NOTES FROM THE 2022 SUMMER SCHOOL ON GLOBAL SYMMETRIES

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JUNE 13, 2022

These notes were taken at the GCS2022 School in June 2022 at the Perimeter Institute of Theoretical Physics. I live-TEXed them and there may be typos or mistakes; please send questions, comments, complaints, and corrections to adebray@purdue.edu.

Contents

Part 1. Kantaro Ohmori, Introduction to symmetries in quantum field theory

1. Extended operators and defects in functorial QFT: 6/13/22

One of the goals of this workshop is to discuss generalized symmetries in quantum field theory. For us, this means topological operators and defects. Here's a short outline of these four talks:

(1) Quantum field theory as a functor, and how to work with extended operators and defects in this formalism
(2) Topological operators and symmetries
(3) One-form symmetries in gauge theories and confinement
(4) "Non-invertible" symmetries

Let's get started. Quantum field theory means a lot of different things to a lot of different people; today, we will only focus on relativistic Euclidean QFT. When we say a d-dimensional QFT, d refers to the dimension of spacetime.

In the non-topological setting, it's not yet completely clear how to define a quantum field theory as a functor, so the following definition will be a little heuristic.

Definition 1.1. A quantum field theory is a symmetric monoidal functor $Z\colon \mathcal{B}ord_S^{\langle d,d-1\rangle} \to \mathcal{V}ect$.

Here $\mathcal{V}ect$ is the symmetric monoidal category of vector spaces with tensor product and $\mathcal{B}ord_S^{\langle d,d-1\rangle}$ is a bordism category. S refers to some kind of geometric structure we want to endow spacetime with: for example, we could ask for just a smooth structure, or a spin structure, or a Riemannian metric, or a principal G-bundle with a connection, or so on. A manifold with S-structure is called an S-manifold.

The objects of $\mathcal{B}ord_S^{\langle d,d-1\rangle}$ are closed, $(d-1)$-dimensional manifolds with an S-structure. The set of morphisms between $(d-1)$-dimensional S-manifolds M and N is the set of (diffeomorphism classes rel boundary) of S-bordisms from M to N. An *S-bordism* X from M to N is a compact, d-dimensional manifold with an identification of S-manifolds $\partial X \xrightarrow{\cong} M \amalg \overline{N}$. Here $\overline{N}$ denotes N with the opposite orientation.

To define a category, we need to compose morphisms; this is accomplished by gluing bordisms. (TODO: picture).

Remark 1.2. We haven't said precisely how to define S, so one might wonder whether it depends on d. For example, there's a difference between a framing of a manifold M (a trivialization of TM) and a stable framing (a trivialization of $TM \oplus \underline{\mathbb{R}}^k$ for some k); the former depends on d and the latter does not, and the two notions are not the same.

Freed–Hopkins [FH21] have shown that for reflection-positive topological field theories, many S-structures that appear to depend on d in fact stabilize and are independent of the dimension. ◀

If $Z\colon \mathcal{B}ord_S^{\langle d,d-1\rangle}$ is a quantum field theory, then for a closed $(d-1)$-dimensional S-manifold M, $Z(M)$ is a vector space. This is called the *state space* of M. If X is a bordism from M to N, then $Z(X)$ is a linear map from the state space of M to the state space of N. We often think of this map as *time evolution* of states. The fact that Z is symmetric monoidal means that $Z(M_1 \amalg M_2) \cong Z(M_1) \otimes Z(M_2)$.

Let $\tau \in (0,\infty)$. Then $M \times [0,\tau]$ is a bordism from M to M (its boundary is $M \amalg \overline{M}$). Let $U_M(\tau) \coloneqq Z(M \times [0,\tau])$, which we can think of as time evolution on M for time τ; in a Hamiltonian system we think of

$$U_M(\tau) = \exp(-\tau H_M), \tag{1.3}$$

where $H_M\colon Z(M) \to Z(M)$ is the Hamiltonian on M. Gluing bordisms implies $U_M(\tau_1) \circ U_M(\tau_2) = U_M(\tau_1+\tau_2)$.

Remark 1.4. We would like to think of these operators as unitary, like in Hamiltonian quantum mechanics; making this precise from the functorial perspective is an area of active research. See for example a recent proposal of Kontevich–Segal [KS21].

To discuss unitarity we need some kind of inner product, but we did not ask for our state spaces to come with inner products. There is a one-parameter family of bilinear pairings around: the cylinder $M \times [0,\tau]$, thought of as a bordism from $M \amalg \overline{M} \to \varnothing$, induces a map $Z(M) \otimes Z(\overline{M}) \to Z(\varnothing) = \mathbb{C}$.[1] ◂

Example 1.5 (Finite gauge theory)**.** (Untwisted) finite gauge theory is a d-dimensional *topological* quantum field theory: the S-structure is topological, rather than geometric. Specifically, it is no structure at all.

Fix a finite group G and $p \in \{0,1,\dots,d-1\}$. If $p>0$, we ask that G is abelian. Therefore we can make sense of $H^{p+1}(M;G)$ when M is a closed manifold: when G is nonabelian and $p=0$, this is the set of isomorphism classes of principal G-bundles on M. For compact M, $H^{p+1}(M;G)$ is finite.

Let M be a closed $(d-1)$-manifold; we define the state space of finite gauge theory on M to be the vector space spanned by the finite set $H^{p+1}(M;G)$.

If W is a bordism from M to N, we define the linear map $Z(W)$ as a form of "finite path integral" — we can't make sense of the path integral in general for gauge theories, but becaue G is finite we can in this case. Fix $A \in H^{p+1}(M;G)$ and let $i_M\colon M \hookrightarrow W$ and $i_N\colon N \hookrightarrow W$ be the inclusions. Define

$$Z(W)|A\rangle \coloneqq c(W) \sum_{\substack{B \in H^{p+1}(W;G)\\ i_M^* B = A}} |i_N^* B\rangle, \tag{1.6}$$

where $c(W) \in \mathbb{R}$ is a mormalization constant that appears so that this definition is functorial when we glue bordisms. ◂

Exercise 1.7. Say $d=2$, $p=0$, and $G=\mathbb{Z}/n$. Calculate $Z(S^1)$, $Z(\Sigma)$, and $Z(\Sigma')$, where Σ is the pair of pants regarded as a bordism from $S^1 \amalg S^1 \to S^1$ and Σ' is Σ in the opposite direction. For bordisms, only calculate the maps up to normalization constants, since we didn't specify those constants in (1.6).

Now let's talk about extended quantum field theory. If M is a closed d-dimensional manifold, it may be regarded as a bordism $\varnothing \to \varnothing$. Applying Z, we obtain a linear map $\mathbb{C} \to \mathbb{C}$, since $Z(\varnothing) = \mathbb{C}$. This map is determined by its value on 1, which is a complex number called the *partition function* of M.

Associated to a closed $(d-1)$-manifold we have a state space. In extended QFT, we assign higher-categorical invariants to manifolds in lower dimensions; for example, to a closed $(d-2)$-dimensional manifold we assign something called a "2-vector space," which is something like a $\mathbb{C}$-linear category; and in general on a closed $(d-k)$-manifold we assign a "k-vector space," some kind of higher catgory. We will not define k-vector spaces precisely here, and indeed different researchers use different definitions.

We are especially interested in $Z(S^{q-1})$ as q varies; this is some sort of higher category. The objects of this category are the codimension-q defects or extended operators (to us, these two words mean the same thing) in the QFT Z. In non-topological theories, we need to specify the radius r of S^{q-1}; the codimension-q defects are the limit of $Z(S^{q-1}(r))$ as $r \to 0$.

There is another formalism for higher-codimension defects or operators, given by something called *decorated bordisms*. This latter approach may be easier to digest from the physics point of view. The two approaches are expected to be equivalent.

Part 2. Mike Hopkins, Lattice systems and topological field theories

Part 3. Clay Córdova, Introduction to anomalies in quantum field theory

[1] The fact that $Z(\varnothing) = \mathbb{C}$ is another consequence of symmetric monoidality.

References

[FH21] Daniel S. Freed and Michael J. Hopkins. Reflection positivity and invertible topological phases. *Geom. Topol.*, 25(3):1165–1330, 2021. https://arxiv.org/abs/1604.06527. 1

[KS21] Maxim Kontsevich and Graeme Segal. Wick rotation and the positivity of energy in quantum field theory. *Q. J. Math.*, 72(1-2):673–699, 2021. https://arxiv.org/abs/2105.10161. 2