GEOMETRY AND STRING THEORY SEMINAR: FALL 2018

ARUN DEBRAY
SEPTEMBER 6, 2018

Contents

1. Moduli of flat SL_3-connections and exact WKB

The first talk this semester was given by Andy Neitzke.

Let C be a thrice-punctured $\mathbb{CP}^3$, say punctured at $\{1, \omega, \omega^2\}$, and let $\mathcal{M}$ denote the moduli space of flat $\mathrm{SL}_3(\mathbb{C})$-connection over C with unipotent holonomy around the punctures; this is an example of a *character variety*. This talk will discuss Andy's work (in progress) with Lotte Hollands on constructing nice coordinate systems on this space, using ideas coming from physics.

Let's start with the simpler case of SL_2, and consider the *Mathieu equation*, a Schrödinger equation with periodic potential. Let $\hbar > 0$; then, the Mathieu equation is

$$\left(-\frac{\hbar^2}{2}\partial_x^2 + \cos x - E\right)\psi(x) = 0. \tag{1.1}$$

Parallel transport (i.e. evolution of solutions) of this equation defines a flat $\mathrm{SL}_2(\mathbb{R})$-connection ∇ on $\mathbb{R}$. You might think it's $\mathrm{GL}_2(\mathbb{R})$, because there are two solutions, but they're related by the Wronskian. Since the potential is periodic, this is a connection on $\mathbb{R}/2\pi\mathbb{Z} = S^1$; now we can ask about its monodromy, or about its eigenvalues (which are easier to write down without making additional choices). In physics, the eigenvalues are known as *quasi-momenta* for a particle moving with respect to this potential.

Let ψ be an eigenfunction with eigenvalue λ. If $E \gg 1$, then $\cos x$ is small, so

$$\psi_\pm(x) := \exp\left(\pm i\frac{\sqrt{2E}}{\hbar}x\right) \tag{1.2}$$

is a basis of the solutions. The eigenvalues are

$$\lambda_\pm = \exp\left(\pm 2\pi\frac{\sqrt{2E}}{\hbar}\right), \tag{1.3}$$

and the trace is $2\cos(2\pi\sqrt{2E}/\hbar)$. Then $|\lambda_\pm| = 1$ for all E.

So the trace is periodic in $\sqrt{E}$. If this is close to ± 2, we're in a region called the "gap": ΔE is exponentially small, and so solutions are stable. When the absolute value of the trace is smaller, we're in the "band," where the monodromy is complex. This means that solutions exponentially blow up or exponentially decay.

Remark 1.4. In solid-state physics, one example of periodic potentials are crystals. One can show that bands and gaps correspond to conducting and insulating states. ◀

Because of this application, physicists have developed lots of techniques for studying these systems, which we can adapt to geometry to study the monodromy.

First, let's complexify: let $z = e^{ix}$; then we have a complex Schrödinger equation

$$\big(\hbar^2\partial_z^2 + P(z)\big)\psi = 0, \tag{1.5}$$

where

$$P(z) = \frac{1}{z^3} - \frac{2E - \hbar^2/4}{z^2} + \frac{1}{z}. \tag{1.6}$$

The $\hbar^2/4$ correction isn't that important.

Remark 1.7. You can do this on any Riemann surface as long as P is a holomorphic quadratic differental; this requires choosing a complex projective structure. But the ideas can be gotten across in coordinates. ◄

To understand the monodromy, we need to get at the solutions. The exact WKB method constructs solutions of the form

$$\psi(z) = \exp\left(\frac{1}{\hbar}\int_{z_*}^{z} \lambda \, \mathrm{d}z\right). \tag{1.8}$$

In order to satisfy (1.5), λ must satisfy the *Riccati equation*

$$\lambda^2 + P + \hbar\partial_z\lambda = 0. \tag{1.9}$$

This is easier to solve than the original equation. Namely, to leading order in $\hbar$, $\lambda^2 + P = 0$. We will then plug this back in to get at higher orders in $\hbar$. Specifically, we get

$$\lambda = \sqrt{-P} - \hbar\frac{P'}{4P} + \hbar^2\sqrt{-P}\frac{5(P')^2 - 4PP''}{32P^3} + \cdots \tag{1.10}$$

This naturally lives on the *spectral curve* for the equation, i.e. the Riemann surface for $\sqrt{-P}$, $\Sigma \coloneqq \{y^2+P(z) = 0\}$, a double cover of the original surface.

This isn't the end of the story, though: solutions will have monodromy around the zeros of P. But we also can't have monodromy (TODO: I missed why). Looking more closely at (1.10), it doesn't actually converge: it's just an asymptotic series. But it's still useful; it admits Borel summation for $\hbar > 0$ and away from a locus called the *Stokes graph* $W(P)$.[1]

The Stokes graph cuts the Riemann surface into domains; inside each domain, everything works, and you learn a lot about the solutions. But you can't do anything in a neighborhood of a zero of P, which prevents the paradox we chanced upon earlier. The upshot is that in each domain, there's a canonical basis (up to scaling) of the solution space: the solutions are a line bundle over the spectral curve, together with a connection ∇^{ab} represented by $\hbar\lambda \, \mathrm{d}z$. And there'sa canonical way to glue these line bundles over $W(P)$, to obtain a line bundle $L \to \Sigma$ together with a flat connection. It's almost flat (the monodromy around branch points might be -1).

It's natural to compute the holonomy $X_\gamma \in \mathbb{C}^\times$ around a curve γ, and this has nice properties. As $\hbar \to 0$, the asymptotic series of this is computable, e.g. $X_\gamma \sim \exp(\hbar^{-1}Z_\gamma)$, where $Z_\gamma = \oint_\gamma \sqrt{-P}\, \mathrm{d}z$. In a given example (choose P, draw the spectral network, fix a loop), this is completely concrete. The trace is almost an eigenvalue of the monodromy, but it has to cross one of the lines in $W(P)$, and the formula shows that. Specifically, one gets a term for the cosine and a term responsible for the gaps (and hence can be studied to learn about the gaps).

So any particular picture/problem comes with its own picture and defines a coordinate system.

What changes for SL_3? We need a higher-rank analogue of the Schrödinger equation, which will have two potentials P_2 and P_3:

$$\left(\partial_z^3 + \hbar^{-2}P_2(z)\partial_z + \left(\hbar^{-3}P_3(z) + \frac{1}{2}\hbar^{-2}P_2'(z)\right)\right)\psi(z) = 0. \tag{1.11}$$

There's a higher WKB method to deal with such equations, but let's look at a specific example, in which

$$P_3 = -\frac{u}{(z^3-1)^2} \qquad \text{and} \qquad P_2 = \frac{9\hbar^2}{(z^3-1)^2}. \tag{1.12}$$

The $9\hbar^2$ term in P_2 won't matter for the spectral curve, though we can't completely ignore higher-order terms in $\hbar$.

Now parallel transport of solutions gives us (I think?) a flar SL_3-connection on C. We want to study the connections with $u > 0$. The higher WKB machinery gives you a basis $\{\psi_1, \psi_2, \psi_3\}$ inside a chamber (the Stokes graph divides the Riemann surface into two chambers), and ther three monodromies around points A, B, and C must satisfy

$$C\psi_1, B^{-1}\psi_2 \in \operatorname{span}\{\psi_1, \psi_2\}, \tag{1.13}$$

[1] The same locus appears in $\mathcal{N} = 2$ supersymmetry, where it's called a *spectral network*, but its origin is older.

along with all cyclic permutations of this condition. This is an algebraic geometry question, and has a cool answer: A, B, and C are unipotent, and in this case there's a continuous family of solutions (not as interesting) plus four exceptional ones, and WKB produces one of these.