GROMOV-WITTEN THEORY LEARNING SEMINAR

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Contents

1. An Overview of Gromov-Witten Theory: 1/29/18

Today, Jonathan spoke, delivering an overview of Gromov-Witten theory and how associativity of quantum cohomology leads to applications in enumerative geometry. Today we always work over $\mathbb{C}$, and follow Fulton-Pandharipande's notes [FP96].

Classically, if X is a nonsingular projective variety and $\beta \in H_2(X;\mathbb{Z})$, we want to know how many algebraic curves in X represent the class β. This relates to very classical questions, such as: if you have $3d-1$ points in $\mathbb{P}^n$, how many degree-d curves pass through them?

Definition 1.1. To simplify notation, let $A_d(X) := H_{2d}(X;\mathbb{Z})$, and similarly $A^d(X) := H^{2d}(X;\mathbb{Z})$.

The moduli space of stable maps. Another important ingredient, whose construction we will punt on, is the *moduli space of stable maps*. Here we summarize its definition. Let X be a smooth projective variety and $\beta \in A_1(X)$. The moduli space of stable maps, denoted $\mathcal{M}_{g,n}(X,\beta)$ is the moduli space of isomorphism classes of pointed maps

$$u\colon (C, p_1, \dots, p_n) \longrightarrow X \tag{1.2}$$

where C is a projective nonsingular curve of genus g, $p_1, \dots, p_n$ are distinct marked points in C, and $u_*([c]) = \beta$. We must impose a stability condition which ensures these maps have finitely many automorphisms, where an automorphism $(C, p_1, \dots, p_n) \to (C', p_1], \dots, p_n')$ must send $p_i \mapsto p_i'$ and commute with the maps to X.

This is all right, but we really want something compact, and therefore will have to consider stable maps which are slightly worse. The compactification $\overline{\mathcal{M}}_{g,n}(X,\beta) \supset \mathcal{M}_{g,n}(X,\beta)$ is the space of stable maps as in (1.2), subject to the following conditions.

- C is a projective, connected, reduced, genus-g curve with at worst nodal singularities, and the p_j are distinct smooth points.
- Stability: for every irreducible compact $E \subset C$ such that if $E \simeq \mathbb{P}^1$ and $u(E) = \{\mathrm{pt}\}$, then E contains at least 3 of the points p_i.
- If E is genus 1 and $u(E) = \{\mathrm{pt}\}$, then E contains at least one of the points p_i.

Why is this a compactification? The idea is that if $u\colon (C, p_1, \dots, p_n) \to X$ is a smooth curve, we can let two points collide. In the compactified moduli space, the collision is avoided by adding another $\mathbb{P}^1$ to C intersecting near the collision point; then, the two points can live in distinct irreducible components.

The next question is: what's the dimension of $\overline{\mathcal{M}}_{g,n}(X,\beta)$ be? Naïvely, the expected dimension is

$$n + \int_\beta c_1(X) + (\dim X - 3)(1-g) = n + 3g - 3 + \chi(TX|_C). \tag{1.3}$$

Here c_1 is the first Chern class, and $\int_\beta c_1(X)$ represents the cap product pairing $A_1(X) \otimes A^1(X) \to \mathbb{Z}$, and $\chi(TX|_C)$ denotes its Euler characteristic:

$$\chi(TX|_C) := h^0(C; TX|_C) - h^1(C; TX_|C). \tag{1.4}$$

This does not depend on the choice of C representing β, which is a fun fact about characteristic classes.

Why is (1.3) a reasonable guess? Here's what's going on.

- The $3g-3$ represents the dimension of the moduli space of the curve C, hence representing how C can change on its own.
- The $\chi(TX|_C)$ represents how C can deform in X.
- The n is the extra data corresponding to the marked points.

We said "naïve," and indeed (1.3) is not the dimension of $\overline{\mathcal{M}}_{g,n}(X,\beta)$ in all cases. But it is true in nice cases, and then you can do some cool stuff.

Gromov-Witten invariants. There are natural *evaluation maps* $p_i\colon \overline{\mathcal{M}}_{g,n}(X,\beta) \to X$ sending

$$(u\colon (C, p_1, \dots, p_n) \to X) \longmapsto u(p_i). \tag{1.5}$$

We can pull back cohomology classes along these maps: suppose $\gamma_1, \dots, \gamma_n \in A^*(X)$. Then, let

$$I_\beta(\gamma_1, \dots, \gamma_n) := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]} p_1^*(\gamma_1) \smile \cdots \smile p_n^*(\gamma_n). \tag{1.6}$$

This is called a *Gromov-Witten invariant* for X. The thing that we're integrating over requires some very technical work to define in general, but for spaces which are "nice" (convex and homogeneous, which we'll discuss later), it's not so bad. $\mathbb{P}^n$ is an example of such a space.

Suppose $\gamma_1, \dots, \gamma_n$ have the correct dimensions such that (1.6) is a number. Then there's an enumerative interpretation of (1.6) (in the convex case): the number of pointed maps $u\colon \Sigma_g \to X$ such that $u_*([\Sigma_g]) = \beta$ and if Γ_i is a subvariety representing the Poincaré dual to γ_i,[1] then $u(p_i) \in \Gamma_i$. Here Σ_g is a curve of genus g.

Important properties.

Proposition 1.7. *If $\beta = 0$, the only nonzero Gromov-Witten invariants occur when $n = 3$.*

Proof sketch. If $\beta = 0$, there's an identification $\overline{\mathcal{M}}_{0,n}(X,\beta) \cong \overline{\mathcal{M}}_{0,n} \times X$, which carries all of the evaluation maps to projection onto X, where $\overline{\mathcal{M}}_{0,n}$ is the (compactified) moduli space of genus-0 curves with n marked points. Call this map π. Then,

$$\begin{aligned} I_\beta(\gamma_1, \dots, \gamma_n) &= \int_{\overline{\mathcal{M}}_{0,n}(X,0)} p_1^*(\gamma_1) \smile \cdots \smile p_n^*(\gamma_n) \\ &= \int_{\overline{\mathcal{M}}_{0,n} \times X} \pi^*(\gamma_1 \smile \cdots \smile \gamma_n) \\ &= \int_{\pi_*([\overline{\mathcal{M}}_{0,n} \times X])} \gamma_1 \smile \cdots \smile \gamma_n. \end{aligned}$$

If $n < 3$, $\mathcal{M}_{0,n}$ is empty, because any choice of n points in $\mathbb{P}^1$ doesn't have a finite automorphism group. For $n > 3$, π has positive-dimensional fibers. ⊠

If $n = 3$, then

$$I_0(\gamma_1, \gamma_2, \gamma_3) = \int_X \gamma_1 \smile \gamma_2 \smile \gamma_3, \tag{1.8}$$

so this Gromov-Witten invariant isn't too hard to calculate.

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[1] We can dodge the Steenrod realizability problem because every even-degree homology class of $\mathbb{P}^n$ is represented by a complex subvariety.

Proposition 1.9. *Suppose $\gamma_1 = 1 \in A^0(X)$. Then, $I_\beta(1, \gamma_2, \dots, \gamma_n)$ is nonzero only when $\beta = 0$ and $n = 3$.*

Proof sketch. If $\beta \neq 0$, $p_1^*(1) \smile \cdots \smile p_n^*(\gamma_n)$ is the pullback of a class in $\overline{\mathcal{M}}_{0,n-1}(X, \beta)$ along the map

$$\overline{\mathcal{M}}_{0,n}(X, \beta) \longrightarrow \overline{\mathcal{M}}_{0,n-1}(X, \beta) \tag{1.10}$$

which forgets the first point. There's a projection formula which then finishes the proof in a similar way to Proposition 1.7. ⊠

In the case $\beta = 0$ and $n = 3$, there's a similar formula to (1.8):

$$I_0(1, \gamma_2, \gamma_3) = \int_X \gamma_2 \smile \gamma_3. \tag{1.11}$$

Proposition 1.12. *If $\gamma_1 \in A^1(X)$, then*

$$I_\beta(\gamma_1, \dots, \gamma_n) = \left(\int_\beta \gamma_1\right) I_\beta(\gamma_2, \dots, \gamma_n).$$

Since $\int_\beta \gamma_1$ is the number of choices for $p_i \in C$ to map to Γ_1, where Γ_1 is a Poincaré dual to γ_1. The proof idea has something to do with the pushforward map (1.10) again.

Next time we'll talk about the quantum cohomology ring, and show that its associativity provides recursive formulas for enumerative invariants.

2. Quantum Cohomology: 2/5/18

Today, Jonathan spoke again, discussing quantum cohomology and an explicit example of how its associativity produces enumerative data on convex varieties.

Recall that last time, we discussed the moduli spaces of stable maps $\overline{\mathcal{M}}_{0,n}(X, \beta)$ given a variety X, a $\beta \in A_1(X)$, and an $n \geq 0$. We can use this moduli space, and the evaluation maps $p_i \colon \overline{\mathcal{M}}_{0,n}(X, \beta) \to X$, to define Gromov-Witten invariants as in (1.6). We then discussed three important properties of Gromov-Witten invariants, namely Propositions 1.7, 1.9 and 1.12; they will be useful when we do calculations.[2]

Now we'll define quantum cohomology in a restricted setting. Some of our notation will be redundant today, but will be useful when we discuss the general case. Fix $X = \mathbb{P}^r$ and $T_0 = 1 \in A^0(X)$. Let $T_1, \dots, T_p$ be a basis for $A^1(X)$ and $T_{p+1}, \dots, T_m$ be a basis for the rest of $A^*(X)$. For $\beta \in A_1(X)$ and $n_{p+1}, \dots, n_m \in \mathbb{N}$, let

$$N(n_{p+1}, \dots, n_m; \beta) := I_\beta(T_{p+1}^{n_{p+1}}, \dots, T_m^{n_m}). \tag{2.1}$$

For $0 \leq i, j \leq m$, define

$$g_{ij} := \int_X T_i \smile T_j \tag{2.2}$$

and g^{ij} be the entries of the matrix inverse to (g_{ij}). By (1.8),

$$T_i \smile T_j = \sum_{e,f} I_0(T_i, T_j, T_e) g^{ef} T_f. \tag{2.3}$$

Definition 2.4. The *quantum potential* of a $\gamma \in A^*(X)$ is

$$\Phi(\gamma) := \sum_{n \geq 3} \sum_{\beta \in H_2(X;\mathbb{Z})} \frac{1}{n!} I_\beta(\gamma^n).$$

The summand is nonzero for only finitely many β for a given n, so this converges. Moreover, if $\gamma = \sum y_i T_i$,

$$\Phi(y_0, \dots, y_n) := \Phi(\gamma) = \sum_{n_0 + \cdots + n_m \geq 3} \sum_\beta I_\beta(T_0^{n_0}, \dots, T_m^{n_m}) \frac{y_0^{n_0}}{n_0!} \cdots \frac{y_m^{n_m}}{n_m!}. \tag{2.5}$$

[2] Kontsevich and Manin [KM94] take these properties as *axioms* for Gromov-Witten theory.

This is a formal power series in $y_0, \ldots, y_n$, and hence one may define

$$\Phi_{ijk} := \frac{\partial^3 \Phi}{\partial y_i \partial y_j \partial y_k} = \sum_{n \geq 0} \sum_{\beta} \frac{1}{n!} I_\beta(\gamma^n, T_i, T_j, T_k). \tag{2.6}$$

Definition 2.7. The *quantum cup product* is

$$T_i * T_j := \sum_{e,f} \Phi_{ije} g^{ef} T_f.$$

Remark. This definition is kind of unenlightening — it's not clear what it's doing. Hopefully through examples we can figure out why it's defined in this way. ◀

Theorem 2.8. *$A^*(X)$ with the quantum cup product is associative, commutative, and has T_0 as a unit.*

The hardest part is associativity, requiring a full page of calculations. We're not going to do that today, but we'll talk about what it implies. Writing everything out,

$$\begin{aligned}(T_i * T_j) * T_k &= \sum_{e,f} \Phi_{i,e} g^{ef} T_f * T_k \\ &= \sum_{e,f} \sum_{c,d} \Phi_{ije} g^{ef} \Phi_{fkc} g^{cd} T_d.\end{aligned}$$

Similarly,

$$T_i * (T_j * T_k) = \sum_{e,f} \sum_{c,d} \Phi_{jke} g^{ef} \Phi_{ifc} g^{cd} T_d.$$

Therefore associativity is equivalent to

$$\Phi_{ije} g^{ef} \Phi_{fkc} = \Phi_{jke} g^{ef} \Phi_{ifc}, \tag{2.9}$$

so if we define

$$F(i, j \mid k, \ell) := \sum_{e,f} \Phi_{ije} g^{ef} \Phi_{fk\ell}, \tag{2.10}$$

then associativity is equivalent to $F(i, j \mid k, \ell) = F(j, k \mid i, \ell)$ for all i, j, k, ℓ.

We can split the quantum potential into two pieces: the "classical" part $\Phi_{\text{classical}}$, given by $\beta = 0$, and the "quantum" part Φ_{quantum}, for which $\beta \neq 0$. Then $\Phi = \Phi_{\text{classical}} + \Phi_{\text{quantum}}$, and using Proposition 1.7,

$$\Phi_{\text{classical}} = \sum_{n_1 + \cdots + n_m = 3} \int_X T_0^{n_0} \smile \cdots \smile T_m^{n_m} \prod_{i=1}^m \frac{y_i^{n_i}}{n_i!}. \tag{2.11}$$

TODO: then there was a big formula for $\Gamma(y)$ whose relation to the story was unclear to me.

Example 2.12. Let's actually do this on $X = \mathbb{P}^2$. For $i = 0, 1, 2$, let $T_i \in H^{2i}(\mathbb{P}^2)$ be the generators corresponding to the orientation coming from the complex structure. That is, T_0 is Poincaré dual to $\mathbb{P}^2$, T_1 to a embedded $\mathbb{P}^1$, and T_2 to a point. Recall that $g_{ij} = \int_{\mathbb{P}^2} T_i \smile T_j$, so

$$g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{2.13}$$

and $g^{-1} = g$.

Associativity of the quantum cup product implies that $F(1, 1 \mid 2, 2) = F(1, 2 \mid 1, 2)$, i.e.

$$\sum_{e,f} \Phi_{11e} g^{ef} \Phi_{f22} = \sum_{e,f} \Phi_{12e} g^{ef} \Phi_{f12}. \tag{2.14}$$

Since $g^{ef} \neq 0$ only when $e + f = 2$, this sum simplifies to

$$\Phi_{110}\Phi_{222} + \Phi_{111}\Phi_{122} + \Phi_{112} + \Phi_{022} = \Phi_{120}\Phi_{212} + \Phi_{121}\Phi_{112} + \Phi_{122}\Phi_{012}. \tag{2.15}$$

Now we have to actually compute some of these things.

$$\Phi_{110} = \sum_{n\geq 0}\sum_{\beta} I_\beta(\gamma^n \cdot T_1 \cdot T_0). \tag{2.16}$$

Most of these are zero for degree reasons, and the only nonzero contribution is from $\int_X T_1^2$.

TODO: then there was another thing I didn't follow…

◄

3. The moduli space of stable maps: 2/12/18

4. The little quantum product: 2/19/18

Today, Yixian spoke about associativity, the little quantum product, and more, finishing up the talks from [FP96].

We will continue to use notation from previous sections, in particular for Gromov-Witten invariants and the ingredients in the quantum product.

Let's suppose our target X is really nice: it's a projective, nonsingular, convex variety.

Definition 4.1. Let $\beta \in H_2(X)$. We call β an *effective class* if there is a stable map realizing β, i.e. $\overline{\mathcal{M}}_{0,n}(X,\beta)$ is nonempty.[3]

The idea of a boundary divisor is to split a reducible stable map into two components.

Definition 4.2. Let $\mu\colon (C, p_1, \dots, p_n) \to X \in \overline{\mathcal{M}}_{0,n}(X,\beta)$ be a stable map such that the domain curve C is reducible. The *boundary divisor* $D(A,B;\beta_1,\beta_2) \subset \overline{\mathcal{M}}_{0,n}(X,\beta)$ is the locus of stable maps which admit the following data:

- a partition $[n] = A \cup B$,[4] and
- effective classes β_1, β_2 such that $\beta_1 + \beta_2 = \beta$,

such that:

1. If $\beta_1 = 0$, $|A| \geq 2$, and if $\beta_2 = 0$, $|B| \geq 2$.
2. There are curves C_A, C_B such that $C_A \cup C_B = C$ and $C_A \cap C_B = \{\mathrm{pt}\}$.
3. The markings in A lie in C_A and the markings in B lie in C_B.
4. $\mu([C_A]) = \beta_1$ and $\mu([C_B]) = \beta_2$.

Theorem 4.3. *Let $e_1\colon \overline{\mathcal{M}}_{0,A\cup\{\mathrm{pt}\}}(X,\beta) \to X$ be the evaluation map at the extra point, and define $e_2\colon \overline{\mathcal{M}}_{0,B\cup\{\mathrm{pt}\}}(X,\beta) \to X$ analogously. Let $D(A,B;\beta_1,\beta_2)$ be a boundary divisor and define*

$$\widetilde{K} := \overline{\mathcal{M}}_{0,A\cup\{\mathrm{pt}\}}(X,\beta_1) \times_X \overline{\mathcal{M}}_{0,B\cup\{\mathrm{pt}\}}(X,\beta_2) \tag{4.4}$$

along e_1 and e_2. If A and B are nonempty, then $\widetilde{K} \cong D(A,B;\beta_1,\beta_2)$.

Let $i,j,k,\ell \in [n]$. We define a divisor

$$D(i,j \mid k,\ell) := \sum_{\substack{i,j\in A\\ k,\ell\in B}} D(A,B;\beta_1,\beta_2). \tag{4.5}$$

Then $D(i,j\mid k,\ell) = D(i,k \mid j,\ell)$.[5]

Recall that associativity of the quantum product, as defined in a previous lecture, is equivalent to (2.9): we have to prove some equalities about quantum potentials.

Lemma 4.6. *Let*

$$\iota\colon D(A,B,\beta_1,\beta_2) \longrightarrow \overline{\mathcal{M}}_{0,A\cup\{\mathrm{pt}\}}(X,\beta_1) \times \overline{\mathcal{M}}_{0,B\cup\{\mathrm{pt}\}}(X,\beta_2)$$

and

$$\alpha\colon D(A,B;\beta_1,\beta_2) \longrightarrow \overline{\mathcal{M}}_{0,n}(X,\beta)$$

[3] TODO: just genus zero?

[4] Here, $[n] := \{1,\dots,n\}$.

[5] TODO: why?

denote inclusion. For $\gamma_1, \dots, \gamma_m \in A^*(X)$,

$$\iota_* \circ \alpha^*(\rho_1^*(\gamma_1) \smile \cdots \smile \rho_n^*(\gamma_n)) = \sum_{e,f} g^{ef} \left(\prod_{a \in A} \rho_a^*(\gamma_a) \rho_{\mathrm{pt}}^*(T_e) \right) \left(\prod_{b \in B} \rho_b^*(\gamma_b) \rho_{\mathrm{pt}}^*(T_f) \right).$$

Define

$$\begin{aligned} G(q, r \mid s, t) &:= \sum_{\substack{q,r \in A \\ s,t \in B}} g^{ef} I_{\beta_1} \left(\prod_{a \in A} \gamma_a T_e \right) I_{\beta_2} \left(\prod_{\beta \in B} \gamma_b T_f \right) \\ &= \sum_{\substack{A \cup B = [n] \\ \beta_1 + \beta_2 = \beta}} \int_{D(A,B,\beta_1,\beta_2)} \rho_1^*(\gamma_1) \smile \cdots \smile \rho_n^*(\gamma_n) \\ &= \int_{D(q,r \mid s,t)} \rho_1^*(\gamma_1) \smile \cdots \smile \rho_n^*(\gamma_n). \end{aligned}$$

Then associativity of the quantum product is asking whether

$$G(q, r \mid s, t) \stackrel{?}{=} G(q, s \mid r, t). \tag{4.7}$$

Next we define

$$F(i, j \mid k, \ell) := \sum_{e,f} \Phi_{ije} g^{ef} \Phi_{fk\ell} = \sum_{\substack{\beta_1 + \beta_2 = \beta \\ e,f,n_1,n_2}} I_{\beta_1}(\gamma^{n_1} \cdot T_i T_j T_\ell) g^{ef} I_{\beta_2}(\gamma^{n_2} \cdot T_k T_e T_f), \tag{4.8}$$

and associativity would imply this is equal to $F(i, k \mid j, \ell)$.

Remark. We're not going to attach intrinsic geometric meaning to F and G; they are tools in the proof of associativity. However, the notes suggestively use Feynman-diagram-like notation for them, which suggests that these things have an interpretation in physics. $\blacktriangleleft$

Recall that the quantum product is defined on a basis by

$$T_i * T_j := \sum_{e,f} \Phi_{ije} g^{ef} T_f. \tag{4.9}$$

We can use this to define the *quantum cohomology ring* $QH^*(X)$ as the algebra generated by $A^*(X)$ under this product. This is naturally a $\mathbb{Q}[[y]]$-algebra (where y acts by the extra factor of γ that has appeared in everything), and if $V := A^*(X) \setminus 0$, it's also a $\mathbb{Q}[[V]]$-algebra, which is a more coordinate-free way to say it. That is, $\mathbb{Q}[[V]]$ is the completion of

$$\bigoplus_{i=0}^{\infty} \mathrm{Sym}^i(V) \otimes \mathbb{Q}$$

at its unique maximal ideal.

The embedding map $A^*(X) \to QH^*(X)$ is a group homomorphism, but not a ring homomorphism.

If X is a homogeneous variety, there's an isomorphism

$$QH^*(X) \cong A^*_{\mathbb{Q}}(X) \otimes \mathbb{Q}[[y]].$$

In general this is not true.

TODO: more stuff happened, including a calculation on $\mathbb{P}^2$, but I didn't get it down.

5. Moduli stacks: 2/26/18

Today, Rok spoke about moduli stacks in the context of Gromov-Witten theory.

Often in algebraic geometry, objects we're interested in have nontrivial automorphisms. This can be frustrating, because it makes it much less likely that they're accurately represented by schemes (i.e. there is often no fine moduli space for a moduli problem with automorphisms).

The reason, from the functor-of-points view, is that the functor of points of a moduli space $\mathcal{M}$ is a functor $\mathcal{M} \colon \mathsf{Sch}^{\mathsf{op}} \to \mathsf{Set}$, and sets don't encode automorphisms. The idea of a set with automorphisms is encoded in a *groupoid*, so we're led to the notion of functors of points valued in groupoids, but this comes with its own technical issues.

- Since groupoids have automorphisms, stacks have two kinds of morphisms, and therefore have an inherent 2-categoricity. If you're willing to work with stacks as 2-functors, this is not a problem, but not everyone is willing to do that, and working around this comes with its own issues.
- To get everything strictly, rather than up to automorphisms, one has to work with categories fibered in groupoids. This is a somewhat technical condition related to straightening and unstraightening constructions, but it makes everything works.

Now that we've broadened our world from set-valued functors to groupoid-valued ones, we need to decide which functors are representable.

Definition 5.1. A functor $\mathcal{X}\colon \mathsf{Sch}^{\mathsf{op}} \to \mathsf{Gpd}$ is a *stack* if it satisfies descent with respect to a given topology of interest. That is, if $\mathfrak{U}$ is a cover of a scheme S in the given topology, then the diagram

$$\mathcal{X}(S) \longrightarrow \prod_{U\in\mathfrak{U}} \mathcal{X}(U) \rightrightarrows \prod_{U,V\in\mathfrak{U}} \mathcal{X}(U\cap V) \Rrightarrow \prod_{U,V,W\in\mathfrak{U}} \mathcal{X}(U\cap V\cap W) \tag{5.2}$$

is a 2-pullback diagram in the 2-category of groupoids.[6]

Since we're at categorical level 2, we need to consider triple intersections, rather than double ones. If you need to care about higher stacks you'll consider higher-order intersections.

We're deliberately ambiguous about what topology to use, because it might depend on the application, but generally the étale topology is a good one to use.

Definition 5.3. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of stacks, i.e. a natural transformation. Then, f is *representable* if for all schemes S and maps $g\colon S \to \mathcal{Y}$, i.e. natural transformations $g\colon \mathrm{Hom}_{\mathsf{Sch}}(-, Y) \Rightarrow \mathcal{Y}$, if T denotes the pullback

$$\begin{tikzcd} T \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ S \arrow[r, "g"] & \mathcal{Y}, \end{tikzcd} \tag{5.4}$$

then T is a scheme.

Using representability, we can extend definitions of schemes to stacks: for example, we say that a map of stacks $f\colon \mathcal{X} \to \mathcal{Y}$ is *étale* if, with notation as in (5.4), the induced map $T \to S$ on all pullbacks to schemes is étale. Hence we can make sense of the following notion.

Definition 5.5. A stack is *algebraic* if it's covered by a scheme.

Definition 5.6. A stack $\mathcal{X}$ is *Artin* (resp. *Deligne-Mumford*) if

- the diagonal $\Delta\colon \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable, and
- there exists a scheme X and a smooth (resp. étale) surjection $X \to \mathcal{X}$.

Remark. Representability of the diagonal is equivalent to a simpler condition: for all morphisms $S \to \mathcal{X}$ and $T \to \mathcal{X}$, where S and T are schemes, the pullback $S \times_{\mathcal{X}} T$ is a scheme. The reason this arises is that the diagram

$$\begin{tikzcd} S \times_{\mathcal{X}} T \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\Delta"] \\ S \times T \arrow[r] & \mathcal{X} \times \mathcal{X} \end{tikzcd} \tag{5.7}$$

is a pullback diagram. ◂

On the other hand, the existence of a surjection from a scheme is hard to check. There are some theorems which can help you, e.g. the Artin representability theorem, but there's no silver bullet.

[6] The objects are groupoids, the morphisms are functors, and the 2-morphisms are natural transformations.

Example 5.8. Let X be a scheme and G be an algebraic group acting on X. Then, we can define the *quotient stack* (sometimes also *stacky quotient*) $[X/G]$ by its functor of points: given a scheme S, $[X/G](S)$ is the groupoid of *G-torsors* $T \to S$ (i.e. G-schemes T such that $T/G \cong S$) together with G-equivariant maps $T \to X$; the morphisms are automorphisms of G-torsors[7] intertwining the maps to X.

Of course, one has to check this is a stack, but this is easier than in general, and Artin representability follows from the natural quotient map $X \to [X/G]$.

For example, if $X = \mathrm{pt}$ and G acts trivially, $[\mathrm{pt}/G]$ is called the *classifying space* of G and denoted BG; it is the moduli space of G-torsors, as maps $S \to BG$ are in natural bijection with G-torsors over S. ◂

Remark. Another perspective on stacks is that they're quotients of groupoids in schemes. For example, if $X \to \mathcal{X}$ is an étale cover of a Deligne-Mumford stack, then the diagram

$$X \times_{\mathcal{X}} X \rightrightarrows X \longrightarrow \mathcal{X} \tag{5.9}$$

is a groupoid object in Sch. This is nice, but it absolutely depends on X, which we can think of as an atlas on $\mathcal{X}$. ◂

Key examples of stacks are moduli spaces of curves, because they tend to have automorphisms.

Definition 5.10. The *moduli stack of stable curves* $\overline{\mathcal{M}}_{g,n}$ is the Deligne-Mumford stack whose functor of points sends a scheme S to the groupoid of maps $f\colon C \to S$ together with n sections $p_1, \dots, p_n$ such that f is proper, flat, has geometric fibers, is dimension 1, is connected, is reduced, has at most nodal singularities, and whose fibers have finitely many automorphisms fixing the images of the sections (i.e. the fibers are stable marked curves). A lot of work goes into showing that this is a Deligne-Mumford stack! It's also proper and smooth.

Remark. Without the finite automorphisms condition, such curves are called *prestable curves*. There is a moduli stack $\overline{\mathcal{M}}^{\mathrm{pre}}_{g,n}$, but it's only an Artin stack, which is not as cool. ◂

One way to construct $\overline{\mathcal{M}}_{g.n}$ is to start with the *Hilbert scheme of points* $\mathrm{Hilb}^1(\mathbb{P}^N)$, the fine moduli space of N points on $\mathbb{P}^1$, which is representable as a scheme; then one constructs an action of a GL_M on it, and the stacky quotient is equivalent to $\overline{\mathcal{M}}_{g.n}$.

By general moduli theory, applying the identity map $\overline{\mathcal{M}}_{g,n} \to \mathcal{M}_{g,n}$, there's a universal stable curve $\overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ such that any family of stable curves $C \to S$ arises as a pullback of $\overline{\mathcal{C}}_{g,n}$. In fact, $\overline{\mathcal{C}}_{g,n} \simeq \overline{\mathcal{M}}_{g,n+1}$ and the map forgets the last point. This is somewhat surprising, and is nontrivial.

Example 5.11. More generally, we're in the moduli stack of stable maps: fix a target X and a $\beta \in A_1(X)$; then the stack is called $\overline{\mathcal{M}}_{g,n}(X, \beta)$. In this case the functor of points sends a scheme S to the groupoid of maps $f\colon C \to S$ with n sections $p_1, \dots, p_n$ with the same conditions as above, except that the curves are *pre*stable, together with a map $\mu\colon C \to X$ such that C is stable with respect to μ.

$\overline{\mathcal{M}}_{g,n}(X, \beta)$ is a Deligne-Mumford stack. The proof is not easy, of course; one proof can be found in Olsson's book [Ols16]. Another approach, due to Kontsevich [Kon95], assumes that representability of the diagonal has already been shown (which is fine, because that's the easier part), then constructs a map

$$\overline{\mathcal{M}}_{g,n}(X, \beta) \longrightarrow \mathrm{Hom}(\overline{\mathcal{C}}_{g,n}, X) \tag{5.12}$$

sending $(C \to X, C \to S) \mapsto C \to X$. This is great, except that it's prestable, so we need to fix it. Kontsevich shows (5.12) is an étale-locally closed immersion, and then there is a map $\mathrm{Hom}(\overline{\mathcal{C}}_{g,n}, X) \hookrightarrow \mathrm{Hilb}^{1\prime}(\overline{\mathcal{C}}_{g,n} \times X)$ sending f to its graph. That this works is highly nontrivial, and depends on nice properties of everything around. Anyways, now we've constructed an étale-local embedding into a scheme, so there must be an étale cover by a scheme.

There's a third construction, due to Toën. It begins with the following general theorem from the theory of stacks.

Theorem 5.13. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Artin stacks. If $\mathcal{X}$ is proper and $\mathcal{Y}$ is locally finite type, then $\mathrm{Hom}_{\mathsf{Stack}}(\mathcal{X}, \mathcal{Y})$ is again an Artin stack.*

TODO: I missed what followed. ◂

[7] A morphism of G-torsors is automatically an automorphism, at least in the complex topology. TODO: is this true more generally in algebraic geometry?

Recall that in Gromov-Witten theory, we've been studying maps of punctured algebraic curves $(C, p_1, \dots, p_n)$ into a projective variety X pushing $[C] \in H_2(C;\mathbb{Z})$ to a fixed class $\beta \in H_2(X;\mathbb{Z})$, and such that each p_i lands in a specified algebraic cycle z_i. We allow reducible curves with specified singularities in order to obtain a compact moduli space $\overline{\mathcal{M}}_{g,n}(X)$.

Within the Deligne-Mumford coarse moduli space $\overline{M}_{g.n}$, we can consider curves which admit maps of the form described above. These define a class in $H^*(\overline{M}_{g,n};\mathbb{Q})$. More precisely, we have an *evaluation map*[8]

$$\pi_1\colon \overline{\mathcal{M}}_{g,n}(X,\beta) \longrightarrow X^n \tag{6.1}$$

sending $(f, C, p_1, \dots, p_n) \mapsto (f(p_1), \dots, f(p_n))$ and a map $\pi_1\colon \overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n}$ which is *not* just the forgetful map — the underlying curve of a stable map might not be stable. However, after contracting some components without marked points, it is, and that's what π_2 does.

Therefore we can pull back by π_1, and we can also push forward by π_2 with a Gysin map[9]

$$\pi_{2!}\colon H^*(\overline{\mathcal{M}}_{g,n}(X,\beta);\mathbb{Q}) \longrightarrow H^{2m+*}(\overline{\mathcal{M}}_{g,n};\mathbb{Q}), \tag{6.2}$$

where

$$m := (g-1)\dim X + \int_\beta \omega_X. \tag{6.3}$$

Here ω_X is the canonical class (Chern class of the canonical bundle).

Definition 6.4. The *Gromov-Witten class* is

$$I_{g,n,\beta}(\alpha_1, \dots, \alpha_n) := \pi_{2!}\pi)_1^*(\alpha_1 \smile \dots \smile \alpha_n) \in H^{2m+|\alpha_1|+\dots+|\alpha_n|}(\overline{\mathcal{M}}_{g,n};\mathbb{Q}),$$

where $\alpha_i \in H^*(X)$ is the Poincaré dual to z_i.

In particular, the Gromov-Witten invariant associated to all this data is

$$\langle I_{g,n,\beta}\rangle(\alpha_1, \dots, \alpha_n) := \int_{\overline{\mathcal{M}}_{g,n}} I_{g,n,\beta}(\alpha_1, \dots, \alpha_n). \tag{6.5}$$

We can define the moduli functor $\overline{\mathcal{M}}_{g,n}(X;\beta)\colon \mathsf{Sch}_\mathbb{C} \to \mathsf{Set}$ sending S to isomorphism classes of stable maps over S with genus g and class β.

Theorem 6.6 (Alexeev)**.** *If X is projective, then $\overline{\mathcal{M}}_{g,n}(X;\beta)$ has a coarse moduli space $\overline{M}_{g,n}(X;\beta)$, which is a projective scheme over $\mathbb{C}$.*

Theorem 6.7 (Kontsevich [Kon95])**.** *If X is projective, $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is an algebraic stack which is proper over $\mathbb{C}$. Furthermore, $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,\beta)$ is smooth for any $\beta \in H_2(\mathbb{P}^r;\mathbb{Z})$.*

This story is very nice, but is too naïve: $\overline{M}_{g,n}(X,\beta)$ may have components whose dimension is "too large," i.e. larger than this argument expects. Therefore integrating cohomology classes over it doesn't work in the way we described.

Let ξ be the fundamental class of $\overline{M}_{g,n}(X,\beta)$, which is Poincaré dual to $1 \in H^*(\overline{M}_{g,n}(X,\beta);\mathbb{Q})$, and let $e := \dim \overline{M}_{g,n}(X,\beta)$. Poincaré duality tells us that capping with ξ defines a map

$$\text{–} \frown \xi\colon H^*(\overline{M}_{g,n}(X,\beta);\mathbb{Q}) \longrightarrow H_{2e-*}(\overline{M}_{g,n}(X,\beta);\mathbb{Q}). \tag{6.8}$$

Therefore if $\pi := (\pi_1,\pi_2)\colon \overline{M}_{g.n}(X,\beta) \to X^n \times \overline{M}_{g,n}$ and $\alpha := \alpha_1 \smile \dots \smile \alpha_n$, then

$$\begin{aligned}\pi_{2!}\pi_1^*(\alpha) &= \pi_{2!}PD^{-1}(\pi_1^*\alpha \frown \xi)\\ &= PD^{-1}p_{2*}\pi_*(\pi^*p_1^*\alpha \frown \xi),\end{aligned}$$

where p_1 and p_2 are the projection maps out of $X^n \times \overline{M}_{g,n}$. Therefore we can rewrite (6.5) as

$$\langle I_{g,n,\beta}\rangle(\alpha_1, \dots, \alpha_n) = PD^{-1}p_{2*}(p_1^*\alpha \frown \xi). \tag{6.9}$$

[8]TODO: I might have $\overline{M}_{g.n}$ versus $\overline{\mathcal{M}}_{g,n}$ wrong in the following.

[9]This is true if X is smooth. Otherwise we have to do some more work.

In very nice situations, this makes sense. In general, it makes sense except for ξ, so to make these formula make sense, we need some sort of replacement $\xi \in H_*(\overline{M}_{g,n}(X,\beta);\mathbb{Q})$. This is what the virtual fundamental class does.

Example 6.10 (Normal cone)**.** One explicit example of a virtual fundamental class comes from the normal cone. Let $E \to Y$ be a rank-r vector bundle over a smooth variety Y. For a section s of E, let $Z := Z(s)$ denote its *zero scheme*, the scheme defined by its zero locus.

This varies badly in s, but we can replace it with something which varies better. Let $\mathscr{I}$ denote the sheaf of ideals of the closed embedding $Z \hookrightarrow Y$. The *normal cone* of Z is

$$C_Z Y := \mathscr{S}\!pec_Z\left(\bigoplus_{k=0}^{\infty} \mathscr{I}^k/\mathscr{I}^{k+1}\right). \tag{6.11}$$

Here $\mathscr{S}\!pec_Z$ means to take the relative Spec over Z. This is an affine cone over Z. The structure map to Z comes from the surjection $(\cdot s)\colon \mathscr{O}(E^*) \to \mathscr{I}$, inducing a surjective map

$$\bigoplus +k \operatorname{Sym}^k(\mathscr{O}(E^*)/\mathscr{I}\mathscr{O}(E^*)) \twoheadrightarrow \bigoplus_k \mathscr{I}^k/\mathscr{I}^{k+1}. \tag{6.12}$$

inducing an embedding $C_Z Y \hookrightarrow E|_Z$ with a pushed-forward fundamental class $[C_Z Y] \in A_n(E|_Z)$. Its pullback $s^*[C_Z Y] \in A_{n-r}(Z)$ has the "expected dimension of Z," even in non-generic situations.

Lemma 6.13. *If $i\colon Z \hookrightarrow Y$ denotes inclusion, then $i_*(s^*[C_Z Y]) \in A_{n-r}(Y)$ is the Euler class of E.*

Therefore $s^*[C_Z Y]$ refines the Euler class, and varies nicely with s. This is the kind of approach we'll use. ⋖

There are various approaches to this: one can look at the perfect tangent-obstruction complex, which was done by Li-Tian [LT98], or using perfect obstruction theory, which was done by Behrend-Fantechi [BF97].

Why does obstruction theory enter the story? Here's an analogy: let's consider deformations of a compact complex manifold M. The infinitesimal deformations live in $H^1(M;\Theta_M)$ (where Θ_M denotes the *tangent sheaf*), and the obstructions live in $H^2(M;\Theta_M)$. Kuranishi theory says that the moduli space of complex structure on M is locally the zero locus of a map $U \to H^2(M;\Theta_M)$, where $U \subset H^1(M;\Theta_M)$ is an open subset.

For us, instead of the tangent space we have something relative. Let $\vec{q} := \sum p_i$ and consider the sheaves of differentials $\Omega^1_C(\vec{q})$ and $f^*\Omega^1_X$ over C; the natural map $f^*\Omega^1_X \to \Omega^1_C(\vec{q})$ induces a sheaf of relative differentials $\Omega^1_{C\hookrightarrow X}$ and we have a long exact sequence of sheaf Exts:

$$0 \longrightarrow \mathscr{E}\!xt^0_C(\Omega^1_{C/\hookrightarrow X}, \mathscr{O}_C) \longrightarrow \mathscr{E}\!xt^0_C(\Omega^1_C(q), \mathscr{O}_C) \longrightarrow \operatorname{Ext}^0_C(f^*\Omega^1_X, \mathscr{O}_C)(\Omega^1_{C\hookrightarrow X}, \overset{\operatorname{Ext}}{\underset{C}{\mathscr{O}}}{}_C) \longrightarrow \cdots \tag{6.14}$$

The analogue of the tangent sheaf is

$$\mathscr{E}\!xt^1_C(\Omega^1_{C\hookrightarrow X}), \mathscr{O}_C) = \mathscr{H}^1(C;\Theta^1_C \to f^*\Theta_X). \tag{6.15}$$

Here $\Theta^1_C \to f^*\Theta_X)$ denotes the tangent sheaf of vector fields on C which vanish at each p_i. The obstructions live in $\mathscr{E}\!xt^2$, hence in $\mathscr{H}^2(C;\Theta^1_C \to f^*\Theta_X)$. Therefore the expected dimension is

$$d_{C,X} := \dim \mathscr{H}^1(C;\Theta^1_C \to f^*\Theta_X) - \dim \mathscr{H}^2(C;\Theta^1_C \to f^*\Theta_X). \tag{6.16}$$

Anyways, using the long exact sequence, you can actually calculate what this is: it will be

$$\begin{aligned} d_{C,X} &= \chi(f^*\Theta_X) + \dim \mathscr{E}\!xt^1(\Omega^1_C(\vec{q}),\mathbb{Q}) - \dim \mathscr{E}\!xt^0(\Omega^1_C(\vec{q}),\mathbb{Q}) \\ &= -\int_\beta \omega_X + (1-g)\dim X + (3g-3) + n \\ &= n - \int_\beta \omega_X + (1-g)(\dim X - 3), \end{aligned}$$

which is what we saw in the first talk.

Li-Tian's approach [LT98] works more generally. One can start with a functor $\mathcal{F}\colon \mathsf{Sch}^{\mathsf{op}}_{\mathbb{C}} \to \mathsf{Set}$, such as the stacky $\overline{\mathcal{M}}_{g,n}(X,\beta)$, an affine scheme S over $\mathbb{C}$, and an $\mathscr{O}_S$-module $\mathscr{N}$. If $S_{\mathscr{N}} = \operatorname{Spec}(\Gamma(\mathscr{O}_S) \oplus \Gamma(\mathscr{N}))$, then we have an infinitesimal extension $S \to S_{\mathscr{N}}$.

Definition 6.17. The *tangent functor* $T\mathcal{F}$ to $\mathcal{F}$ is the functor which, for all $\alpha \in \mathcal{F}(S)$, defines the functor $T\mathscr{F}(\alpha)\colon \mathsf{Mod}_{\mathscr{O}_S} \to \mathsf{Set}$ sending $\mathcal{N}$ to the elements of $\mathcal{F}(S_{\mathcal{N}})$ which restrict to α under pullback $\mathcal{F}(S_{\mathcal{N}}) \to \mathcal{F}(S)$.

7. More virtual fundamental classes: 3/26/18

Today, Jonathan spoke about virtual fundamental classes, including some examples.

In the convex case, we saw that the obstruction was zero, so the virtual fundamental class is the actual fundamental class, and therefore a lot of this is irrelevant. But convex varieties are very special.

Recall that we've been studying Gromov-Witten invariants $I_\beta(\gamma_1, \dots, \gamma_n)$, which we naïvely defined to be an integral over $\overline{\mathcal{M}}_{g,n}(X,\beta)$ of $p_1^*(\gamma_1) \smile \cdots \smile p_n^*(\gamma_n)$, where $p_i\colon \overline{\mathcal{M}}_{g.n}(X,\beta)$ is the i^{th} projection map.

In general, though, we might not have a fundamental class: $\overline{\mathcal{M}}_{g,n}(X,\beta)$ isn't always smooth, and might not have the expected dimension. Last time, we computed the expected dimension of the moduli space, which *is* the dimension in the convex case, and is the dimension which would allow Gromov-Witten invariants to be defined and make sense. We would like to construct a replacement, called the virtual fundamental class, in this dimension.

As a guiding example of what we want to generalize, we'll look at the normal cone. Let Y be a smooth scheme and E be a rank-r vector bundle over Y with a section s and zero scheme $Z \subset Y$ of s. In general, Z behaves weirdly in families, e.g. its dimension can jump, so we want to replace Z with a class in the correct degree. To do this, we constructed the normal cone $C_Z Y$ with an embedding $C_Z Y \hookrightarrow E|_Z$, and then the virtual fundamental class $s^*[C_Z Y] \in A_{n-r}(Z)$. We then saw that in the case of a quintic threefold, this virtual fundamental class, which has the expected dimension, is very helpful.

There are two definitions of the virtual fundamental class, one due to Li-Tian, and a more commonly used one by Behrend-Fantechi. It's stacky, so we'll provide the simpler Li-Tian construction today.

The idea is to endow $\overline{\mathcal{M}}_{g,n}(X,\beta)$, understood through its functor of points $\mathsf{Sch}_{\mathbb{C}}^{\mathsf{op}} \to \mathsf{Set}$, with some extra structure. We defined the tangent functor of such a functor above. More generally, given a functor $\mathcal{F}\colon \mathsf{Sch}_{\mathbb{C}}^{\mathsf{op}} \to \mathsf{Set}$, the tangent functor fits into a tangent-obstruction complex $T^1\mathcal{F} \to T^2\mathcal{F}$, where $T^1\mathcal{F}$ is the tangent functor and $T^2\mathcal{F}$ is some obstructon functor: given (α, S, N) it produces an obstruction class; the vanishing of this class is a necessary and sufficient condition for the existence of an $\overline{\alpha} \in \mathcal{F}(S_N)$ which extends α.

Remark. TODO: what exactly does "vanish" mean? I think (but am not certain) the naturality of this construction in S and N is expressed in some natural transformation, and we can ask whether this is nontrivial. ◀

Definition 7.1. The tangent-obstruction complex is *perfect* if for each (α, S), there's a 2-term complex of locally free sheaves $\mathscr{O}_S$-modules $\mathcal{E}^1 \to \mathcal{E}^2$ such that $T^i\mathcal{F}(\alpha)(N)$ is the i^{th} sheaf cohomology of $\mathcal{E}^\bullet \otimes_{\mathcal{O}_S} \mathcal{N}$.

Again, there's got to be something additive going on. The tangent-obstruction complex ought to be valued in some kind of linear category, in the sense that tangent things tend to be linearizations, but it's not entirely clear what happened. Probably they're valued in $\mathscr{O}_S$-modules, but it's not immediately clear how.

Let's make the strong assumption that $\mathcal{F}$ is represented by a scheme Z, so that $\mathcal{E}^i$ passes to a vector bundle $E^i := \mathrm{Spec}_S(\mathrm{Sym}^\bullet(\mathcal{E}_i))$ (this is relative Spec); equivalently, $\mathcal{O}(E^i) = \mathcal{E}^i$. If $\widehat{E}_1$ denotes the formal completion of E_1 along its zero section, Li-Tian constructed a *Kuranishi map* $F\colon \widehat{E}^1 \to E^2$ (which is non-unique). This has a zero scheme $\widehat{Z}$, and a virtual normal cone C^{ε^-}, which is its restriction to Z, and embeds in E^2. Then, the virtual fundamental class is $s^*[C^{\varepsilon^-}]$, where s is the zero section of E^2. We had to choose $\mathcal{E}^1$ and $\mathcal{E}^2$, but Li-Tian show that the class we get doesn't depend on this choice.

This is a generalization of the normal cone; if Y is smooth, $E \to Y$ is a vector bundle, and s is a section with zero scheme Z, then the virtual normal cone reduces to the normal cone we discussed last time.

Example 7.2. Let $V \subset \mathbb{P}^4$ be a smooth quintic and $d > 0$. To count lines in it, we're interested in the virtual fundamental class $[\overline{\mathcal{M}}_{0,0}(V, d\ell)]^{\text{virt}}$. A line in V comes from a line in $\mathbb{P}^4$, so we have a map

$$\overline{\mathcal{M}}_{0,0}(V, d\ell) \hookrightarrow \overline{\mathcal{M}}_{0,0}(\mathbb{P}^4, d).$$

Not everything here is smooth, so the former is a stack. Let $V_d \to \overline{\mathcal{M}}_{0,0}(V, d\ell)$ be the vector bundle whose fiber over an $f\colon C \to \mathbb{P}^4$ is the space $H^0(C; f^*\mathscr{O}_{\mathbb{P}_4}(5))$.

Fix an $s \in \mathscr{O}_{\mathbb{P}^4}(5)$ that defines V as its zero scheme; then s induces a section $\bar{s}$ of V_d, and there is a sense in which $\overline{\mathcal{M}}_{0,0}(V, d\ell)$ is the zero locus of $\bar{s}$.

TODO: then I had to leave early... ◀

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