M392C NOTES: A COURSE ON SEIBERG-WITTEN THEORY AND 4-MANIFOLD TOPOLOGY

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These notes were taken in UT Austin's M392C (A course on Seiberg-Witten theory and 4-manifold topology) class in Spring 2016, taught by Tim Perutz. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Any mistakes in the notes are my own.

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Lecture 1.

Classification problems in differential topology: 1/18/18

"This is my opinion, but it's the only reasonable opinion on this topic."

This course will be on gauge theory; specifically, it will be about Seiberg-Witten theory and its applications to the topology of 4-manifolds. The course website is `https://www.ma.utexas.edu/users/perutz/GaugeTheory.html`; consult it for the syllabus, assignments, etc.

The greatest mystery in geometric topology is: *what is the classification of smooth, compact, simply-connected four-manifolds up to diffeomorphism?* The question is wide open, and the thoery behaves very differently than the theory in any other dimension.

There's a fascinating bit of partial information known, mostly via PDEs coming from gauge theory, e.g. the *instanton equation* $F_A^+ = 0$ as studied by Donaldson, Uhlenbeck, Taubes, and others. More recently, people have also studied the *Seiberg-Witten equations*

$$D_A\psi = 0 \tag{1.1a}$$

$$\rho(F_A^+) = (\psi \otimes \psi^*)_0. \tag{1.1b}$$

Even without defining all of this notation, it's evident that the Seiberg-Witten equations are more complicated than the instanton equation, and indeed they were discovered later, by Seiberg and Witten in 1994. However, they're much easier to work with — after their discovery, the results of Donaldson theory were quickly reproven, and more results were found, within the decade after their discovery. This course will focus on results from Seiberg-Witten theory.

In some sense, this is a closed chapter: the stream of results on 4-manifolds has slowed to a trickle. But Seiberg-Witten theory has in the meantime found new applications to 3-manifolds, contact topology (including the remarkable proof of the Weinstein conjecture by Taubes), knots, high-dimensional topology, Heegaard-Floer homology, and more. Throughout this constellation of applications, there are many results whose only known proofs use the Seiberg-Witten equations.

The central problem in differential topology is to classify manifolds up to diffeomorphism. To make the problem more tractable, let's restrict to smooth, compact, and boundaryless. An ideal solution would solve the following four problems for some class of manifolds (e.g. compact of a particular dimension, and maybe with some topological constraints).

(1) Write down a set of "standard manifolds" $\{X_i\}_{i\in I}$ such that each manifold is diffeomorphic to precisely one X_i. For example, a list of diffeomorphism classes of closed oriented connected surfaces is given by the sphere and the n-holed torus for all $n \geq 0$.

(2) Given a description of a manifold M, a way to compute invariants to decide for which $i \in I$ $M \cong X_I$. For example, if M is a closed, connected, oriented surface, we can completely classify it by its Euler characteristic.

   A variant of this problem asks for an explicit algorithm to do this when M is encoded with finite information, e.g. a solution set to polynomial equations in $\mathbb{R}^N$ with rational coefficients.

(3) Given M and M', compute invariants to decide whether M is diffeomorphic to M'; once again, there's an algorithmic variant to that problem.

(4) Understand families (fiber bundles) of manifolds diffeomorphic to M. In some sense, this means understanding the homotopy type of the topological group $\mathrm{Diff}(M)$ of self-diffeomorphisms of M.

This is an ambitious request, but much is known in low dimensions. In dimension 1, the first three questions are trivial, and the last is nontrivial, but solved.

Example 1.2. For compact, orientable, connected surfaces, we have a complete solution: a list of diffeomorphism classes is the sphere and $(T^2)^{\# g}$ for all $g \geq 0$, and the Euler characteristic $\chi \coloneqq 2 - 2g$ is a complete invariant which is algorithmically computable from any reasonable input data, solving the second and third questions. Here, "reasonable input data" could include a triangulation, a good atlas (meaning nonempty intersections are contractible), or monodromy data for holomorphic a branched covering map $\Sigma \to S^2$, where here we're thinking of surfaces as Riemann surfaces, with chosen complex structures. Here, the Riemann-Hurwitz formula can be used to compute the Euler characteristic.

For the fourth question, let $\mathrm{Diff}^+(\Sigma)$ denote the topological group of orientation-preserving self-diffeomorphisms of Σ.

Theorem 1.3 (Earle-Eells)**.**

- *The inclusion* $\mathrm{SO}_3 \hookrightarrow \mathrm{Diff}^+(S^2)$ *is a homotopy equivalence.*

- *The identification* $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$ *defined a map* $T^2 \hookrightarrow \mathrm{Diff}^+(T^2)$ *as translations; this map is a homotopy equivalence into the connected component of the identity in* $\mathrm{Diff}^+(T^2)$*, and* $\pi_0 \mathrm{Diff}^+(T^2) \cong \mathrm{SL}_2(\mathbb{Z})$*.*
- *If* $g > 1$*, every connected component of* $\mathrm{Diff}^+(\Sigma_g)$ *is contractible, and the* mapping class group $\mathrm{MCG}(\Sigma_g) := \pi_0 \mathrm{Diff}^+(\Sigma_g)$ *is a finitely presented infinite group which acts with finite stabilizers on a certain contractible manifold called* Teichmüller space.[1]

So all four questions have satisfactory answers, though understanding the mapping class groups of surfaces is still an active area of research. ◀

Example 1.4. The classification of compact, orientable 3-manifolds looks remarkably similar to the classification of surfaces (albeit much harder!), through a vision of Thurston, realized by Hamilton and Perelman. The solution is almost as complete. The proof uses geometry, and nice representatives are quotients by groups acting on hyperbolic space.

As for invariants, the fundamental group is very nearly a complete invariant.[2] ◀

In higher dimensions, there are a few limitations. Generally, the index set I will be uncountable. For example, there are an uncountable number of smooth 4-manifolds homeomorphic to $\mathbb{R}^4$! So there will be no nice list, and no nice moduli space either. But restricted to compact manifolds, there are countably many classes, which follows from triangulation arguments or work of Cheeger in Riemannian geometry.

The next obstacle involves the fundamental group. If M is presented as an *n-handlebody* (roughly, a CW complex with cells of dimension at most n), there is an induced presentation of $\pi_1(M)$, and if M is compact, this is a finite presentation (finitely many generators, and finitely many relations).

Fact. For each $n \geq 4$, all finite presentations arise from compact n-handlebodies (namely, closed n-manifolds). ◀

This is pretty cool, but throws a wrench in our classification goal.

Theorem 1.5 (Markov)**.** *There is no algorithm that decides whether a given finite group presentation gives the trivial group.*

The proof shows that an algorithm which could solve this problem could be used to construct an algorithm that solves the halting problem for Turing machines.

Corollary 1.6. *There is no algorithm to decide whether a given n-handlebody, $n \geq 4$, is simply connected.*

This means that a general classification algorithm cannot possibly work for $n \geq 4$; thus, we will have to restrict what kinds of manifolds we classify.

A third issue in higher-dimensional topology is that in dimension $n \leq 3$, there are existence and uniqueness theorems of "optimal" Riemannian metrics (e.g. constraints on their isometry groups), but for $n \geq 5$, this is not true for any sense of optimal; some choices fail existence, and others fail uniqueness. This is discussed further (and more precisely) in Shmuel Weinberger's "Computers, Rigidity, and Moduli," which has some very interesting things to say about the utility of Riemannian geometry to classify manifolds (or lack thereof).

So four dimensions is special, but for many reasons, not just one.

Those setbacks notwithstanding, we can still say useful things.

- We will restrict to closed manifolds.
- We will focus on the simply-connected case, eliding Markov's theorem.[3]

With these restrictions, we have good answers to the first three questions.

Example 1.7. There is a countable list of compact, simply-connected 5-manifolds, and invariants (cohomology, characteristic classes) which distinguish any two. ◀

Example 1.8. Kervaire-Milnor produced a classification of homotopy spheres in dimensions $5 \leq n \leq 18$, and a conceptual answer in higher dimensions, and further work has applied this in higher dimensions. ◀

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[1]Heuristically, but not literally, Teichmüller space is a classifying space for this group.

[2]The fundamental group cannot distinguish lens spaces, and that's pretty much the only exception.

[3]More generally, one could pick some fixed group G and ask for a classification of closed n-manifolds with $\pi_1(M) \cong G$; people do this, but we won't worry about it.

There is a wider range of conceptual answers to all four questions, more or less explicit, through *surgery theory*, when $n \geq 5$ (surgery theory fails radically in dimension 4). This gives an answer to the following questions.

- Given a finite, n-dimensional CW complex X (where $n \geq 5$), when is it the homotopy type of a compact n-manifold?
- Given a simply-connected compact manifold M, what are the diffeomorphism types of manifolds homotopy equivalent to M? (Again, we need $\dim M \geq 5$.)

Here are necessary and sufficient conditions for the existence question.

- X must be an n-dimensional *Poincaré duality space*, i.e. there is a fundamental class $[X] \in H_n(X;\mathbb{Z})$ which implements the Poincaré duality isomorphism. This basic fact about closed manifolds gets you an incredibly long way towards the answer.
- Next, X must have a tangent bundle — but it's not clear what this means for a general Poincaré duality space. Here we mean a rank-n vector bundle $T \to X$ which is associated to the homotopy type in a certain precise sense: the unit sphere bundle of the stablization of T, considered as a spherical fibration, has to be manifest in X in a certain way.
- If $n \equiv 0 \bmod 4$, there's another obstruction — a certain $\mathbb{Z}$-valued invariant must vanish, interpreted as asking that $T \to X$ satisfies the Hirzebruch signature theorem: the signature of the cup product form on $H^{n/2}(X)$ must be determined by the Pontrjagin classes of T.
- If $n \equiv 2 \bmod 4$, the obstruction is a similar $\mathbb{Z}/2$-valued invariant related to the Arf invariant of the intersection form.
- If n is odd, there are no further obstructions.

That's it. Uniqueness is broadly similar — once you specify a tangent bundle, there are only finitely many diffeomorphism types!

Now we turn to dimension 4, the hardest case. We want to classify smooth, closed, simply-connected 4-manifolds. The first basic invariant (even of 4-dimensional Poincaré duality spaces) is the intersection form Q_P, which we'll begin studying in detail next week. You can realize it as a unimodular matrix modulo integral equivalence. That is, it's a symmetric square matrix over $\mathbb{Z}$ with determinant ± 1, and integral equivalence means up to conjugation by elements of $\mathrm{GL}_b(\mathbb{Z})$.

Theorem 1.9 (Milnor)**.** *The intersection form defines a bijection from the set of homotopy classes of 4-dimensional simply-connected Poincaré spaces to the set of unimodular matrices modulo equivalence.*

So this form captures the entire homotopy type! That's pretty cool.

Theorem 1.10 (Freedman)**.** *The intersection form defines a bijection from the set of homeomorphism classes of 4-dimensional simply-connected topological manifolds to the set of unimodular matrices modulo equivalence.*

Thus this completely classifies (closed, simply-connected) topological four-manifolds. This theorem won Freedman a Fields medal.

The next obstruction, having a tangent bundle, is a mild constraint told to us by Rokhlin.

Theorem 1.11 (Rokhlin)**.** *Let X be a closed 4-manifold. If Q_X has even diagonal entries, then its signature is divisble by* 16*.*

The signature is the number of positive eigenvalues minus the number of negative eigenvalues. Algebra tells us this is already divisible by 8, so this is just a factor-of-2 obstruction, which is not too bad.

But the rest of the story of surgery theory is just wrong in dimension 4. This is where analysis of an instanton moduli space comes in.

Theorem 1.12 (Donaldson's diagonalizability theorem)**.** *Let X be a compact, simply-connected 4-manifold. If Q_X is positive definite, i.e. $xQ_Xx > 0$ for all nonzero $x \in \mathbb{Z}^b$, then Q_X is equivalent to the identity matrix.*

Donaldson proved this theorem as a second-year graduate student!

There's a huge number of unimodular matrices which are positive definite, but not equivalent to the identity; the first example is known as E_8. So this is a strong constraint on their realizability by 4-manifolds.

In subsequent years, Donaldson devised invariants distinguishing infinitely many diffeomorphism types within a single homotopy class. Then, from 1994 onwards, there came new proofs of these results via

Seiberg-Witten theory, which tended to be simpler,[4] and to provide sharper, more general results. We will prove several of these in the second half of the class.

Lecture 2.

Review of the algebraic topology of manifolds: 1/23/18

Though today might be review for some students, it's important to make sure we're all on the same page, and we'll get to the good stuff soon enough. We won't do too many examples today, but will see many in the future.

Cup products. Cup products make sense in a more general sense than manifolds. Let X and Y be CW complexes; then, there is a canonical induced CW structure on $X \times Y$: the product of a pair of discs is homeomorphic to a disc, and we take the cells of $X \times Y$ to be the products of cells of X and cells of Y.

Recall that the *cellular chain complex* $C_*(X)$ is the free abelian group on the set of cells, and the *cellular cochain complex* is the dual: $C^*(X) \coloneqq \mathrm{Hom}(C_*(X), \mathbb{Z})$.

Proposition 2.1 (Künneth formula)**.** *Let X and Y be CW complexes. There is a canonical isomorphism*

$$C^*(X \times Y) \cong C^*(X) \otimes C^*(Y). \tag{2.2}$$

This follows because the cells of $X \times Y$ are the products of those in X and those in Y. There is an analogue of the Künneth formula for pretty much any kind of (ordinary) cohomology theory.

The *diagonal map* $\Delta\colon X \to X \times X$ sending $x \mapsto (x, x)$ is, annoyingly, *not* a cellular map (i.e. it does not preserve the k-skeleton). However, it is homotopic to a cellular map $\delta\colon X \to X \times X$.

Definition 2.3. The *cup product of cochains* is the map $\smile\colon C^*(X) \otimes C^*(X) \to C^*(X)$ which is the composition

$$C^*(X) \otimes C^*(X) \xrightarrow[(2.2)]{\cong} C^*(X \times X) \xrightarrow{\delta^*} C^*(X).$$

We haven't said anything about coboundaries, but the cup product plays well with them, and therefore induces a cup product on cellular cohomology, $\smile\colon H^*(X) \otimes H^*(X) \to H^*(X)$. This is an associative map, and it's *graded*, meaning it sends $H^i(X) \otimes H^j(X)$ into $H^{i+j}(X)$. It's unital and *graded commutative*, meaning

$$x \smile y = (-1)^{|x||y|} y \smile x. \tag{2.4}$$

This turns $H^*(X)$ into a graded commutative ring.

Remark 2.5. The cup product is *not* graded commutative on the level of cochains. However, there are coherent homotopies between $x \smile y$ and $(-1)^{|x||y|} y \smile x$. ◂

The fact that we had to choose $\Delta \simeq \delta$ is annoying, since it's non-explicit and non-canonical. The cup product in singular cohomology does not have this problem, as you can just work with Δ itself, but the tradeoff is that the Künneth formula is less explicit.

There are a few other incarnations of the cup product which are more geometrically transparent, and this will be useful for us when studying manifolds. These have other drawbacks, of course.

1. Čech cohomology is a somewhat unintuitive way to define cohomology, but has the advantage of providing a completely explicit formula for the cup product.
2. de Rham cohomology provides a model for the cup product which is graded-commutative on cochains, but only works with $\mathbb{R}$ coefficients.
3. The intersection theory of submanifolds is a beautiful model for the cup product, but is not always available.

We'll discuss these in turn.

[4] That said, Donaldson's original proof of the diagonalizability theorem stands as one of the most beautiful things in gauge theory.

Čech cohomology. Let M be a manifold,[5] and $\mathfrak{U} = \{U_i\}_{i\in I}$ be an open cover of M. For $J \subset I$, write

$$U_J := \bigcap_{i\in J} U_i.$$

Definition 2.6. We say that $\mathfrak{U}$ is a *good cover* if it is locally finite and all U_J, $J \neq \varnothing$, are empty or contractible.

In particular, on a compact manifold, a good cover is finite.

Lemma 2.7. *Any manifold admits a good cover.*

There are two standard proofs of this — one chooses small geodesic balls around each point for a Riemannian metric on M, and the other chooses an embedding $M \hookrightarrow \mathbb{R}^N$ and then uses the intersections of small balls in $\mathbb{R}^N$ with M.

There is also a uniqueness (really cofinality) statement.

Lemma 2.8. *Any two good covers of a manifold M admit a good common refinement.*

For $k \in \mathbb{Z}_{\geq 0}$, let $[k] := \{0,\dots,k\}$.

Definition 2.9. Let $k \in \mathbb{Z}_{\geq 0}$. A *$k$-simplex* of $\mathfrak{U}$ is a way of indexing a k-fold intersection in $\mathfrak{U}$; specifically, it is an injective map $\sigma\colon [k] \hookrightarrow I$ such that $\mathfrak{U}_{\sigma([k])}$ is nonempty. The set of k-simplices of $\mathfrak{U}$ is denoted $S_k(\mathfrak{U})$.

There is a *boundary map* $\partial_i\colon S_k(\mathfrak{U}) \to S_{k-1}(\mathfrak{U})$ which deletes $\sigma(i)$.

Definition 2.10. Let A be a commutative ring. The *Čech cochain complex valued in A* is the cochain complex $\check{C}^*(M,\mathfrak{U};A)$ defined by

$$C^k(M,\mathfrak{U};A) := \prod_{S_k} A$$

and with differential $\delta\colon \check{C}^k(M,\mathfrak{U};A) \to \check{C}^{k+1}(M,\mathfrak{U};A)$ defined by

$$(\delta\eta)(\sigma) := \sum_{i=0}^{k+1} (-1)^{i+1}\eta(\partial_i\sigma),$$

where η is a cochain and $\sigma\colon [k] \hookrightarrow I$.

One can show that $\delta^2 = 0$, hence define the *Čech cohomology groups* $\check{H}^*(M,\mathfrak{U};A) := \ker(\delta)/\operatorname{Im}(\delta)$.

Proposition 2.11. *Let $\mathfrak{U}$ and $\mathfrak{V}$ be good covers of a manifold M. Then, there is an isomorphism $\check{H}^*(M,\mathfrak{U};A) \cong \check{H}^*(M,\mathfrak{V};A)$.*

Proof idea. By Lemma 2.8, $\mathfrak{U}$ and $\mathfrak{V}$ admit a common refinement $\mathfrak{W}$; then, check that a refinement map of good covers induces an isomorphism in Čech cohomology. ⊠

Thus the Čech cohomology is often denoted $\check{H}^*(M;A)$.

In Čech cohomology, there is a finite, combinatorial model for the cup product: let $\alpha \in \check{C}^i$, $\beta \in \check{C}^j$, and $\sigma\colon [i+j] \hookrightarrow I$. Then, we let

$$(\alpha \smile \beta)(\sigma) := \alpha(\text{beginning of } \sigma)\cdot\beta(\text{end of } \sigma). \tag{2.12}$$

To be sure, this works in a more general setting (and indeed is the definition of cup product in singular cohomology), but the finiteness of Čech cochains on a compact manifold makes it a lot nicer in this setting. However, it's not at all transparent that the cup product is graded commutative on cohomology.

Theorem 2.13. *There is an isomorphism of graded rings $\check{H}^*(M;A) \cong H^*(M;A)$ (where the latter means cellular cohomology).*

[5]Čech cohomology works on a more general class of spaces, but we work with manifolds for simplicity.

de Rham cohomology. Recall that $\Omega^k(M)$ denotes the space of differential k-forms on a manifold M, and

$$\Omega^*(M) := \bigoplus_{k\geq 0} \Omega^k(M).$$

There is an exterior derivative $\mathrm{d}\colon \Omega^*(M) \to \Omega^{*+1}(M)$ with $\mathrm{d}^2 = 0$, so we can define the *de Rham cohomology* $H^*_{\mathrm{dR}}(M) := \ker(\mathrm{d})/\operatorname{Im}(\mathrm{d})$ in the usual way.

In this case, the cup product is induced by the wedge product of differential forms

$$\wedge\colon \Omega^i(M) \otimes \Omega^j(M) \to \Omega^{i+j}(M).$$

Proposition 2.14. *The wedge product is graded commutative on differential forms, hence makes $\Omega^*(M)$ into a DGA (differential graded algebra).*

This is really nice, but can only occur in characteristic zero; if you tried to do this over a field of positive characteristic, you would run into obstructions called Steenrod squares to defining a functorial graded-commutative cochain model for cohomology.

Theorem 2.15 (de Rham)**.** *Let $\mathfrak{U}$ be a good cover of a manifold M. Then there is a natural isomorphism of graded $\mathbb{R}$-algebras $H^*_{\mathrm{dR}}(M) \cong \check{H}^*(M, \mathfrak{U}; \mathbb{R})$.*

There are several different ways of proving this. One is to show that they both satisfy the Eilenberg-Steenrod axioms with $\mathbb{R}$ coefficients, and that up to natural isomorphism there is a single cohomology theory satisfying these isomorphisms. Another is to observe that Čech cohomology is a model for sheaf cohomology, and that both Čech and de Rham cohomology are derived functors of the same functor of sheaves on M applied to the constant sheaf valued in $\mathbb{R}$.

An alternative way to prove it, whose details can be found in Bott-Tu's book, is to form the *Čech-de Rham complex*, a double complex $\check{C}^*(M, \mathfrak{U}; \Omega^\bullet)$. Let D^* denote its *totalization*. Then there are quasi-isomorphisms $\check{C}^* \hookrightarrow D^*$ and $\Omega^*(M) \hookrightarrow D^*$ respecting products, hence inducing isomorphisms $\check{H}^* \cong H^*(D^*) \cong H^*_{\mathrm{dR}}(M)$.

Poincaré duality and the fundamental class Poincaré duality is one of the few (relatively) easy facts about topological manifolds, and one of the only things known until the work of Kirby and Siebenmann in the 1970s. Throughout this section, X denotes a nonempty, connected topological manifold of dimension n. For a reference for this section, see May's *A Concise Course in Algebraic Topology*.

Proposition 2.16.

(1) *If $k > n$, $H_k(X) = 0$.*
(2) *$H_n(X) \cong \mathbb{Z}$ if X is compact and orientable, and is 0 otherwise.*

If X is compact, a choice of orientation defines a generator $[X] \in H_n(X)$, called the *fundamental class* of X. A homeomorphism $f\colon X \to Y$ sends $[X] \mapsto [Y]$ if f preserves orientation and $[X] \mapsto -[Y]$ if f reverses orientation. If X is a CW complex with no cells of dimension $> n$ and a single cell e_n in dimension n, then in cellular homology, $[X] = \pm[e_n]$.

There is a *trace map* or *evaluation map* $H^n(X; A) \to A$ sending $c \mapsto \mathrm{eval}(c, [X])$ (that is, evaluate c on $[X]$); in the de Rham model on a smooth manifold, this is the integration map

$$\eta \longmapsto \int_X \eta.$$

The graded abelian group $H_{-*}(X)$ is a graded module over the graded ring $H^*(X)$ via a map called the *cap product*

$$\frown\colon H^k(X) \otimes H_i(X) \longrightarrow H_{i-k}(X).$$

Place a CW structure on X, and recall that $\delta\colon X \to X \times X$ was our cellular approximation to the diagonal. Then, we can give a cellular model for the cap product:

$$C^*(X) \otimes C_*(X) \xrightarrow{\mathrm{id}\otimes\delta_*} C^*(X) \otimes C_*(X) \otimes C_*(X) \xrightarrow{\mathrm{eval}\otimes\mathrm{id}} C_*(X).$$

Let X and Y be smooth n-manifolds, where X is closed and oriented. Then, $- \frown f_*[X]\colon H^n_{\mathrm{dR}}(Y) \to H^n_{\mathrm{dR}}(X)$ has the explicit model

$$\eta \longmapsto \int_X f^*\eta,$$

showing how the cap product relates to the evaluation map.

Theorem 2.17 (Poincaré duality)**.** *For X a closed, oriented manifold, the map*

$$D_X \coloneqq - \frown [X]\colon H^*(X) \to H_{n-*}(X)$$

is an isomorphism.

For a proof, see May. In the case of smooth manifolds, there's a slick proof using Morse theory; but Poincaré duality is true for topological manifolds as well.

We will let $D^X \coloneqq (D_X)^{-1}$.

Intersections of submanifolds. Intersection theory, though not its relation to the cup product, was discussed in the differential topology prelim. Let X, Y, and Z be closed, oriented manifolds of dimensions n, $n-p$, and $n-q$ respectively, and let $f\colon Y \to X$ and $g\colon Z \to X$ be smooth maps. Let $c_Y \coloneqq D^X(f_*[Y]) \in H^p(X)$, and similarly let $c_Z \coloneqq D^X(g_*[Z]) \in H^q(X)$. We will be able to give a nice interpretation of $c_Y \smile c_Z$.

First, let f' be a smooth map homotopic to f and transverse to g; standard theorems in differential topology show that such a map exists. Transversality means that if $y \in Y$ and $z \in Z$ are such that $f'(y) = g(z)$, then

$$T_x X = Df'(T_y Y) + D_g(T_z Z).$$

Let $P \coloneqq Y{}_{f'}\!\times_g X$, which is exactly the space of pairs (y, z) such that $f'(y) = g(z)$; transversality guarantees this is a smooth manifold of codimension $p+q$ in X. The orientations on X, Y, and Z induce one on P, and there is a canonical map $\phi\colon P \to X$ sending $(y,z) \mapsto f'(y) = g(z)$.

Theorem 2.18. *Let $c_P \coloneqq D^X(\phi_*([P]))$. Then, $c_P = c_Y \smile c_Z$.*

If Y and Z are transverse submanifolds of X, P is exactly their intersection. We will use this result frequently.

Intersection of submanifolds gives a geometric realization of the cup product, but only for those classes represented by maps from manifolds; not all homology classes are realized in this way.

Classes of codimension at most 2 always have representatives arising from embedded submanifolds. The idea is that in general, there's a natural isomorphism

$$H^n(X) \cong [X, K(\mathbb{Z}, n)],$$

where brackets denote homotopy classes of maps and $K(\mathbb{Z}, n)$ is an *Eilenberg-Mac Lane space* for $\mathbb{Z}$ in dimension n, i.e. a space whose only nontrivial homotopy group is $\pi_n \cong \mathbb{Z}$. These spaces always exist, and any two models for $K(\mathbb{Z}, n)$ are homotopic.

Usually Eilenberg-Mac Lane spaces are not smooth manifolds, but there are a few exceptions, including $K(\mathbb{Z}, 1) \simeq S^1$. Hence there is a bijection $[X, S^1] \to H^1(X)$. In the de Rham model, this is the map

$$[f] \longmapsto f^*\omega,$$

where $\omega \in H^1(S^1) \cong \mathbb{Z}$ is the generator. Alternatively, you could think of ω as $D^{S_1}[\mathrm{pt}]$, for any choice of $\mathrm{pt} \in S^1$.

Thus, take $f\colon X \to S^1$ to be a smooth map, where X is a closed, oriented manifold. Let $H_t \coloneqq f^{-1}(t) \subset X$, where $t \in S^1$ is a regular value. Then, H_t comes with a co-orientation, hence an orientation, and $[H_t] = D_X(f^*\omega)$. Thus codimension-1 submanifolds are realizable.

In this course, the case of codimension 2 will be more useful.

Proposition 2.19. *Let $\mathbb{CP}^\infty \coloneqq \operatorname{colim}_n \mathbb{CP}^n$ (the union via the inclusions $\mathbb{CP}^n \hookrightarrow \mathbb{CP}^{n+1}$). Then, $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$.*

Hence there is a class $c \in H^2(\mathbb{CP}^\infty)$ and a natural bijection $[X, \mathbb{CP}^\infty] \to H^2(X)$ sending $[f] \mapsto f^*(c)$.

$\mathbb{CP}^\infty$ is not a smooth manifold, but its low-dimensional skeleta are, and this leads to codimension-2 realizability. Specifically, the inclusion $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty$ defines the pullback map $H^2(\mathbb{CP}^\infty) \to H^2(\mathbb{CP}^1) \cong H_0(\mathbb{CP}^1) \cong \mathbb{Z}$. This maps the tautological class c to $[\mathrm{pt}]$ for any $\mathrm{pt} \in \mathbb{CP}^1$.

Let $f\colon \mathbb{CP}^\infty$ be a map, where X is a smooth, oriented, closed manifold, which is homotopic to a smooth map $\mathbb{CP}^N$ followed by the inclusion $\mathbb{CP}^N \hookrightarrow \mathbb{CP}^\infty$. Let $D \subset \mathbb{CP}^N$ be a hyperplane and $H_D \coloneqq g^{-1}(D)$. Assuming $g \pitchfork D$ (which can always be done for some g in the homotopy class of f), then H_D is a codimension-2 oriented submanifold of X, and $[H_D] = D^X(g^*c)$. Thus codimension-2 homology classes are representable. We will most commonly use this in dimension 4, for which any class $a \in H^2(X)$ is represented by an embedding of a closed, oriented surface $\Sigma \hookrightarrow X$.

In general, realizability is controlled by oriented cobordism, which in higher codimension is different from cohomology, governed by maps into Thom spaces rather than Eilenberg-Mac Lane spaces. This was studied in the 1950s by Rene Thom.

Lecture 3.

Unimodular forms: 1/25/18

Today's lecture will be more algebraic in flavor, though with topology in mind; we'll be discussing the algebra that arises in the middle cohomology of even-dimensional manifolds.

Let M be a closed, oriented manifold of dimension $2n$; its middle cohomology $H^n(M)$ carries a bilinear form $\cdot\colon H^n(M) \otimes H^n(M) \to \mathbb{Z}$ sending

$$x, y \longmapsto x \cdot y := \langle x \smile y, [M] \rangle, \tag{3.1}$$

where $\langle -, - \rangle$ denotes evaluation. If n is even (i.e. $4 \mid \dim M$), this is a symmetric form; if n is odd, it's skew-symmetric, which follows directly from the graded-commutativity of the cup product.

Poincaré duality means there are three different ways to think of this product.

- As defined, it's a pairing $H^n \otimes H^n \to \mathbb{Z}$, where the pairing is evaluation and the cup product.
- Using Poincaré duality, we could reinterpret it as a map $H^n \otimes H_n \to \mathbb{Z}$. In this case, the pairing is evaluation. This is because the Poincaré duality map is capping with the fundamental class, so
$$(x \smile y) \frown [M] = x \frown (y \frown [M]) = x \frown D_M(y).$$
- Using Poincaré duality again, it's a pairing $H_n \otimes H_n \to \mathbb{Z}$, which is the intersection product.

For an abelian group A, let $A_{\mathrm{tors}} \subset A$ denote its torsion subgroup and $A' := A/A_{\mathrm{tors}}$. In this case, the form (3.1) descends to a form on $H^n(M)'$, and we usually use this version of the form.

Remark 3.2. If M is 4-dimensional, the universal coefficients theorem guarantees a short exact sequence

$$0 \longrightarrow (H_1(M))_{\mathrm{tors}} \longrightarrow H^2(M) \longrightarrow H^2(M)' \longrightarrow 0,$$

and that it splits, but non-canonically. In particular, if $H_1(M) = 0$, $H^2(M)$ is torsion-free. ◂

Let $\{e_i\}$ be a $\mathbb{Z}$-basis for $H^n(M')$ and $Q = (Q_{ij})$ be the matrix with entries $Q_{ij} := e_i \cdot e_j$. This is a symmetric matrix of n is even, and is skew-symmetric if n is odd.

The universal coefficients theorem also implies that $H^n(M') \cong \mathrm{Hom}(H_n(M), \mathbb{Z})$, where the map sends a cohomology class y to the evaluation pairing of y and a homology class. This, plus the fact that $\cdot$ is dual to evaluation, implies the following proposition.

Proposition 3.3. *The pairing $\cdot$ is nondegenerate on $H^n(M)'$, i.e. the map $H^n(M') \to \mathrm{Hom}(H^n(M)', \mathbb{Z})$ sending $x \mapsto (y \mapsto x \cdot y)$ is an isomorphism of abelian groups.*

Corollary 3.4. $\det Q \in \{\pm 1\}$.

Now we focus on the case of manifolds which are boundaries. Suppose there is a compact oriented $(2n+1)$-dimensional manifold N such that $M = \partial N$, and let $i\colon N \hookrightarrow M$ denote inclusion.

Proposition 3.5. *Let $L = \mathrm{Im}(i^*) \subset H^n(M; \mathbb{R})$. Then,*

1. *L is* isotropic*, i.e. for all $x, y \in L$, $x \cdot y = 0$.*
2. *$\dim L = (1/2) \dim H^n(M; \mathbb{R})$.*

Proof. Perhaps unsurprisingly, this proof uses algebraic topology of manifolds with boundary, namely *Poincaré-Lefschetz duality*, the analogue of Poincaré duality on a compact manifold with boundary.

Part (1) follows from the fact that i^* is a ring homomorphism:

$$\begin{aligned} i^*u \cdot i^*v &= (i^*u \smile i^*v) \frown [M] \\ &= i^*(u \smile v) \frown [M] \\ &= (u \smile v) \frown i_*[M], \end{aligned}$$

but since $M = \partial N$, $[M] = \partial[M, N]$, where $\partial\colon H_{2n+1}(M, N; \mathbb{R}) \to H_{2n}(N; \mathbb{R})$ is the boundary map in the long exact sequence of a pair and $[M, N] \in H_{2n+1}$ is the relative fundamental class. Hence $i_*[M] = 0$, since $i_* \circ \partial = 0$.

For part (2), Poincaré-Lefschetz duality implies the following diagram is commutative with exact rows:

$$\begin{tikzcd} \cdots \arrow[r] & H^n(N;\mathbb{R}) \arrow[r, "i^*"] \arrow[d, "\cong"'] & H^n(M;\mathbb{R}) \arrow[r, "\delta"] \arrow[d, "\cong"', "D_M"] & H^{n+1}(N,M;\mathbb{R}) \arrow[r, "q"] \arrow[d, "\cong"', "D"] & H^{n+1}(N;\mathbb{R}) \arrow[r] \arrow[d, "\cong"', "D"] & \cdots \\ \cdots \arrow[r] & H_{n+1}(N,M;\mathbb{R}) \arrow[r, "\partial"] & H_n(M;\mathbb{R}) \arrow[r, "i_*"] & H_n(M;\mathbb{R}) \arrow[r, "p"] & H_n(N,M;\mathbb{R}) \arrow[r] & \cdots \end{tikzcd} \tag{3.6}$$

Fix a complement K to L in $H^n(M;\mathbb{R})$; it suffices to show that $\dim K = \dim L$. Since $L = \mathrm{Im}(i^*) = \ker(\delta)$, then $K \cong H^n(M;\mathbb{R})/\ker(\delta) \cong \mathrm{Im}(\delta)$. Since the upper row of (3.6) is exact, $\mathrm{Im}(\delta) = \ker(q)$, and by Poincaré-Lefschetz duality this is isomorphic to $\ker(p)$. Since the lower row of (3.6) is exact, this is isomorphic to $\mathrm{Im}(i_*) \subset H_n(N;\mathbb{R})$. However, i_* and i^* are dual (in the sense of $\mathrm{Hom}(-,\mathbb{R})$), and linear algebra tells us that a map and its dual have the same rank.[6] ⊠

Now let's focus on the case when n is even, so $4 \mid \dim(M)$.

Definition 3.7. A *unimodular lattice* (Λ, σ) is a finite-rank free abelian group Λ together with a nondegenerate symmetric bilinear form $\sigma\colon \Lambda \otimes \Lambda \to \mathbb{Z}$.

Therefore a closed, oriented $4m$-manifold defines a unimodular lattice $(H^{2m}, \cdot)$.

Recall that for (V, σ) a symmetric bilinear form on a real vector space V, there is an orthogonal decomposition

$$V = R \oplus V^+ \oplus V^-, \tag{3.8}$$

where R is the subspace orthogonal to everything, V^+ is the subspace on which σ is positive definite, and V^- is the subspace on which σ is negative definite. Clearly (V,σ) determines $\dim V^+$ and $\dim V^-$.

Proposition 3.9 (Sylvester's law of inertia)**.** *If (V,σ) is a symmetric bilinear form on a real vector space, then $\dim R$, $\dim V^+$, and $\dim V^-$ determine (V,σ) up to isomorphism.*

Definition 3.10. The *signature* of (V,σ) is $\tau \coloneqq \dim V^+ - \dim V^-$. If (Λ,σ) is a unimodular lattice, then $\tau(\Lambda,\sigma) \coloneqq \tau(\Lambda\otimes\mathbb{R}, \sigma\otimes \mathrm{id}_{\mathbb{R}})$. If M is a closed, oriented, $4m$-dimensional manifold, its signature is $\tau(M) \coloneqq \tau(H^{2m}(M)', \cdot)$.

Fact. If M is a closed, oriented $4m$-manifold, then M admits an orientation-reversing self-diffeomorphicm iff $\tau(M) = 0$. ◄

Theorem 3.11.

(1) *Let X_1 and X_2 be closed, oriented 4-manifolds and Y be an oriented cobordism between them, i.e. Y is a compact, oriented 5-manifold together with an orientation-preserving diffeomorphism $\partial Y^5 \cong (-X_1) \amalg X_2$.*[7] *Then, $\tau(X_1) = \tau(X_2)$.*

(2) *Conversely, if $\tau(X_1) = \tau(X_2)$, then X_1 and X_2 are cobordant.*

Therefore, in particular, the signature is a complete cobordism invariant.

Partial proof. For part (1), $H^2(-X_1 \amalg X_2)\otimes\mathbb{R}$ admits a middle-dimensional isotropic subspace by Proposition 3.5, hence has signature zero. But

$$\begin{aligned} \tau(-X_1 \amalg X_2) &= \tau((-H^2(X_1)\otimes\mathbb{R}) \oplus (H^2(X_2)\otimes\mathbb{R})) \\ &= \tau(-X_1) + \tau(X_2) \\ &= -\tau(X_1) + \tau(X_2) = 0. \end{aligned}$$ ⊠

We will not give a full proof of part (2). The idea is that cobordism classes of oriented 4-manifolds form an abelian group Ω_4^{SO} under disjoint union, and by part (1), the signature defines a homomorphism

$$\tau\colon \Omega_4^{\mathrm{SO}} \longrightarrow \mathbb{Z}.$$

This map must be surjective, because $\tau(\mathbb{CP}^2) = 1$ (where the orientation is the standard one coming from its complex structure). To prove it's injective, one uses Thom's cobordism theory, which identifies Ω_4^{SO} with a homotopy group of a space called a *Thom space*, then calculates that group using the Hurewicz theorem and calculation of the homology of the Thom space in question.

[6] The matrix version of this statement is that a matrix and its transpose have the same rank.

[7] Here $-X_1$ denotes X_1 with the opposite orientation.

Next we discuss unimodular lattices mod 2.

Definition 3.12. A *characteristic vector* c for a unimodular lattice (Λ, σ) is a $c \in \Lambda$ such that $\sigma(c, x) \equiv \sigma(x, x) \bmod 2$.

Lemma 3.13. *The characteristic vectors form a coset of* 2Λ *in* Λ.

Proof. Let $\lambda = \Lambda/2\Lambda$, which is a vector space over $\mathbb{F}_2$. The freshman's dream mod 2 implies that the map $\lambda \to \mathbb{Z}/2$ sending $[x] \mapsto \sigma(x, x) \bmod 2$ is linear! Hence there is a symmetric bilinear form $\overline{\sigma}$ on λ induced by σ with determinant 1, hence is nondegenerate. Hence there exists a unique $\overline{c} \in \Lambda$ such that $\overline{\sigma}(x, x) = \overline{\sigma}(\overline{c}, x) \in \mathbb{Z}/2$ for all $x \in \lambda$. The characteristic vectors are exactly the lifts of $\overline{c}$ to Λ, hence are a coset of 2Λ. ⊠

Remark 3.14. In the case of a simply connected 4-manifold M, the element $\overline{c} \in H^2(M)/2H^2(M) \cong H^2(M; \mathbb{Z}/2)$ is exactly the second Stiefel-Whitney class $w_2(TM)$. We'll talk about characteristic classes more next lecture. This follows from the *Wu formula*.

Moreover, the characteristic vectors $c \in H^2(M; \mathbb{Z})$ are exactly the first Chern classes of spin^c structures on M, and the Seiberg-Witten invariants arefunctions on the set of characteristic vectors to $\mathbb{Z}$. We'll say more about this later.

Most of this is true even in the case of non-simply-connected manifolds, but is harder. It is not true, however, that $H^2(M)/2H^2(M) \cong H^2(M; \mathbb{Z}/2)$. ◀

Lemma 3.15. *Let c and c' be characteristics for (Λ, σ). Then,*

$$\sigma(c, c) \equiv \sigma(c', c') \bmod 8.$$

Proof. Write $c' - c = 2x$ for some $x \in \Lambda$. Then,

$$\sigma(c', c') = \sigma(c + 2x, c + 2x) = \sigma(c, c) + 4\underbrace{(\sigma(c, x) + \sigma(x, x))}_{(*)},$$

and $(*)$ is even. ⊠

Definition 3.16. Let (Λ, σ) be a unimodular form. Its type $t \in \mathbb{Z}/2$ is even if $\sigma(x, x)$ is even for all $x \in \Lambda$, and otherwise, it's odd.

Theorem 3.17 (Hasse-Minkowski theorem on unimodular forms)**.** *An indefinite unimodular form (Λ, σ) is classified up to isomorphism by three invariants:*

- *its rank* $\dim_{\mathbb{R}}(\Lambda \otimes \mathbb{R})$,
- *its signature $\tau \in \mathbb{Z}$, and*
- *its type $t \in \mathbb{Z}/2$.*

For the (quite nontrivial) proof, see Serre's *A Course in Arithmetic*. The proof idea is to solve the quadratic equation $\sigma(x, x) = 0$ for $x \in (\Lambda \otimes \mathbb{Q}) \setminus 0$. This is achieved via a *local-to-global principle* which says it suffices to find solutions $x_\infty \in \Lambda \otimes \mathbb{R}$ and $x_p \in \Lambda \otimes \mathbb{Q}_p$ (the p-adic numbers), and this can be done using unimodularity.

This is completely different from the positive definite (equivalently, negative definite) case, for which there are finitely many isomorphism classes below a given rank r, though this number grows rapidly with r and is only known in relatively few cases.

Example 3.18. Let $I_+ := (\mathbb{Z}, 1)$ with $\sigma(x, y) = xy$ and $I_- := (\mathbb{Z}, -1)$ with $\sigma(x, y) = -xy$. Then, $I_+^{\oplus m} \oplus I_-^{\oplus n}$ has rank $m + n$, signature $m - n$, and odd type.

There is a characteristic vector $c := (1, \dots, 1)$. In particular, $c^2 = m - n - \tau$, and therefore for any characteristic vector c, $c^2 \equiv \tau \bmod 8$. ◀

Example 3.19. Let

$$U := \left(\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$$

with quadratic form $(a, b)^2 := 2ab$. This is an even unimodular lattice with rank 2 and signature 0. ◀

The Hasse-Minkowski principle implies the following.

Corollary 3.20. *Let Λ be a unimodular lattice with signature τ and c be a characteristic vector for Λ. Then, $c \cdot c \equiv \tau \bmod 8$. In particular, if Λ is even, then $\tau \equiv 0 \bmod 8$.*

Proof. Either $\Lambda \oplus I_+$ or $\Lambda \oplus I_-$ is indefinite. It has odd type, and signature $\tau(\Lambda \oplus I_\pm) = \tau(\Lambda) \pm 1$, and if c is a characteristic vector for Λ, a characteristic vector for $\Lambda \oplus I_\pm$ is $c \oplus 1$ with square $c \cdot c = \pm 1$.

By Theorem 3.17, $\Lambda \oplus I_\pm \cong mI_+ \oplus nI_-$, so $c^2 \pm 1 \equiv \tau(\Lambda) \pm 1 \bmod 8$. ⊠

Hence an even, positive-definite, unimodular lattice has rank $8k$ for some k.

Example 3.21 (E_8 lattice)**.** The basic example is the E_8 lattice, associated to the Dynkin diagram for the exceptional simple Lie group E_8. As a matrix, it has the form

$$\begin{pmatrix} 2 & -1 & & & & & & -1 \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ -1 & & & & & & -1 & 2 \end{pmatrix}. \tag{3.22}$$

Hasse-Minkowski implies that any even unimodular lattice Λ is isomorphic to $mU \oplus (n \pm E_8)$. The intersection forms of interesting 4-manifolds tend to have E_8 terms. ◀

For more on the E_8 lattice, see the professor's class notes.

Lecture 4.

The intersection form and characteristic classes: 1/30/18

Let's start by writing down the homology and cohomology of a closed, oriented 4-manifold. Poincaré duality narrows the search space considerably.

$$\begin{aligned} H_4(X) &\cong H^0(X) = \mathbb{Z} \cdot 1 \\ H_3(X) &\cong H^1(X) = \mathrm{Hom}(\pi_1(X), \mathbb{Z}) \\ H_2(X) &\cong H^2(X) \cong \mathrm{Hom}(H_2(X), \mathbb{Z}) \oplus H_1(X)_{\mathrm{tors}} \\ \pi_1(X)^{\mathrm{ab}} = H_1(X) &\cong H^3(X) \\ \mathbb{Z} \cdot [\mathrm{pt}] = H_0(X) &\cong H^4(X). \end{aligned}$$

Additively, the homology and cohomology are determined up to isomorphism by π_1 and H_2. Together with the intersection form Q_X on H_2 (or its torsion-free quotient), we have a lot of information, but we still don't have everything: there could be cup products from H^1 and H^2 to H^3, and various other questions, e.g. what's the Hurewicz map $\pi_2(X) \to H_2(X)$? What about mod p coefficients?

If X is simply connected, the story is much simpler:

$$\begin{aligned} H_4(X) &\cong H^0(X) = \mathbb{Z} \cdot 1 \\ H_3(X) &\cong H^1(X) = 0 \\ H_2(X) &\cong H^2(X) \cong \mathrm{Hom}(H_2(X), \mathbb{Z}) \\ H_1(X) &\cong H^3(X) = 0 \\ \mathbb{Z} \cdot [\mathrm{pt}] = H_0(X) &\cong H^4(X). \end{aligned}$$

The only real information is H_2, which determines the cohomology additively. The intersection form determines $H^*(X)$ as a graded ring and $H_*(X)$ as a graded $H^*(X)$-module. All mod p cohomology classes are reductions of integral classes, which follows from the universal coefficient theorem. The natural map $\pi_2(X) \to H_2(X)$ is an isomorphism, by the Hurewicz theorem, which applies to any simply connected space. You can think about higher homotopy groups, and next time we'll think about the homotopy type of X, but thus far everything we can see has been determined by Q_X. (In fact, next time we'll see that it determines the homotopy type of X.)

So that knocks out the homotopy type, but we want further invariants. The next step is to investigate the tangent bundle, a distinguished rank-4 vector bundle. In particular, it has characteristic classes.

Remark 4.1. Spoiler alert: all characteristic classes we discuss today are determined by Q_X, so they don't define any new invariants.[8] Nonetheless, they provide useful tools for computing the intersection form, and therefore will be useful to us. ◂

We now review some of the theory of characteristic classes.

Example 4.2. Let $V \to X$ be a finite-rank real vector bundle over an arbitrary topological space X. The *Stiefel-Whitney classes* are characteristic classes of V in mod 2 cohomology with the following properties.

- The i^{th} *Stiefel-Whitney class* $w_i(V) \in H^i(X;\mathbb{Z}/2)$ for $i \geq 0$. The *total Stiefel-Whitney class* is
$$w(V) := w_0(V) + w_1(V) + \cdots$$
- $w_0(V) = 1$.
- If $i > \operatorname{rank} V$, $w_i(V) = 0$. Hence the total Stiefel-Whitney class is a finite sum.

It's a theorem that the Stiefel-Whitney classes are characterized by the following properties.

(1) If $f\colon Y \to X$ is continuous, $w_i(f^*V) = f^*w_i(V)$ for all i.
(2) If $i > \operatorname{rank} V$, $w_i(V) = 0$.
(3) The *Whitney sum formula*: if $U, V \to X$ are vector bundles, then $w(U \oplus V) = w(U)w(V)$.[9]
(4) If $L \to \mathbb{RP}^1$ denotes the *tautological line bundle* whose fiber over a point $\ell \in \mathbb{RP}^1$ is the line ℓ itself, $w_1(L) \neq 0$ in $H^1(\mathbb{RP}^1;\mathbb{Z}/2) \cong \mathbb{Z}/2$.

For a proof, see Hatcher's notes on vector bundles and K-theory, or Milnor-Stasheff, which provides a somewhat baffling construction in terms of Steenrod algebra on the Thom space. There are various differing constructions which make various facts about Stiefel-Whitney classes easier to prove. ◂

If X is path-connected, the isomorphism $H^1(X;\mathbb{Z}/2) \cong \operatorname{Hom}(\pi_1(X);\mathbb{Z}/2)$ sends $w_1(V)$ to the orientation character of V. In particular, $w_1(V) = 0$ iff V is orientable, so if M is a manifold, M is orientable iff $w_1(TM) = 0$.

Now suppose M is a closed manifold and $V \to M$ is a rank-r vector bundle. Its top Stifel-Whitney class $w_r(V) \in H^r(M;\mathbb{Z}/2) \cong H_{n-r}(M;\mathbb{Z}/2)$ maps to some homology class, which admits a representation by some codimension-r cycle. This cycle has an explicit description: let s be a section of V transverse to the zero section; then $s^{-1}(0)$ is codimension r and represents the homology class which is Poincaré dual to $w_r(V)$.

If V is an orientable bundle on a closed submanifold, its top Stiefel-Whitney class is the mod 2 reduction of its Euler class.

Proposition 4.3. *Let $H \in H^1(\mathbb{RP}^2;\mathbb{Z}/2) \cong \mathbb{Z}/2$ denote the generator, which is Poincaré dual to a hyperplane $\mathbb{RP}^{n-1} \subset \mathbb{RP}^n$. Then $w(T\mathbb{RP}^n) = (1+H)^{n+1}$.*

For a proof, see Milnor-Stasheff or the professor's notes.

Now we specialize to 4-manifolds.

Theorem 4.4 (Wu)**.** *Let X be a closed 4-manifold and $w := w_1^2(TX) + w_2(TX)$. Then w is the* characteristic element *of $H^2(X;\mathbb{Z}/2)$, i.e. for all $u \in H^2(X;\mathbb{Z}/2)$, $w \smile u = u \smile u$.*

There is a more general statement of Wu's theorem for manifolds in other dimensions. See Milnor-Stasheff.

For simply-connected 4-manifolds, the Stiefel-Whitney classes are determined by information we already have.

- If X is a closed, simply-connected 4-manifold, then $w_1(TX)$ vanishes, since H^1 does.
- Hence $w_2(TX) \cup u = u \smile u$ for all $u \in H^2(X;\mathbb{Z}/2)$. In this case, $H^2(X;\mathbb{Z}/2) = H^2(X)/2H^2(X)$; therefore $w_2(TX)$ is the mod 2 reduction of any characteristic vector for Q_X. Therefore w_2 provides no new information.
- $w_4(TX)$ is Poincaré dual to the zeroes of a vector field, so $w_4(TX) \frown [X]$ is the number of zeroes mod 2 of a generic vector field, i.e. the Euler characteristic mod 2. This is again not new information, since it can be read off $H^*(X)$.
- Finally, since H^3 vanishes, so must w_3. It's a theorem of Hirzenbuch and Hopf that $w_3(TX)$ vanishes for any closed, orientable 4-manifolds, and this is quite relevant for our class.

[8] In fact, Q_X determines the homotopy type of the classifying map for TX!

[9] If X is paracompact, every short exact sequence of vector bundles over X splits, so we may replace $U \oplus V$ with an extension of U by V.

Remark 4.5. The more general version of Wu's theorem shows that on any closed, oriented 4-manifold X, $w_3(TX) = \mathrm{Sq}^1 w_2(TX)$. Here Sq^1 is the first *Steenrod square*, a cohomology operation, which has an explicit identification as the *Bockstein map* which measures whether a mod 2 cohomology class lifts to $\mathbb{Z}$ coefficients. In particular, $w_3(TX) = 0$ iff $w_2(TX)$ is the reduction of an integral class, and such lifts are the first Chern classes of spin^c structures. Thus the Hirzebruch-Hopf theorem is important for us because it implies that all closed, oriented 4-manifolds admit a spin^c structure. Since this theorem is trivial in the simply-connected case, though, we will not prove it. ◀

Example 4.6. *Chern classes* are characteristic classes $c_{2i}(E) \in H^{2i}(X;\mathbb{Z})$ for complex vector bundles $E \to X$; again the *total Chern class*

$$c(E) \coloneqq c_0(E) + c_1(E) + \cdots$$

and again $c_0(E) = 1$ and $c_i(E) = 0$ for $i > \operatorname{rank} E$. The Chern classes are uniquely characterized by similar axioms:

(1) $c_i(E) = 0$ if $i > \operatorname{rank} E$.
(2) $c(E \oplus F) = c(E)c(F)$.
(3) If $L \to \mathbb{CP}^1$ denotes the tautological line bundle $L \to \mathbb{CP}^1$, then the Poincaré dual of $c_1(L) \in H^2(\mathbb{CP}^1;\mathbb{Z})$ is $-1 \in H_0(\mathbb{CP}^1)$ (where we take as a generator any positively oriented point).

◀

Proposition 4.7. *If $H \in H^2(\mathbb{CP}^n)$ denotes the Poincaré dual to a hyperplane, so $H = -c_1$ of the tautological line bundle over $\mathbb{CP}^n$, then $c(T\mathbb{CP}^n) = (1 + H)^{n+1}$. The argument is formally identical to the real case.*

Definition 4.8. Isomorphism classes of complex line bundles on X form a group under tensor product; this is called the *topological Picard group* and denoted $\mathrm{Pic}(X)$.

If M is a closed, oriented manifold and $E \to M$ is a rank-r complex vector bundle, then the Poincaré dual of $c_r(E)$ is the zero locus of a generic section of E (i.e. transverse to the zero section).

Proposition 4.9. *The first Chern class defines a homomorphism $c_1\colon \mathrm{Pic}(X) \to H^2(X;\mathbb{Z})$, and this is an isomorphism.*

In particular, for line bundles L_1 and L_2, $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$. One way to use this is to use the above characterization of the Poincaré dual of the top Chern class; another is to show that $B\mathrm{U}_1$, the classifying space for complex line bundles, is a $K(\mathbb{Z}, 2)$, an Eilenberg-Mac Lane space, hence representing cohomology.

For a reference for the following theorem, see Hatcher's notes on vector bundles and K-theory.

Theorem 4.10. *Let $E \to X$ be a complex vector bundle and $E_{\mathbb{R}}$ denote the underlying real vector bundle.*

- *$w_{2i}(E_{\mathbb{R}})$ is the mod 2 reduction of $c_i(E)$.*
- *$w_{2i+1}(E_{\mathbb{R}}) = 0$.*

In particular, $w_2 = c_1 \bmod 2$.

In order to apply Chern classes to manifolds, we need some sort of complex structure.

Definition 4.11. Let M be an even-dimensional manifold. An *almost complex structure* on M is a $J \in \mathrm{End}(TM)$ such that $J^2 = -\mathrm{id}$.

This structure makes TM into a complex vector bundle, where i acts as J. Then we have access to Chern classes, albeit depending on J.

Remark 4.12. As the notation suggests, complex manifolds are almost complex. ◀

Example 4.13. One example of a complex manifold is a complex hypersurface in $\mathbb{CP}^n$: let F be a degree-d homogeneous polynomial in $x_0, \dots, x_d$ and $X \coloneqq \{F = 0\} \subset \mathbb{CP}^n$. Homogeneity means this makes sense in projective space; if we additionally assume that whenever $F = 0$ at least one partial derivative of F is nonzero, then X is smooth.

To study complex manifolds, we can import tools from algebraic geometry. A holomorphic hypersurface D in a complex manifold M defines an invertible sheaf (i.e. complex line bundle) $\mathscr{O}_M(D)$ whose sections over

$U \subset M$ are meromorphic functions on U with only simple poles along $D \cap U$. Moreover, if $N_{D/M} := TM|_D/TD$ denotes the normal bundle to $D \hookrightarrow M$, there's an isomorphism $\mathscr{O}_M(D)|_D \cong N_{D/M}$.

On $\mathbb{CP}^n$, a holomorphic line bundle E is determined by its degree $d := c_1(E) \in H^2(\mathbb{CP}^n)$, because the holomorphic sheaf cohomology $H^1(\mathbb{CP}^n; \mathscr{O}_{\mathbb{CP}^n}) = 0$. If $L \to \mathbb{CP}^n$ denotes the tautological line bundle, then L^* is the positive generator of $\operatorname{Pic} \mathbb{CP}^n$, so there's an isomorphism $E \cong (L^*)^{\otimes d}$.

It follows that $\mathscr{O}_{\mathbb{CP}^n}(X) \cong (L^*)^{\otimes d}$, so $c_1(\mathscr{O}_{\mathbb{CP}^n}(X)) = dH$, where $H = c_1(L^*)$ as before. Hence if $h = i^*H \in H^2(X)$, where $i\colon X \hookrightarrow \mathbb{CP}^n$ is inclusion, then

$$c_1(N_{X/\mathbb{CP}^n}) = i^*c_1(\mathscr{O}_{\mathbb{CP}^n}(X)) = dh. \tag{4.14}$$

Using the short exact sequence

$$0 \longrightarrow TX \longrightarrow T\mathbb{CP}^n|_X \longrightarrow N_{X/\mathbb{CP}^n} \longrightarrow 0, \tag{4.15}$$

we get that

$$\begin{aligned} i^*c(T\mathbb{CP}^n) &= c(TX)c(N_{X/\mathbb{CP}^n}) \\ (1+h)^{n+1} &= c(TX)(1+dh), \end{aligned}$$

and therefore

$$c_j(TX) + dhc_{j-1}(TX) = \binom{n+1}{j} h^j. \tag{4.16}$$

It's possible to explicitly solve this when you have a specific X; the base case is

$$c_1(TX) = (n+1-d)h. \tag{4.17}$$

◀

Example 4.18. Let $n = 3$, so X is a degree-d complex surface, hence a closed, oriented 4-manifold. In that case

$$\begin{aligned} c_1(TX) &= (4-d)h \\ c_2(TX) &= (d^2 - 4d + 6)h^2. \end{aligned} \tag{4.19}$$

Since $[X] \in H_4(\mathbb{CP}^3) \cong H^2(\mathbb{CP}^3)$ is identified with dH under Poincaré duality, then

$$c_2(TX) \frown [X] = d(d^2 - 4d + 6). \tag{4.20}$$

Since this is again the number of zeros (with orientation) of a generic vector field, this integer is the the Euler characteristic, so we have an explicit formula for the Euler characteristic of a degree-d complex surface:

$$\chi(X) = d(d^2 - 4d + 6). \tag{4.21}$$

◀

Theorem 4.22 (Lefschetz hyperplane theorem)**.** *Suppose $n \geq 3$ and X is a hypersurface in $\mathbb{CP}^n$. Then X is simply connected.*

There is a more general version of this theorem.

Therefore if X is a hypersurface in $\mathbb{CP}^n$, its first and third Betti numbers vanish, so

$$\chi(X) = 1 + b_2(X) + 1, \tag{4.23}$$

so

$$b_2(X) = d(d^2 - 4d + 6) - 2. \tag{4.24}$$

So we know the dimension of H^2. It's possible to write down bases using this information, though we won't get into this.

Example 4.25. The *Pontrjagin classes* $p_i(V) \in H^{4i}(X)$ of a real vector bundle V are defined by

$$p_i(V) := (-1)^i c_{2i}(V \otimes \mathbb{C}).$$

◀

Pontrjagin classes satisfy very similar axioms to Stiefel-Whitney and Chern classes; however, be aware that which ones vanish might be tricky. For example, the complexification of a rank-3 vector bundle is a rank-3 complex vector bundle, hence only has access to c_0 and c_2 for defining Pontrjagin classes.

If X is a closed oriented 4-manifold, TX has only one nonvanishing Pontrjagin class, which is $p_4(TX) \in H^4(X) \cong \mathbb{Z}$.

Lemma 4.26. *Let X be a closed, oriented 4-manifold. Then $\sigma(X) := p_1(TX) \frown [X] \in \mathbb{Z}$ is an oriented cobordism invariant.*

The basic idea is that if W is a 5-manifold bounding X, then $TW = TM \oplus \underline{\mathbb{R}}$, which implies $p_1(W) = i^* p_1(M)$, where $i\colon M \hookrightarrow W$ is inclusion.

Lemma 4.27. *If $V \to X$ is a complex tangent bundle, $p_1(V_{\mathbb{R}}) = c_1(V)^2 - 2c_2(V)$.*

For a proof, see the notes.

Recall that this cobordism group $\Omega_4^{\mathrm{SO}} \cong \mathbb{Z}$, and that the signature $\tau\colon \Omega_4^{\mathrm{SO}} \to \mathbb{Z}$ is an isomorphism. Hence $\sigma(X)$ must be proportional to $\tau(X)$.

Theorem 4.28 (Hirzebruch signature theorem)**.** *$\sigma = 3\tau$, i.e. on a closed, oriented 4-manifold X, $p_1(X) \frown [X] = 3\tau(X)$.*

Proof. The proof is corollary of Thom's work: we just have to check on a generator of Ω_4^{SO}, such as $\mathbb{CP}^2$, which has signature 1. Then we use Lemma 4.27: $c_1(T\mathbb{CP}^2) = 3H$ and $c_2(T\mathbb{CP}^2) = \chi(\mathbb{CP}^1) = 3$, so

$$p_1(T\mathbb{CP}^2) = c_1^2(\mathbb{CP}^2) - c_2(\mathbb{CP}^2) = 9 - 2 \cdot 3 = 3. \tag{4.29}$$ ⊠

This also implies that if X_d is a degree-d hypersurface, its signature is

$$\begin{aligned}\tau(X_d) &= \frac{1}{3}(c_1^2(TX_d) - c_2(X_d)) \frown [X_d]\\ &= -\frac{1}{3}(d-2)d(d+2).\end{aligned}$$

Certainly $(d-2)d(d+2)$ is divisible by 3, so this produces an interger, even if we didn't already know that. Moreover,

$$\begin{aligned}w_2(TX_d) &= c_1(TX_d) \bmod 2\\ &= (4-d)h \bmod 2\\ &= dh \bmod 2.\end{aligned}$$

Hence d is even iff $w_2(TX_d) = 0$ iff X_d has even type (i.e. its intersection form has even type). For $d \geq 2$, the intersection form is indefinite, so Theorem 3.17 tells us there's a unique intersection form in this class.

Example 4.30. For $d = 4$, X_d is a *K3 surface*, and one concludes that $c_1 = 0$, $b_2 = 24$, $\tau = -16$, and the intersection form has even type. Hence by Theorem 3.17 the intersection form is $3U \oplus 2(-E_8)$. ◀

The point is that we can explicitly compute intersection forms in examples of interest, and how characteristic classes made this a bit easier.

Lecture 5.

Tangent bundles of 4-manifolds: 2/1/18

This lecture has two goals. The first is to show that if X is a closed, simply-connected manifold, then its tangent bundle is essentially determined by w_2, τ, and χ. The second is to discuss the following theorem.

Theorem 5.1 (Rokhlin)**.** *Let X be a closed, oriented 4-manifold. Then $16 \mid \tau(M)$.*

There are different proofs of this; today we're going to focus on its equivalence to the following fact about stable homotopy groups of the spheres.

Proposition 5.2. *Let $k \geq 5$. Then $\pi_{3+k}(S^k) \cong \mathbb{Z}/24$.*

Obstruction theory For a reference for this part of the lecture, see Hatcher's notes on vector bundles and K-theory.

Obstruction theory is about the following question: let $\pi\colon E \to X$ be a fiber bundle where X is a CW complex. When is there a section of X?

At first, we will assume X is simply connected; we will be able to weaken this hypothesis later. Let F denote the typical fiber of π, i.e. the fiber over a chosen $x \in X$ (which we'll assume is in the 0-skeleton). We'll construct the section s inductively as a series of sections $s^k\colon X^{(k)} \to E|_{X^{(k)}}$ (where $X^{(k)}$ denotes the k-skeleton). If we're given s^k, when does it *not* extend to s^{k+1}?

Let $\Phi\colon (D^{k+1}, \partial D^{k+1}) \to (X^{(k+1)}, X^{(k)})$ denote the inclusion of a $(k+1)$-cell and $\phi := \Phi|_{\partial D^{k+1}} : S^k \to X^{(k)}$ be the attaching map. Then $\phi^*(s^k) := s^k \circ \phi$ is a section of the fiber bundle $\phi^* E \to S^k$. This bundle sits inside of $\Phi^* E \to D^{k+1}$, which is trivially trivial (i.e. canonically trivialized), because D^{k+1} is contractible. Therefore you can think of $\phi^* s^k$ as a map to a fiber: $S^k \to E_{\Phi(0)}$, and canonically up to homotopy, $E_{\Phi(0)} \cong F$ since X is simply connected.

Therefore we have a map $\{(k+1)\text{-cells}\} \to \pi_k F$ sending Φ to the map $\phi^* s^k : S^k \to F$. Since $\pi_k(F)$ is an abelian group (here $k > 1$), this map defines a cellular cochain

$$o^{k+1}(E, S^k) \in C^{k+1}(X; \pi_k(F)).$$

Let $\mathfrak{o}^{k+1} \in H^{k+1}(X; \pi_k(F))$ denote the cohomology class of o^{k+1}; this depends only on the homotopy class of s^k, and if $\mathfrak{o}^{k+1}$ vanishes, then s^k extends to a section s^{k+1} on the $(k+1)$-skeleton.[10]

Definition 5.3. Suppose $\pi_i F = 0$ for $i < k$. Then the *primary obstruction* for E is $\mathfrak{o}^{k+1}(X; \pi_k(F))$.

This is an invariant of the fiber bundle, hence is easier to understand than the higher obstructions, which depend on the choices of s^k that ones makes.

Remark 5.4. If we relaxed the assumption that X is simply connected, then we'd have to use local coefficients with the action of $\pi_1(X)$ on $\pi_k(F)$. So if $\pi_1(X)$ acts trivially on $\pi_i F$ for $i \le k$, the story continues in almost the same way. ◀

Stiefel-Whitney classes as primary obstructions. Though they are usually presented differently, Stiefel-Whitney classes were historically discovered as obstructions to collections of sections of vector bundles.

Definition 5.5. Let $E \to X$ be a real, rank-n vector bundle with a Euclidean metric. Then $V_k(E) \to E$ denotes the *Stiefel bundle*, the fiber bundle whose fiber at x is $V_k(E_x)$, the *Stiefel manifold* of orthonormal k-frames for E_x; standard fiber-bundle methods construct $V_k(E) \to X$ in a canonical fashion.

The typical fiber of $V_k(E)$ is $V_k(\mathbb{R}^n)$. A section of $V_k(E)$ is a k-tuple of orthonormal, hence linearly independent, sections of $E \to X$.

Stiefel and Whitney applied obstruction theory to $V_k(E)$ to understand when E admits linearly independent sections.

Proposition 5.6. *$V_k(\mathbb{R}^n)$ is $(n-k-1)$-connected, and*

$$\pi_{n-k}(V_k(\mathbb{R}^n)) \cong \begin{cases} \mathbb{Z}, & n-k \text{ even or } k = 1, \\ \mathbb{Z}/2, & \text{otherwise.} \end{cases}$$

For example, when $n = 1$, this is telling us that the first homotopy group of S^k is $\pi_k S^k = \mathbb{Z}$. We're not going to prove Proposition 5.6; the idea is to use the long exact sequence of homotopy groups associated to a fiber bundle.

Remark 5.7. It's worth thinking through what information is needed to identify a homotopy group with $\mathbb{Z}$; there are two choices, so this is something like requiring an orientation. ◀

Hence the primary obstruction to finding a section of $V_k(E) \to X$ is an $\mathfrak{o}^{n-k+1} \in H^{n-k+1}(X; \pi_{n-k}(V_k(\mathbb{R}^n)))$. Thus we have characteristic classes for E,

$$\mathfrak{o}_n^k(E) \in \begin{cases} H^{n-k+1}(X; \mathbb{Z}), & n-k \text{ even or } k = 1, \\ H^{n-k+1}(X; \mathbb{Z}/2), & \text{otherwise.} \end{cases}$$

The canonical nature of this construction means these are natural under pullback of vector bundles, hence are indeed characteristic classes.

In either case, there is a mod 2 characteristic class $\overline{\mathfrak{o}}_k^n \in H^{n-k+1}(X; \mathbb{Z}/2)$, and this actually makes sense irrespective of $\pi_1 X$, because $\mathrm{Aut}(\mathbb{Z}/2) = 1$, so all $\mathbb{Z}/2$ local systems are trivial.

Theorem 5.8. $\overline{\mathfrak{o}}_k^n(E) = w_{n-k+1}(E)$.

This perspective on Stiefel-Whitney classes shows that some of them come with canonical integral lifts. For $k = 1$, $\mathfrak{o}_1^n(E) \in H^n(X; \mathbb{Z})$ is the Euler class of E, which requires a choice of orientation on E (unless you use coefficients twisted by the orientation bundle of E). You can take this as the definition of the Euler class if you wish.

[10] One may have to replace s^k by a section homotopic to it.

Applications to 4-manifolds. For a reference, see the classical papers by Dold-Whitney and Hirzebruch-Hopf.

When $E \to X$ is a rank-4 real vector bundle, Theorem 5.8 tells us that $w_2(E)$ is the obstruction to finding 3 linearly independent sections of E over $X^{(2)}$. If ℓ denotes an orthogonal complement to these three sections given a metric on E, then $\ell \to X$ is a line subbundle of E, which is trivial iff $w_1(E) = 0$.

Corollary 5.9. *Let X be a closed, oriented 4-manifold. Then $w_2(TX)$ is the complete obstruction to trivializing TX over $X \setminus \mathrm{pt}$.*

Such an X is called *almost parallelizable.*

Proof. Choose a metric g for TX and a CW structure for $Y = X \setminus \mathrm{pt}$. Over $Y^{(2)}$, we can find orthonormal vector fields $v_1, v_2, v_3 \in \Gamma(TX|_Y)$ by Theorem 5.8, and also a unit vector $v_4 \in (v_1, v_2, v_3)^\perp$. This is because the line bundle $\ell \subset TX$ is trivial: X is orientable, so $w_1(TX) = 0$.

Let $P \to Y$ denote the principal SO_4-bundle of oriented orthonormal frames for Y; then, (v_1, v_2, v_3, v_4) is a section of $P|_{Y^2} \to Y^2$. The obstruction to extending this to Y^3 lies in $H^3(Y; \pi_2\mathrm{SO}_4)$. We'll see later that the universal cover for SO_4 is $S^3 \times S^3$, so

$$\pi_2(\mathrm{SO}_4) = \pi_2(S^3 \times S^3) = \pi_2(S^3) \times \pi_2(S^3) = 0. \tag{5.10}$$

(It's actually true that for any Lie group G, $\pi_2(G) = 0$. But here we don't need the full, harder result.) The upshot is, the obstruction vanishes, so $(v_1, \dots, v_4)$ extends to a section on $Y^{(3)}$. And the cohomology of a punctured 4-manifold vanishes above dimension 3, so all further obstructions vanish. ⊠

Theorem 5.11. *Let X be a closed, oriented 4-manifold, and suppose that $T, T' \to X$ are two rank-4 oriented vector bundles such that $w_2(T) = w_2(T') = 0$.*[11] *Then*

1. *$T \oplus \underline{\mathbb{R}} \cong T' \oplus \underline{\mathbb{R}}$ iff $p_1(T) = p_1(T')$, and*
2. *$T \cong T'$ as oriented vector bundles iff $p_1(T) = p_1(T')$ and $e(T) = e(T')$.*

Recall that the Euler class is the Poincaré dual to the zero set of a generic section.

To prove this, we'll need a result that will be useful again.

Lemma 5.12. *There exists a rank-4 vector bundle $E \to S^4$ with $\langle p_1(E), [S^4]\rangle = -2$ and $\langle e(E), [S^4]\rangle = 1$.*

Proof sketch. One can produce this bundle as a representative of the Bott element $\beta \in K^0(S^4)$. A more explicit construction: if $\mathbb{HP}^1$ denotes the *quaternionic projective line*, then $\mathbb{HP}^1 \cong S^4$ (for the same reason $\mathbb{RP}^1 \cong S^1$ and $\mathbb{CP}^1 \cong S^2$), and $\mathbb{HP}^1$ admits a tautological line bundle $\Lambda \to \mathbb{HP}^1$, which has the correct Pontrjagin and Euler classes. ⊠

Proof of Theorem 5.11. We'll use *Pontrjagin-Thom collapse* to construct a map $f\colon X \to S^4$ which is smooth (at least near an $x \in X$) of degree 1 with $D_x f$ an isomorphism such that $f^{-1}(f(x)) = x$. The idea is to product a map $X \to D^4$ which sends everything outside a disc neighborhood of X to ∂D^4, then collapse by the identification $D^4/\partial D^4 \cong S^4$.

Since $w_2(T) = 0$, there's some ball $B \subset X$ containing x such that T is trivial over $X \setminus B$. Therefore $T \cong f^*U$ for some rank-4 oriented bundle $U \to S^4$. Thus $p_1(T) = f^*p_1(U)$ and $e(T) = f^*e(U)$, and of course the same is true for T' and a $U' \to S^4$. Therefore it suffices to prove the theorem on S^4.

Bundles on the 4-sphere aren't so complicated. Let D_+ and D_- denote the two hemispheres, so $S^4 = D_+ \cup_{S^3} D_-$. $U, U' \to D_\pm$ are canonically trivialized, but the two trivializations of U over D_+ and D_- need not agree; instead, they are related by a map $S^3 \to \mathrm{SO}_4$ called the *clutching function*, which defines an element of $\pi_3\mathrm{SO}_4$.

Conversely, given a $[\gamma] \in \pi_3\mathrm{SO}_4$, one can construct a vector bundle $E_\gamma \to \mathrm{SO}_4$ by gluing the trivial bundles on D_+ and D_- by any representative γ for $[\gamma]$. In the same way, rank 5 vector bundles on S^4 correspond to elements of $\pi_3\mathrm{SO}_5$.

[11] So we could say that T and T' are two spin vector bundles.

Now we prove part (1). By what we've shown so far, $U \oplus \underline{\mathbb{R}} \cong U' \oplus \underline{\mathbb{R}}$ iff they have the same clutching function in $\pi_3 \mathrm{SO}_5 \cong \mathbb{Z}$.[12] The first Pontrjagin number defines a homomorphism

$$\begin{aligned} \pi_3 \mathrm{SO}(5) &\longrightarrow \mathbb{Z} \\ [\gamma] &\longmapsto \langle p_1(E_\gamma), [S^4] \rangle, \end{aligned}$$

and by Lemma 5.12, there exists a vector bundle on S^4 with nonzero Pontrjagin number. Hence this map is injective.

To prove part (2), we look at $\pi_3 \mathrm{SO}_4 \cong \pi_3(S^3 \times S^3) = \pi_3 S^3 \times \pi_3 S^3 = \mathbb{Z}^2$. Hence we can define a map

$$\begin{aligned} \pi_3(\mathrm{SO}_4) \cong \mathbb{Z}^2 &\longrightarrow \mathbb{Z}^2 \\ [\gamma] &\longmapsto (\langle p_1(E_\gamma), [S^4] \rangle, \langle e(E_\gamma), [S^4] \rangle). \end{aligned}$$

Again, using Lemma 5.12, there's a bundle E whose image under the above map is $(-2, 1)$. For TS^4, $p_1 = 0$ and $e = 2$, so the above map is given by the matrix

$$\begin{pmatrix} -2 & 0 \\ 1 & 2 \end{pmatrix},$$

hence is injective. ⊠

Remark 5.13. The theorem is true more generally assuming that $w_2(T) = w_2(T') \neq 0$: p_1 determines their stable isomorphism class and (p_1, e) determines their unstable isomorphism class. This requires a more sophisticated use of obstruction theory, and the proof is more involved. ◂

Corollary 5.14. *Suppose that X and X' are closed, oriented, simply-connected 4-manifolds and $f\colon X' \to X$ is a homotopy equivalence. Then $f^*TX \cong TX'$ as oriented vector bundles.*

Proof. The intersection form Q_X determines $p_1 = 3\tau$ and $e = \chi$, hence the isomorphism classes of f^*TX and TX' coincide. ⊠

Thus TX knows no more than the (oriented) homotopy type of a closed, oriented, simply-connected manifold.

Next time we'll turn to Rokhlin's theorem.

Lecture 6.

Rokhlin's theorem and the homotopy theorem: 2/6/18

Today we'll relate Rokhlin's theorem to the homotopy-theoretic fact that $\pi_8(S^5) \cong \mathbb{Z}/24$. We will then prove a theorem of Milnor that the intersection form of a simply-connected 4-manifold determines its homotopy type (which also knows the tangent bundle, as we saw last time).

Remark 6.1. Rokhlin was Russian, and there are at least three different ways of spelling his name in English. Keep this in mind when looking for references on this theorem. ◂

Theorem 6.2 (Rokhlin)**.** *Let X be a closed, oriented, smooth manifold with $w_2(TX) = 0$. Then $16 \mid \tau(X)$.*

We've seen that the signature of an even unimodular form is 0 mod 8, so the key is getting another factor of 2. By the Hirzenbruch signature theorem, Rokhlin's theorem is equivalent to $48 \mid \langle p_1(TX), [X] \rangle$. This is why the $\mathbb{Z}/24$ that we mentioned arises.

Theorem 6.3. *Rokhlin's theorem is equivalent to the theorem that $\pi_8(S^5) \cong \mathbb{Z}/24$.*

We will later directly prove Rokhlin's theorem using index theory.

Remark 6.4. This is a *stable homotopy group*: for $k \geq 5$, the suspension map $\pi_{3+k}(S^k) \to \pi_{3+k+1}(S^{k+1})$ is an isomorphism, which is a nontrivial theorem in homotopy theory. For $k < 5$, different groups can arise. ◂

[12] Again, if G is any simple Lie group, $\pi_3 G = \mathbb{Z}$.

Remark 6.5. One can compute $\pi_8(S^5)$ using methods internal to homotopy theory. Rokhlin was the first to do this, albeit with a more geometric flavor. Before this, Serre invented a localization method for calculating the first p-torsion of $\pi_*(S^k)$ for p prime, which shows that $\pi_8(S^5)$ has order $2^k \cdot 3$ for some k. One can characterize the 2-primary part using, e.g., the Adams spectral sequence.

Alternatively, there's a comprehensive story due to Adams and Quillen on the image of the J-homomorphism in $\pi_*^s(S^0)$, which touches on deep and difficult ideas. This is the first nontrivial example, and in this dimension it's surjective. ◀

The proof of Theorem 6.3 proceeds via the Pontrjagin-Thom construction. This is an awesome thing, and you should definitely fill in the details and/or read the references.

Definition 6.6. Consider closed k-dimensional manifolds M embedded in $\mathbb{R}^{k+m}$. Assume we're in the *stable range*, which means $m > k$. A *normal framing* for M is a trivialization of the normal bundle $\nu_M \to M$, an isomorphism $\phi\colon \nu_M \to \underline{\mathbb{R}}^m$. Typically only the homotopy class of ϕ is important.

Definition 6.7. Let (M_0, ϕ_0) and (M_1, ϕ_1) be normally framed submanifolds of $\mathbb{R}^{m+k}$. A *framed cobordism* is data (P, Φ), where P is a compact submanifold $P \subset \mathbb{R}^{m+k} \times [0,1]$ with boundary $\partial P \subset \mathbb{R}^{m+k} \times \{0,1\}$ such that

$$\partial P = (M_0 \times \{0\}) \amalg (M_1 \times \{1\}). \tag{6.8}$$

We assume a transversality condition; see the notes for details. The second piece of data is a framing $\Phi\colon \nu_P \to \underline{\mathbb{R}}^m$ which extends ϕ_0 and ϕ_1.

The set of equivalence classes of framed submanifolds of $\mathbb{R}^{m+k}$ under framed cobordism forms an abelian group under disjoint union.[13] This group is called the *framed cobordism group*[14] and denoted Ω_k^{fr}.

Lemma 6.9. *As the notation suggests, the inclusion $\mathbb{R}^{m+k} \hookrightarrow \mathbb{R}^{m+k+1}$ induces an isomorphism on framed bordism groups for normally framed k-manifolds.*

This is not too hard to see; the upshot is that there's not too much homotopical information contained in a normal framing in high codimension.

Definition 6.10. There is a group homomorphism $PT\colon \pi_{k+m}(S^m) \to \Omega_k^{fr}$ called the *Pontrjagin-Thom homomorphism* defined as follows: given an $[f] \in \pi_{k+m}(S^m)$, represent it by a smooth map $f\colon S^{k+m} \to S^m$. Then, $PT(f) \coloneqq f^{-1}(x)$, where x is a regular value. The framing comes from the framing of a neighborhood of x, which is diffeomorphic to $\mathbb{R}^n$, and this framing pulls back to $f^{-1}(x)$.

There is a lot to check here, including:

- Any map $f\colon S^{k+m} \to S^m$ is homotopic to a smooth map.
- The framed cobordism class of $PT(f)$ doesn't depend on the choice of smooth representative of $[f]$.
- The framed cobordism class of $PT(f)$ doesn't depend on the regular value chosen. This is because any two regular values in S^n are connected by a regular path.

Theorem 6.11 (Pontrjagin-Thom)**.** *The Pontrjagin-Thom map is an isomorphism.*

The rough idea is to construct an inverse map, called *Pontrjagin-Thom collapse*.

Example 6.12. For $k = 0$, this is a fact you likely already know: homotopy classes of maps $S^m \to S^m$ are classified by their degree, and the degree is defined to be the number of preimages of a regular value. ◀

Another ingredient that we need for Theorem 6.3 is the J-homomorphism, a map

$$J_k^n\colon \pi_k(\mathrm{SO}_m) \longrightarrow \pi_{k+m}(S^m). \tag{6.13}$$

Suppose $[\theta] \in \pi_k(\mathrm{SO}_m)$, and let $\theta\colon (S^k, *) \to (\mathrm{SO}_m, I)$ be a map; in particular, it's a map of pointed spaces. Thus for each $x \in S^k$, $\theta(x)\colon \mathbb{R}^m \to \mathbb{R}^m$ is an isometry, hence restricts to a map

$$\theta(x)\colon (\overline{D}^n, \partial\overline{D}^n) \longrightarrow (\overline{D}^n, \partial\overline{D}^n), \tag{6.14}$$

[13]The fact that cobordism is an equivalence relation comes from the fact that one can glue cobordisms if they share a boundary component.

[14]Sometimes it's instead called a *framed bordism group*.

where $\overline{D}^n \subset \mathbb{R}^n$ denotes the closed unit disc. Let

$$\widetilde{J}(\theta)\colon (S^k \times \overline{D}^m, S^k \times \partial\overline{D}^m) \longrightarrow (\overline{D}^m, \partial\overline{D}^m) \tag{6.15}$$

send

$$(x, y) \longmapsto \theta(x)(y). \tag{6.16}$$

This map descends to a map on the quotients, a map

$$J(\theta)\colon (S^k \times \overline{D}^m)/(S^k \times \partial\overline{D}^m) \cong S^{k+m} \longrightarrow \overline{D}^m/\partial\overline{D}^m \cong S^m. \tag{6.17}$$

We made choices, and one can show that the homotopy class of $J(\theta)$ is independent of those choices.

Remark 6.18. In the stable range $k > m+1$, the inclusion $\mathrm{SO}_m \hookrightarrow \mathrm{SO}_{m+1}$ induces an isomorphism on π_k, and suspension induces an isomorphism $\pi_{k+m}(S^m) \to \pi_{k+m+1}(S^{m+1})$; the J-homomorphism is compatible with these isomorphisms. ◀

The Pontrjagin-Thom theorem provides a geometric interpretation of the homotopy groups of the spheres; in view of that, we would like a geometric interpretation of the J-homomorphism. $\pi_k(\mathrm{SO}_m)$ acts simply transitively on the set of framings of $S^k \subset \mathbb{R}^{k+m}$, and there is a basepoint, namely the framing coming from the usual framing on $\mathbb{R}^n$. Therefore the set of framings of $S^k \subset \mathbb{R}^{k+m}$ is canonically identified with $\pi_k(\mathrm{SO}_m)$.

Proposition 6.19. *Under the identifications of $\pi_k(\mathrm{SO}_m)$ with the framings of $S^k \subset \mathbb{R}^m$ and $\pi_{m+k}(S^m) \cong \Omega_k^{fr}$, the J-homomorphism is the map sending S^k with a framing to its framed cobordism class.*

This was left as an exercise. It's not clear who originally came up with this, but it is used heavily by Kervaire and Milnor.

Now we specialize to $k = 3$ and $m = 5$, so we're interested in framings of $S^3 \subset \mathbb{R}^8$, and more generally of 3-manifolds in $\mathbb{R}^8$.

Theorem 6.20. *J_3 is surjective.*

The proof uses spin cobordism, which we haven't discussed; a sketch is in the notes. Therefore we have a sequence

$$\pi_3\mathrm{SO}_5 \cong \mathbb{Z} \xrightarrow{J_3} \Omega_3^{fr} \cong \pi_8(S^5) \longrightarrow 0,$$

as π_3 of any simple Lie group is $\mathbb{Z}$.

Proof sketch of Theorem 6.3. First, we'll assume $\pi_8(S^5) \cong \mathbb{Z}/24$, so $\ker(J_3) = 24\mathbb{Z}$. Let X be a closed, oriented 4-manifold with $w_2(TX) = 0$. By Corollary 5.9, TX is trivial after any disc D is removed from X. By the Whitney theorem, we can embed $X \hookrightarrow \mathbb{R}^9$, and over $X \setminus D$, we can choose a trivialization $\Phi\colon N_{X\setminus D} \to \underline{\mathbb{R}}^5$

In general, we can't extend Φ over D; the obstruction $\mathfrak{o}(\nu, \Phi) \in \pi_3(\mathrm{SO}_5)$ is the element represented by Φ on $\partial(X \setminus D) = S^3$. In particular, $\Phi|_{S^3}$ is a framing of S^3 that lies in $\ker(J_3)$, because it's a framing of a 3-manifold that extends to a framed 4-manifold $X \setminus D$. Thus $\mathfrak{o}(\nu, \Phi) \in 24\mathbb{Z}$.

Recall from Lemma 5.12 that there's a rank-4 bundle $E \to S^4$ with $\langle p_1(E), [S^4]\rangle = -2$ and $\langle e(E), [S^4]\rangle = 1$. This implies that the clutching function of $E \oplus \underline{\mathbb{R}}$ is a generator of $\pi_3(\mathrm{SO}_5)$. The upshot (there's a step or two to fill in here) is that

$$2\mathfrak{o}(\nu, \Phi) = \pm p_1(X). \tag{6.21}$$

Since $\langle \mathfrak{o}(\nu, \Phi), [X]\rangle$ is divisible by 24, then $\langle \pi_1(X), [X]\rangle$ is divisible by 48.

Conversely, assume Rokhlin's theorem and let ϕ be a framing for S^3, and suppose it represents a class $[a] \in \ker(J_3) \subset \mathbb{Z}$, so there is a framed 4-manifold (P, Φ) bounding (S^3, ϕ). Let $X \coloneqq P \cup_{S^3} D^4$; since $w_2(TP) = 0$, $w_2(TX) = 0$. Rokhlin's theorem implies $48 \mid \langle p_1(X), [X]\rangle$, and this is twice the obstruction a, so a is divisible by 24. Hence $\ker(J_3) \subset 24\mathbb{Z}$.

To get that it's equal, you need an example. Let X be a K3 surface (a quartic in $\mathbb{CP}^3$); one can show that $\langle p_1(X), [X]\rangle = -48$, which suffices. ⊠

Now we'll have an interlude on relative homotopy groups. Let (X, A, x) be a based pair of topological spaces, i.e. a topological space X, a subspace A, and a basepoint $x \in A$.

Definition 6.22. The n^{th} *relative homotopy set* of (X, A, x) is the set $\pi_n(X, A, x)$ of based maps

$$(D^n, \partial D^n, 0) \longrightarrow X, A, x)$$

This is a pointed set, whose basepoint is the class of any map landing in A: since D^n is contractible, these are all homotopic.

For $n \geq 2$, this pointed set is a group with identity element e. The idea is that there is a collapsing map $c\colon D^n \to D^n \vee D^n$ (pinch the sides of the disc inward), which is a based relative map

$$(D^n, \partial D^n) \longrightarrow (D^n \vee D^n, \partial D^n \vee \partial D^n). \tag{6.23}$$

Therefore we can define $f * g \coloneqq (f \vee g) \circ c$, and this defines a group structure on $\pi_n(X, A, x)$. If $n \geq 3$, this group is abelian, but it might not be for $n = 2$, for the same reason that fundamental groups need not be abelian.

We now summarize a few properties of these groups.

Proposition 6.24 (Long exact sequence of a pair)**.** *If (X, A, x) is a based pair of spaces, there is a long exact sequence of pointed sets*

$$\cdots \longrightarrow \pi_n(A, x) \longrightarrow \pi_n(X, x) \longrightarrow \pi_n(X, A, x) \xrightarrow{\delta} \pi_{n-1}(A, x) \longrightarrow \cdots.$$

For $n \geq 3$ this is a long exact sequence of abelian groups; for $n \geq 2$ this is a long exact sequence of groups.

Definition 6.25. The *Hurewicz map*

$$h\colon \pi_n(X, A, x) \longrightarrow H_n(X, A)$$

is defined to send $[f] \mapsto f_*[D^n, \partial D^n]$.

This is a group homomorphism for $n \geq 2$.

Theorem 6.26. *The Hurewicz maps induce a commutative diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_n(A, x) \arrow[r] \arrow[d, "h"] & \pi_n(X, x) \arrow[r] \arrow[d, "h"] & \pi_n(X, A, x) \arrow[r, "\delta"] \arrow[d, "h"] & \pi_{n-1}(A, x) \arrow[r] \arrow[d, "h"] & \cdots \\
\cdots \arrow[r] & H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X, A) \arrow[r, "\delta"'] & H_{n-1}(A) \arrow[r] & \cdots
\end{tikzcd}$$

Definition 6.27. A pair of spaces (X, A) is *n-connected* if for all $i \leq n$, every map $(D^i, \partial D^i) \to (X, A)$ is homotopic to a map $(D^i, \partial D^i) \to (A, A)$ (as these are all homotopic).

This implies that $\pi_i(X, A, x) = 0$ for $1 \leq i \leq n$ and all $x \in A$.

The key piece of this that we'll need next time is a relative version of the Hurewicz theorem.

Theorem 6.28 (Relative Hurewicz)**.** *Let (X, A) be an $(n-1)$-connected pair, where $n \geq 2$ and A is nonempty and simply connected. Then,*

1. *$H_i(X, A) = 0$ for all $i < n$, and*
2. *the Hurewicz map $h\colon \pi_n(X, A) \to H_n(X, A)$ is an isomorphism.*

That is, the first nontrivial homotopy and homology groups coincide. This is useful in similar ways to the classical Hurewicz theorem.

Next time, we'll define an isomorphism between $\pi_3((S^2)^{\vee n})$ and the $n \times n$ symmetric matrices over $\mathbb{Z}$, which is the key step in the homotopy theorem.

Lecture 7.

The homotopy theorem: 2/8/18

Note: I was out of town and missed this lecture; see the professor's lecture notes instead: https://www.ma.utexas.edu/users/perutz/Gauge%20Theory/L7.pdf.

Lecture 8.

The Hodge star: 2/13/18

Note: I was out of town and missed this lecture; consult the professor's lecture notes (https://www.ma.utexas.edu/users/perutz/Gauge%20Theory/L8.pdf) or George Torres' lecture notes (https://www.ma.utexas.edu/users/gdavtor/notes/gauge_notes.pdf, §2.6) instead.

Lecture 9.

Hodge theory and self-duality in 4 dimensions: 2/15/18

Today we're going to tallk about self-duality in dimension 4 from the perspective of Hodge theory and global differential geometry. We'll start with the general story, then specialize to dimension 4.

The Hodge theorem. Let (M, g) be a closed, oriented Riemannian manifold of dimension n. Recall from last time that this structure induces a linear map called the *Hodge star* $\star\colon \Lambda^k(T_x^*M) \to \Lambda^{n-k}(T_x^*M)$ for any $x \in M$. Globally this induces a map of vector bundles $\star\colon \Lambda^k(T^*M) \to \Lambda^{n-k}(T^*M)$.

Definition 9.1. The *co-differential* $\mathrm{d}^*: \Omega^{k+1}(M) \to \Omega^k(M)$ is the map

$$\mathrm{d}^* \coloneqq (-1)^{k+1}\star^{-1}\mathrm{d}\star = (-1)^{nk+1}\star\mathrm{d}\star.$$

Since $\mathrm{d}^2 = 0$ and $\star^2 = \pm 1$, then $(\mathrm{d}^*)^2 = 0$.

We don't need compactness to define d^*, but if M is compact, something nice happens.

Proposition 9.2. d^* *is the formal adjoint to* d *with respect to the* L^2 *inner product on* $\Omega^k(M)$ *defined by*

$$\langle \alpha, \beta\rangle_{L^2} \coloneqq \int_M g(\alpha, \beta)\,\mathrm{d}x. \tag{9.3}$$

In particular, the definition of d^* was completely local, but its identification as the adjoint is global.

Proof. Applying Stokes' theorem to the identity

$$\mathrm{d}(\alpha\wedge\beta) = \mathrm{d}\alpha\wedge\beta + (-1)^{|\alpha|}\alpha\wedge\mathrm{d}\beta \tag{9.4}$$

shows that

$$\int_M \mathrm{d}\alpha\wedge\beta = (-1)^{|\alpha|+1}\int_M \alpha\wedge\mathrm{d}\beta, \tag{9.5}$$

so letting $\beta = \star\gamma$, we get

$$\int_M g(\mathrm{d}\alpha, \gamma)\,\mathrm{d}x = (-1)^{k+1}\int_M g(\alpha, \star^{-1}\mathrm{d}\star\gamma)\,\mathrm{d}x. \tag{9.6}$$

That is, $\langle \mathrm{d}\alpha, \beta\rangle_{L^2} = \langle \alpha, \mathrm{d}^*\beta\rangle_{L^2}$. ⊠

Definition 9.7. Let (M, g) be a closed, oriented Riemannian manifold. The *Hodge Laplacian* on X is the map $\Delta\colon \Omega^k(M) \to \Omega^k(M)$ defined by

$$\Delta \coloneqq \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d} = (\mathrm{d} + \mathrm{d}^*)^2.$$

The kernel of Δ is called the space of *harmonic k-forms* and denoted $\mathcal{H}_g^k$.

In general, $\mathcal{H}_g^k$ is some infinite-dimensional space; clearly, it contains $\ker(\mathrm{d} + \mathrm{d}^*)$.

Proposition 9.8. *If M is compact,* $\mathcal{H}_g^k = \ker(\mathrm{d} + \mathrm{d}^*)$.

Proof. It suffices to prove the forward inclusion. Since

$$\begin{aligned}\langle \alpha, \Delta\alpha\rangle_{L^2} &= \langle \alpha, \mathrm{d}^*\mathrm{d}\alpha + \mathrm{d}\mathrm{d}^*\alpha\rangle_{L^2}\\ &= \langle \mathrm{d}\alpha, \mathrm{d}\alpha\rangle_{L^2} + \langle \mathrm{d}^*\alpha, \mathrm{d}^*\alpha\rangle_{L^2}\\ &= \|\mathrm{d}\alpha\|_{L^2}^2 + \|\mathrm{d}^*\alpha\|_{L^2}^2,\end{aligned}$$

then if $\Delta\alpha = 0$, then $(\mathrm{d} + \mathrm{d}^*)\alpha = 0$. ⊠

Therefore on a compact manifold, harmonic forms are also solutions to a first-order differential equation. $\mathrm{d}+\mathrm{d}^*$ is a kind of Dirac operator, which implies lots of nice things about harmonic forms.

There's also a variational characterization of harmonic forms. Harmonic forms are closed, hence have classes in de Rham cohomology.

Lemma 9.9. *Harmonic forms strictly minimize $\|\cdot\|_{L^2}^2$ of closed k-forms in their de Rham cohomology class. That is, if $\eta \in \mathcal{H}_g^k$ and $\widetilde{\eta}$ is cohomologous to η, then $\|\eta\|_{L^2}^2 \le \|\widetilde{\eta}\|_{L^2}^2$, and equality occurs iff $\widetilde{\eta} = \eta$.*

Proof. Let $\gamma \in \Omega^{k-1}(M)$. Then

$$\begin{aligned}\|\eta + \mathrm{d}\gamma\|_{L^2}^2 - \|\eta\|_{L^2}^2 &= 2\langle \eta, \mathrm{d}\gamma\rangle_{L^2} + \|\mathrm{d}\gamma\|_{L^2}^2 \\ &= 2\langle \mathrm{d}^*\eta, \gamma\rangle_{L^2} + \|\mathrm{d}\gamma\|_{L^2}^2 \\ &= \|\mathrm{d}\gamma\|_{L^2}^2,\end{aligned}$$

so the difference is nonzero if γ is nonzero. ⊠

Corollary 9.10. *Every de Rham cohomology class contains at most one harmonic representative.*

Lemma 9.9 admits a converse.

Lemma 9.11. *Let $\eta \in \Omega^k(M)$ be closed. If η minimizes $\|\cdot\|_{L^2}^2$ in its de Rham cohomology class, then η is harmonic.*

Proof. Since η is a minimizer, then for any $\gamma \in \Omega^{k-1}(M)$,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \|\eta + t\,\mathrm{d}\gamma\|_{L^2}^2 = 0. \tag{9.12}$$

Thus $2\langle \eta, \mathrm{d}\gamma\rangle_{L^2} = 0$. Since γ is arbitrary, choose $\gamma = \mathrm{d}^*\eta$. Hence

$$0 = \langle \eta, \mathrm{d}\mathrm{d}^*\gamma\rangle_{L^2} = \|\mathrm{d}^*\eta\|_{L^2}^2. \tag{9.13}$$

⊠

Remark 9.14. Even if η isn't closed, minimizing the norm still implies something interesting: that it's *co-closed* (i.e. $\mathrm{d}^*\eta = 0$). ◀

Inside $\Omega^k(M)$, $\ker(\mathrm{d}) \perp \mathrm{Im}(\mathrm{d}^*)$ under the L^2 inner product, and in fact $\ker(\mathrm{d}) = (\mathrm{Im}(\mathrm{d}^*))^\perp$. This is because if $\langle \eta, \mathrm{d}^*\omega\rangle_{L^2} = 0$ for all ω, then $\langle \mathrm{d}\eta, \omega\rangle_{L^2} = 0$, so $\|\mathrm{d}\eta\|_{L^2}^2 = 0$, so $\mathrm{d}\eta = 0$.

The next theorem is powerful, and also requires more nontrivial analysis.

Theorem 9.15 (Hodge theorem)**.** *Let (M, g) be a compact, oriented Riemannian manifold. Then there exists an L^2-orthogonal decomposition*

$$\begin{aligned}\Omega^k(M) &= \ker(\mathrm{d}) \oplus \mathrm{Im}(\mathrm{d}^*) \\ \ker(\mathrm{d}) &= \mathcal{H}_g^k \oplus \mathrm{Im}(\mathrm{d}).\end{aligned}$$

The quotient map $\mathcal{H}_g^k \to H_{\mathrm{dR}}^k(M)$ is an isomorphism.

Proof idea. The key part of the proof is the existence of a harmonic representative in any cohomology class. This problem breaks down into two steps.

(1) First, one finds a *weak solution* to $\Delta\eta = 0$, meaning a solution lying in some Sobolev space L_ℓ^2.
(2) Second, one uses *elliptic regularity* to show that $\eta \in \bigcap_\ell L_\ell^2 = C^\infty$, and hence the solution is smooth.

Both steps use elliptic estimates to show that Δ is a bounded operator between certain Sobolev spaces. ⊠

So instead of working with quotient spaces of infinite-dimensional vector spaces, which can be unwieldy, you can work with finite-dimensional vector spaces, and this is sometimes nicer.

Remark 9.16. In general, the wedge product of two harmonic forms isn't harmonic, so it's difficult to see the ring structure of cohomology using Hodge theory. ◀

We'll have more to say about this later, but for now we'll assume the Hodge theorem and specialize to dimension 4.

(Anti)-self-dual harmonic 2-forms. Let (X, g) be a closed, oriented, Riemannian 4-manifold. Then $\star\colon \Lambda^2(T^*X) \to \Lambda^2(T^*X)$ squares to 1. Therefore, as in the last lecture, there is a splitting

$$\Lambda^2(T^*X) = \Lambda^+_{[g]} \oplus \Lambda^-_{[g]}, \tag{9.17}$$

where $\Lambda^\pm_{[g]}$ is the ± 1-eigenspace of $\star$ on $\Lambda^2(T^*X)$. Let

$$\Omega^\pm_{[g]} := C^\infty(X, \Lambda^\pm_{[g]}); \tag{9.18}$$

these are called the *self-dual* (for 1) or *anti-self-dual* (for -1) 2-forms on X.

Definition 9.19. Let M be a manifold. A *conformal structure* on M is an equivalence class of Riemannian metrics, where $g_1 \sim g_2$ if $g_1 = f g_2$ for some function $f\colon M \to \mathbb{R}_{>0}$.

That is, a conformal structure is a metric up to rescaling.

Last lecture's results apply wholesale: in particular, $\star$ only depends on the conformal class of the metric. This is special to $k = 2$: in general scaling the metric by λ scales $\star$ by $\lambda^{k-n/2}$.

Since $\mathrm{d}^* = -\star\mathrm{d}\star$ in this dimension, then if $\eta \in \Omega^2(X)$, $\eta \in \ker(\mathrm{d} + \mathrm{d}^*)$ iff $\star\eta \in \ker(\mathrm{d} + \mathrm{d}^*)$. Hence if $\eta \in \mathcal{H}^2_g$, then its components

$$\eta^\pm := \frac{1}{2}(1 + \star)\eta \in \Omega^\pm_{[g]} \tag{9.20}$$

are also harmonic. Therefore if $\mathcal{H}^\pm_g := \mathcal{H}^2_g \cap \Omega^\pm_{[g]}$, there's a splitting

$$\mathcal{H}^2_g = \mathcal{H}^+_g \oplus \mathcal{H}^-_g. \tag{9.21}$$

Suppose $\eta \in \mathcal{H}^+_g \setminus 0$. Then

$$\int_X \eta \wedge \eta = \int \eta \wedge \star\eta = \int |\eta|^2 \,\mathrm{d}V > 0, \tag{9.22}$$

and similarly, if $\omega \in \mathcal{H}^-_g \setminus 0$, $\int_X \omega \wedge \omega < 0$.

Therefore the conformal class of g determines maximally positive and negative definite subspaces of the quadratic form $\eta \mapsto \int_X \eta \wedge \eta$ on $H^2_{\mathrm{dR}}(X) \cong \mathcal{H}^2_g$, namely $\mathcal{H}^\pm_g$. Let $b^\pm(X) := \dim \mathcal{H}^\pm_g$, which is the dimension of the maximally positive (resp. negative) subspace of the intersection pairing on de Rham cohomology.

In general, $b^+ \neq b^-$, and in fact their difference is the signature.

Remark 9.23. Let η be self-dual (resp. anti-self-dual). Then η is harmonic iff it's closed; this is because $\mathrm{d}^* = -\star\mathrm{d}\star$, so $\mathrm{d}^*\eta = -\star\mathrm{d}(\star\eta)$ and $\star\eta = \pm\eta$. ◀

Definition 9.24. Let (X, g) be as above and

$$\mathrm{d}^\pm := \frac{1}{2}(1 \pm \star) \circ \mathrm{d}.$$

The *signature complex*[15] is the cochain complex

$$0 \longrightarrow \Omega^0(X) \xrightarrow{\ \mathrm{d}\ } \Omega^1(X) \xrightarrow{\ \mathrm{d}^+\ } \Omega^+_g \longrightarrow 0.$$

We will denote the i^{th} term of this complex by $\mathcal{E}^i$ and the differentials by D, and will denote the whole complex by $(\mathcal{E}^*, D)$.

Theorem 9.25. *There are isomorphisms*

$$H^i(\mathcal{E}) \cong \begin{cases} H^i_{\mathrm{dR}}(X), & i = 0, 1 \\ \mathcal{H}^+_g(X), & i = 2. \end{cases}$$

[15] This is only half of the signature complex; it's all that we need, but in other sources the signature complex generally refers to something larger.

Proof. The case $i = 0$ is true by definition. For $i = 1$, it suffices to prove that if $\alpha \in \Omega^1(X)$ is in $\ker(\mathrm{d}^+)$, then it's closed. Since $\mathrm{d}\alpha = \mathrm{d}^+\alpha + \mathrm{d}^-\alpha$, then

$$\int_X \mathrm{d}\alpha \wedge \mathrm{d}\alpha = \int_X |\mathrm{d}^+\alpha|^2 \,\mathrm{d}V - \int_X |\mathrm{d}^-\alpha|^2 \,\mathrm{d}V. \tag{9.26}$$

Since M is closed,

$$\int_X \mathrm{d}\alpha \wedge \mathrm{d}\alpha = \int_X \mathrm{d}(\alpha \wedge \mathrm{d}\alpha) = 0, \tag{9.27}$$

so for all α, $\|\mathrm{d}^+\alpha\|_{L^2}^2 = \|\mathrm{d}^-\alpha\|_{L^2}^2$. Therefore if $\mathrm{d}^+\alpha = 0$, then $\mathrm{d}^-\alpha = 0$ too, so $\mathrm{d}\alpha = 0$.

For $i = 2$, we'll show that $\Omega^+_{[g]}/\operatorname{Im}(\mathrm{d}^+) \cong \mathcal{H}^+_g$, or more precisely, that the composition

$$\mathcal{H}^+_g \hookrightarrow \Omega^+_{[g]} \longrightarrow \Omega^+_{[g]}/\operatorname{Im}(\mathrm{d}^+) \tag{9.28}$$

is an isomorphism.

Let $\omega \in \Omega^+_{[g]}$; then, it has a *Hodge decomposition*

$$\omega = \omega_h + \mathrm{d}\alpha + \star\mathrm{d}\beta, \tag{9.29}$$

where $\omega_h \in \mathcal{H}^2_g$, $\mathrm{d}\alpha \in \operatorname{Im}(\mathrm{d})$, and $\star\mathrm{d}\beta \in \operatorname{Im}(\mathrm{d}^*)$. The components are unique; hence, if ω is self-dual, $\star\omega_h = \omega_h$ and $\mathrm{d}\beta = \mathrm{d}\alpha$. Thus

$$\omega = \underbrace{\omega_h}_{\mathcal{H}^+_g} + \underbrace{\mathrm{d}\alpha + \star\mathrm{d}\alpha}_{2\mathrm{d}^+\alpha \in \operatorname{Im}(\mathrm{d}^+)}. \tag{9.30}$$

⊠

This calculation will be relevant for computing the dimensions of Seiberg-Witten moduli spaces, as will another involving Euler characteristics that we will discuss later.

Definition 9.31. Fix an $r \geq 3$ and let $\operatorname{Conf} X$ denote the space of conformal structures on X, i.e. the space of C^r-Riemannian metrics modulo the space of positive C^r functions.

In the future, we will need to do some analysis for which C^r regularity, for which things are Banach spaces, are better than C^∞ regularity, where we merely have a Fréchet space.

Fix a $[g_0] \in \operatorname{Conf} X$ and let $\Lambda^\pm := \Lambda^\pm_{[g_0]}$. This provides an identification of $\operatorname{Conf} X$ with the space of C^r bundle maps $m\colon \Lambda^+ \to \Lambda^0-$ such that TODO: given a conformal structure $[g]$, $\Lambda^-_{[g]} = \Gamma_m$, the graph of m, for some such m, and $\Lambda^+_{[g]} = \Gamma_{m^*}$; this was discussed last time.

Hence $\operatorname{Conf} X$ is an open subset of a Banach space, and is particular a Banach manifold. We have an identification of its tangent spaces as Banach spaces:

$$T_{[g_0]} \operatorname{Conf} X = C^r(X, \operatorname{Hom}(\Lambda^-, \Lambda^+)). \tag{9.32}$$

Definition 9.33. The *period map* is the map $P\colon \operatorname{Conf} X \to \operatorname{Gr}_{b^-(X)} H^2_{\mathrm{dR}}(X)$ sending $[g] \mapsto \mathcal{H}^-_g \subset \mathcal{H}^2_g \cong H^2_{\mathrm{dR}}(X)$.

Studying the period map informs us how the space of anti-self-dual forms changes as the metric changes. The first thing we should do is compute the derivative of P.

Since $[g_0] \in \operatorname{Conf} X$ has a neighborhood which is a neighborhood of 0 in $C^r(X, \operatorname{Hom}(\Lambda^-, \Lambda^+))$, then a neighborhood of $\mathcal{H}^-_{[g_0]}$ is identified with a neighborhood of 0 in $\operatorname{Hom}(\mathcal{H}^-_{[g_0]}, \mathcal{H}^+_{[g_0]})$: a subspace is sent to a map which has it as its graph.

Thus, near $[g_0]$, we may think of P as a map $C^r(X, \operatorname{Hom}(\Lambda^-, \Lambda^+)) \to \operatorname{Hom}(\mathcal{H}^-_{g_0}, \mathcal{H}^+_{g_0})$. Therefore its derivative is a map

$$D_{[g_0]}P\colon C^r(X, \operatorname{Hom}(\Lambda^-, \Lambda^+)) \longrightarrow \operatorname{Hom}(\mathcal{H}^-_{g_0}, \mathcal{H}^+_{g_0}). \tag{9.34}$$

Proposition 9.35. *Under this identification, if m is a bundle map and $\alpha^- \in \mathcal{H}^-_{g_0}$,*

$$(D_{[g_0]}P)(m)(\alpha^-) = (m(\alpha^-))_h.$$

Corollary 9.36. *The period map is a submersion.*

Thus you can move around the space of anti-self-dual forms pretty freely. This will be useful for transversality theory.

Lecture 10.

Covariant derivatives: 2/20/18

"I hate signs."

Today we return to local differential geometry, studying connections on vector bundles. Next time we will cover local aspects. Two references on this that may be useful:

- Donaldson-Kronheimer, "The geometry of 4-manifolds," which is terse but useful.
- Berline-Getzler-Vergne, "Heat kernels and Dirac operators," chapter 1. This is not aimed at Seiberg-Witten theory but is useful nonetheless.

Let $E \to M$ be a complex vector bundle. We will let $\Gamma(M, E)$ denote the complex vector space of smooth sections of $E \to M$.

Definition 10.1. Let $E \to M$ be a complex vector bundle and $\langle -, - \rangle$ be a Hermitian inner product on E. A *covariant derivative* or a *connection* in E is a $\mathbb{C}$-linear map

$$\nabla \colon \Gamma(M, E) \longrightarrow \Omega^1_M(E) \coloneqq \Gamma(M, T^*M \otimes_{\mathbb{R}} E)$$

obeying the *Leibniz rule*

$$\nabla(f, s) = \mathrm{d}f \otimes s + f\nabla s, \tag{10.2}$$

where $f \in C^\infty(M)$ and $s \in \Gamma(M, E)$. The connection ∇ is called *unitary* if

$$\mathrm{d}(s_1, s_2) = (\nabla s_1, s_2) + (s_1, \nabla s_2). \tag{10.3}$$

We will let $\mathcal{C}(E)$ denote the space of covariant derivatives on $E \to M$.

Lemma 10.4. *Covariant derivatives are* local operators*, i.e.* $(\nabla s)(x)$ *only depends on the germ of* s *near* x*.*

Proof. Let U be a neighborhood of x and $s_1, s_2 \in \Gamma(M, E)$ be such that $s_1|_U = s_2|_U$. Let χ be a *cutoff functrion* for U, i.e. a smooth function supported in U and equal to 1 in a neighborhood of x. Hence $\chi s_1 = \chi s_2$. Applying (10.2), $\nabla(\chi s_i)(x) = (\nabla s_i)(x)$, so $(\nabla s_0)(x) = (\nabla s_1)(x)$. ⊠

This does not generalize to other forms of geometry, e.g. complex analytic geometry, real analytic geometry, or algebraic geometry, as cutoff functions don't exist. There locality is part of the definition.

Example 10.5. Consider a trivialized line bundle $\underline{\mathbb{C}} \to M$. Then the exterior derivative is a covariant derivative, as it satisfies the Leibniz rule. This defines the *trivial connection*, and is still available in higher-rank vector bundles: let V be a finite-dimensional complex vector space and consider $\underline{V} \coloneqq V \times M \twoheadrightarrow M$. Then we define the trivial connection by

$$\mathrm{d} \coloneqq \mathrm{d} \otimes \mathrm{id}_V \colon \Gamma(M, V) \longrightarrow \Gamma(M, T^*M \otimes V).$$

The trivial connection is unitary with respect to a trivialized Hermitian inner product on V. ◀

Let ∇, ∇' be connections on a vector bundle $E \to M$. Then (10.2) implies $\nabla - \nabla'$ is $C^\infty(M)$-linear, and in fact it can be an arbitrary $C^\infty(M)$-linear map. Hence $\mathcal{C}(E)$ is an affine space over $\Omega^1_M(\operatorname{End} E) \coloneqq \Gamma(M, T^*M \otimes_{\mathbb{R}} \operatorname{End}_{\mathbb{C}}(E))$: if you choose one reference connection ∇, every other connection ∇' is equal to $\nabla + \omega$ for some $\omega \in \Omega^1_M(\operatorname{End} E)$.

If you think through it a bit, the space of unitary connections is in the same way an affine space modeled on $\Omega^1_M(\mathfrak{u}(E))$, where $\mathfrak{u}(E)_x$ is the bundle of skew-Hermitian endomorphisms of E_x for $x \in M$.

Example 10.6. On a trivialized vector bundle $\underline{V} \to M$, we have a canonical trivial connection d, hence $\mathcal{C}(\underline{V})$ is canonically isomorphic to $\Omega^1_M(\operatorname{End} V)$: any connection ∇ satisfies $\nabla = \mathrm{d} + A$ for some $\operatorname{End}_{\mathbb{C}}(V)$-valued 1-form A. ◀

Lemma 10.7. *Covariant derivatives are first-order operators, in that* $(\nabla s)(x)$ *depends only on* $s(x)$ *(zeroth-order information) and* $D_x s \colon T_x M \to T_{s(x)} E$ *(first-order information).*

Proof. Since $(\nabla s)(x)$ is local, we may assume E is trivialized, so $\nabla = \mathrm{d} + A$, which is first-order. ⊠

If $E \to M$ is a vector bundle and $f \colon M' \to M$ is smooth, then a connection ∇ on E pulls back to the pullback bundle $f^*E \to M'$ in the usual way. This satisfies $(g \circ f)^*\nabla = f^*(g^*\nabla)$, as it should.

Lemma 10.8. *Connections exist on any vector bundle $E \to M$, as do unitary connections.*

Proof. Let $\mathfrak{U}$ be an open cover of M which trivializes E, i.e. we have trivializations of $E|_U \to U$ for all $U \in \mathfrak{U}$. Then we have connections ∇_U on $E|_U$. Let $\{\rho_U\}_{U\in\mathfrak{U}}$ be a partition of unity subordinate to $\mathfrak{U}$; then

$$\nabla := \sum_{U\in\mathfrak{U}} \rho_U \nabla_U \tag{10.9}$$

is a connection on E. ⊠

Later, when you have multiple different definitions and perspectives on connections, hold onto this one: it's concrete and straightforward, which can be important when you're confused.

Associated to any connection $\nabla \in \mathcal{C}(E)$ is a *coupled exterior derivative*

$$\begin{aligned} d_\nabla \colon \Omega^k_M(E) &\longrightarrow \Omega^{k+1}_M(E) \\ d_\nabla(\eta\otimes s) &:= (-1)^k \eta\wedge\nabla s + \mathrm{d}\eta\otimes s, \end{aligned} \tag{10.10}$$

where $\eta \in \Omega^k_M(E)$ and $s \in \Gamma(M,E)$; then, we extend $\mathbb{C}$-linearly. This obeys a Leibniz rule: if $f \in C^\infty(M)$,

$$d_\nabla \colon (f\eta\otimes s) = \mathrm{d}f\wedge\eta\otimes s + f d_\nabla(\eta\otimes s). \tag{10.11}$$

In the case $k=0$, d_∇ is just ∇.

Lemma 10.12. $d^2_\nabla \colon \Omega^*_M(E) \to \Omega^{*+2}_M(E)$ *is linear over $C^\infty(M)$ and over Ω^*_M.*

Proof. Suppose $\eta \in \Omega^*_M$ and $s \in \Gamma(M,E)$. Then

$$\begin{aligned} d_\nabla\circ d_\nabla(\eta\otimes s) &= d_\nabla((-1)^k\eta\wedge \underbrace{\nabla s}_{d_\nabla s} + \mathrm{d}\eta\otimes s) \\ &= (-1)^{2k}\eta\wedge d_\nabla\circ d_\nabla s + (-1)^k \mathrm{d}\eta\wedge\nabla s + \mathrm{d}^2\nabla\otimes s + (-1)^{k+1}\mathrm{d}\nabla\wedge\nabla s \\ &= \eta\wedge d^2_\nabla s. \end{aligned}$$

The same calculation applies when s is instead an E-valued differential form. ⊠

Definition 10.13. The *curvature* of a connection ∇, denoted $F_\nabla \in \Omega^2_M(\operatorname{End} E)$ is defined by

$$d_\nabla \circ d_\nabla s = F_\nabla \wedge s.$$

When you're doing calculations, it's important to know what's going on in a forest of confusing operators; it's important to remember where things live. For example, keep in mind that the curvature is an endomorphism-valued 2-form.

If ∇ is unitary, $F_\nabla \in \Omega^2_M(\mathfrak{u}(E))$. Since the definition of the curvature is completely intrinsic, it follows immediately that it's natural: $F_{f^*\nabla} = f^*F_\nabla$.

There are many perspectives on curvature, and we're favoring one of them: that the curvature is the obstruction to $d^2_\nabla = 0$.

Definition 10.14. A connecion is *flat* if its curvature vanishes.

For example, since $\mathrm{d}^2 = 0$, the trivial connection is flat.

A connection ∇ on $E \to M$ induces a connection on $\operatorname{End} E \to M$, as well as its coupled exterior derivatives: we define $d_\nabla \colon \Omega^k_M(\operatorname{End} E) \to \Omega^{k+1}_M(\operatorname{End} E)$ by

- $d_\nabla\alpha := [d_\nabla, \alpha]$ on $\Omega^0_M(\operatorname{End} E)$, and in general
- $(d_\nabla\alpha)(s) := d_\nabla(\alpha s) - \alpha(d_\nabla s)$.

Given a connection ∇ and an endomorphism-valued 1-form A, we have another connection $\nabla + A$.

Proposition 10.15.

$$F_{\nabla+A} = F_\nabla + d_\nabla A + A\wedge A,$$

where $A\wedge A$ denotes a combination of wedge of forms and composition of endomorphisms.

In particular, on a trivial bundle,

$$F_{\mathrm{d}+A} = \mathrm{d}A + A\wedge A. \tag{10.16}$$

Example 10.17. We're going to care the most about connections on line bundles. In this case, the quadratic term $A \wedge A$ disappears, because it's in Ω^2 of something of dimension 1. Hence $F_{\nabla+A} = F_\nabla + d_\nabla A$. if $L \to M$ is Hermitian, then $A \in \Omega^1_M(\mathfrak{u}_1) = i\Omega^1_M$, since $\mathfrak{u}_1 = i\mathbb{R}$. ◀

Let v be a vector field. Then, we will let $\nabla_v : \Gamma(E) \to \Gamma(E)$ denote the contraction of ∇ by v. Thus if $f \in C^\infty(M)$, $\nabla_{fv} = f\nabla_v$.

Let $(x_1, \ldots, x_n)$ be local coordinates on M and $\partial_i := \frac{\partial}{\partial x_i}$ be the coordinate vector fields; let $\nabla_i := \nabla_{\partial_i}$. We can locally trivialize $E \to M$, hence write a connction $\nabla = \mathrm{d} + A$. Hence

$$\nabla = \mathrm{d} + \sum_k A_k \otimes \mathrm{d}x_k,$$

where $A_k(x) \in \mathrm{End}(\mathbb{C}^r)$, i.e. they're matrices.

Similarly, there are $F_{ij}(x) \in \mathrm{End}(\mathbb{C}^r)$ such that the curvature satisfies

$$F_{\mathrm{d}+A} = \sum_{i,j} F_{ij}\, \mathrm{d}x_i \wedge \mathrm{d}x_j. \tag{10.18}$$

Then (10.16) simplifies to

$$F_{ij} = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} + [A_i, A_j]. \tag{10.19}$$

This is what curvature is, concretely. In particular,

$$[\nabla_i, \nabla_j] = \left[\frac{\partial}{\partial x_i} + A_i, \frac{\partial}{\partial x_j} + A_j\right] = F_{ij}, \tag{10.20}$$

so curvature computes the failure of ∇_i to commute.

Lemma 10.21. *Let u and v be vector fields. As elements of $\Gamma(M, \mathrm{End}\, E)$,*

$$F_\nabla(u, v) = [\nabla_u, \nabla_v] - \nabla_{[u,v]}.$$

Thus we have another perspective on curvature: it measures the failure of ∇ to commute with the Lie bracket of vector fields.

Proof. If u and v are coordinate vector fields, this is (10.20). Since every vector field is locally a linear combination of coordinate vector fields, it suffices to show that the right-hand side is $C^\infty(M)$-bilinear (since we already know the left-hand side is), which is just a computation. ⊠

Proposition 10.22 (Bianchi identity)**.** *The curvature is* covariantly closed*, i.e.* $d_\nabla F_\nabla = 0$.

The proof is in the professor's notes.

Gauge transformations.

Definition 10.23. A *gauge transformation* of a (Hermitian) vector bundle $E \to M$ is a (unitary) bundle automorphism $u : E \to E$, hence a smoothly varying family $u_x : E_x \xrightarrow{\cong} E_x$. These form a group which we'll denote $\mathcal{G}_E$.

If $\mathrm{GL}(E)$ denotes the bundle of Lie groups whose fiber at x is $\mathrm{GL}(E_x)$, then $\mathcal{G}_E = \Gamma(M, \mathrm{GL}(E))$.[16]

There's an action of $\mathcal{G}_E$ on $\mathcal{C}(E)$ by $u \cdot \nabla := u^*\nabla$, i.e.

$$(u^*\nabla)(s) = u\nabla(u^{-1}s). \tag{10.24}$$

The curvature transforms by

$$F_{u^*\nabla} = u^*F_\nabla = uF_\nabla u^{-1}. \tag{10.25}$$

Since $u^*\nabla$ and ∇ are both connections on the same bundle, we can compare them and obtain an endomorphism-valued 1-form:

$$u^*\nabla - \nabla = -(d_\nabla u)u^{-1}. \tag{10.26}$$

Here to take $d_\nabla u$ we regard u as a section of $\mathrm{End}\, E$.

[16] $\mathrm{GL}(E)$ is *not* a principal bundle, as it doesn't have a canonical right action of $\mathrm{GL}_{\mathrm{rank}\, E}(\mathbb{C})$; rather, it's an associated bundle of the principal frame bundle by the adjoint action.

On the trivial bundle we have

$$au^*(\mathrm{d}+A)-\mathrm{d} = -(\mathrm{d}u)u^{-1}+uAu^{-1}. \tag{10.27}$$

This has been a lot of formalism; now let's prove something.

Theorem 10.28. *Flat bundles are locally trivial; that is, if $F_\nabla = 0$, then for any $x \in M$ there's a local trivialization of E near x in which $\nabla = \mathrm{d}$.*

It suffices to prove the following.

Proposition 10.29. *Let $H := (-1,1)^n \subset \mathbb{R}^n$ and $\underline{\mathbb{C}}^r \to H$ be a trivial bundle. If ∇ is a flat (unitary) connection in $\underline{\mathbb{C}}^r$ then there's a unitary gauge transformation u such that $u^*\nabla$ is trivial.*

Proof. Write

$$\nabla = \mathrm{d}+A = \mathrm{d}+\sum_k A_k\,\mathrm{d}x_k, \tag{10.30}$$

where $A_k\colon H \to \mathrm{End}(\underline{\mathbb{C}}^r)$ (and is skew-Hermitian in the unitary case). Since ∇ is flat,

$$\partial_i A_j - \partial_j A_i + [A_i, A_j] = 0. \tag{10.31}$$

Inductively, we assume we've gauge-transformed to $A_i = 0$ for $i = 1,\dots,m$. The base case of $m=0$ is vacuous. We want to find a u such that $u^*\nabla$ has $A_i = 0$ for $i = 1,\dots,m+1$.

Well

$$u^*\nabla = \mathrm{d}+\sum_k B_k\,\mathrm{d}x_k, \tag{10.32}$$

where

$$B_k = -(\partial_k u)u^{-1} + uA_k u, \tag{10.33}$$

and several of the A_k are already zero. So we want to solve

$$\begin{gathered}\partial_i u = 0, \qquad i = 1,\dots,m\\ \partial_{m+1}u + uA_{m+1} = 0.\end{gathered} \tag{10.34}$$

This is a system of ODEs linear in x_{m+1}, independent of $x_1,\dots,x_m$ (because $\partial_i A_{m+1} = 0$ by flatness), and whose coefficients depend smoothly on $x_{m+1},\dots,x_m$. Therefore it has a unique solution u which is the identity when x_{p+1} vanishes, which is smooth in all x_i, and is independent of $x_1,\dots,x_m$.

For unitarity, compute that $\frac{\partial}{\partial x_{m+1}}(uu^\dagger) = 0$. ⊠

Lecture 11.

Instantons in line bundles: 2/22/18

Today, we'll specialize to U_1-connections, i.e. in complex line bundles. This leads to the story of instantons in line bundles, which can be completely solved, and is very important in Seiberg-Witten theory.

Let ∇ be a connection on $E \to M$. We can encode it in a collection of local data: let $\mathfrak{U}$ be an open cover such that E is trivialized over each $U_\alpha \in \mathfrak{U}$. Then we can give transition functions $\tau_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}_r(\mathbb{C})$. These satisfy various *cocycle properties*, e.g. $\tau_{\beta\alpha} = \tau_{\alpha\beta}^{-1}$.

On each U_α, $\nabla = \mathrm{d}+A_\alpha$ for some $A_\alpha \in \Omega^1_{U_\alpha}(\mathrm{End}\,\mathbb{C}^r)$, so you have explicit matrix-valued 1-forms. They satisfy a *defining condition*

$$A_\beta = \tau_{\alpha\beta}A_\alpha\tau_{\beta\alpha} - (\mathrm{d}\tau_{\alpha\beta})\tau_{\beta\alpha}, \tag{11.1}$$

which comes from the rule for how covariant derivatives transform under gauge transformations.

If ∇ is flat, Theorem 10.28 implies it's locally trivial, so we may assume $A_\alpha = 0$ for all $U_\alpha \in \mathfrak{U}$. Then (11.1) simplifies to $\mathrm{d}\tau_{\alpha\beta} = 0$ for all α, β.

Corollary 11.2. *A vector bundle with a flat connection determines and is determined by a vector bundle with locally constant transition functions.*

A vector bundle with locally constant transition functions is also called a *local system*. There are other ways to think of them, including locally constant sheaves of vector spaces or as associated to representations of $\pi_1(M)$. Since our goal in this course is to gain insight into simply connected 4-manifolds, we're not going to adopt this perspective very much.

Seiberg-Witten theory has U_1-connections. Let $L \to M$ be a Hermitian line bundle and $\mathcal{A}_L$ denote the space of unitary covariant derivatives on L. If you pick a connection $\nabla \in \mathcal{A}_L$, then $\mathcal{A}_L = \nabla + \Omega^1_M(i\mathbb{R})$, because $\mathfrak{u}(L) \cong i\mathbb{R}$ insude $\mathrm{End}(L) \cong \underline{\mathbb{C}}$.

The gauge group $\mathcal{G}_L = \Gamma(M, \mathrm{U}(L)) = C^\infty(M, \mathrm{U}_1)$, because $\mathrm{U}(L)$ is canonically the trivial principal U_1-bundle. This is unusual: usual the gauge transformations aren't a trivial bundle, because the endomorphism bundle is usually nontriival.

$\mathcal{A}_L$ is an affine Fréchet space, with a C^∞ topology induced from a family of C^r norms. Similarly, $\mathcal{G}_L$ has a topology inherited from the Fréchet space $\Gamma(M, \mathrm{End}\, L)$. The action of $\mathcal{G}_L$ on $\mathcal{A}_L$ is continuous, and we may therefore form the space $B_L := \mathcal{A}_L/\mathcal{G}_L$, the set of gauge orbits of connections.

It's not clear from its definition that B_L is a reasonable space, e.g. is it Hausdorff? For higher-rank bundles, it's true but difficult to prove; the proof exhibits a metric.

Theorem 11.3. *Let L be a line bundle. Then there is a homeomorphism*

$$B_L \cong (S^1)^{b_1(M)} \times F, \tag{11.4}$$

where F is a Fréchet space.

First, let's see what Chern-Weil theory has to say about this. If $\nabla \in \mathcal{A}_L$, then $iF_\nabla \in \Omega^2(M)$ is closed, because it's locally exact: locally $\nabla = \mathrm{d} + A$ and $F_\nabla = \mathrm{d}A$. The class $c_L := [iF_\nabla] \in H^2_{\mathrm{dR}}(M)$ is independent of ∇: any other connection is $\nabla + iA$ for some $A \in \Omega^1(M)$, and its curvature is

$$F_{\nabla + iA} = F_\nabla + i\,\mathrm{d}A. \tag{11.5}$$

Remark 11.6. In the higher-rank case, there's a quadratic term in this expression, which makes things a bit harder. ◂

Proposition 11.7. *There's a universal $\lambda \in \mathbb{R}$ (i.e. not depending on M) such that $c_L = \lambda c_1(L)$.*

Proof. As we discussed last time, curvature pulls back along a smooth map $f\colon N \to M$, which implies $f^*(c_L) = c_{f^*L}$. Chern classes are also natural under pullback: $c_1(f^*L) = f^*c_1(L)$.

All line bundles on manifolds arise as the pullback of the tautological bundle $\Lambda_N \to \mathbb{CP}^N$ along a smooth map $f\colon M \to \mathbb{CP}^N$ for some large N. (This is because $\mathbb{CP}^\infty$ is a $B\mathrm{U}_1$.) Hence it suffices to prove this for $\mathbb{CP}^N$.

The inclusion $i\colon \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^N$ induces an isomorphism on H^2, and $i^*\Lambda_M \cong \Lambda_1$. Hence it suffices to show that $c_{\Lambda_1} = \lambda c_1(\Lambda_1) \in H^2(\mathbb{CP}^1; \mathbb{R}) \cong \mathbb{R}$. Thus this boils down to an explicit calculation over S^2, which is found, e.g. in Bott-Tu.[17] ⊠

In fact, $\lambda = 2\pi$, i.e.

$$\frac{1}{2\pi} c_L = \left[\frac{i}{2\pi} F_\nabla\right] = c_1(L). \tag{11.8}$$

The action of $\mathcal{G}_L$ on $\mathcal{A}_L$ is by

$$u^*\nabla = \nabla - (\mathrm{d}u)u^{-1}, \tag{11.9}$$

where ∇ is a unitary connection and $u\colon M \to \mathrm{U}_1$ is a gauge transformation.

Definition 11.10. Let a group G act on a space X. The action is *semifree* if there is a normal subgroup $N \trianglelefteq G$ which acts trivially on X and such that the induced action of G/N on X is free.

Proposition 11.11. *The normal subgroup $\mathrm{U}_1 \trianglelefteq \mathcal{G}_L$ of constant gauge transformations acts trivially, and hence the $\mathcal{G}_L$-action on $\mathcal{A}_L$-action is semifree.*

Partial proof. If u is constant, $\mathrm{d}u = 0$; so (11.9) simplifies to $u^*\nabla = \nabla$. ⊠

[17] Bott-Tu use the Euler class, but in this setting the two agree.

Proof of Theorem 11.3. We have $\pi_0\mathcal{G}_L \cong [M, S^1]$, which is naturally isomorphic to $H^1(M;\mathbb{Z})$. The isomorphism is very concrete: send a map $f\colon M \to S^1$ to what it pulls back a generator of $H^1(S^1)$ to.

The identity component $\mathcal{G}_L^\circ$ of $\mathcal{G}_L$ is $\{u = e^{i\xi} \mid \xi \in C^\infty(M;\mathbb{R})\}$, the space of gauge transformations with well-defined logarithms. In this case, (11.9) specializes to

$$(e^{i\xi})^*\nabla = \nabla - i\,\mathrm{d}\xi. \tag{11.12}$$

Hence after fixing a connection ∇, we get an identification $\mathcal{A}_L/\mathcal{G}_L^\circ \cong i(\Omega^1(M)/\mathrm{d}\Omega^0(M))$. Let $S \coloneqq \{\nabla + ia \mid d^*a = 0\}$, which is called a *Coulomb gauge slice*. Hodge theory identifies an isomorphism $\Omega^1(M)/\mathrm{d}\Omega^0(M) \cong \ker(\mathrm{d}^*)$, so the inclusion map $S \to \mathcal{A}_L/\mathcal{G}_L^\circ$ is a homeomorphism. $\mathcal{A}_L/\mathcal{G}_L^\circ$ is an infinite-dimensional affine space modeled on the Fréchet space of co-closed 1-forms.

Let's next describe the action of $\pi_0(\mathcal{G}_L) \cong H^1(M;\mathbb{Z})$ on $S \cong \mathcal{A}_L/\mathcal{G}_L^\circ$; given a $u \in \mathcal{G}_L$, (11.9) implies

$$u^*\nabla = \nabla - \mathrm{d}(\log u). \tag{11.13}$$

This logarithm may not be uniquely defined, but $\mathrm{d}(\log u)$ is, and is a closed 1-form with periods in $2\pi i\mathbb{Z}$. Hence, given such a u, there's a unique cohomologous 1-form $\mathrm{d}(\log u) + \mathrm{d}\xi$ such that

$$\nabla + \mathrm{d}(\log u) + \mathrm{d}\xi \in S. \tag{11.14}$$

Hence $\mathrm{d}(\log u) + \mathrm{d}\xi \in \ker(\mathrm{d}) \cap \ker(\mathrm{d}^*)$, so it's a harmonic 1-form.

Since $\ker(\mathrm{d}^*) = \mathcal{H}_g^1 \oplus \mathrm{Im}(\mathrm{d}^*)$, then

$$\begin{aligned} B_L \cong S/\pi_0\mathcal{G}_L &= \frac{\mathcal{H}_g^1}{2\pi(\text{a } \mathbb{Z}\text{-lattice})} \times \mathrm{Im}(\mathrm{d}^*) \\ &\cong \frac{H^1(X;\mathbb{R})}{2\pi H^1(X;\mathbb{Z})} \times \mathrm{Im}(\mathrm{d}^*). \end{aligned}$$

Let $P \coloneqq H^1(X;\mathbb{R})/2\pi H^1(X;\mathbb{Z})$, which is called the *Picard torus*; it's diffeomorphic to $(S^1)^{b_1(X)}$, and $\mathrm{Im}(\mathrm{d}^*)$ is a Fréchet space, as we wanted. ⊠

Remark 11.15. Gauge theory tends to be complicated because we're working in a quotient space, but for rank 1, the techniques we just saw allow us to replace the quotient by something nicer, so the proofs are easier. ◂

Next we'll discuss instantons, which are a specifically 4-dimensional story.

Definition 11.16. Let X be a compact 4-manifold with a conformal structure $[g]$, and let $E \to X$ be a vector bundle with a Hermitian metric. An *instanton* or an *anti-self-dual connection* is a unitary connection ∇ such that $(F_\nabla)^+ = 0$.

Here $(-)^+ \coloneqq (1/2)(1+\star)$, the projection onto self-dual forms. This doesn't affect the endomorphism part, and is only about 2-forms.

Remark 11.17. The term "instanton," or more properly "Yang-Mills instanton," comes to us from physics: Yang-Mills theory is present in the background of this story, and from that perspective there are particle-physics reasons to call these instantons. ◂

Let $u \in \mathcal{G}_E$. Then

$$(F_{u^*\nabla})^+ = (uF_\nabla u^{-1})^+ = uF_\nabla^+ u^{-1}, \tag{11.18}$$

so $\mathcal{G}_E$ sends instantons to instantons. The equation $(F_{\nabla+A})^+ = 0$ is a first-order PDE in A.

Donaldson theory, as developed in the 1980s and 1990s, is the study of instantons, chiefly in rank-2 bundles. Today we're going to focus on the rank-1 case (U_1-instantons).

Proposition 11.19. *Suppose $L \to X$ is a line bundle with an instanton. Then $c_1(L) \in \mathcal{H}_{[g]}^-(\mathbb{Z}) \coloneqq \mathcal{H}_{[g]}^- \cap H^2(X;\mathbb{Z}) \subset H^2_{\mathrm{dR}}(X)$.*

Proof. If $F_\nabla^+ = 0$, then $(i/2\pi)F_\nabla$ is anti-self-dual; since it's also closed, then it's harmonic. ⊠

If $c_1(L) \in \mathcal{H}_{[g]}^-(\mathbb{Z})$, we can represent it by an anti-self-dual form ω. Then any connection ∇ with $(i/2\pi)F_\nabla = \omega$ is an instanton.

In general, $H^2_{\mathrm{dR}}(X)$ is a vector space with two complementary subspaces $\mathcal{H}^+_{[g]}$ and $\mathcal{H}^-_{[g]}$, and an integer lattice $H^2(X;\mathbb{Z})$. These generically don't satisfy any relations, so that the only element of $H^2(X;\mathbb{Z}) \cap \mathcal{H}^-_{[g]}$ is the Chern class of the trivial bundle. Sometimes, however, there are others.

Now let's talk about uniqueness.

Proposition 11.20. *Suppose $\nabla \in \mathcal{A}_L$ is an instanton and $a \in \Omega^1(M)$. Then $\nabla + ia$ is an instanton iff $(F_\nabla + i\,\mathrm{d}a)^+ = 0$.*

$(F_\nabla + i\,\mathrm{d}a)^+ = 0$ means that $\mathrm{d}^+ a = 0$ (this was defined in Definition 9.24). We saw in Theorem 9.25 that

$$\frac{\ker(\mathrm{d}^+)}{\mathrm{Im}(\mathrm{d})} = \frac{\ker(\mathrm{d})}{\mathrm{Im}(\mathrm{d})} = H^1_{\mathrm{dR}}(X), \tag{11.21}$$

So if $\mathcal{M}_L$ denotes the space of instantons in L modulo the action of $\mathcal{G}_L$, $\mathcal{M}_L \subset B_L$. We'll pick this up next time; uniqueness for instantons is governed by the operator $\mathrm{d}^+ + \mathrm{d}^*$.

Lecture 12.

Instantons do not exist generically: 2/27/18

Today we'll finish up the discussion of U_1 instantons, and then discuss generalities of differential operators on manifolds.

Like last time, let (X, g) be a closed, oriented Riemannian manifold and $L \to X$ be a Hermitian line bundle with unitary connection ∇. Assume ∇ is an instanton, i.e. $F^+_\nabla = 0$.

Last time, we proved an existence criterion for instantons, Proposition 11.19: there exists an instanton in L iff $c_1(L) \in \mathcal{H}^-_{[g]}(\mathbb{Z})$. This generically does not hold.

If we do have an instanton ∇, we can ask about uniqueness; we stated Proposition 11.20 last time, but we're actually more interested in instantons modulo gauge equivalence. Recall that $\mathcal{A}_L/\mathcal{G}^0_L \cong \Omega^1/\mathrm{d}\Omega^0$ by the map sending $[\nabla + ia] \mapsto [a]$. Let $\mathcal{I}_L$ denote the space of instantons inside $\mathcal{A}_L$; then $\mathcal{I}_L/\mathcal{G}^0_L \cong \ker(\mathrm{d}^+)/\mathrm{d}\Omega^0 = H^1_{\mathrm{dR}}(X)$. Since $\pi_0\mathcal{G}_L \cong H^1(X;\mathbb{Z})$, then

$$\mathcal{I}_L/\mathcal{G}_L \cong \frac{H^1_{\mathrm{dR}}(X)}{2\pi H^1(X;\mathbb{Z})} = P. \tag{12.1}$$

That is, the moduli space of instantons modulo gauge equivalence, if nonempty, is the Picard torus. In particular, it's a finite-dimensional smooth manifold, which is quite nice, and even a little surprising.

Why should this be the case? Let's work in the Coulomb gauge slice $\mathcal{S} := \nabla + i\ker(\mathrm{d}^*)$, which is diffeomorphic to $\mathcal{A}_L/\mathcal{G}^0_L$. Then $\mathcal{I}_L \cap S = \nabla + i\ker(\mathrm{d}^* \oplus \mathrm{d}^+)$, so we're interested in the operator $\mathrm{d}^* \oplus \mathrm{d}^+$. The ellipticity of this operator is important, but we'll return to that later.

Recall that $\mathrm{d}^*\colon \Omega^0 \hookrightarrow \Omega^1$ and $\mathrm{d}^+\colon \Omega^1 \to \Omega^+_g$. d^+ respects the splittings

$$\begin{aligned} \Omega^1 &= \mathcal{H}^1_g \oplus \mathrm{d}\Omega^- \oplus \mathrm{d}^+\omega^2 \\ \Omega^+_g &= \mathcal{H}^+_{[g]} \oplus \mathrm{d}^+\Omega^1, \end{aligned} \tag{12.2}$$

in that $\mathrm{d}^+\colon \mathrm{d}^*\Omega_2 \to \mathrm{d}^+\Omega^1$ is an isomorphism. Moreover, $\mathrm{d}^*\colon \mathrm{d}\Omega 0 \to \mathrm{d}^)\Omega_1 = \Omega^0_0$, the functions whose integral is zero, is an isomorphism.

Putting all these together, the map

$$\mathrm{d}^* \oplus \mathrm{d}^+ \colon \Omega^1 \longrightarrow \Omega^)_0 \oplus \Omega^+_{[g]} \tag{12.3}$$

has kernel $\mathcal{H}^1_g$ and cokernel $\mathcal{H}^+_{[g]}$. Hence the instanton moduli space $\mathcal{I}_L/\mathcal{G}_L$ is cut out by a function whose derivative $\mathrm{d}^* \oplus \mathrm{d}^+$ is not surjective, but has a cokernel of constant rank $b^+(X)$. Hence the moduli space is cut out cleanly, if not transversely, by its defining equations.

Remark 12.4. We're used to thinking about manifolds arising as preimages of regular values, but must be careful here: we're in the infinite-dimensional setting. The crucial ingredient is the inverse function theorem — which still holds for infinite-dimensional Banach manifolds, but not infinite-dimensional Fréchet manifolds. Hence, to technically set up the moduli space of instantons and make related constructions, one must start with connections of some given Sobolev regularity, though then the story proceeds similarly. ◀

We've mentioned a few times that generically, instantons don't exist on a line bundle, and the idea is that we're intersecting a fixed lattice with a generic subspace, so generically cannot expect any intersection points. Let's make this idea precise.

Theorem 12.5. *For $k < b^+(X)$, $r \geq 3$, and any C^r family of conformal structures $[g_t]_{t\in T}$ parameterized by a k-dimensional Riemannian manifold T, there exist perturbations $\widehat{g}_t$ to g_t, arbitrarily close to $[g_t]$ in the C^r-norm induced by the Riemannian metrics on X and T, such that $\mathcal{H}^-_{[\widehat{g}_t]}(\mathbb{Z}) = 0$ for all $t \in T$.*

Proof. Let C_X denote the space of conformal structures of class C^r, where $r \geq 3$ is fixed. We identified this with an open set on $\Gamma(X, \mathrm{Hom}(\Lambda^-, \Lambda^+))$, where $\Lambda^\pm = \Lambda^\pm_{[g_0]}$ for some reference metric g_0. In particular, $[g] \mapsto m$ where $\Lambda^-_{[g]} = \Gamma_m$. This defines the period map $P\colon C_X \to \mathrm{Gr}^-$, the Grassmannian of negative-definite subspaces of dimension $b^-(X)$ in $H^2_{\mathrm{dR}}(X)$: $P[g] = \mathcal{H}^-_{[g]}$.

Then, $D_{[g]}P\colon T_{[g]}C_x = \Gamma(M, \mathrm{Hom}(\Lambda^-, \Lambda^+)) \to T_{\mathcal{H}^-_{[g]}}\mathrm{Gr}^- = \mathrm{Hom}(\mathcal{H}^-, \mathcal{H}^+)$ is the map

$$(DP)(m)(\alpha_-) = m(\alpha_-)_{\mathrm{har}}, \tag{12.6}$$

where $\alpha_- \in \mathcal{H}^-$.

We'll need one more tool: let $c \in H^2_{\mathrm{dR}}(X) \setminus 0$ and

$$S_c := \{H \in \mathrm{Gr}^- \mid c \in H\}, \tag{12.7}$$

which is a closed subset of the Grassmannian. Here's why: if $H \in S_c$, choose an H' such that $H^2(X) = H \oplus H'$. Then, for any linear map $\mu\colon H \to H'$, $c \in \Gamma_\mu$ iff $\mu(c) = 0$, which is a closed condition. Moreover, $S_c \subset \mathrm{Gr}^-$ is a closed submanifold, and its tangent space is

$$T_H S_c = \{\mu \in \mathrm{Hom}(H, H') \mid \mu(c) = 0\}. \tag{12.8}$$

This identification identifies the normal space $N_H S_c$ with H', sending $[\mu] \mapsto \mu(c)$. Therefore, assuming transversality, S_c has codimension $b^+(X)$.

Lemma 12.9. *The period map P is transverse to S_c.*

Proof. We'll show that if $P[g] \in S_c$, then $\mathrm{Im}(D_{[g]}P)$ spans $N_{P[g]}S_c$. Let's unravel what this means. We want to show that if $\alpha^- \in \mathcal{G}^-_{[g]}$ represents c, then for all $\alpha^+ \in \mathcal{H}_{[g]}+$ there's a map $m\colon \Lambda^- \to \Lambda^+$ such that $m(\alpha^-)_{\mathrm{har}} = \alpha^+$.

If this were not true, there would exist an $\alpha^+ \in \mathcal{H}^+_{[g]}$ which is L^2-orthogonal to $m(\alpha^-)$ for all m, i.e. for all m,

$$\begin{aligned} 0 &= \langle \alpha^+, m(\alpha^-)_{\mathrm{har}} \rangle_{L^2} \\ &= \langle \alpha^+, m)(\alpha^-) \rangle_{L^2} \end{aligned}$$

Choose an $x \in X$ such that $\alpha^+(x) \neq 0$ and $\alpha^-(x) \neq 0$. Such an x exists because harmonic forms are nonvanishing on an open dense subset, by analytic continuation. Let B be a small geodesic ball about x and choose an $m_0\colon \Lambda^-_x \to \Lambda^+_x$ such that $m_0(\alpha^-)_x = \alpha^+_x$. Choose a cutoff function χ supported in $(1/2)B$ and identically 1 in a neighborhood of x.

In B, we can extend m_0 to an m such that $\langle m(\alpha^-), \alpha^+ \rangle \neq 0$; then

$$\langle (\chi m)(\alpha^-), \alpha^+ \rangle_{L^2} = \int_B \chi \langle m(\alpha^-), \alpha^+ \rangle \,\mathrm{dvol} \neq 0. \tag{12.10}$$ ⊠

Therefore we can assume for any $c \neq 0$ in $H^2_{\mathrm{dR}}(X)$ $P^{-1}(S_c) \subset C_X$ is a submanifold of codimension $b^+(X)$. Here we have used the inverse function theorem for Banach spaces. All we have to do is avoid these.

Let $\{[g_t]\}_{t\in T}$ be defined by a smooth map $G\colon T \to C_X$, and let $\mathcal{G}$ denote the space of C^r maps $T \to C_X$ lying with a fixed distance of G with respect to a C^r norm.[18] Then, there's an open subset $U_c \subset \mathcal{G}$ of mans $\widehat{G}$ transverse to $P^{-1}(S_c)$. By dimensionality reasons, $\widehat{G} \pitchfork P^{-1}(S_c)$ means they do not intersect, because $\dim T < b^+(X)$.

Now we elimininate the choice of c: $\bigcap_{c\in H^2(X;\mathbb{Z})'} U_c$ is a countable intersection of open dense subsets, hence by the Baire category theorem is dense, since we're in a complete metric space. Therefore any $\widehat{G}$ in this intersection does the job. ⊠

[18]The norm is a choice here, but all choices are equivalent for the goals of this proof.

The technicalities in transversality arguments take some getting used to, but are grounded in concrete differential topology. There are a thousand other transversality arguments in gauge theory, pseudoholomorphic curves, etc., and they follow very similar lines.

In the remaining minutes of this lecture, we'll discuss differential operators. The major point is that we can do this in a coordinate-free way. Let M be a manifold and $E, F \to M$ be real vector bundles. First-order linear differential operators should be certain maps $D\colon \Gamma(E) \to \Gamma(F)$. In local coordinates $(x_1, \dots, x_n)$ trivializing both E and F, a section s of E looks like a column vector $(s_1, \dots, s_p)$, and a section s' of F is represented by a column vector $(s'_1, \dots, s'_q)$. Then we want our operator D to satisfy a formula like

$$(Ds)_\alpha = \sum_{i,\beta} P^i_{\alpha\beta}(x) \frac{\partial}{\partial x_i} s_\beta + \sum_\beta Q_{\alpha\beta}(x) s_\beta(x). \tag{12.11}$$

Here P and Q are smooth, and Latin indices refer to coordinates on the manifold and Greek indices refer to coordinates on the vector bundle.

The notation in (12.11) is not pleasing to look at, and is worse to work with. Let's try to do it invariantly.

Definition 12.12. A *zeroth-order operator* $L\colon \Gamma(M, E) \to \Gamma(M, F)$ is a $C^\infty(M)$-linear function. The vector space of zeroth-order operators is denoted $\mathcal{D}_0(E, F)$.

There is an isomorphism

$$\mathcal{D}_0(E, F) \xrightarrow{\sigma} \Gamma(M, \mathrm{Hom}(E, F)), \tag{12.13}$$

called the *symbol*, defined by

$$\sigma(L)(x, e) = (Ls)(x), \tag{12.14}$$

where s is an arbitrary section with $s(x) = e \in E_x$; one checks this is well-defined, which is not hard.

Definition 12.15. A *first-order differential operator* is an $\mathbb{R}$-linear map $D\colon \Gamma(M, E) \to \Gamma(M, F)$ such that for all function $f \in C^\infty(M)$, the commutator $[f, D]$ is a zeroth-order operator. The vector space of first-order operators is denoted $\mathcal{D}_1(E, F)$.

This is cleaner than (12.11), though it turns out to be equivalent. In particular, it implies zeroth-order operators are also first-order operators.

Example 12.16. Any covariant derivative $\nabla\colon \Gamma(M, E) \to \Gamma(M, T^*M \otimes E)$ is a first-order operator, because by definition $[f, D]s = \mathrm{d}f \otimes s$. This is algebraic in s, hence $C^\infty(M)$-linear. ◂

Lecture 13.

Differential operators: 3/1/18

Last time, we introduced differential operators between vector bundles $E, F \to M$.

- The zeroth-order differential operators $\mathcal{D}_0(E, F)$ are the $C^\infty(M)$-linear functions $L\colon \Gamma(E) \to \Gamma(F)$, which form an $\mathbb{R}$-vector space isomorphic to $\Gamma(\mathrm{Hom}(E, F))$.
- The first-order differential operators $\mathcal{D}_1(E, F)$ are the $\mathbb{R}$-linear functions $D\colon \Gamma(E) \to \Gamma(F)$ such that for all $f \in C^\infty(M)$, $[D, f] \in \mathcal{D}_0(E, F)$.

The basic example of a first-order operator is the covariant derivative in E, since $[\nabla, f] = \mathrm{d}f \otimes -$.

We discussed what this would look like in coordinates, namely (12.11), but we can also recast this story in terms of jet bundles.

Definition 13.1. The *1-jet bundle* $J_1(E) \to M$ is a vector bundle whose fiber at an $x \in M$ is the space of *1-jets of sections of* E at x, i.e. equivalence classes of $s \in \Gamma(M, E)$ where $s \sim s'$ if $s(x) = s'(x)$ and $s - s'$ is tangent to 0 at x, i.e. $D_x(s - s')\colon T_xM \to T_{(x,0)}E$ maps to the tangent space of the zero section of E.

That is, we locally identify sections which agree to first order. In local coordinates which trivialize E, this implies s and s' have the same Taylor expansion up to quadratic terms.

Remark 13.2. There's a general notion of n^{th}*-order jets* generalizing this directly, where we take equivalence classes of sections which agree to n^{th} order. ◂

Given a 1-jet $[s]$, its value at any $x \in M$ does not depend on the choice of representative, so we have an evaluation map $\mathrm{ev}\colon J^1E \to E$. This fits into a short exact sequence

$$0 \longrightarrow (T^*M) \otimes E \longrightarrow J^1E \xrightarrow{\mathrm{ev}} E \longrightarrow 0. \tag{13.3}$$

There's a tautological map $j^1\colon \Gamma(M,E) \to \Gamma(M, J^1E)$ sending $s \mapsto (x \mapsto j^1_x(s) \coloneqq [s]_x)$.

Definition 13.4. A *first-order jet operator* is a map $D\colon \Gamma(E) \to \Gamma(F)$ of the form $(Ds)(x) = L(j^1_x(s))$, where $L \in \Gamma(\mathrm{Hom}(J^1E, F))$. These form a vector space $\mathcal{D}_1(E,F)_{\mathrm{jet}} \cong \Gamma(\mathrm{Hom}(J^1E,F))$.

In local coordinates which trivialize E, a first-order jet operator D looks like (12.11), which is not terribly complicated: just spell out what such an operator can do.

Theorem 13.5 (Grothendieck)**.** $\mathcal{D}_1(E,F) = \mathcal{D}_1(E,F)_{\mathrm{jet}}$.

Remark 13.6. The fact that Grothendieck was interested in this suggests that it secretly applies to algebraic geometry, and indeed this perspective is quite useful there. ◀

Proof. First we'll prove that a first-order jet operator D is a first-order differential operator. We have $D = L \circ j^1$ for some $L\colon \Gamma(J^1E) \to \Gamma(F)$. If $f \in C^\infty(M)$ and $s \in \Gamma(M,E)$,

$$j^1_x(fs) = f(x)j^1_x(s) + \underbrace{\mathrm{d}f(x) \otimes s(x)}_{(*)}. \tag{13.7}$$

The part marked $(*)$ is in $T^*_xM \otimes E_x$, so by (13.3) is in $(J^1M)_x$. In particular,

$$[f,D](s) = L(\mathrm{d}f \otimes s). \tag{13.8}$$

Next, we compute

$$\begin{aligned}
[D,f](gs) &= L \circ j^1(fgs) - fL(j^1(gs))\\
&\overset{(13.7)}{=} L \circ (df \otimes gs + fj^1(gs)) - fL(j^1(gs))\\
&= gL(\mathrm{d}f \otimes s)\\
&\overset{(13.8)}{=} g[D,f].
\end{aligned}$$

Therefore D is a differential operator.

Before we go the other direction, (13.3) means there's a restriction map ρ fitting into a short exact sequence[19]

(13.9)

$$0 \longrightarrow \Gamma(\mathrm{Hom}(E,F)) \longrightarrow \mathcal{D}_1(E,F)_{\mathrm{jet}} = \Gamma(\mathrm{Hom}(J^1E,F)) \xrightarrow{\rho} \Gamma(\mathrm{Hom}(T^*M \otimes E, F)) \longrightarrow 0.$$

This map ρ is called the *principal symbol* of D; in the coordinate form (12.11), it captures the data of $\{P^i_{\alpha\beta}\}$.

It's also possible to define the symbol σ^1_D of a first-order differential operator D by

$$\sigma^1_D(f) \coloneqq \sigma^0_{[D,f]} \in \Gamma(\mathrm{Hom}(E,F)). \tag{13.10}$$

One can check that

$$\sigma^1_D(fg) = f\sigma^1_D(g) + g\sigma^1_D(f). \tag{13.11}$$

Lemma 13.12. *If $D \in \mathcal{D}(E,F)_1$, $f(x) = 0$, and $\mathrm{d}f|_x = 0$, then $\sigma^1_D(f)_x = 0$.*

Proof sketch. Near x, such an f is a sum of products g_ih_i where $g_i(x) = h_i(x) = 0$. Then, apply the product rule (13.11) to each g_ih_i. ⊠

Moreover, if c is constant near x, then $\sigma^1_D(c)_x = 0$, since $[D,c]_x = 0$. The point of this is that the symbol only depends on $\mathrm{d}f_x \in T^*_xM$.[20] Therefore we can define $\sigma^1_D(\xi)_x$ for a $\xi \in T^*_xM$ to be $\sigma^1_D(f)$ for any f with $\mathrm{d}f_x = \xi$.

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[19]The fact that Γ is exact follows because we're taking smooth sections, rather than holomorphic sections of some complex vector bundle. Therefore local sections can always be extended globally and we don't have to worry about derived functors.

[20]If you prefer to think of the cotangent space as $\mathfrak{m}_x/\mathfrak{m}_x^2$, where $\mathfrak{m}_x$ is the unique maximal ideal of the local ring of germs of functions near x, which is the more common perspective in algebraic geometry, then $\mathrm{d}f_x = [f - f(x)] \in \mathfrak{m}_x/\mathfrak{m}_x^2$.

The comparison of these two symbol operators allows us to finish the proof: we already know about the inclusions

$$\begin{tikzcd} \mathcal{D}_1(E,F) \arrow[r, "\sigma_D^1"] \arrow[d, hook] & \Gamma(\mathrm{Hom}(T^*M \otimes E, F)), \\ \mathcal{D}_1(E,F)_{\mathrm{jet}} \arrow[ur, hook, "\rho"'] & \end{tikzcd} \tag{13.13}$$

so just from definitions, if $D \in \mathcal{D}_1(E,F)$, then $\sigma_D^1 \circ j^1 \in \mathcal{D}_1(E,F)_{\mathrm{jet}}$, and $D - \sigma_D^1 j^1 \in \mathcal{D}(E,F)_0$. Therefore $D \in \mathcal{D}_1(E,F)_{\mathrm{jet}}$ also. ⊠

Once you absorb this formalism, it generalizes straightforwardly to higher-order differential operators.

Definition 13.14. We inductively define the *n^{th}-order linear differential operators* $\mathcal{D}_n(E,F)$ to be the space of $\mathbb{R}$-linear functions $D\colon \Gamma(E) \to \Gamma(F)$ such that $[D,f] \in \mathcal{D}_{n-1}(E,F)$ for all smooth functions f.

We define the *n-jet bundle* $J^nE \to M$to be the space of equivalence classes of sections where $s \sim s'$ if their Taylor expansions agree to order n. Then (13.3) generalizes to

$$0 \longrightarrow \mathrm{Sym}^n(T^*X) \otimes E \longrightarrow J^nE \longrightarrow J^{n-1}E \longrightarrow 0, \tag{13.15}$$

and a tautological map j^n as before, so we can define the space of *n^{th}-order jet operators* $\mathcal{D}_n(E,F)_{\mathrm{jet}}$ to be operators of the form $L \circ j^n$, where $L \in \mathrm{Hom}(J^nE, F)$.

Theorem 13.5 generalizes to $\mathcal{D}_n(E,F) = \mathcal{D}_n(E,F)_{\mathrm{jet}}$, and restricting n^{th}-order jet operators to $J^{n-1}E$ defines the symbol

$$\sigma^n\colon \mathcal{D}_n(E,F)/\mathcal{D}_{n-1}(E,F) \xrightarrow{\cong} \Gamma(\mathrm{Hom}(\mathrm{Sym}^n(T^*M) \otimes E, F)). \tag{13.16}$$

Proposition 13.17. *Let D_1 be an m^{th}-order differential operator and D_2 be an n^{th}-order differential operator. Then $D_1 \circ D_2$ is an $(m+n)^{\text{th}}$-order differential operator and*

$$\sigma^{m+n}_{D_1 \circ D_2} = \sigma^m_{D_1} \circ \sigma^n_{D_2}.$$

The proof uses the composition in $\mathrm{Sym}^\bullet(T^*M)$, which is identified with the polynomial functions on T^*M.

If $\mathcal{D}(E,F) \coloneqq \bigcup_n \mathcal{D}_n(E,F)$, then $\mathcal{D}(E,F)$ is filtered by the order of an operator. We can take its *associated graded* $\mathrm{gr}\mathcal{D}(E,F) \coloneqq \bigoplus_n \mathcal{D}_n/\mathcal{D}_{n+1}$; if $E = F$ composition makes it into a graded ring.

Proposition 13.18. *The symbol defines an isomorphism of graded rings*

$$\sigma\colon \mathrm{gr}\mathcal{D}(E,E) \xrightarrow{\cong} \Gamma(\mathrm{Hom}(\mathrm{Sym}^\bullet(T^*M), \mathrm{End}\,E)).$$

We'll now discuss some examples of this abstract formalism.

Example 13.19. The exterior derivative $\mathrm{d}\colon \Omega^k(M) \to \Omega^{k+1}(M)$ is a first-order differential operator, because $[\mathrm{d}, f] = \mathrm{d}f \wedge -$. Hence the symbol $\sigma^1_{\mathrm{d}}\colon T^*M \to \mathrm{Hom}(\Lambda^k(T^*M), \Lambda^{k+1}(T^*M))$ is the map $\sigma^1_{\mathrm{d}}(\xi)(\alpha) = \xi \wedge \alpha$. ◀

Example 13.20. Let's next look at a coupled exterior derivative $d_\nabla\colon \Omega^k_M(E) \to \Omega^{k+1}_M(E)$, where ∇ is a covariant derivative on E. Then $[d_\nabla, f] = \mathrm{d}f \wedge -$, so the symbol is again $\sigma^1_{d_\nabla}(\xi) = \xi \wedge -$. ◀

Example 13.21. Now suppose M is compact and $E, F \to M$ have metrics. Let $D \in \mathcal{D}_1(E,F)$ and $D^* \in \mathcal{D}_1(F,E)$. We say that D and D^* are *formal adjoints* if

$$\langle s', Ds\rangle_{L^2(F)} = \langle D^*s, s'\rangle_{L^2(E)} \tag{13.22}$$

for all $s \in \Gamma_{L^2}(M,E)$ and $s' \in \Gamma_{L^2}(M,F)$.

One can quickly check (by integration by parts) that if D and D^* are formal adjoints, then $[D^*, f] = -[D,f]^*$, and therefore $\sigma^1_{D^*}(\xi) = -\sigma^1_D(\xi)^*$.

For example, if M is an oriented Riemannian manifold, $\sigma^1_{\mathrm{d}^*}$ is the adjoint to $\xi \wedge -$. You can check this is contraction ι_ξ. The story is simplar for d^*_∇. ◀

Example 13.23. Next let's look at the Hodge Laplacian $\Delta := \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d} = (\mathrm{d}+\mathrm{d}^*)^2$. Proposition 13.17 tells us that

$$\sigma_\Delta(\xi,\xi) = \sigma_{\mathrm{d}}(\xi)\circ\sigma_{\mathrm{d}^*}(\xi) + \sigma_{\mathrm{d}^*})\xi)\circ\sigma_{\mathrm{d}}*(\xi), \tag{13.24}$$

which sends

$$\alpha \longmapsto -\xi\wedge\iota_\xi\alpha - \iota_\xi(\xi\wedge\alpha) = -|\xi|^2\alpha. \tag{13.25}$$

In particular, $\sigma_\Delta(\xi,\xi) = -|\xi|^2\mathrm{id}$.

You can calculate the symbol on two different cotangent vectors using a polarization. ◀

Definition 13.26. An *elliptic operator* is a linear differential operator $D\in\mathcal{D}_n(E,F)$ such that for all $x\in M$ and $\xi\in T_x^*M\setminus 0$, the symbol $\sigma_D^n(\xi,\dots,\xi)$ is an isomorphism.

In particular, the Hodge Laplacian is elliptic. There are plenty of interesting elliptic operators.

Spelling it out a bit, if $\xi = \mathrm{d}f|_x$, then

$$\sigma_D^n(\xi,\dots,\xi) = \frac{1}{n!}[\dots[[D,f],f]\dots,f]. \tag{13.27}$$

Definition 13.28.

- A *generalized Laplacian* is a second-order differential operator $\Delta\in\mathcal{D}_2(E,E)$ such that $\sigma_\Delta^2(\xi,\xi) = -|\xi|^2\mathrm{id}_E$. By (13.27), this is equivalent to

$$\frac{1}{2}[[\Delta,f],f] = -|\mathrm{d}f|^2. \tag{13.29}$$

- A *Dirac operator* is a first-order differential operator $D\in\mathcal{D}_1(E,E)$ such that D^2 is a generalized Laplacian. Equivalently, $\sigma_D(\xi)^2 = -|\xi|^2\mathrm{id}_E$.

Example 13.30. Let (X,g) be an oriented, 4-dimensional Riemannian manifold. Then $\mathrm{d}^*\oplus\mathrm{d}^+\colon\Omega^1(X)\to\Omega^0(X)\oplus\Omega_g^+$ is elliptic: its symbol $\sigma^1(\xi)\colon T_x^*X\to\mathbb{R}\oplus\Lambda^+$ sends

$$\sigma^1(\xi)(a) = -\iota_\xi a + (\xi\wedge a)^+. \tag{13.31}$$

For $\xi\neq 0$, this is an isomorphism. We saw last time that $\mathrm{d}^*\oplus\mathrm{d}^+$ has finite-dimensional kernel and cokernel. This is a manifestation of its ellipticity. ◀

Example 13.32. Consider $\mathrm{d}^*\oplus\mathrm{d}\colon\Omega^\bullet(M)\to\Omega^\bullet(M)$. Its square is the Hodge Laplacian, so it's a Dirac operator. ◀

Remark 13.33. Dirac operators usually come out of spin geometry, but our perspective sheds a different light on it. Nonetheless, it's there in the background: the relation $\sigma_D^1(\xi) = -|\xi|^2\mathrm{id}$ for a Dirac operator D, called the *Clifford relation*, shows that the map $\sigma_D\colon T^*X\to\operatorname{End}E$ extends to a Clifford algebra representation $\sigma\colon\ \mathrm{C}\ell(T_x^*X)\to\operatorname{End}(E_x)$.

Recall that the *Clifford algebra* on an inner product space $(V,\langle\cdot,\cdot\rangle)$ is the free associative algebra $\mathrm{C}\ell(V)$ on V with unit 1 modulo the relation $\xi\cdot\xi = -\|\xi\|^2\cdot 1$. By construction, this satisfies the universal property that any linear map σ into an algebra with $\sigma(\xi)^2 = -|\xi|\cdot 1$ extends to an algebra homomorphism out of the Clifford algebra.

The upshot is that spin geometry is unavoidable, though this is not a bad thing! ◀

Lecture 14.

Clifford algebras, spinors, and spin groups: 3/6/18

Seiberg-Witten theory makes heavy use of spin geometry. Since it's focused in dimension 4, and $\mathrm{Spin}_4\cong\mathrm{SU}_2\times\mathrm{SU}_2$, it's possible to do everything concretely and explicitly — at the expense of losing sight of an interesting general story. We will spend this lecture covering the algebra behind spin geometry, and next lecture on spin geometry.

There are many references for this material; one nice one is Deligne's notes on spinors in the Quantum Fields and Strings book. Like everything Deligne writes, it's terse, yet packed with insights. Our approach will be terse in a different way: we have a lot to get through, so some proofs will be deferred to references.

Throughout today's lecture, fix a commutative ring A containing 1/2. Usually we'll only use $A=\mathbb{R}$ or $A=\mathbb{C}$, but there are times when the increased generality will be useful.

Definition 14.1. A *quadratic A-module* is an A-module M together with a quadratic form $q\colon M \to A$.

In this case, there is an A-bilinear form

$$\langle m, n\rangle \coloneqq \frac{1}{2}(q(u+v) - q(u) - q(v)), \tag{14.2}$$

and $q(v) = \langle v, v\rangle$. (Typically we'll take $A = \mathbb{R}$, $M = \mathbb{R}^n$, and $q = |\cdot|^2$.)

Definition 14.3. Given a quadratic A-module (M, q), its *Clifford algebra* $\mathrm{C}\ell(M, q)$ is the free unital associative A-algebra on M such that for all $m \in M$, $m \cdot m = -q(m) \cdot 1$.

Remark 14.4. The choice of $-q(m) \cdot 1$ instead of $q(m) \cdot 1$ is a convention; some authors use the other convention. ◂

There's an explicit construction using the tensor algebra of M, as with all algebras defined with generators and relations.

Proposition 14.5. *The Clifford algebra $\mathrm{C}\ell(M, q)$ satisfies a universal property: if B is a unital associative A-algebra and $f\colon M \to B$ is an A-linear map with $f(m)^2 = -q(m)1_B$ for all $m \in M$, then there's a unique $\widetilde{f}\colon \mathrm{C}\ell(M, q) \to B$ such that the following diagram commutes:*

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d] & B. \\
\mathrm{C}\ell(V, q) \arrow[ur, dashed, "\exists!", "\widetilde{f}"']
\end{tikzcd}$$

Since $m^2 = -q(m) \cdot 1$, then for all $m, n \in M$,

$$\{m, n\} \coloneqq mn + nm = -2\langle m, n\rangle \cdot 1. \tag{14.6}$$

Formation of Clifford algebras is compatible with extension of scalars $A \to B$. With A fixed, it's also functorial in (M, q), and in particular, the action of the orthogonal group $\mathrm{O}(M, q) \to \mathrm{Aut}(M)$ extends to an action $\mathrm{O}(M, q) \to \mathrm{Aut}\,\mathrm{C}\ell(M, q)$.

Because of the defining relation, the length of a monomial $m_1 \cdots m_\ell \in \mathrm{C}\ell(M, q)$, with $m_i \in M$, is not well-defined, but the length mod 2 is well-defined. Therefore $\mathrm{C}\ell(M, q)$ splits as a direct sum

$$\mathrm{C}\ell(M, q) = \mathrm{C}\ell^0(M, q) \oplus \mathrm{C}\ell^1(M, q), \tag{14.7}$$

where $\mathrm{C}\ell^i(M, q)$ is generated by the monomials with length equal to i mod 2. Alternatively, since $-I \in \mathrm{O}(M, q)$ squares to 1, its eigenvalues are in $\{\pm 1\}$, and $\mathrm{C}\ell^i(M, q)$ is the $(-1)^i$-eigenspace of $-I$. Hence $\mathrm{C}\ell(M, q)$ is a $\mathbb{Z}/2$-graded algebra, or a *superalgebra*.

Definition 14.8. Let B be an A-algebra. Then the *opposite algebra* B^{op} is the A-algebra with the same underlying A-module as B, but with multiplication $a \cdot^{\mathrm{op}} b \coloneqq b \cdot a$.

This is in the ungraded sense.

Lemma 14.9. *There is a unique A-algebra isomorphism $\beta\colon \mathrm{C}\ell(M, q) \to \mathrm{C}\ell(M, q)^{\mathrm{op}}$ extending id_M.*

Proof. Check the universal property (Proposition 14.5) for id_M. ⊠

This map β, regarded as an automorphism of $\mathrm{C}\ell(M, q)$, is called the *prime anti-automorphism* of $\mathrm{C}\ell(M, q)$, and is characterized as the automorphism such that

$$\beta(m_1 m_2 \cdots m_\ell) = m_\ell \cdots m_2 m_1. \tag{14.10}$$

Remark 14.11. Many facts about Clifford algebras appear ad hoc from the algebra perspective, but make more sense using superalgebra, which naturally encodes all of the signs that appear. ◂

Definition 14.12. Let B be an A-superalgebra. Then, its *opposite superalgebra* B^{sop} is the A-superalgebra with the same underlying $\mathbb{Z}/2$-graded A-module as B, but with multiplication

$$a \cdot^{\mathrm{sop}} b \coloneqq (-1)^{|a||b|} b \cdot a. \tag{14.13}$$

Proposition 14.14. *There's an isomorphism $\mathrm{C}\ell(M, q)^{\mathrm{sop}} \cong \mathrm{C}\ell(M, -q)$.*

Generally $\mathrm{C}\ell(M,q)$ and $\mathrm{C}\ell(M,-q)$ are not isomorphic.

Let $F^\ell\,\mathrm{C}\ell(M,q)$ denote the submodule spanned by monomials of length at most ℓ. Then $F^\ell\cdot F^m\subset F^{\ell+m}$, so the Clifford algebra is filtered by length.

Proposition 14.15. *The associated graded of* $\mathrm{C}\ell(M,q)$ *is isomorphic to* Λ^*M *as graded* A*-algebras.*

Proof sketch. If $m\in M$, then $[m]\in F^1/F^0$, so $[m]^2=0$, which is the defining relation for an exterior algebra, so we have an algebra map $i^0\colon\Lambda^*M\to\mathrm{gr}^*\,\mathrm{C}\ell(M,q)$ sending $v\mapsto[v]$, and one can check this is an isomorphism of graded algebras. ⊠

Corollary 14.16. *If* M *is free of rank* r*, then* $\mathrm{C}\ell(M,q)$ *is free of rank* 2^r*, and* $\mathrm{C}\ell^0(M,q)$ *is free of rank* 2^{r-1}.

Definition 14.17. Let B_1 and B_2 be A-superalgebras. Then, their *super-tensor product* $B_1\otimes B_2$ is their tensor product as A-modules with the A-algebra structure satisfying

$$(a\otimes b)\cdot(a'\otimes b')=(-1)^{|b'||a'|}aa'\otimes bb'.$$

Now we have an ambiguity when we write $\otimes$, which we resolve as follows: *we only take the super-tensor product* of superalgebras unless otherwise specified.

Proposition 14.18. *There's an isomorphism of* A*-superalgebras* $\mathrm{C}\ell(M_1\oplus M_2,q_1\oplus q_2)\cong\mathrm{C}\ell(M_1,q_1)\otimes\mathrm{C}\ell(M_2,q_2)$.

This is one of the advantages of using superalgebra for Clifford algebras, rather than ungraded algebra.

Spinors.

Definition 14.19. Let $U=U_0\oplus U_1$ be a *supermodule* over A (i.e. a $\mathbb{Z}/2$-graded module). Its *endomorphism algebra* $\mathrm{SEnd}(U)$ is the A-superalgebra of endomorphisms of U, with the grading $\mathrm{SEnd}^0(U)$ the parity-preserving endomorphisms and $\mathrm{SEnd}^1(U)$ the parity-reversing ones.

For example, if we realize endomorphisms as matrices, $\mathrm{SEnd}^0(U)$ is those which are block diagonal for U_0 and U_1, and $\mathrm{SEnd}^1(U)$ is those which are block anti-diagonal.

Now we specialize to a field k and an extension $k\hookrightarrow K$.

Definition 14.20. Let (V,q) be a quadratic vector space over k, where V is even-dimensional[21] and q is nondegenerate. A *spinor module* (defined over K) is a $\mathrm{C}\ell(V,q)$-supermodule $S=S^+\oplus S^-$, i.e. a representation $\rho\colon\mathrm{C}\ell(V,q)\to\mathrm{SEnd}_K(S)$, whose K-linear extension $\rho_K\colon\mathrm{C}\ell(V,q)\otimes_k K\to\mathrm{SEnd}_K(S)$ is an isomorphism of superalgebras.

Definition 14.21. Let (U,Q) be a quadratic K-vector space, where as before U is even-dimensional and Q is nondegenerate. A *polarization* for (U,Q) is a pair of subspaces $L,L'\subset U$ such that $U=L\oplus L'$, $Q|_L=0$, and $Q|_{L'}=0$. In this case L and L' are called *Lagrangian subspaces*.

The form Q induces an isomorphism $L'\cong L^*$, which induces $(U,Q)\cong(L\oplus L',\mathrm{ev})$, where ev denotes evaluation.

Example 14.22. Over $\mathbb{C}$, all quadratic vector spaces of a given dimension are isomorphic, and a polarization exists for the standard one, so polarizations always exist. Over $\mathbb{R}$, a polarization exists iff the signature is zero. ◀

The point is that a polarization $P=(L,L')$ induces a spinor module $S_P:=\Lambda^*(L^*)$, with the $\mathbb{Z}/2$-grading as the even- and odd-graded parts of $\Lambda^*(L^*)$. In this case, $\mu\in L^*$ acts by the *creation operator*

$$c(\mu):=\mu\wedge -\in\mathrm{SEnd}^1(S),\tag{14.23a}$$

and $\lambda\in L$ acts by the *annihilation operator*

$$a(\lambda):=\iota_\lambda\in\mathrm{SEnd}^1(S).\tag{14.23b}$$

[21]One can work this theory out in odd dimensions, but it's clunkier, and we won't need it.

Proposition 14.24. *These satisfy the* canonical anticommutation relations

$$\{c(\mu), c(\mu')\} = 0 \tag{14.25a}$$

$$\{a(\lambda), a(\lambda')\} = 0 \tag{14.25b}$$

$$\{c(\mu), a(\lambda)\} = \mu(\lambda) \cdot \mathrm{id}. \tag{14.25c}$$

In particular, $\mathrm{C}\ell(V, q)$ acts on S by $\rho\colon \mathrm{C}\ell(L \oplus L^*, \mathrm{ev}) \to \mathrm{SEnd} S$ defined by

$$\rho(\lambda, \mu) := c(\mu) - a(\lambda). \tag{14.26}$$

Proposition 14.27. (S, ρ) *is a spinor module.*

Proof sketch. Write $L = L_1 \oplus \cdots \oplus L_d$ as a sum of lines, so $L^* = L_1^* \oplus \cdots \oplus L_d^*$ and as quadratic vector spaces,

$$(L \oplus L^*, \mathrm{ev}) = \bigoplus_i (L_i \oplus L_i^*, \mathrm{ev}), \tag{14.28}$$

so if $S_i = \Lambda^* L_i^*$, then $S = \bigotimes_i S_i$, and

$$\mathrm{C}\ell(L \oplus L^*, \mathrm{ev}) = \bigotimes_i \mathrm{C}\ell(L_i \oplus L_i^*, \mathrm{ev}). \tag{14.29}$$

(Here we're using the super tensor product.) Now, there are two things to check, both quick: that ρ is the tensor product of $\rho_i\colon \mathrm{C}\ell(L_i \oplus L_i^*) \to \mathrm{SEnd}(S_i)$, and that the proposition is true for lines. ⊠

Hence, in particular, $(\mathbb{R}^{2d}, |\cdot|^2)$ has a spinor module over $\mathbb{C}$.

Definition 14.30. Let A be a superalgebra and $M = M^0 \oplus M^1$ be a supermodule for A. The *parity change* of M, denoted ΠM, is the A-supermodule with even part M^1 and odd part M^0.

Corollary 14.31. *Suppose* (V, q) *is a quadratic* k*-vector space with a polarization* P*.*

1. *Any finite-dimensional indecomposable* $\mathrm{C}\ell(V, q)$*-module is isomorphic to* S_P *as ungraded modules.*
2. *Any finite-dimensional indecomposable* $\mathrm{C}\ell(V, q)$*-supermodule is isomorphic to* S_P *or* ΠS_P*.*

This is reminiscent of matrix algebras, whose only finite-dimensional indecomposables are isomorphic to the defining representations given by matrix multiplication. Thus polarized Clifford algebras are some kind of $\mathbb{Z}/2$-graded versions of matrix algebras.

Projective actions. Let S be a spinor bundle (over K) for $\mathrm{C}\ell(V, q)$ (over k). Then, there's a *projective action* of $\mathrm{O}(V, q)$ on S, i.e. a map

$$\Theta\colon \mathrm{O}(V) \longrightarrow \mathrm{PGL}_K(S) := \mathrm{Aut}_k(S)/K^\times. \tag{14.32}$$

We construct this in two steps. First, we have the map $\mathrm{O}(V, q) \to \mathrm{Aut}(\mathrm{C}\ell(V, q)) \otimes K$, which we denoted by $g \mapsto \mathrm{cl}(g)$. Akin to the fact that all automorphisms of a matrix algebra are inner, given a $g \in \mathrm{O}(V, q)$, there's an $F(g) \in \mathrm{C}\ell(V, q)^\times$ such that for all a,

$$\mathrm{cl}(g)(a) = F(g) a F(g)^{-1}, \tag{14.33}$$

and $F(g)$ is well-defined modulo scalars. Hence we have a map $F\colon \mathrm{C}\ell(V, q) \to \mathrm{C}\ell(V, q)^\times / K^\times$ such that $\mathrm{cl}(g) = \mathrm{Ad}\, F(g)$. Then, we define $\Theta = \rho \circ F\colon \mathrm{O}(V, q) \to \mathrm{Aut}_K(S)/K^\times$.

Correspondingly, there's a projective action of the orthogonal Lie algebra. This time, the intermediary is the Lie algebra $\mathrm{Der}\,\mathrm{C}\ell(V, q)$ of *derivations*:[22]

$$\delta\colon \mathfrak{o}(V, q) \longrightarrow \mathrm{Der}\,\mathrm{C}\ell(V, q). \tag{14.34}$$

Just as $\mathrm{cl}(g) \in \mathrm{Aut}\,\mathrm{C}\ell(V, q)$ is inner for a $g \in \mathrm{O}(V, q)$, $\delta(\xi)$ for $\xi \in \mathfrak{o}(V, q)$ is also inner: there's some $f(\xi) \in \mathrm{C}\ell(V, q)$ such that $\delta(\xi) = [f(\xi), -]$,[23] and again $f(\xi)$ is well-defined mod K. Normalizing, we have a map

$$f\colon \mathfrak{o}(V) \longrightarrow [\mathrm{C}\ell(V, q), \mathrm{C}\ell(V, q)] \cong \mathrm{C}\ell^0(V, q)/K \tag{14.35}$$

characterized by $\xi(v) = [f(\xi), v]$ for $v \in V$ and $\xi \in \mathfrak{o}(V)$, and f is uniquely defined.

[22]This is elaborated on in the professor's notes.

[23]The Lie bracket on $\mathrm{C}\ell(V, q)$, as in any associative algebra, is $[A, B] := AB - BA$.

Theorem 14.36. *Let* $(e_1, \dots, e_{2n})$ *be a basis for* V *and* $\xi \in \mathfrak{o}(V)$*. Then,*

$$f(\xi) = \frac{1}{4} \sum_{i,j} \xi_{ij} e_i \cdot e_i \in \mathrm{C}\ell^0(V, q).$$

This formula will come back in a crucial way later when we want to show the Lichnerowicz formula which guarantees the compactness of the Seiberg-Witten moduli space, and is the reason the $1/4K$ appears in it.

Spin groups. Let (V, q) be a quadratic vector space over a field k, with q nondegenerate. Then, $\mathrm{C}\ell(V, q)^\times$ acts on $\mathrm{C}\ell(V, q)$ by conjugation: $u \cdot v = uvu^{-1}$.

Definition 14.37. The *Clifford group* G is the normalizer of $V \subset \mathrm{C}\ell(V, q)^\times$ in $\mathrm{C}\ell(V, q)$.

Explicitly, $G = \{g \in \mathrm{C}\ell(V, q)^\times \mid gvg^{-1} \in V \text{ for all } v \in V\}$. The action map $G \to \operatorname{Aut} V$ actually factors through a map $\alpha\colon G \to \mathrm{O}(V, q)$: if $u \in V$ and $q(u) \neq 0$, then $-\alpha(u)$ is reflection across the hyperplane $u^\perp$. In particular, since $\mathrm{O}(V, q)$ is generated by reflections, α is surjective, and fits into a short exact sequence

$$1 \longrightarrow K^\times \longrightarrow G \xrightarrow{\alpha} \mathrm{O}(V, q) \longrightarrow 1. \tag{14.38}$$

Let $G^+ := G \cap \mathrm{C}\ell^-(V, q)$, products of even numbers of vectors that are in G. In this case α factors through $\mathrm{SO}(V, q)$ and fits into a short exact sequence

$$1 \longrightarrow K^\times \longrightarrow G^+ \xrightarrow{\alpha} \mathrm{SO}(V, q) \longrightarrow 1. \tag{14.39}$$

This is nearly the spin group. If β denotes the prime anti-automorphism of $\mathrm{C}\ell(V, q)$, then

$$\beta(g)g = v_r \cdots (v_1 v_1) \cdots v_r \in K^\times. \tag{14.40}$$

Definition 14.41. The *spinor norm* $\nu\colon G \to K^\times$ is the map sending $g \mapsto \beta(g)g$.

Definition 14.42. The *spin group* $\mathrm{Spin}(V, q) := G^+ \cap \ker \nu$.

This is a central extension

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}(V, q) \xrightarrow{\alpha} \mathrm{SO}(V, q) \longrightarrow 1. \tag{14.43}$$

That is, this is a short exact sequence and $\{\pm 1\} \in Z(\mathrm{Spin}(V, q))$.

Since we've defined $\mathrm{Spin}(V, q)$ by algebraic equations, it's an algebraic group. Therefore there's a corresponding Lie algebra $\mathfrak{spin}(V, q)$, and $D(\alpha)\colon \mathfrak{spin}(V, q) \to \mathfrak{so}(V, q)$ is an isomorphism. In fact, you can realize $\mathfrak{spin}(V, q)$ inside $(\mathrm{C}\ell(V, q), [-,-])$ as the image of $f\colon \mathfrak{o}(V) \to \mathrm{C}\ell^(V, q)$, just as $\mathfrak{o}(V, q)$ sits inside the matrix algebra for V.

Recall that the projective representation $\Theta\colon \mathrm{SO}(V, q) \to \operatorname{Aut}_K(S)/K^\times$ doesn't lift. But it does lift to a representation ρ of the spin group: the following diagram commutes.

$$\begin{tikzcd} \mathrm{Spin}(V, q) \arrow[r, "\rho"] \arrow[d] & \operatorname{Aut}_K(S) \arrow[d] \\ \mathrm{SO}(V, q) \arrow[r, "\Theta"] & \operatorname{Aut}_K(S)/K^\times. \end{tikzcd} \tag{14.44}$$

In some sense, this is the reason to consider the spin group.

Lecture 15.

Spin groups and spin structures: 3/8/18

Let V be a real vector space of dimension n and $\langle \cdot, \cdot \rangle$ be a positive definite inner product on V. Last time, we constructed the group $\mathrm{Spin}(V) \subset \mathrm{C}\ell^0(V, \|\cdot\|^2)^\times$, as the group of even-length products of unit vectors in V.

Since this is cut out by algebraic equations, it's an algebraic group, but it's also a Lie group: $\mathrm{C}\ell^0(V, \|\cdot\|^2)^\times$ is a Lie group, and $\mathrm{Spin}(V)$ is a closed Lie subgroup.

Last time, near the end, we constructed its Lie algebra $\mathfrak{spin}(V, \|\cdot\|^2)$ as a subalgebra of $\mathrm{C}\ell^0(V, \|\cdot\|^2)$ with the commutator and an action $f\colon \mathfrak{o}(V) \to \mathfrak{spin}(V, \|\cdot\|^2)$, describing the infinitesimal action of a $\xi \in \mathfrak{o}(V)$ by derivations on the Clifford algebra.

The spin group acts on V through inner automorphisms of $\mathrm{C}\ell(V, \|\cdot\|^2)$, and this defines a short exact sequence (14.43). Since the inner product is positive definite, $\mathrm{SO}(V)$ is compact, and therefore $\mathrm{Spin}(V)$ is a compact Lie group, since it's a two-fold cover of another compact Lie group.

Inside $\mathrm{O}(V)$, we can consider the subgroup of things hit by the exponential map

$$\exp\xi \coloneqq \sum_{n=0}^{\infty} \frac{1}{n!}\xi^n \in \operatorname{End} V. \tag{15.1}$$

The image of exp is precisely $\mathrm{SO}(V)$.

Proposition 15.2. *In a similar way,* $\exp\colon \mathfrak{spin}(V) \to \mathrm{Spin}(V)$ *is an isomorphism.*

Proof sketch. By (14.43), $\mathfrak{spin}(V) \cong \mathfrak{so}(V)$, and the only obstruction is the kernel of α. One has to show that $-1 \in \operatorname{Im}(\exp)$, but if e_1 and e_2 are orthonormal in V, then $-1 = \exp((\pi/2)[e_1, e_2])$, so we're good. ⊠

Corollary 15.3. $\mathrm{Spin}(V)$ *is connected, and* $\alpha\colon \mathrm{Spin}(V) \to \mathrm{SO}(V)$ *admits no continuous section. Therefore* $\mathrm{Spin}(V)$ *is characterized as the unique connected Lie group* G *(up to isomorphism) admitting a double cover Lie group homomorphism* $G \twoheadrightarrow \mathrm{SO}(V)$*.*

So you could skip all of the Clifford algebra stuff and use this to *define* the spin group; then you have a handle on the topology of the spin group, but when you want to do differential geometry Clifford algebra is unavoidable. Uniqueness follows because $\pi_1(\mathrm{SO}_n) = \mathbb{Z}$ when $n = 2$ and $\mathbb{Z}/2$ when $n > 2$.

We'll let $\mathrm{Spin}_n \coloneqq \mathrm{Spin}(\mathbb{R}^n)$.

Spinors. Suppose $n = 2m$. Last time, we constructed a representation $\rho\colon\ \mathrm{C}\ell(V, \|\cdot\|^2) \otimes \mathbb{C} \to \mathrm{SEnd}S$, with $S = S^+ \oplus S^-$ and $\dim S^\pm = 2^{m-1}$. Therefore we get two representations

$$\rho^\pm\colon \mathrm{Spin}_{2m} \longrightarrow \mathrm{Aut}_{\mathbb{C}}(S^\pm). \tag{15.4}$$

These representations are irreducible.

If $n = 2m+1$, then we have a single, non-super representation $\rho\colon\ \mathrm{C}\ell(V, \|\cdot\|^2) \otimes \mathbb{C} \to \operatorname{End} S$, and therefore a single irreducible representation of the spin group in this dimension. We can't pick a polarization, but we can split $V_{\mathbb{C}} = \mathbb{C} \oplus (L \oplus L^*)$, a polarization plus an extra line. If $S' \coloneqq \Lambda^* L^*$, then

$$\mathrm{C}\ell(V) \cong \mathrm{C}\ell(\mathbb{C}) \otimes \mathrm{SEnd}(S'). \tag{15.5}$$

Explicitly, $D \coloneqq \mathrm{C}\ell(\mathbb{C}) = \mathbb{C}[\varepsilon]/(\varepsilon^2 + 1)$, where ε is odd. Therefore

$$\mathrm{C}\ell(V) \cong D \otimes \mathrm{SEnd}(S') \cong \mathrm{SEnd}_D(D \otimes S'). \tag{15.6}$$

Therefore if $S \coloneqq D \otimes S'$, the representation is via endomorphisms of a spinor module, but we're over $\mathrm{C}\ell(\mathbb{C})$ instead of $\mathbb{C}$.

Remark 15.7. The spinor representations are the unique irreducible representations of the Clifford algebra over $\mathbb{C}$, but there are additional distinct ones over $\mathbb{R}$. This is related to orientations. ◀

Definition 15.8. Let (S, ρ) be a (super)representation of $\mathrm{C}\ell(V, \|\cdot\|^2)$. A Hermitian inner product $\langle \cdot, \cdot \rangle$ on S is called *spin* if for all $v \in V$, $\rho(v) \in \mathfrak{u}(S)$, i.e.

$$\langle \rho(v)s_1, s_2 \rangle + \langle s_1, \rho(v)s_2 \rangle = 0.$$

Proposition 15.9. *If* $\langle \cdot, \cdot \rangle$ *is a spin inner product, then* $\rho\colon \mathrm{Spin}(V) \to \mathrm{SEnd}(S)$ *is a unitary representation with respect to* $\langle \cdot, \cdot \rangle$*.*

Proof. Suppose $g = v_1 \cdots v_{2r} \in \mathrm{C}\ell^0(V, \|\cdot\|^2)$. Since the inner product is spin,

$$\langle \rho(g)s_1, s_2 \rangle = \langle \rho(v_1 \cdots v_{2r})s_1, s_2 \rangle = -\langle \rho(v_2 \cdots v_{2r})s_1, \rho(v_1)s_2 \rangle. \tag{15.10}$$

Iterating, we get $\langle \rho(g)s_1, s_2 \rangle = \langle s_1, \rho(\beta g)s_2 \rangle$. Therefore $\langle \rho(g)s_1, \rho(g)s_2 \rangle = \langle s_1, \rho(g \cdot \beta g)s_2 \rangle$. Since $g \in \mathrm{Spin}(V)$, $g \cdot \beta g = 1$. ⊠

Corollary 15.11. *Spin inner products always exist.*

Remark 15.12. We could also allow products of odd numbers of elements in $\mathrm{C}\ell^0(V, \|\cdot\|^2)$, defining an analogue of the spin group called the *pin group* $\mathrm{Pin}^-(V)$. Then the above proof still applies, but with a few minus signs throughout, and one mas to average an inner product over this to obtain a spin inner product. ◀

Example 15.13 ($n = 2$)**.** There's a double cover $\mathrm{SO}_2 \to \mathrm{SO}_2$ defined by angle-doubling, and SO_2 is connected, so by Corollary 15.3, $\mathrm{Spin}_2 = \mathrm{SO}_2$. There are standard representations S^+, S^- of $\mathrm{SO}_2 = \mathrm{U}_1$ on $\mathbb{C}$, where $e^{i\theta} \in \mathrm{U}_1$ acts by $e^{\pm i\theta}$; then $S^- \cong (S^+)^*$. We will show that these are the spinors.

Let $S := S^+ \oplus S^-$ with the standard Hermitian inner product. The Clifford map is a map $\rho\colon \mathbb{R}^2 \to \mathfrak{u}(S)$ defined by

$$\rho(v) = \begin{pmatrix} 0 & -\rho^-(v) \\ \rho^+(v) & 0 \end{pmatrix}, \tag{15.14}$$

where $\rho^-(v) = \rho^+(v)^\dagger$ and $\rho^+(v)^\dagger\rho^+(v) = \|v\|^2 \mathrm{id}_{S^+}$. Showing this boils down to understanding $\rho^+\colon \mathbb{C} \to \mathrm{Hom}_{\mathbb{C}}(S^+, S^-) = (S^-)^{\otimes 2}$, which is a $\mathbb{C}$-linear isometry.

Definition 15.15. Define a *spin structure* on a 2-dimensional oriented inner product space V (i.e. a *Hermitian line*) is a Hermitian line L and a $\mathbb{C}$-linear isometry $\rho\colon V \to L^{\otimes 2}$. That is, it's a "square root" of V.

Given a spin structure, set $S^- := L$, $S^+ = L^*$, and construct a Clifford action of V on $S^+ \oplus S^-$ as above. One can check that S is still a spinor representation. In particular, there's an identification $\mathrm{Spin}(V) \cong \mathrm{U}(L)$, and $\alpha\colon \mathrm{U}(L) \to \mathrm{SO}(V)$ is the map sending $g \mapsto \rho \circ g^{\otimes 2} \circ \rho^{-1}$.

This definition is coordinate-free, so applies equally well to a rank-2 oriented vector bundle with a metric $V \to M$. ◀

Example 15.16 ($n = 3$)**.** Let Sp_1 denote the group of unit quaternions in $\mathbb{H}$, i.e. those quaternions q with $q\overline{q} = 1$. This acts on $\mathbb{H}$ by left multiplication; there's an identification $\mathrm{Sp}_1 \cong \mathrm{SU}_2$ carrying this to the defining representation of SU_2.

There is a map $\beta\colon \mathrm{Sp}_1 \to \mathrm{SO}(\mathrm{Im}(\mathbb{H}))$ (here $\mathrm{Im}(\mathbb{H}) := \mathrm{span}_{\mathbb{R}}\{i, j, k\}$) sending q to the map $x \mapsto qxq^{-1}$. The kernel is $\{\pm 1\}$, and this is a map of three-dimensional Lie groups, so it's a local isomorphism, and in particular a double cover. Since Sp_1 is connected, this tells us that $\mathrm{Spin}_3 \cong \mathrm{SU}_2 \cong \mathrm{Sp}_1$.

If we think of Spin_3 as SU_2, it's the symmetries of $\mathbb{C}^2$ with a Hermitian inner product (unitary) and a complex volume form $\Omega \in \Omega^2(\mathbb{C}^2)^* \setminus 0$ (special). In this case, the spinors are $S = \mathbb{C}^2$, which Spin_3 acts on via the defining representation of SU_2. If you think of Spin_3 as Sp_1, the spinor representation is the defining action of Sp_1 on $\mathbb{H}$.

To describe the Lie algebra representation $\rho\colon \mathbb{R}^3 \to \mathfrak{u}(S)$, we'll hypothesize that this is actually valued in $\mathfrak{su}(S)$, the trace-free endomorphisms. We also want the Clifford relations to hold, and a little bit of thought tells you that this is equivalent to ρ being an isometry, where on $\mathfrak{su}(S)$,

$$\langle a, b\rangle := -\frac{1}{2}\,\mathrm{tr}(ab) = \frac{1}{2}\,\mathrm{tr}(a^\dagger b). \tag{15.17}$$

So, given an isometry $\rho\colon \mathbb{R}^3 \to \mathfrak{su}(S)$, we get a Clifford module, and you can check that it's the spinor module.

Let (e_1, e_2, e_3) be an orthonormal, positively oriented basis for $\mathbb{R}^3$ and $\omega := e_1e_2e_3 \in C\ell(\mathbb{R}^3)$. Then ω is central and $\omega^2 = 1$, so it acts by $\pm I$ on S. We will also hypothesize that $\rho(\omega) = 1$, which is an orientation condition.[24]

Definition 15.18. Let V be a 3-dimensional oriented real inner product space. A *spin structure* on V is the data of a 2-dimensional Hermitian vector space S, a unit-length volume form $\Omega \in \Lambda^2(S^*)$, and an isometry $\rho\colon V \to \mathfrak{su}(S)$ such that if (e_1, e_2, e_3) is a positively oriented orthonormal basis of V, then $\rho(e_1)\rho(e_2)\rho(e_3) = I$.

Again, this applies without change to oriented 3-dimensional real vector bundles with a metric.

We can also compute $\mathrm{Spin}V$, the symmetries of the spin structure. It's the set of pairs $(g, \widetilde{g})$ with $g \in \mathrm{SO}(V)$ and $\widetilde{g} \in \mathrm{SU}(S)$ such that ρ intertwines the actions of $g\colon V \to V$ and $\mathrm{Ad}(\widetilde{g})\colon \mathfrak{su}(S) \to \mathfrak{su}(S)$. ◀

Example 15.19 ($n = 4$)**.** There's an isomorphism $\mathrm{Spin}_4 \cong \mathrm{SU}_2 \times \mathrm{SU}_2 \cong \mathrm{Sp}_1 \times \mathrm{Sp}_1$. The identification comes as always through Corollary 15.3: there's a map $\gamma\colon \mathrm{Sp}_1 \times \mathrm{Sp}_1 \to \mathrm{SO}(\mathbb{H})$ sending

$$(q_1, q_2) \longmapsto (x \longmapsto q_1 x q_2^{-1}). \tag{15.20}$$

This exists because Sp_1 acts on $\mathbb{H}$ on the left and on the right by multiplication, and these multiplications commute, so it's a representation of the product. Since $\mathrm{Sp}_1 \times \mathrm{Sp}_1$ is connected, we map to $\mathrm{SO}(\mathbb{H})$, even though *a priori* we were only on $\mathrm{O}(\mathbb{H})$.

[24] This is an instance of the warning from earlier, that there are distinct irreducible representations of real Clifford algebras.

What's the kernel of γ? We need $q_1 x = x q_2$ for all $x \in \mathbb{H}$; setting $x = 1$, we need $q_1 = q_2$, so we're conjugating by a single quaternion, and $\ker(\gamma) = \{\pm I\}$ as in the three-dimensional case. Since both $\mathrm{Sp}_1 \times \mathrm{Sp}_1$ and $\mathrm{SO}(\mathbb{H})$ are six-dimensional, then γ is a two-to-one cover and exhibits $\mathrm{Spin}_4 \cong \mathrm{Sp}_1 \times \mathrm{Sp}_1$.

The spinors are $S^+ = \mathbb{H}$, viewed as a two-dimensional complex vector space, and $S^- = \mathbb{H}$. $\mathrm{Sp}_1 \times \mathrm{Sp}_1$ acts on S^+, resp. S^- through projection onto the first, resp. second coordinate in the group. These actions preserve the quaternionic structure and the Hermitian metrics.

Now we need the Clifford map $\rho\colon \mathbb{R}^4 \longrightarrow \mathrm{Hom}_{\mathbb{C}}(S^+, S^-)$. As before, we'll set $\rho^-(v) := -\rho(v)^\dagger$.

The space $\mathrm{Hom}_{\mathbb{H}}(S^+, S^-)$ has dimension 4 over $\mathbb{R}$, and carries a norm (the operator norm) and an orientation. As before, we'll make the ansatz that $\rho^+\colon \mathbb{R}^4 \to \mathrm{Hom}_{\mathbb{H}}(S^+, S^-)$ is an isometry, and then check that this gives you spinors.

Definition 15.21. A *spin structure* on an oriented, real, 4-dimensional vector space V is a pair of rank-2 Hermitian vector bundles S^+ and S^- with quaternionic structure, and an oriented isometry $\rho^+\colon V \to \mathrm{Hom}_{\mathbb{H}}(S^+, S^-)$.

As before, this applies to oriented real 4-dimensional vector bundles with a metric.

If $\mathrm{Spin}V$ denotes the symmetries of the spin structure, then it's identified with pairs $(g, \widetilde{g})$ such that $g \in \mathrm{SO}(V)$, and $\widetilde{g}$ has the form

$$\widetilde{g} = \begin{pmatrix} \mathrm{SU}(S^+) & 0 \\ 0 & \mathrm{SU}(S^-) \end{pmatrix} \subset \mathrm{SU}(S^+ \oplus S^-), \tag{15.22}$$

such that ρ intertwines the actions of g and $\mathrm{Ad}(\widetilde{g})$. ◂

Lecture 16.

Spin and spin^c structures: topology: 3/20/18

The professor will be out of town next week, so we will have to make up those two lectures at some point; details will be in an email.

We've thus far studied spin structures from a differential-geometric perspective; today we will address the topological side.

Definition 16.1. Let $V \to M$ be a rank-n real vector bundle. A *spin structrure* on V is a pair $\mathfrak{s} = (P_{\mathrm{Spin}}(V), \tau)$, where $P_{\mathrm{Spin}}(V) \to M$ is a principal Spin_n-bundle and τ is an isomorphism of vector bundles

$$\tau\colon P_{\mathrm{Spin}}(V) \times_{\mathrm{Spin}_n} \mathbb{R}^n \xrightarrow{\cong} V. \tag{16.2}$$

A *spin structure on* M is a spin structure on T^*M.

Recall that if G is a Lie group, a *principal G-bundle* $\pi\colon P \to X$ is data of a space P with a smooth, free, right G-action and such that π is the quotient map onto the orbit space. The construction in (16.2) is an example of the *mixing construction*: $P_{\mathrm{Spin}}(V) \times_{\mathrm{Spin}_n} \mathbb{R}^n$ is defined to be the quotient of $P_{\mathrm{Spin}}(V) \times \mathbb{R}^n$ by the equivalence relation $(p, v) \sim (p \cdot g, g^{-1} v)$ for all $g \in \mathrm{Spin}_n$; here Spin_n acts on $\mathbb{R}^n$ through the map to SO_n.

A spin structure $\mathfrak{s}$ induces both an orientation and a Euclidean metric on V: there's a principal SO_n-bundle whose fiber at x is the SO_n-torsor of oriented orthonormal bases for V_x, namely $P_{\mathrm{Spin}}(V) \times_{\mathrm{Spin}_n} \mathrm{SO}_n$, where Spin_n acts on SO_n through the usual map. However, one often thinks of the orientation and metric as given in advance, in which case a spin structure amounts to a choice of a lift of the principal SO_n-bundle of oriented orthonormal frames to a principal Spin_n-bundle.

Definition 16.3. Let $(V, \mathfrak{s} = (P_{\mathrm{Spin}}(V), \tau))$ be a vector bundle with a spin structure. Then its *spinor bundle* is the complex vector bundle $\mathbb{S} := P_{\mathrm{Spin}}(V) \times_{\mathrm{Spin}_n} S$, where S is the spinor representation for Spin_n.

If $\operatorname{rank} V$ is even, there's a Hermitian splitting $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ and a Clifford reprsentation $\rho\colon V \to \mathfrak{u}(\mathbb{S})$ which is odd: it splits as

$$\rho^\pm := \rho|_{\mathbb{S}^\pm}\colon \mathbb{S}^\pm \to \mathbb{S}^\mp, \tag{16.4}$$

and $\rho^-(v) = -\rho^+(v)^\dagger$.

Last time, in Definitions 15.15, 15.18 and 15.21, we gave more geometric definitions of spin structures in low dimensions. These definitions apply to vector bundles more generally, and agree with the more topological definition given today.

Example 16.5. Let $V \to M$ be an oriented rank-2 bundle, which is equivalent to a Hermitian line bundle. Then a spin structure is equivalent to a square root of V as a line bundle. This implies in particular that $\deg V$ must be even, or equivalently, $w_2(V) = 0$: the degree is $\langle c_1(V), [M]\rangle$, and the mod 2 reduction of $c_1(V)$ is $w_2(V)$. ◄

For Definition 15.18, the rank-2 Hermitian vector bundle we ask for is precisely the spinor bundle $\mathbb{S} \to M$, and the oriented isometry $\rho\colon V \to \mathfrak{su}(\mathbb{S})$ is the spinor representation we mentioned above. For Definition 15.21, the rank-2 Hermitian vector bundles we ask for are $\mathbb{S}^+$ and $\mathbb{S}^-$.

Another perspective on the case $n = 4$, which also uses the isomorphism $\mathrm{Spin}_4 \cong \mathrm{SU}_2 \times \mathrm{SU}_2$, is that the fiber of $P_{\mathrm{Spin}}(V)$ at $x \in M$ is a set $\{\theta, \theta^+, \theta^-\}$, where $\theta\colon \mathbb{R}^4 \to V_x$ is an oriented isometry and $\theta^\pm\colon \mathbb{H} \to \mathbb{S}^\pm$ intertwine ρ^+. These quaternionic structures give you the data of an $\mathrm{SU}_2 \times \mathrm{SU}_2$-principal bundle.

Definition 16.6. Let $\mathfrak{s}, \mathfrak{s}'$ be spin structures on V. An *isomorphism of spin structures* $\mathfrak{s} \to \mathfrak{s}'$ is an isomorphism of their principal Spin_n-bundles lying over the identity map of the principal SO_n-bundle of oriented orthonormal frames.

Proposition 16.7. *Suppose V has a spin structure. Then the isomorphism classes of spin structures for V form an $H^1(M;\mathbb{Z}/2)$-torsor, i.e. $H^1(M;\mathbb{Z}/2)$ acts transitively on the set of spin structures for V.*

Proof. Let $\mathfrak{s}, \mathfrak{s}'$ be spin structures on V. We want a Spin_n-equivariant isomorphism $\varphi\colon P_{\mathrm{Spin}}(V) \to P'_{\mathrm{Spin}}(V)$ fitting into the triangle

$$\begin{tikzcd} P_{\mathrm{Spin}}(V) \arrow[rr, "\varphi", "\cong"'] \arrow[dr] & & P'_{\mathrm{Spin}}(V) \arrow[dl] \\ & P_{\mathrm{SO}}(V). & \end{tikzcd} \tag{16.8}$$

In the fibers over an $x \in M$, (16.8) asks for an isomorphism of double covers over $(P_{\mathrm{SO}}(V))_x$, and there are two of these. Varying x, we get a double cover $\mathrm{iso}(\mathfrak{s}, \mathfrak{s}')$, and a global isomorphism is a section of this cover. The obstruction to this is the monodromy of this double cover, which is a map $\pi_1(M) \to \mathbb{Z}/2$, hence an element $\mathfrak{s}' - \mathfrak{s} \in H^1(M;\mathbb{Z}/2)$.

Next we define the freely transitive action of $H^1(M;\mathbb{Z}/2)$ on the set of spin structures. Let $\mathfrak{s}$ be a spin structure, $\mathfrak{U}$ be a good cover of M, and $\delta \in \check{C}^1(\mathfrak{U}; \{\pm 1\})$ be a Čech cocycle. If we trivialize $P_{\mathrm{Spin}}(V)$ over each $U \in \mathfrak{U}$, we obtain transition functions $\chi_{UV}\colon U \cap V \to \mathrm{Spin}_n$ for every pair of intersecting $U, V \in \mathfrak{U}$. Then, the collection $\delta_{UV}\chi_{UV}$ defines transition functions for a new principal Spin_n-bundle, which defines an $H^1(M;\mathbb{Z}/2)$-action. The previous argument showsd that it's freely transitive. ⊠

Now we address existence.

Theorem 16.9. *Let $V \to M$ be an oriented vector bundle, $\mathfrak{U}$ be a good cover of M, and $\omega \in \check{C}^2(\mathfrak{U};\mathbb{Z}/2)$ be a representative for $w_2(V)$. Then a spin structure on V is the same thing as a trivialization of w_2, meaning that*

1. *a spin structure exists iff $w_2(V) = 0$,*
2. *a 1-cochain η such that $\delta\eta = \omega$ defines a spin structure $\mathfrak{s}_\eta$, and*
3. *if η, η' are 1-cochains with $\delta\eta = \omega = \delta\eta'$, then $\mathfrak{s}_\eta - \mathfrak{s}_{\eta'} = [\eta - \eta'] \in H^1(M;\mathbb{Z}/2)$. Moreover, a 0-cochain ζ with $\delta\zeta = \eta - \eta'$ determines an isomorphism $\mathfrak{s}_\eta \xrightarrow{\cong} \mathfrak{s}_{\eta'}$.*

We won't give the proof in detail, but we should say something about why the obstruction is w_2. Locally there's no obstruction, so we can find a spin structure on $V|_U$ for each $U \in \mathfrak{U}$, which is unique up to isomorphism (since $H^1(U;\mathbb{Z}/2) = 0$). We would like to assemble them by choosing isomorphisms θ_{UW} on intersectiosn $U \cap W$, and set $\theta_{WU} = \theta_{UW}^{-1}$. For coherence, we need

$$\omega_{U_1U_2U_3} \coloneqq \theta_{U_3U_1}\theta_{U_2U_3}\theta_{U_1U_3} \tag{16.10}$$

to be a 2-cocycle.

Once we do that, we can check that the cohomology class of ω is natural under pullback and is stable under adding a trivial bundle, which implies it's a stable characteristic class. In $H^2(B\mathrm{SO};\mathbb{Z}/2)$, there's only 0 and w_2, and we saw that there are bundles without spin structures, so it's w_2.

Remark 16.11. If you restrict to tangent bundles, this proof is a lot harder — but working with all vector bundles gives us access to the characteristic-class argument outlined above, which is really nice. ◂

Now we'll talk about spin^c structures, and why they exist on tangent bundles to 4-manifolds.

Definition 16.12. Let $(V, \langle\cdot,\cdot\rangle)$ be a real inner product space (so $\langle\cdot,\cdot\rangle$ is positive definite). The *spin^c group* $\mathrm{Spin}^c(V, \langle\cdot,\cdot\rangle)$ is the subgroup of $\mathrm{C}\ell^0(V\otimes\mathbb{C})^\times$ generated by the scalars U_1 and $\mathrm{Spin}(V, \langle\cdot,\cdot\rangle)$.

If $V = \mathbb{R}^n$ with the usual inner product, we let $\mathrm{Spin}_n^c := \mathrm{Spin}^c(\mathbb{R}^n, \cdot)$.

Inside the complexified Clifford algebra, $\mathrm{Spin}(V, \langle\cdot,\cdot\rangle)\cap \mathrm{U}_1 = \{\pm 1\}$, so

$$\mathrm{Spin}^c(V, \langle\cdot,\cdot\rangle) = (\mathrm{Spin}(V, \langle\cdot,\cdot\rangle)\times \mathrm{U}_1)/\{\pm 1\}. \tag{16.13}$$

Thus, for example, we have a concrete model of Spin_4^c as $(\mathrm{SU}_2\times\mathrm{SU}_2\times\mathrm{U}_1)/\{\pm1\}$. There is also a short exact sequence

$$1 \longrightarrow \mathrm{U}_1 \longrightarrow \mathrm{Spin}^c(V, \langle\cdot,\cdot\rangle) \longrightarrow \mathrm{SO}(V, \langle\cdot,\cdot\rangle) \longrightarrow 1. \tag{16.14}$$

Definition 16.15. Let $V\to M$ be a rank-n real vector bundle, a *spin^c structure* $\mathfrak{s} = (P_{\mathrm{Spin}^c}(V), \tau)$ is a principal spin^c-bundle $P_{\mathrm{Spin}^c}(V)\to M$ together with an isomorphism

$$\tau\colon P_{\mathrm{Spin}^c}(V)\times_{\mathrm{Spin}_n^c}\mathbb{R}^n \overset{\cong}{\to} V. \tag{16.16}$$

Why study spin^c structures? We'll see that there are more of them, and that the obstruction is weaker. This makes invariants defined using spin^c structures easier to work with than those requiring spin structures. For example, on any simply-connected 4-manifold, there's at most one spin structure (up to isomorphism), and frequently there are none.

Given a spin^c structure, there's a *spinor bundle*

$$\mathbb{S} := P_{\mathrm{Spin}^c}(V)\times_{\mathrm{Spin}_n^c} S, \tag{16.17}$$

and if n is even, it splits as $\mathbb{S}^+\oplus\mathbb{S}^-$; there is a Clifford representation $\rho\colon V\to\mathfrak{u}(\mathbb{S})$, which has odd parity when n is even.

Conversely, if $n = 2m$, then we can use this to define a spin^c structure: given Hermitian vector bundles $\mathbb{S}^+$ and $\mathbb{S}^-$ of rank 2^{m-1} and a map

$$\rho\colon V\longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathbb{S}^+, \mathbb{S}^-) \tag{16.18}$$

such that for all $x\in M$ and $e, f\in V_x$,

$$\rho(e)^\dagger\rho(f) + \rho(f)^\dagger\rho(e) = 2\langle e, f\rangle \mathrm{id}_{\mathbb{S}^+}, \tag{16.19}$$

then we can recover a spin^c structure on V. The reason this works is that if V is a finite-dimensional oriented inner product space, then $\mathrm{Spin}^c(V)$ is the group of pairs $(\theta, \widetilde\theta)$ where $\theta\in\mathrm{SO}(V)$ and $\widetilde\theta\in\mathfrak{u}(\mathbb{S})$, such that the diagram

$$\begin{tikzcd} V \arrow[r, "\rho"] \arrow[d, "\theta"'] & \mathfrak{u}(\mathbb{S}) \arrow[d, "\mathrm{Ad}(\widetilde\theta)"] \\ V \arrow[r, "\rho"] & \mathfrak{u}(\mathbb{S}) \end{tikzcd} \tag{16.20}$$

commutes.

Example 16.21. Let's look at $n = 2$. A spin^c structure on a rank-2 vector bundle V is a pair of Hermitian line bundles L_+ and L_- together with a $\mathbb{C}$-linear isometry $V\overset{\cong}{\to}\mathrm{Hom}_{\mathbb{C}}(L_+, L_-)$. It's in addition a spin structure if we can take $L_+ = (L_-)^*$. ◂

The story of existence and uniqueness of spin^c structures is similar to that of spin structures, but we can reuse some of what we've already done.

Proposition 16.22. *If $V\to M$ admits a spin^c structure, then the set of isomorphism classes of its spin^c structures is an $H^2(M;\mathbb{Z})$-torsor.*

One way to see this is that $H^2(M;\mathbb{Z})$ is isomorphic to the group of isomorphism classes of complex line bundles on M. The action of a complex line bundle L on $\mathfrak{s}$ is by twisting the spinor bundle: $\mathbb{S} \mapsto L \otimes \mathbb{S}$.

Let $\lambda\colon \mathrm{Spin}_n^c \to \mathrm{U}_1$ send $(g,z) \mapsto z^2$; since z is only defined up to ± 1, we have to take z^2 here. Then the map $(\alpha,\lambda)\colon \mathrm{Spin}_n^c \to \mathrm{SO}_n \times \mathrm{U}_1$ is a double cover. Since $\mathrm{U}_1 = \mathrm{SO}_2$, then we have an inclusion as block diagonals $\mathrm{SO}_n \times \mathrm{U}_1 \hookrightarrow \mathrm{SO}_{n+2}$.

Proposition 16.23. Spin_n^c *fits into a pullback square*

$$\begin{tikzcd}
\mathrm{Spin}_n^c \arrow[r, "2:1"] \arrow[d] & \mathrm{SO}_n \times \mathrm{U}_1 \arrow[d, hook] \\
\mathrm{Spin}_{n+2} \arrow[r] & \mathrm{SO}_{n+2}.
\end{tikzcd}$$

Given a spin^c structure $\mathfrak{s}$ on V, we can define an associated line bundle $L_{\mathfrak{s}} \coloneqq P_{\mathrm{Spin}^c}(V) \times_{\mathrm{Spin}_n^c} \mathbb{C}$, where Spin_n^c acts on $\mathbb{C}$ through λ.

Corollary 16.24. *Let $L \to M$ be an oriented, rank-2 vector bundle. Then, a spin^c structure $\mathfrak{s}$ on V with $L_{\mathfrak{s}} \cong L$ is equivalent to a spin structure on $V \oplus L$, equivalently to a trivialization of $w_2(V \oplus L)$.*

Corollary 16.25. *Hence, V admits a spin^c structure iff there's a line bundle L such that $w_2(V) = w_2(L)$, or $w_2(V) = c_1(L) \bmod 2$. Equivalently, V admits a spin^c structure iff there's a $c \in H^2(M;\mathbb{Z}/2)$ such that $w_2(V) \equiv c \bmod 2$.*

Here is our obstruction to a spin^c structure: an integer lift of $w_2(V)$. This is easier than the obstruction for a spin structure (which requires $w_2(V) = 0$).

Theorem 16.26 (Hirzebruch-Hopf)**.** *If X is a closed, oriented 4-manifold, then $w_2(T^*X)$ admits an integral lift, and hence X has a spin^c structure.*

Remark 16.27. If X is simply connected (or more generally, $H_1(X)$ has no 2-torsion), this is trivial, because $H^2(X) \twoheadrightarrow H^2(X;\mathbb{Z}/2)$. It's also true for non-closed 4-manifolds (this is due to Teichner and someone else). ◀

We now sketch the proof of Theorem 16.26.

Lemma 16.28. *Let Z be a finite CW complex and r denote reduction mod 2. Then there is a short exact sequence of abelian groups*

$$0 \longrightarrow r(H^k(Z)_{\mathrm{tors}}) \longrightarrow H^k(Z;\mathbb{Z}/2) \xrightarrow{ev} \mathrm{Hom}(H_k(Z),\mathbb{Z}/2) \longrightarrow 0.$$

The proof follows, with a little care, from the universal coefficient theorem.

Proposition 16.29. *Let M be a compact oriented n-manifold. Under the cup product pairing $\smile\colon H^k(M;\mathbb{Z}/2)\otimes H^{n-k}(M;\mathbb{Z}/2) \to \mathbb{Z}/2$, $r(H^k(M)_{\mathrm{tors}})$ and $r(H^{n-k}(M))$ are mutual annihilators.*

Proof sketch. This follows fairly quickly from Lemma 16.28 and Poincaré duality. ⊠

Proof of Theorem 16.26. The Wu formula guarantees that $w_2(T^*X)\cdot x = x^2$ for all $x \in H^2(X;\mathbb{Z}/2)$. Hence if t is a torsion integral class, $w_2(T^*X)\cdot r(t) = r(t)^2 = r(t^2) = 0$ (because $H^4(X)$ has no torsion). Therefore $w_2(T^*X)$ annihilates all torsion classes, and therefore has an integral lift. ⊠

Lecture 17.

Dirac operators and the Lichnerowicz formula: 3/22/18

"Every good differential geometer has their own spelling of Weitzenböck... you occasionally see it spelled correctly."

The Lichnerowicz formula that we discuss today is somewhat involved, but very useful — it's important in establishing compactness of Seiberg-Witten moduli spaces, but also for various other things in differential geometry. Therefore we'll look at it in detail.

Recall that on a Riemannian manifold (M, g), there's a distinguished connection ∇ in TM called the *Levi-Civita connection*, uniquely characterized by the properties that it's orthogonal (i.e. compatible with the metric), meaning

$$\langle \nabla u, v\rangle + \langle u, \nabla v\rangle = d\langle u, v\rangle; \tag{17.1a}$$

and it's torsion-free, i.e.

$$\nabla_u v = \nabla_v u - [u, v] = 0. \tag{17.1b}$$

The concept of torsion of a connection only makes sense on the tangent bundle.

The Levi-Civita connection has a curvature tensor $R = \nabla \circ \nabla \in \Omega^2_M(\mathfrak{so}(TM))$; it lands in $\mathfrak{so}(TM)$ because it's orthogonal. It has an explicit formula,

$$R_{u,v} \coloneqq \nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u,v]}. \tag{17.2}$$

With respect to a local coordinate system $x_1, \dots, x_n$, the curvature has components

$$R_{ijk\ell} \coloneqq g(R_{\partial_i, \partial_j}(\partial_k), \partial_\ell). \tag{17.3}$$

The group S_4 acts on these components by permuting the indices; under this action, $R_{ijk\ell}$ transforms according to the sign representation $S_4 \to \{\pm 1\}$. There's also a cyclic symmetry

$$R_{i(jk\ell)} \coloneqq R_{ijk\ell} + R_{ik\ell j} + R_{i\ell jk} = 0, \tag{17.4}$$

and there's a precise sense in which these are all of the symmetries of $R_{ijk\ell}$ on a general Riemannian manifold.

Definition 17.5. With (M, g) and ∇ as above, let E be a complex vector bundle with a Hermitian connection and $\rho\colon T^*M \to \mathfrak{u}(E)$ is a Clifford map, i.e. $\rho(e)^2 = -|e|^2 \mathrm{id}_E$. A *Clifford connection* is a unitary connection $\widetilde{\nabla}$ on E for which ρ is parallel, i.e. for any local section e of T^*M,

$$[\widetilde{\nabla}_v, \rho(e)] = \rho(\nabla_v e).$$

Proposition 17.6. *If $\mathbb{S}$ is the spinor bundle of a spin structure on M, there is a distinguished Clifford connection.*

Proof sketch. We work in a local trivialization U_α of the principal Spin_n-bundle associated to the spin structure on M. This induces a trivialization of T^*M, so $\nabla = \mathrm{d} + A_\alpha$ for some $A_\alpha \in \Omega^1_{U_\alpha}(\mathfrak{so}(T^*M))$. Since the map $\mathfrak{spin}(T^*M) \to \mathfrak{so}(T^*M)$ is an isomorphism of Lie algebras, we can lift A_α to a connection on $P_{\mathrm{Spin}}(T^*M)$, and it satisfies the equations needed for it to be a connection. Then, one checks that it's a Clifford connection. ⊠

Remark 17.7. There are several different, but equivalent, perspectives on connections. For example, one can think of them as one-forms in principal bundles. From this perspective the proof of Proposition 17.6 is much easier. ◀

We would like a formula for this connection. We need as an ingredient a formula for $f\colon \mathfrak{so}(T_x^*M) \to \mathfrak{spin}(T_x^*M) \subset \mathrm{C}\ell^0(T_x^*M, g)$. Choosing an oriented orthonormal basis $\{e_1, \dots, e_n\}$ for T_x^*M, Theorem 14.36 tells us that for a $\xi \in \mathfrak{so}(T_x^*M)$,

$$f(\xi) = \frac{1}{4}\sum_{i,j} \langle \xi e_i, e_j\rangle e_i e_j. \tag{17.8}$$

In local coordinates $x_1, \dots, x_n$, let $\nabla_{\partial_i} \coloneqq \partial_i + A_i$, for $A_i(x) \in \mathfrak{so}(n)$. Then

$$\begin{aligned} \nabla^{\mathrm{Spin}}_{\partial_i} &= \partial_i + f(A_i) \\ &= \partial_i + \frac{1}{4}\sum_{\alpha,\beta} A_i^{\alpha\beta} \rho_\alpha \rho_\beta, \end{aligned} \tag{17.9}$$

where $\rho_\alpha \coloneqq \rho(\partial_\alpha)$ and $A_i^{\alpha\beta}$ denotes the matrix coefficients for A_i.

Though the formula in (17.9) might look mysterious, it's really just the consequence of the explicit isomorphism between $\mathfrak{so}(T_x^*M)$ and $\mathfrak{spin}(T_x^*M)$ that we established in Theorem 14.36.

Proposition 17.10. *Let $\mathfrak{s}$ be a spin^c structure on M and $\mathbb{S}$ be its spinor bundle. Then the Clifford connections form an affine space modeled on $\Omega^1_M(i\mathbb{R})$.*

In particular, the distinguished connection of Proposition 17.6 isn't unique.

Proof. Let $\widetilde{\nabla}$ be a Clifford connection on $\mathbb{S}$. Any other unitary connection is of the form $\widetilde{\nabla} + A$ for an $A \in \Omega^1_M(\mathfrak{u}(\mathbb{S}))$. If ρ is parallel, then $A_v \in \mathfrak{u}(\mathbb{S})$ is an infinitesimal element of the spinors as a representation of $\mathrm{C}\ell^0(T^*M, g)$. This is an irreducible representation, and its automorphisms are scalars; hence its infinitesimal automorphisms are the Lie algebra $i\mathbb{R}$.

It remains to establish existence, but locally we can choose $\widetilde{\nabla} + a \cdot \mathrm{id}_{\mathbb{S}}$ for a a purely imaginary 1-form, then use a partition of unity to globalize. ⊠

Recall that given a spin^c structure $\mathfrak{s}$, we get an associated line bundle $L_{\mathfrak{s}}$. A Clifford connection on $\mathbb{S}$ is equivalent data to a unitary connection $\widetilde{\nabla}^0$ on $L_{\mathfrak{s}}$, but their local descriptions are different: because the map $\mathrm{Spin}_n^c \to \mathrm{U}_1$ is sends $(g, z) \mapsto z^2$ instead of to z, the local description is

$$(\widetilde{\nabla} + a)^0 = \widetilde{\nabla}^0 + 2a. \tag{17.11}$$

Similarly, the curvature transforms as

$$F_{(\widetilde{\nabla}+a)^0} = F_{\widetilde{\nabla}^0} + 2a. \tag{17.12}$$

Conversely, we can determine the curvature of $\widetilde{\nabla}$ from that of $\widetilde{\nabla}^0$:

$$F_{\widetilde{\nabla}} = f \circ R + \frac{1}{2} F_{\widetilde{\nabla}^0} \otimes \mathrm{id}_{\mathbb{S}}. \tag{17.13}$$

This can be computed locally, where $\mathfrak{s}$ refines to a spin structure, and we can compare $\widetilde{\nabla}$ to ∇^{Spin}.

Dirac operators Let $\mathfrak{s}$ be a spin^c structure on a Riemannian manifold (M, g) with Levi-Civita connection ∇ and a Clifford connection $\widetilde{\nabla}$.

Definition 17.14. The *Dirac operator* for $\widetilde{\nabla}$ is $D := \rho \circ \widetilde{\nabla} \colon \Gamma(\mathbb{S}) \to \Gamma(\mathbb{S})$, which is a first-order differential operator.

Explicitly, $\widetilde{\nabla}$ turns a section of $\mathbb{S}$ into an $\mathbb{S}$-valued 1-form (a section of $T^*M \otimes \mathbb{S}$), and ρ turns cotangent vectors into elements of $\mathrm{End}(\mathbb{S})$, so we get back a section of $\mathbb{S}$ by $f \otimes \phi \mapsto f(\phi)$.

If $e^1, \dots, e^n$ is a local orthonormal frame for T^*M, then the Dirac operator has the formula

$$D\phi = \sum_{i=1}^n \rho(e^i) \circ \widetilde{\nabla}_{e_i} \phi. \tag{17.15}$$

Remark 17.16.

(1) If M is even-dimensional, so that $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ is $\mathbb{Z}/2$-graded, then the Dirac operator is odd, i.e. $D \colon \Gamma(\mathbb{S}^\pm) \to \Gamma(\mathbb{S}^\mp)$, because $\rho(e)$ is also odd.

(2) D is *formally self-adjoint*, meaning that if ϕ is compactly supported, then

$$\int_M g(D\phi, \chi)\,\mathrm{dvol} = \int_M g(\phi, D\chi)\,\mathrm{dvol}. \tag{17.17}$$

It therefore follows from a general result in index theory that D must have index zero, which is a little bit uninteresting. Therefore we will consider "half of D:" $D|_{\Gamma(\mathbb{S}^-)}$ is formally self-adjoint to $D|_{\Gamma(\mathbb{S}^+)}$, and these may have interesting indices. ◀

The symbol of D is $\sigma_D(e) = \rho(e)$, just Clifford multiplication. This means that D satisfies the abstract definition of a (generalized) Dirac operator, and deserves the name we gave it.

The Lichnerowicz formula. We continue to use the Riemannian manifold (M, g) with a spin^c structure $\mathfrak{s}$. In this case the Dirac operator satisfies a nice formula.

Proposition 17.18.

$$D^2 = \widetilde{\nabla}^* \widetilde{\nabla} + \frac{1}{4} K \mathrm{id}_{\mathbb{S}} + \frac{1}{2} \rho(F_{\widetilde{\nabla}^0}),$$

where K denotes the scalar curvature and $\widetilde{\nabla}_v^* \colon \Gamma(\mathbb{S}) \to \Gamma(\mathbb{S})$ is the formal adjoint to $\widetilde{\nabla}_v$.

We also need to define $\rho(F_{\widetilde{\nabla}^0})$; it arises from an action $\rho\colon \Lambda^2(M) \to \operatorname{End}\mathbb{S}$ of 2-forms on spinors by

$$\rho(e \wedge f) := \frac{1}{2}[\rho(e), \rho(f)]. \tag{17.19}$$

In particular, when $\mathfrak{s}$ arises from a spin structure and $\widetilde{\nabla} = \nabla^{\mathrm{Spin}}$,

$$D^2 = (\nabla^{\mathrm{Spin}})^* \nabla^{\mathrm{Spin}} + \frac{1}{4}K, \tag{17.20}$$

since $F_{(\nabla^{\mathrm{Spin}})^0} = 0$. Scalar curvature comes up a lot in Seiberg-Witten theory, and this is why.

Remark 17.21. The formula in Proposition 17.18 is known as a *Weitzenböck formula*: D^2 is *a priori* a second-order operator, but with carefully chosen D and $\widetilde{\nabla}$, $D^2 - \widetilde{\nabla}^*\widetilde{\nabla}$ is zeroth-order! Weitzenböck formulas like this one are common in differential geometry. ◀

Proof of Proposition 17.18. We will work locally near $x = 0$ in a local geodesic coordinate system $(x_1, \dots, x_n)$, letting $e_i := \mathrm{d}x_i(0)$. Let $\widetilde{\nabla}_i := \widetilde{\nabla}_{\partial/\partial i}$ and $\rho_i := \rho(\mathrm{d}x_i)$, so that

$$\widetilde{\nabla} = \sum_i \widetilde{\nabla}_i \otimes \mathrm{d}x_i. \tag{17.22}$$

It's a fact that we won't prove that

$$\widetilde{\nabla}^* = -\sum_i \widetilde{\nabla}_i \otimes \iota\left(\frac{\partial}{\partial x_i}\right) \colon \Gamma(T^*M \otimes \mathbb{S}) \to \Gamma(\mathbb{S}). \tag{17.23}$$

Since we're in geodesic coordinates, $\nabla_i(\mathrm{d}x_j)(0) = 0$. We also have the formulas

$$D = \sum_i \rho_i \nabla_i \tag{17.24}$$

$$\widetilde{\nabla}^*\widetilde{\nabla} = -\sum_i \widetilde{\nabla}_i \widetilde{\nabla}_i. \tag{17.25}$$

The equation (17.25) is another instance of a general fact: the Laplacian as it naturally arises for us has a minus sign, as it tends to in geometry; for applications in analysis the Laplacian has no minus sign.

Now we compute. We'll use the notation $\{A, B\} := AB + BA$ for the anticommutator.[25] At 0,

$$\begin{aligned}
-\widetilde{\nabla}^*\nabla + D^2 &= \sum_i \widetilde{\nabla}_i\widetilde{\nabla}_i + \sum_{i,j} \rho_i \widetilde{\nabla}_i \circ \rho_j \widetilde{\nabla}_j \\
&= \sum_i \widetilde{\nabla}_i\widetilde{\nabla}_i + \frac{1}{2}\sum_{i,j}\{\rho_i, \rho_j\}\widetilde{\nabla}_i\widetilde{\nabla}_j + \sum_{i,j}\rho_i[\widetilde{\nabla}_i, \rho_j]\widetilde{\nabla}_j + \frac{1}{2}\sum_{i,j}\rho_i\rho_j[\widetilde{\nabla}_i, \widetilde{\nabla}_j].
\end{aligned}$$

Using the Clifford relations,

$$\begin{aligned}
&= \sum_i \widetilde{\nabla}\widetilde{\nabla}_i - \sum_{i,j}\delta_{ij}\widetilde{\nabla}_i\widetilde{\nabla}_j + \sum_{i,j}\rho_i\rho(\underbrace{\nabla_i\,\mathrm{d}x_j}_{=0})\widetilde{\nabla}_j + \sum_{i<j}\rho_i\rho_j(F_{\widetilde{\nabla}})_{ij} \\
&= \sum_{i<j}(F_{\widetilde{\nabla}})_{ij}\rho_i\rho_j.
\end{aligned}$$

From (17.13),

$$F_{\widetilde{\nabla}} = f(R) + \frac{1}{2}F_{\widetilde{\nabla}^0} \otimes \mathrm{id}_{\mathbb{S}}, \tag{17.26}$$

and therefore

$$\frac{1}{2}\sum_{i<j}(F^0)_{ij}\rho_i\rho_j = \frac{1}{2}\rho(F^0), \tag{17.27}$$

[25] From the perspective of superalgebra, we could subsume all commutators and anticommutators in this calculation with supercommutators: we understand the commutator of two odd elements to pick up a sign, and hence become an anticommutator. Here ρ_i is odd and $\widetilde{\nabla}_j$ is even.

as in the formula. It remains to establish

$$\sum_{i<j} f(R_{ij})\rho_i\rho_j = \frac{1}{4}K\mathrm{id}_{\mathbb{S}}. \tag{17.28}$$

Expanding the left-hand side, we have

$$\begin{aligned}\sum_{i<j} f(R_{ij})\rho_i\rho_j &= \frac{1}{4}\sum_{\substack{i<j\\k,\ell}} R_{\ell kij}\rho_i\rho_j\rho_k\rho_\ell\\ &= \frac{1}{8}\sum_{i,j,k,\ell} R_{\ell kij}\rho_i\rho_j\rho_k\rho_\ell.\end{aligned} \tag{17.29}$$

Knowing that $R_{\ell(kij)} = 0$, let's fix ℓ and permute the other coordinates. We define

$$S_{ijk\ell} = \sum_{\sigma\in S_3} R_{\ell\sigma(k)\sigma(i)\sigma(j)}\rho_{\sigma(i)}\rho_{\sigma(j)}\rho_{\sigma(k)}, \tag{17.30}$$

and therefore

$$\sum_{i,j,k,\ell} R_{\ell kij}\rho_i\rho_j\rho_k\rho_\ell = \sum_\ell \sum_{i<j<k} S_{ijk\ell}\rho_\ell + \sum_\ell \sum_{i<j=k} S_{ijk\ell}\rho_\ell. \tag{17.31}$$

This would probably be the most opaque step in the proof if you were trying to reproduce it yourself — why is it this symmetry of R that matters?

To understand (17.31), we'll consider its two pieces as two cases. If $i<j<k$,

$$\begin{aligned}S_{ijk\ell} &= \sum_{\sigma\in S_3} R_{\ell\sigma(k)\sigma(i)\sigma(j)}\rho_{\sigma(i)}\rho_{\sigma(j)}\rho_{\sigma(k)}\\ &= \left(\sum_{\sigma\in S_3}\mathrm{sign}(\sigma)R_{\ell\sigma(k)\sigma(i)\sigma(j)}\right)\rho_i\rho_j\rho_k\\ &= \big(R_{\ell(kij)} - R_{\ell(ijk)}\big)\rho_i\rho_j\rho_k = 0.\end{aligned}$$

For $i<j=k$, $\rho_{\sigma(i)}\rho_{\sigma(j)}\rho_{\sigma(k)} = -\rho_{\sigma(i)}$, so

$$S_{ijk\ell} = -\sum_{\sigma\in S_3} R_{\ell\sigma(k)\sigma(i)\sigma(k)}\rho_{\sigma(i)}. \tag{17.32}$$

Therefore

$$\sum_{i,j,k,\ell} R_{\ell kij}\rho_i\rho_j\rho_k\rho_\ell = \sum_\ell\sum_{i<k} S_{ikk\ell}\rho_\ell.$$

Skipping a few steps, this comes out as

$$\begin{aligned}&= -2\sum_{i,k,\ell} R_{\ell iki}\rho_k\rho_\ell\\ &= -2\sum_{i,k} R_{ikik},\end{aligned}$$

which is exactly the scalar curvature. ⊠

Lecture 18.

The Seiberg-Witten equations: 4/3/18

"This is the Agatha Christie approach [to index theory]..."

Since we didn't have class last week, there will be two makeup classes this and next Friday at 3:30.

Before we discuss the Seiberg-Witten equations, we'll say a little more about spin^c structures in dimension 4. If (X,g) is an oriented Riemannian 4-manifold, a spin^c structure is data $\mathfrak{s} = (\mathbb{S},\rho)$ where $\mathbb{S} = \mathbb{S}^+\oplus\mathbb{S}^-$ is the spinor bundle, and $\mathbb{S}^\pm$ are rank-2 Hermitian bundles; and $\rho\colon T^*X\to\mathfrak{u}(\mathbb{S})$ is the Clifford map: it exchanges $\mathbb{S}^+$ and $\mathbb{S}^-$ and $\rho(e)^2 = -|e|^2\mathrm{id}$.

The data we've just specified is almost enough to specify the spin^c structure, but we also need an orientation condition. Let (e_1, e_2, e_3, e_4) be an orthonormal oriented basis for T_x^*X and

$$\omega := -e_1e_2e_3e_4 \in \mathrm{C}\ell^0(T_x^*X). \tag{18.1}$$

This doesn't depend on the choice of oriented orthonormal basis, and if $v \in T_x^*X$, $\omega \cdot v = -v \cdot \omega$. Therefore ω is (super)central in $\mathrm{C}\ell^0(T_x^*X)$, hence squares to 1. Therefore in its action on $\mathbb{S}$, it has $\{\pm 1\}$ eigenspaces preserved by $\mathrm{C}\ell^0$, but exchanged by T_x^*X. Therefore the eigenspaces are $\mathbb{S}^\pm$. We impose the condition that ω acts by ± 1 on $\mathbb{S}^\pm$; with this condition, the data above defines a spin^c structure.

Lemma 18.2. *The line bundles $\Lambda^2\mathbb{S}^+$, $\Lambda^2\mathbb{S}^-$, and $L_\mathfrak{s}$ are isomorphic.*

Proof. We will use the identification

$$\begin{aligned}\mathrm{Spin}_4^c &\cong \frac{\mathrm{SU}_2 \times \mathrm{SU}_2 \times \mathrm{U}_1}{\pm(1,1,1)}\\ &\cong G := \{(A, B) \in \mathrm{U}_2 \times \mathrm{U}_2 \mid \det A = \det B\},\end{aligned} \tag{18.3}$$

via the map $(A, B, z) \mapsto (zA, zB)$. Let $p_+, p_- \colon G \twoheadrightarrow \mathrm{U}_2$ be the projections onto the first and second factors, respectively. The spin^c structure on X induces a principal spin^c-bundle of frames $\mathcal{B}_{\mathrm{Spin}^c}(X) \to X$ for the cotangent bundle, and the components of the spinor bundle are

$$\mathbb{S}^\pm = \mathcal{B}_{\mathrm{Spin}^c}(X) \times_{G,p_\pm} \mathbb{C}^2, \tag{18.4}$$

and therefore

$$\Lambda^2\mathbb{S}^\pm \cong \mathcal{B}_{\mathrm{Spin}^c}(X) \times_{G,\det\circ p_\pm} \mathbb{C}. \tag{18.5}$$

If $\lambda\colon \mathrm{Spin}_4^c \to \mathrm{U}_1$ is the map $\lambda(A, B, z) := z^2$, then $\det\circ p_\pm = \lambda$, and since

$$L_\mathfrak{s} := \mathcal{B}_{\mathrm{Spin}^c}(X) \times_{\mathrm{Spin}_4^c,\lambda} \mathbb{C}, \tag{18.6}$$

then these three line bundles are isomorphic. ⊠

We define $\det\mathfrak{s} := \Lambda^2\mathbb{S}^+$ (which, thanks to the above lemma, is also $\Lambda^2\mathbb{S}^-$ and $L_\mathfrak{s}$).

Self-duality We can extend ρ to act on 2-forms through a map

$$\begin{aligned}\rho\colon \Lambda^2(T^*X) &\longrightarrow \mathrm{End}^0\,\mathbb{S}\\ \rho(e\wedge f) &= \frac{1}{2}[\rho(e), \rho(f)].\end{aligned} \tag{18.7}$$

Recall that $\mathrm{End}^0(\mathbb{S})$ is the space of even endomorphisms of $\mathbb{S}$, i.e. those sending $\mathbb{S}^\pm \to \mathbb{S}^\pm$.

The map constructed in (18.7) is a composite

$$\Lambda^2(T^*X) \xrightarrow[\cong]{g} \mathfrak{so}(T^*X) \xrightarrow{f} \mathfrak{spin}(T^*X) \xrightarrow{\rho} \mathrm{End}\,\mathbb{S}, \tag{18.8}$$

where

- $g\colon e\wedge f \mapsto (x \mapsto \langle x, f\rangle e - \langle x, e\rangle f)$ and
- f is the map we constructed before.

Since $\rho(\Lambda^2(T^*X)) \subset [\mathfrak{u}(\mathbb{S}), \mathfrak{u}(\mathbb{S})] = \mathfrak{su}(\mathbb{S})$, we land in $\mathrm{End}^0(\mathbb{S})$.[26] We also have the action of the Hodge star on $\Lambda^2(T^*X)$, which ρ carries to the action of ω on $\mathbb{S}$:

The upshot is that $\rho(\Lambda^\pm) \subset \mathfrak{su}(\mathbb{S}^\pm)$ (and is 0 on $\mathbb{S}^\mp$), and in fact this is an isomorphism, so ρ identifies self-dual 2-forms with skew-adjoint trace-free endomorphisms of $\mathbb{S}^+$.

Let $\mathcal{A}_{\mathrm{cl}}(\mathbb{S}^+)$ denote the space of Clifford connections ∇ on $\mathbb{S}^+$. The map $A \mapsto A^o$ defines a bijection $\mathcal{A}_{\mathrm{cl}}(\mathbb{S}^+) \cong \mathcal{A}(\det\mathfrak{s})$; more concretely, if $a \in i\Omega^1(X)$, then the local description of this is

$$A = a\cdot \mathrm{id}_{\mathbb{S}^+} = A^0 + 2a. \tag{18.9}$$

The gauge group $\mathcal{G}$ (or $\mathcal{G}(\mathfrak{s})$ if we need to disambiguate) of automorphisms of the fibers $\mathbb{S}_x$ commuting with ρ is also isomorphic to $\mathcal{G}(\det\mathfrak{s}) = C^\infty(X, \mathrm{U}_1)$ via the map $u \mapsto \det(u|_{\mathbb{S}^+})$.

Therefore the configuration space we consider is $\mathcal{A}_{\mathrm{cl}}(\mathbb{S}^+) \times \Gamma(\mathbb{S}^+)$, which has a left $\mathcal{G}$-action by

$$u\cdot(A, \phi) = (u^*A, u\cdot\phi). \tag{18.10}$$

[26] TODO: maybe I misheard this.

This is where the Seiberg-Witten equations live.

(1) The first equation, the *Dirac equation*, is for $(A, \phi) \in \mathcal{A}_{\mathrm{cl}}(\mathbb{S}^+) \times \Gamma(\mathbb{S}^+)$:

$$D_A^+ \phi = 0. \tag{18.11}$$

Here D_A is the Dirac operator $\rho \circ \nabla_A$, and $D_A^+ := D_A|_{\Gamma(\mathbb{S}^+)}$, which maps into $\Gamma(\mathbb{S}^-)$.[27] This equation had already been studied by mathematicians for a while. For fixed A, the Dirac operator is a linear elliptic operator on spinors, with symbol ρ, and in a sense, it leaves A unconstrained: you can pick whatever A you want, then look at the vector space of solutions.

The gauge group action preserves the Dirac equation, which is a quick calculation:

$$\begin{aligned} D^+_{u^*A} &= \rho \circ \nabla_{u^*A} \\ &= \rho \circ (u \nabla_A u^{-1}) \\ &= u \circ \rho \circ \nabla_A \circ u^{-1}, \end{aligned}$$

and therefore

$$D^+_{u^*A}(u \circ \phi) = u\rho\nabla_A \phi = u D_A^+ \phi. \tag{18.12}$$

Therefore $D_A^+\phi = 0$ iff $D^+_{u^*A}\phi = 0$.

(2) The second equation, called the *curvature equation*, is the one that surprised mathematicians:

$$\frac{1}{2}\rho(F(A^0)^+) - (\phi\phi^*)_0 = 0. \tag{18.13}$$

This constrains the gauge orbit of A. Here $F(A^0)^+$ denotes the self-dual piece of the curvature of A^0, which lives in $i\Omega^+(X)$. Therefore ρ carries it to $i \cdot \mathfrak{su}(\mathbb{S}^+)$, the trace-free self-adjoint endomorphisms of $\mathbb{S}^+$.

Since $\phi\phi^*$ is self-adjoint, it lives in $\mathrm{End}(\mathbb{S}^+)$. It acts on χ by

$$(\phi\phi^*)(\chi) = \langle \chi, \phi \rangle \phi, \tag{18.14}$$

where the inner product is antilinear on the right. $(-)_0$ denotes taking the trace-free part: if $\theta \in \mathrm{End}(\mathbb{S}^+)$,

$$\theta_0 := \theta - \frac{1}{2}\operatorname{tr}(\theta)\mathrm{id}_{\mathbb{S}^+}. \tag{18.15}$$

Hence $(\phi\phi^*)_0$ is also in $i\mathfrak{su}(\mathbb{S}^+)$. Solutions to (18.13) are also gauge-invariant: if $\psi = u\phi$, then

$$\psi\psi^* = |u|^2 \phi\phi^* = \phi\phi^*, \tag{18.16}$$

and $F((u^*A)^0) = F(A^0)$, because A^0 is a connection for an abelian Lie group.

We can package both equations together: the left-hand sides of the two equations define a map

$$\mathcal{F}\colon \mathcal{A}_{\mathrm{cl}}(\mathbb{S}^+) \times \Gamma(\mathbb{S}^+) \longrightarrow \Gamma(i\mathfrak{su}(\mathbb{S}^+)) \times \Gamma(\mathbb{S}^-), \tag{18.17}$$

and the Seiberg-Witten equations say that $\mathcal{F} = 0$.

We can also perturb (18.13) by an $\eta \in \Omega^+(X)$ to

$$\frac{1}{2}\rho(F(A^0)^+ - 4i\eta) - (\phi\phi^*)_0 = 0. \tag{18.18}$$

This is also gauge-invariant, and we can package it together with the Dirac equation to get a perturbed Seiberg-Witten equation $\mathcal{F}_\eta = 0$.

We can also linearize these equations, by considering what we get if we add $a + \mathrm{id}_{\mathbb{S}^+}$ to A and χ to ϕ. The result is

$$\mathcal{F}_\eta(A + a \cdot \mathrm{id}_{\mathbb{S}^+}, \phi + \chi) - \mathcal{F}_\eta(A, \phi) = \begin{bmatrix} \rho(\mathrm{d}^+ a) - (\phi\chi^* + \chi\phi^* + \chi\chi^*)_0 \\ D_A^+\chi + (1/2)\rho(a)(\phi + \chi) \end{bmatrix}. \tag{18.19}$$

Then we can explicitly take the linear piece of this.

$$(D_{(A,\phi)}\mathcal{F}_\eta)\begin{bmatrix} a \\ \chi \end{bmatrix} = \begin{bmatrix} \rho \circ \mathrm{d}^+ & 0 \\ 0 & D_A^+ \end{bmatrix}\begin{bmatrix} a \\ \chi \end{bmatrix} + \begin{bmatrix} -(\phi\chi^* + \chi\phi^*)_0 \\ (1/2)\rho(a)\phi \end{bmatrix}. \tag{18.20}$$

[27] Similarly, $D_A^- := D_A|_{\Gamma(\mathbb{S}^-)}$ is the formal adjoint to D_A^+ and maps into $\Gamma(\mathbb{S}^+)$.

This is not elliptic, because we have room to freely choose the connection. To fix this, "gauge-fix," introducing a third equation called the *Coulomb gauge equation.* Given a reference Clifford connection $A_0 \in \mathcal{A}_{\mathrm{cl}}$, any connection differs from A_0 by $a \cdot \mathrm{id}_{\mathbb{S}^+}$; we ask that

$$\mathrm{d}^* a = 0. \tag{18.21}$$

We can append this to the Seiberg-Witten equations, defining a new operator

$$\widetilde{\mathcal{F}}_\eta(A, \phi) := \begin{bmatrix} (1/2)(F(A^0)^+ - 4i\eta) - \rho^{-1}(\phi\phi^*)_0 \\ \mathrm{d}^* a \\ D_A^+ \phi \end{bmatrix}, \tag{18.22}$$

and its linearization is

$$(D_{(A,\phi)}\widetilde{\mathcal{F}}_\eta) \begin{bmatrix} a \\ \chi \end{bmatrix} = \begin{bmatrix} \mathrm{d}^+ & 0 \\ \mathrm{d}^* & 0 \\ 0 & D_A^+ \end{bmatrix} \begin{bmatrix} a \\ \chi \end{bmatrix}, \tag{18.23}$$

plus some zeroth-order terms, similarly to (18.20).

At the level of symbols, $D\widetilde{\mathcal{F}}_\eta$ is the direct sum of

$$\mathrm{d}^+ \oplus \mathrm{d}^* \colon i\Omega^1(X) \longrightarrow i(\Omega^+(X) \oplus \Omega^0(X)) \tag{18.24a}$$

and

$$D_A^+ \colon \Gamma(\mathbb{S}^+) \longrightarrow \Gamma(\mathbb{S}^-), \tag{18.24b}$$

and both of these are elliptic operators, so $D\widetilde{\mathcal{F}}_\eta$ is elliptic! This unlocks some very useful tools.

Some index theory. Now we use some tools from index theory, giving a brief preview of a more detailed account in a future lecture. Let $E, F \to M$ be vector bundles with Euclidean metrics, where M is a closed manifold, and let $\delta \colon \Gamma(E) \to \Gamma(F)$ be an elliptic operator with formal adjoint $\delta^* \colon \Gamma(F) \to \Gamma(E)$.[28] Then $\ker(\delta^*) \cong \operatorname{coker}(\delta)$, and, as an operator $\Gamma_{C^k}(E) \to \Gamma_{C^{k+1}}(F)$, it's *Fredholm*, meaning it has closed image and its kernel and cokernel are both finite-dimensional. Therefore we can make the following definition.

Definition 18.25. The *index* of a Fredholm operator A is

$$\operatorname{ind} A := \dim\ker A - \dim\operatorname{coker} A.$$

The index of δ only depends on its symbol, and ellipticity implies that $\ker(\delta)$ and $\ker(\delta^*)$ consist of smooth sections.

The Atiyah-Singer index theorem provides a formula for $\operatorname{ind}(\delta)$. Recall that a Dirac operator is one which squares to a generalized Laplacian, and its symbol defines a Clifford representation, so if we have $E = E^+ \oplus E^-$ a complex Clifford module over (T^*M, g) and a Dirac operator

$$D^\pm \colon \Gamma(E^\pm) \longrightarrow \Gamma(E^\mp), \tag{18.26}$$

then the Dirac operator is a first-order formally self-adjoint operator.

Let $W \to M := \operatorname{End}_{C\ell(T^*M)} E$; in the case of a spin^c structure, this is a line bundle. In this case, the Atiyah-Singer theorem says that

$$\operatorname{ind}_{\mathbb{C}}(D^+ \colon \Gamma(E^+) \to \Gamma(E^-)) = \int_M \widehat{A}(T^*M) \cdot ch(W). \tag{18.27}$$

That is, it's the integral of a cohomology class.

- $\widehat{A}$ is a certain series in the Pontrjagin classes of M, which begins

$$\widehat{A} = 1 - \frac{1}{24}p_1 + \cdots \in H^{4*}(M;\mathbb{Q}). \tag{18.28}$$

- *ch* is the *Chern character*, a series in the Chern classes of W, which begins

$$\operatorname{rank}\cdot 1 + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \cdots \in H^{2*}(M;\mathbb{Q}). \tag{18.29}$$

[28] Today, we'll work with C^k sections of E and F, for some $k \gg 0$. In practice, though, one always works with a Sobolev space.

The class $\widehat{A}(T^*M)\cdot ch(W)$ need not be homogeneous, so to integrate, we pick out the top-degree part.

In the case of a spin Dirac operator D^{Spin} on a spin 4-manifold X, $W\cong\underline{\mathbb{C}}$, and therefore

$$\operatorname{ind}_{\mathbb{C}} D^{\mathrm{Spin}} = \int_X \widehat{A}(T^*X) = -\frac{1}{24}\langle p_1(T^*X),[X]\rangle = -\frac{1}{8}\tau(X), \tag{18.30}$$

where τ denotes the signature.

We can twist this spin structure $\mathfrak{s}$ by a line bundle L to obtain a spin^c structure $L\otimes\mathfrak{s}$. In this case $W\cong L$, and

$$\operatorname{ind}_{\mathbb{C}} D_A^+ = \int_X \left(1+c_1(L)+\frac{1}{2}c_1(L)^2\right)\left(1-\frac{1}{24}p_1(T^*X)\right) = \frac{1}{8}\big(\langle c_1(\mathbb{S}^+)^2,[M]\rangle - \tau(X)\big). \tag{18.31}$$

This formula is also valid for any closed spin^c 4-manifold, even if it's not spinnable. Next time, we'll discuss the full Seiberg-Witten index.

Lecture 19.

Bounds to solutions to the Seiberg-Witten equations: 4/5/18

We've been discussing the index of the linearized Seiberg-Witten operator $D_{(A,\phi)}(\mathcal{F}_\eta, d^*)$. Last time, we saw that the symbol us that of the direct sum of $D_A^+\colon \Gamma(\mathbb{S}^+)\to\Gamma(\mathbb{S}^+)$ and $\mathrm{d}^*\oplus(\rho\circ\mathrm{d}^+)\colon i\Omega^1(X)\to i(\Omega^0(X)\oplus\Omega^+(X))$. Therefore its index is the sum of $\operatorname{ind} D_A^+$ and $\operatorname{ind}(\mathrm{d}^*\oplus(\rho\circ\mathrm{d}^+))$.

We then saw that, by the Atiyah-Singer index theorem,

$$\operatorname{ind}_{\mathbb{C}} D_A^+ = \frac{1}{8}\big(\langle c_1(\mathfrak{s})^2,[X]\rangle - \tau(X)\big). \tag{19.1}$$

The index is always an integer, but you could have concluded that the right-hand side is an integer *a priori*: since the mod 2 reduction of $c_1(\mathfrak{s})$ is $w_2(X)$, $c_1(\mathfrak{s})$ is characteristic, and we proved in Example 3.18 that for all characteristic vectors c, $c^2\equiv\tau \bmod 8$. This is the index as a complex vector bundle; we'll have to double it to get the index of a real vector bundle.

If $\mathcal{E}^\bullet$ denotes the self-duality complex $\Omega^0(X)\to\Omega^1(X)\to\Omega^+(X)$, then

$$\ker(\mathrm{d}^*\oplus(\rho\circ\mathrm{d}^+)) \cong H^{\mathrm{odd}}(\mathcal{E}^\bullet)\cong H^1_{\mathrm{dR}}(X) \tag{19.2a}$$

$$\operatorname{coker}(\mathrm{d}^+\oplus(\rho\circ\mathrm{d}^+)) \cong H^{\mathrm{even}}(\mathcal{E}^\bullet)\cong \mathbb{R}\oplus\mathcal{H}^+_g(X). \tag{19.2b}$$

Therefore

$$\operatorname{ind}(\mathrm{d}^*\oplus(\rho\circ\mathrm{d}^+)) = b_1(X)-(1+b_2^+(X)). \tag{19.3}$$

Therefore

$$\begin{aligned}\operatorname{ind}_{\mathbb{R}} D_{(A,\phi)}(\mathcal{F}_\eta,d^*) &= \frac{1}{4}\big(\langle c_1(\mathfrak{s})^2,[X]\rangle+\tau(X)\big)+b_1(X)-1-b_2^+(X)\\ &= \frac{1}{4}\big(\langle c_1(\mathfrak{s})^2,[X]\rangle-2\chi(X)-3\tau(X)\big).\end{aligned} \tag{19.4}$$

Definition 19.5. The quantity in (19.4) is called the *Seiberg-Witten index* and denoted $d(\mathfrak{s})$.

This number will be fundamental in Seiberg-Witten theory.

Remark 19.6. One can show that $d(\mathfrak{s})$ is equal to the *Euler number* $\langle e(\mathbb{S}^+),[X]\rangle$ of $\mathbb{S}^+$, which is equal to the number of zeros of a generic section of $\mathbb{S}^+$ (generic in the sense that it intersects transversely with the zero section). One can prove this directly with an analytic argument by showing that asymptotically, almost all of the information in computing it is localized to a small neighborhood of the zero set of ϕ.

One can reverse-engineer this into a different proof of the index formula for a Dirac operator. ◂

We'll learn more about index theory next lecture; now we'll discuss bounds for solutions to the Seiberg-Witten equations, with the eventual goal of proving compactness of the Seiberg-Witten moduli space, in the following way.

Theorem 19.7. *Let $\{(A_i,\phi_i)\}$ be a sequence of solutions to $\mathcal{F}_\eta(A,\phi)=0$. Then there exist $u_i\in\mathcal{G}$ such that $(u_i^*A_i,u_i^*\phi_i)$ converges to a smooth limiting solution (A_∞,ϕ_∞).*

Along the way we'll need to make both some pointwise estimates and some global estimates. We'll start with an inequality about Laplacians.

Let (M, g) be an oriented Riemannian manifold, $E \to M$ be a real vector bundle with a Euclidean metric, and $\nabla\colon \Gamma(E) \to \Gamma(T^*M \otimes E)$ be (the covariant derivative of) an orthogonal connection on E. It has a formal adjoint $\nabla^*\colon \Gamma(T^*M \otimes E) \to \Gamma(E)$.

Proposition 19.8. *For any $s \in \Gamma(E)$,*

$$\frac{1}{2}\mathrm{d}^*\mathrm{d}(|s|)^2 = \langle \nabla^*\nabla s, s\rangle - |\nabla s|.$$

Remark 19.9. If you integrate both sides of this equation, you'll get zero. The point is that it's pointwise, hence stronger. The left-hand side is a kind of divergence. ◀

Corollary 19.10.

$$\frac{1}{2}\mathrm{d}^*\mathrm{d}|s|^2 \le \langle \nabla^*\nabla s, s\rangle.$$

This will be surprisingly helpful when we're playing with the Seiberg-Witten equations.

Lemma 19.11. *If p is a local maximum for $f \in C^\infty(M)$, then $(\mathrm{d}^*\mathrm{d}f)(p) \ge 0$.*

Proof idea. On Euclidean space $\mathbb{E}^n$ (i.e. $\mathbb{R}^n$ with the usual metric) with $p = 0$,

$$\mathrm{d}^*(a_i\,\mathrm{d}x_i) = -\frac{\partial a_i}{\partial x_i}. \tag{19.12}$$

Hence $\mathrm{d}^*\mathrm{d}f = -\sum \partial_i^2 f$, so the lemma follows from the second derivative test.

In general, one can choose coordinates near p such that $p \mapsto 0$ and

$$g = g_{\mathbb{E}^n} + \sum_{i,j} h_{ij}x_ix_j, \tag{19.13}$$

and using the fact that $\mathrm{d}^* = \pm{\star}\mathrm{d}{\star}$, one can check that

$$(\mathrm{d}^*\mathrm{d}f)(0) = -\left(\sum_i \partial_i^2 f\right)(0), \tag{19.14}$$

so the result holds for all metrics. ⊠

Next we'll introduce some *a priori* bounds for the Seiberg-Witten equations.

Lemma 19.15.

(1) *For $\phi, \chi \in \Gamma(\mathbb{S}^+)$,*

$$\langle (\phi\phi^*)_0\chi, \chi\rangle = |\chi|^2\langle \chi, \phi\rangle - \frac{1}{2}|\chi|^2|\phi|^2, \tag{19.16}$$

and hence $\langle (\phi\phi^*)_0\phi, \phi\rangle = (1/2)|\phi|^4$.

(2) *For an $\eta \in \Omega^2(X)$ and a $\phi \in \Gamma(\mathbb{S})$,*

$$\langle \rho(\eta)\phi, \phi\rangle \le |\eta||\phi|^2. \tag{19.17}$$

Theorem 19.18 (Basic pointwise estimate)**.** *If $\mathcal{F}_\eta(A, \phi) = 0$, then*

$$\mathrm{d}^*\mathrm{d}(|\phi|^2) + \frac{1}{2}(\mathrm{scal}_g - 8|\eta|)|\phi|^2 + |\phi|^4 \le 0.$$

Here scal_g is the scalar curvature of X.

Remark 19.19. Why are we considering the deformed Seiberg-Witten equations anyways? There are different reasons, but one is that in a very general setting the Seiberg-Witten invariants will not depend on η, and in some cases there are very convenient choices of η: for example, if X is a symplectic 4-manifold, it's helpful to let η be the symplectic form, and if X is Kähler, one can choose it to be the Kähler form. ◀

Proof of Theorem 19.18. let A be a connection compatible with the metric and ∇_A denote its covariant derivative. By Corollary 19.10,

$$\begin{aligned} \frac{1}{2}\mathrm{d}^*\mathrm{d}(|\phi|^2)) &\le \langle \nabla_A^*\nabla_A\phi, \phi\rangle \\ &= \left\langle D_A^- D_A^+\phi - \frac{1}{4}\mathrm{scal}_g\phi - \frac{1}{2}\rho(F(A^0))\phi, \phi \right\rangle \end{aligned} \tag{19.20}$$

by the Lichnerowicz formula. Imposing the Seiberg-Witten equations,

$$\begin{aligned} \frac{1}{2}\mathrm{d}^*\mathrm{d}|\phi|^2 &\le -\frac{1}{4}\mathrm{scal}_g|\phi|^2 - \langle 2i\rho(\eta)\phi + (\phi\phi^*)_0\phi, \phi\rangle \\ &\le -\frac{1}{4}\mathrm{scal}_g|\phi|^2 + 2|\eta||\phi|^2 - \frac{1}{2}|\phi|^4. \end{aligned}$$ ⊠

The next theorem is really at the heart of the compactness theorem.

Theorem 19.21 (Pointwise bound on $|\phi|$). *Assume X is compact. If*

$$s := \max(8|\eta| - \mathrm{scal}_g, 0)$$

and $\mathcal{F}_\eta(A, \phi) = 0$, then

$$\max|\phi|^2 \le \frac{1}{2}\max s.$$

Proof. By Theorem 19.18,

$$\mathrm{d}^*\mathrm{d}(|\phi|^2) + |\phi|^4 \le \frac{s}{2}|\phi|^2. \tag{19.22}$$

Since X is compact, $|\phi|^2$ achieves its maximum at a point x. Hence $\mathrm{d}^*\mathrm{d}|\phi|^2(x) \ge 0$, so $|\phi|^4 \le (s/2)|\phi|^2$ at x. If $\phi = 0$, the result follows, and if not, then

$$|\phi|^2(x) \le \frac{s(x)}{2} \le \frac{1}{2}\max s. \tag{19.23}$$ ⊠

Remark 19.24. The sign of $(\phi\phi^*)_0$ is crucial; it was wrong in a preprint of the earlier papers and the results completely collapse. It took some effort to track down the sign change. This reinforces the point that these aren't just some random equations, but are quite special. ◂

Corollary 19.25. *If $\eta = 0$ and $\mathrm{scal}_g \ge 0$, then the only solutions to $\mathcal{F} = 0$ are those with $\phi = 0$ identically. In this case we get $F(A^0)^+ = 0$, giving us an abelian instanton.*

Proposition 19.26. *If $\mathcal{F}_\eta(A, \phi) = 0$, then*

$$\left|\mathcal{F}(A^0)^+ - 4i\eta\right| \le \frac{1}{4}\max s.$$

Proof. Since $\rho(F(A^0)^+ - 4i\eta) = (\phi\phi^*)_0$, then

$$\begin{aligned} |F(A^0)^+ - 4i\eta| \le |\rho(F(A^0)^+ - 4i\eta)|_{\mathrm{op}} \\ &= |(\phi\phi^*)_0| \\ &= \frac{1}{2}|\phi|^2 \le \frac{1}{4}\max s. \end{aligned}$$ ⊠

This is about all we can do without some integral (global) estimates; primed with some elliptic theory, we'll tackle them next week. But we do have a finiteness result.

Proposition 19.27. *Among the spin^c structures $\mathfrak{s}$ whose index $d(\mathfrak{s}) \ge d_0$ for some d_0, only finitely many isomorphism classes admit solutions to the Seiberg-Witten equations for which $\eta = 0$.*

Here the metric is fixed. Since spin^c structures are an $H^2(X;\mathbb{Z})$-torsor, there are generally not finitely many spin^c structures without this condition; moreover, this is very false when $\eta \neq 0$.

Proof. Set $F \coloneqq iF(A^0) \in \Omega^2(X)$; $\mathrm{d}F = 0$ and by Chern-Weil theory, $(1/2\pi)[F] = c_1(\mathfrak{s})$. Therefore

$$\begin{aligned}\langle c_1(\mathfrak{s}), [X]\rangle &= \frac{1}{4\pi^2}\int_X F \wedge F \\ &= \frac{1}{4\pi^2}(F^+ + F^-) \wedge (F^+ \wedge F^-) \\ &= \frac{1}{4\pi^2}\int_X (F^+)^2 - (F^-)^2 \\ &= \frac{1}{4\pi^2}\int_X (|F^+|^2 - |F^-|^2)\,\mathrm{vol}_g.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{4\pi^2}\int_X |F|^2\,\mathrm{vol}_g &= \frac{1}{4\pi^2}\int_X \left(|F^+|^2 + |F^-|^2\right)\mathrm{vol}_g \\ &= -\langle c_1(\mathfrak{s})^2, [X]\rangle + \frac{1}{2\pi^2}\int_X |F^+|^2\,\mathrm{vol}_g.\end{aligned} \tag{19.28}$$

Let $S \coloneqq \max s$. If $\mathcal{F}(A, \phi) = 0$, then $|F^+| \le s/4$ by Proposition 19.26. Therefore

$$\int_X |F^+|^2\,\mathrm{vol}_g \le \frac{S^2}{16}\mathrm{vol}(X). \tag{19.29}$$

Therefore

$$\begin{aligned}\frac{1}{4\pi^2}\int_X |F|^2\,\mathrm{vol}_X &\le -\langle c_1(\mathfrak{s})^2, [X]\rangle + \frac{S^2\mathrm{vol}(X)}{32\pi^2} \\ &\le 4d(\mathfrak{s}) - 2\chi(X) - 3\tau(X) + \frac{S^2\mathrm{vol}(X)}{32\pi^2}.\end{aligned}$$

Let $C(X, g) \coloneqq -2\chi(X) - 3\tau(X) + S^2\mathrm{vol}(X)/32\pi^2$. Then,

$$\le -4d_0 + C(X, g),$$

since we specified $d(\mathfrak{s}) \ge d_0$.

There is a norm $\|\cdot\|$ on $H^2_{\mathrm{dR}}(X)$ defined by $\|c\| \coloneqq \min_{\omega\colon [\omega]=c}\|\omega\|_{L^2}$. This is equal to the norm of the harmonic representative of c. Then we have

$$\begin{aligned}\|c_1(\mathfrak{s})\|^2 = \frac{1}{2\pi}\|[F]\|^2 &\le \frac{1}{4\pi^2}\int_X |F|^2\,\mathrm{vol}_g \\ &\le -4d_0 + C(X, g),\end{aligned} \tag{19.30}$$

so

$$c_1(\mathfrak{s}) \in H^2(X; \mathbb{Z}) \cap B_{-4d_0 + C(X,g)}(0), \tag{19.31}$$

which is a finite set. The map $\mathfrak{s} \mapsto c_1(\mathfrak{s})$ has fibers which are noncanonically in bijection with $H^1(X; \mathbb{Z}/2)$, which is finite, so only finitely many spin^c structures meet this criterion. ⊠

Remark 19.32. Let's recall why the fibers of the map $\mathfrak{s} \mapsto c_1(\mathfrak{s})$ has fibers which we can identify with $H^1(X; \mathbb{Z}/2)$. Recall that the set of spin^c structures on X admits a freely transitive action of $H^2(X; \mathbb{Z})$ (the Picard group of complex line bundles). The action of c maps $c_1(\mathfrak{s})$ to $2c + c_1(\mathfrak{s})$, so the fibers are the 2-torsion of $H^2(X)$, which a Bockstein identifies with $H^1(X; \mathbb{Z}/2)$. ◀

Lecture 20.

: 4/6/18

Lecture 21.

The compactness theorem: 4/12/18

Today we want to prove the key theorem about compactness of the Seiberg-Witten moduli space.

Theorem 21.1. *Let (X, g) be a closed, oriented, Riemannian 4-manifold and $\mathfrak{s}$ be a spin^c structure on M. Let $\eta \in \Omega^+(X)$ and (A_j, ϕ_j) be a sequence of solutions to the Seiberg-Witten equations $\mathcal{F}_\eta = 0$. Writing $A_j = A_0 + a_j \cdot \mathrm{id}_{\mathbb{S}^+}$, assume that*

(1) $\mathrm{d}^* a_j = 0$ *(the Coulomb gauge), and*
(2) *the sequence $(a_j)_{\mathrm{harm}}$ is bounded in the L^2-norm on $\mathcal{H}^1(X)$.*

Then a subsequence of (A_j, ϕ_j) converges in C^∞ to a smooth limiting solution (A, ϕ).

Both of the assumptions are always achievable via gauge transformations.

Remark 21.2. There are many different references for proofs of this theorem, though they all have a lot in common. We'll follow a proof by Kronheimer-Mrowka, which generalizes nicely to 4-manifolds with boundary. ◂

We're going to make considerable use of the quadratic bounds we established a few lectures ago; nonetheless, the proof is still subtle.

The proof will use Sobolev spaces and equations with quadratic terms, so we'll need to know how to multiply Sobolev functions.

Lemma 21.3. *Over X, multiplication of C^∞ functions extends to a bounded linear map $L^2_k(X) \otimes L^2_\ell(X) \to L^2_\ell(X)$ provided $k \geq 3$ and $k \geq \ell$. In particular, $L^2_k(X)$ is an algebra for $k \geq 3$.*

We'll skip the proof, but it follows from thinking about the various Sobolev embeddings you have, and how derivatives behave under a product.

Corollary 21.4. *If $E, F \to X$ are Euclidean vector bundles on X, there is a bounded linear multiplication map*

$$L^2_k(E) \otimes L^2_\ell(F) \longrightarrow L^2_\ell(E \otimes F)$$

provided that $k \geq 3$ and $k \geq \ell$.

Lemma 21.5. *We also have bounded linear multiplication maps*

$$L^2_1(X) \otimes L^3_1(X) \longrightarrow L^3(X) \tag{21.6a}$$

$$L^2_2(X) \otimes L^3_1(X) \longrightarrow L^2_1(X) \tag{21.6b}$$

$$L^2_3(X) \otimes L^2_2(X) \longrightarrow L^2_2(X). \tag{21.6c}$$

The proof is similar to the proof of Lemma 21.3.

We next formulate the Seiberg-Witten equations in Sobolev spaces. Lemma 21.3 implies that spin^c L^2_k gauge transformations form a topological group $\mathcal{G}_k$. We can identify this with the group of L^2_k maps $X \to S^1 \subset \mathbb{C}$ under pointwise product.

The space of L^2_k *configurations* $(A = A_0 + a, \phi)$ is akin to the usual space of configurations, but we now ask for $A_0 \in C^\infty$ and $a \in L^2_k$. Lemma 21.3 implies that $\mathcal{G}_k$ acts continuously on the space of L^2_k configurations.

The Seiberg-Witten equations make sense in L^2_k as long as $k \geq 3$.

- The differential term $(D^+_A \phi)$ maps L^2_k to L^2_{k-1}.
- The quadratic terms $(\phi\phi^*)_0$ and $\rho(a)\phi$ map L^2_k to L^2_k.

Building in the Coulomb gauge, we obtain a map

$$\mathcal{F}'_\eta \colon L^2_k(T^*X) \times L^2_k(\mathbb{S}^+) \longrightarrow L^2_{k-1}(X)_0 \times L^2_{k-1}(i\mathfrak{su}(\mathbb{S}^+)) \times L^2_{k-1}(\mathbb{S}), \tag{21.7}$$

where $L^2_{k-1}(X)_0$ means the L^2_{k-1} functions with mean zero. Explicitly,

$$\mathcal{F}'_\eta(a, \phi) = (\mathrm{d}^* a, \mathcal{F}_\eta(A_0 + a \cdot \mathrm{id}, \phi)). \tag{21.8}$$

We want to analyze solutions to $\mathcal{F}'_\eta = 0$.

As before, it's useful to write $\mathcal{F}'_\eta$ as a sum of constant, linear, and quadratic parts:

$$\mathcal{F}'_\eta = \mathcal{D} + q + c. \tag{21.9}$$

$\mathcal{D}$ is the linear part:

$$\mathcal{D}\begin{bmatrix} a \\ \phi \end{bmatrix} = \begin{bmatrix} \mathrm{d}^* a \\ \rho(\mathrm{d}^+ a) \\ D^+_{A_0}\phi \end{bmatrix}. \tag{21.10a}$$

This is a first-order linear differential operator, and is elliptic.

q is the quadratic part:

$$q\begin{bmatrix} a \\ \phi \end{bmatrix} = \begin{bmatrix} 0 \\ (\phi\phi^*)_0 \\ (1/2)\rho(a)\phi \end{bmatrix}. \tag{21.10b}$$

Finally, c is the constant part:

$$c = \begin{bmatrix} 0 \\ (1/2)\rho(F(A_0^0)^+) - 4i\eta \\ 0 \end{bmatrix}. \tag{21.10c}$$

Here A_0^0 is the connection in $\Lambda^2\mathbb{S}^+$ induced from the reference connection A^0 in $\mathbb{S}^+$.

We're now going to exploit ellipticity to build bounds on increasingly regular solutions, in a positive feedback loop. This technique is also called *elliptic bootstrapping*.

Proposition 21.11. *Fix $k \geq 3$, and consider a set S of solutions $\gamma = (a, \phi)$ to $\mathcal{F}'_\eta(a, \phi) = 0$, such that S is bounded in L_k^2: there's a c_k such that $\|\gamma\|_{L_k^2} \leq c_k$. Then there's a c_{k+1} such that $\|\gamma\|_{L_{k+1}^2} \leq c_{k+1}$.*

Proof. As $\mathcal{D}$ is an elliptic operator, it has an elliptic estimate: for some C_k,

$$\|\gamma\|_{L_{k+1}^2} \leq C_k\Big(\|D\gamma\|_{L_k^2} + \|\gamma\|_{L_k^2}\Big). \tag{21.12}$$

Since γ is a solution to the Seiberg-Witten equations,

$$\begin{aligned} \|\gamma\|_{L_{k+1}^2} &\leq C_k\Big(\|q(\gamma) + c\|_{L_k^2} + \|\gamma\|_{L_k^2}\Big) \\ &\leq C_k\Big(\|q(\gamma)\|_{L_k^2} + C'_k + \|\gamma\|_{L_k^2}\Big) \\ &\leq C_k\Big(\|q(\gamma)\|_{L_k^2} + C' + c_k\Big) \\ &\leq C''_k\Big(1 + \|q(\gamma)\|_{L_k^2}\Big). \end{aligned}$$

Using Lemma 21.3,

$$\leq C''(1 + C''' \underbrace{\|\gamma\|_{L_k^2}^2}_{\leq c_k^2}).$$

⊠

Corollary 21.13. *If $\{\gamma_j = (a_j, \phi_j)\}_j$ is a sequence of solutions to $\mathcal{F}'_\eta = 0$ converging to (a, ϕ) in L_3^2, then (a, ϕ) is C^∞.*

Proof. It suffices to show (a, ϕ) is C^ℓ for every ℓ. This follows because for $k \gg \ell$ (the precise value depending on the Sobolev embedding theorem), $\|\psi\|C^\ell \leq k_{k,\ell}\|\psi\|_{L_k^2}$ for some $k_{k,\ell}$. ⊠

Next we're going to need some L_1^p bounds. Our *a priori* bounds from two lectures ago tell us that there's a κ (depending on X, g, and η) such that if $\mathcal{F}_\eta(A, \phi) = 0$, then $\|\phi\|_{C^0} < \kappa$ and $\|F(A^0)^+\|_{C^0} < \kappa$.

If we write $A = A_0 + a \cdot \mathrm{id}$ and suppose that $\mathcal{F}'_\eta(a, \phi) = 0$, then $\mathrm{d}^*a = 0$. We can decompose $a = a_{\mathrm{harm}} + a'$, where a_{harm} is harmonic and $a \in \mathrm{Im}(\mathrm{d}^*)$.

Lemma 21.14. *With notation as above, for all $p > 1$, there's a κ_p depending on X, g, η, and A_0, such that for all (A, ϕ) such that $\mathcal{F}_\eta(A, \phi) = 0$, $\|a'\|_{L_1^p} \leq \kappa_p$.*

Proof. Since

$$F(A^0)^+ = F(A_0^0)^+ + 2\mathrm{d}^+a, \tag{21.15}$$

then by our *a priori* estimates, $\|\mathrm{d}^+a\|_{C^0} \leq \kappa'$ for some κ'.

Now we'll use the sharp elliptic estimate for the elliptic differential operator $\mathrm{d}^* \oplus \mathrm{d}^+$. This is the only time we'll need the stronger version. Specifically, for $b \in (\mathcal{H}^1)^{\perp, L^2}$ (hence orthogonal to $\ker(\mathrm{d}^* \oplus \mathrm{d}^+)$),

$$\|b\|_{L_1^p} \leq C\|(\mathrm{d}^* \oplus \mathrm{d}^+)b\|_{L^p}. \tag{21.16}$$

Hence for $a' \in \mathrm{Im}(\mathrm{d}^*) \subset (\mathcal{H}^1)^\perp$,

$$\|a'\|_{L^p_1} \le C\|\mathrm{d}^+a'\|_{L^p} = C\|\mathrm{d}^+a\|_{L^p} \le C\kappa'\mathrm{vol}(X)^{1/p}. \tag{21.17}$$ ⊠

This is the point where Seiberg-Witten theory becomes much nicer than Donaldson theory: in the latter we only have an estimate like this for $p = 1$, and for the former we have estimates for all p. When Witten introduced the Seiberg-Witten equations to mathematical gauge theorists, this was the point where they recognized everything would be easier — much of the rest of the story is similar to other elliptic estimates, and researchers in the field could figure it all out very quickly.

Now we need to go from L^p_1 bounds to L^2_3 convergence. These are of course very different things; we'll pass through the intermediate step of L^2_1 convergence, showing L^p_1 bounds prodice L^2_1 convergence, then use that to get L^2_3 convergence.

Lemma 21.18. *Let $\{\gamma_j = (a_j, \phi_j)\}_j$ be a sequence of solutions to $\mathcal{F}'_j = 0$ converging in L^2_1 to an L^2_1 limit $\gamma = (a, \phi)$. Then $\gamma_j \to \gamma$ in L^2_3.*

Proof. We will use the bounded multiplication maps in Lemma 21.5 to successively imprve from L^2_1 to L^3_1 to L^2_2 to L^2_3.

Using the elliptic estimate for $\mathcal{D}$,

$$\|\gamma_i - \gamma_j\|_{L^3_1} \le C(\|\mathcal{D}\gamma_i - \mathcal{D}\gamma_j\|_{L^3} + \|\gamma_i - \gamma_j\|_{L^3}). \tag{21.19}$$

Then, since γ_i and γ_j are both solutions to the Seiberg-Witten equations,

$$= C(\|q(\gamma_i) + c - q(\gamma_j) - c\|_{L^3} + \|\gamma_i - \gamma_j\|_{L^3}). \tag{21.20}$$

Since q is quadratic, there's a bilinear form b such that $q(\gamma) = b(\gamma, \gamma)$, and therefore

$$q(\gamma_i) - q(\gamma_j) = b(\gamma_i - \gamma_j, \gamma_i + \gamma_j). \tag{21.21}$$

Since the γ_i converge in L^2_1, then for all $\varepsilon > 0$ there's an i_0 such that for all $i \ge i_0$, $\|\gamma_i - \gamma_{i_0}\|_{L^2_1} \le \varepsilon$. Since

$$b(\gamma_i - \gamma_j, \gamma_i + \gamma_j) = b(\gamma_i - \gamma_j, \gamma_i + \gamma_j - 2\gamma_{i_0}) + 2b(\gamma_i - \gamma_j, \gamma_{i_0}), \tag{21.22}$$

then

$$\|q(\gamma_i) - q(\gamma_j)\|_{L^3} \le \|b(\gamma_i - \gamma_j, \gamma_i + \gamma_j - 2\gamma_{i_0})\|_{L^3} + 2c\|\gamma_{i_0}\|_{C^\infty} \cdot \|\gamma_i - \gamma_j\|_{L^3}.$$

Using Sobolev multiplication,

$$\begin{aligned} &\le C\Big(\|\gamma_i - \gamma_j\|_{L^3_1} \cdot \|\gamma_i + \gamma_j - 2\gamma_{i_0}\|_{L^2_1} + \|\gamma_{i_0}\|_{C^0} \cdot \|\gamma_i - \gamma_j\|_{L^3}\Big) \\ &\le C\Big(2\varepsilon\|\gamma_i - \gamma_j\|_{L^3_1} + \|\gamma_{i_0}\|_{C^0} \cdot \|\gamma_i - \gamma_j\|_{L^3}\Big). \end{aligned}$$

Thus

$$\|\gamma_i - \gamma_j\|_{L^3} \le C\Big(2\varepsilon\|\gamma_i - \gamma_j\|_{L^3_1} + (1 + \|\gamma_{i_0}\|_{C^0})\|\gamma_i - \gamma_j\|_{L^3}\Big). \tag{21.23}$$

Rearranging,

$$(1 - 2\varepsilon C)\|\gamma_i - \gamma_j\|_{L^3_1} \le C(1 + \|\gamma_{i_0}\|_{C^0})\|\gamma_i - \gamma_j\|_{L^3}. \tag{21.24}$$

If we choose $\varepsilon = 1/4C$, so $1 - 2\varepsilon C = 1/2$, we conclude that

$$\|\gamma_i - \gamma_j\|_{L^3_1} \le C\|\gamma_i - \gamma_j\|_{L^3} \le C'\|\gamma_i - \gamma_j\|_{L^2_1} \tag{21.25}$$

via the embedding $L^2_1 \hookrightarrow L^3$. Therefore (γ_j) is Cauchy in L^3_1. A similar argument gets us to L^2_2, then to L^2_3, and we're done. ⊠

Thus, if our sequence converges in L^2_1, it converges in C^∞, and $\|a'\|_{L^p_1}$ is bounded.

The last step is to show L^2_1 convergence. If we assume a_{harm} is bounded, then we in fact get bounds for $\|a\|_{L^p_1}$ for all p.

Proof sketch of the compactness theorem. We need to produce a subsequence converging in L_1^2. Boundedness is easy: we have an L_1^2 bound on a_j, and it's not too hard to produce a bound on ϕ_j (using the bound on $(a_j)_{\mathrm{harm}}$), so $\|\gamma_j\|_{L_1^2}$ is bounded above.

Given a bounded sequence in a separable Hilbert space, we know a subsequence converges weakly, i.e. there's a $\gamma \in L_1^2$ such that $\langle \gamma - \gamma_j \rangle_{L_1^2} \to 0$ for all β. If we can show that

$$\|\gamma\|_{L_1^2} = \limsup \|\gamma_j\|_{L_1^2}, \tag{21.26}$$

then this would be strong convergence, so we're going to prove (21.26).

We proved using Chern-Weil theory that

$$\|F(A_j^0)\|_{L^2}^2 = -4\pi^2\langle c_1(\mathfrak{s})^2, [X]\rangle + 32\|F(A_j^0)^+\|_{L^2}^2. \tag{21.27}$$

We have a pointwise bound on $\|F(A_j^0)\|$, which leads to an L^2 bound and then to similar arguments to show that γ has the correct norm. ⊠

Lecture 22.

Transversality: 4/17/18

Last time, we discussed compactness of the Seiberg-Witten solution space; the professor's lecture notes are slightly more streamlined, and contain a slightly stronger result. Today we discuss transversality, which ensures the solution space is a manifold.

Let (X, g) be a Riemannian 4-manifold with a spin^c structure $\mathfrak{s}$. Then, the solutions to the Seiberg-Witten equations $(A, \phi) \in \mathcal{A}_{\mathrm{cl}}(\mathbb{S}^+) \times \Gamma(\mathbb{S}^+)$ form a space which we'll denote $C(\mathfrak{s})$, and we'll let $C(\mathfrak{s})_k$ denote those of class L_k^2. We will only take $k \geq 4$, so that solutions are continuous.

The gauge group $\mathcal{G}_{k-1}$ acts continuously on $C(\mathfrak{s})_k$; in what follows, we will let $C(\mathfrak{s})$ denote $C(\mathfrak{s})_k$ and $\mathcal{G}$ denote $\mathcal{G}_{k-1}$. There are two kinds of orbits for the $\mathcal{G}$-action on $C(\mathfrak{s})$.

- The *irreducible locus* $C^{\mathrm{irr}}(\mathfrak{s}) := \{(A, \phi) \mid \phi \text{ is not identically } 0\}$. $\mathcal{G}$ acts freely on these: if $u \in \mathcal{G}$ and $u^*A = A$, then u is constant, and if $u \neq 1$, then $u\phi \neq \phi$.
- The *reducible locus* is everything else, $C^{\mathrm{red}}(\mathfrak{s}) := \{(A, 0)\}$. The stabilizer of the $\mathcal{G}$-action is the subgroup of constant gauge transformations, which is identified with U_1.

Given an $x \in X$, we can define the *based gauge group* $\mathcal{G}^x$, the subgroup of gauge transformations u such that $u(x) = 1$. $\mathcal{G}^x$ acts freely on $C(\mathfrak{s})$, and in fact $\mathcal{G} = \mathcal{G}^x \times \mathrm{U}_1$. Here $k \geq 4$ is important: we need continuity of solutions for this to make sense.

U_1 acts *semi-freely* on $C(\mathfrak{s})/\mathcal{G}^x$; that is, on each orbit it acts either freely or trivially. It acts freely on the irreducible locus, and trivially on the reducible locus.

Remark 22.1. This is another reason Seiberg-Witten theory is easier than nonabelian instanton theory: the stabilizer U_1 is much easier to deal with, rather than groups such as SU_2 which appear in other theories. ◂

Now let's look at reducible solutions $(A, 0)$. As usual, we may have a nonzero η, and use Coulomb gauge-fixing. Recalling our study of U_1-instantons, with a minor adaptation to allow for η, and recalling our study of the self-dual complex, $\eta = \eta_{\mathrm{harm}} + \eta'$, where η_{harm} is harmonic and $\eta' \in \mathrm{Im}(\mathrm{d}^+)$, and one can prove the following proposition.

Proposition 22.2. *There exists a reducible solution to $\mathcal{F}'_\eta = 0$ iff*

$$c_1(\mathfrak{s}) + \frac{1}{\pi}[\eta_{\mathrm{harm}}] \in \mathcal{H}^-_{[g]} \subset H^2_{\mathrm{dR}}(X). \tag{22.3}$$

If this is true, the reduced solutions modulo $\mathcal{G}^x$ are a torsor for the Picard torus $H^1(X;\mathbb{R})/H^1(X;\mathbb{Z})$.

See, for example, (12.1).

Proposition 22.2, combined with generic nonexistence for U_1-instantons (Theorem 12.5), yields the following.

Theorem 22.4. *Suppose $b^+(X) > 0$, $[g]$ is a conformal structure on X, and $\eta \in \Omega^2_{[g]}$. Then $[g]$ can be approximated in C^r by C^r-conformal structures $[g_i]$ for a fixed r, such that for each $[g_i]$, there are no reducible solutions to the Seiberg-Witten equations for any spin^c structure on X.*

Remark 22.5. With some harder work, one can replace C^r with C^∞. ◂

This is telling us that reducible solutions generically do not exist. Again, the intuition is that we asked for a line in a real vector space to intersect an integer lattice.

Definition 22.6. Let $\mathfrak{s}$ be a spin^c structure on X. Denote by $\mathcal{W}(\mathfrak{s})$ the pairs $([g], \eta)$ where $[g]$ is a conformal structure on X and $\eta \in \Omega^2_{[g]}$ such that reducible solutions to the Seiberg-Witten solutions exist for $[g]$ and η,

The assignment $([g], \eta) \mapsto [g]$ defines a map $\mathcal{W}(\mathfrak{s}) \to \mathrm{Conf}_X$, and the fiber at a given $[g]$ is a $B^+(X)$-dimensional affine subspace of $\Omega^+_{[g]}$, since it's cut out by an affine equation (22.3); therefore $\mathcal{W}(\mathfrak{s}) \to \mathrm{Conf}_X$ is an affine bundle, a subbundle of the bundle of pairs $([g], \eta)$. In particular, when $b^+(X) = 1$, $\mathcal{W}(\mathfrak{s})$ is a codimension-1 subbundle, and in this setting we'll call it the *wall*. Its complement has two connected components, which we'll call *chambers*.[29]

We had a transversality result for U_1-instantons, which implies the following result.

Proposition 22.7. *Suppose $b^+(X) > 0$ and fix $([g_0], \eta_0)$ and $([g_1], \eta_1)$ not in $\mathcal{W}(\mathfrak{s})$.*

(1) *If $b^+(X) > 1$, any interpolating path $([g_t], \eta_t)$ can be approximated by one which avoids $\mathcal{W}(\mathfrak{s})$.*
(2) *If $b^+(X) = 1$, any interpolating path $([g_t], \eta_t)$ can be approximated by one transverse to $\mathcal{W}(\mathfrak{s})$.*

Remark 22.8. We haven't seen any irreducible solutions yet! They do exist, and we'll talk about them in just a second, but one cool fact about them is that in many contexts, you can use a *wall-crossing formula* to start with a reducible solution and obtain an irreducible one. This was used by Kronheimer-Mrowka to prove Thom's conjecture. ◀

We now turn to transversality of irreducible solutions. We'll need one analytic theorem which we quote without proof, though you can read it in Donaldson's book.

Theorem 22.9 (Unique continuation)**.** *Let L be a linear elliptic operator over a connected manifold X. Suppose $Lu = 0$ and u is identically 0 on an open $U \subset L$. Then u is identically 0.*

We'll allow η to vary; for a given η, write $\eta = \omega + \eta'$, where $\omega = \eta_{\mathrm{harm}}$ is harmonic and $\eta' \in \mathrm{Im}(\mathrm{d}^+)$. More specifically, we'll leave ω fixed and let η' vary. In this setting we have a *parameterized Seiberg-Witten map*

$$\mathcal{F}^{\mathrm{par}}_\omega \colon \mathrm{Im}(\mathrm{d}^+) \times L^2_k(iT^*X) \times L^2_k(\mathbb{S}^+) \longrightarrow L^2_{k-1}(X)_0 \times L^2_{k-1}(i\mathfrak{su}(\mathbb{S}^+)) \times L^2_{k-1}(\mathbb{S}^-) \tag{22.10}$$

defined by

$$\mathcal{F}^{\mathrm{per}}_\omega(\eta', a, \phi) \coloneqq (\mathrm{d}^* a, \mathcal{F}'_{\omega+\eta'}(a, \phi)). \tag{22.11}$$

Theorem 22.12. *If $\mathcal{F}^{\mathrm{per}}_\omega(\eta', a, \phi) = 0$, then $D_{\eta', a, \phi}\mathcal{F}^{\mathrm{per}}_\omega$ is surjective.*

This implies the parameteric space $\{\mathcal{F}^{\mathrm{per}}_\omega = 0\}$ is cut out transversely. It's not finite-dimensional, but that's OK for now.

Proof. Let $D \coloneqq D_{(\eta', A, \phi)}\mathcal{F}^{\mathrm{per}}_\omega$. Let $\delta \in \mathrm{Im}(\mathrm{d}^+)$, $a \in \Omega^1(X)$, and χ be a spinor; then

$$D \begin{bmatrix} \delta \\ a \\ \chi \end{bmatrix} = \begin{bmatrix} \mathrm{d}^* a \\ \rho(\mathrm{d}^+ a + 2i\delta) - (\chi\phi^* + \phi\chi^*)_0 \\ D^+_A \chi + (1/2)\rho(a)\phi \end{bmatrix}. \tag{22.13}$$

This is a Fredholm linear map of Hilbert spaces $D \colon L^2_k \to L^2_{k-1}$. We claim that the L^2 orthogonal complement to $\mathrm{Im}(D)$ is $\{0\}$. This will imply that $\mathrm{Im}(D)$ is dense in L^2 and dense in L^2_{k-1}. Because D is Fredholm, its image is a closed subspace, so its image is everything.

Now let's prove the claim. Take $(f, \alpha, \psi) \in (\mathrm{Im}\, D)^\perp$, so f is a function, α is a 1-form, and ψ is a spinor. Plugging into (22.13),

$$D \begin{bmatrix} \delta \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2i\rho(\delta) \\ 0 \end{bmatrix}. \tag{22.14}$$

Hence for all δ, $\langle \rho(\delta, \alpha\rangle_{L^2} = 0$.

If δ were allowed to be an arbitrary self-dual 2-form, then $\rho(\delta)$ could be arbitrary in $\Gamma(i\mathfrak{su}(\mathbb{S}^+))$, but our assumption that $\delta \in \mathrm{Im}(\mathrm{d}^+)$ amounts to a finite-codimension linear constraint on $\rho(\delta)$. Thus in fact $\alpha = 0$.

[29] This is evident fiberwise; it's also true globally, but this is not immediate. One approach would be to think about metrics instead of conformal structures, for which it's more evident.

Using (22.13) again,

$$D\begin{bmatrix}0\\a\\0\end{bmatrix} = \begin{bmatrix}\mathrm{d}^*a\\ \rho(\mathrm{d}^+a)\\ (1/2)\rho(a)\phi\end{bmatrix}. \tag{22.15}$$

Taking $a = \mathrm{d}^*b$, $\mathrm{d}^*a = (\mathrm{d}^*)^2 b = 0$, so

$$D\begin{bmatrix}0\\ \mathrm{d}^*b\\0\end{bmatrix} = \begin{bmatrix}0\\ \rho(\mathrm{d}^+\mathrm{d}^*b)\\ (1/2)\rho(\mathrm{d}^*b)\phi\end{bmatrix}. \tag{22.16}$$

Since $\rho(\mathrm{d}^*b)\phi \perp \psi$ for arbitrary b, $D(0, \mathrm{d}^*b, 0) \perp (f, 0, \psi)$.

Since (A, ϕ) is irreducible, there's an $x \in X$ such that $\phi(x) \neq 0$, so we can produce a b such that $\rho(\mathrm{d}^*b)\phi(x) \neq 0$, or better yet, can choose b_1, b_2 such that

$$(\rho(\mathrm{d}^*b_1)\phi(x), \rho(\mathrm{d}^*b_2)\phi(x)) \tag{22.17}$$

is an orthonormal basis for $\mathbb{S}^+_x$. By choosing b_i supported near x, we conclude $\psi = 0$ in an open neighborhood of X. Since

$$D\begin{bmatrix}0\\0\\ \chi\end{bmatrix} = \begin{bmatrix}0\\ -(\phi\chi^* + \chi\phi^*)\\ D_A^+\chi\end{bmatrix}, \tag{22.18}$$

and therefore for all χ, $\langle D_A^+\chi, \psi\rangle_{L^2} = 0$, i.e. $\langle \chi, D_A^-\psi\rangle_{L^2} = 0$, and therefore $D_A^-\psi = 0$. Since $\psi = 0$ on an open set, Theorem 22.9 implies $\psi = 0$.

Now we know $\alpha = 0$ and $\chi = 0$, so conclude that $\langle f, \mathrm{d}^*a\rangle_{L^2} = 0$ for all a. Since this is also $\langle \mathrm{d}f, a\rangle_{L^2}$, then $\mathrm{d}f = 0$, so f is constant. We assumed f is mean-zero, so $f = 0$. ⊠

The inverse function theorem for Banach spaces now implies solutions to $\mathcal{F}^{\mathrm{per}}_\omega = 0$, and they form a smooth infinite-dimensional submanifold of the domain. Next time we'll map this to the space of parameters and see what this says about the Seiberg-Witten moduli space.

We're closing in on the point where we can say something nontrivial about 4-manifolds; in particular, we're only a few lectures out from a proof of the diagonalization theorem.

Lecture 23.

Generic transversality for irreducible Seiberg-Witten solutions: 4/19/18

We continue our discussion of transversality in the case of irreducible solutions (A, ϕ) (i.e. ϕ isn't identically zero). We fixed a g-self-dual harmonic form ω and want to verify transversality of solutions to $\mathcal{F}'_{\omega+\eta'} = 0$ when η' is a generic element of $\mathrm{Im}(\mathrm{d}^+)$.

We then rephrased this in parametric terms, defining $\mathcal{F}^{\mathrm{per}}_\omega(\eta', a, \phi) := \mathcal{F}'_{\omega+\eta'}(a, \phi)$, where $A = A_0 + a$ for a reference connection A_0. In Theorem 22.12 we showed that 0 is a regular value for $\mathcal{F}^{\mathrm{per}}_\omega$ restricted to irreducible solutions. This already incorporates the Coulomb condition $\mathrm{d}^*a = 0$, and the codomain was set up such that d^*a is the mean-zero functions.

The condition $\mathcal{F}'_{\omega+\eta'}(a, \phi) = 0$ is equivalent to

$$\mathcal{F}'_\omega(a, \phi) = \begin{bmatrix}0\\ \eta'\\ 0\end{bmatrix}, \tag{23.1}$$

so we've shown that $(0, \eta', 0)$ is a regular value for $\mathcal{F}'_\omega$.

Here's our schematic argument: let U, V, and P be Banach spaces. This might seem like strange notation, but P will be a parameter space. Let $F = (f_1, f_2)\colon U \to V \times P$ be a smooth map, and for all $p \in P$, let $M_p := F^{-1}(0, p)$. (For us, $F = \mathcal{F}'_\omega$, $P = \mathrm{Im}(\mathrm{d}^+)$, and $p = \eta'$.)

We want to show that M_p is cut out transversely, i.e. that $(0, p)$ is regular for F, for "generic" $p \in P$. To do this, we let $F^{\mathrm{per}}\colon U \times P \to V \times P$ send

$$(x, p) \longmapsto (f_1(x), f_2(x) - p). \tag{23.2}$$

Suppose we can show $(0,0)$ is a regular value for F^{per} and let M^{per} be the preimage of $(0,0)$. Then the inverse function theorem for Banach spaces implies M^{per} is a Banach submanifold of $U \times P$, and there's a smooth projection $\Pi\colon M^{\mathrm{per}} \to P$ with $M_p \coloneqq \Pi^{-1}(p)$.

Lemma 23.3. *If $F(x) = (0,p)$ (i.e. $F^{\mathrm{per}}(x,p) = (0,0)$), then*

1. $\ker(D_x F) = \ker(D_{(x,p)}\Pi)$,
2. $\operatorname{coker}(D_x F) \cong \operatorname{coker}(D_{(x,p)}\Pi)$, *and*
3. *therefore $(0,p)$ is a regular value for F iff p is a regular value for Π.*

Proof. Part (1) is a matter of unwinding definitions. Part (2) is a bit more interesting. Since M^{per} is cut out transversely, then given $(x,p) \in M^{\mathrm{per}}$ and a tangent vector (v,q) at $F(x,p)$, the equations

$$\begin{aligned}(D_x f_1)(\dot{x}) &= v\\ (D_x f_2)(\dot{x}) - \dot{p} &= q\end{aligned} \tag{23.4}$$

have a solutions $(\dot{x},\dot{p})$. This is ultimately because $D_x f_1$ surjects. In general, if $L_1\colon U \to V$ and $L_2\colon U \to P$ are linear with L_1 surjective, then the inclusion $P \hookrightarrow V \times P$ induces an isomorphism

$$P/L_1(\ker(L_1)) \xrightarrow{\cong} \operatorname{coker}(L_1, L_2) = (V \times P)/(\operatorname{Im}(L_1, L_2)), \tag{23.5}$$

which is a quick linear-algebraic check. Taking $L_i = D_x f_i$, we conclude

$$\operatorname{coker} D\Pi = P/(Df_2(\ker(Df_1))) \cong \operatorname{coker} DF. \tag{23.6}$$

⊠

So finding transverse M_p is equivalent to finding regular values of Π. And how do we find regular values? Sard's theorem.

Theorem 23.7 (Sard's theorem (version 1))**.** *Let $f\colon M \to N$ be a C^∞ map of finite-dimensional manifolds. Then, the critical values of f are measure zero in N; equivalently, the regular values are full measure.*

Remark 23.8. You don't need very much measure theory to make sense of this statement; choosing a countable atlas for N, we can ask for the image of the critical values in each chart to be measure zero in the Lesbegue measure in $\mathbb{R}^n$, i.e. it can be covered by balls with total volume less than ε, for any $\varepsilon > 0$. Since a countable union of measure-zero sets remains measure-zero, this doesn't go wrong on the manifold, and though measures are not invariant under changes of charts, the notion of measure-zero is. ◂

The problem is that in infinite-dimensions, the statement is meaningless, because "measure-zero" is meaningless. Smale provides us a solution.

Definition 23.9. Let T be a space and $S \subset T$. Then T is a *Baire* (or *residual*, or sometimes *second-category*) subset if it's the intersection of countably many open dense subsets.

Theorem 23.10 (Baire category theorem)**.** *If the topology on T comes from a complete metric, then Baire subsets are dense.*

The notion of genericity we will use is that of being a Baire subset. The countable intersection of Baire subspaces is again Baire, by definition, which is a very useful fact: you can impose countably many generic conditions and points such that all of them are true are still generic.

Theorem 23.11 (Sard's theorem (version 2))**.** *With notation as in Theorem 23.7, the regular values of f are a Baire subspace of N, hence dense.*

Even though N might not have a topology induced by a complete metric, we can check density locally, and locally N looks like $\mathbb{R}^n$, which is complete.

Remark 23.12. Though it's not possibly to directly deduce either of Theorems 23.7 and 23.11 from each other, the proof ideas of both are similar to the point where you could write one down after looking at the other. ◂

In particular, if U, V, and P are finite-dimensional, we get M_p cut out transversally for a Baire (hence dense) set of parameters $p \in P$. This does not generalize to infinite dimensions, though Banach spaces are admit complete metrics. This is good for $L^p_k(E)$ and $C^r(E)$, given a vector bundle $E \to M$ with a Euclidean metric, and Fréchet spaces also admit complete metrics, so we also get spaces like $C^\infty(E)$. Therefore the Baire category theorem applies to Fréchet spaces and even Fréchet manifolds.

Definition 23.13. Let Y and Z be (second-countable) Banach manifolds. A C^∞ map $\Phi\colon Y \to Z$ is called *Fredholm* if $D_y\Phi\colon T_yY \to T_{\Phi(y)}Z$ is a Fredholm map of Banach spaces for all $y \in Y$.

Recall that a linear map is Fredholm iff its kernel and cokernel are finite-dimensional.

Theorem 23.14 (Sard-Smale)**.** *If $\Phi\colon Y \to Z$ is a Fredholm map of Banach manifolds, its regular values are a Baire subspace of Z.*

We're going to prove this at some point, because it's vital. The proof is particularly interesting, constructing finite-dimensional approximations to Φ, and the theorem applies in plenty of other cases of elliptic PDE in geometry.

Remark 23.15. There are versions of Sard's theorem and the Sard-Smale theorem for C^r regularity, though they do impose some additional conditions. ◀

Now we return to $F\colon U \to V \times P$. Let's assume F is Fredholm. Lemma 23.3 means that Π is also Fredholm.[30] Then Theorem 23.14 gives us a Baire set of parameters $p \in P$ such that M_p is transverse. Thus we conclude:

Theorem 23.16. *For a Baire subspace B of $\mathrm{Im}(\mathrm{d}^+)$ and $\eta' \in B$, the space of irreducible solutions to $\mathcal{F}'_{\omega+\eta'} = 0$ is cut out transversely.*

For such η', let $\eta \coloneqq \omega + \eta$ and define

$$\widetilde{M}_\eta^{\mathrm{irr}} \coloneqq \{(a,\phi) \in C(\mathfrak{s})^{\mathrm{irr}} \mid \mathcal{F}'_\eta(a,\phi) = 0\}. \tag{23.17}$$

U_1 acts freely on $\widetilde{M}_\eta^{\mathrm{irr}}$ by constant gauge transformations, and we'll let M_η^{irr} denote the quotient. It has dimension

$$\dim M_\eta^{\mathrm{irr}} = d(\mathfrak{s}) = \frac{1}{4}\big(\langle c_1(\mathfrak{s})^2, [X]\rangle \cdot 2\chi - 3\tau\big), \tag{23.18}$$

and $\dim \widetilde{M}_\eta^{\mathrm{irr}} = \dim M_\eta^{\mathrm{irr}} + 1 = d(\mathfrak{s}) + 1$.

If you're comparing this with the index-theory calculation we made, keep in mind that we took $\mathrm{d}^*\colon \Omega^1 \to \Omega^0$ there, but here we take the codomain to be the mean-0 functions.

Corollary 23.19. *Let $b^+(X) > 0$. Then, for generic (i.e. a Baire subspace of) metrics g and generic g-self-dual 2-forms $\eta' \in \mathrm{Im}(\mathrm{d}^+)$, if ω_g is the g-self-dual harmonic representative of a fixed $w \in H^2_{\mathrm{dR}}(X)$, then the Seiberg-Witten moduli space $\widetilde{M}_{\omega_g+\eta'} \coloneqq \{\mathcal{F}'_{\omega_g+\eta'} = 0\}$ consists only of irreducible solutions, and is a compact $(d(\mathfrak{s})+1)$-dimensional manifold admitting a free U_1-action by constant gauge transformations.*

When $b^+(X) = 1$, there are the two chambers which we discussed last time, and hence two possible values of the Seiberg-Witten invariant. This relates to the fact that we haven't dealt with all of the gauge transformations, just $\pi_0\mathcal{G}$.

In the rest of the lecture, we'll discuss the Sard-Smale theorem. Let $\Phi\colon X \to Y$ be a smooth Fredholm map between Banach manifolds (always assumed to be second-countable). To make minimal choices, we use *tangential charts*, i.e. embeddings $T' \to Z$ where T' is an open subspace of T_zZ for some z. Let $i'\colon T' \to Z$ be such a chart near $z = \Phi(y)$, so $i'(0) = \Phi(y)$ and $D_0 i' = \mathrm{id}$, and do the same for $i\colon T \to Y$ near y. Thus, viewed in these charts, Φ passes to a map $\widetilde{\Phi}\colon T \to T'$ such that $D_0\widetilde{\Phi} = D_y\Phi$.

Let $K \coloneqq D_y\Phi \subset T_yY$, $V \coloneqq \mathrm{Im}\, D_y\Phi \subset T_{\Phi(y)}Z$, and choose complementary subspaces U, C such that

$$T_yY = U \oplus K \tag{23.20}$$

$$T_{\Phi(y)}Z = V \oplus C. \tag{23.21}$$

Hence K and C, the kernel and cokernel respectively, are finite-dimensional.

We've now rephrased the problem to considering a map $\widetilde{\Phi}\colon U \oplus K \to V \oplus C$ with $\widetilde{\Phi}(0) = 0$.

[30]There's one additional detail, that the image of Π is closed, but this follows because $D\Pi$ is a projection. This is also redundant if one works with Hilbert spaces.

Lemma 23.22. *Given the choices of U, C, and the tangential chart near y, there exists a tangential chart near $\Phi(y)$ such that $\widetilde{\Phi}\colon U \oplus K \to V \oplus C$ is of the form*

$$\widetilde{\Phi}(x, k) = (Lx, \phi(x, k)), \tag{23.23}$$

where $L := D_y\Phi|_U : U \to V$ is a linear isometry.

The proof is unenlightening to hear in lecture, and wasn't given. It uses the inverse function theorem for Banach spaces. Nonetheless, this local form for Fredholm maps turns out to be useful.

From this lemma, one deduces Φ is locally closed, and on a chart T, the regular values of Φ are exactly those of $\phi|_K : K \to C$, which is a smooith map of finite-dimensional manifolds. Then one can use the Baire version of Sard's theorem.

Lecture 24.

The diagonalization theorem: 4/24/18

We're now in the position to prove interesting things about 4-manifolds.

Theorem 24.1 (Diagonalization theorem)**.** *Let X be a closed, oriented 4-manifold with positive-definite intersection form Q_X. Then Q_X is diagonalizable over $\mathbb{Z}$, i.e. $Q_X \cong \langle 1\rangle^{b_2(X)}$.*

With an assumption on $\pi_1(X)$, this is due to Donaldson, in a very pretty application of the Yang-Mills equations to topology. We'll prove the full theorem, without restrictions on π_1, using Seiberg-Witten theory; this is due to Kronheimer-Mrowka.

Remark 24.2.

- It's equivalent to prove that negative definite implies diagonalizable for Q_X.
- We can reduce to $b_1(X) = 0$: if $b_1(X) > 0$, there's a 4-manifold Y with $b_1(Y) < b_1(X)$ and $Q_Y \cong Q_X$. Use *surgery* to explicitly construct such a Y: if $h \in H_1(X;\mathbb{Z})$ is nontorsion and *primitive* (not a multiple of another class except by ± 1), then let γ be an embedded loop representing h. If N_γ denotes a tubular neighborhood of γ, it's diffeomorphic to $S^! \times D^3$, with $\gamma = S^1 \times \{0\}$ inside it.
  Let $X^0 := X \setminus N_\gamma$, so that $\partial X_0 \cong S^1 \times S^2$. Since $\partial(D^2 \times S^2) \cong S^1 \times S^2$, we can let
  $$X_\gamma := X^0 \cup_{S^1\times S^2} (D^2 \times S^2), \tag{24.3}$$
  and by construction, $[\gamma] = 0$ in X_γ. Moreover, one can show that $H_1(X_\gamma) = H_1(X)/[\gamma]$, so the Betti number decreases by 1, and $Q_{X_\gamma} \cong Q_X$. ◀

Hence we can assume $b_1(X) = 0$ and $b^+ = 0$, so that Q_X is negative definite. We use this version because $\mathrm{d}^* \oplus \mathrm{d}^+ : \Omega^1(X) \to \Omega^0(X) \oplus \Omega^+(X)$ has kernel $\mathcal{H}^1(X) = 0$ and cokernel $\mathbb{R} \oplus \mathcal{H}^+ = \mathbb{R}$ in this case, and if we land in mean-0 functions, the cokernel is zero. Since this operator appears all the time in Seiberg-Witten theory, this is quite a helpful simplifying assumption.

Fix a spin^c structure $\mathfrak{s}$ on X and let $c := c_1(\mathfrak{s})$. The index of the Dirac operator $D_A^+ : \Gamma(\mathbb{S}^+) \to \Gamma(\mathbb{S}^-)$ is

$$\operatorname{ind} D_A^+ = \frac{1}{4}\big(\langle c^2, [X]\rangle - \tau_X\big) = \frac{1}{4}\big(\langle c^2, [X]\rangle + b_2\big). \tag{24.4}$$

Therefore

$$d(\mathfrak{s}) = \frac{1}{4}\big(\langle c^2, [X]\rangle + b_2\big) - 1. \tag{24.5}$$

In particular, $d(\mathfrak{s})$ is odd: c is characteristic, so $\langle c^2, [X]\rangle \equiv \tau \bmod 8$.

We've shown that for generic η, $\widetilde{M}_\eta^{\mathrm{irr}}$ is cut out transversely of dimension $d(\mathfrak{s}) + 1$, and hence $M_\eta^{\mathrm{irr}}(\mathfrak{s})$ is a $d(\mathfrak{s})$-dimensional manifold. We fix such an η.

If $d(\mathfrak{s}) < 0$, then M_η^{irr} is empty, which is boring, so let's suppose $d(\mathfrak{s}) \geq 0$, i.e. $\langle c^2, [X]\rangle + b_2(X) > 0$. Let $d(\mathfrak{s}) = 2k - 1$.

Since $b^+ = b_1 = 0$, there is a unique gauge orbit of reducible solutions $[A_0, 0]$, with $F(A_0^0)^+ = 2i\eta$, and we can't get rid of it by perturbing η. In fact, it will be useful for us.

The full Seiberg-Witten moduli space $\widetilde{M}_\eta = \{[A_0, 0]\} \cup \widetilde{M}_\eta^{\mathrm{irr}}$ is a compact $2k$-manifold with a basepoint $R := [A_0, 0]$. The U_1-action is trivial on R and free everywhere else.

We're going to study the structure of $\widetilde{M}_\eta$ near R. Letting $D \coloneqq D_{(A_0,0)}\mathcal{F}'_\eta$,

$$D\begin{bmatrix} b \\ \chi \end{bmatrix} = \begin{bmatrix} \mathrm{d}^*b \\ \rho(\mathrm{d}^+b) \\ D^+_{A_0}\chi \end{bmatrix}, \tag{24.6}$$

as we've set the problem up such that everything else vanishes. Since $\mathrm{d}^*b \in L^2_{k-1}(X)_0$ (which follows from the divergence theorem), we can take the target to be mean-zero functions; then $\ker(D) = \ker(D^+_{A_0})$, and $\operatorname{coker} D = \operatorname{coker}(D^+_{A_0}) = \ker(D^-_{A_0})$.

Let's hypothesize that R is *regular*, i.e. $\operatorname{coker}(D) = 0$. We'll come back and justify this assumption later, but it has some impressive consequences: $\widetilde{M}_\eta$ is a $2k$-manifold even at R, and therefore irreducible solutions exist!

Since U_1 fixes R, it acts on $T_R\widetilde{M}_\eta = \ker(D^+_{A_0})$; this is the action of $\mathrm{U}_1 \subset \mathbb{C}$ by scamar multiplication on a complex vector space.

Lemma 24.7. *Let Q be a $2k$-manifold together with a U_1-action fixing some $q \in Q$. If the induced U_1-action on T_qQ has a single weight $N \in \mathbb{Z}$, i.e.*

$$T_qQ \cong \mathbb{R}^k \otimes \mathbb{C}^{\otimes N} \tag{24.8}$$

as a U_1-representation, where U_1 acts trivially on $\mathbb{R}$ and by multiplication on $\mathbb{C}$, then q has a local neighborhood equiariantly modeled on a neighborhood of 0 in T_qQ.

Proof. Averaging over U_1, we have a U_1-invariant metric g near q; then, such a chart is given by the exponential map for g at q. ⊠

The point is that R has a neighborhood modeled on $\mathbb{C}^k$, where U_1 acts by scalar multiplication. Therefore we can remove a small ball around R fixed by U_1, yielding a compact $2k$-manifold $\widetilde{N}$ with $\partial\widetilde{N} = S^{2k-1}$ with a free U_1-action restricting to the standard action on S^{2k-1}. Therefore the quotient N is a compact $(2k-1)$-manifold with boundary $\mathbb{CP}^{n-1}$.

If $k-1$ is even, this is a problem: $\mathbb{CP}^{k-1}$ has odd Euler characteristic, therefore cannot bound a compact manifold. This is because the mod 2 Euler characteristic is equal to the top Stiefel-Whitney number, which is an unoriented cobordism invariant. This is a contradiction.

But if we are in the case where k is even, then we have a principal U_1-bundle $\widetilde{N} \to N$, and its associated line bundle has a first Chern class restricting to ∂N as the boundary of $H^2(\mathbb{CP}^{k-1})$.

We haven't proven that any of these spaces are orientable (except $\mathbb{CP}^j$, I guess), so we're going to reduce to mod 2 coefficients, which behave better for unoriented manifolds. Since $w_2(E) = c_1(E)$ mod 2 for a complex vector bundle E, then

$$w_2(\widetilde{N})|_{\partial N} \neq 0 \in H^2(\mathbb{CP}^{k-1};\mathbb{Z}/2) = \mathbb{Z}/2, \tag{24.9}$$

and in particular, $(w_2|_{\partial N})^{k-1} \neq 0$, using the structure of the ring $H^*(\mathbb{CP}^{k-1};\mathbb{Z}/2)$. Therefore, if $i\colon \partial N \hookrightarrow N$ is inclusion,

$$\langle w_2^{k-1}, i_*[\partial N]\rangle = \langle (w_2|_{\partial N})^{k-1}, [\partial N]\rangle \neq 0. \tag{24.10}$$

But $i_*[\partial N] = 0$, so this is a contradiction too! The idea is that $\mathbb{CP}^{k-1}$ can bound, but the line bundle corresponding to the positive generator of $H^2(\mathbb{CP}^{k-1})$ cannot extend.

One of our hypotheses must fail – we'll show that regularity isn't the problem, so therefore there's no spin^c structure $\mathfrak{s}$ such that $\langle c_1(\mathfrak{s})^2, [X]\rangle + b_2(X) > 0$. That is, there's no characteristic vector c such that $\langle c^2, [X]\rangle + b_2 > 0$. Rephrasing, the positive-define, unimodular lattice Q_X of rank b_2 admits no "short" characteristic vectors c, i.e. those c for which $c^2 < b_2$.

For example, $-Q_X$ can't be even (since this would imply 0 is characteristic). But more can be extracted from this inequality: Kronheimer and Mrowka asked Noam Elkies, who provided the following answer.

Theorem 24.11 (Elkies)**.** *Let Λ be a positive-definite, rank-N, unimodular lattice. If all characteristic vectors c for Λ have $|c|^2 \geq N$, then $\Lambda \cong \langle 1\rangle^N$, i.e. it's diagonalizable.*

The diagonalizability theorem follows, though we haven't yet discussed regularity or the proof of Theorem 24.11. We don't have time for both, but we'll cover regularity.

We want to show that $D^+_{A_0}$ is surjective.

Lemma 24.12. *Fix A_0 and $k \geq 3$, and consider the* parametric Dirac operator

$$D^{\mathrm{par}} \colon L^2_l(iT^*X) \times L^2_k(\mathbb{S}^+) \longrightarrow L^2_{k-1}(\mathbb{S}^-) \tag{24.13a}$$

defined by

$$D^{\mathrm{par}}(a, \phi) \coloneqq D^+_{A_0+a}\phi = D^+_{A_0}\phi + \rho(a)\phi. \tag{24.13b}$$

Restricted to the domain where ϕ isn't identically zero, D^{par} has 0 as a regular value.

Proof. We compute

$$\begin{aligned}(D_{(a,\phi)}D^{\mathrm{part}})(b, \chi) &= D^+_{A_0}\chi + \rho(a)\chi + \rho(b)\phi\\ &= D^+_{A_0+a}\chi - \rho(b)\chi.\end{aligned} \tag{24.14}$$

To show this is surjective, we'll show that if ψ is L^2-orthogonal to its image, then $\psi = 0$.

Taking $b = 0$, we get that $\psi \perp D^+_{A_0+a}\chi$ for all χ, and therefore $\chi \in \ker(D^-_{A_0+a})$. By unique continuation, it therefore suffices to show ψ vanishes on an open set. Take $\chi = 0$ and ϕ not identically zero, so $\psi \perp \rho(b)\phi$ for all b, and there's an $x \in X$ with $\phi(x) \neq 0$, so that

$$\int_X \langle \rho(b)\phi, \psi\rangle \,\mathrm{vol} = 0 \tag{24.15}$$

for all b. Let U be a neighborhood of x on which ϕ doesn't vanish, so there's a b with $\rho(b)\psi = \psi$ on U. If σ is a cutoff function for U and $b' \coloneqq \sigma b$, then $\int \sigma|\psi|^2 = 0$, so $\psi|_U = 0$. ⊠

Therefore $M \coloneqq (D^{\mathrm{par}})^{-1}(0)$ is a Banach manifold, and the projection $\Pi \colon M \to L^2_k(iT^*X)$ sending $(a, \phi) \mapsto a$ is Fredholm. Its fibers are

$$\Pi^{-1}(a) = \{\phi \not\equiv 0 \mid D^+_{A_0+a}\phi = 0\} = \ker(D^+_{A_0+a}) \setminus 0. \tag{24.16}$$

The reguler values of Π are those for which $\mathrm{coker}(D^+_{A_0+a}) = 0$, which by Theorem 23.14, are Bairs subspaces. Therefore for generic a, $D^+_{A_0+a}$ has trivial cokernel.

Let $A \coloneqq A_0 + a$, so

$$\eta(a) = \frac{1}{2i}F(A^0)^+ \in \mathrm{Im}(\mathrm{d}^+), \tag{24.17}$$

since $\mathcal{H}^+ = 0$. We ask for two more conditions on a, which are both Baire for similar reasons: that $\mathrm{coker}(D^+_{A_0+a}) = 0$, and that 0 is a regular value for $\mathcal{F}_\eta = 0$ (note: this is *not* the Coulomb gauge!). Then R is regular and $\widetilde{M}^{\mathrm{irr}}_\eta$ is cut out transversely.

Remark 24.18. The Coulomb gauge is inconvenient for this proof, since d^+a might be nonzero. In this case it's easier to consider a gauge which involves solutions to $\mathcal{F}'_\eta = \mathrm{const}$. The analysis applies unchanged. ◂

Lecture 25.

Seiberg-Witten invariants: 4/26/18

"Then we find a contradiction and the whole thing disappears in a puff of logic, as Douglas Adams would say it."

Before we define Seiberg-Witten invariants, we need a few preliminaries.

Definition 25.1. Let X be a closed, oriented 4-manifold. A *homology orientation* for X is an equivalence class of triples $(H_+, H_-, \mathfrak{o})$, where

- $H^2_{\mathrm{dR}}(X) = H_+ \oplus H_-$ such that the intersection form is positive (resp. negative) definite on H_+ (resp. H_-),
- $\mathfrak{o}$ is an orientation for the vector space $H^1_{\mathrm{dR}}(X) \oplus H^0_{\mathrm{dR}}(X) \oplus H^*_+$, i.e. a positive ray in

$$\mathrm{Det}(H^1_{\mathrm{dR}}(X) \oplus H^0_{\mathrm{dR}}(X) \oplus H^*_+) = \mathrm{Det}(H^1_{\mathrm{dR}}(X)) \otimes \mathrm{Det}(H^0_{\mathrm{dR}}(X)) \otimes \mathrm{Det}(H_+)^*, \tag{25.2}$$

- and $(H_+, H_-, \mathfrak{o})$ is equivalent to $(K_+, K_-, \mathfrak{o}')$ if there is an isomorphism $H_+ \cong K_+$ and the maps $H_+ \hookrightarrow H^2_{\mathrm{dR}}(X) \twoheadrightarrow K_+$ (the latter the projection coming from the splitting) induces an orientation-preserving map between $\mathfrak{o}$ and $\mathfrak{o}'$.

There's a 2-element set $\mathfrak{o}_X$ of homology orientations on X. An orientation-preserving map $\phi\colon X \to X'$ induces a map $\mathfrak{o}_\phi\colon \mathfrak{o}_{X'} \to \mathfrak{o}_X$, so this is contravariantly functorial.

This is a lot of rigamarole for the notion of an orientation of $H^1_{\mathrm{dR}}(X) \oplus H^0_{\mathrm{dR}}(X) \oplus H^*_+$, which comes up because H_+ is not canonically determined. Homology orientations first arose in the theory of instantons, for the same reason they appear here.

We're also going to need conjugate spin^c structures. The idea is that complex conjugation is an involution on Spin^c_n, and therefore acts on spin^c structures too.

Definition 25.3. Let $\mathfrak{s} = (\mathbb{S}, \rho)$ be a spin^c structure on a Riemannian 4-manifold (X, g). Its *conjugate* is the spin^c structure $\overline{\mathfrak{s}} := (\overline{\mathbb{S}}, \rho)$, where $\overline{\mathbb{S}}$ is the same real bundle, but with $\mathbb{C}$ acting via the complex conjugate: $i \cdot \overline{\mathbb{S}} := -i \cdot \mathbb{S}$.

On Chern classes,

$$c_1(\overline{\mathfrak{s}}) = c_1(\Lambda^2\overline{\mathbb{S}}) = -c_1(\mathfrak{s}). \tag{25.4}$$

In particular, conjugation is the identity on a spin^c structure induced from a spin structure. The Seiberg-Witten invariants will be invariant under conjugation, which is an additional useful symmetry.

Now we can formulate Seiberg-Witten invariants, assume $b^+(X) > 1$. Then the invariants are a map

$$sw_{X,\sigma}\colon \mathrm{Spin}^c(X) \to \mathbb{Z}, \tag{25.5}$$

where $\mathrm{Spin}^c(X)$ denotes the $H^2(X;\mathbb{Z})$-torsor of isomorphism classes of spin^c structures on X, and σ is a homology orientation on X. These will have the following basic properties.

(1) $sw_{X,-\sigma} = -sw_{X,\sigma}$.
(2) The Seiberg-Witten invariants are invariant under orientation-preserving diffeomorphisms $f\colon X' \to X$:

$$sw_{X',\mathfrak{o}_f(\sigma)}(f^*\mathfrak{s}) = sw_{X,\sigma}(\mathfrak{s}).\text{[31]} \tag{25.6}$$

(3) $sw_{X,\sigma}$ has finite support, i.e. the invariants are zero away from finitely many spin^c structures.
(4) Conjugation-invariance:

$$sw_{X,\sigma}(\overline{\mathfrak{s}}) = (-1)^{1-b_1+b^+} sw_{X,\sigma}(\mathfrak{s}). \tag{25.7}$$

(5) If $sw_{X,\sigma} \neq 0$, then $d(\mathfrak{s})$ is nonnegative and even.

Unfortunately, if the Seiberg-Witten moduli space is odd-dimensional, the Seiberg-Witten invariants don't tell us anything useful. If it's *zero*-dimensional, which turns out to be the main case, then $sw_{X,\sigma}(\mathfrak{s})$ is a signed count of the finite set of solutions to the equations modulo gauge equivalence.

The simple type conjecture expresses the idea that dimension zero is generic.

Conjecture 25.8 (Simple type conjecture)**.** *If X is simply connected, $b^+(X) > 1$, and $sw_{X,\sigma}(\mathfrak{s}) \neq 0$, then $d(\mathfrak{s}) = 0$.*

It's totally open in general, but is true when X is symplectic.

We can also define Seiberg-Witten invariants when $b^+(X) = 1$. Recall that the space of conformal structures on X, Conf_X, is contractible, and let

$$\mathcal{V} := \{([g], \eta) \mid [g] \in \mathrm{Conf}_X, \eta \in \Omega^+_{[g]}\}, \tag{25.9}$$

which is a vector bundle over Conf_X. Given a spin^c structure $\mathfrak{s}$, let $\mathcal{W}(\mathfrak{s}) \subset \mathcal{V}$ denote the locus for which reducible solutions exist: $c_1(\mathfrak{s}) \in -2i\eta_{\mathrm{harm}}$ and $\mathcal{H}^-_{[g]} \subset H^2_{\mathrm{dR}}(X)$. This is an affine subbundle of $\mathcal{V} \to \mathrm{Conf}_X$ of codimension 1, and therefore $\mathcal{V} \setminus \mathcal{W}(\mathfrak{s})$ has two components, called *chambers*.

Let $\mathrm{Spin}^c(X)_{\mathrm{ch}}$ denote the set of pairs $(\mathfrak{s}, c)$, where $\mathfrak{s} \in \mathrm{Spin}^c(X)$ and c is a chamber for $\mathfrak{s}$. This is a double cover of $\mathrm{Spin}^c(X)$; when $b^+(X) = 1$, the Seiberg-Witten invariants are a map

$$sw_{X,\sigma}\colon \mathrm{Spin}^c(X)_{\mathrm{ch}} \longrightarrow \mathbb{Z}. \tag{25.10}$$

All of the properties enumerated above hold for these Seiberg-Witten invariants, *except* (3), finite support. Conjugation-invariance looks a little different: we need it to send $([g], \eta) \mapsto ([g], -\eta)$. But finite support really fails, and the resulting theory has striking differences.

In this case, there's an additional property.

[31] There's no relation between the Seiberg-Witten invariants of X and $\overline{X}$, as often these are completely different manifolds. Thus you get additional interesting invariants of X, namely those of $\overline{X}$.

Proposition 25.11 (Wall-crossing formula)**.** *Suppose $b^+(X) = 1$ and, for simplicitly, $b_1(X) = 0$. If $d(\mathfrak{s})$ is nonnegative and even and $c_\pm$ are the chambers of $\mathfrak{s}$, then*

$$|sw_{X,\sigma}(\mathfrak{s}, c_+) - sw_{X,\sigma}(\mathfrak{s}, c_-)| = 1.$$

The proof of this is somewhat funny: last time, we made a roundabout argument ending in a contradiction to prove something else. But in the case $b^+(X) = 1$, it actually works, and produces a real moduli space, and yields the wall-crossing formula! Wall-crossing formulas were known in Donaldson theory already, and therefore this one was almost immediately deduced from the Seiberg-Witten equations by the experts.

Now we construct the invariants. Let $C := C(\mathfrak{s}) = \{(A, \phi)\}$ be the configuration space for the equations. (Really, we want L^2_k configurations for $k \geq 4$.) It has a few important quotients:

- Given an $x \in X$, let $B_x := C/\mathcal{G}^x$, where $\mathcal{G}^x$ is the gauge transformations with $u(x) = 1$.
- $B := B_x/\mathrm{U}_1 = C/\mathcal{G}$.
- if $\mathcal{G}^x_0$ denotes the identity component of $\mathcal{G}^x$, then let $\widehat{B}_x := C/\mathcal{G}^x_0$. This is isomorphic to the *Coulomb slice*

$$\{(A = A_0 + a, \phi) \mid \mathrm{d}^* a = 0\}. \tag{25.12}$$

- $\widehat{B} = C/\mathcal{G}^0 = \widehat{B}_x/\mathrm{U}_1$.

Also, if C^{irr} denotes the space of irreducible components, we can define B^{irr}, etc. The projection $B^{\mathrm{irr}}_x \twoheadrightarrow B^{\mathrm{irr}}$ is a principal U_1-bundle, hence has a first Chern class $c := c_1(B^{\mathrm{irr}}_x) \in H^2(B^{\mathrm{irr}})$.

Recall that the $\mathcal{G}^x_0$-orbits of *connections*, which is also the Coulomb slice again, is isomorphic to $H^1_{\mathrm{dR}}(X) \times \mathrm{Im}(\mathrm{d}^*)$, and that the $\mathcal{G}^x$-orbits are isomorphic to $H^1(X;\mathbb{R})/H^1(X;\mathbb{Z}) \times \mathrm{Im}(\mathrm{d}^*)$. Therefore

$$\widehat{B}_x \cong H^1(X;\mathbb{R}) \times \mathrm{Im}(\mathrm{d}^*) \times \Gamma(\mathbb{S}^+), \tag{25.13}$$

and similarly for other variants. In particular,

$$B^{\mathrm{irr}} = C^{\mathrm{irr}}/\mathcal{G} \cong (H^1(X;\mathbb{R})/H^1(X;\mathbb{Z})) \times \mathrm{Im}(\mathrm{d}^*) \times \mathbb{P}\Gamma(\mathbb{S}^+) \times (0,\infty). \tag{25.14}$$

The description (25.14) is important: now we have an explicit description of this configuration space. We're going to use it to compute its cohomology. First, it deformation retracts to something diffeomorphic to

$$(H^1(X;\mathbb{R})/H^1(X;\mathbb{Z})) \times \mathbb{P}H, \tag{25.15}$$

where $H = L^2_k(\mathbb{S}^+)$ is a separable complex Hilbert space.

Lemma 25.16. *Let $\mathbb{CP}^\infty := \varinjlim_{n\to\infty} \mathbb{CP}^n$ across the standard inclusions $\mathbb{CP}^n \hookrightarrow \mathbb{CP}^{n+1}$. Then the inclusion $\mathbb{CP}^\infty \hookrightarrow \mathbb{P}(H)$ is a weak homotopy equivalence.*

That is, it's a map inducing isomorphisms on all homotopy groups.

Proof. Let's look at spheres. If $S(H)$ denotes the unit sphere in H and $S^\infty := \varinjlim_{n\to\infty} S^n$ in the same way as $\mathbb{CP}^\infty$, then both $S(H)$ and S^∞ are contractible (not just weakly contractible), and therefore the map $S^\infty \hookrightarrow S(H)$ is a homotopy equivalence. Therefore we have a commutative diagram

$$\begin{tikzcd} \mathrm{U}_1 \arrow[r] \arrow[d, equal] & S^\infty \arrow[r] \arrow[d, hook] & \mathbb{CP}^\infty \arrow[d, "i"] \\ \mathrm{U}_1 \arrow[r] & S(H) \arrow[r] & \mathbb{P}H, \end{tikzcd} \tag{25.17}$$

where the rows are fibrations. Therefore there are compatible maps between the long exact sequences of homotopy groups associated to their fibrations, from which one can see i is an isomorphism on homotopy groups. ⊠

Corollary 25.18. *There is a ring isomorphism*

$$H^*(B^{\mathrm{irr}}) \cong \Lambda H^1(X;\mathbb{Z})^* \otimes \mathbb{Z}[c].$$

This follows from the Künneth formula.

Now assume $b^+(X) > 0$. For generic (g, η), we obtain a compact $d(\mathfrak{s})$-dimensional manifold $M_\eta \subset B^{\mathrm{irr}}$ of solutions to $\mathcal{F}_\eta = 0 \bmod \mathcal{G}$. (Here we chose (g, η) to preempt the existence of reducible solutions.)

Proposition 25.19. *There is a real line bundle* $\mathrm{Det}\,\underline{\mathrm{ind}} \to B$ *and, whenever* M_η *is regular, a canonical isomorphism*

$$\mathrm{Det}\,\underline{\mathrm{ind}}|_{M_\eta} \cong \mathrm{Det}(TM_\eta).$$

Moreover, a choice of homology orientation determines an orientation of $\mathrm{Det}\,\underline{\mathrm{ind}}$.

The upshot is that σ orients TM_η, so M_η has a fundamental class $[M_\eta]_\sigma \in H_{d(\mathfrak{s})}(B^{\mathrm{irr}})$ whose sign depends on σ.

Definition 25.20. Given a homology orientation σ and a spin^c structure X, the *Seiberg-Witten invariant* is

$$sw_{X,\sigma}(\mathfrak{s}) = \begin{cases} \langle c^{d(\mathfrak{s})/2}, [M_\eta]_\sigma\rangle, & d(\mathfrak{s}) \text{ nonnegative and even} \\ 0, & \text{otherwise.} \end{cases}$$

That is: *this is computing the fundamental class of the space of Seiberg-Witten solutions inside the space of irreducible configurations modulo gauge transformations.* However, it only sees the $\mathbb{CP}^\infty$ part of B^{irr}, and therefore loses some of the information about the fundamental class. It is possible to define a more elaborate invariant valued in $\Lambda H^1(X)$ that sees all of $[M_\eta]_\sigma$.

We made the choice of (g, η) in Definition 25.20, but want our invariants to be independent of choice. Any two choices (g_0, η_0) and (g_1, η_1) can be joined by a smooth path $(g_t, \eta_t)_{t\in[0,1]}$, defining a parameteric space $M_{\underline{\eta}}^{\mathrm{par}} \subset [0,1] \times B$ of pairs $(t \in [0,1], (A,\phi) \bmod \mathcal{G})$ such that $\mathcal{F}_{\eta_t}(A,\phi) = 0$.

Genericity is a little more complicated: if $b^+(X) > 1$, then for generic perturbations of η_t rel endpoints, $M_{\underline{\eta}}^{\mathrm{par}} \subset [0,1] \times B^{\mathrm{irr}}$ and it's a smooth, compact submanifold of dimension $d(\mathfrak{s}) + 1$ with boundary $M_{\eta_0} \amalg M_{\eta_1}$. This can be oriented in the same way as before, extending $\mathrm{Det}\,\underline{\mathrm{ind}}$ in the time direction, and as oriented manifolds, we have

$$\partial M_{\underline{\eta}}^{\mathrm{per}} = (-M_{\eta_0}) \amalg M_{\eta_1}. \tag{25.21}$$

Therefore $[M_{\eta_0}]_\sigma = [M_{\eta_1}]_\sigma$, ensuring the Seiberg-Witten invariants are well-defined.

If $b^+(X) = 1$, such a path exists iff $([g_0], \eta_0)$ and $([g_1], \eta_1)$ are in the same chamber, and then the argument goes as usual.

Now we can address the properties.

- Since we used the orientation to define a fundamental class, sign change under homology orientation reversal is obvious.
- Diffeomorphism invariance is also fairly evident: one can pull back *all* of the data (the metric, η, ...) and therefore an orientation-preserving diffeomorphism of base manifolds induces an orientation-preserving diffeomorphism of their Seiberg-Witten moduli spaces.
- Finite support uses Proposition 19.27: that for a fixed g and η, only finitely many $\mathfrak{s}$ admit irreducible Seiberg-Witten solutions.
- Conjugation maps $C(\mathfrak{s})$ to $C(\overline{\mathfrak{s}})$, and one can check that it preserves Seiberg-Witten moduli spaces. It also modifies homology orientations in a way to produce the somewhat mysterious-looking sign change.
- The dimensionality constraint is also true by definition.

Wall-crossing will be postponed to a future lecture.

We'll conclude today's class with a few words about the determinant index bundle $\mathrm{Det}\,\underline{\mathrm{ind}} \to B$. It actually starts life somewhere else: if H_0 and H_1 are Hilbert spaces (both over $\mathbb{R}$ or over $\mathbb{C}$), there is a space $\mathrm{Fred}(H_0, H_1)$ of Fredholm operators, which is an open subset of the Hilbert space of bounded linear maps $H_0 \to H_1$. It has a locally constant function $\mathrm{ind}\colon \mathrm{Fred}(H_0, H_1) \to \mathbb{Z}$ sending a Fredholm operator to its index.

There is a well-defined virtual vector bundle (i.e. a formal difference of two vector bundles) $\underline{\mathrm{ind}} \to \mathrm{Fred}(H_0, H_1)$ whose fiber over an $F\colon H_0 \to H_1$ is stably isomorphic to the virtual vector space $\ker(F) - \mathrm{coker}(F)$. Then, $\mathrm{Det}\,\underline{\mathrm{ind}} \to \mathrm{Fred}(H_0, H_1)$ is simply its determinant line bundle: its fiber at F is

$$\mathrm{Det}\,\underline{\mathrm{ind}}_F \cong \mathrm{Det}(\ker F) \otimes (\mathrm{Det}(\mathrm{coker}\, F))^*. \tag{25.22}$$

Hence if F is surjective, $\mathrm{Det}\,\underline{\mathrm{ind}}_F = \mathrm{Det}(\ker F)$.

The configuration space C carries a family of linear Fredholm operators $D_{(a,\phi)}$, such that when $\mathcal{F}_\eta(a,\phi) = 0$, $D_{(a,\phi)} = D_{(a,\phi)}\mathcal{F}'_\eta$ (and we've seen these explicit formulas already), so we get a map $D\colon \{\mathcal{F}_\eta = 0\} \to$

$\mathrm{Fred}(H_0, H_1)$ for certain Hilbert spaces H_0 and H_1. Therefore $\mathrm{Det}\,\underline{\mathrm{ind}}$ pulls back over the space of solutions, and at regular points the fiber is identified with the determinant of the tangent space. (There's a little bit more work to get it to descend to B, though.)

Lecture 26.

Symplectic 4-manifolds: 5/1/18

There will be an additional lecture on Friday at 4pm.

This week is symplectic 4-manifold week; today and Thursday, we'll state and prove Taubes' constraints on the Seiberg-Witten invariants of a symplectic 4-manifold. These are the only nonobvious constraints we know of to the existence of symplectic structures. On Friday, we'll discuss the symplectic Thom conjecture on the minimal genus of an embedded surface representing a given homology class.

There's plenty more to be said about Seiberg-Witten theory on symplectic 4-manifolds, e.g. further work of Taubes on relations to Gromov-Witten theory. The stuff we'll discuss on Friday is a gateway into gluing theory and Floer homology, which is a major subject of active research in this field.

First, let's discuss the canonical spin^c structure associated to a complex vector space (hence also vector bundle). Let $(V, \langle\cdot,\cdot\rangle)$ be a $2n$-dimensional real inner product space, and let $J \in \mathrm{SO}(V)$ be a complex structure on V, i.e. $J^2 = -\mathrm{id}$. This makes V into a complex vector space, where $i \cdot v := Jv$ and we extend $\mathbb{R}$-linearly.

If V^* denotes the real dual of V, there is a polarization

$$V^* \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) = V^{1,0} \oplus V^{0,1}, \tag{26.1}$$

where $V^{1,0} := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ and $V^{0,1}$ is the space of antilinear maps; that is, $V^{1,0}$ is the i-eigenspace of J and $V^{0,1}$ is the $(-i)$-eigenspace. Both $V^{1,0}$ and $V^{0,1}$ are isotropic with respect to the complexified inner product, so $V^* \otimes \mathbb{C}$ is polarized.

Let $S := \Lambda^* V^{0,1}$, which is a spinor representation for $C\ell(V^* \otimes \mathbb{C})$, arising from the representation $\rho\colon V^* \to \mathrm{End}_{\mathbb{C}} S$ defined by

$$\rho(e) := \sqrt{2}\big(\varepsilon_{e^{0,1}} - \iota_{\overline{e^{1,0}}}\big). \tag{26.2}$$

Here $\varepsilon_\omega(\eta) := \omega \wedge \eta$ is exterior multiplication, and ι_ω denotes interior multiplication (i.e. contraction with the metric). Then there's a splitting $S = S^+ \oplus S^-$ where $S^+ := \Lambda^{\mathrm{even}} V^{0,1}$ and $S^- := \Lambda^{\mathrm{odd}} V^{0,1}$.

The global story is similar: if (M, g) is a $2n$-dimensional Riemannian manifold with an *almost complex structure*, i.e. a $J \in \Gamma(\mathrm{SO}(T^*M))$ such that $J^2 = -\mathrm{id}$, then it determines an orientation for TM and a Clifford module

$$\rho\colon T^*M \to \mathrm{End}_{\mathbb{C}}(\Lambda^{0,\bullet}(T^*M)) \tag{26.3}$$

just as in (26.2). Here $\Lambda^{0,\bullet}(T^*M) := (\Lambda^\bullet_{\mathbb{C}}(T^*M))^{0,1}$. This is a spin^c structure $\mathfrak{s}_J$, and equivalent almost complex structures (i.e. homotopic J) induce equivalent spin^c structures.

Another way to understand this is that the standard map $\mathrm{U}_n \to \mathrm{SO}_{2n}$ lifts to a map $\mathrm{U}_n \to \mathrm{Spin}^c_{2n}$ followed by the standard map $\mathrm{Spin}^c_{2n} \to \mathrm{SO}_{2n}$.

Lemma 26.4. *Let X be a closed, oriented 4-manifold with an almost complex structure J compatible with the orientation. Then $d(\mathfrak{s}_J) = 0$.*

So we get the nicest possible Seiberg-Witten invariants from almost complex manifolds.

Proof. Let $\mathbb{S}^+$ denote the positive spinor bundle for $\mathfrak{s}_J$. Then $\mathbb{S}^+ = \Lambda^{0,0} \oplus \Lambda^{0,2}_J$, and $\Lambda^{0,0} \cong \mathbb{C}$ and $\Lambda^{0,2}_J = (\Lambda^2_{\mathbb{C}}(T^*X))^{0,1}$. Thus

$$\Lambda^2 \mathbb{S}^+ = \Lambda^{0,2}_J = (\Lambda^{2,0}_J)^* = \Lambda^2_{\mathbb{C}}(TX, J). \tag{26.5}$$

Therefore

$$c_1(\Lambda^2 \mathbb{S}) = c_1(\Lambda^2_{\mathbb{C}}(TX, J)) = c_1(TX, J), \tag{26.6}$$

and one can calculate $p_1(TX)$ similarly, and therefore

$$3\tau(X) = \langle p_1(TX), [X]\rangle = \langle c_1(\mathfrak{s}_J)^2, [X]\rangle - 2\chi(X), \tag{26.7}$$

so $d(\mathfrak{s}_J) = 0$. ⊠

The converse is also true.

If (M, ω) is a *symplectic manifold*,[32] then it admits a compatible almost complex structure J. Then $g(u, v) := \omega(u, Jv)$ defines a Riemannian metric. Such compatible complex structures form a contractible space; this is because Sp_n deformation retracts onto its maximal compact subgroup, which is U_n, and is a fundamental lemma in symplectic geometry. In particular, (M, ω) has a canonical spin^c structure $\mathfrak{s}_{\mathrm{can}} = \mathfrak{s}_J$ for J compatible.

Let $K :=$ *TODO*, and $k := c_1(K_M) \in H^2(X)$. Let $\mathbb{S}^\pm$ be the spinors for $\mathfrak{s}_{\mathrm{can}}$: explicitly, $\mathbb{S}^+ = \Lambda^{0,0} \oplus \Lambda^{0,2}$ and $\mathbb{S}^- = \Lambda^{0,1}$.

Theorem 26.8 (Taubes)**.** *There is a canonical solution $(A_{\mathrm{can}}, \phi_{\mathrm{can}})$ to the Dirac equation $D_A^+ = 0$ for the spin^c structure $\mathfrak{s}_{\mathrm{can}}$. Moreover, for $\tau > 0$, the* Taubes monopole $(A_{\mathrm{can}}, \sqrt{\tau}\phi_{\mathrm{can}})$ *is a solution to the Seiberg-Witten equations $\mathcal{F}_{\eta(\tau)} = 0$, where*

$$\eta(\tau) := iF(A^0_{\mathrm{can}})^+ + \frac{1}{2}\tau\omega.$$

τ is known as the *Taubes parameter*. The Taubes monopole comes up a lot in this field, though sometimes it's disguised.

We can trivialize the $H^2(X)$-torsor $\mathrm{Spin}^c(X)$ by taking $\mathfrak{s}_{\mathrm{can}}$ as an origin; then, the Seiberg-Witten invariants are a map

$$sw_{X,\sigma} \colon H^2(X) \longrightarrow \mathbb{Z}. \tag{26.9}$$

Conjugation-invariance says that

$$sw_{X,\sigma}(k - e) = \pm sw_{X,\sigma}(e). \tag{26.10}$$

Theorem 26.11 (Taubes constraints)**.** *Let (X, ω) be a symplectic manifold with $b^+(X) > 1$. Then there's a canonical homology orientation σ such that:*

(1) *The Taubes monopole is the unique solution (modulo gauge equivalence) to $\mathcal{F}_{\eta(\tau)} = 0$ for τ sufficiently large, it's regular, and $sw_{\lambda,\sigma}(0) = 1$, so $sw_{X,\sigma}(k) = (-1)^{1-b_1(X)+b^+(X)}$.*

(2) *If $sw_{X,\sigma}(e) \neq 0$, then*

$$0 \le e \cdot [\omega] \le k \cdot [\omega], \tag{26.12}$$

and equality holds in either of these iff $e = 0$ or $e = k$.

There's a statement for $b^+(X) = 1$, but it's slightly more complicated, as usual.

Here's one interesting application. Suppose M is a closed, oriented, m-dimensional manifold, where m is even, and $w \in H^2(M; \mathbb{R})$. Does there exist a symplectic form ω on M representing w? There are two immediate constraints:

(1) $\langle w^n, [M] \rangle > 0$, and
(2) there is an almost complex structure J compatible with the orientation on M.

For $m \ge 6$, these are the only known constraints; for $m = 4$, we also have the Taubes constraints on the Seiberg-Witten invariants. When X is simply connected, these are the only known constraints – otherwise, there are knot-theoretic things you can do.

Corollary 26.13. *If (X, ω) is a symplectic manifold with $b^+(X) > 1$, then $k \cdot [\omega] \ge 0$.*

This is the starting point for the classification of symplectic 4-manifold with $k \cdot [\omega] < 0$: the underlying complex surface is either a blowup of $\mathbb{CP}^2$ or a ruled surface.

Taubes found a deeper reason for Corollary 26.13 to be true. A *symplectic surface* inside a symplectic 4-manifold is a surface S with $\omega|_S > 0$.

Theorem 26.14 (Taubes)**.** *With X as in Corollary 26.13, there is a symplectic surface $S \subset X$ whose homology class is the Poincaré dual to k.*

In particular, if S is nonempty,

$$k \cdot [\omega] = \int_S \omega > 0. \tag{26.15}$$

We're not going to prove Theorem 26.14: it involves detailed analysis of the Seiberg-Witten equations.

[32] This means M is a manifold and $\omega \in \Omega^2(M)$ is a closed, nondegenerate 2-form.

Corollary 26.16. *Let X be a K3 surface or a 4-torus (so that k is trivial). Then $sw(0) = 1$ and $sw(e) = 0$ for $e \neq 0$.*

This was known to Witten, who described a way to relate the Seiberg-Witten invariants of Kähler surfaces to invariants defined using complex geometry.

Now we get to work. Before proving Theorem 26.11, we need to study the geometry of almost complex manifolds in some detail. This isn't necessarily hard, but there's no way around it. Let (M, g, J) be a Riemannian manifold with an almost complex structure $J \in \mathrm{SO}(TM)$. Writing $T^*M \otimes \mathbb{C} \cong T^{1,0}M \oplus T^{0,1}M$ as above, define the projections

$$\pi^{1,0} \coloneqq \frac{1}{2}(1 - iJ)\colon T^*M \xrightarrow{\cong} T^{1,0}M \tag{26.17a}$$

$$\pi^{0,1} \coloneqq \frac{1}{2}(1 + iJ)\colon T^*M \xrightarrow{\cong} T^{0,1}M. \tag{26.17b}$$

Both of these are $\mathbb{C}$-linear.

We can also define (p, q)-forms; you may have seen this for complex manifolds, but the story goes through in the almost complex case as well:

$$\begin{aligned} \Lambda^k_{\mathbb{R}}(T^*M) \otimes \mathbb{C} &= \Lambda^k_{\mathbb{C}}(T^*M \otimes \mathbb{C}) \\ &= \Lambda^k_{\mathbb{C}}(T^{1,0}M \oplus T^{0,1}M) \\ &= \bigoplus_{p+q=k} \Lambda^{p,q}, \end{aligned}$$

where

$$\Lambda^{p,q} \coloneqq \Lambda^p_{\mathbb{C}} T^{1,0} \otimes \Lambda^q_{\mathbb{C}} T^{0,1}. \tag{26.18}$$

The space of *(p, q)-forms* is $\Omega^{p,q}(X) \coloneqq \Gamma(\Lambda^{p,q})$, and $\Omega^k(X)$ is a direct sum of $\Omega^{p,q}(X)$ over $k = p + q$.

The de Rham derivative also splits as $\mathrm{d} = \partial_J + \overline{\partial}_J$, where

$$\partial_J \coloneqq \frac{1}{2}(1 - iJ) \circ \mathrm{d}\colon \Omega^{p,q}(X) \longrightarrow \Omega^{p+1,q}(X) \tag{26.19a}$$

$$\overline{\partial} \coloneqq \frac{1}{2}(1 + iJ) \circ \mathrm{d}\colon \Omega^{p,q}(X) \longrightarrow \Omega^{p1,q+1}(X). \tag{26.19b}$$

Definition 26.20. The *Nijenhuis tensor* $N_J\colon \Lambda^2(TX) \to TX$ is defined on vector fields by

$$N_J(u, v) \coloneqq [Ju, Jv] - [u, v] - J[Ju, v] - J[u, Jv].$$

This is C^∞-bilinear, and arises as an obstruction to the *integrability* of J, i.e. lifting it to a complex structure (namely holomorphic coordinates). Why is this? If we complexify everything, obtaining a map

$$N_J\colon \Lambda^2_{\mathbb{C}}(TX \otimes \mathbb{C}) \longrightarrow TX \otimes \mathbb{C}, \tag{26.21}$$

then one can check that $N_J(\Lambda^2(T^{0,1})) \subset T^{1,0}$. Dualizing, we obtain a map $N_J^*\colon \Lambda^{1,0} \to \Lambda^{0,2}$.

Lemma 26.22. *Let f be a function on X. Then,*

$$\overline{\partial}_J^2 f = -\frac{1}{4} N_J^* \circ \partial_J f.$$

We won't give a proof; there's one in the notes, and probably one in Kobayashi-Nomizu.

In the integrable case, we have holomorphic coordinates z_i, and therefore antiholomorphic coordinates $\overline{z}_j$. In this case

$$\overline{\partial} f = \sum_i \frac{\partial f}{\partial \overline{z}_i}\, \mathrm{d}\overline{z}_i, \tag{26.23a}$$

and therefore

$$\overline{\partial}^2 f = 0. \tag{26.23b}$$

In particular, integerability implies the Nijenhuis tensor vanishes.

Theorem 26.24 (Neulander-Nirenberg)**.** *The converse is true: $N_J = 0$ implies that J is integrable, lifting to a complex structure.*

We also have to talk about symplectic manifolds. Let X be a closed 4-manifold with a "compatible triple" (g, J, ω), where g is a Riemannian metric, J is an almost complex structure, and ω is a symplectic form, subject to the compatibility condition

$$g(u, v) = \omega(u, Jv) \tag{26.25}$$

for all $u, v \in TM$. In this case, at any $x \in X$, there's an oriented basis (e_1, e_2, e_3, e_4) for T_x^*X such that $Je_1 = e_2$, $He_3 = e_4$, and

$$\omega(x) = e_1 \wedge e_2 + e_3 \wedge e_4 = e_1 \wedge Je_1 + e_3 \wedge Je_3. \tag{26.26}$$

Therefore $|\omega|_g = 2$ and $\star_g \omega = \omega$. In particular, the symplectic form is self-dual and harmonic. One can check that

$$\begin{aligned} \Lambda_g^+ \otimes \mathbb{C} &= \Lambda^{2,0} \oplus \mathbb{C} \cdot \omega \oplus \Lambda^{0,2} \\ \Lambda_g^- \otimes \mathbb{C} &= \Lambda_0^{1,1}, \end{aligned} \tag{26.27}$$

and $\Lambda_0^{1,1} = \omega^\perp$ in $\Lambda^{1,1}$.

If $L \coloneqq \varepsilon_\omega \colon \Lambda^k \to \Lambda^{k+2}$ and L^* is its adjoint, then $L^*\omega = 2$ and $L^*(\Lambda_0^{1,1}) = 0$. If $\eta \in \Omega_g^+$ (here we're over $\mathbb{R}$), then

$$\eta = \eta^{2,0} + \left(\frac{1}{2} L^*\eta\right)\omega + \overline{\eta^{2,0}}. \tag{26.28}$$

In particular, η is equivalent data to $\eta^{2,0}$ and $L^*\eta$.

Let $\mathbb{S}^\pm$ be the pieces of the spinor bundle for $\mathfrak{s}_{\mathrm{can}}$. Then $\mathbb{S}^+ = \Lambda^{0,0} \oplus \Lambda^{0,2}$ and $\Lambda^{0,0} \cong \underline{\mathbb{C}}$, and $\mathbb{S}^- = \Lambda^{0,1}$. If ρ denotes the Clifford map, then

$$\rho(\omega) = \frac{1}{2}[\rho(e_1), \rho(e_2)] + \frac{1}{2}[\rho(e_3), \rho(e_4)], \tag{26.29}$$

so $\rho(\omega) \in \mathfrak{su}(\mathbb{S}^+)$, and its 0 in $\mathbb{S}^-$. In particular, in the block form coming from $\mathbb{S}^+ = \underline{\mathbb{C}} \oplus \Lambda^{0,2}$,

$$\rho(\omega) = \begin{bmatrix} 2i & 0 \\ 0 & 2i \end{bmatrix}. \tag{26.30}$$

Therefore we can use the action of $\rho(\omega)$ on a spinor to distinguish $\underline{\mathbb{C}}$ from $\Lambda^{0,2}$.

If $\beta \in \Lambda^{0,2}$, then $\rho(\beta) = 2\left[\begin{smallmatrix} 0 & 0 \\ \beta & 0 \end{smallmatrix}\right]$ and $\rho(\overline{\beta}) = 2\left[\begin{smallmatrix} 0 & \overline{\beta} \\ 0 & 0 \end{smallmatrix}\right]$.

Next time, we'll study the Dirac operator in this context.

Lecture 27.

Taubes' constraints: 5/3/18

Before we prove Taubes' constraints (Theorem 26.11), we'll finish the geometric background for it that we started last time.

Definition 27.1. Recall the notion of a compatible triple (g, J, ω) on a $2n$-dimensional manifold M: a Riemannian metric g, an almost complex structure J, and a nondegenerate 2-form ω such that $g(u, v) = \omega(u, Jv)$.

- If $d\omega = 0$, this triple is called an *almost Kähler structure* on M. In this case ω is a symplectic form and $\omega \in \Omega_J^{1,1}(X)$, since $\omega(Ju, Jv) = \omega(u, v)$.
- If in addition J is integrable, this triple is called a *Kähler structure*.

On an almost Kähler manifold M (i.e. a manifold together with an almost Kähler structure), at any $x \in M$ there's a basis $(e_1, \dots, e_n, f_1, \dots, f_n)$ for T_x^*M such that $Je_j = f_j$ and

$$\omega_x = \sum_{j=1}^n e_j \wedge f_j. \tag{27.2}$$

Fix a Hermitian vector bundle $(E \to M, \langle \cdot, \cdot \rangle_E)$, where M is almost Kähler.

Lemma 27.3. *Let A be a unitary connection in E and*

$$\overline{\partial}_A \colon \Omega^{p,q}_M(E) \longrightarrow \Omega^{p,q+1}_M(E) \tag{27.4}$$

denote the projection of $d_A|_{\Omega^{p,q}_M(E)}$ to $\Omega^{p,q+1}_M(E)$. Then on $\Omega^{0,0}_M(E) = \Gamma(E)$,

$$\overline{\partial}_A^2 = F_A^{0,2} - \frac{1}{4} N_J^* \circ \partial_A. \tag{27.5}$$

The proof is essentially the same as Lemma 26.22 (which we also didn't prove).

Lemma 27.6 (Almost Kähler identities)**.** *If $L_\omega \colon \Omega^{p,q}_M(E) \to \Omega^{p+1,q+1}_M(E)$ denotes exterior multiplication with ω and L_ω^* is its adjoint in g, then*

(1) *on $\Omega^{0,1}_M(E)$,*

$$\overline{\partial}_A^* = iL_\omega^* \circ \partial_A, \tag{27.7a}$$

(2) *and on $\Omega^{1,0}_M(E)$,*

$$\partial_A^* = -iL_\omega^* \circ \overline{\partial}_A. \tag{27.7b}$$

Here ∂_A^* and $\overline{\partial}_A^*$ denote formal adjoints.

We're also not going to prove this; Donaldson-Kronheimer provide a hint after which there's some calculation. If M is Kähler these identities are standard, and admit a proof in the spirit of differential geometry: one checks that the identities hold on $\mathbb{C}^n$ with its usual Kähler structure, then that a Kähler structure is one that can be arranged to agree with the standard one on $\mathbb{C}^n$ up to second-order. Since the identities in Lemma 27.6 are first-order, this suffices.

We care about these identities in order to derive a Weitzenböck formula.

Proposition 27.8 (Weitzenböck formula)**.**

$$\frac{1}{2}\nabla_A^*\nabla_A = \overline{\partial}_A^*\overline{\partial}_A + iL_\omega^*(F_A).$$

Proof. Since $\nabla_A = \partial_A + \overline{\partial}_A$, then by (27.7),

$$\nabla_A^* = \partial_A^* + \overline{\partial}_A^* = iL_\omega^*(-\overline{\partial}_A + \partial_A), \tag{27.9}$$

and so

$$\begin{aligned} \nabla_A^*\nabla_A &= iL_\omega^*(-\overline{\partial}_A + \partial)(\partial + \overline{\partial}_A) \\ &= iL_\omega^* \circ [\partial_A, \overline{\partial}_A]. \end{aligned} \tag{27.10}$$

Since

$$F_A^{1,1} = \overline{\partial}_A\partial_A + \partial_A\overline{\partial}_A, \tag{27.11}$$

then

$$iL_\omega^*(F_A) = 2iL_\omega^* \circ \partial_A \circ \overline{\partial}_A = L_\omega^*\big(\overline{\partial}_A\partial_A + \partial_A\overline{\partial}_A,\big), \tag{27.12}$$

and the result follows by comparing terms with (27.10). ⊠

The canonical solution to the Dirac equation. Let (X, ω, J, g) be an almost Kähler 4-manifold, with its canonical spin^c structure $\mathfrak{s}_{\mathrm{can}}$ induced from J. Last time we talked about the spinor bundle of $\mathfrak{s}_{\mathrm{can}}$: $\mathbb{S}^+ = \underline{\mathbb{C}} \oplus \Lambda_J^{0,2}$, and $\mathbb{S}^- = \Lambda_J^{0,1}$.

There is a distinguished spinor $\phi_{\mathrm{can}} := 1 \in \Gamma(\underline{\mathbb{C}})$ and a distinguished Clifford connection $A_{\mathrm{can}} \in \mathcal{A}_{\mathrm{cl}}(\mathbb{S}^+)$. For any $A \in \mathcal{A}_{\mathrm{cl}}(\mathbb{S}^+)$ and $a \in i\Omega^1(X)$, $\nabla_A\phi_{\mathrm{can}} \in \Omega^1_X(\Lambda^{0,0} \oplus \Lambda^{0,2})$ and

$$\nabla_{A + a\cdot\mathrm{id}}(\phi_{\mathrm{can}}) = \nabla_A\phi_{\mathrm{can}} + \underbrace{a \otimes 1_X}_{\in \Omega^1_X(\Lambda^{0,0})}. \tag{27.13}$$

Then, A_{can} is characterized by

$$\nabla_{A_{\mathrm{can}}}\phi_{\mathrm{can}} \in \Omega^1_X(\Lambda^{0,2}). \tag{27.14}$$

Theorem 27.15. *Let $D^+ := D^+_{A_{\mathrm{can}}}$. Then,*

(1) *$D^+\phi_{\mathrm{can}} = 0$, and*

(2)

$$D^+ = \sqrt{2}\Big(\overline{\partial}_J \oplus \overline{\partial}_J^*\Big). \tag{27.16}$$

D^+ is a Dirac operator – you can check that it has the right symbol.

We'll need another lemma whose proof is a fiddly differential-geometric calculation and wasn't provide (but is in the professor's notes). Recall that if ∇^{LC} denotes the Levi-Civita connection, then

$$\nabla_v(\rho(e)\phi) - \rho(e)\nabla_v\phi = \rho(\nabla_v^{\mathrm{LC}} e)\phi. \tag{27.17}$$

One can define an action of differential forms on spinors by averaging the action of 1-forms over the symmetric group: if $e_1, \ldots, e_k$ are 1-forms,

$$\rho(e_1 \wedge \cdots \wedge e_k)\phi := \frac{1}{k!}\sum_{\sigma\in S_k} \operatorname{sign}(\sigma)\rho(e_{\sigma(1)})\cdots\rho(e_{\sigma(k)})\phi. \tag{27.18}$$

Lemma 27.19. *If $\gamma \in \Omega^k(X)$, $\phi \in \Gamma(\mathbb{S})$, and $\delta := \mathrm{d} + \mathrm{d}^*$, then as functions $T^*X \otimes \mathbb{S} \to \mathbb{S}$,*

$$\underbrace{\widetilde{\rho}(\nabla\rho(\gamma)\phi)}_{D(\rho(\gamma)\phi)} - \rho(\delta\gamma)\phi = \widetilde{\rho}(\rho(\gamma)\nabla\phi).$$

Proof of Theorem 27.15. Let $\Omega := (1/2i)\rho(\omega)\colon \mathbb{S}^+ \to \mathbb{S}^+$. From last time we know this is diagonal with respect to the splitting $\Lambda^{0,0} \oplus \Lambda^{0,2}$, and acts as -1 on the first summand and 1 on the second. Thus $\Omega\phi_{\mathrm{can}} = -\phi_{\mathrm{can}}$ and $\Omega(\nabla_v\phi_{\mathrm{can}}) = \nabla_v\phi_{\mathrm{can}}$.

Since $\star\omega = \omega$, then

$$\delta\omega = \mathrm{d}\omega + \mathrm{d}^*\omega = 0. \tag{27.20}$$

Thus

$$\begin{aligned} -D^+(\phi_{\mathrm{can}}) = D^+(\Omega\phi_{\mathrm{can}}) &= \widetilde{\rho}(\nabla\Omega\phi_{\mathrm{can}}) \\ &= \widetilde{\rho}(\Omega\nabla\phi_{\mathrm{can}}) = \widetilde{\rho}(\nabla\phi_{\mathrm{can}}) \\ &= D^+\phi_{\mathrm{can}}. \end{aligned}$$

Perhaps this is a little mysterious.

For the second part, both sides of (27.16) have the same symbol, and the difference is algebraic. Both kill ϕ_{can} by the first part, and ϕ_{can} generates $\Gamma(\Lambda^{0,0})$, so we only have to check the effect on a $\beta \in \Gamma(\Lambda^{0,2})$. Last time we saw that $\rho(\beta) = \left[\begin{smallmatrix} 0 & 0 \\ 2\beta & 0 \end{smallmatrix}\right]$, so

$$\beta = \frac{1}{2}\rho(\beta)\phi_{\mathrm{can}} \tag{27.21}$$

$$\rho(\beta)\nabla_v\phi_{\mathrm{can}} = 0. \tag{27.22}$$

Therefore

$$\begin{aligned} D^+\beta &= \widetilde{\rho}\nabla\beta \\ &= \frac{1}{2}\widetilde{\rho}\nabla(\rho(\beta)\phi_{\mathrm{can}}) \\ &= \frac{1}{2}\rho(\delta\beta)\phi_{\mathrm{can}} + \frac{1}{2}\widetilde{\rho}(\rho(\beta)\nabla\phi_{\mathrm{can}}), \end{aligned} \tag{27.23}$$

and the second term vanishes. On the right-hand side,

$$\rho(\mathrm{d}^*\beta)\phi_{\mathrm{can}} = \sqrt{2}(\mathrm{d}^*\beta)^{0,1} = \sqrt{2}\overline{\partial}^*\beta. \tag{27.24}$$

Since β is self-dual, then $\delta\beta = (2\mathrm{d}^*\beta)^\dagger$, so $D^+\beta = \sqrt{2}\overline{\partial}^*\beta$. ⊠

This is the end of the relatively intricate spin geometry that we've needed to prove these lemmas — now we can return to Seiberg-Witten theory. Let $L \to X$ be a line bundle and $\mathfrak{s} := L \otimes \mathfrak{s}_{\mathrm{can}}$. The spinor bundle for $\mathfrak{s}$ splits as

$$\begin{aligned} \mathbb{S}^+ &= L \otimes (\Lambda^{0,0} \oplus \Lambda^{0,2}) = L \oplus (L \otimes \Lambda^{0,2}) \\ \mathbb{S}^- &= L \otimes \Lambda^{0,1}. \end{aligned} \tag{27.25}$$

If $A \in \mathcal{A}_{\mathrm{cl}}(\mathbb{S}^+)$, we can write it as

$$\nabla = \mathrm{id}_L \otimes \nabla_{\mathrm{can}} + \nabla_B \otimes \mathrm{id}, \tag{27.26}$$

where β is a unitary connection on L. By Theorem 27.15, the associated Dirac operator is

$$D_B^+ = \sqrt{2}\Big(\overline{\partial}_B \oplus \overline{\partial}_B^*\Big). \tag{27.27}$$

The curvature of A is

$$F(A^0) = F(\nabla^0_{\mathrm{can}}) + 2F(B). \tag{27.28}$$

Therefore the Seiberg-Witten curvature equation involves $F^+ = F(\nabla^0_{\mathrm{can}})^+ + 2F_B^+$. We don't actually care what the last term is – we're going to be adding in η anyways, so we might as well make it $\eta - 2F_B^+$.

Since

$$(iF)^+ = iF^{2,0} + \frac{i}{2}(L_\omega^* F)\omega + \overline{iF^{2,0}}, \tag{27.29}$$

then F^+ is equivalent data to $L_\omega^*(iF)$ and $F^{2,0}$.

Let $\phi = \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$, where $\alpha \in \Lambda^{0,0}$ and $\beta \in \Lambda^{0,2}$. Then

$$\phi\phi^* = \begin{bmatrix}\alpha\\ \beta\end{bmatrix}\begin{bmatrix}\overline{\alpha} & \overline{\beta}\end{bmatrix} = \begin{bmatrix}|\alpha|^2 & \alpha\overline{\beta}\\ \overline{\alpha}\beta & |\beta|^2\end{bmatrix}, \tag{27.30}$$

so

$$(\phi\phi^*)_0 = \begin{bmatrix}(1/2)(|\alpha|^2 - |\beta|^2) & \alpha\overline{\beta}\\ \overline{\alpha}\beta & (1/2)(|\beta|^2 - |\alpha|^2)\end{bmatrix}. \tag{27.31}$$

To formulate the curvature equation, recall $\rho(\omega) = \left[\begin{smallmatrix}-2i & 0\\ 0 & 2i\end{smallmatrix}\right]$, $\rho(\beta) = \left[\begin{smallmatrix}0 & 0\\ 2\beta & 0\end{smallmatrix}\right]$, and so on.

The curvature equation is $(1/2)\rho(F^+ - 4i\eta) = (\phi\phi^*)_0$, where $\eta \in \Omega^+(X)$. Supposing that $\eta = F(\nabla^0_{\mathrm{can}})^+/4i + \eta_0$, then the curvature equation becomes

$$\rho(F_B^+ - 2i\eta_0) = (\phi\phi^*)_0, \tag{27.32}$$

which is nicer because F_B^+ is the curvature for a line bundle (specifically, L). And we have a lot of latitude in choosing η_0.

Taubes chooses $\eta_0 \coloneqq (-1/4)\tau\omega$, where $\tau \gg 0$ is the Taubes parameter. In this case, the Seiberg-Witten equations simplify into

$$\overline{\partial}_B\alpha = -\overline{\partial}_B^*\beta \tag{27.33a}$$

$$F_B^{0,2} = \frac{1}{2}\overline{\alpha}\beta \tag{27.33b}$$

$$L_\omega^*(iF_B) = \frac{1}{4}\Big(|\beta|^2 - |\alpha|^2 + \tau\Big). \tag{27.33c}$$

(27.33a) is the Dirac equation, and the other two come from the curvature equation.

The *Taubes monopole* is a canonical solution: $L = \underline{\mathbb{C}}$, B is the trivial connection, $\alpha = \sqrt{\tau}$, and $\beta = 0$. Then $\left[\begin{smallmatrix}\alpha & \beta\end{smallmatrix}\right] = \sqrt{\tau}\phi_{\mathrm{can}}$, and the modified Seiberg-Witten equations (27.33) are somewhat trivially satisfied.

Taubes' constraints largely follow from the following result, which provides cases in which the Taubes monopole is essentially the only solution.

Proposition 27.34. *There is a constant C (depending on X, g, and J) such that if $\langle c_1(L), [\omega]\rangle \le 0$ and (B, α, β) is a solution to (27.33) with parameter $\tau > C$, then $L \cong \underline{\mathbb{C}}$ and in a suitable gauge, B is trivial, $\beta = 0$, and $\alpha = \sqrt{\tau}$.*

Proof. Let (B, α, ϕ) be a solution with parameter τ. Then[33]

$$\frac{1}{2}\|\nabla_B\alpha\|_{L^2}^2 = \frac{1}{2}\langle\nabla_B^*\nabla_B\alpha, \alpha\rangle_{L^2}.$$

By (27.8),

$$= \langle\overline{\partial}_B^*\overline{\partial}_B\alpha, \alpha\rangle_{L^2} + \int_X L_\omega^*(iF_N)|\alpha|^2.$$

[33]All integrals here are with repsect to the Lesbegue measure.

By (27.33a) and (27.33c),

$$= -\langle \overline{\partial}_B^* \overline{\partial}_B^* \beta, \alpha\rangle_{L^2} + \frac{1}{2}\int_X (|\beta|^2 - |\alpha|^2 + \tau)|\alpha|^2$$

$$= -\langle \beta, F_B^{0,2}\alpha\rangle_{L^2} + \frac{1}{4}\int_X \langle \beta, N_J^* \circ \partial_B \alpha\rangle + \frac{1}{2}\int_X \Big(|\beta|^2 - |\alpha|^2 + \tau\Big)|\alpha|^2.$$

By (27.33b),

$$= -\frac{1}{2}\int_X |\alpha|^2|\beta|^2 + \frac{1}{4}\int_X \langle \beta, N_J^* \circ \partial_B \alpha\rangle + \frac{1}{2}\int_X \Big(|\beta|^2 - |\alpha|^2 + \tau\Big)|\alpha|^2$$

$$= \frac{1}{4}\int_X \langle \beta, N_J^* \circ \partial_B \alpha\rangle - \frac{1}{4}\int_X \Big((\tau - |\alpha|^2)^2 + \mathit{TODO}.\Big)$$

We assumed $\langle c_1(L), [\omega]\rangle \le 0$, so since

$$\langle c_1(L), [\omega]\rangle = \frac{1}{2\pi}\int_X iF_B \wedge \omega = \frac{1}{4\pi}\int L_\omega^*(iF_B)\omega\wedge\omega, \tag{27.35}$$

then $\int L_\omega^*(iF_B) \le 0$. Therefore by (27.33c),

$$\int_X |\beta|^2 - |\alpha|^2 + \tau \le 0. \tag{27.36}$$

Therefore

$$\begin{aligned}\frac{1}{2}\|\nabla_B\alpha\|_{L^2}^2 + \frac{1}{4}\int_X \Big((\tau - |\alpha|^2)^2 + \tau|\beta|^2 + 2|\alpha|^2|\beta|^2\Big) &\le \frac{1}{4}\int \langle \beta, N_J^* \circ \partial_B \alpha\rangle \\ &\le C\|\beta\|\|\nabla_B\alpha\| \\ &\le \frac{C}{2}\|\beta\|^2 + \frac{1}{2}\|\nabla_B\alpha\|,\end{aligned}$$

and I didn't get the last two lines. ⊠