M390C NOTES: GEOMETRIC LANGLANDS

ARUN DEBRAY
SEPTEMBER 8, 2016

These notes were taken in UT Austin's Math 390C (Geometric Langlands) class in Fall 2016, taught by David Ben-Zvi. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to `a.debray@math.utexas.edu`. Thanks to Feng Ling and Yuri Sulyma for correcting a few typos.

Contents

1. The Fourier Transform in Representation Theory: 8/25/16 — 1
2. Representation Theory as Gauge Theory: 8/25/16 — 5
3. Group Algebras and Convolution: 8/30/16 — 8
4. Spectral Decomposition: 9/1/16 — 11
5. Projection-Valued Measures: 9/6/16 — 15
6. Quantum Mechanics: 9/8/16 — 17
References — 19

1. The Fourier Transform in Representation Theory: 8/25/16

"One of the traditions we have at UT is we always have to mention Tate."

The initial conception of this class was going to be more akin to a learning seminar about the geometric Langlands program, but this changed: it's now going to be an actual class, but about geometric representation theory and topological field theory. The goal is for this to turn into good lecture notes and even a book, so the class isn't the entire intended audience. As such, feedback is even more helpful than usual.

It's not entirely clear what the prerequisites for this class are; the level of background will grow as the class goes on. The actual amount of technical background needed to state things precisely is huge, and not a reasonable requirement. As such, the class will be more of a sketch and overview of the ideas and how to think about the main characters[1] in this subject. The professor's seminar (Fridays, from 2 to 4, in the same room) is probably a good place to start understanding this material more rigorously.

There will be an introduction to this class this afternoon at geometry seminar.

The Fourier transform. Do you remember Fourier series? The statement is that for L^2 functions $f : S^1 \to \mathbb{C}$,

$$f(\theta) = \sum_{n\in\mathbb{Z}} \widehat{f}(n) e^{2\pi i n\theta}.$$

This is probably the last precise formula we're going to see in this class, which may reassure you or bother you. We also will identify $S^1 \cong \mathrm{U}(1)$. The Fourier coefficients are

$$\widehat{f}(n) = \int_{S^1} f(\theta) e^{-2\pi i n\theta}\, \mathrm{d}\theta.$$

Representation theory starts with this formula.

Relatedly, for an L^2 function $f : \mathbb{R} \to \mathbb{C}$, we have a continuous combination of exponentials with coefficients $\widehat{f}(t)$:

$$f(x) = \int_{\mathbb{R}} \widehat{f}(t) e^{2\pi i x t}\, \mathrm{d}t, \tag{1.1}$$

[1] Pun intended?

where

$$\widehat{f}(t) = \int_{\mathbb{R}} f(x) e^{-2\pi i x t}\, \mathrm{d}x.$$

How should we think of these formulas? The exponentials $e^{2\pi i t -}$ are complex-valued functions on $\mathrm{U}(1)$ and $\mathbb{R}$, respectively. But in fact, they land in $\mathbb{C}^\times$, since they don't hit 0, and in fact they have unit norm, so they are maps into $\mathrm{U}(1)$. Since $e^{a+b} = e^a e^b$, these are homomorphisms of groups. Moreover, these are the only homomorphisms: if $f(\theta_1 + \theta_2) = f(\theta_1) f(\theta_2)$ for an $f : \mathrm{U}(1) \to \mathrm{U}(1)$, then $f(\theta) = e^{2\pi i x \theta}$ for some x, and similarly for functions $\mathbb{R} \to \mathrm{U}(1)$.

In other words, these functions are the *unitary characters* of the domain group: the homomorphisms to $\mathrm{U}(1) \subset \mathrm{GL}(1)$. We can recast these as representations acting through unitary matrices (also *unitary representations*), where an $x \in \mathbb{R}$ acts as multiplication by $e^{2\pi i x t}$ on the (complex) vector space $\mathbb{C}$.

From this viewpoint, we are writing general functions on $\mathrm{U}(1)$ or on $\mathbb{R}$ as linear combinations of characters. This means characters form a "basis." That is, the characters are not strictly a basis, but the space spanned by finite linear combinations of exponentials is dense in any reasonable function space L^2, C^∞, distributions, real analytic functions, L^p spaces, etc. In particular, L^2, smooth, analytic, etc. are conditions on the Fourier coefficients: $f \in L^2(S^1)$ iff $\widehat{f} \in \ell^2$ (the square-integrable sequences of numbers). f is smooth iff its Fourier coefficients are rapidly decreasing (faster than any polynomial).

This is where the analysis of Fourier series takes place: you're interested in different function spaces, and so you're interested in how the coefficients grow. But we're going to ignore it: it's deep and important for analysis, but begins a different track than representation theory. The algebraic content is that algebraic functions (Laurent series) are dense, and we're going to care more about the algebraic side than the analytic side.

Theorem 1.2 (Plancherel)**.** *If $\mathbb{R}$ denotes the x-line and $\widehat{\mathbb{R}}$ denotes the t-line, then the Fourier transform defines a unitary isomorphism $L^2(\mathbb{R}) \xrightarrow{\sim} L^2(\widehat{\mathbb{R}})$.*

This is nice, but doesn't help much for the character-theoretic viewpoint: the exponential $e^{2\pi i x t}$ is not in $L^2(\mathbb{R})$. This is where one uses Schwarz functions.

Definition 1.3. The *Schwarz space* $\mathcal{S}(\mathbb{R})$ is the space of $f \in C^\infty(\mathbb{R})$ such that f and all of its derivatives decrease more rapidly than any polynomial.

The dual space to $\mathcal{S}(\mathbb{R})$, denoted $\mathcal{S}^*$ or $\mathcal{S}'$, is called the space of *tempered distributions*. Our characters $e^{2\pi i x t}$ live in this space, and the Fourier transform extends to a linear homeomorphism $\mathcal{S}'(\mathbb{R}) \cong \mathcal{S}'(\widehat{\mathbb{R}})$.

Thus, it makes sense to define the Fourier transform of the exponential $e^{2\pi i n x}$: we obtain the delta "function" supported at n, δ_n (1 at n and 0 elsewhere), and similarly, the Fourier transform of δ_t is $e^{2\pi i x t}$. That is, the Fourier transform exchanges points and characters; in other words, $\widehat{\mathbb{R}}$ is a sort of moduli space of unitary characters of $\mathbb{R}$.

In some sense, this diagonalizes the group action: if G is either of $\mathbb{R}$ or $\mathrm{U}(1)$, then G acts on itself by translation (both left and right, since G is abelian). Thus, any space of functions on G is acted on by G: an $\alpha \in G$ sends $f \mapsto \alpha * f$ (i.e. $\alpha * f(x) = f(x+\alpha)$). If V is this function space (e.g. $L^2(G)$), then this defines an action of G on V, hence a group homomorphism $G \to \mathrm{End}(V)$. In particular, the exponential $e^{2\pi i x t}$ satisfies

$$\alpha * e^{2\pi i x t} = e^{2\pi i (x+\alpha) t)} = \big(e^{2\pi i x \alpha}\big)\big(e^{2\pi i x t}\big).$$

That is, this exponential is an eigenfunction for $\alpha * -$ for all $\alpha \in G$: characters are joint eigenfunctions, and the Fourier transform is a simultaneous diagonalization.

Succinctly, *the Fourier transform exchanges translation and multiplication*: the translation operator $\alpha * -$ is sent to the multiplication operator $\widehat{f} \mapsto \widehat{\alpha}\widehat{f}$, where $\widehat{\alpha}(t) = e^{2\pi i \alpha t}$. From the perspective of Fourier series, we have a $\mathbb{Z} \times \mathbb{Z}$ matrix with respect to the exponential basis, but only the diagonal entries $\widehat{f}(n) e^{2\pi i n \theta}$ are nonzero.

Before we make this more abstract, let's see what happens to differentiation. Since G is a Lie group, it has a Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}$, in this case $\mathbb{R} \cdot \frac{\mathrm{d}}{\mathrm{d}x}$, the infinitesimal translations at a point. The differential $\frac{\mathrm{d}}{\mathrm{d}x}$ is an infinitesimal translation, and the Fourier transform sends it to a multiplication by $(2\pi i)t$.[2]

[2] To prove this rigorously, one needs to worry about difference quotients.

Pontrjagin duality. We can generalize this to Pontrjagin duality, which is a kind of Fourier transform involving a locally compact abelian topological group (LCA) G, e.g. $\mathbb{R}$, $\mathbb{Z}$, S^1, $\mathbb{Z}/n$, and any finite products of these, including tori, lattices, and finite-dimensional vector spaces. More exotic examples include the p-adics. There will be more interesting examples in the algebraic world.

Definition 1.4. Let G be an LCA group; then, the *(unitary) dual* of G is $\widehat{G} = \mathrm{Hom}_{\mathsf{TopGrp}}(G, \mathrm{U}(1))$, the set of characters of G, with the topology inherited as a subset of the continuous functions $C(G) = \mathrm{Hom}_{\mathsf{Top}}(G, \mathbb{C})$.

We saw that if $G = \mathbb{R}$, then $\widehat{G} = \mathbb{R}$ again, and that if $G = \mathrm{U}(1)$, then $\widehat{G} = \mathbb{Z}$. Conversely, if $G = \mathbb{Z}$, then a homomorphism on G is determined by its value at 1, which can be anything in $\mathrm{U}(1)$, so $\widehat{G} = \mathrm{U}(1)$. If V is a finite-dimensional vector space, then $\widehat{V} = V^*$: any linear functional $\xi \in V^*$ defines a character $v \mapsto e^{2\pi i\langle \xi, v\rangle}$. It's a nice exercise to check that these are all the unitary characters. If $G = \Lambda$ is a lattice, then we obtain its *dual torus* T, and correspondingly a torus goes to its *dual lattice*. Lastly, we have finite abelian groups, e.g. $\mathbb{Z}/n$, which is generated by 1, so we must send 1 to an n^{th} root of unity. Thus, $(\mathbb{Z}/n)^\vee = \mu_n$, the group of n^{th} roots of unity. This is isomorphic to $\mathbb{Z}/n$ again, though in algebraic geometry, where we might not have all roots of unity, things can get more interesting, so it's useful to remember μ_n.

The claim is that the Fourier transform looks exactly the same for any LCA group; maybe we haven't defined too many exciting examples, but this is still noteworthy. We want characters on G to correspond to points on $\widehat{G}$. A point $\chi \in \widehat{G}$ defines a function on G, and correspondingly, a point $g \in G$ defines a function $\widehat{g} : \chi \mapsto \chi(g)$ on $\widehat{G}$, which looks like a nascent Fourier transform. If $g, h \in G$, then $\widehat{gh}(\chi) = \chi(gh) = \chi(g)\chi(h) = \widehat{g}\widehat{h}(\chi)$, so this transform that we're building will start from this duality of the group multiplication and the pointwise product.

One important thing to mention: $\widehat{G}$ is also a group, and in fact is locally compact abelian. The group operation is pointwise product $\chi_1 \cdot \chi_2(g) = \chi_1(g)\chi_2(g)$. This agrees with the group operations for the examples we mentioned.

Theorem 1.5 (Pontrjagin duality)**.** *The natural map $G \mapsto \widehat{\widehat{G}}$ defined by $g \mapsto \widehat{g}$ is an isomorphism of topological groups.*

Hence, this really is a duality. Nonetheless, we'll maintain the distinction between G and $\widehat{G}$: soon we'll try to generalize to nonabelian groups, and then symmetry will break.

Theorem 1.6 (Fourier transform)**.** *If G is an LCA group, then the Fourier transform map*

$$f \longmapsto \widehat{f}(\chi) = \int_G f(g) \cdot \chi(g)\, \mathrm{d}g,$$

where $\mathrm{d}g$ is the Haar measure on G,[3] *defined an isomorphism of Hilbert spaces $L^2(G) \xrightarrow{\sim} L^2(\widehat{G})$.*

Notice that, since the characters on $\mathbb{R}$ are the exponentials and the Haar measure on $\mathbb{R}$ is the usual Lesbegue measure, this generalizes (1.1).

This entire story started in Tate's thesis, which applies Pontrjagin duality to more exotic examples such as $\mathbb{Q}_p$ and $\mathbb{Q}_p^\times$ or even the group $\mathbb{A}^\times$ of *adeles*;[4] see Ramakrishnan-Valenza [1] for a modern take on this subject, including harmonic analysis on LCA groups.

We'll use this to understand all representations of G (well, nice representations). In general, not all representations of G on a space come from functions on G, but we'll be able to use Pontrjagin duality and the group algebra to do something nice.

Function theory. One important philosophy in representation theory is that the action of G on functions on G (nice functions in whichever context we're working in) is the most important, or universal, representation. We'll talk about functions and convolution from a particular perspective that will be useful several times in the class.

Let X be a finite set. Then, $F(X)$, the set of complex-valued functions on X, is unambiguous. The set of measures on X, $M(X)$, is also clear, but there's a natural bijection between them via the counting measure.

Theorem 1.7 (Finite Riesz representation theorem)**.** *There is a natural identification $F(X) = \mathrm{Hom}_{\mathbb{C}}(F(X), \mathbb{C})$.*

[3] This is only unique up to a scalar, so we need to pick one.

[4] Not to be confused with the musician.

This comes from the inner product on $F(X)$

$$\langle f, g\rangle = \sum_{x\in X} f(x)g(x).$$

The more general Riesz representation theorem is about a Hilbert space of functions on $\mathbb{R}$, and is less trivial.

Now, suppose we have two finite sets X and Y. We can form their product, which looks like Figure 1. It's

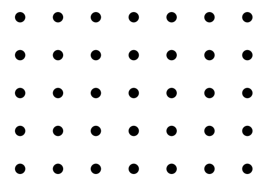

Figure 1. The product of two finite sets.

possible to identify $F(X\times Y) = F(X)\otimes F(Y)$, and via a matrix, or an "integral kernel," this space can be identified with $\mathrm{Hom}_{\mathbb{C}}(F(X), F(Y))$: a kernel $K(x, y) \in F(X \times Y)$ defines an operator $K * - : F(X) \to F(Y)$ defined by

$$K * f(y) = \sum_{x\in X} K(x, y)f(x).$$

In a broader sense, let $\pi_X : X \times Y \to X$ be projection, and define π_Y similarly. Functions can pull back: $\pi_X^* f(x, y) = f(\pi_X(x, y))$, and measures can push forward by integration (or summing, since we're thinking about the counting measure) over the fibers. Thus, we can recast convolution as

$$K * f = \pi_{Y*}(K \cdot \pi_X^* f)(y) = \int_X K(x, y)f(x)\,\mathrm{d}\#.$$

Since $F(X)$ and $F(Y)$ are finite-dimensional vector spaces, K may be identified with a matrix or a linear transform, and this formula is exactly how to multiply a matrix by a vector.

A key desideratum is that, in general, all nice maps between function spaces on X and function spaces on Y come from integral kernels. For example, a map $L^2(\mathbb{R}) \to L^2(\mathrm{pt}) = \mathbb{C}$ is given by a kernel $K \in L^2(\mathbb{R}\times \mathrm{pt}) = L^2(\mathbb{R})$, realized as $f \mapsto \int K \cdot f$, by the Riesz representation theorem for L^2. Another instance of this is the Schwarz kernel theorem.

Theorem 1.8 (Schwarz kernel theorem)**.** *Let X and Y be smooth manifolds. Then,* $\mathrm{Hom}_{\mathsf{Top}}(C_c^\infty(X), \mathrm{Dist}(Y)) \cong \mathrm{Dist}(X \times Y)$*.*

Here, $\mathrm{Dist}(-)$ is the space of distributions, dual to compactly supported smooth functions on the manifold.

If $X = Y$ (back in the world of finite sets), then we can consider δ_Δ, the δ-function of the diagonal. In a basis, this is just the identity matrix, and convolution with K is the identity operator. More generally, if $g : X \to Y$ is a set map, then $g^* : F(Y) \to F(X)$ is represented by the kernel of the graph $\Gamma_g \subset X \times Y$: $K = \delta_{\Gamma_g}$. If this all seems a little silly, the key is that it's easier to understand over finite sets, but will work for "nice" functions in a great variety of contexts.

We can also use this to understand matrix multiplication. Given three finite sets X, Y, and Z, and kernels (functions) $K_1 : F(X) \to F(Y)$ and $K_2 : F(Y) \to F(Z)$, we can compose them. Consider the projections

$$\begin{tikzcd}
 & X\times Y\times Z \arrow[dl, "\pi_{12}"'] \arrow[dr, "\pi_{23}"] \arrow[dd, "\pi_{13}"] & \\
X\times Y & & Y\times Z \\
 & X\times Z. &
\end{tikzcd}$$

Exercise 1.9. Show that the formula for $K_2 \circ K_1$ is

$$\pi_{13*}(\pi_{12}^*(K_1)\cdot \pi_{23}^*(K_2)).$$

Relate this to matrix multiplication.

The distinction between functions and measures is irrelevant in the world of finite sets, so we can push-forward and pull back with impunity, but in a continuous setting, it's important to keep them distinct. This equates to choosing a measure (e.g. choosing a Haar measure, as we did above), and even relates to things like Poincaré duality.

2. Representation Theory as Gauge Theory: 8/25/16

Note: this talk was an overview of the class, presented at the weekly geometry seminar.

2.1. **Representation Theory.**

Representation theory starts from spectral decomposition and the Fourier transform. If G is a locally compact abelian group, we attach its *unitary dual* $\widehat{G}$, the set of irreducible unitary representations of G. These are all one-dimensional, hence described by characters $\chi : G \to \mathrm{U}(1) \subset \mathbb{C}$. The key idea generalizing the Fourier transform is Pontrjagin duality, that the Fourier transform defines an isomorphism $L^2(G) \cong L^2(\widehat{G})$; there are variants for other function spaces.

Example 2.1.

- Finite Fourier series arise from $G = \mathbb{Z}/n$, for which $\widehat{G} = \mathbb{Z}/n$. The dual of $i \mapsto \zeta^i$, where ζ is a primitive root of unity, is a δ-function supported at i.
- Fourier series exchange $G = \mathrm{SO}(2)$ and $\widehat{G} = \mathbb{Z}$.
- The Fourier transform is for $G = \mathbb{R}$ and $\widehat{G} = \mathbb{R}$.

The Fourier transform takes representation theory of G, and turns it into geometry on $\widehat{G}$. For example, characters of G have turned into points of $\widehat{G}$. The group G can act in different translation-like ways: translation, differentiation (infinitesimal translation), and convolution; all of these are simultaneously diagonalized by the Fourier transform, and made into multiplication. Representations of G are turned into families of vector spaces on $\widehat{G}$, in various forms (vector bundles, sheaves, etc.), in a process called *spectral decomposition*.

This is all really nice: the Fourier transform basically solves representation theory for abelian groups. What should we do for nonabelian G?

We'd like to seek a geometry object $\widehat{G}$ parameterizing irreducible representations (unitary or other classes). This $\widehat{G}$ carries a measure, a topology, and even has algebraic geometry; this structure captures notions of families of representations.

Even though we don't know what $\widehat{G}$ is yet, we know that functions on $\widehat{G}$ should act on representations of G in a way that commutes with the G-action.

Example 2.2. If $G = \mathrm{SO}(3)$, then G acts on S^2 and hence also on $L^2(S^2)$ (the Hilbert space of a quantum free particle on a sphere). This action commutes with the spherical Laplacian Δ, and therefore we can decompose $L^2(S^2)$ into Δ-eigenspaces called *spherical harmonics*:

$$L^2(S^2) \cong \bigoplus_{n\in 2\mathbb{Z}_+} V_n.$$

This says a lot about the unitary irreducible representations of $\mathrm{SO}(3)$.

One of the huge goals of representation theory is to produce a nonabelian analogue of the Fourier transform for *arithmetic locally symmetric spaces* $X_\Gamma = \Gamma\backslash G_\mathbb{R}/K$. These are generalizations of the moduli space of elliptic curves: $\mathrm{SL}_2\mathbb{Z}\backslash\mathbb{H} \cong \mathrm{SL}_2\mathbb{Z}\backslash\mathrm{SL}_2\mathbb{R}/\mathrm{SO}_2$, where $\mathbb{H}$ is the upper half-plane.

For every prime p, X_Γ has a hidden p-adic symmetry group $G_{\mathbb{Q}_p}$, along with the manifest $G_\mathbb{R}$ symmetry. This creates a huge amount of symmetry, allowing one to define operators called Hecke operators. At almost all primes, these operators commute, so can we simultaneously diagonalize them? This is, in some sense, a goal of the Langlands program (and access the secrets of the universe, hopefully).

2.2. **Quantum field theory.**

We've just seen representation theory in a nutshell; now, on to quantum field theory in a nutshell.

An n-dimensional quantum field theory $\mathcal{Z}$ attaches to every n-dimensional Riemannian manifold M a Hilbert space $\mathcal{Z}(M)$. It also has *time evolution*: an n-dimensional cobordism $N : M_1 \to M_2$ defines a linear map $\mathcal{Z}(N) : \mathcal{Z}(M_1) \to \mathcal{Z}(M_2)$. Gluing two cobordisms together corresponds to composing their linear maps.

Quantum field theory should be local, and so there's a great deal of structure that can be tracked to understand this condition.

Example 2.3 (Quantum mechanics)**.** Consider a free particle on a manifold X (e.g. $\mathbb{R}^3$), and let $n = 1$. Here, we'll let $\mathcal{Z}(M)$ be a linearization of a space of fields on M, e.g. in the *σ-model*, these fields are maps to X.

In our case, $\mathcal{Z}(\mathrm{pt})$ is the Hilbert space $L^2(X)$, and time evolution is the semigroup defined by the Hamiltonian, which is the Laplacian: $H = \Delta$. Then, the bordism $[0, T]$ is the evolution e^{iH} (so that gluing becomes composition).

Riemannian manifolds are great, but it is sometimes easier to remove the dependence on metrics. A topological field theory removes a dependence on everything but the topology of spacetime. Sometimes, these appear from another source, which is great, but other times, we have to produce these theories by forcing them. To do this, we need to kill the Hamiltonian. Supersymmetry can do this, by making H exact with respect to, e.g. the de Rham operator, and then passing to cohomology.

A *local operator* is a zero-dimensional defect, which labels measurements at a point of any spacetime. Precisely, we take tiny spheres around these points as the sphere shrinks, which defines cobordisms. For example, in quantum mechanics, local operators in quantum mechanics are the operators on the Hilbert space. These operators do not always commute, which is the statement of Heisenberg uncertainty.

We can also consider defects at higher dimensions, or singularities of higher dimensions. A *line operator* is a one-dimensional quantum mechanics living on a one-dimensional submanifold of the spacetime. These also have a huge amount of structure: they form a category.

Scaling this all the way up, a *local boundary condition* is an $(n-1)$-dimensional theory that labels boundaries in $\mathcal{Z}$. These can interface with each other in codimension 2, and there are interfaces between interfaces between interfaces… this creates the algebraic structure of an *$(n-1)$-category.*

But what does this structure buy us? There's a conjecture of Baez-Dolan, now a theorem of Lurie, that it tells us everything.

Theorem 2.4 (Cobordism hypothesis (Lurie))**.** *An n-dimensional topological field theory $\mathcal{Z}$ is uniquely determined by its higher category of boundary conditions.*

The theorem also contains an existence statement, which corresponds to a finiteness condition. You can start with your favorite $(n-1)$-category, whatever that may be, and when you try to germinate it into data on lower- and lower-dimensional manifolds, it might not be "finite enough." There's a more precise sense in the theorem statement.

In particular, a category is a 1-category, so one can think of categories as 2-dimensional TFTs, illuminating a deep geometric perspective on categories. In particular, we're interested in the category of representations of a group, so we should think about the topological field theory it describes.

2.3. **Gauge theory and moduli spaces.** Gauge theory linearizes spaces of G-bundles with connections, in a way invariant under gauge transformations. This has been tremendously influential in low-dimensional topology, producing many invariants of 3- and 4-manifolds arising from cohomology of moduli paces of G-bundles on these manifolds.

An alternate point of view is that n-dimensional gauge theories are to be understood as QFTs whose boundary conditions are $(n-1)$-dimensional QFTs with G-symmetry; that is, *gauge theories are representations of groups on field theories.*

Example 2.5 (2-dimensional Yang-Mills theory)**.** Suppose G is finite or compact. This theory more or less counts G-bundles with a connection on spacetime; the boundary condition is quantum mechanics with symmetry group G (i.e. G acts on $L^2(X)$ and the Hamiltonian).

From the topological setting, one can just declare the boundary conditions to be the category of representations of G, and recover a topological field theory. The local operators are given by functions on G-connections on a very small circle; linearizing this, we get the conjugacy-invariant functions on G, the *class functions* $\mathbb{C}[G/G]$.

There are various ways to compose different operators: *operator product expansion* has these two small circles get closer together and collide. Another alternative is the *little-discs* composition, where we surround two close small circles with a larger circle enclosing them; topologically, this is the same as a pair-of-pants bordism.

Because we can move these small circles around each other, local operators commute.[5] This is surprising: the quantumness of quantum mechanics, its noncommutativity, becomes commutativity in topological field theory.

So the local operators on 2-dimensional Yang-Mills theory are the class functions with convolution, which is a commutative algebra, and is in fact the center of the group algebra $\mathbb{C}[G]$. Abstractly, this is the *Bernstein center* of the category of representations of G (or, of the category of boundary conditions). That is, local operators are precisely functions on the dual $(\mathbb{C}[\widehat{G}], \cdot)$.

Just like we did with quantum mechanics, we might want to model a quantum field theory as a theory of maps to a target. The target is called the *moduli space of vacua* $\mathfrak{M}_{\mathcal{Z}}$, the universal answer to the question "if I realize my theory as a theory of maps, what does it map into?" Local operators are functions on $\mathfrak{M}_{\mathcal{Z}}$.

As in algebraic geometry, we've found a way to obtain a space $\mathfrak{M}_{\mathcal{Z}}$ from a ring (the Bernstein center of the category of boundary conditions). This sends local operators with operator product expansion to functions on $\mathfrak{M}_{\mathcal{Z}}$ with pointwise multiplication, and line defects in $\mathcal{Z}$ with operator product expansion to sheaves on $\mathfrak{M}_{\mathcal{Z}}$ with tensor product; Yang-Mills provides us an analogue of spectral decomposition.

This looks a lot like Fourier theory. However, the moduli space is discrete, which is arguably not exciting, just as there's not to say formally about the representations of compact groups. However, passing to three-dimensional theories produces a continuous moduli space, just akin to passing to representations of noncompact groups.

In this case, we replace the σ-model (a theory of maps) with a gauge model (a theory of connections). Instead of 1-forms, we have 2-forms, and in the abelian case, this is literally a Hodge star operator.

Physics teaches that in the three-dimensional case, there's a great amount of geometry on these moduli spaces:

(1) If two operators travel in linked loops, we obtain a bracket, which is a Poisson bracket: the moduli space is a Poisson variety.
(2) This has a canonical quantization, called the *Nekrasov Ω-background*.
(3) For gauge theories, or those extending to four dimensions, $\mathfrak{M}_{\mathcal{Z}}$ is what's called a *Seiberg-Witten integrable system*.

In other words, *moduli spaces of gauge theories are precisely the modern geometric setting of representation theory*: both are active research areas.

For example, the A-model is a two-dimensional TFT that measures symplectic geometry, and the B-model models complex geometry. Mirror symmetry can be thought of as a Fourier transform between these models. Since the boundary conditions of a three-dimensional TFT are two-dimensional with a group action, they can capture symmetries in symplectic and complex geometry. The analogue of a Fourier transform in this setting leads to the active program called *symplectic duality*.

2.4. **Electric-magnetic duality.** Beilinson-Drinfeld developed a geometric counterpart to the Langlands program, a kind of harmonic analysis taking place on categories of bundles on a Riemann surface. Instead of focusing on a locally symmetric space X_Γ, we focus on a moduli of bundles $\mathrm{Bun}_G(C)$; the Hilbert space of functions on X_Γ is replaced with a category of sheaves on this moduli space. Operators are replaced with functors, and prime numbers are replaced with points of the Riemann surface.

In particular, Hecke operators are functors, and should be some sort of integral operators (convolutions) on sheaves. If $\mathscr{F}$ is a sheaf and $\mathcal{P}$ is a bundle, we'd like to send $\mathscr{F}(\mathcal{P})$ to a "weighted average" of $\mathscr{F}(\mathcal{P}')$ for nearby bundles $\mathcal{P}'$. Specifically, we'd like to modify at a single point x, and keep everything else the same. This lives on the canonical non-Hausdorff space $C \amalg_{C\setminus x} C$, which has two projections down to the two copies of C. This change is called a *Hecke modification*.

Kapustin and Witten realized this can be interpreted via a 4-dimensional gauge theory: Hecke modification is the creation of a magnetic monopole in the bundle. The worldline of this monopole, called a *'t Hooft line*, is a line defect in the theory.

The fundamental question, important for harmonic analysis (or its analogue), is *why do Hecke operators commute?* There was no reason to expect this, but the operators we described do commute — for the same reason as the commutativity of local operators we described above: modifications at two different points don't interact, and modifications at the same point can be dragged off each other, swapped, and slid back

[5]The proof, and the picture, is identical to the picture drawn to show that $\pi_n(X)$ is abelian when $n \geq 2$.

onto each other: since the field theory is topological, these all describe the same operator. This is the same insight as to why higher homotopy groups commute.

Beilinson-Drinfeld axiomatized this multiplication structure, an algebraic structure encoded in collisions of points, into a *factorization algebra*. This provides a geometric theory of operators in conformal field theory (vertex algebras) and quantum field theory.

This was a very fruitful insight: Gaitsgory showed that even if there are singularities (ramified singularities), one recovers the same center as for the ramification. This is a two-dimensional solenoidal defect. This itself had applications by Bezrukavnikov and more, to modular representation theory and more.

Peter Scholze managed to port this back to number theory, producing a geometric source to the commutativity of classical Hecke operators, in the setting of $\operatorname{Spec}\mathbb{Z}$ and $\operatorname{Spec}\mathbb{Z}_p$ (which lies near the prime p in $\operatorname{Spec}\mathbb{Z}$). Very recently, this led to Faurges' conjecture, a physics-inspired conjecture shedding new light on the Langlands conjecture.

In four dimensions, the Hodge star sends 2-forms to 2-forms, so the dual of a gauge theory isn't a σ-model, but rather another gauge theory. This recalls the fact that Maxwell's equations in a vacuum (a gauge theory with gauge group $\mathrm{U}(1)$) is symmetric under Hodge star, which exchanges the roles of electricity and magnetism. A nonabelian generalization, called S-theory, relates a gauge theory with gauge group G to one for its dual group.

Kapustin and Witten interpret geometric Langlands in terms of S-duality: sheaves on the moduli of bundles are boundary conditions for $\mathcal{N} = 4$ super Yang-Mills theory for G, and the Hecke operators correspond to 't Hooft line operators. One can write down an analogue of the Fourier transform.

The physics goes up to eleven! Specifically, M-theory. But the richest known representation-theoretic structure is a six-dimensional theory, known as "theory $\mathcal{X}$."

3. Group Algebras and Convolution: 8/30/16

Last time, we talked about the Fourier transform in the context of a locally compact abelian group G and its unitary dual $\widehat{G} = \mathrm{Hom}_{\mathsf{TopGrp}}(G, \mathrm{U}(1))$. On $G \times \widehat{G}$, there's a *universal character* function $\underline{\chi} : (g, \chi) \mapsto \chi(g)$ evaluating a character on a point. We used this to define the Fourier transform; Theorem 1.5 tells us it induces an isomorphism $L^2(G) \cong L^2(\widehat{G})$.

If π_1 and π_2 are the projections onto the first and second components of $G \times \widehat{G}$, respectively, we can define the Fourier transform as

$$f \longmapsto \pi_{2*}\big(\pi_1^* f \cdot \underline{\chi}\big),$$

or replacing the pushforward with an integral under the Haar measure,

$$\widehat{f}(\chi) = \int_G f(g)\chi(g)\,\mathrm{d}g.$$

In the case $G = \mathbb{R} = \widehat{G}$, this specializes to the usual Fourier transform.

One important takeaway is that this pullback-pushforward formalism applies in many other situations. For example, if X and Y are finite sets, so $F(X)$ denotes the complex-valued functions on X, a $K \in F(X \times Y)$ is an integral kernel, in the sense that we can define a transform $f \mapsto \pi_{2*}(\pi_1^* f \cdot K)$ (where, once again, π_1 and π_2 are the canonical projections). Identifying K with an $|X| \times |Y|$ matrix, this function is just multiplying vectors in $F(X)$ by K. The identity matrix/transform corresponds to the kernel δ_Δ, which is 1 at every diagonal element of $X \times X$ and 0 everywhere else.

We also mentioned that the Fourier transform exchanges convolution and pointwise product. This is an instance of an insight from last time: the Fourier transform is trying to diagonalize the action of a group. Differentiation, a type of infinitesimal translation, is also transformed into a multiplication operator, as is an ordinary translation.

For the rest of this lecture, we will not assume our groups are abelian; we'll return to abelian groups later.

Group algebras. Let G be a group and V be a representation of V; we don't ask for finiteness of G or $\dim V$. This representation is given by a homomorphism $\rho : G \to \mathrm{Aut}(V) \subset \mathrm{End}(V)$. If we pass to $\operatorname{End} V$, we have both multiplication and addition, so we can construct new operators not in the image of ρ. Specifically, we'll take finite linear combinations of $\rho(g_i)$ for various $g_i \in G$. As such, we may as well assume G is finite.

Let ω be a finitely supported measure in G, so

$$\omega = \sum_{g \in G} \omega_g \delta_g,$$

i.e. ω is a finite sum of δ-measures. This defines an endomorphism of V

$$\rho(\omega) = \sum_{g \in G} \omega_g \rho(g).$$

If we assume G is finite, so we may identify functions and measures, then the algebra of such endomorphisms is $\mathbb{C}G$, the associative algebra of functions on G,[6] with multiplication given by convolution, the unique map extending $\delta_g * \delta_h = \delta_{gh}$. Since these δ-measures span $\mathbb{C}G$, every representation V defines a homomorphism $(\mathbb{C}G, *) \to \operatorname{End} V$. The algebra $\mathbb{C}G$ is commutative iff G is abelian.

We can explicitly write down this convolution formula: if $\omega = \sum \omega(g)\delta_g$ and $\tau = \sum \tau(g)\delta_g$, then

$$\begin{aligned}(\omega * \tau)(g) &= \sum_{h \cdot k = g} \omega(h)\tau(k) \\ &= \sum_{h \in G} \omega(h)\tau(h^{-1}g).\end{aligned}$$

This looks a little more like the usual convolution: when $G = \mathbb{R}$, we can convolve L^1 functions by

$$(f * g)(x) = \int_{\mathbb{R}} f(y)g(x-y)\,\mathrm{d}y.$$

We can rethink this in terms of projections from $G \times G$.

$$\begin{tikzcd} & G \times G \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, "\mu"] & \\ G & & G \\ & G. & \end{tikzcd}$$

Here, $\mu : G \times G \to G$ is multiplication. We can define $\omega \boxtimes \tau = \pi_1^*\omega \cdot \pi_2^*\tau$; then, convolution arises from the pushforward $\omega * \tau = \mu_*(\omega \boxtimes \tau)$. Functions pull back, but measures push forward via integration, so this is really about measures, and we don't need to worry about whether G is finite (e.g. we just used $\mathbb{R}$).[7] In fact, all we need is for G to be a measurable group, so that $L^1(G)$ is an algebra under convolution.

If G is a locally compact topological group, then we can take the compactly supported continuous functions $C_c(G)$, which form an associative algebra under convolution. In this setting, $(C_c(G), *)$ is usually called the *group algebra*. This is the continuous analogue of taking linear combinations of group elements and multiplying them, "smearing out" the group multiplication. In this setting, our representations need to have some good notion of integration, so finite-dimensional or a locally convex topological vector space. The action of G on such a representation V induces a homomorphism $C_c(G) \to \operatorname{End}(V)$. In many settings, it's easier to think about modules for an algebra than representations of a group.

If you're coming from algebraic geometry, you might not like measures, as we don't in general know how to integrate/pushforward. In this case, we need to talk about functions. We still have a multiplication map $\mu : G \times G \to G$, so pullback defines a map $\mu^* : F(G) \to F(G \times G)$. There's also a map $F(G) \otimes F(G) \to F(G \times G)$; in nice settings, a Künneth theorem ensures this map is an isomorphism (in algebraic geometry, this is more or less a definition). In analysis, though, this won't literally be true unless you use a completed tensor product.

When this isomorphism holds, we get a map $\Delta : F(G) \to F(G) \otimes F(G)$, called the *coproduct map*. Just like an algebra is a vector space A together with an associative multiplication map $\mu : A \otimes A \to A$, a *coalgebra* is a vector space C together with a *coassociative* map $\Delta : C \to C \otimes C$.

[6] Really, these should be measures: we know the δ-measure at a point, but for infinite G it doesn't make sense to take constant functions.

[7] You might worry that measures can't pull back, but we know how to take the product measure on a product space, and that suffices.

What does it mean to be *co*associative? Abstractly, associativity means that the following diagram is a coequalizer diagram:

$$A \otimes A \otimes A \underset{1\otimes\mu}{\overset{\mu\otimes 1}{\rightrightarrows}} A \otimes A \xrightarrow{\mu} A.$$

That is, $\mu(a, \mu(b,c)) = \mu(\mu(a,b), \mu(c))$. Turn the arrows around, and coassociativity means the following diagram commutes (so is an equalizer diagram):

$$C \xrightarrow{\Delta} C \otimes C \underset{1\otimes\Delta}{\overset{\Delta\otimes 1}{\rightrightarrows}} C \otimes C \otimes C.$$

Since group multiplication is associative, multiplication in the group algebra is associative, or in the function setting, the coproduct in the coalgebra is coassociative. If you don't have a notion of measure, you'll have to work with coalgebras; in the settings we consider, there will be enough duality to not need to worry about this. Specifically, the dual of an algebra A is a coalgebra A^*, and measures go to functions, and vice versa.

Recall that if A is an algebra, then a module for A is a vector space V together with an action $A \otimes V \to V$ compatible with multiplication; if V is finite-dimensional, this is the same as a map $A \to \operatorname{End} V$. Correspondingly, if A^* is a coalgebra, we can define a *comodule* V, which is a space with a map $V \to V \otimes A^*$ compatible with the coproduct.

Matrix Coefficients. These are in a sense two ways of saying the same things, and we can talk about it in a third way: matrix coefficients, which is an extremely useful perspective on representations.

Let V be a finite-dimensional representation of a group G, and V^* be its dual. The *matrix elements* (or *matrix coefficients*) map $V \otimes V^* \to F(G)$ is defined to be the function $g \mapsto \langle w, v \cdot g\rangle$: we pair w and v after acting by g. Dualizing, this is also a map $V \to V \otimes F(G)$, putting w on the other side, or we can take $F(G)^* \otimes V \to V$. This first space is measures on G, hence the group algebra: this map is the usual action $\mathbb{C}G \otimes V \to V$, but thought of in a different way.

Why the name matrix coefficients? Let $\{e_i\}$ be a basis of V and $\{e^j\}$ be the dual basis for V^*. Then, the matrix coefficients map extends uniquely from the assignment $e_i \otimes e^j \mapsto \langle e^j, g \cdot e_i\rangle$, which is the ij^{th} entry of the matrix for g in this basis. Thus, this really is a matrix coefficients map, but stated coordinate-independently.

Definition 3.1. This allows us to define many classes of representations by way of what classes of functions they product under matrix coefficients.

- If G is a Lie group, a *smooth representation* is a representation whose matrix elements are smooth functions (so in $C^\infty(G)$).
- If G is a locally compact group, a *compactly supported representation* is one whose matrix elements are compactly supported continuous functions.
- If G is a Lie group, an *analytic representation* is one whose matrix coefficients are analytic functions.
- If G is a Lie group, a *tempered representation* is one whose matrix coefficients are L^2 functions.
- If G is an algebraic group, an *algebraic representation* is one whose matrix coefficients are algebraic functions.

If V is finite-dimensional, $\operatorname{End} V \cong V \otimes V^*$, and the identity endomorphism is a canonical element of this algebra (in any basis, this is the sum $\sum e_i e^i$). The matrix coefficient associated to this canonical element is called the *character* of V, denoted χ_V.

The matrix coefficients map is first of all a map of vector spaces, but there's a lot more structure: the G-action on V induces an action on V^*, so there is a $(G \times G)$-action on $V \otimes V^*$. Left and right multiplication[8] defines a $(G \times G)$-action on G, hence also on $F(G)$.

Exercise 3.2. Show that matrix coefficients is a $(G \times G)$-equivariant map $V \otimes V^* \to F(G)$.

This carries a lot of structure; for example, this defines a G-action through the diagonal $\Delta : G \to G \times G$: both vectors are acted upon in the same way, and G acts on $F(G)$ by conjugation (since the right action is multiplication by g^{-1}). Under the identification $V \otimes V^* \cong \operatorname{End} V$, this is also conjugation of matrices by those of G. Exercise 3.2 implies that matrix coefficients is invariant under conjugation, or in other words:

[8]In order for these to both be left actions, the right multiplication must be $v \mapsto v \cdot g^{-1}$. This is what we mean by right multiplication here.

Corollary 3.3. *The character of V is a class function: $\chi_V \in (F(G))^G$.*

One of the simplest things we can do with a representation is look at what it fixes.

Definition 3.4. Let V be a representation of G. Then, the *G-invariants* are $V^G = \{v \in V \mid g \cdot v = v \text{ for all } v \in G\}$.

We use the notation $F(G/G)$ for $F(G)^{G_\Delta} = \{f \in F(G) \mid f(h) = f(ghg^{-1}) \text{ for all } g \in G\}$.

Back to abelian groups. This allows us to reinterpret the Fourier transform; we once again suppose that G is a locally compact abelian group. In this case, any notion of the group algebra (compactly supported functions, integrable functions, or all functions in the finite case, or $\mathbb{C}\mathbb{Z} = C_c(\mathbb{Z})$) is commutative.

The fundamental instinct we owe to Gelfand and Grothendieck is that whenever we see a commutative algebra A, we should associate a space X to it, such that $A = F(X)$. This space is called the spectrum, and understanding it is akin to diagonalizing the algebra.

We defined the dual group $\widehat{G}$ by hand, but it turns out to arise naturally as the spectrum of the group algebra, in whichever sense we care about.

Example 3.5. Suppose G is finite, so there aren't many interesting examples, but a lot of different notions of group algebra agree. In fact, let's suppose $G \cong \mathbb{Z}/p$. In this case, $\mathbb{C}[\mathbb{Z}/p] = \bigoplus \mathbb{C} \cdot \chi_n$ for the characters $\chi_n : 1 \mapsto e^{2\pi i n/p}$ provides a basis. If we define $\chi_m \cdot \chi_n = \delta_{mn}$, then this assignment is an algebra isomorphism, and the characters are orthogonal idempotents.

That is, if we think of $\mathbb{C}G$ as functions on a set, then that set is really the set of characters, because the algebra of functions on it with pointwise multiplication is the same as $\mathbb{C}G$. The geometry of this space is a disjoint union of p points.

More generally, for any finite-dimensional G-representation V, we may decompose V as a direct sum of eigenspaces: $V \cong \bigoplus_n V_{\chi_n}$. Alternatively, V is a module for the algebra $\mathbb{C}G$. Over the set of points, the characters $\chi \in \widehat{G}$, we've produced a decomposition into a vector space over each point. That is, *representations of G are the same as vector bundles on $\widehat{G}$.*

This silly when G is finite, but we will apply it in more general situations, and it is our perspective on the Fourier transform. It sends characters to points, L^2 to L^2, and $\mathbb{C}G$ and convolution to $F(\widehat{G})$ under pointwise multiplication: we've diagonalized the action of G. We've only yet justified this in the finite setting, but we'll explain what it means in the infinite setting.

Example 3.6. If $G = \mathbb{Z}$, different notions of the group algebra coincide: $\mathbb{C}\mathbb{Z}$ is the algebra of finite $\mathbb{C}$-linear combinations of formal elements of $\mathbb{Z}$, with convolution for multiplication. This is isomorphic, as rings, to the Laurent polynomials $\mathbb{C}[z, z^{-1}]$, because $z^m \cdot z^n = z^{m+n}$. That is, the group algebra of $\mathbb{Z}$ is the algebra of algebraic (i.e. polynomial) functions on $\mathbb{C}^*$ (since we may invert 0 finitely many times); more succinctly, $\mathrm{Spec}(\mathbb{C}\mathbb{Z}, *) = \mathbb{C}^*$.[9] More generally, any representation of $\mathbb{Z}$ is the same as a module for $\mathbb{C}[z, z^{-1}]$. Later, we will learn to call these the fancy name of "quasicoherent sheaves on $\mathbb{C}^*$," which is the geometric content: this is a geometric structure akin to a vector bundle.

Example 3.7. Let $G = \mathrm{U}(1)$; we'll consider algebraic functions, so finite Fourier series. Here, we get $\mathbb{C}\mathrm{U}(1) \cong \mathbb{C}[z, z^{-1}]$ where $z = e^{2\pi i\theta}$, but the algebra structures are *different*: the convolution product is instead pointwise multiplication of finitely supported functions on $\mathbb{Z}$, so the dual to $\mathrm{U}(1)$ should be $\mathbb{Z}$. Thus, representations of the circle will correspond to vector bundles on the integers.

4. Spectral Decomposition: 9/1/16

Last time, we talked about group algebras; let's remind ourselves what happened.

When G is a finite group, we can define a convolution operation $F(G) \otimes F(G) \to F(G)$ defined by

$$f * g \longmapsto \mu_*(\pi_1^* f \cdot \pi_2^* g).$$

[9] We previously said that $\mathrm{U}(1)$ was dual to $\mathbb{Z}$, but now we obtained $\mathbb{C}^*$. The idea is that every $\lambda \in \mathbb{C}^*$ corresponds to a 1-dimensional $\mathbb{Z}$-representation, where 1 acts as multiplication by λ. Since λ and 1 are both invertible, this indeed defines a representation; but unless $|\lambda| = 1$, this is not a unitary representation. Indeed, if you require unitarity, you obtain $\widehat{\mathbb{Z}} = \mathrm{U}(1)$ as expected.

Here, μ, π_1, and π_2 all fit into the diagram

$$\begin{tikzcd} & G \times G \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, "\mu"] & \\ G & & G \\ & G. & \end{tikzcd}$$

This diagram looks suspiciously like a diagram for matrix multiplication

$$\begin{tikzcd} & X \times X \times X \arrow[dl, "\pi_{12}"'] \arrow[dr, "\pi_2 3"] \arrow[dd, "\pi_{13}"] & \\ X \times X & & X \times X \\ & X \times X, & \end{tikzcd}$$

where the matrix product is $\pi_{13*}(\pi_{12}^* f \cdot \pi_{23}^* g)$.

This is not a coincidence: the group algebra $\mathbb{C}G$, which is the algebra of complex-valued functions on G under convolution, maps into $\mathrm{End}(F(G))$. The action of G on itself by left multiplication defines a representation of G on $F(G)$, hence an action of $\mathbb{C}G$ on $F(G)$.

There's also an action of G on itself from the right. A nice general fact is that a map $T : G \to G$ that commutes with left action arises as a right action by an element of G: $T(gk) = gT(k)$, so T is right multiplication by $T(1)$, as $T(k) = k \cdot T(1)$.[10]

The greater point is that there's only one kind of algebra, which is matrix algebra. We can embed $\mathbb{C}G$ into the functions on $G \times G$ (matrices parameterized by G) invariant under the diagonal action of $g \in G$ acting by (g, g); this map is an isomorphism.

In other words, the group algebra $\mathbb{C}G$ is a subalgebra of the matrix algebra $F(G \times G)$. Inside of this matrix algebra are those elements which are invariant under the diagonal action of G (i.e. those which are conjugation-invatiant), and this is exactly $\mathbb{C}G$. Alternatively, $\mathbb{C}G$ is the operators on $F(G)$ that commute with the right action of G. The point of all these equivalences is that $\mathbb{C}G$ is not just an abstract algebra; it already comes with a pretty concrete representation.

Spectral decomposition. Back to where we were last time.

There are a lot of uses of the word "spectrum" in mathematics: the spectrum in analysis, the spectrum in graph theory, the spectrum in algebraic geometry, and even the spectrum in astrophysics all have the same origin, relating to the set or space of eigenvalues of something.[11]

Last time, we saw that for $G = \mathrm{U}(1)$, the group algebra of finite Fourier series $\mathbb{C}\mathrm{U}(1)_{\mathrm{fin}}$ with convolution is isomorphic to the finitely supported functions on $\mathbb{Z}$. If V is a finite-dimensional (or at least locally convex) vector space, a $\mathrm{U}(1)$-representation on V defines an action of $F_{\mathrm{fin}}(\mathbb{Z})$ on V. These functions are the span of the delta-functions δ_n for $n \in \mathbb{Z}$, which are orthogonal idempotents: $\delta_n \delta_m$ is 0 when $n \neq m$ and 1 when $n = m$. Thus, each δ_n acts as an idempotent operator on V, hence a projection onto $V_n = \mathrm{Im}(\delta_n)$; since the δ_n are orthogonal, so are the V_n, defining an orthogonal direct sum

$$V_{\mathrm{fin}} = \bigoplus_{n \in \mathbb{Z}} V_n \subset V. \tag{4.1}$$

The subspace V_n is the $e^{2\pi i n x}$-eigenspace of the action of $\mathrm{U}(1) \subset V$.

One of the corollaries of Fourier theory is that the inclusion (4.1) is dense: the closure of the span of the V_n is all of V. Why is this? The identity operator on V is an infinite combination

$$\sum_{n \in \mathbb{Z}} \delta_n = 1,$$

hence (after an analytic argument) the limit of finitely supported operators.

[10]This fact, and its proof, apply in more general situations, e.g. the action of a ring on itself by left and right actions.

[11]The usage of spectrum in homotopy theory appears to be unrelated.

One might call V_{fin} the space of U(1)-*finite vectors* of V: these are the vectors that are contained in finite-dimensional subrepresentations of V. This is a really nice discovery by Harish-Chandra: even for noncompact, nonabelian groups, it's possible to extract this algebraic core of the representation.

So to every $n \in \mathbb{Z}$, we've associated a vector space V_n over n. Taking these all at once, we've defined a vector bundle over $\mathbb{Z}$: $\mathbb{Z}$ is discrete, so the dimensions can jump, but the key concept that representations turn into vector bundles will be very important.

Example 4.2. Suppose $G = \mathbb{R}$ acts on a finite-dimensional vector space, or the action is differentiable. In this case, there's an action of $\operatorname{Lie} G = \mathbb{R} \cdot \frac{\mathrm{d}}{\mathrm{d}x}$ on V: an $x \in \mathbb{R}$ acts by $e^{2\pi i x \cdot H}$, where $H : V \to V$ is the action of $\frac{\mathrm{d}}{\mathrm{d}x}$. Here, V is unitary iff H is self-adjoint. This will be very useful next week, when we talk about quantum mechanics.

We've derived data of a vector space V and a single operator H acting on V. A notion called *functional calculus* means we get an action of $\mathbb{C}[t]$ on V, where we think of $\mathbb{C}[t]$ as the algebraic functions on the affine line $\mathbb{A}^1$.[12] We can describe the action explicitly as

$$\sum a_i t^i \longmapsto \sum a_i H^i.$$

The point is, the action of an operator produces the action of a module in one variable. If we kept track of unitarity, then this restricts to an action of the fixed points under conjugation, which are the real affine line $\mathbb{A}^1_{\mathbb{R}}$.

Suppose we have an action of $\mathbb{C}[t]$ on a space V. How should we think of this geometrically? It will be something akin to a vector bundle on $\mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[t]$: specifically, it will be a quasicoherent sheaf. Over every $\lambda \in \mathbb{A}^1$, the fiber V_λ over λ is $V_\lambda/(t-\lambda)$. More formally, $V_\lambda = V \otimes_{\mathbb{C}[t]} \mathbb{C}$, where $\mathbb{C}[t]$ acts on $\mathbb{C}$ by $t \mapsto \lambda$. This is a quotient of V, sort of the "co-eigenspace" because we've quotiented by the λ-eigenspace V^λ; these V_λ fit together into a nice family.

Suppose $U \subset \mathbb{A}^1$ is open; to it, we may associate the vector space $\underline{V}(U) = V \otimes_{\mathbb{C}[t]} \mathbb{C}[U]$. Here, $\mathbb{C}[U]$ is the algebra of operators in $\mathbb{C}[t]$ where we invert those that don't vanish on U. Using this definition, $\underline{V}$ is a quasicoherent sheaf on $\mathbb{A}^1$.

Intuitively, a quasicoherent sheaf is data parametrized by the open subsets of X that satisfies a local-to-global principle.

Definition 4.3. Let X be a variety.[13] A *quasicoherent sheaf* $\mathscr{F}$ on X starts with the following data.

- For every open $U \subset X$, we obtain an abelian group (actually, $\mathscr{O}_X(U)$-module) $\mathscr{F}(U)$.
- For every inclusion of open subsets $V \subset U$, we have a *restriction map* $\rho^U_V : \mathscr{F}(U) \to \mathscr{F}(V)$.

We require it to satisfy the following conditions.

- Restriction maps should compose: a chain of inclusions $W \subset V \subset U$ implies $\rho^V_W \circ \rho^U_V = \rho^U_W$.
- If $\mathfrak{U}$ is an open cover of U, then $\mathscr{F}(U)$ must be determined by $\mathscr{F}(U_i)$ over all $U_i \in \mathfrak{U}$, in the same sense that U is determined by $\mathfrak{U}$. Specifically, the following diagram is a coequalizer diagram:

$$\coprod_{U_i, U_j \in \mathfrak{U}} U_i \cap U_j \rightrightarrows \coprod_{U_i \in \mathfrak{U}} U_i \longrightarrow U.$$

  Thus, we require the following diagram to be an equalizer diagram.

$$\mathscr{F}(U) \longrightarrow \prod_{U_i \in \mathfrak{U}} \mathscr{F}(U_i) \rightrightarrows \prod_{U_i, U_j \in \mathfrak{U}} \mathscr{F}(U_i \cap U_j).$$

- Quasicoherence means there's a semilocal-to-local condition: if $U \subset X$ is affine and $V \subset U$, then $\mathscr{F}(V) = \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{O}_X(V)$.

This is a large and perhaps confusing definition.

[12] More precisely, $\mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[t]$, the spectrum as it arises in algebraic geometry; as a set, $\operatorname{Spec} \mathbb{C}[t]$ is the set of prime ideals of $\mathbb{C}[t]$ with a topological and a geometric structure.

[13] For this class, a variety is something that has a structure sheaf, i.e. for every open $U \subset X$, we have a notion of *algebraic functions* $\mathscr{O}_X(U)$ that behaves well with respect to restriction and gluing.

If $\mathbb{C}[t]$ acts on a vector space V such that V is a finitely generated $\mathbb{C}[t]$-module, the fundamental theorem of finitely generated modules over a PID says that V is the direct sum of its torsion and its free parts, and the torsion part is

$$V_{\text{torsion}} = \bigoplus_{\lambda \in \mathbb{A}^1} V_{\widehat{\lambda}}.$$

where $V_{\widehat{\lambda}}$ is the generalized eigenspace

$$V_{\widehat{\lambda}} = \{v \in V \mid (t-\lambda)^N v = 0\} = \bigcup_{n \geq 1} V_{n\lambda},$$

where $V_{n\lambda}$ is the subspace of V annihilated by $(t-\lambda)^n$. These act by Jordan blocks: $V_{\widehat{\lambda}} \cong \mathbb{C}[t]/(t-\lambda)^n$.

This linear algebra becomes a picture for the spectral decomposition: for every eigenvalue $\lambda \in \mathbb{A}^1$, we have the space $V_{\widehat{\lambda}}$ over V, and these fit together into a quasicoherent sheaf on $\mathbb{A}^1$.

This is the algebraic perspective; let's see how it relates to analysis. Translations define an action of $\mathbb{R}$ on $C^\infty(\mathbb{R})$. The eigenvectors of differentiation are $e^{2\pi i\lambda x}$, since these satisfy $\frac{\mathrm{d}}{\mathrm{d}x} f = (2\pi i\lambda)f$. On the other side, these are Fourier-transformed into the eigenvectors for multiplication, δ_n. These don't live in $L^2(\mathbb{R})$, however.

The generalized eigenvectors for λ acting on distributions include the derivative of δ_λ: the action of t on $\operatorname{span}\{\delta_\lambda, \delta'_\lambda\}$ is a Jordan block. In the same way,

$$\ker\left(\frac{\mathrm{d}}{\mathrm{d}x} - \lambda\right)^2 = \{e^{2\pi i\lambda x}, xe^{2\pi i\lambda x}\},$$

and the action of $\frac{\mathrm{d}}{\mathrm{d}x}$ is a Jordan block in this basis. This is a nontrivial extension of the one-dimensional $\mathbb{R}$-representation $\operatorname{span} e^{2\pi i\lambda x}$ by itself:

$$0 \longrightarrow \operatorname{span} e^{2\pi i\lambda x} \longrightarrow \operatorname{span}\{e^{2\pi i\lambda x}, xe^{2\pi i\lambda x}\} \longrightarrow \operatorname{span} e^{2\pi i\lambda x} \longrightarrow 0.$$

But this does not happen in the world of unitary representations: the existence of an inner product means it's possible by Maschke's theorem to find a complement to any subrepresentation. Thus, all extensions split, and all representations are completely decomposable.

This adds a new entry to the dictionary between the representation theory of G and the geometry of $\widehat{G}$:

- A representation of G goes to a sheaf on $\widehat{G}$.
- Characters on G correspond to points on $\widehat{G}$.
- Extensions of representations correspond to infinitesimals.
- G is compact iff $\widehat{G}$ is discrete. (In this case, there are no nontrivial extensions, and no nontrivial infinitesimals.)

Algebraic geometry in general replaces $\mathbb{C}[t]$ by any commutative ring R, whose representations correspond to any number of commuting operators with specified relations. A representation of R on V will define a quasicoherent sheaf on $\operatorname{Spec} R$, spreading out the representation as a family of vector spaces that algebraic geometry can help us understand. In other words, whenever we have commuting operators, we find geometry.

This whole story applies in several different geometric settings.

- If we start with an algebraic variety, we care about polynomial functions, which form a ring. Spectral theory produces a quasicoherent sheaf (modules in an affine setting).
- If we start with a locally compact topological space, we care about compactly supported continuous functions, which form a C^*-algebra.[14] Spectral theory produces a Hilbert C^*-module.[15] Geometrically, this is the same as a vector bundle with a metric (a fiberwise inner product).
- If X is a measure space, we can talk about essentially bounded functions $L^\infty(X)$, which form something called a *von Neumann algebra*. Spectral theory produces a *projection-valued measure.*

$\mathbb{R}$ has all of these structures, so we can see how different kinds of representations of $\mathbb{R}$ spread out in completely different ways.

Next time, we'll talk about quantum mechanics, which uses all of these rows.

[14]A *C^*-algebra* is a Banach $\mathbb{C}$-algebra (so a $\mathbb{C}$-algebra that is compatibly a Banach space) with a $\mathbb{C}$-antilinear involution $*$ such that $(xy)^* = y^*x^*$ and $\|x\|^2 = \|xx^*\|$. This is an algebraic axiomatization of the notion of continuous complex-valued functions, with $*$ given by pointwise complex conjugation.

[15]This is a Hilbert space that is also a *C^*-module*: a continuous action of the C^*-algebra along with the action of $*$.

5. Projection-Valued Measures: 9/6/16

To a group G we associated a group algebra $\mathbb{C}G$, but this is actually the same notion in different clothes: they all agree in the finite case ($\mathbb{C}G$ is the algebra of functions on G), but in the infinite case, we have different notions. For topological groups, we consider compactly supported continuous functions $C_c(G)$; for measurable functions we consider $L^1(G)$; and for Lie groups we consider $L^2(G)$, all with the convolution product.

In each case, we restrict to "nice" representations in accordance with this specific notion of group algebra and specific kind of group. For example, we consider unitary representations of a locally compact topological group, and in accordance with the C^*-algebra structure on $C_c(G)$, unitary representations may be interpreted as Hilbert C^*-modules, with an inner product valued in $C_c(G)$.

If G is commutative, then any notion of the group algebra $\mathbb{C}G$ is a commutative algebra. The general idea of spectral theory, as espoused by Gelfand and Grothendieck, is that a commutative algebra should be though of as an algebra of functions on a space under pointwise multiplication. There's a dictionary between the algebra and the geometry:

- Starting with a ring,[16] one can obtain an affine algebraic variety X, and from an algebraic variety X, one obtains its ring $\mathbb{C}[X]$ of algebraic functions.
- If we start with a C^*-algebra, we can associate it to a locally compact Hausdorff topological space X, called its spectrum, and recover it as the C^*-algebra of compactly supported continuous $\mathbb{C}$-valued functions on X.
- A von Neumann algebra determines a measure space X, and we can recover it as $L^\infty(X)$.

These three algebraic notions — rings, C^*-algebras, and von Neumann algebras — all come in commutative and noncommutative versions, but here we must restrict to the commutative case.

The key theorem of spectral theory is that this dictionary is an equivalence; since functions pull back, this will be a contravariant equivalence.

- For example, commutative rings are equivalent to (the opposite category of) affine schemes.
- The Gelfand-Neimark theorem says that a commutative C^*-algebra A determines and is determined by a locally compact Hausdorff space $\operatorname{mSpec} A$ called the *Gelfand spectrum*. This spectrum may be characterized as the representations of A, the maximal ideals of A, or the C^*-homomorphisms $(A, *) \to (\mathbb{C}, *)$.
- Similarly, a commutative von Neumann algebra is the same thing as a meausrable space. But isomorphism of measure spaces (*measurable equivalence*) in this context is weak enough that there are only five classes: a finite discrete set $\{1, \ldots, n\}$, an infinite discrete set $\mathbb{N}$, an interval $[0,1]$, an union $[0,1] \cup \{1, \ldots, n\}$, and a union $[0,1] \cup \mathbb{N}$.

So in every case, a commutative group has a commutative group algebra, to which we associate a space $\widehat{G}$, dual to the group in the sense of Fourier theory. The structure on G induces a certain structure on $\widehat{G}$, and the point is that spectral decomposition (again, in a slightly different form in each setting) turns a representation of a commutative G into a module for $\mathbb{C}G$, which is turned into a family of vector spaces over the spectrum $\widehat{G} = \operatorname{Spec} \mathbb{C}G$. As a simple example, if G is finite, $\widehat{G}$ is a finite set, and the family of vector spaces is the joint eigenspaces for all $a \in \mathbb{C}G$.

In each setting, the term "family of vector spaces" means something different. In the algebraic setting, so that $\mathbb{C}G$ is a variety or scheme, then a family of vector spaces is a quasicoherent sheaf (the same thing as a module, in the quasicoherent setting); if G is a locally compact group, the family of vector spaces is a vector bundle on $\widehat{G}$ with a nondegenerate inner product. Finally, in the measurable setting, we obtain something called a projection-valued measure. In each case, the fibers are eigenspaces for the actions of group elements.

Let's elaborate on the situation for projection-valued measures.

Definition 5.1. Let X be a measure space, so we have a σ-algebra M of measurable sets. A *projection-valued measure* is the data of a fixed Hilbert space $\mathscr{H}$ and, for every measurable $U \subset X$, a self-adjoint projection $\pi(U)$ on $\mathscr{H}$, obeying a countable additivity axiom: for all $v, w \in \mathscr{H}$, the function $U \mapsto \langle w, \pi(U)v \rangle \in \mathbb{C}$ should be a complex measure.

[16] Technically, to obtain an algebraic variety, we should start with a finitely-generated reduced $\mathbb{C}$-algebra.

You should think of this as a "measuable family of Hilbert spaces" over X. If $v \in \mathscr{H}$ is a unit vector, $U \mapsto \langle v, \pi(U)v \rangle$ is real and in $[0,1]$, so is a probability measure. Another consequence is that if E and F are disjoint, measurable subsets of X, then their sections $\pi(E)$ and $\pi(F)$ are orthogonal; more generally, $\pi(E) \circ \pi(F) = \pi(E \cap F)$, a version of the presheaf property.

To every measurable characteristic function 1_X we've assigned the projector $\pi(U)$, and this extends to a map $L^\infty(X) \to \mathcal{B}(\mathscr{H})$, the space of bounded operators on $\mathscr{H}$. This allows for a very general formulation of the spectral theorem.

Theorem 5.2 (Spectral theorem (von Neumann))**.** *Let A be a self-adjoint operator acting on a Hilbert space $\mathscr{H}$. Then, there exists a projection-valued measure π on $\mathbb{R}$, supported on the spectrum of A, and such that*

$$A = \int_{\mathbb{R}} x \, \mathrm{d}\pi,$$

i.e. for all $w, v \in \mathscr{H}$,

$$\langle w, A \cdot v \rangle = \int_{\mathbb{R}} \langle w, \pi(A)v \rangle.$$

Note that we're not assuming A is bounded. If it is, though, then this says that A acts as multiplication by x in this measure.

Another way to say this is that an $A \in \operatorname{End} V$ (here V is a vector space) defines a $\mathbb{C}[x]$-module structure on V, where A is the action by x. We've sheafifed this action by making it into a projection-valued measure: $\pi(\lambda)$ is the projection onto the λ-eigenspace.

In the common case where $\mathscr{H} = L^2(\mathbb{R})$, we want to understand $A = \frac{\mathrm{d}}{\mathrm{d}x}$, but this is unbounded; then, the projection-valued measure on $\mathscr{H}$ identifies $\mathscr{H} = L^2(\mathbb{R}_t)$, the Fourier dual to $\mathbb{R}$: $\frac{\mathrm{d}}{\mathrm{d}x}$ acts as a translation by t.

Every $L^2(X)$ carries a natural projection-valued measure over X, where $\pi(U)$ is the projection onto $L^2(U) \subset L^2(X)$. This is akin to the structure sheaf $\mathscr{O}_X$, but in an analytic section. This allows us to reword the spectral theorem: to every self-adjoint operator, we can attach a measure space $X = \operatorname{Spec} A \subset \mathbb{R}$.

The greater point is that sheafification isn't specific to algebraic geometry; it's naturally associated to Fourier theory in any setting, continuous, measurable, differentiable, or algebraic. The fact that Hilbert C^*-modules are associated with vector bundles on X is the beginning of the story of K-theory of C^*-algebras.

All of this was in the setting where A was a commutative algebra. The point of noncommutative geometry is to do this for noncommutative algebras: a noncommutative ring should correspond to some sort of noncommutative algebraic variety; a noncommutative C^*-algebra should define a "noncommutative topological space," and a noncommutative von Neumann algebra (now there's a wealth of good examples) should define a "noncommutative measure space." We might not know what the points of these spaces, but we know what their vector bundles (sheaves, etc.) are, and can compute interesting topological invariants, including K-theory.

If G is a noncommutative group, the dual doesn't always have a nice geometric structure, so we have to be craftier. But we'll still use this dictionary as a guiding philosophy.

Quantum mechanics. Though this feels like a radical transition, quantum mechanics has a lot of similar ingredients: a lot of this spectral theory was developed in order to study quantum mechanics; representation theory has also been used to study quantum mechanics. Since a major point of this class is to reverse the arrow (use ideas from quantum mechanics to understand representation theory), let's briefly discuss quantum mechanics.

The basic data in a quantum-mechanical system is a Hilbert space $\mathscr{H}$, called the *space of states* and a self-adjoint operator H on $\mathscr{H}$, called the *Hamiltonian.* A *pure state* is a nonzero vector in $\mathscr{H}$ up to rescaling (or normalizing), so really an element of $\mathbb{P}\mathscr{H}$, the projective space.

The state space defines time evolution: for any real number T, there's a unitary operator

$$U_T = e^{-iTH/\hbar} \in U(\mathscr{H}).$$

In a sense, we've rotated by 90° to convert a self-adjoint operator into a unitary one. The assignment $U : T \mapsto U_T$ gives a unitary representation of $\mathbb{R}$ on $\mathscr{H}$. The corresponding Lie algebra generator is H, the infinitesimal time evolution.

Let $\mathscr{H}_\lambda$ be the λ-eigenspace of H. Then, we can write that

$$\mathscr{H} = \int^{\oplus} \mathscr{H}_\lambda \, \mathrm{d}_\lambda.$$

Here, the integral denotes a completed direct sum, taking the closure (internally) or completion (abstractly) of the algebraic direct sum of these eigenspaces. The projection $\pi(\lambda) : \mathscr{H} \to \mathscr{H}_\lambda$ is a projection-valued measure on $\mathscr{H}$ over $\mathbb{R}$. If ψ is an *eigenstate* (an eigenvector of H), then $H\psi = \lambda\psi$, so $U_T\psi \in \mathbb{C}\psi$. That is, this state is stationary.

More generally, we can write down time evolution as $\psi_T = U_T\psi$, or

$$-i\hbar\frac{\mathrm{d}\psi(t)}{\mathrm{d}t} = H\psi.$$

This is called the *Schrödinger equation.*

Example 5.3. The main example of a quantum system starts with a Riemannian manifold M (which might as well be standard Euclidean space $\mathbb{R}^3$), $\mathscr{H} = L^2(M)$, and $H = (1/2)\Delta_M$, the Laplacian.

What do we do with this? We want to observe things. The things we observe will be called *observables*, and are the self-adjoint operators on X. On $L^2(\mathbb{R}^3)$, for example, there are two classes of natural operators:

- Multiplication by the i^{th} coordinate defines an unbounded operator $X_i = (x_i \cdot -)$; these are called *position operators.*
- Differentiation in the i^{th} direction is also unbounded: $P_i = -i\hbar\frac{\partial}{\partial x_i}$; these are called *momentum operators.*

Next time, we'll explain how to take measurements (expected values) of an operator $\mathcal{O}$ in a state ψ, which will be $\langle\psi, \mathcal{O}\cdot\psi\rangle \in \mathbb{R}$.

The quantumness of quantum mechanics is that self-adjoint operators are a noncommutative algebra.

6. Quantum Mechanics: 9/8/16

Since we talked about quantum mechanics yesterday, we'll step back and do a lightning review of classical mechanics today.

Classical mechanics starts with a *symplectic manifold* X, i.e. a manifold with a closed nondegenerate 2-form. For example, the *phase space* of a particle moving on a manifold M is the symplectic manifold T^*M with coordinates (q_i, p_i): the q_i coordinates correspond to position coordinates on M and the p_i are momenta in the cotangent direction.

A *pure state* is a point of T^*M, and the *observables* are functions on T^*M, e.g. the coordinates q_i (what is the position of the particle?) and p_i (what is the momentum?). In order to understand the time evolution of this system, we specify a particular observable $H : T^*M \to \mathbb{R}$ called the *Hamiltonian*, such that

$$\frac{\mathrm{d}}{\mathrm{d}t} = \{H, -\}.$$

This is an instance of a more general construction that obtains vector fields from functions on a symplectic manifold: given such a function f, its derivative $\mathrm{d}f$ is a 1-form, and pairing with ω turns this into a vector field called $\{f, -\}$. Time evolution follows the equations

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i} \qquad\qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q_i}.$$

If we have only kinetic energy, and no potential energy, a typical Hamiltonian is

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2,$$

which means $\frac{\mathrm{d}q_i}{\mathrm{d}t} = p_i$ (momentum determines where the particle goes) and $\frac{\mathrm{d}p_i}{\mathrm{d}t} = 0$ (momentum is conserved). If there's some potential energy (e.g. a height function), these equations change.

The analogy with quantum mechanics passes through statistical mechanics, where points are replaced with probability measures on T^*M. In the discrete case, we take a convex combination of points

$$\mu = \sum_{i=1}^{m} \lambda_i \delta_{x_i},$$

with $\lambda_i \geq 0$ summing to 1, so that $\int \mathrm{d}\mu = \sum \lambda_i = 1$. This can be generalized to continuous probability measures: the goal is to replace points with "clouds" specifying where particles are likely to be. The dynamics of this system allow this measure to evolve.

In this probabilistic setting, we need to evaluate functions on measures. One choice is to take the expected value

$$E[f] = \sum \lambda_i f(x_i)$$

(replacing the sum with an integral in the continuous setting), but we could also push the measure forward and obtain a measure on $\mathbb{R}$: $f_*\mu$ is a measure on $\mathbb{R}$ corresponding to all the possible values of f, weighted by likelihood. In the discrete case, this is

$$f_*\mu = \sum_i \lambda_i \delta_{f(x_i)}.$$

The total integral is $E[f]$. This is a nice way to measure measures, but isn't terribly sophisticated: we've collapsed the measure on all of spacetime to $\mathbb{R}$. Don't take this pushforward too seriously.

From this perspective, perhaps quantum mechanics isn't so radical: we don't have points, and focus on the probability measures, but not that much else changes. We start with a Hilbert space $\mathscr{H}$; the *pure states* are vectors $\psi \in \mathscr{H}$, often in *ket notation* $|\psi\rangle$. An *observable* $\mathcal{O}$ is a self-adjoint operator on $\mathscr{H}$, and its *expectation value* on the pure state ψ is

$$\langle \mathcal{O} \rangle_\psi = \frac{\langle \psi \mid \mathcal{O} \mid \psi \rangle}{\langle \psi \mid \psi \rangle} \in \mathbb{C}.$$

That is, we take the inner product weighted by $\mathcal{O}$ and then normalize by $\|\psi\|$.[17] If $\mathcal{O}$ has eigenvectors φ_i with corresponding eigenvalues a_i, then

$$|\psi\rangle = \sum_i \langle \varphi \mid \psi \rangle |\varphi_i\rangle,$$

and therefore the expectation value is

$$\langle \mathcal{O} \rangle_\psi = \sum_i \langle \varphi_i \mid \psi \rangle a_i. \tag{6.1}$$

That is, we understand the expectation value (roughly what we're expecting to observe in state ψ) through the observable's eigenvectors, as long as $\mathcal{O}$ has a discrete spectrum.[18]

The a_i are more or less outcomes, and $\langle \varphi_i \mid \psi \rangle$ are the probabilties: measuring at an eigenvector gives me a pure outcome.

Just as in statistical mechanics, we can recover a measure on $\mathbb{R}$ recording the different outcomes with their weights. This is exactly what we were doing last time: associated to a self-adjoint operator $\mathcal{O}$ on $\mathscr{H}$, we defined a projection-valued measure $\pi_\mathcal{O}$ on $\mathbb{R}$. This is the abstract version of diagonalization, even if we have a continuous spectrum. At a state ψ, this is the measure $U \mapsto \langle \psi, \pi_\mathcal{O}(U) \cdot \psi \rangle \in \mathbb{C}$: over a measurable $U \subset \mathbb{R}$, the image of $\pi_\mathcal{O}(U)$ is the part of $\mathscr{H}$ on which the spectrum of $\mathcal{O}$ lies in U, which generalizes (6.1).

Now, if you ask what you're actually measuring in quantum mechanics, things get confusing, and relate to experimental physics and measurement theory; we're not going to worry too much about that.

Unlike in classical mechanics, the algebra of operators is noncommutative. But if we do have a family R of commuting algebras (which form something like a C^*-algebra or a von Neumann algebra), then they have a common spectrum $M = \operatorname{Spec} R$, and we obtain a probability measure on this space from the state ψ.

For an example, consider a free particle in $\mathbb{R}^3$; classically, we'd use the state space $T^*\mathbb{R}^3$, but the quantum state space is $\mathscr{H} = L^2(\mathbb{R}^3)$. Translation defines *position operators* X_1, X_2, X_3 that act on $\mathscr{H}$ and commute; similarly, we have *momentum operators* $P_i = \frac{\partial}{\partial x_i}$, which commute, but unlike the position operators, these aren't diagonalized already. The momentum eigenvectors are $e^{\langle t, x \rangle}$ for $t \in (\mathbb{R}^3)^*$. Diagonalizing this identifies $\mathscr{H} \cong L^2((\mathbb{R}^3)^*)$, which is exactly the Fourier transform on $\mathbb{R}^3$. This is the *momentum space* picture, or "wave" picture. But we can't simultaneously diagonalize the P_i and the X_i, as they don't commute. This is

[17] At some point, we'll need to explain this notation.

[18] One counterintuitive aspect of functional analysis is that the spectrum is not the set of eigenvalues, but rather the set where the operator $A - \lambda I$ isn't boundedly invertible. Not every point in the spectrum is an eigenvalue, and some operators, e.g. $\frac{\mathrm{d}}{\mathrm{d}x}$, don't have any eigenvectors (as the exponentials aren't L^2), just a continuous spectrum.

the well-known Heisenberg uncertainty principle: since these don't commute, you can't accurately measure both position and momentum at the same time.

One can define the *ring of differential operators* to be $\mathcal{D} = \langle x_i, \frac{\mathrm{d}}{\mathrm{d}x_i} \rangle$ with the relation $[\frac{\mathrm{d}}{\mathrm{d}x_i}, x_j] = \hbar\delta_{ij}$. These act on $L^2(\mathbb{R}^3)$: $\frac{\mathrm{d}}{\mathrm{d}x_i}$ acts as P_i and x_j acts as X_j. Sometimes this is called the *noncommutative cotangent bundle*, but there's no space realizing this noncommutative algebra of functions.

We once again have a distinguished operator called the *Hamiltonian*, which dictates time evolution: the time operator is $U_T = e^{-iTH/\hbar}$. Instead of just talking about expectation operators, we can make measurements of an operator $\mathcal{O}$ somewhere in the middle, at time T_1. That is, we start with ψ, evolve by T_1, then act by $\mathcal{O}$, then let some more time pass: the final correlation function is $\langle \psi \mid U_{T_2}\mathcal{O}U_{T_1} \mid \psi\rangle$. More generally, we can start with a state ψ_{in} and end with a state ψ_{out}, and have a bunch of operators $\mathcal{O}_i$ at times t_i. Then, the *correlation function* of all of these different measurements is

$$\langle \psi_{\mathrm{out}} \mid U_{t_f - t_n}\mathcal{O}_n U_{t_n - t_{n-1}} \cdots U_{t_3 - t_2}\mathcal{O}_2 U_{t_2 - t_1}\mathcal{O}_1 U_{t_1} \mid \psi_{\mathrm{out}}\rangle.$$

This should be read from right to left.

If the operator $\mathcal{O}$ commutes with the Hamiltonian, then it's time-independent (or it doesn't evolve): it's called a *conservation law* or *conserved quantity*. This is because $\langle \psi \mid U_{T_2}\mathcal{O}U_{T_1} \mid \psi\rangle = \langle \psi(T_1 + T_2) \mid \mathcal{O}\psi\rangle$, but $\psi(T_1 + T_2) = U_{T_1+T_2}\psi$.[19]

A *symmetry* of the system is a unitary operator G on $\mathscr{H}$ that preserves the Hamiltonian. For example, if M is a Riemannian manifold, $\mathscr{H} = L^2(M)$, and H is the Laplacian Δ (e.g. on $\mathbb{R}^3$ we take

$$\frac{1}{2}\sum_{i=1}^{3}\left(\frac{\partial}{\partial x_i}\right)^2,$$

which is the usual Laplacian), then we're considering a free particle with kinetic energy, but no potential energy. If G acts on M by isometries, it defines a symmetry of the quantum-mechanical system, since it acts on L^2-functions preserving the inner product and the Laplacian is a coordinate-invariant notion. Another good basic example is the action of SO_3 on $L^2(S^2)$, which commutes with the *spherical Laplacian* Δ_s, the radial part of the Laplacian in $\mathbb{R}^3$.

Passing to the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, a $\xi \in \mathfrak{g}$ defines a self-adjoint operator on $\mathscr{H}$. In a sense, natural observables come from symmetries of the space. The natural example is the free particle: $\mathbb{R}^3$ acts on $L^2(\mathbb{R}^3)$ by translations, and the Lie algebra version is the infinitesimal translations $P_i = -i\hbar\frac{\partial}{\partial x_i}$ (the $-i$ was inserted to make it self-adjoint). These P_i commute with the Hamiltonian, which tells us that momentum is conserved! By contrast, X_i doesn't commute with H, so position is conserved.

Maybe that's not terribly exciting, but there are more exciting symmetries: $\mathrm{SO}_2 \cong \mathrm{U}(1)$ acts on $\mathbb{R}^3$ by rotations around a fixed axis. The derivative of this action is a single operator, the *angular momentum operator*

$$-i\hbar\left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right).$$

This is the key insight behind Noether's theorem: a symmetry of the system corresponds to a conserved quantity, and vice versa.

Next time, we'll understand what happens when we set $H = 0$; symmetries reduce to group representations, which is a useful perspective on group representations.

References

[1] Dinakar Ramakrishnan and Robert J. Valenza. *Fourier Analysis on Number Fields*, volume 186 of *Graduate Texts in Mathematics*. Springer New York, 1999.

[19] This is the origin of the following joke: Two roommates work in a laboratory outside of Hamilton, Ontario. Steve, a regular fellow, and Gork, a literal caveman. He puts on a button-down shirt and tie every day in an attempt to fit in, but he just can't stop being a knuckle-dragging caveman (albeit in a lab coat).

After several years of working there, some of Gork's coworkers are talking during a coffee break. "Gork strikes me as really weird," said one man, "He's been here at the lab for like 6 years, and he never really developed any manners. I figured he would be civilized by now."

Another coworker explains "You really can't expect him to evolve. He commutes with the Hamiltonian."