FALL 2017 GEOMETRIC SATAKE SEMINAR

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1. Overview on the geometric Satake theorem: 9/1/17

This overview was given by David Ben-Zvi.

This semester, we're studying the geometric Satake theorem, one of the most important results in geometric representation theory, and even a central result in the geometric Langlands program.

This theorem involves some potentially unfamiliar words; we'll define them in the course of this seminar.

Theorem 1.1 (Geometric Satake)**.** *Let G be a reductive group over a field k.*[1] *Then, there is a category of $G(\mathscr{O})$-equivariant perverse sheaves on the affine Grassmannian of G, $G(K)/G(\mathscr{O})$, symmetric monoidal under convolution, together with a fiber functor $H^\bullet(-)$, and this is equivalent to $(\mathsf{Rep}^{\mathrm{fd}}_{G^\vee}, \otimes)$ (where $G^\vee$ is the Langlands dual group) as symmetric monoidal categories, with the fiber functor the forgetful map to Vect_k.*

By $\mathsf{Rep}^{\mathrm{fd}}$ we mean the full subcategory of finite-dimensional representations. K will be some local field, and $\mathscr{O}$ is its ring of integers. For example, if $k = \mathbb{C}$, $K = \mathbb{C}((t))$ and $\mathscr{O} = \mathbb{C}[[t]]$, and over $\mathbb{F}_p$, you have $K = \mathbb{F}_p((t))$ and $\mathscr{O} = \mathbb{F}_p[[t]]$.[2]

Okay, first what's a reductive group? For $k = \mathbb{C}$, these are complexifications of compact groups. For example: GL_n, SL_n, PGL_n, SU_n, Sp_n, and E_7.

Now this theorem is saying that we start with one reductive group and we get another, $G^\vee$. This relationship is such that if $G = T$ is a torus, i.e. $(\mathbb{C}^\times)^k$, its Langlands dual is the dual torus $T^\vee$: if T is the quotient of $\mathbb{C}^n$ by a lattice, $T^\vee$ is the quotient of $(\mathbb{C}^n)^*$ by the dual lattice.

Theorem 1.1 is a kind of Fourier transform, a quite fancy one. For example, if $G = \mathrm{GL}_1$, the affine Grassmannian is a (scheme which behaves more or less like) $\mathbb{Z}$: $\mathrm{GL}_1(\mathbb{C}((t)) = \mathbb{C}((t))^\times$ and $\mathrm{GL}_1(\mathbb{C}[[t]])$ is the group of power series with nonzero constant term. When you mod these out, the leading term of the Laurent series becomes the only important thing, in a sense. The next ingredient is the equivariant perverse sheaves, but ends up being vector bundles over Gr in this case, so we get (modulo some reducedness which doesn't come into play here) the category of graded vector spaces. In this case, Theorem 1.1 says the category of graded vector spaces is equivalent to the category of representations of $\mathbb{G}_m$, just like the Fourier transform exchanges functions on $\mathbb{Z}$ with representations of S^1.

You can interpret the geometric Satake theorem as the source of the Langlands dual group: it admits a definition in terms of tori and root data, but it feels somewhat ad hoc, and one is left wondering: where did it all come from? Instead, by the Tannakian perspective on representation theory, Theorem 1.1 is telling us that

[1] You can let k be a ring R, the coefficients. The algebraic geometry we do will still be over $\mathbb{C}$, though; the representations you get end up also being representations over the ring R.

[2] In particular, they will never be $\mathbb{Q}_p$ and $\mathbb{Z}_p$. However, the geometric Satake theorem is a living piece of mathematics, and only in the past year Peter Scholze proved a version of this for the p-adics.

the category of $G(\mathcal{O})$-equivariant sheaves with its fiber functor is canonically the category of representations of a group, and in fact gives us enough information to reconstruct the group! So the geometric Satake theorem is a bridge from G to $G^\vee$, and is one of the only bridges.

Example 1.2. Langlands duality is often somewhat surprising: G and $G^\vee$ don't look like each other, and it's not clear how to obtain one from the other. Of course, $(G^\vee)^\vee \cong G$.

$$\begin{aligned} \mathrm{GL}_n &\longleftrightarrow \mathrm{GL}_n \\ \mathrm{SL}_n &\longleftrightarrow \mathrm{PGL}_n \\ \mathrm{SO}_{2n+1} &\longleftrightarrow \mathrm{Sp}_{2n} \\ \mathrm{SO}_{2n} &\longleftrightarrow \mathrm{SO}_{2n}. \end{aligned}$$ ◄

You can also use the geometric Satake theorem to explain some things which at first appear to not be geometric. For example, $H^*(\mathbb{CP}^n) \cong \mathbb{Z}[x]/(x^{n+1})$ with $|x| = 2$ is acted on by SL_2 by raising and lowering operators, which comes out of complex geometry, but this does not arise from an SL_2-action on $\mathbb{CP}^n$ itself. More generally, SL_k acts on $H^*(\mathrm{Gr}_k(\mathbb{C}^n))$ in a similar way, and is similarly mysterious.

But Theorem 1.1 identifies it with an equivariant perverse sheaf for the action of PGL_k on a Grassmannian, and the action of PGL_k on a Grassmannian is more evident. So we've obtained either new interesting representations, or geometric models for representations of your group, and solved the mysteries of this representation.

It also flows the other direction: if you take a reductive group over R, you recover information about $\mathsf{Rep}_{G^\vee,R}$, the category of representations over R. This is an active topic of research, and people including Geordie Williamson have used it to uncover interesting consequences in modular representation theory.

Of course, the geometric Satake theorem is also entangled with the geometric Langlands conjectures in interesting ways.

We're going to first discuss affine Grassmannians, then perverse sheaves (which generalize cohomology of smooth projective varieties, and could be an entire seminar unto themselves), then their convolution (the fact that it's symmetric monoidal is very deep, and related to commutativity of Hecke algebras). Finally, we'll talk about Tannakian reconstruction, a beautiful abstract story that allows us to extract $G^\vee$ from the theorem.

Here's a very provisional schedule:

- First, a few lectures on the affine Grassmannian: Rustam (next week), and then Richard H. (the week after).
- Then, perverse sheaves (and also intersection cohomology): Arun, Sebastian, and Yan.
- Then, convolution and its commutativity: Richard W. and Vaibhav.
- Then, Tannakian reconstruction: Isabelle and Nicky.
- Finally, putting it together into a proof of the geometric Satake correspondence: Rok and probably also someone else.

There's a lot of further topics and cool applications if anyone is interested after that.

2. Bruhat-Tits trees: 9/1/17

In the second part of the first meeting, Tom Gannon spoke about trees (à la Serre).

Throughout this lecture, let K be a complete field (with respect to some norm), together with a discrete valuaton $v\colon K^\times \to \mathbb{Z}$.[3] Let $\mathscr{O}_K$ denote its ring of integers, which is a local ring, and $\mathfrak{m}$ denote its unique maximal ideal. Let π be a *uniformizer*, i.e. a generator of $\mathfrak{p}$.[4] We'll let $q := |\mathscr{O}_K/\mathfrak{m}|$, and assume that q is finite, so that the residue field $\mathscr{O}_K/\mathfrak{m} \cong \mathbb{F}_q$.

Though we didn't define $v(0)$, we think of it as ∞: the valuation values how many times you can divide an element by π, and for 0 you can do this infinitely often.

Let K be a field with a discrete valuation $v\colon K^\times \to \mathbb{Z}$ and $c \in (0,1)$ be fixed. Then, the map $\|\cdot\|_c\colon K \to [0,\infty)$ with $|x|_c := c^{v(x)}$ is a norm, and moreover is non-Archimedean, satisfying a stronger form of the

[3] Recall that a *discrete valuation* is a surjective group homomorphism $K^\times \to \mathbb{Z}$. You can think of it as measuring how many times a uniformizer π divides a given field element.

[4] You can think of this as a coordinate on the curve.

triangle inequality:

$$|x+y|_c \le \max\{|x|_c, |y|_c\}.$$

Proposition 2.1. *With notation as above, the set $\{x \in K \mid |x| \le 1\}$ is a ring, and in fact a discrete valuation ring; its unique maximal ideal is $\{x \in K \mid |x|_c < 1\}$.*

That it's a discrete valuation ring means the unique maximal ideal is principal. This ring is called the *associated ring of integers* of K. Let's pick a specific value of c, which is $1/q$.

Example 2.2 (2-adic rationals)**.** The 2-adic rationals, $\mathbb{Q}_2$, form a complete field with a discrete valuation. One way to think about this is that there's a norm on $\mathbb{Q}$ given by how many 2s you can factor out; completing it with respect to that norm defines $\mathbb{Q}_2$.

There's also a lower-brow way to think of this, as Laurent series in 2: an element of $\mathbb{Q}_2$ is something like

$$2^{-4} + 2^{-3} + 2^{-1} + 2 + 2^3 + 2^5 + \cdots .$$

Equality is termwise, and addition and multiplication are like those of Laurent series. The coefficients are mod 2, so if you consider p-adics for $p > 2$, you have more options. In this case, the valuation is the smallest N such that the N-coefficient is nonzero.

There's also an algebraic interpretation of $\mathbb{Z}_2$ and $\mathbb{Q}_2$. ◄

Example 2.3. Another example if $K = \mathbb{F}_p((t))$ with $\mathscr{O}_K = \mathbb{F}_p[[t]]$. The valuation is the minimal power of t that appears with a nonzero coefficient, like for $\mathbb{Q}_2$. ◄

Now we'll discuss the Bruhat-Tits tree for $\mathrm{SL}_2(K)$. There's not a lot of motivation, except that this stuff is awesome.

The tree will be a set of vertices and edges; its vertices will be a set of lattices in K^2.

Definition 2.4. A *lattice* in K^2 is an $\mathscr{O}_K$-submodule Λ of K^2 such that $\Lambda \otimes_{\mathscr{O}_K} K = K^2$.

Concretely, these are subsets of K^2 of the form $\mathscr{O}_K \cdot v_1 + \mathscr{O}_K \cdot v_2$, where $\{v_1, v_2\}$ is a basis for K^2. These correspond to the usual lattices in $\mathbb{R}^2$.

Since $\mathrm{GL}_2(K)$ acts on the set of bases of K^2, it acts on the set of lattices. The stabilizer of each lattice is $\mathrm{GL}_2(\mathscr{O}_K)$, and therefore the space of lattices is naturally isomorphic to $\mathrm{GL}_2(K)/\mathrm{GL}_2(\mathscr{O}_K)$. Hey, that space appeared in the statement of the geometric Satake isomorphism!

Theorem 2.5 (Principal divisor theorem)**.** *Let L_1 and L_2 be lattices. Then, there's a basis $\{e, f\}$ for L_1 and $m, n \in \mathbb{Z}$ such that $\{\pi^m e, \pi^n f\}$ is a basis for L_2.*

The proof is linear algebra, spiced up somewhat by the fact that it's over discrete valuation rings. It's also the only place where we assume the residue field is finite.

Remark. If you consider GL_1 instead of GL_2, you get the statement we discussed before, that $\mathrm{GL}_1(K)/\mathrm{GL}_1(\mathscr{O}_K)$ is the integers (and therefore representations of $\mathbb{G}_m$ are equivalent to graded vector spaces). ◄

Now, say that two lattices L_1 and L_2 are equivalent if $L_1 = \pi^\ell L_2$ for some ℓ. The space of equivalence classes is $\mathrm{PGL}_2(K).\mathrm{PGL}_2(\mathscr{O}_K)$. We define the vertices of the Bruhat-Tits tree to be this set.

Now we should talk edges. Let v and w be two vertices, and L_1 and L_2 be lattice representatives for v and w, respectively. By Theorem 2.5, there are m and n carrying a basis for L_1 to a basis for L_2, and we add an edge iff $|m-n| = 1$.[5] Equivalently, we add an edge if there's a rescaling of L_1 called L_1' such that $L_2 \supsetneq L_1' \supsetneq \pi L_2$.

Call this graph G. We'll eventually show it's a tree.

Proposition 2.6. *G is a connected graph.*

Proof. Let L_1 and L_2 be lattices. Then, there are $e, f \in L_1$ and $m, n \in \mathbb{Z}$ such that $\{e, f\}$ is a basis for L_1 and $\{\pi^m e, \pi^m f\}$ is a basis for L_2. Without loss of generality, assume $m \ge n$. Then, there's an edge from L_2 to $\mathscr{O}_K \cdot \pi^{m-n} e + \mathscr{O}_K \cdot f$. Continuing in this way, we must eventually reach L_1. □

A string of points produced by this method is called an *apartment.* More generally, any path of vertices which is finite or half-infinite is called a *chain.* A *simple chain* is one where you never step forward and then back (or vice versa).[6]

[5] There's enough uniqueness in the proof for this to be well-defined, even if m and n aren't unique.

[6] For example, the basic steps of salsa define chains, but not a simple chain; the basic steps of waltz, which return to the same point but after more than one step, are a simple chain.

Remark. These lattices satisfy a Noetherian-esque property: if you have an infinite chain of vertices $w_0 - w_1 - \cdots$, then there exist representative lattices L_i for w_i such that for all i, $L_i \supsetneq L_{i+1} \supsetneq \pi L_i$. $\blacktriangleleft$

Proposition 2.7. *For any simple chain C, there's a $g \in \mathrm{GL}_2(K)$ such that $g \cdot C$ is the chain*

$$\mathscr{O}_K \cdot e_1 + \mathscr{O}_K \cdot e_2 \supsetneq \mathscr{O}_K \cdot \pi e_1 + \mathscr{O}_K \cdot \pi e_2 \supsetneq \mathscr{O}_K \cdot \pi^2 e_1 + \mathscr{O}_K \cdot \pi^2 e_2 \supsetneq \cdots,$$

where $\{e_1, e_2\}$ is the standard basis for K^2.

Corollary 2.8. *G is actually a tree.*

Proof. Assume C is a simple chain that's a cycle in G. Then, Proposition 2.7 preserves connectiveness, but replaces it with something which could not possibly be a cycle. $\square$

Proof sketch of Proposition 2.7. Let the starting chain be $C = L_0 \supsetneq L_1 \supsetneq L_2 \supsetneq \ldots$. We know there's a $g \in \mathrm{GL}_2(K)$ such that $g_0 L_0$ is the starting vertex $\mathscr{O}_K \cdot e_1 + \mathscr{O}_K \cdot e_2$. So g_0 is our candidate. But we don't know whether $g_0 L_1$ is the same as $\mathscr{O}_K \cdot \pi e_1 + \mathscr{O}_K \cdot \pi e_2$, but there's a g_1 in the stabilizer of $g_0 L_0$ that moves it to $\mathscr{O}_K \cdot \pi e_1 + \mathscr{O}_K \cdot \pi e_2$. Then, inductively, one can assume there exists an element in the stabilizer of the first i that brings the next element of the chain into position, and so on.[7] This inductive argument is a little delicate, and uses the fact that the resiude field is finite.

You may have to do this infinitely many times, which is actually fine: you can conjugate the g_i such that

$$g_i = \begin{pmatrix} 1 & x_i \\ 0 & 1‘ \end{pmatrix}$$

for $x_i \in \mathfrak{m}^i$; then, the infinite product is

$$\begin{pmatrix} 1 & \sum x_i \\ 0 & 1 \end{pmatrix},$$

and, using the local topology of K, you can show this sum converges. $\square$

If you act by $\mathrm{SL}_2(K)$ (through the inclusion in $\mathrm{GL}_2(K)$), the parity of $m - n$ is preserved, so you can decompose the tree into a bipartite tree. From the geometric Satake perspective, this says that the affine Grassmannian for $\mathrm{PGL}_2(K)$ has two connected components.

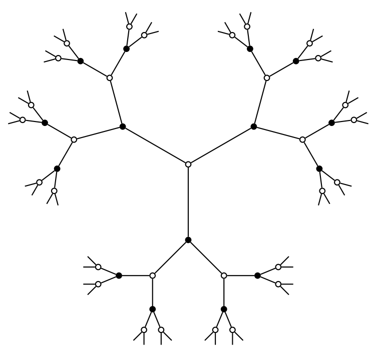

FIGURE 1. The Bruhat-Tits tree for $\mathrm{SL}_2(\mathbb{Q}_3)$. The two parity classes of vertices are in black and white. Source: `https://tex.stackexchange.com/a/135764`.

Another fun fact is that $\mathrm{PSL}_2(\mathscr{O}_K)$ acts on the tree by graph automorphisms, and the double coset space

$$\mathrm{PSL}_2(\mathscr{O}_K) \backslash \mathrm{PSL}_2(K) / \mathrm{PSL}_2(\mathscr{O}_K)$$

is in bijection with the positive integers — well actually, the highest weights, or the irreducible representations of SL_2. This already looks Langlandsy, and more of the story appears: you can define Hecke operators on the tree: for each $n \in \mathbb{N}$, let

$$T_n f(v) = \frac{1}{n} \sum_{|w-v|=n} f(w).$$

[7] One way to think of this is that $\mathrm{GL}_2(K)$ is filtered by the discrete valuation; g_0 is the first piece, $g_1 g_0$ is the second piece, and so forth.

That is, the Hecke operator acts on the space of functions on the tree averages over things that are distance n away.

Theorem 2.9 ((Classical) Satake theorem)**.** *These Hecke operators T_n commute, and generate an algebra isomorphic to the representation ring of* SL_2.

The geometric Satake theorem is a categorified analogue of this theorem. Indeed, the Bruhat-Tits tree appears in the story of the geometric Satake theorem as well, helping us understand the geometry of the affine Grassmannian in the case $G = \mathrm{PGL}_2$.

3. An introduction to the affine Grassmannian: 9/8/17

"That's an affine Grassmannian you've got there. Shame if something happened to it."

Today Rustam spoke about the affine Grassmannian, beginning with the uniformization of G-bundles as motivation from the topological case, then move to the algebraic world.

Uniformization of G-bundles, topological setting. Let Σ be a compact, connected, oriented (real) surface and G be a connected topological group. Let $x \in \Sigma$ and $\mathbb{D}$ be a small disc around it. Let $\mathbb{D}^0 := \mathbb{D} \setminus x$.

Definition 3.1. The *loop groups* of G are:

- the *positive loop group* $L^{\mathrm{top},+}$, the continuous maps $f\colon \mathbb{D} \to G$, under pointwise multiplication;
- the *loop group* $L^{\mathrm{top}}(G)$, the continuous maps $f\colon \mathbb{D}^0 \to G$, and
- $L_X(G) := \{f\colon X \setminus x \to G\}$.

There's a restriction map $L^{\mathrm{top},+}(G) \to L^{\mathrm{top}}(G)$.

Proposition 3.2. *There's a bijection of sets*

$$L_X(G)\backslash L^{\mathrm{top}}(G)/L^{\mathrm{top},+}(G) \cong \mathrm{Bun}_G(X).$$

The idea is that $X \setminus x$ is homotopic to a graph, and since isomorphism classes of principal bundles are classified by maps to BG, and BG is simply connected because G is connected. Hence all principal G-bundles on $X \setminus x$ are trivial, so it matters crucially how we put x back in, and then quotient by redundant data.

Uniformization of G-bundles, algebraic setting. Now, let X be smooth, connected, projective curve over $\mathbb{C}$ and G be a semisimple algebraic group. Let $x \in X$ be a $\mathbb{C}$-point and $\mathbb{D}_x := \mathrm{Spec}(\widehat{\mathscr{O}}_{X,z}) = \mathrm{Spec}(\mathbb{C}[[t]])$ be the formal disc around x, so $x \in \mathbb{D}_x \hookrightarrow X$. Hence, the punctured disc $\mathbb{D}_x \setminus x = \mathrm{Spec}\,\mathbb{C}((t))$.

Definition 3.3. The *loop groups* of G are given by the following functors of points.[8]

- The *positive loop group* $L^+(G)(R) := \mathrm{Hom}(\mathbb{D}_R, G)$.
- The *loop group* is $L(G)(R) := \mathrm{Hom}(\mathbb{D}_x \setminus x, G)$.
- $L_X(G) := \mathrm{Hom}(X_R \setminus x \to G)$.

Hence $L(G)(\mathbb{C}) = G(\mathbb{C}((t)))$ and $L^+(G)(\mathbb{C}) = G(\mathbb{C}[[t]])$.

Theorem 3.4 (Beauville-Laszlo)**.** *There's an equivalence of stacks*

$$L_XG\backslash LG/L^+G \cong \mathrm{Bun}_G(X).$$

This is a hard proof: it doesn't follow from the usual descent arguments. Semisimplicity is crucial, and means that you can't apply this to GL_1.

Let $\mathscr{O}_K := \mathbb{C}[[t]]$, which corresponds to the disc, and $K := \mathbb{C}((t))$, which corresponds to the punctured disc. Recall that the *affine Grassmannian* is $\mathrm{Gr}_G := L(G)/L^+(G) = G(K)/G(\mathscr{O}_K)$.[9]

If $G = \mathrm{GL}_n$, this is the same as the set of (full-rank) lattices on K^n: $G(K)$ acts transitively on all lattices, and the stabilizer is $G(\mathscr{O}_K)$.[10] For other groups, there's a similar description in terms of lattices.

[8]Two of them are not representable as schemes; $L^+(G)$ is a scheme, and $L(G)$ is an ind-scheme but not a formal scheme. An *ind-scheme* is a filtered colimit of schemes, where all maps are closed embeddings, e.g. $\mathbb{A}^\infty$ or $\mathbb{P}^\infty$. $\mathbb{A}^\infty = \mathrm{Spec}\,\mathbb{C}[x_1, x_2, \dots]$ is also a scheme.

[9]The name comes from the affine Weyl group because it has translations in it; the affine Grassmannian is not affine in any usual sense. For this reason, it's sometimes referred to as the *infinite Grassmannian* or the *loop Grassmannian*.

[10]These are just like lattices in $\mathbb{R}$: a lattice Λ is an $\mathscr{O}_K$-submodule of K^n such that $\Lambda \otimes_{\mathscr{O}_K} K = L^n$.

This definition of the Grassmannian characterizes it as a set. Our first goal will be to give it a topology, at least in the case where $G = \mathrm{GL}_n$.

The first tool we'll use is a valuation: if $\vec{v}_1, \dots, \vec{v}_n$ is a basis for a lattice Λ, then $\det(\vec{v}_1, \dots, \vec{v}_n) \in K^\times$. This is not an invariant of Λ, but can be made into one.

Definition 3.5. Let Λ be a lattice. Its *valuation* $v(\Lambda) \in \mathbb{N}$ is the minimum n such that t^n is the determinant of a basis of Λ.

For the trivial lattice $\Lambda^0 := (\mathscr{O}_K)^n$, $n = 2$. Also, if you do this for $G = \mathrm{GL}_1$, the affine Grassmannian is literally $\mathbb{Z}$; you can think of the valuation as a kind of determinant map from $\mathrm{GL}_n \to \mathrm{GL}_1$, and hence to $\mathrm{Gr}_{\mathrm{GL}_1} = \mathbb{Z}$.

Our next tool will be to compare lattices with the standard lattice.

Lemma 3.6. *For all lattices Λ, there's an $a \in \mathbb{N}$ such that*

$$t^a \Lambda^0 \subseteq \Lambda \subseteq t^{-a}\Lambda^0. \tag{3.7}$$

This will allow us to get the ind-structure.

Definition 3.8. Let $\mathrm{Gr}_{\mathrm{GL}_n}^{\ell,a}$ be the set of lattices $\Lambda \in \mathrm{Gr}_{\mathrm{GL}_n}$ such that $v(\Lambda) = \ell$ and (3.7) is satisfied.

Then $\mathrm{Gr}^{\ell,a} \hookrightarrow \mathrm{Gr}^{\ell,b}$ if $b \geq a$, and

$$\mathrm{Gr}_{\mathrm{GL}_n} = \bigcup_{\ell,a} \mathrm{Gr}_{\mathrm{GL}_n}^{\ell,a}.$$

Let $J_{a,n}$ denote the space of $(na - k)$-dimensional subspaces of $t^{-a}\Lambda^0/t^a\Lambda^0 \cong \mathbb{C}^{2an}$. Then, $\mathrm{Gr}_{\mathrm{GL}_n}^{\ell,a}$ embeds into $J_{a,n}$ by taking the quotient by $t^a\Lambda^0$, and $J_{a,n}$ is a Grassmannian! By the Plücker embedding, it's a projective variety.

Definition 3.9. An *ind-projective ind-variety* is an ind-scheme $X = \operatorname{colim}_i X_i$ such that each X_i is projective variety.

Remark. In schemes, there's a significant difference between limits and colimits. Limits of affine schemes always exist, because colimits of rings do. For example, $\mathbb{A}^\infty = \operatorname{Spec} \mathbb{C}[x_1, x_2, \dots] \cong \varprojlim \mathscr{O}_K/t^n\mathscr{O}_K$. But ind-schemes are not always schemes. For example, K, the functor $R \mapsto R((t))$, is the colimit of $K^{\geq -N} \cong \mathbb{A}^\infty$. This is the case where you're only allowed to have finitely many coordinates. Said another way, the positive part of the Laurent series in the affine Grassmannian is fine, albeit infinite-dimensional; the negative tails produce the ind-ness.

For a concrete example, $\mathrm{Gr}_{\mathrm{GL}_1} \cong \mathbb{Z}$, and this is an infinite disjoint union of points, a nice ind-projective ind-variety. You might try to realize it as Spec of an infinite direct product of $\mathbb{C}$s, but these are not isomorphic! The correspondence between coproducts of schemes and products of rings only works fully at the finite level. ◀

Theorem 3.10. *The affine Grassmannian $\mathrm{Gr}_{\mathrm{GL}_n}$ is an ind-projective ind-variety.*

The idea is to use the $J_{a,n}$.

There's an important stratification of the affine Grassmannian: using Gauss-Jordan elimination, the orbits of $\mathrm{GL}_n(\mathscr{O}_K)$ on $\mathrm{Gr}_{\mathrm{GL}_n}$ are identified with coweights $\mathbb{C}^\times \to T$ (where T is a maximal torus for our group). The set of coweights is often denoted $X_\bullet(T)$. Specifically,

$$\mathrm{GL}_n(K) = \coprod_{\substack{\lambda=(a_1,\dots,a_n)\in\mathbb{Z} \\ a_1 \geq \dots \geq a_n}} \mathrm{GL}_n(\mathscr{O}_K) \begin{pmatrix} t^{a_1} & & \\ & \ddots & \\ & & t^{a_n} \end{pmatrix} \mathrm{GL}_n(\mathscr{O}_K). \tag{3.11}$$

The idea: using row reduction, you can get rid of everything except for powers of t (since you're only using Taylor series, not Laurent series). There's a similar perspective for other groups G, for which one might write

$$G(K) = \coprod_{\lambda \in X_\bullet(T)_+} G(\mathscr{O}_K) t^\lambda G(\mathscr{O}_K). \tag{3.12}$$

One can identify $X_\bullet(T)_+ \cong X_\bullet(T)/W$, where W is the Weyl group for G.[11]

This decomposition is really nice: the orbits are all projective varieties. There's a nice Morse-theoretic approach to all this.

The decomposition (3.12) is sort of a "set-theoretic Satake theorem:" the Langlands dual is a little implicit, but tells us that finite-dimensional irreducible representations of $G^\vee$ are indexing this decomposition of the affine Grassmannian. For $G = \mathrm{GL}_n$, which is Langlands self-dual, (3.11) can also be interpreted as indexed by the representations of GL_n.

The algebro-geometric approach to the affine Grassmannian. The affine Grassmannian admits a functor-of-points definition, as the functor $\mathrm{Gr}\colon \mathsf{Alg}_\mathbb{C} \to \mathsf{Set}$ sending a $\mathbb{C}$-algebra R to the set of finitely-generated projective $R[[t]]$-submodules Λ of $R((t))^n$ such that

$$\Lambda \otimes_{R[[t]]} R((t)) = R((t))^n.$$

Theorem 3.13. *This is represented by an ind-projective ind-scheme.*

This scheme is isomorphic to the one described in Theorem 3.10. You'd prove Theorem 3.13 by finding a cover by subfunctors that are represented by projective varieties and whose colimit is Gr again.

We also get a functor-of-points approach to the orbit decomposition (3.11): $\mathrm{Gr}_{\mathrm{GL}_n}(R)$ is the set of (Σ, β) where Σ is a vector bundle on $\mathbb{D}_R$ and $\beta\colon \Sigma|_{\mathbb{D}_R^0} \to \underline{\mathbb{C}}^n$ is an isomorphism with the trivial bundle. If $Y \subset X$, let $\mathrm{Bun}_G(X, Y)$ denote the set of principal G-bundles on X together with data of a trivialization on Y, then we can replace vector bundles with principal G-bundles to obtain a more general description:

$$\mathrm{Gr}_G(R) = \mathrm{Bun}_G(\mathbb{D}_R, \mathbb{D}_R^0).$$

Remark. Unlike in algebraic topology, defining principal G-bundles is tricky: if you try things which are Zariski-locally G-torsors, you get the wrong thing. Keeping in mind that it should always be possible to form an associated vector bundle from a principal G-bundle and a G-representation, followed by some messing around with Grothendieck topologies, leads to the right notion. $\blacktriangleleft$

For general G, Gr_G is an ind-projective ind-scheme; the proof idea is to embed $G \hookrightarrow \mathrm{GL}_n$.

The R-points of $L(G)$ are pairs $(\Sigma, \beta) \in \mathrm{Bun}_G(\mathbb{D}_R, \mathbb{D}_R^0)$ together with a trivialization $\varepsilon\colon \Sigma \to \underline{G}$ on all of $\mathbb{D}_R$. Forgetting ε defines the quotient map to Gr_G; if you think about it, you'll find that this is actually the quotient by $L^+(G)$.

Back to topology. There's a topological version of this story — in topology, the Grassmannian arises in a very different way. Let G be a complex Lie group and K be its maximal compact subgroup, so K is homotopic to G.

Let ΩK denote the (based) loop space of K, the space of basepoint-preserving, continuous maps $S^1 \to K$.[12] Then, there's a model for the affine Grassmannian Gr_G that's homotopic to ΩK.

Another way to think of this is to let $LK = \mathrm{Map}(S^1, K)$ be the free loop space. Then, $\Omega K = LK/K$, and this is like taking the quotient of $L(G)/L^+(G)$.

Now, we want this to be a moduli space of something. Well, $\Omega K \simeq \Omega^2 BK$, where BK is the homotopy type such that $\mathrm{Map}(X, BK)$ is naturally $\mathrm{Bun}_K(X)$ (as a set). This is representable, which is a theorem.

Two-fold loops in Y are identified with maps from a disc into Y such that (a small neighborhood of) the boundary maps to the basepoint. Hence $\Omega K \simeq \Omega^2 BK$ is the set of K-bundles on $\mathbb{C} = \mathbb{R}^2$ that are pointed, in that there's extra data of a trivialization on $\mathbb{C} \setminus \mathbb{D}$, where $\mathbb{D}$ is a small disc. This is precisely what we said the affine Grassmannian was: G-bundles trivialized at a point. So the algebraic geometry and the homotopy theory are the same – since $BK \simeq BG$, you could also take the set of G-bundles on a disc trivialized at a point.

The fact that ΩK is a 2-fold loop space means that it's a homotopical kind of abelian group, which is key – it means the affine Grassmannian is in some sense an abelian group. For example, for $G = \mathrm{GL}_1$, $\Omega^2 BS^1 = \mathbb{Z}$, which *is* a group. This groupiness (E_2-structure) will be crucial to the geometric Satake theorem.

The description of the affine Grassmannian as based loops on a compact Lie groups is what enabled Bott and others to attack it with Morse theory.

[11] $X_\bullet(T)_+$ is the set of *dominant weights*: given a chamber C, the set of weights that pair positively with the elements of that chamber.

[12] See Pressley-Segal for details.