# GEOMETRY AND STRING THEORY SEMINAR: SPRING 2018

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Contents



## 1. Comments on Global Symmetry, Anomalies, and Duality in $(2+1)d$: 1/24/18

Today's talk was given by Val Zakharevich, on the paper [BHS17].

**Definition 1.1.** Let $A$ and $B$ be UV theories which, under renormalization group flow, flow to the same IR theory $C$. Then we'll say that theories $A$ and $B$ are dual.

**Example 1.2.** Let $N_f \leq N$. Then there is a conjectured duality between $\mathrm{SU}(2)_k$-Chern-Simons theory with $N_f$ scalars, also known as *Wilson-Fischer theory*, and $\mathrm{U}(k)_{-N+N_f/2}$-Chern-Simons theory with $N_f$ fermions. ◀

The paper [BHS17] computes the higher symmetries and anomalies of both sides of this duality and of several others; 't Hooft anomaly matching tells us that these should be the same.

This is related to our overarching goal of understanding $\mathrm{QCD}_4$ with a single fermion $\psi$, which has a Lagrangian

$$\mathcal{L} = \mathrm{tr}(F \wedge \star F) + \overline{\psi}\not{D}\psi + m\overline{\psi}\psi, \tag{1.3}$$

where $m \in \mathbb{C}$ is a parameter whose phase diagram we're interested in. Let $\overline{m}$ denote the mass of the domain wall theory, which is a 3D QCD theory, so $\overline{m}$ is real. If $m$ is real and negative, there's a phase transition: for $\overline{m} \ll 0$, the low energy theory is believed to be trivial, and for $\overline{m} \gg 0$ ($m$ negative of larger magnitude), the low-energy theory is believed to be $\mathrm{SU}(N)$-Chern-Simons theory at level 1. The transition point, at $\overline{m} = 0$, should be described by $\mathrm{SU}(N)_{1/2}$ with a single fermion.

### 1.1. Level-rank duality.

*Level-rank duality* is the conjecture that $\mathrm{SU}(N)_k$-Chern-Simons theory and $\mathrm{U}(k)_{-N}$-Chern-Simons theory are isomorphic. A natural generalization is to consider $\mathrm{SU}(N)_k$ together with $N_f$ scalar fields of mass $m$, where $m \in \mathbb{R}$ and $N_f < N$.

- If $m \ll 0$, the Higgs mechanism implies this should be the $\mathrm{SU}(N-N_f)_k$ theory.
- if $m \gg 0$, we should expect the $\mathrm{SU}(N)_k$ theory again.

On the dual side, let's consider $\mathrm{U}(k)_{-N-N_f/2}$ with $N_f$ fermions of mass $m \in \mathbb{R}$.

- If $m \gg 0$, we expect to get $\mathrm{U}(k)_{-N+N_f/2}$.
- If $m \ll 0$, we expect to get $\mathrm{U}(k)_{-N}$, with Lagrangian shifted by $N_f$:

$$\mathcal{L} = \frac{-N+N_f}{4\pi}\,\mathrm{tr}\left(b\,\mathrm{d}b - \frac{2i}{3}b^3\right) + \overline{\psi}\not{D}\psi + m\psi\overline{\psi} + \text{c.c.} \tag{1.4}$$

Level-rank duality switches positive-mass scalars and negative-mass fermions, promising dualities between $\mathrm{SU}(N-N_f)_k \longleftrightarrow \mathrm{U}(k)_{-N+N_f/2}$ and $\mathrm{SU}(N)_k \longleftrightarrow \mathrm{U}(k)_{-N}$.

1.2. **Symmetries.** We now see what symmetries these theories have. First, $\mathrm{SU}(N)_k$ with $N_f$ scalars. On a 3-manifold $M$, the fields are triples $(P, A, \Phi)$, where

- $P \to M$ is a principal $\mathrm{SU}(N)$-bundle with connection,
- $A$ is a connection on $P$, and
- $\varphi \in \Gamma(P \times_{\mathrm{SU}(N)} (\mathbb{C}^N \otimes \mathbb{C}^{N_f}))$ is the $N$ scalar fields.

The Lagrangian is

$$\mathcal{L}(A, \Phi) = \frac{k}{4\pi} \operatorname{tr}\left( A \wedge \mathrm{d}A + \frac{2}{3} A^3 \right) + |D_A \varphi|^2 + m|\varphi|^2 + \lambda |\varphi|^4. \tag{1.5}$$

As usual, we have an $\mathrm{SU}(N)$-gauge symmetry, and there's also a $\mathrm{U}(N_f)$-symmetry acting on $\mathbb{C}^{N_f}$, in which $e^{2\pi i/N}\mathbf{1}$ acts by a gauge symmetry. Hence the global symmetry group (for these symmetries) is $\mathrm{U}(N_f)/(\mathbb{Z}/N)$.

**Ansatz 1.6.** Let $G$ be a compact Lie group and $k$ be a level for $G$, and let $\mathcal{L}_{G_k}$ denote the Lagrangian for Chern-Simons theory with group $G$ and level $k$. Let

$$1 \longrightarrow G \xrightarrow{\ \rho\ } H \xrightarrow{\ \sigma\ } L \longrightarrow 1 \tag{1.7}$$

be a short exact sequence of Lie groups. Then, we take as an ansatz that coupling the $G_k$ theory to a principal $L$-bundle (i.e. given a principal $L$-bundle $P \to M$, we sum over the groupoid of all principal $H$-bundles which quotient to $L$) produces a classical gauge theory for $H$ with Lagrangian $\mathcal{L}_{\widetilde{k}}$ such that

$$\mathcal{L}_{G_k}(P_G, A_G) = \mathcal{L}_{\widetilde{k}}((P_G, A_G) \times_G H). \tag{1.8}$$

When $G$ is finite (so we're in the setting of Dijkgraaf-Witten theory) this is studied in [KT14].

In our setting, (1.7) specializes to $G = \mathrm{SU}(N)$, $H = (\mathrm{SU}(N) \times U(N_f))/(\mathbb{Z}/N)$, and $L = U(N_f)/(\mathbb{Z}/N)$. Chern-Simons theories for $G$ are labeled by $H^4(BG; \mathbb{Z})$, and the map $\rho\colon G \to H$ defines a pullback

$$\rho^*\colon H^4(BH; \mathbb{Z}) \longrightarrow H^4(BG; \mathbb{Z}).$$

Given a $k \in H^4(BG; \mathbb{Z})$, we want to know whether we can implement the theory with a global $L$-symmetry; hence we want to know whether $k \in \mathrm{Im}(\rho^*)$; the theory is anomalous iff this is not true.

If the theory is anomalous, we'd like to compute the anomaly. Suppose that we have a $\widehat{k}_{\mathbb{R}} \in H^4(BH; \mathbb{R})$ such that $\rho^*(\widehat{k}_{\mathbb{R}}) = k_{\mathbb{R}}$ (i.e. the image of $k$ in real cohomology). Then, we can't eliminate the anomaly, but we can couple to a bulk theory: suppose that we can extend $(P, A) \to M$ to $(P_H, A_H) \to X$, where $X$ is a compact 4-manifold with $\partial X = M$. Then, we have an action

$$\text{“}S_{\widehat{k}_{\mathbb{R}}}(A_H)\text{”}\colon ((P_H, A_H) \to X) \longmapsto \int_X \widehat{k}_{\mathbb{R}}(F_H), \tag{1.9}$$

where $F_H$ is the curvature of $A_H$.

This depends on the choice of $X$ and $P_H \to X$ extending $P$, but we can hope that the dependence goes away after exponentiating the action. Let $X'$ be another compact 4-manifold bounding $M$, and let $(P'_H, A') \to X'$ be another extension of $(P, A)$. Let $\widehat{X} \coloneqq X \cup_M X'$; then, $(P_H, A_H)$ and $(P'_H, A'_H)$ glue to a principal $G$-bundle $\widetilde{P}_H \to \widetilde{X}$ with connection $\widetilde{A}$. Then we have that

$$e^{2\pi i S_{\widehat{k}_{\mathbb{R}}}(\widetilde{P}_H, \widetilde{A}_H)} = S_{\hat{\hat{k}}}(\widetilde{P}_H \times_H L) \in \mathbb{R}/2\pi i\mathbb{Z} \tag{1.10}$$

for some $\hat{\hat{k}} \in H^4(BL; \mathbb{R}/\mathbb{Z})$. This $\hat{\hat{k}}$ tells us the anomaly, so we're interested in computing it. Ultimately, this comes from a question purely in algebraic topology: we have a big commutative diagram

$$\begin{tikzcd}
H^4(BL; \mathbb{R}/\mathbb{Z}) \arrow[r] & H^4(BH; \mathbb{R}/\mathbb{Z}) \arrow[r] & H^4(BG; \mathbb{R}/\mathbb{Z}) \\
& H^4(BH; \mathbb{R}) \arrow[r] \arrow[u] & H^4(BG; \mathbb{R}) \arrow[u] \\
& H^4(BH; \mathbb{Z}) \arrow[r] \arrow[u] & H^4(BG; \mathbb{Z}) \arrow[u]
\end{tikzcd} \tag{1.11}$$

Then we have $\hat{\hat{k}}$ in the upper left, $\widehat{k}_{\mathbb{R}}$ in the middle, and $k$ in the lower right. In this case the anomaly theory is purely topological. The computation for the dual theory follows a similar story, but is harder.

To actually calculate this, you can use the Leray-Serre spectral sequence; $k \in H^4(BG;\mathbb{Z})$ transgresses to something in $H^5(BL;\mathbb{Z})$, which tells you which component $\hat{\hat{k}}$ is in.

## 2. On Gauging Finite Subgroups: 1/31/18

Today's talk was given by Dan Freed, on Tachikawa's paper [Tac17].

Let's start with classical electromagnetism on $n$-dimensional Minkowski spacetime $\mathbb{M}$. Choose an $A \in \Omega^1_{\mathbb{M}}/\mathrm{d}\Omega^0_{\mathbb{M}}$, and let $F_A = \mathrm{d}A$. Maxwell's laws tell us that

$$\begin{aligned}\mathrm{d}F_A &= 0\\ \mathrm{d}{\star}F_A &= 0,\end{aligned}$$

but more generally we could let $\mathrm{d}F_A = j_B \in \Omega^3$, a *magnetic current*, and $\mathrm{d}{\star}F_A = j_E \in \Omega^{n-1}$, an *electric current*. If both $j_E$and $j_B$ are nonzer, the theory has an anomaly.

We next consider the quantum theory, by doing some things such as Wick rotation, charge quantization, and downshifting the degree of $A$.[1] The Wick-rotated quantum theory is formulated on an oriented[2] Riemannian manifold $X$, and $A$ is a map $X \to \mathbb{RZ}$, or its exponentiated version $\lambda\colon X \to \mathbb{T}$.

If we introduce point charges $p_1, \ldots, p_m \in X$ with charges $k_1, \ldots, k_m \in \mathbb{Z}$,[3] then the electric current, inserted in the exponentiated action, is

$$\prod_{j=1}^m \lambda(p_j)^{k_j} = \exp\left(2\pi \sum_{j=1}^m i k_j A(p_j)\right), \tag{2.1}$$

which has degree $n$ with $\mathbb{Z}$ coefficients.

The magnetic current is defined using a circle bundle $P \to X$ with connection $\Theta$; one can think of $\lambda$ as a section of $P$, and this data is used to define the kinetic term. This is a degree-2 term with $\mathbb{Z}$ coefficients.

If $L \coloneqq P \times_{\mathbb{T}} \mathbb{C}$ is the complex line bundle associated to $P$, then the *electric coupling* is

$$\prod_{i=1}^m \lambda(p_i)^{k_i} \in \bigotimes_{i=1}^m L_{p_i}^{\otimes k_i}. \tag{2.2}$$

From this perspective, the anomaly is "$\int_X j_B \cdot j_E$."

*Remark.* The term $\lambda$ only exists if $P \to X$ is topologically trivializable.

This is akin to something that happens in topological field theory. Let $Z$ denote the 4D oriented TQFT defined by summing the trivial theory over spin structures. Then $Z(\mathbb{CP}^2) = 0$, since $\mathbb{CP}^2$ admits no global spin structure. But if one varies the manifold in a family, interesting things may nonetheless happen. ◂

*Remark.* One could also replace $\mathbb{T}$ by any finite abelian group $A$. In this case a lot of things are still the same, though we don't choose a connection for $P$. In this case, the magnetic current lives in $H^1(X;A)$ rather than $H^2(X;\mathbb{Z})$ (and we could have thought of it as $H^1(X;\mathbb{T})$ for the $\mathbb{T}$-theory). However, the electric current lives in $H^n(X;A^\vee)$, where $A^\vee \coloneqq \mathrm{Hom}(A,\mathbb{T})$ denotes the Pontrjagin dual of $A$.

We could think of the magnetic current as a map $X \to BA$; in this case the electric current is a map $X \to B^nA^\vee \coloneqq K(A^\vee, n)$. In the rest of this talk, we will adopt this more abstract approach, but you should keep the rigid, geometric approach that we started with in mind for intuition or an example. ◂

From this perspective, the anomaly is a map

$$BA \times B^nA^\vee \longrightarrow B^{n+1}\mathbb{T},$$

which is induced from the pairing $A \otimes A^\vee \to \mathbb{T}$. On a closed, oriented manifold $X$ with an electric and magnetic current we can pull this back to $X$ and integrate it; this is the partition function for the anomaly theory.

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[1] TODO: I have no idea what just happened.

[2] One could impose time-reversal symmetry and study the theory more generally on unoriented manifolds, but for our purposes this will not be necessary.

[3] TODO: I may have gotten this wrong.

**Gauging.** Suppose $T$ is some kind of theory (here we probably mean a Wick-rotated field theory on Riemannian manifolds, perhaps with extra structure and background fields), and suppose it has a (global) $\Gamma$-symmetry,[4] where $\Gamma$ is a finite group. This means that we can couple the theory to $\Gamma$-bundles, formulating it on manifolds with the above data and a background principal $\Gamma$-bundle.

Sometimes this symmetry gets tangled up with other symmetries, e.g. if $T$ is a $\sigma$-model to a space $X$ and $\Gamma$ is a symmetry of $X$. Then, depending on how we implement the symmetry, we might end up with sections of some associated bundle.

But if this is not the case (in a $\sigma$-model sense, if $B\Gamma$ splits off from the target), then we can sum over the maps to $B\Gamma$. This process is called *gauging*.

Now suppose $A \coloneqq \Gamma$ is abelian. Then the gauged theory has a higher symmetry akin to electromagnetism, a $B^{n-2}A^\vee$ symmetry, and we can couple the theory to a background $B^{n-2}A^\vee$-field, which is exactly putting in the electric current. If you try to gauge this symmetry, you'll end up back where you started with, which is a kind of Fourier transform.

On a compact oriented manifold $X$, the electric coupling lives in $H^1(X;A) \times H^{n-1}(X;A^\vee)$; there's a product map

$$H^1(X;A) \times H^{n-1}(X;A^\vee) \longrightarrow H^n(X;\mathbb{T});$$

then we can evaluate on the fundamental class to obtain an element of $\mathbb{T}$, which is what one inserts into the action. This exhibits the two cohomology groups as Pontrjagin duals of each other, so the Fourier transform is an isomorphism between spaces of functions on them.

Turning to the material in the paper, let

$$1 \longrightarrow A \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

be a short exact sequence of groups, where $A$ is abelian. In particular, $A$ is normal in $\Gamma$; we do not assume it is central. An example (in which $A$ is not central) is

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow S_3 \longrightarrow \mathbb{Z}/2 \longrightarrow 1.$$

We consider the situation of a theory $T$ with a $\Gamma$-symmetry, hence an $A$-symmetry, and we assume we can gauge $A$. What happens when we do this?

The new theory should have a $G$-symmetry, which arises as follows: given a principal $G$-bundle $Q \to X$, we can sum over pairs $(P \to X, \varphi)$, where $P \to X$ is a principal $\Gamma$-bundle and

$$\begin{tikzcd} Q \arrow[rr, "\varphi", "\cong"'] \arrow[dr, "G"'] & & P/A \arrow[dl, "G"] \\ & X & \end{tikzcd}$$

is an isomorphism of principal $G$-bundles. In this case the magnetic current arises from the map $BG \to B^2A$ coming from extending the fiber sequence $BA \to B\Gamma \to BG$.

There's another symmetry associated to $B^{n-2}A^\vee$, which arises in a similar way as before. So, following what we did with electromagnetism, if you try to couple the theory to $BG \times B^{n-1}A^\vee$, two things might happen (and are not mutually exclusive).

(1) The two symmetries might interact nontrivially, producing an extension

$$B^{n-1}A^\vee \longrightarrow \mathcal{X} \longrightarrow BG.$$

(2) There may be an anomaly $\mathcal{X} \to B^{n+1}\mathbb{T}$.

For example, $\mathcal{X}$ might be $B\mathcal{G}$, where $\mathcal{G}$ is an extension of $G$ by $B^{n-2}A^\vee$, so an extension of a group by a higher group.

First suppose $\Gamma$ is anomaly-free, so that we can gauge it. Then, the anomaly for the theory coupled to $G$-symmetry is the compositions of the maps

$$BG \times B^{n-1}A^\vee \longrightarrow B^2A \times B^{n-1}A^\vee \longrightarrow B^{n+1}\mathbb{T},$$

where the first map comes from the connecting map $BG \to B^2A$ and the second is the Pontrjagin dual pairing.

[4]"Global symmetry" is redundant, because there is no other kind of symmetry.

Now suppose the $\Gamma$ theory has an anomaly. There are different ways to produce anomalies, such as beginning with a map $MSO \wedge (B\Gamma)_+ \to B^{n+1}\mathbb{T}$, which produces a *gauge-gravity anomaly*, but let's begin with just $B\Gamma \to B^{n+1}\mathbb{T}$, or a *pure gauge anomaly*. This data is equivalent to a cohomology class $[\alpha] \in H^{n+1}(B\Gamma; \mathbb{T})$.

The presence or absence of the anomaly arises from a filtration on $H^{n+1}(B\Gamma; \mathbb{T})$ induced from the fiber sequence $BA \to B\Gamma \to BG$: $BG$ has a cell structure, and we can ask whether a cohomology class over the basepoint extends over the $n$-skeleton. We will discuss what happens in the case when the cohomology class is in the last two pieces of the filtration, which are simpler.

To compute this, one uses the *Leray-Serre spectral sequence*

$$E_2^{p,q} := H^p(G; H^q(A; \mathbb{T})) \Longrightarrow \mathrm{gr}_F H^{p+q}(\Gamma; \mathbb{T}).$$

Here $H^q(A; \mathbb{T})$ is a nontrivial $G$-module from the residual $G$-action induced by $\Gamma$. Now let's look at the two cases.

(1) Suppose $[\alpha]$ lives in the highest filtered part; then, there's an $\overline{\alpha}\colon BG \to B^{n+1}\mathbb{T}$ lifting $\alpha$ across the map $B\Gamma \to BG$. In this case, $\overline{\alpha}$ is the anomaly: if you gauge the $A$-symmetry, the theory couples to $BG \times B^{n-1}A^\vee$, and the anomaly is a sum of the electromagnetic anomaly and $\overline{\alpha}$. This corresponds to a transgression in the spectral sequence.

(2) If $[\alpha]$ lives in the next highest filtered part, we get something in $H^n(G; \underline{A}^\vee)$ (the underline representing a nontrivial $G$-action). The paper [Tac17] considers only the special case, where $A$ is central and we get a bundle

$$B\Gamma \times_{BG} \mathcal{X},$$

where $B\Gamma \to BG$ is a $BA$-bundle and $\mathcal{X} \to BG$ is a $B^{n-1}A^\vee$ bundle. Then we have maps

$$BG \to B^2A \times B^nA^\vee \to B^{n+2}\mathbb{T},$$

and we consider the case where this composite is null. This means it lifts to a map

$$\beta\colon B\Gamma \times_{BG} \mathcal{X} \to B^{n+1}\mathbb{T}.$$

Then, we take $\beta|_{B\Gamma}$ is the anomaly in $\Gamma$. If this was the original anomaly in $\Gamma$, then the theory couples to the extension $B^{n-1}A^\vee \to \mathcal{X} \to BG$, corresponding to some kind of extension theory, and the anomaly for this theory is $\beta|_X$. Then you could gauge the subgroup $B^{n-1}A^\vee$ and go back to the original theory.

(3) Suppose $[\alpha]$ lives in the $n^{\text{th}}$ piece of the filtration, and assume $A$ is central. Then the map $B\Gamma \to B\mathbb{T}^{n+1}$ doesn't descend to $BG$, but does descend to a map $BG \to E$, where $E$ fits into a sequence $B^{n+1}\mathbb{T} \to E \to B^nA^\vee$. This corresponds to an anomaly in a generalized cohomology theory, albeit a relatively simple one. We could try to use this to build $\mathcal{X}$. But you also have to add the electromagnetic anomaly, and this is a little bit unclear.

## 3. Theta, time-reversal, and temperature I: 2/7/18

Today's talk was given by Andy Neitzke, on the paper [GKKS17].

The goal of this paper is to say something new about the phases of 4D Yang-Mills theory with group $G = \mathrm{SU}_N$. There's no supersymmetry here.

Before we tackle this, however, let's look at a simpler example, a 2D $\mathrm{U}_1$ gauge theory with $N$ charged scalar fields $z^i$, $i = 1, \dots, N$, subject to the constraint that $\sum |z_i|^2 = 1$. This is equivalent data to a map to $\mathbb{CP}^{N-1}$, and hence this model is sometimes thought of as a $\sigma$ model with target $\mathbb{CP}^{N-1}$. The $\mathrm{U}_1$ gauge field is denoted $a$, and its curvature is $F_a$. The action on a surface $\Sigma$ is

$$S = \int_\Sigma \sum_i |D_a z^i|^2 + \frac{i\theta}{2\pi}\int_\Sigma F_a + \frac{1}{g^2}\int_\Sigma F_a \wedge \star F_a. \tag{3.1}$$

Here $\theta$ is a real parameter, but we typically think of the $\theta$ theory as equivalent to the $\theta + 2\pi$ theory, because on a closed surface $\Sigma$, $\int_\Sigma F_a = 2\pi n \in 2\pi\mathbb{Z}$, so after exponentating, $e^{in\theta} = e^{in(\theta+2\pi)}$.

The first question you might ask is: what's the (global) symmetry of this theory? The answer is $\mathrm{PSU}_N$, which might be a surprise: there's a $\mathrm{U}_N$ symmetry on the fields, but we gauged the $\mathrm{U}_1$ subgroup, so only $\mathrm{PSU}_N := \mathrm{U}_N/\mathrm{U}_1$ acts faithfully on the (gauge-invariant) local operators.

**Example 3.2.** The operator $z^1$ transforms nontrivially under the $\mathrm{U}_1$ symmetry, hence is not gauge invariant. The operator $z^1\overline{z}^1$ is gauge invariant. ◀

The more modern way of saying that this theory has a $\mathrm{PSU}_N$ symmetry is to say that it couples to background $\mathrm{PSU}_N$ gauge fields. Let $A_{\mathrm{bkgd}}$ be a $\mathrm{PSU}_N$-connection; we want to include $A_{\mathrm{bkgd}}$ in the theory such that if it's the trivial connection, we get back (3.1).

Somewhat like what we did last time, instead of $\mathrm{U}_1$-connections, we'll consider lifts $A$ of $A_{\mathrm{bkgd}}$ to a $\mathrm{U}_N$-connection. Locally $A = a \cdot I_N + A_{\mathrm{bkgd}}$. When writing the action, the first term doesn't change much, but the $\theta$ term is interesting, since we no longer have a $\mathrm{U}_1$-connection.

If $A_{\mathrm{bkgd}} = 0$, then $A = a \cdot I_N$, and $F_a = (1/N)\operatorname{tr}(F_A)$. Hence we'll replace $F_a$ with $(1/N)\operatorname{tr}(F_A)$, even when $A_{\mathrm{bkgd}} \neq 0$. That is, the new $\theta$ term is

$$\frac{i\theta}{2\pi N}\int_\Sigma \operatorname{tr}(F_A). \tag{3.3}$$

We do something analogous for the kinetic term. But we pay a price — since we divided by $N$, this term is not invariant under $\theta \mapsto \theta + 2\pi$. Instead, it's invariant under $\theta \mapsto \theta + 2\pi N$. If we shift $\theta \mapsto \theta + 2\pi$, the exponentiated action changes by

$$\exp\left(\frac{2\pi i}{N}\int_\Sigma w_2(A_{\mathrm{bkgd}})\right). \tag{3.4}$$

In particular, the change only depends on the background field. Therefore this is also true for the partition functions:

$$Z(\theta + 2\pi, A_{\mathrm{bkgd}}) = Z(\theta, A_{\mathrm{bkgd}}) \cdot \exp\left(\frac{2\pi i}{N}\int_\Sigma w_2(A_{\mathrm{bkgd}})\right). \tag{3.5}$$

This exhibits a mixed anomaly between the shift symmetry $\theta \mapsto \theta + 2\pi$ of $\mathbb{Z}$ and the $\mathrm{PSU}_N$ symmetry. You can't gauge both of them at once.

*Remark.* From the perspective of symmetries as coupling to background bundles, we want to express this shift symmetry with a background $\mathbb{Z}$-bundle. The mixed anomaly is saying that $\theta$ becomes a section of an associated bundle, such that $e^{i\theta}$ is still a function to $\mathrm{U}_1$, but not a constant function. ◂

Another option is to add some function of the background fields to the action; this is called a *counterterm*. One natural choice is, for some $p \in \mathbb{Z}$,

$$p\frac{2\pi i}{N}\int w_2(A_{\mathrm{bkgd}}) \in (2\pi i\mathbb{Z})/(2\pi i N\mathbb{Z}). \tag{3.6}$$

If we exponentiate this, it's an $N^{\mathrm{th}}$ root of unity, and therefore the theory with parameters $\theta$ and $p$ should be equivalent to the theory with parameters $\theta + 2\pi$ and $p + 1$.

One thing we can do with this is implement time reversal, thought of as reversing the orientation of $\Sigma$. This acts on $S$ by $\theta \mapsto -\theta$. In the absence of a background $\mathrm{PSU}_N$ field, this is only a symmetry when $\theta = 0$ or $\theta = 2\pi$, in which case the time-reversed theory is equivalent.

With a background field and the counterterm (3.6), time-reversal acts as $(\theta, p) \mapsto (-\theta, -p)$. Therefore we can get time-reversal symmetry for $\theta = 0$ and $p = 0$. And what about $\theta = \pi$? In this case, $(-\pi, -p) \simeq (\pi, -p+1)$, and to get this equivalent to $(\pi, p)$, we need $p \equiv -p + q \bmod N$, i.e. $2p \equiv 1 \bmod N$. Therefore if $N$ is even, there's no way to preserve time-reversal at $\theta = \pi$.

This means the theory has a mixed anomaly between time-reversal symmetry and a $\mathrm{PSU}_N$ symmetry. Anomaly matching means this tells us something about the infrared theory. There are two possibilities.

- The IR theory "has the same anomaly," in that it can be coupled to $\mathrm{PSU}_N$ background fields in such a way that, if we shift the background coupling by $p$, then $p \mapsto -1 + p$ under time reversal.
- At least one of the $\mathrm{PSU}_N$ or time-reversal symmetries is broken in the infrared. This is a little weird, and is an example of spontaneous symmetry breaking.

What actually happens is the second option, and specifically time-reversal symmetry is broken.[5] The theory is gapped, and therefore the IR theory should be an explicit topological field theory we could look at.

[5] More generally, any continuous symmetry cannot be spontaneously broken in 2D.

**Back to Yang-Mills theory.** Remember that we're really interested in Yang-Mills theory with $G = \mathrm{SU}_N$, whose action is

$$aS = \frac{1}{g^2}\int \operatorname{tr}(F \wedge \star F) + \frac{i\theta}{8\pi^2}\int \operatorname{tr}(F\wedge F). \tag{3.7}$$

Again, after exponentiating the action, we have an equivalence between the theory with parameter $\theta$ and the theory with parameter $\theta + 2\pi$. Now there's a $\mathbb{Z}/N$ one-form symmetry, or a $B\mathbb{Z}/N$ symmetry, so we should be able to couple the theory to background $B\mathbb{Z}/N$ fields. Whatever these are, they should have a characteristic class $[B] \in H^2(X; \mathbb{Z}/N)$.

As before, this can be done, and the price is that the theory is not periodic in $\theta$. And as before, there's a specific term expressing that failure: assume $N$ is even. When we shift $\theta \mapsto \theta + 2\pi$, $e^S$ shifts by

$$\exp\left(\frac{2\pi i(N_1)}{2N}\int_X \mathfrak{P}_2(B)\right),$$

where $\mathfrak{P}_2 \colon H^2(X;\mathbb{Z}/N) \to H^4(X;\mathbb{Z}/2N)$ is a cohomology operation called the *Pontrjagin square*. Therefore we can regard this as an element of $\mathbb{Z}/N$, and introduce a counter-term as above: $(\theta, p) \sim (\theta + 2\pi, p + N - 1)$. Therefore one can calculate that time-reversal symmetry cannot hold when $N$ is even at $\theta = \pi$.

## 4. Theta, time-reversal, and temperature II: 2/14/18

Today, Andy Neitzke continued to speak on [GKKS17].

Today, instead of focusing on a toy model, we will look at the case of interest, 4D Yang-Mills theory with gauge group $\mathrm{SU}_N$, whose action on $X$ has a term of the form

$$\frac{i\theta}{8\pi^2}\int \operatorname{tr}(F\wedge F).$$

Before coupling to background fields, the theory with parameter $\theta$ is (believed to be) equivalent to the theory with parameter $\theta + 2\pi$. We will write this as $\theta \simeq \theta + 2\pi$.

This theory has a $B\mathbb{Z}/N$ symmetry, i.e. a $\mathbb{Z}/N$ one-form symmetry; here $\mathbb{Z}/N$ arises as the center of $\mathrm{SU}_N$. At $\theta = 0, \pi$, there's also time-reversal symmetry $T$: $T(\theta) = -\theta \simeq \theta$ iff $\theta = 0, \pi \bmod 2\pi$. When we try to implement both of them simultaneously, we'll discover a mixed anomaly.

This theory couples to a background field, a $\mathbb{Z}/N$-gerbe $B$, which is classified by its characteristic class $[B] \in H^2(X;\mathbb{Z}/N)$.[6]

*Remark.* In [GKKS17], this coupling is described explicitly, and they provide a nice model for the background $B\mathbb{Z}/N$-field, a pair $(B, C)$ where $B \in \Omega^2(X)$ and $C$ is the connection of a line bundle, such that $NB = F_C$, where $F_C$ denotes the curvature of $C$. This is something kind of like $\mathbb{Z}/N$ de Rham differential forms, but instead of something being exact, we want it to be $N$-torsion up to exact things.

In this case, if $\lambda \in \Omega^1(X)$, the symmetry sends $B \mapsto B + \mathrm{d}\lambda$ and $C \mapsto C + N\lambda$. ◂

You can also add counterterms depending purely on the background field $B$ (or its characteristic class), such as

$$\frac{2\pi i p}{2N}\int \mathfrak{P}_2([B]), \tag{4.1}$$

where $\mathfrak{P}_2 \colon H^2(X;\mathbb{Z}/N) \to H^4(X;\mathbb{Z}/2N)$ is the Pontrjagin square, a cohomology operation, and $p \in \mathbb{Z}/2N$, so that this makes sense after exponentiation.

The Pontrjagin square has an abstract definition, but in the explicit model described above,

$$\int \mathfrak{P}_2([B]) = \int \mathrm{d}C \wedge \mathrm{d}C. \tag{4.2}$$

Now there is an equivalence of theories

$$(\theta, p) \simeq (\theta + 2\pi, p + N - 1) \tag{4.3}$$

and time-reversal symmetry acts by $T(\theta, p) = (-\theta, -p)$.

[6] If you don't really know what a gerbe is, that's OK; we're mostly just going to use its characteristic class.

So at $\theta = \pi$, we need $p = -p + N - 1$ to implement both time-reversal and the $B\mathbb{Z}/N$ symmetry. If this is not the case (e.g. $N$ is even), time-reversal symmetry is broken by coupling to $B\mathbb{Z}/N$ fields, exhibiting a mixed anomaly between these two symmetries.

This story has been an UV story so far; 't Hooft anomaly matching posits that there are IR consequences, which could include

(1) the vacuum supporting a TQFT which has the same anomaly (i.e. coupling the background fields in the same way),
(2) the theory is gapless (which is regarded as unlikely, and would imply a low-energy conformal field theory rather than a topological field theory),
(3) the theory is gapped, and the $B\mathbb{Z}/N$-symmetry is broken, or
(4) the theory is gapped, and the time-reversal symmetry is broken.

So we expect there's a TQFT (case (2) is considered unlikely), but its nature is unclear — is it invertible? Case (1) is also considered unlikely, especially for large $N$. Case (3) is also unlikely, which we'll say more about later, which implies that time-reversal symmetry is probably broken in IR. This implies at $\theta = \pi$, time-reversal symmetri is broken and there are two vacua, exchanged by $T$.

*Remark.* In a different theory, $\mathcal{N} = 1$ supersymmetric Yang-Mills theory, it's known (at a physical level of rigor) that there are $N$ vacua. By adding a mass term $m \in \mathbb{R}_+$, one can perturb the theory slightly, breaking supersymmetry. For $\theta \neq 0, \pi$, you get two vacua, but at $\theta = 0, \pi$, there are two degenerate vacua and time-reversal symmetry is broken.

This parallel story is analogous to ours, and suggests that we're on the right track. ◀

You can draw a phase diagram for these theories in $\theta$: at $\theta = \pi$, it looks like a quartic with two global minima, and elsewhere it looks like a quartic with one global minimum and two local minima. At $\theta = 0, 2\pi$, the quartic has a unique local minimum. So there must be a phase transition at $\pi$.

We promised to discuss why the $B\mathbb{Z}/N$ symmetry is unbroken, which has something to do with confinement. The theory has line defects with charges $(a, b) \in (\mathbb{Z}/N)^\vee \times \mathbb{Z}/N$ ($(\mathbb{Z}/N)^\vee \cong \mathbb{Z}/N$ abstractly, of course, but this illustrates how it arises). The first component is an electric charge, and the second is magnetic. But these aren't usual line defects: for $b = 0$ they are, but for $b \neq 0$, we need additional data to formulate the defect on a closed 1-manifold $\ell$, which is the topological data of a surface which bounds $\ell$.

*Remark.* If we had started with $\mathrm{PSU}_N$, the roles of $a$ and $b$ would be switched, with $a \neq 0$ defects arising as line defects on boundaries of surfaces. ◀

If $W$ is a Wilson line given by a representation $V$ of $\mathrm{SU}_N$, then its charge $(a, b)$ is $a = V|_{\mathbb{Z}/N}$ and $b = 0$, though of as charges of "probe particles" we can insert in the theory.

The basic question here is: which line bundles are confined? Here, $Q$ is confined iff a defect $L$ of charge $Q$ has

$$\langle L(\mathrm{loop})\rangle \sim \exp(-\alpha \Delta X \Delta T) \tag{4.4}$$

asymptotically, i.e. the energy cost of creating the loop is linear in $\Delta X$. Heuristically, this means that correlation functions with an insertion of $L$ on a line vanish. The higher-symmetry way to say this is that the charge $Q$ transforms nontrivially under the unbroken 1-form symmetry. The idea is that invariant plus transforming nontrivially forces it to be zero.

So the question of who's confined is related to the question of which symmetries remain unbroken in a given vacuum. There have been many numerical simulations of $\mathrm{SU}_N$-Yang-Mills suggesting that at any $\theta$, all Wilson lines are confined. In the language of higher symmetries, this implies the $B\mathbb{Z}/N$ symmetry is unbroken.

## 5. Theta, time-reversal, and temperature III: 2/21/18

Today, Fei Yan gave the third and last talk on [GKKS17], today focusing on temperature — the point is to use all the material that has been developed to study phases of 4D $\mathrm{SU}_N$ Yang-Mills theory.

The action of this theory is given by

$$S = \int_X \left( -\frac{1}{4g^2} \operatorname{tr}(F \wedge \star F) + \frac{i\theta}{8\pi^2} \operatorname{tr}(F \wedge F) \right). \tag{5.1}$$

We've talked about a $B\mathbb{Z}/N$ symmetry arising because $\mathbb{Z}_N = Z(\mathrm{SU}_N)$, and, at $\theta = 0, \pi$, there's a time-reversal symmetry $T\colon \theta \mapsto -\theta$.

For even $N$, at $\theta = \pi$, there is a mixed anomaly between these two symmetries, and there is no counterterm which resolves this. For odd $N$ at $\theta = \pi$, the anomaly can be resolved with a counterterm, but there is no counterterm that works for both $\theta = 0$ and $\theta = \pi$.

In the IR limit, one of these symmetries is spontaneously broken, and there's an argument that it's time-reversal symmetry for $\theta = \pi$ and both even and odd $N$. Hence we can draw a phase diagram at zero temperature, as we discussed last time, and there is a first-order phase transition at $\theta = \pi$.

Now we turn to finite temperature. That is, take a 3-manifold $Y$ and formulate the theory on $Y \times S^1_\beta$ (here $S^1_\beta$ is a circle with circumference $\beta$), where we regard $S^1$ as time. As $\beta \to \infty$, this recovers the zero-temperature case, and the canonical partition function at temperature $T \coloneqq 1/\beta$ is given by the partition function of the theory on $Y \times S^1_\beta$, which justifies calling this the finite-temperature setting.[7]

Let's regard this as a 3D theory in $Y$. What are its symmetries?

- If $\theta \neq 0, \pi$, the $B\mathbb{Z}/N$ symmetry splits as $B\mathbb{Z}/N$ and $\mathbb{Z}/N$ symmetries (based on whether the one-form symmetry is in the $Y$ direction or the $S^1_\beta$ direction).
- If $\theta = 0, \pi$, we have $B\mathbb{Z}/N$ and $\mathbb{Z}/N$ symmetries as before, and the time reversal symmetry $T$ passes to a global $\mathbb{Z}/2$ symmetry by reversing the orientation on $S^1_\beta$.[8]

Recall that in the 4D theory, we had line operators charged by $(a, b) \in (\mathbb{Z}/N)^\vee \times \mathbb{Z}/N$, and the genuine line operators are those with $b \equiv 0 \bmod N$ (i.e. not coupled to some surface), which are Wilson lines. When $b \not\equiv 0 \bmod N$, the line has to bound a surface. When we dimensionally reduce, we could wrap a genuine line operator around $S^1$, obtaining a point operator, or we could not wrap, getting a line operator. But if $b \neq 0$, things are slightly different: you could not wrap around the $S^1$, getting the same thing back, or we could wrap around $S^1$, getting a point operator attached to a line.

There are special local operators, called *Polyakov loops* $P$, which are Wilson loops wrapped around $S^1$. These detect confinement at finite temperature, which happens iff $\langle P \rangle = 0$. As you increase the temperature, the theory deconfines, which is a phase transition called (unsurprisingly) *the confinement-deconfinement phase transition.*

Akin to the mixed $B\mathbb{Z}/N$ and $T$ symmetry in 4D, the 3D theory has a mixed anomaly between the $B\mathbb{Z}/N$ symmetry, the $\mathbb{Z}/N$ symmetry, and the $\mathbb{Z}/2$ symmetry. We've already studied this in the low-temperature limit, so let's turn to the high-temperature limit, where $\beta \ll \Lambda_{\mathrm{YM}}^{-1}$ (here $\Lambda_{\mathrm{YM}}^{-1}$ is the *dynamical scale* of the Yang-Mills theory). The high-temperature limit of Yang-Mills was studied in the 1980s, which is convenient for us.

Let $N = 2$. Then $\mathbb{Z}/2$ acts in two different ways; let $(\mathbb{Z}/2)^C$ denote the symmetry arising through the center of $\mathrm{SU}_2$ and $(\mathbb{Z}/2)^T$ denote the symmetry coming from time-reversal on $S^1$. Let $g^2_{3d} \sim g^2_{\mathrm{YM},4d}\beta^{-1}$ be an $\mathrm{SU}_2$ gauge field, and $\Phi$ be an adjoint scalar. Let

$$U = Pe^{i\oint_{S^1} A} \coloneqq e^{i\beta\Phi}, \tag{5.2}$$

where $P = \operatorname{tr} U$, and pick a gauge

$$\Phi = \begin{pmatrix} \phi(x) & 0 \\ 0 & -\phi(x) \end{pmatrix}, \tag{5.3}$$

where $\phi \sim \phi + 2\pi/\beta$.

There are two minima of $V^\phi_{\mathrm{eff}}$: $\beta\phi = 0$ ($U = I$) and $\beta\phi = \pi$ ($U = -I$). In this case $(\mathbb{Z}/2)^c$ sends $\phi \mapsto \phi/\beta$ and $U \mapsto -U$, and the $B\mathbb{Z}/2$ and $(\mathbb{Z}/2)^T$ symmetries are unbroken (the latter when $\theta = 0, \pi$). The argument uses Polyakov loops: since $\operatorname{tr} U$ is the expectation of a Polyakov loop, but is nonzero, then the $(\mathbb{Z}/2)^C$ symmetry is broken. The argument is similar for general $N$.

To study phase transitions, let's organize this all into a table.

- The $B\mathbb{Z}/N$ symmetry is unbroken at all values of $\theta$ for the high-temperature and low-temperature cases.

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[7] This is an instance of a very general fact about quantum mechanics: the partition function $Z = \operatorname{tr}(e^{-\beta H})$ is the partition function for the system at temperature $1/\beta$.

[8] There's another symmetry given by orientation-reversal on $Y$ (after Wick rotation?), but it doesn't enter the discussion.

- The $\mathbb{Z}/N$ symmetry is broken for all $\theta$ in the high-temperature case, and is preserved in the low-temperature case.
- The $(\mathbb{Z}/2)^T$ symmetry is unbroken at $\theta = 0, \pi$ in the high-temperature case, but is only unbroken at $\theta = 0$ in the low-temperature case.

This makes the mixed anomaly apparent: at $\theta = \pi$, in both the high-temperature and low-temperature cases, at least one symmetry is broken in the IR theory.

This tells us information about a two-dimensional phase diagram in $\theta$ and $T = 1/\beta$. We know at low temperature there's a phase transition at $\theta$, and at some high temperature there's a confinement-deconfinement phase transition. For $N = 2$ confinement-deconfinement is second-order, and for $N > 2$ it's first-order.[9]

What happens when these two phase transitions meet? Let's specialize to $N = 2$ for a bit. To answer the question, we can gauge the $B\mathbb{Z}/2$ symmetry in 3D. A general fact about higher-form (global) symmetry is that if you gauge a discrete $q$-form symmetry, what you get has a $(d - q - 2)$-form symmetry. With $d = 3$ and $q = 1$, we expect a 0-form $\mathbb{Z}/2$-symmetry, which we'll call $(\mathbb{Z}/2)^B$.

However, this does *not* happen everywhere. It does happen when $\theta \neq 0, \pi$, so we obtain a $(\mathbb{Z}/2)^C \times (\mathbb{Z}/2)^T$ synmmetry. At $\theta = 0$, we have $(\mathbb{Z}/2)^C \times (\mathbb{Z}/2)^B \times (\mathbb{Z}/2)^T$, and at $\theta = \pi$, we get a $D_8$ symmetry, arising through the extension

$$1 \longrightarrow (\mathbb{Z}/2)^C \times (\mathbb{Z}/2)^T \longrightarrow D_8 \longrightarrow (\mathbb{Z}/2)^B \longrightarrow 1. \tag{5.4}$$

The appearance of this $D_8$ is a bit of a surprise, and has something to do with the anomaly. There are two proofs present in [GKKS17].

Let $c$ be a generator of the $(\mathbb{Z}/2)^C$ symmetry, and define $b$ and $t$ similarly. The Polyakov loop is given by the loop with charge $(1, 0)$. We're also interested in the twisted sectors $A := (0, 1)$ and $B := (1, 1)$ — $A$ generates the $B\mathbb{Z}/2$ symmetry, which we've wrapped around the circle, and similarly with $B$.

To see why we get a $D_8$, we'll compute how $c$, $b$, and $t$ act on $A$ and $B$.

- Explicitly, $c$ is the non-identity central element of $\mathrm{SU}_2$. Thus it sends $A \mapsto A$ and $B \mapsto -B$.
- Since both $A$ and $B$ were twisted operators attached to a line, $b$ maps $A \mapsto -A$ and $B \mapsto -B$.
- Since $t$ comes from time-reversal, we recall that $\theta \mapsto \theta + 2\pi$ shifts your line operators by $(a, b) \mapsto (a + b, b)$. The upshot is that $t$ exchanges $A$ and $B$.

Therefore $tct = cb$, $tbt = b$, and $bc = bc$, so if you think of $A, B, -A, -B$ as the vertices of a square, you get all of the symmetries of the square from $c$ and $b$, and therefore get a $D_8$.

We want to use this to understand the phase diagram. The analysis of this $D_8$-action indicates that $(\mathbb{Z}/2)^B$ must be spontaneously broken near the intersection of the two phase transitions.

- Below the confinement-deconfinement transition at $\theta < \pi$, we have two vacua spanned by $A$ and $B$, with $c$ acting by $A \mapsto -A$.
- Below the confinement-deconfinement transition at $\theta > \pi$, we have the same, but with $cb = bc$ acting by $B \mapsto -B$.
- Above the confinement-deconfinement transition, these two vacua separate into four vacua $(\pm A, \pm B)$.

If the two phase transitions didn't intersect, you would be able to show that the $D_8$ symmetry is completely unbroken, so they must intersect, and the $\theta = \pi$ phase transition comes up to meet the confinement-deconfinement one. For $N = 2$, at least, it goes slightly higher, but in general we don't know whether it just meets it.

*Remark.* Some of these arguments are definitely true for large $N$, but require an assumption on the breaking of the $D_8$ symmetry for smaller $N$. You could relax that assumption and end up with a more exotic phase diagram. ◀

## 6. 2-groups: 2/28/18

These are Arun's prepated notes for his talk in GST, on 2-groups. Today, all categories are small categories.

[9] This was concluded using lattice simulations; it's not clear if there's a continuum argument for it.

6.1. **2-groups and crossed modules.** There are many ways to define 2-groups and a web of equivalences between them. We'll discuss a few of them in this part of the talk.

Various notions of 2-groups were introduced and compared by Whitehead [Whi46], Mac Lane-Whitehead, Brown-Spencer [BS76], Hoàng [Hoà75], Joyal-Street, and Baez-Lauda [BL04]. The definition given here, following Baez-Lauda, encodes the philosophy that a group is a monoid in which every element is invertible.

**Definition 6.1.** A *monoidal category* $(\mathsf{C}, \otimes, 1, \alpha, \ell, r)$ is:

- a category $\mathsf{C}$,
- a functor $\otimes\colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}$,
- an *identity object* $1 \in \mathsf{C}$,
- and natural isomorphisms

$$\alpha_{x,y,z}\colon (x \otimes y) \otimes z \xrightarrow{\cong} x \otimes (y \otimes z), \tag{6.2a}$$

$$\ell_x\colon 1 \otimes x \xrightarrow{\cong} x, \tag{6.2b}$$

$$r_x\colon x \otimes 1 \xrightarrow{\cong} x, \tag{6.2c}$$

subject to two coherence conditions that we won't write down (but can be found in [BL04, §2]).

In general, categorification turns conditions into data: associativity is implemented by choosing $\alpha$, and similarly with identity. Here's another example.

**Definition 6.3.** Let $(\mathsf{C}_1, \otimes_1)$ and $(\mathsf{C}_2, \otimes_2)$ be monoidal categories. A *monoidal functor* is a functor $F\colon \mathsf{C}_1 \to \mathsf{C}_2$ together with natural isomorphisms $F(x) \otimes_2 F(y) \cong F(x \otimes_1 y)$ and $F(1_{\mathsf{C}_1}) \cong 1_{\mathsf{C}_2}$ such that (some coherence conditions in [BL04, §2]).

There is a similar notion of a *monoidal natural transformation*; again see [BL04, §2].

**Definition 6.4.** A *2-group* $\mathbb{G}$ is a monoidal category such that for every object $x \in \mathbb{G}$, there's a $y \in \mathbb{G}$ such that $x \otimes y \cong 1$ and $y \otimes x \cong 1$.

A morphism of 2-groups is a monoidal functor. We also have monoidal natural transformations, which means that there's a *2-category* of 2-groups. Approximately what this means is that there objects and morphisms as usual, but given morphisms $f, g\colon x \to y$, there can be "2-morphisms" $H\colon f \Rightarrow g$.[10]

**Example 6.5.** Since 2-groups are supposed to describe mixed 0-form and 1-form symmetries, they should specialize to ordinary groups if one of the symmetries is trivial.

(1) Given a group $G$, let $\mathsf{G}$ be the category whose objects are the elements of $G$ and with only identity morphisms. Group multiplication defines a monoidal structure on $\mathsf{G}$, making it into a 2-group. Heuristically, we've forgotten about the level-1 information, leaving just level-0 information.
(2) Dually, given a group $G$, we can consider the category $BG$ with a single object and $\mathrm{Hom}_{BG}(*, *) = G$ (as a set, then with composition defined by group multiplication). This also defines a 2-group (e.g. the associator uses the associativity of $G$), which we can think of as having a trivial level-0 part and $G$ for its level-1 part. ◀

An alternative perspective on 2-groups is to switch the role of the group and the category. If $\mathsf{C}$ is a small category, let $C_0$ denote its set of objects and $C_1$ its set of morphisms. Then we have two maps $s, t\colon C_1 \rightrightarrows C_0$ sending an arrow to its source, resp. target, and a map $i\colon C_0 \to C_1$ sending $x \mapsto \mathrm{id}_x$. It's possible to encode the axioms of a category (composition, identity, etc.) into properties of $s$, $t$, and $i$.

If $\mathbb{G}$ is a 2-group with objects $G_0$ and $G_1$, then $G_0$ and $G_1$ are groups under tensor product, and $s$, $t$, and $i$ are group homomorphisms, so in some sense the diagrams that encode $(\mathbb{G}, \otimes)$ are formulated entirely in the category of groups! Such a diagram is called an *internal category* in $\mathsf{Grp}$.

However, we now have access to more structure: we can define a *Lie 2-group* to be a pair of Lie groups $G_0$ and $G_1$ and Lie group homomorphisms $s, t, \colon G_1 \to G_0$ and $i\colon G_0 \to G_1$ satisfying the relations encoding associativity, etc.

[10] Making this precise, and even precisely defining 2-categories, requires some effort. We will avoid such questions while noting that good solutions exist.

**Proposition 6.6.** *This construction extends to an equivalence of 2-categories between 2-groups and internal categories in* $\mathsf{Grp}$*.*

See [FB02, §3] for a proof. This in particular implies that every diagram of groups and group homomorphisms $(G_0, G_1, i, s, t)$ satisfying the axioms defining a category arises in this way from a 2-group $\mathbb{G}$.

**Example 6.7.** Let $X$ be a topological space with basepoint $x$. The *fundamental 2-group* of $X$, denoted $\pi_{1,2}(X, x)$ is the 2-group whose objects are loops in $X$ based at $x$ and whose morphisms are equivalence classes of homotopies between paths (where two homotopies are equivalent if they are themselves homotopic). The monoidal structure arises by composition of paths. Every 2-group arises in this way. ◂

6.2. **Classifying spaces and 2-gauge fields.** A third perspective on 2-groups in algebraic topology is that they describe spaces with only two nontrivial homotopy groups, through their classifying spaces. This leads to notions of principal 2-bundles and connections on them (though we won't have a lot to say about this). Fix a 2-group $\mathbb{G}$ (i.e. a discrete 2-group: we want $G_0$ and $G_1$ to be discrete).

We will describe a connected space $B\mathbb{G}$ such that $\pi_1(B\mathbb{G}) \cong H_1 := \pi_0\mathbb{G}$ and $\pi_2(B\mathbb{G}) \cong H_2 := \mathrm{Aut}(1_{\mathbb{G}})$. One such choice is $K(H_0, 1) \times K(H_1, 2)$,[11] but this is wrong: heuristically, it's telling us that the two symmetries don't mix at all. In particular, a map from a space $X$ to this space is data of a principal $H_0$-bundle and an $H_1$-gerbe, corresponding in physics to an $H_0$ symmetry and a $BH_1$-symmetry, but they don't mix at all.

Instead, the mixing of these two symmetries is encoded by making $B\mathbb{G}$ a fiber bundle over $K(H_0, 1)$ with fiber $K(H_1, 2)$. We specify the fiber bundle $p\colon B\mathbb{G} \to K(H_0, 1)$ by its homotopy cofiber, a map $k\colon K(H_0, 1) \to \Sigma K(H_1, 2) \simeq K(H_1, 3)$ called the *k-invariant* of the space $B\mathbb{G}$. Namely, the associator of $\mathbb{G}$ defines a map $H_1 \times H_1 \times H_1 \to H_2$, which is a cocycle for $H^3(H_0; H_1) = H^3(BH_0; H_1)$. Since $BH_0 \simeq K(H_0, 1)$ and cohomology is represented by maps into Eilenberg-Mac Lane spaces, we have a natural identification $H^3(H_0; H_1) \cong [K(H_0, 1), K(H_1, 3)]$ sending the associator of $\mathbb{G}$ to the $k$-invariant of $B\mathbb{G}$.

$B\mathbb{G}$ is an example of a *homotopy 2-type*, i.e. a homotopy type with only two nontrivial homotopy groups. This generalizes the fact that if $G$ is a discrete group, $\pi_i(BG)$ is only nontrivial when $i = 1$.

**Proposition 6.8.** *Every homotopy 2-type is the classifying space of some 2-group, and there is an equivalence of 2-categories between 2-groups and homotopy 2-types.*

If $G$ is an (ordinary) compact Lie group, its classifying space $BG$ is a moduli space for principal $G$-bundles, meaning that every principal $G$-bundle $P \to X$ is the pullback of the universal bundle $EG \to BG$ along a map $X \to BG$, together with a uniqueness condition. We would like something similar to be true for 2-groups, but this is a place in which categorification makes things much harder: this is spelled out by Bartels [Bar04], Baez-Scheiber [BS04], and Baez-Stevenson [BS08], but the interactions between category theory and topology were complicated enough that I didn't figure out anything useful.

As a substitute, if $\mathbb{G}$ is finite, we can use the Postnikov tower of $B\mathbb{G}$ to describe principal $\mathbb{G}$-bundles. Here's what we want.

- Over any space $X$, isomorphism classes of principal $\mathbb{G}$-bundles are in bijection with $[X, B\mathbb{G}]$.
- If the 2-group symmetry is really just an $G_0$-symmetry for some group $G_0$, principal $\mathbb{G}$-bundles should be the same as principal $G_0$-bundles.
- If the 2-group symmetry is really just a $BG_1$-symmetry, then principal $\mathbb{G}$-bundles should be the same as $G_1$-gerbes.

We also want the $k$-invariant to appear somehow.

Given a map $X \to B\mathbb{G}$, we can compose with the projection and obtain a map $X \to BH_0$, so we get a principal $H_0$-bundle $P$. But we can also pull back $B\mathbb{G} \to BH_0$ to $X$, producing a fiber bundle with fiber $K(H_1, 2)$, i.e. an $H_1$-gerbe $Q$. The $k$-invariant provides a constraint on how these are related — unfortunately, I wasn't able to figure out how this goes, but it's going to go something like this: given three loops $\ell_1, \ell_2, \ell_3$ in $X$ with a common basepoint, they have monodromies $h_1, h_2, h_3 \in H_0$ for $P$. The $k$-invariant, as a cocycle for group cohomology, defines a $g = k(h_1, h_2, h_3) \in H_1$, and this should be some kind of monodromy around a higher-dimensional sphere for $Q$ that's related to $\ell_1, \ell_2$, and $\ell_3$.

[11] Here, $K(G, n)$ is an *Eilenberg-Mac Lane space*, which we've been more frequently denoting $B^nG$ in this seminar; the notation $K(G, n)$ is more common in algebraic topology.

6.3. **2-group symmetries in physics.** In the remainder of this talk we will discuss examples of 2-group symmetries in physics.

**Example 6.9** (The Yetter model)**.** For any finite 2-group $\mathbb{G}$, there's a TQFT called the Yetter model with a $\mathbb{G}$-symmetry, defined in much the same way as Dijkgraaf-Witten theory.[12] This model was developed by Yetter [Yet93], Birmingham-Rakowski [BR96], and Mackaay [Mac99], and is a special case of a general construction of Quinn (TODO: cite).

Fix a finite 2-group $\mathbb{G}$ and a cohomology class $\alpha \in H^n(B\mathbb{G}; \mathbb{R}/\mathbb{Z})$. Then $\alpha$ defines a characteristic class for principal $\mathbb{G}$-bundles: if $P \to M$ is a principal $\mathbb{G}$-bundle, it defines up to homotopy a classifying map $f_P \colon M \to B\mathbb{G}$; we let $\alpha(P) \coloneqq f_P^*\alpha \in H^n(M; \mathbb{R}/\mathbb{Z})$.

The Yetter model is the $n$-dimensional, oriented TQFT with a fluctuating $\mathbb{G}$-gauge field $P$ and whose action is $\alpha(P)$. Thus its partition function on a closed $n$-manifold $M$ is summed over $\mathrm{Bun}_{\mathbb{G}}(M)$ using the "2-groupoid measure," so that

$$Z_{\mathbb{G},\alpha}(M) = \text{``}\int_{\mathrm{Bun}_{\mathbb{G}} M} e^{i\pi\langle \alpha(P), [M]\rangle}\, \mathrm{d}P\text{''} = \sum_{P \in \pi_0 \mathrm{Bun}_{\mathbb{G}}(M)} \frac{|^2\mathrm{Aut}(P)|}{|\mathrm{Aut}(P)|} \exp(i\pi\langle \alpha(P), [M]\rangle). \tag{6.10}$$

◀

*Remark.* I wanted to say something about the QED-like theories with 2-group symmetries discussed in [CDI18], but ran out of time, and also didn't completely understand the examples given. ◀

## References


[Bar04] Toby Bartels. Higher gauge theory I: 2-bundles. 2004. https://arxiv.org/pdf/math/0410328.pdf. 12

[BHS17] Francesco Benini, Po-Shen Hsin, and Nathan Seiberg. Comments on global symmetries, anomalies, and duality in $(2+1)d$. *Journal of High Energy Physics*, 2017(4):135, Apr 2017. https://arxiv.org/pdf/1702.07035.pdf. 1

[BL04] John C. Baez and Aaron D. Lauda. Higher-dimensional algebra. V: 2-groups. *Theory and Applications of Categories*, 12:423–491, 2004. https://arxiv.org/pdf/math/0307200.pdf. 11

[BR96] Danny Birmingham and Mark Rakowski. On Dijkgraaf-Witten-type invariants. *Letters in Mathematical Physics*, 37(4):363–374, Aug 1996. https://arxiv.org/pdf/hep-th/9402138.pdf. 13

[BS76] Ronald Brown and Christopher B Spencer. $\mathcal{G}$-groupoids, crossed modules and the fundamental groupoid of a topological group. *Indagationes Mathematicae (Proceedings)*, 79(4):296–302, 1976. 11

[BS04] John C. Baez and Urs Schreiber. Higher gauge theory: 2-connections on 2-bundles. 2004. https://arxiv.org/pdf/hep-th/0412325.pdf. 12

[BS08] John C. Baez and Danny Stevenson. The classifying space of a topological 2-group. 2008. https://arxiv.org/pdf/math/0801.3843.pdf. 12

[CDI18] Clay Cordova, Thomas T. Dumitrescu, and Kenneth Intriligator. Exploring 2-group global symmetries. 2018. https://arxiv.org/abs/1802.04790. 13

[FB02] Magnus Forrester-Barker. Group objects and internal categories. 2002. https://arxiv.org/pdf/math/0212065.pdf. 12

[GKKS17] Davide Gaiotto, Anton Kapustin, Zohar Komargodski, and Nathan Seiberg. Theta, time reversal and temperature. *Journal of High Energy Physics*, 2017(5):91, May 2017. https://arxiv.org/pdf/1703.00501.pdf. 5, 7, 8, 10

[Hoà75] Xuân Sính Hoàng. *Gr-categories*. PhD thesis, Université Paris VII, 1975. 11

[KT14] Anton Kapustin and Ryan Thorngren. Anomalous discrete symmetries in three dimensions and group cohomology. *Phys. Rev. Lett.*, 112(23):231602, Jun 2014. https://arxiv.org/pdf/1404.3230.pdf. 2

[Mac99] Marco Mackaay. Spherical 2-categories and 4-manifold invariants. *Advances in Mathematics*, 143(2):288–348, 1999. https://arxiv.org/pdf/math/9805030.pdf. 13

[Tac17] Yuji Tachikawa. On gauging finite subgroups. 2017. https://arxiv.org/pdf/1712.09542.pdf. 3, 5

[Whi46] J. H. C. Whitehead. Note on a previous paper entitled on adding relations to homotopy groups. *Annals of Mathematics*, 47(4):806–810, 1946. https://www.jstor.org/stable/pdf/1969237.pdf?refreqid=excelsior:bda15bb501371a33e1492799f7db4f8d. 11

[Yet93] David N. Yetter. TQFT's from homotopy 2-types. *Journal of Knot Theory and Its Ramifications*, 02(01):113–123, 1993. 13


[12] This is not the same thing as the Crane-Yetter model!