2019 PCMI PREPARATORY LECTURES

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These notes were taken at UT Austin as part of a learning seminar in preparation for PCMI's 2019 graduate summer school. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

Contents

1. BPS states: 5/21/19

Today, Shehper talked about BPS states in 4D $\mathcal{N}=2$ supersymmetric theories. This is not the only place you can have BPS states, but this is probably the one most relevant to our interests. For a reference, check out Moore's PITP lectures on BPS states.[1]

First, 4D means the dimension of the theory: we have three space coordinates and one time coordinate. There's an underlying symmetry group called the *Poincaré group* of $\mathbb{R}^{1,3}$, whose Lie algebra is

$$\mathfrak{iso}_{1,3} \cong \mathfrak{so}^+_{1,3} \ltimes \mathbb{R}^{1,3}. \tag{1.1}$$

The "+" means that we want transformations to preserve the arrow of time. That is, these transformations correspond to changes between different reference frames. In one, we have local coordinates (t,x,y,z), and from one reference frame to another, the time coordinate t is scaled by something; we want this to be a nonnegative number. The transformations coming from $\mathfrak{so}^+_{1,3}$ are called *(orthochronous special) Lorentz transformations*, but we'll call them Lorentz transformations.

Another way to describe the Poincaré group is as the group of isometries of $\mathbb{R}^{1,3}$.

Now we should clarify what "underlying symmetry" means. This is a statement about QFT, which means we have to indicate how to actually discuss or work with QFT. There are a few different formalisms, e.g. the *Hamiltonian formalism* or *canonical quantization formalism*; or the *path integral formalism*, which comes with the following data:

- A space of *field configurations* $\mathcal{F}$.
- An *action*, a function $S\colon \mathcal{F}\to\mathbb{R}$.
- A set of *local operators*.

From this data one can compute correlation functions associated to local operators $\Phi_1,\dots,\Phi_n$ at points $x_1,\dots,x_n$ in spacetime via the path integral

$$\langle \Phi_1(x_1)\cdots\Phi_n(x_n)\rangle = \int_{\mathcal{F}} \mathcal{D}\varphi\, e^{-S(\varphi)}\Phi_1\cdots\Phi_n. \tag{1.2}$$

Of course, this is not mathematically well-defined in general, but physicists have ways of working with it which agree extremely well with experimental data.

The Hamiltonian formalism in a d-dimensional quantum field theory associates to a $(d-1)$-manifold a Hilbert space $\mathcal{H}$. Elements of $\mathcal{H}$ are called *states*, because they represent states of the physical system. Inside $\mathfrak{iso}_{1,3}$, there's an element P_τ which is time translation by τ: explicitly, under the isomorphism (1.1), these are the elements in $\mathbb{R}\cdot t\subset\mathbb{R}^{1,3}$. This element acts on $\mathcal{H}$ by the Hamiltonian, and this is how the system evolves under time. An eigenvector for the Hamiltonian with eigenvalue λ is said to have *energy* λ.

Assumption 1.3. There is a unique vector $|v\rangle\in\mathcal{H}$, called the *vacuum*, with minimum energy.

[1] The lecture notes can be found at http://www.sns.ias.edu/pitp2/2010files/Moore_LectureNotes.rev3.pdf.

There's a sense in which the vacuum generates all of the states: one can act by local operators to obtain the other states. And in this formalism, the correlation functions are given by

$$\langle \Phi_1 \cdots \Phi_n \rangle := \langle v \mid \Phi_1 \cdots \Phi_n \mid v \rangle. \tag{1.4}$$

Explicitly, assume that $\phi(x)$ is a *Lorentz scalar*, which means it's a field transforming in the trivial representation of $\mathfrak{so}_{1,3}^+$. Here x is position, i.e. the coordinate in the spacetime manifold.

Remark 1.5. A field is not an operator, but it does determine a local operator, e.g. ϕ, as a scalar field (function), has a value at a point x. We will think of ϕ as a local operator sometimes in what follows. ◂

How do we use this to create states in $\mathcal{H}$? The first step is to Fourier transform ϕ, leading to $\widetilde{\phi}(p)$. Now this depends on the momentum p. We can act on $|v\rangle$ by $\widetilde{\phi}$ to obtain other states in $\mathcal{H}$.[2] There are things which have positive momenta and with negative momenta; these should be thought of as particle creation $\widetilde{\phi}^\dagger$, resp. particle annihilation operators $\widetilde{\phi}$ on the space of states. This is analogous to the raising and lowering operators on $\mathfrak{su}_2$-representations.

The physical interpretation is that the vacuum has no particles and no momentum. Acting by one creation operator creates a single particle with a prescribed momentum. Acting by another means two particles, and so on.

Remark 1.6. All of this is in a *free theory*, meaning the action is quadratic in the fields. In general, the story is a little more complicated. ◂

Anyways, back to "underlying symmetry." This means the following.

- The fields are all in representations of $\mathfrak{iso}_{1,3}$ (i.e. governing how it transforms under a change of coordinates).
- The Hilbert space is a unitary representation of $\mathfrak{iso}_{1,3}$. Additionally, we want every operator to be unitary, i.e. $U^\dagger U = \mathbf{1}$.

This means that Poincaré symmetries do not change the norm of states, which is important.

Example 1.7. Here are some irreducible representations of $\mathfrak{so}_{1,3}^+$.

- The *trivial* or *scalar representation* $\mathbb{C}$.
- The *vector representation*, which is the defining representation of $\mathfrak{so}_{1,3}^+$ on $\mathbb{R}^{1,3} \otimes \mathbb{C}$.
- The *tensor representations*, which are obtained from the vector representation by symmetric or exterior powers.
- The *spinor representations*, two 2-dimensional representations which are complex conjugates of each other, but are not isomorphic. In physics these are also called *Weyl spinors*; there's a different thing called a *Dirac spinor*, which transforms in the direct sum of the two spinor representations. ◂

So we've discussed what 4D QFT is. What does $\mathcal{N} = 2$ mean? This is specifying "how much supersymmetry" is present in the theory. Supersymmetry means that we extend the Poincaré algebra to a $\mathbb{Z}/2$-graded Lie algebra (sometimes called a *Lie superalgebra*) $\underline{\mathfrak{g}} = \underline{\mathfrak{g}}^0 \oplus \underline{\mathfrak{g}}^1$. In our situation ($\mathcal{N} = 2$), we'd like

$$\underline{\mathfrak{g}}^0 = \mathfrak{iso}_{1,3} \oplus \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_R \oplus \mathbb{C}, \tag{1.8}$$

where $\mathfrak{su}(2)_R$ denotes $\mathfrak{su}(2)$, but we write "R" to denote that this tracks something called *R-symmetry*, and likewise for $\mathfrak{u}(1)_R$ – $\mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_R$ is the *R-symmetry algebra*. Then, $\mathbb{C}$ is generated by an element Z called the *central charge* of the theory.

Then, we want $\underline{\mathfrak{g}}^1$ to be a spinor representation of $\underline{\mathfrak{g}}^0$; specifically, for $\mathcal{N} = 2$,

$$\underline{\mathfrak{g}}^1 = (2, 1; 2)_{+1} \oplus (1, 2; 2)_{-1}. \tag{1.9}$$

The notation $(a, b; c)_d$ means the irreducible $\mathfrak{so}^+(1, 3)$-representation given by (a, b), the irreducible $\mathfrak{su}(2)_R$-representation of dimension c, and the irreducible $\mathfrak{u}(1)_R$-representation of weight d (i.e. the corresponding to the Lie group representation $\mathrm{U}_1 \to \mathrm{U}_1$ sending $z \mapsto z^d$). $\mathbb{C} \cdot Z$ and $\mathbb{R}^{1,3}$ act trivially.

Since $\underline{\mathfrak{g}}^1$ is odd, the Lie bracket restricted to $\underline{\mathfrak{g}}^1 \times \underline{\mathfrak{g}}^1$ is actually an anticommutator (or Poisson bracket), so it lives in $\mathrm{Sym}^2(\underline{\mathfrak{g}}^1)$.

[2] Contextualizing this, and why we can think of this as associated to position and momentum, is really related to how quantum field theory arises via quantization from classical field theory.

Let $\{Q^A_\alpha\}$ be a basis of $(2,1;2)_{+1}$, where $\alpha \in \{1,2\}$ and $A \in \{1,2\}$; similarly, let $\{\overline{Q}_{\dot\alpha A}\}$ be a basis for $(1,2;2)_{-1}$. So we have eight basis elements in total; they're called *supercharges*.

Remark 1.10. The two-dimensional irreducible representation of $\mathfrak{su}(2)_R$ is pseudoreal. There's a notion of a complex representation being *real*, which means that it's self-conjugate – or at least, the representation and its conjugate are related through a symmetric matrix. A representation is *pseudoreal* if instead we have an antisymmetric matrix: $(M^a)^\dagger = \epsilon^{ab} M_b$ (here ϵ is the Levi-Civita tensor). ◀

The point is that complex conjugation identifies some of these basis vectors, so we have to impose the relation

$$(Q^A_\alpha)^\dagger = \overline{Q}_{\dot\alpha A}. \tag{1.11}$$

Once we've imposed this, we have a real 8-dimensional representation.

We can specify the commutation relations between the supercharges:

$$\{Q^A_\alpha, \overline{Q}_{\dot\beta B}\} = 2\sigma^m_{\alpha\dot\beta} P_m \delta^A{}_B \tag{1.12a}$$

$$\{Q^A_\alpha, Q^B_\beta\} = 2\epsilon_{\alpha\beta}\epsilon^{AB}\overline{Z} \tag{1.12b}$$

$$\{\overline{Q}_{\dot\alpha A}, \overline{Q}_{\dot\beta B}\} = -2\epsilon_{\dot\alpha\dot\beta}\epsilon_{AB} Z. \tag{1.12c}$$

Once we understand $\mathrm{Sym}^2\,\underline{\mathfrak{g}}^1$ as a representation, we can analyze this and learn, e.g. that $\sigma^m_{\alpha\dot\beta} P_m$ transforms in the $(2,2)$ representation of $\mathfrak{so}^+_{1,3}$.

Definition 1.13. A *4D $\mathcal{N} = 2$ supersymmetric quantum field theory* is a QFT with an underlying symmetry algebra $\underline{\mathfrak{g}}$.

To construct BPS states, we need some representations of $\underline{\mathfrak{g}}$. We'll do this by finding an analogue of the Casimir operator inside $\mathfrak{iso}_{1,3}$ – an operator which commutes with all other operators. Explicitly, it's

$$P^2 := -P_0^2 + P_1^2 + P_2^2 + P_3^2. \tag{1.14}$$

This mimics the $\mathfrak{su}_2$ story, where the Casimir is the sum of the squares of the three Pauli matrices. In physics, P^2 is also thought of as the mass squared. For example, if the momentum is zero, this relates to the familiar equation $E^2 = M^2c^2$ – in general momentum changes this.

Now one can choose a particular basis in which $P = (M,0,0,0)$, called the *rest frame*. One place you might want this is if you want a state with particular momenta $M^\mu = (P^0, P^1, P^2, P^3)$, and can obtain it from the rest frame by a Lorentz transformation.

Anyways, once you have $(M,0,0,0)$, you can act on it by $\mathfrak{so}_3$ in the last three coordinates, which produces more things of the same mass. So to create "massive" irreducible representations of $\mathfrak{iso}_{1,3}$ with a fixed mass $M > 0$, we need to look for representations of $\mathfrak{so}_3 \oplus \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_R \oplus \mathbb{C}$ as follows: we want eight generators R^A_α and T^A_α such that $\{R,R\} \neq 0$, $\{T,T\} \neq 0$, $\{R,T\} = 0$, such as

$$\{R^A_\alpha, R^B_\beta\} = 4(M - |Z|)\epsilon_{\alpha\beta}\epsilon^{AB} \tag{1.15a}$$

$$\{T^A_\alpha, T^B_\beta\} = -4(M + |Z|)\epsilon_{\alpha\beta}\epsilon^{AB} \tag{1.15b}$$

$$\{R^A_\alpha, T^B_\beta\} = 0. \tag{1.15c}$$

So we have two copies of a Clifford algebra. Explicitly, if $\zeta \in \mathfrak{u}(1) \setminus 0$,

$$R^A_\alpha := \zeta^{-1} Q^A_\alpha + \zeta\sigma^0_{\alpha\dot\beta}\overline{Q}^{\dot\beta A} \tag{1.16a}$$

$$T^A_\alpha := \zeta^{-1} Q^A_\alpha - \zeta\sigma^0_{\alpha\dot\beta}\overline{Q}^{\dot\beta A}. \tag{1.16b}$$

These have reality constraints coming from those of the supercharges, e.g. $(R^1_1)^\dagger = -R^2_2$ and $(R^2_1)^\dagger = R^1_2$. This means

$$(R^1_1 = (R^1_1)^\dagger)^2 = (R^2_1 + (R^2_1)^\dagger)^2 = 4\left(M + \mathrm{Re}\left(\frac{Z}{\zeta^{-2}}\right)\right). \tag{1.17}$$

This is important for unitarity: we want $A^\dagger = A$: we want $\|A|\psi\rangle\|^2 > 0$ if $\psi \neq 0$, so we want $A^2 \geq 0$.

Suppose we choose $\zeta^{-2} = -Z/|Z|$; then the right-hand side of (1.17) simplifies to $4(M - |Z|)$. Therefore we want $M \geq |Z|$, which is called the *BPS bound*. (Other choices of ζ give you weaker constraints.) That is, in any state in a 4D $\mathcal{N} = 2$ supersymmetric theory, the mass of any state is at least $|Z|$.

There are two cases: $M = |Z|$, which is called a *BPS state*, and $M > |Z|$, which is called a *non-BPS state*. If $M = |Z|$, $\{R^A_\alpha, R^B_\beta\} = \pm 4(M - |Z|) = 0$, which acts trivially, so for BPS states, we only get one copy of the Clifford algebra (called a *short representation* rather than the usual *long representation* with two copies). In particular, the T^A_α split into creation and annihilation operators, and we get four states: $|v\rangle$, $T^\beta_\alpha |v\rangle$, $T^\delta_\gamma |v\rangle$, and $T^\beta_\alpha T^\delta_\gamma |v\rangle$. In a non-BPS state, then we'd be able to create eight states instead of four.

Great, and why do we care about BPS states? In QFT, a lot of things can happen – QFTs usually come in families, meaning there are various parameters in a quantum field theory that one can adjust. In general these parameters vary over a moduli space. If you try to move in this moduli space, short representations do not usually combine into long representations, and usually stay as they are. So BPS states are relatively rigid – or said in other words, the Hilbert space of states can change, but the spectrum of BPS states is generally invariant. Moreover, we can compute it in important situations (which is not true for the general Hilbert space), thanks to work of Gaiotto-Moore-Neitzke. In mathematics, the ways of computing BPS states have to do with things called spectral networs, which are tied to the geometry of Riemann surfaces.

The BPS representations are generally of the form $\rho \otimes s$, where ρ is the representation of $\mathfrak{so}_3 \oplus \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_R$ that we began with, and s is a short representation; similarly, the non-BPS states are in ρ tensored with a long representation. So giving representations of $\mathfrak{su}(2)_R$ and $\mathfrak{u}(1)_R$ gives you new BPS representations.