M392C NOTES: BRIDGELAND STABILITY

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These notes were taken in UT Austin's M392C (Bridgeland Stability) class in Spring 2019, taught by Benjamin Schmidt. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Any mistakes in the notes are my own.

Contents

Lecture 1.

Introduction and quiver representations: 1/22/19

This class will be on Bridgeland stability, though we won't get to that topic specifically for about a month. We'll follow lecture notes of Macrì-Schmidt [MS17], which are on the arXiv.

If you're pre-candidacy, make sure to do at least two exercises in this class, at least one from March or later; otherwise just make sure to show up. (If you're an undergrad who's signed up for this class, please do at least four exercises, at least two from March or later.)

Now let us enter the world of mathematics. We'll begin with two well-known theorems in algebraic geometry; we'll eventually be able to prove these using stability conditions.

Theorem 1.1 (Kodaira vanishing)**.** *Let X be a smooth projective complex variety and L be an ample line bundle. Then for all $i > 0$, $H^i(X; L \otimes \omega_X) = 0$.*

We'll eventually give an approach in the setting where $\dim X \leq 2$. It won't be very hard once the setup is in place. In fact, there are probably plenty of other vanishing theorems one could prove using stability conditions, including some which aren't known yet.

The other theorem is over a century ago, from the Italian school of algebraic geometry.

Theorem 1.2 (Castelnuovo)**.** *Working over an algebraically closed field, let $C \subset \mathbb{P}^3$ be a smooth curve not contained in a plane. Then $g \leq d^2/4 - d + 1$, where g is genus of C and d is its degree.*

Another goal we'll work towards:

Problem 1.3. Explicitly describe some moduli spaces of vector bundles or sheaves.

Here's a concrete outline of the course.

(1) Before we discuss any algebraic geometry, we'll study quiver theory, focusing on moduli spaces of quiver representations. We don't need stability conditions to do this, but these spaces make great simple examples of the general story.
(2) Next, we'll study vector bundles on curves. Bridgeland stability is a generalization of what we can say here for higher dimensions.
(3) A crash course on derived categories and Bridgeland stability. This is pretty formal.
(4) A crash course on intersection theory, which will be necessary for what comes later.
(5) Surfaces.
(6) Threefolds (if we have time).

These are all mostly independent pieces, only coming together in the end, so if you get lost somewhere there's no need to panic; you'll probably be able to pick the course back up soon enough.

And now for the moduli of quiver representations. For this stuff, we'll follow King [Kin94], which is accessible and nice to read. Let k be an algebraically closed field.

Definition 1.4. A *quiver* is the representation theorist's word for a finite directed graph. Explicitly, a quiver Q consists of two finite sets Q_0 and Q_1 of vertices and edges, respectively, together with *tail* and *head* maps $t, h\colon Q_1 \to Q_0$.

Example 1.5. The *Kronecker quiver* is

$$\bullet \rightrightarrows \bullet.$$

The quiver of *type* D_4 is

$$\begin{tikzcd} & & \bullet \arrow[dl] \\ \bullet & \bullet \arrow[l] & \\ & & \bullet. \arrow[ul] \end{tikzcd}$$

We can also consider a quiver with a single vertex v and a single edge $e\colon v \to v$. $\blacktriangleleft$

Definition 1.6. A *representation* W of a quiver Q is a collection of k-vector spaces W_v for each $v \in Q_0$ and linear maps $\phi_e\colon W_{v_1} \to W_{v_2}$ for each edge $e\colon v_1 \to v_2$ in Q_1. The vector $(\dim W_v)_{v \in Q_0}$ inside $\mathbb{C}[Q_0]$ is called the *dimension* of W.

Example 1.7. First, some trivial example. For example, here's a representation of the Krokecker quiver: $(\cdot 1, \cdot 2)\colon k \rightrightarrows k$. A representation of the quiver with one vertex and one edge is a vector space with an endomorphism, e.g. $\mathbb{C}^2$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. $\blacktriangleleft$

Definition 1.8. Let Q be a quiver. A *morphism* of Q-representations $f\colon (W_v, \phi_e) \to (U_v, \psi_e)$ is a collection of linear maps $f_v\colon W_v \to U_v$ for each $v \in Q_0$ such that for all edges e,

$$f_{h(e)} \circ \phi_e = \psi_e \circ f_{t(e)}.$$

If all of these linear maps are isomorphisms, f is called an *isomorphism*.

That is, data of a quiver representation includes a bunch of linear maps, and we want a morphism of quiver representations to commute with these maps.

Representations theorists want to classify quiver representations. This is really hard, so let's specialize to irreducible representations (those not a direct sum of two other ones). This is still really hard! There are classical theorems originating from the French school proving that most quivers do not admit nice classifications of their irreducible representations: some have finitely many, and some have infinitely many but nice parameterizations, and these are uncommon.

One way to make headway on these kinds of problems is to consider a moduli space of quiver representations, which may be more tractable to study.

Problem 1.9. Can you classify the (isomorphism classes of) quiver representations of the quiver with a single vertex and single edge?

Our first, naïve approach to constructing the moduli of quiver representations is to fix a dimension vector $\alpha \in \mathbb{C}[Q_0]$ and define

$$R(Q,\alpha) := \bigoplus_{e\in Q_1} \mathrm{Hom}(W_{t(e)}, W_{h(e)}). \tag{1.10}$$

This is too big: the same isomorphism class appears at more than one point. We can mod out by a symmetry: let

$$\mathrm{GL}(\alpha) := \prod_{v\in Q_0} \mathrm{GL}(W_v) \tag{1.11}$$

act on $R(Q,\alpha)$ by a change of basis on each vector space and on ϕ_e as

$$(g\phi)_e = g_{h(e)}\phi_e g_{t(e)}^{-1}. \tag{1.12}$$

Then as a set the quotient $R(Q,\alpha)/\mathrm{GL}(\alpha)$ contains one element for each isomorphism class. But putting a geometric structure on quotients of varieties is tricky. We'll come back to this point.

Example 1.13. Let Q be the Kronecker quiver and $\alpha = (1,1)$, so that $\mathrm{GL}(\alpha) = k^\times \times k^\times$. Pick $(t,s) \in \mathrm{GL}(\alpha)$; the action on a Q-representation $(\lambda,\mu)\colon k \rightrightarrows k$ produces $(s\lambda t^{-1}, s\mu t^{-1})\colon k \rightrightarrows k$. So if $s = t$, the action is trivial. Quotienting out by the diagonal $s = t$ in $k^\times \times k^\times$, we get $k^\times\colon (s,t) \mapsto s/t$, and this acts on $R(Q,\alpha) = k^2$ by scalar multiplication.

This is an action we know well: the quotient is the space of lines in k^2, also known as $\mathbb{P}^1_k$ – and the zero orbit. This orbit makes life more of a headache: you can't just throw it out, because then you don't get a good map to the quotient, preimages of closed things aren't always closed, etc. But the action on the zero orbit is not free. This phenomenon will appear a lot, and we'll in general have to think about what to remove. After some hard work we'll be able to take the quotient in a reasonable way and get $\mathbb{P}^1$. ◀

A crash course on (linear) algebraic groups. If you want to learn more about algebraic groups, especially because we're not going to give proofs, there are several books called *Linear Algebraic Groups*: the professor recommends Humphreys' book [Hum75] with that title, and also those of Borel [Bor91] and Springer [Spr98].

Definition 1.14. An *algebraic group* is a variety G together with a group structure such that multiplication and taking inverses are morphisms of varieties.

You can guess what a morphism of algebraic groups is: a group homomorphism that's also a map of varieties.

Example 1.15. GL_n is an algebraic group. Inside the space of all $n \times n$ matrices, which is a vector space over k, GL_n is the set of matrices with nonzero determinant. This is an open condition, and the determinant can be written in terms of polynomials, so GL_n is an algebraic group.

Other examples include SL_n and elliptic curves, and we can take products, so $\mathrm{GL}(\alpha)$ is also an algebraic group. ◀

Definition 1.16. A *linear algebraic group* is an algebraic group that admits a closed embedding $G \hookrightarrow \mathrm{GL}_n$ which is also a group homomorphism.

This does not include the data of the embedding. It turns out (this is in, e.g. Humphreys) that any affine algebraic group is linear, but this is not particularly easy to show.

Exercise 1.17. Show that any algebraic group is also a smooth variety.

This does not generalize to group schemes!

We care about groups because they act. We added structure to algebraic groups, and thus care about actions which behave nicely under that structure.

Definition 1.18. A *group action* of an algebraic group G on a variety X is a morphism $\varphi\colon G \times X \to X$ such that for all $g, h \in G$ and $x \in X$,

(1) $\varphi(g, \varphi(h, x)) = \varphi(gh, x)$, and
(2) $\varphi(e, x) = x$.

Example 1.19. $k^\times$ acts on $\mathbb{A}_k^{n+1}$ by scalar multiplication. What's the quotient? We want $\mathbb{P}_k^n$, but there's also the zero orbit, and no other orbit is closed. This makes us sad; we're going to use geometric invariant theory (GIT) to address these issues and become less sad. ◀

Definition 1.20. Let G be an algebraic group.

- A *character* of G is a morphism of algebraic groups $\chi\colon G \to k^\times$. These form a group under pointwise multiplication, and we'll denote this group $X(G)$.
- A *one-parameter subgroup* of G, also called a *cocharacter*, is a morphism of algebraic groups $\lambda\colon k^\times \to G$.

Example 1.21. Since $\det(AB) = \det A \det B$, the determinant defines a character of GL_n. One example of a cocharacter is $\lambda\colon k^\times \to \mathrm{GL}_2$ sending $t \mapsto \left(\begin{smallmatrix} t & 0 \\ 0 & t^2 \end{smallmatrix}\right)$. This cocharacter factors through the diagonal matrices in GL_n; this turns out to be a general fact. ◀

Here are a few nice facts about characters and cocharacters.

Theorem 1.22.

(1) The map $\mathbb{Z} \to X(\mathrm{GL}_n)$ sending $m \mapsto \det^m$ is an isomorphism.
(2) If G and H are algebraic groups, the map $X(G) \times X(H) \to X(G \times H)$ sending

$$(\chi_1, \chi_2) \longmapsto ((g, h) \longmapsto \chi_1(g)\chi_2(h)) \tag{1.23}$$

is an isomorphism.
(3) Up to conjugation, every cocharacter of GL_n lands in the subgroup of diagonal matrices, hence sends $t \mapsto \mathrm{diag}(t^{a_1}, \dots, t^{a_n})$ for $a_1, \dots, a_n \in \mathbb{Z}$.

We're not going to prove these: this would require a considerable detour into the theory of algebraic groups to get to, and you can read the proofs in Humphreys.

Exercise 1.24. Without using the above theorem, show that any morphism of algebraic groups $k^\times \to k^\times$ is of the form $t \mapsto t^n$ for some $n \in \mathbb{Z}$.

Lecture 2.

Geometric invariant theory: 1/24/19

Today we'll discuss some more geometric invariant theory and how to take quotients.

Nagata reinterpreted Hilbert's 14th problem as follows.

Problem 2.1. Let G be a linear algebraic group acting linearly on a finite-dimensional k-vector space V. Is the *ring of invariants*

$$\mathscr{O}(V)^G = \{f \in \mathscr{O}(V) \mid f(gx) = f(x) \text{ for all } g \in G, x \in V\}$$

finitely generated?

The elements of $\mathscr{O}(V)^G$ are called the *invariant polynomials* or *invariant functions* on V.

Nagata proved that this is not always true, though there is a positive answer with some assumptions on G. For example, GL_n and products of general linear groups satisfy this property.

Definition 2.2. A linear algebraic group G is *geometrically reductive* if for every linear action of G on a finite-dimensional vector space V (i.e. a map of algebraic groups $\varphi\colon G \to \mathrm{GL}(V)$) and every fixed point $v \in V$ of the G-action, there is an invariant homogeneous nonconstant polynomial f with $f(v) = 0$.

Remark 2.3. There is a different notion of a reductive group, and it is different. Sorry about that. ◀

Theorem 2.4 (Nagata [Nag63])**.** *If G is geometrically reductive, Problem 2.1 has a positive answer.*

If $\mathrm{char}(k) = 0$, basic facts from the theory of algebraic groups allow one to prove GL_n is geometrically reductive, and in fact in characteristic zero reductive implies geometrically reductive. This is also true in positive characteristic, but is significantly harder!

Remark 2.5. In fact, in characteristic zero, the polynomial f in the definition of geometrically reductive can be chosen such that $\deg(f) = 1$. This property is called *linearly reductive*, so in characteristic zero, reductive, geometrically reductive, and linearly reductive coincide. This is not true in positive characteristic, which is ultimately because of everyone's favorite fact about modular representation theory: representations of a group in positive characteristic need not be semisimple. ◀

Mumford conjectured the following.

Theorem 2.6 (Haboush [Hab75])**.** *If k is algebraically closed and G is reductive, then G is geometrically reductive.*

The difficulty was in positive characteristic.

This led to the first idea of a better quotient: take Spec of the ring of invariants; by this theorem, this gives you a variety. But sometimes this is too small: for $\mathbb{C}^\times$ acting on $\mathbb{C}^n$, this tells you the closed orbits. The only closed orbit is the zero orbit, so we don't get $\mathbb{P}^{n-1}$, alas.

To abrogate this, we'll introduce a numerical criterion. Let G be a geometrically reductive group acting linearly on a finite-dimensional vector space V. Recall that $\mathscr{O}(V)$ is also denoted $k[V]$, the ring of polynomials on V.

Definition 2.7. Let $\chi \in X(G)$ be a character of G.

(1) An $f \in \mathscr{O}(V)$ is *relatively invariant* of *weight* χ if $f(gx) = \chi(g)f(x)$ for all $x \in V$ and $g \in G$. We let $\mathscr{O}(V)^{G,\chi}$ denote the vector space of relatively invariant functions of weight χ, so that $\mathscr{O}(V)^{G,\chi^0} = \mathscr{O}(V)^G$.

(2) Define

$$V /\!/ (G,\chi) := \mathrm{Proj}\left(\bigoplus_{n\geq 0} \mathscr{O}(V)^{G,\chi^n}\right). \tag{2.8}$$

We let $V /\!/ G := \mathrm{Spec}(\mathscr{O}(V)^G)$.

One can check quickly that the product of relatively invariant functions of weights χ^m and χ^n is relatively invariant of weight χ^{m+n}, so the graded abelian group in (2.8) is in fact a graded ring.

Theorem 2.9. *There's a natural map $V /\!/ (G,\chi) \to V /\!/ G$, and this map is projective.*

Example 2.10. Consider $k^\times$ acting on k^{m+1} by scalar multiplication and $\chi\colon k^\times \to k^\times$. Then $k[x_0,\dots,x_m]^{k^*,\chi^n}$ is exactly the vector space of degree-n homogeneous polynomials. Then

$$k^{m+1} /\!/ (k^\times, \mathrm{id}) = \mathrm{Proj}(k[x_0,\dots,x_m]) = \mathbb{P}^m, \tag{2.11}$$

where we give $k[x_0,\dots,x_m]$ its usual grading.

However, if you use other characters, you'll get something different: for $\chi = 1$ you get a single point, and for $\chi = -\mathrm{id}$ the quotient is empty. ◀

We've been calling $V /\!/ (G,\chi)$ a "quotient," but is it really one? We'd like to say it has nice properties that a quotient should have, but in the above example, there isn't a nice map $k^{m+1} \twoheadrightarrow \mathbb{P}^m$. In general we get a nice map like that on an open subset; let's figure out what map that is.

Let Δ be the kernel of $\varphi\colon G \to \mathrm{GL}(V)$.[1]

Definition 2.12.

(1) An $x \in V$ is called *χ-semistable* for a character χ if there is an $f \in \mathscr{O}(V)^{G,\chi^n}$ for some $n \geq 1$ such that $f(x) \neq 0$. The locus of χ-semistable points is denoted V^{ss}_χ.

(2) If x is χ-semistable and we can choose f such that the orbit $G\cdot x \subset \{x \in V \mid f(x) \neq 0\}$ is closed, and $\dim G\cdot x = \dim G - \dim \Delta$, we call x *χ-stable*. The locus of χ-stable points is denoted V^s_χ.

[1] In many references, φ is assumed to be injective.

Stability means that the orbit of x has the largest possible dimension.

Lemma 2.13. *V_χ^{ss} and V_χ^s are Zariski open subsets of V.*

The main theorem of geometric invariant theory in this setting[2] is:

Theorem 2.14 (Mumford)**.** *There is a surjective morphism $\phi\colon V_\chi^{ss} \to V /\!/ (G,\chi)$ such that if $x, y \in V_\chi^{ss}$,*

(1) if $x, y \in V_\chi^s$, then $\phi(x) = \phi(y)$ iff $y \in G \cdot x$, and

(2) in general, $\phi(x) = \phi(y)$ iff $\overline{G\cdot x} \cap \overline{G \cdot y}$ is nonempty, where closures are taken inside V_χ^{ss}.

You can think of ϕ as the map from the original space to the quotient, but we can only see a subset of the original space. For stable points, this actually parameterizes orbits, but this isn't quite true for merely semistable points, and the problem occurs when orbits aren't closed.

Definition 2.15. If $\overline{G\cdot x} \cap \overline{G \cdot y}$ is nonempty, we say x and y are *S-equivalent*.

Remark 2.16. S is for Seshadri, who was one of the developers of this theory. ◂

The numerical criterion we alluded to earlier is a way to find semistable points.

Definition 2.17. Let $\chi\colon G \to k^\times$ be a character and $\lambda\colon k^\times \to G$ be a cocharacter. The composition $\chi \circ \lambda\colon k^\times \to k^\times$ sends $t \mapsto t^n$ for some $n \in \mathbb{Z}$; we denote $\langle \chi, \lambda \rangle := n$.

Theorem 2.18 (Mumford's numerical criterion)**.**

(1) An $x \in V$ is χ-semistable iff $\chi(\Delta) = 1$ and for all cocharacters $\lambda\colon k^\times \to G$ such that $\lim_{t\to 0} \lambda(t)x$ exists, then $\langle \chi, \lambda\rangle \geq 0$.

(2) x is χ-stable iff it's χ-semistable and if λ is as above and $\langle \chi, \lambda \rangle > 0$, then $\lambda(k^\times) \subset \Delta$.

That limit works fine in $\mathbb{C}$, but what about over other fields? It's obvious in formulas, and in general you can define it in terms of trying to extend to a map of varieties $k \to G$.

Proposition 2.19.

(1) The orbit $G \cdot x$ is closed in V_χ^{ss} if for every cocharacter λ with $\langle \chi, \lambda \rangle = 0$ such that the limit $\lim_{t\to 0} \lambda(t)x$ exists, then the limit is in $G \cdot x$.

(2) If $x, y \in V_\chi^{ss}$, then x and y are S-equivalent iff there are cocharacters λ_1, λ_2 with $\langle \chi, \lambda_1 \rangle = \langle \chi, \lambda_2\rangle = 0$ such that $\lim_{t\to 0} \lambda_1(t)x$ and $\lambda_{t\to 0}\lambda_2(t)y$ both exist and are in the same orbit.

Example 2.20. Consider $G = \mathrm{GL}_2$ acting on the space V of 4×2 matrices: to obtain a left action by g, we multiply on the right by g^{-1}. Let $\chi\colon \mathrm{GL}_2 \to k^\times$ be $\det^{-1}$.

What do we expect to parameterize in the quotient? A 4×2 matrix is a linear map $k^2 \to k^4$, and we're parameterizing them up to change of basis of the domain. This should morally parameterize two-dimensional subspaces of k^4, though we never stipulated that our maps are injective. Maybe, hopefully, the open subset of semistable points are the injective maps and we'll get the Grassmannian $\mathrm{Gr}_2(k^4)$.

We claim this is actually the case, and will use the numerical criterion to prove it. Since GL_2 acts faithfully on V, $\Delta = 1$ and the situation simplifies somewhat. We can use a group action to make the cocharacter simpler, or to make a general element of V simpler, but not both. So we'll do the former: let $\lambda = \left(\begin{smallmatrix} t^n & 0 \\ 0 & t^m \end{smallmatrix}\right)$ and

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix}, \quad \text{so that} \quad \lambda(t)A = \begin{pmatrix} a_{11}t^{-n} & a_{12}t^{-m} \\ a_{21}t^{-n} & a_{22}t^{-m} \\ a_{31}t^{-n} & a_{32}t^{-m} \\ a_{41}t^{-n} & a_{42}t^{-m} \end{pmatrix}. \tag{2.21}$$

Since $\det^{-1}(\lambda(t)) = t^{-n-m}$, $\langle \det^{-1}, \lambda \rangle = -n - m$. Therefore $\lim_{t\to 0} \lambda(t)$ exists iff either

(1) $A = 0$, which isn't semistable, because the limit exists for every cocharacter; or
(2) $a_{11} = \cdots = a_{41} = 0$ and $a_{j2} \neq 0$ for some j, and $m \leq 0$, which is also unstable (e.g. $m = 0$, $n = 1$); or
(3) $a_{2i} = 0$ for all i, and $a_{j1} \neq 0$ for some j, and $n \leq 0$, which is again unstable; or

[2] Mumford showed a version where G can act on any quasiprojective variety.

(4) $a_{i1} \neq 0$ for some i and $a_{j2} \neq 0$ for some j, and $m, n \leq 0$, so $\langle \chi, \lambda \rangle = -n - m \geq 0$, and these A are stable.

Now, let's look at an arbitrary cocharacter. This involves changing basis/looking at full orbits of points we found were unstable. When $A = 0$ (case (1)), this is the whole orbit, and it's unstable. For (2) and (3), A has rank 1 in the entire orbit, and therefore these are all unstable. All matrices of rank 2 are stable. ◂

Lecture 3.

Constructing moduli spaces of quiver representations: 1/29/19

Today, we're going to leverage the GIT theory we surveyed in the last lecture to define moduli spaces of quiver representations.

We begin with a quick review of Mumford's numerical criterion, since it will be an important actor today. Let G be an algebraic group acting on a k-vector space V, and let Δ denote the kernel of the associated map $\rho\colon G \to \mathrm{GL}(V)$. Let $\chi\colon G \to k^\times$ be a character. Then, in Theorem 2.18, we saw that $x \in V$ is χ-semistable iff $\chi(\Delta) = 1$ and for all cocharacters $\lambda\colon k^\times \to G$ such that $\lim_{t\to 0} \lambda(t)\cdot x$ exists, $\langle \chi, \lambda \rangle = 0$. Moreover, x is χ-stable if in addition whenever $\langle \chi, \lambda\rangle = 0$, then $\lambda(k^\times) \subset \Delta$.

Now back to quivers. Consider a quiver Q with a set Q_0 of vertices, Q_1 of edges, and head and tail maps $h, t\colon Q_1 \rightrightarrows Q_0$. Let $\alpha \in \mathbb{C}[Q_0]$ be a dimension vector and vector spaces W_v of dimension $\alpha(v)$ for each $v \in Q_0$. We constructed the space

$$R(Q,\alpha) := \prod_{e\in Q_1} \mathrm{Hom}(W_{t(e)}, W_{h(e)}), \tag{3.1}$$

but this is too big to be a moduli space of quiver representation: it contains different points that correspond to isomorphic representations. Therefore we want to take the quotient by $\mathrm{GL}(\alpha)$ as we described, so let's apply GIT to this action and understand stability.

In this setting, the kernel Δ is the *long diagonal* $\{(t\mathrm{id}, \dots, t\mathrm{id}) \in \mathrm{GL}(\alpha) \mid t \in k^\times\}$, which is isomorphic to $k^\times$. If $\theta \in \mathbb{Z}[Q_0]$, it defines a character $\chi\colon \mathrm{GL}(\alpha) \to k^\times$ by

$$\chi_\theta(g) := \prod_{v\in Q_0} g_v^{\theta(v)}. \tag{3.2}$$

All characters of GL_n can be written in this way.

Definition 3.3. Let A be an abelian category. Its *Grothendieck group* is

$$K_0(\mathsf{A}) := \bigoplus_{A\in\mathsf{A}} \mathbb{Z}[A]/\sim, \tag{3.4}$$

where we quotient by an equivalence relation: for all short exact sequences $0 \to A \to B \to C \to 0$, we say $[B] \sim [A] + [C]$.

The $\theta \in \mathbb{Z}[Q_0]$ as above can also be thought of as a function on the Grothendieck group of Rep_Q, the category of finite-dimensional representations of Q. Specifically, $\theta\colon K_0(\mathsf{Rep}_Q) \to \mathbb{Z}$ is defined to send

$$M = (\{M_v\}, \{\psi_e\}) \longmapsto \sum_{v\in Q_0} \theta(v)\cdot \dim(M_v). \tag{3.5}$$

We want there to be semistable points, which means we will only consider θ such that

$$\chi_\theta(\Delta) = \left\{ \prod_{v\in Q_0} t^{\theta(v)\alpha(v)} \mid t \in k^\times \right\} = \{1\}, \tag{3.6}$$

i.e. such that

$$\sum_{v\in Q_0} \theta(v)\alpha(v) = 0. \tag{3.7}$$

Among other things, this means that $\theta(M) = 0$ if $\dim M = \alpha$.

Now we'll perform the GIT analysis. Let $\lambda\colon k^\times \to \mathrm{GL}(\alpha)$ be a cocharacter and $M = (\{M_v\}, \{\phi_e\}) \in \mathsf{Rep}_Q$ have dimension α. The first step will be to construct a descending $\mathbb{Z}$-indexed filtration

$$M \supseteq \cdots \supseteq M^{(n)} \supseteq M^{(n+1)} \supseteq \cdots. \tag{3.8}$$

For each $v \in Q_0$, pick a decomposition

$$M_v = \bigoplus_{n\in\mathbb{Z}} M_v^{(n)}, \tag{3.9}$$

such that $\lambda(t)$ acts on $M_v^{(n)}$ by multiplication by t^n. (Recall that any cocharacter of GL_n has diagonal image up to conjugation, so this makes sense.) Define

$$M_v^{(\geq n)} := \bigoplus_{m\geq n} W_v^{(m)}. \tag{3.10}$$

For each $e \in Q_1$, let $\phi_e^{(m,n)}$ denote the composition

$$M_{t(e)}^{(n)} \hookrightarrow M_{t(e)} \xrightarrow{\phi_e} M_{h(e)} \twoheadrightarrow M_{h(e)}^{(m)}, \tag{3.11}$$

where the projection comes from the decomposition (3.9). Then

$$\lambda(t)\cdot \phi_e^{(m,n)} = t^m \phi_e^{(m,n)} t^{-n} = t^{m-n}\phi_e^{(m,n)}, \tag{3.12}$$

which means the following are equivalent:

- $\lim_{t\to 0} \lambda(t)\phi_e$ exists,
- $\phi_e^{(m,n)} = 0$ whenever $m \leq n$, and
- ϕ_e maps $M_{t(e)}^{(\geq n)}$ into $M_{h(e)}^{(\geq n)}$.

The third condition means that $M_n := (M^{(\geq n)}, \phi_e|_{M^{(\geq n)}})$ is a subrepresentation of M, so if the limit exists it induces the desired filtration of M (3.8). In this case $W_v^{(n)} = W_v^{(\geq n)}/W_v^{(\geq (n+1))}$.

Conversely, given a filtration of M as in (3.8), we can produce a cocharacter $\lambda\colon k^\times \to \mathrm{GL}(\alpha)$: define $\lambda(t)$ to act by t^n on $W_v^{(n)}$.

Since

$$\lim_{t\to 0} \lambda(t)\phi_e^{(m,n)} = \lim_{t\to 0} t^{m-n}\phi_e^{(m,n)} = 0 \tag{3.13}$$

for $m > n$, then

$$\lim_{t\to 0} \lambda(t)\phi_e\colon M_{t(e)}^{(n)} \longrightarrow M_{h(e)}^{(n)} \subset M_{h(e)}^{(\geq n)}. \tag{3.14}$$

Thus the limit is the associated graded:

$$\lim_{t\to 0} \lambda(t)\cdot M = \bigoplus_{n\in\mathbb{Z}} M_n/M_{n+1}. \tag{3.15}$$

Remark 3.16. There's an n such that $M_n = 0$ and $M_n \neq M$. In other words, this filtration isn't the trivial one. This is because $\lambda(k^\times) \subset \Delta$. ◂

Now let's discuss (semi)stability. It will turn out to be equivalent to the following notion.

Definition 3.17. Let $M \in \mathsf{Rep}_Q$ have dimension α and be such that $\theta(M) = 0$. Then M is *θ-semistable* (resp. *θ-stable*) iff for all nonzero proper subrepresentations $N \subset M$, $\theta(N) \geq 0$ (resp. $\theta(N) > 0$).

Theorem 3.18 (King [Kin94])**.** *A point $M \in R(Q,\alpha)$ is χ_θ-semistable (resp. χ_θ-stable) iff $M \in \mathsf{Rep}_Q$ is θ-semistable (resp. θ-stable).*

Proof. First, let's show θ-(semi)stability implies GIT (semi)stability. We assumed $\sum_v \theta(v)\alpha(v) = 0$, which implies

$$\langle \chi_\theta, \lambda\rangle = \sum_{v\in Q_0} \theta(v) \sum_{n\in\mathbb{Z}} n \dim M_v^{(n)}. \tag{3.19}$$

We can change the order of summation because only finitely many $W_v^{(n)}$ are nonzero for v fixed, so

$$\langle \chi_\theta, \lambda \rangle = \sum_{n\in\mathbb{Z}} n \sum_{v\in Q_0} \theta(v) \dim M_v^{(n)} \tag{3.20a}$$

$$= \sum_{n\in\mathbb{Z}} n\theta(M_n/M_{n+1}). \tag{3.20b}$$

Since θ factors through the Grothendieck group, this is

$$= \sum_{n\in\mathbb{Z}} n(\theta(M_n) - \theta(M_{n-1})) \tag{3.20c}$$

$$= \sum_{n\in\mathbb{Z}} \theta(M_n), \tag{3.20d}$$

unwinding the telescoping series. This is nonnegative if M is θ-semistable and positive if M is θ-stable, by definition.

Conversely, suppose M is χ_θ-semistable and let N be a subrepresentation of M. Then $N \subset M$ is a filtration, hence defines a cocharacter $\lambda\colon k^\times \to \mathrm{GL}(\alpha)$ such that

$$0 \overset{(<)}{\le} \langle \chi_\theta, \lambda \rangle = \theta(M) + \theta(N) \tag{3.21}$$

(parentheses for stability), so $\theta(N) \ge -\theta(M) = 0$ (or $>$ for stability). ⊠

Now we have semistable points, and even strictly semistable points. What does S-equivalence look like in this context?

Definition 3.22. Given θ as above, let $\mathsf{P}_\theta \subset \mathsf{Rep}_Q$ denote the full subcategory of θ-semistable representations with $\theta(M) = 0$ (so, those objects, and all of the morphisms between them).

Lemma 3.23. *P_θ is an abelian subcategory of Rep_Q. That is, let $\varphi\colon M \to N$ be a morphism in P_θ, i.e. $\theta(M) = \theta(N) = 0$ and M and N are θ-semistable. Then, $A := \ker(\varphi)$ and $B = \mathrm{coker}(\varphi)$, where the kernel and cokernel are taken in Rep_Q, are in P_θ.*

Proof. The kernel and cokernel fit into an exact sequence

$$0 \longrightarrow A \longrightarrow M \longrightarrow N \longrightarrow B \longrightarrow 0. \tag{3.24}$$

Since θ is additive under short exact sequences, it in fact satisfies

$$\theta(A) + \underbrace{\theta(N)}_{=0} = \underbrace{\theta(M)}_{=0} + \theta(B), \tag{3.25}$$

so $\theta(A) = \theta(B)$. Since M is θ-semistable, $\theta(A) \ge 0$. If $K := \ker(N \to B) \hookrightarrow N$, and N is θ-semistable, then $\theta(K) \ge 0$. Therefore

$$\theta(A) = \theta(B) = \theta(N) - \theta(K) = -\theta(K) \le 0. \tag{3.26}$$

Thus $\theta(A) = \theta(B) = 0$.

Now for semistability. Briefly, if $C \subset A$ is a subrepresentation, then it's also a subrepresentation of M, so $\theta(C) \ge 0$. The argument for B is similar. ⊠

Lecture 4.

Examples of quiver varieties: 1/31/19

Fix a quiver Q. Last time we explained how, given a $\theta \in \mathbb{Z}[Q_0]$, we obtain a function on objects of Rep_Q additive on short exact sequences: $\theta(M) := \sum_{v\in Q_0} \theta_v \cdot \dim(M_v)$, and we also get a character χ_θ of $\mathrm{GL}(\alpha)$, which has weight $\theta(v)$ on the component indexed by v. In Theorem 3.18, we provided a criterion for semistability: a quiver representation M is χ_θ-semistable iff $\theta(M) = 0$ and for all $N \subsetneq M$, $\theta(N) \ge 0$. (If $\theta(N) > 0$, M is χ_θ-stable.)

We then constructed an abelian category P_θ of χ_θ-semistable objects.

Proposition 4.1. *P_θ is a* finite-length category, *i.e. all of its objects are both Noetherian and Artinian.*

Theorem 4.2 (Jordan-Hölder)**.** *Let* A *be a finite-length abelian category. Then any object* $M \in \mathsf{A}$ *has a filtration* $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ *such that* M_i/M_{i-1} *is simple for all* i*. Moreover, the associated graded*

$$\operatorname{gr}(M) := \bigoplus_i M_i/M_{i-1} \tag{4.3}$$

is unique up to isomorphism.

This filtration satisfies a weak uniqueness condition – it's not unique, but is close to it. For example, if A and B are two simple objects, $A \oplus B$ has two filtrations $0 \to A \to A \oplus B$ and $0 \to B \to A \oplus B$. Such a filtration is called a *Jordan-Hölder filtration* of M.

Exercise 4.4. In P_θ, M is simple iff it's θ-stable.

Proposition 4.5.

(1) A θ*-semistable* Q*-representation* M *corresponds to a closed* $\mathrm{GL}(\alpha)$ *orbit in* $R^{ss}_{\chi_\theta}(Q, \alpha)$ *iff* M *is semisimple (i.e. a direct sum of simple objects). Equivalently,* $M \cong \operatorname{gr}(M)$*.*

(2) Two θ*-semistable* Q*-representations* M *and* N *are S-equivalent iff* $\operatorname{gr}(M) \cong \operatorname{gr}(N)$*.*

The proof was left as an exercise, but is not too difficult given how we proved things last class (and is a good way to see if you understood the proof).

We will let $M_\theta(Q, \alpha)$ denote the GIT quotient by $\mathrm{GL}(\alpha)$ for the character χ_θ.

Example 4.6. Consider the quiver

$$Q = \begin{tikzcd} \bullet \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \bullet \end{tikzcd}. \tag{4.7}$$

Let's choose $\alpha = (2, 1)$, so $R(Q, \alpha) = \operatorname{Hom}(k^2, k)^{\oplus 4}$. Then $\mathrm{GL}(\alpha) = \mathrm{GL}_2 \times k^\times$. If you choose $\theta = (-1, 1)$, then $\chi_\theta = \det^{-1}$.

The long diagonal of $\mathrm{GL}(\alpha)$ acts trivially, so let's pass to the quotient via the map $\varphi\colon \mathrm{GL}(\alpha) \to \mathrm{GL}_2$ by $(g, t) \mapsto gt^{-1}$ (here, $g \in \mathrm{GL}_2$ and $t \in k^\times$). Now we have the same scenario as in Example 2.20 – so we leverage our work there and conclude the quiver moduli space is $\mathrm{Gr}_2(k^4)$. ◂

Example 4.8. Now let's consider a slightly more interesting quiver,

$$Q = \begin{tikzcd} \bullet \arrow[r, shift left=1] \arrow[r, shift right=1] & \bullet & \bullet \arrow[l, shift left=1] \arrow[l, shift right=1] \end{tikzcd}. \tag{4.9}$$

Let $\alpha = (1, 1, 1)$, so $R(Q, \alpha) = \operatorname{Hom}(k, k)^2 \times \operatorname{Hom}(k, k)^2$. Choose the character $\theta = (a, b, c) = (a, -a-c, c)$, where $a + b + c = 0$.

Because α is small, there aren't many subrepresentations. For example, the trivial representation $k \rightrightarrows k \leftleftarrows k$ has as a subrepresentation $S_2 := (0 \rightrightarrows k \leftleftarrows 0)$. Since $\theta(S_2) = -a - c$, we must have $-a - c \geq 0$ or all representations are unstable.

By defining more representations, we can infer more about what constraints to put on θ to have a good moduli space. For example, given $x, y, z, w \in k$, let

$$N_1(x, y) := \begin{tikzcd} k \arrow[r, shift left=1, "x"] \arrow[r, shift right=1, "y"'] & k & 0 \arrow[l, shift left=1] \arrow[l, shift right=1] \end{tikzcd} \tag{4.10a}$$

$$N_2(z, w) := \begin{tikzcd} 0 \arrow[r, shift left=1] \arrow[r, shift right=1] & k & k \arrow[l, shift right=1, "z"'] \arrow[l, shift left=1, "w"] \end{tikzcd} \tag{4.10b}$$

$$M(x, y, z, w) := \begin{tikzcd} k \arrow[r, shift left=1, "x"] \arrow[r, shift right=1, "y"'] & k & k \arrow[l, shift right=1, "z"'] \arrow[l, shift left=1, "w"] \end{tikzcd}. \tag{4.10c}$$

Thus $N_1(x, y)$ and $N_2(z, w)$ are both suprepresentations of $M(x, y, z, w)$. Because

$$\theta(N_1(x, y)) = a + (-a - c) = -c \geq 0 \tag{4.11a}$$

$$\theta(N_2(x, y)) = -a \geq 0, \tag{4.11b}$$

we know both a and c must be negative.

There are a few more potential subrepresentations,

$$S_1 := \; k \rightrightarrows 0 \leftleftarrows 0 \tag{4.12a}$$

$$S_2 := \; 0 \rightrightarrows 0 \leftleftarrows k \tag{4.12b}$$

$$N_3 := S_1 \oplus S_2. \tag{4.12c}$$

Using these, we observe that

- $S_1 \hookrightarrow M(x,y,z,w)$ iff $x = y = 0$, and in this case, we need $\theta(S_1) = a \geq 0$, so $a = 0$;
- $S_2 \hookrightarrow M(x,y,z,w)$ iff $z = w = 0$, and in this case, we need $\theta(S_3) = c \geq 0$, so $c = 0$; and
- $N_3 \hookrightarrow M(x,y,z,w)$ iff $x = y = z = w = 0$, and in this case we need $a = c = 0$.

In summary:

(1) If $a < 0$ and $c < 0$, then $M(x,y,z,w)$ is θ-stable iff $(x,y) \neq 0$ and $(z,w) \neq 0$. Moreover, there are no *strictly semistable* (i.e. semistable but not stable) representations. At this point you might guess that the GIT quotient is $\mathbb{P}^1 \times \mathbb{P}^1$.
(2) If $a = 0$ and $c < 0$, then $M(x,y,z,w)$ is θ-semistable iff $(z,w) \neq 0$, and there are no θ-stable representations. Now suppose $M(x,y,z,w)$ is such that $(z,w) \neq 0$; then, $M(x,y,z,w)$ and $M(x',y',z',w')$ are S-equivalent iff $(z',w') = \lambda(z,w)$ for some $\lambda \in k^\times$. In this case the GIT quotient will be a $\mathbb{P}^1$, some sort of boundary where things get mushed together.
(3) In the same way, if $c = 0$ and $a < 0$, then $M(x,y,z,w)$ is θ-semistable iff $(x,z) \neq 0$, and there are no θ-stable representations. Now suppose $M(x,y,z,w)$ is such that $(x,y) \neq 0$; then, $M(x,y,z,w)$ and $M(x',y',z',w')$ are S-equivalent iff $(x',y') = \lambda(x,y)$ for some $\lambda \in k^\times$. Again we get a $\mathbb{P}^1$.
(4) If $a = c = 0$, then all M are strictly semistable, and all of them are S-equivalent. The GIT quotient will be a point.

Even in this small case, things are complicated.

We haven't shown the statements above about S-equivalence, so let's do that. There's nothing in case (1), so let's look at (2). In this case $M(x,y,z,w) \sim M(x',y',z',w')$ iff $N(z,w) \sim N(z',w')$, and it's not hard to check that we can act precisely by scalars, so this is true iff (z',w') is a nonzero scalar multiple of (z,w). (3) is the same. For (4), we have a three-stage Jordan-Hölder filtration:

$$0 \subset S_2 \subset N_1(x,y) \subset M(x,y,z,w), \tag{4.13}$$

and the pieces of the associated graded are $N_1(x,y)/S_2 \cong S_1$ and $M/N_1(x,y) \cong S_3$. Therefore $\mathrm{Gr}(M(x,y,z,w)) \cong S_1 \oplus S_2 \oplus S_3$ for any (x,y,z,w), so they're all S-equivalent.

We know to expect $\mathbb{P}^1 \times \mathbb{P}^1$ if $a, c < 0$, or $\mathbb{P}^1$ if exactly one is zero, or a point if they're both zero. You can think of letting $a \to 0$ as projecting onto the first $\mathbb{P}^1$, and letting $c \to 0$ as projecting onto the second $\mathbb{P}^1$. But we haven't proven any of these yet! We don't even know that they're varieties *a priori*, but this is the correct answer, and it's possible to prove it. ◀

Remark 4.14. It turns out all quiver varieties are rational! You can get irrational varieties by imposing composition relations between arrows in the quiver; in fact, any projective variety arises in this way. ◀

The last thing we'll do is study some more general properties of quiver varieties.

Theorem 4.15. *If Q is acyclic quiver (i.e. it has no loops), then $M_\theta(Q,\alpha)$ is projective.*

Proof. $M_\theta(Q,\alpha) := R(Q,\alpha)/\!/\mathrm{GL}(\alpha)$ always has a projective morphsm to $R(Q,\alpha)/\!/\mathrm{GL}(\alpha)$. That is, by definition, we have a projective morphism

$$\mathrm{Proj}\left(\bigoplus_{n\geq 0} k[R(Q,\alpha)]^{\mathrm{GL}(\alpha),\chi_\theta^n}\right) \longrightarrow \mathrm{Spec}\Big(k[R(Q,\alpha])^{\mathrm{GL}(\alpha)}\Big). \tag{4.16}$$

It therefore suffices to prove the codomain is a point. Since $R(Q,\alpha)/\!/\mathrm{GL}(\alpha) = M_0(Q,\alpha)$, all representations have $\theta(M) = 0$.

Exercise 4.17. Show that if Q is acyclic and $M \in \mathsf{Rep}_Q$ has dimension α, then $\mathrm{gr}(M) = \bigoplus_{v\in Q_0} S_v$, where S_v is the "δ-function", a simple representation with k at v and 0 elsewhere.

Therefore they're all S-equivalent, and the quotient is a point. ☒

Remark 4.18. The empty set is a projective variety, and may occur as the moduli space associated to an acyclic quiver. ◂

Lecture 5.

Moduli spaces and slope stability: 2/5/19

"19th*-century math was a different world.*"

So we've constructed some GIT quotient $M_Q(\alpha,\theta)$ and claimed it's the moduli space of representations of the quiver Q with dimension α, and with (semi)stability tracked by the character θ of $\mathrm{GL}(\alpha)$. But why does this deserve to be called a moduli space anyways? We'll begin by talking about this.

Definition 5.1. Let $\mathcal{M}\colon \mathsf{Sch}_k^{\mathsf{op}} \to \mathsf{Set}$ be a functor.

(1) A k-scheme M is a *fine moduli space* for $\mathcal{M}$ if there's a natural isomorphism of functors $\mathcal{M} \overset{\cong}{\Rightarrow} \mathrm{Hom}(-,M)$.[3] One says that M *represents* $\mathcal{M}$.
(2) A k-scheme M is a *coarse moduli space* for $\mathcal{M}$ if there's a natural transformation $\iota_M\colon \mathcal{M} \Rightarrow \mathrm{Hom}(-,M)$ such that any other natural transformation $\iota_N\colon \mathcal{M} \Rightarrow \mathrm{Hom}(-,N)$ for a k-scheme N factors through ι_M. That is, we ask that there's a unique map $f\colon M \to N$ such that the diagram

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\iota_M"] \arrow[dr, "\iota_N"'] & \mathrm{Hom}(-,M) \arrow[d, "f\circ -"] \\ & \mathrm{Hom}(-,N). \end{tikzcd} \tag{5.2}$$

We'll write down a functor related to quiver representations, and $\mathcal{M}_Q(\alpha,\theta)$ will be a coarse or fine moduli space, depending on whether we have semistable points. A fine moduli space would be the nicest thing, but we won't always get it.

If B is a k-scheme and $\mathcal{M}$ is a moduli space of doodads (whatever a doodad is), we want $\mathcal{M}(B)$ to be the set of "families of doodads parameterized by B." What this means precisely depends on the details of the specific setting. Here's what it is for quiver representations.

Definition 5.3. Let Q be a quiver and B be a k-scheme. A *family of Q-representations on B* is data of a locally free sheaf $\mathscr{W}_v$ for each $v \in Q_0$ and morphisms $\phi(e)\colon \mathscr{W}_{t(a)} \to \mathscr{W}_{s(a)}$ for all $e \in Q_1$. The *rank* of a family of representations is as before: the element of $\mathbb{N}[Q_0]$ sending $v \mapsto \operatorname{rank}\mathscr{W}_v$.

If $\alpha \in \mathbb{N}[Q_0]$ and θ is a character of $\mathrm{GL}(\alpha)$, then a rank-α Q-representation is *θ-semistable* if all of its fibers are.

Given a family of Q-representations $\mathscr{W}$ over B and a line bundle $L \to B$, we can tensor them together to obtain a family of Q-representation $\mathscr{W}\otimes V$ with $(\mathscr{W}\otimes V)_v \coloneqq \mathscr{W}_v \otimes V$ and $\phi_{\mathscr{W}\otimes V}(e) \coloneqq \phi_{\mathscr{W}} \otimes \mathrm{id}_V$.

Therefore we have a functor $\mathcal{M}_Q(\alpha,\theta)\colon \mathsf{Sch}_k \to \mathsf{Set}$ which sends a scheme B to the set of equivalence classes of families of θ-semistable Q-representations with rank α, where we say $\mathscr{W}$ and $\mathscr{W}'$ are equivalent if there is a line bundle $L \to B$ and an isomorphism $\mathscr{W}' \cong \mathscr{W}\otimes L$. Given a map of schemes $\phi\colon B \to B'$, we get a map between these sets by pulling back each $\mathscr{W}_v$ and ϕ.

Remark 5.4. Why this equivalence relation? Well first we want to pass to isomorphism classes anyways, but the idea is that $\mathscr{W}$ and $\mathscr{W}\otimes L$ aren't really very different with respect to the representation theory of Q, so we identify them. ◂

Theorem 5.5 ([Kin94])**.** *Fix a quiver Q, dimension vector α, and character θ for* $\mathrm{GL}(\alpha)$.

(1) $M_Q(\alpha,\theta)$ is a coarse moduli space for $\mathcal{M}_Q(\alpha,\theta)$.
(2) If α is indivisible in the ring $\mathbb{Z}[Q_0]$, then $M_Q(\alpha,\theta)$ is a fine moduli space, and a smooth projective variety.

[3] We need to specify $\mathcal{M}$ rather than just saying "this is a moduli space," because every scheme X is a moduli space for $\mathrm{Hom}(-,X)$. In general, though, when it comes to moduli spaces, you "know it when you see it," which is why people sometimes don't say what $\mathcal{M}$ is when it's clear from context.

The proof isn't particularly enlightening, so we're going to skip it.

The reason we don't expect a fine moduli space in general is that semistable points can have automorphisms, which leads to a moduli stack, rather than a fine moduli space. When α is indivisible, it turns out that all representations are either θ-stable or θ-unstable, and θ-stable points have no automorphisms, making the stackiness go away.

Remark 5.6. If you restrict the functor $\mathcal{M}_Q(\alpha, \theta)$ to families of θ-stable representations, we also get a representable functor. In general, it's an open subfunctor of $\mathcal{M}_Q(\alpha, \theta)$, and often but not always is the smooth locus. ◀

∽ · ∼

We're now done with quivers, and will move to other moduli spaces now. Bridgeland stability conditions, the theme of this class, will provide a general way to produce nicer moduli spaces – often you cannot make a fine moduli space of all of your objects, but you can do this for objects satisfying some kind of stability condition.

Our next class of examples will be moduli spaces of vector bundles on curves, which relates to something called slope stability. In this section, we assume the base field is $\mathbb{C}$; probably these results hold in more generality, but the references were difficult to find.

Exercise 5.7. Let C be a smooth, projective curve and E be a coherent sheaf on C. Show that there is a unique exact sequence

$$0 \longrightarrow T_E \longrightarrow E \longrightarrow F_E \longrightarrow 0, \tag{5.8}$$

where F_E is locally free and T_E is a torsion sheaf, and that this sequence noncanonically splits.

So over a curve, any coherent sheaf can be written as a direct sum of a vector bundle and a rank-zero, torsion sheaf. For this reason, on curves one often considers moduli spaces of vector bundles, where in higher dimensions one thinks about moduli spaces of sheaves.

The following theorem was robably known in the 19$^{\text{th}}$ century, though in its modern form it was proven by Grothendieck.

Theorem 5.9 (Grothendieck)**.** *Let E be a vector bundle on $\mathbb{P}^1$. Then there are unique integers $a_1, \dots, a_n$ with $a_1 > \cdots > a_n$ and unique nonzero vector spaces $V_1, \dots, V_n$ such that*

$$E \cong \bigoplus_{i=1}^{n} \mathscr{O}_{\mathbb{P}^1}(a_i) \otimes V_1.$$

In other words, any vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles.

Proof. First, existence. We'll induct on $r := \operatorname{rank} E$. For $r = 1$ there's nothing to show. For $r > 1$, we'll show how to split off a line bundle, reducing to the inductive assumption. Serre duality tells us that

$$\operatorname{Hom}(\mathscr{O}(a), E) \cong \operatorname{Ext}^1(E, \mathscr{O}(a) \otimes \omega_{\mathbb{P}^1}) = H^1(E^\vee \otimes \mathscr{O}(a-2)) = 0 \text{ if } a \gg 0 \tag{5.10}$$

by Serre vanishing. Here we're also using the fact that $\omega_{\mathbb{P}^1} = \mathscr{O}(-2)$. Let a be the largest integer such that $\operatorname{Hom}(\mathscr{O}(a), E) \neq 0$, and fix a nonzero homomorphism $\phi\colon \mathscr{O}(a) \to E$.

Since $\mathscr{O}(a)$ is a line bundle and E is locally free, all map between them are either zero or injective; thus ϕ is injective, so we have a short exact sequence

$$0 \longrightarrow \mathscr{O}(a) \xrightarrow{\phi} E \longrightarrow \operatorname{coker}(\phi) \longrightarrow 0. \tag{5.11}$$

It suffices to show $F := \operatorname{coker}(\phi)$ is locally free and (5.11) splits. Clearly this was not the same proof given in the 19$^{\text{th}}$ century!

First, that F is locally free. Using Exercise 5.7, we have a short exact sequence

$$0 \longrightarrow T \longrightarrow F \longrightarrow G \longrightarrow 0, \tag{5.12}$$

where T is torsion and G is locally free. Assume T is nonzero. This means, because $\mathbb{P}^1$ is one-dimensional, T is supported in dimension zero, so there's some $x \in \mathbb{P}^1$ such that $\mathbb{C}(x)$ injects into T, where $\mathbb{C}(x)$ denotes the skyscraper sheaf with stalk $\mathbb{C}$ at x; in particular, since T is a subsheaf of F, $\operatorname{Hom}(\mathbb{C}(x), F) \neq 0$.

We can pass from $\mathscr{O}(a)$ to $\mathscr{O}(a+1)$ by multiplying by an equation that cuts out x (since x has codimension 1, we only need one equation). So we have a short exact sequence

$$0 \longrightarrow \mathscr{O}(a) \longrightarrow \mathscr{O}(a+1) \longrightarrow \mathbb{C}(x) \longrightarrow 0. \tag{5.13}$$

Applying $\mathrm{Hom}(\mathbb{C}(x), –)$ to (5.12), we obtain a long exact sequence

$$\underbrace{\mathrm{Hom}(\mathbb{C}(x), E)}_{=0} \longrightarrow \underbrace{\mathrm{Hom}(\mathbb{C}(x), F)}_{\neq 0} \longrightarrow \mathrm{Ext}^1(\mathbb{C}(x), \mathscr{O}(a)) \xrightarrow{(*)} \mathrm{Ext}^1(\mathbb{C}(x), E). \tag{5.14}$$

We want to show that $\mathrm{Hom}(E, \mathbb{C}(x)) \to \mathrm{Hom}(\mathscr{O}(a), \mathbb{C}(x))$ is surjective in order to derive a contradiction; it suffices to show that $(*)$ is injective. Now weave in a piece of the long exact sequence given by applying $\mathrm{Hom}(–, \mathscr{O}(a))$ and $\mathrm{Hom}(–, E)$ to (5.13):

$$\begin{tikzcd}
 & & \mathrm{Hom}(\mathscr{O}(a), \mathscr{O}(a)) \arrow[r, hook, "f"] \arrow[d, "g"] & \mathrm{Hom}(\mathscr{O}(a), E) \arrow[d, hook, "h"] \\
\underbrace{\mathrm{Hom}(\mathbb{C}(x), E)}_{=0} \arrow[r] & \underbrace{\mathrm{Hom}(\mathbb{C}(x), F)}_{\neq 0} \arrow[r] & \mathrm{Ext}^1(\mathbb{C}(x), \mathscr{O}(a)) \arrow[r, "(*)"] \arrow[d] & \mathrm{Ext}^1(\mathbb{C}(x), E) \\
 & & \mathrm{Ext}^1(\mathscr{O}(a+1), \mathscr{O}(a)). &
\end{tikzcd} \tag{5.15}$$

The reason that f is injective is that it sends $\mathrm{id}_{\mathscr{O}(a)}$ to the inclusion map $\mathscr{O}(a) \hookrightarrow E$ we specified. Since $\mathrm{Ext}^1(\mathscr{O}(a+1), \mathscr{O}(a)) = H^1(\mathbb{P}^1, \mathscr{O}(-1)) = 0$, then g is an isomorphism. Therefore $h \circ f$ is injective, so $(*)$ must be too.

Thus we've proven that F is locally free: by induction, $F = \bigoplus_j \mathscr{O}(b_j)^{\oplus r_j}$. We have left to show that $b_j \leq a$ for all j. Assume the opposite, that there's a j_0 such hat $b_{j_0} > a$. Then the composition of the maps

$$\mathscr{O}_{\mathbb{P}^1}(a+1) \hookrightarrow \mathscr{O}_{\mathbb{P}^1}(b_j) \hookrightarrow F \tag{5.16}$$

is nonzero, so throwing $\mathrm{Hom}(\mathscr{O}(a+1), –)$ at (5.11), we obtain a long exact sequence

$$\underbrace{\mathrm{Hom}(\mathscr{O}(a+1), E)}_{\neq 0} \longrightarrow \underbrace{\mathrm{Hom}(\mathscr{O}(a+1), F)}_{\neq 0} \longrightarrow \mathrm{Ext}^1(\mathscr{O}(a+1), \mathscr{O}(a)) = H^1(\mathscr{O}(-1)) = 0, \tag{5.17}$$

which causes a contradiction. Uniqueness follows from another induction on r, using the fact that $\mathrm{Hom}(\mathscr{O}(a), \mathscr{O}(b)) = 0$ if $a > b$.

Briefly, why does $0 \to \mathscr{O}(a) \to E \to \bigoplus_j \mathscr{O}(b_j)^{r_j}$ split? This is because the obstruction is $\mathrm{Ext}^1(\mathscr{O}(b_j), \mathscr{O}(a)) = H^1(\mathscr{O}(a - b_j) = 0$ whenever $a - b_j > 0$. ⊠

Lecture 6.

Slope stability: 2/7/19

The first part of the lecture involved fixing a mistake in the proof of Theorem 5.9. I included those fixes in the notes for the previous lecture; in particular, the proof that's written there should be correct.

In particular, we understand vector bundles on $\mathbb{P}^1$: line bundles and what you can make out of them. But on other curves, vector bundles are more complicated, bringing in the notion of slope stability.

Let C be a smooth projective curve over $\mathbb{C}$ (though most of this still works over any algebraically closed field).

Definition 6.1.

(1) Let $E \to C$ be a vector bundle. The *degree* $d(E)$ of E is the degree of $\mathrm{Det}(E) := \Lambda^{\mathrm{rank}(E)} E$ as a line bundle.
(2) The *degree* of a torsion sheaf T, denoted $d(T)$, is the length of the scheme-theoretic support of T.

(3) The *degree* of a coherent sheaf E is $d(E) := d(T_E) + d(F_E)$ (using Exercise 5.7).
(4) The *rank* of a coherent sheaf E is the rank of its locally free part.

What's going on in the second definition? The idea is that any torsion sheaf on a curve is a successive extension of skyscrapers, and the length of this sequence is always the same. Both of these degrees are additive in short exact sequences.

Definition 6.2.

(5) The *slope* of a coherent sheaf E on C is $\mu(E) = d(E)/r(E)$, if $r(E)$ is positive, and is ∞ otherwise.
(6) A coherent sheaf E is *semistable*, resp. *stable*, if for any nonzero subsheaf $F \subsetneq E$, $\mu(F) \leq \mu(E)$ resp. $\mu(F) < \mu(E)$.

As for quiver representations, it will turn out that you can't make a moduli space out of all vector bundles, but restricting to (semi)stable ones you can.

Exercise 6.3. Show that if $0 \to F \to E \to G \to 0$ is a short exact sequence of coherent sheaves on C, then $d(E) = d(F) + d(G)$ and $r(E) = r(F) + r(G)$.

Exercise 6.4.

(1) Let E be a coherent sheaf on C. Show that if E is semistable, then E is torsion or locally free.
(2) Show that E is stable, resp. semistable if for all surjective, non-isomorphic maps $E \twoheadrightarrow G$, $\mu(E) < \mu(G)$, resp. $\mu(E) \leq \mu(G)$.

Remark 6.5. The notion of degree is the first instance of Chern classes, which exist in a more general setting. ◀

Example 6.6. Let's consider rank-2, degree-zero bundles on $\mathbb{P}^1$. These include $\mathscr{O}(-a) \oplus \mathscr{O}(a)$ for any $a \in \mathbb{Z}$, and that's it, so you get $\mathbb{Z}_{\geq 0}$. This is not a finite-type scheme, which is sad – but $\mathscr{O}(-a) \oplus \mathscr{O}(a)$ is unstable unless $a = 0$, so the moduli space of just (semi)stable bundles will end up being OK. ◀

Fact. This is completely trivial from the definitions we just made, but if E is a nonzero sheaf with rank zero, then $d(E) > 0$. This will create all of the issues in higher dimensions. ◀

But I hear you saying, you don't just care about coherent sheaves which are semistable. This is some arbitrary condition. You care about *all* coherent sheaves. Fortunately, we can approximate any coherent sheaf by semistable ones.

Theorem 6.7. *Let E be a coherent sheaf on C. Then there is a unique filtration (up to isomorphism), called the* Harder-Narasimhan filtration, *$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$, such that E_i/E_{i-1} is semistable for each i and $\mu(E_1/E_0) \geq \mu(E_2/E_1) \geq \cdots \geq \mu(E_n/E_{n-1})$.*

We will prove this later in Proposition 8.18, when we want it to hold in a general setting; the proof is the same, so there's no need to see it twice. It's a very nice proof, and you should look forward to it; it uses pretty much just the fact that $\mathsf{Coh}(C)$ is Noetherian, and some numerical stuff.

Example 6.8. On $\mathbb{P}^1$, we can see this in Theorem 5.9. We have a unique way to write

$$E \cong \mathscr{O}(a_1)^{\oplus r_1} \oplus \cdots \oplus \mathscr{O}(a_n)^{\oplus r_n} \tag{6.9}$$

such that $a_1 > \cdots > a_n$. For $1 \leq i \leq n$, let

$$E \cong \mathscr{O}(a_1)^{\oplus r_1} \oplus \cdots \oplus \mathscr{O}(a_i)^{\oplus r_i}, \tag{6.10}$$

so that $E_i/E_{i-1} = \mathscr{O}(a_i)^{r_i}$, and $\mu(E_i/E_{i-1}) = r_i a_i / r_i = a_i$. ◀

Example 6.11. Let C be an elliptic curve and $p \in C$. Then

$$\operatorname{Ext}^1(\mathscr{O}_C, \mathscr{O}_C) = \operatorname{Hom}(\mathscr{O}_C, \mathscr{O}_C) = \mathbb{C} \tag{6.12a}$$

and

$$\operatorname{Ext}^1(\mathscr{O}_C(p), \mathscr{O}_C) = \operatorname{Hom}(\mathscr{O}_C, \mathscr{O}_C(p)) = \mathbb{C}. \tag{6.12b}$$

You can compute the second one by using the short exact sequence $0 \to \mathscr{O}_C \to \mathscr{O}_C(p) \to \mathscr{O}_p \to 0$ and that $H^0(\mathscr{O}_C) = H^1(\mathscr{O}_C) = \mathbb{C}$.

$\mathrm{Ext}^1(A, B)$ parameterizes extensions of B by A; the middle object may have automorphisms, and in both of these cases there's a $\mathbb{C}^\times$, so we can speak of the (isomorphism type) of the unique nontrivial extensions

$$0 \longrightarrow \mathscr{O}_C \longrightarrow E_1 \longrightarrow \mathscr{O}_C \longrightarrow 0 \tag{6.13a}$$

$$0 \longrightarrow \mathscr{O}_C \longrightarrow E_2 \longrightarrow \mathscr{O}_C(p) \longrightarrow 0. \tag{6.13b}$$

Claim 6.14. E_1 is semistable but not stable.

Proof. We know $r(\mathscr{O}_C) = 1$, $d(\mathscr{O}_C) = 0$, and $\mu(\mathscr{O}_C) = 0$, so using Exercise 6.3, $r(E_1) = 2$, $d(E_1) = 0$, and $\mu(E_1) = 0$. Since $\mathscr{O}_C \subset \mathscr{O}_E$ is nonzero and a proper subsheaf, but $\mu(\mathscr{O}_C) = \mu(E_1)$, then E_1 is not stable.

Now let $F \subset E$ be the first quotient E_1/E_0 in the Harder-Narasimhan filtration, which is nonzero, and assume $\mu(F) > \mu(E) = 0$. Applying $\mathrm{Hom}(F, –)$ to (6.13a), we have a left exact sequence

$$0 \longrightarrow \mathrm{Hom}(F, \mathscr{O}_C) \longrightarrow \mathrm{Hom}(F, E) \longrightarrow \mathrm{Hom}(F, \mathscr{O}_C). \tag{6.15}$$

Thus $\mathrm{Hom}(F, \mathscr{O}_C) \neq 0$, so there's a nonzero map $\varphi\colon F \to \mathscr{O}_C$; let $K := \ker(\varphi)$. Since F is semistable, $\mu(F) \leq \mu(F/K)$, but $\mu(\mathscr{O}_C) = 0$ and $\mathscr{O}_C$ is semistable, so $F/K \hookrightarrow \mathscr{O}_C$, so $\mu(F/K) < 0$ and $\mu(F/K) > \mu(F) > 0$, which is a contradiction. ⊠

Exercise 6.16. Show that E_2 is stable.

◂

Lecture 7.

The moduli space of vector bundles on curves: 2/12/19

Last time, we defined the slope $\mu(E)$ of a coherent sheaf E on a curve C, which is the degree divided by the rank. For a vector bundle, the degree is the class of $\mathrm{Det}\, E$ in $\mathrm{Pic}(C) \cong \mathbb{Z}$, and for a torsion sheaf it's the length of the scheme-theoretic support. The idea is that if you plot a line through $(0, 0)$ and $(r(E), d(E))$, the slope of E is literaly the slope of the line, which is something I wish someone had told me earlier.

Anyways, we then defined (semi)stability for E: E is (semi)stable if for all nonzero proper subsheaves $F \subsetneq E$, $\mu(F) < \mu(E)$ ($\mu(F) \leq \mu(E)$). Then we talked about the Harder-Narasimhan filtration of a coherent sheaf, which is unique up to isomorphism, and such that the quotients are semistable. In fact, we can construct a different filtration.

Theorem 7.1. *Fix a slope $\mu \in \mathbb{Q}$ and let $\mathsf{P}(\mu)$ denote the full subcategory of $\mathsf{Coh}(C)$ of semistable sheaves E with $\mu(E) = \mu$ is a finite-length abelian category, and its simple objects are the stable sheaves.*

We will defer this proof to later, where we'll prove it in a more general setting.

Corollary 7.2. *If E is a semistable sheaf, then there is a Jordan-Hölder filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

such that the quotients E_i/E_{i-1} are all stable, and $\mu(E_i/E_{i-1}) = \mu(E)$. Moreover, the associated graded of this filtration is unique up to isomorphism.

The uniqueness statement is equivalent to the quotients E_i/E_{i-1} being unique up to permutation.

Definition 7.3. Let E and E' be two semistable coherent sheaves on C with the same slope. We say E and E' are *S-equivalent* if $\mathrm{gr}(E) \cong \mathrm{gr}(E')$.

Now let's talk about moduli spaces of vector bundles on curves. For $\mathbb{P}^1$, the fact that vector bundles split as direct sums of line bundles means that everything is pretty easy: if you fix the degree and the rank, the moduli space is empty or a point. This doesn't mean that the whole category is semisimple, though.

We again let C be a smooth curve over $\mathbb{C}$; probably most of this works over a more general algebraically closed field, but the references are harder to find. The cases $g(C) = 1$ and $g(C) \geq 2$ behave very different. We're going to focus on the $g(C) \geq 2$ case; if you'd like to understand the genus 1 case, consult Atiyah [Ati57] or Polishchuk [Pol03].

Thus, we now assume $g(C) \geq 2$. Fix a positive integer r, a $d \in \mathbb{Z}$, and a degree-d line bundle L; we'll try to construct the moduli space of degree-d, rank-r coherent sheaves E with top exterior power isomorphic to L.

Definition 7.4. Given a $\mathbb{C}$-scheme B, define $\mathcal{M}_C(r,d)(B)$ to be the set of vector bundles E on $C \times B$ such that for all $b \in B(\mathbb{C})$, $E|_{C\times\{b\}}$ is semistable of rank r and degree d, modulo the equivalence relation where $E \sim E'$ if $E \cong E' \otimes \pi^* M$ for some $M \in \operatorname{Pic}(B)$, where $\pi\colon E \times B \to B$ is projection onto the second factor.

Pullbacks preserve this equivalence relation, defining a functor $\mathcal{M}_C(r,d)\colon \mathsf{Sch}_{\mathbb{C}}^{\mathsf{op}} \to \mathsf{Set}$.

Definition 7.5. Given a $\mathbb{C}$-scheme B, define $\mathcal{M}_C(r,L)(B)$ to be the set of vector bundles E on $C \times B$ such that for all $b \in B(\mathbb{C})$, $E|_{C\times\{b\}}$ is semistable of rank r and $\operatorname{Det}(E|_{C\times\{b\}}) \cong L$, modulo the equivalence relation where $E \sim E'$ if $E \cong E' \otimes \pi^* M$ for some $M \in \operatorname{Pic}(B)$, where $\pi\colon E \times B \to B$ is projection onto the second factor.

Pullbacks preserve this equivalence relation, defining a functor $\mathcal{M}_C(r,L)\colon \mathsf{Sch}_{\mathbb{C}}^{\mathsf{op}} \to \mathsf{Set}$.

These moduli functors were studied by many people in around the 1970s.

Theorem 7.6 (Drézet, Mumford, Narasimhan, Ramanan, Seshadri, …)**.**

(1) *There is a coarse moduli space $M_C(r,d)$ for $\mathcal{M}_C(r,d)$ parameterizing equivalence classes of semistable vector bundles on C, which has the following properties.*
  (a) *It's nonempty.*[4]
  (b) *It's an integral, normal, factorial, and projective variety over $\mathbb{C}$, with dimension $r^2(g-1)+1$.*
  (c) *There is an open (sometimes empty) subset $M_C^s(r,d) \subset M_C(r,s)$ which parameterizes stable vector bundles.*
  (d) *Except for the case when $g=2$, $r=2$, and d is even, $M_C(r,d) \setminus M_C^s(r,d)$ is precisely the set of singular points. In particular, if there are no semistable-but-not-stable bundles, the moduli space is smooth.*
  (e) *There's an isomorphism $\operatorname{Pic}(M_C(r,d)) \cong \operatorname{Pic}(\operatorname{Pic}^d(C)) \times \mathbb{Z}$.*
  (f) *If $\gcd(r,d)=1$, then $M_C(r,d) = M_C^s(r,d)$, and this is a fine moduli space.*

(2) *There are similar statements for $M_C(r,L)$, except:*
  (a) *Its dimension is $(r^2-1)(g-1)$.*
  (b) *$\operatorname{Pic}(M_C(r,L)) = \mathbb{Z}\cdot\theta$, where θ is ample.*
  (c) *If $u := \gcd(r,d)$, then $\omega_{M_C(r,L)} = -2u\theta$.*

Here $\operatorname{Pic}^d(C)$ is the *Jacobian*, a moduli space of degree-d line bundles. Also, keep in mind that if (r,d) share a common factor p, you can always find a semistable object that's not stable: using nonemptiness of the moduli space $M_C(r/p, d/p)$, choose an element E; then $E^{\oplus p}$ is semistable but not stable for r and d.

The proof would take us a long time to go through in complete detail, though in broad strokes it's similar to the one for quiver representations. Two good references for the proof are Le Portier [Por97] and Newstead [New11].

Abelian categories. We'll discuss general stability conditions on abelian categories. In order to do this, we'll first discuss some generalities on abelian categories. Maybe you've seen some of this before, but probably just for categories of modules and such. Instead, though, we'll deal with more general examples, where injective is not necessarily the same as monomorphic. The Stacks project [Sta19, Tags 09SE and 00ZX] is a good reference for this material.

Definition 7.7. A category A is *additive* if

- for all objects $A,B,C \in \mathsf{A}$, $\operatorname{Hom}_{\mathsf{A}}(A,B)$ is an abelian group, and the composition map $\operatorname{Hom}(A,B) \times \operatorname{Hom}(B,C) \to \operatorname{Hom}(A,C)$ sending $f,g \mapsto g\circ f$ is bilinear;
- there is an object $0 \in \mathsf{A}$ such that $\operatorname{Hom}_{\mathsf{A}}(0,0) = \{0\}$; and
- for all $A,B \in \mathsf{A}$, the coproduct $A \oplus B$ and product $A \times B$ exist and are isomorphic.

Exercise 7.8. Show that 0 is the *zero object*, meaning it's both initial and terminal: for any $A \in \mathsf{A}$, there are unique maps $0 \to A$ and $A \to 0$.

Lots of categories are additive – vector spaces over a field, vector bundles over a base, sheaves over a base, etc. The category of schemes is not additive: binary products and coproducts need not coincide.

From now on, A is an additive category.

[4] There are choices of rank and degree in genus 0 and 1 where this is not true! It's also a significant headache in higher dimensions.

Definition 7.9. Let $f\colon A \to B$ be a morphism in A.

(1) A *kernel* of f is an object $K \in \mathsf{A}$ together with a morphism $i\colon K \hookrightarrow A$ such that $f \circ i = 0$, and such that for any map $i'\colon K' \to A$ such that $f \circ i' = 0$ factors uniquely through i.
(2) A *cokernel* of f is an object $C \in \mathsf{A}$ together with a morphism $\pi\colon B \to C$ such that $\pi \circ f = 0$ and such that any map $\pi'\colon B \to C'$ such that $\pi' \circ f = 0$ factors uniquely through π.

Kernels and cokernels need not exist, but if they do they're unique up to unique isomorphism, as is standard for universal properties. Therefore one usually speaks of *the* kernel and *the* cokernel, which are denoted $\ker(f)$ and $\operatorname{coker}(f)$, respectively.

Definition 7.10. Let f be as in Definition 7.9.

(3) If $\ker(f)$ exists, then the *coimage* of f, denoted $\operatorname{coim}(f)$, is $\operatorname{coker}(i)$, where $i\colon \ker(f) \to A$ is the map specified in the definition of a kernel.
(4) If $\operatorname{coker}(f)$ exists, the *image* of f, denoted $\operatorname{Im}(f)$, is $\ker(\pi)$, where $\pi\colon B \to \operatorname{coker}(f)$ is the map specified in the definition of a cokernel.

Exercise 7.11. Suppose both the kernel and cokernel of f exist. Then show that there exists a unique morphism $h\colon \operatorname{coim}(f) \to \operatorname{Im}(f)$ such that the following diagram commutes.

$$\begin{tikzcd} \ker(f) \arrow[r, "i"] & A \arrow[r, "f"] \arrow[d] & B \arrow[r] & \operatorname{coker}(f) \\ & \operatorname{coim}(f) \arrow[r, dashed, "h", "\exists!"'] & \operatorname{Im}(f) \arrow[u] & \end{tikzcd} \tag{7.12}$$

Definition 7.13. An additive category A is *abelian* if the kernel and cokernel of every morphism exists, and the natural map $\operatorname{coim}(f) \to \operatorname{Im}(f)$ is always an isomorphism.

The usual theorems of homological algebra (five lemma, snake lemma, etc.) hold in abelian categories.

Example 7.14.

(1) If R is any ring (not necessarily commutative), then Mod_R is abelian. These are the standard examples of abelian categories; conversely, every abelian category is a subcategory of a category of modules, but the embedding may be terrible.
(2) The categories of coherent or quasicoherent sheaves over any scheme.
(3) The category of representations of a quiver. ◀

The categories of vector bundles over a positive-dimensional manifold as well as the derived category of an abelian category are examples of additive but not abelian categories.

Lecture 8.

Stability in an abelian category: 2/14/19

Let A be an abelian category.

Definition 8.1. Let $Z\colon K_0(\mathsf{A}) \to \mathbb{C}$ be an additive homomorphism. Suppose that for all nonzero $E \in \mathsf{A}$,

(1) $\operatorname{Im}(Z(E)) \geq 0$, and
(2) $\operatorname{Im}(Z(E)) = 0$ implies $\operatorname{Re}(Z(E)) < 0$.

Then Z is called a *stability function*, $R(E) := \operatorname{Im}(Z(E))$ is called the *generalized rank* of E, and $D(E) := -\operatorname{Re}(Z(E))$ is called the *generalized degree* of E. Then $M(E) = R(E)/D(E)$ is called the *generalized slope* of E. Sometimes $Z(E)$ is called the *central charge* of E.

For example, for coherent sheaves E on a smooth projective curve, $ir(E) - d(E)$ is a stability function, because degree and rank are additive in short exact sequences, and the zero bundle is the only coherent sheaf with both rank and degree zero.

Definition 8.2. An object $E \in \mathsf{A}$ is *Z-stable* (resp. *Z-stable*) if for all nonzero $F \subsetneq E$, $M(F) < M(E)$ (resp. $M(F) \leq M(E)$).

Example 8.3. Another example of this is when A is the category of representations of a quiver Q. Given a $\theta \in \mathbb{Z}[Q_0]$, we get a stability function $Z_\theta(E) := \theta(E) + i \dim E$, where we define $\dim E := \sum_{v \in Q_0} \dim(E_v)$. ◀

This is not quite the same notion as we looked at before for quiver representations, as we haven't fixed a dimension α. But if $\theta(E) = 0$, then E is θ-semistable iff E is Z_θ-semistable, since $M(F) = -\theta(F)/\dim F \leq M(E) = 0$ iff $\theta(F) > 0$, so in this case our two notions of stability coincide. But the more general notion of (semi)stability is useful in Harder-Narasimhan filtrations, in which not all objects have the same slope.

We've defined (semi)stability in terms of subobjects; there's also an equivalent formulation in terms of quotients.

Exercise 8.4. Show that $E \in \mathsf{A}$ is Z-semistable (resp. Z-stable) iff for all quotients $E \twoheadrightarrow G$ with nonzero kernel, $M(E) \leq M(G)$ (resp. $M(E) < M(G)$).

Lemma 8.5. *Let $A, B \in \mathsf{A}$ be Z-semistable objects with $M(A) > M(B)$. Then $\mathrm{Hom}_{\mathsf{A}}(A, B) = 0$.*

Proof. Let $f\colon A \to B$ be an A-morphism. Then it factors as $A \twoheadrightarrow \mathrm{Im}(f) \hookrightarrow B$, but if $\mathrm{Im}(f) \neq 0$, then by semistability of A and B,

$$M(A) \leq M(\mathrm{Im}(f)) \leq M(B) < M(A). \tag{8.6}$$

⊠

So far Harder-Narasimhan filtrations have been very useful, so we're going to ask for them in the general setting. Under some mild hypotheses, we'll show that they exist in general.

Definition 8.7. The pair (A, Z) as above is called a *stability condition* if any nonzero object has a *Harder-Narasimhan filtration* much like before: a filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$ such that E_i/E_{i-1} is Z-semistable and $\mu(E_{i+1}/E_i) > \mu(E_i/E_{i-1})$ for all i.

Exercise 8.8. Show that Harder-Narasimhan filtrations are unique up to unique isomorphism if they exist.

To get this condition, we need a few assumptions on A.

Definition 8.9. An abelian category A is *Noetherian* if for any ascending chain $A_0 \subset A_1 \subset \cdots \subset A$, we have $A_i = A_{i+1}$ for sufficiently large i.[5] The analogous condition with ascending chains replaced by descending chains is called *Artinian*.

For example, a category of representations of a quiver is finite-length, hence both Noetherian and Artinian. The category of coherent sheaves on a curve is Noetherian, but the category of quasicoherent sheaves on a variety is not.

Lemma 8.10. *Let $Z\colon K_0(\mathsf{A}) \to \mathbb{C}$ be a stability function. Assume A is Noetherian and $R\colon K_0(\mathsf{A}) \to \mathbb{R}$ has discrete image. Then for any $E \in \mathsf{A}$, the generalized degrees of subobjects of E are bounded above.*

Proof. We induct on the generalized rank. If $R(E) = 0$, then $R(F) = 0$ for all $F \subset E$, and

$$0 < D(F) = D(E) - D(E/F) < D(E), \tag{8.11}$$

so we can let $D_E := D(E)$ be the upper bound.

More generally, assume $R(E) > 0$, and suppose $F_N \subset E$ be a sequence of objects with $\lim_{n\to\infty} D(F_n) = \infty$. If for some n, $R(F_n) = R(E)$, then $R(E/F_n) = 0$ and $D(E/F_n) \geq 0$, so

$$D(F_n) = D(E) - D(E/F_n) \leq D(E), \tag{8.12}$$

so we may without loss of generality assume $R(F_n) < R(E)$ for all n. We will use these assumptions to construct an increasing sequence $(n_k)_{k\in\mathbb{N}}$ such that

$$D(F_{n_1} + \cdots + F_{n_k}) > k + D(E) \tag{8.13a}$$

$$R(F_{n_1} + \cdots + F_{n_k}) < R(E). \tag{8.13b}$$

Here "+" refers to the internal sum: the smallest subobject of E containing the given subobjects. By assumption, we can choose some n_1 such that $D(F_{n_1}) \geq 1 + D(E)$. Now inductively assume we have constructed $n_1, \ldots, n_{k-1}$. For $n > n_{k-1}$ there's a short exact sequence

(8.14)

$$0 \longrightarrow (F_{n_1} + \cdots + F_{n_{k-1}}) \cap F_n \longrightarrow (F_{n_1} + \cdots + F_{n_{k-1}}) \oplus F_n \longrightarrow F_{n_1} + \cdots + F_{n_{k-1}} + F_n \longrightarrow 0,$$

[5] Here the notation $\subset$ means the kernel is zero.

and therefore

$$D(F_{n_1} + \cdots + F_{n_{k-1}} + F_n) = D(F_{n_1} + \cdots + F_{n_{k-1}}) + D(F_n) - D((F_{n_1} + \cdots + F_{n_k}) \cap F_n). \tag{8.15}$$

By our first inductive assumption, $D((F_{n_1} + \cdots + F_{n_{k-1}}) \cap F_n)$ is bounded above, so

$$\lim_{n\to\infty} D(F_{n_1} + \cdots + F_{n_{k-1}} + F_n) = \infty. \tag{8.16}$$

Therefore we can choose some $n_k > n_{k-1}$ satisfying (8.13a). To check (8.13b), suppose that $R(F_{n_1} + \cdots + F_{n_k}) = R(E)$ (it's not possible for it to be greater); then,

$$k + D(E) < D(F_{n_1} + \cdots + F_{n_k}) < D(E), \tag{8.17}$$

which of course is a contradiction. ⊠

Now that that's out of the way, we can get at Harder-Narasimhan filtrations.

Proposition 8.18. *As above, let $Z\colon K_0(\mathsf{A}) \to \mathbb{C}$ be a stability function such that A is Noetherian and the image of the generalized rank function is discrete. Assume in addition that the image of D is discrete.*[6] *Then Harder-Narasimhan filtrations exist, i.e. (A, Z) is a stability condition.*

Proof. Let E be a nonzero object of A and let $\mathcal{H}(E)$ denote the convex hull of $\{Z(F) \mid F \subset E\}$ in $\mathbb{C}$. By Lemma 8.10, $\mathcal{H}(E)$ is bounded from the left.

Let $\mathcal{H}_\ell$ denote the half-plane to the left of the line frpom 0 to $Z(E)$. If E is semistable, then $0 \subset E$ is a Harder-Narasimhan filtration and we're done. Otherwise, $P(E) := \mathcal{H}(E) \cap H_\ell$ is a finite-vertex, bounded polygon; let $v_0 = 0, v_1, \ldots, v_n = Z(E)$ be the vertices of this polygon in increasing order. Choose $F_i \subset E$ wth $Z(F_i) = v_i$ for $i = 0, \ldots, n-1$. We claim

(1) $F_{i-1} \subset F_i$ for $i = 1, \ldots, n$ (where $F_n = E$),
(2) $G_i := F_i / F_{i-1}$ is semistable, and
(3) $\mu(G_i) > \mu(G_{i+1})$ for each i.

For (1), by definition $Z(F_{i-1} \cap F_i)$ and $Z_(F_{i-1} + F_i)$ are in $\mathcal{H}(E)$ by definition. Moreover, we know that

$$R(F_{i-1} \cap F_i) \le R(F_{i-1}) < R(F_i) \le R(F_{i-1} + F_i), \tag{8.19}$$

so using that generalized degree and rank are additive in the short exact sequence

$$0 \longrightarrow F_{i-1} \cap F_i \longrightarrow F_{i-1} \oplus F_i \longrightarrow F_{i-1} + F_i \longrightarrow 0, \tag{8.20}$$

$Z(F_{i-1} \cap F_i) + Z(F_{i-1} + F_i) = v_{i-1} + v_i$. Therefore

$$Z(F_{i-1} \cap F_i) - Z(F_{i-1} + F_i) = (v_{i-1} - Z(F_{i-1} + F_i)) + (v_i - Z(F_{i-1} + F_i)). \tag{8.21}$$

This can't work unless $Z(F_{i-1} + F_i) = v_i$ and $Z(F_{i-1} \cap F_i) = v_i - 1$.

We'll finish parts (2) and (3) next time.

Lecture 9.

Triangulated categories: 2/19/19

We're in the middle of proving that Harder-Narasimhan filtrations exist for stability conditions $Z\colon K_0(\mathsf{A}) \to \mathbb{C}$ in a general abelian category A exist, as long as

(1) A is abelian,
(2) the image of $R\colon K_0(\mathsf{A}) \to \mathbb{R}$ is discrete, and
(3) the image of $D\colon K_0(\mathsf{A}) \to \mathbb{R}$ discrete.

Though condition (3) is necessary in our proof, it's actually superfluous. However, especially once we study vector bundles on surfaces, we'll see plenty of examples of stability conditions for which the generalized rank and degree maps is not discrete, and we will have to work around this.

[6] This hypothesis, though not strictly necessary, makes the proof much simpler.

Continuation of the proof of Proposition 8.18. Recall that we defined $\mathcal{H}(E)$ to be the convex hull of $\{Z(F) \mid F \subset E\}$, where $E \in \mathsf{A}$. This has infinite area in general, so we cut it off with a line $\mathcal{H}_\ell$ to obtain a genuine polygon $P(E)$ with vertices $v_1, \dots, v_n$, under assumptions (2) and (3) above. This also allows us to choose $F_i \subset E$ with $Z(F_i) = v_i$.

It suffices to prove that $F_{i-1} \subset F_i$ for each i (which we did last time), that the quotients $G_i := F_i/F_{i-1}$ are semistable (which we'll do now), and that $M(G_i) > M(G_{i+1})$ for each i (which we'll do now). Fortunately, the hardest part is already behind us.

Let $\overline{A} \subset G_i$ be a nonzero subobject, and $A \subset F_i$ be its preimage under the quotient $F_i \twoheadrightarrow G_i$. Then $Z(A) \in \mathcal{H}(E)$ and $R(F_{i-1}) \le R(A) \le R(F_i)$. Then $Z(\overline{A}) = Z(A) - Z(F_{i-1})$, which has smaller or equal slope than $Z(G_i) = Z(F_i) - Z(F_{i-1})$.

The last thing we need to prove, that the slopes decrease, is because we're going clockwise around a convex polygon with a vertex at 0, from a vertex above and to the left of 0; therefore you can see that the slopes decrease. ⊠

The fact that the polygon is bounded from the left is the most difficult part; after that it's basically convex geometry.

Exercise 9.1. If you're interested, give a proof without assumption (3).

In particular, we've proven the existence of Harder-Narasimhan filtrations in the settings we promised to: for coherent sheaves on a smooth projective curve (Theorem 6.7) and for quiver representations.

Definition 9.2. A *very weak stability function*[7] is a homomorphism $Z\colon K_0(\mathsf{A})$ such that for all objects E of A, $\operatorname{Im}(Z(E)) \ge 0$ and if $\operatorname{Im}(Z(E)) = 0$, then $\operatorname{Re}(Z(E)) \le 0$.

So it's exactly the same as a stability condition, except that we can have generalized degree zero. We still consider the slope of something with generalized rank and degree zero to be infinity.

Remark 9.3. Proposition 8.18 still holds in this setting, but there's no longer a unique choice for $F_i \subset E$ with $Z(F_i) = v_i$; instead, we must choose the largest one, which is fine, because A is Noetherian. ◂

We will encounter very weak stability conditions when studying slopes of vector bundles on surfaces. In general, stability conditions will form a complex manifold, once formulated in the setting of derived categories, and the geometry of this complex manifold will inform us about stability conditions.

∽ · ∼

To continue with stability conditions, we will need to work with the derived category of an abelian category. The notion of a derived category is due to Verdier [Ver96] from his thesis; we will (concisely) follow Huybrechts [Huy06].

Let A be an abelian category. We would like to construct out of A another category $D(\mathsf{A})$ with two properties:

(1) the objects of $D(\mathsf{A})$ are complexes of objects in A, and
(2) if $F^\bullet \to A$ is a "resolution" (in whatever sense) of an object $A \in \mathsf{A}$, then in $D(\mathsf{A})$, we want $F^\bullet \simeq A$ (here A is regarded as a complex concentrated in degree zero).

As an illustration of the philosophy behind the second point, recall that to compute derived functors of f on A, you resolve A by a complex $F^\bullet$ of nice objects, and apply f to this complex. We'd like to say this is the same as applying the derived functors to A.

One's first guess would be the category of complexes in A, with morphisms commuting with the complex maps, but this doesn't satisfy the second condition. In fact, what we get won't even be abelian, though it will be additive, and with the extra structure of a triangulated category.

Definition 9.4. Let D be a (small) additive category. The structure of a *triangulated category* on D is given by an additive equivalence $T\colon \mathsf{D} \to \mathsf{D}$,[8] and a set of *distinguished triangles*, which are sequences of objects $A \to B \to C \to T(A)$ in D satisfying the axioms **TR1** through **TR4** below.

[7]There is a preexisting notion of a weak stability function, and this is different.

[8]In other settings, T is sometimes denoted $[1]$ or Σ.

First, though, we need some notation: if $A, B \in \mathsf{D}$ and $f\colon A \to B$ is a morphism in D, we let $A[n] := T^n A$ and $f[n]\colon A[n] \to B[n]$ denote $T^n f$. A *morphism* of distinguished triangles from $A \to B \to C \to A[1]$ to $A' \to B' \to C' \to A'[1]$ is morphisms $f\colon A \to A'$, $g\colon B \to B'$, and $h\colon C \to C'$ such that the following diagram commutes:

(9.5)
$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & A[1] \arrow[d, "f[1]"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1]. \end{tikzcd}$$

If f, g, and h are isomorphisms, we call this an *isomorphism* of distinguished triangles.

TR1. (a) For every object $A \in \mathsf{D}$, the triangle

(9.6)
$$A \xrightarrow{\mathrm{id}} A \longrightarrow 0 \longrightarrow A[1]$$

is distinguished.

(b) The set of distinguished triangles is closed under isomorphisms.

(c) Any map $f\colon A \to B$ can be completed to a distinguished triangle $A \xrightarrow{f} B \to C \to A[1]$. Sometimes C is called the *cone* of f, and denoted $C(f)$.

TR2. The triangle

(9.7a)
$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

is distinguished if and only if the triangle

(9.7b)
$$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{f[1]} B[1]$$

is distinguished.

TR3. Suppose $A \to B \to C \to A[1]$ and $A' \to B' \to C' \to A'[1]$ are distinguished triangles and $f\colon A \to A'$ and $g\colon B \to B'$ are maps such that the diagram

(9.8)
$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] & A[1] \arrow[d, "f[1]"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd}$$

commutes. Then there exists an $h\colon C \to C'$ such that when h is inserted in the above diagram, it still commutes.

TR4. The *octahedral axiom*: given $f\colon A \to B$ and $g\colon B \to C$ in D, there is a distinguished triangle

(9.9)
$$C(f) \xrightarrow{\alpha} C(g \circ f) \xrightarrow{\beta} C(g) \xrightarrow{\gamma} C(f)[1]$$

such that the following diagram is commutative:

(9.10)
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r] \arrow[d, "g"] & C(f) \arrow[r] \arrow[d, "\alpha"] & A[1] \arrow[d, equal] \\ A \arrow[r, "g\circ f"] \arrow[d] & C \arrow[r] \arrow[d] & C(g \circ f) \arrow[r] \arrow[d, "\beta"] & A[1] \arrow[d] \\ 0 \arrow[r] \arrow[d] & C(g) \arrow[r, equal] \arrow[d] & C(g) \arrow[r] \arrow[d, "\gamma"] & 0 \arrow[d] \\ A[1] \arrow[r, "f[1]"] & B[1] \arrow[r] & C(f)[1] \arrow[r] & A[2]. \end{tikzcd}$$

Distinguished triangles are the analogue of exact sequences in a triangulated category. You can think of requirement (**TR1**c) as saying that we can take cokernels, even though we're not in an abelian category. Condition **TR3** posits existence but not uniqueness, which is very unusual in category theory; thus some category theorists suggest that perhaps we should be using something different, but this definition is in fact flexible enough to serve our purposes. The octahedral axiom is somewhat tricky to write down, but will be essential when we discuss abelian categories inside our triangulated category, via t-structures and hearts.[9]

This is a long definition, but the intuition you should keep in mind is that we're trying to have an abelian category but we don't quite have one. So, for example, to gain intuition for the octahedral axiom you can make sense of it in an abelian category A: let $f\colon A \hookrightarrow B$ and $g\colon B \hookrightarrow C$ be two morphisms in A; then $C(f) = B/A$, $C(g) = C/B$, and $C(g\circ f) = C/A$. Then the analogue of **TR4** is the existence of a short exact sequence

$$0 \longrightarrow B/A \longrightarrow C/A \longrightarrow C/B \longrightarrow 0, \tag{9.11}$$

which certainly exists. But here it's canonical, whereas in a triangulated category one has to choose the maps α, β, and γ.

We will not prove all of the basic facts about homological algebra, but instead reference Huybrechts' book mentioned above. But there are a few important facts to remember.

Exercise 9.12. If $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow A[1]$ is a distinguished triangle, then $g \circ f = 0$.

Proposition 9.13. *If $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow A[1]$ is a distinguished triangle and $A_0 \in \mathsf{D}$, then there are exact sequences of abelian groups*

$$\mathrm{Hom}(A_0, A) \xrightarrow{f\circ -} \mathrm{Hom}(A_0, B) \xrightarrow{g\circ -} \mathrm{Hom}(A_0, C) \tag{9.14a}$$

$$\mathrm{Hom}(C, A_0) \xrightarrow[-\circ g]{} \mathrm{Hom}(B, A_0) \xrightarrow[-\circ f]{} \mathrm{Hom}(A, A_0). \tag{9.14b}$$

Proof. We will prove (9.14a); the proof of (9.14b) is similar, but with the arrows in the other direction.

Given $a\colon A_0 \to A$, its image in $\mathrm{Hom}(A_0, C)$ is $g \circ f \circ a = 0$ by Exercise 9.12, and therefore $\mathrm{Im}(f \circ -) \subset \ker(g \circ -)$. Now suppose $b\colon A_0 \to B \in \ker(g \circ -)$. Using **TR3**, we can choose some a such that the diagram

$$\begin{tikzcd}
A_0[-1] \arrow[r] \arrow[d, "{b[-1]}"] & 0 \arrow[r] \arrow[d] & A_0 \arrow[r, equal] \arrow[d, "a"] & A_0 \arrow[d, "b"] \\
B[-1] \arrow[r] & C[-1] \arrow[r] & A \arrow[r] & B
\end{tikzcd} \tag{9.15}$$

commutes, and then we have that $b = g \circ a$, so it's in the image as we desired. ⊠

Exercise 9.16. If $A \to B \to C \to A[1]$ is distinguished, show $A \to B$ is an isomorphism iff $C \cong 0$.

Lecture 10.

Derived categories: 2/21/19

Before we get on to derived categories, there's a few more facts about triangulated categories to mention. But once we get them out of the way, we'll only focus on derived categories, never general triangulated ones.

Exercise 10.1 (Five lemma for triangulated categories)**.** In a triangulated category D, consider a morphism of distinguished triangles

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & A[1] \arrow[d, "{f[1]}"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1].
\end{tikzcd} \tag{10.2}$$

[9]However, it's currently open whether there are any categories which satisfy **TR1** through **TR3** but not **TR4**!

Show that if two of f, g, and h are isomorphisms, then so is the third.

Definition 10.3. An additive functor[10] $F\colon \mathsf{D} \to \mathsf{D}'$ is *exact* if it maps distinguished triangles to distinguished triangles and $F \circ [1]_{\mathsf{D}} = [1]_{\mathsf{D}'} \circ F$.

These are the structure-preserving functors for triangulated categories.

Remark 10.4. In all of the examples we care about, $\mathrm{Hom}_{\mathsf{D}}(A, B)$ is more than an abelian group, but is naturally a k-vector space (meaning composition is k-linear). Then in all definitions we can replace abelian groups with k-vector spaces, obtaining the notions of *k-linear abelian categories* and *k-linear triangulated categories*. This extra structure will be important later when we construct moduli spaces and want them to be k-schemes. ◂

Now, on to the crucial example, and why we care about all this triangulated abstraction anyways.

Definition 10.5. Let A be an abelian category.

- A complex $A^\bullet$ of objects in A is called *bounded* if $A^i = 0$ for $i \ll 0$ and $i \gg 0$.
- We let $\mathsf{Kom}^b(\mathsf{A})$ denote the full subcategory of bounded complexes (so: objects are bounded complexes, and we allow all morphisms between them). This is an abelian category.
- A morphism $A^\bullet \to B^\bullet$ in $\mathsf{Kom}^b(\mathsf{A})$ is a *quasi-isomorphism* if the induced map $\mathcal{H}^i(A^\bullet) \to \mathcal{H}^i(B^\bullet)$[11] is an isomorphism for all i.
- The *bounded derived category* of D, denoted $D^b(\mathsf{A})$, is the localization of $\mathsf{Kom}^b(\mathsf{A})$ at the subcategory of quasi-isomorphisms.[12]

Theorem 10.6 (Verdier [Ver96])**.** *There is a category $D^b(\mathsf{A})$ and a functor $Q\colon \mathsf{Kom}^b(\mathsf{A}) \to D^b(\mathsf{A})$ such that*

(1) if $f\colon A^\bullet \to B^\bullet$ is a quasi-isomorphism in $\mathsf{Kom}^b(\mathsf{A})$, then Qf is an isomorphism, and
(2) any functor $F\colon \mathsf{A} \to \mathsf{C}$ such that F of any quasi-isomorphism is invertible factors uniquely (up to natural isomorphisms) through Q.

This doesn't just mean adding inverses to quasi-isomorphisms; in general one has to add more arrows. But in the actual construction, we don't add any objects.

Corollary 10.7. *If $f\colon A^\bullet \to B^\bullet$ is an isomorphism in $D^b(\mathsf{A})$, then $f_*\colon \mathcal{H}^i(A^\bullet) \to \mathcal{H}^i(B^\bullet)$ is the identity map. Therefore $\mathcal{H}^i\colon D^b(\mathsf{A}) \to \mathsf{A}$ is a functor.*

Corollary 10.8. *The full subcategory of $A^\bullet \in D^b(\mathsf{A})$ with the property that $\mathcal{H}^i(A^\bullet) = 0$ for $i \neq 0$ is equivalent to A.*

We think of this as embedding A into $D^b(\mathsf{A})$.

So what are the morphisms in $D^b(\mathsf{A})$? A general morphism $A^\bullet \to B^\bullet$ factors as two maps in $\mathsf{Kom}^b(\mathsf{A})$, $A^\bullet \leftarrow C^\bullet \to B^\bullet$, where the first map is a quasi-isomorphism. Defining composition is a little involved, though.

We would like to make $D^b(\mathsf{A})$ into a triangulated category. First we must define the shift functor, and then the distinguished triangles. The shift functor is easy: let $[1]\colon D^b(\mathsf{A}) \to D^b(\mathsf{A})$ be defined by $E[1]^i := E^{i+1}$.

Recall that an abelian category *has enough injectives* if every object embeds into an injective object.

Lemma 10.9. *Assume either that A has enough injectives[13] or $\mathsf{A} = \mathsf{Coh}(X)$. Then for $A, B \in \mathsf{A}$, there's a natural isomorphism $\mathrm{Hom}_{D^b(\mathsf{A})}(A, B[n]) = \mathrm{Ext}^1_{\mathsf{A}}(A, B)$.*

---

[10] Recall that in an additive category, the hom-sets are actually abelian groups; for an additive functor $F\colon \mathsf{C} \to \mathsf{D}$, the map $\mathrm{Hom}_{\mathsf{C}}(A, B) \to \mathrm{Hom}_{\mathsf{D}}(F(A), F(B))$ must be a group homomorphism.

[11] We'll use $\mathcal{H}^*$ to denote the cohomology of a complex and H^* to denote sheaf cohomology.

[12] This means that we want to consider, in a precise sense, the smallest category containing $\mathsf{Kom}^b(\mathsf{A})$ but such that all quasi-isomorphisms have inverses. Doing this naïvely leads to set-theoretic difficulties, but there is a way to make it work correctly.

[13] If this isn't true, the way to work around this is to embed in some other abelian category which does have enough injectives. For example, $\mathsf{Coh}(X)$ has neither enough projectives nor injectives in general, so we work inside $\mathsf{QCoh}(X)$.

This might make distinguished triangles $A \to B \to C \to A[1]$ make more sense. The map $C \to A[1]$ is a little weird, maybe, and the point is that $\mathrm{Hom}(C, A[1]) = \mathrm{Ext}^1(C, A)$, which may be easier to get a handle on.

Definition 10.10. Let $f\colon A^\bullet \to B^\bullet$ be a morphism in $\mathsf{Kom}^b(\mathsf{A})$. We define its *cone* $\mathrm{C}(f) \in \mathsf{Kom}^b(\mathsf{A})$ by $\mathrm{C}(f)^i = A^{i+1} \oplus B^i$ with the differential

$$\begin{pmatrix} -d_A^{i+1} & 0 \\ f & d_B^i \end{pmatrix}. \tag{10.11}$$

The morphisms $B^i \hookrightarrow A^{i+1} \oplus B^i \twoheadrightarrow A^i$ induce maps of complexes

$$A^\bullet \xrightarrow{f} B^\bullet \longrightarrow \mathrm{C}(f)^\bullet \longrightarrow A^\bullet[1]. \tag{10.12}$$

Definition 10.13. A sequence of morphisms $F \to G \to H \to F[1]$ in $D^b(\mathsf{A})$ is a *distinguished triangle* if it is isomorphic in $D^b(\mathsf{A})$ to (the functor Q applied to) a sequence of morphisms of the form (10.12) for some $f \in \mathrm{Hom}_{\mathsf{Kom}^b(\mathsf{A})}(A, B)$.

In particular, a short exact sequence of complexes defines a distinguished triangle. In practice, this is where the triangles we care about come from.

Theorem 10.14. *The distinguished triangles from Definition 10.13 and the shift functor above provide the structure of a triangulated category on $D^b(\mathsf{A})$.*

The proof involves a *lot* of gory details and diagram chases. You can find a complete proof in Gelfand and Manin's homological algebra book [GM03].

Remark 10.15. There are several variants on this construction; for example, you can start with complexes which are unbounded below but bounded above (so $A^i = 0$ for $i \gg 0$), defining a triangulated category $D^-(\mathsf{A})$; or complexes which are bounded below but not above, defining a triangulated category $D^+(\mathsf{A})$; or complexes which need not be bounded in either direction, defining a triangulated category $D(\mathsf{A})$. All of these are somewhat relevant, though the latter less so; complexes unbounded in both directions are pretty annoying to work with.

The categories we will consider all have finite homological dimension, which means that we essentially don't have to worry about the differences between these. ◀

Now we will discuss derived functors. The basic idea is essentially the same as in your homological algebra class, but instead of a sequence of derived functors on A, we will obtain one derived functor on $D^\pm(\mathsf{A})$. Certain properties of derived functors that are complicated to state from the former perspective, such as Grothendieck-Verdier duality, are much nicer and more concice in the language of derived categories.

Lemma 10.16 (Existence of injective resolutions)**.** *Let A be an abelian category with enough injectives. Then any $\mathsf{A}^\bullet \in D^+(\mathsf{A})$ is isomorphiuc to an $I^\bullet \in D^+(\mathsf{A})$, where I^i is injective for all i.*

We actually need something slightly stronger, which is that there is a construction to replace $A^\bullet$ with $I^\bullet$ that defines a functor $\mathit{fr}\colon D^+(\mathsf{A}) \to D^+(\mathsf{A})$.

Definition 10.17. Let A and B be abelian categories and $F\colon \mathsf{A} \to \mathsf{B}$ be left exact. Then the *right derived functor* of F, denoted $\mathbf{R}F\colon D^+(\mathsf{A}) \to D^+(\mathsf{B})$, is $F \circ \mathit{fr}$.

That is, replace $A^\bullet$ with an injective resolution and apply F to it.

Remark 10.18. Dually, if A has enough projectives and $F\colon \mathsf{A} \to \mathsf{B}$ is right exact, there is a *left exact functor* $\mathbf{L}F\colon D^-(\mathsf{A}) \to D^-(\mathsf{B})$ which takes a functorial projective resolution, then applies F. ◀

Problem: the category of coherent sheaves on a smooth projective variety X in general has neither enough projectives nor enough injectives! This happens even if X is a curve. The latter problem can be satisfied by working in $\mathsf{QCoh}(X)$, and then hoping that whatever you get out is coherent. But $\mathsf{QCoh}(X)$ still doesn't have enough projectives, which is a problem. The issue is really projectivity – for affines, this is the same as thinking about modules over a ring, which is fine. For example, projectives in $\mathsf{QCoh}(X)$ are free; locally free does not imply projective.

But we still want functors on derived categories of coherent sheaves, and even more, we want them to be on $D^b(\mathsf{Coh}(X))$, rather than on $D^{\pm}(\mathsf{Coh}(X))$. This presents an additional challenge.

Let X be a smooth projective variety over k,[14] and let $D^b(X) := D^b(\mathsf{Coh}(X))$.

Proposition 10.19. $\mathsf{QCoh}(X)$ *has enough injectives.*

Lecture 11.

Derived categories of coherent sheaves: 2/26/19

Let X be a smooth projective variety over an algebraically closed field k. Last time, we defined the bounded derived category of X, $D^b(X)$, to be $D^b(\mathsf{Coh}(X))$. This behaves poorly if we remove niceness hypotheses on X, and even in this setting we have neither enough projectives nor injectives, but $\mathsf{QCoh}(X)$ has enough injectives, at least. So if we want to define right derived functors, we can work in the derived category of $\mathsf{QCoh}(X)$ – but we might land in $D^+(\mathsf{QCoh}(X))$. So typically we'd have to run the general story and then check that what we end up with is small enough to land in $D^b(X)$.

Today, we're going to study several examples of derived functors, which are all triangulated functors (i.e. the analogue of exactness but in a triangulated category) as first studied by Verdier, and use them to cleanly state a duality theorem which is inaccessible at the level of cohomology.

Proposition 11.1. *The inclusion functor $D^b(\mathsf{QCoh}(X)) \to D^+(\mathsf{QCoh}(X))$ induces a triangulated equivalence onto the full subcategory of complexes whose cohomology is bounded.*

"Cohomology is bounded" means that $\mathcal{H}^i(A) = 0$ for $i \gg 0$ (since we're in bounded-below complexes, we already know this for $i \ll 0$).

Proposition 11.2. *The functor $D^b(X) \to D^b(\mathsf{QCoh}(X))$ induces an equivalence onto the full subcategory of complexes whose cohomology sheaves are coherent.*

With these two propositions, we'll be able to figure out when we're back in $D^b(X)$.

Now, we'll give several examples of important derived functors and when they satisfy some finiteness conditions (e.g. landing in$D^b(X)$ rather than $D^+(\mathsf{QCoh}(X))$). There is a lot of hard work in basic scheme theory that goes into these proofs.

Theorem 11.3 (Serre duality)**.** *Suppose $n := \dim X$. Then there's a natural isomorphism*

$$\mathrm{Hom}(A^\bullet, B^\bullet[i]) \cong \mathrm{Hom}(B^\bullet, A^\bullet \otimes \omega_X[n-i])^\vee$$

for all $A^\bullet, B^\bullet \in D^b(X)$.

We also need a way to pass from cohomology sheaves to cohomology groups. Specifically, the map $X \to \operatorname{Spec} k$ induces a pushforward/global sections map $\Gamma\colon \mathsf{QCoh}(X) \to \mathsf{Vect}_k$, which is left exact but not exact in general. Hence it has a right derived functor

$$\mathbf{R}\Gamma\colon D^+(\mathsf{QCoh}(X)) \longrightarrow D^+(\mathsf{Vect}_k), \tag{11.4}$$

and the *sheaf cohomology* of $\mathscr{F} \in D^+(\mathsf{QCoh}(X))$ is defined to the sequence of vector spaces $H^i(\mathscr{F}) := \mathcal{H}^i(\mathbf{R}\Gamma(\mathscr{F}))$.

Theorem 11.5 (Grothendieck)**.** *For all $\mathscr{F} \in \mathsf{QCoh}(X)$, if $i > \dim X$, $H^i(X, \mathscr{F}) = 0$.*

Theorem 11.6 (Serre)**.** *If $\mathscr{F} \in \mathsf{Coh}(X)$, then $H^i(X, \mathscr{F})$ is finite-dimensional for all i.*

This crucially uses the fact that X is projective. Even $H^0(\mathscr{F}) = \Gamma(\mathscr{F})$ may be infinite-dimensional on an affine scheme.

So our first example is $\mathbf{R}\Gamma$ restricted to $D^b(X)$, which is a functor

$$H^*\colon D^b(X) \longrightarrow D^b(\mathsf{Vect}_k^{\mathrm{fd}}), \tag{11.7}$$

[14] If you care about more general things, particularly for the case when X isn't smooth, consult Huybrechts [Huy06].

where the target is the bounded derived category of finite-dimensional vector spaces.

Our next example is the direct image. Let Y be another smooth projective k-scheme and $f\colon X \to Y$ be a morphism. Then we have a *direct image* functor $f_*\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(Y)$: if $\mathscr{F} \in \mathsf{QCoh}(X)$ and $U \subset Y$, then $f_*\mathscr{F}(U) := \mathscr{F}(f^{-1}(U))$. This is left exact but not exact, so we have a derived functor

$$\mathbf{R}f_*\colon D^+(\mathsf{QCoh}(X)) \longrightarrow D^+(\mathsf{QCoh}(Y)). \tag{11.8}$$

Theorem 11.9. *If f is proper, $\mathbf{R}f_*$ restricts to a functor $D^b(X) \to D^b(Y)$.*

The third example is sheaf Hom. Recall that if $\mathscr{F}$ and $\mathscr{G}$ are two quasicoherent sheaves on X, we can define $\mathscr{H}\!om(\mathscr{F},\mathscr{G})$, whose value on an open U is $\mathrm{Hom}(\mathscr{F}(U),\mathscr{G}(U))$. Therefore we have a functor $\mathscr{H}\!om(\mathscr{F},\text{–})\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(X)$, which is left exact. Therefore we obtain a right derived functor

$$\mathbf{R}\,\mathscr{H}\!om(\mathscr{F},\text{–})\colon D^+(\mathsf{QCoh}(X)) \longrightarrow D^+(\mathsf{QCoh}(X)). \tag{11.10a}$$

If $\mathscr{F}$ and $\mathscr{G}$ are coherent, so is $\mathscr{H}\!om(\mathscr{F},\mathscr{G})$, so when $\mathscr{F}$ is coherent, this is actually a functor

$$\mathbf{R}\,\mathscr{H}\!om(\mathscr{F},\text{–})\colon D^+(\mathsf{Coh}(X)) \longrightarrow D^+(\mathsf{Coh}(X)). \tag{11.10b}$$

Moreover, using (crucially) that X is smooth, this is actually a map

$$\mathbf{R}\,\mathscr{H}\!om(\mathscr{F},\text{–})\colon D^+(X) \longrightarrow D^+(X). \tag{11.10c}$$

This is the derived-categorical version of Ext.

Definition 11.11. The *derived dual* of an $\mathscr{F} \in \mathsf{Coh}(X)$ is $\mathscr{F}^\vee := \mathbf{R}\,\mathscr{H}\!om(\mathscr{F},\mathscr{O}_X)$.

The next example is the derived tensor product.

Proposition 11.12 (Existence of locally free resolutions)**.** *Any complex $A^\bullet \in D^b(X)$ is isomorphic to some complex $F^\bullet \in D^b(X)$ such that F^i is locally free for each i.*

This proposition crucially uses smoothnes of X.

Locally free sheaves are flat, so we can use a locally free resolution to define the left derived functor of $A^\bullet \otimes \text{–}\colon \mathsf{Coh}(X) \to \mathsf{Coh}(X)$ as a functor

$$A^\bullet \otimes^{\mathbf{L}} \text{–}\colon D^b(X) \longrightarrow D^b(X). \tag{11.13}$$

This therefore generalizes to taking the tensor product with any $A^\bullet \in D^b(X)$. This is the derived-categorical version of Tor.

The final example we need for now is pullback: given a map $f\colon X \to Y$, where Y is as above, we get a map $f^*\colon \mathsf{Coh}(Y) \to \mathsf{Coh}(X)$. This is in general right exact, but it's exact on locally free sheaves, so we can take a locally free resolution and define a left derived functor

$$\mathbf{L}f^*\colon D^b(Y) \longrightarrow D^b(X). \tag{11.14}$$

These functors satisfy a whole bunch of compatibility conditions. Normally these involve a bunch of different hypotheses on the sheaves in question, but we've skirted around that by putting fairly strict hypotheses on X.

Proposition 11.15. *Let $f\colon X \to Y$ be a map of smooth projective varieties (as above), A, B, and $C \in D^b(X)$, and E and $F \in D^b(Y)$.*

1. *$\text{–} \otimes^{\mathbf{L}} \text{–}$ is associative and commutative up to natural isomorphism.*
2. *(Projection formula) There's a natural isomorphism*

$$\mathbf{R}f_*(A) \otimes^{\mathbf{L}} E \simeq \mathbf{R}f_*(A \otimes^{\mathbf{L}} \mathbf{L}f^*(E)). \tag{11.16a}$$

3. *There's a natural isomorphism*

$$\mathbf{L}f^*(E \otimes^{\mathbf{L}} F) \simeq \mathbf{L}f^*(E) \otimes^{\mathbf{L}} \mathbf{L}f^*(F). \tag{11.16b}$$

4. *$\mathbf{L}f^*$ and $\mathbf{R}f_*$ are adjoint functors, i.e. there's a natural isomorphism*

$$\mathrm{Hom}(\mathbf{L}f^*(E), A) \simeq \mathrm{Hom}(E, \mathbf{R}f_*(A)). \tag{11.16c}$$

5. *There's a natural isomorphism*

$$\mathbf{R}\,\mathscr{H}\!om(A,B) \otimes^{\mathbf{L}} C \simeq \mathbf{R}\,\mathscr{H}\!om(A, B \otimes^{\mathbf{L}} C). \tag{11.16d}$$

(6) $-\otimes^{\mathbf{L}} B$ *and* $\mathbf{R}\mathscr{H}om(B,-)$ *are adjoint functors, and moreover there's a natural isomorphism*

$$\mathbf{R}\mathscr{H}om(A\otimes^{\mathbf{L}} B, C) \simeq \mathbf{R}\mathscr{H}om(A, \mathbf{R}\mathscr{H}om(B,C)). \tag{11.16e}$$

(7) *There's a natural isomorphism*

$$\mathbf{R}\mathscr{H}om(A, B\otimes^{\mathscr{L}} C) \simeq \mathbf{R}\mathscr{H}om(\mathbf{R}\mathscr{H}om(B,A),C). \tag{11.16f}$$

(8) *There's a natural isomorphism* $A^\vee \otimes^{\mathbf{L}} B \simeq \mathbf{R}\mathscr{H}om(A,B)$.

These will be helpful for computations, even though we haven't given their proofs. If you'd like to prove them, start with the version for modules over a ring, where things are easier, and try to say the same proof for coherent sheaves.

In this setting we have a very general duality theorem.

Theorem 11.17 (Grothendieck-Verdier duality)**.** *Let* $f\colon X\to Y$ *be a morphism of smooth projective schemes of relative dimension* $\dim f := \dim X - \dim Y$. *Let* $\omega_f := \omega_f \otimes f^*\omega_Y^\vee$, $A\in D^b(X)$, *and* $B\in D^b(Y)$. *Then there is a natural isomorphism*

$$\mathbf{R}f_*\mathbf{R}\mathscr{H}om(A, \mathbf{L}f^*(B)\otimes\omega_f[\dim(f)]) \simeq \mathbf{R}\mathscr{H}om(\mathbf{R}f_*A, B). \tag{11.18}$$

Despite its abstract-looking nature, this is occasionally useful for computations.

Now we'll be able to begin discussing Bridgeland stability, though we won't be able to give a definition until the next lecture. This is a notion of stability conditions on a triangulated category, rather than an abelian category. Bridgeland's original article [Bri07] is very readable, and a good reference.

But before we get to that, we need to discuss a little more triangulated category theory, specifically the notion of a heart of a t-structure. We're not going to discuss the notion of a t-structure in complete generality; if you'd like to read it, consult Beĭlinson-Bernstein-Deligne-Gabber [BBDG83], though they write in French.

The idea of (the heart of) a t-structure on a triangulated category is motivated by the following question.

Question 11.19. Suppose A and B are abelian categories and $F\colon D^b(\mathsf{A})\to D^b(\mathsf{B})$ is a triangulated equivalence. This does *not* necessarily restrict to an equivalence $\mathsf{A}\to\mathsf{B}$ on complexes concentrated in degree zero, so what is the structure of B in $B^b(\mathsf{A})$?

We'll skirt the generality of bounded t-structures by defining the heart directly.

Definition 11.20. Let C be a triangulated category. The *heart of a bounded t-structure* in C is a full additive subcategory $\mathsf{C}^\heartsuit$ such that

(1) if $A,B\in\mathsf{C}^\heartsuit$ and $i,j\in\mathbb{Z}$ with $i>j$, then

$$\mathrm{Ext}^{j-i}(A,B) := \mathrm{Hom}_{\mathsf{C}}(A[i],B[j]) = 0. \tag{11.21}$$

(2) For all $E\in\mathsf{C}$, there are integers $k_1>\cdots>k_m$, objects $E_0,\dots,E_m\in\mathsf{C}$, and a collection of distinguished triangles $E_{i-1}\to E_i\to A_i[k_i]\to E_{i-1}[1]$, where $A_i\in\mathsf{C}^\heartsuit$, $E_0=0$, and $E_m=E$.

The second condition is a kind of Harder-Narasimhan filtration, albeit in the derived sense. It turns out that these two axioms are enough to guarantee that $\mathsf{C}^\heartsuit$ is an abelian category!

Lecture 12.

The heart of a bounded t-structure: 2/28/19

Last time, we defined the heart $\mathsf{C}^\heartsuit$ of a bounded t-structure on a a triangulated category C – even though we didn't define bounded t-structures.

Example 12.1. Today, we'll begin with the standard example, an abelian category A sitting inside its bounded derived category as complexes concentrated in degree zero. When you learn about t-structures, this will be the heart of the *standard t-structure* on $D^b(\mathsf{A})$.

We need to show two things. First, that $\mathrm{Hom}_{D^b(\mathsf{A})}(A[i], B[j]) \neq 0$ for any $i > j$ and $A, B \in \mathsf{A}$. This is true, because we have two complexes concentrated at i and j, respectively, but $i \neq j$ so there are no maps between them.

The second thing we need to check is the existence of a Harder-Narasimhan-esque filtration of objects in $D^b(\mathsf{A})$ by objects in A. Let $E^\bullet \in D^b(\mathsf{A})$, so by definition $\mathcal{H}^i(E^\bullet)$ vanishes for $i \gg 0$ and $i \ll 0$. If $E^\bullet$ has vanishing cohomology, there's nothing to prove. Let r be the length of the largest interval $[i, j] \subset \mathbb{R}$ such that $\mathcal{H}^i(E^\bullet) \neq 0$ and $\mathcal{H}^j(E^\bullet) \neq 0$; we will induct on r. When $r = 0$, there's a single integer k such that $\mathcal{H}^{-k}(E^\bullet) \neq 0$. Define a map of complexes $A^\bullet \to E^\bullet$ by

$$\begin{tikzcd}
\cdots \arrow[r] & E^{-k-1} \arrow[r, "d^{-k-1}"] & E^{-k} \arrow[r, "d^{-k}"] & E^{-k+1} \arrow[r] & \cdots \\
\cdots \arrow[r] & E^{-k-1} \arrow[r] \arrow[u, equal] & \ker(d^{-k}) \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots ,
\end{tikzcd} \tag{12.2}$$

which commutes (there's not much to check), and moreover, is a quasi-isomorphism! We know the cohomology is zero above degree $-k$; at $-k$, we still get $\ker(d^{-k}) / \mathrm{Im}(d^{-k-1})$; and in lower degrees the complexes are the same. So we've cut off the right side of $E^\bullet$; now let's cut off the left by defining another quasi-isomorphism $A^\bullet \xrightarrow{\simeq} \mathcal{H}^{-k}(E^\bullet)[k]$:

$$\begin{tikzcd}
\cdots \arrow[r] & E^{-k-1} \arrow[r] \arrow[d] & \ker(d^{-k}) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \mathcal{H}^{-k}(E^\bullet) \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd} \tag{12.3}$$

In all degrees except for $-k$, this induces the zero map on cohomology, but cohomology is zero, so this is an isomorphism. In degree $-k$, the map is modding out by the image, which also is an isomorphism on cohomology. Thus, $E^\bullet \in \mathsf{A}[k]$ and we're set.

Now, for $r > 0$, let k be the smallest integer with $\mathcal{H}^{-k}(E^\bullet) \neq 0$. We will again "cut off" part of $E^\bullet$, though not quite via quasi-isomorphisms. Consider the two morphisms of complexes

$$\begin{tikzcd}
F^\bullet = \Big(\arrow[d, hook] & \cdots \arrow[r] & E^{-k-2} \arrow[r] \arrow[d, equal] & \ker(d^{-k-1}) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \Big) \\
E^\bullet = \Big(\arrow[d, two heads] & \cdots \arrow[r] & E^{-k-2} \arrow[r] \arrow[d] & E^{-k-1} \arrow[r] \arrow[d] & E^{-k} \arrow[r] \arrow[d] & E^{-k+1} \arrow[r] \arrow[d] & \cdots \Big) \\
\mathcal{H}^{-k}(E^\bullet) \simeq \Big(& \cdots \arrow[r] & 0 \arrow[r] & E^{-k-1} / \ker(d^{-k-1}) \arrow[r] & E^{-k} \arrow[r] & E^{-k-1} \arrow[r] & \cdots \Big)
\end{tikzcd} \tag{12.4}$$

The map $F^\bullet \to E^\bullet$ is *not* a quasi-isomorphism, but this *is* a short exact sequence of complexes, which means it induces a distinguished triangle $F^\bullet \to E^\bullet \to \mathcal{H}^{-k}(E^\bullet)[k]$, and $\mathcal{H}^{-k}(E^\bullet)[k]$ lives in $\mathsf{A}[k]$ as we want. Therefore it suffices to apply this to F, but F has one fewer nonzero cohomology group than E so we apply the inductive assumption.

In fact, we've proven something stronger than the statement about the filtration in Definition 11.20, namely that we can choose the pieces of the filtration to be cohomology groups of $E^\bullet$. Slicing off pieces of the heart! Sounds painful. ◀

Remark 12.5. If you take A inside $D^\pm(\mathsf{A})$, this argument doesn't work, which is what the "bounded" in "bounded t-structure" is all about. ◀

Exercise 12.6. Show that Harder-Narasimhan filtrations for Example 12.1 are unique, in the usual sense. Though this is not asked for by the axioms of Definition 11.20, it is implied by them.

Definition 12.7. Let $\mathsf{C}^\heartsuit$ be the heart of a bounded t-structure for a triangulated category C. Then the *cohomology with respect to* $\mathsf{C}^\heartsuit$ of an object $E^\bullet \in \mathsf{C}$ can be defined as follows: let $E_0 \to \cdots \to E_n$ be the Harder-Narasimhan filtration of $E^\bullet$, with the distinguished triangles $E_{i-1} \to E_i \to A_i[k_i] \to E_{i-1}[1]$. Then, define

$$\mathcal{H}^k_{\mathsf{C}^\heartsuit}(E^\bullet) = \begin{cases} A_i, & \text{if } k = k_i \text{ for some } i \\ 0, & \text{otherwise.} \end{cases}$$

Remark 12.8. **Warning:** we originally motivated t-structures as describing equivalences $D^b(\mathsf{A}) \to D^b(\mathsf{B})$ that don't restrict to equivalences $\mathsf{A} \to \mathsf{B}$. You might think, "oh, in that setting, the notion of cohomology with respect to the heart is the usual cohomology in the other category" but that is not true! ◀

Theorem 12.9. *The heart $\mathsf{C}^\heartsuit$ of a bounded t-structure is an abelian category. If $A \to B \to C \to A[1]$ is a distinguished triangle in C such that $A, B, C \in \mathsf{C}^\heartsuit$, then $0 \to A \to B \to C \to 0$ is a short exact sequence in $\mathsf{C}^\heartsuit$, and conversely.*

Proof sketch. Recall that if $f\colon A \to B$ is a morphism in $\mathsf{C}^\heartsuit$, and $C(f)$ denotes the cone on f, then the triangle $A \to B \to C(f) \to A[1]$ is distinguished. Then $C(f)$ is not always in $\mathsf{C}^\heartsuit$, but we claim it has a two-term Harder-Narasimhan filtration, which makes it accessible.

(1) First, show that $\mathcal{H}^i_{\mathsf{C}^\heartsuit}(C(f)) = 0$ unless $i = -1$ or $i = 0$. This uses the long exact sequence in cohomology associated to a distinguished triangle.
(2) Next, show that $\mathcal{H}^{-1}_{\mathsf{C}^\heartsuit}(C(f)) \cong \ker(f)$ and that $\mathcal{H}^0_{\mathsf{C}^\heartsuit}(C(f)) = \operatorname{coker}(f)$.

So if $C(f)$ *is* in $\mathsf{C}^\heartsuit$, we conclude that $\mathcal{H}^{-1}_{\mathsf{C}^\heartsuit}(C(f)) = 0$, so f is injective, and $\operatorname{coker}(f) = \mathcal{H}^0_{\mathsf{C}^\heartsuit}(C(f)) = C(f)$, so we have exactness at B. Running a similar argument again for the distinguished triangle $B \to C(f) \to A[1] \to B[1]$ shows that the map $B \to C(f)$ is surjective. ⊠

Let's now look at an example that's different from Example 12.1.

Example 12.10. Let A denote the smallest full additive subcategory of $D^b(\mathbb{P}^1)$ containing both $\mathscr{O}_{\mathbb{P}^1}(1)$ and $\mathscr{O}_{\mathbb{P}^1}[2]$ and which is *closed under extensions*, i.e. if $A \to B \to C \to A[1]$ is a distinguished triangle with $A, C \in \mathsf{A}$, then $B \in \mathsf{A}$ too.

We claim that there are no nontrivial extensions, i.e.

$$\mathsf{A} = \{\mathscr{O}_{\mathbb{P}^1}^{\oplus a}[2] \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus b} \mid a, b \in \mathbb{Z}_{\geq 0}\}. \tag{12.11}$$

This is a statement about sheaves not admitting extensions, so we need to compute some Ext groups:

$$\operatorname{Ext}^1(\mathscr{O}_{\mathbb{P}^1}(1), \mathscr{O}_{\mathbb{P}^1}[2]) = H^3(\mathbb{P}^1, \mathscr{O}(-1)) = 0 \tag{12.12}$$

because cohomology on a curve vanishes above degree 1, and

$$\operatorname{Ext}^1(\mathscr{O}_{\mathbb{P}^1}[2], \mathscr{O}_{\mathbb{P}^1}(1)) = H^{-1}(\mathbb{P}^1, \mathscr{O}(1)) = 0. \tag{12.13}$$

Exercise 12.14. Show that A is the heart of a bounded t-structure. Hint: use Theorem 5.9: that any $E \in \mathsf{Coh}(\mathbb{P}^1)$ splits as $T \oplus \mathscr{O}(a_1) \oplus \cdots \oplus \mathscr{O}(a_n)$, where T is torsion and $a_1, \ldots, a_n \in \mathbb{Z}$. You will also want to use the fact that the *Euler sequence*

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^1}(-2) \xrightarrow{\binom{x}{y}} \mathscr{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \longrightarrow 0 \longrightarrow 0, \tag{12.15}$$

where x and y are the two homogeneous coordinates on $\mathbb{P}^1$, is short exact (and then by tensoring around, you can shift this to other $\mathscr{O}(n)$).

Next, we claim that $\mathsf{A} \simeq \mathsf{Vect}_k \oplus \mathsf{Vect}_k$. This is clear on objects by (12.11), and for morphisms we need to check there are no "cross morphisms" $\mathscr{O}_{\mathbb{P}^1}[2] \rightleftarrows \mathscr{O}_{\mathbb{P}^1}$. This is again a fact about cohomology:

$$\operatorname{Hom}(\mathscr{O}_{\mathbb{P}^1}[2], \mathscr{O}_{\mathbb{P}^1}(1)) = H^{-2}(\mathbb{P}^1, \mathscr{O}(1)) = 0 \tag{12.16a}$$

$$\operatorname{Hom}(\mathscr{O}_{\mathbb{P}^1}(1), \mathscr{O}_{\mathbb{P}^1}[2]) = H^2(\mathbb{P}^1, \mathscr{O}(-2)) = 0, \tag{12.16b}$$

and the claim follows.

And most interestingly, we'll show there's no equivalence $D^b(\mathsf{A}) \to D^b(\mathbb{P}^1)$ that restricts to the identity on A. This time, we use Ext: compare

$$\mathrm{Ext}_{D^b(\mathbb{P}^1)}(\mathscr{O}_{\mathbb{P}^1}[2], \mathscr{O}_{\mathbb{P}^1}(1)) = H^0(\mathscr{O}_{\mathbb{P}^1}(1)) \cong k^2, \tag{12.17}$$

but since A is semisimple (it's a direct sum of two copies of Vect_k), all of its extensions vanish. In particular, this is not equivalent to the standard t-structure on $D^b(\mathbb{P}^1)$. ◂

Definition 12.18. Let C be a triangulated category. Its *Grothendieck group* $K_0(\mathsf{C})$ is defined to be the quotient of the free abelian group on the objects of C by the triangles: if $A \to B \to C \to A[1]$ is a distinguished triangle in C, impose the re;lation $[A] + [C] = [B]$ in $K_0(\mathsf{C})$.

Exercise 12.19. Show that if $\mathsf{C}^\heartsuit$ is the heart of a bounded t-structure for a triangulated category C, then the inclusion map $\mathsf{C}^\heartsuit \hookrightarrow \mathsf{C}$ induces an isomorphism $K_0(\mathsf{C}^\heartsuit) \xrightarrow{\cong} K_0(\mathsf{C})$.

In particular, no matter what heart you choose, its Grothendieck group is the same. When we define stability conditions, this means the map $K_0(\mathsf{C}) \to \mathbb{C}$ always has the same domain.

The next definition looks a lot like the definition of a heart of a bounded t-structure, and is indeed very similar, but is different: instead of indexing by $\mathbb{Z}$, we index by $\mathbb{R}$.

Definition 12.20. Let C be a triangulated category. A *slicing* $\mathcal{P}$ of C is a collection of full additive subcategories $\mathcal{P}(\phi) \subset \mathsf{C}$ for all $\phi \in \mathbb{R}$, such that

(1) $\mathcal{P}(\phi)[1] = \mathcal{P}(\phi + 1)$;
(2) if $\phi_1 > \phi_2$ and $A \in \mathcal{P}(\phi_1)$ and $B \in \mathcal{P}(\phi_2)$, then $\mathrm{Hom}_{\mathsf{C}}(A, B) = 0$; and
(3) for all $E \in \mathsf{C}$ there are real numbers $\phi_1 > \phi_2 > \cdots > \phi_n$ and objects $E_i \in \mathsf{C}$ and $A_i \in \mathcal{P}(\phi_i)$ for $i = 1, \dots, n$, together with distinguished triangles $E_i \to E_{i+1} \to A_i \to E_i[1]$, such that $E_0 = 0$ and $E_n = E$.

Definition 12.21. The filtration $0 = E_0 \to E_1 \to \cdots \to E_n$ is again called a *Harder-Narasimhan filtration*. We will write $\phi^+(E) := \phi_1$ and $\phi^-(E) := \phi_n$.

As always, Harder-Narasimhan filtrations are unique.

Proposition 12.22. *For any $a \in \mathbb{R}$, the subcategory $\mathcal{P}((a, a+1])$, defined to be the full subcategory of objects $E \in \mathsf{C}$ such that $\phi^+(E) \le a + 1$ and $\phi^-(E) > a$, is the heart of a bounded t-structure on C.*

We're almost at the definition of a Bridgeland stability condition, and will get there early next lecture. But first, two more exercises.

Exercise 12.23. Let $\mathsf{C}^{\heartsuit_1}, \mathsf{C}^{\heartsuit_2}$ be two hearts of bounded t-structures on a triangulated category C, and suppose $\mathsf{C}^{\heartsuit_1} \subset \mathsf{C}^{\heartsuit_2}$ (that is, as a full subcategory, though in this case fullness is automatic from the definition of the heart). Then $\mathsf{C}^{\heartsuit_1} = \mathsf{C}^{\heartsuit_2}$.

Exercise 12.24. Show similarly that if $\mathcal{P}_1$ and $\mathcal{P}_2$ are two slicings of a triangulated category C such that $\mathcal{P}_1(\phi) \subset \mathcal{P}_2(\phi)$ for all $\phi \in \mathbb{R}$, then $\mathcal{P}_1 = \mathcal{P}_2$.

Lecture 13.

Bridgeland stability conditions: 3/5/19

Today, we're going to define Bridgeland stability conditions! Get excited.

Recall that a slicing $\mathcal{P}$ of a triangulated category C is a collection of full additive subcategories $\mathcal{P}(\phi) \subset \mathsf{C}$ indexed by $\phi \in \mathbb{R}$ such that

(1) $\mathcal{P}(\phi)[1] = \mathcal{P}(\phi + 1)$;
(2) if $\phi_1 > \phi_2$, $A \in \mathcal{P}(\phi_1)$, and $B \in \mathcal{P}(\phi_2)$, then $\mathrm{Hom}(A, B) = 0$; and
(3) for all $E \in \mathsf{C}$ there are real numbers $\phi_1 > \cdots > \phi_n$, $E_i \in \mathsf{C}$, and $A_i \in \mathcal{P}(\phi_i)$ together with a *Harder-Narasimhan filtration* of E by $E_0 \to E_1 \to \cdots \to E_n = E$ such that the cone of the map $E_{i-1} \to E_i$ is A_i.

So the information on the slicing really only depends on $\phi \in [0,1]$.

The definition of a Bridgeland stability condition that we give is a slight modification of Bridgeland's original definition [Bri07] that has been found to be a little easier to work with.

Recall that $K_0(\mathsf{C})$ need not be a finitely generated infinite group; an excellent example is $\mathsf{Coh}(\Sigma_1)$, where Σ_1 is a genus-one curve. Therefore we fix a finite-rank lattice and a surjective group homomorphism $v\colon K_0(\mathsf{C}) \to \Lambda$.

Example 13.1.

(1) If $\mathsf{C} = D^b(\mathsf{Coh}(C))$, where C is a smooth projective curve, we can choose $\Lambda = \mathbb{Z}^2$ with $v = (\mathrm{rank}, \deg)$.

(2) If $\mathsf{C} = D^b(\mathsf{Rep}_Q)$, where Q is a quiver, we can let $\Lambda = \mathbb{Z}[Q_0]$ and v be the dimension vector. ◀

Now fix a norm $\|\cdot\|$ on $\Lambda \otimes \mathbb{R}$.

Definition 13.2. A *Bridgeland stability condition* on a triangulated category C with respect to Λ and v is a pair $\sigma = (\mathcal{P}, Z)$, where $\mathcal{P}$ is a slicing and $Z\colon \Lambda \to \mathbb{C}$ is a group homomorphism, such that for all $E \in P(\phi)$, the argument of $Z(v(E))$ in $\mathbb{C}$ is $e^{i\pi\phi}$, and the *support property*[15] holds: that

$$C_\sigma := \inf\left\{ \frac{|Z(v(E))|}{\|v(E)\|} : E \in \mathcal{P}(\phi) \setminus 0, \phi \in \mathbb{R} \right\} > 0.^{16} \tag{13.3}$$

Objects in $\mathcal{P}(\phi)$ are called *σ-semistable* of *phase ϕ*, and Z is called the *central charge*.

We will want this notion of semistability to match in the examples we've already considered, and it will (but we haven't shown that yet).

Remark 13.4. A $\sigma = (\mathcal{P}, Z)$ such that Definition 13.2 holds except for the support condition is called a *prestability condition*. ◀

We'll discuss some examples in a moment, and will first go over some properties.

Lemma 13.5. *Let C be a triangulated category and $\mathsf{C}^\heartsuit$ be the heart of a bounded t-structure on C. Then, Specifying a stability condition $\sigma = (\mathcal{P}, Z_1)$ on C is equivalent to specifying a stability condition $Z_2\colon K_0(\mathsf{C}^\heartsuit) \to \mathbb{C}$ in the sense of Definition 8.1 on $\mathsf{C}^\heartsuit$ such that*

$$\inf\left\{ \frac{|Z_1(v(E))|}{\|v(E)\|} : E \in \mathsf{C}^\heartsuit \setminus 0 \text{ is } Z\text{-semistable} \right\} > 0. \tag{13.6}$$

Proof sketch. By Proposition 12.22, $\mathcal{P}((0,1])$ is the heart of a bounded t-structure on C, so call it $\mathsf{C}^\heartsuit$. By definition, Z_1 maps $\mathsf{C}^\heartsuit$ to complex numbers with argument $\pi/2 \le \theta \le \pi$, so Z_1 restricts to a stability condition on $\mathsf{C}^\heartsuit$.

Conversely, suppose $Z_2\colon K_0(\mathsf{C}^\heartsuit) \to \mathbb{C}$ is a stability condition on $\mathsf{C}^\heartsuit$. For $\phi \in (0,1]$, define

$$\mathcal{P}(\phi) := \{E \in \mathsf{C}^\heartsuit \mid E \text{ is } Z_2\text{-semistable}\}. \tag{13.7}$$

Then for $\phi \in \mathbb{R}$, let $n := \lceil \phi \rceil - 1$ and $\mathcal{P}(\phi) := \mathcal{P}(\phi - n)[n]$. Then (and there's stuff to check here) $(\mathcal{P}, Z_2$ is a stability condition on C. ⊠

From now on, we'll use "stability condition" to mean "Bridgeland stability condition."

Remark 13.8. Bridgeland stability came out of physics! Or, at least, homological mirror symmetry. Douglas wrote some stuff about it, and Bridgeland stability came out of Bridgeland's work to make everything mathematically precise. ◀

Lemma 13.9. *If $\sigma = (\mathcal{P}, Z)$ is a stability condition, then $P(\phi)$ is a finite-length abelian category.*

This is a really nice thing to have (e.g. we used it both for quivers and curves), and is the reason we impose the support property. Bridgeland instead asked for Lemma 13.9 to hold directly, which is a "local finiteness" condition; this suffices but is harder to work with. For example, this lemma allows us to define stable objects as simple objects in $P(\phi)$.

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[15] This property wasn't part of the original definition, and was added by Kontsevich-Soibelman [KS08].

[16] The choice of C_σ de[pends on the norm we chose on $\Lambda \otimes \mathbb{R}$, but the fact that there exists such a C does not depend on the norm.

Proof sketch of Lemma 13.9. For $\phi > 1$ or $\phi \le 0$, of course, we can shift ϕ to land in $(0,1]$, so without loss of generality, $\phi \in (0,1]$. Then $\mathcal{P}(\phi) \subset \mathcal{P}((0,1])$, which is abelian, so it suffices to show that $\mathcal{P}(\phi)$ is closed under kernels and cokernels. This is essentially the same argument as we gave in Lemma 3.23 to show that the semistable objects in a quiver representation category form an abelian category.

Now we need to show that $\mathcal{P}(\phi)$ has finite length, meaning that it's both Noetherian and Artinian. First we'll show that it's Noetherian. Given $E \in \mathcal{P}(\phi)$ and an ascending chain $E_0 \subset E_1 \subset \cdots \subset E$, let $A_i := E_i/E_{i-1}$, then for each $n \ge 0$, we have

$$Z(v(E)) = \sum_{i=1}^{n} Z(v(A_i)) + Z(v(E/E_n)). \tag{13.10}$$

Since all $Z(v(A_i))$ and $Z(E/E_n)$ lie on the same ray,

$$\begin{aligned} |Z(v(E))| &= \sum_{i=1}^{n} |Z(v(A_i))| + |Z(v(E/E_n))| \\ &\ge \underbrace{\sum_{i=1}^{n} |Z(v(A_i))|}_{=:s_n}. \end{aligned} \tag{13.11}$$

In particular, s_n is monotonically increasing in n, yet bounded, so it converges. Therefore

$$\lim_{i\to\infty} |Z(v(A_i))| = 0. \tag{13.12}$$

The support property says that there's a constant C such that $|Z(v(A_i))| \ge C\|v(A_i)\|$ for all i, and therefore $\lim_{i\to\infty}\|v(A_i)\| = 0$ too. But $v(A_i) \in \Lambda \subset \Lambda \otimes \mathbb{R}$, so since it converges in a discrete space, $v(A_i) = 0$ for i large enough, which means $Z(v(A_i)) = 0$ for i large enough, which means $A_i = 0$ for i large enough.

The Artinian part is the same, but with the arrows reversed. ⊠

Definition 13.13. A *σ-stable object* of *phase ϕ* is a simple object of $\mathcal{P}(\phi)$.

It's not hard to convince yourself that this is the same as we had before: a nontrivial, proper subobject of the same phase is the same thing as a nontrivial, proper subobject with the same slope.

Definition 13.14. Two objects $E, E' \in \mathcal{P}(\phi)$ are *S-equivalent* if their Jordan-Hölder filtrations have the same factors.

So if you want any hope of making a moduli space, you'd have to parameterize S-equivalent objects, just as we saw in the beginning of the semester.

The support condition is sometimes a little hard to check directly, so here's a helpful criterion.

Lemma 13.15. *Let $\sigma = (\mathcal{P}, Z)$ be a prestability condition. Then σ satisfies the support property iff there is a symmetric bilinear form Q on $\Lambda \otimes \mathbb{R}$ such that*

(1) for all σ-semistable objects E, $Q(v(E), v(E)) \ge 0$, and
(2) for all nonzero $v \in \Lambda \otimes \mathbb{R}$ with $Z(v) = 0$, $Q(v,v) < 0$.

We'll prove this next time, but we'll use it right now to discuss an example.

Example 13.16. Let C be a smooth projective curve and $v = (\mathrm{rank}, \deg)\colon K_0(C) \to \mathbb{Z}^2$ with central charge $Z = -d + ir$. This defines a prestability condition, and we can use Lemma 13.15 to check that it is in fact a stability condition: Z is injective, so we could take the standard quadratic form on $\mathbb{R}^2 = \mathbb{Z}^2 \otimes \mathbb{R}$, or even the zero quadratic form! ◀

So of course the hard work goes into the proof of Lemma 13.15. But for stability conditions for sheaves on surfaces, finding the quadratic form is also nontrivial, and on threefolds it's the crux of the whole problem!

Exercise 13.17. Let $\mathsf{A} := \mathsf{Rep}_Q$, where Q is a quiver, $\theta \in \mathbb{Z}[Q_0]$, and let $v\colon K_0(\mathsf{Rep}_Q) \to \mathbb{Z}[Q_0]$ be the dimension vector and $\dim\colon \mathbb{Z}[Q_0] \to \mathbb{R}$ be the ℓ^1-norm (total dimension of a representation). If $Z := -\theta + i\dim$, show that (A, Z) is a stability condition. You can do this either by directly checking the support condition or by finding a quadratic form as in Lemma 13.15.

Remark 13.18. Existence and properties of moduli spaces of Bridgeland semistable objects is an active area of research; see the lecture notes for details. ◂

Lecture 14.

Bridgeland's deformation theorem: 3/7/19

The first thing we're going to do is prove Lemma 13.15, as promised last time; this is a useful way to check whether the support property holds for a given prestability condition. It's really useful, and by conservation of effort, the proof is a little involved. In fact, we already used it in Example 13.16 to exhibit a stability condition on $D^b(\mathsf{Coh}(C))$, where C is a smooth projective curve.

Proof of Lemma 13.15. Let $(\text{–},\text{–})$ denote the standard bilinear form on $\mathbb{C} = \mathbb{R}^2$.

First let's assume $\sigma = (\mathcal{P}, Z)$ is a stability condition, meaning the support condition holds; we want to construct Q. Let $\langle \text{–},\text{–}\rangle$ be a symmetric, positive definite bilinear form on $\Lambda \otimes \mathbb{R}$, and $\|\cdot\|$ be the induced norm. It doesn't matter which one we use, because all norms are equivalent, and the support property holds in one iff it holds in any.

The support property says that there's a $C_\sigma > 0$ such that for all nonzero semistable $E \in \mathsf{C}$, $\|Z(v(E))\| \geq C_\sigma \|v(E)\|$. Define a bilinear form

$$Q(w, w') := \frac{1}{C_\sigma^2}\big(Z(w), Z(w')\big) - \langle w, w'\rangle. \tag{14.1}$$

Hence

$$Q(w, w) = \frac{1}{C_\sigma^2}|Z(w)|^2 - \|w\|^2. \tag{14.2}$$

First, we want to check that if E is σ-semistable, then $Q(v(E), v(E)) > 0$. Well,

$$Q(v(E), v(E)) = \frac{1}{C_\sigma^2}|Z(v(E))|^2 - \|v(E)\|^2 \geq 0, \tag{14.3}$$

as the last inequality is equivalent to the support condition. Next, we want to check that if $w \neq 0$ but $Z(w) = 0$, then $Q(w, w) < 0$ – but this is just $-\|v\|^2$, which of course is negative.

Now the other direction: given a quadratic form Q, which might not be symmetric, we need a norm. We know that if $Q(w, w) > 0$, then $|Z(w)|^2 > 0$. Let $B \subset \Lambda \otimes \mathbb{R}$ denote the closed unit ball. It's compact, so there is a $C > 0$ such that for all $v \in B$,

$$\frac{1}{C^2}|Z(w)|^2 - Q(w, w) > 0. \tag{14.4}$$

Now we can define a bilinear form

$$\langle w, w'\rangle := \frac{1}{C^2}(Z(w), Z(w')) - Q(w, w'). \tag{14.5}$$

Any $w \in \Lambda \otimes \mathbb{R}$ is λ times some $v \in B$. This means $\langle w, w\rangle = \lambda^2\langle v, v\rangle \geq 0$, and is strictly positive if $w \neq 0$. Therefore $\langle \text{–},\text{–}\rangle$ is positive definite, so $\|w\| := \sqrt{\langle w, w\rangle}$ is a norm on $\Lambda \otimes \mathbb{R}$. Then, for any σ-semistable E,

$$|Z(v(E), v(E)| = C^2\|v(E)\|^2 + C^2 Q(v(E), v(E)) \geq C^2\|v(E)\|^2, \tag{14.6}$$

where we use the fact that $Q(v(E), v(E)) \geq 0$. The upshot is that $|Z(v(E))|/\|v(E)\| \geq C$, which is the support property. ⊠

We have two examples of Bridgeland stability conditions so far: for quiver representations and for vector bundles on a curve. We are interested in more examples, particularly for coherent sheaves on surfaces (the first example where Bridgeland stability is really necessary), but first we'll discuss a few more theorems about stability conditions in general, and then do a crash course in intersection theory.

The first general fact will discuss is Bridgeland's deformation theorem, following Bayer's article [Bay16], which is a slightly simpler version of Bridgeland's original proof. We want to study the topology of stability conditions. The function Z lives in $\mathrm{Hom}(\Lambda, \mathbb{C})$, whose topology we understand: it's a real vector space. The more involved bit is placing a topology on the space of slicings.

Let C be a triangulated category and $\mathcal{P}$ be a slicing on C. For any $E \in \mathsf{C}$, let $\{E_i\}$ be its Harder-Narasimhan filtration; then E_1 is semistable, so it has a phase $\phi(E_1) \in [0, 2\pi)$. We similarly let $\phi(E_n/E_{n-1})$ denote the phase of E_n/E_{n-1}, and then define $\phi^+_{\mathcal{P}}(E) := \phi(E_1)$ and $\phi^-_{\mathcal{P}}(E) := \phi(E_n/E_{n-1})$.

Let $\mathrm{Slice}(\mathsf{C})$ denote the set of slicings on C and $\mathrm{Stab}_\Lambda(\mathsf{C})$ denote the set of stability conditions on C factoring through a map $v\colon K_0(\mathsf{C}) \to \Lambda$ (chosen but not specified in the notation). We know $\mathrm{Slice}(\mathsf{C})$ is nonempty, but it's less clear for $\mathrm{Stab}_\Lambda(\mathsf{C})$.

Let's define a "metric" $d\colon \mathrm{Slice}(\mathsf{C}) \times \mathrm{Slice}(\mathsf{C}) \to [0, \infty]$ by

$$d(\mathcal{P}_1, \mathcal{P}_2) := \sup\{|\phi^+_{\mathcal{P}_1}(E) - \phi^+_{\mathcal{P}_2}(E)|, |\phi^-_{\mathcal{P}_1}(E) - \phi^-_{\mathcal{P}_2}(E)| : E \in \mathsf{C}\}. \tag{14.7}$$

This is reflexive and symmetric, and satisfies the triangle inequality, so the only reason this isn't an actual metric is that it can attain the value ∞. Nonetheless, that is not a problem for defining an induced topology.

Definition 14.8. The topology on $\mathrm{Stab}_\Lambda(\mathsf{C})$ is the coarsest topology such that the forgetful maps $\mathrm{Stab}_\Lambda(\mathsf{C}) \to \mathrm{Slice}(\mathsf{C})$ and $\mathrm{Stab}_\Lambda(\mathsf{C}) \to \mathrm{Hom}(\Lambda, \mathbb{C})$ are continuous.

Eventually, we'll prove that the latter map is a local homeomorphism, which implies in particular that (14.7) isn't always infinite.

Theorem 14.9 (Bridgeland's deformation theorem)**.** *The forgetful map* $\mathrm{Stab}_\Lambda(\mathsf{C}) \to \mathrm{Hom}(\Lambda, \mathbb{C})$ *is a local homeomorphism, and in particular induces the structure of a complex manifold*[17] *on* $\mathrm{Stab}_\Lambda(\mathsf{C})$.

Exercise 14.10. Show that $d(\mathcal{P}_1, \mathcal{P}_2) = \sup\{\phi^+_{\mathcal{P}_2}(E) - \phi, \phi - \phi^-_{\mathcal{P}_2}(E) : \text{ for all } \phi \in \mathbb{R}, E \in \mathcal{P}_1(\phi) \setminus 0\}$.

This is a useful lemma for computing distances, because it there is less to check.

This next lemma is going to encompass a significant part of the theorem.

Lemma 14.11. *Let* $\sigma_1 = (\mathcal{P}_1, Z_1)$ *and* $\sigma_2 = (\mathcal{P}_2, Z_2)$ *be two prestability conditions and* Q *be a symmetric bilinear form on* $\Lambda \otimes \mathbb{R}$*. Suppose that*

(1) σ_1 *satisfies the support property with respect to* Q*;*
(2) there is an $\varepsilon > 0$ *such that for all* $v \in \Lambda$ *with* $Q(v) \geq 0$*,*

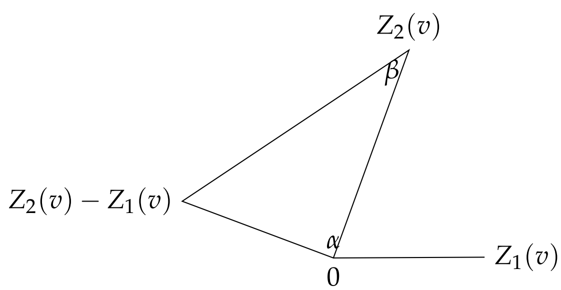
$$\frac{|Z_2(v) - Z_1(v)|}{|Z_1(v)|} < \sin(\pi\varepsilon); \tag{14.12}$$

and either $d(\mathcal{P}_1, \mathcal{P}_2) < 1/4$ *or* $\mathcal{P}_1((0,1]) = \mathcal{P}_2((0,1])$*.*

Then $d(\mathcal{P}_1, \mathcal{P}_2) < \varepsilon$*.*

Proof. Consider the triangle in $\mathbb{C}$ with vertices 0, $Z_2(v)$, and $Z_2(v) - Z_1(v)$:

(14.13)



Let α and β be the angles marked in the diagram. It is a theorem from trigonometry that

$$\frac{|Z_2(v) - Z_1(v)|}{|Z_1(v)|} = \frac{\sin\beta}{\sin\alpha} \overset{(2)}{<} \sin(\pi\varepsilon). \tag{14.14}$$

Therefore

$$\sin\beta < \sin(\pi\varepsilon)\sin\alpha < \sin(\pi\varepsilon), \tag{14.15}$$

so the phases of $Z_1(v)$ and $Z_2(v)$ differ by at most ε.

[17] As stated, there are issues with second countability: there may be uncountably many connected components. But each connected component is really a complex manifold.

Let $E \in \mathcal{P}_1(\phi)$. If $d(\mathcal{P}_1, \mathcal{P}_2) < 1/4$, then

$$E \in \mathcal{P}_2\left(\phi - \frac{1}{4}, \phi + \frac{1}{4}\right) \subset \underbrace{\mathcal{P}_2\left(\phi - \frac{1}{2}, \phi + \frac{1}{2}\right]}_{=:\mathsf{A}}. \tag{14.16}$$

By Proposition 12.22, A is the heart of a bounded t-structure on C, and in particular is abelian.
Alternatively, if $d(\mathcal{P}_1, \mathcal{P}_2) \geq 1/4$, we set

$$\mathsf{A} := \mathcal{P}_1((n, n+1]) = \mathcal{P}_2((n, n+1]) \tag{14.17}$$

for some $n \in \mathbb{Z}$ with $n < \phi \leq n+1$. This is also abelian, again by Proposition 12.22.

Now let $E_1 \subset E$ be the first term in the Harder-Narasimhan filtration of E with respect to σ_2. Then $E_1 \in \mathsf{A}$. Let F_i be the Harder-Narasimhan factors of E_1 with respect to σ_1. Since E is σ_1-stable, $\phi_{\mathcal{P}_1}(F_i) \leq \phi^+_{\mathcal{P}_1}(E_1) \leq \phi_{\mathcal{P}_1}(E)$, and by (2), $\phi_{\mathcal{P}_2}(F_i) - \varepsilon \leq \phi_{\mathcal{P}_1}(F_i)$. Therefore for all i, $\phi_{\mathcal{P}_2}(F_i) \leq \phi + \varepsilon$, so

$$\phi^+_{\mathcal{P}_2}(E) = \phi_{\mathcal{P}_2}(E_1) \leq \phi + \varepsilon. \tag{14.18}$$

Now you can do something similar with $E \twoheadrightarrow E_n/E_{n-1}$ to conclude that $\phi_2^-(E) \geq \phi - \varepsilon$, and then we're done by the lemma. ⊠

Lecture 15.

Bridgeland's deformation theorem, II: 3/26/19

Lecture 16.

Intersection theory (or, variations on a theme of Schubert): 3/28/19

Last time, we discussed the deformation theorem: that given a stability condition $\sigma = (Z, \mathcal{P})$, if we deform Z a little bit to Z', there's a unique slicing $\mathcal{P}'$ such that $(Z', \mathcal{P}')$ is a stability condition. But we don't know how much 'a little bit" means, so let's do that first.

Let $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}_\Lambda(\mathsf{C})$ satisfy the support condition with respect to a quadratic form Q on Λ. The set $\overline{U}$ of $Z' \in \mathrm{Hom}(\Lambda, \mathbb{C})$ such that Q is positive definite on $\ker(Z')$ is open, so let $U \subset \overline{U}$ be the connected component containing Z and V be connected component of the preimage of U in $\mathrm{Stab}_\Lambda(\mathsf{C})$ containing σ.

Theorem 16.1. *The map $V \to U$ sending $(Z', \mathcal{P}') \mapsto Z'$ is a covering map. Moreover, any $\sigma' \in V$ satisfies the support property with respect to Q.*

See Bayer's notes for a proof. This immediately implies that the set of stability conditions is a complex manifold.

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In the last part of the class, we'll discuss Bridgeland stability conditions on surfaces. In order to do that, we'll need some interesting functions from the Grothendieck group, which necessitates some intersection theory. This is a great subject. and could've been the subject of a whole course. Here are some references.

- Eisenbud-Harris, "3264 and all that" [EH16]. This book is long but very readable, with nice exercises, etc. The title merits some explanation: 3264 is the number of conics tangent to five general conics. Moreover, the title is a parody of "1066 and all that," itself a parody history (1066 CE is when the Battle of Hastings took place, beginning the modern English monarchy). There are video lectures by Harris corresponding to the book online.
- The maximal reference is Fulton, "Intersection theory." The statements are all there, and the proofs are not as easy of a read, but this book contains everything. Before this book, though, there were no proofs: people just assumed the statements were true, and sometimes they were right.
- For a quick reference, Hartshorne contains an appendix on this stuff.

Intersection theory has an interesting history. Schubert did a number of computations of some impressive-looking numerical solutions to enumerative questions in geometry. His methods were not quite as rigorous as today's, but he managed to get the right answers. Here's an example.

Definition 16.2. A *twisted cubic* is a map $\mathbb{P}^1 \to \mathbb{P}^3$ which, possibly after a coordinate change, takes the form $[s:t] \mapsto [s^3 : s^2t : st^2 : t^3]$.

Theorem 16.3 (Schubert)**.** *The number of twisted cubics tangent to twelve general quadrics in* $\mathbb{P}^3$ *is* 5819539783680*.*

This is a surprising number; we're used to such numbers being small, or zero, or infinite. But here we are. A modern (and rigorous) proof works in the moduli space of twisted cubics and considers the subvariety of twisted cubics tangent to a given quadric (suitably general). Then, one can compute using intersection-theoretic methods what happens when one intersects this with "itself" twelve times (corresponding to twelve general quadrics).

Hilbert's 15th problem was to make Schubert's calculations rigorous, and develop a theoretical framework for these kinds of enumerative problems; this led to enormous progress in geometry and topology, including homology and cohomology, Grassmannians, and eventually intersection theory in algebraic geometry.

Anyways, on to intersection theory. In the next several lectures, a "scheme" will always be separated of finite type over $\mathbb{C}$, so a variety is the same thing as an integral scheme.

Question 16.4. Let X be a scheme with subschemes Y and Z such that $\dim Y = \operatorname{codim} Z$, how many points are in $Y \cap Z$?

This is the basic question of intersection theory.

The first thing we will discuss is the Chow ring. This involves extracting algebra out of divisors. There are way too many divisors, so we'll mod out by a useful equivalence relation generalizing rational equivalence.

Definition 16.5. For a scheme X, let $Z(X)$ be the free abelian group generated by closed subvarieties of X. This is $\mathbb{Z}$-graded by dimension; let $Z_k(X)$ denote the subgroup of pure dimension k subvarieties.

If $Y \subset X$ is a subscheme with irreducible components $Y_1, \ldots. Y_n$, let $\ell_i := \dim \mathscr{O}_{Y,Y_i}$; because X is Noetherian, this is finite. Define the divisor (Y) by

$$(Y) := \sum_{i=1}^n \ell_i Y_i \in Z(X). \tag{16.6}$$

Definition 16.7. Let $\operatorname{Rat}(X) \subset Z(X)$ be the subgroup generated by elements of the form

$$(\Phi \cap \{0\} \times X) - (\Phi \cap \{1\} \times X), \tag{16.8}$$

where $\Phi \subset \mathbb{P}^1 \times X$ is a subvariety not contained in $\{t\} \times X$ for any $t \in \mathbb{P}^1$.

Then, define the *Chow group* $A(X) := Z(X)/\operatorname{Rat}(X)$. The class of a subvariety $Y \subset X$ in $A(X)$ is denoted $[Y]$.

Sometimes the Chow group is denoted $CH(X)$, but $A(X)$ is used in Fulton (and by Grothendieck and Serre).

We'll show in Lemma 16.11 that the elements (16.8) that we quotient by are all homogeneous, which implies that $A(X)$ is also graded by dimension; we'll let $A_k(X)$ denote the component coming from $Z_k(X)$.

Definition 16.9. If $\alpha, \beta \in Z(X)$ are such that $\alpha - \beta \in \operatorname{Rat}(X)$, we say α and β are *rationally equivalent*.

When X is a smooth variety, we will define a ring structure on $A(X)$ called (surprise) the Chow ring. Just as the Chow group is reminiscent of homology, this will be reminiscent of cohomology; however, on a smooth oriented manifold Poincaré duality tells us we can get cohomology by relabeling homology, and the complex points of a smooth variety over $\mathbb{C}$ are a smooth, oriented manifold, so we're good.

Definition 16.10. Suppose X is a smooth variety of dimension n. Then we define $A^k(X) := A_{n-k}(X)$ and $A^*(X) := \bigoplus_k A^k(X)$.

This will be the Chow ring, but we'll have to figure out what multiplication is.

Lemma 16.11. *Let* $\alpha \in Z_k(X) \setminus 0$ *and* $\beta \in Z_m(X) \setminus 0$*. If* $\alpha - \beta \in \operatorname{Rat}(X)$*, then* $k = \ell$*.*

Proof. We will show that the generators of $\operatorname{Rat}(X)$ have pure dimension. Let $\Phi \subset \mathbb{P}^1 \times X$ be a subvariety not contained in $\{t\} \times X$ for any t. Let $p\colon \Phi \to \mathbb{P}^1$ be the projection and $\Phi_t := p^{-1}(t)$ for $t \in \mathbb{P}^1$.

If Φ_t is nonempty, then

$$\dim \Phi_t \geq \dim \Phi - \dim \mathbb{P}^1 = \dim \Phi - 1. \tag{16.12}$$

If $\dim \Phi_t = \dim \Phi$, we claim $\Phi_t = \Phi$, since Φ is a variety and Φ_t is a subvariety, and of course this contradicts our assumption that Φ isn't contained in a single fiber.

Therefore Φ_0 and Φ_1 are both codimension 1 in Φ, hence live in $Z_{\dim \Phi - 1}(X)$, as desired. ⊠

The ring structure on the Chow ring will be intersection, mostly. Two things can intersect in weird ways, so we'll restrict to nice intersections.

Definition 16.13. Let Y and Z be subvarieties of X. We say they intersect *transversely* at some $p \in Y \cap Z$ if $T_pY + T_pZ = T_pX$. If Y and Z intersect transversely on a dense subset of $Y \cap Z$, we say they intersect *generically transversely*.

"Nice intersections" mean those which are generically transverse. So we'll say that if X and Y intersect generically transversely, then their product in the Chow ring will be $[X] \cdot [Y] = [X \cap Y]$. In general, what one needs to do is adjust X or Y, replacing them by rationally equivalent X' and Y' which do intersect generically transversely, and then use their intersections. There's plenty to check here (can you always do that? does the result depend on the choice?), which we'll discuss in a moment.

Example 16.14. The intersection $\{x = 0\} \cap \{y = 0\}$ in $\mathbb{A}^2$ is transverse. But $\{y = 0\} \cap \{y = x^2\}$ is not transverse at its single intersection point (so it's also not generically transverse). ◂

The fact that the product structure exists was assumed by many authors, but was eventually rigorously proven by Fulton.

Theorem 16.15 (Fulton)**.** *Let X be a smooth, quasiprojective variety. Then there is a unique product structure on $A^*(X)$ such that*

(1) *if Y and Z are subvarieties of X intersecting generically transversely, $[Y] \cdot [Z] = [Y \cap Z]$, and*
(2) *$A^*(X)$ is an associative, commutative ring graded by codimension.*

The key big things to do are show that for general subvarieties Y and Z of a smooth, projective variety X, one can always replace them with rationally equivalent varieties Y' and Z' intersecting generically transversely, and the rational equivalence class of $Y' \cap Z'$ doesn't depend on our choice of Y' and Z'.

This fact is called the "moving lemma." Fulton actually derived it as a corollary of the above theorem, which seems a little weird – and he uses a lot of complicated machinery along the way. Eisenbud and Harris set out to right this wrong, since one ought to be able to prove the moving lemma and then use it to get the ring structure, but this seems to not work.

Showing that there exist rationally equivalent subvarieties which intersect generically transversely is not so hard, and is given in an appendix of Eisenbud-Harris. The second part is much harder! Before Fulton, there were claimed proofs, and there's a reasonable-sounding argument that's not that bad, but it's not correct. Eisenbud-Harris provide this incorrect argument (noting that it's incorrect) in an appendix.

Now let's do some basic computations.

Lemma 16.16. *The map $\mathbb{Z} \to A^0(X)$, sending $m \mapsto m[X]$, is an isomorphism.*

This crucially uses that X is a variety, e.g. a scheme may have multiple components.

Proof. The map $\mathbb{Z} \to Z^0(X) := Z_{\dim X}(X)$ sending $m \mapsto mX$ is an isomorphism: the only codimension-zero (closed) subvariety of X is x itself, and the map $\mathbb{Z} \to Z^0(X)$ is a map between two free abelian groups of the same rank, and it sends the generator to the generator.

Therefore all we have left to do is show that $\mathrm{Rat}^0(X) = 0$. This is the subgroup of $\mathrm{Rat}(X)$ generated by $\Phi \subset \mathbb{P}^1 \times X$ with $\dim \Phi_t = \dim X$ or $\Phi_t = \varnothing$ for all $t \in \mathbb{P}^1$, and such that Φ isn't contained in any single fiber. Thus $\dim \Phi = \dim X + 1$, and since $\mathbb{P}^1 \times X$ is an irreducible variety with dimension $\dim X + 1$, we conclude $\Phi = \mathbb{P}^1 \times X$. This doesn't impose an interesting equivalence relation: it tells us $\Phi_0 - \Phi_1 = (X) - (X) = 0$, which we already knew. ⊠

This is the only thing we can do in general; otherwise it depends on the variety. Next time we'll talk about A^1, which is basically the Picard group, and then the Chow groups of affine and projective spaces.

Lecture 17.

Chow rings: 4/2/19

We've begun discussing intersection theory and the Chow ring: divisors modulo rational equivalence. The ring structure is given by intersection of generically transverse representatives of classes in the ring; the existence of such representative for a pair of classes is guaranteed by the moving lemma. This is a key lemma which we didn't prove, but whose proof can be found in Fulton.

Last time, we required all schemes to come with a few hypotheses; we will continue that assumption. Also, everything involving the Chow ring involves closed subvarieties, whether we say this explicitly or not.

Proposition 17.1. *$A^1(X)$ is isomorphic to the group of Weil divisors of X.*

Well, if you don't know what a Weil divisor is, you can take this as a definition, and the proof is pretty direct.

Proposition 17.2. *$A(\mathbb{A}^n) = \mathbb{Z}\cdot[\mathbb{A}^n]$ concentrated in degree zero.*

Proof. Let $Y \subsetneq \mathbb{A}^n$ be a closed subvariety; we'd like to show that $[Y] = 0$ in the Chow ring. There is a rational equivalence between Y and any translate of it, so we may without loss of generality assume $0 \notin Y$.

Consider the set

$$W^0 = \{(t, ty) \in \mathbb{A}^1 \setminus 0 \times \mathbb{A}^n \mid y \in Y\}, \tag{17.3}$$

which has a projection $p\colon W^0 \to \mathbb{A}^1$ and a map $\varphi\colon \mathbb{A}^1 \setminus 0 \times Y \to W^0$, which sends $(t, y) \mapsto (t, ty)$. Then in fact φ is an isomorphism, with inverse $(t, z) \mapsto (t, z/t)$, which we can do because $t \neq 0$. Because Y is a variety and $\mathbb{A}^1 \setminus 0$ is irrecucible, then $\mathbb{A}^1 \setminus 0 \times Y$ is irreducible and therefore W^0 is too. This means $\overline{W^0}$ is also irreducible.[18] Evidently $p^{-1}(1) = \{1\} \times Y$, so it suffices to show that $p^{-1}(x)$ is empty, where x is any other point of $\mathbb{A}^1$, or to extend to a map $\overline{W^0} \to \mathbb{P}^1$ and show the preimage of ∞ is empty.

If $\mathscr{I}_Y$ denotes the ideal sheaf on $\mathbb{A}^n$ cutting out Y (so, since $\mathbb{A}^n$ is affine, just the ideal I_Y of functions $f \in k[x_1, \dots, x_n]$ that vanish on Y), then $W^0 = V(\{f(z/t) \mid f \in I_Y\})$. Let $W := W_0 \subset \mathbb{P}^1 \times \mathbb{A}^n$. Since $0 \notin Y$, there's some $g \in I_Y$ such that $g(0) \neq 0$, and the function $G(t, z) := g(z/t)$ vanishes on W. At $t = \infty$, $G(t, z)$ extends by

$$\lim_{t\to\infty} g\left(\frac{z}{t}\right) = g(0) \neq 0, \tag{17.4}$$

so $p^{-1}(\infty) = \varnothing$. Therefore $[Y] = [p^{-1}(1)] = [p^{-1}(\infty)] = 0$ in $A^*(\mathbb{A}^n)$. ⊠

Now we'll mention some useful facts for computing Chow rings, reminiscent of the basic theorems about singular homology in algebraic topology.

Proposition 17.5. *Let X be a scheme.*

Mayer-Vietoris: *Let $i_1\colon X_1 \hookrightarrow X$ and $i_2\colon X_2 \hookrightarrow X$ be subschemes. Then the sequence*

$$A(X_1 \cap X_2 \xrightarrow{(i_1, -i_2)} A(X_1) \oplus A(X_2) \xrightarrow{\text{sum}} A(X_1 \cup X_2) \longrightarrow 0 \tag{17.6}$$

is exact as groups.

Excision: *Let $Y \subset X$ be a subscheme and $U = X \setminus Y$. Then the sequence*

$$A(Y) \longrightarrow A(X) \longrightarrow A(U) \longrightarrow 0 \tag{17.7}$$

is exact as groups. If X is a smooth variety, the pullback $A(X) \to A(U)$ is a ring homomorphism.

These are not hard to prove; they just take time and aren't our main goal.

Definition 17.8. Let X a scheme and $\mathfrak{U}$ be a finite collection of locally closed subschemes. Then $\mathfrak{U}$ is an *affine stratification* if

- for all $U \in \mathfrak{U}$, there's an isomorphism $U \cong \mathbb{A}^n$ for some n,
- the closure of each $U \in \mathfrak{U}$ is a union of some collection of $V \in \mathfrak{U}$, and
- if $U, V \in \mathfrak{U}$ are such that $\overline{U} \cap V \neq \varnothing$, then $U \subset \overline{V}$.

[18] This uses a general fact about the Zariski topology: if a dense open set is irreducible, then its closure is also irreducible.

For most schemes these do not exist, but there are stratifications of nice spaces we do care about.

Remark 17.9. Chow rings are different from cohomology rings – given some horrible set of equations in projective space, we can generally find the cohomology, but the Chow ring is often intractable. It is also often less nice than the cohomology ring, e.g. it may be infinitely generated. ◂

We're not going to use the next theorem, but it's pretty, and the fact that it's so recent is a hint at its difficulty.

Theorem 17.10 (Totaro (2014) [Tot14])**.** *If $\mathfrak{U}$ is an affine stratificatin of X, then $\{[U] : U \in \mathfrak{U}\}$ is a basis of $A(X)$.*

Exercise 17.11. Without assuming the previous theorem, show that $\{[U] : U \in \mathfrak{U}\}$ is a generating set of $A(X)$. (Hint: use excision.)

Functoriality is an important property of cohomology. There is an analogue for Chow groups.

Theorem 17.12. *Let $f\colon X \to Y$ be a proper morphism of schemes. Then there is a homomorphism $f_*\colon A_k(X) \to A_k(Y)$, called* pushforward, *such that for all subvarieties $Z \subset Y$,*

- *if $\dim f(Z) < \dim Z$, then $f_*[Z] = 0$,*
- *if $\dim f(Z) = \dim Z$ and $f|_Z$ has degree n, then $f_*[A] = n[f(A)]$, and*
- *pushforward is functorial: $(f \circ g)_* = f * \circ g_*$.*

We know $f|_Z$ has a well-defined degree because it's proper and preserves dimension, hence finite. This degree can depend on Z, though. The real content of this theorem is that the second condition is well-defined under rational equivalence. We're not going to prove it, but it's in Eisenbud-Harris.

Corollary 17.13. *If X is proper, the proper map $X \to \operatorname{Spec} k$ induces a map $\deg\colon A_0(X) \to A(\operatorname{Spec} k) = \mathbb{Z}$ which sends $[p] \mapsto 1$ for any $p \in X$.*

This looks silly but is very useful: if Y and Z are subvarieties of X whose dimensions sum to that of X, then we can compute their intersection number, which is $\deg([X] \cdot [Y])$. This is often tractable, even when we don't know much about the Chow ring.

We'd also like to define a pullback map: given $f\colon Y \to X$, we'd like to say $f^*([Z]) = [f^{-1}(Z)]$ for $Z \subset X$, but this doesn't always make sense: there are problems if f isn't flat. Nonetheless, this issue can often be avoided.

Definition 17.14. Let X and Y be smooth varieties and $f\colon Y \to X$ be a map. A subvariety $Z \subset X$ is *generically transverse* to f if $f^{-1}(Z)$ is generically reduced (i.e. the generic point is reduced, or equivalently there's an open dense locus which is reduced), and $\operatorname{codim}_Y f^{-1}(Z) = \operatorname{codim}_X A$.

We have to restrict to quasi-projective varieties, which isn't that awful.

Theorem 17.15. *Let X and Y be smooth quasi-projective varieties and $f\colon Y \to X$ be a map.*

(1) There exists a unique group homomorphism $f^*\colon A^k(X) \to A^k(Y)$ such that for all $Z \subset X$ generically transverse to f, $f^*[Z] = [f^{-1}(Z)]$.
(2) The equality $f^*[Z] = [f^{-1}(Z)]$ also holds if $\operatorname{codim}_Y f^{-1}(Z) = \operatorname{codim}_X Z$ and Z is Cohen-Macaulay.[19]
(3) $f^*\colon A(X) \to A(X)$ is a ring homomorphism.
(4) (Push-pull formula) If f is proper, $\alpha \in A^k(X)$, and $\beta \in A_\ell(Y)$, then

$$f_*(f^*\alpha \cdot \beta) = \alpha \cdot f_*\beta \in A_{\ell-k}(X). \tag{17.16}$$

There are varieties which are not quasi-projective, though it's not easy to write down examples. Set-theoretically, the push-pull formula is very simple, following from the general fact about set maps that $f(f^{-1}(A) \cap B) = A \cap f(B)$, though then one has to check that all the multiplicities work out.

Remark 17.17. This theory doesn't require the base field to be algebraically closed. There are applications even in complex geometry, for proving that certain varieties aren't rational, which is pretty surprising and neat. ◂

[19]That is, Z has an affine cover by spectra of Cohen-Macaulay rings. It suffices for Z to be smooth.

We'd like to compute the Chow ring of projective space, using 17.11. The embeddings $\mathbb{A}^k \hookrightarrow \mathbb{A}^{k+1}$ induce embeddings $\mathbb{P}^{k-1} \hookrightarrow \mathbb{P}^k$, and $\mathbb{P}^k \setminus \mathbb{P}^{k-1} \cong \mathbb{A}^k$. Therefore letting $U_k := \mathbb{P}^k \setminus \mathbb{P}^{k-1}$, $\mathfrak{U} := \{U_1, \ldots, U_n\}$ is an affine stratification of $\mathbb{P}^n$.

Theorem 17.18. *There is a ring homomorphism $A(\mathbb{P}^n) \cong \mathbb{Z}[x]/(x^{n+1})$, where x has degree 1 and $x^{n-i} = [\mathbb{P}^i]$.*

Proof. First, using Exercise 17.11, $[\mathbb{P}^i]$, $i = 0, \ldots, n$ generate $A(\mathbb{P}^n)$. For the ring structure, we'll induct on n; when $n = 0$, $A(\mathbb{P}^0) = \mathbb{Z} \cdot [\mathbb{P}^0]$ is easy to check by hand. For $n > 0$, use excision:

$$A(\mathbb{P}^{n-1}) \longrightarrow A(\mathbb{P}^n) \longrightarrow A(U_{n-1}) = A(\mathbb{A}^{n-1}) = \mathbb{Z} \cdot [A^{n-1}] \longrightarrow 0. \tag{17.19}$$

We don't yet know whether the first map is injective, but can compute that

$$[\mathbb{P}^i] \cdot [\mathbb{P}^j] = [\mathbb{P}^{i+j-n}]. \tag{17.20}$$

The copies of $\mathbb{P}^i$ and $\mathbb{P}^j$ that we found aren't generically transverse, but we can move $\mathbb{P}^i$ around by choosing a different subspace $\mathbb{A}^{i+1} \subset \mathbb{A}^{n+1}$ through the origin, and in this way can make $\mathbb{P}^i$ and $\mathbb{P}^j$ generically transverse.

The upshot of (17.20) is that $\mathbb{Z}[x]/(x^{n+1})$ surjects onto $A(\mathbb{P}^n)$. We want to show this map is also injective, so it suffices to show each $[\mathbb{P}^i]$ is nonzero and not torsion. For this one computes $\deg(a \cdot [\mathbb{P}^i] \cdot [\mathbb{P}^{n-i}])$, moving $\mathbb{P}^i$ in the same way to make it generically transverse; the answer is $a \deg([\mathbb{P}^0]) = a \neq 0$. ⊠

Lecture 18.

Enumerative problems via Chow rings: 4/4/19

"It seems like you're actually trying to understand the material… [but] for you I don't know."

Last time, we computed the Chow ring of $\mathbb{P}^n$; today, we'll do some applications, including answering questions that don't require knowing what Chow rings are.

Definition 18.1. Let $X \subset \mathbb{P}^n$ be a subvariety (closed as always), of dimension k. Then in the Chow ring, $[X] = \lambda \cdot [\mathbb{P}^k]$ for some $\lambda \geq 1$. This λ is called the *degree* of X.

Here by $[\mathbb{P}^k]$ we mean the class of any linear embedding $\mathbb{P}^k \hookrightarrow \mathbb{P}^n$ (i.e. cut out by linear equations). You can directly show these are all rationally equivalent. There are nonlinear embeddings, though, e.g. a degree-3 embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$.

To compute the degree of $X \subset \mathbb{P}^n$, you can compute its intersection with $\mathbb{P}^{n-k}$; if X is cut out by explicit equations that you have, this is pretty tractable. This is the reason that $d \geq 1$; there must be some intersection with a generic hyperplane.

In cohomology there's a Künneth formula (at least over a field). There is no analogue for Chow rings, even after tensoring with a field; the infinite-dimensional things that can appear in Chow rings mess it up.

Theorem 18.2. *The natural map $A(\mathbb{P}^r) \otimes A(\mathbb{P}^s) \to A(\mathbb{P}^r \times \mathbb{P}^s)$ (i.e. $[X] \otimes [Y] \mapsto [X \times Y]$) is an isomorphism. More precisely, if $\alpha = [H_1 \times \mathbb{P}^s]$ and $\beta = [\mathbb{P}^r \times H_2]$, where H_1, resp. H_2, is a codimension-1 hyperplane in $\mathbb{P}^r$, resp. $\mathbb{P}^s$, then*

$$A(\mathbb{P}^r \times \mathbb{P}^s) \cong \mathbb{Z}[\alpha, \beta]/(\alpha^{r+1}, \beta^{s+1}). \tag{18.3}$$

Moreover, if $X \subset \mathbb{P}^r \times \mathbb{P}^s$ is cut out by a bihomogeneous equation of bidegree (d, e), then $[X] = d\alpha + e\beta$.

The way to prove this is to begin with A_1, showing it's generated by $\alpha^r \beta^{s-1}$ and $\alpha^{r-1}\beta^s$, using some actual geometry. Then A^1 is generated by α, β, and this is the dual basis to $\{\alpha^r\beta^{s-1}, \alpha^{r-1}\beta^s\}$. You can also use the affine stratifications we constructed on $\mathbb{P}^r$ and $\mathbb{P}^s$ to define an affine stratification on the product, by taking the products of the strata; then using excision one can discover the ring structure. But the proof of linear independence is harder.

Remark 18.4. More generally, if X is rational, $A(X \times Y) \cong A(X) \otimes A(Y)$. ◀

Question 18.5. Now let's discuss an application. Let $F_0, F_1, F_2 \in \mathbb{C}[x, y, z]$ be three general homogeneous polynomials. Up to scalars, how many linear combinations $t_0F_0 + t_1F_1 + t_2F_2$ for $[t_0 : t_1 : t_2] \in \mathbb{P}^2$, where the linear combination is regarded in $\mathbb{P}(\mathbb{C}[x, y, z]_3)$ factor?

Here $\mathbb{C}[x,y,z]_n$ is the space of homogeneous degree-n polynomials in three variables. This is an enumerative question, i.e. it's about counting certain things in geometry.

The space of linear (homogeneous) polynomials in three vairables is a $\mathbb{P}^2$, given by (the projectivization of) $\langle x,y,z\rangle$. The space of quadratics is a $\mathbb{P}^5$, given by the projectivization of

$$\langle x^2, xy, xz, y^2, yz, z^2\rangle. \tag{18.6}$$

In general, the space of degree-k homogeneous polynomials is a projective space of dimension $\binom{n+k}{k}-1$ (the dimension drops when you projectivize), so there's a $\mathbb{P}^9$ of cubics.

So maybe we can describe the class of cubics that split in the Chow ring, and the subvariety of all things that look like $t_0F_0 + t_1F_1 + t_2F_2$, and maybe that will be dimension zero and we can count it and obtain an answer in the question. If we choose general F_0, F_1, F_2, the intersection will be generically transverse and we don't have to worry about moving them around.

So consider the map $\tau\colon \mathbb{P}^2\times\mathbb{P}^5 \to \mathbb{P}^9$, thought of as linear, conics $\mapsto$ cubics: explicitly, $(\ell, q)\mapsto \ell\cdot q$. The image of τ is precisely the reducible cubics. Now introduce some notation.

- Let ζ generate $A(\mathbb{P}^9)$, so $A(\mathbb{P}^9) = \mathbb{Z}[\zeta]/(\zeta^{10})$.
- Let $p\colon \mathbb{P}^2\times\mathbb{P}^5\to\mathbb{P}^2$ and $q\colon \mathbb{P}^2\times\mathbb{P}^5\to\mathbb{P}^5$ be the projection maps.
- Let $\alpha := p^*(H_1)$ and $\beta := q^*(H_2)$, where $H_1\subset\mathbb{P}^2$ and $H_2\subset\mathbb{P}^5$ are hyperplanes. Hence by Theorem 18.2, $A(\mathbb{P}^2\times\mathbb{P}^5)\cong\mathbb{Z}[\alpha,\beta]/(\alpha^3,\beta^6)$.

Now, τ is bilinear in $\mathbb{P}^2\times\mathbb{P}^5$, and if $H\subset\mathbb{P}^9$ is a plane, then $\tau^{-1}(H)$ is cut out by a bidegree-$(1,1)$ equation. Therefore $\tau^*(\zeta) = \alpha+\beta$.

By unique factorization, τ maps birationally onto its image, which we'll call Γ.[20] Now, since $1 = [\mathbb{P}^2\times\mathbb{P}^5]$ in $A(\mathbb{P}^2\times\mathbb{P}^5)$,

$$\begin{aligned}\deg(\Gamma) &= \deg(\tau_*(1)\cdot\zeta^7) = \deg(1\cdot\tau^*(\zeta)^7)\\ &= \deg((\alpha+\beta)^7) = \deg(21\alpha^2\beta^5) = 21.\end{aligned}$$

That is, $[\Gamma] = 21\zeta^2$.

Next, we want to compute the class of all $\{t_0F_0+t_1F_1+t_2F_2\}$. This is a linear embedding of a $\mathbb{P}^2$ inside $\mathbb{P}^9$ – and since we chose F_0, F_1, and F_2 generically, this is a generic $\mathbb{P}^2$, and hence represents the class ζ^2. And therefore

$$\deg([\mathbb{P}^2\cap\Gamma]) = \deg(\zeta^7\cdot 21\zeta^2) = 21. \tag{18.7}$$

Therefore exactly 21 cubics factor, up to rescaling!

Exercise 18.8. Let $F_0,\dots,F_3$ be four general cubics in $\mathbb{C}[x,y,z]$. How many $t_0F_0+\cdots+t_3F_3$, for $[t_0:t_1:t_2:t_3]\in\mathbb{P}^3$, factor as three linear polynomials?

So now we have a map $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\to\mathbb{P}^9$ – but we haven't computed $A(\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1)$. So you could, in view of Remark 18.4, use the Künneth formula to compute

$$A(\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1)\cong\mathbb{Z}[\alpha,\beta,\gamma]/(\alpha^2,\beta^2,\gamma^2). \tag{18.9}$$

Remark 18.10. Observe that an affine stratification is the same as a CW structure on your variety! So whenever you have an affine stratification, the Chow ring will be isomorphic to the cohomology – but this is not often true (and many varieties don't have affine stratifications). For example, K3 surfaces have different cohomology and Chow rings. ◀

So why do we care about all of this intersection theory? Well for Bridgeland stability, we'll need to know what a Chern character is. You can do this purely in cohomology, but then you'd have to use some arguments in differential topology to show that cup product in cohomology is given by intersection (well, after a homotopy to make things transverse), which is more familiar to yours truly, but is not a trivial theorem.

Anyways, on to the Chow-theoretic Chern character. Given a normal variety X and $L\in\operatorname{Pic}(X)$, we have a Weil divisor $D\in A^1(X)$.

[20] TODO: I didn't catch why this follows.

Definition 18.11. The *first Chern class* of L is $D \in A^1(X)$.

There will be higher Chern classes, but for line bundles they all vanish. For more general vector bundles they can be nonzero.

Lecture 19.

Chern classes, the Chern character, and the Todd class: 4/9/19

Today, we're going to continue with Chern classes and the Chern character. Well, we barely started last time, but we're going to keep going.

Chern characters are homomorphisms from the Grothendieck group to the Chow ring, and will therefore be useful for defining Bridgeland stability conditions. Chern classes came first and have easier geometric interpretations, but they're multiplicative, not additive, in short exact sequences so we have to reword them as Chern characters.

So let X be a normal[21] variety. Then $L \in \mathrm{Pic}(X)$ defines a Weil divisor $D \in A^1(X)$. Last time, we defined the first Chern class of L, denoted $c_1(L)$, to be that same element of $A^1(X)$.

Remark 19.1. The first Chern class is additive in short exact sequences, and in fact the first Chern character will just be c_1. But this will not generalize to higher Chern classes. ◂

Definition 19.2. Let X be a smooth variety and $E \to X$ be a rank-r vector bundle. The *projectivization* or *projectivized bundle* of E, denoted $\pi\colon \mathbb{P}(E) \to X$, is the fiber bundle whose fiber at $x \in X$ is $\mathbb{P}(E_x)$, the space of lines through the origin in E_x.

This isn't quite a definition in algebraic geometry; really what one has to do is come up with a sheaf of algebras and take its relative Proj. This means we get a relative tautological bundle, i.e. a line bundle $S_E \to \mathbb{P}(E)$, which when restricted to any $\mathbb{P}(E_x)$ is the tautological bundle of $\mathbb{P}(E_x)$. Let $\zeta := c_1(S_E^*)$.

Theorem 19.3.

(1) *The pullback $\pi^*\colon A(X) \to A(\mathbb{P}(E))$ is injective.*
(2) *$\zeta \in A(P(\mathbb{E}))$ satisfies a unique monic polynomial $f(\zeta) = 0$ of degree r and with coefficients in* $\mathrm{Im}(\pi^*)$.
(3) *$A(\mathbb{P}(E)) \cong A(X)[\zeta]/(f(\zeta))$.*

In particular, ζ cannot satisfy any polynomial of smaller degree than f.

Definition 19.4. The *Chern classes* $c_i \in A^i(X)$ are the unique classes such that

$$f(\zeta) = \sum_{i=0}^{r} \pi^* c_{r-i}(E)\zeta^i. \tag{19.5}$$

The *Chern polynomial* $c_t(E) \in A(X)[t]$ is

$$c_t(E) := \sum_{i=1}^{r} c_i(E) t^i. \tag{19.6}$$

where t is a formal variable.

Since f is monic, $c_0(E) = 1$.

Here are some basic properties of Chern classes. Like the previous theorem, we're not going to prove them.

Theorem 19.7.

(1) *If L is a line bundle, $c_t(L) = 1 + c_1(L)t$.*
(2) *If $0 \to F \to E \to G \to 0$ is a short exact sequence of line bundles, then $c_t(E) = c_t(F)c_t(G)$.*
(3) *If $f\colon X \to Y$ is a map of smooth varieties, then $f^*(c_t(E)) = c_t(f^*E)$.*

The first part is something you can more or less do by hand: the projectivization of a line bundle $L \to X$ is just X again, and the tautological bundle is L. (Really, to prove this one needs more precise definitions than the ones we've written down, but you can find them in Hartshorne.) Also, we've only done this for vector bundles, not general sheaves, which is something that we'll have to deal with later.

Because of part (3), one sometimes says that Chern classes are functorial.

[21] Normality might not be necessary; certainly if you're following Hartshorne it is, but Fulton might have a workaround.

Exercise 19.8. Compute $c_t(T_{\mathbb{P}^n})$. Hint: the tangent bundle fits into a short exact sequence called the Euler sequence whose other terms are built from line bundles, and you can use that to compute.

Next, we'll define the Chern character and Todd class, which are particular combinations of Chern classes. To do so, first assume the Chern polynomial factors:

$$c_t(E) = \prod_{i=1}^{r}(1 + a_i t). \tag{19.9}$$

This may not be true in $A(X)[t]$, but we're going to assume that we can work in a bigger ring where this is true. In the end, we'll only end up with actual elements of $A(X)[t]$, so it will be OK.[22]

Definition 19.10. The *Chern character* of E is

$$\mathrm{ch}(E) := \sum_{i=1}^{r} e^{a_i} \in A(X) \otimes \mathbb{Q},$$

where we interpret the exponential via its power series (which will be zero after some finite step).

Definition 19.11. The *Todd class* of E is

$$\mathrm{td}(E) = \prod_{i=1}^{r} \frac{a_i}{1 - e^{-a_i}} \in A(X) \otimes \mathbb{Q}.$$

Again, we interpret this function as its power series.

The theorem is that the coefficients involve combinations of these a_i which are actually in $A(X)$.

Lemma 19.12.

$$\mathrm{ch}(E) = \mathrm{rank}(E) + c_1(E) + \frac{1}{2}(c_1(E)^2 + c_2(E)) + \frac{1}{6}\Big(c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E)\Big) + \cdots$$
$$\mathrm{td}(E) = 1 + \frac{1}{2}c_1(E) + \frac{1}{12}(c_1(E)^2 + c_2(E)) + \frac{1}{24}c_1(E)c_2(E) + \cdots$$

It turns out the Chern character determines the Chern classes; this is not true for the Todd class, as you can see above: on a 3-fold, we can't see c_3.

Now, let X be a smooth, projective variety and $E^\bullet \in D^b(X)$. Then $E^\bullet \simeq F^\bullet$ where each F^i is locally free. (This heavily uses that X is smooth.)

Definition 19.13. The Chern character of $E^\bullet$ is

$$\mathrm{ch}(E^\bullet) := \sum_i (-1)^i \mathrm{ch}(F^i).$$

This definition is also kind of a proposition – there's plenty to prove here, including that if you choose a different choice of $F^\bullet$, you get the same answer, and that the Chern character is additive in short exact sequences. The intuition for this is that we took something multiplicaive and exponentiated it, so it's additive. That's not a real proof, but it gives you the idea of what happened; the real proof has lots of indices and isn't so deep.

Example 19.14. For $X = \mathbb{P}^n$, let's compute $\mathrm{ch}(\mathscr{O}(m))$. We already know $c_1(\mathscr{O}(m)) = mH$, where H is a hyperplane class, so $c_t(\mathscr{O}(n)) = 1 + mHt$. Here we don't need to factor anything, so

$$\mathrm{ch}(\mathscr{O}(m)) = e^{mH} = 1 + mH + \frac{m^2}{2}H^2 + \cdots + \frac{m^n}{n!}H^n. \tag{19.15}$$ ◂

Suppose E is a vector bundle on $\mathbb{P}^2$; we will use the notation $\mathrm{ch}(E) = (r, c, d)$ to mean $\mathrm{ch}(E) = r + cH + dH^2$. Since H generates $A(\mathbb{P}^2)$ and $H^3 = 0$, this works for E arbitrary.

[22] There are different ways to do this and also sleep comfortably at night: for example, one can actually produce such a bigger ring with these roots by passing to the flag manifold of $E \to X$. Alternatively, there's a principle called the *splitting principle* for line bundles, which can be formulated as a theorem, that allows one to reduce definitions or theorems about vector bundles to the case of line bundles. And for line bundles, there's only one root, and it's already in $A(X)$.

Example 19.16. Let $p \in \mathbb{P}^2$ and $\mathscr{I}_p$ denote the ideal sheaf of the closed embedding $\{p\} \hookrightarrow \mathbb{P}^2$. Then p is cut out by two linear equations ℓ_1 and ℓ_2. There is a locally free resolution

$$0 \longrightarrow \mathscr{O}(-2) \xrightarrow{\begin{pmatrix} \ell_2 \\ -\ell_1 \end{pmatrix}} E \xrightarrow{(\ell_1,\ell_2)} E \longrightarrow 0, \tag{19.17}$$

where $\mathscr{O}(-1)^{\oplus 2}$ arises because we cut p out by two lines, and $\mathscr{O}(-2)$ because these two lines satisfy the relation $\ell_2 \cdot \ell_1 - \ell_1 \cdot \ell_2 = 0$. Therefore we get

$$\begin{aligned} \mathrm{ch}(\mathscr{I}_p) &= 2\mathrm{ch}(\mathscr{O}(-1)) - \mathrm{ch}(\mathscr{O}(-2)) \\ &= 2 \cdot \left(1, -1, \frac{1}{2}\right) - (1, -2, 2) \\ &= (1, 0, -1). \end{aligned}$$

Therefore, using the short exact sequence

$$0 \longrightarrow \mathscr{I}_p \longrightarrow \mathscr{O}_{\mathbb{P}^2} \longrightarrow \mathscr{O}_p \longrightarrow 0, \tag{19.18}$$

we have

$$\begin{aligned} \mathrm{ch}(\mathscr{O}_p) &= \mathrm{ch}(\mathscr{O}_{\mathbb{P}^2}) - \mathrm{ch}(\mathscr{I}_p) \\ &= (1, 0, 0) - (1, 0, -1) \\ &= (0, 0, 1) = [P]. \end{aligned}$$

This last fact holds in more generality: it is true on any variety, not just $\mathbb{P}^2$, as we'll see in Example 19.20 (assuming Theorem 19.19). ◂

Grothendieck tells us to look at everything in a relative way, specifically replacing varieties with nice morphisms. This way of thinking leads to a generalization of the Riemann-Roch theorem in algebraic geometry; the analogue in differential geometry is the Atiyah-Singer index theorem.

Theorem 19.19 (Grothendieck-Riemann-Roch)**.** *Let $f\colon X \to Y$ be a proper morphism, where X and Y are smooth. For all $\alpha \in K_0(X)$,*

$$\mathrm{ch}(f_*\alpha)\mathrm{td}(T_Y) = f_*(\mathrm{ch}(\alpha))\mathrm{td}(T_X).$$

This is the most important reason to care about Todd classes.

To recover the classical Riemann-Roch theorem, let X be a curve and $Y = \operatorname{Spec} \mathbb{C}$, though then there's some more work to do. If you use a more general variety for X, you recover the Hirzebruch-Riemann-Roch theorem (Theorem 19.22).

Example 19.20. Let X be a smooth variety and $p \in X$; we're going to compute $\mathrm{ch}(\mathscr{O}_p)$ using the Grothendieck-Riemann-Roch theorem.

Let $f\colon \{p\} \to X$ be the inclusion map; then $T_{\{p\}}$ is trivial, so $\mathrm{td}(T_{\{p\}}) = 1$. The Todd class of X may be complicated, but at least we know its constant coefficient is 1. Then

$$\mathrm{ch}(f_*\mathscr{O}_p) \cdot \mathrm{td}(T_X) = f_*(\mathrm{ch}(\mathscr{O}_p)\mathrm{td}(T_{\{p\}}) = f_*(1) = [p], \tag{19.21}$$

and this must be $(\mathrm{ch}_0 + \mathrm{ch}_1 + \cdots)(1 + \cdots)$, so comparing degree-by-degree, $\mathrm{ch}_0 = 0$, as $[p]$ has nothing in top degree, and so on, until the last one, which is nonzero: we get $\mathrm{ch}_{\mathrm{top}} \cdot 1 = [p]$. Hence $\mathrm{ch}(\mathscr{O}_p) = [p]$. ◂

This is true for zero-dimensional subschemes, but not generally in higher dimensions.

Theorem 19.22 (Hirzebruch-Riemann-Roch)**.** *Let X be a smooth proper variety and $E \in D^b(X)$. Then*

$$\chi(E) = \deg(\mathrm{ch}(E) \cdot \mathrm{td}(T_X)).$$

This is a corollary of the Grothendieck-Riemann-Roch theorem, but it came first.

Exercise 19.23. Let $C \subset \mathbb{P}^3$ be a genus-g, degree-d curve, and let $\mathscr{I}_C$ denote its sheaf of ideals. Compute $\mathrm{ch}(\mathscr{I}_C)$.

Example 19.24. Let's look at $\mathbb{P}^1$. We have $H^0(\mathscr{O}_{\mathbb{P}^1}) = \mathbb{C}$ and $H^1(\mathscr{O}_{\mathbb{P}^1}) = 0$, so $\chi(\mathscr{O}_{\mathbb{P}^1}) = 1 - 0 = 1$. By Theorem 19.22,

$$1 = \deg(\mathrm{ch}(\mathscr{O}_{\mathbb{P}^1}) \cdot \mathrm{td}(T_{\mathbb{P}^1})) = \deg((1,0) \cdot (1, \mathrm{td}_1)) = \deg((1, 0 \cdot 1 + 1 \cdot \mathrm{td}_1)) = \mathrm{td}_1. \tag{19.25}$$

Hence $\mathrm{td}(T_{\mathbb{P}^1}) = (1,1)$. ◂

Remark 19.26. This trick works on any variety X, allowing you to determine the highest-degree piece of the Todd class just from $\chi(\mathscr{O}_X)$. On $\mathbb{P}^1$, we could compute it directly because we have a good understanding of the tangent bundle, but this may not be true in general. Anyways, this trick may be useful for understanding Todd classes of curves, in view of Exercise 19.23. ◂

Example 19.27. Working on $\mathbb{P}^1$ again, and using Example 19.24 to tell us the Todd class, Theorem 19.22 tells us $\chi(\mathscr{O}(n)) = \deg((1,n) \cdot (1,1)) = \deg((1, n+1)) = n+1$. And this recovers what we know from cohomology: if $n \geq 0$, $h^0(\mathscr{O}(n)) = n+1$ and $h^1(\mathscr{O}(n)) = 0$. When $n = -1$, $h^0(\mathscr{O}(-1)) = 0$ and $h^1(\mathscr{O}(-1)) = 0$. For $n \leq -2$, $h^0(\mathscr{O}(n)) = 0$ and $h^1(\mathscr{O}(n)) = h^0(\mathscr{O}(-n-2)) = -n-1$ by Serre duality. ◂

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