TOPOLOGICAL AND GEOMETRIC METHODS IN QFT

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These notes were taken at the NSF-CBMS conference on topological and geometric Methods in QFT at Montana State University in summer 2017. Most of the lectures were given by Dan Freed. I live-TEXed these notes using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to `a.debray@math.utexas.edu`; any mistakes in the notes are my own.

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Day 1. July 31

1. Dan Freed: Bordism and TFT

"Quantization is an art, not a functor."

The first lecture will be about topology, specifically bordism; we'll talk about the grand plan near the end.

Definition 1.1. Let Y_0 and Y_1 be closed d-manifolds. Then, Y_0 and Y_1 are *bordant* if there exists a compact $(d+1)$-manifold X such that $\partial X = Y_0 \amalg Y_1$.

`placeholder`

Figure 1. A bordism between $(S^1)^{\amalg 3}$ and $(S^1)^{\amalg 2}$.

The empty set is a manifold of any dimension, and the disc is a bordism between S^1 and $\emptyset$.

Bordism is an equivalence relation: reflexivity and symmetry are apparent, and transitivity comes from gluing. The set of equivalence classes is a group under disjoint union, denoted Ω_d and called the *bordism group* of d-dimensional manifolds.

The idea of bordism dates back to Poincaré, who tried to use it to define a homology theory of maps of manifolds into a space. He ended up using simplicies, and we got the homology we're familiar with.

Example 1.2. In dimension 0, a single point is not cobordant to an empty set. This comes from one of the most basic theorems in differential topology, that a compact 1-manifold has an even number of boundary points. However, two points are cobordant to an empty set, so the number of points mod 2 defines an isomorphism $\Omega_0 \to \mathbb{Z}/2$. $\blacktriangleleft$

Example 1.3. It's also true that $\Omega_2 \cong \mathbb{Z}/2$. The complete invariant is a nice exercise in differential topology à la Guilleman and Pollack: let $\operatorname{Det} TY$ denote the determinant line bundle of the tangent bundle of Y and s be a section of $\operatorname{Det} TY$ transverse to the zero section. If $S := s^{-1}(0)$, then S is a codimension-1 submanifold of Y, and the mod-2 intersection number of S with itself defines an isomorphism $\Omega_2 \cong \mathbb{Z}/2$. ◀

Definition 1.4. A *bordism invariant* is a homomorphism $\Omega_d \to \mathbb{Z}$.

You can replace $\mathbb{Z}$ with other abelian groups, as we did above in Examples 1.2 and 1.3.

Example 1.5.

(1) One can consider bordism of oriented manifolds, with oriented cobordisms between them. This is again an abelian group, denoted $\Omega_d(\mathrm{SO})$. If $d = 4k$, the signature of the intersection pairing defines a bordism invariant $\Omega_{4k}(\mathrm{SO}) \to \mathbb{Z}$.

(2) Manifolds with a U_n-structure (we'll discuss these and other structures in a little bit) form a cobordism group called $\Omega_d(\mathrm{U})$. The Todd genus $\mathrm{td} : \Omega_{2k}(\mathrm{U}) \to \mathbb{Z}$ is a bordism invariant.

(3) Spin manifolds have an $\hat{A}$-genus $\hat{A} \colon \Omega_{4k}(\mathrm{Spin}) \to \mathbb{Z}$. ◀

The systematic investigation of genera and bordism invariants was undertakenm by Hirzebruch. Notice that the bordism invariants $\operatorname{Hom}(\Omega_d, \mathbb{Z})$ is an abelian group.

We'll now do something called categorification, a specific example of a process that adds additional structure to things: sets or vector spaces are replaced with categories, and functions with functors. Throughout this lecture (and following lectures), let $n := d + 1$.

Definition 1.6. The *bordism category* $\mathsf{Bord}_{\langle n-1,n\rangle}$ is the symmetric monoidal category specified by the following data.

- The objects are closed $(n-1)$-manifolds.
- The hom-set $\mathsf{Bord}_{\langle n-1,n\rangle}(Y_0, Y_1)$ is the set of diffeomorphism classes of bordisms $Y_0 \to Y_1$.
- Composition is gluing of bordisms.
- The identity $\mathrm{id}_Y : Y \to Y$ is the cylinder $Y \times [0, 1]$.
- The monoidal product is disjoint union.
- The monoidal unit is the empty set, regarded as an $(n-1)$-manifold.

There are many ways to think of categories, some more philosophical than others; we're in the business of treating them as algebraic structures like groups or rings. You might imagine a bunch of points with arrows between them. But unlike when we defined bordism groups, these bordisms now have a direction: each bordism X comes with a locally constant function $\partial X \to \{0, 1\}$ choosing which boundary components are incoming and outgoing. Gluing must glue the outgoing component of one bordism to the incoming component of the other. Thus you might imagine each $(n-1)$-manifold M to have a *collar*, a neighborhood of it in these cobordisms diffeomorphic to $M \times [0, 1)$, and cobordisms should respect this collar. You can think of this collar as an infinitesimal thickening in the direction of cobordisms.

We can apply the monoidal product (disjoint union) to both objects and morphisms. It's symmetric, meaning that there's a natural isomorphism $M \amalg N \cong N \amalg M$, which is the maximally symmetric tensor structure one can apply in this case. It's the categorification of the fact that Ω_d is an abelian group.

Our central definition is the categorification of Definition 1.4. We also need a categorification of $\mathbb{Z}$, and we choose $\mathsf{Vect}_{\mathbb{C}}$, the category of complex vector spaces and linear maps, and we'll choose $\otimes$ to be the monoidal structure (you could also choose $\oplus$, but we will not). The "decategorification" from $\mathsf{Vect}_{\mathbb{C}}$ to $\mathbb{Z}$ is the dimension.

Definition 1.7 (Atiyah [Ati88])**.** A *topological field theory* (TFT) is a symmetric monoidal functor

$$F \colon \mathsf{Bord}_{\langle n-1,n\rangle} \longrightarrow (\mathsf{Vect}_{\mathbb{C}}, \otimes).$$

You could ask whether the bordism invariants we discussed lift; that they're integer-valued is an interesting hint, which Atiyah and Segal wondered about (leading to Dirac operators and all sorts of wonderful geometry). You may be wondering where the physics is, given the physics-sounding name of a topological field theory. We'll certainly get there.

The definition of a topological field theory is relatively new, stemming from attempty to understand Chern-Simons theory and related phenomena in the 1980s. As such, it's not as set in stone as other mathematical definitions, and we'll certainly consider variants along the way. So maybe it's better to think of Atiyah's definition as an axiom system, rather than a complete mathematical characterization of physical phenomena.

Topological field theories have stringent finiteness condition.

Definition 1.8. Let C be a symmetric monoidal category and $y \in \mathsf{C}$. *Duality data* for y is a triple $(y^\vee, e, c)$, where $y^\vee \in \mathsf{C}$ and $c\colon 1 \to y \otimes y^\vee$ and $e\colon y^\vee \otimes y \to 1$ are C-morphisms satisfying axioms called the *S-diagrams*. y is *dualizable* if it has duality data; then, $y^\vee$ is called its *dual*, e is called *evaluation*, and c is called *coevaluation*.

In $\mathsf{Vect}_{\mathbb{C}}$, $Y^\vee$ is the usual vector-space dual $\mathrm{Hom}(Y, \mathbb{C})$: evaluation applies a functional to a vector, and its adjoint is coevaluation. But this can only be written as a finite sum of basis vectors if Y is finite-dimensional. Thus a vector space is dualizable iff it's finite-dimensional.

Lemma 1.9. *Every object in $\mathsf{Bord}_{\langle n-1,n\rangle}$ is dualizable.*

Corollary 1.10. *Since a symmetric monoidal functor sends dualizable objects to dualizable ones, $F(Y)$ is a finite-dimensional vector space for any closed manifold Y and TFT F.*

Proof sketch of Lemma 1.9. Let Y be a closed $(n-1)$-manifold and $Y^\vee := Y$. Then, evaluation will be the "outgoing cylinder" $Y \amalg Y \to \emptyset$, and coevaluation is the "incoming cylinder" $\emptyset \to Y \amalg Y$, and these satisfy the necessary axioms. ⊠

`placeholder`

FIGURE 2. The evaluation and coevaluation morphisms in $\mathsf{Bord}_{\langle n-1,n\rangle}$.

That the state spaces are finite-dimensional is striking, and certainly not true for quantum mechanics and quantum field theory in general. So to get to physics we're going to have to leave the purely topological world.

There are many examples, some in Dan's lecture notes.

Example 1.11 (Finite gauge theory [DW90, FQ93])**.** Fix a finite group G, which we'll call the *gauge group* of this theory. Let $\mathrm{Bun}_G(S)$ denote the groupoid of principal G-bundles on a space S; that is, principal G-bundles on S form a category, but all morphisms are invertible. Since G is finite, these are Galois covering spaces of S with covering group G. You can imagine a groupoid with dots and arrows again, but this time every arrow is double-headed.

How should we turn this into a field theory? Principal G-bundles pull back, so given a cobordism $X\colon Y_0 \to Y_1$, we obtain a correspondence diagram

$$\begin{tikzcd} & \mathrm{Bun}_G(X) \arrow[dl, "s"'] \arrow[dr, "t"] & \\ \mathrm{Bun}_G(Y_0) & & \mathrm{Bun}_G(Y_1). \end{tikzcd}$$

This is highly nonlinear, yet a TFT is a linear thing. We'll linearize it by taking functions: if $\mathcal{G}$ is a groupoid, $\mathsf{Fun}_{\mathbb{C}}(\mathcal{G})$ denote the vector space of complex-valued functions on the set of isomorphism classes of $\mathcal{G}$. Since X, Y_0, and Y_1 are compact, their groupoids of principal G-bundles have finitely many isomorphism classes of objects, so we can both pull functions back and push them forward (summing over the fibers), hence defining a linear map

$$t_* \circ s^*\colon \mathsf{Fun}_{\mathbb{C}}(\mathrm{Bun}_G(Y_0)) \longrightarrow \mathsf{Fun}_{\mathbb{C}}(\mathrm{Bun}_G(Y_1)).$$

Thus we obtain a functor F_G, assigning Bun_G to objects and this push-pull formula to morphisms. To a closed n-manifold X (a bordism from $\emptyset$ to itself), we obtain the number $F_G(X) = \#\,\mathrm{Bun}_G(X)$, summing over the groupoid of bundles — but this is a groupoid, not a set, so we have to weight by the number of automorphism groups:

$$F_G(X) = \#\,\mathrm{Bun}(X) = \sum_{[P]\in\pi_0 \mathrm{Bun}_G(X)} \frac{1}{\#\,\mathrm{Aut}(P)}.$$

This already models the physical case: the principal G-bundles are examples of *fluctuating fields*, introduced to define the theory but summed over. The groupoid sum is a simple example of the path integral! ◂

The category of TFTs in dimension n, denoted $\mathsf{TFT}_n \coloneqq \mathrm{Hom}^{\otimes}(\mathsf{Bord}_{\langle n-1,n\rangle}, \mathsf{Vect}_{\mathbb{C}})$, has a composition law that's done pointwise: $(F_1 \otimes F_2)(M) \coloneqq F_1(M) \otimes F_2(M)$, and similarly for bordisms. This will be useful when we try to classify TFTs, providing extra structure useful to us.

Tangential structures. We'll hear more about tangential structures from a geometric perspective later today. Right now, we'll adopt a more homotopical approach. We've just been talking about bare manifolds, but often one introduces additional structure: orientation, spin, and more. Tangential structures are a way to capture a large class of such structures (broadly, the topological ones).

The tangent bundle of an n-manifold M defines a classifying map $M \to B\mathrm{GL}_n(\mathbb{R})$, which lifts to a pullback

$$\begin{tikzcd} TM \arrow[r] \arrow[d] & W_n \arrow[d] \\ M \arrow[r] & B\mathrm{GL}_n(\mathbb{R}). \end{tikzcd}$$

To define a tangential structure, we'll consider Lie group homomorphisms $\rho_n \colon H_n \to \mathrm{GL}_n(\mathbb{R})$ (e.g. inclusion of SO_n, projection down from Spin_n, and so forth). This lifts to a map $B\rho_n \colon BH_n \to B\mathrm{GL}_n(\mathbb{R})$. An H_n*-structure* is a lift of the classifying map

$$\begin{tikzcd} & & B\rho_n^* W_n \arrow[dl] \arrow[d] \\ TM \arrow[r] \arrow[d] \arrow[urr, dashed] & W_n \arrow[d] & BH_n \arrow[dl, "B\rho_n"] \\ M \arrow[r] \arrow[urr, dashed] & B\mathrm{GL}_n(\mathbb{R}). & \end{tikzcd} \tag{1.12}$$

For example, an SO_n-structure is the same thing as an orientation. You will have to reconcile this definition with the more familiar, geometric one.

Hence we have a general definition of what we need.

Definition 1.13. A *tangential structure* is a fibration $\rho\colon \mathcal{X}_n \to B\mathrm{GL}_n(\mathbb{R})$. An $\mathcal{X}_n$-structure on an n-manifold M is a lift of the classifying map along ρ as in (1.12).

For example, an orientation is specified by the map $B\mathrm{SO}_n \to B\mathrm{GL}_n(\mathbb{R})$, and if $\mathcal{X}_n = B\mathrm{GL}_n(\mathbb{R}) \times S$, you get cobordism of manifolds with a map to S.

Path of future lectures.

(1) Bordism and TFT, as we just saw.
(2) Quantum mechanics
(3) An axiom system for Wick-rotated quantum field theory.
(4) Another advantage of axiom systems is they allow you to consider classification theorems.
(5) We'll expand to variations on Definition 1.7, including in particular an extended notion of locality.
(6) Invertibility in TFT, and hence some stable homotopy theory.
(7) The Wick-rotated analogue of unitarity
(8) Extended positivity for invertible TFTs
(9) Non-topological invertible theories
(10) Computations for some electron systems in condensed-matter physics.

We're roughly following the material in [FH16], which will also be useful to keep in mind throughout the week.

2. Dave Morrison: Geometry and Physics: An Overview

"The most powerful method of advance that can be suggested at present is to employ all the resources of pure mathematics in attempts to perfect and generalize the mathematical formalism that forms the existing basis of theoretical physics, and after each success in this direction, to try to interpret the new mathematical features in terms of physical entities."
– Paul Dirac

The title is an impossibly large topic to tackle in an hour, but we'll do what we can to introduce the interaction between geometry, topology, and physics in its modern form. It will be impressionistic and historical.

Maxwell's equations for electricity and magnetism are beautifully symmetric between electricity and magnetism — almost. We add a source term for the electricity term, an electron. But we don't for magnetism, because experimentalists have not discovered a magnetic analogue, a hypothetical *magnetic monopole*.

Dirac's monopoles. In 1931, Dirac asked, what if there was a magnetic monopole m? As an electrically charged particle moves in the presence of a magnetic monpole, there's a singularity if the path hits the monopole, and otherwise is locally constant, but can depend on the path. In particular, if two paths π_1 and π_2 differ only by going different ways around m, the difference in their actions is $I_2 - I_1 = \hbar eg$. In particular, if the particle travels in a loop ℓ, the action depends on the winding number $n(\ell)$ of the loop:

$$I_\ell = n(\ell)\hbar eg.$$

This is a topological invariant, and a discrete one: we exponentiate $e^{2\pi i I_\ell}$, hence $\hbar eg \in \mathbb{Z}$! This is the first instance of topology appearing in physics.

Dirac thought of this in a surprisingly prescient way, chopping up the integral into a lot of little pieces and integrating over paths, long before the notion of a path integral was ever dreamed of.

Interlude. The beginning of quantum field theory, as discovered by Schwinger, Dyson, Feynman, and Tomonaga, was understood reasonably well from the physical perspective, but they weren't able to put it on mathematical foundations. This was particularly true for Feynman's formalism of the path integral. Impressively, the theoretical methods they developed anyways managed to agree with experiment to a stunning degree of accuracy, coming to a zenith in quantum electrodynamics (QED).

As such, the physicists drifted away from mathematics: they couldn't and didn't use math to shore up their theoretical physics, and didn't need to in order to get amazingly accurate results. They abandoned Dirac's manifesto, and in a sense math and physics divorced until the 1970s.

Yang-Mills theory. Around the 1950s, Yang and Mills wrote down nonabelian gauge theory to understand elementary particles with nonabelian gauge symmetry (e.g. SU_2 or SU_3). This wasn't taken so seriously at first; it took an approach different from the S-matrix philosophy popular at the time. This lasted until about the 1970s, when t' Hooft and others quantized it and managed to make it predictive of the experiments coming from particle accelerators. This began the shift in popularity from the S-matrix-dominant perspective to the prevalence of gauge theory that exists today.

Gauge theory is the quantum theory of principal G-bundles and connections. Mathematicians had also been working on these, but in parallel, and so produced different words for the same concepts.[1] In the 1970s, Simons and Yang were both at Stony Brook, and realized after talking to each other that they had such different words for the same concepts, leading to a paper [WY75] of Wu and Yang that was a dictionary between the two fields!

The Atiyah-Singer index theorem. A third interesting interaction between geometry and physics is the Atiyah-Singer index theorem from the early 1960s. This was all developed in and with mathematics: principal G-bundles, characteristic classes, Dirac operators on manifolds, and more.

The physicists and mathematicians were brought together again by the theory of Yang-Mills instantons. For a Lie group G, one considers a principal G-bundle on a 4-manifold M and its curvature F. Then, one can take the Lie-algebra-valued trace: one is interested in the spaces of solutions related to

$$\mathcal{L} = \int_M \mathrm{tr}(F \wedge (\star F)). \tag{2.1}$$

To understand this properly, one need to understand both the mathematical and physical phenomena behind it. There's also interplay between Euclidean and Minkowski signature — one important input is action-minimizing solutions to Euclidean Yang-Mills in $\mathbb{R}^4$ that either vanish at infinity or have bounded growth of some sort.

[1] Both sets of words are still in vogue, even though the mathematicians and physicists are talking to each other again.

The ADHM construction. Atiyah, Drinfel'd, Hitchin, and Manin [AHDM78], four mathematicians, found all of the solutions for $G = \mathrm{SU}_2$. This is impressive on its own, but they used some surprisingly fancy mathematics (Penrose's twistor transform and some algebraic geometry) that was previously not known to be connected to physics. Subsequently, Atiyah gave the Loeb lectures in the Harvard physics department, and this was big news: a mathematician was using geometry to talk to physicists! Even though the Harvard math and physics buildings were near each other, there hadn't been a lot of discussion between the two departments at the time, barring some more traditional mathematical study of PDEs arising in physics.

One surprising fact about these solutions is that even though we want the solutions to be strongly controlled at infinity, the connection does not need to be. You can get a topological invariant called the *instanton number* from the degree of a map from a large S^3 in $\mathbb{R}^4$ to $\mathrm{SU}_2 \cong S^3$. Since $\pi_3(\mathrm{SU}_2) = \pi_3(S^3) \cong \mathbb{Z}$, the homotopy class of this map, written as an integer k, is called the instanton number of the solution. You can also compute it geometrically:

$$8\pi k = \int \mathrm{tr}(F \wedge F).$$

ADHM constructed solutions with arbitrary instanton number.

Since the Lagrangian (2.1) looks very similar to $k/8\pi$, and for a 2-form F, $\star F = \pm F$, you could ask whether your solutions are *self-dual* ($\star F = F$) or *anti-self-dual* ($\star F = -F$). It turns out there's always a decompositioin

$$F = F_{\mathrm{sd}} + F_{\mathrm{asd}},$$

and

$$\begin{aligned} \|F\|^2 &= \|F_{\mathrm{sd}}\|^2 + \|F_{\mathrm{asd}}\|^2 \\ 8\pi^2 k &= \|F_{\mathrm{sd}}\|^2 - \|F_{\mathrm{asd}}\|^2, \end{aligned}$$

so the minimal-action solutions are either self-dual or anti-self-dual.

Anomalies. The next interaction between physics and mathematics arose in the study of anomalies. These are symmetries of the field theory that do not preserve the integration measure in the path integral. The fields are sections of some bundle built from the tangent bundle or spinor bundles (for fermionic theories), or self-dual fields. But in the case of spinor bundles, anomalies popped up.

This led to a question which looks very mathematical: suppose we have a bundle $E \to M \times S^1$, which we can understand as using a symmetry of M to glue $M \times [0, 1]$. Choose a B such that $\partial B = M \times S^1$, and we want to extend this structure to B. The anomaly ends up stated in terms of characteristic classes and invariant polynomials of this structure on B. There are specific steps which determine how this acts on the measure, and if they don't vanish, the symmetry of the classical theory is not a symmetry of the quantum theory, and you have an anomaly. This is okay, but there are some where you really need the symmetry to be present at the quantum level, and for these checking the anomaly is an important and useful tool. This differential-geometric perspective on manipulations of the path integral is due to Zumino and collaborators.

In a spinor theory, matter is essentially a section of a spinor bundle tensored with a gauge bundle. Hence it's potentially subject to an anomaly, but one of the remarkable early discoveries in this field is that the anomaly cancels. When people generalized to supersymmetry, this anomaly vanishes for trivial reasons, and has interesting ramifications on 12-manifolds for the type IIB theory. This leads to the famous Green-Schwarz mechanism. In string theory, there are other ways for the anomalies to cancel.

Donaldson's work on Yang-Mills. The ADHM construction works on $\mathbb{R}^4$ and S^4; Donaldson generalized it to arbitrary compact 4-manifolds to produce remarkable results in topology. This is in some ways the opposite to Dirac's manifesto, taking physics and using it to understand mathematics. At least topology, this was probably the first time understanding flowed in that direction.

In 1988, Witten [Wit88] found a physical interpretation of Donaldson's solutions, but strangely, it didn't depend on the metric, leading to the definition of a topological field theory. From the perspective of something like quantum gravity, the absence of metric dependence is crazy, but it has been extremely useful. With more physics input, Seiberg and Witten took a new approach to the Donaldson-Witten TQFT [SW94a, SW94b] which has made some of the computations more straightforward.

These days, there's also the large overlap between the mathematics and physics of topological phases of matter, kicked off by Haldane and Wen's work. Wen was a string theorist before he did condensed-matter, which is probably where he picked up the perspective of geometric methods.

This ping-pong between math and physics is a great perspective to adopt, and hopefully future research in this area will continue to use input from math to understand physics and physics to understand math.

3. Dan Freed: An axiomatic system for quantum mechanics

First, Dan encouraged all of us to look at the notes he posts online: they contain lots more examples of TFTs, and exercises that will probably generate interesting discussion.

Axiom systems for quantum mechanics have been considered for a long time, starting with Dirac, but mathematicial physicists have considered myriad variations on these axioms. The ones we consider will be useful for considerations on Wick rotation that we'll see in later lectures.

We start with a Riemannian manifold (M, g) together with a potential function $V\colon M \to \mathbb{R}$. This at least seems to model a single particle moving on M, but if, e.g. $M = (\mathbb{R}^n)^k$, this system tracks k particles moving in $\mathbb{R}^n$.

We also have time $\mathbb{M}^1$, which is an affine space modeled on the Euclidean line $\mathbb{E}^1$.[2]

The *Lagrangian* of the system is a density representing the total energy of the system: if we let the system evolve from t_0 to t_1, we get a map $\phi\colon \mathbb{M}^1 \to M$ encoding the trajectory of the particle, and the Lagrangian is

$$L = \left(\frac{1}{2}|\dot{\phi}|^2 - \phi^* V\right)|\mathrm{d}t|.$$

From this we derive both classical and quantum physics. Classically, we apply the Euler-Lagrange equations (which in this case reduce to Newton's equations of motion) to determine which geodesics are permitted, leading to the solution space $\mathcal{N} \subset \mathrm{Map}(\mathbb{M}^1, M)$, which obtains a symplectic form from the Lagrangian density.

Quantum mechanics does something different, integrating over the trajectories. There's a space $\mathcal{S}$ of *states*, which are points of $\mathcal{N}$, or more generally probability distributions on $\mathcal{N}$. There's also a space $\mathcal{O}$ of observables. In general, $\mathcal{S}$ is a convex set containing the *pure states* $\mathcal{S}_0$ (the probability distributions concentrated at a point); the rest are called *mixed states*. The observables $\mathcal{O}$ form a complex vector space with a real structure, and in the same way that $\mathcal{N}$ acquires a symplectic form, $\mathcal{O}$ contains a Lie algebra $\mathcal{O}^\infty$; the bracket is called the *Poisson bracket*.

There will also be a particular observable $H \in \mathcal{O}^\infty_{\mathbb{R}}$ called the *Hamiltonian*. Observing an observable in a given state defines a map from $\mathcal{O}_{\mathbb{R}} \times \mathcal{S}$ to the space of probability measures on $\mathbb{R}$. One can take the expected value of such a measure, and this is the expected or average value of that observable in that stete. Moreover, the Hamiltonian defines a semigroup of automorphisms of $\mathcal{S}$ and $\mathcal{O}$, which describes the time evolution of this system. There are different perspectives on this, some of which are dual (e.g. the Heisenberg picture vs. the Schrödinger picture).

It turns out that, with this mathematical data, $\mathcal{O}$ is also an associative algebra, even a Poisson algebra, but there doesn't seem to be physical meaning to the multiplication. It's more helpful to think of $\mathcal{O}$ as a vector bundle over $\mathbb{M}^1$; given A_{t_i} in $\mathcal{O}_{t_i}$ (the fiber over time t_i), one can form the correlator $\langle A_{t_1} \cdots A_{t_k}\rangle$, which is an important invariant, often with physical meaning.

Using this, we can formulate an axiom system.

Definition 3.1. A *quantum system* is the following data.

- A complex Hilbert space $\mathcal{H}$.
- The Hamiltonian, a self-adjoint operator $H\colon \mathcal{H} \to \mathcal{H}$.
- The space of pure states $\mathcal{S}_0 = \mathbb{P}\mathcal{H}$, and the space of mixed states

$$\mathcal{S} = \{\rho\colon \mathcal{H} \to \mathcal{H} \mid \rho \geq 0, \mathrm{tr}(\rho) = 1\}.$$

- The space of observables $\mathcal{O}_{\mathbb{R}}$, the self-adjoint operators on $\mathcal{H}$.
- Time evolution, a semigroup law

$$t \longmapsto U_t = e^{-itH/\hbar}\colon \mathcal{H} \longrightarrow \mathcal{H}.$$

[2] You might think the distinction between affine space and a vector space is fussy, but it's different to say "this lecture ends in an hour" and "this lecture ends at 1:00," especially since it ends at 3.

The observation map comes from von Neumann's spectral theorem: given a self-adjoint operator A, one obtains a projection-valued measure π_A on the line. Hence the map sends A and ρ to the probability measyre

$$E \subset \mathbb{R} \longmapsto \operatorname{tr}(\pi_A(E) \circ \rho).$$

With our Riemannian manifold (M, g) as above, you should think of $\mathcal{H} = L^2(M)$ and $H = \Delta_g$.

Example 3.2 (Toric code)**.** This example is relevant to what we'll be thinking about this week. It was introduced by Kitaev [Kit03], albeit not quite in this form. Throughout, d denotes the space dimension.

Let Y be a closed manifold with the structure as a finite CW complex, i.e. finite sets of i-cells Δ^i for each $0 \leq i \leq d$. Let Y^i denote the i-skeleton, the cells of dimensions at most i; then $Y^0 \subset Y^1 \subset \cdots$, and this is a filtration. Let Δ^i denote the set of i-cells of Y.

We'll consider the (discrete) groupoid of "relative principal G-bundles" $\mathrm{Bun}_G(Y^1, Y^0)$, pairs (P, s) where $P \to Y^1$ is a principal G-bundle and $s\colon Y^0 \to P|_{Y^0}$ is a section of P on the 0-skeleton. As a set, this is a product of copies of G indexed by the edges of Y.

Now we can incorporate this system into our axiomatic framework. The complex Hilbert space of states is actually finite-dimensional:

$$\mathcal{H} \coloneqq \mathrm{Map}(\mathrm{Bun}_G(Y^1, Y^0); \mathbb{C}) \cong \bigotimes_{e \in \Delta^1} \mathrm{Map}(G, \mathbb{C}).$$

The Hamiltonian is

$$H \coloneqq \sum_{v \in \Delta^0} H_v + \sum_{f \in \Delta^2} H_f,$$

where H_v and H_f are terms corresponding to 0- and 2-cells respectively: given a vertex v, let $\varphi_v\colon \mathrm{Bun}_G(Y^1, Y^0) \to \mathrm{Bun}_G(Y^1, Y^0)$ send $(P, s) \mapsto P(P, s_v)$, where

$$s_v(v') = \begin{cases} s(v), v \neq v' \\ 1 + s(v), v = v'. \end{cases}$$

Then,

$$H_v \psi \coloneqq \frac{1}{2}(\psi - \varphi_v^* \psi),$$

and

$$H_f \psi \coloneqq \mathrm{Hol}_{\partial f}(P) \cdot \psi.$$

That is, take the holonomy of P around the boundary of f, which is either -1 or 1, and multiply by that.

From this definition, it's evident that $\operatorname{Spec} H \subset \mathbb{Z}^{\geq 0}$. The space of ground states is $\mathcal{H}_0 = \mathrm{Map}(\mathrm{Bun}_G(Y); \mathbb{C})$. Why is this? We have a correspondence diagram

$$\mathrm{Bun}_G(Y^1, Y^0) \longrightarrow \mathrm{Bun}_G(Y^1) \longleftarrow \mathrm{Bun}_G(Y);$$

if $H_v \psi = 0$, then $\varphi_v^* \psi = \psi$, so ψ cannot depend on the value of the section s at v; dually, if $H_f \psi = 0$, then $\psi = 0$ on all bundles P which have nontrivial holonomy around f. Thus, requiring $H_v \psi = 0$ for all v pushes us forward to $\mathrm{Bun}_G(Y^1)$, and requiring $H_f \psi = 0$ pulls us back to $\mathrm{Bun})G(Y)$. ◀

Relativity tells us that certain approximations of these systems are the same: since $\hbar$ has units of ML^2/T, then low-energy behavior is the same thing as long-time behavior, and using the speed of light c, which has units of L/T, then this is also the same thing as long-range (long-distance). Much of the interesting qualitative behavior of the system (e.g. ergodicity) fits into one of these paradigms, so understanding this behavior (e.g. via the space of ground states) is important, and is something we'll see later this week. One suprising phenomenon is that, though the toric code depends strongly on the lattice, its space of ground states is a purely topological invariant. This is expected behavior of *gapped systems*, those whose Hamiltonians have a gap between their two smallest eigenvalues. Another example of a gapped system is a particle moving on a compact Riemannian manifold, using spectral theory of the Laplacian; compactness is necessary here.

We want to consider families of systems, e.g. for classifying them. This involves forming a moduli space, a space parameterizing geometric objects. Here's a simple example.

Example 3.3. Let V be a real, two-dimensional vector space, so that $\mathrm{Sym}^2 V^*$ is the space of symmetric bilinear forms $V \times V \to \mathbb{R}$. Such a form has a signature: there's a cone of forms with signature ± 2, and the rest have signature 0, along with some degenerate forms Δ. Thus, the moduli space of nondegenerate bilinear forms is $\mathcal{M} := \mathrm{Sym}^2 V^* \setminus \Delta$, and its set of connected components, also called the *deformation classes* for the original moduli problem, is given by the signature $\sigma\colon \pi_0\mathcal{M} \to \{-2, 0, 2\}$, and is a bijection. $\blacktriangleleft$

In general, you have to fix some discrete invariants: signature or Euler characteristic of a geometric object, dimension, etc.

We'll want to form a moduli space of quantum-mechanical systems and determine the deformation classes. In general, this is set up by fixing some data (e.g. dimension), then considering all systems and removing some singularities. The singularities are those where the Hamiltonians are gapless, and are *phase transitions* (exactly as in the phase transitions from ice to water to gas). There are two kinds: in a *first-order phase transition*, one of the eigenvalues is brought down to zero, but the spectrum is still discrete and even gapped: the dimension of the ground state jumps. In a *second-order phase transition*, the energy gap closes, and the ground state is part of the continuous spectrum. For water, all phase transitions are first-order except for the triple point, which is second-order.

So we throw out the phase transitions and, given a dimension d and a symmetry group I, we'd get a moduli space $\mathcal{M}(d, I)$ of lattice systems in dimension d with I-symmetry. We want to compute the set of deformation classes $\pi_0\mathcal{M}(d, I)$.

But there's a lot more to do yet — we haven't defined these lattice models, let alone the moduli space. More concretely, to attack this physical problem mathematically, we need to make a mathematical model F from it, and justify why we believe this is a good model for the physical problem. After this, we can prove theorems about F, then try to apply these theorems to the original problem.

Though we won't construct moduli space, we do get mathematical models and enough information to compute. The approach proceeds by producing a (not yet completely well-defined) map from $\mathcal{M}(d, I)$ to a moduli space of field theories $\mathcal{M}'(d+1, H)$, where H is some other symmetry group. This map is expected to exist for physical reasons, and we can use $\mathcal{M}'(d+1, H)$, which we understand better, to make progress on the original problem.

Wick rotation. Let's change gears a bit for the last few minutes.

Recall that time evolution defines for every point $t \in \mathbb{R}$ the unitary operator $U_t = e^{-itH/\hbar}$. Because the Hamiltonian H should be a positive definite operator, we can formally extend this to $\mathbb{C}_-$, the semigroup of complex numbers with nonpositive imaginary part. The function $t \mapsto e^{-it\lambda}$, $\lambda > 0$, conformally maps $\mathbb{C}_-$ into the (closed) unit disc. We end up with a holomorphic semigroup whose limit on the boundary is the unitary group, and it acts by "small" operators (in a sense that they're analytically easy to control). This is a problem-solving technique in much the same way that one uses contour integration to understand problems that are formulated entirely on the real line.

Now, if you look at the ray through $-i$, you get a real contracting semigroup $\tau \mapsto e^{-\tau H\hbar}$, whose "imaginary time" is easier to analytically understand. One might wonder whether restricting to imaginary time is sufficient to understand the system, and for quantum mechanics a little operator theory shows this to be the case. The axiom system we discuss in a few lectures uses Wick rotation in a crucial way.

Axioms for quantum mechanics. Let $\mathsf{Bord}_{\langle 0,1\rangle}(\mathrm{SO}^\nabla)$ be the bordism category of oriented Riemannian 0-manifolds (with collars), and $\mathsf{tVect}_{\mathbb{C}}$ be the category of complex topological vector spaces. Then, one could try to think of quantum mechanics as a symmetric monoidal functor

$$F\colon \mathsf{Bord}_{\langle 0,1\rangle}(\mathrm{SO}^\nabla) \longrightarrow \mathsf{tVect}_{\mathbb{C}}.$$

How do we see this? We want to send $\mathrm{pt} \mapsto \mathcal{H}$, and the interval $[a, b]$ to time evolution by $\tau = -i(b-a)$, which is $e^{-\tau H/\hbar}\colon \mathcal{H} \to \mathcal{H}$. The observables also have a geometric interpretation: to observe at x, cut out a small ball around x, producing a bordism starting at the S^0 around x. Hence we get something roughly like $\mathcal{H}^* \otimes \mathcal{H}$, and evaluation defines the observable. (There are some missing words here: we really should let the neighborhood of x shrink to 0 and take a limit, and think about distributions on $\mathcal{H}$.) More generally, to calculate a Wick-rotated correlation function, excise several points, producing maps from $\mathcal{H}^k \otimes (\mathcal{H}^*)^k \to \mathbb{C}$, which gives you the correlation function in question (modulo the same caveats).

We'll generalize this to arbitrary functions to get the story for Wick-rotated quantum field theory in general, and then go back to discuss the relativistic physics that underlines it. For a good reference on all this, see Segal's lectures on this material from about five years ago.

4. Robert Bryant: Symmetries and G-structures

"Sorry... that's the only physics joke I'll make."

The idea for this lecture is that there is a whole collection of geometric structures: complex, almost complex, symplectic, almost symplectic, CR, and more, and we can treat them in a unified way that extends what you've learned about Riemannian geometry. The idea is to look at local invariants and symmetry groups. This perspective was known to Cartan a century ago, but the examples are often newer.

Throughout this lecture, we'll consider geometric structures on an m-manifold M. It'll often be useful to have an auxiliary vector space $\mathfrak{m}$ around, which is a real m-dimensional vector space which we'll think of as a generic tangent space to M.

The bundle of *principal coframes* $\pi\colon \mathscr{F}_M(\mathfrak{m}) \to M$ is the bundle whose fiber at an $x \in M$ is the space of isomorphism $u\colon T_xX \to \mathfrak{m}$. This space is a right $\mathrm{GL}(\mathfrak{m})$-torsor (hence a $\mathrm{GL}_m(\mathbb{R})$-torsor), where if $A \in \mathrm{GL}(\mathfrak{m})$, $u \circ A = A^{-1} \circ u$, so for any Lie subgroup $H \subseteq \mathrm{GL}(\mathfrak{m})$, we can consider a subspace $B \to M$ which is a principal right H-subbundle. An *H-structure* is a section of the bundle $\mathscr{F}_M(\mathfrak{m})/H$.

This formalism captures many different kinds of geometric structures on manifolds.

Example 4.1. Let q be a quadratic form on $\mathfrak{m}$ and $H = \mathrm{O}(\mathfrak{m}, q)$, the orthogonal group preserving q. Then, a point in the coframe bundle $u\colon T_xM \to B$ that's in a principal H-subbundle determines and is determined by a nondegenerate, smoothly varying quadratic form on TM, i.e. a section of $\mathrm{Sym}^2(T^*M)$. Thus, an H-structure is a Riemannian metric. ◀

Example 4.2. Now suppose $J_0\colon \mathfrak{m} \to \mathfrak{m}$ is a complex structure on $\mathfrak{m}$. If we take $H = \mathrm{GL}(J_0, \mathfrak{m})$, then choosing an H-subbundle $H \to B \to M$ is equivalent to choosing an almost complex structure on M. ◀

Example 4.3. Similarly, if $\beta \in \Lambda^2(\mathfrak{m}^*)$ is nondegenerate, then letting $H = \mathrm{Sp}(\beta, \mathfrak{m})$ (the symplectic group preserving this form) we find that H-structures are symplectic structures on M. ◀

The assignment $M \to \mathscr{F}_M(\mathfrak{m})$ is functorial: for diffeomorphisms $f\colon M_1 \to M_2$, we get a map $f_*\colon \mathscr{F}_{M_1}(\mathfrak{m}) \to \mathscr{F}_{M_2}(\mathfrak{m})$ which sends $u \mapsto u \circ (f'(\pi_{M_1}(u)))^{-1}$. This generalizes to H-structures as long as f preserves the H-structure.

The purpose of this talk will be to show why this is interesting and useful. We won't really talk about when H-structures exist: there are topological obstructions, and most even-dimensional manifolds aren't almost complex or symplectic. For a given manifold, it's often not easy to determine when an almost complex structure integrates to a complex structure.

However, homogeneous spaces provide a family of examples with H-structures. Let P be a closed subgroup of a Lie group G and $\eta\colon TG \to \mathfrak{g}$ be the left-invariant 1-form such that $\eta_e = \mathrm{id}_{\mathfrak{g}}$. If $\pi\colon G \twoheadrightarrow G/P$ is the quotient map, then its derivative maps $T_gG \to T_{gP}(G/P)$, and we get a commutative diagram of short exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & T_g(gP) \arrow[r] \arrow[d, "\eta"] & T_gG \arrow[r] \arrow[d, "\eta"] & T_{gP}(G/P) \arrow[r] \arrow[d, "u(g)"] & 0 \\ 0 \arrow[r] & \mathfrak{p} \arrow[r] & \mathfrak{g} \arrow[r] & \mathfrak{g}/\mathfrak{p} = \mathfrak{m} \arrow[r] & 0. \end{tikzcd}$$

In this case, G/P has an H-structure, where $H = \mathrm{Ad}_{\mathfrak{g}/\mathfrak{p}}(P) \subset \mathrm{Aut}(\mathfrak{m})$.

Example 4.4. Let $G = \mathrm{SU}(n+1)/\mathrm{U}(n)$, where we map $\mathrm{U}(n) \to \mathrm{SU}(n+1)$ through the map

$$A \longmapsto \begin{pmatrix} (\det A)^{-1} & 0 \\ 0 & A \end{pmatrix}.$$

Let $P = \mathrm{U}(n)$. Then, $G/P = \mathbb{CP}^n$, though this isn't an injective map: the kernel of the embedding is $Z(\mathrm{SU}(n+1)) = \mathbb{Z}/(n+1)$. Hence there's a fiber bundle $G/(\mathbb{Z}/(n+1)) \to B \to \mathbb{CP}^n$, and $\pi_1(B) \cong \mathbb{Z}/(n+1)$. ◀

This is an example where H isn't a subgroup of $\mathrm{GL}(\mathfrak{m})$, though it is a covering group of a subgroup of $\mathrm{GL}(\mathfrak{m})$. One might call these extended H-structures $\widetilde{H} \to H \hookrightarrow \mathrm{GL}(\mathfrak{m})$, where the first map is a finite cover, and we have an $\widetilde{H}$-bundle $\widetilde{B} \to M$ together with a map $\varphi\colon \widetilde{B} \to B \hookrightarrow \mathscr{F}_M(\mathfrak{m})$, where again the first map is a finite cover.

Example 4.5. There are two common choices of $\widetilde{H}$ common in physics: $\mathrm{Spin}(\mathfrak{m})$, which is a double cover of $\mathrm{SO}(\mathfrak{m})$; and $\mathrm{Pin}^+(\mathfrak{m})$ and $\mathrm{Pin}^-(\mathfrak{m})$, which are double covers of $\mathrm{O}(\mathfrak{m})$. ⋖

You could use a compact Lie group fiber instead of a finite cover, and these are the more interesting cases, though a few things have to change. In general, using a finite cover at least doesn't really change this story with regards to calculating local symmetries or invariants.

Another fun example is $G = \mathrm{G}_2$ and $P = \mathrm{SU}_3$. In this case $G/P = S^6$, and you can use this to get an SU_3-structure on S^6. The inclusion $\mathrm{SU}_3 \hookrightarrow \mathrm{U}_3$ produces the standard almost complex strucure on S^6.

Distinguishing different H-structures locally. Though you might know how to do this for Riemannian geometry, we're going to talk about a uniform way to do this for all groups. The key topological information is the *soldering form*: if $\pi\colon \mathscr{F}_M(\mathfrak{m}) \to M$ is the projection map, then at a $u \in \pi^{-1}(x)$ in $\mathscr{F}_M(\mathfrak{m})$, then we're provided with an isomorphism $T_xX \to \mathfrak{m}$, so the projection map $T_u\mathscr{F}_M(\mathfrak{m}) \twoheadrightarrow T_xM$ defines a smoothly varying assignment to $\mathfrak{m}$, hence a smooth $\mathfrak{m}$-valued 1-form $\omega \in \Omega^1_{\mathscr{F}_M(\mathfrak{m})}(\mathfrak{m})$, called the soldering form. The same construction serves to define a soldering form ω_H for a manifold with H-structure.

Lemma 4.6. *Let $f\colon M_1 \to M_2$ be a diffeomorphism.*
1. *Then, $f^*(\omega_2) = \omega_1$, where ω_i is the soldering form on M_i.*
2. *If f is in addition an isomorphism of H-structures, then $f^*(\omega_{2,H}) = \omega_{1,H}$.*
3. *If H is connected, the converse to (2) is true.*

Example 4.7. Let $H = \{e\}$, corresponding to a parallelization. Then, $\pi\colon B \to M$ is a diffeomorphism, and so $\omega\colon TM \to \mathfrak{m}$ is preserved by a unique $C\colon M \to \mathfrak{m} \otimes \Lambda^2\mathfrak{m}^*$ satisfying

$$\mathrm{d}\omega = C(\omega \wedge \omega),$$

or in indices,

$$\mathrm{d}\omega^i = \frac{1}{2} C^i_{jk} \omega^j \wedge \omega^k.$$

That is, if $f\colon M \to M$ satisfies $f^*\omega = \omega$, then $f^*(C) = C$. You can also check that if C' satisfies $\mathrm{d}C = C'(\omega)$, then $f^*(C') = C'$. These relate to the codimension of the symmetry group: the number of independent such functions is equal to the codimension of the symmetry group. ⋖

This procedure allows you to discover what the symmetries of an H-structure are, which by Noether's theorem is a powerful tool for understanding conservation laws in physics.

So we've solved the problem in the trivial case. Great! Now we'll try to reduce nontrivial cases to the trivial cases, following Cartan.

Suppose we have $H \to B \to M$ and $\omega\colon TB \to \mathfrak{m}$ now has a kernel V, then vertical tangent space. Each fiber can be parallelized individually, because they're canonically identified with $\mathfrak{h}$ through left translation $\tau\colon V \to \mathfrak{h}$. So what we need to do is stitch these together into a form.

Definition 4.8. A *pseudo-connection* on B is an $\mathfrak{h}$-valued 1-form $\Theta\colon TB \to \mathfrak{h}$ such that over each $u \in B$, $\Theta|_{\ker(\omega_u)} = \tau_u$.

Cartan just calls this a connection, but because we haven't asked Θ to be H-equivariant, it's not quite what we're looking for.

Definition 4.9. A pseudo-connection Θ is a *connection* if $R_h^*(\Theta) = \mathrm{Ad}(h^{-1})(\Theta)$ for all $h \in H$. (Here R_h is right translation by h.)

This is the standard definition. But to make Cartan's algorithm work, we need to work with pseudo-connections (or restrict to semisimple groups).

First of all, pseudo-connections always exist (assuming M is paracompact and stuff like that), because connections always exist. But we don't just want some connection, we want one guaranteed to be preserved by our notion of equivalence, like C was in the framed case. This motivates us to write down the *structure equation* for a pseudo-connection Θ:

$$\mathrm{d}\omega = -\Theta \wedge \omega + C(\omega \wedge \omega), \tag{4.10}$$

where C depends on Θ. We want to find a way to choose Θ such that C is preserved by any isomorphism of H-manifolds. So if $\overline{\Theta} = \Theta - p\omega$ is some other pseudo-connection, where $p\colon B \to \mathfrak{h} \otimes \mathfrak{m}^*$ is any smooth map, then

$$\begin{aligned} -\overline{\Theta} \wedge \omega + \overline{C}(\omega \wedge \omega) &= -\Theta \wedge \omega + C(\omega \wedge \omega) \\ (\overline{C} - C)(\omega \wedge \omega) &= -(p\omega) \wedge \omega = (\delta p)(\omega \wedge \omega). \end{aligned}$$

To describe δ, observe that $\mathfrak{h} \subset \mathfrak{m} \otimes \mathfrak{m}^*$, and the composition

$$\mathfrak{h} \otimes \mathfrak{m} \hookrightarrow (\mathfrak{m} \otimes \mathfrak{m}^*) \otimes \mathfrak{m}^* \longrightarrow \mathfrak{m} \otimes \Lambda^2 \mathfrak{m}^*$$

is our δ. If $\mathfrak{h}^{(1)} := \ker(\delta)$ and $H^{0,2}(\mathfrak{h}) := \operatorname{coker}(\delta)$, then we obtain an exact sequence

$$0 \longrightarrow \mathfrak{h}^{(1)} \longrightarrow \mathfrak{h} \otimes \mathfrak{m}^* \xrightarrow{\delta} \mathfrak{m} \otimes \Lambda^2 \mathfrak{m}^* \longrightarrow H^{0,2}(\mathfrak{h}) \longrightarrow 0. \tag{4.11}$$

This is the key: the kernel and cokernel determine existence and uniqueness of connections satisfying the structure equation: the cokernel determines whether you can modify the pseudo-connection without modifying C, and the kernel controls existence.

Definition 4.12. The map $T\colon B \to H^{0,2}(\mathfrak{h})$ is called the *intrinsic torsion* of B.

For example, if $H = \mathrm{O}(n)$, then one can identify $\mathfrak{h} \subset \mathfrak{m}^* \otimes \mathfrak{m}^*$ with $\Lambda^2(\mathfrak{m}^*)$. This means δ is an isomorphism, so $\mathfrak{h}^{(1)}$ and $H^{0,2}(\mathfrak{h})$ vanish. This tells us something familiar.

Corollary 4.13 (Fundamental theorem of Riemannian geometry)**.** *On any Riemannian manifold (M, g), there exists a unique pseudo-connection Θ_0 such that $\mathrm{d}\omega = -\Theta_0 \wedge \omega$, and in fact Θ is a connection.*

Hence we get everything local in Riemannian geometry: (ω, Θ_0) is a canonical choice of coframings, and

$$\mathrm{d}\Theta_0 = -\Theta_0 \wedge \Theta_0 + R(\omega \wedge \omega),$$

for some $R\colon B \to \mathfrak{h} \otimes \Lambda^2 \mathfrak{m}^*$. This is more familiarly known as the Riemann curvature tensor.

Geometrically, T is the obstruction to being able to choose a flat H-structure to first order, i.e. the first derivatives that don't vanish under changes of coordinates. The second-order terms show up in R. Moreover, if a metric is flat to second-order at every point (so $R = 0$), then it's flat.

Example 4.14. Let $H = \mathrm{Sp}(\beta, \mathfrak{m})$, where β is a nondegenerate 2-form on $\mathfrak{m}$. This defines an isomorphism $\mathfrak{m} \cong \mathfrak{m}^*$ allowing us to lower indices, so we can define $\mathfrak{h}^\flat := \operatorname{Sym}^2 \mathfrak{m}^* \subset \mathfrak{m}^* \otimes \mathfrak{m}^*$. Hence δ is a map

$$\delta\colon \operatorname{Sym}^2(\mathfrak{m}^*) \otimes \mathfrak{m}^* \longrightarrow \mathfrak{m}^* \otimes \Lambda^2 \mathfrak{m}^*.$$

This is the exterior derivative of a degree-2 polynomial, which is linear. Your quadratic is the derivative of a cubic function, and so the kernel is $\mathfrak{h}^1 = \operatorname{Sym}^3(\mathfrak{m})$. The cokernel is $H^{0,2}(\mathfrak{m}) \cong \Lambda^3(\mathfrak{m}^*)$. So the obstruction to uniqueness is a 3-form, and the only one we have is $\mathrm{d}\beta$. Similarly, a nontrivial kernel means there's no way to choose a canonical connection. If β is closed, you can at least get a flat space, and Darboux's theorem offers a converse. This story is unusual: usually $\mathfrak{h}^{(1)} = 0$, and for semisimple groups, (4.11) splits equivariantly, so you can use this to choose canonical connections (e.g. for an almost Hermitian structure),[3] not just pseudo-connections, for most structures you will run across in real life.

In the symplectic case, $H^{1,2}(\mathfrak{m}) = 0$ implies $H^{*,2}(\beta) = 0$ for all orders: flatness to first order implies flatness to all orders. ◀

If you do this with a unitary group, you'll discover that it does not carry a unique connection.

If you do this with an extended H-structure $\widetilde{H} \to H$, then the invariants arise as pullbacks of those for H, and similarly for the coframe bundle.

Example 4.15. The simplest example where you need a pseudoconnection instead of a connection is in dimension 3 is the group of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & a & 1 \end{pmatrix}.$$

[3] There are in fact multiple choices for canonical connections in the Hermitian case; they're all functorial for diffeomorphisms. Two examples include the Chern connection and the Kähler connection.

This is an abelian group isomorphic to $\mathbb{R}^2$, but is not reductive as a subgroup of $\mathrm{GL}_3(\mathbb{R})$. In this case, $\mathfrak{h}^{(1)} = 0$, but (4.11) does not split equivariantly, so you get a unique pseudo-connection which is not a connection in general.

Geometrically, this structure is the structure of a full flag $0 \subsetneq L_1 \subsetneq L_2 \subsetneq TM$ plus an isomorphism $L_2/L_1 \cong TM/L_2$. There are connections which match the intrinsic torsion, but they're not canonical, unless the intrinsic torsion vanishes, which it does not always do. ◀

5. Question session

"I'm not going to say what a quantum field theory is. Maybe tomorrow."

After the lectures, we had a discussion/question section about things that confused us. Today, that was gauge theory, classical bordism invariants, the Euler TQFT, Wick rotation, particles and symmetry groups in quantum mechanics, and Lagrangians. People then reviewed several of these topics.

5.1. **Dan Freed: Classical bordism invariants and the Euler TQFT.** Any compact manifold Y has a well-defined Euler characteristic $\chi(Y)$. Is this a bordism invariant? Clearly not: the sphere (nonzero Euler characteristic) is bordant to an empty set (zero Euler characteristic). However, you can check that the mod 2 Euler characteristic does define a bordism invariant $\Omega_d \to \mathbb{Z}/2$. In general, this is not an isomorphism; in some dimensions, it's neither injective nor surjective. We might ask if it's possible to categorify this invariant into a TQFT.

Definition 5.1. The symmetric monoidal category of *super vector spaces* $\mathsf{sVect}_\mathbb{C}$ is the category given by the following data.

- The objects are complex vector spaces with a $\mathbb{Z}/2$-grading $V = V^0 \oplus V^1$. Equivalently, you could ask for an involution $\varepsilon\colon V \to V$; then, the $\mathbb{Z}/2$-grading comes from its (± 1)-eigenvalues.
- The maps are the even linear maps.
- The monoidal structure is the tensor product. But the symmetric monoidal structure uses the grading, the map $V \otimes V' \to V' \otimes V$ sending
$$v \otimes v' \longmapsto (-1)^{|v||v'|} v' \otimes v.$$
Here, $|v|$ is 0 if it's in V^0 and 1 if it's in V^1.

The *dimension* of a super-vector space is $\dim V :- \dim V^0 - \dim V^1$.[4]

Thus, a super-vector space is a categorification of the Euler characteristic: instead of just the ranks of homology groups, we remember the groups themselves, sorting them into odd and even pieces.

But we can also turn it into a TQFT.

Example 5.2. Let $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}^\times$. Then, there is an n-dimensional TQFT called the *Euler TQFT*
$$\varepsilon_\lambda\colon \mathsf{Bord}_{\langle n-1,n\rangle} \longrightarrow \mathsf{Vect}_\mathbb{C}$$
which assigns $\mathbb{C}$ to every $(n-1)$-manifold and to any cobordism X multiplication by $\lambda^{\chi(X)}$. This is a simple and important example of a TQFT, and is invertible, in that it factors through the maximal subgroupoid of $\mathsf{Vect}_\mathbb{C}$, called $\mathsf{Line}_\mathbb{C}$, the groupoid of complex lines and nonzero maps; we'll discuss invertible field theories more later.

This theory satisfies gluing (i.e. is a functor) because the Mayer-Vietoris formula controls how the Euler characteristic changes when you glue across a common boundary. ◀

If you try to generalize this to other bordism invariants, you'll run into a roadblock: the Euler characteristic can be easily defined on a manifold with boundary, but this is less true for things like the signature, Todd genus, etc.

You can also try to use other characteristic-class invariants. For example, you could define for a closed 2-manifold X, the action $\alpha(X) = (-1)^{\langle w_1(X)^2, [X]\rangle}$, but by the Wu formula $w_1^2 = w_2$, so this is again an Euler theory ε_{-1}!

But there are interesting theories. Consider an oriented 8-dimensional invertible theory $\alpha_\lambda\colon \mathsf{Bord}_{\langle 7,8\rangle}(\mathrm{SO}) \to \mathsf{Line}_\mathbb{C}$ which to a closed 8-manifold assigns the number
$$\alpha_\lambda(X) \coloneqq \lambda^{\langle p_1(X)^2 - 17 p_2(X), [X]\rangle},$$

[4] This agrees with the abstract notion of dimension in a symmetric monoidal category.

which comes from its signature.

5.2. **Andy Neitzke and Dave Morrison: What is gauge theory?** There's a whole mathematical subject called gauge theory, which roughly means anything related to principal G-bundles and equations on them. This was kicked off by the work of Donaldson. A typical problem is to study the moduli space of G-bundles on M with connection whose curvature F_∇ satisfies $F_\nabla = \star F_\nabla$.

In physics, gauge theory is a quantum field theory, which is in a sense a quantization of the above; we want to integrate over the principal G-bundles. The trick is, we need to mod out by gauge equivalence, but somehow this whole story is a little unphysical: a modern perspective on quantum field theory eschews the perspective of "quantize a classical field theory" as possibly missing some or having artifacts. The gauge equivalences come up in computations, but the gauge symmetry somehow isn't observable, and this is fundamental. In particular, there are theories which have two descriptions, one gauge-theoretic and one not, so the gauge-theoretic description cannot be invariant.

This was kicked off because in the past 10 or 15 years or so, people have found theories that we can't write down in terms of Lagrangians. These came from other kinds of physics (e.g. string theory). Maybe we'll find better ways of writing down Lagrangians, but people are wondering whether there's a completely different way to characterize these theories which will make the whole business of Lagrangians historical and quaint. Of course, nobody knows how this might go. There are arguments that it's not even possible to write down Lagrangians for them (e.g. that they don't have couplings), but their arguments aren't watertight.

Related to this is the question whether every quantum field theory has a Poisson bracket on its observables. The answer is no, because quantum field theories aren't determined by their point operators.

One example is a six-dimensional example with a self-dual 3-form. This self-dual 3-form is what makes it hard to write a Lagrangian description. However, if you formulate this on $S^1 \times \mathbb{R}^{1,4}$ and do a Fourier expansion on the S^1 and make an effective theory, this turns into an ordinary theory: the self-dual 2-form becomes a connection valued in the Lie algebra of a nonabelian Lie group. But we don't know how to promote this to a self-dual 2-form in 6 dimensions.

5.3. **Andy Neitzke: Why are particles representations of the Poincaré group?** So you have a quantum field theory, and hence a huge Hilbert space acted on by the Poincaré group. There's one state invariant under the action of the Poincaré group, which is (by definition) the vacuum. There's another subspace corresponding to one-particle states, and these are irreducible representation: each type of particle is an irreducible component, because any two universes containing a single particle of the same kind are related by a transformation of the Poincaré group. Such representations are almost always infinite-dimensional. There are lots of other representations (multiple-particle states and so on), of course.

You can also think of the vacuum and one-particle states as the discrete part of the spectrum, and the multi-particle states as a continuum.

In a topological theory, everything is finite-dimensional, so there aren't really particles. Extended notions of locality remember some facts about the excitations, though.

Day 2. August 1

6. Dan Freed: An Axiom System for Wick-rotated QFT

Yesterday, we defined an axiom system for topological field theory, as symmetric monoidal functors

$$F\colon \mathsf{Bord}_{\langle n-1,n\rangle} \longrightarrow \mathsf{Vect}_{\mathbb{C}}.$$

We then described an axiom system for Wick-rotated quantum mechanics, which considered Riemannian manifolds and topological vector spaces:

$$F\colon \mathsf{Bord}_{\langle 0,1\rangle}(\mathrm{SO}^\nabla) \longrightarrow \mathsf{tVect}_{\mathbb{C}}.$$

To define an axiom system for Wick-rotated quantum field theory, we simply combine them.

Definition 6.1 (G. Segal)**.** A *Wick-rotated quantum field theory* is a symmetric monoidal functor

$$F\colon \mathsf{Bord}_{\langle n-1,n\rangle}(\mathcal{X}_n^\nabla) \longrightarrow \mathsf{tVect}_{\mathbb{C}},$$

where $\mathcal{X}_n^\nabla$ is a geometric analogue of a tangential structure.

This seems surprisingly sparse, but works surprisingly well.

We're not going to precisely define the structures $\mathcal{X}_n^\nabla$, but instead give several examples.

- Tangential structures like we discussed yesterday: orientation, spin structure, pin$^+$-structure, etc.
- More geometric structures such as a Riemannian metric, a conformal structure, a volume form, a principal K-bundle with connection (where K is a compact Lie group), and so on.
- Maps to a space M, sections of a fiber bundle, and so on.

In physics, these are all called fields. In this context, they're *background fields*: unlike the fluctuating fields we considered yesterday, they are not integrated over. Though we won't define fields precisely, the key requirement is a sheaf condition: you must be able to glue local data of a field into global data, and all of the examples above satiafy that condition.

One particular special case is when H_n is a Lie group and $\rho_n\colon H_n \to \mathrm{GL}_n(\mathbb{R})$ is a map of Lie groups. For topological field theory, it's important for this map to have finite fibers, though there are some examples for which this doesn't hold, e.g. $H_n = \mathrm{Spin}_n^c$.

We'll see several examples in a later lecture, and for some of them it could be useful to try to fit them into this axiomatic framework, at least heuristically. Until then, we'll show how to extract the usual ingredients of a QFT from this axiom system.

- Let Y be a closed $(n-1)$-manifold with $\mathcal{X}_n^\nabla$-structure, so it's an object in $\mathsf{Bord}_{\langle n-1,n\rangle}(\mathcal{X}_n^\nabla)$. Since $\mathcal{X}_n^\nabla$ is not just a topological structure, we need to consider Y with a two-sided collar. This makes composition (gluing) make sense, and allows you to think of everything as n-dimensional. The collar represents an infinitesimal slice of time, and Y represents space. You might imagine undoing the Wick rotation to obtain something with Lorentz signature and quantizing to produce a state space, which is some topological vector space, and this is what $F(Y)$ is.
- If X is an n-manifold and $x \in X$, we can ask about the quantities near x that can be measured. Physicists call these *local observables*, but you can also call them *point observables.* To see these from the functorial perspective, let $S_\varepsilon(x)$ denote the sphere of radius ε around x in X. Then, $S_\varepsilon(x)$ is a manifold with $\mathcal{X}_n^\nabla$-structure, so $F(S_\varepsilon(x))$ is a vector space of data "near x." To make it "at x," we take the inverse limit:
$$\mathcal{O}_x = \varprojlim_{\varepsilon\to 0} F(S_\varepsilon(x)).$$
- Correlation functions have a similar description: a k-point correlation function $\Phi\colon \mathcal{O}_{x_1}\otimes\cdots\otimes\mathcal{O}_{x_k} \to \mathbb{C}$ comes as the inverse limit as $\varepsilon \to 0$ of this data of spheres of radius ε around $x_1, \dots, x_k$. We can think of $X \setminus (B_\varepsilon(x_1) \cup \cdots \cup B_\varepsilon(x_k))$ as a bordism from $S_\varepsilon(x_1) \amalg \cdots \amalg S_\varepsilon(x_k) \to \varnothing$, and applying F and taking the limit produces Φ.

If the theory doesn't depend on the metric, meaning it's conformal or topological, you can ignore the limit, because $F(S_\varepsilon(x))$ doesn't depend on ε. This would mean that the space of operators is a state space, a phenomenon called *operator-state correspondence.*

Generally, though, these theories are not scale-independent. Rescaling everything by some constant defines a functor from $\mathsf{Bord}_{\langle n-1,n\rangle}(\mathcal{X}_n^\nabla)$ to itself, and this is the action of the renormalization group. One would like for this to have short-range or long-range limits; the short-range limit, if it exists, would still be scale-independent, and hence a conformal field theory. The long-range limit will be useful in the classification of phases.

Let $\mathbb{M}^n$ denote Minkowski spacetime, an affine space modeled on $\mathbb{R}^{1,n-1}$, which acts on $\mathbb{M}^n$ by translations. $\mathbb{R}^{1,n-1}$ is $\mathbb{R}^n$ with the Lorentz metric $x^0 = ct$ and coordinates $x^1, \dots, x^{n-1}$ with metric
$$\mathrm{d}s^2 = (\mathrm{d}x^0)^2 - (\mathrm{d}x^1)^2 - \cdots - (\mathrm{d}x^{n-1})^2.$$

There is a short exact sequence
$$1 \longrightarrow \mathbb{R}^{1,n-1} \longrightarrow \mathrm{Isom}(\mathbb{M}^n) \longrightarrow \mathrm{O}_{1,n-1} \longrightarrow 1.$$

The middle group has four connected components, and $\pi_0\,\mathrm{Isom}(\mathbb{M}^n) \cong \mathbb{Z}/2\times\mathbb{Z}/2$. One of these copies of $\mathbb{Z}/2$ asks whether a given isomorphism preserves or reverses orientation; the other asks whether it preserves or reverses time. There's a group called the *Poincaré group* $\mathcal{P}_n$, which is a double cover of the component of $\mathrm{Isom}(\mathbb{M}^n)$ containing the identity; this is thought of as the symmetry group of the theory.

For non-relativistic quantum field theory, we had a semigroup law $\mathbb{R} \to \mathrm{U}(\mathcal{H})$, telling us how time evolution acts on the state space by unitary operators. In the relativistic case, we have additional symmetries, so ask for a map $U\colon \mathcal{P}_n \to \mathrm{U}(\mathcal{H})$. We can also ask for our Hilbert space to be $\mathbb{Z}/2$-graded: $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$. In physics, one says the statistics of particles is controlled by this splitting; $\mathcal{H}^0$ is for bosons, and $\mathcal{H}^1$ is for fermions.

Inside $\mathcal{P}_n$, there's a copy of translations $\mathbb{R}^{1,n-1}$. In the dual picture $(\mathbb{R}^{1,n-1})^*$, the two directions are energy and momentum; there's a lightcone in the energy direction coming from the Lorentz-signature metric on $(\mathbb{R}^{1,n-1})^*$, and we can precisely say that positive-energy means being in the positive (upper) part of the lightcone C_+^*.

There are multiple approaches to axiomatizing quantum field theory; both the Schrödinger and Heisenberg approaches to quantum mechanics generalize to quantum field theory; Heisenberg's approach becomes in the modern picture the theory of factorization algebras applied to quantum physics, but we're focusing on the Schrödinger formalism.

Wick rotation begins with the observation that $U|_{\mathbb{R}^{1,n-1}}$ is the boundary value of a contracting holomorphic semigroup on $\mathbb{R}^{1,n-1} \oplus \sqrt{-1}\mathbb{R}^{1,n-1}_- \subset \mathbb{C}^n$, the generalization of the lower half of the plane we discussed yesterday.

Positive energy allows you to extend this to a complexified domain $\mathcal{D}$, a complexification of $\mathbb{M}^n$, and once in $\mathcal{D}$, we can restrict to a Euclidean space $\mathbb{E}^n$. Thus we obtain a positive definite metric. This may have felt vague, but there's a mathematically rigorous theory underlying this.

Symmetry groups. The symmetry group $\mathcal{G}_n$ of our relativistic quantum field theory must act on $\mathbb{M}^n$ by isometries. Thus we know we have a homomorphism $q\colon \mathcal{G}_n \to \mathrm{Isom}(\mathbb{M}^n)$. This is not how it's usually thought about: especially in older groups, one reads that the Poincaré group is a subgroup of $\mathcal{G}_n$, but from our perspective it's more natural as a quotient. Relativistic invariance of the theory means the identity component $\mathrm{Isom}(\mathbb{M})^0$ is in $\mathrm{Im}(q)$.

We're going to make three assumptions on q. Some of these are strict and throw out interesting theories.

(1) If $K := \ker(q)$, we ask that K is a compact Lie group. Segal considered some noncompact groups, and there are interesting examples, but we're not going to consider them. There are also other kinds of symmetries we're ignoring: both supersymmetries (those that exchange the grading, and make K into a super-group), and higher symmetries (more homotopical things, making K into a 2-group or 3-group).

(2) $\mathbb{R}^{1,n-1}$ should lift to a normal subgroup of $\mathcal{G}_n$. This is in line with Klein's *Erlangen* program: we want translation-invariance in our theory, and therefore ask for translations in our symmetry group. With this assumption, we can define $G_n := \mathcal{G}_n/\mathbb{R}^{1,n-1}$.

Since $\mathcal{G}_n$ contains $\mathrm{Isom}(\mathbb{M}^n)^0$, which is noncompact, $\mathcal{G}_n$ isn't compact. But after the quotient by translations, we have an exact sequence

$$1 \longrightarrow K \longrightarrow G_n \longrightarrow O_{1,n-1}.$$

Wick rotation first tells us to complexify this, producing a morphism of group extensions for $\mathcal{D}$:

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d, hook] & G_n \arrow[r] \arrow[d, hook] & O_{1,n-1} \arrow[d, hook] \\ 1 \arrow[r] & K(\mathbb{C}) \arrow[r] & G_n(\mathbb{C}) \arrow[r] & \mathrm{O}_n(\mathbb{C}). \end{tikzcd}$$

The top row is the real forms of the groups in the bottom row that fix a Lorentz metric. But when we restrict to $\mathbb{E}^n$, we choose different real forms, those fixing a Euclidean metric:

$$\begin{tikzcd} 1 \arrow[r] & K(\mathbb{C}) \arrow[r] & G_n(\mathbb{C}) \arrow[r] & \mathrm{O}_n(\mathbb{C}) \\ 1 \arrow[r] & K \arrow[r] & H_n \arrow[r, "\rho_n"] \arrow[u, hook] & \mathrm{O}_n. \arrow[u, hook] \end{tikzcd}$$

Notably, all the groups on the bottom row are compact. Compact Lie groups are rigid, and so we can try to enumerate the possibilities.

The first question is to determine the image of ρ_n. Because $\mathrm{O}_{1,n-1}$ contains the image of $\mathrm{Isom}(\mathbb{M}^n)^0$, but $\mathrm{O}_n(\mathbb{C})$ and O_n only have two connected components, one can show that $\mathrm{Im}(\rho_n)$ is either SO_n or O_n. This is determined by whether the system has time-reversal symmetry, which is a particularly important symmetry for condensed-matter systems.

Let's write down some analogues of SO_n and Spin_n.

Definition 6.2. Let SH_n and $\widetilde{SH}_n$ be the Lie groups that fit into the following pullback diagrams:

$$\begin{tikzcd}
\widetilde{SH}_n \arrow[r] \arrow[d] & \mathrm{Spin}_n \arrow[d] \\
SH_n \arrow[r] \arrow[d] & \mathrm{SO}_n \arrow[d, hook] \\
H_n \arrow[r, "\rho_n"] & \mathrm{O}_n.
\end{tikzcd}$$

The following theorem says, in a few ways, that the symmetry group splits.

Theorem 6.3 ([FH16]). *Assume $n \geq 3$.*

(1) *If $\mathfrak{h}_n$ is the Lie algebra of H_n and $\mathfrak{k}$ is that of K, there's a splitting*

$$\mathfrak{h}_n \cong \mathfrak{o}_n' \oplus \mathfrak{k}$$

together with a Lie algebra isomorphism

$$\dot{\rho}_n \colon \mathfrak{o}_n' \xrightarrow{\cong} \mathfrak{o}_n.$$

(2) *$\widetilde{SH}_n \cong \mathrm{Spin}_n \times K$, and there's a $k_0 \in K$ with $k_0^2 = 1$ such that*

$$SH_n \cong (\mathrm{Spin}_n \times K)/\langle(-1, k_0)\rangle.$$

(3) *There's a canonical map $\mathrm{Spin}_n \to H_n$ sending $-1 \mapsto k_0$.*

This is an analogue of the Coleman-Mandula theorem.

There's also a stabilization theory which says that these symmetry groups fit into families: thus we can say spin theory, pin$^-$-theory, etc., rather than a Spin_n-theory, Pin_n^--theory, and so on.

Theorem 6.4 ([FH16]). *For each $m \geq n$, there is a compact Lie group H_m and homomorphisms $i_m \colon H_m \hookrightarrow H_{m+1}$ and $\rho_m \colon H_m \to \mathrm{O}_m$ fitting into a commutative diagram*

$$\begin{tikzcd}
H_n \arrow[r, hook, "i_n"] \arrow[d, "\rho_n"] & H_{n+1} \arrow[r, hook, "i_{n+1}"] \arrow[d, "\rho_{n+1}"] & H_{n+2} \arrow[r, hook] \arrow[d, "\rho_{n+2}"] & \cdots \\
\mathrm{O}_n \arrow[r, hook] & \mathrm{O}_{n+1} \arrow[r, hook] & \mathrm{O}_{n+2} \arrow[r, hook] & \cdots
\end{tikzcd}$$

such that for each m, $\ker(\rho_m) = K$ and each square is a pullback square.

We can therefore for the colimit $\rho \colon H \to \mathrm{O}$, and H_n is the pullback of ρ along the inclusion $\mathrm{O}_n \hookrightarrow \mathrm{O}$.

This stable version of the symmetry group is called the (H, ρ) *symmetry type*, and similarly we speak of the unstable version, the (H_n, ρ_n) *symmetry type*. In Robert Bryant's lecture, we considered maps $\rho_n \colon H_n \to \mathrm{GL}_n(\mathbb{R})$ with finite fibers, but we're looking at Lorentz symmetry, and hence some things become nicer: the real form after Wick rotation is compact, so ρ_n factors through the inclusion $\mathrm{O}_n \hookrightarrow \mathrm{GL}_n(\mathbb{R})$.

Definition 6.5. Let (H_n, ρ_n) be a symmetry type and X be a smooth manifold. Then, a *differential H_n-structure* on X is the data (P, Θ, φ), where $P \to X$ is a principal H_n-bundle, Θ is a connection on P, and $\varphi \colon \mathcal{B}(X) \to \rho(P)$ is an isomorphism carrying Θ to the Levi-Civita connection.

We'll thus consider Wick-rotated theories with differential H_n-structure. The relevant bordism category is denoted $\mathsf{Bord}_{\langle n-1,n\rangle}(H_n^\nabla)$.

This axiom system is radical from the physics perspective: we've restricted to compact manifolds, and therefore one might guess this precludes its use to study long-range behavior. But as we'll see, this is not true.

Condensed-matter physicists study something that could be called "quantum zoology," classifying and understanding states and quantum matter. Today we're going to focus on bosonic/spin systems; in the real world you want electrons and hence fermionic systems, but the generalization is not as hard as you'd expect.

We'll start with a lattice system in any dimension; there are *sites* (dots in the lattice, or 0-cells in a implicial structure). This is in a manifold which represents space; there is no time present in this system yet. At each site i, there's a Hilbert space $\mathcal{H}_i$, which is a copy of $\mathbb{C}^m$. The total Hilbert space for the problem is

$$\mathcal{H} := \bigotimes_{i\in\Delta^0} \mathcal{H}_i.$$

You can think of this as an approximation of the continuum system, a quantum field theory, which is why we can get away with finite-dimensional Hilbert spaces. For the purposes of low-energy physics, this approximation is especially useful.

The Hamiltonian is a local sum: at each site i, there is some operator O_i which acts on finitely many sites in the vincinity of i (and acting by the identity on all other sites).[5] We'd like these operators to all be the same, in that if φ is an automorphism of the manifold and lattice carrying $i \mapsto j$, then $\varphi^* O_j = O_i$. Then, the Hamiltonian is

$$H = \sum_{i\in\Delta^0} O_i.$$

Example 7.1. Suppose $m = 2$, so at each site i we have a qubit $\mathbb{C}^2 = \mathrm{span}\{e_1, e_2\}$ (the standard basis). One choice for the operators O_i produces the Hamiltonian

$$H = -h \sum_{i\in\Delta^0} \sigma_i^x - J \sum_{\substack{e\in\Delta^1\\ \partial e=\{i,j\}}} \sigma_i^z \sigma_j^z \tag{7.2}$$

for some $h, J > 0$.

- The first term is some kind of magnetic term.
- The second term is a spin-spin interaction between nearest neighbors.

The σ_i-terms are *Pauli operators*, the generators of $\mathfrak{su}_2$:

$$\sigma^x = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}, \qquad \sigma^y = \begin{pmatrix} 0 & -i\\ i & 0\end{pmatrix}, \qquad \sigma^z = \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}.$$

Writing σ_i^x means applying σ^x to the $\mathbb{C}^2$ at the site i. These satisfy the commutation relations

$$[\sigma_i^a, \sigma_j^b] = 2i\epsilon^{abc}\delta_{ij}.$$

◀

Let's impose periodic boundary conditions, so this system is on a torus of length $\vec{L}$ (i.e., $\vec{L} = (L_1, \dots, L_d)$, and the length in the x_i-coordinate is L_i). To understand the excitations, consider the Schrödinger equation

$$H|\psi\rangle = E|\psi\rangle.$$

Suppose that there's a gap between the smallest eigenvalue E_0 (corresponding to the ground states) and the second-smallest E_1 (corresponding to the lowest-energy excitations), as in Figure 3. Moreover, this is stable under refinement, in that

$$\lim_{L\to\infty} (E_1 - E_0) = \Delta > 0.$$

Thus, this gap is not an artifact of the discretization, but is inherent in the system.

An example ground state $|0\rangle$ is when all of the sites have the same spin, and an example of an excited state with lowest possible energy $|1\rangle$ is when all of the sites but one have the same spin, and the remaining site has opposite spin. When $J = 0$, the energy gap between $|0\rangle$ and $|1\rangle$ is $2h$, which is clearly independent of the length; if you "turn on J" (meaning make it a small positive number), this gap persists, though now it's $\Delta = 2h + O(J)$. This is hard to rigorously prove.

The other possibility is that the first p eigenvalus $E_0, \dots, E_{p-1}$ are 0, and then E_p is nonzero, with a gap Δ, as in Figure 4. This is called a *gapped, degenerate* system. We ask that

$$\lim_{L\to\infty} (E_\alpha - E_\beta) = 0$$

[5] You can generalize these to those that decay quickly, but we're not going to worry about this today.

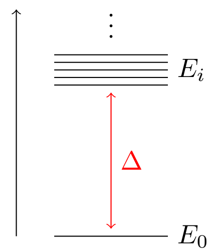

Figure 3. The spectrum of a nondegenerately gapped Hamiltonian.

for $\alpha, \beta \in \{0, \dots, p-1\}$, and that

$$\lim_{L\to\infty} (E_p - E_0) = \Delta.$$

This can arise accidentally, e.g. if you have two eigenvalues whose values don't coincide generally. If you perturb this system, it returns to a nondegenerate gapped system, and hence is the less interesting case. For example, if $h = 0$, $|0\rangle$ has all spins pointing up, and $|1\rangle$ has all spins pointing down, and both of these have the same energy. The next state $|2\rangle$ will look like $\downarrow\downarrow\uparrow\uparrow\uparrow\cdots$, and its energy is $E_2 = E_0 + 4J$. But if you perturb by a magnetic field in the z-direction, producing

$$H = -J \sum_{\partial e = \{i,j\}} \sigma_i^z \sigma_j^z - g \sum_i \sigma_i^z$$

for some small g, E_0 is no longer equal to E_1.

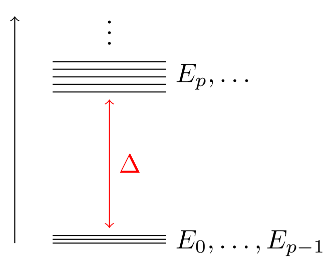

Figure 4. The spectrum of a degenerately gapped Hamiltonian.

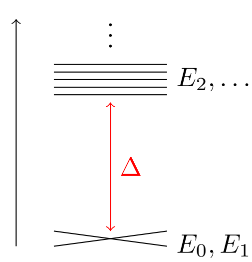

Figure 5. A degenerately gapped system can arise "accidentally," in that a small perturbation in some parameter (here on the x-axis) produces a nondegenerately gapped system.

Definition 7.3. Two degenerate ground states α and β are *locally indistinguishable* if for any local operator A_i,

$$\lim_{L\to\infty} \langle \alpha \mid A_i \mid \beta \rangle = C_a \delta_{\alpha\beta},$$

i.e. it's diagonal.

The toric code has local indistinguishability: in dimension d on a torus, its degeneracy (dimension of the space of ground states) is 2^d, and these states cannot be locally distinguished. More generally, this is true for (intrinsically) topologically ordered states. The examples that we know, and which we think might be all examples, are anyon models. In (spacetime) dimension 3, these are well-understood: the Levin-Wen construction [LW05] produces such a model from a modular tensor category C. The idea is that if you have two anyons in this model and braid them once around each other, the wavefunction changes by some number which is dictated by the data of C. This seems bizarre, but is realized in nature by the fractional quantum Hall state.

This is the zoology: you have a general picture of what can and can't happen.

The third possibility for the spectrum is that it's gapless: $\lim_{L\to\infty}(E_\alpha - E_0) = 0$ for infinitely many α. Often, the low-energy field theory is a conformal field theory.

Phases. We've talked a little bit about perturbing the Hamiltonian. When does this change the physics of the system?

Definition 7.4. Two Hamiltonians H_0 and H_1 belong to the same *gapped phase* if there is a smooth path of Hamiltonians $H(t)$ for $t \in [0, 1]$ such that $H(0) = H_0$, $H(1) = H_1$, and $H(g)$ is gapped for every g.

We allow degenerately gapped systems.

There is a privileged phase, called the *trivial phase* or *product phase*, represented by the Hamiltonian (7.2) for $J = 0$. More generally, if $J \ll h$, the Hamiltonian belongs to the trivial phase. The name "product phase" highlights that each site is in the same state ψ, i.e. $|0\rangle = |\psi\rangle \otimes \cdots \otimes |\psi\rangle$.

Definition 7.4 is a nice definition, but doesn't allow us to change the Hilbert space. Let's stabilize it by permitting one to throw away degrees of freedom corresponding to a trivial phase. Namely, if $H_0\colon \mathcal{H}_0 \to \mathcal{H}_0$ and $H_1\colon \mathcal{H}_1 \to \mathcal{H}_1$ are two systems such that H_0 is in the trivial phase, we can couple them together (also called "stacking") and get a new system $H \coloneqq H_0 + H_1$ acting on $\mathcal{H} \coloneqq \mathcal{H}_0 \otimes \mathcal{H}_1$. We say that H_1 and H are in the same phase.[6]

When $\mathcal{H}_0$ and $\mathcal{H}_1$ are states for different lattices, defining coupling is a little more complicated, but can still be done.

Invertible gapped phases. Let H be a gapped Hamiltonian that is not in the trivial phase.

Definition 7.5. H is in an *invertible phase* if there's a Hamiltonian $\overline{H}$ such that the system produced by coupling H and $\overline{H}$ together is in the trivial phase, i.e. $H + \overline{H}$ acting on $\mathcal{H} \otimes \overline{\mathcal{H}}$ is in a trivial phase.

The nice thing about invertible gapped phases is that they form an abelian group, under the group operation of stacking. The identity is the trivial phase, and the inverse of H is $\overline{H}$ as above.

Another nice thing about (nondegenerately gapped) invertible phases is that they have a unique ground state on every closed manifold, in particular on the torus. This is because when you stack H and $\overline{H}$, it deforms to a trivial phase, which has a single ground state, so since we deformed without closing the energy gap, there has to only be one ground state before deformation. In particular, the anyon models are generally not invertible.

In the real world, though, we can't really impose periodic boundary conditions, and thus we have to consider systems with boundaries. There are lots of choices for terminating the Hamiltonian at the boundary, leading to notions of edge modes. At least for invertible $(2 + 1)D$ systems (without symmetry), no matter how you terminate the boundary, it's gapless, which is somewhat disconcerting. In other words, for infinitely many $\alpha > 0$, it's possible to get $|\alpha\rangle = A|0\rangle$ for a local operator A living at the boundary. So you end up with a conformal field theory on the boundary that's chiral (with left and right central charges, whose difference vanishes mod 8). In $(1 + 1)D$, by contrast, the chiral central charge is always 0. One says that (back in $(2 + 1)D$) the boundary is anomalous. This is sort of the meaning of an anomaly: an anomalous system is one that can only live on the boundary of a higher-dimensional bulk.

The groups of invertible phases in low dimensions have been calculated, and some are given in Table 1. The generators for the nonzero groups are known:

- The generator of the group of fermionic phases in $(1 + 1)D$ is the Majorana wire, which has been realized physically.

[6]When we say $H_0 + H_1$, we mean more precisely $H_0 \otimes 1 + 1 \otimes H_1$, which does actually act on $\mathcal{H}_0 \otimes \mathcal{H}_1$.

- The generator of the group of fermionic phases in $(2+1)D$ is the $p+ip$ superconductor. Twice the generator is in the same phase as the $\nu = 1$ integer Quantum hall effect.
- The generator of the group of bosonic phases in $(2+1)D$ is the $\nu = 8$ integer quantum Hall effect.

	$(1+1)D$	$(2+1)D$	$(3+1)D$
bosons	0	$\mathbb{Z}$	0
fermions	$\mathbb{Z}/2$	$\mathbb{Z}$	0

Table 1. Groups of invertible phases. These have been calculated in several different ways, and one reference is [FH16], which also computes examples with symmetry groups.

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