M390C NOTES: GEOMETRIC LANGLANDS

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These notes were taken in UT Austin's Math 390c (Geometric Langlands) class in Fall 2016, taught by David Ben-Zvi. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to `a.debray@math.utexas.edu`. Thanks to Adrian Clough, Feng Ling, and Yuri Sulyma for correcting a few typos, to Tom Mainiero for the caveat about localizable measure spaces, and to Jay Hathaway for the description of electron orbitals in terms of spherical harmonics.

CONTENTS

1. THE FOURIER TRANSFORM IN REPRESENTATION THEORY: 8/25/16

"One of the traditions we have at UT is we always have to mention Tate."

The initial conception of this class was going to be more akin to a learning seminar about the geometric Langlands program, but this changed: it's now going to be an actual class, but about geometric representation theory and topological field theory. The goal is for this to turn into good lecture notes and even a book, so the class isn't the entire intended audience. As such, feedback is even more helpful than usual.

It's not entirely clear what the prerequisites for this class are; the level of background will grow as the class goes on. The actual amount of technical background needed to state things precisely is huge, and not a reasonable requirement. As such, the class will be more of a sketch and overview of the ideas and how to

think about the main characters[1] in this subject. The professor's seminar (Fridays, from 2 to 4, in the same room) is probably a good place to start understanding this material more rigorously.

There will be an introduction to this class this afternoon at geometry seminar.

The Fourier transform. Do you remember Fourier series? The statement is that for L^2 functions $f : S^1 \to \mathbb{C}$,

$$f(\theta) = \sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{2\pi i n \theta}.$$

This is probably the last precise formula we're going to see in this class, which may reassure you or bother you. We also will identify $S^1 \cong \mathrm{U}(1)$. The Fourier coefficients are

$$\widehat{f}(n) = \int_{S^1} f(\theta) e^{-2\pi i n \theta} \, \mathrm{d}\theta.$$

Representation theory starts with this formula.

Relatedly, for an L^2 function $f : \mathbb{R} \to \mathbb{C}$, we have a continuous combination of exponentials with coefficients $\widehat{f}(t)$:

(1.1) $$f(x) = \int_{\mathbb{R}} \widehat{f}(t) e^{2\pi i x t} \, \mathrm{d}t,$$

where

$$\widehat{f}(t) = \int_{\mathbb{R}} f(x) e^{-2\pi i x t} \, \mathrm{d}x.$$

How should we think of these formulas? The exponentials $e^{2\pi i t-}$ are complex-valued functions on $\mathrm{U}(1)$ and $\mathbb{R}$, respectively. But in fact, they land in $\mathbb{C}^\times$, since they don't hit 0, and in fact they have unit norm, so they are maps into $\mathrm{U}(1)$. Since $e^{a+b} = e^a e^b$, these are homomorphisms of groups. Moreover, these are the only homomorphisms: if $f(\theta_1 + \theta_2) = f(\theta_1) f(\theta_2)$ for an $f : \mathrm{U}(1) \to \mathrm{U}(1)$, then $f(\theta) = e^{2\pi i x \theta}$ for some x, and similarly for functions $\mathbb{R} \to \mathrm{U}(1)$.

In other words, these functions are the *unitary characters* of the domain group: the homomorphisms to $\mathrm{U}(1) \subset \mathrm{GL}(1)$. We can recast these as representations acting through unitary matrices (also *unitary representations*), where an $x \in \mathbb{R}$ acts as multiplication by $e^{2\pi i x t}$ on the (complex) vector space $\mathbb{C}$.

From this viewpoint, we are writing general functions on $\mathrm{U}(1)$ or on $\mathbb{R}$ as linear combinations of characters. This means characters form a "basis." That is, the characters are not strictly a basis, but the space spanned by finite linear combinations of exponentials is dense in any reasonable function space L^2, C^∞, distributions, real analytic functions, L^p spaces, etc. In particular, L^2, smooth, analytic, etc. are conditions on the Fourier coefficients: $f \in L^2(S^1)$ iff $\widehat{f} \in \ell^2$ (the square-integrable sequences of numbers). f is smooth iff its Fourier coefficients are rapidly decreasing (faster than any polynomial).

This is where the analysis of Fourier series takes place: you're interested in different function spaces, and so you're interested in how the coefficients grow. But we're going to ignore it: it's deep and important for analysis, but begins a different track than representation theory. The algebraic content is that algebraic functions (Laurent series) are dense, and we're going to care more about the algebraic side than the analytic side.

Theorem 1.2 (Plancherel). *If $\mathbb{R}$ denotes the x-line and $\widehat{\mathbb{R}}$ denotes the t-line, then the Fourier transform defines a unitary isomorphism $L^2(\mathbb{R}) \xrightarrow{\sim} L^2(\widehat{\mathbb{R}})$.*

This is nice, but doesn't help much for the character-theoretic viewpoint: the exponential $e^{2\pi i x t}$ is not in $L^2(\mathbb{R})$. This is where one uses Schwartz functions.

Definition 1.3. The *Schwartz space* $\mathcal{S}(\mathbb{R})$ is the space of $f \in C^\infty(\mathbb{R})$ such that f and all of its derivatives decrease more rapidly than any polynomial.

The dual space to $\mathcal{S}(\mathbb{R})$, denoted $\mathcal{S}^*$ or $\mathcal{S}'$, is called the space of *tempered distributions*. Our characters $e^{2\pi i x t}$ live in this space, and the Fourier transform extends to a linear homeomorphism $\mathcal{S}'(\mathbb{R}) \cong \mathcal{S}'(\widehat{\mathbb{R}})$.

Thus, it makes sense to define the Fourier transform of the exponential $e^{2\pi i n x}$: we obtain the delta "function" supported at n, δ_n (1 at n and 0 elsewhere), and similarly, the Fourier transform of δ_t is $e^{2\pi i x t}$.

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[1]Pun intended?

That is, the Fourier transform exchanges points and characters; in other words, $\widehat{\mathbb{R}}$ is a sort of moduli space of unitary characters of $\mathbb{R}$.

In some sense, this diagonalizes the group action: if G is either of $\mathbb{R}$ or $\mathrm{U}(1)$, then G acts on itself by translation (both left and right, since G is abelian). Thus, any space of functions on G is acted on by G: an $\alpha \in G$ sends $f \mapsto \alpha * f$ (i.e. $\alpha * f(x) = f(x + \alpha)$). If V is this function space (e.g. $L^2(G)$), then this defines an action of G on V, hence a group homomorphism $G \to \mathrm{End}(V)$. In particular, the exponential $e^{2\pi i x t}$ satisfies

$$\alpha * e^{2\pi i x t} = e^{2\pi i (x+\alpha)t} = \left(e^{2\pi i x \alpha}\right)\left(e^{2\pi i x t}\right).$$

That is, this exponential is an eigenfunction for $\alpha * -$ for all $\alpha \in G$: characters are joint eigenfunctions, and the Fourier transform is a simultaneous diagonalization.

Succinctly, *the Fourier transform exchanges translation and multiplication*: the translation operator $\alpha * -$ is sent to the multiplication operator $\widehat{f} \mapsto \widehat{\alpha}\widehat{f}$, where $\widehat{\alpha}(t) = e^{2\pi i \alpha t}$. From the perspective of Fourier series, we have a $\mathbb{Z} \times \mathbb{Z}$ matrix with respect to the exponential basis, but only the diagonal entries $\widehat{f}(n)e^{2\pi i n \theta}$ are nonzero.

Before we make this more abstract, let's see what happens to differentiation. Since G is a Lie group, it has a Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}$, in this case $\mathbb{R} \cdot \frac{\mathrm{d}}{\mathrm{d}x}$, the infinitesimal translations at a point. The differential $\frac{\mathrm{d}}{\mathrm{d}x}$ is an infinitesimal translation, and the Fourier transform sends it to a multiplication by $(2\pi i)t$.[2]

Pontrjagin duality. We can generalize this to Pontrjagin duality, which is a kind of Fourier transform involving a locally compact abelian topological group (LCA) G, e.g. $\mathbb{R}$, $\mathbb{Z}$, S^1, $\mathbb{Z}/n$, and any finite products of these, including tori, lattices, and finite-dimensional vector spaces. More exotic examples include the p-adics. There will be more interesting examples in the algebraic world.

Definition 1.4. Let G be an LCA group; then, the *(unitary) dual* of G is $\widehat{G} = \mathrm{Hom}_{\mathsf{TopGrp}}(G, \mathrm{U}(1))$, the set of characters of G, with the topology inherited as a subset of the continuous functions $C(G) = \mathrm{Hom}_{\mathsf{Top}}(G, \mathbb{C})$.

We saw that if $G = \mathbb{R}$, then $\widehat{G} = \mathbb{R}$ again, and that if $G = \mathrm{U}(1)$, then $\widehat{G} = \mathbb{Z}$. Conversely, if $G = \mathbb{Z}$, then a homomorphism on G is determined by its value at 1, which can be anything in $\mathrm{U}(1)$, so $\widehat{G} = \mathrm{U}(1)$. If V is a finite-dimensional vector space, then $\widehat{V} = V^*$: any linear functional $\xi \in V^*$ defines a character $v \mapsto e^{2\pi i \langle \xi, v \rangle}$. It's a nice exercise to check that these are all the unitary characters. If $G = \Lambda$ is a lattice, then we obtain its *dual torus* T, and correspondingly a torus goes to its *dual lattice*. Lastly, we have finite abelian groups, e.g. $\mathbb{Z}/n$, which is generated by 1, so we must send 1 to an nth root of unity. Thus, $(\mathbb{Z}/n)^{\vee} = \mu_n$, the group of nth roots of unity. This is isomorphic to $\mathbb{Z}/n$ again, though in algebraic geometry, where we might not have all roots of unity, things can get more interesting, so it's useful to remember μ_n.

The claim is that the Fourier transform looks exactly the same for any LCA group; maybe we haven't defined too many exciting examples, but this is still noteworthy. We want characters on G to correspond to points on $\widehat{G}$. A point $\chi \in \widehat{G}$ defines a function on G, and correspondingly, a point $g \in G$ defines a function $\widehat{g} : \chi \mapsto \chi(g)$ on $\widehat{G}$, which looks like a nascent Fourier transform. If $g, h \in G$, then $\widehat{gh}(\chi) = \chi(gh) = \chi(g)\chi(h) = \widehat{g}\widehat{h}(\chi)$, so this transform that we're building will start from this duality of the group multiplication and the pointwise product.

One important thing to mention: $\widehat{G}$ is also a group, and in fact is locally compact abelian. The group operation is pointwise product $\chi_1 \cdot \chi_2(g) = \chi_1(g)\chi_2(g)$. This agrees with the group operations for the examples we mentioned.

Theorem 1.5 (Pontrjagin duality). *The natural map $G \mapsto \widehat{\widehat{G}}$ defined by $g \mapsto \widehat{g}$ is an isomorphism of topological groups.*

Hence, this really is a duality. Nonetheless, we'll maintain the distinction between G and $\widehat{G}$: soon we'll try to generalize to nonabelian groups, and then symmetry will break.

Theorem 1.6 (Fourier transform). *If G is an LCA group, then the Fourier transform map*

$$f \longmapsto \widehat{f}(\chi) = \int_G f(g) \cdot \chi(g) \, \mathrm{d}g,$$

where $\mathrm{d}g$ is the Haar measure on G,[3] defined an isomorphism of Hilbert spaces $L^2(G) \xrightarrow{\sim} L^2(\widehat{G})$.

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[2]To prove this rigorously, one needs to worry about difference quotients.

[3]This is only unique up to a scalar, so we need to pick one.

Notice that, since the characters on $\mathbb{R}$ are the exponentials and the Haar measure on $\mathbb{R}$ is the usual Lesbegue measure, this generalizes (1.1).

This entire story started in Tate's thesis, which applies Pontrjagin duality to more exotic examples such as $\mathbb{Q}_p$ and $\mathbb{Q}_p^\times$ or even the group $\mathbb{A}^\times$ of *adeles*;[4] see Ramakrishnan-Valenza [9] for a modern take on this subject, including harmonic analysis on LCA groups.

We'll use this to understand all representations of G (well, nice representations). In general, not all representations of G on a space come from functions on G, but we'll be able to use Pontrjagin duality and the group algebra to do something nice.

Function theory. One important philosophy in representation theory is that the action of G on functions on G (nice functions in whichever context we're working in) is the most important, or universal, representation. We'll talk about functions and convolution from a particular perspective that will be useful several times in the class.

Let X be a finite set. Then, $F(X)$, the set of complex-valued functions on X, is unambiguous. The set of measures on X, $M(X)$, is also clear, but there's a natural bijection between them via the counting measure.

Theorem 1.7 (Finite Riesz representation theorem). *There is a natural identification $F(X) = \mathrm{Hom}_{\mathbb{C}}(F(X), \mathbb{C})$.*

This comes from the inner product on $F(X)$

$$\langle f, g \rangle = \sum_{x \in X} f(x) g(x).$$

The more general Riesz representation theorem is about a Hilbert space of functions on $\mathbb{R}$, and is less trivial.

Now, suppose we have two finite sets X and Y. We can form their product, which looks like Figure 1. It's
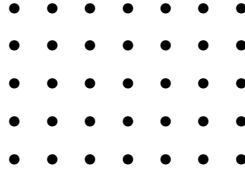

FIGURE 1. The product of two finite sets.

possible to identify $F(X \times Y) = F(X) \otimes F(Y)$, and via a matrix, or an "integral kernel," this space can be identified with $\mathrm{Hom}_{\mathbb{C}}(F(X), F(Y))$: a kernel $K(x, y) \in F(X \times Y)$ defines an operator $K * - : F(X) \to F(Y)$ defined by

$$K * f(y) = \sum_{x \in X} K(x, y) f(x).$$

In a broader sense, let $\pi_X : X \times Y \to X$ be projection, and define π_Y similarly. Functions can pull back: $\pi_X^* f(x, y) = f(\pi_X(x, y))$, and measures can push forward by integration (or summing, since we're thinking about the counting measure) over the fibers. Thus, we can recast convolution as

$$K * f = \pi_{Y*}(K \cdot \pi_X^* f)(y) = \int_X K(x, y) f(x) \, \mathrm{d}\#.$$

Since $F(X)$ and $F(Y)$ are finite-dimensional vector spaces, K may be identified with a matrix or a linear transform, and this formula is exactly how to multiply a matrix by a vector.

A key desideratum is that, in general, all nice maps between function spaces on X and function spaces on Y come from integral kernels. For example, a map $L^2(\mathbb{R}) \to L^2(\mathrm{pt}) = \mathbb{C}$ is given by a kernel $K \in L^2(\mathbb{R} \times \mathrm{pt}) = L^2(\mathbb{R})$, realized as $f \mapsto \int K \cdot f$, by the Riesz representation theorem for L^2. Another instance of this is the Schwartz kernel theorem.

Theorem 1.8 (Schwartz kernel theorem). *Let X and Y be smooth manifolds. Then, $\mathrm{Hom}_{\mathsf{Top}}(C_c^\infty(X), \mathrm{Dist}(Y)) \cong \mathrm{Dist}(X \times Y)$.*

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[4]Not to be confused with the musician.

Here, Dist$(-)$ is the space of distributions, dual to compactly supported smooth functions on the manifold.

If $X = Y$ (back in the world of finite sets), then we can consider δ_Δ, the δ-function of the diagonal. In a basis, this is just the identity matrix, and convolution with K is the identity operator. More generally, if $g : X \to Y$ is a set map, then $g^* : F(Y) \to F(X)$ is represented by the kernel of the graph $\Gamma_g \subset X \times Y$: $K = \delta_{\Gamma_g}$. If this all seems a little silly, the key is that it's easier to understand over finite sets, but will work for "nice" functions in a great variety of contexts.

We can also use this to understand matrix multiplication. Given three finite sets X, Y, and Z, and kernels (functions) $K_1 : F(X) \to F(Y)$ and $K_2 : F(Y) \to F(Z)$, we can compose them. Consider the projections

$$
\begin{array}{ccc}
& X \times Y \times Z & \\
{}^{\pi_{12}}\swarrow & \Big\downarrow & \searrow{}^{\pi_{23}} \\
X \times Y & {}_{\pi_{13}}\Big\downarrow & Y \times Z \\
& X \times Z.
\end{array}
$$

Exercise 1.9. Show that the formula for $K_2 \circ K_1$ is

$$\pi_{13*}(\pi_{12}^*(K_1) \cdot \pi_{23}^*(K_2)).$$

Relate this to matrix multiplication.

The distinction between functions and measures is irrelevant in the world of finite sets, so we can push-forward and pull back with impunity, but in a continuous setting, it's important to keep them distinct. This equates to choosing a measure (e.g. choosing a Haar measure, as we did above), and even relates to things like Poincaré duality.

2. Representation Theory as Gauge Theory: 8/25/16

Note: this talk was an overview of the class, presented at the weekly geometry seminar.

2.1. Representation Theory.
Representation theory starts from spectral decomposition and the Fourier transform. If G is a locally compact abelian group, we attach its *unitary dual* $\widehat{G}$, the set of irreducible unitary representations of G. These are all one-dimensional, hence described by characters $\chi : G \to \mathrm{U}(1) \subset \mathbb{C}$. The key idea generalizing the Fourier transform is Pontrjagin duality, that the Fourier transform defines an isomorphism $L^2(G) \cong L^2(\widehat{G})$; there are variants for other function spaces.

Example 2.1.
- Finite Fourier series arise from $G = \mathbb{Z}/n$, for which $\widehat{G} = \mathbb{Z}/n$. The dual of $i \mapsto \zeta^i$, where ζ is a primitive root of unity, is a δ-function supported at i.
- Fourier series exchange $G = \mathrm{SO}(2)$ and $\widehat{G} = \mathbb{Z}$.
- The Fourier transform is for $G = \mathbb{R}$ and $\widehat{G} = \mathbb{R}$.

The Fourier transform takes representation theory of G, and turns it into geometry on $\widehat{G}$. For example, characters of G have turned into points of $\widehat{G}$. The group G can act in different translation-like ways: translation, differentiation (infinitesimal translation), and convolution; all of these are simultaneously diagonalized by the Fourier transform, and made into multiplication. Representations of G are turned into families of vector spaces on $\widehat{G}$, in various forms (vector bundles, sheaves, etc.), in a process called *spectral decomposition*.

This is all really nice: the Fourier transform basically solves representation theory for abelian groups. What should we do for nonabelian G?

We'd like to seek a geometry object $\widehat{G}$ parameterizing irreducible representations (unitary or other classes). This $\widehat{G}$ carries a measure, a topology, and even has algebraic geometry; this structure captures notions of families of representations.

Even though we don't know what $\widehat{G}$ is yet, we know that functions on $\widehat{G}$ should act on representations of G in a way that commutes with the G-action.

Example 2.2. If $G = \mathrm{SO}(3)$, then G acts on S^2 and hence also on $L^2(S^2)$ (the Hilbert space of a quantum free particle on a sphere). This action commutes with the spherical Laplacian Δ, and therefore we can

decompose $L^2(S^2)$ into Δ-eigenspaces called *spherical harmonics*:

$$L^2(S^2) \cong \bigoplus_{n \in 2\mathbb{Z}_+} V_n.$$

This says a lot about the unitary irreducible representations of SO(3).

One of the huge goals of representation theory is to produce a nonabelian analogue of the Fourier transform for *arithmetic locally symmetric spaces* $X_\Gamma = \Gamma \backslash G_\mathbb{R}/K$. These are generalizations of the moduli space of elliptic curves: $\mathrm{SL}_2\mathbb{Z}\backslash\mathbb{H} \cong \mathrm{SL}_2\mathbb{Z}\backslash\mathrm{SL}_2\mathbb{R}/\mathrm{SO}_2$, where $\mathbb{H}$ is the upper half-plane.

For every prime p, X_Γ has a hidden p-adic symmetry group $G_{\mathbb{Q}_p}$, along with the manifest $G_\mathbb{R}$ symmetry. This creates a huge amount of symmetry, allowing one to define operators called Hecke operators. At almost all primes, these operators commute, so can we simultaneously diagonalize them? This is, in some sense, a goal of the Langlands program (and access the secrets of the universe, hopefully).

2.2. **Quantum field theory.** We've just seen representation theory in a nutshell; now, on to quantum field theory in a nutshell.

An n-dimensional quantum field theory $\mathcal{Z}$ attaches to every n-dimensional Riemannian manifold M a Hilbert space $\mathcal{Z}(M)$. It also has *time evolution*: an n-dimensional cobordism $N : M_1 \to M_2$ defines a linear map $\mathcal{Z}(N) : \mathcal{Z}(M_1) \to \mathcal{Z}(M_2)$. Gluing two cobordisms together corresponds to composing their linear maps.

Quantum field theory should be local, and so there's a great deal of structure that can be tracked to understand this condition.

Example 2.3 (Quantum mechanics). Consider a free particle on a manifold X (e.g. $\mathbb{R}^3$), and let $n = 1$. Here, we'll let $\mathcal{Z}(M)$ be a linearization of a space of fields on M, e.g. in the σ-*model*, these fields are maps to X.

In our case, $\mathcal{Z}(\mathrm{pt})$ is the Hilbert space $L^2(X)$, and time evolution is the semigroup defined by the Hamiltonian, which is the Laplacian: $H = \Delta$. Then, the bordism $[0, T]$ is the evolution e^{iH} (so that gluing becomes composition).

Riemannian manifolds are great, but it is sometimes easier to remove the dependence on metrics. A topological field theory removes a dependence on everything but the topology of spacetime. Sometimes, these appear from another source, which is great, but other times, we have to produce these theories by forcing them. To do this, we need to kill the Hamiltonian. Supersymmetry can do this, by making H exact with respect to, e.g. the de Rham operator, and then passing to cohomology.

A *local operator* is a zero-dimensional defect, which labels measurements at a point of any spacetime. Precisely, we take tiny spheres around these points as the sphere shrinks, which defines cobordisms. For example, in quantum mechanics, local operators in quantum mechanics are the operators on the Hilbert space. These operators do not always commute, which is the statement of Heisenberg uncertainty.

We can also consider defects at higher dimensions, or singularities of higher dimensions. A *line operator* is a one-dimensional quantum mechanics living on a one-dimensional submanifold of the spacetime. These also have a huge amount of structure: they form a category.

Scaling this all the way up, a *local boundary condition* is an $(n-1)$-dimensional theory that labels boundaries in $\mathcal{Z}$. These can interface with each other in codimension 2, and there are interfaces between interfaces between interfaces. . . this creates the algebraic structure of an $(n-1)$-*category*.

But what does this structure buy us? There's a conjecture of Baez-Dolan [1], now a theorem of Lurie [7], that it tells us everything.

Theorem 2.4 (Cobordism hypothesis (Lurie)). *An n-dimensional topological field theory $\mathcal{Z}$ is uniquely determined by its higher category of boundary conditions.*

The theorem also contains an existence statement, which corresponds to a finiteness condition. You can start with your favorite $(n-1)$-category, whatever that may be, and when you try to germinate it into data on lower- and lower-dimensional manifolds, it might not be "finite enough." There's a more precise sense in the theorem statement.

A category is a 1-category, so one can think of categories as 2-dimensional TFTs, illuminating a deep geometric perspective on categories. We're interested in the category of representations of a group, so we should think about the topological field theory it describes.

2.3. **Gauge theory and moduli spaces.** Gauge theory linearizes spaces of G-bundles with connections, in a way invariant under gauge transformations. This has been tremendously influential in low-dimensional topology, producing many invariants of 3- and 4-manifolds arising from cohomology of moduli paces of G-bundles on these manifolds.

An alternate point of view is that n-dimensional gauge theories are to be understood as QFTs whose boundary conditions are $(n-1)$-dimensional QFTs with G-symmetry; that is, *gauge theories are representations of groups on field theories.*

Example 2.5 (2-dimensional Yang-Mills theory). Suppose G is finite or compact. This theory more or less counts G-bundles with a connection on spacetime; the boundary condition is quantum mechanics with symmetry group G (i.e. G acts on $L^2(X)$ and the Hamiltonian).

From the topological setting, one can just declare the boundary conditions to be the category of representations of G, and recover a topological field theory. The local operators are given by functions on G-connections on a very small circle; linearizing this, we get the conjugacy-invariant functions on G, the *class functions* $\mathbb{C}[G/G]$.

There are various ways to compose different operators: *operator product expansion* has these two small circles get closer together and collide. Another alternative is the *little-discs* composition, where we surround two close small circles with a larger circle enclosing them; topologically, this is the same as a pair-of-pants bordism.

Because we can move these small circles around each other, local operators commute.[5] This is surprising: the quantumness of quantum mechanics, its noncommutativity, becomes commutativity in topological field theory.

So the local operators on 2-dimensional Yang-Mills theory are the class functions with convolution, which is a commutative algebra, and is in fact the center of the group algebra $\mathbb{C}[G]$. Abstractly, this is the *Bernstein center* of the category of representations of G (or, of the category of boundary conditions). That is, local operators are precisely functions on the dual $(\mathbb{C}[\widehat{G}], \cdot)$.

Just like we did with quantum mechanics, we might want to model a quantum field theory as a theory of maps to a target. The target is called the *moduli space of vacua* $\mathfrak{M}_{\mathcal{Z}}$, the universal answer to the question "if I realize my theory as a theory of maps, what does it map into?" Local operators are functions on $\mathfrak{M}_{\mathcal{Z}}$.

As in algebraic geometry, we've found a way to obtain a space $\mathfrak{M}_{\mathcal{Z}}$ from a ring (the Bernstein center of the category of boundary conditions). This sends local operators with operator product expansion to functions on $\mathfrak{M}_{\mathcal{Z}}$ with pointwise multiplication, and line defects in $\mathcal{Z}$ with operator product expansion to sheaves on $\mathfrak{M}_{\mathcal{Z}}$ with tensor product; Yang-Mills provides us an analogue of spectral decomposition.

This looks a lot like Fourier theory. However, the moduli space is discrete, which is arguably not exciting, just as there's not to say formally about the representations of compact groups. However, passing to three-dimensional theories produces a continuous moduli space, just akin to passing to representations of noncompact groups.

In this case, we replace the σ-model (a theory of maps) with a gauge model (a theory of connections). Instead of 1-forms, we have 2-forms, and in the abelian case, this is literally a Hodge star operator.

Physics teaches that in the three-dimensional case, there's a great amount of geometry on these moduli spaces:

(1) If two operators travel in linked loops, we obtain a bracket, which is a Poisson bracket: the moduli space is a Poisson variety.
(2) This has a canonical quantization, called the *Nekrasov Ω-background.*
(3) For gauge theories, or those extending to four dimensions, $\mathfrak{M}_{\mathcal{Z}}$ is what's called a *Seiberg-Witten integrable system.*

In other words, *moduli spaces of gauge theories are precisely the modern geometric setting of representation theory*: both are active research areas.

For example, the A-model is a two-dimensional TFT that measures symplectic geometry, and the B-model models complex geometry. Mirror symmetry can be thought of as a Fourier transform between these models. Since the boundary conditions of a three-dimensional TFT are two-dimensional with a group action, they

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[5]The proof, and the picture, is identical to the picture drawn to show that $\pi_n(X)$ is abelian when $n \geq 2$.

can capture symmetries in symplectic and complex geometry. The analogue of a Fourier transform in this setting leads to the active program called *symplectic duality*.

2.4. **Electric-magnetic duality.** Beilinson-Drinfeld developed a geometric counterpart to the Langlands program, a kind of harmonic analysis taking place on categories of bundles on a Riemann surface. Instead of focusing on a locally symmetric space X_Γ, we focus on a moduli of bundles $\mathrm{Bun}_G(C)$; the Hilbert space of functions on X_Γ is replaced with a category of sheaves on this moduli space. Operators are replaced with functors, and prime numbers are replaced with points of the Riemann surface.

In particular, Hecke operators are functors, and should be some sort of integral operators (convolutions) on sheaves. If $\mathscr{F}$ is a sheaf and $\mathcal{P}$ is a bundle, we'd like to send $\mathscr{F}(\mathcal{P})$ to a "weighted average" of $\mathscr{F}(\mathcal{P}')$ for nearby bundles $\mathcal{P}'$. Specifically, we'd like to modify at a single point x, and keep everything else the same. This lives on the canonical non-Hausdorff space $C \amalg_{C \setminus x} C$, which has two projections down to the two copies of C. This change is called a *Hecke modification*.

Kapustin and Witten realized this can be interpreted via a 4-dimensional gauge theory: Hecke modification is the creation of a magnetic monopole in the bundle. The worldline of this monopole, called a *'t Hooft line*, is a line defect in the theory.

The fundamental question, important for harmonic analysis (or its analogue), is *why do Hecke operators commute?* There was no reason to expect this, but the operators we described do commute — for the same reason as the commutativity of local operators we described above: modifications at two different points don't interact, and modifications at the same point can be dragged off each other, swapped, and slid back onto each other: since the field theory is topological, these all describe the same operator. This is the same insight as to why higher homotopy groups commute.

Beilinson-Drinfeld axiomatized this multiplication structure, an algebraic structure encoded in collisions of points, into a *factorization algebra*. This provides a geometric theory of operators in conformal field theory (vertex algebras) and quantum field theory.

This was a very fruitful insight: Gaitsgory showed that even if there are singularities (ramified singularities), one recovers the same center as for the ramification. This is a two-dimensional solenoidal defect. This itself had applications by Bezrukavnikov and more, to modular representation theory and more.

Peter Scholze managed to port this back to number theory, producing a geometric source to the commutativity of classical Hecke operators, in the setting of $\mathrm{Spec}\,\mathbb{Z}$ and $\mathrm{Spec}\,\mathbb{Z}_p$ (which lies near the prime p in $\mathrm{Spec}\,\mathbb{Z}$). Very recently, this led to Faurges' conjecture, a physics-inspired conjecture shedding new light on the Langlands conjecture.

In four dimensions, the Hodge star sends 2-forms to 2-forms, so the dual of a gauge theory isn't a σ-model, but rather another gauge theory. This recalls the fact that Maxwell's equations in a vacuum (a gauge theory with gauge group $\mathrm{U}(1)$) is symmetric under Hodge star, which exchanges the roles of electricity and magnetism. A nonabelian generalization, called S-theory, relates a gauge theory with gauge group G to one for its dual group.

Kapustin and Witten interpret geometric Langlands in terms of S-duality: sheaves on the moduli of bundles are boundary conditions for $\mathcal{N} = 4$ super Yang-Mills theory for G, and the Hecke operators correspond to 't Hooft line operators. One can write down an analogue of the Fourier transform.

The physics goes up to eleven! Specifically, M-theory. But the richest known representation-theoretic structure is a six-dimensional theory, known as "theory $\mathcal{X}$."

3. Group Algebras and Convolution: 8/30/16

Last time, we talked about the Fourier transform in the context of a locally compact abelian group G and its unitary dual $\widehat{G} = \mathrm{Hom}_{\mathsf{TopGrp}}(G, \mathrm{U}(1))$. On $G \times \widehat{G}$, there's a *universal character* function $\underline{\chi} : (g, \chi) \mapsto \chi(g)$ evaluating a character on a point. We used this to define the Fourier transform; Theorem 1.5 tells us it induces an isomorphism $L^2(G) \cong L^2(\widehat{G})$.

If π_1 and π_2 are the projections onto the first and second components of $G \times \widehat{G}$, respectively, we can define the Fourier transform as

$$f \longmapsto \pi_{2*}\big(\pi_1^* f \cdot \underline{\chi}\big),$$

or replacing the pushforward with an integral under the Haar measure,

$$\widehat{f}(\chi) = \int_G f(g)\chi(g)\,\mathrm{d}g.$$

In the case $G = \mathbb{R} = \widehat{G}$, this specializes to the usual Fourier transform.

One important takeaway is that this pullback-pushforward formalism applies in many other situations. For example, if X and Y are finite sets, so $F(X)$ denotes the complex-valued functions on X, a $K \in F(X \times Y)$ is an integral kernel, in the sense that we can define a transform $f \mapsto \pi_{2*}(\pi_1^* f \cdot K)$ (where, once again, π_1 and π_2 are the canonical projections). Identifying K with an $|X| \times |Y|$ matrix, this function is just multiplying vectors in $F(X)$ by K. The identity matrix/transform corresponds to the kernel δ_Δ, which is 1 at every diagonal element of $X \times X$ and 0 everywhere else.

We also mentioned that the Fourier transform exchanges convolution and pointwise product. This is an instance of an insight from last time: the Fourier transform is trying to diagonalize the action of a group. Differentiation, a type of infinitesimal translation, is also transformed into a multiplication operator, as is an ordinary translation.

For the rest of this lecture, we will not assume our groups are abelian; we'll return to abelian groups later.

Group algebras. Let G be a group and V be a representation of V; we don't ask for finiteness of G or $\dim V$. This representation is given by a homomorphism $\rho : G \to \operatorname{Aut}(V) \subset \operatorname{End}(V)$. If we pass to $\operatorname{End} V$, we have both multiplication and addition, so we can construct new operators not in the image of ρ. Specifically, we'll take finite linear combinations of $\rho(g_i)$ for various $g_i \in G$. As such, we may as well assume G is finite.

Let ω be a finitely supported measure in G, so

$$\omega = \sum_{g \in G} \omega_g \delta_g,$$

i.e. ω is a finite sum of δ-measures. This defines an endomorphism of V

$$\rho(\omega) = \sum_{g \in G} \omega_g \rho(g).$$

If we assume G is finite, so we may identify functions and measures, then the algebra of such endomorphisms is $\mathbb{C}G$, the associative algebra of functions on G,[6] with multiplication given by convolution, the unique map extending $\delta_g * \delta_h = \delta_{gh}$. Since these δ-measures span $\mathbb{C}G$, every representation V defines a homomorphism $(\mathbb{C}G, *) \to \operatorname{End} V$. The algebra $\mathbb{C}G$ is commutative iff G is abelian.

We can explicitly write down this convolution formula: if $\omega = \sum \omega(g)\delta_g$ and $\tau = \sum \tau(g)\delta_g$, then

$$(\omega * \tau)(g) = \sum_{h \cdot k = g} \omega(h)\tau(k)$$

$$= \sum_{h \in G} \omega(h)\tau(h^{-1}g).$$

This looks a little more like the usual convolution: when $G = \mathbb{R}$, we can convolve L^1 functions by

$$(f * g)(x) = \int_{\mathbb{R}} f(y)g(x - y)\, \mathrm{d}y.$$

We can rethink this in terms of projections from $G \times G$.

Here, $\mu : G \times G \to G$ is multiplication. We can define $\omega \boxtimes \tau = \pi_1^* \omega \cdot \pi_2^* \tau$; then, convolution arises from the pushforward $\omega * \tau = \mu_*(\omega \boxtimes \tau)$. Functions pull back, but measures push forward via integration, so this is really about measures, and we don't need to worry about whether G is finite (e.g. we just used $\mathbb{R}$).[7] In fact, all we need is for G to be a measurable group, so that $L^1(G)$ is an algebra under convolution.

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[6] Really, these should be measures: we know the δ-measure at a point, but for infinite G it doesn't make sense to take constant functions.

[7] You might worry that measures can't pull back, but we know how to take the product measure on a product space, and that suffices.

If G is a locally compact topological group, then we can take the compactly supported continuous functions $C_c(G)$, which form an associative algebra under convolution. In this setting, $(C_c(G), *)$ is usually called the *group algebra*. This is the continuous analogue of taking linear combinations of group elements and multiplying them, "smearing out" the group multiplication. In this setting, our representations need to have some good notion of integration, so finite-dimensional or a locally convex topological vector space. The action of G on such a representation V induces a homomorphism $C_c(G) \to \mathrm{End}(V)$. In many settings, it's easier to think about modules for an algebra than representations of a group.

If you're coming from algebraic geometry, you might not like measures, as we don't in general know how to integrate/pushforward. In this case, we need to talk about functions. We still have a multiplication map $\mu : G \times G \to G$, so pullback defines a map $\mu^* : F(G) \to F(G \times G)$. There's also a map $F(G) \otimes F(G) \to F(G \times G)$; in nice settings, a Künneth theorem ensures this map is an isomorphism (in algebraic geometry, this is more or less a definition). In analysis, though, this won't literally be true unless you use a completed tensor product.

When this isomorphism holds, we get a map $\Delta : F(G) \to F(G) \otimes F(G)$, called the *coproduct map*. Just like an algebra is a vector space A together with an associative multiplication map $\mu : A \otimes A \to A$, a *coalgebra* is a vector space C together with a *coassociative* map $\Delta : C \to C \otimes C$.

What does it mean to be *co*associative? Abstractly, associativity means that the following diagram is a coequalizer diagram:

$$ A \otimes A \otimes A \underset{1 \otimes \mu}{\overset{\mu \otimes 1}{\rightrightarrows}} A \otimes A \overset{\mu}{\longrightarrow} A. $$

That is, $\mu(a, \mu(b, c)) = \mu(\mu(a, b), \mu(c))$. Turn the arrows around, and coassociativity means the following diagram commutes (so is an equalizer diagram):

$$ C \overset{\Delta}{\longrightarrow} C \otimes C \underset{1 \otimes \Delta}{\overset{\Delta \otimes 1}{\rightrightarrows}} C \otimes C \otimes C. $$

Since group multiplication is associative, multiplication in the group algebra is associative, or in the function setting, the coproduct in the coalgebra is coassociative. If you don't have a notion of measure, you'll have to work with coalgebras; in the settings we consider, there will be enough duality to not need to worry about this. Specifically, the dual of an algebra A is a coalgebra A^*, and measures go to functions, and vice versa.

Recall that if A is an algebra, then a module for A is a vector space V together with an action $A \otimes V \to V$ compatible with multiplication; if V is finite-dimensional, this is the same as a map $A \to \mathrm{End}\, V$. Correspondingly, if A^* is a coalgebra, we can define a *comodule* V, which is a space with a map $V \to V \otimes A^*$ compatible with the coproduct.

Matrix Coefficients. These are in a sense two ways of saying the same things, and we can talk about it in a third way: matrix coefficients, which is an extremely useful perspective on representations.

Let V be a finite-dimensional representation of a group G, and V^* be its dual. The *matrix elements* (or *matrix coefficients*) map $V \otimes V^* \to F(G)$ is defined to be the function $g \mapsto \langle w, v \cdot g \rangle$: we pair w and v after acting by g. Dualizing, this is also a map $V \to V \otimes F(G)$, putting w on the other side, or we can take $F(G)^* \otimes V \to V$. This first space is measures on G, hence the group algebra: this map is the usual action $\mathbb{C}G \otimes V \to V$, but thought of in a different way.

Why the name matrix coefficients? Let $\{e_i\}$ be a basis of V and $\{e^j\}$ be the dual basis for V^*. Then, the matrix coefficients map extends uniquely from the assignment $e_i \otimes e^j \mapsto \langle e^j, g \cdot e_i \rangle$, which is the ij^{th} entry of the matrix for g in this basis. Thus, this really is a matrix coefficients map, but stated coordinate-independently.

Definition 3.1. This allows us to define many classes of representations by way of what classes of functions they product under matrix coefficients.

- If G is a Lie group, a *smooth representation* is a representation whose matrix elements are smooth functions (so in $C^\infty(G)$).
- If G is a locally compact group, a *compactly supported representation* is one whose matrix elements are compactly supported continuous functions.
- If G is a Lie group, an *analytic representation* is one whose matrix coefficients are analytic functions.
- If G is a Lie group, a *tempered representation* is one whose matrix coefficients are L^2 functions.

- If G is an algebraic group, an *algebraic representation* is one whose matrix coefficients are algebraic functions.

If V is finite-dimensional, $\operatorname{End} V \cong V \otimes V^*$, and the identity endomorphism is a canonical element of this algebra (in any basis, this is the sum $\sum e_i e^i$). The matrix coefficient associated to this canonical element is called the *character* of V, denoted χ_V.

The matrix coefficients map is first of all a map of vector spaces, but there's a lot more structure: the G-action on V induces an action on V^*, so there is a $(G \times G)$-action on $V \otimes V^*$. Left and right multiplication[8] defines a $(G \times G)$-action on G, hence also on $F(G)$.

Exercise 3.2. Show that matrix coefficients is a $(G \times G)$-equivariant map $V \otimes V^* \to F(G)$.

This carries a lot of structure; for example, this defines a G-action through the diagonal $\Delta : G \to G \times G$: both vectors are acted upon in the same way, and G acts on $F(G)$ by conjugation (since the right action is multiplication by g^{-1}). Under the identification $V \otimes V^* \cong \operatorname{End} V$, this is also conjugation of matrices by those of G. Exercise 3.2 implies that matrix coefficients is invariant under conjugation, or in other words:

Corollary 3.3. *The character of V is a class function: $\chi_V \in (F(G))^G$.*

One of the simplest things we can do with a representation is look at what it fixes.

Definition 3.4. Let V be a representation of G. Then, the *G-invariants* are $V^G = \{v \in V \mid g \cdot v = v \text{ for all } v \in G\}$.

We use the notation $F(G/G)$ for $F(G)^{G_\Delta} = \{f \in F(G) \mid f(h) = f(ghg^{-1}) \text{ for all } g \in G\}$.

Back to abelian groups. This allows us to reinterpret the Fourier transform; we once again suppose that G is a locally compact abelian group. In this case, any notion of the group algebra (compactly supported functions, integrable functions, or all functions in the finite case, or $\mathbb{C}\mathbb{Z} = C_c(\mathbb{Z})$) is commutative.

The fundamental instinct we owe to Gelfand and Grothendieck is that whenever we see a commutative algebra A, we should associate a space X to it, such that $A = F(X)$. This space is called the spectrum, and understanding it is akin to diagonalizing the algebra.

We defined the dual group $\widehat{G}$ by hand, but it turns out to arise naturally as the spectrum of the group algebra, in whichever sense we care about.

Example 3.5. Suppose G is finite, so there aren't many interesting examples, but a lot of different notions of group algebra agree. In fact, let's suppose $G \cong \mathbb{Z}/p$. In this case, $\mathbb{C}[\mathbb{Z}/p] = \bigoplus \mathbb{C} \cdot \chi_n$ for the characters $\chi_n : 1 \mapsto e^{2\pi i n/p}$ provides a basis. If we define $\chi_m \cdot \chi_n = \delta_{mn}$, then this assignment is an algebra isomorphism, and the characters are orthogonal idempotents.

That is, if we think of $\mathbb{C}G$ as functions on a set, then that set is really the set of characters, because the algebra of functions on it with pointwise multiplication is the same as $\mathbb{C}G$. The geometry of this space is a disjoint union of p points.

More generally, for any finite-dimensional G-representation V, we may decompose V as a direct sum of eigenspaces: $V \cong \bigoplus_n V_{\chi_n}$. Alternatively, V is a module for the algebra $\mathbb{C}G$. Over the set of points, the characters $\chi \in \widehat{G}$, we've produced a decomposition into a vector space over each point. That is, *representations of G are the same as vector bundles on $\widehat{G}$.*

This silly when G is finite, but we will apply it in more general situations, and it is our perspective on the Fourier transform. It sends characters to points, L^2 to L^2, and $\mathbb{C}G$ and convolution to $F(\widehat{G})$ under pointwise multiplication: we've diagonalized the action of G. We've only yet justified this in the finite setting, but we'll explain what it means in the infinite setting.

Example 3.6. If $G = \mathbb{Z}$, different notions of the group algebra coincide: $\mathbb{C}\mathbb{Z}$ is the algebra of finite $\mathbb{C}$-linear combinations of formal elements of $\mathbb{Z}$, with convolution for multiplication. This is isomorphic, as rings, to the Laurent polynomials $\mathbb{C}[z, z^{-1}]$, because $z^m \cdot z^n = z^{m+n}$. That is, the group algebra of $\mathbb{Z}$ is the algebra of algebraic (i.e. polynomial) functions on $\mathbb{C}^*$ (since we may invert 0 finitely many times); more succinctly,

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[8]In order for these to both be left actions, the right multiplication must be $v \mapsto v \cdot g^{-1}$. This is what we mean by right multiplication here.

$\mathrm{Spec}(\mathbb{C}\mathbb{Z}, *) = \mathbb{C}^*$.[9] More generally, any representation of $\mathbb{Z}$ is the same as a module for $\mathbb{C}[z, z^{-1}]$. Later, we will learn to call these the fancy name of "quasicoherent sheaves on $\mathbb{C}^*$," which is the geometric content: this is a geometric structure akin to a vector bundle.

Example 3.7. Let $G = \mathrm{U}(1)$; we'll consider algebraic functions, so finite Fourier series. Here, we get $\mathbb{C}\mathrm{U}(1) \cong \mathbb{C}[z, z^{-1}]$ where $z = e^{2\pi i\theta}$, but the algebra structures are *different*: the convolution product is instead pointwise multiplication of finitely supported functions on $\mathbb{Z}$, so the dual to $\mathrm{U}(1)$ should be $\mathbb{Z}$. Thus, representations of the circle will correspond to vector bundles on the integers.

4. SPECTRAL DECOMPOSITION: 9/1/16

Last time, we talked about group algebras; let's remind ourselves what happened.

When G is a finite group, we can define a convolution operation $F(G) \otimes F(G) \to F(G)$ defined by

$$f * g \longmapsto \mu_*(\pi_1^* f \cdot \pi_2^* g).$$

Here, μ, π_1, and π_2 all fit into the diagram

$$
\begin{array}{ccc}
 & G \times G & \\
\pi_1 \swarrow & \downarrow \mu & \searrow \pi_2 \\
G & & G \\
 & \downarrow & \\
 & G. &
\end{array}
$$

This diagram looks suspiciously like a diagram for matrix multiplication

$$
\begin{array}{ccc}
 & X \times X \times X & \\
\pi_{12} \swarrow & \downarrow \pi_{13} & \searrow \pi_2 3 \\
X \times X & & X \times X \\
 & \downarrow & \\
 & X \times X, &
\end{array}
$$

where the matrix product is $\pi_{13*}(\pi_{12}^* f \cdot \pi_{23}^* g)$.

This is not a coincidence: the group algebra $\mathbb{C}G$, which is the algebra of complex-valued functions on G under convolution, maps into $\mathrm{End}(F(G))$. The action of G on itself by left multiplication defines a representation of G on $F(G)$, hence an action of $\mathbb{C}G$ on $F(G)$.

There's also an action of G on itself from the right. A nice general fact is that a map $T : G \to G$ that commutes with left action arises as a right action by an element of G: $T(gk) = gT(k)$, so T is right multiplication by $T(1)$, as $T(k) = k \cdot T(1)$.[10]

The greater point is that there's only one kind of algebra, which is matrix algebra. We can embed $\mathbb{C}G$ into the functions on $G \times G$ (matrices parameterized by G) invariant under the diagonal action of $g \in G$ acting by (g, g); this map is an isomorphism.

In other words, the group algebra $\mathbb{C}G$ is a subalgebra of the matrix algebra $F(G \times G)$. Inside of this matrix algebra are those elements which are invariant under the diagonal action of G (i.e. those which are conjugation-invariant), and this is exactly $\mathbb{C}G$. Alternatively, $\mathbb{C}G$ is the operators on $F(G)$ that commute with the right action of G. The point of all these equivalences is that $\mathbb{C}G$ is not just an abstract algebra; it already comes with a pretty concrete representation.

Spectral decomposition. Back to where we were last time.

There are a lot of uses of the word "spectrum" in mathematics: the spectrum in analysis, the spectrum in graph theory, the spectrum in algebraic geometry, and even the spectrum in astrophysics all have the same origin, relating to the set or space of eigenvalues of something.[11]

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[9]We previously said that $\mathrm{U}(1)$ was dual to $\mathbb{Z}$, but now we obtained $\mathbb{C}^*$. The idea is that every $\lambda \in \mathbb{C}^*$ corresponds to a 1-dimensional $\mathbb{Z}$-representation, where 1 acts as multiplication by λ. Since λ and 1 are both invertible, this indeed defines a representation; but unless $|\lambda| = 1$, this is not a unitary representation. Indeed, if you require unitarity, you obtain $\widehat{\mathbb{Z}} = \mathrm{U}(1)$ as expected.

[10]This fact, and its proof, apply in more general situations, e.g. the action of a ring on itself by left and right actions.

[11]The usage of spectrum in homotopy theory appears to be unrelated.

Last time, we saw that for $G = \mathrm{U}(1)$, the group algebra of finite Fourier series $\mathbb{C}\mathrm{U}(1)_{\mathrm{fin}}$ with convolution is isomorphic to the finitely supported functions on $\mathbb{Z}$. If V is a finite-dimensional (or at least locally convex) vector space, a $\mathrm{U}(1)$-representation on V defines an action of $F_{\mathrm{fin}}(\mathbb{Z})$ on V. These functions are the span of the delta-functions δ_n for $n \in \mathbb{Z}$, which are orthogonal idempotents: $\delta_n \delta_m$ is 0 when $n \neq m$ and 1 when $n = m$. Thus, each δ_n acts as an idempotent operator on V, hence a projection onto $V_n = \mathrm{Im}(\delta_n)$; since the δ_n are orthogonal, so are the V_n, defining an orthogonal direct sum

$$(4.1) \qquad V_{\mathrm{fin}} = \bigoplus_{n \in \mathbb{Z}} V_n \subset V.$$

The subspace V_n is the $e^{2\pi i n x}$-eigenspace of the action of $\mathrm{U}(1) \subset V$.

One of the corollaries of Fourier theory is that the inclusion (4.1) is dense: the closure of the span of the V_n is all of V. Why is this? The identity operator on V is an infinite combination

$$\sum_{n \in \mathbb{Z}} \delta_n = 1,$$

hence (after an analytic argument) the limit of finitely supported operators.

One might call V_{fin} the space of $\mathrm{U}(1)$-*finite vectors* of V: these are the vectors that are contained in finite-dimensional subrepresentations of V. This is a really nice discovery by Harish-Chandra: even for noncompact, nonabelian groups, it's possible to extract this algebraic core of the representation.

So to every $n \in \mathbb{Z}$, we've associated a vector space V_n over n. Taking these all at once, we've defined a vector bundle over $\mathbb{Z}$: $\mathbb{Z}$ is discrete, so the dimensions can jump, but the key concept that representations turn into vector bundles will be very important.

Example 4.2. Suppose $G = \mathbb{R}$ acts on a finite-dimensional vector space, or the action is differentiable. In this case, there's an action of $\mathrm{Lie}\, G = \mathbb{R} \cdot \frac{\mathrm{d}}{\mathrm{d}x}$ on V: an $x \in \mathbb{R}$ acts by $e^{2\pi i x \cdot H}$, where $H : V \to V$ is the action of $\frac{\mathrm{d}}{\mathrm{d}x}$. Here, V is unitary iff H is self-adjoint. This will be very useful next week, when we talk about quantum mechanics.

We've derived data of a vector space V and a single operator H acting on V. A notion called *functional calculus* means we get an action of $\mathbb{C}[t]$ on V, where we think of $\mathbb{C}[t]$ as the algebraic functions on the affine line $\mathbb{A}^1$.[12] We can describe the action explicitly as

$$\sum a_i t^i \longmapsto \sum a_i H^i.$$

The point is, the action of an operator produces the action of a module in one variable. If we kept track of unitarity, then this restricts to an action of the fixed points under conjugation, which are the real affine line $\mathbb{A}^1_{\mathbb{R}}$.

Suppose we have an action of $\mathbb{C}[t]$ on a space V. How should we think of this geometrically? It will be something akin to a vector bundle on $\mathbb{A}^1 = \mathrm{Spec}\,\mathbb{C}[t]$: specifically, it will be a quasicoherent sheaf. Over every $\lambda \in \mathbb{A}^1$, the fiber V_λ over λ is $V_\lambda/(t - \lambda)$. More formally, $V_\lambda = V \otimes_{\mathbb{C}[t]} \mathbb{C}$, where $\mathbb{C}[t]$ acts on $\mathbb{C}$ by $t \mapsto \lambda$. This is a quotient of V, sort of the "co-eigenspace" because we've quotiented by the λ-eigenspace V^λ; these V_λ fit together into a nice family.

Suppose $U \subset \mathbb{A}^1$ is open; to it, we may associate the vector space $\underline{V}(U) = V \otimes_{\mathbb{C}[t]} \mathbb{C}[U]$. Here, $\mathbb{C}[U]$ is the algebra of operators in $\mathbb{C}[t]$ where we invert those that don't vanish on U. Using this definition, $\underline{V}$ is a quasicoherent sheaf on $\mathbb{A}^1$.

Intuitively, a quasicoherent sheaf is data parametrized by the open subsets of X that satisfies a local-to-global principle.

Definition 4.3. Let X be a variety.[13] A *quasicoherent sheaf* $\mathscr{F}$ on X starts with the following data.

- For every open $U \subset X$, we obtain an abelian group (actually, $\mathscr{O}_X(U)$-module) $\mathscr{F}(U)$.
- For every inclusion of open subsets $V \subset U$, we have a *restriction map* $\rho^U_V : \mathscr{F}(U) \to \mathscr{F}(V)$.

We require it to satisfy the following conditions.

- Restriction maps should compose: a chain of inclusions $W \subset V \subset U$ implies $\rho^V_W \circ \rho^U_V = \rho^U_W$.

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[12]More precisely, $\mathbb{A}^1 = \mathrm{Spec}\,\mathbb{C}[t]$, the spectrum as it arises in algebraic geometry; as a set, $\mathrm{Spec}\,\mathbb{C}[t]$ is the set of prime ideals of $\mathbb{C}[t]$ with a topological and a geometric structure.

[13]For this class, a variety is something that has a structure sheaf, i.e. for every open $U \subset X$, we have a notion of *algebraic functions* $\mathscr{O}_X(U)$ that behaves well with respect to restriction and gluing.

- If $\mathfrak{U}$ is an open cover of U, then $\mathscr{F}(U)$ must be determined by $\mathscr{F}(U_i)$ over all $U_i \in \mathfrak{U}$, in the same sense that U is determined by $\mathfrak{U}$. Specifically, the following diagram is a coequalizer diagram:

$$\coprod_{U_i, U_j \in \mathfrak{U}} U_i \cap U_j \rightrightarrows \coprod_{U_i \in \mathfrak{U}} U_i \longrightarrow U.$$

Thus, we require the following diagram to be an equalizer diagram.

$$\mathscr{F}(U) \longrightarrow \prod_{U_i \in \mathfrak{U}} \mathscr{F}(U_i) \rightrightarrows \prod_{U_i, U_j \in \mathfrak{U}} \mathscr{F}(U_i \cap U_j).$$

- Quasicoherence means there's a semilocal-to-local condition: if $U \subset X$ is affine and $V \subset U$, then $\mathscr{F}(V) = \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{O}_X(V)$.

This is a large and perhaps confusing definition.

If $\mathbb{C}[t]$ acts on a vector space V such that V is a finitely generated $\mathbb{C}[t]$-module, the fundamental theorem of finitely generated modules over a PID says that V is the direct sum of its torsion and its free parts, and the torsion part is

$$V_{\text{torsion}} = \bigoplus_{\lambda \in \mathbb{A}^1} V_{\widehat{\lambda}}.$$

where $V_{\widehat{\lambda}}$ is the generalized eigenspace

$$V_{\widehat{\lambda}} = \{v \in V \mid (t - \lambda)^N v = 0\} = \bigcup_{n \geq 1} V_{n\lambda},$$

where $V_{n\lambda}$ is the subspace of V annihilated by $(t - \lambda)^n$. These act by Jordan blocks: $V_{\widehat{\lambda}} \cong \mathbb{C}[t]/(t - \lambda)^n$.

This linear algebra becomes a picture for the spectral decomposition: for every eigenvalue $\lambda \in \mathbb{A}^1$, we have the space $V_{\widehat{\lambda}}$ over V, and these fit together into a quasicoherent sheaf on $\mathbb{A}^1$.

This is the algebraic perspective; let's see how it relates to analysis. Translations define an action of $\mathbb{R}$ on $C^\infty(\mathbb{R})$. The eigenvectors of differentiation are $e^{2\pi i \lambda x}$, since these satisfy $\frac{\mathrm{d}}{\mathrm{d}x} f = (2\pi i \lambda) f$. On the other side, these are Fourier-transformed into the eigenvectors for multiplication, δ_n. These don't live in $L^2(\mathbb{R})$, however.

The generalized eigenvectors for λ acting on distributions include the derivative of δ_λ: the action of t on $\text{span}\{\delta_\lambda, \delta_\lambda'\}$ is a Jordan block. In the same way,

$$\ker\left(\frac{\mathrm{d}}{\mathrm{d}x} - \lambda\right)^2 = \{e^{2\pi i \lambda x}, x e^{2\pi i \lambda x}\},$$

and the action of $\frac{\mathrm{d}}{\mathrm{d}x}$ is a Jordan block in this basis. This is a nontrivial extension of the one-dimensional $\mathbb{R}$-representation $\text{span}\, e^{2\pi i \lambda x}$ by itself:

$$0 \longrightarrow \text{span}\, e^{2\pi i \lambda x} \longrightarrow \text{span}\{e^{2\pi i \lambda x}, x e^{2\pi i \lambda x}\} \longrightarrow \text{span}\, e^{2\pi i \lambda x} \longrightarrow 0.$$

But this does not happen in the world of unitary representations: the existence of an inner product means it's possible by Maschke's theorem to find a complement to any subrepresentation. Thus, all extensions split, and all representations are completely decomposable.

This adds a new entry to the dictionary between the representation theory of G and the geometry of $\widehat{G}$:

- A representation of G goes to a sheaf on $\widehat{G}$.
- Characters on G correspond to points on $\widehat{G}$.
- Extensions of representations correspond to infinitesimals.
- G is compact iff $\widehat{G}$ is discrete. (In this case, there are no nontrivial extensions, and no nontrivial infinitesimals.)

Algebraic geometry in general replaces $\mathbb{C}[t]$ by any commutative ring R, whose representations correspond to any number of commuting operators with specified relations. A representation of R on V will define a quasicoherent sheaf on $\text{Spec}\, R$, spreading out the representation as a family of vector spaces that algebraic geometry can help us understand. In other words, whenever we have commuting operators, we find geometry.

This whole story applies in several different geometric settings.

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- If we start with an algebraic variety, we care about polynomial functions, which form a ring. Spectral theory produces a quasicoherent sheaf (modules in an affine setting).
- If we start with a locally compact topological space, we care about compactly supported continuous functions, which form a C^*-algebra.[14] Spectral theory produces a Hilbert C^*-module.[15] Geometrically, this is the same as a vector bundle with a metric (a fiberwise inner product).
- If X is a measure space,[16] we can talk about essentially bounded functions $L^\infty(X)$, which form something called a *von Neumann algebra*. Spectral theory produces a *projection-valued measure*.

$\mathbb{R}$ has all of these structures, so we can see how different kinds of representations of $\mathbb{R}$ spread out in completely different ways.

Next time, we'll talk about quantum mechanics, which uses all of these rows.

5. Projection-Valued Measures: 9/6/16

To a group G we associated a group algebra $\mathbb{C}G$, but this is actually the same notion in different clothes: they all agree in the finite case ($\mathbb{C}G$ is the algebra of functions on G), but in the infinite case, we have different notions. For topological groups, we consider compactly supported continuous functions $C_c(G)$; for measurable functions we consider $L^1(G)$; and for Lie groups we consider $L^2(G)$, all with the convolution product.

In each case, we restrict to "nice" representations in accordance with this specific notion of group algebra and specific kind of group. For example, we consider unitary representations of a locally compact topological group, and in accordance with the C^*-algebra structure on $C_c(G)$, unitary representations may be interpreted as Hilbert C^*-modules, with an inner product valued in $C_c(G)$.

If G is commutative, then any notion of the group algebra $\mathbb{C}G$ is a commutative algebra. The general idea of spectral theory, as espoused by Gelfand and Grothendieck, is that a commutative algebra should be though of as an algebra of functions on a space under pointwise multiplication. There's a dictionary between the algebra and the geometry:

- Starting with a ring,[17] one can obtain an affine algebraic variety X, and from an algebraic variety X, one obtains its ring $\mathbb{C}[X]$ of algebraic functions.
- If we start with a C^*-algebra, we can associate it to a locally compact Hausdorff topological space X, called its spectrum, and recover it as the C^*-algebra of compactly supported continuous $\mathbb{C}$-valued functions on X.
- A von Neumann algebra determines a measure space X, and we can recover it as $L^\infty(X)$.

These three algebraic notions — rings, C^*-algebras, and von Neumann algebras — all come in commutative and noncommutative versions, but here we must restrict to the commutative case.

The key theorem of spectral theory is that this dictionary is an equivalence; since functions pull back, this will be a contravariant equivalence.

- For example, commutative rings are equivalent to (the opposite category of) affine schemes.
- The Gelfand-Neimark theorem says that a commutative C^*-algebra A determines and is determined by a locally compact Hausdorff space $\mathrm{mSpec}\, A$ called the *Gelfand spectrum*. This spectrum may be characterized as the representations of A, the maximal ideals of A, or the C^*-homomorphisms $(A, *) \to (\mathbb{C}, *)$.
- Similarly, a commutative von Neumann algebra is the same thing as a measurable space.[18] But isomorphism of measure spaces (*measurable equivalence*) in this context is weak enough that there are only five classes: a finite discrete set $\{1, \ldots, n\}$, an infinite discrete set $\mathbb{N}$, an interval $[0, 1]$, an union $[0, 1] \cup \{1, \ldots, n\}$, and a union $[0, 1] \cup \mathbb{N}$.

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[14]A C^*-*algebra* is a Banach $\mathbb{C}$-algebra (so a $\mathbb{C}$-algebra that is compatibly a Banach space) with a $\mathbb{C}$-antilinear involution $*$ such that $(xy)^* = y^* x^*$ and $\|x\|^2 = \|xx^*\|$. This is an algebraic axiomatization of the notion of continuous complex-valued functions, with $*$ given by pointwise complex conjugation.

[15]This is a Hilbert space that is also a C^*-*module*: a continuous action of the C^*-algebra along with the action of $*$.

[16]Caveat: X must be a localizable measure space, though this includes pretty much any measure space you've ever used or heard of.

[17]Technically, to obtain an algebraic variety, we should start with a finitely-generated reduced $\mathbb{C}$-algebra.

[18]Again, we must restrict to *localizable* measure spaces; see `http://mathoverflow.net/a/137900`, or [11] for a proof.

So in every case, a commutative group has a commutative group algebra, to which we associate a space $\widehat{G}$, dual to the group in the sense of Fourier theory. The structure on G induces a certain structure on $\widehat{G}$, and the point is that spectral decomposition (again, in a slightly different form in each setting) turns a representation of a commutative G into a module for $\mathbb{C}G$, which is turned into a family of vector spaces over the spectrum $\widehat{G} = \operatorname{Spec}\mathbb{C}G$. As a simple example, if G is finite, $\widehat{G}$ is a finite set, and the family of vector spaces is the joint eigenspaces for all $a \in \mathbb{C}G$.

In each setting, the term "family of vector spaces" means something different. In the algebraic setting, so that $\mathbb{C}G$ is a variety or scheme, then a family of vector spaces is a quasicoherent sheaf (the same thing as a module, in the quasicoherent setting); if G is a locally compact group, the family of vector spaces is a vector bundle on $\widehat{G}$ with a nondegenerate inner product. Finally, in the measurable setting, we obtain something called a projection-valued measure. In each case, the fibers are eigenspaces for the actions of group elements.

Let's elaborate on the situation for projection-valued measures.

Definition 5.1. Let X be a measure space, so we have a σ-algebra M of measurable sets. A *projection-valued measure* is the data of a fixed Hilbert space $\mathscr{H}$ and, for every measurable $U \subset X$, a self-adjoint projection $\pi(U)$ on $\mathscr{H}$, obeying a countable additivity axiom: for all $v, w \in \mathscr{H}$, the function $U \mapsto \langle w, \pi(U)v \rangle \in \mathbb{C}$ should be a complex measure.

You should think of this as a "measurable family of Hilbert spaces" over X. If $v \in \mathscr{H}$ is a unit vector, $U \mapsto \langle v, \pi(U)v \rangle$ is real and in $[0,1]$, so is a probability measure. Another consequence is that if E and F are disjoint, measurable subsets of X, then their sections $\pi(E)$ and $\pi(F)$ are orthogonal; more generally, $\pi(E) \circ \pi(F) = \pi(E \cap F)$, a version of the presheaf property.

To every measurable characteristic function 1_X we've assigned the projector $\pi(U)$, and this extends to a map $L^\infty(X) \to \mathcal{B}(\mathscr{H})$, the space of bounded operators on $\mathscr{H}$. This allows for a very general formulation of the spectral theorem.

Theorem 5.2 (Spectral theorem (von Neumann))**.** *Let A be a self-adjoint operator acting on a Hilbert space $\mathscr{H}$. Then, there exists a projection-valued measure π on $\mathbb{R}$, supported on the spectrum of A, and such that*

$$A = \int_{\mathbb{R}} x \, \mathrm{d}\pi,$$

i.e. for all $w, v \in \mathscr{H}$,

$$\langle w, A \cdot v \rangle = \int_{\mathbb{R}} \langle w, \pi(A)v \rangle.$$

Note that we're not assuming A is bounded. If it is, though, then this says that A acts as multiplication by x in this measure.

Another way to say this is that an $A \in \operatorname{End} V$ (here V is a vector space) defines a $\mathbb{C}[x]$-module structure on V, where A is the action by x. We've sheafified this action by making it into a projection-valued measure: $\pi(\lambda)$ is the projection onto the λ-eigenspace.

In the common case where $\mathscr{H} = L^2(\mathbb{R})$, we want to understand $A = \frac{\mathrm{d}}{\mathrm{d}x}$, but this is unbounded; then, the projection-valued measure on $\mathscr{H}$ identifies $\mathscr{H} = L^2(\mathbb{R}_t)$, the Fourier dual to $\mathbb{R}$: $\frac{\mathrm{d}}{\mathrm{d}x}$ acts as a translation by t.

Every $L^2(X)$ carries a natural projection-valued measure over X, where $\pi(U)$ is the projection onto $L^2(U) \subset L^2(X)$. This is akin to the structure sheaf $\mathscr{O}_X$, but in an analytic section. This allows us to reword the spectral theorem: to every self-adjoint operator, we can attach a measure space $X = \operatorname{Spec} A \subset \mathbb{R}$.

The greater point is that sheafification isn't specific to algebraic geometry; it's naturally associated to Fourier theory in any setting, continuous, measurable, differentiable, or algebraic. The fact that Hilbert C^*-modules are associated with vector bundles on X is the beginning of the story of K-theory of C^*-algebras.

All of this was in the setting where A was a commutative algebra. The point of noncommutative geometry is to do this for noncommutative algebras: a noncommutative ring should correspond to some sort of non-commutative algebraic variety; a noncommutative C^*-algebra should define a "noncommutative topological space," and a noncommutative von Neumann algebra (now there's a wealth of good examples) should define a "noncommutative measure space." We might not know what the points of these spaces, but we know what their vector bundles (sheaves, etc.) are, and can compute interesting topological invariants, including K-theory.

If G is a noncommutative group, the dual doesn't always have a nice geometric structure, so we have to be craftier. But we'll still use this dictionary as a guiding philosophy.

Quantum mechanics. Though this feels like a radical transition, quantum mechanics has a lot of similar ingredients: a lot of this spectral theory was developed in order to study quantum mechanics; representation theory has also been used to study quantum mechanics. Since a major point of this class is to reverse the arrow (use ideas from quantum mechanics to understand representation theory), let's briefly discuss quantum mechanics.

The basic data in a quantum-mechanical system is a Hilbert space $\mathscr{H}$, called the *space of states* and a self-adjoint operator H on $\mathscr{H}$, called the *Hamiltonian*. A *pure state* is a nonzero vector in $\mathscr{H}$ up to rescaling (or normalizing), so really an element of $\mathbb{P}\mathscr{H}$, the projective space.

The state space defines time evolution: for any real number T, there's a unitary operator

$$U_T = e^{-iTH/\hbar} \in U(\mathscr{H}).$$

In a sense, we've rotated by $90°$ to convert a self-adjoint operator into a unitary one. The assignment $U : T \mapsto U_T$ gives a unitary representation of $\mathbb{R}$ on $\mathscr{H}$. The corresponding Lie algebra generator is H, the infinitesimal time evolution.

Let $\mathscr{H}_\lambda$ be the λ-eigenspace of H. Then, we can write that

$$\mathscr{H} = \int^\oplus \mathscr{H}_\lambda \, \mathrm{d}_\lambda.$$

Here, the integral denotes a completed direct sum, taking the closure (internally) or completion (abstractly) of the algebraic direct sum of these eigenspaces. The projection $\pi(\lambda) : \mathscr{H} \to \mathscr{H}_\lambda$ is a projection-valued measure on $\mathscr{H}$ over $\mathbb{R}$. If ψ is an *eigenstate* (an eigenvector of H), then $H\psi = \lambda\psi$, so $U_T\psi \in \mathbb{C}\psi$. That is, this state is stationary.

More generally, we can write down time evolution as $\psi_T = U_T\psi$, or

$$-i\hbar\frac{\mathrm{d}\psi(t)}{\mathrm{d}t} = H\psi.$$

This is called the *Schrödinger equation*.

Example 5.3. The main example of a quantum system starts with a Riemannian manifold M (which might as well be standard Euclidean space $\mathbb{R}^3$), $\mathscr{H} = L^2(M)$, and $H = (1/2)\Delta_M$, the Laplacian.

What do we do with this? We want to observe things. The things we observe will be called *observables*, and are the self-adjoint operators on X. On $L^2(\mathbb{R}^3)$, for example, there are two classes of natural operators:

- Multiplication by the ith coordinate defines an unbounded operator $X_i = (x_i \cdot -)$; these are called *position operators.*
- Differentiation in the ith direction is also unbounded: $P_i = -i\hbar\frac{\partial}{\partial x_i}$; these are called *momentum operators.*

Next time, we'll explain how to take measurements (expected values) of an operator $\mathcal{O}$ in a state ψ, which will be $\langle \psi, \mathcal{O} \cdot \psi \rangle \in \mathbb{R}$.

The quantumness of quantum mechanics is that self-adjoint operators are a noncommutative algebra.

6. QUANTUM MECHANICS: 9/8/16

Since we talked about quantum mechanics yesterday, we'll step back and do a lightning review of classical mechanics today.

Classical mechanics starts with a *symplectic manifold* X, i.e. a manifold with a closed nondegenerate 2-form. For example, the *phase space* of a particle moving on a manifold M is the symplectic manifold T^*M with coordinates (q_i, p_i): the q_i coordinates correspond to position coordinates on M and the p_i are momenta in the cotangent direction.

A *pure state* is a point of T^*M, and the *observables* are functions on T^*M, e.g. the coordinates q_i (what is the position of the particle?) and p_i (what is the momentum?). In order to understand the time evolution of this system, we specify a particular observable $H : T^*M \to \mathbb{R}$ called the *Hamiltonian*, such that

$$\frac{\mathrm{d}}{\mathrm{d}t} = \{H, -\}.$$

This is an instance of a more general construction that obtains vector fields from functions on a symplectic manifold: given such a function f, its derivative $\mathrm{d}f$ is a 1-form, and pairing with ω turns this into a vector field called $\{f, -\}$. Time evolution follows the equations

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i} \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q_i}.$$

If we have only kinetic energy, and no potential energy, a typical Hamiltonian is

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2,$$

which means $\frac{\mathrm{d}q_i}{\mathrm{d}t} = p_i$ (momentum determines where the particle goes) and $\frac{\mathrm{d}p_i}{\mathrm{d}t} = 0$ (momentum is conserved). If there's some potential energy (e.g. a height function), these equations change.

The analogy with quantum mechanics passes through statistical mechanics, where points are replaced with probability measures on T^*M. In the discrete case, we take a convex combination of points

$$\mu = \sum_{i=1}^{m} \lambda_i \delta_{x_i},$$

with $\lambda_i \geq 0$ summing to 1, so that $\int \mathrm{d}\mu = \sum \lambda_i = 1$. This can be generalized to continuous probability measures: the goal is to replace points with "clouds" specifying where particles are likely to be. The dynamics of this system allow this measure to evolve.

In this probabilistic setting, we need to evaluate functions on measures. One choice is to take the expected value

$$E[f] = \sum \lambda_i f(x_i)$$

(replacing the sum with an integral in the continuous setting), but we could also push the measure forward and obtain a measure on $\mathbb{R}$: $f_*\mu$ is a measure on $\mathbb{R}$ corresponding to all the possible values of f, weighted by likelihood. In the discrete case, this is

$$f_*\mu = \sum_i \lambda_i \delta_{f(x_i)}.$$

The total integral is $E[f]$. This is a nice way to measure measures, but isn't terribly sophisticated: we've collapsed the measure on all of spacetime to $\mathbb{R}$. Don't take this pushforward too seriously.

From this perspective, perhaps quantum mechanics isn't so radical: we don't have points, and focus on the probability measures, but not that much else changes. We start with a Hilbert space $\mathscr{H}$; the *pure states* are vectors $\psi \in \mathscr{H}$, often in *ket notation* $|\psi\rangle$. An *observable* $\mathcal{O}$ is a self-adjoint operator on $\mathscr{H}$, and its *expectation value* on the pure state ψ is

$$\langle \mathcal{O} \rangle_\psi = \frac{\langle \psi \mid \mathcal{O} \mid \psi \rangle}{\langle \psi \mid \psi \rangle} \in \mathbb{C}.$$

That is, we take the inner product weighted by $\mathcal{O}$ and then normalize by $\|\psi\|$.[19] If $\mathcal{O}$ has eigenvectors φ_i with corresponding eigenvalues a_i, then

$$|\psi\rangle = \sum_i \langle \varphi \mid \psi \rangle |\varphi_i\rangle,$$

and therefore the expectation value is

(6.1) $$\langle \mathcal{O} \rangle_\psi = \sum_i \langle \varphi_i \mid \psi \rangle a_i.$$

That is, we understand the expectation value (roughly what we're expecting to observe in state ψ) through the observable's eigenvectors, as long as $\mathcal{O}$ has a discrete spectrum.[20]

The a_i are more or less outcomes, and $\langle \varphi_i \mid \psi \rangle$ are the probabilites: measuring at an eigenvector gives me a pure outcome.

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[19]At some point, we'll need to explain this notation.

[20]One counterintuitive aspect of functional analysis is that the spectrum is not the set of eigenvalues, but rather the set where the operator $A - \lambda I$ isn't boundedly invertible. Not every point in the spectrum is an eigenvalue, and some operators, e.g. $\frac{\mathrm{d}}{\mathrm{d}x}$, don't have any eigenvectors (as the exponentials aren't L^2), just a continuous spectrum.

Just as in statistical mechanics, we can recover a measure on $\mathbb{R}$ recording the different outcomes with their weights. This is exactly what we were doing last time: associated to a self-adjoint operator $\mathcal{O}$ on $\mathscr{H}$, we defined a projection-valued measure $\pi_{\mathcal{O}}$ on $\mathbb{R}$. This is the abstract version of diagonalization, even if we have a continuous spectrum. At a state ψ, this is the measure $U \mapsto \langle \psi, \pi_{\mathcal{O}}(U) \cdot \psi \rangle \in \mathbb{C}$: over a measurable $U \subset \mathbb{R}$, the image of $\pi_{\mathcal{O}}(U)$ is the part of $\mathscr{H}$ on which the spectrum of $\mathcal{O}$ lies in U, which generalizes (6.1).

Now, if you ask what you're actually measuring in quantum mechanics, things get confusing, and relate to experimental physics and measurement theory; we're not going to worry too much about that.

Unlike in classical mechanics, the algebra of operators is noncommutative. But if we do have a family R of commuting algebras (which form something like a C^*-algebra or a von Neumann algebra), then they have a common spectrum $M = \operatorname{Spec} R$, and we obtain a probability measure on this space from the state ψ.

For an example, consider a free particle in $\mathbb{R}^3$; classically, we'd use the state space $T^*\mathbb{R}^3$, but the quantum state space is $\mathscr{H} = L^2(\mathbb{R}^3)$. Translation defines *position operators* X_1, X_2, X_3 that act on $\mathscr{H}$ and commute; similarly, we have *momentum operators* $P_i = \frac{\partial}{\partial x_i}$, which commute, but unlike the position operators, these aren't diagonalized already. The momentum eigenvectors are $e^{\langle t, x \rangle}$ for $t \in (\mathbb{R}^3)^*$. Diagonalizing this identifies $\mathscr{H} \cong L^2((\mathbb{R}^3)^*)$, which is exactly the Fourier transform on $\mathbb{R}^3$. This is the *momentum space* picture, or "wave" picture. But we can't simultaneously diagonalize the P_i and the X_i, as they don't commute. This is the well-known Heisenberg uncertainty principle: since these don't commute, you can't accurately measure both position and momentum at the same time.

One can define the *ring of differential operators* to be $\mathcal{D} = \langle x_i, \frac{\mathrm{d}}{\mathrm{d}x_i} \rangle$ with the relation $[\frac{\mathrm{d}}{\mathrm{d}x_i}, x_j] = \hbar \delta_{ij}$. These act on $L^2(\mathbb{R}^3)$: $\frac{\mathrm{d}}{\mathrm{d}x_i}$ acts as P_i and x_j acts as X_j. Sometimes this is called the *noncommutative cotangent bundle*, but there's no space realizing this noncommutative algebra of functions.

We once again have a distinguished operator called the *Hamiltonian*, which dictates time evolution: the time operator is $U_T = e^{-iTH/\hbar}$. Instead of just talking about expectation operators, we can make measurements of an operator $\mathcal{O}$ somewhere in the middle, at time T_1. That is, we start with ψ, evolve by T_1, then act by $\mathcal{O}$, then let some more time pass: the final correlation function is $\langle \psi \mid U_{T_2} \mathcal{O} U_{T_1} \mid \psi \rangle$. More generally, we can start with a state ψ_{in} and end with a state ψ_{out}, and have a bunch of operators $\mathcal{O}_i$ at times t_i. Then, the *correlation function* of all of these different measurements is

$$\langle \psi_{\mathrm{out}} \mid U_{t_f - t_n} \mathcal{O}_n U_{t_n - t_{n-1}} \cdots U_{t_3 - t_2} \mathcal{O}_2 U_{t_2 - t_1} \mathcal{O}_1 U_{t_1} \mid \psi_{\mathrm{out}} \rangle.$$

This should be read from right to left.

If the operator $\mathcal{O}$ commutes with the Hamiltonian, then it's time-independent (or it doesn't evolve): it's called a *conservation law* or *conserved quantity*. This is because $\langle \psi \mid U_{T_2} \mathcal{O} U_{T_1} \mid \psi \rangle = \langle \psi(T_1 + T_2) \mid \mathcal{O}\psi \rangle$, but $\psi(T_1 + T_2) = U_{T_1 + T_2} \psi$.[21]

A *symmetry* of the system is a unitary operator G on $\mathscr{H}$ that preserves the Hamiltonian. For example, if M is a Riemannian manifold, $\mathscr{H} = L^2(M)$, and H is the Laplacian Δ (e.g. on $\mathbb{R}^3$ we take

$$\frac{1}{2} \sum_{i=1}^{3} \left(\frac{\partial}{\partial x_i} \right)^2,$$

which is the usual Laplacian), then we're considering a free particle with kinetic energy, but no potential energy. If G acts on M by isometries, it defines a symmetry of the quantum-mechanical system, since it acts on L^2-functions preserving the inner product and the Laplacian is a coordinate-invariant notion. Another good basic example is the action of SO_3 on $L^2(S^2)$, which commutes with the *spherical Laplacian* Δ_s, the radial part of the Laplacian in $\mathbb{R}^3$.

Passing to the Lie algebra $\mathfrak{g} = \operatorname{Lie}(G)$, a $\xi \in \mathfrak{g}$ defines a self-adjoint operator on $\mathscr{H}$. In a sense, natural observables come from symmetries of the space. The natural example is the free particle: $\mathbb{R}^3$ acts on $L^2(\mathbb{R}^3)$ by translations, and the Lie algebra version is the infinitesimal translations $P_i = -i\hbar \frac{\partial}{\partial x_i}$ (the $-i$ was

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[21]This is the origin of the following joke: Two roommates work in a laboratory outside of Hamilton, Ontario. Steve, a regular fellow, and Gork, a literal caveman. He puts on a button-down shirt and tie every day in an attempt to fit in, but he just can't stop being a knuckle-dragging caveman (albeit in a lab coat).

After several years of working there, some of Gork's coworkers are talking during a coffee break. "Gork strikes me as really weird," said one man, "He's been here at the lab for like 6 years, and he never really developed any manners. I figured he would be civilized by now."

Another coworker explains "You really can't expect him to evolve. He commutes with the Hamiltonian."

inserted to make it self-adjoint). These P_i commute with the Hamiltonian, which tells us that momentum is conserved! By contrast, X_i doesn't commute with H, so position is conserved.

Maybe that's not terribly exciting, but there are more exciting symmetries: $\mathrm{SO}_2 \cong \mathrm{U}(1)$ acts on $\mathbb{R}^3$ by rotations around a fixed axis. The derivative of this action is a single operator, the *angular momentum operator*

$$-i\hbar\left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right).$$

This is the key insight behind Noether's theorem: a symmetry of the system corresponds to a conserved quantity, and vice versa.

Next time, we'll understand what happens when we set $H = 0$; symmetries reduce to group representations, which is a useful perspective on group representations.

7. Spherical Harmonics and Topological Fields: 9/13/16

We're in the midst of a discussion of quantum mechanics. Recall that we had a Hilbert space $\mathscr{H}$ and a Hamiltonian H: for example, to study the free particle on a Riemannian manifold M, $\mathscr{H} = L^2(M)$ and H is the Laplacian $(1/2)\Delta_M$. The observables are the self-adjoint operators on H, and conserved quantities (or conservation laws), which are operators invariant under time evolution $U_T = e^{iTH/\hbar}$.

In the case of the free particle on M, we had two natural classes of operators: the functions on M, and, if M is flat, differential operators with constant coefficients. Both of these form commutative algebras, and hence are simultaneously diagonalized by the Fourier transform.

We're most interested in the relation to representation theory, where we have a group G acting on $\mathscr{H}$ as a unitary representation in a way commuting with the Hamiltonian. For example, we can let G act on M by isometries, which necessarily preserve the Laplacian. A large source of representations "in nature" arise in this fashion.

We can use this to simplify $\mathscr{H}$. For example, if $M = \mathbb{R}^3$ with the usual metric, we can let $G = \mathrm{SO}_3$ act on M by rotations. This restricts to an action of G on S^2, which is compact. Thus, we obtain a unitary representation of G on $\mathscr{H} = L^2(S^2)$. We can differentiate it to obtain a representation of $\mathfrak{so}_3 = \mathrm{Lie}(\mathrm{SO}_3)$ on $\mathscr{H}$. The Lie algebra $\mathfrak{so}_3$ is generated by the unit vectors i, j, and k, and the Lie bracket $[\xi, \eta]$ is just the cross product $\xi \times \eta$. Each basis vector is the angular momentum around its axis (e.g. i is the angular momentum observable around the x-axis). These operators are noncommutative, though, which is a little unfortunate.

We can also define the *total angular momentum* or *Casimir operator*

$$C = -\frac{1}{2}(i^2 + j^2 + k^2),$$

which is an operator on $\mathscr{H}$. It may be realized in the *universal enveloping algebra* $U(\mathfrak{so}_3)$, approximately defined to be the minimal noncommutative algebra generated by i, j, and k with their usual relations extending to, e.g. $ij = ji + i \times j$. Then, the unitary representation extends to a homomorphism $U(\mathfrak{so}_3)$ to the self-adjoint operators on $\mathscr{H}$.

The Casimir operator is related to the Laplacian: $(1/2)\Delta_{S^2}$, and in fact it commutes with i, j, and k. As an element of $U(\mathfrak{so}_3)$, it's therefore central (so it commutes with i, j, and k in every representation); moreover, it generates the center of $U(\mathfrak{so}_3)$,[22] which is $Z(\mathfrak{so}_3) = Z(U(\mathfrak{so}_3)) \cong \mathbb{C}[C]$ as an algebra.[23] The action of C extends to an action of $\mathbb{C}[C]$ on $\mathscr{H}$, and since this is commutative, we can spectrally decompose. Since M is compact, the spectrum is discrete, and therefore we can decompose

$$\mathscr{H} = \overline{\bigoplus_{\ell \in \mathbb{Z}_+} \mathscr{H}_{\ell(\ell+1)}}.$$

Each eigenspace is a finite-dimensional unitary representation of SO_3.

Claim.

(1) Each eigenspace $\mathscr{H}_\lambda$ is an irreducible representation of SO_3.
(2) In fact, these are all of the irreducible representations of SO_3. This is very special.

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[22]Generally, Lie algebras don't have interesting centers, so when we write $Z(\mathfrak{so}_3)$, we'll almost always mean $Z(U(\mathfrak{so}_3))$.

[23]For more general G, there may be more than one generator; this case is somewhat special.

The eigenfunctions are called *spherical harmonics*, and they have direct physical meaning, such as studying spherically symmetric quantum-mechanical problems, including in particular electron shells; see Figure 2 for an example that might be familiar from high-school chemistry. Many physicists were bothered by the intrusion of representation theory into their quantum mechanics, calling it the *Gruppenpest* (well into the 1970s!); this course may be the "revenge of the Gruppenpest," flipping this on its head to use physics to understand representation theory.

FIGURE 2. Eigenfunctions of a representation of SO_3 on $L^2(S^2)$, the *spherical harmonics*, which arise in chemistry as electron orbitals. Source: `https://en.wikipedia.org/wiki/Spherical_harmonics`. More precisely, the coordinate functions define three position operators X, Y, and Z, which are unbounded operators. The spectral theory of each operator gives a projection-valued measure on $\mathbb{R}$, and taking the product yields a projection-valued measure P on $\mathbb{R}^3$. Now, given a special eigenfunction $Y_{l,m}$, taking $\langle Y_{l,m}, P Y_{l,m} \rangle$ gives a probability density on R^3, which roughly gives the atomic orbitals.

These are also related to the irreducible representations of $SL_2\mathbb{C}$, which are actions on homogeneous polynomials in two variables of a given degree $\mathrm{span}_{\mathbb{C}}\{x^n, x^{n-1}y, x^{n-2}y^2, \ldots, xy^{n-1}, y^n\}$. This is because $PSL_2\mathbb{C}$ is the complexification of SO_3; we're technically looking at the complexification and representations of SU_2, but these are closely related.

After separating variables, the eigenfunctions of Δ_{S^2} extend uniquely to *harmonic functions* on $\mathbb{R}^3 \setminus 0 = S^2 \times \mathbb{R}_+$, i.e. the functions f such that $\Delta_{\mathbb{R}^3} f = 0$. In particular, $\mathbb{R}[x, y, z]$ can be decomposed as the radial functions $\mathbb{R}[x_1^2 + x_2^2 + x_3^2]$ tensored with the harmonic polynomials.

Topological quantum mechanics. We have not been terribly precise about analytical issues in this class, and we'll turn to topological issues where we can speak more precisely.

Though the theory on S^2 is beautiful, understanding $L^2(M)$ and Δ_M is quite hard in general. Time evolution $U_T = e^{iT\Delta/\hbar}$ models heat flow $f(t) = e^{-t\Delta} f_0$, or $\frac{\partial}{\partial t} = -\Delta f$. To simplify this, let's kill some time: precisely, let's set $H = 0$, which simplifies a lot of quantum mechanics, albeit removing a lot of interesting parts.[24]

There's a way to do this that's less silly, and ultimately leads to the notion of a topological field theory. This is motivated by supersymmetry, in which we pass from $L^2(M)$ to the differential forms $\Omega^\bullet(M)$,[25] in this context called the $\mathcal{N} = 2$ SUSY algebra. Moreover, there are more operators than just H: there's a graded

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[24]Alternatively, one could pass to $\ker H$, but for $(L^2(M), \Delta)$, where M is compact, this is just constant functions, which seems silly.

[25]Technically, we should consider L^2 *differential forms*, but soon it won't matter.

Lie algebra[26] of three operators H, Q, and Q^*, with degrees 0, 1, and -1 respectively, and the relations

$$[Q, Q] = [Q^*, Q^*] = 0$$
$$[Q, H] = [Q^*, H] = 0$$
$$[Q, Q^*] = H.$$

So each of Q and Q^* squares to 0, and commutes with H.

On $\Omega^\bullet(M)$, there's really only one thing we can get: H is the Laplacian Δ, Q is the *de Rham differential* d, and Q^* is its adjoint d*, i.e. $\star$d$\star$ (where $\star$ is the Hodge-$\star$ operator). From the relations, we have $\Delta = \mathrm{dd}^* + \mathrm{d}^*\mathrm{d}$: that is, there's a chain homotopy from Δ to 0. This means Q^* makes H Q-exact, so $H = 0$ on Q-cohomology $H^*_{\mathrm{dR}}(M)$.

If M is instead a complex manifold, a similar story provides another way to set H to 0: we let $H = \Delta_{\overline\partial} = \Delta_{\mathrm{d}}$, $Q = \overline\partial$ and $Q^* = \overline\partial^*$. Then, $\Delta = \overline\partial\overline\partial^* + \overline\partial^*\overline\partial$, so it vanishes on the cohomology $H^\bullet(M, \mathscr{O})$, where $\mathscr{O}$ is the sheaf of holomorphic functions on M. This is part of the Dolbeault cohomology of M.

The de Rham case is called the *A-type*, and the complex case the *B-type*. This all comes from Witten's paper [13] on Morse theory.

We don't need to pass to cohomology, though; we can actually work with the derived category of $(\mathscr{H}, Q)$-chain complexes. This is why setting $H = 0$ isn't as dumb as it looks: there are actual, physical systems where one might care about a Hamiltonian exact with respect to a differential operator.

Let's try to abstract what "topological quantum mechanics" actually means. Fix a target manifold M; we'd like to study a "particle on M," whose time-evolution is given by maps from one-dimensional spacetimes to M. Then, $L^2(M)$ can be thought of as functions on maps from a point to M.

To a 0-manifold X_0, we'd like to attach a space of functions on $\mathrm{Maps}(X_0, M)$. To avoid analytical issues, we have two options.

- One choice is to make M a finite set. This is a little silly, but it's immediately clear what functions are.
- Alternatively, let M be a compact manifold, and by "functions," we mean de Rham cohomology.

For example, if X_0 is three points, we want to take cohomology of maps from three points to M, i.e. $H^*(M^3) = H^*(M)^{\otimes 3}$.

Now, let's think about time evolution: if X_0 and X_1 are two 0-manifolds (space), we'll model time evolution as a cobordism (spacetime) Y between them, which may be nontrivial. We want this to induce a map on cohomology $Y : H^*(M^{X_0}) \to H^*(M^{X_1})$ (here M^X is functions $X \to M$). If M is a finite set, we can use *spans*:

$$
\begin{array}{ccc}
 & M^Y & \\
\pi_1 \swarrow & & \searrow \pi_2 \\
M^{X_0} & & M^{X_1},
\end{array}
$$

where we send $f \mapsto \pi_{2*}\pi_1^* f$. This is an integral transform as we discussed earlier, and is our version of the path integral.

In general, if we have maps $\pi_1 : Z \to X$ and $\pi_2 : Z \to Y$, we get a span

$$
\begin{array}{ccc}
 & Z & \\
\swarrow \quad \downarrow p & & \searrow \\
X \longleftarrow & X \times Y & \longrightarrow Y
\end{array}
$$

with induced product maps; the universal property of $X \times Y$ guarantees this diagram commutes, and induces the map $p : Z \to X \times Y$. This allows one to think of correspondences as producing matrices.

But the point is that we can replace finite sets with cohomology of compact manifolds: if M and N are compact oriented manifolds, then the (de Rham) cohomology obeys a Künneth formula

$$H^*(M \times N) = H^*(M) \otimes H^*(N),$$

---

[26]Sometimes, this is called a *Lie superalgebra*, but the $\mathbb{Z}$-grading is more information that's useful to have around.

and Poincaré duality defines an isomorphism $H^*(M) \xrightarrow{\sim} H^*(M)^*$. This allows us to identify

$$H^*(M \times N) \cong H^*(M) \otimes H^*(N) \cong (H^*(M))^* \otimes H^*(N) = \mathrm{Hom}(H^*(M), H^*(N)).$$

The point is that maps between cohomology rings can be represented by cohomology classes on the product: if $K \in H^*(M \times N)$, akin to an integral kernel, the map $H^*(M) \to H^*(N)$ is realized by

$$f \longmapsto \pi_{2*}(\pi_1^*(f \smile K)).$$

The cup product arises from pulling back by the diagonal map $\Delta : X \to X \times X$.

In order for cohomology to push forward in general, one needs a proper map and orientations, but in this particular case we're OK. If we restrict to cobordisms where at least one manifold is nonempty,[27] a cobordism between 0-dimensional manifolds X_0 and X_1 defines a map $H^*(M^{X_0}) \to H^*(M^{X_1})$.

The interval defines the identity cobordism, and induces the identity map on cohomology. But what if we move one point around to get a cobordism $\varnothing \to \{\bullet, \bullet\}$? This arises as a span

$$\begin{array}{ccc} & M & \\ \swarrow & & \searrow \\ \bullet & & M \times M \end{array}$$

and therefore the induced map is $\mathbb{C} = H^*(\mathrm{pt}) \to H^*(M \times M)$ sends $1 \mapsto [\Delta]$, the cohomology of the class of the diagonal. Similarly, turning around this cobordism defines a *trace map* $H^*(M \times M) \to \mathbb{C}$.

8. Topological Quantum Mechanics: 9/15/16

We're going to talk a little more about topological quantum mechanics today. Recall that the classical Hamiltonian perspective begins with a symplectic manifold T^*M and a Hamiltonian $H : T^*M \to |R$, such that infinitesimal time evolution is $\frac{\mathrm{d}}{\mathrm{d}t} = \{H, -\}$, the Poisson bracket. In quantum mechanics, the symplectic manifold is replaced with the Hilbert space $L^2(M)$ with a Hamiltonian H, and time evolution becomes $\frac{\mathrm{d}}{\mathrm{d}t} = (i/\hbar)H$.

There's also a Lagrangian perspective, we start with a Riemannian manifold M, and the Hamiltonian is

$$H = \frac{1}{2} \sum_i p_i^2 = \frac{1}{2} \langle p, p \rangle_{g^{-1}}.$$

That is, H is one-half of the Laplacian. The Riemannian metric g defines an isomorphism between TM and T^*M, so $\frac{\mathrm{d}}{\mathrm{d}t} = \{H, -\}$ defines a geodesic flow, which is a flow along geodesics. The Lagrangian is a functional $\mathcal{L} : TM \to \mathbb{R}$ which, under the Legendre transform, is sent to the Hamiltonian $H : T^*M \to \mathbb{R}$. This defines the action of a path (its classical energy)

$$S(\gamma) = \int_I \mathcal{L}(\gamma, \dot{\gamma}) \, \mathrm{d}t.$$

The classical trajectories are critical points of S, which are solutions to the Euler-Lagrange equations. These are geodesics.

There's also a Lagrangian approach to quantum mechanics, which leads to the usual perspective on (topological) quantum field theories: the Feynman path integral. A classical particle follows a geodesic, but a quantum particle is thought to take all possible paths, weighted with some probability measure, and so given a path $\gamma : I \to M$, there's an action functional $S(\gamma)$ which represents the weight attached to a given path. In other words, we take the vanilla measure $D\gamma$ on the space of paths and weight it by $e^{-S(\gamma)}$, more or less. In the limit, this becomes a δ-measure on geodesics.

Of course, this is a sketch of an idea — there are difficulties defining the path integral in general. But we can think of the space M^I of paths on M as a correspondence between M and M

$$\begin{array}{ccc} & M^I & \\ {\scriptstyle \gamma \mapsto \gamma(0)} \swarrow & & \searrow {\scriptstyle \gamma \mapsto \gamma(T)} \\ M & & M, \end{array}$$

---

[27]This avoids an infinite-dimensionality issue; in particular, though, it restricts us from using closed manifolds as cobordisms $\varnothing \to \varnothing$.

so we can set up an integral transform as usual, with the weighting $e^{-S(\gamma)}$ as the integral kernel. Given a function $f \in L^2(M)$ (our Hilbert space), we describe its time evolution $U_T f$ at an $x \in M$ by

$$(8.1) \qquad (U_T f)(x) = \int_{\substack{M_T \\ \varphi(T)=x}} f(\varphi(0)) e^{-iS(\varphi)/\hbar} \, D\varphi.$$

In general, of course, this doesn't make sense. But where it does, Feynman proved that it's the same time evolution operator U_T as for the Hamiltonian formulation of quantum mechanics.

We can rewrite (8.1) as an integral transform

$$(U_T f)(x) = \pi_{2*}\left(\pi_1^* f \cdot e^{-S(\varphi)} D\varphi\right).$$

This is perhaps a lower-level description of where the Hamiltonian comes from. This and (8.1) may not always make sense, but they are how we should think of defining field theories: rather than making time an exponentiation operator, it's turned into something geometric. We can imagine replacing the interval I of time into another manifold, such as a circle, since the action is completely local for paths. Moreover, the semigroup property $U_{T_2} U_{T_1} = U_{T_1+T_2}$ is replaced with a composition of correspondences

$$(8.2)$$

$$\begin{array}{ccccc}
& & M^{I_{T_1+T_2}} & & \\
& \swarrow & & \searrow & \\
& M^{I_{T_1}} & & M^{I_{T_2}} & \\
\swarrow & & \searrow \quad \swarrow & & \searrow \\
M. & & M & & M
\end{array}$$

The top path space is $M^{I_{T_1+T_2}} = M^{I_{T_1}} \times_M M^{I_{T_2}}$: the fiber product gives us paths that agree at the end of T_1 and the beginning of T_2.

We aim to topologize this. Let's start with the toy example where M is a finite set, so $\mathscr{H}$ is the Hilbert space of functions on M (or of a particle on M). Then, $\mathscr{H} \otimes \mathscr{H}$ is the space of functions on $M \times M$, or the space of 2 particles on M. We should take the completed tensor product $L^2(M) \widehat{\otimes} L^2(M)$ in order for it to agree with $L^2(M \times M)$.

So to any 0-manifold X, we'll attach the vector space $\mathscr{H}^{\otimes X}$ (the state space for particles indexed by X), and disjoint unions map to tensor products. Time evolution should act on these Hilbert spaces, and we can encode this as a functor: a time evolution from X_0 to X_1 is a bordism $Y : X_0 \to X_1$, as in Figure 3.
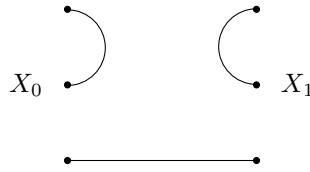

FIGURE 3. A bordism $Y : X_0 \to X_1$, which is a morphism in Bord_1.

To be precise, we're defining a category Bord_1 whose objects are 0-manifolds X and whose morphisms $\mathrm{Hom}_{\mathrm{Bord}_1}(X_0, X_1)$ are (diffeomorphism classes of) bordisms $Y : X_0 \to X_1$.

Given such a bordism, we define a correspondence

$$\begin{array}{ccc}
& M^Y & \\
{}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\
M^{X_0}, & & M^{X_1}
\end{array}$$

and this defines a map $U_Y : f \mapsto \pi_{2*}\pi_1^* f$ from $\mathscr{H}^{\otimes X_0} \to \mathscr{H}^{\otimes X_1}$.

Exercise 8.3. Show that this assignment defines a functor $Z : \mathrm{Bord}_1 \to \mathsf{Vect}_{\mathbb{C}}$. In fact, this sends disjoint unions to tensor products, so it's even a symmetric monoidal functor.
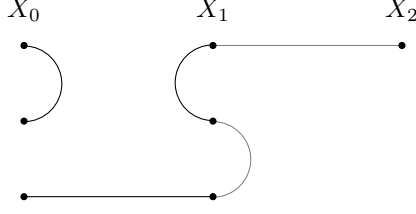
FIGURE 4. Gluing two bordisms.

The key is that we can glue a bordism $Y_0 : X_0 \to X_1$ with a bordism $Y_1 : X_1 \to X_2$ into a bordism $Y_0 \amalg_{X_1} Y_1 : X_0 \to X_2$ as in Figure 4, which appears at the top of a composition of correspondences as in (8.2).

For example, for any manifold M, the identity bordism is $M \times [0,1] : M \to M$; this is mapped to the identity map $\mathrm{id} : \mathscr{H} \to \mathscr{H}$. In this sense, the Hamiltonian has been set to 0: only exotic time evolutions have an effect. For example, a semicircle defines a bordism from two points to the empty set, which is mapped to a function $\mathscr{H} \otimes \mathscr{H} \to \mathbb{C}$.

This functor Z factors through another category. The category of *correspondences of sets*, denoted $\mathrm{Corr}(\mathsf{Set})$, is the category whose objects are sets and whose morphisms are isomorphism classes of *spans*

$$\mathrm{Hom}_{\mathrm{Corr}(\mathsf{Set})}(A, B) = \left\{ \begin{array}{c} C \\ \swarrow \quad \searrow \\ A \qquad B \end{array} \right\} / \cong .$$

In other words, these are the sets over the product $A \times B$. Correspondences compose like in (8.2):

$$
\begin{array}{c}
D \times_B E \\
\swarrow \qquad \searrow \\
D \qquad\qquad E \\
\swarrow \quad \searrow \quad \swarrow \quad \searrow \\
A. \qquad\quad B \qquad\quad C
\end{array}
$$

Now, our functor $Z : \mathrm{Bord}_1 \to \mathsf{Vect}_{\mathbb{C}}$ factors as $\mathrm{Bord}_1 \to \mathrm{Corr}(\mathsf{Set}) \to \mathsf{Vect}_{\mathbb{C}}$: the first functor takes maps to M for every object and morphism of Bord_1, and the latter takes functions to $\mathbb{C}$.

(A) The evaluation bordism
$E : \{\bullet, \bullet\} \to \varnothing$.

(B) The coevaluation bordism
$C : \varnothing \to \{\bullet, \bullet\}$.

FIGURE 5

An "outgoing" semicircle is a bordism $E : \{\bullet, \bullet\} \to \varnothing$ as in Figure 5a, which corresponds to the span

$$
\begin{array}{c}
M \\
{\scriptstyle \Delta}\swarrow \qquad \searrow \\
M \times M \qquad *.
\end{array}
$$

This induces an *evaluation map* $\mathscr{H} \otimes \mathscr{H} \to \mathbb{C}$, which we can think of in three different ways:

- Given $f, g \in \mathscr{H}$, we can take their product $f \boxtimes g \in \mathscr{H} \otimes \mathscr{H}$, and then evaluate: if $fg = \Delta^*(f \boxtimes g)$, the evaluation map is

$$f, g \longmapsto \int_X fg(x)\,\mathrm{d}x = \int_X f(x)g(x)\,\mathrm{d}x.$$

- Alternatively, we have an identification $\mathscr{H} \otimes \mathscr{H} \cong \mathrm{End}\,\mathscr{H}$; under this identification, the evaluation map is the trace $\mathrm{Tr} : \mathrm{End}\,\mathscr{H} \to \mathbb{C}$.

- Finally, the reason for the name: $\mathscr{H} \otimes \mathscr{H} \cong \mathscr{H} \otimes \mathscr{H}^*$, and we can evaluate an element of $\mathscr{H}^*$ on an element of $\mathscr{H}$, which also recovers the evaluation map as a map $\mathscr{H} \otimes \mathscr{H}^* \to \mathbb{C}$.

Dually, the "incoming" semicircle is a bordism $C : \varnothing \to \{\bullet, \bullet\}$ as in Figure 5b, corresponding to the span

$$
\begin{array}{ccc}
 & M & \\
 \swarrow & & \searrow{\scriptstyle\Delta} \\
* & & M \times M.
\end{array}
$$

This defines a map $\mathbb{C} \to \mathscr{H} \otimes \mathscr{H} \cong \operatorname{End} \mathscr{H}$, called *coevaluation*, which sends $1 \mapsto \delta_\Delta$ (the function equal to 1 on the diagonal and 0 elsewhere), which maps to the identity endomorphism.

Suppose $f : M \to M$ is a function, and let $K = \delta_{\Gamma_f}$, the δ-function supported on the graph of f. Then, $K \in \mathscr{H} \otimes \mathscr{H}$ is a kernel, and the evaluation map is

$$
\operatorname{Tr}(\delta_{\Gamma_f}) = \sum_{x \in X} K(x, x),
$$

which is the number of fixed points of f.

The circle defines a bordism $S^1 : \varnothing \to \varnothing$, which can be cut into coevaluation followed by evaluation. Thus, this functor Z sends this bordism to the map $Z(S^1) : \mathbb{C} \to \mathbb{C}$ composing coevaluation and evaluation, so $1 \mapsto \operatorname{id} \mapsto \operatorname{Tr}(\operatorname{id}) = \dim \mathscr{H}$.

Returning to Tuesday's approach to topological quantum mechanics, we can replace the finite set M with a compact oriented manifold and lt $\mathscr{H} = H^*(M)$. The homotopy equivalence $M^I \simeq M$ defines the identity span arising from the identity bordism: $H^*(M \times \{0\}) \cong H^*(M^I) \cong H^*(M \times \{1\})$.

The evaluation bordism defines the map $H^*(M) \to H^*(M) \to \mathbb{C}$ given by

(8.4)
$$
f, g \longmapsto \int_X \Delta^*(f \boxtimes g) \, \mathrm{d}x.
$$

Under Poincaré duality, $H^*(M) \cong H^*(M)^*$, and (8.4) agrees with the usual evaluation pairing $H^*(M) \otimes H^*(M)^* \to \mathbb{C}$. Similarly, coevaluation defines a map $\mathbb{C} \to H^*(M) \otimes H^*(M)$ sending $1 \mapsto [\Delta]$, the class of the diagonal under the Künneth identification $H^*(M) \otimes H^*(M) \cong H^*(M \times M)$.

We can also ask about the circle. Thinking about paths, this gives us the loop space $\mathcal{L}M$, and we will recover a map $\mathbb{C} \to \mathbb{C}$ sending $1 \mapsto \operatorname{Tr}(\operatorname{id})$. However, $\mathscr{H} = \bigoplus \mathscr{H}^i$ is a graded vector space, and everything is graded-commutative, so it's important to define the trace in a graded sense: if $K : \mathscr{H} \to \mathscr{H}$, its trace is

$$
\operatorname{Tr}_{\mathscr{H}} K = \sum_i (-1)^i \operatorname{Tr}_{\mathscr{H}^i} K
$$

(where the right-hand traces are the usual, ungraded trace). Therefore

$$
Z(S^1) = \operatorname{Tr}_{\mathscr{H}}(\operatorname{id}) = \sum_i (-1)^i \operatorname{Tr}_{\mathscr{H}^i} \operatorname{id} = \sum_i (-1)^i \dim H^i(M) = \chi(M),
$$

so we recover the Euler characteristic. We thought of the trace as counting the fixed points (by intersecting with the diagonal, just as in differential topology), but the Lefschetz fixed-point formula tells us that, modulo transversality issues, the fixed points recover the Euler characteristic.

Of course, if you try to this in analysis, the path integral becomes an integral over $\mathcal{L}M$, which is kind of nonsense, but at least you'd have the Hamiltonian to help you: the circle has a length, and we obtain something that depends on its length. Specifically, rather than $\operatorname{Tr}(\operatorname{id})$, you get the trace of the heat operator

$$
Z(T) = \operatorname{Tr}(e^{-T\Delta}).
$$

This is called the *partition function*. On a compact manifold (so the Laplacian has a discrete spectrum), this is more explicitly

$$
Z(T) = \sum_{\lambda_i \text{ eigenvalues of } \Delta} e^{-\lambda_i T} \dim \mathscr{H}_{\lambda_i}.
$$

From a statistical mechanics perspective, this is a sum of possibilities weighted by the expectation of finding the system in a given state.

We're going to ignore these analytical issues, but the point is that there's lots of interesting reasons to care about them.

Next time, we're going to go from quantum mechanics to (topological) quantum field theory, by replacing 1-dimensional manifolds (time) with higher-dimensional ones (spacetime).

9. Quantum Field Theory: 9/20/16

Note: there's no class next Tuesday, and next Thursday, Andy Neitzke will give the lecture.

We're in the midst of defining an abstract setting for topological quantum mechanics. The setting is a *bordism category* $\mathrm{Bord}_1^{\mathrm{or}}$ or $\mathrm{Bord}_1^{\mathrm{fr}}$, the 1-dimensional oriented (resp. framed) bordism category. In this dimension, the two are equivalent.

The objects are (1-)oriented zero-dimensional manifolds, and the morphisms are oriented bordisms: a morphism from X_0 to X_1 is an oriented one-manifold Y with boundary $Y \cong X_0 \amalg X_1$ as oriented manifolds. This category is symmetric monoidal under disjoint union.

In this abstract picture, topological quantum mechanics is a symmetric monoidal functor $Z : (\mathrm{Bord}_1^{\mathrm{or}}, \amalg) \to (\mathsf{Vect}_{\mathbb{C}}, \otimes)$: zero-manifolds go to complex vector spaces, bordisms go to linear maps, and disjoint unions map to tensor products.

All oriented 0-manifolds are disjoint unions of the positively oriented point pt_+ and the negative oriented point pt_-. Let $V = Z(\mathrm{pt}_+)$ and $W = Z(\mathrm{pt}_-)$, which are the "Hilbert spaces" associated to these points. The basic cobordisms are the identity bordisms $\mathrm{pt}_+ \times [0,1] \mapsto \mathrm{id}_V$ and $\mathrm{pt}_- \times [0,1] \mapsto \mathrm{id}_W$, but also an evaluation bordism $E : \mathrm{pt}_+ \amalg \mathrm{pt}_- \to \varnothing$ as in Figure 5a, which Z maps to an evaluation map $\varepsilon : V \otimes W \to \mathbb{C}$, and correspondingly a coevaluation $C : \varnothing \to \mathrm{pt}_+ \amalg \mathrm{pt}_-$ in Figure 5b, which Z transforms into $\eta : \mathbb{C} \to W \otimes V$.

We have one other piece of information, the *S-diagram* or the *mark of Zorro*, which is the composition of the bordisms $(\mathrm{id}_{\mathrm{pt}_+} \amalg E) \circ (C \amalg \mathrm{id}_{\mathrm{pt}_+})$, which maps to $(\varepsilon \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes \eta) : V \to V$. But since the S-diagram is diffeomorphic to the identity bordism, then $(\varepsilon \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes \eta) = \mathrm{id}_V$.

Claim. This forces an isomorphism $V \cong W^*$ and forces V to be finite-dimensional.

Let's compose coevaluation and evaluation to obtain a bordism $S^1 : \varnothing \to \varnothing$, which Z maps to a diagram

$$\mathbb{C} \xrightarrow{\eta} V \otimes V^* \xrightarrow{\varepsilon} \mathbb{C}$$

with $1 \mapsto \mathrm{id}_V$, $\wr$, Tr, to $\mathrm{End}\, V$.

That coevaluation and evaluation give you the identity of V and the trace come from the S-diagram. But the composite is $\mathrm{Tr}(\mathrm{id}_v) = \dim V$. That is, $Z(S^1) = \dim V$.

We can think of matrix multiplication as an operator like in usual quantum mechanics: the identity bordism id_V sends a vector $v \in V$ to itself, but if we modify it (thought of as a small local modification), replacing a piece with $E \amalg \mathrm{id}_V$, this turns into $\varepsilon \otimes \mathrm{id}_V$, which is matrix multiplication $V \otimes V^* \otimes V = V \otimes \mathrm{End}\, V : (v, M) \mapsto Mv$. Pictorially, this bordism is diffeomorphic to the interval minus a small interval, so the operator given by removing an interval around a point is matrix multiplication. This is reminiscent of inserting a measurement $\mathcal{O}(t)$ at a time t; if it seems weird, it will make more sense in higher dimension.

This same picture shows how to multiply operators (which is just matrix multiplication): we want a bordism that goes to $\mathrm{End}\, V \otimes \mathrm{End}\, V \to \mathrm{End}\, V$, which is the bordism $\mathrm{id}_{\mathrm{pt}_+} \amalg C \amalg \mathrm{id}_{\mathrm{pt}_-}$. If you glue this into an interval, this is just the removal of two intervals, so it corresponds to two composing the two matrix multiplication operators. In quantum mechanics, this is mostly true, but the amount of time between them matters, as there is nontrivial time evolution. Here, though, matrix multiplication is associative. We can recover this from the topology: consider the two bordisms $(\mathrm{pt}_+ \sqcup \mathrm{pt}_-)^{\amalg 3} \to (\mathrm{pt}_+ \sqcup \mathrm{pt}_-)$ inducing the maps $\mathrm{End}\, V \otimes \mathrm{End}\, V \otimes \mathrm{End}\, V \to \mathrm{End}\, V$ that are $M_1 \otimes M_2 \otimes M_3 \mapsto (M_1 M_2) M_3$ and $M_1 \otimes M_2 \otimes M_3 \mapsto M_1 (M_2 M_3)$. Drawing a picture (TODO) shows these bordisms are diffeomorphic, so the maps are equal.

Before we increase the dimension, there's one more aspect of quantum mechanics we can port over to this setting. Evaluation is a map $\varepsilon\, \mathrm{End}\, V \to \mathbb{C}$, so given an $\mathcal{O} \in \mathrm{End}\, V$, we get a number $\varepsilon(\mathcal{O}) = \mathrm{Tr}(\mathcal{O})$. This is the partition function of the circle where we've inserted $\mathcal{O}$ as a local operator: this breaks it into the interval which goes to evaluation. If we take multiple local operators on the circle, we recover that the trace is invariant under cyclic permutations, coming once again from diffeomorphisms of bordisms. In fact, from this perspective, this is why trace is cyclic; organizing algebra into pictures illustrates why these properties must be true: trace is cyclic because it's something attached to a circle, and unlike other arguments, this proof propagates to more sophisticated settings. There's a sense in which these are expected values.

Quantum field theory, quickly. Though we'll soon pass to the topological case, we're going to quickly talk about (Euclidean) quantum field theory. We start with an n-dimensional Riemannian manifold M, and in a *Lagrangian quantum field theory*, one attaches a *space of fields* $\mathcal{F}(M)$. Fields are local expressions in M, e.g. functions on M, maps to a fixed target, sections of a vector bundle, or even vector bundles themselves, or spinors... this is in general a large space.

Then, we take a measure on the space of fields, written $e^{-S(\varphi)} D\varphi$. This is the part that's not always rigorous, since the space of fields may be infinite-dimensional, but the measure should respect the locality. This leads to analytic technicalities that we won't have to worry about in the topological case. S is called the *action*, and on a path φ, we want to integrate a Lagrangian

$$S(\varphi) = \int_M \mathscr{L}(\varphi(x)) \, \mathrm{d}x.$$

Again, this is a caricature in the general situation. But physicists learn to make sense of it in their own mysterious ways. In settings where this makes sense, one can define the *partition function*

$$Z(M) = \int_{\mathcal{F}(M)} e^{-S(\varphi)} D\varphi,$$

which is thought of as the volume of the space of fields.

The states on an $(n-1)$-dimensional manifold form a Hilbert space, which are the functions on $\mathcal{F}(N)$, and an n-dimensional manifold with boundary defines a bordism form the incoming component to the outgoing component. These states are functionals, akin to the state space $L^2(X)$ for a free particle on X in quantum mechanics.

Technically, we haven't defined the fields on a time-slice, aka $(n-1)$-manifold; instead, we mean a "germ" of this manifold, which is a collar $N \times [-\varepsilon, \varepsilon]$. So, really, all manifolds in n-dimensional QFT are n-manifolds; it's just that some of them look like collars around $(n-1)$-manifolds.

Suppose M is a bordism between the (collars of) $(n-1)$-manifolds N_{in} and N_{out}. Then, the QFT defines a "time evolution" $Z(M) : Z(N_{\mathrm{in}}) \to Z(N_{\mathrm{out}})$. The formula is that if $f \in Z(N_{\mathrm{in}})$, $Z(M)(f)$ is a function on $\mathcal{F}(N_{\mathrm{out}})$ defined by

$$Z(M)(f)(\varphi_{\mathrm{out}}) = \int_{\varphi|_{N_{\mathrm{out}}} = \varphi_{\mathrm{out}}} f(\varphi|_{N_{\mathrm{in}}}) e^{-S(\varphi)} D\varphi.$$

This, again, doesn't make sense in the general case, and even in specific cases is extremely difficult (which is why we'll pass to the topological setting). However, we can write it as a correspondence: restriction defines functions π_1 and π_2 in

(9.1)
$$
\begin{array}{ccc}
 & \mathcal{F}(M) & \\
 {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\
\mathcal{F}(N_{\mathrm{in}}), & & \mathcal{F}(N_{\mathrm{out}})
\end{array}
$$

and time evolution is a kernel transform

$$Z(M)(f) = \pi_{2*}\left(\pi_1^* f e^{-S(\varphi)} D\varphi\right).$$

Let's translate this into categories. A QFT is, in this context, a kind of functor:

- The domain is $\mathrm{Bord}_{n-1,n}^{\mathrm{Riemannian}}$ whose objects are (collars of) $(n-1)$-dimensional Riemannian manifolds and whose morphisms and whose morphisms are Riemannian bordisms.
- The codomain is $\mathsf{TopVect}_{\mathbb{C}}$: since these vector spaces are infinite-dimensional, we need some sort of topology. This is symmetric monoidal under completed tensor product $\widehat{\otimes}$.
- The functorial QFT will factor through a category of correspondences that look like (9.1), which govern how the fields glue.

Functoriality means that we must be able to compose bordisms. This is already subtle: why can you glue Riemannian manifolds along a codimension-1 boundary? This is why we required collars: if the boundary already comes with a collar, then we're just gluing manifolds of the same dimension, which is fine. Categorically, this gluing is a pushout, and for the space of fields, this becomes a fiber product:

$$\mathcal{F}(M \amalg_{N_2} M') = \mathcal{F}(M) \times_{\mathcal{F}(N_2)} \mathcal{F}(M').$$

Composition of these spans informs us how to glue fields. This is akin to the Fubini theorem, expressing the locality of the measure (across codimension 1, at least).

If M is a closed manifold, so that it defines a bordism $M : \varnothing \to \varnothing$, it defines a correspondence $* \leftarrow \mathscr{F}(M) \to *$, which is mapped to the function $Z(M) : \mathbb{C} \to \mathbb{C}$, which is identified with a number, the partition function of M. We'll de-emphasize the role of closed n-manifolds in the topological case, however.

One aspect we've ignored is the dependence of $Z(M)$ on M: the Riemannian $(n-1)$-manifolds form a space, and the spaces we obtain glue together into a vector bundle. Should we require this to be smooth? To simplify everything, we'll require it to even be locally constant, which will lead to topological field theory.

10. TOPOLOGICAL FIELD THEORY AND GROUPOIDS: 9/22/16

We've been discussing the functorial approach to QFT, which envisions a QFT as a monoidal functor
$$Z : \mathrm{Bord}_{n-1,n}^{\mathrm{Riemannian}} \longrightarrow \mathsf{TopVect}_{\mathbb{C}},$$
where the domain category has for its objects collars of compact $(n-1)$-dimensional Riemannian manifolds and for its morphisms bordisms between them. Morphisms must compose, and this arises from gluing of bordisms.

Most of our examples come from Lagrangian field theories, which factor this functor through a category of fields, which is some sort of category of correspondences. These glue by the fiber product across correspondences as in (8.2), which is, in a sense, the path integral.

If we fix two $(n-1)$-dimensional manifolds $N_{\mathrm{in}}, N_{\mathrm{out}}$ and consider $\mathrm{Hom}(N_{\mathrm{in}}, N_{\mathrm{out}})$, it's more than a set: it's some kind of space. In particular, we'd like for it to be a family of Riemannian over a space S, so a submersion $\mathcal{M} \to S$ of Riemannian manifolds with a connection. We're not going to dwell on this, but the point is there is a natural notion of family, so we can ask how Z varies as the bordism does. Specifically, for N_{in} and N_{out} fixed, Z defines an element of
$$\mathrm{Fun}(\mathcal{M}, \mathrm{Hom}(Z(N_{\mathrm{in}}), Z(N_{\mathrm{out}}))).$$
We can ask for this to be continuous, differentiable, etc.

Similarly, if $\mathcal{N}$ is the space of compact, n-collared, $(n-1)$-manifolds, the assignment $N \mapsto Z(N)$ defines a vector space over every point in $\mathcal{N}$. We can ask for this to be a vector bundle, akin to imposing continuity, or ask for a connection, akin to differentiability.

To pass to the world of topological field theory, we're going to ask for something much stronger: for these functions to be locally constant, i.e. constant on isotopy classes of manifolds. This boils down to something much simpler: we only see the topology of manifolds, as the spaces of metrics tend to be contractible. On $\mathcal{N}$, the locally constant condition means that the vector bundle should be a vector bundle with an integrable connection, hence constant on contractible system. This is a *local system*, a representation $\pi_1(\mathcal{N}, [N]) \to \mathrm{GL}(Z(N))$ at any particular manifold N. So there are isomorphisms between any two manifolds on the same connected component of $\mathcal{N}$, but the isomorphisms aren't canonical. Typically, these arise by adding differentials and passing to cohomology (except for maybe one or two examples); in physics, this is the domain of supersymmetry.

We know locally constant $\mathbb{C}$-valued functions by another name, $H^0(\mathcal{M}, \mathbb{C})$. But then, why just 0? Let's consider the total cohomology H^*, or even cochains, valued in $\mathrm{Hom}(Z(N_{\mathrm{in}}), Z(N_{\mathrm{out}}))$; this is where a lot of interesting things happen.

This is a process whose output is a topological field theory. The bordism category depends on something ultimately topological: instead of Riemannian manifolds, we consider, for example, oriented manifolds and oriented cobordism. In fact, the tubular neighborhood theorem makes the collar question a lot simpler: we can let the objects of $\mathrm{Bord}_{n-1,n}^{\mathrm{SO}}$ be honest oriented $n-1$-manifolds N, but with an orientation of the tangent bundle of the collar, which is $TN \oplus \mathbb{R}$. This is what allows us to glue. In the same way, last time we discussed the 1-oriented points pt_+ and pt_-, which come from the orientations on $T\mathrm{pt} \oplus \mathbb{R} \cong \mathbb{R}$, thought of as "incoming" and "outgoing."

We can generalize orientation to any kind of *tangential structure*. This means a reduction of the structure group of $TN \oplus \mathbb{R}$ from $\mathrm{O}(n)$ to some subgroup $H \subset \mathrm{O}(n)$.[28] Orientations correspond to the reduction $\mathrm{SO}(n) \subset \mathrm{O}(n)$; it's also possible to consider unoriented TQFTs, using $H = \mathrm{O}(n)$. We can also reduce to the trivial subgroup $H = 1$, so that all tangent bundles are trivialized, producing *framed bordism*.

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[28]More generally, we may consider a homomorphism of Lie groups $H \to \mathrm{O}(n)$, e.g. spin structure.

So in this setting, we have a category $\mathrm{Bord}^{\mathrm{SO}}_{n-1,n}$ whose objects are $(n-1)$-manifolds N with an orientation on $TN \oplus \mathbb{R}$, whose morphisms are bordisms, and whose composition is once again gluing.[29]

Thus, an *oriented topological field theory* is a symmmetric monoidal functor

$$Z : (\mathrm{Bord}^{\mathrm{SO}}_{n-1,n}, \mathrm{II}) \longrightarrow (\mathsf{Vect}_{\mathbb{C}}, \otimes),$$

where the words "symmetric monoidal" means that disjoint unions map to tensor products.

As for QFTs, a lot of examples will factor through a category of correspondences, such as those of sets.

Example 10.1. We'll start with a silly example, and then work harder to get an interesting one.

Fix a finite set X. We'll let the space of fields on any $(n-1)$-manifold N be the space X^N of continuous maps from N to X; a bordism $M : N_{\mathrm{in}} \to N_{\mathrm{out}}$ defines a correspondence

$$
\begin{array}{ccc}
 & X^M & \\
 \swarrow & & \searrow \\
X^{N_{\mathrm{in}}} & & X^{N_{\mathrm{out}}}.
\end{array}
$$

Then, we can let $Z(N) = \mathrm{Fun}(X^N, \mathbb{C})$. This is an example of a *σ-model*: the fields are maps to a target.

What kinds of things can we calculate with this theory? A closed n-manifold is a bordism $M : \varnothing \to \varnothing$, so X^M is a correspondence from $*$ to $*$, and so $Z(M) : \mathbb{C} \to \mathbb{C}$ is a number, equal to the number of points in X^M, which is $|X|^{|\pi_0(M)|}$.

Example 10.2 (Untwisted Dijkgraaf-Witten theory). Our first real example will still be simple, but it gets a name: it's an n-dimensional unoriented field theory attached to a finite group G. This isn't a σ-model, but rather a *gauge theory*: the fields are principal G-bundles on M. Since G is finite, this is the same thing as a Galois covering of M with covering group G.[30] If M is connected, this is the set of representations $\pi_1(M, *) \to G$. Thus, this is insensitive to the geometry of M, just its topology.

There's a subtlety here, though: we want the fields to be the set of G-bundles on M, which means we really should consider the set of isomorphism classes of G-bundles on M. This is bad, because it doesn't glue: every G-bundle is locally trivial, so if we could glue along codimension 1 submanifolds, we'd only recover a globally trivial bundle.

To fix this, you have to let the fields be the *groupoid* of principal G-bundles on M, the collection of G-bundles with isomorphisms and automorphisms made explicit.

Groupoids. The simplest model for a groupoid is an equivalence relation on a set X, which is an inclusion $i : E \hookrightarrow X \times X$ that is reflexive, symmetric, and transitive: reflexivity means the diagonal is in $\mathrm{Im}(i)$; symmetry means if $x \sim y$, then $y \sim x$, and transitivity means we can compose arrows: $x \sim y$ and $y \sim z$ force $x \sim z$. But we rarely ever care about E: instead, we think about it as the equivalence classes X/E.

A *groupoid* is a way to present this quotient. A groupoid $\mathcal{G}$ is a pair of sets X_0, X_1 with two maps $s, t : X_1 \rightrightarrows X_0$ called the *source* and *target* maps, respectively: we think of X_0 as the objects and X_1 as the arrows or equivalence relations. The key axioms are that if $t(e_1) = s(e_2)$, then we can compose $e_2 \circ e_1$, which is another arrow in X_1, and that every arrow has an inverse, composing to an identity.

This is equivalent to defining a category with a set of objects and all of whose morphisms are invertible.

Now we ask, when are two groupoids (or categories) the same? There's a definition of isomorphism of categories, a functor $F : \mathsf{C} \to \mathsf{D}$, which should require an inverse functor $G : \mathsf{D} \to \mathsf{C}$ such that $FG, GF = \mathrm{id}$, but this isn't helpful: in a category, it's pointless to ask whether two objects are equal. Instead, we have to ask about isomorphism classes of objects, which leads to the much better notion of an *equivalence of categories*, where we merely require that FG and GF are naturally isomorphic to the identity.

This is also the correct notion of equivalence for groupoids: we care about the isomorphism classes of objects and their automorphisms, not about the actual number of things in an isomorphism class.

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[29]Question TODO: is it precisely true that the isotopy classes of Riemannian manifolds refine to this category of oriented manifolds? Or is it unoriented manifolds, or something else?

[30]Usually, one considers principal G-bundles with a connection or with a flat connection, but since G is finite, there's a canonical one. Nonetheless, for the physics, remembering the connection is helpful.

Example 10.3. Let M be a topological space. The *fundamental groupoid* or *Poincaré groupoid* $\pi_{\leq 1}(M)$ is the groupoid whose objects are the points $x \in M$ and whose morphism set $\mathrm{Hom}_{\pi_{\leq 1}(M)}(x, y)$ is the set of homotopy classes of paths $x \to y$. We don't care about the set of objects, but if $\bar{M}$ is contractible, then $\pi_{\leq 1}(M) \simeq \pi_{\leq 1}(*)$. More generally, a homotopy equivalence induces an equivalence of groupoids, not an isomorphism.

Another way to understand this is to draw a picture: $\pi_{\leq 1}(X)$ has for its objects $\pi_0(M)$, and for each connected component X, the arrows of X are $\pi_1(M, x)$, where $x \in X$. In fact, any groupoid $\mathcal{G}$ is the fundamental groupoid of a space:

$$\mathcal{G} \simeq \pi_{\leq 1}\left(\coprod_{m \in \pi_0(\mathcal{G})} K(\pi_1(\mathcal{G}, m), 1) \right).$$

If a groupoid acts on a space X, we can take the quotient $X/\mathcal{G}$ (equivalence classes), but what we mean is to keep track of the automorphisms. If X is a geometric object, this really should be a stack, but we won't worry about that right now.

We can apply this to define Dijkgraaf-Witten theory: to a manifold M, we assign the groupoid of fields $\mathcal{F}(M)$ to be the groupoid of principal G-bundles on M where we keep track of isomorphisms. To understand this, let's turn to the Seifert-Van Kampen theorem, which tells us how the fundamental group changes when we glue spaces together. Combining this with the connected components, this can be concisely stated as

$$\pi_{\leq 1}(X \amalg_U X') = \pi_{\leq 1}(X) \amalg_{\pi_{\leq 1}(U)} \pi_{\leq 1}(X').$$

To make this correct, we need to understand the fiber product of groupoids: if $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{K}$ are groupoids and we have maps $f : \mathcal{G} \to \mathcal{K}$ and $g : \mathcal{H} \to \mathcal{K}$, the objects of $\mathcal{G} \times_{\mathcal{K}} \mathcal{H}$ are the pairs of $x \in \pi_0 \mathcal{G}$ and $y \in \pi_0 \mathcal{H}$ such that $f(x) = g(y)$ *along with a choice of isomorphism* $\gamma : f(x) \xrightarrow{\sim} g(y)$; the morphisms $(x, y, \gamma) \to (x', y', \gamma')$ are the data of morphisms (φ, ψ) such that $\varphi(x) = x'$, $\psi(y) = y'$, and $\psi \circ \gamma \circ \varphi^{-1} = \gamma'$. This is an example of a homotopy pullback.

Example 10.4. Let G be a group and consider $\mathcal{K} = \bullet/G$ (sometimes called BG, but this can be confusing), the groupoid with $\pi_0 \mathcal{K} = \{\bullet\}$ and $\pi_1(\mathcal{G}, \bullet) = G$. Then, let $\mathcal{G} = \mathcal{H} = \bullet$, the groupoid with a single object and a single identity morphism. The fiber product $\bullet \times_{\mathcal{K}} \bullet$ has for its objects the isomorphisms $\bullet \to \bullet$, which are the points of G, and for its morphisms, the actions of G. In a sense, all actions are free!

Example 10.5. Let G be a finite group and $H, K \leq G$ be subgroups. Thus, we can define the groupoids $\bullet/H$ and $\bullet/K$, and inclusion of arrows defines maps $\bullet/H \to \bullet/G$ and $\bullet/K \to \bullet/G$; the fiber product is the groupoid of double cosets $H \backslash G / K$. This will come up again, so it's worth thinking through.

Next time, we'll talk about local systems and how the fiber product of groupoids correctly glues fields on manifolds, defining a nontrivial example of a field theory.

11. "Ask a Physicist" Day: 9/29/16

Today, Andy Neitzke gave the lecture.

First, a question from Prof. Neitzke about extended Dijkgraaf-Witten theory: for a compact oriented 3-manifold, we get a number, and for a closed 2-manifold, we get a vector space. There should be a category associated to a closed 2-manifold, but what is it? We didn't know. The answer is probably in Freed's paper "Higher algebraic structures and quantization."

There should be a simpler question attached to classical Chern-Simons theory: for an (oriented) 3-manifold, a principal G-bundle P, and a connection ∇ on P, then there's an associated number $\mathrm{CS}(\nabla)$. For example, if $P = M \times G$ and $\nabla = \mathrm{d} + A$ for an $A \in \Omega^1(\mathfrak{g})$, the number we obtain is

$$\mathrm{CS}(\nabla) = \exp\left(\frac{1}{4\pi i} \int_M \mathrm{Tr}\left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A \right) \right).$$

If we define a bordism category of manifolds with a principal G-bundle, this is a simple example of a TQFT on that bordism category. What happens when we extend it?

For $1 + 1$-dimensional Dijkgraaf-Witten theory, there's a category associated to the point, which is the category of representations of G.

Question 11.1. The Lagrangian formulation of quantum field theory includes data of a path integral associated to a bordism of Riemannian manifolds. In class, we were told this is a richer theory than the Hamiltonian perspective; are they not equivalent?

It's possible to formulate both the Lagrangian and Hamiltonian perspectives precisely enough to make them equivalent, but they might not be equally easy to formalize. In any case, many physicists prefer the Lagrangian formalism because the Hamiltonian formalism requires choosing a time direction, a breaking of symmetry, while the Lagrangian formalism does not.

Imagine that your spacetime M^n is literally a product $M^{n-1} \times I$ of space and time. Both formalisms attach a Hilbert space $V = Z(M^{n-1})$ to M^{n-1}, and time evolution is a linear map $Z(M^n) : V \to V$. But if we're in the Lorentzian or Riemannian world, the interval has a length: $I_t = [0, t]$, and $M_t^n = M^{n-1} \times I_t$. The operator we obtain $U_t = Z(M_t^n)$ depends on this length, on how much we evolve time by. Time evolution has a composition law $U_t U_{t'} = U_{t+t'}$, so if you're optimistic about your functional analysis, you can say that $U_t = e^{tH}$ for a Hamiltonian H; then, V and H define the Hamiltonian formalism of the QFT.

Suppose we're literally in quantum mechanics, where the classical Lagrangian is $L(q, \dot{q})$, and relates the the classical Hamiltonian $H(q, p)$ by a Legendre transform. In quantum mechanics, the path integral is rigorously defined, and is called *Weiner measure*.

Question 11.2. What's the progress on the formal definition of the path integral/measure in higher dimensions?

There's actually a nonexistence theorem: you can list some formal properties that you'd like the path integral to satisfy, and then show that no measure can satisfy all of them. So the path integral will not exist as literally stated. There's a theorem that, in a sense, most of the contribution to the path integral is from really spiky paths. Also, it doesn't seem to be the case that anyone is working on it right now.

Question 11.3. What about the algebraic QFT formalism?

Physicists, at least, haven't gleaned anything from the program — there hasn't been much progress. In particular, the algebraic formalism doesn't address the path integral. There's a book by Costello called "Renormalization and Effective Field Theory" which addresses perturbative quantum field theory and the mathematical foundations thereof; it's not a first introduction, but is a good read.

Question 11.4. In an extended (T)QFT, one assigns categories and then higher categories to manifolds of higher codimension. What does this actually mean in physics? Relatedly, what does it mean that the Standard Model, which is a QFT, describes the universe?

First of all, there's currently no mathematical definition of what a quantum field theory is. From the phyicist's perspective, we generally believe there is a Hilbert space $\mathscr{H} = Z_{sm}(\mathbb{R}^3)$ which describes our universe. What is this vector space like?

Inside $\mathscr{H}$, there's one element Ω called the *vacuum*. It's *not* the zero vector — it corresponds to the state where nothing is there. All vectors in $\mathscr{H}$ correspond to possible configurations of the universe. Some QFTs have multiple vacuum states, but the Standard Model doesn't.

There are also *one-particle states*, which correspond physically to states in the universe where there's one particle propagating through space.

Let's step back into QED, which is a component of the Standard Model where the only force is electromagnetism. There are particles called photons, which are the quanta of this system. The Hilbert space $Z_{\text{QED}}(\mathbb{R}^3)$ is infinite-dimensional, and even the subspace of one-particle states is infinite-dimensional: for every $k \in \mathbb{R}^{3,1}$ with $\|k\|^2 = 0$, one obtains a one-particle state $|k\rangle \in Z_{\text{QED}}(\mathbb{R}^3)$ — and all of these states are linearly independent. Each state corresponds to a particle in a location with a specified energy. These are akin to δ-functions in $L^2(\mathbb{R})$, even with the analytical issues: you have to pick an approximation to a δ-function, since they're not literally L^2.

One weird aspect of this interpretation is that, like in quantum mechanics, the states can be understood as rays in the space, forming the projective space $\mathbb{P}\mathscr{H}$: no measurement will distinguish $|k\rangle$ and $i|k\rangle$. But to do calculations, the linear structure is very useful, and you'll never need to think about the projective space unless you're making this interpretation.

In addition to one-particle states, there are 2-particle states, 3-particle states, etc., so inside $\mathscr{H}$ there's the symmetric algebra $\text{Sym}^*(\mathscr{H}_{1-\text{par.}})$. This algebra is called *Fock space*; to many people, this is all you need to worry about, and their model of the universe involves finitely many particles propagating.

In this perspective, a field isn't directly observable, but relates to operators. QED has a classical action

$$(11.5) \qquad S = \int_M F \wedge (\star F),$$

where F is the curvature of some principal U(1)-bundle. (On $\mathbb{R}^4$, they're all equivalent, so you may as well fix the trivial bundle.) If $\mathcal{C}$ denotes the set of connections on the trivial U(1)-bundle, then to any $x \in \mathbb{R}^4$ we obtain a function $F(x) : \mathcal{C} \to \Lambda^2(\mathbb{R}^4)$. The connection isn't gauge-invariant: under a symmetry of the system, we can trivialize it at a point. So a field is a measurement of the *curvature* of the connection, which is gauge-invariant — an electromagnetic potential is a connection on a principal U(1)-bundle, and an electromagnetic field is the curvature on that connection.

The components of the electromagnetic field fit together into a skew-symmetric matrix

$$\begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & B_x \\ -E_y & -B_z & 0 & B_y \\ -E_z & -B_x & -B_y & 0 \end{pmatrix}.$$

Here, E_x is the electric field in the x-direction, B_y is the magnetic field in the y-direction, etc.

Particle creation is an operator $\mathcal{H} \to \mathcal{H}$ carrying the vacuum state into a one-particle state. Canonical quantization is the process of taking a map on the configuration space $F : \mathcal{C} \to \mathbb{R}$ and turning it into an operator $\mathcal{O}_F : \mathcal{H} \to \mathcal{H}$, which is how we obtain this particle creation map.

Given an action (11.5), we can try to find the equations of motion corresponding to that action. For QED (or even just quantized Maxwell theory, so there aren't electrons), the critical points of S are those for which $\mathrm{d} \star F = 0$. This is really nice, because it's linear. The solutions are of the form

$$A_\mu(x) = \varepsilon_\mu e^{ikx},$$

where $\mu = 0, 1, 2, 3$. This looks essentially like a plane wave propagating in some direction. The restriction that $\|k\|^2 = 0$ means it must propagate in a null direction, meaning that photons propagate at the speed of light.

ε_μ is called the *polarization vector*, and while solving this, one discovers that $\varepsilon \cdot k = 0$. One says that EM waves may be polarized vertically or horizontally (akin to wearing polarized glasses); here, there are four dimensions of polarization, but forcing $\varepsilon \cdot k = 0$ kills one of them. This means that the polarization must be perpendicular to the direction of travel. We need to eliminate one more degree of freedom, and this comes from gauge transformations: two solutions ε, ε' are *gauge equivalent* if $\varepsilon' = \varepsilon + \alpha k$. We quotient by equivalences and obtain the two-dimensional space of polarizations.

Somewhat related to this is the problem of quantization of a symplectic manifold: given a symplectic manifold (X, ω), we want to introduce a Hilbert space $\mathcal{H}$ such that functions on X correspond to operators on $\mathcal{H}$. The commutative algebra of functions on X is replaced with noncommutative algebra of operators: $[\mathcal{O}_f, \mathcal{O}_g] = \hbar\{f, g\} + \cdots$. In a sense, as $\hbar \to 0$, we recover the commutative case.

Physically, to any quantum field theory there's a symplectic manifold of classical solutions; then, you may want to quantize it. Unfortunately, there's no totally general quantization procedure (this is actually a theorem), but you can always make a *deformation quantization*: given a Poisson manifold (so not even necessarily symplectic), there's a one-parameter family of noncommutative algebras deforming the algebra of functions, but it's not guaranteed that these algebras act as operators on a given Hilbert space, and this second step is the harder step. Kontsevich's paper "Deformation quantization of Poisson manifolds" is the big reference.

The generality notwithstanding, in this particular example, one can produce the quantization by hand, assisted by the convenience that everything is linear.

Example 11.6. The simplest example is a two-dimensional Poisson algebra $\mathbb{C}[p, q]$ with $\{p, q\} = 1$. We may deform this to a family of noncommutative algebras of operators which act on $\mathcal{H} = L^2(\mathbb{R})$, where $\mathcal{O}_q$ is multiplication by x, and $\mathcal{O}_p$ is $\hbar \frac{\mathrm{d}}{\mathrm{d}x}$; thus, the bracket is $[\mathcal{O}_f, \mathcal{O}_g] = \hbar\{f, g\}$ as desired.

In practice, you want to compute correlation functions, create a particle at $F(x_1)$ in vacuum and destroy it at $F(x_2)$; the correlation function, sometimes called the *two-point function*, is

$$\langle \Omega \mid F(x_2)F(x_1) \mid \Omega \rangle.$$

If you just create a particle, the resulting state $F_1(x)\Omega$ is orthogonal to the vacuum state, so the correlation function would be zero.

You can also compute a four-way correlation function $\langle \Omega \mid F(x_4)F(x_3)F(x_2)F(X_1) \mid \Omega \rangle$, which measures in a sense the probability of a scattering process involving four particles. When you take a QFT course, you spend the first year learning how to compute these kinds of quantities.

A free particle has a choice of momentum, giving an infinite-dimensional state space. So a topological theory can't care about scale-dependent quantities such as momentum, and indeed always has finite-dimensional state spaces.

Question 11.7. Do the exotic spacetime evolutions in TQFT have any physical meaning, given that our spacetime does not appear to undergo such evolutions?

The exotic spacetime evolutions in topological field theories aren't useful for the application to, say, the Standard Model, but for applications of TQFT and QFT to condensed matter, where you have a small exotic-matter system in your lab and can actually put it into a torus configuration or whatever, these are more useful. That said, for something like the pair-of-pants bordism, it's going to be difficult to find a physical system where that needs to be accounted for; but in a two-dimensional theory, suppose we have a plate where we're making two probes at one time, and one probe at another.

In the language of topological field theory, instead of making a probe at a point, one probes by removing a small circle around that point (and take the limit as the radius goes to zero). When we bring the two probes together, there's some relation between the disturbance caused by two probes and the disturbance caused when they're brought together. Rather than the pair of pants literally describing the topology, it's there behind the scenes governing this relation. Extended field theory also shows up here.

12. Local Systems: 10/4/16

Professor Ben-Zvi is back today.

The penultimate class, we talked about groupoids, including our favorite example $\bullet/G$; we discussed that the fiber product of two subgroups $\bullet/K \times_{\bullet/G} \bullet/H$, which is the double coset space $K \backslash G/H$. We'll see this coset space a lot in the next few weeks.

Definition 12.1. Let X be a topological space and G be a group, Then, the *groupoid of G-local systems* is the groupoid
$$\mathrm{Loc}_G X = \mathrm{Hom}_{\mathsf{Grpd}}(\pi_{\leq 1}(X), \bullet/G).$$

That is, the objects are functors from $\pi_{\leq 1}(X)$ to $\bullet/G$, and the morphisms are natural transformations. Just as functors beteween two categories are themselves a category, maps between a groupoid form a groupoid (all natural transformations are natural isomorphisms).

Remark. The groupoid of local systems only depends on the fundamental groupoid of X, so it's possible to extend the definition and consider G-local systems on groupoids.

If X is connected, $\mathrm{Loc}_G(X)$ is the quotient groupoid $\mathrm{Hom}_{\mathsf{Grp}}(\pi_1(X), G)/G$: we remember the orbits.

We can think of local systems as fields; that is, they define a functor $\mathrm{Loc}_G : \mathrm{Bord}_{n-1,n} \to \mathrm{Corr}(\mathsf{Grpd})$: a bordism $M : N_{\mathrm{in}} \to N_{\mathrm{out}}$ defines inclusion maps $i_0 : N_{\mathrm{in}} \hookrightarrow M$ and $i_1 : N_{\mathrm{out}} \hookrightarrow M$; when we apply Loc_G we obtain the span

$$
\begin{array}{ccc}
 & \mathrm{Loc}_G M & \\
{}_{i_0^*}\swarrow & & \searrow{}^{i_1^*} \\
\mathrm{Loc}_G(N_{\mathrm{in}}) & & \mathrm{Loc}_G(N_{\mathrm{out}}).
\end{array}
$$

Though we care about this example the most, it can be generalized to any *cospan* of topological spaces $N_{\mathrm{in}} \hookrightarrow M \hookleftarrow N_{\mathrm{out}}$, and we can replace $\bullet/G$ with other groupoids.

Example 12.2.
 (1) If $X = \bullet$ (or more generally, is contractible), then $\mathrm{Loc}_G \bullet = \bullet/G$.
 (2) If $X = S^1$, there's one piece of monodromy, which acts by conjugation. Thus, $\mathrm{Loc}_G S^1$ is the groupoid of G acting on itself by conjugation, which we'll denote G/G. This relates to S^1 being the classifying space $B\mathbb{Z}$ of $\mathbb{Z}$.

(3) On the torus T^2, there are two directions of monodromy, and they commute. Thus, we obtain the *commutator groupoid*
$$[G, G] = \{g, h \in G : gh = hg\}/G,$$
where the quotient means we remember the G-orbits, producing a groupoid.

(4) More generally, if Σ_g is a connected, compact Riemann surface,
$$\operatorname{Loc}_G \Sigma_g = \left\{ A_i, B_i \in G, i = 1, \ldots, g \mid \prod_{i=1}^{g} [A_i, B_i] = 1 \right\}/G.$$

(5) If X is any space, $X \times S^1$ has an additional monodromy (additional copy of $\mathbb{Z}$ in its fundamental group), so a local system on $X \times S^1$ is the data of a local system on X and an automorphism of it (which is the monodromy around S^1), i.e. $\operatorname{Loc}_G(X \times S^1) = \{\rho \in \operatorname{Loc}_G X, \gamma \in \operatorname{Aut}(\rho)\}$.

This last construction is important, and more general.

Definition 12.3. If Y is a groupoid, then its *inertia groupoid* or *loop groupoid* $IY = \mathcal{L}Y$ is the groupoid $\{y \in Y, \gamma \in \operatorname{Aut}(y)\}$.

That is, local systems in $X \times S^1$ are the loop groupoid of local systems on X.

If X is a space and Y is a groupoid, we'll use the notation $[X, Y] = \operatorname{Hom}_{\mathsf{Grpd}}(\pi_{\leq 1} X, Y)$ for the groupoid of maps (since if Y is also $\pi_{\leq 1}$ of a space Z, then the components of $[X, Y]$ are the homotopy classes of maps from X to Z).

In this notation, $\operatorname{Loc}_G X = [X, \bullet/G]$, and the loop groupoid is $[S^1, Y] = \operatorname{Hom}_{\mathsf{Grpd}}(\pi_{\leq 1} S^1, Y)$. Since we're in the world of groupoids, it's probably better to think of $\pi_{\leq 1} S^1$ as $B\mathbb{Z} = \bullet/\mathbb{Z}$. This "classifying space" is a bit unusual: to a homotopy theorist, a classifying space is the quotient of a contractible space by a free G-action. For groupoids, we're allowed to take groupoid quotients, so all actions are free — and the point is certainly contractible! This may feel like a dodge, but these two viewpoints are reconcilable.

Example 12.4. Suppose a group G acts on a space X. Then, we can form the *quotient groupoid* X/G, which is presented by the maps $X \Leftarrow G \times X$, where the top map is the G-action $(g, x) \mapsto g \cdot x$, and the bottom map is $(g, x) \mapsto x$. The objects of this groupoid are the points of X, and the morphisms are $g : x \to g \cdot x$ for all $g \in G$ and $x \in X$.

In other words, we've re-encoded the group action: $\pi_0(X/G)$ is the set of orbits, and $\pi_1(X/G, x)$ is the stabilizer of x.

Groupoids are categories, but we try not to think about sets of objects, since they're not invariant under equivalence; instead, we have the set of components and the group of automorphisms at a point.

Since every topological space defines a groupoid and every groupoid is the fundamental groupoid of a space, we can think of groupoids as approximations to topological spaces. Just as a continuous map from a topological space into a discrete set only depends on π_0, a map into a groupoid depends only on $\pi_{\leq 1}$. We called these "homotopy classes," and we'll eventually relate these to *bona fide* homotopy classes.

Unlike for maps to discrete sets, for groupoids maps out of S^1 are nontrivial, which allows us to define the loop groupoid. We saw that $\mathcal{L}(\bullet/G) = \operatorname{Loc}_G S^1 = G/G$, the quotient *groupoid* of G acting on itself by conjugation. More generally, if we present a groupoid Y as its set of connected components $\{\Gamma_i\}$ with their automorphisms attached, then $\mathcal{L}Y$ has for its connected components Γ_i/Γ_i.

In other words, $\operatorname{Loc}_G X \times S^1 = [X \times S^1, \bullet/G] = [S^1, \operatorname{Loc}_G X]$. We'll be applying $- \times S^1$ a lot in topological field theory.

We also often think of S^1 as two copies of the interval glued at their endpoints. This gives another understanding of the loop groupoid. The maps of two points into a groupoid Y are just $Y \times Y$, but when we add two connecting intervals between then, we add to each point $(y_1, y_2) \in Y \times Y$ the data of two automorphisms $\gamma_1, \gamma_2 : y_1 \xrightarrow{\sim} y_2$. That is, we're taking the intersection of the diagonal with itself — as a set, this isn't new, but as a groupoid, this is $Y \times_{Y \times Y} Y$. This has a different presentation, so is (canonically) equivalent to the inertia groupoid. We'll see this again: self-intersection with the diagonal is surprisingly fundamental.

Recall that we have a functor $\operatorname{Loc}_G : \operatorname{Bord}_{n-1,n} \to \operatorname{Corr}(\mathsf{Grpd})$ (or more generally, we can use maps to some fixed groupoid). These are the fields; let's linearize by taking "functions" on the groupoid to obtain the vector space.

Remark. A *finite groupoid* is a groupoid $\mathcal{G}$ such that $\pi_0\mathcal{G}$ is finite and for every object $x \in \mathcal{G}$, its automorphisms $\pi_1(\mathcal{G}, x)$ are finite. When we refer to functions on a groupoid or to the measure on a groupoid, we will generally need to assume that our groupoids are finite; however, groupoids of principal bundles or G-local systems on compact spaces are finite, so this is no obstruction to us.

This means we want to define "functions on a groupoid" in a way that's functorial and invariant under equivalence. That is, given a presentation $\mathcal{G} \rightrightarrows X$ for a groupoid, we want to define functions to be those fixed by $\mathcal{G}$, i.e. $\mathbb{C}[\mathcal{G} \rightrightarrows X] = \mathbb{C}[X]^{\mathcal{G}} = \mathbb{C}[\pi_0(X/\mathcal{G})]$. These are the functions on X that are constant on orbits.

Like any good notion of function, this should pull back, and indeed it does: if $f : \mathcal{G} \to \mathcal{H}$ is a morphism of groupoids, we obtain a pullback map $f^* : \mathbb{C}[\mathcal{H}] \to \mathbb{C}[\mathcal{G}]$. But we also wish to push functions forward, so that correspondences become honest morphisms of vector spaces, and this is more subtle. On finite sets, we just used the counting measure, but for groupoids this isn't the correct way to count: just as in combinatorics, we weight objects by their automorphisms. This means $|\bullet/G| = 1/|G|$. If you think of the set $\pi_0\mathcal{G}$, this defines a measure and doesn't depend on choice of presentation for $\mathcal{G}$.

It turns out this is essentially unique, forced by our desire to functorially map from correspondences to $\mathsf{Vect}_\mathbb{C}$. Suppose we're given a composition of correspondences, which is a fiber product:

$$
\begin{array}{ccc}
& X \times_Z W & \\
\swarrow & & \searrow \\
X & & W \\
\swarrow \quad \searrow & & \swarrow \quad \searrow \\
Y \qquad\quad & Z & \qquad\quad S.
\end{array}
$$

If we wish to push forward, there are two ways to compose: going along the top, or going through Z. Functoriality means these must agree. For example, let's consider the constant function 1 on $\bullet$ and propagate it through

$$
\begin{array}{ccc}
& G & \\
\swarrow & & \searrow \\
\bullet & & \bullet \\
\searrow & & \swarrow \\
& \bullet/G. &
\end{array}
$$

If we're thinking about sets, 1 pulls back to 1 on G, and then we sum over the fiber to get $|G|$. Thus, this must pull back to $|G|$ on $\bullet/G$. In order for $1 \mapsto 1$, we have to weight $\bullet \twoheadrightarrow \bullet/G$ by $1/|G|$.[31]

Thus, for $\pi : X \to Y$ a map of groupoids, we have the formula

$$
\pi_* f(y) = \sum_{x : \pi(x) \xrightarrow{\sim} y} \frac{\pi_1(Y, y)}{\pi_1(X, x)} f(x).
$$

We can define an inner product on these spaces: the unique map $p : X \to \bullet$ induces an integration map $p_* = \int_X : \mathbb{C}[X] \to \mathbb{C}[\bullet] = \mathbb{C}$. Then, for $f, g \in \mathbb{C}[X]$, we let

$$
\langle f, g \rangle_X = \int_X \overline{f} g.
$$

This is like the L^2 inner product for functions on a finite set, but we're weighting differently, using the measure we defined.

Claim. $(\pi^*, \pi_*) : \mathbb{C}[X] \rightleftarrows \mathbb{C}[Y]$ are adjoint maps in this inner product.

That is, $\langle \pi^* x, y \rangle_Y = \langle x, \pi^* y \rangle_X$.

There are some finiteness assumptions we need, but for bordism everything is compact and satisfies these constructions.

We now have all the ingredients we need to map $\mathrm{Bord}_{n-1,n} \to \mathrm{Corr}(\mathsf{Grpd}) \to \mathsf{Vect}_\mathbb{C}$, and this defines a topological quantum field theory, which is a kind of Dijkgraaf-Witten theory.

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[31]For a careful exposition of this story, check out [14].

Group algebras. We can productively interpret group algebras in this language.

Let $X \to Y$ be a map of groupoids. Then, $\mathbb{C}[X \times_Y X]$ has an associative algebra structure coming from the diagram

$$
\begin{array}{ccc}
 & X \times_Y X \times_Y X & \\
\overset{\pi_{13}}{\swarrow} & \Big| & \overset{\pi_{23}}{\searrow} \\
X \times_Y X & \Big| & X \times_Y X, \\
 & \Big\downarrow {\scriptstyle \pi_{12}} & \\
 & X \times_Y X &
\end{array}
$$

where given two functions on the left and right, we pull them back to the center, then push forward to a function on $X \times_Y X$. If we loosen what we mean by "groupoids" or "functions" one of the big themes of geometric representation theory is that many algebras arise in this way.

Suppose X and Y are sets; then, $X \times_Y X$ is the subset of $X \times X$ of elements mapping to the same thing in Y. Thus, we recover (if X and Y are finite) the algebra of block diagonal matrices on X, where the blocks are indexed by Y.

If $X = \bullet$ and $Y = \bullet/G$, then the fiber product is $\bullet \times_{\bullet/G} \bullet = G$, and $\mathbb{C}[G]$ is in fact the group algebra with the convolution product. If instead $X = \bullet/K$, we get $\mathbb{C}[K\backslash G/K]$, functions on the K-double cosets of G. The associative algebra structure on $\mathbb{C}[K\backslash G/K]$ is called the *Hecke algebra* for G and K. This is isomorphic to the subalgebra of $\mathbb{C}G$ consisting of K-bi-invariant functions.[32]

Example 12.5 ((1+1)-dimensional Dijkgraaf-Witten theory)**.** The objects of (1+1)-dimensional Dijkgraaf-Witten theory are compact 1-manifolds, which are disjoint unions of copies of the circle; thus, for objects it suffices to know what it does to S^1.

S^1 is sent to functions on $\mathrm{Loc}_G\, S^1 = G/G$, so we get the algebra of class functions $\mathbb{C}[G/G]$, functions on the conjugacy classes of G. The incoming disc defines a bordism $D_i : \varnothing \to S^1$, and therefore induces a map $\mathbb{C} \to \mathbb{C}[G/G]$. This map arises from the span

$$
\begin{array}{ccc}
 & \bullet/G & \\
 & \swarrow \quad \searrow & \\
\bullet & & G/G,
\end{array}
$$

and therefore sends 1 to the δ-function at the identity, an assignment called the *unit* map.

Dually, the outgoing disc $D_o : S^1 \to \varnothing$ defines a span

$$
\begin{array}{ccc}
 & \bullet/G & \\
 & \swarrow \quad \searrow & \\
G/G & & \bullet,
\end{array}
$$

which is sent to the *trace map* $f \mapsto f(e)/|G|$.

Next time, we'll talk about what structure a pair of pants introduces to this example.

13. Local Operators: 10/6/16

"So I think of this [operator] as the Death Star, and we're approaching it."

We're going to start seeing more and more structure in these topological field theories; the first example of these structures is local operators.

In quantum mechanics, there are states and operators (observables): there are two perspectives on quantum mechanics, the *Schrödinger perspective* and the *Heisenberg perspective*, which formalize time evolution as acting on the states or acting on the operators.

In quantum field theory, we still have states: associated to a compact $(n-1)$-dimensional Riemannian manifold N, we have a Hilbert space $Z(N)$ of states. This is a lot of data: fully describing a field theory, even a topological one, would require understanding the space of states for all $(n-1)$-manifolds, and for $n > 2$, this is complicated. Next time, we will learn how to simplify this problem.

---

[32]This isomorphism involves multiplication by $|K|$, so if we replace $\mathbb{C}$ by a field whose characteristic divides $|K|$, this will not be true.

As for operators, though, imagine we have a bordism $M : N_{\text{in}} \to N_{\text{out}}$. For any $x \in M$, we can define an operator at x, $\mathcal{O}_x$, which in the path integral formalism is evaluation of the field at a point. Given a field ψ on N_{in}, we define

$$(13.1) \qquad \langle \mathcal{O}(x) \rangle = \int_{\varphi|_{N_{\text{in}}} = \psi} \mathcal{O}(\varphi(x)) e^{-iS(\varphi)} \, D\varphi.$$

For example, we might look at Taylor series or even the residue class of a singularity. This can be approximated using small balls around x; one says that the local operators in QFT are defined by

$$(13.2) \qquad \varprojlim_{\varepsilon \to 0} Z(S_\varepsilon^{n-1}),$$

so we cut a tiny ball out of M and measure fields on that sphere. This is the kin dof object you can stick into a correlation function and make measurements for.

In topological field theory, local operators are simpler. In this case, there's no need for ε in (13.2), so we define the *space of local operators* of a TQFT Z to simply be $Z(S^{n-1})$.

We call these operators, but what do they operate on? The same picture applies: at an $x \in M$, we cut out a ball around x (the size doesn't matter). This manifold is now diffeomorphic to a bordism $N_{\text{in}} \amalg S^{n-1} \to N_{\text{out}}$. That is, given a bordism $M : N_{\text{in}} \to N_{\text{out}}$ and an $x \in M$, we obtain a new bordism $N_{\text{in}} \amalg S^{n-1} \to N_{\text{out}}$, and therefore a map $Z(S^{n-1}) \otimes Z(N_{\text{in}}) \to Z(N_{\text{out}})$. Sometimes this is called the *ultraviolet picture*, since it takes place at the small scale; a larger-scale use of S^n can be called the *infrared picture*.

For example, on the most boring bordism $N \times I : N \to N$, this defines a linear map $Z(S^{n-1}) \otimes Z(N) \to Z(N)$, which is naturally identified with a map $Z(S^{n-1}) \to \text{End}(Z(N))$: these are literally operators. You can also do this by inserting other manifolds into a bordism, but S^{n-1} is the simplest; these will show up when we study defects later.

How does this picture relate to the integral (13.1)? We think of states as functionals on the space of fields, and the functions on $N_{\text{in}} \amalg S^{n-1}$ are the tensor product of those on N_{in} and those on S^{n-1}. Fixing some operator $\mathcal{O} \in Z(S^{n-1})$, if we evaluate the path-integral by the kernel transform weighted by $e^{-is(\varphi)} D\varphi$ on the fields of the bordism, we end up multiplying the function in $Z(N_{\text{in}})$ by the operator $\mathcal{O}(\varphi(x))$ on S^{n-1}, which leads to (13.1).

One important point to make is that these are operators that don't extend over the disc. In physics there are two kinds of operators, *order operators* and *disorder operators*; in the latter, we're allowed to have some kinds of singularities, e.g. 't Hooft operators. In a non-Lagrangian field theory, this doesn't make sense; for TQFT, we ask questions that don't preserve the Lagrangian formulation, and hence all operators must consider singularities. This has implications on non-perturbative dualities, which usually exchange order operators (such as Wilson operators) and disorder operators (e.g. 't Hooft operators). Electromagnetic duality is the most famous example. We'll get to this later in the course.

Recall that for topological quantum mechanics, operators were defined by excising an interval from a bordism $[a, b]$. These operators compose by matrix multiplication on the Hilbert space, which is associative but not commutative.

For TQFTs of dimension greater than 1, local operators are a commutative algebra! This is surprising, because the noncommutativity of operators on quantum mechanics is famous. This noncommutativity is absent in TQFT, or at least hidden further inside.

Before we discuss commutativity, we need to define the algebra structure on operators. Suppose I have two operators $\mathcal{O}_1$ at $x_1 \in M$ and $\mathcal{O}_2$ at $x_2 \in M$; assume M is connected, or at least x_1 and x_2 lie in the same connected component of M. The dependence of an operator on the location of a point is locally constant, so in this case, I can surround x_1 and x_2 by small spheres to define $\mathcal{O}_1$ and $\mathcal{O}_2$; then, their product is the operator defined by a sphere surrounding these two spheres. This is a pair-of-pants bordism, which induces the composition map $* : Z(S^{n-1}) \otimes Z(S^{n-1}) \to Z(S^{n-1})$, called *operator product expansion*.

Commutativity then follows: since $n > 1$, we're allowed to rotate x_1 and x_2 around each other, which doesn't change the pair-of-pants bordism. Thus, the algebra of operators is commutative, for the same reason that for $n \geq 2$, the homotopy group $\pi_n(X)$ is abelian. If we had more structure around, e.g. a map of chain complexes, we might notice that a space being simply connected or not has implications on this algebra structure, but we'll get to that later.

This algebra acts as operators on $Z(N)$ for any other $(n-1)$-manifold, so $Z(N)$ is a $Z(S^{n-1})$-module.

Example 13.3. Let's return to Dijkgraaf-Witten theory for a finite group G. We already saw in Example 10.2 that $Z(S^1) = \mathbb{C}[\mathrm{Loc}_G S^1] = \mathbb{C}[G/G]$, the space of class functions on G.

A local system on the pair-of-pants is determined by the two essential curves g and h, so requires data of monodromy on g, h, and gh. The span of groupoids is $(G/G) \times (G/G) \leftarrow (G \times G)/G \to G/G$, and on functions this defines the map

$$f_1, f_2 \longmapsto f_1 * f_2(g) = \sum_{h \cdot k]g} \frac{|G_g|}{|G_{h,k}|} f_1(h) f_2(g).$$

That is, this algebra structure is a normalized convolution by orders of stabilizers. This factor $|G_g/G_{h,k}|$ is the number of conjugacy classes of pairs (h, k) such that $h \cdot k = g$; we sort of can only see as finely as conjugacy classes.

The commutativity of class functions follows from the topology, but it's also possible to explicitly prove that inside the entire group algebra $(\mathbb{C}G, *)$, the class functions are the center. We'll be able to say this explicitly from TQFT soon.

In fact, we can extract a finer algebraic invariant.

2-dimensional TQFTs and Frobenius algebras. This is a story that everyone tells in TQFT, almost by common law: on $Z(S^1)$ we have much more structure than a commutative algebra.

FIGURE 6. Three two-dimensional bordisms that induce structure on $Z(S^1)$. Left: the incoming disc $D_i : \varnothing \to S^1$ distinguishing the unit $\delta_e \in \mathbb{C}[G/G]$. Center: the outgoing disc bordism $D_o : S^1 \to \varnothing$ which is a trace. Right: the "elbow macaroni" bordism defines an inner product.

First, the "incoming disc" (Figure 6, left) defines a bordism $D_i : \varnothing \to S^1$, hence a map $\mathbb{C} \to \mathbb{C}[G/G]$. This sends 1 to the unit δ_e of the algebra $\mathbb{C}[G/G]$, because if you cap off one disc in the pair of pants, the result is the identity bordism.

Dually, the "outgoing disc" (Figure 6, center) defines a bordism $D_o : S^1 \to \varnothing$, hence a map $\mathbb{C}[G/G] \to \mathbb{C}$, which is a trace, evaluation at $1/|G|$. Again, this comes from capping off one leg of the pair-of-pants.

Definition 13.4. A *commutative Frobenius algebra* over $\mathbb{C}$ is a commutative algebra A with a nondegenerate trace $\mathrm{Tr} : A \to \mathbb{C}$.

This induces an inner product $\langle \cdot, \cdot \rangle : A \otimes A \to \mathbb{C}$ given by $\mathrm{Tr} \circ \mu$. The trace here comes from capping off the pair of pants to get an "elbow macaroni" bordism $S^1 \amalg S^1 \to \varnothing$ (Figure 6, right), which is multiplication composed with the trace. By "Zorro's lemma" (see Figure 7), this trace is nondegenerate.
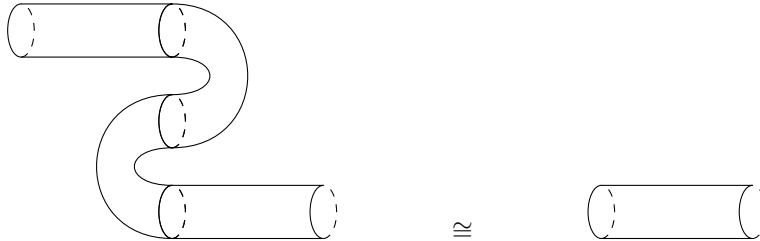

FIGURE 7. "Zorro's lemma," that these two bordisms are equivalent, is used to prove the nondegeneracy of the inner product on $Z(S^1)$.

The following is a folk theorem, though it appeared in Dijkgraaf's thesis.

Theorem 13.5. *A 2-dimensional oriented TQFT Z is equivalent to the data of a commutative Frobenius algebra $Z(S^1)$.*

The idea is that, using Morse theory, any 2-manifold-with-boundary can be cut into incoming and outgoing pairs of pants and incoming and outgoing discs. The inner product allows us to take the adjoint of multiplication, so we know what the map associated to incoming discs and pairs of pants are. The axioms of a Frobenius algebra are exactly the data needed to ensure this is consistent.

Let's assume we're working with a 2-dimensional TQFT Z such that $Z(S^1)$ is semisimple. By Wedderburn's theorem, this is true for $\mathbb{C}[G/G]$: it's isomorphic to a direct sum

$$\mathbb{C}[G/G] \cong \bigoplus_{\text{irreps.}} \mathbb{C} \cdot \chi_V,$$

i.e. an orthogonal direct sum of lines over all isomorphism classes of irreducible representations of G. The idea is that the entire group algebra $\mathbb{C}G$ is isomorphic to the direct sum of all $V \otimes V^* = \operatorname{End} V$ over these irreducible representations, and inside that $\mathbb{C} \cdot \operatorname{id}_V$ maps to $\mathbb{C}[G/G]$. Up to maybe some scaling, this is an algebra homomorphism.

The primitive idempotents for this decomposition are

$$(13.6) \qquad\qquad e_v = \frac{\dim V}{|G|}\chi_V,$$

again indexed by the irreducible representations of G: $e_V^2 = e_V$, $e_V e_W = 0$ for $V \neq W$, and $\sum e_V = 1$.

More generally, if $Z(S^1)$ is semisimple, we have a decomposition indexed by primitive idempotents

$$Z(S^1) \cong \bigoplus_{i \in I} \mathbb{C}e_i$$

such that $e_i e_j = \delta_{ij} e_i$. If we take $\operatorname{Spec}(Z(S^1))$, we just get I: as a commutative algebra, this is nothing more than the functions on I, but as a Frobenius algebra, we need to know the trace. This is determined on the e_i, so let $\lambda_i = \operatorname{Tr}(e_i)$, so we have λ_i living over the point i in I.

For $\mathbb{C}[G/G]$, the characters are defined by $\chi_V(g) = \operatorname{Tr}_V(g)$, so for the primitive idempotents (13.6),

$$(13.7) \qquad\qquad \lambda_V = \operatorname{Tr}(e_V) = \left(\frac{\dim V}{|G|}\right)^2.$$

That is, we consider functions on I weighted by λ_i at the point $i \in I$, the so-called *Plancherel measure*.

We've completely described the Frobenius algebra associated to 2-dimensional Dijkgraaf-Witten theory, and we can go back to the topology using *Mednykh's formula*. See [14] for a reference on this material, or [12] for a different proof using lattice TQFTs.

By the general theory, we know $Z(\Sigma_g) = \# \operatorname{Loc}_G(\Sigma)$; we will count this in two ways. Let X be the bordism $S^1 \to S^1$ by composing an incoming, then an outgoing pair of pants. As a manifold, X is a twice-punctured torus.

Exercise 13.8. Show that on $\mathbb{C}e_i \subset \mathbb{C}[G/G]$, the map $Z(X)$ sends $\mathbb{C} \cdot e_i \to \mathbb{C} \cdot e_i$ by $e_i \mapsto e_i/\lambda_i$.

This is useful because a closed, oriented surface of genus g decomposes as a bordism as a cap, then g copies of X, then a cap. Thus, the function $Z(\Sigma_g) : \mathbb{C} \to \mathbb{C}$ sends

$$1 \longmapsto \sum e_V \longmapsto \sum \frac{e_V}{\lambda_V} \longmapsto \cdots \longmapsto \sum \frac{e_V}{\lambda_V^g} \longmapsto \sum (\lambda_V)^{1-g}.$$

By (13.7), we get Mednykh's formula

$$(13.9) \qquad\qquad \# \operatorname{Loc}_G \Sigma_g = \sum_{V \text{ irrep.}} \left(\frac{\dim V}{|G|}\right)^{2-2g}.$$

This encodes a lot of the representation theory of finite groups.

Corollary 13.10. *Applying (13.9) to the sphere ($g = 0$), we recover that*

$$|G| = \sum (\dim V)^2,$$

the squares of the dimensions of the classes of representations is the order of the group.

Corollary 13.11. *Applying* (13.9) *to the torus* ($g = 1$), *we recover that the number of conjugacy classes of G is equal to the number of isomorphism classes of irreducible representations of G.*

This TQFT extends to unoriented surfaces, and applying it to $\mathbb{RP}^2$ recovers other formulas in representation theory.

More generally, given a TQFT (of dimension greater than 1), we get a commutative ring $Z(S^{n-1})$, and the instinct of algebraic geometry is to realize this ring as a ring of functions on the space $\mathcal{M}_Z = \operatorname{Spec} Z(S^{n-1})$, called the *moduli space of vacua* of Z. We'll elaborate on this space later.

In our case, $\mathcal{M}_Z$ was the set of isomorphism classes of irreducible representations of G with the Plancherel measure, and this is exactly $\widehat{G}$. This ties to the spectral theory that the class began with: (13.9) is a spectral decomposition, where the number $\# \operatorname{Loc}_G(\Sigma_g)$ sums over the points in $\widehat{G}$, with a contribution from each point. This is a major goal in this class: we'll use TQFT to encode structures in representation theory, and then take Spec and obtain the dual.

Next time, we'll talk about seeing not just characters, but the whole category of representations through an extended TQFT! There are versions of this for G a compact group, called 2D (topological) Yang-Mills; though there are convergence problems, the formulas are similar. There are many questions in representation theory, including the geometric Langlands program, that can be encoded in fancier versions of this story.

14. Fancier Local Operators: 10/11/16

Last time, we discussed 2D Yang-Mills theory Z with a finite gauge group G, which assigns to a circle the space $Z(S^1) = \mathbb{C}[G/G] = \mathbb{C}[\operatorname{Loc}_G S^1]$: on the circle, monodromy induces conjugation by group elements. If Σ is a surface with n incoming components and m outgoing components, then each boundary component is an S^1, so $Z(\Sigma)$ defines a linear map $Z(\Sigma) : Z(S^1)^{\otimes n} \to Z(S^1)^{\otimes m}$ which comes from a push-pull map $\operatorname{Loc}_G \partial_{\text{in}} M \leftarrow \operatorname{Loc}_G M \to \operatorname{Loc}_G \partial_{\text{out}} M$.[33]

If Σ is a closed surface, the span of groupoids is $\bullet \leftarrow \operatorname{Loc}_G \Sigma \to \bullet$, and $Z(\Sigma)$ is identified with the number

$$\# \operatorname{Loc}_G \Sigma = \int_{\operatorname{Loc}_G \Sigma} 1 \, d\mu,$$

where μ is the usual measure on a finite groupoid.

We also discussed local operators in this setting: for any 1-manifold, cutting a small sphere out of a bordism defines a map $Z(S^1) \to \operatorname{End} Z(N)$. When $N = S^1$, this defines an algebra structure on $Z(S^1)$, which is isomorphic to $(\mathbb{C}[G/G], *)$, the class function algebra with convolution.

The *moduli space* of a TQFT is $\mathcal{M}_Z = \operatorname{Spec} Z(S^{n-1})$, and in this case $\mathcal{M}_Z = \operatorname{Spec}(\mathbb{C}[G/G], *) = \widehat{G}$, the (discrete) set of irreducible representations of G. Since $Z(S^1)$ is a Frobenius algebra, then it has an inner product, and therefore $\widehat{G}$ has a measure: for an irreducible G-representation $V \in \widehat{G}$, the weight of V is $(\dim V/|G|)^2$.

There are orthogonal idempotents in $Z(S^1)$ defined for an irreducible representation V by

$$e_V = \chi_V \frac{\dim V}{|G|},$$

so that $e_V e_W = \delta_{VW} e_V$. Equivalently, we can write 1 (the δ-function at the identity of G) as a sum of these idempotents:

$$1 = \sum_{\text{irreps. } V} e_V.$$

This is a Plancherel formula, which identifies $(\mathbb{C}[G/G], *) \cong L^2(\widehat{G}, \mu)$; this is a kind of Fourier transform.

If you look the Plancherel formula up on Wikipedia, it's off by a factor of $|G|$: it uses the fact that

$$\sum_{\text{irreps. } V} (\dim V)^2 = |G|,$$

---

[33]To push forward, we need to identify functions with measures, or "integrate over the fiber." We do this by choosing the groupoid measure, as we discussed previously. In the corresponding cohomological version, this is a "wrong-way" map (also called a Gysin map, shriek map, transfer map, or a surprise map). [4] makes explicit the connection between the cohomological picture and the groupoid picture.

and therefore
$$\sum_V \frac{(\dim V)^2}{|G|} = 1.$$
This is a probability measure, which is good! But when we generalize to infinite groups, having a probability measure will be less helpful than the measure that the trace forced upon is.

The points of $\mathcal{M}_Z = \widehat{G}$ define *superselection sectors* for Z, akin to a decomposition into a direct sum independent parts Z_V indexed by the isomorphism classes of irreducible representations. Indeed,
$$Z(S^1) = \bigoplus_{V \in \widehat{G}} \mathbb{C} \cdot e_V = \bigoplus_{V \in \widehat{G}} \mathbb{C} \cdot \chi_V.$$
The state space is a direct sum of Hilbert spaces, and the invariants associated to a closed surface are similarly a sum: if Σ_g is the closed surface of genus g, then
$$Z(\Sigma) = \sum_{V \in \widehat{G}} \left(\frac{\dim V}{|G|} \right)^{2-2g}.$$
The geometry of this theory, $\mathrm{Spec}(\mathbb{C}[G/G])$, tells us that this theory breaks up as an orthogonal direct sum of "sub-theories."

You can play interesting games with this (e.g. Hausel and Rodriguez-Villegas), such as counting local systems on a punctured surface. Let Σ be a closed surface and $\widetilde{\Sigma} = \Sigma \setminus \bigcup_i D_i$ for a few small discs D_i in Σ. Restricting a local system to a small loop around the disc D_i defines a map $\mathrm{Loc}_G(\widetilde{\Sigma}) \to \mathrm{Loc}_G(S^1)$; since $\mathrm{Loc}_G \widetilde{\Sigma}$ is identified with the groupoid of maps $\pi_1(\widetilde{\Sigma}) \to G$ up to conjugation by G, this says that we have a distinguished conjugacy class, and we can ask where it goes. Thus, we're trying to count the punctured local systems $\# \mathrm{Loc}_G(\widetilde{\Sigma}, \{C_i\})$, so we're tracking the monodromy around D_i. (Here, the C_i are conjugacy classes in G identified by loops around each D_i.)

Under the identification of $\mathrm{Loc}_C S^1 \cong \mathbb{C}[G/G]$, $C_i \mapsto \delta_{C_i}$. Thinking of $\widetilde{\Sigma}$ as a bordism with $\bigcup D_i$ as its incoming boundary and $\varnothing$ as an outgoing boundary, it defines a map $Z(\widetilde{\Sigma}) : Z(S^1)^{\otimes n} \to \mathbb{C}$, which sends
$$\delta_{C_1} \otimes \cdots \otimes \delta_{C_k} \longmapsto \# \mathrm{Loc}_G(\Sigma, \{C_i\}).$$
The operation sending $\Sigma \mapsto \widetilde{\Sigma}$ is a disorder operator: it changes the fields, introducing singularities.

From this point of view, there's nothing special about conjugacy classes, and in fact there's even something "unspecial" about them. You can mark several points on a Riemann surface and insert a local system $f_i \in \mathbb{C}[G/G]$ at each point. As for $\widetilde{\Sigma}$, thinking of this marked Riemann surface as a bordism identifies a number: the number of representations of (the fundamental group of) the punctured surface, weighted by automorphisms
$$\sum_{\rho \in \mathrm{Loc}_G \Sigma \setminus \bigcup D_i} \frac{\prod_i f_i(\rho|_{\partial D_i})}{|\mathrm{Aut}(\rho)|}.$$
The way to solve this is to write f_i in the basis of idempotents e_V, or moreover, insert the idempotents as local operators themselves. On a 2-punctured surface, if we attach e_V to one puncture and e_W to the other, but $V \not\cong W$, then the number we obtain is 0. Puncturing once and inserting an e_V identifies the V-summand in $Z(\Sigma)$. These idempotents are Fourier dual to the δ-functions (coming from characters).

Extended field theory. What follows is a big pivot, where we try to add more structure to field theories. There are many ways to explain this, and we'll start with some examples.

The space $\mathbb{C}[G/G]$ of class functions has a more rigid structure than we have sussed out so far: it has a canonical basis of characters χ_V for $V \in \widehat{G}$. If we identify $\mathbb{C}[G/G]$ with the *representation ring* or *Grothendieck ring* generated by isomorphism classes of representations under direct sum and tensor product, these are positive elements (i.e. corresponding to actual representations, not virtual ones). This structure actually arises from the field theory: it relates to *boundary conditions* of Z, which are heuristically machines that produce states $x \in Z(N)$ for *any* compact $(n-1)$-manifold N. (Thus far, this hasn't been very interesting, because all 1-manifolds are disjoint unions of circles.) We'll investigate how to produce these boundary conditions from something geometric.

First, let's try to construct representations geometrically. Suppose a group G acts on X; then, functions on X is a G-representation, and this will be a natural source of representations of G. Thus, instead of vector

spaces, we can think of spaces X that G acts on. And we can identify G-sets X with groupoids $\mathcal{X}$ with maps $\mathcal{X} \to \bullet/G$.

In one direction, given an action of G on X, let $\mathcal{X} = X/G$; the unique map $X \to \bullet$ is G-equivariant, so it induces a morphism of groupoids $\mathcal{X} = X/G \to \bullet/G$. In the other direction, given such a morphism $\mathcal{X} \to \bullet/G$, we can pull back with the quotient map $\bullet \to \bullet/G$:

$$
\begin{array}{ccc}
X & \longrightarrow & \bullet \\
\downarrow & \lrcorner & \downarrow \\
\mathcal{X} & \longrightarrow & \bullet/G.
\end{array}
$$

Exercise 14.1. What is the G-action on X? We have $X = \mathcal{X} \times_{\bullet/G} \bullet$ and $G = \bullet \times_{\bullet/G} \bullet$ as groupoids, so to define a map $G \times X \to X$ we can try to make sense of what $\bullet \times_{\bullet/G} \bullet \times_{\bullet/G} \mathcal{X}$ is.

Geometrically, this is the identification of the groupoid of covering spaces of a space with the representations of the fundamental group into G, modulo conjugation.

In a gauge theory, the fields on M are local systems $\mathrm{Loc}_G M = \mathrm{Hom}_{\mathsf{Grpd}}(\pi_{\leq 1} M, \bullet/G)$. We also defined the σ-model, which fixes a set Y and defines the fields to be $\mathrm{Hom}_{\mathsf{Set}}(M, Y)$. You could even mix these together and consider field theories where we fix a (finite) groupoid $\mathcal{Y}$ and the fields on M are the groupoid

$$
\mathcal{G}_{\mathcal{Y}}(M) = \mathrm{Hom}_{\mathsf{Grpd}}(\pi_{\leq 1} M, \mathcal{Y}).
$$

A physicist might call this theory a *gauged σ-model*; we'll call it $Z_{\mathcal{Y}}$. The technical details are exactly the same as before: if M is a manifold with boundary $\partial M = N_{\mathrm{in}} \amalg N_{\mathrm{out}}$, restriction defines a span

$$
\mathrm{Hom}_{\mathsf{Grpd}}(\pi_{\leq 1} M, \mathcal{Y})
$$

$$
\mathrm{Hom}_{\mathsf{Grpd}}(\pi_{\leq 1} N_{\mathrm{in}}, \mathcal{Y}) \qquad \mathrm{Hom}_{\mathsf{Grpd}}(\pi_{\leq 1} N_{\mathrm{out}}, \mathcal{Y}).
$$

This defines a map $Z_{\mathcal{Y}}(N_{\mathrm{in}}) \to Z_{\mathcal{Y}}(N_{\mathrm{out}})$ between the spaces of functions on these groupoids of maps to $\mathcal{Y}$. For example, when $n = 2$, $Z_{\mathcal{Y}}(S^1)$ is the functions on the inertia groupoid (or loop groupoid) $\mathbb{C}[\mathcal{L}\mathcal{Y}]$.

We also get some new local operators: let $\mathcal{X} \hookrightarrow \mathcal{Y}$ be any subgroupoid. Then, for any $(n-1)$-manifold N_{in} we have a distinguished element $\delta_{\mathcal{G}_{\mathcal{X}}(N_{\mathrm{in}})} \in \mathbb{C}[\mathcal{G}_{\mathcal{Y}}(N_{\mathrm{in}})] = Z(N_{\mathrm{in}})$. If $M : N_{\mathrm{in}} \to N_{\mathrm{out}}$ is a bordism, this local operator acts on a $\varphi \in Z(N_{\mathrm{out}})$ by

$$
(14.2) \qquad Z(M)(\delta_{\mathcal{G}_{\mathcal{X}}(N_{\mathrm{in}})})(\varphi_{\mathrm{out}}) = \sum_{\substack{\varphi \in \mathcal{G}_{\mathcal{Y}}(M) \\ \varphi|_{N_{\mathrm{in}}} \text{ lands in } \mathcal{X} \\ \varphi|_{N_{\mathrm{out}}} = \varphi_{\mathrm{out}}}} \frac{1}{|\mathrm{Aut}\,\varphi_{\mathrm{out}}|}.
$$

This is a complicated, but explicit formula, for the general push-pull map. It seems reasonable to abbreviate it as a path integral: for any $f \in Z(N_{\mathrm{in}})$ and field φ on N_{out},

$$
Z(M)(f)(\varphi) = \int_{\varphi|_{N_{\mathrm{out}}} = \varphi_{\mathrm{out}}} f(\varphi|_{N_{\mathrm{in}}}) e^{-S(\varphi)}\, D\varphi.
$$

If f is a δ-function on a subspace $\widetilde{\mathcal{F}}$ of fields on N_{in}, we measure just the volume of the space of fields φ which become fields on $\widetilde{\mathcal{F}}$ when we restrict to N_{in}.

That is, we can express a boundary condition as a subset $\widetilde{\mathcal{F}}$ of the fields $\mathcal{F}(N_{\mathrm{in}})$ on N_{in}; then, the push-pull construction (14.2) defines a functional on $\mathcal{G}_{\mathcal{Y}}(N_{\mathrm{out}})$, which is integrating over the space of fields $\varphi \in \mathcal{G}_{\mathcal{Y}}(M)$ such that $\varphi|_{N_{\mathrm{in}}} \in \widetilde{\mathcal{F}}$.

We can also think of this as a composition of spans:

(14.3)

$$
\begin{array}{ccccc}
& \mathcal{G}_{\mathcal{X}}(N_{\mathrm{in}}) & & \mathcal{G}_{\mathcal{Y}}(M) & \\
\swarrow & & \searrow \swarrow & & \searrow \\
\bullet & & \mathcal{G}_{\mathcal{Y}}(N_{\mathrm{in}}) & & \mathcal{G}_{\mathcal{Y}}(N_{\mathrm{out}}).
\end{array}
$$

The constant function on $\mathcal{G}_{\mathcal{X}}(N_{\mathrm{in}}) = \mathrm{Hom}_{\mathsf{Grpd}}(\pi_{\leq 1} N_{\mathrm{in}}, \mathcal{X})$ passes to $\delta_{\mathcal{X}} \in Z(N_{\mathrm{in}})$, and pushing and pulling again, we obtain the local operator we defined above.

When you see a diagram like (14.3), there is an urge to take the fiber product $\mathcal{G}_{\mathcal{X}}(N_{\mathrm{in}}) \times_{\mathcal{G}_{\mathcal{Y}}(N_{\mathrm{in}})} \mathcal{G}_{\mathcal{Y}}(M)$, and this is the space of morphisms of groupoids from $\pi_{\leq 1} M$ to $\mathcal{Y}$ that restrict to $\mathcal{X}$ on N_{in}.

Example 14.4. Suppose, as before, we let $\mathcal{Y} = \bullet/G$. Let X be a G-set and take $\mathcal{X} = X/G$, which has a G-equivariant map to $\mathcal{Y}$ as described above. The groupoid of maps $\pi_{\leq 1} N \to X/G$ is identified with the groupoid of pairs (P, f) where P is a principal G-bundle on N and $f : P \to X$ is a G-equivariant map. Equivalently, we can identify it with the groupoid of data (ρ, s), where $\rho \in \mathrm{Loc}_G N$ and s is a section of the associated X-bundle. Next time, we'll explain these words in more detail and how they relate to representation theory: X is a source of representations.

We'll also talk about what an extended field theory actually is, axiomatizing some of what we've seen before. Groupoids technically form a 2-category (the 2-morphisms are equivalences of groupoids, but we'll go over this more clearly), so we should try to make the equivalences more explicit. This also allows representations to enter: vector bundles on the groupoid $\bullet/G$ are representations of G.

15. Boundary Conditions and Extended TQFTs: 10/13/16

Fix a finite group G and consider 3-dimensional Dijkgraaf-Witten theory — to a closed 3-manifold M, we associate the number $\# \mathrm{Loc}_G M$, and to a surface Σ, we attach the vector space $Z(\Sigma) = \mathbb{C}[\mathrm{Loc}_G \Sigma]$. This is also a functorial TQFT, and can be defined on bordisms, etc.

In this case, the local operators are $Z(S^2)$, which act by cutting out a little ball in a bordism on $Z(\Sigma)$ for all Σ. However, since S^2 is simply connected, $\mathrm{Loc}_G S^2$ is trivial, and so $Z(S^2) = \mathbb{C}$. There are no interesting local operators, but there are other operators, called *Verlinde loop operators*. The picture is almost the same, but instead of excising a point, take a loop $\gamma \subset \Sigma$ and excise a tubular neighborhood of it inside the bordism.[34] After excising the loop, this is a bordism $T^2 \amalg \Sigma \to \Sigma$, so the loop operator acts by $Z(T^2) \times Z(\Sigma) \to Z(\Sigma)$.

Topologically, you might as well contract the bordism. What we obtain is a version of Σ doubled along γ, a non-Hausdorff picture $\Sigma \amalg_{\Sigma \setminus \gamma} \Sigma$. Though this isn't Hausdorff, we can still understand its local systems: there's a funny homotopical version of a circle over every point in γ. These versions of S^1 can act on this as well: the action of $Z(T^2)$ depends on the choice of the loop (homotopy class of) γ. $\mathrm{Loc}_G T^2 = [G, G]$, and this contains G/G by considering the local systems trivial transverse to γ. Thus, we have a map $\mathbb{C}[G/G] \hookrightarrow \mathbb{C}[[G, G]]$, and therefore an action $\mathbb{C}[G/G] \otimes \mathbb{C}[\mathrm{Loc}_G \Sigma] \to \mathbb{C}[\mathrm{Loc}_G \Sigma]$. This ring $\mathbb{C}[\mathrm{Loc}_G \Sigma]$ has a ring structure, which we haven't seen before; we'll be able to derive it from TQFT, but not from this one. The point is that the action of $\mathbb{C}[G/G]$ will be multiplication by certain elements: if V is a representation of G and γ is a loop, then $\rho \in \mathrm{Loc}_G \Sigma$ acts by multiplication by $\mathrm{Tr}_V \rho(\gamma)$, the monodromy. That is, these functions, which show up when one studies surface groups, arise naturally in TQFT. If we try this over $\mathrm{SL}_2(\mathbb{C})$ (which isn't a finite group, so we'd have to work more), then these functions are called *Goldman functions* and form an integrable system called the *Goldman integrable system*.

This loop operator is also a push-pull operator: when we removed the torus, we had up to homotopy $\Sigma \amalg_{\Sigma \setminus \gamma}$. The two copies of Σ in this double define a cospan $\Sigma \to \Sigma \amalg_{\Sigma \setminus \gamma} \leftarrow \Sigma$, and therefore a span of local

---

[34]Like local operators, these are examples of defect operators, for which one excises a more general submanifold; we'll talk more about these later.

systems

$$\text{Loc}_G \Sigma \amalg_{\Sigma \backslash \gamma} \Sigma$$

$$\text{Loc}_G \Sigma \qquad \text{Loc}_G \Sigma.$$

Moreover, $\text{Loc}_G \Sigma \amalg_{\Sigma \backslash \gamma} \Sigma \cong \text{Loc}_G \Sigma \times_{G/G} \text{Loc}_G T^2$; this is how $\mathbb{C}[\text{Loc}_G T^2]$ acts on $Z(\Sigma)$.

Towards extended field theory. We'll approach this from a few different angles. The goal is to express the locality of the path integral.

One of the axioms for a field theory was that it turns gluing of bordisms into composition of functions. This comes from gluing fields: we've expressed locality in time (bordisms), as we can cut a bordism into a sequence of bordisms to evaluate it. But the path integral should also be local in space: $Z(N)$ should be local in N. This does not come out of the axioms we wrote down, so we'll have to upgrade our definition.

The extreme approach is to talk about boundary conditions. Heuristically, a boundary condition $\mathcal{B}$ is a machine for producing states $\mathcal{B}(N) \in Z(N)$ for all $(n-1)$-manifolds N. This means $\mathcal{B}$ has to make sense on any such manifold; thinking of it acting on manifolds with boundary, it could be a condition on the behavior of a field on the boundary of the manifold. It's akin to a state, but that makes sense locally, hence on any manifold: instead of just being a $\psi \in Z(N)$ for some N, it defines an element in every state space.

In the one-dimensional case, any element of $\mathscr{H} = Z(\bullet)$ is a boundary condition, since it defines an element of every 0-manifold. This is a little silly; in 2 dimensions, it's more interesting.

A less extreme version would be to try to extend Z to compact $(n-2)$-manifolds. This would mean that we've cut an $(n-1)$-manifold N along some $(n-2)$-manifold P into pieces, e.g. $N = N_+ \amalg_P N_-$. We'd like to define Z on noncompact $(n-1)$-manifolds with boundary and compact $(n-2)$-manifolds, and allow the algebra to reflect this gluing: $Z(N) = \langle Z(N_+), Z(N_-) \rangle_{Z(P)}$.

This pairing-like formula doesn't mean anything yet, but it reflects the story one dimension higher: let M be a closed n-manifold cut into two bordisms M_+ and M_- along a codimension-1 submanifold N; then, the number $Z(M)$ can be obtained as $Z(M) = \langle Z(M_+), Z(M_-) \rangle_{Z(N)}$, where this notation means the pairing in the Hilbert space $Z(N)$.

This idea is due to Dan Freed in [3]; there were a series of papers in the mid-1990s which espoused this idea of "extended locality" by Lawrence, Freed, Freed-Quinn, etc. Subsequently, Baez-Dolan conjectured the structure in [1].

We'd like the pairing for $Z(N)$ along $Z(P)$ to reflect this; abstractly, it should be some kind of delooping of the category of vector spaces. In category theory, Hom acts as a kind of inner product, and we need it to be a $\mathbb{C}$-vector space.

Definition 15.1. A $\mathbb{C}$-*linear category* C is a category whose spaces of morphisms $\text{Hom}_{\mathsf{C}}(X, Y)$ are all complex vector spaces, and such that the composition map $\text{Hom}_{\mathsf{C}}(X, Y) \times \text{Hom}_{\mathsf{C}}(Y, Z) \to \text{Hom}_{\mathsf{C}}(X, Z)$ is $\mathbb{C}$-bilinear. One also says C is *enriched* over $\mathbb{C}$. The category of small $\mathbb{C}$-linear categories is denoted $\mathsf{Cat}_{\mathbb{C}}$.

In this case, Hom defines a pairing $\mathsf{C} \times \mathsf{C}^{\mathsf{op}} \to \mathsf{Vect}_{\mathbb{C}}$. That is, to the $(n-2)$-manifold P, we attach a $\mathbb{C}$-linear category $Z(P)$, and to an $(n-1)$-manifold with boundary N_+ we associate an object $Z(N_+) \in Z(\partial N_+)$, just as an n-manifold with boundary M defines an element $Z(M) \in Z(\partial M)$.

We can also bring bordism down one dimension as well: given three $(n-2)$-manifolds P_1, P_2, and P_3 and two bordisms $N_1 : P_1 \to P_2$ and $N_2 : P_2 \to P_3$, we obtain ($\mathbb{C}$-linear) functors $Z(N_1) : P_1 \to P_2$ and $Z(N_2) : P_2 \to P_3$, and they compose.

Since we're considering $(n-1)$-manifolds with boundary, we can also take bordisms of these. That is, in dimension n, we must consider bordisms of bordisms, which are manifolds with (codimension 2) corners. If N_1 and N_2 are $(n-1)$-manifolds with boundary, $Z(N_1)$ and $Z(N_2)$ are functors, and so a bordism $M : N_1 \to N_2$ defines a natural transformation $Z(M) : Z(N_1) \Rightarrow Z(N_2)$. This forms the structure of a *bicategory*.[35]

Let's gather all this data up.

Definition 15.2. $\text{Bord}_{n-2,n}$ is the 2-category defined by the following data.

---

[35]In category theory and especially higher category theory, it's important to carefully distinguish equality and isomorphism; this is why we have a bicategory here, rather than a 2-category. In general, we try to minimize strictness. TODO be precise or at least consistent.

- The objects are compact $(n-2)$-manifolds.
- The 1-morphisms $P_1 \to P_2$ are the $(n-1)$-dimensional bordisms $N : P_1 \to P_2$.
- The 2-morphisms $N_1 \to N_2$ are n-dimensional bordisms with corners $M : N_1 \to N_2$.

As usual, we can take any other geometric structure, e.g. an orientation or a framing.

Notice that if $P = \varnothing$, this encapsulates the structure we had before: a closed $(n-1)$-manifold is a 1-morphism $\varnothing \to \varnothing$, and the bordisms between these are 2-morphisms.

Definition 15.3. A *(2-)extended TQFT* is a symmetric monoidal 2-functor $Z : \mathrm{Bord}_{n-2,n} \to \mathsf{Cat}_{\mathbb{C}}$.

Here, the 2-category structure on $\mathsf{Cat}_{\mathbb{C}}$ is "categories, functors, natural transformations:" the objects are small $\mathbb{C}$-linear categories, the 1-morphisms $\mathrm{Hom}_{\mathsf{Cat}_{\mathbb{C}}}(\mathsf{C}, \mathsf{D})$ are $\mathbb{C}$-linear functors $\mathsf{C} \to \mathsf{D}$, and the 2-morphisms between two functors are the $\mathbb{C}$-linear natural transformations between them.

The symmetric monoidal condition implies $Z(\varnothing) = \mathsf{Vect}_{\mathbb{C}}$, and therefore Z contains the data of a nonextended field theory in the 1- and 2-morphisms over $\varnothing \to \varnothing$.

One caveat is that, like the story for Riemannian bordisms, we're not really changing dimension: $(n-1)$-manifolds are really understood with collars (a direction of noncompactness), and similarly $(n-2)$-manifolds have two directions of noncompactness. If we can consider all noncompact directions (an n-collar of a point), we would achieve full locality, and $Z(\bullet)$ will be called the *category of boundary conditions*.

This sounds like a lot of work, but thanks to a trick called *dimensional reduction* (also *compactification*), we only need to think about 2-extended field theories for now. Let Z be an n-dimensional field theory and N be a fixed $(n-1)$-manifold. Then, there's a new 1-dimensional field theory $Z|_N$ defined by $Z|_N(R) = Z(N \times R)$ for any 1-manifold R. We classified 1-dimensional field theories: they correspond to finite-dimensional vector spaces, and $Z|_N$ is identified with $Z(N)$.

The point is, we can do this trick in any dimension. Let S be a compact k-manifold; then, in the same way, we can define an $(n-k)$-dimensional theory $Z|_S$ by $Z|_S(T) = Z(S \times T)$. This is strictly less data than Z, of course, but if you want to understand a 4-dimensional field theory, you can start by observing that it defines a 2-dimensional TQFT for every surface, and these already tell you a lot about the field theory.

Let's put this into play for the field theory defined by maps into a groupoid $\mathcal{X}$: for a 1-manifold N, $Z(N) = \mathbb{C}[\mathrm{Hom}_{\mathsf{Grpd}}(\pi_{\leq 1}N, \mathcal{X})]$. A morphism of groupoids $\mathcal{Y} \to \mathcal{X}$ defines a boundary condition: intuitively, if $\mathcal{X}$ is like a space X, then this boundary condition restricts fields on the boundary to a subspace $Y \subset X$. If $\mathcal{X}$ is like a group, then this is akin to requiring the group to act on the boundary conditions in a specified way.

So a boundary condition is a machine for producing elements $\mathcal{B}(N) \in Z(N)$ for all 1-manifolds N. This comes from a span

$$
\begin{array}{ccc}
 & \mathrm{Hom}_{\mathsf{Grpd}}(\pi_{\leq 1}N, \mathcal{Y}) & \\
\swarrow & & \searrow {\scriptstyle \pi_N} \\
\bullet & \mathrm{Hom}_{\mathsf{Grpd}}(\pi_{\leq 1}N, \mathcal{X}), &
\end{array}
$$

and indeed $\mathcal{B}(N) = \pi_{N*}1 \in Z(N)$.

If we tack on the map $\mathrm{Hom}_{\mathsf{Grpd}}(\pi_{\leq 1}N, \mathcal{X}) \to \bullet$, we obtain the number $\# \mathrm{Hom}_{\mathsf{Grpd}}(\pi_{\leq 1}N, \mathcal{Y})$, which is the output of an $(n-1)$-dimensional TQFT $Z_{\mathcal{Y}}$ associated to $\mathcal{Y}$.

That is, $\mathcal{X}$ defines an n-dimensional TQFT Z, and $\mathcal{Y}$ defines an $(n-1)$-dimensional TQFT $Z_{\mathcal{Y}}$. A map $\mathcal{Y} \to \mathcal{X}$ defines a boundary condition $\mathcal{B} : N \mapsto \mathcal{B}(N) \in Z(N)$, which is something like an $(n-1)$-dimensional field theory, but refines it: instead of producing numbers, it defines vectors. Physicists would call this $\mathcal{B}$ an $(n-1)$-dimensional field theory *coupled to* Z.

If $\mathcal{X}$ is a discrete groupoid, so just a set X, and $Y \to X$ is a map from another set, then topological quantum mechanics on Y is a 1-dimensional field theory sending $\bullet \mapsto \mathbb{C}[Y]$ and $S^1 \mapsto \dim(\mathbb{C}[Y]) = \# Y$. There's also a 2-dimensional TQFT associated to X in which $Z(S^1) = \mathbb{C}[\mathrm{Hom}_{\mathsf{Grpd}}(\pi_{\leq 1}, X)] = \mathbb{C}[X]$. Along

a span
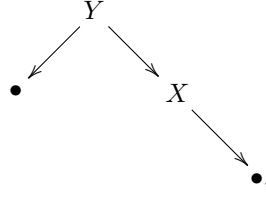
the function 1 maps to 1_Y maps to $\pi_* 1_Y = \delta_Y$ and then maps to $\#Y$. Thus, we've inserted a refinement: defining $\mathcal{B}(S^1) = \delta_Y \in Z(S^1) = \mathbb{C}[Y]$.

We want $Z(\bullet)$ to be the category (or space) of all boundary conditions, so that $\mathbb{C}[Y]$ is an object of $Z(\bullet)$. Here, it's easy to guess what we get; if you take the trivial vector bundle $\underline{\mathbb{C}}_Y$ on Y, and push it forward to X, the result $\pi_* \underline{\mathbb{C}}_Y \to X$ attaches to every $x \in X$ the vector space $\mathbb{C}[\pi^{-1}(x)]$. Instead of looking at all functions on X, we can break them up into where they live over X.

The proposal is that $Z(\bullet)$ is the category of vector bundles on X. Inside this category, $\mathcal{B}(S^1) \in Z(S^1)$ maps $x \mapsto \#(\pi^{-1}(x)) = \dim \mathbb{C}[\pi^{-1}(x)]$: since it's a function valued in $\mathbb{N}$, we can refine it into dimensions of vector spaces. Thus, $\mathcal{B}(S^1)$ is the vector bundle $\mathbb{C}[\pi^{-1}(x)] \to x$: a boundary condition is a family of vector spaces over X, a statement about quantum mechanics on X. Next time, we'll talk about vector bundles on groupoids, and what a boundary condition is when $X = \bullet/G$: we'll recover $Z(\bullet) = \mathsf{Rep}_G$, and boundary conditions will recover characters of representations. This is still just for finite groups, but as soon as we understand this for finite groups, we'll break out the topology.

16. 2-Extended Field Theory: 10/20/16

> *"What is G/K? G/K is G... mod K."*

Recall that we've been talking about extended TQFT in a "two-tier" theory as a functor $Z : \mathrm{Bord}_{n-2,n} \to \mathsf{Cat}$ between 2-categories. The objects of $\mathrm{Bord}_{n-2,n}$ are $(n-2)$-manifolds, and the 1-morphisms are $(n-1)$-dimensional bordisms between them. The 2-morphisms are n-dimensional bordisms between $(n-1)$-dimensional bordisms.

The codomain category is the 2-category of small categories: the objects are small categories, the 1-morphisms are functors $F : \mathsf{C} \to \mathsf{D}$, and the 2-morphisms are natural transformations $\eta : f \Rightarrow G$ of functors. We can restrict to particular kinds of categories (e.g. $\mathbb{C}$-linear ones).

All the field theories we've discussed thus far are Lagrangian, meaning Z factors through a category of fields. In this context, these fields will be some 2-category of groupoids. In the nonextended case, we linearized by taking a groupoid $\mathcal{X}$ to its functions $\mathbb{C}[\mathcal{X}] = \{\mathcal{X} \to \mathbb{C}\}$. Now, we categorify this, moving one level up: a groupoid $\mathcal{X}$ is also a category, so we can consider the category $\mathsf{Vect}(X) = \mathrm{Functors}(\mathcal{X}, \mathsf{Vect}_\mathbb{C})$, the category of functors from $\mathcal{X}$ to the category of complex vector spaces. That is, a vector bundle on a groupoid $\mathcal{X}$ is defined to be a functor $\mathcal{X} \to \mathsf{Vect}_\mathbb{C}$. This linearization process is also called *quantization*.

For example, suppose a group G acts on a set X, and $\mathcal{X} = X/G$ is the action groupoid. Then, the category of vector bundles on X is the category of G-equivariant vector bundles on X; if $\mathcal{X} = X$ is a discrete groupoid (no non-identity morphisms), then vector bundles on $\mathcal{X}$ are vector bundles on X. If $\mathcal{X} = \bullet/G$, then $\mathsf{Vect}\mathcal{X}$ is the category of representations of G.[36]

Most generally, if $\mathcal{X}$ is presented by two maps $s, t : \mathcal{G} \rightrightarrows X$, the category of vector bundles on $\mathcal{X}$ is the category of data $\{V \in \mathsf{Vect}(X), \gamma : s^* V \xrightarrow{\sim} t^* V\}$ satisfying an associativity condition reminiscent of the beginning of the Čech complex:

$$X \times_\mathcal{X} X \times_\mathcal{X} X = \mathcal{G} \times_X \mathcal{G} \Rrightarrow X \times_\mathcal{X} X = \mathcal{G} \underset{t}{\overset{s}{\rightrightarrows}} X.$$

The condition we want is that if $a : x \to y$ and $b : y \to z$, then $(ab)^* V_x = b^* a^* V_x$.

Like vector bundles on spaces, vector bundles on groupoids should pull back. But this is obvious: a map of groupoids is a functor $\pi : \mathcal{X} \to \mathcal{Y}$, and a vector bundle $\mathcal{Y}$ is a functor $V : \mathcal{Y} \to \mathsf{Vect}_\mathbb{C}$, so the pullback of V to $\mathcal{X}$ is $\pi^* V = V \circ \pi : \mathcal{X} \to \mathsf{Vect}_\mathbb{C}$.

This determines pushforward. In the nonextended case, functions on a groupoid pull back and push forward, and the pullback and pushforward maps are adjoint in the sense of Hilbert spaces. One categorical

---

[36]If you're getting frustrated with all the finite sets and discreteness, in a week or so we'll introduce geometry.

level up, we might expect pushforward π_* to be an adjoint to pullback $\pi^* : \mathsf{Vect}\mathcal{Y} \to \mathsf{Vect}\mathcal{X}$. For categories, one must distinguish left and right adjoints, but in this case the left and right adjoints are the same — this is an *ambidextrous adjunction*. One adjoint is akin to pulling back functions, and the other is akin to pushing forward measures. TODO: is some kind of finiteness necessary for this?

Explicitly, given $\pi : X \to Y$ and a vector bundle $V \in \mathsf{Vect}X$, we can define over a $y \in Y$

$$\pi_* V|_y = \bigoplus_{\pi(x)=y} V|_x = \prod_{\pi(x)=y} V|_y.$$

In this finite case, these are equivalent vector bundles; the direct sum is the colimit/left adjoint perspective, and the product is the limit/right adjoint perspective.

Under the unique map $X \to \bullet$, this simplifies: π_* is the global sections functor (as a left adjoint) or $\mathrm{Hom}(\mathsf{C}, \bullet)$ (as a right adjoint). Under the "quotient map" $\bullet \to \bullet/G$, the pullback map $\mathsf{Vect}\,\bullet\,/G \to \mathsf{Vect}(\bullet) = \mathsf{Vect}_{\mathbb{C}}$ is the forgetful functor $\mathsf{Rep}_G \to \mathsf{Vect}_{\mathbb{C}}$.

Specializing, consider the map $\bullet/G \to \bullet$. In this case, the pushforward is $\mathrm{Hom}_{\mathsf{Rep}_G}(\mathbb{C}, -)$, which defines both the invariants (left adjoint) and the coinvariants (right adjoint) of a representation! This is a nice discovery. In the other direction, pullback sends $\mathbb{C} \to \bullet$ to the regular representation. In some sense, this isn't surprising: there are only two representations you can construct *a priori* for any group, and if you push forward the trivial representation.

TODO: I got confused. Here's what I think I know:

- For the map $\bullet \to \bullet/G$, pullback is the forgetful functor $\mathsf{Rep}_G \to \mathsf{Vect}_{\mathbb{C}}$. Pushforward sends $\mathbb{C} \to \bullet$ to the regular representation.
- For the map $\bullet/G \to \bullet$, pullback sends $\mathbb{C} \to \bullet$ to the trivial representation, and pushforward is both invariants and covariants.

Now, suppose $K \subset G$ is a subgroup; then, inclusion induces a map $\bullet/K \to \bullet/G$. Then, any G-representation is a K-representation, and the pullback of vector bundles is the forgetful functor $\mathsf{Rep}_G \to \mathsf{Rep}_K$. The pushforward is induction of representations (in this case, it's also coinduction, because the left and right adjoints agree). This is an example of *Frobenius reciprocity*.

The fiber product of $\bullet/K \to \bullet/G$ with a point is G/K:

$$
\begin{array}{ccc}
G/K & \longrightarrow & \bullet \\
\downarrow & \lrcorner & \downarrow \\
\bullet/K & \longrightarrow & \bullet/G.
\end{array}
$$

IF W is a vector bundle on $\bullet/K$, it pulls back to an associated bundle $\underline{W} \to G/K$, which we may understand as a principal K-bundle on G/K or as a G-equivariant bundle on G/K. If $K \subset G$ comes to us from representation theory, this perspective on representations of K can be extremely useful.

Well, what is this principal bundle? Suppose G acts on a space X and $\mathcal{P} \to M$ is a principal G-bundle. Then, we can form the associated-bundle construction $(X)_{\mathcal{P}} = \mathcal{P} \times_G X$,[37] defined to be $\mathcal{P} \times X/((pg, x) = (p, gx))$ (equivalently, we could require $(pg^{-1}, gx) = (p, x)$).

Choose any $y \in M$; then, the fiber over y is noncanonically identified with X:

$$
\begin{array}{ccc}
X & \longrightarrow & (X)_{\mathcal{P}} \\
\downarrow & \lrcorner & \downarrow \\
y & \hookrightarrow & M.
\end{array}
$$

To make this identification canonical, one must choose a $p \in \pi^{-1}(y)$. This associated-bundle construction carries with it any structure on X preserved by G: if X has a G-invariant inner product, $(X)_{\mathcal{P}}$ has a G-invariant metric, and so on.
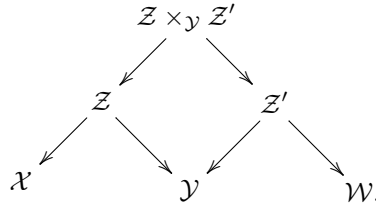
If $K \subset G$ is a subgroup and W is a K-representation, then we can start with $\mathcal{P} = G$ over G/K; the associated bundle is $\underline{W} = (W)_K \to G/K$ defined by $\bigoplus_{\gamma \in G/K} \underline{W}|_\gamma$. The ambidexterity inherent in this discussion means we could also take the direct product.

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[37]This notation, though standard for associated bundles, is unfortunate, because this is not actually a fiber product.

This ambidexterity arises in a different, topological situation, and this is not a coincidence: there's ambidexterity in the oriented bordism 2-category. Given any bordism $N : P \to P'$, we can form the same bordism with the opposite orientation, $N^{op} : P' \to P$. Since we're in a 2-category, we're allowed to consider bordisms of bordisms, and so N is cobordant to N^{op}: specifically, consider the cylinder $N \times I : N \to N$, and pull the outgoing boundary back to become another incoming boundary. This defines a bordism $N \amalg N^{op} \to \varnothing$. This can be generalized: if N is a pair of pants, let P be its "ankle" boundary (the boundary that's $S^1 \amalg S^1$). Then, $N \times I$ is a bordism from N to N, and pulling the outgoing boundary back to an incoming boundary, we obtain a bordism $N \amalg_P N \to S^1 \times [0,1]$ to the cylinder.

Without this ambidexterity, there would be no hope for turning this groupoid story into a field theory. But we can do it, and we will. Overall, we'd like to define a field theory as a composition of functors $\mathrm{Bord}_{n-2,n} \to \mathsf{Grpd} \to \mathsf{Cat}$.

Vector bundles are functorial not just for maps of groupoids, but for correspondences of groupoids: given a correspondence of groupoids $\mathcal{X} \leftarrow \mathcal{Z} \to \mathcal{Y}$, we can define a functor $\mathcal{Z} : \mathsf{Vect}\mathcal{X} \to \mathsf{Vect}\mathcal{Y}$ by pushforward-pullback. To claim functoriality, we need to know that it behaves well under base change (compositions of correspondences):

$$
\begin{array}{ccccc}
 & & \mathcal{Z} \times_{\mathcal{Y}} \mathcal{Z}' & & \\
 & \swarrow & & \searrow & \\
 \mathcal{Z} & & & & \mathcal{Z}' \\
 \swarrow & & \searrow & \swarrow & & \searrow \\
 \mathcal{X} & & & \mathcal{Y} & & \mathcal{W}.
\end{array}
$$

But as everything is 2-categorical, we can try to define a 2-category of correspondences $\mathsf{2Corr}$.

- The objects of $\mathsf{2Corr}$ groupoids.
- The morphisms $\mathrm{Hom}(\mathcal{X}, \mathcal{Y})$ are correspondences $\mathcal{X} \leftarrow \mathcal{Z} \to \mathcal{Y}$.
- The 2-morphisms are correspondences of correspondences

(16.1)

$$
\begin{array}{ccccc}
 & & \mathcal{Z} & & \\
 & {}^{p_{\mathcal{Z}}}\nearrow \; {}^{\pi_{\mathcal{Z}}}\uparrow & & \nwarrow {}^{q_{\mathcal{Z}}} & \\
 \mathcal{X} & \leftarrow & \mathcal{W} & \rightarrow & \mathcal{Y} \\
 & {}_{p_{\mathcal{Z}'}}\searrow \; {}_{\pi'_{\mathcal{Z}}}\downarrow & & \swarrow {}_{p_{\mathcal{Z}}} & \\
 & & \mathcal{Z}'. & &
\end{array}
$$

To make everything suitably functorial, we need to turn (16.1) into a natural transformation between the functors defined by $\mathcal{Z}$ and by $\mathcal{Z}'$. These are, respectively, $q_{\mathcal{Z}*} p_{\mathcal{Z}}^* : \mathsf{Vect}\mathcal{X} \to \mathsf{Vect}\mathcal{Y}$ and $q_{\mathcal{Z}'*} p_{\mathcal{Z}'}^* : \mathsf{Vect}\mathcal{X} \to \mathsf{Vect}\mathcal{Y}$.

How does this work? Well, since pullback and pushforward are adjoints, monadicity defines a natural transformation $\mathrm{id} \to \pi_{\mathcal{Z}*} \pi_{\mathcal{Z}}^*$ (and similarly for $\pi_{\mathcal{Z}}$). This induces natural transformations

$$
q_{\mathcal{Z}*} p_{\mathcal{Z}}^* \longrightarrow q_{\mathcal{Z}*} \pi_{\mathcal{Z}*} \pi_{\mathcal{Z}}^* p_{\mathcal{Z}}^*
$$

and

$$
q_{\mathcal{Z}'*} p_{\mathcal{Z}'}^* \longleftarrow q_{\mathcal{Z}'*} \pi_{\mathcal{Z}'*} \pi_{\mathcal{Z}'}^* p_{\mathcal{Z}'}^*.
$$

Since the diagram (16.1) commutes, then $q_{\mathcal{Z}*} \pi_{\mathcal{Z}*} \pi_{\mathcal{Z}}^* p_{\mathcal{Z}}^* = q_{\mathcal{Z}'*} \pi_{\mathcal{Z}'*} \pi_{\mathcal{Z}'}^* p_{\mathcal{Z}'}^*$, so the natural transformation induced by the 2-morphism (16.1) is

$$
\begin{array}{ccc}
q_{\mathcal{Z}*} p_{\mathcal{Z}}^* & \longrightarrow & q_{\mathcal{Z}*} \pi_{\mathcal{Z}*} \pi_{\mathcal{Z}}^* p_{\mathcal{Z}}^* \\
 & & \| \\
q_{\mathcal{Z}'*} p_{\mathcal{Z}'}^* & \longleftarrow & q_{\mathcal{Z}'*} \pi_{\mathcal{Z}'*} \pi_{\mathcal{Z}'}^* p_{\mathcal{Z}'}^*.
\end{array}
$$

Thus, given a 2-functor $\mathrm{Bord}_{n-2,n} \to \mathsf{2Corr}(\mathsf{Grpd})$, we can define a 2-extended field theory by taking vector bundles on groupoids. This defines a lot of field theories! For example, we could take the fields (the 2-functor) to be $\mathrm{Hom}_{\mathsf{Grpd}}(-, \mathcal{X})$.

If $\mathcal{X} = \bullet/G$, this defines 3-dimensional Dijkgraaf-Witten theory, as described in [5, 3].

- Given a closed 3-manifold M, we obtain the number of G-local systems on M, as in the nonextended theory.
- Given a closed 2-manifold Σ, we obtain the vector space $\mathbb{C}[\mathrm{Loc}_G \Sigma]$, functions on the groupoid of G-local systems on Σ.
- To S^1, we get the category of vector bundles on $\mathrm{Loc}_G S^1$.
- To a point, we get the category of representations of G.

17. More Boundary Conditions: 10/25/16

Recall that we're in the middle of discussing boundary conditions in extended field theory: we set up the bordism 2-category $\mathsf{Bord}_{n-2,n}$ and the 2-category of categories Cat. A field theory is a 2-functor $Z : \mathsf{Bord}_{n-2,n} \to \mathsf{Cat}$, and we want it to factor through a 2-category of correspondences of groupoids. One example is the theory of maps into a groupoid $\mathcal{X}$ (so the field theory is the category of vector bundles on the groupoid of maps to $\mathcal{X}$). A boundary condition for this theory arises from a morphism of groupoids $\mathcal{Y} \to \mathcal{X}$.

If P is an $(n-2)$-manifold, composition defines a map $\pi_P : \mathrm{Hom}_{\mathsf{Grpd}}(\pi_{\leq 1} P, \mathcal{Y}) \to \mathrm{Hom}_{\mathsf{Grpd}}(\pi_{\leq 1} P, \mathcal{X})$, which we can think of as a span with the trivial vector bundle $\underline{\mathbb{C}}$ over a point. Pushing forward, we obtain a vector bundle $\pi_{P*}\underline{\mathbb{C}} \in \mathsf{Vect}(\mathrm{Hom}_{\mathsf{Grpd}}(\pi_{\leq 1} P, \mathcal{X}))$, i.e. from the map $\mathcal{Y} \to \mathcal{X}$, we obtain an object in $Z(P)$.

If N is an $(n-1)$-manifold, then composition defines a map $\pi_N : \mathrm{Hom}(\pi_{\leq 1} N, \mathcal{Y}) \to \mathrm{Hom}(\pi_{\leq 1} N, \mathcal{X})$ in the same way, which defines a span with the trivial function $\underline{1}$. Pushing forward, we obtain a function $\pi_{N*}\underline{1} \in \mathbb{C}[\mathrm{Hom}(\pi_{\leq 1} N, \mathcal{X})]$, i.e. an element in $Z(N)$.

Concretely, if $\mathcal{X} = X/G$, then $\mathrm{Hom}_{\mathsf{Grpd}}(\pi_{\leq 1} N, \mathcal{X})$ is equivalent to the groupoid of principal G-bundles $\mathcal{P} \to N$ along with a section of $(X)_{\mathcal{P}} \to N$. These are twisted G-maps $\mathcal{P} \to X$, as in the diagram

$$
\begin{array}{ccc}
\mathcal{P} & \longrightarrow & X \\
\downarrow{\scriptstyle G} & & \downarrow{\scriptstyle G} \\
N & \longrightarrow & \mathcal{X} \\
& \searrow & \downarrow \\
& & \bullet/G.
\end{array}
$$

Let's specialize further, to where $n = 2$ and $\mathcal{X} = \bullet/G$. Then, a map $\mathcal{Y} \to \bullet/G$ is equivalent to data of a G-action on a set Y, which is an object of the category $Z(\bullet) = \mathsf{Vect}\,\bullet\,/G = \mathsf{Rep}_G$. The boundary condition associated to a G-action on a set Y is the representation $\mathbb{C}[Y]$.

On the circle, $Z(S^1) = \mathbb{C}[G/G] = \mathbb{C}[\mathrm{Hom}(\pi_{\leq 1} S^1, \bullet/G)]$. Given the boundary condition $\mathcal{Y}$, we obtain the morphism $\mathrm{Hom}(\pi_{\leq 1} S^1, \mathcal{Y}) \to \mathrm{Hom}(\pi_{\leq 1} S^1, \bullet/G) = G/G$. Recall that $\mathrm{Hom}(\pi_{\leq 1} S^1, \mathcal{Y})$ is the inertia groupoid $\{g \in G, y \in Y^g\}/G$.[38] This maps to G/G by forgetting y, sending $(g, y) \mapsto \{g\}/G$.

In dimension 1, linearization yields functions, so the pushforward of $\underline{1}$ under this map is some function $\chi_V \in \mathbb{C}[G/G]$. This is the character of the representation $\mathbb{C}[Y]$ that $\mathcal{Y}$ defines: $\chi_V(g) = \#\pi^{-1}(g) = \#Y^g = \mathrm{Tr}_{\mathbb{C}[Y]}(g)$. This is an example of the *Frobenius character formula*.

This method attaches the character in $Z(S^1)$ to any representation (in $Z(\bullet)$): this illustrates the idea that a boundary condition is a local beast which produces a state on every $(n-1)$-manifold.

Example 17.1. Given a group G, there are two canonical examples for representations of G.

(1) If $Y = \bullet$, we have $\mathcal{Y} = \bullet/G$, and the map to $\mathcal{X} = \bullet/G$ is the identity. The associated boundary condition is called the *Neumann boundary condition*. The associated representation is the trivial representation $\mathbb{C}[Y] = \mathbb{C}$, whose character $\chi_{\mathbb{C}}$ is the constant function 1 on G/G, arising as the pushforward of the constant function $\underline{1} \in \mathbb{C}[G/G]$ under the identity map $G/G \to G/G$.

(2) We also could take $Y = G$, so $\mathcal{Y} = \bullet$ and we pushforward the trivial representation along $\bullet \to \bullet/G$, obtaining the regular representation. This is a *Dirichlet boundary condition*. On the circle, the inertia map is $\bullet \to \bullet/G$, and we pushforward $\underline{1}$. The character we obtain is $\chi = |G|\delta_e \in \mathbb{C}[G/G]$: the image of the inertia map only contains the identity.

---

[38] Here, Y^g denotes the fixed point set of $g \in G$; we could also phrase this as $\{y \in Y, g \in G_y\}/G$, where G_y is the stabilizer subgroup of y, but then we have a tricker conjugation action to mod out by.

For a more general 2-dimensional TQFT Z, a boundary condition defines an object $\mathcal{B} \in Z(\bullet)$, and we'll soon explain how this defines a "character" $\chi_{\mathcal{B}} \in Z(S^1)$. We saw this explicitly for $\mathcal{X} = \bullet/G$, where it's literally the characters of representations, and for $\mathcal{X} = X$, where the character is the rank function on a vector bundle. We can put these together to describe these generalized characters for all groupoids.

Another way to think of this is that given a boundary condition /, we want to extend the TQFT Z from $\mathrm{Bord}_{0,2}$ to bordisms with boundary components indexed by /. A Dirichlet boundary condition, where $Y = G$ and $\mathcal{Y} = \bullet$, then for a surface Σ, the fields are the groupoid $\mathrm{Loc}_G \Sigma$. But for a surface Σ' with a boundary indexed by /, the fields are the G-local systems on Σ' which are trivial on the orange boundary component.

For example, let's consider local systems on an interval $[0, 1]$.

- If there are no orange boundary components, then local systems are trivial, but have automorphisms: $\mathrm{Loc}_G([0, 1]) = \bullet/G$.
- If one boundary component is marked orange, then we have to trivialize on the boundary, so $\mathrm{Loc}_G([0, 1]) = \bullet$.
- If both components are marked orange, then we must trivialize on both endpoints, so $\mathrm{Loc}_G([0, 1]) = G$.

If we apply this to the evaluation bordism $\bullet \amalg \bullet \to \varnothing$, we usually have the pairing $\mathrm{Hom}: \mathsf{Rep}_G \otimes \mathsf{Rep}_G \to \mathsf{Vect}$, but if we mark one of the inputs as orange, this is the functor $\mathrm{Hom}(\mathbb{C}[G], -): \mathsf{Rep}_G \to \mathsf{Vect}$.

So we've extended these functors from manifolds with boundary to manifolds with orange markings. There will be a version of the cobordism hypothesis for these field theories: everything is determined by what we attach to a point.

Example 17.2 (Frobenius character formula). Let $K < G$ be a subgroup and $Y = G/K$. Then, consider the groupoid $\mathcal{Y} = G \backslash Y = \bullet/K \to \bullet/G$, which is the induction functor: $\underline{\mathbb{C}} \mapsto \mathrm{Ind}_K^C \mathbb{C}$. The inertia groupoid of $\mathcal{Y}$ is K/K, which maps to G/G to determine the character attached to the induced representation. $K/K = \{g \in G, x \in (G/K)^g\}/G$, so the character formula counts the fiber:

$$\chi_{\mathrm{Ind}_K^G}(g) = \#(G/K)^g = \#\{k \in K \mid G \text{ conjugate to } g\}/K.$$

You have to count them "correctly," but this is how the correct notion of counting arises: it's very geometric.

If $K = B$ is a Borel subgroup, e.g. in the case $G = \mathrm{GL}_n(\mathbb{F}_q)$, so B is the subgroup of upper triangular matrices, then $G/B = \mathrm{F}\ell_n(\mathbb{F}_q)$, the flag variety, because B is the stabilizer of the flag $k \subset k^2 \subset k^3 \subset \cdots \subset K^n$. In this case, the induced representation is the *Springer resolution*

$$\{g \in G, x \in (\mathrm{F}\ell_n)^g\} \longrightarrow G.$$

The Springer resolution is one of a few things that explain everything in representation theory.

In this case, it's more customary to think of the marked boundaries as red, so the fields on a surface with marked boundary Σ are G-local systems on Σ along with B-reductions, i.e. choices of flags, on the red boundary.

Another way to think of this is that on the red boundary $\partial \Sigma$, we have a lift of the map $\Sigma \to \mathcal{X}$ along the map $\mathcal{Y} \to \mathcal{X}$:

$$
\begin{array}{ccc}
\partial \Sigma & \longrightarrow & \mathcal{Y} = \bullet/B \\
\downarrow & & \downarrow \\
\Sigma & \longrightarrow & \mathcal{X} = \bullet/G.
\end{array}
$$

We have a G-local system $\mathcal{P}$ on Σ plus a section of $(G/B)_{\mathcal{P}}$ on $\partial \Sigma$; this is also equivalent to the data of a G-local system $\mathcal{P}_G$ on Σ and a B-local system $\mathcal{P}_B$ on the red boundary, plus the data of an isomorphism $\mathrm{Ind}_B^G \mathcal{P}_B \cong \mathcal{P}_G$.

Consider the interval with two red boundary points. Z applied to this interval has an algebraic structure induced from the *pair of chaps* bordism (TODO: add picture), which defines a map $\mathbb{C}[K \backslash G/K] \otimes \mathbb{C}[K \backslash G/K] \to \mathbb{C}[K \backslash G/K]$. This algebra is the *Hecke algebra* for G and K; if $K = \bullet$, this is just the group algebra again.

Now, we'll step back and attach an algebra to a groupoid X: modules over this algebra will be the same as vector bundles over X. This is inspired by algebraic geometry: if X were an affine variety rather than a groupoid, we think about quasicoherent sheaves on X (the analogue of vector bumdles) in terms of modules on $\mathscr{O}(X)$.

If $\mathcal{X} = X$ is a set, we can think of the algebra $\mathbb{C}[X]$ (where the multiplication is pointwise), and indeed, $\mathsf{Vect}X = \mathsf{Mod}_{\mathbb{C}[X]}$. If $\mathcal{X} = \bullet/G$, then $\mathsf{Vect}\,\bullet/G = \mathsf{Rep}_G = \mathsf{Mod}_{\mathbb{C}[G]}$.

We'll generalize this by starting with a presentation $\mathcal{X} = [\mathcal{G} \rightrightarrows X]$. Then, $\mathbb{C}\mathcal{G}$ is the functions on the arrows $\mathcal{G}$, with the condition that if $g, h \in \mathcal{G}$ are composable and $f \in \mathbb{C}\mathcal{G}$, then $f(g \cdot h) = f(g)f(h)$. For example, if we get a groupoid $\mathcal{G}$ from finite sets $X \times_Y X \rightrightarrows X \to Y$, then $\mathbb{C}\mathcal{G}$ is the algebra of block diagonal $|X| \times |X|$ matrices with the blocks indexed by $X \to Y$, and it injects into the algebra of block diagonal matrices on X.

In general, given a groupoid presentation $\mathcal{G} \rightrightarrows X$, $\mathbb{C}\mathcal{G}$ is an algebra object in the category of $\mathbb{C}[X]$ $\mathbb{C}[X]$-bimodules: the bimodule structure is induced by the diagonal map $\Delta : X \to X \times_\mathcal{G} X$, followed by left and right multiplication. This means that the composition map $\mathbb{C}\mathcal{G} \times \mathbb{C}\mathcal{G} \to \mathbb{C}\mathcal{G}$ is $\mathbb{C}[X]$-bilinear, and hence descends to a map $\mathbb{C}\mathcal{G} \otimes_{\mathbb{C}[X]} \mathbb{C}\mathcal{G} \to \mathbb{C}\mathcal{G}$. There's a lot of different ways to describe this: all of the following are equivalent descriptions.

- $\mathbb{C}[X]$-bimodules.
- $\mathbb{C}[X] \otimes \mathbb{C}[X]$-modules.
- $\mathbb{C}[X \times X]$-modules.
- Endomorphisms of the category $\mathsf{Mod}_{\mathbb{C}[X]}$.

Moreover, vector bundles on $X \times X$ are $\mathbb{C}[X]$-bimodules are functors from $\mathsf{Vect}X$ to $\mathsf{Vect}X$, where given a vector bundle K, the functor we obtain is a kernel transform

$$V \longmapsto \pi_{2*}(\pi_1^* V \otimes K).$$

This is a categorified version of the kernel transform we began the class with.

More abstractly, this is a monad: we understand it on $\mathcal{X}$ by pulling it up to X and understanding the two maps to it from $\mathcal{G}$: the (category of modules over the) groupoid algebra is a nice way to present the category of vector bundles on $\mathcal{X}$. But here something interesting happens: given two equivalent presentations of groupoids, their groupoid algebras may not be equivalent, but their categories of modules *are* equivalent. This is a kind of equivalence called *Morita equivalence*. We'll use this to form a 2-category of algebras, algebra homomorphisms, and Morita equivalences. Realizing an algebra as a groupoid algebra is a useful way to capture its Morita equivalence class, and is a common technique in geometric representation theory.

One example is $n \times n$ matrices: the groupoid algebra is Morita equivalent to $\mathbb{C}$. Similarly, the block diagonal matrices we discussed above define an algebra Morita equivalent to $\mathbb{C}[Y]$.

18. Integrable lattice models from extended field theories: 10/26/16

Note: David Ben-Zvi gave this talk at UT's geometry and string theory seminar; it's not part of the class, but is closely related.

The talk follows a survey article of Costello [2] about spin chains and gauge theory, as well as Witten's article on the same subject.

Spin chains are a model for $1+1$-dimensional quantum mechanics, or, equivalently, a 2-dimensional lattice model for statistical mechanics. Consider a lattice $\mathbb{Z}/N$ for the circle, so N particles on the circle. Each particle has a state space $V = \mathrm{span}_\mathbb{C}\{\uparrow, \downarrow\} \cong \mathbb{C}^2$, and we let the Hilbert space for the system be $\mathscr{H} = V^{\otimes N}$, the tensor product of the state spaces at each lattice point.

Rather than giving the Hamiltonian, we'll give the *transfer matrix*, the exponential of the Hamiltonian, which defines time evolution. This is an operator $T : \mathscr{H} \to \mathscr{H}$. We imagine the particles moving in the time direction, and at some point they interact with a particle W, represented by a circular time-slice in $S^1 \times [0, 1]$.

When one of the lattice points interacts with W, the interaction is governed by an *R-matrix* $R : V \otimes W \to W \otimes V$.[39] In bases for V and W, this is determind by four indices $R_{ik}^{j\ell}$. For example, for the XXX spin chain, so $V = W = \mathbb{C}^2$, the R-matrix is

$$R(v \otimes w) = w \otimes v + \frac{c}{z} w \otimes v,$$

where c is the quadratic Casimir operator associated to $\mathrm{SL}_2(\mathbb{C})$. It'll be important later that this depends on an additional parameter z.

---

[39]Historically, this is usually written as a map $V \otimes W \to V \otimes W$, and in that framework it might be better to label ours $\check{R}$.

When we consider all lattice points separately, we tensor their operators together, defining an operator

$$R^{\otimes N} \in (\operatorname{End} V)^{\otimes N} \otimes (\operatorname{End} W)^{\otimes N}.$$

When we consider how the particle W interacts with all lattice points jointly, we obtain an operator

$$R^{\circ N} \in (\operatorname{End} V)^{\otimes N} \otimes \operatorname{End} W.$$

$R^{\circ N}$ is obtained from $R^{\otimes N}$ by composing the operators in $(\operatorname{End} W)^{\otimes N}$) together. Now, to obtain the transfer matrix, we have to close the particle W off, so the formula is

$$T = \operatorname{Tr}_W R^{\circ N}.$$

Suppose we time-evolve M times (so we have M interactions with the particle) and then close off. Now, the lattice is $L = \mathbb{Z}/N \times \mathbb{Z}/M$, and the partition function for this system is

$$Z(L, R) = \operatorname{Tr}_{V^{\otimes N}} T^{\otimes M}.$$

This partition function can be determined from local data (maybe this isn't a surprise if you're used to statistical mechanics).

Lemma 18.1.
$$Z(L, R) = \sum_{\text{spin configurations } \sigma} \prod_{\text{vertices } v} R(v, \sigma).$$

Here, the configurations are those in a square-lattice model for ice: suppose the lattice is of water molecules and the oxygen molecules are at the vertices; then, the configurations are choices of the directions the hydrogens point in. This is a choice of assigning the indices i, j, k, and ℓ to the four outgoing edges to a vertex, and is an example of the *six-vertex model*. Then, $R(v, \sigma) = R_{ik}^{j\ell}$ with i, j, k, and ℓ determined by this configuration; this is called the *Boltzmann weight* of the system.

When these were first studied, it was realized they're integrable, meaning we should be able to obtain exact solutions to them. Thus, we want to search for lots of (actually infinitely many) operators commuting with T. In particular, since R and T depend on a *spectral parameter* z and

(18.2) $$[T(z), T(w)] = 0,$$

then the Laurent coefficients of $T(z)$ define operators on $\mathscr{H}$ that commute with T. This can be used to solve this model in practice.

The *Yang-Baxter equation* is a local condition on R that guarantees that the above argument works to show the system is integrable. First suppose $r, p, q \in \mathbb{C}$ and we're given families of vector spaces V_r, W_p depending on r and p, respectively. Then, we define

$$\mathcal{L} = R(V_r, W_p) : V_r \otimes W_p \longrightarrow W_p \otimes V_r$$
$$\mathcal{R} = R(W_p, W_q) : W_p \otimes W_q \longrightarrow W_q \otimes W_p.$$

The Yang-Baxter equation can be concisely written as

(18.3) $$\mathcal{R}\mathcal{L}\mathcal{L} = \mathcal{L}\mathcal{L}\mathcal{R}.$$

This can be written as a braid diagram (TODO: add).

The Yang-Baxter equation implies (18.2), and so it's a more fundamental explanation for the integrability. This has been a rich theory for several decades; this is the origin of quantum groups, though the spectral parameter is not the same parameter usually given for a quantum group.

Field-theoretic perspective. We're going to show how this can arise from field theory in a surprisingly natural way.

We'll start by thinking of a vector space as a topological quantum-mechanical system: it is the Hilbert space for some system, but we have no Hamiltonian. We'd like to realize vector spaces V and W as defects in a topological field theory, associated to allowing singularities in spacetime. The familiar example is Wilson lines in Chern-Simons theory. From a condensed-matter perspective, we're looking at a system with a one-dimensional bulk. These line defects may be thought of as operators.

There's an OPE (operator product expansion) between two line operators as we move them together; since we're doing topological field theory, the distance doesn't matter. From what Costello calls the Atiyah-Segal-Freed axioms for a topological field theory, given an n-dimensional TQFT Z and a line defect, we consider

the link L of this line, which is an S^{n-2}, so $Z(L) = Z(S^2)$. Since this is codimension 2, it's a category C, and (if $n \geq 2$) this is a monoidal category, meaning that given two line defects V and W, we can collide them to a third $V * W$, and $*$ is associative.[40]

In 3 or more dimensions, there's more room to move around: you can move the operators around each other, defining an isomorphism $V * W \overset{\sim}{\to} W * V$. However, there's no reason for this isomorphism to square to the identity, since the lines may be tangled around each other in a nontrivial way. However, they do satisfy the *braid relations*, which are the relations given by twisting braids around each other. This defines a braided monoidal category structure on C. (In dimensions 4 and higher, there's even more room to move, and C is symmetric monoidal.)

Braids are nice, because we already know how to define the Yang-Baxter equation in terms of braids. But everything is one categorical level up, braiding vector spaces instead of operators, so we don't quite know how to write down an R-matrix.

Nonetheless, the line operators are operators: they (and C) act on whatever $Z(M)$ is (when M is an $(n-2)$-manifold) in the same way local operators do: the bordism $I \times M$: at a point $p \in M$, we insert the line, defining a bordism that Z maps to an action $\mathsf{C} \times Z(M) \to Z(M)$. Even when $n = 3$, there's other things happening: we can consider surface defects by attaching surfaces, and these result in interesting actions by interesting categories.

To simplify our lives, suppose one of these modules $Z(M)$ (where M is $(n-2)$-dimensional) is boring: it's the category of vector spaces. Then, C acts on Vect, meaning there is a functor $\mathsf{C} \to \operatorname{End}(\mathsf{Vect}) = \mathsf{Vect}$. Another way to say this is that we have a monoidal functor $(\mathsf{C}, *) \to (\mathsf{Vect}, \otimes)$, called the *fiber functor*, from "interesting OPE to ordinary tensor product."[41]

This is where R-matrices come from: the braiding $\eta : V * W \overset{\sim}{\to} W * V$ on C passes to vector spaces, defining an identification $R : V \otimes W \overset{\sim}{\to} W \otimes V$. Here, V and W are vector spaces, and since the braiding isn't the one squaring to the identity, R is nontrivial, and we recover the R-matrix from before.

A reconstruction theorem implies that, as monoidal categories, C is equivalent to the category of H-modules where H is a *quasitriangular* Hopf algebra (here, "quasitriangular" is an annoying way to say "coming from a braided monoidal category"): the Hopf algebra structure ensures $V \otimes W$ is also an H-module, and quasitriangularity ensures that $R(V, W)$ satisfies the Yang-Baxter equation (ultimately coming from the braiding).

Example 18.4. One example of a quasitriangular Hopf algebra is given by a quantum group: given a finite-dimensional Lie algebra, $\mathfrak{g}$ (e.g. $\mathfrak{sl}_2$) and a root of unity q, we can form a quantum group $U_q(\mathfrak{g})$ which is a quasitriangular Hopf algebra.

The spin chain comes from drawing a few pictures of the same kind: we need to find the Hilbert space and its time evolution.

The Hilbert space is a vector space, so we imagine it associated to an $(n-1)$-manifold. Given an $(n-2)$-manifold M and a point $p \in M$, consider the circle $\{p\} \times S^1 \subset M \times S^1$. Take N points on this circle and attach the local state space V — if $Z(M) = \mathsf{Vect}$, then the state space here is $\mathscr{H} = V^{\otimes N}$.

To obtain the transfer matrix, consider another line defect W and wrap it around a small circle, which defines a local operator that acts on $\mathscr{H}$; this will be the transfer matrix T.

Precisely, on $I \times S^1 \times M$, we insert lines with space V on $I \times S^1 \times \{p\}$ (in the I-direction), and over another point q, we insert an operator on a loop $\{t\} \times S^1 \times \{q\}$ acting as W. The partition function arises from closing this cylinder off (in the I-direction).

Unfortunately, we've reconstructed everything except for the interesting part, the spectral parameter that gave us integrability (as well as motivation). The picture shows us that $[T(q), T(r)] = 0$ is true (if q and r are in the same connected component), but obvious: since this theory is topological, it's locally constant.

However, Costello's insight is that there are lots of field theories where the dependence on the $(n-2)$-manifold M is not topological: this whole story still works when the dependence on S^1 and I is topological, but on M is not. Therefore, we can look at four-dimensional QFTs, e.g. $\mathcal{N} = 1$ (or 2, or 4) gauge theories, made into QFTs that are topological in two directions and, e.g. holomorphic in the remaining two directions. In particular, we can consider 4-manifolds of the form $\Sigma_1 \times \Sigma_2$, where Σ_1 is a topological surface, and Σ_2 is a Riemann surface.

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[40]To be precise, $*$ is associative up to natural isomorphism.

[41]You might hope for this to be faithful, but it's not important for today's talk.

When you run through the same setup, the dependence of T on p, q, and r is holomorphic rather than topological, with poles where they collide. Costello shows that for $\mathcal{N} = 1$, one recovers spin chains as we talked about them, and, just as 3-dimensional Chern-Simons theory recovers Drinfeld-Jimbo quantum groups $U_q(\mathfrak{g})$, for 4-dimensional $\mathcal{N} = 1$ theory, what you get is both stronger and weaker: quantum groups depending on affine algebras: $U_q(\mathfrak{g}[z, z^{-1}])$, the *Yangian* $U_h(\mathfrak{g}[z])$, and quantum groups associated to an elliptic curve $E_q(\mathfrak{g})$. Though these are more complicated, they came first historically. The Yangian comes from the structure of line operators in these mixed field theories, and is a kind of vertex algebra.

19. Algebras, Categories, 2D TQFTs, and Groupoids: 10/27/16

For the next few lectures, we're going to discuss how to relate groupoids, algebras, categories, and two-dimensional TQFTs: given a groupoid, we can obtain an algebra, then a category, then finally a TQFT. We'll work over $\mathbb{C}$, but this works over any field k.

Let A be an associative $\mathbb{C}$-algebra. Then, the category Mod_A of A-modules is a $\mathbb{C}$-linear abelian category, and it's pointed: A is a distinguished object. The endomorphisms of A as a left A-module are $\mathrm{End}_{\mathsf{Mod}_A}(A)$, which are just the actions of A on itself from the right: $\mathrm{End}_{\mathsf{Mod}_A}(A) = A^{\mathsf{op}}$. That is, we can recover A from some structure on its category of modules.

Conversely, let C be a $\mathbb{C}$-linear category, and choose an object M; then, we can recover an algebra $A = \mathrm{End}_{\mathsf{C}}(M)^{\mathsf{op}}$. We'd like this to be an inverse to $A \mapsto \mathsf{Mod}_A$, but if we choose M badly (e.g. $M = 0$), it doesn't quite work.

We can extract a tensor-Hom adjunction from this:

$$\mathrm{Hom}_{\mathsf{C}}(M, -) : \mathsf{C} \rightleftarrows \mathsf{Vect} : M \otimes_{\mathbb{C}} -,$$

and this factors through Mod_A:

$$
\begin{array}{ccc}
\mathsf{C} & \underset{M \otimes_{\mathbb{C}} -}{\overset{\mathrm{Hom}_{\mathsf{C}}(M,-)}{\rightleftarrows}} & \mathsf{Vect}_{\mathbb{C}} \\
& M \otimes_A - \searrow & \uparrow \;\; \text{Forget} \; \Vert \; - \otimes_{\mathbb{C}} A \\
& & \mathsf{Mod}_A.
\end{array}
$$

We'd like to be able to carry generators and relations from Mod_A to C: given a sequence

$$A^{\oplus q} \longrightarrow A^{\oplus p} \longrightarrow N \longrightarrow 0,$$

we can carry it to a sequence

$$M^{\oplus q} \longrightarrow M^{\oplus p} \longrightarrow M \otimes_A N \longrightarrow 0.$$

So when is $\mathrm{Hom}(M, -)$ an equivalence from C to Mod_A? We need two conditions on M:

- M must be projective.
- M must be a *generator*, meaning $\mathrm{Hom}(M, N) = 0$ iff $N = 0$. One says $\mathrm{Hom}(M, -)$ is *conservative*.

This is a special case of a very general theorem about adjoints, the Barr-Beck theorem. The related Tannakian formalism is also a consequence of the Barr-Beck theorem.

Remark. The Mitchell embedding theorem states that every abelian category is a full subcategory of some category of modules over an algebra, and in particular there's not always an equivalence. This doesn't contradict what we just said: not every abelian category has a projective generator.

Now, we'd like to think about (nice) categories C and D as algebras: $\mathsf{C} = \mathsf{Mod}_A$ and $\mathsf{D} = \mathsf{Mod}_B$, and pointed functors $F : \mathsf{C} \to \mathsf{D}$ as morphisms $f : A \to B$. (Certainly, we can go in the other direction: $f : A \to B$ defines a pointed functor $\mathsf{Mod}_A \to \mathsf{Mod}_B$). In fact, we have an analogue of the integral transform.

Let M be an (B, A)-bimodule, meaning a left B-module and a right A-module such that these two actions commute. This will often be written ${}_B M_A$. This bimodule determines a functor $F : \mathsf{Mod}_A \to \mathsf{Mod}_B$ defined by $F = {}_B M \otimes_A -$. All nice functors from Mod_A to Mod_B are of this form, where "nice" means it commutes with colimits (one says it's *continuous*) and is right-exact.

Theorem 19.1 (Eilenberg-Watts). *Let $F : \mathsf{Mod}_A \to \mathsf{Mod}_B$ be a continuous functor. Then, $F = {}_BM \otimes_A -$ for some (B, A)-bimodule M.*

This is a direct analogue of the integral transform, albeit categorified.

Definition 19.2. Let A and A' be $\mathbb{C}$-algebras. A *Morita morphism* $A \to A'$ is an (A, A')-bimodule, hence inducing a functor $\mathsf{Mod}_A \to \mathsf{Mod}_{A'}$ as above.

Geometrically, over finite sets X and Y, we can think of $\mathbb{C}[X]$-modules as vector bundles on X and $\mathbb{C}[Y]$ as vector bundles on Y. Then, $\mathsf{Vect}(X \times Y) = {}_{\mathbb{C}[X]}\mathsf{Mod}_{\mathbb{C}[Y]} = \mathsf{Mod}_{\mathbb{C}[X \times Y]}$. If $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are affine schemes, then $\mathsf{Mod}_A = \mathsf{QC}(X)$ and $\mathsf{Mod}_B = \mathsf{QC}(Y)$, and we recover a kernel transform for quasicoherent sheaves: a continuous functor $F : \mathsf{QC}(X) \to \mathsf{QC}(Y)$ is given by a kernel $K \in \mathsf{QC}(X \times Y) = \mathsf{Mod}_{A \otimes B}$.

Definition 19.3. The *Morita 2-category* Alg is defined by the following data.
  - The objects are associative algebras A.
  - The morphisms $A \to B$ are (A, B)-bimodules. Given an (A, B)-bimodule ${}_AM_B$ and a (B, C)-bimodule ${}_BM_C$, composition is the tensor product
$$({}_BN_C) \circ ({}_AM_B) = {}_AM \otimes_B N_C.$$
  - The 2-morphisms between ${}_AM_B$ and ${}_AM'_B$ are the (A, B)-bimodule homomorphisms $M \to M'$.

From this perspective, Theorem 19.1 says that sending $A \mapsto \mathsf{Mod}_A$ defines a full embedding of Alg into the 2-category of $\mathbb{C}$-linear abelian categories, continuous functors, and natural transformations. The image is a subcategory of the categories with projective generators.

Categories don't always come with canonical generators. If X is a finite set or an affine variety, so that we can think of $\mathsf{Mod}_{\mathbb{C}[X]}$ as $\mathsf{Vect}(X)$ or $\mathsf{QC}(X)$, we can consider the constant vector bundle or sheaf $\underline{\mathbb{C}}$, and this is a projective generator.

More generally, suppose $\mathcal{X} = [\mathcal{G} \rightrightarrows X]$ is a groupoid; we want to construct an interesting functor $\mathsf{Vect}\mathcal{X} \to \mathsf{Vect}$. Global sections is a functor, but isn't interesting, as it forgets too much of the structure, e.g. if $\mathcal{X} = \bullet/G$, then Γ takes G-invariants, which loses a lot of structure. We'd like to get the regular representation instead.

Given the presentation $\mathcal{G} \rightrightarrows X$, we have a projection map $\pi : X \to \mathcal{X}$. Now, we have a more interesting functor, $\operatorname{Hom}(\pi_*\underline{\mathbb{C}}, -)$.

Exercise 19.4. Show that $\mathsf{Vect}\mathcal{X}$ is Morita equivalent to the category of modules over $\operatorname{End}(\pi_*\underline{\mathbb{C}})$, and that this algebra is $\mathbb{C}\mathcal{G} = \operatorname{End}(\underline{\mathbb{C}}_{\mathcal{G}})$.

The point is that vector bundles over a groupoid canonicallu form a pointed category as soon as we choose a presentation.

If $\mathcal{X} = \bullet$, it admits a presentation as $\mathcal{G} = X \times X \rightrightarrows X \to \bullet$, where X is some finite set. Then, $\mathcal{X} = \mathsf{Vect}_{\mathbb{C}}$, so we've established a Morita equivalence from $\mathsf{Vect}\mathcal{X}$ to $\mathsf{Mod}_{\mathbb{C}\mathcal{G}}$. The groupoid algebra is $\mathbb{C}\mathcal{G} = \operatorname{End}(\pi_*\underline{\mathbb{C}}) = \operatorname{Mat}_{n \times n}$. That is, we've discovered a Morita equivalence from $\mathbb{C}$-vector spaces to $\operatorname{Mat}_{n \times n}$-modules.

To be precise, $\mathbb{C}^n$ is a $(\operatorname{Mat}_{n \times n}, \mathbb{C})$-bimodule, where the left action is matrix multiplication and the right action is scalar multiplication.

Definition 19.5. Two algebras A and B are *Morita equivalent* if they're equivalent in Alg, i.e. there is an (A, B)-bimodule M and a (B, A)-bimodule N such that ${}_AM \otimes_B N_A$ is naturally isomorphic to $\operatorname{id}_{\mathsf{Mod}_A}$ and ${}_BN \otimes_A M_B$ is naturally isomorphic to $\operatorname{id}_{\mathsf{Mod}_B}$.

We're trying to think of functors as bimoduls; the identity functor is represented by the *diagonal bimodule* ${}_AA_A$.[42] Thus, a Morita equivalence is an isomorphism of an (A, B)-bimodule with a diagonal bimodule.

Using a tensor-hom adjunction, there's a dual way to write this, which is the usual definiton of Morita equivalence: A and B are Morita equivalent if there's a projective generator $P \in \mathsf{Mod}_A$ such that $\operatorname{End}_A P = B^{\mathsf{op}}$. This is the object in Mod_A that corresponds to the basepoint $B \in \mathsf{Mod}_B$; since B is a projective generator, then P has to be too, and similarly $\operatorname{End}_A(P) = \operatorname{End}_B(B) = B^{\mathsf{op}}$. Thus, we know how B acts on P by endomorphisms, so it has an (A, B)-bimodule structure, and therefore defines a Morita morphism.

This has all been in the world of noncommutative rings; what happens if A and B are commutative?

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[42]If A is commutative, this is literally the (A, A)-bimodule of functions on the diagonal of $A \times A$.

Proposition 19.6. *Let A and B be commutative rings that are Morita-equivalent. Then, $A \cong B$.*

Proof. We'll show that given Mod_A for a commutative ring A, we can functorially recover A. This is a useful tool.

Since A is commutative, $A \cong Z(A) = \{a \in A \mid ab = ba \text{ for all } b \in A\}$. We rewrite the center in categorical terms: $Z(A) = \mathrm{Hom}_{A,A}(A, A)$: multiplication by an $a \in A$ commutes with the left and right action of A on itself by multiplication. Thus, $Z(A) = \mathrm{End}_{A,A}({}_A A_A) = \mathrm{End}(\mathrm{id}_{\mathsf{Mod}_A})$: the algebra of endomorphisms of the identity functor on Mod_A.

What is this saying? This says that for any associative algebra B, we can recover $Z(B)$ from Mod_B in a canonical way, so since A is commutative, we canonically recover A. $\qquad\square$

Definition 19.7. Let C be a category. Then, its *Bernstein center* is $Z(\mathsf{C}) = \mathrm{End}(\mathrm{id}_\mathsf{C})$, the endomorphisms of the identity functor.

We've just seen that if $\mathsf{C} = \mathsf{Mod}_A$ for an associative algebra A, then $Z(\mathsf{Mod}_A) = Z(A)$. Often, the first thing you'll want to do with a category is find its center.

Associated to an (A, A)-bimodule P, the A-module

$$\mathrm{Hom}_{A,A}({}_A A_A, {}_A P_A) = \mathrm{Hom}_{\mathsf{Mod}_A \to \mathsf{Mod}_A}(\mathrm{id}_{\mathsf{Mod}_A}, P \otimes -)$$

is called the 0th *Hochschild cohomology* of A with coefficients in M (or just the *center* of P), and is denoted $HH^0(A, {}_A P_A)$. There is a sense in which this is a trace of the functor $P \otimes -$. For example, $HH^0(A, A) = Z(A)$.

If M and N are left A-modules, then $\mathrm{Hom}_\mathbb{C}(M, N)$ is an (A, A)-bimodule, with the left and right A-actions induced from N and M, respectively. We can ask for the A-linear homomorphisms inside this, and in fact

$$\mathrm{Hom}_A(M, N) = HH^0(A, \mathrm{Hom}_\mathbb{C}(M, N)).$$

We saw that $Z(\mathrm{Mat}_{n,n}) = \mathbb{C}$ (which it must be, since it's Morita equivalent to $\mathbb{C}$); more generally, the algebra $\mathbb{C}[X \times_Y X]$ of block diagonal matrices induced by a map $f : X \to Y$ is Morita equivalent to $\mathbb{C}[Y] = Z(\mathbb{C}[X \times_Y X])$.

Next time, we'll recast all of this in the language of TQFTs, but we'll say a little about it today. We want one of these categories C to define a two-dimensional oriented field theory Z such that $Z(\bullet) = \mathsf{C}$. The idenity functor corresponds to the closed interval $[0, 1] : \mathsf{C} \to \mathsf{C}$, and we can also draw the picture of the identity natural transformation I.

More interestingly, we can remove a small circle from the bordism defining I, which defines a ring homomorphism $Z(S^1) \to \mathrm{End}(\mathrm{id}_\mathsf{C})$; we'll discover that this map is an isomorphism.[43] Since $\mathrm{End}(\mathrm{id}_\mathsf{C}) = HH^0(\mathsf{C}) = Z(A)$ (if $\mathsf{C} = \mathsf{Mod}_A$ for some algebra A), then we can apply this to our examples: if $\mathsf{C} = \mathsf{Rep}_G$ for a finite group G, then its center is $Z(\mathbb{C}G)$, the algebra of class functions, which is what Dijkgraaf-Witten theory assigns to S^1.

Intuitively, the center of the category is the operations one can perform on every object; from this perspective, it's less surprising that local operators recover the center, since this is exactly what they were defined to do. And indeed, this is a general feature of TQFTs: in general, $Z(S^{n-1})$ acts on $Z(M)$ for any $(n-1)$-manifold M in an n-dimensional TQFT by excising a small sphere from $M \times I$, even if we're not thinking about bordisms with corners. But when we can draw corners (extended or fully extended field theories), we have more pictures to choose from.

The idea is that local operators of a TQFT form a kind of center. We'll define the center of a TQFT, and it'll end up being the same. For example, we can ask for the center of Z to be the algebra acting on the state spaces for all $(n-1)$-manifolds compatible with the maps coming from bordism. This is exactly how $Z(S^{n-1})$ acts as local operators, and if the TQFT comes from an algebra, then we tend to recover the center of this algebra.

20. Hochschild Homology: 11/1/16

We've been discussing 2-extended field theories $\mathrm{Bord}_{0,2} \to \mathsf{Cat}$. As usual, it factors through the 2-category of correspondences of groupoids, but we also saw that it's useful to factor this further through the Morita 2-category of algebras. Today will be the last underived class: soon we will replace finite sets and finite groupoids with actual geometry.

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[43]Here it's important that we're considering oriented TQFTs rather than framed TQFTs.

There's a foundational result in this field called the cobordism hypothesis which has multiple forms. In the form below, it's due to Schommer-Pries [10], with related statements due to Moore-Segal [8]; the derived and more general versions are due to Costello and Lurie [7]. We'll return to this later.

Theorem 20.1 (Cobordism hypothesis, special case (Schommer-Pries [10]))**.** *Given any finite-dimensional semisimple algebra A, there exists a unique 2-extended* framed *field theory $Z : \mathrm{Bord}_{0,2}^{\mathrm{fr}} \to \mathsf{Alg} \to \mathsf{Cat}$ (i.e. that factors through the Morita 2-category Alg) such that $Z(\mathrm{pt}^+) = \mathsf{Mod}_A$.*[44] *Conversely, given any such 2-extended TQFT $Z : \mathrm{Bord}_{0,2}^{\mathrm{fr}} \to \mathsf{Alg} \to \mathsf{Cat}$, there exists a finite-dimensional, semisimple algebra A such that $Z(A) \simeq \mathsf{Mod}_A$.*

The point is, these TQFTs are determined by their values on a point. The algebras that we've been thinking about are group algebras $\mathbb{C}G$ and functions on a set $\mathbb{C}X$.

Note that Z must be symmetric monoidal; in particular, there's a tensor product on the category of small $\mathbb{C}$-linear categories, called the *Deligne tensor product*, such that $\mathsf{Mod}_A \otimes \mathsf{Mod}_B = \mathsf{Mod}_{A \otimes B}$.

Recall that a Frobenius algebra A is an algebra with a trace $\mathrm{Tr} : A \to \mathbb{C}$. We say that A is *symmetric* (distinct from commutative) if $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$. This is useful to understand oriented field theories: they have extra striucture, which is encoded in a Frobenius algebra.

Theorem 20.2 (Cobordism hypothesis, special case (Schommer-Pries [10]))**.** *The data of a finite-dimensiona, semisimple, symmetric Frobenius algebra A is equivalent to that of an oriented 2-extended TQFT $Z : \mathrm{Bord}_{0,2}^{\mathrm{or}} \to \mathsf{Alg} \to \mathsf{Cat}$ through the identification $Z(\bullet) = A$.*

In this case, $Z(S^1)$ is the center of $Z(\bullet)$, i.e. $\mathrm{End}(\mathrm{id}_{Z(\bullet)})$, which is the center of the Frobenius algebra A. We saw this for Dijkgraaf-Witten theory: to a point we assigned the category of representations (modules over $\mathbb{C}G$), and to S^1 we assigned its center, the class functions.

Remark. The cobordism hypothesis can be vastly generalized. It applies to fully extended field theories of n-manifolds, meaning they're defined on k-manifolds for $0 \leq k \leq n$ where the codomain is any symmetric monoidal n-category $(\mathsf{C}, \otimes)$. Framed fully extended, n-dimensional field theories are fully determined by their values on a point, and the value on a point must satisfy a strong finiteness condition called *n-dualizability*. To replace the field theory with an oriented field theory, we also need these objects to be $\mathrm{SO}(n)$-fixed. For $n = 2$, the $\mathrm{SO}(n)$-fixed objects of Alg are Frobenius algebras.

Traces. We're going to talk about traces in a very general setting. Recall that if Z is an n-dimensional field theory and N is an $(n-1)$-manifold, then $Z(N \times S^1) = \dim Z(N)$. This is because we can cut S^1 into a coevaluation followed by an evaluation, which Z sends to a map $\mathbb{C} \to Z(N) \otimes Z(N)^* \to \mathbb{C}$. Identifying $Z(N) \otimes Z(N)^* \cong \mathrm{End}\,V$, this is $1 \mapsto \mathrm{id}$ followed by the trace. The "mark of Zorro" diagram shows that this map isn't zero, and therefore defines an isomorphism $Z(N) \cong Z(N)^*$. In particular, $Z(N)$ must be finite-dimensional.

Now, let's repeat the story one level higher.

Definition 20.3. Let C be a category and Z be a TQFT such that for an $(n-2)$-manifold P, $Z(P) \cong \mathsf{C}$. Then, the *dimension* of C is $Z(P^{n-2} \times S^1)$.

This is well-defined, which perhaps is surprising. The idea is that no matter which Z and P you pick, this comes from two symmetric monoidal functors $\mathsf{Vect} \to \mathsf{C} \otimes \mathsf{C}'$ and $\mathsf{C} \otimes \mathsf{C}' \to \mathsf{Vect}$ satisfying a similar mark of Zorro condition. In particular, we can always map $\mathsf{C} \otimes \mathsf{C}^{\mathsf{op}} \to \mathsf{Vect}$ using Hom. When $\mathsf{C} = \mathsf{Mod}_A$, we recover $\mathsf{C}' = (\mathsf{Mod}_A)^{\mathsf{op}} = \mathsf{Mod}_{A^{\mathsf{op}}}$. Then, $\mathsf{Mod}_A \otimes \mathsf{Mod}_{A^{\mathsf{op}}} = \mathsf{Mod}_{A \otimes A^{\mathsf{op}}}$, which is the category of (A, A)-bimodules. In particular, the map to $\mathsf{Vect}_{\mathbb{C}}$ is just A as an (A, A)-bimodule, and the map from $\mathsf{Vect}_{\mathbb{C}}$ is also A as an (A, A)-bimodule.

Since $\mathsf{Mod}_A \otimes \mathsf{Mod}_{A^{\mathsf{op}}} = \mathrm{End}(\mathsf{Mod}_A)$ in the Morita 2-category, then we think of (A, A)-bimodules as endofunctors of Mod_A, and the trace map is the map to $\mathsf{Vect}_{\mathbb{C}}$. This trace is called *Hochschild homology* HH_0.[45]

Specifically, suppose B is an (A, A)-bimodule. Then, the trace is tensoring with ${}_A A_A$, the diagonal bimodule, so the trace of B is $HH_0(A, B) = A \otimes_{A \otimes A^{\mathsf{op}}} B$ as a $\mathbb{C}$-vector space, the Hochschild homology of A with coefficients in the bimodule B.

[44]Here, pt^+ is the point with the standard framing.

[45]Higher Hochschild homology arises by replacing the tensor product with a derived tensor product.

For example, $HH_0(A, A) = A/(ab - ba)$. What does this mean? It's the target of the universal trace: any trace map $\mathrm{Tr} : A \to M$ ought to satisfy $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$, and this uniquely factors through $A/(ab - ba)$. This is called the *trace* or *cocenter* of A; calling it the abelianization is tempting, but it's not an algebra.

So the dimension of Mod_A, the invariant that we attach to a circle, is $\mathrm{Tr}(\mathrm{id}_{\mathsf{Mod}_A}) = HH_0(A, A)$.

Suppose M_A is a right A-module and $_AN$ is a left A-module. Then, $M_A \otimes_A \, _AN$ is an (A, A)-bimodule, and $HH_0(A, M_A \otimes_A \, _AN) = M \otimes_A N = (M \otimes N)/(ma \otimes n - m \otimes an)$. If A is commutative, then we think of A-modules M and N as quasicoherent sheaves $\mathscr{M}$, $\mathscr{N}$ on $\mathrm{Spec}\,A$, we're thinking of their external product $\mathscr{M} \boxtimes \mathscr{N}$ on $\mathrm{Spec}\,A \times \mathrm{Spec}\,A$. Modding out by the ideal $(a \otimes 1 - 1 \otimes a)$ is the restriction of this sheaf to the diagonal. Finally, we just remember the vector space of global sections.

More poetically, this arises from the diagonal correspondence:

$$
\begin{array}{ccc}
 & X & \\
\overset{\Delta}{\swarrow} & & \searrow \\
X \times X & & \bullet,
\end{array}
$$
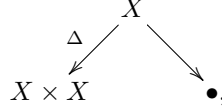
so we send $\mathscr{M} \boxtimes \mathscr{N} \mapsto \Delta^*(\mathscr{M} \boxtimes \mathscr{N})$, and then take global sections.

For example, we have a 2-extended field theory Z associated to a finite set X (i.e. $Z(S^1) = \mathbb{C}[X]$). We obtain a correspondence $\bullet \leftarrow X \to X \times X \leftarrow X \to \bullet$ by cutting the circle into two pieces; this sends $\mathbb{C}$ to the constant vector bundle $\underline{\mathbb{C}} \to X$, then to $\underline{\mathbb{C}}_\Delta \to X \times X$ ($\mathbb{C}$ on the diagonal, trivial elsewhere), then back to $\underline{\mathbb{C}} \to X$, and then to $\mathbb{C}[X] \to \bullet$. Tucked in here is the trace map $\mathsf{Vect}(X \times X) \to \mathsf{Vect}$, which is Hochschild homology: restrict to the diagonal and then take global sections.

We conclude $\dim \mathsf{Vect}X = \mathbb{C}[X]$. If $\mathcal{X}$ is a groupoid, the same argument shows $\dim \mathsf{Vect}(\mathcal{X}) = \mathbb{C}[I\mathcal{X}]$, the functions on the inertia groupoid. For example, $\mathrm{Tr}(\mathrm{id}_{\mathsf{Rep}_G}) = \mathbb{C}[I(\bullet/G)]$, which is the space of class functions. Alternatively, you can write this as $HH_0(\mathbb{C}G, \mathbb{C}G) = \mathbb{C}[G/G]$.

Another example: let X be a finite set and $\gamma : X \to X$ be a function, which induces a map $\gamma^* : \mathsf{Vect}X \to \mathsf{Vect}X$. The trace of γ^* counts intersections with the diagonal, and so $\mathrm{Tr}(\gamma^*) = \mathbb{C}[X^\gamma]$, the functions on the fixed points of X.

For $\mathbb{C}G$, we also saw that $\mathbb{C}[G/G]$ is the 0th Hochschild cohomology; this is because $\mathbb{C}G$ is a Frobenius algebra. For any Frobenius algebra A, dualizing defines an isomorphism $HH_0(A, A) = A \otimes_{A \otimes A^{\mathrm{op}}} A \cong \mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(A, A) = HH^0(A, A)$. So we can identify the center and cocenter, which is cool.

Don't forget about the cocenter and trace, though; they're the natural home of characters. Given a projective A-module M, we would like to obtain a character $\mathrm{ch}(M) \in \dim(\mathsf{Mod}_A)$. For example, if $A = \mathbb{C}X$ and $M \in \mathsf{Vect}X$, we obtain its rank function $\mathrm{rank}\,M \in \mathbb{C}X$. When we add geometry, this will turn into a Chern character in cohomology associated to a vector bundle. For another example, if $A = \mathbb{C}G$, and M is a representation of G, then its character is the usual character in representation theory, $\chi_M \in \mathbb{C}[G/G]$.

In general, if M is a projective A-module, then M is a direct summand of a free module $A^{\oplus N}$, so there is a projection $\pi \in \mathrm{Mat}_{N \times N}\,A$ onto M. The usual trace of these matrices is $\mathrm{Tr} : \mathrm{Mat}_{N \times N}(A) \to HH_0(\mathrm{Mat}_{N \times N}\,A)$, but since A is Morita equivalent to $\mathrm{Mat}_{N \times N}\,A$, then $\mathrm{Tr}(\mathrm{id}_{\mathsf{Mod}_A}) = \mathrm{Tr}(\mathrm{id}(\mathsf{Mod}_{\mathrm{Mat}_{N \times N}\,A}))$, so they have the same Hochschild homology. In particular, the trace of the projection π lives in $HH_0(A)$. In general, characters and traces live in Hochschild homology.

If we know M is finite-dimensional and A is semisimple, then we can bypass much of the Morita theory: we know by the Peter-Weyl theorem, any representation can be obtained from the regular representation, and then the usual trace suffices. The character of a projective module is naturally an element of Hochschild homology, but the character of a finite-dimensional representation naturally lives in $(HH_0(A))^*$.

Another caveat: if we have merely an algebra, rather than a Frobenius algebra, then we should be thinking of framed TQFTs, not oriented TQFTs. In particular, there's more than one isomorphism class of framings on $S^1 \times \mathbb{R}$:[46] the *blackboard framing* arising from an annulus, inheriting the framing from $\mathbb{R}$, and the *cylinder framing* which inherits the usual trivialization of TS^1 and the usual framing on $\mathbb{R}$. There are more framings,

---

[46]When we attach structure to a manifold in a TQFT, we need to think of it as top-dimensional, so the point is really $\mathbb{R}^2$ and the circle is really $S^1 \times \mathbb{R}^2$, when it comes to assigning orientations, framings, etc.

in fact one for every integer, but these two are the most interesting: the blackboard framing comes from $HH^0(A)$ and the cylindrical framing comes from $HH_0(A)$.[47]

This perspective has a lot to say about the algebra of HH_0 and HH^0: operator product expansion (covering two smaller circles by a larger one) defines a ring structure on $HH^0(A)$, but there's no natural ring structure on $HH_0(A)$. However, rotating the circle doesn't preserve the framing on the annulus, but it does for the cylinder, so $HH_0(A)$ has an S^1-action, and $HH^0(A)$ doesn't. From the physics perspective, $HH^0(A)$ is the local operators, and $HH_0(A)$ is the states. The *operator-state correspondence* applies in oriented bordism, but breaks for framed bordism: local operators aren't states. All of our theories have been oriented, so this subtlety didn't show up until now.

In the derived world, the Frobenius condition corresponds to asking a manifold to be Calabi-Yau.

Using these framings, we can write down some interesting bordisms: consider a long cylinder with a small circle removed. This is a bordism from the cylindrical framing and the annular framing to the cylindrical framing, and algebraically it gives us an action of $HH^0(A)$ on $HH_0(A)$.

Next time, we'll talk about Hecke algebras, and then the B-model, replacing finite sets by algebraic varieties. The pictures are the same, but we'll say the word "derived" a lot. Then, we'll go to 3 and 4 dimensions, where geometric Langlands arises.

21. The Hecke Algebra: 11/3/16

"If you don't like Weyl groups, well... maybe you took the wrong class."

Let's start by summarizing where we are using a pictorial calculus for categories. This is part of a very general idea in category theory that conditions in category theory can be expressed with certain pictures and diagrams, and the idea that these can be encoded with topological field theories is due to Moore and Segal [8]. The cobordism hypothesis figures greatly into these ideas.

Recall that if C is a category satisfying a suitable finiteness condition, there exists a 2D TQFT Z such that $Z(\bullet) = \mathsf{C}$. We also saw that for every $(n-1)$-dimensional manifold N, $Z(N)$ is finite-dimensional, because $S^! \times N$ defines a bordism $\bullet \to \bullet \amalg \bullet \to \bullet$, and therefore a map $\mathbb{C} \to V \otimes V^* \to \mathbb{C}$ which sends $1 \mapsto \mathrm{Tr}(\mathrm{id}_V) = \dim V$. This also provides the duality condition on C, that there's a pairing $ev : \mathsf{C} \otimes \mathsf{C}^{\mathsf{op}} \to \mathsf{Vect}$. This isn't too strong a condition: if $\mathsf{C} = \mathsf{Mod}_A$, the (A,A)-bimodule ${}_A A_A$ realizes this pairing.

Another, stronger restriction is that cobordism categories have lots of adjoints. TODO: saddle bordism shows that ev has to have both left and right adjoints. For $\mathsf{C} = \mathsf{Mod}_A$, this is equivalent to A being a projective $A \otimes A^{\mathsf{op}}$-module; if this holds we call A *separable*. In characteristic 0, this is equivalent to A being semisimple. We're going to derive everything soon, and one reason for this is so that oiur categories have more adjoints.

If Z is a framed TQFT, then recall that the annulus N and the cylinder C are different framed surfaces; $Z(N) = HH^0(A)$ and $Z(C) = HH_0(A)$; these are called the center and cocenter of A, respectively. In general, crossing with S^1 will produce a trace-like operation; this inuition is often used in physics, where it corresponds to wrapping time back around to the beginning. For example, if the torus T^2 has the standard framing as a quotient $\mathbb{R}^2/\mathbb{Z}^2$, then $Z(T^2) = \dim Z(S^1) = \dim HH_0(A)$.

An object $P \in \mathsf{C}$ (which will have to be projective) defines a boundary condition for Z, which means we extend the domain of Z to bordisms with boundary components labeled by M. For example, the coevaluation bordism $\bullet \amalg \bullet \to \varnothing$ defines the pairing $\mathsf{C} \otimes \mathsf{C}^{\mathsf{op}} \to \mathsf{Vect}$, but labeling the endpoints with objects P and Q means we obtain the vector space $\mathrm{Hom}(P,Q)$.

For example, the interval with both ends marked with P defines the vector space $\mathrm{End}\,P$, and the algebraic structure is manifest in the topology, with the "pair-of-chaps" bordism we defined earlier, which defines the composition map $\mathrm{End}\,P \otimes \mathrm{End}\,P \to \mathrm{End}\,P$. More generally, given two intervals labeled by (P,Q) and by (Q,R), this bordism recovers the composition map $\mathrm{Hom}(P,Q) \otimes \mathrm{Hom}(Q,R) \to \mathrm{Hom}(P,R)$. This was the origin of category theory in physics: if you start by labeling boundary conditions, which is a very physical idea, you end up with a composition law and a category of boundary conditions anyways. These were originally called *open-closed field theories* with *open strings* and *closed strings* as coming from string theory.

---

[47]The other framings define other algebraic objects, which are all $HH_0(A, ?)$, where the ? is a twisted version of the bimodule structure on A; sometimes these are called *higher Hochschild homology*, but have neither the S^1-symmetry nor the product structure, so are less useful. These can also be constructed from HH^0.

In fact, $\operatorname{End} P$ is an associative Frobenius algebra. This is spelled out in [8] with a nice picture argument. We need a map $\operatorname{End} P \to \mathbb{C}$, which comes from the half-disc (with the circle boundary, but not the line boundary, colored). Similarly, the counit map $\mathbb{C} \to \operatorname{End} P$ comes from the dual bordism.

The *whistle bordism* (TODO) is a 2-manifold with corners. Z sends this to a trace map $\operatorname{End} P \to Z(S^1)$, which maps $\operatorname{id}_P \mapsto \chi_P$, the character of P. This is the state associated to the boundary condition P, called the *boundary state* in physics.

Example 21.1 (Hecke algebras)**.** This formalism provides us a nice pictoral way to think about Hecke algebras.

Let $K \subset G$ be a subgroup, which defines a map of groupoids $\bullet/K \to \bullet/G$. In the TQFT defined with groupoids of G-local systems, we'll let a marked boundary component go to a reduction to a K-local system.

The span

$$
\begin{array}{ccc}
 & \bullet/K & \\
\swarrow & & \searrow{\scriptstyle \pi} \\
\bullet & & \bullet/G
\end{array}
$$

sends $\underline{\mathbb{C}} \to \bullet$ to $\pi_* \underline{\mathbb{C}} \to \bullet/G$, which is the induced representation $\operatorname{Ind}_K^G \mathbb{C} \in \mathsf{Rep}_G$, defined by $\operatorname{Ind}_K^G \mathbb{C} = \mathbb{C}G \otimes_{\mathbb{C}K} \mathbb{C}$.[48] The induced representation represents the functor $\mathsf{Rep}_G \to \mathsf{Vect}$ of taking K-invariants, because of the hom-tensor adjunction:

$$
\operatorname{Hom}_{\mathbb{C}G}(\mathbb{C}G \otimes_{\mathbb{C}K} \mathbb{C}, V) = \operatorname{Hom}_{\mathbb{C}K}(\mathbb{C}, V).
$$

The Hecke algebra[49] $\mathcal{H}_{G,K}$, which arises from the interval with both endpoints marked, can be realized in a couple equivalent ways.

- We can think of it as the groupoid algebra $\mathbb{C}[K\backslash G/K]$. This double coset space arises as the fiber product

$$
\begin{array}{ccc}
K\backslash G/K & \longrightarrow & \bullet/K \\
\downarrow & \lrcorner & \downarrow \\
\bullet/K & \longrightarrow & \bullet/G.
\end{array}
$$

- The TFT picture says that the Hecke algebra is the space of functions on the groupoid of G-local systems on an interval that are reduced to K-local systems on the endpoints.
- The induced representation $\operatorname{Ind}_K^G \mathbb{C} = \mathbb{C}[G/K]$, so using its functor of points the Hecke algebra is $(\mathbb{C}[G/K])^K$.
- Thus, the Hecke algebra is also the algebra of endomorphisms of the functor of K-invariants $\mathsf{Rep}_G \to \mathsf{Vect}$.
- Intuitively, the Hecke algebra is the algebra of things that act on V^K for any G-representation V. This is why the Hecke algebra appears so much in representation theory.
- Up to Morita equivalence, it suffices to describe $\mathcal{H}_{G,K}$ by its category of modules. $\operatorname{Ind}_K^G \mathbb{C}$ is a projective representation (well, for a finite group all representations are projective), but isn't a generator. Instead, the category of $\mathcal{H}_{G,K}$-modules is equivalent to the category of Rep_G generated by (spanned by) $\operatorname{Ind}_K^G \mathbb{C}$. In particular, the Hecke algebra is a Frobenius algebra, so it defines a TFT, but it doesn't see the whole theory that G does. One might call this a sub-TFT; in physics, this is known as a *superselection sector*.

For a concrete case, suppose $G = \operatorname{GL}_n \mathbb{F}_q$ and $K = B$ is the Borel subgroup, the subgroup of upper triangular matrices. In this case, Bruhat decomposition shows that the double coset space $B\backslash G/B$ is in bijection with the Weyl group for $\operatorname{GL}_n \mathbb{F}_q$, which is just S_n (by Gaussian elimination). Thus, $\mathcal{H}_{G,B} = \mathbb{C}S_n$ as vector spaces, but the multiplication is different. It's generated by T_α where α are simple reflections (switching i and $i+1$) with relations $(T_\alpha + 1)(T_\alpha - q) = 0$ and *braid relations* $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$. This algebra is called the *finite Hecke algebra* and denoted $\mathcal{H}_q$.

(It's worth working this out explicitly for $G = \operatorname{SL}_2 \mathbb{F}_q$ and its Borel subgroup; then, $G/B \cong \mathbb{P}^1_{\mathbb{F}_q}$, and the Hecke algebra is generated by 1 and T with $(T+1)(T-q) = 0$; the Weyl group S_2.)

---

[48]In this case, this is the same as the coinduced representation, which is written with a Hom.

[49]The last e in "Hecke" is not silent.

The finite Hecke algebra is the G-equivariant endomorphisms of $\mathbb{C}[G/B]$; G/B is the flag variety, and the functions on it define a representation called the *unipotent principal series representation*. The modules over $\mathcal{H}_q$ are the representations appearing in the decomposition of $\mathbb{C}[G/B]$, which are in a sense the easiest G-representations to get access to; they're called the *unipotent principal series*.

Another way to say this is that (isomorphism classes of) irreducible representations of G that appear in $\mathbb{C}[G/B]$ are in bijection with (isomorphism classes of) irreduicble representations of $\mathcal{H}_q$, thanks to all this abstract Morita theory.

The space G/B is the variety of flags in $(\mathbb{F}_q)^n$, and we have a natural correspondence from the double coset space, which is a pair of G/B terms modded out by the diagonal action, so that we can ask G-equivariant questions on the flag variety $G/B = \mathrm{F}\ell_n$. Specifically,

$$G\backslash(G/B \times G/B) = B\backslash G/B$$

$$B\backslash(G/B) \qquad G\backslash(G/B)$$

$$G\backslash\bullet.$$

These are called *Hecke correspondences*.

When $G = \mathrm{GL}_n\mathbb{F}_q$, the Hecke algebra looks like functions on S_n as a vector space, so suppose σ is your favorite permutation. We want to make this correspondence between G/B and G/B explicit, and we obtain set groupoid of flags $F_1, F_2 \in \mathrm{F}\ell_n$ that are in relative position with respect to σ, and these project down to F_1 and F_2 in the two copies of the flag variety.

This whole story began by inducing the trivial representation from B to G. But what if you start with a different representation of a maximal torus T? Then, you obtain a different Hecke algebra, and its category of representations is different, but still fairly accessible, and is called a *principal series*. This determines a correspondence

$$\bullet/B$$

$$\bullet/T \qquad \bullet/G,$$

called *parabolic induction*. Almost everything in geometric representation theory comes from a diagram that looks like this one.

22. More Hecke Algebras: 11/8/16

"Okay, let's try to do some math."

Today, we're going to talk some more about Hecke algebras. Fix a group G and a subgroup K.

One natural question to ask is, given a G-representation V, what acts on its K-invariants? Or rather, over all G-representations, what acts universally on all K-invariants? The Hecke algebra $\mathcal{H}_{G,K} = \mathrm{End}((-)^K : \mathsf{Rep}_G \to \mathsf{Vect}) = \mathbb{C}[K\backslash G/K]$ is the answer.

Suppose X is a set with a G-action. Then, it's also natural to ask what acts on X/K. Then, $\mathbb{C}[X/K] = \mathbb{C}[X]^K$, the K-invariants of the G-representation of functions on X. There's a stacky/groupoid way to think of this: X/G is a groupoid with the presentation $X \times G \rightrightarrows X$. To take the K-invariants, we take a fiber product with K to obtain $X \times_K G/K \rightrightarrows X/K$. The G-action doesn't descend to a group action, but rather a groupoid action; when you descend to functions on these groupoids, though, you do get an action of $\mathbb{C}[K\backslash G/K]$ on $\mathbb{C}[X/K]$. Associated to this is a G/K-bundle on the quotient groupoid X/K; we're looking at the K-invariants of this bundle.

Here's a big diagram which encodes some of this information.

$$X \times_K G/K$$

$$X/K \qquad K\backslash G/K \qquad X/K$$

$$X/G \qquad \bullet/K \qquad \bullet/K \qquad X/G$$

$$\bullet/G.$$
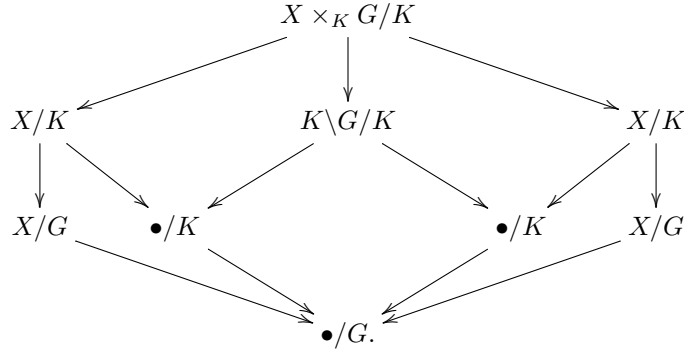
Every square in this diagram is a pullback.

This "stacky," abstract-nonsense perspective seems a little much, but the key is that everything works if you replace "functions on a groupoid" with something else: it's also useful to take vector bundles on groupoids, and since that's also functorial, the whole formalism goes through without change, and indeed the category $\mathsf{Vect}(K\backslash G/K)$ acts on the category $\mathsf{Vect}(X/K)$. Moreover, if $\mathcal{P}$ is a principal K-bundle over a groupoid Y, then G acts on $\mathcal{P}$, and the action passes to an action of $\mathcal{H}_{G,K}$ on the base Y, *once we linearize* (by taking functions, vector bundles, etc.).

We also talked about a great example, where G is a reductive subgroup of $\mathrm{GL}_n(\mathbb{F}_q)$ (e.g. $\mathrm{GL}_n(\mathbb{F}_q)$ works) and B is its Borel subgroup (for $G = \mathrm{GL}_n(\mathbb{F}_q)$, the group of upper triangular matrices). Then, the Hecke algebra is, as a vector space, naturally identified with the functions on the Weyl group (S_n in this case), but the multiplication is different, in a sense deformed by q.

This has a lot to say about highest weight theory of Lie groups. Last time, we mentioned that the category of $\mathcal{H}_{G,B}$-modules is equivalent to the full subcategory of Rep_G generated by $\mathbb{C}[G/B]$. Let T be a maximal torus inside B, e.g. the subgroup of diagonal matrices $(\mathbb{F}_q^*)^n$. The correspondence $T \leftarrow B \hookrightarrow G$ is a correspondence of groupoids

(22.1)
$$\bullet/B$$
$$\bullet/T \qquad \bullet/G,$$

so after linearizing (taking the category of vector bundles), we have maps

$$\mathsf{Rep}_B$$
$$\mathsf{Rep}_T \rightleftarrows \mathsf{Rep}_G.$$

The functor $\mathsf{Rep}_T \to \mathsf{Rep}_G$ is called *parabolic induction*, and the functor $\mathsf{Rep}_G \to \mathsf{Rep}_T$ is called *parabolic restriction*. The representations generated by the image of parabolic induction are called *principal series*.

Let N be the subgroup of upper triangular matrices with 1s on the diagonal, so $N = [B, B]$. Then, $\bullet/B = T\backslash(N\backslash G)/G$, and T normalizes N. Recall that $\bullet/G$ is the stacky "classifying space" for G-actions, so we can promote the maps in (22.1) to G-equivariant maps of sets with G-actions:

$$N\backslash G$$
$$\bullet/N \qquad G/B$$
$$\bullet \qquad \bullet/B \qquad \bullet$$
$$\bullet/T \qquad \bullet/G.$$
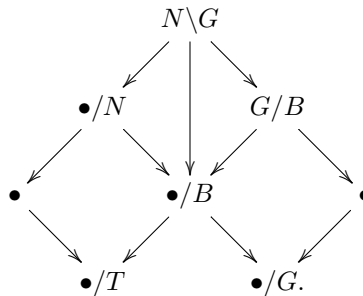
Here, again, the squares are fiber diagrams. One interesting consequence is that the principal series is exactly the representations appearing in $\mathbb{C}[G/N]$.

Explicitly, if $G = \mathrm{SL}_2(k)$, then these subgroups are

$$T = \left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \right\} \qquad B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right\} \qquad N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\}.$$

In particular, $T \cong k^\times$ and $N \cong k$. Then, $G/N = k^2 \setminus 0$, which has both an $\mathrm{SL}_2(k)$-action and a $k^\times$-action; the quotient of the latter action is $\mathbb{P}^1$.

Back to TFT. You may have been wondering how this relates to topological field theory. We're going to consider *bipartite surfaces*, which are surfaces cut along a curve into two disjoint pieces. We color the curve red, one piece white, and the other piece blue.

Given such a bipartite surface Σ, consider the groupoid of G-local systems with choices of a reduction to a T-local system on the blue domain and a reduction to a B-local system on the red domain (the curve).

More generally, suppose $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are (finite) groupoids, so they define field theories $Z_\mathcal{X}$, $Z_\mathcal{Y}$, and $Z_\mathcal{Z}$ where the space of fields is the maps to the groupoid in question. If we have a correspondence $\mathcal{X} \leftarrow \mathcal{Z} \to \mathcal{Y}$, we can define a "domain wall between $Z_\mathcal{X}$ and $Z_\mathcal{Y}$ using bipartite surfaces, a TFT on these bipartite surfaces:

- On the white part of a bipartite surface Σ, the fields are maps to $\mathcal{X}$.
- On the blue part of Σ, the fields are maps to $\mathcal{Y}$.
- On the red part, the fields are maps to $\mathcal{Z}$.

Bipartite surfaces and bipartite bordisms define a bordism category, which allows us to make this definition precise.

Definition 22.2. A *domain wall* between (oriented) TFTs Z_1 and Z_2 is a field theory on the bordism category of bipartite surfaces that restricts to Z_1 on the white part and Z_2 on the blue part.

A boundary condition on a field theory Z is an example of a domain wall between Z and the trivial theory that assigns Vect to a point and $\mathbb{C}$ (as the trivial Frobenius algebra) to S^1. The red curve in the bipartite surface corresponds to the colored boundary component for the boundary condition.

There's a version of the cobordism hypothesis for domain walls.

Theorem 22.3 (Cobordism hypothesis witrh singularities)**.** *Let Z_1 and Z_2 be two-dimensional oriented TFTs. Then, the category of domain walls between Z_1 and Z_2 is equivalent to the category of adjoint pairs functors $F : Z_1(\bullet) \leftrightarrows Z_2(\bullet) : G$ such that G is both a left and a right adjoint for F.*

Part of the beauty of the many forms of the cobordism hypothesis is that the "implementation" of the field theories doesn't matter: we've been working with Lagrangian field theories, but if they're presented in any other ways, it still applies.

You do need some sort of finiteness assumption, however.

Example 22.4. This example shows how the cobordism hypothesis can fail without the finiteness condition. Even so, it's still a useful perspective.

Let G be a reductive complex group, B be a Borel subgroup, and T be a maximal torus. We still have a diagram (22.1); let's apply it to the bipartite cylinder $S^1 \times [0,1]$ that's blue on $S^1 \times [0,1/2)$, white on $S^1 \times (1/2,1]$, and red on $S^1 \times \{1/2\}$. Thus we get a map $\mathbb{C}[T/T] \to \mathbb{C}[G/G]$, which is the Frobenius character formula: if V is a T-representation, its character χ_V is mapped to $\chi_{\mathrm{Ind}_T^G V} \in \mathbb{C}[G/G]$. More generally, one can consider the correspondence $T/T \leftarrow B/B \to G/G$, so we can consider more interesting functions. This leads to the Springer correspondence.

If $G = \mathrm{GL}_n\mathbb{C}$, then we can choose B to be the subgroup of upper triangular matrices and T to be the subgroup of diagonal matrices. Then, $T \cong (C^*)^n$, and Rep_G is equivalent to the category of representations of its maximal compact subgroup G_c, in this case U_n. As before, N is the subgroup of upper triangular matrices with 1s on the diagonal. If V is a G-representation, its highest weight can be obtained from this as the T-representation V^N. If V is irreducible, one can show V^N is an irreducible T-representation; since T is abelian, then V^N is one-dimensional, so we can define its weight by where 1 goes.

You can also obtain the Weyl group by staring at this diagram and thinking about monads. This is all explained in Sam Gunningham's thesis [6].

Geometrically, for $G = \mathrm{GL}_2$ and given a weight λ, $\mathbb{C}[G/N] = \Gamma(G/B, \mathscr{O}(\lambda))$, and $G/N = \mathbb{A}^2 \setminus 0$.

One can use this to define a TFT which assigns to a point the category $\mathsf{Rep}_G = \mathsf{Rep}_{G_c}$; this TFT is called two-dimensional *topological Yang-Mills theory* for G_c. However, it doesn't fit into the framework of the cobordism hypothesis: Rep_{G_c} has infinitely many symbols, and in fact breaks into a direct sum of Vect over all of the positive roots $\lambda \in \Lambda^+$.

Nonetheless, it makes sense to assign to the circle the space

$$Z(S^1) = \bigoplus_{\substack{V_\lambda \text{irreducible} \\ \lambda \in \Lambda^+}} C \cdot e_\lambda.$$

The distinction between the direct sum and the direct product matters when G isn't discrete; in physics, it's common to solve this by taking a completion, obtaining $L^@(G_c/G_c)$ by Peter-Weyl. Since this is infinite-dimensional, $Z(T^2) = \dim Z(S^1)$ doesn't make sense, so this isn't really a topological field theory. However, many aspects of this theory make sense. For example, to a surface Σ_g of genus g we associate

$$Z(\Sigma_g) = \sum_{V \text{ irrep.}} (\dim V)^{2-2g},$$

which converges for any $g > 1$.

This is useful to both physicists and representation theorists: it looks a lot like the kinds of field theories physicists actually care about, and the whole story of parabolic induction applies in this case, and in particular, all representations of G_c are in the principal series.

Topological 2D Yang-Mills theory is a limit of area-dependent Yang-Mills theories which aren't topological, but which also have interesting things to say about Casimir elements and such. The partition function doesn't always converge: if $\mathcal{P}$ is a principal G-bundle with a specified connection ∇,

$$Z = \int_{\mathcal{P}, \nabla} e^{-\mathrm{YM}(\mathcal{P}, \nabla)}.$$

Of course, to the physicists, this is a toy model for 4D Yang-Mills, which describes (part of) the world around us.

Physicists also define a field theory by specifying its Hilbert space and its Hamiltonian: the Hilbert space is $\mathscr{H} = L^2(G_c/G_c)$ with an unusual measure such that this is naturally identified with $L^2(T_c)^W$, and the Hamiltonian is the Casimir operator.

Witten wrote some amazing papers on this in the late 1980s, most notably "2D Yang-Mills revisited." This led to a flurry of applications: for example, local operators in this theory can be used for calculations in intersection theory on moduli spaces associated to Riemann surfaces. The analysis in here is interesting, but not easy.

This was an interesting example, but it wasn't a topological field theory, strictly speaking. We're going to expand our scope to think about interesting things, even if they may have convergence issues.

Recall that a QFT Z is a functor out of a Riemannian bordism category, but there is a giant moduli space of Riemannian manifolds, $\mathcal{M}_n$. The field theory Z induces a function on $\mathcal{M}_n$: when Z is a topological field theory, we asked only for Z to be locally constant, which implied it was only dependent on the topology of the manifold. That is, on closed n-manifolds, a TFT is a map $Z : \pi_0 \mathcal{M}_n \to \mathbb{C}$, and on $(n-1)$-manifolds, we have a map $: \mathcal{M}_{n-1} \to \mathsf{Vect}$ that's locally constant, i.e. a local system on $\mathcal{M}_{n-1}$. In particular, if N is an $(n-1)$-manifold, $Z(N)$ is naturally a representation of the mapping class group $\mathrm{MCG}(N) = \pi_0(\mathrm{Diff}(N)) = \pi_1(B\,\mathrm{Diff}(N))$. This gets a little more interesting for 2-manifolds.

More generally, if $M : N_{\mathrm{in}} \to N_{\mathrm{out}}$ is a cobordism, the map $Z(N_{\mathrm{in}}) \to Z(N_{\mathrm{out}})$ is a map of local systems, i.e. a locally constant map.

In supersymmetric (SUSY) quantum field theory, we obtain something much richer. These field theories arise from cohomological operations: there will be a "full" (i.e. nontopological) QFT, along with extra operators $Q, Q^\dagger$ such that $[Q, Q^\dagger]$ is the Hamiltonian, and such that $Q^2 = 0$. Then, we can think of Q as a differential such that $H = 0$ on Q-cohomology, thus defining something independent of the metric. In higher dimensions, there's something called the *stress tensor* S which governs the dependence of a QFT on the metric.

So rather than asking for Z to be a locally constant function, we'll ask for Z to be a cochain in $C^*(\mathcal{M}_n, \mathbb{C})$, i.e. a chain map $C_*(\mathcal{M}_n) \to \mathbb{C}$. In this class, we'll often switch between the chain level and the homology level. Since Z is invariant under Q, then it's closed, and the dependence on the metric is encoded in the

cohomology class of Z; if we force this to be a class of degree 0, then we conclude that Z is locally constant as before.

The idea is to replace the vector space $Z(N)$ with a chain complex $(Z(N), Q)$, such that if $M : N_{\mathrm{in}} \to N_{\mathrm{out}}$ is a bordism, $Z(M) : Z(N_{\mathrm{in}}) \to Z(N_{\mathrm{out}})$ is a chain map of degree 0. If $\mathcal{M}_n(N_{\mathrm{in}}, N_{\mathrm{out}})$ denotes the space of Riemannian bordisms from $N_{\mathrm{in}} \to N_{\mathrm{out}}$, then $Z(M)$ for various bordisms M defines a cochain $Z \in C^*(\mathcal{M}_n(N_{\mathrm{in}}, N_{\mathrm{out}}), \mathrm{Hom}(Z(N_{\mathrm{in}}), Z(N_{\mathrm{out}})))$. That is, Z is a function on the space of cobordisms, valued in the space of linear maps, and it's a cochain for Q. We also ask for it to be a cocycle.

Equivalently, one can think of Z as a chain map $Z : C_*(\mathcal{M}_n(N_{\mathrm{in}}, N_{\mathrm{out}})) \to \mathrm{Hom}(Z(N_{\mathrm{in}}), Z(N_{\mathrm{out}}))$ that's a chain map of degree 0. In other words, instead of asking for it to be completely locally constant, we ask that its dependence be specified with respect to the differential. This is the kind of structure that we're going to see. This structure is often called a *topological conformal field theory* (because of the relationship with the moduli space of Riemann surfaces), and these days they may just be called topological field theories.

Such examples arise very naturally from algebraic varieties: if the fields are maps to a fixed variety, the chain complex arises very naturally. This will lead to interesting new operations.

23. COHOMOLOGICAL FIELD THEORIES: 11/10/16

Last time, we talked about something that was more confusing and complicated than it needs to be. Let a group G act on a space X, and let $K \le G$ be a subgroup. Then, we can consider the groupoid $X \times_K G/K$, which X/G acts on through the descent diagram

$$X/K \times_{X/G} X/K \Longrightarrow X/K \longrightarrow X/G.$$

We care about this because X/G is usually horrible, and this perspective helps avoid some of that horror.

In general, X will have topology or geometry, meaning that the quotients X/G and X/K are stacks. A typical example is $\mathrm{SL}_2\mathbb{Z} \backslash \mathrm{SL}_2\mathbb{R} / \mathrm{SO}_2 \cong \mathbb{R}_+$: the quotient stack is $\mathcal{M}_{1,1}$, the moduli stack of elliptic curves. In this case, X is the upper half-plane. The Hecke operators are a kind of descent data: an r in $\mathbb{R}_+$ is an operator on $\mathbb{C}[\mathcal{M}_{1,1}]$ which acts by

$$f \longmapsto r * f(z) = \int_{\mathrm{dist}(x,z)=r} f(x)\,\mathrm{d}x.$$

There's another description of this in terms of p-adics:

$$\mathcal{M}_{1,1} = \mathrm{SL}_2\mathbb{Q} \backslash \prod_{p \text{ prime}} \mathrm{SL}_2\mathbb{Q}_p / \left(\prod_{p \text{ prime}} \mathrm{SL}_2\mathbb{Z}_p \times \mathrm{SO}_2 \right).$$

Taking away one of the factors does something interestring: for any prime p, let $K = \mathrm{SL}_2\mathbb{Z}_p \subset \mathrm{SL}_2\mathbb{Q}_p$. Then, if $\mathcal{M}_{1,1} = X/K$, $\mathrm{SL}_2\mathbb{Q}_p$ acts on X, and the Hecke algebra $\mathbb{C}[\mathrm{SL}_2\mathbb{Z}_p \backslash \mathrm{SL}_2\mathbb{Q}_p / \mathrm{SL}_2\mathbb{Z}_p]$ acts on $\mathbb{C}[\mathcal{M}_{1,1}]$. There's good geometry in here: $\mathrm{SL}_2\mathbb{Q}_p / \mathrm{SL}_2\mathbb{Z}_p$ can be identified with a rooted infinite p-ary tree, and therefore the Hecke algebra is identified with a polynomial algebra in a single generator $\mathbb{C}[T_p]$, where T_p^r acts on a function f on this tree by averaging f over all nodes of distance r from the root. The continuous averaging operator was turned into a discrete operator. This is the fundamental example of Hecke operators: we hope to get to the geometric Langlands version of this in a couple of weeks.[50]

Anyways, let's move forward from Tuesday.

We've seen that there are three worlds of topological field theories. First was a bordism category. Second was the codomain category, vector spaces or vector bundles or the Morita 2-category. A TFT functor will factor through fields, which will be some sort of correspondences of groupoids.

We'll study the derived version of this story in cohomological field theory, which will require upgrading all three of these pieces.

First, we'll upgrade the bordism category. Recall that if N_{in} and N_{out} are $(n-1)$-dimensional manifolds, their hom-set $\mathrm{Bord}_{n-1,n}(N_{\mathrm{in}}, N_{\mathrm{out}})$ is the set of isotopy classes of bordisms $M : N_{\mathrm{in}} \to N_{\mathrm{out}}$, which was really a set of connected components of the space of Riemannian bordisms. Instead of a discrete set of morphisms, we're going to consider this underlying space which parameterizes familes of bordisms. We must choose what

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[50]Hecke algebras appear in very different places in the world: the professor mentioned that, while in grad school, one friend learned about them in this context while he learned how to use them in the deformation theory of representations of the symmetric group. It turned out they were looking at the same things!

kind of bordisms we want: Riemannian, topological, etc. Once we choose this space, though, we will only consider its homotopy type: there are lots of choices we can make, but we will ignore contractible choices (as we still only care about topological questions). For example, a choice of a metric is contractible, so does not affect the homotopy type.

The space we obtain is

$$\mathbf{Bord}_{n-1,n}(N_{\mathrm{in}}, N_{\mathrm{out}}) = \coprod_{\text{diff. classes of } M} B \operatorname{Diff} M.$$

What is the space on the right? It's the classifying space of the diffeomorphism group of M.[51] This is the moduli space such that for any space S, submersions $E \to S$ with constant fiber M are in natural bijection with homotopy classes of maps $[S, B \operatorname{Diff}(M)]$. If M is a bordism, then the family over S is a map from S into the space of bordisms diffeomorphic to M, assigning to each point $p \in S$ its fiber, a bordism from N_{in} to N_{out}. Since we care about submersions, these are always locally trivial, and therefore these are $\operatorname{Diff}(M)$-principal bundles. (This local triviality is particular to algebraic topology, and would be untrue in algebraic geometry).

If you want to preserve some extra structure, encode it into $\operatorname{Diff}(M)$: for example, to get the space of oriented bordisms, we instead consider $\operatorname{Diff}^+(M)$, the group of orientation-preserving diffeomorphisms of M.

Example 23.1. This works very nicely in 1+1 dimensions. Let's consider the space $\mathbf{Bord}_{1,2}^{\mathrm{or}}((S^1)^k, (S^1)^{n-k})$. The diffeomorphism type of a bordism from k circles to $n-k$ circles is determined by its genus g: it must be the oriented surface $\Sigma_{g,n}$ with genus g and n boundary components. Thus,

$$\mathbf{Bord}_{1,2}^{\mathrm{or}}((S^1)^k, (S^1)^{n-k}) = \coprod_g B \operatorname{Diff}(\Sigma_{g,n}).$$

If $g > 1$, or $g \geq 1$ and $n \geq 1$, or $g = 0$ and $n \geq 3$, then $\Sigma_{g,n}$ is hyperebolic. In this case, $B \operatorname{Diff} \Sigma_{g,n} = B(\pi_0(\operatorname{Diff} \Sigma_{g,n}))$: this group, as a topological group, is only interesting on π_0.

Definition 23.2. The group of connected components of $\operatorname{Diff} \Sigma_{g,n}$ is the *mapping class group* of $\Sigma_{g,n}$, called $\Gamma_{g,n}$ or $\operatorname{MCG}_{g,n}$.

In this case, $B\Gamma_{g,n}$ has a nice description: if $\operatorname{Teich}_{g,n}$ denotes *Teichmüller space* for (g,n), the moduli of Riemann surfaces with genus g, n boundary components, and markings of π_1, then $\operatorname{Teich}_{g,n}$ is contractible and $\operatorname{Teich}_{g,n}/\Gamma_{g.n}$ models $B\Gamma_{g.n}$. This action is not free, so there's something funny going on: to say this, you have to recognize $B\Gamma_{g,n}$ as an orbifold or stack. But this is also the moduli $\mathcal{M}_{g,n}$ of curves of genus g and n marked points, so

$$\mathbf{Bord}_{1,2}^{\mathrm{or}}((S^1)^k, (S^1)^{n-k}) = \coprod_{\text{hyperbolic}} \mathcal{M}_{g.n} \amalg \cdots.$$

For example, $\operatorname{Diff} T^2 \simeq \operatorname{PSL}_2(\mathbb{Z}) \times T^2 \cong \operatorname{PSL}_2\mathbb{Z} \times B\mathbb{Z} \times B\mathbb{Z}$, and $\operatorname{Diff} S^2 \simeq \operatorname{SO}_3$. These aren't trivial, but they're not scary: diffeomorphism groups of surfaces are tractable. In higher dimensions, this is harder.

The takeaway is that topological field theory in dimension 2 is very similar to conformal field theory: spaces are very close to having complex structures. Cohomological field theory was first discovered in this case, and thus had the misleading name "topological conformal field theory" (TCFT), but the complex geometry is more of a coincidence, a useful model for $B \operatorname{Diff} \Sigma_{g.n}$.

So now that we've defined $\mathbf{Bord}_{n-1,n}(N_{\mathrm{in}}, N_{\mathrm{out}})$ and seen that it's not always terrible, we'll begin asking questions about it, such as its cohomology, and what they imply for topological field theory.

First, what kind of beast is $\mathbf{Bord}_{n-1,n}$? This is a category whose hom-spaces are topological spaces, but we only care about them up to homotopy: we will never pass to homotopy classes, but we will concern ourselves with homotopical questions. One might call this a category *enriched* over Top, or a *topological category*. Worrying about associativity up to homotopy (or higher homotopy) leads us to realize this space as an ∞-*category* or an $(\infty,1)$-*category*. This seems like a lot of higher category theory, but the point is that we're considering a category where set-theoretic notions are replaced with homotopical ones.

This is an acceptable definition of an ∞-category, but there are better ones. In any case, what matters is when two of them are the same, which is when there's a natural homotopy equivalence of hom-spaces. This

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[51]Diffeomorphisms must be the identity on the boundary.

is precisely what we were looking for: we consider honest hom-spaces, and then only care about homotopical questions.

Now that we have a bordism ∞-category, we're going to do something radical: we're going to linearize it. This reduces our questions from homotopical to $\mathbb{C}$-linear. We want a covariant functor from spaces to $\mathbb{C}$-vector spaces, and the natural choice is $H_*(-;\mathbb{C}) : \mathsf{Top} \to \mathsf{GrVect}$. There's an intermediate choice, to take chains on X, which form a differential graded (dg) vector space, and then take homology. So instead of hom-spaces, we'll replace them with chains or homology classes. Chains satisfy a nice universal property for mapping the bordism category into a linear category.

The spaces we consider are nice enough that the choices of singular vs. CW vs. simplicial homology don't matter, and singular or simplicial or CW chains will be the same in the derived category. This arose from the idea that if Δ is a triangulation on a space X and Δ' is a refinement, then the simplicial chains aren't the same, but there is a quasi-isomorphism $C_{*,\Delta}(X) \to C_{*,\Delta'}(X)$. More generally, any two triangulations share a common refinement, so there's a chain of quasi-isomorphisms. This is a kind of stabilization, like in homotopy theory, but less scary.

In summary, we started with $\mathrm{Bord}_{n-1,n} = \pi_0(\mathbf{Bord}_{n-1,n})$. Though we don't want to work with the whole $(\infty,1)$-category of bordisms, we can recover more information with $H_*\mathbf{Bord}_{n-1,n}$, or better the chains $C_*\mathbf{Bord}_{n-1,n}$. That is, the morphisms $\mathrm{Hom}_{C_*\mathbf{Bord}_{n-1,n}}(N_{\mathrm{in}}, N_{\mathrm{out}}) = C_*(\mathbf{Bord}_{n-1,n}(N_{\mathrm{in}}, N_{\mathrm{out}}))$. This category is a category enriched over chain complexes,[52] also known as a *differential graded (dg) category*.

In this framework, a topological field theory is a dg functor (the correct notion of a morphism of dg categories) from $C_*\mathbf{Bord}_{n-1,n}$ to some linear category; we'll want it to preserve some symmetric monoidal structure. The easiest choice of codomain category is $\mathsf{dgVect}_{\mathbb{C}}$, the category of differential graded vector spaces, with the graded tensor product $\otimes$. To every $(n-1)$-manifold N, we get a chain complex $Z(N)$, and to every chain in $C_*\mathbf{Bord}_{n-1,n}(N_{\mathrm{in}}, N_{\mathrm{out}})$, we get a dg linear map $Z(N_{\mathrm{in}}) \to Z(N_{\mathrm{out}})$. In particular, taking π_0 or H_0, there's a chain map of degree 0 between $Z(N_{\mathrm{in}})$ and $Z(N_{\mathrm{out}})$, which in particular induces a map on homology.

Today we'll get a lot of the abstract nonsense out of the way, and give some examples next lecture.

There are three ways to make sense of dgVect; it'll be important for us to take the right one.

(1) The first is the most naïve: the objects are pairs $(V^\bullet, \partial)$ of graded vector spaces and differentials $\partial : V^\bullet \to V^{\bullet-1}$ such that $\partial^2 = 0$. The maps are chain maps of degree 0.

(2) The second approach is through $D(\mathsf{Vect})$, the derived category of the category of vector spaces. The objects are the same, but we invert quasi-isomorphisms (the maps inducing isomorphisms on homology): for example if I have maps $(W, \partial_W) \xleftarrow{\sim} (V, \partial_V) \to (U < \partial_U)$, we invert the first map and obtain a morphism $(W, \partial_W) \to (U, \partial_U)$. One advantage of this approach is that chains on a space X is a functor into the derived category, and it doesn't matter what model you choose for chains.

(3) The most sophisticated model is the dg category version of $D(\mathsf{Vect})$. As this is a version of $D(\mathsf{Vect})$, quasi-isomorphisms are inverted, so that we don't have to keep track of two different resolutions of the same object. But in $D(\mathsf{Vect})$, the morphisms were homotopy classes of degree-0 chain maps, which is anathema to the whole idea of a derived category, where we invert rather than passing to homotopy classes.

The key is that if (V, ∂_V) and (W, ∂_W) are dg vector spaces, then there's a dg vector space of morphisms between them, $\mathrm{Hom}^\bullet((V, \partial_V), (W, \partial_W))$: the kth graded piece (for $k \in \mathbb{Z}$) is the space of degree-k chain maps, and there is a differential $\partial\varphi = \varphi \circ \partial_v + \partial_W \circ \varphi$. The H_0 of this complex is the homotopy classes of these morphisms, since $\varphi \simeq \varphi'$ iff $\varphi - \varphi' = \partial H$ for some other map H.

We'll prefer the third approach, and use dgVect to refer to the dg category whose object are chain complexes and whose morphisms are these hom complexes, such that we invert quasi-isomorphisms.

This is an enhanced version of the derived category: $H_0(\mathsf{dgVect}) = D(\mathsf{Vect})$ (taking homology of the category means taking homology of its hom-complexes). That said, for most of the things we do in this class, the derived category will suffice.

The derived category is an example of a *triangulated category*. This definition has a complicated set of axioms, and bad properties, but the notion of a dg category is both stronger (exact triangles of the triangulated category can be recovered on H_0) and simpler.

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[52]This means that the morphisms are chain complexes, and composition is a morphism of chain complexes. The hard part of the definition is that quasi-isomorphism should be the right notion of "sameness."

Next time, we'll state some very concrete things that come out of this setting; the basic source of field theories was the assignment of functions on a set X, but now, given a *space* X, we can take chains on X (or homology). If X is an algebraic variety, we can take $H^{\bullet}(X, \mathscr{O}_X)$, a derived version of functions. These are the A- and B-models, respectively. Everything will look the same, but be much more interesting.

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