M392C NOTES: RATIONAL HOMOTOPY THEORY

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These notes were taken in UT Austin's Math 392C (rational homotopy theory) class in Fall 2015, taught by Jonathan Campbell. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

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Lecture 1.

Postnikov Towers and Principal Fibrations: 8/27/15

First, we'll outline some aspects of the course.

$X \to Y$ is a rational equivalence if $\pi_*(X)\otimes\mathbb{Q} \xrightarrow{\cong} \pi_*(Y)\otimes\mathbb{Q}$. The goal is to define a category $\mathrm{Ho}(\mathsf{Top}_\mathbb{Q})$ where, more or less, the isomorphisms are rational equivalences. The point is that this is a purely algebraic category, equivalent to a category of differential graded algebras, $\mathrm{Ho}(\mathsf{CDGA}_\mathbb{Q})$.

The first half of the course will deal with something called Sullivan's method: we'll get our hands on rational equivalence, and produce the rationalization functor $X \mapsto X_\mathbb{Q}$. We're developing it as it "could have been done," with some computations to show that things get a lot easier over $\mathbb{Q}$ (e.g. homology of Eilenberg-MacLane spaces is the same as for spheres).

Then, we'll have to talk about model categories, which is a good way of producing homotopy categories or homotopy theories for more than just topological categories. Intuitively, a model category is a category in which one can do homotopy theory. Using this, we'll talk about the homotopy theory of commutative, differential algebras over $\mathbb{Q}$.

This isn't how it was originally done by Sullivan et al., and so we'll also discuss the classical construction. We'll also produce functors from simplicial sets to differential graded $\mathbb{Q}$-algebras and topological spaces, with adjoints and so on. One of these, turning a simplicial set into a differential graded $\mathbb{Q}$-algebra, will resemble the functor Ω^* of differential forms, but is more combinatorial.

This will enable us to prove equivalence, with all sorts of cool consequences: Whitehead products appear in the differential graded algebras category; automorphisms of CDGAs correspond to automorphisms of $\mathsf{Top}_\mathbb{Q}$, which relate to automorphisms of topological spaces nicely, and so on.

The rest of the course will discuss Quillen's model, which relates differential graded Lie algebras to rational spaces. That might not mean anything right now, and we'll have to learn a little more machinery for it. Thus, this course will cover some classical and some modern algebraic topology, making the useful notion of model categories nice and concrete.

Here are some good references for this subject.

- Griffiths-Morgan, *Rational Homotopy Theory and Differential Forms*; it's all right, and geometric (they use simplicial complexes, rather than simplicial sets, and therefore don't get as nice of a result). The second edition came out a year ago, but is similar to the first edition. The beginning has a beautiful exposition of algebraic topology in general.
- There's a GTM by Felix, Halperin, and Thomas, called *Rational Homotopy Theory*. It's pretty beefy, and the one gripe the professor has is that it doesn't use model categories at all, making things opaque. But there's definitely a bootlegged copy…
- Katherine Hess, who is a great writer, has a survey paper, about 20 pages, called *Rational Homotopy Theory*.

Those are the only expository works, but there are also some papers.

- Sullivan, "Infinitesimal Computations in Algebraic Topology." Sullivan is crazy, and the paper is very hard to read. Hopefully after the course everything is easier to read.
- Quillen, "Rational Homotopy Theory." This paper also isn't that easy to read.

There are a few other sources; things will be well cited in this class.

Now for some math.

Definition. Let X be a connected topological space. A *Postnikov tower* is a sequence

$$\begin{tikzcd} & X_n \arrow[d] \\ & \vdots \arrow[d] \\ & X_2 \arrow[d] \\ X \arrow[r] & X_1 \end{tikzcd}$$

such that

(1) there are maps $X \to X_i$,
(2) $\pi_i(X) \cong \pi_i(X_n)$ for $i \leq n$, and
(3) $\pi_i(X_n) = 0$ for $i > n$.

As a consequence of the three properties, the homotopy fiber $X_n \to X_{n-1}$ is a $K(\pi_n(X), n)$, i.e. an Eilenberg-MacLane space. In some sense, this is a "co-cellular" way of building a space out of Eilenberg-MacLane spaces.

Theorem 1.1. *Postnikov towers exist.*

The proof is easy: just attach cells to X to kill homotopy above a given degree. But that's not so useful of a characterization. We want to know: what information in stage n determines stuff in stage $(n+1)$?

To produce spaces with certain fibers, classifying maps are useful. Suppose X_{n+1} arises as a (homotopy) pullback: if $\star$ denotes a contractible space, this would look like

$$\begin{tikzcd} X_{n+1} \arrow[r] \arrow[d] & \star \arrow[d] \\ X_n \arrow[r] & K(\pi_{n+1}(X), n+2). \end{tikzcd}$$

It would be nice if fibrations with fiber $K(G, n)$ were classified by maps $X \to K(G, n+1)$, because then we could work with the cohomology group $H^{n+2}(X, \pi_{n+1}(X))$. It's not very easy to compute stuff in this cohomology group, however.

In any case, not every fibration is even classified in this manner!

Definition. A fibration $K(\pi, n) \to E \to B$ is *principal* if it arises as a pullback of a path fibration as follows.

$$\begin{tikzcd} K(\pi, n) \arrow[r, equal] \arrow[d] & K(\pi, n) \arrow[d] \\ E \arrow[r] \arrow[d] & P \arrow[d] \\ B \arrow[r] & K(\pi, n+1) \end{tikzcd}$$

There's an equivalent, less useful, formulation in the lecture notes. The reason we like our formulation is the following theorem.

Theorem 1.2. *A connected CW complex X with $\pi_1(X)$ acting trivially on $\pi_n(X)$ has a Postnikov tower composed of principal fibrations.*

As a consequence, X_{n+1} is determined from X_n by a map $k_n : X_n \to K(\pi_{n+1}(X), n+2)$; this determines a class $[k_n] \in H^{n+2}(X_n, \pi_{n+1}(X))$, called a *k-invariant*. This is why we care about Postnikov towers: they are built up nicely in stages, using cohomology classes that, in nice cases, we can compute. And so in rational homotopy theory, where the homotopy groups are nicer, the k-invariants are nicer.

We'll use spectral sequences in this class; an introduction to them can be found in the professor's lecture notes.

Another takeaway from these results is that Eilenberg-MacLane spaces are pretty fundamental building blocks. Though they have nice homotopy, their cohomology groups are generally pretty nasty, leading to computations called Steenrod operations. But rationally, there's a nice result.

Theorem 1.3.
$$H^*(K(\mathbb{Z}, n); \mathbb{Q}) = \begin{cases} \mathbb{Q}[x], & n \text{ even} \\ \Lambda_{\mathbb{Q}}(x), & n \text{ odd}, \end{cases},$$
where the generators x have degree n.

These are the simplest differential graded $\mathbb{Q}$-algebras, and suggest that all spaces' rational homotopy will be built out of them (which is true).

Proof. As a base case, $H^*(K(\mathbb{Z}, 1); \mathbb{Q}) = H^*(S^1; \mathbb{Q}) = \Lambda_{\mathbb{Q}}(x)$, which is fine. More generally, we'll use the fibration
$$K(\mathbb{Z}, n-1) \longrightarrow \star \longrightarrow K(\mathbb{Z}, n).$$
By induction, if n is odd, then $H^*(K(\mathbb{Z}, n-1); \mathbb{Q}) = \mathbb{Q}[x]$, with $\deg x = n-1$, since $n-1$ is even. Let's use the Serre spectral sequence, for which
$$E_2^{p,q} = H^p(K(\mathbb{Z}, n-1); \mathbb{Q}) \otimes H^q(K(\mathbb{Z}, n-1); \mathbb{Q}).$$
For example, when $n = 3$, we have
$$\begin{array}{ccccccc}
\mathbb{Q}y^3 & & & & & & \\
0 & & & & & & \\
\mathbb{Q}y^2 & & & & & & \\
0 & & & & & & \\
\mathbb{Q}y & 0 & 0 & \mathbb{Q}x \otimes \mathbb{Q}y & & & \\
0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & \mathbb{Q}x & H^4 & H^5 & \cdots
\end{array}$$
Here, degree increases from 0 to the right and going upwards. This also uses the Hurevitch theorem. Then, we remark that d_2, from $(0, 2)$ to $(3, 0)$, has to be an isomorphism, because the E_∞-page is 0.[1] But since the Serre spectral sequence is linear, we also have isomorphisms from $(4, 0)$ to $(2, 2)$ to $(0, 4)$; specifically, $d_3 : \mathbb{Q}y^2 \mapsto \mathbb{Q}x \otimes \mathbb{Q}y$, so $H^6(K(\mathbb{Z}, 3); \mathbb{Q}) = 0$. And this means that $x^2 = 0$. Then, we continue by induction to show that all higher $H^q(K(\mathbb{Z}, 3); \mathbb{Q})$ are zero. Thus, $H^*(K(\mathbb{Z}, 3); \mathbb{Q}) = \Lambda_{\mathbb{Q}}(x)$, and the case for general n is similar. ⊠

Exercise 1. Handle the case where n is even, which is somewhat similar.

Again, this is suggestive: Eilenberg-MacLane spaces build topological spaces up, and they have differential graded algebras for their rational cohomology groups.

Next lecture, we'll discuss Serre theory, the tricks that Serre used to compute the rational homotopy groups of the spheres. These are strong clues that, rationally, things are much nicer.

After that, we'll discuss rational equivalence, and then CDGAs and their homotopy theory, necessitating a discussion of model categories. The course will get less computational at this point.

Lecture 2.

Serre Theory: 9/1/15

"Whistle guy has really got me off my game!"

Last lecture may have gone a little fast, so we'll reboot and say some things that we didn't mean to assume. Then, we'll start Serre theory.

Definition. If G is a group, an *Eilenberg-MacLane space* $K(G, n)$ is a space whose homotopy groups are
$$\pi_i(K(G, n)) = \begin{cases} G, & i = n \\ 0, & i \neq n. \end{cases}$$

[1] Though the first few lectures will use spectral sequences, they won't be very important after that, so don't drop the course if this is the only thing making you uncomfortable.

For example, S^1 is a $K(\mathbb{Z},1)$, $\mathbb{C}P^\infty$ is a $K(\mathbb{Z},2)$, and $\mathbb{R}P^\infty$ is a $K(\mathbb{Z}/2,1)$. Also, $K(G,n-1)=\Omega K(G,n)$ (i.e. the space of loops). One can think of building spaces out of these, because they're the simplest spaces from the perspective of homotopy theory.

Another of the reasons they're important is the following theorem, which is hard to prove.

Theorem 2.1. $[X,K(G,n)]=\widetilde{H}^n(X;G)$.

In particular, the cohomolgy functor is representable.

This is why, last time, a map $k : X\to K(\pi_{n+1}(X),n+2)$ corresponded to $H^{n+2}(X;\pi_{n+1}(X))$: k-invariants arise from the representability of cohomology.

Postnikov towers are a way of using Eilenberg-MacLane spaces to build a space up, one homotopy group at a time.

Theorem 2.2. *Eilenberg-MacLane spaces exist for all n and all G, where G is abelian if $n>1$.*

Recall by the Eckmann-Hilton argument that $\pi_n(X)$ is always abelian when $n>1$.

Proof idea. When $n=1$, one can produce a fiber bundle where the total bundle is trivial and the fiber is G, with the discrete topology, thus producing a sequence $G\to EG\to BG$, and one can show that $\pi_1(BG)=G$ and $\pi_i(BG)=0$ for $i\geq 2$. This is done in chapter 1 of Hatcher.

If $n>1$, take a resolution for G:

$$\mathbb{Z}[r_\beta]\longrightarrow \mathbb{Z}[g_\alpha]\longrightarrow G,$$

where the g_α are generators and r_β are relations. Then, consider a bouquet of spheres $\bigvee_\alpha S^n_\alpha$; each relation gives a map $S^n\to\bigvee_\alpha S^n_\alpha$ using the degree of the relation, so glue cells onto this bouquet via the relations, forming pushouts of the form

$$\begin{tikzcd}
\bigvee_\beta S^n \arrow[r] \arrow[d] & \bigvee_\alpha S^n_\alpha \\
\bigvee_\beta D^n \arrow[r] & X \arrow[u, hook]
\end{tikzcd}$$

Then, the n-skeleton $X^{(n)}$ is given by the generators, and so we have an exact sequence

$$\pi_{n+1}(X,X^{(n)})\longrightarrow \pi_n(X^{(n)})\longrightarrow \pi_n(X)\longrightarrow \pi_n(X,X^{(n)}).$$

Then, $\pi_{n+1}(X,X^{(n)})=\mathbb{Z}[r_\beta]$, $\pi_n(X^{(n)})=\mathbb{Z}[g_\alpha]$, and $\pi_n(X,X^{(n)})=0$, so $\pi_n(X)=G$. There are only n-cells, so there's no lower homotopy, and we can kill all higher homotopy in a standardized manner. ⊠

Anyways, last time we talked about how $H^*(K(\mathbb{Z},n),\mathbb{Q})$ is $\mathbb{Q}[x]$ when n is even and $\Lambda_\mathbb{Q}(x)$ when n is odd; this suggests that since the constituents of spaces have simple rational homotopy, then maybe rational homotopy provides some insights.

We proved this with the Serre spectral sequence, and we'll use it a few more times to get nice short exact sequences, so let's carefully state what's going on.

Theorem 2.3 (Homological Serre sequence)**.** *Let $F\to E\to B$ be a fibration with $\pi_1(B)$ acting trivially on $H^*(F,G)$. Then, there is a spectral sequence $\{E^r_{p,q},\mathrm{d}_r\}$ such that:*

(1) $\mathrm{d}_r : E^r_{p,q}\to E^r_{p-r,q-r+1}$, i.e. d_r is of degree $(-r,r-1)$.
(2) $E^\infty_{n,n-p}=F^p_n/F^{p-1}_n$, where $F^\bullet_n$ is some filtration of $H_n(E;G)$, i.e. $H_n(E;G)=F^n_n\supset F^{n-1}_n\supset\cdots\supset F^0_n=0$.
(3) $E^2_{p,q}=H_p(B;H_q(F;G))$.

Remark. If G is a field, then Künneth gives us the nicer result that $E^2_{p,q}=H_p(B)\otimes H_q(E)$.

Theorem 2.4 (Cohomological Serre sequence)**.** *With the same setup as Theorem 2.3, there exists a spectral sequence $(E^{p,q}_r,\mathrm{d}_r)$ such that:*

(1) $\mathrm{d}_r : E^{p,q}_r\to E^{p+q,q-r+1}_r$.
(2) $E^{p,n-p}_\infty=F^n_p/F^n_{p-1}$, where $F^n_\bullet$ is some filtration of $H^n(E;G)$.
(3) $E^{p,q}_2\cong H^p(B:H^q(F;G))$.

The cohomological Serre sequence is multiplicative: there's a product that ultimately comes from the cup product. This lends some rigidity to the cohomological theory that is often very useful.

Example 2.5. Suppose we're in a case of the homological Serre spectral sequence such that $E^2_{p,q}$ for $0 < q < n$. It turns out that in situations like this, you can leverage your knowledge of the sequence to obtain useful exact sequences.

In this case, the first differential that does anything interesting is $\mathrm{d}_{n+1} : E^2_{n+1,0} \to E^2_{0,n}$ (previous differentials all map to zero). This means that $E^3_{n+1,0} = E^2_{n+1,0}/\operatorname{Im}(\mathrm{d}_{n+1})$, and nothing else hits this, so this is also $E^\infty_{n+1,0}$. Thus, we have a sequence

$$E^2_{n+1,0} \longrightarrow E^2_{0,n} \longrightarrow E^\infty_{0,n} \longrightarrow 0. \tag{9.1.1}$$

Furthermore, there is a filtration

$$H_n(E) = F^n_n \supset F^{n-1}_n \supset \cdots \supset F^0_n,$$

with $E^\infty_{p,n-p} = F^p_n/F^{p-1}_n$. In particular, $E^\infty_{0,n} = F^0_n$, $F^\infty_{1,n} = 0$, and $E^\infty_{n,0} = F^n_n/F^{n-1}_n = H_n(X)/E^\infty_{0,n}$. That is, we have a sequence

$$0 \longrightarrow E^\infty_{0,n} \longrightarrow H_n(X) \longrightarrow E^\infty_{n,0} \longrightarrow 0,$$

which we can join to (9.1.1) to produce

$$E^2_{n+1,0} \longrightarrow E^2_{0,n} \longrightarrow E^\infty_{0,n} \longrightarrow H_n(E) \longrightarrow E^\infty_{n,0} \longrightarrow 0.$$

This may seem a little contrived, but it happens, for example, when a fiber in a fibration is n-connected.

Serre Theory. This was sometimes called Serre's thesis. It will use some very abstract computations to determine the rational homotopy groups of the spheres.

Definition. Let $\mathcal{C}$ be one of the three classes (the *Serre classes*): FG of finitely generated abelian groups, $\mathcal{T}_P$, the torsion abelian groups with orders drawn from a set P, and $\mathcal{F}_P$, the finite groups in $\mathcal{T}_P$.

Lemma 2.6. *The classes $\mathcal{C}$ are closed under extension: that is, if $A, C \in \mathcal{C}$ and $0 \to A \to B \to C \to 0$ is short exact, then $B \in \mathcal{C}$. Moreover, for any $A, B \in \mathcal{C}$, $A \otimes B \in \mathcal{C}$ and $\operatorname{Tor}(A, B) \in \mathcal{C}$.*

The point is that the disgusting machinery of the Serre spectral sequence mostly leaves a Serre class intact, which will be useful for proving the Hurewicz theorem mod $\mathcal{C}$. First, though, we'll need more lemmas.

Lemma 2.7. *Let $F \to E \to B$ be a filtration satisfying the hypotheses of the homological Serre spectral sequence, and assume F, E, and B are all path-connected. If any two of $H_*(B)$, $H_*(E)$, and $H_*(F)$ are in $\mathcal{C}$, then so is the third.*

Proof. We'll show that if $H_*(B)$ and $H_*(F)$ are in $\mathcal{C}$, then $H_*(E)$ is. Recall that

$$\begin{aligned} E^2_{p,q} &= H_p(B; H_q(F)) \\ &= H_p(B;\mathbb{Z}) \otimes H_q(F;\mathbb{Z}) \oplus \operatorname{Tor}(H_{p-1}(B), H_q(F)), \end{aligned}$$

by the universal coefficient theorem. The first two terms are in $\mathcal{C}$ by assumption, and Lemma 2.6 implies the last one is. Thus, $E^2_{p,q} \in \mathcal{C}$, so all subsequent pages must be too (since homology is just kernels and images, which don't pop us out of $\mathcal{C}$). Since $H_n(X)$ has successive filtrations whose quotients are $E^\infty_{p,q}$, which are all in $\mathcal{C}$, then $H_n(E) \in \mathcal{C}$. ⊠

Lemma 2.8. *If $\pi \in \mathcal{C}$, then $H_k(K(\pi, n), \mathbb{Z}) \in \mathcal{C}$ for all k.*

At this point, there aren't many guesses for tools we can use: the Serre sequence is basically the only tool we have for homotopy groups.

Proof. Recall that we have a fibration $K(\pi, n-1) \to * \to K(\pi, n)$; we'll apply the Serre sequence. By induction, it's sufficient to consider the case $n = 1$.

For right now, we'll consider $\mathcal{C} = FG$. By Künneth, it's sufficient to show for $K(\mathbb{Z}, 1)$ and $K(\mathbb{Z}/m, 1)$, which are S^1 and lens spaces, for which this is true. ⊠

Lemma 2.9. *Let X be simply connected. Then, $\pi_n(X) \in \mathcal{C}$ implies $H_n(X) \in \mathcal{C}$.*

Remark. The converse also holds (i.e. if we know this for all n, then X must be simply connected), but we won't show that yet.

Proof of Lemma 2.9. We'll use a Postnikov tower.

$$\begin{tikzcd} K(\pi_n(X), n) \arrow[r] & X_n \arrow[d] \\ & X_{n-1} \end{tikzcd} \tag{9.1.2}$$

From Lemma 2.8, we know that $H_k(K(\pi_n(x), n), \mathbb{Z}) \in \mathcal{C}$, so once again using the homological Serre sequence, $H_*(X_n)$ is computed using $E^2_{p,q} = H_p(X_{n-1}, H_q(K(\pi_n(x), n)))$, and therefore $H_n(X_n; \mathbb{Z}) \in \mathcal{C}$. But this is $H_n(X)$. ⊠

There seems to be a duality, where things with complicated homology seem to have uncomplicated homotopy, and vice versa.

Now we can get to the reason we're doing this abstract nonsense.

In the following theorems, *isomorphic mod* $\mathcal{C}$ means that the kernel and cokernel of the map are both in $\mathcal{C}$.

Theorem 2.10 (Mod $\mathcal{C}$ Hurewicz)**.** *If* X *has* $\pi_i(X) \in \mathcal{C}$ *for* $i < n$*, then* $h : \pi_n(X) \to H_n(X)$ *is an isomorphism mod* $\mathcal{C}$*.*

Corollary 2.11. *If* $\pi_i(X) \in \mathcal{C}$ *for all* i*, then* $\pi_n(X) \to H_n(X)$ *is isomorphic mod* $\mathcal{C}$ *for all* i*.*

Corollary 2.12. *If* $\pi_i(X) \in FG$*, then* $\pi_i(X) \otimes \mathbb{Q} \to H_i(X) \otimes \mathbb{Q}$ *is an isomorphism.*

Proof of Theorem 2.10. $\pi_n(X) \to H_n(X)$ is the same as $\pi_n(X_n) \to H_n(X_n)$, where $\{X_i\}$ is the Postnikov tower. Look at the fibration (9.1.2), and use the five-term exact sequence

$$0 \longrightarrow H_{n+1}(X_{n-1}) \longrightarrow H_n(K(\pi_n(X), n)) \longrightarrow E^{\infty}_{0,n} \longrightarrow H_n(X_n) \longrightarrow H_n(X_{n-1}).$$

The composition of the middle maps is the inclusion of the fiber, so we get a four-term exact sequence

$$0 \longrightarrow H_{n+1}(X_{n-1}) \longrightarrow H_n(K(\pi_n(X), n)) \longrightarrow H_n(X_n)H_n(X_{n-1}) \longrightarrow 0.$$

By induction, the first and last terms are in $\mathcal{C}$, which is exactly what we need for the middle arrow to be an isomorphism mod $\mathcal{C}$.

Now, we can put that into the following diagram.

$$\begin{tikzcd}
H_n(K(\pi_n(X), n)) \arrow[r, "\cong"] \arrow[d, "h"', "\cong"] & \pi_n(X_n) \arrow[d] \\
H_n(K(\pi_n(X), n)) \arrow[r, "\cong \text{ mod } \mathcal{C}"'] & H_n(X_n)
\end{tikzcd}$$

The isomorphism on the left follows from the usual Hurewicz theorem, and the one on the top is from the construction of the Postnikov tower. Then, we just showed the result for the bottom arrow, so the result is that the arrow on the right is an isomorphism mod $\mathcal{C}$. ⊠

You can see the game: Postnikov tower, fibration, Serre sequence, and then hope and pray that there's an equivalence (e.g. one coarse enough such as $\mathcal{C}$).

One corollary here is that homology and homotopy don't differ much over $\mathbb{Q}$, which is pretty nice.

Lecture 3.

Rational Homotopy Groups of Spheres: 9/3/15

"I looked through my notes the other day, and I had at least five spellings of Hurewicz."

Though we'll do some significant computations with rational homotopy groups today, we'll start by clarifying a few things from last time.

Proposition 3.1. *Let* X *be a space with a Postnikov tower* $\{X_i\}$*. Then,* $H_n(X) = H_n(X_n)$*.*

Proof. There's a map $X \to X_n$; consider it as a fibration, so we get a fiber $X^{>n} \to X \to X_n$. So we can either use the long exact sequence of a fibration or the Serre spectral sequence, and the former isn't so useful here.

We have $E_2^{p,q} = H_p(X_n, H_q(X^{>n}, \mathbb{Z}))$. We know $X^{>n}$ doesn't have homotopy in degrees n or lower, and therefore it doesn't have homology there either. This implies that $E_\infty^{0,n} = H_n(X_n)$, and for $p+q = n$ otherwise, $E_n^{p,q} = 0$. Since $E_\infty^{p,n-p}$ filters $H_n(X)$ and there's only one nonzero term in the filtration, $H_n(X) = H_n(X_n)$. ⊠

This is a common technique with spectral sequences: things work because there end up being large gaps. We've even seen it a few times before.

Another loose end from last time is the following corollary of Hurewicz mod $\mathcal{C}$.

Corollary 3.2. *Suppose that* X *is a space such that* $\pi_1(X)$ *acts trivially on* X *(i.e.* X *has a Postnikov tower). Then,* $H_*(X; \mathbb{Q}) = 0$ *iff* $\pi_*(X) \otimes \mathbb{Q} = 0$*.*

Now, we may compute the rational homotopy groups of spheres. It's pretty amazing that knowledge of the homotopy of Eilenberg-MacLane spaces and the Serre spectral sequence will do it. It's really clever (well of course it is; it's Serre).

We know $\pi_n(S^n) = \mathbb{Z}$, and therefore $\pi_n(S^n) \otimes \mathbb{Q} = \mathbb{Q}$.

Theorem 3.3. *$\pi_i(S^n)$ for $i > n$ are finite except when $n = 2k$, in which case $\pi_{4k+1}(S^{2k}) = \mathbb{Z} \oplus A$ for a finite A.*

Proof. The whole thing is based on the following: consider a map $S^n \to K(\mathbb{Z}, n)$ inducing an isomorphism on π_n, i.e., this is the generator of $\pi_n(K(\mathbb{Z}, n)) = \mathbb{Z}$. As usual, we'll convert it into a fibration $F \to S^n \to K(\mathbb{Z}, n)$. Repeating, we get a fibration $K(\mathbb{Z}, n-1) \to F \to S^n$.[2] The point is, $\pi_i(F)$ and $\pi_i(S^n)$ agree when $i > n$, and F fits nicely into these two fibrations.

Obviously, we're going to use the Serre spectral sequence on this, but first we need to understand the cohomology of $K(\mathbb{Z}, n-1)$.

When n is odd, $E^2_{p,q} = H^p(S^n; \mathbb{Q}) \otimes H^q(K(\mathbb{Z}, n-1), \mathbb{Q}) = \Lambda(x) \otimes \mathbb{Q}[y]$, where $\deg x = n$ and $\deg y = n-1$. We computed this in the first lecture. This means that in row 0, there's only a $\mathbb{Q}$ at $(0,0)$ and a $\mathbb{Q}x$ at $(0,n)$; in $n-1$, we have $\mathbb{Q}y$ at column 0 and $\mathbb{Q}xy$ in column n, and then y^2 in place of y at row $2n-2$, and so on.

Since F is $(n-1)$-connected, then $H^{n-1}(F; \mathbb{Q})$ must vanish. Thus, $\mathbb{Q}y$ has to die, and this can only happen if $\mathrm{d}_{n-1} : \mathbb{Q}y \to \mathbb{Q}x$ is an isomorphism. Then, by multiplicativity, the other differentials have to be isomorphisms: if $y \mapsto x$, then $\mathbb{Q}y^2 \xrightarrow{\sim} \mathbb{Q}xy$, and so on. Thus, $H^*(F; \mathbb{Q}) = 0$, and therefore that $\pi_i(F) \otimes \mathbb{Q} = 0$ for all i. Thus, $\pi_i(F)$ (and therefore $\pi_i(S^n)$) is finite for $i > n$ when n is odd.

When n is even, then E^2 page has only $\mathbb{Q}$ at $(0,0)$, $\mathbb{Q}x$ at $(0,n)$, $\mathbb{Q}y$ at $(n-1,0)$, and $\mathbb{Q}xy$ at $(n-1,n)$. As before, $H^{n-1}(F; \mathbb{Q}) = 0$, so $\mathrm{d}_{n-1}\mathbb{Q}y \to \mathbb{Q}x$ is an isomorphis. That means that $H^*(F; \mathbb{Q}) = H^*(S^{2n-1}; \mathbb{Q})$, i.e. there's a generaotr at degree $2n-1$, and nothing else. This tells us that $\pi_{2n-1}(S^n) \otimes \mathbb{Q} = \mathbb{Q}$ by Hurewicz, and that everything below it vanishes.

Finally, we have to address $i > 2n-1$, where we want the homotopy to vanish. Let $F \to F^{<2n-1}$ be obtained by killing all homotopy above $i \geq 2n-1$. Turn this into a fibration and take the fiber to get $F^{\geq 2n-1} \to F \to F^{<2n-1}$.

We know $\pi_i(F^{<2n-1})$ is finite, since it agrees with $\pi_i(F)$ on this range, and therefore $H^*(F^{<2n-1}; \mathbb{Q}) = 0$. Using the Serre spectral sequence, we have that $H^*(F^{\geq 2n-1}; |Q) = H^*(F; \mathbb{Q}) = H^*(S^{2n-1}; \mathbb{Q})$. Now, consider $F^{\geq 2n-1} \to K(\mathbb{Z}, 2n-1)$ which induces an isomorphism mod FG on π_{2n-1}. Once again, we'll take the fiber to get $K(\mathbb{Z}, 2n-2) \to \widetilde{F} \to F^{\geq 2n-1}$, and as in the previous case, the homotopy of $\widetilde{F}$ and of $F^{\geq 2n-1}$ are equal in degrees $2n$ and above.

Once again, we use the Serre spectral sequence: $E_2^{p,q} = H^p(F^{\geq 2n-1}; \mathbb{Q}) \otimes H^q(K(\mathbb{Z}, n-2); \mathbb{Q})$, but the first part of the tensor product is $H^p(S^{2n-1}; \mathbb{Q})$, so we have exactly the same case as when n was odd. Thus, $H^*(\widetilde{F}; \mathbb{Q})$ vanishes for the same reason. ⊠

Though this proof looks frightful, what happened was that we set up the Serre spectral sequence, and then took the one fact we knew, that F is $(n-1)$-connected, and used it to set up just enough calculations to figure things out.

Again, it's incredible, even though the Serre spectral sequence is a big machine, it's human-understandable (and even admits a really quick proof with simplicial sets), and surprisigly few techniques are needed. We do need that the homotopy groups of the spheres are finitely generated, but this isn't too bad either.

Rational Spaces and Localization. Now that we've done some rational calculations and seen that they're easier than regular ones, we can see that rational equivalence might be a nicer idea (which is true; the category ends up being completely algebraic).

Almost all of the theory from here on out will deal with simply connected spaces. This is because π_1 of non-simply-connected groups might not be abelian, and so we usually can't tensor it with $\mathbb{Q}$. You can sort of do this with nilpotent groups, but we won't delve into that.

Definition. A simply connected space X is called $\mathbb{Q}$-*local* or a $\mathbb{Q}$-*space* if

(1) $\pi_*(X)$ is a $\mathbb{Q}$-vector space, or
(2) $H_*(X; \mathbb{Z})$ is a $\mathbb{Q}$-vector space.

Proposition 3.4. *The above two criteria are equivalent.*

Proof. Once again, we'll use Postnikov towers and the Serre spectral sequence. It's almost boring, but it's also interesting how useful they are.

For homology implying homotopy, start induction at $X_2 = K(\pi_2(X), 2)$. We know $H_*(X_2, \mathbb{Z})$ is a $\mathbb{Q}$-vector space, by computation of $H^*(K(\mathbb{Q}, n); \mathbb{Z})$.

The inductive step looks at the fibration

$$\begin{tikzcd} K(\pi_n(X), n) \arrow[r] & X_n \arrow[d] \\ & X_{n-1}. \end{tikzcd}$$

[2] This is because a fibration fits into a longer sequence $\cdots \to \Omega E \to \Omega B \to F \to E \to B$.

Then, $E_2^{p,q} = H_p(X_{n-1}; H_q(K(\pi_n(X), n)))$, but all of these are $\mathbb{Q}$-vector spaces, and therefore $H_*(X_n)$ is a $\mathbb{Q}$-vector space; then, take the limit.

The other direction isn't any more interesting; we assume $H_*(X;\mathbb{Z})$ is a $\mathbb{Q}$-vector space. Suppose we know that $\pi_*(X_{n-1})$ is a $\mathbb{Q}$-vector space. Then, $H_*(X_{n-1};\mathbb{Z})$ is, and so is $H_*(X, X_{n-1})$, and $\pi_n(X) = H_{n+1}(X, X_{n-1})$. ⊠

This is what we mean by $\mathbb{Q}$-local. In other words, there's only field-level information, and there's no torsion.

Definition. For "nice" topological spaces X and Y, we say that X and Y are $\mathbb{Q}$-*equivalent* if there's a map $X \to Y$ that induces either

(1) $H_*(X;\mathbb{Q}) \xrightarrow{\sim} H_*(Y;\mathbb{Q})$ or
(2) $\pi_*(X) \otimes \mathbb{Q} \xrightarrow{\sim} \pi_*(Y) \otimes \mathbb{Q}$.

Equivalently, the fiber of $X \to Y$ is $\mathbb{Q}$-*trivial* (also called $\mathbb{Q}$-*acyclic*, meaning it has no rational homology or homotopy).

We'll want to introduce the notion of a localization, producing a map from a space into a simpler topological space that preserves all the information over $\mathbb{Q}$.

Definition. A map $j : X \to X_{\mathbb{Q}}$ with $X_{\mathbb{Q}}$ a $\mathbb{Q}$-local space is a *localization* if one of the following is true:

(1) $j_* : H_*(X;\mathbb{Q}) \to H_*(X_{\mathbb{Q}};\mathbb{Q})$ is an isomorphism;
(2) $j_* : \pi_*(X) \otimes \mathbb{Q} \to \pi_*(X_{\mathbb{Q}}) \otimes \mathbb{Q}$ is an isomorphism; or
(3) j is universal in that if Y is $\mathbb{Q}$-local and $f : X \to Y$ is continuous, then there exists a unique $f' : X_{\mathbb{Q}} \to Y$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "j"'] & & Y \\ & X_{\mathbb{Q}} \arrow[ur, "f'"'] & \end{tikzcd}$$

We'll be able to construct a localization for any X by inducting on a Postnikov tower, tensoring everything with $\mathbb{Q}$.

Lemma 3.5. *Let G be an abelian group and $K(G,n)$ be an Eilenberg-MacLane space for it. Then, its localization is $K(G,n) \to K(G \otimes \mathbb{Q}, n)$.*

Proof. Follows from the definitions. ⊠

Theorem 3.6. *If X is a simply connected topological space, then there exists a localization $X \to X_{\mathbb{Q}}$.*

Proof. We'll induct on Postnikov towers; assume we've created a localization $X_{n-1} \to (X_{n-1})_{\mathbb{Q}}$, and we'd like to construct an $(X_n)_{\mathbb{Q}}$. How do we do that? We know how to construct X_n out of X_{n-1}: it fits into a homotopy fiber square

$$\begin{tikzcd} X_n \arrow[d] & \\ X_{n-1} \arrow[r, "k_n"'] & K(\pi_n(X), n+1), \end{tikzcd}$$

where k_n is a k-invariant. Compose this map with our localization of $K(\pi_n(X), n+1)$, so by the universal property of localizations, we get maps $X_{n-1} \to (X_{n-1})_{\mathbb{Q}} \to K(\pi_n(X) \otimes \mathbb{Q}, n+1)$; then, this pulls back up the fiber diagram, giving us an $(X_n)_{\mathbb{Q}}$. ⊠

At the start of the next lecture, knowing that localization of spheres exists produces a much more conceptual calculation of the homotopy groups of spheres.

Example 3.7. Chern classes give us maps $\varphi : BU \to \prod_k K(\mathbb{Z}, 2k)$; representability says that a map $BU \to K(\mathbb{Z}, 2k)$ is equivalent to a class in $H^{2k}(BU)$. Thus, computation of $H^*(BU, \mathbb{Q})$ tells us that φ^* is an isomorphism on cohomology, and therefore $BU_{\mathbb{Q}} \simeq \prod_k K(\mathbb{Q}, 2k)$. In particular, $\Omega^2 BU_{\mathbb{Q}} \simeq BU_{\mathbb{Q}}$. In other words, this is rational Bott periodicity.

Lecture 4.

Commutative Differential Graded $\mathbb{Q}$-algebras and Model Categories: 9/8/15

First, we'll talk a little bit about localization; then, we'll move to something completely different.

Proposition 4.1. *$S^{2k+1}_{\mathbb{Q}}$ is a $K(\mathbb{Q}, 2k+1)$, and there's a fibration $K(\mathbb{Q}, 4k-1) \to S^{2k}_{\mathbb{Q}} \to K(\mathbb{Q}, 2k)$.*

Once again, we have an even and an odd case.

Proof. $S^{2k+1}_{\mathbb{Q}} \to K(\mathbb{Q}, 2k+1)$ induces an isomorphism on homology, and therefore rational homotopy.

For the fibration, consider $S^{2k}_{\mathbb{Q}} \to K(\mathbb{Q}, 2k)$ acting on $H_{2k}(-)$. Then, we get a fiber $F \to S^{2k}_{\mathbb{Q}} \to K(\mathbb{Q}, 2k)$; to figure out what the fiber is, we'll use the Serre spectral sequence.

$$\begin{array}{r|cccc}
4k-1 & \mathbb{Q} & \mathbb{Q} & & \\
 & & & & \\
 & \mathbb{Q} & \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \\
\hline
 & 0 & 2k & 4k & 6k
\end{array}$$

Then, $E^2_{p,q} = H^*(F) \otimes H^*(K(\mathbb{Q}, 2k))$, so $H^*(F; \mathbb{Q}) = H^*(S^{4k-1}_{\mathbb{Q}})$. That is, F is an $M(\mathbb{Q}, 4k-1)$, but Moore spaces and Eilenberg-MacLane spaces are the same in rational homotopy. ⊠

Remark. A *Moore space* is the homology analogue of an Eilenberg-MacLane space; an $M(G, n)$ has homology equal to G in degree n and 0 otherwise.

Right now, we have lots of nice notions of equivalence in topology, including homotopy equivalence, weak homotopy equivalence, and rational homotopy equivalence. We don't have that going on for commutative differential graded algebras. The reason to introduce the abstraction of model categories is to allow for very natural notions of homotopy in categories such as this one.

Definition. A *commutative differential graded algebra* is a graded algebra A^* over a field k, i.e. $A^* = \bigoplus_{p \geq 0} A^p$ with

(1) a differential $d : A^* \to A^{*+1}$ with $d^2 = 0$ and $d(ab) = da \cdot b + (-1)^{\deg a} a \cdot db$, and
(2) a multiplication $A^p \otimes A^q \to A^{p+q}$ such that $ab = (-1)^{\deg a \deg b} ba$.

You could formalize this as a chain complex, but we've chosen cochain complexes. The idea is that the multiplication and differential should both respect the grading, and otherwise it's exactly what the name suggests. Moreover, a morphism of CDGAs is a morphism of k-algebras commiting with the differential and grading. Thus, we get a category of k-CDGAs.

In our case, of course, we'll usually take $k = \mathbb{Q}$. In this case, we'll more or less get a categorical equivalence between this category and the category of rational homotopy types, which is incredible!

Example 4.2.

(1) An important example over $\mathbb{R}$ is $\Omega^*(X; \mathbb{R})$ when X is a manifold, the algebra of differential forms. Here, multiplication is given by the wedge product.
(2) Over $\mathbb{Q}$, we could take $\mathbb{Q}[x] \otimes \Lambda(y)$, where $dx = y$, $\deg x = 1$, and $\deg y = 2$. Then, $d(x^n y) = 0$, which you can check, and that $H^*(-) = k$ (i.e. taken with d), and this field k has degree 0. This algebra is, in some mysterious sense we'll clarify, "the interval;" notice at least that it has similar cohomology.
(3) The Koszul complex $K(x_1, \ldots, x_n) = \Lambda(x_1) \otimes \cdots \otimes \Lambda(x_n)$, with a complicated mess of differentials.
(4) Another great example, but one that is important, is cohomology $H^*(X; \mathbb{Q})$ with trivial differential. This is important because equivalences between CDGAs will be quasi-isomorphisms, which can send nontrivial differentials to trivial ones. Things quasi-isomorphic to CDGAs with trivial differential are called *formal*, and include important classes such as the rational homology of Kähler manifolds.

A significant non-example is $C^*(X; \mathbb{Q})$. This is because multiplication isn't commutative. In some cases, e.g. finite fields, Steenrod operations have nice structure, and one generally hopes that cochains are a good representative for your space, containing lots of information about your space; so it would've been nice if there was a commutative structure akin to $\Omega^*(M; \mathbb{R})$ for manifolds M. Thus, it doesn't work, but its failure to work is motivational.

Moreover, there isn't really a "free" CDGA, only semi-free ones; the issue is that you cannot freely choose the differential. But given a graded vector space V^*, the semi-free CDGA is

$$F(V^*) = k[V_{\mathrm{even}}] \otimes \Lambda(V_{\mathrm{odd}}).$$

We have a forgetful functor from CDGAs over $\mathbb{Q}$ to $\mathbb{Q}$-chains and then to $\mathbb{Q}$-graded vector spaces; this isn't adjoint to the latter, but to the former. Note that the category of CDGAs is probably abelian; certainly, it has kernels, and all of the structure that we need.

We implicitly used the cohomology in Example 4.2; here's the explicit definition.

Definition.

- The *cohomology* of $A^\bullet$, denoted $H^*(A^\bullet)$, is defined by $H^n(A^\bullet) = (\ker d : A^n \to A^{n+1})/(\operatorname{Im} d : A^{n-1} \to A^n)$.

- Let C^* and D^* be CDGAs, with a morphism $f : C^* \to D^*$, and define $M_f^n = C^n \oplus D^{n-1}$ with differential $d_M : M_f^n \to M_f^{n+1}$ given by $\begin{pmatrix} d_C & 0 \\ f & -d_D \end{pmatrix}$. Then, the *relative cohomology* is $H^*(D^\bullet, C^\bullet) = H^*(M_f^\bullet)$.

Notice that we have things like mapping cylinders, relative cohomology, etc., so CDGAs do look somewhat like topological spaces, which will be nice.

Lemma 4.3. *There exists a long exact sequence*

$$\cdots \longrightarrow H^n(C^\bullet) \longrightarrow H^n(D^\bullet) \longrightarrow H^{n+1}(D^\bullet, C^\bullet) \longrightarrow H^{n+1}(C^\bullet) \longrightarrow \cdots$$

Definition. $f : C^* \to D^*$ is a *quasi-isomorphism* if $H^*(f)$ is an isomorphism.

What we'll do now is spend some time codifying what it means for a category to have homotopy theory; that both rational homotopy and CDGAs live in the same world. This is because both topological spaces and CDGAs are model categories.

If you do much more homotopy theory, model categories will be really useful. Even where they're not explicitly talked about, they'll clarify ideas and thoughts in papers and books.

A model category can be thought of as a model for homotopy; in such a category, there will be things that act like cylinders (or, dually, paths), and therefore we can speak of when objects are homotopic. This will follow from some impressively minimal axioms, because Quillen was extremely clever. We'll define the category, and then show that these induce notions of homotopy equivalence and weak equivalence, and some correspondences between the two.

First, recall a category-theoretic definition.

Definition. Let $f : A \to B$ and $g : A' \to B'$ be morphisms in a category $\mathcal{C}$. Then, f is said to be a *retract* of g if the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] \arrow[rr, bend left, "\mathrm{id}"] & A' \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\
B \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & B' \arrow[r] & B
\end{tikzcd}$$

Definition. A *model category* is a category $\mathcal{C}$ closed under small[3] limits and colimits, together with three subcategories:

- $\mathrm{co}(\mathcal{C})$ (sometimes denoted $\mathrm{cof}(\mathcal{C})$), the subcategory of *cofibrations*,
- $\mathrm{fib}(\mathcal{C})$, the subcategory of *fibrations*, and
- $w(\mathcal{C})$, the subcategory of *weak equivalences*.[4]

We'll also call $w(\mathcal{C}) \cap \mathrm{co}(\mathcal{C})$ the *trivial cofibrations* (as in homotopically trivial), and $w(\mathcal{C}) \cap \mathrm{fib}(\mathcal{C})$ the *trivial cofibrations*. If $f : A \to B$ is a morphism in $\mathrm{co}(\mathcal{C})$, then it will often be denoted $f : A \hookrightarrow B$, and similarly if $f \in \mathrm{fib}(\mathcal{C})$, then it'll be denoted $f : A \twoheadrightarrow B$. Weak equivalences will be denoted with $\simeq$.

This data is subject to the following axioms.

1. The *two-out-of-three axiom*: if $f : A \to B$ and $g : B \to C$ are morphisms in $\mathcal{C}$ such that two of gf, g, and gf are in $w(\mathcal{C})$, then the third is.
2. Retracts of cofibrations, fibrations, and weak equivalences are cofibrations, fibrations, and retracts, respectively; that is, each of these three categories is closed under retracts.
3. The *lifting axiom*: let $i : A \to B$ be a trivial cofibration and $p : X \twoheadrightarrow Y$ be a fibration. If the following diagram commutes, then the following dotted arrow exists.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, hook, "i", "\simeq"'] & X \arrow[d, two heads, "p"] \\
B \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

Similarly, if i is a cofibration and p is a trivial fibration, then the following dotted arrow exists.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, hook, "i"'] & X \arrow[d, two heads, "p"', "\simeq"] \\
B \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

[3] We really don't care about set-theoretic issues in this class, as evidenced when the first day didn't begin with "In the beginning, there was a Grothendieck universe." We'll primarily use finite limits and colimits, and requiring pushouts to exist is good enough for us.

[4] Sometimes $w(\mathcal{C})$ is called the "water closet," albeit not in serious literature.

(4) *Functorial factorization*: Every morphism $f : X \to Y$ can be factored as $X \hookrightarrow Y' \overset{\sim}{\twoheadrightarrow} Y$, where the first map is in $\mathrm{co}(\mathcal{C})$ and the second in $w(\mathcal{C}) \cap \mathrm{fib}(\mathcal{C})$; it can also be factored as $X \overset{\simeq}{\hookrightarrow} X' \twoheadrightarrow Y$, where the first map is a trivial fibration and the second is a cofibration,. Moreover, these factorizations are functorial in that (for the first factorization; the second is similar) there exists a functor $(F, G) : \operatorname{Mor}\mathcal{C} \to \operatorname{Mor}\mathcal{C} \times \operatorname{Mor}\mathcal{C}$ such that $F(f)$ is a cofibration, $G(f)$ is a trivial fibration, and $G(f) \circ F(f) = f$.

Remark. $\mathcal{C}$ has an initial and final object, since pushouts exist. These will be denoted 0 and $*$, respectively.

The lifting axiom looks pretty specifically homotopical, but it's still surprisingly minimal for all of this homotopy theory to exist in a category. And functorial factorization is basically cellular approximation.

It should ease your mind that topological spaces form a model category, with fibrations given by fibrations, cofibrations given by cofibrations, and weak equivalences given by weak homotopy equivalences. This is nonetheless hard to show; simplicial sets are also a category, but this is even harder to show.

Definition.

- An object B is called *cofibrant* if $0 \to B$ is a cofibration.
- Similarly, Y is called *fibrant* if $Y \to *$ is a fibration.

These objects tend to have nicer properties. Moreover, every object is weakly equivalent to a cofibrant and a fibrant object.

Definition. Given any object B, $0 \to B$ factors as $0 \hookrightarrow QB \overset{\sim}{\twoheadrightarrow} B$, so that $QB \simeq B$ and QB is cofibrant. QB is called the *cofibrant replacement* of B.

Similarly, $Y \to *$ factors as $Y \overset{\sim}{\hookrightarrow} RY \twoheadrightarrow *$, with $RY \simeq Y$ and RY fibrant; this RY is called the *fibrant replacement* of Y.

There are lots of terms here, but we'll need to use all of them to talk about stuff. Topological spaces are already all fibrant, so intuition can be challenging here.

The lifting property means that fibrations and cofibrations mutually define each other, so there's sort of too much information in these problems.

Definition. Let $X \in \mathcal{C}$ be an object. Then, a *cylinder* for X, denoted $\mathrm{Cyl}(X)$, is a factorization of $X \sqcup X \to X$ as $X \sqcup X \hookrightarrow \mathrm{Cyl}(X) \overset{\sim}{\to} X$.

Remark. The above factorization need not be the functorial factorization guaranteed by our category. But we can use that, and therefore there's a cylinder functor.

The abstract definition can be realized topologically as something weakly equivalent to X (which is true of $X \times [0, 1]$), but that fills in the area between two copies of X.

We'll see that everything in model categories has a dual. The notion of a path object is dual to the cylinder.

Definition. If $Y \in \mathcal{C}$, then a *path object* for Y, denoted PY, is one that arises from a factorization $Y \overset{\sim}{\to} PY \to\!\!\!\!\to Y \times Y$ of the diagonal map $Y \to Y \times Y$.

Again, this can be made functorial. This notion, making a space into a path space, also appears in homotopy theory. But we can go the other way, making avatars of our familiar notions in homotopy theory inside a more general model category. For example, we'll define a homotopy!

Note: next lecture is next Tuesday, so don't come to class on Thursday.