GEOMETRY AND STRING THEORY SEMINAR: SPRING 2019

ARUN DEBRAY
FEBRUARY 20, 2019

These notes were taken in UT Austin's geometry and string theory seminar in Spring 2019. I live-TeXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

Contents

1. Anomalies and extended conformal manifolds: 1/23/19

These are Arun's prepared notes for his talk, on the paper "Anomalies of duality groups and extended conformal manifolds" [STY18] by Seiberg, Tachikawa, and Yonekura.

1.1. Generalities on anomalies.

As we've seen previously in this seminar, if you ask four people what an anomaly in QFT is, you'll probably get four different answers. Here are some of them.

- An anomaly means the action isn't invariant under the gauge group.
- An anomaly is an obstruction to coupling the theory to a background G-symmetry, or to gauging such a symmetry.
- An anomaly is realized in the nonvanishing of an anomaly polynomial.
- An anomaly as a *relative field theory* as advocated by Freed-Teleman [FT14]: within the framework of functorial QFT, consider an invertible $(n+1)$-dimensional QFT $\alpha\colon \mathsf{Bord}_n \to \mathsf{C}$; a QFT relative to α is a morphism $Z\colon \mathbf{1} \to \tau_{\le n}\alpha$ (i.e., truncate α). The upshot is that the partition function of on a closed n-manifold X isn't a number, but rather an element of the line $\alpha(X)$, and so on.
- Building on this is the idea that every quantum field theory has an anomaly, and if the anomaly is trivial, trivializing it is data that manifests in choices in studying the theory.

Seiberg, Tachikawa, and Yonekura introduce another perspective! But fortunately they relate it to most of the perspectives above. The idea is to consider a QFT with a parameter space $\mathcal{M}$, or a family of QFTs over $\mathcal{M}$. For example, you might have a parameter in a space $\widehat{\mathcal{M}}$ acted on by a group G, and then $\mathcal{M} = \widehat{\mathcal{M}}/G$. Fixing a spacetime manifold X, one expects the partition function to be a function on $\mathcal{M}$, but for an anomalous theory, this doesn't quite work (e.g. if the partition function as a function on $\widehat{\mathcal{M}}$ isn't G-invariant).

One way to fix this is to consider a space $\mathcal{F}$ of counterterms; the total space $\mathcal{N}$ will then be a fiber bundle over $\mathcal{M}$ with fiber $\mathcal{F}$. If constructed correctly, the partition function is then a function on $\mathcal{N}$, and the anomaly manifests in the fact that $\mathcal{N} \to \mathcal{M}$ isn't the trivial $\mathcal{F}$-bundle, and the function doesn't descend to $\mathcal{M}$. Alternatively, one can descend it as a section of a line bundle on $\mathcal{M}$ rather than a function.

The point is: these perspectives are all related. Today, we'll follow Seiberg-Tachikawa-Yonekura as they discuss an $\mathrm{SL}_2(\mathbb{Z})$-anomaly on 4D Maxwell theory from several of these perspectives.

1.2. **An anomalous $\mathrm{SL}_2(\mathbb{Z})$-symmetry in 4D Maxwell theory.** Now, we'll study these ideas as applied specifically to Maxwell theory in dimension 4. Throughout, we will restrict to spin 4-manifolds; everything still works when generalized to oriented manifolds, but the details are more complicated. Consult Seiberg-Tachikawa-Yonekura to learn what changes.

Maxwell theory is pure 4D U_1 gauge theory. The action on a 4-manifold X is

$$S = \frac{1}{g^2}\int_X F\wedge\star F + \frac{i\theta}{8\pi^2}\int_X F\wedge F, \tag{1.1}$$

where F is the curvature of the U_1 gauge field. Here g and θ are real-valued parameters, though θ is 2π-periodic. In dimension 4 only, we can rewrite this in terms of the self-dual and anti-self-dual pieces of F:

$$S = \frac{i\overline{\tau}}{4\pi}\int_X \|F_+\|^2 - \frac{i\tau}{4\pi}\int_X \|F_+\|^2, \tag{1.2}$$

where

$$\tau := \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}. \tag{1.3}$$

That is, our single parameter τ is valued in $\mathbb{H}$, the upper half-plane.

1.2.1. *The anomaly as variance under a symmetry.* Now $\mathrm{SL}_2(\mathbb{Z}) = \langle S, T \mid S^4 = 1, (ST)^3 = S^2\rangle$ acts on $\mathbb{H}$ by $S\tau = -1/\tau$ and $T\tau = \tau + 1$. We'd like to quotient by this action and obtain a parameter space $\mathcal{M} := \mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$. This action has stabilizer, though: at $\tau = i$, the stabilizer is $\mathbb{Z}/2$, and at $e^{i\pi/3}$, it's $\mathbb{Z}/3$. Thus it's helpful to think of $\mathcal{M}$ as the quotient *stack*, which means just that we remember the $\mathbb{Z}/2$ at i and the $\mathbb{Z}/3$ at $e^{i\pi/3}$.[1]

TODO: picture of the stack.

However, the action (1.2) is not invariant: Witten [Wit95] uses physics arguments to show that

$$Z_{T\cdot\tau}(X) = Z_\tau(X) \tag{1.4a}$$

$$Z_{S\cdot\tau}(X) = \tau^u\overline{\tau}^v Z_\tau(X), \tag{1.4b}$$

where $u = (1/4)(\chi(X)+\sigma(X))$ and $v = (1/4)(\chi(X)-\sigma(X))$.

Remark 1.5. Because Maxwell theory is a free QFT, making the argument rigorous is probably easier than for general QFTs (this does not mean "easy"!). ◄

In other words, the theory is anomalous for this $\mathrm{SL}_2(\mathbb{Z})$-action: on $\mathcal{M}$, the partition function isn't a well-defined function.

One might attempt to remedy this by throwing counterterms into the action. Our two options are the signature and the Euler characteristic, so such a counterterm would look like

$$f(\tau,\overline{\tau})\chi(X) + g(\tau,\overline{\tau})\sigma(X).\text{[2]} \tag{1.6}$$

This does not save us: consider $\tau = e^{i\pi/3}$, which has a $\mathbb{Z}/3$ stabilizer generated by ST^{-1}. This acts on the partition function by $e^{i\pi\sigma(X)/3}$, but does not change the counterterm (1.6), so this factor cannot be canceled. This is a nice application of the stacky perspective on $\mathcal{M}$.

However, a counterterm can simplify the anomaly. Letting $\eta\colon \mathbb{H}\to\mathbb{C}$ be the Dedekind η-function, $f(\tau,\overline{\tau}) := \mathrm{Re}\log\eta(\tau)$ and $g(\tau,\overline{\tau}) := i\,\mathrm{Im}\log\eta(\tau)$. The new partition function satisfies

$$Z'_\tau(X) = \eta(\tau)^{-(\chi(X)+\sigma(X))/2}\eta(-\overline{\tau})^{-(\chi(X)-\sigma(X))/2}Z_\tau(X), \tag{1.7}$$

and it transforms under the $\mathrm{SL}_2(\mathbb{Z})$-action as $Z'_{T\cdot\tau}(X) = Z'_\tau(X)$ and $Z'_{S\cdot\tau}(X) = \exp(-i\pi\sigma(X)/3)Z'_\tau(X)$.

This is the first description of this anomaly, as expressing how the partition function changes under $\mathrm{SL}_2(\mathbb{Z})$. It involves the signature, which is a "gravitational" term (meaning an invariant of the underlying manifold), and $\mathrm{SL}_2(\mathbb{Z})$, so it's a mixed anomaly. There could also be a pure $\mathrm{SL}_2(\mathbb{Z})$ anomaly, but to investigate it one should couple the theory to a background principal $\mathrm{SL}_2(\mathbb{Z})$-bundle, which the paper doesn't do.

---

[1] Really this is the quotient stack $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$; the reason we're using all of $\mathrm{SL}_2(\mathbb{Z})$ is that its center will act nontrivially later.

[2] The notation $(\tau,\overline{\tau})$ indicates these functions need not be holomorphic.

1.2.2. *Extending the parameter space.* Next we'll describe a fiber bundle $\mathcal{N} \to \mathcal{M}$ such that the partition function is a function on $\mathcal{N}$, and interpret it as a section of a line bundle over $\mathcal{M}$.

The gravitational term $\theta_{\mathrm{grav}} \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$; we'll consider an extended space of parameters, namely $\mathbb{H} \times S^1$, and define an action of $\mathrm{SL}_2(\mathbb{Z})$ on both of them in a way which gets rid of the anomaly.

Definition 1.8. The character group of $\mathrm{SL}_2(\mathbb{Z})$ is cyclic of order 12: given a $k \in \mathbb{Z}/12$, define the character $\chi_k \colon \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{U}_1$ by $\chi_k(S) := e^{-i\pi k/2}$ and $\chi_k(T) := e^{-i\pi k/6}$.

Of course, one should check these satisfy the relations of $\mathrm{SL}_2(\mathbb{Z})$, hence actually define a character. We'll single out χ_8: on T it's 1 and on S it's $e^{2i\pi/3}$, which looks a lot like the anomaly we saw above.

Now define the $\mathrm{SL}_2(\mathbb{Z})$-action on $\mathbb{H} \times S^1$ by

$$g \cdot (\tau, \theta_{\mathrm{grav}}) := (g \cdot \tau, \chi_8(g) \cdot \theta_{\mathrm{grav}}) \tag{1.9}$$

(where U_1 acts on S^1 by the standard representation). The partition function is

$$Z'_{\tau,\theta_{\mathrm{grav}}}(X) = Z'_\tau(X) e^{i\theta_{\mathrm{grav}}\sigma/16}, \tag{1.10}$$

so if $g \in \mathrm{SL}_2(\mathbb{Z})$,

$$Z'_{g\cdot(\tau,\theta_{\mathrm{grav}})}(X) = Z'_{(\tau,\theta_{\mathrm{grav}})}(X); \tag{1.11}$$

the factors of $e^{i\pi/3}$ and $e^{2i\pi/3}$ cancel out. The partition function is invariant, so descends to a function on the *extended conformal manifold* $\mathcal{N} := (\mathbb{H} \times S^1)/\mathrm{SL}_2(\mathbb{Z})$, which is an S^1-bundle over $\mathcal{M}$. This is a key idea of their paper: the partition function of an anomalous theory is only a function on this extended parameter space $\mathcal{N}$.

An alternative perspective is to use a line bundle to encode a twist. A function on $\mathcal{N}$ is a section of the trivial line bundle. We can ask whether it descends to $\mathcal{M}$, not necessarily as a function, but as a section of a line bundle. This is governed by *descent data*: rotating the S^1 defines a U_1-action on $\mathcal{N}$ whose quotient is $\mathcal{M}$. Then, an equivariant line bundle L' on $\mathcal{N}$ descends to a line bundle L on $\mathcal{M}$ (nonequivariant – in taking the quotient we've "used up" the equivariance), and an equivariant section of L' descends to a section of L.

So let's give the trivial line bundle $\underline{\mathbb{C}} \to \mathcal{N}$ a nontrivial U_1-action: given $(\tau, \theta_{\mathrm{grav}}, w)$ with $(\tau, \theta_{\mathrm{grav}}) \in \mathcal{N}$ and $w \in \underline{\mathbb{C}}_{(\tau,\theta_{\mathrm{grav}})}$, and given a $z \in \mathrm{U}_1$, define

$$z \cdot (\tau, \theta_{\mathrm{grav}}, w) := \left(\tau, z \cdot \theta_{\mathrm{grav}}, \exp\left(\frac{\sigma(X)}{16}\right) zw\right). \tag{1.12}$$

This defines an equivariant line bundle $L' \to \mathcal{N}$ such that a function Z on $\mathcal{N}$ such that $Z(\tau, z \cdot \theta_{\mathrm{grav}}) = e^{\sigma(X)/16} Z(\tau, \theta_{\mathrm{grav}})$, such as the partition function, is an equivariant section of this line bundle. Descending, we obtain a nonequivariant line bundle $L \to \mathcal{M}$, and the partition function is a section.

Ok, so which line bundle do we get? We know the U_1-equivariant line bundle on $\mathcal{N}$ that we began with, hence a $\mathrm{U}_1 \times \mathrm{SL}_2(\mathbb{Z})$-bundle on $\mathbb{H} \times S^1$, and then we can quotient by U_1 to obtain an $\mathrm{SL}_2(\mathbb{Z})$-equivariant line bundle on $\mathbb{H}$, then quotient by $\mathrm{SL}_2(\mathbb{Z})$ to get back $L \to \mathcal{M}$. The point is, passing through $\mathbb{H} \twoheadrightarrow \mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ may be easier to think about, and we can compare to known line bundles.

Definition 1.13. The *Hodge bundle* $L_H \to \mathcal{M}$ is the quotient of the $\mathrm{SL}_2(\mathbb{Z})$-equivariant line bundle $L'_H \to \mathbb{H}$ which is nonequivariantly trivial and whose $\mathrm{SL}_2(\mathbb{Z})$-action is defined by $g \cdot (\tau, z) = (g \cdot \tau, \chi_1(g) \cdot z)$.

The 12^{th} tensor power of L_H is trivial, ultimately because the abelianization of $\mathrm{SL}_2(\mathbb{Z})$ is $\mathbb{Z}/12$.

If you run this argument, you get that the line bundle L arising from Maxwell theory on X is $L_H^{\otimes(\sigma/2)}$. The intuition is that we have $\exp(\sigma/16)$ in the U_1-action, and $\mathrm{SL}_2(\mathbb{Z})$ acts on U_1 by eight times the generator, giving us $\sigma/2$.

1.2.3. *Anomaly polynomials.* Seiberg-Tachikawa-Yonekura also discuss anomaly polynomials. In general, suppose we have a family of $2k$-dimensional manifolds $\mathcal{X} \to \mathcal{M}$, and write the fiber at $m \in \mathcal{M}$ as X_m.[3] The partition function $Z(X_m)$ is a section of a line bundle $L \to \mathcal{M}$. The goal of the anomaly polynomial is to determine L, or equivalently its first Chern class. Therefore the anomaly polynomial $\mathbb{A}_{2k+2} \in H^{2k+2}(\mathcal{X})$ is defined to satisfy

$$c_1(L) = \int_{X_m} \mathbb{A}_{2k+2}, \tag{1.14}$$

[3]Despite the similar notation, this time X is varying and the QFT is constant; previously, it was the other way around.

here denoting integration along the fiber, the pushforward map $H^*(\mathcal{X}) \to H^{*-2k}(\mathcal{M})$.[4]

Seiberg-Tachikawa-Yonekura compute $\mathbb{A}_6$ for Maxwell theory in an interesting way: they realize it as the dimensional reduction of a 6D theory Z_6 along a torus T. This theory also has an anomaly polynomial $\mathbb{A}_8 \in H^8(T \times \mathcal{X})$, and integrating over the fibers $T \times X_m$ produces $c_1(L)$ again.

Therefore we can compute $c_1(L)$ in two ways: first integrating along the fiber of $T \times \mathcal{X} \to \mathcal{X}$, then $\mathcal{X} \to \mathcal{M}$ as above, or by first integrating along the fiber of $T \times \mathcal{X} \to T \times \mathcal{M}$, then $T \times \mathcal{M} \to \mathcal{M}$.

Since I don't have a whole lot of time, and because I didn't fully understand the arguments in this section, I'm going to skip over the computations, which is unfortunate, because they look mathematically interesting. The summary is that knowing the 6D anomaly polynomial, and knowing the anomaly polynomial of the 2D theory $Z_6(- \times X_m)$, allows one to pin down the anomaly polynomial of the 4D theory in terms of the central charge of the 2D theory. In the case of Maxwell theory, the central charge is $c = \sigma(X)$, and using arguments from 2D CFT that I could not follow, the corresponding line bundle $L \to \mathcal{M}$ is $L_H^{c/2}$, which agrees with what we saw above.

1.2.4. *Relative field theory.* This is the most topological perspective on anomalies, so I love it.

So with that in mind, we expect Z to really be a QFT relative to an invertible TFT α in dimension 5, and with the same background fields. That is, the symmetry type is

- spin 5-manifolds, since we began with spin 4-manifolds; together with
- an $\mathrm{SL}_2(\mathbb{Z})$-bundle, since we're considering an $\mathrm{SL}_2(\mathbb{Z})$ symmetry: even though we didn't couple to a background $\mathrm{SL}_2(\mathbb{Z})$-bundle, this $\mathrm{SL}_2(\mathbb{Z})$-symmetry still appears in the anomaly.

The anomaly theory α is topological, because it is a finite-order, unitary invertible field theory.[5] Therefore it cannot see τ, θ, or θ_{grav}, so this is the entire symmetry type; moreover, its partition function is a bordism invariant, an element of

$$\mathrm{Hom}(\Omega_5^{\mathrm{Spin}}(B\mathrm{SL}_2(\mathbb{Z})), \mathrm{U}_1). \tag{1.15}$$

Seiberg-Tachikawa-Yonekura [STY18] show this is abstractly isomorphic to $\mathbb{Z}/36$, but they don't produce an isomorphism, so it's difficult to get one's hands on this.

TODO: their results

Here are some other things we can say about the computation of α.

(1) First, we can only expect to know the answer modulo "pure $\mathrm{SL}_2(\mathbb{Z})$" theories; let's discuss what that means. The action of α can include terms which are characteristic classes for $\mathrm{SL}_2(\mathbb{Z})$-bundles as well as "gravitational" terms which depend on the underlying manifold itself. A theory whose action has no gravitational terms is called a pure $\mathrm{SL}_2(\mathbb{Z})$-theory.

We haven't studied Maxwell theory coupled to a background principal $\mathrm{SL}_2(\mathbb{Z})$-bundle, so we're not going to be able to distinguish any of the pure $\mathrm{SL}_2(\mathbb{Z})$-theories. However, these are a subgroup of the group of all 5D invertible TFTs with symmetry type $\mathrm{Spin} \times \mathrm{SL}_2(\mathbb{Z})$, so we can ask whether we can identify the anomaly in the quotient.

Seiberg-Tachikawa-Yonekura show that the pure $\mathrm{SL}_2(\mathbb{Z})$ theories form a $\mathbb{Z}/6$ inside this $\mathbb{Z}/36$. They make this argument using the Atiyah-Hirzebruch spectral sequence

$$E^2_{p,q} = H_p(B\mathrm{SL}_2(\mathbb{Z}); \Omega_q^{\mathrm{Spin}}(\mathrm{pt})) \Longrightarrow \Omega_{p+q}^{\mathrm{Spin}}(B\mathrm{SL}_2(\mathbb{Z})). \tag{1.16}$$

The pure $\mathrm{SL}_2(\mathbb{Z})$-theories are those on the line $q = 0$, so they only see the homology of $B\mathrm{SL}_2(\mathbb{Z})$ and not the spin bordism groups. The argument that $E^\infty_{5,0} = \mathbb{Z}/6$ is a fun but elaborate spectral sequence proof, chaining together three instances of the Atiyah-Hirzebruch spectral sequence and playing them off of each other.

Let A denote the quotient of $\mathrm{Hom}(\Omega_5^{\mathrm{Spin}}(B\mathrm{SL}_2(\mathbb{Z})), \mathrm{U}_1)$ by the pure $\mathrm{SL}_2(\mathbb{Z})$-theories, so that $A \cong \mathbb{Z}/6$. We've seen that when we don't couple to principal $\mathrm{SL}_2(\mathbb{Z})$-bundles, the anomaly has order 3; this means it has order 3 in A.

(2) If we had an explicit description of the elements of $\mathrm{Hom}(\Omega_5^{\mathrm{Spin}}(B\mathrm{SL}_2(\mathbb{Z})), \mathrm{U}_1)$, we could do more, using the anomaly TFT to determine information about the line bundle $L \to \mathcal{M}$. The idea of a QFT relative to some invertible TFT α is that the partition function of X is an element of the line $\alpha(X)$;

[4] To do this, we need the relative tangent bundle of the map $\mathcal{X} \to \mathcal{M}$ to be oriented.

[5] This is a quite nontrivial theorem of Freed-Hopkins [FH16].

applying this to the family of theories Z_τ in $\mathcal{M}$, $\alpha(X)$ defines a line bundle over $\mathcal{M}$, and the partition function is a section of this line bundle.

At $\tau = e^{i\pi/3}$, we have a $\mathbb{Z}/3$-symmetry on L_τ which we've determined just by studying Maxwell theory, and we can compare this with another $\mathbb{Z}/3$-symmetry on $\alpha(X)$ that can be determined just by studying α, and these must match, which could help determine α in general.

More explicitly, consider $\tau = e^{i\pi/3} \in \mathcal{M}$ with its $\mathbb{Z}/3$ stabilizer; this is a $\mathrm{pt}/(\mathbb{Z}/3)$. A strong form of $\mathbb{Z}/3$ symmetry is being able to extend to a family over this space, so let's try to do this. Take a spin 4-manifold X and the trivial $\mathrm{SL}_2(\mathbb{Z})$-bundle $P_{\mathrm{triv}} \to X$ and extend to (X, P) over $\mathrm{pt}/(\mathbb{Z}/3)$, where the monodromy of P around an $a \in \mathbb{Z}/3 \subset \mathrm{SL}_2(\mathbb{Z})$ is right multiplication by a^{-1} on $\mathrm{SL}_2(\mathbb{Z})$. We can then evaluate α on this family of manifolds, to produce a line bundle over $\mathrm{pt}/\mathbb{Z}/3$ (namely, a line with a $\mathbb{Z}/3$-action), and we can compare this with the fiber of L over $e^{i\pi/3}$ with its $\mathbb{Z}/3$-action. These should be isomorphic, and given a concrete description of these invertible TFTs, one could use this to learn more information about the anomaly of Maxwell theory even though we haven't coupled to $\mathrm{SL}_2(\mathbb{Z})$-bundles. We can also do this with the $\mathbb{Z}/2$ stabilizer at $\tau = i$.

2. Integrability and 4D gauge theory I: 1/30/19

Today, Sebastian spoke about work of Costello-Witten-Yamazaki [CWY17, CWY18] on integrability and gauge theory. Next week's talk will continue this story, with calculations supporting the ideas presented today.

Integrability is roughly the same thing as being exactly solvable. Across a wide variety of mathematical disciplines (knot theory, differential equations, statistical mechanics, ...), this is understood to mean satisfying the *Yang-Baxter equation*. There's a nice, very readable paper of Perk-Au-Yang [PAY06] which explains this unifying perspective.

Consider some particles in a 2D system, with quantum numbers in some vector space V with basis $\{e_i\}$, and suppose there's a "spectral parameter" $z \in \mathbb{C}$. Consider some particles participating in a collision as in Figure 1: the incoming particles have quantum numbers i and j, and spectral parameters z_1 and z_2, and the outcoming particles have quantum numbers k and ℓ. This defines a map $R(z_1, z_2)\colon V \otimes V \to V \otimes V$; assume $R(z_1, z_2)$ only depends on $z_1 - z_2$.

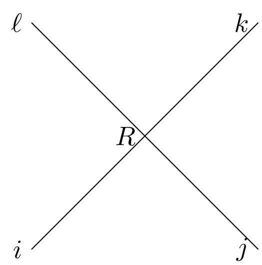

Figure 1. The definition of the R-matrix.

The Yang-Baxter equation encodes the ability to move a line behind such a picture. We'll use the notation $z_{ij} \coloneqq z_i - z_j$, and $R_{ij}\colon V^{\otimes 3} \to V^{\otimes 3}$ to act by R on the i and j^{th} copies of V and by the identity on the remaining copy. Then the Yang-Baxter equation (YBE) is

$$R_{23}(z_{23})R_{13}(z_{13})R_{12}(z_{12}) = R_{12}(z_{12})R_{13}(z_{13})R_{23}(z_{23}). \tag{2.1}$$

The picture is in Figure 2.

One in addition often asks for solutions to satisfy *unitarity*, i.e. $R_{12}(z_{12})R_{21}(z_{21}) = \mathrm{id}$. The string diagram perspective is that one can separate two strands which cross twice. Unitary solutions to the Yang-Baxter equation are still too complicated to classify, so people usually introduce more structure.

Another variant is the *quasi-classical Yang-Baxter equation*: one introduces another complex parameter $\hbar$, and near $\hbar = 0$, we ask

$$R_\hbar(z) = \mathbf{1} + \hbar r(z) + O(\hbar^2), \tag{2.2}$$

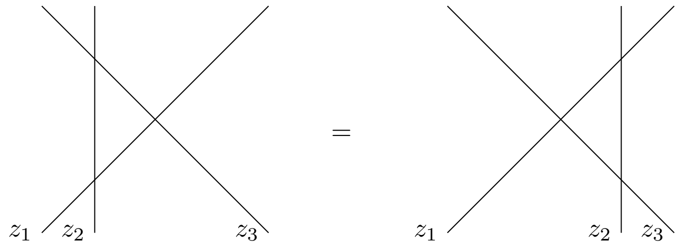

FIGURE 2. The string diagram version of the Yang-Baxter equation.

where $r(z)$ is the *classical R-matrix*, which up to $O(\hbar^2)$ satisfies the *classical Yang-Baxter equation*

$$[r_{12}(z_{12}), r_{13}(z_{13}) + r_{23}(z_{23})] + [r_{13}(z_{13}), r_{23}(z_{23})] = 0. \tag{2.3}$$

Belavin-Drinfeld [BD82] classified solutions associated to a complex Lie algebra $\mathfrak{g}$ associated to a real Lie group G. In this case $r(z) \in \mathfrak{g} \otimes \mathfrak{g}$. Choosing a basis $\mathfrak{g} = \langle t^a \rangle$, we can write

$$r(z) = \sum_{a,b} r_{a,b}(z) t^a \otimes t^b, \tag{2.4}$$

where $\det(r_{a,b}) \neq 0$. The poles of $r(z)$ span a lattice Γ of rank ≤ 2, and three situations emerge.

- If Γ is rank 0, this is the *rational* setting, and we think of solutions as representations of a *Yangian*. In this case $\mathbb{C}/\Gamma = \mathbb{C}$.
- If Γ is rank 1, this is the *trigonometric* setting, and solutions are representations of a *quantum affine algebra*. $\mathbb{C}/\Gamma \cong \mathbb{C}^\times$.
- If Γ is rank 2, this is the *elliptic* setting, and solutions are representations of an *elliptic algebra*. The quotient is an elliptic curve.

The goal of Costello-Witten-Yamazaki is to relate this story to 4D gauge theory. So let's consider 4D gauge theory on $\mathbb{R}^2 \times \mathbb{C}$, with coordinates $(x, y, z, \overline{z})$, and fields

$$A = A_x \,\mathrm{d}x + A_y \,\mathrm{d}y + A_{\overline{z}} \,\mathrm{d}\overline{z}. \tag{2.5}$$

In particular, each A_i is not required to be holomorphic in z. On our Lie algebra $\mathfrak{g}$, choose an invariant nondegenerate quadratic form, such as the Killing form, and let t_a be an orthonormal basis.

The action is

$$S = \frac{1}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{C}} \mathrm{d}z \wedge CS(A) = -\frac{1}{2\pi} \int z \operatorname{tr}(F \wedge F), \tag{2.6}$$

where the Chern-Simons term is

$$CS(A) := \operatorname{tr}\left(A \wedge \mathrm{d}A + \frac{2}{3} A^3 \right). \tag{2.7}$$

The diffeomorphism group of $\mathbb{R}^2$ preserves this action, where the action is by

$$A_i \longmapsto g^{-1} A_i g + g^{-1} \partial_i g, \tag{2.8}$$

where $i = x, y, \overline{z}$.

The classical equations of motion are $F_{xy} = 0$ and $F_{x\overline{z}} = F_{y\overline{z}} = 0$, meaning solutions are a flat bundle over $\mathbb{R}^2$ and vary holomorphically in $\mathbb{C}$. Therefore all gauge-invariant quantities coming from A must vanish. Correlation functions look like

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}A \, \mathcal{O} \exp(iS/\hbar)}{\int \mathcal{D}A \, \exp(iS/\hbar)}. \tag{2.9}$$

If the length is $\hbar$, the theory is not renormalizable by power counting, though all possible counterterms vanish by the equations of motion. Therefore one can quantize using perturbation theory, and the quantum theory is IR free.

Remark 2.10. This theory has a framing anomaly, similarly to ordinary Chern-Simons theory. We're not going to worry much about this today, though we'll learn more about it next week. ◀

So now we can formulate this QFT on curved manifolds $\Sigma \times C$, where Σ is a topological surface and C is a Riemann surface. Choose a closed holomorphic 1-form ω;[6] the action is

$$S = \frac{1}{2\pi} \int \omega \wedge CS(A). \tag{2.11}$$

We want to study this with perturbation theory; the action depends on $\omega/\hbar$. This causes a problem: zeros of ω correspond to poles of $\hbar$. At least we can understand poles of ω as zeros of $\hbar$, and they spend some time on this. Therefore let's assume ω has no zeros.

Knowledge of the genus of C allows us to use the Riemann-Roch theorem. Since ω has no zeros, this tells us

$$-\#\text{poles} = 2g(C) - 2. \tag{2.12}$$

This cuts down the possibilities pretty drastically. There are only a few options (up to rescaling, etc.):

- $C = \mathbb{C}$ with $\omega = \mathrm{d}z$, so there's a double pole at infinity; this is the rational case.
- $C = \mathbb{C}^\times$ with $\omega = \mathrm{d}z/z$, so there are simple poles at 0 and ∞; this is the trigonometric case.
- C is an elliptic curve with $\omega = \mathrm{d}z$, and there are no poles; this is the elliptic case.

These correspond to the three cases we mentioned earlier.

Remark 2.13. It's no coincidence we get an abelian group in all three cases: since ω has no zeros, we can invert it and get a vector field, and that vector field generates the group action. ◀

The framing anomaly puts constraints on Σ; it must be 2-framed, so it must be a torus.[7]

The equations of motion tell us there are no local observables, so the easiest operators to look at are Wilson lines. Choose a loop $K \subset \Sigma \times C$ and a G-representation ρ; then the Wilson line operator is

$$W_\rho(K) = \mathrm{tr}_\rho\, P\exp\left(\oint_K A_i\,\mathrm{d}x^i\right). \tag{2.14}$$

That is, we get a group element g by walking around K using the connection A, and then compute the trace of $\rho(g)$. However, since A had no $\mathrm{d}z$ term, we can't parallel transport in the z-direction, which means K must be a loop solely in Σ, with a fixed value z_0 in C.

Actually, we can do something different, using a representation $\widehat{\rho}$ of the Lie algebra

$$\mathfrak{g}[[z]] = \bigoplus_{n\ge 0} \mathfrak{g}\otimes\mathbb{C}\cdot z^n. \tag{2.15}$$

(More generally, we could use $\mathfrak{g}[[z-z_0]]$.) If $\mathfrak{g}$ is spanned by t_a, then $\mathfrak{g}[[z]]$ is spanned by $t_{a,n}(z) \coloneqq t_a z^n$, which you can think of as "matrices." Write

$$[t_{a,n}(z), t_{b,m}(z)] = f_{ab}{}^c t_{c,m+n}. \tag{2.16}$$

We'd like finite-dimensional representations with $t_{a,n} = 0$ for $n > n_0$. For example, if $n_0 = 2$, this tells us that $[t_{a,1}, t_{b,1}] = 0$, $\{t_{a,0}\}$ span $\mathfrak{g}$, and $[t_{a,0}, t_{b,1}] = f_{ab}^c t_{c,1}$ means that we just get the adjoint representation of $\mathfrak{g}$.

Because of the IR freeness of the theory, the fields must vanish at infinity, and this allows one to understand the holonomy itself, rather than just its trace, as a meaningful observable.

How does this relate to the Yang-Baxter equations? Looking back at Figure 2, we want this diagram to mean a triple of Wilson lines labeled in representations. When you move one across the others, the $\mathrm{Diff}(\mathbb{R}^2)$-symmetry is broken at the middle, but we still expect the Yang-Baxter equation, and the unitary equation, to be true, since these can be at different heights (so we can really move one behind the others).

3. Integrability and 4D gauge theory II: 2/6/19

Today, Ivan spoke, continuing the story of the work of Costello-Witten-Yamazaki [CWY17, CWY18] on how solutions of the Yang-Baxter equation arise from a 4D gauge theory.

Last time, we considered a system of particles in 2D spacetime, with a vector space V of internal states and a spectral parameter $z \in \mathbb{C}$. Particle-particle interactions are governed by the R-matrix $R(z_1, z_2) \in \mathrm{End}(V_1 \otimes V_2)$, and we asked for it to satisfy the Yang-Baxter equation (2.1), which you can think of in terms of the picture Figure 2.

---

[6] Or meromorphic if you want to work on compact manifolds.

[7] This is different from standard Chern-Simons theory, which imposes a weaker constraint.

People don't really solve just the Yang-Baxter equation without introducing more structure. For example, one can let the R-matrices depend holomorphically on another parameter $\hbar$, in which case we want to satisfy the quasi-classical Yang-Baxter equation (2.2). Working in a fixed semisimple Lie algebra $\mathfrak{g}$, the solutions fall into three kinds: rational, trigonometric, and elliptic, corresponding to representations of a Yangian $Y_\hbar(\mathfrak{g})$, a quantum group $U_{q,\hbar}(\mathfrak{g})$, or an elliptic algebra $E_{\tau,q,\hbar}(\mathfrak{g})$, respectively.

We will relate this to a 4D gauge theory on $M \coloneqq \Sigma \times C$, where Σ is an oriented smooth surface and C is a Riemann surface. The gauge group G is a semisimple complex Lie group. We took for fields the partial G-connections $A = A_x\,\mathrm{d}x + A_y\,\mathrm{d}y + A_{\overline{z}}\,\mathrm{d}\overline{z}$; $g \in G$ acts on A_i by $gA_ig^{-1} + g\partial_i g^{-1}$ for $i = x, y, \overline{z}$.

The action (2.6) can be generalized by replacing $\mathrm{d}z$ with a meromorphic closed 1-form with no zeros. This constrains C by Riemann-Roch to either $\mathbb{C}$, $\mathbb{C}^\times$, or an elliptic curve, corresponding respectively to the rational, trigonometric, and elliptic situations.

The next question is: how does the R-matrix arise in this theory? The answer is to look at Wilson lines $K \subset \Sigma \times \{z\}$; these correspond in the Yang-Baxter picture to worldlines labeled by $z \in \mathbb{C}$. This still doesn't explain why there's an R-matrix. The answer is that the theory we're looking at it IR-free, so there's a local picture, obtained by a scaling limit in Σ. In this case, the R-matrix picture (two-particle interactions) looks a lot like Feynman diagrams for the perturbation theory in this scaling limit.

Now, let $C = \mathbb{C}$, $\omega = \mathrm{d}z$, and $\Sigma = \mathbb{R}^2$ (so we're in the rational setting). Our goal is to compute the $O(\hbar)$ contribution to the R-matrix. It turns out the angle of the two worldlines doesn't matter.

Remark 3.1. The theory does have a framing anomaly, and higher-order terms in $\hbar$ do have an angle dependence expressed in terms of this anomaly. But just for first-order it's OK. ◀

We end up getting

$$\hbar t_{a,p} \otimes t_{b,p'} \int \mathrm{d}x\,\mathrm{d}y'\, P^{ab}(x - x', y - y', z - z', \overline{z} - \overline{z}'), \tag{3.2}$$

where $\{t_a\}$ is an orthonormal basis for $\mathfrak{g}$ with respect to the Killing form, $t_{a,n} \coloneqq t_a z^n \in \mathfrak{g}[[z]]$, and P denotes the propagator for the free theory. Then the equations of motion are fairly simple:

$$\mathrm{d}z \wedge F_A = 0 \tag{3.3a}$$

$$F_{xy} = F_{x\overline{z}} = F_{y\overline{z}} = 0, \tag{3.3b}$$

and the linearized equations of motion are even simpler:

$$\mathrm{d}z \wedge \mathrm{d}A = 0. \tag{3.4}$$

Fix the gauge

$$\partial_X A_x + \partial_y A_y + 4\partial_z A_{\overline{z}} = 0. \tag{3.5}$$

In this case the propagator P satisfies

$$\frac{i}{2\pi}\,\mathrm{d}z \wedge \mathrm{d}P = \delta^4(x, y, z, \overline{z})\mathbf{1} \tag{3.6}$$

$$(\partial_x i u_x + \partial_y i_y + \partial z i_{\partial\overline{z}})P = 0. \tag{3.7}$$

Therefore

$$P^{ab}(x, y, z, \overline{z}) = \frac{\delta^{ab}}{2\pi}\,\frac{x\,\mathrm{d}y \wedge \mathrm{d}\overline{z} y\,\mathrm{d}\overline{z} \wedge \mathrm{d}x + z\overline{z}\,\mathrm{d}x \wedge \mathrm{d}y}{(x^2 + y^2 + z\overline{z})^2}. \tag{3.8}$$

Plugging this into (3.2), we get for the $O(\hbar)$ contribtution

$$(3.2) = \hbar t_{a,p} \otimes t_{b,p'} \int \mathrm{d}x\,\mathrm{d}y\,\frac{\delta^{ab} 2\overline{z}_1 - \overline{z}_2}{(x^2 + y^2 + |z_1 - z_2|^2)^2} \tag{3.9}$$

$$= \frac{\hbar(\sum_a t_{a,p} \otimes t_{a,p'})}{z_1 - z_2} \tag{3.10}$$

$$= \frac{\hbar C_{p,p'}}{z_1 - z_2}. \tag{3.11}$$

That is,

$$R_\hbar(z_1 - z_2) = \mathbf{1} + \frac{\hbar C_{p,p'}}{z_1 - z_2} + O(\hbar^2), \tag{3.12}$$

which recovers what we expect in the rational setting.

This is related to the Yangian, in that if V_1 and V_2 are representations of the Yangian, then $R_\hbar(z) = \mathrm{End}(V_1 \otimes V_2)$ should be an intertwiner. We can interpret this using OPEs of Wilson lines. Consider two Wilson lines K, K_ε supported on $\Sigma \times \{0\}$ and $\Sigma \times \{\varepsilon\}$. Letting $\varepsilon \to 0$, one has

$$\lim_{\varepsilon\to 0} \widetilde{W}_\rho(K_\varepsilon) \otimes \widetilde{W}_{\rho'}(K_\varepsilon) = \widetilde{W}_{\rho\otimes\rho'}(K) - \hbar t_{a,p} \otimes t_{b,p'} f_{abc} \int \mathrm{d}x\, m\partial_z A_x^C(x,y,0,0) + O(\hbar^2). \tag{3.13}$$

Last time, we said tht if $\widehat{\rho}$ is a representation of $\mathfrak{g}[[z]]$ with $\widehat{\rho}(z^n t_a) = t_{a,n}$, then

$$\widetilde{W}_{\widehat{\rho}}(K) = P\exp\left(\oint_K \sum_{n=0}^\infty \frac{\partial}{\partial z^n} A_i^c(x,y,0,0) t_{c,n}\, \mathrm{d}x^i\right). \tag{3.14}$$

Therefore it seems that the fused Wilson line is associated to a $\mathfrak{g}[[z]]$-representation with

$$\begin{aligned} t_{a,0} &= t_{a,p} \otimes 1 + 1 \otimes t_{a,p'} \\ t_{c,1} &= -\hbar t_{a,p} \otimes t_{b,p'} f_{abc}. \end{aligned} \tag{3.15}$$

But this is actually not true: there are further identities that don't hold. On $\mathfrak{g}[[z]]$, $[t_{a,1}, t_{b,2}] = f_{abc}t_{c,2}$ plus the Jacobi identity imply that $f_{\mu\nu a}[t_{a,1}, t_{b,1}]$ is equal to a cyclic permutation of $\mu\nu a$ in this. But instead, their sum is $Q_{\mu\nu b}(t_{a,0})$, which is nonzero. The answer is that the fused Wilson line is associated to a representation of a one-parameter deformation of $\mathcal{U}(\mathfrak{g}[[z]])$.

4. From Gaussian measures to factorization algebras: 2/13/19

Today Charlie spoke about the second chapter of Costello-Gwilliam [**?**], with the goal of motivating (pre)factorization algebras via Gaussian integrals on finite-dimensional vector spaces.

Let $Q\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a positive-definite bilinear form and $p\colon \mathbb{R}^n \to \mathbb{R}$ be a polynomial function. We're interested in computing the expectation value

$$\langle p \rangle \coloneqq \int_{\mathbb{R}^n} p(x) \exp\left(-\frac{1}{2}Q(x,x)\right) \mathrm{d}^n x. \tag{4.1}$$

We're going to try to compute this without having to evaluate any integrals!

Definition 4.2. We will use $\mathrm{Vect}(\mathbb{R}^n)$ to denote the space of vector fields on $\mathbb{R}^n$ whose component functions are polynomials and $P(\mathbb{R}^n)$ to denote the space of polynomial functions on $\mathbb{R}^n$.

Definition 4.3. Given a polynomial vector field v on $\mathbb{R}^n$, its *divergence* $\mathrm{Div}\, v \in P(\mathbb{R}^n)$ is defined to act on a volume form ω by $(\mathrm{Div}\, V)\omega = \mathcal{L}_v\omega = \mathrm{d}\iota_v\omega$ (this last equality follows from Cartan's magic formula).

Here are a few important facts about the divergence.

Lemma 4.4.

(1) *For any polynomial vector field v and volume form ω, $\int (\mathrm{Div}\, v)\omega = 0$.*

(2)

$$\begin{aligned} (\mathrm{Div}\, v)\omega &= \nabla\left(\exp\left(-\frac{1}{2}Q(x,x)v\right)\right) \mathrm{d}x^n \\ &= \left(v\exp\left(-\frac{1}{2}Q(x,x)\right) + \nabla\cdot v \exp\left(-\frac{1}{2}Q(x,x)\right)\right) \mathrm{d}x^n \\ &= (-Q(v,x) + \nabla\cdot v)\omega. \end{aligned}$$

(3) *$P(\mathbb{R}^n)/\mathrm{Im}(\mathrm{Div})$ is one-dimensional, and we have a canonical identification with $\mathbb{R}$ provided by integration.*

The last piece is particularly crucial: the quotient is still one-dimensional in a certain infinite-dimensional setting, where we'll use an identification with $\mathbb{R}$ as a definition of integration.

Now we'll apply this to physics. Consider the free scalar quantum field theory on a Riemannian manifold (M, g), which is spacetime. The space of fields is $C^\infty(M)$, and the action is

$$S(\phi) \coloneqq \frac{1}{2}\int_M \phi(\Delta_g + m^2)\phi\, \mathrm{dvol}_g, \tag{4.5}$$

where m is the mass of the theory and Δ_g is the Laplacian. We would like to define correlation functions

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle \text{ ``:=''} \int_{C^\infty(M)} \phi(x_1)\cdots\phi(x_n)e^{-S(\phi)}\,\mathrm{d}\phi, \tag{4.6}$$

but of course the measure $\mathrm{d}\phi$ doesn't exist on most M.

Our strategy to abrogate this is to put the observables into a chain complex: for an open $U\subset M$, take

$$\mathrm{Obs}^{-3}(v) \longrightarrow \mathrm{Obs}^{-2}(v) \longrightarrow \mathrm{Vect}(C^\infty(U)) \xrightarrow{\mathrm{Div}} P(C^\infty(U)). \tag{4.7}$$

We would like this to have the following properties:

1. H^0 of this complex should be the space of physically distinguishable observables on U, and
2. if U_1 and U_2 are disjoint open subsets of an open $V\subset X$, an observable on U_1 and an observable on U_2 should together define an observable on V, and this should define a map $H^0(\mathrm{Obs}^*(U_1))\otimes H^0(\mathrm{Obs}^*(U_2))\to H^0(\mathrm{Obs}^*(V))$.

To make this into real math, we have to define $P(C^\infty(U))$, $\mathrm{Vect}(C^\infty(U))$, and Div.

Definition 4.8. Given an $F\in C_c^\infty(U^n)$, define a function $F\colon C^\infty(U)\to\mathbb{R}$ by

$$\phi\longmapsto \int_{U^n} F(x_1,\dots,x_n)\phi(x_1)\cdots\phi(x_n)\,\mathrm{dvol}^n. \tag{4.9}$$

The symmetric group S_n acts on these functions by permuting the n inputs. We then define $P(C^\infty(U))$ as

$$P(C^\infty(U)) \coloneqq \bigoplus_{n=0}^\infty C_c^\infty(U^n)_{S_n}. \tag{4.10}$$

Definition 4.11. An $F\in C_c^\infty(U^{n+1})$ defines a vector field on U^n by the rule

$$\phi\longmapsto \int_{U^n} F(x_1,\dots,x_{n+1})\phi(x_1)\cdots\phi(x_n)\,\mathrm{dvol}^n. \tag{4.12}$$

(Precisely, this is a rule for differentiating functions into one-forms, which is specified by a vector field.) The symmetric group again acts, and we define $\mathrm{Vect}(C^\infty(U))$ by

$$\mathrm{Vect}(C^\infty(U)) \coloneqq \bigoplus_{n=0}^\infty C_c^\infty(U^{n+1})_{S_n}. \tag{4.13}$$

Defining the divergence is a little trickier.

Definition 4.14. Let $X\in\mathrm{Vect}(C^\infty(U))$ be homogeneous, i.e. a homogeneous element in the direct sum (4.13), and suppose $\deg(X)=n$. Then we define

$$\mathrm{Div}(X) \coloneqq -\big(\Delta_{g,n+1}+m^2\big)X(x_1,\dots,x_{n+1}) + \sum_{i=1}^n \int_U X(x_1,\dots,x_i,\dots,x_n,X_i)\,\mathrm{dvol}, \tag{4.15}$$

and extend to inhomogeneous elements by linearity.

Now we have some structure.

- Given $P\in P(C^\infty(U))$ and $X\in\mathrm{Vect}(C^\infty(U))$, we can multiply them to obtain another vector field.
- Given $P\in P(C^\infty(U)))$, if $U\subset V$, we can extend by 0 to obtain $\epsilon_0(P)\in P(C^\infty(V))$.

Lemma 4.16. *Let $P\in P(C^\infty(U))$ and $X\in\mathrm{Vect}(C^\infty(U))$. Then $\mathrm{Div}(PX)=P\,\mathrm{Div}\,X+X(P)$.*

Now we've got everything we need to make (4.7) make sense. Given $U_1,U_2\subset V$ all opens in M, there is a map

$$\begin{aligned} \widetilde{m}_V^{U_1,U_2}\colon P(C^\infty(U_1))\otimes P(C^\infty(U_2)) &\longrightarrow P(C^\infty(V))\\ P_1\otimes P_2 &\longmapsto \epsilon_0(P_1)\cdot\epsilon_0(P_2). \end{aligned} \tag{4.17}$$

Lemma 4.18. *The map $\widetilde{m}_V^{U_1,U_2}$ descends to a map $m_V^{U_1,U_2}\colon H^0(\mathrm{Obs}^*(U_1))\otimes H^0(\mathrm{Obs}^*(U_2))\to H^0(\mathrm{Obs}^*(V))$ if and only if U_1 and U_2 are disjoint.*

Proof. Let $P_W := P(C^\infty(W))$ and consider the diagram

$$\begin{tikzcd}
(\mathrm{Im}(\mathrm{Div}_{U_1}) \otimes P_{U_2}) \oplus (P_{U_1} \otimes \mathrm{Im}(\mathrm{Div}_{U_2})) \arrow[r, hook] & P_{U_1} \otimes P_{U_2} \arrow[r] \arrow[d, "\widetilde{m}_V^{U_1,U_2}"] & H^0(\mathrm{Obs}^*(U_1)) \otimes H^0(\mathrm{Obs}^*(U_2)) \\
\mathrm{Im}(\mathrm{Div}_V) \arrow[r] & P_V \arrow[r] & H^0(\mathrm{Obs}^*(V)).
\end{tikzcd} \tag{4.19}$$

We would like to move the vertical arrow from the second column to the third column. Since the rows are exact, it suffices to show that $\widetilde{m}_V^{U_1,U_2}$ applied to anything coming from the upper left is zero in cohomology. Well, if $P \in P_{U_1}$ and $X \in \mathsf{Vect}(C^\infty(U_2))$, then using Lemma 4.16,

$$P\,\mathrm{Div}(X) = \mathrm{Div}_V(PX) + X(P). \tag{4.20}$$

U_1 and U_2 are disjoint iff $X(P)$ vanishes for all X and all P, and in this case, $P\,\mathrm{Div}(X) \in \mathrm{Im}(\mathrm{Div}_V)$, so when we pass to $H^0(\mathrm{Obs}^*(V))$ along the bottom row, it's zero by exactness. The case when $P \in P_{U_2}$ and $X \in \mathrm{Vect}(C^\infty(U_1))$ is the same. ⊠

With the remaining time, let's talk a bit about the classical limit. In this case we consider $e^{-S/\hbar}$; to say this more explicitly,

$$\mathrm{Div}_\hbar(X) := -\frac{1}{\hbar}\big(\Delta_{g,n+1} + m^2\big)X + \nabla \cdot X. \tag{4.21}$$

The limit of the image of $\mathrm{Div}_\hbar$ as $\hbar \to 0$ is the ideal in $P(C^\infty(U))$ generated by linear functionals, and therefore $\mathrm{Obs}^0(U)$ becomes functions on the critical locus of the Euler-Lagrange equations. The upshot is that multiplication is commutative, so we don't just get a prefactorization algebra, but in fact have a commutative one.

5. Classical free scalar field theory: 2/20/19

Today, Mario spoke about chapter 2, with a perspective rooted a little more in physics. The goal is to discuss how some of the formalism from last week can be used to recover some physics statements about free theories.

Let's recall some facts about classical field theory on a Riemannian manifold (M, g). There is some action S, which is a function on smooth functions on M, typically expressed as an integral of a Lagrangian density:

$$S(\phi) = \int_M \mathrm{dvol}\,\mathcal{L}(\phi). \tag{5.1}$$

This functional is minimized when ϕ satisfies the *Euler-Lagrange equations*

$$\partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi} - \frac{\delta\mathcal{L}}{\delta\phi} = 0. \tag{5.2}$$

The *classical observables* are the functions on the space of solutions to the Euler-Lagrange equations (the critical locus of S).

In quantum field theory, observables are no longer functions, but are instead operators Φ which act on the Hilbert space. Speaking precisely, these are operator-valued distributions. For example, we could choose a $U \subset M$ and a smooth function f supported in U; then we define

$$\Phi(f) := \int_M \mathrm{dvol}\, f(x)\Phi(x). \tag{5.3}$$

Now let's discuss why we want the divergence operator and the other tools from last time. Strictly speaking, you cannot observe $\Phi(f)$, but instead its expectation values on states $\langle\Phi(f_1)\cdots\Phi(f_n)\rangle$. Heuristically, this is defined via the path integral

$$\langle\Phi(f_1)\cdots\Phi(f_n)\rangle := \frac{\int \mathcal{D}\phi\,\langle f_1\Phi\rangle\cdots\langle f_n\Phi\rangle e^{-S/\hbar}}{\int \mathcal{D}\phi\, e^{-S/\hbar}}. \tag{5.4}$$

This normalization is by the expectation value of just the vacuum, which is appropriately called the *vacuum expectation value* (vev). However, this is not a rigorous definition; the motivation for what we did last week is to be able to replace (5.4) with something mathematical.

Recall that we defined $H^0(\mathrm{Obs}^*(U))$ for $U \subset M$, the observables of a (free) quantum field theory, to be the polynomial functions on $C^\infty(U)$ modulo the image of the divergence operator Div_μ, where μ is a measure. Specifically, Div_μ is a map $\mathrm{Vect}(\mathbb{R}^n) \to P(\mathbb{R}^n)$ which, given a vector v, returns $\mathcal{L}_v(\mu) = \mathrm{d}(i_v\mu)$.

If you're in the image of Div_μ, then you have zero expectation value: following directly from the definition,

$$\int \mathrm{Div}_\mu(v)\,\mathrm{d}\mu = 0. \tag{5.5}$$

Therefore observables whose difference is in $\mathrm{Im}(\mathrm{Div}_\mu)$ can't be distinguished by observations, which is why we mod out by this space.

Recall that $P(C^\infty(U))$ is the direct sum over $n \geq 0$ of $C_c^\infty(U^n)_{S_n}$, where S_n acts by permuting the indices. If P is given by $f_1, \ldots, f_n \in \mathrm{Sym}^n(C_c^\infty(U))$, we can evaluate P on Φ by

$$P(\Phi) = \int \mathrm{dvol}\, f_1\Phi \cdots \int \mathrm{dvol}\, f_n\Phi. \tag{5.6}$$

Last time, we also define $\mathrm{Vect}_c(C^\infty(U))$ in a similar way, but this time we take the direct sum of $C^\infty(U^{n+1})_{S_n}$. Then $\mathrm{Div}_\mu \colon \mathrm{Vect}_c(C^\infty(U)) \to P(C^\infty(U))$ by

$$\mathrm{Div}_\mu\left(f_1, \ldots, f_n, \frac{\partial}{\partial\phi}\right) = -f_1 \cdots f_n \frac{\partial S}{\partial \phi} + \sum_i f_1 \ldots \widehat{f_i} \ldots f_n \int \mathrm{dvol}\, f_i \cdot \phi. \tag{5.7}$$

Here the notation $\widehat{f_i}$ means that f_i is missing from the product.

If for example $S(\phi) = (1/2)\int \mathrm{dvol}\,\Phi(\Delta + m^2)\Phi$ is the action for the free scalar field theory with mass m, then

$$\frac{\partial S}{\partial \Phi}(\Phi) = (\Delta + m^2)\Phi. \tag{5.8}$$

Now, we can recover Wick's theorem by looking at the cokernel of the divergence operator. Specifically, let $G(x,y)\colon U \times U \to \mathbb{R}$ be the Green's function for $\Delta_x + m^2$, i.e.

$$(\Delta_x + m^2)G(x,y) = \delta(x-y), \tag{5.9}$$

and given functions f_1 and f_2, let $\widetilde{\Phi} \coloneqq \int \mathrm{d}y\, G(x,y) f_2(y)$. Then

$$\mathrm{Div}\left(f\frac{\partial}{\partial\widetilde{\Phi}}\right) = -f_1(\Delta + m^2)\widetilde{\Phi} + \int \mathrm{d}x\, f_1(x)\widetilde{\Phi}(x) \tag{5.10}$$

$$= -f_1(x)f_2(y) + \int \mathrm{d}x\,\mathrm{d}y\, f_1(x)G(x-y)f_2(y), \tag{5.11}$$

which implies that

$$\langle \Phi(f_1)\Phi(f_2)\rangle = \int \mathrm{d}x\,\mathrm{d}y\, f_1(x)G(x,y)f_2(y). \tag{5.12}$$

Here $\frac{\partial}{\partial\widetilde{\Phi}}(\widetilde{f}) = \int \mathrm{dvol}\,\widetilde{\Phi}\widetilde{f}$. And the conclusion is, the two-point correlation functions are exactly what we want them to be, which is a special case of Wick's theorem. And more generally, if we have four functions $f_1, f_2.f_3.f_4$, then letting $\widetilde{\Phi} \coloneqq \int \mathrm{d}y\, G(x,y)f_4(x)$, we have the general case of Wick's theorem:

$$\mathrm{Div}\left(f_1f_2f_3\frac{\partial}{\partial\widetilde{\Phi}}\right) = -f_1f_2f_3f_4 + f_1f_2\int \mathrm{d}x\, f_3(x)G(x,y)f_4(x) + \cdots \tag{5.13}$$

As we discussed last time, we do not have a map $H^0(\mathrm{Obs}^*(V)) \otimes H^0(\mathrm{Obs}^*(V)) \to H^0(\mathrm{Obs}^*(V))$, which is saying that we don't quite have an algebra; instead, we can pair observables together iff they live on disjoint opens. This leads to the definition of a prefactorization algebra,

References

[BD82] A. A. Belavin and V. G. Drinfel'd. Solutions of the classical Yang-Baxter equation for simple Lie algebras. *Functional Analysis and Its Applications*, 16(3):159–180, Jul 1982. 6

[CWY17] Kevin Costello, Edward Witten, and Masahito Yamazaki. Gauge theory and integrability, I. 2017. http://arxiv.org/abs/1709.09993. 5, 7

[CWY18] Kevin Costello, Edward Witten, and Masahito Yamazaki. Gauge theory and integrability, II. 2018. http://arxiv.org/abs/1802.01579. 5, 7

[FH16] Daniel S. Freed and Michael J. Hopkins. Reflection positivity and invertible topological phases. 2016. https://arxiv.org/abs/1604.06527. 4

[FT14] Daniel S. Freed and Constantin Teleman. Relative quantum field theory. *Communications in Mathematical Physics*, 326(2):459–476, Mar 2014. https://arxiv.org/abs/1212.1692. 1

[PAY06] J.H.H. Perk and H. Au-Yang. Yang-Baxter equations. In Jean-Pierre Françoise, Gregory L. Naber, and Tsou Sheung Tsun, editors, *Encyclopedia of Mathematical Physics*, pages 465–473. Academic Press, Oxford, 2006. https://arxiv.org/abs/math-ph/0606053. 5

[STY18] Nathan Seiberg, Yuji Tachikawa, and Kazuya Yonekura. Anomalies of duality groups and extended conformal manifolds. *Progress of Theoretical and Experimental Physics*, 2018(7):073B04, 2018. https://arxiv.org/abs/1803.07366. 1, 4

[Wit95] E. Witten. On S-duality in abelian gauge theory. *Selecta Mathematica*, 1(2):383–410, Sep 1995. https://arxiv.org/abs/hep-th/9505186. 2