M383C NOTES

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SEPTEMBER 2, 2015

These notes were taken in UT Austin's Math 383c class in Fall 2015, taught by Todd Arbogast. I live-TeXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

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Lecture 1.

General Remarks: 8/26/15

Though the course name is "Methods of Applied Mathematics," this is a misnomer; the course is really about functional analysis.

The course will use the Canvas website (http://canvas.utexas.edu/), and office hours will be after class (modulo lunch), Mondays and Wednesdays from 12:30 to 1:50. Under UT Direct, there's also a CLIPS page, but that's less central to the course.

The textbook is a set of course notes; it hasn't changed much since 2013, so if you have that version, you'll be fine. They'll be ready at the copy center by Friday or Monday.

Homework will be due every week, assigned one Friday, and due the next. The first assignment will be due in a little over a week. We're encouraged to work in groups, but must write up our own individual proofs. Midterms will be weeks 7 and 12, probably, and will be topical; the final, at the end of the semester, will be comprehensive.

In this course, we'll cover chapters 2 – 5 of the lecture notes. Some elementary topology and Lesbegue integration (the first chapter) will be assumed.

Now, for some math. The professor is an applied mathematician, doing numerical analysis, and more specifically, approximation of differential equations. Functional analysis is useful for that, but also plenty of other fields, even including abstract algebra! Nonetheless, the course will be presented from an applied perspective.

The background is that we're trying to solve a problem of the form $T(u) = f$. Here, T is a model or differential equation; it's some kind of operator. f is the data that we're given, and we want to find the solution u. We use the framework of functional analysis to understand the nature of the functions u and f: their properties and what classes of functions they live in. We also want to know the nature of the operator T. In particular, we'll focus on cases where T is linear, since anything nonlinear can usually be locally approximated with a linear one. Thus, we should start with the linear case.

The set of all functions is a vector space, of course, so we're led to study vector spaces. At the undergraduate level, one studies finite-dimensional spaces, but here we'll use infinite-dimensional ones. Vector spaces also give us the required linearity. But since we also have questions of convergence, we'll introduce topology, so this course combines algebra and topology.

In this class, $\mathbb{F}$ will denote a field, either $\mathbb{R}$ or $\mathbb{C}$ (a lot of the time, the stuff we're doing won't depend on which).

Definition. Let X be a vector space over $\mathbb{F}$. Then, X is a *normed linear space* (henceforth NLS) if it has a *norm*, a function $\|\cdot\| : X \to \mathbb{R}^+ = [0, \infty)$ such that for every $x, y \in X$ and $\lambda \in \mathbb{F}$,

- $\|\lambda x\| = |\lambda|\|x\|$,
- $\|x\| = 0$ iff $x = 0$, and

- $\|x+y\| \le \|x\| + \|y\|$.

The last stipulation is called the *triangle inequality*.

These conditions on the norm mean it's a measure of size: stretching a vector stretches the norm, the only thing with size 0 is the origin, and the triangle inequality corresponds to the familiar geometric one. It turns out these are the only properties we need to measure size.

Example 1.1.

(1) d-dimensional *Euclidean space* $\mathbb{F}^d$ comes with a familiar norm: if $x = (x_1, \dots, x_n)$ for $x_j \in \mathbb{F}$, then
$$\|x\| = \sqrt{\sum_{j=1}^{d} |x_j|^2}.$$
Sometimes, this is simply denoted $|x|$. Thus, whenever we talk about $\mathbb{F}^d$, we really mean $(\mathbb{F}^d, \|\cdot\|)$, the normed linear space.

(2) If $a < b$, where $a, b \in [-\infty, \infty]$, let $C([a,b])$ denote the space of continuous functions $f : [a,b] \to \mathbb{F}$ such that $\sup_{x\in[a,b]} |f(x)|$ is finite.[1] This is indeed a vector space; then, it turns to a normed linear space with the norm
$$\|f\| = \sup_{x\in[a,b]} |f(x)|.$$
Notice that the norm must be finite, which is satisfied here. The first two properties are clearly satisfied, and because the absolute value is a norm on $\mathbb{R}$, then the triangle equality is also satisfied.

(3) We can pair $C([a,b])$ with a different norm $\|\cdot\|_{L^1}$, defined by
$$\|f\|_{L^1} = \int_a^b |f(x)|\,\mathrm{d}x.$$
The integral certainly exists, since f is continuous, but it might be infinite; thus, we assume that a and b are finite, so $[a,b]$ is compact, and
$$\int_a^b |f(x)|\,\mathrm{d}x \le (b-a) \sup_{x\in[a,b]} |f(x)|,$$
so we're bounded. It's also not that hard to show that $\|\cdot\|_{L^1}$ is a norm, as the integral is linear.

We now have two norms on $C([a,b])$; are they "the same?" Though the underlying vector spaces are the same, the measures of size are different, so as normed linear spaces they are not the same.

We can find more examples sitting inside other NLSes.

Proposition 1.2. *Let $(X, \|\cdot\|)$ be an NLS and $V \subseteq X$ be a linear subspace. Then, $(V, \|\cdot\|)$ is an NLS.*

It's easy to check that the three requirements are still met.

We can measure size, so since we're in a vector space, we can measure distance. In general, we have a metric. Specifially, if $(X, \|\cdot\|)$ is an NLS, define $d : X \times X \to \mathbb{R}^+$ by $d(x,y) = \|x - y\|$. Why is this a metric? It has to satisfy the following three properties for all $x, y, z \in X$.

(1) $d(x,y) = 0$ iff $x = y$.
(2) $d(x,y) = d(y,x)$.
(3) $d(x,y) + d(y,z) \le d(x,z)$.

It's easy to check that the d induced from the norm is indeed a metric; each metric property follows from one of the norm properties.

And now that we can measure distance, we have a topology; specifically a metric topology, the simplest of all topologies. That is, a normed linear space is a metric space. To be specific, define the *ball of radius r about x*, where $r > 0$ and $x \in X$, is
$$B_r(x) = \{y \in X \mid d(x,y) < r\}.$$
This is an open ball, so the distance must be strictly less than r.

[1] Recall that the *supremum* of a set is its least upper bound: for example, $\sup(0,1) = 1$, even though 1 isn't part of the set. This distinguishes the supremum from the maximum.

The topology is defined by setting $U \subseteq X$ to be *open* if for every $x \in U$, there exists an $r > 0$ such that $B_r(x) \subseteq U$. In other words, an open set doesn't contain its boundary. A set $F \subseteq X$ is *closed* if the complement $F^c = X \setminus F$ is open.

Definition. A subset F of a metric space X is *sequentially closed* if whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in F converging to an $x \in X$ (in the sense of the metric, i.e. $d(x_n, x) \to 0$), then $x \in F$.

In a metric space (this is *not* true in general!), F is closed iff F is sequentially closed.

Now, we have algebra (the vector space), the metric (giving us convergence, compactness, etc.), and the norm. How are they related?

Proposition 1.3. *In an NLS X, addition, scalar multiplication, and the norm are all continuous functions.*

Proof. We'll prove this for addition and the norm; scalar multiplication is analogous to addition.

Addition is a function $+ : X \times X \to X$. Let $\{x_n\} \subseteq X$ with $x_n \to x$ and $\{y_n\} \subseteq X$ with $y_n \to y$. Continuity is equivalent to $\{x_n + y_n\} \to x + y$ for all such sequences. That is, I need $d(x_n + y_n, x + y) \to 0$, but that's equivalent to $\|(x_n + y_n) - (x + y)\| \to 0$.

Since $x_n \to x$ and $y_n \to y$, then $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$. It looks like we should use the triangle inequality.

$$\begin{aligned} \|(x_n + y_n) - (x + y)\| &= \|(x_n - x) + (y_n - y)\| \\ &\le \|x_n - x\| + \|y_n - y\| \to 0. \end{aligned}$$

The norm is a little different. Suppose $x_n \to x$, which means we need to show that $\|x_n\| \to \|x\|$. Well,

$$\begin{aligned} \|x\| &= \|x - x_n + x_n\| \\ &\le \|x - x_n\| + \|x_n\| \\ &\le 2\|x - x_n\| + \|x\|. \end{aligned}$$

Since we've sandwiched $\|x - x_n\|$, then $\lim \|x_n\| = \|x\|$.[2] ⊠

Lecture 2.

Banach Spaces: 8/28/15

Recall that if $(X, \|\cdot\|)$ is an NLS, we have a metric $d(x, y) = \|x - y\|$ and a topology. More generally, if (X, d) is a metric space, $x_n \to x$ is the same as $d(x_n, x) \to 0$. In our case, this means that $\|x_n - x\| \to 0$.

Definition. A sequence $\{x_n\}_{n=1}^{\infty}$ is a *Cauchy sequence* if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.

Here, n and m go to infinity independently, which might be confusing; an alternate way to phrase this is that $\{x_n\}$ is Cauchy if for all $\varepsilon > 0$, there exists an $N = N_\varepsilon > 0$ such that $d(x_n, x_m) \le \varepsilon$ whenever $m, n \ge N$.

In a Cauchy sequence, the terms get closer and closer together, but do they converge? Consider $(0, \infty)$ and $x_n = 1/n$. This is Cauchy, but would converge to 0, which isn't part of our set; in a sense, it's a "hole" in our set. This is annoying.

Definition.

- A metric space X is *complete* if every Cauchy sequence on X converges in X.
- A complete NLS is called a *Banach space*.

We'll also give some properties of subspaces of NLSes.

Definition. Let X be an NLS. A set $M \subseteq X$ is *bounded* if there exists an $R > 0$ such that $M \subseteq \overline{B_R(0)} = \{x : \|x\| \le R\}$.

Equivalently, M is bounded if there's a finite R such that $\|x\| \le R$ for all $x \in M$.

Proposition 2.1. *Every Cauchy sequence in an NLS is bounded.*

[2]This was all that the professor said about the proof that the norm is continuous. Here's an alternate proof in case you, like me, didn't get it: since $x_n \to x$, then for any $n \in \mathbb{N}$, there's an N_n such that if $m \ge N_n$, then $x_m - x \in B_{1/n}(0)$. But that means that $\|x_m - x\| < 1/n$. Since $1/n \to 0$, then $\|x_n - x\| \to 0$ as well.

Proof. The idea is that all but a finite number of points in a sequence are within distance 1 of each other.

Let $\{x_n\}_{n=1}^\infty$ be a Cauchy sequence in an NLS X. By definition (using $\varepsilon = 1$), there's an $N > 0$ such that $\|x_n - X_N\| \le 1$ for all $n \ge N$. Using the triangle inequality, $\|x_n\| \le \|x_N\| + 1$ for all $n \ge N$.

Now, let $M = \max\{\|x_1\|, \dots, \|x_{N-1}\|\}$ and $R = \max\{\|x_N\| + 1, M\}$; both of these are finite sets, and therefore have maxima. Thus, $\|x_n\| \le R$ for all n. ⊠

Even if the limit isn't there, the sequence is still bounded, which is nice. Also, notice how we used the norm; boundedness in metric spaces maybe isn't so interesting.

Example 2.2. Let's give some examples of Banach spaces.

(1) $\mathbb{R}^d$ and $\mathbb{C}^d$, as we learned in elementary real analysis.
(2) $C([a,b])$ with $\|f\| = \sup_{x\in[a,b]}|f(x)|$ is Banach, because a sequence $\{f_n\}$ is Cauchy iff it converges uniformly, and we know the uniform limit of continuous functions is continuous.

$C([a,b])$ with norm

$$\|f\|_{L^1} = \int_a^b |f(x)|\,\mathrm{d}x$$

is *not* complete, and therefore not Banach! This will verify the statement we made last lecture, that these spaces aren't the same. This is interesting behavior, because it doesn't happen in finite dimensions, and is an example of the subtle differences in behavior between finite-dimensional and infinite-dimensional vector spaces.

We'll let $a = -1$ and $b = 1$, though by suitable rescaling or translation this works for any $[a,b]$ with a and b finite.

Let $f_n(x)$ be 1 on $[-1,0]$, then decrease linearly on $[0,1/n]$, and then be 0 on $[1/n,1]$. Then,

$$\begin{aligned}\|f_n - f_m\|_{L^1} &= \int_{-1}^1 |f_n(x) - f_m(x)|\,\mathrm{d}x \\ &= \int_0^1 |f_n(x) - f_m(x)|\,\mathrm{d}x \\ &\le \int_0^1 (|f_n(x)| + |f_m(x)|)\,\mathrm{d}x \\ &= \frac{1}{2n} + \frac{1}{2m}.\end{aligned}$$

This goes to 0, so $\{f_n\}$ is Cauchy. But it converges to the step function

$$f(x) = \begin{cases} 1, & x < 0 \\ 0, & x > 0. \end{cases}$$

This is because

$$\begin{aligned}\|f_n - f\|_{L^1} &= \int_{-1}^1 |f_n(x) - f(x)|\,\mathrm{d}x \\ &= \int_0^1 |f_n(x)|\,\mathrm{d}x = \frac{1}{2n},\end{aligned}$$

which goes to 0, so $f_n \to f$ after all.

This means that when we talk about $C([a,b])$, unless otherwise specified, we'll use the other norm, which makes it into a Banach space.

This situation, where the same vector space has two norms with different topological properties, is actually fairly common.

Definition. Let X be a vector space and $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on X. One says that the two norms are *equivalent* if there exist $c, d > 0$ such that for all $x \in X$, $c\|x\|_1 \le \|x\|_2 \le d\|x\|_1$.

This means that, though they might not agree precisely, the vague notions of "small" and "large" are the same in both norms.

We'll see eventually that all norms on a finite-dimensional space are equivalent, even though we already know that $\|\cdot\|$ and $\|\cdot\|_{L^1}$ are inequivalent on $C([a,b])$. We do know, however, that for $f \in C([0,1])$, $\|f\|_{L^1} \le \|f\|$,[3] but the other bound fails: there is no constant C such that $\|f\| \le C\|f\|_{L^1}$. We'll see this using the sequence $\{f_n\}$, where f_n increases linearly from 0 to n on $[0,1/n]$, decreases on $[1/n,2/n]$, and is 0 elsewhere. This sweeps out a triangle, so $\|f_n\| = n$, but $\|f_n\|_{L^1} = 1$ for all n, and thus no such C exists.

Proposition 2.3. *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two equivalent norms on X. Then, their induced topologies are the same.*

To be precise, the collections of open sets $\mathscr{O}_1$ and $\mathscr{O}_2$ induced from $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively, are identical.

Proof. We'll let $B_r^1(x)$ denote the ball of radius r around x in $\|\cdot\|_1$, and define $B_r^2(x)$ similarly.

Since $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, there exist c and d such that for any x and r, $B^1_{r/d}(x) \subseteq B^2_r(x) \subseteq B^1_{r/c}(x)$. Thus, if O_2 is any open set in $\mathscr{O}_2$, then for any $x \in O_2$, there's an r such that $B^2_r(x) \subseteq O_2$, and therefore $B^1_{r/d}(x) \subseteq O_2$, and so O_2 is open in $\mathscr{O}_1$, and the argument in the other direction is similar. ⊠

Convexity. Convexity is an important notion because it allows us to talk about the line joining two points.

Definition. Let X be a vector space over $\mathbb{F}$. Then, a set $C \subseteq X$ is convex if whenever $x, y \in C$, the line $\{tx + (1-t)y : 0 \le t \le 1\}$ is contained in C.

Proposition 2.4. *In any NLS, $B_r(x)$ is convex.*

Proof. Let $y, z \in B_r(x)$ and $t \in [0,1]$. We want to show that $ty + (1-t)z \in B_r(x)$. We'll have to write x as $x + tx - tx$ and then use the triangle inequality. Specifically,

$$\begin{aligned} \|ty + (1-t)z - x\| &= \|t(y-x) + (1-t)(z-x)\| \\ &\le t\|y-x\| + (1-t)\|z-x\| \\ &< tr + (1-t)r = r. \end{aligned}$$ ⊠

This is more interesting than it looks, because in some spaces that are otherwise similar to NLSes, there exist balls that are non-convex.

Even in finite dimensions, balls aren't necessarily round; they can even be square! But that doesn't make much of a difference.

Linear Operators. We'll talk about linear operators in order to manipulate and transform functions.

Definition. A *linear operator* is a function $T : X \to Y$ of vector spaces X and Y such that

(1) $T(x+y) = T(x) + T(y)$, and
(2) $T(\lambda x) = \lambda T(x)$.

The idea is that scalar multiplication and addition in X and Y (which are *a priori* very different) are considered the same by T, which commutes with them.

Definition. A linear operator $T : X \to Y$, where X and Y are NLSes, is *bounded* if it takes bounded sets to bounded sets.

That is, if $C \subseteq X$ is bounded, then $T(C) = \{y : y = T(x) \text{ for some } x \in C\}$.

The definition is nice, but everybody thinks of bounded operators by the following characterization.

Proposition 2.5. *Let X and Y be normed linear spaces and $T : X \to Y$ be linear. Then, T is bounded iff there exists an $C > 0$ such that $\|Tx\|_Y \le C\|x\|_X$ for all $x \in X$.*

Proof. First, suppose T is bounded. Then, the image of $B_1(0)$ (in X) is some bounded set, and therefore contained in a ball $B_R(0)$ for some R. In particular, if $y \in B_1(0)$, then $\|Ty\|_Y \le R$.

Given $x \in X$, if $x = 0$ then $Tx = 0$, so we're good. If $x \ne 0$, let $y = (1/2\|x\|_X) \cdot x$, so that $\|y\| = 1/2$, and therefore $y \in B_1(0)$, and therefore $\|Ty\| \le R$. That is,

$$\left\| T\left(\frac{1}{2\|x\|} \|x\| \right) \right\| = \frac{1}{2\|x\|} \|Tx\| \le R,$$

and therefore $\|Tx\| \le 2R\|x\|$, so with $C = 2R$ we're done.

Conversely, suppose there exists a $C > 0$ such that $\|Tx\| \le C\|x\|$ for all $x \in X$. Let $M \subseteq X$ be bounded; then, $M \subseteq B_R(0)$ for some R. For an $x \in M$, $\|Tx\| \le C\|x\| \le CR$, so $T(X) \subseteq B_{CR}(0)$ in Y, and thus T is bounded. ⊠

[3] More generally, on $C([a,b])$, $\|f\|_{L^1} \le (b-a)\|f\|$.

Lecture 3.

Bounded Linear Operators: 8/31/15

Let X and Y be normed linear spaces; the maps between them that we'll consider are linear operators $T : X \to Y$, as in the previous lecture.

If T is one-to-one and onto, then we should have an inverse $T^{-1} : Y \to X$. It's easy to check that T^{-1} is linear; you probably checked this as an undergraduate. In this situation, we have structure preservation: it doesn't matter whether you check addition in X or in Y, or scalar multiplication. Thus, in the sense of linear algebra, X and Y look the same; they have the same addition and scalar multiplication. In this case, we say that X and Y are *isomorphic*; they may be unequal as sets (e.g. sequences or functions), but identical from the perspective of linear algebra.

For vector spaces, these maps are pretty cool, but for topology, we care about continuous maps $f : X \to Y$. Thus, as you might guess, when studying normed linear spaces, we care about maps $X \to Y$ that are both linear and continuous.

Definition. If X and Y are NLSes, then $B(X,Y)$ denotes the set of functions $f : X \to Y$ that are both linear and continuous.

Continuity means that for all $\varepsilon > 0$ there exists a $\delta > 0$ depending on x and ε such that when $d(x,y) < \delta$, then $d(f(x), f(y)) \le \varepsilon$. But since there's a norm defining the metric, this is equivalent to stating that when $\|x-y\| < \delta$, then $\|f(x)-f(y)\| \le \varepsilon$. And if $f = T$ is a linear operator, then $\|T(x)-T(y)\| < \varepsilon$ is equivalent to requiring $\|T(x-y)\| \le \varepsilon$. In other words, this doesn't depend on x at all: letting $z = x - y$, continuity of a linear $T : X \to Y$ means that when $\|z\| < \delta$, then $\|Tz\| \le \varepsilon$.

In other words, if you know what a linear map does around 0, you know what it looks like everywhere.

Proposition 3.1. *Let X and Y be NLSes and $T : X \to Y$ be linear. Then, the following are equivalent:*

(1) T is continuous.
(2) T is continuous at some $x_0 \in X$.
(3) T is bounded.

This is why we used the notation $B(X,Y)$: it stands for "bounded." And we can now talk about bounded linear maps, with continuity understood.

Proof. Clearly, (1) $\implies$ (2). For (2) $\implies$ (3), suppose T is continuous at some $x_0 \in X$. With $\varepsilon = 1$, this means there's a $\delta > 0$ such that $\|x - x_0\| \le \delta$ implies $\|Tx - Tx_0\| \le 1$, i.e. $\|T(x-x_0)\| \le 1$. In other words, with $z = x - x_0$, when $\|z\| \le \delta$, we have $\|Tz\| \le 1$.

For $x = 0$ boundedness is clear, but if $x \ne 0$, then

$$\begin{aligned}\|Tx\|_Y &= \left\| \frac{\|x\|}{\delta} T\left(\frac{\delta x}{\|x\|}\right)\right\|_Y \\ &= \frac{\|x\|}{\delta}\left\| T\left(\frac{\delta x}{\|x\|}\right)\right\| \le \frac{1}{\delta}\|x\|_X,\end{aligned}$$

so with $C = 1/\delta$, T is a bounded operator.

For (3) $\implies$ (1), we know $\|Tx\|_Y \le C\|x\|_X$ for some fixed C and all $x \in X$. Let $\varepsilon > 0$ and pick any $x_0 \in X$. Then, if $\delta = \varepsilon/C$ and $\|x - x_0\| \le \delta$, then

$$\|T(x-x_0)\| \le C\|x-x_0\| \le C\delta = \varepsilon,$$

so T is continuous at x_0 and therefore everywhere. ⊠

It turns out $B(X,Y)$ is a vector space itself, with $(f+g)(x) = f(x) + g(x)$ and $(\lambda \cdot f)(x) = \lambda \cdot (f(x))$, which is little surprise. But we do have to check that if $f = T$ and $g = S$ are linear, $f + g$ and λf are also linear, i.e. $(T+S)(x+y) = (T+S)(x) + (T+S)(y)$, and similarly for scalar multiplication.

What makes this more interesting is that $B(X,Y)$ is an NLS itself. What's the norm, you ask? Excellent question. The norm is

$$\|T\| = \|T\|_{B(X,Y)} = \sup_{x \in B_1(0)} \|Tx\|_Y.$$

Since T is continuous and bounded, $T(B_1(0))$ is a bounded set. Then, the norm of T is the radius of the smallest ball that contains $T(B_1(0))$, which is the supremum of the amount that T scales any point in the unit ball. Since T is bounded, the norm is a finite, nonnegative number.

Note that, even though we called this a norm, we still have to check that it's a norm!

Proposition 3.2. *Let X and Y be NLSes. Then, $\|\cdot\|_{B(X,Y)}$ is a norm on $B(X,Y)$. Moreover, if $T \in B(X,Y)$,*

$$\|T\| = \sup_{\|x\|_X \le 1} \|Tx\|_Y = \sup_{\|x\|_X = 1} \|Tx\|_Y = \sup_{x \ne 0} \frac{\|Tx\|_X}{\|x\|_X}.$$

Furthermore, if Y is Banach, then $B(X,Y)$ is too.

This last point is quite interesting: completeness follows when the range is complete, but the domain doesn't matter.

Proof. First, that $\|\cdot\|$ is a norm: we have three properties to show.

- We need $\|T\| = 0$ iff $T = 0$. Clearly, if $T = 0$ (i.e. $T(x) = 0$ for all x), then $\|T\| = \sup_{x \in B_1(0)} \|Tx\| = \|0\| = 0$. Conversely, if we assume $\|T\| = 0$, then for any $x \in B_1(0)$, $\|Tx\| = 0$, so $Tx = 0$. Thus, $T|_{B_1(0)} = 0$. For general x, we'll scale $x = 2\|x\|(x/2\|x\|)$, so
$$Tx = 2\|x\| T\left(\frac{x}{2\|x\|}\right) = 2\|x\| \cdot 0 = 0,$$
since $x/2\|x\| \in B_1(0)$. Thus, $T = 0$.
- For linearity of the norm,
$$\|\lambda T\| = \sup_{x \in B_1(0)} \|\lambda Tx\| = \sup_{x \in B_1(0)} |\lambda| \|Tx\| = |\lambda| \sup_{x \in B_1(0)} \|Tx\| = |\lambda| \|T\|.$$

Exercise. Finish the proof that this is a norm by addressing the triangle inequality, which isn't too complicated.

Next, we have the different ways of calculating the norm. The idea is that since T is continuous, the supremum shouldn't depend on whether the boundary is present or not. One interesting corollary of the formulas for calculating $\|T\|$ is that for any $x \in X$, $\|Tx\| \le \|T\| \|x\|$.

The last part does require care. Let $\{T_n\}_{n=1}^\infty$ be a Cauchy sequence. That is, given an $\varepsilon > 0$, there's an $N > 0$ such that if $m, n \ge N$, then $\|T_n - T_m\|_{B(X,Y)} \le \varepsilon$. Thus, given an $x \in X$, $\|T_n x - T_m x\|_Y \le \|T_n - T_m\| \|x\|_X$. The right-hand side goes to 0 as a Cauchy sequence in m and n, and therefore the left-hand side does too. That is, $\{T_n x\}_{n-1}^\infty \subset Y$ is a Cauchy sequence. Since Y is Banach, this means there's a limit $\lim_{n\to\infty} T_n x = T(x) \in Y$. This defines a map $T : X \to Y$; we need to prove that it's bounded linear and that $T_n \to T$.

First, let's look at linearity.

$$T(x+y) = \lim_{n\to\infty} T_n(x+y) = \lim_{n\to\infty} (T_n x + T_n y).$$

Since addition is continuous, we can break this up as

$$= \lim_{n\to\infty} T_n x + \lim_{n\to\infty} T_n y = Tx + Ty.$$

Similarly, since scalar multiplication is continuous,

$$T(\lambda x) = \lim_{n\to\infty} T_n(\lambda x) = \lambda T(x).$$

Next, let's check that T is bounded. Since the norm is continuous,

$$\begin{aligned} \|Tx\|_Y &= \left\| \lim_{n\to\infty} T_n x \right\|_Y \\ &= \lim_{n\to\infty} \|T_n x\|_Y. \end{aligned}$$

However, this limit *a priori* might not exist, so we have to use the lim sup.

$$\begin{aligned} &\le \limsup_{n\to\infty} \|T_n\| \|x\|_X \\ &= M\|x\|_X. \end{aligned}$$

Here, M is an upper bound on $\|T_n\|$, because $\{T_n\}$ is Cauchy and therefore bounded. Thus, we know $T \in B(X,Y)$.

Finally, to show $T_n \to T$, we need to be careful: limits depend on the topology that we're using, and so we should be careful that we're using the topology defined by $\|\cdot\|_{B(X,Y)}$.

Let $x \in B_1(0)$. Then,

$$\begin{aligned} \|Tx - Ty\|_Y &= \lim_{m\to\infty} \|T_m x - T_n x\| \\ &= \lim_{m\to\infty} \|(T_m - T_n)x\| \\ &\le \limsup_{m\to\infty} \|T_m - T_n\|\|x\|. \end{aligned}$$

Since $\{T_n\}$ is Cauchy, then for any $\varepsilon > 0$, $\|T_m - T_n\| \le \varepsilon$ when m, n are sufficiently large, and therefore the lim sup goes to 0 as $n \to \infty$, and so $T_n \to T$. ⊠

There's one particularly important case, in which $Y = \mathbb{F}$.

Definition. The *dual space* of an NLS X is $X^* = B(X, \mathbb{F})$.

By Proposition 3.2, X^* is always a Banach space.

Though $B(X, Y)$ can be complicated for general Y, one can often understand it more easily using X^*.

Example 3.3. We can connect this with finite-dimensional linear algebra that we're more familiar with, and see that it's actually quite special.

Let X be a d-dimensional vector space over $\mathbb{F}$ with basis $\{e_n\}_{n=1}^d$. Thus,

$$\begin{aligned} X &= \operatorname{span}\{e_1, \dots, e_d\} \\ &= \{\alpha_1 e_1 + \cdots + \alpha_d e_d \mid \alpha_i \in \mathbb{F}\}, \end{aligned}$$

and we can write $x = x_1 e_1 + \cdots + x_d e_d \in X$. The map $T : X \to \mathbb{F}^d$ sending $x \mapsto (x_1, \dots, x_d)$ is one-to-one, onto, and linear, so all finite-dimensional vector spaces over a specified field are isomorphic. Moreover, we showed that all norms over a finite-dimensional vector space are equivalent, so as NLSes, they're all isomorphic too! There are many norms, which may still be interesting, but there's only one topology.

Lecture 4.

ℓ^p-norms: 9/2/15

Recall that we were looking at examples of Banach spaces, and that the first examples we saw (Example 3.3) were finite-dimensional vector spaces. If $d = \dim X$ is finite, so that $X = \operatorname{span}\{e_1, \dots, e_n\}$ (which is a basis for X), then the map $T : X \to \mathbb{F}^d$ sending $(x_1 e_1 + \cdots + x_d e_d) \mapsto (x_1, \dots, x_d)$ is an isomorphism of vector spaces, and the claim is that these maps define the same topology as well.

But first, let's define some norms on $\mathbb{F}^d$. Let $1 \le p \le \infty$, and define

$$\|x\|_{\ell^p} = \begin{cases} \left(\displaystyle\sum_{n=1}^d |x_n|^p \right)^{1/p}, & p < \infty \\ \max_n |x_n|, & p = \infty. \end{cases}$$

Sometimes, these are denoted $\|x\|_{\ell_p}$. Also, the case $p = 2$ is our familiar Euclidean norm $\|x\|_{\ell^2} = |x|$.

We do have to show that these are norms. When $p = 1, \infty$, it's a straightforward check, and when $1 < p < \infty$, the first two properties are pretty simple, but the triangle inequality is harder.

Lemma 4.1 (Young's inequality[4]). *Let $1 < p < \infty$ and q be the* conjugate exponent *defined such that $1/p + 1/q = 1$. If $a, b \ge 0$, then $ab \le a^p/p + b^q/q$, with equality iff $a^p = b^q$. Moreover, for all $\varepsilon > 0$, there exists a C depending on p and ε such that $ab \le \varepsilon a^p + C b^q$.*

Proof. The proof is easy once you know the trick, to look at the right function. Let $u : [0, \infty) \to \mathbb{R}$ send

$$u(t) = \frac{t^p}{p} + \frac{1}{q} - t.$$

Its derivative is well-defined: $u'(t) = t^{p-1} - 1$, so $u'(0) = 1$. In particular, $u(0) = 1/q$, and $u(1) = 0$ is a strict minimum.

[4] Young's inequality technically refers to a more general statement; this could be called "Young's inequality for products."

We'll apply this to $t = ab^{-q/p}$:

$$\begin{aligned} 0 \le u(ab^{-q/p}) &= \frac{a^p}{pb^q} + \frac{1}{q} - \frac{a}{b^{q/p}} \\ &= \frac{1}{b^q}\left(\frac{a^p}{p} + \frac{b^q}{q} - \frac{ab^q}{b^{q/p}}\right), \end{aligned}$$

but $b^q/b^{q/p} = b$, since $q - q/p = q(1-1/p) = 1$. Thus, $0 \le a^p/p + b^q/q - ab$, and equality holds iff $t = ab^{-q/p} = 1$, where $u(t)$ is equal to 0.

For the second part, we can write

$$ab = \left((\varepsilon p)^{1/p} a\right)\left((\varepsilon p)^{-1/p} b\right) \le \frac{\varepsilon p a^p}{p} + \frac{(\varepsilon p)^{-q/p}}{q} b^q. \qquad \boxtimes$$

For conjugate exponents, we have the convention that the conjugate of 1 is ∞, and vice versa.

Theorem 4.2 (Hölder's inequality). *Let $1 \le p \le \infty$ and q be its conjugate exponent. If $x, y \in \mathbb{F}^d$, then*

$$\sum_n |x_n y_n| \le \|x\|_{\ell^p} \|y\|_{\ell^q}.$$

When $p = 2$, this is also known as the *Cauchy-Schwarz inequality*.

Proof. The cases $p = 1, \infty$ are trivial or $x, y = 0$; expand their definitions out. Thus, we're left with $1 < p < \infty$, so we can use Lemma 4.1.

Let $a = |x_n|/\|x\|_{\ell^p}$ and $b = |y_n|/\|y\|_{\ell^q}$. Then, by Lemma 4.1,

$$\frac{|x_n|}{\|x\|_{\ell^p}} \frac{|y_n|}{\|y\|_{\ell^q}} \le \frac{|x_n|^p}{p\|x\|_{\ell^p}^p} + \frac{|y_n|^q}{q\|y\|_{\ell^q}^q},$$

so summing all n entries,

$$\frac{\sum_n |x_n y_n|}{\|x\|_{\ell^p}\|y\|_{\ell^q}} \le \frac{1}{p} + \frac{1}{q} = 1. \qquad \boxtimes$$

TODO: this went by a little too fast.

Now, we can use this to prove the triangle inequality for $\|\cdot\|_{\ell^p}$. We'll need two things for the Hölder inequality, so just take one term out of the p^{th} power:

$$\begin{aligned} \|x+y\|_{\ell^p}^p &= \sum_{n=1}^d |x_n + y_n|^p \\ &\le \sum_{n=1}^d |x_n + y_n|^{p-1}(|x_n| + |y_n|) \\ &\le \left(\sum_{n=1}^d |x_n + y_n|^{(p-1)q}\right)^{1/q} (\|x\|_{\ell^p} + \|y\|_{\ell^q}). \end{aligned}$$

Since p and q are conjugate, $p = (p-1)q$, so the first term is $\|x - y\|_{\ell^p}^{p/q}$. Thus,

$$\|x+y\|_{\ell^p}^{p-p/q} \le \|x\|_{\ell^p} + \|y\|_{\ell^p},$$

and $p - p/q = 1$, so we're done.

Moreover, all these norms are equivalent.

Proposition 4.3. *Let $1 \le p \le \infty$. Then, for all $x \in \mathbb{F}^d$,*

$$\|x\|_{\ell^\infty} \le \|x\|_{\ell^p} \le d^{1/p}\|x\|_{\ell^\infty}.$$

These estimates are sharp: the first at $x = (1, 0, 0 \dots, 0)$, and the second at $x = (1, 1, \dots, 1)$. TODO: add a proof?

Notice that this proof method fails horribly in infinite dimensions.

It turns out that on all finite-dimensional vector spaces, all norms are equivalent.

Proposition 4.4. *All norms on a finite-dimensional NLS are equivalent. Moreover, a $K \subset X$ is compact iff it is closed and bounded.*

That means there's only one topology.

Proof. Let $d = \dim X$ and $\{e_n\}_{n=1}^d$ be a basis. Then, let $T : X \to \mathbb{F}^d$ be the coordinate map defined above. Let $\cong$ denote an isomorphism of NLSes.

We'll define a norm $\|\cdot\|_1$ on x by $\|x\|_1 = \|Tx\|_{\ell^1}$: of the three properties, the last two are trivial (since T is linear), so we just need to prove that $\|x\|_1 = 0$ iff $x = 0$. But T is one-to-one and onto, so this follows, and $\|\cdot\|_1$ is in fact a norm.[5]

Thus, $(X, \|\cdot\|_1) \cong (\mathbb{F}^d, \|\cdot\|_{\ell^1})$, so they really are the "same" space. This is because $T : X \to \mathbb{F}^d$ is a bounded map, with $C = 1$, and therefore continuous, and T^{-1} is also linear and continuous. Thus, T is an isomorphism of vector spaces and a homeomorphism of topological spaces, so we can take results in $\mathbb{F}^d$ and apply them to X.

The Heine-Borel theorem from undergraduate real analysis tells us that $K \subset \mathbb{F}^d$ is closed and bounded iff it's compact. But since X and $\mathbb{F}^d$ have the same topology, then this is also true in X. In particular, $S_1^1 = \{x \in X : \|x\|)1 = 1\}$ is also compact.

Now, for any norm $\|\cdot\|$ on X and $x \in X$,

$$\|x\| = \left\| \sum_{n=1}^d x_n e_n \right\| \le \sum_{n=1}^d |x_n| \|e_n\| \le C\|x\|_1,$$

where $C = \max_n \|e_n\|$. Notice that this step won't work in infinite dimensions. Our upper bound implies that $(Top)_{\|\cdot\|} \subseteq (Top)_{\|\cdot\|_1}$, so the former topology is said to be stronger. We'll prove the two are equal by providing a lower bound.

We have a continuous map $\|\cdot\| : (X, \|\cdot\|_1) \to \mathbb{R}$. It's also continuous as a map $\|\cdot\| : (X, \|\cdot\|) \to \mathbb{R}$. Let $a = \inf_{x \in S_1^1} \|x\|$; since S^1 is compact and the norm is continuous, the minimum is attained, and it must be positive (because $0 \notin S_1^1$).

Thus, for any $x \in X$, $\|x/\|x\|_1\| \ge a$, so $\|x\| \ge a\|x\|_1$, which is our desired lower bound. ⊠

Corollary 4.5. *If X is a d-dimensional NLS, then $X \cong \mathbb{F}^d$.*

Corollary 4.6. *If X and Y are NLSes and X is finite-dimensional, then every linear $T : X \to Y$ is bounded and $X^* = \mathbb{F}^d$, given by $T(x) = y \cdot x$.*

[5] A great way to create a new norm is to map from one space to another (or the same one) and pull the norm back.