INTRODUCTION TO SPECTRAL SEQUENCES

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Contents

1. Introduction to the general formalism: 5/8/17

Today, Adrian spoke about what a spectral sequence is and where they come from. The next four lectures will be interesting examples, even if today is somewhat dry.

Definition 1.1. A **(homological) spectral sequence** is the data of

- modules over a ring[1] $E^r_{p,q}$ indexed by $r \geq N$ for some positive N and $p, q \in \mathbb{Z}$, and
- maps $d_r \colon E^r_{p,q} \to E^r_{p-r,q-1+r}$, called **differentials**,

subject to the following conditions:

- $d_r^2 = 0$, and
- for all p, q, and r, $E^{r+1}_{p,q}$ is the homology of the chain complex $(E^r_{p-r\bullet,q-1+r\bullet}, d_r)$ at $E^r_{p,q}$.

The way in which the differentials affect the grading is pretty opaque, so let's see what it looks like for small r.

$$\begin{tikzcd} E^0_{p,q} \arrow[d, "d_0"] \\ E^0_{p,q-1} \end{tikzcd} \qquad \begin{tikzcd} E^1_{p-1,q} & E^1_{p,q} \arrow[l, "d_1"'] \end{tikzcd} \qquad \begin{tikzcd} E^2_{p-2,q+1} & & \\ & & E^2_{p,q}. \arrow[ull, "d_2"'] \end{tikzcd}$$

The differentials swing from downward to leftward, and comes closer and closer to pointing northwest.

This is a lot of structure, and one usually visualizes it as a book, with **pages** $E^r_{\bullet,\bullet}$, and each page is thought of as a lattice with the differentials:

$$\begin{tikzcd} & \vdots & \vdots & \vdots & \\ \cdots & E^r_{p+1,q-1} & E^r_{p+1,q} & E^r_{p+1,q+1} & \cdots \\ \cdots & E^r_{p,q-1} & E^r_{p,q} \arrow[ull] & E^r_{p,q+1} \arrow[ull] & \cdots \arrow[ull] \\ \cdots & E^r_{p-1,q-1} & E^r_{p-1,q} \arrow[ull] & E^r_{p-1,q+1}. \arrow[ull] & \cdots \arrow[ull] \\ & \vdots & \vdots & \vdots & \end{tikzcd}$$

The point of this heavy machinery is that there's a machine which takes filtered objects and functors satisfying an excision property to spectral sequences, and such pairs arise in many contexts in algebra, topology, and geometry.

[1]In the general setup, one has to be somewhat agnostic about what these are: in any context where one can do homological algebra, one can define spectral sequences: abelian groups, modules over a ring, objects in an abelian category. . .

Definition 1.2. Let $\mathbb{Z}$ denote the **poset category** of the integers, i.e. there's a unique arrow $m \to n$ iff $m \leq n$. Then, a **filtered object** in a category C is a functor $X : \mathbb{Z} \to \mathsf{C}$.

The idea is a topological space X together with inclusions $X_i \hookrightarrow X_{i+1}$, such that X is the union of all of the X_i. More generally, one can let X be the colimit over i of $X(i)$. One example is the CW filtration of a CW complex X, where $X(n)$ is the n-skeleton of X.

Definition 1.3. Let C be either Top_*, the category of pointed topological spaces, or $\mathsf{Ch}(\mathsf{Mod}_A)$, the category of chain complexes of A-modules for a ring A.

- Let $f : X \to Y$ be a C-morphism, so that we can take its mapping cone C_f and obtain a sequence $X \to Y \to C_f$. If we iterate this construction, $C_{Y \to C_f}$ is weakly equivalent to ΣX, and the mapping cone of this is weakly equivalent to ΣY, so we obtain a sequence
$$X \xrightarrow{f} Y \longrightarrow C_f \longrightarrow \Sigma X \xrightarrow{\Sigma f} \Sigma Y \longrightarrow \Sigma C_f \longrightarrow \dots$$
Such a sequence is called a **cofiber sequence**.[2]
- A **functor satisfying excision** is a covariant or contravariant functor $\mathsf{C} \to \mathsf{Ab}$ taking cofiber sequences to long exact sequences.[3]

To see why $C_{Y \to C_f} \simeq \Sigma X$, one can work with particularly nice maps, so that $Y \to C_f$ is an injection, and its mapping cone crushes Y to a point, producing ΣX. The cofiber C_f is the topological analogue of the quotient Y/X.

Example 1.4. Here are some examples of these functors. First, let $\mathsf{C} = \mathsf{Top}_*$:

(1) Covariant functors $\mathsf{Top}_* \to \mathsf{Ab}$ with excision include homology functors H_n.
(2) For covariant functors sending fiber sequences to long exact sequences, we have homotopy groups π_i.
(3) Contravariant functors with excision include cohomology functors H^n.

For the category of chain complexes, cofiber and fiber sequences are the same thing.

(4) Covariant functors include homology and covariant derived functors such as $\operatorname{Ext}^i(M,–)$ and $\operatorname{Tor}_i(M,–)$.
(5) Contravariant functors include cohomology and contravariant derived functors such as $\operatorname{Ext}^i(–,M)$. ◂

From here, one can draw picture of the argument for why such a functor defines a spectral sequence:

`(Diagram to be made later.)`

From this diagram, one can see how the differentials arise, and they have the grading for the E_2 page. In particular, given the filtration $\{X_p\}$ of X, we can let $E^2_{p,q} \coloneqq H_{p+q}(X_p)$.[4] Thus the E^1 page is

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
H_2(X_0) & \xleftarrow{d_1} & H_3(X_1) & \xleftarrow{d_1} & H_4(X_2) & \longleftarrow & \cdots \\
H_1(X_0) & \xleftarrow{d_1} & H_2(X_1) & \xleftarrow{d_1} & H_3(X_2) & \longleftarrow & \cdots \\
H_0(X_0) & \xleftarrow{d_1} & H_1(X_1) & \xleftarrow{d_1} & H_2(X_2). & \longleftarrow & \cdots
\end{array}$$

The key is explaining how the differentials occur. Let h be a homology theory, $X = \{X_i\}$ be a filtration, and $C_i \coloneqq X_i/X_{i-1}$ be the cofibers. Then we have a diagram

$$\begin{tikzcd}
& h(C_1) & h(C_2) \arrow[l] & h(C_3) \arrow[l] & \\
h(X_0) \arrow[r] & h(X_1) \arrow[r] \arrow[u] & h(X_2) \arrow[r] \arrow[u] & h(X_3) \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

[2]You may prefer to call this a **cofibre sequence**.

[3]There's a version of this for functors taking fiber sequnences to long exact sequences, but we won't need to use it.

[4]Technically, we started only with one functor H, but we can define $H_{n-1}(X) \coloneqq H_n(\Sigma X)$ and extend to a family of functors, just as for homology.

Any pair $\to, \uparrow$ fits into a long exact sequence with connecting morphism $\delta\colon h(C_i) \to h(\Sigma X_{i-1})$:

$$\begin{tikzcd}
 & h(C_1) \arrow[dl, dashed, "\delta"'] & h(C_2) \arrow[l] \arrow[dl, dashed, "\delta"'] & h(C_3) \arrow[l] \arrow[dl, dashed, "\delta"'] & \\
h(X_0) \arrow[r] & h(X_1) \arrow[r] \arrow[u] & h(X_2) \arrow[r] \arrow[u] & h(X_3) \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

This is how the first differentials arise: take the connecting morphism δ, then map back $h(X_{i-1}) \to h(C_{i-1})$. Considering longer sequences of maps after taking homology gives you the higher-order differentials.

What follows was a complicated diagram chase that was hard to live-TEX.

We had the E^1 page and differentials, and after taking homology, we get the E^2 page:

$$\begin{tikzcd}
 & E^2_{0,2} & E^2_{1,2} & E^2_{2,2} \\
 & E^2_{0,1} & E^2_{1,1} \arrow[ull] & E^2_{2,1} \arrow[ull] \\
 & E^2_{0,0} & E^2_{1,0} \arrow[ull] & E^2_{2,0} \arrow[ull]
\end{tikzcd}.$$

2. The Atiyah-Hirzebruch spectral sequence: 5/9/17

Today, I'm going to talk about the Atiyah-Hirzebruch spectral sequence. Last time, we talked about how to construct a spectral sequence from a filtration of a topological space; today, we'll black-box that construction and use it to compute some stuff. Namely, we'll use the CW fibration associated to any CW complex.

Let E^* be a generalized cohomology theory and X be a CW complex. The **Atiyah-Hirzebruch spectral sequence** is a spectral sequence

$$E_2^{p,q} = H^p(X; E^q(\mathrm{pt})) \Longrightarrow E^{p+q}(X).$$

We'll explain what all this actually means.

Convergence. Sometimes you're reading a book and it feels like it goes on forever. It's nice when spectral sequences don't do that. As an example, we'll look at a **first-quadrant spectral sequence**, one where $E_2^{p,q} = 0$ when $p < 0$ or $q < 0$. In this setup, if you pick any (p, q), then after finitely many pages, the differentials are so long that they leave the first quadrant, so you get a sequence $0 \to E^r_{p,q} \to 0$, and therefore when you take homology, nothing changes. Thus it makes sense to say what the end of the spectral sequence is.

Definition 2.1. Whenever it makes sense, we'll define the E_∞**-page** of the spectral sequence to be $E_\infty^{p,q} = E^r_{p,q}$ for $r \gg 0$. One says $E_r^{p,q}$ **converges** or **abuts** to $E_\infty^{p,q}$.

Typically this is something interesting we want to calculate.

Definition 2.2. Let $A_\bullet$ be a graded abelian group together with an exhaustive filtration $\{F_p\}$.

- The **associated graded** of the filtration $\{F_i\}$ is
$$(\mathrm{gr}\, A)_{p,q} \coloneqq F_p A_{p+q} / F_{p-1} A_{p+q}.$$
- A spectral sequence $E_r^{p,q}$ **converges (weakly)** to $A_\bullet$, written
$$E_r^{p,q} \Longrightarrow A_\bullet,$$
if it has an E_∞ page and the E_∞ page is the associated graded of $A_\bullet$.

Remark. There is a notion of **conditional convergence**, due to Boardman, which essentially means "not always weakly convergent, but converges under hypotheses often met in practice." Unfortunately, defining this precisely would be a huge digression. ◀

Generalized cohomology theories. The Atiyah-Hirzebruch spectral sequence is used to compute things which behave like homology or cohomology, but are slightly different: they satisfy all of the Eilenberg-Steenrod axioms except for the dimension axiom. These generalized cohomology theories have been a huge area of focus in algebraic topology in the last half century.

Definition 2.3. A **generalized cohomology theory** (also **extraordinary cohomology theory**) is a collection of functors $h^n\colon \mathsf{Top}_* \to \mathsf{Ab}$ such that:

- Given a map $f\colon A \to X$, let X/A denote its cofiber. There is a natural transformation $\delta\colon h^n(X/A) \to h^{n+1}(A)$ such that the following sequence is long exact:
$$\cdots \longrightarrow h^n(A) \xrightarrow{h^n(f)} h^n(X) \longrightarrow h^n(X/A) \xrightarrow{\delta} h^{n+1}(A) \longrightarrow \cdots$$
- h^n takes wedge sums to direct sums: if $X = \bigvee_i X_i$, then the natural map
$$\bigoplus h^n(X_i) \longrightarrow h^n(X)$$
is an isomorphism.

The dual notion of a **generalized homology theory** is the same, except the differentials go in the other direction. This defines a reduced homology theory, i.e. one for spaces with basepoints.

Example 2.4 (K-theory)**.** Let X be a compact Hausdorff space. Then, the set of isomorphism classes of complex vector bundles on X is a semiring, so we can take its group completion and obtain a ring $K^0(X)$.

The following theorem is foundational and beautiful.

Theorem 2.5 (Bott periodicity)**.** $K^0(\Sigma^2 X) \cong K^0(X)$.

This allows us to promote K^* into a **2-periodic** generalized cohomology theory K^*, called **complex K-theory**, by setting $K^{2n}(X) = K^0(X)$ and $K^{2n+1}(X) = K^0(\Sigma X)$.[5]

Like cohomology, K-theory is **multiplicative**, i.e. it spits out $\mathbb{Z}$-graded rings. However, $K^i(X)$ is often nonzero for negative i.

Exercise 2.6. For example, show that as graded abelian groups, $K^*(\mathrm{pt}) = \mathbb{Z}[t, t^{-1}]$, where $|t| = 2$.

K-theory admits a few variants.

- If you use real vector bundles instead of complex vector bundles, everything still works, but Bott periodicity is 8-fold periodic. Thus we obtain a periodic, multiplicative cohomology theory called **real K-theory**, denoted $KO^*(X)$. Its value on a point is encoded in the **Bott song**.
- Sometimes it will be simpler to consider a smaller variant where we only keep the negative-degree elements. This is called **connective K-theory**, and is denoted ku^* (for complex K-theory) or ko^* (for real K-theory). These are also multiplicative. ◀

Example 2.7 (Bordism)**.** Let X be a space and define $\Omega_n^{\mathrm{O}}(X)$ to be the set of equivalence classes of maps of n-manifolds $M \to X$, where $[f_0\colon M \to X] \sim [f_1\colon N \to X]$ if there's a cobordism $Y\colon M \to N$ and a map $F\colon Y \to X$ extending f_0 and f_1. This is an abelian group under disjoint union, and the collection $\{\Omega_n^{\mathrm{O}}\}$ defines a generalized homology theory called **unoriented bordism**.[6]

The following theorem was the beginning of differential topology.

Theorem 2.8 (Thom)**.** *As graded abelian groups,* $\Omega_n^{\mathrm{O}}(\mathrm{pt}) \cong \mathbb{F}_2[x_2, x_4, x_5, x_6, \dots] = \mathbb{F}_2[x_i \mid i \neq 2^j - 1]$. *Moreover,* Ω_*^{O} *is a direct sum of (suspended) ordinary cohomology theories.*

There's a lot of variations, based on whatever flavors of manifolds you consider. Using oriented manifolds produces **oriented bordism** Ω_*^{SO}, spin manifolds produce **spin bordism** Ω_*^{Spin}, and so forth. These are not direct sums of ordinary cohomology theories in general. ◀

[5]Extending from compact Hausdorff spaces to all of Top is possible, but then one loses the vector-bundle-theoretic description.

[6]The corresponding cohomology theory is called **cobordism**.

2.1. **The definition.** Recall that if X is a CW complex, it has a **CW filtration** in which X_n is the n**-skeleton**, the union of all cells of dimension $\leq n$. Then, X_n/X_{n-1} is a wedge of n-spheres indexed by the n-cells of X. Using this formalism we can define some spectral sequences.

Definition 2.9.

- Let E_* be a generalized homology theory and X be a CW complex. Then, the CW filtration on X induces a spectral sequence of homological type that strongly converges, called the **Atiyah-Hirzebruch spectral sequence**:
$$E^2_{p,q} = H_p(X; E_q(\mathrm{pt})) \Longrightarrow E_{p+q}(X).$$
- Let E^* be a generalized cohomology theory and X be a CW complex. Then, the CW filtration on X induces a spectral sequence of cohomological type that *conditionally* converges, called the **Atiyah-Hirzebruch spectral sequence**:
$$E_2^{p,q} = H^p(X; E^q(\mathrm{pt})) \Longrightarrow E^{p+q}(X).$$

Calculations.

Example 2.10. We'll use the Atiyah-Hirzebruch spectral sequence to compute $K^*(\mathbb{CP}^n)$. Recall that
$$H^p(\mathbb{CP}^k; A) = \begin{cases} A, & p \text{ even} \\ 0, & \text{odd.} \end{cases}$$
Hence
$$E_2^{p,q} = \begin{cases} \mathbb{Z}, & p, q \text{ even, } 0 \leq p \leq 2k \\ 0, & \text{otherwise.} \end{cases}$$
Thus all the differentials are zero! So $E_2^{p,q} \cong E_\infty^{p,q}$. Hence the E_∞ page has no torsion, and therefore $K^*(\mathbb{CP}^n)$ is isomorphic to its associated graded.
$$K^i(\mathbb{CP}^n) = \begin{cases} \mathbb{Z}^{n+1}, & i \text{ even} \\ 0, & \text{otherwise.} \end{cases}$$ ◀

Exercise 2.11. Let Σ be a genus-g orientable closed surface. Compute $K^*(\Sigma_g)$.

Exercise 2.12. What changes when you replace K^* with KO^*?