FALL 2017 GOODWILLIE CALCULUS SEMINAR

ARUN DEBRAY
SEPTEMBER 13, 2017

These notes were taken in Andrew Blumberg's student seminar in Fall 2017. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

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1. Introduction: 9/13/17

Today, Nicky gave an overview of Goodwillie calculus, following Nick Kuhn's notes.

The setting of Goodwillie calculus is to consider two topologically enriched,[1] based model categories C and D and a functor $F\colon \mathsf{C} \to \mathsf{D}$ between them.

Example 1.1.

(1) Top, the category of topological spaces.
(2) Sp, the category of spectra.
(3) If Y is a topological space, we can also consider ${}_{Y\backslash}\mathsf{Top}_{/Y}$, the category of spaces over and under Y, i.e. the diagrams $Y \to X \to Y$ which compose to the identity. ◀

We want F to satisfy some kind of Mayer-Vietoris property, or excision. Hence, we assume C and D are *proper*, in that the pushout of a weak equivalence along a cofibration is also a weak equivalence. We'll also ask that in D, sequential colimits of homotopy Cartesian cubes are again homotopy Cartesian, and we'll elaborate on what this means.

We also place a condition on F: Goodwillie calls it "continuous," meaning that it's an enriched functor: the induced map

$$\mathrm{Map}_{\mathsf{C}}(X, Y) \longrightarrow \mathrm{Map}_{\mathsf{D}}(F(X), F(Y))$$

is a continuous map between topological spaces (or a morphism of simplicial sets; for the rest of this section, we'll let V denote the choice of Top_* or sSet_* that we made). If $X \in \mathsf{C}$ and $K \in \mathsf{V}$, then we have a tensor-hom adjunction

$$\mathsf{C}(X \otimes K, Y) \cong \mathsf{V}(K, \mathsf{C}(X, Y)).$$

From this, F produces the *assembly map*

$$F(X) \otimes K \longrightarrow F(X \otimes K).$$

We'll also require F to be weakly homotopical in that it sends homotopy equivalences to homotopy equivalences.

The idea of Goodwillie calculus is to approximate F by a tower of functors, akin to Postnikov truncations, $\cdots \to P_2 \to P_1 \to P_1 \to P_0$. The fiber D_i of P_i, akin to the i^{th} Postnikov section, is like the i^{th} term in a Taylor series:

$$\begin{aligned}
P_0(X) &\simeq P_0(*) \\
D_1(X) &\simeq D_1(*) \otimes X \\
D_2(X) &\simeq (D_2(X) \otimes X \otimes X)_{h\Sigma_2},
\end{aligned}$$

[1] As usual, we can take them to be enriched either over Top or over sSet. This has the important consequence that C and D are tensored and cotensored over Top_*, resp. sSet_*.

where Σ_2 acts by switching the two copies of X, and so on. Each P_i will satisfy more and more of the Mayer-Vietoris property. This is akin to the first three terms in a Taylor series for f: $f(a)$, $xf'(a)$, and $x^2 f''(a)/2$.

Weak natural transformations. We'll also need to know what a weak equivalence of functors is. This would allow us to study the homotopy category of $\mathsf{Fun}(\mathsf{C},\mathsf{D})$.

Definition 1.2. A *weak natural transformation* $F \Rightarrow G\colon \mathsf{C} \to \mathsf{D}$ is one of the two zigzags

$$F \xleftarrow{\sim} H \longrightarrow G \qquad\qquad \text{or} \qquad\qquad F \longleftarrow H \xrightarrow{\sim} G,$$

where $F \xrightarrow{\sim} G$ means an objectwise weak equivalence.

Commutativity of a diagram of weak natural transformations is computed in $\mathsf{ho}(\mathsf{D})$.[2] You can also form spectra in D in the usual way (inverting suspension, etc).

Diagrams[3]. Let S be a finite set. We'll let $\mathcal{P}(S)$ denote its power set, made into a poset category under inclusion. Similarly, we'll let $\mathcal{P}_0(S) \coloneqq \mathcal{P}(S)\setminus\{\varnothing\}$ and $\mathcal{P}_1(S) \coloneqq \mathcal{P}(S)\setminus\{S\}$, again regarded as poset categories.

Definition 1.3.

(1) A *d-cube* in C is a functor $\chi\colon \mathcal{P}(S) \to \mathsf{C}$, where $|S| = d$.
(2) A d-cube χ is *Cartesian* if

$$\chi(\varnothing) \xrightarrow{\sim} \operatorname*{holim}_{T\in\mathcal{P}_0(S)} \chi(T).$$

(3) A d-cube χ is *co-Cartesian* if

$$\chi(S) \xrightarrow{\sim} \operatorname*{hocolim}_{T\in\mathcal{P}_1(S)} \chi(T).$$

(4) A d-cube χ is *strongly co-Cartesian* if $\chi|_{\mathcal{P}(T)}\colon \mathcal{P}(T) \to \mathsf{C}$ is co-Cartesian for all $T \in \mathcal{P}(S)$ with $|T| \geq 2$.

Example 1.4.

(1) If $d = 0$, a (Cartesian or co-Cartesian) 0-cube is something weakly equivalent to the initial object.
(2) A (Cartesian or co-Cartesian) 1-cube is an equivalence.
(3) A 2-cube is something of the form

$$\begin{tikzcd} \mathrm{fib}_f \arrow[r] \arrow[d] & \mathrm{fib}_g \arrow[d] \\ A \arrow[r] \arrow[d, "f"] & B \arrow[d, "g"] \\ C \arrow[r] & D. \end{tikzcd}$$

We let $\partial\chi$ denote the *boundary* of χ, the top row; the middle row is $\chi_\top$, and the bottom row is $\chi_\perp$.[3] In this case, Cartesian and co-Cartesian correspond to (homotopy) pushout and pullback squares, which explains their names in the general case. ◀

There's a way to produce co-Cartesian cubes canonically from a finite set. Let $\phi\colon X^{\amalg T} \to X$ denote the fold map.

Definition 1.5. Let T be a finite set and $X \in \mathsf{C}$, and let

$$X \star T \coloneqq \mathrm{cofib}\left(\phi\colon \coprod_T X \to X\right).$$

Now, for $T \subset [d]$, the assignment $T \mapsto X \star T$ defines a co-Cartesian $(d+1)$-cube.

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[2]There are more worked-out expositions of the homotopy theory of functors, in case this one looks ad hoc, but we don't need the entire background.

[3]These are also written χ_{top} and χ_{bottom}.

For example, when $d=1$, this is the homotopy pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d] & CX \simeq * \arrow[d] \\ CX \simeq * \arrow[r] & \Sigma X. \end{tikzcd}$$

This formalism allows us to define the analogue of the Mayer-Vietoris principle that we'll need for the Goodwillie tower.

Definition 1.6. An $F\colon \mathsf{C} \to \mathsf{D}$ with F, C, and D as above is *d-excisive* if for all strongly co-Cartesian $(d+1)$-cubes χ, $F(\chi)$ is a Cartesian $(d+1)$-cube in D.

Example 1.7.

(1) 0-excisive functors are homotopy constant.
(2) 1-excisive functors are those that satisfy the Mayer-Vietoris property. In Sp, $\mathrm{Map}_{\mathsf{Sp}}(C,-)$ and L_E are both 1-excisive. ◀

There are some nice properties about how d-excisive functors behave with respect to cofibration sequences, etc., and we will return to them in due time. We will now construct Taylor towers.

Fix an $X \in \mathsf{C}$, and let

$$T_dF(X) \coloneqq \operatorname*{holim}_{T\in\mathcal{P}_0([d+1])} F(X \star T).$$

Remark. There is a natural map $t_dF\colon F \to T_dF$, and by definition, this is an equivalence if F is d-excisive. ◀

Set $P_dF\colon \mathsf{C} \to \mathsf{D}$ to be the functor sending

$$X \longmapsto \operatorname{hocolim}\left(F(X) \xrightarrow{t_dF} T_dF(X) \xrightarrow{t_dt_dF} T_dT_dF(X) \longrightarrow \cdots \right).$$

For example, if $F(*) \simeq *$, then $T_1F(X)$ is the homotopy pullback of

$$\begin{tikzcd} & F(CX) \simeq * \arrow[d] \\ * \simeq F(CX) \arrow[r] & F(\Sigma X), \end{tikzcd}$$

and hence is $\Omega F(\Sigma X)$. In this case

$$P_1F(X) = \operatorname*{hocolim}_{n\to\infty} \Omega^n F \Sigma^n X.$$

For example, if $F = \mathrm{id}$ and $\mathsf{C} = \mathsf{D}$, then $P_1(\mathrm{id}) = \Omega^\infty\Sigma^\infty$, which is cool: the "first derivative" of the identity tells us information of stable homotopy! The calculation of the second derivative will be harder.