

The Chronicles of IIBordia: Dualities, Bordisms, and the Swampland

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Abstract

In this work we investigate the Swampland Cobordism Conjecture in the context of type IIB string theory geometries with non-trivial duality bundle. Quite remarkably, we find that many non-trivial bordism classes with duality bundles in $Mp(2, \mathbb{Z})$, a double cover of $SL(2, \mathbb{Z})$ related to fermions, correspond to asymptotic boundaries of well-known supersymmetric F-theory backgrounds. These include $[p, q]$ -7-branes, non-Higgsable clusters, S-folds, as well as various lower-dimensional generalizations. These string theoretic objects break the global symmetries associated to the non-trivial bordism groups, providing a strong test of the Cobordism Conjecture. Further including worldsheet orientation reversal promotes the duality group to the Pin^+ cover of $GL(2, \mathbb{Z})$. The corresponding bordism groups require a new non-supersymmetric “reflection 7-brane” and its compactifications to ensure the absence of global symmetries, thus providing an interesting prediction of the Cobordism Conjecture for non-supersymmetric type IIB backgrounds.

A major component of the present work is the explicit derivation of the involved bordism groups as well as their generators, which correspond to asymptotic boundaries of explicit string theory backgrounds. The main tool is the Adams spectral sequence, to which we provide a detailed introduction. We anticipate that the same techniques can be applied in a wide variety of settings.



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Part I

Introduction and summary

1 Introduction

One of the big challenges in the study of quantum gravity is that while various long distance / weak coupling limits are by now well understood, the development of a UV complete microscopic formulation still remains elusive. In the absence of a full theory of quantum gravity, it is natural to develop constraints from various complementary “bottom up” methods, including the general principles of dualities and symmetries.

Broadly speaking, the Swampland program [1, 2] (see [3–5] for reviews) is an ambitious “bottom up” approach aimed at addressing precisely these issues. This has led to an interconnected web of conjectural results, and has also provided a new perspective on many previously considered string theory backgrounds. With this in mind, it is natural to ask whether Swampland considerations can provide additional insight into the structure of non-perturbative dualities. Conversely, we can use well-known string theoretic dualities to test various Swampland conjectures.

The interplay between symmetries and quantum gravity has a long history, including the “folk theorem” that there are no global symmetries (see also [6–8]). In the context of the Swampland conjectures, this was significantly sharpened as the Cobordism Conjecture (CC) of McNamara and Vafa [9] which asserts that the bordism group of quantum gravity is trivial:

$$\Omega_k^{\text{QG}} = 0. \quad (1.1)$$

In physical terms, bordism theory arises because we are considering field configurations in the gravitational path integral which tend towards asymptotic boundary conditions.¹ The statement of the Cobordism Conjecture then is that in quantum gravity one can *always* connect two asymptotic field configurations. Of course, the precise definition of $\Omega_k^{\text{QG}} = 0$ would in some sense require one to first specify a definition of quantum gravity and all its long distance limits. That being said, the elements of Ω_k^{QG} should be viewed as the equivalence classes of k -dimensional compactifications of quantum gravity that can be connected to each other via a domain wall. The statement above implies that the Landscape of all quantum gravities is in fact connected via these domain walls. The reason is that if the bordism group Ω_k^{QG} does not vanish, the non-trivial bordism groups would act as conserved global charges.

Now, given a putative symmetry \mathcal{G} of our theory, we can equip our spacetime with a

¹An important point to stress here is that a priori, one does not even need to specify a smooth interior geometry. All that is required is that we have a notion of asymptotically defined field configurations, as in the S-matrix of asymptotic flat space, or the AdS/CFT correspondence.

\mathcal{G} -bundle², and in practice, it often happens that $\Omega_k^{\mathcal{G}} \neq 0$. In other words, there are field configurations in the gravitational path integral which cannot be topologically deformed into one another. In such situations the CC makes a sharp prediction: In order for Ω_k^{QG} to be trivial, there must exist new defects (i.e., some source for the symmetry \mathcal{G}) which supplement the spectrum of objects in the theory. At this point, the conjecture has passed a number of non-trivial checks (see e.g., [9–23]), providing a “bottom up” approach to the construction of defects such as p -branes and particular gravitational instantons (see e.g., [24]). In all known physical applications, the relevant bordism groups is simply a finite sum of free and torsional pieces:

$$\Omega_k^{\mathcal{G}} \simeq \mathbb{Z}^n \oplus \bigoplus_i (\mathbb{Z}/m_i \mathbb{Z})^{n_i}, \quad (1.2)$$

in the obvious notation.³

Now, bordism groups specify a generalized homology theory and as such are intrinsically Abelian objects. It is natural to ask whether these considerations are compatible with the existence of non-Abelian duality groups, such as those which frequently arise in string theory and various quantum field theories. In particular, reference [12] showed that the Cobordism Conjecture, when combined with the famous $\text{SL}(2, \mathbb{Z})$ duality group of type IIB strings / F-theory successfully predicts the existence of $[p, q]$ 7-branes, one of the core ingredients in non-perturbative F-theory vacua. In more technical terms, these are associated with Ω_1^{QG} , namely asymptotic one-dimensional boundary geometries.

Given this, we can ask whether the CC correctly predicts the known spectrum of objects of IIB / F-theory, and conversely, whether we can use the CC to *predict* new non-perturbative objects which cannot be accessed via other (usually supersymmetric methods). The aim of the present paper will be to produce a systematic classification of possible objects compatible with the basic duality of type IIB / F-theory. The main tool we use to accomplish this is the calculation of the corresponding bordism groups associated to twisted Spin structures involving the duality bundle.

Of course, this begs the question as to the precise notion of “duality bundle” which we will need to consider in this paper. To begin, even specifying the duality group of IIB turns out to be remarkably subtle. It is well-known that the type IIB supergravity action is invariant under an $\text{SL}(2, \mathbb{R})$ group, and taking into account quantization of fluxes, this is reduced to $\text{SL}(2, \mathbb{Z})$, which is also the group of large diffeomorphisms of a torus, the starting point for the geometric formulation of type IIB strings via F-theory [25–27]. As noted in [28, 29], the duality group must also be extended to an appropriate action on the fermions of IIB supergravity, and this leads to $\text{Mp}(2, \mathbb{Z})$, the metaplectic cover of $\text{SL}(2, \mathbb{Z})$.⁴ Even this is not

²Here, \mathcal{G} in general contains information about internal symmetries, the tangential structure of the manifold, as well as their mixtures to twisted tangential structures.

³Compared with much of the physics literature, we have opted to write $\mathbb{Z}/m\mathbb{Z}$ to denote the cyclic group with m elements. This is to avoid any confusions with other notation used in the text, including \mathbb{Z}_p for the p-adic integers, for example.

⁴It is analogous to the extension of $\text{SO}(d)$ to $\text{Spin}(d)$.

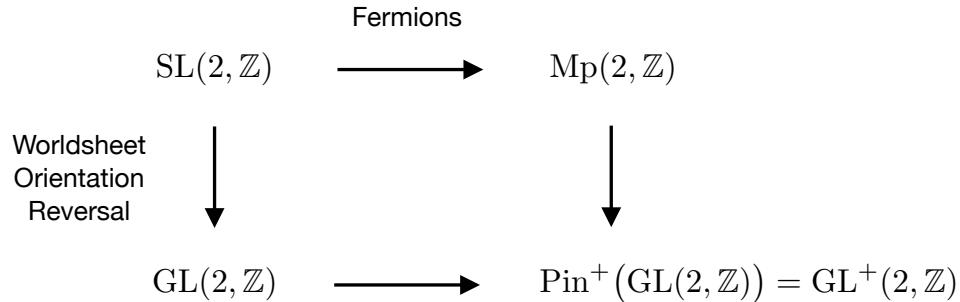


Figure 1: Various extensions of the duality group of type IIB string theory.

quite the full duality group, because one must also allow for the IIB worldsheet orientation reversal and $(-1)^{F_L}$ left-moving spacetime fermion parity transformations. Geometrically, these act as reflections on the F-theory torus. Indeed, for M- / F-theory duality to hold, one must allow for such reflections, since M-theory implicitly makes sense on Pin^+ backgrounds. Taking this into account, it follows (see [29]) that the IIB duality group is most accurately specified as the Pin^+ cover of $\text{GL}(2, \mathbb{Z})$. See Figure 1 for a depiction.

In physical terms, the duality group $\text{SL}(2, \mathbb{Z})$ tells us about the bosonic / non-chiral sector of IIB strings / F-theory. Taking into account fermions we arrive at $\text{Mp}(2, \mathbb{Z})$. For supersymmetric F-theory backgrounds specified by genus-one fibered Calabi-Yau spaces, the appearance of fermions can always be viewed as “implicit,” and so one expects the relevant objects predicted by the CC for $\text{Mp}(2, \mathbb{Z})$ to typically be supersymmetric. On the other hand, $\text{GL}^+(2, \mathbb{Z})$ includes reflections which, among other things can send a D3-brane to an anti-D3-brane. As such, IIB field configurations which include such symmetries provides us access to *non-supersymmetric* objects. Said differently, we can use $\text{Mp}(2, \mathbb{Z})$ -duality bordisms to test the CC, and $\text{GL}^+(2, \mathbb{Z})$ -duality bordisms to make predictions for new, possibly non-supersymmetric objects!

Given all of this, there are clearly three different notions of “duality bundle”-bordisms which will be of interest to us in the present work. First of all, we can consider the bordism groups associated with $\text{SL}(2, \mathbb{Z})$ dualities, and in which the interpolating geometries also have a Spin structure. Formally these are captured by the bordism groups:

$$\Omega_k^{\text{Spin}}(B\text{SL}(2, \mathbb{Z})) , \quad (1.3)$$

(see also [30]), where the “ B ” refers to the classifying space of $\text{SL}(2, \mathbb{Z})$.

Accounting for the appearance of fermions, this naturally enlarges to $\text{Mp}(2, \mathbb{Z})$ duality transformations. Here, an important subtlety is that—as is well-known in many F-theory constructions—there is no need for the spacetime to retain a Spin structure. Rather, we can instead have a twisted Spin structure which is correlated with duality group transformations,⁵

⁵This is not an issue for $\text{SL}(2, \mathbb{Z})$ transformations since it is only sensitive to the bosonic sector of the theory.

namely we consider manifolds with $(\text{Spin}(k) \times \text{Mp}(2, \mathbb{Z})) / (\mathbb{Z}/2\mathbb{Z})$ structure. In this case, the relevant bordism groups are given by:

$$\Omega_k^{\text{Spin-Mp}(2, \mathbb{Z})}(\text{pt}), \quad (1.4)$$

see also [31, 32]. Since $\text{Mp}(2, \mathbb{Z})$ is in some sense the “minimal” extension of $\text{SL}(2, \mathbb{Z})$ which accommodates supersymmetry, it is natural to expect that for each such bordism generator, there is a corresponding supersymmetric F-theory background which can be viewed as filling in the bulk of these bordism generators. We indeed find that this is the case, namely at least for k odd (so that the bulk space is even-dimensional), each generator of $\Omega_k^{\text{Spin-Mp}(2, \mathbb{Z})}(\text{pt})$ matches to a genus-one fibered Calabi-Yau threefold $X \rightarrow \mathcal{B}$, where $\partial\mathcal{B}$ can be viewed as specify a generator of $\Omega_k^{\text{Spin-Mp}(2, \mathbb{Z})}(\text{pt})$ with prescribed duality bundle.

The outcome of this analysis is that at least for $\text{Spin-Mp}(2, \mathbb{Z})$ bordisms, the main building blocks of many F-theory compactifications, including $[p, q]$ 7-branes, non-Higgsable clusters, S-folds, as well as lower-dimensional variants are all successfully predicted by the CC. This amounts to an extremely non-trivial test of the CC, and it is remarkable that such minimal topological inputs reconstruct these intricate structures.

Turning finally to the full IIB duality group $\text{GL}^+(2, \mathbb{Z})$, we also determine the corresponding $\text{Spin-GL}^+(2, \mathbb{Z})$ bordism groups:

$$\Omega_k^{\text{Spin-GL}^+(2, \mathbb{Z})}(\text{pt}). \quad (1.5)$$

In this case, the appearance of reflections means that we should not expect supersymmetry to be retained for these generators, and in general we do not expect there to be any genus-one fibered Calabi-Yau space which fills in the interior of such boundary geometries. In this case, the CC amounts to a set of *predictions* for objects which need not preserve supersymmetry. Quite remarkably, we find that the generators are, in this case, primarily inherited from $\text{Spin-Mp}(2, \mathbb{Z})$ (namely they are supersymmetric objects), but that there is one additional non-supersymmetric “reflection 7-brane” [23], as detected by $\Omega_1^{\text{Spin-GL}^+(2, \mathbb{Z})}(\text{pt})$. Starting from these ingredients, we find that nearly all of the other generators are in some sense inherited from further compactification of these higher-dimensional objects.⁶

In a ten-dimensional spacetime, the CC tells us about the spectrum of defects associated with Ω_k^G for $1 \leq k \leq 9$. The bordism groups for $k = 10$ and $k = 11$ also specify important physical data. For $k = 10$, Ω_{10}^G detects parameters such as discrete theta angles, and for $k = 11$, Ω_{11}^G classifies candidate contributions to anomalies of the ten-dimensional theory. A detailed analysis of type IIB duality anomalies was presented in [33], where it was shown that a subtle duality anomaly in IIB backgrounds can be cancelled by adding an additional topological term to the standard type IIB action.

⁶There are a few outlier cases at low dimensions which do not appear to be related in an obvious way to any of these other objects, but this is the exception rather than the rule.

The main computational tool we use to determine these bordism groups is the Adams spectral sequence. Another aim of this work will be to provide an introduction to this tool. The Adams spectral sequence has an (undeserved) reputation for being somewhat unwieldy, but in the case at hand it turns out to be rather tractable, once a number of intermediate simplifications are implemented. In particular, one of the key ideas undergirding our results is the fact that the non-Abelian duality groups of interest can be decomposed into corresponding amalgamated products, namely they can be decomposed into some simpler generators, each of which has a physical interpretation.⁷ The procedure of simplifying the bundle structure is accomplished through a procedure known as “shearing”, and with these elements in place, the calculation of the Adams spectral sequence vastly simplifies. Finding explicit bordism generators is also somewhat challenging, but in many cases we can verify that we have found a primitive generator by determining a suitable combination of η -invariants of the manifold (which in the cases of interest is a bordism invariant).

While we primarily emphasize the application of these bordism groups in the study of IIB strings / F-theory and the Swampland program, it is clear that these results have broader applications both within string theory and quantum field theory, as well as in pure math. In the context of string compactification / quantum gravity, one can reinterpret the different Ω_k^G 's as classifying anomalies of a $k - 1$ dimensional gravitational theory. Similarly, in the context of quantum field theory, these same Ω_k^G 's specify mixed gravitational-duality anomalies, a topic which has recently been studied for example in [30, 31, 34, 35]. In the context of pure math, many of the calculations we present are new, and can be viewed as the starting point for developing a more systematic treatment of Ω_k^G for all k (not just “small values”).

This paper is huge because we provide all details behind the calculations. It is our hope that this reference can be efficiently used by others to learn the techniques behind the Adams spectral sequence and the calculation of bordism groups in general.

To make the paper manageable, we have organized it into three main parts, as well as a few Appendices:

- In the remainder of this Introductory part, we first give a brief introduction to bordism theory, since it forms a core component of all of what follows. We then give a brief summary of the results and their physical interpretation. We then turn to our conclusions and future directions. We also include a map to the contents of the rest of the paper.
- In Part II, we give a physical interpretation of each of the generators of Ω_k^G ; we defer the derivation of these groups to Part III. The CC asserts that there is a corresponding spacetime defect associated with each such generator, and this leads us to the (re)discovery of various interesting backgrounds of type IIB string theory.

⁷For example, 7-branes with a fixed value of the axio-dilaton.

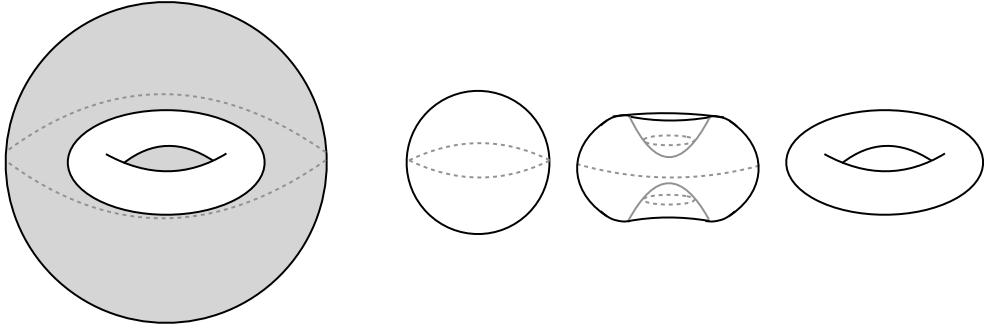


Figure 2: Oriented bordism between a 2-sphere and a 2-torus (can be understood as a 3-ball with an interior torus cut out).

- Part III introduces the necessary mathematical tools to calculate the various bordism groups discussed above. One of the main tools in this computation is the Adams spectral sequence which is introduced in great detail. These techniques are consecutively used in order to derive the bordism groups $\Omega_k^{\text{Spin}}(B\text{SL}(2, \mathbb{Z}))$, $\Omega_k^{\text{Spin-Mp}(2, \mathbb{Z})}(\text{pt})$ and $\Omega_k^{\text{Spin-GL}^+(2, \mathbb{Z})}(\text{pt})$ for $k \leq 11$. Moreover, we describe the generating manifolds for each of the non-trivial group factors. Wherever possible we also give a physical interpretation of these mathematical manipulations.
- In the Appendices we give some additional technical details on the structure of the duality groups, η -invariants, and the May-Milgram theorem. Additionally, we also comment on the use of the Smith homomorphism which connects different duality bundle structures separated by different dimensionality; the physical interpretation of this can be viewed as duality defects inherited from a higher-dimensional bulk.

2 Bordism: What is it?

We now proceed to a brief introduction to bordism theory. This forms the core mathematical structure which we use in what follows. Bordism is an equivalence relation defined on the set of k -dimensional manifolds equipped with a given tangential structure, e.g., Spin structure, orientation, or similar notions (see e.g., [36] for a mathematical introduction and [37, 9] for a description in a physical context). Two k -dimensional manifolds X_1 and X_2 are in the same equivalence class, i.e., define the same element in Ω_k^G if $X_1 \sqcup (-X_2)$ arises as the boundary of a $(k+1)$ -dimensional manifold over which the structure extends, see Figure 2. Here, the minus sign indicates an orientation reversal for X_2 .

The disjoint union of manifolds (or equivalently, the connected sum since the two are bordant) naturally defines an Abelian additive operation turning Ω_k^G into an Abelian group. Elements on the trivial class correspond to those manifolds that are the boundary of a manifold in one higher dimension, see Figure 3 for examples. If some bordism class generates

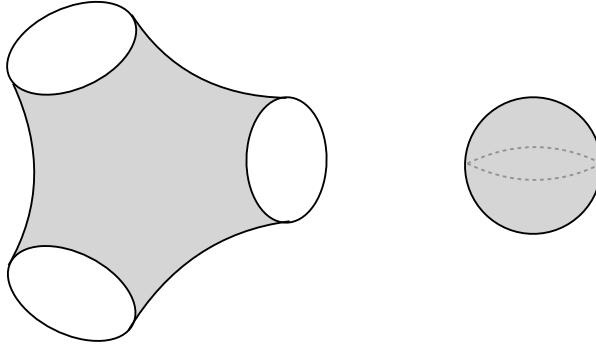


Figure 3: Left: Depiction of a bordism group factor $\mathbb{Z}/3\mathbb{Z}$, where three copies of a generator bound a manifold. Right: Null-bordism of the 2-sphere, i.e., a 3-ball.

a finite subgroup $\mathbb{Z}/n\mathbb{Z}$, this implies that while a single copy of a manifold representing a generating bordism class is not a boundary, the disjoint union of n copies of that manifold is, see Figure 3.

If \mathcal{G} in addition to tangential structures ξ also includes information about an internal symmetry G in the theory, the defining data of the associated bordism group contains a map from the spacetime manifold X into the classifying space of the symmetry group BG specifying the associated gauge field backgrounds. The continuous deformation of the spacetime manifold with a given structure is then complemented with a continuous deformation of this classifying map and the associated bordism group is denoted as

$$\Omega_k^\xi(BG). \quad (2.1)$$

In general the tangential structure, e.g., the Spin structure, can mix with the internal symmetry group, defined by quotients of the form

$$\frac{\text{Spin}(k) \times G}{\mathbb{Z}/2\mathbb{Z}} \quad (2.2)$$

leading to twisted Spin structures, which we denote as

$$\Omega_k^{\text{Spin-}G}(\text{pt}), \quad (2.3)$$

which do appear in our analysis once one includes fermions.

Importantly, bordism groups can be understood as a generalized homology theory [38]. In practice this means that the bordism groups are described as homotopy groups of a smash product (a homotopical analogue of a tensor product) of the manifold X under investigation with a spectrum E :

$$h_*(X) = \pi_*(X \wedge E). \quad (2.4)$$

For Spin bordism groups $\Omega_k^{\text{Spin}}(\text{pt})$ the relevant spectrum is the Thom spectrum denoted by *MTSpin*, which in the presence of internal symmetries or twisted tangential structure needs to be generalized accordingly. In order to derive various bordism groups one hence needs a good understanding of *MTSpin* and its generalizations. For that it is often useful to decompose the spectrum into easier parts, all of which contain portions of the complete information. The main simplification is that we use *p-local equivalences* of *MTSpin* with other spectra, which probes the *p*-torsion part of the Spin bordism group. In the present analysis we are faced with the analysis of bordism groups of the form

$$\Omega_k^{\text{SO}}(B\mathbb{Z}/3\mathbb{Z}), \quad \Omega_k^{\text{SO}}(BD_6), \quad \Omega^{\text{Spin}}(B\mathbb{Z}/4\mathbb{Z}), \quad \Omega_k^{\text{Spin-Z}/8\mathbb{Z}}(\text{pt}), \quad \Omega_k^{\text{Spin-D}_{16}}(\text{pt}), \quad (2.5)$$

where D_{16} denotes the dihedral group with 16 elements.⁸ In several cases we can extract the needed information from isomorphisms to other generalized (co)homology theories, such as *BP*-homology and *ku*-homology groups. In many other cases, however, we will need to derive the results using spectral sequence techniques, most prominently the Adams spectral sequence. Especially for large k this becomes increasingly complicated and we will present the details of these calculations explicitly in Part III.

Once the bordism groups are derived, we are faced with another challenge; the determination of the associated generators, i.e., a set of k -manifolds with the given structure that serve as representatives of all the bordism classes we have found. To find these manifolds, we are aided by *bordism invariants*: Group homomorphisms from the bordism group to \mathbb{Z} or $\mathbb{Z}/n\mathbb{Z}$, which can help with the identification of generators. For instance, $\Omega_4^{\text{Spin}}(\text{pt}) = \mathbb{Z}$ has a bordism invariant given by integrating the first Pontrjagin class over M_4 , a given four-manifold

$$\int_{M_4} p_1, \quad (2.6)$$

which is often referred to as the first Pontrjagin number. This means that we can take as a generator of $\Omega_4^{\text{Spin}}(\text{pt})$ any Spin 4-manifold with minimal Pontrjagin number. Many bordism invariants are like the Pontrjagin number, integrals of characteristic classes of the tangent bundle (as well as the gauge bundle, if present). However, this does not apply to all of them; some are detected by more exotic quantities, such as η -invariants of Dirac operators (see e.g. [39, 37]), and it is not always easy to identify the representative. Thus, the detection as well as the construction of the relevant manifolds is an art more than a science, but we will also describe it in great detail. It is noteworthy that certain types of manifolds naturally appear as generators of Spin- D_{16} bordisms which can be described as fiber bundles of real projective spaces or lens spaces over a base real projective base. One certain class of these

⁸Here, we really refer to the dihedral group and not the binary dihedral group. It contains an element associated to reflections R of order two and an element associated to rotations U of order n . They obey $RUR^{-1} = U^{-1}$. In particular, the reflection does not appear as a certain power of the rotation such as for the binary dihedral case. See also Appendix B.

manifolds already appeared in [33] and was referred to as ‘Arcanum XI’ and will appear here in certain lower-dimensional variants that we also refer to as Arcana.⁹

3 Summary of our results

In this Section we briefly summarize the computational results of Part III concerning the bordism groups

$$\Omega_k^{\text{Spin}}(B\text{SL}(2, \mathbb{Z})), \quad \Omega_k^{\text{Spin-Mp}(2, \mathbb{Z})}(\text{pt}), \quad \Omega_k^{\text{Spin-GL}^+(2, \mathbb{Z})}(\text{pt}). \quad (3.1)$$

For $1 \leq k \leq 9$, the CC predicts a corresponding defect which “fills in the bulk” associated with the corresponding bounding manifold equipped with an appropriate duality bundle. In the case of $k = 10$, the bordism groups are associated with discrete theta angles, and in the case of $k = 11$ these provide a coarse classification of possible contributions to duality anomalies. Our primary focus will be on Spin-Mp($2, \mathbb{Z}$) and Spin-GL $^+$ ($2, \mathbb{Z}$) bordisms since these are the cases where we also account for the contributions from fermionic degrees of freedom. Finally, for $1 \leq k \leq 9$ we give a physical interpretation of defects that allow these non-trivial bordism backgrounds to be bounded.

The bordism groups are given in Table 1. It is apparent that they are far from trivial and thus demand the existence of various defects to ensure that Ω_k^{QG} vanishes. To find these defects, it is useful to analyze representatives of the generators. For the Spin-Mp($2, \mathbb{Z}$) classes they are given in Table 2. The generators are given by

- Lens spaces: $L_k^{2n-1} = S^{2n-1}/(\mathbb{Z}/k\mathbb{Z})$

They naturally come equipped with a $\mathbb{Z}/k\mathbb{Z}$ bundle by the fibration

$$\begin{array}{ccc} \mathbb{Z}/k\mathbb{Z} & \hookrightarrow & S^{2n-1} \\ & \downarrow & \\ & & L_k^{2n-1} \end{array} \quad (3.2)$$

The duality bundle can be characterized in terms of this bundle by embedding $\mathbb{Z}/k\mathbb{Z}$ into the duality group. Moreover, some of the lens spaces allow for various choices for tangential structures, such as Spin- $\mathbb{Z}/8\mathbb{Z}$ structure for the L_4^5 . The different choices are indicated via a tilde. When these different tangential structures cannot be deformed into each other we expect several generating manifolds with a similar geometric structure. This can be verified explicitly by calculating associated bordism invariants, such as η -invariants for Dirac and Rarita-Schwinger fields.

⁹Following the book of Thoth (see the summary in https://en.wikipedia.org/wiki/Major_Arcana), the relevant arcana for us will be: Arcanum V (The Hierophant), Arcanum IX (The Hermit), Arcanum XI (Lust).

k	$\Omega_k^{\text{Spin}}(B\text{SL}(2, \mathbb{Z}))$	$\Omega_k^{\text{Spin-Mp}(2, \mathbb{Z})}(\text{pt})$	$\Omega_k^{\text{Spin-GL}^+(2, \mathbb{Z})}(\text{pt})$
	Section 12	Section 13	Section 14
0	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}
1	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})$	$(\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$
2	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	0	$\mathbb{Z}/2\mathbb{Z}$
3	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})$	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 3} \oplus (\mathbb{Z}/3\mathbb{Z})$
4	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}
5	$(\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/9\mathbb{Z})$	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/32\mathbb{Z}) \oplus (\mathbb{Z}/9\mathbb{Z})$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$
6	0	0	0
7	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/32\mathbb{Z}) \oplus (\mathbb{Z}/9\mathbb{Z})$	$(\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/9\mathbb{Z})$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 3} \oplus (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/9\mathbb{Z})$
8	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$
9	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 3} \oplus (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z})$ $\oplus (\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/27\mathbb{Z})$	$(\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/128\mathbb{Z})$ $\oplus (\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/27\mathbb{Z})$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 8}$
10	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 5}$	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 4}$
11	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2} \oplus (\mathbb{Z}/8\mathbb{Z})^{\oplus 2} \oplus (\mathbb{Z}/128\mathbb{Z})$ $\oplus (\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/27\mathbb{Z})$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2} \oplus (\mathbb{Z}/8\mathbb{Z})$ $\oplus (\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/27\mathbb{Z})$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 9} \oplus (\mathbb{Z}/8\mathbb{Z})$ $\oplus (\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/27\mathbb{Z})$

Table 1: Bordism groups of the three approximations to the structure of IIB quantum gravity.

- Lens space bundles: Q_4^{2n-1}

The spaces denoted as Q_4^{2n-1} (not to be confused with quaternionic lens spaces) are lens space bundles in the sense that they are the total spaces of the fibrations

$$\begin{array}{ccc} L_4^{2n-3} & \hookrightarrow & Q_4^{2n-1} \\ & & \downarrow \\ & & \mathbb{CP}^1 \end{array} \quad (3.3)$$

The non-trivial fibration structure is obtained by embedding the covering $(2n - 3)$ -sphere of L_4^{2n-3} in the fiber of a sum of line bundles $\mathcal{L}_1 \oplus \dots \oplus \mathcal{L}_{n-1}$ over \mathbb{CP}^1 . For our purposes it will be enough to consider the line bundles

$$\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \dots \oplus \mathcal{L}_{n-1} = H^{\pm 2} \oplus \underline{\mathbb{C}} \oplus \dots \oplus \underline{\mathbb{C}}, \quad (3.4)$$

where $H \cong \mathcal{O}(1)$ denotes the hyperplane bundle of \mathbb{CP}^1 and $\underline{\mathbb{C}} \cong \mathcal{O} \cong \mathcal{O}(0)$ the trivial line bundle. Again, these spaces are equipped with a Spin- $\mathbb{Z}/8\mathbb{Z}$ structure inherited

k	$\Omega_k^{\text{Spin-Mp}(2,\mathbb{Z})}(\text{pt})$	Generators
0	\mathbb{Z}	pt_+
1	$(\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})$	(L_4^1, L_3^1)
2	0	
3	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})$	(L_4^3, L_3^3)
4	\mathbb{Z}	E
5	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/32\mathbb{Z}) \oplus (\mathbb{Z}/9\mathbb{Z})$	$(\tilde{L}_4^5, L_4^5, L_3^5)$
6	0	
7	$(\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/9\mathbb{Z})$	(Q_4^7, L_3^7)
8	$\mathbb{Z} \oplus \mathbb{Z}$	(B, \mathbb{HP}^2)
9	$(\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/128\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/27\mathbb{Z})$	$(\tilde{L}_4^9, \mathbb{HP}^2 \times L_4^1, L_4^9, \mathbb{HP}^2 \times L_3^1, L_3^9)$
10	$\mathbb{Z}/2\mathbb{Z}$	X_{10}
11	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2} \oplus (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/27\mathbb{Z})$	$(\mathbb{HP}^2 \times L_4^3, X_{10} \times L_4^1, Q_4^{11}, \mathbb{HP}^2 \times L_3^3, L_3^{11})$

Table 2: Generators of Spin-Mp(2, \mathbb{Z}) bordism groups. Generators are listed in the same order as the decomposition of the group.

from the lens space and demand a non-trivial duality bundle.

- Enriques surface: E

It can be obtained by a fixed-point free $\mathbb{Z}/2\mathbb{Z}$ action on a K3 surface.

- Bott manifold: B

An arbitrary class in $\Omega_8^{\text{Spin}}(\text{pt})$ is specified by the value of the index of the Dirac operator and its signature. We shall specify a “Bott manifold” as any Spin 8-manifold with Dirac index equal to 1 (see also [40]). An example is an 8-manifold with a Spin(7) structure.¹⁰

- Quaternionic projective space: \mathbb{HP}^2

This 8-manifold is the analog of the more familiar \mathbb{CP}^2 , replacing the complex numbers \mathbb{C} by the quaternions \mathbb{H} in the construction. It has Dirac index 0, and signature 1.

- Milnor surface: X_{10}

This is any closed 10-dimensional Spin manifold having non-trivial $w_4 w_6$, where w_i

¹⁰As a brief aside, one can often produce Spin(7) spaces by taking a $\mathbb{Z}/2\mathbb{Z}$ quotient of a suitable Calabi-Yau fourfold [41], and the Dirac index on the Calabi-Yau fourfold is 2.

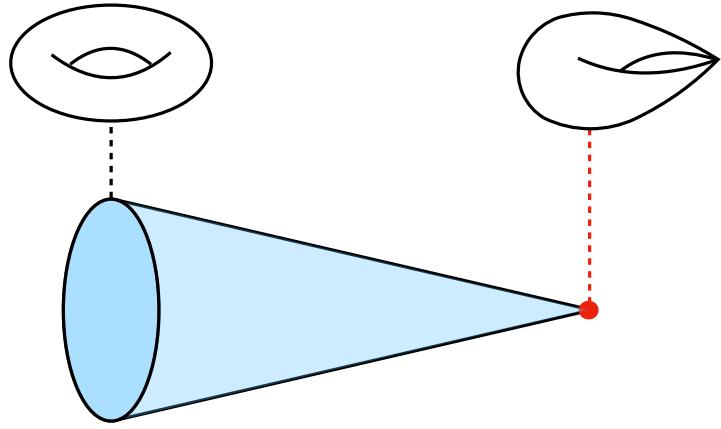


Figure 4: The odd-dimensional Spin-Mp($2, \mathbb{Z}$) manifolds often appear as the asymptotic boundary of the base manifold of a genus-one fibered, non-compact, singular Calabi-Yau manifold.

denotes the i th Stiefel-Whitney class of the tangent bundle. It also appears as one of the generators of $\Omega_{10}^{\text{Spin}}(\text{pt})$. For an explicit description of a Milnor surface, see [42, §3].

As we will see below, for the construction of many of the above backgrounds it is useful to geometrize the duality bundle using an auxiliary torus. This directly leads us to F-theory [25–27] (see [43–45] for reviews). In particular the Spin-Mp($2, \mathbb{Z}$) manifolds in odd dimensions described above can often be realized as the boundary $\partial\mathcal{B}$ of the base manifold of a genus-one fibration $X \rightarrow \mathcal{B}$, whose total space is given by a local, singular Calabi-Yau manifold. Thus, one naturally generates conical geometries with a central singularity indicating the presence of various defects, see Figure 4. From this it is also apparent why we expect these backgrounds to preserve at least part of the supersymmetry.

All of the manifolds discussed so far actually only contain duality bundles in $\text{Mp}(2, \mathbb{Z}) \subset \text{GL}^+(2, \mathbb{Z})$, i.e., they correspond to bordism classes not involving reflections. Generators for those classes involving $\text{GL}^+(2, \mathbb{Z})$ in their duality bundle are given in Table 3.

In addition to the generators relevant for Spin-Mp($2, \mathbb{Z}$) bordism, we also find:

- Circle with reflection bundle: S_R^1

This describes a circle with a non-trivial duality bundle implemented by a transition function with the reflection in $\text{GL}^+(2, \mathbb{Z})$. The two classes that we find at degree one tell us that there are two distinct compactifications of type IIB on a circle with a duality bundle; these were recently described in [46], and are related to each other by turning on a discrete theta angle in the RR sector.

- Circle with periodic boundary conditions: S_p^1

Described by a circle without duality bundle but with periodic boundary conditions for fermions. This is the generator of $\Omega_1^{\text{Spin}}(\text{pt}) = \mathbb{Z}/2\mathbb{Z}$.

k	$\Omega_k^{\text{Spin-GL}^+(2,\mathbb{Z})}(\text{pt})$	Generators
0	\mathbb{Z}	pt_+
1	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	(L_4^1, S_R^1)
2	$\mathbb{Z}/2\mathbb{Z}$	$S_p^1 \times S_R^1$
3	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 3} \oplus (\mathbb{Z}/3\mathbb{Z})$	$(L_4^3, \mathbb{RP}^3, \widetilde{\mathbb{RP}}^3, L_3^3)$
4	\mathbb{Z}	E
5	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	(L_4^5, X_5)
6	0	
7	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 3} \oplus (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/9\mathbb{Z})$	$(\mathbb{RP}^7, \widetilde{\mathbb{RP}}^7, W_2^7, W_1^7, L_3^7)$
8	$\mathbb{Z} \oplus \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$	$(W_{1,8}, \mathbb{HP}^2, W_1^7 \times S_p^1)$
9	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 8}$	$(W_1^7 \times S_p^1 \times S_p^1, X_9, \tilde{X}_9, L_4^9, W_1^9, B \times L_4^1, \mathbb{HP}^2 \times L_4^1, \mathbb{HP}^2 \times S_R^1)$
10	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 4}$	$(B \times L_4^1 \times S_p^1, W_1^9 \times S_p^1, \mathbb{HP}^2 \times L_4^1 \times S_p^1, X_{10})$
11	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 9} \oplus (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/27\mathbb{Z})$	$(\mathbb{RP}^{11}, \widetilde{\mathbb{RP}}^{11}, X_{11}, \tilde{X}_{11}, \mathbb{HP}^2 \times L_4^3, \mathbb{HP}^2 \times \mathbb{RP}^3, \mathbb{HP}^2 \times \widetilde{\mathbb{RP}}^3, X_{10} \times L_4^1, X_{10} \times S_R^1, Q_4^{11}, \mathbb{HP}^2 \times L_3^3, L_3^{11})$

Table 3: Generators of $\text{Spin-GL}^+(2,\mathbb{Z})$ bordism groups. Generators are listed in the same order as the decomposition of the group, just as in Table 2.

- Real projective spaces: \mathbb{RP}^{2n-1} and $\widetilde{\mathbb{RP}}^{2n-1}$

These are the usual real projective space derived as $S^{2n-1}/(\mathbb{Z}/2\mathbb{Z})$. They come equipped with natural $\mathbb{Z}/2\mathbb{Z}$ bundles via the fibration

$$\begin{array}{ccc} \mathbb{Z}/2\mathbb{Z} & \hookrightarrow & S^{2n-1} \\ & & \downarrow \\ & & \mathbb{RP}^{2n-1} \end{array} \tag{3.5}$$

For \mathbb{RP}^{2n-1} the $\mathbb{Z}/2\mathbb{Z}$ -bundle is associated to reflections in $\text{GL}^+(2,\mathbb{Z})$ whereas for $\widetilde{\mathbb{RP}}^{2n-1}$ it is a combination of rotation and reflection. These are described by two different embeddings of $\mathbb{Z}/2\mathbb{Z}$ into $\text{GL}^+(2,\mathbb{Z})$ as described below in Section 14.3.2 and Appendix B.2. A convenient way to implement this is an identification with the $\mathbb{Z}/2\mathbb{Z}$ action given by worldsheet reflections $\hat{\Omega}$ in \mathbb{RP}^{2n-1} and an identification with the $\mathbb{Z}/2\mathbb{Z}$ action given by worldsheet reflection in combination with an S-duality transformation \hat{S} for $\widetilde{\mathbb{RP}}^{2n-1}$.

- Arcana: X_i and \tilde{X}_i ($i \neq 10$)

The Arcana are spaces that can be defined as

$$X_i = (S^{2k-1} \times S^{i+1-2k})/D_4. \quad (3.6)$$

Because $D_4 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, the D_4 -action on S^{i+1-2k} is equivalent to a pair of $\mathbb{Z}/2\mathbb{Z}$ -actions, which make X_i into a fiber bundle of a real projective space over another real projective space:

$$\begin{array}{ccc} \mathbb{RP}^{2k-1} & \hookrightarrow & X_i \\ & & \downarrow \\ & & \mathbb{RP}^{i+1-2k} \end{array} \quad (3.7)$$

Again, the non-trivial duality bundle described by the natural D_4 -bundle of X_i can be implemented by either an action of worldsheet reflections $\hat{\Omega}$ for X_i or $\hat{S}\hat{\Omega}$ for \tilde{X}_i implemented by two different embeddings of D_4 into D_{16} , see Section 14.3.3 and Appendix B.2. For $i \in \{5, 9\}$ one has $k = 2$ and for $i = 11$ one has $k = 3$.

- Lens space bundles over real projective space: W_1^{2n-1}

These spaces can be obtained from the lens space bundles over spheres, including Q_4^{2n-1} and the trivial bundle, by a further $\mathbb{Z}/2\mathbb{Z}$ quotient. In this way W_1^7 can be identified as “half Q_4^7 ”. Since $\mathbb{Z}/2\mathbb{Z}$ acts as sign reversal in the base the resulting spaces are given by

$$\begin{array}{ccc} L_4^k & \hookrightarrow & W_1^n \\ & & \downarrow \\ & & \mathbb{RP}^m \end{array} \quad (3.8)$$

The associated duality bundles for these spaces contain both reflections as well as rotations. We use two examples: W_1^7 , for which $k = 5$ and $m = 2$; and W_1^9 , for which $k = 5$ and $m = 4$.

- Prism space bundle over $(T^2 \times S^2)/(\mathbb{Z}/2)$: W_2^7

In dimension 7, we have an additional generator, which is described as a bundle of the prism manifold S^3/Γ , where Γ is the Dyciclic group of order 16:

$$\begin{array}{ccc} S^3/\Gamma & \hookrightarrow & W_2^7 \\ & & \downarrow \\ & & \frac{T^2 \times S^2}{\mathbb{Z}/2} \end{array} \quad (3.9)$$

Here, the base is the quotient of $T^2 \times S^2$ by the combined involution of the involution

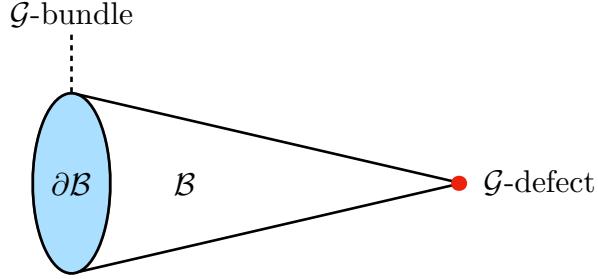


Figure 5: Bulk / Boundary characterization of bordism generators and the corresponding defects predicted by the CC. Viewing the bordism generator for manifolds equipped with a structure defined by \mathcal{G} as the boundary of some bulk space \mathcal{B} equipped with a \mathcal{G} -bundle, the CC predicts the existence of a \mathcal{G} -defect which allows the class to be trivialized in Ω_k^{QG} .

that leads to the Klein Bottle on T^2 and antipodal mapping on S^2 . The associated duality bundle also includes reflections, just as in the previous case.

- Half-Bott manifold: $W_{1,8}$

The half Bott manifold can be obtained by a $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$ quotient of a complete intersection manifold in \mathbb{CP}^7 . We refer to this as a “half-Bott” manifold because quotienting by just one of the two $\mathbb{Z}/2\mathbb{Z}$ ’s would produce a Bott manifold. The half-Bott is obtained after a further $\mathbb{Z}/2\mathbb{Z}$ quotient, leading to a non-Spin manifold on which $\int[\hat{A}]_8$ (the class that measures the Dirac index on a Spin manifold) is equal to 1/2. Although the manifold is not Spin, it carries a Spin- D_{16} structure.

For more details of the generators and their duality bundles see the explicit constructions in Sections 13 and 14.

With all the bordism groups and their generators determined we can try to relate the specific backgrounds to (extended) string theory objects. The CC demands that all these classes are actually boundaries of some IIB backgrounds, which necessarily involve singular geometries and duality bundles. The generators can be viewed as the asymptotic geometry around the respective defects, see Figure 5 for a depiction. In the 10-dimensional string theory we therefore find that non-trivial bordism classes $\Omega_k^{\mathcal{G}}$ are related to defects of codimension $(k+1)$. Since type IIB string theory contains many objects of codimension 2ℓ with $\ell \in \{1, \dots, 4\}$ it is tempting to expect that some of these objects are related to the defects predicted by the Cobordism Conjecture. This is indeed correct as we will carefully analyze in Part II, with one potential caveat: while the backgrounds we construct are certainly consistent at the level of supergravity, it may be possible that type IIB string theory does not make sense on some of them. This will happen if, for instance, the worldvolume of a probe brane has an anomaly in the backgrounds, similarly to the situation in [47]. In the language of [9], we would say that the bordism class is “gauged”, and remove it from the discussion. We do not know if any of the classes we construct is “gauged” in this way, as a systematic analysis has not been carried out, unlike in the M-theory case [40]. In this paper, we will

assume that all the classes we construct uplift to IIB string theory backgrounds, and find the objects that would become their boundary if this is the case. It is worth noticing that in known examples, whenever a bordism class is gauged in the sense of [9] (killing the class), it is often the case that a closely related “cousin” exists, having the same geometry but e.g., different flux, and which remains alive. An example is the class of \mathbb{RP}^4 in M-theory, which is itself inconsistent, but the inconsistency is removed if one includes a half-integer G_4 flux [47]. So while the reader should keep in mind that we have not checked all anomalies of probe branes in the backgrounds we discuss, we believe these are unlikely to do much more than force the introduction of some fluxes.

For Spin-Mp(2, \mathbb{Z}) classes and odd $k \leq 9$, we find the identifications in Table 4. Many of the Spin-Mp(2, \mathbb{Z}) configurations can be associated with supersymmetric backgrounds for F-theory containing interesting objects, such as non-Higgsable clusters [48] (see also [49]) and S-folds [50, 51] (see also [52–54]), as well as generalizations thereof.

k	$\Omega_k^{\text{Spin-Mp}(2, \mathbb{Z})}(\text{pt})$	Generators	Defects
1	$(\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})$	(L_4^1, L_3^1)	$[p, q]$ -7-branes
3	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})$	(L_4^3, L_3^3)	non-Higgsable clusters
5	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/32\mathbb{Z}) \oplus (\mathbb{Z}/9\mathbb{Z})$	$(\tilde{L}_4^5, L_4^5, L_3^5)$	S-folds
7	$\mathbb{Z}/4\mathbb{Z}$	Q_4^7	twisted compactification of S-folds
	$\mathbb{Z}/9\mathbb{Z}$	L_3^7	
9	$(\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/128\mathbb{Z}) \oplus (\mathbb{Z}/27\mathbb{Z})$	$(\tilde{L}_4^9, L_4^9, L_3^9)$	S-instantons
	$(\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})$	$(\mathbb{HP}^2 \times L_4^1, \mathbb{HP}^2 \times L_3^1)$	compactifications of $[p, q]$ -7-branes

Table 4: Defects which source asymptotic Spin-Mp(2, \mathbb{Z}) bordism classes.

For Spin-GL⁺(2, \mathbb{Z}) bordism classes, not all the defects can be associated to generalizations of known string theory objects. This is simply because any putative bulk geometry would necessarily break supersymmetry, and most objects understood in string theory typically preserve some supersymmetry. Given this, we find it very remarkable that the inclusion of essentially one new physical object suffices to generate the remaining bordism classes. This new object is a 7-brane (codimension two in the 10d spacetime) that implements reflections in the GL⁺(2, \mathbb{Z}) duality group, and we refer to it as a reflection 7-brane or “R7-brane” for short [23]. The R7-brane trivializes the class of a circle with a reflection bundle, and compactifications involving wrapped R7-branes are enough to trivialize all the other classes at lower degrees. The full results are summarized in Table 5.

The even-dimensional generators are conceptually different, since they usually descend directly from the generators of Spin manifolds and thus, with a few exceptions, do not

k	$\Omega_k^{\text{Spin-GL}^+(2, \mathbb{Z})}(\text{pt})$	Generators	Defects
1	$\mathbb{Z}/2\mathbb{Z}$	L_4^1	$[p, q]$ -7-branes
	$\mathbb{Z}/2\mathbb{Z}$	S_R^1	R7-brane
3	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})$	(L_4^3, L_3^3)	non-Higgsable clusters
	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	$(\mathbb{RP}^3, \widetilde{\mathbb{RP}}^3)$	O5-plane, compactified R7-brane
5	$\mathbb{Z}/2\mathbb{Z}$	L_4^5	S-folds
	$\mathbb{Z}/2\mathbb{Z}$	X_5	twisted compactification of O5 and R7
7	$\mathbb{Z}/9\mathbb{Z}$	L_3^7	S-string
	$\mathbb{Z}/4\mathbb{Z}$	W_1^7	double-twisted compactification of S-fold
	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	$(\mathbb{RP}^7, \widetilde{\mathbb{RP}}^7)$	O1-plane, compactified R7-brane
	$\mathbb{Z}/2\mathbb{Z}$	W_2^7	twisted compactification of a D-type Du Val singularity
9	$\mathbb{Z}/2\mathbb{Z}$	L_4^9	S-instantons
	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 4}$	$(W_1^7 \times S_p^1 \times S_p^1, B \times L_4^1,$ $\mathbb{HP}^2 \times L_4^1, \mathbb{HP}^2 \times S_R^1)$	compactifications of 7-branes
	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	(X_9, \tilde{X}_9)	twisted compactifications of O5 and R7
	$\mathbb{Z}/2\mathbb{Z}$	W_1^9	double-twisted compactifications of S-folds

Table 5: Defects which source $\text{Spin-GL}^+(2, \mathbb{Z})$ bordism classes. Here ‘‘R7-brane’’ refers to a codimension two defect (i.e., a 7-brane) in which monodromy around it leads to a reflection on the corresponding F-theory torus.

carry a non-trivial duality bundle. While the particular string backgrounds with these even-dimensional manifolds as a boundary still need to be identified, they are closely related to the cases which appeared in the analysis of [9].

4 Conclusions and future directions

In this section we present our conclusions and future directions. Readers interested in a more complete account will find it in the subsequent parts of this paper.

The main theme in much of this paper is the interplay between the non-Abelian duality symmetry of type IIB strings / F-theory and the Cobordism Conjecture. We have seen that for $\text{Spin-Mp}(2, \mathbb{Z})$ bordisms, the CC passes many non-trivial checks. Conversely, the CC predicts the existence of some genuinely new non-supersymmetric backgrounds, some

of which will be explored in future work. It is important to stress that at no point did our analysis of bordism groups actually rely on supersymmetry. In this sense it is a rather robust tool which is complementary to many existing methods. A central component of our analysis has been the computation of all of the relevant bordism groups via the Adams spectral sequence. The results we have obtained are more broadly applicable in the study of quantum field and strings, and in the remainder of this section we discuss some further potential avenues for investigation.

We have seen that the CC predicts various new backgrounds and objects for type IIB string theory. In the case of $\text{Spin-Mp}(2, \mathbb{Z})$ bordism, these have a lift to F-theory geometries that preserve part of the supersymmetry. These singular configurations usually contain localized interacting degrees of freedom. For example, we have found non-Higgsable clusters that realize a simple class of six-dimensional superconformal field theories (SCFTs) as well as S-folds. It would therefore be interesting to analyze the localized degrees of freedom and their dynamics for the other backgrounds as well, e.g., the S-strings. The same is true for the more challenging $\text{Spin-GL}^+(2, \mathbb{Z})$ setups, which do not have a conventional F-theory interpretation.

Turning the discussion around, we can also use the $\text{Spin-GL}^+(2, \mathbb{Z})$ bordisms to generalize our notion of F-theory itself. Instead of an $\text{SL}(2, \mathbb{Z})$ action on the fiber, it is rather clear that we should also permit reflection symmetries generating $\text{GL}(2, \mathbb{Z})$ actions on the geometry of the total space. It would be very interesting to see whether one can find configurations that preserve part of the supersymmetry while still explicitly containing duality transition functions with determinant (-1) . These likely do not contain R7-branes since these objects appear to be non-supersymmetric [23].

From this perspective it is natural to ask whether the appearance of reflections could also be used to construct metastable non-supersymmetric vacua, a topic of clear phenomenological relevance. From this perspective it is natural to ask whether R7-branes, in tandem with other string theoretic ingredients, could be used to construct such backgrounds.

Type IIB string theory is of course not isolated and can be related to other string theories as well as M-theory via various dualities. This motivates a further investigation into the fate of the predicted new backgrounds under these dualities. Using the usual dictionary, a circle compactification of an F-theory background is related to M-theory of the same torus-fibered geometry (without the extra circle). In this way we can track the $\text{Spin-GL}^+(2, \mathbb{Z})$ structure to a Pin^+ structure of the M-theory spacetime [29]. Since the M-theory geometry does not have to preserve the torus fiber, it might be that several classes can be trivialized without resorting to singular objects. However, at least a subset of the novel IIB configurations must survive and will thus have a direct M-theory counterpart.

A particularly fascinating implication of our results is that we have uncovered four discrete θ -angles for IIB string theory in ten dimensions. Some of them were already discussed in the literature long ago; for instance, the θ -angles that do not vanish on the product of a Bott manifold with a two-torus do not vanish on exotic 10-spheres, and so were implicitly

discussed in [55]. It remains an important open question to elucidate whether these and other θ -angles, such as the ones proposed in [33] (see also [40, 56, 57] as well as [46]), are part of the Landscape or the Swampland.

The computation of bordism groups with a prescribed duality bundle structure has many other potential applications. As an example, in the context of a $(k - 1)$ -dimensional theory of quantum gravity, the bordism group Ω_k^G characterizes possible anomalies of a given theory with non-trivial dualities. In particular, we can apply this for all of the different refinements of IIB duality considered in this paper, namely $\text{SL}(2, \mathbb{Z})$, $\text{Mp}(2, \mathbb{Z})$ and $\text{GL}^+(2, \mathbb{Z})$. Moreover, this also provides non-trivial constraints on theories with possibly large duality symmetries. It would be interesting to investigate other supergravity theories and their U-duality symmetries, perhaps along the lines of [33, 58].

Conversely, it is also natural to consider situations in which the actual duality group is a proper subgroup of the ones considered in this paper. This arises in various supergravity and supersymmetric quantum field theories (see e.g., [59] for a recent discussion), and so it is natural to ask about the spectrum of objects predicted by the CC in such cases.

One can also consider limits in which the effects of gravity are switched off, namely in the study of quantum field theories with a known duality symmetry. In that context, one can also consider mixed gravitational-duality anomalies, much as in [30, 31, 34, 35]. Again, the bordism group calculations provided here amount to a characterization of possible anomalies which can arise in a given theory.

Aside from these physical applications, there are also a number of interesting directions to pursue on purely mathematical grounds. To give one example, here we have focused on the case of bordism groups Ω_k^G for $k \leq 11$. It would be quite interesting to develop methods which also work at “large” values of k .

A fortuitous feature of our calculations is that all of the relevant data is concentrated at low primes, in particular primes $p = 2, 3$. Turning the question around, one could also ask whether there is a natural sense in which higher primes could arise in this context. A potentially promising route in this vein would likely to be consider similar calculations for the congruence subgroups of $\text{SL}(2, \mathbb{Z})$ such as $\Gamma(p)$, $\Gamma_0(p)$ and $\Gamma_1(p)$, as well as their metaplectic and Pin^+ covers.

Another intriguing direction to develop would be the relation between the bordism theories considered here and related constructions in mathematics. For example, many of the generators we produced for Spin- $\text{Mp}(2, \mathbb{Z})$ bordism can be thought of as the boundary geometries of genus-one fibered Calabi-Yau spaces. Since there is also a purely algebro-geometric approach to bordism developed by Levine-Morel in [60–64] (see also [65]), it is natural to ask whether there is a more direct connection between Levine-Morel’s bordism theory and Spin- $\text{Mp}(2, \mathbb{Z})$ bordism. In a different direction, there could also be connections with chromatic homotopy theory: the duality bundles we consider can often be “geometrized” in terms of a genus-one curve / elliptic curve, suggesting connections to elliptic cohomology theories (see,

e.g., [66]); indeed, there are notions of $\mathrm{SL}(2, \mathbb{Z})$ -equivariant elliptic cohomology or related objects [67] believed to be related to string theory [68–70]. More directly, work of Tachikawa, Yamashita, and Yonekura [71–73] relates the theory of topological modular forms, a sort of universal elliptic cohomology theory, to anomaly cancellation in heterotic string theory, and it would be interesting to pursue analogous questions in type IIB string theory. We believe that these questions are all ripe for further exploration, and look forward to exploring some of them in the near future.

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5 Final words of encouragement

We hope that this brief introduction and summary of our results is sufficient for those only interested in the final results of our work and their physical implications.

For those brave enough to undertake the journey to understand how these results were obtained: The road is long and arduous, yet the rewards are more than worth it. You will learn bordism theory, F-theory, topology, index theory, the intricacies of IIB string theory, and above all, the Dark Art of Spectral Sequences. This arcane tool will allow you to tackle questions and problems which would be beyond reach otherwise, in String Theory and beyond. The Map of the Lands of IIBordia, conjured forth through the sorcerous arts of Paco Giudice, is depicted at the beginning of the paper to aid you in your quest. Any road taken from Introduction Camp will lead somewhere interesting, and they can all be taken largely independently from each other.¹¹ The legend goes that only the traveler who has explored all corners of the Land of IIBordia will achieve full enlightenment. We hope it is of some help should you lose your way. Godspeed.

¹¹The paths of the physicist warrior and mathemagician are closer than they might first appear; we do not specify a metric on IIBordia (the metric is still unknown).

Part II

Type IIB backgrounds and defects

Having given an overview to the main results of this paper, we now turn to an analysis of the defects predicted by the Cobordism Conjecture [9]. As mentioned in the introduction, the Cobordism Conjecture states that for a consistent theory of quantum gravity in D dimensions, the lower bordism groups must be trivial

$$\Omega_k^{\text{QG}} = 0, \quad \text{for } 0 < k < D, \quad (5.1)$$

in order to avoid associated global symmetries. Since the bordism groups under investigation are far from trivial (see Table 1), we are left with two options:

- There are more ingredients in the full theory of quantum gravity than is captured by the duality bundle and Spin structure. These can often be identified with extended objects in the corresponding string theory. After the inclusion of these objects the spacetime can be deformed to a point and thus becomes null-bordant. This process can be regarded as a breaking of the global symmetry induced by non-trivial bordism classes.
- The backgrounds associated to non-trivial bordism classes are forbidden in the full theory of quantum gravity. Constraints of this kind might be imposed by subtle tadpole cancellation conditions. This can be interpreted as a gauging of the associated global symmetry.

In the following we will assume that the first of the above possibilities is realized and describe the necessary objects whose inclusion trivializes the bordism groups under consideration. For small k this will lead to the (re)discovery of various type IIB configurations and objects. Moreover, since type IIB contains spacetime fermions we focus on the two variants $\text{Spin-Mp}(2, \mathbb{Z})$ and $\text{Spin-GL}^+(2, \mathbb{Z})$.

Before we start our analysis, we point out that the bordism groups under consideration are only sensitive to the spacetime topology and the discrete duality bundle. In particular our analysis does not include the various higher-form Ramond-Ramond (RR) and Neveu-Schwarz (NS) fluxes of type IIB. A consequence of this is that there can in principle be multiple string backgrounds which have the same asymptotic duality bundle structure but which correspond to physically distinct flux backgrounds. Wherever possible we are interested in the supersymmetric variants of the objects breaking the global symmetry induced by the non-trivial bordism classes which can be related to known string theory objects. Therefore, we allow for induced fluxes, as necessary.

6 Spin-Mp defects

In this section we focus on the non-trivial bordism classes in $\Omega_k^{\text{Spin-Mp}(2,\mathbb{Z})}(\text{pt})$. We will find that these classes are closely related to certain F-theory backgrounds. For these, the value of the axio-dilaton

$$\tau = C_0 + ie^{-\phi}, \quad (6.1)$$

with C_0 the RR 0-form and dilaton ϕ , is encoded in an auxiliary torus $T^2 \simeq \mathbb{C}/\Lambda_\tau$ defined by the lattice $\Lambda_\tau = \langle 1, \tau \rangle$ fibered over the physical spacetime. Thus, F-theory geometries are described by torus fibrations [25–27] (see [43, 45] for reviews). If the total space of the fibration is Calabi-Yau, some of the supersymmetry will be preserved and the background is guaranteed to solve the Einstein equations. The Mp(2, \mathbb{Z}) bundle of the type IIB background is directly encoded in the information associated to this fibration. Moreover, the total space of the elliptic fibrations, as long as they are smooth, allow for a Spin structure in the case of the Spin-Mp(2, \mathbb{Z}) bordism classes, which already hints at their relevance in the supersymmetric setups.

The specific strategy for odd dimensional Spin-Mp(2, \mathbb{Z}) bordism generators is as follows. If the generator is given by a lens space it can be described as the asymptotic boundary of the quotient $\mathbb{C}^n/(\mathbb{Z}/k\mathbb{Z})$. The duality bundle is then specified as the holonomy generated by traversing the non-trivial 1-cycle of the lens space, i.e., to interpolate between two points identified by the $\mathbb{Z}/k\mathbb{Z}$ action. Using the geometrization of the duality bundle described by F-theory, this can be realized by specifying the action of $\mathbb{Z}/k\mathbb{Z}$ on the T^2 fiber, thus obtaining a space of the form

$$(\mathbb{C}^n \times T^2)/(\mathbb{Z}/k\mathbb{Z}). \quad (6.2)$$

For an appropriate $\mathbb{Z}/k\mathbb{Z}$ action on the fiber and base coordinate the total space is described by a non-compact, singular Calabi-Yau $(n+1)$ -fold, indicating that part of the supersymmetry is preserved, with the generator of the bordism group as the asymptotic boundary of the base manifold, see Figure 4. The constructed F-theory geometries with singular central fiber then provide a natural class of backgrounds which have an asymptotic boundary corresponding to a candidate generator in the associated bordism group. For the $\mathbb{Z}/k\mathbb{Z}$ action on the central fiber to be well-defined, the complex structure of the torus might be restricted to special values leading to particular fiber degenerations summarized in Table 6, see also Appendix B. At this level we only specified the bosonic version of the duality bundle, i.e., the $\text{SL}(2, \mathbb{Z})$ bundle. The full Spin-Mp(2, \mathbb{Z}) structure is specified by choosing a Spin structure on the total space.

We summarize the involved objects associated to $\Omega_k^{\text{Spin-Mp}(2,\mathbb{Z})}(\text{pt})$ for odd $k \leq 9$ in Table 4 above and will now investigate each dimension individually. As mentioned previously, the even-dimensional asymptotic backgrounds are closely related to backgrounds that appeared

Monodromy	Fixed value of τ	Kodaira type	Gauge algebra
S	$\tau = i$	III^*	\mathfrak{e}_7
S^2	τ arbitrary	I_0^*	$\mathfrak{so}(8)$
S^3	$\tau = i$	III	$\mathfrak{su}(2)$
U^2	$\tau = e^{2\pi i/3}$	IV^*	\mathfrak{e}_6
U^4	$\tau = e^{2\pi i/3}$	IV	$\mathfrak{su}(3)$

Table 6: Fixed value of the complex structure τ of the fiber torus depending on $\mathrm{SL}(2, \mathbb{Z})$ bundle and associated Kodaira type and gauge algebra of the central singularity.

in [9].

6.1 Codimension-two defects

Let us start with

$$\Omega_1^{\mathrm{Spin-Mp}(2, \mathbb{Z})}(\mathrm{pt}) = (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z}) \simeq \mathbb{Z}/24\mathbb{Z}, \quad (6.3)$$

generated by circles with non-trivial $\mathrm{Mp}(2, \mathbb{Z})$ duality bundle. These are classified by the transition functions and thus the non-trivial bordism classes are in one-to-one correspondence with the Abelianization of the duality group

$$\mathrm{Ab}(\mathrm{Mp}(2, \mathbb{Z})) = \mathbb{Z}/24\mathbb{Z}. \quad (6.4)$$

In terms of the duality group generators presented in Appendix B the transition function generating $\Omega^{\mathrm{Spin-Mp}(2, \mathbb{Z})}(\mathrm{pt})$ is given by \hat{T} , the Spin lift of the T generator in $\mathrm{SL}(2, \mathbb{Z})$. To break this bordism group we therefore need to introduce a defect in real codimension two, which induces a non-trivial duality monodromy given by \hat{T} . Luckily such a defect is well-known in type IIB string theory and F-theory and is given by the D7-brane.¹² Hence, we see that the requirement of vanishing bordism classes forces us to include defects that can be associated to known stringy objects, namely $[p, q]$ -7-branes (see e.g., the review in [45]).¹³

The associated F-theory geometry can be constructed as follows. The non-trivial duality bundle on the circle translates to a monodromy of the F-theory fiber torus around the

¹²Note that one can also choose different combinations of 7-branes that break the full bordism group. These can be interpreted as stacks of more general $[p, q]$ -7-branes, see below.

¹³In order to recover the non-Abelian braiding of general $[p, q]$ -7-branes [12] suggested an alternative formulation involving (in the supersymmetric case) the moduli spaces of genus-one curves.

spacetime circle, leading to a geometry of the type

$$(S^1 \times T^2)/(\mathbb{Z}/k\mathbb{Z}), \quad (6.5)$$

where the $\mathbb{Z}/k\mathbb{Z}$ acts by a rotation by $\frac{2\pi}{k}$ and as the associated $Mp(2, \mathbb{Z})$ transformation on the fiber. Now, filling the base S^1 to obtain a disc, one encounters at least one singular fiber in the interior which indicates the presence of $[p, q]$ -7-branes. This can be seen by interpreting (6.5) as the boundary of

$$(\mathbb{C} \times T^2)/(\mathbb{Z}/k\mathbb{Z}), \quad (6.6)$$

with the base coordinate $z \mapsto e^{2\pi i/k} z$, having a singular central fiber at $z = 0$. In this way all the relevant defects can be described as local patches of elliptically-fibered K3 manifolds, demonstrating that half of the supersymmetry is preserved by the associated defects and only BPS-objects are needed to break the global symmetry associated to the bordism group.

Decomposing $\mathbb{Z}/24\mathbb{Z}$ in (6.3) into the parts $(\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})$ we see that instead of a D7-brane one can use brane stacks generating the monodromies \hat{S} and \hat{U}^2 , see Appendix B, respectively, to break the individual summands. The brane stack associated to the monodromy \hat{S} is given in terms of a type III^* F-theory fiber corresponding to a \mathfrak{e}_7 stack; similarly, \hat{U}^2 corresponds to a type IV^* fiber, i.e., an \mathfrak{e}_6 stack, see Table 6 and e.g., [74, 45].

As an aside we now want to give an intuitive explanation for why the monodromy of 24 D7-branes leads to the trivial class of the bordism group. For that we use F-theory intuition and encode the duality bundle in the geometry of an elliptic fibration. The 24 7-branes translate to 24 I_1 fibers of the elliptic fibration. Since we know that a K3 surface can be constructed as an elliptic fibration over \mathbb{CP}^1 with 24 I_1 fibers, we have found the bounding manifold in an F-theory setting. Forgetting about the Spin structure, 12 I_1 fibers suffice to obtain a trivial $SL(2, \mathbb{Z})$ monodromy and the periodicity of the duality part of $\Omega_1^{\text{Spin}}(B\text{SL}(2, \mathbb{Z}))$, i.e., the $\mathbb{Z}/12\mathbb{Z}$, factor can be explained by the existence of an elliptic fibration for dP_9 or half-K3, see also [75].

6.2 Codimension-three defects

We have that $\Omega_2^{\text{Spin-Mp}(2, \mathbb{Z})}(\text{pt}) = 0$, and so every two-dimensional manifold with a $\text{Spin-Mp}(2, \mathbb{Z})$ structure is trivial in bordism. This is to be contrasted with $\Omega_2^{\text{Spin}}(\text{pt}) \cong \mathbb{Z}/2\mathbb{Z}$, but it is similar to $\Omega_2^{\text{SO}}(\text{pt}) = 0$ for oriented bordism. One way to understand this vanishing is to look ahead to Lemmas 13.4 and 13.10, where we learn that the 2-torsion subgroup of $\Omega_2^{\text{Spin-Mp}(2, \mathbb{Z})}$ is isomorphic to the 2-torsion subgroup of $\Omega_2^{\text{Spin-Z}/8\mathbb{Z}}$, and that the 3-torsion subgroup is isomorphic to the 3-torsion subgroup of $\Omega_2^{\text{Spin}}(B\mathbb{Z}/3\mathbb{Z})$. Then, since $\tilde{\Omega}_2^{\text{Spin}}(B\mathbb{Z}/3\mathbb{Z}) = 0$, $\Omega_2^{\text{Spin-Mp}(2, \mathbb{Z})}(\text{pt})$ is essentially equivalent to $\Omega_2^{\text{Spin-Z}/8\mathbb{Z}}(\text{pt})$, corresponding to manifolds with $\text{Spin-Z}/8\mathbb{Z}$ structure, and this bordism group also vanishes. We will put this seemingly trivial result to interesting physical use in Section 7.2.

6.3 Codimension-four defects

Next, let us discuss the necessary real codimension-four defects associated to

$$\Omega_3^{\text{Spin-Mp}(2,\mathbb{Z})}(\text{pt}) = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z}), \quad (6.7)$$

with generators given by the two lens spaces L_4^3 and L_3^3 equipped with non-trivial duality bundles. As above these can be conveniently lifted to F-theory geometries described as the boundaries of

$$(\mathbb{C}^2 \times T^2)/(\mathbb{Z}/k\mathbb{Z}). \quad (6.8)$$

The $\mathbb{Z}/k\mathbb{Z}$ action on the base coordinates is given by

$$(z_1, z_2) \mapsto (\omega z_1, \omega z_2), \text{ with } \omega = e^{2\pi i/k}. \quad (6.9)$$

The action on the fiber is the natural $\text{Mp}(2, \mathbb{Z})$ action with an element of order k .

For $k = 4$ the $\text{Mp}(2, \mathbb{Z})$ element is given by \hat{S}^2 . In $\text{SL}(2, \mathbb{Z})$ this is associated to the element

$$S^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (6.10)$$

Denoting the complex coordinate on the torus $T^2 \simeq \mathbb{C}/\Lambda_\tau$ by λ this action can be described as

$$\lambda \mapsto -\lambda = \omega^2 \lambda. \quad (6.11)$$

This geometry

$$(\mathbb{C}^2 \times T^2)/(\mathbb{Z}/4\mathbb{Z}): \quad (z_1, z_2, \lambda) \mapsto (\omega z_1, \omega z_2, \omega^2 \lambda), \quad (6.12)$$

is also well-known in the F-theory literature, see e.g., [76, 48, 49, 77, 78]. It is the superconformal limit of the non-Higgsable cluster over a curve of self-intersection (-4) , which has been shrunk to zero size in the geometry (6.12). On the tensor branch, i.e., at finite size of the (-4) -curve, one has a gauge theory with gauge algebra $\mathfrak{so}(8)$ localized on a 7-brane stack wrapping this compact curve. This background preserves $\mathcal{N} = (1, 0)$ supersymmetry in six dimensions. Observe that the local $(3, 0)$ -form

$$\Omega_{(3,0)} = dz_1 \wedge dz_2 \wedge d\lambda, \quad (6.13)$$

is invariant under the $\mathbb{Z}/4\mathbb{Z}$ action, so the quotient is a singular Calabi-Yau.

A very similar approach works for the second generator, whose associated $\text{Mp}(2, \mathbb{Z})$ ele-

ment is given by \hat{U}^4 with

$$U^4 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}). \quad (6.14)$$

As opposed to the background above a well-defined action on the lattice Λ_τ fixes the axio-dilaton to

$$\tau = e^{2\pi i/3}, \quad (6.15)$$

for the central fiber and the geometry is given by

$$(\mathbb{C}^2 \times T^2)/(\mathbb{Z}/3\mathbb{Z}): \quad (z_1, z_2, \lambda) \mapsto (\omega z_1, \omega z_2, \omega \lambda), \text{ with } \omega = e^{2\pi i/3}. \quad (6.16)$$

Again, this is the singular limit of a non-Higgsable cluster, this time the collapsed curve is given by a curve of self-intersection (-3) . The induced gauge algebra hosted by 7-branes wrapping the curve is given by $\mathfrak{su}(3)$ associated to a type *IV* singularity, see Table 6. The configuration preserves the same amount of supersymmetry as above. Correspondingly, the local $(3, 0)$ -form of equation (6.13) is invariant under the $\mathbb{Z}/3\mathbb{Z}$ action.

We see that while no new stringy objects need to be introduced in order to break the bordism group, the codimension-four defects are associated to very rich and interesting F-theory backgrounds related to six-dimensional superconformal field theories, see [79, 80] for recent reviews.

6.4 Codimension-five defects

The group $\Omega_4^{\mathrm{Spin-Mp}(2, \mathbb{Z})}(\mathrm{pt})$ is free and generated by a single generator E , the Enriques surface. This can be understood as a refinement of $\Omega_4^{\mathrm{Spin}}(\mathrm{pt}) = \mathbb{Z}$ generated by K3 to manifolds with Spin- $\mathbb{Z}/8\mathbb{Z}$ structure; the corresponding class is generated, in the F-theory literature, by the Enriques Calabi-Yau (see e.g. [81]). Thus we need a type IIB background that allows the collapse of E to a point. Such a background will necessarily be non-supersymmetric. We also expect it will be strongly coupled since the spectrum of IIB on K3 as well as E is chiral. A detailed study of this defect is outside of the scope of this paper.

6.5 Codimension-six defects

Defects of codimension-six become necessary due to the non-vanishing bordism group

$$\Omega_5^{\mathrm{Spin-Mp}(2, \mathbb{Z})}(\mathrm{pt}) = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/32\mathbb{Z}) \oplus (\mathbb{Z}/9\mathbb{Z}), \quad (6.17)$$

generated by lens spaces. As in codimension four we can lift the configurations to an F-theory background by considering¹⁴

$$(\mathbb{C}^3 \times T^2)/(\mathbb{Z}/k\mathbb{Z}), \quad (6.18)$$

which describes so-called S-fold backgrounds [50, 51], see also [82, 53, 54]. The fact that there are two different S-folds with $k = 4$ can be traced back to the fact that there exist two different Spin- $\mathbb{Z}/8\mathbb{Z}$ structures on L_4^5 . This leads to interesting consequences for the 4d SCFTs localized on the singular point of these backgrounds, which will be discussed in [83]. Here, we will focus on their general properties. For that, note that η -invariants in 5d are bordism invariants and thus can be used to analyze bordism relations between different $\mathbb{Z}/k\mathbb{Z}$ action of the lens spaces.

For $k = 3$ the lens space L_3^5 is Spin and can be interpreted as the asymptotic boundary of $\mathbb{C}^3/(\mathbb{Z}/3\mathbb{Z})$, where the discrete group action on the complex coordinates of \mathbb{C}^3 takes the form

$$z_i \mapsto e^{2\pi i j_i/k} z_i \text{ with } j_i \in \{1, 2\}, \quad (6.19)$$

see also Appendix C.1. This group action introduces an isolated singularity at $z_i = 0$. Using the analysis in [84], this singularity is terminal if the action on the base z_i takes the form

$$(z_1, z_2, z_3) \mapsto (\zeta z_1, \zeta^a z_2, \zeta^{-a} z_3) \quad (6.20)$$

with ζ a primitive 3rd root of unity and a co-prime with respect to 3.¹⁵

To get a IIB background, we equip this with a duality bundle so that the total space in the F-theory model is a Calabi-Yau four-fold. This implicitly involves picking a choice of complex structure. In the local patch near the origin, we thus get a singularity of the form $\mathbb{C}^4/(\mathbb{Z}/k\mathbb{Z})$ where the action on the local coordinates is

$$(z_1, z_2, z_3, \lambda) \mapsto (\zeta z_1, \zeta^a z_2, \zeta^{-a} z_3, \zeta^{-1} \lambda), \quad (6.21)$$

and it follows from references [84, 85] that this is an isolated terminal singularity of the local four-fold.¹⁶

In fact, this is just the celebrated $\mathbb{Z}/3\mathbb{Z}$ S-fold in [50]. Calculating the η -invariant of a Dirac fermion with charge $q = 1$ in this background (see Appendix C.1), we find

$$\eta_{\frac{1}{2}}^D(L_3^5(1, -1, 1)) = \frac{1}{9}, \quad (6.22)$$

¹⁴The complex structure of the singular central fiber is fixed to be $\tau = e^{2\pi i/3}$ for $k = 3$ and $\tau = i$ for $k = 4$.

¹⁵Note also that this interior geometry is not a Calabi-Yau threefold since the determinant is ζ rather than 1.

¹⁶Note that this bulk geometry is a Calabi-Yau fourfold since the determinant is 1.

where we explicitly denoted the $\mathbb{Z}/3\mathbb{Z}$ action on the geometry by $L_3^5(j_1, j_2, j_3)$, in the obvious notation. Thus, the asymptotic boundary of this S-fold is a generator for the $\mathbb{Z}/9\mathbb{Z}$ bordism group factor. One can also consider more general lens spaces. Taking the group action to be uniform on all the z_i , we observe that

$$\eta_{\frac{1}{2}}^D(L_3^5(1, 1, 1)) = -\frac{1}{9}, \quad (6.23)$$

and so is related to the generator via:

$$8[L_3^5(1, -1, 1)] \sim [L_3^5(1, 1, 1)], \quad (6.24)$$

with \sim indicating that the two sides are bordant.

In the corresponding F-theory description, both the asymptotic lens space $L_3^5(1, -1, 1)$ and the lens space $L_3^5(1, 1, 1)$ can preserve supersymmetry, but this is accomplished in different ways. In the case of the S-fold with asymptotic boundary $L_3^5(1, -1, 1)$, we require a non-trivial elliptic fibration to retain supersymmetry. On the other hand, $L_3^5(1, 1, 1)$ is the asymptotic boundary of the local Calabi-Yau threefold $\mathcal{O}(-3)$ over \mathbb{CP}^2 , and so is compatible with a trivial elliptic fibration.

To see this in more detail, introduce the local $(4, 0)$ -form:

$$\Omega_{(4,0)} = dz_1 \wedge dz_2 \wedge dz_3 \wedge d\lambda, \quad (6.25)$$

where λ is the complex local coordinate on the F-theory torus. For $L_3^5(1, -1, 1)$ we see that λ needs to transform as

$$\lambda \rightarrow \zeta^{-1}\lambda, \quad (6.26)$$

in order for $\Omega_{(4,0)}$ to be invariant. This precisely indicates the non-trivial duality bundle, which is translated to a non-trivial torus bundle in the F-theory geometry. Since, the supercharges also transform under the duality, one finds that this background preserves part of the supersymmetry, famously leading to D3-brane probe theories with $\mathcal{N} = 3$ supersymmetry. Summarizing, for the supersymmetric F-theory background the central singularity corresponds to a $\mathbb{Z}/3\mathbb{Z}$ S-fold.

A general comment here is that the above analysis has favored a specific complex structure on the bulk geometry. Observe that we could have complex conjugated, for example z_2 to \bar{z}_2 . Doing so, it might at first appear that the roles of the lens spaces have switched roles. It is important to note, however, that we have also implicitly fixed a choice of duality bundle. The η -invariants detect topological data and are insensitive to these holomorphic geometry issues.

A very similar discussion can be performed for $k = 4$ in which case there are two inequivalent Spin- $\mathbb{Z}/8\mathbb{Z}$ structures as we discuss in more detail in [83]. For our purposes, it

will suffice to consider the lens spaces $L_4^5(1, 1, 1)$. Since there are two classes, we also need two bordism invariants in order to establish relations between manifolds. We can choose these to be the η -invariant for a Dirac fermion as well as a Rarita-Schwinger field, respectively. Distinguishing between the two different Spin- $\mathbb{Z}/8\mathbb{Z}$ structures by a tilde, we find (see also [32]),

	$\eta_{1/2}^D$	$\eta_{1/2}^{RS}$	(6.27)
$L_4^5(1, 1, 1)$	$-\frac{5}{32}$	$\frac{11}{32}$	
$\tilde{L}_4^5(1, 1, 1)$	$-\frac{3}{32}$	$-\frac{3}{32}$	

As pointed out in [32], the following two combinations generate the individual factors in the bordism group

$$\mathbb{Z}/32\mathbb{Z} : [L_4^5(1, 1, 1)], \quad \mathbb{Z}/2\mathbb{Z} : [\tilde{L}_4^5(1, 1, 1)] + 9 [L_4^5(1, 1, 1)]. \quad (6.28)$$

With respect to this choice of complex structure, we observe that the corresponding bulk geometries do not lead to terminal singularities in $\mathbb{C}^3/(\mathbb{Z}/4\mathbb{Z})$, whereas variants of the form $L_4^5(1, -1, 1)$ and $\tilde{L}_4^5(1, -1, 1)$ would.¹⁷ However, for $\mathbb{Z}/4\mathbb{Z}$ both of the variants can preserve some supersymmetry since the covering space holomorphic $(4, 0)$ -form is invariant under both group actions. We give an extensive analysis of these type of backgrounds in [83]. In order to make sure that this leads to the known $\mathbb{Z}/4\mathbb{Z}$ S-fold backgrounds, we determine the set of bordism invariants in Table (6.27) for $(j_1, j_2, j_3) = (1, -1, 1)$

	$\eta_{1/2}^D$	$\eta_{1/2}^{RS}$	(6.29)
$L_4^5(1, -1, 1)$	$\frac{5}{32}$	$-\frac{11}{32}$	
$\tilde{L}_4^5(1, -1, 1)$	$\frac{3}{32}$	$\frac{3}{32}$	

We see that only the sign of the η -invariants flip, so indeed the full bordism group is taken care of by including the associated S-fold backgrounds in type IIB.

6.6 Codimension-seven defects

Consider next codimension-seven defects. By inspection of Table 1, we see that the bordism group with $k = 6$ vanishes for $SL(2, \mathbb{Z})$, Spin-Mp($2, \mathbb{Z}$) and Spin-GL $^+(2, \mathbb{Z})$ structures. Therefore, there are no singular codimension-seven defects predicted by the Cobordism Conjecture in these cases. To give an explanation for this, we can compare to the similar group with

¹⁷Comparing [50] with [51, 32], one is considering a different choice of complex structure to present the geometry, which in turn influences the presentation of the cyclic group action on the ambient $\mathbb{R}^6 \times T^2$. For example, the $\mathbb{Z}/4\mathbb{Z}$ S-fold defined by the local group action $(z_1, z_2, z_3, \lambda) \mapsto (\zeta z_1, \zeta^{-1} z_2, \zeta z_3, \zeta^{-1} \lambda)$ of [50] is instead presented as the local group action $(z_1, z_2, z_3, \lambda) \mapsto (\zeta z_1, \zeta z_2, \zeta z_3, \zeta \lambda)$ in [51, 32]. The physical content, however, is fixed by the choice of a prescribed duality bundle structure, and this is what is detected by the η -invariants.

Spin structure alone, which also vanishes, $\Omega_6^{\text{Spin}}(\text{pt}) = 0$. For instance, for any Calabi-Yau manifold, if the SYZ conjecture is true [86], one expects to have a T^3 fibration, and since T^3 is itself a boundary, a boundary for the whole Calabi-Yau can be constructed fiberwise.¹⁸ In the cases under consideration, this does not work directly, because the corresponding three-dimensional bordism groups do not vanish. But it may be that a more detailed analysis shows that the non-trivial bordism classes can never appear as fibers, recovering the result that the bordism groups vanish. At any rate, it would be interesting to develop a detailed understanding of the smooth geometries that act as null-bordisms for arbitrary 6-manifolds, but this is beyond the scope of the paper.

6.7 Codimension-eight defects

The defects in codimension-eight are associated with the generators of

$$\Omega_7^{\text{Spin-Mp}(2,\mathbb{Z})}(\text{pt}) = (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/9\mathbb{Z}). \quad (6.30)$$

The asymptotic geometries are given by a seven-dimensional lens space L_3^7 and a manifold denoted Q_4^7 which can be described as a fibration of L_4^5 over \mathbb{CP}^1 .

Consider L_3^7 first. As in the lower codimension cases, this can be lifted to the boundary of a singular F-theory geometry given by

$$(\mathbb{C}^4 \times T^2)/(\mathbb{Z}/3\mathbb{Z}), \quad (6.31)$$

fixing the complex structure of the torus on the central fiber to $\tau = e^{2\pi i/3}$. With the variant given by $L_3^7(1, 1, 1, 1)$, we retain some supersymmetry for the explicit group action:

$$(z_1, z_2, z_3, z_4, \lambda) \mapsto (\zeta z_1, \zeta z_2, \zeta z_3, \zeta z_4, \zeta^{-1} \lambda), \quad (6.32)$$

in the obvious notation. In particular, the existence of a supersymmetric background requires a non-trivial elliptic duality bundle structure. By contrast, for the asymptotic boundary $L_3^7(1, -1, 1, -1)$, the base of the F-theory model is already a Calabi-Yau fourfold so supersymmetry is compatible with a trivial duality bundle structure.¹⁹

We now ask whether these backgrounds provide generators for the corresponding bordism group factors. For that we determine $\eta_1^D(L_3^7) - \eta_0^D(L_3^7)$ in both cases. Note that we consider the difference of two η -invariants with different charges, since a single η -invariant is not a bordism invariant due to potential contributions of the index density on 8-manifolds.

$$\begin{aligned} \eta_1^D(L_3^7(1, 1, 1, 1)) - \eta_0^D(L_3^7(1, 1, 1, 1)) &= \frac{1}{9}, \\ \eta_1^D(L_3^7(1, -1, 1, -1)) - \eta_0^D(L_3^7(1, -1, 1, -1)) &= \frac{1}{9}. \end{aligned} \quad (6.33)$$

¹⁸To realize this argument carefully, it would be necessary to take into account degeneracies of the fibration.

¹⁹Observe that the base of the F-theory model is a terminal singularity in the base.

Indeed, these two lens spaces are bordant. We therefore find new type IIB backgrounds that are described by lower-dimensional versions of the S-folds above, which we call S-strings. Focusing on the variant with $(j_1, j_2, j_3, j_4) = (1, 1, 1, 1)$ we can explicitly determine the transformation properties of the supercharges, see also the analysis in [83]. This is compatible with an exotic $\mathcal{N} = (8, 2)$ supersymmetry²⁰ in the two dimensional worldsheet theory of the S-string, but it is possible that in an actual string background with additional fluxes and brane sources switched on that this is reduced further. We will leave a full analysis, as well as potential physical applications for future work.

Consider next the second generator Q_4^7 . This can be presented as the total space of a lens space bundle over \mathbb{CP}^1

$$\begin{array}{ccc} L_4^5 & \hookrightarrow & Q_4^7 \\ & & \downarrow \\ & & \mathbb{CP}^1 \end{array} \quad (6.34)$$

as was described in Section 3. We therefore see that it is not a genuinely new background but the compactification of a $\mathbb{Z}/4\mathbb{Z}$ S-fold on a \mathbb{CP}^1 .

We now argue that there is a IIB background with asymptotic geometry Q_4^7 which preserves some supersymmetry. In the description of the manifold in Section 3 we have focused on the embedding of the lens space in $H^{\pm 2} \oplus \underline{\mathbb{C}} \oplus \underline{\mathbb{C}}$ over \mathbb{CP}^1 . This means that the first complex coordinate z_1 of the ambient space of the fiber gets twisted, i.e., transforms as a non-trivial line bundle over \mathbb{CP}^1 . Since zero modes on a sphere are only possible for fields transforming as scalars under rotations on the base \mathbb{CP}^1 , preserved supercharges need to respect that property. One can view this as specifying a topological twist [87], which for the space above happens with respect to a $\text{Spin}(2) \simeq \text{U}(1)$ subgroup of the full R-symmetry associated to rotations in the internal space acting on z_1 . Depending on the bundle H^2 or H^{-2} , as a section of which z_1 transforms, this leads to an (anti)-correlation of Spin-components in the respective two-dimensional subspace described by the non-trivial line bundle of the fiber, i.e., in the Spin component on \mathbb{CP}^1 and \mathbb{C}_{z_1} . We see that in general this preserves half of the components of supercharges of L_4^5 . Since we have seen that these backgrounds preserve supersymmetry, so do the twisted compactification on Q_4^7 .

It would be interesting to analyze the associated 2d supersymmetric quantum field theories (SQFTs) described by these backgrounds, which potentially realize exotic supersymmetries, see also [88]. Thus, we discover new type IIB configurations which we called S-strings, as well as supersymmetry preserving compactifications of the original 4d S-folds.

²⁰As usual we indicate the supersymmetry charges in the smallest spinor representation, which in two dimensions is Majorana-Weyl.

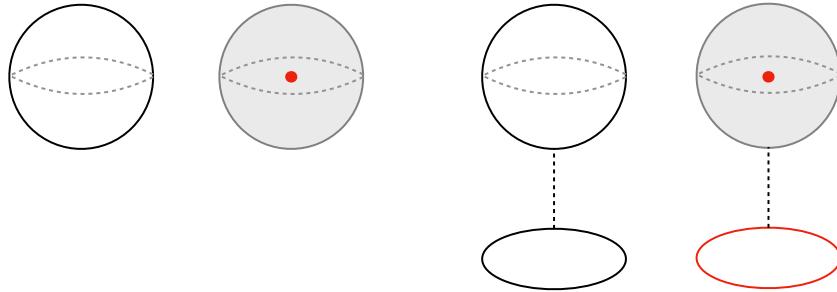


Figure 6: Knowing how to bound a certain bordism class allows us to bound a higher-dimensional variant in which the class appears as the fiber of a fibration.

6.8 Fibration generators

In the previous section we observed that some of the asymptotic geometries are obtained via the process of ‘‘compactification,’’ in the sense that they descend from a bordism generator in a lower-dimensional asymptotic geometry.

Indeed, by inspection of the generators in Table 2 and 3, several of the higher-dimensional bordism group generators can be understood as fibrations, for which the fiber is related to generators in lower dimension. In general consider the fibration

$$\begin{array}{ccc} \mathcal{F} & \hookrightarrow & X \\ & \downarrow & \\ & & \mathcal{B} \end{array} \quad (6.35)$$

where X generates part of $\Omega_k^G(\text{pt})$ and the fiber \mathcal{F} is related to an element in $\Omega_\ell^G(\text{pt})$, with $\ell = k - \dim(\mathcal{B})$. We already know how to describe the fiber as the boundary of a space containing certain defects and we can generalize this to the full fibration by wrapping the corresponding defect on \mathcal{B} , see Figure 6.

In the context of an F-theory compactification, the appearance of a non-trivial fibration is actually required to retain supersymmetry in the lower-dimensional vacua. Indeed, given an elliptically fibered n -fold $X_n \rightarrow \mathcal{B}_{n-1}$ for some base \mathcal{B}_{n-1} , we have demand a fixed duality bundle structure on the asymptotic geometry $\partial\mathcal{B}_{n-1}$. If we compactify further by fibering the original background over a space \mathcal{C}_m , the condition that the same bundle structure is retained upon further compactification means, in tandem with the Ricci flatness condition, that the resulting Calabi-Yau $(n+m)$ -fold $X_{n+m} \rightarrow \mathcal{B}_{n+m-1}$ has a base \mathcal{B}_{n+m-1} which is itself fibered over \mathcal{C}_m , namely we have $\mathcal{B}_{n-1} \rightarrow \mathcal{B}_{n+m-1} \rightarrow \mathcal{C}_m$.

In many cases, this can be interpreted in gauge theory terms as the implementation of a corresponding (partial) topological twist of the sort considered in reference [87] (see also [89]). To see why, recall that in many F-theory models, a singularity in the bulk can be interpreted

as geometrically engineering a gauge theory, as obtained from branes wrapped on cycles of the base of the F-theory model. Now, the condition that we retain supersymmetry in such models amounts to the requirement that the brane worldvolume theory is topologically twisted [90]. This partial twisting has been carried out for F-theory on Calabi-Yau threefolds (implicitly) in [91], for Calabi-Yau fourfolds in [92, 93], and Calabi-Yau fivefolds in [94, 95]. See also [96–98] for related analyses in the context of geometric engineering in M-theory backgrounds. From this perspective, it is natural to interpret some of our asymptotic geometries as obtained from a further compactification of a higher-dimensional vacuum.

Of course, we will also be interested in backgrounds where supersymmetry may not be preserved, but even here there is a correlation between the bundle structure in the fiber and base, so we will continue to refer to these as “twisted compactifications”. In physical terms, these cases are obtained by starting from a higher-dimensional defect and wrapping it over a cycle in the new base. In general, the duality bundle will also involve the base of the fibration. This happens for example for the Spin- $\mathrm{GL}^+(2, \mathbb{Z})$ generators W_1^7 and W_1^9 , as well as for the Arcana. Moreover, in general the duality bundle will also involve reflections. To construct the defect for the fiber bundle, we would wrap the defect in the fiber along the base, where there is too a duality bundle. In order for these configurations to be well-defined, the worldvolume degrees of freedom of the defects wrapped around \mathcal{B} need to be compatible with these generalized duality bundles, that is, reflections should be a global symmetry of the worldvolume theory of the fiber. Since the defects involve the reflection 7-brane (R7-brane) whose worldvolume theory is hard to identify (see however [23]) we assume that the duality bundle is compatible with the defects and postpone a detailed analysis for future work. An encouraging hint is that the R7-brane is a defect for a circle compactification with a monodromy in the reflection symmetry, a stringy generalization of a vortex, and in general, the symmetry that is protecting a vortex is unbroken at its core (as an example, consider vortex strings in the abelian Higgs model).

6.9 Codimension-nine defects

As in the case of codimension-five defects, these classes correspond to Spin- $\mathbb{Z}/8\mathbb{Z}$ generalizations of the non-trivial Spin 8-manifolds specified by the value of p_1^2 and p_2 . In the case of the quaternionic projective space \mathbb{HP}^2 , we can view this as the boundary geometry of a linear dilaton background for a gauged WZW model. Indeed, since we have the coset construction $\mathbb{HP}^2 = \mathrm{Sp}(3)/(\mathrm{Sp}(2) \times \mathrm{Sp}(1))$, it is enough to begin with the WZW model for the Lie group $\mathrm{Sp}(n+1)$ and gauge the appropriate subgroup. Solving the string theory equations of motion then requires a dilaton gradient. See e.g., [99] for further discussion. The corresponding geometry will have a singularity as the dilaton runs to strong coupling, but the point of view that we take here is that existence of the boundary (as demanded by the Cobordism Conjecture) in fact requires that this singularity be allowed. The Cobordism Conjecture has been used in the past to argue for the consistency of strong coupling singularities whose fate

and consistency are hard to determine otherwise, see [13].

The other bordism generator we have encountered is the Bott manifold B , which we can take to be a manifold of Spin(7) metric holonomy. Unlike in the previous case, there is no simple way to describe the bounding geometry of this manifold in the worldsheet, although presumably it would also involve a singularity where the dilaton runs to strong coupling. It would be very interesting to elucidate the physics of the corresponding non-supersymmetric defect, but this lies beyond the scope of this paper.

6.10 Codimension-ten defects

Codimension-ten defects are objects localized in spacetime and can be associated to exotic instanton backgrounds. However, for

$$\Omega_9^{\text{Spin-Mp}(2,\mathbb{Z})}(\text{pt}) = (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/128\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/27\mathbb{Z}), \quad (6.36)$$

only the first three summands $(\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/128\mathbb{Z}) \oplus (\mathbb{Z}/27\mathbb{Z})$ are associated to localized defects, which we call S-instantons, whereas $(\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})$ can be understood as the 7-brane stacks of Section 6.1 wrapped around \mathbb{HP}^2 . We therefore focus on the three factors generated by L_k^9 , which can be described as the projection to the base of F-theory backgrounds of the form

$$(\mathbb{C}^5 \times T^2)/(\mathbb{Z}/k\mathbb{Z}). \quad (6.37)$$

For $k = 3$ the central singularity is canonical and terminal for all consistent values of the j_i . For $L_3^9(1, 1, 1, 1, 1)$ there is the possibility of describing the geometry above in terms of a non-compact singular Calabi-Yau 6-fold, for which the elliptic fiber coordinate needs to transform under $\mathbb{Z}/3\mathbb{Z}$, $\lambda \mapsto e^{2\pi i/3}\lambda$. It is tempting to speculate about preserved supersymmetries in the effective zero-dimensional system, ‘living’ on the S-instanton defect.

For $k = 4$ the two generators differ once more by the choice of Spin- $\mathbb{Z}/8\mathbb{Z}$ structure and can be detected by the η -invariants

	$\eta_{1/2}^D$	$\eta_{1/2}^{\text{RS}}$
$L_4^9(1, 1, 1, 1, 1)$	$\frac{9}{128}$	$-\frac{19}{128}$
$\tilde{L}_4^9(1, 1, 1, 1, 1)$	$\frac{7}{128}$	$\frac{3}{128}$

(6.38)

which suggests the two generators

$$\mathbb{Z}/128\mathbb{Z} : [L_4^9(1, 1, 1, 1, 1)], \quad \mathbb{Z}/4\mathbb{Z} : 7[L_4^9(1, 1, 1, 1, 1)] - 9[\tilde{L}_4^9(1, 1, 1, 1, 1)]. \quad (6.39)$$

The above geometries can be realized as boundaries of a local and singular elliptically fibered Calabi-Yau six-fold. All of the central singularities are canonical and terminal, both in the base and total space of the associated F-theory model. Again, the analysis of the bordism

groups points towards interesting new backgrounds of type IIB string theory described by S-instantons.

This concludes our discussion of Spin-Mp(2, \mathbb{Z}) bordism defects.

7 Spin-GL⁺ defects

We now include the reflections and their Pin⁺ lift in the duality group and analyze the defects associated with the generators of the bordism groups $\Omega_k^{\text{Spin-GL}^+(2, \mathbb{Z})}(\text{pt})$. Comparing these bordism groups with the ones for Spin-Mp(2, \mathbb{Z}) manifolds in Table 1, we see that the groups in dimension $k = (4n + 1)$ are reduced while the groups in dimension $k = (4n + 3)$ are typically increased. The reason is based on two competing effects. First, since every Spin-Mp(2, \mathbb{Z}) manifold is also a Spin-GL⁺(2, \mathbb{Z}) manifold, one naively expects to find more generators. However, the Spin-GL⁺(2, \mathbb{Z}) manifolds also allow a larger class of deformations and thus tend to identify formerly distinct generators.

The fact that this identification predominantly happens in dimensions $k = (4n + 1)$, can be understood as follows. In many cases the Spin-Mp(2, \mathbb{Z}) generators are associated to characteristic classes of the duality bundle. These characteristic classes switch sign under the action of the reflection operator exactly in dimension $k = (4n + 1)$ and thus the associated bundles can be deformed into each other on the level of Spin-GL⁺(2, \mathbb{Z}) manifolds. In dimension $k = (4n + 3)$, however, the characteristic classes are even under reflections, and can not be deformed into each other. We will demonstrate this explicitly for $n = 0$, i.e., $k = 1$, below.

As for Spin-Mp(2, \mathbb{Z}) bordisms we encounter several interesting new classes and backgrounds for type IIB string theory as well as known objects. We summarize these for the odd-dimensional bordism groups in Table 5.

7.1 Codimension-two defects

The codimension-two defects predicted by

$$\Omega_1^{\text{Spin-GL}^+(2, \mathbb{Z})}(\text{pt}) = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}), \quad (7.1)$$

seem to be more sparse than in the Spin-Mp(2, \mathbb{Z}) case discussed in Section 6.1. The reason, as pointed out above, is that the additional reflections allow for a deformation of several distinct Spin-Mp(2, \mathbb{Z}) structures into each other, thus, identifying the associated manifolds via Spin-GL⁺(2, \mathbb{Z}) bordisms. We demonstrate this for the subgroup $\mathbb{Z}/3\mathbb{Z}$ of $\Omega_1^{\text{Spin-Mp}(2, \mathbb{Z})}(\text{pt})$.

The $\mathbb{Z}/3\mathbb{Z}$ factor is generated for example by a circle with a transition function associated to $g = \hat{U}^4$, see Appendix B.²¹ The disjoint union of three copies of this generator is null-

²¹Technically this is twice the generator L_3^1 defined above, that also generates the full $\mathbb{Z}/3\mathbb{Z}$ summand,

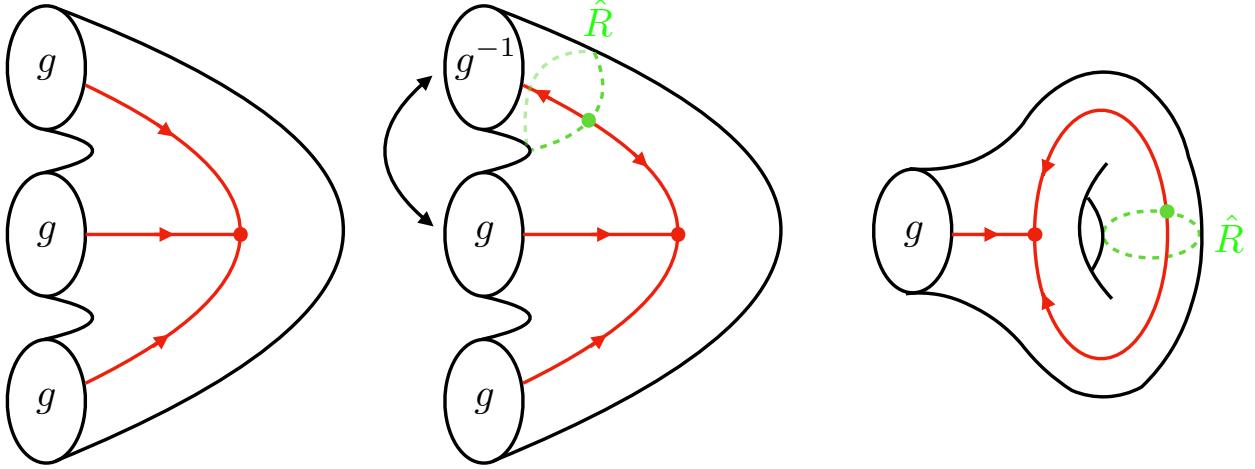


Figure 7: Depiction of the null-bordism for $2[L_3^1]$ in $\Omega_1^{\text{Spin-GL}^+(2,\mathbb{Z})}(\text{pt})$. The red lines denote $\mathbb{Z}/3\mathbb{Z}$ transition functions with the arrow distinguishing between g and g^{-1} , where g generates $\mathbb{Z}/3\mathbb{Z}$. The green dotted line depicts a reflection transition function \hat{R} , which effectively flips the arrow.

bordant, see the left-hand side of Figure 7. However, one cannot glue two of the three boundaries because the transition functions do not match. Once we include reflections, however, one can change the transition function according to the group law

$$\hat{R}\hat{U}^4\hat{R}^{-1} = \hat{U}^{-4} = \hat{U}^8. \quad (7.2)$$

Including such a reflection transition function in the two-dimensional bulk corresponds to switching on a non-trivial background for the reflection part of the symmetry. This allows us to glue two of the boundary components. This means that $2[L_3^1]$ bounds as a $\text{Spin-GL}^+(2,\mathbb{Z})$ manifold. Since on $\mathbb{Z}/3\mathbb{Z}$, both non-trivial elements of the group are twice each other, all bordism classes in this factor are trivialized. This is summarized pictorially in Figure 7.

The same happens for the $\mathbb{Z}/8\mathbb{Z}$ factor generated by L_4^1 . The reflection acts on the group element encoding the transition function as

$$\hat{R}\hat{S}\hat{R} = \hat{S}^{-1} = \hat{S}^7. \quad (7.3)$$

This demonstrates that each element in $\mathbb{Z}/8\mathbb{Z} \subset \Omega_1^{\text{Spin-Mp}(2,\mathbb{Z})}(\text{pt})$ gets mapped to its inverse under reflection, and by the same argument as above, every class corresponding to an element of $\mathbb{Z}/8\mathbb{Z}$ which is a multiple of two becomes trivial in cobordism. This kills the elements of order 2, and 4, and the remaining non-trivial class is a circle with a holonomy given by the \hat{S} generator, whose boundary is given by a stack of 7-branes inducing an \mathfrak{e}_7 singularity in the fiber, i.e., fiber type III^* . This is the generator of one of the $\mathbb{Z}/2\mathbb{Z}$ factors, see also the

but is more convenient here because of the amalgam structure.

arguments in [24].

The other generator in (7.1) is given by a circle with a transition function given by \hat{R} , denoted as S_R^1 . This requires the addition of a new type of 7-brane in type IIB string theory that induces a monodromy of determinant (-1) , i.e., including reflections, which we refer to as reflection 7-brane or R7-brane. See reference [23] for further analysis on the physics of R7-branes. Two variants of this reflection brane will be particularly important, see also Appendix B and Figure 10. One, the Ω -brane, induces a monodromy given by the worldsheet orientation reversal. The other, the F_L -brane, instead generates fermion parity in the left-moving sector of the string worldsheet. They are S-dual to each other and can therefore be deformed into each other by brane moves involving the usual $[p, q]$ -7-branes, similar to the transitions described in [100–102].

We want to emphasize that the mechanism that we have discussed here for constructing a geometric boundary of some bordism classes in $\text{Spin-Mp}(2, \mathbb{Z})$ means that we have geometrized the corresponding 7-branes. Ordinarily, $[p, q]$ -7-branes are not described by a smooth geometry, and fields become singular at their core. The geometric defects described above allow us to construct completely smooth objects with the same duality monodromy as e.g., a type IV^* singularity, but that are completely smooth – just a torus glued to flat space with a particular monodromy. A similar situation arises e.g., in heterotic string theory, where NS5-brane charge can be carried by a singular brane or a K3 geometry (or a gauge bundle). Just as in that case, we should entertain the possibility that the smooth and singular supersymmetric configurations are two different “phases” or configurations of a single underlying object, specified by its monodromy charges. The topologically non-trivial, non-supersymmetric smooth 7-brane background we constructed should be viewed as a highly excited state of the supersymmetric 7-brane, which, being BPS, is the lightest object with the specified monodromy.

Finally, we have seen that an inclusion of reflections allows deformations of many of the different $\text{Spin-Mp}(2, \mathbb{Z})$ classes into each other. This was possible because they reverse the transition functions, or alternatively the characteristic classes of the duality background. Hence we expect such reductions of the bordism group to happen in dimensions where the typical duality background transforms with a minus sign under reflection. This is the case in dimension $k = (4n + 1)$ (see the characterization of the duality bundle in [33]) and precisely corresponds to the reductions of the bordism groups in Table 1. In dimensions $k = (4n + 3)$ the duality backgrounds typically are invariant under reflections and one does not encounter this reduction. Instead, one often obtains new backgrounds involving transition functions that contain reflections. In even dimensions the duality structure is usually not what supports the non-trivial bordism classes and we also do not expect a reduction, verifying the observations in Table 1.

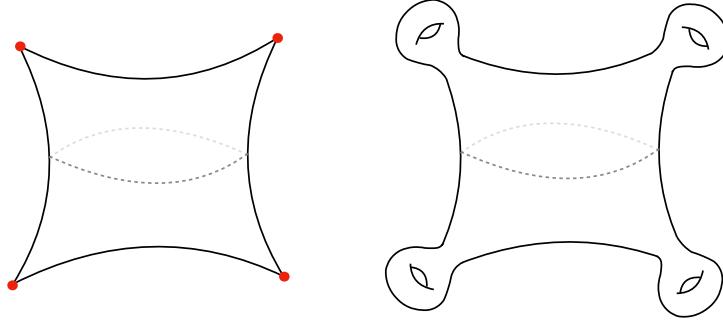


Figure 8: Null-bordism for F-theory on K3 via a transition from a brane stack to a smooth $\text{Spin-GL}^+(2, \mathbb{Z})$ geometry. Here, small discs around the orientifold planes are replaced by 2-tori with a non-trivial duality bundle containing reflections.

7.2 Deforming F-theory on K3 to nothing

There is an interesting application of the bordism groups and branes described so far, which we felt merits its own subsection. It is related to one of the outstanding questions left open in [9], namely constructing a boundary for F-theory on K3. A K3 surface has a limit in which it can be described as an elliptic fibration over \mathbb{CP}^1 with four I_0^* fibers, which is related to the description as a $T^4/(\mathbb{Z}/2\mathbb{Z})$ orbifold. This is known as Sen's limit in the F-theory literature [9]. F-theory on K3 in Sen's limit is a perturbative IIB configuration, which can be described as a $T^2/(\mathbb{Z}/2\mathbb{Z})$ compactification of 10-dimensional IIB where there are four singularities (of Kodaira type I_0^*) at the fixed points of the action, as depicted in the left panel of Figure 8. Each of these singularities can be described as a brane stack composed of four D7-branes on top of an O7⁻ orientifold plane.

An I_0^* singularity is specified by a linking circle with monodromy \hat{S}^2 , which is one of the classes which we argued is trivial in $\text{Spin-GL}^+(2, \mathbb{Z})$ bordism in the previous subsection. As a result, the brane stack that describes its boundary can be replaced, at the level of monodromies, by a completely smooth configuration without any branes, as depicted in Figure 9. As we emphasized near the end of the previous Subsection, we cannot be sure that the I_0^* singularity is the same as the smooth configuration depicted to the right in Figure 9. But they have the same asymptotic charge, so we will now assume that the two configurations can be deformed into each other and explore the consequences. What this means is that F-theory on K3 is cobordant to the configuration depicted on the right panel of Figure 8, where each of the singularities has been replaced by a glued torus with the appropriate duality monodromy. We have now replaced F-theory on K3 by a completely smooth IIB configuration, which is however non-supersymmetric. The interesting point is that since $\Omega_2^{\text{Spin-GL}^+(2, \mathbb{Z})}(\text{pt}) = 0$, this configuration is trivial in bordism of smooth manifolds with duality bundle.

In other words, we have reduced the global question of finding the boundary of F-theory on K3 to the *local* question of establishing the equivalence between the I_0^* singularity and

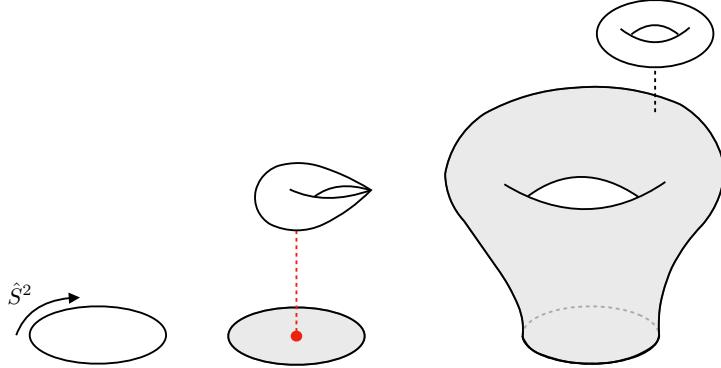


Figure 9: Bordism for a circle with transition function \hat{S}^2 , by singular configuration involving a brane stack or non-singular configuration containing reflections.

the smooth bordism we constructed.

7.3 Codimension-three defects

There is a single generator of

$$\Omega_2^{\text{Spin-GL}^+(2,\mathbb{Z})}(\text{pt}) = \mathbb{Z}/2\mathbb{Z}, \quad (7.4)$$

associated to a codimension-three defect. However, since it is given by a product of the circle with periodic boundary conditions on the fermions without a duality bundle, i.e., the generator of $\Omega_1^{\text{Spin}}(\text{pt})$, and the generator of $\Omega_1^{\text{Spin-GL}^+(2,\mathbb{Z})}(\text{pt})$ involving the reflection brane, we do not need to introduce additional defects. Indeed, the inclusion of an R7-brane allows us to fill one of the circles. Thus the associated string theory background corresponds to an R7-brane wrapped on an S^1 .

7.4 Codimension-four defects

For codimension-four objects we have

$$\Omega_3^{\text{Spin-GL}^+(2,\mathbb{Z})}(\text{pt}) = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}). \quad (7.5)$$

The first two factors are identical to $\text{Spin-Mp}(2,\mathbb{Z})$ with the exact same generators and we need to include the same non-Higgsable clusters as in Section 6.3. The remaining two $\mathbb{Z}/2\mathbb{Z}$ factors both involve the action of reflections and have an asymptotic \mathbb{RP}^3 geometry. It is not difficult to identify these objects as O5-planes, or their S-duals [103] if we are interested in the background with F_L -branes. Once more we stress that since our analysis is not sensitive to RR- and NS-fluxes in the boundary we cannot further specify the discrete torsion allowed for these objects, nor can we match their $[p,q]$ -5-brane charges. While the background with

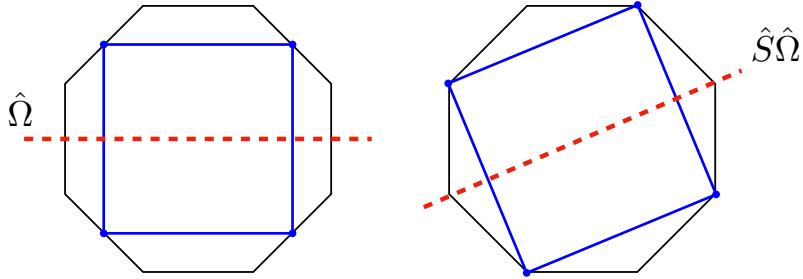


Figure 10: Two embeddings of D_4 and D_8 into D_{16} with the action of $\hat{\Omega}$ and $\hat{S}\hat{\Omega}$ indicated.

reflection branes likely breaks supersymmetry completely, the orientifold backgrounds can preserve part of the supersymmetry. It would be interesting to explore whether also the second generator can be bounded by a supersymmetric background.

We can also provide an alternative construction for a related defect, as a background of the form

$$(\mathbb{C}^2 \times T^2)/(\mathbb{Z}/2\mathbb{Z}), \quad (7.6)$$

where the $\mathbb{Z}/2\mathbb{Z}$ action of the fiber now also involves a reflection R . The two different variants are associated to the two possible embeddings of D_4 or D_8 into D_{16} , see also [33] and Appendix B.2, illustrated in Figure 10. We see that for the two variants the reflections can be described by $\hat{\Omega}$ and $\hat{S}\hat{\Omega}$, respectively. Note that instead of $\hat{\Omega}$ one could also use its S-dual given by $(-1)^{\hat{F}_L}$. Therefore, the base geometry is given by the asymptotic boundary of

$$\mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z})_{\hat{\Omega}} \quad \text{or} \quad \mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z})_{\hat{S}\hat{\Omega}}, \quad (7.7)$$

where the subscript on the $\mathbb{Z}/2\mathbb{Z}$ indicates which symmetries are acting on the background. Picturing the real projective spaces as asymptotic boundaries of the complex line bundle $\mathcal{O}(-2)$ over \mathbb{CP}^1 , we see that this configuration can be obtained by wrapping an Ω -brane and a combination of an Ω -brane with a stack of $[p, q]$ -7-branes generating \hat{S} on the base \mathbb{CP}^1 . This is parallel to the construction of the non-Higgsable clusters above, with the difference that the curve of self-intersection (-2) is now additionally wrapped by an R7-brane. Thus, the defect killing this bordism class can be realized by either ordinary O-planes or by a wrapped R7-brane. This suggests that the compactification of an R7-brane on \mathbb{RP}^2 is a non-supersymmetric object with the same charges as an O5-plane.

7.5 Codimension-five defects

As for $\text{Spin-Mp}(2, \mathbb{Z})$ the bordism group in dimension four is also a free \mathbb{Z} factor generated by the Enriques surface E . As above we do not know how to properly bound this manifold in type IIB string theory. However, since it is the same generator as in Section 6.4, we do not expect that new ingredients or backgrounds involving R7-branes will be necessary.

7.6 Codimension-six defects

Much as in the case of codimension-two defects considered in Section 7.1, in codimension-six (an increase in dimension of four), we inherit bordism generators from the related $\text{Spin-Mp}(2, \mathbb{Z})$ backgrounds

$$\Omega_5^{\text{Spin-GL}^+(2, \mathbb{Z})}(\text{pt}) = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}). \quad (7.8)$$

The only surviving configuration is L_4^5 generating $\mathbb{Z}/2\mathbb{Z}$ which can be bounded by including the $\mathbb{Z}/4\mathbb{Z}$ S-fold background discussed in Section 6.5, with either Spin structure.

The other generator X_5 , Arcanum V, can be described as an \mathbb{RP}^3 -fibration over \mathbb{RP}^2

$$\begin{array}{ccc} \mathbb{RP}^3 & \hookrightarrow & X_5 \\ & & \downarrow \\ & & \mathbb{RP}^2 \end{array} \quad (7.9)$$

The fiber \mathbb{RP}^3 can then be identified with the asymptotic boundary of $\mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z})$, and can be bounded by the configuration in Section 7.4 – an O5-plane. Similar to the twisted compactification involving Q_4^7 in Section 6.7, we can describe the boundary of the Arcanum V class by wrapping an O5-plane on the base \mathbb{RP}^2 . Although the O5-plane does not have localized degrees of freedom, one could in principle consider a O5-D5 brane stack, which would also be a boundary. For such configurations, there is a potential question as to whether it is indeed consistent to compactify the theory on \mathbb{RP}^2 , which is not a Spin manifold, similar to other classes we discussed before. We will now see that this is the case more explicitly. As described in Section 14.3.3, the Arcanum V manifold is a quotient of $S^2 \times S^3$ by two $\mathbb{Z}/2\mathbb{Z}$ actions, one of which is just the antipodal mapping on S^3 (with a certain duality bundle). So the boundary of this is precisely an $\mathbb{R}^4/(\mathbb{Z}/2\mathbb{Z})$ singularity with an action of Ω (an O5-plane compactified on $S^2/(\mathbb{Z}/2\mathbb{Z}) = \mathbb{RP}^2$, where the $(\mathbb{Z}/2\mathbb{Z})$ action is a composition of the antipodal mapping on S^2 with an action of $(-1)^{F_L}$ and the reflection of three coordinates of the normal bundle of the O5. These additional three reflections ensure that the action commutes with the one defining the O5-plane, and that it is a symmetry of the worldvolume theory. In terms of worldvolume fields, one is quotienting by a $\mathbb{Z}/2\mathbb{Z}$ involving the antipodal mapping together with a combination of $(-1)^{F_L}$ and a $\mathbb{Z}/2\mathbb{Z}$ on the $\text{SO}(4)$ normal bundle directions, which is a symmetry of the IIB background. Furthermore, because it involves

only perturbative symmetries, this defect should be amenable to a IIB worldsheet description too, as a particular orbifold of an orientifold.

To sum up, once more, the defects necessary to generate the bordism groups can all be obtained by (non-trivially fibered) compactifications involving the reflections of real coordinates in the ambient space of the fiber.

7.7 Codimension-seven defects

Just as in Section 6.6, the bordism group with $k = 6$ vanishes, and all classes can be killed geometrically.

7.8 Codimension-eight defects

Defects in codimension eight are associated to elements in

$$\Omega_7^{\text{Spin-GL}^+(2,\mathbb{Z})}(\text{pt}) = (\mathbb{Z}/2\mathbb{Z})^{\oplus 3} \oplus (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/9\mathbb{Z}). \quad (7.10)$$

As predicted in dimension $(4n+3)$ we discover the full set of Spin-Mp($2, \mathbb{Z}$) classes bounded by S-strings and twisted compactifications of the S-folds discussed in Section 6.7. We further find two real projective spaces, for which the bordism defects are simply the O1-plane and its S-dual. These singularities can be described as $\mathbb{C}^4/(\mathbb{Z}/2\mathbb{Z})$, where the $\mathbb{Z}/2\mathbb{Z}$ group action is generated by an appropriate combination of duality generators:

$$\mathbb{C}^4/(\mathbb{Z}/2\mathbb{Z})_{\hat{\Omega}} \quad \text{and} \quad \mathbb{C}^4/(\mathbb{Z}/2\mathbb{Z})_{S\hat{\Omega}}, \quad (7.11)$$

associated to the two D_8 embeddings depicted in Figure 10. The subscript on the $\mathbb{Z}/2\mathbb{Z}$ indicates the generator for this group. Much as in the case of the defects in Section 7.4, objects with the same bordism charge can be constructed by wrapping reflection branes or a combination of reflection and conventional $[p,q]$ -7-branes on the base \mathbb{CP}^3 . Once more it would be interesting to add fluxes and study the associated string charges as well as supersymmetry properties.

The defects for the two remaining generators are slightly more subtle. The defect for W_1^7 can be obtained from the Spin-Mp($2, \mathbb{Z}$) generator by an additional $\mathbb{Z}/2\mathbb{Z}$ action involving reflections. As for Q_4^7 , we can interpret its defect as a compactification (with non-trivial fibration) of the S-fold in codimension-six, with the subtle difference that the fibration does not only involve discrete rotations encoded in a Spin- $\mathbb{Z}/8\mathbb{Z}$ structure but also involves reflections captured by the more general Spin- D_{16} structure of W_1^7 . We therefore refer to these theory as double-twisted compactifications in Table 5 and once more point out the assumption discussed in Section 6.8 concerning the consistency of the duality action on the worldvolume fields.

The last generator, W_2^7 , has a slightly different interpretation. It originates from a manifold with $\text{Spin-GL}^-(2, \mathbb{Z})$ structure via a Smith homomorphism. It can be understood as a quotient of $S^3/Q_{16} \times T^2 \times S^2$, with Q_{16} the binary dihedral group of order 16. The discrete quotient introduces a non-trivial fibration of S^3/Q_{16} over \mathbb{RP}^2 , whose total space is in turn fibered over a Klein bottle, see Section 14.3.7 for further details. As discussed in 6.8 we can interpret the corresponding defect as the twisted compactification of an object with S^3/Q_{16} as an asymptotic boundary, i.e., a D-type Du Val singularity in type IIB string theory. Therefore, this defect does not require the introduction of any further string theory object, but once more points towards interesting spacetime configurations (here, twisted compactifications of ADE singularities). As in other cases above it would be very interesting to determine the behavior of the localized degrees of freedom of this background under the twist introduced by the fibration structure.

We see that these backgrounds are natural generalizations of those found in 6.7, which induces a different class of string like defects and twisted compactifications.

7.9 Codimension-nine defects

The codimension-nine defects are induced by

$$\Omega_8^{\text{Spin-GL}^+(2, \mathbb{Z})}(\text{pt}) = \mathbb{Z} \oplus \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}). \quad (7.12)$$

Let us first focus on the $\mathbb{Z}/2\mathbb{Z}$ generator given by $W_1^7 \times S_p^1$. We see that a generator of the bordism group in dimension seven appears and one can interpret this as a further circle compactification of the twisted theory discussed in the previous subsection. Therefore, we also do not need to include more defects to break the symmetries associated to this class. Similarly, we also know how to describe S_p^1 as a boundary and one can complete the second factor to a disc with several 7-branes wrapping around W_1^7 .

The remaining two generators associated to the free \mathbb{Z} subgroups are closely related to the generators of Spin bordisms. We find the quaternionic projective space \mathbb{HP}^2 that also appeared for the Spin-Mp($2, \mathbb{Z}$) groups, see Section 6.9. The second generator $W_{1,8}$ is a refinement of the Bott manifold B to Spin- D_{16} manifolds and due to the fact that its Pontrjagin numbers are half of those of the Bott manifold could also be called half-Bott, which might also be realized in terms of a linear dilaton background.

7.10 Codimension-ten defects

Last but not least we also analyze the defects in codimension ten associated to

$$\Omega_9^{\text{Spin-GL}^+(2, \mathbb{Z})}(\text{pt}) = (\mathbb{Z}/2\mathbb{Z})^{\oplus 8}. \quad (7.13)$$

Once more many of the Spin-Mp(2, \mathbb{Z}) classes are trivial due to the action of reflections on the various backgrounds in dimension $k = (4n + 1)$.

Four of the generators, namely

$$W_1^7 \times S_p^1 \times S_p^1, \quad B \times L_4^1, \quad \mathbb{HP}^2 \times L_4^1, \quad \mathbb{HP}^2 \times S_R^1, \quad (7.14)$$

are related to the compactification of classes above and thus are already taken care of by the higher-dimensional defects. Another generator L_4^9 describes the surviving Spin-Mp(2, \mathbb{Z}) configuration associated to an S-instanton discussed in Section 6.10.

This leaves three genuinely new backgrounds. Two of them are described by nine-dimensional versions of the Arcana X_9 and \tilde{X}_9 . These backgrounds have the same underlying manifold, which is an \mathbb{RP}^3 -bundle over \mathbb{RP}^6 , with two different Spin- D_8 structures associated to the two distinct embeddings into D_{16} in Figure 10. Since the fibers are given by the generators in $k = 3$, we can generate the corresponding bordism classes by wrapping R7-branes, without the need for any additional new objects. More concretely, since the defect of \mathbb{RP}^3 is just a D_4 ALE singularity in IIB string theory, the additional background can be obtained as a $\mathbb{Z}/2\mathbb{Z}$ quotient of the resulting worldvolume geometry.

This leaves the final generator W_1^9 described by a L_4^5 lens space bundle over \mathbb{RP}^4 . Since we know the defect that has L_4^5 as an asymptotic boundary, the most direct way to realize the generator for this $\mathbb{Z}/2\mathbb{Z}$ factor is via wrapping the corresponding S-fold defect over \mathbb{RP}^4 with a twist involving reflections. Again, we emphasize that we have not checked the consistency of this wrapping; although reflections are indeed a symmetry of the S-fold worldvolume theory at the bosonic level (since τ is fixed to i , and reflections flip the real, but not the imaginary part of τ), it could be that this compactification is inconsistent at the quantum level. To study this more systematically would also require a detailed analysis of anomalies in the worldvolume theory. We defer this issue to future work.

This concludes our analysis of Spin-GL⁺(2, \mathbb{Z}) bordism groups and their associated defects. By the application of the Cobordism Conjecture we indeed find a new string theory defect that was not discussed before, the R7-brane. The inclusion of this codimension-two object then also allows the higher-dimensional backgrounds to appear as boundaries by wrapping the R7-branes on compact curves in the bulk. While the presence of R7-branes is expected to break supersymmetry, it is interesting that in some situations there are supersymmetry preserving orientifold planes which also trivialize the corresponding bordism groups. Including the asymptotic Ramond-Ramond fluxes in the discussion is a promising future direction to distinguish these backgrounds.

8 Ω_{10} and discrete θ -angles

So far we have discussed bordism groups in dimensions $k \leq 9$, which are associated to defects. The bordism groups

$$\Omega_{10}^{\text{Spin-Mp}(2,\mathbb{Z})}(\text{pt}) = \mathbb{Z}/2\mathbb{Z}, \quad \Omega_{10}^{\text{Spin-GL}^+(2,\mathbb{Z})}(\text{pt}) = (\mathbb{Z}/2\mathbb{Z})^{\oplus 4}, \quad (8.1)$$

describe potential discrete θ -angles, topological couplings that can be added to the 10d action without changing their low-energy physics.

For example the generator in $\text{Spin-Mp}(2,\mathbb{Z})$ bordisms is inherited from a generator of $\Omega_{10}^{\text{Spin}}(\text{pt})$ and is described by a manifold X_{10} with non-trivial $w_4 w_6$ composed of Stiefel-Whitney classes. This opens up the possibility of adding a term of the form

$$-\Delta S = i\pi \int w_4 w_6, \quad (8.2)$$

to the action. This implies that all spacetimes that are in the bordism class of X_{10} are weighted with an additional minus sign in the path integral, whereas manifolds that are trivial in bordism enter with a plus sign. See [104] for examples of discrete θ -angles in field theory and [40, 46] for examples in string and M-theory.

Once one can write down a topological term such as (8.2) to the supergravity action, two possibilities arise: either the value of the θ -angle is frozen (to either 0 or π) in the case of (8.2) by some consistency condition, or it is dynamical. In the latter case, it leads to two physically distinct versions of type IIB string theory, such as is the case in nine dimensions [46], or the proposed alternative anomaly cancellation mechanisms in [33]. As an example of the former, consider equation (8.2) in type IIA string theory. Since IIA is secretly the small radius limit of M-theory on S^1 , and $\Omega_{11}^{\text{Spin}}(\text{pt}) = 0$, the discrete θ -angle is absent. It is likely that this fact can be used to argue that (8.2) is also absent from the IIB supergravity action via T-duality, but this argument must be done carefully. We leave it to future work.

Moving on to $\text{Spin-GL}^+(2,\mathbb{Z})$ θ -angles, the manifold X_{10} also generates one of the $\mathbb{Z}/2\mathbb{Z}$ factors of the ten-dimensional $\text{Spin-GL}^+(2,\mathbb{Z})$ bordism group. The remaining generators are all of the form $M \times S^1$, with M a generator of one of the nine-dimensional bordism classes discussed above, similarly to the M-theory discrete θ -angle proposed in [40]. Just like the above, these discrete θ -angles are captured by η -invariants of fermions, and cannot be given local expressions in terms of cohomology classes. In fact, the background $B \times S_R^1 \times S_p^1$ has the F-theory description²² of $B \times KB \times T^2$, which under M- / F-theory duality corresponds to $B \times KB \times S^1$, which is the background charged under the θ -angle proposed in [40]. Thus, we see the θ -angle described in [40], plus additional contributions. It is likely that the M- / F-theory duality map can give additional consistency conditions that freeze some of these

²²There KB denotes the Klein bottle and B denotes a Bott manifold.

θ -angles: for instance, another one of our θ -angles detects $M = \mathbb{H}\mathbb{P}^2 \times S_R^1$. While this is a non-trivial bordism class, $\mathbb{H}\mathbb{P}^2 \times KB \times S^1$ is trivial in \mathfrak{m}^c bordism, and there is no corresponding θ -angle. This probably means that there is a valid F-theory bordism, that we do not detect with our analysis, that kills the corresponding IIB θ -angle. It would be interesting to study this in more detail.

9 Duality anomalies

We are left with the bordism groups in dimension eleven. They parametrize the potential non-perturbative anomalies of the duality as well as duality mixing with gravity of a ten-dimensional theory. In order to explore these anomalies one has to define an invertible topological field theory in eleven dimensions [105], the anomaly theory \mathcal{A} , that reproduces the phase of the partition function of the 10d theory when placed on a manifold with boundary.

For type IIB the anomaly theory \mathcal{A}_{IIB} receives contributions from the dilatini, the gravitini, as well as the chiral 4-form (with self-dual five-form field strength), and a detailed derivation thereof was presented in [33]. Quite surprisingly, the duality anomaly does not vanish. While the Spin- $\text{GL}^+(2, \mathbb{Z})$ manifolds with reflections are not problematic from the anomaly viewpoint, certain backgrounds inherited from Spin- $\text{Mp}(2, \mathbb{Z})$ are. For example one has

$$\exp(2\pi i \mathcal{A}[L_3^{11}]) = e^{2\pi i/3}, \quad (9.1)$$

indicating a mod 3 anomaly for the duality. In order to restore the duality one therefore is forced to make modifications to the theory.

One possibility is to cancel the anomaly using a topological version of the Green-Schwarz mechanism, [106]. This introduces new discrete fields of various form degrees whose higher-form gauge transformations depend on the duality background. While the discrete, topological nature of these fields does not introduce any new local degrees of freedom, the completeness hypothesis [107] suggests the presence of defects coupling electrically and magnetically to them. Also these defects will be discrete, in that they are able to decay to the vacuum once one stacks a sufficient number of them and therefore do not appear in a perturbative description. The implementation of this mechanism is not unique and in principle several versions are imaginable. This would lead to theories that differ in the spectrum of non-local objects and opens up the possibility of a discrete landscape in ten dimensions. It might however also be the case that unexplored quantum gravity constraints affect this anomaly cancellation mechanism.

Alternatively, there is the (less radical, and thus more plausible) possibility of cancelling the duality anomalies without the introduction of new topological sectors. For this to work one couples the chiral 4-form field to the duality and gravitational background in a certain way, leading to a new topological term of the type IIB action. Once more this term only

depends on discrete/torsional information and will not affect the local low-energy dynamics. However, this coupling leads to an additional contribution to the anomaly theory, via the quadratic refinement \tilde{Q} necessary for the description of the self-dual 4-form, that is capable of compensating the anomalous phase. It is very intriguing that the anomalies we find are precisely of the type that can be cancelled by this mechanism, see also [108] for a six-dimensional realization of such a discrete Green-Schwarz mechanism. Moreover, up to a small number of prefactors, that we believe to be fixed by the analysis of certain special type IIB backgrounds this cancellation is unique.

To illustrate this discrete anomaly cancellation “miracle” let us come back to the anomaly on $[L_3^{11}]$. We need

$$\mathcal{A}_{\text{IIB}}[L_3^{11}] - \tilde{Q}[\check{c}] = 0 \bmod \mathbb{Z}, \quad (9.2)$$

where \check{c} indicates the differential cohomology element describing the duality background. It turns out that on L_3^{11} the only possibility is

$$\tilde{Q}[\check{c}] = \frac{1}{3}, \quad (9.3)$$

which shows that the anomaly can only be cancelled in such a way if $\mathcal{A}_{\text{IIB}}[L_3^{11}] = 1/3$ which is precisely what we find. Since $[L_3^{11}]$ describes a $\mathbb{Z}/27\mathbb{Z}$ factor of the bordism group, there is a 1 in 26 chance of being able to cancel the anomaly (with uniform prior). A similar discussion holds for the other anomalous generators.

We see that the investigation of discrete anomalies heavily uses the determination of bordism group and their cancellation demand the modification of the low energy theory in subtle but interesting ways.

This concludes the physical interpretation of the defects predicted by the Cobordism Conjecture. We now turn to the computation of IIB duality bordism groups.

Part III

Bordism classes for type IIB

In this part of the paper we compute $\Omega_k^{\text{Spin}}(B\text{SL}(2, \mathbb{Z}))$, $\Omega_k^{\text{Spin-Mp}(2, \mathbb{Z})}(\text{pt})$, and $\Omega_k^{\text{Spin-GL}^+(2, \mathbb{Z})}(\text{pt})$ for $k \leq 11$. For $k \leq 5$, $\Omega_k^{\text{Spin}}(B\text{SL}(2, \mathbb{Z}))$ and $\Omega_k^{\text{Spin-Mp}(2, \mathbb{Z})}(\text{pt})$ were known before, but the rest of our computations are new. The results of our computations are presented in Tables 1, 2 and 3.

Our primary tool for making these computations is the Adams spectral sequence. Though we are far from the first in the physics literature to use it, this method has a reputation for being difficult to understand. Therefore we start in Section 10 by introducing some definitions and concepts that will be helpful for understanding the calculations we make in the next few sections. In Section 11 we summarize the essential facts needed to follow our Adams spectral sequence arguments: how to compute the $\mathcal{A}(1)$ -module structure on cohomology, how to determine the E_2 -page from this data, and a few standard tricks for resolving differentials and extension questions. Then we march through the three main computations.

- In Section 12, we determine $\Omega_k^{\text{Spin}}(B\text{SL}(2, \mathbb{Z}))$. We show how to determine this in terms of $\Omega_k^{\text{Spin}}(B\mathbb{Z}/4\mathbb{Z})$ and $\Omega_k^{\text{Spin}}(B\mathbb{Z}/3\mathbb{Z})$, both of which can quickly be reduced to things already in the mathematics literature.
- In Section 13, we compute $\Omega_k^{\text{Spin-Mp}(2, \mathbb{Z})}(\text{pt})$. The result is the same, except that instead of $\Omega_k^{\text{Spin}}(B\mathbb{Z}/4\mathbb{Z})$, we get $\Omega_k^{\text{Spin-Z}/8\mathbb{Z}}(\text{pt})$. This bordism theory has been studied in dimensions 5 and below by Campbell [109], Hsieh [110], and Davighi-Lohitsiri [111]; we extend their computations to dimension 11.
- In Section 14, we address $\Omega_k^{\text{Spin-GL}^+(2, \mathbb{Z})}(\text{pt})$. In this case, $\Omega_k^{\text{Spin}}(B\mathbb{Z}/3\mathbb{Z})$ is replaced with $\Omega_k^{\text{Spin}}(BD_6)$, which we compute with the Atiyah-Hirzebruch spectral sequence; and $\Omega_k^{\text{Spin-Z}/8\mathbb{Z}}(\text{pt})$ is replaced with a bordism theory we call $\Omega_k^{\text{Spin-D}_{16}}(\text{pt})$, which we compute with the Adams spectral sequence.

With this very brief overview of the upcoming sections we will now go into the details of the derivation and the construction of the generating manifolds. Since this part contains many mathematical results that are of interest beyond their applications in physics, we will also adopt a more mathematical style in this part. Wherever possible, we try to provide a paragraph that is intended to add some physical intuition to the more technical procedures and which is indicated by a heading “**Physics intuition**”.²³

²³Of course this should neither dissuade mathematicians from reading these paragraphs, nor physicists from delving into the full details of the construction.

10 Preliminaries on computations

The bordism computations we undertake require some tools and concepts which are standard in algebraic topology, but are not as well-known in physics. In this subsection, we briefly review these ideas and point the interested reader to more complete references.

10.1 Classifying spaces

In this subsection we go over some basics on classifying spaces, their tautological bundles, and Lashof's approach to different kinds of bordism. We do this both to provide background to the reader and to standardize notation.

Let G be a topological group. Then there is a space BG , called the *classifying space* of G , which has the following properties.

- BG is the quotient of a contractible space EG by a free G -action. Said differently, $EG \rightarrow BG$ is a principal G -bundle with EG contractible. This principal G -bundle is called the *universal* or *tautological* principal G -bundle.
- For any space X , isomorphism classes of principal G -bundles $P \rightarrow X$ are equivalent to homotopy classes of maps $f: X \rightarrow BG$; in one direction, this equivalence is achieved by setting $P := f^*(EG)$.

Cohomology classes for BG define characteristic classes for principal G -bundles: given $c \in H^*(BG)$, set $c(P) := f^*(c)$ if f is the classifying map for $P \rightarrow X$.

BG is only well-defined up to homotopy type. Generally classifying spaces are large: for example, the homotopy type $B\mathbb{Z}/2\mathbb{Z}$ can be realized by infinite-dimensional real projective space, but not by any finite-dimensional CW complex.

When G is a matrix group, it has a canonical representation on a vector space V , so we can define a *tautological vector bundle* $\mathcal{T}_G \rightarrow BG$ by the following “mixing construction:”

$$\mathcal{T}_G := EG \times_G V := EG \times G / ((x \cdot g, v) \simeq (x, g \cdot v)). \quad (10.1)$$

For example, $\mathcal{T}_{O(n)} \rightarrow BO(n)$ is a rank- n real vector bundle; $\mathcal{T}_{U(1)} \rightarrow BU(1)$ is a complex line bundle; and so on. We use the notation $\sigma \rightarrow \mathbb{RP}^n$ to denote the tautological line bundle on \mathbb{RP}^n , which is the pullback of $\mathcal{T}_{B\mathbb{Z}/2\mathbb{Z}} \rightarrow B\mathbb{Z}/2\mathbb{Z}$ by the inclusion $\mathbb{RP}^n \hookrightarrow \mathbb{RP}^\infty = B\mathbb{Z}/2\mathbb{Z}$.

Let $O := \varinjlim_n O(n)$. Concretely, one can think of this as the union of all orthogonal groups of all sizes, where we include $O(n)$ in O_{n+1} by sending $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. This is a topological group, and its classifying space BO classifies *rank-zero virtual vector bundles*, i.e., pairs of vector bundles V and W such that $\text{rank}(V) = \text{rank}(W)$. We think of this data as “ $V - W$,” in much the same way that the integers are defined as equivalence classes of

pairs of natural numbers (p, q) and thought of as $p - q$. That is, a map $X \rightarrow BO$ is equivalent data to a rank-zero virtual vector bundle $V - W \rightarrow X$. As before, this correspondence arises as the pullback of a tautological bundle $\mathcal{T}_{BO} \rightarrow BO$. Analogues of this story hold for other families of Lie groups, defining spaces $B\mathrm{SO}$, $B\mathrm{Spin}$, etc.

Any actual vector bundle $V \rightarrow X$ of rank n defines a rank-zero virtual vector bundle $V - \underline{\mathbb{R}}^n$, hence a map to BO . This construction is *stable* in the sense that V and $V \oplus \underline{\mathbb{R}}$ define isomorphic virtual vector bundles.

Definition 10.2. A *tangential structure* is a map $\xi: B \rightarrow BO$. A ξ -structure on a vector bundle $V \rightarrow X$ is data of a lift of the classifying map $f: X \rightarrow BO$ to a map $\tilde{f}: X \rightarrow B$ such that $\xi \circ \tilde{f} = f$:

$$\begin{array}{ccc} & B & \\ & \swarrow \tilde{f} \quad \downarrow \xi & \\ X & \xrightarrow{f} & BO \end{array} \tag{10.3}$$

If M is a manifold, a ξ -structure on M means a ξ -structure on TM .

For example, if ξ is $B\mathrm{SO} \rightarrow BO$, a ξ -structure is equivalent to an orientation; for $B\mathrm{Spin} \rightarrow BO$, we obtain a Spin structure; and for $BO \times BG \rightarrow BO$, we obtain a principal G -bundle.

Suppose M is a manifold with boundary ∂M , and let $\nu \rightarrow \partial M$ be the normal bundle to the inclusion $\partial M \hookrightarrow M$. Then $TM|_{\partial M} \cong T(\partial M) \oplus \nu$.²⁴ Moreover, the outward unit normal vector field trivializes ν , so $T(\partial M) \oplus \underline{\mathbb{R}} \cong TM|_{\partial M}$. Thus the virtual vector bundles defined by $T(\partial M)$ and $TM|_{\partial M}$ are isomorphic, so a ξ -structure on M induces a ξ -structure on ∂M .

Because ξ -structures restrict to boundaries, we can define bordism of manifolds equipped with ξ -structures: a closed manifold M with ξ -structure is trivial in ξ -bordism if M bounds a compact manifold W with ξ -structure, such that the identification $M \cong \partial W$ can be made compatible with the ξ -structures on both sides. This general formalism is due to Lashof [112]; the bordism group of n -manifolds with ξ -structure is denoted Ω_n^ξ . It is often the case that ξ is a map $BG \rightarrow BO$, e.g., oriented and Spin bordism; in that case, the ξ -bordism groups are typically denoted Ω_*^G . For example, Ω_*^{SO} refers to the oriented bordism groups.

Remark 10.4 (Multiplicative structure). The product of two oriented manifolds has an induced orientation, and this is compatible with the bordism equivalence relation, making Ω_*^{SO} into a \mathbb{Z} -graded commutative ring. This is likewise true for many other bordism theories, including G -bordism for $G = \mathrm{O}$, Spin, Spin^c , String, and U, though it is not always true, e.g., for $G = \mathrm{Pin}^\pm$. When this is true, the multiplication lifts from bordism rings to ring structures on the corresponding Thom spectra, which can be thought of as expressing the naturality of this multiplication with respect to ξ -bordism groups of spaces. Determining the ring structure, when present, was classically an important part of bordism theory.

²⁴To obtain such a splitting, one needs to choose a Riemannian metric on ∂M , but the space of such metrics is contractible. Therefore from the perspective of homotopy theory, which is all that is needed in this section, the specific choice does not matter.

When ξ -bordism is not a ring, it is often a module over some other kind of bordism which is a ring. For example, the product of a Spin manifold and a Pin $^+$ manifold has a canonical Pin $^+$ structure, making $\Omega_*^{\text{Pin}^+}$ into an Ω_*^{Spin} -module.

From a physics point of view, these ring and module structures provide information about compactifications. In the framework of the cobordism conjecture, if a class $x \in \Omega_k^\xi$ can be represented by a product $[M \times N]$ of ξ -manifolds,²⁵ then a natural candidate for the defect that the cobordism conjecture predicts for the class $[M \times N]$ is the compactification on M of the defect predicted for N , although there may be obstructions to wrapping the defect on M , as described briefly in Section 6.8 and various other places of the Chronicles.

There is a similar interpretation of ring and module structures in the bordism classification of invertible field theories [114, 105]: a unitary invertible n -dimensional topological field theory on ξ -manifolds is determined by its partition function Z , which is a bordism invariant $Z: \Omega_n^\xi \rightarrow \mathbb{C}^\times$. Compactifying this theory on a k -dimensional ξ -manifold M yields another unitary invertible TFT, classified by the bordism invariant

$$\Omega_{n-k}^\xi \xrightarrow{-\times[M]} \Omega_n^\xi \xrightarrow{Z} \mathbb{C}^\times. \quad (10.5)$$

Here we assumed ξ -bordism is a ring, but the story generalizes to modules too.

▷ **Physics intuition:** Classifying spaces BG are also well-known in physics where they define the various physically distinct gauge backgrounds, the isomorphism classes of principal G -bundles, of a gauge theory with gauge group G . A simple example is a theory with gauge group U(1) in which case the classifying space is given by

$$BU(1) = \mathbb{CP}^\infty = K(\mathbb{Z}, 2), \quad (10.6)$$

where $K(2, \mathbb{Z})$ denotes the Eilenberg-Mac Lane space with $\pi_2(K(\mathbb{Z}, 2)) = \mathbb{Z}$ and all other homotopy groups π_i with $i \neq 2$ vanishing. The cohomology of $BU(1)$ is generated by a single element in degree 2 which pulls back on the spacetime to the first Chern class, i.e., the field strength,

$$c_1 = \frac{1}{2\pi} F \in H^2(M; \mathbb{Z}), \quad (10.7)$$

of the gauge bundle specified by the classifying map f from spacetime M into $BU(1)$. One can also specify the background of higher-form gauge fields by maps into higher classifying spaces. For example, a 1-form symmetry gauge theory with gauge group U(1) (as specified

²⁵Here we are being a little imprecise about which tangential structures are placed on M , N , and $M \times N$. In this paper M has a Spin structure, and N as well as $M \times N$ have ξ -structures in a setting where ξ -bordism is a module over Spin bordism. The full details of the relationship between compactifications and tangential structures are subtle; see Schommer-Pries [113, §9] for a careful analysis.

by a 2-form gauge potential) is described by maps into

$$B^2\mathrm{U}(1) = BK(\mathbb{Z}, 2) = K(\mathbb{Z}, 3), \quad (10.8)$$

which once more is described by an Eilenberg-Mac Lane space containing information about fluxes in $H^3(M; \mathbb{Z})$. \triangleleft

10.2 Working one prime at a time

Fix a prime p , and let $\mathbb{Z}_{(p)}$ denote the ring of rational numbers whose denominators are not divisible by p . Tensoring an Abelian group with $\mathbb{Z}_{(p)}$ throws out information that is prime to p . For example, if A is a finitely generated Abelian group, there is an isomorphism

$$A \cong \mathbb{Z}^r \oplus (\mathbb{Z}/p_1^{e_1}\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/p_n^{e_n}\mathbb{Z}) \quad (10.9)$$

for some primes p_1, \dots, p_n , and nonnegative integers r, e_1, \dots, e_n . Then $A \otimes \mathbb{Z}_{(p)}$ is a direct sum of $\mathbb{Z}_{(p)}^r$ and the $\mathbb{Z}/p_i^{e_i}\mathbb{Z}$ summands for which $p_i = p$. In particular, given $A \otimes \mathbb{Z}_{(p)}$ for all primes p , A is determined up to isomorphism.

Let E and F be generalized homology theories. A map $E \rightarrow F$ is a *p -local equivalence* if for all spaces X , $E_*(X) \otimes \mathbb{Z}_{(p)} \rightarrow F_*(X) \otimes \mathbb{Z}_{(p)}$ is an isomorphism. In homotopy theory, it is very common to calculate E -homology by finding p -local equivalences to simpler theories at different primes, then putting everything back together afterwards. We will use this approach, both directly in Section 10.5 and 10.6, and also indirectly in analyzing the p -torsion in bordism groups for different primes p separately. When we say “ p -locally” or “the p -primary part,” we mean working up to p -local equivalences, or equivalently throwing out torsion that is prime to p .²⁶

10.3 Bluff your way through spectra

A few steps in our computations require saying the word “spectrum,” in the sense of homotopy theory. The purpose of this subsection is not to give a definition or proper introduction, but to provide just enough information so that a reader without a background in algebraic topology can follow those steps in our computations. For more in-depth references, see Freed-Hopkins [105, Section 6.1] or Beaudry-Campbell [115, Section 2], or Schwede [116] for a more homotopical and in-depth perspective.

▷ **Physics intuition:** Spectra are essential in the study of generalized (co)homology theories such as bordism. Generalized (co)homology classes of X are described by constructing maps

²⁶“Equivalently” is not quite true, but it is true for all spaces or spectra whose homology groups are finitely generated Abelian groups. Essentially all situations one could reasonably encounter in these sorts of computations meet this criterion.

to the associated spectrum, then taking homotopy classes.

For generalized cohomology theories the cohomology groups of a space X are given by homotopy classes of maps from X to the respective spectrum. This is very similar to what classifying spaces do to define the various inequivalent principal G -bundles as we discussed above. However, spectra know more than classifying spaces. For example they can have homotopy groups in negative degree, which means that they are not necessarily given by topological spaces. Since they are naturally very big (remember that already $BU(1) = \mathbb{CP}^\infty$ is infinite-dimensional), it is often easier to split them into pieces. That can be done by working with other spectra that are p -locally equivalent to the original spectrum and hence contains the same information at the prime p .

A relatively simple example is the Eilenberg-Mac Lane spectrum (Example 10.12) HA with A an Abelian group, for which one has

$$H^k(X; A) = [X, \Sigma^k HA], \quad (10.10)$$

i.e., the cohomology with coefficients in A . Here Σ^k is the k -fold suspension operation. Interpreting the classes in $H^k(X; A)$ as field strengths of various (higher-form) gauge fields,²⁷ we see that, in a sense, the spectrum HA does not only have information about BA , the classifying space of A , but simultaneously about all higher classifying spaces $B^n A$ as well as their relations. Likewise, more complicated spectra contain information about classifying spaces of Abelian n -groups, and in fact there is an equivalence between the category of spectra whose homotopy groups are only nonzero in degrees $0 \leq k \leq n - 1$ and a category of Abelian n -groups [117]. See [118, 119] for some applications of this idea in the mathematical physics literature.

If one can decompose A into simpler Abelian groups, one can also decompose HA : for example, if $A = \mathbb{Z}/6\mathbb{Z} = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})$, $H\mathbb{Z}/6\mathbb{Z} \simeq (H\mathbb{Z}/2\mathbb{Z}) \vee (H\mathbb{Z}/3\mathbb{Z})$, inducing a direct sum on cohomology groups:

$$\begin{aligned} H^k(X; \mathbb{Z}/6\mathbb{Z}) &= [X, \Sigma^k H\mathbb{Z}/6\mathbb{Z}] = [X, \Sigma^k H\mathbb{Z}/2\mathbb{Z} \vee \Sigma^k H\mathbb{Z}/3\mathbb{Z}] \\ &= H^k(X; \mathbb{Z}/2\mathbb{Z}) \oplus H^k(X; \mathbb{Z}/3\mathbb{Z}). \end{aligned} \quad (10.11)$$

Therefore at $p = 2$, we would only have to worry about $H\mathbb{Z}/2\mathbb{Z}$, and at $p = 3$ we would only have to worry about $H\mathbb{Z}/3\mathbb{Z}$. This is a somewhat simple example, but we will take advantage of analogous p -local simplifications of bordism spectra to reduce the evaluation of bordism groups to either known or more tractable calculations, as we discuss below in Section 10.5 and 10.6. We are, however, not only interested in Spin bordisms but more general tangential structures such as Spin-Mp(2, \mathbb{Z}) and Spin-GL⁺(2, \mathbb{Z}). Again there is a trick to relate these to Spin bordism called “shearing,” which we discuss in Section 10.4. \triangleleft

Spectra in the sense of homotopy theory were invented and named by Lima [120, 121]; they

²⁷See the “Physics intuition” part on classifying spaces around equation (10.6).

are etymologically unrelated to spectra in algebraic geometry, functional analysis, quantum mechanics, etc. Here are two key facts about spectra:

- Spectra behave very much like topological spaces: they have homotopy, homology, and cohomology groups, and one can perform operations such as the suspension of a spectrum, or the spectrum of maps between a space and a spectrum, or two spectra; there is also a notion of homotopy equivalence of spectra.
- Spectra represent generalized (co)homology theories. For every spectrum E , there is a generalized homology theory $h_*(X) := \pi_*(X \wedge E)$ and a generalized cohomology theory $h^*(X) = \pi_{-*}(\mathrm{Map}(X, E))$,²⁸ and every generalized homology or cohomology theory arises from a spectrum in this way.

There are a few important differences between spaces and spectra: spectra can have homotopy groups in negative degrees, and the suspension operation is invertible on spectra. That is, taking Σ^{-1} of a spectrum is a sensible operation even though taking Σ^{-1} of a space does not make sense. Also, there is not generally a cup product on the cohomology of a spectrum.

It is possible to take direct sums, etc., of generalized cohomology theories, and this lends spectra an algebraic flavor. It can be useful to think of spectra as akin to Abelian groups.

Example 10.12 (Eilenberg-Mac Lane spectra). Given an Abelian group A , there is a spectrum HA whose corresponding homology theory is $H_*(-; A)$ and whose cohomology theory is $H^*(-; A)$. HA is called an *Eilenberg-Mac Lane spectrum*.

Example 10.13 (Suspension spectra). Given a topological space X with chosen basepoint in X , there is a spectrum $\Sigma^\infty X$, called the *suspension spectrum* of X , whose corresponding homology theory is $\lim_{k \rightarrow \infty} \pi_{*+k}(- \wedge \Sigma^k X)$. When $X = S^0$, its suspension spectrum is denoted \mathbb{S} and called the *sphere spectrum*; its homology theory evaluated on a space X are the stable homotopy groups of X .

The sphere spectrum is an important object in stable homotopy theory, playing a role akin to the integers in the land of Abelian groups. Given an Abelian group A , there is a natural isomorphism $\mathbb{Z} \otimes A \xrightarrow{\cong} A$, making Abelian groups naturally into \mathbb{Z} -modules. Similarly, for every spectrum E , there is a natural homotopy equivalence $\mathbb{S} \wedge E \xrightarrow{\cong} E$, and there is a sense in which this makes spectra naturally into modules over \mathbb{S} . This has the concrete consequence that $\pi_*(\mathbb{S})$, called the *stable homotopy groups of the spheres*, is a graded ring, and for every spectrum E , $\pi_*(E)$ is a $\pi_*(\mathbb{S})$ -module. We use this fact a few times, e.g., in Lemma 13.22.

²⁸The minus sign here is an artifact of homological versus cohomological grading: stable homotopy groups have long exact sequences whose differentials lower degree, but in a generalized cohomology theory, we want the differential in a long exact sequence to raise degree. This nuance is not crucial for understanding the mathematics in this paper.

Example 10.14 (Thom spectra). Let X be a space and $V \rightarrow X$ be a virtual vector bundle, and let $f_V: X \rightarrow BO$ be the classifying map of V . Then there is a spectrum X^V called a *Thom spectrum* with two key properties: its cohomology resembles that of X , and its homotopy groups are bordism groups. Specifically:

- The *Thom isomorphism theorem* produces a natural isomorphism

$$U: H^*(X; \mathbb{Z}_{w_1(V)}) \xrightarrow{\cong} \widetilde{H}^{*+\text{rank}(V)}(X^V; \mathbb{Z}), \quad (10.15)$$

where $\mathbb{Z}_{w_1(V)}$ denotes the local system given by the orientation bundle of V . Often one sees the technically ambiguous notation $U := U(1)$; this class is called the *Thom class*. The analogous isomorphism is true for other coefficient groups, and for $\mathbb{Z}/2\mathbb{Z}$ cohomology the local coefficient system $(\mathbb{Z}/2\mathbb{Z})_{w_1(V)}$ is trivial, so the Thom isomorphism uses untwisted cohomology on both sides.

- The *Pontrjagin-Thom theorem* [122–125] identifies $\pi_{*+\text{rank}(V)}(X^V)$ with the bordism groups of manifolds with a *normal* f_V -structure, i.e. a lift of the classifying map $f_\nu: M \rightarrow BO$ of the stable normal bundle to a map $\tilde{f}_\nu: M \rightarrow X$ such that $f_\nu = f_V \circ \tilde{f}_\nu$.²⁹

Because of the shifts by $\text{rank}(V)$, it is common to replace V with the rank-zero virtual vector bundle $V - \underline{\mathbb{R}}^{\oplus \text{rank}(V)}$, often denoted $V - \text{rank}(V)$ for short, which allows one to drop the shifts in the descriptions above. We often do this in this paper.

When V is trivial and rank-0, X^V is the suspension spectrum $\Sigma^\infty X$ from Example 10.13. Thus given a space X with basepoint $i: \text{pt} \hookrightarrow X$ and a vector bundle $V \rightarrow X$, we obtain a map $\text{pt}^{i^*V} \rightarrow X^V$, i.e. a map $\mathbb{S} \rightarrow X^V$. We will use this fact a few times.

If G is a topological group with a map to the infinite orthogonal group O , we would like to obtain a Thom spectrum whose homotopy groups are bordism groups of manifolds with a G -structure on TM , rather than on the stable normal bundle. To do this, compose with the map $-1: BO \rightarrow BO$ before taking the Thom spectrum³⁰. The resulting spectrum is called a *Madsen-Tillman spectrum* and is denoted MTG . We will most frequently use $MTSO$, whose homotopy groups are the bordism groups of oriented manifolds; $MTSpin$, whose homotopy groups are Spin bordism groups; and variants of $MTSpin$.

10.4 Shearing

It is not always obvious how to write the symmetry type of interest in terms of Spin structures. For example, if one studies a theory with fermions and a $U(1)$ symmetry with an

²⁹Pontrjagin and Thom focused on a few specific normal structures; the idea to consider them in general is due to Lashof [112].

³⁰The intuition being that the normal and tangent bundles add up to a trivial bundle, so their stable equivalence classes add up to zero.

additional selection rule that bosonic fields have even charges and fermionic fields have odd charges, the symmetry type is Spin^c . Our Adams spectral sequence techniques compute Spin bordism groups, so we would like to express Spin^c structures in terms of Spin structures. “Shearing” is a general approach to describing a tangential structure ξ as equivalent to some sort of “twisted ξ' -structure,” where ξ' is easier to understand; for us, ξ' will always be Spin. Once we find this equivalence, Corollary 10.19 describes the ξ -bordism groups as ξ' -bordism groups of a Thom spectrum, giving us access to the spectral sequence techniques we discuss in the next section.

For example, a Spin^c structure on a vector bundle $E \rightarrow M$ is equivalent to a complex line bundle $L \rightarrow M$ and a Spin structure on $E \oplus L$; we will see how to pass this through Corollary 10.19 to deduce that

$$\Omega_*^{\text{Spin}^c}(\text{pt}) \cong \Omega_*^{\text{Spin}}((\text{BU}(1))^{\mathcal{T}_{\text{U}(1)} - 2}). \quad (10.16)$$

There are different perspectives on shearing, phrased more or less abstractly. Though some amount of homotopy theory is needed for the proofs, we have tried to express the statements of Definition 10.17 and Corollaries 10.19 and 10.23 in terms of groups, representations, and vector bundles, so that they are hopefully easier to use.

▷ **Physics intuition:** Shearing is also familiar in physical constructions in which fermions are charged under an internal symmetry. While uncharged fermions might be forbidden, charged fermions are allowed if the internal bundle compensates for the missing Spin structure. One of the most familiar examples is that of a Spin^c structure, for which one only allows fermions of odd charge under an internal $\text{U}(1)$ symmetry. The obstruction to a Spin structure, given by the second Stiefel-Whitney class w_2 , is compensated by a non-trivial field strength of the $\text{U}(1)$ gauge field. Extending the tangent bundle of spacetime with a complex line bundle associated to the $\text{U}(1)$ principal bundle one can define a Spin structure on this extended bundle, the Spin^c structure on spacetime. From this it is also clear that the twisted Spin structures will lead to a correlation between w_2 and the characteristic classes of the bundles describing the internal symmetries.

For example, \mathbb{CP}^2 has $w_2 \neq 0$, so it has no Spin structure, but it has a Spin^c structure in which the $\text{U}(1)$ -bundle is the unit circle bundle inside the canonical bundle.

For the cases of interest for our later discussion of type IIB string theory, the twisted Spin structures have a nice interpretation in terms of F-/M-theory. On the one hand, a $\text{Spin-Mp}(2, \mathbb{Z})$ structure on a manifold corresponds to a Spin structure on an associated 2-torus fibration over this manifold. It thus induces a Spin structure on the total space in F- and M-theory. A $\text{Spin-GL}^+(2, \mathbb{Z})$ structure, on the other hand, allows for orientation reversal and thus can be interpreted as a Pin^+ structure on a torus fibration, where the orientation reversal only happens in the fiber. This can be further lifted by the arguments above to a

Spin structure on a 5-torus fibration,³¹ where the additional three circles are invariant under $\mathrm{SL}(2, \mathbb{Z})$ transformations but flip orientation under application of the reflection operator. \triangleleft

Definition 10.17. Let X be a space and $V \rightarrow X$ be a virtual vector bundle. Then an (X, V) -twisted Spin structure on a vector bundle $E \rightarrow M$ is a map $f: M \rightarrow X$ and a Spin structure on $E \oplus f^*V \rightarrow M$.

With X and V fixed, (X, V) -twisted Spin structures define a symmetry type and thus a notion of bordism. We let $\Omega_{k+V}^{\mathrm{Spin}}(X)$ denote the Abelian group of bordism classes of k -manifolds with an (X, V) -twisted Spin structure.

Lemma 10.18 (Shearing). *Let $\mathcal{T}_{\mathrm{Spin}} \rightarrow B\mathrm{Spin}$ denote the tautological stable vector bundle, defined in Section 10.1. The symmetry type for (X, V) -twisted Spin structures is $\phi: B\mathrm{Spin} \times X \rightarrow BO$ with the map given by the vector bundle $-\mathcal{T}_{\mathrm{Spin}} \oplus V \rightarrow B\mathrm{Spin} \times X$.*

Proof. Given an (X, V) -twisted Spin structure on $E \rightarrow M$, namely a map $f: M \rightarrow X$ and a Spin structure on $E \oplus f^*V$, we obtain a map $M \rightarrow B\mathrm{Spin} \times X$ given by $(E \oplus f^*V, f)$, and using that homotopy classes of maps to $B\mathrm{Spin}$ are naturally identified with stable virtual Spin vector bundles. Conversely, given a map $(\psi, f): M \rightarrow B\mathrm{Spin} \times X$, take $\phi \circ \psi$ to obtain a Spin vector bundle E' , and let $E := E' - f^*V$; then we have a canonical (X, V) -twisted Spin structure on E . These two operations are inverses up to isomorphism, so every (X, V) -twisted Spin structure on any vector bundle on any space pulls back from $B\mathrm{Spin} \times X$ in a unique way up to homotopy, which is what we wanted to prove. \square

The Pontrjagin-Thom theorem then implies

Corollary 10.19. *There is a natural isomorphism $\Omega_{*+V}^{\mathrm{Spin}}(X) \cong \Omega_*^{\mathrm{Spin}}(X^{V-\mathrm{rank}(V)})$, where $X^{V-\mathrm{rank}(V)}$ is the Thom spectrum of the virtual bundle $V - \underline{\mathbb{R}}^{\mathrm{rank}(V)} \rightarrow X$.*

That is, twisted Spin bordism is Spin bordism of something — and as we discussed in Example 10.14, that “something” has relatively easy-to-understand cohomology. This suggests that tools such as the Atiyah-Hirzebruch and Adams spectral sequences, which take that cohomology as input, can be used to compute twisted Spin bordism groups.

One way to produce vector bundles is to choose a space with a principal G -bundle $P \rightarrow M$ and a G -representation ρ on a vector space V . Then the *associated bundle*

$$P_\rho := P \times V / ((p \cdot g, v) \simeq (p, \rho(g) \cdot v)) \quad (10.20)$$

is a vector bundle with rank equal to $\dim(V)$. Often $P \rightarrow M$ is the universal bundle $EG \rightarrow BG$; in this case we sometimes write ρ as shorthand for $(EG)_\rho$.

³¹Note that a Pin^+ structure on the tangent bundle of a manifold TM is equivalent to a Spin structure on $TM \oplus \mathrm{Det}(TM)^{\oplus 3}$.

Lemma 10.21. *Let G and H be topological groups and $\rho: G \rightarrow \mathrm{SO}(d)$ be a representation. If there exists a pullback diagram*

$$\begin{array}{ccc} H & \longrightarrow & \mathrm{Spin} \\ \downarrow p_1 & & \downarrow p_2 \\ \mathrm{SO} \times G & \xrightarrow{(\mathrm{id}, \rho)} & \mathrm{SO} \times \mathrm{SO}(d) \xrightarrow{\oplus} \mathrm{SO}, \end{array} \quad (10.22)$$

then H -structures are naturally equivalent to $(BG, (EG)_\rho)$ -twisted Spin structures.

Proof. That (10.22) is a pullback square implies that a lift of a map $f: M \rightarrow \mathrm{SO} \times G$ across p_1 is equivalent to a lift of $\oplus \circ (\mathrm{id}, \rho) \circ f$ across p_2 . The former is the data on transition functions of an oriented vector bundle $E \rightarrow M$ and a principal G -bundle $P \rightarrow M$ to define an H -structure, and the latter is the data on transition functions to define a Spin structure on $E \oplus P_\rho \rightarrow M$, where $P_\rho \rightarrow M$ is the vector bundle associated to P and ρ . \square

Corollary 10.23. *Let*

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1 \quad (10.24)$$

be a central extension corresponding to the cohomology class $\omega \in H^2(BG; \mathbb{Z}/2\mathbb{Z})$. Let $\rho: G \rightarrow \mathrm{SO}(d)$ be a representation with $w_2(\rho) = \omega$. Then $\mathrm{Spin} \times_{\mathbb{Z}/2\mathbb{Z}} \tilde{G}$ -structures are naturally equivalent to $(BG, (EG)_\rho)$ -twisted Spin structures.

Proof. By Lemma 10.21, we just have to produce a diagram (10.22), where $H = \mathrm{Spin} \times_{\mathbb{Z}/2\mathbb{Z}} \tilde{G}$, and show it is a pullback diagram. The first step is to produce a map $\phi: \mathrm{Spin} \times_{\mathbb{Z}/2\mathbb{Z}} \tilde{G} \rightarrow H$. Let $g \in \tilde{G}$ denote the nonzero element of the $\mathbb{Z}/2\mathbb{Z} \subset \tilde{G}$ defined by the central extension (10.24). By taking Spin covers, the map

$$(\mathrm{id}, \rho) \circ p_1: \mathrm{Spin} \times_{\mathbb{Z}/2\mathbb{Z}} \tilde{G} \longrightarrow \mathrm{SO} \quad (10.25)$$

lifts to a map

$$\tilde{\phi}: \mathrm{Spin} \times \tilde{G} \longrightarrow \mathrm{Spin}; \quad (10.26)$$

we will show it descends to a map ϕ with domain $\mathrm{Spin} \times_{\mathbb{Z}/2\mathbb{Z}} \tilde{G}$ by showing that $(-1, g) \in \mathrm{Spin} \times \tilde{G}$ is sent to the identity; therefore $\tilde{\phi}$ descends to the quotient $\mathrm{Spin} \times_{\mathbb{Z}/2\mathbb{Z}} \tilde{G}$.

Now to show $\tilde{\phi}(-1, g) = 1$. Restricted to $\mathrm{Spin} \subset \mathrm{Spin} \times \tilde{G}$, $\tilde{\phi}$ is the identity,³² so $\tilde{\phi}(-1, 1) = -1$. And restricted to \tilde{G} , this map is the Spin cover of $\rho: G \rightarrow \mathrm{SO}$, so by definition it sends $g \mapsto -1$. Therefore $\tilde{\phi}(-1, g) = (-1)^2 = 1$ and we can descend to $\mathrm{Spin} \times_{\mathbb{Z}/2\mathbb{Z}} \tilde{G}$.

³²One way to think about this is that, downstairs on the map $\mathrm{SO} \rightarrow \mathrm{SO}$, this map takes the direct sum with an identity matrix. The group SO consists of equivalence classes of special orthogonal matrices of any size, with A and $A \oplus I_n$ identified. So this map, and its Spin cover, are the identity.

Finally, we have to check that (10.22) is a pullback diagram, i.e., that the pullback of the two maps to SO is $\mathrm{Spin} \times_{\mathbb{Z}/2\mathbb{Z}} \tilde{G}$. We can again check on the two factors: it suffices to pull back further to SO and G and check that we obtain $\mathrm{Spin} \rightarrow \mathrm{SO}$ and $\tilde{G} \rightarrow G$. For $\mathrm{Spin} \rightarrow \mathrm{SO}$, this is tautological: pull back $\mathrm{Spin} \rightarrow \mathrm{SO}$ by the identity $\mathrm{SO} \rightarrow \mathrm{SO}$ and obtain $\mathrm{Spin} \rightarrow \mathrm{SO}$ again. For pulling back to G , and asking whether we get $\tilde{G} \rightarrow G$, this is asking precisely that $\tilde{G} \rightarrow G$ is the Spin cover of G for the representation ρ . This occurs precisely when $w_2(\rho)$ is the class of the central extension (10.24). \square

Remark 10.27. There are a few other approaches to proving shearing theorems. Freed-Hopkins [105, Section 10] identify classifying spaces of twisted Spin bordism groups with $B\mathrm{Spin} \times X$ by showing both spaces are homotopy pullbacks of the same diagram; their approach is also used in [109, 126–131]. Another approach uses a result of Beardsley [132], who works with Thom spectra directly.

▷ **Physics intuition:** Let us apply the general discussion above to the simple specific example of $\mathrm{Spin}\text{-}\mathbb{Z}/8\mathbb{Z}$, that will turn out to be useful in the following applications. First, we can define the central extension

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/8\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow 1, \quad (10.28)$$

defined by $\mathrm{Ext}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \simeq H^2(B\mathbb{Z}/4\mathbb{Z}; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, whose non-trivial element we will denote by ω . Now we choose a complex one-dimensional representation of $\mathbb{Z}/4\mathbb{Z}$, i.e., a map $\rho : \mathbb{Z}/4\mathbb{Z} \rightarrow \mathrm{U}(1)$, such that its associated line bundle over $B\mathbb{Z}/4\mathbb{Z}$, which we also denote by ρ , has

$$w_2(\rho) = \omega \in H^2(B\mathbb{Z}/4\mathbb{Z}; \mathbb{Z}/2\mathbb{Z}). \quad (10.29)$$

These are precisely the representations with odd charge $q \in \{1, 3\}$ in the sense that the map ρ is defined as

$$\rho : a \longmapsto e^{2\pi i qa/4}, \quad a \in \mathbb{Z}/4\mathbb{Z}. \quad (10.30)$$

We thus find that $\mathrm{Spin}\text{-}\mathbb{Z}/8\mathbb{Z}$ structures can be described as twisted $(B\mathbb{Z}/4\mathbb{Z}, \rho)$ -Spin structures. Using the Pontrjagin-Thom theorem we have

$$\Omega_*^{\mathrm{Spin}\text{-}\mathbb{Z}/8\mathbb{Z}}(\mathrm{pt}) \simeq \widetilde{\Omega}_*^{\mathrm{Spin}}((B\mathbb{Z}/4\mathbb{Z})^{\rho-2}), \quad (10.31)$$

and we successfully reduced the computation to the theory of Spin bordisms to which we can apply the Adams spectral sequence. To further simplify the evaluation we can split the bordism groups to their prime parts, as we will discuss next. \triangleleft

10.5 Splitting Spin bordism at odd primes

The homotopy theorists' philosophy of working one prime at a time makes for an effective way to compute Spin bordism: at each prime p , Spin bordism decomposes as a direct sum of easier-to-understand homology theories, and this information can be calculated separately and then put together into the final answer. For all odd primes p , the p -local story is similar; the 2-local description is more complicated, and we will go over it in the next subsection.

The map $\Omega_*^{\text{Spin}} \rightarrow \Omega_*^{\text{SO}}$ is a p -local equivalence, ultimately because $\text{Spin}(n) \rightarrow \text{SO}(n)$ is a double cover.³³ Brown-Peterson [134] show that for any odd prime p , Ω_*^{SO} splits p -locally as a sum of copies of *Brown-Peterson homology*, denoted BP .³⁴ The BP -homology of a point is the polynomial ring

$$BP_*(\text{pt}) \cong \mathbb{Z}_{(p)}[v_1, v_2, v_3, \dots], \quad (10.32)$$

where v_i has degree $2(p^i - 1)$. Work of Thom [122], Averbuh [135], and Milnor [136] showed that

$$\Omega_*^{\text{SO}} \otimes \mathbb{Z}_{(p)} \cong \mathbb{Z}_{(p)}[x_4, x_8, x_{12}, \dots], \quad (10.33)$$

where x_{4i} is in degree $4i$, i.e. it is a bordism class of $4i$ -dimensional manifolds. So to determine the precise way in which oriented bordism splits into copies of BP -homology at p , one just has to compare these two graded polynomial rings: begin with BP in degree 0, and then move upward, adding another copy of BP for each monomial not already accounted for.

For example, if $p = 3$, v_1 is in degree 4, so the 3-local coefficient groups of both BP_* and Ω_*^{SO} have a $\mathbb{Z}_{(3)}$ in degrees 0 and 4, and nothing else below degree 8. That is, for $k \leq 7$, $\Omega_k^{\text{SO}}(X) \otimes \mathbb{Z}_{(3)} \cong BP_k(X)$.

Continuing in this way, we find that for $k \leq 15$,

$$\Omega_k^{\text{SO}}(X) \otimes \mathbb{Z}_{(3)} \cong BP_k(X) \oplus BP_{k-8}(X) \oplus BP_{k-12}(X). \quad (10.34)$$

However, for $p \geq 5$, v_1 is in degree at least 8, so $BP_4(\text{pt}) = 0$. Therefore we need an additional BP_{k-4} summand to capture $\Omega_4^{\text{SO}} \cong \mathbb{Z}$:

$$\Omega_k^{\text{SO}}(X) \otimes \mathbb{Z}_{(5)} \cong BP_k(X) \oplus BP_{k-4}(X) \oplus BP_{k-8}(X) \oplus \dots \quad (10.35)$$

In this paper, we only need $p = 3$.

³³In a little more detail: the short exact sequence $1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \text{Spin} \rightarrow \text{SO} \rightarrow 1$ gives us a fiber bundle $\pi: B\text{Spin} \rightarrow B\text{SO}$ with fiber $B\mathbb{Z}/2\mathbb{Z}$. If p is an odd prime, $H^*(B\mathbb{Z}/2\mathbb{Z}; \mathbb{Z}/p\mathbb{Z})$ is trivial, meaning the map $H^*(B\text{SO}; \mathbb{Z}/p\mathbb{Z}) \rightarrow H^*(B\text{Spin}; \mathbb{Z}/p\mathbb{Z})$ is an isomorphism, e.g. using the Serre spectral sequence. The Thom isomorphism shows that the induced map of Thom spectra is an isomorphism in $\mathbb{Z}/p\mathbb{Z}$ cohomology, and the p -local stable Whitehead theorem [133, Chapitre III, Théorème 3] turns this into the desired p -local equivalence.

³⁴There is one kind of BP for each prime p . Unfortunately, standard notation is for p to be implicit.

10.6 Splitting Spin bordism 2-locally

When $p = 2$, BP does not appear as a summand in Spin bordism. Instead, we obtain ordinary mod 2 homology as well as two generalized homology theories related to K -theory.

- *Connective real K -theory*, denoted $ko_*(X)$. This is a variant of real K -theory which vanishes in negative degrees. Its homotopy groups follow the usual Bott periodicity $\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, 0, \mathbb{Z}, 0, 0, 0$, and then repeating, but starting in degree 0 and continuing upwards only: if $k < 0$, $ko_k = 0$.
- Another theory denoted $ko\langle 2 \rangle_*(X)$. This is built from KO -theory in a similar way to ko , but instead of asking for everything in degrees $k < 0$ to vanish, we ask for everything in degrees $k < 2$ to vanish, then shift down by 2 so that the lowest nonzero homotopy group is in degree 0. Thus the homotopy groups begin with $\mathbb{Z}/2\mathbb{Z}$ in degree 0, then $0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, 0, \mathbb{Z}, \dots$

Theorem 10.36 (Anderson-Brown-Peterson [137]). *There is a 2-primary isomorphism from Spin bordism to a sum of shifts of ko -theory, $ko\langle 2 \rangle$ -theory, and mod 2 homology. For $k < 16$, this isomorphism is of the form*

$$\Omega_k^{\text{Spin}}(X) \otimes \mathbb{Z}_{(2)} \cong (ko_k(X) \oplus ko_{k-8}(X) \oplus ko\langle 2 \rangle_{k-10}(X)) \otimes \mathbb{Z}_{(2)}. \quad (10.37)$$

Remark 10.38. In principle, one can extract the explicit decomposition generalizing (10.37) to $k \geq 16$ from Anderson-Brown-Peterson's paper, but they do not give a closed form. Freed-Hopkins [105, Figure 7] draw a picture which may be helpful for seeing the next several summands.

We therefore want to compute ko -homology and $ko\langle 2 \rangle$ -homology at the prime 2. For ko , there is a convenient and well-established trick: the Adams spectral sequence is especially simple. We will discuss this in Section 11.

For $ko\langle 2 \rangle$ -homology, we only need to know it in degrees 0 and 1, which is used in 10- and 11-dimensional Spin bordism, and there it is especially simple.

Lemma 10.39. *For $k \leq 1$, and X a space or connective spectrum, $ko\langle 2 \rangle_k(X) \cong H_k(X; \mathbb{Z}/2\mathbb{Z})$.*

Proof. The Postnikov truncation map $ko\langle 2 \rangle \rightarrow \tau_{\leq 1}(ko\langle 2 \rangle)$ is 1-connected by definition,³⁵ and $\tau_{\leq 1}(ko\langle 2 \rangle)$ has only one nonzero homotopy group, $\mathbb{Z}/2\mathbb{Z}$ in degree 0, so it is equivalent to $H\mathbb{Z}/2\mathbb{Z}$. So $ko\langle 2 \rangle$ has a 1-connected map to $H\mathbb{Z}/2\mathbb{Z}$, which implies the theorem statement. \square

In higher degrees, one can compute $ko\langle 2 \rangle$ -homology similar to ko -homology. See Freed-Hopkins [105, Section D.1] for how to do this with the Adams spectral sequence.

³⁵A map of spaces or spectra $X \rightarrow Y$ is n -connected if it induces an isomorphism on all homotopy groups in degrees n and below. This Postnikov truncation map is the universal example of such a 1-connected map.

Remark 10.40. There is a complex analogue of ko -theory, called ku ; it can be built from KU in an analogous way, displaying Bott periodicity only in nonnegative degrees: the homotopy groups of ku are $\mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}, \dots$. Anderson-Brown-Peterson's decomposition of Spin bordism in terms of ko -homology has an analogue: Spin c bordism decomposes as a sum of shifts of ku -homology and mod 2 homology [137–139].

There are several useful maps involving ko and ku , and the ways in which they interact occasionally help address differentials or extension questions in spectral sequences computing ko - or ku -homology. We use four maps.

- The *complexification* homomorphism $c: ko_n(X) \rightarrow ku_n(X)$ is the connective cover of the map $KO \rightarrow KU$ which complexifies a real vector bundle.
- *Realification*, or forgetting from a complex vector bundle to a real one, induces a map $R: ku_n(X) \rightarrow ko_{n-2}(X)$. The degree shift is related to complex Bott periodicity.
- A map $\eta: ko_n(X) \rightarrow ko_{n+1}(X)$, given by multiplying by the nonzero class in $ko_1(\text{pt}) \cong \mathbb{Z}/2\mathbb{Z}$; this lifts in Spin bordism to taking the product with S_p^1 .
- A map $b: ku_n(X) \rightarrow ku_{n+2}(X)$ which is the connective cover of the Bott periodicity isomorphism.

These maps interact quite nicely. We will use the following two facts.

- The composition $R \circ b \circ c: ko_n(X) \rightarrow ko_n(X)$ is multiplication by 2 (see [140, Theorem 1]).
- The three maps η , c , and R fit together into a long exact sequence [141, Section 12]³⁶

$$\dots \xrightarrow{R} ko_n(X) \xrightarrow{\eta} ko_{n+1}(X) \xrightarrow{c} ku_{n+1}(X) \xrightarrow{R} ko_{n-1}(X) \xrightarrow{\eta} \dots \quad (10.41)$$

11 What you need to know about the Adams spectral sequence

Some of our computations use the Adams spectral sequence. This is a standard tool in homotopy theory, but not (yet) in theoretical physics, so we summarize a few important facts here to allow the reader to follow along at a high level. For a more complete introduction including some things we gloss over, see Beaudry-Campbell's excellent paper [115].

³⁶Bott only considers the case of periodic K -theory, though his result implies the connective version we use. See Bruner-Greenlees [142, Section 4.1.B].

▷ **Physics intuition:** Before we describe the Adams spectral sequence explicitly, we give a brief introduction to spectral sequences themselves. Spectral sequences are used in order to determine certain generalized (co)homology groups. They do this via increasingly precise approximations described by the pages of a spectral sequence. The starting point is the second page, which usually can be obtained by ordinary (co)homology or something like it, which often can be found in the literature or determined by other means (as we will describe for the Adams spectral sequence for ko -homology below). Each page is bigraded, parametrized by two non-negative integers s and t . Moreover, on each page there are maps, the differentials d_r , that map from the (s, t) entry on the r^{th} page to the $(s + r, t + r - 1)$ entry.³⁷ These differentials fit together into chain complexes, and the next page of the spectral sequence, i.e., the next better approximation, is the homology of these chain complexes. Iterating this procedure one obtains the ∞ page of the spectral sequence $E_\infty^{s,t}$. Now the desired (co)homology groups h_n arise as an extension of the entries $E_\infty^{s,t}$ with $s + t = n$. In other words these entries form a filtration of the homology groups of interest. We therefore see that the spectral sequences can be seen as an iterative procedure to approximate and finally to determine generalized (co)homology groups. The specification of the action of the differentials as well as the evaluation of the extension problems are highly non-trivial problems, for which one can utilize several tricks, some of which will appear in the following. The Adams spectral sequence, for which it is more challenging to determine the second page, has the advantage that many of the differentials and extension questions can be accessed more straightforwardly. For the Atiyah-Hirzebruch spectral sequence, for which the second page reduces to ordinary (co)homology, it can be harder to formulate the differentials and extension questions. ◁

Though there are more general versions of the Adams spectral sequence, we focus on the one for computing ko -homology, which takes the form

$$E_2^{s,t} = \text{Ext}_{\mathcal{A}(1)}^{s,t}(H^*(X; \mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \implies ko_*(X)_2^\wedge. \quad (11.1)$$

The notation $(-)_2^\wedge$ means that this spectral sequence computes the 2-completion of the ko -homology of X . “2-completed” and “2-localized” are not exactly the same thing, but for finitely generated Abelian groups the 2-completion and 2-localization determine each other. All bordism groups we need to worry about in our computations are finitely generated, so this distinction will not cause any problems for us.

The E_2 -page is a little more complicated: for any space or spectrum X , $H^*(X; \mathbb{Z}/2\mathbb{Z})$ is a module over the *Steenrod algebra* \mathcal{A} of stable cohomology operations. $\mathcal{A}(1)$ is the subalgebra generated by Sq^1 and Sq^2 . Ext is a functor classifying extensions of $\mathcal{A}(1)$ -modules. The purpose of this section is to go over how to define and work with these algebraic objects. In the cases we consider in this paper, the $\mathcal{A}(1)$ -action on $H^*(X; \mathbb{Z}/2\mathbb{Z})$ and the corresponding

³⁷Different spectral sequences have different grading conventions; this is how differentials in the Adams spectral sequence work. The same is true for extension questions.

Ext groups can more or less be looked up, but the reader who wants to learn more on how to compute them should consult Beaudry-Campbell [115].

▷ **Physics intuition:** As discussed above, the Adams spectral sequence determines groups which appear in the full bordism group at prime 2. The starting point of the Adams spectral sequence, its second page, relies on extensions of the mod 2 cohomology of the space or spectrum under investigation. Moreover it is sensitive to an algebra structure of this extension generated by the Steenrod operators Sq^1 and Sq^2 . The big advantage of the Adams spectral sequence compared to other approaches, such as for example the Atiyah-Hirzebruch spectral sequence, is that the extra structure present determines a large number of differentials and extension questions for free. Analogous computations are notoriously difficult in many of the alternative spectral sequences.

While some of the pieces in the Adams spectral sequence as well as other spectral sequences have a physical interpretations, see e.g., [56, 143, 22], at this point we do not have a good intuition concerning the full picture. In the following we thus mainly restrict to a mathematical description of the involved techniques and their application for the cases under consideration and leave a physical analysis of the Adams spectral sequence for future work. See however [22] for a physical interpretation of the E_1 page of the Adams spectral sequence.
 □

11.1 The Steenrod algebra

The goal of this subsection is to explain what the Steenrod algebra \mathcal{A} and its subalgebra $\mathcal{A}(1)$ are, and to provide some techniques for computing $\mathcal{A}(1)$ -module structures on mod 2 cohomology. See [115, Section 3] for more information.

A *stable cohomology operation* (in mod 2 cohomology) is a natural transformation of Abelian groups $H^*(-; \mathbb{Z}/2\mathbb{Z}) \rightarrow H^{*+k}(-; \mathbb{Z}/2\mathbb{Z})$ which commutes with the suspension isomorphism. Under composition, the set of stable cohomology operations is a graded $\mathbb{Z}/2\mathbb{Z}$ -algebra, denoted \mathcal{A} and called the *Steenrod algebra*; its action on cohomology makes the mod 2 cohomology of any space or spectrum into an \mathcal{A} -module. Steenrod showed that \mathcal{A} is generated by operators called *Steenrod squares* $\text{Sq}^k: H^*(-; \mathbb{Z}/2\mathbb{Z}) \rightarrow H^{*+k}(-; \mathbb{Z}/2\mathbb{Z})$ subject to some relations. In this paper, we only need $\mathcal{A}(1)$, the subalgebra of \mathcal{A} generated by Sq^1 and Sq^2 .

▷ **Physics intuition:** We offer a potential physical interpretation of Sq^1 and Sq^2 . As discussed in [56], the Wu formula characterizes many Steenrod squares in the cohomology of a manifold by describing certain information associated to submanifolds and their normal bundles, with Sq^i roughly given by $w_i(N) \cup -$. Since we are only concerned with Spin and related structures, we only need to worry about $i = 1, 2$. This suggests that the $\mathcal{A}(1)$ -module structure on the mod 2 cohomology of a space knows about the realization of fermions in the space X as well as its subspaces, and perhaps there is even a direct interpretation in terms of possible brane worldvolumes. □

The definitions of Steenrod squares are fairly abstract, making it difficult to compute the \mathcal{A} - or $\mathcal{A}(1)$ -module structures on cohomology from the definition. Instead, one can generally determine these module structures using a few key properties. Let X be a space.

1. If $x \in H^k(X; \mathbb{Z}/2\mathbb{Z})$, then $\text{Sq}^k(x) = x^2$.
2. If $x \in H^n(X; \mathbb{Z}/2)$ and $n < k$, then $\text{Sq}^k(x) = 0$.
3. (Naturality) given a map $f: X \rightarrow Y$ and $x \in H^n(Y; \mathbb{Z}/2\mathbb{Z})$, $\text{Sq}^k(f^*x) = f^*\text{Sq}^k(x)$.
4. (Cartan formula) If X is a space and $x, y \in H^*(X; \mathbb{Z}/2\mathbb{Z})$, then

$$\text{Sq}^k(xy) = \sum_{i+j=k} \text{Sq}^i(x)\text{Sq}^j(y). \quad (11.2)$$

Here we allow $i, j = 0$: $\text{Sq}^0 = \text{id}$.

These properties actually uniquely characterize the Steenrod squares.

Remark 11.3. A few caveats: for spectra, most of these are false, in part because the cohomology groups of spectra do not carry a cup product. Steenrod squares are still natural in the above sense, though.

Another unrelated caveat is that the Cartan formula shows that the \mathcal{A} -module structure on the cohomology of a space is related to the cup product, but the cup product does not make cohomology into an \mathcal{A} -algebra.

Here are a few other useful facts about Steenrod squares. First, a formula for the Steenrod squares of Thom spectra (recall the introduction of Thom spectra and the Thom isomorphism U from Example 10.14).

Proposition 11.4. *Let $V \rightarrow X$ be a vector bundle and $Ux \in \tilde{H}^*(X^V; \mathbb{Z}/2\mathbb{Z})$. Then*

$$\text{Sq}^k(Ux) = \sum_{i+j=k} Uw_i(V)\text{Sq}^j(x). \quad (11.5)$$

That is, if U denotes the Thom class, i.e. the image of $1 \in H^0(X; \mathbb{Z}/2\mathbb{Z})$ under the Thom isomorphism, then $\text{Sq}^k(U) = Uw_k(V)$, and so one can compute Steenrod squares of Ux using the Cartan formula. This property is specific to Thom spectra.

Second, the Wu formula computes Steenrod squares of Stiefel-Whitney classes of vector bundles:

$$\text{Sq}^k(w_j(V)) = \sum_{i=0}^k \binom{j-k+i-1}{i} w_{k-i}(V)w_{j+i}(V). \quad (11.6)$$

So computing Steenrod squares is relatively routine:

- Proposition 11.4 computes Steenrod squares for a Thom spectrum X^V in terms of the Steenrod squares for X and the Stiefel-Whitney classes of V .
- The Cartan formula (11.2) determines the Steenrod squares of all of $H^*(X; \mathbb{Z}/2\mathbb{Z})$ in terms of the Steenrod squares of the generators of the cohomology ring.
- For a specific generator x , one can deduce some Steenrod squares automatically from items (1) and (2) above. Otherwise, one can try to realize $x = w_n(V)$ for some vector bundle V and use the Wu formula, or use naturality: find a space Y that is easier to understand, together with a map $f: X \rightarrow Y$ such that $x = f^*(y)$, where we understand $\text{Sq}^k(y)$.

Often one can gain a complete understanding of the $\mathcal{A}(1)$ -module structure on the mod 2 cohomology of a space or Thom spectrum by combining these techniques with computations already in the literature. For example, this suffices for all computations we need in this paper.

Example 11.7. Let $k > 1$; then $H^*(B\mathbb{Z}/2^k\mathbb{Z}; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[x, y]/(x^2)$, where the degrees of x and y are respectively $|x| = 1$ and $|y| = 2$ [144, Proposition 4.5.1]. We will determine the $\mathcal{A}(1)$ -module structure on this cohomology ring. Because x is degree 1, (2) implies $\text{Sq}^i(x) = 0$ for $i \geq 2$. Sq^0 is the identity, so $\text{Sq}^0(x) = x$, and since $|x| = 1$, $\text{Sq}^1(x) = x^2 = 0$ by (1).

For y , we know $\text{Sq}^0(y) = y$, $\text{Sq}^2(y) = y^2$ by (1), and $\text{Sq}^i(y) = 0$ for $i \geq 3$ by (2), so only Sq^1 is left. In Lemma 13.12, we prove that if $\rho: \mathbb{Z}/2^k\mathbb{Z} \rightarrow \text{U}(1)$ denotes the standard one-dimensional complex representation of $\mathbb{Z}/2^k\mathbb{Z}$ by rotations and $V_\rho := E\mathbb{Z}/2^k\mathbb{Z} \times_{\mathbb{Z}/2^k\mathbb{Z}} \mathbb{C} \rightarrow B\mathbb{Z}/2^k\mathbb{Z}$ is the associated complex line bundle, then $y = w_2(V_\rho)$. Because V_ρ is a complex line bundle, it is the pullback of the tautological bundle $\mathcal{T}_{\text{U}(1)} \rightarrow \text{BU}(1)$ by some map $f: B\mathbb{Z}/2^k\mathbb{Z} \rightarrow \text{BU}(1)$.

Because both Steenrod squares and Stiefel-Whitney classes are natural under pullback, $\text{Sq}^1(y) = f^*\text{Sq}^1(w_2(\mathcal{T}_{\text{U}(1)}))$, and $\text{Sq}^1(w_2(\mathcal{T}_{\text{U}(1)})) \in H^3(\text{BU}(1); \mathbb{Z}/2\mathbb{Z}) = 0$, so $\text{Sq}^1(y) = f^*(0) = 0$, finishing our calculation of Steenrod squares of y .

For a general element of $H^*(B\mathbb{Z}/2^k\mathbb{Z}; \mathbb{Z}/2\mathbb{Z})$ we can use the Cartan formula. For example, $\text{Sq}^1(xy) = \text{Sq}^1(x)y + x\text{Sq}^1(y) = 0$, and

$$\text{Sq}^2(xy) = \text{Sq}^2(x)y + \text{Sq}^1(x)\text{Sq}^1(y) + x\text{Sq}^2(y) = 0 + 0 + xy^2. \quad (11.8)$$

We will use this example in the proof of Theorem 13.15.

See Beaudry-Campbell [115, Section 3.4] for some more worked examples.

It is traditional to describe $\mathcal{A}(1)$ -modules pictorially, rather than just algebraically: one uses a straight vertical line from x to y to indicate $\text{Sq}^1(x) = y$, and a curve from z to w to indicate $\text{Sq}^2(z) = w$. This convention goes back at least to Mahowald-Milgram [145, Section 4] and is standard in the field. We give some examples in Figure 11.

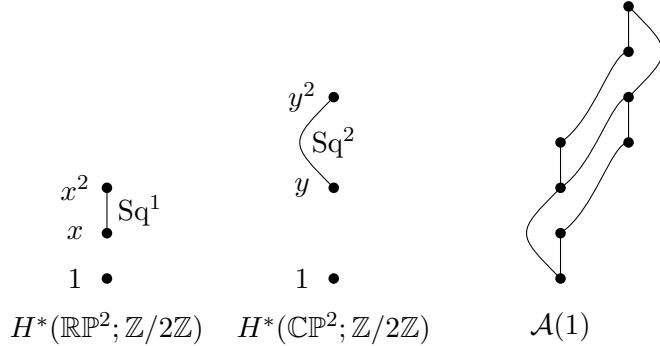


Figure 11: Drawing the $\mathcal{A}(1)$ -module structures on mod 2 cohomology. Left: $H^*(\mathbb{R}\mathbb{P}^2; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[x]/(x^3)$ with $|x| = 1$; the vertical line indicates that $\text{Sq}^1(x) = x^2$. Center: $H^*(\mathbb{C}\mathbb{P}^2; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[y]/(y^3)$ with $|y| = 2$; the curve indicates $\text{Sq}^2(y) = y^2$. Right: $\mathcal{A}(1)$ as a module over itself. All straight lines denote Sq^1 -actions and all curves denote Sq^2 -actions.

11.2 Determining Ext groups

The input to the Adams spectral sequence is the Ext groups of the $\mathcal{A}(1)$ -module structure on cohomology. If M is an $\mathcal{A}(1)$ -module, $\text{Ext}_{\mathcal{A}(1)}^{s,t}(M, \mathbb{Z}/2\mathbb{Z})$ has a lot of additional structure, and the most important takeaway of this section is what that structure is and how we display it in Ext charts; it will be important in determining differentials and extensions in the Adams spectral sequence.

These Ext groups are Abelian groups of equivalence classes of certain extensions of $\mathcal{A}(1)$ -modules, but one rarely if ever needs to work directly with the definition. In examples, the Ext groups needed for an Adams spectral sequence calculation of twisted Spin bordism can usually be looked up or calculated using a computer program, and if one needs to compute by hand, it is possible to use a long exact sequence to quickly reduce to computations in the literature. As always, Beaudry-Campbell [115, Section 4] is a good reference for additional information that we do not provide here.

Since $\mathcal{A}(1)$ is a graded algebra, we will work with graded $\mathcal{A}(1)$ -modules. If M is an $\mathcal{A}(1)$ -module and $k \in \mathbb{Z}$, we let $\Sigma^k M$ denote the $\mathcal{A}(1)$ -module which is the same underlying ungraded module, but with the gradings of all elements increased by k . It is common to think of Σ^k as k applications of a “shift operator” Σ , and as such one writes $\Sigma M := \Sigma^1 M$.

If M and N are $\mathcal{A}(1)$ -modules, $\text{Ext}_{\mathcal{A}(1)}^{s,t}(M, N)$ is an Abelian group of equivalence classes of extensions of the form

$$0 \longrightarrow \Sigma^t N \longrightarrow P_1 \longrightarrow \cdots \longrightarrow P_s \longrightarrow M \longrightarrow 0, \quad (11.9)$$

where P_1, \dots, P_s are $\mathcal{A}(1)$ -modules and all maps are $\mathcal{A}(1)$ -module maps. Two of these sequences $(P_i), (P'_i)$ are equivalent if there is a third sequence (Q_i) and $\mathcal{A}(1)$ -module maps $(P_i) \leftarrow (Q_i) \rightarrow (P'_i)$ commuting with the maps in $(P_i), (P'_i)$, and (Q_i) such that the maps on

M and on $\Sigma^t N$ are the identity (see [146, Tag 06XP, Definition 06XT]). Defining the Abelian group structure on $\mathrm{Ext}_{\mathcal{A}(1)}^{s,t}(M, N)$, called *Baer sum*, is a little involved, and we will not need to know the definition in this paper; we point the interested reader to Mac Lane [147, §III.5]. This perspective on Ext is due to Yoneda [148, Section 3.4]. For $s = 0$, there is an easier description of Ext: $\mathrm{Ext}_{\mathcal{A}(1)}^{0,t}(M, N) = \mathrm{Hom}_{\mathcal{A}(1)}(M, \Sigma^t N)$.

Remark 11.10. Given an extension of the form (11.9), which represents a class in $\mathrm{Ext}^{s,t}(M, N)$, together with a map $L \rightarrow M$, we can pull back (11.9) by taking the fiber products of the maps $P_i \rightarrow M$ with $L \rightarrow M$, and thereby obtain an extension representing a class in $\mathrm{Ext}^{s,t}(L, N)$. That is, when the second argument is held constant, Ext is a contravariant functor.

There is a product on Ext groups called the *Yoneda product* [148, Section 4] with signature

$$\mathrm{Ext}_{\mathcal{A}(1)}^{s_1, t_1}(L, N) \times \mathrm{Ext}_{\mathcal{A}(1)}^{s_2, t_2}(N, M) \longrightarrow \mathrm{Ext}_{\mathcal{A}(1)}^{s_1+s_2, t_1+t_2}(L, M). \quad (11.11)$$

The idea of this product is that, given two extensions of the form (11.9) with matching ends, one can glue them together. See [148, Section 4.2] or [115, Section 4.2] for the full story. Letting $L = N = M = \mathbb{Z}/2\mathbb{Z}$, the Yoneda product makes $\mathrm{Ext}_{\mathcal{A}(1)}^{*,*}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ into a $(\mathbb{Z} \times \mathbb{Z})$ -graded commutative $\mathbb{Z}/2\mathbb{Z}$ -algebra, and letting $L = N = \mathbb{Z}/2\mathbb{Z}$, the Yoneda product makes $\mathrm{Ext}_{\mathcal{A}(1)}^{*,*}(M, \mathbb{Z}/2\mathbb{Z})$ into a $(\mathbb{Z} \times \mathbb{Z})$ -graded $\mathrm{Ext}_{\mathcal{A}(1)}^{*,*}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ -module. We will refer to these as $\mathrm{Ext}(\mathbb{Z}/2\mathbb{Z})$ and $\mathrm{Ext}(M)$ respectively. The $\mathrm{Ext}(\mathbb{Z}/2\mathbb{Z})$ -module structure on $\mathrm{Ext}(M)$ provides quite a bit of information in the Adams spectral sequence; in cases where the Adams spectral sequence is more powerful than the Atiyah-Hirzebruch spectral sequence, this module structure is often the reason.

Here is what $\mathrm{Ext}(\mathbb{Z}/2\mathbb{Z})$ looks like.

Theorem 11.12 (Liulevicius [149, Theorem 3]).

$$\mathrm{Ext}(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[h_0, h_1, v, w]/(h_0h_1, h_1^3, vh_1, h_0^2w - v^2),$$

where $\deg(h_0) = (1, 1)$, $\deg(h_1) = (1, 2)$, $\deg(v) = (3, 7)$, and $\deg(w) = (4, 12)$.

Ext groups are traditionally displayed in a diagram, and this will make the algebra structure in Theorem 11.12 easier to visualize when we draw it in Figure 12. The standard conventions for drawing Ext of $\mathcal{A}(1)$ -modules are as follows. Each $\mathbb{Z}/2\mathbb{Z}$ summand in $\mathrm{Ext}^{s,t}$ is displayed as a dot at coordinates $(t-s, s)$; $t-s$ is called the *topological degree* and s the *filtration*. With this convention, the action of h_0 does not change the topological degree and increases the filtration by 1, so we use a vertical line connecting two dots to indicate that h_0 carries the lower-filtration $\mathbb{Z}/2\mathbb{Z}$ isomorphically to the higher-filtration $\mathbb{Z}/2\mathbb{Z}$. Likewise, the action of h_1 increases both the topological degree and the filtration by 1, so we depict it with a diagonal line. It is less common to draw the actions of v and w in Ext charts. With these conventions, we can draw $\mathrm{Ext}(\mathbb{Z}/2\mathbb{Z})$ in Figure 12.

Remark 11.13 (Using preexisting calculations of Ext groups). There are many calculations of Ext groups of $\mathcal{A}(1)$ -modules in the literature, and often they suffice for whatever calculations

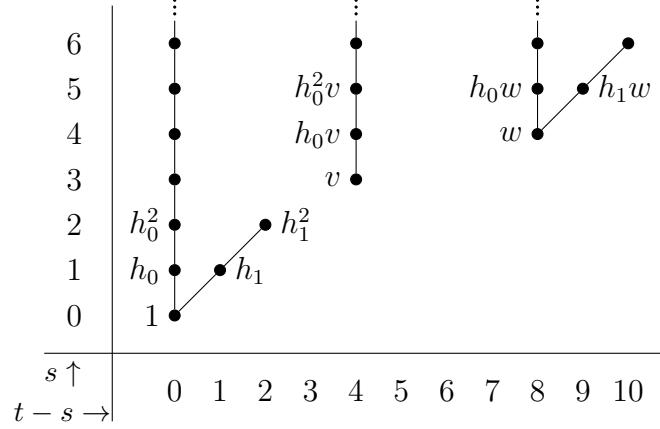


Figure 12: $\text{Ext}_{\mathcal{A}(1)}^{*,*}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$, with some elements labeled. Multiplication by w is injective, and the diagram continues in a similar way to the right. One can see some of the relations in Theorem 11.12 using this diagram: for example, $h_1^2 \neq 0$ and $h_1^3 = 0$, because traveling two steps from 1 on a diagonal line leads to a nonzero element, but traveling one more step leads to an empty square, corresponding to the zero group. The relations $h_0 h_1 = 0$ and $v h_1 = 0$ follow from the absence of a diagonal line (h_1 -action) out of h_0 and v . The relation $h_0^2 w = v^2$ is not depicted in this diagram.

one might need; this is the case for this paper. Beaudry-Campbell [115] provide calculations of Ext of many of the most common $\mathcal{A}(1)$ -modules; some more calculations can be found in [150–155, 109, 156, 157, 127, 130]. There are also computer programs for computing Ext written by Bruner [158] and Chatham-Chua [159].

The last thing we should mention here is that when it *is* necessary to know the Ext groups of a module and you cannot find them in the literature, one good trick is to use that a short exact sequence of $\mathcal{A}(1)$ -modules $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ induces a long exact sequence in Ext of the form

$$\dots \longrightarrow \text{Ext}^{s,t}(N) \longrightarrow \text{Ext}^{s,t}(M) \longrightarrow \text{Ext}^{s,t}(L) \xrightarrow{\delta} \text{Ext}^{s+1,t}(N) \longrightarrow \dots \quad (11.14)$$

The proof is purely formal, relying on a different but equivalent characterization of Ext as the derived functors of Hom . It is common to compute with this long exact sequence by drawing $\text{Ext}(L)$ and $\text{Ext}(N)$ both on the same chart; the boundary map increases the filtration by 1 and lowers the topological degree by 1. All maps in (11.14) commute with the $\text{Ext}(\mathbb{Z}/2\mathbb{Z})$ -action, which typically reduces the computation of the boundary map to a small number of calculations. We will give an example below; there are many more examples in [115], and a few more in [156, 130].

Example 11.15. Let $C\eta := \Sigma^{-2} \tilde{H}^*(\mathbb{CP}^2; \mathbb{Z}/2\mathbb{Z})$: it consists of two $\mathbb{Z}/2\mathbb{Z}$ summands in degrees 0 and 2, joined by a nonzero Sq^2 -action. One could equivalently define $C\eta =$

$\mathcal{A}(1)/(Sq^1, Sq^3)$. There is a short exact sequence of $\mathcal{A}(1)$ -modules

$$0 \longrightarrow \Sigma^2\mathbb{Z}/2\mathbb{Z} \longrightarrow C\eta \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0, \quad (11.16a)$$

which looks like this:

$$\begin{array}{ccccc} & \bullet & \longrightarrow & \bullet & \\ & \downarrow & & \nearrow & \\ \Sigma^2\mathbb{Z}/2\mathbb{Z} & & C\eta & & \mathbb{Z}/2\mathbb{Z} \end{array} \quad (11.16b)$$

We will use the induced long exact sequence in Ext to compute $\text{Ext}(C\eta)$. This can be looked up (e.g. [115, Figure 22]), but in the proof of Proposition D.13, we will need to know the action of $v \in \text{Ext}(\mathbb{Z}/2\mathbb{Z})$, which is not discussed in the cited reference.

To run the long exact sequence, we need to know $\text{Ext}(\Sigma^2\mathbb{Z}/2\mathbb{Z})$ and $\text{Ext}(\mathbb{Z}/2\mathbb{Z})$. The latter is Theorem 11.12; for the former, shifting an $\mathcal{A}(1)$ -module simply shifts the t -grading on its Ext by the same amount. Therefore we can draw the long exact sequence in Ext associated to (11.16) in Figure 13, top.

For most of the boundary maps, either their source or their target vanishes; in the range visible in Figure 13, top, only the four boundary maps pictured (the solid and dashed arrows) could be nonzero. Moreover, because the long exact sequence commutes with the action of $\text{Ext}(\mathbb{Z}/2\mathbb{Z})$, the three dashed boundary maps are determined by the solid boundary map, via the actions of h_1 , w , and h_1w . We will show the solid boundary map is an isomorphism, so that the dashed boundary maps are too.

To show the solid boundary map is an isomorphism, it suffices by exactness to show that $\text{Ext}^{0,2}(C\eta) = 0$, since we already know $\text{Ext}^{1,2}(\Sigma^2\mathbb{Z}/2\mathbb{Z}) = 0$. To show $\text{Ext}^{0,2}(C\eta)$ vanishes, use that it is identified with $\text{Hom}_{\mathcal{A}(1)}(C\eta, \Sigma^2\mathbb{Z}/2\mathbb{Z})$, which vanishes: since $C\eta$ is a cyclic $\mathcal{A}(1)$ -module, maps out of $C\eta$ are determined by their values on the generator, which is in degree 0; since $\Sigma^2\mathbb{Z}/2\mathbb{Z}$ has no nonzero elements in degree 0, a map $C\eta \rightarrow \Sigma^2\mathbb{Z}/2\mathbb{Z}$ must vanish. This finishes the calculation; we draw the final answer in Figure 13, bottom. There are two things we want to mention about the result of the computation.

- For all n , the v -action on $\text{Ext}(C\eta)$ carries the blue tower in topological degree $4n$ injectively into the blue tower in topological degree $4n + 4$, and likewise carries the red tower in topological degree $4n + 2$ injectively into the red tower in topological degree $4n + 6$. This is because the maps in the long exact sequence respect the $\text{Ext}(\mathbb{Z}/2\mathbb{Z})$ -action, so it suffices to know this fact for $\text{Ext}(\Sigma^2\mathbb{Z}/2\mathbb{Z})$ and $\text{Ext}(\mathbb{Z}/2\mathbb{Z})$; here, the $\text{Ext}(\mathbb{Z}/2\mathbb{Z})$ -action is multiplication.
- In a short exact sequence of $\mathcal{A}(1)$ -modules, the $\text{Ext}(\mathbb{Z}/2\mathbb{Z})$ -action on Ext of the middle term cannot always be determined from the long exact sequence: there may be “missing actions.” For example, acting by h_0 is an isomorphism $\text{Ext}^{2,6}(C\eta) \xrightarrow{\cong} \text{Ext}^{3,7}(C\eta)$ (and similarly in topological degree 12), but looking at Figure 13, our method cannot see

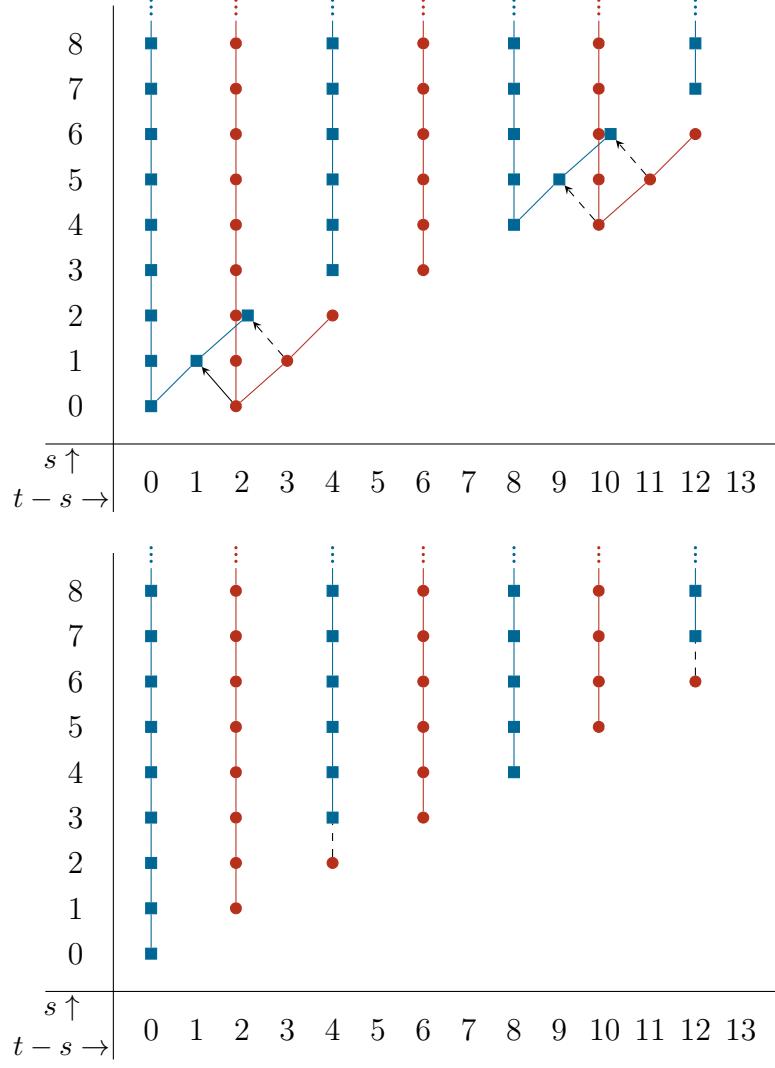


Figure 13: Top: the long exact sequence in Ext associated to the short exact sequence of $\mathcal{A}(1)$ -modules (11.16). The solid and dashed arrows are boundary maps. Bottom: $\text{Ext}(C\eta)$ as calculated by this long exact sequence. The dashed lines indicate h_0 -actions not visible to the long exact sequence, which must be calculated another way; see Example 11.15 for more details.

this. To check for missing actions, one has to compute another way, such as fitting the module into another long exact sequence. Beaudry-Campbell [115, Example 4.5.6] calculate $\text{Ext}(C\eta)$ in a different way, and can see these hidden actions.

11.3 Differentials

The differentials on the E_r -page of the Adams spectral sequence have signature $d_r: E_r^{s,t} \rightarrow E_r^{s+r, t+r-1}$. Thus when the Adams E_r -page is displayed with $(t-s, s)$ coordinates as is standard, this differential goes one tick to the left and r ticks upwards.

In general, Adams spectral sequence differentials are very difficult: for example, in the Adams spectral sequence computing the stable homotopy groups of the spheres, entire papers are dedicated to computing single differentials! See [160–162] for some examples. The “Mahowald uncertainty principle” is the belief that any single calculation to compute Adams differentials will leave infinitely many unaddressed.

However, Adams differentials are also extremely constrained: there is a lot of structure on the E_r -page of the Adams spectral sequence, and differentials are compatible with this structure in various ways. This has the consequence that determining a differential in the Adams spectral sequence tends to either be very easy or very hard.

In our situation, things are good: there is not enough room for differentials to be too difficult in degrees 12 and below, and the simpler Adams spectral sequence for ko -theory tends to have easier differentials as well. We will go over a few standard tools for computing differentials; they will suffice to determine all Adams differentials in this paper.

11.3.1 Differentials are equivariant for the $\mathrm{Ext}(\mathbb{Z}/2\mathbb{Z})$ -action

The first, and most useful, fact is that differentials are equivariant for the $\mathrm{Ext}(\mathbb{Z}/2\mathbb{Z})$ -action on the E_r -page. That is, if $a \in \mathrm{Ext}(\mathbb{Z}/2\mathbb{Z})$ and $x \in E_r^{s,t}$, $d_r(a \cdot x) = a \cdot d_r(x)$. For example, this can be used to show that all differentials in the Adams spectral sequence for $ko_*(\mathrm{pt})$ vanish: the E_2 -page is $\mathrm{Ext}(\mathbb{Z}/2\mathbb{Z})$. Looking at Figure 12, it looks like there could be a d_r differential from h_1 to h_0^{r+1} , but $h_0 h_1 = 0$ and $h_0(h_0^{r+1}) \neq 0$. If $d_r(h_1) \neq 0$, then $d_r(h_0 h_1) = h_0(d_r(h_1)) = h_0(h_0^{r+1})$, so d_r sends 0 to something nonzero, which is a contradiction.

In summary, the only way for a differential to kill elements in an h_0 -tower is with another h_0 -tower. In a similar way, if $h_1 x = 0$ and $h_1 y \neq 0$, then $d_r(x)$ cannot equal y . We will use this fact often.

11.3.2 Margolis' theorem

Margolis' theorem is another very useful tool.

Theorem 11.17 (Margolis [163]). *A splitting $H^*(X; \mathbb{Z}/2\mathbb{Z}) \cong \Sigma^k \mathcal{A}(1) \oplus M$ as $\mathcal{A}(1)$ -modules lifts to a splitting $ko \wedge X \simeq \Sigma^k H\mathbb{Z}/2\mathbb{Z} \vee X'$ of spectra.*

This is a bit abstract, so we go over some more explicit consequences. $\mathrm{Ext}(\Sigma^k \mathcal{A}(1))$ consists of a single $\mathbb{Z}/2\mathbb{Z}$ summand in bidegree $s = 0, t = k$. This is a direct summand of the E_2 -page of the Adams spectral sequence computing $ko_*(X)$; Margolis' theorem says this splitting comes from a splitting of spectra, so the entire Adams spectral sequence, including all differentials and extension problems, splits into the single $\mathbb{Z}/2\mathbb{Z}$ coming from $\Sigma^k \mathcal{A}(1)$ and everything else. That is:

Corollary 11.18. *Let $S \subset \text{Ext}(H^*(X; \mathbb{Z}/2\mathbb{Z}))$ be a $\mathbb{Z}/2\mathbb{Z}$ summand coming from a $\Sigma^k \mathcal{A}(1)$ summand in $H^*(X; \mathbb{Z}/2\mathbb{Z})$. Then all differentials to and from S vanish, and no element of S participates in a nonsplit extension on the E_∞ -page.*

This will zero out a lot of differentials in the Adams spectral sequence for $\text{Spin-GL}^+(2, \mathbb{Z})$ bordism for us.

11.3.3 Miscellaneous tricks

The Adams spectral sequence is natural, which is abstract nonsense, but helps us deduce differentials. In a little more detail, given a map of spectra $f: X \rightarrow Y$, there is a map from the E_r -page of the Adams spectral sequence of X to the Adams E_r -page of Y , and this map commutes with differentials. When $r = 2$, this map has another description: cohomology is contravariantly functorial, giving us a map $f^*: H^*(Y; \mathbb{Z}/2\mathbb{Z}) \rightarrow H^*(X; \mathbb{Z}/2\mathbb{Z})$, and $\text{Ext}(-, \mathbb{Z}/2\mathbb{Z})$ is contravariantly functorial (as we discussed in Remark 11.10), giving us a map $f_{\sharp}: \text{Ext}(H^*(X; \mathbb{Z}/2\mathbb{Z})) \rightarrow \text{Ext}(H^*(Y; \mathbb{Z}/2\mathbb{Z}))$, and f_{\sharp} is the map on Adams E_2 -pages.

To apply this, one lets X be a spectrum whose Adams spectral sequence is well-understood, and Y one whose Adams spectral sequence is not as understood — or sometimes vice versa. In either case, differentials that one already knows might map to differentials one wants to understand. This is particularly useful for us for the map from $\text{Spin-Mp}(2, \mathbb{Z})$ bordism to $\text{Spin-GL}^+(2, \mathbb{Z})$ bordism: we determine all differentials in the Adams spectral sequence for $\Omega_*^{\text{Spin-Mp}(2, \mathbb{Z})}(\text{pt})$ using the May-Milgram theorem (see below), then use the map $\Omega_*^{\text{Spin-Mp}(2, \mathbb{Z})}(\text{pt}) \rightarrow \Omega_*^{\text{Spin-GL}^+(2, \mathbb{Z})}(\text{pt})$ to deduce many differentials in the Adams spectral sequence for $\text{Spin-GL}^+(2, \mathbb{Z})$ bordism. See Figures 23 and 24. This is a common technique when computing with spectral sequences: see, e.g., [130, 164–169, 162, 170–173, 40] for other Adams examples.

The May-Milgram theorem [174] is another helpful tool for computing differentials: it expresses differentials between towers of elements linked by h_0 -actions in terms of Bockstein homomorphisms on cohomology, which are typically much easier to compute. This is an essential piece of our calculation of $\text{Spin-}\mathbb{Z}/8\mathbb{Z}$ bordism in Section 13.2, and we say more about the May-Milgram theorem there.

There are various other techniques for computing Adams differentials in relatively systematic ways. A lot of information can be gained from secondary operations called Massey products and Toda brackets [175, 176]. Recently, Gheorghe, Isaksen, Wang, and Xu have used methods originating in motivic stable homotopy theory to make great progress on the Adams spectral sequence for the sphere spectrum [170, 177, 178].

Of course, bordism groups have geometric meaning, and it is sometimes possible to deduce differentials in an Adams spectral sequence calculation of a bordism group by finding a generator for that bordism group. This technique is used in, e.g., [179, Appendix F].

▷ **Physics intuition:** Some aspects of the Adams spectral sequence admit a physical in-

terpretation. The core object which enters in the E_2 page of the Adams spectral sequence involves objects of the form $E_2^{s,t} = \text{Ext}_{\mathcal{A}(1)}^{s,t}(M, N)$, where M and N are $\mathcal{A}(1)$ modules. To unpack the definitions, the extension problem of interest involves the long exact sequence:

$$0 \longrightarrow \Sigma^t N \longrightarrow P_1 \longrightarrow \cdots \longrightarrow P_s \longrightarrow M \longrightarrow 0, \quad (11.19)$$

where P_1, \dots, P_s are $\mathcal{A}(1)$ -modules and all maps are $\mathcal{A}(1)$ -module maps, namely we look for sequences involving s terms in the resolution of M , and $\Sigma^t N$ is the (reduced) suspension of N . Interpreting N as a CW complex, recall that the reduced suspension ΣN involves sweeping out N from a “starting point” to a “final point”. Said differently, one can introduce an auxiliary timelike direction and track the evolution of N along this auxiliary time. This is a rather natural operation in the context of a path integral for a topological quantum field theory where one specifies an initial and final time evolution step, and provides a physical starting point for understanding the $t = 0$ and $t = 1$ terms of the E_2 page. Treating $\Sigma N = \Sigma^1 N$ itself as another CW complex, observe that we can repeat the entire process, arriving at $\Sigma^2 N$. Consider next the grading by the integer s . These sorts of resolutions figure prominently in the study of B-branes on Calabi-Yau threefolds (see e.g., [180–182] and [183] for a review). In that context, B-branes are objects of the (bounded) derived category of coherent sheaves, and a long exact sequence can be interpreted as specifying a bound state of branes undergoing tachyon condensation (via morphisms between coherent sheaves). This suggests that in the term with P_1, \dots, P_s , each one of these objects should be viewed as specifying as an intermediate object appearing in the bound state fusing $\Sigma^t N$ to M . Lastly, let us turn to the differentials of the spectral sequence. The case which is somewhat “easier” to understand from a physical perspective just involves differentials d_1 which act on the bigrading as $(s, t) \rightarrow (s+1, t)$, namely we simply add on another term in the long exact sequence. In related contexts such as the derived category of coherent sheaves, such a differential can be interpreted as a supercharge action, and taking its cohomology returns the supersymmetric bound states (up to stability). The interpretation of d_2 and the higher differentials in principle follows from this more primitive d_1 object, but we leave the details of how this works and how to interpret the physical systems described above in the context of the Adams spectral sequence in detail for future investigation. \triangleleft

11.4 Extensions and the E_∞ -page

Once we have computed all the differentials in the Adams spectral sequence, can we read off the ko -homology groups? Almost, but not quite; the purpose of this section is to finish the job.

Many spectral sequences arise from a *filtration* on the algebraic object A we want to compute. This means something like data of subobjects $A_1 \subset A_2 \subset \cdots \subset A$ such that

$\bigcup_i A_i = A$.³⁸ What the spectral sequence computes is the *associated graded* of the filtration, i.e. the quotients A_{i+1}/A_i . This is a good approximation of A , but it is not always enough: for example, $A = \mathbb{Z}/4\mathbb{Z}$ and $A = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$ both admit two-term filtrations with $A_1 = \mathbb{Z}/2\mathbb{Z}$ and $A_2 = A$, and in both cases the associated gradeds are two copies of $\mathbb{Z}/2\mathbb{Z}$. Therefore if a spectral sequence outputs $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$, one has to determine whether A is in fact $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$, or whether $A \cong \mathbb{Z}/4\mathbb{Z}$. This is known as an *extension question*.

Extensions in the Adams spectral sequence behave a lot like differentials: there is a lot of structure that handles most of them, but when the structure does not help, extensions can be quite difficult, and we have a variety of ad hoc techniques to address them.

11.4.1 General structure of extension questions in the Adams spectral sequence

The $(t-s, s)$ -grading convention for the Adams spectral sequence has the advantage that extension questions are vertical. Specifically, suppose that for topological degree k fixed, the E_∞ -page for the Adams spectral sequence computing $ko_*(Y)$ is nonzero only at $s = 0, 1, 3$, and 4; these are the groups $E_\infty^{0,k}$, $E_\infty^{1,k+1}$, $E_\infty^{3,k+3}$, and $E_\infty^{4,k+4}$. We draw a picture of this in Figure 14.

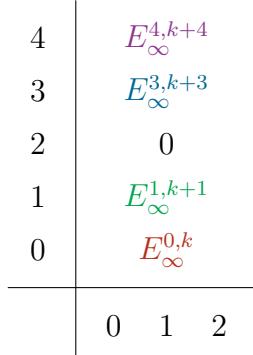


Figure 14: A piece of an E_∞ -page in fixed topological degree. This scenario leads to the three extension questions in (11.20).

Given this data, there are abelian groups A_1 and A_2 and extensions

$$0 \longrightarrow E_\infty^{1,k+1} \longrightarrow A_1 \longrightarrow E_\infty^{0,k} \longrightarrow 0 \tag{11.20a}$$

$$0 \longrightarrow E_\infty^{3,k+3} \longrightarrow A_2 \longrightarrow A_1 \longrightarrow 0 \tag{11.20b}$$

$$0 \longrightarrow E_\infty^{4,k+4} \longrightarrow ko_k(Y) \longrightarrow A_2 \longrightarrow 0. \tag{11.20c}$$

That is, we work upwards, extending what we have already computed by the next group in the E_∞ -page.

³⁸In general, one uses a colimit rather than a union, but the idea is the same. Additionally, sometimes one sees variants of this notion of filtration, e.g. we might encounter $A_i \supset A_{i+1}$, etc.

If there are only finitely many nonzero groups on the E_∞ -page in fixed topological degree, then this process stops once we have incorporated the last one, but it is possible to have infinitely many nonzero groups, and therefore infinitely many steps to take in this part of the computation. This is quite common: it occurs in both the computations for Spin-Mp(2, \mathbb{Z}) bordism and Spin-GL⁺(2, \mathbb{Z}) bordism, and in most of the examples seen in [105, 109, 115, 130].

Fortunately, this infinite-seeming question has a finite answer — the Atiyah-Hirzebruch spectral sequence can be used to show that unless X is overwhelmingly large,³⁹ the (X, V) -twisted Spin bordism groups are finitely generated Abelian groups. Therefore in an extension question with infinitely many steps, all but finitely many of the groups must combine into a free Abelian group.⁴⁰

Remark 11.21. Similar extension questions arise in other spectral sequences; in particular, we have to confront extension questions in an Atiyah-Hirzebruch spectral sequence in the proof of Theorem 14.3. The broad shape of the question is the same: the final answer we want to compute is an iterated extension of all elements on the E_∞ -page in a given total degree. However, this time the grading is different: rather than working vertically, all elements of total degree k lie on the diagonal line $p + q = k$ on the E^∞ -page. Our proof of Theorem 14.3 does not need to get into the details of this kind of extension question; see [37, Section 2.2.2, Section 3] for more information and more examples of extension problems in Atiyah-Hirzebruch spectral sequences.

11.4.2 What we can infer from the $\text{Ext}(\mathbb{Z}/2\mathbb{Z})$ -action

Because the $\text{Ext}(\mathbb{Z}/2\mathbb{Z})$ -action on the E_2 -page of the Adams spectral sequence that we introduced in Section 11.2 commutes with differentials, it passes to an $\text{Ext}(\mathbb{Z}/2\mathbb{Z})$ -action on each page of the spectral sequence, including the E_∞ -page. This extra data automatically resolves a large quantity of extension problems, especially when computing ko -homology in physically relevant degrees; this fact is probably the biggest competitive advantage of the Adams spectral sequence over the Atiyah-Hirzebruch spectral sequence.⁴¹

The key fact is:

³⁹Specifically, we require that X is homotopy equivalent to a CW complex with finitely many cells in each dimension. This is true not just for every spectral sequence computation in this paper, but also all of the spectral sequence computations in mathematical physics that we cite.

⁴⁰Under the hood, what is happening here is that in an infinite-steps extension problem, the final answer is the limit of the extensions at each finite step. This is where 2-completion assists us: a \mathbb{Z} summand in $ko_k(Y)$ becomes a \mathbb{Z}_2 -summand in $ko_k(Y)_2^\wedge$, where \mathbb{Z}_2 denotes the 2-adic integers, and $ko_k(Y)_2^\wedge$ is what the Adams spectral sequence actually computes. Since $\mathbb{Z}_2 = \varprojlim_n \mathbb{Z}/2^n\mathbb{Z}$, then infinite-step extension questions can produce copies of \mathbb{Z}_2 ; since we know $ko_k(Y)$ is finitely generated, any infinite-step extension question must combine all but finitely many pieces into $\mathbb{Z}_2^{\oplus r}$, which arises from \mathbb{Z}^r by 2-completing.

⁴¹As an example, Pin[−] structures are equivalent to $(B\mathbb{Z}/2\mathbb{Z}, \sigma)$ -twisted Spin structures, where $\sigma \rightarrow B\mathbb{Z}/2\mathbb{Z}$ is the tautological line bundle. Therefore one can compute Pin[−] bordism by computing $\Omega_*^{\text{Spin}}((B\mathbb{Z}/2\mathbb{Z})^{\sigma-1})$, which one can study with either the Adams or Atiyah-Hirzebruch spectral sequences. In the Adams spectral sequence, the $\text{Ext}(\mathbb{Z}/2\mathbb{Z})$ -action solves all extension problems [109, Example 6.3], but in the Atiyah-Hirzebruch spectral sequence, we have almost no information about extensions without additional work.

Lemma 11.22. *In the Adams spectral sequence computing $ko_*(Y)$,⁴² suppose there are classes $\bar{x} \in E_\infty^{s,t}$ and $\bar{y} \in E_\infty^{s+1,t+1}$ such that $h_0\bar{x} = \bar{y}$. Then there are elements $x, y \in ko_{t-s}(Y)$ whose images in the E_∞ -page are \bar{x} , resp. \bar{y} , such that $2x = y$.*

That is, an h_0 -action forces the extension $0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow A \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$ to have $A = \mathbb{Z}/4\mathbb{Z}$, not $A = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})!$ Likewise, longer sequences of elements on the E_∞ -page linked by h_0 -actions correspond to copies of $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/8\mathbb{Z}$, and so on in ko -homology. See Figure 15 for a picture. The converse is false: not every non-trivial extension is detected by h_0 , as we discuss in Section 11.4.3.

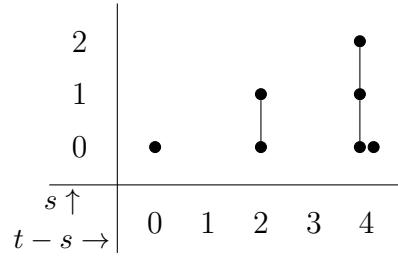


Figure 15: The E_∞ -page for the Adams spectral sequence computing Pin^c bordism. There are extension questions in degrees 2 and 4; the action of h_0 in degree 2 forces $\Omega_2^{\text{Pin}^c} \cong \mathbb{Z}/4\mathbb{Z}$, rather than $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$, and the h_0 -actions in degree 4 mean both extensions are nonsplit, forcing $\Omega_4^{\text{Pin}^c} \cong (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$. Figure adapted from Beaudry-Campbell [115, Figure 42]; see there for more information.

The other generators of $\text{Ext}(\mathbb{Z}/2\mathbb{Z})$ correspond to actions by other elements of ko_* .

- The analogue of Lemma 11.22 for h_1 is that if $\bar{x} \in E_\infty^{s,t}$, $\bar{y} \in E_\infty^{s+1,t+2}$, and $h_1x = y$, then there are classes $x \in ko_{t-s}(X)$ and $y \in ko_{t-s+1}(X)$ corresponding to \bar{x} , resp. \bar{y} , such that $\eta \cdot x = y$, where η is the nonzero element of ko_1 . When we pass to Spin bordism, η is the class of S_p^1 , so if x is represented by a manifold M , $y = [S_p^1 \times M]$. We use this fact quite a bit in our search for generators in Section 14.3.
- The action by $v \in \text{Ext}_{\mathcal{A}(1)}^{3,7}(\mathbb{Z}/2\mathbb{Z})$ lifts in Spin bordism to taking the product with the K3 surface.
- The action by $w \in \text{Ext}_{\mathcal{A}(1)}^{4,12}(\mathbb{Z}/2\mathbb{Z})$ lifts in Spin bordism to taking the product with the Bott manifold.

11.4.3 Hidden extensions and some tricks

Though the presence of an h_0 -action on the E_∞ -page indicates multiplication by 2 in ko -homology, the converse is not true: there are multiplications by 2 in some ko -homology

⁴²This fact, along with most of the facts in this subsection, generalize to Adams spectral sequences over any subalgebra of the Steenrod algebra computing other generalized homology groups.

groups that do not correspond to h_0 -actions on the E_∞ -page. The same thing can happen with other elements of ko_* and their images in the Adams E_∞ -page, notably h_1 and η . Extensions that are not detected by the $\text{Ext}(\mathbb{Z}/2\mathbb{Z})$ -action are called *hidden extensions*.

Hidden extensions are relatively uncommon, especially in physically relevant dimensions: for example, there are none present in the Adams spectral sequence computations done by Freed-Hopkins [105] and Beaudry-Campbell [115]. However, hidden extensions are not so uncommon that we can ignore them, and in fact we will discover non-trivial hidden extensions in Spin- $\mathbb{Z}/8\mathbb{Z}$ and Spin- D_{16} bordism.

One nice technique to address extension questions is to map to or from some other group we already understand. From the induced map on E_∞ -pages of Adams spectral sequences we can often deduce that the original map was injective or surjective, which can suffice: for example, if we are trying to differentiate $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$ from $\mathbb{Z}/4\mathbb{Z}$, it suffices to produce an injective map from $\mathbb{Z}/4\mathbb{Z}$. We use this technique to resolve an Adams hidden extension in Lemma 13.22 and some Atiyah-Hirzebruch extensions in Theorem 14.3.

Another useful technique when applying the Adams spectral sequence to bordism questions is to compute subgroups of bordism groups in different ways. For example, in Section 13.2 we see that the Adams spectral sequence implies $\Omega_5^{\text{Spin-}\mathbb{Z}/8\mathbb{Z}}(\text{pt})$ is an Abelian group of order 64, but does not fully determine which one we obtain. In Appendix C.3, we show that a combination of η -invariants is a bordism invariant $\Omega_5^{\text{Spin-}\mathbb{Z}/8\mathbb{Z}}(\text{pt}) \rightarrow \mathbb{R}/\mathbb{Z}$ and on a lens space takes on the value $-5/32$, so $\Omega_5^{\text{Spin-}\mathbb{Z}/8\mathbb{Z}}(\text{pt})$ must be isomorphic to either $(\mathbb{Z}/32\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$ or $\mathbb{Z}/64\mathbb{Z}$; another η -invariant computation shows we obtain the former.

In the Adams spectral sequence for Spin- D_{16} bordism, we will make frequent use of the following result. Recall that η is the nonzero element of $ko_1(\text{pt})$, and in twisted Spin bordism η represents the circle with non-bounding boundary conditions for fermions S_p^1 .

Lemma 11.23. *Let $x \in ko_n(X)$ be such that $\eta x \neq 0$. Then there is no $y \in ko_n(X)$ such that $2y = x$.*

Proof. Suppose such a y exists. Then $\eta(2y) \neq 0$, but this is also equal to $(2\eta)y$ and $2\eta = 0$ in $ko_*(\text{pt})$. \square

Typically when we use this, such as in Lemmas 14.29 and 14.31, we deduce $\eta x \neq 0$ from an h_1 -action on the E_∞ -page.

The final tool we rely on heavily in this paper is Margolis' theorem (Corollary 11.18): we previously used it to kill differentials on elements in the E_r -page coming from free $\mathcal{A}(1)$ -module summands in cohomology, but this theorem also prevents those elements from participating in hidden extensions.

Remark 11.24. There are many other tricks people use to determine extensions in Adams spectral sequences. One that might be useful in related questions is that the multiplication-

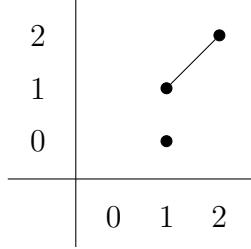


Figure 16: A picture of the E_∞ -page in the scenario in Lemma 11.23. The h_1 -action in this figure implies that $ko_1(X) \cong (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$, rather than $\mathbb{Z}/4\mathbb{Z}$.

by-2 map on ko -theory factors as

$$ko \xrightarrow{c} ku \xrightarrow{b} \Sigma^2 ku \xrightarrow{R} ko, \quad (11.25)$$

where c is the complexification map, b is the Bott map, and R is a “realification” map obtained from forgetting the complex structure on a vector bundle (see Section 10.6). Occasionally this is helpful for splitting extensions, e.g., if b acts by 0 on $ku_n(X)$, one can conclude that multiplication by 2 is the zero map on $ko_n(X)$. We use this in Proposition E.16.

Like with differentials, recently developed techniques involving synthetic spectra have been used on extension questions in [177, 184–186].

12 Computation of $\Omega_*^{\text{Spin}}(B\text{SL}(2, \mathbb{Z}))$

In this section we undertake the first of the three bordism computations in this paper: $\Omega_*^{\text{Spin}}(B\text{SL}(2, \mathbb{Z}))$ in dimensions 11 and below. This is the easiest of the three: we can assemble everything we need from results already in the literature. The technique to do so is a generalization of a better-known technique to calculate group cohomology: if a group G factors as an amalgamated product $G \cong H_1 *_K H_2$, then there is a Mayer-Vietoris sequence computing the cohomology of G in terms of that of H_1 , H_2 , and K :

$$\longrightarrow H^k(BK) \longrightarrow H^k(BH_1) \oplus H^k(BH_2) \longrightarrow H^k(BG) \longrightarrow H^{k+1}(BK) \longrightarrow \quad (12.1)$$

There is an isomorphism $\text{SL}(2, \mathbb{Z}) \cong (\mathbb{Z}/4\mathbb{Z}) *_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{Z}/6\mathbb{Z})$, as we review in Appendix B, and (12.1) allows for a complete calculation of $H^*(B\text{SL}(2, \mathbb{Z}))$. Note also that the two factors in the amalgamated product correspond to the two finite subgroups, generated by S and U , fixing the axio-dilaton τ to the special values indicated in Table 6.

An analogous Mayer-Vietoris sequence exists for any generalized homology or cohomology theory. This reduces the computation of $\Omega_*^{\text{Spin}}(B\text{SL}(2, \mathbb{Z}))$ to the computation of $\Omega_*^{\text{Spin}}(B\mathbb{Z}/\ell\mathbb{Z})$ for $\ell = 2, 4, 6$. By localizing at $p = 2$ and $p = 3$, as described in Section 10.2, we will be able to simplify further, so that we only need to know $\Omega_*^{\text{Spin}}(B\mathbb{Z}/\ell\mathbb{Z})$ for $\ell = 3$ and

$\ell = 4$. We then compute these groups using closely related computations that are already in the literature. In particular, we do not need any spectral sequences in this section.

In Section 12.1, we compute the 2-local spin bordism groups of $BSL(2, \mathbb{Z})$ in dimensions 11 and below; in Section 12.2 we work at odd primes. Then, in Section 12.3, we find a set of generators for $\Omega_k^{\text{Spin}}(BSL(2, \mathbb{Z}))$ for $k \leq 11$.

12.1 Working at $p = 2$

First, we simplify $BSL(2, \mathbb{Z})$.

Lemma 12.2. *The inclusion $\mathbb{Z}/4\mathbb{Z} \hookrightarrow SL(2, \mathbb{Z})$ induces a map*

$$\Omega_k^{\text{Spin}}(B\mathbb{Z}/4\mathbb{Z}) \longrightarrow \Omega_k^{\text{Spin}}(BSL(2, \mathbb{Z})), \quad (12.3)$$

which is an isomorphism after tensoring both sides with $\mathbb{Z}_{(2)}$.

The trick we use to prove this comes up a few more times later, so we extract it as a lemma. First, recall that for an amalgamated product, $G = H_1 *_K H_2$, there is a Mayer-Vietoris sequence for any generalized homology theory E :

$$\cdots \longrightarrow E_n(BK) \longrightarrow E_n(BH_1) \oplus E_n(BH_2) \longrightarrow E_n(BG) \longrightarrow E_{n-1}(BK) \longrightarrow \cdots \quad (12.4)$$

Lemma 12.5. *The Mayer-Vietoris sequence (12.4) is natural in amalgamated products of groups. That is, if $f: H_1 *_K H_2 \rightarrow H'_1 *_K' H'_2$ satisfies $f(H_i) \subset H'_i$ and $f(K) \subset K'$, then the maps it induces on E -homology commute with (12.4).*

Proof. We use the following model for the classifying space BG of a discrete group G : let C_G denote the category with a single object pt and with $\text{End}_{C_G}(\text{pt}) = G$. Then let BG be the geometric realization of the nerve of C_G . This is a simplicial complex whose n -simplices are indexed by n -tuples of elements of G .

This model has the advantage that if $G_0 \subset G$, $BG_0 \subset BG$ as the union of the simplices of n -tuples of elements of G_0 . In particular, for an amalgamated product $G := H_1 *_K H_2$, $BH_1 \cap BH_2 = BK$ inside BG , yielding (12.4). And with this model, the induced map $f: B(H_1 *_K H_2) \rightarrow B(H'_1 *_K' H'_2)$ satisfies $f(BH_i) \subset BH'_i$ and $f(BK) \subset BK'$, which is the condition needed to commute with a Mayer-Vietoris sequence. \square

Proof of Lemma 12.2. We can write $SL(2, \mathbb{Z}) = (\mathbb{Z}/4\mathbb{Z}) *_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{Z}/6\mathbb{Z})$ and $\mathbb{Z}/4\mathbb{Z} = (\mathbb{Z}/4\mathbb{Z}) *_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})$, and with these descriptions the inclusion $f: \mathbb{Z}/4\mathbb{Z} \hookrightarrow SL(2, \mathbb{Z})$ is compatible with these descriptions as amalgamated products: it is the identity on the first factor, inclusion $\mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{Z}/6\mathbb{Z}$ on the second factor, and the identity on the common $\mathbb{Z}/2\mathbb{Z}$ subgroup.

By Lemma 12.5, we obtain a commutative diagram of Mayer-Vietoris sequences for Spin bordism:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \Omega_n^{\text{Spin}}(B\mathbb{Z}/2\mathbb{Z}) & \longrightarrow & \Omega_n^{\text{Spin}}(B\mathbb{Z}/4\mathbb{Z}) \oplus \Omega_n^{\text{Spin}}(B\mathbb{Z}/6\mathbb{Z}) & \longrightarrow & \Omega_n^{\text{Spin}}(B\mathbb{Z}/4\mathbb{Z}) \longrightarrow \cdots \\
& & \downarrow a & & \downarrow (b,c) & & \downarrow d \\
\cdots & \longrightarrow & \Omega_n^{\text{Spin}}(B\mathbb{Z}/2\mathbb{Z}) & \longrightarrow & \Omega_n^{\text{Spin}}(B\mathbb{Z}/4\mathbb{Z}) \oplus \Omega_n^{\text{Spin}}(B\mathbb{Z}/6\mathbb{Z}) & \longrightarrow & \Omega_n^{\text{Spin}}(B\text{SL}(2, \mathbb{Z})) \longrightarrow \cdots
\end{array} \tag{12.6}$$

As f is the identity on the amalgamating $\mathbb{Z}/2\mathbb{Z}$, the vertical arrow a is the identity map; likewise for the first factor $\mathbb{Z}/4\mathbb{Z}$ and the map b . For c , we use the fact that the map $H_*(B\mathbb{Z}/2\mathbb{Z}) \rightarrow H_*(B\mathbb{Z}/6\mathbb{Z})$ is an isomorphism at the prime 2, which implies (e.g. by naturality of the Atiyah-Hirzebruch spectral sequence) that $c: \Omega_n^{\text{Spin}}(B\mathbb{Z}/2\mathbb{Z}) \rightarrow \Omega_n^{\text{Spin}}(B\mathbb{Z}/6\mathbb{Z})$ is also a 2-local isomorphism. By the five lemma, the remaining arrow, $d: \Omega_n^{\text{Spin}}(B\mathbb{Z}/4\mathbb{Z}) \rightarrow \Omega_n^{\text{Spin}}(B\text{SL}(2, \mathbb{Z}))$, is also a 2-local isomorphism.⁴³ \square

So we need to determine $\Omega_k^{\text{Spin}}(B\mathbb{Z}/4\mathbb{Z})$ for $k \leq 11$, and by Theorem 10.36, we need $ko_k(B\mathbb{Z}/4\mathbb{Z})$ for $k \leq 11$ and $ko \langle 2 \rangle_k(B\mathbb{Z}/4\mathbb{Z})$ for $k \leq 1$. The groups $ko_*(B\mathbb{Z}/4\mathbb{Z})$ are computed by Bruner-Greenlees [142, Example 7.3.3],⁴⁴ and for $ko \langle 2 \rangle_*(B\mathbb{Z}/4\mathbb{Z})$, use Lemma 10.39.

We summarize the results in Table 7, which specifies $ko_k(B\text{SL}(2, \mathbb{Z}))$ at the prime 2, which

k	$ko_k(B\mathbb{Z}/4\mathbb{Z})$	$ko \langle 2 \rangle_k(B\mathbb{Z}/4\mathbb{Z})$
0	\mathbb{Z}	$\mathbb{Z}/2\mathbb{Z}$
1	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z})$	$\mathbb{Z}/2\mathbb{Z}$
2	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	
3	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z})$	
4	\mathbb{Z}	
5	$\mathbb{Z}/4\mathbb{Z}$	
6	0	
7	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/32\mathbb{Z})$	
8	\mathbb{Z}	
9	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2} \oplus (\mathbb{Z}/8\mathbb{Z})$	
10	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	
11	$(\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/128\mathbb{Z})$	

Table 7: The groups $ko_k(B\mathbb{Z}/4\mathbb{Z})$ for $k \leq 11$ and $ko \langle 2 \rangle_k(B\mathbb{Z}/4\mathbb{Z})$ for $k \leq 1$.

can be assembled together with $ko_{k-8}(B\mathbb{Z}/4\mathbb{Z})$ and $ko \langle 2 \rangle_{k-10}(B\mathbb{Z}/4\mathbb{Z})$ into $\Omega_k \text{Spin}(B\mathbb{Z}/4\mathbb{Z})$ as in Theorem 10.36.

⁴³In a little more detail, we want to prove that after tensoring everything in (12.6) with $\mathbb{Z}_{(2)}$, the rightmost vertical arrow is an isomorphism. To use the five lemma, the rows must be exact, which is true because $\mathbb{Z}_{(p)}$ is a flat \mathbb{Z} -module for any prime p .

⁴⁴Beware that there is a typo in Bruner-Greenlees' calculations in the table on p. 135 in the row $n = 10$: $ko_{10}(B\mathbb{Z}/4\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$, not $\mathbb{Z}/2\mathbb{Z}$.

12.2 Working at odd primes

Lemma 12.7. *The inclusion $\mathbb{Z}/3\mathbb{Z} \hookrightarrow \mathrm{SL}(2, \mathbb{Z})$ induces a map*

$$\Omega_*^{\mathrm{Spin}}(B\mathbb{Z}/3\mathbb{Z}) \longrightarrow \Omega_*^{\mathrm{Spin}}(\mathrm{BSL}(2, \mathbb{Z})), \quad (12.8)$$

which is an isomorphism after tensoring with $\mathbb{Z}/p\mathbb{Z}$ for any odd prime p .

Proof. The proof is essentially the same as for Lemma 12.2. The inclusion $\mathbb{Z}/6\mathbb{Z} \hookrightarrow \mathrm{SL}(2, \mathbb{Z})$ can be restated as $(\mathbb{Z}/2\mathbb{Z}) *_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{Z}/6\mathbb{Z}) \hookrightarrow (\mathbb{Z}/4\mathbb{Z}) *_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{Z}/6\mathbb{Z})$, inducing maps commuting with Mayer-Vietoris sequences. The map $H_*(B\mathbb{Z}/2\mathbb{Z}) \rightarrow H_*(B\mathbb{Z}/4\mathbb{Z})$ is a p -local equivalence at any odd prime p , so the maps on Spin bordism are also p -local equivalences, allowing us to conclude that $\Omega_*^{\mathrm{Spin}}(B\mathbb{Z}/6\mathbb{Z}) \rightarrow \Omega_*^{\mathrm{Spin}}(\mathrm{BSL}(2, \mathbb{Z}))$ is a p -local equivalence. Finally, because $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, the map $B\mathbb{Z}/3\mathbb{Z} \rightarrow B\mathbb{Z}/6\mathbb{Z}$ induces a p -local isomorphism on homology, hence also on spin bordism, so we can replace $B\mathbb{Z}/6\mathbb{Z}$ with $B\mathbb{Z}/3\mathbb{Z}$ to obtain the theorem statement. \square

For $p > 3$, $H^\ell(B\mathbb{Z}/3\mathbb{Z}; \mathbb{Z}/p\mathbb{Z})$ vanishes for $\ell > 0$ [187, Corollary 10.2], so the map $B\mathbb{Z}/3\mathbb{Z} \rightarrow \mathrm{pt}$ induces an isomorphism on mod p cohomology, hence by a version of Whitehead's theorem [133, Chapitre III, Théorème 3], is a stable p -local equivalence. Thus it suffices to understand the 3-local story. Recall from (10.34) the decomposition of $\Omega_*^{\mathrm{SO}} \otimes \mathbb{Z}_{(3)}$ into copies of BP -homology: when $k \leq 15$, this decomposition takes the form

$$\Omega_k^{\mathrm{SO}}(X) \otimes \mathbb{Z}_{(3)} \cong BP_k(X) \oplus BP_{k-8}(X) \oplus BP_{k-12}(X). \quad (12.9)$$

Bahri-Bendersky-Davis-Gilkey [188, Theorem 1.2(a)] compute $BP_*(B\mathbb{Z}/3\mathbb{Z})$ in terms of the ℓ -homology of $B\mathbb{Z}/3\mathbb{Z}$, where ℓ is a generalized homology theory called the 3-local Adams summand of ku (so that $\ell_*(X)$ refers to the ℓ -homology groups of X).

Theorem 12.10 (Bahri-Bendersky-Davis-Gilkey [188, Theorem 1.2(a)]).

$$BP_*(B\mathbb{Z}/3\mathbb{Z}) \cong \ell_*(B\mathbb{Z}/3\mathbb{Z}) \otimes_{\mathbb{Z}_{(3)}} \mathbb{Z}_{(3)}[v_2, v_3, \dots]. \quad (12.11)$$

So we must determine $\ell_*(B\mathbb{Z}/3\mathbb{Z})$, which we do using ku -homology. There is an isomorphism⁴⁵

$$ku_k(X) \otimes \mathbb{Z}_{(3)} \cong \ell_k(X) \oplus \ell_{k-2}(X). \quad (12.12)$$

Here ku is *connective complex K-theory* (see Remark 10.40).

Theorem 12.13 (Hashimoto [191, Theorem 3.1]). *There is an isomorphism*

$$\widetilde{ku}_n(B\mathbb{Z}/3\mathbb{Z}) \cong (\mathbb{Z}/3^{j+1}\mathbb{Z})^{\oplus s} \oplus (\mathbb{Z}/3^j\mathbb{Z})^{\oplus 2-s}, \quad (12.14)$$

⁴⁵The p -local decomposition of ku into a sum of shifts of ℓ was a folk theorem. Adams [189, Lecture 4, Corollary 8] proved the analogous decomposition for KU ; for ku the earliest statement we could find is [190, Proposition 2.7].

if $n = 4j + 2s - 1$ for $0 < s \leq 2$. Otherwise, $\widetilde{ku}_n(B\mathbb{Z}/3\mathbb{Z}) = 0$.

Explicitly, in low degrees,

$$\begin{aligned}\widetilde{ku}_1(B\mathbb{Z}/3\mathbb{Z}) &\cong \mathbb{Z}/3\mathbb{Z}, \\ \widetilde{ku}_3(B\mathbb{Z}/3\mathbb{Z}) &\cong (\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z}), \\ \widetilde{ku}_5(B\mathbb{Z}/3\mathbb{Z}) &\cong (\mathbb{Z}/9\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z}), \\ \widetilde{ku}_7(B\mathbb{Z}/3\mathbb{Z}) &\cong (\mathbb{Z}/9\mathbb{Z}) \oplus (\mathbb{Z}/9\mathbb{Z}), \\ \widetilde{ku}_9(B\mathbb{Z}/3\mathbb{Z}) &\cong (\mathbb{Z}/27\mathbb{Z}) \oplus (\mathbb{Z}/9\mathbb{Z}), \\ \widetilde{ku}_{11}(B\mathbb{Z}/3\mathbb{Z}) &\cong (\mathbb{Z}/27\mathbb{Z}) \oplus (\mathbb{Z}/27\mathbb{Z}),\end{aligned}\tag{12.15}$$

and the pattern continues. Feeding this to (12.12),

$$\begin{aligned}\widetilde{\ell}_1(B\mathbb{Z}/3\mathbb{Z}) &\cong \mathbb{Z}/3\mathbb{Z}, \\ \widetilde{\ell}_3(B\mathbb{Z}/3\mathbb{Z}) &\cong \mathbb{Z}/3\mathbb{Z}, \\ \widetilde{\ell}_5(B\mathbb{Z}/3\mathbb{Z}) &\cong \mathbb{Z}/9\mathbb{Z}, \\ \widetilde{\ell}_7(B\mathbb{Z}/3\mathbb{Z}) &\cong \mathbb{Z}/9\mathbb{Z}, \\ \widetilde{\ell}_9(B\mathbb{Z}/3\mathbb{Z}) &\cong \mathbb{Z}/27\mathbb{Z}, \\ \widetilde{\ell}_{11}(B\mathbb{Z}/3\mathbb{Z}) &\cong \mathbb{Z}/27\mathbb{Z},\end{aligned}\tag{12.16}$$

and so on. Even-degree reduced ℓ -homology of $B\mathbb{Z}/3\mathbb{Z}$ vanishes.

Now feed this to Theorem 12.10. Because $|v_2| = 2(3^2 - 1) = 16$, in degrees 15 and below, Theorem 12.10 tells us there is no difference between the BP -homology and the ℓ -homology of $B\mathbb{Z}/3\mathbb{Z}$.

Finally, (10.34) gets us back to (reduced) oriented bordism.

$$\begin{aligned}\widetilde{\Omega}_1^{\text{SO}}(B\mathbb{Z}/3\mathbb{Z}) \otimes \mathbb{Z}_{(3)} &\cong \mathbb{Z}/3\mathbb{Z}, \\ \widetilde{\Omega}_3^{\text{SO}}(B\mathbb{Z}/3\mathbb{Z}) \otimes \mathbb{Z}_{(3)} &\cong \mathbb{Z}/3\mathbb{Z}, \\ \widetilde{\Omega}_5^{\text{SO}}(B\mathbb{Z}/3\mathbb{Z}) \otimes \mathbb{Z}_{(3)} &\cong \mathbb{Z}/9\mathbb{Z}, \\ \widetilde{\Omega}_7^{\text{SO}}(B\mathbb{Z}/3\mathbb{Z}) \otimes \mathbb{Z}_{(3)} &\cong \mathbb{Z}/9\mathbb{Z}, \\ \widetilde{\Omega}_9^{\text{SO}}(B\mathbb{Z}/3\mathbb{Z}) \otimes \mathbb{Z}_{(3)} &\cong (\mathbb{Z}/27\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z}), \\ \widetilde{\Omega}_{11}^{\text{SO}}(B\mathbb{Z}/3\mathbb{Z}) \otimes \mathbb{Z}_{(3)} &\cong (\mathbb{Z}/27\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z}).\end{aligned}\tag{12.17}$$

With this we have all the ingredients we need to determine $\Omega_*^{\text{Spin}}(\text{BSL}(2, \mathbb{Z}))$, which we summarize in Table 8.

k	$\Omega_k^{\text{Spin}}(B\text{SL}(2, \mathbb{Z}))$	Generators
0	\mathbb{Z}	pt
1	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})$	(S_p^1, L_4^1, L_3^1)
2	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	$(S_p^1 \times S_p^1, S_p^1 \times L_4^1)$
3	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})$	$(S_p^1 \times S_p^1 \times L_4^1, L_4^3, L_3^3)$
4	\mathbb{Z}	K3
5	$(\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/9\mathbb{Z})$	(Q_4^5, L_3^5)
6	0	
7	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/32\mathbb{Z}) \oplus (\mathbb{Z}/9\mathbb{Z})$	$(\tilde{L}_4^7, L_4^7, L_3^7)$
8	$\mathbb{Z} \oplus \mathbb{Z}$	(B, \mathbb{HP}^2)
9	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 3} \oplus (\mathbb{Z}/4\mathbb{Z})$ $\oplus (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/27\mathbb{Z})$	$(B \times L_4^1, B \times S_p^1, \mathbb{HP}^2 \times S_p^1, \mathbb{HP}^2 \times L_4^1,$ $Q_4^9, \mathbb{HP}^2 \times L_3^1, L_3^9)$
10	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 5}$	$(B \times S_p^1 \times S_p^1, \mathbb{HP}^2 \times S_p^1 \times S_p^1, B \times S_p^1 \times L_4^1,$ $\mathbb{HP}^2 \times S_p^1 \times L_4^1, X_{10})$
11	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ $\oplus (\mathbb{Z}/8\mathbb{Z})^{\oplus 2} \oplus (\mathbb{Z}/128\mathbb{Z})$ $\oplus (\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/27\mathbb{Z})$	$(X_{10} \times L_4^1, \mathbb{HP}^2 \times L_4^1 \times S_p^1 \times S_p^1,$ $\mathbb{HP}^2 \times L_4^3, \tilde{L}_4^{11}, L_4^{11},$ $\mathbb{HP}^2 \times L_3^3, L_3^{11})$

Table 8: Bordism groups $\Omega_k^{\text{Spin}}(B\text{SL}(2, \mathbb{Z}))$ and their generators (in the same order as the group summands or described by linear combinations of them) for $k \leq 11$.

12.3 Finding generators

We focus on $\tilde{\Omega}_k^{\text{Spin}}(B\text{SL}(2, \mathbb{Z}))$, i.e., the subgroups of the bordism groups for which the principal $\text{SL}(2, \mathbb{Z})$ -bundle is non-trivial — when the bundle is trivial, this is just Spin bordism, whose low-degree generators are standard. We discuss these generators in Appendix A.

Now the more interesting part. As a consequence of Lemmas 12.2 and 12.7, the inclusions $\mathbb{Z}/3\mathbb{Z} \rightarrow \text{SL}(2, \mathbb{Z})$ and $\mathbb{Z}/4\mathbb{Z} \rightarrow \text{SL}(2, \mathbb{Z})$ induce an isomorphism

$$\tilde{\Omega}_k^{\text{Spin}}(B\mathbb{Z}/4\mathbb{Z}) \oplus \tilde{\Omega}_k^{\text{Spin}}(B\mathbb{Z}/3\mathbb{Z}) \xrightarrow{\cong} \tilde{\Omega}_k^{\text{Spin}}(B\text{SL}(2, \mathbb{Z})). \quad (12.18)$$

So we will describe the generators of $\tilde{\Omega}_k^{\text{Spin}}(B\text{SL}(2, \mathbb{Z}))$ in terms of Spin manifolds with principal $\mathbb{Z}/3\mathbb{Z}$ - and $\mathbb{Z}/4\mathbb{Z}$ -bundles. We are not the first to study generators of $\tilde{\Omega}_k^{\text{Spin}}(B\mathbb{Z}/k\mathbb{Z})$, and previous work suggests that we should try lens spaces and lens space bundles for generators,

and use η -invariants to detect them.

We start with the $\mathbb{Z}/3\mathbb{Z}$ case: it is easier, because the lens spaces L_3^{2k-1} all have unique Spin structures. A theorem of Rosenberg [192, Proof of Theorem 2.12] shows that the group $\Omega_*^{\text{Spin}}(B\mathbb{Z}/3\mathbb{Z})$ is generated by lens spaces and products of lens spaces with Spin manifolds. We use η -invariants to find specific generators of specific groups, as formulas for η -invariants of twisted Dirac operators on lens spaces are readily available. We go over these formulas in Appendix C.1, and in Table 16, we compute some η -invariants which are bordism invariants on L_3^{2k-1} for $2 \leq k \leq 6$. This gives us the following generators; in all cases the duality bundle is induced from the principal $\mathbb{Z}/3\mathbb{Z}$ -bundle $S^{2k-1} \rightarrow L_3^{2k-1}$ by the inclusion $\mathbb{Z}/3\mathbb{Z} \hookrightarrow \text{SL}(2, \mathbb{Z})$.

- $L_3^1 \cong S^1$ generates the $\mathbb{Z}/3\mathbb{Z}$ summand in $\Omega_1^{\text{Spin}}(\text{BSL}(2, \mathbb{Z}))$.
- L_3^3 generates the $\mathbb{Z}/3\mathbb{Z}$ summand in $\Omega_3^{\text{Spin}}(\text{BSL}(2, \mathbb{Z}))$.
- L_3^5 generates the $\mathbb{Z}/9\mathbb{Z}$ summand in $\Omega_5^{\text{Spin}}(\text{BSL}(2, \mathbb{Z}))$.
- L_3^7 generates the $\mathbb{Z}/9\mathbb{Z}$ summand in $\Omega_7^{\text{Spin}}(\text{BSL}(2, \mathbb{Z}))$.
- L_3^9 generates the $\mathbb{Z}/27\mathbb{Z}$ summand in $\Omega_9^{\text{Spin}}(\text{BSL}(2, \mathbb{Z}))$.
- L_3^{11} generates the $\mathbb{Z}/27\mathbb{Z}$ summand in $\Omega_{11}^{\text{Spin}}(\text{BSL}(2, \mathbb{Z}))$.

If M is a Spin manifold of dimension $4m$, $\eta^D(M \times N) = \text{Index}^D(M) \eta^D(N)$, where D refers to the Dirac operator; similar results hold for twisted Dirac operators. This gives us the last two 3-torsion generators we need.

- $\mathbb{HP}^2 \times L_3^1 \cong \mathbb{HP}^2 \times S^1$ generates the $\mathbb{Z}/3\mathbb{Z}$ summand in $\Omega_9^{\text{Spin}}(\text{BSL}(2\mathbb{Z}))$.
- $\mathbb{HP}^2 \times L_3^3$ generates the $\mathbb{Z}/3\mathbb{Z}$ summand in $\Omega_{11}^{\text{Spin}}(\text{BSL}(2\mathbb{Z}))$.

In both cases, the duality bundle is trivial on \mathbb{HP}^2 and induced from $S^{2k-1} \rightarrow L_3^{2k-1}$, as above, on the lens component of the product.

Now for the 2-torsion. An analogue of Rosenberg's result above is still true, but now the generators are a little more complicated: lens spaces and lens space bundles over S^2 . Specifically, regard S^2 and \mathbb{CP}^1 and consider a vector bundle which is a sum of complex line bundles $\mathcal{L}_1 \oplus \dots \oplus \mathcal{L}_{m-1} \rightarrow \mathbb{CP}^1$. Act by $\mathbb{Z}/2^\ell\mathbb{Z}$ as multiplication by the 2^ℓ th roots of unity on the unit sphere bundle of this vector bundle; this is a free action and the quotient is an $L_{2^\ell}^{2m-3}$ -bundle over S^2 . We will specifically let $Q_{2^\ell}^{2m-1}$ denote the case that $\mathcal{L}_1 = \mathcal{O}(2)$ and the remaining line bundles are trivial. See Appendix C.2 for more information.

Theorem 12.19 (Botvinnik-Gilkey-Stolz [193, Section 5]). *For values $\ell \geq 2$, the subgroup $ko_{2m-1}(B\mathbb{Z}/2^\ell\mathbb{Z})$ of $\Omega_{2m-1}^{\text{Spin}}(B\mathbb{Z}/\ell\mathbb{Z})$ is generated by lens spaces and lens space bundles over S^2 of the form Q_4^{2m-1} described above, and is detected by η -invariants.*

This will be good enough — in Table 7, we saw that if k is even and less than 11, $\widetilde{ko}_k(B\mathbb{Z}/4\mathbb{Z})$ vanishes except in degrees 2 and 10, where it is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. These two $\mathbb{Z}/2\mathbb{Z}$ summands are the products of lens spaces with Spin manifolds, specifically $L_1^4 \times S_p^1$ and $L_1^4 \times S_p^1 \times \mathbb{H}\mathbb{P}^2$. Thus we know how to find the entire ko_k summand of $\widetilde{\Omega}_k^{\text{Spin}}(B\mathbb{Z}/4\mathbb{Z})$.⁴⁶ The generators of the ko_{k-8} summands are the direct products of the generators of the ko_k summands with $\mathbb{H}\mathbb{P}^2$. And for $ko\langle 2 \rangle_{k-10}$, Lemma 10.39 (identifying $ko\langle 2 \rangle_k(X) \cong H_k(X; \mathbb{Z}/2\mathbb{Z})$ for $k \leq 1$ and X connective) implies that we can detect generators using mod 2 cohomology characteristic classes, giving us $X_{10} \times L_1^4$ generating $ko\langle 2 \rangle_1(B\mathbb{Z}/4\mathbb{Z}) \hookrightarrow \Omega_{11}^{\text{Spin}}(B\mathbb{Z}/4\mathbb{Z})$.

Once again we refer to Appendix C for formulas and calculations of η -invariants of lens spaces and lens space bundles. We learn from Table 19 the following generators; in all cases the duality bundles are induced from $S^{2k-1} \rightarrow L_4^{2k-1}$ and the inclusion $\mathbb{Z}/4\mathbb{Z} \hookrightarrow \text{SL}(2, \mathbb{Z})$ like in the 3-torsion case. This time there is the subtlety that L_4^{2k-1} has two Spin structures for k even and zero Spin structures for $k > 1$ and odd.

- $L_4^1 \cong S^1$ with either of its Spin structures generates the remaining $\mathbb{Z}/4\mathbb{Z}$ summand of $\Omega_1^{\text{Spin}}(\text{BSL}(2, \mathbb{Z}))$.
- L_4^3 with either of its Spin structures generates the $\mathbb{Z}/8\mathbb{Z}$ summand of $\Omega_3^{\text{Spin}}(\text{BSL}(2, \mathbb{Z}))$.
- The lens space bundle Q_4^5 with either of its Spin structures generates the $\mathbb{Z}/4\mathbb{Z}$ summand of $\Omega_5^{\text{Spin}}(\text{BSL}(2, \mathbb{Z}))$.
- L_4^7 with its two Spin structures generates the remaining $(\mathbb{Z}/32\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$ summands of $\Omega_7^{\text{Spin}}(\text{BSL}(2, \mathbb{Z}))$.
- The lens space bundle Q_4^9 with either of its Spin structures generates the $\mathbb{Z}/8\mathbb{Z}$ summand of $\Omega_9^{\text{Spin}}(\text{BSL}(2, \mathbb{Z}))$. Similarly to the 3-torsion case, $\mathbb{H}\mathbb{P}^2 \times L_4^1$ generates the $\mathbb{Z}/4\mathbb{Z}$ summand in $\Omega_9^{\text{Spin}}(\text{BSL}(2, \mathbb{Z}))$.
- L_4^{11} with its two Spin structures generates the two summands $(\mathbb{Z}/128\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z})$ of $\Omega_{11}^{\text{Spin}}(\text{BSL}(2, \mathbb{Z}))$. Similarly to the 3-torsion case, $\mathbb{H}\mathbb{P}^2 \times L_4^3$ generates the remaining $\mathbb{Z}/8\mathbb{Z}$ summand in $\Omega_{11}^{\text{Spin}}(\text{BSL}(2, \mathbb{Z}))$.

13 Computation of $\Omega_*^{\text{Spin-Mp}(2, \mathbb{Z})}(\text{pt})$

The defining representation of $\text{SL}(2, \mathbb{Z})$ is not Spin; its Spin cover is called the *metaplectic group* $\text{Mp}(2, \mathbb{Z})$. Taking the quotient by the central $\mathbb{Z}/2\mathbb{Z}$ subgroup of the Spin group defines a short exact sequence

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \text{Mp}(2, \mathbb{Z}) \longrightarrow \text{SL}(2, \mathbb{Z}) \longrightarrow 1 . \quad (13.1)$$

⁴⁶Alternatively, Botvinnik-Gilkey [155, Theorem 4.5] tell us how to detect all of $ko_k(B\mathbb{Z}/2^\ell\mathbb{Z})$.

The S-duality symmetry of type IIB string theory, a priori an $\mathrm{SL}(2, \mathbb{Z})$ symmetry, mixes with fermion parity to form the extension (13.1), so to study duality defects in type IIB string theory, as we do in Section 6, we should compute the bordism groups of manifolds with structure group

$$\mathrm{Spin}\text{-}\mathrm{Mp}(2, \mathbb{Z}) := \mathrm{Spin} \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Mp}(2, \mathbb{Z}). \quad (13.2)$$

In this section, we compute $\mathrm{Spin}\text{-}\mathrm{Mp}(2, \mathbb{Z})$ bordism groups in dimensions 11 and below. Our approach is similar to how we determined $\Omega_*^{\mathrm{Spin}}(\mathrm{BSL}(2, \mathbb{Z}))$ in Section 12: we use the amalgamated product decomposition (B.6)

$$\mathrm{Mp}(2, \mathbb{Z}) \xrightarrow{\cong} (\mathbb{Z}/12\mathbb{Z}) *_{(\mathbb{Z}/4\mathbb{Z})} (\mathbb{Z}/8\mathbb{Z}) \quad (13.3)$$

to express $\mathrm{Spin}\text{-}\mathrm{Mp}(2, \mathbb{Z})$ in simpler terms at $p = 2$ and at odd primes. We tackle odd primes in Section 13.1, where we find that for any odd prime p , $\mathrm{Spin}\text{-}\mathrm{Mp}(2, \mathbb{Z})$ bordism is p -locally isomorphic to $\Omega_*^{\mathrm{Spin}}(B\mathbb{Z}/3\mathbb{Z})$ (Lemma 13.4), so we can reuse our computations from Section 12.2. In Section 13.2, we localize at 2 and learn that $\mathrm{Spin}\text{-}\mathrm{Mp}(2, \mathbb{Z})$ bordism is 2-locally isomorphic to bordism for the group $\mathrm{Spin}\text{-}\mathbb{Z}/8\mathbb{Z} := \mathrm{Spin} \times_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/8\mathbb{Z}$. For $k > 5$, $\Omega_k^{\mathrm{Spin}\text{-}\mathbb{Z}/8\mathbb{Z}}(\mathrm{pt})$ is not in the literature. We compute it up to dimension 11 using the Adams spectral sequence, following a strategy of Campbell [109, Section 7.9] and Davighi-Lohitsiri [111, Appendix A.4].

In Section 13.3, we produce generators for the $\mathrm{Spin}\text{-}\mathrm{Mp}(2, \mathbb{Z})$ bordism groups we computed, following the same strategy that we did in Section 12.3 for $\Omega_*^{\mathrm{Spin}}(\mathrm{BSL}(2, \mathbb{Z}))$. Then in Section 13.4, we discuss the ring structure on $\Omega_*^{\mathrm{Spin}\text{-}\mathbb{Z}/8\mathbb{Z}}(\mathrm{pt})$.

13.1 Working at odd primes

Lemma 13.4. *Let p be an odd prime. The inclusion $\mathbb{Z}/12\mathbb{Z} \hookrightarrow \mathrm{Mp}(2, \mathbb{Z})$ induces a p -local equivalence*

$$\Omega_k^{\mathrm{Spin}}(B\mathbb{Z}/3\mathbb{Z}) \longrightarrow \Omega_k^{\mathrm{Spin}\text{-}\mathrm{Mp}(2, \mathbb{Z})}(\mathrm{pt}). \quad (13.5)$$

Proof. Just as in the proofs of Lemmas 12.2 and 12.7, the inclusion $\mathbb{Z}/12\mathbb{Z} \hookrightarrow \mathrm{Mp}(2, \mathbb{Z})$ can be written as

$$\mathbb{Z}/12\mathbb{Z} = (\mathbb{Z}/4\mathbb{Z}) *_{\mathbb{Z}/4\mathbb{Z}} (\mathbb{Z}/12\mathbb{Z}) \longrightarrow (\mathbb{Z}/8\mathbb{Z}) *_{\mathbb{Z}/4\mathbb{Z}} (\mathbb{Z}/12\mathbb{Z}) = \mathrm{Mp}(2, \mathbb{Z}), \quad (13.6)$$

so the induced map on $\mathbb{Z}_{(p)}$ -homology is an isomorphism. Therefore the map

$$B(\mathrm{Spin} \times_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/12\mathbb{Z}) \longrightarrow B(\mathrm{Spin} \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Mp}(2, \mathbb{Z})) \quad (13.7)$$

also induces an isomorphism on $\mathbb{Z}_{(p)}$ -homology; this map also intertwines the maps down to

BO given by forgetting $\mathbb{Z}/12\mathbb{Z}$ or $Mp(2, \mathbb{Z})$, so it induces a map of Thom spectra

$$MT(\text{Spin} \times_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/12\mathbb{Z}) \longrightarrow MT(\text{Spin} \times_{\mathbb{Z}/2\mathbb{Z}} Mp(2, \mathbb{Z})), \quad (13.8)$$

which by the Thom isomorphism is also an isomorphism on $\mathbb{Z}_{(p)}$ -homology, hence by the stable Whitehead theorem [133, Chapitre III, Théorème 3] is also an isomorphism on p -local homotopy groups.

Next, the map $i: \mathbb{Z}/6\mathbb{Z} \hookrightarrow \mathbb{Z}/12\mathbb{Z}$ is an isomorphism on p -local homology, because under the isomorphisms $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/12\mathbb{Z} \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, i can be identified with the inclusion $\mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{Z}/4\mathbb{Z}$ and the identity on $\mathbb{Z}/3\mathbb{Z}$. Therefore the induced map of Thom spectra

$$MT(\text{Spin} \times_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}) \longrightarrow MT(\text{Spin} \times_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/12\mathbb{Z}), \quad (13.9)$$

induces an isomorphism on $\mathbb{Z}_{(p)}$ -homology, hence also on p -local homotopy groups, just like (13.8).

Lastly, because $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, $\text{Spin} \times_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z} \cong \text{Spin} \times \mathbb{Z}/3\mathbb{Z}$. \square

So at odd primes we just need $\Omega_k^{\text{Spin}}(B\mathbb{Z}/3\mathbb{Z})$, which we determined in Section 12.2. The generators are the same as the 3-torsion generators we found in Section 12.3.

13.2 Working at $p = 2$

In this section, we run the Adams spectral sequence for the 2-primary part of $\Omega_*^{\text{Spin-Mp}(2, \mathbb{Z})}(\text{pt})$. There are several hidden extensions we have to resolve, and we resolve them by finding generators for the bordism groups. We find several non-split hidden extensions.

We let $\text{Spin-Z}/8\mathbb{Z}$ denote the symmetry type $\text{Spin} \times_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/8\mathbb{Z}$ with the map to O projection onto the first factor. $\text{Spin-Z}/8\mathbb{Z}$ bordism has been studied in [109, 110, 37, 111, 32, 130], but only in dimensions 5 and below. Our computations in dimensions 6-11 are new.

Lemma 13.10. *The inclusion $\mathbb{Z}/8\mathbb{Z} \hookrightarrow Mp(2, \mathbb{Z})$ induces a 2-local equivalence*

$$\Omega_*^{\text{Spin-Z}/8\mathbb{Z}}(\text{pt}) \longrightarrow \Omega_*^{\text{Spin-Mp}(2, \mathbb{Z})}(\text{pt}). \quad (13.11)$$

The proof follows the same line of reasoning as Lemma 13.4, except that we cannot untwist like in the last line of that proof.

Next, we shear $\text{Spin-Z}/8\mathbb{Z}$ bordism.

Lemma 13.12. *$H^*(B\mathbb{Z}/4\mathbb{Z}; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[x, y]/(x^2)$, where $|x| = 1$ and $|y| = 2$. The class y can be characterized as follows:*

1. *Let $\rho: \mathbb{Z}/4\mathbb{Z} \rightarrow \text{SO}(2)$ denote the rotation representation and $(E\mathbb{Z}/4\mathbb{Z})_\rho \rightarrow B\mathbb{Z}/4\mathbb{Z}$ denote the associated rank-2 vector bundle; then $w_2((E\mathbb{Z}/4\mathbb{Z})_\rho) = y$.*

2. The cohomology class of the central extension

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/8\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow 0 \quad (13.13)$$

is equal to y .

Proof. The cohomology ring is standard; see, e.g. [144, Proposition 4.5.1]. Since the group $H^2(B\mathbb{Z}/4\mathbb{Z}; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, there is only one nonzero element, so for the rest of the lemma it suffices to show that $w_2((E\mathbb{Z}/4\mathbb{Z})_\rho)$ and (13.13) are non-trivial. For the former, this follows because ρ does not lift to $\text{Spin}(2)$, and for the latter, this follows because (13.13) is not split. \square

Thus Corollary 10.23 tells us that $\text{Spin-}\mathbb{Z}/8\mathbb{Z}$ structures are naturally equivalent to $(B\mathbb{Z}/4\mathbb{Z}, (E\mathbb{Z}/4\mathbb{Z})_\rho)$ -twisted Spin structures. For ease of reading, we will use ρ to denote both the representation and the associated vector bundle that we have been calling $(E\mathbb{Z}/4\mathbb{Z})_\rho$; the specific meaning will be clear from context. Thus $\text{Spin-}\mathbb{Z}/8\mathbb{Z}$ structures are naturally equivalent to $(B\mathbb{Z}/4\mathbb{Z}, \rho)$ -twisted Spin structures, and Lemma 10.18 implies

$$\Omega_*^{\text{Spin-}\mathbb{Z}/8\mathbb{Z}}(\text{pt}) \cong \Omega_*^{\text{Spin}}((B\mathbb{Z}/4\mathbb{Z})^{\rho-2}). \quad (13.14)$$

By (10.37), we need to compute $ko_*(((B\mathbb{Z}/4\mathbb{Z})^{\rho-2})$ in degrees 11 and below as well as $ko\langle 2 \rangle_*((B\mathbb{Z}/4\mathbb{Z})^{\rho-2})$ in degrees 1 and below. Lemma 10.39 and the mod 2 Thom isomorphism take care of the latter: we learn $ko\langle 2 \rangle_k((B\mathbb{Z}/4\mathbb{Z})^{\rho-2})$ is $\mathbb{Z}/2\mathbb{Z}$ for $k = 0$ and $k = 1$.

Theorem 13.15. *The first several ko-homology groups of $(B\mathbb{Z}/4\mathbb{Z})^{\rho-2}$ are:*

$$\begin{aligned} ko_0((B\mathbb{Z}/4\mathbb{Z})^{\rho-2}) &\cong \mathbb{Z} & ko_6((B\mathbb{Z}/4\mathbb{Z})^{\rho-2}) &\cong 0 \\ ko_1((B\mathbb{Z}/4\mathbb{Z})^{\rho-2}) &\cong \mathbb{Z}/8\mathbb{Z} & ko_7((B\mathbb{Z}/4\mathbb{Z})^{\rho-2}) &\cong \mathbb{Z}/4\mathbb{Z} \\ ko_2((B\mathbb{Z}/4\mathbb{Z})^{\rho-2}) &\cong 0 & ko_8((B\mathbb{Z}/4\mathbb{Z})^{\rho-2}) &\cong \mathbb{Z} \\ ko_3((B\mathbb{Z}/4\mathbb{Z})^{\rho-2}) &\cong \mathbb{Z}/2\mathbb{Z} & ko_9((B\mathbb{Z}/4\mathbb{Z})^{\rho-2}) &\cong (\mathbb{Z}/128\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z}) \\ ko_4((B\mathbb{Z}/4\mathbb{Z})^{\rho-2}) &\cong \mathbb{Z} & ko_{10}((B\mathbb{Z}/4\mathbb{Z})^{\rho-2}) &\cong 0 \\ ko_5((B\mathbb{Z}/4\mathbb{Z})^{\rho-2}) &\cong (\mathbb{Z}/32\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) & ko_{11}((B\mathbb{Z}/4\mathbb{Z})^{\rho-2}) &\cong \mathbb{Z}/8\mathbb{Z}. \end{aligned} \quad (13.16)$$

Barrera-Yanez [194, Theorem 3.1] computes $ko_{2k+1}((B\mathbb{Z}/4\mathbb{Z})^{\rho-2})$ using analytic methods. We nonetheless work through the spectral sequence computation because we will need it when we study $\text{Spin-GL}^+(2, \mathbb{Z})$ bordism in Section 14.

Proof. For ease of reading, let $X := (B\mathbb{Z}/4\mathbb{Z})^{\rho-2}$. We use the Adams spectral sequence, as we reviewed in Section 11. The steps of the problem are:

1. Determine the $\mathcal{A}(1)$ -module structure on $H^*(X; \mathbb{Z}/2\mathbb{Z})$ using the techniques we reviewed in Section 11.1.
2. Use this to compute $\text{Ext}_{\mathcal{A}(1)}^{s,t}(H^*(X; \mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z})$, which is the E_2 -page of the Adams spectral sequence. We discussed Ext and how to determine it in Section 11.2.
3. Compute the differentials in the spectral sequence. We discussed differentials in general Adams spectral sequences in Section 11.3; in this example, (a variant of) the May-Milgram theorem computes all the differentials for us. We discuss this tool in more detail in Appendix D.
4. Finally, we need to resolve some extension questions; we discussed the generalities of extension questions in Section 11.4. To resolve the extension questions in the calculation of $ko_*(X)$, we use calculations of η -invariants of lens spaces and lens space bundles from Appendix C.3.

First, the $\mathcal{A}(1)$ -module structure on cohomology. The Thom isomorphism (10.15) provides an isomorphism $U: H^*(B\mathbb{Z}/4\mathbb{Z}; \mathbb{Z}/2\mathbb{Z}) \rightarrow H^*(X; \mathbb{Z}/2\mathbb{Z})$; the class $U := U(1)$ is called the *Thom class*.

Proposition 11.4 computes $\text{Sq}(U\alpha)$ in terms of $\text{Sq}(\alpha)$ and $w(\rho)$. We just need Sq^1 and Sq^2 , so we just need w_1 and w_2 . Since ρ has image contained in $\text{SO}(2)$, w_1 vanishes, and Lemma 13.12 tells us $w_2 = y$. We calculated the Steenrod squares of classes in $H^*(B\mathbb{Z}/4\mathbb{Z}; \mathbb{Z}/2\mathbb{Z})$ in Example 11.7. We learn $\text{Sq}^1(x) = 0$, $\text{Sq}^2(x) = 0$, $\text{Sq}^1(y) = 0$, and $\text{Sq}^2(y) = y^2$. Using Proposition 11.4, which tells us $\text{Sq}^1(U\alpha) = \text{Sq}^1(U)\alpha + USq^1(\alpha)$ and $\text{Sq}^2(U\alpha) = \text{Sq}^2(U)\alpha + \text{Sq}^1(U)\text{Sq}^1(\alpha) + USq^2(\alpha)$, we learn that Sq^1 vanishes on all classes in $H^*(X; \mathbb{Z}/2\mathbb{Z})$, and that

$$\text{Sq}^2(U\alpha) = U\alpha y + USq^2(\alpha). \quad (13.17)$$

Using this, we can completely describe the $\mathcal{A}(1)$ -module structure on $H^*(X; \mathbb{Z}/2\mathbb{Z})$. This module splits as a direct sum of modules which are two-dimensional vector spaces over $\mathbb{Z}/2\mathbb{Z}$. Specifically, if $C\eta$ denotes the $\mathcal{A}(1)$ -module consisting of $\mathbb{Z}/2\mathbb{Z}$ summands in degrees 0 and 2 connected by a Sq^2 , then

$$H^*((B\mathbb{Z}/4\mathbb{Z})^{\rho-2}; \mathbb{Z}/2\mathbb{Z}) \cong \textcolor{red}{C\eta} \oplus \Sigma \textcolor{brown}{C\eta} \oplus \Sigma^4 \textcolor{brown}{C\eta} \oplus \Sigma^5 \textcolor{green}{C\eta} \oplus \Sigma^8 \textcolor{teal}{C\eta} \oplus \Sigma^9 \textcolor{teal}{C\eta} \oplus \Sigma^{12} \textcolor{violet}{C\eta} \oplus P, \quad (13.18)$$

where P is 12-connected, i.e. has no nonzero classes in degrees 12 or below. We only care about the Adams spectral sequence in degrees 12 and below, which means we can (and will) ignore P . We draw the $\mathcal{A}(1)$ -module structure on $H^*(X; \mathbb{Z}/2\mathbb{Z})$ in Figure 17.

Now to determine the Adams E_2 -page. The E_2 -page of the Adams spectral sequence splits as the direct sum of the Ext groups of all of the summands we identified in (13.18). For a single $C\eta$ summand, one can look this up in, e.g., Beaudry-Campbell [115, Figure 22], so we can draw the E_2 -page in Figure 18, top.

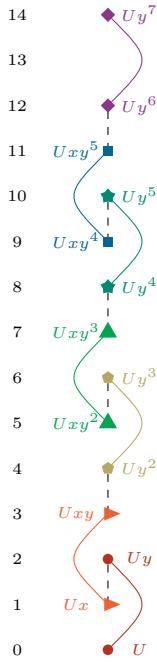


Figure 17: The mod 2 cohomology of $H^*((B\mathbb{Z}/4\mathbb{Z})^{\rho-2}; \mathbb{Z}/2\mathbb{Z})$ in low degrees. Different colors/shapes represent different summands, and the curved lines represent Sq^2 -actions as described in Section 11.1. The dashed lines indicate the presence of a non-trivial Bockstein $\beta: H^k((B\mathbb{Z}/4\mathbb{Z})^{\rho-2}; \mathbb{Z}/4\mathbb{Z}) \rightarrow H^{k+1}((B\mathbb{Z}/4\mathbb{Z})^{\rho-2}; \mathbb{Z}/4\mathbb{Z})$ from a preimage of the lower class to a preimage of the higher class. This figure is complete in degrees 12 and below.

There are plenty of differentials that we cannot rule out using the h_i -action. One can quickly check that many of them are nonzero, e.g. by computing $ko_*(X) \otimes \mathbb{Q}$ using the fact that, after tensoring with \mathbb{Q} , the Atiyah-Hirzebruch spectral sequence always collapses [195, Bemerkung 14.18].

The good news is, the E_2 -page has enough structure to allow us to determine the differentials without too much work.

Definition 13.19. An h_0 -tower in the E_r -page of the Adams spectral sequence is an infinite sequence $x_i \in E_r^{s+i, t-s-i}$ such that $h_0 x_i = x_{i+1}$.

Looking at the E_2 -page in Figure 18, top, we see lots of h_0 -towers in our spectral sequence. Because differentials commute with h_0 -actions, if we know d_r on one element of an h_0 -tower, we know d_r on the entire tower; for this reason it is common to refer to differentials between h_0 -towers, instead of just elements.

May-Milgram [174] relate differentials between h_0 -towers to Bockstein homomorphisms in cohomology. The latter are typically not so hard to compute, suggesting an approach to solving our Adams spectral sequence: compute Bocksteins in $H^*(X)$ and use them to determine Adams differentials. We discuss the May-Milgram theorem in greater detail in

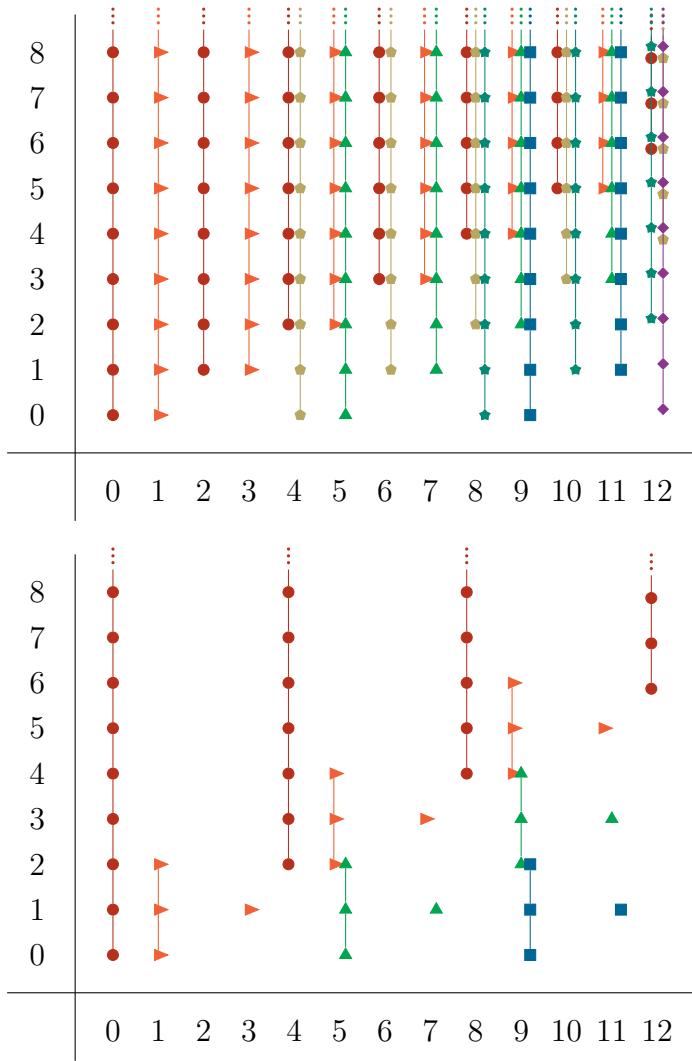


Figure 18: The Adams spectral sequence computing $ko_*((B\mathbb{Z}/4\mathbb{Z})^{\rho-2}; \mathbb{Z}/2\mathbb{Z})$. Top: the E_2 -page. Bottom: the E_∞ -page.

Appendix D.

We will actually use a variant of the May-Milgram theorem, Proposition D.13, tailored for spaces/spectra whose cohomology is a direct sum of copies of $C\eta$. Let $\beta_r: H^n(-; \mathbb{Z}/2^r\mathbb{Z}) \rightarrow H^{n+1}(-; \mathbb{Z}/2^r\mathbb{Z})$ be the Bockstein associated to the short exact sequence

$$0 \longrightarrow \mathbb{Z}/2^r\mathbb{Z} \longrightarrow \mathbb{Z}/2^{2r}\mathbb{Z} \longrightarrow \mathbb{Z}/2^r\mathbb{Z} \longrightarrow 0 ; \quad (13.20)$$

then Proposition D.13 says that if $\alpha \in H^n(X; \mathbb{Z}/2^r\mathbb{Z})$ and $\beta_r(\alpha) \in H^{n+1}(X; \mathbb{Z}/2^r\mathbb{Z})$ both generate $\mathbb{Z}/2^r$ summands and the mod 2 reductions of α and $\beta_r(\alpha)$ belong to $C\eta$ $\mathcal{A}(1)$ -module summands M_0 , resp. M_1 of $H^*(X; \mathbb{Z}/2\mathbb{Z})$, then there is a d_r differential between the h_0 -towers in $\text{Ext}_{\mathcal{A}(1)}(X)$ in degrees $n + 4k$ and $n + 4k + 1$ coming from M_0 and M_1 , and conversely differentials imply such Bockstein relations. We will use Proposition D.13 as a black box in this section and direct the curious reader to read the proof and context in Appendix D.

The Thom isomorphism commutes with the Bockstein β_r when $r > 1$,⁴⁷ so we can compute the action of β_r in $H^*(X; \mathbb{Z}/2^r\mathbb{Z})$ using the action in $H^*(B\mathbb{Z}/4\mathbb{Z}; \mathbb{Z}/2^r\mathbb{Z})$. It is a standard fact⁴⁸ that there is a class α in degree n satisfying the conditions of Proposition D.13 outlined in the previous paragraph exactly when $r = 2$ and n is odd, and that the mod 2 reductions of α and $\beta_r(\alpha)$ are Uxy^k and Uy^{k+1} where $k = (n - 1)/2$; thus, in Figure 17, we represent these Bocksteins with dotted lines.

Invoking Proposition D.13 on these classes, we learn the following differentials.

1. Suppose $x \in E_2^{s,t}$ is an element of a red h_0 -tower, i.e. coming from the copy of $C\eta \subset H^*(X; \mathbb{Z}/2\mathbb{Z})$. If $t - s \equiv 0 \pmod{4}$, $d_r(x) = 0$ for all r . If $t - s \equiv 2 \pmod{4}$, $d_2(x)$ is nonzero, and is an element of the h_0 -tower in degree $t - s - 1$ coming from $\text{Ext}(\Sigma C\eta)$ (right-pointing triangles). These differentials follow from the absence of any Bockstein in degree 0 and the presence of a Bockstein on $\mathbb{Z}/4\mathbb{Z}$ -cohomology lifts of Ux and Uy .
2. All differentials emerging from the h_0 -towers coming from $\text{Ext}(\Sigma C\eta)$ (right-pointing triangles) vanish, because $\mathbb{Z}/4\mathbb{Z}$ -cohomology preimages of nonzero elements of $\Sigma C\eta$ are not in the image of β_2 . An analogous statement is true for $\Sigma^5 C\eta$ (upward-pointing triangles) and $\Sigma^9 C\eta$ (squares).
3. If $x \in E_2^{s,t}$ is an element of an h_0 -tower coming from $\text{Ext}(\Sigma^4 C\eta)$ (pentagons), $d_2(x) \neq 0$. If $t - s \equiv 0 \pmod{4}$, $d_2(x) \in \text{Ext}(\Sigma C\eta)$ (right-pointing triangles), and if $t - s \equiv 2 \pmod{4}$, $d_2(x) \in \text{Ext}(\Sigma^5 C\eta)$ (upward-pointing triangles). These differentials follow from Bocksteins in $\mathbb{Z}/4\mathbb{Z}$ -valued cohomology whose mod 2 reductions are $Uxy \mapsto Uy^2$, resp. $Uxy^2 \mapsto Uy^3$.

⁴⁷For $r = 1$, one must be careful with twisted versus untwisted coefficients.

⁴⁸To prove this standard fact, recall from Appendix D (specifically, around (D.5)) that β_r acts non-trivially on $H^{n-1}(X; \mathbb{Z}/2^r\mathbb{Z})$ if and only if $H^n(X; \mathbb{Z})$ has 2^r -torsion; then compute using the identification $H^*(B\mathbb{Z}/4\mathbb{Z}; \mathbb{Z}) \cong \mathbb{Z}[y]/4y$, with $|y| = 2$.

4. If $x \in E_2^{s,t}$ is an element of an h_0 -tower coming from $\text{Ext}(\Sigma^8 C\eta)$ (stars), $d_2(x) \neq 0$. If $t-s \equiv 0 \pmod{4}$, $d_2(x) \in \text{Ext}(\Sigma^5 C\eta)$ (upward-pointing triangles), and if $t-s \equiv 2 \pmod{4}$, $d_2(x) \in \text{Ext}(\Sigma^9 C\eta)$ (squares). These differentials follow from Bocksteins in $\mathbb{Z}/4\mathbb{Z}$ -valued cohomology whose mod 2 reductions are $Uxy^3 \mapsto Uy^4$, resp. $Uxy^4 \mapsto Uy^5$.
5. If x is an element of the h_0 -tower in topological degree 12 coming from $\text{Ext}(\Sigma^{12} C\eta)$ (diamonds), then $d_2(x) \in \text{Ext}(\Sigma^9 C\eta)$ (squares), which follows from a β_2 whose mod 2 reduction sends $Uxy^5 \mapsto Uy^6$.

One could continue, but we do not need to. This is all of the differentials in the range we care about, and we display the E_∞ -page in Figure 18, bottom. From this we can immediately deduce all of the ko -homology groups in range except in degrees 5, 7, 9, and 11, where there could be extension problems. Specifically:

- In degree 5, we could have $(\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z})$, $(\mathbb{Z}/16\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z})$, or $(\mathbb{Z}/32\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$, depending on how the green pieces (upward-pointing triangles) interact with the orange pieces (rightward-pointing triangles) in the Adams filtration.
- In degree 7, we could have $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$ or $\mathbb{Z}/4\mathbb{Z}$.
- In degree 9, there are a lot of options. The answer must be an Abelian group of order 2^9 and must have either two or three cyclic summands. No element has order greater than 2^7 . For example, we could have $(\mathbb{Z}/8\mathbb{Z})^{\oplus 3}$, $(\mathbb{Z}/128\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$, $(\mathbb{Z}/128\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z})$, or other options.
- In degree 11, we could have $(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$, $(\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$, or $\mathbb{Z}/8\mathbb{Z}$.

These extension questions have been studied by Hsieh [110, Section 2.2] and Hsieh-Tachikawa-Yonekura [32, Section 8.4] in degree 5 and by Barrera-Yanez [194, Theorem 3.1] in all odd degrees. Their proofs compute bordism invariants built from η -invariants. For example, they show that a particular η -invariant evaluated on L_4^5 is equal to $1/32 \pmod{\mathbb{Z}}$, and that the value of this η -invariant in \mathbb{R}/\mathbb{Z} is a bordism invariant of Spin- $\mathbb{Z}/8\mathbb{Z}$ manifolds, implying that $\Omega_5^{\text{Spin-}\mathbb{Z}/8\mathbb{Z}}(\text{pt})$ has a $\mathbb{Z}/32\mathbb{Z}$ subgroup, hence must be $(\mathbb{Z}/32\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$. We resolve the extension problems in a similar way to Barrera-Yanez, using formulas for η -invariants calculated by Atiyah-Patodi-Singer [196, Proposition 2.12], Donnelly [197, Section 4], Botvinnik-Gilkey-Stolz [193], and Hsieh-Tachikawa-Yonekura [32, Appendix C]. Specifically, in Section C.3, if L_4^{2k-1} denotes the lens space $S^{2k-1}/(\mathbb{Z}/4\mathbb{Z})$ and Q_4^{2k-1} is the L_4^{2k-3} -bundle over S^2 defined in Section 12.3, then

$$\begin{aligned} \eta_{1/2}^D(L_4^5) &= -\frac{5}{32} & \eta_{1/2}^D(L_4^9) &= \frac{9}{128} \\ \eta_{3/2}^D(\tilde{L}_4^5) &= -\frac{3}{32} & \eta_{3/2}^D(\tilde{L}_4^9) &= -\frac{7}{128} \\ (\eta_{3/2}^D - \eta_{1/2}^D)(Q_4^7) &= \frac{1}{4} & (\eta_{3/2}^D - \eta_{1/2}^D)(Q_4^{11}) &= \frac{1}{8}. \end{aligned} \tag{13.21}$$

In order to say this we must specify the Spin- $\mathbb{Z}/8\mathbb{Z}$ structures on these lens space bundles; we do this in Appendix C, so see there for details.

Since the values of these η -invariants mod \mathbb{Z} are bordism invariants, these imply that $ko_5(X) \cong (\mathbb{Z}/32\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$, $ko_7(X) \cong \mathbb{Z}/4$, $ko_9(X) \cong (\mathbb{Z}/128\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z})$, and $ko_{11}(X) \cong \mathbb{Z}/8\mathbb{Z}$.

Though all previous calculations of these groups used analytic methods, we can address the extension problem in degree 5 purely with algebra, providing a check on the η -invariant computations. It would be interesting to try to generalize this to the extension questions in dimensions 7, 9, and 11.

Lemma 13.22. $ko_5((B\mathbb{Z}/4\mathbb{Z})^{\rho-2}) \cong (\mathbb{Z}/32\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$.

Proof. Recall from Remark 10.40 the η , c , R long exact sequence (10.41); we will study this long exact sequence for the ko - and ku -homology of $X = (B\mathbb{Z}/4\mathbb{Z})^{\rho-2}$. Since $V \rightarrow B\mathbb{Z}/4\mathbb{Z}$ is a complex vector bundle, it is in particular Spin^c . The generalized cohomology theory ku is oriented for Spin^c vector bundles [198, 199],⁴⁹ so there is a Thom isomorphism $ku_*((B\mathbb{Z}/4\mathbb{Z})^{\rho-2}) \cong ku_*(B\mathbb{Z}/4\mathbb{Z})$. Hashimoto [191] shows $ku_5(B\mathbb{Z}/4\mathbb{Z}) \cong (\mathbb{Z}/16\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$.

Our three options for $ko_5((B\mathbb{Z}/4\mathbb{Z})^{\rho-2})$ are $(\mathbb{Z}/8\mathbb{Z})^{\oplus 2}$, $(\mathbb{Z}/16\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z})$, and $(\mathbb{Z}/32\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$. None of these Abelian groups can map injectively into $(\mathbb{Z}/16\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$, so $c: ko_5((B\mathbb{Z}/4\mathbb{Z})^{\rho-2}) \rightarrow ku_5(B\mathbb{Z}/4\mathbb{Z})$ is not injective. Exactness of (10.41) implies that the map $\eta: ko_4((B\mathbb{Z}/2\mathbb{Z})^{\rho-2}) \rightarrow ko_5((B\mathbb{Z}/2\mathbb{Z})^{\rho-2})$ is nonzero,⁵⁰ since $2\eta = 0$ in $\pi_1(\mathbb{S})$, $2\eta(x) = 0$ for any $x \in ko_4((B\mathbb{Z}/4\mathbb{Z})^{\rho-2})$. Because $ko_4((B\mathbb{Z}/4\mathbb{Z})^{\rho-2})$ has a single generator, the image of η is a $\mathbb{Z}/2\mathbb{Z}$ subgroup of $ko_5((B\mathbb{Z}/4\mathbb{Z})^{\rho-2})$, and exactness of (10.41) then implies

$$c/\ker(c): ko_5((B\mathbb{Z}/4\mathbb{Z})^{\rho-2})/(\mathbb{Z}/2\mathbb{Z}) \longrightarrow ku_5((B\mathbb{Z}/4\mathbb{Z})^{\rho-2}) \cong (\mathbb{Z}/16\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}, \quad (13.23)$$

is injective. If $ko_5((B\mathbb{Z}/4\mathbb{Z})^{\rho-2})$ were isomorphic to $(\mathbb{Z}/8\mathbb{Z})^{\oplus 2}$, this could not happen: after quotienting out by $\mathbb{Z}/2\mathbb{Z}$, there would be too many elements x with $2x \neq 0$.

With a little more work, we can also eliminate the possibility $(\mathbb{Z}/16\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z})$: since η has Adams filtration 1 in the Adams spectral sequence for the sphere spectrum, multiplication by η must raise Adams filtration by at least 1. Looking at Figure 18, bottom, the generator x of $ko_4((B\mathbb{Z}/4\mathbb{Z})^{\rho-2})$ has Adams filtration at least 2, because its image in the E_∞ -page is in filtration 2. Thus ηx has filtration at least 3 and $2\eta = 0$; looking at the 5-line of the E_∞ -page in Figure 18, bottom, this can only occur if the image of ηx in the E_∞ -page is the generator of $E_\infty^{4,9} \cong \mathbb{Z}/2\mathbb{Z}$. In particular, the h_0 -actions on the 5-line imply there is some class $y \in ko_5((B\mathbb{Z}/4\mathbb{Z})^{\rho-2})$ such that $4y = \eta$. Thus if there is an isomorphism

⁴⁹Alternatively, one can use the ku -orientation for complex vector bundles constructed by Conner-Floyd [200, Section 5].

⁵⁰One expects to see the action of η in the E_∞ -page of the Adams spectral sequence, as it is detected by $h_1 \in \text{Ext}(\mathbb{Z}/2\mathbb{Z})$. This η -action is not seen by the E_∞ -page, which makes it an example of a “hidden multiplicative extension.” See Section 13.4.

$ko_5((B\mathbb{Z}/4\mathbb{Z})^{\rho-2}) \cong (\mathbb{Z}/16\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z})$, ηx is identified with either $(8, 0)$ or $(8, 2)$. Thus the quotient by ηx is $(\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z})$, which does not inject into $(\mathbb{Z}/16\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$.

The only remaining option is $ko_5((B\mathbb{Z}/4\mathbb{Z})^{\rho-2}) \cong (\mathbb{Z}/32\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$. \square

This finishes the calculation of Spin- $\mathbb{Z}/8\mathbb{Z}$ bordism groups in dimensions 11 and below. \square

Collecting all the pieces we find the Spin-Mp($2, \mathbb{Z}$) bordism groups summarized in Table 9. For convenience we already include the various generating manifolds that we will explore

k	$\Omega_k^{\text{Spin-Mp}(2, \mathbb{Z})}(\text{pt})$	Generators
0	\mathbb{Z}	pt_+
1	$(\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})$	(L_1^4, L_3^1)
2	0	
3	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})$	(L_4^3, L_3^3)
4	\mathbb{Z}	E
5	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/32\mathbb{Z}) \oplus (\mathbb{Z}/9\mathbb{Z})$	$(\tilde{L}_4^5, L_4^5, L_3^5)$
6	0	
7	$(\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/9\mathbb{Z})$	(Q_4^7, L_3^7)
8	$\mathbb{Z} \oplus \mathbb{Z}$	(B, \mathbb{HP}^2)
9	$(\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/128\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/27\mathbb{Z})$	$(\tilde{L}_4^9, \mathbb{HP}^2 \times L_4^1, L_4^9, \mathbb{HP}^2 \times L_3^1, L_3^9)$
10	$(\mathbb{Z}/2\mathbb{Z})$	X_{10}
11	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2} \oplus (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/27\mathbb{Z})$	$(\mathbb{HP}^2 \times L_4^3, X_{10} \times L_4^1, Q_4^{11}, \mathbb{HP}^2 \times L_3^3, L_3^{11})$

Table 9: Bordism groups $\Omega_k^{\text{Spin-Mp}(2, \mathbb{Z})}(\text{pt})$ and their generators (in the same order as group summands) for $k \leq 11$.

next.

13.3 Finding generators

At odd primes, there is no difference between $\Omega_*^{\text{Spin}}(B\text{SL}(2, \mathbb{Z}))$ and $\Omega_*^{\text{Spin-Mp}(2, \mathbb{Z})}(\text{pt})$, so we can recycle the generators from Section 12.3. And at prime 2 we do not have to modify our strategy much: Botvinnik-Gilkey [155, Theorem 4.5] show that the \hat{A} -genus and η -invariants completely detect the ko_* summand of Spin- $\mathbb{Z}/2^k\mathbb{Z}$ bordism when $k \geq 3$. We already determined the specific lens spaces and η -invariants for this in and around (13.21),

so we have found the generators coming from the 2-torsion in ko_* . For ko_{*-8} , much like we did for $\Omega_*^{\text{Spin}}(\text{BSL}(2, \mathbb{Z}))$, take the product of \mathbb{HP}^2 with the corresponding generator of $ko_*((B\mathbb{Z}/4)^{\rho-2})$; and for $ko\langle 2 \rangle_{*-10}$, Lemma 10.39 means we can detect generators with mod 2 cohomology classes in the degrees we need, giving us the generators X_{10} in degree 10 and $X_{10} \times L_4^1$ in degree 11.

13.4 Multiplicative structure

Because of the use of products in studying compactifications, as we discussed in Remark 10.4, we discuss how one can compute the product structure on Spin- $\mathbb{Z}/8\mathbb{Z}$ bordism in this subsection.

The product of two Spin- $\mathbb{Z}/2k\mathbb{Z}$ manifolds (M, P) and (N, Q) (here $P \rightarrow M$ and $Q \rightarrow N$ are the associated $\mathbb{Z}/k\mathbb{Z}$ -bundles) has an induced Spin- $\mathbb{Z}/2k\mathbb{Z}$ structure, given in informal terms by adding the characteristic classes of P and Q . In a little more detail, let \otimes denote the tensor product of $\mathbb{Z}/k\mathbb{Z}$ -bundles, which lifts the addition operation of their characteristic classes in $H^1(-; \mathbb{Z}/k\mathbb{Z})$. If $\pi_1: M \times N \rightarrow M$ and $\pi_2: M \times N \rightarrow N$ are the projections onto the first, respectively second factor, then there are isomorphisms

$$\begin{aligned} T(M \times N) \oplus (\pi_1^* P \otimes \pi_2^* Q)_\rho &\xrightarrow{\cong} \pi_1^* TM \oplus \pi_2^* TN \oplus \pi_1^* P_\rho \oplus \pi_2^* Q_\rho \\ &\cong \pi_1^*(TM \oplus P_\rho) \oplus \pi_2^*(TN \oplus Q_\rho), \end{aligned} \quad (13.24)$$

so Spin structures on $TM \oplus P_\rho$ and $TN \oplus Q_\rho$, i.e. Spin- $\mathbb{Z}/k\mathbb{Z}$ structures on (M, P) and (N, Q) , induce a Spin structure on $T(M \times N) \oplus (\pi_1^* P \otimes \pi_2^* Q)_\rho$, i.e. a Spin- $\mathbb{Z}/k\mathbb{Z}$ structure on $(M \times N, \pi_1^* P \otimes \pi_2^* Q)$. Allowing P and/or Q to be trivial shows this product is compatible with the product on Spin bordism, implying that Spin- $\mathbb{Z}/2k\mathbb{Z}$ bordism is not just a ring, but also an Ω_*^{Spin} -algebra.

We briefly describe how to make the ring structure on Spin- $\mathbb{Z}/8\mathbb{Z}$ bordism explicit in low dimensions. In high dimensions, determining the complete ring structure on Ω_*^{Spin} is still open (see, e.g., [201]), and the ring structure on Spin- $\mathbb{Z}/8\mathbb{Z}$ bordism is likely to be yet more complicated, so we focus on dimensions 11 and below. Most products we might care about vanish for degree reasons, or because torsion cannot map non-trivially to free summands. The only interesting products are of the form

$$\Omega_{4k}^{\text{Spin-}\mathbb{Z}/8\mathbb{Z}} \times \Omega_{4\ell}^{\text{Spin-}\mathbb{Z}/8\mathbb{Z}} \longrightarrow \Omega_{4(k+\ell)}^{\text{Spin-}\mathbb{Z}/8\mathbb{Z}} \quad (13.25a)$$

and

$$\Omega_{4k}^{\text{Spin-}\mathbb{Z}/8\mathbb{Z}} \times \Omega_{2\ell+1}^{\text{Spin-}\mathbb{Z}/8\mathbb{Z}} \longrightarrow \Omega_{4k+2\ell+1}^{\text{Spin-}\mathbb{Z}/8\mathbb{Z}}. \quad (13.25b)$$

For products of the first type, both bordism groups are free Abelian, and so one can compute products by tensoring with \mathbb{Q} and using rational characteristic classes including Pontrjagin

classes.⁵¹

For products of type (13.25b), Botvinnik-Gilkey [155] show that $\Omega_{4k+2\ell+1}^{\text{Spin-Z}/8\mathbb{Z}}$ is detected by η -invariants, so one can compute the product structure in Spin- $\mathbb{Z}/8\mathbb{Z}$ bordism by computing η -invariants of product manifolds, aided by decomposition results of the form $\eta(M \times N) = \text{Index}(M)\eta(N)$.

Most of the Ω_*^{Spin} -algebra structure on Spin- $\mathbb{Z}/8\mathbb{Z}$ bordism is determined by the action of $\text{Ext}(\mathbb{Z}/2\mathbb{Z})$ on the E_∞ -page corresponding to Spin- $\mathbb{Z}/8\mathbb{Z}$ bordism, together with the extensions we found. However, there can also be extensions involving not just multiplication by 2^m , but also products with various elements of Ω_*^{Spin} . For example, $\eta = [S_p^1]$ corresponds to $h_1 \in \text{Ext}(\mathbb{Z}/2\mathbb{Z})$, but there can be products of the form $y = \eta x$ such that, if \bar{x} and \bar{y} are the images of x and y in the E_∞ -page, $h_1 \bar{x} = 0$.

This happens in Spin- $\mathbb{Z}/8\mathbb{Z}$ bordism, and in the lowest degree possible! Looking at Figure 18, bottom, h_1 acts trivially on the 0-line, suggesting that S_p^1 bounds as a Spin- $\mathbb{Z}/8\mathbb{Z}$ manifold, but this is false! Recall from the proof of Lemma 13.22 that $\eta: \Omega_4^{\text{Spin-Z}/8\mathbb{Z}} \rightarrow \Omega_5^{\text{Spin-Z}/8\mathbb{Z}}$ is nonzero, and sends the generator to $(16, 0) \in \Omega_5^{\text{Spin-Z}/8\mathbb{Z}} \cong (\mathbb{Z}/32\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$. The only way for this to be compatible with the multiplication of classes in degrees 0, 4, and 8 is for $\eta: \Omega_{4k}^{\text{Spin-Z}/8\mathbb{Z}} \rightarrow \Omega_{4k+1}^{\text{Spin-Z}/8\mathbb{Z}}$ to always be nonzero. With $k = 0$, we learn that the Spin- $\mathbb{Z}/8\mathbb{Z}$ bordism class of S_p^1 is non-trivial and is divisible by 4.⁵² We display all of the hidden extensions that we have unearthed in Figure 19.

A similar story holds for Spin- $\mathbb{Z}/2k\mathbb{Z}$ bordism for all even k : the class of S_p^1 is nonzero, and is divisible by k . When $k \equiv 0 \pmod{4}$, this is a hidden multiplicative extension; when $k \equiv 2 \pmod{4}$, this extension comes from an h_1 -action on the E_∞ -page.⁵³

14 Computation of $\Omega_*^{\text{Spin-GL}^+(2,\mathbb{Z})}(\text{pt})$

The full duality group of type IIB string theory is the Pin^+ cover of $\text{GL}(2, \mathbb{Z})$, which we denote by $\text{GL}^+(2, \mathbb{Z})$. Thus $\text{GL}^+(2, \mathbb{Z})$ contains a canonical $\mathbb{Z}/2\mathbb{Z}$ subgroup, the kernel of the quotient $\text{GL}^+(2, \mathbb{Z}) \rightarrow \text{GL}(2, \mathbb{Z})$. The nonzero element of this $\mathbb{Z}/2\mathbb{Z}$ subgroup acts by fermion parity, so the spacetime symmetry type is $\text{Spin-GL}^+(2, \mathbb{Z}) := \text{Spin} \times_{\mathbb{Z}/2\mathbb{Z}} \text{GL}^+(2, \mathbb{Z})$. In this section, we determine the bordism groups and their generators for this symmetry type in dimensions 11 and below; just like for $\text{Spin} \times \text{SL}(2, \mathbb{Z})$ and $\text{Spin-Mp}(2, \mathbb{Z})$, we make heavy

⁵¹In high enough dimensions, one has to also use analogues of Pontrjagin classes in KO -theory, which is a theorem of Anderson-Brown-Peterson [202]. See Freed-Hopkins [40, Appendix B] for an example computation with these characteristic classes.

⁵²There is also a more direct proof by studying the image of $[S_p^1] \in \Omega_1^{\text{Spin}}$ under the map of Atiyah-Hirzebruch spectral sequences induced by the unit map $\mathbb{S} \rightarrow (B\mathbb{Z}/4\mathbb{Z})^{\rho-2}$.

⁵³This h_1 -action appears in [150, Section 2] but was left out by [109, Figure 7.4] and [115, Figures 25 and 26]. To see it, compute the long exact sequence in Ext associated to the short exact sequence of $\mathcal{A}(1)$ -modules $0 \rightarrow \Sigma R_0 \rightarrow R_1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$, where R_0 and R_1 are as defined in [115, Section 4.6]. Campbell [109, Section 7.8] shows that the cohomology of the Thom spectrum corresponding to Spin- $\mathbb{Z}/4\mathbb{Z}$ bordism is isomorphic to R_1 .

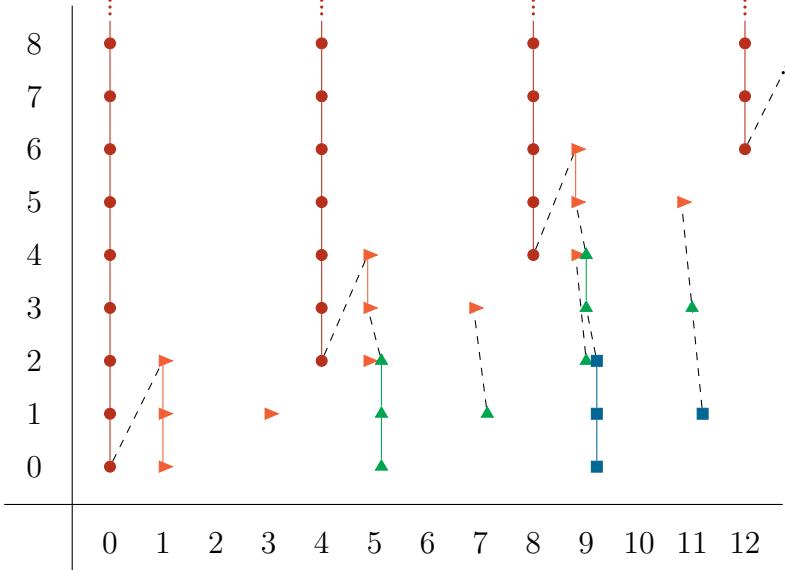


Figure 19: Adding hidden extensions to the E_∞ -page for $k o_*((B\mathbb{Z}/4\mathbb{Z})^{p-2})$; compare Figure 18, bottom. In this figure, all lines that do not shift the topological degree (x -coordinate) represent multiplication by 2 in Spin-\$\mathbb{Z}/8\mathbb{Z}\$ bordism, and all lines that increase the topological degree by 1 represent multiplication by $[S^1_p]$. The dashed lines are hidden extensions that we cannot see from the Adams spectral sequence.

use of an amalgamated product decomposition for $\mathrm{GL}^+(2, \mathbb{Z})$, which we prove in Appendix B:

$$\mathrm{GL}^+(2, \mathbb{Z}) \xrightarrow{\cong} D_{16} *_{D_8} D_{24}. \quad (14.1)$$

We compute at odd primes in Section 14.1 using the Atiyah-Hirzebruch spectral sequence and at $p = 2$ in Section 14.2 using the Adams spectral sequence. In Section 14.3 we determine generators of the bordism groups we calculate; this is the lengthiest part of the calculation. We identify families of generators coming from Spin-\$\mathrm{Mp}(2, \mathbb{Z})\$ bordism, Spin bordism of $B\mathbb{Z}/2\mathbb{Z}$, and Spin-\$D_8\$ bordism in Sections 14.3.1, 14.3.2, and 14.3.3 respectively; after that we still have four generators left to find, and we construct each one in turn in Sections 14.3.4, 14.3.5, 14.3.6, and 14.3.7. Finally, in Section 14.4, we discuss multiplicative structures on Spin-\$D_{16}\$ bordism.

14.1 Working at odd primes

Because the map $\mathrm{Spin} \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{GL}^+(2, \mathbb{Z}) \rightarrow \mathrm{SO} \times \mathrm{GL}(2, \mathbb{Z})$ is a double cover, the map $B(\mathrm{Spin}-\mathrm{GL}^+(2, \mathbb{Z})) \rightarrow B\mathrm{SO} \times B\mathrm{GL}(2, \mathbb{Z})$ is a fibration with fiber $B\mathbb{Z}/2\mathbb{Z}$. This implies that if p is an odd prime, the forgetful map $\Omega_*^{\mathrm{Spin}-\mathrm{GL}^+(2, \mathbb{Z})} \rightarrow \Omega_*^{\mathrm{SO}}(B\mathrm{GL}(2, \mathbb{Z}))$ is a p -local isomorphism; the proof is analogous to the argument we gave in Section 10.5 that $\Omega_*^{\mathrm{Spin}} \rightarrow \Omega_*^{\mathrm{SO}}$ is an odd-primary equivalence.

We will therefore compute $\Omega_*^{\text{SO}}(B\text{GL}(2, \mathbb{Z}))$ at the prime 3; at other odd primes, the map $B\text{GL}(2, \mathbb{Z}) \rightarrow \text{pt}$ induces an isomorphism on $\mathbb{Z}_{(p)}$ cohomology, hence also on p -local bordism.⁵⁴ In (B.10), we factored $\text{GL}(2, \mathbb{Z})$ as an amalgamated product. Using this and (12.4), we can show that the map $\Omega_*^{\text{SO}}(BD_{12}) \rightarrow \Omega_*^{\text{SO}}(B\text{GL}(2, \mathbb{Z}))$ is a 3-local isomorphism, with essentially the same proof as Lemma 12.2. There is an isomorphism $D_{12} \cong \mathbb{Z}/2\mathbb{Z} \times D_6$;⁵⁵ since the mod 3 cohomology of $B\mathbb{Z}/2\mathbb{Z}$ vanishes in positive degrees, $D_6 \hookrightarrow D_{12}$ induces an odd-primary equivalence on classifying spaces. Thus we should compute $\Omega_*^{\text{SO}}(BD_6)$. Since $|D_6|$ is only divisible by 2 and 3, this is trivial at primes 5 and above, so we focus on 3 as usual.

As we discussed in Section 12.2 (see especially (10.34)), for $k \leq 15$ there is a 3-local isomorphism

$$\Omega_k^{\text{SO}}(BD_6) \otimes \mathbb{Z}_{(3)} \xrightarrow{\cong} BP_k(BD_6) \oplus BP_{k-8}(BD_6) \oplus BP_{k-12}(BD_6). \quad (14.2)$$

Theorem 14.3.

$$\widetilde{BP}_k(BD_6) \cong \begin{cases} \mathbb{Z}/3\mathbb{Z}, & k = 3 \\ \mathbb{Z}/9\mathbb{Z}, & k = 7 \\ \mathbb{Z}/27\mathbb{Z}, & k = 11 \\ 0, & \text{all other } k \leq 11. \end{cases} \quad (14.4)$$

Proof. We compute $\widetilde{BP}_*(BD_6)$ using the Atiyah-Hirzebruch spectral sequence (AHSS)⁵⁶

$$E_{p,q}^2 = H_p(BD_6; BP_q) \Longrightarrow BP_{p+q}(BD_6). \quad (14.5)$$

Handel [203] computes $H^*(BD_{2n}; \mathbb{Z})$, and his result together with the universal coefficient theorem tells us $H_*(BD_6; \mathbb{Z}_{(3)})$:

$$H_k(BD_6; \mathbb{Z}_{(3)}) \cong \begin{cases} \mathbb{Z}_{(3)}, & k = 0 \\ \mathbb{Z}/3\mathbb{Z}, & k \equiv 3 \pmod{4} \\ 0, & \text{otherwise.} \end{cases} \quad (14.6)$$

Thus we can draw the E^2 -page of the Atiyah-Hirzebruch spectral sequence in Figure 20. All differentials vanish by degree reasons: in the homological AHSS, differentials go up and to the left, and decrease total degree by 1, so all such differentials are either to or from a zero

⁵⁴To see this claim about cohomology, use the Mayer-Vietoris sequence associated to the amalgamated product description of $\text{GL}(2, \mathbb{Z})$ given in (B.10), together with the fact that if G is a finite group and p does not divide the order of G , then $H^*(BG; \mathbb{Z}/p\mathbb{Z})$ vanishes in positive degrees [187, Corollary 10.2]. Once we know the map is an isomorphism on $\mathbb{Z}_{(p)}$ cohomology, the Atiyah-Hirzebruch spectral sequence also allows us to conclude it for p -local oriented bordism.

⁵⁵Inscribe the Star of David in a hexagon.

⁵⁶Compared with the Adams spectral sequence, we indicate the page by a superscript $E_{*,*}^2$ to emphasize that this sequence arises from homology.

group.

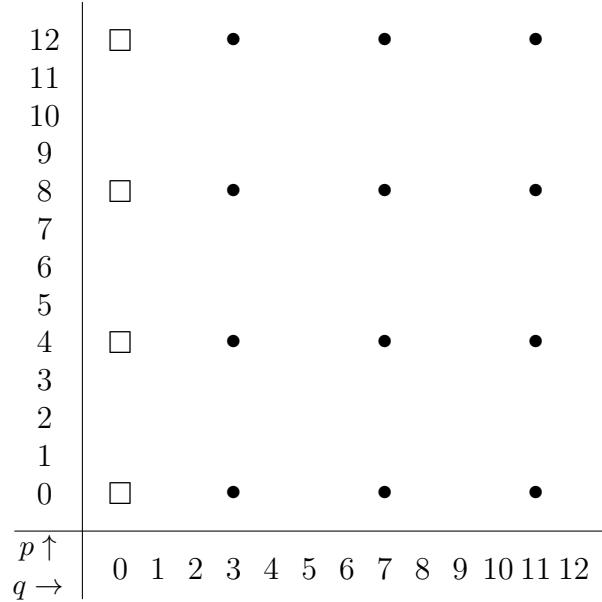


Figure 20: The E^2 -page of the Atiyah-Hirzebruch spectral sequence computing $\Omega_*^{SO}(BD_6) \otimes \mathbb{Z}_{(3)}$. Here $\square := \mathbb{Z}_{(3)}$ and $\bullet := \mathbb{Z}/3\mathbb{Z}$.

For reduced BP -homology, we quotient out by the $\mathbb{Z}_{(3)}$ s along the vertical line $p = 0$. So we have a $\mathbb{Z}/3\mathbb{Z}$ in degree 3 and two extension questions in degrees 7 and 11, and the remaining reduced BP -homology groups in the range we care about vanish.

The inclusion $\mathbb{Z}/3\mathbb{Z} \hookrightarrow D_6$ induces a map of Atiyah-Hirzebruch spectral sequences for $BP_*(B\mathbb{Z}/3\mathbb{Z}) \rightarrow BP_*(BD_6)$. The homology of $B\mathbb{Z}/3\mathbb{Z}$ is \mathbb{Z} in degree 0, $\mathbb{Z}/3\mathbb{Z}$ in all odd degrees, and 0 in positive even degrees, and the map of spectral sequences is an isomorphism on the E^2 -page in total degrees 3, 7, and 11. For the same reason as for BD_6 , this spectral sequence collapses for $B\mathbb{Z}/3\mathbb{Z}$; therefore the map on the E^∞ -pages of the AHSSes is an isomorphism in degrees $p + q = 7$ and $p + q = 11$, which implies the map $BP_k(B\mathbb{Z}/3\mathbb{Z}) \rightarrow BP_k(BD_6)$ is an isomorphism for $k = 7, 11$.

In Section 12.2, we saw that $BP_7(B\mathbb{Z}/3\mathbb{Z}) \cong \mathbb{Z}/9\mathbb{Z}$ and $BP_{11}(B\mathbb{Z}/3\mathbb{Z}) \cong \mathbb{Z}/27\mathbb{Z}$, resolving the extension question. \square

Remark 14.7. Another way to make this calculation is to use the ‘‘Leray-Serre-Atiyah-Hirzebruch spectral sequence’’

$$E_{p,q}^2 = H_p(B\mathbb{Z}/2\mathbb{Z}; BP_q(B\mathbb{Z}/3\mathbb{Z})) \implies BP_{p+q}(BD_6). \quad (14.8)$$

We found working through this variant to be a nice exercise — the spectral sequence collapses on the E^2 -page and yields an isomorphism $BP_*(B\mathbb{Z}/n\mathbb{Z})_{\mathbb{Z}/2\mathbb{Z}} \cong BP_*(BD_6)$.⁵⁷ Then we

⁵⁷There are at least two additional, and quite different, proofs of this fact due to Kamata-Minami [204, Theorem 1.3] and Kamata [205, Theorem 1.3].

directly compute this action and see what the coinvariants are.

Surjectivity of the map $\Omega_*^{\text{SO}}(B\mathbb{Z}/3\mathbb{Z}) \rightarrow \Omega_*^{\text{SO}}(BD_6)$ in the range we care about implies that the generators of the 3-torsion subgroup of $\Omega_*^{\text{Spin-GL}^+(2,\mathbb{Z})}$ in this range can be chosen to be the generators we found for $\Omega_k^{\text{SO}}(B\mathbb{Z}/3\mathbb{Z})$ for $k = 3, 7$, and 11 . We produced these generators in Section 12.3: L_3^3 in dimension 3, L_3^7 in dimension 7, and L_3^{11} and $\mathbb{HP}^2 \times L_3^3$ in dimension 11.

14.2 Adams spectral sequence at $p = 2$

Using the amalgamated product descriptions of $\text{Mp}(2, \mathbb{Z})$ and $\text{GL}^+(2, \mathbb{Z})$ in (B.6) and (B.12), there is a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}/8\mathbb{Z} & \xrightarrow{a} & \text{Mp}(2, \mathbb{Z}) \\ \downarrow & & \downarrow \\ D_{16} & \xrightarrow{b} & \text{GL}^+(2, \mathbb{Z}), \end{array} \quad (14.9)$$

where the vertical arrows are the standard inclusions, a is the Spin cover of the standard inclusion $\mathbb{Z}/4\mathbb{Z} \hookrightarrow \text{SL}(2, \mathbb{Z})$, and b is the Pin^+ cover of the standard inclusion $D_8 \hookrightarrow \text{GL}(2, \mathbb{Z})$. Now apply $B(\text{Spin} \times_{\mathbb{Z}/2\mathbb{Z}} -)$ to (14.9); then the horizontal arrows induce equivalences on mod 2 cohomology, which follows from a Mayer-Vietoris argument similar to Lemmas 12.2, 12.7, 13.4 and 13.10, using the amalgamated product descriptions of $\text{Mp}(2, \mathbb{Z})$ and $\text{GL}^+(2, \mathbb{Z})$ referenced above. Thus, in particular:

Corollary 14.10. *The inclusion b in (14.9) induces a map $\Omega_*^{\text{Spin} \times_{\mathbb{Z}/2\mathbb{Z}} D_{16}} \rightarrow \Omega_*^{\text{Spin-GL}^+(2,\mathbb{Z})}$, and this map is a 2-local equivalence, where $\mathbb{Z}/2\mathbb{Z} \subset D_{16}$ is generated by a half turn.*

Another consequence of (14.9) is that this equivalence is compatible with the 2-primary equivalences from $\text{Spin-Z}/8\mathbb{Z}$ bordism to $\text{Spin-Mp}(2, \mathbb{Z})$ bordism from Lemma 13.10. We will use this to compute the 2-torsion subgroup of the image of the map $\Omega_k^{\text{Spin-Mp}(2,\mathbb{Z})} \rightarrow \Omega_k^{\text{Spin-GL}^+(2,\mathbb{Z})}$ for $k \leq 11$ in Section 14.3.1.

First, we need to shear the Spin- D_{16} symmetry type. To do so, we will need to know the mod 2 cohomology of BD_8 .

Lemma 14.11 (Handel [203, Theorem 5.5]). *When $k \equiv 0 \pmod{4}$,*

$$H^*(BD_{2k}; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[x, y, w]/(xy + y^2) \quad (14.12)$$

with $|x| = |y| = 1$ and $|w| = 2$. If V is the standard two-dimensional real representation of D_{2k} and χ_y is the one-dimensional representation which is 1 on reflections and -1 on a generating rotation, then $x = w_1(V)$, $y = w_1(\chi_y)$, and $w = w_2(V)$.

We also need the Steenrod squares: $\text{Sq}(x) = x + x^2$ and $\text{Sq}(y) = y + y^2$ are forced by the axioms, and $\text{Sq}(w) = w + wx + w^2$ follows from the Wu formula (11.6).

Lemma 14.13. *The cohomology class of the central extension*

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow D_{16} \longrightarrow D_8 \longrightarrow 1 \quad (14.14)$$

is equal to w_2 of the standard representation $V: D_8 \rightarrow \text{O}(2)$.

Here and in the proof of the lemma, when we say “ w_k of a representation,” we mean w_k of its associated vector bundle.

Proof. First, observe that the extension (14.14) is split when pulled back by the inclusion of any reflection $\mathbb{Z}/2\mathbb{Z} \hookrightarrow D_8$, but non-trivial when pulled back to the rotation subgroup $\mathbb{Z}/4\mathbb{Z} \hookrightarrow D_8$. This is because the inclusion $\mathbb{Z}/2\mathbb{Z} \hookrightarrow D_8$ defining any reflection lifts to an inclusion $\mathbb{Z}/2\mathbb{Z} \hookrightarrow D_{16}$ defining another reflection, but the preimage of $\mathbb{Z}/4\mathbb{Z}$ in D_{16} is the rotation subgroup $\mathbb{Z}/8\mathbb{Z}$, and the quotient $\mathbb{Z}/8\mathbb{Z} \rightarrow \mathbb{Z}/4\mathbb{Z}$ does not split.

We will finish by showing $w_2(V)$ is the unique element of $H^2(BD_8; \mathbb{Z}/2\mathbb{Z})$ which is trivial when pulled back by the inclusion of a reflection subgroup, but non-trivial when pulled back to $B\mathbb{Z}/4\mathbb{Z}$. Since the classification of extensions by elements in H^2 is natural, this will force the class of (14.14) to be $w_2(V)$.

To compute pullback of $w_2(V)$ along the inclusion of a subgroup $H \hookrightarrow D_8$, it suffices to compute w_2 of the vector bundle induced by the restriction of the standard representation of D_8 to H .

1. If $i: \mathbb{Z}/2\mathbb{Z} \hookrightarrow D_8$ is a subgroup generated by a reflection, then the reflection leaves a line invariant, so $i^*V \cong \mathbb{R} \oplus \chi$ for some one-dimensional representation χ . Stiefel-Whitney classes do not change when one adds a trivial summand, so

$$i^*(w_2(V)) = w_2(\mathbb{R} \oplus \chi) = w_2(\chi) = 0, \quad (14.15)$$

because w_2 vanishes on one-dimensional representations.

2. The restriction of V to $\mathbb{Z}/4\mathbb{Z}$ is the standard rotation representation of $\mathbb{Z}/4\mathbb{Z}$, and as we discussed in Lemma 13.12, the standard rotation representation is not Spin. Therefore $w_2(V)$ is nonzero when pulled back to $\mathbb{Z}/4\mathbb{Z}$.

Using Lemma 14.11, the other seven elements of $H^2(BD_8; \mathbb{Z}/2\mathbb{Z})$ are either 0 or of the form $aw + bx^2 + cy^2$ with $b \neq 0$ or $c \neq 0$. The zero class is trivial when pulled back to $\mathbb{Z}/4\mathbb{Z}$; if we can show that x^2 and y^2 are non-trivial when pulled back to some reflection subgroup, then we will have shown that w is the only class which is trivial when pulled back to reflection subgroups and non-trivial when pulled back to the rotation subgroup, which is what we wanted to show.

The Whitney sum formula implies $x^2 = w_2(2\text{Det}(V))$, and $\text{Det}(V)$ can be identified with the representation $D_8 \xrightarrow{q} \mathbb{Z}/2\mathbb{Z} = O(1)$ given by quotienting by the subgroup of rotations. Thus $2\text{Det}(V)$ and its characteristic classes pull back from $\mathbb{Z}/2\mathbb{Z}$ by q . For any reflection T in D_8 , the composition $\mathbb{Z}/2\mathbb{Z} \cong \langle T \rangle \hookrightarrow D_8 \xrightarrow{q} \mathbb{Z}/2\mathbb{Z}$ is the identity, so induces the identity map on characteristic classes, so $w_2(2\text{Det}(V))$ must pull back to a nonzero class.

For y^2 , the story is subtler. Recall from Appendix B.2 that there are two classes of reflection subgroups in D_8 ; the two embeddings i_4 and \tilde{i}_4 of $\mathbb{Z}/2\mathbb{Z}$ into D_8 (see the appendix for their definitions) represent the two classes, and do not induce the same map on cohomology. Specifically, the one-dimensional representation L from Lemma 14.11 is non-trivial when pulled back along \tilde{i}_4 , so w_1 of the associated line bundle must be nonzero, hence it is the nonzero element of $H^1(B\mathbb{Z}/2\mathbb{Z}; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Thus the square of that element, which is the pullback of y^2 along \tilde{i}_4 , is also nonzero. \square

V is not orientable, so we cannot invoke Corollary 10.23 to shear just yet — instead, consider the representation $V \oplus 3\text{Det}(V)$. One can check with the Whitney sum formula that $w_1(V \oplus 3\text{Det}(V)) = 0$ and $w_2(V \oplus 3\text{Det}(V)) = w_2(V)$, so using Corollary 10.23 we discover:

Corollary 14.16. *Spin- D_{16} structures are naturally equivalent to $(BD_8, V \oplus 3\text{Det}(V))$ -twisted Spin structures.*

Therefore, by Corollary 10.19, if $X := (BD_8)^{V+3\text{Det}(V)-5}$, the Spin- D_{16} bordism groups are the Spin bordism groups of X .⁵⁸ In degree 4, this bordism group and its generator were studied by Pedrotti [206, Theorem 9.0.11], and bordism for a closely related structure, where D_8 is replaced with $O(2)$, is studied by Guillou-Marin [207], Kirby-Taylor [208, Section 6], and Stehouwer [131, Section 4].⁵⁹

Lemma 14.17. $ko \langle 2 \rangle_0(X) \cong \mathbb{Z}/2\mathbb{Z}$ and $ko \langle 2 \rangle_1(X) \cong (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$.

Proof. Lemma 10.39 reduces this to computing the mod 2 homology of X ; then the result follows by passing Lemma 14.11 through the universal coefficient theorem and the Thom isomorphism. \square

⁵⁸The -5 is so that there is no degree shift in the Thom isomorphism and Pontrjagin-Thom theorem; see Example 10.14.

⁵⁹Spin- $O(2)$ structures, but not their bordism groups, also appear in work of Han-Huang-Liu-Zhang [209, Section 6] and Lazaroiu-Shahbazi [210].

Theorem 14.18. *The first several ko-homology groups of $(BD_8)^{V+3\text{Det}(V)-5}$ are:*

$$\begin{aligned}
ko_0((BD_8)^{V+3\text{Det}(V)-5}) &\cong \mathbb{Z} & ko_6((BD_8)^{V+3\text{Det}(V)-5}) &\cong 0 \\
ko_1((BD_8)^{V+3\text{Det}(V)-5}) &\cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2} & ko_7((BD_8)^{V+3\text{Det}(V)-5}) &\cong (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})^{\oplus 3} \\
ko_2((BD_8)^{V+3\text{Det}(V)-5}) &\cong \mathbb{Z}/2\mathbb{Z} & ko_8((BD_8)^{V+3\text{Det}(V)-5}) &\cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}) \\
ko_3((BD_8)^{V+3\text{Det}(V)-5}) &\cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 3} & ko_9((BD_8)^{V+3\text{Det}(V)-5}) &\cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 6} \\
ko_4((BD_8)^{V+3\text{Det}(V)-5}) &\cong \mathbb{Z} & ko_{10}((BD_8)^{V+3\text{Det}(V)-5}) &\cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2} \\
ko_5((BD_8)^{V+3\text{Det}(V)-5}) &\cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2} & ko_{11}((BD_8)^{V+3\text{Det}(V)-5}) &\cong (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})^{\oplus 4}.
\end{aligned} \tag{14.19}$$

Proof. We will use the Adams spectral sequence. The first step is to compute the E_2 -page, which is the Ext over $\mathcal{A}(1)$ of $H^*(X; \mathbb{Z}/2\mathbb{Z})$. Therefore we need to determine how $\mathcal{A}(1)$ acts on $H^*(X; \mathbb{Z}/2\mathbb{Z})$. Proposition 11.4 implies that for $a \in H^*(BD_8; \mathbb{Z}/2\mathbb{Z})$,

$$\text{Sq}^1(Ua) = U(w_1(V \oplus 3\text{Det}(V))a + \text{Sq}^1(a)) \tag{14.20a}$$

$$\text{Sq}^2(Ua) = U(w_2(V \oplus 3\text{Det}(V))a + w_1(V \oplus 3\text{Det}(V))\text{Sq}^1(a) + \text{Sq}^2(a)). \tag{14.20b}$$

In Lemma 14.11, we learned $w_1(V) = x$ and $w_2(V) = w$, so using the Whitney sum formula, $w_1(V \oplus 3\text{Det}(V)) = 0$ and $w_2(V \oplus 3\text{Det}(V)) = w$. Thus we have determined the $\mathcal{A}(1)$ -action on $H^*(X; \mathbb{Z}/2\mathbb{Z})$ in terms of the $\mathcal{A}(1)$ -action on $H^*(BD_8; \mathbb{Z}/2\mathbb{Z})$, at least in principle. To make it explicit, first use the Cartan formula (11.2) to reduce to the computation of Steenrod squares of the generators x , y , and w . We gave the Steenrod squares of these elements right before the statement of Lemma 14.13. Now we have a completely explicit description of the $\mathcal{A}(1)$ -module structure on the cohomology of X ; for example, $\text{Sq}^1(Ux) = Ux^2$ and $\text{Sq}^2(U) = Uwx$. One can continue in this vein to determine the $\mathcal{A}(1)$ -action for all classes in degrees 12 and below; we used a computer program to verify the following computation.

Proposition 14.21. *There is an isomorphism of $\mathcal{A}(1)$ -modules*

$$\begin{aligned}
H^*((BD_8)^{V+3\text{Det}(V)-5}; \mathbb{Z}/2\mathbb{Z}) &\cong \textcolor{red}{R}_2 \oplus \textcolor{brown}{\Sigma J} \oplus \textcolor{brown}{\Sigma^4 \mathcal{O}} \oplus \textcolor{teal}{\Sigma^7 \mathbb{Z}/2\mathbb{Z}} \oplus \textcolor{teal}{\Sigma^8 R_2} \oplus \textcolor{teal}{\Sigma^9 J} \oplus \textcolor{violet}{\Sigma^{12} \mathcal{O}} \\
&\quad \oplus \textcolor{violet}{\Sigma^3 \mathcal{A}(1)} \oplus \textcolor{violet}{\Sigma^3 \mathcal{A}(1)} \oplus \textcolor{violet}{\Sigma^5 \mathcal{A}(1)} \oplus \textcolor{violet}{\Sigma^5 \mathcal{A}(1)} \oplus \textcolor{violet}{\Sigma^7 \mathcal{A}(1)} \\
&\quad \oplus \textcolor{violet}{\Sigma^7 \mathcal{A}(1)} \oplus \textcolor{violet}{\Sigma^9 \mathcal{A}(1)} \oplus \textcolor{violet}{\Sigma^9 \mathcal{A}(1)} \oplus \textcolor{violet}{\Sigma^{11} \mathcal{A}(1)} \oplus \textcolor{violet}{\Sigma^{11} \mathcal{A}(1)} \\
&\quad \oplus \textcolor{violet}{\Sigma^{11} \mathcal{A}(1)} \oplus \textcolor{violet}{\Sigma^{11} \mathcal{A}(1)} \oplus P
\end{aligned} \tag{14.22}$$

where P is 13-connected. The free $\mathcal{A}(1)$ summands are generated by the classes Ux^3 , Uy^3 , Uw^2x , Uw^2y , Ux^7 , Uy^7 , Uw^2x^5 , Uw^2y^5 , Ux^{11} , Uy^{11} , Uw^4x^3 , and Uw^4y^3 . We draw this in Figure 21.

These component $\mathcal{A}(1)$ -modules are defined as follows.

- R_2 is the kernel of the unique nonzero map $\Sigma^{-1}\mathcal{A}(1) \rightarrow \Sigma^{-1}\mathbb{Z}/2\mathbb{Z}$.

- $J := \mathcal{A}(1)/\text{Sq}^3$. This module is called the *Joker*.
- $\mathcal{O} := \mathcal{A}(1)/(\text{Sq}^1, \text{Sq}^2 \text{Sq}^3)$. This is called the *upside-down question mark*.
- $\mathbb{Z}/2\mathbb{Z}$ is simply a one-dimensional $\mathbb{Z}/2\mathbb{Z}$ -vector space with a trivial $\mathcal{A}(1)$ -action.

We will not worry about P : its Ext is concentrated in degrees 13 and above, and hence will not affect the computation. For the rest of these $\mathcal{A}(1)$ -modules M , $\text{Ext}_{\mathcal{A}(1)}(M)$ is known: see Theorem 11.12 for $\mathbb{Z}/2\mathbb{Z}$ and [115, Figure 29] for R_2 , J , and \mathcal{O} .

With this in hand, we draw the E_2 -page of the Adams spectral sequence for $ko_*(X)$ in Figure 22, top. The black dots are the portion coming from the $\Sigma^k \mathcal{A}(1)$ summands in $H^*(X; \mathbb{Z}/2\mathbb{Z})$; Margolis' theorem (in the form of Corollary 11.18) implies they do not participate in nonzero differentials or non-trivial extensions.

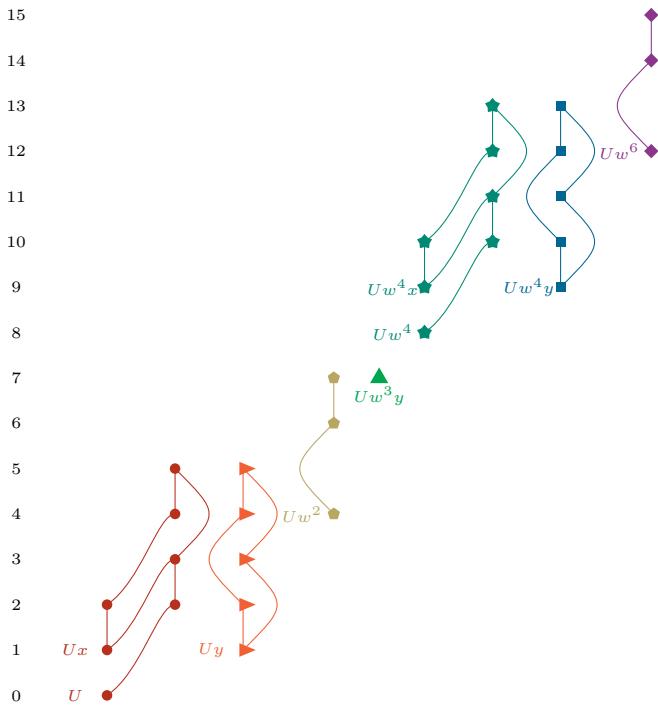


Figure 21: Part of the mod 2 cohomology of $(BD_8)^{V+3\text{Det}(V)-5}$ in low degrees, as stated in Proposition 14.21. The pictured summand contains all elements in degrees 12 and below except for those contained in $12 \Sigma^k \mathcal{A}(1)$ summands which are not drawn to avoid cluttering the diagram.

There is potential for many differentials. We will deduce some of them using our previous work on Spin-Mp(2, \mathbb{Z}) bordism: the map $\mathbb{Z}/4\mathbb{Z} \hookrightarrow D_8$ induces a map of Thom spectra $\Phi: (B\mathbb{Z}/4\mathbb{Z})^{\rho-2} \rightarrow (BD_8)^{V+3\text{Det}(V)-5}$. Upon taking Spin bordism, this map has a geometric description: sending a Spin- $\mathbb{Z}/8\mathbb{Z}$ manifold to the bordism class of its canonical Spin- D_{16} structure.

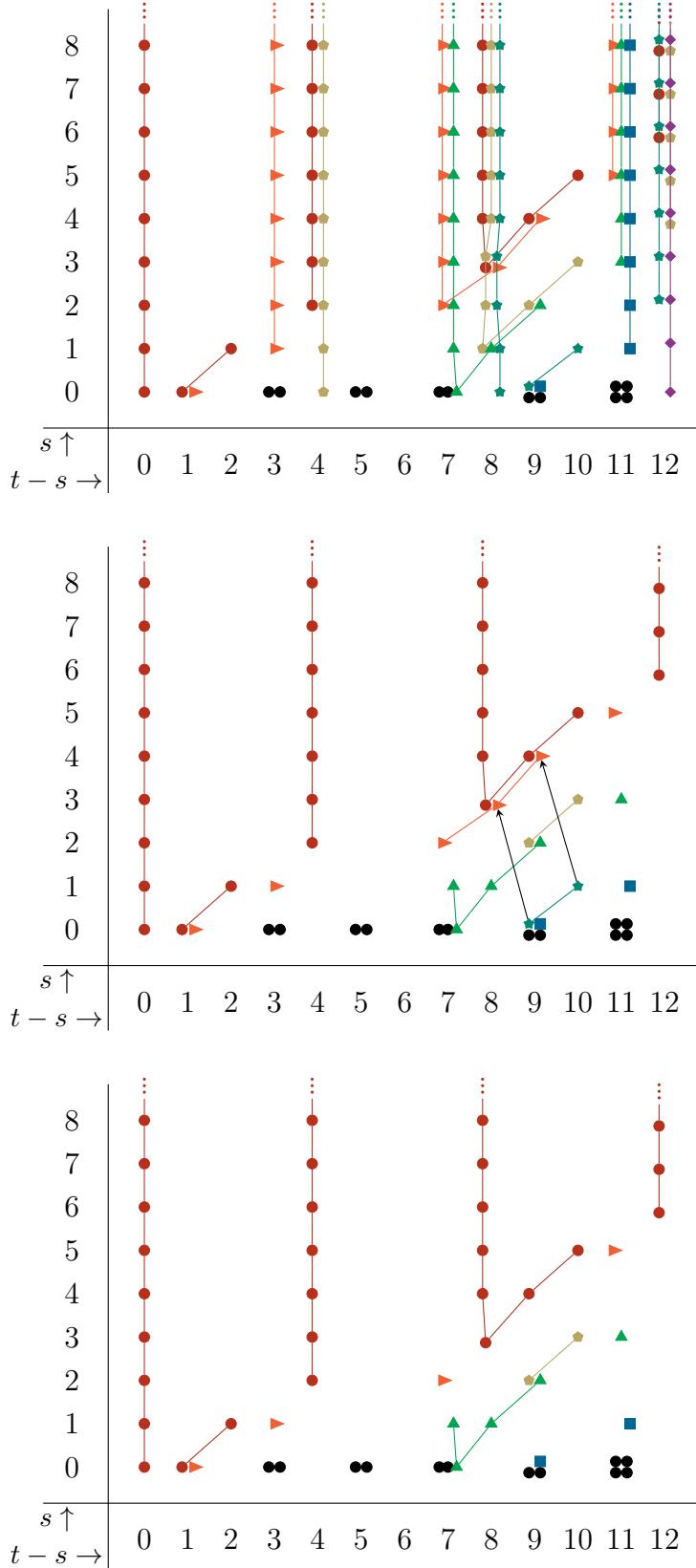


Figure 22: The Adams spectral sequence computing $ko_*(((BD_8)^{V+3\text{Det}(V)} - 5))$. Top: the E_2 -page. There are many nonzero d_2 s, as we will establish in Lemma 14.24. Middle: the E_3 -page; we will resolve these differentials in Lemma 14.26. Bottom: the $E_4 = E_\infty$ -page.

For $2 \leq r \leq \infty$, Φ induces a map Φ_* from the E_r -page of the Adams spectral sequence for $ko_*((B\mathbb{Z}/4\mathbb{Z})^{r-2})$ to the E_r -page of the Adams spectral sequence for $ko_*(X)$, and this map commutes with differentials. We will compute the image of Φ_* and see that it is large enough for us to infer most of the differentials we need.

To compute Φ_* , we can use a trick: we have decomposed the cohomology of the Thom spectra $(B\mathbb{Z}/4\mathbb{Z})^{r-2}$ and X into summands, and the map between them mostly respects this decomposition. For each summand, the map is part of a short exact sequence of $\mathcal{A}(1)$ -modules, so we can compute using the long exact sequence in Ext. See [115, Section 4.6] for more information and examples of this technique. We run this computation one summand at a time in Figure 23.

Definition 14.23. We will say that a differential *vanishes for easy reasons* if it can be shown to vanish because either its domain or codomain is zero; it must vanish in order to be equivariant for the h_i -action on the E_2 -page; or it is killed by Margolis' theorem because it comes from a $\Sigma^k \mathcal{A}(1)$ summand in cohomology (see Corollary 11.18).

Lemma 14.24. *In the range $t - s \leq 12$, all d_2 s either vanish for easy reasons or are in the image of Φ_* .*

Proof. We computed $\text{Im}(\Phi_*)$ in Figure 23; then one just checks the lemma directly. Specifically, one observes that all d_2 s that do not vanish for easy reasons are either between h_0 -towers or from the 11-line to the 10-line. All h_0 -towers are in the image of Φ_* ,⁶⁰ as is the entire 10-line, and the entire 11-line except for the summands arising from $\Sigma^{11} \mathcal{A}(1)$ summands in $H^*((BD_8)^{V+3\text{Det}(V)-5}; \mathbb{Z}/2\mathbb{Z})$, and Margolis' theorem kills differentials emerging from these latter summands. Thus d_2 s from the 11-line to the 10-line either vanish for easy reasons or are in the image of Φ_* . \square

Therefore we know all the d_2 s in range: the differentials between h_0 -towers are the same as they were for $ko_*((B\mathbb{Z}/4\mathbb{Z})^{r-2})$, and all other d_2 s vanish. We draw the E_3 -page in Figure 22, middle. Most d_3 differentials vanish for easy reasons, except for $d_3: E_3^{0,9} \rightarrow E_3^{3,11}$ and $d_3: E_3^{1,11} \rightarrow E_3^{4,13}$. A combination of Margolis' theorem and h_i -equivariance means that the first d_3 is either 0 or carries the teal summand to the orange summand and vanishes on the remaining generators of $E_3^{0,9}$; h_1 -equivariance then forces the latter d_3 to either vanish or surject onto the orange summand in $E_3^{4,13}$, and one vanishes iff the other does.

To address these d_3 s, we need to show that they are the last possible differentials that can be nonzero. All longer differentials vanish for easy reasons with the exception of $d_4: E_2^{1,12} \rightarrow E_2^{5,15}$.

⁶⁰ Φ_* is not always surjective on h_0 -towers, e.g. the green h_0 -tower in topological degree 7. But it is surjective enough: for every h_0 -tower in range, the cokernel of Φ_* on that h_0 -tower is finite-dimensional. Since h_0 -towers are generated by applying h_0 to some element of the E_2 -page and differentials must be equivariant for h_0 -actions, this “not quite surjectivity” is good enough: it determines the values of differentials on all elements of the h_0 -tower.

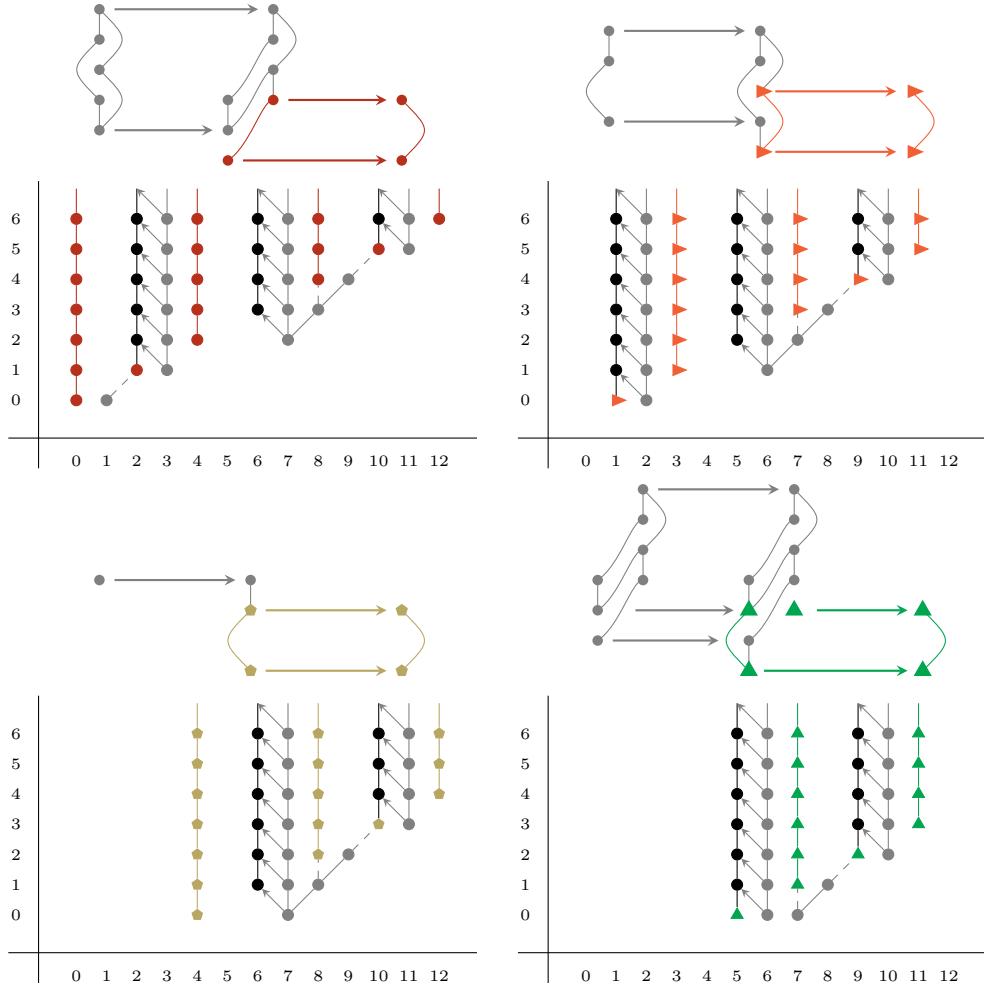


Figure 23: Part of the computation of the image of the map on the E_2 -page induced from the map from Spin- $\mathbb{Z}/8\mathbb{Z}$ bordism to Spin- D_{16} bordism. Color is the image; black is the kernel. In the lower-right (green triangles) figure, both green summands in degree 2 map non-trivially to the degree-2 summand in $\mathcal{C}\eta$; the degree-two class in the kernel ΣR_2 maps to their sum. The arguments for the teal star, blue square, and purple diamond summands are analogous to the arguments for the red circle, orange triangle, and yellow pentagon summands, respectively.

Lemma 14.25. *This $d_4: E_4^{1,12} \rightarrow E_4^{5,15}$ vanishes.*

Proof. The generator x of $E_4^{1,12} \cong \mathbb{Z}/2\mathbb{Z}$ is in $\text{Im}(\Phi_*)$. Since d_4 commutes with Φ_* , $d_4(x) \in \text{Im}(\Phi_*)$ too. However, in the proof of Theorem 13.15, we saw that in the Adams spectral sequence for $ko_*((B\mathbb{Z}/4\mathbb{Z})^{\rho-2})$, $E_4^{5,15} = 0$, so $d_4(x) = 0$. \square

See Figure 24 for a picture of this proof.

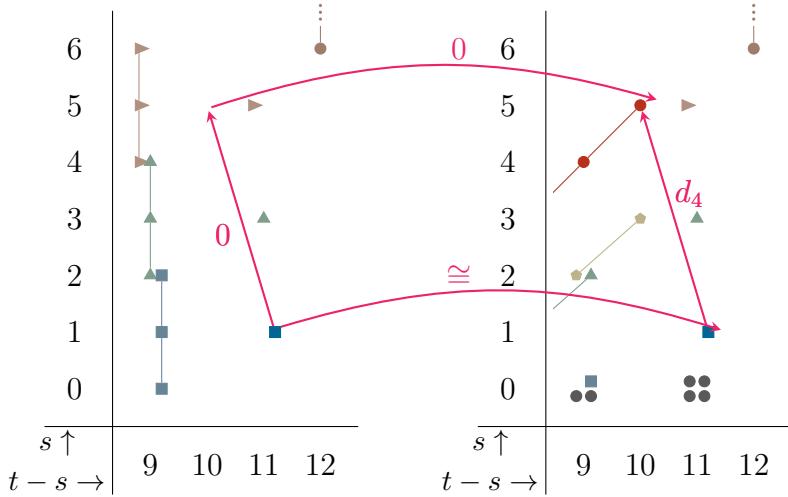


Figure 24: Left: the E_4 -page for $ko_*((B\mathbb{Z}/4)^{V-2})$ (the calculation for $\text{Spin-Mp}(2, \mathbb{Z})$ bordism). Right: the E_4 -page for $ko_*((BD_8)^{V+3\text{Det}(V)-5})$ (the calculation for $\text{Spin-GL}^+(2, \mathbb{Z})$ bordism). The horizontal pink arrows are the map between these spectral sequences induced by Φ . This map commutes with differentials, so the pictured d_4 must vanish. This is a picture proof of Lemma 14.25.

Now we understand the E_∞ -page with the exception of the two indicated d_3 differentials.

Lemma 14.26. *The d_3 s starting at $E_3^{0,9}$ and $E_3^{1,11}$ do not vanish.*

Proof. $E_3^{4,13} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, but because Adams differentials commute with h_1 , the image of $d_3: E_3^{1,11} \rightarrow E_3^{4,13}$ is contained in the orange triangle $\mathbb{Z}/2\mathbb{Z}$ summand, as opposed to the red circle. Therefore this differential vanishes if and only if the nonzero element of that summand lives to the E_∞ -page. Call that nonzero element α . Since differentials commute with h_1 , the destinies of the differentials starting in $E_3^{0,9}$ and $E_3^{1,11}$ are bonded: one vanishes if and only if the other does. Thus, to prove this lemma, we will prove that $\alpha \in E_3^{4,13}$ cannot survive to the E_∞ -page.

Suppose conversely that α *does* survive. Looking at Figure 23, we see that α is in the image of the map Φ_* of spectral sequences induced by Φ ; choose the preimage $\tilde{\alpha}$ which is the nonzero element of the orange triangle $\mathbb{Z}/2\mathbb{Z}$ summand of $E_3^{4,13}$ for $ko_*((B\mathbb{Z}/4\mathbb{Z})^{\rho-2})$. Let $\tilde{\beta}$ be the nonzero element of the green triangle $\mathbb{Z}/2\mathbb{Z}$ summand of $E_3^{2,11}$ for $ko_*((B\mathbb{Z}/4\mathbb{Z})^{\rho-2})$ and β be the nonzero element of the green triangle $\mathbb{Z}/2\mathbb{Z}$ summand of $E_3^{2,11}$ for $ko_*(X)$; then

$\Phi_*(\tilde{\beta}) = \beta$. By our assumption, all four of these classes survive to their respective E_∞ -pages, so define elements nonzero a , b , \tilde{a} , and \tilde{b} in $ko_9((B\mathbb{Z}/4)^{\rho-2})$ and $ko_*(X)$, where a lifts α , b lifts β , \tilde{a} lifts $\tilde{\alpha}$, and \tilde{b} lifts $\tilde{\beta}$, such that $\Phi(\tilde{a}) = a$ and $\Phi(\tilde{b}) = b$. The hidden extension we deduced in the proof of Theorem 13.15 (see Figure 19) implies we can choose $\tilde{a} = 2\tilde{b}$, and therefore $a = 2b$ as well.

Both α and β are h_1 times some other classes γ and δ , respectively, and by our assumption both γ and δ survive to the E_∞ -page, hence lifting to define classes $c, d \in ko_8(X)$ with $\eta c = a$ and $\eta d = b$. That is, $a = 2\eta d$, but this is a contradiction, because $2\eta = 0$ and $a \neq 0$. \square

So $E_4 = E_\infty$ for $t - s < 12$; we draw this in Figure 22, bottom.

Now we have to address extension questions. By inspection, the only place a hidden extension could occur for $t - s < 6$ is in dimension 3, and Margolis' theorem (Corollary 11.18) rules out a hidden extension there. In topological degrees 7 through 11, however, we must address extension problems.

Lemma 14.27. *The extension*

$$0 \longrightarrow \textcolor{red}{\mathbb{Z}/2\mathbb{Z}} \longrightarrow ko_7(X) \longrightarrow (\textcolor{green}{\mathbb{Z}/4\mathbb{Z}}) \oplus (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) \longrightarrow 0 \quad (14.28)$$

splits, so $ko_7(X) \cong (\textcolor{green}{\mathbb{Z}/4\mathbb{Z}}) \oplus (\textcolor{red}{\mathbb{Z}/2\mathbb{Z}}) \oplus (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$.

Proof. Margolis' theorem (Corollary 11.18) splits off the black $\mathbb{Z}/2\mathbb{Z}$ summands. If (14.28) does not split, then the orange rightward-pointing triangle $\textcolor{red}{\mathbb{Z}/2\mathbb{Z}}$ and green upward-pointing triangles $\textcolor{green}{\mathbb{Z}/4\mathbb{Z}}$ combine into a $\mathbb{Z}/8\mathbb{Z}$. Let $x \in ko_7(X)$ be an element whose image in the E_∞ -page is a generator of the green upward-pointing triangle $\textcolor{green}{\mathbb{Z}/2\mathbb{Z}} \subset E_\infty^{0,7}$; then (14.28) splits iff $4x = 0$.

The computation in Figure 23 shows that $2x \in \text{Im}(\Phi_*)$. Let $y \in ko_7((B\mathbb{Z}/4)^{\rho-2})$ be a preimage of $2x$. Then $4x = \Phi_*(2y)$, and $2y = 0$, so (14.28) splits. \square

Lemma 14.29. $ko_8(X) \cong \textcolor{red}{\mathbb{Z}} \oplus (\textcolor{green}{\mathbb{Z}/2\mathbb{Z}})$.

Proof. There are infinitely many extension questions to address here, but the isomorphism type of the final answer is either $\textcolor{red}{\mathbb{Z}} \oplus (\textcolor{green}{\mathbb{Z}/2\mathbb{Z}})$, or \mathbb{Z} , and the latter case can only occur if the first extension

$$0 \longrightarrow \textcolor{red}{E}_\infty^{3,11} \longrightarrow A \longrightarrow \textcolor{blue}{E}_\infty^{1,9} \longrightarrow 0 \quad (14.30)$$

is non-trivial, but the non-trivial h_1 -action $\textcolor{red}{E}_\infty^{3,11} \rightarrow \textcolor{red}{E}_\infty^{4,13}$ lifts to a nonzero action by η in $ko_*(X)$, which by Lemma 11.23 splits this extension. \square

Lemma 14.31. $ko_9(X) \cong (\textcolor{red}{\mathbb{Z}/2\mathbb{Z}}) \oplus (\textcolor{brown}{\mathbb{Z}/2\mathbb{Z}}) \oplus (\textcolor{green}{\mathbb{Z}/2\mathbb{Z}}) \oplus (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$.

Proof. We need to determine the extension

$$0 \longrightarrow \textcolor{green}{\mathbb{Z}/2\mathbb{Z}} \longrightarrow A \longrightarrow \textcolor{red}{\mathbb{Z}/2\mathbb{Z}} \longrightarrow 0 . \quad (14.32)$$

A priori, the extension could also involve the red circle and yellow pentagon summands, but Lemma 11.23 implies that restricted to those summands, the extension splits, so all we have to determine is (14.32). Then, argue as in the proof of Lemma 14.26 to conclude that if x is a preimage of the generator of the blue square summand, then $2x = 0$. This splits (14.32). \square

Lemma 14.33. $ko_{10}(X) \cong (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$.

Proof. We must split the extension

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow A \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0. \quad (14.34)$$

On the E_∞ -page, the action of h_1 from the 9-line to the 10-line is surjective. This lifts to imply that the action of $\eta \in \pi_1(\mathbb{S})$, which defines a map $ko_9(X) \rightarrow ko_{10}(X)$, is surjective. This means the orders of the elements in $ko_{10}(X)$ are bounded above by the order of the largest element in $ko_9(X)$, and $2ko_9(X) = 0$, so there can be no order-4 elements in $ko_{10}(X)$. \square

Lemma 14.35. $ko_{11}(X) \cong (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})^{\oplus 4}$.

Proof. The four $\mathbb{Z}/2\mathbb{Z}$ summands in filtration 0 split off by Margolis' theorem (Theorem 11.17), because they came from $\Sigma^{11}\mathcal{A}(1)$ summands in $H^*(X; \mathbb{Z}/2\mathbb{Z})$. That leaves the orange, green, and blue summands in higher Adams filtration; they are in the image of the map Φ from the E_∞ -page for Spin- $\mathbb{Z}/8\mathbb{Z}$ bordism to Spin- D_{16} bordism. For Spin- $\mathbb{Z}/8\mathbb{Z}$ bordism, we saw in the proof of Theorem 13.15 that these three $\mathbb{Z}/2\mathbb{Z}$ summands combined to a $\mathbb{Z}/8\mathbb{Z}$, so that is also true here. \square

Remark 14.36. Using this, we can give another proof of Lemma 14.27. Recall from Theorem 11.12 the element $v \in \text{Ext}^{3,7}(\mathbb{Z}/2\mathbb{Z})$, which acts on the pages of the Adams spectral sequence for $ko_*(X)$ for a space or spectrum X . In Spin bordism, this corresponds to taking the direct product with a K3 surface.

One can show that v carries the green upward-pointing triangle $\mathbb{Z}/2\mathbb{Z} \subset E_\infty^{0,7}$ isomorphically onto $E_\infty^{3,14} \cong \mathbb{Z}/2\mathbb{Z}$, and likewise carries $E_\infty^{2,9} \cong \mathbb{Z}/2\mathbb{Z}$ isomorphically onto $E_\infty^{5,16} \cong \mathbb{Z}/2\mathbb{Z}$.⁶¹ Let $x \in ko_7(X)$ be a class whose image in the E_∞ -page is the generator of the green upward-pointing triangle $\mathbb{Z}/2\mathbb{Z} \subset E_\infty^{0,7}$, and suppose that the extension in (14.28) does not split. As we noted in the proof of Lemma 14.27, Margolis' theorem implies the black $\mathbb{Z}/2\mathbb{Z}$ summands split off, so the only way for (14.28) to not split is for the green and orange summands to combine, which would imply that the image of $4x$ in the E_∞ -page is the nonzero element of the orange triangle $\mathbb{Z}/2\mathbb{Z} \subset E_\infty^{2,9}$.

Now act by v . Lemma 14.35 means the colored summands in the 11-line assemble to a $\mathbb{Z}/8\mathbb{Z}$ with generator y , and a generator of the green upward-pointing triangle $\mathbb{Z}/2\mathbb{Z} \subset E_\infty^{3,14}$

⁶¹This comes from a computation of the $\text{Ext}(\mathbb{Z}/2\mathbb{Z})$ -action on $\text{Ext}(\mathbb{Z}/2\mathbb{Z})$ and $\text{Ext}(\mathcal{J})$. For $\text{Ext}(\mathbb{Z}/2\mathbb{Z})$ the claimed v -action is a consequence of the ring structure, which we gave in Theorem 11.12; for $\text{Ext}(\mathcal{J})$ one can produce the claimed v -action using the long exact sequence in Ext groups associated to a short exact sequence of $\mathcal{A}(1)$ -modules, similarly to how we computed some v -actions in Example 11.15.

lifts to $2y \in \mathbb{Z}/8\mathbb{Z}$. The v -action from the 7- to the 11-line we noted above implies $K3 \cdot x = 2y$,⁶² so $4(K3 \cdot x) = 0$. But the non-trivial v -action on the orange summands that we noted above implies $K3 \cdot (4x) \neq 0$, which is a contradiction.

This finishes our proof of Theorem 14.18. \square

Collecting all the pieces we find the Spin-GL $^+(2, \mathbb{Z})$ bordism groups summarized in Table 10. Again, we already include the generators which we will derive next.

k	$\Omega_k^{\text{Spin-GL}^+(2, \mathbb{Z})}(\text{pt})$	Generators
0	\mathbb{Z}	pt_+
1	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$	(L_1^4, S_R^1)
2	$\mathbb{Z}/2\mathbb{Z}$	$S_p^1 \times S_R^1$
3	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 3} \oplus (\mathbb{Z}/3\mathbb{Z})$	$(L_4^3, \mathbb{RP}^3, \widetilde{\mathbb{RP}}^3, L_3^3)$
4	\mathbb{Z}	E
5	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	(L_4^5, X_5)
6	0	
7	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 3} \oplus (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/9\mathbb{Z})$	$(\mathbb{RP}^7, \widetilde{\mathbb{RP}}^7, W_2^7, W_1^7, L_3^7)$
8	$\mathbb{Z} \oplus \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$	$(W_{1,8}, \mathbb{HP}^2, W_1^7 \times S_p^1)$
9	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 8}$	$(W_1^7 \times S_p^1 \times S_p^1, X_9, \tilde{X}_9, L_4^9, W_1^9,$ $B \times L_4^1, \mathbb{HP}^2 \times L_4^1, \mathbb{HP}^2 \times S_R^1)$
10	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 4}$	$(B \times L_4^1 \times S_p^1, W_1^9 \times S_p^1, \mathbb{HP}^2 \times L_4^1 \times S_p^1, X_{10})$
11	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 9} \oplus (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/27\mathbb{Z})$	$(\mathbb{RP}^{11}, \widetilde{\mathbb{RP}}^{11}, X_{11}, \tilde{X}_{11}, \mathbb{HP}^2 \times L_4^3, \mathbb{HP}^2 \times \mathbb{RP}^3,$ $\mathbb{HP}^2 \times \widetilde{\mathbb{RP}}^3, X_{10} \times L_4^1, X_{10} \times S_R^1, Q_4^{11}, \mathbb{HP}^2 \times L_3^3, L_3^{11})$

Table 10: Bordism groups $\Omega_k^{\text{Spin-GL}^+(2, \mathbb{Z})}(\text{pt})$ and their generators (in the same order as group summands) for $k \leq 11$.

14.3 Determining the generators at $p = 2$

The map from Spin-Mp $(2, \mathbb{Z})$ bordism to Spin-GL $^+(2, \mathbb{Z})$ bordism is surjective on 3-torsion, as we saw in Section 14.1. This map is not surjective on 2-torsion, and we need to find generators for everything in the (quite large) cokernel.

⁶²There is more than one choice of generator y ; a more careful way to say this is that we can choose a generator y such that $K3 \cdot x = 2y$.

14.3.1 Generators coming from $\text{Spin-Mp}(2, \mathbb{Z})$ bordism

We used the map from $\text{Spin-Mp}(2, \mathbb{Z})$ bordism to $\text{Spin-GL}^+(2, \mathbb{Z})$ bordism already to solve the Adams spectral sequence for $\text{Spin-GL}^+(2, \mathbb{Z})$ bordism; we will use this map again to find some of our generators. We already saw in Section 14.1 that 3-locally, this map is surjective, so for the rest of this subsection we focus on the situation at the prime 2, where this is the map from $\text{Spin-}\mathbb{Z}/8\mathbb{Z}$ bordism to $\text{Spin-}D_{16}$ bordism. Looking at Figure 23, we can characterize the image of this map.

- In dimensions 0 and 4, the map is surjective, so we learn that pt_+ generates $\Omega_0^{\text{Spin-}D_{16}}(\text{pt})$ and the Enriques surface generates $\Omega_4^{\text{Spin-}D_{16}}(\text{pt})$.
- In dimensions 1 and 3, this map sees the orange triangle $\mathbb{Z}/2\mathbb{Z}$ summands, giving us lens spaces L_4^k as generators of those summands in $\text{Spin-GL}^+(2, \mathbb{Z})$ bordism; however, there are other $\mathbb{Z}/2\mathbb{Z}$ summands that are not in the image.
- In dimensions 2 and 10, the image of the map is 0.
- $\Omega_5^{\text{Spin-}D_{16}}(\text{pt}) \cong (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$; Margolis' theorem tells us these summands are detected by the mod 2 characteristic classes w^2x and w^2y . The w^2y summand is in the image of the map from $\text{Spin-}\mathbb{Z}/8\mathbb{Z}$ bordism, and as such is generated by the lens space L_4^5 .
- In dimension 7, the image of the map contains none of the generators, though it does contain twice the generator of the green triangle $\mathbb{Z}/4\mathbb{Z}$ summand.
- In dimension 8, one of the generators is \mathbb{HP}^2 , which is in the image of this map. The other generator X is not in the image, but $2X$, which is bordant to the Bott manifold, is.
- In dimension 9, $\Omega_9^{\text{Spin-}D_{16}}(\text{pt})$ consists of eight $\mathbb{Z}/2\mathbb{Z}$ summands. Three are in the image of the map from $\text{Spin-}\mathbb{Z}/8\mathbb{Z}$ bordism: the green triangle $\mathbb{Z}/2\mathbb{Z}$ and blue square $\mathbb{Z}/2\mathbb{Z}$ summands in ko_9 , and the orange triangle $\mathbb{Z}/2\mathbb{Z}$ in ko_1 , giving us three generators: L_4^9 , \tilde{L}_4^9 , and $\mathbb{HP}^2 \times L_4^1$, respectively.
- In dimension 11, the image of the map contains the $\mathbb{Z}/8\mathbb{Z}$ summand, giving us Q_4^{11} as a generator, but not the four $\mathbb{Z}/2\mathbb{Z}$ summands in ko_{11} which are detected by the mod 2 cohomology classes x^{11} , y^{11} , w^4x^3 , and w^4y^3 . On ko_3 it sees one of the three $\mathbb{Z}/2\mathbb{Z}$ summands like in dimension 3, giving us the generator $\mathbb{HP}^2 \times L_4^3$. On $ko\langle 2 \rangle_1$, it sees one of the generators, giving us $X_{10} \times L_4^1$.

See Figure 25 for a depiction of this map.

From the perspective of F-theory, classes in $\Omega_*^{\text{Spin-Mp}(2, \mathbb{Z})}(\text{pt})$ correspond to defects that preserve supersymmetry, and classes in $\Omega_*^{\text{Spin-GL}^+(2, \mathbb{Z})}(\text{pt})$ can break supersymmetry. Thus

the image of the map $\Omega_*^{\text{Spin-Mp}(2,\mathbb{Z})} \rightarrow \Omega_*^{\text{Spin-GL}^+(2,\mathbb{Z})}$ consists of the classes which, even though we do not care about preserving supersymmetry, can nevertheless be matched to supersymmetry-preserving defects. See Section 7.1 for more information. For example, in dimension 1, the map $\Omega_1^{\text{Spin-Mp}(2,\mathbb{Z})} \rightarrow \Omega_1^{\text{Spin-GL}^+(2,\mathbb{Z})}$ is a map $\mathbb{Z}/24\mathbb{Z} \rightarrow (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$: each generator of $\mathbb{Z}/24\mathbb{Z}$ is sent to the generator of one $\mathbb{Z}/2\mathbb{Z}$ summand, and the second $\mathbb{Z}/2\mathbb{Z}$ summand is new.

- The $\mathbb{Z}/24\mathbb{Z}$ summand of $\Omega_1^{\text{Spin-Mp}(2,\mathbb{Z})}(\text{pt})$ is generated by the lens space L_{12}^1 , corresponding in string theory to a D7-brane (see Section 6.1). Twice the image of this class in $\text{Spin-GL}^+(2,\mathbb{Z})$ bordism vanishes; this means that instead of needing a stack of 24 D7-branes to be trivial, it suffices to take just two.
- The novel $\mathbb{Z}/2\mathbb{Z}$ summand in $\Omega_1^{\text{Spin-GL}^+(2,\mathbb{Z})}(\text{pt})$, generated by S_R^1 , corresponds to a new defect that we predict to not preserve supersymmetry. This is the reflection 7-brane, as we discuss in Section 7.1 (see also [23]).

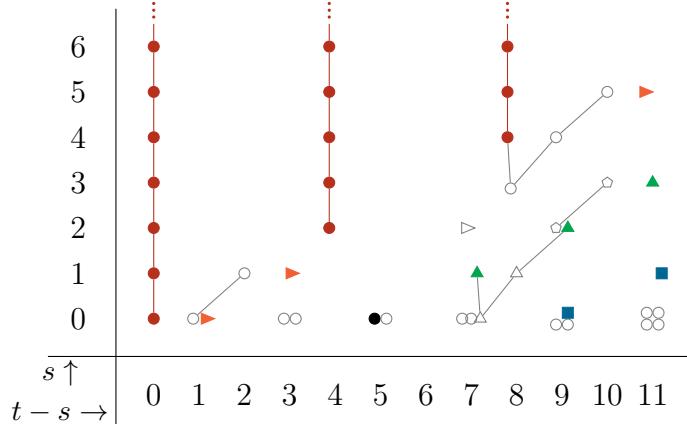


Figure 25: Part of the image of the map $\Omega_*^{\text{Spin-Mp}(2,\mathbb{Z})}(\text{pt}) \rightarrow \Omega_*^{\text{Spin-GL}^+(2,\mathbb{Z})}(\text{pt})$: specifically, the part coming from ko_* . The ko_{*-8} piece is similar; the $ko \langle 2 \rangle_{*-10}$ piece is straightforward, thanks to Lemma 10.39, and the 3-local part is surjective, as we show in Section 14.1.

14.3.2 Generators coming from $\Omega_k^{\text{Spin}}(B\mathbb{Z}/2\mathbb{Z})$

Let $i_4, \tilde{i}_4: D_4 \hookrightarrow D_{16}$ be the two inclusions defined in Appendix B.2: specifically, these maps send rotations to rotations and reflections to reflections, but i_4 sends the standard generating reflection s of D_4 to the standard generating reflection of D_8 , and \tilde{i}_4 sends s to a different reflection. The maps i_4 and \tilde{i}_4 induce two maps from Spin- D_4 manifolds to Spin- D_{16} manifolds. Since $D_4 \cong (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$, $\text{Spin} \times_{\mathbb{Z}/2\mathbb{Z}} D_4 \cong \text{Spin} \times \mathbb{Z}/2\mathbb{Z}$ as symmetry types: a Spin- D_4 manifold is the same thing as a Spin manifold with a principal $\mathbb{Z}/2\mathbb{Z}$ -bundle. The generators of $\Omega_k^{\text{Spin}}(B\mathbb{Z}/2\mathbb{Z})$ are well understood, so we will use them to find more Spin- D_8 manifolds. See Figure 26 for a depiction of what this method can see.

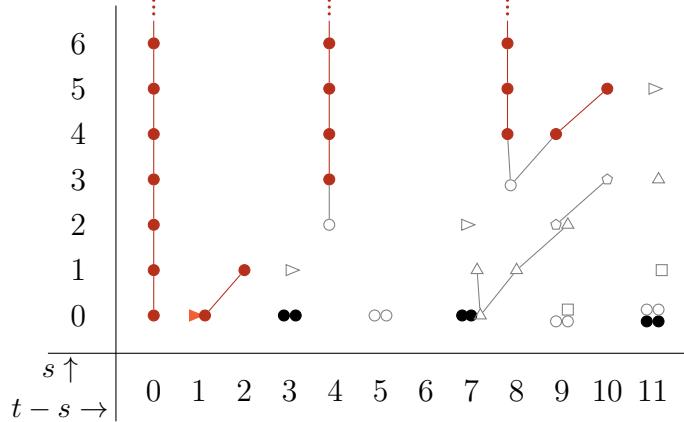


Figure 26: The colored and black summands are the image of $\Omega_*^{\text{Spin}}(B\mathbb{Z}/2\mathbb{Z}) \rightarrow \Omega_*^{\text{Spin-}D_{16}}(\text{pt})$ under the two maps i_4, \tilde{i}_4 . See Section 14.3.2 for more information. Like in Figure 25, this picture just encodes the summands coming from ko_* ; ko_{*-8} is analogous and $ko \langle 2 \rangle_{*-10}$ is straightforward thanks to Lemma 10.39.

Given a Spin manifold M with a principal $\mathbb{Z}/2\mathbb{Z}$ -bundle, we will let M denote its Spin- D_{16} structure induced by i_4 and \tilde{M} denote its Spin- D_{16} structure induced by \tilde{i}_4 . We will find several examples where one generator of $\Omega_*^{\text{Spin}}(B\mathbb{Z}/2\mathbb{Z})$ gives us two different generators of $\Omega_*^{\text{Spin-}D_{16}}(\text{pt})$ under i_4 and \tilde{i}_4 .

Recall that $H^*(B\mathbb{Z}/2\mathbb{Z}; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[a]$, with a in degree 1.

Lemma 14.37. *The pullback map $i_4^*: H^*(BD_8; \mathbb{Z}/2\mathbb{Z}) \rightarrow H^*(B\mathbb{Z}/2\mathbb{Z}; \mathbb{Z}/2\mathbb{Z})$ sends $x \mapsto a$, $y \mapsto 0$, and $w \mapsto 0$. \tilde{i}_4^* sends $x \mapsto a$, $y \mapsto a$, and $w \mapsto 0$.*

Proof. Recall that we defined x , y , and w as Stiefel-Whitney classes of vector bundles associated to representations of D_8 . Therefore to understand i_4^* and \tilde{i}_4^* , it suffices to restrict those representations to $\mathbb{Z}/2\mathbb{Z} \subset D_8$ and compute the Stiefel-Whitney classes of the associated vector bundles over $B\mathbb{Z}/2\mathbb{Z}$.

- For x and w , we used the standard representation V of D_8 on \mathbb{R}^2 by rotations and reflections. For both i_4 and \tilde{i}_4 , the pullback of V is $\mathbb{R} \oplus \sigma$, where σ is the sign representation, so the pullback of x is $w_1(\sigma) = a$ and the pullback of w is $w_2(\sigma) = 0$.
- For y , we used the real character χ_y of D_8 in which a quarter-turn is sent to -1 and the reflection s is sent to 1 . Therefore $i_4^*(\chi_y)$ is trivial, so $i_4^*(y) = 0$ — and since $\chi_y(rs) = -1$, $\tilde{i}_4^*(y)$ is non-trivial! Hence it is the unique nonzero character of $\mathbb{Z}/2\mathbb{Z}$, which has $w_1 = a$, so $\tilde{i}_4^*(y) = a$. \square

In particular, we hypothesize that if c is a mod 2 cohomology class corresponding to some generator of $\Omega_*^{\text{Spin-}D_{16}}(\text{pt})$ in Adams filtration 0, and $i_4^*(c)$ or $\tilde{i}_4^*(c)$ is nonzero, then perhaps that generator can be realized by an element of $\Omega_*^{\text{Spin}}(B\mathbb{Z}/2\mathbb{Z})$. This guess applies to $\textcolor{brown}{x}$, $\textcolor{brown}{y}$,

$x^3, y^3, x^7, y^7, w_4^2x, w_4^2y, x^{11}, y^{11}, w_4^2x^3, w_4^2y^3, w_4w_6x$, and w_4w_6y — and in all of these cases, we can find a generator.

- For $\textcolor{red}{x}$, $\textcolor{blue}{y}$, $x^3, y^3, x^7, y^7, x^{11}$, and y^{11} , the generator is the real projective space of the appropriate dimension, with principal $\mathbb{Z}/2\mathbb{Z}$ -bundle $S^k \rightarrow \mathbb{RP}^k$. For x^k , use \mathbb{RP}^k , and for y^k , use $\widetilde{\mathbb{RP}}^k$. We use S_R^1 to denote \mathbb{RP}^1 , as we often think of this generator in Part II as a circle with duality bundle induced by the Möbius bundle and a map $\mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathrm{GL}^+(2, \mathbb{Z})$ defined by a reflection.
- For w_4^2x and w_4^2y , use $\mathbb{HP}^2 \times S_R^1$, resp. $\mathbb{HP}^2 \times \widetilde{S}_R^1$, with principal $\mathbb{Z}/2\mathbb{Z}$ -bundle the Möbius bundle on the S^1 factor. Likewise, for $w_4^2x^3$ and $w_4^2y^3$, use $\mathbb{HP}^2 \times \mathbb{RP}^3$ and $\mathbb{HP}^2 \times \widetilde{\mathbb{RP}}^3$, with principal $\mathbb{Z}/2\mathbb{Z}$ -bundle $\mathbb{HP}^2 \times S^3 \rightarrow \mathbb{HP}^2 \times \mathbb{RP}^3$.
- For w_4w_6x , use $X_{10} \times S_R^1$, and for w_4w_6y , use $X_{10} \times \widetilde{S}_R^1$; in both cases the principal $\mathbb{Z}/2\mathbb{Z}$ -bundle is the Möbius bundle on the S_R^1 factor.

We can also use $\Omega_*^{\mathrm{Spin}}(B\mathbb{Z}/2\mathbb{Z})$ to see three classes in positive Adams filtration. The h_1 -action $E_\infty^{0,1} \rightarrow E_\infty^{1,3}$ lifts to the product with S_p^1 , as we discussed in Section 11.4.2, implying that $\Omega_2^{\mathrm{Spin}-D_{16}}(\mathrm{pt})$ is generated by $S_p^1 \times S_R^1$, with the principal $\mathbb{Z}/2\mathbb{Z}$ -bundle non-trivial on the second factor; it does not matter whether we use i_4 or \tilde{i}_4 here. The action of the Bott element $w \in \mathrm{Ext}(\mathbb{Z}/2\mathbb{Z})$ defines an isomorphism $E_\infty^{0,1} \rightarrow E_\infty^{4,13}$; as we mentioned in Section 11.4.2, this means the red circle summand $\textcolor{red}{\mathbb{Z}/2\mathbb{Z}}$ of $\Omega_9^{\mathrm{Spin}-D_{16}}(\mathrm{pt})$ (corresponding to $E_\infty^{4,13}$) is generated by $B \times S_R^1$, with the principal $\mathbb{Z}/2\mathbb{Z}$ -bundle non-trivial on the second factor. Finally, the h_1 -action $E_\infty^{4,13} \rightarrow E_\infty^{5,15}$ lifts to tell us that the red circle $\textcolor{red}{\mathbb{Z}/2\mathbb{Z}}$ summand of $\Omega_{10}^{\mathrm{Spin}-D_{16}}(\mathrm{pt})$ is generated by $B \times S_p^1 \times S_R^1$, with the principal $\mathbb{Z}/2\mathbb{Z}$ -bundle non-trivial on the last factor.

14.3.3 Spin- D_8 generators and the Arcana

We can do better by using D_4 instead of $\mathbb{Z}/2\mathbb{Z}$. Analogous to i_4 and \tilde{i}_4 , there are two maps $i_8, \tilde{i}_8: D_4 \hookrightarrow D_8$, which we also discuss in Appendix B.2. Since $D_4 \cong (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$, $H^*(BD_4; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[a, b]$ with $|a| = |b| = 1$.

Lemma 14.38. $i_8^*(x) = \tilde{i}_8^*(x) = \tilde{i}_8^*(y) = a$, $i_8^*(y) = 0$, and $i_8^*(w) = \tilde{i}_8^*(w) = ab + b^2$.

Proof. Recall that we defined x , y , and w as Stiefel-Whitney classes of vector bundles associated to D_8 -representations. Therefore their pullbacks to BD_4 are the corresponding Stiefel-Whitney classes of the bundles associated to the restrictions of those representations to D_4 . Since D_4 is Abelian, these representations split into sums of one-dimensional representations, allowing one to calculate their Stiefel-Whitney classes in terms of a and b . \square

We say that a *Spin- D_8 structure* is analogous to a Spin- D_{16} structure, but with D_8 in place of D_{16} . Giambalvo [154] and Pedrotti [206, Section 6] study Spin- D_8 bordism, and

Barkeshli-Chen-Hsin-Manjunath [211, Section VII.D], Ning-Qi-Gu-Wang [212, Section II], and Manjunath-Calvera-Barkeshli [213] apply this symmetry type to physics. Lemma 14.38 says that a Spin- D_8 structure is data of two principal $\mathbb{Z}/2\mathbb{Z}$ -bundles (equivalent to a principal $(D_4 = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}))$ -bundle) and a trivialization of $w_2(TM) - ab - b^2$, where a and b are the degree-1 characteristic classes of the principal $\mathbb{Z}/2\mathbb{Z}$ -bundles. Given a Spin- D_8 manifold M , i_8 and \tilde{i}_8 endow it with two Spin- D_{16} structures, which we denote M and \widetilde{M} , respectively.

The shearing argument in Corollary 14.16 applies *mutatis mutandis* with D_4 in place of D_8 : if V is the (associated vector bundle to the) standard two-dimensional real representation of D_4 , we obtain a homotopy equivalence

$$MT(\text{Spin-}D_8) \xrightarrow{\sim} MT\text{Spin} \wedge (BD_4)^{V+3\text{Det}(V)-5}. \quad (14.39)$$

So as usual, we need information about the $\mathcal{A}(1)$ -module structure on cohomology of $M_4 := (BD_4)^{V+3\text{Det}(V)-5}$ to run the Adams spectral sequence for Spin- D_8 bordism. By pulling back from $H^*(BD_8; \mathbb{Z}/2\mathbb{Z})$, one learns that in $H^*(BD_4; \mathbb{Z}/2\mathbb{Z})$, $w_1(V + 3\text{Det}(V) - 5) = 0$ and $w_2(V + 3\text{Det}(V) - 5) = ab + b^2$. Therefore in $H^*(M_4; \mathbb{Z}/2\mathbb{Z})$, $\text{Sq}^1(U) = 0$ and $\text{Sq}^2(U) = U(ab + b^2)$. The Steenrod squares of a and b can be determined directly from the axioms to be $\text{Sq}(a) = a + a^2$ and $\text{Sq}(b) = b + b^2$. Therefore using the Cartan formula we can compute Steenrod squares of arbitrary elements of $H^*(M_4; \mathbb{Z}/2\mathbb{Z})$.

Lemma 14.40. *Suppose $x \in H^k(M_4; \mathbb{Z}/2\mathbb{Z})$ is such that $\text{Sq}^2\text{Sq}^2\text{Sq}^2(x) \neq 0$. Then there is a $\mathbb{Z}/2\mathbb{Z}$ summand in $\Omega_k^{\text{Spin-}D_8}(\text{pt})$ detected by x .*

Proof. The fact that $\text{Sq}^2\text{Sq}^2\text{Sq}^2(x) \neq 0$ implies that x generates a free rank-1 summand in $H^*(M_4; \mathbb{Z}/2\mathbb{Z})$ [105, Lemma D.8]. Margolis' theorem, in the form of Corollary 11.18, then implies the element of $\text{Ext}^{0,k}(H^*(M_4; \mathbb{Z}/2\mathbb{Z}))$ corresponding to this summand survives to the E_∞ -page and generates a $\mathbb{Z}/2\mathbb{Z}$ summand of $\Omega_k^{\text{Spin-}D_8}(\text{pt})$. Finally, classes on the E_∞ -page in filtration 0 are detected by the corresponding classes in mod 2 cohomology (see [40, Section 8.4]), so this summand is detected by x . \square

We will use this several times below to show that various characteristic classes detect elements in Spin- D_8 bordism; in all cases, we verified $\text{Sq}^2\text{Sq}^2\text{Sq}^2(x) \neq 0$ using a computer, though these computations are not too tedious to do by hand.

Giambalvo [154, Lemma 4.2] shows that the Adams spectral sequence for Spin- D_8 bordism collapses.⁶³ Now we characterize the image of Spin- D_8 bordism in Spin- D_{16} bordism. Giambalvo's computations [154, Section 4] show that there is an R_2 summand in $H^*(M_4; \mathbb{Z}/2\mathbb{Z})$, and the $\textcolor{red}{R}_2$ summand in $H^*((BD_8)^{V+3\text{Det}(V)-5}; \mathbb{Z}/2\mathbb{Z})$ pulls back isomorphically to this R_2 , so the map on the E_2 -pages between the Ext groups of these summands is an isomorphism. The rest of the E_2 -page of the Adams spectral sequence is in filtration

⁶³What Giambalvo computes is slightly different: the characteristic class condition is $w_2(M) = ab$, rather than $w_2(M) = ab + b^2$. These two conditions are related by an automorphism of $D_4 = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$, so their notions of bordism are the same.

0 [154, Section 4], which is canonically a subspace of mod 2 cohomology, so we can just compute how mod 2 cohomology classes pull back along the two maps $D_4 \rightarrow D_8$.

- Since i_4 and \tilde{i}_4 factor through $D_4 \hookrightarrow D_8$ followed by $i_8, \tilde{i}_8: D_8 \rightarrow D_{16}$, the image of Spin- D_8 bordism inside Spin- D_{16} bordism contains everything we found in Section 14.3.2.
- In degree 5, we want to detect the cohomology classes w^2x and w^2y . We get $i_8^*(w^2x) = \tilde{i}_8^*(w^2x) = a^3b^2 + ab^4$, $i_8^*(w^2y) = 0$, and $\tilde{i}_8^*(w^2y) = a^3b^2 + ab^4$. One can calculate that $\text{Sq}^2\text{Sq}^2\text{Sq}^2(U(a^3b^2 + ab^4)) \neq 0$, so by Lemma 14.40, there is a closed Spin- D_8 5-manifold detected by $a^3b^2 + ab^4$. Applying i_8^* and \tilde{i}_8^* , X_5 is the generator corresponding to w^2x and \tilde{X}_5 is the generator corresponding to w^2y .
- In degree 7, we want to detect the cohomology class w^3y (we already hit x^7 and y^7 using $\Omega_7^{\text{Spin}}(B\mathbb{Z}/2\mathbb{Z})$). Even though $i_8^*(w^3y) = 0$, $\tilde{i}_8^*(w^3y) \neq 0$:

$$\tilde{i}_8^*(w^3y) = a^4b^3 + a^3b^4 + a^2b^5 + ab^6, \quad (14.41)$$

but $\text{Sq}^2\text{Sq}^2\text{Sq}^2$ vanishes on this class. In fact, the bordism invariant $\Omega_7^{\text{Spin-}D_8}(\text{pt}) \rightarrow \mathbb{Z}/2\mathbb{Z}$ given by integrating this class vanishes! One can deduce this a priori by noting: $2\Omega_7^{\text{Spin-}D_8}(\text{pt}) = 0$, and if M is a closed Spin- D_{16} 7-manifold with $w^3y \neq 0$ then $2[M] \neq 0$, as $[M]$ generates the green triangle $\mathbb{Z}/4\mathbb{Z} \subset \Omega_7^{\text{Spin-}D_{16}}(\text{pt})$. So if $\tilde{i}_8^*(w^3y)$ were nonzero on some manifold M , then $2[M] = 0$ in $\Omega_7^{\text{Spin-}D_8}(\text{pt})$ and $2[\tilde{M}] \neq 0$ in $\Omega_7^{\text{Spin-}D_{16}}(\text{pt})$, which would be no good.

- In degree 9, we want to detect w^2x^5 and w^2y^5 . As usual, $i_8^*(w^2y^5) = 0$ and

$$i_8^*(w^2x^5) = \tilde{i}_8^*(w^2x^5) = \tilde{i}_8^*(w^2y^5) = a^7b^2 + a^5b^4. \quad (14.42)$$

$\text{Sq}^2\text{Sq}^2\text{Sq}^2(U(a^7b^2 + a^5b^4)) \neq 0$, so by Lemma 14.40, some closed Spin- D_8 manifold X_9 is detected by $a^7b^2 + a^5b^4$, and X_9 , respectively \tilde{X}_9 give us w^2x^5 , respectively w^2y^5 .

There is another summand of $\Omega_9^{\text{Spin-GL}^+(2,\mathbb{Z})}(\text{pt})$ detected by the mod 2 cohomology class w^4y . One can use Wu's theorem to show that this class vanishes on all Spin- D_8 manifolds. This class is in the image of the map from Spin-Mp(2, \mathbb{Z}) bordism and we already know a representative for it, so we will not worry about it.

- Finally, degree 11. We want to detect the classes w^4x^3 and w^4y^3 , which pull back as $i_8^*(w^4y^3) = 0$ and

$$i_8^*(w^4x^3) = \tilde{i}_8^*(w_4x^3) = \tilde{i}_8^*(w_4y^3) = a^7b^4 + a^3b^8. \quad (14.43)$$

As usual, check that $\text{Sq}^2\text{Sq}^2\text{Sq}^2$ of this is nonzero, which by Lemma 14.40 implies there is a closed Spin- D_8 11-manifold such that X_{11} and \tilde{X}_{11} are detected by w^4x^3 and w^4y^3 . This fulfills a promise we made in [33, Appendix C].

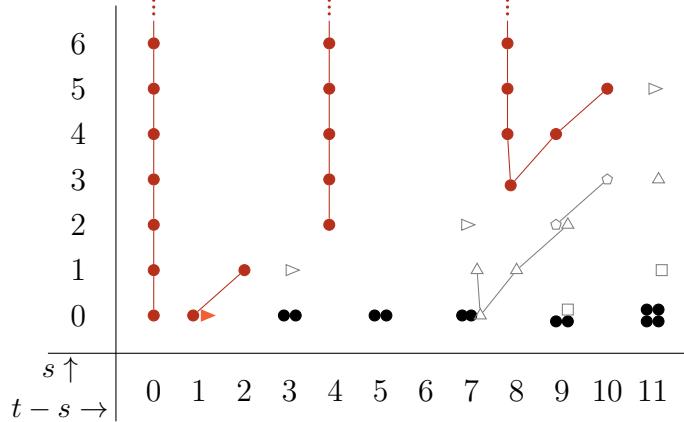


Figure 27: The image of Spin- D_8 bordism in Spin- D_{16} bordism; see Section 14.3.3 for more information. Like in Figure 25, this picture just encodes the summands coming from ko_* ; ko_{*-8} is analogous and we already accounted for $ko \langle 2 \rangle_{*-10}$ in §14.3.2.

Now we need to actually write down X_5 , X_9 , and X_{11} . We call these manifolds ‘‘Arcana manifolds.’’ We wrote down X_{11} in [33, Appendix C.2], but the other two are new. The Arcana are generalized Dold manifolds in the sense of Nath-Sankaran [214], and are a special case of a construction studied by Sarkar-Zvengrowski [215, Example 3.1].

Definition 14.44. The ‘‘Arcanum XI’’ manifold X_{11} is $(S^6 \times S^5)/((\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}))$, where if we regard $S^6 \times S^5 \subset \mathbb{R}^7 \times \mathbb{R}^6$ with coordinates $(x_1, \dots, x_7, y_1, \dots, y_6)$, the generators α and β of $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$ act by

$$\begin{aligned}\alpha(x_1, \dots, x_7, y_1, \dots, y_6) &= (-x_1, \dots, -x_7, -y_1, y_2, \dots, y_6) \\ \beta(x_1, \dots, x_7, y_1, \dots, y_6) &= (x_1, \dots, x_7, -y_1, \dots, -y_6).\end{aligned}\tag{14.45}$$

In [33, Appendix C.2], we show this is a Spin- D_8 manifold with $a^7b^4 + a^3b^8 \neq 0$, so that it is the manifold we were looking for in dimension 11.

We define X_5 and X_9 similarly.

Definition 14.46. The ‘‘Arcanum V’’ manifold X_5 is a quotient $(S^2 \times S^3)/((\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}))$. As above, we embed $S^2 \times S^3 \subset \mathbb{R}^3 \times \mathbb{R}^4$ with coordinates $(x_1, x_2, x_3, y_1, y_2, y_3, y_4)$ and let the generators α and β of $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$ act by

$$\begin{aligned}\alpha(x_1, x_2, x_3, y_1, y_2, y_3, y_4) &= (x_1, x_2, x_3, -y_1, -y_2, -y_3, -y_4) \\ \beta(x_1, x_2, x_3, y_1, y_2, y_3, y_4) &= (-x_1, -x_2, -x_3, y_1, -y_2, -y_3, -y_4).\end{aligned}\tag{14.47}$$

The ‘‘Arcanum IX’’ manifold X_9 is a quotient $(S^6 \times S^3)/((\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}))$. As above, we embed $S^6 \times S^3 \subset \mathbb{R}^7 \times \mathbb{R}^4$ with coordinates $(x_1, \dots, x_7, y_1, y_2, y_3, y_4)$ and let the generators

α and β of $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$ act by

$$\begin{aligned}\alpha(x_1, \dots, x_7, y_1, y_2, y_3, y_4) &= (x_1, \dots, x_7, -y_1, -y_2, -y_3, -y_4) \\ \beta(x_1, \dots, x_7, y_1, y_2, y_3, y_4) &= (-x_1, \dots, -x_7, y_1, -y_2, -y_3, -y_4).\end{aligned}\tag{14.48}$$

Projection onto the x_i coordinates makes X_5 and X_9 into fiber bundles over \mathbb{RP}^2 and \mathbb{RP}^6 , respectively, with fiber \mathbb{RP}^3 in both cases.

Remark 14.49. All three of these manifolds are projectivizations of vector bundles: if $\sigma \rightarrow \mathbb{RP}^n$ denotes the tautological line bundle, then there are diffeomorphisms $X_5 \cong \mathbb{P}(\underline{\mathbb{R}} \oplus \sigma^{\oplus 3}) \rightarrow \mathbb{RP}^2$, $X_9 \cong \mathbb{P}(\underline{\mathbb{R}} \oplus \sigma^{\oplus 3}) \rightarrow \mathbb{RP}^6$, and $X_{11} \cong \mathbb{P}(\underline{\mathbb{R}}^5 \oplus \sigma) \rightarrow \mathbb{RP}^6$.

Now we want to show that X_5 and X_9 are Spin- D_8 manifolds which represent the bordism classes that we are looking for. To perform this computation, we will *stably split* the tangent bundles of the Arcana manifolds: after adding on two trivial summands, TX_5 , TX_9 , and TX_{11} become isomorphic to direct sums of line bundles, allowing an easier calculation of their Stiefel-Whitney classes. We will use this technique several times in the next few sections, so we take a moment to go over how this technique works.

Recall that the normal bundle to $S^n \hookrightarrow \mathbb{R}^{n+1}$ is trivial, with trivialization specified by the outward unit normal vector field, and this specifies a stable splitting

$$TS^n \oplus \underline{\mathbb{R}} \xrightarrow{\cong} T\underline{\mathbb{R}}^{n+1}|_{S^n} = \underline{\mathbb{R}}^{n+1}.\tag{14.50}$$

But better than that, the normal bundle is *equivariantly* trivial for the $O(n+1)$ -action on S^n , lifting (14.50) to an isomorphism of equivariant vector bundles. If $G \subset O(n+1)$ acts freely on S^n , we can therefore descend (14.50) to the quotient S^n/G .

Example 14.51 (Real projective spaces). Let $\mathbb{Z}/2\mathbb{Z}$ act as $\{\pm 1\}$ on \mathbb{R}^{n+1} , which restricts to the antipodal action on S^n . The outward unit normal vector field is invariant under this action, so (14.50) upgrades to an equivalence of $\mathbb{Z}/2\mathbb{Z}$ -equivariant vector bundles, where $\mathbb{Z}/2\mathbb{Z}$ acts trivially on the normal bundle and non-trivially on $\underline{\mathbb{R}}^{n+1}$: specifically, on each $\underline{\mathbb{R}}$ summand, $\mathbb{Z}/2\mathbb{Z}$ acts by -1 .

Therefore this stable splitting descends to the quotient: if $\sigma \rightarrow \mathbb{RP}^n$ denotes the quotient by $\mathbb{Z}/2\mathbb{Z}$ of one of the $\underline{\mathbb{R}}$ summands on the right of (14.50), then we have obtained an isomorphism of (non-equivariant) vector bundles

$$T\mathbb{RP}^n \oplus \underline{\mathbb{R}} \xrightarrow{\cong} \sigma^{\oplus(n+1)}.\tag{14.52}$$

This is a common way to calculate the Stiefel-Whitney classes of \mathbb{RP}^n : since σ is a non-trivial line bundle, $w(\sigma) = 1 + x$, where $x \in H^1(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z})$ is the generator, and then the Whitney sum formula tells us

$$w(T\mathbb{RP}^n) = w(T\mathbb{RP}^n \oplus \underline{\mathbb{R}}) = w(\sigma^{\oplus(n+1)}) = (1 + x)^{n+1}.\tag{14.53}$$

Pretty much by definition, $\sigma \rightarrow \mathbb{R}\mathbb{P}^n$ is the line bundle associated to the principal $\mathbb{Z}/2\mathbb{Z}$ -bundle $S^n \rightarrow \mathbb{R}\mathbb{P}^n$ and the $\mathbb{Z}/2\mathbb{Z}$ -representation we put on \mathbb{R} to make $\underline{\mathbb{R}}$ equivariant. This will not be the last time something like that happens.

Example 14.54 (Lens spaces). For lens spaces, we want $S^{2n-1} \subset \mathbb{C}^n$, so (14.50) looks instead like $TS^{2n-1} \oplus \underline{\mathbb{R}} \cong \underline{\mathbb{C}}^n$. Now choose a primitive k^{th} root of unity ζ and let $\mathbb{Z}/k\mathbb{Z}$ act on \mathbb{C}^n by having the generator act as multiplication by ζ . As above, this leaves invariant the outward unit normal vector field, so the stable splitting is $\mathbb{Z}/k\mathbb{Z}$ -equivariant, and therefore it descends to the quotient [216, 217]:

$$TL_k^{2n-1} \oplus \underline{\mathbb{R}} \xrightarrow{\cong} \mathcal{L}^{\oplus n}, \quad (14.55)$$

where $\mathcal{L} \rightarrow L_k^{2n-1}$ is the quotient of $\underline{\mathbb{C}} \rightarrow S^{2n-1}$ by the $\mathbb{Z}/k\mathbb{Z}$ -action, i.e. the associated complex line bundle bundle to the principal $\mathbb{Z}/k\mathbb{Z}$ -bundle $S^{2n-1} \rightarrow L_k^{2n-1}$ and the standard representation of $\mathbb{Z}/k\mathbb{Z}$ on \mathbb{C} .

Now back to the show.

Lemma 14.56. *There is a stable splitting of the tangent bundle $\pi: \mathbb{P}(\underline{\mathbb{R}}^p \oplus \sigma^{\oplus q}) \rightarrow \mathbb{R}\mathbb{P}^n$ as*

$$T\mathbb{P}(\underline{\mathbb{R}}^p \oplus \sigma^{\oplus q}) \oplus \underline{\mathbb{R}}^2 \xrightarrow{\cong} \pi^*\sigma^{\oplus(n+1)} \oplus \tau^{\oplus p} \oplus (\pi^*\sigma \otimes \tau)^{\oplus q}, \quad (14.57)$$

where $\tau \rightarrow \mathbb{P}(\underline{\mathbb{R}}^p \oplus \sigma^{\oplus q})$ is the tautological line bundle in the fiber direction (see Section 10.1).

Our argument follows that of [215, Section 5.1].

Proof. First consider the sphere bundle $S(\underline{\mathbb{R}}^p \oplus \sigma^{\oplus q}) \rightarrow \mathbb{R}\mathbb{P}^n$; choosing a connection for this fiber bundle splits $TS(\underline{\mathbb{R}}^p \oplus \sigma^{\oplus q}) \cong V \oplus \pi^*(T\mathbb{R}\mathbb{P}^n)$, where $V \rightarrow S(\underline{\mathbb{R}}^p \oplus \sigma^{\oplus q})$ is the vertical tangent bundle. The fiberwise normal bundle is trivial, as usual thanks to the outward unit normal vector field in the fiber direction, so we have a splitting

$$TS(\underline{\mathbb{R}}^p \oplus \sigma^{\oplus q}) \oplus \underline{\mathbb{R}} \xrightarrow{\cong} \underline{\mathbb{R}}^p \oplus \pi^*\sigma^{\oplus q} \oplus \pi^*(T\mathbb{R}\mathbb{P}^n). \quad (14.58)$$

Use Example 14.51 to simplify:

$$TS(\underline{\mathbb{R}}^p \oplus \sigma^{\oplus q}) \oplus \underline{\mathbb{R}}^2 \xrightarrow{\cong} \underline{\mathbb{R}}^p \oplus \pi^*\sigma^{\oplus q} \oplus \pi^*\sigma^{\oplus(n+1)}. \quad (14.59)$$

Now add in the $\mathbb{Z}/2\mathbb{Z}$ antipodal action on the fibers. This is trivial in the horizontal direction, in particular on $\pi^*\sigma^{\oplus(n+1)}$, and the fiberwise normal bundle is equivariantly trivial like in the examples above. This $\mathbb{Z}/2\mathbb{Z}$ acts by -1 on each summand of $\underline{\mathbb{R}}^p$ and $\pi^*\sigma^{\oplus q}$. Therefore when we take the quotient, each of these bundles is tensored with τ , producing the decomposition in (14.57). \square

Corollary 14.60. *X_5 and X_9 have Spin-D₈ structures whose D₄-bundles are their universal covers $S^p \times S^q \rightarrow X_n$.*

Proof. For $m = 5, 9$, let $a \in H^1(X_m; \mathbb{Z}/2\mathbb{Z})$ be the class dual to $\alpha \in (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) = \pi_1(X_m)$ under the identification $H^1(X_m; \mathbb{Z}/2\mathbb{Z}) \cong \text{Hom}(\pi_1(X_m), \mathbb{Z}/2\mathbb{Z})$, and likewise let b be the class dual to β . The way the $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$ -actions were defined on X_m implies that taking the quotient by β produces the sphere bundle $S(\mathbb{R}^p \oplus \sigma^{\oplus q}) \rightarrow \mathbb{RP}^n$, and then taking the quotient by α is the fiberwise antipodal map; therefore $w(\pi^*\sigma) = 1 + b$ and $w(\tau) = 1 + a$.

Lemma 14.56 tells us

$$TX_m = \pi^*\sigma^{\oplus(m-3)} \oplus \tau \oplus (\pi^*\sigma \otimes \tau)^{\oplus 3}, \quad (14.61)$$

so $w(X_m) = (1+b)^{m-3}(1+a)(1+a+b)^3$; expanding out, $w_1(X_m) = 0$ and $w_2(X_m) = ab + b^2$, which is what we needed. \square

Lemma 14.62. *The class $a^3b^2 + ab^4$ is nonzero in $H^5(X_5; \mathbb{Z}/2\mathbb{Z})$, and the class $a^7b^2 + a^5b^4$ is nonzero in $H^9(X_9; \mathbb{Z}/2\mathbb{Z})$.*

Therefore X_5 and X_9 represent the Spin- D_8 bordism classes that we have been looking for.

Proof. The proof resembles that of [33, Lemma C.12]: we want to compute cup products, so we pass through Poincaré duality and compute the intersections of submanifolds of X_m Poincaré dual to the cohomology classes we want to multiply together. There is a transversality condition which is satisfied for generic choices of submanifold representatives. The Arcana manifolds themselves are a little bit difficult to visualize, so we work upstairs in $S^p \times S^q$, intersecting submanifolds which are sent to themselves by the $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$ -action and then taking the quotient.

For example, consider $N := \{x_1 = 0\} \subset S^p \times S^q$. This is a codimension-1 submanifold and $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$ sends N to itself. Though N is null-homologous in $S^p \times S^q$, there is no null-homology that is equivariant for the $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$ -action, and therefore the quotient of N in X_m is not null-homologous, and represents some nonzero class in $H^1(X_m; \mathbb{Z}/2\mathbb{Z})$. Specifically, because α acts trivially on N , but β acts non-trivially, N is Poincaré dual to b . Similarly, $\{x_i = 0\}$ is Poincaré dual to b for each i ; $\{y_1 = 0\}$ is Poincaré dual to a , and $\{y_i = 0\}$ for $i > 1$ is Poincaré dual to $a + b$.

On $S^2 \times S^3$ consider the set of solutions W to the equations $x_1 = x_2 = y_1 = y_2 = y_3 = 0$, which is the intersection of the five codimension-1 submanifolds $\{x_1 = 0\}, \{x_2 = 0\}, \dots, \{y_3 = 0\}$. The images of these submanifolds in X_5 are transverse, so W is a zero-dimensional submanifold, and the Poincaré dual of W is the cup product of the classes these submanifolds define. Specifically, W is Poincaré dual to $b^2a(a+b)^2 = a^3b^2 + ab^4$, so we can check whether $a^3b^2 + ab^4$ is zero in $H^5(X_5; \mathbb{Z}/2\mathbb{Z})$ by checking whether W is null-homologous. Since W is zero-dimensional, this amounts to counting whether W has an odd or even number of points.

Upstairs, the set of solutions to $x_1 = x_2 = y_1 = y_2 = y_3 = 0$ has four solutions in $S^2 \times S^3$: $\{x_3 = \pm 1, y_4 = \pm 1, \text{ all other } x_i, y_i = 0\}$. Thus when we take the quotient by $(\mathbb{Z}/2\mathbb{Z}) \oplus$

$(\mathbb{Z}/2\mathbb{Z})$, these four solutions are identified, so there is one solution in X_5 , and we conclude that $a^3b^2 + ab^4 \neq 0$ in $H^5(X_5; \mathbb{Z}/2\mathbb{Z})$.

For X_9 , the argument is the same; the equations we want are $x_1 = x_2 = y_1 = y_2 = y_4 = 0$ and $x_3y_3 = x_4y_3 = x_5y_3 = x_6y_3 = 0$. Again, on $S^6 \times S^3$, the solutions are $\{x_7 = \pm 1, y_3 = \pm 1\}$, so there is one solution on X_9 , and this corresponds to the cohomology class $b^2(a)(a+b)^2a^4 = a^7b^2 + a^5b^4$. \square

The computation that $a^7b^4 + a^5b^4 \neq 0$ on X_{11} , so that X_{11} and \tilde{X}_{11} represent the 11-dimensional Spin- D_{16} bordism classes we want to detect, is given in [33, Proposition C.7].

At this point we have only four classes left to detect: generators of the green upward-pointing triangle $\mathbb{Z}/4\mathbb{Z}$ and orange rightward-pointing triangle $\mathbb{Z}/2\mathbb{Z}$ summands in dimension 7; the generator of the \mathbb{Z} in dimension 8; and the generator of the yellow pentagon $\mathbb{Z}/2\mathbb{Z}$ in dimension 9. All remaining classes are products of these four classes and non-bounding circles or tori.

14.3.4 The generator $W_{1,8}$ in dimension 8

First we tackle the generator of the red \mathbb{Z} in dimension 8. As we established in Section 14.3.3 (see especially Figure 27), this bordism class is in the image of the map $i_8: \Omega_8^{\text{Spin-}D_8} \rightarrow \Omega_8^{\text{Spin-}D_{16}}$, so we can find a representative whose Spin- D_{16} structure comes from a Spin- D_8 structure. Moreover, looking at the inclusion of Spin- $\mathbb{Z}/8\mathbb{Z}$ bordism into Spin- D_{16} bordism, we learn that two copies of our desired generator must be bordant to the Bott manifold; thus it suffices to construct a Spin- D_8 manifold whose Pontrjagin numbers are half those of the Bott manifold. For example, a connected Calabi-Yau fourfold X_4 has $\hat{A}(M) = 2$, which is twice $\hat{A}(\text{Bott})$, so its Pontrjagin numbers are twice those of a Bott manifold, so any quotient of X_4 by an orientation-preserving action of a group of order 4 has the correct Pontrjagin numbers to be our generator. We will construct such a quotient $W_{1,8}$ of a specific choice of X_4 , and show $W_{1,8}$ is Spin- D_8 , so that it is the generator we want.

Definition 14.63. Let X_4 be the complete intersection on \mathbb{CP}^7 given by choosing projective coordinates

$$(x_1 : x_2 : x_3 : x_4 : x_5 : x_6 : y_1 : y_2) \text{ for } \mathbb{CP}^7, \quad (14.64)$$

and consider the complete intersection specified by the three equations

$$\sum_{i=1}^6 x_i^2 = x_1^2 + P(y) = x_2^2 + P'(y) = 0, \quad (14.65)$$

where P and P' are cubic polynomials. For generic choices of P and P' , the intersection is complete, and we obtain a smooth Calabi-Yau fourfold X_4 .⁶⁴

⁶⁴The criterion for a complete intersection to be Calabi-Yau is that the sums of the degrees of the defining polynomials add up to one plus the dimension of the ambient projective space. See, e.g., [218].

Consider the following action of $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$ on \mathbb{CP}^7 :

$$\begin{aligned}\alpha(x_1 : \cdots : x_6 : y_1 : y_2) &= (-x_1 : \cdots : -x_6 : y_1 : y_2) \\ \beta(x_1 : \cdots : x_6 : y_1 : y_2) &= (x_1^* : \cdots : x_6^* : y_1 : y_2).\end{aligned}\tag{14.66}$$

This action leaves the equations (14.65) invariant, and so defines an action of $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$ on X_4 . One can check that both α and β are orientation-preserving and, restricted to X_4 , lack fixed points: in a fixed point for α , the last two equations would not have a solution for y , while in a fixed point for β , all the x_i are real, and then the first equation implies they all vanish. Thus the quotient $W_{1,8} := X_4 / ((\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}))$ is an oriented smooth manifold. $W_{1,8}$ cannot be Spin: if it were, its \hat{A} -genus would be $1/2$, but the \hat{A} -genus is an integer on Spin manifolds. We call this manifold the ‘‘half Bott manifold’’ because $W_{1,8} \sqcup W_{1,8}$ is bordant to a Bott manifold.

Proposition 14.67. $W_{1,8}$ admits a Spin- D_8 structure.

Proof. Consider the standard Spin structure on the ambient \mathbb{CP}^7 . Equivariant spinors on \mathbb{CP}^7 invariant under the action (14.66) will descend to spinors on $W_{1,8}$. To construct spinors on \mathbb{CP}^n explicitly, we can start with Dirac spinors Ψ on $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$. For $z \in \mathbb{C}^{n+1}$, the transformation $z \rightarrow \lambda z$ that we quotient by to get complex projective space induces an infinitesimal action on spinor fields containing a rotation and a scaling transformation [219]:

$$\delta_\lambda \Psi = (M - \text{id})\Psi, \quad M = \sum_{i=1}^n \Gamma_i \Gamma_{i+1}, \tag{14.68}$$

where the first part of the above transformation implements the infinitesimal rotation involved in a complex rescaling, and the part proportional to the identity matrix implements the rescaling. Spinors on \mathbb{CP}^n correspond precisely to those spinors on \mathbb{C}^{n+1} which satisfy $\delta_\lambda \Psi = 0$. As is clear from (14.68), this will only be possible if the matrix M has a $+1$ eigenvalue. According to the analysis around equation (D.6) of [32], this is only possible if $n + 1$ is even. This is a very elementary way to see that we only have a Spin structure on \mathbb{CP}^n when n is odd.

Taking now $n = 7$, spinors on $W_{1,8}$ correspond to spinors on \mathbb{C}^8 satisfying $\delta_\lambda \Psi = 0$ and which are also invariant under the Spin lifts of (14.66),

$$\Psi = \left(\prod_{i=1}^{12} \Gamma_i \right) \Psi, \quad \Psi = \left(\prod_{i=0}^5 \Gamma_{2i+1} \right) \Psi. \tag{14.69}$$

For this to work, the two matrices

$$M_1 = \prod_{i=1}^{12} \Gamma_i, \quad M_2 = \prod_{i=0}^5 \Gamma_{2i+1} \tag{14.70}$$

should generate the group $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$ when restricted to spinors with $\delta_\lambda \Psi = 0$. But one can explicitly compute that instead the commutation relations are

$$M_1^2 = \text{id}, \quad M_2^2 = -\text{id}, \quad M_1 M_2 M_1 = -M_2, \quad (14.71)$$

i.e. they generate a D_8 group. This is why there is no Spin structure on $W_{1,8}$. However, it is also clear that, by tensoring the above matrices with an appropriate representation of D_8 , two wrongs can make a right, and we can obtain D_4 representations. So $W_{1,8}$ admits a Spin- D_8 structure, as claimed. \square

14.3.5 The generator W_1^9 in dimension 9

The next generator we address corresponds to the yellow pentagon $\mathbb{Z}/2\mathbb{Z}$ summand in $E_\infty^{2,11}$. The manifold we find resembles the generators we give for the remaining classes in dimension 7, but this one is simpler, so we start in dimension 9 as a warm-up.

The $\mathbb{Z}/2\mathbb{Z}$ class under investigation is not in the images of the maps from Spin- $\mathbb{Z}/8\mathbb{Z}$ or Spin- D_8 bordism, so we need its representative W_1^9 to have a fundamental group at least as complicated as D_8 . We will define W_1^9 momentarily; the key results in this section are Proposition 14.74, which shows that W_1^9 is Spin- D_{16} , and Proposition 14.77, which shows that it is linearly independent from the other generators we have found.

Definition 14.72. Let ζ be a primitive k^{th} root of unity and L_k^{2n-1} denote the lens space which is the quotient of $S^{2n-1} \subset \mathbb{C}^n$ by the $\mathbb{Z}/k\mathbb{Z}$ -action on \mathbb{C}^n which is multiplication by ζ , which preserves the unit sphere. Complex conjugation exchanges ζ with another primitive k^{th} root of unity, and therefore the image of a $\mathbb{Z}/k\mathbb{Z}$ -orbit of S^{2n-1} under complex conjugation is another $\mathbb{Z}/k\mathbb{Z}$ -orbit. Therefore this involution descends to an involution on L_k^{2n-1} , which we also call *complex conjugation*.

Since complex conjugation is orientation-reversing on \mathbb{C} , it is orientation-preserving on \mathbb{C}^{2m} and orientation-reversing on \mathbb{C}^{2m+1} . The $\mathbb{Z}/k\mathbb{Z}$ -action on S^{2n-1} is orientation-preserving, so we can descend to L_k^{2n-1} and infer that complex conjugation on L_k^{2n-1} is orientation-preserving when n is even and is orientation-reversing when n is odd.

Definition 14.73. Let $\mathbb{Z}/2\mathbb{Z}$ act on $L_4^5 \times S^4$ by complex conjugation on the lens space and the antipodal map on S^4 . This is a free action; we define W_1^9 to be the quotient.

As far as we can tell, this manifold was first studied by Kamata-Minami [204, Section 2], where it is called $D(5, 4)$.

The first thing to check is whether W_1^9 is orientable, or equivalently, whether the involution we used to define it is orientation-preserving. This is true: the antipodal map is orientation-reversing on S^{2n} , and we just saw that complex conjugation is orientation-reversing on L_k^5 . Therefore the combination of these two involutions is orientation-preserving. In fact, because L_4^5 and S^4 come with canonical orientations, W_1^9 is also canonically oriented.

The complex conjugation involution passes to inversion on $\pi_1(L_4^5) = \mathbb{Z}/4\mathbb{Z}$, so $\pi_1(W_1^9) = D_8$; concretely, W_1^9 is also the quotient of $S^5 \times S^4$ by the D_8 -action generated by two diffeomorphisms r and s : $r = (m, \text{id})$ is multiplication by i on S^5 and the identity on S^4 and s is reflection on S^5 and the antipodal map on S^4 .

Proposition 14.74. W_1^9 with its D_8 -bundle $S^5 \times S^4 \rightarrow W_1^9$ has a Spin- D_{16} structure.

To do this, we will stably split TW_1^9 in order to compute its low-degree Stiefel-Whitney classes, following Fujino [220, Proposition 3.2].

Proof. Using Examples 14.51 and 14.54, there is an isomorphism of vector bundles

$$T(L_4^5 \times S^4) \oplus \underline{\mathbb{R}}^2 \xrightarrow{\cong} \mathcal{L}^{\oplus 3} \oplus \underline{\mathbb{R}}^5. \quad (14.75)$$

Let $\mathbb{Z}/2\mathbb{Z}$ act on $L_4^5 \times S^4$ by complex conjugation on L_4^5 and the antipodal map on S^4 , so that the quotient is W_1^9 . Then, Example 14.51 together with the analogous fact about complex conjugation tells us that (14.75) upgrades to an isomorphism of $\mathbb{Z}/2\mathbb{Z}$ -equivariant vector bundles, where:

- $\mathbb{Z}/2\mathbb{Z}$ acts trivially on the $\underline{\mathbb{R}}^2$ on the left-hand side of (14.75).
- $\mathbb{Z}/2\mathbb{Z}$ acts by complex conjugation on each \mathcal{L} summand on the right-hand side.
- $\mathbb{Z}/2\mathbb{Z}$ acts by -1 on each factor of $\underline{\mathbb{R}}$ on the right-hand side.

Therefore when we take the quotient, we reprove a theorem of Fujino [220, Proposition 3.2]: that there is an isomorphism of vector bundles

$$TW_1^9 \oplus \underline{\mathbb{R}}^2 \xrightarrow{\cong} V^{\oplus 3} \oplus \sigma^{\oplus 5}, \quad (14.76)$$

where V and σ are as follows: if $P \rightarrow W_1^9$ denotes the quotient $S^5 \times S^4 \rightarrow W_1^9$, which is a principal D_8 -bundle, then σ is associated to P and the sign representation $D_8 \rightarrow O(1)$ sending rotations to 1 and reflections to -1 , and V is associated to the standard representation $D_8 \rightarrow O(2)$ as rotations and reflections on \mathbb{R}^2 .⁶⁵ Thus $\sigma = \text{Det}(V)$, so $w_1(V) = w_1(\sigma)$.

Throwing the Whitney sum formula at (14.76), and using the fact that $w_1(V) = w_1(\sigma)$ and $w_2(\sigma) = 0$, because σ is a line bundle, we learn that $w_1(TW_1^9) = 0$ and that $w_2(TW_1^9) = w_2(V)$. Since V is associated to a principal D_8 -bundle via the standard representation $D_8 \rightarrow O(2)$, this is precisely the condition for W_1^9 to have a Spin- D_{16} structure. \square

We have found all but two of the generators of $\Omega_9^{\text{Spin-}D_{16}}(\text{pt})$. The remaining two each generate $\mathbb{Z}/2\mathbb{Z}$ summands. We will show that W_1^9 generates the yellow $\mathbb{Z}/2\mathbb{Z}$ summand, but at present we also do not know a generator for the green triangle $\mathbb{Z}/2\mathbb{Z}$ summand. However,

⁶⁵What we call V , Fujino [220, Proposition 3.1] calls η_1 .

looking at the E_∞ -page (Figure 22, bottom), we know that there is some Spin- D_{16} 7-manifold W_1^7 such that this green $\mathbb{Z}/2\mathbb{Z}$ is generated by $W_1^7 \times S_p^1 \times S_p^1$. For now, choose such a manifold W_1^7 ; we will not need to know anything more about it right now. We will choose an explicit example of W_1^7 later in Section 14.3.6.

Proposition 14.77. *There is a bordism invariant $\mu: \Omega_9^{\text{Spin-}D_{16}}(\text{pt}) \rightarrow \mathbb{Z}/2\mathbb{Z}$ which is 1 on W_1^9 and vanishes on the other generators (which are L_4^9 , $W_1^7 \times S_p^1 \times S_p^1$, Bott $\times S_R^1$, $\mathbb{HP}^2 \times S_R^1$, $\mathbb{HP}^2 \times \widetilde{S}_R^1$, X_9 , and \widetilde{X}_9).*

Proof. Let M be a Spin- D_{16} manifold with associated D_8 -bundle $P \rightarrow M$. The quotient of P by $D_4 \subset D_8$ is a Spin- D_8 manifold \widehat{M} which is a double cover of M : it is Spin- D_8 because the D_8 -action on P lifts to a D_{16} -action on its spinor bundle \mathbb{S} , and pulling back along $D_4 \hookrightarrow D_8$, we get the subgroup $D_8 \subset D_{16}$ acting on \mathbb{S} . The double cover $\widehat{M} \rightarrow M$ corresponds to the class $y(P) \in H^1(M; \mathbb{Z}/2\mathbb{Z})$. The assignment $M \mapsto \widehat{M}$ is compatible with taking boundaries, and therefore defines a map

$$D: \Omega_k^{\text{Spin-}D_{16}}(\text{pt}) \longrightarrow \Omega_k^{\text{Spin-}D_8}(\text{pt}). \quad (14.78)$$

We use notation for Spin- D_8 characteristic classes from Section 14.3.3: a and b are the two generators of $H^*(BD_4; \mathbb{Z}/2\mathbb{Z})$, both in degree 1. The class a is detected by a reflection and b is detected by a generating rotation. Our invariant μ is defined to be

$$\mu: \Omega_9^{\text{Spin-}D_{16}}(\text{pt}) \xrightarrow{D} \Omega_9^{\text{Spin-}D_8}(\text{pt}) \xrightarrow{f(a^5b^4 + a^3b^6)} \mathbb{Z}/2\mathbb{Z}. \quad (14.79)$$

Recall the maps $i_8, \tilde{i}_8: D_8 \rightarrow D_{16}$ that we used to define two Spin- D_{16} structures on a Spin- D_8 manifold M . The composition $D \circ i_8 = 0$, because $i_8^*(y) = 0$, and therefore $\mu \circ i_8 = 0$. For \tilde{i}_8 , $\tilde{i}_8^*(y) = \tilde{i}_8^*(x) = a$, so manifolds in the image of the map $D \circ \tilde{i}_8$ have $a(P) = 0$, so $\mu \circ \tilde{i}_8 = 0$ too. Therefore $\mu(N) = 0$ when N is one of the following generators: Bott $\times S_R^1$, $\mathbb{HP}^2 \times S_R^1$, $\mathbb{HP}^2 \times \widetilde{S}_R^1$, X_9 , and \widetilde{X}_9 .

Next L_4^4 . $D(L_4^4) = \mathbb{RP}^9$, which is Spin- D_8 but not Spin. Therefore $ab + b^2 \neq 0$ for \mathbb{RP}^9 , so $a \neq b$; since $H^1(\mathbb{RP}^9; \mathbb{Z}/2\mathbb{Z})$ has only one nonzero element, one of a and b vanishes, and therefore $a^5b^4 + a^3b^6 = 0$ and $\mu(L_4^4) = 0$.

Finally, $W_1^7 \times S_p^1 \times S_p^1$. The associated D_4 -bundle is trivial in the torus directions, so its characteristic classes pull back across the projection $\text{pr}_1: W_1^7 \times S_p^1 \times S_p^1 \rightarrow W_1^7$, and the same is true of $D(W_1^7 \times S_p^1 \times S_p^1) \cong D(W_1^7) \times S_p^1 \times S_p^1$. The degree-9 class $a^5b^4 + a^3b^6$ vanishes on the 7-manifold $D(W_1^7)$, so $\mu(W_1^7 \times S_p^1 \times S_p^1) = 0$.

That accounts for all of the previously discovered generators, so it suffices to show that $\mu(W_1^9) = 1$. First, we should characterize $D(W_1^9)$. This is a projective bundle over a projective space, like the Arcana; as in Section 14.3.3, it is an \mathbb{RP}^5 -bundle over \mathbb{RP}^4 which we specify as a $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$ quotient of $S^5 \times S^4$ inside $\mathbb{R}^6 \times \mathbb{R}^5$ with coordinates $(x_1, \dots, x_6, y_1, \dots, y_5)$. Specifically, letting α and β be the generators of $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$,

act as follows:

$$\begin{aligned}\alpha(x_1, x_2, x_3, x_4, x_5, x_6, y_1, \dots, y_5) &= (-x_1, -x_2, -x_3, -x_4, -x_5, -x_6, y_1, \dots, y_5), \\ \beta(x_1, x_2, x_3, x_4, x_5, x_6, y_1, \dots, y_5) &= (-x_1, x_2, -x_3, x_4, -x_5, x_6, -y_1, \dots, -y_5).\end{aligned}\quad (14.80)$$

Why is this? $D(W_1^9)$ is the quotient of $\mathbb{RP}^5 \times S^4$ by an involution; β is this involution, and α is the antipodal action on S^5 to get us to $\mathbb{RP}^5 \times S^4$.

As in Lemma 14.62, one can check that $w(D(W_1^9)) = ab + b^2$ and that the set of solutions to the equations $x_2 = x_3 = x_4 = x_5 = x_6 = y_1 = y_2 = y_3 = y_4 = 0$ has four solutions $\{x_1 = \pm 1, y_5 = \pm 1\}$ on $S^5 \times S^4$, hence one solution on $D(W_1^9)$, so the corresponding cohomology class is nonzero. Tracking the $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$ -actions on these coordinates like in Lemma 14.62, the class we get is $(a+b)^2 a^3 b^4 = a^5 b^4 + a^3 b^6$, so $\mu(W_1^9) = 1$. \square

Remark 14.81. The manifolds $(\mathbb{CP}^m \times S^n)/(\mathbb{Z}/2\mathbb{Z})$, where $\mathbb{Z}/2\mathbb{Z}$ acts by complex conjugation on \mathbb{CP}^m and by the antipodal map on S^n , are called *Dold manifolds*. Dold studied them in [221], showing that they generate the unoriented bordism ring. These manifolds are quotients of $S^{2m+1} \times S^n$ by a free $O(2)$ -action, and in this regard W_1^9 is an analogue of the Dold manifolds with $O(2)$ replaced with D_8 . From this perspective, it is not so surprising that a manifold like W_1^9 shows up in our list of generators. Kamata-Minami [204] and Kamata [205] use a family of manifolds including W_1^9 to study $\Omega_*^U(BD_{2k})$, providing an additional hint that W_1^9 , or something like it, would be a nonzero element of $\Omega_9^{\text{Spin-}D_{16}}(pt)$.

14.3.6 The generator W_1^7 in dimension 7

Next, the $\mathbb{Z}/4\mathbb{Z}$ in dimension 7. Recall the definition of Q_4^7 from Appendix C.2: the quotient of the unit sphere bundle in $V := O(2) \oplus \mathbb{C}^3 \rightarrow \mathbb{CP}^1$ by the $\mathbb{Z}/4\mathbb{Z}$ -action given by multiplication by i . This is a fiber bundle over $S^2 = \mathbb{CP}^1$ with fiber L_4^5 .

Definition 14.82. Let τ denote the the involution on Q_4^7 which is complex conjugation on the lens space fibers and the antipodal map on S^2 . This is a free involution; let W_1^7 be the quotient.

W_1^7 is a fiber bundle over \mathbb{RP}^2 with fiber L_4^5 . Reflection on L_4^5 is orientation-reversing, and the antipodal map on S^2 is orientation-reversing, so W_1^7 is orientable.

Complex conjugation acts on $\pi_1(Q_4^7) \cong \mathbb{Z}/4\mathbb{Z}$ by -1 , so $\pi_1(W_1^7) \cong D_8$. The universal cover is $S(V) \rightarrow W_1^7$, and this is a principal D_8 -bundle.

Proposition 14.83. W_1^7 , with principal D_8 -bundle $S(V) \rightarrow W_1^7$, has a Spin- D_{16} structure.

Proof. First, like in Section 14.3.5, we split the stable tangent bundle of W_1^7 . The splitting (14.50) behaves well in families; that is, if $\pi: E \rightarrow M$ is a Euclidean vector bundle over

a manifold M and $i: S(E) \hookrightarrow E$ denotes the unit sphere bundle, then the normal bundle to the inclusion i is trivializable using the outward unit normal vector field. That is,

$$TS(E) \oplus \underline{\mathbb{R}} \xrightarrow{\cong} TE|_{S(E)} \xrightarrow{\cong} (\pi^*TM \oplus \pi^*E)|_{S(E)}, \quad (14.84)$$

the latter isomorphism given by choosing a connection on V . We care about the vector bundle $V := \underline{\mathbb{C}}^2 \oplus \mathcal{O}(2) \rightarrow S^2$; from (14.84) we learn

$$TS(V) \oplus \underline{\mathbb{R}} \xrightarrow{\cong} \pi^*(TS^2) \oplus \underline{\mathbb{C}}^2 \oplus \pi^*\mathcal{O}(2). \quad (14.85)$$

Add on the normal bundle to $S^2 \subset \mathbb{R}^3$ to stably trivialize TS^2 :

$$TS(V) \oplus \underline{\mathbb{R}}^2 \xrightarrow{\cong} \underline{\mathbb{R}}^3 \oplus \underline{\mathbb{C}}^2 \oplus \pi^*\mathcal{O}(2). \quad (14.86)$$

Now we descend to Q_4^7 , which means adding $\mathbb{Z}/4\mathbb{Z}$ -actions to (14.86). $\mathbb{Z}/4\mathbb{Z}$ acts on $TS(V)$ by differentiating the $\mathbb{Z}/4\mathbb{Z}$ -action on Q_4^7 . The $\mathbb{Z}/4\mathbb{Z}$ -action on the normal bundle to $S(V) \subset V$ is equivariantly trivial, just as in (14.54), and the action on the normal bundle to $S^2 \subset \mathbb{R}^3$ is trivial, because $\mathbb{Z}/4\mathbb{Z}$ acts trivially on the base S^2 . This takes care of the left-hand side of (14.86); for the right-hand side, $\mathbb{Z}/4\mathbb{Z}$ acts trivially on $\underline{\mathbb{R}}^3$, because this summand came from the base and $\mathbb{Z}/4\mathbb{Z}$ acts fiberwise; and on each summand of $\underline{\mathbb{C}}^2 \oplus \pi^*\mathcal{O}(2)$, $\mathbb{Z}/4\mathbb{Z}$ acts fiberwise by the standard one-dimensional complex representation. Therefore when we take the quotient, we obtain a stable splitting

$$TQ_4^7 \oplus \underline{\mathbb{R}}^2 \xrightarrow{\cong} \underline{\mathbb{R}}^3 \oplus \mathcal{L}_1^{\oplus 2} \oplus \mathcal{L}_2, \quad (14.87)$$

where $\mathcal{L}_1, \mathcal{L}_2 \rightarrow Q_4^7$ are complex line bundles. Explicitly, \mathcal{L}_1 is associated to the principal $\mathbb{Z}/4\mathbb{Z}$ -bundle $S(V) \rightarrow Q_4^7$ via the rotation representation $\rho: \mathbb{Z}/4\mathbb{Z} \rightarrow \mathrm{U}(1)$ and $\mathcal{L}_2 \cong \mathcal{L}_1 \otimes p^*\mathcal{O}(2)$, where $p: Q_4^7 \rightarrow S^2$ is the bundle map. Using these descriptions and the Serre spectral sequence for the fiber bundle $p: Q_4^7 \rightarrow S^2$ one can compute the Chern classes of \mathcal{L}_1 and \mathcal{L}_2 , similarly to the calculation in the proof of Lemma 14.89 below.

Next we want to descend to W_1^7 , so we incorporate $\mathbb{Z}/2\mathbb{Z}$ -actions into (14.87). Again, the normal bundles on the left-hand side are equivariantly trivial. On the right-hand side, $\mathbb{Z}/2\mathbb{Z}$ acts on $\underline{\mathbb{R}}^3$ by the antipodal map, which is -1 . On $\mathcal{L}_1^{\oplus 2}$ and \mathcal{L}_2 , $\mathbb{Z}/2\mathbb{Z}$ acts by complex conjugation. Thus we have a stable splitting analogous to (14.76):

$$TW_1^7 \oplus \underline{\mathbb{R}}^2 \xrightarrow{\cong} p^*\sigma^{\oplus 3} \oplus E_1^{\oplus 2} \oplus E_2, \quad (14.88)$$

where $p: W_1^7 \rightarrow \mathbb{RP}^2$ is induced from $\pi: Q_4^7 \rightarrow S^2$, $E_1, E_2 \rightarrow W_1^7$ are real, rank-two vector bundles, and $\sigma \rightarrow \mathbb{RP}^2$ is the tautological line bundle. By keeping track of which elements of D_8 act orientation-reversingly on each summand, we learn $w_1(p^*\sigma) = w_1(E_1) = w_1(E_2)$. Call this element x . Using this, and the fact that $w_2(\sigma) = 0$, apply the Whitney sum formula to (14.88). We learn $w_1(TW_1^7) = 0$ (indeed, we already checked that W_1^7 is orientable) and

that $w_2(TW_1^7) = w_2(E_2) + x^2$.

To proceed we need to better understand the mod 2 cohomology ring of W_1^7 .

Lemma 14.89. *The Serre spectral sequence for the fiber bundle $L_4^5 \rightarrow W_1^7 \rightarrow \mathbb{RP}^2$ collapses, providing an isomorphism*

$$H^*(W_1^7; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[x, y, w]/(x^3, xy + y^2, w^3), \quad (14.90)$$

where $|x| = |y| = 1$ and $|w| = 2$.

The class $x \in H^1(W_1^7; \mathbb{Z}/2\mathbb{Z})$ is the same x we met above.

Proof. Draw the Serre spectral sequence in Figure 28 and see what happens! For degree reasons, the only differentials that can be nonzero are of the form $d_2: E_2^{0,q} \rightarrow E_2^{2,q-1}$. Since $E_2^{0,*} = H^*(L_4^5; \mathbb{Z}/2\mathbb{Z})$ is generated as a ring by $y \in H^1(L_4^5; \mathbb{Z}/2\mathbb{Z})$ and $w \in H^2(L_4^5; \mathbb{Z}/2\mathbb{Z})$, the Leibniz rule tells us that all d_2 s out of the $p = 0$ line are determined by their values on y and w . We will show $d_2(y) = 0$ and $d_2(w) = 0$.

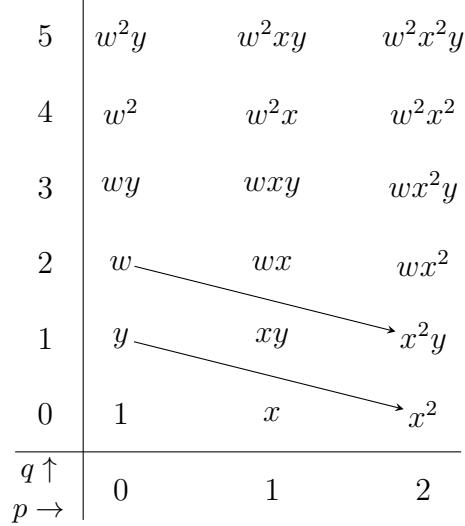


Figure 28: The Serre spectral sequence computing $H^*(W_1^7; \mathbb{Z}/2\mathbb{Z})$. In Lemma 14.89, we show the two pictured differentials vanish, and that this implies the entire spectral sequence collapses.

Suppose $d_2(y) \neq 0$. Then on the E_∞ -page, there is a single $\mathbb{Z}/2\mathbb{Z}$ summand in total degree 1, forcing $H^1(W_1^7; \mathbb{Z}/2\mathbb{Z})$ to be $\mathbb{Z}/2\mathbb{Z}$. However, we already know

$$H^1(W_1^7; \mathbb{Z}/2\mathbb{Z}) = \text{Hom}(\pi_1(W_1^7), \mathbb{Z}/2\mathbb{Z}) = \text{Hom}(D_8, \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}), \quad (14.91)$$

so $d_2(y) = 0$.

Now suppose $d_2(w) \neq 0$. This, together with $d_2(y) = 0$, implies $d_2(wy) = wyx^2$, $d_2(w^2) = 0$, and $d_2(w^2y) = 0$. But then on the E_∞ -page, there are two $\mathbb{Z}/2\mathbb{Z}$ summands in total degree 2 and three $\mathbb{Z}/2\mathbb{Z}$ summands in total degree 5, which violates Poincaré duality for W_1^7 .

The multiplicative structure is clear except for the relation $xy + y^2 = 0$. This follows from the fact that x and y pull back from BD_8 , and $xy + y^2 = 0$ in $H^*(BD_8; \mathbb{Z}/2\mathbb{Z})$. \square

Recall that $w_2(TW_1^7) = w_2(E_2) + x^2$, and that we want to show this equals w_2 of the associated bundle to the principal D_8 -bundle $S(V) \rightarrow W_1^7$ and the standard two-dimensional real representation of D_8 . Call this bundle E .

Lemma 14.89 tells us that $w_2(E_2) = a_1x^2 + a_2xy + a_3z$ and $w_2(E) = b_1x^2 + b_2xy + b_3z$.

- To compute a_1 and b_1 , let \widetilde{W} be the quotient of $S(V)$ by $\mathbb{Z}/2\mathbb{Z}$ acting by the antipodal map on the base and complex conjugation on the fibers. Then $\pi: \widetilde{W} \rightarrow W$ is a principal $\mathbb{Z}/4\mathbb{Z}$ -bundle, and x^2 pulls back to the nonzero class in $H^2(\widetilde{W}; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. One can check that π^*E_2 is Spin, but π^*E is not, so $a_1 = 0$ and $b_1 = 1$.
- To compute a_2 and b_2 , first pull back $W_1^7 \rightarrow \mathbb{RP}^2$ along $\mathbb{RP}^1 \hookrightarrow \mathbb{RP}^2$; then pull back along the inclusion $L_4^1 \hookrightarrow L_4^5$, which is compatible with the fiber bundle structure. The resulting space is the Klein bottle KB , as a $L_4^1 = S^1$ -bundle over $\mathbb{RP}^1 = S^1$. For both E_2 and E , $\pi_1(KB) \cong \mathbb{Z}$ acts in a way that may be lifted to the spinor bundle, so $a_2 = b_2 = 0$.
- Finally, $a_3 = b_3 = 1$ by restricting E_2 and E to the fiber; as bundles over lens spaces, neither is Spin.

Thus $w_2(E_2) + x^2 = w_2(E)$. \square

Recall that $\Omega_7^{\text{Spin-}D_{16}}(\text{pt}) \cong (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$. We are trying to generate the green $\mathbb{Z}/4\mathbb{Z}$ summand (corresponding to the upward-pointing triangle pieces in the E_∞ -page).

If M is a Spin- D_{16} manifold with associated principal D_8 -bundle $P \rightarrow M$, let $\widehat{M} := P/\mathbb{Z}/4\mathbb{Z}$; then, \widehat{M} is a Spin- $\mathbb{Z}/8\mathbb{Z}$ manifold which is a double cover of M : the Spin- $\mathbb{Z}/8\mathbb{Z}$ structure comes from $P \rightarrow \widehat{M}$, which is a $\mathbb{Z}/4\mathbb{Z}$ bundle, and the action on spinors lifts to the pullback of $D_{16} \rightarrow D_8$ along $\mathbb{Z}/4\mathbb{Z} \hookrightarrow D_8$, giving us $\mathbb{Z}/8\mathbb{Z}$ as claimed. The double cover $\widehat{M} \rightarrow M$ is classified by $x(P) \in H^1(M; \mathbb{Z}/2\mathbb{Z})$. Like the Spin- D_8 double cover we used in Section 14.3.5, this assignment defines a bordism invariant

$$\varpi: \Omega_k^{\text{Spin-}D_{16}}(\text{pt}) \longrightarrow \Omega_k^{\text{Spin-}\mathbb{Z}/8\mathbb{Z}}(\text{pt}). \quad (14.92)$$

Proposition 14.93. $\varpi(W_1^7)$ is a generator of $\Omega_7^{\text{Spin-}\mathbb{Z}/8\mathbb{Z}}(\text{pt}) \cong \mathbb{Z}/4\mathbb{Z}$.

This implies that W_1^7 represents either $(1, 0)$ or $(1, 1)$ in $\Omega_7^{\text{Spin-}D_{16}}(\text{pt}) \cong (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$, which is what we wanted.

Proof. Essentially by definition, the double cover of W_1^7 we get from ϖ is Q_4^7 , and Q_4^7 with any of its Spin- $\mathbb{Z}/8\mathbb{Z}$ structures is order 4 in $\Omega_7^{\text{Spin-}\mathbb{Z}/8\mathbb{Z}}(\text{pt})$. \square

14.3.7 The orange generator in dimension 7

We are missing one generator in dimension 7, corresponding in the E_∞ -page to the orange (rightward-pointing triangle) $\mathbb{Z}/2\mathbb{Z}$ summand in topological degree 7. Let Q_{16} be the Pin[−] cover of D_8 with reference to the standard, two-dimensional representation; that is, Q_{16} is the pullback of $D_8 \hookrightarrow O(2)$ by the double cover $\text{Pin}^-(2) \rightarrow O(2)$.⁶⁶ Q_{16} is called the *generalized quaternion group* of order 16, or the *binary dihedral group* $2D_8$, or the *dicyclic group* Dic_{16} . Q_{16} acts freely on S^3 ; the quotient is called a *prism manifold*.

Inside Q_{16} there are classes \hat{r} and j such that Q_{16} has the presentation

$$Q_{16} = \langle \hat{r} \mid \hat{r}^8 = \text{id}, \ j^2 = \hat{r}^4, \ j^{-1} \hat{r} j = \hat{r}^{-1} \rangle. \quad (14.94)$$

Under the double cover $Q_{16} \rightarrow D_8$, \hat{r} is sent to a rotation and j is sent to a reflection.

Let ζ be the involution on the quaternions \mathbb{H} which is 1 in the 1 and i directions and -1 in the j and k directions. Then, up to a factor of -1 , ζ commutes with the standard action of Q_{16} on \mathbb{H} : if we think of \mathbb{H} as \mathbb{C}^2 , $\zeta(z, w) = (z, -w)$. This commutes with multiplication by a unit complex number on both factors (the action of \hat{r}), but $j \cdot (z, w) = (w, -z)$, so $\zeta \circ j = -j \circ \zeta$. Since ζ is an isometry, it restricts to an involution on $S^3 \subset \mathbb{H}$. We would like to descend it to an involution on S^3/Q_{16} ; a priori, this is not possible, because ζ does not commute with the Q_{16} -action on S^3 , but the two actions commute up to a sign, and since S^3/Q_{16} is a quotient of \mathbb{RP}^3 , sign discrepancies do not matter, allowing ζ to induce a (very much non-free) $\mathbb{Z}/2\mathbb{Z}$ -action $\bar{\zeta}$ on S^3/Q_{16} .

The final generator, which we call W_2^7 , is the quotient of

$$(S^3/Q_{16}) \times T_p^2 \times S^2 \quad (14.95)$$

where $T_p^2 = S_p^1 \times S_p^1$, by the involution which is:

- $\bar{\zeta}$ on S^3/Q_{16} ,
- the involution on T_p^2 whose quotient is the Klein bottle KB , and
- the antipodal action on S^2 .

The quotient is an $S^3/Q_{16} \times T^2$ -bundle over \mathbb{RP}^2 ; it can also be realized as a fiber bundle over KB whose fiber is a S^3/Q_{16} -bundle over \mathbb{RP}^2 . The quotient is oriented. Using the long

⁶⁶Another way Q_{16} arises is as the spin cover of the representation $V \oplus \text{Det}(V): D_8 \rightarrow \text{SO}(3)$, i.e. the pullback of this representation by the double cover $\text{SU}(2) \rightarrow \text{SO}(3)$.

exact sequence in homotopy groups associated to a fibration, we learn

$$\pi_1(W_2^7) \cong (Q_{16} \times \mathbb{Z}^2) \rtimes \mathbb{Z}/2\mathbb{Z}, \quad (14.96)$$

where $\mathbb{Z}/2\mathbb{Z}$ acts separately on Q_{16} and on \mathbb{Z}^2 ; therefore \mathbb{Z}^2 is a normal subgroup of $\pi_1(W_2^7)$.

Definition 14.97. We define a principal D_8 -bundle $P \rightarrow W_2^7$ by specifying a map from the fundamental group $\lambda: \pi_1(W_2^7) \rightarrow D_8$; then P is defined so that its monodromy around a loop γ in W_2^7 is $\lambda([- \gamma])$, which uniquely specifies a principal D_8 -bundle up to isomorphism.

First, let $q_1: \pi_1(W_2^7) \rightarrow Q_{16} \rtimes \mathbb{Z}/2\mathbb{Z}$ be the quotient by the normal $\mathbb{Z}^2 \subset \pi_1(W_2^7)$ noted above. Then let q_2 be the quotient by $-1 \in Q_{16}$; because ζ is multiplication by -1 in some coordinate directions, once we take this quotient, the semidirect product untwists, so q_2 is a homomorphism

$$Q_{16} \rtimes \mathbb{Z}/2\mathbb{Z} \rightarrow D_8 \times \mathbb{Z}/2\mathbb{Z}. \quad (14.98)$$

Finally, let $q_3: D_8 \times \mathbb{Z}/2\mathbb{Z} \rightarrow D_8$ be the identity on the first factor and the inclusion of a reflection subgroup on the second factor.⁶⁷ Then define

$$\lambda := q_3 \circ q_2 \circ q_1. \quad (14.99)$$

To show W_2^7 is the last generator, we must show it is Spin- D_{16} with the principal D_8 -bundle in Definition 14.97, and that the bordism class of (W_2^7, P) is linearly independent of that of W_1^7 . The first fact, which we prove in Proposition 14.101, is similar to our proofs for W_1^7 and W_1^9 in Propositions 14.74 and 14.83: stably split the tangent bundle and compute. Our approach showing that W_2^7 is linearly independent from W_1^7 , however, is different from previous approaches: we use a *Smith homomorphism* $S_{x^2}: \Omega_7^{\text{Spin-}D_{16}} \rightarrow \Omega_5^{\text{Spin-}Q_{16}}$ and show that the images of W_1^7 and W_2^7 are linearly independent. For background information on the Smith homomorphism, see Appendix E.1.

Because we constructed the principal D_8 -bundle $P \rightarrow W_2^7$ using Q_{16} , it will be helpful in the rest of this section to know $H^*(BQ_{16}; \mathbb{Z}/2\mathbb{Z})$, so that we can assign characteristic classes to principal Q_{16} -bundles.

Proposition 14.100 ([222, Lemma 4.1.1b] and [223, Theorem 4.40ii]). *There is an isomorphism $H^*(BQ_{16}; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[\hat{x}, \hat{y}, p]/(\hat{x}\hat{y} + \hat{y}^2, \hat{x}^3)$, where \hat{x} and \hat{y} are the pullbacks of x , respectively y , by the map $Q_{16} \rightarrow D_8$, so have degree 1; and $|p| = 4$.*

Proposition 14.101. *W_2^7 with its principal D_8 -bundle $P \rightarrow W_2^7$ from Definition 14.97 has a Spin- D_{16} structure.*

Proof. As usual, we begin by analyzing the $\mathbb{Z}/2\mathbb{Z}$ -action on the tangent bundle of $S^3/Q_{16} \times T_p^2 \times S^2$. The tangent bundle to T_p^2 is trivial, because T^2 admits a Lie group structure;

⁶⁷The choice of specific reflection subgroup does not matter here, even though it does elsewhere in this paper.

the tangent bundle to S^2 is trivial after appending on a single trivial summand, as we saw in (14.50). The tangent bundle of S^3/Q_{16} is actually trivializable, like for any compact, oriented 3-manifold, but for us it is more useful to have an explicit isomorphism to a different vector bundle.

Like in Example 14.54, consider the Q_{16} -action on $TS^3 \oplus \underline{\mathbb{R}} \cong \underline{\mathbb{C}}^2$. This action is the standard representation of Q_{16} on \mathbb{C}^2 , and the outward unit normal vector field is invariant, so we learn that $T(S^3/Q_{16}) \oplus \underline{\mathbb{R}}$ is a rank-two complex vector bundle $E \rightarrow S^3/Q_{16}$ induced from the principal Q_{16} -bundle $S^3 \rightarrow S^3/Q_{16}$.

Now the $\mathbb{Z}/2\mathbb{Z}$ -actions. From Example 14.51 we know that $\mathbb{Z}/2\mathbb{Z}$ acts on $TS^2 \oplus \underline{\mathbb{R}}$ by -1 . For T_p^2 , use the fact that the tangent space of the Klein bottle splits as $\underline{\mathbb{R}}$ plus a non-trivial line bundle to infer how $\mathbb{Z}/2\mathbb{Z}$ acts on $T(T_p^2)$. And for $T(S^3/Q_{16}) \oplus \underline{\mathbb{R}} \cong E$, $\mathbb{Z}/2\mathbb{Z}$ acts by ζ . Therefore

$$TW_2^7 \oplus \underline{\mathbb{R}}^2 \xrightarrow{\cong} E' \oplus \underline{\mathbb{R}} \oplus \sigma^{\oplus 4}, \quad (14.102)$$

where $E' \rightarrow W_2^7$ is a real, rank-4 vector bundle and $\sigma \rightarrow W_2^7$ is the pullback of the tautological bundle on \mathbb{RP}^2 along $W_2^7 \rightarrow \mathbb{RP}^2$. Applying the Whitney sum formula to (14.102), we learn that $w_1(TW_2^7) = w_1(E')$ and $w_2(TW_2^7) = w_2(E')$. The action of ζ on \mathbb{H} is orientation-preserving, so $w_1(E') = 0$, so W_2^7 is orientable. To compute w_2 , first observe that $E \rightarrow S^3/Q_{16} \times T_p^2 \times S^2$ is Spin, because the action of Q_{16} on \mathbb{H} is Spin; thus we just have to ask how $\mathbb{Z}/2\mathbb{Z}$ acts on E . In coordinates, we obtain two trivial actions and two reflection actions on the four real coordinates, and this can be used to show $w_2(E') = \hat{x}^2$. Thus $w_2(TW_2^7) = \hat{x}^2$.

Next we need to compute $w(P)$, which we do by tracing w through the pullbacks by q_3 , q_2 , and then q_1 . In Lemma 14.37 we proved that for any inclusion $\mathbb{Z}/2\mathbb{Z} \hookrightarrow D_8$ given by a reflection, w pulls back to 0, so $q_3^*(w) = w$; a priori one has to verify there is no term such as $x\hat{x}$ or $y\hat{x}$ coming from the cross term in the Künneth formula, but all such terms vanish in $q_3^*(w)$. Next, since Q_{16} is the Pin^- cover of D_8 with respect to the standard representation V , $w_2(V)$ and $w_1(V)^2$ are identified in $H^2(BQ_{16}; \mathbb{Z}/2\mathbb{Z})$. Therefore $q_2^*(w) = q_2^*(x^2) = \hat{x}^2$. Finally, q_1^* pulls back $\hat{x} \mapsto \hat{x}$, so $w(P) = \lambda^*(w) = \hat{x}^2$.

We have shown that W_2^7 is orientable and $w_2(TW_2^7) = w(P)$, so (W_2^7, P) is Spin- D_{16} . \square

Now we show that W_1^7 and W_2^7 are linearly independent in $\Omega_7^{\text{Spin-}D_{16}}(\text{pt}) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Since we already established that W_1^7 has order 4 in Proposition 14.93, this will suffice to show that W_1^7 and W_2^7 together generate $\Omega_7^{\text{Spin-}D_{16}}(\text{pt})$.

Let $\xi: H \rightarrow O$ be a tangential structure and $c \in H^k(BH; \mathbb{Z}/2\mathbb{Z})$, so we may interpret c as a characteristic class for manifolds with H -structure. Let M be an H -manifold and $N \subset M$ be a closed, $(n-k)$ -dimensional submanifold such that the image of the mod 2 fundamental class of N in $H_{n-k}(M; \mathbb{Z}/2\mathbb{Z})$ is Poincaré dual to $c(M) \in H^k(M; \mathbb{Z}/2\mathbb{Z})$. In this case we say that N is a *smooth representative of the Poincaré dual to $c(M)$* . In some settings, N acquires a canonical H' -structure, where $H' \rightarrow O$ is some other tangential structure. In this case, the class of N in $\Omega_{n-k}^{H'}$ depends only on the class of M in Ω_n^H , and this construction lifts to a homomorphism $S_c: \Omega_n^H \rightarrow \Omega_{n-k}^{H'}$. This is called a *Smith homomorphism*.

We will say more about Smith homomorphisms in Appendix E; what we need to know right now is that there is a Smith homomorphism $S_{x^2}: \Omega_n^{\text{Spin-}D_{16}} \rightarrow \Omega_{n-2}^{\text{Spin-}Q_{16}}$ defined by taking a smooth representative of the Poincaré dual of $x(P)^2$, where $P \rightarrow M$ is the principal D_8 -bundle associated to a Spin- D_{16} structure and $\text{Spin-}Q_{16} := \text{Spin} \times_{\mathbb{Z}/2} Q_{16}$. We prove this in Appendix E.3.

The following three propositions jointly imply W_1^7 and W_2^7 are linearly independent in $\Omega_7^{\text{Spin-}D_{16}}(\text{pt})$.

Proposition 14.103. *Let $\nu: \Omega_7^{\text{Spin-}D_{16}}$ be the bordism invariant*

$$\nu: \Omega_7^{\text{Spin-}D_{16}} \xrightarrow{S_{x^2}} \Omega_5^{\text{Spin-}Q_{16}} \xrightarrow{\int w^2 y} \mathbb{Z}/2. \quad (14.104)$$

Then $\nu(W_1^7) = 1$ and $\nu(W_2^7) = 0$.

Proposition 14.105. *$S_{x^2}(W_2^7) \neq 0$ in $\Omega_5^{\text{Spin-}Q_{16}}$.*

Proposition 14.106. *$\Omega_5^{\text{Spin-}Q_{16}} \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus k}$ for some k .*

We prove Proposition 14.106 in Appendix E.3 in the form of Proposition E.16; we prove Propositions 14.103 and 14.105 here. First, though, we compute $x(P)^2 \in H^2(W_2^7; \mathbb{Z}/2\mathbb{Z})$.

Lemma 14.107. *Let $\alpha \in H^1(W_2^7; \mathbb{Z}/2\mathbb{Z})$ be the element of $H^1(W_2^7; \mathbb{Z}/2\mathbb{Z})$ corresponding to the principal $\mathbb{Z}/2\mathbb{Z}$ -bundle $S^3/Q_{16} \times T_p^2 \times S^2 \rightarrow W_2^7$. Then $x(P)^2 = \hat{x}^2 + \hat{x}\alpha + \alpha^2$.*

A quick look at the Serre spectral sequence for the fiber bundle

$$S^3/Q_{16} \times T_p^2 \longrightarrow W_2^7 \longrightarrow \mathbb{RP}^2 \quad (14.108)$$

shows that the three monomials \hat{x}^2 , $\hat{x}\alpha$, and α^2 are linearly independent in $H^2(W_2^7; \mathbb{Z}/2\mathbb{Z})$.

Proof of Lemma 14.107. The characteristic class $x(P)$ is the pullback of the nonzero class in $H^1(B\mathbb{Z}/2\mathbb{Z}; \mathbb{Z}/2\mathbb{Z})$ by the composition $W_2^7 \rightarrow BD_8 \rightarrow B\mathbb{Z}/2\mathbb{Z}$ in which the first map is the classifying map for P and the second map is induced from the quotient by the subgroup of rotations. So we need to track how λ pulls back $x \in H^1(BD_8; \mathbb{Z}/2\mathbb{Z})$. Applying q_3^* , if α' is the generator of $H^1(B\mathbb{Z}/2\mathbb{Z}; \mathbb{Z}/2\mathbb{Z})$ for the $\mathbb{Z}/2\mathbb{Z}$ summand in $BD_8 \times B\mathbb{Z}/2\mathbb{Z}$, then $q_3^*(x) = x + \alpha'$, because x is nontrivial on reflections (as we discussed in Lemma 14.37). Then q_2^* sends $x \mapsto \hat{x}$ and q_1^* does not affect this class, and we learn $q_1^* q_2^*(\alpha') = \alpha$. Therefore $x(P) = \hat{x} + \alpha$ and $x(P)^2 = \hat{x}^2 + \hat{x}\alpha + \alpha^2$. \square

Now the propositions we promised.

Proof of Proposition 14.103. First we show $\nu(W_1^7) = 1$. For this, it suffices to show $S_{x^2}(W_1^7) = L_4^5$ with the duality bundle induced from $S^5 \rightarrow L_4^5$, because for this duality bundle $\int_{L_4^5} w^2 y =$

1. To see this, use that a smooth submanifold $i: N \hookrightarrow M$ is a smooth representative of the Poincaré dual of a class α if for any $\beta \in H^*(M; \mathbb{Z}/2\mathbb{Z})$,

$$\int_N i^*(\beta) = \int_M \alpha\beta. \quad (14.109)$$

This equation is only interesting if $|\beta| = \dim(N)$ and $|\alpha| = \dim(M) - \dim(N)$; otherwise both sides vanish. Let $N \subset W_1^7$ be the L_4^5 fiber for $M = W_1^7$ over some point in \mathbb{RP}^2 ; we want to verify (14.109) in this example. This is taken care of by Lemma 14.89: the only nonzero case is $\beta = w^2y$, and

$$\int_{L_4^5} w^2y = \int_{W_1^7} w^2x^2y = 1. \quad (14.110)$$

The second half of this proposition is showing that $\nu(W_2^7) = 0$. We first evaluate the Smith homomorphism on W_2^7 ; Lemma 14.107 tells us to compute the Poincaré dual of $\hat{x}^2 + \hat{x}\alpha + \alpha^2$. We may do this one monomial at a time and take the disjoint union.

- Because \hat{x}^2 comes from the fiber in (14.108), the Poincaré dual of \hat{x}^2 can also be taken to be a fiber bundle, specifically one of the form $S^1 \times T^2 \rightarrow Y_1 \rightarrow \mathbb{RP}^2$. One finds that the duality bundle is trivial when restricted to any fiber S^1 , and that these S^1 s carry the nonbounding Spin structure; therefore Y_1 bounds by filling in the fiberwise S^1 s with copies of D^2 .
- Since α^2 is a top-degree cohomology class for the base \mathbb{RP}^2 , the Poincaré dual of α^2 is the fiber $S^3/Q_{16} \times T_p^2$; the argument is essentially the same as the way we calculated $S_{x^2}(W_1^7)$ above. The duality bundle on $S^3/Q_{16} \times T_p^2$ pulls back across the quotient $S^3/Q_{16} \times T_p^2 \rightarrow S^3/Q_{16}$, so its characteristic classes also pull back from S^3/Q_{16} and therefore vanish in degrees above 3. Thus $\int_{S^3/Q_{16} \times T^2} w^2y = 0$.
- Finally, one representative of the Poincaré dual of $x\alpha$ is a manifold which factors as a fiber bundle $\Sigma \times T_p^2 \rightarrow Y_2 \rightarrow \mathbb{RP}^1$, where Σ is a closed, unorientable surface. The class w^2 on this manifold pulls back from a 3-manifold obtained by quotienting Y_2 by T_p^2 , so $w^2 = 0$ for Y_2 , and therefore $\int_{Y_2} w^2y = 0$ too.

Therefore $\nu(W_2^7) = 0$. □

Proof of Proposition 14.105. In the proof of the previous proposition, we found a representative of $S_{x^2}(W_2^7)$ that is a disjoint union of three 5-manifolds $S^3/Q_{16} \times T_p^2$, Y_1 , and Y_2 , and we saw that Y_1 bounds as a Spin- Q_{16} manifold. In Proposition E.13, we prove that $[S^3/Q_{16} \times T_p^2] \neq 0$ in $\Omega_5^{\text{Spin-}Q_{16}}$; here is where it is crucial that we gave T_p^2 the nonbounding Spin structure when we defined W_2^7 , so that we obtain $S^3/Q_{16} \times T_p^2$ after the Smith homomorphism. Finally, Y_2 is not bordant to $S^3/Q_{16} \times T_p^2$, so we can conclude that $S_{x^2}(W_2^7) \neq 0$. □

This finishes the determination of the generators of $\text{Spin-GL}^+(2, \mathbb{Z})$ bordism in dimensions 11 and below.

14.4 Multiplicative structure

We saw in Section 13.4 that $\text{Spin-}\mathbb{Z}/8\mathbb{Z}$ bordism is a ring. $\text{Spin-}D_{16}$ bordism is not a ring, ultimately because there is no non-trivial way to tensor together two D_8 -bundles into a single D_8 -bundle. $\text{Spin-}D_{16}$ bordism is still an Ω_*^{Spin} -module, just as for any kind of twisted Spin bordism, and most of this information can be read off of the E_∞ -page of the Adams spectral sequence.

One surprise is that even though $\text{Spin-}\mathbb{Z}/8\mathbb{Z}$ bordism is a ring, and we have a map from $\text{Spin-}\mathbb{Z}/8\mathbb{Z}$ bordism to $\text{Spin-}D_{16}$ bordism, $\text{Spin-}D_{16}$ bordism is not a module over $\text{Spin-}\mathbb{Z}/8\mathbb{Z}$ bordism! This is because in $\Omega_*^{\text{Spin-}\mathbb{Z}/8\mathbb{Z}}(\text{pt})$, the class of S_p^1 is divisible by 4. However, we saw that the generator x of $\Omega_2^{\text{Spin-}D_{16}}(\text{pt}) \cong \mathbb{Z}/2\mathbb{Z}$ is S_p^1 times another class, but x is not divisible by 4. Therefore there is not a natural $\text{Spin-}D_{16}$ structure on the product of a $\text{Spin-}\mathbb{Z}/8\mathbb{Z}$ manifold and a $\text{Spin-}D_{16}$ manifold. This is similar to (and related to) the fact that the inclusion $U(1) \rightarrow O(2)$ defines a map from Spin^c bordism to $\text{Spin-O}(2)$ bordism, and Spin^c bordism is a ring, but $\text{Spin-O}(2)$ bordism is not a module over this ring (in this case, because taking the product with S_p^1 is not always zero, which cannot happen in any $\Omega_*^{\text{Spin}^c}$ -module).

This concludes our computations. Additional details are presented in the Appendices.

A Collection of bordism groups

In this Appendix we collect some of the bordism groups calculated in this paper up to dimension eleven. We also list a set of generators.

k	$\Omega_k^{\text{Spin}}(\text{pt})$	Generators
0	\mathbb{Z}	pt_+
1	$\mathbb{Z}/2\mathbb{Z}$	S_p^1
2	$\mathbb{Z}/2\mathbb{Z}$	$S_p^1 \times S_p^1$
3	0	
4	\mathbb{Z}	K3
5	0	
6	0	
7	0	
8	$\mathbb{Z} \oplus \mathbb{Z}$	(B, \mathbb{HP}^2)
9	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$	$(B \times S_p^1, \mathbb{HP}^2 \times S_p^1)$
10	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$	$(B \times S_p^1 \times S_p^1, \mathbb{HP}^2 \times S_p^1 \times S_p^1, X_{10})$
11	0	

Table 11: Bordism groups for Spin manifolds in dimension smaller than twelve and a full set of generators.

We list some of the generators and explain the notation.

- S_p^1 is a circle with Spin structure induced from the Lie group framing, i.e., periodic boundary condition for fermions.
- K3 denotes a K3 surface and E denotes an Enriques surface.
- B describes a Bott manifold and \mathbb{HP}^2 quaternionic projective space. The name “Bott manifold” does not fix B up to diffeomorphism; we never need to pick a specific Bott manifold, but explicit examples are constructed in [224, Section 7] and [40, Section 5.3].
- X_{10} is a Milnor hypersurface.
- L_k^{2n-1} are lens spaces, see also Appendix C.1. The discrete $\mathbb{Z}/k\mathbb{Z}$ -bundle is induced by the fibration $S^{2n-1} \rightarrow L_k^{2n-1}$. The notation \tilde{L}_k^{2n-1} denotes the choice of a distinct Spin

k	$\Omega_k^{\text{Spin}}(B\mathbb{Z}/3\mathbb{Z})$	Generators
0	\mathbb{Z}	pt_+
1	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})$	(S_p^1, L_3^1)
2	$\mathbb{Z}/2\mathbb{Z}$	$S_p^1 \times S_p^1$
3	$\mathbb{Z}/3\mathbb{Z}$	L_3^3
4	\mathbb{Z}	K3
5	$\mathbb{Z}/9\mathbb{Z}$	L_3^5
6	0	
7	$\mathbb{Z}/9\mathbb{Z}$	L_3^7
8	$\mathbb{Z} \oplus \mathbb{Z}$	(B, \mathbb{HP}^2)
9	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/27\mathbb{Z})$	$(B \times S_p^1, \mathbb{HP}^2 \times S_p^1, \mathbb{HP}^2 \times L_3^1, L_3^9)$
10	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$	$(B \times S_p^1 \times S_p^1, \mathbb{HP}^2 \times S_p^1 \times S_p^1, X_{10})$
11	$(\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/27\mathbb{Z})$	$(\mathbb{HP}^2 \times L_3^3, L_3^{11})$

Table 12: Bordism groups for Spin manifolds with $\mathbb{Z}/3\mathbb{Z}$ bundle in dimension smaller than twelve and a full set of generators. This contains the Spin bordism groups above.

or Spin- $\mathbb{Z}/8\mathbb{Z}$ structure, whose realization can be extracted from the lift of the $\mathbb{Z}/k\mathbb{Z}$ action on fermions given in (C.3).

- The spaces Q_k^{2n-1} are lens space bundles of L_k^{2n-3} over \mathbb{CP}^1 and $\mathbb{Z}/k\mathbb{Z}$ bundle on the lens space fiber, see also Appendix C.2.
- S_R^1 is a circle with a duality bundle involving a reflection.
- \mathbb{RP}^{3+4m} and $\widetilde{\mathbb{RP}}^{3+4m}$ are real projective spaces with $\mathbb{Z}/2\mathbb{Z}$ duality bundle associated to different embeddings of the reflection into D_8 (or its lift to D_{16}).
- W_1^7 is a $\mathbb{Z}/2\mathbb{Z}$ quotient of Q_4^7 .
- W_2^7 is a $\mathbb{Z}/2\mathbb{Z}$ quotient, involving reflections, of $S^3/Q_{16} \times S^2 \times T^2$, where the quotient acts via reflections in the S^3/Q_{16} factor, as antipodal mapping on the S^2 , and as the fixed-point-free isometry that would yield the Klein Bottle when acting on T^2 .
- $W_{1,8}$ is a half Bott manifold that can be constructed as a $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$ quotient of a Calabi-Yau 4-fold.
- W_1^9 is a $\mathbb{Z}/2\mathbb{Z}$ quotient of $L_4^5 \times S^4$.

k	$\Omega_k^{\text{Spin}}(B\mathbb{Z}/4\mathbb{Z})$	Generators
0	\mathbb{Z}	pt_+
1	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z})$	(S_p^1, L_4^1)
2	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$	$(S_p^1 \times S_p^1, S_p^1 \times L_4^1)$
3	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z})$	$(S_p^1 \times S_p^1 \times L_4^1, L_4^3)$
4	\mathbb{Z}	K3
5	$\mathbb{Z}/4\mathbb{Z}$	Q_4^5
6	0	
7	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/32\mathbb{Z})$	(\tilde{L}_4^7, L_4^7)
8	$\mathbb{Z} \oplus \mathbb{Z}$	(B, \mathbb{HP}^2)
9	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 3} \oplus (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z})$	$(B \times S_p^1, \mathbb{HP}^2 \times S_p^1, B \times L_4^1, \mathbb{HP}^2 \times L_4^1, Q_4^9)$
10	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 5}$	$(B \times S_p^1 \times S_p^1, \mathbb{HP}^2 \times S_p^1 \times S_p^1, X_{10},$ $(B \times S_p^1 \times L_4^1, \mathbb{HP}^2 \times S_p^1 \times L_4^1)$
11	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ $\oplus (\mathbb{Z}/8\mathbb{Z})^{\oplus 2} \oplus (\mathbb{Z}/128\mathbb{Z})$	$(X_{10} \times L_4^1, \mathbb{HP}^2 \times S_p^1 \times S_p^1 \times L_4^1,$ $\mathbb{HP}^2 \times L_4^3, \tilde{L}_4^{11}, L_4^{11})$

Table 13: Bordism groups for Spin manifolds with $\mathbb{Z}/4\mathbb{Z}$ bundle in dimension smaller than twelve and a full set of generators. This contains the Spin bordism groups above. In dimension 9, to get a set of generators which produce the given direct-sum decomposition, as opposed to merely a generating set, replace $B \times L_4^1$ with $(B \times L_4^1) \# (-2Q_4^9)$.

- X_5 , X_9 , and X_{11} and their tilded versions are fibrations of real projective spaces over real projective spaces with appropriate duality bundle distinguished by the embedding of D_8 into D_{16} , that we refer to as Arcana.

k	$\Omega_k^{\text{Spin-}(\mathbb{Z}/8\mathbb{Z})}(\text{pt})$	Generators
0	\mathbb{Z}	pt_+
1	$\mathbb{Z}/8\mathbb{Z}$	L_4^1
2	0	
3	$\mathbb{Z}/2\mathbb{Z}$	L_4^3
4	\mathbb{Z}	E
5	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/32\mathbb{Z})$	(\tilde{L}_4^5, L_4^5)
6	0	
7	$\mathbb{Z}/4\mathbb{Z}$	Q_4^7
8	$\mathbb{Z} \oplus \mathbb{Z}$	(B, \mathbb{HP}^2)
9	$(\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/128\mathbb{Z})$	$(\tilde{L}_4^9, \mathbb{HP}^2 \times L_4^1, L_4^9)$
10	$\mathbb{Z}/2\mathbb{Z}$	X_{10}
11	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z})$	$(\mathbb{HP}^2 \times L_4^3, X_{10} \times L_4^1, Q_4^{11})$

Table 14: Bordism groups for Spin- $\mathbb{Z}/8\mathbb{Z}$ manifolds with dimension lower than twelve and a full set of generators.

B Duality groups

In this Appendix we summarize the various duality groups and their individual group structure, see also [33].

The first group we discuss is $\text{SL}(2, \mathbb{Z})$ which can be understood as the bosonic duality group of type IIB. It is generated by two elements

$$U = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (\text{B.1})$$

of finite order

$$S^4 = \text{id}, \quad U^6 = \text{id}. \quad (\text{B.2})$$

It can be defined as

$$\text{SL}(2, \mathbb{Z}) = \langle U, S | S^4 = \text{id}, \quad S^2 = U^3 \rangle = (\mathbb{Z}/6\mathbb{Z}) *_{(\mathbb{Z}/2\mathbb{Z})} (\mathbb{Z}/4\mathbb{Z}), \quad (\text{B.3})$$

which makes the structure as an amalgamated free product evident, see also [30–32]. The

k	$\Omega_k^{\text{Spin-}D_{16}}(\text{pt})$	Generators
0	\mathbb{Z}	pt_+
1	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$	(L_4^1, S_R^1)
2	$\mathbb{Z}/2\mathbb{Z}$	$S_p^1 \times S_R^1$
3	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$	$(L_4^3, \mathbb{RP}^3, \widetilde{\mathbb{RP}}^3)$
4	\mathbb{Z}	E
5	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$	(L_4^5, X_5)
6	0	
7	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 3} \oplus (\mathbb{Z}/4\mathbb{Z})$	$(\mathbb{RP}^7, \widetilde{\mathbb{RP}}^7, W_2^7, W_1^7)$
8	$\mathbb{Z} \oplus \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$	$(W_{1,8}, \mathbb{HP}^2, W_1^7 \times S_p^1)$
9	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 8}$	$(W_1^7 \times S_p^1 \times S_p^1, X_9, \tilde{X}_9, L_4^9, W_1^9, B \times L_4^1, \mathbb{HP}^2 \times L_4^1, \mathbb{HP}^2 \times S_R^1)$
10	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 4}$	$(B \times L_4^1 \times S_p^1, W_1^9 \times S_p^1, \mathbb{HP}^2 \times L_4^1 \times S_p^1, X_{10})$
11	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 9} \oplus (\mathbb{Z}/8\mathbb{Z})$	$(\mathbb{RP}^{11}, \widetilde{\mathbb{RP}}^{11}, X_{11}, \tilde{X}_{11}, \mathbb{HP}^2 \times L_4^3, \mathbb{HP}^2 \times \mathbb{RP}^3, \mathbb{HP}^2 \times \widetilde{\mathbb{RP}}^3, X_{10} \times L_4^1, X_{10} \times S_R^1, Q_4^{11})$

Table 15: Bordism groups for Spin- D_{16} manifolds with dimension lower than twelve and a full set of generators.

usual generator T of infinite order is obtained as follows

$$T = S^{-1}U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (\text{B.4})$$

The Abelianization of $\text{SL}(2, \mathbb{Z})$ is given by

$$\text{Ab}(\text{SL}(2, \mathbb{Z})) = \mathbb{Z}/12\mathbb{Z}, \quad (\text{B.5})$$

which is generated by the image of T in $\text{Ab}(\text{SL}(2, \mathbb{Z}))$.

After we include fermions we need to lift the action of the duality group to the fermionic degrees of freedom. This leads to the second group $\text{Mp}(2, \mathbb{Z})$ the metaplectic group, see [28, 32]. Defining the generators as \hat{U} (order 12) and \hat{S} (order 8) it is given by

$$\text{Mp}(2, \mathbb{Z}) = \langle \hat{U}, \hat{S} | \hat{S}^8 = \text{id}, \hat{S}^2 = \hat{U}^3 \rangle = (\mathbb{Z}/12\mathbb{Z}) *_{(\mathbb{Z}/4\mathbb{Z})} (\mathbb{Z}/8\mathbb{Z}). \quad (\text{B.6})$$

The Spin lift is encoded in

$$\hat{S}^4 = \hat{U}^6 = (-1)^F. \quad (\text{B.7})$$

Its Abelianization is given by

$$\text{Ab}(\text{Mp}(2, \mathbb{Z})) = \mathbb{Z}/24\mathbb{Z}, \quad (\text{B.8})$$

generated by the image of $\hat{T} = \hat{S}^{-1}\hat{U}$.

Finally, we also include the reflection operator

$$R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (\text{B.9})$$

to define the group

$$\text{GL}(2, \mathbb{Z}) = \langle U, S, R \mid S^4 = \text{id}, RSR^{-1} = S^{-1}, RUR^{-1} = U^{-1}, S^2 = U^3 \rangle = D_{12} *_{D_4} D_8, \quad (\text{B.10})$$

where D_{2n} denotes the dihedral (not binary dihedral) group of $2n$ elements. For example, one has

$$D_8 = \langle S, R \mid S^4 = \text{id}, R^2 = \text{id}, RSR^{-1} = S^{-1} \rangle. \quad (\text{B.11})$$

The Pin^+ cover of $\text{GL}(2, \mathbb{Z})$ is defined as

$$\begin{aligned} \text{GL}^+(2, \mathbb{Z}) &= \langle \hat{U}, \hat{S}, \hat{R} \mid \hat{S}^8 = \text{id}, \hat{R}^2 = \text{id}, \hat{R}\hat{S}\hat{R}^{-1} = \hat{S}^{-1}, \hat{R}\hat{U}\hat{R}^{-1} = \hat{U}^{-1}, \hat{S}^2 = \hat{U}^3 \rangle \\ &= D_{16} *_{D_8} D_{24}, \end{aligned} \quad (\text{B.12})$$

where \hat{R} denotes the Pin^+ lift of R . Note that the Pin^+ cover precisely means that the lift of the reflections \hat{R} squares to id and not $(-1)^F$ as would be the case for the Pin^- cover. In both cases the Abelianization is given by

$$\text{Ab}(\text{GL}(2, \mathbb{Z})) = \text{Ab}(\text{GL}^+(2, \mathbb{Z})) = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}), \quad (\text{B.13})$$

generated by the images of R, S and \hat{R}, \hat{S} , respectively.

B.1 Fermion parity and worldsheet orientation reversal

One can also consider the action of worldsheet orientation reversal Ω given by

$$\Omega = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{B.14})$$

This can be deduced from the action on the 2-form fields of type IIB that transform as a $\text{GL}(2, \mathbb{Z})$ doublet $(C_2, B_2)^T$. Similarly, one finds the representation of the left-handed fermion parity $(-1)^{F_L}$ in terms of a $\text{GL}(2, \mathbb{Z})$ element

$$(-1)^{F_L} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (\text{B.15})$$

With this one can see that

$$S \Omega S^{-1} = (-1)^{F_L}, \quad (\text{B.16})$$

implying that Ω and $(-1)^{F_L}$ are S-dual. The relation with the reflection R is given by

$$S R = (-1)^{F_L}, \quad R S = \Omega, \quad (\text{B.17})$$

from which we can define the associated Spin-lifts to elements in $\text{GL}^+(2, \mathbb{Z})$

$$\hat{S} \hat{R} = (-1)^{\hat{F}_L}, \quad \hat{R} \hat{S} = \hat{\Omega}, \quad (\text{B.18})$$

which both square to the identity because of (B.12).

Both these elements leave the dilaton field invariant but produce a sign flip for the RR 0-form so that on the axio-dilaton, we have the transformation rule

$$\Omega \text{ or } (-1)^{F_L} : \quad \tau \rightarrow -\bar{\tau}. \quad (\text{B.19})$$

The fact that this involves complex conjugation generates obstructions for the definition of local Calabi-Yau manifolds involving reflections.

Note also that while the chiral 4-form C_4 of type IIB string theory is invariant under $\text{SL}(2, \mathbb{Z})$ transformations, it changes sign under the reflection in $\text{GL}(2, \mathbb{Z})$. Moreover, as mentioned above the two 2-form field of type IIB transform as doublet under the $\text{GL}(2, \mathbb{Z})$ symmetry and therefore (p, q) -strings and their magnetic dual 5-branes will be affected by non-trivial backgrounds.

B.2 Embeddings of dihedral groups

Some of the generators for $\text{Spin-GL}^+(2, \mathbb{Z})$ manifolds originate from $\text{Spin-}D_4$ or $\text{Spin-}D_8$ manifolds via embedding of these groups in $\text{GL}^+(2, \mathbb{Z})$. There is more than one possible embedding of D_4 and D_8 into D_{16} ; each will lead to different $\text{Spin-}D_{16}$ structures and consequently to different $\text{Spin-GL}^+(2, \mathbb{Z})$ manifolds. This results in different bordism classes realized by the same underlying manifold with different bundles on top. In this Appendix we describe these embeddings and how to characterize them.

Let us first focus on the embedding of $D_4 \cong (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$ into D_{16} . The latter can be realized as the symmetry group of the octagon. When embedding $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$ here, one $\mathbb{Z}/2\mathbb{Z}$ will always be associated with a 180° rotation, which we will call s_4 , while the other $\mathbb{Z}/2\mathbb{Z}$ describes reflections and is denoted by r_4 . The first embedding i_4 is given by

$$i_4: \quad s_4 \mapsto \hat{S}^4, \quad r_4 \mapsto \hat{\Omega}. \quad (\text{B.20})$$

One can easily show that this has the correct group structure by applying the defining identities in (B.12). Geometrically, this is embedding the group as that generated by reflection along one of the octagon sides and rotation by 180° . The second embedding modifies the reflections as follows

$$\tilde{i}_4: \quad s_4 \mapsto \hat{S}^4, \quad r_4 \mapsto \hat{S} \hat{\Omega}, \quad (\text{B.21})$$

with the correct group structure

$$r_4^2 = \hat{S} \hat{\Omega} \hat{S} \hat{\Omega} = \hat{S} \hat{S}^{-1} = \text{id}, \quad s_4 r_4 = \hat{S}^5 \hat{\Omega} = \hat{S} \hat{\Omega} \hat{\Omega} \hat{S}^4 \hat{\Omega} = \hat{S} \hat{\Omega} \hat{S}^{-4} = \hat{S} \hat{\Omega} \hat{S}^4 = r_4 s_4, \quad (\text{B.22})$$

where we use that

$$\hat{\Omega} \hat{S} \hat{\Omega} = \hat{R} \hat{S}^2 \hat{R} \hat{S} = \hat{S}^{-1}, \quad (\text{B.23})$$

and therefore can replace \hat{R} in the definition of the dihedral group. This second embedding is similar, but sends r_4 to a reflection along an octagonal diagonal, rather than a side.

In similar fashion, we find two different embeddings of D_8 into D_{16} , given by

$$i_8: \quad s_8 \mapsto \hat{S}^2, \quad r_8 \mapsto \hat{\Omega}, \quad (\text{B.24})$$

and

$$\tilde{i}_8: \quad s_8 \mapsto \hat{S}^2, \quad r_8 \mapsto \hat{S} \hat{\Omega}, \quad (\text{B.25})$$

with reflection r_8 and rotation s_8 , respectively. For \tilde{i}_8 we verify the defining property

$$\tilde{r}_8 \tilde{s}_8 \tilde{r}_8 = \hat{S} \hat{\Omega} \hat{S}^3 \hat{\Omega} = \hat{S}^{-2} = \tilde{s}_8^{-1}. \quad (\text{B.26})$$

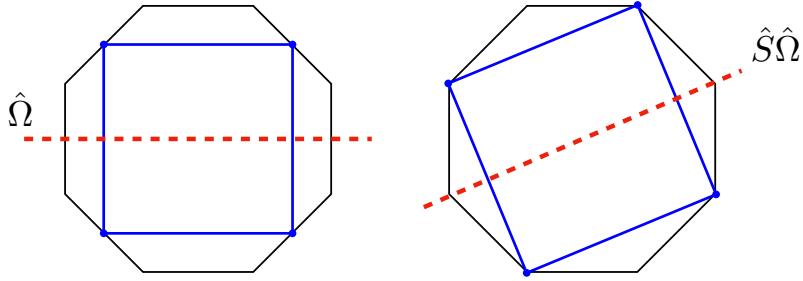


Figure 29: Two embeddings of D_8 into D_{16} with the action of $\hat{\Omega}$ and $\hat{S}\hat{\Omega}$ indicated.

Of course we can also define intermediate maps of the form

$$\begin{aligned} s_4 &\mapsto s_8^2 \mapsto \hat{S}^4, \quad r_4 \mapsto r_8 \mapsto \hat{\Omega}, \\ s_4 &\mapsto s_8^2 \mapsto \hat{S}^4, \quad r_4 \mapsto r_8 \mapsto \hat{S}\hat{\Omega}, \end{aligned} \tag{B.27}$$

to define embeddings of D_4 into D_8 .

Interpreting D_{2n} as the symmetries of an n -gon we can picture the different embeddings of the reflection as a reflection with respect to a line through two sides or two corners. This is depicted in Figure 10, which we reproduce here (see Figure 29). From this we also see that all other embeddings that map the reflection to an element $\hat{S}^k\hat{\Omega}$ can be derived from the embeddings above by conjugation with a rotation, i.e., shifting the axis of reflection by a rotation defined by a multiple of \hat{S} . As we have seen this is the case for $(-1)^{\hat{F}_L}$ as S-dual of $\hat{\Omega}$.

C η -invariants

In this Appendix we summarize the calculation of η -invariants on lens spaces and lens space bundles. These are often useful in order to guarantee that certain manifolds generate the non-trivial bordism classes we are looking for and can resolve certain extension questions in spectral sequence computation. They can also be used in order to demonstrate whether different lens spaces, i.e., lens spaces derived from the quotient of a different group action, are bordant. For that we need the η -invariants to be bordism invariants, which is the case for dimension $(4n + 1)$ but not in dimension $(4n - 1)$, since the $(4n)$ -dimensional index density has pure gravitational contributions. For that reason we use the difference of two η -invariants with different charges in dimension $(4n - 3)$ as bordism invariants for which the pure gravitational contribution to the index density vanishes, see also the discussion in Section C.3 below.

C.1 η -invariants for lens spaces

We start with the discussion of η -invariants for lens spaces. We will closely follow the description in [32] with generalizations to include the η -invariants for Rarita-Schwinger fields of spin $\frac{3}{2}$ as well as general $\mathbb{Z}/k\mathbb{Z}$ actions. In most of the discussion we are interested in the fractional part of the η -invariant only and the results are understood mod \mathbb{Z} .

In general we define the lens spaces as

$$L_k^{2n-1}(j_1, \dots, j_n) = S^{2n-1}/(\mathbb{Z}/k\mathbb{Z}), \quad (\text{C.1})$$

where j_i are integers co-prime to k ; these manifolds can be understood as the asymptotic boundary of $\mathbb{C}^n/(\mathbb{Z}/k\mathbb{Z})$. The $\mathbb{Z}/k\mathbb{Z}$ acts on the complex coordinates z_i of \mathbb{C}^n as

$$z_i \mapsto \exp(2\pi i j_i/k) z_i. \quad (\text{C.2})$$

This can be summarized in an action on the tangent bundle of $\mathbb{R}^{2n} \cong \mathbb{C}^n$ by a $(2n \times 2n)$ -matrix τ with

$$\det(\text{id} - \tau(\ell)) = |1 - e^{2\pi i \ell j_1/k}|^2 \dots |1 - e^{2\pi i \ell j_n/k}|^2 \quad (\text{C.3})$$

Moreover, this naturally defines a $\mathbb{Z}/k\mathbb{Z}$ bundle given by the fibration structure

$$\begin{array}{ccc} \mathbb{Z}/k\mathbb{Z} & \hookrightarrow & S^{2n-1} \\ & \downarrow & \\ & & L_k^{2n-1}(j_1, \dots, j_n) \end{array} \quad (\text{C.4})$$

which in many of our generators can be interpreted as a non-trivial duality bundle. Dirac fermions ψ_q of charge q under this $\mathbb{Z}/k\mathbb{Z}$ bundle transform as follows

$$\mathbb{Z}/k\mathbb{Z} : \quad \psi_q \mapsto e^{-2\pi i q/k} \exp(-\frac{\pi}{k}(j_1 \Gamma^1 \Gamma^2 + \dots + j_n \Gamma^{2n-1} \Gamma^{2n})) \psi_q. \quad (\text{C.5})$$

defining a Spin lift of the group action. Here, Γ^ℓ are γ -matrices on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and we further define

$$\bar{\Gamma} = i^{-n} \Gamma^1 \dots \Gamma^{2n}. \quad (\text{C.6})$$

For this action to be well-defined one must have

$$e^{-2\pi i q} \exp(-\pi(j_1 \Gamma^1 \Gamma^2 + \dots + j_n \Gamma^{2n-1} \Gamma^{2n})) = 1, \quad (\text{C.7})$$

which demands q to potentially be half-integer. This can also be summarized in a matrix $\rho(\ell)$ acting on the Spin bundle.

Spin structures on lens spaces

Before we describe the calculation of various η -invariants for lens spaces in detail we want to describe the different Spin structures on them. For that we look at the lens spaces of the form

$$L_4^{4n-1} := L_4^{4n-1}(1, \dots, 1). \quad (\text{C.8})$$

These are Spin, but allow for two different Spin structures. These can be identified by considering the lift of the rotational action as discussed in (C.5) to spinor fields. Switching off the duality bundle for now (i.e., setting $q = 0$), we see that there is more than one lift that defines a well-defined $\mathbb{Z}/4\mathbb{Z}$ action

$$\psi \mapsto \alpha \exp\left(-\frac{\pi}{4}(\Gamma^1\Gamma^2 + \dots + \Gamma^{4n-1}\Gamma^{4n})\right)\psi, \quad (\text{C.9})$$

where α takes into account an ambiguity that was absorbed in the duality piece $e^{-2\pi iq/k}$, in (C.5). In order to define a $\mathbb{Z}/4\mathbb{Z}$ action α has to satisfy

$$\alpha^4 = 1 \rightarrow \alpha = e^{-2\pi i \kappa/4}. \quad (\text{C.10})$$

The choice of κ defines the different possible lifts of the rotational action to the spinor bundle, i.e., the different Spin structures. Reinstating the duality bundle we further see that this choice can be incorporated in the shift of the charge of the associated fermion. On a manifold with Spin structure specified by κ a fermion of charge q thus transforms as if it had charge $\kappa + q$ on the manifold specified by Spin structure $\kappa = 0$. Since the Spin structures are classified by $H^1(L_4^{4n-1}; \mathbb{Z}/2\mathbb{Z})$ one further has to restrict to α 's that are ± 1 , i.e., only the two values $\kappa = \{0, 2\}$ correspond to distinct Spin structures.

Very similar statements hold for lens space bundles as well as lens spaces that do not allow for a Spin structure but a Spin- $\mathbb{Z}/8\mathbb{Z}$ structure, as for example L_4^5 . In all of these cases the various structures can be probed by varying the effective charge of the spinors, which in the case of twisted Spin structures might be subject to consistency constraints.

Dirac fields

Let us start the discussion with the usual Dirac fields with spin $\frac{1}{2}$, whose η -invariants we denote as η^D . The equivariant index theorem [197, 32] allows to deduce the fractional part of the η invariants in terms of transformation property of the fermions at the singular point in $\mathbb{C}^n/(\mathbb{Z}/k\mathbb{Z})$

$$\eta_q^D(L_k^{2n-1}) = -\frac{1}{k} \left(\sum_{\ell=1}^{k-1} \frac{\text{tr}(\bar{\Gamma}\rho(\ell))}{\det(\text{id} - \tau(\ell))} \right), \quad (\text{C.11})$$

where the matrices τ and ρ are the ones defined above. For the action (C.5) one has

$$\text{tr}(\bar{\Gamma}\rho(\ell)) = e^{-2\pi i \ell q/k} (e^{-\pi i \ell j_1/k} - e^{\pi i \ell j_1/k}) \dots (e^{-\pi i \ell j_n/k} - e^{\pi i \ell j_n/k}). \quad (\text{C.12})$$

Together with (C.3) this implies

$$\eta_q^D(L_k^{2n-1}) = -\frac{1}{(2i)^n k} \sum_{\ell=1}^{k-1} \frac{e^{-2\pi i \ell q/k}}{\sin(\pi \ell j_1/k) \dots \sin(\pi \ell j_n/k)}, \quad (\text{C.13})$$

which coincides with the formula in [32] for all j_i equal.

Rarita-Schwinger fields

We present two ways to determine η_q^{RS} , the η -invariant for the Rarita-Schwinger operators of $\mathbb{Z}/k\mathbb{Z}$ charge q . One can deduce it from the fact that a Rarita-Schwinger field Ψ_μ transforms in the tensor product of the Spin and the tangent bundle. This modifies the matrix $\rho_{\text{RS}}(\ell)$ to also incorporate the rotation of the vector index, giving

$$\begin{aligned} \text{tr}(\bar{\Gamma}\rho_{\text{RS}}(\ell)) &= e^{-2\pi i \ell q/k} (e^{-\pi i \ell j_1/k} - e^{\pi i \ell j_1/k}) \dots (e^{-\pi i \ell j_n/k} - e^{\pi i \ell j_n/k}) \\ &\times (2 \cos(2\pi \ell j_1/k) + \dots + 2 \cos(2\pi \ell j_n/k) - 1). \end{aligned} \quad (\text{C.14})$$

The minus one in the last factor roughly accounts for the fact that the tangent space of L_k^{2n-1} is one dimension smaller than that of \mathbb{C}^n , see also (C.16) below. More formally, calculations using the equivariant index theorem such as these essentially compute the η invariant by finding a one-higher dimensional manifold with boundary in which the index can be computed via the APS index theorem. A Rarita-Schwinger operator in $(d+1)$ dimensions restricts to a d -dimensional Rarita-Schwinger + a Dirac contribution. The extra -1 in (C.14) is precisely removing the additional Dirac contribution. Putting everything together, the η -invariant for the Rarita-Schwinger operator is then given by

$$\begin{aligned} \eta_q^{\text{RS}}(L_k^{2n-1}) &= -\frac{1}{(2i)^n k} \sum_{\ell=1}^{k-1} \left(\frac{e^{-2\pi i \ell q/k}}{\sin(\pi \ell j_1/k) \dots \sin(\pi \ell j_n/k)} \right. \\ &\quad \left. \times (2 \cos(2\pi \ell j_1/k) + \dots + 2 \cos(2\pi \ell j_n/k) - 1) \right). \end{aligned} \quad (\text{C.15})$$

Alternatively, one can derive the RS η -invariant by noting that the complexification of the tangent bundle of the lens space decomposes as

$$(TL_k^{2n-1} \otimes \mathbb{C}) \oplus \mathbb{C} \simeq \bigoplus_{i=1}^n (\mathcal{L}^{j_i} \oplus \mathcal{L}^{-j_i}), \quad (\text{C.16})$$

where \mathcal{L} is the complex line bundle associated to a $\mathbb{Z}/k\mathbb{Z}$ bundle of the form defined in (C.4) with all j_i set to 1. Tensoring with \mathcal{L}^q , see also [33], we have

$$(TL_k^{2n-1} \otimes \mathcal{L}^q) \oplus \mathcal{L}^q \simeq \bigoplus_{i=1}^n (\mathcal{L}^{q+j_i} \oplus \mathcal{L}^{q-j_i}), \quad (\text{C.17})$$

which implies

$$\eta_q^{\text{RS}}(L_k^{2n-1}) = \sum_{i=1}^n \left(\eta_{q+j_i}^D(L_k^{2n-1}) + \eta_{q-j_i}^D(L_k^{2n-1}) \right) - \eta_q^D(L_k^{2n-1}). \quad (\text{C.18})$$

Plugging in (C.13) we recover the result in (C.15). Setting all j_i to 1 we once more find a perfect agreement with the formulae in [33].

Signature operator

For completeness we also state the formula for the η -invariant associated to the signature, which can be derived from the bi-spinor field, as discussed in [32].

The ρ matrix for the signature with charge q is again modified according to its action on bi-spinor fields and one has

$$\text{tr}(\bar{\Gamma}\rho_{\text{sig}}(\ell)) = e^{-2\pi i \ell q/k} \prod_{i=1}^n (e^{-\pi i \ell j_1/k} - e^{\pi i \ell j_1/k})(e^{-\pi i \ell j_1/k} + e^{\pi i \ell j_1/k}), \quad (\text{C.19})$$

with which one finds

$$\eta_q^{\text{sig}}(L_k^{2n-1}) = -\frac{1}{(i)^n k} \sum_{\ell=1}^{k-1} \frac{e^{2\pi i \ell q/k}}{\tan(\pi \ell j_1/k) \dots \tan(\pi \ell j_n/k)}, \quad (\text{C.20})$$

in agreement with results for $j_i = 1$.

C.2 η -invariants for lens space bundles

We can also describe the fractional part of η -invariants for lens space bundles described as manifolds Q_k^{2n-1} in the main text. Let us briefly recall the construction of these lens space bundles.

Starting with the base manifold $B = \mathbb{CP}^1$ we define the direct sum of $(n-1)$ complex line bundles, which can be described as tensor product of the hyperplane bundle $H = \mathcal{O}(1)$

$$\mathcal{L}_1 \oplus \dots \oplus \mathcal{L}_{n-1} = H^{m_1} \oplus \dots H^{m_{n-1}}. \quad (\text{C.21})$$

At each point on \mathbb{CP}^1 the fiber is therefore given by \mathbb{C}^{n-1} . Next, we take the sphere bundle

of this sum of line bundles and define a $\mathbb{Z}/k\mathbb{Z}$ action as usual, acting fiberwise as

$$\mathbb{Z}/k\mathbb{Z} : \quad z_i \mapsto e^{2\pi i j_i/k} z_i, \quad (\text{C.22})$$

with z_i describing the fiber coordinate of the i^{th} line bundle. The η -invariant of such a space is given by [139, 110]

$$\begin{aligned} \eta_q^D(Q_k^{2n-1}(j_1, \dots, j_{n-1}; m_1, \dots, m_{n-1})) = & -\frac{1}{k(-2i)^n} \sum_{\ell=1}^{k-1} \frac{e^{2\pi iq\ell/k}}{\sin(\pi\ell j_1/k) \dots \sin(\pi\ell j_{n-1}/k)} \\ & \times \sum_{r=1}^{n-1} m_r \cot(\pi\ell j_r/k). \end{aligned} \quad (\text{C.23})$$

Applied to the case $(m_1, \dots, m_{n-1}) = (\pm 2, 0, \dots, 0)$ this yields

$$\eta_q^D(Q_k^{2n-1}(j_1, \dots, j_{n-1})) = \mp \frac{2}{k(-2i)^n} \sum_{\ell=1}^{k-1} \frac{e^{2\pi iq\ell/k} \cos(\pi\ell j_1/k)}{\sin(\pi\ell j_1/k)^2 \sin(\pi\ell j_2/k) \dots \sin(\pi\ell j_{n-1}/k)}, \quad (\text{C.24})$$

with which we can resolve some extension problems.

Alternative methods to evaluate these η -invariants include adiabatic limits of equivariant η -invariants, see e.g., [225–228].

C.3 Evaluation of η -invariants to resolve extension questions

Since the spectral sequences describe a filtration of the groups under investigation one is often left with extension problems. One way to resolve them is to find bordism invariants such as (the difference of) η -invariants that are sufficiently fine to account for the non-trivial extensions. For that we evaluate the bordism invariants on our candidate for the generator, if one obtains a result of the form $1/k$ one can be certain that this manifold generates at least a subgroup of order k , i.e., $\mathbb{Z}/k\mathbb{Z}$, which often fixes the group completely.

As mentioned above to apply this techniques it is crucial to use bordism invariants. From the APS index theorem [229, 196, 230] one sees that for $X = \partial Y$

$$\eta_q^{\text{op}}(X) = \text{Index}_q^{\text{op}}(Y) - \int_Y I^{\text{op}}, \quad (\text{C.25})$$

where op can stand for D, RS, or sig, and I^{op} denotes the usual index density of the associated operator. This means that the fractional part of the η -invariants for two bordant manifolds, i.e., described as boundaries of a manifold in one higher dimension Y , can differ by contributions from the index density. Since all background symmetries besides those for the tangent bundle are discrete, the index density is given in terms of pure gravitational contributions. Since these gravitational contributions are non-vanishing only in dimension

$4n$, $n \in \mathbb{Z}$, we see that η -invariants are bordism invariants in dimension other than $4n - 1$. In dimension $4n - 1$, one can consider the difference of two η -invariants with different charges for which the index density contribution cancels.

We can apply this to the generators of $\Omega_k^{\text{Spin}}(B\mathbb{Z}/3\mathbb{Z})$ generated by the lens spaces L_3^{2n-1} described above. For even n one has to consider the difference of two η -invariants, since a single one can have contributions from \hat{A} in $4m$ dimensions. For odd n one further needs to use charges of the form $q = Q + \frac{1}{2}$, with $Q \in \mathbb{Z}$ to account for a properly defined $\mathbb{Z}/3\mathbb{Z}$ action on the fermions. The results are summarized in Table 16.

Generator	Bordism invariant
L_3^3	$\eta_1^D - \eta_0^D = -\frac{1}{3}$
L_3^5	$\eta_{1/2}^D = -\frac{1}{9}$
L_3^7	$\eta_1^D - \eta_0^D = \frac{1}{9}$
L_3^9	$\eta_{1/2}^D = \frac{1}{27}$
L_3^{11}	$\eta_1^D - \eta_0^D = -\frac{1}{27}$

Table 16: Bordism invariants of some lens spaces of the form L_3^{2n-1} which can be used to determine some extension problems in finding $\Omega_k^{\text{Spin}}(B\mathbb{Z}/3\mathbb{Z})$.

With these we see that the lens spaces indeed generate the full summands in the associated bordism groups.

The same consideration can be applied for the lens spaces L_4^{2n-1} that generate part of the bordism groups at prime 2. The corresponding bordism invariants are summarized in Table 17. Note that in five and nine dimensions we have two different Spin structures. In

Generator	Bordism invariant
L_4^3	$\eta_1^D - \eta_0^D = -\frac{3}{8}$
L_4^5	$\eta_{1/2}^D = -\frac{5}{32}, \quad \eta_{1/2}^{\text{RS}} = \frac{11}{32}$
\tilde{L}_4^5	$\eta_{3/2}^D = -\frac{3}{32}, \quad \eta_{3/2}^{\text{RS}} = -\frac{3}{32}$
L_4^9	$\eta_{1/2}^D = \frac{9}{128}, \quad \eta_{1/2}^{\text{RS}} = -\frac{19}{128}$
\tilde{L}_4^9	$\eta_{3/2}^D = \frac{7}{128}, \quad \eta_{3/2}^{\text{RS}} = \frac{3}{128}$

Table 17: Bordism invariants of some lens spaces of the form L_4^{2n-1} which can be used to determine some extension problems in finding $\Omega_k^{\text{Spin-Mp}(2,\mathbb{Z})}(\text{pt})$ at prime 2.

order to form two linearly independent generators one needs a second bordism invariant,

e.g., the Rarita-Schwinger invariant, as done in Sections 6.5 and 6.10. From the values of the η -invariants above one can already see that the bordism group contains a $\mathbb{Z}/32\mathbb{Z}$ and $\mathbb{Z}/128\mathbb{Z}$ summand, respectively, which already solves the extension problem in Section 13 for dimension five. In dimension nine one can define a linearly independent bordism invariant that evaluates to zero on one of the Spin structures on the lens spaces and demonstrates that the other Spin structure generators a $\mathbb{Z}/4\mathbb{Z}$ as needed; this is done in the final paragraphs of Section 6.10.

We are left with the generators Q_4^{2n-1} in dimension seven and eleven. Evaluating associ-

Generator	Bordism invariant
Q_4^7	$\eta_{3/2}^D - \eta_{1/2}^D = \frac{1}{4}$
Q_4^{11}	$\eta_{3/2}^D - \eta_{1/2}^D = -\frac{1}{8}$

Table 18: Bordism invariants of some lens space bundles of the form Q_4^{2n-1} which can be used to determine some extension problems in finding $\Omega_k^{\text{Spin-Mp}(2,\mathbb{Z})}(\text{pt})$ at prime 2.

ated bordism invariants, see Table 18, we are again able to fix the extension question.

For the determination of the bordism groups of Spin manifolds with $\text{SL}(2,\mathbb{Z})$ bundle we also need some further bordism invariants which are summarized in Table 19 (the tilde on \tilde{L}_4^7 denotes the alternative Spin structure for lens space as discussed above), which can be

Generator	Bordism invariant
Q_4^5	$\eta_1^D = -\frac{1}{4}$
L_4^7	$\eta_1^D - \eta_0^D = \frac{5}{32}, \quad \eta_2^D - \eta_0^D = \frac{1}{4}$
\tilde{L}_4^7	$\eta_3^D - \eta_2^D = -\frac{3}{32}, \quad \eta_0^D - \eta_2^D = -\frac{1}{4}$
Q_4^9	$\eta_1^D = \frac{1}{8}$
L_4^{11}	$\eta_1^D - \eta_0^D = -\frac{9}{128}, \quad \frac{1}{2}(\eta_2^D - \eta_0^D) = -\frac{1}{16}$
\tilde{L}_4^{11}	$\eta_3^D - \eta_2^D = \frac{7}{128}, \quad \frac{1}{2}(\eta_0^D - \eta_2^D) = \frac{1}{16}$

Table 19: Further bordism invariants for the determination of $\Omega_k^{\text{Spin}}(B\text{SL}(2,\mathbb{Z}))$.

evaluated by the application of the same formulas as above.⁶⁸

⁶⁸Note that in twelve dimensions the index of a fermion in a real representation is even, see e.g. [32], and therefore we see that $\frac{1}{2}(\eta_2^D - \eta_0^D) \bmod \mathbb{Z}$ is indeed a bordism invariant.

D The May-Milgram theorem

In this Appendix, we discuss the May-Milgram theorem, a key tool for determining differentials in Adams spectral sequences. We highlight an important nuance that occurs when applying this theorem to the Adams spectral sequence over $\mathcal{A}(1)$; this nuance has not always been treated carefully in recent such calculations, and we hope this appendix provides a clear guide for how to work with the May-Milgram theorem in the setting of the Adams spectral sequence over $\mathcal{A}(1)$.⁶⁹

Definition D.1. Choose an $r \geq 1$. The short exact sequence

$$0 \longrightarrow \mathbb{Z}/2^r\mathbb{Z} \xrightarrow{\cdot 2^r} \mathbb{Z}/2^{2r}\mathbb{Z} \xrightarrow{\text{mod } 2^r} \mathbb{Z}/2^r\mathbb{Z} \longrightarrow 0 \quad (\text{D.2})$$

induces a long exact sequence in cohomology:

$$\longrightarrow H^*(-; \mathbb{Z}/2^r\mathbb{Z}) \longrightarrow H^*(-; \mathbb{Z}/2^{2r}\mathbb{Z}) \longrightarrow H^*(-; \mathbb{Z}/2^r\mathbb{Z}) \xrightarrow{\beta_r} H^{*+1}(-; \mathbb{Z}/2^r\mathbb{Z}) \longrightarrow \dots \quad (\text{D.3})$$

We let β_r be the connecting homomorphism in this long exact sequence; it is called a *Bockstein homomorphism*.

(D.2) is compatible with the multiplication-by- 2^r short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2^r} \mathbb{Z} \xrightarrow{\text{mod } 2^r} \mathbb{Z}/2^r\mathbb{Z} \longrightarrow 0, \quad (\text{D.4})$$

so there is a commutative diagram of long exact sequences

$$\begin{array}{ccccccc} \dots & \longrightarrow & H^k(-; \mathbb{Z}) & \xrightarrow{\cdot 2^r} & H^k(-; \mathbb{Z}) & \xrightarrow{\text{mod } 2^r} & H^k(-; \mathbb{Z}/2^r\mathbb{Z}) \longrightarrow H^k(-; \mathbb{Z}) \longrightarrow \dots \\ & & \downarrow \text{mod } 2^r & & \downarrow \text{mod } 2^{2r} & & \downarrow \text{mod } 2^r \\ \dots & \longrightarrow & H^k(-; \mathbb{Z}/2^r\mathbb{Z}) & \xrightarrow{\cdot 2^r} & H^k(-; \mathbb{Z}/2^{2r}\mathbb{Z}) & \xrightarrow{\text{mod } 2^r} & H^k(-; \mathbb{Z}/2^r\mathbb{Z}) \xrightarrow{\beta_r} H^k(-; \mathbb{Z}/2^r\mathbb{Z}) \longrightarrow \dots \end{array} \quad (\text{D.5})$$

Using this diagram, one can show that $\beta_r \neq 0$ acts on $H^{k-1}(X; \mathbb{Z}/2^r\mathbb{Z})$ precisely when $H^k(X; \mathbb{Z})$ has 2^r -torsion elements. We want to use this to detect direct summands.

Definition D.6. A *key Bockstein class for 2^r* is an element $x \in H^*(X; \mathbb{Z}/2^r\mathbb{Z})$ in the image of β_r , and such that for all $s < r$, $x \bmod s$ is not in the image of β_s .

Lemma D.7. *Degree- k key Bockstein classes for 2^r are in bijective correspondence with $\mathbb{Z}/2^r\mathbb{Z}$ direct summands in $H^{k+1}(X; \mathbb{Z})$.*

⁶⁹Though we focused on $\mathcal{A}(1)$ in this paper, there are other subalgebras of the Steenrod algebra such that the Adams spectral sequences over those subalgebras compute interesting generalized homology theories, such as 2-completed ku - or tmf -homology: see Beaudry-Campbell [115, Remark 3.2.2]. The nuance we highlight regarding the May-Milgram theorem and the techniques we use to resolve it should apply for these more general examples as well.

The proof is a diagram chase, so we give a quick summary. If x is in the image of β_r , the preimage of x in \mathbb{Z} cohomology generates a $\mathbb{Z}/2^r\mathbb{Z}$ subgroup. This subgroup might not be a direct summand, but the mod s condition fixes that problem: if $x = 2y$, then $x \bmod 2 = 0$, and $0 = \beta_2(0)$. Thus key Bockstein classes correspond to direct summands.

Definition D.8. An h_0 -tower in an Adams spectral sequence is an infinite sequence of nonzero elements $x_i \in E_2^{s+i, t+i}$ such that $h_0 x_i = x_{i+1}$. We do not require this sequence to begin at $i = 0$.

For example, looking at $\text{Ext}_{\mathcal{A}(1)}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ in Figure 12, there are h_0 -towers in all degrees $4k$.

Because differentials commute with the h_0 -action on the E_r -page, the behavior of differentials on h_0 -towers is very restricted. If T is an h_0 -tower, one of two things occurs:

1. every element of T survives to the E_∞ -page, or
2. there is a single number r such that d_r differentials kill all but finitely many elements of T .

In the second situation, d_r goes between T and exactly one other h_0 -tower T' . Because of these two cases, we often refer to d_r s between h_0 -towers or to entire h_0 -towers surviving or not surviving to the E_∞ -page. Now we can state the main theorem:

Theorem D.9 (May-Milgram [174]). *Consider the Adams spectral sequence computing the 2-completed stable homotopy groups of a space or spectrum X . There is a bijection between nonzero d_r differentials out of h_0 -towers in topological degree k and key Bockstein classes for 2^r in $H^k(X; \mathbb{Z}/2^r\mathbb{Z})$.*

The slogan is “Bocksteins tell us everything about differentials between h_0 -towers.”

Remark D.10. The May-Milgram theorem is usually stated differently, identifying Adams differentials between h_0 -towers and differentials in the *Bockstein spectral sequence* associated to the short exact sequence (D.4). The pathway to the way we state it in Theorem D.9 is to solve said Bockstein spectral sequence, as in [231, Section 24.2].

We would like to use the May-Milgram theorem in the computation of twisted Spin bordism, or, thanks to Anderson-Brown-Peterson’s theorem [137], ko -homology. Thus we should take a look at Bocksteins in $H^*(ko \wedge X; \mathbb{Z}/2^r\mathbb{Z})$. This is the important nuance we mentioned above: previous work in the mathematical physics literature applying the May-Milgram theorem, such as [130], has not always been clear about passing from Bocksteins in $H^*(X)$ to Bocksteins in $H^*(ko \wedge X)$.

The Künneth formula says that $H^*(ko \wedge X; \mathbb{Z}/2^r\mathbb{Z})$ is at least as complicated as the combination $H^*(ko; \mathbb{Z}/2^r\mathbb{Z}) \otimes H^*(X; \mathbb{Z}/2^r\mathbb{Z})$. Mahowald-Milgram [145, Corollary 1.3] compute $H^*(ko; \mathbb{Z}_{(2)})$, which is complicated enough to make direct computation of Bocksteins

in $H^*(ko \wedge X; \mathbb{Z}/2^r\mathbb{Z})$ look imposing. Bayen [232, Chapter 1] provides a general approach using the Hopf algebra structure of \mathcal{A} , but we can get away with an ad hoc approach.

Let $\mathcal{A}(0)$ be the subalgebra of \mathcal{A} generated by Sq^1 ; then $\mathcal{A}(0) \cong \mathbb{Z}/2\mathbb{Z}[Sq^1]/((Sq^1)^2)$, i.e. this is an exterior algebra. There is an isomorphism⁷⁰

$$H^*(H\mathbb{Z}; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\cong} \mathcal{A} \otimes_{\mathcal{A}(0)} \mathbb{Z}/2\mathbb{Z}, \quad (\text{D.11})$$

so just like the change-of-rings theorem provides an Adams spectral sequence beginning with Ext over $\mathcal{A}(1)$ and computing ko -homology, there is an even simpler Adams spectral sequence of the form⁷¹

$$E_2^{s,t} = \text{Ext}_{\mathcal{A}(0)}^{s,t}(H^*(X; \mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \implies H_{t-s}(X; \mathbb{Z})_2^\wedge. \quad (\text{D.12})$$

The idea behind our ad hoc approach to the May-Milgram theorem is that differentials between Adams towers are easy to compute over $\mathcal{A}(0)$, because we generally know the integral homology of X ; then we pull them back by the map of Adams spectral sequences induced by the quotient $\mathcal{A}(1) \rightarrow \mathcal{A}(0)$.

If M is an $\mathcal{A}(0)$ -module which is a finite-dimensional $\mathbb{Z}/2\mathbb{Z}$ -vector space in each degree, then M is a direct sum of shifts of the two $\mathcal{A}(0)$ -modules $\mathbb{Z}/2\mathbb{Z}$ and $\mathcal{A}(0)$. The Ext groups of these modules are simple to compute.

- $\text{Ext}_{\mathcal{A}(0)}(\mathcal{A}(0), \mathbb{Z}/2\mathbb{Z})$ consists of a single $\mathbb{Z}/2\mathbb{Z}$ in bidegree $(0, 0)$, as is always the case for Ext of an algebra over itself.
- $\text{Ext}_{\mathcal{A}(0)}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ consists of a single h_0 -tower in topological degree 0, which is a consequence of Koszul duality [115, Example 4.5.5].

Therefore a d_r differential between h_0 -towers in the $\mathcal{A}(0)$ -Adams spectral sequence for X corresponds to a $\mathbb{Z}/2^r\mathbb{Z}$ summand in the homology of X , as we draw in Figure 30, and as we noted above such a summand is equivalent data to a key Bockstein class.

Recall that $C\eta$ denotes the \mathcal{A} -module consisting of two $\mathbb{Z}/2\mathbb{Z}$ summands in degrees 0 and 2 linked by a Sq^2 .

Proposition D.13 (May-Milgram for copies of $C\eta$). *Let X be a space or spectrum and $\textcolor{red}{M}_0$ and $\textcolor{teal}{M}_1$ be two $\mathcal{A}(1)$ -module summands of $H^*(X; \mathbb{Z}/2\mathbb{Z})$ isomorphic to shifts of $C\eta$. Suppose there exist $x \in \textcolor{red}{M}_0$ and $y \in \textcolor{teal}{M}_1$ of degrees n and $n+1$, respectively, and classes $\tilde{x}, \tilde{y} \in H^*(X; \mathbb{Z}/2^r\mathbb{Z})$ such that $\tilde{x} \bmod 2 = x$, $\tilde{y} \bmod 2 = y$, $\beta_r(\tilde{x}) = \tilde{y}$, and \tilde{y} is a key Bockstein class. Then for all $k \geq 0$, there is a d_r differential from the h_0 -tower in degree $n+4k+1$ coming from $\text{Ext}_{\mathcal{A}(1)}(\textcolor{teal}{M}_1)$ to the h_0 -tower in degree $n+4k$ coming from $\text{Ext}_{\mathcal{A}(1)}(\textcolor{red}{M}_0)$.*

⁷⁰We do not know who originally proved this. Ravenel [233, Lemma 3.1.11] remarks that it is “standard.”

⁷¹In fact, this Adams spectral sequence is isomorphic to a certain presentation of the Bockstein spectral sequence [234, Section 1.4], and May-Milgram’s proof of their theorem can be interpreted from this point of view.

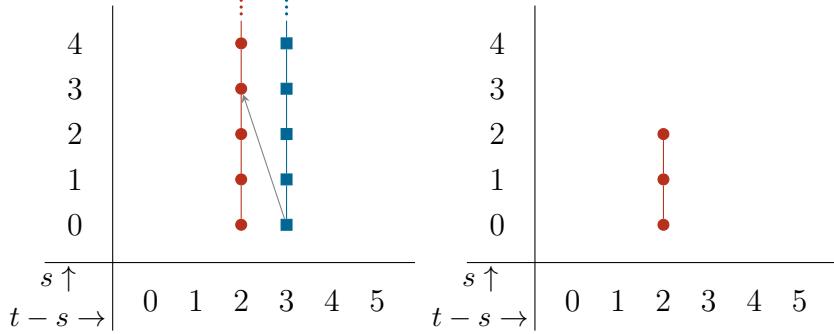


Figure 30: A differential in an Adams spectral sequence over $\mathcal{A}(0)$ between two h_0 -towers, arising from two $\mathbb{Z}/2\mathbb{Z}$ summands in the $\mathcal{A}(0)$ -module structure on cohomology. The d_r -differential on the E_r -page (left) leaves a $\mathbb{Z}/2^{r}\mathbb{Z}$ summand in the E_∞ -page (right). Here $r = 3$.

The existence of these h_0 -towers in degrees $n + 4k$ and $n + 4k + 1$ follows from the computation we did in Example 11.15.

Proof. Recall from Example 11.15 that the h_0 -towers in $\text{Ext}_{\mathcal{A}(1)}(\Sigma^n C\eta)$ in topological degrees $n + 4k$ and $n + 4k + 4$ are linked by the action of $v \in \text{Ext}_{\mathcal{A}(1)}^{3,7}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$: all but finitely many elements of the higher-degree tower are in the image of this v -action applied to the lower-degree tower. This v -action commutes with differentials, so if we have proven the theorem for k we can deduce the theorem for $k + 1$. Thus in the rest of the proof we assume $k = 0$.

As an $\mathcal{A}(0)$ -module, $C\eta \cong \mathbb{Z}/2\mathbb{Z} \oplus \Sigma^2 \mathbb{Z}/2\mathbb{Z}$, so $\text{Ext}_{\mathcal{A}(0)}(C\eta, \mathbb{Z}/2\mathbb{Z})$ consists of h_0 -towers in degrees 0 and 2. The map $\mathcal{A}(1) \rightarrow \mathcal{A}(0)$ induces a map $\phi: \text{Ext}_{\mathcal{A}(1)}(C\eta, \mathbb{Z}/2\mathbb{Z}) \rightarrow \text{Ext}_{\mathcal{A}(0)}(C\eta, \mathbb{Z}/2\mathbb{Z})$; thinking of elements of Ext as equivalence classes of extensions of $\mathcal{A}(1)$ -modules, like we discussed in Section 11.2, ϕ takes an extension of $\mathcal{A}(1)$ -modules and forgets the Sq^2 -action to obtain the same extension but of $\mathcal{A}(0)$ -modules. This model for ϕ lends itself to computations: for example, in Figure 31, we show that

$$\phi^{1,3}: \text{Ext}_{\mathcal{A}(1)}^{1,3}(C\eta, \mathbb{Z}/2\mathbb{Z}) \longrightarrow \text{Ext}_{\mathcal{A}(0)}^{1,3}(C\eta, \mathbb{Z}/2\mathbb{Z}) \quad (\text{D.14})$$

is nonzero; since both of these Ext groups are isomorphic to $\mathbb{Z}/2\mathbb{Z}$, this means $\phi^{1,3}$ is an isomorphism. From the Yoneda product description of the action by h_0 (11.11), it is possible to show that ϕ commutes with h_0 -actions, so the calculation in Figure 31 implies that in degrees $s \geq 1$, ϕ maps the h_0 -tower in topological degree 2 in $\text{Ext}_{\mathcal{A}(1)}(C\eta)$ isomorphically onto the h_0 -tower in topological degree 2 in $\text{Ext}_{\mathcal{A}(0)}(C\eta)$. The same is true for the h_0 -towers in topological degree 0, which we leave as an exercise for the reader.⁷² Therefore there is a d_r differential from the h_0 -tower in degree $n + 1$ in $\text{Ext}_{\mathcal{A}(1)}(\mathcal{M}_1)$ to the h_0 -tower in degree n

⁷²Another way to compute ϕ in both degrees 0 and 2 is to observe that $\Sigma^2 C\eta \cong \tilde{H}^*(\mathbb{CP}^2; \mathbb{Z}/2\mathbb{Z})$, so ϕ is the morphism of Adams spectral sequences corresponding to the map $\psi: ko_*(\mathbb{CP}^2) \rightarrow H_*(\mathbb{CP}^2)$, and ψ can be computed with the Atiyah-Hirzebruch spectral sequence.

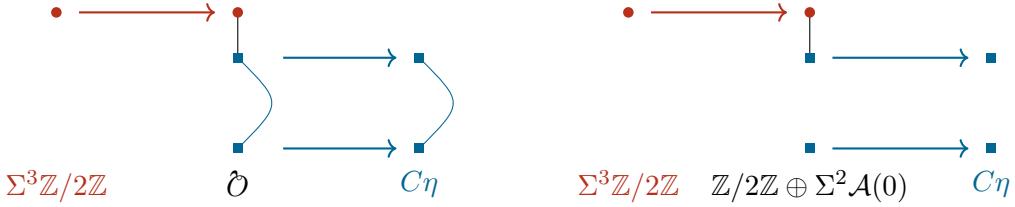


Figure 31: Left: an extension of $\mathcal{A}(1)$ -modules representing the nonzero element x of $\text{Ext}_{\mathcal{A}(1)}^{1,3}(C\eta, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Right: if we only remember the action of $\mathcal{A}(0)$, this extension is still nonsplit, so $\phi(x) \neq 0$. This computation is part of the proof of Proposition D.13.

in $\text{Ext}_{\mathcal{A}(1)}(\mathcal{M}_0)$ exactly when the same is true for Ext over $\mathcal{A}(0)$, and the latter is true when preimages of x and y in $\mathbb{Z}/2^r\mathbb{Z}$ cohomology are linked by a β_r Bockstein and the lift of y is a key Bockstein class. \square

See Figure 32 for a picture of the key idea in the proof of Proposition D.13.

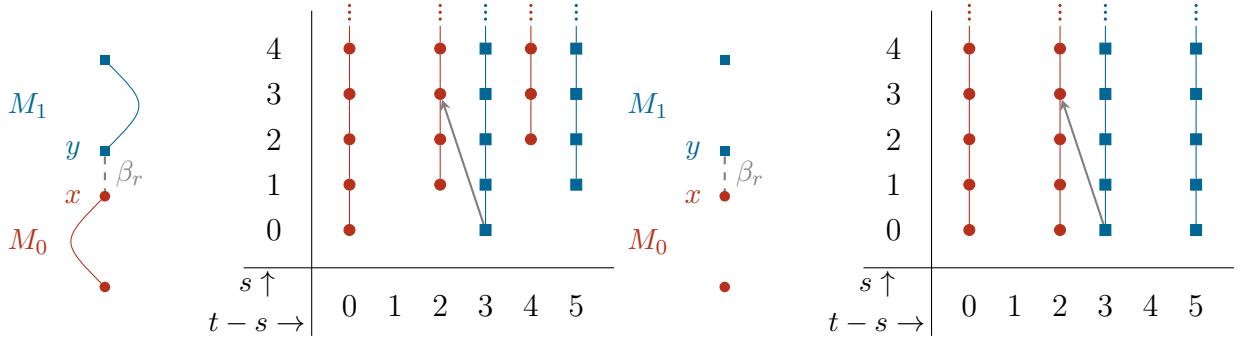


Figure 32: Part of the proof of Proposition D.13. The left-hand side is part of the module structure and Adams spectral sequence over $\mathcal{A}(1)$ for $\mathcal{M}_0 \oplus \mathcal{M}_1 \subset H^*(X; \mathbb{Z}/2\mathbb{Z})$; the right-hand side is the corresponding data over $\mathcal{A}(0)$. The gray d_r differential on the right can be deduced from the existence of a β_r Bockstein, and implies the gray d_r differential on the left.

Remark D.15. Proposition D.13 is quite narrow in scope, but simple to prove. We suggest a similar approach to other applications of the May-Milgram theorem to twisted spin bordism questions: compute differentials between h_0 -towers over $\mathcal{A}(0)$, where they correspond to Bocksteins in the cohomology of X , then pull them back to Ext over $\mathcal{A}(1)$ to learn about $ko_*(X)$.

E Some Smith homomorphisms

Smith homomorphisms are certain maps between bordism groups in which both the dimension and the tangential structure vary. Specialized to the cases of Spin- $M_p(2, \mathbb{Z})$ and Spin- $GL^+(2, \mathbb{Z})$ bordism, there are a few Smith homomorphisms that are interesting and

helpful, and the purpose of this appendix is to discuss them. In Appendix E.1 we introduce Smith homomorphisms and generalities; in Appendix E.2 we discuss a sequence of Smith maps between $\Omega_k^{\text{Spin}}(BSL(2, \mathbb{Z}))$ and $\Omega_k^{\text{Spin-Mp}(2, \mathbb{Z})}(\text{pt})$; and in Appendix E.3 we discuss a Smith map for Spin- $GL^+(2, \mathbb{Z})$ bordism. This last Smith map plays a key role in our proof in Section 14.3.7 that $[W_2^7] \in \Omega_7^{\text{Spin-GL}^+(2, \mathbb{Z})}(\text{pt})$ is linearly independent from $[W_1^7]$.

E.1 This charming man(ifold): generalities on the Smith(homomorphism)s

In this subsection we review the general setup for Smith homomorphisms in twisted Spin bordism, so that we can study examples involving Spin-Mp(2, \mathbb{Z}) and Spin- $GL^+(2, \mathbb{Z})$ bordism in the rest of this appendix. See [235–237] for more detail with an eye towards physics applications.

Recall from Definition 10.17 that for a space X and vector bundle $V \rightarrow X$, an (X, V) -twisted Spin structure on a vector bundle $E \rightarrow M$ is a map $f: M \rightarrow X$ and a Spin structure on $E \oplus f^*V$. Suppose M is a manifold and TM has an (X, V) -twisted Spin structure. Choose another vector bundle $W \rightarrow X$ of rank r and let $N \subset M$ be a *smooth representative* for the Poincaré dual of $w_r(f^*W)$; that is, the image of the mod 2 fundamental class of N in $H_*(M; \mathbb{Z}/2\mathbb{Z})$ is Poincaré dual to $w_r(f^*W) \in H^*(M; \mathbb{Z}/2\mathbb{Z})$.

The key fact underlying the Smith homomorphism is that in this situation, N has a canonical $(X, V \oplus W)$ -twisted Spin structure, and that this construction factors through bordism classes to define a map

$$S_W: \Omega_{n+V}^{\text{Spin}}(X) \longrightarrow \Omega_{n-r+V \oplus W}^{\text{Spin}}(X). \quad (\text{E.1})$$

Now the explanation: if N is a smooth representative of the Poincaré dual of the top Stiefel-Whitney class of a vector bundle $E \rightarrow M$, then the normal bundle ν_N of $N \hookrightarrow M$ is isomorphic to $E|_N$. Using the isomorphism $TN \oplus \nu_N \cong TM|_N$, the (X, V) -twisted Spin structure on M induces a Spin structure on

$$(TM \oplus f^*V)|_N \cong TN \oplus \nu_N \oplus f^*V|_N \cong TN \oplus E|_N \oplus f^*V|_N, \quad (\text{E.2})$$

and plugging in $E = f^*W$, we have found an $(X, V \oplus W)$ -twisted Spin structure on N .

A theorem of [237] puts the Smith homomorphism into a long exact sequence, making calculating kernels and images of Smith homomorphisms easier.

Theorem E.3 ([237]). *Let $p: S(V) \rightarrow X$ be the sphere bundle of the vector bundle $V \rightarrow X$. Then there is a long exact sequence*

$$\cdots \longrightarrow \Omega_{k+p^*W}^{\text{Spin}}(S(V)) \xrightarrow{p_*} \Omega_{k+V}^{\text{Spin}}(X) \xrightarrow{S_W} \Omega_{k-r+V \oplus W}^{\text{Spin}}(X) \longrightarrow \Omega_{k-1+p^*W}^{\text{Spin}}(S(V)) \longrightarrow \cdots, \quad (\text{E.4})$$

Often, $S(V)$ is something like a relatively simple classifying space and the entire long

exact sequence can be worked out using techniques similar to the ones in this paper. We discuss a few examples below.

E.2 Smith homomorphisms for Spin-Mp(2, \mathbb{Z}) bordism

Recall from Section 13 that Spin-Mp(2, \mathbb{Z}) structures are equivalent in a natural way to $(BSL(2, \mathbb{Z}), V)$ -twisted Spin structures, where $V \rightarrow BSL(2, \mathbb{Z})$ is induced from the standard two-dimensional real representation of $SL(2, \mathbb{Z})$.⁷³ Therefore there are codimension-2 Smith homomorphisms exchanging Spin bordism of $BSL(2, \mathbb{Z})$ and Spin-Mp(2, \mathbb{Z}) bordism.

To compute these Smith homomorphisms, we would like to identify the third term in the long exact sequence (E.4). The sphere bundle of $V \rightarrow SL(2, \mathbb{Z})$ is homotopy equivalent to $SL(2, \mathbb{R})/SL(2, \mathbb{Z})$, which is a little complicated to describe, so instead we will work at the primes 2 and 3 separately, where it is easier.

For $p = 2$, this Smith homomorphism pulls back to a Smith homomorphism exchanging Spin- $\mathbb{Z}/8\mathbb{Z}$ bordism and the Spin bordism of $B\mathbb{Z}/4\mathbb{Z}$. The sphere bundle of $V \rightarrow B\mathbb{Z}/4\mathbb{Z}$ is homotopy equivalent to S^1 [237], and every vector bundle trivializes when pulled back to its sphere bundle, so there are long exact sequences

$$\cdots \longrightarrow \Omega_k^{\text{Spin}}(S^1) \longrightarrow \Omega_k^{\text{Spin-Z}/8\mathbb{Z}}(\text{pt}) \xrightarrow{S_V} \Omega_{k-2}^{\text{Spin}}(B\mathbb{Z}/4\mathbb{Z}) \longrightarrow \Omega_{k-1}^{\text{Spin}}(S^1) \longrightarrow \cdots \quad (\text{E.5a})$$

$$\cdots \longrightarrow \Omega_k^{\text{Spin}}(S^1) \longrightarrow \Omega_k^{\text{Spin}}(B\mathbb{Z}/4\mathbb{Z}) \xrightarrow{S_V} \Omega_{k-2}^{\text{Spin-Z}/8\mathbb{Z}}(\text{pt}) \longrightarrow \Omega_{k-1}^{\text{Spin}}(S^1) \longrightarrow \cdots \quad (\text{E.5b})$$

Lemma E.6. *For any generalized homology theory E , $E_k(S^1) \cong E_k(\text{pt}) \oplus E_{k-1}(\text{pt})$.*

Proof sketch. This is equivalent to the collapse of the Atiyah-Hirzebruch spectral sequence computing the E -homology of S^1 and the absence of hidden extensions. One can show this by considering the maps $\text{pt} \rightarrow S^1 \rightarrow \text{pt}$ and their effect on differentials and extension questions. \square

Using this lemma and the calculations of $\Omega_*^{\text{Spin}}(B\mathbb{Z}/4\mathbb{Z})$ and $\Omega_*^{\text{Spin-Z}/8\mathbb{Z}}(\text{pt})$ we made in Table 7 and Section 13.2, we can make the long exact sequences (E.5) explicit. For example:

Proposition E.7. *The Smith homomorphism $S_V: \Omega_5^{\text{Spin-Z}/8\mathbb{Z}}(\text{pt}) \rightarrow \Omega_3^{\text{Spin}}(B\mathbb{Z}/4\mathbb{Z})$, which is a map $(\mathbb{Z}/32\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) \rightarrow (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$, is surjective, sending $(x, y) \mapsto (x \bmod 8, y)$.*

Proof. Once we know this map is surjective, its value on generators follows, possibly after an automorphism of $(\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$. The next map after our Smith homomorphism S_V in the long exact sequence is a map $\Omega_3^{\text{Spin}}(B\mathbb{Z}/4\mathbb{Z}) \rightarrow \Omega_4^{\text{Spin}}(S^1)$; the domain of this map is torsion, as we computed in Section 13.2, and the codomain is free, so this map is 0, so by exactness S_V is surjective. \square

⁷³Strictly speaking, we only proved this after localizing at 2 and at 3, but the same argument building on Corollary 10.23 holds before localizing.

One could alternatively prove this by computing Poincaré duals of characteristic classes for generators of $\Omega_5^{\text{Spin-Z}/8\mathbb{Z}}(\text{pt})$; we find it helpful to have both methods available.

For $p = 3$, the argument will have a similar feel: the sphere bundle of $V \rightarrow B\mathbb{Z}/3\mathbb{Z}$ is again homotopy equivalent to S^1 [237], and 3-locally, both $\Omega_*^{\text{Spin}}(B\text{SL}(2, \mathbb{Z}))$ and $\Omega_*^{\text{Spin-Mp}(2, \mathbb{Z})}(\text{pt})$ are equivalent to $\Omega_*^{\text{SO}}(B\mathbb{Z}/3\mathbb{Z})$, which we proved in Lemmas 12.7 and 13.4, respectively. Therefore our Smith long exact sequence has the form

$$\cdots \longrightarrow \Omega_k^{\text{SO}}(S^1) \longrightarrow \Omega_k^{\text{SO}}(B\mathbb{Z}/3\mathbb{Z}) \xrightarrow{S_V} \Omega_{k-2}^{\text{SO}}(B\mathbb{Z}/3\mathbb{Z}) \longrightarrow \Omega_{k-1}^{\text{SO}}(S^1) \longrightarrow \cdots \quad (\text{E.8})$$

Lemma E.6 and (12.17) allow us to populate this long exact sequence. We learn, for example, that the Smith homomorphism $\Omega_3^{\text{SO}}(B\mathbb{Z}/3\mathbb{Z}) \rightarrow \Omega_1^{\text{SO}}(B\mathbb{Z}/3\mathbb{Z})$ is an isomorphism $\mathbb{Z}/3\mathbb{Z} \rightarrow \mathbb{Z}/3\mathbb{Z}$, because the two terms surrounding this map in the long exact sequence are $\Omega_k^{\text{SO}}(S^1)$ for $k = 2, 3$, which both vanish. In a similar way one can show that $\tilde{\Omega}_5^{\text{SO}}(B\mathbb{Z}/3\mathbb{Z}) \rightarrow \tilde{\Omega}_3^{\text{SO}}(B\mathbb{Z}/3\mathbb{Z})$ is a surjective map $\mathbb{Z}/9\mathbb{Z} \rightarrow \mathbb{Z}/3\mathbb{Z}$, because the next map in the long exact sequence is from a torsion Abelian group to a free one, and hence vanishes.

E.3 Smith homomorphisms for Spin-GL⁺(2, \mathbb{Z}) bordism

As we noted above, the possible Smith homomorphisms out of $\Omega_k^{\text{Spin-GL}^+(2, \mathbb{Z})}(\text{pt})$ are given by the isomorphism classes of vector bundles over $B\text{GL}(2, \mathbb{Z})$, and this gives us a lot of options, even in codimension 1 and 2. We focus on a particular example which we need in Section 14.3.7: a codimension-2 map exchanging the Pin⁺ and Pin⁻ covers of $\text{GL}(2, \mathbb{Z})$.

Let $\text{GL}^-(2, \mathbb{Z})$ denote the Pin⁻ cover of $\text{GL}(2, \mathbb{Z})$ and $\text{Spin-GL}^-(2, \mathbb{Z}) := \text{Spin} \times_{\{\pm 1\}} \text{GL}^-(2, \mathbb{Z})$. Our arguments above studying Spin-GL⁺(2, \mathbb{Z}) bordism also apply in this case, allowing one to show that:

- The map $\Omega_k^{\text{Spin-GL}^-(2, \mathbb{Z})}(\text{pt}) \rightarrow \Omega_k^{\text{SO}}(BD_{12})$ is a p -local isomorphism for any odd prime p . The proof is the same as the argument given at the beginning of Section 14.1.
- The Pin⁻ cover of D_8 is the quaternion group Q_{16} , implying the central extension $0 \rightarrow \mathbb{Z}/2 \rightarrow Q_{16} \rightarrow D_8 \rightarrow 0$ is classified by $w + x^2 \in H^2(BD_8; \mathbb{Z}/2\mathbb{Z})$. The inclusion $D_8 \hookrightarrow \text{GL}(2, \mathbb{Z})$ lifts to an inclusion $Q_{16} \hookrightarrow \text{GL}^-(2, \mathbb{Z})$ inducing a 2-local isomorphism $\Omega_k^{\text{Spin-}Q_{16}}(\text{pt}) \rightarrow \Omega_k^{\text{Spin-GL}^-(2, \mathbb{Z})}(\text{pt})$.
- A Spin- Q_{16} structure on a vector bundle $V \rightarrow M$ is equivalent data to a $(BD_8, V \oplus \text{Det}(V))$ -twisted Spin structure, i.e. a principal D_8 -bundle $P \rightarrow M$ and data of an identification $w_2(V) = w(P) + x(P)^2$. The proof is similar to that of Corollary 14.16 with $V \oplus \text{Det}(V)$ in place of $V \oplus 3\text{Det}(V)$ and $w + x^2$ in place of w .

We consider the Smith homomorphism given by the vector bundle $2\text{Det}(V)$, which lowers the dimension by 2. This goes from $(B\text{GL}(2, \mathbb{Z}), V \oplus \text{Det}(V))$ -twisted Spin bordism

to $(B\mathrm{GL}(2, \mathbb{Z}), V \oplus 3\mathrm{Det}(V))$ -twisted Spin bordism, i.e. $\mathrm{Spin}\text{-}\mathrm{GL}^-(2, \mathbb{Z})$ bordism to $\mathrm{Spin}\text{-}\mathrm{GL}^+(2, \mathbb{Z})$ bordism. We can also use it to go from $(B\mathrm{GL}(2, \mathbb{Z}), V \oplus 3\mathrm{Det}(V))$ -twisted Spin bordism to $(B\mathrm{GL}(2, \mathbb{Z}), V \oplus 5\mathrm{Det}(V))$ -twisted Spin bordism, but since four copies of any vector bundle is Spin, this can be identified with $(B\mathrm{GL}(2, \mathbb{Z}), V \oplus \mathrm{Det}(V))$ -twisted Spin bordism, and the Smith homomorphism goes from $\mathrm{Spin}\text{-}\mathrm{GL}^+(2, \mathbb{Z})$ bordism to $\mathrm{Spin}\text{-}\mathrm{GL}^-(2, \mathbb{Z})$ bordism.⁷⁴

Like in the previous section, it is easier to work at $p = 2$ and $p = 3$ separately. After 3-localizing, this Smith homomorphism is a map $\Omega_k^{\mathrm{SO}}(BD_{12}) \rightarrow \Omega_{k-2}^{\mathrm{SO}}(BD_{12})$, but in Theorem 14.3 we saw that at least for all k within the range we care about, at least one of $\tilde{\Omega}_k^{\mathrm{SO}}(BD_{12})$ and $\tilde{\Omega}_{k-2}^{\mathrm{SO}}(BD_{12})$ vanishes, so this Smith homomorphism is trivial. Using unreduced bordism does not make the map much more interesting.

At $p = 2$, though, this Smith homomorphism is more useful. The whole story with $\mathrm{GL}^\pm(2, \mathbb{Z})$ pulls back: the Smith homomorphism exchanges Spin- D_{16} and Spin- Q_{16} bordism. However, the third term in these Smith long exact sequences is a little complicated: twisted Spin bordism groups of $B(\mathbb{Z} \ltimes \mathbb{Z}/4\mathbb{Z})$, where \mathbb{Z} acts on $\mathbb{Z}/4\mathbb{Z}$ by $\alpha \cdot \beta = (-1)^\alpha \beta$, and twisted by the pullback of $V \oplus \mathrm{Det}(V)$. Proving this amounts to calculating the sphere bundle of $2\mathrm{Det}(V) \rightarrow BD_8$, which we found a fun exercise similar to some of the sphere bundle calculations in [237], and to [238, Lemma 7].

These twisted Spin bordism groups are not so hard to calculate using similar techniques to the ones in this paper, and we invite the reader to try some of these computations.⁷⁵

We use the map $S_{x^2}: \Omega_7^{\mathrm{Spin}-D_{16}}(\mathrm{pt}) \rightarrow \Omega_5^{\mathrm{Spin}-Q_{16}}(\mathrm{pt})$ in Section 14.3.7, so we take the opportunity here to investigate $\Omega_5^{\mathrm{Spin}-Q_{16}}(\mathrm{pt})$. Spin- Q_{16} bordism has also been studied in dimension 4 by Pedrotti [206, Theorem 9.0.14] using other methods; he proves that $\Omega_4^{\mathrm{Spin}-Q_{16}}(\mathrm{pt}) \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$, which we will rediscover during the proof of Proposition E.13.

The group Q_{16} acts freely on S^3 , where Q_{16} is considered as a subgroup of $\mathrm{SU}(2)$ acting on the unit sphere in \mathbb{C}^2 . The quotient S^3/Q_{16} is called a *prism manifold*, and the quotient $\mathbb{RP}^3 \rightarrow S^3/Q_{16}$ is a principal D_8 -bundle.

Lemma E.9. *S^3/Q_{16} with the D_8 -bundle $P := \mathbb{RP}^3 \rightarrow S^3/Q_{16}$ admits a Spin- Q_{16} structure, and $\int_{S^3/Q_{16}} w(P)y(P) = 1$.*

Before we prove this, we need to better understand the mod 2 cohomology of S^3/Q_{16} . Recall from Proposition 14.100 that $H^*(BQ_{16}; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[\hat{x}, \hat{y}, p]/(\hat{x}\hat{y} + \hat{y}^2, \hat{x}^3)$, where $|\hat{x}| = |\hat{y}| = 1$ and $|p| = 4$, and \hat{x} , resp. \hat{y} are the pullbacks of $x, y \in H^1(BD_8; \mathbb{Z}/2\mathbb{Z})$ under $BQ_{16} \rightarrow BD_8$.

⁷⁴Kirby-Taylor [238, Lemma 7] study a similar pair of Smith homomorphisms which exchange Pin^- and Pin^+ bordism.

⁷⁵Alternately, one could apply the Smith homomorphism for just $\mathrm{Det}(V)$, and do it twice, e.g. getting one long exact sequence for $V \oplus \mathrm{Det}(V)$ to $V \oplus 2\mathrm{Det}(V)$ and a second one for $V \oplus 2\mathrm{Det}(V)$ to $V \oplus 3\mathrm{Det}(V)$. In these cases the third term in the long exact sequence is $\Omega_k^{\mathrm{Spin}}(B\mathbb{Z}/4\mathbb{Z})$.

Lemma E.10 (Tomoda-Zvengrowski [239, Section 2 and Theorem 2.2(1)]). *There is an isomorphism $H^*(S^3/Q_{16}; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[\hat{x}, \hat{y}] / (\hat{x}\hat{y} + \hat{y}^2, \hat{x}^3)$, and if $f: S^3/Q_{16} \rightarrow BQ_{16}$ is the classifying map of the principal Q_{16} -bundle $S^3 \rightarrow S^3/Q_{16}$, then $f^*: H^*(BQ_{16}; \mathbb{Z}/2\mathbb{Z}) \rightarrow H^*(S^3/Q_{16}; \mathbb{Z}/2\mathbb{Z})$ is the quotient by (p) , i.e. it sends $\hat{x} \mapsto \hat{x}$, $\hat{y} \mapsto \hat{y}$, and $p \mapsto 0$.*

Proof of Lemma E.9. The relations in the cohomology ring in Lemma E.10 imply that $\hat{x}^2\hat{y} = \hat{x}\hat{y}^2 = \hat{y}^3$ is the nonzero element in the top degree mod 2 cohomology of S^3/Q_{16} .

Recall that the extension

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\alpha} Q_{16} \xrightarrow{\beta} D_8 \longrightarrow 1 \quad (\text{E.11})$$

is classified by $w+x^2 \in H^2(BD_8; \mathbb{Z}/2\mathbb{Z})$, so $\beta^*: BD_8 \rightarrow BQ_{16}$ sends $w(ED_8) + x(ED_8)^2 \mapsto 0$. Thus for the D_8 -bundle $\beta^*ED_8 \rightarrow BQ_{16}$, $w = \hat{x}^2$. This bundle is the fiber $\alpha: B\mathbb{Z}/2\mathbb{Z} \rightarrow BQ_{16}$ of β , and when we pull back further along $f: S^3/Q_{16} \rightarrow BQ_{16}$, β^*ED_8 pulls back to $\mathbb{RP}^3 \rightarrow Q_{16}$. Therefore $w(\mathbb{RP}^3) = \hat{x}^2$. Likewise, $y \in H^1(BD_8; \mathbb{Z}/2\mathbb{Z})$ pulls back to $\hat{y} \in H^1(S^3/Q_{16}; \mathbb{Z}/2\mathbb{Z})$, so

$$\int_{S^3/Q_{16}} w(P)y(P) = \int_{S^3/Q_{16}} \hat{x}^2\hat{y} = 1. \quad (\text{E.12}) \quad \square$$

We will use this result in Section 14.3.7.

Proposition E.13. $[S^3/Q_{16} \times T^2] \neq 0$ in $\Omega_5^{\text{Spin-}Q_{16}}(\text{pt})$.

Proof. We use the Adams spectral sequence. Since we remain below degree 8, there is no difference between Spin bordism and *ko*-homology. Similarly to the case of Spin- D_{16} bordism, the fact that a Spin- Q_{16} structure is equivalent to a $(BD_8, V \oplus \text{Det}(V))$ -twisted Spin structure allows us to identify $\Omega_k^{\text{Spin-}Q_{16}}(\text{pt}) \cong \Omega_k^{\text{Spin}}((BD_8)^{V+\text{Det}(V)-3})$. In the cohomology of the Thom spectrum $(BD_8)^{V+\text{Det}(V)-3}$, $\text{Sq}^1(U) = 0$ and $\text{Sq}^2(U) = U(w+x^2)$; using this, there is an $\mathcal{A}(1)$ -module isomorphism

$$H^*((BD_8)^{V+\text{Det}(V)-3}; \mathbb{Z}/2\mathbb{Z}) \cong \color{red}{\mathcal{O}} \oplus \color{green}{\Sigma^3 \mathbb{Z}/2\mathbb{Z}} \oplus \color{blue}{\Sigma^4 R_2} \oplus \color{purple}{\Sigma^5 J} \oplus \Sigma \mathcal{A}(1) \oplus \Sigma \mathcal{A}(1) \oplus \Sigma^5 \mathcal{A}(1) \oplus \Sigma^5 \mathcal{A}(1) \oplus P, \quad (\text{E.14})$$

where P is concentrated in degrees 7 and above. We draw this in Figure 33, left. We already know the Ext of each of these summands already (except for P , which as usual does not matter in this proof), so we can draw the E_2 -page of the Adams spectral sequence in Figure 33, right.

In the range depicted, most differentials vanish either for degree reasons or by h_i -equivariance. The exceptions are the differentials from the red and blue towers in the 4-line to the green tower in the 3-line. Like in the case of Spin- $\mathbb{Z}/8\mathbb{Z}$ bordism that we treated

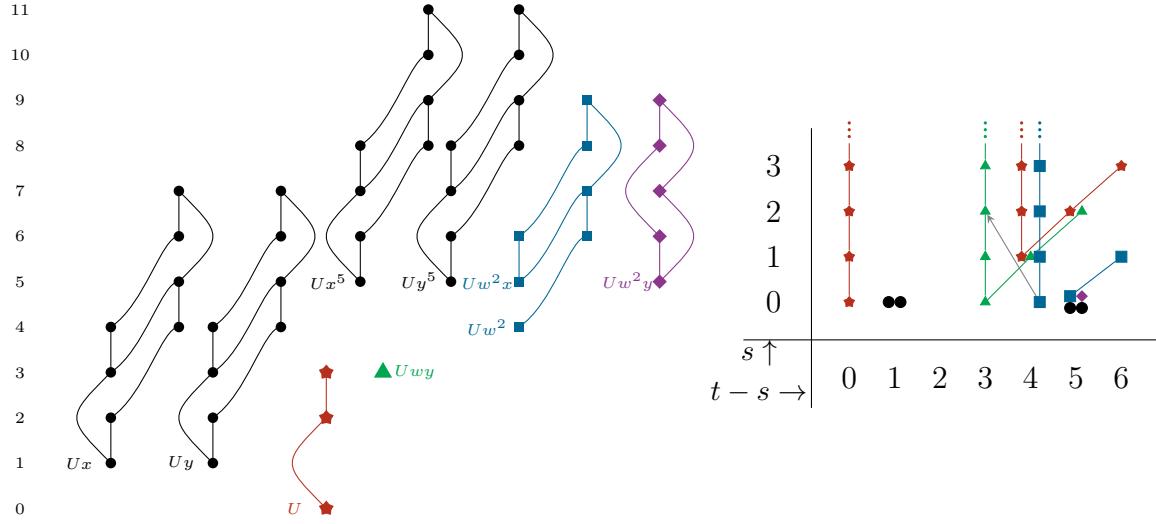


Figure 33: Left: the $\mathcal{A}(1)$ -module structure on $H^*((BD_8)^{V \oplus \text{Det}(V)-3}; \mathbb{Z}/2\mathbb{Z})$ in low degrees. The pictured submodule contains all elements in degrees 6 and below. Right: the E_2 -page of the corresponding Adams spectral sequence computing $ko_*((BD_8)^{V \oplus \text{Det}(V)-3})$; in Proposition E.13, we use this to study Spin- Q_{16} bordism.

in Section 13.2, the May-Milgram theorem implies there is a d_2 from the blue tower to the green tower.

Therefore we learn that the green triangle $\mathbb{Z}/2\mathbb{Z}$ summands in $E_2^{0,3}$, $E_2^{1,5}$, and $E_2^{2,7}$ all survive to the E_∞ -page. They are linked by h_1 -actions, so we have a diagram

$$\begin{array}{ccccccc}
 \mathbb{Z}/4\mathbb{Z} & \xrightarrow{\eta} & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\eta} & \mathbb{Z}/2\mathbb{Z} \\
 \downarrow \cong & & \downarrow & & \downarrow \\
 \Omega_3^{\text{Spin-}Q_{16}}(\text{pt}) & \xrightarrow{\eta} & \Omega_4^{\text{Spin-}Q_{16}}(\text{pt}) & \xrightarrow{\eta} & \Omega_5^{\text{Spin-}Q_{16}}(\text{pt}),
 \end{array} \tag{E.15}$$

with the takeaway that $\eta^2: \Omega_3^{\text{Spin-}Q_{16}}(\text{pt}) \rightarrow \Omega_5^{\text{Spin-}Q_{16}}(\text{pt})$ is nonzero precisely on the odd elements of $\Omega_3^{\text{Spin-}Q_{16}}(\text{pt}) \cong \mathbb{Z}/4\mathbb{Z}$. The image of a generator in the E_∞ -page is in Adams filtration zero, where it corresponds to the mod 2 cohomology class Uwy , which has the convenient consequence that $\int wy$ is also nonzero exactly on the odd elements of $\Omega_3^{\text{Spin-}Q_{16}}(\text{pt})$. So if we can show that this integral is nonzero on S^3/Q_{16} , then we also deduce that $\eta^2 \cdot [S^3/Q_{16}]$, i.e. $[S^3/Q_{16} \times T_p^2]$, is nonzero, which is what we want to prove.

By Lemma E.9, S^3/Q_{16} is an odd element of $\Omega_3^{\text{Spin-}Q_{16}}(\text{pt})$, so $[S^3/Q_{16} \times T_p^2] \neq 0$ in $\Omega_5^{\text{Spin-}Q_{16}}(\text{pt})$, as we wanted. \square

Our Adams computation in the proof of Proposition E.13 leaves open the possibility of a hidden extension in degree 5. We will show that there is no such hidden extension, so that $\Omega_5^{\text{Spin-}Q_{16}}$ is a direct sum of copies of $\mathbb{Z}/2\mathbb{Z}$; we stated but did not prove this in Section 14.3.7

as Proposition 14.106.

Proposition E.16. $\Omega_5^{\text{Spin-}Q_{16}} \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 6}$.

Proof. We let $X'_8 := (BD_8)^{V \oplus \text{Det}(V) - 3}$ for readability. The multiplication-by-2 map $2: ko_k(X) \rightarrow ko_k(X)$ factors through ku -theory; specifically, as we discussed in Remark 10.40, $2 = R \circ b \circ c$. We will show that $b \circ c: ko_5(X'_8) \rightarrow ku_7(X'_8)$ vanishes, which implies multiplication by 2 vanishes on $ko_5(X'_8)$. To do this, we need to study $ku_*(X'_8)$ with the Adams spectral sequence.

The change-of-rings trick that enables one to work over $\mathcal{A}(1)$ for ko -theory has an analogue for ku -theory. The catalyst is Adams' calculation [240] that as an \mathcal{A} -module, $H^*(ku; \mathbb{Z}/2\mathbb{Z}) \cong \mathcal{A} \otimes_{\mathcal{E}(1)} \mathbb{Z}/2\mathbb{Z}$, where $\mathcal{E}(1)$ is the algebra generated by $Q_0 := \text{Sq}^1$ and $Q_1 := \text{Sq}^2 \text{Sq}^1 + \text{Sq}^1 \text{Sq}^2$. Using the change-of-rings theorem [115, Section 4.5], the Adams E_2 -page for $ku_*(X)$ simplifies:

$$\begin{aligned} E_2^{s,t} &= \text{Ext}_{\mathcal{A}}^{s,t}(H^*(ku \wedge X; \mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \\ &\cong \text{Ext}_{\mathcal{A}}^{s,t}(H^*(ku; \mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}/2} H^*(X; \mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \\ &\cong \text{Ext}_{\mathcal{A}}^{s,t}(\mathcal{A} \otimes_{\mathcal{E}(1)} \mathbb{Z}/2 \otimes_{\mathbb{Z}/2} H^*(X; \mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \\ &\cong \text{Ext}_{\mathcal{E}(1)}^{s,t}(H^*(X; \mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}). \end{aligned} \tag{E.17}$$

As $\mathcal{E}(1)$ is an exterior algebra on two generators, this reformulation of the E_2 -page is much easier to calculate. This technique also appears in [232, 130].

Lemma E.18. *There are isomorphisms of $\mathcal{E}(1)$ -modules*

1. $\textcolor{teal}{R}_2 \cong \mathcal{O} \oplus \Sigma \mathcal{E}(1)$,
2. $\textcolor{violet}{J} \cong \mathcal{E}(1) \oplus \Sigma^2 \mathbb{Z}/2\mathbb{Z}$, and
3. $\mathcal{A}(1) \cong \mathcal{E}(1) \oplus \Sigma^2 \mathcal{E}(1)$.

One can prove this by directly checking how Q_0 and Q_1 act on these $\mathcal{A}(1)$ -modules, though this also appears in [130, Section 4.4.3] (specifically in the proof of Theorem 4.53). Using this, there is an isomorphism of $\mathcal{E}(1)$ -modules

$$\begin{aligned} H^*(X'_8; \mathbb{Z}/2\mathbb{Z}) &\cong \textcolor{red}{O} \oplus \textcolor{green}{\Sigma^3 \mathbb{Z}/2\mathbb{Z}} \oplus \textcolor{blue}{\Sigma^4 \mathcal{O}} \oplus \Sigma \mathcal{E}(1) \oplus \Sigma \mathcal{E}(1) \oplus \Sigma^3 \mathcal{E}(1) \oplus \Sigma^3 \mathcal{E}(1) \oplus \\ &\quad \Sigma^5 \mathcal{E}(1) \oplus \Sigma^5 \mathcal{E}(1) \oplus \textcolor{teal}{\Sigma^5 \mathcal{E}(1)} \oplus \textcolor{violet}{\Sigma^5 \mathcal{E}(1)} \oplus P', \end{aligned} \tag{E.19}$$

where P' is concentrated in degrees 7 and above (hence is irrelevant for our computations). We draw this decomposition in Figure 34: straight lines denote Q_0 -actions and dashed curved lines denote Q_1 -actions.

To determine the E_2 -page of the Adams spectral sequence, we need to compute $\text{Ext}_{\mathcal{E}(1)}$ of the summands in (E.19) as modules over $\text{Ext}_{\mathcal{E}(1)}(\mathbb{Z}/2\mathbb{Z})$; the formal structure of the argument

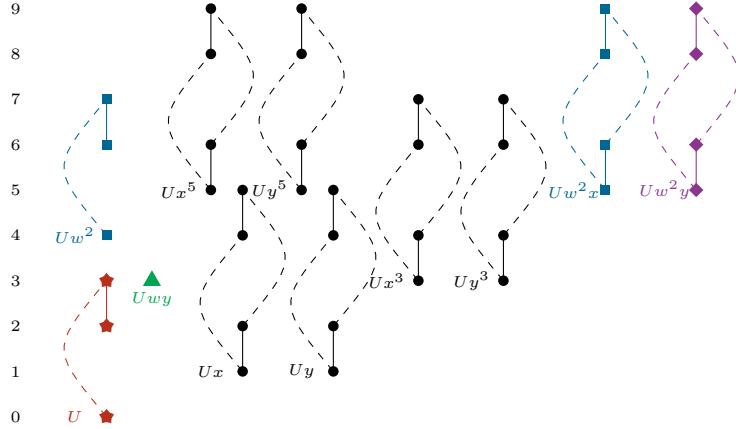


Figure 34: The $\mathcal{E}(1)$ -module structure on $H^*(X'_8; \mathbb{Z}/2\mathbb{Z})$ in low degrees. The pictured submodule contains all elements in degrees 6 and below. See the proof of Proposition E.16 for more information.

is the same as over $\mathcal{A}(1)$, but the specific computations are different, leading to ku -homology rather than ko -homology.

So, there is an isomorphism

$$\mathrm{Ext}_{\mathcal{E}(1)}^{s,t}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[h_0, v_1] \quad (\text{E.20})$$

with $|h_0| = (1, 1)$ and $|v_1| = (1, 3)$ [115, Example 4.5.6], and this algebra acts canonically on $\mathrm{Ext}_{\mathcal{E}(1)}^{s,t}(H^*(X; \mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z})$, providing information about the ku_* -action on $ku_*(X)$. Specifically, h_0 -actions on the E_∞ -page lift to multiplication by 2, and v_1 -actions on the E_∞ -page lift to an action by the Bott element $b \in ku_2$. Just as in the $\mathcal{A}(1)$ case, there can be ‘‘hidden extensions,’’ where 2 or b act non-trivially on $ku_*(X)$, but in a way not detected by h_0 or v_1 acting on the E_∞ -page.

We want to know the Ext groups of \mathcal{O} and $\mathcal{E}(1)$ as modules over this algebra. These are known: $\mathrm{Ext}_{\mathcal{E}(1)}^{s,t}(\mathcal{O}, \mathbb{Z}/2\mathbb{Z})$ is in [153, Section 3], and $\mathrm{Ext}_{\mathcal{E}(1)}^{s,t}(\mathcal{E}(1), \mathbb{Z}/2\mathbb{Z})$ consists of a single $\mathbb{Z}/2\mathbb{Z}$ in bidegree $(0, 0)$, since $\mathcal{E}(1)$ is a free $\mathcal{E}(1)$ -module. Now we can draw the E_2 -page of the Adams spectral sequence for $ku_*(X'_8)$, which we do in Figure 35, left. Like over $\mathcal{A}(1)$, h_0 -actions are written with vertical lines; v_1 -actions are drawn with lighter diagonal lines.

There is potential for differentials, and indeed they are present. All of them are either zeroed out by Margolis’ theorem or are determined by the differentials from the 4-line to the 3-line using that the $\mathrm{Ext}_{\mathcal{E}(1)}(\mathbb{Z}/2\mathbb{Z})$ -action commutes with differentials. And there is a differential between the h_0 -towers on the 4- and 3-lines, which we deduce from the corresponding differential in the ko -homology Adams spectral sequence (Figure 33, right), or alternatively by comparing with the Atiyah-Hirzebruch spectral sequence. Once we take this differential, there can be no others in range; we display the E_∞ -page in Figure 35, right. We see from this E_∞ -page that $ku_5((BD_8)^{V \oplus \mathrm{Det}(V)-3}) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 5}$, with a four-dimensional subspace \mathcal{S}

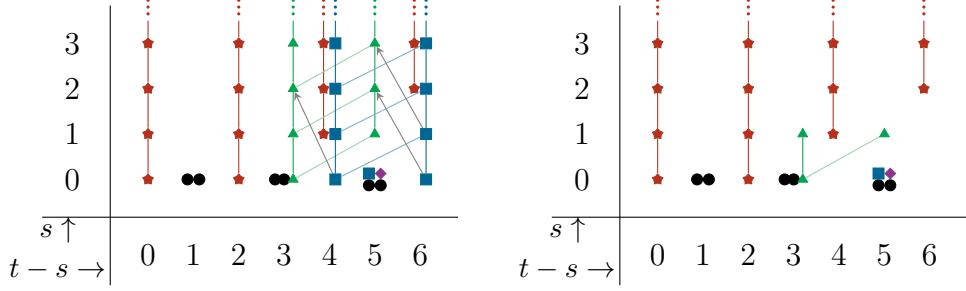


Figure 35: The Adams spectral sequence computing $ku_*(X'_8)$. Some v_1 -actions are hidden for readability. Left: the E_2 -page. Right: the $E_3 = E_\infty$ -page. Margolis' theorem lifts $E_\infty^{0,5} \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 4}$ to a $(\mathbb{Z}/2\mathbb{Z})^{\oplus 4}$ subspace \mathcal{S} of $ku_5(X'_8)$.

coming from the $(\Sigma^5 \mathcal{E}(1))^{\oplus 4}$ summand in $H^*((BD_8)^{V \oplus \text{Det}(V)-3}; \mathbb{Z}/2\mathbb{Z})$ and Margolis' theorem.

Lemma E.21. \mathcal{S} is precisely the kernel of

$$R: ku_5((BD_8)^{V \oplus \text{Det}(V)-3}) \longrightarrow ko_3((BD_8)^{V \oplus \text{Det}(V)-3}). \quad (\text{E.22})$$

Proof. Recall $ko_3(X'_8) \cong \mathbb{Z}/4\mathbb{Z}$, and let x be a generator. Then $\eta(kx) \neq 0$ iff k is odd, which we discovered in the proof of Proposition E.13 (specifically see (E.15)). Recall from the proof of Lemma 13.22 that the maps η , c , and R fit together into a long exact sequence; thus if $R(z) \neq 0$, $R(z) \notin \text{Im}(\eta)$, so $R(z) = 2x$.

The connective cover map $ku \rightarrow KU$ induces a map $i_c: ku_*(X) \rightarrow KU_*(X)$ which is the localization of the ku_* -module $ku_*(X)$ away from the ideal generated by the Bott element b . That is, for every $x \in ku_*(X)$, if $b^m x = 0$ for some m , $i_c(x) = 0$; if $b^m x$ is never zero, then $i_c(x) \neq 0$, and $i_c(x)$ is divisible by arbitrarily high powers of b . In the former case, x is called *b-torsion*; in the latter case, x is called *b-periodic*. An analogous statement is true with $i_r: ko_*(X) \rightarrow KO_*(X)$ in place of i_c ; the Bott element w in ko_* is the image of the Bott manifold B under the Atiyah-Bott-Shapiro map $MTSpin \rightarrow ko$.

The class $2x \in ko_3(X'_8)$ is w -periodic. One way to see this is to lift to its preimage in Spin bordism, which is represented by $2S^3/Q_{16}$, as we showed in the proof of Proposition E.13. This manifold is detected by some kind of twisted η -invariant. w -periodicity of this class is equivalent to $B^m \times (2S^3/Q_{16})$ being nonbounding for all m , which one can prove by computing the same η -invariant on that class and using the factorization $\eta(M \times N) = \text{Index}(M)\eta(N)$ when m is 4ℓ -dimensional. Thus $i_r(2x) \neq 0$. If $z \in \mathcal{S}$, $bz = 0$ by Margolis' theorem,⁷⁶ so

⁷⁶There is a subtlety here that we want to be careful about. Margolis proved that free \mathcal{A} -module summands in $H^*(X; \mathbb{Z}/2\mathbb{Z})$ lift to split copies of $H\mathbb{Z}/2\mathbb{Z}$ off of X as a spectrum, which in particular prevents non-trivial differentials and hidden extensions involving those $H\mathbb{Z}/2\mathbb{Z}$ factors in the Adams spectral sequence. In particular, $\pi_*(\mathbb{S})_2^\wedge$ acts trivially on the copies of $\mathbb{Z}/2\mathbb{Z}$ in $\pi_*(X)$ coming from these $H\mathbb{Z}/2\mathbb{Z}$ summands. Using the change-of-rings theorem, one learns that free $\mathcal{E}(1)$ -module summands in $H^*(X; \mathbb{Z}/2\mathbb{Z})$ correspond to $H\mathbb{Z}/2\mathbb{Z}$ summands in $ku \wedge X$, but a priori we do not know that this splitting is as ku -modules, only as spectra; in principle $b \in ku_2$ could act non-trivially.

$i_c(z) = 0$.

Suppose $z \in \mathcal{S}$ is such that $R(z) \neq 0$; then $R(z) = 2x$. The map R is the connective cover of the realification map $R: KU \simeq \Sigma^2 KU \rightarrow KO$, implying there is a commutative diagram

$$\begin{array}{ccc} ku_5(X'_8) & \xrightarrow{R} & ko_3(X'_8) \\ i_c \downarrow & & i_r \downarrow \\ KU_5(X'_8) & \xrightarrow{R} & KO_3(X'_8). \end{array} \quad (\text{E.23})$$

Traveling along the upper right, $i_r(R(z)) = i_r(2x) \neq 0$, but along the lower left, $R(i_c(z)) = 0$, which is a contradiction, so $\mathcal{S} \subset \ker(R)$. To turn this into an equality, use the η, c, R long exact sequence to show $\ker(R)$ is a four-dimensional $\mathbb{Z}/2\mathbb{Z}$ -vector space. \square

By exactness of the η, c, R sequence, we conclude $\text{Im}(c: ko_5(X'_8) \rightarrow ku_5(X'_8)) = \mathcal{S}$. Finally, Margolis' theorem (or a slight strengthening, as we discussed in Footnote 76) implies b acts trivially on classes in $ku_*(X'_8)$ coming from free $\mathcal{E}(1)$ -module summands in cohomology. Therefore $b \circ c = 0$ on $ko_5(X'_8)$ as promised, so multiplication by 2 is the zero map on $ko_5(X'_8)$. \square

This proof, elaborate as it was, is the simplest argument we found. We would be interested in learning of a simpler way to address this extension question.

Fortunately, though, this problem does not occur: b acts trivially on the $\mathbb{Z}/2\mathbb{Z}$ summands in $ku_*(X)$ coming from free $\mathcal{E}(1)$ -submodules of $H^*(X; \mathbb{Z}/2\mathbb{Z})$. A reference for this is Bruner-Greenlees [234, Section 2.1].

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