FALL 2016 ALGEBRAIC GEOMETRY SEMINAR

ARUN DEBRAY
SEPTEMBER 2, 2016

These notes were taken in a learning seminar on algebraic groups in Fall 2016. Thanks to Tom Gannon for finding and fixing a few errors.

Contents

1. What is a scheme?: 8/31/16

Today, Tom provided an introduction and overview, with the goal of understanding what a *scheme*, the central object of study in algebraic geometry, is. We'll start with sheaves, a way of understanding locality of things in geometry, then discuss locally ringed spaces, the spectrum of a ring, and finally schemes, their properties, and a little bit about morphisms of schemes. Finally, we'll learn a little about varieties.

1.1. Sheaves.

Definition 1.1. A *presheaf of rings*[1] $\mathscr{F}$ on a topological space X consists of the data

(1) for every open $U \subset X$, a ring $\mathscr{F}(U)$, and
(2) for every inclusion of open sets $V \subset U$, a ring homomorphism $\rho_V^U : \mathscr{F}(U) \longrightarrow \mathscr{F}(V)$ called the *restriction map*,

such that for every nested inclusion of opens $W \subset V \subset U$, the restriction maps compose: $\rho_W^U = \rho_W^V \circ \rho_V^U$.

Elements of $\mathscr{F}(U)$ are called *sections*.

The idea is that $\mathscr{F}(U)$ is some collection of data on U, such as the continuous real-valued functions on U, which define a ring. Given such a function, we can restrict it to a $V \subset U$, and this is exactly what the restriction map does. If I want to further restrict to another subset, it doesn't matter whether I restrict to V first.

Presheaves have some problems, and we define sheaves to fix these problems.

Definition 1.2. A *sheaf* $\mathscr{F}$ on a space X is a presheaf such that

(1) sections can be computed locally: if $U \subset X$ is open, $\mathfrak{U}$ is an open cover of U, and $s \in \mathscr{F}(U)$, then if $\rho_{U_i}^U(s) = 0$ for all $U_i \in \mathfrak{U}$, then $s = 0$.
(2) compatible sections can be glued: with U and $\mathfrak{U}$ as above, suppose we have data of $s_i \in \mathscr{F}(U_i)$ for each $U_i \in \mathfrak{U}$ such that for all $U_i, U_j \in \mathfrak{U}$, $\rho_{U_i \cap U_j}^{U_i}(s_i) = \rho_{U_i \cap U_j}^{U_j}(s_j)$, then there is a section $s \in \mathscr{F}(U)$ such that $\rho_{U_i}^U(s) = s_i$ for all $U_i \in \mathfrak{U}$.

To understand this intuitively, think about continuous real-valued functions, which can be uniquely determined from local data, and can be glued together from compatible functions on an open cover.

Example 1.3. Let X be a space. We've already been referring to the sheaf C_X of continuous $\mathbb{R}$-valued functions: $C_X(U) = \{f : U \longrightarrow \mathbb{R} \text{ continuous}\}$. Restriction of functions defines a restriction map, functions are determined by local data, and compatible functions may be glued together.

This is a good example of sheaves for your intuition: sheaves in general behave a lot like a sheaf of functions, and it's convenient to think of the restriction map as actual restriction of functions.

[1] One can talk about presheaves of sets, groups, or of any other category, by replacing "rings" in this definition by "sets," "groups," or whatever you're using.

Example 1.4. Let X be a manifold; then, we can define the sheaf C_X^∞ of smooth functions: $C_X^\infty(U)$ is the ring of smooth functions $U \to \mathbb{R}$. This is very similar, but it's interesting that this sheaf uniquely determines the smooth structure on the manifold X.

That is, smooth structure is determined by what you call smooth functions. This is a rule that applies more generally in geometry: a geometric structure is determined by the sheaf of functions to some base that we allow.

Remark. The empty set is an open subset of a space X. You can prove or define (depending on your taste for empty arguments) that for any sheaf $\mathscr{F}$ on X, $\mathscr{F}(\varnothing) = 0$.

Definition 1.5. Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves on a space X. Then, a morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$ is the data of for all open $U \subset X$, a ring homomorphism $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ that commutes with restriction in the following sense: for all inclusions of open sets $V \subset U$, the following diagram commutes:

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\rho_V^U"] & \mathscr{G}(U) \arrow[d, "\rho_V^U"] \\ \mathscr{F}(V) \arrow[r, "\varphi(V)"] & \mathscr{G}(V). \end{tikzcd}$$

That is, we want to map in a way that doesn't affect how we restrict. A sheaf is data parametrized by a topological space, and we want a morphism of sheaves to respect this parametrization.

Definition 1.6. Let $\mathscr{F}$ be a sheaf on X and $p \in X$. Then, the *stalk* of $\mathscr{F}$ at p is

$$\mathscr{F}_p = \{(s, U) \mid U \subset X \text{ is open}, s \in \mathscr{F}(U)\}/\sim,$$

where $(s, U) \sim (t, V)$ if there's an open $W \subset U \cap V$ containing p such that $\rho_W^U(s) = \rho_W^V(t)$.

That is, we define two functions to be equivalent if they agree on any neighborhood of the point. These are sort of infinitesimal data of functions near the point p.

1.2. **Locally ringed spaces.**

Definition 1.7. A *local ring* is a ring A with a unique maximal ideal $\mathfrak{m} \subset A$, often denoted $(A, \mathfrak{m})$.

This is the same as saying $A^\times = A \setminus \mathfrak{m}$: everything outside the maximal ideal is invertible.

Definition 1.8. A *locally ringed space* is a pair $(X, \mathscr{O}_X)$, where X is a topological space and $\mathscr{O}_X$ is a sheaf of rings, such that all stalks $\mathscr{O}_{X,p}$ are local rings.

Example 1.9. Manifolds are examples of locally ringed spaces: if X is a manifold, let $\mathscr{O}_X = C_X^\infty$, the smooth, real-valued functions. Let $p \in X$ and $\mathfrak{m}_{X,p}$ be the functions vanishing at p inside $\mathscr{O}_{X,p}$, which is an ideal. Then, any $f \in \mathscr{O}_{X,p} \setminus \mathfrak{m}_{X,p}$ is a unit: since it doesn't vanish at p, there's an open neighborhood U of p on which f doesn't vanish, so $1/f$ is smooth on U, and therefore defines an inverse to f in $\mathscr{O}_{X,p}$.

This locally ringed formalism is surprisingly useful: the maximal ideal of a stalk will always be functions vanishing at a point, even in weirder situations.

Of course, we want to understand morphisms of locally ringed spaces.

Definition 1.10. A *morphism of locally ringed spaces* $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is the data $(\varphi, \varphi^\#)$ of a continuous map $\varphi : X \to Y$ and a morphism of sheaves $\varphi^\# : \varphi_*\mathscr{O}_Y \to \mathscr{O}_X$ such that the induced map on stalks preserves the notion of vanishing at a point, i.e. for every $p \in X$, the preimage of the maximal ideal $\mathfrak{m}_{X,p}$ is contained in $\mathfrak{m}_{Y,\varphi(p)}$.

Here, $\varphi_*\mathscr{O}_X$ is the *pushforward* of $\mathscr{O}_X$, which attaches to every open $U \subset Y$ the ring $\varphi_*\mathscr{O}_X(U) = \mathscr{O}_X(\varphi^{-1}(U))$: since φ is continuous, this is again an open set.

The pushforward is an important definition in its own right. It's necessary to check that it actually defines a sheaf, but this isn't too complicated.

As an example, a smooth map $\varphi : X \to Y$ of manifolds defines a morphism of locally ringed spaces: φ is continuous, and a continuous map $f : V \to \mathbb{R}$ is sent to the map $f \circ \varphi : \varphi^{-1}(V) \to \mathbb{R}$. This is called the *pullback* of f. This is curious: we could have started with a merely continuous function that sends smooth functions to smooth functions, and it's forced to be smooth. Thus, the geometry of smooth manifolds is determined entirely by their structure as locally ringed spaces! Similarly, we'll define schemes to be certain kinds of locally ringed spaces.

1.3. **The spectrum of a ring.** A scheme is a particular kind of locally ringed space, locally isomorphic to $\operatorname{Spec} A$ for rings A, in the same way that a manifold is locally $\mathbb{R}^n$. Let's discuss the local model better.

Definition 1.11. The *spectrum* of a (commutative) ring A is $\operatorname{Spec} A = \{\mathfrak{p} \subset A \mid \mathfrak{p} \text{ is prime}\}$.

Let's briefly recall localization of rings.

Definition 1.12. If A is a ring and $S \subset A$ is a sub*set* such that $1 \in S$ and whenever $x, y \in S$, then $xy \in S$, we call S a *multiplicative subset.* Then, we can define the *localization* $S^{-1}A$ to be the ring of fractions $\{a/s \mid a \in A, s \in S\}$, where $a/s = b/s'$ iff there exists a $t \in S$ such that $t(s'b - sa) = 0$.

This is strongly reminiscent of the field of fractions of an integral domain, for which $S = A \setminus 0$; the equivalence relation is what allows us to know that $1/2 = 2/4$. For example, if $A = \mathbb{Z}$ and $S = \mathbb{Z} \setminus 0$, then $S^{-1}A = \mathbb{Q}$. In the same sense, a more general localization is akin to formally adding inverses of S.

Example 1.13. Let $\mathfrak{p} \subset A$ be a prime ideal. Then, $S = A \setminus \mathfrak{p}$ is multiplicative, since if $x \notin \mathfrak{p}$ and $y \notin \mathfrak{p}$, then $xy \notin \mathfrak{p}$. The localization $S^{-1}A$ is denoted $A_\mathfrak{p}$, the set of fractions a/s where $a \in A$ and $s \notin \mathfrak{p}$, with some equivalence relation. This makes everything except $\mathfrak{p}$ for units, so the image of $\mathfrak{p}$ is maximal in $A_\mathfrak{p}$.

Similarly, if $f \in A$, we can define $S = (f)$. The localization $S^{-1}A$ is denoted A_f, fractions of the form a/f^n; this makes f into a unit.

We need to define $\operatorname{Spec} A$ as a topological space, and then place a sheaf structure on it. With this structure, $\operatorname{Spec} A$ will be an *affine scheme.*

Definition 1.14. Let $I \subset A$ be an ideal. Then, let $D(I) \subset \operatorname{Spec} A$ be the set of prime ideals not containing I; if $I = \{f\}$, $D(I) = D(f) = \{\mathfrak{p} \mid f \notin \mathfrak{p}\}$. We define the topology on $\operatorname{Spec} A$ to have as its open sets $D(I)$ for all ideals I.

One has to check that these are closed under finite intersection and arbitrary union, but this is true, so $\operatorname{Spec} A$ is indeed a topological space.

Example 1.15. For example, $\operatorname{Spec} \mathbb{Z}$ as a set is the set of prime numbers and 0, since these account for all the ideals. The topology is curious: $(0) \subset \mathfrak{p}$ for all prime ideals $\mathfrak{p} \subset \mathbb{Z}$, so the zero ideal "lives everywhere."

The open sets are $D(a)$, the set of primes not dividing a, unless $a = 0$, in which case we get $\emptyset$.

The open set $D(f)$ is actually isomorphic as a topological space to $\operatorname{Spec}(A_f)$; for this reason, it's called a *distinguished affine open.*

Now, we just need to define the structure sheaf $\mathscr{O}_A$: what are the functions on $\operatorname{Spec} A$? We define $\mathscr{O}_A(U)$ to be the ring of functions $f : U \longrightarrow \coprod_{\mathfrak{p} \in U} A_\mathfrak{p}$ such that $f(\mathfrak{p}) \in A_\mathfrak{p}$ and for all $\mathfrak{p} \in U$, there's an $a/s \in A_\mathfrak{p}$ and an open $V \subset U$ such that for all $\mathfrak{q} \in V$, $f(\mathfrak{q}) = a/s$.

There are a bunch of equivalent definitions, but this is one of the most concrete: a section is a function to a weird space, but other definitions don't explicitly make the structure sheaf a sheaf of functions, and so it's harder to prove that the structure sheaf is, in fact, a sheaf.

Distinguished opens are particularly nice, in that $\mathscr{O}_X(D(f)) \cong A_f$. Moreover, for any $\mathfrak{p} \in \operatorname{Spec} A$, one can show $\mathscr{O}_{A,\mathfrak{p}} \cong A_\mathfrak{p}$. $A_\mathfrak{p}$ is a local ring, with (the image of) $\mathfrak{p}$ as its unique maximal ideal.

1.4. **Examples.** First, let's understand $\operatorname{Spec} \mathbb{Z}$ as a scheme, not just a topological space. $D(6)$ is the set of all primes except 2 and 3, plus the zero ideal. The acceptable functions on it are isomorphic to $\mathbb{Z}_6 = \{a/6^n \mid n \geq 0, a \in \mathbb{Z}\} = \mathbb{Z}[1/6]$. Thinking of these as functions, the function $21/6$ has value $21/6$ — but in different rings. Over (5), $21/6$ takes the value $21/6 \in \mathbb{Z}_{(5)}$; at (7), $21/6$ takes the value $21/6 \in \mathbb{Z}_{(7)}$. Here, $\mathbb{Z}_{(5)}$ is the ring of fractions whose denominators aren't divisible by 5. We can make sense of this for all primes except 2 or 3, and the function $21/6$ can't exist there (since dividing by zero is bad). At (0), the value is $21/6 \in \mathbb{Z}_{(0)} = \mathbb{Q}$.

Next, we'll do a more geometric example.

Example 1.16. Let k be a field (if you like, $k = \mathbb{C}$ makes for good geometric intuition). We define *affine n-space* $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \dots, x_n]$. All prime ideals of $\mathbb{A}^1_k$ look like (f) for some $f \in k[x]$; this prime ideal is prime iff f is irreducible. If k is algebraically closed, e.g. $k = \mathbb{C}$, this is only the case when $f(x) = x - a$ or $f(x) = 0$.

We associate the point $(x - a)$ to the point $a \in \mathbb{C}$, so we have a complex line of points plus the zero ideal, which is weird: it somehow lives everywhere.

$\mathbb{A}^2_\mathbb{C}$ is a little stranger: not only do we have a $\mathbb{C}^2$ worth of points (a, b) corresponding to $(x - a, y - b)$, and (0) which is once again everywhere, there are additional prime ideals: $(y - x^2)$ is a prime ideal, and it somehow lives at the entire curve $\{y = x^2\} \subset \mathbb{C}^2$. This is disorienting, but sometimes is useful.