

How to Build Anomalous (3+1)d Topological Quantum Field Theories

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We develop a systematic framework for constructing (3+1)-dimensional topological quantum field theories (TQFTs) that realize specified anomalies of finite symmetries, as encountered in gauge theories with fermions or fermionic lattice systems. Our approach generalizes the Wang–Wen–Witten symmetry-extension construction to the fermionic setting, building on two recent advances in the study of fermionic TQFTs and related homotopy theory. The first is the categorical classification of anomalous TQFTs in (3+1)d. The second, which we develop further in a planned sequel to this paper, is a *hastened Adams spectral sequence* for computing supercohomology groups, closely paralleling techniques from cobordism theory. By integrating supercohomology and cobordism methods within the recently developed categorical framework of fusion 2-categories, we provide a concrete and systematic route to constructing fermionic TQFTs with specified anomalies, thereby establishing a conceptual bridge between anomaly realization, cobordism, and higher-categorical structures.

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I. Introduction

A central question in high-energy and condensed matter physics is whether a given phase can be realized as the infrared (IR) description of some ultraviolet (UV) theory, where the UV theory may be either a weakly coupled gauge theory or a lattice model. In particular, when a UV theory carries a specified anomaly, the corresponding IR dynamics are forbidden to be *trivial* and difficult to analyze directly. Anomaly matching has proven to be a powerful tool in addressing this question. In (2+1)d, anomaly matching for ordinary symmetries and categorical symmetries, has played a central role in mapping out phase diagrams [1–6], conjecturing dualities among distinct UV gauge theories with matter [7–10], and providing a complete classification of the possible lattice realizations of a given UV theory [11, 12]. This success naturally motivates extending the perspective to higher-dimensional theories.

In this paper we focus on symmetries of theories in (3+1)d. As a concrete example of a theory that exhibits some of the symmetries we consider, consider a gauge theories with N_f left-handed chiral fermions transforming in some representation R of the gauge group. Classically, there is a chiral U(1) symmetry rotating the fermions,

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but the ABJ anomaly reduces this symmetry to

$$\mathrm{U}(1) \rightarrow \mathbb{Z}/(N_f \cdot I_R) \quad (\text{I.1})$$

where I_R is the Dynkin index of the representation. Given the strongly-coupled nature of such theories, it is natural to ask whether they flow to a nontrivial TQFT in the IR. Questions of this kind frequently arise in the study of the dynamics of more general UV gauge theories [13, 14] and, notably, in understanding aspects of the Standard Model [15, 16].

Assuming that the symmetry remains unbroken in the IR, which can be justified either numerically or through theoretical results such as the Vafa–Witten theorem [17, 18], the works of Wang, Wen, and Witten [19] (see also [20]) introduced the *symmetry extension* procedure. Given a specified anomaly, this procedure produces candidate IR topological orders through explicit constructions realized by topological quantum field theories (TQFTs). In this paper, we focus on (3+1)d fermionic theories¹ and ask:

Question I.2. *How can one construct a (3+1)d fermionic TQFT that saturates a given anomaly associated to a finite G -symmetry?*

Recent advances in higher category theory have significantly deepened our understanding of the classification of TQFTs and their symmetry enrichments in (3+1)d. In the bosonic case, such TQFTs are built from nondegenerate braided fusion 2-categories [21, 22], while in the fermionic case they arise from **2sVect**-enriched nondegenerate braided fusion 2-categories [23]. Furthermore, symmetry enrichments and the associated “obstructions” for enrichment, for both bosonic and fermionic (3+1)d TQFTs, have been systematically studied in [23] (see also [24]). This establishes the categorical foundation of our work, which we will leverage for concrete constructions of anomalous TQFTs.

In this work, we incorporate the perspective of anomaly matching and the categorical formulation of (3+1)d TQFTs to answer Question I.2 in some examples. We do so by explicitly providing the construction for the anomalous TQFTs. Specifically, given an anomaly (potentially associated to some specific UV theory), we ask how it can be trivialized by a bigger symmetry group by symmetry extension. This naturally gives the symmetry extension following [19] in the fermionic context. Then this data is fed into the machinery of [23] and we are thus able to construct a candidate (3+1)d fermionic TQFT with the given anomaly.

I.1. Fermionic symmetries and supercohomology

Notably, the classification of (3+1)d TQFTs uses the data of *supercohomology* in an explicit way. Before we state the main results of our paper, we first review mathematical formulations of fermionic symmetries and supercohomology.

A fermionic symmetry [25, §7] is given by a symmetry group G_f , and two additional pieces of data: (1) a map $\rho: G_f \rightarrow \mathbb{Z}/2$ such that the symmetry element is antiunitary or unitary if the image under ρ is 1 or 0, respectively, and (2) a central $\mathbb{Z}/2$ subgroup $\langle (-1)^F \rangle \subset G_f$ in the kernel of ρ generated by fermionic parity. This motivates describing the fermionic symmetry using the following three pieces of data: (1) a (bosonic) symmetry group $G := G_f / \langle (-1)^F \rangle$, (2) a class $s \in H^1(BG; \mathbb{Z}/2)$, corresponding to ρ , and (3) a class $\omega \in H^2(BG; \mathbb{Z}/2)$, classifying the extension $G_f \rightarrow G$.

As a more specific restatement of what was said in the last section, we generalize the work of [19] to the fermionic case and give a procedure for constructing a fermionic (3+1)d TQFT which has a particular G -anomaly valued in the twisted supercohomology group denoted as $SH^5(BG, s, \omega)$. Supercohomology² is a generalized cohomology theory first proposed in [28, 29] for classifying fermionic SPTs, and thus it can be thought of as a “simplification” of the spin cobordism that classifies fermionic SPTs in general; we discuss it in further technical detail in §II.2.

In [28], supercohomology is defined as the cohomology of an explicit chain complex: the n -cochains are triples (a, b, c) as follows:

- a cochain $a \in C^{n-2}(BG; \mathbb{Z}/2)$, called the Majorana layer.
- a cochain $b \in C^{n-1}(BG; \mathbb{Z}/2)$, called the Gu–Wen layer.
- a cochain $c \in C^n(BG; \mathbb{C}^\times)$, called the Dijkgraaf–Witten layer.

The differential mixes together information from different layers, so that cocycles satisfy certain equations relating a , b , and c . These equations were derived in [28, 29], and we review them in §II.2. There we also discuss twisted supercohomology, introduced by [30], which incorporates the data of (s, ω) into those equations.

For any fermionic symmetry (G, s, ω) , it is possible to choose a set of generators of $SH^n(BG, s, \omega)$, such that each generator has a cocycle representative with exactly one of a , b , or c nonzero. Accordingly, we will say that the

¹ In this paper, a fermionic theory refers to a field theory – topological or non-topological – whose definition requires a *twisted* spin structure, which may, but need not, coincide with the ordinary spin structure.

² Confusingly, there are two closely related generalized cohomology theories called “supercohomology:” the *restricted supercohomology* of [26, 27], and the *extended supercohomology* of [28–30]. In this paper we will exclusively use SH to denote the latter, and use rSH for the former.

generator is in the Majorana ($a \neq 0$), Gu–Wen ($b \neq 0$), or Dijkgraaf–Witten ($c \neq 0$) layer as part of our descriptions of supercohomology groups in the main results section.

I.2. Notions for Anomalies of 3+1d Fermionic TQFTs

We now make an important clarification regarding the nature of the obstruction associated with a fermionic $(3+1)$ d TQFTs equipped with a G -symmetry. The full obstruction is fundamentally a *categorical G -obstruction*, which we define as the obstruction in faithfully equipping a **2sVect**-enriched nondegenerate braided fusion 2-category with a G -symmetry, analogous to the bosonic setting in [31].

On the other hand, supercohomology, $SH^5(BG, s, \omega)$, only serves as a subset of this full categorical G -obstruction. The complete structure of the categorical obstruction for fermionic $(3+1)$ d TQFTs also includes contributions from the *super-Witt group* [32], which is the piece that lies beyond the reach of supercohomology.

Unfortunately, understanding the precise structure of the super-Witt group is difficult, and the associated layer in the categorical G -obstruction does not admit a known simple closed-form expression like supercohomology. In contrast, the cocycle formula for supercohomology makes an explicit state-sum construction of the fermionic TQFTs possible, and we are thus able to generalize the constructions developed in [19] to the fermionic setting.

The categorical G -obstruction is also related to, but not the same as, the “usual” notion of ’t Hooft anomalies³ for G -symmetry classified by the cobordism group $I_{\mathbb{Z}}MSpin^n(BG, s, \omega)$ [33], the Anderson dual of (twisted) spin bordism, which we also denote as $U_{Spin}^n(BG, s, \omega)$. The missing piece is the so-called $p + ip$ layer represented by a cochain in $C^{n-3}(BG; \mathbb{Z})$. Interestingly, an anomaly with a nontrivial element in this layer is identified as an obstruction to the construction of fermionic TQFTs saturating the given anomaly in recent works [14, 16]. This piece also does not appear explicitly in the higher-categorical framework of $(3+1)$ d TQFTs [23, 24]. Nevertheless, supercohomology can also be thought of as an approximation to the ’t Hooft anomaly.

Motivated by the utility of cocycles, and by the fact that supercohomology approximates both the full categorical obstruction and usual ’t Hooft anomalies in continuous field theories in the UV, we employ supercohomology throughout our analysis to construct $(3+1)$ -dimensional

topological theories via the fermionic Wang–Wen–Witten construction. Since the ’t Hooft anomalies in the UV theories take values in $U_{Spin}^n(BG, s, \omega)$, we will only consider the elements in supercohomology that remain nontrivial under the natural map from supercohomology to spin bordism.

In §A.2, we will clarify the relation between the categorical G -obstruction, ’t Hooft anomalies (of continuum field theories) with G -symmetry, and supercohomology, in greater details.

I.3. Main Results

As discussed in Subsection I.2, we start with a fermionic symmetry, written as (BG, s, ω) , and construct a $(3+1)$ d topological theory that saturates a particular obstruction valued in supercohomology $SH^5(BG, s, \omega)$. Our construction involves the following steps. We find a group H , with $p: H \rightarrow G$, such that the generator for the group $SH^5(BG, s, \omega)$ trivializes when pulled back to $SH^5(BH, s', \omega')$, where $s' = p^*(s)$ and $\omega' = p^*(\omega)$. The trivialization gives a torsor over $SH^4(BH, s', \omega')$, which, as we will review in §II.3, gives rise to a $(3+1)$ d fermionic G-SET. Suppose there is a subgroup $K \hookrightarrow H$ such that

$$1 \longrightarrow K \longrightarrow H \longrightarrow G \longrightarrow 1; \quad (\text{I.3})$$

then by gauging K we obtain a K -gauge theory, with the Lagrangian description given by a class in $SH^4(BK, s', \omega')$, equipped with a G -symmetry and anomaly $SH^5(BG, s, \omega)$.

We will focus on the symmetry types given by the following examples, and classify all possible anomalies. We will then provide the data of an extension H that is needed to construct $(3+1)$ d topological theories that can saturate each anomaly.

Disclaimer I.4. Our objective is to construct topological quantum field theories that saturate an anomaly that a given UV theory may possess. We do not, however, assert that the theories obtained in this way necessarily arise as the IR limit of the UV theory in question. In particular, there may exist physical mechanisms – beyond the scope of our present analysis – that drive the IR dynamics to a gapless phase. Moreover, alternative choices in the construction we present could lead to distinct yet equally reasonable TQFTs saturating the same anomaly.

Disclaimer I.5. Because we are interested in RG invariants and anomaly matching, we study anomalies as *deformation* classes of reflection-positive invertible field theories (IFTs) with (twisted) spin structure. Freed–Hopkins [33, §5.4] and Grady [34] show that deformation classes of these IFTs are classified by how that deformation classes are described by the Anderson dual $\Sigma I_{\mathbb{Z}}(-)$ of spin bordism. However, classifications of fusion 2-categories most naturally use the Pontryagin dual $I_{\mathbb{C}^\times}(-)$, e.g. in [24, 35–39], and so supercohomology is built using $I_{\mathbb{C}^\times}$ as well.

³ In this paper, by *anomaly* of a TQFT, we refer to the categorical G -obstruction or supercohomology as its approximation. On the other hand, by ’t Hooft anomaly, we refer to the notion of “obstruction to gauging” in a continuum quantum field theory, classified by higher-dimension symmetry-protected topological states (SPTs) through the mechanism of anomaly inflow. See §A.2 for detailed clarification of the relevant concepts.

Though the distinction between $I_{\mathbb{C}^\times}$ and $\Sigma I_{\mathbb{Z}}$ is conceptually important in general, it does not come into play in this paper: for anomalies of finite-group symmetries of 4-dimensional theories, the Pontryagin-to-Anderson map is an isomorphism. We will therefore not dwell on this difference.

Example I.6. We consider fermionic theories with a $G = \mathbb{Z}/n$ symmetry, with the following classification for potential obstructions.

- If n is odd, the map from \mathbb{C}^\times -cohomology to supercohomology is an isomorphism, so $SH^5(BG) \cong \mathbb{Z}/n$.
- If $n = 2$: by Proposition B.11, $SH^5(B\mathbb{Z}/2) = 0$.
- If $n = 2^k$ and $k \geq 2$: by Proposition B.17 $SH^5(BG) \cong \mathbb{Z}/2^{k-1}$.

We now compute groups for H , such that generator for each group that classifies anomalies trivializes when pulled back to $SH^5(BH)$.

Theorem I.7.

1. For n odd, pullback by the map p in the short exact sequence

$$1 \longrightarrow \mathbb{Z}/n \longrightarrow \mathbb{Z}/n^2 \xrightarrow{p} \mathbb{Z}/n \longrightarrow 1, \quad (\text{I.8})$$

is the zero map $SH^5(B\mathbb{Z}/n) \rightarrow SH^5(B\mathbb{Z}/n^2)$.

2. For $k \geq 2$ and $m \geq k/2$, pullback by the map p in the short exact sequence

$$1 \longrightarrow \mathbb{Z}/2^m \longrightarrow \mathbb{Z}/2^{k+m} \xrightarrow{p} \mathbb{Z}/2^k \longrightarrow 1, \quad (\text{I.9})$$

is the zero map $SH^5(B\mathbb{Z}/2^k) \rightarrow SH^5(B\mathbb{Z}/2^{k+m})$.

In this example, each of the summands in $SH^5(BG)$ is accounted for in $\mathcal{U}_{\text{Spin}}^5(BG)$, which classifies SPTs. When we use the term ‘‘SPT’’ we specifically mean an invertible TQFT formulated in the continuum, which is a natural object to associate to gauge theories. Hence by gauging the $\mathbb{Z}/2$ -subgroup of $\mathbb{Z}/2n$, or the $\mathbb{Z}/2^m$ -subgroup of $\mathbb{Z}/2^{k+m}$ we see that:

Corollary I.10. Any class $\alpha \in SH^5(BG)$, where G is one of the groups in Example I.6, can be realized as the anomaly of a $(3+1)d$ gauge theory by gauging the \mathbb{Z}/n , resp. $\mathbb{Z}/2^m$ subgroups of a \mathbb{Z}/n^2 , resp. $\mathbb{Z}/2^{k+m}$ symmetry as in Equations (I.8) and (I.9).

Example I.11. We now consider theories with $G = \mathbb{Z}/2n$ symmetry, where the $\mathbb{Z}/2$ subgroup is generated by fermion parity. The symmetry algebra is given by:

$$g^n = (-1)^F, \quad (\text{I.12})$$

where g is the generator for G , and $(-1)^F$ is fermion parity. We consider the following cases for the obstruction associated to this symmetry, which in general is presented as a twisted supercohomology group.

- If n is odd, the twist is trivial, and $SH^5(B\mathbb{Z}/n) \cong \mathbb{Z}/n$ as in Example I.6.
- If $n = 2$: by Proposition B.21, $SH^5(B\mathbb{Z}/2, 0, x^2) \cong \mathbb{Z}/8$, where $x \in H^1(B\mathbb{Z}/2; \mathbb{Z}/2)$, and the generator of this $\mathbb{Z}/8$ resides in the Majorana layer.
- If $n = 2^k$ and $k \geq 2$: by Proposition B.25, $SH^5(B\mathbb{Z}/2^k, 0, y) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2^{k+1}$. This isomorphism may be chosen so that a generator of $\mathbb{Z}/2^{k+1}$ is in the Gu–Wen layer, and the generator for $\mathbb{Z}/2$ is in the Majorana layer.

In each case above, the map from twisted supercohomology to twisted spin cobordism is injective in degree 5. Therefore these twisted supercohomology classes give rise to SPTs.

Theorem I.13.

1. The map p in the short exact sequence

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z}/4 \xrightarrow{p} \mathbb{Z}/2 \longrightarrow 1, \quad (\text{I.14})$$

induces a map $p^*: \mathbb{Z}/8 \cong SH^5(B\mathbb{Z}/2, 0, x^2) \rightarrow SH^5(B\mathbb{Z}/4, 0, y) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/8$ whose kernel is $2\mathbb{Z}/8$. The map q in

$$1 \longrightarrow \mathbb{Z}/4 \longrightarrow \mathbb{Z}/8 \xrightarrow{q} \mathbb{Z}/2 \longrightarrow 1, \quad (\text{I.15})$$

induces the zero map $SH^5(B\mathbb{Z}/2, 0, x^2) \rightarrow SH^5(B\mathbb{Z}/8, 0, 0)$.

2. If $m \geq (k-1)/2$, then pullback along the map p in the sequence

$$1 \longrightarrow \mathbb{Z}/2^{m+1} \longrightarrow \mathbb{Z}/2^{k+m+1} \xrightarrow{p} \mathbb{Z}/2^k \longrightarrow 1, \quad (\text{I.16})$$

is the zero map $SH^5(B\mathbb{Z}/2^k, 0, y) \rightarrow SH^5(B\mathbb{Z}/2^{k+m+1}, 0, 0)$.

The case of odd n was already covered by Theorem I.7.

Corollary I.17. Any class in $SH^5(B\mathbb{Z}/2, 0, x^2)$, resp. $SH^5(B\mathbb{Z}/2^k, 0, y)$, can be realized as the anomaly of a $(3+1)d$ gauge theory by gauging the $\mathbb{Z}/4$, resp. $\mathbb{Z}/2^{m+1}$, subgroup of a $\mathbb{Z}/8$, resp. $\mathbb{Z}/2^{k+m+1}$ symmetry as in Equations (I.15) and (I.16).

Remark I.18. We have not checked whether the number m in Theorem I.13, part (2), is minimal. If it is not, there would be more efficient constructions of the TQFTs in Corollary I.17.

Example I.19. We also give one example that involves time-reversal, fermion parity, and the chiral symmetry all interacting. Let T be the generator of a time-reversal symmetry. We consider the following symmetry algebra, where $k \geq 2$:

$$g^{2^k} = T^2 = (-1)^F. \quad (\text{I.20})$$

This corresponds to the twist $s = x_1$, $\omega = y$ for the group $\mathbb{Z}/2 \times \mathbb{Z}/2^k$, where x_1 generates $H^1(B\mathbb{Z}/2; \mathbb{Z}/2)$ and y generates $H^2(B\mathbb{Z}/2^k; \mathbb{Z}/2)$. We compute the corresponding twisted supercohomology group in Proposition B.37:

$$SH^5(B\mathbb{Z}/2 \times B\mathbb{Z}/2^k, x_1, y) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4, \quad (\text{I.21})$$

We also describe how to choose this isomorphism such that the classes α_{Maj} , α_{DW} , and α_{GW} mapping to $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ under (I.21), respectively, are in the Majorana, Dijkgraaf–Witten, and Gu–Wen layers respectively, and we show that the kernel of the map to spin bordism is the subgroup generated by α_{Maj} .

Remark I.22. If x denotes the generator of $H^1(B\mathbb{Z}/2^k; \mathbb{Z}/2)$, then the twists (x_1, y) and $(x_1, x^2 + y)$ over $B\mathbb{Z}/2 \times B\mathbb{Z}/2^k$ are equivalent: as we describe in Appendix B.5, they are exchanged by an automorphism of $B\mathbb{Z}/2 \times B\mathbb{Z}/2^k$. This corresponds to switching whether $T^2 = g^{2^k}$ equals 1 or $(-1)^F$. We will work with (x_1, y) in this paper.

Theorem I.23. *If p denotes the map in the short exact sequence*

$$1 \rightarrow \mathbb{Z}/4 \times \mathbb{Z}/2 \rightarrow \mathbb{Z}/8 \times \mathbb{Z}/2^{k+1} \xrightarrow{p} \mathbb{Z}/2 \times \mathbb{Z}/2^k \rightarrow 1, \quad (\text{I.24})$$

then α_{GW} and α_{DW} are in the kernel of

$$\begin{aligned} p^*: SH^5(B\mathbb{Z}/2 \times B\mathbb{Z}/2^k, x_1, y) &\longrightarrow \\ &SH^5(B\mathbb{Z}/8 \times B\mathbb{Z}/2^{k+1}, x_1, 0). \end{aligned} \quad (\text{I.25})$$

As in Remark I.18, it is in principle possible that a smaller extension could kill α_{GW} and α_{DW} .

We would like to continue the story of Corollaries I.10 and I.17 using Theorem I.23, but the technical framework we use has not been developed in the case of time-reversal symmetries. Importantly, the fusion 2-categories used to construct G -SETs in [23] have not yet been shown to admit a higher unitary, or “higher dagger” structure, in the sense of [40]. For SETs with time-reversal symmetry, the appropriate framework should be given by objects in the category of higher Hilbert spaces [41], which can be viewed heuristically as unitary fusion 2-categories. However, since the theory of unitary fusion 2-categories has not yet been systematically developed, we state the following result only as a conjecture – although we believe that a complete and rigorous formulation should be achievable.

Conjecture I.26. *If $\beta \in SH^5(B\mathbb{Z}/2 \times B\mathbb{Z}/2^k, x_1, y)$ is in the subgroup spanned by α_{DW} and α_{GW} , there is a $(3+1)d\mathbb{Z}/4 \times \mathbb{Z}/2$ gauge theory with anomaly β obtained by generalizing the constructions in Corollaries I.10 and I.17.*

We summarize the results stemming from the last three examples compactly in Table I. While supercohomology is theoretically speaking the obstruction theory that make the most sense to use, it also has its own challenges that we will elaborate on in §A.2. The main one being

that computing these groups using the Atiyah–Hirzebruch spectral sequence (AHSS) in higher degrees is challenging, especially when focusing on 0-form symmetries as in this paper. It is common to solve these problems with the Adams spectral sequence, but for supercohomology that becomes complicated quickly, even for the relatively small groups we study in this paper. Thus, we develop a hastened Adams spectral sequence in [42] for performing computations with supercohomology, as well as for some related generalized cohomology theories. In particular, for the computation in Example I.19, the Atiyah–Hirzebruch spectral sequence leaves ambiguous a difficult extension problem, but using the hastened Adams spectral sequence it can be efficiently resolved.

I.4. Outline

The structure of this paper is as follows. In §II.1 we review the Wang–Wen–Witten construction. In §II.2 we explain how to realize the twists of supercohomology. In §II.3 we explain the classification data for $(3+1)d$ G -SETs, which is an essential ingredient for the construction of anomalous fermionic TQFTs. In §III we perform the computations of degree 5 supercohomology for the examples we considered in §I.3, and show how to trivialize the generators for these groups. In Appendix A we give a comprehensive account of the three cohomology theories that appear, discuss the merits of each, and justify our choice of using supercohomology for constructing TQFTs. Finally, in Appendices B and C we fill in the technical details regarding the computations needed for §I.3.

II. Preliminaries

II.1. Symmetry Extension By Wang–Wen–Witten

We now review the symmetry extension procedure of Wang–Wen–Witten [19], who construct a bosonic n -dimensional TQFT with G -symmetry that is the boundary of a bulk invertible field theory, labeled by a class $\pi \in H^{n+1}(BG; \mathbb{C}^\times)$. We denote by $\tilde{\pi} \in Z^{d+1}(BG; \mathbb{C}^\times)$ a cocycle lift of π .⁴ In particular, Wang–Wen–Witten show how to construct a K -gauge theory with anomalous G -symmetry, provided the following data.

- A group H such that

$$1 \longrightarrow K \longrightarrow H \xrightarrow{p} G \longrightarrow 1 \quad (\text{II.1})$$

is a short exact sequence, where K is a finite abelian group.

⁴ While the classification of bosonic invertible field theories goes beyond simply group cohomology, we will only consider those that are classified by cohomology in this review.

G	s	ω	SH^5	H	s	ω	SH^5	Trivialized	H'	s	ω	SH^5	Trivialized
$\mathbb{Z}/k, k \text{ odd}$	0	0	($\mathbb{Z}/k, \text{DW}$)	\mathbb{Z}/k^2	0	0	($\mathbb{Z}/k^2, \text{DW}$)	($\mathbb{Z}/k, \text{DW}$)	—	—	—	—	—
$\mathbb{Z}/2^k, k \geq 2$	0	0	($\mathbb{Z}/2^{k-1}, \text{DW}$)	$\mathbb{Z}/2^{k+m}, k \geq 2$	0	0	($\mathbb{Z}/2^{k+m-1}, \text{DW}$)	($\mathbb{Z}/2^{k-1}, \text{DW}$)	—	—	—	—	—
$\mathbb{Z}/2$	0	x^2	($\mathbb{Z}/8, \text{Maj}$)	$\mathbb{Z}/4$	0	0	($\mathbb{Z}/2, \text{DW}$)	($\mathbb{Z}/4, \text{Maj}$)	$\mathbb{Z}/8$	0	0	($\mathbb{Z}/4, \text{DW}$)	($\mathbb{Z}/2, \text{DW}$)
$\mathbb{Z}/2^k, k \geq 2$	0	y	($\mathbb{Z}/2^{k+2}, \text{GW}$) $\oplus (\mathbb{Z}/2, \text{Maj})$	$\mathbb{Z}/2^{k+1}$	0	0	($\mathbb{Z}/2^k, \text{DW}$)	($\mathbb{Z}/2, \text{GW}$) $\oplus (\mathbb{Z}/2, \text{Maj})$	$\mathbb{Z}/2^{k+m+1}$	0	0	($\mathbb{Z}/2^{k+m}, \text{DW}$)	($\mathbb{Z}/2^k, \text{DW}$)
$\mathbb{Z}/2 \times \mathbb{Z}/2^k, k \geq 2$	x_1	$y/y + x_1^2$	$\mathbb{Z}/2 \oplus (\mathbb{Z}/2, \text{DW})$ $(\mathbb{Z}/4, \text{GW})$	$\mathbb{Z}/2 \times \mathbb{Z}/2^{k+1}$	x_1	0	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4$ Order at Most 32	($\mathbb{Z}/2, \text{DW}$)	—	—	—	—	—
$\mathbb{Z}/2 \times \mathbb{Z}/2^k, k \geq 2$	x_1	$y/y + x_1^2$	$\mathbb{Z}/2 \oplus (\mathbb{Z}/2, \text{DW})$ $(\mathbb{Z}/4, \text{GW})$	$\mathbb{Z}/8 \times \mathbb{Z}/2^{k+1}$	x_1	y	Order at Most 32	($\mathbb{Z}/4, \text{GW}$)	—	—	—	—	—

TABLE I: On the left we consider the groups G with their associated twists $s \in H^1(BG; \mathbb{Z}/2)$ and $\omega \in H^2(BG; \mathbb{Z}/2)$.

In the columns titled “ SH^5 ”, each item within a box gives the direct summands of the full group $SH^5(X, s, \omega)$.

Alongside each summand we provide the layer in which the generator for that group resides: either the Majorana (Maj), Gu–Wen (GW), or Dijkgraaf–Witten (DW) layer. The generator in red indicates that it does not appear in twisted spin bordism. On the right we consider the extended group (H and H') with associated twists. In the first column titled “Trivialized” we show which generators of $SH^5(BG, s, \omega)$ are trivialized when pulled to $SH^5(BH, s, \omega)$. If the whole group is not trivialized, then the second column titled “Trivialized” explains how the remainder is trivialized when pulled back to $SH^5(BH', s, \omega)$. In the column H , $m \geq \frac{k}{2}$, and in the column H' , $m \geq \frac{k-1}{2}$.

- A class $\lambda \in H^n(BH; \mathbb{C}^\times)$ parametrizing H -invertible TQFTs on the n -dimensional boundary that realizes the class π in the bulk.
- A class in $H^n(BK; \mathbb{C}^\times)$ obtained by restricting λ to K , giving the Dijkgraaf–Witten action for the K -gauge theory.

The first step is to choose H such that $p^*\tilde{\pi} = d\tilde{\lambda}$, with $\tilde{\lambda} \in C^n(BH; \mathbb{C}^\times)$, i.e. π is trivializable upon pulling back. Let N be a $(n+1)$ -dimensional manifold with $\partial N = M$, and $P \rightarrow N$ be a principal G -bundle whose restriction to M lifts to a principal H -bundle $Q \rightarrow M$. Let $\int_N \pi(Q)$ be the action for the invertible TQFT on N (i.e. pull back π by the classifying map for Q). Provided

$$g^*\tilde{\pi} = d(h^*\tilde{\lambda}) \quad (\text{II.2})$$

is satisfied on the boundary, we can construct an almost trivial boundary theory with partition function

$$\mathcal{Z}(M) = \frac{1}{\text{Aut}(P)} \cdot \sum_{P \in \pi_0 \mathcal{B}\text{un}_H(M)} \exp \left(-2\pi i \int_M \tilde{\lambda}(P) \right). \quad (\text{II.3})$$

Here $\tilde{\lambda}(P)$ denotes the pullback of $\tilde{\lambda}$ by the classifying map of P .

This theory couples to the bulk theory, since

$$\int_N g^*\tilde{\pi} - \int_M h^*\tilde{\lambda} \quad (\text{II.4})$$

is well defined. By taking the restriction $p^*\tilde{\pi}|_K = d\tilde{\lambda}|_K$, we find that $p^*\tilde{\pi}|_K$ trivializes on K , due to exactness. Hence, it is possible to gauge the K -symmetry. This results in a K -gauge theory with an anomalous G -symmetry whose anomaly is given by the cocycle $\tilde{\pi}$.

To make the construction more explicit, we spell out the steps to produce the K -gauge theory with G -symmetry. Let $\widehat{K} = \text{Hom}(K, \text{U}(1))$. Given a K -valued m -cochain ω and a \widehat{K} -valued n -cochain θ , we will let $\omega \cup \theta$ denote the

logarithm of the Pontryagin pairing of ω and θ , so that it is an element of $C^{m+n}(-; \mathbb{R}/\mathbb{Z})$.

We choose K so that $\pi = e \cup z$ for $e \in Z^2(BG; K)$, and $z \in Z^{n-1}(BG; \widehat{K})$. Such a choice of K is shown in [43, §2.7] to always exist when $n \geq 3$.

We take $a \in C^1(BG; K)$ such that $da = e$, and hence $\tilde{\lambda} = p^*(-a \cup z)$, which satisfies $d\tilde{\lambda} = p^*\tilde{\pi}$. By substituting $p^*(-a \cup z)$ into the exponent in Equation (II.3) we get

$$\exp \left(2\pi i \int_M h^*a \cup h^*z \right) \quad (\text{II.5})$$

where we will leave implicit the pullback by p^* . We will take $h^*z = g^*z$, by restricting h^* to the part that pulls back cocycles valued in BG , and treat z as a BG cocycle. We let $a' = h^*a \in C^1(M; K)$ so that $da' = g^*e$. By adding in the boundary term $d(a' \cup b)$ into the exponential, the full partition function can be written as⁵

$$\mathcal{Z} \sim \sum_{\substack{a' \in C^1(M; K) \\ b \in C^{n-2}(M; \widehat{K})}} \exp \left(2\pi i \int_M a' \cup g^*z + g^*e \cup b + a' \cup db \right) \quad (\text{II.6})$$

where the coupling to G -symmetry is through the terms g^*z and g^*e . Though we did not write it to avoid clutter, there is also a normalization in (II.6).

A natural next step is to generalize the Wang–Wen–Witten construction to the fermionic case, to produce fermionic TQFTs that saturate some anomaly. In [44], Kobayashi–Ohmori–Tachikawa make progress in this direction by writing down a path integral for a boundary fermionic TQFT, for which the bulk SPT is classified by a cocycle pair (β, γ) with $\gamma \in C^{d+1}(BG; \mathbb{C}^\times)$ and $\beta \in C^d(BG; \mathbb{Z}/2)$ is a beyond-cohomology layer. As we

⁵ Though this looks like an infinite sum, it is nonzero for only finitely many choices of a and b , hence is well-defined.

mentioned in the introduction (and we will say more about in §II.2), the full obstruction should have contributions in supercohomology which notably includes a third layer, the *Majorana layer*. However, a path integral description of the K -gauge theory that generalizes [44] to include the third layer remains elusive.

Therefore it remains unclear what the Wang–Wen–Witten construction using supercohomology, i.e. pulling back a supercohomology class in such a way that it trivializes on a larger group, actually yields in the fermionic setting. A priori it is only a formal manipulation. To ameliorate this situation, we will explain in §II.3 how a fermionic Wang–Wen–Witten construction naturally arises when axiomatizing anomalous (3+1)d fermionic theories with fusion 2-categories. Specifically, we review the classification (3+1)d G -SETs, which uses fusion 2-categories and twisted supercohomology. The data required to implement the Wang–Wen–Witten construction is precisely what the classification of (3+1)d G -SETs provides. This offers a conceptual foundation for why even without a path integral presentation in terms of cocycles, it is possible to construct a well-defined (3+1)d fermionic topological quantum field theory that is the boundary for an invertible fermionic topological field theory.

In cases where the symmetry group involves time-reversal that mixes nontrivially with a finite unitary symmetry and fermion parity, one would expect to construct a boundary TQFT from a class in $SH^5(X, s, \omega)$ with $s \neq 0$. However, the theory of fusion 2-categories so far does not accommodate twists of supercohomology that arise when G has antiunitary generators, and thus the categorical description for G -SETs involving time reversal is not fully fledged. While a complete formulation of the corresponding category with the appropriate unitarity structures has yet to be established, we do not anticipate any fundamental obstructions to its construction. We thus conjecture, by means of a physically reasonable extrapolation to the unitary setting, that there is an extension of the theory of fusion 2-categories to not-necessarily-unitary twists, which agrees with the Wang–Wen–Witten construction applied to $SH^5(X, s, \omega)$.

II.2. Twisted Supercohomology in Two Ways

There are two ways of realizing supercohomology that will be important for this work: the first is as the Pontryagin dual of the spectrum $\tau_{\leq 2} ko$, which fits into the following fiber sequence:

$$\tau_{\geq 4} ko \longrightarrow ko \longrightarrow \tau_{\leq 2} ko. \quad (\text{II.7})$$

The homotopy groups and k -invariants of SH can thus be read off of those of ko : see [45, Proof of Lemma 5.6] for the latter.

The second realization of supercohomology is in terms of the Picard 2-groupoid $\mathbf{2sVect}^\times$. From this definition of supercohomology, we see that the homotopy groups

are given by [46, (1.39)]

$$\pi_{-2}(SH) = \mathbb{Z}/2, \quad \pi_{-1}(SH) = \mathbb{Z}/2, \quad \pi_0(SH) = \mathbb{C}^\times. \quad (\text{II.8})$$

The k -invariants are calculated in (*ibid.*, (1.42)). The k -invariant connecting the two copies of $\mathbb{Z}/2$ is

$$\text{Sq}^2: H^*(-; \mathbb{Z}/2) \rightarrow H^{*+2}(-; \mathbb{Z}/2) \quad (\text{II.9})$$

and the k -invariant connecting $\mathbb{Z}/2$ with \mathbb{C}^\times is

$$(-1)^{\text{Sq}^2}: H^*(-; \mathbb{Z}/2) \rightarrow H^{*+2}(-; \mathbb{C}^\times). \quad (\text{II.10})$$

A mild variant of an argument of Gaiotto–Johnson–Freyd [47, §5.4] proves these two definitions of supercohomology coincide.

Both the homotopical and the categorical perspectives generalize to twisted supercohomology. From the homotopical point of view, the map $ko \rightarrow \tau_{\leq 2} ko$ induces a map of twisting data, so we can use twists of ko -theory to twist SH . Given a space X , choose $s \in H^1(X; \mathbb{Z}/2)$ and $\omega \in H^2(X; \mathbb{Z}/2)$. The data (s, ω) defines a twist of ko -theory over X [48], hence also define a twist of SH over X . We denote by $SH^n(X, s, \omega)$ the corresponding degree- n twisted supercohomology group. When $s = 0$, this is sometimes written $SH^{n+\omega}(X)$, e.g. in [24, 37, 39]; see those papers for some examples of applications of twisted supercohomology to fusion 2-category theory.

To use the Wang–Wen–Witten construction, we also need a cocycle description of twisted supercohomology. We now present the following conditions that the cocycles of §I.1 must satisfy in the twisted setting:

- the cochain $a \in C^{n-2}(BG; \mathbb{Z}/2)$ solves $da = 0$,
- the cochain $b \in C^{n-1}(BG; \mathbb{Z}/2)$ solves $db = (\text{Sq}^2 + \omega)a$, and
- the cochain $c \in C^n(BG; \mathbb{C}^\times)$ solves $dc = (-1)^{(\text{Sq}^2 + \omega)b} \cdot f_\omega(a)$.⁶

Using the homotopical definition of supercohomology, one can calculate supercohomology groups using Atiyah–Hirzebruch spectral sequence (AHSS). In [42], we also develop a complementary tool, the *hastened Adams spectral sequence* (HASS), helps resolve many extensions in the AHSS. Importantly, for almost all the examples in Table I we will need to use the hastened Adams spectral sequence to compute the value of the degree 5 supercohomology. In order to turn the symmetries we consider in the examples of §I.3 into a computation involving twisted supercohomology, we first realize the symmetry as a twisted spin structure for the background manifold. By using the relationship between twisted spin bordism and twisted ko , followed by the relationship between twisted

⁶ The cochain $f_\omega(a)$ represents the failure of $(\text{Sq}^2 + \omega)b$ to be closed.

ko with twisted $\tau_{\leq 2} ko$, we can transfer the data defining the twisted spin structure to the data for a twisted supercohomology computation.

Our second perspective on twisted supercohomology makes contact with applications in the fusion 2-categories literature [23, 24, 35, 37–39, 49]. The standard way in which twisted supercohomology arises in the context of fusion 2-categories (see, e.g., [37, §4]) is in terms of local systems with fiber $\mathbf{2sVect}^\times$. Specifically, the homotopy groups of this Picard 2-groupoid are

$$\begin{aligned}\pi_0 \mathbf{2sVect}^\times &= \mathbb{Z}/2, & \pi_1 \mathbf{2sVect}^\times &= \mathbb{Z}/2, \\ \pi_2 \mathbf{2sVect}^\times &= \mathbb{C}^\times,\end{aligned}\tag{II.11}$$

with the unique nontrivial Postnikov invariants connecting the groups [46, 50]. Therefore the spectrum corresponding to $\mathbf{2sVect}^\times$ under the stable homotopy hypothesis [51, 52] is $I_{\mathbb{C}^\times}(\tau_{\leq 2} ko) = SH$, as it has isomorphic homotopy groups and Postnikov invariants. Thus, the abelian group of homotopy classes of maps

$$X \longrightarrow B^{n-2} \mathbf{2sVect}^\times\tag{II.12}$$

is naturally isomorphic to $SH^n(X)$.

Like for twisted ordinary cohomology, we will use automorphisms of $\mathbf{2sVect}^\times$ to twist supercohomology. The automorphisms of interest to us are:

Fermion parity: tensor a 1-morphism with the odd line.
This defines a $B\mathbb{Z}/2$ -action.

Duality: send objects, 1-morphisms, and 2-morphisms to their duals. This almost defines a $\mathbb{Z}/2$ -action.

The Koszul sign rule means that duality does not square to the identity, but rather participates in an abelian 2-group extension with fermion parity:

$$0 \rightarrow B\mathbb{Z}/2 \rightarrow \mathbb{A} \rightarrow \mathbb{Z}/2 \rightarrow 0.\tag{II.13}$$

2-group extensions of the form (II.13) are classified by $H^3(B\mathbb{Z}/2; \mathbb{Z}/2) \cong \mathbb{Z}/2$ [53, Theorem 1], so the extension \mathbb{A} of duality by fermion parity is uniquely specified up to isomorphism by the fact that it is non-split.

Thus, given a space X and a map $f: X \rightarrow B\mathbb{A}$, we can form the associated bundle

$$\begin{array}{ccc}(B^{n-2} \mathbf{2sVect}^\times) \times_{\mathbb{A}} f^*(E\mathbb{A}) & & \\ \downarrow & & \\ X. & & \end{array}\tag{II.14}$$

Then $SH^{n+f}(X)$ is the abelian group of homotopy classes of sections of (II.14).

Though \mathbb{A} is not split, there is a homotopy equivalence of spaces $B\mathbb{A} \simeq B\mathbb{Z}/2 \times B^2\mathbb{Z}/2$, so we will identify a twist of supercohomology by a triple (X, s, ω) , where $s \in H^1(X; \mathbb{Z}/2)$ and $\omega \in H^2(X; \mathbb{Z}/2)$, matching the homotopical definition of twisted supercohomology.

Hence if X is a space equipped with a map $\omega: X \rightarrow B^2\mathbb{Z}/2$, the ω -twisted n -th supercohomology of X is the

group of homotopy classes of $B\mathbb{Z}/2$ -equivariant maps from X to $B^{n-2} \mathbf{2sVect}^\times$. In the companion paper [42], we show that the two notions of (X, s, ω) -twisted supercohomology that we have introduced are naturally isomorphic.

Remark II.15. In the context of fusion 2-categories the s -twist in the first definition of twisted supercohomology has not previously appeared in the literature. One reason for this is because the TQFTs that fusion 2-categories were made to construct are oriented [54]. It would be interesting to have a definition of fusion 2-categories with a unitary structure, to parallel what exists for fusion 1-categories; symmetries of unitary fusion 2-categories could potentially correspond to twists with $s \neq 0$.⁷

Remark II.16. The space of homotopy equivalences $\phi: B\mathbb{A} \xrightarrow{\sim} B\mathbb{Z}/2 \times B^2\mathbb{Z}/2$ is not connected, implying there is an ambiguity in how we identified the data (s, ω) with a twist of supercohomology. There are a few ways to address this, which we will discuss in more detail in [42]. We choose the (standard) convention that, if $a \in H^1(B\mathbb{Z}/2; \mathbb{Z}/2)$ denotes the unique nonzero class, that $(B\mathbb{Z}/2, a, 0)$ -twisted supercohomology maps to the twist of spin cobordism which is pin^- cobordism, rather than pin^+ cobordism.

This ambiguity does not affect twists (X, s, ω) for which $s = 0$, so it will not play a major role in this paper.

II.3. Anomalous Topological Orders

We now summarize how the classification of (3+1)d fermionic G -SETs, as well as their anomalies, is formulated in terms of fusion 2-categories and twisted supercohomology. We then explain how this classification naturally integrates into the framework of the fermionic Wang–Wen–Witten construction, and why it is useful for constructing fermionic topological theories that saturate an anomaly.

The classification of (3+1)d fermionic G -SETs is conducted by first starting with a “closed” topological order i.e. one without symmetry, and then enriching by a finite G -symmetry. This enrichment is made precise categorically in [23], and physically it amounts to adding in G -symmetry defects into the theory. Let $\mathcal{Z}(\mathfrak{C})$ denote the Drinfeld center of a fusion 2-category \mathfrak{C} [60], which is a nondegenerate braided fusion 2-category. The closed (3+1)d topological orders were classified in [21, 22, 61], and separated into three cases:

1. When all the excitations are bosonic, the category that describes the topological order takes the form $\mathcal{Z}(\mathbf{2Vect}_K^\pi)$, where $\mathbf{2Vect}_K^\pi$ denotes the fusion 2-category of K -graded 2-vector spaces with pentagonator twisted by a class $\pi \in H^4(BK; \mathbb{C}^\times)$ [54, Construction 2.1.16].⁸

⁷ See [40, 41, 55–59] for recent progress towards unitary higher categories.

2. When the spectrum contains an emergent fermion, the category that describes the topological order takes the form $\mathcal{Z}(\mathbf{2sVect}_K^\varpi)$, where $\mathbf{2sVect}_K^\varpi$ is the fusion 2-category of K -graded 2-super vector spaces with pentagonator twisted by (a cocycle representative of) $\varpi \in SH^4(BK, \omega)$ [61].
3. When the spectrum contains a local fermion, the theories are classified by classes in $SH^4(BK)$ [22, Corollary V.5].

For the symmetry structures given in Table I, whose SH^5 obstruction we would like to saturate by a (3+1)d topological theory, our primary focus will be on the topological orders corresponding to the second and third cases. We hence use the term “fermionic” for theories in either the second or third item above. We note that the categories in the first and second item of the list describes (almost all)⁸ of the nondegenerate braided fusion 2-categories, while the categories in the third item are nondegenerate **2sVect**-enriched braided fusion 2-categories. Analogous to bosonic G -SETs in (2+1)d, bosonic (3+1)d G -SETs are, categorically, nondegenerate faithfully graded G -crossed braided fusion 2-categories. The work of [23] shows that in the case when the SET has a local fermion we have:

Theorem II.17 ([23, Proposition 4.4]). *(3+1)d G -SETs with local fermions are equivalent to nondegenerate **2sVect**-enriched braided fusion 2-categories with a fully faithful braided 2-functor from **2Rep**(G).*

In the case where the G -SET has an emergent fermion Theorem II.17 can be generalized following [23, Remark 4.6].

Theorem II.18. *(3+1)d fermionic G -SETs with emergent fermions are equivalent to nondegenerate fermionic braided fusion 2-categories with a fully faithful braided 2-functor from **2Rep**(G_f).*

Here, G_f denotes a fermionic symmetry in the sense of §I.1. Nondegenerate fermionic braided fusion 2-categories are classified by the following data [23, Theorem 3.20]: a finite group K , a class $\kappa \in H^2(B^2\mathbb{Z}/2, \mathbb{Z}/2)$, a class $\varsigma \in SH^5(B^2\mathbb{Z}/2, 0, \kappa)$, $\tau \in H^2(BK; \mathbb{Z}/2)$, and $\varpi \in SH^4(BK, 0, \tau)$. For the constructions that are relevant for the examples we present in this paper, we can take those nondegenerate fermionic braided fusion 2-categories where ς is trivial. Such categories, with a fully faithful braided 2-functor from **2Rep**(G_f) are thus classified by the following data:

- A group H fitting into a short exact sequence,
- $$1 \rightarrow K \rightarrow H \rightarrow G \rightarrow 1. \quad (\text{II.19})$$
- A class $\varpi' \in SH^4(BH, 0, \omega')$, where $\omega' \in H^2(BH; \mathbb{Z}/2)$, such that $\omega'|_K = \tau$ and $\varpi'|_K = \varpi$.

Using the classification in Theorem II.18, our generalized Wang–Wen–Witten construction involves the following data.

Ansatz II.20. A (3+1)d fermionic topological theory with G -symmetry and $\varphi \in SH^5(BG, 0, \omega)$ obstruction is realized as a K -gauge theory from the following data:

- A group H such that

$$1 \longrightarrow K \longrightarrow H \xrightarrow{p} G \longrightarrow 1 \quad (\text{II.21})$$

is a short exact sequence.

- A class in $\varpi \in SH^4(BH, 0, \omega')$ parametrizing H -invertible TQFTs on the 4-dimensional boundary that realize the obstruction φ in the bulk. If the obstruction in the bulk actually vanishes, then gauging the entirety of the H -symmetry would lead to a G -SET described by Theorem II.18.
- A class in $SH^4(BK, 0, \omega'|_K)$ giving the Dijkgraaf–Witten action for the K -gauge theory, from gauging the non-anomalous subgroup K of H .

In summary, the rigorous definition of fermionic G -SETs ensures that the corresponding topological order can be obtained by gauging the full H -symmetry described in the second item above. Consequently, gauging any subgroup $K \subset H$ yields a well-defined K -gauge theory, even in the absence of an explicit path-integral construction. In the local fermion case, the data is essentially the same, except we only consider supercohomology and not twisted supercohomology whenever it appears.

We note that since the obstruction φ is valued in twisted supercohomology with $\omega \in H^2(BG; \mathbb{Z}/2)$, if one wants to trivialize the pullback $p^*\varphi$ then it is necessary to pick $\omega' \in H^2(BH; \mathbb{Z}/2)$ such that it is equivalent to the pullback of ω to H . The equation that needs to be solved to trivialize $p^*\varphi$ is given by

$$p^*\tilde{\varphi} = d\tilde{\omega}. \quad (\text{II.22})$$

Here $\tilde{\varphi}$ is a cocycle representative of φ , and we let $\tilde{\omega} = (a, b, c)$ be a cocycle representative of ϖ and $\tilde{\omega}'$ be a cocycle representative of ω' . We see that

$$d\tilde{\omega} = \left(da, db + (\mathrm{Sq}^2 + \tilde{\omega}')a, dc + (-1)^{(\mathrm{Sq}^2 + \tilde{\omega}')b} \cdot f_{\omega'}(a) \right). \quad (\text{II.23})$$

Furthermore, taking the cocycle $\tilde{\varphi} = (\alpha, \beta, \gamma)$ we find that Equation (II.22) becomes the following system of equations:

$$p^*\alpha = da \quad (\text{II.24})$$

$$p^*\beta = db + (\mathrm{Sq}^2 + \tilde{\omega}')a$$

$$p^*\gamma = dc + (-1)^{(\mathrm{Sq}^2 + \tilde{\omega}')b} \cdot f_{\tilde{\omega}'}(a),$$

⁸ To define an actual fusion 2-category, one must choose a cocycle representative of π , but the Morita class, and therefore the topological order, does not depend on this choice.

⁹ The exception is (3+1)d theory \mathcal{T} in [49], which has a gravitational anomaly given by $\int w_2 w_3$.

in which solving for (a, b, c) would allow us to construct the theory with anomaly cocycle $\tilde{\varphi}$.

III. Constructing the Relative TQFTs

We now construct (3+1)d G -SETs with respect to the symmetry structures given in Examples I.6, I.11, and I.19. We do this by explaining how to trivialize generators of $SH^5(BG, s, \omega)$ that parametrize the G -anomaly by pulling back to a larger group. This will establish Theorems I.7, I.13, and I.23. Once an element of $SH^5(BG, s, \omega)$ has been trivialized, one can gauge a subgroup symmetry, following the Wang–Wen–Witten construction, in order to obtain a TQFT realizing that anomaly class.

III.1. Computations for $\mathbb{Z}/n \times \mathbb{Z}/2^F$ (Example I.6)

Proof of Theorem I.7. Let $n = p_1^{k_1} p_2^{k_2} \dots p_\ell^{k_\ell}$ be the prime factorization of n . Then the map $r_j: B\mathbb{Z}/n \rightarrow B\mathbb{Z}/p_j^{k_j}$ induced by the mod $p_j^{k_j}$ reduction map $\mathbb{Z}/n \rightarrow \mathbb{Z}/p_j^{k_j}$ is a homotopy equivalence after localizing at p , so all p -primary torsion in $SH^5(B\mathbb{Z}/n)$ is in the image of r_j^* . Thus, it suffices to consider the case when $n = p^k$, as trivializations in these cases induce trivializations for all n .

When $n = p^k$ is odd, the AHSS is only nontrivial in the Dijkgraaf–Witten layer, because the $\mathbb{Z}/2$ -valued cohomology of $B\mathbb{Z}/n$ vanishes in positive degrees. Therefore, there is a canonical isomorphism $SH^5(B\mathbb{Z}/n) \cong H^5(B\mathbb{Z}/n; \mathbb{C}^\times)$, which has a canonical isomorphism to the group $\mu_n \subset \mathbb{C}^\times$ of n^{th} roots of unity. We now consider how to trivialize the generator $e^{2\pi i/n}$ by pulling back from \mathbb{Z}/n to a larger group. The generator of $H^5(B\mathbb{Z}/n; \mathbb{C}^\times)$ is the image of $xy^2 \in H^5(B\mathbb{Z}/p^k; \mathbb{Z}/p)$ under the exponential map

$$\begin{aligned} \mathbb{Z}/p &\rightarrow \mathbb{C}^\times \\ \ell &\mapsto \exp(2\pi i \ell/p), \end{aligned} \tag{III.1}$$

where $x \in H^1(B\mathbb{Z}/p^k; \mathbb{Z}/p) \cong \mathbb{Z}/p$ and $y \in H^2(B\mathbb{Z}/p^k; \mathbb{Z}/p) \cong \mathbb{Z}/p$ are the standard generators. When we pull back to $H^5(B\mathbb{Z}/p^{2k}; \mathbb{Z}/p)$, y trivializes, so xy^2 pulls back to 0 as well.

When $n = 2$, Proposition B.11 shows that $SH^5(B\mathbb{Z}/2) = 0$.

In the case when $G = \mathbb{Z}/2^k$ for $k \geq 2$, the mod 2 cohomology is given by the ring

$$H^*(B\mathbb{Z}/2^k; \mathbb{Z}/2) = \mathbb{Z}/2[x, y]/(x^2), \quad |x| = 1, \quad |y| = 2. \tag{III.2}$$

Proposition B.17 shows that $SH^5(B\mathbb{Z}/2^k) = \mathbb{Z}/2^{k-1}$, and the generator is in the Dijkgraaf–Witten layer. This means we can express it as a class in $H^5(B\mathbb{Z}/2^k; \mathbb{C}^\times) \cong H^6(B\mathbb{Z}/2^k; \mathbb{Z})$. We pull back the generator of $H^6(B\mathbb{Z}/2^k; \mathbb{Z})$ to $H^6(B\mathbb{Z}/2^{k+m}; \mathbb{Z})$ along the sequence

$$1 \longrightarrow \mathbb{Z}/2^m \longrightarrow \mathbb{Z}/2^{k+m} \longrightarrow \mathbb{Z}/2^k \longrightarrow 1 \tag{III.3}$$

where $y^3 \in H^6(B\mathbb{Z}/2^k; \mathbb{Z})$ pulls back to $8^m y^3$. Therefore the generator of $H^6(B\mathbb{Z}/2^k; \mathbb{Z})$ given by y^3 trivializes when pulled back to $H^6(B\mathbb{Z}/2^{k+m}; \mathbb{Z})$ if $m \geq \frac{k}{2}$. This establishes Theorem I.7. \square

We compare these supercohomology groups with the corresponding groups of SPTs (mathematically, reflection-positive invertible field theories).

- For $n = p^k$ with p odd, $U_{\text{Spin}}^5(B\mathbb{Z}/n)$ is isomorphic to $\mathbb{Z}/n \oplus \mathbb{Z}/n$ if $p \geq 5$ or $\mathbb{Z}/3^{k-1} \oplus \mathbb{Z}/3^{k+1}$ if $p = 3$.¹⁰ In both cases, one can show the map from supercohomology to U_{Spin}^5 is injective.
- For $\mathbb{Z}/2$, we have $SH^5(B\mathbb{Z}/2) = 0$. Similarly, Manhowald’s computation of $ko_*(B\mathbb{Z}/2)$ [65, Lemma 7.3] implies $U_{\text{Spin}}^5(B\mathbb{Z}/2) = 0$.
- For $\mathbb{Z}/2^k$, $k \geq 2$, $SH^5(B\mathbb{Z}/2^k) \cong \mathbb{Z}/2^{k-1}$, and $U_{\text{Spin}}^5(B\mathbb{Z}/2^k) \cong \mathbb{Z}/2^k \oplus \mathbb{Z}/2^{k-2}$ [66].¹¹ One can show the map from SH^5 to U_{Spin}^5 is injective, so that supercohomology realizes a subgroup of the group of (4+1)d $\mathbb{Z}/2^k$ -SPTs.

III.2. Computations for $\mathbb{Z}/(2n)^F$ (Example I.11)

Since the group structure in this example has the unitary \mathbb{Z}/n symmetry mixing with fermion parity, the computations involve twisted supercohomology where the twist arises from a degree 2 cohomology class in $H^2(B\mathbb{Z}/n; \mathbb{Z}/2)$. In the case where n is odd, the twist is trivial, and we can apply the same computation as those in Example I.6.

We first treat the case when $n = 2$, so that

$$g^2 = (-1)^F. \tag{III.4}$$

This corresponds to giving spacetime a G -structure where $G = \text{Spin} \times_{\{\pm 1\}} \mathbb{Z}/4$ [68], which is equivalent to a $(B\mathbb{Z}/2, 0, x^2)$ -twisted spin structure: see [69, §7.8] and [70, Example 6.23]. When $n = 2^k$, the story is similar: we obtain a $\text{Spin} \times_{\{\pm 1\}} \mathbb{Z}/2^{k+1}$ -structure, which is equivalent to a $(B\mathbb{Z}/2^k, 0, y)$ -twisted spin structure:

¹⁰ These isomorphisms follow from computations of Bahri–Bendersky–Davis–Gilkey [62, Theorem 1.2(a)] and Hashimoto [63, Theorem 3.1]: see [64, §12.2] for the details, as well as [16] for a closely related physical construction.

¹¹ This calculation is not stated explicitly in *loc. cit.*, so we spell out the details here. As usual it is equivalent to show that $ko_5(B\mathbb{Z}/2^k) \cong \mathbb{Z}/2^k \oplus \mathbb{Z}/2^{k-2}$; [66, Theorem 2.5] shows that the order of this group is indeed 2^{2k-2} , and (*ibid.*, Proposition 5.1(a)) constructs an element of order 2^k in $ko_5(B\mathbb{Z}/2^k)$. The h_0 -action on the E_∞ -page of the Adams spectral sequence for $ko_*(B\mathbb{Z}/2^k)$ (see [67, Theorem 13.36, Proposition 13.38, and Figure 1]) implies that $ko_5(B\mathbb{Z}/2^k)$ cannot be decomposed into a direct sum of three nontrivial abelian groups, and that no element has order more than 2^k .

see [13] and [70, Example 6.23]. These pass to the corresponding twists of supercohomology as described in §II.2. The obstructions associated to these symmetry structures are captured by $SH^5(B\mathbb{Z}/2, 0, x^2) = \mathbb{Z}/8$ and $SH^5(B\mathbb{Z}/2^k, 0, y) = \mathbb{Z}/2^k \oplus \mathbb{Z}/2$, respectively, which we calculated in Propositions B.21 and B.25.

Proof of Theorem I.13. In the case when n is odd, the same proof as that given for Theorem I.7 applies to trivialize the generator. To trivialize a generator for $SH^5(B\mathbb{Z}/2, 0, x^2) \cong \mathbb{Z}/8$, we consider the symmetry extension sequence given by

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z}/4 \xrightarrow{p} \mathbb{Z}/2 \longrightarrow 1. \quad (\text{III.5})$$

Lemma III.6. $p^*(x^2) = 0$.

Proof. Because x^2 is the unique nontrivial class in $H^2(B\mathbb{Z}/2; \mathbb{Z}/2)$, it classifies any nonsplit central extension of $\mathbb{Z}/2$ by $\mathbb{Z}/2$, such as (III.5). When we pull this extension back along p , it tautologically splits. \square

Thus the twist $(0, x^2)$ over $B\mathbb{Z}/2$ pulls back to the trivial twist $(0, 0)$ over $B\mathbb{Z}/4$. Therefore the map p in Equation

(III.5) induces a pullback map $SH^5(B\mathbb{Z}/2, 0, x^2) \rightarrow SH^5(B\mathbb{Z}/4, 0, 0)$. We now show that this trivializes the $\mathbb{Z}/4$ subgroup $2\mathbb{Z}/8 \subset \mathbb{Z}/8$. To perform this computation in supercohomology, we will study the map $(\tau_{\leq 2} ko)_5(B\mathbb{Z}/2, 0, x^2) \rightarrow (\tau_{\leq 2} ko)_5(B\mathbb{Z}/4, 0, 0)$, and then take the dual with respect to $I_{\mathbb{C}^\times}$. To finish the computation, we will need to use something called the *Smith long exact sequence*. We refer the reader to Theorems B.48 and B.50 for background material and references about this long exact sequence. We first apply the Smith long exact sequence to the analogous twisted spin bordism groups to obtain the map $\Omega_5^{\text{Spin}}(B\mathbb{Z}/2, 0, x^2) \rightarrow \Omega_5^{\text{Spin}}(B\mathbb{Z}/4, 0, 0)$ induced by p . Then by naturality, mapping $M\text{Spin} \rightarrow ko \rightarrow \tau_{\leq 2} ko$, we obtain the map $(\tau_{\leq 2} ko)_5(B\mathbb{Z}/2, 0, x^2) \rightarrow (\tau_{\leq 2} ko)_5(B\mathbb{Z}/4, 0, 0)$.

Let σ be the tautological line bundle over $B\mathbb{Z}/2$; we will also write σ for the pullback bundle across $p: B\mathbb{Z}/4 \rightarrow B\mathbb{Z}/2$. In particular, over both $B\mathbb{Z}/2$ and $B\mathbb{Z}/4$, $w_1(\sigma) = x$. Consider the following two Smith long exact sequences in spin bordism, the first one constructed in [71, (A.29)],¹² and the second constructed in [77, Theorem 3.1], where it is attributed to Stong.¹³

$$\dots \longrightarrow \Omega_5^{\text{Spin}}(B\mathbb{Z}/2, 0, 0) \longrightarrow \Omega_5^{\text{Spin}}(B\mathbb{Z}/4, 0, 0) \xrightarrow{\text{sm}_\sigma} \Omega_4^{\text{Spin}}(B\mathbb{Z}/4, x, 0) \longrightarrow \Omega_4^{\text{Spin}}(B\mathbb{Z}/2, 0, 0) \longrightarrow \dots \quad (\text{III.7a})$$

$$\dots \longrightarrow \Omega_5^{\text{Spin}} \longrightarrow \Omega_5^{\text{Spin}}(B\mathbb{Z}/2, 0, x^2) \xrightarrow{\text{sm}_\sigma} \Omega_4^{\text{Spin}}(B\mathbb{Z}/2, x, x^2) \longrightarrow \Omega_4^{\text{Spin}} \longrightarrow \dots \quad (\text{III.7b})$$

We can form the following commuting square involving the two sequences in Equation (III.7), by looking at the middle entries of each:

$$\begin{array}{ccc} \Omega_5^{\text{Spin}}(B\mathbb{Z}/4, 0, 0) & \xrightarrow{\text{sm}_\sigma} & \Omega_4^{\text{Spin}}(B\mathbb{Z}/4, x, 0) \\ \cong \mathbb{Z}/4 \text{ [82, 7.3.3]} & & \cong \mathbb{Z}/4 \text{ [71, A.28]} \\ \downarrow & & \downarrow 1 \mapsto 4 \\ \Omega_5^{\text{Spin}}(B\mathbb{Z}/2, 0, x^2) & \xrightarrow{\text{sm}_\sigma} & \Omega_4^{\text{Spin}}(B\mathbb{Z}/2, x, x^2) \\ \cong \mathbb{Z}/16 \text{ [83, §3]} & & \cong \mathbb{Z}/16 \text{ [77, §2]} \end{array} \quad (\text{III.8})$$

The entries in the bottom row were originally not phrased in this way; see [70, Example 6.23] for the connection to (s, ω) -twisted spin bordism. The right vertical map in

Equation (III.8) was computed in [71, Proposition A.35 (1)], and the horizontal maps are isomorphisms (see [71, Theorem A.28], resp. [80, §3.4]). This implies the left vertical map takes $1 \mapsto 4$.

¹² $(B\mathbb{Z}/4, x, 0)$ -twisted spin bordism, sometimes called *epin bordism*, is also studied in [72–75]. See also [76] for a closely related symmetry type in a physics application.

¹³ This long exact sequence is studied more systematically, as part of a family of related Smith long exact sequences, in [68, 70, 75, 78–81].

We would like to understand the analogous square in $\tau_{\leq 2} ko$ -homology. We calculate the left two entries in Propositions B.17 and B.21. For the entries on the right, we will show both horizontal arrows sm_σ are isomorphisms. This is because they belong to long exact sequences just as in (III.7), but with Ω_*^{Spin} replaced with $(\tau_{\leq 2} ko)_*$. Thus, to show that the horizontal maps in (III.9) are isomorphisms, it suffices to know that $\tau_{\leq 2} ko_\ell(B\mathbb{Z}/2, 0, 0) = 0$ and $\tau_{\leq 2} ko_\ell(\text{pt}) = 0$ for $\ell = 4, 5$. The former is Proposition B.11 and the latter follows from degree considerations.

Thus we have the commuting square

$$\begin{array}{ccc}
 \boxed{(\tau_{\leq 2} ko)_5(B\mathbb{Z}/4, 0, 0)} & \xrightarrow{\text{sm}_\sigma} & \boxed{(\tau_{\leq 2} ko)_4(B\mathbb{Z}/4, x, 0)} \\
 \downarrow \scriptstyle \cong \mathbb{Z}/2 \text{ (B.17)} & & \downarrow \scriptstyle 1 \mapsto 4 \\
 \boxed{(\tau_{\leq 2} ko)_5(B\mathbb{Z}/2, 0, x^2)} & \xrightarrow{\text{sm}_\sigma} & \boxed{(\tau_{\leq 2} ko)_4(B\mathbb{Z}/2, x, x^2)} \\
 \downarrow \scriptstyle \cong \mathbb{Z}/8 \text{ (B.21)} & & \downarrow \scriptstyle \cong \mathbb{Z}/8
 \end{array} \quad (\text{III.9})$$

That the rightmost map sends $1 \mapsto 4$, just as in (III.8), follows from an AHSS calculation similar to the proof of [71, Proposition A.35 (1)].

By naturality, we can form the following commuting cube that maps between the squares in Equation (III.8) and Equation (III.9):

$$\begin{array}{ccccc}
 & \Omega_5^{\text{Spin}}(B\mathbb{Z}/4, 0, 0) & & \Omega_4^{\text{Spin}}(B\mathbb{Z}/4, x, 0) & \\
 \swarrow & \downarrow & \searrow & \downarrow \scriptstyle 1 \mapsto 4 & \\
 (\tau_{\leq 2} ko)_5(B\mathbb{Z}/4, 0, 0) & \xrightarrow{\quad} & (\tau_{\leq 2} ko)_4(B\mathbb{Z}/4, x, 0) & \xleftarrow{\quad} & \\
 \downarrow & \downarrow & \downarrow \scriptstyle 1 \mapsto 4 & \downarrow \scriptstyle 1 \mapsto 4 & \\
 \Omega_5^{\text{Spin}}(B\mathbb{Z}/2, 0, x^2) & \xrightarrow{\quad} & \Omega_4^{\text{Spin}}(B\mathbb{Z}/2, x, x^2) & \xleftarrow{\quad} & \\
 \swarrow & \searrow & \swarrow & \searrow & \\
 (\tau_{\leq 2} ko)_5(B\mathbb{Z}/2, 0, x^2) & \xrightarrow{\quad} & (\tau_{\leq 2} ko)_4(B\mathbb{Z}/2, x, x^2) & \xleftarrow{\quad} &
 \end{array}$$

We see that the left vertical map in the cube, $(\tau_{\leq 2} ko)_5(B\mathbb{Z}/4, 0, 0) \rightarrow (\tau_{\leq 2} ko)_5(B\mathbb{Z}/2, 0, x^2)$, also takes $1 \mapsto 4$. After dualizing, we find that pulling back across $SH^5(B\mathbb{Z}/2, 0, x^2) \rightarrow SH^5(B\mathbb{Z}/4, 0, 0)$ trivializes the $\mathbb{Z}/4$ subgroup $2\mathbb{Z}/8 \subset \mathbb{Z}/8$. What remains nontrivial in the pullback is a group $\mathbb{Z}/2$, with generator in the Dijkgraaf–Witten layer, expressible as $y^3 \in H^6(B\mathbb{Z}/4; \mathbb{Z})$. Using the same reasoning as the proof of Theorem I.7 given in §III.1, we can trivialize this obstruction by a further extension

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z}/8 \longrightarrow \mathbb{Z}/4 \longrightarrow 1. \quad (\text{III.10})$$

Therefore, the entire group $SH^5(B\mathbb{Z}/2, 0, x^2)$ is trivializable.

The last case to study in this example is when $k \geq 2$. We consider the symmetry extension sequence given by

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2^{k+1} \longrightarrow \mathbb{Z}/2^k \longrightarrow 1. \quad (\text{III.11})$$

The twist y pulls back to the trivial element in $H^2(B\mathbb{Z}/2^{k+1}; \mathbb{Z}/2)$, and therefore we will look to trivialize a generating set of $SH^5(B\mathbb{Z}/2^k, 0, y)$ by pulling back to $B\mathbb{Z}/2^{k+m+1}$ for m sufficiently large – specifically, a class γ_{GW} of order 2^{k+1} , whose image in the E_∞ -page of the AHSS is in the Gu–Wen layer, and a class γ_{Maj} of order 2, whose corresponding image is in the Majorana layer. The details of how each generator is trivialized is given in Appendix C. With these details in place, this concludes the proof of Theorem I.13. \square

Remark III.12. In [16], Cheng–Wang–Yang explicitly construct the TQFT state which realizes the anomaly $(1, 2^{k-2}) \in \mathbb{Z}/2 \oplus \mathbb{Z}/(2^{k+1})$, with the help of the crystalline equivalence principle. Our results match their

results. In particular, we also confirm that $\mathbb{Z}/2$ gauge theory is not enough and the minimal gauge group K has to be $\mathbb{Z}/4$.

III.3. Computations for $\mathbb{Z}/(2^{k+1})^F \times \mathbb{Z}/2^T$ (Example I.19)

In this example the symmetry algebra not only includes fermion parity and a $\mathbb{Z}/2^{k+1}$ unitary symmetry in which the generator g satisfies $g^k = (-1)^F$, but also a time-reversal symmetry, which reverses the orientation of the background manifold. This corresponds to a G -structure for the group $G = \text{Pin}^+ \times_{\{\pm 1\}} \mathbb{Z}/2^{k+1}$.¹⁴ When $k = 1$, this is the “pin– $\mathbb{Z}/4$ structure” studied by Montero–Vafa [84] and Krulewski–Stehouwer [85]; in general, this structure is analogous to a pin^c structure, with U_1 replaced by $\mathbb{Z}/2^{k+1}$. Thus, analogously to how a pin^c structure is equivalent to a $(B\mathbb{Z}/2 \times BU_1, x_1, c_1)$ -twisted spin structure [33, §10], where $x_1 \in H^1(B\mathbb{Z}/2; \mathbb{Z}/2)$ and $c_1 \in H^2(BU_1; \mathbb{Z}/2)$ are the generators, $\text{Pin}^+ \times_{\{\pm 1\}} \mathbb{Z}/2^{k+1}$ structures are equivalent to $(B\mathbb{Z}/2 \times B\mathbb{Z}/2^k, x_1, y)$ -twisted spin structures.

For the rest of this subsection, assume $k > 1$. By Proposition B.37, there is an isomorphism $SH^5(B\mathbb{Z}/2 \times$

¹⁴ This structure is equivalent to $\text{Pin}^- \times_{\{\pm 1\}} \mathbb{Z}/2^{k+1}$ via an automorphism of $\mathbb{Z}/2 \times \mathbb{Z}/2^{k+1}$, analogously to how Pin^c is isomorphic to both $\text{Pin}^+ \times_{\{\pm 1\}} U_1$ and $\text{Pin}^- \times_{\{\pm 1\}} U_1$. Thus, depending on one’s choice of generator T for the time-reversal symmetry, one could have $T^2 = 1$ or $T^2 = (-1)^F$.

$B\mathbb{Z}/2^k, x_1, y) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4$, and we may choose the isomorphism such that

- the class $\alpha_{\text{Maj}} := (1, 0, 0)$ is in the Majorana layer,
- the class $\alpha_{\text{DW}} := (0, 1, 0)$ is in the Dijkgraaf–Witten layer, and
- the class $\alpha_{\text{GW}} := (0, 0, 1)$ is in the Gu–Wen layer.

Moreover, it follows from Lemma B.44, part (3) that α_{Maj} generates the kernel of the map to $U_{\text{Spin}}^5(B\mathbb{Z}/2 \times B\mathbb{Z}/2^k, x_1, y)$, so we will focus on α_{DW} and α_{GW} .

Proof of Theorem I.23. Since we do not know which of $(-1)^{x_1^4 x}$, $(-1)^{xy^2}$ corresponds to α_{DW} , we will trivialize all of the classes on the E_∞ -page that could correspond to α_{DW} and α_{GW} by pulling back to $B\mathbb{Z}/8 \times B\mathbb{Z}/2^{k+1}$. These classes are $x_1^3 x \in E_\infty^{4,1}$, $(-1)^{x_1^4 x} \in E_\infty^{5,0}$, $(-1)^{xy^2} \in E_\infty^{5,0}$, and linear combinations of them. Thus it suffices to trivialize these three classes.

To trivialize $x_1^3 x$ and $(-1)^{x_1^4 x}$, first pull back to $SH^5(B\mathbb{Z}/4 \times B\mathbb{Z}/2^k, x_1, y)$, so $x_1^2 \mapsto 0$. This implies that for the Dijkgraaf–Witten layer, $(-1)^{x_1^4 x} \mapsto 0$ as well, but it does not suffice to trivialize α_{GW} (corresponding to $x_1^3 x$) – all we know is that it pulls back to some class in the Dijkgraaf–Witten layer.

Thus, to trivialize α_{GW} , we may pull back to $B\mathbb{Z}/4 \times B\mathbb{Z}/2^k$, then work in x_1 -twisted \mathbb{C}^\times -cohomology.

Lemma III.13. *For $p, q \geq 2$, $2 = 0$ in $H^*(B\mathbb{Z}/2^p \times B\mathbb{Z}/2^q; \mathbb{C}_{x_1}^\times)$.*

Proof. Use the long exact sequence associated to $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathbb{C}^\times \rightarrow 0$ as usual to reduce to the analogous claim with \mathbb{Z}_{x_1} coefficients. The result then follows from the Künneth formula for twisted cohomology and the calculations of $H^*(B\mathbb{Z}/2^p; \mathbb{Z})$ and $H^*(B\mathbb{Z}/2^p; \mathbb{Z}_x)$, which can be found in Lemma C.2 and [71, Lemma A.12], respectively. \square

Lemma III.14. *Let $\beta \in H^6(B\mathbb{Z}/4 \times B\mathbb{Z}/2^k; \mathbb{Z}/2)$ be a class in the image of the twisted mod 2 reduction map $\tilde{r}_2: H^6(-; \mathbb{Z}_{x_1}) \rightarrow H^6(-; \mathbb{Z}/2)$. Then the pullback of β to $H^6(B\mathbb{Z}/8 \times B\mathbb{Z}/2^{k+1}; \mathbb{Z}/2)$ vanishes.*

Proof. The set $\{y_1^3, y_1^2 y_2, y_1 y_2^2, y_2^3, x_1 x_2 y_1^2, x_1 x_2 y_1 y_2, x_1 x_2 y_2^2\}$ is a basis for $H^6(B\mathbb{Z}/4 \times B\mathbb{Z}/2^k; \mathbb{Z}/2)$, where x_1 and y_1 come from $B\mathbb{Z}/4$ and x_2 and y_2 come from $B\mathbb{Z}/2^k$. Thus every class is either y_1 or y_2 times some degree-4 class. But y_1 and y_2 pull back to 0 for $\mathbb{Z}/8 \times \mathbb{Z}/2^{k+1}$, as follows from Lemma C.2 after mod 2 reduction, so $\beta \mapsto 0$. \square

By Lemma III.14, when we pull α_{GW} back to $H^5(B\mathbb{Z}/8 \times B\mathbb{Z}/2^{k+1}; \mathbb{C}_{x_1}^\times)$, its mod 2 reduction vanishes, but by Lemma III.13, this implies the pullback of α_{GW} is 0.

This leaves $(-1)^{xy^2}$. Pull back to $SH^5(B\mathbb{Z}/2 \times B\mathbb{Z}/2^{k+1}, x_1, 0)$, in which y trivializes. Thus if we pull back to $B\mathbb{Z}/8 \times B\mathbb{Z}/2^{k+1}$, all three of these classes map to 0. \square

IV. Conclusion and Discussion

Our goal in this project was to construct (3+1)-d fermionic TQFTs that realize prescribed 't Hooft anomalies; we focused on anomalies of certain UV gauge theories with a G -symmetry (Question I.2). Using the framework of fusion 2-categories and the cocycle description of supercohomology, we extended the Wang–Wen–Witten symmetry extension procedure to unitary symmetries in the fermionic setting.

Our construction crucially relies on the use of twisted supercohomology $SH^5(BG, s, \omega)$ to classify anomalies. This choice is motivated by the fact that supercohomology is directly related to fusion 2-categories and the categorical obstruction, that supercohomology admits a concrete cocycle formulation suitable for state sum constructions and, in the cases examined, that it provides a good approximation of the classification of invertible TQFTs (SPTs), given by the twisted spin cobordism group $U_{\text{Spin}}^5(BG, s, \omega)$.

- We explicitly computed $SH^5(BG, s, \omega)$ groups for $G = \mathbb{Z}/n$, both untwisted and twisted by fermion parity (Examples I.6 and I.11).
- We found group extensions $H \twoheadrightarrow G$ that trivialize the anomaly classes $\omega \in SH^5(BG, s, \omega)$ of interest, as detailed in Theorems I.7 and I.13.
- Each trivialization provides a torsor over $SH^4(BH, s', \omega')$, from which the (3+1)d fermionic G -Symmetry Enriched Topological phase (G -SET) is constructed. This construction demonstrates a concrete path to realizing these anomalous gapped phases.

A significant technical contribution announced in this paper is the development of a *hastened Adams spectral sequence* for computing supercohomology groups. This tool made the computations in Examples I.6 and I.11 tractable. Further development and refinement of these spectral sequence techniques will be essential for classifying anomalies of more complex symmetry groups, such as non-abelian groups or symmetries with non-trivial s and ω twists, thus expanding the reach of the fermionic symmetry extension procedure [42].

In summary, our results provide a systematic, mathematically rigorous path for constructing candidate IR topological orders that can saturate a given UV anomaly, offering a powerful tool for studying strongly-coupled fermionic gauge theories.

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us his insights into the anomalies of topological orders, which helped shape Appendix A.

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A. Anomalies of topological orders from obstruction theory

In principle, an anomaly is a fundamental concept that can be applied to any quantum system, including those described by a continuum quantum field theory (QFT) [86] or a lattice system [87–92]. Specifically, a ’t Hooft anomaly is defined as the “obstruction to gauging” in a QFT, classified by symmetry-protected topological states (SPTs) in one dimension higher through the mechanism of anomaly inflow. Similar notions of anomalies, or obstructions, have been rigorously defined in lattice systems recently. Given a topological order presented as a higher category, it is anticipated that there is also a well-defined notion of anomaly based on its categorical/algebraic data [6, 22, 31, 35, 93, 94]. In this appendix, we provide a mathematically rigorous definition of anomaly in the specific setting of a (3+1)d topological order described by a fusion 2-category.

The classification that emerges from this definition differs from the familiar notions of ’t Hooft anomaly in a continuum QFT. Nevertheless, the two are related by a natural map between their underlying spectra, which we will discuss in detail. We emphasize that there is no universal definition of anomaly that applies uniformly across all physical contexts – whether in QFTs, lattice systems, or categorical descriptions of topological orders. Instead, the appropriate classification framework depends on the physical setting under consideration.

We propose that a broad class of these different notions of anomalies, particularly those connected to the ’t Hooft anomaly of a QFT, can be systematically organized using the language of generalized cohomology theory, reviewed in Appendix A.1. Furthermore, physical processes may give rise to maps between generalized cohomology theories, such as renormalization group flow connecting theories described by algebraic data in higher category theory, i.e. topological order, to a TQFT. This perspective is reminiscent of the perspective in [90–92] in the context of lattice systems.

A.1. Generalized cohomology theory

Classical cohomology theories like singular cohomology, de Rham cohomology, and sheaf cohomology share common axiomatic properties but capture different topological and geometric information. Generalized cohomology theory provides a unifying framework that encompasses these classical theories while allowing for new, exotic cohomology theories with applications throughout mathematics and physics. A standard textbook for generalized cohomology theory is [95].

We start with the definition of a spectrum, which is a homotopical object representing a generalized homology or cohomology theory. An Ω -spectrum¹⁵ E is a sequence of pointed topological spaces $\{E_n\}_{n \in \mathbb{Z}}$ together with structure maps, which are homotopy equivalences:

$$\sigma_n : E_n \xrightarrow{\sim} \Omega E_{n+1} \tag{A.1}$$

where Ω denotes the based loop space functor. These structure maps encode the fundamental relationships between different degrees of the cohomology theory.

One can think of a spectrum as encoding “stable” homotopy-theoretic information. While individual spaces E_n may have complicated unstable behavior, the spectrum captures what remains after we have “stabilized” by taking suspensions.

Given an Ω -spectrum $E = \{E_n, \sigma_n\}$, we can define a *generalized cohomology theory* by setting:

$$E^n(X) := [X, E_n] \tag{A.2}$$

where $[X, E_n]$ denotes the set of homotopy classes of pointed maps from X to E_n .

The structure maps σ_n induce *suspension isomorphisms*

$$E^n(X) \xrightarrow{\cong} E^{n+1}(\Sigma X). \tag{A.3}$$

This is the key property that makes the theory “stable” and gives it the structure of a cohomology theory: the generalized cohomology theory E^* satisfies the Eilenberg-Steenrod axioms [100] except the dimension axiom. The **coefficient groups** $E^n(*) := E^n(S^0)$ are the cohomology groups of a point, and these can be computed as:

$$E^n(*) = \pi_{-n}(E) := \text{colim}_k \pi_{k-n}(E_k), \tag{A.4}$$

where $\pi_{-n}(E)$ denotes the n -th stable homotopy group of the spectrum E .

Some standard examples of spectra and their related generalized cohomology theories are as follows:

¹⁵ There are many different yet equivalent ways to define spectra; see for example [96]. We use Ω -spectra because they tend to appear in physics applications: see, for example, [47, 90, 97–99].

1. **Eilenberg–Mac Lane spectra and ordinary cohomology:** The Eilenberg–Mac Lane spectrum is built from $E_n = K(\mathbb{Z}, n)$, the Eilenberg–Mac Lane spaces. This gives usual singular cohomology $H^*(X; \mathbb{Z})$.
2. **KO -theory, K -theory and the connective cover ko :** The spectrum KO gives rise to real K -theory, a generalized cohomology theory denoted $KO^*(X)$, which classifies real vector bundles and their formal differences up to stable equivalence. The KO -cohomology groups exhibit 8-fold Bott periodicity, with coefficient groups KO^n cycling through the pattern $\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, 0$ as n increases from 0.

The connective cover ko is obtained by truncating KO below degree 0, and yields the connective real K -theory $ko^*(X)$. The geometric meaning of ko -theory is less obvious than for KO , but it provides a computationally convenient approximation to the spectrum of fermionic invertible field theories.

3. **Thom spectra and cobordism theories:** The Thom construction associates to each vector bundle E over a base space X a pointed space called the Thom space $\text{Th}(E)$, formed by taking the one-point compactification of the total space of E . When applied to universal bundles over classifying spaces of classical Lie groups, this construction yields the standard Thom spectra that govern the relevant cobordism theory. Various flavors of cobordism (oriented, unoriented, complex, etc.) arise from these Thom spectra. In particular, $M\text{Spin}$ is built from the universal bundle over $B\text{Spin}$ and yields the spin cobordism theory relevant in the classification of fermionic invertible field theories.

In an n -dimensional quantum system with global symmetry G , the kinds of anomalies we consider are classified by $E^*(BG)$ with $*$ the degree depending on the dimension under consideration.¹⁶ We will examine three different generalized cohomology theories, corresponding to three different spectra.

- The first is the target $I_{\mathbb{Z}}M\text{Spin}$, which classifies d -dimensional reflection-positive invertible field theories (IFTs) with spin structure [33, 34]. Maps $BG \rightarrow I_{\mathbb{Z}}M\text{Spin}$ then correspond to reflection-positive invertible field theories protected by G -symmetry. By anomaly inflow, such invertible topological field theories can be used to cancel the anomalies for fermionic theories with G -symmetry in $(d - 1)$ -dimensions.
- The second generalized cohomology theory is associated to what we call a *categorical obstruction*, which represents the obstruction for the G -crossed extension of the underlying category. This gives a mathematically rigorous definition of anomalies of topological orders, based on their categorical description using higher fusion categories in this work. See [23, Definition 4.25] for a precise definition of the categorical anomaly in the context of (3+1)d topological order with G -symmetry. The associated spectrum is closely related to the super-Witt group $s\mathcal{W}$ [32], and hence will be denoted by $s\mathcal{W}$.
- Supercohomology SH . Supercohomology is first proposed in [27] for classifying fermionic SPTs. For our purpose, supercohomology theory was defined in two equivalent ways in §II.2, and the corresponding spectrum can be thought of as the spectrum of $I_{\mathbb{Z}}M\text{Spin}$ truncated to only degrees 0, 1, 2. Similar truncations also appear in e.g. classifying mixed-state SPTs [101].

In the rest of this appendix, we give a detailed account of the relationship between $I_{\mathbb{Z}}M\text{Spin}$ and $s\mathcal{W}$, and explain how supercohomology is a useful middle ground between the two.

A.2. IFTs, categorical obstruction, and supercohomology

Anomaly inflow postulates that 't Hooft anomalies are classified by IFTs in one higher dimension, which can cancel the anomaly of the theory living on the boundary; see Freed–Teleman [102] and Freed [98, 103] for the connection to IFTs. This is an example of the more general classification of anomalies in terms of generalized cohomology: Freed–Hopkins [33] and Grady [34] showed that, for fermionic theories, the spectrum in question is $I_{\mathbb{Z}}M\text{Spin}$.

While this perspective is sufficiently general across different quantum systems, it may not be able to capture all the algebraic information of the underlying quantum system associated to symmetries. For fermionic topological orders in (3+1)d which have a fusion 2-categorical description [22, 23], we may be able to define anomalies purely in terms of the interaction of the symmetry and the categorical data. We define the *anomaly* for a G -action on a **2sVect**-enriched nondegenerate braided fusion 2-category \mathfrak{B} (which was called the categorical G -obstruction in the main text), to be the failure to construct a **2sVect**-enriched nondegenerate faithfully graded G -crossed braided fusion 2-category extending

¹⁶ Depending on different twists s and ω , BG should also be twisted accordingly like in the discussion of twisted supercohomology in §II.2. Here and in the rest of Appendix A, we omit the twist for simplicity.

the G -action on \mathcal{B} . As explained in [23, Section 4.4], this perspective of anomaly is equivalent to an anomaly for a (3+1)d fermionic G -SET. This gives the full algebraic data that characterizes the interplay between symmetries and the underlying categorical data.

To be more specific, let us first specialize to (2+1)d, and review the classical 1-categorical result in [31]. In [31], Etingof–Nikshych–Ostrik–Meir constructed faithfully graded G -crossed braided extensions of a braided fusion 1-category \mathcal{B} . Such G -crossed braided extensions are parametrized by the homotopy classes of maps

$$BG \longrightarrow B\mathcal{P}ic(\mathcal{B}), \quad (\text{A.5})$$

where $\mathcal{P}ic(\mathcal{B})$ is the Picard groupoid of \mathcal{B} , given by the space of invertible \mathcal{B} -modules. See [104, §2.2] for a physical introduction and an example of how the extension theory proceeds. One way to think of this extension is to imagine a specific case when \mathcal{B} is nondegenerate and represents a (2+1)d TQFT. If \mathcal{B} has a G -symmetry, i.e. a map $\rho: G \rightarrow \mathcal{A}ut^{br}(\mathcal{B})$, then to form a G -crossed braided extension of \mathcal{B} is to insert G -defects into \mathcal{B} such that the fusion and associativity relations respect the group multiplication of G [105]. The result is a (2+1)d G -SET, i.e. a nondegenerate G -crossed braided fusion 1-category, that incorporates extra data such as the symmetry fractionalization of objects in \mathcal{B} under G . The different G -crossed extensions parametrize SET phases. We define the *categorical obstruction* to be the complete obstruction, in the sense of obstruction theory in algebraic topology, to the existence of a lift

$$\begin{array}{ccc} & B\mathcal{P}ic(\mathcal{B}) & \\ \nearrow \gamma & \downarrow & \\ BG & \longrightarrow & B\mathcal{A}ut^{br}(\mathcal{B}). \end{array} \quad (\text{A.6})$$

In other words, the obstruction corresponds to the inability to define a topological phase in which symmetry fractionalization is non-anomalous and the G -crossed braided consistency conditions, like the heptagon equations in [105], are satisfied. Maps to $B\mathcal{A}ut^{br}(\mathcal{B})$ that factor through $B\mathcal{P}ic(\mathcal{B})$ are precisely those G -actions on \mathcal{B} that are non-anomalous.

We can generalize the obstruction to higher dimensional theories, and moreover provide a space which classifies anomalies. Let \mathbf{C} be a fusion n -category, which can be loosely defined inductively via delooping and Karoubi completing as in [106].¹⁷ There is a fiber sequence of spaces given in [104, Theorem 5.2.24], which follows from unpublished work by Jones–Reutter:

$$B\mathbf{C}^\times \longrightarrow B\mathcal{A}ut^\otimes(\mathbf{C}) \longrightarrow B\mathbf{Bimod}(\mathbf{C})^\times, \quad (\text{A.7})$$

where $(-)^{\times}$ denotes only taking the invertible parts of a symmetric monoidal category. The rightmost entry parametrizes obstructions to lifting a map $X \rightarrow B\mathcal{A}ut^\otimes(\mathbf{C})$ to $X \rightarrow B\mathbf{C}^\times$. There is an analogous sequence in the fermionic case, when each entry is a category enriched in super (n) -vector spaces:¹⁸

$$B\mathcal{S}\mathbf{C}^\times \longrightarrow B\mathcal{S}\mathcal{A}ut^\otimes(\mathbf{C}) \longrightarrow B\mathcal{S}\mathbf{Bimod}(\mathbf{C})^\times. \quad (\text{A.8})$$

Example A.9. Let \mathbf{C} in (A.7) be a connected fusion 2-category of the form $\mathbf{Mod}(\mathcal{B})$ where \mathcal{B} is a nondegenerate braided fusion 1-category. For background on the foundations of fusion 2-categories, we recommend [54]. Then we get the sequence

$$B\mathcal{P}ic(\mathcal{B}) \longrightarrow B\mathcal{A}ut^{br}(\mathcal{B}) \longrightarrow B\mathbf{Bimod}(\mathbf{Mod}(\mathcal{B}))^\times, \quad (\text{A.10})$$

and hence the 3-groupoid $B\mathbf{Bimod}(\mathbf{Mod}(\mathcal{B}))^\times$ parametrizes obstructions. What is commonly referred to as the “ G -anomaly” for bosonic topological theories in (2+1)d, is an obstruction to an extension that is associative [31], and given by a class in $H^4(BG; \mathbb{C}^\times)$. A categorical understanding of topological order affords us a deeper understanding of the obstructions which go beyond ordinary cohomology. In particular, in (2+1)d there is an additional contribution to the total obstruction coming from the Witt class of \mathcal{B} [108], and a map $\Sigma^4 H\mathbf{C}^\times \rightarrow B\mathbf{Bimod}(\mathbf{Mod}(\mathcal{B}))^\times$ implementing the comparison of the total obstruction with the obstruction that lives in ordinary cohomology.

¹⁷ See [104, Section 3.1] for an explanation of a crucial technical assumption, that must be made with our current understanding of condensation, in order for the inductive construction to be valid at for all values of n . For the contents of this paper, we will not require those assumptions. See [107, Section 4.1] for a treatment of higher fusion categories in terms of Cauchy completion.

¹⁸ Analogously to the construction of higher fusion categories, we obtain super (n) -vector spaces via condensation completion, beginning with the fusion 1-category of super vector spaces \mathbf{sVect} .

As discussed in §II.3, (3+1)d fermionic topological orders are described by nondegenerate **2sVect**-enriched braided fusion 2-categories \mathfrak{B} . When one takes $\mathbf{C} = \mathbf{Mod}(\mathfrak{B})$ in (A.7), then the categorical obstruction to performing a faithfully G -crossed braided extension is parametrized by the 4-groupoid $BsWitt := B\mathcal{S}\mathbf{Bimod}(\mathbf{Mod}(\mathfrak{B}))^\times = B4s\mathbf{Vect}^\times$. The details of the enrichment over **2sVect** and the appearance of this groupoid are presented in [23, Section 4].

The homotopy groups of $sWitt$ were computed in [23], and given by:

$$\begin{aligned} \pi_0 sWitt &= s\mathcal{W}, & \pi_1 sWitt &= 0, & \pi_2 sWitt &= \mathbb{Z}/2, \\ \pi_3 sWitt &= \mathbb{Z}/2, & \pi_4 sWitt &= \mathbb{C}^\times, \end{aligned} \tag{A.11}$$

where $s\mathcal{W}$ is the super-Witt group of braided fusion categories \mathcal{B} with Müger center **sVect**, given in [32]. Such categories are also referred to as slightly degenerate braided fusion categories. By [32, Proposition 5.18] we have

$$s\mathcal{W} = s\mathcal{W}_{pt} \oplus s\mathcal{W}_2 \oplus s\mathcal{W}_\infty, \tag{A.12}$$

where $s\mathcal{W}_{pt}$ is generated by the Witt classes of Abelian super MTCs, $s\mathcal{W}_2$ is an elementary Abelian 2-group, and $s\mathcal{W}_\infty$ is a free group of countable rank. Determining the k -invariants of the space $sWitt$ is an important open question, especially in the context of this work for computing categorical obstructions.

Definition A.13. Let \mathcal{SW}^* denote the generalized cohomology theory corresponding to the spectrum whose n -th space is $B^{n-4}sWitt$.

Thus $\mathcal{SW}^n(BG)$ parametrizes homotopy classes of maps

$$BG \rightarrow B^{n-4}sWitt. \tag{A.14}$$

In the relevant range for our applications, \mathcal{SW} has the following homotopy groups:

$$\begin{aligned} \pi_{-4} \mathcal{SW} &= s\mathcal{W}, & \pi_{-3} \mathcal{SW} &= 0, & \pi_{-2} \mathcal{SW} &= \mathbb{Z}/2, \\ \pi_{-1} \mathcal{SW} &= \mathbb{Z}/2, & \pi_0 \mathcal{SW} &= \mathbb{C}^\times. \end{aligned} \tag{A.15}$$

The categorical obstruction given by \mathcal{SW} resembles the more familiar 't Hooft anomalies that are classified by $I_\mathbb{Z}M\text{Spin}$, which in the relevant range, has homotopy groups

$$\begin{aligned} \pi_{-4} I_\mathbb{Z}M\text{Spin} &= 0, & \pi_{-3} I_\mathbb{Z}M\text{Spin} &= \mathbb{Z}, \\ \pi_{-2} I_\mathbb{Z}M\text{Spin} &= \mathbb{Z}/2, & \pi_{-1} I_\mathbb{Z}M\text{Spin} &= \mathbb{Z}/2, \\ \pi_0 I_\mathbb{Z}M\text{Spin} &= 0. \end{aligned} \tag{A.16}$$

But \mathcal{SW} notably differs from $I_\mathbb{Z}M\text{Spin}$ in its -4 homotopy group. Nevertheless, it is conjectured [109] that there exists a map from the categorical obstruction to the 't Hooft anomaly, i.e. a map

$$p: \mathcal{SW} \rightarrow I_\mathbb{Z}M\text{Spin}. \tag{A.17}$$

which maps nondegenerate braided fusion (n)-categories enriched in super (n)-vector spaces, to reflection positive invertible spin TQFTs. **In the rest of Appendix A, we assume this conjecture.** We summarize a heuristic construction for part of this map due to what we learned in [109]. Comparing the homotopy groups of the spectrum \mathcal{SW} and the spectrum of $I_\mathbb{Z}M\text{Spin}$, we have

π_*	\mathcal{SW}	$I_\mathbb{Z}M\text{Spin}$
+1	0	\mathbb{Z}
0	\mathbb{C}^\times	0
-1	$\mathbb{Z}/2$	$\mathbb{Z}/2$
-2	$\mathbb{Z}/2$	$\mathbb{Z}/2$
-3	0	\mathbb{Z}
-4	$s\mathcal{W}$	0

In degrees $-2, \dots, +1$, the two spectra are determined (noncanonically) by their homotopy groups together with the fact that the Postnikov k -invariants of consecutive homotopy groups are all nontrivial [47, Section 5]. In this range of degree, \mathcal{SW} looks like $I_\mathbb{Z}M\text{Spin}$, except that \mathbb{C}^\times is replaced with \mathbb{Z} in one degree higher. Indeed, after truncating to degrees -2 and above, the map (A.17) “is” the cofiber of the exponential map $\mathbb{C} \rightarrow \mathbb{C}^\times$, in that the fiber of (A.17) is the Eilenberg–Mac Lane spectrum HC . This says that the map (A.17) is very close to being an equivalence: in degrees -1 and below, it is an isomorphism on homotopy groups, and in degrees 0 and 1, it is a Bockstein.

In these degrees, it is possible to describe the map (A.17) field-theoretically: in principle, this map describes how every invertible object of $\Omega^2 \mathbf{4sVect}^\times \simeq \mathbf{sAlg}^\times$, the Morita 2-category of superalgebras, gives rise to a two-dimensional reflection-positive invertible spin TFT. This is standard: the unit in \mathbf{sAlg}^\times gives rise to the trivial theory, and the unique nontrivial Morita class, represented by the Clifford algebra $\mathcal{C}\ell_1$, gives rise to the Arf theory [110].

It remains to address the maps in degrees -3 and -4 . The existence of such a map was communicated to us in [109], and progress on mapping the torsion part of $s\mathcal{W}$ to the degree -3 entry in $I_{\mathbb{Z}}M\text{Spin}$ has been announced in [111].

In the setting of our paper, we would like to understand the relationship between SH , \mathcal{SW} , an $d\mathcal{U}_{\text{Spin}}$ in degree 5 when applied to BG for a finite group G . Thus consider the maps

$$SH^5(BG) \xrightarrow{\mathcal{I}} \mathcal{SW}^5(BG) \xrightarrow{p} \mathcal{U}_{\text{Spin}}^5(BG), \quad (\text{A.18})$$

where the map $\mathcal{I}: SH \rightarrow \mathcal{SW}$ is the Postnikov (-3) -connected cover.

Lemma A.19. *If the map $(p \circ \mathcal{I})_*: SH^5(BG) \rightarrow \mathcal{U}_{\text{Spin}}^5(BG)$ is an isomorphism for a given group G , then there is a subgroup A of $H^1(BG; s\mathcal{W})$ and a splitting $\mathcal{SW}^5(BG) \cong SH^5(BG) \oplus A$ of the map $\mathcal{I}_*: SH^5(BG) \rightarrow \mathcal{SW}^5(BG)$.*

Proof. The map $\mathcal{I}: SH \rightarrow \mathcal{SW}$ of spectra is an isomorphism on homotopy groups in all degrees -3 and above, so its cofiber is the Postnikov quotient $\tau_{\leq(-4)}\mathcal{SW}$. As this spectrum has only one nonzero homotopy group $\pi_4(\tau_{\leq(-4)}\mathcal{SW}) \cong s\mathcal{W}$, it must be an Eilenberg–Mac Lane spectrum: $\tau_{\leq(-4)}\mathcal{SW} \cong \Sigma^{-4}Hs\mathcal{W}$. That is, we have a fiber sequence

$$SH \xrightarrow{\mathcal{I}} \mathcal{SW} \xrightarrow{\tau_{\leq(-4)}} \Sigma^{-4}Hs\mathcal{W}. \quad (\text{A.20})$$

Combining the induced long exact sequence from (A.20) with the data from the lemma statement, we have the following commutative diagram, where the top row is exact:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H^0(BG; s\mathcal{W}) & \xrightarrow{\delta^0} & SH^5(BG) & \xrightarrow{\mathcal{I}} & \mathcal{SW}^5(BG) \xrightarrow{\tau_{\leq(-4)}} H^1(BG; s\mathcal{W}) \xrightarrow{\delta^1} \cdots \\ & & & & \searrow \cong & & \downarrow p \\ & & & & & & \mathcal{U}_{\text{Spin}}^5(BG) \end{array} \quad (\text{A.21})$$

Since $p \circ \mathcal{I}: SH^5(BG) \rightarrow \mathcal{U}_{\text{Spin}}^5(BG)$ is an isomorphism by hypothesis, it provides a section of $\mathcal{I}: SH^5(BG) \rightarrow \mathcal{SW}^5(BG)$. Thus \mathcal{I} is a split injection. Because the sequence in (A.21) is exact, $A := \ker(\delta^1) \subset H^1(BG; s\mathcal{W})$ is a complementary summand to the image of \mathcal{I} , which finishes the proof. \square

Furthermore, if $SH^5(BG) \rightarrow I_{\mathbb{Z}}M\text{Spin}$ is surjective then $SH^5(BG)$ captures a subgroup of $\mathcal{SW}^5(BG)$.

We now summarize the arguments for and against using these two types of obstructions, as well as supercohomology, to build a (3+1)d topological theory:

- While the correct obstruction to G -crossed braided extensions of a **2sVect**-enriched nondegenerate braided fusion 2-category are classes in $\mathcal{SW}^5(BG)$, computing this group is hard because we do not know about higher differentials in the Atiyah–Hirzebruch spectral sequence that computes $\mathcal{SW}^5(BG)$; in particular, this is tied to the fact that the k -invariants of $Bs\text{Witt}$ are not fully determined. Even assuming the existence of the map $\mathcal{SW} \rightarrow I_{\mathbb{Z}}M\text{Spin}$ does not mean we can necessarily pull back differentials, as some elements may be sent to zero.
- Spin cobordism is usually tractable to compute by the Adams spectral sequence. It also classifies anomalies for continuous quantum field theories. However, it is less directly related to the categorical obstructions. Like $\mathcal{SW}^5(BG)$, spin cobordism also has a contribution that goes beyond cohomology starting in dimension 4, which we do not know of a good cocycle description for.¹⁹
- Using the hastened Adams spectral sequence for supercohomology that we develop in [42], supercohomology is roughly as computable as $I_{\mathbb{Z}}M\text{Spin}$; see Appendix B. Supercohomology also has a cocycle description [27, 28]. Therefore it is both possible in theory and tractable in practice to apply the fermionic Wang–Wen–Witten construction on supercohomology, with the hopes of writing down a state-sum that generalizes [44]. Furthermore, we know that (3+1)d fermionic topological orders are classified by degree 4 supercohomology classes [22, Corollary V.4]. Supercohomology is an approximation not only to \mathcal{SW} , but also to spin cobordism in low degrees using the first definition of supercohomology in §II.2. Hence $SH^5(BG)$ may contain classes which map to 0 in $\mathcal{U}_{\text{Spin}}^5(BG)$,

¹⁹ See Brumfiel–Morgan [112, 113] for cocycle descriptions of $I_{\mathbb{C}^\times}M\text{Spin}$ in lower degrees.

however the two may at times also coincide. In the case when they do coincide, $SH^5(BG)$ really does have an interpretation in terms of classifying fermionic G -SPTs. See Example I.6 for an example when the two groups coincide, and Example I.19 for an example where the two groups do not coincide.

We do not know exactly how much $SH^5(BG)$ misses of the full categorical anomaly given by $s\mathcal{W}^5(BG)$. To fully answer this question we would need to understand how to compute $s\mathcal{W}^5(BG)$, which is a difficult open problem. Finding a cocycle description of this group is expected to be even harder. Thus, we will ignore the bottom layer with $s\mathcal{W}$ in our approximation to the categorical obstruction.²⁰

Remark A.22. There is the natural question of what it actually means to give a state sum construction for a TQFT whose Lagrangian description involves a class in $s\mathcal{W}^5(BG)$, which contains the group $s\mathcal{W}$. We believe this question to be related to realizing discrete invertible phases with “SPT index” valued in $\tilde{\mathcal{U}}_{\text{Spin}}^5(BG)$. In spacetime dimension three or lower, one could define an SPT index valued in $\mathcal{U}_{\text{Spin}}^3(BG)$ via the cocycles (α, β, γ) of supercohomology. But it is not known how to go to higher dimensions. In particular, one should provide an answer for how to work with a “cocycle” valued in $s\mathcal{W}$. Such a cocycle should have the interpretation as the super Witt class of a (2+1)d topological order with a G -symmetry. Such Witt classes are defined in [104, Definition 5.2.3]. Trivializing a cocycle upon pulling back to a group H would mean that the (2+1)d topological order with a G -symmetry is Witt trivial in the class of (2+1)d topological order with a H -symmetry.

We now discuss how these three obstructions come together in an example involving (2+1)d fermionic TQFTs.

Example A.23. In analogy to Example A.9, the categorical obstruction for a G -crossed braided extension of a slightly degenerate braided fusion category \mathcal{A} is given by an element in $SH^4(BG)$, as shown in [24]. However, this again misses the anomaly given by the Witt class $[\mathcal{A}] \in s\mathcal{W}$. Taking the anomaly from the Witt class into account would make this example line up with the conjecture that there is a map from $s\mathcal{W} \rightarrow I_{\mathbb{Z}} M\text{Spin}$ with properties as described above. In the case where G is a unitary symmetry, we have a match between $SH^4(BG)$ and $\tilde{\mathcal{U}}_{\text{Spin}}^4(BG)$, where the latter denotes reduced spin cobordism, and $s\mathcal{W}^4(BG)$ splits as $SH^4(BG) \oplus s\mathcal{W}$.

B. Spectral Sequence Computations

In this appendix, we provide the technical computations involving the hastened Adams and Atiyah–Hirzebruch spectral sequences used in §III to prove the main theorems.

Throughout this appendix, we make a technical assumption: that for all (X, a, b) -twisted supercohomology groups that we consider, there is a vector bundle $V \rightarrow X$ such that $w_1(V) = a$ and $w_2(V) = b$. This is true, and straightforward to verify, for all examples appearing in this paper.²¹

First, we provide details about the AHSS for the groups we consider in this paper. For a fermionic symmetry group given by (G, s, ω) such as in Table I, the entries of the AHSS on the E_2 -page are given by:

$$E_2^{i,j} = \begin{array}{c|ccccccc} j & & & & & & & \\ \hline & 0 & 1 & 2 & 3 & 4 & 5 & 6 & i \\ \hline E_2^{i,2} & H^0(BG; \mathbb{Z}/2) & H^1(BG; \mathbb{Z}/2) & H^2(BG; \mathbb{Z}/2) & H^3(BG; \mathbb{Z}/2) & \dots & & & \\ E_2^{i,1} & H^0(BG; \mathbb{Z}/2) & H^1(BG; \mathbb{Z}/2) & H^2(BG; \mathbb{Z}/2) & H^3(BG; \mathbb{Z}/2) & H^4(BG; \mathbb{Z}/2) & \dots & & \\ E_2^{i,0} & H^0(BG; \mathbb{C}_s^\times) & H^1(BG; \mathbb{C}_s^\times) & H^2(BG; \mathbb{C}_s^\times) & H^3(BG; \mathbb{C}_s^\times) & H^4(BG; \mathbb{C}_s^\times) & H^5(BG; \mathbb{C}_s^\times) & H^6(BG; \mathbb{C}_s^\times) & \dots \end{array} \quad (\text{B.1})$$

The rows for $j = 2$ and $j = 1$ come from the $\mathbb{Z}/2$ coefficient ring of the space BG , and we will write the entries/generators in terms of generators of the $\mathbb{Z}/2$ coefficient ring. The subscript s in the $j = 0$ row is an indicator that G has nontrivial action on the \mathbb{C}^\times module determined by s . We can write the elements in the $j = 0$ row using elements in $H^*(BG; \mathbb{Z}/2)$ with the help of the map $\mathbb{Z}/2 \rightarrow \mathbb{C}^\times$. This allows us to present the entries as $(-1)^x$ where x is an element in $\mathbb{Z}/2$ cohomology. When it is not possible, we will only write down the explicit group of the entry without giving a name to the corresponding generator of the entry.

Since supercohomology is a Postnikov truncation of the Pontryagin dual of ko , the d_2 differentials in the supercohomology Atiyah–Hirzebruch spectral sequence follow from the respective differentials in the ko -AHSS, which were

²⁰ It would be interesting to expand the definition of fusion 2-categories to incorporate unitarity, and compare if the analogous obstructions with and without restriction from unitarity.

²¹ This assumption is not true in general: see [114–119] for counterexamples where X is the classifying space of a compact Lie group. For the (hastened) Adams spectral sequence, this assumption is unnecessary [42, 120]; for the Atiyah–Hirzebruch spectral sequence, this assumption is used to prove the formulas (B.3) for differentials. We conjecture that these formulas hold even without this assumption, but this is not in the literature to our knowledge.

computed by Bott [121].²² First, define twisted Steenrod squares acting on $H^*(X; \mathbb{Z}/2)$ by

$$\text{Sq}_s^1(x) := \text{Sq}^1(x) + sx \quad (\text{B.2a})$$

$$\text{Sq}_{s,\omega}^2(x) := \text{Sq}^2(x) + s\text{Sq}^1(x) + \omega x. \quad (\text{B.2b})$$

Then the Atiyah–Hirzebruch d_2 s have the formula

$$d_2 : E_2^{i,2} \rightarrow E_2^{i+2,1} \quad X \mapsto \text{Sq}_{s,\omega}^2(X), \quad (\text{B.3a})$$

$$d_2 : E_2^{i,1} \rightarrow E_2^{i+2,0} \quad X \mapsto (-1)^{\text{Sq}_{s,\omega}^2(X)}. \quad (\text{B.3b})$$

There is also potentially a nontrivial d_3 differential

$$d_3 : E_3^{i,2} \rightarrow E_3^{i+3,0}. \quad (\text{B.4})$$

We do not know an explicit formula for general i .²³ On the E_∞ -page we must resolve potential extension problems, i.e., resolve how different entries in different rows are assembled together to give the full supercohomology group. This can be especially tricky, especially when the total degree is higher than 3.

At this point, it is traditional in the mathematical physics literature to turn to the *Adams spectral sequence*, whose structure makes many extension problems easier, and which admits a remarkable simplification for computing twisted spin bordism (see [124]). However, we need to compute twisted supercohomology, for which the standard Adams spectral sequence is messier. Instead, we use a variant called the *hastened Adams spectral sequence* (HASS). HASSes were introduced in [125] and systematized in [126] associated to the general data of a map of spectra; in a companion paper [42] we apply this to supercohomology and study a number of examples. Here, we give an overview of the HASS for $\tau_{\leq 2} ko$, then apply it in several examples.

Let $\mathcal{A}(1)$ denote the subalgebra $\langle \text{Sq}^1, \text{Sq}^2 \rangle$ inside the Steenrod algebra \mathcal{A} of mod 2 stable cohomology operations, and let $H_{s,\omega}^*(X; \mathbb{Z}/2)$ be the $\mathcal{A}(1)$ -module whose underlying graded vector space is $H^*(X; \mathbb{Z}/2)$, but where Sq^1 acts by Sq_s^1 and Sq^2 acts by $\text{Sq}_{s,\omega}^2$ (see (B.2)).²⁴ Then the input data to the Adams spectral sequence computing (X, s, ω) -twisted spin bordism is

$$E_2^{s,t} = \text{Ext}_{\mathcal{A}(1)}^{s,t}(H_{s,\omega}^*(X; \mathbb{Z}/2), \mathbb{Z}/2), \quad (\text{B.5})$$

where Ext is a functor classifying extensions of $\mathcal{A}(1)$ -modules of different lengths. In this paper, when we write $\text{Ext}(M)$ we mean $\text{Ext}_{\mathcal{A}(1)}^{*,*}(M, \mathbb{Z}/2)$.

In the hastened Adams spectral sequence for (the dual of) supercohomology, most of (B.5) is the same, but Ext is replaced with a different functor \mathcal{Q} , which one can think of as a “difference of two Ext’s.” The following theorem makes this precise.

Theorem B.6 ([126, Proposition 12.33], [42]). *Let $\hat{\mathcal{O}}$ denote the $\mathcal{A}(1)$ -module $\mathcal{A}(1)/(\text{Sq}^1, \text{Sq}^2 \text{Sq}^3)$.*

1. *There is a map of \mathbb{Z}^2 -graded $\text{Ext}(\mathbb{Z}/2)$ -modules*

$$g_4 : \text{Ext}_{\mathcal{A}(1)}^{s,t}(\hat{\mathcal{O}}, \mathbb{Z}/2) \longrightarrow \text{Ext}_{\mathcal{A}(1)}^{s+3,t+2}(\mathbb{Z}/2, \mathbb{Z}/2) \quad (\text{B.7})$$

which is induced from the Postnikov cover map $\tau_{\geq 4} ko \rightarrow ko$.

2. *There is a functor $\mathcal{Q}^{*,*}$ from $\mathcal{A}(1)$ -modules to \mathbb{Z}^2 -graded $\text{Ext}(\mathbb{Z}/2)$ -modules which commutes with direct sums and such that for all $\mathcal{A}(1)$ -modules M , there is a long exact sequence*

$$\cdots \rightarrow \text{Ext}_{\mathcal{A}(1)}^{s,t}(\hat{\mathcal{O}} \otimes M, \mathbb{Z}/2) \xrightarrow{g_4} \text{Ext}_{\mathcal{A}(1)}^{s+3,t+2}(M, \mathbb{Z}/2) \rightarrow \mathcal{Q}^{s,t}(M) \rightarrow \text{Ext}_{\mathcal{A}(1)}^{s+1,t}(\hat{\mathcal{O}} \otimes M, \mathbb{Z}/2) \xrightarrow{g_4} \cdots \quad (\text{B.8})$$

3. *Let X be a space of finite type,²⁵ $s \in H^1(X; \mathbb{Z}/2)$, and $\omega \in H^2(X; \mathbb{Z}/2)$. Then the HASS for $\tau_{\leq 2} ko_*(X, s, \omega)$ converges strongly and has signature*

$$E_2^{s,t} = \mathcal{Q}^{s,t}(H_{s,\omega}^*(X; \mathbb{Z}/2)) \Longrightarrow \tau_{\leq 2} ko_{t-s}(X, s, \omega)_2^\wedge. \quad (\text{B.9})$$

The map $ko_(X, s, \omega) \rightarrow \tau_{\leq 2} ko_*(X, s, \omega)$ lifts to a map from the ordinary Adams spectral sequence to the HASS.*

Because \mathcal{Q} commutes with direct sums and fits into the sequence (B.8), it is straightforward to compute it on $\mathcal{A}(1)$ -modules of interest. In [42], we compute \mathcal{Q} on many common $\mathcal{A}(1)$ -modules, and we use this to compute the E_2 -pages of the HASSes we use below. Once we have done this, running the HASS is just as in the usual Adams spectral sequence.

²² For an explicit statement of these differentials, see Anderson–Brown–Peterson [45, Proof of Lemma 5.6]. In addition, see [71, Lemma A.23] for the details on passing the differentials through Anderson duality.

²³ Results for low degrees based on physical constructions can be found in [30, 122]; see also [123].

²⁴ It is not immediately obvious that Sq_s^1 and $\text{Sq}_{s,\omega}^2$ satisfy the Adem relations and thus define an $\mathcal{A}(1)$ -action; this was shown in [120, Lemma 2.38(3)].

²⁵ The finite-type hypothesis appears for technical reasons and holds in all circumstances one might reasonably encounter in mathematical physics.

B.1. Example: $SH^5(B\mathbb{Z}/2)$

We first compute $SH^5(B\mathbb{Z}/2)$, which we use in Example I.6. The $\mathbb{Z}/2$ cohomology ring of $B\mathbb{Z}/2$ is given by

$$H^*(B\mathbb{Z}/2; \mathbb{Z}/2) = \mathbb{Z}/2[x], \quad |x| = 1 \quad (\text{B.10})$$

where x is the nontrivial generator of $H^1(B\mathbb{Z}/2; \mathbb{Z}/2)$.

Proposition B.11. *The group $SH^\ell(B\mathbb{Z}/2) = 0$ for $\ell = 4, 5$.*

For $\ell = 4$ this is due to Décoppet [37, Example 4.13]; for $\ell = 5$ this is new.

Proof. The AHSS which computes this group has the following E_2 -page:

$$E_2^{i,j} = \begin{array}{c|ccccccc} j & & & & & & & \\ \hline 2 & 1 & x & x^2 & x^3 & \dots & & \\ 1 & 1 & x & x^2 & x^3 & x^4 & \dots & \\ 0 & \mathbb{C}^\times & (-1)^x & 0 & (-1)^{x^3} & 0 & (-1)^{x^5} & 0 \dots \\ \hline & 0 & 1 & 2 & 3 & 4 & 5 & 6 & i \end{array} \quad (\text{B.12})$$

The d_2 differentials are given by:

$$d_2 : E_2^{i,2} \rightarrow E_2^{i+2,1} \quad X \mapsto \text{Sq}^2 X, \quad (\text{B.13})$$

$$d_2 : E_2^{i,1} \rightarrow E_2^{i+2,0} \quad X \mapsto (-1)^{\text{Sq}^2 X}. \quad (\text{B.14})$$

After resolving the d_2 differential, the E_3 -page is given as follows:

$$E_3^{i,j} = \begin{array}{c|ccccccc} j & & & & & & & \\ \hline 2 & 1 & x & 0 & 0 & \dots & & \\ 1 & 1 & x & x^2 & 0 & 0 & \dots & \\ 0 & \mathbb{C}^\times & (-1)^x & 0 & (-1)^{x^3} & 0 & 0 & 0 \dots \\ \hline & 0 & 1 & 2 & 3 & 4 & 5 & 6 & i \end{array} \quad (\text{B.15})$$

In particular, $SH^\ell(B\mathbb{Z}/2)$ is trivial for $\ell = 4, 5$. \square

See Wang–Gu [28, Table II], Gaiotto–Johnson–Freyd [127, §4], and Yu [128, §2.8] for $SH^\ell(B\mathbb{Z}/2)$ when $\ell < 4$.

B.2. Example: $SH^5(B\mathbb{Z}/2^k), k \geq 2$

We now compute $SH^5(B\mathbb{Z}/2^k)$ for $k \geq 2$. This is also relevant in Example I.6. The $\mathbb{Z}/2$ cohomology ring of $B\mathbb{Z}/2^k$ is given by

$$H^*(B\mathbb{Z}/2^k; \mathbb{Z}/2) = \mathbb{Z}/2[x, y]/(x^2), \quad |x| = 1, |y| = 2. \quad (\text{B.16})$$

Proposition B.17. *For $k \geq 2$, the group $SH^5(B\mathbb{Z}/2^k) \cong \mathbb{Z}/2^{k-1}$, with generator in the Dijkgraaf–Witten layer.*

Proof. The E_2 -page of the AHSS that compute this group is given as follows:

$$E_2^{i,j} = \begin{array}{c|ccccccl} j & & & & & & & \\ \hline 2 & 1 & x & y & xy & \dots & & \\ 1 & 1 & x & y & xy & y^2 & \dots & \\ 0 & \mathbb{C}^\times & \mathbb{Z}/2^k & 0 & \mathbb{Z}/2^k & 0 & \mathbb{Z}/2^k & 0 \dots \\ \hline & 0 & 1 & 2 & 3 & 4 & 5 & 6 & i \end{array} \quad (\text{B.18})$$

with the same d_2 differential as in Equations (B.13) and (B.14). After resolving the d_2 differential, the E_3 -page is given as follows:

$$E_3^{i,j} = \begin{array}{c|ccccccc} j & & & & & & & \\ \hline 2 & 1 & x & 0 & 0 & \dots & & \\ 1 & 1 & x & y & 0 & 0 & \dots & \\ 0 & \mathbb{C}^\times & \mathbb{Z}/2^k & 0 & \mathbb{Z}/2^k & 0 & \mathbb{Z}/2^{k-1} & 0 \dots \\ \hline 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & i \end{array} \quad (\text{B.19})$$

We see that $SH^5(B\mathbb{Z}/2^k) = \mathbb{Z}/2^{k-1}$, $k \geq 2$ and the whole group is in the Dijkgraaf–Witten layer. \square

See Wang–Gu [28, Table II] for $SH^\ell(B\mathbb{Z}/2^k)$ when $\ell < 4$ and Décoppet [37, Example 4.13] for $\ell = 4$. Specifically, Wang–Gu’s work resolves the extension question in total degree 3 in (B.19), which we will need to use later in this article.

Proposition B.20 (Wang–Gu [28, Table II]). *For $k \geq 2$, $SH^3(B\mathbb{Z}/2^k) \cong \mathbb{Z}/2^{k+1} \oplus \mathbb{Z}/2$.*

B.3. Example: $SH^5(B\mathbb{Z}/2, 0, x^2)$

We now combine the hastened Adams spectral sequence and the Atiyah–Hirzebruch spectral sequence to compute $SH^5(B\mathbb{Z}/2, 0, x^2)$ and determine the filtration of its generators. We will use these results in Example I.11. Recall $H^*(B\mathbb{Z}/2; \mathbb{Z}/2)$ from (B.10).

Proposition B.21. *$SH^\ell(B\mathbb{Z}/2, 0, x^2)$ is isomorphic to $\mathbb{Z}/2$ for $\ell = 4$ and $\mathbb{Z}/8$ for $\ell = 5$. In the latter case, there is a generator of the $\mathbb{Z}/8$ residing in the Majorana layer.*

The case $\ell = 4$ verifies a prediction of Décoppet [37, Example 4.13].

Proof. The E_2 -page is the same as Equation (B.12), with the twisted d_2 differentials given by

$$d_2 : E_2^{i,2} \rightarrow E_2^{i+2,1} \quad X \mapsto \text{Sq}^2 X + x^2 X, \quad (\text{B.22})$$

$$d_2 : E_2^{i,1} \rightarrow E_2^{i+2,0} \quad X \mapsto (-1)^{\text{Sq}^2 X + x^2 X}. \quad (\text{B.23})$$

After resolving the d_2 differentials, the E_3 -page is given as follows:

$$E_3^{i,j} = \begin{array}{c|ccccccc} j & & & & & & & \\ \hline 2 & 0 & 0 & x^2 & x^3 & \dots & & \\ 1 & 1 & 0 & 0 & 0 & x^4 & \dots & \\ 0 & \mathbb{C}^\times & (-1)^x & 0 & 0 & 0 & (-1)^{x^5} & 0 \dots \\ \hline 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & i \end{array} \quad (\text{B.24})$$

As we do not know about potential d_3 differentials or extensions in this spectral sequence, we turn to the hastened Adams spectral sequence. In this and future HASS arguments, we assume some background with the ordinary Adams spectral sequence; Beaudry–Campbell’s article [124] is an excellent introduction covering everything we assume.

In Figure 1, left, we display the $\mathcal{A}(1)$ -module $R_1 := H_{0,x^2}^*(B\mathbb{Z}/2; \mathbb{Z}/2)$, which was calculated by Campbell [69, Figure 7.2]. By Theorem B.6, part 3, the E_2 -page of the HASS for $(B\mathbb{Z}/2, 0, x^2)$ -twisted $\tau_{\leq 2} ko$ -homology is $\mathcal{Q}^{*,*}(H_{0,x^2}^*(B\mathbb{Z}/2; \mathbb{Z}/2))$. Using the long exact sequence (B.8), we calculate this E_2 -page in [42], and we display it in Figure 1, right. In degree 5, there is room for a d_2 differential $d_2(m_2) = \mu_2$. In fact, though, this differential vanishes, because m_2 is in the image of the map of Adams spectral sequences induced by $ko \rightarrow \tau_{\leq 2} ko$ [42], and in the Adams spectral sequence for the corresponding twist of $B\mathbb{Z}/2$ over ko , $d_2(m_2) = 0$ [69, §7.8]. Therefore on the E_∞ -page we have $(\tau_{\leq 2} ko)_4(B\mathbb{Z}/2, 0, x^2) = \mathbb{Z}/2$ and $(\tau_{\leq 2} ko)_5(B\mathbb{Z}/2, 0, x^2) = \mathbb{Z}/8$. The corresponding twisted supercohomology is the Pontryagin dual group. Thus the d_3 mentioned previously in the AHSS vanishes.

Combining with the Atiyah–Hirzebruch spectral sequence, we see that the generator of $\mathbb{Z}/8$ lies in the Majorana layer. \square

See Wang–Gu [30, Table III] and Zhang–Wang–Yang–Qi–Gu [129] for $SH^\ell(B\mathbb{Z}/2, 0, x^2)$ for $\ell < 4$.

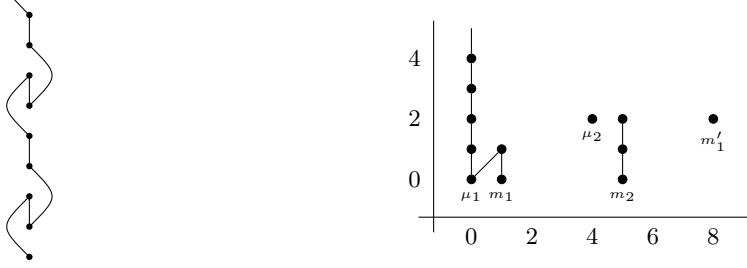


FIG. 1: Left: The $\mathcal{A}(1)$ -module structure on $R_1 \cong H_{0,x^2}^*(B\mathbb{Z}/2; \mathbb{Z}/2)$ [69, Figure 7.2]. Right: $\mathcal{Q}(R_1)$, computed in [42].

B.4. Example: $SH^5(B\mathbb{Z}/2^k, 0, y), k \geq 2$

We compute $SH^5(B\mathbb{Z}/2^k, 0, y)$ for $k \geq 2$, which we use in Example I.11.

Proposition B.25.

1. For $k \geq 2$, the group $SH^\ell(B\mathbb{Z}/2^k, 0, y)$ is isomorphic to $\mathbb{Z}/2$ for $\ell = 4$ and to $\mathbb{Z}/2^{k+1} \oplus \mathbb{Z}/2$ for $\ell = 5$.
2. The isomorphism $SH^5(B\mathbb{Z}/2^k, 0, y) \cong \mathbb{Z}/2^{k+1} \oplus \mathbb{Z}/2$ may be chosen so that the class γ_{GW} mapping to $(1, 0)$ has image in the Gu–Wen layer of the E_∞ -page of the AHSS, and the class γ_{Maj} mapping to $(0, 1)$ has image in the Majorana layer.

The case $\ell = 4$ verifies a prediction of Décoppet [37, Example 4.13].

Proof. This will again require the HASS. The E_2 -page of the AHSS is given by Equation (B.18), with the twisted d_2 differentials given by

$$d_2: E_2^{i,2} \rightarrow E_2^{i+2,1} \quad X \mapsto \text{Sq}^2 X + yX, \quad (\text{B.26})$$

$$d_2: E_2^{i,1} \rightarrow E_2^{i+2,0} \quad X \mapsto (-1)^{\text{Sq}^2 X + yX}. \quad (\text{B.27})$$

After resolving the d_2 differentials, the E_3 -page is given as follows:

$$E_3^{i,j} = \begin{array}{c|ccccccc} j & & & & & & & \\ \hline & 0 & 0 & y & xy & \cdots & & \\ 2 & 1 & 0 & 0 & 0 & y^2 & \cdots & \\ 1 & 0 & \mathbb{C}^\times & \mathbb{Z}/2^k & \mathbb{Z}/2^{k-1} & 0 & \mathbb{Z}/2^k & 0 \dots \\ 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & i \end{array} \quad (\text{B.28})$$

There is room for a nontrivial d_3 differential in total degree 5, and hence we turn to the hastened Adams spectral sequence.

The input to the (usual or hastened) Adams spectral sequence is the $\mathcal{A}(1)$ -module $H_{0,y}^*(B\mathbb{Z}/2^k; \mathbb{Z}/2)$ from [120, Definition 2.31(3)]. Let $V \rightarrow B\mathbb{Z}/2^k$ be the vector bundle associated to the rotation representation of $\mathbb{Z}/2^k$ on \mathbb{R}^2 . Then $(0, y) = (w_1(V), w_2(V))$, (i.e. this is a *vector bundle twist* of supercohomology, in the language of [120]), so there is an $\mathcal{A}(1)$ -module isomorphism (see [120])

$$H_{0,y}^*(B\mathbb{Z}/2^k; \mathbb{Z}/2) \cong H^*((B\mathbb{Z}/2^k)^{V-2}; \mathbb{Z}/2), \quad (\text{B.29})$$

where for a space X with bundle $V \rightarrow X$ with rank r_V , we denote by X^{V-r_V} the associated *Thom spectrum*, which is the suspension spectrum of the Thom space.

The $\mathcal{A}(1)$ -module structure on $H^*((B\mathbb{Z}/2^k)^{V-2}; \mathbb{Z}/2)$ is computed in [64, 69, 130].²⁶ Given an $\mathcal{A}(1)$ -module M , let $\Sigma^k M$ denote the same $\mathcal{A}(1)$ -module with grading increased by k ; we let $\Sigma M := \Sigma^1 M$. For example, define $C\eta := \Sigma^{-2} \tilde{H}^*(\mathbb{CP}^2; \mathbb{Z}/2)$. Then there is an $\mathcal{A}(1)$ -module isomorphism

$$H^*((B\mathbb{Z}/2^k)^{V-2}; \mathbb{Z}/2) \cong C\eta \oplus \Sigma C\eta \oplus \Sigma^4 C\eta \oplus \Sigma^5 C\eta \oplus F, \quad (\text{B.30})$$

²⁶ The references [69, 130] appear to use a different vector bundle than V , but this is a typo.

where F is concentrated in degrees 8 and above (and thus we may ignore it). The $\Sigma^{2k}C\eta$ summand is spanned by Uy^k and Uy^{k+1} , where U is the Thom class, and the $\Sigma^{2k+1}C\eta$ summand is spanned by Uxy^k and Uxy^{k+1} .

By Theorem B.6, \mathcal{Q} commutes with direct sums and suspensions and vanishes in topological degrees below the minimum degree of a bounded-below $\mathcal{A}(1)$ -module, so we can ignore F and only need $\mathcal{Q}(C\eta)$. We compute this in [42] and give the result in Figure 2, left (compare $\text{Ext}_{\mathcal{A}(1)}(C\eta)$, displayed in [124, Figure 22]). Using this, we can draw the E_2 -page of the HASS in Figure 2, center. Differentials can be computed by comparing to the corresponding Adams spectral sequence for twisted spin bordism, as in [69, §7.9] or [64, §13.2]: except for on the E_k -page, all differentials vanish. Thus we obtain the $E_{k+1} = E_\infty$ -page in Figure 2, right. As in the usual Adams spectral sequence, vertical lines represent h_0 -multiplication, which lifts to multiplication by 2, so we deduce that $(\tau_{\leq 2} ko)_5(B\mathbb{Z}/2^k, 0, y) \cong \mathbb{Z}/2^{k+1} \oplus \mathbb{Z}/2$, and the corresponding twisted supercohomology is the Pontryagin dual group. Thus the Atiyah–Hirzebruch d_3 mentioned above vanishes.

Comparing the E_∞ -page of the AHSS with the answer we found by the HASS, we see there is a hidden extension in total degree 5 in the AHSS. It must be an extension of the Dijkgraaf–Witten layer by either the Gu–Wen layer or the Majorana layer.

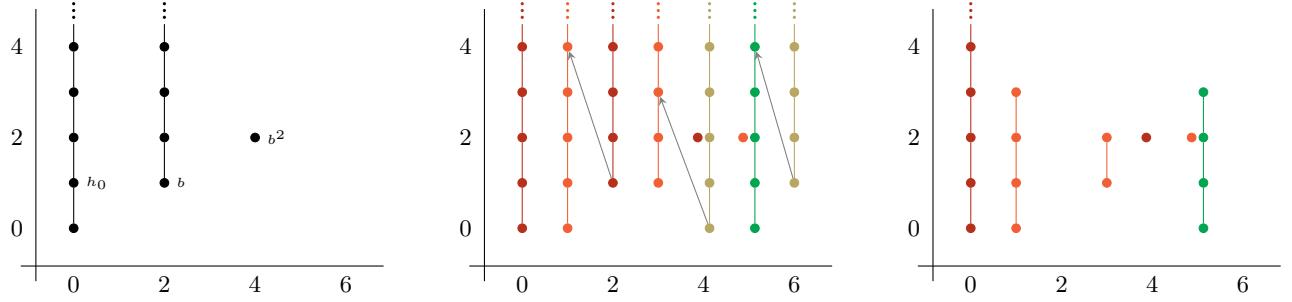


FIG. 2: Left: $\mathcal{Q}(C\eta)$, computed in [42]. Center: the E_k -page of the HASS computing $\tau_{\leq 2} ko(B\mathbb{Z}/2^k, 0, y)$ (here $k = 3$). Right: the $E_{k+1} = E_\infty$ -page.

Lemma B.31. *The hidden extension in degree 5 of the AHSS is between the Dijkgraaf–Witten and Gu–Wen layers; thus, the isomorphism $\phi: SH^5(B\mathbb{Z}/2^k, 0, y) \xrightarrow{\cong} \mathbb{Z}/2^{k+1} \oplus \mathbb{Z}/2$ may be chosen so that $\gamma_{GW} := \phi^{-1}(1, 0)$ has image in the E_∞ -page of the AHSS in the Gu–Wen layer and $\gamma_{Maj} := \phi^{-1}(0, 1)$ has image in the Majorana layer.*

Proof. Let $\iota: \mathbb{Z}/2 \rightarrow \mathbb{Z}/2^k$ be the map sending $1 \mapsto 2^{k-1}$; we will also let ι denote the induced map on classifying spaces. Recall that $\iota^*(y) = x^2 \in H^2(B\mathbb{Z}/2^k; \mathbb{Z}/2)$,²⁷ so we have a map $SH^*(B\mathbb{Z}/2^k, 0, x^2) \rightarrow SH^*(B\mathbb{Z}/2^k, 0, y)$, and therefore a map of AHSSes computing these supercohomology groups. This map is compatible with the extension problems on the E_∞ -pages in the following sense: each extension is a short exact sequence from a group on the E_∞ -page to the corresponding quotient of supercohomology, and the map ι induces a commutative diagram of short exact sequences. Thus, in particular, if $rSH^n(X, a, b)$ denotes the quotient of (X, a, b) -twisted supercohomology by the Majorana layer, so that $rSH \cong I_{\mathbb{C}^\times}(\tau_{\leq 1} ko)$,²⁸ then $SH^n(X, a, b)$ is an extension of $rSH^n(X, a, b)$ by the Majorana layer $E_\infty^{2,n-2}$, and specializing to the map ι we get a commutative diagram of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & {}^kE_\infty^{5,0} & \longrightarrow & rSH^5(B\mathbb{Z}/2^k, 0, y) & \longrightarrow & {}^kE_\infty^{4,1} & \longrightarrow 0 \\ & & \downarrow \iota^* & & \downarrow \iota^* & & \downarrow \iota^* & \\ 0 & \longrightarrow & {}^1E_\infty^{5,0} & \longrightarrow & rSH^5(B\mathbb{Z}/2^k, 0, x^2) & \longrightarrow & {}^1E_\infty^{4,1} & \longrightarrow 0, \end{array} \quad (\text{B.32})$$

where ${}^{\ell}E_r^{p,q}$ denotes the AHSS for the twisted supercohomology of $B\mathbb{Z}/2^\ell$. To prove the lemma, it would suffice to show that the upper central term of (B.32), $rSH^5(B\mathbb{Z}/2^k, 0, y)$, is isomorphic to $\mathbb{Z}/2^{k+1}$, as this plus the HASS computation would force the rSH -to-Majorana extension to split. Therefore our next task is to fill in the entries of (B.32). We computed ${}^1E_\infty^{5-j,j}$ in (B.24) (there we claim it is the E_3 -page, but in the proof of Proposition B.21 we show that d_3 vanishes going to or from total degree 5), and we computed ${}^kE_\infty^{5-j,j}$ in (B.28) (again, this was the E_3 -page, and we used the HASS to show this equals E_∞ in degree 5). Because y pulls back to x^2 , the map

²⁷ Because y is w_2 of the standard rotation representation ρ of $B\mathbb{Z}/2^k$, it suffices to show that restricting ρ to $\mathbb{Z}/2$ yields the representation 2σ ; then $w_2(2\sigma) = x^2$ by the Whitney sum formula.

²⁸ rSH is Gu–Wen restricted supercohomology [26, 27].

$\iota^*: {}^k E_\infty^{4,1} \rightarrow {}^1 E_\infty^{4,1}$ is an isomorphism $\mathbb{Z}/2 \xrightarrow{\cong} \mathbb{Z}/2$. The map on $E_\infty^{5,0}$ can be computed by finding its image under the Bockstein $H^5(-; \mathbb{C}^\times) \rightarrow H^6(-; \mathbb{Z})$; there it is a map

$$\iota^*: \mathbb{Z}/2^k \cong H^6(B\mathbb{Z}/2^k; \mathbb{Z}) \longrightarrow H^6(B\mathbb{Z}/2; \mathbb{Z}) \cong \mathbb{Z}/2. \quad (\text{B.33})$$

To show this map is nonzero (which uniquely determines it), use the universal coefficient theorem to show that it suffices to show that the image in mod 2 cohomology is nonzero; there we already know the map sends $y^3 \mapsto x^6$, hence is nonzero.

We have thus filled in most of (B.32); only the middle column remains. Because $SH^5(B\mathbb{Z}/2, 0, x^2) \cong \mathbb{Z}/8$ (Proposition B.21) and $rSH^5(B\mathbb{Z}/2, 0, x^2)$ is a quotient of this $\mathbb{Z}/8$ by the $\mathbb{Z}/2$ in the Majorana layer, we have $rSH^5(B\mathbb{Z}/2, 0, x^2) \cong \mathbb{Z}/4$. Thus (B.32) becomes the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}/2^k & \longrightarrow & rSH^5(B\mathbb{Z}/2^k, 0, y) & \longrightarrow & \mathbb{Z}/2 & \longrightarrow 0 \\ & & \downarrow \scriptstyle 1 \mapsto 1 & & \downarrow \scriptstyle \iota^* & & \downarrow \scriptstyle \cong & \\ 0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/4 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow 0, \end{array} \quad (\text{B.34})$$

and one can quickly check that this is only possible when $rSH^5(B\mathbb{Z}/2^k, 0, y) \cong \mathbb{Z}/2^{k+1}$. As noted above, this finishes the proof of the lemma. \square

Looking at the E_∞ -page of the HASS (Figure 2, right), we also see that $SH^4(B\mathbb{Z}/2^k, 0, y) \cong \mathbb{Z}/2$. \square

See Wang–Gu [30, Table III] and Zhang–Wang–Yang–Qi–Gu [129] for $SH^\ell(B\mathbb{Z}/2^k, 0, y)$ for $\ell < 4$.

B.5. Example: $SH^5(B\mathbb{Z}/2 \times B\mathbb{Z}/2^k, x_1, y), k \geq 2$

We now compute $SH^5(B\mathbb{Z}/2 \times B\mathbb{Z}/2^k, x_1, y), k \geq 2$, which we use in Example I.19. The $\mathbb{Z}/2$ cohomology ring of $B\mathbb{Z}/2 \times B\mathbb{Z}/2^k$ is

$$H^*(B\mathbb{Z}/2 \times B\mathbb{Z}/2^k; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_1, x, y]/(x^2), \quad |x_1| = |x| = 1, |y| = 2. \quad (\text{B.35})$$

There is an isomorphism $SH^5(B\mathbb{Z}/2 \times B\mathbb{Z}/2^k, x_1, y) \cong SH^5(B\mathbb{Z}/2 \times B\mathbb{Z}/2^k, x_1, y + x_1^2)$, because the twists (x_1, y) and $(x_1, y + x_1^2)$ are related by an automorphism of $B\mathbb{Z}/2 \times B\mathbb{Z}/2^k$. Consider

$$f: \mathbb{Z}/2 \times \mathbb{Z}/2^k \rightarrow \mathbb{Z}/2 \times \mathbb{Z}/2^k, \quad (1, 0) \mapsto (1, 2^{k-1}), \quad (0, 1) \mapsto (0, 1), \quad (\text{B.36})$$

under which we have $f^*(x_1) = x_1$, $f^*(x) = x$ and $f^*(y) = y + x_1^2$.

Proposition B.37. *There is an isomorphism $SH^5(B\mathbb{Z}/2 \times B\mathbb{Z}/2^k, x_1, y) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$ such that the generators α_{GW} , α_{DW} , and α_{Maj} , corresponding to $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ respectively, have the following properties.*

1. *The images of α_{DW} , α_{GW} , and α_{Maj} in the E_∞ -page of the AHSS are in the Dijkgraaf–Witten, Gu–Wen, and Majorana layers, respectively.*
2. *The kernel of the map $SH^5 \rightarrow U_{\text{Spin}}^5$ is spanned by α_{Maj} .*

Lemma B.38. *The following hold for the E_3 -page of the Atiyah–Hirzebruch spectral sequence computing $SH^*(B\mathbb{Z}/2 \times B\mathbb{Z}/2^k, x_1, y)$.*

1. *There are exactly 16 classes in total degree 4.*
2. *A basis for total degree 5 consists of $(-1)^{x_1^4 x}$ (DW layer), $(-1)^{xy^2}$ (DW layer), $x_1^3 x$ (GW layer), $x_1^3 + x_1 y$ (Majorana layer), and xy (Majorana layer).*
3. *In the corresponding spectral sequence for U_{Spin}^* -cohomology, $x_1^3 + x_1 y \in E_2^{3,2}$ is in the image of d_2 .*

Proof. As usual, the twisted d_2 differentials are given by

$$d_2: E_2^{i,2} \rightarrow E_2^{i+2,1} \quad X \mapsto \text{Sq}_{x_1,y}^2(X) := \text{Sq}^2 X + x_1 \text{Sq}^1 X + y X, \quad (\text{B.39})$$

$$d_2: E_2^{i,1} \rightarrow E_2^{i+2,0} \quad X \mapsto (-1)^{\text{Sq}^2 X + x_1 \text{Sq}^1 X + y X}. \quad (\text{B.40})$$

We assemble these ingredients and give the E_2 -page as follows:

$$E_2^{i,j} = \begin{array}{c|ccccccccc} j & & & & & & & & \\ \hline 2 & 1 & x_1, x & x_1^2, x_1x, y & x_1^3, x_1^2x, x_1y, xy & x_1^4, x_1^3x, x_1^2y, x_1xy, y^2 & \dots & & \\ 1 & 1 & x_1, x & x_1^2, x_1x, y & x_1^3, x_1^2x, x_1y, xy & x_1^4, x_1^3x, x_1^2y, x_1xy, y^2 & \dots & & \\ 0 & -1 & (-1)^x & (-1)^{x_1^2}, (-1)^y & (-1)^{x_1^2x}, (-1)^{xy} & (-1)^{x_1^4}, (-1)^{x_1^2y}, (-1)^{y^2} & (-1)^{x_1^4x}, (-1)^{x_1^2xy}, (-1)^{xy^2} & \dots & \\ \hline 0 & 0 & 1 & 2 & 3 & 4 & 5 & \dots & i \end{array} \quad (\text{B.41})$$

After resolving the d_2 differentials, the E_3 -page is given by:

$$E_3^{i,j} = \begin{array}{c|ccccccccc} j & & & & & & & & \\ \hline 2 & 0 & 0 & y & x_1^3 + x_1y, xy & \dots & & & \\ 1 & 0 & x_1 & x_1x & x_1^3x & \dots & & & \\ 0 & -1 & (-1)^x & (-1)^{x_1^2} & (-1)^{x_1^2x} & (-1)^{x_1^4}, (-1)^{y^2} & (-1)^{x_1^4x}, (-1)^{xy^2} & (-1)^{x_1^6}, (-1)^{x_1^2y^2} & \dots \\ \hline 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & i \end{array} \quad (\text{B.42})$$

This proves items (1) and (2) of the lemma statement. There could be nontrivial d_3 differentials $d_3: E_3^{2,2} \rightarrow E_3^{5,0}$ and $d_3: E_3^{3,2} \rightarrow E_3^{6,0}$, as well as potentially a hidden extension between different layers in degree 5; we will in a moment turn to the HASS to solve these problems.

Lastly we prove part (3). In this spectral sequence, $d_2: E_2^{i,3} \rightarrow E_2^{i+2,2}$ is identified with the map $H^i(-; \mathbb{Z}) \rightarrow H^{i+2}(-; \mathbb{Z}/2)$ which is reduction modulo 2 followed by Sq^2 [121]. (See also Footnote 22.)

Let $\tilde{e} \in H^1(B\mathbb{Z}/2 \times B\mathbb{Z}/2^k; \mathbb{Z}_{x_1})$ be the twisted Euler class of σ_1 , the tautological line bundle over $B\mathbb{Z}/2$, pulled back to the product $B\mathbb{Z}/2 \times B\mathbb{Z}/2^k$ (see [131, Lemma 1]). Then $\tilde{e} \bmod 2 = w_1(\sigma_1) = x_1$ (*ibid.*), so

$$d_2(\tilde{e}) = \text{Sq}^2_{x_1,y}(x_1) = x_1^3 + x_1y, \quad (\text{B.43})$$

which proves part (3). \square

Lemma B.44. *The following facts hold for the E_∞ -page of the HASS computing $(B\mathbb{Z}/2 \times B\mathbb{Z}/2^k, x_1, y)$ -twisted $\tau_{\leq 2} ko$ -homology.*

1. *There are exactly 16 classes in topological degree 4.*
2. *There are classes b , c , and e in topological degree 5 such that $\{b, c, h_0c, e\}$ is a basis for topological degree 5.*
3. *The cokernel of the map of E_∞ -pages from the ko -Adams SS to the $\tau_{\leq 2} ko$ -HASS in topological degree 5 is $\mathbb{Z}/2$, spanned by e .*
4. *There are no hidden extensions in topological degree 5, so $\tau_{\leq 2} ko_5(B\mathbb{Z}/2 \times B\mathbb{Z}/2^k, x_1, y) \cong \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{\oplus 2}$.*

We will postpone the proof of Lemma B.44 in order to first see how it helps us.

Proof of Proposition B.37, assuming Lemma B.44. Comparing Lemmas B.38 and B.44 in total degree 5, there appears to be a discrepancy: there are 32 classes in E_3 of the AHSS and 16 in E_∞ of the HASS. (These two spectral sequences compute SH -cohomology, resp. $\tau_{\geq 2} ko$ -homology, which are Pontryagin dual and therefore abstractly isomorphic whenever they are finite.) This means that there must be a d_r , $r \geq 3$, in the AHSS that kills some class in total degree 5. Since the numbers of elements in total degree 4 match between these two spectral sequences, this differential must go from total degree 5 to total degree 6. The only option for this differential is $d_3: E_3^{3,2} \rightarrow E_3^{6,0}$. Moreover, this differential will be preserved by the map into the U_{Spin}^* -AHSS, where it must vanish on $x_1^3 + x_1y$, so that $d_3^2 = 0$; thus $d_3(xy) \neq 0$. For degree reasons there can be no more nonzero differentials in total degree 5 for the AHSS, so we know that the E_∞ -page is spanned by $(-1)^{x_1^4x}$, $(-1)^{xy^2}$, x_1^3x , and $x_1^3 + x_1y$, in the DW, DW, GW, and Majorana layers respectively.

To finish, we resolve the extensions on the E_∞ -page of the AHSS. The HASS calculations in Lemma B.44 imply we must answer the following two questions,

1. $SH^5(B\mathbb{Z}/2 \times B\mathbb{Z}/2^k, x_1, y) \cong \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{\oplus 2}$, but the E_∞ -page of the AHSS has four $\mathbb{Z}/2$ summands in total degree 5. Where is the hidden extension?
2. What is the filtration in the AHSS of the class that is killed when one maps to U_{Spin}^* ?

Question (2) is easier: by the previous paragraph, $x_1^3 + x_1y \in E_\infty^{3,2}$ is killed by the map of AHSSes to $\mathcal{U}_{\text{Spin}}^*$. This class lifts to a class α_{Maj} which is killed when one passes to $\mathcal{U}_{\text{Spin}}^*$.

Now (1). The HASS analysis implies that the hidden extension is between two classes that are not in the kernel of the map to $\mathcal{U}_{\text{Spin}}^*$, and these two classes are necessarily in two different layers of the AHSS filtration. This uniquely forces it to be an extension of a $\mathbb{Z}/2$ subgroup of the Dijkgraaf–Witten layer by the unique $\mathbb{Z}/2$ in the Gu–Wen layer (spanned by x_1^3x), giving a generator α_{GW} in the Gu–Wen layer generating a $\mathbb{Z}/4$. A complementary subgroup to the image of $2\alpha_{\text{GW}}$ in $E_\infty^{5,0}$ lifts to the generator α_{DW} . \square

Proof of Lemma B.44. As in the previous example (and all examples in this paper), the twist (x_1, y) is a vector bundle twist: it is $(w_1(W), w_2(W))$ for the vector bundle $W := \sigma_1 \boxplus V$, where σ_1 is the tautological bundle over $B\mathbb{Z}/2$ and V is the bundle associated to the standard representation of $\mathbb{Z}/2^k$ on \mathbb{C} . The notation \boxplus denotes external direct sum, i.e. pull these bundles back to the product $B\mathbb{Z}/2 \times B\mathbb{Z}/2^k$, then direct sum them. Thus $H_{x_1,y}^*(B\mathbb{Z}/2 \times B\mathbb{Z}/2^k; \mathbb{Z}/2) \cong H^*((B\mathbb{Z}/2 \times B\mathbb{Z}/2^k)^{W-3}; \mathbb{Z}/2)$, like in the previous example.

The Thom spectrum associated to an external direct sum splits as a smash product, so the Künneth formula calculates its cohomology:

$$\begin{aligned} H_{x_1,y}^*(B\mathbb{Z}/2 \times B\mathbb{Z}/2^k; \mathbb{Z}/2) &\cong H^*((B\mathbb{Z}/2 \times B\mathbb{Z}/2^k)^{W-3}; \mathbb{Z}/2) \\ &\cong H^*((B\mathbb{Z}/2)^{\sigma-1} \wedge (B\mathbb{Z}/2^k)^{V-2}; \mathbb{Z}/2) \\ &\cong H^*((B\mathbb{Z}/2)^{\sigma-1}; \mathbb{Z}/2) \otimes H^*((B\mathbb{Z}/2^k)^{V-2}; \mathbb{Z}/2) \\ &\stackrel{(B.30)}{\cong} P \otimes (\textcolor{red}{C}\eta \oplus \Sigma C\eta \oplus \textcolor{brown}{\Sigma^4 C}\eta \oplus \textcolor{teal}{\Sigma^5 C}\eta \oplus F). \end{aligned} \quad (\text{B.45})$$

Here $P := H^*((B\mathbb{Z}/2)^{\sigma-1}; \mathbb{Z}/2)$. Letting $R_6 := P \otimes C\eta$,

$$H_{x_1,y}^*(B\mathbb{Z}/2 \times B\mathbb{Z}/2^k; \mathbb{Z}/2) \cong \textcolor{red}{R}_6 \oplus \textcolor{brown}{\Sigma R}_6 \oplus \textcolor{brown}{\Sigma^4 R}_6 \oplus \textcolor{teal}{\Sigma^5 R}_6 \oplus F' \quad (\text{B.46})$$

for some $\mathcal{A}(1)$ -module F' concentrated in degrees 8 and above. We compute $\mathcal{Q}(R_6)$ in [42] (compare $\text{Ext}_{\mathcal{A}(1)}(R_6)$ in [124, Figure 41]) and draw the result in Figure 3, left. Using this, we draw the E_2 -page of the HASS for $(B\mathbb{Z}/2 \times B\mathbb{Z}/2^k, x_1, y)$ -twisted $\tau_{\leq 2} ko$ -homology in Figure 3, center. The differentials $d_2: E_2^{0,5} \rightarrow E_2^{2,6}$ and $d_2: E_2^{0,6} \rightarrow E_2^{2,7}$ could be nonzero; all other differentials in range vanish because their source or target is the zero group. To describe the differentials more carefully, we name the following classes.

1. $\mathcal{Q}^{s,t}(\textcolor{red}{R}_6) \cong \mathbb{Z}/2$ for each of $(s, t) = (0, 4), (2, 7)$, and $(0, 6)$; let a, e , and f be the nonzero elements of each of these groups, respectively. Thus, through the split inclusion $\textcolor{red}{R}_6 \hookrightarrow H_{x_1,y}^*(B\mathbb{Z}/2 \times B\mathbb{Z}/2^k; \mathbb{Z}/2)$ in (B.46), we obtain classes a, e , and f in $E_2^{0,4}, E_2^{2,7}$, and $E_2^{0,6}$, respectively.
2. Repeat this procedure to define $c \in \mathcal{Q}^{0,5}(\textcolor{brown}{\Sigma R}_2) \hookrightarrow E_2^{0,5}$, $g \in \mathcal{Q}^{0,6}(\textcolor{brown}{\Sigma^4 R}_2) \hookrightarrow E_2^{0,6}$, and $b \in \mathcal{Q}^{0,5}(\textcolor{teal}{\Sigma^5 R}_2) \hookrightarrow E_2^{0,5}$ as the unique nonzero elements in their respective $\mathcal{Q}^{s,t}$ groups, then included into the E_2 -page.

These classes are labeled in Figure 3, center.

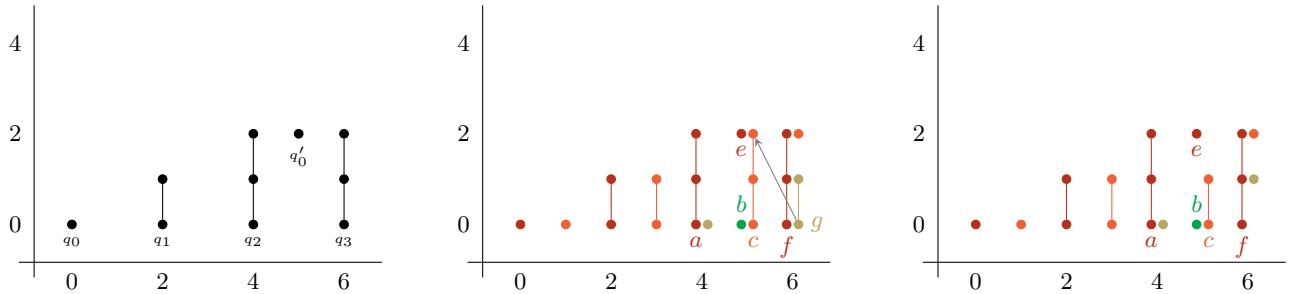


FIG. 3: Left: $\mathcal{Q}(R_6)$, computed in [42]. Center: the E_2 -page of the HASS computing $\tau_{\leq 2} ko_*(B\mathbb{Z}/2 \times B\mathbb{Z}/2^k, x_1, y)$. We calculate the d_2 s in range in Lemma B.47. Right: the E_∞ -page.

Thus $d_2: E_2^{0,5} \rightarrow E_2^{2,6}$ sends b, c , or both to 0 or $h_0^2 a$, and $d_2: E_2^{0,6} \rightarrow E_2^{2,7}$ sends f and g to elements of $\{0, e, h_0^2 c, e + h_0^2 c\}$.

The map $\tau_{\leq 2}: ko \rightarrow \tau_{\leq 2} ko$ induces a map of Adams spectral sequences; a, b, c, f , and g are in the image of this map, so their differentials are as well, but e is *not* in the image of this map, as can be seen by comparing $\text{Ext}_{\mathcal{A}(1)}(R_6)$

(see [124, Figure 41]) and $\mathcal{Q}(R_6)$. This proves part (3) of the lemma statement. Thus $d_2(f)$ and $d_2(g)$ are either 0 or $h_0^2 c$.

To finish the proofs of parts (1), (2), and (4) of the lemma statement, we prove the following lemma.

Lemma B.47. $d_2(b) = d_2(c) = 0$, $d_2(g) = h_0^2 c$, and $d_2(f) = \lambda h_0^2 c$ for some $\lambda \in \mathbb{Z}/2$. Equivalently, $(\tau_{\leq 2} ko)_n(B\mathbb{Z}/2 \times B\mathbb{Z}/2^k, x_1, y)$ is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/8$ for $n = 4$ and $\mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{\oplus 2}$ for $n = 5$.

Proof. Rather than directly compute these differentials, we will use a different technique, the *Smith long exact sequence*, to compute these twisted $\tau_{\leq 2} ko$ -homology groups. See [64, 70, 71, 78, 132–138] for more examples of this technique.

Theorem B.48 (James). *Let $V, W \rightarrow X$ be vector bundles of ranks r_V, r_W , respectively, and $p: S(W) \rightarrow X$ be the sphere bundle of W . For any generalized homology theory E_* , there is a long exact sequence*

$$\cdots \rightarrow E_k(S(W)^{p^*V-r_V}) \xrightarrow{p_*} E_k(X^{V-r_V}) \xrightarrow{\text{sm}_W} E_{k-r_W}(X^{V+W-(r_V+r_W)}) \rightarrow E_{k-1}(S(W)^{p^*V-r_V}) \rightarrow \cdots \quad (\text{B.49})$$

Theorem B.50 ([70]). *With notation as in Theorem B.48, suppose $E = \Omega^\xi$ is a bordism homology theory for a tangential structure ξ . Then, under the identification of $\Omega_k^\xi(X^{V-r_V})$ as the abelian group of bordism classes of (X, V) -twisted n -dimensional ξ -manifolds,²⁹ sm_W is the Smith homomorphism, which sends the bordism class of an (X, V) -twisted ξ -manifold $(M, f: M \rightarrow X)$ to the bordism class of the Poincaré dual of the Euler class of $f^*(W)$.^{30,31}*

See also [68, 135, 137–140] for applications and interpretations of the Smith long exact sequence in quantum physics.

To apply Theorem B.48, let $E_* = \tau_{\leq 2} ko_*((B\mathbb{Z}/2)^{\sigma-1} \wedge \dots)$, $X = B\mathbb{Z}/2^k$, and both V and W be the complex line bundle associated to the rotation representation of $\mathbb{Z}/2^k$. By [70, Example 7.28], the map $S(V) \rightarrow B\mathbb{Z}/2^k$ can be identified up to homotopy with the modulo 2^k reduction map $S^1 \simeq B\mathbb{Z} \rightarrow B\mathbb{Z}/2^k$. For any generalized homology theory E , $E_n(S^1) \cong E_n \oplus E_{n-1}$, as can be shown by using the Atiyah–Hirzebruch spectral sequence for the reduced E -homology of S^1 . Letting $M := (B\mathbb{Z}/2)^{\sigma-1}$ for brevity, we have the following long exact sequence:

$$\cdots \rightarrow (\tau_{\leq 2} ko)_n(M) \oplus (\tau_{\leq 2} ko)_{n-1}(M) \rightarrow (\tau_{\leq 2} ko)_n(M \wedge (B\mathbb{Z}/2^k)^{V-2}) \rightarrow (\tau_{\leq 2} ko)_{n-2}(M \wedge (B\mathbb{Z}/2^k)_+) \xrightarrow{\partial} \cdots \quad (\text{B.51})$$

Lemma B.52. $(\tau_{\leq 2} ko)_n((B\mathbb{Z}/2)^{\sigma-1})$ is isomorphic to $\mathbb{Z}/2$ for $n = 0, 1, 5$, $\mathbb{Z}/8$ for $n = 2, 6$, and 0 for $n = 3, 4, 7$.

Wang–Gu [30, Table III] study the corresponding supercohomology groups in degrees 4 and below.

Proof sketch. This can be computed using the HASS in the same way as we computed $(\tau_{\leq 2} ko)_*(B\mathbb{Z}/2, 0, x^2)$ in §B.3. See Figure 4, left, for a picture of the $\mathcal{A}(1)$ -module structure on $P := H^*((B\mathbb{Z}/2)^{\sigma-1}; \mathbb{Z}/2)$ and Figure 4, right, for $E_2 = \mathcal{Q}(P)$, which is calculated in [42]. The spectral sequence collapses, mostly for degree reasons. The only remaining differential is the d_2 from degree 6 to degree 5. This differential is in the image of the map of Adams spectral sequences induced by $ko \rightarrow \tau_{\leq 2} ko$: in the Adams spectral sequence for $ko_*((B\mathbb{Z}/2)^{\sigma-1})$, whose E_2 -page is calculated in [141, §2], this differential does vanish, so we are done. We draw the $E_2 = E_\infty$ -page of the Adams spectral sequence for $ko_*((B\mathbb{Z}/2)^{\sigma-1})$ in Figure 4, center.

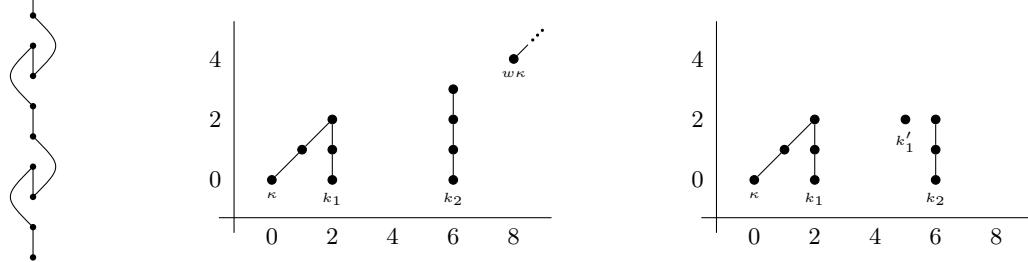


FIG. 4: Left: the $\mathcal{A}(1)$ -module structure on $P := H^*((B\mathbb{Z}/2)^{\sigma-1}; \mathbb{Z}/2)$. Center: $\text{Ext}_{\mathcal{A}(1)}(P)$, the E_2 -page of the Adams spectral sequence computing $ko_*((B\mathbb{Z}/2)^{\sigma-1})$. Right: $\mathcal{Q}(P)$, the E_2 -page of the HASS computing $(\tau_{\leq 2} ko)_*((B\mathbb{Z}/2)^{\sigma-1})$. The classes κ , k_1 , and k_2 are in the image of the map of E_2 -pages induced by the truncation $ko \rightarrow \tau_{\leq 2} ko$. We use this in the proof of Lemma B.52.

²⁹ Given a vector bundle $V \rightarrow X$, an (X, V) -twisted ξ -structure [68, §4] on a vector bundle $E \rightarrow M$ is the data of a map $f: M \rightarrow X$ and a ξ -structure on $E \oplus f^*(V)$. The bordism groups of manifolds whose tangent bundles have (X, V) -twisted ξ -structures are naturally isomorphic to the ξ -bordism groups of the Thom spectrum $X^{V-\text{rank}(V)}$ [64, Corollary 10.19].

³⁰ It is true, yet nontrivial, that the Poincaré dual carries a canonical $(X, V \oplus W)$ -twisted ξ -structure and that its bordism class does not depend on the choice of M .

³¹ Depending on ξ , one may have to use a generalized cohomology Euler class in Theorem B.50; see [70, Appendix B]. This detail will not play a role in this paper.

Lemma B.53. $\mathcal{M}_n := (\tau_{\leq 2} ko)_n((B\mathbb{Z}/2)^{\sigma-1} \wedge (B\mathbb{Z}/2^k)^{V-2})$ is isomorphic to $\mathbb{Z}/2$ for $n = 0, 1$ and $\mathbb{Z}/4$ for $n = 2, 3$. In higher degrees:

- \mathcal{M}_4 is isomorphic to either $\mathbb{Z}/8 \oplus \mathbb{Z}/2$, if $d_2(b) = d_2(c) = 0$, or to $\mathbb{Z}/4 \oplus \mathbb{Z}/2$, if at least one of $d_2(b)$ or $d_2(c)$ is nonzero.
- If $d_2(b) = d_2(c) = 0$, then \mathcal{M}_5 is isomorphic to either $\mathbb{Z}/8 \oplus (\mathbb{Z}/2)^{\oplus 2}$, if $d_2(f) = d_2(g) = 0$, or to $\mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{\oplus 2}$.

Proof. These follow from the computation of the E_2 -page of the HASS for these groups in Figure 3, as well as the observation we made that $e \notin \text{Im}(d_2)$. In principle, there could be a hidden extension from b or $h_0 c$ to e in degree 5, but because b and c are in the image of the map of spectral sequences induced by $\tau_{\leq 2}$, and e is not, this cannot occur. \square

Lemma B.54. Let $\mathcal{N}_n := (\tau_{\leq 2} ko)_n((B\mathbb{Z}/2)^{\sigma-1} \wedge (B\mathbb{Z}/2^k)_+)$. Then $\mathcal{N}_0 \cong \mathbb{Z}/2$ and $\mathcal{N}_1 \cong (\mathbb{Z}/2)^{\oplus 2}$. In higher degrees:

- \mathcal{N}_2 is isomorphic to either $\mathbb{Z}/8 \oplus (\mathbb{Z}/2)^{\oplus 2}$ or $\mathbb{Z}/8 \oplus \mathbb{Z}/4$.
- $\mathcal{N}_3 \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2$.

Our proof is an adaptation of the ideas in [132, §7.2.2], which are used there to compute $\Omega_*^{\text{Pin}^-}(B\mathbb{Z}/4)$ in low degrees. We replace $B\mathbb{Z}/4$ with $B\mathbb{Z}/2^k$ and truncate spin bordism to $\tau_{\leq 2} ko$, but the outline of the proof is not very different.

Proof. For any spaces X and Y and generalized cohomology theory E , there is a natural isomorphism

$$E_*(X \wedge Y_+) \xrightarrow{\cong} \tilde{E}_*(X) \oplus \tilde{E}_*(X \wedge Y), \quad (\text{B.55})$$

which is exactly the splitting of the $E_*(X \wedge -)$ -homology of Y into the $E_*(X \wedge -)$ -homology of a point and the reduced $E_*(X \wedge -)$ -homology of Y . Therefore \mathcal{N}_n is the direct sum of $(\tau_{\leq 2} ko)_n((B\mathbb{Z}/2)^{\sigma-1})$, which we computed in Lemma B.53, and $\tilde{\mathcal{N}}_n := (\tau_{\leq 2} ko)_n((B\mathbb{Z}/2)^{\sigma-1} \wedge (B\mathbb{Z}/2^k))$. We will focus on the latter, then implicitly direct-sum on $(\tau_{\leq 2} ko)_n((B\mathbb{Z}/2)^{\sigma-1})$ to obtain the groups in the lemma statement.

We attack $\tilde{\mathcal{N}}_n$ with the hastened Adams spectral sequence. The E_2 -page is \mathcal{Q} applied to the $\mathcal{A}(1)$ -module

$$\tilde{H}^*((B\mathbb{Z}/2)^{\sigma-1} \wedge B\mathbb{Z}/2^k; \mathbb{Z}/2) \cong \tilde{H}^*((B\mathbb{Z}/2)^{\sigma-1}; \mathbb{Z}/2) \otimes_{\mathbb{Z}/2} \tilde{H}^*(B\mathbb{Z}/2^k; \mathbb{Z}/2). \quad (\text{B.56})$$

There is an isomorphism

$$\tilde{H}^*(B\mathbb{Z}/2^k; \mathbb{Z}/2) \cong \Sigma \mathbb{Z}/2 \oplus \Sigma^2 C\eta \oplus \Sigma^3 C\eta \oplus \overline{F}, \quad (\text{B.57})$$

where \overline{F} is concentrated in degrees 6 and above (see, e.g., [69, Figure 7.5] or [67, Proposition 13.20]). Recalling from around (B.45) that $R_6 := P \otimes C\eta$, we get

$$\tilde{H}^*((B\mathbb{Z}/2)^{\sigma-1} \wedge B\mathbb{Z}/2^k; \mathbb{Z}/2) \cong \Sigma P \oplus \Sigma^2 R_6 \oplus \Sigma^3 R_6 \oplus F, \quad (\text{B.58})$$

where F is concentrated in degrees 6 and above (and so we can ignore it). We obtained $\mathcal{Q}(P)$ in Figure 4 and $\mathcal{Q}(R_6)$ in Figure 3, left, so we can draw the HASS E_2 -page in Figure 5, left. For degree reasons, there is only one possible nonzero differential in this range, $d_2: E_2^{0,4} \rightarrow E_2^{2,5}$. Moreover, by inspecting the E_2 -page, the value of \mathcal{N}_3 claimed in the lemma statement is equivalent to the claim that the differential in question is nonzero.

Looking at Figure 5, left, the source of this differential, $E_2^{0,4}$, is isomorphic to $\mathbb{Z}/2$. Let $\phi \in E_2^{0,4}$ be the nonzero element. If M is an ℓ -connected $\mathcal{A}(1)$ -module (i.e. it vanishes in degrees ℓ and below), exactness of (B.8) implies the map $t: \text{Ext}_{\mathcal{A}(1)}(M) \rightarrow \mathcal{Q}(M)$ is an isomorphism for $t-s \leq 4+\ell$; therefore, since $\tilde{H}^*((B\mathbb{Z}/2)^{\sigma-1} \wedge B\mathbb{Z}/2^k; \mathbb{Z}/2)$ is 0-connected, all classes in topological degree ≤ 4 are in the image of t . This includes ϕ and all possible values of $d_2(\phi)$, so if $\tilde{\phi}$ is the unique preimage of ϕ in $\text{Ext}_{\mathcal{A}(1)}^{0,4}$, then $d_r(\phi) \neq 0$ if and only if $d_r(\tilde{\phi}) \neq 0$ for all $r \geq 2$. Thus, it suffices to show $\tilde{\phi}$ does not survive to the E_∞ -page in the Adams spectral sequence for ko -homology: since $\tilde{\phi}$ is in filtration 0, it cannot be in the image of a differential, and the only differential it could possibly support is a d_2 , for degree reasons.

Since $\tilde{\phi}$ is in filtration 0, it corresponds uniquely to an $\mathcal{A}(1)$ -module homomorphism

$$\Phi: \tilde{H}^*((B\mathbb{Z}/2)^{\sigma-1} \wedge B\mathbb{Z}/2^k; \mathbb{Z}/2) \longrightarrow \mathbb{Z}/2. \quad (\text{B.59})$$

Since $E_2^{0,4} \cong \mathbb{Z}/2$, there must be a unique nonzero such $\mathcal{A}(1)$ -module homomorphism, and a straightforward calculation shows that such a homomorphism is nonzero on Ux_1^2y .

The behavior of filtration-0 classes in an Adams spectral sequence for bordism is standard: see [142, §8.4]. In particular, the following are equivalent.

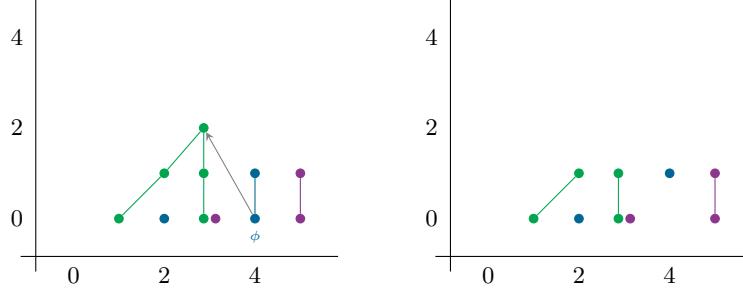


FIG. 5: Left: the E_2 -page of the HASS computing $(\tau_{\leq 2} ko)_*((B\mathbb{Z}/2)^{\sigma-1} \wedge (B\mathbb{Z}/2^k)^{V-2})$. We use this spectral sequence in the proof of Lemma B.54, where we show that the pictured d_2 is nonzero. Right: the $E_3 = E_\infty$ -page.

1. $\tilde{\phi}$ survives to the E_∞ -page.
2. There is a closed, 4-dimensional $(B\mathbb{Z}/2 \times B\mathbb{Z}/2^k, \sigma \boxplus V)$ -twisted spin manifold (see Footnote 29) N with $\int_N x_1^2 y \neq 0$.

Moreover, using the Whitney sum formula and the definition of an (X, V) -twisted spin structure, one can show that the notion of twisted spin structure appearing in item 2 above is the data of a pin $^-$ structure and a principal $\mathbb{Z}/2^k$ -bundle $P \rightarrow N$, and that $x = w_1(N)$. Thus $\int_N x_1^2 y = \int_N w_1(N)^2 y(P)$. Since we want to show that $d_2(\phi) \neq 0$ to finish the proof of the lemma, it will therefore suffice to show that there is no closed pin $^-$ 4-manifold N with principal $\mathbb{Z}/2^k$ -bundle $P \rightarrow N$ with $\int_N w_1(N)^2 y(P) \neq 0$.

Now consider the Smith homomorphism from Theorem B.50 associated to the data $\xi = \text{Spin}$, $X = B\mathbb{Z}/2 \times B\mathbb{Z}/2^k$, $V = 0$, and $W = \sigma$. Because the sphere bundle of $\sigma \rightarrow B\mathbb{Z}/2$ is contractible, the long exact sequence in Theorem B.48 simplifies to an isomorphism

$$\text{sm}_\sigma: \widetilde{\Omega}_k^{\text{Spin}}(B\mathbb{Z}/2 \wedge B\mathbb{Z}/2^k) \longrightarrow \widetilde{\Omega}_{k-1}^{\text{Pin}^-}(B\mathbb{Z}/2^k). \quad (\text{B.60})$$

Thus this map is called a *Smith isomorphism*. It is a special case of a general family of Smith isomorphisms discussed in [70, §7.1]; other examples in this family include the Smith isomorphisms discussed in [68, 132, 136, 139, 143–149]. The example in (B.60) was first studied in [132, §7.2.2].

Recall that we have reduced the proof of the lemma to the assertion that there is no closed pin $^-$ 4-manifold N and principal $\mathbb{Z}/2^k$ -bundle $P \rightarrow N$ such that $\int_N w_1(N)^2 y(P) \neq 0$. We can pull this back across (B.60): it suffices to show that there is no closed, spin 5-manifold W with principal $\mathbb{Z}/2$ -bundle $Q_1 \rightarrow W$ and principal $\mathbb{Z}/2^k$ -bundle $Q_2 \rightarrow W$ such that $\int_{\text{sm}_\sigma(W)} w_1^2 y \neq 0$. By Theorem B.50, any smooth submanifold representative of the Poincaré dual of $x(Q_1)$ (i.e. the Euler class of the line bundle associated to Q_1) represents the bordism class $\text{sm}_\sigma(W)$. That is, we want to show that for all (W, Q_1, Q_2) as above,

$$\int_{\text{PD}(x(Q_1))} w_1(\text{PD}(x_1(Q_1))^2 \cdot y(Q_2|_{\text{PD}(x_1(Q_1))})) = 0. \quad (\text{B.61})$$

where PD means any choice of submanifold representative of the Poincaré dual; the integral does not depend on this choice.

It is standard that if $i: N \hookrightarrow M$ is a smooth representative of the Poincaré dual of the Euler class $e(E)$ of a vector bundle $E \rightarrow M$, then the normal bundle ν of $N \subset M$ is isomorphic to $E|_N$, and that if $z \in H^*(M; \mathbb{Z}/2)$, then

$$\int_N i^*(z) = \int_M e(E)i^*(z). \quad (\text{B.62})$$

In the situation at hand, M is oriented, so $w_1(TN) = w_1(\nu)$ by the Whitney sum formula. Thus, applying (B.62) to (B.61), we obtain

$$\int_{\text{PD}(x_1(Q_1))} w_1(\text{PD}(x_1(Q_1))^2 \cdot y(Q_2|_{\text{PD}(x_1(Q_1))})) = \int_W x_1(Q_1)^3 y(Q_2). \quad (\text{B.63})$$

To finish the proof of the lemma, we will show this vanishes. Since W is a closed, oriented 5-manifold, the Wu formula implies

$$\int_W x_1(Q_1)^3 y(Q_2) = \int_W \text{Sq}^1(x_1(Q_1)^2 y(Q_2)) = \int_W w_1(W)x_1(Q_1)^2 y(Q_2) = 0. \quad (\text{B.64})$$

We draw the E_∞ -page of this HASS in Figure 5, right. \square

Remark B.65. There is a potential hidden extension by 2 in degree 2 which our proof does not address; this is why \mathcal{N}_2 is left ambiguous in the statement of Lemma B.54. It is possible to show that this extension splits, so that $\mathcal{N}_2 \cong \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^{\oplus 2}$. For $k = 2$ this follows from [132, Theorem 17]. One way to prove that the extension splits for all k is to pull back across $\tau_{\leq 2}$ and answer the equivalent question in ko -homology, using that the multiplication-by-2 map factors as

$$ko_n(X) \xrightarrow{c} ku_n(X) \xrightarrow{b} ku_{n+2}(X) \xrightarrow{R} ko_n(X), \quad (\text{B.66})$$

where c is complexification, b is the complex Bott periodicity map, and R is obtained from the realification map (see [150, Theorem 1]). By studying the effects of c , b , and R on the corresponding Adams spectral sequences, one can show that their composition must vanish, so that $\widetilde{\mathcal{N}}_2$ contains no elements of order 4.

Using Lemmas B.52, B.53, and B.54, we write down the Smith long exact sequence (B.51) in low degrees in Figure 6. Some of the maps are determined up to isomorphism by exactness; we also depict those in Figure 6. These maps are calculated starting in degrees 0 and 1, and then propagating that information upwards in order to degrees 2, 3, and 4 using exactness of the sequence.

$$\begin{array}{ccccccc}
k & (\tau_{\leq 2} ko)_n(M) \oplus (\tau_{\leq 2} ko)_{n-1}(M) & (\tau_{\leq 2} ko)_n(M \wedge (B\mathbb{Z}/2^k)^{V-2}) & (\tau_{\leq 2} ko)_{n-2}(M \wedge (B\mathbb{Z}/2^k)_+) \\
\\
0 & \mathbb{Z}/2 & \xrightarrow{\cong} & \mathbb{Z}/2 & & 0 \\
\\
1 & \begin{bmatrix} 1 \\ 0 \end{bmatrix} \curvearrowright \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \xrightarrow{[0 \ 1]} & \mathbb{Z}/2 & & 0 \\
\\
2 & \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \curvearrowright \mathbb{Z}/8 \oplus \mathbb{Z}/2 & \xrightarrow{[1 \ 0]} & \mathbb{Z}/4 & \xrightarrow{0} & \mathbb{Z}/2 \curvearrowright \\
\\
3 & \phi_1 \curvearrowright \mathbb{Z}/8 & \xrightarrow{1} & \mathbb{Z}/4 & \xrightarrow{0} & (\mathbb{Z}/2)^{\oplus 2} \curvearrowright \\
\\
4 & 0 & & \mathcal{M}_4 & \xleftarrow{\phi_2} & \mathcal{N}_2 \curvearrowright \\
\\
5 & \mathbb{Z}/2 & \xrightarrow{\phi_3} & \mathcal{M}_5 & \xrightarrow{\phi_4} & \mathbb{Z}/4 \oplus \mathbb{Z}/2
\end{array}$$

FIG. 6: The long exact sequence (B.51). We calculated the $\tau_{\leq 2} ko$ -homology groups appearing in this sequence in Lemmas B.52, B.53, and B.54; \mathcal{N}_2 , \mathcal{M}_4 , and \mathcal{M}_5 were not completely determined by those lemmas. We use this long exact sequence in the proof of Lemma B.47.

Since $\text{Im}(\phi_1) = \ker(1: \mathbb{Z}/8 \rightarrow \mathbb{Z}/4) = 4\mathbb{Z}/8 \cong \mathbb{Z}/2$, we obtain a short exact sequence

$$0 \longrightarrow \mathcal{M}_4 \xrightarrow{\phi_2} \mathcal{N}_2 \xrightarrow{\phi_1} \text{Im}(\phi_1) \cong \mathbb{Z}/2 \longrightarrow 0. \quad (\text{B.67})$$

Recall from Lemma B.53 that \mathcal{M}_4 is isomorphic to either $\mathbb{Z}/8 \oplus \mathbb{Z}/2$ or $\mathbb{Z}/4 \oplus \mathbb{Z}/2$, and from Lemma B.54 that \mathcal{N}_2 is isomorphic to either $\mathbb{Z}/8 \oplus (\mathbb{Z}/2)^{\oplus 2}$ or $\mathbb{Z}/8 \oplus \mathbb{Z}/4$. Of the four possible options, only the two with $\mathcal{M}_4 \cong \mathbb{Z}/8 \oplus \mathbb{Z}/2$ are compatible with exactness of (B.67). Lemma B.53 then tells us that $d_2(b) = d_2(c) = 0$ and that \mathcal{M}_5 is isomorphic to one of $\mathbb{Z}/8 \oplus (\mathbb{Z}/2)^{\oplus 2}$ or $\mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{\oplus 2}$. In particular, $N := |\mathcal{M}_5|$ is either 16 or 32. Since ϕ_4 is surjective, $\text{Im}(\phi_4)$ has order 8, so $\ker(\phi_4) = \text{Im}(\phi_3)$ has order $N/8$. Since the domain of ϕ_3 is $\mathbb{Z}/2$, $\text{Im}(\phi_3)$ has order at most 2, so $N/8 \leq 2$, or $N \leq 16$, implying $\mathcal{M}_4 \cong \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{\oplus 2}$. \square

Dualizing the results of this lemma, we get twisted supercohomology groups:

- $SH^4(B\mathbb{Z}/2 \times B\mathbb{Z}/2^k, x_1, y) \cong \mathbb{Z}/8 \oplus \mathbb{Z}/2$.
- $SH^5(B\mathbb{Z}/2 \times B\mathbb{Z}/2^k, x_1, y) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4$.

Thus the $d_3: E_3^{2,2} \rightarrow E_3^{5,0}$ in the AHSS of Equation (B.42) vanishes. Consulting the E_∞ -page of the same AHSS that computes $SH^5(B\mathbb{Z}/2 \times B\mathbb{Z}/2^k, x_1, y)$, we see that there must be a generator of $\mathbb{Z}/2$ that resides in the DW layer. The class e in Figure 3 does not appear in the analogous twisted spin cobordism computation, and so this $\mathbb{Z}/2$ generator must be in the Majorana layer as that is the only layer that can differ between supercohomology and spin cobordism. Therefore, the generator for $\mathbb{Z}/4$ must be in the Gu–Wen layer. This establishes Proposition B.37. \square

C. Trivializing Supercohomology Generators

Recall from Proposition B.25 that $SH^5(B\mathbb{Z}/2^k, 0, y) \cong \mathbb{Z}/2^{k+1} \oplus \mathbb{Z}/2$, with the two summands generated by classes γ_{GW} , resp. γ_{Maj} . The images of these two classes in the E_∞ -page of the AHSS are in the Gu–Wen, resp. Majorana layers. In this appendix, we will show how to trivialize both of these classes by pulling back to a larger group, finishing the proof of Theorem I.13.

Throughout this section, let $p: \mathbb{Z}/2^{k+1} \rightarrow \mathbb{Z}/2^k$ be the mod 2^k reduction map and

$$p^*: SH^5(B\mathbb{Z}/2^{k+1}, 0, 0) \longrightarrow SH^5(B\mathbb{Z}/2^k, 0, y) \quad (\text{C.1})$$

be the induced map on supercohomology. To see that the codomain of p^* is indeed the untwisted supercohomology of $B\mathbb{Z}/2^{k+1}$, it suffices to show that p pulls back $(0, y) \mapsto (0, 0)$ in ordinary cohomology, which one can check directly. Alternatively, follows from the following lemma by mod 2 reduction.

Lemma C.2. *For any m , let V_ρ denote the representation of \mathbb{Z}/m on \mathbb{C} in which $1 \in \mathbb{Z}/m$ acts by $e^{2\pi i/m}$; we will also write $V_\rho \rightarrow B\mathbb{Z}/m$ for the associated complex line bundle.³² Let $c_{1,\ell} := c_1(V_\rho) \in H^2(B\mathbb{Z}/2^\ell; \mathbb{Z})$.*

1. *There is a ring isomorphism $H^*(B\mathbb{Z}/2^\ell; \mathbb{Z}) \cong \mathbb{Z}[c_{1,\ell}]/(2^\ell \cdot c_{1,\ell})$.*
2. *The map $p^*: H^*(B\mathbb{Z}/2^k; \mathbb{Z}) \rightarrow H^*(B\mathbb{Z}/2^{k+1}; \mathbb{Z})$ sends $c_{1,k} \mapsto 2c_{1,k+1}$.*
3. *The map $p^*: H^{2n-1}(B\mathbb{Z}/2^k; \mathbb{C}^\times) \cong \mathbb{Z}/2^k \rightarrow H^{2n-1}(B\mathbb{Z}/2^{k+1}; \mathbb{C}^\times) \cong \mathbb{Z}/2^{k+1}$ sends a generator of the domain to 2^n times a generator of the codomain.*

Proof. Part (1) is standard: for example, the cohomology ring is computed by Eilenberg–Mac Lane in [151–153]. For part (2), it suffices to understand p^* in degree 2, as that degree generates both cohomology rings. The universal coefficient theorem provides a natural isomorphism

$$\text{Ext}(H_1(B\mathbb{Z}/n; \mathbb{Z}), \mathbb{Z}) \xrightarrow{\cong} H^2(B\mathbb{Z}/n; \mathbb{Z}), \quad (\text{C.3a})$$

and the Hurewicz theorem canonically identifies $\mathbb{Z}/n \cong H_1(B\mathbb{Z}/n; \mathbb{Z})$. Finally, the long exact sequence in $\text{Ext}(-, \mathbb{Z})$ associated to the exponential short exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathbb{C}^\times \rightarrow 0$ collapses to a natural isomorphism

$$\text{Hom}(A, \mathbb{C}^\times) \xrightarrow{\delta} \text{Ext}(A, \mathbb{Z}) \quad (\text{C.3b})$$

for any finite abelian group A . Unwinding all of these natural isomorphisms, to show that $p^*(c_{1,k}) = 2c_{1,k+1}$, it suffices to show that p itself sends $1 \mapsto 1$, which is true.

Finally (3). For any finite group G , $H^*(BG; \mathbb{R})$ vanishes in positive degrees, so the Bockstein $H^m(BG; \mathbb{C}^\times) \rightarrow H^{m+1}(BG; \mathbb{Z})$ is a natural isomorphism for $m \geq 1$. Since the domain and codomain of p are both finite groups, this reduces part (3) to part (2). \square

We will use two Smith long exact sequences in this proof. First take Theorem B.48 with $E_* = \tau_{\leq 2} ko_*$, $X = B\mathbb{Z}/2^k$, and $V = W = V_\rho$. There is a homotopy equivalence $S(V_\rho) \simeq S^1$, which stably splits as $\mathbb{S} \vee \Sigma \mathbb{S}$ [70, Example 7.28]. Thus we have a long exact sequence

$$\dots \rightarrow (\tau_{\leq 2} ko)_n(\mathbb{S} \vee \Sigma \mathbb{S}) \rightarrow (\tau_{\leq 2} ko)_n((B\mathbb{Z}/2^k)^{V_\rho}) \xrightarrow{\text{sm}_{V_\rho}} (\tau_{\leq 2} ko)_{n-2}(B\mathbb{Z}/2^k) \rightarrow \dots \quad (\text{C.4})$$

Lemma C.5. *Under the map $p: B\mathbb{Z}/2^{k+1} \rightarrow B\mathbb{Z}/2^k$, the bundle $V_\rho \rightarrow B\mathbb{Z}/2^k$ pulls back to $V_\rho \otimes V_\rho \rightarrow B\mathbb{Z}/2^{k+1}$.*

Proof. It suffices to show this at the level of representations of $\mathbb{Z}/2^{k+1}$; since this is a cyclic group, it suffices to check on a generator. Specifically, for both $p^*(V_\rho)$ and $V_\rho \otimes V_\rho$, it is straightforward to see that $1 \in \mathbb{Z}/2^{k+1}$ acts by $e^{2\pi i/k}$. \square

³² We do not record m in the notation for V_ρ , as it will always be clear from context.

The other Smith long exact sequence we need uses $E = \tau_{\leq 2} ko$ again; this time $X = B\mathbb{Z}/2^{k+1}$, $V = 0$, and $W = V_\rho \otimes V_\rho$:

$$\cdots \rightarrow (\tau_{\leq 2} ko)_n(S(V_\rho \otimes V_\rho)) \rightarrow (\tau_{\leq 2} ko)_n(B\mathbb{Z}/2^{k+1}) \xrightarrow{\text{sm}_{V_\rho \otimes V_\rho}} (\tau_{\leq 2} ko)_{n-2}(B\mathbb{Z}/2^{k+1}) \rightarrow \cdots , \quad (\text{C.6})$$

A priori the rightmost term in (C.6) is a $(B\mathbb{Z}/2^{k+1}, V_\rho \otimes V_\rho)$ -twisted $\tau_{\leq 2} ko$ -homology group, but $V_\rho \otimes V_\rho$ has a canonical spin structure, so we obtain untwisted $\tau_{\leq 2} ko$ -homology. In a little more detail, a spin structure on a complex line bundle is equivalent to a choice of square root with respect to tensor product [154], and for $V_\rho \otimes V_\rho$, we have the square root V_ρ .

The sphere bundle $S(V_\rho \otimes V_\rho)$ fits in the following diagram, where both squares are pullback squares:

$$\begin{array}{ccccc} S(V_\rho \otimes V_\rho) & \longrightarrow & S(V_\rho) & \longrightarrow & E\mathbb{C}^\times \\ \downarrow & & \downarrow & & \downarrow \\ B\mathbb{Z}/2^{k+1} & \longrightarrow & B\mathbb{Z}/2^k & \longrightarrow & B\mathbb{C}^\times . \end{array} \quad (\text{C.7})$$

We identified $S(V_\rho) \simeq S^1 = B\mathbb{Z}$ above and need to compute $S(V \otimes V)$.

Lemma C.8. *There is a homotopy equivalence $S(V_\rho \otimes V_\rho) \simeq B\mathbb{Z} \times B\mathbb{Z}/2$ under which*

1. *the map $S(V_\rho \otimes V_\rho) \rightarrow B\mathbb{Z}/2^{k+1}$ in (C.7) is B of the map $\mathbb{Z} \times \mathbb{Z}/2 \rightarrow \mathbb{Z}/2^{k+1}$ sending $(c, d) \mapsto c + 2^k d$, and*
2. *the map $S(V_\rho \otimes V_\rho) \rightarrow S(V_\rho)$ is identified with the map $B\mathbb{Z} \times B\mathbb{Z}/2 \rightarrow B\mathbb{Z}$ which is projection onto the first factor.*

Proof. Conveniently, $S(V_\rho \otimes V_\rho)$ is the pullback of the diagram $B\mathbb{Z}/2^{k+1} \rightarrow B\mathbb{Z}/2 \leftarrow B\mathbb{Z}$, which is the result of applying the classifying space functor to the following diagram of groups:

$$\begin{array}{ccc} \mathbb{Z} & & \\ \downarrow \text{mod } 2^k & & \\ \mathbb{Z}/2^{k+1} & \xrightarrow[\text{mod } 2^k]{} & \mathbb{Z}/2^k \end{array} \quad (\text{C.9})$$

The bar construction model for the classifying space functor preserves pullbacks, so $S(V_\rho)$ is homotopy equivalent to the classifying space of the group which is the pullback of (C.9). In the category of groups, there is an explicit formula for the pullback of the diagram $H \xrightarrow{f} G \xleftarrow{g} K$ [155, Tag 0020], namely

$$H \times_G K \cong \{(h, k) \in H \times K : f(h) = g(k)\}. \quad (\text{C.10})$$

The maps to H and K are projection onto the first, resp., second factor.

Applying this to (C.9), we see that the pullback group is $\mathbb{Z} \times \mathbb{Z}/2$, with the map to $\mathbb{Z}/2^{k+1}$ sending $(c, d) \mapsto c + 2^k d$ and the map to \mathbb{Z} sending $(c, d) \mapsto c$. Applying the classifying space functor, we have $S(V_\rho \otimes V_\rho) \cong B\mathbb{Z} \times B\mathbb{Z}/2$ as well as the maps to $S(V_\rho)$ and to $B\mathbb{Z}/2^{k+1}$. \square

The Smith long exact sequence (B.49) is by construction natural in the data X , V , and W , so from (C.7) we obtain the following commutative diagram, whose rows are exact; cross-references indicate where we have already determined some of the entries in this diagram:

$$\begin{array}{ccccccc} (\tau_{\leq 2} ko)_5(B\mathbb{Z} \times B\mathbb{Z}/2) & \longrightarrow & (\tau_{\leq 2} ko)_5(B\mathbb{Z}/2^{k+1}, 0, 0) & \xrightarrow{\text{sm}_{V_\rho \otimes V_\rho}} & (\tau_{\leq 2} ko)_3(B\mathbb{Z}/2^{k+1}, 0, 0) & \longrightarrow & (\tau_{\leq 2} ko)_4(B\mathbb{Z} \times B\mathbb{Z}/2) \\ \downarrow & & \downarrow \text{\scriptsize{$\cong \mathbb{Z}/2^k$ (B.17)}} & & \downarrow \text{\scriptsize{$\cong \mathbb{Z}/2^{k+2} \oplus \mathbb{Z}/2$ (B.20)}} & & \downarrow \\ (\tau_{\leq 2} ko)_5(\mathbb{S} \vee \Sigma \mathbb{S}) & \longrightarrow & (\tau_{\leq 2} ko)_5(B\mathbb{Z}/2^k, 0, y) & \xrightarrow{\text{sm}_{V_\rho}} & (\tau_{\leq 2} ko)_3(B\mathbb{Z}/2^k, 0, 0) & \longrightarrow & (\tau_{\leq 2} ko)_4(\mathbb{S} \vee \Sigma \mathbb{S}) . \\ \downarrow \text{\scriptsize{$\cong \mathbb{Z}/2^{k+1} \oplus \mathbb{Z}/2$ (B.25)}} & & \downarrow & & \downarrow \text{\scriptsize{$\cong \mathbb{Z}/2^{k+1} \oplus \mathbb{Z}/2$ (B.20)}} & & \end{array} \quad (\text{C.11})$$

Thus we would like to find $(\tau_{\leq 2} ko)_\ell(X)$ for $\ell = 4, 5$ and $X = \mathbb{S} \vee \Sigma \mathbb{S}$ and $B\mathbb{Z} \times B\mathbb{Z}/2$. A straightforward Atiyah–Hirzebruch spectral sequence calculation shows $(\tau_{\leq 2} ko)_\ell(\mathbb{S} \vee \Sigma \mathbb{S}) \cong 0$ whenever $\ell \geq 4$. It is also true that $(\tau_{\leq 2} ko)_5(B\mathbb{Z} \times$

$B\mathbb{Z}/2$) vanishes: to see this, use the stable splitting $B\mathbb{Z} \simeq \mathbb{S} \vee \Sigma\mathbb{S}$ to reduce to showing $(\tau_{\leq 2}\tilde{k}o)_\ell(B\mathbb{Z}/2)$ vanishes for $\ell = 4, 5$. This is equivalent to $SH^\ell(B\mathbb{Z}/2)$ vanishing for these values of ℓ , which we showed in Proposition B.11. Thus in (C.11), $\text{sm}_{V_\rho \otimes V_\rho}$ is injective and sm_{V_ρ} is an isomorphism, so their $I_{\mathbb{C}^\times}$ duals are surjective, resp. an isomorphism.

We now determine the maps in the middle square of (C.11). We consider the $I_{\mathbb{C}^\times}$ dual sequences in supercohomology, where there are maps f , g , and h , yet to be determined, so that the central square of (C.11) has the form

$$\begin{array}{ccc} \mathbb{Z}/2^k & \xleftarrow{g} & \mathbb{Z}/2^{k+2} \oplus \mathbb{Z}/2 \\ f \uparrow & & \uparrow h \\ \mathbb{Z}/2^{k+1} \oplus \mathbb{Z}/2 & \xleftarrow{\cong} & \mathbb{Z}/2^{k+1} \oplus \mathbb{Z}/2. \end{array} \quad (\text{C.12})$$

In a moment, we will choose an isomorphism $SH^3(B\mathbb{Z}/2^k) \cong \mathbb{Z}/2^{k+1} \oplus \mathbb{Z}/2$; once we do so, choose the isomorphism $SH^5(B\mathbb{Z}/2^k, 0, y) \cong \mathbb{Z}/2^{k+1} \oplus \mathbb{Z}/2$ so that the bottom isomorphism between the two in (C.12) becomes the identity $\mathbb{Z}/2^{k+1} \oplus \mathbb{Z}/2 \rightarrow \mathbb{Z}/2^{k+1} \oplus \mathbb{Z}/2$.

Lemma C.13. *There are choices of the isomorphisms $SH^3(B\mathbb{Z}/2^\ell) \cong \mathbb{Z}/2^{\ell+1} \oplus \mathbb{Z}/2$ from Proposition B.20 such that, with respect to those isomorphisms, the map $h: SH^3(B\mathbb{Z}/2^k) \rightarrow SH^3(B\mathbb{Z}/2^{k+1})$ in (C.12) is given by the matrix $\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$.*

Lemma C.14. *There are $\lambda_1 \in (\mathbb{Z}/2^k)^\times$ and $\lambda_2 \in \mathbb{Z}/2$ such that, with respect to the isomorphism $SH^3(B\mathbb{Z}/2^{k+1}) \cong \mathbb{Z}/2^{k+2} \oplus \mathbb{Z}/2$ chosen in Lemma C.13 and the isomorphism $SH^5(B\mathbb{Z}/2^{k+1})$ chosen in Proposition B.17, $(a, b) \mapsto \lambda_1 a + \lambda_2 2^{k-1} b$.*

Proof. This description of g is true for any surjective map $\mathbb{Z}/2^{k+2} \oplus \mathbb{Z}/2 \rightarrow \mathbb{Z}/2^k$: surjectivity guarantees $f(1, 0)$ is a unit, so we can define $\lambda_1 := f(1, 0)^{-1}$; then $\lambda_2 := (1/2^{k-1})f(0, 1)$. \square

Proof of Lemma C.13. Recall that h corresponds to the map $p^*: SH^3(B\mathbb{Z}/2^k) \rightarrow SH^3(B\mathbb{Z}/2^{k+1})$, so we begin by calculating the effect of p^* on the E_∞ -page of the supercohomology AHSSes. We computed these E_∞ -pages in (B.19); for the reader's convenience, we reproduce these two E_∞ -pages in Figure 7. Like in the proof of Lemma B.31, we will let ${}^\ell E_\infty^{p,q}$ denote the $E_\infty^{p,q}$ entry of the AHSS computing $SH^*(B\mathbb{Z}/2^\ell)$. In particular, ${}^\ell E_\infty^{\bullet, 3-\bullet}$ consists of the following three summands:

- ${}^\ell E_\infty^{3,0} \cong H^3(B\mathbb{Z}/2^\ell; \mathbb{Z}) \cong \mathbb{Z}/2^\ell$,
- ${}^\ell E_\infty^{2,1} \cong H^2(B\mathbb{Z}/2^\ell; \mathbb{Z}/2) \cong 2 \cdot y$, and
- ${}^\ell E_\infty^{1,2} \cong H^1(B\mathbb{Z}/2^\ell; \mathbb{Z}/2) \cong 2 \cdot x$.

It is straightforward to check that in mod 2 cohomology, p pulls back $y \mapsto 0$ and $x \mapsto x$. Thus $p^*: {}^k E_\infty^{3-j,j} \rightarrow {}^{k+1} E_\infty^{3-j,j}$ is an isomorphism for $j = 1, 2$. On ${}^k E_\infty^{3,0}$, Lemma C.2, part (3) computes p^* on $H^3(-; \mathbb{C}^\times)$.

To finish, we need to lift from the E_∞ -page to the actual supercohomology groups. We have an extension problem to resolve for $SH^3(B\mathbb{Z}/2^\ell)$, where $\ell \geq 2$: $\mathbb{Z}/2^\ell$ in the Dijkgraaf–Witten layer, $\mathbb{Z}/2$ in the Gu–Wen layer, and $\mathbb{Z}/2$ in the Majorana layer combine to $\mathbb{Z}/2^{\ell+1} \oplus \mathbb{Z}/2$ (Proposition B.20). Thus we have a nonsplit extension of either the Gu–Wen layer or the Majorana layer by the Dijkgraaf–Witten layer. In fact, the extension is between the DW and GW layers; to see this, first note that this is equivalent to the corresponding extension in *restricted* supercohomology $rSH^3(B\mathbb{Z}/2^\ell)$ being nonsplit, just as in the proof of Lemma B.31. Gu–Wen [27, (F12)] computed $rSH^3(B\mathbb{Z}/2^\ell) \cong \mathbb{Z}/2^{\ell+1}$ for $\ell \geq 2$, implying a nonsplit extension in restricted supercohomology, and therefore an extension between the GW and DW layers in supercohomology.

Since p^* is an isomorphism on the Majorana layer, and the Majorana layer splits off for all $\ell \geq 2$, choose any splitting of the Majorana layer off of the GW and DW layers for $\ell = 2$; for $\ell > 2$, inductively choose the splitting that makes the pullback map p^* diagonal. Thus we have chosen isomorphisms $SH^3(B\mathbb{Z}/2^\ell) \cong rSH^3(B\mathbb{Z}/2^\ell) \oplus \mathbb{Z}/2$ such that p^* is a diagonal matrix whose $(2, 2)$ entry is 1 and whose $(1, 1)$ entry is to be determined.

We have a commutative diagram of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & {}^k E_\infty^{3,0} = \mathbb{Z}/2^k & \xrightarrow{1 \mapsto 2} & rSH^3(B\mathbb{Z}/2^k) \cong \mathbb{Z}/2^{k+1} & \xrightarrow{\text{mod } 2} & {}^k E_\infty^{2,1} = \mathbb{Z}/2 & \longrightarrow 0 \\ & & \downarrow 1 \mapsto 4 & & \downarrow p^* & & \downarrow 1 \mapsto 0 & \\ 0 & \longrightarrow & {}^{k+1} E_\infty^{3,0} \cong \mathbb{Z}/2^{k+1} & \xrightarrow{1 \mapsto 2} & rSH^3(B\mathbb{Z}/2^{k+1}) \cong \mathbb{Z}/2^{k+2} & \xrightarrow{\text{mod } 2} & {}^{k+1} E_\infty^{2,1} = \mathbb{Z}/2 & \longrightarrow 0 \end{array} \quad (\text{C.15})$$

which is the map induced by p^* between the extensions of the Dijkgraaf–Witten and Gu–Wen layers. The leftmost and rightmost vertical arrows follow from our calculation of p^* applied to $E_\infty^{3,0}$ and $E_\infty^{2,1}$; commutativity then forces the middle vertical arrow to send $1 \mapsto 4$. Here we have been cavalier about the choice of isomorphism $rSH^3(B\mathbb{Z}/2^\ell) \cong \mathbb{Z}/2^{\ell+1}$, but this is easily fixed: choose any such isomorphism for $\ell = 2$, then inductively define it for larger ℓ so that the middle vertical arrow in (C.15) sends $1 \mapsto 4$. \square

Corollary C.16. *With respect to the isomorphisms chosen above, the map f in (C.12) sends $(1, 0) \mapsto 4\lambda_1$ and $(0, 1) \mapsto \lambda_2$.*

Proof. Directly compute the maps h and g using Lemmas C.13 and C.14. \square

Now we can return to the last case of Theorem I.13.

Proof of Theorem I.13, case $n = \mathbb{Z}/2^k$ for $k \geq 2$. By Corollary C.16, a generating set of $SH^5(B\mathbb{Z}/2^k, 0, y)$ is sent to elements of order at most 2^{k-1} in $SH^5(B\mathbb{Z}/2^{k+1}, 0, 0) \cong \mathbb{Z}/2^{k+2}$. By Proposition B.17, these images are in the Dijkgraaf–Witten layer of $SH^5(B\mathbb{Z}/2^{k+1}, 0, 0)$, so to pull them back further, it suffices to work in \mathbb{C}^\times cohomology. Specifically, if we pull back further to $SH^5(B\mathbb{Z}/2^{k+m+1}, 0, 0)$, the map on the DW layer is multiplication by 8^m by Lemma C.2, part (3), so the composition

$$SH^5(B\mathbb{Z}/2^k, 0, y) \longrightarrow SH^5(B\mathbb{Z}/2^{k+1}, 0, 0) \longrightarrow SH^5(B\mathbb{Z}/2^{k+m+1}, 0, 0) \quad (\text{C.17})$$

sends a generating set to multiples of $4 \cdot 8^m$, which equals 0 in $H^5(B\mathbb{Z}/2^{k+m+1}; \mathbb{C}^\times)$ if $m \geq \frac{k-1}{2}$, and this establishes Theorem I.13. \square

$$E_\infty^{i,j} = \begin{array}{c|ccccccc} j & & & & & & & \\ \hline 2 & 1 & x & 0 & 0 & \dots & & \\ 1 & 1 & x & y & 0 & 0 & \dots & \\ 0 & \mathbb{C}^\times & \mathbb{Z}/2^{k+1} & 0 & \mathbb{Z}/2^{k+1} & 0 & \mathbb{Z}/2^k & 0 \dots \\ \hline 0 & 1 & 2 & 3 & 4 & 5 & 6 & i \end{array} \quad (\text{C.18})$$

$$E_\infty^{i,j} = \begin{array}{c|ccccccc} j & & & & & & & \\ \hline 2 & 1 & x & 0 & 0 & \dots & & \\ 1 & 1 & x & y & 0 & 0 & \dots & \\ 0 & \mathbb{C}^\times & \mathbb{Z}/2^k & 0 & \mathbb{Z}/2^k & 0 & \mathbb{Z}/2^{k-1} & 0 \dots \\ \hline 0 & 1 & 2 & 3 & 4 & 5 & 6 & i \end{array} \quad (\text{C.19})$$

FIG. 7: In total degree 5 and below, the above diagrams give the E_∞ -page of $SH^3(B\mathbb{Z}/2^{k+1}, 0, 0)$ on the left, and $SH^3(B\mathbb{Z}/2^k, 0, 0)$ on the right.

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