M390C NOTES: GEOMETRIC LANGLANDS

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MARCH 4, 2021

These notes were taken in UT Austin's M390C (Geometric Langlands) class in Spring 2021, taught by David Ben-Zvi. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Any mistakes in the notes are my own. Thanks to Tom Gannon, Charlie Reid, Saad Slaoui, and Jackson Van Dyke for finding and fixing a few errors.

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Lecture 1.

Overview and a perspective on modular forms: 1/19/21

This is a class on the geometric Langlands program from a particular perspective, incorporating its relationship to electric-magnetic duality. The class is over Zoom.

The geometric Langlands program lies halfway between number theory and physics. Maybe we are Odysseus and trying to navigate back home, between the two perils of Charybdis (physics, classically the big whirlpool) and Scylla (number theory, classically the monster). You can probably extend the analogy further, e.g. derived algebraic geometry is the Calypso islands. Extending the analogy further is left as an exercise.

There isn't any particular recommended background for this class — in particular, you don't need to know physics or number theory. Mackey [Mac80] wrote a nice overview of a perspective on the relationship between symmetry and harmonic analysis which could be fun to read. In this and the next lecture, we'll talk about modular forms and some relationships to physics; after that, we will begin the course properly: in a sense, the geometric Langlands program is a vast generalization of the Fourier transform, so we will begin with the Fourier transform, in a way that will be helpful when we do generalize.

Modular and automorphic forms, and physics We're not going to be super technical about number theory. The idea of modular and automorphic forms is to do a kind of harmonic analysis or quantum mechanics on *arithmetic locally symmetric spaces.* As an example, the upper half-plane $\mathbb{H}$ has a model as $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2$. The *modular group* $\Gamma \coloneqq \mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}$, with a fundamental domain $\Gamma\backslash\mathbb{H}$ (TODO: picture): the fundamental domain is noncompact, and goes off to infinity along the y-axis, and there are a couple of orbifold points, where the Γ-action has stabilizer.

More generally, we might consider a Lie group G with maximal compact $K \subset G$ and a lattice $\Gamma \subset G$. Then we consider the space $\Gamma\backslash G/K$ and study the space of functions on it. You might imagine a particle moving on this locally symmetric space, so we're interested in $L^2(\Gamma\backslash G/K)$, with a Laplacian Δ acting on this, and we can decompose the space of functions in terms of subspaces of eigenfunctions. This is one way in which modular forms can arise.

There are myriad variants of this. You can yeet K out of the story and study $L^2(\Gamma\backslash G)$ with its K-action. (TODO: something about the unit tangent bundle.) Plus, there's no need to restrict ourselves to linearizing with functions: you can use forms or sections of other vector bundles, such as $\Gamma(\Gamma\backslash\mathbb{H}, \omega^{\otimes k/2})$. De Rham says this is related to the cohomology of $\Gamma\backslash\mathbb{H}$, possibly with twisted coefficients. All of these variants are examples of things related to modular forms.

Now maybe you're thinking that if you pass to cohomology, you're no longer doing quantum mechanics, but in fact this is the domain of something called *topological quantum mechanics*; for example, this is discussed by Witten in his paper [Wit82] on supersymmetric quantum mechanics and its relationship to Morse theory.

Automorphic forms follow a similar story, but G is a more general Lie group. For example, pick your favorite reductive algebraic group such as GL_n or Sp_n, let G be the real points of this group, Γ be the integral points of this group, and K be the maximal compact of G. There is a long history of studying spaces of functions on $\Gamma\backslash G/K$ via harmonic analysis, and thinking of it as quantum mechanics. For example, if we started with Sp_{2n}, we get $\mathrm{Sp}_{2n}(\mathbb{Z})\backslash\mathrm{Sp}_{2n}(\mathbb{R})/\mathrm{U}_n$.

But there's a *lot* more structure here than in a typical quantum-mechanical setup. You can see this already for modular forms ($G = \mathrm{SL}_2(\mathbb{R})$). Namely, there's an additional variable: we can generalize from $\mathbb{Z}$ to other rings of integers in number fields. That is, given the field $\mathbb{Q}$, we think of $\mathbb{Z}$ as $\mathcal{O}_\mathbb{Q}$, the ring of integers of this number field, and obtain the group $\mathrm{SL}_2(\mathcal{O}_\mathbb{Q})$. Now we can replace $\mathbb{Q}$ with any finite extension $F/\mathbb{Q}$ and let $\mathcal{O}_F$ be the ring of integers of F, and consider a new lattice $\mathrm{SL}_2(\mathcal{O}_F)$. To make this completely precise, one has to fiddle with $\mathrm{SL}_2(\mathbb{R})$, since F may have more than one place at infinity, but this is the kind of technical detail we'll avoid for now.

And there is another way to vary the data: fix F, say $F = \mathbb{Q}$. Then we can vary the *conductor* or the *ramification data*. That is, the fundamental domain of Γ on $\mathbb{H}$ has a lot of covering spaces $\Gamma'\backslash\mathbb{H}$, where $\Gamma' \subset \mathrm{SL}_2(\mathbb{Z})$ is a *congruence subgroup*. One example of a congruence subgroup is, given $N \in \mathbb{Z}$, the subgroup

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \mathrm{id} \bmod N \right\}. \tag{1.1}$$

A variant is $\Gamma_0(N)$, the subgroup of matrices which are upper triangular mod N.

As is common in number theory, we can look at different places for primes in $\mathcal{O}_F$. For example, with $F = \mathbb{Q}$, this means looking at the local data at a prime p, which involves looking at $\mathrm{SL}_2(\mathbb{Q}_p)$. So the Hilbert space that we wanted to produce in the end depends on all this data: G and Γ and K, but also possibly F and the congruence subgroup and the prime.

Anyways, we'll get a Hilbert space and can study the spectral theory of the Laplacian. Maybe surprisingly, the eigenspaces are usually not one-dimensional. This "degeneracy" is because of *Hecke operators*, which are a crucial part of this story. At a high level, the Laplacian fits into a large family of commuting operators, and if p is a prime not dividing N, this family has a member T_p called the *Hecke operator*, giving an action of $\mathbb{C}[T_p]$ on $L^2(\Gamma\backslash G/K)$. And these all commute, so the tensor product of all of these $\mathbb{C}[T_p]$ over all primes acts on the Hilbert space, preserving the eigenspaces.

From the quantum mechanics perspective, this amount of commuting operators is unusual. You can think of this as an *integrable system*, with lots of conserved quantities. Usually (TODO: if I understood correctly), integrable systems are the opposite of chaos, but these arithmetic systems are studied as good examples of quantum chaos! This is a feature of the arithmetic story, and "arithmetic quantum chaos" behaves a lot more like quantum integrable systems than one might expect.

In this system, there is a special collection of measurements/states for a modular (or automorphic) form, called *periods*. One basic example is, given a modular function f on the fundamental domain $\Gamma/\backslash H$, integrate it:

$$\int_{i\mathbb{R}_+} f. \tag{1.2}$$

We will study modular functions/forms with these invariants. Hecke used L-functions to produce examples of these invariants.

Definition 1.3. A *Maass form* is an eigenfunction for the Laplacian on $L^2(\Gamma\backslash G/K)$. Specifically, *modular forms* are the holomorphic sections of $\omega^{\otimes k/2}$.

Modular forms can also arise by looking at the (twisted) cohomology of $\Gamma\backslash\mathbb{H}$; this is what's called *Eichler-Shimura theory*.

Our emphasis in this class will be more about topological quantum mechanics, rather than quantum mechanics; we care mostly about ground states. Modular forms are sort of like ground states here.

And there's one more piece of essential structure, to add to our already large pile of structures. There are these mysterious operators that allow you to vary the group! That is, these Hilbert spaces for different groups talk to each other! This is called *Langlands functoriality*. Part of the goal of this class is to explain this structure within physics.

But what the Langlands program itself does is to take these automorphic forms and spectrally decompose them in a prism (TODO: picture of the prism from Dark Side of the Moon, or maybe because this has something to do with physics, Dark Side of the Muon?). Automorphic forms enter on the left, and the prism spectrally decomposes them under the Hecke algebra (the algebra of all these commuting Hecke operators). And the different "colors" (eigenvalues) are given by representations of Galois groups of number fields, which is a surprising and magical statement. Moreover, there is a duality: these Galois representations are not into the complex points of G, but instead into $G^\vee_{\mathbb{C}}$, where $G^\vee$ is a dual group under something called Langlands duality.

For example, if $G = \mathrm{PSL}_2(\mathbb{R})$, then $G^\vee = \mathrm{SL}_2(\mathbb{C})$. A relatively explicit way to see how this enters is that if E is an elliptic curve over $\mathbb{Q}$, then $H^1(E)$ is a two-dimensional vector space carrying a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ action, and this is part of how modular forms appear in the story. This is one of the "colors" (eigenvalues) in this theory. The representation morally has image in $\mathrm{SL}_2(\mathbb{R})$, though to see this idea precisely requires setting things up a little more carefully. Elliptic curves appear in two ways in this, as the moduli space of elliptic curves is an arithmetic locally symmetric space. This isn't necessary to see the high-level overview, but it's crucial for actually proving anything! It's useful to know that elliptic curves have covers which are automorphic curves, and this provides a bridge between the two sides of the Langlands program. This is useful, but only applies for GL_2 — in general, you don't have this bridge, and the two sides are very far apart. For example, $\mathrm{GL}_3(\mathbb{Z})\backslash\mathrm{GL}_3(\mathbb{R})/\mathrm{O}_3$ is not a moduli space of anything: it's not a manifold. And this makes your proofs much harder and the duality more mysterious: why should function theory on these spaces have anything to do with Galois representations?[1]

One of the major goals of this class is to show how the (geometric) Langlands program arises in physics, not in quantum mechanics, but in four-dimensional (topological) field theory. Rather than beginning with a quantum mechanics system, we replace it with something much richer and more complicated — and scary. The key adjective "topological" helps mollify this: we throw out dynamics and look at ground states of the Laplacian, like looking only at harmonic forms rather than everything. We will try to match the structure of the Langlands program with the structure of this TFT.

Why 4? Quantum mechanics seems canonical enough, but 4d physics seems less so. We introduce another adjective, *arithmetic quantum field theory*, following the paradigm of arithmetic topology. This is an idea making an analogy between number fields and geometric objects that arise in physics, often manifolds. With a robust enough analogy, you can envision constructions with manifolds as having meaning in the world of number fields. The basic tenets of this theory are outlines in Weil's Rosetta stone (TODO: cite), which establishes a dictionary between number fields, function fields, and Riemann surfaces.

- Given a number field $F/\mathbb{Q}$, we consider $\mathrm{Spec}(\mathcal{O}_F)$, which has points labeled by primes in $\mathcal{O}_F$.
- The analogy between number fields and functional fields is older and better understood. We replace F with a (smooth, projective) curve C over a finite field $\mathbb{F}_q$. The field of rational functions $\mathbb{F}_q(C)$ on C has a lot of structure reminiscent of F, and the ring of regular functions $\mathbb{F}_q[C]$ resembles $\mathcal{O}_F$ (e.g. they're both Dedekind domains). The points of C are like the primes in $\mathcal{O}_F$.
- But why stop at $\mathbb{F}_q$? Let Σ be a compact Riemann surface. Points of Σ are the analogues of primes, in Weil's dictionary, and one can try to make geometric analogues of number-theoretic questions. The field of meromorphic functions on Σ is analogous to F and $\mathbb{F}_q(C)$, and the analogue of the ring

[1] Another technical detail to not worry about: when we replace $\mathbb{C}$ with $\overline{\mathbb{Q}_\ell}$, which is necessary for making some of these things precise, one must use étale cohomology instead of singular/Zariski cohomology. But that's not crucial for the point of this lecture.

of integers is a little complicated — Σ has no nonconstant entire functions, so we have to remove some points, analogues of points at infinity in the number field setting.

The crucial change in the arithmetic topology analogy is to replace Riemann surfaces with 3-manifolds. The reason behind this surprising change is that Riemann surfaces has strong similarities to curves over algebraically closed fields of positive characteristic. When you study a point $\operatorname{Spec}\mathbb{F}_q \hookrightarrow C$, you should remember the internal structure given by the Galois group action. Étale topology tells us to think of $\operatorname{Spec}(\mathbb{F}_q)$ as sort of like a circle, because the étale fundamental group of $\operatorname{Spec}(\mathbb{F}_q)$ is $\operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \cong \widehat{\mathbb{Z}}$. The Frobenius is a topological generator of this fundamental group. So there's too much structure here to match with a Riemann surface. If you base-change to $\overline{\mathbb{F}}_q$, replacing these "circles" with their "universal covers," we obtain an extra line direction which topology/cohomology doesn't see, because it's contractible, but now we obtain something that feels a little more like a Riemann surface.

So from the point of view of Galois theory, function fields of curves over $\mathbb{F}_q$ feel less like Riemann surfaces and more like bundles of Riemann surfaces over S^1. This is equivalent data to a Riemann surface Σ and a diffeomorphism $\phi\colon \Sigma \to \Sigma$. We then build the bundle as

$$\Sigma \times [0,1]/((x,0) \sim (\phi(x),1)), \tag{1.4}$$

the construction called the *mapping torus*. On the function-field side, we think of the Frobenius as ϕ. There's a difference here, in that we don't have a canonical choice of ϕ on the Riemann surface side (the identity is boring so let's not use that one), so we think of ϕ as "generic."

And so we arrive at the arithmetic topology dictionary, built by many workers, including Mumford writing to Mazur, Mazur, Kapranov, Reznakov, Morijita, and Kim. This is also known as the "knots and primes" dictionary: number fields are analogues of 3-manifolds. This is not something totally born out of nowhere; it's a refinement of Weil's dictionary.

Just how not all number fields have unique Frobenii (Frobeniuses?), but rather different ones at different primes, our 3-manifolds Y are not just surface bundles over curves. Primes on the number field side now correspond to embedded circles in Y, i.e. knots. Local fields, such as $\mathbb{Q}_p$, look like 2-manifolds. There aren't a lot of 2-manifolds fibered over the circle, but that's okay. And there are many more aspects of the analogy, such as the relationship between Legendre symbols and linking numbers, and more. The nLab page on arithmetic topology has a great list.

This is not an incredibly precise dictionary, and don't make the mistake of trying to associate specific primes to specific knots. For example, if yous said $\mathbb{Q}$ is the sphere, then you'd discover the Poincaré conjecture is false in the number-field setting, which is unfortunate. Rather, let's imagine that number fields are a new class of examples of 3-manifolds, with some commonalities and some other properties, and function fields are another family. So we can then study our new, rich class of examples.

Returning to the question of why four-dimensional topological field theory, well, first we have to discuss exactly what a topological field theory is, but we will see that one of the basic invariants of such a creature is that to ever $(n-1)$-manifold, one obtains a vector space. So the Langlands program assigns vector spaces (or things related to it, such as graded vector spaces, or chain complexes) to function and number fields, which are 3-dimensional in our analogy, and therefore we expect a four-dimensional story in physics.

More generally, an n-dimensional quantum field theory has dynamics: you get in addition to your vector space on an $(n-1)$-manifold, a Hilbert space structure and a Hamiltonian. Again, you might have something like a chain complex instead of a vector space. But the Hamiltonian makes this more like a quantum mechanics problem on your codimension-1 manifold. In topological theories, the Hamiltonian is 0.

Now, back to locally symmetric spaces: if F is a number field, we think of the field theory as assigning to it the vector space $L^2(\Gamma_{\mathcal{O}_F}\backslash G/K)$. The space $\Gamma\backslash G/K$ does not directly appear; instead, it is a moduli space of solutions to certain relevant equations on a 3-manifold.

Other parts of the story carry over too. Turning on the conductor/ramification N, we have not just a 3-manifold, but also a knot or link inside it, which we think of as the locus along which singularities can appear. In physics, these are called "codimension-2 defects," an important piece of data in general QFT.

To recap: we went very quickly today, and will go quickly on Thursday, but the class will mostly go more slowly, starting next week, where we more carefully keep track of the structures on both sides of this story, trying to stay on the safe, geometric tightrope between these two paradigms.

Thursday we will dig into more of the physics analogues of the variables we can twiddle on the number-theoretic side: what happens if we vary the conductor, if we vary the number field F, if we play with functoriality, etc.

Lecture 2.

A tale of two TFTs: 1/21/21

Last time, we talked about a perspective on modular forms (or automorphic forms): pick your favorite reductive algebraic group or matrix group, such as GL_n or PSL_2,[2] and let $F/\mathbb{Q}$ be a number field. You can let $F = \mathbb{Q}$ if you want. Let $\mathcal{O}_F$ be the ring of integers of F; if $F = \mathbb{Q}$, $\mathcal{O}_F = \mathbb{Z}$.

Today we will discuss what happens when we vary F, and how this affects a moduli space of principal bundles (TODO: missed this). We obtained a locally symmetric space by taking the real points of the group and taking the double quotient by an "arithmetic" lattice and the maximal compact. For example, we can take $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathrm{PSL}_2(\mathbb{R})/\mathrm{SO}_2$. This is for $S = \mathbb{Q}$.

Now let's consider more general F. We have $\mathrm{PSL}_2(\mathcal{O}_F)$ with no issue, but what should replace $\mathrm{PSL}_2(\mathbb{R})$ and its maximal compact? Instead we consider $F \otimes_{\mathbb{Q}} \mathbb{R}$, which is a ring of the form

$$F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}, \tag{2.1}$$

where F has r_1 embeddings into $\mathbb{R}$ and r_2 pairs of conjugate embeddings into $\mathbb{C}$. Then we replace $\mathrm{PSL}_2(\mathbb{R})$ with $\mathrm{PSL}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$.

Example 2.2. Suppose $F = \mathbb{Q}(\sqrt{d})$, where d is squarefree.

- If $d > 0$, so this is a real quadratic extension, $r_1 = 2$ and $r_2 = 0$.
- If $d < 0$, so this is an imaginary quadratic extension, $r_1 = 0$ and $r_2 = 1$. $\blacktriangleleft$

Then we can take the maximal compact K of $\mathrm{PSL}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$ as normal, and obtain a locally symmetric space. If F is a real quadratic field, this leads us to *Hilbert modular forms*, via $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$, acting on $\mathbb{H} \times \mathbb{H}$. In the imaginary quadratic case, we get $\mathrm{PSL}_2(\mathbb{C})/\mathrm{SO}_3 \cong \mathbb{H}^3$, hyperbolic 3-space, and there are connections to hyperbolic geometry. Usually these double quotients are not algebraic varieties, as this example demonstrates.

For other F, we'll get products of $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{C})$; what's most interesting is the arithmetic lattice $\mathrm{PSL}_2(\mathcal{O}_F)$.

Once we have these arithmetic locally symmetric spaces $\mathcal{M}$, we want to produce vector spaces out of them, including $L^2(\mathcal{M})$, $H^*(\mathcal{M})$, and twisted versions thereof. Importantly for the geometric Langlands program, these vector spaces carry an action of a huge commutative algebra, which is a tensor product over all the primes in F of a polynomial ring in $\mathrm{rank}(G)$ generators.

One could also allow ramification, obtaining generalizations $\mathcal{M}_{G,F,N}$, where $N \in \mathrm{O}_F$, and you replace $\mathrm{PSL}_2(\mathcal{O}_F)$ with a congruence subgroup Γ_N in which we impose conditions on our matrices mod N. These are a few different conditions you might impose (e.g. identity mod N, or upper triangular mod N). The arithmetic locally symmetric space is $\Gamma_N\backslash G_{\mathbb{R}}/K$, and the large commutative algebra is now "only" a tensor product over the primes p not dividing N.

This kind of idea, of a vector space associated to a number field, or maybe a vector space associated to a number field and some primes, is reminiscent under the arithmetic topology analogy to the state spaces in a 4d topological field theory. As we discussed last time, this is a refinement of Weil's Rosetta stone, where $\operatorname{Spec}\mathbb{Z}$ feels like a curve with points $\operatorname{Spec}\mathbb{F}_p$ associated to primes p, and $\operatorname{Spec}\mathbb{Z}_p$ as a small disc around this point. Inside that there is the punctured disc $\operatorname{Spec}\mathbb{Q}_p$. This is analogous to having a smooth, reduced algebraic curve over a finite field $\mathbb{F}_q$, which locally looks like $\operatorname{Spec}\mathbb{F}_q[t]$. Here the points are $\operatorname{Spec}\mathbb{F}_q$, and around this is the disc $\operatorname{Spec}\mathbb{F}_q[[t]]$ with the punctured disc $\operatorname{Spec}\mathbb{F}_q((t))$.

Now we look at the étale topology of $\operatorname{Spec}\mathbb{Z}$, which is a fancy way to say we care about the cohomology of Galois groups. From this perspective, the Rosetta stone isn't quite rich enough: $\operatorname{Spec}\mathbb{F}_p \hookrightarrow \operatorname{Spec}\mathcal{O}_F$ is a point, but not étale-topologically: the étale topos tells us that this "point" has a whole bunch of interesting covering spaces, such as $\operatorname{Spec}(\mathbb{F}_{p^n}) \to \operatorname{Spec}(\mathbb{F}_p)$. Its étale fundamental group is $\widehat{\pi}_1^{\text{ét}}(\operatorname{Spec}\mathbb{F}_p) = \mathrm{Gal}(\mathbb{F}_p) \cong \widehat{\mathbb{Z}}$. You can think of this profiniteness as not really there: we can only see finite extensions or, said differently,

[2] We're not thinking specifically of these groups as over a specific field, such as $\mathrm{GL}_n(\mathbb{R})$, but a machine for assigning to a field k a group $\mathrm{GL}_n(k)$. This technicality is important because F varies today.

finite covering spaces, and at this level there's no way to distinguish $\mathbb{Z}$ and $\widehat{\mathbb{Z}}$. This is a common feature of étale fundamental groups.

So the point is that the point $\operatorname{Spec}\mathbb{F}_p$ behaves a lot like a circle if you want to do étale things. (TODO: picture of $\operatorname{Spec}\mathbb{Z}$ with a circle at each prime). Therefore $\operatorname{Spec}\mathbb{Z}$, and its siblings $\operatorname{Spec}\mathcal{O}_F$, feel more like 3-manifolds than Riemann surfaces. And there are other ways to make this fuzzy analogy less fuzzy: there is a version of Poincaré duality, for example, with the correct dimension.

The monodromy around these circles is the Frobenius, but different Frobenii at different primes don't talk to each other. Because the curve $C/\mathbb{F}_q$ maps to $\operatorname{Spec}(\mathbb{F}_q)$, which is sort of like a circle, we think of these 3-manifolds as fibered over a circle.

Given this perspective, what is $\operatorname{Spec}(\overline{\mathbb{F}}_q)$? Étale-topologically, this is actually a point, but if you want a good dictionary between covering spaces and Galois representations, it should be a covering space of the circle, and in fact the universal one, $\mathbb{R}$. This is fine: for the purposes of topology and cohomology, $\mathbb{R}$ is a fine stand-in for a point. Now base-change C to $\overline{C} := C_{\overline{\mathbb{F}}_q} := C \times_{\operatorname{Spec}\mathbb{F}_q} \operatorname{Spec}\overline{\mathbb{F}}_q$; now we have something which feels like a bundle of Riemann surfaces, i.e. curves over algebraically closed fields, fibered over the real line $\operatorname{Spec}\overline{\mathbb{F}}_q$.

As we discussed last time, the monodromy around the circle is the Frobenius, so we can think of these surface bundle analoues as mapping tori for the Frobenius.

Let's discuss one more piece of evidence for this arithmetic topology dictionary: what happens with $\operatorname{Spec}\mathbb{Z}$? Let p be prime so we get a "circle" $\operatorname{Spec}\mathbb{F}_p$ in $\operatorname{Spec}\mathbb{Z}$. The neighborhood $\operatorname{Spec}\mathbb{Q}_p$ now behaves like a tubular neighborhood of this circle inside $\operatorname{Spec}\mathbb{Z}$. More generally, we can work with $\operatorname{Spec}\mathcal{O}_F$ and a prime $p \in F$ and a place v to complete $\mathcal{O}_F$ at, and obtain a local field F_v; then we might expect $\operatorname{Spec}(F_v)$ to be a tubular neighborhood of the circle $\operatorname{Spec} F/p$ — though (TODO) the place has to know something about p.

If F_v is a non-Archimedean local field, such as $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$, then $\operatorname{Gal}(\overline{F}_v/F_v)$ surjects onto $\mathbb{Z}_\ell \rtimes \widehat{\mathbb{Z}}$. Fun fact for those interested in group theory: this semidirect product is an example of a *Baumslag-Solitar group*

$$\mathrm{BS}(1,p) := \langle \sigma, u \mid \sigma u \sigma = u^p \rangle. \tag{2.3}$$

Here σ is the Frobenius and u is a generator of $\mathbb{Z}_\ell$. This interpolates between $\mathbb{Z}\times\mathbb{Z} = \pi_1(T^2)$, for $p = 1$, and $p = -1$, which is π_1 of the Klein bottle. So this does feel sort of like a torus neighborhood of a knot in a 3-manifold. So primes look like circles, and local fields look like 2-manifolds — not just any 2-manifolds, but 2-manifolds fibered over the circle.

In general, the Galois group $\operatorname{Gal}(\overline{F}_v/F_v)$ can be assembled from three pieces: the Galois group of the residue field $\widehat{\mathbb{Z}}\cdot\sigma$, the *tame part*, which is a product of $\mathbb{Z}_\ell$s for $\ell \neq p$ (here p is the characteristic of the residue field), and the *wild part*, which is a p-group.

This analogy is nice and important, tying arithmetic and geometric Langlands together, but we will spend the most time in places where it is the most concrete. Let's summarize the analogy.

- The following objects are thought of as three-dimensional: number fields $\operatorname{Spec}\mathcal{O}_F$ and function fields of curves $C/\mathbb{F}_q$, and mapping tori of diffeomorphisms $\phi\colon \Sigma \to \Sigma$ of Riemann surfaces. The first two of these are related to the *global Langlands program*, and we refer to the "global arithmetic setting."
- Here are some two-dimensional objects: local fields $F_v/\mathbb{Q}_p$ and their spectra, which are lik punctured discs; and $\mathbb{F}_q((t))$, which is also sort of a punctured disc. This is the setting of the *local Langlands program* in the "local arithmetic setting". There are two more kinds of 2-dimensional objects: a curve over an algebraically closed field of positive characteristic $\overline{C}/\overline{\mathbb{F}}_q$, or a closed Riemann surface Σ. These latter two objects form the "global geometric setting."
- One-dimensional objects: $\operatorname{Spec}\mathbb{F}_q$ and $\operatorname{Spec}\mathbb{C}((t))$ both are analogues of circles. The latter is a punctured disc, so not exactly one-dimensional, but it's close enough to be useful; it is the "local geometric setting."
- And lastly, zero-dimensional objects: $\operatorname{Spec}\overline{\mathbb{F}}_q$ and $\operatorname{Spec}\mathbb{C}$.

One major theme in this class is to apply a 4d topological field theory to these objects. If you complain that there aren't any 4-manifolds, that's a good question, but we will only consider a few 4-manifolds, such as products of 3-manifolds with circles or more generally mapping tori.

The Langlands program is an equivalence of 4d "arithmetic topological field theories." Arithmetic TFTs are not an entirely well-defined object, but we have much of the data that such a definition would need. We pick a group G and get a dual group $G^\vee$; then the two arithmetic TFTs are called $\mathcal{A}_G$ and $\mathcal{B}_{G^\vee}$; $\mathcal{A}_G$ is called the *automorphic* or *magnetic side*, and $\mathcal{B}_{G^\vee}$ is called the *spectral* or *electric side*. There is a sense in which this is a 4-dimensional version of mirror symmetry (which is usually a story about 2d QFT). These two TFTs $\mathcal{A}_G$ and $\mathcal{B} + G^\vee$ are fully extended, in that they assign higher-categorical objects to lower-dimensional objects. That is, we will be able to assign things to two-, one-, and maybe zero-dimensional objects in the above dictionary: a two-manifold gets a category, a 1-manifold gets a 2-category, and if you're very ambitious, a 0-manifold gets a 3-category.

$\mathcal{A}_G$ is a machine for taking a 3-manifold M and attaching a vector space $\mathcal{A}_G(M)$, which we will see is built from functions on arithmetic locally symmetric spaces. For example, $\operatorname{Spec} \mathcal{O}_F$ gets some sort of functions on $\mathcal{M}_{G,F}$. This is a large amount of structure, and one advantage is that it will explain some of the weird properties of modular forms. We will also spend some time on the $\mathcal{B}$-side, which is easier to describe.

There's an interesting tradeoff involving dimension, category number, and difficulty: making sense of what these arithmetic TFTs assign to 4-manifolds is very difficult: there are infinities and difficult renormalizations to deal with, and analyis that is beyond the scope of the course. Vector spaces are nicer and easier to make, but 3-manifolds are difficult. Dimension 2 is the sweet spot: categories aren't that bad, and 2-manifolds are pretty tractable. By the time we get to 1-manifolds, we have to work the category theory harder, and understanding what happens in dimension 0 is almost entirely open. We will not solve this open question in this class.

Now a TFT has additional structure: you can take a manifold with some additional structure, called defects, and assign algebraic data to this too. These bells and whistles line up very nicely on the arithmetic and topological sides: for example, number theorists will tell you the importance of allowing ramification/congruence subgroups in defining your arithmetic locally symmetric spaces. Under the arithmetic topology dictionary, this corresponds to studying what your TFT assigns to a 3-manifold with an embedded link with a label. In physics language, the link is a *codimension 2 defect*. The additional data of the link gets the modified vector space built using the congruence subgroup.

The large commutative algebra acting on the space of functions on $\mathcal{M}$ contains elements called *Hecke operators*, and these correspond to codimension 3 defects, or *line operators*. Physics-wise, you can think of these as "creating magnetic monopoles." And periods of automorphic forms correspond to boundary conditions, which are a codimension 1 phenomenon. This is related to recent work and work in progress of the professor!

Finally, there is Langlands functoriality, which also fits into this picture: more than just boundaries, there are codimension 1 phenomena called *domain walls*, which you can think of as an interface between two regions on a manifold which have two different theories on them.

So to summarize this, all the bells and whistles in the theory of automorphic forms belong to this QFT story.

The $\mathcal{B}$-side. Here we've taken the theory of automorphic forms and passed it through a prism to decompose it into "colors" related to Galois representations. Number-theoretically, this side is very very hard, because Galois groups of number fields are complicated, and the $\mathcal{A}$-side is often used to gain information about the $\mathcal{B}$-side. Geometrically, fundamental groups of Riemann surfaces are much easier, so the $\mathcal{B}$-side is used to learn about the $\mathcal{A}$-side.

But at least the $\mathcal{B}$-side is easier to state: we study the algebraic geometry of spaces of Galois representations; the $\mathcal{A}$-side has to do with the topology of the arithmetic locally symmetric space $\mathcal{M}$, by contrast. Geometrically, we might fix a manifold M and consider the space $\mathrm{Loc}_n(M)$ of representations $\pi_1(M) \to \mathrm{GL}_n(\mathbb{C})$. These are nice objects, called *character varieties*, and you can study the algebra of functions on them. You also don't just have to restrict to $\mathrm{GL}_n(\mathbb{C})$: we in particular care about $\mathrm{Loc}_{G^\vee}(M) := \{\pi_1(M) \to G^\vee\}$. The vector space that $\mathcal{B}_{G^\vee}$ assigns to a 3-manifold is the vector space of functions on $\mathrm{Loc}_{G^\vee}(M)$, and in fact defining this theory as an extended TFT is considerably easier than for the $\mathcal{A}$-side. Back on the arithmetic side, $\pi_1(M)$ is analogous to a Galois group, so in the arithmetic setting, we are looking at varieties of Galois representations.

The conjectured equivalence of these two (arithmetic) TFTs is something amazing: the huge amount of structure on the $\mathcal{A}$-side is equivalent to the simpler-to-define $\mathcal{B}$-side, and all of the structure passes back and forth. But "conjectured" is a very big word here: in both the arithmetic and geometric settings, there's a lot left to do, and even to define, to make these analogies precise. In the geometric setting, more is known, but there's still plenty of work in progress, including work of the professor, Arinkin, Gaitsgory, Raskin, and many more. The number field story is the work of Lafforgue and many others, but not a lot of this is proven.

The dictionary is not just nice to look at: you can use work done in the geometric setting to learn about what you should be working towards in the arithmetic setting, for example.

Lecture 3.

Spectral theory and sheaf theory: 1/26/21

Today we begin going more slowly and deeply; in the next several lectures, we'll focus on the picture of the prism, in which automorphic forms enter in on the left and are spectrally decomposed into Galois representations. Before we get into spectral decomposition, though, what's a spectrum?

We start with geometry, which might mean different things precisely to different people, but in geometry there is some notion of spaces, and given a space there is a commutative ring of functions, with multiplication taken pointwise. This is the basic starting point for algebraic geometry: to study spaces, study their algebra of functions. In fact, we can think of taking the ring of functions as a contravariant functor $\mathcal{O}$ from spaces to commutative rings: given a map $X \to Y$ of spaces, we want a pullback map of functions which preserves pointwise addition and multiplication.

The fundamental idea of a spectrum is to go backwards: begin with commutative algebra and build a space out of it. Category theory clarifies precisely what we're trying to do: there should be a spectrum functor Spec from commutative algebra to spaces, which satisfies a universal property concisely summarized by asking for it to be a right adjoint of $\mathcal{O}$. Maybe to be completely precise, we need to say what classes of spaces and functions we care about, and there are different options, but in those different situations you have this basic question.

Explicitly, saying that Spec is a right adjoint to $\mathcal{O}$ says that for every commutative algebra R, the set of maps of spaces $X \to \operatorname{Spec} R$ is naturally isomorphic to $\operatorname{Hom}_{\mathcal{R}ing}(R, \mathcal{O}(X))$. You can produce a "weak solution" in the functor category $\mathcal{F}un(\mathcal{S}paces, \mathcal{S}et)$, where $\operatorname{Spec} R$ sends $X \mapsto \operatorname{Hom}_{\mathcal{R}ing}(R, \mathcal{O}(X))$, and we can ask whether this is a true solution, in that it's represented by an actual space. We also want $\mathcal{O}(\operatorname{Spec} R) = R$, where "=" means "naturally isomoprhic".

There are several different settings in which this works pretty well.

Example 3.1. Suppose "space" means "finite set," and "function" means k-valued functions, for your favorite field k. (My favorite field is $\mathbb{C}$. What's yours?) Now, $\mathcal{O}(X)$ is the ring of functions $\{X \to k\}$, i.e. $\prod_{x \in X} k$, or the algebra of diagonal $|X| \times |X|$ matrices. So in this setting, there are plenty of algebras, such as $k[x]/(x^2)$, which are not the k-algebras of functions on spaces. ◀

Example 3.2 (Gelfand)**.** By "spaces" we mean locally compact Hausdorff spaces and by "functions" we mean $\mathbb{C}$-valued continuous functions vanishing at infinity: $\mathcal{O}(X) \coloneqq C_0(X)$. This has the structure of a *commutative C^*-algebra*, with the $*$-operation given by complex conjugation.[3]

In this setting, there is a nice Spec functor from commutative C^*-algebras to l.c. Hausdorff spaces, the *Gelfand spectrum.* Given such an algebra A, $\operatorname{mSpec} A$ is defined to be the space of maximal ideals of A. These are identified with the set $\operatorname{Hom}_{C^*}(A, \mathbb{C})$, and this can be profitably thought of as the "points" of A: $\mathbb{C}$ is the functions on a point, so this is heuristically the maps $\mathrm{pt} \to \operatorname{mSpec} A$. Alternatively, these are the unitary one-dimensional representations of A.

Theorem 3.3 (Gelfand-Naimark)**.** $(\mathcal{O}, \operatorname{mSpec})$ *are contravariant equivalences of categories from locally compact Hausdorff spaces to commutative C^*-algebras.*

So the world of continuous topology is completely known by algebra.

Under this equivalence, compact Hausdorff spaces (i.e. the nice ones) are exchanged with *unital* commutative C^*-algebras (i.e. the nice ones). This is because the constant function with value 1 is bounded in the C^* norm iff the domain is compact. ◀

Example 3.4. As an even coarser example, we can let "spaces" mean measure spaces and "functions" mean $L^\infty(X)$, which lands in the world of commutative von Neumann algebras. There should be a few more words here to make everything precisely. There is again a contravariant equivalence of categories, and this time there's not very many isomorphism classes of objects: finite unions of points, countable unions of points, intervals, and unions of intervals and some points. ◀

[3] See [aHRW09] for some discussion on this duality.

Example 3.5. Algebraic geometry is the best-studied example, but it doesn't work quite as well as some of these other examples. In this case, spaces means *locally ringed spaces*, i.e. topological spaces X together with a sheaf of rings $\mathcal{O}_X$ with a property that we're not going to go into here, and rings means commutative rings. Taking global sections of $\mathcal{O}_X$ defines a contravariant functor to commutative rings, and there is a right adjoint Spec, the spectrum of a ring. It is not essentially surjective: things in the image are called *affine schemes*, and on the full subcategory of affine schemes, $(\mathcal{O}, \operatorname{Spec})$ are contravariant equivalences of categories.

Unfortunately, this doesn't capture lots of important examples: locally ringed spaces which locally look like affine schemes. Lots of important objects in algebraic geometry, such as projective lines, are built out of affine schemes this way, but are not themselves affine. So you expand your notion of geometry a little bit, but from this perspective there are useful things which are weak but not strong solutions to representing a functor to sets, and at that point why not just do geometry with said weak solutions? ◀

Example 3.6 (Quillen-Sullivan rational homotopy theory)**.** In homotopy theory, you might want to study the *homotopy category*, a category built out of (locally compact weakly Hausdorff) topological spaces by inverting homotopy equivalences.[4] Here "rings" means graded commutative rings, and the functor is $H^*(-;\mathbb{Z})$. Cohomology is nice but this isn't quite flexible enough to set up a good spectral theory.

You can get a better correspondence by remembering the entire category of (locally compact weakly Hausdorff) topological spaces and letting the functor be rational cochains, which is valued in the category of commutative differential graded $\mathbb{Q}$-algebras, or CDGAs. Quillen-Sullivan showed that this defines a nice spectral theory: restricting to simply connected spaces, this defines an equivalence of categories from spaces modulo rational homotopy equivalences to simply connected $\mathbb{Q}$-CDGAs.[5]

There are analogous statements by Mandell p-adically, and by Allen Yuan [Yua19] very recently integrally, albeit using cochains with a little more structure. ◀

Remark 3.7. Most of the uses of "spectrum" in math – algebro-geometric, operator-theoretic, even mathematical-physicsy — are all related. The one exception is the homotopy theorists' spectrum, which means something different. Beware this common source of confusion. In this class, "spectrum" will mostly mean integrally. ◀

Let's get back to spectral decomposition. Let R be a commutative ring (often a k-algebra; for us, always a k-algebra, where $k = \mathbb{C}$) acting on a module (for us, a complex vector space) V. This is data of an algebra homomoprhism $R \to \operatorname{End}(V)$. Our perspective on spectral decomposition is that we want to sheafify, or localize, or spread out, or spectrally decompose this module, over $\operatorname{Spec} R$. To do this, we will use that $\mathcal{M}od_R$ is *symmetric monoidal*: we have a tensor product $\otimes_R \colon \mathcal{M}od_R \times \mathcal{M}od_R \to \mathcal{M}od_R$. Thus we can define the sheaf $\underline{V}$ associated to the module V to be

$$\underline{V}(U) \coloneqq V \otimes_R \mathcal{O}(U). \tag{3.8}$$

One immediate consequence is that you can talk about the *support* of an element $v \in V$, which is a subset $\operatorname{supp}(v) \subset X$.

For example, if X is a finite set, so $R = \prod_{x\in X} \mathbb{C}$, then

$$V = \bigoplus_{x\in X} V_x, \tag{3.9}$$

It will be useful for the sheaves produced by this construction to have a name.

Definition 3.10. A *quasicoherent sheaf* on $\operatorname{Spec} R$ is one obtained from an R-module in this way. If the R-module is finitely generated, the sheaf is called a *coherent sheaf*.

Not all schemes are affine, so we say that (quasi)coherent sheaves are those which are locally of this form.

TODO: example I missed, where the spectral decomposition is the usual one of a vector space into eigenspaces. Is this $\mathbb{C}[x]$ with x acting by the matrix in question? Or more of the finite set example? This example is a little basic: points are open and closed, and so the eigenspaces are both a hom and a tensor. This is not true in general.

Now, how does the spectral theorem appear in this context? Let V be a vector space (not necessarily finite-dimensional, though in general you need some nice topology here) and $M \in \operatorname{End}(V)$. We think of this

[4] You have to do this carefully, to avoid set-theoretic issues, but it can be done.

[5] You can generalize slightly to *nilpotent* spaces, but you must have some sort of condition on π_1. This is sort of like the non-affineness in algebraic geometry, though in practice it behaves a little differently.

matrix as a map of algebras: $\mathrm{Hom}_{\mathcal{S}et}(\mathrm{pt}, \mathrm{End}(V))$. Now $\mathrm{End}(V)$ has a lot of additional structure — it's an associative $\mathbb{C}$-algebra, though not commutative. So there should be an adjunction

$$\mathrm{Hom}_{\mathcal{S}et}(\mathrm{pt}, \mathrm{End}(V)) = \mathrm{Hom}_{\mathcal{A}lg_{\mathbb{C}}}(\mathrm{Free}, \mathrm{End}(V)), \tag{3.11}$$

where Free is the free associative algebra on one generator, which is $\mathbb{C}[x]$. So rather than the matrix M, we will think about the data of V being a $\mathbb{C}[x]$-module. In some contexts, this is called "functional calculus" — once you have a matrix, you can act by polynomials in this matrix. Our approach here is to think of all of these together, rather than just M. In fact, if you have nice enough topology, you can complete, and make sense of things like functions on $\mathbb{R}$, not just polynomials.

Anyways, the idea is that $M \in \mathrm{End}(V)$ is equivalent to V being the global sections of some quasicoherent sheaf on the *affine line* $\mathbb{A}^1 := \operatorname{Spec} \mathbb{C}[x]$. This will be our basic case of spectral decomposition: the simplest case of a family of commuting operators is a single operator.

We want to study how V spreads out over $\mathbb{A}^1$. There is a short exact sequence

$$0 \longrightarrow V_{\mathrm{tors}} \longrightarrow V \longrightarrow V_{\mathrm{free}} \longrightarrow 0, \tag{3.12}$$

using the structure theory of modules over PIDs, and in fact this splits. So $V_{\mathrm{free}} \cong \mathbb{C}[x]^{\oplus r}$, and the torsion part is a direct sum

$$V_{\mathrm{tors}} \cong \bigoplus_{\lambda \in \mathrm{Spec}(V)} V_{\widehat{\lambda}}. \tag{3.13}$$

The free part we will call the *continuous spectrum*, and the torsion part the *discrete spectrum*. Specifically, if V is supported at $\lambda \in \mathbb{A}^1$, this is saying λ is a generalized eigenvalue, and $V_{\widehat{\lambda}}$ is (data equivalent to) the Jordan block for M.

Eigenvectors $Mv = xv = \lambda v$ are equivalent to elements of $\mathrm{Hom}_{\mathbb{C}[x]}(\mathbb{C}_\lambda, V)$, and this has to do with a quotient, rather than a sub (TODO: missed something here). And if you look in the continuous spectrum, there are no eigenvectors, which is to say that over $\mathbb{A}^1$, the free part looks like sections of the trivial bundle, and the discrete spectrum is skyscrapers at points. In both cases we can take fibers (quotients). (TODO: I should draw a picture of something like this.)

Remark 3.14. There are many versions of the spectral dictionary. If we talk about von Neumann algebras and measurable spaces, the corresponding spectral theorem is von Neumann's spectral theorem: $M = A$ is a self-adjoint operator on a Hilbert space V. This theorem, reinterpreted sheafily, says there's a "sheaf," i.e. a projection-valued measure on $\mathbb{R}$ ($\mathbb{R}$ is the spectrum), and the operator A can be reconstructed as a direct integral

$$A = \int_{\mathbb{R}} x \, \mathrm{d}\pi_A, \tag{3.15}$$

with respect to the projection-valued measure π_A.

To make sense of this, let's say what a projection-valued measure is. This is an assignment from every measurable subset $U \subset \mathbb{R}$ to a projection operator $\pi_A(U)$ on V. Different projections should commute. So it's sort of a sheaf of Hilbert spaces, in a particularly weak sense. You can think of the space of sections on U to be $\mathrm{Im}(\pi_A(U))$, and there is a countable additivity property that

$$U \longmapsto \langle w, \pi_A(U)(v) \rangle \tag{3.16}$$

must be a $\mathbb{C}$-valued measure on $\mathbb{R}$. The spectrum of A is the support of π_A. If V is finite-dimensional, the direct integral (3.15) is a direct sum, and is a decomposition of V into A-eigenspaces. In general, the direct integral takes the continuous spectrum into account. ◀

And there are versions with similar pictures in the other spectral settings we discussed, such as Hilbert C^*-modules for C^*-algebras, etc. There's even a homotopical version of this: if $R = C^*(X)$ for a space X, then $\mathcal{M}od_R$ injects into $\mathcal{L}oc(X)$, the category of locally constant (complexes of) sheaves on X, and a module over R, finitely generated in the right sense, has a corresponding (complex of) sheaves on X, and it is finitely generated in the same way. This has to do with the fact that $C^*(X) = \mathrm{End}(k_X)$, where k_X is the constant local system, and $\underline{M}(U) = M \otimes_{C^*(X)} C^*(X)$. There are some homotopical details to fill in here, but everything can be made precise; the point is that there is an analogue of the story here too: you have algebras R as associated to spaces $\operatorname{Spec} R$, and modules over R spread out over $\operatorname{Spec} R$.

Spectral decomposition shines where there are lots of examples of algebras, spaces, and modules for us to work with. For this reason, we turn to Fourier theory.

In this setting, we want distributions to be the linear dual $\mathrm{Hom}(V, \mathbb{C})$ to V. There are lots of weak eigenvectors, such as δ-functions, which might not be actual eigenvectors (for example, the problem with the continuous spectrum we saw above). For example, if $V = L^2(\mathbb{R})$, with the operator $M = x$, you can't make sense of "a function supported at x" in L^2-land. You can do this for distributions, though. Dually, if $M^\vee = \frac{\mathrm{d}}{\mathrm{d}x}$, we have natural eigenvectors $e^{i\lambda x}$ for λ, but these are not L^2: they live in something bigger, controlled by a different norm. This is what the continous spectrum often looks like: you have a direct integral, which is different than direct sum. The things you're integrating aren't actually subsets. For example, functions on $\mathbb{A}^1$ are functions on a point, directly assembled together, but functions at a point in $\mathbb{A}^1$ aren't a subset of functions on $\mathbb{A}^1$, but instead a quotient. This behaves a little better when there's no torsion, but that obscures the general story. And this general story isn't a weird analysis fact, because it appears for polynomials in algebra too.

Next time, we'll talk about Fourier theory, or abelian duality, from this perspective: as a natural source for commuting operators. Abelian groups G acting on vector spaces V are a good place to look for large algebras of commuting operators. We will spectrally decompose V using these operators. The aim of the class, and in some sense the broader aim of the Langlands program, is a nonabelian generalization of this, and we will spend a few weeks on the abelian story.

Lecture 4.

Some Fourier theory: 1/28/21

Today, we'll say a bit more about spectral decomposition before diving into Fourier theory.

We begin with a commutative algebra A, and build a geometric object $\operatorname{Spec} A$. What precisely these things are depends on your specific formalism, e.g. if you care about C^*-algebras and topological spaces, or commutative rings and affine schemes, or other possibilities.

Last time we saw how an A-module M "spreads out" in a spectral decomposition over $\operatorname{Spec} A$, defining a sheaf on it. You can think of this with physics: there is a physical system with an algebra A of observables and a space M of states. $\operatorname{Spec} A$ has the defining universal property that maps $A \to \mathcal{O}(X)$ are in natural bijetion with maps $X \to \operatorname{Spec} A$. In physics, you might think of making observations as a way of understanding the geometry of X, and observations might be something like functions to a line (so $\operatorname{Spec} A$ is the line here). So maybe X fibers over the line. Observables on the line now tell us something about X.

Modules and states linearize this story: we took M and sheafified it into a sheaf $\mathcal{M} \to \operatorname{Spec} A$. A single function on X is a map to $\mathbb{A}^1 = \operatorname{Spec} k[x]$, and likewise a single matrix (endomorphism) of a vector space gave us a sheaf on $\mathbb{A}^1$ via the Jordan form.

This is exactly how observations happen in quantum mechanics: we don't have a classical phase space like in classial mechanics, only its linearization, the Hilbert space $\mathcal{H}$ of states of the system. Observables are self-adjoint operators on $\mathcal{H}$; it is also useful to talk about the (noncommutative) algebra $\mathrm{End}(\mathcal{H})$ of all operators on $\mathcal{H}$. Let $\mathcal{O}$ be an observable; then, just as for a vector space and an endomorphism, we can sheafify $\mathcal{O}$ into a projection-valued measure (the analogue of a sheaf) on $\mathbb{R}$, where $\mathbb{R}$ is the spectrum of the algebra generated by a single operator.

From this perspective, a state $|\varphi\rangle \in \mathcal{H}$ is a section of the sheaf, i.e. projections of each vector onto the "fibers," which are the images of the projection-valued measures. Given a section, you can ask where it is supported, i.e. you made a measurement, where does it live? That's the support. We can also do something more precise: use the norm. Now $\|\varphi\|^2$ is a complex-valued measure on $\mathbb{R}$: take φ, project onto a fiber, and then take the norm. Suitably normalized, this is a probability measure on $\mathbb{R}$, which tells you where you expect this measurement to live, and ask questions such as what its expectation value is, as

$$\int_{-\infty}^{\infty} x \|\mathrm{proj}_{\mathcal{H}_\lambda} |\varphi\rangle\|^2 \, \mathrm{d}\lambda. \tag{4.1}$$

This is a continuous version of the fact that the expected value over a finite probability space S is a sum:

$$\frac{1}{\langle\varphi \mid \varphi\rangle}\sum_{\lambda\in S}\lambda\langle\psi_i \mid \varphi\rangle|\varphi\rangle, \tag{4.2}$$

where $\{\psi_i\}$ is an S-indexed basis of eigenvectors.

The main observable that is part of the data of a quantum mechanics system is the *Hamiltonian*, a particular self-adjoint operator H. It is the energy functional: its eigenstates are the steady states of the system, fixed by time-evolution, and their eigenvalues measure the energy for each state.

The perspective of algebras as observables, modules as spaces of states, and sheaves as spectral decomposition will come up again and again in the class, even though we will mostly see commutative algebras (which glue a lot better: you can try C^*-algebras, but Gelfand-Naimark tells us that all spaces are affine, so why glue?). The context and language will be fancier, but this philosophy will still shine through.

Let's talk about Fourier theory and abelian duality, which will carry us through the next few weeks. We will think of this as a special case of spectral decomposition, where the source of commuting operators is an abelian group G.

So, fix an abelian group G and a representation $T\colon G\to \mathrm{Aut}(V)\subset \mathrm{End}\,V$. We'll write that the action of $g\in G$ on V is called T_g. Often, we can cook up good examples of these representations by considering a G-action on a space X, and then taking functions on X. Then G acts by pullback. As we've said before a few times in this class, there are different levels of regularity you can do this in, and the specific one you pick isn't that important right now. If this worries you, sprinkle in the word "finite" where it helps.

The most canonical space that G acts on is itself, giving us the *regular representation.* We want to decompose, so we need to figure out what the spectrum is. If G acts on a one-dimensional vector space, we get a character $\chi\colon G\to\mathbb{C}^\times\subset\mathbb{C}$, and these are the eigenvalues that can arise.

Let $\widehat{G} \coloneqq \mathrm{Hom}_{\mathcal{G}rp}(G,\mathbb{C}^\times)$ be the set of all of the characters; if you make G topological or algebraic or something, also impose that condition on the characters (e.g. continuous or smooth or polynomial). In fact, for a moment imagine G is a finite abelian group — and let's try not to think about the classification of finite abelian groups, because the story will go through in greater generality.

Because not all matrices are invertible, $\mathrm{End}(V)$ is only a monoid, and $G\to\mathrm{Aut}(V)\subset\mathrm{End}(V)$ is a monoid map. This is not a lot of structure; we want more. Specifically, $\mathrm{End}(V)$ is an algebra that we've forgotten down to a monoid. So this looks like one half of an adjunction:

$$\mathrm{Hom}_{\mathcal{M}onoid}(G,\mathrm{For}(\mathrm{End}(V))) = \mathrm{Hom}_{\mathcal{A}lg_{\mathbb{C}}}(\mathbf{?},\mathrm{End}\,V). \tag{4.3}$$

Filling in the $\mathbf{?}$ is the free algebra on a monoid, i.e. the *group algebra* $\mathbb{C}[G]$, which is the algebra generated by symbols δ_g for $g\in G$ with multiplication $\delta_g\cdot\delta_h=\delta_{gh}$. So you can think of this as formal linear combinations of elements of G — or you can say this is the algebra of functions $G\to\mathbb{C}$, whence the notation δ_g. And this is a suggestion that $\mathbb{C}[G]$ should really be thought of as dual to functions, as measures. The canonical counting measure on a finite set identifies functions and measures for us, but this won't generalize.

For general functions $f_1,f_2\colon G\to\mathbb{C}$, the product is convolution:

$$(f_1 * f_2)(k) = \sum_{g\in G}\left(\sum_{gh=k} f_1(g)f_2(h)\right) = \sum_{g\in G} f_1(g)f_2(kg^{-1}). \tag{4.4}$$

If you've studied convolution in an analysis class, this ought to look familiar. Also, so far we have not needed G to be abelian! And you can recast this in terms of pushforward and pullback.

There are three maps $G\times G\to G$: project onto the first and second factors π_1, resp. π_2, but also multiplication μ. So we can define an *external product*

$$f_1\boxtimes f_2 \coloneqq \pi_1^*f_1\pi_1^*f_2\colon G\times G\longrightarrow\mathbb{C}, \tag{4.5}$$

and because G is finite, we can push this forward along μ, summing over the fibers, and obtain the usual convolution; indeed, this is what (4.4) is telling us. If you care about infinite groups with some sort of regularity, that regularity can buy you the pushforward map in that setting.

The group algebra $\mathbb{C}[G]$ is commutative iff G is abelian, and G acting on V induces a $\mathbb{C}[G]$-action on G, more or less by that adjunction. The key fact is that

$$\operatorname{Spec}(\mathbb{C}[G], *) = \widehat{G}, \tag{4.6}$$

i.e. functions on $\widehat{G}$ are the group algebra. Why is this true? A point of the spectrum is a map $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{C}[G]$, so unwinds to a map $\mathbb{C}[G] \to \mathbb{C}$, under multiplication, which gives us a character.

So we have a dictionary

$$(\mathbb{C}[G], *) \overset{\sim}{\longleftrightarrow} (\mathcal{O}(\widehat{G}), \cdot), \tag{4.7}$$

and characters on the left are exchanged with δ-functions on the right. Moreover, $g \in G$ acts by translation on the left, and on the right it defines a function $\widehat{g}$, and translation is exchanged with multiplication by $\widehat{g}$.

This is a form of the Fourier transform. If f is a function on G, then there is an inversion formula

$$f = \sum_{t \in \widehat{G}} \widehat{f}(t) \cdot \chi_t, \tag{4.8}$$

where $\widehat{f}(t)$ is the coefficient of f in an orthonormal basis. And $\widehat{(f)}$ corresponds to $(\widehat{f}\cdot)$. All these statements are restatements of each other, and of the key statement that the group algebra is functions on the dual with pointwise multiplication, as well as functions on G under convolution.

There is an additional symmetry: $\widehat{G}$ is not just a set, but is itself an abelian group.[6] This is because you can pointwise multiply functions on G: the product of two characters is still a character. So $\widehat{G}$ is really the dual group. For this to deserve the name "dual," we need $\widehat{\widehat{G}}$ to be naturally isomorphic to G: we want an assignment for every $g \in G$ a function from characters to $\mathbb{C}^\times$, which of course is $\chi \mapsto \chi(g)$.

There is a more symmetric way to say this: there are two projections

$$\begin{tikzcd} G \times \widehat{G} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \widehat{G} \\ G, \end{tikzcd} \tag{4.9}$$

and on $G \times \widehat{G}$, there is a canonical function

$$K(g, t) \coloneqq \chi_t(g) = \chi_g(t), \tag{4.10}$$

which you can think of as a multiplicative kernel. Here χ_g is the character on $\widehat{G}$ given by g. So this setup is more symmetric (and may also remind you of some kernels from a functional analysis class). Now we can describe the Fourier transform as a pullback-pushforward:

$$\widehat{f}(t) = \pi_{2*}(\pi_1^* f \cdot \chi)(t) = \sum_g f(g)\overline{K(g, t)}. \tag{4.11}$$

We get a complex conjugate because there was one in the inner product. And it means the inverse Fourier transform looks slightly different:

$$f(g) = \sum_t \widehat{f}(t) K(g, t). \tag{4.12}$$

Great, we've spectrally decomposed functions on G — said differently, we've simultaneously diagonalized the action of G on the space of functions on G. "Simultaneous diagonalization" is a reminder how important being abelian is to the whole story.

Now this is a lot of work for just one representation, but because all G-representations are $\mathbb{C}[G]$-representations, then for any representation $\mathbb{C}[G] \to \operatorname{End} V$, we get a spectral decomposition of V over the spectrum $\widehat{G}$, i.e.

$$V = \bigoplus_{t \in \widehat{G}} V_{\chi_t}. \tag{4.13}$$

[6]If G is finite, the classification of finite abelian groups shows $\widehat{G}$ is noncanonically isomorphic to G. This is not true in general, so don't use it for your intuition any more than you need to.

That is, the fiber at t is the χ_t-isotypic component of V. This is sort of overkill for finite abelian groups, but generalizes.

Let G be a locally compact topological abelian group (LCA). There are lots of good examples that are not finite: $\mathbb{Z}$, U_1, $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{Z}_p$, $\mathbb{Q}_p^\times$, and many more. The dual is

$$\widehat{G} = \mathrm{Hom}_{\mathcal{T}op\mathcal{G}rp}(G, \mathrm{U}_1), \tag{4.14}$$

i.e. the continuous, unitary characters of G. If G is finite, this is exactly what we had already: all characters are valued in roots of unity.[7]

It was crucial for us that the dual $\widehat{G}$ was the spectrum of the group algebra. There are different ways to implement that in the LCA setting; the way we'll do it is to endow G with a measure. Specifically, G carries a *Haar measure*, which is a left-invariant measure with respect to G acting on itself by multiplication, and is unique up to scaling. If G is compact, this measure is bi-invariant and can be normalized to total measure 1, but we will often care about noncompact groups.

Anyways, our stand-in for $\mathbb{C}[G]$ is the C^*-algebra $L^1(G)$ (with respect to a chosen Haar measure), with convolution

$$(f_1 * f_2)(h) = \int_G f_1(g) f_2(g^{-1}h)\, \mathrm{d}g. \tag{4.15}$$

You can extract this from a kernel transform just as in the finite-group case. With this definition, $\mathrm{mSpec}(L^1(G), *) \cong \widehat{G}$, not just as topological spaces, but also group structures, and this is a version of the Fourier transform. This is again a duality, often called *Pontrjagin duality*, where the natural map $G \to \widehat{\widehat{G}}$ is an isomorphism of topological abelian groups.

The Fourier transform again has the formula

$$f \longmapsto \pi_{2*}(\pi_1^* f \cdot K), \tag{4.16}$$

where K is the kernel, with the same formula as before. The thing that's new here is that you have to do analysis to see what regularity appears on the other side. For L^1, you get an isomorphism $(L^1(G), *) \cong (C_v(\widehat{G}), \cdot)$, where C_v denotes the space of continuous functions vanishing at infinity. The Plancherel theorem gives you $(L^2(G), *) \cong (L^2(\widehat{G}), \cdot)$. And for both of those, translations by group elements are exchanged with multiplication. (To say this completely precisely, you may need to work with distributions, so that you have δ-functions.) Characters are exchanged with points; this is symmetric, but in the geometric Langlands program and the related topological field theory, the two sides are not symmetric.

Example 4.17 (Fourier series)**.** Say $G = \mathrm{U}_1$. Then $\widehat{G} = \mathrm{Hom}(\mathrm{U}_1, \mathrm{U}_1)$, which can be identified with $\mathbb{Z}$ via the map

$$n \longmapsto (x \longmapsto \exp(2\pi i n x)). \tag{4.18}$$

That is, $\mathbb{Z} \cong \widehat{\mathrm{U}_1}$.

The theory of Fourier series identifies $L^2(\mathrm{U}_1) \cong L^2(\mathbb{Z})$; the latter is often called ℓ^2. One decomposes a periodic function (a function on U_1) into its Fourier modes. There are versions of this for other kinds of regularity.

To think of this as a kernel transform, there is a function on $\mathrm{U}_1 \times \mathbb{Z}$ sending

$$x, n \longmapsto \exp(2\pi i n x), \tag{4.19}$$

and characters $\mathbb{Z} \to \mathrm{U}_1$ are identified by where 1 goes, which is anywhere, and therefore $\widehat{\mathbb{Z}} = \mathrm{U}_1$. ◂

This illuminates a nice fact about Pontrjagin duality: G is compact iff $\widehat{G}$ is discrete.

One nice reference for all this is Ramakrishnan-Valenza [RV99].

[7] If you care about algebraic geometry, you might be used to saying $\mathbb{G}_m$ instead of U_1. There is a version of this story for $\mathbb{G}_m$, too, but this particular kind of regularity requires U_1.

Lecture 5.

Pontrjagin and Cartier duality: 2/2/21

We spent some time the other day discussing Pontrjagin duality. To review, choose a locally compact abelian group G; its dual is

$$\widehat{G} := \mathrm{Hom}_{\mathcal{A}b}(G, \mathrm{U}_1), \tag{5.1}$$

which has canonically the structure of a locally compact abelian groups. This has many properties that resemble a Fourier transform, including that L^1 functions on one side are identifed with continuous functions on the other side that vanish at infinity, as well as L^2 functions on one side passing to L^2 functions on the other side. Characters on one side exchange with points on the other, and translation and convolution exchange with multiplication. G is compact iff $\widehat{G}$ is discrete (and, of course, vice versa).

Examples:

- The dual of U_1 is $\mathbb{Z}$: the characters $\mathrm{U}_1 \to \mathrm{U}_1$ are the maps $z \mapsto z^n$, indexed by $n \in \mathbb{Z}$. Correspondingly, the dual of $\mathbb{Z}$ is U_1.
- Let T be a compact torus, which is defined as a quotient $\mathbb{R}^d/\Lambda = \Lambda \otimes_{\mathbb{Z}} \mathrm{U}_1$ for a full-rank lattice $\Lambda \subset \mathbb{Z}^d$. This has an associated *dual lattice* $\Lambda^\vee \subset \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$, and the dual of T is $\Lambda^\vee$.
- $\mathbb{R}^d$ is self-dual — or, more precisely, the dual of $\mathbb{R}^d$ is $(\mathbb{R}^d)^*$. More generally, for a finite-dimensional real vector space V, the dual is V^*, the usual linear dual.

This last identification recovers the theory of the Fourier transform. Say x is the coordinate on the primal $\mathbb{R}$ and t is the coordinate on the dual $\mathbb{R}$; then the canonical character on $\mathbb{R}_x \times \mathbb{R}_t$ is $\exp(2\pi i x t)$, and when one writes down the transform as a pullback-pushforward, one recovers the usual Fourier transform.

More generally, if we began with V and obtained the Pontrjagin dual V^* (i.e. also the usual linear dual), the canonical pairing is $\exp(2\pi i\langle x, t\rangle)$, and the Fourier transform is

$$f(x) = \int_V \widehat{f}(t) e^{2\pi i \langle x, t\rangle}\, \mathrm{d}t. \tag{5.2}$$

In the Fourier transform, we know that differentiation by x (on the primal side) is exchanged with multiplication by t (on the dual side), just like translation by group elements. This makes sense: differentiation is an infinitesimal transfomation. It is a general example of how group theory on one side is exchanged with geometry on the other side.

And in general, if G is an abelian Lie group, then its Lie algebra $\mathfrak{g}$ maps to the Lie algebra of vector fields on G, which sits inside the algebra of differential operators on G, and there is an adjunction between forgetting the structure of an associative algebra to a Lie algebra and a free functor building the *universal enveloping algebra* $\mathcal{U}(\mathfrak{g})$ out of a Lie algebra $\mathfrak{g}$. Since $\mathfrak{g}$ is abelian, $\mathcal{U}(\mathfrak{g})$ is the usual symmetric algebra $\mathrm{Sym}^* \mathfrak{g}$.

By the adjunction, $\mathcal{U}(\mathfrak{g})$ acts on $C^\infty(G)$ as differential operators, and so we can sheafify over $\mathrm{Spec}(\mathcal{U}(\mathfrak{g})) = \mathfrak{g}^*$. So for $G = \mathbb{R}_x$, for example, $\mathfrak{g}^* = \mathbb{R}_t$ can be identified with $\mathrm{Spec}\,\mathbb{R}[t]$, where t *is* $\frac{\mathrm{d}}{\mathrm{d}x}$, which explains why differentiation becomes multiplication by t. So every aspect of the group theory on the primal side is simultaneously diagonalized, because everything is commutative.

Speaking of commutative things, let's talk about quantum mechanics, which is famously commutative.[8] This is a place the Fourier transform will happen. For the classical mechanics of a particle moving on a manifold M, the phase space is T^*M, the cotangent space, with local coordinates q (position) and p (momentum), with the momenta pointing in the bundle direction. These generate the algebra of observables.

In quantum mechanics, we replace T^*M with L^2 functions on half of the variables: just the positions. That is, the Hilbert space is $\mathcal{H} := L^2(M)$, and the observables include differential operators on M, including both functions on M and tangent vectors, with $p_j = i\frac{\mathrm{d}}{\mathrm{d}q_j}$. If $\xi \in TM$ and $f\colon M \to \mathbb{R}$ is a function, there is a commutator

$$\xi f = f\xi + \hbar f'. \tag{5.3}$$

Here $\hbar$ is Planck's constant.

We want to say that "states look like the square root of the observables," which has to do with the fact that we threw out half of the observables classically in order to quantize. The Fourier transform reenters the story because we want to diagonalize the momentum operators and their derivatives. This isn't really meaningful

[8] This is a joke.

in general, but if M is an abelian Lie group — often $\mathbb{R}^n$ — then we have a natural basis of commuting vector fields, and a Fourier transform exchanging $L^2(\mathbb{R}^n_q) \cong L^2(\mathbb{R}^n_p)$. This ultimately leads to an interesting nontrivial isomorphism between quantum mechanics on G and on $\widehat{G}$, and crucially, this isomorphism cannot be seen at the classical level. This is sort of a very simple version of abelian duality or mirror symmetry in 1d, and we will see echoes of this story in higher-dimensional, less trivial settings.

In the next few weeks, we will discuss Cartier duality, which is the algebraic version of this story, and then pass to its main manifestations in physics and in number theory: electromagnetic duality and class field theory. One takeaway will be that both of these can be thought of as kinds of Fourier transforms. These are abelian cases of the Langlands program.

Cartier duality is about algebraic groups. You can think of these in a few ways: things like in Lie groups but in algebraic geometry, such as group objects in the category of varieties, but it will also be helpful to think of algebraic groups through their functors of points. A variety X defines a functor $\mathcal{A}lg_k \to \mathcal{S}et$ by $X(R) \coloneqq \mathrm{Hom}(\operatorname{Spec} R, X)$: the set of "$R$-points" of X. Here $\mathcal{A}lg_k$ denotes commutative k-algebras.

If X is a group object in the category of varieties, that is equivalent structure to factoring the functor of points through $\mathcal{G}rp \to \mathcal{S}et$. That is, $\{\operatorname{Spec} R \to X\}$, the R-valued points of X, is a group, and these are compatible as R varies. And through the Yoneda lemma, X is determined by its functor of points.

Example 5.4. Consider the functor $R \mapsto \mathrm{GL}_n(R)$. This is representable, defining an algebraic group GL_n. You can play this game with other familiar groups such as SL_n. ◂

As with Pontrjagin duality, we care more about abelian algebraic groups. By the magic of the Yoneda lemma, this is the same thing as asking for the functor of points to factor through $\mathcal{A}b \hookrightarrow \mathcal{G}rp \to \mathcal{S}et$.

Example 5.5.

(1) The *additive group* $\mathbb{G}_a \coloneqq \mathbb{A}^1$, which sends an algebra R to the abelian group of functions on $\operatorname{Spec} R$.
(2) The *multiplicative group* $\mathbb{G}_m \coloneqq \mathbb{A}^1 \setminus 0$. This is $\mathrm{Aut}(k)$. Its functor of points is $\mathbb{G}_m(R) = R^\times$. This is our analogue of U_1: characters are functions to $\mathbb{G}_m$.
(3) Another slightly weirder example: the constant functor valued in $\mathbb{Z}$. ◂

Now given an abelian algebraic group G, we can let $\widehat{G} \coloneqq \mathrm{Hom}(G, \mathbb{G}_m)$, where this is homomorphisms of algebraic groups (if you like, natural transformations of $\mathcal{G}rp$-valued functors of points).

Useful examples.

- $\widehat{\mathbb{Z}} = \mathbb{G}_m$, just as in the topological case: a map out of $\mathbb{Z}$ is determined by where 1 goes, and 1 can go anywhere.
- Conversely, $\widehat{\mathbb{G}}_m = \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$.
- If we start with $\mathbb{Z}/n$, we get μ_n, the group scheme of n^{th} roots of unity.

Say Λ is a lattice in a complex vector space V. Then the dual of Λ is the dual torus $T^\vee = \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{G}_m$, where $\Lambda^* \subset V^*$ is the dual lattice. And the torus V/Λ has Cartier dual $\Lambda^\vee$. This is the analogue of Fourier series in the algebraic geometry setting.

To avoid regularity issues, we assume G is finite for now (finite flat group scheme, or spectrum of an Artinian algebra. . .).[9] Then $\widehat{G} = \mathrm{Spec}(\mathcal{O}(G)^*)$. The algebra $\mathcal{O}(G)$ has additional structure coming from G, namely a *coalgebra structure* given by the pullback of the multiplication map, called *comultiplication*:

$$\Delta \coloneqq \mu^* \colon \mathcal{O}(G) \longrightarrow \mathcal{O}(G) \otimes_k \mathcal{O}(G). \tag{5.6}$$

This plays the role of convolution here. There is similarly a *counit map* $\mathcal{O}(G) \to k$ arising as pullback of the inclusion of the identity $\mathrm{id} \colon \operatorname{Spec} k \to G$.

So $\mathcal{O}(G)$ has an algebra and a coalgebra structure. Do they play nice together? Yes they do! There are a few commutation relations that are satisfied and this data together gives a *commutative, cocommutative Hopf algebra*. (The abelian assumption on G is required for cocommutativity). And in fact, the theory of finite

[9] These technicalities arise only in positive characteristic, where you can have nonregular finite group schemes, such as the kernel of the Frobenius $\varphi \colon \mathbb{A}^1 \to \mathbb{A}^1$, which has no closed points yet is nonreduced of length p. Another (counter)example is the group scheme of the n^{th} roots of unity, which over $\mathbb{C}$ is just $\mathbb{Z}/n$ as you might expect, but in positive-characteristic can look different.

abelian group schemes over k is contravariantly equivalent to the theory of commutative, cocommutative, finite-dimensional Hopf algebras.

Duality defines an involution on the category of commutative, cocommutative, finite-dimensional Hopf algebras, and when you pass this through Spec, you get Cartier duality for finite abelian group schemes. If you want to study infinite algebraic groups, you need to worry about regularity, because duals of infinite-dimensional vector spaces are a little trickier.

Remark 5.7. Why should comultiplication be convolution? Say G is a finite abelian group, without any algebraic geometry aroud. Then $(\mathbb{C}[G], *)$ can be identified with the algebra of measures on G, and convolution is identified with the pushforward of measures under multiplication. ◀

Now let's see what happens with the Fourier transform here. As in the topological case, we expect $\mathbb{G}_a$ to pass to $\mathbb{G}_a$, but the Cartier dual of $\mathbb{G}_a$ is actually something different, the *formal completion* of $\mathbb{G}_a$, which, confusingly, is also written $\widehat{\mathbb{G}}_a$. For now, I will use $\widehat{\mathbb{G}}_a$ to denote the Cartier dual and $\mathbb{G}_a^\wedge$ to denote the formal completion. What this actually is is the union of all infinitesimal neighborhoods of the origin:

$$\mathbb{G}_a^\wedge = \bigcup_n \operatorname{Spec} k[t]/(t^n). \tag{5.8}$$

Let's assume k has characteristic zero, so we don't have to worry about defining the character of $\mathbb{G}_a = \operatorname{Spec} k[x]$ to be

$$e^{xt} = \sum_n \frac{(xt)^n}{n!}. \tag{5.9}$$

To allow this in algebraic geometry, this sum is required to be finite, which is why we passed to formal completions: t, the dual coordinate, is nilpotent, so this is fine.

More generally, if V is a finite-dimensional k-vector space, which defines an abelian algebraic group, its Cartier dual is $(V^*)^\wedge$, the formal completion of the dual near 0, and the duality is again given by $\exp(\langle x, t\rangle)$, which makes sense formally (i.e. in this formal neighborhood of the origin).

Spectral theory is asking about G-representations, which are identified with $\mathcal{O}(G)$-comodules, i.e. $\mathcal{O}(\widehat{G})$-modules, or quasicoherent sheaves on $\widehat{G}$. For this to be literally true G has to be finite, but there is a version of this story in general. This is the way in which Cartier duality gives us decompositions of representations.

For example, $\mathcal{R}ep(\mathbb{Z})$ is identified with $k[z, z^{-1}]$-modules: $\mathcal{O}(\mathbb{G}_m) = k[z, z^{-1}]$. So a single matrix (vector space and endomorphism) is identified with a quasicoherent sheaf on the line, i.e. on $\mathbb{A}^1 = \operatorname{Spec} k[z]$, and requiring the matrix to be invertible is saying, well, 0 cannot be an eigenvalue, so you are a sheaf on $\mathbb{A}^1 \setminus 0 = \mathbb{G}_m$, the Cartier dual of $\mathbb{Z}$ as expected.

In the other direction, a representation of $\mathbb{G}_m$ is the same thing as a $\mathbb{Z}$-graded vector space, or a sheaf on the discrete abelian group $\mathbb{Z}$. That is, if V is a $\mathbb{G}_m$-representation, we can write

$$V = \bigoplus_{n\in\mathbb{Z}} V_n, \tag{5.10}$$

where V_n is the eigenspace in which $z \in \mathbb{G}_m$ acts by z^n, and this gives us the $\mathbb{Z}$-grading. This is how the Fourier series takes a representation of the multiplicative group and spits out a sheaf on a discrete set.

Example 5.11. Here's another algebraic example of Fourier series (lattices and tori exchanged) which takes a little effort to set up, but is nice. This example is in topology. Let M be a 3-manifold and G be the Picard group of M, the group of isomorphism classes of complex line bundles on M up to isomorphism. The group structure is tensor product. We can equivalently use principal U_1-bundles, or $\pi_0(\mathrm{Map}(M, B\mathrm{U}_1))$. One can use as a model for $B\mathrm{U}_1$ the space $\mathbb{CP}^\infty$ or an Eilenberg-Mac Lane space $K(\mathbb{Z}, 2)$.

Because $B\mathrm{U}_1 = K(\mathbb{Z}, 2)$, homotopy classes of maps to $B\mathrm{U}_1$ are naturally identified with $H^2(M;\mathbb{Z})$ as abelian groups, with the map given by sending a line bundle to its first Chern class. For now, assume $H^2(M;\mathbb{Z})$ is free, though we can tell a version of this story in the presence of torsion.

Now $G \coloneqq H^2(M;\mathbb{Z})$ is abelian; let's calculate its Cartier dual. This is a dual torus

$$\begin{aligned}\Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{G}_m &= \mathrm{Hom}(H_1(M), \mathbb{G}_m)\\ &\cong \mathrm{Hom}(\pi_1(M), \mathbb{G}_m).\end{aligned}$$

This can be identified with the group of isomorphism classes of flat $\mathbb{C}^\times$-bundles on M, where given a bundle we use the flat connection to define a monodromy map $\pi_1(M) \to \mathbb{C}^\times$. So we obtain a duality between

$\operatorname{Pic}(M)$ and $\operatorname{Loc}_{\mathbb{C}^\times}(M)$ — this isn't super deep, since it's just exchanging $\mathbb{Z}^n$ and $(\mathbb{C}^\times)^n$. You can make this independent of basis by replacing line bundles with T-bundles, where T is a torus. Let $T^\vee$ denote the dual torus.

Now the same story allows us to identify $\mathcal{B}un_T(M)$ and $\mathcal{L}oc_{T^\vee}(M)$, where the latter is the group of isomorphism classes of flat $T^\vee$-bundles. Choosing a basis these are $(\mathbb{C}^\times)^n$-bundles. Again a flat connection is determined by its monodromy. Both of these can be thought of as categories, but we're not using that, just thinking of the sets of isomorphism classes.

So again this is just lattice-torus Cartier duality, but it looks a little more suggestive: bundles and local systems are more geometric, and indeed this examples appears when one studied electromagnetic duality on a 3-manifold. And if you apply this idea to arithmetic topology, thinking of a number field as a 3-manifold, you get an instance of Cartier duality and you get the statement of class field theory!

This abelian duality is a four-dimensional story, and has avatars in all lower dimensions. This is part of the 3-manifold story. And in this story, you have more than just the set of isomorphism classes, but considerably more structure. ◀

Lecture 6.

Electric-magnetic duality: 2/4/21

Last time we discussed Cartier duality, an algebro-geometric version of Pontrjagin duality, e.g. with $\mathbb{G}_m$ (or $\mathbb{C}^\times$ for those of you who always work over $\mathbb{C}$) instead of U_1. But the algebraic and analytic stories do have useful things to say to each other.

There are different versions of representation theory of G, which correspond to different notions of group algebra in the dual world. For example, if we consider the group algebra as a von Neumann algebra, the dual $\widehat{G}$ has the structure of a measure space. If we consider the group C^* algebra, $\widehat{G}$ is a topological space. And if we just consider a discrete algebra (so G is very small), then $\widehat{G}$ is an algebraic variety. You should think of this as one dual object $\widehat{G}$, even if the precise mathematical objects are not the same.

And now for some examples, though they're really the same example as Fourier series, between $\mathbb{Z}$ and the circle, said with a little more flair.

Example 6.1 (Pontrjagin-Poincaré duality)**.** Let M be a compact n-dimensional manifold and G be a locally compact abelian group, so we can use M as a source of abelian groups, specifically $H^i(M;G)$. Pontrjagin-Poincaré duality is the existence of a perfect pairing

$$H^i(M;G)\otimes H^{n-i}(M;\widehat{G}) \longrightarrow H^n(M;G\otimes\widehat{G}) \longrightarrow H^n(M;\mathrm{U}_1) \longrightarrow \mathrm{U}_1, \tag{6.2}$$

hence a natural identification of $H^{n-i}(M;\widehat{G})$ as the Pontrjagin dual of $H^i(M;G)$. In homotopy theory, this is related to a fairly general abstract version of Pontrjagin duality called *Brown-Comenetz duality.*

(TODO: if G has a topology, what happens? I want to make sure I understand this.)

Specializing to dimension 1, we get duality from $\mathbb{Z}$ (or maps $\mathrm{pt}\to\mathbb{Z}$) to $\mathbb{C}^\times$ (or maps $\mathrm{pt}\to\mathbb{C}^\times$). Not terribly exciting. But this can be thought of as an equivalence between quantum mechanics on $\mathbb{Z}$ and quantum mechanics on $\mathbb{C}^\times$.

What about dimension 2? Now we can consider $H^0(S^1;\mathbb{Z})$, which we learned has dual $H^1(S^1;\mathbb{C}^\times)$. In particular, we have a Fourier transform identifying functions on these ableian groups. We can think of $H^0(S^1;\mathbb{Z})$ as $[S^1,\mathbb{Z}]$, the homotopy classes of maps to $\mathbb{Z}$. For $H^1(S^1;\mathbb{C}^\times)$, we want to call it $\mathcal{L}oc_{\mathbb{C}^\times}(S^1)$, but this should really mean the stacky version, so let's just let $\operatorname{Loc}_{\mathbb{C}^\times}(S^1)$ denote the underlying space. ◀

Let's talk about local systems. A G-local system is a sheaf that's locally isomorphic to the constant sheaf $\underline{G}$, and in which the gluing maps are identified with G acting on itself by multiplication. In some sense, this doesn't use the topology or geometry of G, just the underlying set. For example, a local system has a monodromy map $\pi_1(X)\to G$, and it is determined up to isomorphism by this map. Said a little differently, isomorphism classes of local systems are given by $H^1(X;G)$ — even if G is nonabelian! Usually we're used to only defining cohomology for abelian groups, but for nonabelian groups you can define singular or Čech cohomology in degrees 0 and 1 only, and H^1 recovers isomorphism classes of principal bundles.

Local systems have automorphisms, and it is often important to remember that, but right now we just see isomorphism classes. When we consider just the underlying space $\operatorname{Loc}_{\mathbb{C}^\times}(S^1)$, this is the space $\mathbb{C}^\times$, so we once again get the Cartier duality between $\mathbb{Z}$ and $\mathbb{C}^\times$. This seems boring but now that we're one dimension higher and seeing principal bundles, this duality has something to do with gauge theory for U_1.

Example 6.3 (Another 2d example)**.** Now let's consider $H^1(S^1;\mathbb{Z})$, which we call the *A side*. We call the dual $H^0(S^1;\mathbb{C}^\times)$ the *B side*. This is not about albums!

On the B side, we are looking at maps to $\mathbb{C}^\times$, where $\mathbb{C}^\times$ has the discrete topology. In physics this is an exampe of a *scalar field*. On the A side, we're looking at $[S^1, B\mathbb{Z}]$; since $B\mathbb{Z} = K(\mathbb{Z},1) = S^1$, we're looking at maps to S^1. This is another scalar field, and you could think of this as a very very special case of T-duality or mirror symmetry. In T-duality, one exchanges radii $R \leftrightarrow 1/R$ for these two circles.

If you want to soup this up a little bit, you can replace $\mathbb{Z}$ with a lattice Λ, so that you don't have a canonical basis and the duality is a little more interesting. Then the dual is the dual complex torus $T_{\mathbb{C}}^\vee$, but $H^1(S^1;\Lambda) = [S^1, T]$, where T is the compact torus $T \coloneqq \Lambda_{\mathbb{Z}} S^1$. Again this is an example of T-duality. ◀

Example 6.4. Now we step up to 3d. Let Σ be a compact, oriented 2-manifold. Then Pontrjagin-Poincaré duality exchanges $H^1(\Sigma;\mathbb{Z})$ and $H^1(\Sigma;\mathbb{C}^\times)$. On the left we have a scalar field $[\Sigma, B\mathbb{Z}]$, or $[\Sigma, S^1]$, and on the right we get $\mathrm{Loc}_{\mathbb{C}^\times}(\Sigma)$, which is about 3d gauge theory.

Another approach is to exchange $H^2(\Sigma;\mathbb{Z})$ and $H^0(\Sigma;\mathbb{C}^\times)$. Now $H^2(\Sigma;\mathbb{Z}) \cong H^1(\Sigma;\mathrm{U}_1) = [\Sigma, B\mathrm{U}_1]$, since $B\mathrm{U}_1 = K(\mathbb{Z},2)$. This is about gauge theory for principal U_1-bundles. On the right side, we get $[\Sigma, \mathbb{C}^\times]$, a theory of a scalar field. Again if you want this to be more interesting, you can upgrade to tori and lattices, and you get isomorphisms of vector spaces. ◀

Example 6.5. Now a four-dimensional example: $H^2(M;\mathbb{Z})$ and $H^2(M;\mathbb{C}^\times)$. You can make this work with torsion but for ease of exposition let's assume $H^2(M;\mathbb{Z})$ is torsion-free. The cool thing about dimension 4 is that both sides of this correspondence can be identified with gauge theory: on the A side, $H^2(M;\mathbb{Z}) = [M, K(\mathbb{Z},2)] = [M, B\mathrm{U}_1]$ so we get principal U_1-bundles. Said differently, $[M, B\mathrm{U}_1] = \pi_0(\mathrm{Map}(M, \mathbb{CP}^\infty))$.

There is an equivalence between principal U_1-bundles and complex line bundles, given in one direction by taking the associated line bundle to a principal U_1-bundle and in the other direction by choosing a metric and taking the unit circle bunde. So we can think of $H^2(M;\mathbb{Z})$ as the Picard group of M, though this might be a strange version of it if you're used to the Picard group with more structure (e.g. a variety).

On the B side we get $\mathbb{C}^\times$-valued local systems, at least without stackiness.

This is an instance of electric-magnetic duality. See Witten's lectures "Abelian duality I" and "Abelian duality II" in the quantum fields and strings books, as well as Freed [Fre00] and Freed-Moore-Segal [FMS07] for more mathematical approaches.

(TODO: something I missed) and we want to study solutions to equations $\mathrm{d}u = 0$ and $\mathrm{d}{\star}u = 0$, where $u = \mathrm{d}\varphi$. The first condition is automatic, because u is exact. The second is equivalent to ${\star}\mathrm{d}{\star}u = 0$, or equivalently φ is harmonic.

But this reformulation of the harmonic condition shows that we have a pretty symmetry: $\{\mathrm{d}u = 0, \mathrm{d}{\star}u = 0\}$ is symmetric under the Hodge star. Since ${\star}u$ is closed, it is at least locally $\mathrm{d}\varphi^\vee$. So the theory of a U_1-valued scalar is dual to a theory with another U_1-valued scalar. Or, if you consider maps to one torus, the dual side is about maps to a dual torus. When you introduce metrics so that you can talk about harmonicity better, you will see that if R is the radius of the A side circle, the radius of the B side cicle is $1/R$.

Now let's see what's happening in dimension 3. Let $\varphi\colon \Sigma \to S^1$ be a smooth map and $u = \mathrm{d}\varphi$. Imposing $\mathrm{d}u = 0$ and $\mathrm{d}{\star}u = 0$ is dual to asking for $F = {\star}u = dA$.

But the symmetry in 4d is more interesting. Quick crash course in electromagnetism: one has an *electric field* $E \in \Omega^1(\mathbb{R}^3)$ and a *magnetic field* $B \in \Omega^2(\mathbb{R}^3)$. If you want to work relativistically on Minkowski space, we instead collate these together into the *field strenth*

$$F \coloneqq B - \mathrm{d}t \wedge E \in \Omega^2(\mathbb{M}^4), \tag{6.6}$$

where $\mathbb{M}^4$ denotes Minkowksi space.

Maxwell's equations in a vacuum tell us that $\mathrm{d}F = 0$ and $\mathrm{d}{\star}F = 0$. This is of course symmetric under $F \leftrightarrow {\star}F$. If we break relativistic symmetry and write

$${\star}F = B^\vee - \mathrm{d}t \wedge E^\vee, \tag{6.7}$$

then $B^\vee = -{\star}_3 E$ and $E^\vee = {\star}_3 B$, where ${\star}_3$ means to take the Hodge star on a time slice, which is a 3-manifold. (That is, space is 3-dimensional, and spacetime is 4-dimensional.)

If you throw in electromagnetic charges with currents, these equations modify, and we now have electric and magnetic currents j_E and j_B, respectively, which are 3-forms, and we ask that $\mathrm{d}F = j_B$ and $\mathrm{d}\star F = j_E$. On a time-slice M, $\int_M j_B = Q_B$ and $\int_M j_E = Q_E$, the magnetic, resp. electric charges. And Stokes' theorem tells you Gauss' law: if $\Sigma \subset M$ is closed, the *magnetic flux*

$$b_\Sigma = \int_\Sigma F = -Q_B. \tag{6.8}$$

This has an interpretation as the average number of field lines leaving Σ. Similarly, the total *electric flux* is

$$e_\Sigma = \int_\Sigma \star F = Q_E. \tag{6.9}$$

(TODO: don't assume Arun's minus signs are correct!)

In nature, there are no magnetic monopoles, so $Q_B = 0$, and Gauss' law says that if you integrate over a closed surface, you get 0. This allows us to introduce a new object, the *electromagnetic potential* A, and writing $F = \mathrm{d}A$. This breaks the symmetry between electricity and magnetism. Given this data, $\nabla \coloneqq \mathrm{d} + A$ is a connection on a principal U_1-bundle on M, and the gauge symmetries

$$\begin{aligned} \nabla &\longmapsto g^{-1}\nabla g \\ A &\longmapsto A + g^{-1}\,\mathrm{d}g, \end{aligned} \tag{6.10}$$

where $g\colon M \to \mathrm{U}_1$ is smooth, does not affect F.

This is a very convenient trick to play but is it meaningful in physics? Yes! There is an experimentally measured principle called the *Aharonov-Bohm effect*, which gives meaning in physics to A. Even when $F = 0$ (the connection is flat), charge particles acquire phases along loops, which measures the holonomy of the flat connection. So A really exists (albeit only up to gauge transformations.)

Now that we've replaced F with $\mathrm{d}A$ (and introduced A) on 3-manifolds, let's also see what happens when we do this in dimension 4. This choice now implements Dirac charge quantization. For $M = N \times \mathbb{R}$, where N is 3-dimensional and M is 4-dimensional, say that ∇ is a connection on the principal U_1-bundle $P \to M$. We still have fluxes even without charged particles, as we can integrate

$$b_\Sigma = \frac{1}{2\pi i}\int_\Sigma F. \tag{6.11}$$

This is no longer zero, and in fact using Chern-Weil theory this is $\langle c_1(P), [\Sigma]\rangle$. This is an integer, rather than an arbitrary complex number, which is Dirac charge quantization. There is an analogue of this for the electirc flux

$$e_\Sigma = \frac{1}{2\pi i}\int_\Sigma \star F. \tag{6.12}$$

This is all of the classical story for now, and we turn to the quantum story. We will discuss it in the Hamiltonian formalism. Quantum *field* theory means that instead of doing quantum mechanics with point particles, we do it with fields. These are things which have a locality property and sheafify (this is not a precise definition!): maps to a space, sections of vector bundles, principal bundles or vector bundles and connections, and so on.

As a heuristic, QFT on $M^{d-1} \times \mathbb{R}$ should feel lie quantum mechanics on a space of fields on M^{d-1}. This is often infinite-dimensional. This is very much like a standard procedure in analysis where you regard a PDE as an infinite-dimensional ODE.

We will have a Hilbert space of states $\mathcal{H}$, which is heuristically L^2 of the space of fields on M. Part of what quantum field theory does is to attach Hilbert spaces to time-slices. For now we're not going to worry about regularity issues: L^2 means "functions that are a Hilbert space," but we note that good solutions exist. So we want to consider the space $\mathcal{C}(M)$ of isomorphism classes of principal line bundles $L \to M$ with unitary connection. Connections always have isomorphisms and we are going to ignore this for now.

So we will try doing quantum mechanics with the Hilbert space $L^2(\mathcal{C}(M))$. $\mathcal{C}(M)$ has a nice structure: it is an infinite-dimensional analogue of a Lie group under tensor product of line bundles and connections, and there is a sense in which it splits as the product of a lattice, a torus, and an infinite-dimensional vector space. So we have both sides of the abelian duality we've discussed today, as well as the vector space where we want to study Fourier series.

The lattice comes from a discrete invariant, the first Chern class $c_1\colon \mathcal{C}(M)\to H^2(M;\mathbb{Z})$; in physics this measures the magnetic flux. Geometrically, one can also recover this by taking the curvature of the connection $F\colon \mathcal{C}(M)\to\Omega^2(M)$. But it's not just any 2-form: it is closed and has integral periods (which means that it talks to $H^2(M;\mathbb{Z})$). The kernel of the curvature map is the space $\mathcal{C}_\flat(M)$ of flat connections, which are the same thing as U_1-valued local systems on M (i.e. principal U_1-bundles, where U_1 has the discrete topology, and we take isomorphism classes of these bundles). We discussed how you can think of local systems up to isomorphism as $H^1(M;\mathbb{R}/\mathbb{Z})$, identifying $\mathbb{R}/\mathbb{Z}\cong\mathrm{U}_1$, and $H^1(M;\mathbb{R}/\mathbb{Z})$ is a torus $\mathrm{U}_1\otimes\mathbb{Z}H^1(M;\mathbb{Z})$, assuming there's no torsion in $H^1(M;\mathbb{Z})$.

The lattice $\Lambda := H^2(M;\mathbb{Z})$ is called the *lattice of magnetic fluxes*, and inside $\mathcal{C}(M)$ we have the torus $T=\mathrm{Loc}_{\mathrm{U}_1}(M)$ of local systems, and there's a vector space part too. So we know what L^2 looks like: we have Fourier series for T and Λ, and for L^2 of the vector space, we have a Fourier transform, even though it's infinite-dimensional.

Electric-magnetic duality from this perspective will apply Pontrjagin duality on $\mathcal{C}(M)$, as we will discuss next time. We will be able to write the same Hilbert space in two ways, and magnetic measurements on one side are electric measurements on the other side. ◂

Lecture 7.

Lagrangian quantum field theory: 2/9/21

Today we'll discuss Euclidean quantum field theory (QFT) from the Lagrangian perspective.

Fix a d-dimensional manifold M; here d is the dimension of spacetime. To M we attach a space of fields $\mathcal{F}(M)$. Fields are things which can sheafify, i.e. they can be assembled from local data which glues: for example, principal bundles and connections, functions, sections of vector bundles, differential forms, and so on. Associated to a field $\varphi\in\mathcal{F}(M)$, there is an *action* $S(\varphi)$, which is a function $S\colon\mathcal{F}(M)\to\mathbb{R}$.

The "quantum" in quantum field theory means that rather than computing explicit measurements we get a probability distribution, which we obtain by integrating over the space of fields. This is the notorious path integral, which is not always mathematically well-defined, but we're not going to worry about that for now. Physicists have ways of calculating path integrals.

If M is closed, the most fundamental measurement we can make is the volume of the space of fields, the *partition function*

$$Z(M) := \int_{\mathcal{F}(M)} e^{-S(\varphi)/\hbar}\,\mathcal{D}\varphi. \tag{7.1}$$

We also want to calculate expectation values of operators. If $\mathcal{O}_x$ is an operator at a point $x\in M$, it defines a functional on $\mathcal{F}(M)$ which physically one can interpret as making a measurement at x. The measurement is calculated by a modified path integral

$$Z(M) := \int_{\mathcal{F}(M)} \mathcal{O}_x(\varphi)e^{-S(\varphi)/\hbar}\,\mathcal{D}\varphi. \tag{7.2}$$

Among these are the *disorder operators*, for which inserting $\mathcal{O}_x$ means to look at fields that might possibly be singular at x, e.g. by looking at nice fields within $\mathcal{F}(M\setminus x)$.

We're not going to make this rigorous, but it will be helpful for us understanding time evolution and the Hamiltonian perspective from last time. Let M be a bordism from ∂M_{in} to $\partial M_{\mathrm{out}}$. Because fields sheafify, we can restrict them to the incoming and outgoing boundaries, defining maps $\mathcal{F}(\partial M_{\mathrm{in}})\leftarrow\mathcal{F}(M)\to\mathcal{F}(\partial M_{\mathrm{out}})$, and we have (more or less) the path integral measure $\mathcal{D}\varphi$ on $\mathcal{F}(M)$, so we can perform an integral transform, at least heuristically, from functionals on $\mathcal{F}(\partial M_{\mathrm{in}})$ to functionals on $\mathcal{F}(\partial M_{\mathrm{out}})$. Let $\mathcal{H}_{\mathrm{in}}$ and $\mathcal{H}_{\mathrm{out}}$ be these spaces of functionals. There's lots of details to worry about here, including regularity and defining the path integral measure.

ANyways, from all this data we get a linear map $\mathcal{H}_{\mathrm{in}}\to\mathcal{H}_{\mathrm{out}}$. Its value on a functional $f\in\mathcal{H}_{\mathrm{in}}$ and a field $\varphi\in\mathcal{F}(\partial M_{\mathrm{out}})$ can be described by considering the subspace T of $\mathcal{F}(M)$ of the fields together with data of restriction to φ on $\partial M_{\mathrm{out}}$. Then integrate, using the path integral measure for ∂M_{in}, the value of f on the restrictions of the fields in T to ∂M_{in}.

This is the relationship between the Lagrangian and Hamiltonian perspectives. If M is a cylinder interpreted as the identity bordism, we want to think of this map as time evolution by $e^{itH/\hbar}$, where t is the length of the cylinder. The Lagrangian perspective uncovers a lot of additional rich structure which we will find useful.

Example 7.3. Say $d = 1$ and we want to study quantum mechanics on a Riemannian manifold X. Then the fields on a 1-manfold C are $\mathcal{F}(C) \coloneqq \mathrm{Map}(C, X)$; the critical points of the action are geodesics, and $\mathcal{H} = L^2(\mathrm{Map}(\mathrm{pt}, X)) = L^2(X)$, and the Hamiltonian is the Laplacian. ◀

However, if we try to do this in higher dimensions, the problems that occur because of our infinite-dimensional spaces of fields get worse. But let's continue forwards anyways, and see what we can learn even though it's not mathematically precise.

Example 7.4 (Quantum Maxwell theory)**.** This is a four-dimensional field theory. The fields on M are the (gauge equivalence classes of) principal U_1-bundles on M with a connection $\mathrm{d} + A$. Let F denote the curvature of the connection. The classical equations (i.e. the equations whose solutions are critical points of the action) are $\mathrm{d}F = 0$ and $\mathrm{d}{\star}F = 0$. The action is

$$S = \frac{g}{2\pi i}\int_M F \wedge \star F + \theta \int F \wedge F \tag{7.5}$$

where g and θ are parameters, and there might be some constants to fix. Note that by Chern-Weil theory, $F \wedge F$ is a fixed multiple of $c_1(P)^2$, where $P \to M$ is the principal U_1-bundle.

The Hilbert space $\mathcal{H}$ on a closed 3-manifold N is the functions on the space of fields $\mathcal{C}(N)$, specifically L^2 of the space of connections on principal U_1-bundles, mod gauge.

Because U_1 is abelian, we'd like to think of $\mathcal{C}(N)$ as some sort of infinite-dimensional Lie group. Taking the first Chern class surjects $\mathcal{C}(N) \twoheadrightarrow \Lambda \coloneqq H^2(N : \mathbb{Z})$, and the kernel of this map is the space of flat connections $T \coloneqq H^1(N; \mathrm{U}_1) \cong BH^1(N; \mathbb{Z})$. The space of fields splits as $\Lambda \times T \times V$ for some infinite-dimensional vector space V.

The lattice Λ acts on $\mathcal{H}$, related to Dirac monopoles, though in the nonabelian case this is related to *'t Hooft operators*. Specifically, given $\lambda \in \Lambda$, there is a line bundle L with $c_1(P) = \lambda$. Then the action of Λ on $\mathcal{C}(N)$ is to tensor with L, and this induces an action on $\mathcal{H}$.

This creates magnetic monopoles in M! Let γ be a smooth closed curve whose homology class is Poincaré dual to λ. (TODO: physics goes here — this has something to do with a magnetic monopole on γ.)

Now if N is the cylinder bordism for N with an embedded knot γ at time $t = 1/2$, our equations ruled out magneti monpoles, so we need to consider the space of fields on $(M \times I) \setminus \gamma$. The link of this knot is $S^2 \times S^1$. Pick a point on γ where our would-be monopole would be, and let S be the linking 2-sphere. Then the magnetic charge inside S is nonzero.

Let's look at the space $\mathcal{C}_{c=1}$ of connections on $(M \times I) \setminus \gamma$ such that $(1/2\pi i)\int_{S^2} F = 1$, i.e. inside the 2-sphere there is 1 unit of magnetic charge. This space of connections is mathematically well-defined, and we get a correspondence

$$\mathcal{C}(N_{\mathrm{in}}) \longleftarrow \mathcal{C}_{c=1} \longrightarrow \mathcal{C}(N_{\mathrm{out}}). \tag{7.6}$$

Therefore by the usual integral transform, we get a map $\mathcal{H} \to \mathcal{H}$ whose effect is to shift by $\lambda = c_1(P)$. This is an example of a *disorder operator*: we're not measuring our fields, but instead looking at a different space of fields. Our measurement makes the fields singular along γ. The field-theoretic description is complicated, even though the action on $\mathcal{H}$ is simple.

Wilson operators are correspondingly simple to describe field-theoretically. Let $\gamma \subset N$ be a simple closed curve. Even classically, there's a function $W_\gamma \colon \mathcal{C}(N) \to \mathbb{C}$ sending (L, ∇) to the holonomy of ∇ around γ. This is valued in $\mathrm{U}_1 \subset \mathbb{C}$. Once again we can put γ inside $[0,1] \times M$ at $t = 1/2$ to define an integral transform $\mathcal{H} \to \mathcal{H}$. We're no longer measuring at a point, but instead along a loop. And the action is to multiply functionals in $\mathcal{H}$ by W_γ.

These W_γ are eigenfunctions for the action of T on $L^2(\mathcal{C}(M))$, multiplying by the holonomy. This is something one should check, but it's Fourier theory. This provides a decomposition

$$\mathcal{H} = \bigoplus_{e \in H^(M;\mathbb{Z})} \mathcal{H}_e, \tag{7.7}$$

the *electric fluxes*.

This is the first part of electric-magnetic duality, which you can think of as doing Fourier analysis on $\mathcal{C}(N)$, which we decomposed into a lattice, a torus, and a vector space. The Fourier transform identifies $L^2(\mathcal{C}_{\mathrm{U}_1}(M))$ and $L^2(\mathcal{C}_{\mathrm{U}_1^\vee}(M))$, switching the lattice Λ_B with the torus $T_B^\vee$, and conversely T_E to $\Lambda_E^\vee$, and the vector space goes to the vector space.

On one side, the Wilson operators are diagonal; on the other side, the 't Hooft operators are diagonal. This is the hallmark of electric-magnetic duality.

Electric-magnetic duality is inherently quantum: you cannot see it from the classical Lagrangian perspective. You have to quantize to see one half of the operators. The magnetic operator from the Lagrangian perspective destroys a connection by making it singular along the loop. Since electric-magnetic duality switches this with operators that make sense in the classical perspective, we need a different perspective to see it.

Later, when we talk about the geometric Satake isomorphism, we will interpret it as telling us what the 't Hooft operators are in nonabelian gauge theories. ◂

Example 7.8 (Abelian duality in $d = 2$)**.** This will be a simple instance of T-duality. Let M be a 2-manifold; the fields are $\mathrm{Map}(M, S^1)$, and the Hilbert space on S^1 is $L^2(\mathrm{Map}(S^1, S^1))$. This is graded by $H^1(S^1;\mathbb{Z})$, the winding number of the map.

What T-duality does here is that we can take a dual circle $(S^1)^\vee$; if the primal circle has radius R, the dual has radius $1/R$. Then there's a correspondence between L^2 of these two spaces of maps. Concretely, you can think of $L^2(H^0(M;\mathrm{U}_1))$ in the dual case.

On the primal side, we can shift the winding number, which is an analogue of the 't Hooft operators in 4d Maxwell theory. On the dual side, we can evaluate a function at a point, which is a Wilson analogue. Duality exchanges these operators, and is a very simple case of mirror symmetry. ◂

The quantum field theories we care the most about in this class are the topological ones. Our goal is to cut down from these gigantic Hilbert spaces of functions on infinite-dimensional manifolds to finite-dimensional Hilbert spaces, in a way that is still helpful.

For example, let's look at quantum mechanics. The point is assigned a Hilbert space $\mathcal{H}$, and a time interval $[0, T]$ is assigned a time evolution operator $r^{iTH/\hbar}$. The Hamiltonian is the Laplacian, and the entire spectral theory of the Laplacian appears in the story, and there are subtleties.

In topological quantum mechanics, we kill time. Not in the sense of, let's hang around for a bit, but in the sense of, let's force time-evolution to be the identity map. There are a few ways to do this: you could restrict to the zero eigenspace of the Laplacian, but these are harmonic functions, which are constant on closed Riemannian manifolds by the maximum principle. So whatever theory we get is not super interesting.

Alternatively, following Witten [Wit82], you can kill time in a derived sense by introducing supersymmetry.

(1) First, enlarge $L^2(X)$ to the space of L^2 differential forms on the target X. So we're adding new fields.
(2) Using this, we can identify the (super) Lie group of symmetries.

For $\Omega_X^\bullet$, we have in addition to $\mathfrak{u}_1$ acting by grading, we have two operators $Q \coloneqq \mathrm{d}$ and $\mathbb{Q}^* \coloneqq d^*$. The commutation relations between Q and Q^* and $\mathfrak{u}_1$ (the *R-symmetry*) tell us that we should think of Q and Q^* in degrees 1 and -1, respectively. Then these jointly define a graded Lie algebra called the *one-dimensional $\mathcal{N} = 1$ supersymmetry algebra*, and they all *graded* commute with each other, except $[Q, Q^*] = H$: $\Delta = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$. The other commutation relations use $\mathrm{d}^2 = 0$, etc.

That is, Q^* is a (chain) homotopy from H to 0. Therefore on Q-cohomology (which is ordinary de Rham cohomology), H acts by zero: there's some Hodge theory here, saying harmonic forms define representatives in de Rham cohomology. And yet we have more stuff left: maybe not harmonic functions, but now we have cohomology!

That is, topological quantum mechanics assigns to X the "Hilbert space" $(\Omega_X^\bullet, \mathrm{d})$, or $H^*_{\mathrm{dR}}(X)$, with $H = 0$. And this is a topological invariant in X, indicating that the dependence on Riemannian geometry has gone away. This is very drastic!

Lecture 8.

Topological quantum mechanics, topological Maxwell theory, and class field theory: 2/11/21

As we discussed a little bit last time, in topological quantum mechanics we kill the time-evolution operator in a derived sense: first we added a differential Q, making the Hilbert space a chain complex, and then showing that H is exact, or homotopy to 0. You could then work with cohomology as your Hilbert space, but these days people have realized it's more convenient to work in a derived sense with the entire chain complex and H exact.

The grading (by $\mathbb{Z}$ or $\mathbb{Z}/2$) is part of the structure of the action by the supersymmetry algebra, but in some sense, which came first? Maybe you are studying representations of the algebra through its R-symmetry (here this is $\mathfrak{u}_1$, which is where the grading comes from), and the grading comes afterward; alternatively, you begin with the grading and then produce the algebra action.

There are two different ways we can implement this.

(1) In "A-type topological quantum mechanics," we let $\mathcal{H} \coloneqq \Omega_X^\bullet$, $Q \coloneqq \mathrm{d}$, $Q^* \coloneqq \mathrm{d}^*$, and $H \coloneqq \Delta$, the Hodge Laplacian. This is the example we discused last time.

(2) In "B-type topological quantum mechanics," instead of a Riemannian manifold, we let X be a complex manifold. Then there's a similar complex around (you could call it a complex complex): $\mathcal{H} \coloneqq \Omega_X^{0,\bullet}$, $Q \coloneqq \overline{\partial}$, $Q^* \coloneqq \overline{\partial}^*$, and $H \coloneqq \Delta_{\overline{\partial}}$. The cohomology with respect to Q is $H^{0,*}(X)$, which is quasi-isomorphic to $\mathbf{R}\Gamma(\mathcal{O}_X)$, the derived global sections of the sheaf of functions on X.

Both of these are representations of the one-dimensional $\mathcal{N} = 1$ supersymmetry algebra. Here $\mathcal{N}$ measures "how much supersymmetry there is;" there's a precise way to define this, but right now it's best to think of this as telling you this is the minimal amount of supersymmetry possible in dimension 1.

There is also an $\mathcal{N} = 2$ supersymmetry algebra, and an $\mathcal{N} = 2$ version of topological quantum mechanics. Here we combine the two types above by letting X be a compact Kähler manifold, with bigraded chain complex $\Omega_X^{\bullet,\bullet}$, and we need more operators: $\overline{\partial}$, $\overline{\partial}^*$, ∂, and ∂^*. There is a deep mathematical fact (the hard Lefschetz theorem) that $\mathrm{SL}_2(\mathbb{C})$ acts on the cohomology; the diagonal part gives the grading, and there are also raising and lowering operators. The R-symmetry group is $\mathrm{SU}_2 \subset \mathrm{SL}_2(\mathbb{C})$.

From a mathematical point of view this is a difficult, deep theorem, but in physics it emerges in a more elementary way: you ask what the supersymmetry algebra is for quantum mechanics on a Kähler manifold, and a fact about representation theory (which is not necessarily simpler) tells you that we get $\mathcal{N} = 2$, and therefore an $\mathrm{SL}_2(\mathbb{C})$-action, because that's present in the $\mathcal{N} = 2$ supersymmetry algebra in $d = 1$.

One can push this a little further: if X is compact *hyper*Kähler, the supersymmetry algebra enlarges to $\mathcal{N} = 4$, and instead of $\mathrm{SL}_2(\mathbb{C})$, the cohomology has a Spin_5-action, as part of the $\mathcal{N} = 4$ super Lie algebra. And you can even go all the way to $\mathcal{N} = 8$. These algebras are distinguished by the number of linearly independent odd operators (the *supercharges*): two in $\mathcal{N} = 1$, four in $\mathcal{N} = 2$, eight in $\mathcal{N} = 4$, and sixteen in $\mathcal{N} = 8$. The dimension of the space of odd-order operators is a mathematically better way to think about $\mathcal{N}$, though as you can see this is not literally what $\mathcal{N}$ is.

By adding some more fields to these theories, we'll find some additional supersymmetry, i.e. enlarging the super Lie group that acts on the theory. Our desiderata are that the Hamiltonian $H = [Q, –]$, i.e. it's exact. And for a topological theory, we want to do more: there is in quantum mechanics generally an operator T called the *stress tensor* measuring the dependence on the metric, and we want this to also be Q-exact, to make the theory more topological: nothing depends on the metric on the level of cohomology. This is explained in more detail in Costello-Gwilliam's book.

Topological Maxwell theory Now let's soup this up to dimension 4, taking Maxwell theory, adding fields, and finding supersymmetry to give us a topological field theory in this Q-exact sense. This is an $\mathcal{N} = 4$ theory, which has sixteen supercharges and a whole bunch of data, including a complex line bundle with connection, a 1-form which is called the *Higgs field*, a complex scalar u, and four fermions (giving us odd operators).

There are again two options, indexed by A and B. For the A-twist, the Hilbert space on our 3-manifold M is the cohomology of the space of connections, with some nuances. First of all, connections are a stack, because some connections have stabilizer group actions, and second, the gauge group acts and we take equivariant cohomology. That is, $\mathcal{H} = H^\bullet_{\mathrm{U}_1}(\mathcal{C}(M))$. So this is like doing the A-version of topological quantum mechanics (de Rham theory) on the space of U_1-connections.

Alternatively, in the B-twist, we let ∇ be a connection on a principal $\mathbb{C}^\times$-bundle, rather than a U_1-bundle. Then we do B-type topological quantum mechanics (the Dolbeault story) on the space of $\mathbb{C}^\times$-connections.

Back to the A side for a moment. We will abuse notation slightly to say $\mathrm{Pic}(M)$ is the topological space $\mathcal{C}(M)$. Then $H^0(\mathrm{Pic}(M))$ is the locally constant functions on $\mathcal{C}(M)$. The connected components of $\mathcal{C}(M)$ are the topological types of principal U_1-bundles, which is $H^2(M; \mathbb{Z})$. So we get back what we thought about a few lectures ago, but in a fancier context.

For higher cohomology, we decompose $\mathrm{Pic}(M)$ as a lattice times a torus times a vector space times $B\mathrm{U}_1$, and we need to consider. We see $H^2(M;\mathbb{Z})$, the torus $H^1(M;\mathrm{U}_1)$, and $\mathbb{C}[u]$ coming from the cohomology of $B\mathrm{U}_1$. Under the topological twist, things are much simpler.

On the B-side, we care about $\mathrm{Loc}_{\mathbb{C}^\times}(M)$, which looks like a complex torus $T_{\mathbb{C}}^\vee$, the torus of maps $\pi_1(M) \to \mathbb{C}^\times$. These factor through $H_1(M)$, so this is the torus dual to the lattice $H_1(M)$. So our Hilbert spaces look like $\mathbb{C}[H^2(M;\mathbb{Z})]$ or $\mathbb{C}[T_{\mathbb{C}}^\vee]$, and there is Poincaré-Pontrjagin duality that exchanges them.

One takeaway is that what we saw before about Fourier series on the cohomology of a manifold appears as part of the story in topological Maxwell theory, and there are additional new things going on. The A-twisted super Maxwell theory is studying the topology of the space of connections, and the B-twisted version studies the algebraic geometry of the space of flat connections. Electric-magnetic duality switches these two. These theories are not the same theory with two descriptions, but a duality between two genuinely different theories.

Topology of Pic on the A-side being dual to the algebraic geometry of Loc is a perspective we're going to see again and again.

Behind the curtain, these two topological field theories are topological twists of a supersymmetric QFT called $\mathcal{N} = 4$ super Maxwell theory. This is a supersymmetric version of the theory of light in our world. $\mathcal{N} = 4$ here tells us that we're looking at complex geometry (much like how in dimension 1, $\mathcal{N} = 2$ is a signal that X is Kähler). There's a lot of literature about $\mathcal{N} = 4$ super-Yang-Mills; this is a special case with gauge group U_1, so if you're looking for references, try super-Yang-Mills.

The duality again exchanges Wilson and 't Hooft operators. On the A side, we have 't Hooft operators, which shift on the lattice; on the B side, we have Wilson operators, which multiply by the monodromy along a curve $\gamma \in H_1(M;\mathbb{Z})$. 't Hooft operators create magnetic monopoles. The physics theory has both electricity and magnetism, but by choosing either of these twists, we've broken that symmetry: on the A side, we only have electric measurements, and on the B side we only have magnetic ones.

Now let's introduce defects. These will change the space of fields slightly.

- First, there are "timelike" 't Hooft operators. Choose a point $x \in M$ and consider electromagnetism in the presence of a monopole sitting at x. That is, in $M \times [0,1]$, the line $\{x\} \times [0,1]$ is the worldline of the monopole: it doesn't move. We will then consider electromagnetism in the presence of this monopole, by consider connections on $M \setminus x$. This space splits as a disjoint union

$$\mathcal{C}(M \setminus x) = \amalg_{c_1(P|_{S_\varepsilon(x)})=n} \mathcal{C}(M, nx) \tag{8.1}$$

  i.e. indexed over the Chern class on a linking sphere of x. We think of n as the charge of the monopole. Then we obtain a Hilbert space of L^2 functions on this space of connections.
- On the B side, we want "timelike" Wilson operators on $\mathrm{Loc}_{\mathbb{C}^\times}(M, x)$, defined to be flat $\mathbb{C}^\times$-connections together with a trivialization of the fiber at x. This is a principal $\mathbb{C}^\times$-bundle over $\mathrm{Loc}_{\mathbb{C}^\times}(M)$. This "extra $\mathbb{C}^\times$" is dual to the "extra $\mathbb{Z}$" of charges on the A side, and Fourier series identify the spaces of functions on these extra groups. This is interpreted in physics as creating an electrically charged particle without adding a new field for the particle.

So, electric-magnetic duality tells us that creating a magnetic monopole is dual to creating an electrically charged particle that's heavy enough to break this symmetry (we're not adding a new field corresponding to this particle).

There are yet more defects to consider. The next one, called *ramification* or adding a *surface defect* or *solenoid*, matches in physics what happens when we consider a current running through a wire wrapped tightly around a cylinder to form a tube. Our fields are going to do something funny here; specifically, we remove a curve β from M and consider $\mathrm{Loc}_{\mathbb{C}^\times}(M \setminus \beta)$. Now there's π_1, which means something in the world in the Aharonov-Bohm effect: if you go around a solenoid, the electric field is affected. (TODO: maybe I, Arun, misinterpreted this.)

So we can look at holomorphic functions on this space of local systems (this is the B, or electric, side). On the A, or magnetic side, we look at cohomology of the space of connections on M with a trivialization along β. This is a map $\beta \to U_1 = S^1$, which has a winding number, adding a $\mathbb{Z}$ where dually we had a $\mathbb{C}^\times$ (holonomy around the extra loop). Again, the functions on $\mathbb{Z}$ and $\mathbb{C}^\times$ are identified via a Fourier transform. The mathematics here is all interpretations of Poincaré duality on a 3-manifold, and it's neat how all of these Fourier-theoretic statements admit different physical meaning.

Our physics story here is about geometric Langlands; what's the corresponding arithmetic story? This is a subject in number theory called class field theory. Once again we're just going to get Fourier theory exchanging $\mathbb{Z}$ and $\mathbb{C}^\times$ (or lattices and tori, more generally), but we will find these groups in interesting places and ways. Again the topology of something called Pic is exchanged with the algebraic geometry of something called Loc. At first, "topology" is just going to mean π_0, and "algebraic geometry" means holomorphic functions, but one can extract more structure.

Now the setup. Let F be a finite extension of $\mathbb{Q}$. Number theorists attach an important object to F, the *(ideal) class group* $C\ell(F) := \mathrm{Pic}(\mathrm{Spec}\,\mathcal{O}_F)$. That is, this is telling us about line bundles on $\mathrm{Spec}\,\mathcal{O}_F$, i.e. rank-1 projective $\mathcal{O}_F$-modules, up to isomorphism. In this setting, though, it has another, more elementary interpretation: fractional ideals of $\mathcal{O}_F$ (i.e. sitting inside F) modulo principal ideals. The class group is always finite, which is pretty neat, and doesn't have a clear analogue in other settings. One of the most important invariants of a number field is the *class number* $n_F := |C\ell(F)|$.

(Unramified) class field theory relates the class group to Galois theory. It identifies $\mathbb{C}[C\ell(F)]$ with $\mathbb{C}[\mathrm{Loc}_{\mathbb{G}_m}(\mathcal{O}_F)]$. If you look at a textbook on class field theory, it will use very different notation, but this is used to match with our more geometric/physical story from earlier today. So what is Loc exactly? These are π_1-representations, and our analogue of π_1 is the Galois group. For a manifold, π_1 is the automorphisms of a universal cover over the base, which under our arithmetic topology dictionary would be the automorphisms of a field extension F^{un}/F, where F^{un} is the *maximal unramified abelian extension* of F.

Why this, and not $\overline{F}$? We're thinking of F not as a point $\mathrm{Spec}\,F$, but as rational functions on a scheme, $\mathrm{Spec}(\mathcal{O}_F)$. The algebraic closure corresponds to allowing fields of functions on curves with punctures, but let's not worry about that for the moment, instead thinking about closed curves.

Definition 8.2. A field extension $F \hookrightarrow E$ is *unramified* at the prime $p \in \mathcal{O}_F$ if $\mathcal{O}_E \otimes_{\mathcal{O}_F} \mathcal{O}_F/p$ has no nilpotents.

These nilpotent operators correspond to branch points in a branched cover: some interesting geometry is happening here. Anyways, the poset of unramified, abelian extensions has a maximal element, and that's what F^{un} is. And this is the étale fundamental group of $\mathrm{Spec}\,\mathcal{O}_F$.

There's a technical condition here where we need things to also be unramified at infinite places: what happens when we take $F \otimes_{\mathbb{Q}} \mathbb{R}$? This is isomorphic to $\mathbb{R}^r \times \mathbb{C}^s$, where r and s count embeddings of F into things. We say that E/F is *unramified at infinity* if there are no extensions $\mathbb{R} \hookrightarrow \mathbb{C}$ here. For example, totally imaginary fields (for which $r = 0$) are unramified at infinity.

Anyways, the main theorem of unramified class field theory is that $C\ell(F) \cong \mathrm{Gal}(F^{un}/F)$. This is the group of automorphisms of the *Hilbert class field* F^{un}. We want to interpret this in terms of Pic and Loc. The class group is a Picard group, so that's fine; on the other side, we interpret $\mathrm{Hom}(\mathrm{Gal}(F^{un}/F), \mathbb{C}^\times)$ as $H_1(\mathrm{Spec}(\mathcal{O}_F))$ (TODO: how?). The character dual of the Galois group is also $\mathrm{Hom}(C\ell(F), \mathbb{C}^\times) = C\ell(F)^\vee$.

There is an analogue of Poincaré duality in this setting known as *Artin-Verdier duality*, which applies to spectra of rings of integers. This is part of the arithmetic topology dictionary: another way in which number fields behave like 3-manifolds. The proof of Artin-Verdier duality uses class field theory, so this is sort of upside down from how we argued in the topological setting.

Also, $\mathrm{Spec}(\mathcal{O}_F)$ is more like an unoriented 3-manifold: there is a thing which gives duality, but it's not the constant sheaf. Instead we get something using cyclotomic characters. This has something to do with the difference between $\mathbb{Z}/n$ and the n^{th} roots of unity μ_n, or $\mathbb{Q}/\mathbb{Z}$ and all roots of unity, or $\mathbb{C}^\times$ and $\mathbb{G}_m$. The dualizing object is rather than $\mathbb{Z}$, the *Tate twist* $\mathbb{Z}(1)$, which can be interpreted back in physics as corresponding to accounting for the factor of $2\pi i$ that appears in the Fourier transform. These details make the precise statement of Artin-Verdier duality more confusing, but the key idea is an analogue of Poincaré duality.

Next time, we'll discuss among other things the function-field version of this story.

Lecture 9.

Class field theory and function fields: 2/23/21

It's been a little while since we had class, thanks to the Texas winter storm, so let's review what's happened. We were in the midst of discussing abelian duality, and how it related the topology of principal bundles with the algebraic geometry of local systems. Among the examples of abelian duality we considered, we discussed a (relatively weak) form of class field theory. Let's go over that again.

Let F be a number field (i.e. a finite extension of $\mathbb{Q}$). Class field theory identifies the Galois group of the maximal unramified abelian Galois extension of F with the class group of $\mathcal{O}_F$. Therefore we can identify $\mathbb{C}[C\ell_F] \cong \mathbb{C}[\mathrm{Loc}_1(\operatorname{Spec}\mathcal{O}_F)]$. The right-hand side of this isomorphism can be interpreted as the characters of the Galois group.

This is not always an interesting statement. For example, when $F = \mathbb{Q}$, both sides vanish, even though the Galois theory of $\mathbb{Q}$ is interesting. To see this data, we will have to allow ramifications, which we will discuss later; we'll have to figure out in particular what to replace the class group with. Geometrically, if we pretend that $\operatorname{Spec} F$ is something like a Riemann surface, then class field theory is talking about covers; ramifications allow the cover to branch. If there were actual Riemann surfaces, allowing branching makes the problem much more interesting.

The bulk of today's lecture, though, will focus on the more geometric setting of function fields. Let k be a field (no further conditions for now) and C be a smooth projective curve over k. When you draw pictures, $k = \mathbb{C}$ and C is a compact Riemann surface. To such a curve C, we attach a field $F := k(C)$ of rational functions on C. The abelian group $\mathrm{Pic}(C)$ of divisors fits into a short exact sequence

$$0 \longrightarrow \mathrm{Jac}(C) \longrightarrow \mathrm{Pic}(C) \longrightarrow \mathbb{Z} \longrightarrow 0, \tag{9.1}$$

where the surjection onto $\mathbb{Z}$ is given by the degree, $\sum_{x\in C} n_x x \mapsto \sum n_x$, for $x \in C(k)$. This short exact sequence splits: choose $x \in C(k)$; then the skyscraper $\mathbb{Z}\cdot x$ defines a map $\mathbb{Z} \to \mathrm{Pic}(C)$ which is a section. This is already different from the class field theory case: $\mathrm{Pic}(C)$ is our analogue of the class group, but it always is infinite, and the class group is finite.

Now assume $k = \mathbb{F}_q$ is finite. Then there is a map $\pi_1^{\text{ét}}(C) \twoheadrightarrow \widehat{Z} := \varprojlim_n \mathbb{Z}/n$. This $\widehat{\mathbb{Z}}$ is $\pi_1^{\text{ét}}(\operatorname{Spec} k)$, i.e. $\mathrm{Gal}^{un}(\overline{k}/k)$, topologically generated by the Frobenius.

We will replace $\pi_1^{\text{ét}}(C)$ with the (unramified) Weil group $W^{\mathrm{un}}(C)$, which is the pullback

$$\begin{tikzcd} W^{\mathrm{un}}(C) \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d, hook] \\ \pi_1^{\text{ét}}(C) \arrow[r] & \widehat{\mathbb{Z}}. \end{tikzcd} \tag{9.2}$$

Another thing we can do is take the maximal abelian quotient $W^{ur,ab}(C)$, which surjects onto $\mathbb{Z}$ and still has a section for every k-point of the curve. This looks more like the class group. Thinking of C as like a Riemann surface, the section at a point $x \in C(k)$ is sort of like an embedding of $\operatorname{Spec}(k)$ as a circle in the 3-manifold (the surface bundle).

At this point, it's your choice whether you use the Weil group or the étale fundamental group. Now we can also change Pic. For k finite, the Jacobian $\mathrm{Jac}(C)$ is also a finite abelian group, so all of the "infinite-ness" comes from the degree map onto $\mathbb{Z}$. So we can replace Pic by its profinite completion (the limit over all finite subgroups), which now surjects onto $\widehat{\mathbb{Z}}$.

Theorem 9.3 (Unramified class field theory)**.** *There is a map* $\mathrm{Pic}(C) \to \pi_1^{\text{ét},ab}(C)$ *which is an isomorphism on profinite completions. Equivalently, there is an isomorphism* $\mathrm{Pic}(C) \cong W^{un,ab}(C)$ *respecting the degree homomorphism and its sections.*

So this isomorphism respects quite a lot of structure. This statement is not interesting in all cases: for example, if $C = \mathbb{P}^1$, both sides are $\mathbb{Z}$.

Class field theory is telling us that characters if $\mathrm{Pic}(C)$ are isomorphic to characters of the (unramified, abelian) Weil group. But just like in class field theory, we don't need to abelianize when we take characters: this is the same thing as characters of $W^{un}(C)$, or the characters of $\pi_1^{\text{ét}}(C)$ — here, though, we have to restrict to finite-order characters. So maps into $\mathbb{Q}/\mathbb{Z}$ instead of $\mathbb{C}^\times$. And characters of π_1 deserve to be called rank-one local systems on C (again with some question about coefficients, and certainly true for systems with finite coefficents). So $\mathrm{Pic}^\vee(C) \cong \mathrm{Loc}(C)$.

Inside the functions on $\mathrm{Pic}(C)$ we have the characters: functions which know and respect the multiplication law; and these are identified with functions on the set of local systems. But this is not the only way to think about these.

Recall that we can think of characters via the push-pull diagram, as we discussed a while ago, and this allows you to see that they are precisely the eigenfunctions for the translation action of $\mathrm{Pic}(C)$ on itself. And since $\mathrm{Pic}(C)$ is generated by $\mathbb{Z}x$ for $x \in C(k)$, characters can be thought of as functions on $\mathrm{Pic}(C)$ which are eigenfunctions for the action of $\mathbb{Z}x$ for all $x \in C(k)$. What is this action explicilty? Given $\mathcal{L} \in \mathrm{Pic}(C)$ and $x \in C$, the action sends $\mathcal{L} \mapsto \mathcal{L}(x)$: add $1 \cdot x$ to the divisor of $\mathcal{L}$.

Being an eigenfunction tells you that a function f behaves nicely under this process:

$$f(\mathcal{L}(x)) = \gamma_x \cdot f(\mathcal{L}). \tag{9.4}$$

Now functions on $\mathrm{Pic}(C)$ are identified with functions on $\mathrm{Loc}(C)$, via a Fourier transform. Let ρ be a local system, thought of as a map $\rho\colon \pi_1^{\text{ét}}(C) \to F$, where F is some field of coefficients. This gives us a bunch of eigendata:

$$f(\mathcal{L}(x)) = \rho(\mathrm{Fr}_x) f(\mathcal{L}). \tag{9.5}$$

The operator $\mathcal{L} \mapsto \mathcal{L}(x)$ is the first example of a *Hecke operator*, analogous to the 't Hooft or Dirac monopole operators that we saw in physics, where we identified H^2 of a 3-manifold M with a lattice, and acted by translating on that lattice. And we thought of$H^2(M)$ as $\pi_0(\mathrm{Pic}(M))$.

The analogue of Wilson operators here is $\rho \in \mathrm{Loc}(C) \mapsto \rho(\mathrm{Fr}_x)$: on the 3-manifold M, we sent a local system to multiplication by the monodromy around each loop.

In algebraic geometry, we can't really talkm about local systems as maps from π_1, so we'll avoid infinite covers. Finite covers correspond to maps $\pi_1 \to \mu_n \subset \mathbb{Q}/\mathbb{Z}$. And in characteristic p, the theory only behaves well when $p \nmid n$. So we will work with a prime ℓ different from p; then there is a nice theory of ℓ-adic local systems. Specifically, we consider maps $\pi_1 \to \mathbb{Z}/\ell^n$, which sits inside $\mathbb{Z}_\ell$, or we could even go all the way to $\mathbb{Q}_\ell$ or $\overline{\mathbb{Q}}_\ell$. There is a good theory of ℓ-adic representations of π_1 into, e.g. $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$.

The nice thing about $\overline{\mathbb{Q}}_\ell$ is that there is a (highly noncanonical and difficult-to-describe) isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$, basically because they're both algebraically closed fields of characteristic zero. And the isomorphism certainly doesn't respect the topology!

So what we'll mean by local systems is something like continuous (in the profinite topology) representations from the unramified Galois group of C to $\overline{\mathbb{Q}}_\ell^\times$. And in this situation, everywhere we would've said "complex-valued functions," we now mean $\mathbb{Q}_\ell$-valued functions. The shape of the story is the same, even if the details are different.

Key to our generalizations will be how we identify $\mathrm{Pic}(C)$. We've been thinking of it as the group of divisors mod principal divisors, a quotient

$$\left(\bigoplus_{x\in C} \mathbb{Z}\right) / k(C)^\times. \tag{9.6}$$

This is related to line bundles as follows: given a line bundle $\mathcal{L}$, we can trivialize it generically, meaning there is a smooth section of $\mathcal{L}$ on all but finitely many points $x_1, \dots, x_n$, defining an isomorphism $\mathcal{O}_C|_{C\setminus\{x_i\}} \xrightarrow{\cong} \mathcal{L}|_{C\setminus\{x_i\}}$.

On the other hand, for all $x \in C$, we can trivialize $\mathcal{L}$ very close to x. By this we mean take a disc around x, $D_x \coloneqq \mathrm{Spec}(\mathcal{O}_x)$ ($\mathcal{O}_x$ means the *completed* local ring at x, which a choice of coordinate identifies with $k[[t]]$). The pullback of $\mathcal{L}$ to D_x is trivialized.

The local degree at x of a section s, $\deg_x s$, is nonzero, but by changing the trivialization of $\mathcal{L}$ on D_x, we get "the same" local degree. So we realy should be looking at $K_x^\times/\mathcal{O}_x^\times$. Choosing a coordinate defines an isomorphism $k((t))^\times/k[[t]]^\times$. Concretely, these are equivalence classes of Laurent series with nonzero leading coefficient, where series are equivalent if they agree after multiplying by an invertible Taylor series. This means that we can zero out all but the leading term t^{-N}, so we end up identifying this quotient with $\mathbb{Z}$, which is where degrees ought to live. Great!

Now we want to vary the line bundle. This means we have to vary the locus $U \coloneqq C \setminus \{x_i\}$ where the bundle is trivialized. We will address this by using all points, including huge amounts of data: a line bundle $\mathcal{L}$, a rational section, and a trivialization of $\mathcal{L}|_{D_x}$ for *all* $x \in C$. This is some huge product of $K_x^\times$ for all $x \in C$, but we only see some of the elements of this product. This is the *restricted product*

$$\prod_{x\in C}{}' K_x^\times \subset \prod_{x\in C} K_x^\times, \tag{9.7}$$

meaning that for all but finitely many $x \in C$, we obtain an element of $\mathcal{O}_x^\times \subset K_x^\times$. This is still a lot of data! So let's now reduce the data.

Now if you take line bundles and rational sections, you end up with divisors, which is reassuring: we get

$$\prod{}' K_x^\times / \prod \mathcal{O}_x^\times = \prod{}' (K_x^\times / \mathcal{O}_x^\times) = \prod{}' \mathbb{Z}, \tag{9.8}$$

i.e. finite formal sums of points $x \in C$, which is what divisors are. (TODO: I need to fix these restricted products.)

So we can identify the Picard group with a double quotient

$$F^\times \backslash \prod{}' K_x^\times / \prod \mathcal{O}_x^\times, \tag{9.9}$$

where on the left, $F^\times$ accounts for changes in section, and on the right, the product accounts for changes of trivialization on D_x. This is the group of divisors modulo principal divisors ($F^\times$). In the number field setting this corresponds to the *(unramified) idele class group* of F.

Lecture 10.

: 2/25/21

Lecture 11.

Boundary conditions and extended TFT: 3/2/21

> *"There's a pun here, which has been left as an exercise for Arun"* (Note: I have not yet solved this exercise)

Note: I missed Thursday's lecture. I'll watch the video and update notes soon. In the meantime, check out Jackson's notes.

Today we're going to consider ramification, which corresponds in physics to boundary theories, or considering operators on singularities. (TODO: some stuff I missed about physics examples. In particular, solenoids, and Cartier duality)

So under the equivalence $\mathrm{Pic}(C) \simeq \mathrm{Loc}_1(C)$, we can generalize in two ways: we can put *level structures* on line bundles on the left, adding trivializations along positive-codimension regions. On the right, we can add singularities to our local systems. And the full story of class field theory, which takes into account ramifications, shows that the correspondence upgrades to including these generalizations too.

In field theory, we have a space of fields $\mathcal{F}(M)$ on the manifold M, and there is a restriction map $\pi_{\partial M} \colon \mathcal{F}(M) \to \mathcal{F}(\partial M)$. Boundary data modifies how we attach a vector space to $N \coloneqq \partial M$: specifically, we choose a sheaf $\mathcal{E}_N \to N$, and we use this to write down a different vector space. For example, we could attach to M the vector space $\Gamma_M(\pi_{\partial M}^* \mathcal{E}_N)$.

Example 11.1. In topological Maxwell theory (the A-side), the vector space we attached to M is $H^*(\mathcal{C}(M))$, the cohomology of the space of U_1-connections on M. Recall that at least at the level of components, $\pi_0(\mathcal{C}(M)) \cong H^2(M;\mathbb{Z})$. If we consider a submanifold $N \subset M$ and consider boundary data, we would like to replace this with something that looks like relative cohomology: $H^2(M,N;\mathbb{Z})$.

First, let's remember how we got $H^2(M;\mathbb{Z})$. This arose by showing that $\mathcal{C}(M) \simeq \mathrm{Map}(M, K(\mathbb{Z},2)) = \mathrm{Map}(M, B\mathrm{U}_1)$: the space of connections on a principal U_1-bundle is contractible, and the topological type is classified by $B\mathrm{U}_1 \simeq K(\mathbb{Z},2)$. Now, given a bundle and connection, we can restrict to N, defining a map

$$\mathrm{Map}(M, K(\mathbb{Z},2)) \longrightarrow \mathrm{Map}(N, K(\mathbb{Z},2)) = \mathrm{Map}(N, B\mathrm{U}_1). \tag{11.2}$$

One sheaf we can consider is $\mathcal{E} \coloneqq i_* \underline{\mathbb{C}}$ (where $i \colon \mathrm{pt} \hookrightarrow N$ is an inclusion map); then $H^*(\mathcal{C}(M); \pi_N^* \mathcal{E})$ is the cohomology of the space of line bundles on M with data of a trivialization on N. So this is an example of a *Dirichlet boundary condition.*

On connected components, this is preciely $H^2(M,N;\mathbb{Z})$, namely the group of connected components of the space of maps $f \colon M \to K(\mathbb{Z},2)$ together with null-homotopies of the restriction $f|_N$. This space is the pullback of the diagram

$$\begin{tikzcd} & \mathrm{Map}(M, K(\mathbb{Z},2)) \arrow[d] \\ \mathrm{pt} \arrow[r] \arrow[ur, phantom, ""] & \mathrm{Map}(N, K(\mathbb{Z},2)). \end{tikzcd} \tag{11.3}$$

Dually, we can consider the *Neumann boundary condition*: let M_0 denote the complement of a tubular neighborhood of N in M, so M_0 is a manifold with boundary. Again we take the constant sheaf on $\mathcal{F}(\partial M_0) \leftarrow \mathcal{F}(M_0)$, and pull back (to the constant sheaf again), considering $H^*(\mathcal{F}(M_0), \underline{\mathbb{C}})$. This is the cohomology of the space of fields which can have arbitrary singularities on N. But we could also take the pullback of a skyscraper sheaf, which is equivalent to considering the cohomology of the space of connections on M_0 with trivializations on the boundary. There are plenty of more general conditions you can put here; each sheaf on the boundary is a machine that gives you a new vector space on the bulk manifold. ◂

We'll come back to the number-theoretic aspects of this in a bit, but first, extended topoloical field theory!

Recall that topological field theory attaches to a closed n-manifold some number (the partition function), which can be heuristically thought of as the volume of the space of fields. And to a closed $(n-1)$-manifold N, we attached a complex vector space (often a Hilbert space), which we think of as the space of functionals on the space of fields on M. We arrived at this by breaking relativistic symmetry and thinking of N as being at a single instant of time in its cylinder.

Now we will go further: given a closed $(n-2)$-manifold P, we will assign a category, often something like the category of sheaves on $\mathcal{F}(P)$. In general, this category will be "linear," e.g. Hom-spaces are complex vector spaces; thus this is another kind of linearization of the fields, albeit more categorified.

Traditionally, this arose from the Atiyah-Segal (TODO: citations here) definition of topological field theory, defined as a symmetric monoidal functor

$$Z\colon \mathcal{B}ord_{n-1,n} \longrightarrow \mathcal{V}ect. \tag{11.4}$$

Here $\mathcal{B}ord_{n-1,n}$ is the category whose objects are closed, oriented $(n-1)$-manifolds and whose morphisms $N_0 \to N_1$ are (diffeomorphism classes of)[10] bordisms from N_0 to N_1. "Symmetric monoidal" means that disjoint unions of manifolds are sent to tensor products in $\mathcal{V}ect$. Composition is gluing bordisms.

The idea is that a bordism from N_0 to N_1 gives you a linear map $Z(N_0) \to Z(N_1)$ between the vector spaces assigned to its boundary. And a closed n-manifold M is a bordism $\varnothing \to \varnothing$, hence defines a linear map $\mathbb{C} \to \mathbb{C}$, which we identify with a complex number, which is the partition function of M.

The bordism-theoretic perspective tells us that it's possible to compute the partition function by chopping M up into pieces, specifically bordisms, and then computing what Z assigns to those bordisms, then composing to obtain the partition function. Physically speaking, this expresses the locality of the partition function in spacetime.

But state spaces also ought to be local! Extended TFT is telling us how to repeat this story one dimension lower and one category number higher. If N is a closed $(n-1)$-manifold that factors as a composition of bordisms $N = N_1 \cup_P N_2$, we would like to express

$$Z(N) = \langle Z(N_1), Z(N_2)\rangle_{Z(P)}, \tag{11.5}$$

in some categorified sense of the pairing, just as we computed the partition function by gluing. This perspective was articulated by Freed, Lawrence, and Dolan among others.

To make this precise, we build a bordism 2-category $\mathcal{B}ord_{n-2,n}$, supplying the following data.

- **Objects:** Closed, oriented $(n-2)$-manifolds.
- **Morphisms:** Compact, oriented bordisms, as before.
- **2-morphisms:** In a 2-category, there are now "2-morphisms" between morphisms. Here we allow bordisms between bordisms N_0 and N_1, thought of as manifolds with corners.

For 2-morphisms, $\partial N_0 = \partial N_1$, so rather than thinking of $[0,1]\times[0,1]$ as your prototypical manifold-with-corners, you should think of the eye-shaped quotient $([0,1]\times[0,1])/((0,a)\sim(0,b),(1,a)\sim(1,b))$.

The bordism 2-category is again symmetric monoidal with respect to disjoint union, though the definition of a symmetric monoidal 2-category is a bit more involved.

[10] This parenthetical is important for composition to be associative on the nose.

Definition 11.6. A *two-tier TFT* or *once-extended TFT* is a symmetric monoidal functor of 2-categories

$$Z\colon \mathcal{B}ord_{n-2,n} \longrightarrow \mathcal{C}, \tag{11.7}$$

where $\mathcal{C}$ is some symmetric monoidal 2-category.

In this case, it's not immediately clear what $\mathcal{C}$ should be, so we just leave that as a choice. Generally, though, we want this to generalize the Atiyah-Segal definition of TFT, so we stipulate that $\mathcal{C}$ is a *delooping* of $\mathcal{V}ect$. That is, $\Omega\mathcal{C}$, defined to be $\mathrm{End}(1_{\mathcal{C}})$, is equivalent to $\mathcal{V}ect$ as symmetric monoidal categories.

Now, if N is an $(n-1)$-manifold factored as $N_1 \cup_P N_2$ as before, we can write N as a composition of morphisms in $\mathcal{B}ord_{n-2,n}$, and therefore exploit locality to compute $Z(N)$, at least in principle.

To compute this in practice we ought to choose $\mathcal{C}$. There are multiple choices that work well, but we'll let $\mathcal{C} = \mathcal{C}at_{\mathbb{C}}$, the symmetric monoidal 2-category of $\mathbb{C}$-*linear categories*. Specifically:

Objects: (Small) categories enriched over $\mathcal{V}ect_{\mathbb{C}}$.
Morphisms: Functors which respect this enrichment.
2-morphisms: Natural transformations which respect this enrichment.

And sure enough, $\Omega\mathcal{C}at_{\mathbb{C}} \simeq \mathcal{V}ect$.

Though we didn't define (topologial) Maxwell theory formally as an Atiyah-Segal TFT, it is still useful to think from this perspective: given a closed, oriented 2-manifold Σ, Maxwell theory ought to assign a $\mathbb{C}$-linear category, and if we factor a closed 3-manifold N as $N \cong N_1 \amalg_\Sigma N_2$, we can compute the vector space assigned to N using this decomposition.

What does abelian duality look like in codimension 2? We have several examples to look into (this is vacuous in dimensions 1 and below, so we start at dimension 2).

Example 11.8. In dimension 2, we had two examples of abelian duality: in one, $H^0(S^1;\mathbb{Z}) \leftrightarrow H^1(S^1;\mathbb{C}^\times)$, and in the other, $H^1(S^1;\mathbb{Z}) \leftrightarrow H^0(S^1;\mathbb{C}^\times)$. These are interpreted as different field theories; let's extend them to codimension 2. First the first example:

- On the A-side ($H^0(S^1;\mathbb{Z})$), the fields are maps to $\mathbb{Z}$. Maybe not the most interesting, but we can extend: $\mathcal{F}(\mathrm{pt}) = \mathbb{Z}$, and we linearize by taking sheaves. There're not many options for what kind of sheaves we take here, and in the end $Z(\mathrm{pt}) = \mathcal{S}hv(\mathbb{Z})$, or the category of $\mathbb{Z}$-graded vector spaces. This is $\mathbb{C}$-linear, which is good.
- Now on the B-side, corresponding to $H^1(S^1;\mathbb{C}^\times)$. The fields are $\mathbb{C}^\times$-local systems, which is pretty boring on pt: the set of isomorphism classes of $\mathbb{C}^\times$-local systems is a singleton, and sheaves on this is not just boring, but doesn't match what we got on the A-side. To remedy this we have to introduce… stacks.[11] The space of local systems enhances to a stack $\mathcal{L}oc_{\mathbb{C}^\times}(\mathrm{pt})$ encoding the fact that local systems have symmetries. In this specific case, we have a single isomorphism class of local systems, but it has lots of isomorphisms, a whole $\mathbb{C}^\times$ of them. Therefore this stack is equivalent to $B\mathbb{C}^\times = \mathrm{pt}/\mathbb{C}^\times$. We want to take sheaves on this, which here means sheaves on a point together with a $\mathbb{C}^\times$-action on the (unique) stalk. That is, what we assign to a point in the B-side field theory is the category of representations of $\mathbb{C}^\times$. Again this is $\mathbb{C}$-linear.

Duality means we better get an equivalence of categories between $\mathcal{V}ect_{\mathbb{Z}}$ and $\mathcal{R}ep_{\mathbb{C}^\times}$, and this is given by the isotypic decomposition: if V is a $\mathbb{C}^\times$-representation, it splits as

$$V = \bigoplus_{n\in\mathbb{Z}} V_n, \tag{11.9}$$

where $V_n \subset V$ is the subspace on which $z \in \mathbb{C}^\times$ acts by z^n.

This can be thought of Fourier-theoretically, by thinking of these two categories as the categories of $\mathbb{C}[\mathbb{Z}]$-modules (A-side) or $\mathbb{C}[\mathbb{C}^\times]$-modules (B-side). The Fourier transform requires thinking a little bit about comodules, so we won't get into the details right now.

Now the other example, $H^1(S^1;\mathbb{Z}) \leftrightarrow H^0(S^1;\mathbb{C}^\times)$. Again, the A-side will involve topology, and the B-side will involve complex geometry.

- The A-side used (functions on) $H^1(S^1;\mathbb{Z})$. We can think of this as isomorphism classes of $\mathbb{Z}$-valued local systems, but it is more convenient to map to $S^1 = B\mathbb{Z}$, so periodic functions. Now we try to extend to a point: fields are maps to a point, so we get $\mathcal{F}(\mathrm{pt}) = S^1 = B\mathbb{Z}$. When we take sheaves,

[11] If you don't know anything about stacks, that's OK, and we'll discuss some important things about them next time.

that could mean more than one thing, but there's only one notion from the topological point of view: locally constant sheaves, i.e. $\mathcal{L}oc_{\mathbb{Z}}(S^1)$. These are representations of $\mathbb{Z}$, or sheaves on $B\mathbb{Z}$, where here $\mathbb{Z}$ comes from $\pi_1(S^1)$. So we end up with $\mathbb{C}[\mathbb{Z}]$-modules, with the algebra structure on $\mathbb{C}[\mathbb{Z}]$ coming from convolution.

This has a fancy name: the *wrapped Fukaya category* of T^*S^1. In general this is a difficult object to study for a given symplectic manifold X, but when X is a cotangent bundle it admits a description as local systems.

- On the B-side, we had $H^0(S^1;\mathbb{C}^\times)$. The fields are maps to $\mathbb{C}^\times$ with the discrete topology (we're doing algebraic geometry on this side, not topology). The fields on a point are therefore $\mathbb{C}^\times$, and when we take sheaves, here we use algebraic geometry and mean quasicoherent sheaves on $\mathbb{C}^\times$. This is the category of $\mathbb{C}[\mathbb{C}^\times]$-modules, where the algebra structure is pointwise multiplication. Pontrjagin duality identifies $\mathbb{C}[\mathbb{C}^\times]$ with pointwise multiplication with $\mathbb{C}[\mathbb{Z}]$ with convolution, and therefore we see that $\mathcal{L}oc(S^1) \simeq \mathcal{QC}(\mathbb{C}^\times)$.

  This is a very special case of homological mirror symmetry, which in general relates Fukaya categories on the A-side (in symplectic topology, which is why things were more topological on the A-side) with categories of quasicoherent sheaves on the B-side (hence we care about algebro-geometric notions of sheaves). ◀

You can run a roughly similar game in 3d, but let's jump to the key example: 4d Maxwell theory. On the A-side, we care about the topology of line bundles, and on the B-side, we care about the algebraic geometry of *flat* line bundles. We understood how this works in codimension 1, where we had a Pontrjagin-Poincaré duality between $H^2(M;\mathbb{Z})$ and $H^1(M;\mathbb{C}^\times)$, where M is a closed, oriented 3-manifold; now let's see what this attaches to a closed, oriented topological surface Σ.

On the A-side, we only have topology, and our fields are line bundles with connection. We got H^2 by taking π_0 of this space. So to Σ, we attach the space of fields

$$\mathrm{Pic}(\Sigma) \coloneqq \mathrm{Map}(\Sigma, B\mathrm{U}_1). \tag{11.10}$$

Choose a complex structure on Σ; then there is the more familiar $\mathrm{Pic}(\Sigma)$, the space of holomorphic or algebraic line bundles on Σ, an object in algebraic geometry. This space depends on the complex structure, but its topological type does not, and is in fact homotopy equivalent to $\mathrm{Map}(\Sigma, B\mathrm{U}_1)$. Again this factors as

$$0 \longrightarrow \mathrm{Jac}(\Sigma) \longrightarrow \mathrm{Pic}(\Sigma) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0, \tag{11.11}$$

where $\mathrm{Jac}(\Sigma)$ is the space of degree-0 line bundles. Suppose Σ is connected and has genus g; then, the homotopy type of Jac is a torus $\mathbb{C}^g/\mathbb{Z}^{2g}$ of real dimension $2g$.

Remark 11.12. Line bundles have automorphisms, and therefore $\mathrm{Pic}(\Sigma)$ and $\mathrm{Jac}(\Sigma)$ are really stacks, but we don't need to worry about this at present. ◀

The short exact sequence (11.11) is reminiscent of the *exponential exact sequence* of complex geometry:

$$0 \longrightarrow H^1(\Sigma;\mathcal{O}_\Sigma)/H^1(\Sigma;\mathbb{Z}) \longrightarrow H^1(\Sigma;\mathcal{O}_\Sigma^\times) \longrightarrow H^2(\Sigma;\mathbb{Z}). \tag{11.13}$$

The identifications $H^1(\Sigma;\mathcal{O}_\Sigma^\times) \cong \mathrm{Pic}(\Sigma)$ (again, ignoring stackiness) and $H^2(\Sigma;\mathbb{Z}) \cong \mathbb{Z}$, together with the connecting morphism with the degree map, means that

$$\mathrm{Jac}(\Sigma) \cong H^1(\Sigma;\mathcal{O}_\Sigma)/H^1(\Sigma;\mathbb{Z}). \tag{11.14}$$

This identifies the Jacobian with $K(H^1(\Sigma;\mathbb{Z}),1)$, a torus with fundamental group $H^1(\Sigma;\mathbb{Z})$. This torus can also be described as $S^1 \otimes_{\mathbb{Z}} H^1(\Sigma;\mathbb{Z})$.

What we attach to Σ is the $\mathbb{C}$-linear category $\mathrm{Loc}(\mathrm{Pic}(\Sigma))$.[12] We will see on Thursday how this is related to the category that the B-side theory attaches to Σ.

[12] Stackiness does not affect this abelian category, though it will affect the derived category.

Lecture 12.

Extended TFT and topological Maxwell theory: 3/4/21

As we began discussing last time, extended TFT tells us to not just assign complex numbers to 4-manifolds[13] and vector spaces to 3-manifolds, but also $\mathbb{C}$-linear categories to closed surfaces. And given a 3-manifold Y with boundary Σ, we get a linear functor $Z(\Sigma) \to Z(\varnothing) = \mathcal{V}ect$. This functor assigns a vector space to Y together with data of an object $\mathcal{E} \in Z(\Sigma)$. Dually, we could switch incoming and outgoing, so we get a linear functor $Z(Y)\colon \mathcal{V}ect \to Z(\Sigma)$, giving us a distinguished object $Z(Y)(\mathbb{C})$.

(TODO: an example with sheaves, I think?)

We also discussed some examples last time, most notably topological Maxwell theory. On the A-side, which is about the topology of spaces of U_1-bundles, which we called "Pic," we considered an oriented surface Σ, chose a complex structure, and then considered the space $\mathrm{Pic}(\Sigma)$ of line bundles on Σ. The homotopy type of this space does not depend on the complex structure, and noncanonically it splits as

$$\mathrm{Pic}(\Sigma) \cong \mathbb{Z} \times \mathrm{Jac}(\Sigma) \times B\mathrm{U}_1, \tag{12.1}$$

where the last factor, which we ignore, is about constant maps into $B\mathrm{U}_1$ (giving trivial line bundles). You can think of this as stackiness: $B\mathrm{U}_1$ is trying to be pt/U_1, giving us additional automorphisms of line bundles. But we don't have to worry about it right now.

The space $\mathrm{Jac}(\Sigma) \cong \mathbb{C}^g/\mathbb{Z}^{2g}$, assuming Σ is connected and has genus g. This is because it arises as $H^1(\Sigma; \mathcal{O}_\Sigma)/H^1(\Sigma;\mathbb{Z})$. We care about topological stuff only, and so we end up with a torus, which is also an Eilenberg-Mac Lane space $K(H^1(\Sigma;\mathbb{Z}), 1)$.

We have a space and now should linearize to produce a category, so we will take a category of sheaves. Just as there were different specific kinds of functions you can take when someone says "take functions on a space," which is a functional analysis question, we can consider different kinds of sheaves. We will take locally constant ones, so the category we assign to Σ is $\mathrm{Loc}(\mathrm{Pic}(\Sigma))$. We're not diving into the derived world yet (which means we ignore the $B\mathrm{U}_1$ in (12.1) for now, as $B\mathrm{U}_1$ is simply connected), so $\mathrm{Loc}(\mathrm{Pic}\Sigma)$ decomposes as

$$\mathrm{Loc}(\mathrm{Pic}\Sigma) \simeq \mathcal{V}ect_{\mathbb{Z}} \otimes \mathrm{Loc}(\mathrm{Jac}(\Sigma)), \tag{12.2}$$

where $\mathcal{V}ect_{\mathbb{Z}}$ means $\mathbb{Z}$-graded vector spaces. But because the Jacobian is a torus, local systems on it are canonically identified with representations of $\Lambda \coloneqq H^1(\Sigma;\mathbb{Z})$, and therefore $\mathrm{Loc}(\mathrm{Pic}(\Sigma))$ is identified with the category of $\mathbb{Z}$-graded Λ-representations. This has a monoidal structure under convolution, which is not told to us by the TFT.

Now the B-side, which is about the algebraic geometry of $\mathbb{C}^\times$-local systems. We have our surface Σ again, but we don't even need to choose a complex structure to consider $\mathrm{Loc}_{\mathbb{C}^\times}(\Sigma) = \mathrm{Hom}(\pi_1(\Sigma), \mathbb{C}^\times)/\mathbb{C}^\times$ — but since $\mathbb{C}^\times$ is abelian, it acts trivially by conjugation. And since $H_1(\Sigma) = \pi_1(\Sigma)^{\mathrm{ab}}$, this space of local systems is identified with

$$\mathrm{Hom}(H_1(\Sigma), \mathbb{C}^\times)/\mathbb{C}^\times \cong (H^1(\Sigma;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}^\times)/\mathbb{C}^\times, \tag{12.3}$$

where again $\mathbb{C}^\times$ acts trivially. We should attach a category to this, and something algebraic geometry flavored, such as $\mathcal{QC}$. What do we mean precisely by this? $\mathrm{Loc}_{\mathbb{C}^\times}(\Sigma)$ looks like a torus again, but this extra $\mathbb{C}^\times$-action can't be gotten rid of, and so the category of sheaves is the category of modules for $\mathbb{C}[\mathrm{Loc}_{\mathbb{C}^\times}(\Sigma)]$ under the "multiplication" monoidal structure, tensored with $\mathcal{R}ep_{\mathbb{C}^\times}$.

We want this category to be equivalent to the category of locally constant sheaves on $\mathrm{Pic}(\Sigma)$ that we constructed on the A-side. TODO: I missed this, but the key ingredient is an identification of algebras $\mathbb{C}[\Lambda]$ with convolution and $\mathbb{C}[T_{\mathbb{C}}^\vee]$ under pointwise multiplication, which therefore induces an equivalence of sheaves. This is a first example of *Fourier-Mukai duality*, abelian duality identifying categories of sheaves. We threw in this additional $\mathbb{Z}$-grading on the A-side and the $\mathbb{C}^\times$-action on the B-side, though.

This is the first version of the geometric Langands correspondence, albeit a very simple one. It is specifically the Betti geometric Langlands equivalence for $G = \mathrm{GL}_1$.

Next, we'll connect this to the 3-manifold story, and then explain the number-theoretic analogue. Let Y be a compact, oriented 3-manifold with $\partial Y \cong \Sigma$. We can restrict line bundles from Y to Σ, which defines a map $\pi\colon \mathrm{Pic}(Y) \to \mathrm{Pic}(\Sigma)$; we can then try to push forward or pull back sheaves along π. For example, choosing

[13] This particular aspect of TFT will generally not come up in this class.

$\mathcal{E} \in \mathrm{Loc}(\mathrm{Pic}\Sigma)$, we want to obtain a vector space, which we did by pulling back to $\mathrm{Pic}(Y)$ and taking sections (rather than just taking functions, as we did when Y is closed).

On the B-side, we have $\pi^\vee : \mathrm{Loc}_{\mathbb{C}^\times} Y \to \mathrm{Loc}_{\mathbb{C}^\times}(\Sigma)$, so we can again pull back $\mathcal{E}^\vee \in \mathcal{QC}(\mathrm{Loc}_{\mathbb{C}^\times}(\Sigma))$ and take sections. The Fourier-Mukai transform includes a correspondence $\mathcal{E} \leftrightarrow \mathcal{E}^\vee$ inducing an isomorphism

$$\Gamma(\mathrm{Pic}(Y), \pi^*\mathcal{E}) \cong \Gamma(\mathrm{Loc}_{\mathbb{C}^\times}(Y), (\pi^\vee)^*\mathcal{E}^\vee). \tag{12.4}$$

This all looks fancy, but remember that these spaces are just tori, and everything here is really coming from a combination of Poincaré and character duality.

We can now ramify. Let's consider the subcategory of local systems which when we restrict to Σ land in a particular subset. One of the motiviations for doing this was number-theoretic: ramifications are where everything interesting happens in number theory.

So let's let C be a curve over a finite field $\mathbb{F}_q$. Let x be an $\mathbb{F}_q$-point of C; we will talk about allowing ramification at x. Let K_x be the completion of the field of rational functions at x, so $K_x \cong \mathbb{F}_{q'}((t))$, where $\mathbb{F}_q \hookrightarrow \mathbb{F}_{q'}$ is a finite extension. Then $\operatorname{Spec} K_x$ is a formal disc around $C \setminus x$. Recall that points corresponded to circles in topology, so the boundary of a tubular neighborhood of a knot coresponds to what we're looking at here.

On the B-side (Galois side), we restrict

$$\mathrm{Loc}_1(C \setminus x) \to \mathrm{Loc}_1(D) = \mathrm{Hom}_{\mathrm{cts}}(\mathrm{Gal}(\overline{K}_x/K_x), \overline{\mathbb{Q}}_\ell^\times) = \mathrm{Hom}_{\mathrm{cts}}(\mathrm{Gal}^{\mathrm{ab}} K_x, \overline{\mathbb{Q}}_\ell^\times) \tag{12.5}$$

And what about $\mathrm{Pic}(C)$? We described this as

$$\mathrm{Pic}(C) = F^\times \backslash \prod_{y\in C}{}' K_y^\times / \prod_{y \in C} \mathcal{O}_Y^\times. \tag{12.6}$$

Now we have to remove the point x. We take the restricted products over C and now work over $C \setminus x$, so we obtain

$$\mathrm{Pic}(C)/(\mathbb{Z}_x = K_x^\times/\mathcal{O}_x^\times). \tag{12.7}$$

This is not a fun space, just like if you killed the $\mathbb{Z}$ in $\mathrm{Pic}(\Sigma)$ happening at a point on a Riemann surface Σ. But it will be useful for us, so let's call it $\mathrm{Pic}(C, \widehat{x})/K_x^\times$; this notation is justified by the third isomorphism theorem. $\mathrm{Pic}(C, \widehat{x})$ is line bundles on C with a trivialization on D_x, or equivalently, line bundles on $C \setminus x$ and a trivialization on $D_x^\times$. Taking the quotient by $K_x^\times$ gets rid of that trivialization on the formal disc.

So we don't need to puncture the curve in other to study the punctured cuve: you just have to know how $K_x^\times$ acts there. There is a single isomorphism class of line bundles on the punctured disc $D_x^\times$, but there are lots of functions on $D_x^\times$, so this line bundle has a lot of automorphisms. So as a stack,

$$\mathrm{Pic}(D_x)^\times \cong \mathrm{pt}/K_x^\times, \tag{12.8}$$

and studying this picture is equivalent to studying the action of $K_x^\times$ on $\mathrm{Pic}(C, \widehat{x})$, which feels much less like abstract nonsense and much more like classical number theory. However, number theorists say to not take infinite level structure. Specifically, $K_x^\times$ contains $\mathcal{O}^\times = \mathbb{F}_{q'}[[t]]$, and this contains subrings corresponding to congruence subgroups $\mathcal{O}^{\times(n)}$, functions of the form $1 + t^n(-)$, thought of as a trivialization to level n. So the action of $K_x^\times$ on $\mathrm{Pic}(C, nx)$ (sort of the n^{th} order neighborhood) factors through $K_x^\times/\mathcal{O}^{\times(n)}$.

Until very recently, there weren't great techniques to deal with this, so let's look at smooth representations of $K^\times$, meaning not what you might usually expect over $\mathbb{C}$, but rather something more p-adic: that every vector is fixed by $\mathcal{O}^{\times N}$ for some $N \gg 0$. For example, if we give the object acted on the discrete topology, this imposes continuity, because it is expressing the topology on $K_x^\times$.

TODO: something I missed, comparing this to a surface bounding a 3-manifold. But analogously to moving sheaves between the bulk and the boundary there, we push sheaves around on $C \setminus x$ and $D_x^\times$.

How do these disparate pieces of data talk to each other? On the A-side we had representations of $K_x^\times$ coming from $\mathrm{Pic}(C, \widehat{x})$, and on the B-side we have sheaves on $\mathrm{Loc}_1(\mathcal{O}^\times)$, a module for functions on $\mathcal{O}^\times$. We need an analogue of the Fourier-Mukai theorem to get between these two things that are like categories of sheaves. Local class field theory is all about this — though it does not usually get expressed in this equivalence-of-categories manner. Just like global class field theory is usually posited as an isomorphism between the maximal unramified abelian Galois group with the completed Picard group, as we discussed, and then reinterpreted in an abelian duality manner using Pontrjagin duality, we will do something similar with local class field theory, an isomorphism

$$\widehat{K}_x^\times \xrightarrow{\cong} \mathrm{Gal}^{\mathrm{ab}}(\overline{K}_x/K_x). \tag{12.9}$$

Here we profinitely complete the left-hand side. Anyways this mere isomorphism of abelian groups tells us that the group ring of $K_x^\times$ is Pontrjagin dual to $\mathrm{Hom}(\mathrm{Gal}^{\mathrm{ab}}\,K, \overline{\mathbb{Q}}_\ell^\times)$, which we're calling "Loc." Again on one side we have a group algebra with convolution, and on the other we have a group algebra with multiplication. Instead of $\mathbb{C}$, our group algebras are over $\overline{\mathbb{Q}}_\ell$. One has to be careful with topology here.

Now, pass to modules, which tells us that representations of $K_x^\times$ are equivalent to quasicoherent sheaves on Loc_1. Making this precise takes a lot of work, to be clear. But the point is that representations of $K_n^\times$ on one side is identified with the algebraic geometry of a space of local systems, just as in the topological story.

We want to think of (12.9) as identifying representations of $K_x^\times$ with modules for functions on GL_1 Galois representations. We don't have a moduli-theoretic interpretation of $\mathrm{Pic}(C, \widehat{x})$ here, which is why we take adeles:

$$F^\times \backslash {\prod_v}' K_v^\times \big/ \prod_{v\notin S} \mathcal{O}_v^\times, \tag{12.10}$$

which is the ramified idele[14] class group, and which is acted on by a large abelian group, namely $\prod_{v\in S} K_v^\times$.

Next time, we'll finish up abelian groups, understanding the duality we discussed today on Riemann surfaces as a spectral decomposition or Fourier transform. This will pave the way to the nonabelian generalization that is the goal of the course.

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[14] TODO: adele? idele?