denote the category of schemes.[9] There is a functor $F\colon \mathcal{S}ch \to \mathcal{S}et$ sending a scheme T to the set of isomorphism classes of flat proper maps $Z \to T$ where $Z \subset T \times \mathbb{P}^N$ is closed. That is, this is the set of families of closed subschemes of Z parametrized by T. That $\mathrm{Hilb}(\mathbb{P}^N)$ is the moduli space for such data is encoded in the theorem that it *represents* the functor F, i.e. F is naturally isomorphic to $\mathrm{Hom}_{\mathcal{S}ch}(-, \mathrm{Hilb}(\mathbb{P}^N))$. Plugging in $\mathrm{Hilb}(\mathbb{P}^N)$ and its identity map, we obtain an element of $F(\mathrm{Hilb}(\mathbb{P}^N))$, which is exactly the universal family $\mathcal{Z}$, which you can check.

This is a nice picture — too nice, in fact, for some very nice moduli spaces, such as families of curves! The reason for this failure is that curves have automorphisms. This wasn't an issue for the Hilbert scheme, but in general you might be interested in moduli problems for objects which have automorphisms. For example, there are families of curves whose fibers are all isomorphic (so the map to the moduli space is constant) but the fiber bundle itself is nontrivial (so the map should be nonconstant). We will see an example next time.

Lecture 6.

Stacks: 9/17/19

"French [is] almost English, right? It's a little hard if you insist on English pronunciation, though."

In addition to these lectures on stacks/moduli spaces, it may be helpful to keep some references in hand:

(1) Dan Edidin's notes [Edi98] on the construction of the moduli space of curves.
(2) The original article by Deligne and Mumford [DM69] on the irreducibility of the moduli space of curves.

Last time, in Example 5.13, we discussed a moduli space of subvarieties of $\mathbb{P}^n$. We would like to repeat this story to construct a moduli space of complete[10] curves of a fixed genus g. To do so, we need a notion of families of curves.

Definition 6.1. Let S be a scheme. A *curve of genus g* over S is a morphism of schemes $\pi\colon C \to S$ such that

[8] There are more general examples also called Hilbert schemes.
[9] It actually suffices to use affine schemes, if you'd prefer to do that.
[10] This property is the analogue of compactness in differential geometry.

(1) π is proper and flat.
(2) Each *geometric fiber* $C_S := \operatorname{Spec} k \times_S C$, where k is an algebraically closed field and $\operatorname{Spec} k \to S$ is a k-point, is reduced,[11] connected, one-dimensional, and has *arithmetic genus* equal to g, i.e. $h^1(C_S, \mathcal{O}_{C_S}) = g$.

You can think of this as follows: S is a space of parameters, and we're thinking about a family of curves parametrized by S. Some of the conditions tell us this is a suitably nice family. One can also impose additional restrictions, e.g. imposing that C is nonsingular — though it was a major insight of Deligne and Mumford that if you want a compact space, you have to allow certain singularities, such as nodes.

However, there is a fundamental and serious problem obstructing the construction of a moduli space: the functor $F_g \colon \mathcal{S}ch \to \mathcal{S}et$ sending S to the set of isomorphism classes of (nonsingular) genus-g curves over S is not representable. Ultimately this is because there are *isotrivial* families of curves (i.e. trivial after a finite base change) that are nontrivial: if M_g is the hypothetical representing object and $\varphi\colon S \to M_g$ determines an isotrivial family, then we have a finite map $T \to S$ such that the composition with φ is constant. Hence φ is also constant – but there are examples of nonconstant φ, so M_g can't exist.

Example 6.2. Let's explicitly construct one of these families in any genus $g > 0$. Let C_0 be a curve with a nontrivial automorphism, such as a hyperelliptic curve, which is a branched double cover of $\mathbb{P}^1$. One explicit example is the projective closure of

$$y^2 = (x-g-1)(x-g)\cdots(x-1)(x+1)\cdots(x+g+1) = 0. \tag{6.3}$$

This has genus g. The nontrivial automorphism ϕ flips the sign of y. See Figure 1 for a picture.

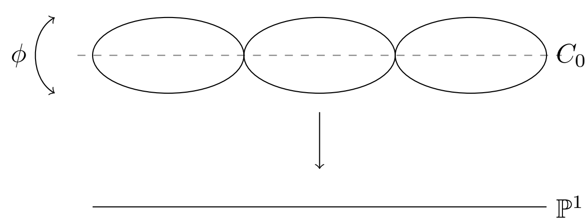

Figure 1. A hyperelliptic curve is a branched double cover of $\mathbb{P}^1$.

Recall that $\mathbb{G}_m := \operatorname{Spec} \mathbb{C}[t, t^{-1}]$, which as a space is $\mathbb{C}^\times$. Define a $\mathbb{Z}/2$-action on $C_0 \times \mathbb{G}_m$ by asking the nontrivial element of $\mathbb{Z}/2$ to act by $(\phi, -1)$. Then $(C_0 \times \mathbb{G}_m)/(\mathbb{Z}/2)$ is a family of genus-g curves over $\mathbb{G}_m$, and is nontrivial, but is trivialized by pulling back by the multiplication-by-2 map $\mathbb{G}_m \to \mathbb{G}_m$. ◀

So there must be some kind of internal structure of the moduli … thingy which keeps track of this data, and indeed we'll see how that happens.

Remark 6.4. Well it seems like the problem is automorphsims, so why don't we just remove curves which have automorphisms? This actually works; it just isn't useful. The resulting space isn't compact: in a sense you've just poked a bunch of holes in it, and moreover, some of the holes you've poked through were things you wanted to study. ◀

The upshot is that we will solve the moduli problem by constructing an *algebraic stack*. Algebraic stacks are a generalization of the notion of schemes with automorphisms baked into the story from the very beginning. As we go along, the book *Champs algébriques* [LMB00] by Laumon and Moret-Bailly is a great reference. It's in French, but is a great enough reference that it's still worth recommending.[12]

We want to first formalize the notion of a family of objects parameterized by a scheme, along with fiberwise automorphisms. Fix a base scheme S (you can think $S = \mathbb{C}$ for concreteness), and recall that the category $\mathcal{S}ch_S$ of *schemes over* S is the category whose objects are pairs of a scheme T and a map of schemes $f_T \colon T \to S$, and whose morphisms $(T_1, f_{T_1}) \to (T_2, f_{T_2})$ are maps $T_1 \to T_2$ of schemes intertwining f_{T_1} and f_{T_2}.

[11] This rules out a problem which can only happen in algebraic geometry, i.e. nilpotent elements of the ring of functions. Thus C_S is a variety.

[12] One thing which might be confusing: in French, *champ* means both "stack" and "field", the latter both in the algebraic sense and the physics sense.

Let $\mathcal{S} := \mathcal{S}ch_S$; thus the following definitions can all be extended to categories, groupoids, etc. over arbitrary categories, though we won't need that level of generality.

Definition 6.5.

(1) A *category over* $\mathcal{S}$ is a category $\mathcal{C}$ together with a functor $p_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{S}$. If $T \in \mathcal{S}$, its *fiber category* $\mathcal{C}(T)$ is the subcategory of $\mathcal{C}$ whose objects are those $x \in \mathcal{C}$ with $p_{\mathcal{C}}(x) = T$ and whose morphisms are those morphisms in $\mathcal{C}$ whose image under $p_{\mathcal{C}}$ is the identity on T.

(2) A category $\mathcal{C}$ over $\mathcal{S}$ is a *groupoid over* $\mathcal{S}$, or a *fibered groupoid*, if it satisfies the following conditions.
   (a) For all $f \colon B' \to B$ in $\mathcal{S}$ and objects $X \in \mathcal{C}$, there is a map $\phi \colon X' \to X$ in $\mathcal{C}$ such that $p_{\mathcal{C}}(\phi) = f$; in particular, $p_{\mathcal{C}}(X) = B$ and $p_{\mathcal{C}}(X') = B'$.
   (b) Suppose we have $X, X', X'' \in \mathcal{C}$, maps $\phi' \colon X' \to X$ and $\phi'' \colon X'' \to X$, and a map $h \colon p_{\mathcal{C}}(X') \to p_{\mathcal{C}}(X'')$ such that $p_{\mathcal{C}}(\phi') = p_{\mathcal{C}}(\phi'') \circ h$. Then, there is a *unique* map $\chi \colon X' \to X''$ such that $\phi' = \phi'' \circ \chi$.

Remark 6.6. If $\mathcal{C}$ is a groupoid over $\mathcal{S}$, all of its fiber categories are groupoids, but in general this is a stronger notion. We also see that:

- $\phi \colon X' \to X$ is an isomorphism if and only if $p_{\mathcal{C}}(\phi)$ is an isomorphism.
- Given a map $f \colon B' \to B$ in $\mathcal{S}$ and an $X \in \mathcal{C}$ with $p_{\mathcal{C}}(X) = B$, there is a unique $X' \in \mathcal{C}$ (up to unique isomorphism) X' with a map $\phi \colon X' \to X$ such that $p_{\mathcal{C}}(X') = B'$ and $p_{\mathcal{C}}(\phi) = f$. We call X' the *pullback* of X, and denote it f^*X. Pullback is natural enough to define a functor $f^* \colon \mathcal{C}(B) \to \mathcal{C}(B')$. ◂

Example 6.7. Let $F \colon \mathcal{S}^{op} \to \mathcal{S}et$ be a contravariant functor. This yields a groupoid $\mathcal{F}$ over $\mathcal{S}$:

Objects: the pairs (B, β) such that $B \in \mathcal{S}$ and $\beta \in F(B)$.
Morphisms: the maps between (B, β) and (B', β') are the maps $f \colon B \to B'$ such that $F(f)(\beta') = \beta$.

The fiber categories don't have any nontrivial morphisms, but this is still useful: any S-scheme X defines such a functor via its *functor of points*: the functor $\mathcal{S}^{op} \to \mathcal{S}et$ sending $B \mapsto \mathrm{Hom}_{\mathcal{S}}(B, X)$. This seems a little silly, but will later tell us that schemes embed into stacks, and therefore stacks extend things we already think about. ◂

The next example is "more stacky," i.e. the fiber categories are more interesting.

Example 6.8. Let X be a scheme over S with an action of a (flat) group scheme G over S. For example, you could let $S = \operatorname{Spec} \mathbb{C}$, $G = \mathrm{GL}_n(\mathbb{C})$, and $X = \mathbb{A}^n_{\mathbb{C}}$ via the standard representation. The *quotient groupoid* $[X/\!\!/G]$ is the groupoid over $\mathcal{S}$ given by the following data.

- The objects are pairs $(\pi \colon E \to B, f \colon E \to X)$, where $\pi \colon E \to B$ is a principal G-bundle and f is G-equivariant.
- The morphisms (π, E, B, f, X) to (π', E', B', f', X') are pairs $\phi \colon E' \to E$ and $\psi \colon B' \to B$ such that $f \circ \phi = f'$ and

$$\begin{tikzcd} E' \arrow[r, "\phi"] \arrow[d, "\pi'"] & E \arrow[d, "\pi"] \\ B' \arrow[r, "\psi"] & B \end{tikzcd} \tag{6.9}$$

  is Cartesian.

It turns out that if G acts freely on X and X/G exists as a scheme, then $[X/\!\!/G] \simeq X/G$ (where we regard X/G as a groupoid over $\mathcal{S}$ as in Example 6.7). ◂

Remark 6.10. If $X = \mathrm{pt}$, this tells us that $[\mathrm{pt}/\!\!/G]$ is the moduli "space" of principal G-bundles. It's worth comparing to the analogue in algebraic topology, which is called BG; this is at least an actual space, but doesn't have nearly as nice of a canonical construction. ◂

Lecture 7.

The 2-category of groupoids: 9/19/19

Recall from last time that $\mathcal{S}$ denotes the category of schemes over a base S. An S-scheme X defines a groupoid fibered over $\mathcal{S}$ by $\underline{X}(T) := \mathrm{Hom}_{\mathcal{S}}(T, X)$. We also saw tht a group acting on X induces a quotient groupoid $[X /\!\!/ G]$. There is a functor $F_g\colon \mathcal{S} \to \mathcal{S}et$ sending an S-scheme T to the groupoid of (families of) curves over T.

Now we define the *moduli groupoid of curves* of genus g, $\mathcal{M}_g$. Its objects are curves $X \to B$ of genus g, where B is any scheme and for all geometric points s of B, X_s is nonsingular. The morphisms from $X' \to B'$ to $X \to B$ are pairs of maps $X' \to X$ and $B' \to B$ such that the diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ B' \arrow[r] & B \end{tikzcd} \tag{7.1}$$

commutes and is Cartesian, i.e. $X' \cong B' \times_B X$. This is *not* the groupoid associated to F_g.

Similarly, one can define the *universal curve* $\mathcal{C}_g$ over $\mathcal{M}_g$, whose objecrs are pairs of a smooth curve $X \to B$ of genus g (with the same conditions as before) and a section $\sigma\colon B \to X$, and whose morphisms include compatibility of the sections under pullback.

We'd like to say that forgetting the section defines a map $\mathcal{C}_g \to \mathcal{M}_g$, and to do this we need to know what maps of groupoids are.

Definition 7.2. Let $p_1\colon \mathcal{C}_1 \to \mathcal{S}$ and $p_2\colon \mathcal{C}_2 \to \mathcal{S}$ be groupoids over $\mathcal{S}$. A functor $\varphi\colon \mathcal{C}_1 \to \mathcal{C}_2$ is a *morphism of groupoids over* $\mathcal{S}$ if $p_1 = p_2 \circ \varphi$. Here we ask for equality of functors, not just up to natural isomorphism.

Thus forgetting the section indeed specifies a map $\mathcal{C}_g \to \mathcal{M}_g$.

We would also like to say that a morphism $f\colon X \to Y$ of schemes is equivalent to a morphism $\underline{f}\colon \underline{X} \to \underline{Y}$ of groupoids over $\mathcal{S}$; this would tell us that $\mathcal{S}$ embeds into the category of groupoids over $\mathcal{S}$, and allow us to think of X and $\underline{X}$ as the same.

- In the forward direction, given f, we can define $\underline{f}$ on objects by

$$\underline{f}\colon \big(B \overset{u}{\to} X\big) \longmapsto \big(B \overset{f\circ u}{\to} Y\big), \tag{7.3}$$

  and on morphisms by

$$\underline{f}\colon \left(\begin{tikzcd} B \arrow[dr, "u"] \arrow[d, "s"'] & \\ B' \arrow[r, "u'"'] & X \end{tikzcd}\right) \longmapsto \left(\begin{tikzcd} B \arrow[dr, "f\circ u"] \arrow[d, "s"'] & \\ B' \arrow[r, "f\circ u'"'] & X \end{tikzcd}\right). \tag{7.4}$$

- Conversely, given $\underline{f}$ the map of groupoids over $\mathcal{S}$, $\underline{f}(\mathrm{id}\colon X \to X)$ is some map $X \to Y$, which we call f.

You can unwind the diagrams to check that these are mutually inverse operations, which is exactly what is happening in Yoneda's lemma; $\underline{f}$ can be viewed as a natural transformation between the two functors $\mathrm{Hom}_{\mathcal{S}}(-, X)$ and $\mathrm{Hom}_{\mathcal{S}}(-, Y)$ from $\mathcal{S}$ to $\mathcal{S}et$.

Remark 7.5. Similarly, if $B \in \mathcal{S}$ and $\mathcal{C}$ is a groupoid over $\mathcal{S}$, there's a natural isomorphism $\mathrm{Hom}(\underline{B}, \mathcal{C}) \overset{\cong}{\to} \mathcal{C}(B)$ by $p \mapsto p(\mathrm{id}_B)$. ◀

Example 7.6. This is a somewhat silly example, but for any groupoid $\mathcal{C}$ over $\mathcal{S}$, the structure map $p_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{S}$ is a morphism of groupoids $\mathcal{C} \to \underline{S}$. ◀

Example 7.7. Let X be a scheme over S with an action of a group S-scheme G; this yields a quotient map $q\colon \underline{X} \to [X /\!\!/ G]$. Recall that an object of $[X /\!\!/ G]$ is a scheme E with a G-action and an identification of the quotient with B, and a G-equivariant map $E \to X$.

The map q sends the object $s\colon B \to X$ to $G \times B$, which carries the left G-action, and whose quotient is identified with B in the standard way. The map $G \times B \to X$ sends $g, b \mapsto g \cdot s(b)$, and this is G-equivariant.

On morphisms, q is the map

$$\left(\begin{tikzcd} B' \arrow[r, "f"] \arrow[dr, "s'"'] & B \arrow[d, "s"] \\ & X \end{tikzcd}\right) \longmapsto \left(\begin{tikzcd} G \times B' \arrow[r, "\mathrm{id}\times f"'] \arrow[d] \arrow[rr, bend left] & G \times B \arrow[r] \arrow[d] & X \\ B' \arrow[r, "f"'] & B & \end{tikzcd}\right). \tag{7.8}$$

◀

We now have groupoids over $\mathcal{S}$ and morphisms thereof, so it seems reasomable to ask whether two groupoids over $\mathcal{S}$ are isomorphic. But this is a subtle issue, as with all questions involving isomorphisms of categories, because in addition to objects (categories, or in this case groupoids over $\mathcal{S}$) and morphisms (suitable functors), we have natural transformations, and in order to even formulate commutativity of diagrams in a useful way, we need to take natural transformations into account.

Specifically, $\mathcal{S}$-groupoids form what's called a *2-category*: roughly speaking, instead of just a set of morphisms between objects X and Y, we have a category; the objects in this category are called 1-*morphisms* and the maps are called 2-*morphisms*. For $\mathcal{S}$-groupoids:

- The objects are groupoids over $\mathcal{S}$.
- The 1-morphisms are the functors we considered in Definition 7.2, so functors commuting with the functors to $\mathcal{S}$.
- The 2-morphisms are the natural isomorphisms between these functors.

Proposition 7.9. *Let X and Y be schemes. Then X and Y are isomorphic if and only if $\underline{X}$ and $\underline{Y}$ are isomorphic in this 2-category $\mathcal{G}pd_{\mathcal{S}}$.*

Here, "isomorphic in $\mathcal{G}pd_{\mathcal{S}}$" means that there are 1-morphisms $\underline{f}\colon \underline{X} \to \underline{Y}$ and $g\colon \underline{Y} \to \underline{X}$ and 2-morphisms (natural isomorphisms) $h_1\colon \underline{f} \circ \underline{g} \to \mathrm{id}$ and $h_2\colon \underline{g} \circ \underline{f} \to \mathrm{id}$.[13] TODO: I didn't write down the proof in time. The key point, though, is that we can't upgrade this into "an isomorphism determines an isomorphism" because $\underline{f} \circ \underline{g}$ and $\underline{g} \circ \underline{f}$ might not be *equal* to id, just *naturally isomorphic* to it.

Anyways, the upshot is that schemes over S embeds as a full subcategory of $\mathcal{G}pd_{\mathcal{S}}$, even in this 2-categorical setting. Therefore we will no longer distinguish X and $\underline{X}$, etc., nor S and $\mathcal{S}$.

With our enriched understanding of the structure of $\mathcal{G}pd_S$, we can define fiber products and Cartesian diagrams of groupoids over S.

Definition 7.10. Let $F, G, H \in \mathcal{G}pd_S$ and $f\colon F \to G$ and $h\colon G$ be morphisms of S-groupoids. The *fiber product* $F \times_G H$ is the S-groupoid with the following data.

- The objects over $B \in \mathcal{S}$ are triples (x, y, ψ) with $x \in F(B)$, $y \in H(B)$, and $\psi\colon f(X) \xrightarrow{\cong} h(y)$ is an isomorphism in $G(B)$.
- The morphisms $(x, y, \psi) \to (x', y', \psi')$ over a map $B \to B'$ are the pairs $(\alpha\colon x' \to x, \beta\colon y' \to y)$ such that $\psi \circ f(\alpha) = h(\beta) \circ \psi'$.

You're probably thinking of this as sitting in a square

$$\begin{tikzcd} F \times_G H \arrow[r, "q"] \arrow[d, "p"] & H \arrow[d, "h"] \\ F \arrow[r, "f"] & G, \end{tikzcd} \tag{7.11}$$

where p is remembers x and forgets y and ψ, and q remembers y and forgets x and ψ, but *this diagram is not commutative*, as

$$fp(x, y, \psi) = f(x) \neq h(y) = gq(x, y, \psi). \tag{7.12}$$

However, there is a natural isomorphism $fp \simeq hq$, meaning (7.11) is 2-*commutative* (i.e. commutative up to 2-morphisms, which in this 2-category are isomorphisms). Explicitly, the natural isomorphism is $\eta_{(x,y,\psi)} \coloneqq \psi\colon f(x) \xrightarrow{\cong} h(y)$.

This allows us to formulate the universal property of the pullback: if

$$\begin{tikzcd} T \arrow[r, "\alpha"] \arrow[d, "\beta"'] & H \arrow[d, "h"] \\ F \arrow[r, "f"] & G \end{tikzcd} \tag{7.13}$$

[13] Compare to the definition of a homotopy equivalence of topological spaces.

is a 2-commutative diagram in $\mathcal{G}pd_S$, then there is a 1-morphism (i.e. functor over S) $\phi\colon T \to F \times_G H$, unique up to 2-morphsims (i.e. natural isomorphisms) such that the following diagram is 2-commutative:

$$\begin{tikzcd}
T \arrow[dr, "\phi", "\exists!"'] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "\beta"'] & & \\
& F \times_G H \arrow[r, "q"] \arrow[d, "p"] & H \arrow[d, "h"] \\
& F \arrow[r, "f"] & G.
\end{tikzcd} \tag{7.14}$$

Example 7.15. Because $\mathcal{S}ch_S$ embeds in $\mathcal{G}pd_S$ as a full subcategory, if X, Y, and Z are schemes and $f\colon X \to Z$ and $g\colon Y \to Z$ are maps of schemes over S, then $\underline{X} \times_{\underline{Z}} \underline{Y} \simeq \underline{X \times_Z Y}$. ◀

Example 7.16. We can use pullback to define base change. Let F be a groupoid over S and $T \to S$ be a map of schemes. The $T \times_S F$ is a groupoid over T. In fact, for all schemes B over T, $F(B)$ and $(T \times_S F)(B)$ are naturally isomorphic as functors $\mathcal{S}ch_T^{op} \to \mathcal{S}et$. ◀

This last example allows us to (finally!) define stacks, though we'll need some more material to define algebraic stacks.

Definition 7.17. Let (F, p_F) be a groupoid over S, B be a scheme over S, and $X, Y \in F(B)$. The *iso-functor* determined by this data is the functor $\underline{\mathrm{Iso}}_B\colon \mathcal{S}ch_B^{op} \to \mathcal{S}et$ which

- sends an object $f\colon B' \to B$ to the set of isomorphisms $\{\phi\colon f^*X \xrightarrow{\cong} f^*Y\}$, and
- sends the morphism $h\colon B'' \to B'$ to the morphism

$$(\phi\colon f^*X \to f^*Y) \longmapsto (h^*\phi\colon h^*f^*X \to h^*f^*Y). \tag{7.18}$$

Theorem 7.19 (Deligne-Mumford)**.** *If X and Y are curves over B of the same genus $g \geq 2$, then $\underline{\mathrm{Iso}}_B(X,Y)$ is represented by a scheme $\mathrm{Iso}_B(X,Y)$.*

Proof sketch. Since $g \geq 2$, then $\omega_{X/B}$ and $\omega_{Y/B}$ are ample, and therefore for any map $f\colon B' \to B$, any isomorphism $f^*X \to f^*Y$ preserves the polarization. Then the representing object is the relative Hilbert scheme for the graph of $f^*X \to f^*Y$. ⊠

Remark 7.20. $\mathrm{Iso}_B(X,Y)$ is finite and unramified over B, which relates to the fact that automorphism groups of genus-≥ 2 curves are finite. However, $\mathrm{Iso}_B(X,Y) \to B$ is not in general flat; the cardinalities of the fibers can jump. ◀

We're almost at the definition of a stack: we just need $\underline{\mathrm{Iso}}_B$ to satisfy a sheaf condition and a gluing condition. We'll do this next time.

References

[CDGP91] Philip Candelas, Xenia C. De La Ossa, Paul S. Green, and Linda Parkes. A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory. *Nuclear Physics B*, 359(1):21–74, 1991. 2, 3

[DM69] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.*, (36):75–109, 1969. 13

[Edi98] Dan Edidin. Notes on the construction of the moduli space of curves. 1998. http://arxiv.org/abs/math/9805101. 13

[LMB00] Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques*, volume 39 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2000. 14