

FALL 2017 GEOMETRIC SATAKE SEMINAR

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1. OVERVIEW ON THE GEOMETRIC SATAKE THEOREM: 9/1/17

This overview was given by David Ben-Zvi.

This semester, we're studying the geometric Satake theorem, one of the most important results in geometric representation theory, and even a central result in the geometric Langlands program.

This theorem involves some potentially unfamiliar words; we'll define them in the course of this seminar.

Theorem 1.1 (Geometric Satake). *Let G be a reductive group over a field k .¹ Then, there is a category of $G(\mathcal{O})$ -equivariant perverse sheaves on the affine Grassmannian of G , $G(K)/G(\mathcal{O})$, symmetric monoidal under convolution, together with a fiber functor $H^\bullet(-)$, and this is equivalent to $(\text{Rep}_{G^\vee}^{\text{fd}}, \otimes)$ (where G^\vee is the Langlands dual group) as symmetric monoidal categories, with the fiber functor the forgetful map to Vect_k .*

By Rep^{fd} we mean the full subcategory of finite-dimensional representations. K will be some local field, and \mathcal{O} is its ring of integers. For example, if $k = \mathbb{C}$, $K = \mathbb{C}((t))$ and $\mathcal{O} = \mathbb{C}[[t]]$, and over \mathbb{F}_p , you have $K = \mathbb{F}_p((t))$ and $\mathcal{O} = \mathbb{F}_p[[t]]$.²

Okay, first what's a reductive group? For $k = \mathbb{C}$, these are complexifications of compact groups. For example: GL_n , SL_n , PGL_n , SU_n , Sp_n , and E_7 .

Now this theorem is saying that we start with one reductive group and we get another, G^\vee . This relationship is such that if $G = T$ is a torus, i.e. $(\mathbb{C}^\times)^k$, its Langlands dual is the dual torus T^\vee : if T is the quotient of \mathbb{C}^n by a lattice, T^\vee is the quotient of $(\mathbb{C}^n)^*$ by the dual lattice.

Theorem 1.1 is a kind of Fourier transform, a quite fancy one. For example, if $G = \text{GL}_1$, the affine Grassmannian is a (scheme which behaves more or less like) \mathbb{Z} : $\text{GL}_1(\mathbb{C}((t))) = \mathbb{C}((t))^\times$ and $\text{GL}_1(\mathbb{C}[[t]])$ is the group of power series with nonzero constant term. When you mod these out, the leading term of the Laurent series becomes the only important thing, in a sense. The next ingredient is the equivariant perverse sheaves, but ends up being vector bundles over Gr in this case, so we get (modulo some reducedness which doesn't come into play here) the category of graded vector spaces. In this case, Theorem 1.1 says the category of graded vector spaces is equivalent to the category of representations of \mathbb{G}_m , just like the Fourier transform exchanges functions on \mathbb{Z} with representations of S^1 .

¹You can let k be a ring R , the coefficients. The algebraic geometry we do will still be over \mathbb{C} , though; the representations you get end up also being representations over the ring R .

²In particular, they will never be \mathbb{Q}_p and \mathbb{Z}_p . However, the geometric Satake theorem is a living piece of mathematics, and only in the past year Peter Scholze proved a version of this for the p -adics.

You can interpret the geometric Satake theorem as the source of the Langlands dual group: it admits a definition in terms of tori and root data, but it feels somewhat ad hoc, and one is left wondering: where did it all come from? Instead, by the Tannakian perspective on representation theory, Theorem 1.1 is telling us that the category of $G(\mathcal{O})$ -equivariant sheaves with its fiber functor is canonically the category of representations of a group, and in fact gives us enough information to reconstruct the group! So the geometric Satake theorem is a bridge from G to G^\vee , and is one of the only bridges.

Example 1.2. Langlands duality is often somewhat surprising: G and G^\vee don't look like each other, and it's not clear how to obtain one from the other. Of course, $(G^\vee)^\vee \cong G$.

$$\begin{aligned} \mathrm{GL}_n &\longleftrightarrow \mathrm{GL}_n \\ \mathrm{SL}_n &\longleftrightarrow \mathrm{PGL}_n \\ \mathrm{SO}_{2n+1} &\longleftrightarrow \mathrm{Sp}_{2n} \\ \mathrm{SO}_{2n} &\longleftrightarrow \mathrm{SO}_{2n}. \end{aligned}$$

◀

You can also use the geometric Satake theorem to explain some things which at first appear to not be geometric. For example, $H^*(\mathbb{CP}^n) \cong \mathbb{Z}[x]/(x^{n+1})$ with $|x| = 2$ is acted on by SL_2 by raising and lowering operators, which comes out of complex geometry, but this does not arise from an SL_2 -action on \mathbb{CP}^n itself. More generally, SL_k acts on $H^*(\mathrm{Gr}_k(\mathbb{C}^n))$ in a similar way, and is similarly mysterious.

But Theorem 1.1 identifies it with an equivariant perverse sheaf for the action of PGL_k on a Grassmannian, and the action of PGL_k on a Grassmannian is more evident. So we've obtained either new interesting representations, or geometric models for representations of your group, and solved the mysteries of this representation.

It also flows the other direction: if you take a reductive group over R , you recover information about $\mathrm{Rep}_{G^\vee, R}$, the category of representations over R . This is an active topic of research, and people including Geordie Williamson have used it to uncover interesting consequences in modular representation theory.

Of course, the geometric Satake theorem is also entangled with the geometric Langlands conjectures in interesting ways.

We're going to first discuss affine Grassmannians, then perverse sheaves (which generalize cohomology of smooth projective varieties, and could be an entire seminar unto themselves), then their convolution (the fact that it's symmetric monoidal is very deep, and related to commutativity of Hecke algebras). Finally, we'll talk about Tannakian reconstruction, a beautiful abstract story that allows us to extract G^\vee from the theorem.

Here's a very provisional schedule:

- First, a few lectures on the affine Grassmannian: Rustam (next week), and then Richard H. (the week after).
- Then, perverse sheaves (and also intersection cohomology): Arun, Sebastian, and Yan.
- Then, convolution and its commutativity: Richard W. and Vaibhav.
- Then, Tannakian reconstruction: Isabelle and Nicky.
- Finally, putting it together into a proof of the geometric Satake correspondence: Rok and probably also someone else.

There's a lot of further topics and cool applications if anyone is interested after that.

2. BRUHAT-TITS TREES: 9/1/17

In the second part of the first meeting, Tom Gannon spoke about trees (à la Serre).

Throughout this lecture, let K be a complete field (with respect to some norm), together with a discrete valuation $v: K^\times \rightarrow \mathbb{Z}$.³ Let \mathcal{O}_K denote its ring of integers, which is a local ring, and \mathfrak{m} denote its unique maximal ideal. Let π be a *uniformizer*, i.e. a generator of \mathfrak{p} .⁴ We'll let $q := |\mathcal{O}_K/\mathfrak{m}|$, and assume that q is finite, so that the residue field $\mathcal{O}_K/\mathfrak{m} \cong \mathbb{F}_q$.

Though we didn't define $v(0)$, we think of it as ∞ : the valuation values how many times you can divide an element by π , and for 0 you can do this infinitely often.

³Recall that a *discrete valuation* is a surjective group homomorphism $K^\times \rightarrow \mathbb{Z}$. You can think of it as measuring how many times a uniformizer π divides a given field element.

⁴You can think of this as a coordinate on the curve.

Let K be a field with a discrete valuation $v: K^\times \rightarrow \mathbb{Z}$ and $c \in (0, 1)$ be fixed. Then, the map $\|\cdot\|_c: K \rightarrow [0, \infty)$ with $|x|_c := c^{v(x)}$ is a norm, and moreover is non-Archimedean, satisfying a stronger form of the triangle inequality:

$$|x + y|_c \leq \max\{|x|_c, |y|_c\}.$$

Proposition 2.1. *With notation as above, the set $\{x \in K \mid |x| \leq 1\}$ is a ring, and in fact a discrete valuation ring; its unique maximal ideal is $\{x \in K \mid |x|_c < 1\}$.*

That it's a discrete valuation ring means the unique maximal ideal is principal. This ring is called the *associated ring of integers* of K . Let's pick a specific value of c , which is $1/q$.

Example 2.2 (2-adic rationals). The 2-adic rationals, \mathbb{Q}_2 , form a complete field with a discrete valuation. One way to think about this is that there's a norm on \mathbb{Q} given by how many 2s you can factor out; completing it with respect to that norm defines \mathbb{Q}_2 .

There's also a lower-brow way to think of this, as Laurent series in 2: an element of \mathbb{Q}_2 is something like

$$2^{-4} + 2^{-3} + 2^{-1} + 2 + 2^3 + 2^5 + \dots$$

Equality is termwise, and addition and multiplication are like those of Laurent series. The coefficients are mod 2, so if you consider p -adics for $p > 2$, you have more options. In this case, the valuation is the smallest N such that the N -coefficient is nonzero.

There's also an algebraic interpretation of \mathbb{Z}_2 and \mathbb{Q}_2 . ◀

Example 2.3. Another example if $K = \mathbb{F}_p((t))$ with $\mathcal{O}_K = \mathbb{F}_p[[t]]$. The valuation is the minimal power of t that appears with a nonzero coefficient, like for \mathbb{Q}_2 . ◀

Now we'll discuss the Bruhat-Tits tree for $\mathrm{SL}_2(K)$. There's not a lot of motivation, except that this stuff is awesome.

The tree will be a set of vertices and edges; its vertices will be a set of lattices in K^2 .

Definition 2.4. A *lattice* in K^2 is an \mathcal{O}_K -submodule Λ of K^2 such that $\Lambda \otimes_{\mathcal{O}_K} K = K^2$.

Concretely, these are subsets of K^2 of the form $\mathcal{O}_K \cdot v_1 + \mathcal{O}_K \cdot v_2$, where $\{v_1, v_2\}$ is a basis for K^2 . These correspond to the usual lattices in \mathbb{R}^2 .

Since $\mathrm{GL}_2(K)$ acts on the set of bases of K^2 , it acts on the set of lattices. The stabilizer of each lattice is $\mathrm{GL}_2(\mathcal{O}_K)$, and therefore the space of lattices is naturally isomorphic to $\mathrm{GL}_2(K)/\mathrm{GL}_2(\mathcal{O}_K)$. Hey, that space appeared in the statement of the geometric Satake isomorphism!

Theorem 2.5 (Principal divisor theorem). *Let L_1 and L_2 be lattices. Then, there's a basis $\{e, f\}$ for L_1 and $m, n \in \mathbb{Z}$ such that $\{\pi^m e, \pi^n f\}$ is a basis for L_2 .*

The proof is linear algebra, spiced up somewhat by the fact that it's over discrete valuation rings. It's also the only place where we assume the residue field is finite.

Remark. If you consider GL_1 instead of GL_2 , you get the statement we discussed before, that $\mathrm{GL}_1(K)/\mathrm{GL}_1(\mathcal{O}_K)$ is the integers (and therefore representations of \mathbb{G}_m are equivalent to graded vector spaces). ◀

Now, say that two lattices L_1 and L_2 are equivalent if $L_1 = \pi^\ell L_2$ for some ℓ . The space of equivalence classes is $\mathrm{PGL}_2(K)/\mathrm{PGL}_2(\mathcal{O}_K)$. We define the vertices of the Bruhat-Tits tree to be this set.

Now we should talk edges. Let v and w be two vertices, and L_1 and L_2 be lattice representatives for v and w , respectively. By Theorem 2.5, there are m and n carrying a basis for L_1 to a basis for L_2 , and we add an edge iff $|m - n| = 1$.⁵ Equivalently, we add an edge if there's a rescaling of L_1 called L'_1 such that $L_2 \supsetneq L'_1 \supsetneq \pi L_2$.

Call this graph G . We'll eventually show it's a tree.

Proposition 2.6. *G is a connected graph.*

Proof. Let L_1 and L_2 be lattices. Then, there are $e, f \in L_1$ and $m, n \in \mathbb{Z}$ such that $\{e, f\}$ is a basis for L_1 and $\{\pi^m e, \pi^n f\}$ is a basis for L_2 . Without loss of generality, assume $m \geq n$. Then, there's an edge from L_2 to $\mathcal{O}_K \cdot \pi^{m-n} e + \mathcal{O}_K \cdot f$. Continuing in this way, we must eventually reach L_1 . ◻

⁵There's enough uniqueness in the proof for this to be well-defined, even if m and n aren't unique.

A string of points produced by this method is called an *apartment*. More generally, any path of vertices which is finite or half-infinite is called a *chain*. A *simple chain* is one where you never step forward and then back (or vice versa).⁶

Remark. These lattices satisfy a Noetherian-esque property: if you have an infinite chain of vertices $w_0 - w_1 - \dots$, then there exist representative lattices L_i for w_i such that for all i , $L_i \supsetneq L_{i+1} \supsetneq \pi L_i$. \blacktriangleleft

Proposition 2.7. *For any simple chain C , there's a $g \in \mathrm{GL}_2(K)$ such that $g \cdot C$ is the chain*

$$\mathcal{O}_K \cdot e_1 + \mathcal{O}_K \cdot e_2 \supsetneq \mathcal{O}_K \cdot \pi e_1 + \mathcal{O}_K \cdot \pi e_2 \supsetneq \mathcal{O}_K \cdot \pi^2 e_1 + \mathcal{O}_K \cdot \pi^2 e_2 \supsetneq \dots,$$

where $\{e_1, e_2\}$ is the standard basis for K^2 .

Corollary 2.8. *G is actually a tree.*

Proof. Assume C is a simple chain that's a cycle in G . Then, Proposition 2.7 preserves connectiveness, but replaces it with something which could not possibly be a cycle. \square

Proof sketch of Proposition 2.7. Let the starting chain be $C = L_0 \supsetneq L_1 \supsetneq L_2 \supsetneq \dots$. We know there's a $g \in \mathrm{GL}_2(K)$ such that $g_0 L_0$ is the starting vertex $\mathcal{O}_K \cdot e_1 + \mathcal{O}_K \cdot e_2$. So g_0 is our candidate. But we don't know whether $g_0 L_1$ is the same as $\mathcal{O}_K \cdot \pi e_1 + \mathcal{O}_K \cdot \pi e_2$, but there's a g_1 in the stabilizer of $g_0 L_0$ that moves it to $\mathcal{O}_K \cdot \pi e_1 + \mathcal{O}_K \cdot \pi e_2$. Then, inductively, one can assume there exists an element in the stabilizer of the first i that brings the next element of the chain into position, and so on.⁷ This inductive argument is a little delicate, and uses the fact that the residue field is finite.

You may have to do this infinitely many times, which is actually fine: you can conjugate the g_i such that

$$g_i = \begin{pmatrix} 1 & x_i \\ 0 & 1 \end{pmatrix}$$

for $x_i \in \mathfrak{m}^i$; then, the infinite product is

$$\begin{pmatrix} 1 & \sum x_i \\ 0 & 1 \end{pmatrix},$$

and, using the local topology of K , you can show this sum converges. \square

If you act by $\mathrm{SL}_2(K)$ (through the inclusion in $\mathrm{GL}_2(K)$), the parity of $m - n$ is preserved, so you can decompose the tree into a bipartite tree. From the geometric Satake perspective, this says that the affine Grassmannian for $\mathrm{PGL}_2(K)$ has two connected components.

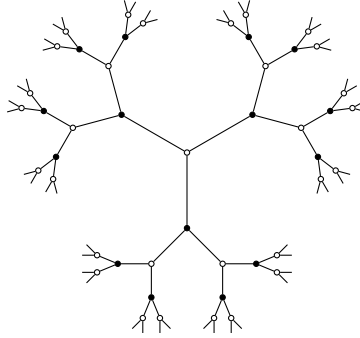


FIGURE 1. The Bruhat-Tits tree for $\mathrm{SL}_2(\mathbb{Q}_3)$. The two parity classes of vertices are in black and white. Source: <https://tex.stackexchange.com/a/135764>.

Another fun fact is that $\mathrm{PSL}_2(\mathcal{O}_K)$ acts on the tree by graph automorphisms, and the double coset space

$$\mathrm{PSL}_2(\mathcal{O}_K) \backslash \mathrm{PSL}_2(K) / \mathrm{PSL}_2(\mathcal{O}_K)$$

⁶For example, the basic steps of salsa define chains, but not a simple chain; the basic steps of waltz, which return to the same point but after more than one step, are a simple chain.

⁷One way to think of this is that $\mathrm{GL}_2(K)$ is filtered by the discrete valuation; g_0 is the first piece, $g_1 g_0$ is the second piece, and so forth.

is in bijection with the positive integers — well actually, the highest weights, or the irreducible representations of SL_2 . This already looks Langlandsy, and more of the story appears: you can define Hecke operators on the tree: for each $n \in \mathbb{N}$, let

$$T_n f(v) = \frac{1}{n} \sum_{|w-v|=n} f(w).$$

That is, the Hecke operator acts on the space of functions on the tree averages over things that are distance n away.

Theorem 2.9 ((Classical) Satake theorem). *These Hecke operators T_n commute, and generate an algebra isomorphic to the representation ring of SL_2 .*

The geometric Satake theorem is a categorified analogue of this theorem. Indeed, the Bruhat-Tits tree appears in the story of the geometric Satake theorem as well, helping us understand the geometry of the affine Grassmannian in the case $G = \mathrm{PGL}_2$.

3. AN INTRODUCTION TO THE AFFINE GRASSMANNIAN: 9/8/17

“That’s an affine Grassmannian you’ve got there. Shame if something happened to it.”

Today Rustam spoke about the affine Grassmannian, beginning with the uniformization of G -bundles as motivation from the topological case, then move to the algebraic world.

Uniformization of G -bundles, topological setting. Let Σ be a compact, connected, oriented (real) surface and G be a connected topological group. Let $x \in \Sigma$ and \mathbb{D} be a small disc around it. Let $\mathbb{D}^0 := \mathbb{D} \setminus x$.

Definition 3.1. The *loop groups* of G are:

- the *positive loop group* $L^{\mathrm{top},+}$, the continuous maps $f: \mathbb{D} \rightarrow G$, under pointwise multiplication;
- the *loop group* $L^{\mathrm{top}}(G)$, the continuous maps $f: \mathbb{D}^0 \rightarrow G$, and
- $L_X(G) := \{f: X \setminus x \rightarrow G\}$.

There’s a restriction map $L^{\mathrm{top},+}(G) \rightarrow L^{\mathrm{top}}(G)$.

Proposition 3.2. *There’s a bijection of sets*

$$L_X(G) \backslash L^{\mathrm{top}}(G) / L^{\mathrm{top},+}(G) \cong \mathrm{Bun}_G(X).$$

The idea is that $X \setminus x$ is homotopic to a graph, and since isomorphism classes of principal bundles are classified by maps to BG , and BG is simply connected because G is connected. Hence all principal G -bundles on $X \setminus x$ are trivial, so it matters crucially how we put x back in, and then quotient by redundant data.

Uniformization of G -bundles, algebraic setting. Now, let X be smooth, connected, projective curve over \mathbb{C} and G be a semisimple algebraic group. Let $x \in X$ be a \mathbb{C} -point and $\mathbb{D}_x := \mathrm{Spec}(\widehat{\mathcal{O}_{X,z}}) = \mathrm{Spec}(\mathbb{C}[[t]])$ be the formal disc around x , so $x \in \mathbb{D}_x \hookrightarrow X$. Hence, the punctured disc $\mathbb{D}_x \setminus x = \mathrm{Spec} \mathbb{C}((t))$.

Definition 3.3. The *loop groups* of G are given by the following functors of points.⁸

- The *positive loop group* $L^+(G)(R) := \mathrm{Hom}(\mathbb{D}_R, G)$.
- The *loop group* is $L(G)(R) := \mathrm{Hom}(\mathbb{D}_x \setminus x, G)$.
- $L_X(G) := \mathrm{Hom}(X_R \setminus x \rightarrow G)$.

Hence $L(G)(\mathbb{C}) = G(\mathbb{C}((t)))$ and $L^+(G)(\mathbb{C}) = G(\mathbb{C}[[t]])$.

Theorem 3.4 (Beauville-Laszlo). *There’s an equivalence of stacks*

$$L_X G \backslash L G / L^+ G \cong \mathrm{Bun}_G(X).$$

⁸Two of them are not representable as schemes; $L^+(G)$ is a scheme, and $L(G)$ is an ind-scheme but not a formal scheme. An *ind-scheme* is a filtered colimit of schemes, where all maps are closed embeddings, e.g. \mathbb{A}^∞ or \mathbb{P}^∞ . $\mathbb{A}^\infty = \mathrm{Spec} \mathbb{C}[x_1, x_2, \dots]$ is also a scheme.

This is a hard proof: it doesn't follow from the usual descent arguments. Semisimplicity is crucial, and means that you can't apply this to GL_1 .

Let $\mathcal{O}_K := \mathbb{C}[[t]]$, which corresponds to the disc, and $K := \mathbb{C}((t))$, which corresponds to the punctured disc. Recall that the *affine Grassmannian* is $\mathrm{Gr}_G := L(G)/L^+(G) = G(K)/G(\mathcal{O}_K)$.⁹

If $G = \mathrm{GL}_n$, this is the same as the set of (full-rank) lattices on K^n : $G(K)$ acts transitively on all lattices, and the stabilizer is $G(\mathcal{O}_K)$.¹⁰ For other groups, there's a similar description in terms of lattices.

This definition of the Grassmannian characterizes it as a set. Our first goal will be to give it a topology, at least in the case where $G = \mathrm{GL}_n$.

The first tool we'll use is a valuation: if $\vec{v}_1, \dots, \vec{v}_n$ is a basis for a lattice Λ , then $\det(\vec{v}_1, \dots, \vec{v}_n) \in K^\times$. This is not an invariant of Λ , but can be made into one.

Definition 3.5. Let Λ be a lattice. Its *valuation* $v(\Lambda) \in \mathbb{N}$ is the minimum n such that t^n is the determinant of a basis of Λ .

For the trivial lattice $\Lambda^0 := (\mathcal{O}_K)^n$, $n = 2$. Also, if you do this for $G = \mathrm{GL}_1$, the affine Grassmannian is literally \mathbb{Z} ; you can think of the valuation as a kind of determinant map from $\mathrm{GL}_n \rightarrow \mathrm{GL}_1$, and hence to $\mathrm{Gr}_{\mathrm{GL}_1} = \mathbb{Z}$.

Our next tool will be to compare lattices with the standard lattice.

Lemma 3.6. *For all lattices Λ , there's an $a \in \mathbb{N}$ such that*

$$(3.7) \quad t^a \Lambda^0 \subseteq \Lambda \subseteq t^{-a} \Lambda^0.$$

This will allow us to get the ind-structure.

Definition 3.8. Let $\mathrm{Gr}_{\mathrm{GL}_n}^{\ell,a}$ be the set of lattices $\Lambda \in \mathrm{Gr}_{\mathrm{GL}_n}$ such that $v(\Lambda) = \ell$ and (3.7) is satisfied.

Then $\mathrm{Gr}^{\ell,a} \hookrightarrow \mathrm{Gr}^{\ell,b}$ if $b \geq a$, and

$$\mathrm{Gr}_{\mathrm{GL}_n} = \bigcup_{\ell,a} \mathrm{Gr}_{\mathrm{GL}_n}^{\ell,a}.$$

Let $J_{a,n}$ denote the space of $(na - k)$ -dimensional subspaces of $t^{-a} \Lambda^0 / t^a \Lambda^0 \cong \mathbb{C}^{2an}$. Then, $\mathrm{Gr}_{\mathrm{GL}_n}^{\ell,a}$ embeds into $J_{a,n}$ by taking the quotient by $t^a \Lambda^0$, and $J_{a,n}$ is a Grassmannian! By the Plücker embedding, it's a projective variety.

Definition 3.9. An *ind-projective ind-variety* is an ind-scheme $X = \mathrm{colim}_i X_i$ such that each X_i is projective variety.

Remark. In schemes, there's a significant difference between limits and colimits. Limits of affine schemes always exist, because colimits of rings do. For example, $\mathbb{A}^\infty = \mathrm{Spec} \mathbb{C}[x_1, x_2, \dots] \cong \varprojlim \mathcal{O}_K / t^n \mathcal{O}_K$. But ind-schemes are not always schemes. For example, K , the functor $R \mapsto R((t))$, is the colimit of $K^{\geq -N} \cong \mathbb{A}^\infty$. This is the case where you're only allowed to have finitely many coordinates. Said another way, the positive part of the Laurent series in the affine Grassmannian is fine, albeit infinite-dimensional; the negative tails produce the ind-ness.

For a concrete example, $\mathrm{Gr}_{\mathrm{GL}_1} \cong \mathbb{Z}$, and this is an infinite disjoint union of points, a nice ind-projective ind-variety. You might try to realize it as Spec of an infinite direct product of \mathbb{C} s, but these are not isomorphic! The correspondence between coproducts of schemes and products of rings only works fully at the finite level. \blacktriangleleft

Theorem 3.10. *The affine Grassmannian $\mathrm{Gr}_{\mathrm{GL}_n}$ is an ind-projective ind-variety.*

The idea is to use the $J_{a,n}$.

There's an important stratification of the affine Grassmannian: using Gauss-Jordan elimination, the orbits of $\mathrm{GL}_n(\mathcal{O}_K)$ on $\mathrm{Gr}_{\mathrm{GL}_n}$ are identified with coweights $\mathbb{C}^\times \rightarrow T$ (where T is a maximal torus for our group).

⁹The name comes from the affine Weyl group because it has translations in it; the affine Grassmannian is not affine in any usual sense. For this reason, it's sometimes referred to as the *infinite Grassmannian* or the *loop Grassmannian*.

¹⁰These are just like lattices in \mathbb{R} : a lattice Λ is an \mathcal{O}_K -submodule of K^n such that $\Lambda \otimes_{\mathcal{O}_K} K = L^n$.

The set of coweights is often denoted $X_\bullet(T)$. Specifically,

$$(3.11) \quad \mathrm{GL}_n(K) = \coprod_{\substack{\lambda=(a_1,\dots,a_n)\in\mathbb{Z} \\ a_1\geq\cdots\geq a_n}} \mathrm{GL}_n(\mathcal{O}_K) \begin{pmatrix} t^{a_1} & & \\ & \ddots & \\ & & t^{a_n} \end{pmatrix} \mathrm{GL}_n(\mathcal{O}_K).$$

The idea: using row reduction, you can get rid of everything except for powers of t (since you're only using Taylor series, not Laurent series). There's a similar perspective for other groups G , for which one might write

$$(3.12) \quad G(K) = \coprod_{\lambda\in X_\bullet(T)_+} G(\mathcal{O}_K) t^\lambda G(\mathcal{O}_K).$$

One can identify $X_\bullet(T)_+ \cong X_\bullet(T)/W$, where W is the Weyl group for G .¹¹

This decomposition is really nice: the orbits are all projective varieties. There's a nice Morse-theoretic approach to all this.

The decomposition (3.12) is sort of a “set-theoretic Satake theorem:” the Langlands dual is a little implicit, but tells us that finite-dimensional irreducible representations of G^\vee are indexing this decomposition of the affine Grassmannian. For $G = \mathrm{GL}_n$, which is Langlands self-dual, (3.11) can also be interpreted as indexed by the representations of GL_n .

The algebro-geometric approach to the affine Grassmannian. The affine Grassmannian admits a functor-of-points definition, as the functor $\mathrm{Gr}: \mathrm{Alg}_{\mathbb{C}} \rightarrow \mathrm{Set}$ sending a \mathbb{C} -algebra R to the set of finitely-generated projective $R[[t]]$ -submodules Λ of $R((t))^n$ such that

$$\Lambda \otimes_{R[[t]]} R((t)) = R((t))^n.$$

Theorem 3.13. *This is represented by an ind-projective ind-scheme.*

This scheme is isomorphic to the one described in Theorem 3.10. You'd prove Theorem 3.13 by finding a cover by subfunctors that are represented by projective varieties and whose colimit is Gr again.

We also get a functor-of-points approach to the orbit decomposition (3.11): $\mathrm{Gr}_{\mathrm{GL}_n}(R)$ is the set of (Σ, β) where Σ is a vector bundle on \mathbb{D}_R and $\beta: \Sigma|_{\mathbb{D}_R^0} \rightarrow \underline{\mathbb{C}}^n$ is an isomorphism with the trivial bundle. If $Y \subset X$, let $\mathrm{Bun}_G(X, Y)$ denote the set of principal G -bundles on X together with data of a trivialization on Y , then we can replace vector bundles with principal G -bundles to obtain a more general description:

$$\mathrm{Gr}_G(R) = \mathrm{Bun}_G(\mathbb{D}_R, \mathbb{D}_R^0).$$

Remark. Unlike in algebraic topology, defining principal G -bundles is tricky: if you try things which are Zariski-locally G -torsors, you get the wrong thing. Keeping in mind that it should always be possible to form an associated vector bundle from a principal G -bundle and a G -representation, followed by some messing around with Grothendieck topologies, leads to the right notion. \blacktriangleleft

For general G , Gr_G is an ind-projective ind-scheme; the proof idea is to embed $G \hookrightarrow \mathrm{GL}_n$.

The R -points of $L(G)$ are pairs $(\Sigma, \beta) \in \mathrm{Bun}_G(\mathbb{D}_R, \mathbb{D}_R^0)$ together with a trivialization $\varepsilon: \Sigma \rightarrow \underline{G}$ on all of \mathbb{D}_R . Forgetting ε defines the quotient map to Gr_G ; if you think about it, you'll find that this is actually the quotient by $L^+(G)$.

Back to topology. There's a topological version of this story — in topology, the Grassmannian arises in a very different way. Let G be a complex Lie group and K be its maximal compact subgroup, so K is homotopic to G .

Let ΩK denote the (based) loop space of K , the space of basepoint-preserving, continuous maps $S^1 \rightarrow K$.¹² Then, there's a model for the affine Grassmannian Gr_G that's homotopic to ΩK .

Another way to think of this is to let $LK = \mathrm{Map}(S^1, K)$ be the free loop space. Then, $\Omega K = LK/K$, and this is like taking the quotient of $L(G)/L^+(G)$.

Now, we want this to be a moduli space of something. Well, $\Omega K \simeq \Omega^2 BK$, where BK is the homotopy type such that $\mathrm{Map}(X, BK)$ is naturally $\mathrm{Bun}_K(X)$ (as a set). This is representable, which is a theorem.

¹¹ $X_\bullet(T)_+$ is the set of *dominant weights*: given a chamber C , the set of weights that pair positively with the elements of that chamber.

¹²See Pressley-Segal for details.

Two-fold loops in Y are identified with maps from a disc into Y such that (a small neighborhood of) the boundary maps to the basepoint. Hence $\Omega K \simeq \Omega^2 BK$ is the set of K -bundles on $\mathbb{C} = \mathbb{R}^2$ that are pointed, in that there's extra data of a trivialization on $\mathbb{C} \setminus \mathbb{D}$, where \mathbb{D} is a small disc. This is precisely what we said the affine Grassmannian was: G -bundles trivialized at a point. So the algebraic geometry and the homotopy theory are the same – since $BK \simeq BG$, you could also take the set of G -bundles on a disc trivialized at a point.

The fact that ΩK is a 2-fold loop space means that it's a homotopical kind of abelian group, which is key – it means the affine Grassmannian is in some sense an abelian group. For example, for $G = \mathrm{GL}_1$, $\Omega^2 BS^1 = \mathbb{Z}$, which *is* a group. This groupiness (E_2 -structure) will be crucial to the geometric Satake theorem.

The description of the affine Grassmannian as based loops on a compact Lie groups is what enabled Bott and others to attack it with Morse theory.

4. GEOMETRY OF THE AFFINE GRASSMANNIAN: 9/15/17

“Now I have a blank page, which is where I wrote the examples.”

Last time, we spent some time defining the affine Grassmannian and thinking about it from a functor-of-points perspective. Today we want to focus on its geometry, and how we're going to think about it; this will be more useful than being vaguely afraid that it's some terrifying infinite-dimensional space.

Hecke operators. We saw in Tom's talk that there are Hecke operators which act on the Bruhat-Tits tree, and that they commute. Hecke operators exist in much greater generality, but are usually a hair more complicated.

Fix a curve C and reductive group G ; then,

$$\mathcal{H}ecke := \{(c, P_1 \xrightarrow{G} C, P_2 \xrightarrow{G} C, \beta: P_1|_{C \setminus c} \xrightarrow{\cong} P_2|_{C \setminus c})\}.$$

That is, $\mathcal{H}ecke$ is the pairs of principal G -bundles and isomorphisms between them away from a point.

There is a correspondence

$$\begin{array}{ccc} & \mathcal{H}ecke & \\ \swarrow & & \searrow \\ C \times \mathrm{Bun}_G(C) & & \mathrm{Bun}_G(C), \end{array}$$

by forgetting (P_2, β) , resp. (c, P_1, β) . Let \underline{G} denote the trivial principal G -bundle; then, for any $c \in C$, the point $(c, \underline{G}) \in C \times \mathrm{Bun}_G(C)$. Using this, we can define the affine Grassmannian, and $\mathcal{H}ecke$ contains Gr_G in a way compatible with the forgetful map $\mathcal{H}ecke \rightarrow C \times \mathrm{Bun}_G(C)$.

If you wanted to do this with nontrivial principal G -bundles, you would obtain some kind of twisted version of the affine Grassmannian. Twisted means sheaves, so one might imagine that some class of sheaves on Gr_G act on some class of sheaves on Bun_G . This is true, and important: we'll learn about perverse sheaves this semester, and they will act on \mathcal{D} -modules on Bun_G . In this sense, geometric Satake acts on geometric Langlands.

There is a classical perspective on this. Instead of $\mathcal{H}ecke$, we consider the space of data (c, E, ℓ) , where $c \in C$, $E \rightarrow C$ is a rank- n vector bundle, and ℓ is a line in $E|_c$. This fibers over $C \times \mathrm{Bun}_n(C)$ (where Bun_n is the moduli space of vector bundles), and the fiber is \mathbb{P}^{n-1} , which is contained in the affine Grassmannian. You can think of this \mathbb{P}^{n-1} as all of the modifications you can make to your vector bundle in a neighborhood of a point when you remove that point, and the Hecke-like operators act as averaging, like for the Bruhat-Tits tree. These are called *elementary modifications* in algebraic geometry.

Some (not too hard) combinatorics. This is a lot of data, but it will all be useful soon.

Let G be a reductive group, B be a Borel subgroup, and T be a maximal torus, such that $T \subset B \subset G$.

Example 4.1. If you chose $G = \mathrm{GL}_n$, then one can let B be the subgroup of upper triangular matrices and T be the subgroup of diagonal matrices. \blacktriangleleft

Let Φ denote the set of *roots* of (G, T) , with respect to the adjoint action of T on \mathfrak{g} . That is, G acts on its Lie algebra \mathfrak{g} by conjugation, and the action of $t \in T$ on $x \in \mathfrak{g}$ is denoted $x \mapsto \mathrm{Ad}_t(x)$. Under this action, \mathfrak{g} decomposes as a direct sum

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

Here, \mathfrak{t} is called the *zero-weight space*. Concretely, we realize Φ as a subset of \mathfrak{t}^* .

The choice of torus and Borel subgroup defines for us a decomposition $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}$, where \mathfrak{n} is the Lie algebra of the normalizer of the maximal torus.¹³

Definition 4.2. We introduce the following notation.

- Let $\Phi^+ \subset \Phi$ be the *positive roots*, i.e. those in $\mathfrak{b} := \text{Lie}(B)$.
- $\Phi_s \subset \Phi^+$ will denote the *simple roots*, those which are not sums of other positive roots in a nontrivial way.
- Let $X^*(T) := \text{Hom}(T, \mathbb{G}_m)$, the *character lattice*, which contains Φ .
- Let $X_*(T) := \text{Hom}(\mathbb{G}_m, T)$, the *cocharacter lattice*, which contains Φ^\vee , the coroots (see below).

For each root α , there exists a *coroot* $\alpha^\vee \in X_*(T)$ satisfying

$$\alpha \cdot \alpha^\vee = z^2,$$

where $\alpha \cdot \lambda(z) := z^{\langle \alpha, \lambda \rangle}$.

Definition 4.3. More notation:

- Let $Q := \mathbb{Z}\Phi$ be the *root lattice*.
- Let $Q^\vee := \mathbb{Z}\Phi^\vee$, the *coroot lattice*.

There's a partial order on Q^\vee , extending to $X_*(T)$, where $0 \leq \lambda$ iff λ is a nonnegative combination of positive coroots.

Definition 4.4. The *dominant cocharacters* are

$$X_*(T)_+ := \{\lambda \in X_*(T) \mid \langle \lambda, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Phi^+\}.$$

Finally, and most mysteriously, let

$$\rho := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha,$$

which defines a function $X_*(T) \rightarrow \frac{1}{2}\mathbb{Z}$. The appearance of ρ is a subtle and mysterious aspect of the representation theory of compact Lie groups or algebraic groups; David has a MathOverflow post with a good explanation.¹⁴

Okay, now for some examples.

Example 4.5. Let $G = \text{SL}_2(\mathbb{C})$. We can then choose

$$(4.6) \quad \begin{aligned} B &= \left\{ \begin{pmatrix} \lambda & \mu \\ 0 & \lambda^{-1} \end{pmatrix} \right\} \\ T &= \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \right\}. \end{aligned}$$

The roots are $\Phi = \{\pm 2\}$ corresponding to the functions

$$\pm \alpha: \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix} \mapsto \lambda^{\pm 2}.$$

The character lattice is hence $Q = 2\mathbb{Z} \subset \mathbb{Z} \cong X^*(T)$, where n is sent to the n^{th} power map.

The coroots are $\{\pm 1\}$, corresponding with the functions

$$\pm \alpha^\vee: \lambda \mapsto \begin{pmatrix} \lambda^{\pm 1} & \\ & \lambda^{\mp 1} \end{pmatrix}.$$

In this case, $X_*(T) \cong \mathbb{Z}$ (n sends $\lambda \mapsto (\lambda^n \lambda^{-n})$), and the cocharacter lattice Q^\vee is all of $X_*(T)$, even though the character lattice isn't all of $X^*(T)$!

In this case, the dominant cocharacters are $X_*(T)_+ \cong \mathbb{Z}_{\geq 0}$, with the partial order sent to the usual order. In this case, $\rho = (1/2) \cdot 2 = 1$. ◀

¹³**TODO:** this might be wrong; double-check.

¹⁴There are many perspectives on ρ : spin structures on the flag manifold, Serre duality, the Weyl character formula, and more.

Example 4.7. Now consider $G = \mathrm{PGL}_2(\mathbb{C}) := \mathrm{SL}_2(\mathbb{C})/\mu_2$, where $\mu_2 := \{\pm 1\}$ is the square roots of unity. We can choose B and T to be basically the same as in (4.6). In this case, we obtain a dual-looking setup (which is not a coincidence).

- The character lattice is $Q \cong \mathbb{Z}$, and this is all of

$$X^*(T) = \left\{ \left[\begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix} \right] \mapsto \lambda^{2n} \right\}_{n \in \mathbb{Z}}.$$

- The cocharacter lattice is $Q^\vee \cong 2\mathbb{Z}$ inside

$$X_*(T) = \left\{ \lambda \mapsto \left[\begin{pmatrix} \lambda^{n/2} & \\ & \lambda^{-n/2} \end{pmatrix} \right] \right\}_{n \in \mathbb{Z}}. \quad \blacktriangleleft$$

Applications to geometry of the affine Grassmannian. As before, fix $K := \mathbb{C}((t))$ and $\mathcal{O}_K := \mathbb{C}[[t]]$.

Let $\lambda: \mathbb{G}_m \rightarrow T$ be a cocharacter for G , which determines a map $K^\times \rightarrow T((t)) \subset G(\mathcal{O}_K)$ sending $t \mapsto t^\lambda$.

Let $L_\lambda := t^\lambda \cdot G(\mathcal{O}_K) \in \mathrm{Gr}_G$ (since this is $G(K)/G(\mathcal{O}_K)$), and define $\mathrm{Gr}_\lambda := G(\mathcal{O}_K) \cdot L_\lambda$. This Gr_λ is called a *Schubert cell*. Last time, in (3.11) and (3.12), we saw a decomposition of Gr_G into a disjoint union; this is actually as Schubert cells:

$$\mathrm{Gr}_G = \coprod_{\lambda \in X_*(T)_+} \mathrm{Gr}_\lambda.$$

Let

$$\mathrm{Gr}_{\leq \mu} := \bigcup_{\substack{\lambda \leq \mu \\ \lambda \in X_*(T)_+}} \mathrm{Gr}_\lambda,$$

which is called a (*spherical*) *Schubert variety* containing Gr_μ as an open subvariety.

Proposition 4.8. Gr_μ is a smooth, quasiprojective¹⁵ variety of dimension $\langle 2\rho, \mu \rangle$.

So ρ makes its first surprise appearance (as the sum of all the positive roots).

Proof. Let

$$P_\mu^\alpha := G(\mathcal{O}_K) \cap (t^\mu G(\mathcal{O}_K) t^{-\mu}),$$

which is the stabilizer for $G(\mathcal{O}_K)$ acting on Gr_G : $\mathrm{Gr}_\mu \cong G(\mathcal{O}_K)/P_\mu^a$. Thus Gr_μ is a homogeneous space for a group in characteristic 0, so it's quasiprojective. Smoothness comes from the map

$$G(\mathcal{O}_K)/P_\mu^a \longrightarrow G(K)/G(\mathcal{O}_K)$$

sending $g \mapsto g^{t^\mu}$, so it remains to calculate its dimension.

We have an identification

$$T_{L_\mu} \mathrm{Gr}_\mu \cong \mathfrak{g}(\mathcal{O}_K)/(\mathcal{G}(\mathcal{O}_K) \cap \mathrm{Ad}_{t^\mu}(\mathfrak{g}(\mathcal{O}_K))),$$

and we have the root decomposition

$$\mathfrak{g}(\mathcal{O}_K) = {}^{\mathrm{fr}}(\mathcal{O}_K) \oplus \bigoplus_{\alpha \in \Phi^+} (\mathfrak{g}_\alpha(\mathcal{O}_K) \oplus \mathfrak{g}_{-\alpha}(\mathcal{O}_K)).$$

For \mathfrak{t} , which is abelian, the adjoint action is trivial, and for $x_\alpha \in \mathfrak{g}_\alpha$,

$$\mathrm{Ad}_{t^\mu}(x_\alpha) = t^{\langle \alpha, \mu \rangle} x_\alpha.$$

Therefore we can calculate

$$(4.9) \quad \mathrm{Ad}_{t^\mu} \mathfrak{g}(\mathcal{O}_K) = \mathfrak{t}(\mathcal{O}_K) \oplus \left(\bigoplus_{\alpha \in \Phi^+} t^{\langle \alpha, \mu \rangle} \mathfrak{g}_\alpha(\mathcal{O}_K) \oplus t^{-\langle \alpha, \mu \rangle} \mathfrak{g}_{-\alpha}(\mathcal{O}_K) \right).$$

Hence

$$(4.10) \quad \mathfrak{g}(\mathcal{O}_K) \cap \mathrm{Ad}_{t^\mu}(\mathfrak{g}(\mathcal{O}_K)) = \mathfrak{t}(0) \oplus \bigoplus_{\alpha \in \Phi^+} (\mathfrak{g}_\alpha(\mathcal{O}_K) \oplus \mathfrak{g}_{-\alpha}(\mathcal{O}_K)).$$

¹⁵This means it's an open subvariety of a projective variety

Therefore

$$(4.11) \quad \mathfrak{g}(\mathcal{O}_K)/(\mathfrak{g}(\mathcal{O}_K) \cap \text{Ad}_{t^\mu} \mathfrak{g}(\mathcal{O}_K)) \cong \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha(\mathcal{O}_K)/t^{\langle \alpha, \mu \rangle} \mathfrak{g}_\alpha(\mathcal{O}_K).$$

Since $\dim(\mathfrak{g}_\alpha) = 1$, the dimension of (4.11) is

$$\sum_{\alpha \in \Phi^+} \langle \alpha, \mu \rangle = \langle 2\rho, \mu \rangle. \quad \square$$

Bam, geometry.

We also know its closure, which is a projective variety, though far from smooth, except in very special and very interesting cases.

Proposition 4.12. *The Zariski closure of Gr_μ is $\text{Gr}_{\leq \mu}$, which in particular is a projective variety.*

Proof sketch. Suppose $\lambda \geq \mu$, so that there exists a positive coroot α^\vee such that $\mu - \alpha^\vee$ is dominant and $\lambda \leq \mu - \alpha^\vee \leq \mu$. This means it suffices to prove that $t^{\mu - \alpha^\vee} \in \overline{\text{Gr}_\mu}$. To do this, construct a curve $X \subset \text{Gr}_{\leq \mu}$ such that $t^{\mu - \alpha^\vee} \in X$ and $X \cdot \{t^{\mu - \alpha^\vee}\} \subset \text{Gr}_\mu$. \square

Example 4.13. Let $G = \text{SL}_2$. For any $m \in \mathbb{Z}$,

$$t^{\lambda_m} = \begin{bmatrix} t^m & 0 \\ 0 & 1 \end{bmatrix} \in \text{PGL}_2(K).$$

Let $K_m := \text{Ad}_{t^{\lambda_m}}(\text{SL}_2(\mathcal{O}_K)) \subset \text{SL}_2(K)$. The idea is that we want to use copies of $\text{SL}_2(\mathcal{O}_K)$ to move things around in $\text{SL}_2(K)$.

Let $\{e, h, f\}$ be the standard triple of generators in \mathfrak{sl}_2 , so if \mathfrak{k}_m is the Lie algebra of K_m , $\mathfrak{k}_m = \text{span}_{\mathcal{O}_K} \{t^m e, h, t^{-m} f\}$.

Consider the point

$$\sigma_m := \begin{pmatrix} 0 & -t^m \\ t^m & 0 \end{pmatrix} \in K_m$$

and

$$L^{>0}\text{SL}_2 := \ker(\text{ev}: \text{SL}_2(\mathcal{O}_K) \rightarrow \text{SL}_2(\mathbb{C})).$$

Let $K_m^{(1)} := \text{Ad}_{t^{\lambda_m}}(L^{>0}\text{SL}_2)$.

One can show that $K_m/K_m^{(1)} \cong \text{SL}_2(\mathbb{C})$, so if $i_\alpha: \text{SL}_2(\mathbb{C}) \rightarrow G$ is the canonical homomorphism induced by $\alpha \in \Phi$, the Lie algebra of its image is $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$.

Now, let $m = \langle \mu, \alpha \rangle - 1$. Let

$$C_{\mu, \alpha} := (Li_\alpha)(K_m) \cdot L_\mu.$$

This is a homogeneous space for $\text{SL}_2(\mathbb{C})$, hence must be a \mathbb{P}^1 , and this is the curve we were looking for (and punted on in the proof of Proposition 4.12).

Exercise 4.14. Show that if B is the subgroup of upper triangular matrices inside $\text{SL}_2(\mathbb{C})$, $\text{SL}_2(\mathbb{C})/B \cong \mathbb{P}^1$.

Therefore one can make a *balloon decomposition* of the entire space as a sequence of \mathbb{P}^1 “balloons,” akin to a Morse decomposition of a manifold. There’s a great paper of Goresky-MacPherson which shows how incredibly useful this is. \blacktriangleleft

Parity of Schubert cells. The Schubert cells have a natural even-odd parity directly analogous to the parity in the Bruhat-Tits tree for SL_2 , defined by the map $p: X_*(T) \rightarrow \mathbb{Z}/2$ sending

$$\mu \mapsto (-1)^{\langle 2\rho, \mu \rangle}.$$

If μ is a coroot, so that $\langle \rho, \mu \rangle \in \mathbb{Z}$, this defines a map $X_*(T)/Q^\vee \rightarrow \mathbb{Z}/2$. But $X_*(T)/Q$ is something we’ve seen before: $\pi_1(G)$!¹⁶ This is because $\pi_1(G) \cong \pi_0(\text{Gr}_G)$, because $\pi_1(G) = \pi_0(LG)$, and $\text{Gr}_G = LG L^+ G$ and $L^+ G$ is connected.

Therefore we have a map $p: \pi_0(\text{Gr}_G) \rightarrow \mathbb{Z}/2$, which decomposes Gr_G into even and odd pieces $\text{Gr}_G^{(+)}$ and $\text{Gr}_G^{(-)}$. We say Gr_μ is *even*, resp. *odd*, if it’s in $\text{Gr}_G^{(+)}$, resp. $\text{Gr}_G^{(-)}$. By (4.8), Gr_μ is even (resp. odd) iff $\dim \text{Gr}_\mu$ is even (resp. odd). This means the parity map doesn’t depend on the choice of B or T , which is nice.

¹⁶There are various notions of the fundamental group in algebraic geometry, but we’re over \mathbb{C} , so they coincide here. One concrete definition is the fundamental group of $G(\mathbb{C})$ in the complex topology.

Schubert cells and partial flag varieties. For $\mu \in X_*(T)$, let P_μ be the *parabolic subgroup* of G corresponding to μ . That is, P_μ is generated by the root subgroups $U_\alpha \subset G$ (isomorphic to SL_2) for $\alpha \in \Phi$ satisfying $\langle \alpha, \mu \rangle \leq 0$.

Let $ev: G(\mathcal{O}_K) \rightarrow G$ send $g(t) \mapsto g(0)$ again.

Proposition 4.15. *There is a natural projection*

$$p_\mu: \mathrm{Gr}_\mu \cong G(\mathcal{O}_K) / \underbrace{(G(\mathcal{O}_K) \cap (t^\mu G(\mathcal{O}_K) t^{-\mu}))}_{P_\mu^a} \longrightarrow G/P_\mu$$

whose fibers are affine spaces.

The quotient G/P_μ is called a *partial flag variety*, and generalizes the flag variety G/B , the variety of full flags, filtrations

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = V$$

such that $\dim V_i = i$. But we could just as well take partial flags $0 \subsetneq V_1 \subsetneq V_2 \subsetneq V_n = V$, and this is GL_n over the subgroup of upper triangular matrices. Hence these can and should be called partial flag varieties.

Proof of Proposition 4.15. Recall (4.9) and (4.10), the formulas for $\mathrm{Ad}_{t^\mu} \mathfrak{g}(\mathcal{O}_K)$ and $\mathfrak{g}(\mathcal{O}_K) \cap \mathrm{Ad}_{t^\mu} \mathfrak{g}(\mathcal{O}_K)$, respectively; evaluation defines a map

$$\mathfrak{g}(\mathcal{O}_K) \cap \mathrm{Ad}_{t^\mu} \mathfrak{g}(\mathcal{O}_K) \xrightarrow{ev} \mathfrak{t} \oplus \bigoplus_{\langle \alpha, \mu \rangle \leq 0} \mathfrak{g}_\alpha,$$

and the latter space is the Lie algebra of P_μ . That is, we have a commutative diagram

$$\begin{array}{ccc} G(\mathcal{O}_K) & \xrightarrow{ev} & G \\ \uparrow & & \uparrow \\ P_\mu^a & \xrightarrow{ev_\mu} & P_\mu. \end{array}$$

$\ker(ev) = L^{>0}G$, which is a pro-unipotent group (a limit of unipotent groups). This implies $\ker(ev)/\ker(ev_\mu)$ is a finite-dimensional unipotent group, and these are all affine spaces. \square

Example 4.16. Let $G = \mathrm{PGL}_2(\mathbb{C})$, so as in Example 4.7, $X_*(T) = \mathbb{Z}$ and $Q^\vee = 2\mathbb{Z}$ inside $X^*(T)$. The partial order is

$$\begin{aligned} \cdots &\leq -2 \leq 0 \leq 2 \leq \cdots \\ \cdots &\leq -3 \leq -1 \leq 1 \leq 3 \leq \cdots \end{aligned}$$

and $X_*(T)_+ = \mathbb{Z}_{\geq 0}$.

Now let's make some Schubert cells. The easiest is Gr_0 :

$$\mathrm{Gr}_0 = G(\mathcal{O}_K) \cdot 1 \cdot G(\mathcal{O}_K) = G(\mathcal{O}_K) = \mathrm{pt}.$$

More excitingly,

$$\mathrm{Gr}_1 = \mathrm{Gr}_{\leq 1} = G(\mathcal{O}_K) \cdot \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \cdot G(\mathcal{O}_K),$$

which is a \mathbb{P}^1 . This is all of the modifications one can make in the neighborhood of something, like we saw for the Bruhat-Tits tree.

This demonstrates an interesting phenomenon: the following are equivalence (for any reductive G):

- Gr_μ is smooth.
- $\mathrm{Gr}_\mu = \overline{\mathrm{Gr}_\mu}$.
- μ is such that $\langle \mu, \alpha \rangle \leq 1$, where $\alpha \in \Phi_+$ is a *miniscule* root.

Some groups don't even have miniscule roots, so smoothness is generally very uncommon.

Though it's complicated to write out at the board, the answer for Gr_2 is that $\mathrm{Gr}_2 \cong \mathcal{O}(-n) \rightarrow \mathbb{P}^1$ for some n , which is indeed an affine bundle. If you think in terms of lattices, these are lattices containing some given lattice for Gr_1 . Similarly, Gr_3 will be a fiber bundle over the total space for Gr_2 , which is a variety you may have encountered before. (For general groups, everything is a little more complicated.)

As soon as we understand an orbit, we can calculate its cohomology, and these have natural SL_2 -actions. The cohomology of Gr_1 is the irreducible 1-dimensional representation (the trivial one), and the cohomology of Gr_2 is the two-dimensional one. There's plenty of excellent geometry here. ◀

5. PERVERSE SHEAVES: 9/22/17

"I thought, I understood sheaves, so all I have to understand is one adjective! But it turns out perverse sheaves aren't really sheaves, and they're also not

Today, Sebastian spoke about perverse sheaves and intersection cohomology. Today is intended to be a less scary and more heuristic overview of why we want to consider perverse sheaves, with the technical details left to a future talk. Today, all sheaves are valued in \mathbb{Q} -vector spaces, and we assume all spaces are locally path-connected and locally simply connected, so that we can take universal covers of path-connected spaces.

To begin, let's recall a definition.

Definition 5.1.

- A *locally constant sheaf* \mathcal{L} on a space X is a sheaf such that for every $x \in X$, there's a neighborhood U of x such that $\mathcal{L}(U) \cong \mathcal{L}_x$.
- A *local system* is a locally constant sheaf with finite-dimensional stalks.

Example 5.2.

- (1) Let V be a vector space. Then, the *constant sheaf* \underline{V}_X is a locally constant sheaf, and if V is finite-dimensional, is a local system.
- (2) The *orientation sheaf* O_X , defined by $O_X(U) := H_n(U; \mathbb{Q})^*$, is a locally constant sheaf, and a local system on nice spaces. ◀

Given a locally constant sheaf \mathcal{L} and an $x_0 \in X$, there's a *monodromy representation* of $\pi_1(X, x_0)$ on \mathcal{L}_{x_0} : given a loop ℓ from x_0 to itself, consider an open cover \mathfrak{U} of a neighborhood of ℓ , such that we get a chain of transition maps $\mathcal{L}(U_0) \rightarrow \mathcal{L}(U_1) \rightarrow \cdots \rightarrow \mathcal{L}(U_m) \rightarrow \mathcal{L}(U_0)$; the composition of these is the automorphism we assign to ℓ , which is called its *monodromy*, and one can check this defines a group homomorphism $\pi_1(X, x_0) \rightarrow GL(\mathcal{L}_{x_0})$.

Remark. Really, this defines a functor from the category of local systems on X , as a full subcategory of $Sh(X, \mathbb{Q})$, to the category of finite-dimensional representations of $\pi_1(X, x_0)$, and this functor is an equivalence of categories! ◀

One can also ask about extensions of local systems: if $U \subset X$ is open, where X is a smooth manifold, and \mathcal{L} is a local system on U , can we extend it to X ? Is such an extension unique?

Using the equivalence of categories mentioned above, we can find a few easy-to-understand criteria.

- (1) If ℓ is a loop in U that's contractible in X , the monodromy around ℓ of \mathcal{L} must be trivial.
- (2) We need U to be "big enough," in that $\pi_1(U)$ surjects onto $\pi_1(X)$. This is automatic if $X \setminus U$ has codimension at least 2.

This is great, but what if X is singular? For concreteness, take a stratified space X , and let U be the smooth locus. If you try to extend a local system \mathcal{L} on U to X , you end up with something new — a perverse sheaf. Though these are quite technical to define, you can define an analogue of sheaf cohomology for perverse sheaves, which is good.

A perverse sheaf is, more or less, a collection of local systems on the individual strata, where the behavior on when you pass between strata depends on how severe the singularities are. The slogan is, "perverse sheaves are the singular version of local systems." For a short overview, check out <http://www.ams.org/notices/201005/rtx100500632p.pdf>.

Remark. The extension of a constant sheaf in this setting is in general *not* constant! ◀

You also get different perverse sheaves for spaces which are homotopic but not homeomorphic, which is unusual for things we're used to taking cohomology of.

Let $P(X)$ denote the category of perverse sheaves on X . Let's discuss some of its formal properties:

- $P(X)$ is an abelian category, a full subcategory of $D(X)$, the derived category of X (which itself is a full subcategory of the category of sheaves on X).
- $D(X)$ has a t -structure, and its heart $D(X)^\heartsuit$ is exactly $P(X)$.

- There's a duality endomorphism on $D(X)$, and $P(X)$ is stable under this duality. This is Verdier duality, an extension of Poincaré duality.
- Though $P(X)$ is not in general semisimple,¹⁷ it's still nice: there are finitely many simple objects, which are intersection cohomology complexes, and any object can be obtained as a sequence of extensions of simple objects. One says that $P(X)$ has *finite length*.

So one of the puzzles of perverse sheaves is that they have really nice formal properties, but the definition is a mess. Frequently, one takes them as a black box satisfying the formal properties.

Definition 5.3. A *stratified pseudomanifold* is a space X together with a filtration $X \supset X_0 \supset X_1 \supset \cdots \supset X_n = \emptyset$, such that each $X_i \subset X$ is closed and

- the k -stratum $S_k := X_k \setminus X_{k-1}$ is a topological $(n - k)$ -manifold,
- $S_1 = \emptyset$,
- if $\Sigma = X_1 = X_2$ (the *singular locus*), then $S_0 = X \setminus \Sigma$ is dense in X ,
- and one more technical consideration called local normal triviality.

The Whitney stratification of a complex variety makes it into a stratified pseudomanifold.

We want to compute (co)homology, which means we need a version of chains for these spaces.

Definition 5.4. A *geometric chain* in S_0 means a singular chain ξ in X whose support $|\xi|$ in S_0 is closed in S_0 .

These are less important than what you call cycles, though.

Definition 5.5.

- A *perversity* is a map $p: \{2, 3, \dots, n\} \rightarrow \mathbb{Z}$.
- Given an $i \in \mathbb{Z}$ and a perversity p , a $Z \subseteq X$ is (p, i) -allowable if $\dim Z \leq i$ and for all strata $S_\alpha \subset X$,

$$\dim(S_\alpha \cap Z) \leq i - \text{codim } S_\alpha + p(\alpha).$$

This is a generalization of transversality: how can a 1-chain intersect the singular locus in a reasonably transverse manner?

Definition 5.6. A geometric i -chain ξ in S_0 with coefficients in \mathcal{L} is p -allowable if

- $|\xi| \subset X$ is (p, i) -allowable, and
- $|\partial\xi|$ is $(p, i - 1)$ -allowable.

The space of p -allowable chains is denoted $I^p C_i(X, \mathcal{L})$, and these define a complex

$$\cdots \longrightarrow I^p C_{i+1}(X, \mathcal{L}) \longrightarrow I^p C_i(X, \mathcal{L}) \longrightarrow I^p C_{i-1}(X, \mathcal{L}) \longrightarrow \cdots$$

called the *intersection complex*. The homology of this complex is denoted $I^p H_i(X, \mathcal{L})$, and is called the *intersection cohomology*.

Example 5.7. Take two spheres and identify their north poles together, then their south poles together. This space X , the (unreduced) suspension of $S^1 \amalg S^1$, is a stratified pseudomanifold with singular locus $\{N, S\}$, the (identified) north and south poles. Let a be a meridian in the first sphere and b be a meridian in the second sphere.

Let $\mathcal{L} = \underline{\mathbb{Q}}$ be the constant sheaf. Its (usual) homology is:

- $H_2(X)$ is generated by $[\Sigma a]$ and $[\Sigma b]$.¹⁸
- $H_1(X)$ is generated by $\Sigma a - \Sigma b$.
- $H_0(X)$ is generated by $[a] = [b]$.

But for intersection cohomology, we get something different. For $p = -1$,

- $I^{-1} H_2(X)$ is zero, because nothing can intersect the strata with the correct dimension.
- $I^{-1} H_1(X)$ is spanned by the two equators A and B .
- $I^{-1} H_0(X)$ is generated by $[a]$ and $[b]$, which are not equal.

For $p = 0$,

¹⁷It will be for the affine Grassmannian!

¹⁸In this example, Σ denotes unreduced suspension.

- $I^0 H_2(X)$ is generated by $[\Sigma A]$ and $[\Sigma B]$.
- $I^0 H_1(X)$ is zero.
- $I^0 H_0(X)$ is spanned by $[a]$ and $[b]$, which are not equal.

For $p = 1$,

- $I^1 H_1(X)$ is spanned by $[\Sigma A]$ and $[\Sigma B]$.
- $I^0 H_1(X)$ is spanned by $[\Sigma a]$ and $[\Sigma b]$.
- $I^1 H_0(X)$ is zero. ◀

We'll define a class of spaces on which intersection cohomology interpolates between (Borel-Moore) homology and ordinary cohomology.

Definition 5.8. An n -dimensional stratified pseudomanifold is *normal* if for any $x \in \Sigma$ there's a neighborhood U of x such that $U \setminus \Sigma$ is connected.

Remark. Every stratified pseudomanifold X has a *normalization* $\pi: \tilde{X} \rightarrow X$ such that \tilde{X} is normal, $\pi|_{\pi^{-1}(S_0)}$ is one-to-one, and $\pi|_{\pi^{-1}(\Sigma)}$ is n -to-1, where n is the number of connected components of $U \setminus \Sigma$. ◀

The normalization map defines a map of chain complexes $C_*(\tilde{X}) \rightarrow C_*(X)$, such that if $t: \alpha \mapsto \alpha - 2$, we get a map on intersection cohomology

$$I^t H_*(\tilde{X}) \longrightarrow I^t H_*(X).$$

Proposition 5.9. *If X is normal, then there are chain maps $I^t C_i(X) \rightarrow C_i(X)$ and $C^{n-1}(X) \rightarrow I^0 C_i(X)$ inducing isomorphisms*

$$\begin{aligned} I^t H_*(X) &\longrightarrow H_*(X) \\ H^{n-i}(X) &\longrightarrow I^0 H_i(X). \end{aligned}$$

This is the sense in which for normal spaces, intersection homology generalizes ordinary homology and cohomology.

Recall that Poincaré duality says for a smooth, oriented manifold X ,

$$H_i(X) \otimes H_{i-1}^c(X) \longrightarrow H_0^c(X) \longrightarrow H_0^c(\text{pt}) \cong \mathbb{Q}$$

is a perfect pairing. This fails for singular spaces, but intersection cohomology fixes this!

Definition 5.10.

- Two local systems L and L' are *dual* if there is a map $L \otimes L' \rightarrow \mathcal{O}_{S_0}$ which is a perfect pairing on each fiber.
- Two perversities p and q are *dual* (written $q = p^*$) if $p + q = t$, i.e. for each α , $p(\alpha) + q(\alpha) = \alpha - 2$.

Theorem 5.11 (Goresky-MacPherson). *Let $k = \dim X$ and L and L' be dual local systems. Then, there is a perfect pairing*

$$I: I^p H_i(X, L) \otimes I^{p^*} H_{k-i}(X, L') \longrightarrow \mathbb{Q}.$$

You should think of this as an intersection number.¹⁹

For $U \hookrightarrow X$ open, we get morphisms $C_i(X) \rightarrow C_i(U)$ and $I^p C_i(X) \rightarrow I^p C_i(U)$, but for Borel-Moore chains, we get covariant functoriality: $C_i^c(U) \hookrightarrow C_i^c(X)$. This allows us to define two sheaves on X ,

$$\begin{aligned} \mathcal{D}_X^{-i}(U) &:= C_i(U) \\ I^p \mathcal{C}_X^{-i}(U) &:= I^p C_i(U). \end{aligned}$$

¹⁹It seems like the passage from Poincaré duality to this theorem has followed the usual “fancification functor” of algebraic structures, where something you thought you understood is replaced with something ostensibly geometric, more formal, and considerably more general...

These form cochain complexes $\mathcal{D}_X^\bullet, I^p\mathcal{C}_X^\bullet$ of sheaves; one can take (compactly supported) global sections to recover the original vector spaces:

$$\begin{aligned} H_i^c(X) &= H^{-i}(\Gamma_c(\mathcal{D}_X^\bullet)) \\ I^p H_i^c(X) &= H^{-i}(\Gamma_c(I^p\mathcal{C}_X^\bullet)) \\ H_i(X) &= H^{-i}(\Gamma(\mathcal{D}_X^\bullet)) \\ I^p H_i(X) &= H^{-i}(\Gamma(I^p\mathcal{C}_X^\bullet)). \end{aligned}$$

If you want something cohomological, change the grading.

Definition 5.12. The *intersection cohomology* is the hypercohomology of $I^p\mathcal{C}_X^\bullet$:

$$I^p H^k(X) := \mathbb{H}^k(X, I^p\mathcal{C}_X^\bullet).$$

Thus $I^p H^k(X) = I^p H_{-k}(X)$. So these are concentrated in negative degrees, which is a little strange, but ultimately okay. And what's good about this is that you can do it sheafily, since everything is local.

6. THE DERIVED CATEGORY THEORY SURROUNDING PERVERSE SHEAVES: 9/29/17

These are Arun's prepared notes for his talk.

6.1. The six-functor formalism and Verdier duality. We'll begin with the six-functor formalism as an introduction to Verdier duality. There will be no proofs.

Motivated by the fundamental importance of Poincaré duality in algebraic topology, we're going to try to make it work in algebraic geometry. On smooth varieties, everything is fine — but it does not work in general. We'll present two ways to fix it: the first relaxes to the derived category, but it turns out we'll be able to establish a duality on a curious abelian subcategory of that category, which is the category of perverse sheaves.

Recall that if $f: X \rightarrow Y$ is a morphism of schemes, it defines an adjunction (f^*, f_*) between the direct and inverse image functors:

$$f_*(\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))f^*(\mathcal{G}) = \left(V \mapsto \varinjlim_{f(U) \subseteq V} \mathcal{G}(V) \right)^{\text{sh}},$$

and a third covariant functor $f_!$ which only keeps the $s \in \mathcal{F}(f^{-1}(U))$ for which $f|_{\text{supp}(s)}$ is proper. We'd like this to also have an adjoint, and in order to do so, we must pass to the derived category. In this case, $Rf_!$ has a right adjoint, $Rf^!: D^b(Y) \rightarrow D^b(X)$. Thus we have six functors associated to derived categories of sheaves, which come in three adjoint pairs:

- $(-\otimes^L \mathcal{F}, R\mathcal{H}om(\mathcal{F}, -)),$
- $(Rf^*, f_*),$ and
- $(Rf_!, Rf^!).$

Henceforth we will drop the L s and R s. There's a natural transformation $f_! \rightarrow f_*$, an isomorphism when f is proper, and $f^! = f^*$ if f is an open embedding.

Let $p: X \rightarrow \text{pt}$ denote the crush map and $\omega_X := p^!(\mathbb{C}) \in D^b(X)$. This is in general a complex of sheaves, and is called the *dualizing complex*.

These functors are in particular generalizations of cohomology. Recall that H^* is the right derived functor of the global sections functor $\Gamma = p_*$. We also have

$$\begin{aligned} H^*(X) &= H^*(p_*\mathbb{C}) & H_*^!(X) &= H^*(f_*\omega_X) \\ H_*^*(X) &= H^n(p_!\mathbb{C}) & H_*(X) &= H^{-n}(f_!\omega_X). \end{aligned}$$

Now, define $\mathbb{D} := \mathcal{H}om(-, \omega_X)$, which is an endofunctor of $D^b(X)$.

Theorem 6.1 (Verdier duality).

- (1) *There's a natural isomorphism $\mathbb{D}^2 \simeq \text{id}$.*
- (2) *A map $f: X \rightarrow Y$ induces a natural isomorphism $D_Y \circ f_! = f_* \circ \mathbb{D}_X$.*
- (3) *Verdier duality reduces to Poincaré duality in that if X is smooth and \mathcal{L} is a local system on X , $\mathbb{D}\mathcal{L} \cong \mathcal{L}^\vee[2 \dim X]$.*

This dimension-shifting result is really nice, but it just doesn't work on singular varieties: you might imagine understanding a complex on each stratum of, say, its Whitney stratification, and these strata will have different dimensions. So if we'd like a nice duality result, we'll have to restrict to a class of sheaves which account for this offset.

6.2. The perverse t -structure. We're going to define the category of perverse sheaves using an unusual t -structure on $D^b(X)$.

t -structures are formalizations of two things you've probably already encountered, where shifts of something simpler generate a triangulated category.

- If \mathbf{A} is an abelian category, there's a fully faithful functor $\mathbf{A} \rightarrow D(\mathbf{A})$ sending a complex A to $\cdots \rightarrow 0 \rightarrow A \rightarrow 0 \rightarrow \cdots$, and this sends short exact sequences to distinguished triangles. Moreover, the entire derived category is built from these and "attaching maps."
- In algebraic topology, one can understand a space in terms of its Postnikov decomposition, which builds it as a tower from attaching maps between spaces $K(G, n)$ with a single homotopy group. A fiber sequence defines a long exact sequence on homotopy groups (better: a distinguished triangle in the homotopy category).

Definition 6.2. Let \mathcal{C} be a triangulated category. A t -structure on \mathcal{C} is a pair $\mathcal{C}^{\geq 0}$ and $\mathcal{C}^{\leq 0}$ of full subcategories such that

- if $\mathcal{C}^{\leq i} = \mathcal{C}^{\leq 0}[i]$ and $\mathcal{C}^{\geq i} = \mathcal{C}^{\geq 0}[i]$, then $\text{Hom}(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 1}) = 0$.
- $\mathcal{C}^{-1} \subseteq \mathcal{C}^{\leq 0}$ and $\mathcal{C}^{\geq 1} \subseteq \mathcal{C}^{\geq 0}$.
- any $X \in \mathcal{C}$ belongs to a distinguished triangle

$$A \longrightarrow X \longrightarrow B \longrightarrow A[1],$$

where $A \in \mathcal{C}^{\leq 0}$ and $B \in \mathcal{C}^{\geq 1}$.

In this case, we make the following additional definitions:

- The inclusion $\mathcal{C}^{\leq n}$ has a right adjoint, which we denote $\tau_{\leq n}$.
- Similarly, the inclusion $\mathcal{C}^{\geq n}$ has a left adjoint, which we denote $\tau_{\geq n}$.
- A t -structure is *nondegenerate* if the intersection of all of the subcategories $\mathcal{C}^{\geq n}$ is empty, and the intersection of all of the $\mathcal{C}^{\leq n}$ is empty.
- The *heart* of a t -structure is $\mathcal{C}^{\heartsuit} := \mathcal{C}^{\geq 0} \cap \mathcal{C}^{\leq 0}$.

Definition 6.3. Let \mathcal{C} be a triangulated category and \mathbf{A} be an abelian category. A functor $F: \mathcal{C} \rightarrow \mathbf{A}$ is *cohomological* if it sends a distinguished triangle $A \rightarrow B \rightarrow C \rightarrow A[1]$ to a long exact sequence

$$\cdots \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow F(A[1]) \longrightarrow \cdots$$

The most important example is the i^{th} cohomology sheaf $\mathcal{H}^i(X) := \tau_{\geq 0}\tau_{\leq 0}(X[i])$.

Theorem 6.4. The heart of a t -structure is an abelian category, and the functor $\tau_{\leq 0}\tau_{\geq 0}: \mathcal{C} \rightarrow \mathcal{C}^{\heartsuit}$ is cohomological.

Example 6.5.

- (1) Indeed, if X is a variety, $D^b(X)$ has a t -structure in which $D^{\leq 0}$ is the complexes whose cohomology vanishes in all negative gradings, and $D^{\geq 0}$ is the complexes whose cohomology vanishes in all positive gradings. You can check this is a triangulated structure and that its heart is the category of sheaves on X .
- (2) The stable homotopy category \mathbf{Spc} has a t -structure in which $\mathbf{Spc}^{\geq 0}$ is the *connective spectra* (those whose negative homotopy groups are zero) and $\mathbf{Spc}^{\leq 0}$ is the *coconnective spectra* (positive homotopy groups are zero). The heart of this t -structure is the subcategory of spectra with only a single homotopy group in degree 0, which is the Eilenberg-Mac Lane spectra, and this is equivalent to \mathbf{Ab} . ◀

But we can define another, different t -structure on $D^b(X)$, and its heart is the perverse sheaves.

Let X be a space with a stratification

$$X = \coprod_{\lambda \in \Lambda} X_{\lambda},$$

such that $D^b(X_\lambda)$ is preserved by Verdier duality (e.g. any Whitney stratification). Let $d_\lambda := \dim(X_\lambda)$ and $i_\lambda: X_\lambda \hookrightarrow X$ be inclusion. Let²⁰

$$\begin{aligned} {}^pD_\lambda^{\leq 0} &:= \{\mathcal{F} \in D_{\text{const}}^b(X_\lambda) \mid \mathcal{H}^i(\mathcal{F}) = 0 \text{ for } i > -d_\lambda\} \\ {}^pD_\lambda^{\geq 0} &:= \{\mathcal{F} \in D_{\text{const}}^b(X_\lambda) \mid \mathcal{H}^i(\mathcal{F}) = 0 \text{ for } i < -d_\lambda\}; \end{aligned}$$

the intersection of these two subcategories is $\text{Loc}(X_\lambda)[d_\lambda]$. In particular, this shift is preserved by Verdier duality, and is the only shift on X_λ that is.

Lemma 6.6. $({}^pD_\lambda^{\leq 0}, {}^pD_\lambda^{\geq 0})$ is a t -structure on X_λ .

Proof. Any shift of a t -structure is a t -structure, so we may replace $-d_\lambda$ by 0, and then we have the standard t -structure on $D^b(X_\lambda)$. \square

Definition 6.7. The *perverse t -structure* on $D^b(X)$ is specified by

$$\begin{aligned} {}^pD^{\leq 0} &:= \{\mathcal{F} \in D^b(X) \mid i_\lambda^* \mathcal{F} \in {}^pD_\lambda^{\leq 0} \text{ for all } \lambda \in \Lambda\} \\ {}^pD^{\geq 0} &:= \{\mathcal{F} \in D^b(X) \mid i_\lambda^* \mathcal{F} \in {}^pD_\lambda^{\geq 0} \text{ for all } \lambda \in \Lambda\}. \end{aligned}$$

Its heart, denoted $\mathbf{P}_\Lambda(X)$, is the category of *perverse sheaves* on X with respect to the stratification Λ .

Proposition 6.8. *This is indeed a t -structure, and $\mathbf{P}_\Lambda(X)$ is an abelian category.*

To prove this, we need to use something about geometry.

Theorem 6.9 (Recollement). *Let $i: Z \hookrightarrow X$ be a closed embedding, where X is quasiprojective, $U := X \setminus Z$, and $j: U \hookrightarrow X$. Then,*

- (1) $j^* i_* = 0$, $i^* j_! = 0$, and $i^! j_* = 0$, so $\text{Hom}(j_! A, i_* B) = 0$ and $\text{Hom}(i_* A, j_* B) = 0$.
- (2) For any $A \in D^b(X)$, the adjunction maps define distinguished triangles

$$\begin{aligned} j_! j^! A &\longrightarrow A \longrightarrow i_* i^* A \longrightarrow \\ i_! i^! A &\longrightarrow A \longrightarrow j_* j^* A \longrightarrow \end{aligned}$$

- (3) $i_* = i_!$, and $j^! = j^*$ is fully faithful, so there are natural isomorphisms $i^* i_* \Rightarrow \text{id} \Rightarrow i^! i_!$ and $j^* j_* \Rightarrow \text{id} \Rightarrow j^! j_!$.

Theorem 6.10 (Beilinson-Bernstein-Drinfeld). *Assume the setup of the previous theorem,²¹ and let $(D_Z^{\leq 0}, D_Z^{\geq 0})$ and $(D_U^{\leq 0}, D_U^{\geq 0})$ be t -structures on $D^b(Z)$ and $D^b(U)$ respectively. Then, the full subcategories*

$$\begin{aligned} D^{\leq 0} &= \{X \mid i^* X \in D_Z^{\leq 0} \text{ and } j^* X \in D_U^{\leq 0}\} \\ D^{\geq 0} &= \{X \mid i^* X \in D_Z^{\geq 0} \text{ and } j^* X \in D_U^{\geq 0}\} \end{aligned}$$

defines a t -structure on $D^b(X)$.

Proof of Proposition 6.8. Induct across the stratification Λ : if X_0 denotes the smallest nonempty stratum, then $X_0 \hookrightarrow X$ is closed, so we're in the situation of Theorems 6.9 and (6.10). The perverse t -structure on X is a t -structure if the t -structure $({}^pD_0^{\leq 0}, {}^pD_0^{\geq 0})$ on X_0 is and if the perverse t -structure on $X \setminus X_0$ is, so it suffices by induction to think about X_0 , and we already proved this is a t -structure on X_0 in Lemma 6.6. \square

To eliminate the dependence on the stratification, we do the same thing we did for Čech cohomology: a refinement Λ' of Λ defines a fully faithful embedding $\mathbf{P}_\Lambda(X) \hookrightarrow \mathbf{P}_{\Lambda'}(X)$, and taking the colimit across all of these embeddings, we obtain the category of *perverse sheaves* $\mathbf{P}(X)$. (So a perverse sheaf is a perverse sheaf with respect to some stratification, and two perverse sheaves are isomorphic if they agree on some common refinement.)

²⁰Here \mathcal{H} denotes the i^{th} cohomology, as a sheaf on X , and $D_{\text{const}}^b(X)$ denote the full subcategory of sheaves with locally constant cohomology sheaves.

²¹The proof is categorical, so you could forget about the varieties and assume only that you have triangulated categories satisfying the consequences of that theorem.

6.3. Nonetheless, perverse sheaves have nice properties.

Theorem 6.11. \mathbb{D} exchanges ${}^pD^{\leq 0}$ and ${}^pD^{\geq 0}$, so \mathbb{D} preserves the heart and therefore is an involution on $\mathcal{P}(X)$.

Proposition 6.12. A complex \mathcal{F} is a perverse sheaf iff for all i ,

- (1) $\dim \operatorname{supp} \mathcal{H}^i(\mathcal{F}) \leq i$, and
- (2) $\dim \operatorname{supp} \mathcal{H}^i(\mathbb{D}(\mathcal{F})) \leq -i$.

Proposition 6.13. $\mathcal{P}(X)$ is Noetherian and Artinian, meaning ascending and descending chains of objects in it stabilize.

This is *not* true for constructible sheaves, which are Noetherian but not Artinian.

Proposition 6.14. Let $Y \subset X$ be a smooth, locally closed subvariety and $\mathcal{L} \in \mathcal{L}\mathcal{O}\mathcal{C}(Y)$, and let $d_Y := \dim(Y)$. Then, there is a perverse sheaf $IC(Y, \mathcal{L}) \in \mathcal{P}(X)$ such that the following are true, and it is unique up to unique isomorphism.

- (1) If $i < -d_Y$, $\mathcal{H}^i(IC(Y, \mathcal{L})) = 0$.
- (2) $\mathcal{H}^{-d_Y}(IC(Y, \mathcal{L}))|_Y \cong \mathcal{L}$.
- (3) The inequalities in Proposition 6.12 for $IC(Y, \mathcal{L})$ are strict.

This perverse sheaf is called the *intersection cohomology complex*. Here are a few more nice properties.

Proposition 6.15. Let Y and \mathcal{L} be as above, and $j: Y \hookrightarrow X$ be the embedding.

- (1) $\mathcal{H}^i(IC(Y, \mathcal{L})) = 0$ unless $i \in [-d_Y, 0]$,
- (2) $\mathcal{H}^{-d_Y}(IC(Y, \mathcal{L})) \cong \mathcal{H}^0(j_*\mathcal{L})$, and
- (3) $IC(Y, \mathcal{L}^*) \cong IC(Y, \mathcal{L})^\vee$. (Here $*$ denotes duality of local systems and $^\vee$ denotes Verdier duality).

Proposition 6.16. Let Y and \mathcal{L} be as above and $U \subset Y$ be an open subvariety. The intersection complex $IC(Y, \mathcal{L})$ is the minimal perverse sheaf extending its restriction to U , where “minimal” means that it in $\mathcal{P}(X)$, it has no sub- or quotient object supported on $Y \setminus U$.

Theorem 6.17. The simple objects in $\mathcal{P}(X)$ are the intersection complexes $IC(Y, \mathcal{L})$, where

- $Y \hookrightarrow X$ is a smooth, locally closed subvariety of X , and
- \mathcal{L} is an irreducible, locally constant sheaf on Y .

Corollary 6.18.

- (1) For any $k < 0$ and intersection complexes $IC(Y, \mathcal{L})$ and $IC(Y', \mathcal{L}')$,

$$\operatorname{Ext}_{\mathcal{P}(X)}^k(IC(Y, \mathcal{L}), IC(Y', \mathcal{L}')) = 0.$$

- (2) Suppose \mathcal{L} and \mathcal{L}' are irreducible and locally constant. Then,

$$\operatorname{Hom}_{D^b(X)}(IC(Y, \mathcal{L}), IC(Y', \mathcal{L}')) = \operatorname{Hom}_{\mathcal{P}(X)}(IC(Y, \mathcal{L}), IC(Y', \mathcal{L}')) = 0.$$

Proposition 6.19 (Perverse continuation principle). Let $U \subset X$ be a smooth, Zariski-open subset. Then, any map $a: \mathcal{L} \rightarrow \mathcal{L}'$ of local systems on U uniquely extends to a map $IC(a): IC(X, \mathcal{L}) \rightarrow IC(X, \mathcal{L}')$, and the map

$$IC: \operatorname{Hom}_{\mathcal{L}\mathcal{O}\mathcal{C}(U)}(\mathcal{L}, \mathcal{L}') \longrightarrow \operatorname{Hom}_{\mathcal{P}(X)}(IC(X, \mathcal{L}), IC(X, \mathcal{L}'))$$

is an isomorphism.

Theorem 6.20 (Perverse Artin vanishing). Let P be a perverse sheaf on an affine variety X .

- $\mathcal{H}^i(P) = 0$ if $i \notin [-\dim X, 0]$.
- $\mathcal{H}_c^i(P) = 0$ if $i \notin [0, \dim X]$.

Partial proof. We'll use the Grothendieck composition-of-functors spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{H}^q(P)) \implies H^{p+q}(P).$$

We know $\dim \operatorname{supp} \mathcal{H}^q(P) \leq -q$, and $Y_q := \operatorname{supp} \mathcal{H}^q(P)$ is a closed affine subvariety. Using Artin vanishing for constructible sheaves, $H^p(Y_q, \mathcal{H}^q(P)) = 0$ when $p > -q \geq \dim Y_q$, so if $p + q > 0$, $E_2^{p,q}$ vanishes, and therefore the E_∞ -page must also vanish there, providing the upper bound.

The rest of the proof uses Verdier duality... somehow. \square

Corollary 6.21 (Perverse Lefschetz hyperplane theorem). *Let X be a quasiprojective variety, $P \in \mathcal{P}(X)$, and $X_1 \subseteq X$ be a generic hyperplane section.²² The map $H^i(X; P) \rightarrow H^i(X_1; P|_{X_1})$ is*

- *an isomorphism if $i \leq -2$, and*
- *injective for $i = -1$.*

There's a similar statement for cohomology with compact supports.

Proof when X is projective. Let $U := X \setminus X_1$, $i: X_1 \hookrightarrow X$ be inclusion, and $j: U \hookrightarrow X$ be inclusion. The distinguished triangle $j_* j^! P \rightarrow P \rightarrow i_* i^* P \rightarrow$ (from Theorem 6.9) induces a long exact sequence in cohomology (not cohomology sheaves):

$$\cdots \longrightarrow H^k(X; j_! j^* P) \longrightarrow H^k(X; P) \longrightarrow H^k(X_1; P|_{X_1}) \longrightarrow H^{k+1}(X; j_* j^! P) \longrightarrow \cdots$$

Now, it suffices to show that when $k < 0$,

$$H^k(X, j_! j^* P) = H_c^k(X, j_! j^* P) \cong H_c^k(U; j^* P) = 0,$$

and since U is affine, this is Theorem 6.20. (The equivalence of H and H_c on X is what used the assumption of projectivity.) \square

²²This means: X is a locally closed subset of some \mathbb{P}^N ; choose a generic hyperplane $H \subset \mathbb{P}^N$ and let $X_1 := H \cap X$.