NOTES FROM THE 2022 SUMMER SCHOOL ON GLOBAL SYMMETRIES

ARUN DEBRAY
JUNE 13, 2022

These notes were taken at the GCS2022 School in June 2022 at the Perimeter Institute of Theoretical Physics. I live-TEXed them and there may be typos or mistakes; please send questions, comments, complaints, and corrections to adebray@purdue.edu.

CONTENTS

Part 1. Kantaro Ohmori, Introduction to symmetries in quantum field theory

1. EXTENDED OPERATORS AND DEFECTS IN FUNCTORIAL QFT: 6/13/22

One of the goals of this workshop is to discuss generalized symmetries in quantum field theory. For us, this means topological operators and defects. Here's a short outline of these four talks:

(1) Quantum field theory as a functor, and how to work with extended operators and defects in this formalism
(2) Topological operators and symmetries
(3) One-form symmetries in gauge theories and confinement
(4) "Non-invertible" symmetries

Let's get started. Quantum field theory means a lot of different things to a lot of different people; today, we will only focus on relativistic Euclidean QFT. When we say a d-dimensional QFT, d refers to the dimension of spacetime.

In the non-topological setting, it's not yet completely clear how to define a quantum field theory as a functor, so the following definition will be a little heuristic.

Definition 1.1. A quantum field theory is a symmetric monoidal functor $Z\colon \mathcal{B}ord_S^{\langle d,d-1\rangle} \to \mathcal{V}ect$.

Here $\mathcal{V}ect$ is the symmetric monoidal category of vector spaces with tensor product and $\mathcal{B}ord_S^{\langle d,d-1\rangle}$ is a bordism category. S refers to some kind of geometric structure we want to endow spacetime with: for example, we could ask for just a smooth structure, or a spin structure, or a Riemannian metric, or a principal G-bundle with a connection, or so on. A manifold with S-structure is called an S-manifold.

The objects of $\mathcal{B}ord_S^{\langle d,d-1\rangle}$ are closed, $(d-1)$-dimensional manifolds with an S-structure. The set of morphisms between $(d-1)$-dimensional S-manifolds M and N is the set of (diffeomorphism classes rel boundary) of S-bordisms from M to N. An S-bordism X from M to N is a compact, d-dimensional manifold with an identification of S-manifolds $\partial X \overset{\cong}{\to} M \amalg \overline{N}$. Here $\overline{N}$ denotes N with the opposite orientation.

To define a category, we need to compose morphisms; this is accomplished by gluing bordisms. (TODO: picture).

Remark 1.2. We haven't said precisely how to define S, so one might wonder whether it depends on d. For example, there's a difference between a framing of a manifold M (a trivialization of TM) and a stable framing (a trivialization of $TM \oplus \mathbb{R}^k$ for some k); the former depends on d and the latter does not, and the two notions are not the same.

Freed–Hopkins [FH21] have shown that for reflection-positive topological field theories, many S-structures that appear to depend on d in fact stabilize and are independent of the dimension.   ◄

If $Z\colon \mathcal{B}ord_S^{\langle d, d-1\rangle}$ is a quantum field theory, then for a closed $(d-1)$-dimensional S-manifold M, $Z(M)$ is a vector space. This is called the *state space* of M. If X is a bordism from M to N, then $Z(X)$ is a linear map from the state space of M to the state space of N. We often think of this map as *time evolution* of states. The fact that Z is symmetric monoidal means that $Z(M_1 \amalg M_2) \cong Z(M_1) \otimes Z(M_2)$.

Let $\tau \in (0, \infty)$. Then $M \times [0, \tau]$ is a bordism from M to M (its boundary is $M \amalg \overline{M}$). Let $U_M(\tau) := Z(M \times [0, \tau])$, which we can think of as time evolution on M for time τ; in a Hamiltonian system we think of

$$(1.3) \qquad\qquad U_M(\tau) = \exp(-\tau H_M),$$

where $H_M\colon Z(M) \to Z(M)$ is the Hamiltonian on M. Gluing bordisms implies $U_M(\tau_1) \circ U_M(\tau_2) = U_M(\tau_1 + \tau_2)$.

Remark 1.4. We would like to think of these operators as unitary, like in Hamiltonian quantum mechanics; making this precise from the functorial perspective is an area of active research. See for example a recent proposal of Kontevich–Segal [KS21].

To discuss unitarity we need some kind of inner product, but we did not ask for our state spaces to come with inner products. There is a one-parameter family of bilinear pairings around: the cylinder $M \times [0, \tau]$, thought of as a bordism from $M \amalg \overline{M} \to \varnothing$, induces a map $Z(M) \otimes Z(\overline{M}) \to Z(\varnothing) = \mathbb{C}$.[1]   ◄

Example 1.5 (Finite gauge theory). (Untwisted) finite gauge theory is a d-dimensional *topological* quantum field theory: the S-structure is topological, rather than geometric. Specifically, it is no structure at all.

Fix a finite group G and $p \in \{0, 1, \ldots, d-1\}$. If $p > 0$, we ask that G is abelian. Therefore we can make sense of $H^{p+1}(M; G)$ when M is a closed manifold: when G is nonabelian and $p = 0$, this is the set of isomorphism classes of principal G-bundles on M. For compact M, $H^{p+1}(M; G)$ is finite.

Let M be a closed $(d-1)$-manifold; we define the state space of finite gauge theory on M to be the vector space spanned by the finite set $H^{p+1}(M; G)$.

If W is a bordism from M to N, we define the linear map $Z(W)$ as a form of "finite path integral" — we can't make sense of the path integral in general for gauge theories, but becaue G is finite we can in this case. Fix $A \in H^{p+1}(M; G)$ and let $i_M\colon M \hookrightarrow W$ and $i_N\colon N \hookrightarrow W$ be the inclusions. Define

$$(1.6) \qquad\qquad Z(W)|A\rangle := c(W) \sum_{\substack{B \in H^{p+1}(W;G) \\ i_M^* B = A}} |i_N^* B\rangle,$$

where $c(W) \in \mathbb{R}$ is a mormalization constant that appears so that this definition is functorial when we glue bordisms.   ◄

Exercise 1.7. Say $d = 2$, $p = 0$, and $G = \mathbb{Z}/n$. Calculate $Z(S^1)$, $Z(\Sigma)$, and $Z(\Sigma')$, where Σ is the pair of pants regarded as a bordism from $S^1 \amalg S^1 \to S^1$ and Σ' is Σ in the opposite direction. For bordisms, only calculate the maps up to normalization constants, since we didn't specify those constants in (1.6).

Now let's talk about extended quantum field theory. If M is a closed d-dimensional manifold, it may be regarded as a bordism $\varnothing \to \varnothing$. Applying Z, we obtain a linear map $\mathbb{C} \to \mathbb{C}$, since $Z(\varnothing) = \mathbb{C}$. This map is determined by its value on 1, which is a complex number called the *partition function* of M.

Associated to a closed $(d-1)$-manifold we have a state space. In extended QFT, we assign higher-categorical invariants to manifolds in lower dimensions; for example, to a closed $(d-2)$-dimensional manifold we assign something called a "2-vector space," which is something like a $\mathbb{C}$-linear category; and in general on a closed $(d-k)$-manifold we assign a "k-vector space," some kind of higher catgory. We will not define k-vector spaces precisely here, and indeed different researchers use different definitions.

We are especially interested in $Z(S^{q-1})$ as q varies; this is some sort of higher category. The objects of this category are the codimension-q defects or extended operators (to us, these two words mean the same thing) in the QFT Z. In non-topological theories, we need to specify the radius r of S^{q-1}; the codimension-q defects are the limit of $Z(S^{q-1}(r))$ as $r \to 0$.

There is another formalism for higher-codimension defects or operators, given by something called *decorated bordisms*. This latter approach may be easier to digest from the physics point of view. The two approaches are expected to be equivalent.

Part 2. Clay Córdova, Introduction to anomalies in quantum field theory

2. : 6/13/22

We begin by looking at degenerate ground states in quantum mechanics. The setup has a separable Hilbert space W, e.g. $L^2(\mathbb{R})$. This is the state space; the quantum states are nonzero elements of W modulo phases: we identify ψ and $\lambda\psi$ for $\lambda \in \mathbb{C}^\times$. Time evolution in this system is described using a self-adjoint positive Hermitian operator H,

---

[1]The fact that $Z(\varnothing) = \mathbb{C}$ is another consequence of symmetric monoidality.

called the *Hamiltonian*; $H = H^\dagger$. Some of these assumptions are because we are in the unitary setting. If $\mathcal{O}$ is an operator, it evolves under time as

$$\mathcal{O}(t) = e^{iHt}\mathcal{O}(0)e^{-iHt}. \tag{2.1}$$

Since H is self-adjoint and positive, its eigenvalues are real and nonnegative. Let E_i be the eigenvalues of H in ascending order; they represent the energy levels of the theory. We are particularly interested in the eigenvectors for the smallest eigenvalue. If this eigenspace is more than one-dimensional, we say this system has a *degenerate ground state*.

We are interested in questions related to ground state degeneracy. For example, when is there a degenerate ground state? Is this degeneracy stable under deforming H?

Example 2.2. Consider a system of n particles moving on $\mathbb{R}$ in the presence of a potential $V\colon \mathbb{R}^n \to \mathbb{R}$. Then $W = L^2(\mathbb{R}^n)$ and the Hamiltonian is

$$H := -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + V(x_1, \ldots, x_n) \tag{2.3}$$

Theorem 2.4. *Let $L^2_{loc}(R^n)$ denote the Hilbert space of functions which are square-integrable on compact subsets of $\mathbb{R}^n$. If $W = L^2_{loc}(\mathbb{R}^n)$, $V \geq 0$, and $V \to \infty$ as $|x_i| \to \infty$, then H has a nondegenerate ground state.*

The proof is an exercise, though see Reed–Simon volume IV [RS78, Chapter 8]. The assumptions cover many of the typical examples of quantum-mechanical systems, such as a double well. ◄

The point of introducing Theorem 2.4 here is that it's not easy to produce examples of systems with ground state degeneracy.

One of the goals of this series of lectures is to develop a theory of invariants, called *anomalies*, which imply degenerate ground states in quantum mechanics and quantum field theory which are robust to perturbations.

Symmetry in quantum mechanics plays a key role. There are two kinds of symetries: *unitary transformations*, operators $U\colon W \to W$ such that $[U, W] = 0$, and *antiunitary transformations*, operators $T\colon W \to W$ such that $[H, T]$ acts on operators by conjugation by a unitary operator: $\mathcal{O} \mapsto U\mathcal{O}U^{-1}$. When T is antiunitary and $\lambda \in \mathbb{C}^\times$, $T(\lambda w) = \overline{\lambda}T(w)$.

Example 2.5. Consider a particle on a circle. Our variable x now is 2π-periodic: we identify x and $x + 2\pi$. We write down a Lagrangian

$$L = \frac{1}{2}\dot{x}^2 + \frac{\theta}{2\pi}\dot{x}, \tag{2.6}$$

where $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ is a parameter. From the Lagrangian, one does a canonical transformation to obtain the Hamiltonian; this is a standard trick, and the answer is

$$H = \frac{1}{2}\left(-i\frac{d}{dx} - \frac{\theta}{2\pi}\right)^2, \tag{2.7}$$

acting on the functions on the circle. The Lagrangian and Hamiltonian are quadratic, so it's easy to solve this explicitly for all energy levels at once. When we do this, we will see something interesting.

The eigenfunctions are Fourier modes: $\exp(inx)$ for $n \in \mathbb{Z}$, which has eigenvalue $E_n = (1/2)(n - \theta/2\pi)^2$. Though the energy levels are the same, the states themselves move around, so there's some sort of spectral flow. At $\theta = 0$, the ground state has energy level 0, and excited states (the eigenstates for eigenvalues larger than the smallest eigenvalue) are doubly degenerate: $\pm n$ gives you a two-dimensional eigenspace. But at $\theta = \pi$, all eigenvalues have two-dimensional eigenspaces. See Figure 1 for a picture of the spectrum as θ varies. One might think that this "level-crossing" behavior is generic, but this is not correct.

Exercise 2.8. Determine the codimension of the level-crossing loci for a multiparameter Hamiltonian (TODO: may have misunderstood this exercise?).

This system has symmetries, and let's take advantage of them. Let a be a time-independent constant; then the system is symmetric under the shift $x \mapsto x + a$. This defines unitary operators U_a which generate the Lie group U_1 under composition.

Another symmetry is reflection, $C\colon x \mapsto -x$. This acts on the Hamiltonian, and changes θ; therefore it is only actually a symmetry at $\theta = 0, \pi$. In some sense, generically we only have the U_1 symmetry, but at $\theta = 0, \pi$, the symmetry group enhances.

How do the symmetris act on the eigenstates? For translation, the answer is not so tricky:

$$U_a(w_n) = e^{ina}w_n. \tag{2.9}$$

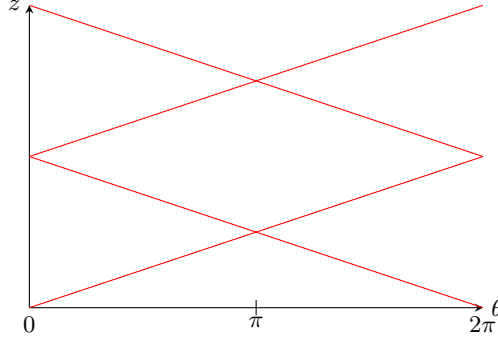**Exercise 2.10.** Convince yourself that at $\theta = 0$, $C(w_N) = w_{-n}$, and at $\theta = \pi$, $C(w_n) = w_{-n+1}$.

FIGURE 1. The spectrum of the particle on a circle considered in Example 2.5. The red lines are the eigenvalues and how they change under θ. Notce the crossing at $\theta = 0, \pi$, where most, resp. all eigenspaces are two-dimensional. For other values of θ, the eigenspces are all one-dimensional.

This leads to a simple-looking question: *what is the group of symmetries generated by U_a and C?* There are two different answers, depending on what exactly you mean.

(1) When acting on operators — operators are generated by x and its derivatives, and compositions thereof, so we obtain the group O_2: U_a forms SO_2, and C acts as reflections on the circle.

(2) When acting on states, however, the answer is not the same at $\theta = \pi$. There we find an additional phase:

$$(2.11) \qquad CU_aC^{-1}(w_n) = CU_a(w_{-n+1}) = C\big(e^{ia(-n+1)}w_{-n+1}\big) = e^{ia}U_{-a}(w_n).$$

If this generated an O_2, we would've just exepcted $U_{-a}(w_n)$.

In the second setting, the phase we obtained tells us that the action of O_2 is in fact a projective representation associated to the double cover

$$(2.12) \qquad 1 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathrm{Pin}_2^- \longrightarrow O_2 \longrightarrow 1.$$

When acting on operators, we see O_2; when acting on states, we see this double cover. ◄

Your first question might be: why is this discrepancy between states and operators even possible? It is possible because states are rays in the Hilbert space of states, not vectors. A symmetry only has to be a projective representation on the Hilbert space. Concretely, we end up with a cocycle $\mu \in Z^2(O_2; U_1)$ that twists the multiplication in the action on states, and μ is not cohomologous to 0.[2] Because operators are acted upon by conjugation, they are blind to this cocycle.

Another key observation: all states have the same value of μ once we fix θ: at $\theta = 0$, $\mu = 0$, and at $\theta = \pi$, we have the same nonzero cocycle.

Each eigenspace of H forms a projective representation of O_2 with given μ. At $\theta = \pi$, where μ is not cohomologous to zero, there are no one-dimensional projective representations — and therefore there is ground state degeneracy!

Projective representations are among the easiest examples of anomalies, and they are also robust under symmetry-preserving deformations: they are classified by discrete cocycles. So if we smoothly modify the Hamiltonian, we will preserve the property of ground state degeneracy.

Exercise 2.13. Once again consider the particle on the circle, and modify the Hamiltonian by adding a potential:

$$(2.14) \qquad H = \frac{1}{2}\left(-i\frac{\mathrm{d}}{\mathrm{d}x} - \frac{\theta}{2\pi}\right)^2 + \frac{\lambda}{2\pi}\cos(2x).$$

(1) Show that for small λ and at $\theta = 0$, $|E_{+1} - E_{-1}| = \lambda + O(\lambda^2)$. The interpretation is that the degeneracy at $\theta = 0$ is a coincidence: we can turn on an arbitrarily small potential, deforming the Hamiltonian by an arbitrarily small amount, and undo the degeneracy.

(2) Show that at $\theta = \pi$, the ground state degeneracy persists for $\lambda \neq 0$.

The system at $\theta = \pi$ with nonzero λ is quite difficult: we don't know how to exactly solve it. But we do know the energy levels.

---

[2]Complex projective representations use U_1-valued cocycles; the fact that we had a $\mathbb{Z}/2$ cover, rather than a U_1 cover, comes from the fact that this cocycle is valued in the $\mathbb{Z}/2$ subgroup $\{\pm 1\} \subset U_1$. TODO: so we only see the image in $Z^2(O_2; U_1)$; how do we tell apart pin^+ and pin^- then?

The $\cos(2x)$ term breaks the shift symmetry from U_1 to $\mathbb{Z}/2$ (we can still shift by half of a period). This is actually all that we need to have the degeneracy.

Example 2.15. Consider a system of real fermions with a time-reversal symmetry. We'll have N fermions $\psi^i(t)$, $i = 1, \dots, N$. For simplicity assume N is even. Classically they're Grassmann variables; when we quantize we obtain a Clifford algebra: these variables' supercommutator is $\{\psi^i, \psi^j\} = 2\delta^{ij}$. We let the time-reversal symmetry T act on these operators as

$$(2.16) \qquad T\psi^i(t)T^{-1} = -\psi^i(-t).$$

The Hilbert space W, a Clifford algebra, is finite-dimensional, and more precisely $\dim(W) = 2^{N/2}$. Choose the zero Hamiltonian; then all states are ground states! That's a lot of ground state degeneracy.

One typically considers a mass term, inducing a quadratic deformation of the Hamiltonian:

$$(2.17) \qquad \Delta H \overset{?}{=} im\psi^1\psi^2,$$

for some $m \in \mathbb{R}$. As $T\Delta HT^{-1} = -\Delta H$, though, this mass term is incompatible with time-reversal. This crucially uses that T is an anti-unitary symmetry.

So quadratic deformations are no help. What about quartic deformations?

To answer this question, it's helpful to group the fermions into complex pairs. Let

$$(2.18) \qquad a_n = \frac{1}{\sqrt{2}}\big(\psi^{2n-1} + i\psi^{2n}\big).$$

Then the creation and annihilation operators for these complex fermions satisfy $\{a_n, a_m^\dagger\} = \delta_{mn}$. Each pair generates a two-component space $W_\pm$, and

$$(2.19) \qquad \begin{aligned} a(w_-) &= 0 & a^\dagger(w_+) &= 0 \\ a^\dagger(w_-) &= w_+ & a(w_+) &= w_-. \end{aligned}$$

TODO: I didn't quite follow this, sorry! In a general state, there are $N/2$ labels $w_{\pm\pm\cdots\pm}$. Consider the quartic deformation

$$(2.20) \qquad \Delta H = 4q\psi^1\psi^2\psi^3\psi^4,$$

where $q > 0$. Then we can factor ΔH as

$$(2.21) \qquad \Delta H = -q\Big(a_1a_1^\dagger - \frac{1}{2}\Big)\Big(a_2a_2^\dagger - \frac{1}{2}\Big).$$

There are aligned states w_{++} and w_{--} with $\Delta E = -q$, and w_{+-} and w_{-+} also are aligned (TODO: ?). Anyways, the upshot is that there is twofold ground state degeneracy, sort of like we saw with quantum mechanics on a circle. ◄

Exercise 2.22. Show that for $N = 8$ there exists a T-invariant quartic deformation leading to a unique ground state.

Consequently, the quantity $N \bmod 8$ is protected by T. It is also an indication that there is another anomaly.

Part 3. **Mike Hopkins, Lattice systems and topological field theories**

REFERENCES

[FH21] Daniel S. Freed and Michael J. Hopkins. Reflection positivity and invertible topological phases. *Geom. Topol.*, 25(3):1165–1330, 2021. https://arxiv.org/abs/1604.06527. 1

[KS21] Maxim Kontsevich and Graeme Segal. Wick rotation and the positivity of energy in quantum field theory. *Q. J. Math.*, 72(1-2):673–699, 2021. https://arxiv.org/abs/2105.10161. 2

[RS78] Michael Reed and Barry Simon. *Methods of modern mathematical physics. IV. Analysis of operators.* Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978. 3