

# FALL 2017 GOODWILLIE CALCULUS SEMINAR

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These notes were taken in Andrew Blumberg’s student seminar in Fall 2017. I live- $\text{\TeX}$ ed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to [a.debray@math.utexas.edu](mailto:a.debray@math.utexas.edu).

## CONTENTS

1. Introduction: 9/13/17	1
2. Interpolating between stable and unstable phenomena: 9/20/17	3
3. Two paths to homotopy colimits: 9/27/17	5
4. The Blakers-Massey theorem: 10/4/17	6
5. Snaith splitting: 10/11/17	9
6. Manifold calculus: 10/18/17	9

## 1. INTRODUCTION: 9/13/17

Today, Nicky gave an overview of Goodwillie calculus, following Nick Kuhn’s notes.

The setting of Goodwillie calculus is to consider two topologically enriched,<sup>1</sup> based model categories  $\mathcal{C}$  and  $\mathcal{D}$  and a functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  between them.

### Example 1.1.

- (1)  $\mathbf{Top}$ , the category of topological spaces.
- (2)  $\mathbf{Sp}$ , the category of spectra.
- (3) If  $Y$  is a topological space, we can also consider  $Y \backslash \mathbf{Top}_Y$ , the category of spaces over and under  $Y$ , i.e. the diagrams  $Y \rightarrow X \rightarrow Y$  which compose to the identity.  $\blacktriangleleft$

We want  $F$  to satisfy some kind of Mayer-Vietoris property, or excision. Hence, we assume  $\mathcal{C}$  and  $\mathcal{D}$  are *proper*, in that the pushout of a weak equivalence along a cofibration is also a weak equivalence. We’ll also ask that in  $\mathcal{D}$ , sequential colimits of homotopy Cartesian cubes are again homotopy Cartesian, and we’ll elaborate on what this means.

We also place a condition on  $F$ : Goodwillie calls it “continuous,” meaning that it’s an enriched functor: the induced map

$$\mathrm{Map}_{\mathcal{C}}(X, Y) \longrightarrow \mathrm{Map}_{\mathcal{D}}(F(X), F(Y))$$

is a continuous map between topological spaces (or a morphism of simplicial sets; for the rest of this section, we’ll let  $\mathbf{V}$  denote the choice of  $\mathbf{Top}_*$  or  $\mathbf{sSet}_*$  that we made). If  $X \in \mathcal{C}$  and  $K \in \mathbf{V}$ , then we have a tensor-hom adjunction

$$\mathcal{C}(X \otimes K, Y) \cong \mathbf{V}(K, \mathcal{C}(X, Y)).$$

From this,  $F$  produces the *assembly map*

$$F(X) \otimes K \longrightarrow F(X \otimes K).$$

We’ll also require  $F$  to be weakly homotopical in that it sends homotopy equivalences to homotopy equivalences.

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<sup>1</sup>As usual, we can take them to be enriched either over  $\mathbf{Top}$  or over  $\mathbf{sSet}$ . This has the important consequence that  $\mathcal{C}$  and  $\mathcal{D}$  are tensored and cotensored over  $\mathbf{Top}_*$ , resp.  $\mathbf{sSet}_*$ .

The idea of Goodwillie calculus is to approximate  $F$  by a tower of functors, akin to Postnikov truncations,  $\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0$ . The fiber  $D_i$  of  $P_i$ , akin to the  $i^{\text{th}}$  Postnikov section, is like the  $i^{\text{th}}$  term in a Taylor series:

$$\begin{aligned} P_0(X) &\simeq P_0(*) \\ D_1(X) &\simeq D_1(*) \otimes X \\ D_2(X) &\simeq (D_2(X) \otimes X \otimes X)_{h\Sigma_2}, \end{aligned}$$

where  $\Sigma_2$  acts by switching the two copies of  $X$ , and so on. Each  $P_i$  will satisfy more and more of the Mayer-Vietoris property. This is akin to the first three terms in a Taylor series for  $f$ :  $f(a)$ ,  $xf'(a)$ , and  $x^2 f''(a)/2$ .

**Weak natural transformations.** We'll also need to know what a weak equivalence of functors is. This would allow us to study the homotopy category of  $\text{Fun}(\mathbf{C}, \mathbf{D})$ .

**Definition 1.2.** A *weak natural transformation*  $F \Rightarrow G: \mathbf{C} \rightarrow \mathbf{D}$  is one of the two zigzags

$$F \xleftarrow{\sim} H \longrightarrow G \quad \text{or} \quad F \longleftarrow H \xrightarrow{\sim} G,$$

where  $F \xrightarrow{\sim} G$  means an objectwise weak equivalence.

Commutativity of a diagram of weak natural transformations is computed in  $\text{ho}(\mathbf{D})$ .<sup>2</sup> You can also form spectra in  $\mathbf{D}$  in the usual way (inverting suspension, etc).

**Diagrams<sup>3</sup>.** Let  $S$  be a finite set. We'll let  $\mathcal{P}(S)$  denote its power set, made into a poset category under inclusion. Similarly, we'll let  $\mathcal{P}_0(S) := \mathcal{P}(S) \setminus \{\emptyset\}$  and  $\mathcal{P}_1(S) := \mathcal{P}(S) \setminus \{S\}$ , again regarded as poset categories.

**Definition 1.3.**

- (1) A  $d$ -cube in  $\mathbf{C}$  is a functor  $\mathcal{X}: \mathcal{P}(S) \rightarrow \mathbf{C}$ , where  $|S| = d$ .
- (2) A  $d$ -cube  $\mathcal{X}$  is *Cartesian* if

$$\mathcal{X}(\emptyset) \xrightarrow{\sim} \text{holim}_{T \in \mathcal{P}_0(S)} \mathcal{X}(T).$$

- (3) A  $d$ -cube  $\mathcal{X}$  is *co-Cartesian* if

$$\mathcal{X}(S) \xrightarrow{\sim} \text{hocolim}_{T \in \mathcal{P}_1(S)} \mathcal{X}(T).$$

- (4) A  $d$ -cube  $\mathcal{X}$  is *strongly co-Cartesian* if  $\mathcal{X}|_{\mathcal{P}(T)}: \mathcal{P}(T) \rightarrow \mathbf{C}$  is co-Cartesian for all  $T \in \mathcal{P}(S)$  with  $|T| \geq 2$ .

**Example 1.4.**

- (1) If  $d = 0$ , a (Cartesian or co-Cartesian) 0-cube is something weakly equivalent to the initial object.
- (2) A (Cartesian or co-Cartesian) 1-cube is an equivalence.
- (3) A 2-cube is something of the form

$$\begin{array}{ccc} \text{fib}_f & \longrightarrow & \text{fib}_g \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \\ \downarrow f & & \downarrow g \\ C & \longrightarrow & D. \end{array}$$

We let  $\partial\mathcal{X}$  denote the *boundary* of  $\mathcal{X}$ , the top row; the middle row is  $\mathcal{X}_\top$ , and the bottom row is  $\mathcal{X}_\perp$ .<sup>3</sup> In this case, Cartesian and co-Cartesian correspond to (homotopy) pushout and pullback squares, which explains their names in the general case.  $\blacktriangleleft$

There's a way to produce co-Cartesian cubes canonically from a finite set. Let  $\phi: X^{\text{IIT}} \rightarrow X$  denote the fold map.

<sup>2</sup>There are more worked-out expositions of the homotopy theory of functors, in case this one looks ad hoc, but we don't need the entire background.

<sup>3</sup>These are also written  $\mathcal{X}_{\text{top}}$  and  $\mathcal{X}_{\text{bottom}}$ .

**Definition 1.5.** Let  $T$  be a finite set and  $X \in \mathbf{C}$ , and let

$$X \star T := \text{cofib} \left( \phi: \coprod_T X \rightarrow X \right).$$

Now, for  $T \subset [d]$ , the assignment  $T \mapsto X \star T$  defines a co-Cartesian  $(d+1)$ -cube.

For example, when  $d = 1$ , this is the homotopy pushout

$$\begin{array}{ccc} X & \longrightarrow & CX \simeq * \\ \downarrow & & \downarrow \\ CX \simeq * & \longrightarrow & \Sigma X. \end{array}$$

This formalism allows us to define the analogue of the Mayer-Vietoris principle that we'll need for the Goodwillie tower.

**Definition 1.6.** An  $F: \mathbf{C} \rightarrow \mathbf{D}$  with  $F$ ,  $\mathbf{C}$ , and  $\mathbf{D}$  as above is *d-excisive* if for all strongly co-Cartesian  $(d+1)$ -cubes  $\mathcal{X}$ ,  $F(\mathcal{X})$  is a Cartesian  $(d+1)$ -cube in  $\mathbf{D}$ .

**Example 1.7.**

- (1) 0-excisive functors are homotopy constant.
- (2) 1-excisive functors are those that satisfy the Mayer-Vietoris property. In  $\mathbf{Sp}$ ,  $\text{Map}_{\mathbf{Sp}}(C, -)$  and  $L_E$  are both 1-excisive.  $\blacktriangleleft$

There are some nice properties about how  $d$ -excisive functors behave with respect to cofibration sequences, etc., and we will return to them in due time. We will now construct Taylor towers.

Fix an  $X \in \mathbf{C}$ , and let

$$T_d F(X) := \text{holim}_{T \in \mathcal{P}_0([d+1])} F(X \star T).$$

*Remark.* There is a natural map  $t_d F: F \rightarrow T_d F$ , and by definition, this is an equivalence if  $F$  is  $d$ -excisive.  $\blacktriangleleft$

Set  $P_d F: \mathbf{C} \rightarrow \mathbf{D}$  to be the functor sending

$$X \mapsto \text{hocolim} \left( F(X) \xrightarrow{t_d F} T_d F(X) \xrightarrow{t_d T_d F} T_d T_d F(X) \longrightarrow \dots \right).$$

For example, if  $F(*) \simeq *$ , then  $T_1 F(X)$  is the homotopy pullback of

$$\begin{array}{ccc} & F(CX) \simeq * & \\ & \downarrow & \\ * \simeq F(CX) & \longrightarrow & F(\Sigma X), \end{array}$$

and hence is  $\Omega F(\Sigma X)$ . In this case

$$P_1 F(X) = \text{hocolim}_{n \rightarrow \infty} \Omega^n F \Sigma^n X.$$

For example, if  $F = \text{id}$  and  $\mathbf{C} = \mathbf{D}$ , then  $P_1(\text{id}) = \Omega^\infty \Sigma^\infty$ , which is cool: the “first derivative” of the identity tells us information of stable homotopy! The calculation of the second derivative will be harder.

## 2. INTERPOLATING BETWEEN STABLE AND UNSTABLE PHENOMENA: 9/20/17

Today, Adrian gave an overview of what we're going to learn about this semester.

**Functors are like functions.** We have an analogy between smooth functions and nice functors from  $\mathbf{Top}_*$  to  $\mathbf{Top}_*$  or  $\mathbf{Sp}$ .<sup>4</sup> This analogy sends

- degree- $n$  polynomials to  $n$ -excisive functors,
- homogeneous degree- $n$  polynomials to homogeneous  $n$ -excisive functors (defined using Cartesian cubes), and
- Taylor series to Taylor towers of functors.

<sup>4</sup>Perhaps more generality is possible, but we'll worry about that later.

In Higher Algebra, Lurie takes the idea that an  $\infty$ -category is like a manifold as an anchor for doing a lot of very interesting mathematics, which is one angle for interpreting this analogy.

Let  $\text{Homog}_n(\mathbf{C}, \mathbf{D})$  denote the category of homogeneous  $n$ -excisive functors  $F: \mathbf{C} \rightarrow \mathbf{D}$ , where  $\mathbf{C}$  and  $\mathbf{D}$  are categories with the assumptions we placed on them last time.

**Theorem 2.1** (Goodwillie, Lurie). *The functor*

$$\Omega^\infty \circ -: \text{Homog}_n(\text{Top}_*, \text{Sp}) \longrightarrow \text{Homog}_n(\text{Top}_*, \text{Top})$$

*is an equivalence.*

Let  $\text{Lin}_n(\mathbf{C}, \mathbf{D})$  denote the category of multilinear functors in  $n$  variables and  $\text{FS}_{\Sigma_n}$  denote the category of *FS-spectra* for  $\Sigma_n$ ,<sup>5</sup> the category of spectra together with an action of  $\Sigma_n$  by automorphisms.

**Theorem 2.2** (Goodwillie, Lurie). *When  $\mathbf{C} = \text{Top}_*$  or  $\text{Sp}$ , the functors*

$$\text{FS}_{\Sigma_n} \xrightarrow{A} \text{Lin}_n(\mathbf{C}, \mathbf{C}) \xrightarrow{B} \text{Homog}_n(\mathbf{C}, \mathbf{C})$$

*are both equivalences, where*

- *A sends  $C_n$  to the multilinear functor*

$$(X_1, \dots, X_n) \longrightarrow (C_n \wedge X_1 \wedge \dots \wedge X_n)_{h\Sigma_n},$$

*and*

- *$B = - \circ \Delta$ , where  $\Delta: X \mapsto (X, \dots, X)$  is the diagonal.*

So there's not really a difference between these different perspectives.

We'd like to push this analogy further: is it true that  $n$ -excisive functors are precisely the things you get by extending  $(n-1)$ -excisive functors by  $n$ -homogeneous excisive functors? Fortunately, this is true, for “nice”  $n$ -excisive functors (where “nice” isn't too restrictive).

Another thing about polynomials is that they're uniquely determined by  $n+1$  points. There's an analogue for functors. Let  $\text{Set}_*^{\leq n+1}$  denote the full subcategory of  $\text{Set}_*$  consisting of sets with cardinality at most  $n+1$  (including the basepoint) and  $i: \text{Set}_*^{\leq n+1} \hookrightarrow \text{Top}_*$  be the usual inclusion.

**Theorem 2.3** (Lurie). *The  $n$ -excisive functors  $F: \text{Top}_* \rightarrow \text{Sp}$  are precisely the functors arising as left Kan extension of a functor  $\tilde{F}: \text{Set}_*^{\leq n+1} \rightarrow \text{Sp}$  along  $i$ .*

**Interpolating between stable and unstable homotopy theory.** Unfortunately, I didn't get everything that happened here, but the idea is to consider the Taylor tower of the identity  $\text{Top}_* \rightarrow \text{Top}_*$ . The first homogeneous piece is  $\Omega^\infty \Sigma^\infty$ , which somehow says that we see stable information, and after that is  $\Omega^\infty(C_2 \wedge X \wedge X)_{\Sigma_2}$  and so on. You can get a spectral sequence out of this.

The Blakers-Massey theorem is another manifestation or maybe explanation of the fact that Goodwillie calculus gets stable phenomena out of unstable ones.

**Theorem 2.4** (Blakers-Massey). *Consider a diagram indexed on the unit  $n$ -cube (the objects are the vertices, interpreted as a poset category using the dictionary order), and assume the map from the space at  $(0, \dots, 0)$  to the space at  $e_i$  is  $k_i$ -connected. Then, the arrow from the homotopy limit of this diagram to the space at  $(0, \dots, 0)$  is  $(-1 + n + \sum k_i)$ -connected.*

So we don't quite have spectra at any finite level, but if you impose higher and higher excisiveness, you can't have bounded connectivity.

**Calculus of embeddings.** Let  $M$  be a manifold, and consider presheaves of topological spaces on it, i.e. functors  $F: O(M)^{\text{op}} \rightarrow \text{Top}$ , where  $O(M)$  is the poset category of open sets on  $M$ , ordered by inclusion. We restrict to the  $F$  such that

- if  $U \subset V$  is an isotopy equivalence, then  $F(U) \rightarrow F(V)$  is a homotopy equivalence, and
- 

$$F\left(\bigcup_i U_i\right) = \text{holim } F(U_i),$$

indexed by the inclusion relations among the  $U_i$ .

<sup>5</sup>This term is due to C. Wu. You might also hear *doubly naïve  $\Sigma_n$ -spectra* or *spectra with a  $\Sigma_n$ -action*.

**Definition 2.5.** Such an  $F$  is an  $n$ -excisive sheaf if for any closed subsets  $A_1, \dots, A_n \subseteq U$ , the homotopy colimit of the “cube” diagram of  $U \setminus \mathcal{A}$  for all  $\mathcal{A} \subset \{A_1, \dots, A_n\}$  is  $F(U)$ .

For  $n = 1$ , this is the same as the usual sheaf condition (which is the strongest condition: the least amount of information is needed to determine it from local information).

### 3. TWO PATHS TO HOMOTOPY COLIMITS: 9/27/17

*“This was recently alluded to in Derived Memes for Spectral Schemes.”*

Today, Adrian spoke again, about two ways to think about homotopy colimits.

Recall that a *relative category* is a pair  $(\mathcal{C}, \mathcal{W})$ , where  $\mathcal{W} \subseteq \mathcal{C}$  is a subcategory containing all isomorphisms. A *relative functor* between relative categories  $(\mathcal{C}, \mathcal{W})$  and  $(\mathcal{C}', \mathcal{W}')$  is a functor  $F: \mathcal{C} \rightarrow \mathcal{C}'$  such that  $F(\mathcal{W}) \subset \mathcal{W}'$ . These are the settings for general abstract homotopy theory.

To really talk about homotopy (co)limits, we need  $\infty$ -categories. But there are five facts about  $\infty$ -categories that might make them easier to digest.

- (1)  $\infty$ -categories generalize ordinary categories. This is true both as a statement to help with intuition, and as an embedding  $\mathbf{Cat} \subset \mathbf{Cat}_\infty$ .
- (2) Any relative category determines an  $\infty$ -category.
- (3) Any relative functor determines an  $\infty$ -functor.<sup>6</sup>
- (4) Let  $(\mathcal{C}, \mathcal{W})$  be a relative category and  $\underline{\mathcal{C}}$  be the  $\infty$ -category it determines. Then, there’s a canonical functor  $L_{\mathcal{C}}: \mathcal{C} \rightarrow \underline{\mathcal{C}}$ .
- (5) In nice cases, the set of relative functors from  $(\mathcal{C}, \mathcal{W})$  to  $(\mathcal{C}', \mathcal{W}')$  determines the space of  $\infty$ -functors  $\underline{\mathcal{C}} \rightarrow \underline{\mathcal{C}'}$ .

Thus we can also work with relative categories, though with some niceness assumptions present.

**Definition 3.1.** Let  $(\mathcal{C}, \mathcal{W})$  be a relative category and  $\mathbf{J}$  be a small category. The homotopy colimit of a functor  $D: \mathbf{J} \rightarrow \mathcal{C}$  is a presentation of  $\varinjlim L_{\mathcal{C}} \circ D$  inside  $\mathcal{C}$ .

Our running examples will be homotopy pushouts (and dually, homotopy pullbacks as homotopy limits).

Another way to think about this comes from the universal property for colimits: if  $\mathcal{C}^{\mathbf{J}}$  denotes the functor category, there’s an adjunction

$$(3.2) \quad \mathcal{C}^{\mathbf{J}} \xrightleftharpoons[\Delta]{\lim} \mathcal{C},$$

where  $\Delta(X)$  is the constant functor  $\mathbf{J} \rightarrow \mathcal{C}$  sending all objects to  $X$  and all morphisms to  $\mathrm{id}_X$ . This is true for any category  $\mathcal{C}$ , but if in addition  $(\mathcal{C}, \mathcal{W})$  is a relative category, we can formally invert the morphisms in  $\mathcal{W}$  to define the homotopy category  $\mathrm{Ho}(\mathcal{C})$ ; then, we have a derived version of (3.2):

$$(3.3) \quad \mathrm{Ho}(\mathcal{C}^{\mathbf{J}}) \xrightleftharpoons[\mathrm{Ho}(\Delta)]{\mathrm{hocolim}} \mathrm{Ho}(\mathcal{C}),$$

One simple idea is that it’s possible to encode  $\infty$ -functors in relative categories, by functors  $F$  that *aren’t* relative, as long as for every relative equivalence  $E: \mathcal{D} \simeq \mathcal{C}$ ,  $F \circ E$  is relative.

**Definition 3.4.** Let  $(\mathcal{C}, \mathcal{W})$  and  $(\mathcal{C}', \mathcal{W}')$  be relative categories, an endofunctor  $Q$  of  $\mathcal{C}$ , and a functor  $F: \mathcal{C} \rightarrow \mathcal{C}'$ , a *left deformation* is a natural transformation  $Q \Rightarrow \mathrm{id}_{\mathcal{C}}$  such that  $F|_{\mathrm{Im} Q}$  is relative.

This includes examples such as (co)fibrant replacement, e.g. in the category of complexes of  $A$ -modules, let  $Q$  be cofibrant replacement (taking a projective resolution), and  $F$  tensoring with something which isn’t necessarily flat over  $A$ . Then,  $F$  behaves badly, but not on projectives.

**Proposition 3.5.** *Given a left deformation  $Q$  such that  $\mathrm{Im}(Q) \simeq \mathcal{C}$  under the natural inclusion, then  $F \circ Q$  is automatically relative.*

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<sup>6</sup> $\infty$ -functors are the correct notion of functor between  $\infty$ -categories; in most situations, these are just called “functors.”

It turns out which left deformation you use doesn't really matter, much like for cofibrant replacement: the natural transformation to the identity means that if  $Q$  and  $Q'$  are left deformations, you have a diagram

$$\begin{array}{ccc} Q'(Q(x)) & \xrightarrow{\sim} & Q(x) \\ \downarrow \wr & & \downarrow \wr \\ Q'(x) & \xrightarrow{\sim} & x, \end{array}$$

where  $\sim$  denotes weak equivalences (i.e. morphisms in  $\mathcal{W}$ ). You can use this to draw a diagram to define the homotopy colimit as a pushout:

$$\begin{array}{ccccc} F(\emptyset) & \longrightarrow & F(0) & & \\ \downarrow & \searrow \sim & \downarrow & \searrow \sim & \\ F(1) & & Q(F(\emptyset)) & \longrightarrow & Q(F(0)) \\ & \searrow \sim & \downarrow & & \downarrow \\ & & Q(F(1)) & \longrightarrow & \text{hocolim } F. \end{array}$$

That is, one way to compute the homotopy colimit is to cofibrantly replace, then compute an ordinary limit.

**Example 3.6.** One concrete model for the (homotopy type of the) homotopy pushout of  $X_0$  and  $X_1$  along maps  $f: X_\emptyset \rightarrow X_0$  and  $g: X_\emptyset \rightarrow X_1$  in topological spaces is a *mapping cylinder*  $X_0 \amalg X_\emptyset \times I \amalg X_1 / \sim$ , where we glue  $X_0$  to  $X_\emptyset \times \{0\}$  using  $f$  and  $X_1$  to  $X_\emptyset \times \{1\}$  using  $g$ .  $\blacktriangleleft$

Another perspective is that this is the same data as a homotopy coherent data  $h_0: X_0 \rightarrow Z$  and  $h_1: X_1 \rightarrow Z$  (where  $Z$  is the mapping cylinder), in that  $h_0 \circ f, h_1 \circ g: X_\emptyset \rightarrow Z$  are homotopic.

One can generalize this to the homotopy colimit over an arbitrary diagram involving a disjoint union indexed over  $n$ -simplices for every composition of  $n$  morphisms in the diagram, modulo an equivalence relation. The idea is that maps out of this space into  $Z$  corresponds exactly to a homotopy coherent diagram indexed by  $\mathbf{J}$ .

It's possible to reconcile this perspective and the more abstract, categorical one, involving a way to replace homotopy colimits with ordinary colimits.

#### 4. THE BLAKERS-MASSEY THEOREM: 10/4/17

Today, Rok spoke on the proof of the Blakers-Massey theorem. All limits (colimits) in today's lecture are homotopy limits (homotopy colimits).

Let's start by recalling some things we already know. Recall that if  $S$  is a set, an  $S$ -cube is a map  $\mathcal{X}: \mathcal{P}(S) \rightarrow S$ , where we denote  $\mathcal{X}(T) = X_T$ . Such a  $\mathcal{X}$  is  $k$ -Cartesian if the natural map

$$X_\emptyset \longrightarrow \varinjlim_{T \neq \emptyset} X_T$$

is  $k$ -connected. The dual notion of  $k$ -co-Cartesian asks for the natural map

$$\varprojlim_{T \subsetneq S} X_T \longrightarrow X_S$$

is  $k$ -connected.  $\mathcal{X}$  is *strongly (homotopy) co-Cartesian* if all of its faces are co-Cartesian (i.e.  $k$ -co-Cartesian for every  $k$ ).

**Lemma 4.1.** *Let  $\mathcal{X}$  and  $\mathcal{Y}$  be  $n$ -cubes. Then,  $f: \mathcal{X} \rightarrow \mathcal{Y}$  is  $k$ -Cartesian as an  $(n+1)$ -cube iff  $\mathcal{F}_y := \text{fib}_y(f)$  is a  $k$ -Cartesian  $n$ -cube for all  $y \in Y_\emptyset$ .*

By the fiber we mean the homotopy fiber.

*Proof.* Let  $\mathcal{Z}$  be  $f: \mathcal{X} \rightarrow \mathcal{Y}$  interpreted as an  $(n+1)$ -cube, and  $\tilde{\mathcal{Y}}$  be  $\text{id}: \mathcal{Y} \rightarrow \mathcal{Y}$  interpreted as an  $(n+1)$ -cube. Therefore we have a diagram

$$\begin{array}{ccc} X_\emptyset & \longrightarrow & \varinjlim_{T \neq \emptyset} \mathcal{Z}_T \\ \downarrow & & \downarrow \\ Y_\emptyset & \xrightarrow{\sim} & \varinjlim_{T \neq \emptyset} \tilde{\mathcal{Y}}_T. \end{array}$$

Therefore we obtain a map

$$(4.2) \quad \text{fib}(X_\emptyset \rightarrow Y_\emptyset) \longrightarrow \text{fib}\left(\varinjlim_{T \neq \emptyset} \mathcal{Z}_T \rightarrow \varinjlim_{T \neq \emptyset} \tilde{\mathcal{Y}}_T\right) \simeq \varinjlim_{\substack{T \neq \emptyset \\ T \subseteq [n+1]}} (\mathcal{Z}_T - \tilde{\mathcal{Y}}_T).$$

But looking at the diagram

$$\begin{array}{ccccc} \text{fib}(\mathcal{X} \rightarrow \mathcal{Y}) & \longrightarrow & \mathcal{X} & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow & & \downarrow \text{id} \\ * & \longrightarrow & \mathcal{Y} & \xrightarrow{\text{id}} & \mathcal{Y}, \end{array}$$

the right-hand side of (4.2) is also weakly equivalent to

$$\varinjlim_{\substack{T \neq \emptyset \\ T \subseteq [n]}} \text{fib}(\mathcal{X}_T - \mathcal{Y}_T),$$

so we're done.  $\square$

We can use this to interpret the Blakers-Massey theorem in terms of more familiar results in algebraic topology.

**Theorem 4.3** (Blakers-Massey, dimension 2). *Suppose  $\mathcal{X}$  is the diagram*

$$(4.4) \quad \begin{array}{ccc} X_\emptyset & \xrightarrow{f_2} & X_2 \\ \downarrow f_1 & & \downarrow \\ X_1 & \longrightarrow & X_{12}, \end{array}$$

*and suppose it is co-Cartesian. If each  $f_i$  is  $k_i$ -connected, then  $\mathcal{X}$  is  $(k_1 + k_2 - 1)$ -Cartesian.*

There's also a dual version. This implies that

$$X_\emptyset \longrightarrow X_1 \times_{X_{12}} X_2$$

is  $(k_1 + k_2 - 1)$ -connected.

**Corollary 4.5** (Freudenthal suspension theorem). *Suppose  $X$  is  $k$ -connected. Then, the map  $X \rightarrow \Omega\Sigma X$  is  $(2k - 1)$ -connected.*

*Proof.* Consider the diagram

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma X. \end{array}$$

The two arrows coming out of  $X$  are  $k$ -connected, so by Theorem 4.3, the map

$$X \longrightarrow * \times_{\Sigma X} * \simeq \Omega\Sigma X$$

is  $(2k - 1)$ -connected.  $\square$

This says that highly connected spaces are close to being stable: taking  $\Omega\Sigma$  of a highly connected space doesn't change it within a large range.

**Definition 4.6.** An *excisive triad*  $(X; A, B)$  is three spaces  $X$ ,  $A$ , and  $B$  such that  $A, B \subset X$ ,  $X = A \cup B$ , and  $A \cap B$  is a nonempty, connected space.

**Corollary 4.7** (Homotopy excision). *Let  $(X; A, B)$  be an excisive triad. Suppose that  $(A, A \cap B)$  is  $k$ -connected and  $(B, A \cap B)$  is  $\ell$ -connected. Then, the inclusion map  $(A, A \cap B) \rightarrow (X, B)$  is  $(k + \ell - 1)$ -connected.*

*Proof.* By Lemma 4.1, it suffices to prove that the map  $A \cap B \rightarrow A \times_X B$  is  $(k + \ell - 1)$ -connected. Then, by Van Kampen's theorem, the diagram

$$\begin{array}{ccc} A \cap B & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & X \end{array}$$

is co-Cartesian, and the arms are  $k$ - and  $\ell$ -connected, so Theorem 4.3 applies and we're done.  $\square$

The proof of the general Blakers-Massey theorem is inductive on the dimension, and Theorem 4.3 will be our base case.

*Proof of Theorem 4.3.* First, let's tackle a special case: we'll show that if  $e^d$  denotes a  $d$ -dimensional cell, the diagram

$$\begin{array}{ccc} X & \longrightarrow & X \cup e^{d_2} \\ \downarrow & & \downarrow \\ X \cup e^{d_1} & \longrightarrow & X \cup e^{d_1} \cup e^{d_2} \end{array}$$

induces a  $(d_1 + d_2 - 3)$ -connected.

This ultimately depends on a transversality argument, which is where the topology sneaks in. The sketch is that if  $p$  is in the interior of  $e^{d_1}$  and  $q$  is in the interior of  $e^{d_2}$ , we want to consider a diagram

$$\begin{array}{ccc} Y \setminus \{p, q\} & \longrightarrow & Y \setminus \{p\} \\ \downarrow & & \downarrow \\ Y \setminus \{q\} & \longrightarrow & Y, \end{array}$$

inducing a map

$$g: (D', \partial D') \longrightarrow (Y \setminus p \times_Y Y \setminus q, Y \setminus \{p, q\}).$$

Let

$$G(x, t_1, t_2) := (g(x_1, t_1), g(x_2, t_2)) \in \check{e}^{d_1} \times \check{e}^{d_2}.$$

This is transverse to  $(p, q)$  if  $i + 2 < d_1 + d_2$ , hence  $(p, q) \notin \text{Im}(G)$  in this range. (Checking transversality is neither trivial nor terrible.)

Now we'll use this to prove the general theorem (still in dimension 2). By CW approximation, we can replace (4.4) with

$$\begin{array}{ccc} X & \longrightarrow & X \cup Y_1 \\ \downarrow & & \downarrow \\ X \cup Y_2 & \longrightarrow & X \cup Y, \end{array}$$

where  $Y_1$  (resp.  $Y_2$ ) is the set of cells of dimension greater than  $k_1$  (resp.  $k_2$ ), and  $Y$  is the set of all of the cells. Since we're interested in the attaching map  $(D^i, \partial D^i) \rightarrow (X \cup Y, X)$ , which necessarily only hits finitely many cells, we can assume we're only attaching a finite number of cells.

This means we can induct over the set of cells, attaching them one at a time, and this is the special case we proved above.  $\square$

Let's also talk about the general case.

**Theorem 4.8** (Blakers-Massey (Goodwillie)). *Let  $\mathcal{X}$  be a strongly co-Cartesian  $n$ -cube, and assume  $\mathcal{X}_{\emptyset} \rightarrow \mathcal{X}_{\{i\}}$  is  $k_i$ -connected. Then,  $\mathcal{X}$  is  $(k_1 + \dots + k_n + 1 - n)$ -Cartesian.*

What does this mean geometrically? We have  $n$  spaces, and we want to do as many pushouts as we can. There's another, more geometric statement, which is the original one



**Theorem 4.9** (Blakers-Whitehead (1953)). *Let  $\mathfrak{U}$  be a finite open cover of  $X$ , and for each  $U \in \mathfrak{U}$ , let*

$$A_{(U)} := \bigcap_{\substack{V \in \mathfrak{U} \\ V \neq U}} U.$$

*If the map  $A_{(U)} \hookrightarrow A_U$  is  $k_U$ -connected, then for  $i < 1 - |\mathfrak{U}| + \sum_{U \in \mathfrak{U}} k_U$ ,  $\pi_i(X; A_U)$  for  $U \in \mathfrak{U}$  is 0.*

*Proof sketch.* Let's assume  $n = |\mathfrak{U}| = 3$ . In this case, we can reduce to a cube of attaching cells as in the proof of Theorem 4.3: we want to prove that

$$\begin{array}{ccccc} X & \xrightarrow{\quad} & X \cup e_1 & & \\ & \searrow & \downarrow & \searrow & \\ & X \cup e_2 & \xrightarrow{\quad} & X \cup e_2 \cup e_1 & \\ \downarrow & & \downarrow & & \downarrow \\ X \cup e_3 & \xrightarrow{\quad} & X \cup e_1 \cup e_3 & & \\ & \searrow & \downarrow & \searrow & \\ & X \cup e_3 \cup e_2 & \xrightarrow{\quad} & X \cup e_1 \cup e_2 \cup e_3 & \end{array}$$

is  $(d_1 + d_2 + d_3)$ -Cartesian (where the attaching map for  $e_i$  is  $d_i$ -connected). To prove this, one applies Theorem 4.3 to each of the three faces containing the vertex  $X$ . This gets you that each face is  $(d_1 + d_2 + d_3 - 1)$ -co-Cartesian, but that's not strong enough — we actually need a stronger version of Theorem 4.3: under the theorem assumptions, if  $\mathcal{X}$  is  $j$ -connected, then it's  $\min\{k_1 + k_2 - 1, j - 1\}$ -Cartesian. This is not hard to prove, and gets you the  $d_1 + d_2 + d_3 - 2$  needed.  $\square$

5. SNAITH SPLITTING: 10/11/17

6. MANIFOLD CALCULUS: 10/18/17

Today, Adrian spoke about manifold calculus.

Recall that if  $F: \mathbf{Top} \rightarrow \mathbf{Sp}$  is a functor preserving filtered colimits, then  $F$  is  $n$ -excisive if it is the left Kan extension of  $F$  restricted to the subcategory of sets of at most  $n$  elements.

**Definition 6.1.** Let  $X$  be a topological space; then,  $\mathbf{Open}_X$  denotes the poset category of open subsets of  $X$ , ordered by inclusion.

Let  $M$  be a manifold (always we will assume smooth and Hausdorff); we'll consider presheaves on  $M$ , functors  $F: \mathbf{Open}_X \rightarrow \mathbf{Top}$ .

**Definition 6.2.** Such a presheaf is an *isotopy functor* if

- (1) it takes filtered homotopy colimits to homotopy limits, and
- (2) for every *isotopy equivalence*  $I: U \hookrightarrow V$  in  $\mathbf{Open}_X$ ,<sup>7</sup> the induced map  $F(V) \rightarrow F(U)$  is a homotopy equivalence.

This feels like a sheaf condition, but isn't. We'll get there.

**Definition 6.3.**  $F$  is *polynomial of degree  $\leq k$*  if for all  $U \in \mathbf{Open}_X$  and pairwise disjoint, closed subsets  $A_1, \dots, A_k \subseteq U$ , the cube defined by the function  $\mathcal{P}([k]) \rightarrow \mathbf{Top}$  defined by

$$S \mapsto F\left(U \setminus \bigcup_{i \in S} A_i\right)$$

is homotopy Cartesian.

For example, when  $k = 1$ , we ask for the square

$$\begin{array}{ccc} F(U) & \xrightarrow{\quad} & F(U \setminus A_0) \\ \downarrow & & \downarrow \\ F(U \setminus A_1) & \xrightarrow{\quad} & F(U \setminus (A_0 \cup A_1)) \end{array}$$

<sup>7</sup>That is, there's an  $F: V \hookrightarrow U$  such that  $i \circ f$  and  $f \circ i$  are isotopic to the identity.

to be a homotopy pullback square.

**Definition 6.4.** For all  $k \in \mathbb{N}$ , let  $\mathbf{Open}_M^k \subset \mathbf{Open}_M$  be the full subcategory on the open subsets of  $M$  diffeomorphic to a disjoint union of  $k$  balls.

**Theorem 6.5.**  $F$  is polynomial of degree  $\leq k$  iff it's the left Kan extension of  $F|_{\mathbf{Open}_M^k}$ .

We will now endow  $\mathbf{Open}_M$  with a family of Grothendieck toposes  $\mathcal{I}_k$ ; in  $\mathcal{I}_k$ , we say that a covering of  $U \in \mathbf{Open}_M$  is a set  $\{V_j\}_{j \in J}$  of open subsets of  $U$  such that for all  $k$ -tuples of points  $a_1, \dots, a_k \in U$ , there is some  $j$  such that  $a_1, \dots, a_k \in V_j$ .

For example, on  $S^2$ ,  $S^2 \setminus (0, 0, 1)$ ,  $S^2 \setminus (1, 0, 0)$ , and  $S^2 \setminus (0, 1, 0)$  is a cover of  $U = S^2$  in  $\mathcal{I}_2$  (any two points must be in one of these three sets). Thus being a cover for  $\mathcal{I}_k$  is harder as  $k$  increases, and hence being a sheaf in the  $\mathcal{I}_k$ -topology is also harder.

**Theorem 6.6.**  $F$  is polynomial of degree  $\leq k$  iff it's an  $\infty$ -sheaf with respect to  $\mathcal{I}_k$ .

**Example 6.7.**

- (1) For  $k = 1$ , consider the functor  $U \mapsto C^\infty(U, X)$ , sending  $U$  to the space of smooth maps from  $U$  to  $X$ .
- (2) For a related example (also  $k = 1$ ), let  $F(U)$  be the space of immersions  $U \hookrightarrow X$ .
- (3) For a functor which is polynomial of degree  $\leq k$ , consider the map  $U \mapsto C^\infty(U^{\amalg k}, M)$ .  $\blacktriangleleft$

Anyways, this formalism will allow us to construct Taylor approximations to isotopy functors  $F: \mathbf{Open}_M \rightarrow \mathbf{Top}$  by restricting to  $\mathbf{Open}_M^k$  and left Kan extending. For an open  $U \subseteq M$ , let

$$T^k F(U) := \operatorname{holim}_{D^n \amalg \dots \amalg D^n \subset U} F(D^n \amalg \dots \amalg D^n),$$

where we take a  $k$ -fold disjoint union.

*Remark.* Let  $\mathbf{Emb}_n$  denote the category of  $n$ -manifolds and embeddings. There's an analogue of  $\mathcal{I}_k$ , which is the first thing you'd write down. For nice functors  $\mathbf{Emb}^{\text{op}} \rightarrow \mathbf{Top}$ , sheafification is the same thing as polynomial approximation.  $\blacktriangleleft$