M390C NOTES: GEOMETRIC LANGLANDS

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These notes were taken in UT Austin's M390C (Geometric Langlands) class in Spring 2021, taught by David Ben-Zvi. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Any mistakes in the notes are my own. Thanks to Tom Gannon, Charlie Reid, and Jackson Van Dyke for finding and fixing a few errors.

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Lecture 1.

Overview and a perspective on modular forms: 1/19/21

This is a class on the geometric Langlands program from a particular perspective, incorporating its relationship to electric-magnetic duality. The class is over Zoom.

The geometric Langlands program lies halfway between number theory and physics. Maybe we are Odysseus and trying to navigate back home, between the two perils of Charybdis (physics, classically the big whirlpool) and Scylla (number theory, classically the monster). You can probably extend the analogy further, e.g. derived algebraic geometry is the Calypso islands. Extending the analogy further is left as an exercise.

There isn't any particular recommended background for this class — in particular, you don't need to know physics or number theory. Mackey [Mac80] wrote a nice overview of a perspective on the relationship between symmetry and harmonic analysis which could be fun to read. In this and the next lecture, we'll talk about modular forms and some relationships to physics; after that, we will begin the course properly: in a sense, the geometric Langlands program is a vast generalization of the Fourier transform, so we will begin with the Fourier transform, in a way that will be helpful when we do generalize.

Modular and automorphic forms, and physics We're not going to be super technical about number theory. The idea of modular and automorphic forms is to do a kind of harmonic analysis or quantum mechanics on *arithmetic locally symmetric spaces*. As an example, the upper half-plane $\mathbb{H}$ has a model as $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2$. The *modular group* $\Gamma \coloneqq \mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}$, with a fundamental domain $\Gamma\backslash\mathbb{H}$ (TODO: picture): the fundamental domain is noncompact, and goes off to infinity along the y-axis, and there are a couple of orbifold points, where the Γ-action has stabilizer.

More generally, we might consider a Lie group G with maximal compact $K \subset G$ and a lattice $\Gamma \subset G$. Then we consider the space $\Gamma\backslash G/K$ and study the space of functions on it. You might imagine a particle moving on this locally symmetric space, so we're interested in $L^2(\Gamma\backslash G/K)$, with a Laplacian Δ acting on this, and we can decompose the space of functions in terms of subspaces of eigenfunctions. This is one way in which modular forms can arise.

There are myriad variants of this. You can yeet K out of the story and study $L^2(\Gamma\backslash G)$ with its K-action. (TODO: something about the unit tangent bundle.) Plus, there's no need to restrict ourselves to linearizing with functions: you can use forms or sections of other vector bundles, such as $\Gamma(\Gamma\backslash\mathbb{H}, \omega^{\otimes k/2})$. De Rham says this is related to the cohomology of $\Gamma\backslash\mathbb{H}$, possibly with twisted coefficients. All of these variants are examples of things related to modular forms.

Now maybe you're thinking that if you pass to cohomology, you're no longer doing quantum mechanics, but in fact this is the domain of something called *topological quantum mechanics*; for example, this is discussed by Witten in his paper [Wit82] on supersymmetric quantum mechanics and its relationship to Morse theory.

Automorphic forms follow a similar story, but G is a more general Lie group. For example, pick your favorite reductive algebraic group such as GL_n or Sp_n, let G be the real points of this group, Γ be the integral points of this group, and K be the maximal compact of G. There is a long history of studying spaces of functions on $\Gamma\backslash G/K$ via harmonic analysis, and thinking of it as quantum mechanics. For example, if we started with Sp_{2n}, we get $\mathrm{Sp}_{2n}(\mathbb{Z})\backslash\mathrm{Sp}_{2n}(\mathbb{R})/\mathrm{U}_n$.

But there's a *lot* more structure here than in a typical quantum-mechanical setup. You can see this already for modular forms ($G = \mathrm{SL}_2(\mathbb{R})$). Namely, there's an additional variable: we can generalize from $\mathbb{Z}$ to other rings of integers in number fields. That is, given the field $\mathbb{Q}$, we think of $\mathbb{Z}$ as $\mathcal{O}_\mathbb{Q}$, the ring of integers of this number field, and obtain the group $\mathrm{SL}_2(\mathcal{O}_\mathbb{Q})$. Now we can replace $\mathbb{Q}$ with any finite extension $F/\mathbb{Q}$ and let $\mathcal{O}_F$ be the ring of integers of F, and consider a new lattice $\mathrm{SL}_2(\mathcal{O}_F)$. To make this completely precise, one has to fiddle with $\mathrm{SL}_2(\mathbb{R})$, since F may have more than one place at infinity, but this is the kind of technical detail we'll avoid for now.

And there is another way to vary the data: fix F, say $F = \mathbb{Q}$. Then we can vary the *conductor* or the *ramification data*. That is, the fundamental domain of Γ on $\mathbb{H}$ has a lot of covering spaces $\Gamma'\backslash\mathbb{H}$, where $\Gamma' \subset \mathrm{SL}_2(\mathbb{Z})$ is a *congruence subgroup*. One example of a congruence subgroup is, given $N \in \mathbb{Z}$, the subgroup

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \mathrm{id} \bmod N \right\}. \tag{1.1}$$

A variant is $\Gamma_0(N)$, the subgroup of matrices which are upper triangular mod N.

As is common in number theory, we can look at different places for primes in $\mathcal{O}_F$. For example, with $F = \mathbb{Q}$, this means looking at the local data at a prime p, which involves looking at $\mathrm{SL}_2(\mathbb{Q}_p)$. So the Hilbert space that we wanted to produce in the end depends on all this data: G and Γ and K, but also possibly F and the congruence subgroup and the prime.

Anyways, we'll get a Hilbert space and can study the spectral theory of the Laplacian. Maybe surprisingly, the eigenspaces are usually not one-dimensional. This "degeneracy" is because of *Hecke operators*, which are a crucial part of this story. At a high level, the Laplacian fits into a large family of commuting operators, and if p is a prime not dividing N, this family has a member T_p called the *Hecke operator*, giving an action of $\mathbb{C}[T_p]$ on $L^2(\Gamma\backslash G/K)$. And these all commute, so the tensor product of all of these $\mathbb{C}[T_p]$ over all primes acts on the Hilbert space, preserving the eigenspaces.

From the quantum mechanics perspective, this amount of commuting operators is unusual. You can think of this as an *integrable system*, with lots of conserved quantities. Usually (TODO: if I understood correctly), integrable systems are the opposite of chaos, but these arithmetic systems are studied as good examples of quantum chaos! This is a feature of the arithmetic story, and "arithmetic quantum chaos" behaves a lot more like quantum integrable systems than one might expect.

In this system, there is a special collection of measurements/states for a modular (or automorphic) form, called *periods*. One basic example is, given a modular function f on the fundamental domain $\Gamma/\backslash H$, integrate it:

$$\int_{i\mathbb{R}_+} f. \tag{1.2}$$

We will study modular functions/forms with these invariants. Hecke used L-functions to produce examples of these invariants.

Definition 1.3. A *Maass form* is an eigenfunction for the Laplacian on $L^2(\Gamma\backslash G/K)$. Specifically, *modular forms* are the holomorphic sections of $\omega^{\otimes k/2}$.

Modular forms can also arise by looking at the (twisted) cohomology of $\Gamma\backslash\mathbb{H}$; this is what's called *Eichler-Shimura theory*.

Our emphasis in this class will be more about topological quantum mechanics, rather than quantum mechanics; we care mostly about ground states. Modular forms are sort of like ground states here.

And there's one more piece of essential structure, to add to our already large pile of structures. There are these mysterious operators that allow you to vary the group! That is, these Hilbert spaces for different groups talk to each other! This is called *Langlands functoriality.* Part of the goal of this class is to explain this structure within physics.

But what the Langlands program itself does is to take these automorphic forms and spectrally decompose them in a prism (TODO: picture of the prism from Dark Side of the Moon, or maybe because this has something to do with physics, Dark Side of the Muon?). Automorphic forms enter on the left, and the prism spectrally decomposes them under the Hecke algebra (the algebra of all these commuting Hecke operators). And the different "colors" (eigenvalues) are given by representations of Galois groups of number fields, which is a surprising and magical statement. Moreover, there is a duality: these Galois representations are not into the complex points of G, but instead into $G^\vee_{\mathbb{C}}$, where $G^\vee$ is a dual group under something called Langlands duality.

For example, if $G = \mathrm{PSL}_2(\mathbb{R})$, then $G^\vee = \mathrm{SL}_2(\mathbb{C})$. A relatively explicit way to see how this enters is that if E is an elliptic curve over $\mathbb{Q}$, then $H^1(E)$ is a two-dimensional vector space carrying a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ action, and this is part of how modular forms appear in the story. This is one of the "colors" (eigenvalues) in this theory. The representation morally has image in $\mathrm{SL}_2(\mathbb{R})$, though to see this idea precisely requires setting things up a little more carefully. Elliptic curves appear in two ways in this, as the moduli space of elliptic curves is an arithmetic locally symmetric space. This isn't necessary to see the high-level overview, but it's crucial for actually proving anything! It's useful to know that elliptic curves have covers which are automorphic curves, and this provides a bridge between the two sides of the Langlands program. This is useful, but only applies for GL_2 — in general, you don't have this bridge, and the two sides are very far apart. For example, $\mathrm{GL}_3(\mathbb{Z})\backslash\mathrm{GL}_3(\mathbb{R})/\mathrm{O}_3$ is not a moduli space of anything: it's not a manifold. And this makes your proofs much harder and the duality more mysterious: why should function theory on these spaces have anything to do with Galois representations?[1]

One of the major goals of this class is to show how the (geometric) Langlands program arises in physics, not in quantum mechanics, but in four-dimensional (topological) field theory. Rather than beginning with a quantum mechanics system, we replace it with something much richer and more complicated — and scary. The key adjective "topological" helps mollify this: we throw out dynamics and look at ground states of the Laplacian, like looking only at harmonic forms rather than everything. We will try to match the structure of the Langlands program with the structure of this TFT.

Why 4? Quantum mechanics seems canonical enough, but 4d physics seems less so. We introduce another adjective, *arithmetic quantum field theory*, following the paradigm of arithmetic topology. This is an idea making an analogy between number fields and geometric objects that arise in physics, often manifolds. With a robust enough analogy, you can envision constructions with manifolds as having meaning in the world of number fields. The basic tenets of this theory are outlines in Weil's Rosetta stone (TODO: cite), which establishes a dictionary between number fields, function fields, and Riemann surfaces.

- Given a number field $F/\mathbb{Q}$, we consider $\mathrm{Spec}(\mathcal{O}_F)$, which has points labeled by primes in $\mathcal{O}_F$.
- The analogy between number fields and functional fields is older and better understood. We replace F with a (smooth, projective) curve C over a finite field $\mathbb{F}_q$. The field of rational functions $\mathbb{F}_q(C)$ on C has a lot of structure reminiscent of F, and the ring of regular functions $\mathbb{F}_q[C]$ resembles $\mathcal{O}_F$ (e.g. they're both Dedekind domains). The points of C are like the primes in $\mathcal{O}_F$.
- But why stop at $\mathbb{F}_q$? Let Σ be a compact Riemann surface. Points of Σ are the analogues of primes, in Weil's dictionary, and one can try to make geometric analogues of number-theoretic questions. The field of meromorphic functions on Σ is analogous to F and $\mathbb{F}_q(C)$, and the analogue of the ring of integers is a little complicated — Σ has no nonconstant entire functions, so we have to remove some points, analogues of points at infinity in the number field setting.

The crucial change in the arithmetic topology analogy is to replace Riemann surfaces with 3-manifolds. The reason behind this surprising change is that Riemann surfaces has strong similarities to curves over algebraically closed fields of positive characteristic. When you study a point $\mathrm{Spec}\,\mathbb{F}_q \hookrightarrow C$, you should remember the internal structure given by the Galois group action. Étale topology tells us to think of $\mathrm{Spec}(\mathbb{F}_q)$

[1] Another technical detail to not worry about: when we replace $\mathbb{C}$ with $\overline{\mathbb{Q}_\ell}$, which is necessary for making some of these things precise, one must use étale cohomology instead of singular/Zariski cohomology. But that's not crucial for the point of this lecture.

as sort of like a circle, because the étale fundamental group of $\mathrm{Spec}(\mathbb{F}_q)$ is $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \cong \widehat{\mathbb{Z}}$. The Frobenius is a topological generator of this fundamental group. So there's too much structure here to match with a Riemann surface. If you base-change to $\overline{\mathbb{F}}_q$, replacing these "circles" with their "universal covers," we obtain an extra line direction which topology/cohomology doesn't see, because it's contractible, but now we obtain something that feels a little more like a Riemann surface.

So from the point of view of Galois theory, function fields of curves over $\mathbb{F}_q$ feel less like Riemann surfaces and more like bundles of Riemann surfaces over S^1. This is equivalent data to a Riemann surface Σ and a diffeomorphism $\phi\colon \Sigma \to \Sigma$. We then build the bundle as

$$\Sigma \times [0,1]/((x,0) \sim (\phi(x),1)), \tag{1.4}$$

the construction called the *mapping torus*. On the function-field side, we think of the Frobenius as ϕ. There's a difference here, in that we don't have a canonical choice of ϕ on the Riemann surface side (the identity is boring so let's not use that one), so we think of ϕ as "generic."

And so we arrive at the arithmetic topology dictionary, built by many workers, including Mumford writing to Mazur, Mazur, Kapranov, Reznakov, Morijita, and Kim. This is also known as the "knots and primes" dictionary: number fields are analogues of 3-manifolds. This is not something totally born out of nowhere; it's a refinement of Weil's dictionary.

Just how not all number fields have unique Frobenii (Frobeniuses?), but rather different ones at different primes, our 3-manifolds Y are not just surface bundles over curves. Primes on the number field side now correspond to embedded circles in Y, i.e. knots. Local fields, such as $\mathbb{Q}_p$, look like 2-manifolds. There aren't a lot of 2-manifolds fibered over the circle, but that's okay. And there are many more aspects of the analogy, such as the relationship between Legendre symbols and linking numbers, and more. The nLab page on arithmetic topology has a great list.

This is not an incredibly precise dictionary, and don't make the mistake of trying to associate specific primes to specific knots. For example, if yous said $\mathbb{Q}$ is the sphere, then you'd discover the Poincaré conjecture is false in the number-field setting, which is unfortunate. Rather, let's imagine that number fields are a new class of examples of 3-manifolds, with some commonalities and some other properties, and function fields are another family. So we can then study our new, rich class of examples.

Returning to the question of why four-dimensional topological field theory, well, first we have to discuss exactly what a topological field theory is, but we will see that one of the basic invariants of such a creature is that to ever $(n-1)$-manifold, one obtains a vector space. So the Langlands program assigns vector spaces (or things related to it, such as graded vector spaces, or chain complexes) to function and number fields, which are 3-dimensional in our analogy, and therefore we expect a four-dimensional story in physics.

More generally, an n-dimensional quantum field theory has dynamics: you get in addition to your vector space on an $(n-1)$-manifold, a Hilbert space structure and a Hamiltonian. Again, you might have something like a chain complex instead of a vector space. But the Hamiltonian makes this more like a quantum mechanics problem on your codimension-1 manifold. In topological theories, the Hamiltonian is 0.

Now, back to locally symmetric spaces: if F is a number field, we think of the field theory as assigning to it the vector space $L^2(\Gamma_{\mathcal{O}_F}\backslash G/K)$. The space $\Gamma\backslash G/K$ does not directly appear; instead, it is a moduli space of solutions to certain relevant equations on a 3-manifold.

Other parts of the story carry over too. Turning on the conductor/ramification N, we have not just a 3-manifold, but also a knot or link inside it, which we think of as the locus along which singularities can appear. In physics, these are called "codimension-2 defects," an important piece of data in general QFT.

To recap: we went very quickly today, and will go quickly on Thursday, but the class will mostly go more slowly, starting next week, where we more carefully keep track of the structures on both sides of this story, trying to stay on the safe, geometric tightrope between these two paradigms.

Thursday we will dig into more of the physics analogues of the variables we can twiddle on the number-theoretic side: what happens if we vary the conductor, if we vary the number field F, if we play with functoriality, etc.

Lecture 2.

A tale of two TFTs: 1/21/21

Last time, we talked about a perspective on modular forms (or automorphic forms): pick your favorite reductive algebraic group or matrix group, such as GL_n or PSL_2,[2] and let $F/\mathbb{Q}$ be a number field. You can let $F = \mathbb{Q}$ if you want. Let $\mathcal{O}_F$ be the ring of integers of F; if $F = \mathbb{Q}$, $\mathcal{O}_F = \mathbb{Z}$.

Today we will discuss what happens when we vary F, and how this affects a moduli space of principal bundles (TODO: missed this). We obtained a locally symmetric space by taking the real points of the group and taking the double quotient by an "arithmetic" lattice and the maximal compact. For example, we can take $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathrm{PSL}_2(\mathbb{R})/\mathrm{SO}_2$. This is for $S = \mathbb{Q}$.

Now let's consider more general F. We have $\mathrm{PSL}_2(\mathcal{O}_F)$ with no issue, but what should replace $\mathrm{PSL}_2(\mathbb{R})$ and its maximal compact? Instead we consider $F \otimes_{\mathbb{Q}} \mathbb{R}$, which is a ring of the form

$$F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}, \tag{2.1}$$

where F has r_1 embeddings into $\mathbb{R}$ and r_2 pairs of conjugate embeddings into $\mathbb{C}$. Then we replace $\mathrm{PSL}_2(\mathbb{R})$ with $\mathrm{PSL}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$.

Example 2.2. Suppose $F = \mathbb{Q}(\sqrt{d})$, where d is squarefree.

- If $d > 0$, so this is a real quadratic extension, $r_1 = 2$ and $r_2 = 0$.
- If $d < 0$, so this is an imaginary quadratic extension, $r_1 = 0$ and $r_2 = 1$. ◀

Then we can take the maximal compact K of $\mathrm{PSL}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$ as normal, and obtain a locally symmetric space. If F is a real quadratic field, this leads us to *Hilbert modular forms*, via $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$, acting on $\mathbb{H} \times \mathbb{H}$. In the imaginary quadratic case, we get $\mathrm{PSL}_2(\mathbb{C})/\mathrm{SO}_3 \cong \mathbb{H}^3$, hyperbolic 3-space, and there are connections to hyperbolic geometry. Usually these double quotients are not algebraic varieties, as this example demonstrates.

For other F, we'll get products of $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{C})$; what's most interesting is the arithmetic lattice $\mathrm{PSL}_2(\mathcal{O}_F)$.

Once we have these arithmetic locally symmetric spaces $\mathcal{M}$, we want to produce vector spaces out of them, including $L^2(\mathcal{M})$, $H^*(\mathcal{M})$, and twisted versions thereof. Importantly for the geometric Langlands program, these vector spaces carry an action of a huge commutative algebra, which is a tensor product over all the primes in F of a polynomial ring in $\mathrm{rank}(G)$ generators.

One could also allow ramification, obtaining generalizations $\mathcal{M}_{G,F,N}$, where $N \in \mathrm{O}_F$, and you replace $\mathrm{PSL}_2(\mathcal{O}_F)$ with a congruence subgroup Γ_N in which we impose conditions on our matrices mod N. These are a few different conditions you might impose (e.g. identity mod N, or upper triangular mod N). The arithmetic locally symmetric space is $\Gamma_N\backslash G_{\mathbb{R}}/K$, and the large commutative algebra is now "only" a tensor product over the primes p not dividing N.

This kind of idea, of a vector space associated to a number field, or maybe a vector space associated to a number field and some primes, is reminiscent under the arithmetic topology analogy to the state spaces in a 4d topological field theory. As we discussed last time, this is a refinement of Weil's Rosetta stone, where $\operatorname{Spec}\mathbb{Z}$ feels like a curve with points $\operatorname{Spec}\mathbb{F}_p$ associated to primes p, and $\operatorname{Spec}\mathbb{Z}_p$ as a small disc around this point. Inside that there is the punctured disc $\operatorname{Spec}\mathbb{Q}_p$. This is analogous to having a smooth, reduced algebraic curve over a finite field $\mathbb{F}_q$, which locally looks like $\operatorname{Spec}\mathbb{F}_q[t]$. Here the points are $\operatorname{Spec}\mathbb{F}_q$, and around this is the disc $\operatorname{Spec}\mathbb{F}_q[[t]]$ with the punctured disc $\operatorname{Spec}\mathbb{F}_q((t))$.

Now we look at the étale topology of $\operatorname{Spec}\mathbb{Z}$, which is a fancy way to say we care about the cohomology of Galois groups. From this perspective, the Rosetta stone isn't quite rich enough: $\operatorname{Spec}\mathbb{F}_p \hookrightarrow \operatorname{Spec}\mathcal{O}_F$ is a point, but not étale-topologically: the étale topos tells us that this "point" has a whole bunch of interesting covering spaces, such as $\operatorname{Spec}(\mathbb{F}_{p^n}) \to \operatorname{Spec}(\mathbb{F}_p)$. Its étale fundamental group is $\widehat{\pi}_1^{\text{ét}}(\operatorname{Spec}\mathbb{F}_p) = \mathrm{Gal}(\mathbb{F}_p) \cong \widehat{\mathbb{Z}}$. You can think of this profiniteness as not really there: we can only see finite extensions or, said differently, finite covering spaces, and at this level there's no way to distinguish $\mathbb{Z}$ and $\widehat{\mathbb{Z}}$. This is a common feature of étale fundamental groups.

So the point is that the point $\operatorname{Spec}\mathbb{F}_p$ behaves a lot like a circle if you want to do étale things. (TODO: picture of $\operatorname{Spec}\mathbb{Z}$ with a circle at each prime). Therefore $\operatorname{Spec}\mathbb{Z}$, and its siblings $\operatorname{Spec}\mathcal{O}_F$, feel more like

[2] We're not thinking specifically of these groups as over a specific field, such as $\mathrm{GL}_n(\mathbb{R})$, but a machine for assigning to a field k a group $\mathrm{GL}_n(k)$. This technicality is important because F varies today.

3-manifolds than Riemann surfaces. And there are other ways to make this fuzzy analogy less fuzzy: there is a version of Poincaré duality, for example, with the correct dimension.

The monodromy around these circles is the Frobenius, but different Frobenii at different primes don't talk to each other. Because the curve $C/\mathbb{F}_q$ maps to $\operatorname{Spec}(\mathbb{F}_q)$, which is sort of like a circle, we think of these 3-manifolds as fibered over a circle.

Given this perspective, what is $\operatorname{Spec}(\overline{\mathbb{F}}_q)$? Étale-topologically, this is actually a point, but if you want a good dictionary between covering spaces and Galois representations, it should be a covering space of the circle, and in fact the universal one, $\mathbb{R}$. This is fine: for the purposes of topology and cohomology, $\mathbb{R}$ is a fine stand-in for a point. Now base-change C to $\overline{C} \coloneqq C_{\overline{\mathbb{F}}_q} \coloneqq C \times_{\operatorname{Spec}\mathbb{F}_q} \operatorname{Spec}\overline{\mathbb{F}}_q$; now we have something which feels like a bundle of Riemann surfaces, i.e. curves over algebraically closed fields, fibered over the real line $\operatorname{Spec}\overline{\mathbb{F}}_q$.

As we discussed last time, the monodromy around the circle is the Frobenius, so we can think of these surface bundle analoues as mapping tori for the Frobenius.

Let's discuss one more piece of evidence for this arithmetic topology dictionary: what happens with $\operatorname{Spec}\mathbb{Z}$? Let p be prime so we get a "circle" $\operatorname{Spec}\mathbb{F}_p$ in $\operatorname{Spec}\mathbb{Z}$. The neighborhood $\operatorname{Spec}\mathbb{Q}_p$ now behaves like a tubular neighborhood of this circle inside $\operatorname{Spec}\mathbb{Z}$. More generally, we can work with $\operatorname{Spec}\mathcal{O}_F$ and a prime $p \in F$ and a place v to complete $\mathcal{O}_F$ at, and obtain a local field F_v; then we might expect $\operatorname{Spec}(F_v)$ to be a tubular neighborhood of the circle $\operatorname{Spec} F/p$ — though (TODO) the place has to know something about p.

If F_v is a non-Archimedian local field, such as $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$, then $\operatorname{Gal}(\overline{F}_v/F_v)$ surjects onto $\mathbb{Z}_\ell \rtimes \widehat{\mathbb{Z}}$. Fun fact for those interested in group theory: this semidirect product is an example of a *Baumslag-Solitar group*

$$\mathrm{BS}(1,p) \coloneqq \langle \sigma, u \mid \sigma u \sigma = u^p \rangle. \tag{2.3}$$

Here σ is the Frobenius and u is a generator of $\mathbb{Z}_\ell$. This interpolates between $\mathbb{Z}\times\mathbb{Z} = \pi_1(T^2)$, for $p = 1$, and $p = -1$, which is π_1 of the Klein bottle. So this does feel sort of like a torus neighborhood of a knot in a 3-manifold. So primes look like circles, and local fields look like 2-manifolds — not just any 2-manifolds, but 2-manifolds fibered over the circle.

In general, the Galois group $\operatorname{Gal}(\overline{F}_v/F_v)$ can be assembled from three pieces: the Galois group of the residue field $\widehat{\mathbb{Z}}\cdot\sigma$, the *tame part*, which is a product of $\mathbb{Z}_\ell$s for $\ell \neq p$ (here p is the characteristic of the residue field), and the *wild part*, which is a p-group.

This analogy is nice and important, tying arithmetic and geometric Langlands together, but we will spend the most time in places where it is the most concrete. Let's summarize the analogy.

- The following objects are thought of as three-dimensional: number fields $\operatorname{Spec}\mathcal{O}_F$ and function fields of curves $C/\mathbb{F}_q$, and mapping tori of diffeomorphisms $\phi\colon \Sigma \to \Sigma$ of Riemann surfaces. The first two of these are related to the *global Langlands program*, and we refer to the "global arithmetic setting."
- Here are some two-dimensional objects: local fields $F_v/\mathbb{Q}_p$ and their spectra, which are lik punctured discs; and $\mathbb{F}_q((t))$, which is also sort of a punctured disc. This is the setting of the *local Langlands program* in the "local arithmetic setting". There are two more kinds of 2-dimensional objects: a curve over an algebraically closed field of positive characteristic $\overline{C}/\overline{\mathbb{F}}_q$, or a closed Riemann surface Σ. These latter two objects form the "global geometric setting."
- One-dimensional objects: $\operatorname{Spec}\mathbb{F}_q$ and $\operatorname{Spec}\mathbb{C}((t))$ both are analogues of circles. The latter is a punctured disc, so not exactly one-dimensional, but it's close enough to be useful; it is the "local geometric setting."
- And lastly, zero-dimensional objects: $\operatorname{Spec}\overline{\mathbb{F}}_q$ and $\operatorname{Spec}\mathbb{C}$.

One major theme in this class is to apply a 4d topological field theory to these objects. If you complain that there aren't any 4-manifolds, that's a good question, but we will only consider a few 4-manifolds, such as products of 3-manifolds with circles or more generally mapping tori.

The Langlands program is an equivalence of 4d "arithmetic topological field theories." Arithmetic TFTs are not an entirely well-defined object, but we have much of the data that such a definition would need. We pick a group G and get a dual group $G^\vee$; then the two arithmetic TFTs are called $\mathcal{A}_G$ and $\mathcal{B}_{G^\vee}$; $\mathcal{A}_G$ is called the *automorphic* or *magnetic side*, and $\mathcal{B}_{G^\vee}$ is called the *spectral* or *electric side*. There is a sense in which this is a 4-dimensional version of mirror symmetry (which is usually a story about 2d QFT). These two TFTs $\mathcal{A}_G$ and $\mathcal{B} + G^\vee$ are fully extended, in that they assign higher-categorical objects to lower-dimensional objects. That is, we will be able to assign things to two-, one-, and maybe zero-dimensional objects in the above dictionary: a two-manifold gets a category, a 1-manifold gets a 2-category, and if you're very ambitious, a 0-manifold gets a 3-category.

$\mathcal{A}_G$ is a machine for taking a 3-manifold M and attaching a vector space $\mathcal{A}_G(M)$, which we will see is built from functions on arithmetic locally symmetric spaces. For example, $\operatorname{Spec} \mathcal{O}_F$ gets some sort of functions on $\mathcal{M}_{G,F}$. This is a large amount of structure, and one advantage is that it will explain some of the weird properties of modular forms. We will also spend some time on the $\mathcal{B}$-side, which is easier to describe.

There's an interesting tradeoff involving dimension, category number, and difficulty: making sense of what these arithmetic TFTs assign to 4-manifolds is very difficult: there are infinities and difficult renormalizations to deal with, and analyis that is beyond the scope of the course. Vector spaces are nicer and easier to make, but 3-manifolds are difficult. Dimension 2 is the sweet spot: categories aren't that bad, and 2-manifolds are pretty tractable. By the time we get to 1-manifolds, we have to work the category theory harder, and understanding what happens in dimension 0 is almost entirely open. We will not solve this open question in this class.

Now a TFT has additional structure: you can take a manifold with some additional structure, called defects, and assign algebraic data to this too. These bells and whistles line up very nicely on the arithmetic and topological sides: for example, number theorists will tell you the importance of allowing ramification/congruence subgroups in defining your arithmetic locally symmetric spaces. Under the arithmetic topology dictionary, this corresponds to studying what your TFT assigns to a 3-manifold with an embedded link with a label. In physics language, the link is a *codimension 2 defect*. The additional data of the link gets the modified vector space built using the congruence subgroup.

The large commutative algebra acting on the space of functions on $\mathcal{M}$ contains elements called *Hecke operators*, and these correspond to codimension 3 defects, or *line operators*. Physics-wise, you can think of these as "creating magnetic monopoles." And periods of automorphic forms correspond to boundary conditions, which are a codimension 1 phenomenon. This is related to recent work and work in progress of the professor!

Finally, there is Langlands functoriality, which also fits into this picture: more than just boundaries, there are codimension 1 phenomena called *domain walls*, which you can think of as an interface between two regions on a manifold which have two different theories on them.

So to summarize this, all the bells and whistles in the theory of automorphic forms belong to this QFT story.

The $\mathcal{B}$-side. Here we've taken the theory of automorphic forms and passed it through a prism to decompose it into "colors" related to Galois representations. Number-theoretically, this side is very very hard, because Galois groups of number fields are complicated, and the $\mathcal{A}$-side is often used to gain information about the $\mathcal{B}$-side. Geometrically, fundamental groups of Riemann surfaces are much easier, so the $\mathcal{B}$-side is used to learn about the $\mathcal{A}$-side.

But at least the $\mathcal{B}$-side is easier to state: we study the algebraic geometry of spaces of Galois representations; the $\mathcal{A}$-side has to do with the topology of the arithmetic locally symmetric space $\mathcal{M}$, by contrast. Geometrically, we might fix a manifold M and consider the space $\mathrm{Loc}_n(M)$ of representations $\pi_1(M) \to \mathrm{GL}_n(\mathbb{C})$. These are nice objects, called *character varieties*, and you can study the algebra of functions on them. You also don't just have to restrict to $\mathrm{GL}_n(\mathbb{C})$: we in particular care about $\mathrm{Loc}_{G^\vee}(M) := \{\pi_1(M) \to G^\vee\}$. The vector space that $\mathcal{B}_{G^\vee}$ assigns to a 3-manifold is the vector space of functions on $\mathrm{Loc}_{G^\vee}(M)$, and in fact defining this theory as an extended TFT is considerably easier than for the $\mathcal{A}$-side. Back on the arithmetic side, $\pi_1(M)$ is analogous to a Galois group, so in the arithmetic setting, we are looking at varieties of Galois representations.

The conjectured equivalence of these two (arithmetic) TFTs is something amazing: the huge amount of structure on the $\mathcal{A}$-side is equivalent to the simpler-to-define $\mathcal{B}$-side, and all of the structure passes back and forth. But "conjectured" is a very big word here: in both the arithmetic and geometric settings, there's a lot left to do, and even to define, to make these analogies precise. In the geometric setting, more is known, but there's still plenty of work in progress, including work of the professor, Arinkin, Gaitsgory, Raskin, and many more. The number field story is the work of Lafforgue and many others, but not a lot of this is proven.

The dictionary is not just nice to look at: you can use work done in the geometric setting to learn about what you should be working towards in the arithmetic setting, for example.

Lecture 3.

Spectral theory and sheaf theory: 1/26/21

Today we begin going more slowly and deeply; in the next several lectures, we'll focus on the picture of the prism, in which automorphic forms enter in on the left and are spectrally decomposed into Galois representations. Before we get into spectral decomposition, though, what's a spectrum?

We start with geometry, which might mean different things precisely to different people, but in geometry there is some notion of spaces, and given a space there is a commutative ring of functions, with multiplication taken pointwise. This is the basic starting point for algebraic geometry: to study spaces, study their algebra of functions. In fact, we can think of taking the ring of functions as a contravariant functor $\mathcal{O}$ from spaces to commutative rings: given a map $X \to Y$ of spaces, we want a pullback map of functions which preserves pointwise addition and multiplication.

The fundamental idea of a spectrum is to go backwards: begin with commutative algebra and build a space out of it. Category theory clarifies precisely what we're trying to do: there should be a spectrum functor Spec from commutative algebra to spaces, which satisfies a universal property concisely summarized by asking for it to be a right adjoint of $\mathcal{O}$. Maybe to be completely precise, we need to say what classes of spaces and functions we care about, and there are different options, but in those different situations you have this basic question.

Explicitly, saying that Spec is a right adjoint to $\mathcal{O}$ says that for every commutative algebra R, the set of maps of spaces $X \to \operatorname{Spec} R$ is naturally isomorphic to $\mathrm{Hom}_{\mathcal{R}ing}(R, \mathcal{O}(X))$. You can produce a "weak solution" in the functor category $\mathcal{F}un(\mathcal{S}paces, \mathcal{S}et)$, where $\operatorname{Spec} R$ sends $X \mapsto \mathrm{Hom}_{\mathcal{R}ing}(R, \mathcal{O}(X))$, and we can ask whether this is a true solution, in that it's represented by an actual space. We also want $\mathcal{O}(\operatorname{Spec} R) = R$, where "=" means "naturally isomoprhic".

There are several different settings in which this works pretty well.

Example 3.1. Suppose "space" means "finite set," and "function" means k-valued functions, for your favorite field k. (My favorite field is $\mathbb{C}$. What's yours?) Now, $\mathcal{O}(X)$ is the ring of functions $\{X \to k\}$, i.e. $\prod_{x \in X} k$, or the algebra of diagonal $|X| \times |X|$ matrices. So in this setting, there are plenty of algebras, such as $k[x]/(x^2)$, which are not the k-algebras of functions on spaces. ◀

Example 3.2 (Gelfand)**.** By "spaces" we mean locally compact Hausdorff spaces and by "functions" we mean $\mathbb{C}$-valued continuous functions vanishing at infinity: $\mathcal{O}(X) \coloneqq C_0(X)$. This has the structure of a *commutative C^*-algebra*, with the $*$-operation given by complex conjugation.[3]

In this setting, there is a nice Spec functor from commutative C^*-algebras to l.c. Hausdorff spaces, the *Gelfand spectrum*. Given such an algebra A, $\mathrm{mSpec} A$ is defined to be the space of maximal ideals of A. These are identified with the set $\mathrm{Hom}_{C^*}(A, \mathbb{C})$, and this can be profitably thought of as the "points" of A: $\mathbb{C}$ is the functions on a point, so this is heuristically the maps $\mathrm{pt} \to \mathrm{mSpec} A$. Alternatively, these are the unitary one-dimensional representations of A.

Theorem 3.3 (Gelfand-Naimark)**.** $(\mathcal{O}, \mathrm{mSpec})$ *are contravariant equivalences of categories from locally compact Hausdorff spaces to commutative C^*-algebras.*

So the world of continuous topology is completely known by algebra.

Under this equivalence, compact Hausdorff spaces (i.e. the nice ones) are exchanged with *unital* commutative C^*-algebras (i.e. the nice ones). This is because the constant function with value 1 is bounded in the C^* norm iff the domain is compact. ◀

Example 3.4. As an even coarser example, we can let "spaces" mean measure spaces and "functions" mean $L^\infty(X)$, which lands in the world of commutative von Neumann algebras. There should be a few more words here to make everything precisely. There is again a contravariant equivalence of categories, and this time there's not very many isomorphism classes of objects: finite unions of points, countable unions of points, intervals, and unions of intervals and some points. ◀

Example 3.5. Algebraic geometry is the best-studied example, but it doesn't work quite as well as some of these other examples. In this case, spaces means *locally ringed spaces*, i.e. topological spaces X together with a sheaf of rings $\mathcal{O}_X$ with a property that we're not going to go into here, and rings means commutative

[3] See [aHRW09] for some discussion on this duality.

rings. Taking global sections of $\mathcal{O}_X$ defines a contravariant functor to commutative rings, and there is a right adjoint Spec, the spectrum of a ring. It is not essentially surjective: things in the image are called *affine schemes*, and on the full subcategory of affine schemes, $(\mathcal{O}, \operatorname{Spec})$ are contravariant equivalences of categories.

Unfortunately, this doesn't capture lots of important examples: locally ringed spaces which locally look like affine schemes. Lots of important objects in algebraic geometry, such as projective lines, are built out of affine schemes this way, but are not themselves affine. So you expand your notion of geometry a little bit, but from this perspective there are useful things which are weak but not strong solutions to representing a functor to sets, and at that point why not just do geometry with said weak solutions? ◂

Example 3.6 (Quillen-Sullivan rational homotopy theory)**.** In homotopy theory, you might want to study the *homotopy category*, a category built out of (locally compact weakly Hausdorff) topological spaces by inverting homotopy equivalences.[4] Here "rings" means graded commutative rings, and the functor is $H^*(-;\mathbb{Z})$. Cohomology is nice but this isn't quite flexible enough to set up a good spectral theory.

You can get a better correspondence by remembering the entire category of (locally compact weakly Hausdorff) topological spaces and letting the functor be rational cochains, which is valued in the category of commutative differential graded $\mathbb{Q}$-algebras, or CDGAs. Quillen-Sullivan showed that this defines a nice spectral theory: restricting to simply connected spaces, this defines an equivalence of categories from spaces modulo rational homotopy equivalences to simply connected $\mathbb{Q}$-CDGAs.[5]

There are analogous statements by Mandell p-adically, and by Allen Yuan [Yua19] very recently integrally, albeit using cochains with a little more structure. ◂

Remark 3.7. Most of the uses of "spectrum" in math – algebro-geometric, operator-theoretic, even mathematical-physicsy — are all related. The one exception is the homotopy theorists' spectrum, which means something different. Beware this common source of confusion. In this class, "spectrum" will mostly mean integrally. ◂

Let's get back to spectral decomposition. Let R be a commutative ring (often a k-algebra; for us, always a k-algebra, where $k = \mathbb{C}$) acting on a module (for us, a complex vector space) V. This is data of an algebra homomoprhism $R \to \operatorname{End}(V)$. Our perspective on spectral decomposition is that we want to sheafify, or localize, or spread out, or spectrally decompose this module, over $\operatorname{Spec} R$. To do this, we will use that $\mathcal{M}od_R$ is *symmetric monoidal*: we have a tensor product $\otimes_R \colon \mathcal{M}od_R \times \mathcal{M}od_R \to \mathcal{M}od_R$. Thus we can define the sheaf $\underline{V}$ associated to the module V to be

$$\underline{V}(U) \coloneqq V \otimes_R \mathcal{O}(U). \tag{3.8}$$

One immediate consequence is that you can talk about the *support* of an element $v \in V$, which is a subset $\operatorname{supp}(v) \subset X$.

For example, if X is a finite set, so $R = \prod_{x\in X} \mathbb{C}$, then

$$V = \bigoplus_{x\in X} V_x, \tag{3.9}$$

It will be useful for the sheaves produced by this construction to have a name.

Definition 3.10. A *quasicoherent sheaf* on $\operatorname{Spec} R$ is one obtained from an R-module in this way. If the R-module is finitely generated, the sheaf is called a *coherent sheaf*.

Not all schemes are affine, so we say that (quasi)coherent sheaves are those which are locally of this form.

TODO: example I missed, where the spectral decomposition is the usual one of a vector space into eigenspaces. Is this $\mathbb{C}[x]$ with x acting by the matrix in question? Or more of the finite set example? This example is a little basic: points are open and closed, and so the eigenspaces are both a hom and a tensor. This is not true in general.

Now, how does the spectral theorem appear in this context? Let V be a vector space (not necessarily finite-dimensional, though in general you need some nice topology here) and $M \in \operatorname{End}(V)$. We think of this matrix as a map of algebras: $\operatorname{Hom}_{\mathcal{S}et}(\mathrm{pt}, \operatorname{End}(V))$. Now $\operatorname{End}(V)$ has a lot of additional structure — it's an associative $\mathbb{C}$-algebra, though not commutative. So there should be an adjunction

$$\operatorname{Hom}_{\mathcal{S}et}(\mathrm{pt}, \operatorname{End}(V)) = \operatorname{Hom}_{\mathcal{A}lg_{\mathbb{C}}}(\mathrm{Free}, \operatorname{End}(V)), \tag{3.11}$$

[4] You have to do this carefully, to avoid set-theoretic issues, but it can be done.

[5] You can generalize slightly to *nilpotent* spaces, but you must have some sort of condition on π_1. This is sort of like the non-affineness in algebraic geometry, though in practice it behaves a little differently.

where Free is the free associative algebra on one generator, which is $\mathbb{C}[x]$. So rather than the matrix M, we will think about the data of V being a $\mathbb{C}[x]$-module. In some contexts, this is called "functional calculus" — once you have a matrix, you can act by polynomials in this matrix. Our approach here is to think of all of these together, rather than just M. In fact, if you have nice enough topology, you can complete, and make sense of things like functions on $\mathbb{R}$, not just polynomials.

Anyways, the idea is that $M \in \operatorname{End}(V)$ is equivalent to V being the global sections of some quasicoherent sheaf on the *affine line* $\mathbb{A}^1 := \operatorname{Spec}\mathbb{C}[x]$. This will be our basic case of spectral decomposition: the simplest case of a family of commuting operators is a single operator.

We want to study how V spreads out over $\mathbb{A}^1$. There is a short exact sequence

$$0 \longrightarrow V_{\text{tors}} \longrightarrow V \longrightarrow V_{\text{free}} \longrightarrow 0, \tag{3.12}$$

using the structure theory of modules over PIDs, and in fact this splits. So $V_{\text{free}} \cong \mathbb{C}[x]^{\oplus r}$, and the torsion part is a direct sum

$$V_{\text{tors}} \cong \bigoplus_{\lambda\in\operatorname{Spec}(V)} V_{\widehat{\lambda}}. \tag{3.13}$$

The free part we will call the *continuous spectrum*, and the torsion part the *discrete spectrum*. Specifically, if V is supported at $\lambda \in \mathbb{A}^1$, this is saying λ is a generalized eigenvalue, and $V_{\widehat{\lambda}}$ is (data equivalent to) the Jordan block for M.

Eigenvectors $Mv = xv = \lambda v$ are equivalent to elements of $\operatorname{Hom}_{\mathbb{C}[x]}(\mathbb{C}_\lambda, V)$, and this has to do with a quotient, rather than a sub (TODO: missed something here). And if you look in the continuous spectrum, there are no eigenvectors, which is to say that over $\mathbb{A}^1$, the free part looks like sections of the trivial bundle, and the discrete spectrum is skyscrapers at points. In both cases we can take fibers (quotients). (TODO: I should draw a picture of something like this.)

Remark 3.14. There are many versions of the spectral dictionary. If we talk about von Neumann algebras and measurable spaces, the corresponding spectral theorem is von Neumann's spectral theorem: $M = A$ is a self-adjoint operator on a Hilbert space V. This theorem, reinterpreted sheafily, says there's a "sheaf," i.e. a projection-valued measure on $\mathbb{R}$ ($\mathbb{R}$ is the spectrum), and the operator A can be reconstructed as a direct integral

$$A = \int_{\mathbb{R}} x \,\mathrm{d}\pi_A, \tag{3.15}$$

with respect to the projection-valued measure π_A.

To make sense of this, let's say what a projection-valued measure is. This is an assignment from every measurable subset $U \subset \mathbb{R}$ to a projection operator $\pi_A(U)$ on V. Different projections should commute. So it's sort of a sheaf of Hilbert spaces, in a particularly weak sense. You can think of the space of sections on U to be $\operatorname{Im}(\pi_A(U))$, and there is a countable additivity property that

$$U \longmapsto \langle w, \pi_A(U)(v)\rangle \tag{3.16}$$

must be a $\mathbb{C}$-valued measure on $\mathbb{R}$. The spectrum of A is the support of π_A. If V is finite-dimensional, the direct integral (3.15) is a direct sum, and is a decomposition of V into A-eigenspaces. In general, the direct integral takes the continuous spectrum into account. ◀

And there are versions with similar pictures in the other spectral settings we discussed, such as Hilbert C^*-modules for C^*-algebras, etc. There's even a homotopical version of this: if $R = C^*(X)$ for a space X, then $\mathcal{M}od_R$ injects into $\mathcal{L}oc(X)$, the category of locally constant (complexes of) sheaves on X, and a module over R, finitely generated in the right sense, has a corresponding (complex of) sheaves on X, and it is finitely generated in the same way. This has to do with the fact that $C^*(X) = \operatorname{End}(k_X)$, where k_X is the constant local system, and $\underline{M}(U) = M \otimes_{C^*(X)} C^*(X)$. There are some homotopical details to fill in here, but everything can be made precise; the point is that there is an analogue of the story here too: you have algebras R as associated to spaces $\operatorname{Spec} R$, and modules over R spread out over $\operatorname{Spec} R$.

Spectral decomposition shines where there are lots of examples of algebras, spaces, and modules for us to work with. For this reason, we turn to Fourier theory.

In this setting, we want distributions to be the linear dual $\mathrm{Hom}(V, \mathbb{C})$ to V. There are lots of weak eigenvectors, such as δ-functions, which might not be actual eigenvectors (for example, the problem with the continuous spectrum we saw above). For example, if $V = L^2(\mathbb{R})$, with the operator $M = x$, you can't make sense of "a function supported at x" in L^2-land. You can do this for distributions, though. Dually, if $M^\vee = \frac{\mathrm{d}}{\mathrm{d}x}$, we have natural eigenvectors $e^{i\lambda x}$ for λ, but these are not L^2: they live in something bigger, controlled by a different norm. This is what the continous spectrum often looks like: you have a direct integral, which is different than direct sum. The things you're integrating aren't actually subsets. For example, functions on $\mathbb{A}^1$ are functions on a point, directly assembled together, but functions at a point in $\mathbb{A}^1$ aren't a subset of functions on $\mathbb{A}^1$, but instead a quotient. This behaves a little better when there's no torsion, but that obscures the general story. And this general story isn't a weird analysis fact, because it appears for polynomials in algebra too.

Next time, we'll talk about Fourier theory, or abelian duality, from this perspective: as a natural source for commuting operators. Abelian groups G acting on vector spaces V are a good place to look for large algebras of commuting operators. We will spectrally decompose V using these operators. The aim of the class, and in some sense the broader aim of the Langlands program, is a nonabelian generalization of this, and we will spend a few weeks on the abelian story.

Lecture 4.

Some Fourier theory: 1/28/21

Today, we'll say a bit more about spectral decomposition before diving into Fourier theory.

We begin with a commutative algebra A, and build a geometric object $\operatorname{Spec} A$. What precisely these things are depends on your specific formalism, e.g. if you care about C^*-algebras and topological spaces, or commutative rings and affine schemes, or other possibilities.

Last time we saw how an A-module M "spreads out" in a spectral decomposition over $\operatorname{Spec} A$, defining a sheaf on it. You can think of this with physics: there is a physical system with an algebra A of observables and a space M of states. $\operatorname{Spec} A$ has the defining universal property that maps $A \to \mathcal{O}(X)$ are in natural bijetion with maps $X \to \operatorname{Spec} A$. In physics, you might think of making observations as a way of understanding the geometry of X, and observations might be something like functions to a line (so $\operatorname{Spec} A$ is the line here). So maybe X fibers over the line. Observables on the line now tell us something about X.

Modules and states linearize this story: we took M and sheafified it into a sheaf $\mathcal{M} \to \operatorname{Spec} A$. A single function on X is a map to $\mathbb{A}^1 = \operatorname{Spec} k[x]$, and likewise a single matrix (endomorphism) of a vector space gave us a sheaf on $\mathbb{A}^1$ via the Jordan form.

This is exactly how observations happen in quantum mechanics: we don't have a classical phase space like in classial mechanics, only its linearization, the Hilbert space $\mathcal{H}$ of states of the system. Observables are self-adjoint operators on $\mathcal{H}$; it is also useful to talk about the (noncommutative) algebra $\mathrm{End}(\mathcal{H})$ of all operators on $\mathcal{H}$. Let $\mathcal{O}$ be an observable; then, just as for a vector space and an endomorphism, we can sheafify $\mathcal{O}$ into a projection-valued measure (the analogue of a sheaf) on $\mathbb{R}$, where $\mathbb{R}$ is the spectrum of the algebra generated by a single operator.

From this perspective, a state $|\varphi\rangle \in \mathcal{H}$ is a section of the sheaf, i.e. projections of each vector onto the "fibers," which are the images of the projection-valued measures. Given a section, you can ask where it is supported, i.e. you made a measurement, where does it live? That's the support. We can also do something more precise: use the norm. Now $\|\varphi\|^2$ is a complex-valued measure on $\mathbb{R}$: take φ, project onto a fiber, and then take the norm. Suitably normalized, this is a probability measure on $\mathbb{R}$, which tells you where you expect this measurement to live, and ask questions such as what its expectation value is, as

$$\int_{-\infty}^{\infty} x \|\mathrm{proj}_{\mathcal{H}_\lambda} |\varphi\rangle\|^2 \, \mathrm{d}\lambda. \tag{4.1}$$

This is a continuous version of the fact that the expected value over a finite probability space S is a sum:

$$\frac{1}{\langle \varphi \mid \varphi \rangle} \sum_{\lambda \in S} \lambda \langle \psi_i \mid \varphi \rangle |\varphi\rangle, \tag{4.2}$$

where $\{\psi_i\}$ is an S-indexed basis of eigenvectors.

The main observable that is part of the data of a quantum mechanics system is the *Hamiltonian*, a particular self-adjoint operator H. It is the energy functional: its eigenstates are the steady states of the system, fixed by time-evolution, and their eigenvalues measure the energy for each state.

The perspective of algebras as observables, modules as spaces of states, and sheaves as spectral decomposition will come up again and again in the class, even though we will mostly see commutative algebras (which glue a lot better: you can try C^*-algebras, but Gelfand-Naimark tells us that all spaces are affine, so why glue?). The context and language will be fancier, but this philosophy will still shine through.

Let's talk about Fourier theory and abelian duality, which will carry us through the next few weeks. We will think of this as a special case of spectral decomposition, where the source of commuting operators is an abelian group G.

So, fix an abelian group G and a representation $T\colon G\to \mathrm{Aut}(V)\subset \operatorname{End} V$. We'll write that the action of $g\in G$ on V is called T_g. Often, we can cook up good examples of these representations by considering a G-action on a space X, and then taking functions on X. Then G acts by pullback. As we've said before a few times in this class, there are different levels of regularity you can do this in, and the specific one you pick isn't that important right now. If this worries you, sprinkle in the word "finite" where it helps.

The most canonical space that G acts on is itself, giving us the *regular representation*. We want to decompose, so we need to figure out what the spectrum is. If G acts on a one-dimensional vector space, we get a character $\chi\colon G\to\mathbb{C}^\times\subset\mathbb{C}$, and these are the eigenvalues that can arise.

Let $\widehat{G} := \mathrm{Hom}_{\mathcal{G}rp}(G,\mathbb{C}^\times)$ be the set of all of the characters; if you make G topological or algebraic or something, also impose that condition on the characters (e.g. continuous or smooth or polynomial). In fact, for a moment imagine G is a finite abelian group — and let's try not to think about the classification of finite abelian groups, because the story will go through in greater generality.

Because not all matrices are invertible, $\mathrm{End}(V)$ is only a monoid, and $G\to\mathrm{Aut}(V)\subset\mathrm{End}(V)$ is a monoid map. This is not a lot of structure; we want more. Specifically, $\mathrm{End}(V)$ is an algebra that we've forgotten down to a monoid. So this looks like one half of an adjunction:

$$\mathrm{Hom}_{\mathcal{M}onoid}(G,\mathrm{For}(\mathrm{End}(V))) = \mathrm{Hom}_{\mathcal{A}lg_{\mathbb{C}}}(\mathbf{?},\operatorname{End} V). \tag{4.3}$$

Filling in the **?** is the free algebra on a monoid, i.e. the *group algebra* $\mathbb{C}[G]$, which is the algebra generated by symbols δ_g for $g\in G$ with multiplication $\delta_g\cdot\delta_h=\delta_{gh}$. So you can think of this as formal linear combinations of elements of G — or you can say this is the algebra of functions $G\to\mathbb{C}$, whence the notation δ_g. And this is a suggestion that $\mathbb{C}[G]$ should really be thought of as dual to functions, as measures. The canonical counting measure on a finite set identifies functions and measures for us, but this won't generalize.

For general functions $f_1,f_2\colon G\to\mathbb{C}$, the product is convolution:

$$(f_1 * f_2)(k) = \sum_{k\in G}\left(\sum_{gh=k} f_1(g)f_2(h)\right) = \sum_{k\in G}\sum_{g\in G} f_1(g)f_2(kg^{-1}). \tag{4.4}$$

If you've studied convolution in an analysis class, this ought to look familiar. Also, so far we have not needed G to be abelian! And you can recast this in terms of pushforward and pullback.

There are three maps $G\times G\to G$: project onto the first and second factors π_1, resp. π_2, but also multiplication μ. So we can define an *external product*

$$f_1\boxtimes f_2 := \pi_1^* f_1 \pi_1^* f_2\colon G\times G\longrightarrow\mathbb{C}, \tag{4.5}$$

and because G is finite, we can push this forward along μ, summing over the fibers, and obtain the usual convolution; indeed, this is what (4.4) is telling us. If you care about infinite groups with some sort of regularity, that regularity can buy you the pushforward map in that setting.

The group algebra $\mathbb{C}[G]$ is commutative iff G is abelian, and G acting on V induces a $\mathbb{C}[G]$-action on G, more or less by that adjunction. The key fact is that

$$\mathrm{Spec}(\mathbb{C}[G],*) = \widehat{G}, \tag{4.6}$$

i.e. functions on $\widehat{G}$ are the group algebra. Why is this true? A point of the spectrum is a map $\operatorname{Spec}\mathbb{C}\to\operatorname{Spec}\mathbb{C}[G]$, so unwinds to a map $\mathbb{C}[G]\to\mathbb{C}$, under multiplication, which gives us a character.

So we have a dictionary

$$(\mathbb{C}[G], *) \xleftrightarrow{\sim} (\mathcal{O}(\widehat{G}), \cdot), \tag{4.7}$$

and characters on the left are exchanged with δ-functions on the right. Moreover, $g \in G$ acts by translation on the left, and on the right it defines a function $\widehat{g}$, and translation is exchanged with multiplication by $\widehat{g}$.

This is a form of the Fourier transform. If f is a function on G, then there is an inversion formula

$$f = \sum_{t\in\widehat{G}} \widehat{f}(t) \cdot \chi_t, \tag{4.8}$$

where $\widehat{f}(t)$ is the coefficient of f in an orthonormal basis. And $\widehat{(f)}$ corresponds to $(\widehat{f}\cdot)$. All these statements are restatements of each other, and of the key statement that the group algebra is functions on the dual with pointwise multiplication, as well as functions on G under convolution.

There is an additional symmetry: $\widehat{G}$ is not just a set, but is itself an abelian group.[6] This is because you can pointwise multiply functions on G: the product of two characters is still a character. So $\widehat{G}$ is really the dual group. For this to deserve the name "dual," we need $\widehat{\widehat{G}}$ to be naturally isomorphic to G: we want an assignment for every $g \in G$ a function from characters to $\mathbb{C}^\times$, which of course is $\chi \mapsto \chi(g)$.

There is a more symmetric way to say this: there are two projections

$$\begin{tikzcd} G \times \widehat{G} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \widehat{G} \\ G, \end{tikzcd} \tag{4.9}$$

and on $G \times \widehat{G}$, there is a canonical function

$$K(g,t) \coloneqq \chi_t(g) = \chi_g(t), \tag{4.10}$$

which you can think of as a multiplicative kernel. Here χ_g is the character on $\widehat{G}$ given by g. So this setup is more symmetric (and may also remind you of some kernels from a functional analysis class). Now we can describe the Fourier transform as a pullback-pushforward:

$$\widehat{f}(t) = \pi_{2*}(\pi_1^* f \cdot \chi)(t) = \sum_g f(g)\overline{K(g,t)}. \tag{4.11}$$

We get a complex conjugate because there was one in the inner product. And it means the inverse Fourier transform looks slightly different:

$$f(g) = \sum_t \widehat{f}(t) K(g,t). \tag{4.12}$$

Great, we've spectrally decomposed functions on G — said differently, we've simultaneously diagonalized the action of G on the space of functions on G. "Simultaneous diagonalization" is a reminder how important being abelian is to the whole story.

Now this is a lot of work for just one representation, but because all G-representations are $\mathbb{C}[G]$-representations, then for any representation $\mathbb{C}[G] \to \operatorname{End} V$, we get a spectral decomposition of V over the spectrum $\widehat{G}$, i.e.

$$V = \bigoplus_{t\in\widehat{G}} V_{\chi_t}. \tag{4.13}$$

That is, the fiber at t is the χ_t-isotypic component of V. This is sort of overkill for finite abelian groups, but generalizes.

Let G be a locally compact topological abelian group (LCA). There are lots of good examples that are not finite: $\mathbb{Z}$, U_1, $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{Z}_p$, $\mathbb{Q}_p^\times$, and many more. The dual is

$$\widehat{G} = \mathrm{Hom}_{\mathcal{T}op\mathcal{G}rp}(G, \mathrm{U}_1), \tag{4.14}$$

[6]If G is finite, the classification of finite abelian groups shows $\widehat{G}$ is noncanonically isomorphic to G. This is not true in general, so don't use it for your intuition any more than you need to.

i.e. the continuous, unitary characters of G. If G is finite, this is exactly what we had already: all characters are valued in roots of unity.[7]

It was crucial for us that the dual $\widehat{G}$ was the spectrum of the group algebra. There are different ways to implement that in the LCA setting; the way we'll do it is to endow G with a measure. Specifically, G carries a *Haar measure*, which is a left-invariant measure with respect to G acting on itself by multiplication, and is unique up to scaling. If G is compact, this measure is bi-invariant and can be normalized to total measure 1, but we will often care about noncompact groups.

Anyways, our stand-in for $\mathbb{C}[G]$ is the C^*-algebra $L^1(G)$ (with respect to a chosen Haar measure), with convolution

$$(f_1 * f_2)(h) = \int_G f_1(g) f_2(g^{-1}h)\,\mathrm{d}g. \tag{4.15}$$

You can extract this from a kernel transform just as in the finite-group case. With this definition, $\mathrm{mSpec}(L^1(G), *) \cong \widehat{G}$, not just as topological spaces, but also group structures, and this is a version of the Fourier transform. This is again a duality, often called *Pontrjagin duality*, where the natural map $G \to \widehat{\widehat{G}}$ is an isomorphism of topological abelian groups.

The Fourier transform again has the formula

$$f \longmapsto \pi_{2*}(\pi_1^* f \cdot K), \tag{4.16}$$

where K is the kernel, with the same formula as before. The thing that's new here is that you have to do analysis to see what regularity appears on the other side. For L^1, you get an isomorphism $(L^1(G), *) \cong (C_v(\widehat{G}), \cdot)$, where C_v denotes the space of continuous functions vanishing at infinity. The Plancherel theorem gives you $(L^2(G), *) \cong (L^2(\widehat{G}), \cdot)$. And for both of those, translations by group elements are exchanged with multiplication. (To say this completely precisely, you may need to work with distributions, so that you have δ-functions.) Characters are exchanged with points; this is symmetric, but in the geometric Langlands program and the related topological field theory, the two sides are not symmetric.

Example 4.17 (Fourier series)**.** Say $G = \mathrm{U}_1$. Then $\widehat{G} = \mathrm{Hom}(\mathrm{U}_1, \mathrm{U}_1)$, which can be identified with $\mathbb{Z}$ via the map

$$n \longmapsto (x \longmapsto \exp(2\pi i n x)). \tag{4.18}$$

That is, $\mathbb{Z} \cong \widehat{\mathrm{U}_1}$.

The theory of Fourier series identifies $L^2(\mathrm{U}_1) \cong L^2(\mathbb{Z})$; the latter is often called ℓ^2. One decomposes a periodic function (a function on U_1) into its Fourier modes. There are versions of this for other kinds of regularity.

To think of this as a kernel transform, there is a function on $\mathrm{U}_1 \times \mathbb{Z}$ sending

$$x, n \longmapsto \exp(2\pi i n x), \tag{4.19}$$

and characters $\mathbb{Z} \to \mathrm{U}_1$ are identified by where 1 goes, which is anywhere, and therefore $\widehat{\mathbb{Z}} = \mathrm{U}_1$. ◀

This illuminates a nice fact about Pontrjagin duality: G is compact iff $\widehat{G}$ is discrete.

One nice reference for all this is Ramakrishnan-Valenza [RV99].

Lecture 5.

Pontrjagin and Cartier duality: 2/2/21

We spent some time the other day discussing Pontrjagin duality. To review, choose a localy compact abelian group G; its dual is

$$\widehat{G} := \mathrm{Hom}_{\mathcal{A}b}(G, \mathrm{U}_1), \tag{5.1}$$

which has canonically the structure of a locally compact abelian groups. This has many properties that resemble a Fourier transform, including that L^1 functions on one side are identifed with continuous functions on the other side that vanish at infinity, as well as L^2 functions on one side passing to L^2 functions on

[7] If you care about algebraic geometry, you might be used to saying $\mathbb{G}_m$ instead of U_1. There is a version of this story for $\mathbb{G}_m$, too, but this particular kind of regularity requires U_1.

the other side. Characters on one side exchange with points on the other, and translation and convolution exchange with multiplication. G is compact iff $\widehat{G}$ is discrete (and, of course, vice versa).

Examples:

- The dual of U_1 is $\mathbb{Z}$: the characters $\mathrm{U}_1 \to \mathrm{U}_1$ are the maps $z \mapsto z^n$, indexed by $n \in \mathbb{Z}$. Correspondingly, the dual of $\mathbb{Z}$ is U_1.
- Let T be a compact torus, which is defined as a quotient $\mathbb{R}^d/\Lambda = \Lambda \otimes_{\mathbb{Z}} \mathrm{U}_1$ for a full-rank lattice $\Lambda \subset \mathbb{Z}^d$. This has an associated *dual lattice* $\Lambda^\vee \subset \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$, and the dual of T is $\Lambda^\vee$.
- $\mathbb{R}^d$ is self-dual — or, more precisely, the dual of $\mathbb{R}^d$ is $(\mathbb{R}^d)^*$. More generally, for a finite-dimensional real vector space V, the dual is V^*, the usual linear dual.

This last identification recovers the theory of the Fourier transform. Say x is the coordinate on the primal $\mathbb{R}$ and t is the coordinate on the dual $\mathbb{R}$; then the canonical character on $\mathbb{R}_x \times \mathbb{R}_t$ is $\exp(2\pi i x t)$, and when one writes down the transform as a pullback-pushforward, one recovers the usual Fourier transform.

More generally, if we began with V and obtained the Pontrjagin dual V^* (i.e. also the usual linear dual), the canonical pairing is $\exp(2\pi i\langle x, t\rangle)$, and the Fourier transform is

$$f(x) = \int_V \widehat{f}(t) e^{2\pi i \langle x,t\rangle}\, \mathrm{d}t. \tag{5.2}$$

In the Fourier transform, we know that differentiation by x (on the primal side) is exchanged with multiplication by t (on the dual side), just like translation by group elements. This makes sense: differentiation is an infinitesimal transfomation. It is a general example of how group theory on one side is exchanged with geometry on the other side.

And in general, if G is an abelian Lie group, then its Lie algebra $\mathfrak{g}$ maps to the Lie algebra of vector fields on G, which sits inside the algebra of differential operators on G, and there is an adjunction between forgetting the structure of an associative algebra to a Lie algebra and a free functor building the *universal enveloping algebra* $\mathcal{U}(\mathfrak{g})$ out of a Lie algebra $\mathfrak{g}$. Since $\mathfrak{g}$ is abelian, $\mathcal{U}(\mathfrak{g})$ is the usual symmetric algebra $\mathrm{Sym}^* \mathfrak{g}$.

By the adjunction, $\mathcal{U}(\mathfrak{g})$ acts on $C^\infty(G)$ as differential operators, and so we can sheafify over $\mathrm{Spec}(\mathcal{U}(\mathfrak{g})) = \mathfrak{g}^*$. So for $G = \mathbb{R}_x$, for example, $\mathfrak{g}^* = \mathbb{R}_t$ can be identified with $\mathrm{Spec}\,\mathbb{R}[t]$, where t *is* $\frac{\mathrm{d}}{\mathrm{d}x}$, which explains why differentiation becomes multiplication by t. So every aspect of the group theory on the primal side is simultaneously diagonalized, because everything is commutative.

Speaking of commutative things, let's talk about quantum mechanics, which is famously commutative.[8] This is a place the Fourier transform will happen. For the classical mechanics of a particle moving on a manifold M, the phase space is T^*M, the cotangent space, with local coordinates q (position) and p (momentum), with the momenta pointing in the bundle direction. These generate the algebra of observables.

In quantum mechanics, we replace T^*M with L^2 functions on half of the variables: just the positions. That is, the Hilbert space is $\mathcal{H} \coloneqq L^2(M)$, and the observables include differential operators on M, including both functions on M and tangent vectors, with $p_j = i\frac{\mathrm{d}}{\mathrm{d}q_j}$. If $\xi \in TM$ and $f\colon M \to \mathbb{R}$ is a function, there is a commutator

$$\xi f = f\xi + \hbar f'. \tag{5.3}$$

Here $\hbar$ is Planck's constant.

We want to say that "states look like the square root of the observables," which has to do with the fact that we threw out half of the observables classically in order to quantize. The Fourier transform reenters the story because we want to diagonalize the momentum operators and their derivatives. This isn't really meaningful in general, but if M is an abelian Lie group — often $\mathbb{R}^n$ — then we have a natural basis of commuting vector fields, and a Fourier transform exchanging $L^2(\mathbb{R}^n_q) \cong L^2(\mathbb{R}^n_p)$. This ultimately leads to an interesting nontrivial isomorphism between quantum mechanics on G and on $\widehat{G}$, and crucially, this isomorphism cannot be seen at the classical level. This is sort of a very simple version of abelian duality or mirror symmetry in 1d, and we will see echoes of this story in higher-dimensional, less trivial settings.

In the next few weeks, we will discuss Cartier duality, which is the algebraic version of this story, and then pass to its main manifestations in physics and in number theory: electromagnetic duality and class field theory. One takeaway will be that both of these can be thought of as kinds of Fourier transforms. These are abelian cases of the Langlands program.

[8] This is a joke.

Cartier duality is about algebraic groups. You can think of these in a few ways: things like in Lie groups but in algebraic geometry, such as group objects in the category of varieties, but it will also be helpful to think of algebraic groups through their functors of points. A variety X defines a functor $\mathcal{A}lg_k \to \mathcal{S}et$ by $X(R) \coloneqq \mathrm{Hom}(\operatorname{Spec} R, X)$: the set of "$R$-points" of X. Here $\mathcal{A}lg_k$ denotes commutative k-algebras.

If X is a group object in the category of varieties, that is equivalent structure to factoring the functor of points through $\mathcal{G}rp \to \mathcal{S}et$. That is, $\{\operatorname{Spec} R \to X\}$, the R-valued points of X, is a group, and these are compatible as R varies. And through the Yoneda lemma, X is determined by its functor of points.

Example 5.4. Consider the functor $R \mapsto \mathrm{GL}_n(R)$. This is representable, defining an algebraic group GL_n. You can play this game with other familiar groups such as SL_n. ◀

As with Pontrjagin duality, we care more about abelian algebraic groups. By the magic of the Yoneda lemma, this is the same thing as asking for the functor of points to factor through $\mathcal{A}b \hookrightarrow \mathcal{G}rp \to \mathcal{S}et$.

Example 5.5.

(1) The *additive group* $\mathbb{G}_a \coloneqq \mathbb{A}^1$, which sends an algebra R to the abelian group of functions on $\operatorname{Spec} R$.
(2) The *multiplicative group* $\mathbb{G}_m \coloneqq \mathbb{A}^1 \setminus 0$. This is $\mathrm{Aut}(k)$. (TODO: describe functor of points) This is our analogue of U_1: characters are functions to $\mathbb{G}_m$.
(3) Another slightly weirder example: the constant functor valued in $\mathbb{Z}$. ◀

Now given an abelian algebraic group G, we can let $\widehat{G} \coloneqq \mathrm{Hom}(G, \mathbb{G}_m)$, where this is homomorphisms of algebraic groups (if you like, natural transformations of $\mathcal{G}rp$-valued functors of points).

Useful examples: $\widehat{\mathbb{Z}} = \mathbb{G}_m$, just as in the topological case: a map out of $\mathbb{Z}$ is determined by where 1 goes, and 1 can go anywhere. If we started with $\mathbb{Z}/n$, we get TODO. And conversely, $\widehat{\mathbb{G}}_m = \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$.

Say Λ is a lattice in a complex vector space V. Then the dual of Λ is the dual torus $T^\vee = \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{G}_m$, where $\Lambda^* \subset V^*$ is the dual lattice. And the torus V/Λ has Cartier dual $\Lambda^\vee$. This is the analogue of Fourier series in the algebraic geometry setting.

To avoid regularity issues, we assume G is finite for now (finite flat group scheme, or spectrum of an Artinian algebra...).[9] Then $\widehat{G} = \mathrm{Spec}(\mathcal{O}(G)^*)$. The algebra $\mathcal{O}(G)$ has additional structure coming from G, namely a *coalgebra structure* given by the pullback of the multiplication map, called *comultiplication*:

$$\Delta \coloneqq \mu^* \colon \mathcal{O}(G) \longrightarrow \mathcal{O}(G) \otimes_k \mathcal{O}(G). \tag{5.6}$$

This plays the role of convolution here. There is similarly a *counit map* $\mathcal{O}(G) \to k$ arising as pullback of the inclusion of the identity $\mathrm{id}\colon \operatorname{Spec} k \to G$.

So $\mathcal{O}(G)$ has an algebra and a coalgebra structure. Do they play nice together? Yes they do! There are a few commutation relations that are satisfied and this data together gives a *commutative, cocommutative Hopf algebra*. (The abelian assumption on G is required for cocommutativity). And in fact, the theory of finite abelian group schemes over k is contravariantly equivalent to the theory of commutative, cocommutative, finite-dimensional Hopf algebras.

Duality defines an involution on the category of commutative, cocommutative, finite-dimensional Hopf algebras, and when you pass this through Spec, you get Cartier duality for finite abelian group schemes. If you want to study infinite algebraic groups, you need to worry about regularity, because duals of infinite-dimensional vector spaces are a little trickier.

Remark 5.7. Why should comultiplication be convolution? Say G is a finite abelian group, without any algebraic geometry aroud. Then $(\mathbb{C}[G], *)$ can be identified with the algebra of measures on G, and convolution is identified with the pushforward of measures under multiplication. ◀

Now let's see what happens with the Fourier transform here. As in the topological case, we expect $\mathbb{G}_a$ to pass to $\mathbb{G}_a$, but the Cartier dual of $\mathbb{G}_a$ is actually something different, the *formal completion* of $\mathbb{G}_a$, which,

[9] These technicalities arise only in positive characteristic, where you can have nonregular finite group schemes, such as the kernel of the Frobenius $\varphi\colon \mathbb{A}^1 \to \mathbb{A}^1$, which has no closed points yet is nonreduced of length p. Another (counter)example is the group scheme of the n^{th} roots of unity, which over $\mathbb{C}$ is just $\mathbb{Z}/n$ as you might expect, but in positive-characteristic can look different.

confusingly, is also written $\widehat{\mathbb{G}}_a$. For now, I will use $\widehat{\mathbb{G}}_a$ to denote the Cartier dual and $\mathbb{G}_a^\wedge$ to denote the formal completion. What this actually is is the union of all infinitesimal neighborhoods of the origin:

$$\mathbb{G}_a^\wedge = \bigcup_n \operatorname{Spec} k[t]/(t^n). \tag{5.8}$$

Let's assume k has characteristic zero, so we don't have to worry about defining the character of $\mathbb{G}_a = \operatorname{Spec} k[x]$ to be

$$e^{xt} = \sum_n \frac{(xt)^n}{n!}. \tag{5.9}$$

To allow this in algebraic geometry, this sum is required to be finite, which is why we passed to formal completions: t, the dual coordinate, is nilpotent, so this is fine.

More generally, if V is a finite-dimensional k-vector space, which defines an abelian algebraic group, its Cartier dual is $(V^*)^\wedge$, the formal completion of the dual near 0, and the duality is again given by $\exp(\langle x, t\rangle)$, which makes sense formally (i.e. in this formal neighborhood of the origin).

Spectral theory is asking about G-representations, which are identified with $\mathcal{O}(G)$-comodules, i.e. $\mathcal{O}(\widehat{G})$-modules, or quasicoherent sheaves on $\widehat{G}$. For this to be literally true G has to be finite, but there is a version of this story in general. This is the way in which Cartier duality gives us decompositions of representations.

For example, $\mathcal{R}ep(\mathbb{Z})$ is identified with $k[z, z^{-1}]$-modules: $\mathcal{O}(\mathbb{G}_m) = k[z, z^{-1}]$. So a single matrix (vector space and endomorphism) is identified with a quasicoherent sheaf on the line, i.e. on $\mathbb{A}^1 = \operatorname{Spec} k[z]$, and requiring the matrix to be invertible is saying, well, 0 cannot be an eigenvalue, so you are a sheaf on $\mathbb{A}^1 \setminus 0 = \mathbb{G}_m$, the Cartier dual of $\mathbb{Z}$ as expected.

In the other direction, a representation of $\mathbb{G}_m$ is the same thing as a $\mathbb{Z}$-graded vector space, or a sheaf on the discrete abelian group $\mathbb{Z}$. That is, if V is a $\mathbb{G}_m$-representation, we can write

$$V = \bigoplus_{n\in\mathbb{Z}} V_n, \tag{5.10}$$

where V_n is the eigenspace in which $z \in \mathbb{G}_m$ acts by z^n, and this gives us the $\mathbb{Z}$-grading. This is how the Fourier series takes a representation of the multiplicative group and spits out a sheaf on a discrete set.

Example 5.11. Here's another algebraic example of Fourier series (lattices and tori exchanged) which takes a little effort to set up, but is nice. This example is in topology. Let M be a 3-manifold and G be the Picard group of M, the group of isomorphism classes of complex line bundles on M up to isomorphism. The group structure is tensor product. We can equivalently use principal U_1-bundles, or $\pi_0(\mathrm{Map}(M, B\mathrm{U}_1))$. One can use as a model for $B\mathrm{U}_1$ the space $\mathbb{CP}^\infty$ or an Eilenberg-Mac Lane space $K(\mathbb{Z}, 2)$.

Because $B\mathrm{U}_1 = K(\mathbb{Z}, 2)$, homotopy classes of maps to $B\mathrm{U}_1$ are naturally identified with $H^2(M;\mathbb{Z})$ as abelian groups, with the map given by sending a line bundle to its first Chern class. For now, assume $H^2(M;\mathbb{Z})$ is free, though we can tell a version of this story in the presence of torsion.

Now $G \coloneqq H^2(M;\mathbb{Z})$ is abelian; let's calculate its Cartier dual. This is a dual torus

$$\begin{aligned}\Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{G}_m &= \mathrm{Hom}(H_1(M), \mathbb{G}_m)\\ &\cong \mathrm{Hom}(\pi_1(M), \mathbb{G}_m).\end{aligned}$$

This can be identified with the group of isomorphism classes of flat $\mathbb{C}^\times$-bundles on M, where given a bundle we use the flat connection to define a monodromy map $\pi_1(M) \to \mathbb{C}^\times$. So we obtain a duality between $\mathrm{Pic}(M)$ and $\mathrm{Loc}_{\mathbb{C}^\times}(M)$ — this isn't super deep, since it's just exchanging $\mathbb{Z}^n$ and $(\mathbb{C}^\times)^n$. You can make this independent of basis by replacing line bundles with T-bundles, where T is a torus. Let $T^\vee$ denote the dual torus.

Now the same story allows us to identify $\mathcal{B}un_T(M)$ and $\mathcal{L}oc_{T^\vee}(M)$, where the latter is the group of isomorphism classes of flat $T^\vee$-bundles. Choosing a basis these are $(\mathbb{C}^\times)^n$-bundles. Again a flat connection is determined by its monodromy. Both of these can be thought of as categories, but we're not using that, just thinking of the sets of isomorphism classes.

So again this is just lattice-torus Cartier duality, but it looks a little more suggestive: bundles and local systems are more geometric, and indeed this examples appears when one studied electromagnetic duality on a 3-manifold. And if you apply this idea to arithmetic topology, thinking of a number field as a 3-manifold, you get an instance of Cartier duality and you get the statement of class field theory!

This abelian duality is a four-dimensional story, and has avatars in all lower dimensions. This is part of the 3-manifold story. And in this story, you have more than just the set of isomorphism classes, but considerably more structure. ◁

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