M392C NOTES: BRIDGELAND STABILITY

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These notes were taken in UT Austin's M392C (Bridgeland Stability) class in Spring 2019, taught by Benjamin Schmidt. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to `a.debray@math.utexas.edu`. Any mistakes in the notes are my own.

Contents

Lecture 1.

Introduction and quiver representations: 1/22/19

This class will be on Bridgeland stability, though we won't get to that topic specifically for about a month. We'll follow lecture notes of Macrì-Schmidt [MS17], which are on the arXiv.

If you're pre-candidacy, make sure to do at least two exercises in this class, at least one from March or later; otherwise just make sure to show up. (If you're an undergrad who's signed up for this class, please do at least four exercises, at least two from March or later.)

Now let us enter the world of mathematics. We'll begin with two well-known theorems in algebraic geometry; we'll eventually be able to prove these using stability conditions.

Theorem 1.1 (Kodaira vanishing)**.** *Let X be a smooth projective complex variety and L be an ample line bundle. Then for all $i > 0$, $H^i(X; L \otimes \omega_X) = 0$.*

We'll eventually give an approach in the setting where $\dim X \leq 2$. It won't be very hard once the setup is in place. In fact, there are probably plenty of other vanishing theorems one could prove using stability conditions, including some which aren't known yet.

The other theorem is over a century ago, from the Italian school of algebraic geometry.

Theorem 1.2 (Castelnuovo)**.** *Working over an algebraically closed field, let $C \subset \mathbb{P}^3$ be a smooth curve not contained in a plane. Then $g \leq d^2/4 - d + 1$, where g is genus of C and d is its degree.*

Another goal we'll work towards:

Problem 1.3. Explicitly describe some moduli spaces of vector bundles or sheaves.

Here's a concrete outline of the course.

(1) Before we discuss any algebraic geometry, we'll study quiver theory, focusing on moduli spaces of quiver conditions. We don't need stability conditions to do this, but these spaces make great simple examples of the general story.
(2) Next, we'll study vector bundles on curves. Bridgeland stability is a generalization of what we can say here for higher dimensions.
(3) A crash course on derived categories and Bridgeland stability. This is pretty formal.
(4) A crash course on intersection theory, which will be necessary for what comes later.
(5)

These are all mostly independent pieces, only coming together in the end, so if you get lost somewhere there's no need to panic; you'll probably be able to pick the course back up soon enough.

And now for the moduli of quiver representations. For this stuff, we'll follow King [Kin94], which is accessible and nice to read. Let k be an algebraically closed field.

Definition 1.4. A *quiver* is the representation theorist's word for a finite directed graph. Explicitly, a quiver Q consists of two finite sets Q_0 and Q_1 of vertices and edges, respectively, together with *tail* and *head* maps $t, h\colon Q_1 \to Q_0$.

Example 1.5. The *Kronecker quiver* is

$$\bullet \rightrightarrows \bullet.$$

The quiver of *type* D_4 is

$$\begin{tikzcd} & & \bullet \arrow[dl] \\ \bullet & \bullet \arrow[l] & \\ & & \bullet. \arrow[ul] \end{tikzcd}$$

We can also consider a quiver with a single vertex v and a single edge $e\colon v \to v$. $\blacktriangleleft$

Definition 1.6. A *representation* W of a quiver Q is a collection of k-vector spaces W_v for each $v \in Q_0$ and linear maps $\phi_e\colon W_{v_1} \to W_{v_2}$ for each edge $e\colon v_1 \to v_2$ in Q_1. The vector $(\dim W_v)_{v \in Q_0}$ inside $\mathbb{C}[Q_0]$ is called the *dimension* of W.

Example 1.7. First, some trivial example. For example, here's a representation of the Krokecker quiver: $(\cdot 1, \cdot 2)\colon k \rightrightarrows k$. A representation of the quiver with one vertex and one edge is a vector space with an endomorphism, e.g. $\mathbb{C}^2$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. $\blacktriangleleft$

Definition 1.8. Let Q be a quiver. A *morphism* of Q-representations $f\colon (W_v, \phi_e) \to (U_v, \psi_e)$ is a collection of linear maps $f_v\colon W_v \to U_v$ for each $v \in Q_0$ such that for all edges e,

$$f_{h(e)} \circ \phi_e = \psi_e \circ f_{t(e)}.$$

If all of these linear maps are isomorphisms, f is called an *isomorphism*.

That is, data of a quiver representation includes a bunch of linear maps, and we want a morphism of quiver representations to commute with these maps.

Representations theorists want to classify quiver representations. This is really hard, so let's specialize to irreducible representations (those not a direct sum of two other ones). This is still really hard! There are classical theorems originating from the French school proving that most quivers do not admit nice classifications of their irreducible representations: some have finitely many, and some have infinitely many but nice parameterizations, and these are uncommon.

One way to make headway on these kinds of problems is to consider a moduli space of quiver representations, which may be more tractable to study.

Problem 1.9. Can you classify the (isomorphism classes of) quiver representations of the quiver with a single vertex and single edge?

Our first, naïve approach to constructing the moduli of quiver representations is to fix a dimension vector $\alpha \in \mathbb{C}[Q_0]$ and define

$$R(Q, \alpha) := \bigoplus_{e \in Q_1} \mathrm{Hom}(W_{t(e)}, W_{h(e)}). \tag{1.10}$$

This is too big: the same isomorphism class appears at more than one point. We can mod out by a symmetry: let

$$\mathrm{GL}(\alpha) := \prod_{v \in Q_0} \mathrm{GL}(W_v) \tag{1.11}$$

act on $R(Q, \alpha)$ by a change of basis on each vector space and on ϕ_e as

$$(g\phi)_e = g_{h(e)} \phi_e g_{t(e)}^{-1}. \tag{1.12}$$

Then as a set the quotient $R(Q, \alpha)/\mathrm{GL}(\alpha)$ contains one element for each isomorphism class. But putting a geometric structure on quotients of varieties is tricky. We'll come back to this point.

Example 1.13. Let Q be the Kronecker quiver and $\alpha = (1,1)$, so that $\mathrm{GL}(\alpha) = k^\times \times k^\times$. Pick $(t,s) \in \mathrm{GL}(\alpha)$; the action on a Q-representation $(\lambda, \mu)\colon k \rightrightarrows k$ produces $(s\lambda t^{-1}, s\mu t^{-1})\colon k \rightrightarrows k$. So if $s = t$, the action is trivial. Quotienting out by the diagonal $s = t$ in $k^\times \times k^\times$, we get $k^\times\colon (s,t) \mapsto s/t$, and this acts on $R(Q,\alpha) = k^2$ by scalar multiplication.

This is an action we know well: the quotient is the space of lines in k^2, also known as $\mathbb{P}^1_k$ – and the zero orbit. This orbit makes life more of a headache: you can't just throw it out, because then you don't get a good map to the quotient, preimages of closed things aren't always closed, etc. But the action on the zero orbit is not free. This phenomenon will appear a lot, and we'll in general have to think about what to remove. After some hard work we'll be able to take the quotient in a reasonable way and get $\mathbb{P}^1$. $\blacktriangleleft$

A crash course on (linear) algebraic groups. If you want to learn more about algebraic groups, especially because we're not going to give proofs, there are several books called *Linear Algebraic Groups*: the professor recommends Humphreys' book [Hum75] with that title, and also those of Borel [Bor91] and Springer [Spr98].

Definition 1.14. An *algebraic group* is a variety G together with a group structure such that multiplication and taking inverses are morphisms of varieties.

You can guess what a morphism of algebraic groups is: a group homomorphism that's also a map of varieties.

Example 1.15. GL_n is an algebraic group. Inside the space of all $n \times n$ matrices, which is a vector space over k, GL_n is the set of matrices with nonzero determinant. This is an open condition, and the determinant can be written in terms of polynomials, so GL_n is an algebraic group.

Other examples include SL_n and elliptic curves, and we can take products, so $\mathrm{GL}(\alpha)$ is also an algebraic group. $\blacktriangleleft$

Definition 1.16. A *linear algebraic group* is an algebraic group that admits a closed embedding $G \hookrightarrow \mathrm{GL}_n$ which is also a group homomorphism.

This does not include the data of the embedding. It turns out (this is in, e.g. Humphreys) that any affine algebraic group is linear, but this is not particularly easy to show.

Exercise 1.17. Show that any algebraic group is also a smooth variety.

This does not generalize to group schemes!

We care about groups because they act. We added structure to algebraic groups, and thus care about actions which behave nicely under that structure.

Definition 1.18. A *group action* of an algebraic group G on a variety X is a morphism $\varphi\colon G \times X \to X$ such that for all $g, h \in G$ and $x \in X$,

(1) $\varphi(g, \varphi(h, x)) = \varphi(gh, x)$, and
(2) $\varphi(e, x) = x$.

Example 1.19. $k^\times$ acts on $\mathbb{A}^{n+1}_k$ by scalar multiplication. What's the quotient? We want $\mathbb{P}^n_k$, but there's also the zero orbit, and no other orbit is closed. This makes us sad; we're going to use geometric invariant theory (GIT) to address these issues and become less sad. $\blacktriangleleft$

Definition 1.20. Let G be an algebraic group.

- A *character* of G is a morphism of algebraic groups $\chi\colon G \to k^*$. These form a group under pointwise multiplication, and we'll denote this group $X(G)$.
- A *one-parameter subgroup* of G, also called a *cocharacter*, is a morphism of algebraic groups $\lambda\colon k^* \to G$.

Example 1.21. Since $\det(AB) = \det A \det B$, the determinant defines a character of GL_n. One example of a cocharacter is $\lambda\colon k^* \to \mathrm{GL}_2$ sending $t \mapsto \left(\begin{smallmatrix} t & 0 \\ 0 & t^2 \end{smallmatrix}\right)$. This cocharacter factors through the diagonal matrices in GL_n; this turns out to be a general fact. $\blacktriangleleft$

Here are a few nice facts about characters and cocharacters.

Theorem 1.22.

(1) *The map* $\mathbb{Z} \to X(\mathrm{GL}_n)$ *sending* $m \mapsto \det^m$ *is an isomorphism.*
(2) *If* G *and* H *are algebraic groups, the map* $X(G) \times X(H) \to X(G \times H)$ *sending*

$$(\chi_1, \chi_2) \longmapsto ((g, h) \longmapsto \chi_1(g)\chi_2(h)) \tag{1.23}$$

is an isomorphism.
(3) *Up to conjugation, every cocharacter of* GL_n *lands in the subgroup of diagonal matrices, hence sends* $t \mapsto \mathrm{diag}(t^{a_1}, \dots, t^{a_n})$ *for* $a_1, \dots, a_n \in \mathbb{Z}$.

We're not going to prove these: this would require a considerable detour into the theory of algebraic groups to get to, and you can read the proofs in Humphreys.

Exercise 1.24. Without using the above theorem, show that any morphism of algebraic groups $k^\times \to k^\times$ is of the form $t \mapsto t^n$ for some $n \in \mathbb{Z}$.

References

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