M392C NOTES: APPLICATIONS OF QUANTUM FIELD THEORY TO GEOMETRY

ARUN DEBRAY
SEPTEMBER 7, 2017

These notes were taken in UT Austin's M392C (Applications of Quantum Field Theory to Geometry) class in Fall 2017, taught by Andy Neitzke. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Thanks to Andy Neitzke for a few corrections.

Contents

1. Donaldson invariants and supersymmetric Yang-Mills theory: 8/31/17 1
2. Zero-dimensional QFT and Feynman diagrams: 9/5/17 5
3. A Little Effective Field Theory: 9/7/17 8

Lecture 1.

Donaldson invariants and supersymmetric Yang-Mills theory: 8/31/17

"The wind blowing on it, well, that's not the worst thing that could happen to a pond! Now imagine you have a laser..."

The course website is https://www.ma.utexas.edu/users/neitzke/teaching/392C-qft-geometry/. There are also lecture notes which are hosted at https://github.com/neitzke/qft-geometry, and are currently a work in progress; if you have contributions or improvements, feel free to contribute them, as a pull request or otherwise. (I'm also taking notes, of course, and if you find problems or typos in my notes, feel free to let me know.) There's also a Slack channel for course-related discussions, which may be easier to use than office hours.

There will be exercises in this course, and you should do at least one-fourth of them for the best grade. Of course, you also want to do them in order to gain understanding. Some worked-out computations could be useful for submitting to the professor's lecture notes.

This course will be relatively wide-ranging; today's prerequisites involve some gauge theory, but the next few lectures won't as much.

Suppose you want to study the topology of smooth manifolds X. Surprisingly, it's really effective to introduce a geometrical gadget, e.g. a Riemannian metric g. Using it, we can define the *Laplace operator* on differential forms $\Delta\colon \Omega^k(X) \to \Omega^k(X)$, which has the formula

$$\Delta := \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d},$$

where $\mathrm{d}\colon \Omega^k(X) \to \Omega^{k+1}(X)$ is the de Rham differential, and $\mathrm{d}^*\colon \Omega^{k+1}(X) \to \Omega^k(X)$ is its adjoint in the L^2-inner product on differential forms induced by the metric. Thus d is canonical, but d^* depends on the choice of metric.

Next we consider the equation

$$\Delta\omega = 0. \tag{1.1}$$

This is a linear equation, so its space of solutions $\mathscr{H}_{k,g} := \ker(\Delta\colon \Omega^k \to \Omega^k)$, called the *space of harmonic k-forms*, is a vector space. If X is compact, it's even a finite-dimensional vector space, which is a consequence of the ellipticity of the Laplace operator.[1] Hence we can define a nonnegative integer

$$b_k(X) := \dim \mathscr{H}_{k,g},$$

[1] For a general differential operator on differential forms, nothing like this is true.

called the k^{th} *Betti number* of X It's a fact that $b_k(X)$ does not depend on the choice of the metric! Thus they are invariants of the smooth manifold X.

In fact, there's even a categorified version of this. This reflects a recent (last decade or so) trend of replacing numbers with vector spaces, sets with categories, etc.

Theorem 1.2. *If X is compact,*[2] *there is a canonical isomorphism $\mathscr{H}_{k,g} \cong H^k(X;\mathbb{R})$, where the latter is the singular cohomology of X with coefficients in $\mathbb{R}$.*

This shows $b_k(X)$ doesn't depend on the smooth structure of X, and is even a homotopy invariant. This will not be true for the Donaldson invariants that we'll discuss later.

Exercise 1.3. Work out some of these spaces of harmonic forms for a metric on S^1 and S^2.

You have to choose a metric, and there are more or less convenient ones to pick. But no matter how you change the metric, there will be a canonical way to identify them.[3]

If X is oriented and $4n$-dimensional, there's a small refinement of the middle Betti number b_{2n} and space of harmonic forms $\mathscr{H}_{2n}$. The *Hodge star operator*

$$\star\colon \Omega^p(X) \longrightarrow \Omega^{\dim X - p}(X)$$

is an involution on $\Omega^{2n}(X)$.

Remark. Let's recall the Hodge star operator. This is an operator on differential forms defined using the Riemannian metric satisfying $\star^2 = 1$ in even dimension, and $[\star, \Delta] = 0$. Hence it acts on harmonic forms. On $\mathbb{R}^2$ with the usual metric, $\star(1) = \mathrm{d}x \wedge \mathrm{d}y$, and $\star(f\,\mathrm{d}x) = f\,\mathrm{d}y$. ◁

Hence we can decompose $\Omega^{2n}(X)$ into the (± 1)-eigenspaces of $\star$: let $\Omega^{2n,\pm}(X)$ denote the ± 1-eigenspace for $\star$. Similarly, $\mathscr{H}_{2n}(X)$ splits into $\mathscr{H}_{2n}^{\pm}(X)$. Thus b_{2n} also splits:

$$b_{2n}(X) = b_{2n}^+(X) + b_{2n}^-(X).$$

These spaces and numbers are also topological invariants, and can be understood in that way.

Exercise 1.4. In dimension $4n+2$, the Hodge star squares to -1. You can still extract topological information from this; what do you get?

Linear equations seem to behave more or less the same in all dimensions. But nonlinear equations behave very differently in different dimensions. In the 1980s, Donaldson used nonlinear equations to produce new and interesting invariants of 4-manifolds. Let X be a connected, oriented 4-manifold with a Riemannian metric g.

Fix a compact Lie group G. For Donaldson, $G = \mathrm{SU}(2)$, and it's probably fine to assume that for much of this class. Fix a principal G-bundle $P \to X$. We'll consider connections on P.

Remark. If you don't know what a connection is, that's OK. Locally, a connection on P is represented by a Lie algebra-valued 1-form $A \in \Omega_X^1(\mathfrak{g})$, and has a *curvature 2-form* $F \in \Omega_X^2(\mathfrak{g}_P)$, which locally is written

$$F = \mathrm{d}A + A \wedge A.$$

Because $\mathrm{SU}(2)$ is nonabelian, $A \wedge A$ isn't automatically zero. ◁

Since F is a 2-form and $\dim X = 4$, we can decompose F into its *self-dual part* F^+ and its *anti-self-dual part* F^-, defined by the splitting of Ω^p by the Hodge star.

Exercise 1.5. Show that if you reverse the orientation of X, F^+ and F^- switch.

Donaldson studied the *anti-self-dual Yang-Mills equation* (ASD YM):

$$F^+ = 0. \tag{1.6}$$

By Exercise 1.5, this is not really different than studing the self-dual Yang-Mills equation; the reason one prefers the ASD version is that it occurs more naturally on certain complex manifolds which were test cases for Donaldson theory.

If G is abelian, e.g. $\mathrm{U}(1)$, (1.6) is linear. But if G is nonabelian, e.g. $\mathrm{SU}(2)$, then (1.6) is nonlinear.

[2] Compactness is really necessary for this.

[3] Interesting question: if you change the metric infinitesimally, how does $\mathscr{H}_k$ change?

Definition 1.7. The *instanton moduli space* is the space $\mathscr{M}$ of equations on P obeying (1.6), modulo the action of the *gauge group* $\mathscr{G}$, the bundle automorphisms of P.[4]

Exercise 1.8. Show that if $G = \mathrm{U}(1)$, then $\mathscr{M}$ is only governed by linear algebra in that

$$\mathscr{M} \cong H^1(M : \mathbb{R})/H^1(X;\mathbb{Z}).$$

So in this case we don't find anything new, though the way we found it is still interesting.

When G is nonabelian, this is not a vector space. It still has some reasonable structure. We now fix $G = \mathrm{SU}(2)$. In this case, (topological) isomorphism classes of principal SU(2)-bundles are classified by the integers, given by the formula

$$k := \int_X c_2(P) \in \mathbb{Z},$$

where c_2 denotes the second Chern class.

This means the moduli of instantons is a disjoint union over $\mathbb{Z}$ of spaces $\mathscr{M}_k$.

Theorem 1.9. *If $k > 0$ and g is chosen generically, $\mathscr{M}_k$ is a finite-dimensional manifold.*

Hence one could learn topological information about X by studing topological properties of $\mathscr{M}_k$. The first idea would be the Betti numbers, but these turn out not to depend on the smooth structure.

Proposition 1.10. *Assuming $k > 0$ and g is generic,*

$$\dim \mathscr{M}_k = 8k - 3(1 - b_1(X) + b_2^+(X)).$$

But there's more to $\mathscr{M}_k$ than the dimension. Donaldson introduced an orientation on $\mathscr{M}_k$, which is canonically defined (and a lot of hard work!), and one can produce classs $\tau_\alpha \in \Omega^*(\mathscr{M}_k)$ labeled by classes $\alpha \in H_*(X)$. Using these, the *Donaldson invariants* are the real numbers

$$\langle \mathscr{O}_{\alpha_1} \cdots \mathscr{O}_{\alpha_\ell} \rangle := \int_{\mathscr{M}} \tau_{\alpha_1} \wedge \cdots \wedge \tau_{\alpha_\ell} \in \mathbb{R}. \tag{1.11}$$

Theorem 1.12. *If $b_2^+(X) > 1$, the Donaldson invariants are independent of g.*

Moreover, they really depend on smooth information: it's not possible to reconstruct them out of algebraic or differential topology, unlike the Betti numbers. So these are very powerful. Their study is called *Donaldson theory*. One good refernece is Donaldson and Kronheimer's book.

Unfortunately, Donaldson theory is technically very hard: the ASD YM equation is hard to study: $\mathscr{M}_k$ is usually noncompact, and (1.11) is an integral over a noncompact space, which is no fun.

What does this have to do with quantum field theory? In 1988, Witten, following a suggestion of Atiyah, found an interpretation of the Donaldson invariants in terms of quantum field theory (hence the suggestive notation in (1.11)).

There are many different quantum field theories: the Standard Model describes three of the four fundamental forces of the universe; quantum electrodynamics describes electromagnetism. Witten interpreted the Donaldson invariants in terms of a specific QFT, called "(a topological twist of) $\mathscr{N} = 2$ supersymmetric Yang-Mills theory (SYM) with gauge group SU(2)."

One imagines X to be a "spacetime" or "universe" whose laws of physics are governed by $\mathscr{N} = 2$ supersymmetric Yang-Mills theory, and to compute the Donaldson invariants, one conducts "experimental measurements" (correlation functions). According to the rules of Lagrangian quantum field theory, this means computing an integral over an infinite-dimensional space (which is alarming, but so it goes):

$$\langle \mathscr{O}_\alpha \rangle = \int_{\mathscr{C}} \mathrm{d}\mu\, \Phi_\alpha e^{-S},$$

where

- $\mathscr{C}$ is the *space of fields*, some sort of infinite-dimensional space akin to the space of functions on X or forms on X,
- $S\colon \mathscr{C} \to \mathbb{R}$ is a functional called the *action*,
- $\Phi_\alpha\colon \mathscr{C} \to \mathbb{R}$ is a (set of) *observables*,

[4] TODO: not sure if I got this right.

- and $\mathrm{d}\mu$ is some measure on $\mathscr{C}$.

In general, computing these correlation functions are very hard,[5] but in $\mathscr{N}=2$ SYM, Witten found localization, a way to reduce it to Donaldson's integrals over finite-dimensional spaces.

This is undoubtedly cool, and brings geometric topology into quantum field theory, but it does not make it much easier to actually compute Donaldson invariants.

The next step was taken in 1995, by Seiberg and Witten, who were interested in a different but related physics problem. They answered a fundamental question about SYM: how it behaves at low energies.

To make an analogy, suppose you have a pond, and you're pond-ering what happens when wind goes across the surface. You're good at physics, so you model the pond as a system of 10^{30} molecules of water and other things, then rent some time on a supercomputer where you model the action on the wind and… somehow this seems wrong. Instead, you model the water and the wind using things like the Navier-Stokes equations. This is not easy, but it's much, much easier.

The idea is there's a "high-energy" description, in terms of 10^{30} particles, but the "low-energy" description[6] involves things like temperature, pressure, liquid, and other things that are hard to define from the high-energy approach. The low-energy picture is very useful for calculations, though if you fire a laser into your pond it wouldn't suffice. Obtaining the description of the low-energy physics from the high-energy physics is typically very hard; in this case, one would have to define temperature and pressure and a lot of things starting from fundamentals. But you just have to do it once, then can apply it to all bodies of water, etc.

Seiberg and Witten applied this to $\mathscr{N}=2$ SYM with gauge group SU(2), and showed that its low-energy description is (roughly) $\mathscr{N}=2$ SYM with gauge group U(1), coupled to matter (sometimes called monopoles). Since the gauge group is abelian, this is much easier. Now, one can imagine that there's an easier description of the Donaldson invariants in terms of the low-energy theory (though, again, this was not the original intent of Seiberg and Witten), and this is given by the *Seiberg-Witten equations*. They look more complicated but are actually vastly simpler.[7]

In the Seiberg-Witten equations, the fields are

- a connection Θ in a U(1)-bundle $\mathscr{E}$, or equivalently a determinant line of a spin^c-structure, and
- a section ψ of S^+, a spinor bundle associated to a spin^c-structure.

In this case, there's a *Dirac operator* $\not{D}$ and a pairing

$$q\colon S^+\otimes S^+\longrightarrow \Lambda^2_+T^*X.$$

Then, the Seiberg-Witten equations are

$$F^+ = q(\psi,\overline{\psi}) \tag{1.13a}$$
$$\not{D}\psi = 0. \tag{1.13b}$$

Let $\widetilde{\mathscr{M}}$ denote the moduli space of pairs (Θ,ψ) satisfying (1.13) modulo the action of some group. For generic g, this is a compact manifold, so understanding its topology is much easier, and the correlation functions for the low-energy theory can be written as integrals over $\widetilde{\mathscr{M}}$, and there's a simple formula relating these to the correlation functions for the high-energy theory. Once this was realized, there was very rapid progress of its use in applications, though understanding precisely why it's the same came more slowly, beginning from a physical argument by Moore and Witten and proceeding to a very different-looking mathematical proof much more recently.

This is an application of QFT to geometry, as we will study in this course. Somehow the most powerful applications involve taking a low-energy limit, and many of them also involve localization in supersymmetric QFT (from an infinite-dimensional integral to a finite-dimensional one).

We will start more slowly: first considering QFT where $\dim X=0$, then $\dim X=1$ (which is quantum mechanics); in these cases, the physics can be made completely rigorous (though it's not necessarily easy). We'll briefly talk about $\dim X=2$, then jump into $\dim X=4$.

[5] Unless $\dim X=0$, where $\mathscr{C}$ is finite-dimensional. We'll talk about this in the next few lectures.

[6] The term "low-energy," despite sounding pejorative, is actually a very useful thing to have.

[7] For a reference, check Morgan's book on the subject.

Lecture 2.

Zero-dimensional QFT and Feynman diagrams: 9/5/17

Last time, we talked about two perspectives on physics, high-energy (or *fundamental*) and low-energy (or *effective*). For example, the high-energy description of a pond is the physics of the 10^{30} or so particles in it, and the low-energy description is the Navier-Stokes equations. We're interested in the relationship between Donaldson theory in the high-energy perspective and Seiberg-Witten theory in the low-energy perspective, which is a story about four-dimensional QFT. But over the next few lectures, we're going to learn about this passage from fundamental to effective in 0-dimensional QFT, one of the few cases where it's known how to make everything rigorous. Nonetheless, it's still an interesting theory, e.g. it has Feynman diagrams.

We also discussed that in the Lagrangian formalism to QFT on a spacetime X, one evaluates integrals over a space $\mathscr{C}(X)$, which is some kind of function space. Hence, it's usually infinite-dimensional, unless $\dim X = 0$. Hence, let's assume $X = \mathrm{pt}$, so $\mathscr{C}(X) = \{X \to \mathbb{R}\} = \mathbb{R}$. There are many choices for $S\colon \mathscr{C} \to \mathbb{R}$,[8] such as

$$S(x) = \frac{m}{2}x^2 + \frac{\lambda}{4!}x^4,$$

where $m, \lambda > 0$. Here m might mean some kind of mass, and λ measures the interaction in the system.

Now we can define something important and fundamental: the *partition function*

$$Z := \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-S(x)}.$$

The *observables* are polynomial functions $f : \mathscr{C} \to \mathbb{R}$, and their *(unnormalized) expectation values* are

$$\langle f \rangle := \int_{-\infty}^{\infty} \mathrm{d}x\, f(x) e^{-S(x)}.$$

We require f to be polynomial so that this integral converges. All of these are functions in m and λ. Also, notice that all of these are completely well-defined; maybe this is a trivial observation, but it won't be true when we ascend to higher dimensions.

Computing these quantities is less trivial. Let's start with Z, or even $Z_0 := Z(m, \lambda = 0)$. This is a Gaussian:

$$Z_0 = \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-mx^2/2} = \sqrt{\frac{2\pi}{m}}.$$

In order for this to be well-defined, we need $m = 0$ of course, but there's a physical reason to throw out this case, as it corresponds to a system with more than one vacuum state and a degenerate critical point of the action.

To compute the partition function for $\lambda > 0$, we're not sure how to directly evaluate the integral, but we can try to expand it out as a Taylor series in λ around 0. This will allow us to understand the system in the presence of weak interactions, which is often exactly what physicists want to know. We'll leave $e^{-mx^2/2}$ alone, since we know how to integrate it exactly. The $\lambda x^4/4!$ term expands to

$$Z(m, \lambda) = \int_{-\infty}^{\infty} \mathrm{d}x \sum_{n=0}^{\infty} \left(-\frac{\lambda}{4!}\right)^n \frac{x^{4n}}{n!} e^{-mx^2/2}.$$

We'd like to switch the sum and integral to obtain

$$= \sum_{n=0}^{\infty} \left(-\frac{\lambda}{4!}\right)^n \underbrace{\int_{-\infty}^{\infty} \frac{x^{4n}}{n!} e^{-mx^2/2}}_{I}, \tag{2.1}$$

but we have to be careful about convergence. If this works, though, the integral I is tractable.

Exercise 2.2. Show that

$$\int_{-\infty}^{\infty} \mathrm{d}x\, x^{2k} e^{-mx^2/2} = \sqrt{\frac{2\pi}{m}} \frac{1}{m^k} \frac{(2k)!}{k! 2^k}.$$

[8] One can also use $\mathbb{C}$-valued actions.

Hence, modulo the assumption we made before, if $\widetilde{\lambda} := \lambda/m^2$,

$$Z(m,\lambda) = \sqrt{\frac{2\pi}{m}} \sum_{n=0}^{\infty} \left(-\frac{1}{96}\right)^n \frac{(4n)!}{n!(2n)!} \widetilde{\lambda}^n \tag{2.3}$$
$$= \sqrt{\frac{2\pi}{m}} \left(1 - \frac{1}{8}\widetilde{\lambda} + \frac{35}{384}\widetilde{\lambda}^2 + \cdots + (1390.1)\widetilde{\lambda}^{10} + \cdots\right).$$

This is called the *perturbation series* for this partition function. Though this partition function is a scalar multiple of a Bessel function, often these series are actually divergent for any $\widetilde{\lambda} > 0$. This means the assumption we made in (2.1) was wrong. There's various ways to think about this — if this function did converge to its Taylor series, it would do so in a neighborhood of 0 in $\mathbb{C}$, hence for negative λ. Physically, this doesn't make sense.

Nonetheless, the perturbation series is still useful in those cases.

Definition 2.4. Let $f : \mathbb{R}_+ \to \mathbb{C}$ be a function and $s := \sum_{n=0}^{\infty} c_n t^n$ be a formal series. We say that s is an *asymptotic series* for f as $t \to 0^+$ if for all $N \geq 0$,

$$\lim_{t\to 0^+} t^{-N} \left| f(t) - \left(\sum_{n=0}^{N} c_n t^n\right)\right| = 0.$$

In this case, we write

$$f(t) \underset{t\to 0^+}{\sim} \sum_{n=0}^{\infty} c_n t^n.$$

In particular, this means that

$$\lim_{t\to 0^+} |f(t) - c_0| = 0$$
$$\lim_{t\to 0^+} \frac{1}{t}|f(t) - c_0 t + c_1| = 0,$$

and so on. So even if s doesn't converge, it's still useful, capturing the limits, linear behavior, quadratic behavior, etc., of f. You have encountered other asymptotic series in your life: Stirling's formula for the factorial is an asymptotic series for the gamma function at ∞: it doesn't actually converge in a sensible way, but it captures a lot of useful information.

Proposition 2.5. *The series* (2.3) *is an asymptotic series for the partition function* $Z(m,\lambda)$ *as* $\lambda \to 0^+$.

So it's not equality, but it's a useful and interesting approximation.

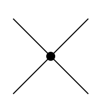
You might wonder whether there's some better series approximating $Z(m,\lambda)$ that actually converges, but this is not true.

Proposition 2.6. *If f has a convergent Taylor series at x_0, then its Taylor series is an asymptoric series for f at x_0.*

Proposition 2.7. *Every smooth function f can have at most one perturbation series as $x \to x_0$.*

Sometimes none exists.

We will interpret (2.3) in terms of Feynman diagrams. The basic object is a vertex with four half-edges attached:



A *Feynman diagram* for (2.3) is a placement of some of these vertices and a way of connecting the half-edges. (Feynman diagrams for other systems may look different.)

placeholder

FIGURE 1. Some Feynman diagrams with one or two vertices.

Let D_n denote the set of diagrams with n vertices.

Proposition 2.8. *The number of ways to pair up $2k$ objects is $(2k)!/k!2^k$.*

Corollary 2.9.

$$|D_n| = \frac{(4n)!}{(2n)!2^{2n}}.$$

There's also a group action of a group $G_n := (S_4)^n \rtimes S_n$ on D_n, where the i^{th} copy of S_4 permutes the half-edges for the i^{th} vertex, and S_n shuffles the n vertices. In other words, we can restate the asymptotic series for the partition function (2.3) in a more combinatorial manner: since $Z_0 = \sqrt{2\pi/m}$.

$$\frac{Z(m,\lambda)}{Z_0} \sim \sum_{n=0}^{\infty} (-\widetilde{\lambda})^n \frac{|D_n|}{|G_n|}.$$

We want to describe $|D_n|/|G_n|$ as the cardinality of some kind of quotient set, but this is only literally true if the G_n-action on D_n is free. The proper thing to do, as suggested by the orbit-stabilizer theorem, is to sum over orbits, weighted by the order of their stabilizers.[9] Thus

$$\frac{Z(m,\lambda)}{Z_0} \sim \sum_{n=0}^{\infty} (-\widetilde{\lambda})^n \sum_{[\Gamma]\in D_n/G_n} \frac{1}{|\operatorname{Aut}\Gamma|}.$$

Since $\widetilde{\lambda} = \lambda/m^2$ and a Feynman diagram in D_n has n^2 edges, we can rewrite (2.3) in a way that is completely a combinatorial sum over Feynman diagrams:

$$\frac{Z(m,\lambda)}{Z_0} \sim \sum_{n\geq 0} \sum_{[\Gamma]\in D_n/G_n} \frac{(-\lambda)^{|V(\Gamma)|}}{m^{|E(\Gamma)|}} \cdot \frac{1}{|\operatorname{Aut}(\Gamma)|}.$$

Here, $V(\Gamma)$ is the set of vertices of Γ, and $E(\Gamma)$ is the set of edges. This leads to the *Feynman rules* for summing over the Feynman diagrams for this theory:

- Draw one representative Γ for each orbit in D_n/G_n.
- Define its weight w_Γ as the product of factors $-\lambda$ for each vertex and $1/m$ for each edge, weighted by $1/|\operatorname{Aut}(\Gamma)|$.

Then,

$$\frac{Z}{Z_0} \sim \sum_{[\Gamma]} w_\Gamma.$$

Example 2.10. Let's calculate some low-order terms.

- The empty Feynman diagram has the weight 1.
- The action of $G_1 \cong S_4$ on D_1 is transitive, so we only need a single representative, such as the "figure-8 diagram." Its stabilizer group has order 8, so there's a contributing factor of $(-\lambda)/8m^2$.
- There are three orbits in D_2/G_2, represented by a graph with zero self-loops, which contributes a term of $\lambda^2/48m^4$, one with one self-loop on each vertex, which contributes $\lambda^2/16m^4$, and one with two self-loops on each vertex, which contributes $\lambda^2/128m^4$.

Thus, the perturbative expansion is

$$\begin{aligned}\frac{Z}{Z_0} &\sim 1 - \frac{\lambda}{8m^2} + \frac{\lambda^2}{48m^4} + \frac{\lambda^2}{16m^4} + \frac{\lambda^2}{128m^4} + O(\lambda^3)\\ &= 1 - \frac{\lambda}{8m^2} + \frac{35}{384}\frac{\lambda^2}{m^4} + O(\lambda^3).\end{aligned}$$

The higher-order terms correspond to diagrams with 3 or more vertices. ◀

If you know the automorphism group of a diagram Γ, then the automorphism group of $\Gamma \amalg \Gamma$ is very similar: a copy of $\operatorname{Aut}(\Gamma)$ for each component, plus the S_2 switching them. If you follow your nose in this line of thought, you can determine the sum in terms of only nonempty, connected diagrams.

Proposition 2.11.

$$\sum_{\Gamma} w_\Gamma = \exp\left(\sum_{\Gamma \text{ connected, nonempty}} w_\Gamma \right).$$

[9] Another way to think about this is to consider the quotient *groupoid* D_n/G_n, and sum over it in the groupoid measure, which amounts to the same thing.

This suggests that $\log(Z/Z_0)$ is an important physical quantity, and indeed, it's called the *free energy* of the system, as in statistical mechanics. We'd like to say that

$$\log\left(\frac{Z(m,\lambda)}{Z_0}\right) \sim \sum_{\Gamma \text{ connected, nonempty}} w_\Gamma,$$

though there's an analysis argument to check here.

Now we want to compute expectation values. Let's start with

$$\langle x^k \rangle := \int_{-\infty}^{\infty} x^n e^{-S}\,\mathrm{d}x.$$

If k is odd this is 0, but for k even, we can compute an asymptotic series for this function with a similar sum over Feynman diagrams, but with different rules:

- In addition to the 4-valent vertices from before, each diagram must have exactly k univalent vertices.
- We only consider automorphisms which fix these vertices.

You can work this out with a similar argument as for Z/Z_0.

To compute the *normalized expectation values* $\langle x^k\rangle/Z$, use the same diagrams, but with the rule that every connected component of Γ must have at least one univalent vertex. You can then draw out the first few diagrams and conclude things such as

$$\frac{\langle x^2\rangle}{2} \sim \frac{1}{m} - \frac{\lambda}{2m^3} + O(\lambda^2).$$

More generally, there's no need to constrain ourselves to a quartic interaction: we can isntead consider the action

$$S = \frac{m}{2}x^2 + \sum_{k=3}^{\infty} \frac{\lambda_k x^k}{k!}. \tag{2.12}$$

In this case, we consider Feynman diagrams with vertices of aribitrary valence ≥ 3, and sum with the rules that an edge contributes $-1/m$ and an n-valent vertex contributes $-\lambda_n$. We can actually carry out the analysis even if (2.12) doesn't converge (in which case we don't get an asymptotic series for a function, but that's OK). Anyways, tabulating the Feynman diagrams we get the beginning of the normalized perturbative expansion

$$\frac{Z}{Z_0} \sim 1 - \frac{\lambda_4}{8m^2} + \frac{\lambda_3^2}{12m^3} + \cdots$$

Yet another generalization is to consider actions on $\mathscr{C} = \mathbb{R}^N$, rather than $\mathbb{R}$, corresponding to considering the theory on N points, rather than one point. Now, the quartic term is some 4-tensor, so (using the Einstein summation convention) the most general action is

$$S = \frac{1}{2}x^i M_{ij} x^j + \frac{1}{4!}C_{ijk\ell}x^i x^j x^k x^\ell,$$

and Z_0 is again a Gaussian:

$$Z_0 = \int_{\mathbb{R}^n} e^{-x^i M_{ij} x^j/2} = \frac{(2\pi)^{N/2}}{\sqrt{\det M}}.$$

In this case, one can compute with Feynman diagrams again, but this time labeling the edges with labels $1,\dots,N$.

Lecture 3.

A Little Effective Field Theory: 9/7/17

Today, we're going to illustrate the passage from the fundamental to the effective using zero-dimensional QFT: the fundamental theory will be an action $S(x,y)$ in two variables, and its effective theory S_{eff} will be a simpler theory in a single variable.

Last time, we discussed the fields $\mathscr{C} = \mathbb{R}^N$ in a zero-dimensional QFT with an action

$$S := \frac{1}{2}x^i M_{ij} x^j + \frac{1}{4!}C_{ijk\ell}x^i x^j x^k x^\ell.$$

As $C \to 0$, one wants to compute the asymptotic series, which amounts to a sum over Feynman diagrams. In this context, one can sum over unlabeled diagrams Γ, but with the weight incorporating the labels of the half-edges in

$\{1,\dots,N\}$. Explicitly, the weight of an edge i to j should be $(M^{-1})^{ij}$, and that of a vertex with half-edges i, j, k, and ℓ is $C_{ijk\ell}$.

More abstractly, if V is a finite-dimensional vector space with a measure μ, you can choose an $M \in \mathrm{Sym}^2 V^*$ and a $C \in \mathrm{Sym}^4 V^*$, and define the action

$$S(x) := \frac{1}{2}M(x,x) + \frac{1}{4!}C(x,x,x,x).$$

Then, one would compute the partition function

$$\int \mathrm{d}\mu\, e^{-S(x)}.$$

Now let's focus on a specific example. We can start with fields $\mathsf{C} = \mathbb{R}^2$ with coordinates x, y and an action

$$S(x,y) := \frac{m}{2}x^2 + \frac{M}{2}y^2, \tag{3.1}$$

which is two uncoupled systems. So let's turn on coupling in (3.1):

$$S(x,y) := \frac{m}{2}x^2 + \frac{M}{2}y^2 + \frac{\mu}{4}x^2y^2. \tag{3.2}$$

Say that we're actually interested in x: we want to compute Z and $\langle x^n\rangle$, but *not* $\langle y\rangle$ or $\langle f(x,y)\rangle$ that depends on y. This might happen in a system which naturally comes with both x and y, but y is some extra degrees of freedom. We'll see this is natural when $M \gg m$.

There are only a few kinds of labels in the Feynman diagram, because M and C in (3.2) have a lot of zeroes: we'll use a solid line for $1/m$ (corresponding to x^2) and a dashed line for $1/M$ (for y^2); all vertices must have two solid half-edges and two dashed half-edges, weighted by $-\mu$.

Let's compute $\log(Z/Z_0)$; by Proposition 2.11, this allows us to only sum over connected diagrams. There is only one diagram with a single vertex (order μ), and three with two vertices (order μ^2). Their respective computations are

$$\log\left(\frac{Z}{Z_0}\right) \sim -\frac{\mu}{4mM} + \frac{\mu^2}{16m^2M^2} + \frac{\mu^2}{16m^2M^2} + \frac{\mu^2}{8m^2M^2} + O(\mu^3).$$

For correlation functions, we must add n univalent vertices for x^n. The μ^0-term (the "tree level") calculates exactly the noninteracting theory. When we enumerate the diagrams for $\langle x^2\rangle$, there's one with zero 4-valent vertices, one for a single 4-valent vertex, and three with two 4-valent vertices, and the sum is

$$\frac{\langle x^2\rangle}{Z} \sim \frac{1}{m} - \frac{\mu}{2m^2M} + \frac{\mu^2}{4m^3M^2} + \frac{\mu^2}{2m^3M^2} + \frac{\mu^2}{4m^3M^2} + O(\mu^3).$$

This is not the logarithm: since we've normalized this calculation, it's a sum over Feynman diagrams for which every connected component contains a univalent vertex.

This explodes more quickly than other ones we considered: to compute $\langle x^4\rangle$, there are a lot of diagrams to sum over, even just at the μ^2. The answer will be

$$\frac{\langle x^4\rangle}{Z} \sim \frac{3}{m^2} - \frac{3\mu}{m^3M} + \frac{33\mu^2}{4m^4M^2} + O(\mu^3).$$

And since we only care about x, there should be some way to simplify this and get all of the dashed lines out of the way first. One idea is: if we only want

$$\langle x^n\rangle = \int_{\mathbb{R}^2} \mathrm{d}x\,\mathrm{d}y\, x^n e^{-S(x,y)},$$

then by Fubini's theorem, we can integrate out the dependence on y, defining S_{eff} such that

$$e^{-S_{\mathrm{eff}}(x)} := \int_{\mathbb{R}} \mathrm{d}y\, e^{-S(x,y)}.$$

Then

$$\langle x^n\rangle = \int_{\mathbb{R}} \mathrm{d}x\, x^n e^{-S_{\mathrm{eff}}(x)}.$$

In this particular example, we can compute S_{eff}, or at least its asymptotic series (which suffices if we want to do the asymptotic series for $\langle x^n\rangle$ in the original theory). The answer for the asymptotic series for $\mu \to 0$ is

$$S_{\mathrm{eff}}(x) \sim \frac{m_{\mathrm{eff}}}{2}x^2 + \sum_{k\ge 3}\lambda_k x^k, \tag{3.3}$$

where m_{eff} is some effective mass. The interacting term is interesting — there are interactions between multiple xs (vertices with four solid edges). These arise because of Feynman diagrams such as the one in Figure 2, where by "ignoring y" we close the gap between these two vertices and obtan an interaction between two copies of x.

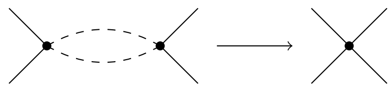


FIGURE 2. Left: a Feynman diagram for the action (3.2). In the effective field theory (3.3), the dashed lines correspond to terms which are integrated out, so this diagram becomes a quartic x-x interaction (on the right).

Specifically, in (3.3), the terms are

$$\begin{aligned} m_{\mathrm{eff}} &= m + \frac{\mu}{2M} \\ \lambda_k &= \begin{cases} 0, & k \text{ odd} \\ -\left(-\dfrac{\mu}{M}\right)^{k/2}\dfrac{1}{2^{k/2+2}k}, & k \text{ even.} \end{cases} \end{aligned}$$

Thus, as $M \to \infty$, $m_{\mathrm{eff}} \to m$: when $M \gg m$, this is a more reasonable approximation.

This is our first baby example of an effective field theory. The fact that we integrated out the degrees of freedom we didn't care about is a useful heuristic to have around.

Symmetries. Let's go back to $\mathscr{C} = \mathbb{R}$ and

$$S = \frac{m}{2}x^2 + \frac{\lambda}{4!}x^4.$$

This is in a sense the simplest nontrivial example: if you had a cubic term instead of a quartic term, $\int e^{-S}$ wouldn't be well-defined (it goes to ∞ as $x \to \pm\infty$).

Proposition 3.4. *$\langle x^n\rangle = 0$ when n is odd.*

Proof.

$$\begin{aligned} \langle x^n\rangle &= \int_{-\infty}^{\infty} \mathrm{d}x\, x^n e^{-S(x)} \\ &= \int_{-\infty}^{\infty} \mathrm{d}(-x)\,(-x)^n e^{-S(-x)} \\ &= (-1)^n \int_{-\infty}^{\infty} \mathrm{d}x\, x^n e^{-S(x)} \\ &= (-1)^n\langle x^n\rangle. \end{aligned}$$

⊠

One takeaway is that this theory is symmetric under the group $\mathbb{Z}/2$ acting on $\mathscr{C}$ as multiplication by $\{\pm 1\}$. This leads to a very general principle.

Proposition 3.5. *Let $S\colon \mathscr{C} \to \mathbb{R}$ and the measure on $\mathscr{C}$ are both G-invariant for a group G, then $\langle \mathscr{O}\rangle = \langle \mathscr{O}^g\rangle$ for any observable $\mathscr{O}\colon \mathscr{C} \to \mathbb{R}$, where $\mathscr{O}^g = g^*\mathscr{O}$.*

If G is a Lie group, we can differentiate this equation: take $g = \exp(tX)$ for some $X \in \mathfrak{g}$: taking

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \big(\langle \mathscr{O}\rangle = \langle \mathscr{O}^{tX}\rangle\big),$$

we conclude that $\langle X(\mathscr{O})\rangle = 0$.

In general, symmetries are an extremely important ingredient in QFT.

Fermions and super-vector spaces. You might remember that we wanted to do something topological, but our computations, as functions in the parameters (m, λ), were not deformation-invariant (you could think of them as nonconstant functions on a moduli space of QFTs). To get things that are, we need one more ingredient: fermions.

The way to do this, which will return again and again in this course, is to replace the manifold $\mathscr{C}$ by a supermanifold! Since we've so far only considered vector spaces, we'll get a slightly gentler introduction in the form of super-vector spaces.

For a reference on this material, check out Etingof's course notes for a class on the mathematics of QFT.[10]

Definition 3.6. A *super-vector space* is a $\mathbb{Z}/2$-graded vector space $V = V^0 \oplus V^1$.

For example, if $V^0 = \mathbb{R}^p$ and $V^1 = \mathbb{R}^q$, V is denoted $\mathbb{R}^{p|q}$. This can be done over any field, but we're only going to consider $\mathbb{R}$ or $\mathbb{C}$.

These are not so terrible. But how we do algebra with them is also different: if you are taking tensor products, super-vector spaces are not the same as $\mathbb{Z}/2$-graded vector spaces![11]

Definition 3.7. The symmetric monoidal category of super-vector spaces $(\mathsf{sVec}, \otimes, s_{-,-})$ is the same as that for ordinary $\mathbb{Z}/2$-graded vector spaces $\mathsf{Vect}^{\mathbb{Z}/2}$, except for the symmetry

$$s_{V,W} : V \otimes W \to W \otimes V.$$

For $\mathsf{Vect}^{\mathbb{Z}/2}$, this is the map $v \otimes w \mapsto w \otimes v$, but in sVec, it's defined on homogeneous v, w by

$$v \otimes w \longmapsto (-1)^{|v||w|} w \otimes v,$$

where $v \in V^{|v|}$ and $w \in W^{|w|}$; non-homogeneous elements are sums of homogeneous ones, so this determines $s_{V,W}$.

So the point is if v or w is in V^1, we multiply by -1:

$$s(v \otimes w) = \begin{cases} -w \otimes v & v \text{ or } w \text{ is in } V^1 \\ w \otimes v, & v, w \in V^0. \end{cases}$$

This category is considerably more useful than it looks. There's a sense in which sVec and $\mathsf{Vect}^{\mathbb{Z}/2}$ are the only two symmetric monoidal structures that can be placed on the monoidal category $(\mathsf{Vect}^{\mathbb{Z}/2}, \otimes)$.

Other algebraic constructions are also different.

Definition 3.8. The *symmetric algebra* on a super-vector space V is the superalgebra ($\mathbb{Z}/2$-graded algebra)

$$\mathrm{Sym}^*(V) := T^*V / \langle v \otimes w - s(w \otimes v) \rangle.$$

Thus, if $V = V^0$, $\mathrm{Sym}^*(V)$ is the usual symmetric algebra, but if $V = V^1$, $\mathrm{Sym}^*(V) = \Lambda^*(V^1)$, the exterior algebra! In general, it'll be a mix of these two things.

We can use this to define polynomial functions: in ordinary algebra, there's a canonical isomorphism between the algebra of polynomials on a vector space V and $\mathrm{Sym}^*(V^*)$.

Definition 3.9. Motivated by this, if $V \in \mathsf{sVec}$, we define its *algebra of polynomial functions* $\mathscr{O}(V)$ to be

$$\mathscr{O}(V) := \mathrm{Sym}^*(V^*).$$

Here $V^* := \mathrm{Hom}_{\mathsf{sVec}}(V, \mathbb{R}^{1|0}) = (V^0)^* \oplus (V^1)^*$. $\mathscr{O}(V)$ is itself a super-vector space, in fact a (super)commutative algebra! That is, $p \cdot q = (-1)^{|p||q|} q \cdot p$.

In physics, the even direction corresponds to bosonic stuff, and the odd direction to fermionic stuff. So $\mathscr{C}$ may be a super-vector space, and we can take the action function $S \in \mathscr{O}^0(\mathscr{C})$.

Example 3.10. Let's consider a purely fermionic theory, such as $\mathscr{C} = \mathbb{R}^{0|2}$. Then, $\mathscr{C}$ has coordinate functions $\psi^1, \psi^2 \in \mathscr{O}^1(\mathscr{C})$, which have odd statistics in the sense that

$$\begin{aligned} \psi^1\psi^2 &= -\psi^2\psi^1 \\ (\psi^1)^2 &= 0 \\ (\psi^2)^2 &= 0. \end{aligned}$$

[10] For supermanifolds specifically, see https://ocw.mit.edu/courses/mathematics/18-238-geometry-and-quantum-field-theory-fall-200 lecture-notes/sec9.pdf.

[11] If the base field has characteristic 2, these two notions are actually the same, which quickly follows from Definition 3.7. But this will not be important to us.

This, $\mathscr{O}^0(\mathscr{C})$ has basis $\{1, \psi^1\psi^2\}$ and $\mathscr{O}^1(\mathscr{C})$ has basis $\{\psi^1, \psi^2\}$. Thus $\mathrm{Sym}^* \mathscr{C}$ is four-dimensional, which is as expected, since it should be $\Lambda^*\mathbb{R}^2$.

Since there's no quartic terms in ψ^1 and ψ^2, we actually can't introduce interactions, so our action functional is

$$S := \frac{1}{2}M\psi^1\psi^2.$$

This is somewhat like a function, but it behaves very weirdly: $S^2 = 0$!

We'd like to make sense of the partition function in this setting. In order to do this, we need rules for integrating over odd variables. To integrate over $\mathbb{R}^{0|1}$ with odd coordinate ψ, the most general function is $a\psi + b$, so we can stipulate that its integral is

$$\int_{\mathbb{R}^{0|1}} \mathrm{d}\psi\,(a\psi + b) := a.$$

We'll define the exponential via its power series, which means it's much simpler than for bosons!

Now, on $\mathbb{R}^{0|k}$, we have to specify order of integration: to compute

$$\int_{\mathbb{R}^{0|k}} \mathrm{d}\psi^1\,\mathrm{d}\psi^2 \cdots \mathrm{d}\psi^k F = \int_{\mathbb{R}^{0|1}} \mathrm{d}\psi^1 \left(\int_{\mathbb{R}^{0|1}} \mathrm{d}\psi^2 \left(\cdots \int_{\mathbb{R}^{0|1}} F \right) \cdots \right),$$

first evaluate the innermost integral, then the next innermost, and so on, ending at the outermost ($\mathrm{d}\psi^1$ in the above equation).

Hence the partition function is

$$\begin{aligned} Z &= \int_{\mathbb{R}^{0|2}} \mathrm{d}\psi^1\,\mathrm{d}\psi^2 e^{-S(\psi^1,\psi^2} \\ &= \int_{\mathbb{R}^{0|2}} \mathrm{d}\psi^1\,\mathrm{d}\psi^2 \left(1 - \frac{1}{2}M\psi^1\psi^2\right) \\ &= -\frac{1}{2}M \int_{\mathbb{R}^{0|2}} \mathrm{d}\psi^1\,\mathrm{d}\psi^2\,\psi^1\psi^2 \\ &= \frac{1}{2}M \int_{\mathbb{R}^{0|2}} \mathrm{d}\psi^1\,\mathrm{d}\psi^2\,\psi^2\psi^1 \\ &= \frac{1}{2}M. \end{aligned}$$

◀

For bosons (i.e. even fields), we had a Gaussian

$$\int_{-\infty}^{\infty} e^{-Mx^2/2}\,\mathrm{d}x = \frac{\sqrt{2\pi}}{\sqrt{M}}.$$

This is suggestive: if you arrange the masses of bosons and fermions right, things might cancel out to produce a theory whose dependence on the mass cancels out and is deformation-invariant.