INTRODUCTION TO SPECTRAL SEQUENCES

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Contents

1. Introduction to the general formalism: 5/8/17

Today, Adrian spoke about what a spectral sequence is and where they come from. The next four lectures will be interesting examples, even if today is somewhat dry.

Definition 1.1. A **(homological) spectral sequence** is the data of

- modules over a ring[1] $E^r_{p,q}$ indexed by $r \geq N$ for some positive N and $p, q \in \mathbb{Z}$, and
- maps $d_r \colon E^r_{p,q} \to E^r_{p-r,q-1+r}$, called **differentials**,

subject to the following conditions:

- $d_r^2 = 0$, and
- for all p, q, and r, $E^{r+1}_{p,q}$ is the homology of the chain complex $(E^r_{p-r\bullet,q-1+r\bullet}, d_r)$ at $E^r_{p,q}$.

The way in which the differentials affect the grading is pretty opaque, so let's see what it looks like for small r.

$$\begin{tikzcd} E^0_{p,q} \arrow[d, "d_0"] \\ E^0_{p,q-1} \end{tikzcd} \qquad \begin{tikzcd} E^1_{p-1,q} & E^1_{p,q} \arrow[l, "d_1"'] \end{tikzcd} \qquad \begin{tikzcd} E^2_{p-2,q+1} & & \\ & & E^2_{p,q}. \arrow[ull, "d_2"'] \end{tikzcd}$$

The differentials swing from downward to leftward, and comes closer and closer to pointing northwest.

This is a lot of structure, and one usually visualizes it as a book, with **pages** $E^r_{\bullet,\bullet}$, and each page is thought of as a lattice with the differentials:

$$\begin{tikzcd} & \vdots & \vdots & \vdots & \\ \cdots & E^r_{p+1,q-1} & E^r_{p+1,q} & E^r_{p+1,q+1} & \cdots \\ \cdots & E^r_{p,q-1} & E^r_{p,q} \arrow[ull] & E^r_{p,q+1} \arrow[ull] & \cdots \arrow[ull] \\ \cdots & E^r_{p-1,q-1} & E^r_{p-1,q} \arrow[ull] & E^r_{p-1,q+1}. \arrow[ull] & \cdots \arrow[ull] \\ & \vdots & \vdots & \vdots & \end{tikzcd}$$

The point of this heavy machinery is that there's a machine which takes filtered objects and functors satisfying an excision property to spectral sequences, and such pairs arise in many contexts in algebra, topology, and geometry.

[1] In the general setup, one has to be somewhat agnostic about what these are: in any context where one can do homological algebra, one can define spectral sequences: abelian groups, modules over a ring, objects in an abelian category…

Definition 1.2. Let $\mathbb{Z}$ denote the **poset category** of the integers, i.e. there's a unique arrow $m \to n$ iff $m \le n$. Then, a **filtered object** in a category C is a functor $X \colon \mathbb{Z} \to \mathsf{C}$.

The idea is a topological space X together with inclusions $X_i \hookrightarrow X_{i+1}$, such that X is the union of all of the X_i. More generally, one can let X be the colimit over i of $X(i)$. One example is the CW filtration of a CW complex X, where $X(n)$ is the n-skeleton of X.

Definition 1.3. Let C be either Top_*, the category of pointed topological spaces, or $\mathsf{Ch}(\mathsf{Mod}_A)$, the category of chain complexes of A-modules for a ring A.

- Let $f \colon X \to Y$ be a C-morphism, so that we can take its mapping cone C_f and obtain a sequence $X \to Y \to C_f$. If we iterate this construction, $C_{Y \to C_f}$ is weakly equivalent to ΣX, and the mapping cone of this is weakly equivalent to ΣY, so we obtain a sequence
$$X \xrightarrow{f} Y \longrightarrow C_f \longrightarrow \Sigma X \xrightarrow{\Sigma f} \Sigma Y \longrightarrow \Sigma C_f \longrightarrow \dots$$
Such a sequence is called a **cofiber sequence**.[2]
- A **functor satisfying excision** is a covariant or contravariant functor $\mathsf{C} \to \mathsf{Ab}$ taking cofiber sequences to long exact sequences.[3]

To see why $C_{Y \to C_f} \simeq \Sigma X$, one can work with particularly nice maps, so that $Y \to C_f$ is an injection, and its mapping cone crushes Y to a point, producing ΣX. The cofiber C_f is the topological analogue of the quotient Y/X.

Example 1.4. Here are some examples of these functors. First, let $\mathsf{C} = \mathsf{Top}_*$:

(1) Covariant functors $\mathsf{Top}_* \to \mathsf{Ab}$ with excision include homology functors H_n.
(2) For covariant functors sending fiber sequences to long exact sequences, we have homotopy groups π_i.
(3) Contravariant functors with excision include cohomology functors H^n.

For the category of chain complexes, cofiber and fiber sequences are the same thing.

(4) Covariant functors include homology and covariant derived functors such as $\mathrm{Ext}^i(M,–)$ and $\mathrm{Tor}_i(M,–)$.
(5) Contravariant functors include cohomology and contravariant derived functors such as $\mathrm{Ext}^i(–,M)$.

From here, one can draw picture of the argument for why such a functor defines a spectral sequence:

`(Diagram to be made later.)`

From this diagram, one can see how the differentials arise, and they have the grading for the E_2 page. In particular, given the filtration $\{X_p\}$ of X, we can let $E^2_{p,q} \coloneqq H_{p+q}(X_p)$.[4] Thus the E^1 page is

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
H_2(X_0) & H_3(X_1) \arrow[l, "d_1"'] & H_4(X_2) \arrow[l, "d_1"'] & \cdots \arrow[l] \\
H_1(X_0) & H_2(X_1) \arrow[l, "d_1"'] & H_3(X_2) \arrow[l, "d_1"'] & \cdots \arrow[l] \\
H_0(X_0) & H_1(X_1) \arrow[l, "d_1"'] & H_2(X_2). \arrow[l, "d_1"'] & \cdots \arrow[l]
\end{tikzcd}$$

The key is explaining how the differentials occur. Let h be a homology theory, $X = \{X_i\}$ be a filtration, and $C_i \coloneqq X_i/X_{i-1}$ be the cofibers. Then we have a diagram

$$\begin{tikzcd}
 & h(C_1) & h(C_2) \arrow[l] & h(C_3) \arrow[l] & \\
h(X_0) \arrow[r] & h(X_1) \arrow[r] \arrow[u] & h(X_2) \arrow[r] \arrow[u] & h(X_3) \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

[2] You may prefer to call this a **cofibre sequence**.

[3] There's a version of this for functors taking fiber sequnences to long exact sequences, but we won't need to use it.

[4] Technically, we started only with one functor H, but we can define $H_{n-1}(X) \coloneqq H_n(\Sigma X)$ and extend to a family of functors, just as for homology.

Any pair $\rightarrow, \uparrow$ fits into a long exact sequence with connecting morphism $\delta\colon h(C_i) \to h(\Sigma X_{i-1})$:

$$\begin{tikzcd}
 & h(C_1) \arrow[dl, dashed, "\delta"'] & h(C_2) \arrow[l] \arrow[dl, dashed, "\delta"'] & h(C_3) \arrow[l] \arrow[dl, dashed, "\delta"'] & \\
h(X_0) \arrow[r] & h(X_1) \arrow[u] \arrow[r] & h(X_2) \arrow[u] \arrow[r] & h(X_3) \arrow[u] \arrow[r] & \cdots
\end{tikzcd}$$

This is how the first differentials arise: take the connecting morphism δ, then map back $h(X_{i-1}) \to h(C_{i-1})$. Considering longer sequences of maps after taking homology gives you the higher-order differentials.

What follows was a complicated diagram chase that was hard to live-TEX.

We had the E^1 page and differentials, and after taking homology, we get the E^2 page:

$$\begin{tikzcd}
 & E^2_{0,2} & E^2_{1,2} & E^2_{2,2} \\
 & E^2_{0,1} & E^2_{1,1} \arrow[ull] & E^2_{2,1} \arrow[ull] \\
 & E^2_{0,0} & E^2_{1,0} \arrow[ull] & E^2_{2,0} \arrow[ull]
\end{tikzcd}.$$