M392C NOTES: A COURSE ON SEIBERG-WITTEN THEORY AND 4-MANIFOLD TOPOLOGY

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These notes were taken in UT Austin's M392C (A course on Seiberg-Witten theory and 4-manifold topology) class in Spring 2016, taught by Tim Perutz. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Any mistakes in the notes are my own.

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Lecture 1.

Classification problems in differential topology: 1/18/18

"This is my opinion, but it's the only reasonable opinion on this topic."

This course will be on gauge theory; specifically, it will be about Seiberg-Witten theory and its applications to the topology of 4-manifolds. The course website is https://www.ma.utexas.edu/users/perutz/GaugeTheory.html; consult it for the syllabus, assignments, etc.

The greatest mystery in geometric topology is: *what is the classification of smooth, compact, simply-connected four-manifolds up to diffeomorphism?* The question is wide open, and the thoery behaves very differently than the theory in any other dimension.

There's a fascinating bit of partial information known, mostly via PDEs coming from gauge theory, e.g. the *instanton equation* $F_A^+ = 0$ as studied by Donaldson, Uhlenbeck, Taubes, and others. More recently, people have also studied the *Seiberg-Witten equations*

$$D_A\psi = 0 \tag{1.1a}$$

$$\rho(F_A^+) = (\psi \otimes \psi^*)_0. \tag{1.1b}$$

Even without defining all of this notation, it's evident that the Seiberg-Witten equations are more complicated than the instanton equation, and indeed they were discovered later, by Seiberg and Witten in 1994. However, they're much easier to work with — after their discovery, the results of Donaldson theory were quickly reproven, and more results were found, within the decade after their discovery. This course will focus on results from Seiberg-Witten theory.

In some sense, this is a closed chapter: the stream of results on 4-manifolds has slowed to a trickle. But Seiberg-Witten theory has in the meantime found new applications to 3-manifolds, contact topology (including the remarkable proof of the Weinstein conjecture by Taubes), knots, high-dimensional topology, Heegaard-Floer homology, and more. Throughout this constellation of applications, there are many results whose only known proofs use the Seiberg-Witten equations.

The central problem in differential topology is to classify manifolds up to diffeomorphism. To make the problem more tractable, let's restrict to smooth, compact, and boundaryless. An ideal solution would solve the following four problems for some class of manifolds (e.g. compact of a particular dimension, and maybe with some topological constraints).

(1) Write down a set of "standard manifolds" $\{X_i\}_{i \in I}$ such that each manifold is diffeomorphic to precisely one X_i. For example, a list of diffeomorphism classes of closed oriented connected surfaces is given by the sphere and the n-holed torus for all $n \geq 0$.
(2) Given a description of a manifold M, a way to compute invariants to decide for which $i \in I$ $M \cong X_I$. For example, if M is a closed, connected, oriented surface, we can completely classify it by its Euler characteristic.
    A variant of this problem asks for an explicit algorithm to do this when M is encoded with finite information, e.g. a solution set to polynomial equations in $\mathbb{R}^N$ with rational coefficients.
(3) Given M and M', compute invariants to decide whether M is diffeomorphic to M'; once again, there's an algorithmic variant to that problem.
(4) Understand families (fiber bundles) of manifolds diffeomorphic to M. In some sense, this means understanding the homotopy type of the topological group $\mathrm{Diff}(M)$ of self-diffeomorphisms of M.

This is an ambitious request, but much is known in low dimensions. In dimension 1, the first three questions are trivial, and the last is nontrivial, but solved.

Example 1.2. For compact, orientable, connected surfaces, we have a complete solution: a list of diffeomorphism classes is the sphere and $(T^2)^{\# g}$ for all $g \geq 0$, and the Euler characteristic $\chi := 2 - 2g$ is a complete invariant which is algorithmically computable from any reasonable input data, solving the second and third questions. Here, "reasonable input data" could include a triangulation, a good atlas (meaning nonempty intersections are contractible), or monodromy data for holomorphic a branched covering map $\Sigma \to S^2$, where here we're thinking of surfaces as Riemann surfaces, with chosen complex structures. Here, the Riemann-Hurwitz formula can be used to compute the Euler characteristic.

For the fourth question, let $\mathrm{Diff}^+(\Sigma)$ denote the topological group of orientation-preserving self-diffeomorphisms of Σ.

Theorem 1.3 (Earle-Eells)**.**

- *The inclusion* $\mathrm{SO}_3 \hookrightarrow \mathrm{Diff}^+(S^2)$ *is a homotopy equivalence.*
- *The identification* $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$ *defined a map* $T^2 \hookrightarrow \mathrm{Diff}^+(T^2)$ *as translations; this map is a homotopy equivalence into the connected component of the identity in* $\mathrm{Diff}^+(T^2)$*, and* $\pi_0 \mathrm{Diff}^+(T^2) \cong \mathrm{SL}_2(\mathbb{Z})$*.*
- *If* $g > 1$*, every connected component of* $\mathrm{Diff}^+(\Sigma_g)$ *is contractible, and the* mapping class group $\mathrm{MCG}(\Sigma_g) := \pi_0 \mathrm{Diff}^+(\Sigma_g)$ *is a finitely presented infinite group which acts with finite stabilizers on a certain contractible manifold called* Teichmüller space.[1]

So all four questions have satisfactory answers, though understanding the mapping class groups of surfaces is still an active area of research. ◂

Example 1.4. The classification of compact, orientable 3-manifolds looks remarkably similar to the classification of surfaces (albeit much harder!), through a vision of Thurston, realized by Hamilton and Perelman. The solution is almost as complete. The proof uses geometry, and nice representatives are quotients by groups acting on hyperbolic space.

As for invariants, the fundamental group is very nearly a complete invariant.[2] ◂

In higher dimensions, there are a few limitations. Generally, the index set I will be uncountable. For example, there are an uncountable number of smooth 4-manifolds homeomorphic to $\mathbb{R}^4$! So there will be no nice list, and no nice moduli space either. But restricted to compact manifolds, there are countably many classes, which follows from triangulation arguments or work of Cheeger in Riemannian geometry.

The next obstacle involves the fundamental group. If M is presented as an *n-handlebody* (roughly, a CW complex with cells of dimension at most n), there is an induced presentation of $\pi_1(M)$, and if M is compact, this is a finite presentation (finitely many generators, and finitely many relations).

Fact. For each $n \geq 4$, all finite presentations arise from compact n-handlebodies (namely, closed n-manifolds). ◂

This is pretty cool, but throws a wrench in our classification goal.

[1] Heuristically, but not literally, Teichmüller space is a classifying space for this group.
[2] The fundamental group cannot distinguish lens spaces, and that's pretty much the only exception.

Theorem 1.5 (Markov)**.** *There is no algorithm that decides whether a given finite group presentation gives the trivial group.*

The proof shows that an algorithm which could solve this problem could be used to construct an algorithm that solves the halting problem for Turing machines.

Corollary 1.6. *There is no algorithm to decide whether a given n-handlebody, $n \geq 4$, is simply connected.*

This means that a general classification algorithm cannot possibly work for $n \geq 4$; thus, we will have to restrict what kinds of manifolds we classify.

A third issue in higher-dimensional topology is that in dimension $n \leq 3$, there are existence and uniqueness theorems of "optimal" Riemannian metrics (e.g. constraints on their isometry groups), but for $n \geq 5$, this is not true for any sense of optimal; some choices fail existence, and others fail uniqueness. This is discussed further (and more precisely) in Shmuel Weinberger's "Computers, Rigidity, and Moduli," which has some very interesting things to say about the utility of Riemannian geometry to classify manifolds (or lack thereof).

So four dimensions is special, but for many reasons, not just one.

Those setbacks notwithstanding, we can still say useful things.

- We will restrict to closed manifolds.
- We will focus on the simply-connected case, eliding Markov's theorem.[3]

With these restrictions, we have good answers to the first three questions.

Example 1.7. There is a countable list of compact, simply-connected 5-manifolds, and invariants (cohomology, characteristic classes) which distinguish any two. ◀

Example 1.8. Kervaire-Milnor produced a classification of homotopy spheres in dimensions $5 \leq n \leq 18$, and a conceptual answer in higher dimensions, and further work has applied this in higher dimensions. ◀

There is a wider range of conceptual answers to all four questions, more or less explicit, through *surgery theory*, when $n \geq 5$ (surgery theory fails radically in dimension 4). This gives an answer to the following questions.

- Given a finite, n-dimensional CW complex X (where $n \geq 5$), when is it the homotopy type of a compact n-manifold?
- Given a simply-connected compact manifold M, what are the diffeomorphism types of manifolds homotopy equivalent to M? (Again, we need $\dim M \geq 5$.)

Here are necessary and sufficient conditions for the existence question.

- X must be an n-dimensional *Poincaré duality space*, i.e. there is a fundamental class $[X] \in H_n(X;\mathbb{Z})$ which implements the Poincaré duality isomorphism. This basic fact about closed manifolds gets you an incredibly long way towards the answer.
- Next, X must have a tangent bundle — but it's not clear what this means for a general Poincaré duality space. Here we mean a rank-n vector bundle $T \to X$ which is associated to the homotopy type in a certain precise sense: the unit sphere bundle of the stablization of T, considered as a spherical fibration, has to be manifest in X in a certain way.
- If $n \equiv 0 \bmod 4$, there's another obstruction — a certain $\mathbb{Z}$-valued invariant must vanish, interpreted as asking that $T \to X$ satisfies the Hirzebruch signature theorem: the signature of the cup product form on $H^{n/2}(X)$ must be determined by the Pontrjagin classes of T.
- If $n \equiv 2 \bmod 4$, the obstruction is a similar $\mathbb{Z}/2$-valued invariant related to the Arf invariant of the intersection form.
- If n is odd, there are no further obstructions.

That's it. Uniqueness is broadly similar — once you specify a tangent bundle, there are only finitely many diffeomorphism types!

Now we turn to dimension 4, the hardest case. We want to classify smooth, closed, simply-connected 4-manifolds. The first basic invariant (even of 4-dimensional Poincaré duality spaces) is the intersection form Q_P, which we'll begin studying in detail next week. You can realize it as a unimodular matrix modulo integral equivalence. That is, it's a symmetric square matrix over $\mathbb{Z}$ with determinant ± 1, and integral equivalence means up to conjugation by elements of $\mathrm{GL}_b(\mathbb{Z})$.

[3]More generally, one could pick some fixed group G and ask for a classification of closed n-manifolds with $\pi_1(M) \cong G$; people do this, but we won't worry about it.

Theorem 1.9 (Milnor)**.** *The intersection form defines a bijection from the set of homotopy classes of 4-dimensional simply-connected Poincaré spaces to the set of unimodular matrices modulo equivalence.*

So this form captures the entire homotopy type! That's pretty cool.

Theorem 1.10 (Freedman)**.** *The intersection form defines a bijection from the set of homeomorphism classes of 4-dimensional simply-connected topological manifolds to the set of unimodular matrices modulo equivalence.*

Thus this completely classifies (closed, simply-connected) topological four-manifolds. This theorem won Freedman a Fields medal.

The next obstruction, having a tangent bundle, is a mild constraint told to us by Rokhlin.

Theorem 1.11 (Rokhlin)**.** *Let X be a closed 4-manifold. If Q_X has even diagonal entries, then its signature is divisble by* 16*.*

The signature is the number of positive eigenvalues minus the number of negative eigenvalues. Algebra tells us this is already divisible by 8, so this is just a factor-of-2 obstruction, which is not too bad.

But the rest of the story of surgery theory is just wrong in dimension 4. This is where analysis of an instanton moduli space comes in.

Theorem 1.12 (Donaldson's diagonalizability theorem)**.** *Let X be a compact, simply-connected 4-manifold. If Q_X is positive definite, i.e. $xQ_Xx > 0$ for all nonzero $x \in \mathbb{Z}^b$, then Q_X is equivalent to the identity matrix.*

Donaldson proved this theorem as a second-year graduate student!

There's a huge number of unimodular matrices which are positive definite, but not equivalent to the identity; the first example is known as E_8. So this is a strong constraint on their realizability by 4-manifolds.

In subsequent years, Donaldson devised invarinats distinguishing infinitely many diffeomorphism types within a single homotopy class. Then, from 1994 onwards, there came new proofs of these results via Seiberg-Witten theory, which tended to be simpler,[4] and to provide sharper, more general results. We will prove several of these in the second half of the class.

Lecture 2.

Review of the algebraic topology of manifolds: 1/23/18

Though today might be review for some students, it's important to make sure we're all on the same page, and we'll get to the good stuff soon enough. We won't do too many examples today, but will see many in the future.

Cup products. Cup products make sense in a more general sense than manifolds. Let X and Y be CW complexes; then, there is a canonical induced CW structure on $X \times Y$: the product of a pair of discs is homeomorphic to a disc, and we take the cells of $X \times Y$ to be the products of cells of X and cells of Y.

Recall that the *cellular chain complex* $C_*(X)$ is the free abelian group on the set of cells, and the *cellular cochain complex* is the dual: $C^*(X) := \mathrm{Hom}(C_*(X), \mathbb{Z})$.

Proposition 2.1 (Künneth formula)**.** *Let X and Y be CW complexes. There is a canonical isomorphism*

$$C^*(X \times Y) \cong C^*(X) \otimes C^*(Y). \tag{2.2}$$

This follows because the cells of $X \times Y$ are the products of those in X and those in Y. There is an analogue of the Künneth formula for pretty much any kind of (ordinary) cohomology theory.

The *diagonal map* $\Delta\colon X \to X \times X$ sending $x \mapsto (x, x)$ is, annoyingly, *not* a cellular map (i.e. it does not preserve the k-skeleton). However, it is homotopic to a cellular map $\delta\colon X \to X \times X$.

Definition 2.3. The *cup product of cochains* is the map $\smile\colon C^*(X) \otimes C^*(X) \to C^*(X)$ which is the composition

$$C^*(X) \otimes C^*(X) \xrightarrow[(2.2)]{\cong} C^*(X \times X) \xrightarrow{\delta^*} C^*(X).$$

[4]That said, Donaldson's original proof of the diagonalizability theorem stands as one of the most beautiful things in gauge theory.

We haven't said anything about coboundaries, but the cup product plays well with them, and therefore induces a cup product on cellular cohomology, $\smile\colon H^*(X)\otimes H^*(X)\to H^*(X)$. This is an associative map, and it's *graded*, meaning it sends $H^i(X)\otimes H^j(X)$ into $H^{i+j}(X)$. It's unital and *graded commutative*, meaning

$$x\smile y=(-1)^{|x||y|}y\smile x. \tag{2.4}$$

This turns $H^*(X)$ into a graded commutative ring.

Remark. The cup product is *not* graded commutative on the level of cochains. However, there are coherent homotopies between $x\smile y$ and $(-1)^{|x||y|}y\smile x$. ◀

The fact that we had to choose $\Delta\simeq\delta$ is annoying, since it's non-explicit and non-canonical. The cup product in singular cohomology does not have this problem, as you can just work with Δ itself, but the tradeoff is that the Künneth formula is less explicit.

There are a few other incarnations of the cup product which are more geometrically transparent, and this will be useful for us when studying manifolds. These have other drawbacks, of course.

(1) Čech cohomology is a somewhat unintuitive way to define cohomology, but has the advantage of providing a completely explicit formula for the cup product.
(2) de Rham cohomology provides a model for the cup product which is graded-commutative on cochains, but only works with $\mathbb{R}$ coefficients.
(3) The intersection theory of submanifolds is a beautiful model for the cup product, but is not always available.

We'll discuss these in turn.

Čech cohomology. Let M be a manifold,[5] and $\mathfrak{U}=\{U_i\}_{i\in I}$ be an open cover of M. For $J\subset I$, write

$$U_J:=\bigcap_{i\in J}U_i.$$

Definition 2.5. We say that $\mathfrak{U}$ is a *good cover* if it is locally finite and all U_J, $J\neq\varnothing$, are empty or contractible.

In particular, on a compact manifold, a good cover is finite.

Lemma 2.6. *Any manifold admits a good cover.*

There are two standard proofs of this — one chooses small geodesic balls around each point for a Riemannian metric on M, and the other chooses an embedding $M\hookrightarrow\mathbb{R}^N$ and then uses the intersections of small balls in $\mathbb{R}^N$ with M.

There is also a uniqueness (really cofinality) statement.

Lemma 2.7. *Any two good covers of a manifold M admit a good common refinement.*

For $k\in\mathbb{Z}_{\geq 0}$, let $[k]:=\{0,\dots,k\}$.

Definition 2.8. Let $k\in\mathbb{Z}_{\geq 0}$. A *k-simplex* of $\mathfrak{U}$ is a way of indexing a k-fold intersection in $\mathfrak{U}$; specifically, it is an injective map $\sigma\colon[k]\hookrightarrow I$ such that $\mathfrak{U}_{\sigma([k])}$ is nonempty. The set of k-simplices of $\mathfrak{U}$ is denoted $S_k(\mathfrak{U})$.

There is a *boundary map* $\partial_i\colon S_k(\mathfrak{U})\to S_{k-1}(\mathfrak{U})$ which deletes $\sigma(i)$.

Definition 2.9. Let A be a commutative ring. The *Čech cochain complex valued in* A is the cochain complex $\check{C}^*(M,\mathfrak{U};A)$ defined by

$$C^k(M,\mathfrak{U};A):=\prod_{S_k}A$$

and with differential $\delta\colon\check{C}^k(M,\mathfrak{U};A)\to\check{C}^{k+1}(M,\mathfrak{U};A)$ defined by

$$(\delta\eta)(\sigma):=\sum_{i=0}^{k+1}(-1)^{i+1}\eta(\partial_i\sigma),$$

where η is a cochain and $\sigma\colon[k]\hookrightarrow I$.

One can show that $\delta^2=0$, hence define the *Čech cohomology groups* $\check{H}^*(M,\mathfrak{U};A):=\ker(\delta)/\operatorname{Im}(\delta)$.

[5] Čech cohomology works on a more general class of spaces, but we work with manifolds for simplicity.

Proposition 2.10. *Let $\mathfrak{U}$ and $\mathfrak{V}$ be good covers of a manifold M. Then, there is an isomorphism $\check{H}^*(M, \mathfrak{U}; A) \cong \check{H}^*(M, \mathfrak{V}; A)$.*

Proof idea. By Lemma 2.7, $\mathfrak{U}$ and $\mathfrak{V}$ admit a common refinement $\mathfrak{W}$; then, check that a refinement map of good covers induces an isomorphism in Čech cohomology. ⊠

Thus the Čech cohomology is often denoted $\check{H}^*(M; A)$.

In Čech cohomology, there is a finite, combinatorial model for the cup product: let $\alpha \in \check{C}^i$, $\beta \in \check{C}^j$, and $\sigma\colon [i+j] \hookrightarrow I$. Then, we let

$$(\alpha \smile \beta)(\sigma) \coloneqq \alpha(\text{beginning of } \sigma) \cdot \beta(\text{end of } \sigma). \tag{2.11}$$

To be sure, this works in a more general setting (and indeed is the definition of cup product in singular cohomology), but the finiteness of Čech cochains on a compact manifold makes it a lot nicer in this setting. However, it's not at all transparent that the cup product is graded commutative on cohomology.

Theorem 2.12. *There is an isomorphism of graded rings $\check{H}^*(M; A) \cong H^*(M; A)$ (where the latter means cellular cohomology).*

de Rham cohomology. Recall that $\Omega^k(M)$ denotes the space of differential k-forms on a manifold M, and

$$\Omega^*(M) \coloneqq \bigoplus_{k \geq 0} \Omega^k(M).$$

There is an exterior derivative $\mathrm{d}\colon \Omega^*(M) \to \Omega^{*+1}(M)$ with $\mathrm{d}^2 = 0$, so we can define the *de Rham cohomology* $H^*_{\mathrm{dR}}(M) \coloneqq \ker(\mathrm{d})/\operatorname{Im}(\mathrm{d})$ in the usual way.

In this case, the cup product is induced by the wedge product of differential forms

$$\wedge\colon \Omega^i(M) \otimes \Omega^j(M) \to \Omega^{i+j}(M).$$

Proposition 2.13. *The wedge product is graded commutative on differential forms, hence makes $\Omega^*(M)$ into a DGA (differential graded algebra).*

This is really nice, but can only occur in characteristic zero; if you tried to do this over a field of positive characteristic, you would run into obstructions called Steenrod squares to defining a functorial graded-commutative cochain model for cohomology.

Theorem 2.14 (de Rham)**.** *Let $\mathfrak{U}$ be a good cover of a manifold M. Then there is a natural isomorphism of graded $\mathbb{R}$-algebras $H^*_{\mathrm{dR}}(M) \cong \check{H}^*(M, \mathfrak{U}; \mathbb{R})$.*

There are several different ways of proving this. One is to show that they both satisfy the Eilenberg-Steenrod axioms with $\mathbb{R}$ coefficients, and that up to natural isomorphism there is a single cohomology theory satisfying these isomorphisms. Another is to observe that Čech cohomology is a model for sheaf cohomology, and that both Čech and de Rham cohomology are derived functors of the same functor of sheaves on M applied to the constant sheaf valued in $\mathbb{R}$.

An alternative way to prove it, whose details can be found in Bott-Tu's book, is to form the *Čech-de Rham complex*, a double complex $\check{C}^*(M, \mathfrak{U}; \Omega^\bullet)$. Let D^* denote its *totalization*. Then there are quasi-isomorphisms $\check{C}^* \hookrightarrow D^*$ and $\Omega^*(M) \hookrightarrow D^*$ respecting products, hence inducing isomorphisms $\check{H}^* \cong H^*(D^*) \cong H^*_{\mathrm{dR}}(M)$.

Poincaré duality and the fundamental class Poincaré duality is one of the few (relatively) easy facts about topological manifolds, and one of the only things known until the work of Kirby and Siebenmann in the 1970s. Throughout this section, X denotes a nonempty, connected topological manifold of dimension n. For a reference for this section, see May's *A Concise Course in Algebraic Topology.*

Proposition 2.15.

(1) *If $k > n$, $H_k(X) = 0$.*
(2) *$H_n(X) \cong \mathbb{Z}$ if X is compact and orientable, and is 0 otherwise.*

If X is compact, a choice of orientation defines a generator $[X] \in H_n(X)$, called the *fundamental class* of X. A homeomorphism $f\colon X \to Y$ sends $[X] \mapsto [Y]$ if f preserves orientation and $[X] \mapsto -[Y]$ if f reverses orientation. If X is a CW complex with no cells of dimension $> n$ and a single cell e_n in dimension n, then in cellular homology, $[X] = \pm[e_n]$.

There is a *trace map* or *evaluation map* $H^n(X;A) \to A$ sending $c \mapsto \mathrm{eval}(c,[X])$ (that is, evaluate c on $[X]$); in the de Rham model on a smooth manifold, this is the integration map

$$\eta \longmapsto \int_X \eta.$$

The graded abelian group $H_{-*}(X)$ is a graded module over the graded ring $H^*(X)$ via a map called the *cap product*

$$\frown : H^k(X) \otimes H_i(X) \longrightarrow H_{i-k}(X).$$

Place a CW structure on X, and recall that $\delta\colon X \to X \times X$ was our cellular approximation to the diagonal. Then, we can give a cellular model for the cap product:

$$C^*(X) \otimes C_*(X) \xrightarrow{\mathrm{id}\otimes\delta_*} C^*(X) \otimes C_*(X) \otimes C_*(X) \xrightarrow{\mathrm{eval}\otimes\mathrm{id}} C_*(X).$$

Let X and Y be smooth n-manifolds, where X is closed and oriented. Then, $- \frown f_*[X]\colon H^n_{\mathrm{dR}}(Y) \to H^n_{\mathrm{dR}}(X)$ has the explicit model

$$\eta \longmapsto \int_X f^*\eta,$$

showing how the cap product relates to the evaluation map.

Theorem 2.16 (Poincaré duality)**.** *For X a closed, oriented manifold, the map*

$$D_X \coloneqq - \frown [X]\colon H^*(X) \to H_{n-*}(X)$$

is an isomorphism.

For a proof, see May. In the case of smooth manifolds, there's a slick proof using Morse theory; but Poincaré duality is true for topological manifolds as well.

We will let $D^X \coloneqq (D_X)^{-1}$.

Intersections of submanifolds. Intersection theory, though not its relation to the cup product, was discussed in the differential topology prelim. Let X, Y, and Z be closed, oriented manifolds of dimensions n, $n-p$, and $n-q$ respectively, and let $f\colon Y \to X$ and $g\colon Z \to X$ be smooth maps. Let $c_Y \coloneqq D^X(f_*[Y]) \in H^p(X)$, and similarly let $c_Z \coloneqq D^X(g_*[Z]) \in H^q(X)$. We will be able to give a nice interpretation of $c_Y \smile c_Z$.

First, let f' be a smooth map homotopic to f and transverse to g; standard theorems in differential topology show that such a map exists. Transversality means that if $y \in Y$ and $z \in Z$ are such that $f'(y) = g(z)$, then

$$T_xX = Df'(T_yY) + D_g(T_zZ).$$

Let $P \coloneqq Y{}_{f'}\!\times_g X$, which is exactly the space of pairs (y,z) such that $f'(y) = g(z)$; transversality guarantees this is a smooth manifold of codimension $p+q$ in X. The orientations on X, Y, and Z induce one on P, and there is a canonical map $\phi\colon P \to X$ sending $(y,z) \mapsto f'(y) = g(z)$.

Theorem 2.17. *Let $c_P \coloneqq D^X(\phi_*([P]))$. Then, $c_P = c_Y \smile c_Z$.*

If Y and Z are transverse submanifolds of X, P is exactly their intersection. We will use this result frequently.

Intersection of submanifolds gives a geometric realization of the cup product, but only for those classes represented by maps from manifolds; not all homology classes are realized in this way.

Classes of codimension at most 2 always have representatives arising from embedded submanifolds. The idea is that in general, there's a natural isomorphism

$$H^n(X) \cong [X, K(\mathbb{Z}, n)],$$

where brackets denote homotopy classes of maps and $K(\mathbb{Z}, n)$ is an *Eilenberg-Mac Lane space* for $\mathbb{Z}$ in dimension n, i.e. a space whose only nontrivial homotopy group is $\pi_n \cong \mathbb{Z}$. These spaces always exist, and any two models for $K(\mathbb{Z}, n)$ are homotopic.

Usually Eilenberg-Mac Lane spaces are not smooth manifolds, but there are a few exceptions, including $K(\mathbb{Z}, 1) \simeq S^1$. Hence there is a bijection $[X, S^1] \to H^1(X)$. In the de Rham model, this is the map

$$[f] \longmapsto f^*\omega,$$

where $\omega \in H^1(S^1) \cong \mathbb{Z}$ is the generator. Alternatively, you could think of ω as $D^{S_1}[\mathrm{pt}]$, for any choice of $\mathrm{pt} \in S^1$.

Thus, take $f\colon X \to S^1$ to be a smooth map, where X is a closed, oriented manifold. Let $H_t \coloneqq f^{-1}(t) \subset X$, where $t \in S^1$ is a regular value. Then, H_t comes with a co-orientation, hence an orientation, and $[H_t] = D_X(f^*\omega)$. Thus codimension-1 submanifolds are realizable.

In this course, the case of codimension 2 will be more useful.

Proposition 2.18. *Let $\mathbb{CP}^\infty \coloneqq \operatorname{colim}_n \mathbb{CP}^n$ (the union via the inclusions $\mathbb{CP}^n \hookrightarrow \mathbb{CP}^{n+1}$). Then, $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$.*

Hence there is a class $c \in H^2(\mathbb{CP}^\infty)$ and a natural bijection $[X, \mathbb{CP}^\infty] \to H^2(X)$ sending $[f] \mapsto f^*(c)$.

$\mathbb{CP}^\infty$ is not a smooth manifold, but its low-dimensional skeleta are, and this leads to codimension-2 realizability. Specifically, the inclusion $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty$ defines the pullback map $H^2(\mathbb{CP}^\infty) \to H^2(\mathbb{CP}^1) \cong H_0(\mathbb{CP}^1) \cong \mathbb{Z}$. This maps the tautological class c to [pt] for any pt $\in \mathbb{CP}^1$.

Let $f\colon \mathbb{CP}^\infty$ be a map, where X is a smooth, oriented, closed manifold, which is homotopic to a smooth map $\mathbb{CP}^N$ followed by the inclusion $\mathbb{CP}^N \hookrightarrow \mathbb{CP}^\infty$. Let $D \subset \mathbb{CP}^N$ be a hyperplane and $H_D \coloneqq g^{-1}(D)$. Assuming $g \pitchfork D$ (which can always be done for some g in the homotopy class of f), then H_D is a codimension-2 oriented submanifold of X, and $[H_D] = D^X(g^*c)$. Thus codimension-2 homology classes are representable. We will most commonly use this in dimension 4, for which any class $a \in H^2(X)$ is represented by an embedding of a closed, oriented surface $\Sigma \hookrightarrow X$.

In general, realizability is controlled by oriented cobordism, which in higher codimension is different from cohomology, governed by maps into Thom spaces rather than Eilenberg-Mac Lane spaces. This was studied in the 1950s by Rene Thom.