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THE GROMOV-LAWSON-ROSENBERG CONJECTURE: THE TWISTED CASE

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ABSTRACT. Let M be a compact smooth orientable manifold with cyclic fundamental group of dimension $m \geq 5$ whose universal cover is spin. We prove that M admits a metric of positive scalar curvature if and only if the A-roof genus of M vanishes; this proves the Gromov-Lawson-Rosenberg conjecture in this special case. We also show that the eta invariant completely detects certain connective K theory groups.

1. INTRODUCTION

Let M be a smooth closed connected orientable manifold. If M is a spin manifold, let $D(M,s)$ be the Dirac operator defined by a spin structure s on M; D is a self-adjoint elliptic partial differential operator on M. If M is simply connected, the spin structure on M is unique but in general there can be inequivalent spin structures on M. If $m \equiv 0 \bmod 4$, we may decompose D into the chiral Dirac operators $D(M,s) = D^+(M,s) \oplus D^-(M,s)$. We define

$$\hat{A}(M,s) := \dim\ker(D^+(M,s)) - \dim\ker(D^-(M,s)) \in \mathbb{Z}$$

as the index of the spin complex; $\hat{A}(M,s) = \hat{A}(M)$ is independent of the particular spin structure chosen. If $m \equiv 1 \bmod 8$, let $\hat{A}(M,s) \in \mathbb{Z}_2$ be the mod 2 reduction of $\dim(\ker(D(M,s)))$. If $m \equiv 2 \bmod 8$, then $\dim(\ker(D(M,s)))$ is even and we let $\hat{A}(M,s) \in \mathbb{Z}_2$ be the mod 2 reduction of $\dim(\ker(D(M,s)))/2$. We set $\hat{A}(M,s) = 0$ in the remaining dimensions.

If M admits a metric of positive scalar curvature, the Bochner-Lichnerowicz [13] formula shows that there are no harmonic spinors so $\hat{A}(M,s) = 0$. Thus if

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$\hat{A}(M,s) \neq 0$, then M does not admit a metric of positive scalar curvature. We will show that the $\hat{A}$ genus provides the only obstruction to the existence of a metric of positive scalar curvature on M provided that $m \geq 5$, that the fundamental group of M is cyclic, that M is orientable, and that the universal cover of M is spin. More precisely:

Theorem 1.1. *Let M be a closed connected orientable manifold of dimension $m \geq 5$ with cyclic fundamental group $\mathbb{Z}_\ell$ whose universal cover $\tilde{M}$ is a spin manifold.*

1) If M is spin, then M admits a metric of positive scalar curvature if and only if $\hat{A}(M,s) = 0$ for every spin structure s on M.

2) If M is not spin, then M admits a metric of positive scalar curvature if and only if $\hat{A}(\tilde{M}) = 0$.

If M is simply connected, then Theorem 1.1 was proved by Stolz [19]. The case ℓ odd follows from results of Kwasik and Schultz [11]; see also Botvinnik and Gilkey [5] for a different proof. We can work one prime at a time to prove Theorem 1.1 so we shall assume $\ell = 2^q$. The case $\ell = 2$ was handled by Rosenberg and Stolz [17] so we shall assume $\ell \geq 4$ henceforth. The case (1) where M is spin is joint work with S. Stolz [6]; the case (2) where M is not spin was not discussed in [6] and is one of the new results of this paper. In this paper we give a unified treatment of both cases using a combination of analysis and topology. After the work reported in Theorem 1.1 was concluded, we were made aware by S. Stolz that work of Kwasik and Schultz [12] could be used to give a different proof of Theorem 1.1 (2) using purely topological methods.

Although Theorem 1.1 is a result in differential geometry, the proof crosses over into several different fields. It uses a surprising amount of algebraic topology: equivariant bordism and connective K theory which we will discuss in §1. It also uses the machinery of elliptic operator theory: the eta invariant as we shall discuss in §2; this analytic tool plays a crucial role in the development. In §3, we will complete the proof of Theorem 1.1. We will also show in Theorem 4.5 that the eta invariant and the $\hat{A}$ genus completely detect the connective K theory groups $ko_m(B\mathbb{Z}_\ell)$ and the twisted connective K theory groups $ko_m(B\mathbb{Z}_\ell;\xi_1)$; this new result is of interest in its own right since it permits an analytic computation of these groups. In §4, we provide certain computations of a homotopy theoretical nature used in §1.

Throughout our work in this area, we have been in contact with S. Stolz and it is a pleasure to acknowledge many helpful conversations with him.

2. Equivariant bordism and connective K theory

We need to define twisted spinors to handle the case in which M is not spin but in which the universal cover of M is spin. Let X be a topological space and let ξ be a real vector bundle over X. Let $T(M)$ be the tangent bundle of a manifold M. The twisted bordism group $\Omega_m(X;\xi)$ consists of bordism classes of triples (M, s, f) where M is a closed manifold of dimension m, where f is a continuous map from M to X, and where s is a spin structure on the bundle $T(M) \oplus f^*(\xi)$. Note that a spin structure on ξ gives a canonical isomorphism between the ordinary spin bordism groups $\Omega_m(X)$ and the twisted spin bordism groups $\Omega_m(X;\xi)$.

We shall be interested in the special case in which $X = B\mathbb{Z}_\ell$ is the classifying space of the cyclic group $\mathbb{Z}_\ell := \{\lambda \in \mathbb{C} : \lambda^\ell = 1\}$ for $\ell = 2^q \geq 4$. A $\mathbb{Z}_\ell$ structure on a manifold M is a map $f : M \to B\mathbb{Z}_\ell$ or equivalently a representation of the fundamental group $\pi_1(M)$ to $\mathbb{Z}_\ell$. If $\pi_1(M) = \mathbb{Z}_\ell$, M has a canonical $\mathbb{Z}_\ell$ structure $f : M \to B\mathbb{Z}_\ell$ with f_* an isomorphism on the fundamental group. Let $w_i(\cdot)$ be the Stiefel-Whitney classes. Let x generate $H^1(B\mathbb{Z}_\ell;\mathbb{Z}_2) = \mathbb{Z}_2$ and y generate $H^2(B\mathbb{Z}_\ell;\mathbb{Z}_2) = \mathbb{Z}_2$. Only the first two Stiefel-Whitney classes of the real vector bundle ξ over $B\mathbb{Z}_\ell$ are relevant. We define bundles ξ_i by requiring:

$$w_1(\xi_0) = 0,\ w_2(\xi_0) = 0,\ w_1(\xi_1) = 0,\ w_2(\xi_1) = y,$$
$$w_1(\xi_2) = x,\ w_2(\xi_2) = 0,\ w_1(\xi_3) = x,\ w_2(\xi_3) = y.$$

We will use $\Omega_m(B\mathbb{Z}_\ell) = \Omega_m(B\mathbb{Z}_\ell;\xi_0)$ to prove Theorem 1.1 (1), and we will use $\Omega_m(B\mathbb{Z}_\ell;\xi_1)$ to prove Theorem 1.1 (2).

The following result provides the link between differential geometry and algebraic topology which we shall be exploiting. We will use it to reduce the question of constructing a metric of positive scalar curvature on M to a question regarding the bordism class $[M]$. We refer to [9, 18] for the proof of the first assertion and to [15, 16] for the proof of the second assertion.

Theorem 2.1.

1) *Let M be a compact manifold which is not necessarily connected but which admits a Riemannian metric of positive scalar curvature. Then any manifold which can be obtained from M by performing surgeries in codimension at least 3 also admits a metric with positive scalar curvature.*

2) *Let M be a connected manifold of dimension $m \geq 5$ with cyclic fundamental group $\mathbb{Z}_\ell$. Suppose that there exists a manifold M_1 which admits a metric of*

positive scalar curvature so that $[(M,s,f)] = [(M_1,s_1,f_1)]$ *in* $\Omega_m(B\mathbb{Z}_\ell;\xi)$. *Then* M *admits a metric of positive scalar curvature.*

Let M be a connected compact oriented manifold of dimension $m \geq 5$ with cyclic fundamental group $\mathbb{Z}_\ell$ where $\ell = 2^q \geq 4$. Assume the universal cover $\tilde{M}$ of M is spin. Suppose first M is spin. Then there are two inequivalent spin structures s_i on M since $H^1(M;\mathbb{Z}_2) = \mathbb{Z}_2$;

$$[(M,s_i,f)] \in \Omega_m(B\mathbb{Z}_\ell;\xi_0) = \Omega_m(B\mathbb{Z}_\ell) \text{ for } i = 1,2.$$

If $m \equiv 1,2 \bmod 8$, we let $\hat{A}(M) = \hat{A}(M,s_1) \oplus \hat{A}(M,s_2) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$. If $m \equiv 0 \bmod 4$, we let $\hat{A}(M) = \hat{A}(M,s_1) = \hat{A}(M,s_2) \in \mathbb{Z}$; this is independent of the spin structure chosen. We let $\hat{A}(M) = 0$ otherwise. Suppose next that M is not spin; let $\tilde{M}$ be the ℓ – fold cover. We define $\hat{A}(M) = \hat{A}(\tilde{M})$ if $m \equiv 0 \bmod 4$ and $\hat{A}(M) = 0$ otherwise. Since $w_1(M) = 0$ and $w_2(M) = f^*(y)$, $w_1(T(M) \oplus f^*(\xi_1)) = 0$ and $w_2(T(M) \oplus f^*(\xi_1)) = 0$ so $T(M) \oplus f^*(\xi_1)$ admits two natural spin structures s_i and $[(M,s_i,f)] \in \Omega_m(B\mathbb{Z}_\ell;\xi_1)$. Inequivalent spin structures are parametrized by representations of $\pi_1(M)$ to $\mathbb{Z}_\ell$; s_2 is s_1 twisted by the non-trivial representation of $\mathbb{Z}_\ell$ to $\mathbb{R}$.

Let $\Omega_m^+(B\mathbb{Z}_\ell;\xi)$ be the subgroup of $\Omega_m(B\mathbb{Z}_\ell;\xi)$ which is generated by classes $[(M,s,f)]$ where M admits a metric of positive scalar curvature. It is an immediate consequence of the index theorem that the $\hat{A}$ genus extends to the untwisted bordism groups $\Omega_m(B\mathbb{Z}_\ell)$ with values in $\mathbb{Z}$ if $m \equiv 0 \bmod 4$ and in $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ if $m \equiv 1,\ 2 \bmod 8$. Similarly, if $m \equiv 0 \bmod 4$, the map $(M,s,f) \to \hat{A}(\tilde{M},\tilde{s}) \in \mathbb{Z}$ extends to a map from $\Omega_m(B\mathbb{Z}_\ell;\xi_1)$ to $\mathbb{Z}$; we set $\hat{A} = 0$ otherwise. The following reformulation of Theorem 1.1 is an immediate consequence of Theorem 2.1:

Lemma 2.2. *Let* $m \geq 5$ *and let* $\ell = 2^q \geq 4$.

1) *If* $\Omega_m^+(B\mathbb{Z}_\ell;\xi_0) = \ker(\hat{A}) \cap \Omega_m(B\mathbb{Z}_\ell;\xi_0)$, *then Theorem 1.1 (1) holds.*

2) *If* $\Omega_m^+(B\mathbb{Z}_\ell;\xi_1) = \ker(\hat{A}) \cap \Omega_m(B\mathbb{Z}_\ell;\xi_1)$, *then Theorem 1.1 (2) holds.*

The twisted spin bordism groups are too large for our purposes and it is convenient to introduce smaller groups to work with. Let $\mathbb{H}P^2$ be the quaternionic projective plane with the usual homogeneous metric of positive scalar curvature. Let $p: E \to B$ be a geometrical fiber bundle with fiber $\mathbb{H}P^2$ where the transition functions lie in the group of isometries $PSp(3)$ of $\mathbb{H}P^2$. Since $\mathbb{H}P^2$ is simply connected, p_* is an isomorphism of fundamental groups so a $\mathbb{Z}_\ell$ structure on E is equivalent to a $\mathbb{Z}_\ell$ structure on B. Let $T_m(B\mathbb{Z}_\ell;\xi) \subseteq \Omega_m(B\mathbb{Z}_\ell;\xi)$ be the subgroup generated by bordism classes (E,s,f_E) where $f_B : B \to B\mathbb{Z}_\ell$ and $f_E = f_B \circ p$. Let $\mathrm{Thom}(\xi)$ be the Thom space of the k dimensional real vector bundle ξ over

$B\mathbb{Z}_\ell$. The groups $\Omega_m(B\mathbb{Z}_\ell)$ and $\tilde{\Omega}_{m+k}(\mathrm{Thom}(\xi))$ are isomorphic via the natural Thom-Pontrjagin isomorphism. We define the *twisted connective K theory* groups by $ko_m(B\mathbb{Z}_\ell;\xi) := \tilde{k}o_{m+k}(\mathrm{Thom}(\xi))$. The following result of Stolz 20 is fundamental to our study:

Theorem 2.3. *There is a 2-local isomorphism between the group $ko_m(B\mathbb{Z}_\ell;\xi)_{(2)}$ and the group $\{\Omega_m(B\mathbb{Z}_\ell;\xi)/T_m(B\mathbb{Z}_\ell;\xi)\}_{(2)}$.*

Remark. We have restricted to the case $\ell = 2^q \geq 4$. The bordism groups in question will be 2-primary so it is not necessary to localize at the prime 2.

We will show in Lemma 3.2 that if $p : E \to B$ is a geometric $\mathbb{H}P^2$ fibration, then E admits a metric of positive scalar curvature. Consequently $T_m(B\mathbb{Z}_\ell;\xi) \subseteq \Omega^+_m(B\mathbb{Z}_\ell;\xi)$ and we may define $ko^+_m(B\mathbb{Z}_\ell;\xi)$ to be the image of $\Omega^+_m(B\mathbb{Z}_\ell;\xi)$ in $ko_m(B\mathbb{Z}_\ell;\xi) = \Omega_m(B\mathbb{Z}_\ell;\xi)/T_m(B\mathbb{Z}_\ell;\xi)$. Since the $\hat{A}$ genus vanishes on $T_m(B\mathbb{Z}_\ell;\xi)$, it extends naturally to $ko_m(B\mathbb{Z}_\ell;\xi)$. The following reformulation of Theorem 1.1 is now immediate:

Lemma 2.4. *Let $m \geq 5$ and let $\ell = 2^q \geq 4$.*

1) *If $ko^+_m(B\mathbb{Z}_\ell;\xi_0) = \ker(\hat{A}) \cap ko_m(B\mathbb{Z}_\ell;\xi_0)$, then Theorem 1.1 (1) holds.*

2) *If $ko^+_m(B\mathbb{Z}_\ell;\xi_1) = \ker(\hat{A}) \cap ko_m(B\mathbb{Z}_\ell;\xi_1)$, then Theorem 1.1 (2) holds.*

These connective K theory groups are much smaller and easier to deal with than are the corresponding bordism groups and therefore Lemma 2.4 is a significant improvement of Lemma 2.2. But what is crucial for our purposes is that their orders can be computed using the Adams spectral sequence. Let

$$\mathcal{E}(m;\xi_0) := |\ker(\hat{A}) \cap ko_m(B\mathbb{Z}_\ell;\xi_0)|,\ \mathcal{E}(m;\xi_1) := |\ker(\hat{A}) \cap ko_m(B\mathbb{Z}_\ell;\xi_1)|,$$

$$\mathcal{E}(m;\xi_2) := |ko_m(B\mathbb{Z}_\ell;\xi_2)|,\text{ and } \mathcal{E}(m;\xi_3) := |ko_m(B\mathbb{Z}_\ell;\xi_3)|.$$

We will postpone the proof of the following Theorem until §4.

Theorem 2.5. *If $\ell = 2^q \geq 4$, then*

	$\mathcal{E}(m,\xi_0)$	$\mathcal{E}(m,\xi_1)$	$\mathcal{E}(m,\xi_2)$	$\mathcal{E}(m,\xi_3)$
$m=8k$	1	1	2^{2k+1}	2^{2k+1}
$m=8k+1$	$(\ell/2)^{2k+1}$	$(2\ell)^{2k+1}$	2	1
$m=8k+2$	1	1	2^{2k+3}	2^{2k+1}
$m=8k+3$	$2(2\ell)^{2k+1}$	$(\ell/2)^{2k+1}$	2	1
$m=8k+4$	1	1	2^{2k+2}	2^{2k+2}
$m=8k+5$	$(\ell/2)^{2k+2}$	$(2\ell)^{2k+2}$	1	1
$m=8k+6$	1	1	2^{2k+2}	2^{2k+2}
$m=8k+7$	$(2\ell)^{2k+2}$	$(\ell/2)^{2k+2}$	1	1

3. The eta invariant

We refer to [8] for details concerning the analytic facts used in this section. Let M be a compact manifold without boundary. Let P be an elliptic self-adjoint first order differential operator. Let $\eta(z,P) := \frac{1}{2}\{Tr_{L^2}(P(P^2)^{-(z+1)/2})\}$; this is holomorphic for $Re(z) > 0$ and has a meromorphic extension to $\mathbb{C}$ which is regular at $z=0$. We define $\eta(P) := \{\eta(z,P) + \dim(\ker(P))\}|_{z=0} \in \mathbb{R}$ as a measure of the spectral asymmetry of P; this is the eta invariant of Atiyah, Patodi, and Singer [2]. Let π be a finite group. If $f : M \to B\pi$ gives M a π structure and if ρ is a representation of π, let $\rho(M,f)$ be the associated flat vector bundle over M. Let P_ρ be the operator P twisted by this flat bundle and let $\eta(P,\rho) = \eta(P_\rho)$ be the associated eta invariant.

The spinor group $Spin(\cdot)$ is the double cover of the special orthogonal group $SO(\cdot)$ and can be described in terms of Clifford algebras; see [1] for details. A Riemannian manifold M is orientable if we can restrict the structure group of the tangent bundle $T(M)$ from $O(m)$ to $SO(m)$; M is spin if we can lift the structure group from $SO(m)$ to $Spin(m)$; this should be thought of as a secondary orientation. We use this lift to define the Dirac operator P on the bundle of spinors over any spin manifold; P is a self-adjoint elliptic first order partial differential operator.

Let $Spin^c(\cdot) := Spin(\cdot) \times S^1/(\theta,\lambda) \sim (-\theta,-\lambda)$ be the complexification of the spinor group. The map $(\theta,\lambda) \to \lambda^2$ defines the determinant representation $\det : Spin^c \to S^1$; see Hitchin [10] for details. Let π be a finite group and let ξ be a real vector bundle over $B\pi$ which admits a spin^c structure with associated flat determinant line bundle; this is the case for $\pi = \mathbb{Z}_\ell$ and $\xi = \xi_i$. If s is a spin structure on $T(M) \oplus f^*(\xi)$, then s induces a spin^c structure on M such that the associated determinant line bundle is flat. Let P be the Dirac operator defined by this structure and let $\eta(M,g,s,f)(\rho)=\eta(P_\rho)\in\mathbb{R}$ be the associated eta invariant. We will often suppress the role of (g,s,f) in the interests of notational simplicity. Since the map $\rho\to\eta(\cdot)(\rho)$ is additive in ρ, it extends to the group representation ring $R(\pi)$. Let $R_0(\pi)$ denote the augmentation ideal of virtual representations of virtual dimension 0.

Theorem 3.1. *Let ξ be a real vector bundle over the classifying space of a finite group π which admits a spin^c structure with flat determinant line bundle. Let $\rho\in R(\pi)$. Let m be odd. If $m\equiv 3$ mod 4, assume* $\dim(\rho)=0$.

1) *The map $(M,s,f)\to\eta(M,g,s,f)$ extends to homomorphisms η_ρ where we have $\eta_\rho:\Omega_m(B\pi;\xi)\to\mathbb{R}/\mathbb{Z}$ and $\eta_\rho: ko_m(B\pi;\xi)_{(2)}\to(\mathbb{R}/\mathbb{Z})_{(2)}$.*

2) *Assume that ξ admits a spin structure. If $m\equiv 3$ mod 8 and if ρ is of real type, or if $m\equiv 7$ mod 8 and if ρ is of quaternion type, then we have $\eta_\rho:\Omega_m(B\pi)\to\mathbb{R}/2\mathbb{Z}$ and $\eta_\rho: ko_m(B\pi)_{(2)}\to(\mathbb{R}/2\mathbb{Z})_{(2)}$.*

Remark. If $\pi=\mathbb{Z}_\ell$ for $\ell=2^q\geq 4$, it is not necessary to localize at the prime 2.

PROOF. We first show η extends to bordism by modifying an argument of Atiyah, Patodi, and Singer [2, II Theorem 3.3]. Suppose that $(M,s,f)=\partial(W,s_W,f_W)$. This means that W is a compact Riemannian manifold of dimension $m+1$ with smooth boundary M and that the structures on M extend over W. Let D_ρ be the operator of the spin^c complex on W with coefficients in $\rho(W,f_W)$. Since the determinant line bundle of the spin^c structure on W is flat, the index theorem for manifolds with boundary yields $\text{index}(D_\rho)=\dim(\rho)\int_W\hat{A}(W)-\eta(M,g,s,f)(\rho)$. If $m\equiv 3$ mod 4, then $\dim(\rho)=0$; if $m\equiv 1$ mod 4, then $\hat{A}(W)=0$. Thus $\eta(M,g,s,f)(\rho)=-\text{index}(D_\rho)\in\mathbb{Z}$ which proves the first assertion.

The extra factor of 2 in (2) comes from a quaternion structure. We use Atiyah, Bott, and Shapiro [1]. Let $C\ell^{\pm}(m)$ be the Clifford algebra on $\mathbb{R}^m$ generated by the relation $v*w+w*v=\pm(v,w)1$. We identify the Lie groups $Pin^{\pm}(m)$ with the subgroup of $C\ell^{\pm}(m)$ generated by the unit sphere of $\mathbb{R}^m$. We identify the Lie group $Spin(m)$ with the subgroup of $Pin^-(m)$ generated by products of an even number of elements of the unit sphere. Let $\nu=16^k$. Recall that $C\ell^+(8k)=M_\nu(\mathbb{R})$ and that $C\ell^+(8k+4)=M_{2\nu}(\mathbb{H})$. If $m=8k+3$, then the group $Spin(m+1)$ has a quaternion representation which is isomorphic to two copies of the fundamental spin representation so the spin complex with coefficients in a real representation admits a quaternion structure and the index of D_ρ is even. Similarly if $m=8k+7$, then the group $Spin(m+1)$ has a real representation whose complexification is isomorphic to the fundamental spin representation and the spin complex with coefficients in a quaternion representation ρ admits a quaternion structure and the index of D_ρ is even. This proves the assertions concerning bordism.

To show that η extends to connective K theory, it suffices to show that η vanishes on geometric fiber bundles $\mathbb{H}P^2 \to E \to B$. If ξ admits a spin structure, this follows from the Adiabatic limit theorem of Bismut and Cheeger [4]. Rather than reprove the Adiabatic limit theorem in the twisted case, we give a direct proof. The remaining assertions of Theorem 3.1 will follow from Theorem 2.3 and from the following Lemma. □

Lemma 3.2. *Let $p : E \to B$ be a geometrical $\mathbb{H}P^2$ fiber bundle. Let π be a finite group and let ξ be a real vector bundle over $B\pi$ which admits a spin^c structure with flat associated determinant line bundle. Let $\rho \in R(\pi)$; if $m \equiv 3$ mod 4, assume* $\dim(\rho) = 0$. *Let $f_B : B \to B\pi$ give B a π structure and let $f_E = f_B \circ p$. Let s_E be a spin structure on $T(E) \oplus f_E^*\xi$. Then there exists a metric g_E on E with positive scalar curvature so that $\eta(E, g_E, s_E, f_E)(\rho) = 0$ in $\mathbb{R}$.*

PROOF. Let g_F be the standard metric of positive scalar curvature on the fiber $\mathbb{H}P^2$ and let g_B be any Riemannian metric on the base B. Let F_x be the fiber of E over $x \in B$. There exists a metric g_E on the total space E so that the induced metric on each F_x is g_F, so that each F_x is totally geodesic, and so that $p : E \to B$ is a Riemannian submersion; see Besse [3, 9.59] for details. Let $\mathcal{V}$ and $\mathcal{H}$ be the vertical and horizontal distributions of the submersion. We define the canonical variation $g_E(t)$ of the metric by imposing the three conditions: $g_E(t)|_\mathcal{V} = tg_F$, $g_E(t)|_\mathcal{H} = p^*(g_B)$, and $g_E(t)(\mathcal{V}, \mathcal{H}) = 0$. Let $\mathcal{K}_F$ and $\mathcal{K}_E(t)$ be the scalar curvature of the metrics on F and on E. Then $\mathcal{K}_E(t) = t^{-1}\mathcal{K}_F + O(1)$; see Besse 3, 9.70. In particular, $\mathcal{K}_E(t) \to \infty$ as $t \downarrow 0$.

Let $\rho \in R_0(\pi)$. We will show there exists $t_0(\rho)$ so that if $0 < t < t_0(\rho)$, we have $\eta(E, g_E(t), s_E, f_E)(\rho) = 0$ in $\mathbb{R}$. Let δ be the regular representation of π, let ρ_0 be the trivial representation of π, and let $\chi = |\pi|\rho_0 - \delta$. Then $Tr(\chi(1)) = 0$ and $Tr(\chi)(\lambda) = |\pi|$ for $\lambda \neq 1$. Thus if $\rho \in R_0(\pi)$, $|\pi|\rho = \chi\rho$. Therefore it suffices to show $\eta(E, g_E(t), s_E, f_E)(\rho\chi) = 0$ in $\mathbb{R}$. Let $\rho = \tilde{\rho}_1 - \tilde{\rho}_2$ where $\tilde{\rho}_i$ are actual representations of π which have the same dimension. Since the bundles $\tilde{\rho}_i(B, f_B)$ admit flat connections, these bundles are rationally trivial. Thus we may choose an integer N so $N\tilde{\rho}_i(B, f_B)$ is isomorphic to a trivial bundle for $i = 1, 2$. We replace $\tilde{\rho}_i$ by $N\tilde{\rho}_i$ to assume $\tilde{\rho}_i(B, f_B)$ is a trivial bundle of dimension ν.

Let ∇_i^B be the flat connections on the trivial bundle of dimension ν over B which are defined by the flat structures $\tilde{\rho}_i$. We define a smooth 1-parameter family of connections on this bundle by setting $\nabla^B(s) := s\nabla_1^B + (1-s)\nabla_2^B$. We pull back these structures to the total space E to define the corresponding structures over E. Since p is a Riemannian submersion for any t and since the curvature tensors $R_E(s)$ of the connections on E are the pull-back of the curvature tensors $R_B(s)$ on B, the norms of these curvature tensors can be bounded uniformly for all $(s,t) \in [0,1] \times \mathbb{R}^+$. Let P be the tangential operator of the spin^c complex. Since the associated determinant line bundle is flat by hypothesis, the curvature of the determinant line bundle is zero and we may use the "Lichnerowicz-Weitzenböch formula" to express the operator of the spin complex in the form $P(s,t) = \nabla(s,t)^*\nabla(s,t) + \mathcal{K}_E(t)/4 + \Psi(p^*R_B(s))$; the error term $\Psi(\cdot)$ depends only on the Clifford module structure of the base B and is therefore uniformly bounded in operator norm for all (s,t). Since $\mathcal{K}_E(t) \to \infty$ as $t \downarrow 0$, $\mathcal{K}_E(t)/4 + \Psi(p^*R_E(s))$ is positive if t is sufficiently small. Thus there are no harmonic twisted spinors and the eta invariant $\eta(P(s,t) \otimes \chi)$ of the operator $P(s,t)$ with coefficients in χ is a well defined real valued function which is smooth in (s,t). The variation of the eta invariant is given by a local formula. Since we are taking coefficients in a flat bundle χ of virtual dimension 0, the variation vanishes and consequently $\eta(P(s,t) \otimes \chi)$ is independent of (s,t). This shows that $\eta(E, g_E(t), s_E, f_E)(\chi\rho) = \eta(P(1,t) \otimes \chi) - \eta(P(0,t) \otimes \chi) = 0$. This completes the proof if $\dim(\rho) = 0$. If $m \equiv 1 \bmod 4$, we must show there exists t_0 so $\eta(E, g_E(t), s_E, f_E)(\rho_0) = 0$ in $\mathbb{R}$ for $0 < t < t_0$. Again, we replace ρ_0 by $|\pi|\rho_0$. Since we have already proved the Lemma for $R_0(\pi)$, we may replace $|\pi|\rho_0$ by the right regular representation δ since we have that $|\pi|\rho_0 - \delta \in \mathbb{R}_0(\pi)$. Thus it suffices to show $\eta(E, g_E(t), s_E, f_E)(\delta) = 0$ in $\mathbb{R}$ for $0 < t < t_0$. Let $\tilde{E}$ be the principal π bundle corresponding to the given π structure. Since the associated π structure on $\tilde{E}$ is trivial, the associated structure on $\tilde{E}$ is a spin structure so $\eta(E, g_E(t), s_E, f_E)(\delta) = \eta(\tilde{E}, g_{\tilde{E}}(t), s_{\tilde{E}}, f_{\tilde{E}})$. Since the metric has positive scalar curvature, there are no harmonic spinors. If $m \equiv 1 \bmod 4$, the tangential operator $\tilde{P}$ of the spin complex over $\tilde{E}$ has spectrum which is symmetric about the origin and $\eta(\tilde{E}, g_{\tilde{E}}(t), s_{\tilde{E}}, f_{\tilde{E}}) = \frac{1}{2}\dim\ker(\tilde{P}) = 0$. □

We conclude this section with a brief discussion of the non-orientable case. We replace the spinor group with the pinor group and change the parity of the dimension to sssume that m is even. If $\rho \in R(\pi)$, let $\eta(M, s, f)(\rho)$ be the eta invariant of the tangential operator of the pin^c complex with coefficients in the flat bundle defined by ρ. Theorem 3.1 and Lemma 3.2 generalize to

Theorem 3.3. *Let ξ be a real vector bundle over the classifying space of a finite group π which admits a pin^c structure with flat determinant line bundle. Let $\rho \in R(\pi)$ and let m be even.*

1) The map $(M, s, f) \to \eta(M, g, s, f)$ extends to homomorphisms η_ρ where we have $\eta_\rho : \Omega_m(B\pi; \xi) \to \mathbb{R}/\mathbb{Z}$ and $\eta_\rho : ko_m(B\pi; \xi)_{(2)} \to (\mathbb{R}/\mathbb{Z})_{(2)}$.

2) Let $p : E \to B$ be a geometrical $\mathbb{H}P^2$ fiber bundle. Let $f_B : B \to B\pi$ give B a π structure and let $f_E = f_B \circ p$. Let s_E be a spin structure on $T(E) \oplus f_E^*\xi$. Then there exists a metric g_E on E with positive scalar curvature so that $\eta(E, g_E, s_E, f_E)(\rho) = 0$ in $\mathbb{R}$.

PROOF. The proof that η_ρ extends to a homomorphism η_ρ from $\Omega_m(B\pi; \xi)$ to $\mathbb{R}/\mathbb{Z}$ is the same as that given in Theorem 3.1 for the case $m \equiv 1 \bmod 4$ if we replace the tangential operator of the spin^c complex by the tangential operator of the pin^c complex since the interior integrand vanishes identically. The proof that η_ρ vanishes on $T_m(B\pi)$ is the similar to that given in Lemma 3.2 for the case $m \equiv 1 \bmod 4$. We replace the tangential operator of the spin^c complex by the tangential operator of the pin^c complex. We note that $\tilde{E}$ is orientable and thus the tangential operator of the pin^c complex is the total operator of the spin^c complex. This shows the spectrum of the operator on $\tilde{E}$ symmetric about the origin. □

Remark. Under certain circumstances, the invariant in Theorem 3.3 (1) lifts from $\mathbb{R}/\mathbb{Z}$ to $\mathbb{R}/2\mathbb{Z}$ and from $(\mathbb{R}/\mathbb{Z})_{(2)}$ to $(\mathbb{R}/2\mathbb{Z})_{(2)}$; we omit details as the discussion is a bit technical.

4. Spherical space forms

Let $\mathbb{Z}_\ell := \{\lambda \in \mathbb{C} : \lambda^\ell = 1\}$ be the cyclic group of order $\ell = 2^q \geq 4$. We define linear representations of $\mathbb{Z}_\ell$ by setting $\rho_\mu(\lambda) = \lambda^\mu$. If $\vec{a} = (a_1, ..., a_k)$ is a collection of odd indices, let $\tau_{\vec{a}} := \rho_{a_1} \oplus ... \oplus \rho_{a_k} : \mathbb{Z}_\ell \to U(k)$; $\tau_{\vec{a}}$ is fixed point free and we let $L^{2k-1}(\ell; \vec{a}) := S^{2k-1}/\tau_{\vec{a}}(\mathbb{Z}_\ell)$ be the resulting quotient manifold. Let $H^{\otimes 2}$ be the tensor square of the Hopf line bundle over the sphere S^2 and let ϵ be the trivial complex line bundle. Let $\mathcal{S}(\mathcal{W}_k)$ be the sphere bundle of the complex vector bundle $\mathcal{W}_k := H^{\otimes 2} \oplus (k-1)\epsilon$ over S^2. We extend $\tau_{\vec{a}}$ to act without fixed points on the fiber spheres of $\mathcal{S}(\mathcal{W}_k)$ and define the lens space bundle $X^{2k+1}(\ell; \vec{a}) := \mathcal{S}(\mathcal{W}_k)/\tau_{\vec{a}}(\mathbb{Z}_\ell)$. We give $L^{2k-1}(\ell; \vec{a})$ and $X^{2k+1}(\ell; \vec{a})$ the natural $\mathbb{Z}_\ell$ structures. The stable tangent bundle of the lens space $L^{2k-1}(\ell; \vec{a})$ is naturally isomorphic to the underlying real vector bundle of the complex vector bundle defined by $\tau_{\vec{a}}$ so $T(L^{2k-1}(\ell; \vec{a}))$ admits a stable almost complex structure. Let $|\vec{a}| = a_1 + ... + a_k$.

The lift described by Hitchin [10] of the unitary group $U(k)$ to the group $Spin^c(2k)$ gives $L^{2k-1}(\ell;\vec{a})$ a natural spin^c structure with flat associated determinant line bundle. This structure can be reduced to a spin structure if and only if we can take the square root of the associated determinant line bundle or equivalently the square root of the determinant representation $\det(\tau_{\vec{a}}) = \rho_{|\vec{a}|}$. This is possible if and only if k is even. The bundle ξ_1 is the underlying real 2-plane bundle of the complex bundle defined by the representation ρ_1 over $B\mathbb{Z}_\ell$. If k is odd, we can take the square root of the representation $\rho_{1+|\vec{a}|}$ so $T(L^{2k-1}(\ell;\vec{a}))\oplus f^*(\xi_1)$ admits a spin structure. This and similar arguments show that:

$$[(L^{4k-1}(\ell;\vec{a}),\cdot)] \in \Omega_{4k-1}(B\mathbb{Z}_\ell),\ [(L^{4k+1}(\ell;\vec{a}),\cdot)] \in \Omega_{4k+1}(B\mathbb{Z}_\ell;\xi_1),$$
$$[(X^{4k+1}(\ell;\vec{a}),\cdot)] \in \Omega_{4k+1}(B\mathbb{Z}_\ell), \text{ and } [(X^{4k+3}(\ell;\vec{a}),\cdot)] \in \Omega_{4k+3}(B\mathbb{Z}_\ell;\xi_1).$$

Lemma 4.1. *If $m \geq 3$, then $L^m(\ell;\vec{a})$ and $X^m(\ell;\vec{b})$ admit metrics of positive scalar curvature.*

PROOF. The natural metric on S^m descends to a metric of constant sectional curvature $+1$ on the lens spaces $L^m(\ell;\vec{a})$ for $m \geq 3$. The spaces $X^m(\ell;\vec{a})$ are geometrical lens space bundles over S^2 with fiber $L^{m-2}(\ell;\vec{a})$. We choose a metric so the fibers are totally geodesic and consider the canonical variation as was done in the proof of Lemma 3.2. Then by 3, 9.70d we have $\mathcal{K}_X = t^{-1}\mathcal{K}_L + \mathcal{K}_S + O(t)$ where $\mathcal{K}_L$ and $\mathcal{K}_S$ are the scalar curvature of $L^{m-2}(\ell;\vec{a})$ and S^2. The Lemma follows since $\mathcal{K}_L \geq 0$ and $\mathcal{K}_S > 0$. □

The eta invariant is combinatorially computable. Define

$$\psi(\vec{a}) := \det(\tau_{\vec{a}})^{-1/2}\det(I-\tau_{\vec{a}}) \in R_0(\mathbb{Z}_\ell)^k \text{ for } k \text{ even},$$
$$\psi(\vec{a}) := (\rho_1\det(\tau_{\vec{a}}))^{-1/2}\det(I-\tau_{\vec{a}}) \in R_0(\mathbb{Z}_\ell)^k \text{ for } k \text{ odd},$$
$$\mathcal{G}_L(\vec{a})(\lambda) = \{\psi(\vec{a})(\lambda)\}^{-1} \text{ for } \lambda \neq 1, \text{ and}$$
$$\mathcal{G}_X(\vec{b})(\lambda) = (1+\lambda^{b_1})(1-\lambda^{b_1})^{-1}\mathcal{G}_L(\vec{b})(\lambda) \text{ for } \lambda \neq 1.$$

Lemma 4.2. *Let $\rho \in R(\mathbb{Z}_\ell)$. Let $\tilde{\Sigma}_\lambda = \Sigma_{\lambda^\ell=1,\lambda\neq 1}$ and $\Sigma_\lambda = \Sigma_{\lambda^\ell=1}$.*
1) We have that $\eta(L^m(\ell;\vec{a}))(\rho) = \ell^{-1}\tilde{\Sigma}_\lambda Tr(\rho(\lambda))\mathcal{G}_L(\vec{a})(\lambda)$ and that $\eta(X^m(\ell;\vec{b}))(\rho) = \ell^{-1}\tilde{\Sigma}_\lambda Tr(\rho(\lambda))\mathcal{G}_X(\vec{a})(\lambda)$.
2) If $\rho \in R_0(\mathbb{Z}_\ell)^j$ and if $2j+1 > m$, then $\eta(L^m(\ell;\vec{a}))(\rho) \in \mathbb{Z}$ and $\eta(X^m(\ell;\vec{b}))(\rho) \in \mathbb{Z}$.

PROOF. We refer to [6] for the proof of (1) if L^m or X^m is spin; this is a straightforward application of formulas of Donnelly [7]. The proof if L^m or X^m has a twisted spin structure is similar. We use (1) to prove (2). If a is an odd integer, then $R_0(\mathbb{Z}_\ell) = (\rho_0 - \rho_a)R(\mathbb{Z}_\ell)$. Thus $R_0(\mathbb{Z}_\ell)^k = \psi(\vec{a})R(\mathbb{Z}_\ell)$ for $\vec{a} = (a_1, ..., a_k)$. If $\rho \in R_0(\mathbb{Z}_\ell)^{k+1}$, then there exists $\gamma \in R_0(\mathbb{Z}_\ell)$ so $\rho = \gamma\psi(\vec{a})$. Thus $\eta(L^{2k-1}(\ell;\vec{a}))(\rho) = \ell^{-1}\tilde{\Sigma}_\lambda Tr(\gamma(\lambda)) = \ell^{-1}\Sigma_\lambda Tr(\gamma(\lambda)) \in \mathbb{Z}$. The proof of the corresponding assertion for $X^{2k+1}(\ell;\vec{a})$ is similar. $\square$

Let $j > 1$. In the free Abelian group generated by lens spaces and lens space bundles, define:

$$\mathcal{B} : L^m(\ell;\vec{a}) \to L^{m+4}(\ell;\vec{a},1,1) - 3L^{m+4}(\ell;\vec{a},1,3)$$
$$\mathcal{B} : X^m(\ell;\vec{a}) \to X^{m+4}(\ell;\vec{a},1,1) - 3X^{m+4}(\ell;\vec{a},1,3)$$
$$K_L^5 := L^5(\ell;1,1,1) - 3L^5(\ell;1,1,3),\ K_L^{4j+1} := \mathcal{B}K_L^{4j-3}$$
$$K_X^5 := X^5(\ell;1,1) - 3X^5(\ell;1,3),\text{ and } K_X^{4j+1} = \mathcal{B}K_X^{4j-3}.$$

Lemma 4.3. *Let $\psi := \psi(1,1) = \rho_{-1}(\rho_0 - \rho_1)^2$, let $j \geq 1$, and let $\ell = 2^q \geq 4$.*

1) *We have $\eta(L^3(\ell;1,1))(\rho_{-1}(\rho_0 - \rho_1)) = \eta(L^5(\ell;1,1,1))(\psi\rho_{-1}) = (\ell-1)/2\ell$.*
2) *We have $\eta(X^3(\ell;1))(\psi) = \eta(X^5(1,1))(\psi(\rho_0 - \rho_1)) = (\ell-2)/\ell$.*
3) *If $\rho \in R(\mathbb{Z}_\ell)$, then $\eta(K_L^{4j+1})(\psi\rho) \in \mathbb{Z}$ and $\eta(K_X^{4j+1})(\psi\rho) \in \mathbb{Z}$.*
4) *There exists $\gamma_{L,j} \in R(\mathbb{Z}_\ell)$ so that $\eta(K_L^{4j+1})(\gamma_{L,j}) = (\ell-1)/2\ell$.*
5) *There exists $\gamma_{X,j} \in R(\mathbb{Z}_\ell)$ so that $\eta(K_X^{4j+1})(\gamma_{X,j}) = (\ell-2)/\ell$.*
6) *$\eta(L^3(\ell;1,1) - 3L^3(\ell;1,3))(\rho_0 - \rho_{\ell/2}) = (-1)^q$.*

PROOF. We use Lemma 4.2 to prove (1) and (2) by computing

$$\begin{aligned}\eta(L^3(\ell;1,1))(\rho_{-1}(\rho_0-\rho_1)) &= \eta(L^5(\ell;1,1,1))(\psi\rho_{-1}) = \ell^{-1}\tilde{\Sigma}_\lambda(1-\lambda)^{-1}\\ &= (2\ell)^{-1}\tilde{\Sigma}_\lambda\{(1-\lambda)^{-1} + (1-\bar{\lambda})^{-1}\} = (2\ell)^{-1}\tilde{\Sigma}_\lambda 1 = (\ell-1)/2\ell\end{aligned}$$
$$\eta(X^3(\ell;1))(\psi) = \eta(X^5(\ell;1,1))(\psi(\rho_0-\rho_1)) = \ell^{-1}\tilde{\Sigma}_\lambda(1+\lambda) = (\ell-2)/\ell.$$

Since $\mathcal{G}_L(\vec{a},1)(\lambda) - 3\mathcal{G}_L(\vec{a},3)(\lambda) = \psi(\lambda)\mathcal{G}_L(\vec{a},3)(\lambda)$, we have

$$\eta(K_L^{4j+1})(\gamma) = \eta(L^{4j+1}(\ell;(j+1)\cdot 1, j\cdot 3))(\psi^j\gamma)$$
$$\eta(K_X^{4j+1})(\gamma) = \eta(X^{4j+1}(\ell;j\cdot 1, j\cdot 3))(\psi^j\gamma).$$

As $\psi^{j+1}\rho \in R_0(\mathbb{Z}_\ell)^{2j+2}$, (3) follows from Lemma 4.2. As $\psi^j R(\mathbb{Z}_\ell) = R_0(\mathbb{Z}_\ell)^{2j}$, we can choose $\gamma_{L,j}$ and $\gamma_{X,j}$ so that $\gamma_{L,j}\psi^j\mathcal{G}_L(\ell;(j+1)\cdot 1, j\cdot 3)(\lambda) = (1-\lambda)^{-1}$ and so that $\gamma_{X,j}\psi^j\mathcal{G}_X(\ell;j\cdot 1, j\cdot 3)(\lambda) = 1+\lambda$. The argument proving (1) and (2) then establishes (4) and (5). We compute

$$\begin{aligned}&\eta(L^3(\ell;1,1) - 3L^3(\ell;1,3))(\rho_0 - \rho_{\ell/2})\\ &= \ell^{-1}\tilde{\Sigma}_\lambda\lambda(1-\lambda)(1-\lambda^{3\ell/2})/(1-\lambda^3)\\ &= \ell^{-1}\tilde{\Sigma}_\ell(\lambda-\lambda^2)(1+\lambda^3+...+\lambda^{3\ell/2-3})\\ &= \ell^{-1}\Sigma_\ell(\lambda-\lambda^2)(1+\lambda^3+...+\lambda^{3\ell/2-1}).\end{aligned}$$

Since $\ell^{-1}\Sigma_\lambda\lambda^\ell = 1$ and since $\ell^{-1}\Sigma_\lambda\lambda^a = 0$ for $1 \le a \le 3\ell/2 - 1$ and $a \ne \ell$, the eta invariant is 1 if q is even so $\ell - 1 = 3\nu$ and -1 if q is odd so $\ell - 2 = 3\nu$. $\square$

The Poincare dual A^* of an Abelian group A is the group of homomorphisms from A to $\mathbb{R}/\mathbb{Z}$. Let $\eta^*(M)$ be the homomorphism which sends ρ to $\eta(M)(\rho)$. By Theorem 3.1, the map which sends M to $\eta^*(M)$ extends to homomorphisms $\eta^* : ko_{4k+1}(\xi_i, B\mathbb{Z}_\ell) \to R(\mathbb{Z}_\ell)^*$ and $\eta_0^* : ko_{4k+3}(\xi_i, B\mathbb{Z}_\ell) \to R_0(\mathbb{Z}_\ell)^*$ for $i = 0, 1$. For $k \ge 0$, we define

$$\begin{aligned}
\mathcal{M}_{4k+3}(\xi_0) &:= \operatorname{span}\{[L^{4k+3}(\ell;\vec{a})]\} \subset ko^+_{4k+3}(B\mathbb{Z}_\ell)\\
\mathcal{M}_{4k+3}(\xi_1) &:= \operatorname{span}\{[X^{4k+3}(\ell;\vec{a})]\} \subset ko^+_{4k+3}(B\mathbb{Z}_\ell;\xi_1),\\
\mathcal{M}_{4j+5}(\xi_0) &:= \operatorname{span}\{[X^{4k+5}(\ell;\vec{a})]\} \subset ko^+_{4k+5}(B\mathbb{Z}_\ell),\\
\mathcal{M}_{45+5}(\xi_1) &:= \operatorname{span}\{[L^{4k+5}(\ell;\vec{a})]\} \subset ko^+_{4k+5}(B\mathbb{Z}_\ell;\xi_1).
\end{aligned}$$

Lemma 4.4. *If $k \ge 0$, then*

1) $|\eta_0^*(\mathcal{M}_{4k+3}(\xi_0))| \ge (2\ell)^{k+1}$ *and* $|\eta_0^*(\mathcal{M}_{4k+3}(\xi_1)| \ge (\ell/2)^{k+1}$.
2) $|\eta^*(\mathcal{M}_{4k+5}(\xi_0))| \ge (\ell/2)^{k+2}$ *and* $|\eta^*(\mathcal{M}_{4k+5}(\xi_1))| \ge (2\ell)^{k+2}$.
3) $|ko^+_{4k+3}(B\mathbb{Z}_\ell;\xi_0)| \ge (2\ell)^{k+1}$ *and* $|ko^+_{4k+3}(B\mathbb{Z}_\ell;\xi_1)| \ge (\ell/2)^{k+1}$.
4) $|ko^+_{4k+5}(B\mathbb{Z}_\ell;\xi_0)| \ge (\ell/2)^{k+2}$ *and* $|ko^+_{4k+5}(B\mathbb{Z}_\ell;\xi_1)| \ge (2\ell)^{k+2}$.
5) $|ko^+_{8k+3}(B\mathbb{Z}_\ell;\xi_0)| \ge 2(2\ell)^{2k+1}$.

PROOF. We first establish (2). If $\xi = \xi_0$, let $c = \ell/2$; if $\xi = \xi_1$, let $c = 2\ell$. Let $\psi := \psi(1,1)$. The homomorphism which sends ρ to $\psi\rho$ defines a dual map ψ^* from $R(\mathbb{Z}_\ell)^*$ to $R(\mathbb{Z}_\ell)^*$. Clearly

$$|\eta^*\mathcal{M}_{4k+5}(\xi)| \ge |\psi^*\eta^*\mathcal{M}_{4k+5}(\xi)| \cdot |\ker(\psi^*) \cap \eta^*\mathcal{M}_{4k+5}(\xi)|.$$

By Lemma 4.3, $|\psi^*\eta^*\mathcal{M}_5(\xi)| \ge c$ and $|\ker(\psi^*) \cap \eta^*\mathcal{M}_{4k+5}(\xi)| \ge c$. This proves (2) if $4k+5 = 5$. Since $\eta(M^{4k+5}(\ell;\vec{a},1,1))(\psi\rho) = \eta(M^{4k+1}(\ell;\vec{a}))(\rho)$ where M is a lens space or lens space bundle, $\eta^*\mathcal{M}_{4k+1}(\xi) \subset \psi^*\eta^*\mathcal{M}_{4k+5}(\xi)$ and (2) now follows by induction.

If $k = 0$, Lemma 4.3 implies assertion (1) so we may assume that $4k+3 \ge 7$ and that $4k+1 \ge 5$. Let $\delta = (\rho_0 - \rho_1)$. If $\rho \in R(\mathbb{Z}_\ell)$ and if M is a lens space or a lens space bundle, then $\eta(M^{4k+3}(\ell;\vec{a},1))(\delta\rho) = \eta(M^{4k+1}(\ell;\vec{a}))(\rho)$. Thus $\eta^*\mathcal{M}_{4k+1}(\xi_0) \subseteq \delta^*\eta_0^*\mathcal{M}_{4k+3}(\xi_1)$ and $\eta^*\mathcal{M}_{4k+1}(\xi_1) \subseteq \delta^*\eta_0^*\mathcal{M}_{4k+3}(\xi_0)$ and (1) for $k > 0$ follows from (2). Assertions (3) and (4) follow from (1) and (2) since the eta invariant extends to connective K theory and since $\mathcal{M}_m(\xi_0)$ and $\mathcal{M}_m(\xi_1)$ are generated by bordism classes of manifolds which admit metrics of positive scalar curvature for $m \ge 3$ by Lemma 4.1. To prove (5), we must squeeze out a single factor of 2 if $m = 8k+3$ and if $\xi = \xi_0$. We use Lemma 4.3 (6). Let $M = (L^3(\ell;1,1) - 3L^3(\ell;1,3)) \times (B^8)^j$ where B^8 is the Bott manifold; B^8 is an 8 dimensional simply connected spin manifold with $\hat{A}(B^8) = 1$. Since the eta invariant is multiplicative,

$$\eta(M)(\rho_0 - \rho_{\ell/2}) = \hat{A}(B^8)^j \eta(L^3(\ell;1,1) - 3L^3(\ell;1,3))(\rho_0 - \rho_{\ell/2}) \neq 0$$

in $\mathbb{R}/2\mathbb{Z}$. If $\rho \in R_0(\mathbb{Z}_\ell)$, then $\psi\rho \in R_0(\mathbb{Z}_\ell)^3$ so $\eta(M)(\rho) = \eta(L^3(\ell;1,3))(\psi\rho) \in \mathbb{Z}$. Thus $M \in \ker(\eta^*)$ and the refined eta invariant of Theorem 3.1 (1) is non-trivial and generates the missing factor of 2. □

PROOF. We now demonstrate Theorem 1.1. The lower bounds of Lemma 4.4 and the upper bounds of Theorem 2.5 show that if $m \geq 3$ is odd, then we have that $ko_m^+(B\mathbb{Z}_\ell;\xi_0) = \ker(\hat{A}) \cap ko_m(B\mathbb{Z}_\ell;\xi)$ and $ko_m^+(B\mathbb{Z}_\ell;\xi_1) = ko_m(B\mathbb{Z}_\ell;\xi_1)$. □

The eta invariant and the $\hat{A}$ genus completely detect the connective K theory groups $ko_m(B\mathbb{Z}_\ell)$ and $ko_m(B\mathbb{Z}_\ell;\xi_1)$. Let $\theta(M) = \eta(M)(\rho_0 - \rho_{\ell/2}) \in \mathbb{R}/2\mathbb{Z}$ if $\xi = \xi_0$ and $m \equiv 3 \mod 8$; set $\theta = 0$ otherwise. We set $\eta^* = 0$ if m is even. The following result follows from Theorem 2.5 and the proof of Lemma 4.4.

Theorem 4.5. *Let $\ell = 2^q \geq 4$. Let $x \in ko_m(B\mathbb{Z}_\ell;\xi_i)$ for $i = 0,1$. Let $m \geq 2$. Then $x = 0$ if and only if $\hat{A}(x) = 0$, $\eta^*(x) = 0$, and $\theta(x) = 0$.*

5. Orders of connective K theory groups

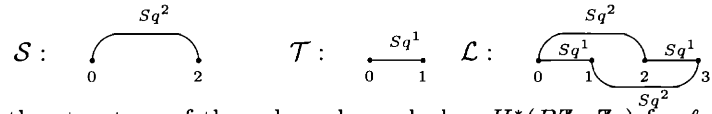
We complete the proof of Theorem 2.5 by computing the orders of the groups $ko_n(B\mathbb{Z}_\ell;\xi_i)$ for $1 \leq i \leq 3$, and $\ell = 2^q \geq 4$; the orders of the groups $ko_n(B\mathbb{Z}_\ell;\xi_0) \cong ko_n(B\mathbb{Z}_\ell)$ were computed in [6]. Let ko denote the spectrum classifying the connective K theory $ko_*(\cdot)$. We shall write $H^*(\cdot)$ for $H^*(\cdot;\mathbb{Z}_2)$. Let $\mathcal{A}$ be the Steenrod algebra. We use here the Adams spectral sequence

$$E_2^{*,*} = \mathrm{Ext}_{\mathcal{A}}^{*,*}(\widetilde{H}^*(ko \wedge X), \mathbb{Z}_2) \Longrightarrow \widetilde{ko}_*(X). \tag{5.1}$$

Let $\mathcal{A}(1)$ be the subalgebra of $\mathcal{A}$ generated by the operations Sq^1 and Sq^2. Recall that $H^*(ko) \cong \mathcal{A}//\mathcal{A}(1)$ as a module over the Steenrod algebra. The Künneth isomorphism $H^*(ko \wedge X) \cong \mathcal{A}//\mathcal{A}(1) \otimes_{\mathcal{A}} H^*(X)$ and the change of rings formula $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}//\mathcal{A}(1) \otimes_{\mathcal{A}} \mathcal{M}, \mathcal{N}) \cong \mathrm{Hom}_{\mathcal{A}(1)}(\mathcal{M}, \mathcal{N})$ (for any modules $\mathcal{M}$, $\mathcal{N}$ over the Steenrod algebra) imply the E_2-term of the Adams spectral sequence (5.1) has the form $\mathrm{Ext}_{\mathcal{A}(1)}^{*,*}(\widetilde{H}^*(X);\mathbb{Z}_2) \Longrightarrow \widetilde{ko}_*(X)$. In particular, it means that we have to compute only the $\mathcal{A}(1)$-module structure of $H^*(X_i)$ to obtain the E_2-term.

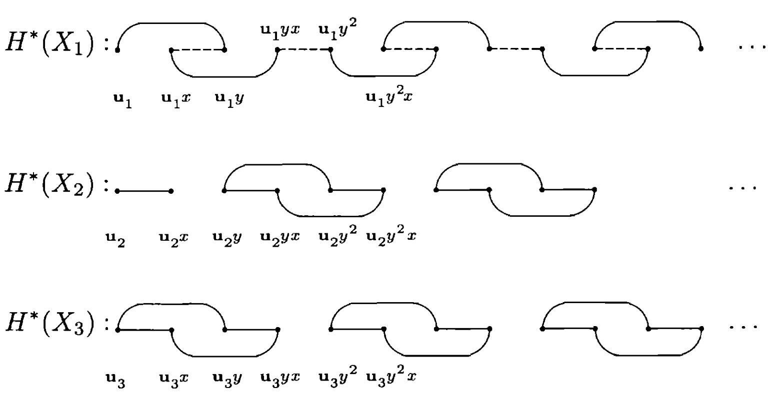
In what follows, we shall omit certain details involving elementary homotopy theoretic methods in the interests of brevity. We shall not compute the Adams spectral sequence in detail nor shall will we prove the statements below concerning the structure of the $\mathcal{A}(1)$-modules $H^*(X_i)$ nor will we compute the corresponding E_2-terms.

We shall need the following three $\mathcal{A}(1)$-modules:

(5.2) 

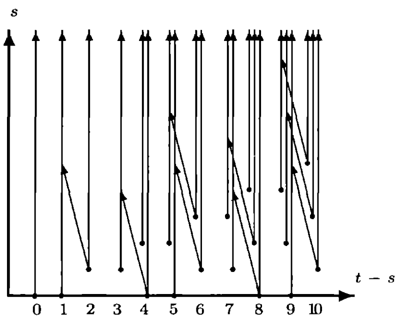
We recall the structure of the cohomology algebra $H^*(B\mathbb{Z}_\ell;\mathbb{Z}_2)$ for $\ell = 2^q \geq 4$. Let x and y generate $H^1(B\mathbb{Z}_\ell;\mathbb{Z}_2) = \mathbb{Z}_2$ and $H^2(B\mathbb{Z}_\ell;\mathbb{Z}_2) = \mathbb{Z}_2$ respectively. Then $H^*(B\mathbb{Z}_\ell;\mathbb{Z}_2) \cong \Lambda(x) \otimes P(y)$. We note that $Sq^i x = 0$ for any $i \geq 1$, that $Sq^2 y = y^2$, and that $\beta^q y = x$ where β^q is the q-th Bockstein operator. Let $M(\xi) = \Sigma^{-k}T(\xi)$ be the corresponding Thom spectrum. In order to compute $H^*(M(\xi);\mathbb{Z}_2)$ as a module over the algebra $\mathcal{A}(1)$, we will use the Wu formula: $Sq^j\mathbf{u} = \mathbf{u}w_j(\xi)$, where $\mathbf{u}$ is the Thom class of the bundle ξ. Let $X_i = M(\xi_i)$, and $\mathbf{u}_i \in H^0(X_i)$, $i = 1, 2, 3$ be the corresponding Thom class. The following statement one may prove by standard computations.

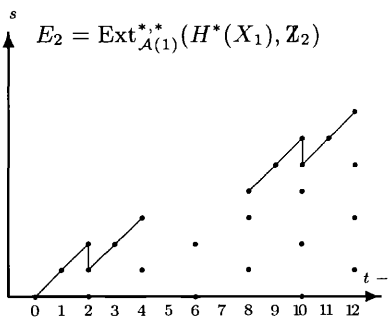
Proposition 5.1. *The stable $\mathcal{A}(1)$-module structure of $H^*(X_i)$ is given by:*

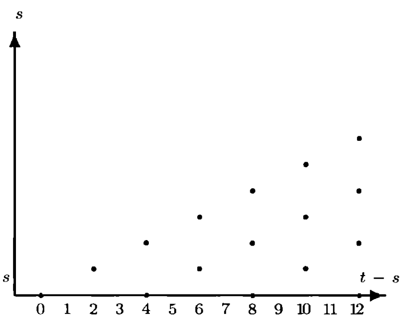


Equivalently there are $\mathcal{A}(1)$-stable isomorphisms: $H^*(X_1) \cong (\Lambda(x) \otimes \mathbb{Z}_2[y^2]) \otimes \mathcal{S}$, $H^*(X_2) \cong \mathcal{T} \oplus (\mathbb{Z}_2[y^2] \otimes \Sigma^{-2}\mathcal{L})$, *and* $H^*(X_3) \cong \mathbb{Z}_2[y^2] \otimes \mathcal{L}$ *where* $\mathcal{S}$, $\mathcal{T}$, $\mathcal{L}$ *are the modules from* (5.2).

The groups $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)}(\mathcal{T},\mathbb{Z}_2)$, $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)}(\mathcal{S},\mathbb{Z}_2)$, and $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)}(\mathcal{S},\mathbb{Z}_2)$ are well-known, see 14. We obtain the following charts for the E_2-terms of the corresponding Adams spectral sequence converging to $ko_*(X_i)$:







$E_2 = E_\infty = \mathrm{Ext}^{*,*}_{\mathcal{A}(1)}(H^*(X_2), \mathbb{Z}_2)$ $\quad$ $E_2 = E_\infty = \mathrm{Ext}^{*,*}_{\mathcal{A}(1)}(H^*(X_3), \mathbb{Z}_2)$

There is a nontrivial differential d_q (it is displayed above for $q = 3$) which is determined by the Bockstein operator β^q in the Adams spectral sequence $E_\infty = \mathrm{Ext}^{*,*}_{\mathcal{A}(1)}(H^*(X_1), \mathbb{Z}_2) \implies ko_*(X_1)$; there are no differentials in the two other Adams spectral sequences by dimensional reasons. Direct computation gives the following formula for $\epsilon(m, X_i) := |ko_m(X_i)|$:

	$\epsilon(m, X_1)$	$\epsilon(m, X_2)$	$\epsilon(m, X_3)$
$m = 8k$	∞	2^{2k+1}	2^{2k+1}
$m = 8k+1$	$(2\ell)^{2k+1}$	2	1
$m = 8k+2$	1	2^{2k+3}	2^{2k+1}
$m = 8k+3$	$(\ell/2)^{2k+1}$	2	1
$m = 8k+4$	∞	2^{2k+2}	2^{2k+2}
$m = 8k+5$	$(2\ell)^{2k+2}$	1	1
$m = 8k+6$	1	2^{2k+2}	2^{2k+2}
$m = 8k+7$	$(\ell/2)^{2k+2}$	1	1

Note that $ko_{4k}(X_1) = \mathbb{Z}$ and this is detected by the A-roof genus. Theorem 2.5 now follows. This establishes all the results of this paper.

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