M392C NOTES: APPLICATIONS OF QUANTUM FIELD THEORY TO GEOMETRY

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These notes were taken in UT Austin's M392C (Applications of Quantum Field Theory to Geometry) class in Fall 2017, taught by Andy Neitzke. I live-TeXed them using vim, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Thanks to Michael Hott and Andy Neitzke for a few corrections.

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Lecture 1.

Donaldson invariants and supersymmetric Yang-Mills theory: 8/31/17

"The wind blowing on it, well, that's not the worst thing that could happen to a pond! Now imagine you have a laser…"

The course website is https://www.ma.utexas.edu/users/neitzke/teaching/392C-qft-geometry/. There are also lecture notes which are hosted at https://github.com/neitzke/qft-geometry, and are currently a work in progress; if you have contributions or improvements, feel free to contribute them, as a pull request or otherwise. (I'm also taking notes, of course, and if you find problems or typos in my notes, feel free to let me know.) There's also a Slack channel for course-related discussions, which may be easier to use than office hours.

There will be exercises in this course, and you should do at least one-fourth of them for the best grade. Of course, you also want to do them in order to gain understanding. Some worked-out computations could be useful for submitting to the professor's lecture notes.

This course will be relatively wide-ranging; today's prerequisites involve some gauge theory, but the next few lectures won't as much.

Suppose you want to study the topology of smooth manifolds X. Surprisingly, it's really effective to introduce a geometrical gadget, e.g. a Riemannian metric g. Using it, we can define the *Laplace operator* on differential forms $\Delta\colon \Omega^k(X) \to \Omega^k(X)$, which has the formula

$$\Delta := \mathrm{dd}^* + \mathrm{d}^*\mathrm{d},$$

where $\mathrm{d}\colon \Omega^k(X) \to \Omega^{k+1}(X)$ is the de Rham differential, and $\mathrm{d}^*\colon \Omega^{k+1}(X) \to \Omega^k(X)$ is its adjoint in the L^2-inner product on differential forms induced by the metric. Thus d is canonical, but d^* depends on the choice of metric.

Next we consider the equation

$$\Delta\omega = 0. \tag{1.1}$$

This is a linear equation, so its space of solutions $\mathscr{H}_{k,g} := \ker(\Delta\colon \Omega^k \to \Omega^k)$, called the *space of harmonic k-forms*, is a vector space. If X is compact, it's even a finite-dimensional vector space, which is a consequence of the ellipticity of the Laplace operator.[1] Hence we can define a nonnegative integer

$$b_k(X) := \dim \mathscr{H}_{k,g},$$

called the k^{th} *Betti number* of X It's a fact that $b_k(X)$ does not depend on the choice of the metric! Thus they are invariants of the smooth manifold X.

In fact, there's even a categorified version of this. This reflects a recent (last decade or so) trend of replacing numbers with vector spaces, sets with categories, etc.

Theorem 1.2. *If X is compact,[2] there is a canonical isomorphism $\mathscr{H}_{k,g} \cong H^k(X;\mathbb{R})$, where the latter is the singular cohomology of X with coefficients in $\mathbb{R}$.*

This shows $b_k(X)$ doesn't depend on the smooth structure of X, and is even a homotopy invariant. This will not be true for the Donaldson invariants that we'll discuss later.

Exercise 1.3. Work out some of these spaces of harmonic forms for a metric on S^1 and S^2.

You have to choose a metric, and there are more or less convenient ones to pick. But no matter how you change the metric, there will be a canonical way to identify them.[3]

If X is oriented and $4n$-dimensional, there's a small refinement of the middle Betti number b_{2n} and space of harmonic forms $\mathscr{H}_{2n}$. The *Hodge star operator*

$$\star\colon \Omega^p(X) \longrightarrow \Omega^{\dim X - p}(X)$$

is an involution on $\Omega^{2n}(X)$.

Remark. Let's recall the Hodge star operator. This is an operator on differential forms defined using the Riemannian metric satisfying $\star^2 = 1$ in even dimension, and $[\star, \Delta] = 0$. Hence it acts on harmonic forms. On $\mathbb{R}^2$ with the usual metric, $\star(1) = \mathrm{d}x \wedge \mathrm{d}y$, and $\star(f\,\mathrm{d}x) = f\,\mathrm{d}y$. ◄

Hence we can decompose $\Omega^{2n}(X)$ into the (± 1)-eigenspaces of $\star$: let $\Omega^{2n,\pm}(X)$ denote the ± 1-eigenspace for $\star$. Similarly, $\mathscr{H}_{2n}(X)$ splits into $\mathscr{H}_{2n}^{\pm}(X)$. Thus b_{2n} also splits:

$$b_{2n}(X) = b_{2n}^+(X) + b_{2n}^-(X).$$

These spaces and numbers are also topological invariants, and can be understood in that way.

Exercise 1.4. In dimension $4n+2$, the Hodge star squares to -1. You can still extract topological information from this; what do you get?

Linear equations seem to behave more or less the same in all dimensions. But nonlinear equations behave very differently in different dimensions. In the 1980s, Donaldson [8] used nonlinear equations to produce new and interesting invariants of 4-manifolds. Let X be a connected, oriented 4-manifold with a Riemannian metric g.

Fix a compact Lie group G. For Donaldson, $G = \mathrm{SU}(2)$, and it's probably fine to assume that for much of this class. Fix a principal G-bundle $P \to X$. We'll consider connections on P.

Remark. If you don't know what a connection is, that's OK. Locally, a connection on P is represented by a Lie algebra-valued 1-form $A \in \Omega^1_X(\mathfrak{g})$, and has a *curvature 2-form* $F \in \Omega^2_X(\mathfrak{g}_P)$, which locally is written

$$F = \mathrm{d}A + A \wedge A.$$

Because SU(2) is nonabelian, $A \wedge A$ isn't automatically zero. ◄

Since F is a 2-form and $\dim X = 4$, we can decompose F into its *self-dual part* F^+ and its *anti-self-dual part* F^-, defined by the splitting of Ω^p by the Hodge star.

[1] For a general differential operator on differential forms, nothing like this is true.

[2] Compactness is really necessary for this.

[3] Interesting question: if you change the metric infinitesimally, how does $\mathscr{H}_k$ change?

Exercise 1.5. Show that if you reverse the orientation of X, F^+ and F^- switch.

Donaldson studied the *anti-self-dual Yang-Mills equation* (ASD YM):

$$F^+ = 0. \tag{1.6}$$

By Exercise 1.5, this is not really different than studing the self-dual Yang-Mills equation; the reason one prefers the ASD version is that it occurs more naturally on certain complex manifolds which were test cases for Donaldson theory.

If G is abelian, e.g. U(1), (1.6) is linear. But if G is nonabelian, e.g. SU(2), then (1.6) is nonlinear.

Definition 1.7. The *instanton moduli space* is the space $\mathscr{M}$ of equations on P obeying (1.6), modulo the action of the *gauge group* $\mathscr{G}$, the bundle automorphisms of P.[4]

Exercise 1.8. Show that if $G = \mathrm{U}(1)$, then $\mathscr{M}$ is only governed by linear algebra in that

$$\mathscr{M} \cong H^1(M : \mathbb{R})/H^1(X;\mathbb{Z}).$$

So in this case we don't find anything new, though the way we found it is still interesting.

When G is nonabelian, this is not a vector space. It still has some reasonable structure. We now fix $G = \mathrm{SU}(2)$. In this case, (topological) isomorphism classes of principal SU(2)-bundles are classified by the integers, given by the formula

$$k := \int_X c_2(P) \in \mathbb{Z},$$

where c_2 denotes the second Chern class.

This means the moduli of instantons is a disjoint union over $\mathbb{Z}$ of spaces $\mathscr{M}_k$.

Theorem 1.9. *If $k > 0$ and g is chosen generically, $\mathscr{M}_k$ is a finite-dimensional manifold.*

Hence one could learn topological information about X by studing topological properties of $\mathscr{M}_k$. The first idea would be the Betti numbers, but these turn out not to depend on the smooth structure.

Proposition 1.10. *Assuming $k > 0$ and g is generic,*

$$\dim \mathscr{M}_k = 8k - 3(1 - b_1(X) + b_2^+(X)).$$

But there's more to $\mathscr{M}_k$ than the dimension. Donaldson introduced an orientation on $\mathscr{M}_k$, which is canonically defined (and a lot of hard work!), and one can produce classs $\tau_\alpha \in \Omega^*(\mathscr{M}_k)$ labeled by classes $\alpha \in H_*(X)$. Using these, the *Donaldson invariants* are the real numbers

$$\langle \mathscr{O}_{\alpha_1} \cdots \mathscr{O}_{\alpha_\ell} \rangle := \int_{\mathscr{M}} \tau_{\alpha_1} \wedge \cdots \wedge \tau_{\alpha_\ell} \in \mathbb{R}. \tag{1.11}$$

Theorem 1.12. *If $b_2^+(X) > 1$, the Donaldson invariants are independent of g.*

Moreover, they really depend on smooth information: it's not possible to reconstruct them out of algebraic or differential topology, unlike the Betti numbers. So these are very powerful. Their study is called *Donaldson theory*. One good refernece is Donaldson and Kronheimer's book [9].

Unfortunately, Donaldson theory is technically very hard: the ASD YM equation is hard to study: $\mathscr{M}_k$ is usually noncompact, and (1.11) is an integral over a noncompact space, which is no fun.

What does this have to do with quantum field theory? In 1988, Witten [17], following a suggestion of Atiyah, found an interpretation of the Donaldson invariants in terms of quantum field theory (hence the suggestive notation in (1.11)).

There are many different quantum field theories: the Standard Model describes three of the four fundamental forces of the universe; quantum electrodynamics describes electromagnetism. Witten interpreted the Donaldson invariants in terms of a specific QFT, called "(a topological twist of) $\mathscr{N} = 2$ supersymmetric Yang-Mills theory (SYM) with gauge group SU(2)."

One imagines X to be a "spacetime" or "universe" whose laws of physics are governed by $\mathscr{N} = 2$ supersymmetric Yang-Mills theory, and to compute the Donaldson invariants, one conducts "experimental measurements"

[4] TODO: not sure if I got this right.

(correlation functions). According to the rules of Lagrangian quantum field theory, this means computing an integral over an infinite-dimensional space (which is alarming, but so it goes):

$$\langle \mathscr{O}_\alpha \rangle = \int_{\mathscr{C}} \mathrm{d}\mu \, \Phi_\alpha e^{-S},$$

where

- $\mathscr{C}$ is the *space of fields*, some sort of infinite-dimensional space akin to the space of functions on X or forms on X,
- $S\colon \mathscr{C} \to \mathbb{R}$ is a functional called the *action*,
- $\Phi_\alpha\colon \mathscr{C} \to \mathbb{R}$ is a (set of) *observables*,
- and $\mathrm{d}\mu$ is some measure on $\mathscr{C}$.

In general, computing these correlation functions are very hard,[5] but in $\mathscr{N} = 2$ SYM, Witten found localization, a way to reduce it to Donaldson's integrals over finite-dimensional spaces.

This is undoubtedly cool, and brings geometric topology into quantum field theory, but it does not make it much easier to actually compute Donaldson invariants.

The next step was taken in 1995, by Seiberg and Witten [14, 15], who were interested in a different but related physics problem. They answered a fundamental question about SYM: how it behaves at low energies.

To make an analogy, suppose you have a pond, and you're pond-ering what happens when wind goes across the surface. You're good at physics, so you model the pond as a system of 10^{30} molecules of water and other things, then rent some time on a supercomputer where you model the action on the wind and... somehow this seems wrong. Instead, you model the water and the wind using things like the Navier-Stokes equations. This is not easy, but it's much, much easier.

The idea is there's a "high-energy" description, in terms of 10^{30} particles, but the "low-energy" description[6] involves things like temperature, pressure, liquid, and other things that are hard to define from the high-energy approach. The low-energy picture is very useful for calculations, though if you fire a laser into your pond it wouldn't suffice. Obtaining the description of the low-energy physics from the high-energy physics is typically very hard; in this case, one would have to define temperature and pressure and a lot of things starting from fundamentals. But you just have to do it once, then can apply it to all bodies of water, etc.

Seiberg and Witten applied this to $\mathscr{N} = 2$ SYM with gauge group SU(2), and showed that its low-energy description is (roughly) $\mathscr{N} = 2$ SYM with gauge group U(1), coupled to matter (sometimes called monopoles). Since the gauge group is abelian, this is much easier. Now, one can imagine that there's an easier description of the Donaldson invariants in terms of the low-energy theory (though, again, this was not the original intent of Seiberg and Witten), and this is given by the *Seiberg-Witten equations*. They look more complicated but are actually vastly simpler.[7]

In the Seiberg-Witten equations, the fields are

- a connection Θ in a U(1)-bundle $\mathscr{E}$, or equivalently a determinant line of a spin^c-structure, and
- a section ψ of S^+, a spinor bundle associated to a spin^c-structure.

In this case, there's a *Dirac operator* $\not{D}$ and a pairing

$$q\colon S^+ \otimes S^+ \longrightarrow \Lambda^2_+ T^*X.$$

Then, the Seiberg-Witten equations are

$$F^+ = q(\psi, \overline{\psi}) \tag{1.13a}$$

$$\not{D}\psi = 0. \tag{1.13b}$$

Let $\widetilde{\mathscr{M}}$ denote the moduli space of pairs (Θ, ψ) satisfying (1.13) modulo the action of some group. For generic g, this is a compact manifold, so understanding its topology is much easier, and the correlation functions for the low-energy theory can be written as integrals over $\widetilde{\mathscr{M}}$, and there's a simple formula relating these to the correlation functions for the high-energy theory. Once this was realized, there was very rapid progress of its use in applications, though understanding precisely why it's the same came more slowly, beginning from a physical

[5] Unless $\dim X = 0$, where $\mathscr{C}$ is finite-dimensional. We'll talk about this in the next few lectures.

[6] The term "low-energy," despite sounding pejorative, is actually a very useful thing to have.

[7] For a reference, see Morgan [13].

argument by Moore and Witten [12] and proceeding to a very different-looking mathematical proof much more recently.

This is an application of QFT to geometry, as we will study in this course. Somehow the most powerful applications involve taking a low-energy limit, and many of them also involve localization in supersymmetric QFT (from an infinite-dimensional integral to a finite-dimensional one).

We will start more slowly: first considering QFT where $\dim X = 0$, then $\dim X = 1$ (which is quantum mechanics); in these cases, the physics can be made completely rigorous (though it's not necessarily easy). We'll briefly talk about $\dim X = 2$, then jump into $\dim X = 4$.

Lecture 2.

Zero-dimensional QFT and Feynman diagrams: 9/5/17

Last time, we talked about two perspectives on physics, high-energy (or *fundamental*) and low-energy (or *effective*). For example, the high-energy description of a pond is the physics of the 10^{30} or so particles in it, and the low-energy description is the Navier-Stokes equations. We're interested in the relationship between Donaldson theory in the high-energy perspective and Seiberg-Witten theory in the low-energy perspective, which is a story about four-dimensional QFT. But over the next few lectures, we're going to learn about this passage from fundamental to effective in 0-dimensional QFT, one of the few cases where it's known how to make everything rigorous. Nonetheless, it's still an interesting theory, e.g. it has Feynman diagrams.

We also discussed that in the Lagrangian formalism to QFT on a spacetime X, one evaluates integrals over a space $\mathscr{C}(X)$, which is some kind of function space. Hence, it's usually infinite-dimensional, unless $\dim X = 0$. Hence, let's assume $X = \mathrm{pt}$, so $\mathscr{C}(X) = \{X \to \mathbb{R}\} = \mathbb{R}$. There are many choices for $S\colon \mathscr{C} \to \mathbb{R}$,[8] such as

$$S(x) = \frac{m}{2}x^2 + \frac{\lambda}{4!}x^4,$$

where $m, \lambda > 0$. Here m might mean some kind of mass, and λ measures the interaction in the system.

Now we can define something important and fundamental: the *partition function*

$$Z := \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-S(x)}.$$

The *observables* are polynomial functions $f : \mathscr{C} \to \mathbb{R}$, and their *(unnormalized) expectation values* are

$$\langle f \rangle := \int_{-\infty}^{\infty} \mathrm{d}x\, f(x) e^{-S(x)}.$$

We require f to be polynomial so that this integral converges. All of these are functions in m and λ. Also, notice that all of these are completely well-defined; maybe this is a trivial observation, but it won't be true when we ascend to higher dimensions.

Computing these quantities is less trivial. Let's start with Z, or even $Z_0 := Z(m, \lambda = 0)$. This is a Gaussian:

$$Z_0 = \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-mx^2/2} = \sqrt{\frac{2\pi}{m}}.$$

In order for this to be well-defined, we need $m = 0$ of course, but there's a physical reason to throw out this case, as it corresponds to a system with more than one vacuum state and a degenerate critical point of the action.

To compute the partition function for $\lambda > 0$, we're not sure how to directly evaluate the integral, but we can try to expand it out as a Taylor series in λ around 0. This will allow us to understand the system in the presence of weak interactions, which is often exactly what physicists want to know. We'll leave $e^{-mx^2/2}$ alone, since we know how to integrate it exactly. The $\lambda x^4/4!$ term expands to

$$Z(m, \lambda) = \int_{-\infty}^{\infty} \mathrm{d}x \sum_{n=0}^{\infty} \left(-\frac{\lambda}{4!}\right)^n \frac{x^{4n}}{n!} e^{-mx^2/2}.$$

[8] One can also use $\mathbb{C}$-valued actions.

We'd like to switch the sum and integral to obtain

$$= \sum_{n=0}^{\infty} \left(-\frac{\lambda}{4!}\right)^n \underbrace{\int_{-\infty}^{\infty} \frac{x^{4n}}{n!} e^{-mx^2/2}}_{I}, \tag{2.1}$$

but we have to be careful about convergence. If this works, though, the integral I is tractable.

Exercise 2.2. Show that

$$\int_{-\infty}^{\infty} \mathrm{d}x\, x^{2k} e^{-mx^2/2} = \sqrt{\frac{2\pi}{m}} \frac{1}{m^k} \frac{(2k)!}{k!2^k}.$$

Hence, modulo the assumption we made before, if $\widetilde{\lambda} := \lambda/m^2$,

$$\begin{aligned} Z(m,\lambda) &= \sqrt{\frac{2\pi}{m}} \sum_{n=0}^{\infty} \left(-\frac{1}{96}\right)^n \frac{(4n)!}{n!(2n)!} \widetilde{\lambda}^n \\ &= \sqrt{\frac{2\pi}{m}} \left(1 - \frac{1}{8}\widetilde{\lambda} + \frac{35}{384}\widetilde{\lambda}^2 + \cdots + (1390.1)\widetilde{\lambda}^{10} + \cdots\right). \end{aligned} \tag{2.3}$$

This is called the *perturbation series* for this partition function. Though this partition function is a scalar multiple of a Bessel function, often these series are actually divergent for any $\widetilde{\lambda} > 0$. This means the assumption we made in (2.1) was wrong. There's various ways to think about this — if this function did converge to its Taylor series, it would do so in a neighborhood of 0 in $\mathbb{C}$, hence for negative λ. Physically, this doesn't make sense.

Nonetheless, the perturbation series is still useful in those cases.

Definition 2.4. Let $f : \mathbb{R}_+ \to \mathbb{C}$ be a function and $s := \sum_{n=0}^{\infty} c_n t^n$ be a formal series. We say that s is an *asymptotic series* for f as $t \to 0^+$ if for all $N \geq 0$,

$$\lim_{t\to 0^+} t^{-N} \left| f(t) - \left(\sum_{n=0}^{N} c_n t^n\right)\right| = 0.$$

In this case, we write

$$f(t) \underset{t\to 0^+}{\sim} \sum_{n=0}^{\infty} c_n t^n.$$

In particular, this means that

$$\lim_{t\to 0^+} |f(t) - c_0| = 0$$
$$\lim_{t\to 0^+} \frac{1}{t} |f(t) - c_0 t + c_1| = 0,$$

and so on. So even if s doesn't converge, it's still useful, capturing the limits, linear behavior, quadratic behavior, etc., of f. You have encountered other asymptotic series in your life: Stirling's formula for the factorial is an asymptotic series for the gamma function at ∞: it doesn't actually converge in a sensible way, but it captures a lot of useful information.

Proposition 2.5. *The series* (2.3) *is an asymptotic series for the partition function* $Z(m,\lambda)$ *as* $\lambda \to 0^+$.

So it's not equality, but it's a useful and interesting approximation.

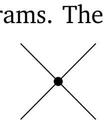
You might wonder whether there's some better series approximating $Z(m,\lambda)$ that actually converges, but this is not true.

Proposition 2.6. *If* f *has a convergent Taylor series at* x_0*, then its Taylor series is an asymptoric series for* f *at* x_0.

Proposition 2.7. *Every smooth function* f *can have at most one perturbation series as* $x \to x_0$.

Sometimes none exists.

We will interpret (2.3) in terms of Feynman diagrams. The basic object is a vertex with four half-edges attached:



A *Feynman diagram* for (2.3) is a placement of some of these vertices and a way of connecting the half-edges. (Feynman diagrams for other systems may look different.)

placeholder

FIGURE 1. Some Feynman diagrams with one or two vertices.

Let D_n denote the set of diagrams with n vertices.

Proposition 2.8. *The number of ways to pair up $2k$ objects is $(2k)!/k!2^k$.*

Corollary 2.9.

$$|D_n| = \frac{(4n)!}{(2n)!2^{2n}}.$$

There's also a group action of a group $G_n := (S_4)^n \rtimes S_n$ on D_n, where the i^{th} copy of S_4 permutes the half-edges for the i^{th} vertex, and S_n shuffles the n vertices. In other words, we can restate the asymptotic series for the partition function (2.3) in a more combinatorial manner: since $Z_0 = \sqrt{2\pi/m}$.

$$\frac{Z(m,\lambda)}{Z_0} \sim \sum_{n=0}^{\infty} (-\widetilde{\lambda})^n \frac{|D_n|}{|G_n|}.$$

We want to describe $|D_n|/|G_n|$ as the cardinality of some kind of quotient set, but this is only literally true if the G_n-action on D_n is free. The proper thing to do, as suggested by the orbit-stabilizer theorem, is to sum over orbits, weighted by the order of their stabilizers.[9] Thus

$$\frac{Z(m,\lambda)}{Z_0} \sim \sum_{n=0}^{\infty} (-\widetilde{\lambda})^n \sum_{[\Gamma]\in D_n/G_n} \frac{1}{|\mathrm{Aut}\,\Gamma|}.$$

Since $\widetilde{\lambda} = \lambda/m^2$ and a Feynman diagram in D_n has n^2 edges, we can rewrite (2.3) in a way that is completely a combinatorial sum over Feynman diagrams:

$$\frac{Z(m,\lambda)}{Z_0} \sim \sum_{n\geq 0} \sum_{[\Gamma]\in D_n/G_n} \frac{(-\lambda)^{|V(\Gamma)|}}{m^{|E(\Gamma)|}} \cdot \frac{1}{|\mathrm{Aut}(\Gamma)|}.$$

Here, $V(\Gamma)$ is the set of vertices of Γ, and $E(\Gamma)$ is the set of edges. This leads to the *Feynman rules* for summing over the Feynman diagrams for this theory:

- Draw one representative Γ for each orbit in D_n/G_n.
- Define its weight w_Γ as the product of factors $-\lambda$ for each vertex and $1/m$ for each edge, weighted by $1/|\mathrm{Aut}(\Gamma)|$.

Then,

$$\frac{Z}{Z_0} \sim \sum_{[\Gamma]} w_\Gamma.$$

Example 2.10. Let's calculate some low-order terms.

- The empty Feynman diagram has the weight 1.
- The action of $G_1 \cong S_4$ on D_1 is transitive, so we only need a single representative, such as the "figure-8 diagram." Its stabilizer group has order 8, so there's a contributing factor of $(-\lambda)/8m^2$.
- There are three orbits in D_2/G_2, represented by a graph with zero self-loops, which contributes a term of $\lambda^2/48m^4$, one with one self-loop on each vertex, which contributes $\lambda^2/16m^4$, and one with two self-loops on each vertex, which contributes $\lambda^2/128m^4$.

[9] Another way to think about this is to consider the quotient *groupoid* D_n/G_n, and sum over it in the groupoid measure, which amounts to the same thing.

Thus, the perturbative expansion is

$$\begin{aligned}\frac{Z}{Z_0} &\sim 1 - \frac{\lambda}{8m^2} + \frac{\lambda^2}{48m^4} + \frac{\lambda^2}{16m^4} + \frac{\lambda^2}{128m^4} + O(\lambda^3)\\ &= 1 - \frac{\lambda}{8m^2} + \frac{35}{384}\frac{\lambda^2}{m^4} + O(\lambda^3).\end{aligned}$$

The higher-order terms correspond to diagrams with 3 or more vertices. ◂

If you know the automorphism group of a diagram Γ, then the automorphism group of $\Gamma \amalg \Gamma$ is very similar: a copy of Aut(Γ) for each component, plus the S_2 switching them. If you follow your nose in this line of thought, you can determine the sum in terms of only nonempty, connected diagrams.

Proposition 2.11.

$$\sum_\Gamma w_\Gamma = \exp\Bigg(\sum_{\Gamma \text{ connected, nonempty}} w_\Gamma\Bigg).$$

This suggests that $\log(Z/Z_0)$ is an important physical quantity, and indeed, it's called the *free energy* of the system, as in statistical mechanics. We'd like to say that

$$\log\left(\frac{Z(m,\lambda)}{Z_0}\right) \sim \sum_{\Gamma \text{ connected, nonempty}} w_\Gamma,$$

though there's an analysis argument to check here.

Now we want to compute expectation values. Let's start with

$$\langle x^k \rangle := \int_{-\infty}^{\infty} x^n e^{-S}\,\mathrm{d}x.$$

If k is odd this is 0, but for k even, we can compute an asymptotic series for this function with a similar sum over Feynman diagrams, but with different rules:

- In addition to the 4-valent vertices from before, each diagram must have exactly k univalent vertices.
- We only consider automorphisms which fix these vertices.

You can work this out with a similar argument as for Z/Z_0.

To compute the *normalized expectation values* $\langle x^k \rangle / Z$, use the same diagrams, but with the rule that every connected component of Γ must have at least one univalent vertex. You can then draw out the first few diagrams and conclude things such as

$$\frac{\langle x^2 \rangle}{2} \sim \frac{1}{m} - \frac{\lambda}{2m^3} + O(\lambda^2).$$

More generally, there's no need to constrain ourselves to a quartic interaction: we can isntead consider the action

$$S = \frac{m}{2}x^2 + \sum_{k=3}^{\infty} \frac{\lambda_k x^k}{k!}. \tag{2.12}$$

In this case, we consider Feynman diagrams with vertices of aribitrary valence ≥ 3, and sum with the rules that an edge contributes $-1/m$ and an n-valent vertex contributes $-\lambda_n$. We can actually carry out the analysis even if (2.12) doesn't converge (in which case we don't get an asymptotic series for a function, but that's OK). Anyways, tabulating the Feynman diagrams we get the beginning of the normalized perturbative expansion

$$\frac{Z}{Z_0} \sim 1 - \frac{\lambda_4}{8m^2} + \frac{\lambda_3^2}{12m^3} + \cdots$$

Yet another generalization is to consider actions on $\mathscr{C} = \mathbb{R}^N$, rather than $\mathbb{R}$, corresponding to considering the theory on N points, rather than one point. Now, the quartic term is some 4-tensor, so (using the Einstein summation convention) the most general action is

$$S = \frac{1}{2}x^i M_{ij} x^j + \frac{1}{4!}C_{ijk\ell}x^i x^j x^k x^\ell,$$

and Z_0 is again a Gaussian:

$$Z_0 = \int_{\mathbb{R}^n} e^{-x^i M_{ij} x^j/2} = \frac{(2\pi)^{N/2}}{\sqrt{\det M}}.$$

In this case, one can compute with Feynman diagrams again, but this time labeling the edges with labels $1, \dots, N$.

Lecture 3.

A Little Effective Field Theory: 9/7/17

Today, we're going to illustrate the passage from the fundamental to the effective using zero-dimensional QFT: the fundamental theory will be an action $S(x,y)$ in two variables, and its effective theory S_{eff} will be a simpler theory in a single variable.

Last time, we discussed the fields $\mathscr{C} = \mathbb{R}^N$ in a zero-dimensional QFT with an action

$$S := \frac{1}{2} x^i M_{ij} x^j + \frac{1}{4!} C_{ijk\ell} x^i x^j x^k x^\ell.$$

As $C \to 0$, one wants to compute the asymptotic series, which amounts to a sum over Feynman diagrams. In this context, one can sum over unlabeled diagrams Γ, but with the weight incorporating the labels of the half-edges in $\{1, \dots, N\}$. Explicitly, the weight of an edge i to j should be $(M^{-1})^{ij}$, and that of a vertex with half-edges i, j, k, and ℓ is $C_{ijk\ell}$.

More abstractly, if V is a finite-dimensional vector space with a measure μ, you can choose an $M \in \mathrm{Sym}^2 V^*$ and a $C \in \mathrm{Sym}^4 V^*$, and define the action

$$S(x) := \frac{1}{2} M(x,x) + \frac{1}{4!} C(x,x,x,x).$$

Then, one would compute the partition function

$$\int \mathrm{d}\mu \, e^{-S(x)}.$$

Now let's focus on a specific example. We can start with fields $\mathsf{C} = \mathbb{R}^2$ with coordinates x, y and an action

$$S(x,y) := \frac{m}{2} x^2 + \frac{M}{2} y^2, \tag{3.1}$$

which is two uncoupled systems. So let's turn on coupling in (3.1):

$$S(x,y) := \frac{m}{2} x^2 + \frac{M}{2} y^2 + \frac{\mu}{4} x^2 y^2. \tag{3.2}$$

Say that we're actually interested in x: we want to compute Z and $\langle x^n \rangle$, but *not* $\langle y \rangle$ or $\langle f(x,y) \rangle$ that depends on y. This might happen in a system which naturally comes with both x and y, but y is some extra degrees of freedom. We'll see this is natural when $M \gg m$.

There are only a few kinds of labels in the Feynman diagram, because M and C in (3.2) have a lot of zeroes: we'll use a solid line for $1/m$ (corresponding to x^2) and a dashed line for $1/M$ (for y^2); all vertices must have two solid half-edges and two dashed half-edges, weighted by $-\mu$.

Let's compute $\log(Z/Z_0)$; by Proposition 2.11, this allows us to only sum over connected diagrams. There is only one diagram with a single vertex (order μ), and three with two vertices (order μ^2). Their respective computations are

$$\log\left(\frac{Z}{Z_0}\right) \sim -\frac{\mu}{4mM} + \frac{\mu^2}{16m^2M^2} + \frac{\mu^2}{16m^2M^2} + \frac{\mu^2}{8m^2M^2} + O(\mu^3).$$

For correlation functions, we must add n univalent vertices for x^n. The μ^0-term (the "tree level") calculates exactly the noninteracting theory. When we enumerate the diagrams for $\langle x^2 \rangle$, there's one with zero 4-valent vertices, one for a single 4-valent vertex, and three with two 4-valent vertices, and the sum is

$$\frac{\langle x^2 \rangle}{Z} \sim \frac{1}{m} - \frac{\mu}{2m^2M} + \frac{\mu^2}{4m^3M^2} + \frac{\mu^2}{2m^3M^2} + \frac{\mu^2}{4m^3M^2} + O(\mu^3).$$

This is not the logarithm: since we've normalized this calculation, it's a sum over Feynman diagrams for which every connected component contains a univalent vertex.

This explodes more quickly than other ones we considered: to compute $\langle x^4 \rangle$, there are a lot of diagrams to sum over, even just at the μ^2. The answer will be

$$\frac{\langle x^4 \rangle}{Z} \sim \frac{3}{m^2} - \frac{3\mu}{m^3M} + \frac{33\mu^2}{4m^4M^2} + O(\mu^3).$$

And since we only care about x, there should be some way to simplify this and get all of the dashed lines out of the way first. One idea is: if we only want

$$\langle x^n \rangle = \int_{\mathbb{R}^2} \mathrm{d}x\,\mathrm{d}y\; x^n e^{-S(x,y)},$$

then by Fubini's theorem, we can integrate out the dependence on y, defining S_{eff} such that

$$e^{-S_{\text{eff}}(x)} := \int_{\mathbb{R}} \mathrm{d}y\; e^{-S(x,y)}.$$

Then

$$\langle x^n \rangle = \int_{\mathbb{R}} \mathrm{d}x\; x^n e^{-S_{\text{eff}}(x)}.$$

In this particular example, we can compute S_{eff}, or at least its asymptotic series (which suffices if we want to do the asymptotic series for $\langle x^n \rangle$ in the original theory). The answer for the asymptotic series for $\mu \to 0$ is

$$S_{\text{eff}}(x) \sim \frac{m_{\text{eff}}}{2} x^2 + \sum_{k\ge 3} \lambda_k x^k, \tag{3.3}$$

where m_{eff} is some effective mass. The interacting term is interesting — there are interactions between multiple xs (vertices with four solid edges). These arise because of Feynman diagrams such as the one in Figure 2, where by "ignoring y" we close the gap between these two vertices and obtan an interaction between two copies of x.

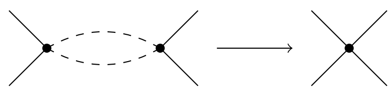


FIGURE 2. Left: a Feynman diagram for the action (3.2). In the effective field theory (3.3), the dashed lines correspond to terms which are integrated out, so this diagram becomes a quartic x-x interaction (on the right).

Specifically, in (3.3), the terms are

$$\begin{aligned} m_{\text{eff}} &= m + \frac{\mu}{2M} \\ \lambda_k &= \begin{cases} 0, & k \text{ odd} \\ -\left(-\dfrac{\mu}{M}\right)^{k/2} \dfrac{1}{2^{k/2+2}k}, & k \text{ even.} \end{cases} \end{aligned}$$

Thus, as $M \to \infty$, $m_{\text{eff}} \to m$: when $M \gg m$, this is a more reasonable approximation.

This is our first baby example of an effective field theory. The fact that we integrated out the degrees of freedom we didn't care about is a useful heuristic to have around.

Symmetries. Let's go back to $\mathscr{C} = \mathbb{R}$ and

$$S = \frac{m}{2}x^2 + \frac{\lambda}{4!}x^4.$$

This is in a sense the simplest nontrivial example: if you had a cubic term instead of a quartic term, $\int e^{-S}$ wouldn't be well-defined (it goes to ∞ as $x \to \pm\infty$).

Proposition 3.4. *$\langle x^n \rangle = 0$ when n is odd.*

Proof.

$$\begin{aligned} \langle x^n \rangle &= \int_{-\infty}^{\infty} \mathrm{d}x\; x^n e^{-S(x)} \\ &= \int_{-\infty}^{\infty} \mathrm{d}(-x)\,(-x)^n e^{-S(-x)} \\ &= (-1)^n \int_{-\infty}^{\infty} \mathrm{d}x\; x^n e^{-S(x)} \\ &= (-1)^n \langle x^n \rangle. \end{aligned}$$

⊠

One takeaway is that this theory is symmetric under the group $\mathbb{Z}/2$ acting on $\mathscr{C}$ as multiplication by $\{\pm 1\}$. This leads to a very general principle.

Proposition 3.5. *Let $S : \mathscr{C} \to \mathbb{R}$ and the measure on $\mathscr{C}$ are both G-invariant for a group G, then $\langle \mathscr{O} \rangle = \langle \mathscr{O}^g \rangle$ for any observable $\mathscr{O} : \mathscr{C} \to \mathbb{R}$, where $\mathscr{O}^g = g^* \mathscr{O}$.*

If G is a Lie group, we can differentiate this equation: take $g = \exp(tX)$ for some $X \in \mathfrak{g}$: taking

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \big(\langle \mathscr{O} \rangle = \langle \mathscr{O}^{tX} \rangle\big),$$

we conclude that $\langle X(\mathscr{O}) \rangle = 0$.

In general, symmetries are an extremely important ingredient in QFT.

Fermions and super-vector spaces. You might remember that we wanted to do something topological, but our computations, as functions in the parameters (m, λ), were not deformation-invariant (you could think of them as nonconstant functions on a moduli space of QFTs). To get things that are, we need one more ingredient: fermions.

The way to do this, which will return again and again in this course, is to replace the manifold $\mathscr{C}$ by a supermanifold! Since we've so far only considered vector spaces, we'll get a slightly gentler introduction in the form of super-vector spaces.

For a reference on this material, check out Etingof's course notes for a class on the mathematics of QFT.[10]

Definition 3.6. A *super-vector space* is a $\mathbb{Z}/2$-graded vector space $V = V^0 \oplus V^1$.

For example, if $V^0 = \mathbb{R}^p$ and $V^1 = \mathbb{R}^q$, V is denoted $\mathbb{R}^{p|q}$. This can be done over any field, but we're only going to consider $\mathbb{R}$ or $\mathbb{C}$.

These are not so terrible. But how we do algebra with them is also different: if you are taking tensor products, super-vector spaces are not the same as $\mathbb{Z}/2$-graded vector spaces![11]

Definition 3.7. The symmetric monoidal category of super-vector spaces $(\mathsf{sVec}, \otimes, s_{-,-})$ is the same as that for ordinary $\mathbb{Z}/2$-graded vector spaces $\mathsf{Vect}^{\mathbb{Z}/2}$, except for the symmetry

$$s_{V,W} : V \otimes W \to W \otimes V.$$

For $\mathsf{Vect}^{\mathbb{Z}/2}$, this is the map $v \otimes w \mapsto w \otimes v$, but in sVec, it's defined on homogeneous v, w by

$$v \otimes w \longmapsto (-1)^{|v||w|} w \otimes v,$$

where $v \in V^{|v|}$ and $w \in W^{|w|}$; non-homogeneous elements are sums of homogeneous ones, so this determines $s_{V,W}$.

So the point is if v or w is in V^1, we multiply by -1:

$$s(v \otimes w) = \begin{cases} -w \otimes v & v \text{ or } w \text{ is in } V^1 \\ w \otimes v, & v, w \in V^0. \end{cases}$$

This category is considerably more useful than it looks. There's a sense in which sVec and $\mathsf{Vect}^{\mathbb{Z}/2}$ are the only two symmetric monoidal structures that can be placed on the monoidal category $(\mathsf{Vect}^{\mathbb{Z}/2}, \otimes)$.

Other algebraic constructions are also different.

Definition 3.8. The *symmetric algebra* on a super-vector space V is the superalgebra ($\mathbb{Z}/2$-graded algebra)

$$\mathrm{Sym}^*(V) := T^*V / \langle v \otimes w - s(w \otimes v) \rangle.$$

Thus, if $V = V^0$, $\mathrm{Sym}^*(V)$ is the usual symmetric algebra, but if $V = V^1$, $\mathrm{Sym}^*(V) = \Lambda^*(V^1)$, the exterior algebra! In general, it'll be a mix of these two things.

We can use this to define polynomial functions: in ordinary algebra, there's a canonical isomorphism between the algebra of polynomials on a vector space V and $\mathrm{Sym}^*(V^*)$.

Definition 3.9. Motivated by this, if $V \in \mathsf{sVec}$, we define its *algebra of polynomial functions* $\mathscr{O}(V)$ to be

$$\mathscr{O}(V) := \mathrm{Sym}^*(V^*).$$

[10] For supermanifolds specifically, see https://ocw.mit.edu/courses/mathematics/18-238-geometry-and-quantum-field-theory-fall-200 lecture-notes/sec9.pdf.

[11] If the base field has characteristic 2, these two notions are actually the same, which quickly follows from Definition 3.7. But this will not be important to us.

Here $V^* := \mathrm{Hom}_{\mathsf{sVec}}(V, \mathbb{R}^{1|0}) = (V^0)^* \oplus (V^1)^*$. $\mathscr{O}(V)$ is itself a super-vector space, in fact a (super)commutative algebra! That is, $p \cdot q = (-1)^{|p||q|} q \cdot p$.

In physics, the even direction corresponds to bosonic stuff, and the odd direction to fermionic stuff. So $\mathscr{C}$ may be a super-vector space, and we can take the action function $S \in \mathscr{O}^0(\mathscr{C})$.

Example 3.10. Let's consider a purely fermionic theory, such as $\mathscr{C} = \mathbb{R}^{0|2}$. Then, $\mathscr{C}$ has coordinate functions $\psi^1, \psi^2 \in \mathscr{O}^1(\mathscr{C})$, which have odd statistics in the sense that

$$\begin{aligned} \psi^1\psi^2 &= -\psi^2\psi^1 \\ (\psi^1)^2 &= 0 \\ (\psi^2)^2 &= 0. \end{aligned}$$

This, $\mathscr{O}^0(\mathscr{C})$ has basis $\{1, \psi^1\psi^2\}$ and $\mathscr{O}^1(\mathscr{C})$ has basis $\{\psi^1, \psi^2\}$. Thus $\mathrm{Sym}^* \mathscr{C}$ is four-dimensional, which is as expected, since it should be $\Lambda^*\mathbb{R}^2$.

Since there's no quartic terms in ψ^1 and ψ^2, we actually can't introduce interactions, so our action functional is

$$S := \frac{1}{2} M \psi^1 \psi^2. \tag{3.11}$$

This is somewhat like a function, but it behaves very weirdly: $S^2 = 0$!

We'd like to make sense of the partition function in this setting. In order to do this, we need rules for integrating over odd variables. To integrate over $\mathbb{R}^{0|1}$ with odd coordinate ψ, the most general function is $a\psi + b$, so we can stipulate that its integral is

$$\int_{\mathbb{R}^{0|1}} \mathrm{d}\psi\, (a\psi + b) := a.$$

We'll define the exponential via its power series, which means it's much simpler than for bosons!

Now, on $\mathbb{R}^{0|k}$, we have to specify order of integration: to compute

$$\int_{\mathbb{R}^{0|k}} \mathrm{d}\psi^1\, \mathrm{d}\psi^2 \cdots \mathrm{d}\psi^k F = \int_{\mathbb{R}^{0|1}} \mathrm{d}\psi^1 \bigg(\int_{\mathbb{R}^{0|1}} \mathrm{d}\psi^2 \Big(\cdots \int_{\mathbb{R}^{0|1}} F \Big) \cdots \bigg),$$

first evaluate the innermost integral, then the next innermost, and so on, ending at the outermost ($\mathrm{d}\psi^1$ in the above equation).

Hence the partition function is

$$\begin{aligned} Z &= \int_{\mathbb{R}^{0|2}} \mathrm{d}\psi^1\, \mathrm{d}\psi^2 e^{-S(\psi^1,\psi^2} \\ &= \int_{\mathbb{R}^{0|2}} \mathrm{d}\psi^1\, \mathrm{d}\psi^2 \left(1 - \frac{1}{2} M \psi^1\psi^2\right) \\ &= -\frac{1}{2} M \int_{\mathbb{R}^{0|2}} \mathrm{d}\psi^1\, \mathrm{d}\psi^2\, \psi^1\psi^2 \\ &= \frac{1}{2} M \int_{\mathbb{R}^{0|2}} \mathrm{d}\psi^1\, \mathrm{d}\psi^2\, \psi^2\psi^1 \\ &= \frac{1}{2} M. \end{aligned}$$

◂

For bosons (i.e. even fields), we had a Gaussian

$$\int_{-\infty}^{\infty} e^{-Mx^2/2}\, \mathrm{d}x = \frac{\sqrt{2\pi}}{\sqrt{M}}.$$

This is suggestive: if you arrange the masses of bosons and fermions right, things might cancel out to produce a theory whose dependence on the mass cancels out and is deformation-invariant.

Lecture 4.

Supersymmetry in zero dimensions: 9/12/17

We've been doing zero-dimensional quantum field theory, and we will continue to do so today. Last time, we introduced supersymmetry, so $\mathscr{C}$ is a super-vector space. We looked at a particular specific example where $\mathscr{C}$ is odd, e.g. $\mathbb{R}^{0|2}$, which has two odd coordinate functions $\psi^1, \psi^2 \in \mathscr{O}^1(\mathscr{C})$. The total coordinate algebra is $\mathscr{O}(\mathscr{C}) = \Lambda^*(\mathbb{R}^2)$, and the even functions are spanned by $1, \psi^1\psi^2 \in \mathscr{O}^0(\mathscr{C})$. Since ψ^1 and ψ^2 are odd, $\psi^1\psi^2 = -\psi^2\psi^1$.

Let's introduce the action (3.11): since there are only odd terms, there can be no interacting terms, because higher-order powers of ψ^1 and ψ^2 vanish! The partition function is

$$Z = \int_{\mathscr{C}} \mathrm{d}\mu\, e^{-S} = \int_{\mathscr{C}} \mathrm{d}\mu \left(1 - \frac{1}{2}M\psi^1\psi^2\right). \tag{4.1}$$

Last time, we discussed a heuristic way to understand the measure $\mathrm{d}\mu$; today we'll be more explicit.

Definition 4.2. The *parity change operator* $\Pi\colon \mathsf{sVec} \to \mathsf{sVec}$ sends a super-vector space $V = V^0 \oplus V^1$ to the super-vector space with even part V^1 and odd part V^0.

That is, Π just switches the odd and even parts of a super-vector space.

Definition 4.3. A *translation-invariant measure* on an odd super-vector space $V = V^1$ is a $\mathrm{d}\mu \in \Lambda^{\mathrm{top}}(\Pi V)$. For an $f \in \mathscr{O}(V) \cong \Lambda^*((\Pi V)^*)$, let $f^{\mathrm{top}} \in \Lambda^{\mathrm{top}}((\Pi V)^*)$ be its top-degree component; then, the integral of f with respect to $\mathrm{d}\mu$ is

$$\int \mathrm{d}\mu\, f := (\mathrm{d}\mu) \cdot (f^{\mathrm{top}}).$$

There's a one-dimensional space of measures, determined up to a scalar.

Exercise 4.4. For $V = \mathbb{R}^{0|1}$ with odd coordinate ψ, show there's a measure $\mathrm{d}\mu$ on V such that for all $a, b \in \mathbb{R}$,

$$\int \mathrm{d}\mu\,(a\psi + b) = a.$$

This measure is called $\mathrm{d}\psi$. Notice that

$$\int \mathrm{d}\psi\,\psi = 1 \qquad \text{and} \qquad \int \mathrm{d}\psi = 0.$$

The fact that the integral of a constant in an odd direction is 0 is one of the striking features of this "Grassmann integration."

We can also give names to some more measures:

Exercise 4.5. For $V = \mathbb{R}^{0|1}$ and $c \in \mathbb{R}$, show there are measures $c\mathrm{d}\psi$ and $\mathrm{d}(c\psi)$ on V such that

$$\int_V (c\mathrm{d}\psi) f(\psi) = c \int \mathrm{d}\psi\, f(\psi)$$
$$\int_V \mathrm{d}(c\psi) f(c\psi) = \int_V \mathrm{d}\psi\, f(\psi).$$

Then prove the Grassmann change-of-variables formula

$$\mathrm{d}(c\psi) = \frac{1}{c}\mathrm{d}\psi,$$

or equivalently that

$$\int_V \mathrm{d}(c\psi)\, c\psi = 1.$$

Similarly, on $\mathbb{R}^{0|q}$, define $\mathrm{d}\boldsymbol{\psi} = \mathrm{d}\psi^1\,\mathrm{d}\psi^2 \cdots \mathrm{d}\psi^q$ to be the unique measure such that

$$\int_{\mathbb{R}^{0|q}} \mathrm{d}\boldsymbol{\psi}\, \psi^q \psi^{q-1} \cdots \psi^1 = 1.$$

This definitely depends on how the ψ^i are ordered; we'll stick with this convention, which is common in physics.

These behave more like measures than top-degree forms: you need no choice of orientation to integrate. These definitions might be strange, but they're forced on you if you want a good change-of-variables formula.

Now, we know how to calculate the partition function (4.1):

$$Z = \int_{\mathbb{R}^{0|2}} \mathrm{d}\psi \left(1 - \frac{1}{2} M \psi^1 \psi^2\right) = \frac{1}{2} M.$$

More fermions. If we add more fermions, we can turn on interactions: if $\mathscr{C}$ is any even-dimensional[12] odd super-vector space with translation-invariant measure $\mathrm{d}\mu$, let

$$M \in \mathrm{Sym}^2(V^*) = \Lambda^2((\Pi V^1)^*)$$
$$C \in \mathrm{Sym}^4(V^*) = \Lambda^4((\Pi V^1)^*).$$

If V is at least four-dimensional, C can be nonzero. In that case we can change the action (3.11) to one with interactions:

$$S := \frac{1}{2} M + \frac{1}{4!} C \in \mathscr{O}(\mathscr{C}).$$

After choosing a basis for ΠV^1, equivalently an isomorphism $V \cong \mathbb{R}^{0|q}$, we can rewrite this in coordinates:

$$S = \frac{1}{2} M_{IJ} \psi^I \psi^J + \frac{1}{4!} C_{IJKL} \psi^I \psi^J \psi^K \psi^L, \tag{4.6}$$

where M_{IJ} is an antisymmetric matrix with real entries, and C_{IJKL} is a totally antisymmetric tensor. Again the partition function is $Z = \int \mathrm{d}\mu\, e^{-S}$, but this time it's possible to evaluate it algebraically.

Exercise 4.7. If $\mathscr{C} = \mathbb{R}^{0|4}$ and

$$S = m\psi^1\psi^2 + m\psi^3\psi^4 + \lambda \psi^1\psi^2\psi^3\psi^4, \tag{4.8}$$

show that

$$Z = m^2 - \lambda.$$

Remark. This is much easier than the bosonic case, where calculations like this flowed through asymptotic series, Feynman diagrams, etc. There is a perturbation-theoretic description of the fermionic case as a Feynman diagram expansion; the rules are quite similar to those for bosons, but with some extra signs. ⋖

In the theory (4.8), $Z_0 = m^2$.

Remark. For the more general theory of the form (4.6), $Z_0 = \mathrm{Pf}(M)$, the *Pfaffian* of the antisymmetric matrix M; this is a number which squares to the determinant. ⋖

To compute Z/Z_0 in (4.8), you can again sum over Feynman diagrams with four-valent vertices, but skew-symmetry introduces a sign rule which forces all Feynman diagrams with more than one vertex to have weight 0.

Bosons and fermions together. Now we consider $\mathscr{C} = V = V^0 \oplus V^1$ with both odd and even parts. We need a theory of integration for such spaces, but that won't be so hard: we'll first integrate over the odd part, then over the even part.

We also want some functions to integrate; polynomials don't have finite integrals on V^0.

Definition 4.9. Let $C^\infty(V) := C^\infty(V^0) \otimes \mathscr{O}(V^1)$.

We also need a measure to integrate with.

Definition 4.10. Let V be a super-vector space.

- The *Berezinian line* of V is
$$\mathrm{Ber}(V) := \Lambda^{\mathrm{top}} V^0 \otimes (\Lambda^{\mathrm{top}}(\Pi V^1))^*.$$
- An *integration measure* is an element of $\mathrm{Ber}(V^*)$.[13]

[12] If $\mathscr{C}$ is odd-dimensional, (4.6) still makes sense, but skew-symmetry forces us to leave out one fermion, so the partition function is 0. However, some correlation functions will be nonzero.

[13] To be completely precise, this would be a measure twisted by the orientation bundle, since measures don't require orientation to integrate.

If V is purely odd, this reduces to the above definition of the space of measures.

Since V now has an even subspace, integration will depend on orientation again.

Definition 4.11. Let V be an oriented super-vector space and $\mathrm{d}\mu = \omega^0 \otimes \omega^1 \in \mathrm{Ber}(V^*)$ be an integration measure. For any $f = f^0 \otimes f^1 \in C^\infty(V)$, its *integral* is

$$\int_V \mathrm{d}\mu\, f := \int_{V^0} \omega^0 f^0 \left(\int_{V^1} \omega^1 f^1 \right).$$

That is: integrate the odd part, then the even part.

On $\mathbb{R}^{p|q}$ there's a canonical measure

$$\mathrm{d}\mu = \mathrm{d}\mathbf{x}\,\mathrm{d}\boldsymbol{\psi} := \left(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^p\right) \otimes \left(\mathrm{d}\psi^1 \cdots \mathrm{d}\psi^q\right).$$

Example 4.12. Take $\mathscr{C} = \mathbb{R}^{1|2}$ with action

$$S(x, \psi^1, \psi^2) := S_1(x) + S_2(x)\psi^1\psi^2. \tag{4.13}$$

Then, the partition function is

$$Z = \int \mathrm{d}x\,\mathrm{d}\boldsymbol{\psi}\, e^{-S} = \int \mathrm{d}x\, S_2(x) e^{-S_1(x)}.$$

In other words, in the purely bosonic theory with action S_1, this is just the correlation function $\langle S_2(x) \rangle$. This can be a helpful perspective, but it also obscures why this is happening. ◀

For general S_1 and S_2, these are not super interesting.[14] But there is a special case that is much better. Fix an $h\colon \mathbb{R} \to \mathbb{R}$ such that as $|x| \to \infty$, $h(x) \to \infty$. Then, set

$$\begin{aligned} S_1(x) &:= \frac{1}{2} h(x)^2 \\ S_2(x) &:= h'(x). \end{aligned}$$

Hence

$$S = \frac{1}{2} h(x)^2 + h'(x)\psi^1\psi^2, \tag{4.14}$$

using the action (4.13). This action is invariant under a certain odd vector field on $\mathscr{C}$; we're going to explain what this means.

Definition 4.15. Let A be a super-commutative super-algebra, a *derivation* D on A with degree $|D|$ is a function $D\colon A \to A$ such that $D(a + a') = D(a) + D(a')$ and

$$D(aa') = (Da)a' + (-1)^{|a||D|} a(Da').$$

The set of all derivations of $\mathscr{O}(V)$ is a super-vector space, which we'll denote $\mathrm{Vect}(V)$.

Exercise 4.16. Show that $\mathrm{Vect}(V)$ is a *super-Lie algebra*, in the same way that vector fields on a vector space are a Lie algebra.[15]

On $\mathbb{R}^{p|q}$, we have the usual derivations/vector fields $\partial_{x_i} \in \mathrm{Vect}^0(V)$, but now also some odd vector fields $\partial_{\psi^I} \in \mathrm{Vect}^1(V)$, defined to satisfy

$$\begin{aligned} \partial_{\psi^I}(x^i) &= 0 \\ \partial_{\psi^I}(\psi^J) &= \delta_I^J. \end{aligned}$$

Hence

$$\partial_{\psi^1}(\psi^1\psi^2) = \psi^2 \qquad \text{and} \qquad \partial_{\psi^1}(\psi^2\psi^1) = -\psi^2.$$

Now we'll discuss the symmetry in the action (4.14). Let

$$\begin{aligned} Q_1 &:= \psi^1 \partial_x + h(x)\partial_{\psi^2} \\ Q_2 &:= \psi^2 \partial_x - h(x)\partial_{\psi_1}. \end{aligned} \tag{4.17}$$

[14] No pun intended.

[15] There's a whole theory of super-manifolds and Lie super-groups and more. But it's possible to go a long way before needing to understand the whole package.

Then,

$$Q_1 S = \psi^1 h'(x)h(x) + h(x)h'(x)\partial_{\psi^2}(\psi^1\psi^2) = 0,$$

and similarly for $Q_2 S$.

Exercise 4.18. This means that if $X = [Q_1, Q_2]$, then X is an even vector field and $XS = 0$. Find X and show this explicitly.

There's also a sense in which Q_1 and Q_2 are divergence-free.

Definition 4.19. Let

$$X := h^i \partial_{x^i} + g^I \partial_{\psi^I}.$$

Then, the *Lie derivative* along X of a section of $\mathrm{Ber}(V^*)$ is

$$\mathscr{L}_X(\mathrm{d}\mathbf{x}\,\mathrm{d}\boldsymbol{\psi}) := \left(\partial_{x^i} h^i + \partial_{\psi^I} g^I\right) \mathrm{d}\mathbf{x}\,\mathrm{d}\boldsymbol{\psi}.$$

If $\mathscr{L}_X(\mathrm{d}\mu) = 0$, we say X is *divergence-free*.

There is a coordinate-free definition of this, which can be found in [19]. Other references on the general theory:

- Deligne-Morgan, "Notes on supersymmetry (following Joseph Bernstein)" [7].
- Witten recently wrote some notes on integration on supermanifolds in [18], which are pretty down-to-Earth.

Lemma 4.20. *Let Q be a divergence-free vector field on an oriented super-vector space V with measure $\mathrm{d}\mu$. For any $f \in C^\infty(V)$,*

$$\int_V \mathrm{d}\mu\, Qf = 0.$$

Proof. We'll compute in coordinates: suppose $Q = h^i \partial_{x^i} + g^I \partial_{\psi^I}$. Then,

$$\begin{aligned}
\int_V \mathrm{d}\mu\, Qf &= \int_{V^0} \mathrm{d}\mathbf{x}\,(Qf)^{\mathrm{top}} \\
&= \int_{V^0} \mathrm{d}\mathbf{x}\left(h^i \partial_{x^i} f + g^I \partial_{\psi^I} f\right)^{\mathrm{top}} \\
&= \int_{V^0} \mathrm{d}\mathbf{x}\left(-(\partial_{x^i} h^i) f + (-1)^{|g^I|}(\partial_{\psi^I} g^I) f\right)^{\mathrm{top}} \\
&= 0,
\end{aligned}$$

because Q is divergence-free. ⊠

Using this, we can show that a certain deformation of these theories is actually constant.

Proposition 4.21. *Let V be an oriented super-vector space with measure $\mathrm{d}\mu$, $S \in C^\infty(V)$, and Q be a divergence-free odd vector field on V with $[Q, Q] = 0$ and $QS = 0$. For any smooth family of odd elements $\{\Psi_t\} \in C_c^\infty(V)$ with $\Psi_0 = 0$, let*

$$S_t := S + Q\Psi_t,$$

which is called a Q-exact deformation *of S. Then, Z_t is independent of t.*

Proof. Let $\Psi_t' := \partial_t \Psi_t$. Since

$$Z_t = \int_{\mathscr{C}} \mathrm{d}\mu\, e^{-(S+Q\Psi_t)},$$

then

$$\begin{aligned}
\partial_t Z_t &= -\int_{\mathscr{C}} \mathrm{d}\mu\,(Q\Psi_t') e^{-(S+Q\Psi_t)} \\
&= -\int_{\mathscr{C}} \mathrm{d}\mu\, Q\left(\Psi_t' e^{-(S+Q\Psi_t)}\right) \\
&= 0
\end{aligned}$$

by Lemma 4.20. ⊠

Lecture 5.

Localization in supersymmetry: 9/14/17

Today, we're going to use the $(0+1)$-dimensional field theory that we've been developing to do something actually topological. Recall that our state space is $\mathscr{C} = \mathbb{R}^{1|2}$, and given a smooth $h\colon \mathbb{R} \to \mathbb{R}$ such that $|h(x)| \to \infty$ as $|x| \to \infty$, we defined the action (4.14), and we'd like to compute its partition function $Z = \int_{\mathscr{C}} e^{-S}$.

Rather than boldly going forward as in previous lectures, we first observed that the partition function is invariant under two symmetries Q_1 and Q_2 (4.17). If

$$Q := Q_1 + Q_2 = (\psi^1 + \psi^2)\partial_x + h(x)(\partial_{\psi^2} - \partial_{\psi^1}),$$

then Q acts on $C^\infty(\mathscr{C})$ and $[Q,Q] = 0$. By Proposition 4.21, for any deformation $\psi_t \in C_c^\infty(\mathscr{C})$. That is, if $S_t := S + Q(\psi_t)$ and $Z_t := \int e^{-S_t}$, then $\partial_t Z_t = 0$. One way to think of this is to take Q as a differential operator and consider "Q-cohomology" — then, Proposition 4.21 tells us that Z_t only depends on the cohomology class of S.

Consider deforming $h(x)$ to a family $h_t(x)$ in a compactly supported manner, which defines a variation S_t of S. Using dots to denote $\frac{\mathrm{d}}{\mathrm{d}t}$,

$$\dot{S}(x) = h(x)\dot{h}(x) + \dot{h}'(x)\psi^1\psi^2.$$

Since $\dot{S}(x) = Q\Psi$ with $\Psi = -\dot{h}(x)\psi^1$, Proposition 4.21 tells us that Z *does not depend on* $h(x)$, as long as you only take compactly supported deformations.

Exercise 5.1. Bootstrap this to show that Z only depends on the behavior at infinity: it's only a function of $\varepsilon_\pm$, where $\lim_{h\to\pm\infty} = \varepsilon_\pm\infty$.

This is in a sense topological; certainly, there's no dependence on the metric.

One way to think of this which will come up again and again is that the action makes the configuration space only care about compact things. If you switch the behavior of h at $\pm\infty$, which requires doing something noncompact, it will change the invariants. Donaldson theory has the same behavior, with chambers in which the invariants do not change (where $b_2^+(X) > 1$), plus "wall-crossing phenomena" on their boundaries (where $b_2(X) = 1$).

Now let's compute Z, using topological invariance and a trick called localization. Since Z doesn't depend on our choice of h, let's do something nice: Z does not depend on λ in the variation $h(x) \to \lambda h(x)$ for $\lambda > 0$, so let's compute the limit as $\lambda \to \infty$. That is, we need to understand the asymptotics of

$$\lim_{\lambda\to\infty} \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-\lambda F(x)}, \tag{5.2}$$

where $F(x) \to \infty$ as $x \to \infty$. The asymptotics are controlled by something called the *method of steepest descent*, which may be surprising at first.

Proposition 5.3. *Assume F has a unique global minimum at x_c.*[16] *Then, as $\lambda \to \infty$,*

$$\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-\lambda F(x)} \sim \sqrt{\frac{2\pi}{\lambda F''(x_c)}}\, e^{-\lambda F(x_c)}. \tag{5.4}$$

That is, neither side of (5.4) has a limit at $\lambda \to \infty$, but their ratio does, and its limit is 1.

The proof is a little bizarre: first you only look at a tiny neighborhood of x_c, and then expand that neighborhood to the whole real line, and each of these contributes an exponentially small amount to the integral. For a full proof, check out [3]; it has a big list of cool tricks for computing asymptotic expansions like this one.

Another way to interpret (5.3) is that it gives us permission to truncate $F(x)$ to quadratic order around F_c. Thus, let's reshape h such that all of its global minima are 0, and make a quadratic approximation of S by summing over all of these local minima. The fermionic part is already quadratic, so we just have to look at the bosonic part. As $\lambda \to \infty$, we get that

$$Z(\lambda) \simeq \sum_{x_c : h(x_c) = 0} \int \mathrm{d}x\, \mathrm{d}\psi\, \exp\!\left(-\frac{1}{2}\lambda^2 h'(x_c)^2 (x - x_c)^2 - \lambda h'(x_c)\psi^1\psi^2\right).$$

[16] Otherwise, there would be a sum over local minima.

This is a Gaussian in the bosonic and in the fermionic parts:

$$\begin{aligned}
&= \sum_{x_c} \left(\sqrt{\frac{2\pi}{\lambda^2 h'(x_c)^2}} (\lambda h'(x_c)) \right) \\
&= \sqrt{2\pi} \sum_{h_c} \frac{h'(x_c)}{|h'(x_c)|} \\
&= \sqrt{2\pi} \sum_{h_c} \mathrm{sign}(h'(x_c)).
\end{aligned}$$

This is actually not so hard to evaluate directly, first integrating out the fermionic part then the bosonic part, but it's a useful example nonetheless.

Now we can look at one function in each deformation class.

- If $\lim_{x\to\pm\infty} h(x) = \pm\infty$, then there are an odd number of points x_c with sign $+1$ and an even number with sign -1, so we get $Z/\sqrt{2\pi} = -1$.
- If $\lim_{x\to\pm\infty} h(x) = \mp\infty$, this reverses: there are an even number with sign $+1$ and an odd number with sign -1, so $Z/\sqrt{2\pi} = -1$ again.
- If $\lim_{x\to\pm\infty} h(x) = \infty$ (or if it goes to $-\infty$), the number of critical points with positive and negative signs are the same, so $Z = 0$.

Hence you can express this in terms of $\varepsilon_\pm$:

$$Z = \sqrt{2\pi}\frac{|\varepsilon_+ - \varepsilon_-|}{2}.$$

Localization in a 0-dimensional σ-model. Let (M,ω) be a compact, $2n$-dimensional symplectic manifold: this means ω is a differential 2-form on M with $\mathrm{d}\omega = 0$ and $\omega^n \neq 0$, and assume there is $\mathrm{U}(1)$-action on M generated by the vector field $Y := \omega^{-1}(\mathrm{d}H)$, where $H\colon M \to \mathbb{R}$ is some function.

Let's assume all the fixed points of Y are isolated,[17] and pick an $\alpha \in \mathbb{R}$. We're going to use all this stuff to prove something cool, a formula for

$$\int_M \frac{\omega^n}{n!} e^{i\alpha H}. \tag{5.5}$$

Example 5.6. In examples, this integral is something people actually care about. Let $M = S^2$ with $\omega := \sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\varphi$ and $H := z = \cos\theta$, so $Y = \partial_\varphi$. Then,

$$\int_M \frac{\omega^n}{n!} e^{i\alpha H} = \int_{S^2} e^{i\alpha\cos\theta} \sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\varphi = 2\pi\int_0^\pi e^{i\alpha\cos\theta}\sin\theta\,\mathrm{d}\theta.$$

This is a Bessel function, and it's also funny to notice it is a great example of the kinds of integrals you teach for u-substitution and never expect to see anywhere else. It is hence easy to solve:

$$\begin{aligned}
2\pi\int_0^\pi e^{i\alpha\cos\theta}\sin\theta\,\mathrm{d}\theta &= -2\pi\int_1^{-1} e^{i\alpha z}\,\mathrm{d}z \\
&= \frac{2\pi}{i\alpha}\left(e^{i\alpha} - e^{-i\alpha}\right) \\
&= 4\pi\frac{\sin\alpha}{\alpha}.
\end{aligned}$$

◀

This answer demonstrates a localization phenomenon: it's a sum of contributions only from the north and south poles. In general, the integral (5.5) is a sum of contributions

$$(\pm)\frac{2\pi}{i\alpha} e^{i\alpha H(x_c)},$$

summed over the fixed points x_c of the $\mathrm{U}(1)$-action. This is an instance of the Duistermaat-Heckman theorem [10], and we're going to prove it using localization in supersymmetry.

To do this, we're going to need a supermanifold that's not a super-vector space, but it's not so bad.

[17] It is possible to excise this assumption, but it's helpful for now.

Definition 5.7. Let $E \to M$ be a vector bundle. Its *parity change* ΠE is a supermanifold whose algebra of functions is $C^\infty(\Pi E) := C^\infty(M, \Lambda^*(E))$.

We won't go into the general theory of supermanifolds here. Concretely, for $E = TM$, in local coordinates on M, we have even coordinates x^i, $i = 1, \dots, 2n$, and odd coordinates ψ^i for $i = 1, \dots, 2n$, and we can translate between functions on $\Pi(TM)$ and differential forms on M by exchanging ψ^i and $\mathrm{d}x^i$.

If we want to write down a zero-dimensional quantum field theory, we ought to have an action. Let $\mathscr{C} := \Pi(TM)$ and take

$$S := -i\alpha(H + \omega),$$

or in coordinates,

$$= -i\alpha\big(H + \omega_{ij}\psi^i\psi^j\big).$$

There's a canonical measure (up to scaling) $\mathrm{d}\mathbf{x}\,\mathrm{d}\boldsymbol{\psi}$ on $\Pi(TM)$, which in local coordinates is exactly $\mathrm{d}\mathbf{x}\,\mathrm{d}\boldsymbol{\psi}$ from before, and is invariant under change-of-charts. This might be surprising. A more abstract way to think of this is that the super-tangent bundle $T\mathscr{C}$ to $\mathscr{C}$ factors into a short exact sequence

$$0 \longrightarrow \Pi(\pi^* TM) \longrightarrow T\mathscr{C} \longrightarrow TM \longrightarrow 0,$$

so $\mathrm{Ber}(T\mathscr{C}) = \mathrm{Ber}(TM) \otimes \mathrm{Ber}(\Pi(TM))$, hence must be trivial. Hence the partition function is

$$Z := \int_{\mathscr{C}} \mathrm{d}\mathbf{x}\,\mathrm{d}\boldsymbol{\psi}\, e^{-S}.$$

If we integrate over fermions first, we get

$$Z = (i\alpha)^n \int_M \frac{\omega^n}{n!} e^{i\alpha H}.$$

We want to compute this by localization. This means first writing down a vector field under which S is invariant. We'll take

$$Q := \mathrm{d} + \iota_Y = \psi^i \partial_{x^i} + Y^j \partial_{\psi^j},$$

where ι_Y is contraction;[18] then, Q is odd and S is invariant under it:

$$\begin{aligned} QS &= (\mathrm{d} + \iota_Y)(H + \omega) \\ &= \mathrm{d}H + \iota_Y \omega \\ &= 0, \end{aligned}$$

Moreover,

$$\begin{aligned} \frac{1}{2}[Q,Q] &= [\mathrm{d}, \iota_Y] = \mathscr{L}_Y \\ &= \psi^i(\partial_{x^i} Y^j)\partial_{\psi^j} + Y^i \partial_{x^i}. \end{aligned}$$

We want to use localization to obtain the fixed points of Y through a perturbation $S \to S + \lambda Q\Psi$ as $\lambda \to \infty$ for some Ψ. To do this, we need to choose a U(1)-invariant metric g on M, which we can always do, and the answer will turn out to not depend on it. Then, let

$$\Psi := g(Y) = g_{ij}\psi^i Y^j = \psi^i Y_i.$$

Exercise 5.8. Check that

$$Q\Psi = g(Y,Y) - d(gY) \tag{5.9}$$

$$Q^2\Psi = 0, \tag{5.10}$$

e.g. by computing in coordinates.[19]

[18] This is the differential in the Cartan model for the U(1)-equivariant cohomology for M, but that's not important right now.

[19] Even though the derivations only form a (super)-Lie algebra, so Q^2 doesn't make sense on that level, it's still acting on vector fields, and we can iterate its action. This differs from $[Q,Q]$ by $1/2$, so it doesn't make a difference.

Thus, we know the perturbation (5.9). As before, Z is independent of λ, which uses (5.10).

Now, you can take $\lambda \to \infty$ and use the method of steepest descent to conclude that

$$Z \sim \sum_{x_c \in M : Y(x_c)=0} e^{i\alpha H(x_c)} (2\pi)^n \frac{(\mathrm{d}(gY)(x_c))^n/n!}{\sqrt{\det(D^2(g(Y,Y))(x_c)}} \tag{5.11}$$

This looks complicated, but just like before, the fermionic and bosonic pieces almost cancel each other out, leaving behind a topological contribution. Here, $g(Y,Y)$ is a real-valued function on M, so we can take its Hessian $D^2(g(Y,Y))$ and evaluate it at the critical point x_c.

Both the numerator and the denominator of the fraction in (5.11) are naturally valued in $\Lambda^{\mathrm{top}} T^*_{x_c} M$. We'll exploit this to calculate their ratio in a local model by diagonalizing the U(1)-action on $T_{x_c}M$. That is, we choose an isomorphism

$$T_{x_c} M \cong \bigoplus_{i=1}^n \mathbb{R}_i^2,$$

where U(1) acts on $\mathbb{R}_i^2$ by

$$\theta \longmapsto \begin{pmatrix} \cos k_i\theta & \sin k_i\theta \\ -\sin k_i\theta & \cos k_i\theta \end{pmatrix}$$

for some weights $k_1, \dots, k_n \in \mathbb{R}$.

Example 5.12. The 2-dimensional case is simplest: take $T_{x_c}M = \mathbb{R}^2$ with weight k. Let $\omega = r\,\mathrm{d}r \wedge \mathrm{d}\theta$ be the symplectic form and the standard metric $g := \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2$ is U(1)-invariant. Then, $Y -= k\partial_\theta$, $H = (1/2)kr^2$, and $g(Y,Y) = k^2r^2$, then

$$\mathrm{d}(gY) = 2kr\,\mathrm{d}r \wedge \mathrm{d}\theta \sqrt{\det(D^2(g(Y,Y)))} = 2k^2 r\,\mathrm{d}r \wedge \mathrm{d}\theta. \tag{5.13}$$

Again, almost everything cancels out, so we get

$$Z = (2\pi)^n \sum_{x_c} \frac{e^{i\alpha H(x_c)}}{\prod_{i=1}^n k_i(x_c)},$$

i.e.

$$\int_M \frac{\omega^n}{n!} e^{i\alpha H} = \left(\frac{2\pi}{i\alpha}\right)^n \sum_{x_c} \frac{e^{i\alpha H(x_c)}}{\prod_{i=1}^n k_i(x_c)}. \tag{5.14}$$ ◀

(5.14) is known as the *Duistermaat-Heckman formula*. We've just given a completely rigorous proof of it, which probably differs greatly from their original proof in [10].

Next time, we'll wrap up this story and begin thinking about higher dimensions.

Lecture 6.

One-dimensional QFT: 9/19/17

Reminder: there are exercises in the professor's notes, and you should try them!

We've been talking about localization in the past week, and we're on the way to thinking about effective field theory. Last time, we discussed the Duistermaat-Heckman formula (5.14) for a compact symplectic manifold (M, ω) (i.e. M is a compact $2n$-dimensional manifold, and $\omega \in \Omega^2(M)$ is a closed form with $\omega^n \neq 0$). We assumed we had a vector field Y which generates a U(1)-action,[20] This Y is generated by a Hamiltonian $H : M \to \mathbb{R}$, in the sense that $Y = \omega^{-1}(\mathrm{d}H)$; we assume H has isolated fixed points.[21]

Then, we showed that the integral

$$\int_M \frac{\omega^n}{n!} e^{i\alpha H}$$

depends only on the fixed points of H, and the precise formula is (5.14). The equation uses the infinitesimal U(1)-action on the tangent space of a fixed point.

[20] This is a strong assumption: a general smooth function $H : M \to \mathbb{R}$ can be taken for a Hamiltonian, and we can let $Y = \omega^{-1}(\mathrm{d}H)$, which generally does not generate a U(1)-action, as $e^{2\pi Y} \neq \mathrm{id}$. So as an equation, we're assuming $e^{2\pi Y} = \mathrm{id}$.

[21] Unlike the previous assumption, this is generically true. We also assumed H is Morse in the final step; this assumption probably can be removed, but the argument will be nontrivial.

Exercise 6.1. Suppose V is a $2n$-dimensional vector space with an orientation and an inner product, i.e. a reduction of the structure group from $\mathrm{GL}_{2n}(\mathbb{R})$ to $\mathrm{SO}(2n)$. Then, define a natural line $\mathrm{Pf}(V)$ with $\mathrm{Pf}(V)^{\otimes 2} = \mathrm{Det}(V) := \Lambda^{2n}(V)$.

Exercise 6.2. We saw what happens for $M = S^2$ last time, in Example 5.6. Try it with $\mathbb{CP}^2$.[22]

Remark. Another way to interpret (5.14) is that "the stationary phase approximation to

$$\int \frac{\omega^n}{n!} e^{i\alpha H}$$

is exact." This is an asymptotic analysis as $\alpha \to \infty$; we've already done this for things like $\int e^{-tF}$, but in this case there's something weirder going on: as $\alpha \to \infty$, the function is oscillating more and more rapidly. The fact that it only depends on the critical points in the end is a manifestation of the fact that these oscillations cancel each other out.

This stationary phase analysis is much like the method of steepest descent that we've been doing: approximate the integrand by its quadratic Taylor expansion around each critical point. There are some tricky technicalities, and you have to make rigorous the idea that you're integrating something only conditionally convergent

The point is, if you hear someone saying the stationary phase approximation is exact, that's a different statement with a different proof than the approach we used. There's a really great exposition of this approach in [2]. ◀

In our proof of the Duistermaat-Heckman formula, we used localization for

$$\int_{\mathscr{C}} \mathrm{d}\mu\, e^{-S},$$

where $\mathscr{C} = \Pi TM$, the parity change of the tangent bundle, and

$$S = -i\alpha(\omega + H) \in C^\infty(\mathscr{C}) = \Omega^*(M).$$

If $Q := \iota_Y + \mathrm{d}$, then $QS = 0$.

There's an interpretation of this in terms of U(1)-equivariant cohomology which allows for a more general formula than (5.14). Namely, we think of Q as an "equivariant differential," and we can generalize to any $S \in C^\infty(\mathscr{C})$ with $QS = 0$, i.e. any *equivariantly closed* form $\alpha \in \Omega^*(M)$ on any compact manifold M with a U(1)-action.

Theorem 6.3 (Atiyah-Bott-Berline-Vergne [2, 5])**.** *Let M be a compact manifold with a* U(1)*-action with isolated fixed points $\{x_c\}$, and let $\beta \in \Omega^*(M)$ be an equivariantly closed form. Then,*

$$\int_M e^\beta = (-2\pi i)^n \sum_{x_c} \frac{e^{\beta^{\mathrm{bot}}(x_c)}}{\prod_{i=1}^n k_i(x_c)}.$$

Here, β^{bot} denotes the piece of β in Ω^0; equivariantly closed forms are generally non-homogeneous.

The equivariant folks also call this theorem "the localization theorem in equivariant cohomology," and like this formulation of it better.

We can also generalize to non-isolated fixed points, and we will need to use this later. In this case, the steepest descent analysis of

$$\int \mathrm{d}\mu\, e^{-S+\lambda Q\Psi}$$

as $\lambda \to \infty$ is now localized on the fixed *set* P, and the integrand is determined by the local structure around P.[23]

Let NP denote the normal bundle, and recall that in the steepest descent analysis, we introduced a U(1)-invariant metric g on M. Since the volume form ω^n on a symplectic manifold defines an orientation, NP is also oriented, and the orientation and the metric g define an $\mathrm{SO}(2n)$-structure.

Definition 6.4. Let X be a manifold with a U(1)-equivariant vector bundle $E \to X$ together with a reduction of its structure group to $\mathrm{SO}(2n)$ compatible with the U(1)-action. Let $Y \in \Omega^0(\mathfrak{so}(E))$ denote the action of U(1), and choose a U(1)-invariant metric g on E and let $F \in \Omega^2(\mathfrak{so}(E))$ denote its curvature form. Then, the *equivariant Euler form* of E is

$$\mathrm{Eul}(E) := \mathrm{Pf}\left(\frac{1}{2\pi}(Y + F)\right).$$

[22] TODO: which circle action?

[23] We may need to make a transversality assumption on P, but it's OK.

In general, $\mathrm{Eul}(E)$ is concentrated in even degrees of $\Omega^*(X)$.[24] If $n = 1$, the Euler form has a simpler formula:

$$\mathrm{Eul}(E) = \frac{1}{2\pi}(ik + F),$$

with k as in Example 5.12.

More generally, the bottom piece of the Euler form is $\prod ik_i/2\pi \neq 0$, so if all $k_i \neq 0$, there's an inverse to the Euler form $1/\mathrm{Eul}(E) \in \Omega^*(X)$, using the fact that

$$\frac{1}{a+x} = \frac{1}{a}\left(\frac{1}{1+a^{-1}x}\right) = \frac{1}{a}\left(1 - a^{-1}x + a^{-2}x^2 - \cdots\right).$$

This leads to the most general version of the ABBV formula, which is one of the coolest things you can do with 0-dimensional supersymmetric quantum field theory.

Theorem 6.5 (Atiyah-Bott-Berline-Vergne [2, 5]). *With M, β, and P as above,*

$$\int_M e^\beta = \int_P \frac{e^\beta}{\mathrm{Eul}(NP)}.$$

Quantum field theory in one dimension. Now we'll move on to the one-dimensional case, which specializes to undergraduate quantum mechanics. Choose a compact Riemannian 1-manifold (X, η): either $X = [0, T]$, or $X \cong S^1$ with circumference T. We'll parametrize X by t, which you can think of as time. Now, the space $\mathscr{C}_X$ of (some kind of generalized) functions on X will be infinite-dimensional.

Let's define a theory. Fix a Riemannian manifold (Y, g), which we'll call the *target*, and $V : Y \to \mathbb{R}$, called the *potential*.

- For $X \cong S^1$, let $\mathscr{C}_{S^1} := \{\phi : S^1 \to Y\}$, the C^∞ maps from S^1 to Y, and
- for $X = [0, T]$, fix $y_0, y_1 \in Y$ and let $\mathscr{C}_{[0,T]_{y_0}^{y_1}} := \{\phi : [0, T] \to Y \mid \phi(0) = t_0, \phi(T) = y_1\}$.

So for S^1 we get loops, and for $[0, T]$ we get paths with chosen endpoints. Let $\mathrm{d}V_X$ denote the volume form on X and R denote the scalar curvature of Y; then, we define the action $S : \mathscr{C}_X \to \mathbb{R}$ to be

$$S(\phi) := \int_X \mathrm{d}V_X \, \frac{1}{2}\left(g(\dot\phi, \dot\phi) + V(\phi) - \frac{1}{3}R(\phi)\right). \tag{6.6}$$

Or in coordinates,

$$= \int_X \mathrm{d}t \sqrt{\eta_{tt}} \left(\frac{1}{2} g_{ij}(\phi(t))\dot\phi^i(t)\dot\phi^j(t)\eta^{tt}(t) + V(\phi(t)) - \frac{1}{3}R(\phi(t))\right). \tag{6.7}$$

If you parametrize X by arc length, $\eta_{tt} = 1$ and this simplifies:

$$= \int_X \frac{1}{2} g_{ij}\dot\phi^i\dot\phi^j + V(\phi(t)) - \frac{1}{3}R(\phi(t)). \tag{6.8}$$

We would like to define the partition function

$$Z_X \text{ ``$=$'' } \int_{\mathscr{C}_X} \mathrm{d}\phi \, e^{-S(\phi)}, \tag{6.9}$$

but here we run aground: $\mathrm{d}\phi$ is now a measure on an infinite-dimensional Banach space. There's no analogue of the Lesbegue measure here: a unit ball contains infinitely many balls of radius 1/4, so there's no consistent way to define the volume of anything to be nonzero and finite. Nonetheless, in a sense $\mathrm{d}\phi$ doesn't exist, but the whole expression (6.9) will exist; it's something that statistical mechanics researchers call the Weiner measure.

Physicists make sense of (6.9) by discretization. For concreteness, set $X = [0, T]$ and fix $y_0, y_1 \in Y$. We'll replace X by a *lattice*: for some $N > 0$, let $t_0, t_1, \ldots, t_N \in X$ such that $\Delta_t = t_j - t_{j-1} = T/N$. The discretized field space $\mathscr{C}_{X;N}$ is the space of piecewise geodesic paths $\phi : X \to Y$ that are smooth on (t_{j-1}, t_j) and such that the path from $\phi(t_{j-1})$ to $\phi(t_j)$ is the unique minimal geodesic between them.[25] The map $\phi \mapsto (\phi(t_0), \ldots, \phi(t_N))$ defines

[24] TODO: Does the cohomology class of the Euler form depend on the metric?

[25] One might be surprised to learn this stuff was formalized and written down surprisingly recently, in the mid-2000s.

an embedding $\mathscr{C}_{X;N} \subset Y^{N+1}$, and this is a finite-dimensional manifold, so we can use the *product measure*

$$\mathrm{d}\mu_N := \frac{1}{(4\pi\Delta t)^{N \dim Y/2}} \prod_{n=1}^{N-1} \mathrm{d}u\,\mathrm{d}y(\phi(t_n)).$$

Then, we can define the discretized partition function

$$Z_{X;N} = \int_{\mathscr{C}_{X;N}} e^{-S}\,\mathrm{d}\mu_N,$$

and try to take the limit as $N \to \infty$. This does exist!

Theorem 6.10. *The limit* $\lim_{N\to\infty} Z_{X;N}$ *exists, and is the heat kernel* $k_T(y_0, y_1)$*.*

Interestingly, it only depends on the endpoints y_0, y_1 and the total length.

Definition 6.11. Fix Y and V as above, For $t \in \mathbb{R}_+$, the *heat kernel (deformed by V)* is a smooth function $k_t \colon Y \times Y \to \mathbb{R}$ satisfying the *heat equation*

$$\partial_t k_t(x,y) + (-\Delta_X + V(x))k_t(x,y) = 0, \tag{6.12}$$

and as a distribution,

$$\lim_{t\to 0} k_t(x,y) = \delta(x,y). \tag{6.13}$$

You can also characterize the heat kernel as the fundamental solution to the heat equation (6.12).

Exercise 6.14. Show that when $Y = \mathbb{R}^n$ and $V = 0$, the heat kernel is

$$k_t(x,y) = \left(\frac{1}{4\pi t}\right)^{n/2} \exp\left(-\frac{1}{4t}\|x-y\|^2\right). \tag{6.15}$$

The heat kernel is the kernel of an integral operator, the operator U_t of heat evolution for time t. This is the operator evolving solutions to (6.12) forward in time. As an integral kernel, this has the formula

$$(U_t f)(x) = \int_M \mathrm{d}u\,\mathrm{d}y\, k_t(x,y) f(y)\,\mathrm{d}y.$$

U_t is a *smoothing* operator: it maps distributions to smooth functions. It also defines a linear operator on $L^2(M)$ which has the formula

$$U_t = e^{-t(-\Delta+V)}.$$

Heuristic proof of Theorem 6.10 when $V = 0$. Let's discretize the heat operator: let $U_T = (U_{\Delta t})^N$ and

$$K_T(y_N, y_0) := \int_{Y^{N-1}} \prod_{n=1}^{N-1} \mathrm{dvol} \prod_{n=0}^{N-1} k_{\Delta t}(y_{n-1}, y_n). \tag{6.16}$$

When Δt is sufficiently small (N is sufficiently large), we have short-time asymptotics of $k_{\Delta t}$:

$$k_{\Delta t}(x,y) \sim \left(\frac{1}{4\pi\Delta t}\right)^{\dim Y/2} \exp\left(-\frac{1}{4t} d(x,y)^2\right).$$

This is the piece that we're not making precise. If you substitute this into (6.16), you get

$$\begin{aligned} k_T(y_N, y_0) &\sim \int_{Y^{N-1}} \prod_{n=1}^{N-1} \mathrm{dvol} \prod_{n=0}^{N-1} \left(\frac{1}{4\pi\Delta t}\right)^{\dim Y/2} \exp\left(-\frac{1}{4\pi\Delta t} d(y_{n+1}, y_n)^2\right) \\ &= \int \mathrm{d}\mu_N \exp\left(-\frac{\Delta t}{4}\left(\frac{d(y_{n+1}, y_n)}{\Delta t}\right)^2\right) \\ &= Z_{X;N}. \end{aligned}$$

⊠

Though this was not a proof, this proof can be made rigorous; see [6]. Exactly where the scalar curvature goes is somewhat of a mystery, though some more careful analysis of the asymptotics above can be found in [4].

Lecture 7.

Local observables: 9/21/17

One thing that came up a few times in the past few lectures about localization is the question of if M is a compact manifold with a U(1)-action with isolated fixed points, why is the infinitesimal action nontrivial? The idea is that in a local model, i.e. $\mathbb{R}^n$ with a single fixed point, the U(1)-action is diffeomorphic to the standard rotation action with the origin as its fixed point, whose infinitesimal action nontrivial.

More rigorously, one can fix a U(1)-invariant Riemannian metric on M, which means the exponential map is U(1)-invariant map between a tubular neighborhood of the fixed-point set and its normal bundle, which implies the action must be nontrivial. Thus we do not need to worry about transversality, etc. Choosing such a metric requires averaging over U(1), and therefore crucially requires U(1) to be compact.

Last time, we also talked about the heat kernel: on a compact Riemannian manifold Y and for $t \in (0,\infty)$, given a potential function $V\colon Y \to \mathbb{R}$, we obtained a heat kernel function $k_t\colon Y\times Y \to \mathbb{R}$ defined to satisfy (6.12) and (6.13), which uniquely characterizes it. If $V = 0$, this is the usual heat equation, and in general it's a perturbation. The idea (when $V = 0$) is that if there's a point source of heat at y at $t = 0$, $k_t(x,y)$ calculates the amount of heat at x at time t, so for t small, it looks like an approximation to a δ-function, and when t is large, heat is spread evenly (on a compact manifold).

We then considered a 1-dimensional quantum field theory whose space of fields $\mathscr{C}_{[0,T]^{y_1}_{y_0}}$ on the interval $[0,T]$ with the usual metric is the space of functions $\phi\colon [0,T] \to Y$ with $\phi(0) = y_0$ and $\phi(T) = y_1$. The action is

$$S = \int_0^T \frac{1}{4} g(\dot\phi,\dot\phi) + V(\phi) - \frac{1}{3}R(\phi),$$

where R is the scalar curvature on Y. Then, in Theorem 6.10, we showed that the partition function

$$Z_{[0,T]^{y_1}_{y_0}} = k_T(y_0,y_1),$$

though defining this as an integral on $\mathscr{C}_{[0,1]^{y_1}_{y_0}}$ doesn't quite make sense; instead, we had to discretize $[0,T]$ and the action and the path integral and take a continuum limit. In each case there is a finite-dimensional space of paths and the integral does make sense, and what you get is kind of a Riemann sum for the path integral.

The proof is trickier than one thinks: if you just use the leading term, you don't get the scalar curvature, and the next term is where the scalar curvature comes from, but there's an extra factor of 2 to account for.

Precisely, the leading term in the asymptotic is

$$k_{\Delta t}(x,y) \underset{\substack{\Delta t\to 0\\ x\to y}}{\sim} \left(\frac{1}{4\pi\Delta t}\right)^{\dim Y/2} \exp\left(-\frac{1}{4\Delta t} d(x,y)^2\right).$$

The scalar curvature comes up in the next term, but there are corrections in Δt *and* $x - y$, and these account for the spurious factor of 2.

Remark. People care about the heat kernel for a lot of different reasons, but this is a good one: the simplest version of quantum mechanics (one-dimensional QFT) has the heat kernel as its partition function, so this is really something fundamental. ◀

One new feature of 1-dimensional QFT is that you can glue intervals together. If you trace this through the argument, this turns into the semigroup law for the heat kernel: the heat equation describes time evolution of something, and composing the intervals $[0,T_1]$ and $[T_1,T_2]$ corresponds to first evolving the system for time T_1, then using that as the initial condition and evolving for $T_2 - T_1$. In higher dimensions there are more and more ways to do this, and therefore there are more and more interesting structures.

Remark. We only gave the proof for $V = 0$; in the case $V \neq 0$ there's an analogous proof using *Trotter's formula*

$$e^{A+B} = \lim_{N\to\infty} \left(e^{A/N}e^{B/N}\right)^N.$$ ◀

You can also formulate this QFT on a sphere. In this case the partition function is a trace:

$$Z_{S^1(T)} = \int_Y k_T(y,y)\,\mathrm{d}y.$$

If $U_T : L^2(Y) \to L^2(Y)$ is the time evolution operator by T, with formula

$$U_T = e^{-(-\Delta+V)T},$$

which is unitary, then U_T is a trace-class operator (meaning that, though it's infinite-dimensional, its trace can still be rigorously defined).

Local observables. There's yet another structure present in the 1-dimensional case that we didn't have in dimension zero. Let $\mathscr{O} : \mathscr{C}_X \to \mathbb{R}$ be a function which only depends on the *k-jet* of ϕ at a $t \in X$, i.e. only depends on the first k derivatives of ϕ at t (including $\phi(t)$).

Example 7.1. One quick example of a local observable would be to choose a function $F : Y \to \mathbb{R}$ and define

$$(\mathscr{O}_F(t))(\phi) := F(\phi(t)). \qquad \blacktriangleleft$$

We can then define its *expectation* using a path integral:

$$\langle \mathscr{O}_F(t) \rangle := \int_{\mathscr{C}_X} \mathrm{d}\phi\, (\mathscr{O}_F(t))(\phi) e^{-S} = \int_{\mathscr{C}_X} \mathrm{d}\phi\, F(\phi(t)) e^{-S(\phi)}. \tag{7.2}$$

Once again, to make this rigorous one must discretize and show that the continuum limit exists, but when one does, there's a nice answer.

Let $\widehat{F} : L^2(Y) \to L^2(Y)$ denote the operator sending $\psi \mapsto F \cdot \psi$. Also let $H := -\Delta + V$.

Theorem 7.3. *$\langle \mathscr{O}_F(t) \rangle_{[0,T]_{y_0}^{y_1}}$ is the kernel representing*

$$e^{-H(T-t)} \widehat{F} e^{-Ht}.$$

Exercise 7.4. Give a heuristic proof for this, similar to the one we gave for Theorem 6.10. (Or a rigorous proof, if you want; it should look very similar to the one in [6].)

So the idea is that we've stuck the operator into the path integral, and this computes a modified heat flow, where we've stuck in a related operator into the time evolution.

If one has multiple local observables $\mathscr{O}_{F_1}(t_1), \dots, \mathscr{O}_{F_k}(t_k)$, there's a similar definition for the expectation $\langle \mathscr{O}_{F_1}(t_1) \cdots \mathscr{O}_{F_k}(t_k) \rangle$, where you stick them all into the path integral. The answer is very similar to the one in Theorem 7.3: if we do this on a circle and assume that $t_1 < \cdots < t_k$, then

$$\langle \mathscr{O}_{F_1}(t_1) \cdots \mathscr{O}_{F_k}(t_k) \rangle = \mathrm{tr}\Big(e^{-(T-t_k)H} \widehat{F}_k \cdots \widehat{F}_2 e^{-(t_2-t_1)H} \widehat{F}_1 e^{-t_1 H}\Big).$$

There's a similar answer on the interval (though without a trace).

The general slogan is *the path integral converts local observables into operators*, and this process is called *path-integral quantization*.

So far, everything has been commutative. But for more general observables, path-integral quantization can turn commutative operators into noncommutative ones!

So let's define a local observable depending on the 1-jet, i.e. a function $F : TY \to \mathbb{R}$. Now we let

$$(\mathscr{O}_F(t))(\phi) := F(\phi'(t)),$$

which defines a function $\mathscr{O}_F : \mathscr{C}_X \to \mathbb{R}$, and define its expectation as in (7.2), heuristically as a path integral and rigorously as a limit over discretizations. Path integral quantization will turn $\mathscr{O}_F$ into an operator $\widehat{F} : L^2(Y) \to L^2(Y)$ in that, for example,

$$\langle \mathscr{O}_F(t) \rangle_{S^1(T)} = \mathrm{tr}\Big(e^{-HT} \cdot \widehat{F}\Big).$$

Exercise 7.5. Let $Y = \mathbb{R}$ with the standard metric and introduce coordinates (x, p) on TY ($x \in \mathbb{R}$, $p \in T_x\mathbb{R}$). We know the operator x will quantize to the operator $\widehat{x}$, multiplication by x on $L^2(\mathbb{R})$,[26] and show that p quantizes to $2\partial_x : L^2(\mathbb{R}) \to L^2(\mathbb{R})$. You can also use the flat metric on S^1 instead of $\mathbb{R}$ if you'd like, though you have to replace x by e^{ix}.

[26] This is not technically an operator $L^2(\mathbb{R}) \to L^2(\mathbb{R})$, but there are ways to work around this: as soon as you apply e^{-HT} it does make sense. You can also work with a compactly supported version if you'd like.

This is very weird: commutative things became noncommutative. Where did this come from?

We'll compute the commutator, which is somehow the most fundamental object to come out of this question. Let $Y = \mathbb{R}$ or S^1. Naïvely,

$$\operatorname{Kernel}\left(e^{-t_3H}\widehat{p}e^{-t_2H}\widehat{x}e^{-t_1H}\right) = \int_{\mathscr{C}} \mathrm{d}\phi\, \phi'(t_1+t_2)\phi(t_1)e^{-S(\phi)} \tag{7.6a}$$

$$\operatorname{Kernel}\left(e^{-t_3H}\widehat{x}e^{-t_2H}\widehat{p}e^{-t_1H}\right) = \int_{\mathscr{C}} \mathrm{d}\phi\, \phi(t_1+t_2)\phi'(t_1)e^{-S(\phi)}. \tag{7.6b}$$

With the path integral defined by discretization, these are actually both literally true. Moreover, as $t_2 \to 0$, they look equal. But what happens when we discretize? Let $y_1 := t_1 - \Delta t$, $y_2 = t_1$, and $y_3 = t_1 + \Delta t$. We know $\phi(t_1) = y_2$, but what about the derivative? We have the two choices

$$\frac{1}{\Delta t}(y_3 - y_2) \qquad \text{or} \qquad \frac{1}{\Delta t}(y_2 - y_1).$$

If you take the continuum limits of (7.6a) and (7.6b), you'll wind up with terms like these.[27] So in the path integral, there are again two possibilities:

$$\frac{1}{\Delta t}\int \mathrm{d}y_2\, y_2(y_3 - y_2)e^{-F} \tag{7.7a}$$

$$\frac{1}{\Delta t}\int \mathrm{d}y_2\, y_2(y_2 - y_1)e^{-F}, \tag{7.7b}$$

where

$$F = \frac{(y_1-y_2)^2 + (y_2-y_3)^2}{(\Delta t)^2}.$$

You can compare (7.7a) and (7.7b) directly by integration by parts:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}y_2}\left(y_2e^{-F}\right) &= e^{-F} - y_2\frac{\mathrm{d}F}{\mathrm{d}y_2}e^{-F}\\ &= e^{-F} - \frac{y_2(-2(y_1-y_2) + 2(y-2-y_3))}{(\Delta t)^2}e^{-F},\end{aligned}$$

so the difference between (7.7a) and (7.7b) is

$$2\int \mathrm{d}y_2 e^{-F},$$

which was the value of the path integral with the operator 2 inserted. Therefore we conclude

$$\lim_{t_2\to 0}\langle\cdots\widehat{p}e^{-t_2H}\widehat{x}\cdots\rangle - \langle\cdots\widehat{x}e^{-t_2H}\widehat{p}\cdots\rangle = 2\langle\cdots 1\cdots\rangle,$$

or

$$[\widehat{p},\widehat{x}] = 2.$$

This is how we got noncommutativity: it's important to be careful when you're doing two things at the same point.

What's next? We've seen that the one-dimensional QFT of maps into Y has to do with heat flow on Y, e.g.

$$Z_{S^1(T)} = \operatorname{tr}(e^{-TH}).$$

This knows all of the eigenvalues of the Laplacian on Y, and is in particular very far from being topological. Next time, we'll cure this just as we did in dimension 0: we'll add fermions to make a supersymmetric quantum field theory. For example, instead of considering maps $S^1 \to Y$, we'll consider maps from the supermanifold ΠTS^1 into Y (so a bosonic part that looks like the circle and additional fermionic directions). This again has to do with heat flow, but for spinors, and the answer will be a super-trace of e^{-TH}, and this is the index of a Dirac operator — all but the zero eigenspace cancel out, and we'll obtain something topological.

[27] You can also average, in which case you'll average (7.6a) and (7.6b).

Lecture 8.

The harmonic oscillator and some partition functions: 9/26/17

"So far, we don't look directly at the sun."

We've been talking about one-dimensional QFT, with data a Riemannian manifold (Y, g) and a potential $V: Y \to \mathbb{R}$. The space of fields is $\mathscr{C}_X = \mathrm{Map}(X, Y)$, where X is an interval, though we only approach it through its discretization.

We're going to modify the action slightly, in order to match the standard quantum-mechanical convention for the Hamiltonian in quantum mechanics,

$$H = -\frac{1}{2}\Delta + V.$$

Consequently the action should be

$$S(\phi) = \int_X \left(\frac{1}{2}g(\dot{\phi}, \dot{\phi}) + V(\phi) - \frac{1}{3}R(\phi)\right) \mathrm{d}x. \tag{8.1}$$

This changes the conventions with the heat kernel, but nothing substantiative is different.

We also saw that the interval of length T is associated with the operator $e^{-HT}: L^2(Y) \to L^2(Y)$, which sends

$$\psi(x) \longmapsto \widetilde{\psi}(x) := \int_Y k_T(x, y)\psi(y)\,\mathrm{dvol}_Y,$$

where k_T is the heat kernel. The reason it has the name e^{-HT} is that if you differentiate it, you get the heat equation:

$$\underbrace{\frac{\partial}{\partial T}\left(e^{-HT}\psi\right)}_{\Psi(T,x)} = -He^{-HT}\psi,$$

and $\psi(T, x)$ satisfies the heat equation, somewhat tautologically. In functional analysis, this is sometimes called *functional calculus*.

In a similar way, the circle with circumference T is associated with the operator $\mathrm{tr}_{L^2(Y)} e^{-HT}$.

Example 8.2 (Harmonic oscillator)**.** The harmonic oscillator is familiar to students of quantum mechanics. In this model we let $Y = \mathbb{R}$ and

$$V(x) = \frac{1}{2}\omega^2 x^2,$$

for some $\omega \in \mathbb{R}$. The idea is that a particle can move around on a line, but is constrained by the potential to stay close to the origin, in that paths far from the origin aren't weighted very heavily.

The Hamiltonian is the operator

$$H = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\omega^2 x^2, \tag{8.3}$$

which isn't *a priori* an operator $L^2(\mathbb{R}) \to L^2(\mathbb{R})$, but it's defined densely enough that the theory still makes sense.

The eigenvalues are $\{1/2 + n \mid n = 0, 1, 2, \dots\}$, and the eigenfunctions look like

$$\begin{aligned}
\psi_0 &= e^{-\omega x^2/2}\\
\psi_1 &= xe^{-\omega x^2/2}\\
\psi_2 &= \left(x^2 - \frac{1}{2\omega}\right)e^{-\omega x^2/2},
\end{aligned}$$

and in general, these are built out of *Hermite polynomials* H_n: up to some fixed constant $C(\omega)$,

$$\psi_n(x) = C(\omega)H_n(x\sqrt{\omega})e^{-\omega x^2/2}.$$

The partition function on S^1 is nice:

$$\begin{aligned}
Z_{S^1(T)} &= \mathrm{tr}\left(e^{-TH}\right)\\
&= \sum_{n=0}^{\infty} \exp\left(-\omega\left(n + \frac{1}{2}\right)T\right),
\end{aligned}$$

which is a geometric series, so its sum is

$$= \frac{1}{2\sinh(\omega T/2)}.$$

◀

Example 8.4. There's another example in which one can obtain things pretty explicitly is where Y is compact, and a simple choice is S^1. When $V = 0$, this is called a σ-model on S^1, or the particle on a circle.

So let $Y = S^1(R)$ (i.e. the circle of circumference R, or $\mathbb{R}/R\mathbb{Z}$) and $V = 0$, so

$$H = -\frac{1}{2}\frac{\partial^2}{\partial x^2}$$

acting on $\mathscr{H} = L^2(S^1(R))$. The eigenfunctions are a natural Fourier basis for $L^2(S^1)$:

$$\begin{aligned}\psi_0(x) &= 1\\ \psi_{2n-1}(x) &= \sin\left(\frac{2\pi n x}{R}\right)\\ \psi_{2n}(x) &= \cos\left(\frac{2\pi n x}{R}\right).\end{aligned}$$

You can check these are periodic with period R. The eigenvalues scale with n^2 (unlike in Example 8.2, where they're evenly spaced): we have 0 with multiplicity 1, and $2\pi^2 n^2/R^2$ with multiplicity 2 for $n \geq 1$ (coming from ψ_{2n-1} and ψ_{2n}).

Thus the partition function on $S^1(T)$ is

$$\begin{aligned}Z_{S^1(T)} &= \mathrm{tr}\left(e^{-TH}\right)\\ &= 1 + 2\sum_{n=1}^{\infty}\exp\left(-\frac{2\pi^2 n^2 T}{R^2}\right)\\ &= \sum_{n=-\infty}^{\infty}\exp\left(-\frac{2\pi^2 n^2 T}{R^2}\right)\\ &= \vartheta\left(\tau = \frac{2\pi i T}{R^2}, z = 0\right).\end{aligned}$$

This $\vartheta(\tau, z)$ is called the *Jacobi ϑ-function*, where $\mathrm{Im}(\tau) > 0$ and $z \in \mathbb{C}$ (regarded as a point on an elliptic curve). This is a little bit of a mystery: why does this function appear in one of the simplest quantum-mechanical model? Is it possible to get the ϑ-function for nonzero z?

To answer that question, we'll use another nice thing about the system: a symmetry. There's an action of U(1) on this theory: for an $\alpha \in \mathrm{U}(1)$, we have an operator $S_\alpha \colon \mathscr{H} \to \mathscr{H}$ such that

$$(S_\alpha\psi)(x) := \psi(x + \alpha).$$

That this is a symmetry of the system means it commutes with the Hamiltonian:

$$[S_\alpha, H] = 0.$$

Remark. For those who are interested in bridges between topology and quantum field theory, it's worth mentioning that this QFT has a "global U(1)-symmetry." One thing you can do to these kinds of theories is formulate them on Riemannian manifolds X (i.e. the spacetime) equipped with a principal U(1)-bundle with connection. This is a general principle, but in this case means to take $X = S^1$ with a flat principal U(1)-bundle, which is characterized by its holonomy $\alpha \in \mathrm{U}(1)$. In this case, instead of integrating over loops, you're integrating over "twisted loops" that don't completely close up, but have a "twisted boundary condition"

$$\phi(x + T) = \phi(x) + \alpha.$$

This intertwines the U(1)-action on the principal bundle and on the target. ◀

Exercise 8.5. Compute $\mathrm{tr}_{\mathscr{H}}\, e^{-TH}S_\alpha$, and show that it's $\vartheta(\tau, z)$ for some nonzero z depending on α.

These days, people such as Dan Freed have been thinking of this perspective, especially in higher dimensions, as spreading the theory out over the moduli space of principal U(1)-bundles. This is not the historical way to think about symmetries, but is interesting and fruitful. ◀

These examples were very nice, in that the Hamiltonian only has a point spectrum. Examples where the Hamiltonian has a continuous spectrum exist and are physically relevant, e.g. scattering phenomena. Often these also have discrete spectra.

Determinants. Recall that if V is a finite-dimensional vector space, $M: V \otimes V \to \mathbb{R}$ is a positive-definite bilinear form, $\mathrm{d}\mu$ is a translation-invariant measure on V, and $c \in \mathbb{R}$, then we had a formula

$$(2\pi c)^{-\dim V/2} \int_V \mathrm{d}\mu\, e^{-M(x,x)/2} = \frac{\mathrm{d}\mu}{\sqrt{\det cM}}. \tag{8.6}$$

Something tricky is going on: $\sqrt{\det cM}$ is a density, hence defines a measure, so the right-hand side is a ratio of two measures, hence a number! This is because of how it transforms under change-of-coordinates: if $M \mapsto A^{\mathrm{T}} MA$, then

$$\det M \longmapsto (\det A)^2 \det M,$$

so $\sqrt{(\det M)} \mapsto |\det A| \sqrt{\det M}$, which is why it's a density.

Remark. Recall that the space of densities on V is $\Lambda^{\mathrm{top}}(V^*) \otimes_{\mathbb{Z}/2} u(V)$, where $u(V)$ is the orientation bundle. That is, a density is a pair (ω, u), where u is an orientation of V and ω is a volume form, such that $(\omega, u) \sim (-\omega, -u)$. And this is fine because if you use u and ω to integrate a function, using $-u$ and $-\omega$ gives you the same answer. Densities form a one-dimensional vector space, and unlike for volume forms, there is a canonical notion of a positive density. ◀

We're going to try to understand this for V infinite-dimensional: because the action (8.1) is quadratic in ϕ, our discretized path integral looks like the left-hand side of (8.6), where $c = \Delta t = T/N$. When we let $N \to \infty$ (so $\dim V \to \infty$), the left-hand side exists, but the right-hand side doesn't make sense, since we can't choose nontrivial translation-invariant measures on an infinite-dimensional vector space.

In the finite-dimensional case, we can avoid talking about the measure by choosing an inner product on V such that $\|\mathrm{d}\mu\| = 1$ (that is, a density specified by a top form with norm 1). This lets us identify $V = V^*$. This simplifies the linear algebra a bit: we can use this identification to replace $M\delta t : V \otimes V \to \mathbb{R}$ with an $A: V \to V$, and

$$\frac{\mathrm{d}\mu}{\sqrt{\det(M\Delta t)}} = \frac{1}{\sqrt{\det A}},$$

where this time, it's the ordinary determinant of a matrix.

Example 8.7. For the harmonic oscillator, the matrix is

$$A = \begin{pmatrix} 2 + \omega^2 T^2/N^2 & -1 & 0 & \cdots & -1 \\ -1 & 2 + \omega^2 T^2/N^2 & -1 & & \\ 0 & -1 & \ddots & & \\ \vdots & & & & \\ -1 & & & & \end{pmatrix}.$$

If A is an $N \times N$ matrix, then pleasantly,

$$\lim_{N\to\infty} \frac{1}{\sqrt{\det A}} = \frac{1}{2\sinh(\omega T/2)},$$

which is what we got for the partition function in Example 8.2 in a totally different way! But it's less obvious how to generalize A itself to infinite dimensions. This was worked out in a recent paper of Ludewig [11].

In the infinite-dimensional case, we'll choose the space V of functions $\phi: S^1 \to \mathbb{R}$ such that a certain norm is finite. That is, using the eigenbasis we found above, we can write

$$\phi(t) = c\sqrt{T} + \sum_{n=1}^{\infty} \frac{\sqrt{2T}}{2\pi n}\left(a_n \sin\left(\frac{2\pi n}{T} t\right) + b_n \cos\left(\frac{2\pi n}{T} t\right)\right). \tag{8.8}$$

Then, we let the norm be

$$\|\phi\|^2 := c^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2.$$

This is not the usual L^2-norm, and looks more like a Sobolev norm (and in fact is equivalent to one). Anyways, we take V to be the space of functions for which this is finite.

Using (8.8), the action

$$S = \frac{1}{2}\int \mathrm{d}t\left(g(\dot\phi(t),\dot\phi(t)) + \omega^2\phi(t)^2\right)$$

becomes

$$S(\phi) = \frac{1}{2}\left(\omega^2 T^2 c62 + \sum_{n=1}^\infty\left(1+\frac{\omega^2T^2}{4\pi^2n^2}\right)\left(a_n^2+b_n^2\right)\right).$$

The eigenvalues of this operator are

$$\lambda = \omega^2T^2, 1+\frac{\omega^2T^2}{4\pi^2n^2},$$

where the latter has multiplicity 2 for each $n>0$, and therefore one can show that

$$\sqrt{\det A} = 2\sinh\left(\frac{1}{2}\omega T\right),$$

as desired. Usually in physics this is heuristically done with some sort of ζ-regularization, but in the one-dimensional case everything can be made rigorous! ◀

Lecture 9.

Symmetries and effective field theory in 1D QFT: 9/28/17

In the last few lectures, we've been learning about one-dimensional QFT, though in a specific class of examples: Lagrangian quantum field theories (so the partition function is an integral) and specifically, a σ-model with action

$$S = \int_X \mathrm{d}t\left(\frac{1}{2}g(\dot\phi,\dot\phi) + V(\phi) - \frac{1}{6}R\right),$$

and we showed that the partition function is about heat flow (on an interval of length T, it's heat flow for time T,[28] and on a circle of circumference T, it's the trace of that operator). There are two ways to think about this (for concreteness, let $X = S^1(T)$):

(1) an integral over loops in the target Y, or
(2) as the trace of e^{-HT} in $L^2(Y)$,

and it's possible to rigorously show these are equal. These are generic in one-dimensional QFT: the Hilbert space might not always be $L^2(Y)$, but the fact that we recover traces of operators on S^1 is a recurring theme.

In Example 8.2 (where $Y=\mathbb{R}$ and $V=\omega^2x^2/2$), these interpretations turn into an infinite-dimensional determinant of $g(\dot\phi,\dot\phi)$ (coming from what is, more or less, an infinite-dimensional Gaussian integral) for (1) and a sum

$$\sum_n e^{-T\lambda_n},$$

where λ_n is the n^{th} eigenvalue of H for (2).

These two perspectives have established names: (1) is called the *Lagrangian formulation*, and (2) is called the *Hamiltonian formulation*. They're supposed to be formally equivalent, though showing this is difficult.

Remark. These two perspectives also exist in classical mechanics, and can be recovered from these by taking a classical limit. Classically, one restricts to the extrema of the action S, and there the proof that the Lagrangian and Hamiltonian formulations are equivalent is easier. ◀

Exercise 9.1. Figure out these two interpretations for the other example we considered, Example 8.4, where the Hamiltonian interpretation produces

$$Z_{S^1(T)} = \vartheta\left(\tau = \frac{2\pi i T}{R^2}, z=0\right).$$

[28] The thing that allows you to get the whole operator on the interval is that it has a boundary, and so you're freely able to choose boundary conditions and therefore understand what the heat kernel does to them.

Show that for the Lagrangian formulation (discretize the path integral), you get

$$Z_{S^1(T)} = \vartheta\left(\tau = \frac{iR^2}{2\pi^2 T}, z = 0\right).$$

These are indeed equal, thanks to the modularity of the ϑ-function, or the Poisson summation formula. So in this case you again recover something mathematically interesting.

We can also add symmetry to the picture. Recall that in the zero-dimensional case, we found a Lie algebra action of vector fields on $\mathscr{C}$ which annihilate S (or, exponentiated, a Lie group action of G on $\mathscr{C}$ preserving S). This produced constraints on the correlation functions: $\langle \mathscr{O} \rangle = \langle \mathscr{O}^g \rangle$ and $\langle X \mathscr{O} \rangle = 0$ if $X \in \mathfrak{g}$.

In the model we've been considering, we can choose two kinds of symmetries:

- isometries of X, or
- isometries of Y that preserve V.

The first exists for any choice of parameters (though there's not much to say for an interval), but the second might be the trivial group for some choices of Y and V.

For $X = S^1(T)$, the isometry group is U(1), acting by $t \mapsto t + c$. This produces a constraint on the correlation functions:

$$\langle \mathscr{O}_1(t_1)\mathscr{O}_2(t_2)\cdots\mathscr{O}_n(t_n)\rangle = \langle \mathscr{O}_1(t_1+c)\mathscr{O}_2(t_2+c)\cdots\mathscr{O}_n(t_n+c)\rangle.$$

Exercise 9.2. Show this from the Hamiltonian perspective (it comes from the cyclicity of the trace).

Any isometry of Y preserving V produces a similar formula.

We'll use this as an opportunity to introduce some useful notation describing how symmetries act on the fields.[29] For an action of U(1), which is connected, it suffices to use the Lie algebra $\mathfrak{u}_1 \cong \mathbb{R}$, so we'll describe the action of the shift $t \mapsto t + \varepsilon$. To first-order in ε, this is

$$\phi(t+\varepsilon) = \phi(t) + \varepsilon\dot{\phi}(t).$$

We represent this by writing

$$\delta\phi \simeq \varepsilon\dot{\phi}. \tag{9.3}$$

It feels like this should be more complicated, but this notation encapsulates the fact that this symmetry is completely local, and therefore actually very simple.

Effective field theory in 1 dimension. Consider a system where $Y = \mathbb{R}^2$ with coordinates (x, y) and potential

$$V = \frac{1}{2}x^2 + \frac{1}{2}\omega^2 y^2 + \frac{\mu}{4}x^2 y^2.$$

Quantum-mechanically, this is telling us about a system with two kinds of particles with a slight coupling between them.

Suppose $\omega \gg 1$, which means that y is oscillating very rapidly. Therefore, if you care about x, you should be able to eliminate y by integrating out over all of those oscillations and replacing them with their average values.[30] Said plainly, this is somewhat crazy: we're going to integrate over one infinite-dimensional space to end up in another. But this will be very helpful.

Suppose we're interested in normalized expectation values involving only x, such as

$$\frac{\langle x(0)x(t)\rangle_{S^1(T)}}{Z_{S^1(T)}} = \frac{1}{Z_{S^1(T)}}\int_{\mathscr{C}_{S^1(T)}} \mathrm{d}x\,\mathrm{d}y\, x(0)x(t)e^{-S(x,y)},$$

where by pullback along the map $S^1(T) \to Y$, $x, y\colon S^1 \to \mathbb{R}$. This is fine because the infinite-dimensional determinants present for the expectation value and for $Z_{S^1(T)}$ cancel each other out. Since this is quadratic, it's possible to attach it explicitly with a change of variables, proceeding somewhat similarly to before, but there's a more general method.

[29] This notation will not work for discrete symmetries; for example, $t \mapsto -t$ on the interval or the circle is an isometry preserving the action, called *time-reversal symmetry*. This is a $\mathbb{Z}/2$-symmetry having to do with orientation-reversal, and many theories do not have it. Nonetheless, we're not going to worry about discrete symmetries for now, and for supersymmetry, which we're going to switch to soon, this notation will be extremely convenient.

[30] In atomic physics, this is called the *Born-Oppenheimer approximation*.

Before we go to the effective field theory, we're going to remember how we used Feynman diagrams for zero-dimensional QFT to calculate asymptotic series. In this case we summed over four-valent vertices with labels i, j, k, ℓ, weighted by $\lambda_{ijk\ell}$, and edges i to j weighted by M_{ij}^{-1} (where M is the quadratic piece of the action).

The basic new ingredient in 1-dimensional QFT is the Green's functions D_x and D_y on S^1, defined to (distributionally) satisfy

$$\begin{aligned}\left(\partial_t^2 - 1\right)D_x(t) &= \delta(t)\\ \left(\partial_t^2 - \omega^2\right)D_y(t) &= \delta(t).\end{aligned}$$

Explicitly,

$$\begin{aligned}D_x(t) &= \frac{1}{2}\sum_{n\in\mathbb{Z}} e^{-|t+nT|}\\ D_y(t) &= \frac{1}{2\omega}\sum_{n\in\mathbb{Z}} e^{-\omega|t+nT|}.\end{aligned}$$

The Feynman diagram expansion is a little more complicated.

- At zeroth-order, we have a segment with endpoints labeled $0, t$, which contributes a factor $D_x(t)$.
- At first order, we put a dashed loop (for y) at $t' \in (0, t)$, for *every* value of t', and attach a Green's function for y there. This involves an integral (thankfully, over a one-dimensional space):

$$\frac{\mu}{2}\int_{S^1(T)} \mathrm{d}t'\, D_x(t')D_x(t-t')D_y(0).$$

- In higher dimensions you'll have more diagrams and more integrals over finite-dimensional spaces.

This is known as *perturbation theory* in quantum mechanics. Each of these things gives you a number, and so while this is complicated, you can calculate it in principle.

The effective field theory description will be easier. We decide to rid ourselves of these dashed lines (corresponding to terms in y). Formally,

$$S_{\mathrm{eff}}(x) = -\log\int \mathrm{d}y\, e^{-S(x,y)}.$$

In this case, at second order we get a Feynman diagram that looks like Figure 2, with edges labeled by t and t (one side) and t' and t' (the other side). We still have integrals, but they're easier.

- The terms that are second-order in x are

$$\int \mathrm{d}t\, \frac{1}{2}\dot{x}(t)^2 + \frac{1}{2}x(t)^2 + \frac{\mu}{2}\int \mathrm{d}t\, x(t)^2 D_y(0),$$

where the second term comes from a dashed loop. We can rewrite this as

$$\int \mathrm{d}t\, \frac{1}{2}\dot{x}(t)^2\left(\frac{1}{2} + \frac{\mu}{2}D_y(0)\right)x(t)^2.$$

- The term that's fourth-order in x (from the diagram like in Figure 2) is

$$\frac{\mu^2}{2}\int \mathrm{d}t\,\mathrm{d}t'\, x(t)^2 x(t')^2 D_y(t-t')^2.$$

One particularly weird consequence of the fourth-order term (and appearing more strongly in higher-order terms) is the presence of nonlocal phenomena, coming from $\mathrm{d}t\,\mathrm{d}t'$. The idea is that x-particles may be connected by y-fields, coupled by $D_y(t-t')$. So

$$S_{\mathrm{eff}} \neq \int \mathrm{d}t\, \dot{x}(t)^2 + V(x(t))$$

and therefore the effective theory is more complicated.

But not all hope is lost: $D_y(t-t')$ decays exponentially away from $t-t'=0$. So we can expand the non-local interaction in powers of $t-t'$:

$$\int \mathrm{d}t\,\mathrm{d}t'\, x(t)^2 x(t')^2 D_t(t-t')^2$$
$$= \int \mathrm{d}t\,\mathrm{d}t' \bigg(x(t)^4 + 2x(t)^3\dot{x}(t)(t-t') + \Big(x(t)^2\dot{x}(t)^2 + \frac{1}{2}x(t)^3\ddot{x}(t)\Big)(t-t')^2 + O((t-t')^3)\bigg) D_y(t-t')^2.$$

The idea is, we can replace the nonlocal term with something local, as long as we're willing to take derivatives of the fields.

Without evaluating in detail, we can learn something about the shape of the answer by integrating over t'. The action you get is

$$S = \int \mathrm{d}t\, c_1 x(t)^4 + \frac{c_2}{\omega}\Big(x^2\dot{x}^2 + \frac{1}{2}x^3\ddot{x}\Big) + \cdots. \tag{9.4}$$

That is, you have an infinite-series of "higher-derivative interactions" suppressed by larger and larger negative powers of ω. Some correlation functions will be dominated by paths where only the first few terms are large enough to really contribute (which is what we'll mean by a low-energy limit), and in this case we have a systematic expansion in terms of powers of ω^{-1}, and the theory looks local after all!

This is interesting: when you integrate out a field, there's no reason to expect that the theory you get is local. Even in this case, where the heuristics suggested a low-energy limit was possible, we had to make some estimates to recover locality, some of which were physically rather than mathematically justified.[31]

Introducing supersymmetry. We'll now begin studying supersymmetry quantum mechanics. There was nothing topological about the model we've been studying: indeed, it computed the trace of the heat kernel. To make it topological, we'll do the same thing that we did in zero dimensions: making a new quantum field theory whose configuration space is

$$\mathscr{C} = \Pi(T\,\mathrm{Map}(X,Y)),$$

the parity change of the tangent space of the infinite-dimensional space of maps from X to Y, an infinite-dimensional supermanifold.

Let's say what this actually means. A point of $\mathrm{Map}(X,Y)$ is a map $\phi\colon X\to Y$ (again, X is an interval with boundary conditions $[0,T]_{y_0}^{y_1}$ or a circle $S^1(T)$). Taking the tangent bundle (and parity change) means also specifying a fermionic term $\psi\in\Pi\Gamma(\phi^*TY)$: pull back the tangent bundle by ϕ, then take a section of it.

Remark. If M is oriented, ΠTM has a canonical measure. This reassures us that we're on relatively safe ground. ◂

Now let's write the action, which depends on these bosonic and fermionic directions (ϕ,ψ). Let ∇_t be the pullback of the Levi-Civita connection to ϕ^*TY by ϕ; then, the action is

$$S(\phi,\psi) = \int \mathrm{d}t\, \frac{1}{2}\big(g(\dot{\phi},\dot{\phi}) + g(\psi,\nabla_t\psi)\big). \tag{9.5}$$

These two terms are qualitatively different: the first is second-order in time derivatives, but the second is only first-order: it looks like $\int \mathrm{d}t\,\psi\dot{\psi}$. This reflects another difference between bosons and fermions: using integrating by parts, for a boson ϕ,

$$\int \mathrm{d}t\,\phi\dot{\phi} = -\int \mathrm{d}t\,\dot{\phi}\phi = -\int \mathrm{d}t\,\phi\dot{\phi},$$

so this term, which might look meaningful locally, is zero. But fermions anticommute and hence pick up an extra sign:

$$\int \mathrm{d}t\,\psi\dot{\psi} = -\int \mathrm{d}t\,\dot{\psi}\psi = \int \mathrm{d}t\,\psi\dot{\psi},$$

So we could have inserted a bosonic term like this one, but it would vanish. Similarly, a term like $\int \dot{\psi}\dot{\psi}$ would have vanished as well.

[31] Mathematical justifications should be possible, just require more thought.

The action S is invariant under time translation again, and we write

$$\begin{aligned}\delta\phi &= \varepsilon\dot\phi\\ \delta\psi &= \varepsilon\dot\psi.\end{aligned}\tag{9.6}$$

There's also an additional odd symmetry (i.e. an odd vector field on the supermanifold),

$$\begin{aligned}\delta\phi &= \varepsilon\psi\\ \delta\psi &= -\varepsilon\dot\phi.\end{aligned}\tag{9.7}$$

To understand what this means, you have to think of ε as having odd parity. More explicitly, one has an odd vector field

$$Q = \int \mathrm{d}t\,\psi(t)\frac{\delta}{\delta\phi(t)} - \dot\phi\frac{\delta}{\delta\psi(t)},$$

and $QS = 0$.

Lecture 10.

Perturbation theory in quantum mechanics and spin structures: 10/3/17

The last computation we did might not be terribly easy to follow, so today we're going to start with something different, but in the same spirit, and that should be a little clearer.

Recall that if you're doing an integral over paths with a Gaussian action, you're going to get an infinite-dimensional determinant. For non-Gaussian actions, you can make an asymptotic expansion in the "non-Gaussianity."

Example 10.1 (Quartic oscillator)**.** As usual, this will be a one-dimensional σ-model, whose fields are maps $X \to Y$. The target Y is $\mathbb{R}$ with the usual metric, and the action is

$$V = \frac{1}{2}\omega^2x^2 + \frac{\lambda}{4!}x^4.\tag{10.2}$$

The Hamiltonian is therefore

$$\begin{aligned}H &:= -\frac{1}{2}\frac{\partial^2}{\partial x^2} + V(x)\\ &= -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\omega^2x^2 + \frac{\lambda}{4!}x^4.\end{aligned}$$

As usual in physics, we'd like to compute the eigenvalues of H acting on $L^2(\mathbb{R})$. In quantum mechanics, you solve this by directly looking at the Hamiltonian, but we're trying to use this as a toy example of a QFT, so we're going to get them out of the partition function

$$Z_{S^1(T)} = \sum_n e^{-E_nT},$$

where

$$H\psi_n = E_n\psi_n,$$

i.e. E_n is the n^{th} eigenvalue.

Let Z_0 denote the partition function for the $\lambda = 0$ theory; then, we can compute $Z_{S^1(T)}/Z_0$ as a sum over Feynman diagrams.

Recall that if W is a finite-dimensional state space and we have an action

$$S = \frac{1}{2}M(x,x) + \frac{1}{4!}C(x,x,x,x),$$

where $M\colon W^{\otimes 2}\to\mathbb{R}$ and $C\colon W^{\otimes 4}\to\mathbb{R}$, we summed over Feynman diagrams with 4-valent vertices (corresponding to $C \in (V^*)^{\otimes 4}$) and with edges weighted by M^{-1}.

Now, though, the state space is infinite-dimensional, the functions $x\colon S^1\to\mathbb{R}$, and the second-order term is

$$M(x,x) = \int_0^T \mathrm{d}t\,\frac{1}{2}\|\dot x(t)\|^2 + \frac{1}{2}\omega^2x^2.$$

Its "inverse"[32] is the Green's function for the Laplace operator:

$$G(t,t') = G(t-t') = \frac{1}{2\omega}\sum_{n\in\mathbb{Z}} e^{-\omega|t-t'+nT|}. \tag{10.3}$$

This inverts M in the sense that

$$M(G(t,t'), f(t)) = f(t').$$

In finite dimensions, recall that

$$M(M^{-1}(\eta,\cdot), \nu) = \eta(\nu),$$

justifying our choice to call it an inverse.

Now, weighting by M^{-1} means weighting by an integral of a Green's function.

- The empty diagram comtributes a factor of 1.
- In first-order, we have a single vertex and the "figure-8 diagram." All half-edges are labeled with time t, and the contribution is
$$-\frac{\lambda}{8}\int_0^T \mathrm{d}t\, G(t,t)G(t,t).$$
- At second-order, two vertices have two different times t and t' associated with their half-edges. For example, the diagram with four edges between the vertices v and v' contributes[33]
$$\frac{\lambda^2}{48}\int_0^T\int_0^T \mathrm{d}t\,\mathrm{d}t'\, G(t-t')^4.$$

Our explicit formula (10.3) for $G(t,t')$ means these can be concretely evaluated, and the answer is, to first order,

$$\log\left(\frac{Z}{Z_0}\right) \sim -\frac{\lambda T}{32\omega^2}\left(\coth\frac{\omega T}{2}\right)^2 + O(\lambda^2).$$

We want to use this to calculate eigenvalues, or at least the zeroth eigenvalue. We know

$$\log Z = \log\left(\sum_n e^{-TE_n}\right),$$

and as $T\to\infty$,

$$\log Z \sim -TE_0(\lambda)$$
$$\log Z_0 \sim -TE_0(\lambda = 0).$$

Therefore

$$\log\left(\frac{Z}{Z_0}\right) \sim -T(E_0(\lambda) - E_0),$$

and a similar method gives you the corrections for higher eigenvalues. Concretely,

$$\begin{aligned} E_0(\lambda) - E_0 &\sim \lambda\left(\lim_{T\to\infty}\frac{1}{32\omega^2}\left(\coth\frac{\omega T}{2}\right)^2\right) \\ &= -\frac{\lambda}{32\omega^2}, \end{aligned}$$

and this is precisely the result you obtain by more conventional quantum-mechanical methods. ◀

Let's return to supersymmetric quantum mechanics. We want to write down a 1-dimensional QFT whose space of fields $\mathscr{C} = \Pi T\,\mathrm{Map}(X,Y)$. Formally, one defines $\mathscr{C}$ as Spec of some $\mathbb{C}$-algebra in supergeometry, so it's difficult to speak precisely of its points, but they should be maps $\phi\colon X\to Y$ (bosonic) and $\psi\in\Pi\Gamma(\phi^*TY)$ (fermionic).

The action $S\in C^\infty(\mathscr{C})$ is given in (9.4). There's a time-translation symmetry (9.6); concretely, this means that in local coordinates on Y,

$$\delta\phi^I = \varepsilon\dot\phi^I$$
$$\delta\psi^I = \varepsilon\dot\psi^I.$$

Here δ is the shift or translation. This symmetry was present in ordinary quantum mechanics.

[32] In finite dimensions, this is an inverse matrix, but not in infinite dimensions.

[33] In Minkowski signature, which is common for relativistic QFT, there are some additional complications due to boundary terms, etc.

But in the supersymmetric case, there's an additional odd symmetry (9.7), which defines a derivation $Q \in \mathrm{Vect}^1(\mathscr{C}_X)$. This will be the engine that makes supersymmetry behave so differently.

Proposition 10.4. *$QS = 0$.*

Proof (when $Y = \mathbb{R}^n$). We just have to compute (though there are a few steps left implicit): in local coordinates,

$$S = \frac{1}{2}\big(\dot\phi^I\dot\phi^I + \psi^I\dot\psi^I\big),$$

and therefore

$$\begin{aligned}\delta S &= \frac{1}{2}\big(2\varepsilon\dot\psi^I\dot\phi^I + \big(-\varepsilon\dot\phi^I\dot\psi^I - \psi^I\ddot\phi^I\big)\big)\\ &= 0,\end{aligned}$$

since we can commute $\dot\psi^I$ and $\dot\phi^I$. ⊠

Exercise 10.5. Show that Q is "a square-root of time-translations," in that

$$\frac{1}{2}[Q,Q] = H.$$

We'd like to obtain something topological out of this, but there's a metric around. It turns out the variation is Q-exact, so the partition function is invariant.[34]

Exercise 10.6. Show that if we vary the metric g on Y under the variation

$$g_{IJ} \longmapsto g_{IJ} + \delta g_{IJ},$$

then the action varies by

$$S \longmapsto S + Q\Psi,$$

where

$$\Psi := \int \mathrm{d}t\left(\frac{1}{2}(\delta g)_{IJ}\dot\phi^I\psi^J\right).$$

Thus we expect $Z_{S^1(T)}$ to be independent of the metric on Y, and indeed this is true, but it *does* depend on a spin structure on Y!

Spin structures and spinors. We now need to discuss spin structures and spinors. For this section, Morgan [13] is a good reference.

Definition 10.7. Let V be a finite-dimensional vector space. Its *tensor algebra* is the free (noncommutative) algebra on V; explicitly, this is the $\mathbb{Z}$-graded algebra

$$T(V) := \bigoplus_{n\ge 0} V^{\otimes n},$$

where $V^{\otimes 0} = \mathbb{C}$, and the multiplication rule is

$$(v_1\otimes\cdots\otimes v_k)\times(v'_1\otimes\cdots\otimes v'_{k'}) = v_1\otimes\cdots\otimes v_k\otimes v'_1\otimes\cdots\otimes v'_{k'}.$$

(Often, this multiplication rule is also denoted $\otimes$.)

This is the progenitor of all sorts of useful algebraic structures, such as symmetric algebras, exterior algebras, and the one we need, Clifford algebras.

Definition 10.8. Let V be a finite-dimensional real (or complex) vector space together with a symmetric, positive-definite quadratic form $\langle\cdot,\cdot\rangle$. Let $\mathscr{I}$ denote the two-sided ideal of $T(V)$ generated by $\{v\otimes v - \langle v,v\rangle \mid v\in V\}$. Then, the *Clifford algebra* of V is

$$\mathrm{Cliff}(V) := T(V)/\mathscr{I}.$$

[34] There are some infinite-dimensional subtleties, but it can be shown rigorously.

The idea is to impose the relation $v^2 = -\langle v, v\rangle$ with minimal other choices. This does not respect the $\mathbb{Z}$-grading on $T(V)$, but it does respect it mod 2, so the Clifford algebra is $\mathbb{Z}/2$-graded.

In the Clifford algebra, we have a relation

$$\frac{1}{2}(vw + wv) = -\langle v, w\rangle,$$

and therefore if $e_1, \dots, e_n$ is a basis of V, then $\mathrm{Cliff}(V)$ has a basis consisting of 1, $e_1, \dots, e_n$, e_ie_j for $i < j$, $e_ie_je_k$ for $i < j < k$, and so on. Thus it's 2^n-dimensional.

Remark. The exterior algebra $\Lambda^*(V)$ is also 2^n-dimensional, leading one to suspect it's related to the Clifford algebra. Indeed, the Clifford algebra can be interpreted as a deformation of $\Lambda^*(V)$. ◂

Definition 10.9. The *pin group* $\mathrm{Pin}(V)$ is the group of all elements $v_1v_2\dots v_m \in \mathrm{Cliff}(V)$ for which each $v_i \in V$ and $\langle v_i, v_i\rangle = 1$.[35] The *spin group* $\mathrm{Spin}(V)$ is $\mathrm{Pin}(V) \cap \mathrm{Cliff}^0(V)$.

The pin and spin groups are Lie groups, and in fact compact. There's a canonical action of $\mathrm{Spin}(V)$ on V given by

$$g \cdot v = gvg^{-1} \in V \subset \mathrm{Cliff}(V),$$

where we interpret gvg^{-1} as multiplication in $\mathrm{Cliff}(V)$. This action preserves the metric on V, so we obtain a map $\mathrm{Spin}(V) \to \mathrm{SO}(V)$. This map is in fact a double cover, and if $\dim V \geq 3$, this is the universal cover of $\mathrm{SO}(V)$.

Notationally, we will let $\mathrm{Cliff}(n)$ denote the Clifford algebra of $\mathbb{R}^n$ with the usual inner product. It is generated by $e_1, \dots, e_n$, with a relation

$$\frac{1}{2}[e_i, e_j] = -\delta_{ij}.$$

Example 10.10. In low dimensions, these are familiar objects.

- $\mathrm{Cliff}(1) \cong \mathbb{C}$, $\mathrm{Cliff}^0(1) \cong \mathbb{R}$, and $\mathrm{Spin}(1) \cong \mathbb{Z}/2$.
- $\mathrm{Cliff}(2) \cong \mathbb{H}$, $\mathrm{Cliff}^0(2) \cong \mathbb{C}$, and $\mathrm{Spin}(2) \cong \mathrm{U}(1)$.
- $\mathrm{Cliff}(3) \cong \mathbb{H} \oplus \mathbb{H}$, $\mathrm{Cliff}^0(3) \cong \mathbb{H}$, and $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$.

In higher dimensions, though, they're harder to explicitly identify with familiar objects. ◂

We'll need some topology of the spin group and some representation theory.

Definition 10.11. Fix an oriented Riemannian manifold Y. The *bundle of orthonormal frames*[36] is the principal $\mathrm{SO}(n)$-bundle $P \to Y$ whose fiber at $p \in Y$ is the space of oriented, orthonormal bases of T_pY. A *spin structure* is a lift of P to a principal $\mathrm{Spin}(n)$-bundle (across the covering map $\mathrm{Spin}(n) \to \mathrm{SO}(n)$).

Not every oriented Riemannian manifold admits a spin structure, and there may be multiple spin structures (isomorphism classes of lifts of P), but we'll say more about that later.

Example 10.12. Let $Y = S^1$ with the standard metric and orientation. Then, $\mathrm{SO}(1)$ is trivial, so $P = Y$ again. Thus a spin structure is a double cover of Y; there are two of these up to isomorphism, hence two spin structures on S^1. ◂

Lecture 11.

Clifford algebras and spin structures: 10/5/17

Last time, we defined and discussed Clifford algebras: given a real, finite-dimensional vector space V with a positive-definite inner product, one can construct its Clifford algebra $\mathrm{Cliff}(V)$, a $\mathbb{Z}/2$-graded associative algebra. Inside this, we constructed a Lie group called $\mathrm{Pin}(V)$, and the intersection of $\mathrm{Pin}(V)$ and $\mathrm{Cliff}^0(V)$ we called the spin group $\mathrm{Spin}(V)$. There's a double cover $\mathrm{Spin}(V) \to \mathrm{SO}(V)$.

Remark.

$$\pi_1(\mathrm{SO}(n)) = \begin{cases} 1, & n = 1 \\ \mathbb{Z}, & n = 2 \\ \mathbb{Z}/2, & n \geq 3. \end{cases}$$

It turns out that $\mathrm{Spin}(n)$ is connected for $n \geq 2$, and therefore is the universal cover for $\mathrm{SO}(n)$ when $n \geq 3$. ◂

[35] For $V = \mathbb{R}^n$ and the usual inner product, this group is usually denoted $\mathrm{Pin}^+(n)$; if we started with a negative definite form, we'd obtain its sibling $\mathrm{Pin}^-(n)$. For $V = \mathbb{C}^n$, there's only one kind, $\mathrm{Pin}^c(n)$.

[36] This definition sounds much scarier than it actually is!

For $V = \mathbb{R}^n$, we denoted these $\mathrm{Spin}(n)$, $\mathrm{Cliff}(n)$, etc.

If (M, g) is an oriented Riemannian manifold, it has a bundle of orthonormal frames $P \to M$, which is a principal $\mathrm{SO}(n)$-bundle; we then defined a spin structure to be a lift of P to a principal $\mathrm{Spin}(n)$-bundle $\widetilde{P}$. Two spin structures $\widetilde{P}$ and $\widetilde{P}'$ are equivalent if there's an isomorphism of principal $\mathrm{Spin}(n)$-bundles $\widetilde{P} \cong \widetilde{P}'$ that commutes with the projections back to P. In Example 10.12 we showed that the circle has two spin structures, which relates to its double covers. There's a more general fact.

Exercise 11.1. Let M be a manifold which admits a spin structure (sometimes called *spinnable*). Show that the set of spin structures on M up to equivalence is a torsor for $H^1(M;\mathbb{Z}/2)$.[37] Idea: given a spin structure Q and a double cover C, one can "twist Q by C" to obtain another spin structure $Q \otimes_{\mathbb{Z}/2} C$, and the abelian group of isomorphism classes of double covers of M is canonically identified with $H^1(M;\mathbb{Z}/2)$.

Not all manifolds are spinnable; in general, this is a codimension 2 phenomenon.

Example 11.2. Let $M = \mathbb{CP}^2$, with the Fubini-Study metric and the usual complex orientation; this manifold does not admit a spin structure![38]

Let H denote a hyperplane in $\mathbb{CP}^2$, i.e. an embedded $\mathbb{CP}^1$. Then, $T\mathbb{CP}^2$, restricted to H, is $\mathscr{O}(1) \oplus \mathscr{O}(2)$: $\mathscr{O}(2)$ is $T\mathbb{CP}^1$ and $\mathscr{O}(1)$ is the normal bundle.[39]

$\mathbb{CP}^1$ has two charts, so we can explicitly write down transition functions for $\mathscr{O}(1) \oplus \mathscr{O}(2)$, which are aps $S^1 \to \mathrm{U}(1) \times \mathrm{U}(1) \subset \mathrm{SO}(4)$. Explicitly, let $R_\theta \in \mathrm{SO}(2)$ denote the matrix which acts through rotation by θ; then one of the transition functions is

$$\theta \longmapsto \begin{pmatrix} R_\theta & \\ & R_{2\theta} \end{pmatrix}. \tag{11.3}$$

A spin structure is a lift of this map to $\mathrm{Spin}(4)$, the universal cover of $\mathrm{SO}(4)$. But the loop defined by (11.3) is the nontrivial element of $\pi_1(\mathrm{SO}(4))$, and therefore this map cannot lift to $\mathrm{Spin}(4)$. ◀

So we see that not every oriented 4-manifold is spinnable. This is minimal.

- Every compact, oriented 2-manifold Σ is spinnable. This ultimately is because $\chi(\Sigma)$ is always even.
- Every compact, oriented 3-manifold M is spinnable, and this is for a more striking reason: the tangent bundle of M is trivial!
- For 4-manifolds, we got stuck on a codimension-2 submanifold with odd intersection number. It turns out that a compact, oriented 4-manifold is spinnable iff the self-intersection number of all embedded surfaces is even.

We're stressing the embedded-loops perspective, rather than a more abstract one, because it is the way spin phenomena will appear in this course.

Proposition 11.4. *There's an irreducible, complex, $\mathbb{Z}/2$-graded representation $S = S^0 \oplus S^1$ of* $\mathrm{Cliff}(2n)$*, with* $\dim S^0 = \dim S^1 = 2^{n-1}$*. Up to isomorphism and shifting the grading, this is the unique irreducible representation of* $\mathrm{Cliff}(2n)$*.*

This in particular means there's an action of $\mathrm{Spin}(2n)$ on S^0 and S^1. For example, when $n = 1$, one can explicitly write down the matrices

$$e_1 \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad e_2 \longmapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

and check that they satisfy the Clifford relations $e_i^2 = -1$ and $e_1 e_2 = -e_2 e_1$. Under the explicitly identification $\mathrm{Cliff}(2) \cong \mathbb{H}$, this is the defining representation of $\mathbb{H}$, i.e. acting on itself by left multiplication.

We'd like to bring this theory to vector bundles over a manifold.

Definition 11.5. Let M be a $2n$-dimensional spun manifold, and let $Q \to M$ be its spin structure. Let

$$SM := Q \times_{\mathrm{Spin}(2n)} S,$$

which is called the *spinor bundle*.

[37] This means that a choice of a spin structure defines an isomorphism of abelian groups from the set of spin structures to $H^1(M;\mathbb{Z}/2)$.

[38] This fact does not depend on the choice of metric or orientation.

[39] A generic section of the normal bundle intersects itself at one point, which is the reason why the normal bundle is $\mathscr{O}(1)$; a similar argument gets you $\mathscr{O}(2)$ for the tangent bundle.

This bundle has some additional structure.

- The Levi-Civita connection induces a connection ∇ on SM.
- There's an action of the tangent bundle, i.e. a map of vector bundles $\rho\colon TM \to \mathrm{End}(SM)$, induced from the action of $\mathrm{Cliff}(n)$ on $\mathbb{R}^n$. This requires checking that this action is equivariant for the action of $\mathrm{Spin}(n)$, which follows because we more or less defined the $\mathrm{Spin}(n)$-action on $\mathbb{R}^n$ using the Clifford algebra action.

The key player in our story will be a canonical first-order differential operator. On the tangent bundle, there's no canonical first-order operator (though the Laplacian is a canonical second-order operator), so this is something new and cool.

Definition 11.6. The *Dirac operator* $\partial\!\!\!/\colon C^\infty(SM) \to C^\infty(SM)$ is defined locally by the formula

$$\partial\!\!\!/ = \sum_{i=1}^{2n} \rho(e_i) \circ \nabla_{e_i},$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal basis for TM.

This is an odd operator, interchanging S^0 and S^1.

Example 11.7. Suppose $M = \mathbb{R}^2$, so there's a unique spin structure, and SM is the trivial bundle $\underline{\mathbb{C}} \to \mathbb{R}$. Then, the Dirac operator is

$$\partial\!\!\!/ = e_1\partial_1 + e_2\partial_2 = \begin{pmatrix} 0 & \partial_1 + i\partial_2 \\ -\partial_1 + i\partial_2 & 0 \end{pmatrix}.$$

If you square it, you get

$$\partial\!\!\!/^2 = \begin{pmatrix} -\partial_1^2 - \partial_2^2 & 0 \\ 0 & -\partial_1^2 - \partial_2^2 \end{pmatrix},$$

which looks a lot like the Laplacian, but acts on the spinor bundle. ◂

Definition 11.8. Let M be an even-dimensional spin manifold.[40] Define the *spinor Laplacian* $\Delta := -\partial\!\!\!/^2\colon C^\infty(SM) \to C^\infty(SM)$.

This is an even operator, preserving S^0 and S^1. In general, the Laplacian is a self-adjoint operator, so a square root abstractly exists, but having a concrete description is part of the magic of spinor bundles.

Back to supersymmetric quantum mechanics. We've been considering a 1-dimensional supersymmetric QFT, whose space of fields is $\mathscr{C}_X = \Pi T\,\mathrm{Map}(X, Y)$, with bosonic fields $\phi\colon X \to Y$ and fermionic fields $\psi \in \Pi\Gamma(\phi^* TY)$; the action is (9.5).

We showed that there's an odd vector field Q on $\mathscr{C}_X$ and an even vector field H on $\mathscr{C}_X$ with $[Q, Q] = H$ and $QS = 0$. Using the theory of spinors we've just developed, we'll associated Q to the Dirac operator and H to the Laplacian.

Let $X = S^1(T)$. We'll try to calculate

$$Z_{S^1(T)} = \int_{\mathscr{C}_X} \mathrm{d}\phi\, \mathrm{d}\psi\, e^{-S(\phi,\psi)}$$

by discretization. The supergeometry adds some nuance, but the general story still works.

Define $\mathscr{C}_{X;N}$ be the space of piecewise geodesic paths $S^1 \to Y$, e.g. ϕ changing direction at $t_1, \ldots, t_n$, along with odd elements $\psi_i \in T_{\phi(t_i)}Y$. We'll then discretize the action, and define

$$Z_{X;N} := \int_{\mathscr{C}_{X;N}} \mathrm{d}\psi\, \mathrm{d}\phi\, e^{-S_{\mathrm{disc}}}.$$

The first apparent obstacle to writing down S_{disc} is: at a turning point, which direction do we apply the Levi-Civita connection in? There's no standard answer in the literature; what we're going to do is use the formula

$$\nabla_t = \frac{\partial}{\partial t} + A_t.$$

This relies on a choice of a local frame, but we have one around: choose the trivialization F_ϕ of $\phi^* TY$ coming from a fixed trivialization of TS^1. It will turn out that the limit of the fermionic integral as $N \to \infty$ exists, but depends on F_ϕ.

[40] There is an analogous story in odd dimensions, which we will not need; there are slightly different statements, though.

On $\mathbb{R}^n$, the fermionic piece of the discretized action $S_{\text{disc}} \in C^\infty(\mathscr{C}_{X;N})$ is

$$S_{\text{disc}} = \sum_i \psi_i^I(\psi_{i+1}^I - \psi_i^I),$$

where $I = 1, \dots, 2n$. On a general Riemannian manifold, if $A_t = \alpha\, \mathrm{d}t$, we would instead have

$$S_{\text{disc}} = \sum_i \psi_i^I\big(\psi_{i+1}^I - \psi_i^I + \alpha_J^I \psi_i^J\big).$$

This looks coordinate-dependent, but will turn out to be okay.

The space of trivializations F_ϕ of ϕ^*TY is a torsor for $\mathscr{L}\mathrm{SO}(2n)$: any two choices differ by a loop. This loop group is disconnected, and its connected components are canonically $\pi_1(\mathrm{SO}(2n))$; let τ be the generator of this group. One can show that the integral over fermions is

$$\omega(F_\phi) = -\omega(\tau F_\phi),$$

which is the precise sense in which it depends on the trivialization, which is topological.

This means we have a problem in defining Z_X: we need an extra structure on Y which picks out a "good" class of trivializations F_ϕ. This is exactly where we'll use a spin structure! Choose a spin structure $Q \to Y$; then, we can restrict to the F_ϕ which lift to the spin structure (i.e. F_ϕ maps into $\mathrm{SO}(2n)$, and we want it to lift across the map $\mathrm{Spin}(2n) \to \mathrm{SO}(2n)$). This cures the sign problem.

Example 11.9. Let's look at a toy model: let Y be a spun surface and $\phi : S^1(T) \to Y$. Let $P \to Y$ be the bundle of oriented frames, a principal SO(2)-bundle with the Levi-Civita connection; then, trivializing $E := \phi^*P$ produces a trivial principal SO(2)-bundle over S^1 with a (perhaps nontrivial) connection

$$\nabla_t = \partial_t + \alpha R,$$

where $\alpha \in \mathbb{R}$ and

$$R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then, we can make an educated guess for the (fermionic part of the) discretized action,

$$S_{\text{disc}} := \frac{1}{2}\sum_{i=1}^N \psi_i^1\psi_{i+1}^1 + \psi_i^2\psi_{i+1}^2 + \frac{\alpha T}{N}\big(\psi_i^1\psi_{i+1}^2 - \psi_i^2\psi_{i+1}^1\big). \tag{11.10}$$

Let $W = \int_{\mathscr{C}_X} e^{-S}$.

Proposition 11.11.

$$\lim_{N\to\infty} W = \frac{1}{2}\sin\!\left(\frac{1}{2}\alpha T\right).$$

This is a fermionic integral, so lots of stuff is nilpotent, and this quickly reduces to an algebraic, then a combinatorial problem. It's probably true that

$$W = \mathrm{Re}\left(1 + \frac{i\alpha T}{2N}\right)^N,$$

and as $N \to \infty$, we get $e^{i\alpha T}$. (There may be details wrong, but we'll sort them out.)

Notice that this changes by a sign under $\alpha T \mapsto \alpha T + 2\pi$, and this is the sign change that we've been concerned by. ◂

Lecture 12.

The index of the Dirac operator: 10/10/17

"Not many people know this, but you have to put your pants on one leg at a time!"

We've been studying supersymmetric quantum mechanics, a theory of "super maps" to a target Y, which is a Riemannian manifold with a spin structure. The space of fields $\mathscr{C}_X$ consists of ordinary maps $\phi : X \to Y$ as well as a fermionic part, $\psi \in \Gamma(\Pi\phi^*TY)$. To compute partition functions, we'll first integrate over fermions, then over bosons.

Remark. There's a slight extension of the story of bosonic (i.e. non-supersymmetric) quantum mechanics that we discussed earlier: in addition to the input data of a compact Riemannian manifold Y and a potential $V\colon Y\to\mathbb{R}$, suppose that one also has a vector bundle $E\to Y$ with a metric (equivalently, one could take its principal $\mathrm{O}(n)$-bundle of frames) together with a compatible connection ∇.

In this situation, one can formulate the heat equation "compatible with E," i.e. a similar-looking equation to (6.12), but for sections of E. Let $f_t\in\Gamma(Y,E)$ vary with C^2 regularity in t; then, the heat equation is

$$\frac{\partial}{\partial t}f_t+Hf_t=0, \tag{12.1}$$

where

$$\Delta=\nabla^*\nabla$$
$$H=-\frac{1}{2}\Delta+V.$$

Previously, the heat kernel (6.15) was a function on $Y\times Y$; in this situation, the heat kernel coupled to E is a section $k_t\in\Gamma(E^*\boxtimes E)$. The bundle $E^*\boxtimes E\to Y\times Y$ is the vector bundle whose fiber over (y_0,y_1) is

$$E^*_{y_0}\otimes E^*_{y_1}=\mathrm{Hom}(E_{y_0},E_{y_1}).$$

This is precisely what we need, for integrating with respect to this kernel maps sections to sections, as time evolution ought to.

Just as we did before, you can get the heat kernel coupled to E out of a one-dimensional quantum field theory: the fields are the same, and the action is the same. But we add something to the partition function

$$Z_{S^1(T)}(E)=\int_{\mathscr{C}_{S^1(T)}}\mathrm{d}\phi\ \mathrm{tr}(\mathrm{Hol}_\phi)e^{-S(\phi)}, \tag{12.2}$$

(and similarly for open boundary conditions). Here, we use the connection on E to define the holonomy Hol_ϕ generated by parallel transport for any loop $\phi\colon S^1\to Y$.[41] Coupling a theory to a bundle is a common technique in physics; if the vector bundle has rank n, this represents having n flavors of particles instead of one, and the nontriviality of the vector bundle encodes particle-particle relations.

One can rigorously show, with a proof whose sketch looks similar to how we got the heat kernel out of the untwisted case, that (12.2) is $\mathrm{tr}(e^{-HT})$, where the trace is now in the space of L^2 sections of E. ◂

We're talking about this today because we'll need it in supersymmetric quantum mechanics. When we integrate over fermions, we'll get an effective field theory of maps $X\to Y$ coupled to the spinor bundle $S\to Y$. More precisely, $S=S^0\oplus S^1$, and we'll find that

$$\begin{aligned}Z_{S^1(T),\mathrm{eff}}&=Z(S^0)-Z(S^1)\\&=\mathrm{tr}_{L^2(S^0)}(e^{-HT})-\mathrm{tr}_{L^2(S^1)}(e^{-HT}).\end{aligned}$$

This quantity is also called the *super-trace* in super-algebra: if V is a super-vector space and $A\in\mathrm{End}\,V$, the super-trace of A is

$$\mathrm{Str}_V A:=\mathrm{tr}_{V^0}A-\mathrm{tr}_{V^1}A.$$

Thus $Z_{S^1(T),\mathrm{eff}}=\mathrm{Str}_{L^2(S)}\,e^{-HT}$.

The Dirac operator $\partial\!\!\!/$ on S has a block form on S^0 and S^1:

$$\partial\!\!\!/=\begin{pmatrix}0&\partial\!\!\!/^1\\\partial\!\!\!/^0&0\end{pmatrix}.$$

Precisely, $\partial\!\!\!/^i\colon S^i\to S^{1-i}$ for $i=0,1$.

Definition 12.3. The *index* of an operator $A\colon V\to V$ on a (possibly infinite-dimensional) vector space V is $\mathrm{ind}(V):=\dim\ker(V)-\dim\mathrm{coker}(V)$.

This is only interesting on infinite-dimensional vector spaces — and even there, many bounded operators do not have finite indices.

[41]This formula (12.2) looks a lot like the expectation for the observable $\mathrm{tr}\,\mathrm{Hol}_\phi$, and indeed you can think of it as an expectation in the original theory. But this other perspective is also useful.

Remark. The Dirac operator $\not{\partial}$ is a first-order *elliptic* differential operator, meaning its principal symbol is invertible. For a more concrete example, in local coordinates the Laplacian on functions has the form

$$\Delta = -\sum_i \frac{\partial^2}{\partial {x_i}^2} + \text{lower-order terms}.$$

Its *symbol* $\sigma_\Delta : M \to \mathbb{R}$,[42] where $\pi : T^*X \to X$ is projection is obtained by replacing $\frac{\partial}{\partial x_i}$ with k_i.

Ellipticity means that σ is invertible off of the zero section. On a compact manifold, the theory of elliptic operators is awesome: the kernel and cokernel of the operator are finite-dimensional, and, no matter what kind of sections you start with, the kenrel consists of C^∞ sections! So the index of the Dirac operator is something interesting. ◁

Since the Dirac operator is self-adjoint, then $\dim \operatorname{coker} \not{\partial}^0 = \dim \ker \not{\partial}^1$.

Proposition 12.4. *The partition function*

$$Z_{S^1(T),\mathrm{eff}} = \operatorname{Str} e^{-HT} = \operatorname{ind}(\not{\partial}^0) = \dim \ker(\not{\partial}^0) - \dim \ker(\not{\partial}^1).$$

It's pretty cool that you get this quantity out of supersymmetric quantum mechanics.

Proof. Since $[\not{\partial}, \not{\partial}] = -\Delta$ (again, this is a supercommutator), we can consider $L^2(S)$ as a unitary representation of the super-Lie algebra generated by $\not{\partial}$ and Δ with that relation, which will have a nice decomposition into irreducibles.

Since Δ is central, we can diagonalize it, so it acts as a scalar E (called the energy) in each $\mathbb{Z}/2$-graded irreducible V. There are three cases.

(1) $\dim V = 1|1$, and $E > 0$. These are the states with $\psi \in V^0$ and $\not{\partial}\psi \in V^1$.
(2) $\dim V = 1|0$ and $E = 0$, for a state ψ with $\not{\partial}\psi = 0$.
(3) $\dim V = 0|1$ and $E = 0$, for a state ψ with $\not{\partial}\psi = 0$.

All irreducibles of the first kind will cancel out in the supertrace, since $\not{\partial}$ switches V^0 and V^1.

If $\Delta\psi = -E\psi$, then $E > 0$ and $\ker \Delta = \ker \not{\partial}$. This is because

$$\begin{aligned} -E\langle \psi, \psi \rangle &= \langle \psi, \Delta\psi \rangle \\ &= \langle \psi, -\Delta^2 \psi \rangle \\ &= -\langle \Delta\psi, \Delta\psi \rangle \le 0. \end{aligned}$$

Now let's see what happens to the supertrace in each of the three cases.

- For representations of type (1),
$$\operatorname{Str}_V e^{T\Delta} = e^{TE} - E^{TE} = 0.$$
- For representations of type (2),
$$\operatorname{Str}_V e^{T\Delta} = \operatorname{Str}_V 1 = 1.$$
- For representations of type (3),
$$\operatorname{Str}_V e^{T\Delta} = -\operatorname{Str}_V 1 = -1.$$

Hence we get a factor of 1 for each piece of $\ker \not{\partial}^0$ and a -1 for each piece of $\ker \not{\partial}^1$, as desired. ⊠

This proof was a toy example of something common in supersymmetry: there's an action of a super-Lie algebra on some super-vector space, and it decomposes into irreducibles, most of which cancel out.

Remark. The super-trace, and hence the partition function, doesn't depend on T! The limit $T \to \infty$ makes it clearer why Proposition 12.4 is true; the limit $T \to 0$ makes heat flow local, which is how we'll approach the index theorem. ◁

[42]More generally, for a differential operator $E \to F$, its symbol lives in $\Gamma(\pi^* \operatorname{Hom}(E, F))$.

To use this to prove interesting mathematics, we'll provide a formula for ind $\partial\!\!\!/$, called the index theorem, which relates it to topology, specifically with characteristic classes.

Let C be a symmetric function on countably many variables $\{y_i\}$. Using C, one can define a characteristic class of principal $\mathrm{SO}(2n)$-bundles $E \to X$.[43] Choose a metric on X and a compatible connection on E with curvature form $F \in \Omega^2(\mathfrak{so}_{2n})$. In local coordinates, we can block-diagonalize

$$F = \bigoplus_{i=1}^{n} \begin{pmatrix} 0 & F_i \\ -F_i & 0 \end{pmatrix},$$

where $F_i \in \Omega^2(X)$.

Now, consider the form $C(\{F_i\}/2\pi i) \in \Omega^*(X)$.

Proposition 12.5. *The cohomology class of $C(\{F_i\}/2\pi i)$ does not depends on the choice of metric or connection.*

There is something to prove here.

Example 12.6 (Pontrjagin classes)**.** Let X be a compact Riemannian manifold and $E = TX$, and let

$$C = \prod_i (1 + y_i^2).$$

The characteristic class associated to C is called the *(total) Pontrjagin class*

$$p(X) \in \Omega^*(X) = 1 + p_1(X) + p_2(X) + \cdots .$$

p_k is called the k^{th} *Pontrjagin class*. ◀

Since C is a function in y_i^2, the Pontrjagin class $p_k \in \Omega^{4k}(X)$. For example,

$$p_1(X) = -\frac{1}{4\pi^2}\operatorname{tr}(F \wedge F).$$

Remark. Strictly speaking, we've defined a representative for the Pontrjagin cohomology class in de Rham cohomology. If one works more topologically, one can define Pontrjagin classes for principal $\mathrm{O}(n)$-bundles in integer cohomology $p_k \in H^{4k}(X;\mathbb{Z})$. ◀

Example 12.7. The lowest-dimensional interesting examples are four-manifolds, where we have

$$\int_{S^4} p_1(S^4) = 0$$
$$\int_{\mathbb{CP}^2} p_1(\mathbb{CP}^2) = 3$$
$$\int_{K3} p_1(K3) = -48.$$

This already implies some interesting topology: suppose that $\mathbb{CP}^2$ had an orientation-reversing diffeomorphism. This would multiply $\int p_1(\mathbb{CP}^2)$ by -1, but since the (cohomology class of the) Pontrjagin classes of a vector bundle don't depend on the orientation, this would force $3 = -3$, which is probably not true. So we deduce that $\mathbb{CP}^2$ and $K3$ admit no orientation-reversing diffeomorphisms! That is, $\mathbb{CP}^2$ is not diffeomorphic to $\overline{\mathbb{CP}^2}$, which is kind of strange: someone living in $\mathbb{CP}^2$ would observe some inherent chirality in their universe. ◀

Remark. The Hirzebruch signature theorem in four dimensions implies that if $\dim X = 4$, $p_1(X) = 3\sigma(X)$, where $\sigma(X)$ is the signature of the intersection pairing in middle cohomology. See, for example, the computations in Example 12.7. ◀

Definition 12.8. Let X be a compact Riemannian manifold. Its *$\hat{A}$-genus*[44] $\widehat{A}(X) \in \Omega^*(X)$ is the characteristic class of TX associated to the symmetric function

$$C = \prod_i \frac{y_i/2}{\sinh(y_i/2)} = \prod_i \left(1 - \frac{y_i^2}{24} + \frac{7y_i^4}{5760} + \cdots\right).$$

[43] This construction is in fact universal, and works for n odd, but the wording is a little different.

[44] This is pronounced "A-hat."

In low dimensions, the $\widehat{A}$-genus is

$$\widehat{A}(X) = 1 - \frac{1}{24}p_1(X) + \frac{7p_1(X)^2 - 4p_2(X)}{5760} + \cdots .$$

We'll use supersymmetric quantum mechanics to derive a formula for the Dirac operator in terms of characteristic classes:

Theorem 12.9 (Atiyah-Singer)**.** *Let X be a closed spin manifold. Then,*

$$\operatorname{ind} \not{\partial}^0 = \int_X \widehat{A}(X).$$

This will explain some surprising divisibility results in the indices of Dirac operators of spin manifolds.

Lecture 13.

: 10/12/17

Though we have been studying and will continue to study supersymmetric quantum mechanics, there are many things that are called that, and ours isn't necessarily the most commonly studied one. To clarify, physicists use a number $\mathcal{N}$, which keeps track of the size of the odd part of the super-Lie algebra of symmetries. In our case, this is the super-Lie algebra generated by Q and H with $[Q, Q] = H$, so there's just one odd piece. Thus, the theory we've been studying would be called $\mathcal{N} = 1$ supersymmetric QFT.[45]

Though there are different flavors of quantum mechanics, which are all theories of maps $X \to Y$, where Y is a Rienannian manifold of any dimension and X is 1-dimensional. All of them have the same formal structure of a one-dimensional quantum field theory: there is a Hilbert space $\mathcal{H}$ associated to a point, and a Hamiltonian $H : \mathcal{H} \to \mathcal{H}$; time evolution on $[0, T]$ acts by the operator $e^{-TH} : \mathcal{H} \to \mathcal{H}$. In this way, the formal structure of a one-dimensional quantum field theory: a functor from the category of 0-dimensional Riemannian manifolds and one-dimensional Riemannian cobordisms to the category of topological vector spaces.

$\mathcal{N}$	$\mathcal{H}$	H	Q
0	$L^2(Y)$	$-(1/2)\Delta$ (Laplace-Beltrami)	n/a
1	$\Gamma_{L^2}(Y, SY)$	$-(1/2)\Delta$ (spinor Laplacian)	$\not{\partial}$
2	$\Omega_{L^2}(Y) := \Gamma_{L^2}(\Lambda^* TY)$	$-(1/2)\Delta$, $\Delta = [\mathrm{d}, \mathrm{d}^*]$	d, d^*
4	$\Omega_{L^2}(Y)$	$-(1/2)\Delta$, $\Delta = [\mathrm{d}, \mathrm{d}^*]$	∂, $\overline{\partial}$, ∂^*, $\overline{\partial}^*$

TABLE 1. Various kinds of supersymmetric quantum mechanics. The $\mathcal{N} = 4$ case requires a Kähler structure on Y; there's a similar $\mathcal{N} = 8$ case when Y is hyperkähler (i.e. Kähler with respect to two different anticommuting complex structures).

See Table 1 for some common examples of supersymmetric quantum mechanics. There are others not in the table, e.g. some for which $\mathcal{N}$ isn't a power of 2. Just as how the $\mathcal{N} = 4$ and $\mathcal{N} = 8$ cases underline interesting geometry on Y, these other models lead to new and interesting geometric structures (e.g. hyperkähler with torsion, etc.). However, the $\mathcal{N} = 2^k$ theories are usually studied the most frequently because they are the dimensional reductions of higher-dimensional theories that we want to understand.

We now return to the story in progress: we're studying $\mathcal{N} = 1$ supersymmetric quantum mechanics, whose space of fields $\mathcal{C}_X$ has bosonic terms $\phi : X \to Y$ and fermionic terms $\psi \in \Pi\Gamma(\phi^* TY)$, and a partition function

$$Z_{S^1(T)} = \int_{\mathcal{C}_{S^1(T)}} \mathrm{d}\phi \, \mathrm{d}\psi \, e^{-S(\phi,\psi)}.$$

We'd like to solve this by integrating out the fermions first, and the claim is that what you get is

$$W(\phi) = e^{-S(\phi)} \operatorname{Str}_S \operatorname{Hol}(\phi^*(TY)),$$

as we mentioned last time.

[45] Some people will call this $\mathcal{N} = 1/2$ supersymmetric QFT: be wary!

Remark. The usual method of proving this would be discretizing the path integral, but that doesn't work, because of something called the *fermion doubling problem*. This is a standard gotcha in lattice field theory: you want to take a continuum limit, but when you do, you get a theory with twice as many fermions as you started with. It should be possible to overcome this, discretize the path integral, and show that what you get agrees with the infinite-dimensional determinant ⊲

So instead we're going to use the other method we've discussed: computing an infinite-dimensional determinant. We want to compute

$$\int \mathrm{d}\psi \, e^{-(1/2)\int g(\psi,\nabla_t\psi)},$$

so we want the Pfaffian for the skew pairing on $\Gamma(\phi^*TY)$ defined by

$$(\psi_1,\psi_2)\longmapsto \int \mathrm{d}t\,\frac{1}{2}\langle\psi_1,\nabla_t\psi_2\rangle. \tag{13.1}$$

As before, the determinant of a bilinear form isn't quite a number, until we choose a metric on $\Gamma(\phi^*TY)$; you'll see where we need it in the proof.

Let's start by looking at ∇_t. Its eigenvalues are

$$\lambda_{k,\pm i} = \frac{2\pi i}{T}\left(k \pm \frac{\alpha_i}{2\pi}\right),$$

where $k\in\mathbb{Z}$ and $e^{\pm i\alpha_i}$ are the eigenvalues of the holonomy operator on $\phi^*(TY)$. You can see this by choosing a local trivialization in which

$$\nabla_t = \partial_t + \begin{pmatrix} i\alpha_1/T & & & & \\ & -i\alpha_1/T & & & \\ & & \ddots & & \\ & & & i\alpha_N/T & \\ & & & & -i\alpha_N/T \end{pmatrix}.$$

Now to get a number for the determinant, we need a metric. Let's try the one that makes Fourier modes an orthonormal basis (the naïve L^2 norm). If we do this, then

$$\det(13.1) = \prod_{k\in\mathbb{Z}}\prod_{i=1}^{n}\frac{2\pi i}{T}\left(k+\frac{\alpha_i}{2\pi}\right)\left(k-\frac{\alpha_i}{2\pi}\right),$$

which diverges. Oops.

Instead, let's take the Sobolev norm

$$\|\psi\|^2 = \int \mathrm{d}t\, g(\psi,\psi) + g(\nabla_t\psi,\nabla_t\psi).$$

Then the norm of the k^{th} Fourier mode is asymptotically about k^2, and these functions are still orthogonal. For large k, the determinant relative to this norm is

$$\det(13.1) \sim \prod_{|k|>M}\prod_{i=1}^{n}\frac{2\pi i}{T}\left(1+\frac{\alpha_i}{2\pi k}\right)\left(1-\frac{\alpha_i}{2\pi k}\right),$$

and this converges.

Therefore the determinant is an honest function of the $\alpha_i\in\mathbb{C}$ with zeros of multiplicity 2 at $\alpha_i = 2\pi k$; moreover, it's periodic under $\alpha_i\mapsto\alpha_i+2\pi$, and it's real for $\alpha_i\in\mathbb{R}$. This pins down the determinant up to a constant multiple: it must be

$$\det(13.1) \propto \prod_{i=1}^{n}\left(2\sin\left(\frac{\alpha_i}{2}\right)\right)^2.$$

This already comes to us as the square of something, so we have an obvious candidate,

$$\prod_{i=1}^{n} 2\sin\left(\frac{\alpha_i}{2}\right).$$