M392C NOTES: REPRESENTATION THEORY

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Lecture 1.

Lie groups and smooth actions: 1/18/17

"I've never even seen this many people in a graduate class... I hope it's good."

Today we won't get too far into the math, since it's the first day, but we'll sketch what exactly we'll be talking about this semester.

This class is about representation theory, which is a wide subject: previous incarnations of the subject might not intersect much with what we'll do, which is the representation theory of Lie groups, algebraic groups, and Lie algebras. There are other courses which cover Lie theory, and we're not going to spend much time on the basics of differential geometry or topology. The basics of manifolds, topological spaces, and algebra, as covered in a first-year graduate class, will be assumed.

In fact, the class will focus on the reductive semisimple case (these words will be explained later). There will be some problem sets, maybe 2 or 3 in total. The problem sets won't be graded, but maybe we'll devote a class midsemester to going over solutions. If you're a first-year graduate student, an undergraduate, or a student in another department, you should turn something in, as per usual.

Time for math.

We have to start somewhere, so let's define Lie groups.

Definition 1.1. A *Lie group* G is a group object in the category of smooth manifolds. That is, it's a smooth manifold G that is also a group, with an operation $m : G \times G \to G$, a C^∞ map satisfying the usual group axioms (e.g. a C^∞ inversion map, associativity).

Though in the early stages of group theory we focus on finite or at least discrete groups, such as the dihedral groups, which describe the symmetries of a polygon. These have discrete symmetries. Lie groups are the objects that describe continuous symmetries; if you're interested in these, especially if you come from physics, these are much more fundamental.

Example 1.2. The group of $n \times n$ invertible matrices (those with nonzero determinant) is called the *general linear group* $\mathrm{GL}_n(\mathbb{R})$. Since the determinant is multiplicative, this is a group; since $\det(A) \neq 0$ is an open condition, as the determinant is continuous, $\mathrm{GL}_n(\mathbb{R})$ is a manifold, and you can check that multiplication is continuous. ◀

Example 1.3. The *special linear group* $\mathrm{SL}_n(\mathbb{R})$ is the group of $n \times n$ matrices with determinant 1. This is again a group, and to check that it's a manifold, one has to show that 1 is a regular value of $\det : M_n(\mathbb{R}) \to \mathbb{R}$. But this is true, so $\mathrm{SL}_n(\mathbb{R})$ is a Lie group. ◀

Example 1.4. The *orthogonal group* $\mathrm{O}(n) = \mathrm{O}(n, \mathbb{R})$ is the group of orthogonal matrices, those matrices A for which $A^t = A^{-1}$. Again, there's an argument here to show this is a Lie group. ◀

You'll notice most of these are groups of matrices, and this is a very common way for Lie groups to arise, especially in representation theory.

We can also consider matrices with complex coefficients.

Example 1.5. The *complex general linear group* $\mathrm{GL}_n(\mathbb{C})$ is the group of $n \times n$ invertible complex matrices. This has several structures.

- For the same reason as $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$ is a Lie group.
- $\mathrm{GL}_n(\mathbb{C})$ is also a *complex Lie group*: it's a complex manifold, and multiplication and inversion are not just smooth, but holomorphic.
- It's also a *algebraic group* over $\mathbb{C}$: a group object in the category of algebraic varieties. This perspective will be particularly useful for us.

We can also define the *unitary group* $\mathrm{U}(n)$, the group of $n \times n$ complex matrices such that $A^\dagger = A^{-1}$: their inverses are their transposes. One caveat is that this is *not* a complex Lie group, as this equation isn't holomorphic. For example, $\mathrm{U}(1) = \{z \in \mathbb{C} \text{ such that } |z| = 1\}$ is topologically S^1, and therefore is one-dimensional as a real manifold! This is also $\mathrm{SO}(2)$ (the circle acts by rotating $\mathbb{R}^2$). More generally, a *torus* is a finite product of copies of $\mathrm{U}(1)$. ◀

There are other examples that don't look like this, *exceptional groups* such as G_2, E_6, and F_4 which are matrix groups, yet not obviously so. We'll figure out how to get these when we discuss the classification of simple Lie algebras.

Here's an example of interest to physicists:

Example 1.6. Let q be a quadratic form of signature $(1, 3)$ (corresponding to Minkowski space). Then, $\mathrm{SO}(1, 3)$ denotes the group of matrices fixing q (origin-fixing isometries of Minkowski space), and is called the *Lorentz group*.

Smooth actions. If one wants to add translations, one obtains the *Poincaré group* $\mathrm{SO}(1, 3) \ltimes \mathbb{R}^{1,3}$. ◀

In a first course on group theory, one sees actions of a group G on a set X, usually written $G \curvearrowright X$ and specified by a map $G \times X \to X$, written $(g, x) \mapsto g \cdot x$. Sometimes we impose additional structure; in particular, we can let X be a smooth manifold, and require G to be a Lie group and the action to be smooth, or Riemannian manifolds and isometries, etc.[1]

It's possible to specify this action by a continuous group homomorphism $G \to \mathrm{Diff}(X)$ (or even smooth: $\mathrm{Diff}(X)$ has an infinite-dimensional smooth structure, but being precise about this is technical).

Example 1.7. $\mathrm{SO}(3) := \mathrm{SL}_3(\mathbb{R}) \cap \mathrm{O}(3)$ denotes the group of rotations of three-dimensional space. Rotating the unit sphere defines an action of $\mathrm{SO}(3)$ on S^2, and this is an *action by isometries*, i.e. for all $g \in \mathrm{SO}(3)$, the map $S^2 \to S^2$ defined by $x \mapsto g \cdot x$ is an isometry. ◀

Example 1.8. Let $\mathbb{H} := \{x + iy \mid y > 0\}$ denote the upper half-plane. Then, $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ by Möbius transformations. ◀

[1] What if X has singular points? It turns out the axioms of a Lie group action place strong constraints on where singularities can appear in interesting situations, though it's not completely ruled out.

Smooth group actions arise in physics: if S is a physical system, then the symmetries of S often form a Lie group, and this group acts on the space of configurations of the system.

Where do representations come into this? Suppose a Lie group G acts on a space X. Then, G acts on the complex vector space of functions $X \to \mathbb{C}$, and G acts by linear maps, i.e. for each $g \in G$, $f \mapsto f(g \cdot -)$ is a linear map. This is what is meant by a representation, and for many people, choosing certain kinds of functions on X (smooth, continuous, L^2) is a source of important representations in representation theory. Representations on $L^2(X)$ are particularly important, as $L^2(X)$ is a Hilbert space, and shows up as the state space in quantum mechanics, where some of this may seem familiar.

Representations.

Definition 1.9. A *(linear) representation* of a group G is a vector space V together with an action of G on V by linear maps, i.e. a map $G \times V \to V$ written $(g, v) \mapsto g \cdot v$ such that for all $g \in G$, the map $v \mapsto g \cdot v$ is linear.

This is equivalent to specifying a group homomorphism $G \to \mathrm{GL}(V)$.[2] Sometimes we will abuse notation and write V to mean V with this extra structure.

If G is in addition a Lie group, one might want the representation to reflect its smooth structure, i.e. requiring that the map $G \to \mathrm{GL}(V)$ be a homomorphism of Lie groups.

The following definition, codifying the idea of a representation that's as small as can be, is key.

Definition 1.10. A representation V is *irreducible* if it has no nontrivial invariant subspaces. That is, if $W \subseteq V$ is a subspace such that for all $w \in W$ and $g \in G$, $g \cdot w \in W$, then either $W = 0$ or $W = V$.

We can now outline some of the goals of this course:

- Classify the irreducible representations of a given group.
- Classify all representations of a given group.
- Express arbitrary representations in terms of irreducibles.

These are not easy questions, especially in applications where the representations may be infinite-dimensional.

Example 1.11 (Spherical harmonics)**.** Here's an example of this philosophy in action.[3]

Let's start with the Laplacian on $\mathbb{R}^3$, a second-order differential operator

$$\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2,$$

which acts on $C^\infty(\mathbb{R}^3)$. After rewriting in spherical coordinates, the Laplacian turns out to be a sum

$$\Delta = \frac{1}{r^2}\Delta_{\mathrm{sph}} + \Delta_{\mathrm{rad}},$$

of spherical and radial parts independent of each other, so Δ_{sph} acts on functions on the sphere. We're interested in the eigenfunctions for this spherical Laplacian for a few reasons, e.g. they relate to solutions to the *Schrödinger equation*

$$\dot{\psi} = \widehat{H}(\psi),$$

where the Hamiltonian is

$$\widehat{H} = -\Delta + V(r),$$

where V is a potential.

The action of $\mathrm{SO}(3)$ on the sphere by rotation defines a representation of $\mathrm{SO}(3)$ on $C^\infty(S^2)$, and we'll see that finding the eigenfunctions of the spherical Laplacian boils down to computing the irreducible components inside this representation:

$$V_0 \oplus V_2 \oplus V_4 \oplus \cdots \overset{\text{dense}}{\subseteq} C^\infty(S^2),$$

where the V_{2k} run through each isomorphism class of irreducible representations of $\mathrm{SO}(3)$. They are also the eigenspaces for the spherical Laplacian, where the eigenvalue for V_{2k} is $\pm k(k+1)$, and this is not a coincidence since the spherical Laplacian is what's known as a Casimir operator for the Lie algebra $\mathfrak{so}(3)$. We'll see more things like this later, once we have more background. ◀

[2]This general linear group $\mathrm{GL}(V)$ is the group of invertible linear maps $V \to V$.

[3]No pun intended.

Lecture 2.

Representations theory of compact groups: 1/20/17

First, we'll discuss some course logistics. There are course notes (namely, the ones you're reading now) and a website, https://www.ma.utexas.edu/users/gunningham/reptheory_spring17.html. We won't stick to one textbook, as indicated on the website, but the textbook of Kirrilov is a good reference, and is available online. The class' office hours will be Monday from 2 to 4 pm, at least for the time being.

The course will fall into two parts.

(1) First, we'll study finite-dimensional representations of things such as compact Lie groups (e.g. $\mathrm{U}(n)$ and $\mathrm{SU}(2)$) and their complexified Lie algebras, reductive Lie algebras (e.g. $\mathfrak{gl}_n(\mathbb{C})$ and $\mathfrak{sl}_2(\mathbb{C})$). There is a nice dictionary between the finite-dimensional representation theories of these objects. The algebra $\mathfrak{sl}_2(\mathbb{C})$ is semisimple, which is stronger than reductive. Every reductive Lie algebra decomposes into a sum of a semisimple Lie algebra and an abelian Lie algebra, and abelian Lie algebras are relatively easy to understand, so we'll dedicate some time to semisimple Lie algebras.

We'll also spend some time understanding the representation theory of reductive algebraic groups over $\mathbb{C}$, e.g. $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{SL}_2(\mathbb{C})$. Again, there is a dictionary between the finite-dimensional representations here and those of the Lie groups and reductive Lie algebras we discussed.

All together, these form a very classical and standard subject that appears everywhere in algebra, analysis, and physics.

(2) We'll then spend some time on the typically infinite-dimensional representations of noncompact Lie groups, such as $\mathrm{SL}_2(\mathbb{R})$ or the Lorentz group $\mathrm{SO}(1,3)$. These groups have interesting infinite-dimensional, but irreducible representations; the classification of these representations is intricate, with analytic issues, yet is still very useful, tying into among other things the Langlands program.

All these words will be defined when they appear in this course.

We'll now begin more slowly, with some basics of representations of compact groups.

Example 2.1. Here are some examples of compact topological groups.

- Finite groups.
- Compact Lie groups such as $\mathrm{U}(n)$.
- The p-adics $\mathbb{Z}_p$ with their standard topology: two numbers are close if their difference is divisible by a large power of p. $\mathbb{Z}_p$ is also a profinite group. ◀

Definition 2.2. Let G be a compact group. A *(finite-dimensional) (continuous) (complex) representation* of G is a finite-dimensional complex vector space V together with a continuous homomorphism $\rho : G \to \mathrm{GL}(V)$.

If you pick a basis, $V \cong \mathbb{C}^n$, so $\mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{C})$. These are the $n \times n$ invertible matrices over the complex numbers, so ρ assigns a matrix to each $g \in G$ in a continuous manner, where $\rho(g_1g_2) = \rho(g_1)\rho(g_2)$, so group multiplication is sent to matrix multiplication. Sometimes it's more natural to write this through the *action map* $G \times V \to V$ sending $(g, v) \mapsto \rho(g) \cdot v$.

The plethora of parentheses in Definition 2.2 comes from the fact that representations may exist over other fields, or be infinite-dimensional, or be discontinuous, but in this part of the class, when we say a representation of a compact group, we mean a finite-dimensional, complex, continuous one.

Example 2.3. Let S_3 denote the *symmetric group on* 3 *letters*, the group of bijections $\{1, 2, 3\} \to \{1, 2, 3\}$ under composition. Its elements are written in *cycle notation*: $(1\ 2)$ is the permutation exchanging 1 and 2, and $(1\ 2\ 3)$ sends $1 \mapsto 2$, $2 \mapsto 3$, and $3 \mapsto 1$. There are six elements of S_3: $S_3 = \{e, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$.

For representation theory, it can be helpful to have a description in terms of generators and relations. Let $s = (1\ 2)$ and $t = (2\ 3)$, so $(1\ 3) = sts = tst$, $(1\ 2\ 3) = st$, and $(1\ 3\ 2) = ts$. Thus we obtain the presentation

$$S_3 = \langle s, t \mid s^2 = t^2 = e, sts = tst\rangle. \tag{2.4}$$

The relation $sts = tst$ is an example of a *braid relation*; there exist similar presentations for all the symmetric groups, and this leads into the theory of Coxeter groups.

There's a representation you can always build for any group called the *trivial representation*, in which $V = \mathbb{C}$ and $\rho_{\mathrm{triv}} : G \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$ sends every $g \in G$ to the identity map $(1 \in \mathbb{C}^\times)$.

To get another representation, let's remember that we wanted to build representations out of functions on spaces. S_3 is a discrete space, so let's consider the space $X = \{x_1, x_2, x_3\}$ (with the discrete topology). Then, S_3 acts on X by permuting the indices; we want to linearize this.

Let $V = \mathbb{C}[X] = \mathbb{C}x_1 \oplus \mathbb{C}x_2 \oplus \mathbb{C}x_3$, a complex vector space with basis $\{x_1, x_2, x_3\}$. We'll have S_3 act on V by permuting the basis; this is an example of a *permutation representation.* This basis defines an isomorphism $\mathrm{GL}(V) \xrightarrow{\sim} \mathrm{GL}_3(\mathbb{C})$, which we'll use to define a representation. Since s swaps x_1 and x_2, but fixes x_3, it should map to the matrix

$$s \longmapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly,

$$t \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(You should check that these assignments satisfy the relations in (2.4).)

Now something interesting happens. When you think of X as an S_3-space, it has no invariant subsets (save itself and the empty set). But this linearization is no longer irreducible: the element $v = x_1 + x_2 + x_3 \in V$ is fixed by all permutations acting on X: $\rho(g) \cdot v = v$ for all $g \in S_3$.

More formally, let W be the subspace of V spanned by v; then, W is a subrepresentation of V. ◀

Let's take a break from this example and introduce some terminology.

Definition 2.5. Let G be a group and V be a G-representation. A *subrepresentation* or *G-invariant subspace* of V is a subspace $W \subseteq V$ such that for all $g \in G$ and $w \in W$, $g \cdot w \in W$. If V has no nontrivial (i.e. not 0 or V) subrepresentations, V is called *irreducible.*

This means the same G-action defines a representation on W.

If $W \subseteq V$ is a subrepresentation, the quotient vector space V/W is a G-representation, called the *quotient representation*, and the G-action is what you think it is: in coset notation, $g \cdot (v + W) = (gv + W)$. This is well-defined because W is G-invariant.

We can always take quotients, but unlike for vector spaces in general, it's more intricate to try to find a complement: does the quotient split to identify V/W with a subrepresentation of V?

Returning to Example 2.3, we found a three-dimensional representation V and a one-dimensional subrepresentation W. Let's try to find another subrepresentation U of V such that, as S_3-representations, $V \cong W \oplus U$. The answer turns out to be $U = \mathrm{span}_{\mathbb{C}}\{x_1 - x_2, x_2 - x_3\}$.

Claim. U is a subrepresentation, and $V = U \oplus W$.

This isn't as obvious, because neither $x_1 - x_2$ or $x_2 - x_3$ is fixed by all elements of S_3. However, for any $g \in S_3$, $g \cdot (x_1 - x_2)$ is contained in U, and similarly for $x_2 - x_3$. Let's set $U \cong \mathbb{C}^2$ with $x_1 - x_2 \mapsto \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $x_2 - x_3 \mapsto \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. Then, we can explicitly describe U in terms of the matrices for s and t in $\mathrm{GL}(U) \cong \mathrm{GL}_2(\mathbb{C})$, where the identification uses this basis.

Since $s = (1\ 2)$, it sends $x_1 - x_2 \mapsto x_2 - x_1 = -(x_1 - x_2)$ and $x_2 - x_3 \mapsto x_1 - x_3 = (x_1 - x_2) + (x_2 - x_3)$, so

$$s \longmapsto \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}.$$

In the same way,

$$t \longmapsto \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}.$$

The general theme of finding interesting representations inside of naturally arising representations will occur again and again in this class.

Lecture 3.

Operations on representations: 1/23/17

"I want to become a representation theorist!"
"Are you Schur?"

Last time, we discussed representations of groups and what it means for a representation to be irreducible; today, we'll talk about some other things one can do with representations. For the time being, G can be any group; we will specialize later.

The first operation on a representation is very important.

Definition 3.1. A *homomorphism* of G-representations $V \to W$ is a linear map $\varphi : V \to W$ such that for all $g \in G$, the diagram

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "g\cdot"] & W \arrow[d, "g\cdot"] \\ V \arrow[r, "\varphi"] & W. \end{tikzcd}$$

This is also called an *intertwiner*, a *G-homomorphism*, or a *G-equivariant* map.

An *isomorphism* of representations is a homomorphism that's also a bijection.[4]

More explicitly, this means φ commutes with the G-action, in the sense that $\varphi(g \cdot v) = g \cdot \varphi(v)$. This is one advantage of dropping the ρ-notation: it makes this definition cleaner.

Remark. If $\varphi : V \to W$ is a G-homomorphism, then $\ker(\varphi) \subseteq V$ is a subrepresentation, and similarly for $\mathrm{Im}(\varphi) \subseteq W$. ◀

The set of G-homomorphisms from V to W is a complex vector space,[5] denote $\mathrm{Hom}_G(V, W)$.

Several constructions carry over from the world of vector spaces to the world of G-representations. Suppose V and W are G-representations.

- The direct sum $V \oplus W$ is a G-representation with the action $g \cdot (v, w) = (g \cdot v, g \cdot w)$. This has dimension $\dim V + \dim W$.
- The tensor product $V \otimes W$ is a G-representation: since it's generated by pure tensors, it suffices to define $g \cdot (v \otimes w)$ to be $(gv) \otimes (gw)$ and check that this is compatible with the relations.
- The dual space $V^* := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is a G-representation: if $\alpha \in V^*$, we define $(g \cdot \alpha)(v) := \alpha(g^{-1}v)$. This might be surprising: you would expect $\alpha(gv)$, but this doesn't work: you want $g \cdot (h\alpha)$ to be $(gh) \cdot \alpha$, but you'd get $(hg) \cdot \alpha$. This is why you need the g^{-1}.
- Since $\mathrm{Hom}_{\mathbb{C}}(V, W)$ is naturally isomorphic to $V^* \otimes W$,[6] it inherits a G-representation structure.

Definition 3.2. Given a G-representation V, the *space of G-invariants* is the space

$$V^G := \{v \in V \mid g \cdot v = v\}.$$

This can be naturally identified with $\mathrm{Hom}_G(\mathbb{C}_{\mathrm{triv}}, V)$, where $\mathbb{C}_{\mathrm{triv}}$ is the trivial representation with action $g \cdot z = z$ for all $z \in \mathbb{C}$. The identification comes by asking where 1 goes to.

These are also good reasons for using the action notation rather than writing $\rho : G \to \mathrm{Aut}(V)$, which would require more complicated formulas.

There are a couple of different ways of stating Schur's lemma, but here's a good one.

Lemma 3.3 (Schur)**.** *Let V and W be irreducible G-representations. Then,*

$$\mathrm{Hom}_G(V, W) = \begin{cases} 0, & \text{if } V \not\cong W \\ \mathbb{C}, & \text{if } V \cong W. \end{cases}$$

"Irreducible" is the key word here.

Remark. Schur's lemma requires us to work over $\mathbb{C}$ (more generally, over any algebraically closed field). It also assumes that V and W are finite-dimensional. There are no assumptions on G; this holds in much greater generality (e.g. over $\mathbb{C}$-linear categories). ◀

[4]If $f : V \to W$ is an isomorphism of representations, then $f^{-1} : W \to V$ is also a G-homomorphism, making this a reasonable definition. This is a useful thing to check, and doesn't take too long.

[5]Recall that we're focusing exclusively on complex representations. If we look at representations over another field k, we'll get a k-vector space.

[6]This depends on the fact that V and W are finite-dimensional.

In general, there's a distinction between "isomorphic" and "equal" (or at least naturally isomorphic); in the latter case, there's a canonical isomorphism, namely the identity. In this case, the second piece of Lemma 3.3 can be restated as saying for any irreducible G-representation V,

$$\mathrm{Hom}_G(V, V) = \mathbb{C} \cdot \mathrm{id}_V.$$

Thus, any G-homomorphism $\varphi : V \to V$ is $\lambda \cdot \mathrm{id}_V$ for some $\lambda \in \mathbb{C}$, and in a basis is a diagonal matrix with every diagonal element equal to λ.

Proof of Lemma 3.3. Suppose $\varphi : V \to W$ is a nonzero G-homomorphism. Thus, $\ker(\varphi) \subset V$ isn't be all of V, so since V is irreducible it must be 0, so φ is injective. Similarly, since $\mathrm{Im}(\varphi) \subset W$ isn't 0, it must be all of W, since W is irreducible, Thus, φ is an isomorphism, so if $V \not\cong W$, the only G-homomorphism is the zero map.

Now, suppose $\varphi : V \to V$ is a G-homomorphism. Since $\mathbb{C}$ is algebraically closed, φ has an eigenvector: there's a $\lambda \in \mathbb{C}$ and a $v \in V$ such that $\varphi(v) = \lambda \cdot v$. Since φ and $\lambda \cdot \mathrm{id}_V$ are G-homomorphisms, so is $\varphi - \lambda \mathrm{id}_V : V \to V$, so its kernel, the λ-eigenspace of φ, is a subrepresentation of V. Since it's nonzero, then it must be all of V, so $V = \ker(\varphi - \lambda \mathrm{id}_V)$, and therefore $\varphi = \lambda \mathrm{id}_V$. ⊠

This is the cornerstone of representation theory, and is one of the reasons that the theory is so much nicer over $\mathbb{C}$.

Corollary 3.4. *If G is an abelian group, then any irreducible representation of G is one-dimensional.*

Proof. Let V be an irreducible G-representation and $g \in G$. Since G is abelian, $v \mapsto gv$ is a G-homomorphism: $g \cdot (hv) = h(g \cdot v)$. By Schur's lemma, the action of g is $\lambda \cdot \mathrm{id}_V$ for some $\lambda \in \mathbb{C}$, so any $W \subseteq V$ is G-invariant. Since V is irreducible, this can only happen when V is 1-dimensional. ⊠

Example 3.5. Let's talk about the irreducible representations of $\mathbb{Z}$. This isn't a compact group, but we'll survive. A representation of $\mathbb{Z}$ is a homomorphism $\mathbb{Z} \to \mathrm{GL}(V)$ for some vector space V; since $\mathbb{Z}$ is a free group, this is determined by what 1 goes to, which can be chosen freely. That is, representations of $\mathbb{Z}$ are in bijection with invertible matrices.

By Corollary 3.4, irreducible $\mathbb{Z}$-representations are the 1-dimensional invertible matrices, which are identified with $\mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$, the nonzero complex numbers. ◂

Our greater-scope goal is to understand all representations by using irreducible ones as building blocks. In the best possible case, your representation is a direct-sum of irreducibles; we'll talk about that case next time.

Lecture 4.

Complete reducibility: 1/25/17

We've discussed what it means for a representation to be irreducible, and irreducible representations are the smallest representations; we hope to build all representations out of irreduicbles. The nicest possible case is complete reducibility, which we'll discuss today.

Suppose G is a group (very generally), V is a representation, and $W \subseteq V$ is a subrepresentation. If $i : W \hookrightarrow V$ denotes the inclusion map, then there's a projection map onto the quotient $V \twoheadrightarrow U := V/W$. This is encoded in the notion of a *short exact sequence*:

$$0 \longrightarrow W \xrightarrow{i} V \xrightarrow{j} U \longrightarrow 0,$$

which means exactly that i is injective, j is surjective, and $\mathrm{Im}(i) = \ker(j)$. The nicest short exact sequence is

$$0 \longrightarrow W \longrightarrow W \oplus U \longrightarrow U \longrightarrow 0,$$

where the first map is inclusion into the first factor and the second is projection onto the second factor. In this case, one says the short exact sequence *splits*. This is equivalent to specifying a projection $V \twoheadrightarrow U$ or an inclusion $W \hookrightarrow V$. Since direct sums are easier to understand, this is the case we'd like to know better.

Example 4.1. We saw last time that a representation of $\mathbb{Z}$ is given by the data of an invertible matrix which specifies the action of 1, akin to a discrete translational symmetry.

Consider the $\mathbb{Z}$-representation V on $\mathbb{C}^2$ given by the matrix

$$A := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

This is not an irreducible representation, because $\binom{1}{0}$ is an eigenvector for A with eigenvalue 1, so

$$W := \left\{ \begin{pmatrix} a \\ 0 \end{pmatrix} \mid a \in \mathbb{C} \right\}$$

is a subrepresentation of V. Since $\binom{1}{0}$ has eigenvalue 1, W is the trivial representation $\mathbb{C}_{\mathrm{triv}}$. Moreover, the quotient V/W is also the trivial representation, so V sits in a short exact sequence

$$0 \longrightarrow \mathbb{C}_{\mathrm{triv}} \longrightarrow V \longrightarrow \mathbb{C}_{\mathrm{triv}} \longrightarrow 0.$$

However, V itself is not trivial, or it would have been specified by a diagonalizable matrix. Thus, V is not a direct sum of subrepresentations (one says it's *indecomposable*), but it's not irreducible! If U is a 1-dimensional subrepresentation of V, then $U = \mathbb{C}v$ for some nonzero $v \in V$. Since $Av \in U$, $Av = \lambda v$ for some $\lambda \in \mathbb{C}$, meaning v is an eigenvector, and in our particular case, $v = e_1$. Thus any direct sum of two nontrivial subrepresentations must be $U + U = U$, which isn't all of V. ◀

We want to avoid these kinds of technicalities on our first trek through representation theory, and fortunately, we can.

Definition 4.2. A representation V of G is *completely reducible* or *semisimple* if every subrepresentation $W \subseteq V$ has a *complement*, i.e. another subrepresentation U such that $V \cong U \oplus W$.

Remark. There are ways to make this more general, e.g. for infinite-dimensional representations, one may want closed subrepresentations. But for finite-dimensional representations, this definition suffices. ◀

A finite-dimensional semisimple representation V is a direct sum of its irreducible subrepresentations. The idea is that its subrepresentations must also be semisimple, so you can use induction.

The terminology "semisimple" arises because *simple* is a synonym for irreducible, in the context of representation theory.

So semisimple representations are nice. You might ask, *for which groups G are all representations semisimple*? To answer this question, we'll need a few more concepts.

Definition 4.3. A representation V of G is called *unitary* if it admits a G-invariant inner product, i.e. a map $B : V \times V \to \mathbb{C}$ that is:

- linear in the first factor and antilinear in the other,[7]
- *antisymmetric*, i.e. $B(v, w)) = \overline{B(w, v)}$,
- *positive definite*, i.e. $B(v, v) \geq 0$ and $B(v, v) = 0$ iff $v = 0$, and
- *G-invariant*, meaning $B(g \cdot v, g \cdot w) = B(v, w)$ for all $g \in G$.

The reason for the name is that if V is a unitary representation with form B, then as a map $G \to \mathrm{GL}(V) \cong \mathrm{GL}(\mathbb{C}^n)$, this representation factors through $\mathrm{U}(n)$, the *unitary matrices*, which preserve the standard Hermitian inner product on $\mathbb{C}^n$. So unitary representations are representations of G into some unitary group.

Proposition 4.4. *Unitary representations are completely reducible.*

Proof sketch. How would you find a complement to a subspace? The usual way to do this is to take an orthogonal complement, and an invariant inner product is what guarantees that the orthogonal complement is a subrepresentation.

Let V be a unitary representation and $B(-,-)$ be its invariant inner product. Let $W \subseteq V$ be a subrepresentation, and let

$$U = W^\perp := \{v \in V \mid B(v, w) = 0 \text{ for all } w \in W\}.$$

Then, U is a subrepresentation (which you should check), and $V = W \oplus U$. ⊠

[7]Some people have the opposite convention, defining the first factor to be antilinear and the second to be linear. Of course, the theory is equivalent. Such a form is called a *Hermitian form*, and if it's antisymmetric and positive definite, it's called a *Hermitian inner product*.

Classifying unitary representations is a hard problem in general. Building invariant Hermitian forms isn't too bad, but making them positive definite is for some reason much harder. In any case, for compact groups there is no trouble.

Proposition 4.5.

(1) *Given an irreducible unitary representation, the invariant form $B(-,-)$ is unique up to multiplication by a positive scalar.*
(2) *If $W_1, W_2 \subseteq V$, where V is unitary and W_1 and W_2 are nonisomorphic irreducible subrepresentations, then W_1 is orthogonal to W_2, i.e. $B(w_1, w_2) = 0$ for all $w_1 \in W_1$ and $w_2 \in W_2$.*

Proof. Part (1) is due to Schur's lemma (Theorem 3.3). Consider the map $B : V \to \overline{V}^*$ defined by $v \mapsto B(v, -)$ (here, $\overline{V}$ is the *conjugate space*, where the action of $a+bi$ on $\overline{V}$ is the action of $a-bi$ on V); in fact, you could use this map to define a unitary structure on a representation. B is a G-isomorphism, so by Schur's lemma, every such isomorphism, derived from every possible choice of Hermitian form, must be a scalar multiple of this one. Since B must be positive definite, this scalar had better be positive. ⊠

One particular corollary of part (1) is that if V is a unitary representation, $V^* \cong \overline{V}$.

So if you care about compact groups (in particular finite groups), this is all you need.

Theorem 4.6 (Maschke)**.** *Any representation of a compact group admits a unitary structure.*

We'll give a first proof for finite groups, then later one for Lie groups; we probably won't prove the most general case.

Proof for finite groups. Let G be a finite group and V be a G-representation. The first step to finding a G-invariant inner product is to find any Hermitian inner prodict, e.g. picking a basis and declaring it to be orthonormal, yielding an inner product B_0 that's probably not G-invariant.

We want to average B_0 over G to obtain something G-invariant, which is why we need finiteness.[8] That is, let

$$B(v, w) \coloneqq \frac{1}{|G|} \sum_{g \in G} B_0(g \cdot v, g \cdot w).$$

Then, B is a unitary structure: it's still positive definite and Hermitian, and since multiplication by an $h \in H$ is a bijection on G, this is G-invariant. ⊠

Lecture 5.

Some examples: 1/27/17

Last time, we claimed that every finite-dimensional representation of a compact group admits a unitary structure (Theorem 4.6), and therefore is completely reducible. We proved it for finite groups; later today, we'll talk about the Haar measure on a Lie group and how to use it to prove Theorem 4.6. The Haar measure also exists on topological groups more generally, but we won't construct it.

Semisimplicity is a very nice condition, and doesn't exist in general, just as most operators aren't self-adjoint, and so generally don't have discrete spectra. This is more than just a metaphor: given a compact group G, let $\widehat{G}$ denote the set of isomorphism classes of irreducible representations of G, sometimes called its *spectrum.* (In some cases, this has topology, but when G is compact, $\widehat{G}$ is discrete.) Using Theorem 4.6, any representation V of G is a direct sum of irreducibles in a unique way (up to permuting the factors):

$$V = \bigoplus_{W \in \widehat{G}} W^{\oplus m_W},$$

where m_W is the *multiplicity* of W in V. The summand $W^{\oplus m_W}$ is called the *W-isotypical component* of V. The analogy with the Fourier transform can be made stronger.

[8] More generally, we could take a compact group, replacing the sum with an integral. Compactness is what guarantees that the integral converges.

Example 5.1. Let $G = S_3$. We've already seen the trivial representation $\mathbb{C}_{\mathrm{triv}}$ and a representation $V = \mathbb{C}(x_1 - x_2) \oplus \mathbb{C}(x_2 - x_3)$, where S_3 acts by permuting the x_i terms; we showed that V is an irreducible representation of dimension 2.

There's a third irreducible representation $\mathbb{C}_{\mathrm{sign}}$, a one-dimensional (therefore necessarily irreducible) representation $\rho_{\mathrm{sign}} : S_3 \to \mathbb{C}^\times$, where $\rho_{\mathrm{sign}}(\sigma)$ is 1 if σ is even and -1 if σ is odd.

Exercise 5.2. Show that $\mathbb{C}_{\mathrm{triv}}$, V, and $\mathbb{C}_{\mathrm{sign}}$ are all of the irreducible representations of S_3 (up to isomorphism).

We'll soon see how to prove this quickly, though it's not too bad to do by hand. ◀

Example 5.3. The *circle group*, variously denoted S^1, U(1), SO(2), $\mathbb{R}/\mathbb{Z}$, or $\mathbb{T}$, is the abelian compact Lie group of real numbers under addition modulo $\mathbb{Z}$. Corollary 3.4 tells us all irreducible representations of S^1 are 1-dimensional.

Since S^1 is a quotient of the additive group $(\mathbb{R}, +)$, we'll approach this problem by classifying the one-dimensional representations of $\mathbb{R}$ and seeing which ones factor through the quotient. Such a representation is a map $(\mathbb{R}, +) \to (\mathbb{C}^\times, \times) \cong \mathrm{GL}_1(\mathbb{C})$; in other words, it turns addition into multiplication.

We know a canonical way to do this: for any $\xi \in \mathbb{C}$, there's a representation $\chi_\xi \colon t \mapsto e^{\xi t}$. And it turns out these are all of the continuous 1-dimensional representations of $\mathbb{R}$ up to isomorphism.

Exercise 5.4. Prove this: that every 1-dimensional representation $\rho : \mathbb{R} \to \mathbb{C}^\times$ is isomorphic to some χ_ξ. Here's a hint:

(1) First reduce to the case where ρ is C^1. (This requires techniques we didn't ask as prerequisites.)
(2) Now, show that ρ satisfies a differential equation $\rho'(t) = \rho'(0)\rho(t)$. As ρ is a homomorphism, this means $\rho(0) = 1$. Then, uniqueness of solutions of ODEs shows $\rho = \chi_{\rho'(0)}$.

There's something interesting going on here: representations of $\mathbb{R}$ are determined by their derivatives at the identitty. This is a shadow of a grander idea: to understand the representations of a Lie group G, look at representations of its Lie algebra $\mathfrak{g}$.

The noncompactness of $\mathbb{R}$ is also reflected in its representation theory: not all of its representations are unitary. U(1) is the group of unit complex numbers under multiplication, so χ_ξ is unitary iff $\xi = i\eta$ for $\eta \in \mathbb{R}$. In this case, $\widehat{\mathbb{R}}$ is used to denote the unitary representations, so $\widehat{\mathbb{R}} = i\mathbb{R}$. In particular, it's a group (since $\mathbb{R}$ is abelian), abstractly isomorphic to $\mathbb{R}$, and not discrete (as $\mathbb{R}$ is not compact).

Great; now, when does χ_ξ descend to a representation of S^1? This means that $\chi_\xi(0) = \chi_\xi(1)$, so ξ is an integer multiple of $2\pi i$. Thus, S^1, a compact group, has a discrete set of irreducible representations, and $\widehat{S}^1 = 2\pi i\mathbb{Z}$. You might be able to see the Fourier series hiding in here. ◀

Example 5.5. Let's look at a nonabelian example, SU(2), the set of matrices

$$\mathrm{SU}(2) \coloneqq \left\{ \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \mid a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}.$$

As a manifold, this is isomorphic to the unit sphere $S^3 \subset \mathbb{C}^2$, and indeed it's a Lie group, compact and non-abelian.

Today we'll write down some representations; later, we can use theory to prove this is all of them. The first one, as for any matrix group, comes for free: the *standard representation* or *defining representation* uses the preexisting embedding $\rho_{\mathrm{std}} \colon \mathrm{SU}(2) \hookrightarrow \mathrm{GL}_2(\mathbb{C})$ sending $A \mapsto A$. We also have the trivial representation.

We'll obtain some other representations as functions on a space SU(2) acts on. Matrix multiplication defines a smooth action of SU(2) on $\mathbb{C}^2$, so consider the vector space P of polynomial functions on $\mathbb{C}^2$. Then, SU(2) acts on P as follows: if $f \in P$ and $A \in \mathrm{SU}(2)$,

$$A \cdot f(\mathbf{x}) \coloneqq f(A^{-1}\mathbf{x}).$$

The A^{-1} term arises because you need composition to go the right way:

$$A(Bf(\mathbf{x})) = f(B^{-1}A^{-1}\mathbf{x}) = f((AB)^{-1}\mathbf{x}) = (AB)f(\mathbf{x}).$$

This is an aspect of a very general principle: if a group acts on the left on a space, then it acts on the right on its space of functions. But you can always turn right actions into left actions using inverses.

This P is an infinite-dimensional representation, but it splits into homogeneous finite-dimensional subrepresentations. Let P_n denote the homogeneous polynomials of degree n, e.g. $P^3 = \mathrm{span}_{\mathbb{C}}\{x^3, x^2y, xy^2, y^3\}$.

Proposition 5.6. *These P_n are irreducible of dimension $n+1$, and form a complete list of isomorphism types of irreducible representations of* $\mathrm{SU}(2)$.

In particular, $\widehat{\mathrm{SU}(2)} = \mathbb{Z}_{\ge 0}$. We get a discrete set of representations, since SU(2) is compact, but it's not a group, because SU(2) isn't abelian. Abelian groups are nice because their representations are 1-dimensional, and compact groups are nice because their representations are discrete. Abelian compact groups are even nicer, but there's not so many of those. ◂

In the last few minutes, we'll say a little about integration. Next time, we'll talk about characters and matrix coefficients.

On a topological group, finding the Haar measure is a delicate matter (but you can do it); on a Lie group, it's simpler, but requires some differential geometry. For the finite-groups case of Maschke's theorem, we averaged an inner product over G so that it was G-invariant. For compact groups, we can't sum in general, since there are infinitely many elements, but you can integrate. So what we're looking for is a measure on a Lie group. This comes naturally from a volume form, and to prove G-invariance, we'd like it to be G-invariant itself.

That is, our wish list is a left-invariant volume form on a Lie group G, an $\omega_g \in \Lambda^{\mathrm{top}}(T_g^* G)$. But you can get this by choosing something at the identity and defining $\omega_g = (\cdot g^{-1})^* \omega_1$ (that is, pull it back under left multiplication by g^{-1}).

If G is compact, then you can use right-invariance to show the space of such forms is trivial, so the form is also unique. We'll return to this a little bit next time.

Lecture 6.

Matrix coefficients and characters: 1/30/17

We'll start with a review of the end of last lecture. Let G be a compact Lie group; then, there's a bi-invariant volume form ω on G, and gives G finite volume. You can normalize so that the total volume is 1. The measure this defines is G-invariant, and is called the *Haar measure*, written $\mathrm{d}g$.

Remark. If G is a finite group, this is $1/|G|$ times the counting measure. You can use this to obtain the finite-groups proof of Maschke's theorem, etc. from the Lie groups one, as the integrals becomes sums. ◂

Example 6.1. Let's consider the circle group U(1) again. We saw in Example 5.3 that its irreducible representations are of the form $\chi_n : t \mapsto e^{2\pi i n t}$ for $t \in \mathbb{R}/\mathbb{Z}$ (or $z \mapsto z^n$ for $z \in \mathrm{U}(1)$), indexed over $n \in \mathbb{Z}$.

We can tie this to Fourier series. Consider $C(\mathrm{U}(1), \mathbb{C}) = C(\mathrm{U}(1))$, the space of complex-valued continuous functions on U(1). We've already seen how a group acts on its space of functions, so $C(\mathrm{U}(1))$ is an infinite-dimensional representation of U(1). We want to decompose this as a sum of irreducibles χ_n. The χ_{-n}-isotypical component $\mathbb{C}\chi_n \subset C(\mathrm{U}(1))$ (since χ_n is itself a continuous, $\mathbb{C}$-valued function on U(1)) is isomorphic to the one-dimensional representation determined by χ_n. So we might hope to decompose $C(\mathrm{U}(1))$ as a sum of these χ_n indexed by the integers.

Consider the Hermitian inner product on $C(\mathrm{U}(1))$ in which

$$(f_1, f_2)_{L^2} := \int_G f_1(g)\overline{f_2(g)}\,\mathrm{d}g \tag{6.2}$$

(i.e. taken in the Haar measure); since U(1) is compact, this converges. The same construction may be made for any compact Lie group G. Now, we can ask how the χ_{-n}-isotypic components fit together inside $C(\mathrm{U}(1))$.

It's quick to check that, with the Haar measure normalized as above,

$$\langle \chi_n, \chi_m \rangle = \begin{cases} 1, & n = m \\ 0, & n \neq m. \end{cases}$$

If $C_{\mathrm{alg}}(\mathrm{U}(1))$ denotes the space of algebraic functions on U(1), then

$$C_{\mathrm{alg}}(\mathrm{U}(1)) = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}\chi_n \subset C(\mathrm{U}(1)).$$

Then, using the Stone-Weierstrass approximation theorem (which we take as a black box), $C_{\mathrm{alg}}(\mathrm{U}(1))$ is uniformly dense in $C(\mathrm{U}(1))$: any continuous function $\mathrm{U}(1) \to \mathbb{C}$ can be approximated by sums of these

χ_n. Also, if $L^2(\mathrm{U}(1))$ denotes the completion of $C(\mathrm{U}(1))$ in this inner product, $C_{\mathrm{alg}}(\mathrm{U}(1))$ is also dense in $L^2(\mathrm{U}(1))$.

This recovers the Fourier-theoretic statement: the χ_n are an orthonormal basis for $L^2(\mathrm{U}(1))$, so every function has a Fourier series, an infinite sum of characters. ◀

We want to do the same thing, but for a general compact Lie group G.

Let V be a finite-dimensional representation of G, and choose a basis of V, which provides an identification $\mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{C})$. Thus, the representation may be considered as a map $\rho_V : G \to \mathrm{GL}(V) \overset{\cong}{\to} \mathrm{GL}_n(\mathbb{C})$. Let $1 \le i, j \le n$; then, taking the ij^{th} component of a matrix defines a map $\mathrm{GL}_n(\mathbb{C}) \to \mathbb{C}$. Composing this map with ρ_V defines the *matrix coefficients map* $m_{V,i,j} : G \to \mathbb{C}$.

Picking a basis is unsatisfying, so let's state this in a more invariant way. Let $v \in V$ and $\alpha \in V^*$; they will play the role of i and j. Then, we have a matrix coefficients map

$$\begin{aligned} m_{V,\alpha,v} : G &\longrightarrow \mathbb{C} \\ g &\longmapsto \alpha(\rho_V(g) \cdot v). \end{aligned}$$

Another way to write this is as a map $m_V : V^* \otimes V \to C(G)$ sending $\alpha, v \mapsto m_{V,\alpha,v}$. Since $V^* \otimes V \cong \mathrm{End}(V)$ canonically, this is determined by where it sends the identity function 1_V. The resulting function, denoted χ_V, is called the *character* of V, and is the trace of $\rho_V(g)$: in a basis, the formula is

$$\chi_V(g) \coloneqq \sum_{i=1}^{n} m_{V,i,i}(g).$$

Characters are extremely useful in representation theory.

Inside $C(G)$, let $C_{\mathrm{alg}}(G)$ denote the union of $\mathrm{Im}(m_V)$ as V ranges over all finite-dimensional representations.

Lemma 6.3. *$C_{\mathrm{alg}}(G)$ is a subring of $C(G)$ under pointwise addition and multiplication.*

Proof sketch. The reason is that if V and W are representations, $v, w \in V$, and $\alpha, \beta \in V^*$, then

$$\begin{aligned} m_{V,\alpha,v} + m_{W,\beta,w} &= m_{V\oplus W,\alpha+\beta,v+w} \\ m_{V,\alpha,v} m_{W,\beta,w} &= m_{V\otimes W,\alpha\otimes\beta,v\otimes w}. \end{aligned}$$ ⊠

If you like algebra, this is very nice: we'll later see that $C_{\mathrm{alg}}(G)$ has the structure of a commutative Hopf algebra, meaning it's the ring of functions of a complex algebraic group.

Proposition 6.4.

(1) *If V is an irreducible representation of G, m_V is injective.*
(2) *If V and W are non-isomorphic irreducible representations of G, then $\mathrm{Im}(m_V) \perp \mathrm{Im}(m_W)$ in the L^2-inner product (6.2).*
(3) *If you restruct this inner product to $V^* \otimes V$ through m_V (where V is irreducible), it equals $(1/\dim V)(\text{–}, \text{–})_{V^* \otimes V}$, where*

$$(\alpha_1 \otimes v_1, \alpha_2 \otimes v_2) \coloneqq (\alpha_1, \alpha_2)_{V^*}(v_1, v_2)_V,$$

for any G-invariant inner product $(\text{–}, \text{–})_V$ and induced dual inner product $(\text{–}, \text{–})_{V^*}$.

Exercise 6.5. Under the canonical identification $V^* \otimes V \cong \mathrm{End}\, V$, what does this inner product do on $\mathrm{End}(V)$?

Now, consider $C(G)$ as an (infinite-dimensional) representation of $G \times G$, where the first factor acts on the left and the second acts on the right.

We'll prove the proposition next time. It turns out that $V^* \otimes V$ is irreducible as a $(G \times G)$-representation, and the matrix coefficients map is $(G \times G)$-equivariant. Thus, we can express all of this in terms of $(G \times G)$-invariant bilinear forms.

Lecture 7.

The Peter-Weyl theorem: 2/1/17

Last time, we introduced $C_{\mathrm{alg}}(G)$, the subspace of continuous (complex-valued) functions on G generated by matrix coefficients of finite-dimensional representations.

Theorem 7.1. *There is an isomorphism of $(G \times G)$-representations*

$$C_{\mathrm{alg}}(G) \cong \bigoplus_{V \in \widehat{G}}^{\perp} (V^* \otimes V),$$

where $\overset{\perp}{\oplus}$ denotes an orthogonal direct sum. This isomorphism preserves an invariant Hermitian form.

Proof sketch. Suppose $V \in \widehat{G}$; we then check $V^* \otimes V$ is an irreducible $(G \times G)$-representation (which is an exercise). Last time, we saw that $m_V : V^* \otimes V \to C(G)$ is a $(G \times G)$-homomorphism, and therefore must be injective (since its kernel is a subrepresentation of $V^* \otimes V$). This implies orthogonality, by a lemma from last week.

It remains to check that $(-,-)_{L^2}$, restricted to $V^* \otimes V$ via m_V, is $(-,-)_{V^* \otimes V} \cdot (1/\dim V)$. This can be computed in coordinates, choosing an orthonormal basis for V. ⊠

This relates to a bunch of related statements called the Peter-Weyl theorem, excellently exposited by Segal and Macdonald in "Lectures on Lie groups and Lie algebras."

Theorem 7.2 (Peter-Weyl)**.** $C_{\mathrm{alg}}(G) \subseteq C(G)$ *is dense in the uniform norm.*

If you only care about compact matrix groups,[9] this density is a consequence of the Stone-Weierstrass theorem: any continuous function on a compact subset of $\mathbb{R}^{n^2}$ can be approximated uniformly by polynomials. Then, one shows that the Peter-Weyl theorem holds for every compact Lie group by showing every compact Lie group has a faithful representation. Maybe we'll return to this point later, when we return to more analytic issues.

There's another consequence involving L^2 functions.

Corollary 7.3. $C_{\mathrm{alg}}(G)$ *is dense in* $L^2(G)$*, i.e.*

$$L^2(G) \cong \widehat{\bigoplus}(V^* \otimes V).$$

Here, $\widehat{\oplus}$ denotes the completion of the direct sum.

You can compare this to the usual situation in Fourier analysis: you can write functions on a circle in terms of exponentials, and this is a generalization — you can write any function in terms of matrix coefficients.

If you restrict to class functions, there's a cleaner result.

Definition 7.4. The *class functions* on a group G are the $L^2(G)$ functions invariant under conjugation. That is, G acts on $L^2(G)$ by $g \cdot f(x) = f(gxg^{-1})$, and we consider the invariants $L^2(G)^G$.

Corollary 7.5. *The class functions decompose as*

$$L^2(G)^G \cong \widehat{\bigoplus}_{V \in \widehat{G}} (V^* \otimes V)^G = \widehat{\bigoplus} \mathbb{C}\chi_V.$$

Here, $(V^* \otimes V)^G = \mathrm{End}_G(V) = \mathbb{C} \cdot 1_V$ *and* $\chi_V(g) := \mathrm{tr}(\rho_V(g))$ *is the character of* V.

Corollary 7.6. *The set $\{\chi_V \mid V \in \widehat{G}\}$ is an orthonormal basis for the class functions $L^2(G)^G$. In particular,*

$$(\chi_V, \chi_W) = \begin{cases} 0, & V \ncong W \\ 1, & V \cong W. \end{cases}$$

This makes it clear that the orthogonality relations arise purely from representation theory.

This has the following surprising corollary.

Corollary 7.7. *The isomorphism class of a representation is determined by its character.*

[9] That is, we think of $G \subseteq \mathrm{U}(N)$ for some N, or equivalently, G admits a faithful, finite-dimensional representation.

Usually you check for things to be isomorphic by finding an isomorphism between them. But in this case, you can just compute a complete invariant, and that's pleasantly surprising.

Proof. Any finite-dimensional representation W can be written

$$W \cong \bigoplus_{V \in \widehat{G}} V^{\oplus m_V}.$$

Thus,

$$\chi_W = \sum_{V \in \widehat{G}} m_V \chi_V,$$

so $m_V = \langle \chi_V, \chi_W \rangle$. ⊠

Maybe this is less surprising if you've already seen some character theory for finite groups.

The Peter-Weyl theorem is also useful for studying functions on homogeneous spaces. For the rest of this lecture, let G be a compact Lie group and H be a closed subgroup. Then, $C(G/H) = C(G)^H$, the H-invariant functions on G,[10] where H has the right action on $C(G)$. The Peter-Weyl theorem says that

$$C(G.H) = C(G)^H \cong \left(\widehat{\bigoplus} V^* \otimes V\right)^H = \widehat{\bigoplus} V^* \otimes V^H.$$

Example 7.8. For example, consider the 2-sphere $S^2 = \mathrm{SO}(3)/\mathrm{SO}(2)$, or $\mathrm{SU}(2)/\mathrm{U}(1)$ through the Hopf fibration, which is the double cover of the previous quotient.

Recall that the irreducible representations of SU(2) are P_n for each $n \geq 0$, where P_n is the space of homogeneous polynomials of degree n in two variabls (which is an $(n+1)$-dimensional space). Then,

$$C(S^2) = C(\mathrm{SU}(2))^{\mathrm{U}(1)} \cong \widehat{\bigoplus_{n \in \mathbb{N}}} P_n^* \otimes (P_n^{\mathrm{U}(1)}).$$

Here, we've switched from L^2 functions to continuous ones, which is overlooking a little analysis, but the analysis isn't that bad in this case, so it's all right.

Anyways, what does it mean for a polynomial to be fixed by U(1)? Well, $\mathrm{U}(1) \hookrightarrow \mathrm{SU}(2)$ through matrices such as $\left(\begin{smallmatrix} t & 0 \\ 0 & \bar{t} \end{smallmatrix}\right)$, and the action is

$$\begin{pmatrix} t & 0 \\ 0 & \bar{t} \end{pmatrix} \cdot x^a y^b = (t^{-1}x)^a (ty)^b = t^{b-a} x^a y^b.$$

Thus, we need $b = a$ and $a + b = n$, so

$$P_n^{\mathrm{U}(1)} \cong \begin{cases} \mathbb{C}, & n \text{ even} \\ 0, & n \text{ odd}. \end{cases}$$

Therefore

$$C(S^2) = \widetilde{\bigoplus}_{n \in 2\mathbb{N}} P_n^*,$$

and you can go even further and figure the decomposition out explicitly. ◀

The point of this example is that, even though we used very little information about the sphere, we put some significant constraints on functions on it, which is part of what makes representation theory cool.

Remark. In physics, these are indexed by $n/2$ instead of by n, and $n/2$ is called the *spin* of the representation. One says that only the integer-spin representation terms appear, not the half-integer ones. ◀

[10] This is a different action than the conjugation action we used to write $C(G)^G$. Unfortunately, there's no easy way to notate these both.

Lecture 8.

Character tables: 2/3/17

There's homework posted on the website; it's due at the end of the month.

Let G be a compact group. Then, $C(G)$, the space of continuous (complex-valued) functions on G, is called the *group algebra*: it has an algebra structure given by pointwise multiplication, but there's another algebra structure given by *convolution*: if $f_1, f_2 \in C(G)$, we define

$$(f_1 * f_2)(g) := \int_G f_1(h) f_2(h^{-1}g)\,\mathrm{d}h,$$

where $\mathrm{d}h$ is the normalized Haar measure. This looks asymmetric, which is kind of strange, but what we're doing is integrating over the pairs of elements whose product equals g:

$$(f_1 * f_2)(g) = \int_{\{h_1h_2=g\}} f_1(h_1)f_2(h_2).$$

The identification comes by $h \mapsto (h, h^{-1}g)$.[11] This product makes $C(G)$ into a noncommutative $\mathbb{C}$-algebra.[12]

If V is a finite-dimensional representation of G, then $C(G)$ acts on V by the formula

$$f * v = \int_G f(g)(g\cdot v)\,\mathrm{d}g$$

for any $v \in V$ and $f \in C(G)$. You can check that this makes V into a module for $C(G)$. A δ-function at some $h \in G$ isn't continuous, unless G is finite, but the integral still makes sense distributionally, and you get $\delta_h * v = h \cdot v$.

You can also restrict to class functions.

Exercise 8.1. Show that class functions are central in $(C(G), *)$, and in fact that $C(G)^G = Z(G)$: the class functions are the center.

In particular, this means the class functions are a commutative algebra, and it sees the representation theory of G: the characters are all in $C(G)^G$, satisfy orthogonality relations, and see isomorphism classes of representations. The irreducible characters also have a nice formula under convolution.

Proposition 8.2. *If $V, W \in \widehat{G}$, then*

$$\chi_V * \chi_W = \begin{cases} \left(\dfrac{1}{\dim V}\right)\chi_V, & V = W \\ 0, & V \neq W. \end{cases}$$

Proof. The trick is to turn the inner product into convolution at the identity:

$$\begin{aligned}(\chi_V, \chi_W)_{L^2} &= \int_G \chi_V(g)\overline{\chi_W(g)}\,\mathrm{d}g \\ &= \int_G \chi_V(g)\chi_W(g^{-1})\,\mathrm{d}g \\ &= (\chi_V * \chi_W)(1).\end{aligned}$$

This proves it at the identity; the rest is TODO. ⊠

Finite groups. In this section, assume G is finite.

There are some particularly nice results for finite groups, since every function on a finite group is continuous. If you've been going to Tamás Hausel's talks this week (and the two more talks next week), his work on computing cohomology of moduli spaces, uses the character theory of finite groups in an essential way.

When G is finite, the group algebra $C(G)$ is usually denoted $\mathbb{C}[G]$. This notation means a vector space with basis G, i.e.

$$\mathbb{C}[G] = \left\{\sum a_g g \mid g \in G, a_g \in \mathbb{C}\right\}.$$

[11] If G isn't compact, e.g. $G = \mathbb{R}^n$, this definition makes sense, but doesn't always converge; in this case, you can restrict to compactly supported functions, though their convolution won't be compactly supported. In this way one recovers the usual convolution operator on $\mathbb{R}^n$.

[12] There's a slight caveat here: there's no identity in this algebra, unless G is discrete.

You can make this definition (finite weighted sums of elements of G) for any group, but it won't be isomorphic to $C(G)$ unless G is discrete. In any case, when G is finite, $\mathbb{C}[G]$ is a finite-dimensional, unital algebra.

Let's think about what the Peter-Weyl theorem says in this context. Since $\mathbb{C}[G]$ is finite-dimensional, we can ignore the completion and obtain an isomorphism

$$\mathbb{C}[G] \cong \bigoplus_{V \in \widehat{G}} (V^* \otimes V). \tag{8.3}$$

This has a number of fun corollaries. First, take the dimension of each side:

Corollary 8.4. *Let G be a finite group. Then, G has finitely many isomorphism classes of irreducible representations, and moreover*

$$|G| = \sum_{V \in \widehat{G}} (\dim V)^2.$$

For example, once you've found the trivial and sign representations of S_3, you're forced to conclude there's one more irreducible 2-dimensional representation or two more one-dimensional representations.

Looking at class functions, every function invariant on conjugacy classes is continuous now, so we have two bases for $\mathbb{C}[G]^G$: $\{\chi_V \mid V \in \widehat{G}\}$ as usual, and the set of δ-functions on conjugacy classes of G.

Corollary 8.5. $|\widehat{G}|$ *is equal to the number of conjugacy classes of G.*

This leads to an organizational diagram called the *character table* for a finite group: across the top are the conjugacy classes $[g]$ and down the left are the irreducible characters χ. The entry in that row and that column is $\chi(g)$. This table says a lot about the representation theory of G. By Corollary 8.5, it's square.

Example 8.6. Let $G = S_3$, so conjugacy classes are cycle types (as in any symmetric group). The trivial

	e	$(1\ 2)$	$(1\ 2\ 3)$
χ_{triv}	1	1	1
χ_{sign}	1	-1	1
χ_V	2	0	-1

Table 1. Character table of S_3.

representation has character 1 on all conjugacy classes; the sign representation has value 1 on e and $(1\ 2\ 3)$ and -1 on $(1\ 2)$. Then, you can compute χ_V where V is the irreducible two-dimensional representation, and see Table 1 for the character table. Alternatively, you can use the orthogonality relations to compute the remaining entries. ◀

This matrix looks like it should be unitary, but isn't quite. Nonetheless, it has some nice properties: notice that the columns are also orthogonal.

Example 8.7. S_4 is only a little more complicated. The conjugacy classes are e, $(1\ 2)$, $(1\ 2\ 3)$, $(1\ 2)(3\ 4)$, and $(1\ 2\ 3\ 4)$.

We know the trivial and sign representations, and there's the defining representation W where S_4 permutes a basis of $\mathbb{C}^4$. This is reducible, since it fixed $(1, 1, 1, 1)$; alternatively, you could use the orthogonality relation to check. So we know there's a copy of the trivial representation, so $\chi_V := \chi_W - \chi_{\mathrm{triv}}$, and you can check it's irreducible. ◀

Lecture 9.

The character theory of SU(2): 2/6/17

Last time, we got stuck on the proof of Proposition 8.2. The key idea that was missing is that if $m_V : V^* \otimes V \to C(G)$ is the matrix coefficients function for the representation V, then $\mathrm{Im}(m_V)$ is a two-sided ideal of $C(G)$ under convolution. This follows from the formula for convolving with a matrix coefficient: given $v \in V$, $\alpha \in V^*$, and an $f \in C(G)$,

$$m_{V,\alpha,v} * f = m_{V,\alpha,v'},$$

where

$$v' = \int_G f(h)h^{-1} \cdot v \, \mathrm{d}h.$$

This is fairly easy to prove.

Characters are examples of matrix coefficients, so if $V, W \in \widehat{G}$, $\chi_V * \chi_W \in_V (V^* \otimes V)^G \cap m_W(W^* \otimes W)^G$. Since $m_V(V^* \otimes V)^G \cong \mathrm{End}_G V$, and similarly for W, this space is trivial when $V \ncong W$ and is one-dimensional when $V \cong W$. So the proof boils down to checking $\chi_V * \chi_V = (1/\dim V)\chi_V$.

This means that the characters are almost orthogonal idempotents. To get actual idempotents, though, we have to normalize: let $e_V \coloneqq \dim(V)\chi_V \in C(G)^G$, so that $e_V * e_V = 1$ and $e_V * e_W = 0$ if $V \ncong W$. If U is any (unitary) representation of G, then $C(G)$ acts on U, and under this action, e_V acts by projection onto the V-isotypical component of U.

For yet another way of thinking about this, we have the matrix coefficients map $m_V : \mathrm{End}_{\mathbb{C}}(V) \to C(G)$, which sends $1_V \mapsto \chi_V$, but also the action map $\mathrm{act}_V : C(G) \to \mathrm{End}(V)$. This sends $e_V \mapsto 1_V$, so these maps aren't literally inverses. They are adjoint with respect to the appropriate Hermitian forms, however.

Last time, we also worked out some examples of representations of finite groups. We managed to translate that problem into a completely different problem, of identifying certain class functions in $C(G)^G$. Today, we're going to try to generalize this to an infinite group, namely SU(2).

Let's fix some notation. Concretely,

$$G = \mathrm{SU}(2) = \left\{ \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \mid |a|^2 + |b|^2 = 1 \right\},$$

and inside this lies the *maximal torus*

$$T = \mathrm{U}(1) = \left\{ \begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix} \mid |z| = 1 \right\}.$$

Some of what we say will generalize, but SU(2) is the nicest case.

We'd like to understand the conjugacy classes of SU(2). Let $\tau \coloneqq (\mathrm{tr}/2) : \mathrm{SU}(2) \to [-1, 1]$, so $\tau\left(\begin{smallmatrix} z & 0 \\ 0 & \overline{z} \end{smallmatrix}\right) = \mathrm{Re}(z)$. The fibers of τ are exactly the conjugacy classes of SU(2). What do they look like? Geometrically, SU(2) is a 3-sphere in $\mathbb{R}^4$, and we're fixing one of the real coordinates to be a particular number, so the level sets are 2-spheres, except for the poles, which are single-point conjugacy classes, and are the central elements of SU(2).

What this means is that a class function is determined by its restriction to the maximal torus T, and for any unit complex number z, $\left(\begin{smallmatrix} z & 0 \\ 0 & \overline{z} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \overline{z} & 0 \\ 0 & z \end{smallmatrix}\right)$ are in the same conjugacy class. Thus, restriction determines an isomorphism

$$C(\mathrm{SU}(2))^{\mathrm{SU}(2)} \xrightarrow{\cong} C(\mathrm{U}(1))^{\mathbb{Z}/2}, \tag{9.1}$$

where $\mathbb{Z}/2$ acts on $C(\mathrm{U}(1))$ by conjugation. This means we can think of class functions on SU(2) in terms of their Fourier coefficients!

Given a representation V of SU(2), we can restrict it to a U(1)-representation V'. We decompose this into *weight spaces*

$$V' = \bigoplus_{n \in \mathbb{Z}} V_n,$$

where $z \in \mathrm{U}(1)$ acts on V_n by z^n. Thus, identifying the character $\chi_V \in C(\mathrm{SU}(2))^{\mathrm{SU}(2)}$ with its image under (9.1), we get

$$\chi_V \longmapsto \sum_{n \in \mathbb{Z}} \dim(V_n) z^n.$$

Part of what made the representation theory of U(1) awesome was integration: we had a nice formula for the integral of a class function. How can we generalize this to SU(2)? The map (9.1) doesn't preserve volume, as its fibers are spheres with different radii.

Recall that the irreducible representations of SU(2) are the spaces P_n of homogeneous polynomials in two variables of degree n. If we restrict P_n to a U(1)-representation and decompose it, we discover

$$P_n = \mathbb{C} \cdot x^n \oplus \mathbb{C} \cdot x^{n-1}y \oplus \cdots \oplus \mathbb{C} \cdot xy^{n-1} \oplus \mathbb{C} \cdot y^n.$$

The term $x^k y^{n-k}$ has weight $n-2k$, i.e. z acts as z^{n-2k} on $\mathbb{C}\cdot x^k y^{n-k}$. Therefore as a U(1)-representation,

$$\begin{aligned}\chi_{P_n}(z) &= z^n + z^{n-2} + \cdots + z^{-n+2} + z^{-n}\\ &= \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}}.\end{aligned}\tag{9.2}$$

(9.2) is called the *Weyl character formula* for SU(2). There will be corresponding formulas for other groups.

We would like to use something like this to do integration: given a class function $f \in C(\mathrm{SU}(2))^{\mathrm{SU}(2)}$, let

$$f(\phi) \coloneqq f\begin{pmatrix} e^{i\phi} & \\ & e^{-i\phi}\end{pmatrix},$$

so $z = e^{i\phi}$. We want to determine a function $J(\phi)$ such that

$$\int_{\mathrm{SU}(2)} f(g)\,\mathrm{d}g = \int_0^{2\pi} f(\phi)J(\phi)\,\mathrm{d}\phi.$$

The thing that powers this is that f is determined by its image under (9.1), so we should be able to determine its integral in those terms. This requires that f is a class function, and is not true in general.

Here, $J(\phi)$ is the area of the conjugacy class whose trace is $2\cos\phi$. This conjugacy class is a sphere of radius $\sqrt{1-\cos^2(\phi)} = |\sin\phi|$. Thus, $J(\phi) = C\sin^2\phi$, where C is such that

$$\int_0^{2\pi} C\sin^2\phi\,\mathrm{d}\phi = 1.$$

This means $C = 1/\pi$, so the integration formula for class functions is

$$\int_{\mathrm{SU}(2)} f(g)\,\mathrm{d}g = \frac{1}{\pi}\int_0^{2\pi} f(\phi)\sin^2\phi\,\mathrm{d}\phi.\tag{9.3}$$

Example 9.4. Let's try to compute this for $n = 1$.

$$\|\chi_{P_1}\|^2 = \frac{4}{\pi}\int_0^{2\pi}\cos^2\phi\sin^2\phi\,\mathrm{d}\phi.$$

This is difficult but tractable, and the answer is

$$= \frac{1}{\pi}\left[\frac{1}{2}\phi - \frac{1}{8}\sin(4\phi)\right]_0^{2\pi} = 1.$$

This is good, because we said P_1 is irreducible, so the norm of its character had better be 1. ◀

Lecture 10.

Representation theory of Lie groups: 2/8/17

Last time, we discussed the character theory of SU(2), using a convenient isomorphism of the algebra of class functions $C(\mathrm{SU}(2))^{\mathrm{SU}(2)}$ with $C(\mathrm{U}(1))^{\mathbb{Z}/2}$. If P_n denotes the irreducible SU(2)-representation of dimension $n+1$, and $z = e^{i\phi}$, the image of χ_{P_n} in $C(\mathrm{U}(1))^{\mathbb{Z}/2}$ is

$$\chi_{P_n}(z) = z^n + z^{n-2} + \cdots + z^{-n+2} + z^{-n}.$$

Then we used the Weyl integration formula for SU(2), (9.3), to show that

$$\begin{aligned}\|\chi_{P_1}\|^2 &= \frac{1}{\pi}\int_0^{2\pi}\sin^2(2\phi)\,\mathrm{d}\phi\\ &= \frac{1}{\pi}\left[\frac{\phi}{2} - \frac{1}{8}\sin(4\phi)\right]_0^{2\pi} = 1.\end{aligned}$$

Thus, P_1 is an irreducible representation!

Exercise 10.1. Show that for every n, $\|\chi_{P_n}\|^2 = 1$, so each P_n is irreducible.

So we've found some irreducible representations, and in a very curious manner, only using their characters. We then need to show there are no additional irreducible representations.

Exercise 10.2. Show that the functions $\{\cos(n\phi)\}_{n\in\mathbb{Z}_{\geq 0}}$ are a basis for $C_{\mathrm{alg}}(\mathrm{U}(1))^{\mathbb{Z}/2}$.

This implies $\{\chi_{P_n}\}$ is also a basis, and are orthonormal, so they account for all irreducible representations. You might next wonder what the isomorphism class of $P_n \otimes P_m$ is, as a direct sum of irreducibles. Recall that $\chi_{V\otimes W} = \chi_V \chi_W$, so if $m \leq n$,

$$\chi_{P_n\otimes P_m} = \chi_{P_n}\chi_{P_m} = \left(\frac{z^{n+1} - z^{n-1}}{z - z^{-1}}\right)\left(z^m + z^{m-2} + \cdots + z^{-m}\right). \tag{10.3}$$

Exercise 10.4 (Clebsch-Gordon rule)**.** Show that (10.3) satisfies

$$\chi_{P_n}\chi_{P_m} = \sum_{k=0}^{m} \chi_{P_{n+m-2k}}.$$

Now that that's settled, let's look at other compact Lie groups. There's a double cover $\mathrm{SU}(2) \twoheadrightarrow \mathrm{SO}(3)$, so $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\pm I$. Thus, the irreducible representations of SO(3) are given as the representations of SU(2) in which $-I$ acts trivially. These are the P_n for even n. In other words, $\mathrm{SU}(2) \cong \mathrm{Spin}(3)$, so the irreducible representations of SO(3) have integer spin, and those of Spin(3) may have half-integer spin.

It's possible to get a little more information from this: there's a double cover $\mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{SO}(4)$, so you can work out the representation theory of SO(4) in a similar way. But there's loads of other interesting groups, including U_n, Sp_n, and many more.

An overview of the theory. Let G be a compact, connected Lie group.[13] Then, G admits a *maximal torus* T, a maximal abelian subgroup (which is necessarily isomorphic to $\mathrm{U}(1)^n$), and any $g \in G$ is conjugate to an element of T. For example, when $G = \mathrm{U}(n)$, T is the subgroup of diagonal unitary matrices, which is isomorphic to $\mathrm{U}(1)^n$; that every $g \in \mathrm{U}(n)$ is conjugate to something in T means that every unitary matrix is diagonalizable. It's also true that any two maximal tori are conjugate to each other.

Thus, just like we did for SU(2) and U(1), we can express a class function on G in terms of its restriction to T, and then use Fourier theory. Let $\widehat{T} = \mathrm{Hom}(T, \mathrm{U}(1))$, which is a lattice $\mathbb{Z}^n$. Our goal is to write the irreducible characters of G as Fourier series on T. There are still some questions, though: how do we do integration? And can we obtain a formula for the characters of the irreducible representations? The Weyl integration formula and Weyl character formula will answer these questions.

Tackling a general compact Lie group with the technology we've developed is complicated. The unitary group is within reach, but it involves knowledge of symmetric functions and representation theory of the symmetric group. So we'll start with some more general theory.

Lie algebras. We want to reduce the representation theory of Lie groups to pure algebra. For example, if G is a finite group, there are finitely many generators and relations, so we can express a representation as a finite set of matrices satisfying the relations. Lie groups aren't finitely (or even countably) generated, so we can't just do algebra.

There are good presentations of Lie groups that are topological, however: SO(3) is generated by rotations about certain angles $R_{x,t}$, $R_{y,t}$, $R_{z,t}$, which form a frame in $\mathbb{R}^3$, and $t \in \mathbb{R}/\mathbb{Z}$ is an angle. So SO(3) can be described in terms of a one-parameter subgroup. We'd like to replace SO(3) with its collection of one-parameter subgroups; these can be identified with tangent vectors at the identity, which is why the somewhat surprising notion of a Lie algebra is introduced.

Definition 10.5. A *one-parameter subgroup* of a Lie group G is a homomorphism $\rho\colon \mathbb{R} \to G$, where $\mathbb{R}$ is a Lie group under addition.

Homomorphisms of Lie groups are always smooth. Given a one-parameter subgroup ρ, we can obtain its derivative at the identity, $\rho'(0) \in T_1G$. We'll let $\mathfrak{g}$ (written `\mathfrak g` in LaTeX) denote T_1G: what we're saying is that a smooth path going through the identity defines a tangent vector.

Lemma 10.6. *The assignment* $\Phi\colon \rho \mapsto \rho'(0)$ *is a bijection* $\mathrm{Hom}_{\mathsf{LieGrp}}(\mathbb{R}, G) \to \mathfrak{g}$.

That is, a direction in $\mathfrak{g}$ gives rise to a unique one-parameter subgroup in that direction.

[13]Focusing on connected groups isn't too restrictive: any compact Lie group G is an extension of its identity component G^0 by the finite group $\pi_0(G)$.

Proof sketch. This is a generalization of the proof that the characters of $\mathbb{R}$ are exponential maps: that also involves showing they're determined by their values at 0 using a differential equation.

Let ρ be a one-parameter subgroup; then, it satisfies the ODE $\rho'(t) = \rho(t)\rho'(0)$ with the initial condition $\rho(0) = 1_G$. Surjectivity of Φ follows from the existence of solutions to ODEs, and injectivity follows from uniqueness. You need a way of passing from a local solution to a global one, but this can be done. ⊠

Now, we can define the *exponential map* $\exp\colon \mathfrak{g} \to G$ sending $A \mapsto \Phi^{-1}(A)(1)$: given a tangent vector, move in that direction for a short time, and then return that element. This map is a local diffeomorphism (by the inverse function theorem), so the Lie algebra encodes the Lie group in some neighborhood of the identity. Then, one asks what algebraic properties of $\mathfrak{g}$ come from the group structure on G.

Lecture 11.

Lie algebras: 2/10/17

"We're the Baker-Campbell-Hausdorff law firm — call us if you can't commute!"

Let G be a Lie group. Last time, we defined its *Lie algebra* $\mathfrak{g} := T_1G$, and a local diffeomorphism $\exp\colon \mathfrak{g} \to G$. A local inverse to the exponential map is called a *logarithm*.

Example 11.1. If $G = \mathrm{GL}_n(\mathbb{R})$, then G is an open submanifold of $M_n(\mathbb{R})$, the vector space of $n \times n$ matrices. Thus, $T_1\mathrm{GL}_n(\mathbb{R}) = T_1M_n(\mathbb{R})$, and the tangent space to a vector space is canonically identified with that vector space. Thus, the Lie algebra $\mathfrak{g}$, also written $\mathfrak{gl}_n(\mathbb{R})$, is $M_n(\mathbb{R})$. More generally, the Lie algebra of $\mathrm{GL}(V)$ is $\mathfrak{gl}(V) = \mathrm{End}(V)$.

In this case, the exponential map is the matrix exponential $\exp\colon M_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ sending

$$A \longmapsto e^A = \sum_{i=0}^{\infty} \frac{A^i}{i!} = 1 + A + \frac{A^2}{2} + \cdots,$$

so a one-parameter subgroup ρ_A in $\mathrm{GL}_n(\mathbb{R})$ is given by $\rho_A(t) = e^{tA}$ for any matrix A. ◀

More generally, if $G \subset \mathrm{GL}_n(\mathbb{R})$ is any Lie subgroup, the exponential map $\mathfrak{g} \to G$ is the restriction of the matrix exponential $M_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ to $\mathfrak{g} \subset M_n(\mathbb{R})$.

Example 11.2. $\mathrm{SL}_n(\mathbb{R})$ is the group of matrices $\{A \in \mathrm{GL}_n(\mathbb{R}) \mid \det(A) = 1\}$ (since 1 is a regular value of the determinant function, this is in fact a Lie group). If you differentiate the determinant condition, you get that $\mathrm{tr}(A) = 0$, because $\exp(\mathrm{tr}(A)) = \det(\exp(A))$, and therefore the Lie algebra of $\mathrm{SL}_n(\mathbb{R})$ is

$$\mathfrak{sl}_n(\mathbb{R}) = \{A \in \mathfrak{gl}_n(\mathbb{R}) \mid \mathrm{tr}(A) = 0\}.$$

In particular, the logarithm gives a local chart from $\mathrm{SL}(n)$ into the vector space $\mathfrak{sl}_n(\mathbb{R})$, which is another way to show $\mathrm{SL}_n(\mathbb{R})$ is a Lie group. (Kirrilov's book uses this to explain why all of the classical groups are Lie groups.)

$\mathrm{O}(n)$ is the group of $n \times n$ matrices with $AA^t = I$, so differentiating this, the Lie algebra is

$$\mathfrak{o}(n) = \{A \mid A + A^t = 0\},$$

the *skew-symmetric matrices*. This is also the Lie algebra for $\mathrm{SO}(n)$: $\mathrm{SO}(n) \subset \mathrm{O}(n)$ is the connected component of the identity, and therefore $T_1\mathrm{O}(n) = T_1\mathrm{SO}(n) = \mathfrak{o}(n)$ is again the Lie algebra of skew-symmetric matrices.

The same does not apply for $\mathrm{SU}(n) \subset \mathrm{U}(n)$, since $\mathrm{U}(n) = \{A \in M_n(\mathbb{C}) \mid AA^\dagger = I\}$ is connected: its Lie algebra is

$$\mathfrak{u}(n) = \{A \mid A + A^\dagger\} = 0,$$

the *skew-Hermitian matrices*. But not all of these are traceless, so $\mathfrak{su}(n)$ is the skew-Hermitian matrices with trace zero. ◀

The basic question is, how much does $\mathfrak{g}$ know about G? Clearly not everything, because $\mathfrak{o}(n) = \mathfrak{so}(n)$. To make this question precise, we need more structure on $\mathfrak{g}$.

To recover information about G, we need to know what $\mathfrak{g}$ says about the multiplication on G. Thus, given $A, B \in \mathfrak{g}$, what is $C(A, B) := \log(\exp(A)\exp(B))$? We can Taylor-expand it around $A = B = 0$. The first term is $A+B$ (essentially by the product rule), and there must be higher-order terms unless G is abelian. Let $b(A,B)$ be twice the second-order term, so

$$C(A,B) = A + B = \frac{1}{2}b(A,B) + O(A^2, AB, B^2).$$

This $b\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ has some important properties.

Proposition 11.3.

(1) *b is bilinear.*
(2) *b is* skew-symmetric*: $b(A,B) = -b(A,B)$*
(3) *b satisfies the* Jacobi rule*:*

$$b(b(A,B),C) + b(b(B,C),A) + b(b(C,A),B) = 0.$$

This allows us to clarify what exactly we mean by a Lie algebra.

Definition 11.4. A *Lie algebra* is a vector space L together with an operation $b\colon L\times L\to L$, called the *Lie bracket*, satisfying the properties in Proposition 11.3. A *Lie algebra homomorphism* is a linear map between Lie algebras that commutes with the Lie bracket.

All of the Lie algebras we've defined, e.g. those in Examples 11.1 and 11.2, will be understood to come with their Lie brackets.

Exercise 11.5. Prove Proposition 11.3. Part of this will involve unpacking what "Taylor series" means in this context. Hint: first, notice that $C(A,0) = A$, $C(0,B) = B$, and $C(-B,-A) = -C(A,B)$ by properties of the exponential map. These together give (1) and (2); the last part will also follow from properties of C.

This is pretty neat. You might wonder what additional information the higher-order terms give you, but it turns out that you don't get anything extra.

Theorem 11.6 (Baker-Campbell-Hausdorff)**.** *The Taylor expansion of $C(A,B)$ can be expressed purely in terms of addition and the Lie bracket. In particular, the formula begins*

$$C(A,B) = A + B + \frac{1}{2}b(A,B) + \frac{1}{12}(b(A,b(A,B)) + b(B,b(B,A))) + \cdots$$

This is already quite surprising, and it turns out you can calulate it explicitly for most Lie groups we care about.

Example 11.7. If $G = \mathrm{GL}_n(\mathbb{R})$ or $\mathrm{GL}_n(\mathbb{C})$, so $\mathfrak{g} = M_n(\mathbb{R})$ (resp. $M_n(\mathbb{C})$), then $b(A,B)$ is the matrix commutator $[A,B] = AB - BA$. The same is true for any matrix group (i.e. Lie subgroup of $\mathrm{GL}_n(\mathbb{R})$ or $\mathrm{GL}_n(\mathbb{C})$): the commutator of two traceless matrices is traceless, so $[,]$ also defined the Lie bracket on $\mathfrak{sl}_n(\mathbb{R})$, $\mathfrak{so}_n$, $\mathfrak{su}_n$, $\mathfrak{u}_n$, and so on. ◀

We've seen that Lie algebras can't distinguish a Lie group and the connected component of the identity. It also can't see covering maps, since the tangent space depends only on local data, so, e.g., $\mathfrak{spin}_n = \mathfrak{so}_n$ through the double cover $\mathrm{Spin}(n) \twoheadrightarrow \mathrm{SO}(n)$. However, these are the only obstructions: the Lie algebra tells you everything else.

Theorem 11.8 (Lie's theorem)**.** *There is an equivalence of categories between the category of connected and simply-connected Lie groups and the category of finite-dimensional Lie algebras, given by sending $G\mapsto\mathfrak{g}$.*