M392C NOTES: RATIONAL HOMOTOPY THEORY

ARUN DEBRAY
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These notes were taken in UT Austin's Math 392C (rational homotopy theory) class in Fall 2015, taught by Jonathan Campbell. I live-TEXed them using vim, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

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Lecture 1.

Postnikov Towers and Principal Fibrations: 8/27/15

First, we'll outline some aspects of the course.

$X \to Y$ is a rational equivalence if $\pi_*(X)\otimes\mathbb{Q} \xrightarrow{\cong} \pi_*(Y)\otimes\mathbb{Q}$. The goal is to define a category $\mathrm{Ho}(\mathsf{Top}_\mathbb{Q})$ where, more or less, the isomorphisms are rational equivalences. The point is that this is a purely algebraic category, equivalent to a category of differential graded algebras, $\mathrm{Ho}(\mathsf{CDGA}_\mathbb{Q})$.

The first half of the course will deal with something called Sullivan's method: we'll get our hands on rational equivalence, and produce the rationalization functor $X \mapsto X_\mathbb{Q}$. We're developing it as it "could have been done," with some computations to show that things get a lot easier over $\mathbb{Q}$ (e.g. homology of Eilenberg-MacLane spaces is the same as for spheres).

Then, we'll have to talk about model categories, which is a good way of producing homotopy categories or homotopy theories for more than just topological categories. Intuitively, a model category is a category in which one can do homotopy theory. Using this, we'll talk about the homotopy theory of commutative, differential algebras over $\mathbb{Q}$.

This isn't how it was originally done by Sullivan et al., and so we'll also discuss the classical construction. We'll also produce functors from simplicial sets to differential graded $\mathbb{Q}$-algebras and topological spaces, with adjoints and so on. One of these, turning a simplicial set into a differential graded $\mathbb{Q}$-algebra, will resemble the functor Ω^* of differential forms, but is more combinatorial.

This will enable us to prove equivalence, with all sorts of cool consequences: Whitehead products appear in the differential graded algebras category; automorphisms of CDGAs correspond to automorphisms of $\mathsf{Top}_\mathbb{Q}$, which relate to automorphisms of topological spaces nicely, and so on.

The rest of the course will discuss Quillen's model, which relates differential graded Lie algebras to rational spaces. That might not mean anything right now, and we'll have to learn a little more machinery for it. Thus, this course will cover some classical and some modern algebraic topology, making the useful notion of model categories nice and concrete.

Here are some good references for this subject.

- Griffiths-Morgan, *Rational Homotopy Theory and Differential Forms*; it's all right, and geometric (they use simplicial complexes, rather than simplicial sets, and therefore don't get as nice of a result). The second edition came out a year ago, but is similar to the first edition. The beginning has a beautiful exposition of algebraic topology in general.
- There's a GTM by Felix, Halperin, and Thomas, called *Rational Homotopy Theory*. It's pretty beefy, and the one gripe the professor has is that it doesn't use model categories at all, making things opaque. But there's definitely a bootlegged copy...
- Katherine Hess, who is a great writer, has a survey paper, about 20 pages, called *Rational Homotopy Theory*.

Those are the only expository works, but there are also some papers.

- Sullivan, "Infinitesimal Computations in Algebraic Topology." Sullivan is crazy, and the paper is very hard to read. Hopefully after the course everything is easier to read.
- Quillen, "Rational Homotopy Theory." This paper also isn't that easy to read.

There are a few other sources; things will be well cited in this class.

Now for some math.

Definition. Let X be a connected topological space. A *Postnikov tower* is a sequence

$$\begin{tikzcd} & X_n \arrow[d] \\ & \vdots \arrow[d] \\ & X_2 \arrow[d] \\ X \arrow[r] & X_1 \end{tikzcd}$$

such that

(1) there are maps $X \to X_i$,
(2) $\pi_i(X) \cong \pi_i(X_n)$ for $i \le n$, and
(3) $\pi_i(X_n) = 0$ for $i > n$.

As a consequence of the three properties, the homotopy fiber $X_n \to X_{n-1}$ is a $K(\pi_n(X), n)$, i.e. an Eilenberg-MacLane space. In some sense, this is a "co-cellular" way of building a space out of Eilenberg-MacLane spaces.

Theorem 1.1. *Postnikov towers exist.*

The proof is easy: just attach cells to X to kill homotopy above a given degree. But that's not so useful of a characterization. We want to know: what information in stage n determines stuff in stage $(n+1)$?

To produce spaces with certain fibers, classifying maps are useful. Suppose X_{n+1} arises as a (homotopy) pullback: if $\star$ denotes a contractible space, this would look like

$$\begin{tikzcd} X_{n+1} \arrow[r] \arrow[d] & \star \arrow[d] \\ X_n \arrow[r] & K(\pi_{n+1}(X), n+2). \end{tikzcd}$$

It would be nice if fibrations with fiber $K(G,n)$ were classified by maps $X \to K(G, n+1)$, because then we could work with the cohomology group $H^{n+2}(X, \pi_{n+1}(X))$. It's not very easy to compute stuff in this cohomology group, however.

In any case, not every fibration is even classified in this manner!

Definition. A fibration $K(\pi,n) \to E \to B$ is *principal* if it arises as a pullback of a path fibration as follows.

$$\begin{tikzcd} K(\pi,n) \arrow[r, equal] \arrow[d] & K(\pi,n) \arrow[d] \\ E \arrow[r] \arrow[d] & P \arrow[d] \\ B \arrow[r] & K(\pi,n+1) \end{tikzcd}$$

There's an equivalent, less useful, formulation in the lecture notes. The reason we like our formulation is the following theorem.

Theorem 1.2. *A connected CW complex X with $\pi_1(X)$ acting trivially on $\pi_n(X)$ has a Postnikov tower composed of principal fibrations.*

As a consequence, X_{n+1} is determined from X_n by a map $k_n : X_n \to K(\pi_{n+1}(X), n+2)$; this determines a class $[k_n] \in H^{n+2}(X_n, \pi_{n+1}(X))$, called a *$k$-invariant*. This is why we care about Postnikov towers: they are built up nicely in stages, using cohomology classes that, in nice cases, we can compute. And so in rational homotopy theory, where the homotopy groups are nicer, the k-invariants are nicer.

We'll use spectral sequences in this class; an introduction to them can be found in the professor's lecture notes.

Another takeaway from these results is that Eilenberg-MacLane spaces are pretty fundamental building blocks. Though they have nice homotopy, their cohomology groups are generally pretty nasty, leading to computations called Steenrod operations. But rationally, there's a nice result.

Theorem 1.3.
$$H^*(K(\mathbb{Z},n);\mathbb{Q}) = \begin{cases} \mathbb{Q}[x], & n \text{ even} \\ \Lambda_{\mathbb{Q}}(x), & n \text{ odd}, \end{cases},$$
where the generators x have degree n.

These are the simplest differential graded $\mathbb{Q}$-algebras, and suggest that all spaces' rational homotopy will be built out of them (which is true).

Proof. As a base case, $H^*(K(\mathbb{Z},1);\mathbb{Q}) = H^*(S^1;\mathbb{Q}) = \Lambda_{\mathbb{Q}}(x)$, which is fine. More generally, we'll use the fibration
$$K(\mathbb{Z}, n-1) \longrightarrow \star \longrightarrow K(\mathbb{Z},n).$$
By induction, if n is odd, then $H^*(K(\mathbb{Z},n-1);\mathbb{Q}) = \mathbb{Q}[x]$, with $\deg x = n-1$, since $n-1$ is even. Let's use the Serre spectral sequence, for which
$$E_2^{p,q} = H^p(K(\mathbb{Z},n-1);\mathbb{Q}) \otimes H^q(K(\mathbb{Z},n-1);\mathbb{Q}).$$
For example, when $n = 3$, we have
$$\begin{array}{ccccccc} \mathbb{Q}y^3 & & & & & & \\ 0 & & & & & & \\ \mathbb{Q}y^2 & & & & & & \\ 0 & & & & & & \\ \mathbb{Q}y & 0 & 0 & \mathbb{Q}x \otimes \mathbb{Q}y & & & \\ 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & \mathbb{Q}x & H^4 & H^5 & \cdots \end{array}$$
Here, degree increases from 0 to the right and going upwards. This also uses the Hurevitch theorem. Then, we remark that d_2, from $(0,2)$ to $(3,0)$, has to be an isomorphism, because the E_∞-page is 0.[1] But since the Serre spectral sequence is linear, we also have isomorphisms from $(4,0)$ to $(2,2)$ to $(0,4)$; specifically, $d_3 : \mathbb{Q}y^2 \mapsto \mathbb{Q}x \otimes \mathbb{Q}y$, so $H^6(K(\mathbb{Z},3);\mathbb{Q}) = 0$. And this means that $x^2 = 0$. Then, we continue by induction to show that all higher $H^q(K(\mathbb{Z},3);\mathbb{Q})$ are zero. Thus, $H^*(K(\mathbb{Z},3);\mathbb{Q}) = \Lambda_{\mathbb{Q}}(x)$, and the case for general n is similar. ☒

[1]Though the first few lectures will use spectral sequences, they won't be very important after that, so don't drop the course if this is the only thing making you uncomfortable.

Exercise 1. Handle the case where n is even, which is somewhat similar.

Again, this is suggestive: Eilenberg-MacLane spaces build topological spaces up, and they have differential graded algebras for their rational cohomology groups.

Next lecture, we'll discuss Serre theory, the tricks that Serre used to compute the rational homotopy groups of the spheres. These are strong clues that, rationally, things are much nicer.

After that, we'll discuss rational equivalence, and then CDGAs and their homotopy theory, necessitating a discussion of model categories. The course will get less computational at this point.

Lecture 2.

Serre Theory: 9/1/15

"Whistle guy has really got me off my game!"

Last lecture may have gone a little fast, so we'll reboot and say some things that we didn't mean to assume. Then, we'll start Serre theory.

Definition. If G is a group, an *Eilenberg-MacLane space* $K(G,n)$ is a space whose homotopy groups are

$$\pi_i(K(G,n)) = \begin{cases} G, & i = n \\ 0, & i \neq n. \end{cases}$$

For example, S^1 is a $K(\mathbb{Z},1)$, $\mathbb{C}P^\infty$ is a $K(\mathbb{Z},2)$, and $\mathbb{R}P^\infty$ is a $K(\mathbb{Z}/2,1)$. Also, $K(G,n-1) = \Omega K(G,n)$ (i.e. the space of loops). One can think of building spaces out of these, because they're the simplest spaces from the perspective of homotopy theory.

Another of the reasons they're important is the following theorem, which is hard to prove.

Theorem 2.1. $[X,K(G,n)] = \widetilde{H}^n(X;G)$.

In particular, the cohomolgy functor is representable.

This is why, last time, a map $k : X \to K(\pi_{n+1}(X), n+2)$ corresponded to $H^{n+2}(X;\pi_{n+1}(X))$: k-invariants arise from the representability of cohomology.

Postnikov towers are a way of using Eilenberg-MacLane spaces to build a space up, one homotopy group at a time.

Theorem 2.2. *Eilenberg-MacLane spaces exist for all n and all G, where G is abelian if $n > 1$.*

Recall by the Eckmann-Hilton argument that $\pi_n(X)$ is always abelian when $n > 1$.

Proof idea. When $n = 1$, one can produce a fiber bundle where the total bundle is trivial and the fiber is G, with the discrete topology, thus producing a sequence $G \to EG \to BG$, and one can show that $\pi_1(BG) = G$ and $\pi_i(BG) = 0$ for $i \geq 2$. This is done in chapter 1 of Hatcher.

If $n > 1$, take a resolution for G:

$$\mathbb{Z}[r_\beta] \longrightarrow \mathbb{Z}[g_\alpha] \longrightarrow G,$$

where the g_α are generators and r_β are relations. Then, consider a bouquet of spheres $\bigvee_\alpha S^n_\alpha$; each relation gives a map $S^n \to \bigvee_\alpha S^n_\alpha$ using the degree of the relation, so glue cells onto this bouquet via the relations, forming pushouts of the form

$$\begin{tikzcd} \bigvee_\beta S^n \arrow[r] \arrow[d] & \bigvee_\alpha S^n_\alpha \arrow[d, hook] \\ \bigvee_\beta D^n \arrow[r] & X \end{tikzcd}$$

Then, the n-skeleton $X^{(n)}$ is given by the generators, and so we have an exact sequence

$$\pi_{n+1}(X,X^{(n)}) \longrightarrow \pi_n(X^{(n)}) \longrightarrow \pi_n(X) \longrightarrow \pi_n(X,X^{(n)}).$$

Then, $\pi_{n+1}(X,X^{(n)}) = \mathbb{Z}[r_\beta]$, $\pi_n(X^{(n)}) = \mathbb{Z}[g_\alpha]$, and $\pi_n(X,X^{(n)}) = 0$, so $\pi_n(X) = G$. There are only n-cells, so there's no lower homotopy, and we can kill all higher homotopy in a standardized manner. ⊠

Anyways, last time we talked about how $H^*(K(\mathbb{Z},n),\mathbb{Q})$ is $\mathbb{Q}[x]$ when n is even and $\Lambda_{\mathbb{Q}}(x)$ when n is odd; this suggests that since the constituents of spaces have simple rational homotopy, then maybe rational homotopy provides some insights.

We proved this with the Serre spectral sequence, and we'll use it a few more times to get nice short exact sequences, so let's carefully state what's going on.

Theorem 2.3 (Homological Serre sequence). *Let $F \to E \to B$ be a fibration with $\pi_1(B)$ acting trivially on $H^*(F,G)$. Then, there is a spectral sequence $\{E^r_{p,q}, \mathrm{d}_r\}$ such that:*

(1) $\mathrm{d}_r : E^r_{p,q} \to E^r_{p-r,q-r+1}$, i.e. d_r is of degree $(-r, r-1)$.
(2) $E^\infty_{n,n-p} = F^p_n/F^{p-1}_n$, where $F^\bullet_n$ is some filtration of $H_n(E;G)$, i.e. $H_n(E;G) = F^n_n \supset F^{n-1}_n \supset \cdots \supset F^0_n = 0$.
(3) $E^2_{p,q} = H_p(B; H_q(F;G))$.

Remark. If G is a field, then Künneth gives us the nicer result that $E^2_{p,q} = H_p(B) \otimes H_q(E)$.

Theorem 2.4 (Cohomological Serre sequence). *With the same setup as Theorem 2.3, there exists a spectral sequence $(E^{p,q}_r, \mathrm{d}_r)$ such that:*

(1) $\mathrm{d}_r : E^{p,q}_r \to E^{p+q,q-r+1}_r$.
(2) $E^{p,n-p}_\infty = F^n_p/F^n_{p-1}$, where $F^n_\bullet$ is some filtration of $H^n(E;G)$.
(3) $E^{p,q}_2 \cong H^p(B : H^q(F;G))$.

The cohomological Serre sequence is multiplicative: there's a product that ultimately comes from the cup product. This lends some rigidity to the cohomological theory that is often very useful.

Example 2.5. Suppose we're in a case of the homological Serre spectral sequence such that $E^2_{p,q}$ for $0 < q < n$. It turns out that in situations like this, you can leverage your knowledge of the sequence to obtain useful exact sequences.

In this case, the first differential that does anything interesting is $\mathrm{d}_{n+1} : E^2_{n+1,0} \to E^2_{0,n}$ (previous differentials all map to zero). This means that $E^3_{n+1,0} = E^2_{n+1,0}/\mathrm{Im}(\mathrm{d}_{n+1})$, and nothing else hits this, so this is also $E^\infty_{n+1,0}$. Thus, we have a sequence

$$E^2_{n+1,0} \longrightarrow E^2_{0,n} \longrightarrow E^\infty_{0,n} \longrightarrow 0. \tag{2.1}$$

Furthermore, there is a filtration

$$H_n(E) = F^n_n \supset F^{n-1}_n \supset \cdots \supset F^0_n,$$

with $E^\infty_{p,n-p} = F^p_n/F^{p-1}_n$. In particular, $E^\infty_{0,n} = F^0_n$, $F^\infty_{1,n} = 0$, and $E^\infty_{n,0} = F^n_n/F^{n-1}_n = H_n(X)/E^\infty_{0,n}$. That is, we have a sequence

$$0 \longrightarrow E^\infty_{0,n} \longrightarrow H_n(X) \longrightarrow E^\infty_{n,0} \longrightarrow 0,$$

which we can join to (2.1) to produce

$$E^2_{n+1,0} \longrightarrow E^2_{0,n} \longrightarrow E^\infty_{0,n} \longrightarrow H_n(E) \longrightarrow E^\infty_{n,0} \longrightarrow 0.$$

This may seem a little contrived, but it happens, for example, when a fiber in a fibration is n-connected.

Serre Theory. This was sometimes called Serre's thesis. It will use some very abstract computations to determine the rational homotopy groups of the spheres.

Definition. Let $\mathscr{C}$ be one of the three classes (the *Serre classes*): *FG* of finitely generated abelian groups, $\mathscr{T}_P$, the torsion abelian groups with orders drawn from a set P, and $\mathscr{F}_P$, the finite groups in $\mathscr{T}_P$.

Lemma 2.6. *The classes $\mathscr{C}$ are closed under extension: that is, if $A, C \in \mathscr{C}$ and $0 \to A \to B \to C \to 0$ is short exact, then $B \in \mathscr{C}$. Moreover, for any $A, B \in \mathscr{C}$, $A \otimes B \in \mathscr{C}$ and $\mathrm{Tor}(A,B) \in \mathscr{C}$.*

The point is that the disgusting machinery of the Serre spectral sequence mostly leaves a Serre class intact, which will be useful for proving the Hurewicz theorem mod $\mathscr{C}$. First, though, we'll need more lemmas.

Lemma 2.7. *Let $F \to E \to B$ be a filtration satisfying the hypotheses of the homological Serre spectral sequence, and assume F, E, and B are all path-connected. If any two of $H_*(B)$, $H_*(E)$, and $H_*(F)$ are in $\mathscr{C}$, then so is the third.*

Proof. We'll show that if $H_*(B)$ and $H_*(F)$ are in $\mathscr{C}$, then $H_*(E)$ is. Recall that

$$\begin{aligned} E^2_{p,q} &= H_p(B;H_q(F)) \\ &= H_p(B;\mathbb{Z}) \otimes H_q(F;\mathbb{Z}) \oplus \operatorname{Tor}(H_{p-1}(B), H_q(F)), \end{aligned}$$

by the universal coefficient theorem. The first two terms are in $\mathscr{C}$ by assumption, and Lemma 2.6 implies the last one is. Thus, $E^2_{p,q} \in \mathscr{C}$, so all subsequent pages must be too (since homology is just kernels and images, which don't pop us out of $\mathscr{C}$). Since $H_n(X)$ has successive filtrations whose quotients are $E^\infty_{p,q}$, which are all in $\mathscr{C}$, then $H_n(E) \in \mathscr{C}$. ⊠

Lemma 2.8. *If $\pi \in \mathscr{C}$, then $H_k(K(\pi,n),\mathbb{Z}) \in \mathscr{C}$ for all k.*

At this point, there aren't many guesses for tools we can use: the Serre sequence is basically the only tool we have for homotopy groups.

Proof. Recall that we have a fibration $K(\pi, n-1) \to * \to K(\pi,n)$; we'll apply the Serre sequence. By induction, it's sufficient to consider the case $n = 1$.

For right now, we'll consider $\mathscr{C} = FG$. By Künneth, it's sufficient to show for $K(\mathbb{Z},1)$ and $K(\mathbb{Z}/m,1)$, which are S^1 and lens spaces, for which this is true. ⊠

Lemma 2.9. *Let X be simply connected. Then, $\pi_n(X) \in \mathscr{C}$ implies $H_n(X) \in \mathscr{C}$.*

Remark. The converse also holds (i.e. if we know this for all n, then X must be simply connected), but we won't show that yet.

Proof of Lemma 2.9. We'll use a Postnikov tower.

$$\begin{tikzcd} K(\pi_n(X),n) \arrow[r] & X_n \arrow[d] \\ & X_{n-1} \end{tikzcd} \tag{2.2}$$

From Lemma 2.8, we know that $H_k(K(\pi_n(x),n),\mathbb{Z}) \in \mathscr{C}$, so once again using the homological Serre sequence, $H_*(X_n)$ is computed using $E^2_{p,q} = H_p(X_{n-1}, H_q(K(\pi_n(x),n)))$, and therefore $H_n(X_n;\mathbb{Z}) \in \mathscr{C}$. But this is $H_n(X)$. ⊠

There seems to be a duality, where things with complicated homology seem to have uncomplicated homotopy, and vice versa.

Now we can get to the reason we're doing this abstract nonsense.

In the following theorems, *isomorphic mod $\mathscr{C}$* means that the kernel and cokernel of the map are both in $\mathscr{C}$.

Theorem 2.10 (Mod $\mathscr{C}$ Hurewicz)**.** *If X has $\pi_i(X) \in \mathscr{C}$ for $i < n$, then $h : \pi_n(X) \to H_n(X)$ is an isomorphism mod $\mathscr{C}$.*

Corollary 2.11. *If $\pi_i(X) \in \mathscr{C}$ for all i, then $\pi_n(X) \to H_n(X)$ is isomorphic mod $\mathscr{C}$ for all i.*

Corollary 2.12. *If $\pi_i(X) \in FG$, then $\pi_i(X) \otimes \mathbb{Q} \to H_i(X) \otimes \mathbb{Q}$ is an isomorphism.*

Proof of Theorem 2.10. $\pi_n(X) \to H_n(X)$ is the same as $\pi_n(X_n) \to H_n(X_n)$, where $\{X_i\}$ is the Postnikov tower. Look at the fibration (2.2), and use the five-term exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H_{n+1}(X_{n-1}) \arrow[r] & H_n(K(\pi_n(X),n)) \arrow[r] & E^\infty_{0,n} \arrow[r] & H_n(X_n) \arrow[r] & H_n(X_{n-1}). \end{tikzcd}$$

The composition of the middle maps is the inclusion of the fiber, so we get a four-term exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H_{n+1}(X_{n-1}) \arrow[r] & H_n(K(\pi_n(X),n)) \arrow[r] & H_n(X_n)H_n(X_{n-1}) \arrow[r] & 0. \end{tikzcd}$$

By induction, the first and last terms are in $\mathscr{C}$, which is exactly what we need for the middle arrow to be an isomorphism mod $\mathscr{C}$.

Now, we can put that into the following diagram.

$$\begin{tikzcd} H_n(K(\pi_n(X),n)) \arrow[r, "\cong"] \arrow[d, "h"', "\cong"] & \pi_n(X_n) \arrow[d] \\ H_n(K(\pi_n(X),n)) \arrow[r, "\cong \bmod \mathscr{C}"'] & H_n(X_n) \end{tikzcd}$$

The isomorphism on the left follows from the usual Hurewicz theorem, and the one on the top is from the construction of the Postnikov tower. Then, we just showed the result for the bottom arrow, so the result is that the arrow on the right is an isomorphism mod $\mathscr{C}$. ☒

You can see the game: Postnikov tower, fibration, Serre sequence, and then hope and pray that there's an equivalence (e.g. one coarse enough such as $\mathscr{C}$).

One corollary here is that homology and homotopy don't differ much over $\mathbb{Q}$, which is pretty nice.

Lecture 3.

Rational Homotopy Groups of Spheres: 9/3/15

"I looked through my notes the other day, and I had at least five spellings of Hurewicz."

Though we'll do some significant computations with rational homotopy groups today, we'll start by clarifying a few things from last time.

Proposition 3.1. *Let X be a space with a Postnikov tower $\{X_i\}$. Then, $H_n(X) = H_n(X_n)$.*

Proof. There's a map $X \to X_n$; consider it as a fibration, so we get a fiber $X^{>n} \to X \to X_n$. So we can either use the long exact sequence of a fibration or the Serre spectral sequence, and the former isn't so useful here.

We have $E_2^{p,q} = H_p(X_n, H_q(X^{>n}, \mathbb{Z}))$. We know $X^{>n}$ doesn't have homotopy in degrees n or lower, and therefore it doesn't have homology there either. This implies that $E_\infty^{0,n} = H_n(X_n)$, and for $p+q=n$ otherwise, $E_n^{p,q} = 0$. Since $E_\infty^{p,n-p}$ filters $H_n(X)$ and there's only one nonzero term in the filtration, $H_n(X) = H_n(X_n)$. ☒

This is a common technique with spectral sequences: things work because there end up being large gaps. We've even seen it a few times before.

Another loose end from last time is the following corollary of Hurewicz mod $\mathscr{C}$.

Corollary 3.2. *Suppose that X is a space such that $\pi_1(X)$ acts trivially on X (i.e. X has a Postnikov tower). Then, $H_*(X;\mathbb{Q}) = 0$ iff $\pi_*(X) \otimes \mathbb{Q} = 0$.*

Now, we may compute the rational homotopy groups of spheres. It's pretty amazing that knowledge of the homotopy of Eilenberg-MacLane spaces and the Serre spectral sequence will do it. It's really clever (well of course it is; it's Serre).

We know $\pi_n(S^n) = \mathbb{Z}$, and therefore $\pi_n(S^n) \otimes \mathbb{Q} = \mathbb{Q}$.

Theorem 3.3. *$\pi_i(S^n)$ for $i > n$ are finite except when $n = 2k$, in which case $\pi_{4k+1}(S^{2k}) = \mathbb{Z} \oplus A$ for a finite A.*

Proof. The whole thing is based on the following: consider a map $S^n \to K(\mathbb{Z}, n)$ inducing an isomorphism on π_n, i.e., this is the generator of $\pi_n(K(\mathbb{Z},n)) = \mathbb{Z}$. As usual, we'll convert it into a fibration $F \to S^n \to K(\mathbb{Z},n)$. Repeating, we get a fibration $K(\mathbb{Z}, n-1) \to F \to S^n$.[2] The point is, $\pi_i(F)$ and $\pi_i(S^n)$ agree when $i > n$, and F fits nicely into these two fibrations.

Obviously, we're going to use the Serre spectral sequence on this, but first we need to understand the cohomology of $K(\mathbb{Z}, n-1)$.

When n is odd, $E^2_{p,q} = H^p(S^n;\mathbb{Q}) \otimes H^q(K(\mathbb{Z},n-1),\mathbb{Q}) = \Lambda(x) \otimes \mathbb{Q}[y]$, where $\deg x = n$ and $\deg y = n-1$. We computed this in the first lecture. This means that in row 0, there's only a $\mathbb{Q}$ at $(0,0)$ and a $\mathbb{Q}x$ at $(0,n)$; in $n-1$, we have $\mathbb{Q}y$ at column 0 and $\mathbb{Q}xy$ in column n, and then y^2 in place of y at row $2n-2$, and so on.

Since F is $(n-1)$-connected, then $H^{n-1}(F;\mathbb{Q})$ must vanish. Thus, $\mathbb{Q}y$ has to die, and this can only happen if $d_{n-1} : \mathbb{Q}y \to \mathbb{Q}x$ is an isomorphism. Then, by multiplicativity, the other differentials have to be isomorphisms: if $y \mapsto x$, then $\mathbb{Q}y^2 \xrightarrow{\sim} \mathbb{Q}xy$, and so on. Thus, $H^*(F;\mathbb{Q}) = 0$, and therefore that $\pi_i(F) \otimes \mathbb{Q} = 0$ for all i. Thus, $\pi_i(F)$ (and therefore $\pi_i(S^n)$) is finite for $i > n$ when n is odd.

[2] This is because a fibration fits into a longer sequence $\cdots \to \Omega E \to \Omega B \to F \to E \to B$.

When n is even, then E^2 page has only $\mathbb{Q}$ at $(0,0)$, $\mathbb{Q}x$ at $(0,n)$, $\mathbb{Q}y$ at $(n-1,0)$, and $\mathbb{Q}xy$ at $(n-1,n)$. As before, $H^{n-1}(F;\mathbb{Q}) = 0$, so $\mathrm{d}_{n-1}\mathbb{Q}y \to \mathbb{Q}x$ is an isomorphis. That means that $H^*(F;\mathbb{Q}) = H^*(S^{2n-1};\mathbb{Q})$, i.e. there's a generaotr at degree $2n-1$, and nothing else. This tells us that $\pi_{2n-1}(S^n)\otimes\mathbb{Q} = \mathbb{Q}$ by Hurewicz, and that everything below it vanishes.

Finally, we have to address $i > 2n-1$, where we want the homotopy to vanish. Let $F \to F^{<2n-1}$ be obtained by killing all homotopy above $i \geq 2n-1$. Turn this into a fibration and take the fiber to get $F^{\geq 2n-1} \to F \to F^{<2n-1}$.

We know $\pi_i(F^{<2n-1})$ is finite, since it agrees with $\pi_i(F)$ on this range, and therefore $H^*(F^{<2n-1};\mathbb{Q}) = 0$. Using the Serre spectral sequence, we have that $H^*(F^{\geq 2n-1};|Q) = H^*(F;\mathbb{Q}) = H^*(S^{2n-1};\mathbb{Q})$. Now, consider $F^{\geq 2n-1} \to K(\mathbb{Z},2n-1)$ which induces an isomorphism mod FG on π_{2n-1}. Once again, we'll take the fiber to get $K(\mathbb{Z},2n-2) \to \widetilde{F} \to F^{\geq 2n-1}$, and as in the previous case, the homotopy of $\widetilde{F}$ and of $F^{\geq 2n-1}$ are equal in degrees $2n$ and above.

Once again, we use the Serre spectral sequence: $E_2^{p,q} = H^p(F^{\geq 2n-1};\mathbb{Q})\otimes H^q(K(\mathbb{Z},n-2);\mathbb{Q})$, but the first part of the tensor product is $H^p(S^{2n-1};\mathbb{Q})$, so we have exactly the same case as when n was odd. Thus, $H^*(\widetilde{F};\mathbb{Q})$ vanishes for the same reason. ⊠

Though this proof looks frightful, what happened was that we set up the Serre spectral sequence, and then took the one fact we knew, that F is $(n-1)$-connected, and used it to set up just enough calculations to figure things out.

Again, it's incredible, even though the Serre spectral sequence is a big machine, it's human-understandable (and even admits a really quick proof with simplicial sets), and surprisigly few techniques are needed. We do need that the homotopy groups of the spheres are finitely generated, but this isn't too bad either.

Rational Spaces and Localization. Now that we've done some rational calculations and seen that they're easier than regular ones, we can see that rational equivalence might be a nicer idea (which is true; the category ends up being completely algebraic).

Almost all of the theory from here on out will deal with simply connected spaces. This is because π_1 of non-simply-connected groups might not be abelian, and so we usually can't tensor it with $\mathbb{Q}$. You can sort of do this with nilpotent groups, but we won't delve into that.

Definition. A simply connected space X is called $\mathbb{Q}$-*local* or a $\mathbb{Q}$-*space* if

(1) $\pi_*(X)$ is a $\mathbb{Q}$-vector space, or
(2) $H_*(X;\mathbb{Z})$ is a $\mathbb{Q}$-vector space.

Proposition 3.4. *The above two criteria are equivalent.*

Proof. Once again, we'll use Postnikov towers and the Serre spectral sequence. It's almost boring, but it's also interesting how useful they are.

For homology implying homotopy, start induction at $X_2 = K(\pi_2(X),2)$. We know $H_*(X_2,\mathbb{Z})$ is a $\mathbb{Q}$-vector space, by computation of $H^*(K(\mathbb{Q},n);\mathbb{Z})$.

The inductive step looks at the fibration

$$\begin{tikzcd} K(\pi_n(X),n) \arrow[r] & X_n \arrow[d] \\ & X_{n-1}. \end{tikzcd}$$

Then, $E_2^{p,q} = H_p(X_{n-1};H_q(K(\pi_n(X),n)))$, but all of these are $\mathbb{Q}$-vector spaces, and therefore $H_*(X_n)$ is a $\mathbb{Q}$-vector space; then, take the limit.

The other direction isn't any more interesting; we assume $H_*(X;\mathbb{Z})$ is a $\mathbb{Q}$-vector space. Suppose we know that $\pi_*(X_{n-1})$ is a $\mathbb{Q}$-vector space. Then, $H_*(X_{n-1};\mathbb{Z})$ is, and so is $H_*(X,X_{n-1})$, and $\pi_n(X) = H_{n+1}(X,X_{n-1})$. ⊠

This is what we mean by $\mathbb{Q}$-local. In other words, there's only field-level information, and there's no torsion.

Definition. For "nice" topological spaces X and Y, we say that X and Y are $\mathbb{Q}$-*equivalent* if there's a map $X \to Y$ that induces either

(1) $H_*(X;\mathbb{Q}) \xrightarrow{\sim} H_*(Y;\mathbb{Q})$ or
(2) $\pi_*(X)\otimes\mathbb{Q} \xrightarrow{\sim} \pi_*(Y)\otimes\mathbb{Q}$.

Equivalently, the fiber of $X \to Y$ is $\mathbb{Q}$-*trivial* (also called $\mathbb{Q}$-*acyclic*, meaning it has no rational homology or homotopy).

We'll want to introduce the notion of a localization, producing a map from a space into a simpler topological space that preserves all the information over $\mathbb{Q}$.

Definition. A map $j : X \to X_{\mathbb{Q}}$ with $X_{\mathbb{Q}}$ a $\mathbb{Q}$-local space is a *localization* if one of the following is true:

(1) $j_* : H_*(X;\mathbb{Q}) \to H_*(X_{\mathbb{Q}};\mathbb{Q})$ is an isomorphism;
(2) $j_* : \pi_*(X) \otimes \mathbb{Q} \to \pi_*(X_{\mathbb{Q}}) \otimes \mathbb{Q}$ is an isomorphism; or
(3) j is universal in that if Y is $\mathbb{Q}$-local and $f : X \to Y$ is continuous, then there exists a unique $f' : X_{\mathbb{Q}} \to Y$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "j"'] & & Y \\ & X_{\mathbb{Q}} \arrow[ur, "f'"'] & \end{tikzcd}$$

We'll be able to construct a localization for any X by inducting on a Postnikov tower, tensoring everything with $\mathbb{Q}$.

Lemma 3.5. *Let G be an abelian group and $K(G,n)$ be an Eilenberg-MacLane space for it. Then, its localization is $K(G,n) \to K(G \otimes \mathbb{Q}, n)$.*

Proof. Follows from the definitions. $\boxtimes$

Theorem 3.6. *If X is a simply connected topological space, then there exists a localization $X \to X_{\mathbb{Q}}$.*

Proof. We'll induct on Postnikov towers; assume we've created a localization $X_{n-1} \to (X_{n-1})_{\mathbb{Q}}$, and we'd like to construct an $(X_n)_{\mathbb{Q}}$. How do we do that? We know how to construct X_n out of X_{n-1}: it fits into a homotopy fiber square

$$\begin{tikzcd} X_n \arrow[d] & \\ X_{n-1} \arrow[r, "k_n"'] & K(\pi_n(X), n+1), \end{tikzcd}$$

where k_n is a k-invariant. Compose this map with our localization of $K(\pi_n(X), n+1)$, so by the universal property of localizations, we get maps $X_{n-1} \to (X_{n-1})_{\mathbb{Q}} \to K(\pi_n(X) \otimes \mathbb{Q}, n+1)$; then, this pulls back up the fiber diagram, giving us an $(X_n)_{\mathbb{Q}}$. $\boxtimes$

At the start of the next lecture, knowing that localization of spheres exists produces a much more conceptual calculation of the homotopy groups of spheres.

Example 3.7. Chern classes give us maps $\varphi : BU \to \prod_k K(\mathbb{Z}, 2k)$; representability says that a map $BU \to K(\mathbb{Z}, 2k)$ is equivalent to a class in $H^{2k}(BU)$. Thus, computation of $H^*(BU, \mathbb{Q})$ tells us that φ^* is an isomorphism on cohomology, and therefore $BU_{\mathbb{Q}} \simeq \prod_k K(\mathbb{Q}, 2k)$. In particular, $\Omega^2 BU_{\mathbb{Q}} \simeq BU_{\mathbb{Q}}$. In other words, this is rational Bott periodicity.

Lecture 4.

Commutative Differential Graded $\mathbb{Q}$-algebras and Model Categories: 9/8/15

First, we'll talk a little bit about localization; then, we'll move to something completely different.

Proposition 4.1. $S^{2k+1}_{\mathbb{Q}}$ *is a* $K(\mathbb{Q}, 2k+1)$*, and there's a fibration* $K(\mathbb{Q}, 4k-1) \to S^{2k}_{\mathbb{Q}} \to K(\mathbb{Q}, 2k)$.

Once again, we have an even and an odd case.

Proof. $S^{2k+1}_{\mathbb{Q}} \to K(\mathbb{Q}, 2k+1)$ induces an isomorphism on homology, and therefore rational homotopy.

For the fibration, consider $S^{2k}_{\mathbb{Q}} \to K(\mathbb{Q},2k)$ acting on $H_{2k}(-)$. Then, we get a fiber $F \to S^{2k}_{\mathbb{Q}} \to K(\mathbb{Q},2k)$; to figure out what the fiber is, we'll use the Serre spectral sequence.

$$\begin{array}{r|cccc}
4k-1 & \mathbb{Q} & \mathbb{Q} & & \\
 & & & & \\
 & \mathbb{Q} & \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \\
\hline
 & 0 & 2k & 4k & 6k
\end{array}$$

Then, $E^2_{p,q} = H^*(F) \otimes H^*(K(\mathbb{Q},2k))$, so $H^*(F;\mathbb{Q}) = H^*(S^{4k-1}_{\mathbb{Q}})$. That is, F is an $M(\mathbb{Q},4k-1)$, but Moore spaces and Eilenberg-MacLane spaces are the same in rational homotopy. ⊠

Remark. A *Moore space* is the homology analogue of an Eilenberg-MacLane space; an $M(G,n)$ has homology equal to G in degree n and 0 otherwise.

Right now, we have lots of nice notions of equivalence in topology, including homotopy equivalence, weak homotopy equivalence, and rational homotopy equivalence. We don't have that going on for commutative differential graded algebras. The reason to introduce the abstraction of model categories is to allow for very natural notions of homotopy in categories such as this one.

Definition. A *commutative differential graded algebra* is a graded algebra A^* over a field k, i.e. $A^* = \bigoplus_{p\geq 0} A^p$ with

(1) a differential $d : A^* \to A^{*+1}$ with $d^2 = 0$ and $d(ab) = da \cdot b + (-1)^{\deg a} a \cdot db$, and
(2) a multiplication $A^p \otimes A^q \to A^{p+q}$ such that $ab = (-1)^{\deg a \deg b} ba$.

You could formalize this as a chain complex, but we've chosen cochain complexes. The idea is that the multiplication and differential should both respect the grading, and otherwise it's exactly what the name suggests. Moreover, a morphism of CDGAs is a morphism of k-algebras commiting with the differential and grading. Thus, we get a category of k-CDGAs.

In our case, of course, we'll usually take $k = \mathbb{Q}$. In this case, we'll more or less get a categorical equivalence between this category and the category of rational homotopy types, which is incredible!

Example 4.2.

(1) An important example over $\mathbb{R}$ is $\Omega^*(X;\mathbb{R})$ when X is a manifold, the algebra of differential forms. Here, multiplication is given by the wedge product.
(2) Over $\mathbb{Q}$, we could take $\mathbb{Q}[x] \otimes \Lambda(y)$, where $dx = y$, $\deg x = 1$, and $\deg y = 2$. Then, $d(x^n y) = 0$, which you can check, and that $H^*(-) = k$ (i.e. taken with d), and this field k has degree 0. This algebra is, in some mysterious sense we'll clarify, "the interval;" notice at least that it has similar cohomology.
(3) The Koszul complex $K(x_1,\dots,x_n) = \Lambda(x_1) \otimes \cdots \otimes \Lambda(x_n)$, with a complicated mess of differentials.
(4) Another great example, but one that is important, is cohomology $H^*(X;\mathbb{Q})$ with trivial differential. This is important because equivalences between CDGAs will be quasi-isomorphisms, which can send nontrivial differentials to trivial ones. Things quasi-isomorphic to CDGAs with trivial differential are called *formal*, and include important classes such as the rational homology of Kähler manifolds.

A significant non-example is $C^*(X;\mathbb{Q})$. This is because multiplication isn't commutative. In some cases, e.g. finite fields, Steenrod operations have nice structure, and one generally hopes that cochains are a good representative for your space, containing lots of information about your space; so it would've been nice if there was a commutative structure akin to $\Omega^*(M;\mathbb{R})$ for manifolds M. Thus, it doesn't work, but its failure to work is motivational.

Moreover, there isn't really a "free" CDGA, only semi-free ones; the issue is that you cannot freely choose the differential. But given a graded vector space V^*, the semi-free CDGA is

$$F(V^*) = k[V_{\mathrm{even}}] \otimes \Lambda(V_{\mathrm{odd}}).$$

We have a forgetful functor from CDGAs over $\mathbb{Q}$ to $\mathbb{Q}$-chains and then to $\mathbb{Q}$-graded vector spaces; this isn't adjoint to the latter, but to the former. Note that the category of CDGAs is probably abelian; certainly, it has kernels, and all of the structure that we need.

We implicitly used the cohomology in Example 4.2; here's the explicit definition.

Definition.

- The *cohomology* of $A^\bullet$, denoted $H^*(A^\bullet)$, is defined by $H^n(A^\bullet) = (\ker d : A^n \to A^{n+1})/(\operatorname{Im} d : A^{n-1} \to A^n)$.
- Let C^* and D^* be CDGAs, with a morphism $f : C^* \to D^*$, and define $M_f^n = C^n \oplus D^{n-1}$ with differential $d_M : M_f^n \to M_f^{n+1}$ given by $\begin{pmatrix} d_C & 0 \\ f & -d_D \end{pmatrix}$. Then, the *relative cohomology* is $H^*(D^\bullet, C^\bullet) = H^*(M_f^\bullet)$.

Notice that we have things like mapping cylinders, relative cohomology, etc., so CDGAs do look somewhat like topological spaces, which will be nice.

Lemma 4.3. *There exists a long exact sequence*

$$\cdots \longrightarrow H^n(C^\bullet) \longrightarrow H^n(D^\bullet) \longrightarrow H^{n+1}(D^\bullet, C^\bullet) \longrightarrow H^{n+1}(C^\bullet) \longrightarrow \cdots$$

Definition. $f : C^* \to D^*$ is a *quasi-isomorphism* if $H^*(f)$ is an isomorphism.

What we'll do now is spend some time codifying what it means for a category to have homotopy theory; that both rational homotopy and CDGAs live in the same world. This is because both topological spaces and CDGAs are model categories.

If you do much more homotopy theory, model categories will be really useful. Even where they're not explicitly talked about, they'll clarify ideas and thoughts in papers and books.

A model category can be thought of as a model for homotopy; in such a category, there will be things that act like cylinders (or, dually, paths), and therefore we can speak of when objects are homotopic. This will follow from some impressively minimal axioms, because Quillen was extremely clever. We'll define the category, and then show that these induce notions of homotopy equivalence and weak equivalence, and some correspondences between the two.

First, recall a category-theoretic definition.

Definition. Let $f : A \to B$ and $g : A' \to B'$ be morphisms in a category $\mathscr{C}$. Then, f is said to be a *retract* of g if the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r] \arrow[rr, bend left, "\mathrm{id}"] \arrow[d, "f"] & A' \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\
B \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & B' \arrow[r] & B
\end{tikzcd}$$

Definition. A *model category* is a category $\mathscr{C}$ closed under small[3] limits and colimits, together with three subcategories:

- $\mathrm{co}(\mathscr{C})$ (sometimes denoted $\mathrm{cof}(\mathscr{C})$), the subcategory of *cofibrations*,
- $\mathrm{fib}(\mathscr{C})$, the subcategory of *fibrations*, and
- $w(\mathscr{C})$, the subcategory of *weak equivalences*.[4]

We'll also call $w(\mathscr{C}) \cap \mathrm{co}(\mathscr{C})$ the *trivial cofibrations* (as in homotopically trivial), and $w(\mathscr{C}) \cap \mathrm{fib}(\mathscr{C})$ the *trivial cofibrations*. If $f : A \to B$ is a morphism in $\mathrm{co}(\mathscr{C})$, then it will often be denoted $f : A \hookrightarrow B$, and similarly if $f \in \mathrm{fib}(\mathscr{C})$, then it'll be denoted $f : A \twoheadrightarrow B$. Weak equivalences will be denoted with $\simeq$.

This data is subject to the following axioms.

(1) The *two-out-of-three axiom*: if $f : A \to B$ and $g : B \to C$ are morphisms in $\mathscr{C}$ such that two of gf, g, and gf are in $w(\mathscr{C})$, then the third is.
(2) Retracts of cofibrations, fibrations, and weak equivalences are cofibrations, fibrations, and retracts, respectively; that is, each of these three categories is closed under retracts.

[3]We really don't care about set-theoretic issues in this class, as evidenced when the first day didn't begin with "In the beginning, there was a Grothendieck universe." We'll primarily use finite limits and colimits, and requiring pushouts to exist is good enough for us.

[4]Sometimes $w(\mathscr{C})$ is called the "water closet," albeit not in serious literature.

(3) The *lifting axiom*: let $i : A \to B$ be a trivial cofibration and $p : X \twoheadrightarrow Y$ be a fibration. If the following diagram commutes, then the following dotted arrow exists.

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook, "i", "\simeq"'] & X \arrow[d, two heads, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Similarly, if i is a cofibration and p is a trivial fibration, then the following dotted arrow exists.

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook, "i"'] & X \arrow[d, two heads, "p"', "\simeq"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

(4) *Functorial factorization*: Every morphism $f : X \to Y$ can be factored as $X \hookrightarrow Y' \overset{\sim}{\twoheadrightarrow} Y$, where the first map is in $\mathrm{co}(\mathscr{C})$ and the second in $w(\mathscr{C}) \cap \mathrm{fib}(\mathscr{C})$; it can also be factored as $X \overset{\simeq}{\hookrightarrow} X' \twoheadrightarrow Y$, where the first map is a trivial fibration and the second is a cofibration,. Moreover, these factorizations are functorial in that (for the first factorization; the second is similar) there exists a functor $(F, G) : \mathrm{Mor}\,\mathscr{C} \to \mathrm{Mor}\,\mathscr{C} \times \mathrm{Mor}\,\mathscr{C}$ such that $F(f)$ is a cofibration, $G(f)$ is a trivial fibration, and $G(f) \circ F(f) = f$.

Remark. $\mathscr{C}$ has an initial and final object, since pushouts exist. These will be denoted 0 and $*$, respectively.

The lifting axiom looks pretty specifically homotopical, but it's still surprisingly minimal for all of this homotopy theory to exist in a category. And functorial factorization is basically cellular approximation.

It should ease your mind that topological spaces form a model category, with fibrations given by fibrations, cofibrations given by cofibrations, and weak equivalences given by weak homotopy equivalences. This is nonetheless hard to show; simplicial sets are also a category, but this is even harder to show.

Definition.

- An object B is called *cofibrant* if $0 \to B$ is a cofibration.
- Similarly, Y is called *fibrant* if $Y \to *$ is a fibration.

These objects tend to have nicer properties. Moreover, every object is weakly equivalent to a cofibrant and a fibrant object.

Definition. Given any object B, $0 \to B$ factors as $0 \hookrightarrow QB \overset{\sim}{\twoheadrightarrow} B$, so that $QB \simeq B$ and QB is cofibrant. QB is called the *cofibrant replacement* of B.

Similarly, $Y \to *$ factors as $Y \overset{\sim}{\hookrightarrow} RY \twoheadrightarrow *$, with $RY \simeq Y$ and RY fibrant; this RY is called the *fibrant replacement* of Y.

There are lots of terms here, but we'll need to use all of them to talk about stuff. Topological spaces are already all fibrant, so intuition can be challenging here.

The lifting property means that fibrations and cofibrations mutually define each other, so there's sort of too much information in these problems.

Definition. Let $X \in \mathscr{C}$ be an object. Then, a *cylinder* for X, denoted $\mathrm{Cyl}(X)$, is a factorization of $X \sqcup X \to X$ as $X \sqcup X \hookrightarrow \mathrm{Cyl}(X) \xrightarrow{\sim} X$.

Remark. The above factorization need not be the functorial factorization guaranteed by our category. But we can use that, and therefore there's a cylinder functor.

The abstract definition can be realized topologically as something weakly equivalent to X (which is true of $X \times [0,1]$), but that fills in the area between two copies of X.

We'll see that everything in model categories has a dual. The notion of a path object is dual to the cylinder.

Definition. If $Y \in \mathscr{C}$, then a *path object* for Y, denoted PY, is one that arises from a factorization $Y \xrightarrow{\sim} PY \twoheadrightarrow Y \times Y$ of the diagonal map $Y \to Y \times Y$.

Again, this can be made functorial. This notion, making a space into a path space, also appears in homotopy theory. But we can go the other way, making avatars of our familiar notions in homotopy theory inside a more general model category. For example, we'll define a homotopy!

Note: next lecture is next Tuesday, so don't come to class on Thursday.

Lecture 5.

Homotopy in Model Categories: 9/15/15

We've defined cylinders and path objects in model categories, and today will define homotopies in two flavors, left and right. The next few lectures will be some relatively boring but useful abstract manipulations in model categories; then we'll get to some cool stuff. But you have to take your medicine.

Throughout this lecture, $\mathscr{C}$ will denote a model category.

Definition. Let $f_0, f_1 : X \to Y$ be maps in $\mathscr{C}$.

(1) f_0 is *left homotopic* to f_1, written $f_0 \simeq_\ell f_1$, if there exists an $H : \mathrm{Cyl}(X) toY$ such that $H \circ i_0 = f_0$ and $H \circ i_1 = f_1$, where $i_0 \sqcup i_1 : X_1 \sqcup X_1 \hookrightarrow \mathrm{Cyl}(X)$.
(2) f_0 is *right homotopic* to f_1, written $f_0 \simeq_r f_1$, if there is a map $K : X \to PY$ such that $p_0 K = f_0$ and $p_1 K = g$, where $p_0 \times p_1 : PY \to Y \times Y$.
(3) f_0 is *homotopic* to f_1, written $f_0 \simeq f_1$, if $f_0 \simeq_\ell f_1$ and $f_0 \simeq_r f_1$.
(4) $f : X \to Y$ is a *homotopy equivalence* if there exists a $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$.

This is at once very abstract and somewhat familiar from topology; we'll show that in nice case left and right homotopy are identical, which is why we don't usually talk about them as distinct in topology. The usual topological notion is usually defined as left homotopy, but homotopies of chain complexes are examples of right homotopies.

Again, this isn't the most earthshattering definition, but it's a useful model to hold in your head. Nonetheless, the following results aren't super exciting to prove.

Lemma 5.1.

(1) Let $f, g : B \to X$ be maps in $\mathscr{C}$. If $f \simeq_\ell g$ and $h : X \to Y$ is any other map, then $h \circ f \simeq_\ell h \circ g$.
(2) If $e : A \to B$ is a map and $f \simeq_r g$, then $f \circ e \simeq_r g \circ e$.

Proof. Since $f \simeq_\ell g$, then there's an $H : \mathrm{Cyl}(B) \to X$ with $H i_0 = f$ and $H i_1 = g$. Then, $h \circ H : \mathrm{Cyl}(B) \to X \to Y$ is a left homotopy between $h \circ f$ and $h \circ g$. (2) is similar. ⊠

A lot of the work we'll be doing will be to get rid of these "handedness issues."

Proposition 5.2. *Suppose $f, g : B \to X$ are two maps in $\mathscr{C}$.*

(1) If X is fibrant, $f \simeq_\ell g$, and $h : A \to B$ is any map, then $f \circ h \simeq_\ell g \circ h$.
(2) If B is cofibrant, $f \simeq_r g$, and $e : X \to Y$ is any map, then $e \circ f \simeq_r e \circ g$.

Proof. We'll prove (1); then, (2) is similar.

We have a left homotopy $H : \mathrm{Cyl}(B) \to X$ where $B \sqcup B \hookrightarrow \mathrm{Cyl}(B) \xrightarrow{\sim} B$. We'd prefer a map $\mathrm{Cyl}(B) \overset{\twoheadrightarrow}{\to} B$ (i.e. a trivial fibration).

Factor $\mathrm{Cyl}(B) \xrightarrow{\sim} B$ as $\mathrm{Cyl}(B) \hookrightarrow \mathrm{Cyl}'(B) \overset{\sim}{\twoheadrightarrow} B$; by the two-out-of-three rule, $\mathrm{Cyl}(B) \hookrightarrow \mathrm{Cyl}'(B)$ is a weak equivalence, and so we have maps

$$\begin{tikzcd} B \sqcup B \arrow[r, hook] \arrow[rr, bend right] & \mathrm{Cyl}(B) \arrow[r, hook] & \mathrm{Cyl}'(B) \arrow[r, two heads, "\sim"] & B, \end{tikzcd}$$

which presents $\mathrm{Cyl}'(B)$ as a cylinder object.

The homotopy H can be considered to be a map $H' : \mathrm{Cyl}'(B) \to X$: by two-out-of-three $\mathrm{Cyl}(B) \overset{\sim}{\hookrightarrow} \mathrm{Cyl}(B')$ is a trivial cofibration. Since X is fibrant, we can draw the diagram

$$\begin{tikzcd} \mathrm{Cyl}(B) \arrow[r] \arrow[d, hook] & X \arrow[d] \\ \mathrm{Cyl}'(B) \arrow[r] \arrow[ur, dashed, "H'"] & *, \end{tikzcd}$$

so by the path lifting property we get the desired H'. All of this was just to move the homotopy to come out of a different object!

Now, we can form the diagram

$$\begin{tikzcd}
A \sqcup A \arrow[r, "h \sqcup h"] \arrow[d, hook] & B \sqcup B \arrow[r] & \mathrm{Cyl}'(B) \arrow[r] \arrow[d, two heads, "\simeq"] & X \\
\mathrm{Cyl}(A) \arrow[r] \arrow[urr, dashed] & A \arrow[r, "h"'] & B.
\end{tikzcd}$$

This completes the proof, because it gives us a map $\mathrm{Cyl}(A) \to X$ through which h factors. ☒

These proofs look very similar: we don't have very many rules. We can factor and compose, and not very much more.

As an aside, we'll prove Ken Brown's lemma.[5] It's invoked a lot, often even without a name. Note that no structure on F is assumed below.

Lemma 5.3 (Ken Brown)**.** *Let $\mathscr{C}$ and $\mathscr{D}$ be model categories and $F : \mathscr{C} \to \mathscr{D}$ be a functor. Suppose F takes trivial cofibrations between cofibrant objects to weak equivalences; then, F takes weak equivaelences between cofibrant objects to weak equivalences.*

This seems small, but is extremely useful: if we know what F does to trivial cofbirations on cofibrant objects, we recover it on all weak equivalences of cofibrant objects. The proof is a fun game of model-categoric axioms.

Proof. Suppose $f : A \xrightarrow{\sim} B$ is a weak equivalence of cofibrant objects. Consider $A \sqcup B \to B$ given by $f \sqcup \mathrm{id}$, and factor it as

$$\begin{tikzcd}
A \sqcup B \arrow[r, hook, "p"'] & C \arrow[r, two heads, "\sim", "q"'] & B.
\end{tikzcd}$$

Consequently, we have

$$\begin{tikzcd}
0 \arrow[r, hook] & A \arrow[r, hook] \arrow[rrr, bend right, "\sim"'] & A \sqcup B \arrow[r, hook] & C \arrow[r, "\simeq"] & B
\end{tikzcd} \tag{5.1}$$

by the 2/3 rule, $A \hookrightarrow C$ is a trivial cofibration, since C is a cofibrant object.

By assumption, $F(A \overset{\sim}{\hookrightarrow} C)$ is a weak equivalence, but $F(5.1) = F(\mathrm{id})$. By 2/3, $B \hookrightarrow C$ is a weak equivalence, and therefore it's a trivial cofibration between cofibrant objects. Then, since $F(B \hookrightarrow C) \cdot F(C \overset{\sim}{\twoheadrightarrow} B) = F(\mathrm{id})$, then $F(C \overset{\sim}{\twoheadrightarrow} B)$ is a weak equivalence. Thus, $F(A \hookrightarrow A \sqcup B \hookrightarrow C \to B) = F(A \xrightarrow{\sim} B)$, so it factors as $F(A \hookrightarrow C) \circ F(C \twoheadrightarrow B)$; both of these are weak equivalences, and therefore the composite is to. ☒

Again, we'll get back to rational homotopy theory soon. But that lemma will be useful.

Lemma 5.4. *Let B be cofibrant in $\mathscr{C}$. Then, left homotopy is an equivalence relation in* $\mathrm{Map}_{\mathscr{C}}(B, X)$

We're working towards a goal: we've defined homotopy equivalence and weak equivalence in unrelated ways, and we want to prove that they're related; in the category of topological spaces, this result is just as consciously proven from differing definitions, and is known as Whitehead's theorem.

Proof. We'll leave reflexivity and symmetry to the reader, and they're not hard.

Suppose $f \simeq_\ell g$ and $g \simeq_\ell h$, where $f, g, h : B \to X$ via homotopies $H_{f,g} : B' \to X$ and $H_{g,h} : B'' \to X$, where B' and B'' are cylinder objects.

We want to "glue" these two cylinders together, and do so in topology. But in category theory we use pushouts. So let's consider the following diagram.

$$\begin{tikzcd}
B \arrow[r, hook, "i''_0"] \arrow[d, hook, "i'_1"'] & B' \arrow[d] & B \arrow[l, "i''_1"'] \\
B' \arrow[r] & L & \\
B \arrow[u, "i'_0"] & &
\end{tikzcd}$$

[5]People older than Ken Brown call it Kenny Brown's lemma. Never Kenny's lemma.

By the universal property of pushouts, we get a map $\widetilde{H} : L \to X$. However, since L might not be a cylinder object, then $\widetilde{H}$ might not be a homotopy. We solve this the only way we know how: by factoring stuff.

We do know that $B \sqcup B \to K \xrightarrow{\sim} B$, and factor $B \sqcup B \to K$ as $B \sqcup B \hookrightarrow L' \tilde{B}$, so L' is a cylinder object and therefore gives us a homotopy $\widetilde{H}' : L' \to X$. ⊠

Lemma 5.5. *Dually, if X is fibrant, then right homotopy is an equivalence relation on* $\mathrm{Map}_{\mathscr{C}}(B,X)$.

We're in the middle of proving that weak equivalences and homotopy equivalences are reasonable things, and behave the way we want them to.

Lemma 5.6. *Let B be cofibrant and $h : X \to Y$ be a trivial fibration* or *a weak equivalence between fibrant objects. Then,* $\mathrm{Map}_{\mathscr{C}}(B,X)/\simeq_\ell \xrightarrow{\cong} \mathrm{Map}(B,Y)/\simeq_\ell$.

Proof. The equivalence of the two hypotheses on h follows from Ken Brown's lemma, so we'll suppose $h : X \overset{\sim}{\twoheadrightarrow} Y$ is a trivial fibration.

For surjectivity, supose $g : B \to Y$ is given. Then, we can lift it in the following diagram.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, hook] & X \arrow[d, two heads, "\sim"] \\
B \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

Injectivity isn't much harder. Suppose $hf \simeq_\ell hg$ for two $f, g : B \to X$, and this left homotopy is via $H : \mathrm{Cyl}(B) \to Y$. Then, we can lift in the following diagram, which provides a homotopy between f and g.

$$\begin{tikzcd}
B \sqcup B \arrow[r, "f \sqcup g"] \arrow[d, hook] & X \arrow[d, two heads, "\sim"] \\
\mathrm{Cyl}(B) \arrow[r, "H"'] \arrow[ur, dashed] & Y
\end{tikzcd}$$

⊠

There's a dual statement, given by switching fibrant and cofibrant, and precomposition and postcomposition. It turns out that model categories are sort of self-dual. The proof is the same, with path objects instead of cylinder objects. This means that a lot of books on model cateogories could often be half as long! Hirshhorn is particularly representative of this.

The next lectures will return to using this in rational homotopy theory, including showing that chain complexes are a model category.

Lemma 5.7. *Let $f, g : B \to X$. Then, if B is cofibrant, then $f \simeq_\ell g$ implies $f \simeq_r g$, and dually, if X is fibrant, then $f \simeq_r g$ implies $f \simeq_\ell g$.*

In topological spaces and chain complexes, everything is fibrant, so we're not used to having two kinds of homotopy, and in simplicial sets, everything is cofibrant. So secretly there's two kinds of homotopy, but we haven't needed to use them.

Another way to think about Lemma 5.7 is as a generalization of the adjunction between cylinders and path spaces on the category of topological spaces.

Definition. A *fibrant-cofibrant* object of $\mathscr{C}$ is, unsurprisingly, an object of $\mathscr{C}$ that's both fibrant and cofibrant. The subcategory of fibrant-cofibrant objects and their maps is denoted $\mathscr{C}_{\mathrm{cf}}$.

These are the "nice" objects, where homotopy equivalence is at its nicest, and all the lemmas we've given so far apply.

The following theorem has a somewhat boring proof, which can be found in the lecture notes, but wasn't given in class.

Theorem 5.8. *If $\mathscr{C}$ is a model category, then a map in $\mathscr{C}_{\mathrm{cf}}$ is a weak equivalence iff it is a homotopy equivalence.*

This is really important, and here are some consequences.

- We can take the category $\mathscr{C}_{\mathrm{cf}}/\sim$, modding out by weak equivalence. There are weird set-theoretic issues in general unless we take an equivalence relation (which, in this case, weak equivalence is).

- The homotopy category $\mathrm{Ho}(\mathscr{C})$, defined by formally inverting weak equivalences, is equivalent as categories to $\mathscr{C}_{\mathrm{cf}}/\sim$. This is why model categories are nice: homotopy theory is in some sense about notions of equivalence, and this equates two of them.

Exercise 2. Read about ordinals and transfinite composition, and be very uncomfortable with the foundations of mathematics, as you should be.

This will be used next time for the small object argument, but it's not fun to talk about in lecture. We'll be able to actually make a model category, though! And you'll realize that attaching cells in infinite constructions in algebraic topology requires ordinals too.

Lecture 6.

Today is the Cofibrantly Generated Model Categories Day: 9/17/15

"Now we get to do the small object argument. It's fun! I can see you share my enthusiasm, but I'm enjoying myself, and that's all that matters here."

Cofibrantly generated model categories are nice; they can be built and realized relatively nicely; by showing that the category of differential graded $\mathbb{Q}$-algebras is a cofibrantly generated model category, we'll be able to do some stuff faster.

We'll start off with some fun stuff about the sizes of... things. We'll consider ordinals as categories by analogy, with $a \to b$ if $a \le b$, so it makes sense to talk about functors from ordinals to categories.

Definition. Let λ be an ordinal. Then, a *λ-sequence* in a category $\mathscr{C}$ is a colimit-preserving functor $X : \lambda \to \mathscr{C}$. Then, the map $X_0 \to \varinjlim_{\beta<\lambda} X_\beta$ is called a *transfinite composition*.

That is, λ has a diagram associated to it, and X preserves this diagram.

It may be boring to play around with set theory like this, but we make infinite constructions like this (e.g. infinite cell complexes) in topology all the time, and we want to know that we're not breaking set theory.

Definition. Let κ be a cardinal. Then, an ordinal λ is *κ-filtered* if

(1) λ is a limit ordinal, and
(2) for any $S \le \lambda$ such that $|S| < \kappa$, $\sup(S) < \lambda$.

This is a weird definition, but if κ is finite, being, for example, 10-filtered means being an infinite limit ordinal (a supremum of 10 things is finite, but could be arbitrarily large).[6]

Definition. Let $\mathscr{C}$ be a category with small colimits (i.e. colimits indexed by sets) and $M \subseteq \mathrm{Mor}(\mathscr{C})$ be a collection of morphisms. We then say that $A \in \mathscr{C}$ is *κ-small with respect to M* if for any κ-filtered ordinal λ and $X : \lambda \to \mathscr{C}$ with $X_\beta \to X_{\beta+1}$ in M, we have that as sets,

$$\mathrm{colim}_{\beta<\lambda} \mathrm{Map}_{\mathscr{C}}(A, X_\beta) \xrightarrow{\cong} \mathrm{Map}_{\mathscr{C}}(A, \mathrm{colim}_{\beta<\lambda} X_\beta).$$

This is really saying that a map $A \to \mathrm{colim}\, X_\beta$ factors through some X_β.

Exercise 3. Think about why this is equivalent to the definition.

Remark. If we don't specify κ, then saying that A is *small* with respect to M implies that there exists some κ making it true, and if $M = \mathrm{Mor}(\mathscr{C})$, we simply say that A is *small* (sometimes *compact*).

This will allow us to get control over colimits, which is important.

Definition. Let I be a class of maps in $\mathscr{C}$. We say that $f \in \mathrm{Mor}(\mathscr{C})$ is *I-injective* if f has the right lifting property with respect to I, i.e. $I \oslash^R f$.[7] Similarly, we say that f is an *I-cofibration* if $f \oslash^L A$ whenever A is I-injective.

That is, $I \oslash^R I$-inj., and I-cof $\oslash^L I$-inj.

The next definition looks reasonably topological, but is abstracted enough to be useful for us.

[6] Recall that we built ordinals by von Neumann's model: $1 = \{\bullet\}$, $2 = \{\bullet, \{\bullet\}\}$, and in general $\mathrm{succ}(S) = S \cup \{S\}$; then, the first infinite ordinal is the union of all of the finite ones, and therefore is called a *limit ordinal*, i.e. it's not the successor of some ordinal.

[7] This should really be a box with a slash, but I get package conflicts when I try to make that work.

Definition. If $I \subseteq \mathrm{Mor}(\mathscr{C})$ has all small colimits, then a *relative I-cell complex* is a transfinite composition of pushouts along I of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\varphi"] & X \arrow[d] \\ B \arrow[r] & X \sqcup_A B, \end{tikzcd}$$

where $\varphi \in I$.

Exercise 4. Show that I-cells are I-cofibrant.

The next argument, though still formal, is at least a little closer to topology or model categories.

Theorem 6.1. *Let $\mathscr{C}$ be a category with all small colimits and $I \subseteq \mathrm{Mor}(\mathscr{C})$, and suppose the domains of I are all small relative I-cells. Then, there is a functorial factorization*

$$(F,G) : \mathrm{Mor}(\mathscr{C}) \longrightarrow \mathrm{Mor}(\mathscr{C}) \times \mathrm{Mor}(\mathscr{C}),$$

such that for all f, $F(f)$ is a relative I-cell and $G(f)$ is I-injective.

That is, every map $A \to B$ factors as $A \to C \to B$, where the first is an I-cell and the second is I-injective.

Proof. This is the celebrated small object argument.

Assume there exists some κ such that all domains are κ-small, and let λ be a κ-filtered ordinal. Our goal will be to define a λ-sequence $Z^\lambda : \lambda \to \mathscr{C}$ such that $X \to \mathrm{colim}_{\beta<\lambda} Z^\lambda \to Y$.

Start with $Z_0 = Y$, and if we're defined Z_α for all $\alpha < \beta$, where β is a limit ordinal, then define $Z_\beta = \mathrm{colim}_{\alpha<\beta} Z_\alpha$. If β is a successor ordinal, let S be the set of *all* diagrams of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\varphi"] & Z_\beta \arrow[d] \\ B \arrow[r] & Y, \end{tikzcd}$$

with $\varphi \in I$, and define $Z_{\beta+1}$ to be the pushout

$$\begin{tikzcd} \coprod_{s\in S} A_s \arrow[r] \arrow[d] & Z_\beta \arrow[d] \\ \coprod_{s\in S} B_s \arrow[r] & Z_{\beta+1}. \end{tikzcd}$$

Then, let Z be the colimit of this sequence. There's obviously an induced $Z \to Y$, so we want to provide a lift in the following diagram.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\in I"'] & Z \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

but our map $A \to Z$ must factor through some Z_β. Then, smallness gives us a map $B \to Z_{\beta+1}$, and therefore into the colimit, which is our desired lift. ⊠

This seems pedantic, but if you want to make infinite constructions of cells, which you do all the time in homotopy theory, you need to make some kind of smallness. This is the category-theoretic statement that if you map a compact space into some cell complex, the image is compact.

The phrase "I permits the small object argument" means that the domains of I are all small relative I-cells (which is what made this argument work).

Corollary 6.2. *If I permits the small object argument and $f : A \to B$ is an I-cofibration, then there is an I-cell $g : A \to C$ such that f is a retract of*

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "g"'] & A \arrow[r, equal] \arrow[d, "f"'] & A \arrow[d, "g"'] \\ C \arrow[r] & B \arrow[r] & C \end{tikzcd}$$

That is, cofibrations are I-cells or retracts thereof. This will be important for characterizing cofibrations later, and the proof is example of an important kind of argument.

Proof. Factor $A \to B$ as $A \to C \to B$, where the intermediate maps are, respectively, an I-cell and I-injective. Then, we get the following diagram.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "I\text{-inj.}"] \\ B \arrow[ur, dashed] \arrow[r, equal] & B \end{tikzcd}$$

⊠

Remark. This is an example of the "retract argument."

Now, we'll give a whole bunch more definitions, avoiding some extremely dry, formal proofs. The upshot will be to show that $\mathbb{Q}$-CDGAs are a cofibrantly generated model category; then, over the next few lectures, we'll begin to build their homotopy theory, which will feel less formal and more topological.

Definition. $\mathscr{M}$ is a *cofibrantly generated model category* (CGMC) if there exist $I, J \subseteq \mathrm{Mor}(\mathscr{M})$ such that

- the domains of I and J are small with respect to the sets of I-cells and J-cells, respectively
- $\mathrm{fib}(\mathscr{M}) = J$-inj.
- $w(\mathscr{M}) \cap \mathrm{co}(\mathscr{M}) = I$-inj.

Such a structure will be denoted $(\mathscr{M}, w(\mathscr{M}), I, J)$. I is called the *generating cofibrations*, and J is called the *generating trivial cofibrations*.

The first point means that we can (and do!) use the small object argument on I and J.

Proposition 6.3. *Let $(\mathscr{M}, w(\mathscr{M}), I, J)$ be a CGMC. Then,*

(1) $\mathrm{co}(\mathscr{M}) = I$*-cof,*
(2) every cofibration is a retract of an I-cell, and
(3) the domains of I are small relative to cofibrations.

Moreover, all three statements hold with "trivial fibrations" in place of "cofibrations," and J replacing I.

Theorem 6.4. *Let $\mathscr{C}$ be a category with subcategories $w(\mathscr{C})$, I, and J. Then, $(\mathscr{C}, w(\mathscr{C}), I, J)$ is a CFMC if the following are true.*

(1) $w(\mathscr{C})$ has the $2/3$ property.
(2) I and J permit the small object argument.
(3) J-cell $\subseteq w(\mathscr{C}) \cap I$-cof.
(4) I-inj $\subseteq w(\mathscr{C}) \cap J$-inj.
(5) Either $w(\mathscr{C}) \cap I$-cof $\subseteq J$-cof or $w(\mathscr{C}) \cap I$-inj $\subseteq J$-inj.

You should look up a proof, though we won't prove it ourselves. The point is, if you prove a bunch of lifting properties, you get a CGMC more or less for free.

Let $\mathsf{Ch}_{\mathbb{Q}}$ denote the category of chain complexes over $\mathbb{Q}$. Let S^{n-1} denote a chain complex with $\mathbb{Q}$ in degree n and nothing else, and D^n denote one with degrees $n+1$ and n with $d : \mathbb{Q} \to \mathbb{Q}$ given by isomoprhism. Then, let $I = \{S^{n-1} \to D^n \mid n \geq 0\}$ and $J = \{0 \to D^n \mid n \geq 0\}$.

Theorem 6.5. $(\mathsf{Ch}_{\mathbb{Q}}, w(\mathsf{Ch}_{\mathbb{Q}}), I, J)$ *has a cofibrantly generated model category structure, where the weak equivalences are given by quasi-isomorphisms.*

Corollary 6.6. *The fibrations in* $\mathsf{Ch}_{\mathbb{Q}}$ *are surjections; the cofibrations are retracts of relative I-cells. In particular, the cofibrant objects are I-cellular objects, and this is illuminating in general: cofibrant objects are built out of small numbers of cells.*

Topological spaces, Top, form another important example. Let $I = \{S^{n-1} \to D^n \mid n \geq 1\}$ and $J = \{D^n \to D^n \times I \mid n \geq 0\}$, and let $w(\mathsf{Top})$ denote the weak homotopy equivalence (i.e. $X \xrightarrow{\sim} Y$ if $\pi_i(X) \cong \pi_i(Y)$ for all i).

Theorem 6.7. $(\mathsf{Top}, w(\mathsf{Top}), I, J)$ *is also a cofibrantly generated model category.*

A different category of topological spaces appears for the Quillen adjunction between topological spaces and simplicial sets.

You really should read the proofs, which are boring but instructive.

One cool thing is that, not only can we use adjunctions to induce model category structures, but in many cases it's really easy.

Theorem 6.8. *Let $F : \mathscr{C} \rightleftarrows \mathscr{D} : U$ be an adjunction between a CDMC $\mathscr{C}$ and $\mathscr{D}$ satisfying some closure axioms (e.g. $\mathscr{D}$ has all small colimits). If FI and FJ permit the small object argument and U takes relative FJ-cell complexes to weak equivalences, then $\mathscr{D}$ is a CGMC with (FI, FJ) as generating (trivial) cofibrations, and weak equivalences those maps $f \in \mathscr{D}$ such that $U(f) \in w(\mathscr{C})$.*

In practice, these conditions are quite easy to check, so we get a model category structure for free.

There is a free forgetful adjunction $\Lambda : \mathsf{Ch}_{\mathbb{Q}} \rightleftarrows \mathrm{CDGA}_{\mathbb{Q}} : U$ given by $\Lambda(C) = \Lambda_{\mathbb{Q}}[C^{\mathrm{odd}}] \otimes \mathbb{Q}[C^{\mathrm{even}}]$, with differential uniquely determined.

Theorem 6.9. *There is a model category structure on $\mathrm{CDGA}_{\mathbb{Q}}$ induced from $\mathsf{Ch}_{\mathbb{Q}}$ such that $I = \{\Lambda S^{n-1} \to \Lambda D^n \mid n \geq 1\}$, $J = \{0 \to \Lambda D^n \mid n \geq 0\}$, and the weak equivalences are quasi-isomorphisms. Moreover, cofibrations are retracts of relative I-cells and fibrations are degree-wise surjections.*

This tells us a lot about homotopies of CDGAs, and in particular the things that are well-behaved. Sometimes, people will avoid this high-level language, and "Sullivan algebra" is a word for a cofibrant object in some model category. Then, a lot of their manipulations to get things to factor come from stuff we've already done in model categories.

Lecture 7.

Homotopy Theory of CDGAs: 9/22/15

Today, we'll leverage all that we did over the past few days and develop the homotopy theory of $\mathbb{Q}$-CDGAs, from a relatively formal viewpoint that involves very little actually thinking about elements. All of this works over an arbitrary field, but we only care about $\mathbb{Q}$ in this class.

Recall that we defined S^{n-1} to be the chain $0 \to \cdots \to 0 \to \mathbb{Q} \to 0 \to \cdots$ and S^n to be $0 \to \cdots \to 0 \to \mathbb{Q} \to \mathbb{Q} \to 0 \to \cdots$, which are in $\mathsf{Ch}_{\mathbb{Q}}$. We also discussed (but didn't prove) that there is a CGMC structure on the category of $\mathbb{Q}$-CDGAs, such that the generating cofibrations are $\Lambda S^{n-1} \to \Lambda D^n$ and $0 \to \Lambda D^n$, with weak equivalences given by quasi-isomorphisms. Cofibrations are retracts of relative I-cell complexes and fibrations are degree-wise surjections.

Corollary 7.1. *The cofibrant $\mathbb{Q}$-CDGAs are exactly the I-cells.*

Corollary 7.2. *All $\mathbb{Q}$-CDGAs are fibrant, since they all map to the final object.*

Definition. A *Sullivan algebra* is a cofibrant object of $\mathrm{CDGA}_{\mathbb{Q}}$.

This is usually stated differently, less categorically. This means it's an I-cell, which means it's inductively built out of a series of iterated pushouts:

$$\begin{tikzcd} \coprod_n \Lambda S^{n-1} \arrow[r] \arrow[d] & \Lambda(k-1) \arrow[d] \\ \coprod_n \Lambda D^n \arrow[r] & \Lambda(k). \end{tikzcd}$$

Sullivan and Felix-Halperin-Thomas originally defined Sullivan algebras as iterated pushouts. The horizontal maps in some sense pick elements, and the vertical maps tell us how the differential is going to work. We may suppress the notation of the differential, so diagrams such as the above are understood to have a differential floating around.

Definition. A *Sullivan model* for a $\mathbb{Q}$-CDGA A is a Sullivan algebra ΛV and a quasi-isomorphism $\Lambda V \xrightarrow{\sim} A$.

This is a cofibrant replacement, pretty much by definition. But this gives us a neat category.

Corollary 7.3. *For all $A \in \mathrm{CDGA}_{\mathbb{Q}}$, there exists a Sullivan model for A.*

Since everything is fibrant, we don't need fibrant replacement, so we should define homotopy and lifting.

Definition. One often sees the notation $\Lambda(t,dt) = \Lambda D^1$.

You should think of this as the interval: it's the data of a $\mathbb{Q}\langle t\rangle$ in degree 0 and a $\mathbb{Q}\langle dt\rangle$ in degree 1, with d providing an isomoprhism between them. Thus, the homology groups vanish, so in some sense this is contractible.

Definition. Two maps $f,g : A \to B$ of $\mathbb{Q}$-CDGAs are *homotopic* if there is a map $H : A \to B\otimes\Lambda(t,dt)$ such that $(\mathrm{id}\otimes\varepsilon_0)H = f$ and $(\mathrm{id}\otimes\varepsilon_1)H = g$.

The following lemma is due to Felix-Alperin-Thomas.

Lemma 7.4 (Lifting lemma)**.** *Suppose we have*

$$\begin{tikzcd} & A \arrow[d, "\simeq"] \\ \Lambda V \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

where ΛV is Sullivan, and $A \to B$ is a surjective quasi-isomorphism. Then, the dotted lift exists.

Proof. This follows from the properties of a model category, since ΛV is cofibrant and $A \to B$ is a weak fibration, so the lift exists. ⊠

Definition. A *relative Sullivan algebra* is a relative I-cell, i.e. given by taking a $\mathbb{Q}$-CDGA and then attaching by pushouts.

Alternatively, we can define it as a CDGA of the form $B\otimes\Lambda V$, where B is a subapgebra, V is built as $V(0)\subset V(1)\subset\cdots$ such that $d : V(0)\to B$ and $d : V(k)\to B\otimes\Lambda(k-1)$.

The latter definition is how it's usually stated, and considerably less clear at least to the professor.

Lemma 7.5. *$B \to B\otimes\Lambda V$ is a cofibration.*

All relative I-cell complexes are cofibrations.

Definition. For any $\mathbb{Q}$-CDGA A, let UA denote its underlying vector space, and define $E(A) = \Lambda(UA\oplus dUA)$, with $d : UA \xrightarrow{\cong} dUA$.

This is a big contractible object, akin to $\Lambda(t,dt)$ from earlier.

Theorem 7.6. *If $f : A\to B$ is a morphism in $\mathrm{CDGA}_{\mathbb{Q}}$, then it factors as $A \xrightarrow{\cong} A\otimes E(B) \to B$, where the first is relative Sullivan and the second is a surjection.*

This is not that hard to do directly, but is even easier with model categories, falling out just from the axioms.

Definition. Let $[A,B]$ denote the homotopy classes of maps $A\to B$.

Proposition 7.7. *If $A \xrightarrow{\simeq} C$ is a quasi-isomorphism and ΛV is Sullivan,then $[\Lambda V, A] \xrightarrow{\cong} [\Lambda V, C]$ is a bijection.*

Proof. Since ΛV is cofibrant, A and B are fibrant, and homotopy is an equivalence relation, this follows from model-categoric abstraction. ⊠

We can also prove another lifting property.

Theorem 7.8. *Consider the diagram*

$$\begin{tikzcd} B \arrow[r] \arrow[d, "f"'] & A \arrow[d, "\simeq"', "g"] \\ B\otimes\Lambda V \arrow[r, "\psi"] \arrow[ur, dashed, "\iota"] & C. \end{tikzcd}$$

If f is a relative Sullivan algebra and g is a quasi-isomorphism, then there's a lift.

Once again, this is clear from the model-categoric viewpoint. You can see why we went through it; it's very useful and makes stuff much easier to remember.

Example 7.9. The one example everyone gives is $\Lambda(a,b,c)$ with $\deg a,b,c = 1$, with $da = bc$, $db = ac$, and $dc = ba$. This is *not* a Sullivan algebra; if you try to realize it as one, you run into degree issues. As a complex, it looks like this:

$$\mathbb{Q} \longrightarrow \langle a,b,c\rangle \longrightarrow \langle ab,ac,bc\rangle \longrightarrow \langle abc\rangle \longrightarrow 0.$$

Thys, $H^3(\Lambda(a,b,c)) = \mathbb{Q}\langle a,b,c\rangle$ and $H^0 = \mathbb{Q}$. Then, define $f : \Lambda(x) \to A$, where there's no differential and $\deg x = 3$. Then, if $f(x) = abc$, one can check that f is a quasi-isomorphism.

We want to assign one algebra for an equivalence class of rational spaces, so we need one best one to assign. This is where minimal Sullivan algebras come into play. We'll think of Sullivan algebras as cell complexes; then, minimal Sullivan algebras can be thought of as akin to CW complexes.

Definition. Let ΛV be a Sullivan algebra; it is said to be *minimal* if $\operatorname{Im} d \subset \Lambda V^+ \cdot \Lambda V^+$, where ΛV^+ denotes the positive-degree elements.

That is, $\operatorname{Im}(d)$ lies in the set of "decomposable" elements. For degree reasons, this means d has to be attached to things of lower dimension, which sounds like CW complexes, and will sound even more once we actually produce minimal models. This notion is useful, but doesn't come from the formalism of model categories.

Lemma 7.10. *Any CDGA of the form $(\Lambda V, d)$ with V only living in degrees 2 or greater, and such that $\operatorname{Im} d \subset \Lambda V^+ \cdot \Lambda V^+$, is a minimal Sullivan algebra.*

Why is this? Consider the degree-k elements, V^k. By definition, $d : V^k \to V^{k+1}$, but by assumption $d : V^k \to \Lambda^+ V \cdot \Lambda^+ V$, and $(\Lambda^+ V \cdot \Lambda^+ V)^k \subseteq \Lambda V^{k+1}$, which means that $\Lambda V^{\le 2} \subset \Lambda V^{\le 3} \subset \dots$ is an okay filtration for building the Sullivan algebra.

That's all for today; next time, we'll build Sullivan algebras. Read about simplicial sets.

Lecture 8.

Minimal Sullivan Models and Simplicial Sets: 9/24/15

Today, we'll talk about minimal Sullivan algebras in the simply connected case, and then about simplicial sets.

Proposition 8.1. *Let A be a $\mathbb{Q}$-CDGA such that $H^0(A) = \mathbb{Q}$ and $H^1(A) = 0$. Then, there is a milimal Sullivan model ΛV for A.*

Proof. Start at $H^2(A)$. This is a rational vector space, so pick a V^2 such that $\Lambda V^2 \to A$ is an isomorphism on H^2.

Then, for the inductive step, assume we've constructed an $f_K : \Lambda V^{\le k} \to A$ such that $H^k(f_k)$ is an isomorphism and $H^{k+1}(f_k)$ is injective, and we want a $\Lambda V^{\le k+1}$ and an $f_{k+1} : \Lambda V^{\le k+1} \to A$ that's an isomorphism in H^{k+1}. So let's take a long exact sequence:

$$H^{k+1}(\Lambda V^{\le k}) \xrightarrow{\;a\;} H^{k+1}(A) \longrightarrow H^{k+1}(A, \Lambda V^{\le k}) \longrightarrow H^{k+2}(\Lambda V^{\le k}) \xleftrightarrow{\;b\;} H^{k+2}(A) \longrightarrow \cdots$$

a fails to be an isomorphism since it's not surjective, so since we're working with vector spaces, we can pick a basis for the stuff not hit by it:

$$H^{k+1}(A) = \operatorname{Im} H^{k+1}(f_k) \oplus \bigoplus_{i \in I} \mathbb{Q}\langle \alpha_i \rangle.$$

b fials to be injective because it has a kernel, so similarly we write

$$\ker H^{k+2}(f_k) = \bigoplus_{j \in J} \mathbb{Q}\langle z_j \rangle.$$

The idea is to add in extra stuff to hit the α_i and make the z_j exact (so that they vanish in homology). Let $V^{k+1} = \langle v'_i, v''_j \rangle_{i \in I, j \in J}$, with $\deg v'_i, v''_j = k+1$. Set $dv'_i = 0$ and $dv''_j = z_j$. Finally, let $\Lambda V^{\le k+1} = \Lambda V^{\le k} \otimes V^{k+1}$.

Next, we need our f_{k+1}, so define $f_{k+1}(V') = \alpha_i$ and $f_{k+1}(v''_j) = b_J$, where $f_k(z_j) = b_j$.

Finally, why does our inductive hypothesis hold? Surjectivity of $H^{k+1}(f_{k+1})$ follows by the construction of f, so to show that it's injective, suppose $f_{k+1}(\alpha) = 0$, so

$$\alpha = \sum \lambda_i v'_i + \sum \lambda_j v''_j + R, \tag{8.1}$$

where $R \in \Lambda V^{\leq k}$ is some lower-order terms. Since α represents homology, then $d\alpha = 0$, and (8.1) becomes

$$d\alpha = \sum \lambda_j z_j + dR = 0.$$

In homology, this is 0, so $\sum \lambda_j [z_j] = 0$; since z_j is a basis, then $\lambda_j = 0$ for each j. Thus, α is only a sum of v_i' terms and a remainder, so we'll show this is all zero in homology too.

We know $f_{k+1}(\alpha) = 0$ in homology, and this is

$$f_{k+1}(\alpha) = \sum \lambda_i \alpha_i + f_{k+1}(R) = \sum \lambda_i \alpha_i + f_k(R).$$

Passing to homology, $\sum \lambda_i [\alpha_i] = 0$, but again, they're a basis, so the $\lambda_i = 0$ too.

Finally, $H^{k+1}(f_{k+1})$ is injective by construction. Finally, the condition given last time gives minimality. ☒

It turns out you can do this for things which aren't simply connected, but it's a little harder, and the correspondence between rational spaces and CDGAs is generated by a correspondence for simply connected objects.

We'll need to do more with $\mathbb{Q}$-CDGAs, and therefore actually construct approximations to spaces. This is where simplicial sets come in; they have a weird kind of differential geometry or differential forms, but we should construct them first. Good references for this is Emily Riehl's notes, and Freedman's notes (which are illustrated!). Avoid May's book, which has no pictures, and probably avoid Goerss-Jardine, which is very hard unless you think natively in terms of functors.

We do have to define simplicial sets to be functors, so we have to define the indexing categories for these functors.

Definition. The *simplex category* Δ has objects ordered sets $[n] = \{0 < 1 < \cdots < n\}$ and morphisms order-preserving maps.

Simplicial sets will be indexed on the opposite category, but first, let's present Δ slightly more simplicially.

Remark. All morphisms in Δ are generated by the following classes.

(1) The *coface maps* $d^i : [n-1] \to [n]$ given by missing the i^{th} element (e.g. $d^2 : [2] \to [3]$ sends $0 \mapsto 0$, $1 \mapsto 1$, and $2 \mapsto 3$).
(2) The *codegeneracy maps* $s^i : [n+1] \to [n]$ that sends $i, i+1 \mapsto i$ and otherwise is injective and surjective.

These satisfy the relations

$$\begin{aligned} d^j \circ d^i &= d^i \circ d^{j-1}, \qquad i < j \\ s^j \circ s^i - s^i \circ s^{j+1}, &\qquad i \leq j \\ s^j \circ d^i &= \begin{cases} d^i \circ s^{j-1}, & i < j \\ \text{id}, & i = j, i = j+1 \\ d^{i-1} \circ s^j, & i > j+1. \end{cases} \end{aligned}$$

You can check this if you want. But the takeaway is, often one only calculates with these maps when working with simplicial sets.

Definition. A *simplicial set* is a functor $X : \Delta^{\text{op}} \to \text{Set}$; the category of simplicial sets (morphisms are natural transformations of functors) is denoted Set_Δ.

This is a terrible definition: these are really geometric objects, and this definition doesn't help us with that at all! But we'll see some geometry and more different definitions, which will be better.

A first example is $\Delta^n = \text{Hom}_\Delta(-, [n])$, which isn't really geometric either. But it has some degrees. $\Delta^2[0] = \text{Hom}_\Delta([0],[2])$, which is given by the three maps $\bullet \to \{\bullet_1, \bullet_2, \bullet_3\}$. $\Delta^2[1] = \text{Hom}_\Delta([1],[2])$. This is generated by the three coface maps, and everything else is degenerate (coming from the 0^{th} degree). In $\Delta^2[2]$, the identity is the only nondegenerate map, and all higher maps are degenerate.

This is all combinatorial nonsense, but we want to draw suggestive pictures. Corresponding to elements of $\Delta^2[0]$ should be points, and of $\Delta^2[1]$ should be lines, and of $\Delta^2[2]$ should be faces. Then, we actually have simplices! And a codegeneracy map sends a face to a boundary line (or a line to a boundary point, etc.), and coface maps embed an edge into a face next to it.

This motivates the following, dual definition.

Definition. A *simplicial set* is a collection of sets X_n, called *n-simplices*, along with *face maps* $d_i : X_n \to X_{n-1}$ and *degeneracy maps* $s_i : X_n \to X_{n+1}$ such that

$$d_i \circ d_j = d_{j-1} \circ d_i, \qquad i < j$$
$$s_i \circ s_j = s_j \circ s_{i-1}, \qquad i \le j$$
$$d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i, & i < j \\ \mathrm{id}, & i = j, j+1 \\ s_j \circ d_{i-1}, & i > j+1. \end{cases}$$

This is getting closer to geometry, of some sort. Thinking of them combinatorially can be weird, so think of X_n as a collection of geometric n-simplices, where the face and degeneracy maps tell us how everything is glued together. A face map $d_1 : X_2 \to X_1$ says that the first face of the first 2-simplex corresponds to the first 1-simplex. This is what this data is encoding: it's telling you how to glue together some geometric object.

Degeneracy maps don't have the same geometric equivalent, and are in fact really confusing before you get used to them.

Remark. We want simplicial sets to model geometry. For example, $\pi : \Delta^2 \to \Delta^1$ should model sending a triangle to one of its faces. But maps on simplicial sets are given degree-wise: $X_n \to Y_n$. Thus, a simplicial set model for $\Delta^2 \to \Delta^1$ sends $\Delta^2_2 \to \Delta^2_1$, but Δ^2_1 is empty — so we need these strange collapsing maps to model projections.

Lemma 8.2. *Let* $\Delta^n = \mathrm{Hom}_\Delta(-,[n])$*; then,* $\mathrm{Hom}_{\mathsf{Set}_\Delta}(\Delta^n, X) = X_n$.

Proof. This is pretty much exactly the statement of Yoneda's lemma. ⊠

This is used without statement really often, so it's nice to have it written down.

Example 8.3. Here's another example of a simplicial set, which can come up in some kinds of work really frequently. Let $\mathscr{C}$ be a category, and define its *nerve* $N(\mathscr{C})$ as follows: let n^{th}-simplicies be $N(\mathscr{C})_N n = \{C_0 \to C_1 \to \cdots \to C_n\}$ (i.e. compositions of n morphisms in $\mathscr{C}$); let the face map d_i compose the i^{th} and $(i+1)^{\text{st}}$) morphisms, and the degeneracy map s_i insert the identity map at stage i.

An alternate presentation: $[n]$ is a category with order-preserving maps as morphisms; then, $N(\mathscr{C})_n$ is the category of functors $[n] \to \mathscr{C}$. This makes it easier to see that the simplicial identities hold.

This lends a geometric structure to categories, which can be surprisingly useful.

The following is also useful, but tends not to be written down.

Definition. Let $[n]^+_-$ be the set whose objects are $-, x_1, \dots, x_n, +$. Then, L^+_- is the category whose objects are these $[n]^+_-$ and whose morphisms are order-preserving maps preserving $-$ and $+$ ($x_i \mapsto \pm$ is acceptable).

Lemma 8.4. $\Delta^{\mathrm{op}} \cong L^+_-$.

Example 8.5. Consider $d^i : [2] \to [3]$. Then, let's define a map $[3]^+_- \to [2]^+_-$: since d^i misses the i^{th} spot, then the i^{th} and $(i+1)^{\text{th}}$ spots in $[3]^+_-$ should hit i in $[2]^+_-$. Similarly, the map corresponding to the codegeneracy $s^i : [3] \to [2]$ is the unique thing that misses only the i^{th} spot in $[2]^+_- \to [3]^+_-$.

This is a covariant presentation of Δ^{op}, which therefore makes it much easier to think of.

Geometric Realization. Simplicial sets have a model-categorical structure. It's a pain to prove, but it'll make our lives a lot easier to state things.

There's also geometric realizations of simplicial sets, which arise from a more general construction. This is a somewhat unsurprising notion: a way of sending a simplicial set to a cellular topological space. In other words, it makes our intuition into an actual topological object.

Definition. Let $\mathscr{C}$ be a category with products and colimits, and let $X : I \to \mathscr{C}$ and $Y : I^{\mathrm{op}} \to \mathscr{C}$. Then, define

$$Y \otimes_I X = \mathrm{coeq}\left(\coprod_{c \to d \in \mathscr{C}} X(c) \times Y(d) \rightrightarrows \coprod_{c \in \mathscr{C}} X(c) \times Y(c) \right).$$

The two arrows are pulling back in the first or second arguments.

This is also equal to

$$\coprod_{c \in \mathscr{C}} X(c) \times U(c) / (x, fg) \sim (xf, g).$$

Check out a good paper by Hollender and Vogt to learn more about this.

Lecture 9.

Simplicial Sets are a Model Category: 9/29/15

Recall that if $X : I \to \mathscr{C}$ and $Y : I^{\mathrm{op}} \to \mathscr{C}$ are functors, and if $\mathscr{C}$ has products and quotients, we defined

$$X \otimes_I Y = \operatorname{coeq}\left(\coprod_{c\to d} X(c)\times Y(d) \rightrightarrows \coprod_c X(c)\times Y(c)\right) = \coprod_{c\in\mathscr{C}} X(c)\times Y(c)/(x, yf) \sim (fx, y).$$

This is a surprisingly useful technique: for example, $Y \otimes_I * = \operatorname{colim} I$, and one can define the *homotopy colimit* $\operatorname{hocolim} X = B(I\downarrow \mathrm{id})\otimes_I X$.

We'll use this to turn a simplicial set into a topological space: specifically, we have $X : \Delta^{\mathrm{op}} \to \mathsf{Set}_\Delta$, so tensoring with some $\Delta \to \mathsf{Top}$ should work.

Definition. $\Delta^\bullet : \Delta \to \mathsf{Top}$, the *cosimplicial space* functor, sends

$$\Delta^n \longmapsto \left\{\sum t_i = 1 \mid t_0,\dots,t_n \ge 0\right\}, \tag{9.1}$$

with maps

$$d^i : \Delta^{n-1} \to \Delta^n \longmapsto ((x_0,\dots,x_{n-1}) \mapsto (x_0,\dots,x_{i-1},0,x_i,\dots,x_{n-1}))$$
$$s^i : \Delta^{n+1} \to \Delta^n \longmapsto ((x_0,\dots,x_{n+1}) \mapsto (x_0,\dots,x_{i-1},x_{i+1},\dots,x_{n+1})).$$

It's easy to check that the cosimplicial identities hold.

Definition. Let $X : \Delta^{\mathrm{op}} \to \mathsf{Set}$ be a simplicial set. Then, let $\widetilde{X} : \Delta^{\mathrm{op}} \to \mathsf{Set} \overset{D}{\hookrightarrow} \mathsf{Top}$, where $D : \mathsf{Set} \hookrightarrow \mathsf{Top}$ gives the discrete topology. Then, the *geometric realization* of X is the topological space

$$|X| = \widetilde{X} \otimes_\Delta \Delta^\bullet = \operatorname{coeq}\left(\coprod_{[m]\to[n]} X_m \times \Delta^n \rightrightarrows \coprod_m X_m \times \Delta^m \right).$$

The coequalizer is telling us how to glue things together.

More geometrically, this is

$$\coprod_m X_m \times \Delta^m / ((x, d^j p) \sim (d_j x, p), (x, s^j p) \sim (s_j x, p)).$$

Examples are reasonably nice; the easiest one that isn't a simplex is taking two 2-simplicies and gluing them along an edge, and seeing what happens.

Simplicial sets also help us realize a concept from more elementary algebraic topology.

Definition. If X is a topological space, define $\mathrm{Sing}(X) \in \mathsf{Set}_\Delta$ with n-simplices $\mathrm{Sing}(X)_n = \mathrm{Map}_{\mathsf{Top}}(\Delta^n, X)$, with face and degeneracy relations given by pulling back d^i and s^i.

Fact. For suitably nice X (at least compactly generated, though compactness suffices), $|\mathrm{Sing}(X)| \simeq X$.

$\mathrm{Sing}(X)$ is an awfully big and inefficient encoding of X, but it is a real simplicial set (no set-theoretic issues). Just don't try to compute nontrivial homology with it.

Example 9.1. The *n-horns* Λ_i^n can be defined as follows: recall that $\Delta^n = \mathrm{Hom}_\Delta(-,[n])$, so let $\Lambda_i^n = \mathrm{Hom}_\Delta(-,[n]) \setminus (\mathrm{id}, i : [n-i] \to [n])$. The idea (and the reason for the notation) is that for a 2-simplex Δ^2, Λ_2^2 is the first and third edges of this simplex, which do make a Λ shape.

We can also define the *simplicial spheres* $\partial\Delta^n = \mathrm{Hom}_\Delta(-,[n]) \setminus \{\mathrm{id}\}$. Since the identity map corresponds to the n-face, the boundary is really analogous to a sphere.

It can be tricky to go back and forth between the combinatorial objects and the geometric meaning. It's neither nice nor intuitive, but there is some intuition! Though it may be obnoxious, Yoneda's lemma makes representable functors really nice, so there is that.

It turns out (surprise, surprise) that simplicial sets form a model category structure. The proof is long and involved, and we won't prove it, though Quillen and a few others did write it up well.

Theorem 9.2. Set_Δ *is a cofibrantly generated model category, with generating fibrations* $I = \{\partial\Delta^n \hookrightarrow \Delta^n \mid n \ge 1\}$, *generating acyclic cofibrations* $J = \left\{\Lambda_i^n \hookrightarrow \Delta^n \mid n \ge 1, i \ge 0\right\}$*, and weak equivalences* $X \to Y$ *if* $|X| \to |Y|$ *is a weak equivalence on* Top.

The generators look a lot like the generators for Top as a CGMC, with a little bit of homotopy thrown in.

Corollary 9.3. *Cofibrations are level-wise injections, and fibrations are exactly the maps $X \to Y$ with the right lifting property with respect to J, i.e. for any horn Λ_i^n and the following maps, then the dotted arrow exists.*

$$\begin{tikzcd} \Lambda_i^n \arrow[r] \arrow[d, "\in J"'] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$

Sometimes, these are called "horn-filling conditions." Note that ∞-categories are defined from simplicial sets in a reasonable way, so maybe this makes them look less scary.

Corollary 9.4. *X is fibrant if the dotted arrow exists in the folllowing diagram for all maps $f : \Lambda_i^n \to X$.*

$$\begin{tikzcd} \Lambda_i^n \arrow[r, "f"] \arrow[d, hook] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & * \end{tikzcd}$$

In this case, X is called a Kan complex.

A quasi-category is just a cofibrant object in some model category, which sounds either really helpful or voodoo. But there's no reason to be afraid of them.

Sing(X) is a good example of a Kan complex.

The homotopy theory of simplicial sets and that of topological spaces are exactly equivalent.

Theorem 9.5. *There is an adjunction* $|-| : \mathsf{Set}_\Delta \rightleftarrows \mathsf{Top} : \mathsf{Sing}(-)$ *inducing an equivalence* $\mathrm{Ho}(\mathsf{Set}_\Delta) \cong \mathrm{Ho}(\mathsf{Top})$.

This is why simplicial sets often stand in for topological spaces in homotopy theory, which is often very important to know.

Now, we want to define an equivalence of homotopy categories between $\mathbb{Q}$-CDGAs and simplicial sets (the latter because they are easier to work with combinatorially). We'll be able to make the important definition by the end of the lecture, and then the rest of the class will delve into proving its properties; we're really starting the core of the course.

Rational Differential Forms.

Definition. A *simplicial object* in a category $\mathscr{C}$ is a functor $X : \Delta^{\mathrm{op}} \to \mathscr{C}$.

For example, simplicial rings; simplicial groups are more or less chain complexes, and simplicial varieties (essential for defining higher Chow groups). It's not clear why simplicial methods are so ubiquitous; perhaps because they encode some incredible general notion of homotopy. There's even a paper associating PDEs to simplicial sets!

We can even talk about simplicial CDGAs, which are functors $\Delta^{\mathrm{op}} \to \mathrm{CDGA}_{\mathbb{Q}}$. (This is as bad as it gets; not worse.) The category of simplicial CDGAs is denoted $\mathrm{CDGA}_{\mathbb{Q},\Delta}$. There are two gradings here, given by the simplicial structure and the degree of the complex, so these are often written $A_\bullet^\bullet$, with the simplicial in the subscript and the chain complex in the superscript. For example, $A_\bullet^p$ is all terms of degree p in the chain complex with simplicial terms varying; this is a simplicial vector space. Similarly, $A_n^\bullet$ is a chain complex.

Remark. By forgetting structure, $A_\bullet^p \in \mathsf{Set}_\Delta$ (a vector space is a set).

Now, we hit an important definition.

Definition. Let $A_\bullet^\bullet \in \mathrm{CDGA}_{\mathbb{Q},\Delta}$ and $K \in \mathsf{Set}_\Delta$. Define the set $A^p(K) = \mathrm{Hom}_{\mathsf{Set}_\Delta}(K, A_\bullet^p)$, so that $A^\bullet(K) \in \mathrm{CDGA}_{\mathbb{Q}}$.

Corollary 9.6. *If* $A_\bullet^\bullet \in \mathrm{CDGA}_{\mathbb{Q},\Delta}$. *Then,* $A_\bullet^\bullet(\Delta^n) = A_n^\bullet$.

The proof is the Yoneda lemma.

We'll use a specific case of this to define and state theorems about some differential forms: we'll want analogues of the Poincaré lemma and an extension lemma, and this is the cleanest language to give it in.

Remark. We can define differential forms on the geometric simplex Δ^n (defined as in (9.1)) by $f(t_0,\ldots,t_n)\mathrm{d}t_0 \wedge \cdots \wedge \mathrm{d}t_n$, where $f : \Delta^n \to \mathbb{R}$, which makes sense, and then use these to define differential forms on simplicial complexes (topological forms glued together from simplicies) by requiring that they restrict appropriately to the faces. We won't be too specific about this, because we're about to define a real definition, but it's still useful intuition to have.

Since Sullivan used simplicial complexes rather than simplicial sets, that is how he actually defined differential forms, which makes a bunch of things more complicated.

Definition.

$$(A_{\mathrm{PL}})_n = \Lambda_{\mathbb{Q}}(t_0,\ldots,t_n,\mathrm{d}t_0,\ldots,\mathrm{d}t_n)/\left(\sum t_i - 1, \sum \mathrm{d}t_i\right),$$

which is a CDGA. Intuitively, these are "differential forms on a simplex."

Of course, we could define this over other fields, but we don't care to.

In fact, $(A_{\mathrm{PL}})_n$ is part of a simplicial $\mathbb{Q}$-CDGA $(A_{\mathrm{PL}})_\bullet$, via defining face and degeneracy maps $d_i : (A_{\mathrm{PL}})_n \to (A_{\mathrm{PL}})_{n-1}$ by

$$d_i(t_k) = \begin{cases} t_k, & k < i \\ 0, & k = i \\ t_{k-1}, & k > i, \end{cases}$$

and $s_j : (A_{\mathrm{PL}})_n \to (A_{\mathrm{PL}})_{n+1}$ by

$$s_j(t_k) = \begin{cases} t_k, & k < j \\ t_k + t_{k+1}, & k = j \\ t_{k+1}, & k > j, \end{cases}$$

and such that they commute with the differential.

We've pushed our differential forms into a giant simplicial complex, but what this is actually encoding is just what happens to differential forms when you restrict to faces.

Now, here is our core definition.

Definition. If $K \in \mathsf{Set}_\Delta$, define $A_{\mathrm{PL}}(K) = (A_{\mathrm{PL}})^\bullet_\bullet(K) \in \mathrm{CDGA}_{\mathbb{Q}}$. These are called the *rational, piecewise-linear differential forms*.

This is basically piecewise differential forms on a set, but notice: it defines an association from simplicial sets to $\mathrm{CDGA}_{\mathbb{Q}}$, which will be crucial.

Corollary 9.7. $A_{\mathrm{PL}}(\Delta^n) = (A_{\mathrm{PL}})_n$.

This is exactly what we had hoped would be true: this topological space corresponds to this simplex, this collection of forms (the algebra).

Like regular differential forms, we use these to integrate or get cohomology; specifically, we'll want versions of the Poincaré lemma and the de Rham theorem.