SUMMER 2016 ALGEBRAIC GEOMETRY SEMINAR

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Contents

1. Separability, Varieties and Rational Maps: 5/16/16

Today's lecture was given by Tom Oldfield, on the first half of chapter 10.

This seminar has a website, located at
`https://www.ma.utexas.edu/users/toldfield/Seminars/Algebraicgeometryreading.html`.

The first half of Chapter 10 is about separated morphisms and varieties; it only took us 10 chapters! Vakil writes that he was very conflicted about leaving a proper treatment of algebraic varieties, a cornerstone of classical algebraic geometry, to so late in the notes. But from a modern perspective, our hands are tied: varieties are defined in terms of properties, which means building those properties out of other properties and out of the large amount of technology you need for modern algebraic geometry. With that technology out of the way, here we are.

One of these properties is separability. Let $\pi : X \to Y$ be a morphism of schemes; then, the **diagonal** is the induced morphism $\delta_\pi : X \to X \times_Y X$ defined by $x \mapsto (x, x)$; this maps into the fiber product because it fits into the diagram

(1.1)
$$\begin{tikzcd}
X \arrow[dr, "\delta_\pi"] \arrow[drr, bend left, "1_X"] \arrow[ddr, bend right, "1_X"'] & & \\
& X \times_Y X \arrow[r, "p_2"'] \arrow[d, "p_1"] & X \arrow[d, "\pi"'] \\
& X \arrow[r, "\pi"] & Y.
\end{tikzcd}$$

Here, p_1 nad p_2 are the projections onto the first and second components, respectively, and 1_X is the identity map on X.

The diagonal has a few nice properties. Suppose $V \subset Y$ is open, and $U, U' \subset \pi^{-1}(V)$ are open subsets of X. Then, $U \times_V U' = p_1^{-1}(U) \cap p_2^{-1}(U')$: we constructed fiber products such that they send open embeddings to intersections. In particular, if $U \cong \operatorname{Spec} A$, $U' \cong \operatorname{Spec} A'$, and $V \cong \operatorname{Spec} B$ are affine, $U \times_V U' \cong \operatorname{Spec}(A \otimes_B A')$. Therefore $\delta_\pi^{-1}(U \times_V U') = \delta_\pi^{-1}(p_1^{-1}(U) \cap p_2^{-1}(U')) = U \cap U'$. That is, the diagonal turns intersections into fiber products.

This argument feels like it takes place in Set, but goes through word-for-word for schemes.

Definition 1.2. A morphism $\pi : X \to Y$ of schemes is a **locally closed embedding** if it factors as $\pi = \pi_1 \circ \pi_2$, where π_2 is a closed embedding and π_1 is an open embedding.

Proposition 1.3. *For any $\pi : X \to Y$, δ_π is locally closed.*

Proof. Let $\{V_i\}$ be an affine open cover of Y, so $V_i \cong \operatorname{Spec} B_i$ for each B, and $\mathfrak{U}_i = \{U_{ij}\}$ be an affine open cover of $\pi^{-1}(V_i)$ for each i. Then, $\{U_{ij} \times_{V_i} U_{ij'} : i, j, j'\}$ covers $X \times_Y X$. More interestingly, $\{U_{ij} \times_{V_i} U_{ij} : i, j\}$ covers $\operatorname{Im}(\delta_\pi)$: this is because if $x \in U_{ij}$, then $\delta_\pi(x) \in p_1^{-1}(U_{ij})$ and in $p_2^{-1}(U_{ij})$, and $p_1^{-1}(U_{ij}) \cap p_2^{-1}(U_{ij}) = U_{ij} \times_{V_i} U_{ij}$.

Now, it suffices to show that $\delta_\pi : \delta_\pi^{-1}(U_{ij} \times_{V_i} U_{ij}) \to U_{ij} \times_{V_i} U_{ij}$ is closed, since the property of being a closed embedding is affine-local. Since each $U_{ij} \cong \operatorname{Spec} A_{ij}$ is affine, then it suffices to understand what's happening ring-theoretically: the diagonal map corresponds to the ring morphism $A_{ij} \otimes_{V_i} A_{ij} \to A_{ij}$ sending $a \otimes a' \mapsto aa'$. This is clearly surjective, which is exactly the criterion for a morphism of schemes to be a closed embedding. ☐

Corollary 1.4. *If X and Y are affine schemes, then δ_π is a closed embedding.*

Corollary 1.5. *If Δ denotes $\operatorname{Im}(\delta_\pi)$, then for any open $V \subset Y$ and $U \subset \pi^{-1}(V)$, $\Delta \cap (U \times_V U') \cong U \cap U'$ is a homeomorphism of topological spaces.*

This follows because a locally closed embedding is homeomorphic onto its image.

These will all be super useful once we define separability, which we'll do now.

Definition 1.6. A morphism $\pi : X \to Y$ is **separated** if $\delta_\pi : X \to X \times_Y X$ is a closed embedding.

This is weird upon first glance: why do we look at the diagonal to understand things about a morphism? The answer is that the diagonal has nice category-theoretic properties, so we can prove some useful properties by doing a few diagram chases.

More geometrically, separability corresponds to the Hausdorff property in topological spaces, and there's a criterion for this in terms of the diagonal.

Proposition 1.7. *If T is a topological space, then T is Hausdorff iff the diagonal morphism $T \to T \times T$ is a closed embedding.*

Equivalently, the image $\Delta \subset T \times T$ is a closed subspace.

Remark. Since schemes are topological spaces, you might think this proves separated schemes are Hausdorff, but this is untrue: fiber products of schemes are generally not fiber products of underlying spaces, and therefore closed embeddings of schemes are not the same as closed embeddings of their underlying spaces.

Separability is a nice property, and is good to have. But like Hausdorfness, we generally won't need to use schemes that aren't separated.

Example 1.8.

(1) By Corollary 1.4, all morphisms of affine schemes are separated.

(2) If we can cover $X \times_Y X$ by the sets $U_{ij} \times_{V_i} U_{ij}$ (with these sets as in the proof of Proposition 1.3), then π is separated.

(3) For a counterexample, let $X = \mathbb{A}^1_{(0,0)}$ be the "line with two origins" over a field k. This isn't a separated scheme: the diagonal is a "line with four origins," and these cannot be separated topologically: every open set containing one contains all of them. So take one affine piece of X, which contains exactly one origin, and therefore its image ought to contain all four, but it doesn't, so $X \to \operatorname{Spec} k$ isn't closed. This might feel a little imprecise, but one can make it fully rigorous.

We want separated morphisms to be nice: we'd like them to be preserved under base change and composition, and we'd like locally closed embeddings to be separated.

Proposition 1.9. *Locally closed embeddings are separated.*

This is the only example of a hands-on proof of a property; it's not hard, but the rest will be less abstract and easier. First, though, let's reframe it:

Proposition 1.10. *Any monomorphism of schemes is separated.*[1]

[1] More is true in general; all you need is that $p_1 = p_2$ in the diagram (1.1), which is analogous to an injectivity condition on π. Hence, it suffices that π is injective as a map of sets, but this is a weird notion for schemes, so we generally phrase it in terms of monomorphisms.

Proof. By point (2) of Example 1.8, it suffices to prove that fiber products $U_{ij} \times_{V_i} U_{ij}$ cover $X \times_Y X$ for our affine covers. So let's look at the fiber diagram (1.1) again; it tells us that $\pi \circ p_1 = \pi \circ p_2$. But since π is a monomorphism, then $p_1 = p_2$, so for any $z \in X \times_Y Z$, $p_1(z) = p_2(z)$; call this point x_z. Then, if $x_z \in U_{ij}$, $z \in p^{-1}(U_{ij})$ and $z \in p_2^{-1}(U_{ij})$, and their intersection is the fiber product. ☐

Since locally closed embeddings are monomorphisms, Proposition 1.9 follows as a corollary.

At this point, we can define varieties, and Vakil does so, but can't do anything with them, so we'll come back to them in a little bit.

Proposition 1.11. *If A is a ring, $\mathbb{P}^n_A \to \operatorname{Spec} A$ is separated.*

The idea of the proof is to compute: we already know a cover of $\mathbb{P}^n_A$ by $n+1$ affine schemes, and can check that the induced map on rings is surjective.

The following proposition gives us an important geometric property of separability.

Proposition 1.12. *If A is a ring and $X \to \operatorname{Spec} A$ is separated, then for any affine open subsets $U, V \subset X$, $U \cap V$ is also affine.*

Proof. The diagonal is a closed embedding, so $\delta : U \times V \to U \times_A V$ is also a closed embedding. Therefore $U \times V$ is isomorphic to a closed subscheme of an affine scheme, and therefore is affine. ☐

It's surprising how useful these arguments with the diagonal are: we got a useful and nontrivial result in one line! In general, you can prove a weirdly large amount of things by factoring them through the diagonal. In fact, le'ts use it to define another property.

Definition 1.13. A morphism $\pi : X \to Y$ is **quasiseparated** if δ_π is quasicompact.

This isn't the same as the other definition we were given, that for all affine $V \subset Y$ and $U, U' \subset \pi^{-1}(V)$, $U \cap U'$ is quasicompact. But it turns out to be equivalent.

Proposition 1.14. *$\pi : X \to Y$ is quasiseparated in the sense of Definition 1.13 iff it's quasiseparated in the sense we defined previously.*

The proof is a diagram chase involving the "magic diagram" for fiber products. This states that if $X_1, X_2 \to Y \to Z$ are maps in some category and the relevant fiber products exist, the diagram

$$\begin{tikzcd} X_1 \times_Y X_2 \arrow[r] \arrow[d] & X_1 \times_Z X_2 \arrow[d] \\ Y \arrow[r, "\delta"] & Y \times_Z Y \end{tikzcd}$$

is a fiber diagram; the proof is a diagram chase following from the associativity of products, or checking the universal property. This diagram is also very ubiquitous for proofs like these.

Proposition 1.15. *Separability and quasiseparability are preserved under base change.*

Proof. Suppose $\pi : X \to Y$ is separated and $\varphi : S \to Y$ is another map of schemes, so there's an induced morphism $\pi' : Z = X \times_Y S \to S$ fitting into the diagram

$$\begin{tikzcd} Z \arrow[r, "\pi'"] \arrow[d, "p_1"] & S \arrow[d, "\varphi"] \\ X \arrow[r, "\pi"] & Y. \end{tikzcd}$$

The magic diagram for this is the fiber diagram

$$\begin{tikzcd} Z \arrow[r, "\delta_{\pi'}"] \arrow[d] & Z \times_S Z \arrow[d] \\ X \arrow[r, "\delta_\pi"] & X \times_Y X. \end{tikzcd}$$

If π is separated, δ_π is closed, and therefore $\delta_{\pi'}$ is closed (since closed embeddings are preserved under base change), so π' is separated. The same argument works with π quasiseparated and δ_π quasicompact. ☐

There are a few related properties that we won't prove, but whose proofs are very similar to the previous one.

Proposition 1.16. *Separability and quasiseparability are*

1. *local on the target,*
2. *closed under composition, and*
3. *closed under taking products: if $\pi : X \to Y$ and $\pi' : X' \to Y'$ are separated morphisms of schemes over a scheme S, then $\pi \times \pi' : X \times_S X' \to Y \times_S Y'$ is separated; if π and π' are merely quasiseparated, so is $\pi \times \pi'$.*

Each of these is a diagram chase with the right diagram, and not a particularly hard one; the last one follows as a general categorical consequence of the others.

Now, though, we can define varieties.

Definition 1.17. Let k be a field. A **k-variety** is a k-scheme $X \to \operatorname{Spec} k$ that is reduced, separated, and of finite type. A **subvariety** of a given variety X is a reduced, locally closed subscheme.

Reducedness is a property of X, but the others are properties of the structure morphism $X \to \operatorname{Spec} k$. Notice that the affine line with doubled origin is reduced and of finite type, so separability is important for avoiding pathologies.

It's nontrivial that a subvariety $Y \subset X$ is itself a variety. X is finite type over $\operatorname{Spec} k$, so it's covered by finitely many affine opens that are schemes of finitely generated k-algebras, which are Noetherian, so X is Noetherian. Hence, $Y \hookrightarrow X$ is a finite-type morphsism into a Noetherian scheme, so Y is finite type; but we do need separability to be preserved under composition, which we just saw how to prove.

We did not require varieties to be irreducible; irreducibility doesn't behave as well as we would like, unless k is particularly nice.

Proposition 1.18. *The product of irreducible varieties over an algebraically closed field k is an irreducible k-variety.*

This follows from the nontrivial fact that if A and B are k-algebras that are integral domains, then $A \otimes_k B$ is an integral domain.

The last important thing we'll discuss today is a big meta-theorem about classes of morphisms.

Theorem 1.19 (Cancellation theorem)**.** *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "\tau"'] & & Y \arrow[dl, "\rho"] \\ & Z, & \end{tikzcd}$$

i.e. $\tau = \rho \circ \pi$, and let P be a property of morphisms preserved under base change and composition. If τ has P and δ_ρ has P, then π also has P.

The name is because we're "cancelling" ρ out of the composition.

The proof uses the notion of the graph of a morphism.

Definition 1.20. Let X and Y be schemes over a scheme S, and $\pi : X \to Y$ be a map of S-schemes. Then, the **graph** of π is the morphism $\Gamma_\pi : X \to X \times_S Y$ defined by $\Gamma_\pi; (1_X, \pi)$.

That is, this sends a point to its image on the graph. We use this because any morphism factors through its graph. Then, since δ_ρ has P, so must Γ_π, which is useful. It seems weirdly abstract and pointless, but the idea is that the nice properties of the diagonal, including locally closed embeddings, can be canceled off. In fact, if Y is separated, we can cancel off properties of closed embeddings, and if Y is quasiseparated, we can cancell off properties of quasicompact morphisms.

Rational Maps. Let's talk about rational maps, which are rational maps defined almost everywhere, and up to almost everywhere agreement. Rational maps are usually only defined on reduced varieties, since it's nearly impossible to get a hold on them otherwise; they're inherently geometric, and geometry tends to involve varieties.

Definition 1.21. A **rational map** $\pi : X \dashrightarrow Y$ is an equivalence class of morphisms $f : U \to Y$, where $U \subset X$ is a dense open subset; (f, U) and (f', U') are considered equivalent if there's a dense open set $V \subset U \cap U'$ if $f|_V = f'|_V$. One says π is **dominant** if its image is dense, or equivalently, for all nonempty opens $V \subseteq Y$, $\pi^{-1}(V) \neq \varnothing$.

Notice that dominance is well-defined, as it's independent of choice of representative.

Proposition 1.22. *Let X and Y are irreducible schemes, then $\pi : X \dashrightarrow Y$ is dominant iff the generic point of X maps to the generic point of Y.*

Proof. In the reverse direction, the generic point η_Y of Y is contained in every open subset of Y, so the preimage contains the generic point η_X of X, and in particular is nonempty.

In the other direction, suppose $\pi(\eta_X) \neq \eta_Y$; let $U = Y \setminus \overline{\pi(\eta_X)}$, which is an open subset. Thus, $\eta_X \notin \pi^{-1}(U)$, which is an open set. Since η_X is dense, it meets every nonempty open, so $\pi^{-1}(U)$ is empty, and therefore π isn't dominant. ☐

This is a pretty useful characterization of dominance. But why do we care about dominance? Because of composition.

Remark. Let $\pi : X \dashrightarrow Y$ and $\rho : Y \dashrightarrow Z$ be rational maps. If π is dominant and X is irreducible, it's possible to make sense of $\rho \circ \pi : X \dashrightarrow Z$ as a rational map, which is dominant iff ρ is.

This is nontrivial: if π isn't dominant, one might discover that the domain of ρ doesn't intersect the image of π; if they do, however, π^{-1} of the domain of definition of ρ is a nonempty open of X; since X is irreducible, it must be dense.

Definition 1.23. A rational map $\pi : X \dashrightarrow Y$ is **birational** if it's dominant and there exists a dominant $\psi : Y \dashrightarrow X$ such that as rational maps, $\pi \circ \psi \sim 1_X$ and $\psi \circ \pi \circ 1_Y$. In this case, one says π and ψ are **birational(ly equivalent)**.

Proposition 1.24. *Let X and Y be reduced schemes; then, X and Y are birational iff there exist dense open subschemes $U \subset X$ and $V \subset Y$ such that $U \cong V$.*

The idea is that we can let U and V be the domains of definition for our rational maps.

The notion of rationality is very specific to algebraic geometry; in the differentiable category, it's complete nonsense. Since any manifold can be triangulated, any two manifolds of the same dimension are birationally equivalent: remove the edges of the triangles, and you get a dense open set; clearly, any two triangles are birational. However, there exist algebraic varieties of the same dimension that aren't birationally equivalent.

Definition 1.25. A variety X over k is **rational** if it's birational to $\mathbb{A}_k^n$ for some n.

For example, $\mathbb{P}_k^n$ is rational. Rationality loses some information, but what it keeps is interesting.

Finally, let's see what dominance means in terms of ring morphisms.

Definition 1.26. Let $\varphi : \operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism of affine schemes and $\varphi^\sharp : B \to A$ be the induced map on global sections. Then, φ is dominant (i.e. as a rational map) iff $\ker(\varphi^\sharp) \subset \mathfrak{N}(A)$.

Here, $\mathfrak{N}(A)$ denotes its nilradical, the intersection of all prime ideals of A (equivalently, the ideal of nilpotent elements). That is, if A and B are reduced, dominance is equivalent to injectivity! Interestingly, this also corresponds to an inclusion of function fields, i.e. a field extension! We've reduced a geometric problem to a problem about algebra. Often, we can go in the other direction, e.g. for varieties. In this setting, birationality means isomorphism on the function fields.

2. Proper Morphisms: 5/19/16

These are Arun's lecture notes on rational maps to separated schemes and proper morphisms, corresponding to sections 10.2 and 10.3 in Vakil's notes. I'm planning on talking about the following topics:

- Rational maps to separated schemes, including the reduced-to-separated theorem and some corollaries.
- The definition of proper morphisms, and that they form a nice class of morphisms. Projective A-schemes are proper over A.

Throughout this lecture, S is a scheme, which will often be the base scheme.

Rational Maps to Separated Schemes. If X and Y are spaces and $\pi, \pi' : X \rightrightarrows Y$ are continuous, it's sometimes useful to talk about the locus where they agree, $\{x \in X : \pi(x) = \pi'(x)\}$. Categorically, this is the equalizer $\mathrm{Eq}(\pi, \pi') \hookrightarrow X$, which is characterized by the property that if $\varphi : W \to X$ is a continuous map such that $\pi \circ \varphi = \pi' \circ \varphi$, then it factors through $\mathrm{Eq}(\pi, \pi')$, i.e. there's a unique $h : W \to \mathrm{Eq}(\pi, \pi')$ such that the following diagram commutes.

$$\begin{tikzcd} W \arrow[d, dashed, "\exists!\, h"'] \arrow[dr, "\varphi"] & & \\ \mathrm{Eq}(\pi,\pi') \arrow[r, hook] & X \arrow[r, shift left, "\pi"] \arrow[r, shift right, "\pi'"'] & Y. \end{tikzcd}$$

So if we can do this for schemes, we'll have a subscheme where two morphisms agree, rather than just a set. The universal property for the equalizer is the same as for the fiber product

$$\begin{tikzcd} \mathrm{Eq}(\pi,\pi') \arrow[r] \arrow[d, hook, "i"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\delta"] \\ X \arrow[r, "{(\pi,\pi')}"] & Y \times Y, \end{tikzcd} \tag{2.1}$$

where δ is the diagonal morphism. We know fiber products of schemes exist, so equalizers do too.

Lemma 2.2 (Vakil ex. 10.2.A)**.** *If $\pi, \pi' : X \rightrightarrows Y$ are two morphisms of schemes over S, then $i : \mathrm{Eq}(\pi, \pi') \hookrightarrow X$ is a locally closed subscheme of X. If Y is separated over S, $\mathrm{Eq}(\pi, \pi')$ is a closed subscheme.*

Proof. Since we're over S, the product in (2.1) should be replaced with $Y \times_S Y$, the product in Sch_S. Since δ is a locally closed embedding, and this is a property preserved under base change, then i is too. If $Y \to S$ is separated, then δ is a closed embedding, and this is also preserved by pullbacks. $\square$

Remark. The locus where two maps agree does not need to be reduced, e.g. if $\pi, \pi' : \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ are defined by $\pi(x) = 0$ and $\pi'(x) = x^2$, then they agree "to first order" at 0, and $\mathrm{Eq}(\pi, \pi') = \operatorname{Spec} \mathbb{C}[x]/(x^2)$.

The central result about these is the reduced-to-separated theorem.

Theorem 2.3 (Reduced-to-separated theorem (Vakil Thm. 10.2.2))**.** *Let $\pi, \pi' : X \rightrightarrows Y$ be two morphisms of S-schemes. If X is reduced, Y is separated over S, and π and π' agree on a dense open subset, then $\pi = \pi'$.*

This is equality in the sense of morphisms of schemes, which is stronger than pointwise equality.

Proof. By Lemma 2.2, $\mathrm{Eq}(\pi, \pi') \hookrightarrow X$ is a closed subscheme, but it contains a dense open set. Since X is reduced, its only closed subscheme containing a dense open set is itself. $\square$

Corollary 2.4. *If X is reduced, Y is separated, and $\pi : X \dashrightarrow Y$ is a rational map, then there is a maximal $U \subset X$ such that $\pi|_U : U \to Y$ is an honest morphism. In particular, this is true for rational functions on reduced schemes.*

This U is called the **domain of definition** of π; its complement is sometimes called the **locus of indeterminacy**.

Proof. We can choose U to be the union of all domains of representatives of π. If $f_1 : V_1 \to Y$ and $f_2 : V_2 \to Y$ are two morphisms representing π, then f_1 and f_2 agree on a dense open subset of $V_1 \cap V_2$, so by the reduced-to-separated theorem agree on all of $V_1 \cap V_2$. Thus, we can glue representing morphisms on their intersection and therefore define π on all of U. $\square$

Next, we need to digress slightly to understand the image of a locally closed embedding. This is from section 8.3 of the notes.

If $\pi : X \to Y$ is a morphism of schemes, it's in particular a continuous function, so its image $\pi(X) \subset Y$ is a subspace. This will be referred to as the **set-theoretic image**. As usual, the topological version of a thing tends to be less well-behaved than the scheme-theoretic one, so we'll define an image of π that's a subscheme of Y. Schemes are locally cut out by equations, so it seems reasonable to say that a closed subscheme $i : Z \hookrightarrow Y$ **contains the image** of π if functions in $\mathscr{O}_Y$ that vanish on Z also vanish when pulled back to X. That is, the composition $\mathscr{I}_{Z/Y} \to \mathscr{O}_Y \to \pi_* \mathscr{O}_X$ is zero, where $\mathscr{I}_{Z/Y} = \ker(i^\sharp : \mathscr{O}_Y \to i_* \mathscr{O}_Z)$ is the sheaf of ideals associated to the closed embedding of Z into Y.

Definition 2.5. The **scheme-theoretic image** $\mathrm{Im}(\pi)$ of π is the intersection of all closed subschemes containing the image of π.[2] If π is a locally closed embedding, $\mathrm{Im}(\pi)$ is also called the **scheme-theoretic closure** of π.

That is, $\mathrm{Im}(\pi)$ is the smallest closed subscheme of Y such that locally vanishing on $\mathrm{Im}(\pi)$ implies locally vanishing when pulled back to X.

Theorem 2.6 (Vakil cor. 8.3.5)**.** *Let $\pi : X \to Y$ be a morphism of schemes. If X is reduced or Y is quasicompact, the closure of the set-theoretic image of π is the underlying set of* $\mathrm{Im}(\pi)$.

We lack the time to prove this, but it follows from the defining properties of closed embeddings.

Just like we defined the graph of a morphism of S-schemes $\pi : X \to Y$ to be $\Gamma_\pi = (\mathrm{id}, \pi) : X \to X \times_S Y$, we can define the graph of a rational map in nice situations.

Definition 2.7. Let $\pi : X \dashrightarrow Y$ be a rational map over S, where X is reduced and Y is separated over S. For any representative morphism $f : U \to Y$ of π, the **graph of the rational map** π, denoted Γ_π, is the scheme-theoretic closure of the map $\Gamma_f \hookrightarrow U \times_S Y \hookrightarrow X \times_S Y$. (The first map is a closed embedding, and the second is an open embedding.)

The following diagram might make this definition clearer.

$$\begin{tikzcd}
\Gamma_\pi \arrow[r, hook, "\text{cl.}"] & X \times_S Y \arrow[dl] \arrow[dr] & \\
X \arrow[u, dashed] \arrow[rr, dashed, "\pi"] & & Y.
\end{tikzcd}$$

A priori this definition depends on the choice of representative, but fortunately, this isn't actually the case.

Proposition 2.8 (Vakil ex. 10.2.E)**.** *The graph of a rational map π is independent of choice of representative.*

Proof. Let $\xi' : U \to Y$ and $\xi : V \to Y$ be two representatives of π. Without loss of generality, we can assume V is the maximal domain of definition for π, so $U \subset V$ and $\xi' = \xi|_U$. Thus, we have a bunch of embeddings fitting into the diagram

$$\begin{tikzcd}
\Gamma_{\xi'} \arrow[r, hook, "\text{cl.}"] \arrow[d, hook] & U \times Y \arrow[r, hook, "\text{op.}"] \arrow[d, hook, "\text{op.}"] & X \times Y \\
\Gamma_\xi \arrow[r, hook, "\text{cl.}"] & V \times Y. \arrow[ur, hook, "\text{op.}"'] &
\end{tikzcd}$$

Thus, $\Gamma_{\xi'}$ factors as a subset of a closed subset of $V \times Y$, so its scheme-theoretic closure, which is just the closure of its underlying set by Theorem 2.6, must factor through this. In particular, the graph of π as defined with respect to ξ' embeds into $V \times Y$. Thus, we can assume $V = X$, since everything takes place inside V. In this case, Γ_π as defined by ξ is just Γ_ξ, and $\Gamma_\xi \cong X$ by projection onto the first factor. This projection restricts to an isomorphism $\Gamma_{\xi'} \cong U$, and carries the embedding $\Gamma_\xi \hookrightarrow \Gamma_{\xi'}$ to the embedding $U \hookrightarrow X$. Finally, to form the graph of π with respect to ξ', we take the closure, and since U is a dense open subset, we get X, or all of Γ_ξ. $\square$

Finally, we discuss one application to effective Cartier divisors. (This is actually an excuse to introduce effective Cartier divisors, since they show up again and again.)

Definition 2.9. A closed embedding $\pi : X \hookrightarrow Y$ is an **effective Cartier divisor** if $\mathscr{I}_{X/Y}$ is locally generated by a single non-zerodivisor. That is, there's an affine open cover $\mathfrak{U}$ of Y such that for each $U_i = \operatorname{Spec} A_i \in \mathfrak{U}$, there's a $t_i \in A$ that is not a zerodivisor and such that $\mathscr{I}_{X/Y}(U) = A_i/(t_i)$.

Proposition 2.10 (Vakil ex. 10.2.G)**.** *Let X be a reduced S-scheme and Y be a separated S-scheme. If $i : D \hookrightarrow X$ is an effective Cartier divisor, there is at most one way to extend an S-morphism $\pi : X \setminus D \to Y$ to all of X.*

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[2] There's something to prove here, that containing the image of π is well-behaved under intersections.

Proof. This is true if we know it on an affine cover, so without loss of generality assume $X = \operatorname{Spec} A$ is affine and $D = V(t)$ for some $t \in A$ that isn't a zerodivisor. If $D(t) = X \setminus D$ is dense in X, then we're done by Theorem 2.3. Since X is reduced, then by Theorem 2.6 this is equivalent to the scheme-theoretic closure of $D(t)$ being all of X. Given a closed subscheme $Z \hookrightarrow X$, we want to understand when functions vanishing on Z pull back to the zero function on $D(t)$. The map $\Gamma(X, \mathscr{O}_X) \to \Gamma(D(t), \mathscr{O}_X)$ is also $A \to A_t$; since t isn't a zerodivisor, this is injective, so a function pulls back to 0 on $D(t)$ iff it vanishes on all of X. Hence, $\operatorname{Im}(D(t) \hookrightarrow X) = X$ as desired. ∎

Proper Morphisms. The next topological notion we introduce to algebraic geometry is that of a proper map. Recall that a continuous map of topological spaces is proper if the preimage of any compact set is compact. Compactness doesn't really behave the same way in algebraic geometry, so we'll have to define properness in a different way, which will satisfy similar properties.

Proper maps are closed maps, meaning the image of a closed set is closed. This would be a reasonable starting point, except that closed maps are not preserved by fiber products. It turns out the right way to fix this is just to pick the ones that behave well.

Definition 2.11. A morphism $\pi : X \to Y$ of schemes is **universally closed** if for all morphisms $Z \to Y$, the pullback $Z \times_Y X \to Z$ is a closed map.

That is, it remains closed under arbitrary base change.

Lemma 2.12. *Universal closure is a "nice" property of schemes, i.e. local on the target, closed under composition, and preserved by base change.*

Proof. Clearly, universal closure is closed under composition, and by definition, it's preserved by fiber products. Being a closed map is local on the target, and therefore so is universal closure. ∎

We use universal closure to define the property we really care about.

Definition 2.13. A morphism $\pi : X \to Y$ is **proper** if it's separated, finite type, and universally closed. If A is a ring, an A-scheme X is said to be **proper over** A if the structure morphism $X \to \operatorname{Spec} A$ is proper.

Example 2.14. Closed embeddings are our first example of proper morphisms: they're affine, and therefore separated. Closed embeddings are closed maps, and since the pullback of a closed embedding is a closed embedding, a closed embedding is universally closed. Finally, closed morphisms are finite type (which boils down the fact that if $B \twoheadrightarrow A$ is a surjective ring map, A is a finitely generated B-algebra).[3]

This agrees with our intuition for topological spaces, which is good.

Proposition 2.15 (Vakil prop. 10.3.4)**.**

(1) *Properness is a "nice" property of schemes (in the sense of Lemma 2.12).*
(2) *Properness is closed under products: if $\pi : X \to Y$ and $\pi' : X' \to Y'$ are proper morphisms of S-schemes, then $\pi \times \pi' : X \times_S X' \to Y \times_S Y'$ is proper.*
(3) *Given a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "\tau"'] & & Y \arrow[dl, "\rho"] \\ & Z, & \end{tikzcd}$$

if τ is proper and ρ is separated, then π is proper.

For example, by (3), any morphism from a proper k-scheme to a separated k-scheme is proper (let $Z = \operatorname{Spec} k$).

Proof. Everything in this proposition comes nearly for free. We already knew finite type and separability to be nice properties of schemes, and by Lemma 2.12, so is universal closure; since properness is having all three at once, it too must be a nice property. (2) is a formal consequence of (1), which is proven for any nice class

[3]The same line of reasoning shows that finite morphisms are proper, which is a generalization: they're affine, hence separated, and closed maps; since they're preserved under base change, they must also be universally closed. Finally, finite morphisms are finite type.

of morphisms in Vakil's ex. 9.4.F. Finally, since closed embeddings are proper, the cancellation theorem from last lecture applies to prove (3). ☐

According to Vakil, the next example is the most important example of proper morphisms.

Theorem 2.16 (Vakil thm. 10.3.5)**.** *If A is a ring and X is a projective A-scheme, $X \to \operatorname{Spec} A$ is proper.*

Proof. Since X is projective, the structure morphism factors as $X \hookrightarrow \mathbb{P}^n_A \to A$, a closed embedding followed by the structure map for $\mathbb{P}^n_A$. Since closed embeddings are proper (Example 2.14), it suffices to show $\mathbb{P}^n_A \to \operatorname{Spec} A$ is proper, because proper morphisms are closed under composition. Projective schemes are finite type, and we proved last time that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is separated, so it remains to check universal closure.

If $\varphi : X \to \operatorname{Spec} A$ is an arbitrary morphism, we would like for the map $\mathbb{P}^n_A \times_A X \to X$ to be closed. Since $\mathbb{P}^n_A = \mathbb{P}^n_{\mathbb{Z}} \times_A \operatorname{Spec} \mathbb{Z}$, then we have the following commutative diagram, in which both squares are pullback squares:

$$\begin{tikzcd}
\mathbb{P}^n_A \times_A X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\varphi"] \\
\mathbb{P}^n_A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Spec} A \arrow[d] \\
\mathbb{P}^n_{\mathbb{Z}} \arrow[r] & \operatorname{Spec} \mathbb{Z}.
\end{tikzcd}$$

By checking the universal property, we see that the outer rectangle is a pullback square too: in other words, $\mathbb{P}^n_A \times_A X = \mathbb{P}^n_X$, so it suffices to show that the structure map $\mathbb{P}^n_X \to X$ is closed for arbitrary X. Being a closed map is a local condition, so we can check on an affine cover of $\mathbb{P}^n_X$; pulling back by $\operatorname{Spec} B \hookrightarrow X$ gives us $\mathbb{P}^n_B \to \operatorname{Spec} B$, so it suffices to know that the structure map is closed for all rings B. This is precisely the fundamental theorem of elimination theory (Thm. 7.4.7 in Vakil's notes), so we're done. ☐

Perhaps surprisingly, the converse is almost true: it's difficult to come up with examples of schemes that are proper, but not projective.

The last thing we'll prove about proper schemes is another analogue of compactness. Recall that if M is a compact, connected complex manifold, all holomorphic functions on M are constant. We'll be able to prove a scheme-theoretic analogue of this.

Proposition 2.17 (Vakil 10.3.7)**.** *Let k be an algebraically closed field and X be a connected, reduced, proper k-scheme. Then $\Gamma(X, \mathscr{O}_X) \cong k$.*

Proof. First, we can naturally identify $\Gamma(X, \mathscr{O}_X)$ with the ring of k-scheme maps $X \to \mathbb{A}^1_k$: using the $(\Gamma, \operatorname{Spec})$ adjunction, $\operatorname{Hom}_{\mathsf{Sch}_k}(X, \mathbb{A}^1_k) = \operatorname{Hom}_{\mathsf{Alg}_k}(k[t], \Gamma(X, \mathscr{O}_X)) = \Gamma(X, \mathscr{O}_X)$, so functions on X are actually a ring of functions, which is nice.

Let $f \in \Gamma(X, \mathscr{O}_X)$, so f corresponds to a morphism $\pi : X \to \mathbb{A}^1_k$. If $i : \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$ is the usual open embedding, let $\pi' = i \circ \pi$. Since X is proper and $\mathbb{P}^1_k$ is separated over k, then π' must be proper, by Proposition 2.15, part 3 (let $Z = \operatorname{Spec} k$). Thus, π' is closed, so the set-theoretic image of π' is a closed, connected subset of $\mathbb{P}^1_k$. Since $\mathbb{P}^1_k$ has the cofinite topology, then $\operatorname{Im}(\pi)$ must be a single closed point p or all of $\mathbb{P}^1_k$, but if the latter, it can't factor through i. Since π' factors through $\mathbb{A}^1_k$, p is a closed point in $\mathbb{A}^1_k$, hence identified with an element of k.

The underlying set of the scheme-theoretic image of π is the closure of the set-theoretic image, so it's just p again; since X is reduced, so is its scheme-theoretic image. Thus, $\pi : X \to \mathbb{A}^1_k$ is a constant map of schemes $x \mapsto p$, and tracing through the adjunction, this corresponds to the constant function $f = p \in \Gamma(X, \mathscr{O}_X)$. ☐

3. Dimension: 5/23/16

Today's lecture was given by Gill Grindstaff, on the first half of Chapter 11. This section relies on a lot of commutative algebra, which can make it difficult.

There are equiuvalent topological and algebraic formulations of the definition of the dimension of a scheme. We'd like this to agree with the intuitive notions of dimension: $\mathbb{A}^n$ should be n-dimensional, for example.

To motivate these definitions, recall that the dimension of a vector space V is the cardinality of some (and therefore any) basis for V. However, it's equivalent to say that the dimension of V is the supremum of lengths of nested chains of subspaces $0 \subsetneq W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_n = V$ (so we don't count 0). The scheme-theoretic definition will resemble this.

Definition 3.1.

- The **Krull dimension of a topological space** X is the supremum of lengths of chains $X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_n = X$ in which each X_i is a closed, irreducible subset of X.
- The **Krull dimension of a ring** A is the supremum of lengths of chains $0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ of prime ideals in A.

Fact. If A is a ring, then $\dim \operatorname{Spec} A = \dim A$. This is because $\mathfrak{p}_i \mapsto V(\mathfrak{p}_i)$ defines an inclusion-reversing bijection between the poset of prime ideals of A and the poset of closed irreducible subspaces of $\operatorname{Spec} A$, and in particular sends chains of nested subsets to chains of nested subsets (in the other direction).

Example 3.2.

(1) Every prime ideal of $\mathbb{Z}$ is of the form $\mathfrak{p} = (p)$ for a prime number p, or is the zero ideal. Hence, the longest chain we can make is $(p) \supset (0)$, so $\dim \mathbb{Z} = 1$.
(2) Similarly, for any field k, in $k[t]$ the longest chains we can make are $(f(t)) \supset (0)$ for f irreducible, so $\dim \mathbb{A}^1_k = 1$.
(3) In $k[x]/(x^2)$, (0) is the only prime ideal, so $\dim k[x]/(x^2) = 0$.

Dimension is *not* local, unlike the dimension of manifolds: consider the space $Z \subset \mathbb{A}^3$ consisting of the union of the xy-plane and the z-axis, which is not irreducible. Then, the dimension of the xy-plane is 2 but the dimension of the z-axis is 1. We do know that $\dim(Z)$ is the maximum of the dimensions of its irreducible subsets, however.

Definition 3.3. A scheme X is **equidimensional** if each of its irreducible components has the same dimension.

An equidimensional scheme of dimension 1 is called a **curve**; an equidimensional scheme of dimension 2 is called a **surface**; and so forth.

In order to get a handle on dimension, we'll need to do come commutative algebra.

Theorem 3.4. *Let* $\pi : \operatorname{Spec} A \to \operatorname{Spec} B$ *be induced from an integral extension* $A \to B$ *of rings. Then,* $\dim \operatorname{Spec} A = \dim \operatorname{Spec} B$.

This follows from an algebraic result.

Theorem 3.5 (Going-up theorem). *Let* $A \hookrightarrow B$ *be an integral extension,* $\mathfrak{p}_1 \subset \mathfrak{p}_2$ *be prime ideals of* A*, and* $\mathfrak{q}_1$ *be a prime ideal of* B *such that* $\mathfrak{q}_1 \cap A = \mathfrak{p}_1$.[4] *Then, there is a prime ideal* $\mathfrak{q}_2 \subset B$ *such that* $\mathfrak{q}_1 \subset \mathfrak{q}_2$ *and* $\mathfrak{q}_2 \cap A = \mathfrak{p}_2$.

This can be inductively extended to chains of prime ideals, which proves one half of Theorem 3.4.

The proof of Theorem 3.5 depends on the following lemma.

Lemma 3.6. *Let* $A \hookrightarrow B$ *be an integral extension. If* B *is a field, then* A *is a field.*

The following exercise is another nice property.

Exercise 3.7. Let $\nu : \widetilde{X} \to X$ be the normalization. Then, $\dim \widetilde{X} = \dim X$.

The **normalization** of X replaces rings with their integral closures on an affine cover; after checking that this behaves well, it defines a nice scheme that X embeds into as an open dense subset. The key to the proof is that the dimension of a ring is the same as the dimension of its integral closure, which follows from Theorem 3.4.

The next thing we'd like to define is codimension, but there are some weird pathologies: recalling Z, the union of the xy-plane and the z-axis, what's the codimension of the z-axis in Z? Should it be 0, since there's nothing above it? Or is it 1, since Z is 2-dimensional and the z-axis is 1-dimensional? There's no good answer, and as a result one only defines codimension inside irreducible schemes.

Definition 3.8. Let X be an *irreducible* topological space and $Y \subset X$. Then, the **codimension** $\operatorname{codim}_X Y$ is the supremum of lengths of chains of irreducible closed subsets $\overline{Y} \subsetneq Z_1 \subsetneq Z_2 \subsetneq \cdots \subsetneq Z_n = X$.

[4]One often says that $\mathfrak{q}_1$ **lies over** $\mathfrak{p}_1$.

Since Y might not be closed, we must start with $\overline{Y}$. This is satisfactory: a closed point has codimension 2 inside $\mathbb{A}^2$.

There's also an algebraic analogue of codimension.

Definition 3.9. The **codimension of a prime ideal** $\mathfrak{p}$ in a ring A, written $\operatorname{codim}_R \mathfrak{p}$, is the supremum of lengths of *decreasing* chains of prime ideals $\mathfrak{p} \supsetneq \mathfrak{q}_1 \supsetneq \mathfrak{q}_2 \supsetneq \cdots \supsetneq \mathfrak{q}_n \supsetneq (0)$.

In particular, this implies that $\operatorname{codim}_R \mathfrak{p} = \dim R_{\mathfrak{p}}$.

Here are some useful results about dimension.

Proposition 3.10. *Let R be a UFD and $\mathfrak{p} \subset R$ be a codimension-one prime ideal. Then, $\mathfrak{p}$ is principal.*

Theorem 3.11. *Let A be a finitely generated k-algebra that's an integral domain. Then,* $\dim \operatorname{Spec} A = \operatorname{tr.deg.} K(A)/k$.

Here, $K(A)$ denotes the field of fractions of A, and $\operatorname{tr.deg.} K/k$ is the **transcendence degree** of a field extension K/k; the idea is: how many transcendental elements do you need to adjoin to get K? This intuition turns out to be correct.

Lemma 3.12 (Noether normalization)**.** *Let S be a finitely generated k-algebra that's an integral domain, such that* $\operatorname{tr.deg.} K(A)/k = n$*. Then, there exist $x_1, \dots, x_n \in A$, algebraically independent*[5] *over k, such that A is a finite extension of $k[x_1, \dots, x_n]$.*

These are very useful because the transcendence degree is much easier to understand than all prime ideals of a ring.

Just as we have a going-up theorem, there's also one in the pther dimension.

Theorem 3.13 (Going-down theorem)**.** *Let $\phi : B \hookrightarrow A$ be a finite extension of rings (i.e. A is finitely-generated as a B-*module*), where B is an integrally closed domain and A is an integral domain. Suppose $\mathfrak{q}_1 \subset \mathfrak{q}_2$ are prime ideals of B and $\mathfrak{p}_2 \subset A$ is a prime ideal such that $\phi^{-1}(\mathfrak{p}_2) = \mathfrak{q}_2$ (so it lies over $\mathfrak{q}_2$). Then, there exists a prime $\mathfrak{p}_1 \subset \mathfrak{p}_2$ such that $\mathfrak{p}_1$ lies over $\mathfrak{q}_1$.*

This requires more assumptions and is harder to prove.

At this point, we talked about a few exercises.

Lemma 3.14. *Let X be a topological space and $U \subset X$ be open. Then, there's a bijection between the irreducible closed subsets of U and the irreducible closed subsets of X meeting U.*

This resembles a theorem from commutative algebra establishing a bijection between prime ideals of B/I and prime ideals of B containing I (where $I \subset B$ is an ideal).

The proof of Lemma 3.14 sets up the bijection by sending an irreducible closed subset $F \subset U$ to $\overline{F} \subset X$, and sends an irreducible $F' \subset X$ to $F \cap U \subset U$.

Exercise 3.15 (Vakil ex. 11.1.B)**.** Show that a scheme has dimension n iff it can be covered by affine open subsets of dimension at most n, where equality is achieved for some affine scheme in the cover.

4. Codimension One: 5/26/16

Today, Richard spoke about sections 11.3 and 11.4, on codimension 1 miracles. We'll skip the last section of chapter 11, because it provides solely algebraic proofs of some of these theorems, and doesn't assist one's geometric intuition.

Definition 4.1. A scheme X is **locally Noetherian** if it has a cover by affine opens $\operatorname{Spec} A_i$ such that each A_i is a local ring. If in addition X is quasicompact, it's called **Noetherian**.

All varieties are locally Noetherian and even Noetherian

One of the codimension 1 miracles is Krull's principal ideal theorem. There are a couple versions.

Theorem 4.2 ((Geometric) Krull's principal ideal theorem)**.** *Let X be a locally Noetherian scheme and $f \in \Gamma(X, \mathscr{O}_X)$. Then, the irreducible components of $V(f)$ are codimension* 0 *or* 1 *in X.*

[5] Just as linear independence means not satisfying any nonzero linear relation, **algebraic independence** means not satisfying any nonzero polynomial relation.

Recall that $V(f)$ is the set of points $x \in X$ such that the stalk $[f] \in \mathscr{O}_{X,x}$ is equal to 0.

Theorem 4.2 follows from the algebraic version.

Theorem 4.3 ((Algebraic) Krull's principal ideal theorem)**.** *Let A be a Noetherian ring and $f \in A$. Then, every prime ideal $\mathfrak{p} \subset A$ minimal among those containing f has codimension at most 1. If f isn't a zerodivisor, the codimension is exactly 1.*

Since we can pass between (co)dimension of prime ideals and (co)dimension of schemes, these two formulations of the theorem are equivalent.

Definition 4.4. Let $X \hookrightarrow Y$ be a closed embedding. Then, X is **locally principal** if there is an affine open cover $\mathfrak{U}$ of Y such that for every $\operatorname{Spec} A \in \mathfrak{U}$, $X \cap \operatorname{Spec} A$ is cut out by a principal ideal of A.

That is, X is locally cut out by a single equation.

Corollary 4.5. *A locally principal closed subscheme has codimension 0 or 1.*

There are a lot of interesting exercises that derive further consequences of this theorem: here are a few.

Proposition 4.6 (Vakil ex. 11.3.C)**.** *Let X be a closed subset of $\mathbb{P}^n_k$ of dimension at least 1. Then, every nonempty hypersurface intersects X.*

This tells us, for example, that there are no parallel hypersurfaces in projective space; in particular, this is not true of affine space. Finding nice hypersurfaces is often a good way to reduce the dimensionality of a question.

Proposition 4.7 (Vakil ex. 11.3.E)**.** *Let $X, Y \subset \mathbb{A}^d_k$ be equidimensional subvarieties of codimensions m and n, respectively. Then, $X \cap Y$ has codimension at most $m + n$.*

Proposition 4.8 (Vakil ex. 11.3.G)**.** *Let A be a Noetherian ring and $f \in A$ be such that f isn't contained in any prime ideal of codimension 1. Then, f is invertible.*

The idea is to consider the dimension of the quotient $A/\mathfrak{p}$ if $f \in \mathfrak{p}$.

Example 4.9. Sometimes, codimension behaves pathologically. Let k be a field and $A = k[x]_{(x)}[t]$: elements of A are expressions of the form

$$\Phi = \sum_{i=1}^{n} \frac{f_i(x)}{g_i(x)} t^i,$$

where $x \nmid g_i(x)$. The ideal $\mathfrak{p} = (xt - 1)$ is prime, and $A/(xt-1) = k[x]_{(x)}[1/x] \cong k(x)$, so $(xt-1)$ is maximal, and hence has dimension 0. By Theorem 4.3, since $xt - 1$ is not a zerodivisor, then $\operatorname{codim}_A \mathfrak{p} = 1$.

Naïvely, we might expect this implies $\dim A = 1$, but in fact there's an irreducible chain of length 2: $(0) \subsetneq (t) \subsetneq (x, t)$, so $\dim A \geq 2$ (and in fact is exactly 2). So codimension is not just the difference in dimension.

Another cool application of dimension is to characterize UFDs (at least among Noetherian rings).

Proposition 4.10 (Vakil 11.3.5)**.** *Let A be a Noetherian integral domain. Then, A is a UFD iff all codimension 1 prime ideals are principal.*

Proof. The forward direction is Proposition 3.10: if $\mathfrak{p}$ is codimension 1, then for any $f \in \mathfrak{p}$, if g is an irreducible prime factor of f, then $(g) \subset \mathfrak{p}$, but since $\operatorname{codim} \mathfrak{p} = 1$, this forces $(g) = \mathfrak{p}$.

Conversely, we want to show that an $a \in A$ is irreducible iff it's prime. If a is irreducible, then by Theorem 4.2, $V(a)$ has a codimension 1 point $[(p)]$, so $a = a'p$ for some a'. Thus, a' must be a unit, so $(a) = (p)$, and hence a is prime. The other direction uses the Noetherian hypothesis. ∎

The next great property of codimension 1 is a generalization of Krull's principal ideal theorems.

Theorem 4.11 (Krull height theorem)**.** *Let $X = \operatorname{Spec} A$, where A is a Noetherian ring and $Z = V(r_1, \ldots, r_\ell)$ be an irreducible subset. Then, $\operatorname{codim}_X Z \leq \ell$.*

Though this looks like it should follow inductively from Theorem 4.2, it's more subtle.

Another nice result is algebraic Hartogs' lemma, analogous to Hartogs' lemma in several complex variables, which is about poles or singularities of holomorphic functions.

Theorem 4.12 (Algebraic Hartogs' lemma)**.** *Let A be an integrally closed Noetherian integral domain. Then, if P denotes the set of prime ideals of A of codimension 1, then $A = \bigcap_{\mathfrak{p}\in P} A_{\mathfrak{p}}$.*

This intersection is understood to take place in the fraction field $K(A)$. The relation to the complex-analytic version is that if $f \in K(A)$, it can be interpreted as a rational function. If $f \notin A_{\mathfrak{p}}$, it's thought of as having a pole at $\mathfrak{p}$, and if it's in $\mathfrak{p}A_{\mathfrak{p}}$, it has a zero at $\mathfrak{p}$. Hartogs' lemma states that we can extend over singularities of codimension 2 or higher, but not necessarily codimension 1.

Dimension of fibers of morphisms of varieties. Recall that the fundamental theorem of elimination theory (which we used to prove Proposition 2.16) states that for every ring A, $\mathbb{P}^n_A \to \operatorname{Spec} A$ is a closed map. This tells us that closed subsets of projective space are cut out by inhomogeneous equations in $n+1$ variables over A.

One therefore wonders about the locus where the solution of the system of $n+1$ inhomogeneous equations is dimension at least d, for some d. This is a closed condition on the coefficients (just as in linear algebra), and therefore this locus is closed (just like in linear algebra).

Proposition 4.13 (Vakil ex. 11.4.A)**.** *Let $\pi : X \to Y$ be a morphism of locally Noetherian schemes, $p \in X$, and $q = \pi(p)$. Then,* $\operatorname{codim}_X p \le \operatorname{codim}_Y q + \operatorname{codim}_{\pi^{-1}(q)} p$.

Example 4.14. For this, it's good to have a picture. Suppose X is the union of the xy-plane and the z-axis inside $\mathbb{A}^3$, and $Y = \mathbb{A}^2$. Let $\pi : X \to Y$ crush the z-axis down to 0, $p = (0,0,1)$, and $q = (0,0)$. Then, $\operatorname{codim}_X p = 1$, $\operatorname{codim}_Y q = 2$, and $\operatorname{codim}_{\pi^{-1}(q)} p = 1$ (since $\pi^{-1}(q)$ is the z-axis). Indeed, $1 \le 2+1$.

Now, we have a result akin to the regular value theorem in differential topology: if $f : X \to Y$ is a smooth map of manifolds and $y \in Y$ is a regular value, then $f^{-1}(y) \subset X$ has codimension equal to the difference of their dimensions, or is empty.

Theorem 4.15 (Vakil 11.4.1)**.** *Let $\pi : X \to Y$ be a morphism of finite type k-schemes,* $\dim X = m$, *and* $\dim Y = n$. *Then, there is an open $U \subseteq Y$ such that for all $q \in U$, the fiber over q has pure dimension $m - n$ or is empty.*

Fiber dimension in general is discontinuous, but curiously, it obeys **upper semicontinuity** (we say that for all $\varepsilon > 0$, there's a $\delta > 0$ such that if $|x_0 - x| < \delta$, then $f(x) \le f(x_0) + \varepsilon$). The intuition is that the value can jump, but then the "upper part" is closed. This is exactly as in real analysis.

Proposition 4.16 (Vakil 11.4.2)**.** *Let $\pi : X \to Y$ be a morphism of finite type k-schemes.*

(1) *The dimension of the fiber of π at a $p \in X$ (specifically, of the largest component of $\pi^{-1}(\pi(p))$ containing p) is upper semicontinuous on X.*
(2) *If in addition π is proper[6], then the dimension of the fiber above a $y \in Y$ is upper semicontinuous on Y.*

Though it's surprising that upper semicontinuity exists, it appears in other places in algebraic geometry. It tells us that the dimension can increase when one takes limits. The dimension of the fiber is never smaller than what you think, but can be bigger, e.g. when we collapsed the z-axis onto the xy-plane in Example 4.14.

5. Regularity: 5/30/16

Tody, Jay Hathaway spoke about sections 12.1–12.3, on regularity, to the sonorous sounds of high schoolers warming up for the Texas State Solo and Ensemble Festival.

First, we'll talk about the Zariski cotangent space. Recall that in differential geometry, a manifold is a locally ringed space (M, C^∞_M), with C^∞_M the sheaf of smooth functions. The cotangent space at an $x \in M$ is linear functionals on germs of smooth functions on x to first order: that is, if $\mathfrak{m}_x$ is the maximal ideal of the local ring $C^\infty_{M,x}$, then the cotangent space is $T^*_x M = \mathfrak{m}_x/\mathfrak{m}_x^2$ (the $\mathfrak{m}_x^2$ term contains all of the higher-order information). This motivates the algebraic definition of a cotangent space.

Definition 5.1. Let $(A, \mathfrak{m})$ be a local ring. Then, the **Zariski cotangent space** of A is the $(A/\mathfrak{m})$-vector space $\mathfrak{m}/\mathfrak{m}^2$. If X is a scheme, the **Zariski cotangent space** at a $p \in X$ is $\mathfrak{m}_p/\mathfrak{m}_p^2$, where $\mathfrak{m}_p$ is the maximal ideal of the local ring $\mathscr{O}_{X,p}$.

[6] Equivalently, it's a closed map, since we already have the other hypotheses.

Just like in differential geometry, tangent vectors correspond to derivations.

Proposition 5.2 (Vakil ex. 12.1.A)**.** *Let X be a scheme and k be the residue field of $\mathscr{O}_{X,p}$ at a $p \in X$. Then, $(\mathfrak{m}_p/\mathfrak{m}_p^2)^\vee \cong \mathrm{Der}_k(\mathscr{O}_{X,p}, \mathscr{O}_{X,p})$.*

Recall that $(\mathfrak{m}_p/\mathfrak{m}_p^2)^\vee = \mathrm{Hom}_k(\mathfrak{m}_p/\mathfrak{m}_p^2, k)$; since this is the dual of the cotangent space, it's reasonable to call it the tangent space.

Partial proof. Let $\nabla : \mathscr{O}_{X,p} \to \mathscr{O}_{X,p}$ be a derivation, and write $f' = \nabla f$ for a germ $f \in \mathscr{O}_{X,p}$. The Leibniz rule tells us that $(fg)' = f'(p)g(p) + f(p)g'(p)$, so the map $f \mapsto f'(p)$ makes sense on $\mathfrak{m}_p$ (the functions that vanish at p) and vanishes on $\mathfrak{m}_p^2$ (functions vanishing to second order at p), so it defines a map $\phi : \mathfrak{m}_p/\mathfrak{m}_p^2 \to k$. $\square$

The other direction is more fiddly, and we'll see a better proof later in the notes. The idea is that we can write $\mathscr{O}_{X,p} = k \oplus \mathfrak{m}_p$ as a split square-zero extension (which is the tricky part, because it's not natural in any sense); then, given a morphism $\phi : \mathfrak{m}_p/\mathfrak{m}_p^2 \to k$ we define a derivation to be 0 on k and ϕ on $\mathfrak{m}_p$, more or less, and this satisfies the Leibniz rule. In a later chapter this is done in greater generality.

Suppose $\pi : X \to Y$ is a map of schemes, $p \in X$, and $q = \pi(p)$. Then, pullback gives us a map of stalks $\pi^\sharp : \mathscr{O}_{Y,q} \to \mathscr{O}_{X,p}$: since a map of schemes is a map of locally ringed spaces, this carries the maximal ideal $\mathfrak{n}_q \subset \mathscr{O}_{Y,q}$ to the maximal ideal $\mathfrak{m}_p \subset \mathscr{O}_{X,p}$. Thus, it descends to a pullback map on the cotangent spaces $\mathfrak{n}_q/\mathfrak{n}_q^2 \to \mathfrak{m}_p/\mathfrak{m}_p^2$. This works for general locally ringed spaces; if you do this for smooth manifolds, the dual of this map is the usual derivative Df of a smooth function f.

Proposition 5.3 (Vakil ex. 12.1.G)**.** *Let $X = \mathrm{Spec}\, k[x_1, \dots, x_n]/(f_1, \dots, f_n)$ for $f_1, \dots, f_n \in k[x_1, \dots, x_n]$. Then, $T_p^* X = \ker \mathrm{Jac}_p(f_1, \dots, f_r)$, which is the Jacobian of $f_1, \dots, f_r$ evaluated at p.*

This is a thing you can sit down and compute; much later in the notes, the sheaf of Kähler differentials can be employed to understand this more cleanly. The idea is that we take the ideal $(x_1, \dots, x_n)$, then mod out by all degree 2 monomials. After this, the f_i decompose into their first-order components.

The cotangent space is the beginning of our understanding of smoothness.

Theorem 5.4. *Let $(A, \mathfrak{m})$ be a Noetherian local ring and $k = A/\mathfrak{m}$ be its residue field. Then,*

$$\dim A \le \dim_k \mathfrak{m}/\mathfrak{m}^2. \tag{5.5}$$

The proof involves Nakayama's lemma.

Definition 5.6. If equality holds in (5.5), one says A is **regular**.

Definition 5.7. Let X be a locally Noetherian scheme.

- If $p \in X$ and $\mathscr{O}_{X,p}$ is a regular local ring, then X is **regular at** p.
- X is **regular** if it's regular at all $p \in X$.
- If X is not regular at p, it's called **singular at** p, and X is called **singular**.

6. An Algebraic Interlude: 6/1/16

Today, Tom reviewed §§7.2 and 7.3, discussing some commutative algebra that is necessary for understanding smoothness, and some finiteness conditions on morphisms.

Definition 6.1. Let $\varphi : B \to A$ be a ring homomorphism.

- An $a \in A$ is **integral** over B if there exists a *monic* polynomial $f \in B[x]$ such that $f(a) = 0$, i.e. there exist $b_0, \dots, b_{n-1} \in B$ such that

$$a^n + \varphi(b_{n-1})a^{n-1} + \cdots + \varphi(b_0) = 0. \tag{6.2}$$

- A is **integral over** B if all $a \in A$ are integral over B. In this case, φ is called **integral**.
- If φ is integral and injective, φ is called an **integral extension**.

Integrality is a generalization of the algebraicity of a field extension.

Definition 6.3. Let $f : X \to Y$ be a morphism of schemes. Then, f is **integral** if for all affine opens $\mathrm{Spec}\, B \subset Y$ and affine opens $\mathrm{Spec}\, A \subset f^{-1}(\mathrm{Spec}\, B)$, the induced map on global sections $f^\sharp : B \to A$ is integral.

This is how we define almost all fancy properties of schemes: take some ring-theoretic property and require it to hold affine-locally.

Definition 6.4.

- A ring homomorphism of schemses $\varphi : B \to A$ is **finite** if it induces a finitely generated B-module structure on A. Often, one says that "A is a finite B-module."
- A morphism $f : X \to Y$ of schemes is **finite** if for all affine opens $\operatorname{Spec} B \subset Y$, the preimage $f^{-1}(\operatorname{Spec} B) \cong \operatorname{Spec} A$ is affine, and the induced map on global sections $f^\sharp : B \to A$ is finite.

Integral and finite morphisms of schemes have the nice properties we require of properties of morphisms.

Proposition 6.5 (Vakil ex. 7.2.A)**.** *Integrality and finiteness can be checked affine-locally.*

The proofs use the same trick that we always use to check this: reduce to the affine communication lemma. It doesn't come for free: one must check that the ring-theoretic statement is true for the ring A iff it's true for each A_{f_i}, where $(f_1, \dots, f_i) = 1$. This is analogous to checking on an open cover. It's good to work this out, albeit not more than once.

Integrality plays well under quotients and localization; intuitively, integrality of morphisms of schemes is well-behaved locally.

Proposition 6.6 (Vakil ex. 7.2.B)**.** *Let $\varphi : B \to A$ be an integral morphism.*

(1) *If $S \subset B$ is a multiplicative set, $S^{-1}\varphi : S^{-1}B \to S^{-1}A$ is integral.*
(2) *If $J \subseteq A$ is an ideal and $I = \varphi^{-1}(J)$, then $B/I \to A/J$ is integral.*
(3) *If $I' \subseteq B$ is an ideal, then $B/I' \to A/I'A$ is integral (here, $I'A$ is the ideal generated by $\varphi(I')$).*

Moreover, (1) and (2) preserve the property that φ is an integral extension.

Surjective ring maps are tautologically integral, but we can do even better: they're finite, and we'll show finiteness implies integrality.

Partial proof. For the first part, suppose $a/s \in S^{-1}A$, so we know there exist b_i satisfying (6.2). When we multiply by s^n, this shows

$$\left(\frac{a}{s}\right)^n + \frac{b_{n-1}}{s}\left(\frac{a}{s}\right)^{n-1} + \cdots + \frac{b_0}{s^n} = 0,$$

so a/s is integral over $S^{-1}B$.

The second part follows from taking the integrality condition (6.2) mod J.

If φ is an integral extension, we just have to check injectivity. For part (1), this follows because localization is an exact functor: you can check that if $0/1 = \varphi(b)/\varphi(s)$ inside $S^{-1}A$, then there's a $t \in S$ such that $\varphi(t) = 0 = \varphi(tb)$, so $tb = 0$, and therefore $b/s = bt/st = 0$, so φ is injective. For (2), injectivity follows more or less by definition of the quotient. $\square$

Lemma 6.7. *Let $\varphi : B \to A$ be a ring homomorphism. Then, $a \in A$ is integral over B iff there's a subalgebra M of A containing a that is finitely generated as a B-module.*

Again, this is a property of algebraicity, and it invites an interesting question: given a monic polynomial satisfying a, and a monic polynomial satisfying a', how do we write down one satisfying $a + a'$? This is tricky, and there is one that exists, but the point of the lemma is that you need not do it directly.

Proof. In the forward direction, suppose a is integral over B. Then, $B[a] \subset A$ is a finitely generated B-module, because it's generated by $1, a, \dots, a^{n-1}$ over B. Conversely, suppose $M = \langle m_1, \dots, m_k\rangle_B$ is a finitely generated B-submodule of A containing a. That is, there are $\lambda_{ij} \in B$ such that

$$am_i = \sum_{j=1}^k \lambda_{ij} m_j.$$

If $\Lambda = (\lambda_{ij})$ is the matrix of these coefficients and $\vec{m} = (m_1, \dots, m_k)^{\mathrm{T}}$, then this says that, as matrices over B, $(aI - \Lambda)\vec{m} = 0$. We'd like to invert this, but we're not over a field. Using the adjugate matrix, which does exist over rings, we have that $\det(aI - \Lambda)$ annihilates $\vec{m}$, and so since A' contains 1 and $(m_1, \dots, m_k)$ generates M, $\det(aI - \Lambda) \cdot 1 = 0$. This is great, because $\det(aI - \Lambda)$ is a monic, degree-k polynomial with coefficients in $\varphi(B)$. $\square$

Extending Lemma 6.7, one can show that $a \in A$ is integral over B iff $B[a]$ is a finitely generated B-module.

Corollary 6.8 (Vakil cor. 7.2.2)**.** *If $\varphi : B \to A$ is a finite ring homomorphism, then it's integral.*

The converse is not true, e.g. the inclusion $\mathbb{Q} \hookrightarrow \overline{\mathbb{Q}}$. Corollary 6.8 is a generalization of the field-theoretic statement that finite extensions are algebraic, but the converse is not true.

Proposition 6.9. *A composition of integral ring homomorphisms is integral.*

Proof. Let $\varphi : B \to A$ and $\psi : C \to B$ be integral ring homomorphisms. Let $a \in A$, so $a \in B[\varphi(b_1), \dots, \varphi(b_n)] \subset A$. We can write each $\varphi(b_i)$ as a polynomial in finitely many $\varphi(\psi(\gamma_{ij}))$, and so a is generated over C by these finitely many λ_{ij}. $\square$

Proposition 6.10 (Vakil ex. 7.2.D)**.** *Let $\varphi : B \to A$ be a ring homomorphism. Then, the elements of A that are integral over B form a B-subalgebra $\overline{B} \subset A$, called the **integral closure** of B in A.*

If the ambient ring A is absent, an integral closure usually refers to the integral closure in the field of fractions.

The idea of the proof is that it reduces to checking that if a and a' are integral over B, then so are $a + a'$ and aa'. We want to look at $B[a + a']$ and $B[aa']$, which are subextensions of $B[a][a']$. We know $B[a][a']$ is integral over $B[a]$, and $B[a]$ is integral over B, so by Proposition 6.9, $B[a][a']$ is integral over A.

Using these smaller results, we can understand a bigger theorem, the lying over theorem.

Theorem 6.11 (Lying over, Vakil thm. 7.2.5)**.** *Let $\varphi : B \hookrightarrow A$ be an integral extension. Then, for any prime ideal $\mathfrak{p} \subset B$, there is a prime ideal $\mathfrak{q} \subset A$ **lying over** $\mathfrak{p}$, i.e. $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$.*

What this also says is that if $\pi : \operatorname{Spec} A \to \operatorname{Spec} B$ is an integral map of schemes, then φ is surjective as a map of sets. This can be useful.

We can extend this to a statement about chains of ideals.

Theorem 6.12 (Going up)**.** *Let $\varphi : B \to A$ be an integral ring homomorphism, $n > m \geq 1$, $\mathfrak{q}_1 \subsetneqq \cdots \subsetneqq \mathfrak{q}_m$ be a chain of strictly increasing prime ideals of A, and $\mathfrak{p}_1 \subsetneqq \cdots \subsetneqq \mathfrak{p}_n$ be a chain of strictly increasing prime ideals of B. If each $\mathfrak{q}_i$ lies over $\mathfrak{p}_i$ for $1 \leq i \leq m$, then we can extend the chain: there exist $\mathfrak{q}_{m+1} \subsetneqq \cdots \subsetneqq \mathfrak{q}_n$ such that $\mathfrak{q}_i$ lies over $\mathfrak{p}_i$ for $1 \leq i \leq n$.*

Geometrically, this states that if $\varphi : \operatorname{Spec} A \to \operatorname{Spec} B$ is an integral morphism of schemes and I have a chain of irreducible subsets $Y_1 \supsetneqq Y_2 \supsetneqq \cdots \supsetneqq Y_m$ of $\operatorname{Spec} A$, a chain of irreducible subsets $X_1 \supsetneqq X_2 \supsetneqq \cdots \supsetneqq X_n$ of $\operatorname{Spec} B$, and $\varphi(Y_i) = X_i$, then we can find irreducible subsets mapping to $X_{m+1}, \dots, X_n$ and preserving the chain relations.

The proof idea is to apply the lying over theorem many times. Interestingly, it tells you how to define the scheme-theoretic fiber, and therefore in some sense motivates the definition of the fiber product: the primes lying over $\mathfrak{p}$ are the fiber $\varphi^{-1}(\mathfrak{p})$ as a set: as a scheme, we take the fiber product with $\operatorname{Spec} k_{\mathfrak{p}}$, where $k_{\mathfrak{p}}$ is the residue field $k_{\mathfrak{p}} = B_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$; this is because the prime ideals lying over $\mathfrak{p}$ are the prime ideals lying over 0 in the residue field. Ring-theoretically, we get the pushout $A \otimes_B k_{\mathfrak{p}}$.

The point is, the setwise fiber has the same underlying set as the scheme-theoretic fiber, which admits the more abstract definition through a universal property. Equating these two viewpoints is nice to know. It also makes it easier to compute stalks of points in a fiber.

Now, let's talk about the various theorems called Nakayama's lemma. Like the cupcakes at the front of the class today, some are better than others.

Lemma 6.13 (Nakayama's lemma 1, Vakil 7.2.8)**.** *Let A be a ring, $I \subseteq A$ be an ideal, and M be a finitely generated A-module such that $M = IM$. Then, there's an $a \in A$ such that $a = 1 \pmod{I}$ and $aM = 0$.*

Proof. Choose a generating set $m_1, \dots, m_k$ for M; since $M = IM$, $M = \langle m_1, \dots, m_k \rangle_I$. That is, there exist $\lambda_{ij} \in I$ for $1 \leq i, j \leq k$ such that

$$m_i = \sum_{j=1}^{k} \lambda_{ij} m_j.$$

If $\vec{m} = (m_1, \dots, m_k)^{\mathrm{T}}$ and $\Lambda = (\lambda_{ij})$ as before, then $(1 - \Lambda)\vec{m} = 0$, and therefore we can choose $a = \det(1 - \Lambda)$, so $am_i = 0$ for each i, and $a = 1 \pmod{I}$ (since Λ is I-valued). $\square$

Recall that the **Jacobson radical** $\mathrm{Jac}(A)$ of a ring A is the intersection of its maximal ideals.

Lemma 6.14 (Nakayama's lemma 2, Vakil 7.2.9)**.** *Let A be a ring, $I \subsetneq A$ be an ideal contained in $\mathrm{Jac}\, A$, and M be a finitely generated A-module such that $M = IM$. Then, $M = 0$.*

Proof. Using Lemma 6.13, we have an $a \in A$ such that $a = 1 \pmod I$ (so $a = 1 + i$ for some $i \in I$) and $aM = 0$. For all maximal ideals $\mathfrak{m} \subset A$, $a \notin \mathfrak{m}$, because $a = 1 + i$ for some $i \in I \subset \mathfrak{m}$. Thus, a is a unit, so $M = aM = 0$. ☐

This is slick, but doesn't show you why M must be finitely generated. A more explicit proof chooses (using Zorn's lemma) a maximal submodule $N \subsetneq M$ and $x \in M \setminus N$. Then, we can define a map $\varphi : A \twoheadrightarrow M/N$ sending $a \mapsto a \cdot [x]$. Hence, $A/\ker\varphi \cong M/N$ as A-modules (there's no good ring structure here), forcing $I \subseteq \ker\theta$. This implies $IM \subseteq N \subsetneq M$, but $IM = M$, so no such N exists, and therefore $M = 0$. So we don't need M to be finitely generated, which is pretty cool.

All the other versions of Nakayama's lemma follow from these two.

Lemma 6.15 (Nakayama's lemma 3)**.** *Let A be a ring, $I \subset A$ be an ideal contained in $\mathrm{Jac}\, A$, M be an A-module, and $N \subset M$ be a submodule. If $N/IN \to M/IM$ is surjective, then $N = M$.*

These are useful for proving various submodules are the whole model, etc., which is useful for showing that xactness is a local condition, e.g. if M is an A-module, $M = 0$ iff $M_\mathfrak{p} = 0$ for all prime ideals $\mathfrak{p} \subset A$ iff $M_\mathfrak{m} = 0$ for all maximal ideals $\mathfrak{m} \subset A$.

7. More Regularity and Smoothness: 6/2/16

Today's lecture was given by Jay and Danny.

Recall that in Proposition 5.2, we said that if $(A, \mathfrak{m})$ is a local ring with reside field k, then $\mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k) = \mathrm{Der}_k(A, k)$. We showed how to obtain a homomorphism given a derivation; let's go in the other direction.

Suppose $\phi : \mathfrak{m}/\mathfrak{m}^2 \to k$ is a homomorphism; then, we can write $A/\mathfrak{m}^2$ as a square-zero extension of k by $\mathfrak{m}/\mathfrak{m}^2$, i.e. as rings, $A/\mathfrak{m}^2 = k \oplus \mathfrak{m}/\mathfrak{m}^2$, sending $f \mapsto (f(p), f - f(p) \pmod{\mathfrak{m}^2})$. Now, we define a derivation ∇: if $\lambda \in k$ and $m \in \mathfrak{m}/\mathfrak{m}^2$, then let $\nabla(\lambda + m) = \phi(m)$. This obeys the Leibniz rule: since $(\mathfrak{m}/\mathfrak{m}^2)^2 = 0$ in the square-zero extension,

$$\nabla((\lambda_1 + m_1)(\lambda_2 + m_2)) = \nabla(\lambda_1\lambda_2 + \lambda_2 m_1 + \lambda_1 m_2) = \lambda_2\phi(m_1) + \lambda_1\phi(m_2),$$

and therefore ∇ is indeed a derivation.

A related result allows us to understand the tangent space to a scheme X as the first-order infinitesimal information at X.

Proposition 7.1. *Let X be a scheme, k be a field, and $p \in X$ be a k-valued point. Then,*

$$\mathrm{Hom}_{\mathsf{Sch}}(\mathrm{Spec}\, k[x]/(x^2), X) = T_pX.$$

Proof. Since the tangent space is locally defined, we may assume $X = \mathrm{Spec}\, A$, where A is a k-algebra, and p represents the maximal ideal $\mathfrak{m})_p \subset A$. A morphism $\pi : \mathrm{Spec}\, k[x]/(x^2) \to X$ therefore induces a ring map $\phi : A \to k[x]/(x^2)$ such that $\phi^{-1}(x) = \mathfrak{m}_p$; hence, it factors through the map $\widetilde{\phi} : A/\mathfrak{m}_p^2 \to k[x]/(x^2)$. Then, the desired tangent vector is the map $\mathfrak{m}_p/\mathfrak{m}_p^2 \to k[x]/(x^2) \to k$, where the first map is $\widetilde{\varphi}|_{\mathfrak{m}_p/\mathfrak{m}_p^2}$ and the second map takes the coefficient of x.

In the other direction, suppose we have a $\phi \in \mathrm{Hom}_k(\mathfrak{m}_p/\mathfrak{m}_p^2, k)$. Once again we can take a square-zero extension $A/\mathfrak{m}_p^2 \cong k \oplus \mathfrak{m}_p/\mathfrak{m}_p^2$, and so the desired morphism of schemes is Spec of the ring map $A \to A/\mathfrak{m}_p^2 = k \oplus \mathfrak{m}_p/\mathfrak{m}_p^2 \to k[x]/(x^2)$, where the latter map sends $\lambda + m \mapsto \lambda + \phi(m)x \pmod{\mathfrak{m}^2}$. One has to check these are inverses, but it follows because they were defined in the same way. ☐

It would also be nice to know whether this bijection depends on the choice of $\mathrm{Spec}\, A \subset X$ containing p; it turns out not to, and also doesn't depend on the splitting, since A is a k-algebra (there's already a natural map $k \to A$), so it's suitably natural.

We can also now prove Theorem 5.4: if $(A, \mathfrak{m})$ is a Noetherian local ring with residue field, then the Krull dimension of A is at most $\dim_k \mathfrak{m}/\mathfrak{m}^2$: the dimension of the tangent space is an upper bound.

Proof of Theorem 5.4. Since A is Noetherian, $\mathfrak{m}$ is finitely generated, and therefore $\mathfrak{m}/\mathfrak{m}^2$ is a finite-dimensional k-vector space, say n-dimensional. Let $\{f_1, \dots, f_n\}$ be a basis of $\mathfrak{m}/\mathfrak{m}^2$. By one of the many versions of Nakayama's lemma (version 4 in Vakil's notes), this lifts to a generating set of $\mathfrak{m}$: there is a lift $\widetilde{f}_i \in \mathfrak{m}$ for each f_i such that $\mathfrak{m} = (\widetilde{f}_1, \dots, \widetilde{f}_n)$. Krull's height theorem says that any ideal containing n elements has height at most n, so the height of $\mathfrak{m}$ is at most n.

Now, we'd like to talk about $\dim A$; since A is a local ring, any chain of its prime ideals is contained entirely in $\mathfrak{m}$; thus, the length of any chains of primes in A is bounded above by the height of $\mathfrak{m}$, i.e. $\dim A \leq n \leq \dim_k \mathfrak{m}/\mathfrak{m}^2$. (In fact, if a chain doesn't contain $\mathfrak{m}$, $\mathfrak{m}$ can be added to it, so $\dim A = n$.) $\square$

Recall that if A is a Noetherian local ring and the inequality in the above proof is an equality, A is called regular, and that a locally Noetherian scheme X is regular at a $p \in X$ if $\mathscr{O}_{X,p}$ is a regular ring.

The intuition for regularity is that at a singularity, there are "too many tangent directions," so the tangent space has too high of a dimension. For example, on the scheme that's the union of the x- and y-axes, the tangent space is one-dimensional everywhere except the origin, where there are two directions, and correspondingly a two-dimensional tangent space. Hence, this scheme is regular everywhere except the origin.

There's a criterion for regularity that will motivate the definition of smoothness.

Proposition 7.2. *Let p be a k-valued point of $X = \operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_r)$. If X is pure dimension d, then X is regular iff* $\operatorname{corank} \operatorname{Jac}(f_1, \dots, f_r)|_p = d$.

Proposition 7.3 (Vakil ex. 12.2.E)**.** *Let k be an algebraically closed field and $X = \operatorname{Spec} k[x_1, \dots, x_n]/(f)$ be a **hypersurface** (meaning it has codimension 1). Then, a closed point $p \in X$ is singular iff* $\operatorname{Jac} f = 0$.

That is, the singular points are cut out by $f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}$. This is useful, because it means they form a closed subset.

Regularity is helpful, but there are a few drawbacks: for example, it's not obvious when the Jacobian criterion is sufficient; we know it's sufficient at k-valued points when k is algebraically closed, but that's somewhat restrictive. The fix is actually to define smoothness.

Definition 7.4. Let X be a finite-type k-scheme of pure dimension d. Then, X is **smooth (of dimension d)** if it can be covered by affine opens of the form $\operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_r)$ with $\operatorname{corank} \operatorname{Jac}(f_1, \dots, f_r)|_p = d$ for all $p \in X$.

One thing that would be nice to know is whether this satisfies the affine communication lemma. We'll return to this much later, when we define the smoothness of a morphism and show it has nice properties. It will be hard to show that this is affine-local in general, since n and r aren't required to be fixed.

The interesting examples are all hard, and we will have to return to them later. However, we can do a few easier examples.

Example 7.5. Consider $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \dots, x_n]$. Then, $\operatorname{Jac}(0) : k \to k^n$ always has corank n, and therefore $\mathbb{A}^n_k$ is smooth of dimension n, which is reassuring.

It's useful to compare regularity and smoothness.

Proposition 7.6 (Vakil ex. 12.2.I)**.** *Let k be an algebraically closed field and X be a finite-type k-scheme. Then, X is smooth iff it's regular at its closed points.*

Proof. Smooth definitely implies regular at closed points, by a previous exercise.

First, we show it for affines: suppose $X = \operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_r)$. The locus L where $\operatorname{Jac}(f_1, \dots, f_r)$ has corank d is a locally closed set, since bounding the rank above is a closed condition, but bounding it below is an open condition. That is, $L = U \cap F$, where U is open adn F is closed. If X is regular at all closed points, this contains all closed points, which are dense in X (since it's a finite type k-scheme: the closed points are those where the residue field is a finite extension of k, hence must be k). Thus, $F = X$, and $L = U$ is an open dense set. Thus, $L = X \setminus V(I)$ for an ideal $I \subseteq k[x_1, \dots, x_n]/(f_1, \dots, f_r)$, but L contains all maximal ideals, so I isn't contained in any maximal ideal, and therefore $I = A$, so $L = X$.

Since both regularity and smoothness can be checked locally, this suffices. $\square$

Here is another comparison between regularity and smoothness. Recall that a field k is **perfect** if $k = k^{\operatorname{char} k}$, including all characteristic 0 fields and finite fields, but excluding fields such as $\mathbb{F}_p(t)$.

Proposition 7.7. *Let k be a field.*

(1) *Every smooth k-scheme is regular.*
(2) *If k is perfect, every regular, finite-type k-scheme is smooth.*

Part 2 begins here, where Danny took over, and talked about §§12.4–12.6.

Discrete valuation rings are an excellent example of dimension 1 magic. We care about Noetherian local rings, because they're the stalks of pretty much every scheme we encounter. Using theorems like Krull's principal ideal theorem, we can induct as long as we understand the dimension 1 case, so let's think about that case.

We have a bunch of nice classes of rings that aren't *a priori* related to the Noetherian property, such as being a PID, being regular, being normal, and so on. In nice cases, we can relate these (which is the "dimension 1 magic" in question).

Theorem 7.8. *Let $(A, \mathfrak{m})$ be a 1-dimensional Noetherian local ring. Then, the following are equivalent.*

(1) *$(A, \mathfrak{m})$ is regular.*
(2) *$\mathfrak{m}$ is principal.*

Proof. To show (1) $\implies$ (2), suppose A is regular, so $\dim A = \dim_k \mathfrak{m}/\mathfrak{m}^2 = 1$, so by Nakayama's lemma, a basis $\{f\}$ of $\mathfrak{m}/\mathfrak{m}^2$ lifts to a generator $\widetilde{f}$ of $\mathfrak{m}$, so $\mathfrak{m} = (\widetilde{f})$. Conversely, if $\mathfrak{m} = (r)$, then $\mathfrak{m}/\mathfrak{m}^2$ is a 1-dimensional vector space, as needed. ☐

Proposition 7.9 (Vakil ex. 12.2.A)**.** *A zero-dimensional regular local ring is a field.*

Note that such a ring is trivially Noetherian.

Proof. Let A be such a ring and $\mathfrak{m}$ be its unique maximal ideal. By Nakayama's lemma, version 2 (Lemma 6.14), since $\mathfrak{m}/\mathfrak{m}^2 = 0$, then $\mathfrak{m} = \mathfrak{m}(\mathfrak{m})$, so $\mathfrak{m} = (0)$. Conversely, a field is zero-dimensional, and its maximal ideal is (0), so $(0)/(0)^2$ is zero-dimensional, so fields are regular. ☐

Theorem 7.10 (Vakil thm. 12.2.13)**.** *If $(A, \mathfrak{m})$ is a finite-dimensional regular local ring, then it's an integral domain.*

Proof. We induct on the dimension. The base case $n = 0$ follows from Proposition 7.9.

For the inductive step, suppose $n > 0$, and we know the result for things of dimension at most n. Let $f \in \mathfrak{m} \setminus \mathfrak{m}^2$, so $A/(f)$ is local and $\dim Z(A/(f)) = n - 1$ (where Z is the Zariski tangent space at $\mathfrak{m}$). Exercise 11.3.B informs us that since $(A, \mathfrak{m})$ is Noetherian, then for any $f \in \mathfrak{m}$, $\dim A/(f) \geq n - 1$, so by Theorem 5.4, $\dim A/(f) = n - 1$, so by the inductive assumption, $A/(f)$ is an integral domain.

We extend this to A: choose a minimal prime $\mathfrak{p}$ of A such that $\dim A/\mathfrak{p} = n$ (we're going to show $\mathfrak{p} = 0$): we can do this because $\dim A = n$, so there is a chain of primes of length n, and we can take $\mathfrak{p}$ to be the lowest prime on the chain. We can check $A/\mathfrak{p}$ is regular: its Zariski tangent space would need to be n-dimensional, and using Theorem 5.4 again, this is indeed the case.

The same argument with $A/\mathfrak{p}$ in place of A shows that $(A/\mathfrak{p})/(f) \cong A/(\mathfrak{p} + (f))$ is a regular local ring of dimension $n - 1$, and hence an integral domain. Thus, $A/(\mathfrak{p} + (f))$ is a quotient map between two maps of the same dimension; the kernel is a prime ideal $\mathfrak{q}$, and extends any chain of prime ideals from the codomain to one strictly longer — unless $\mathfrak{q} = (0)$, so this is an isomorphism: $(A/\mathfrak{p})/(f) \cong A/(\mathfrak{p} + (f))$. In particular, $\mathfrak{p} + (f) = (f)$, so $\mathfrak{p} \subsetneq (f)$, and therefore any $u \in \mathfrak{p}$ can be written as $u = fv$ for some $v \in A$. Since $\dim A?(\mathfrak{p} + (f)) < \dim A/\mathfrak{p}$, then $f \notin \mathfrak{p}$, and therefore $v \in \mathfrak{p}$. Thus, $\mathfrak{p} \subset (f)\mathfrak{p} \subset \mathfrak{p}$, so by Nakayama's lemma, $\mathfrak{p} = (0)$, so A is an integral domain. ☐

The following statement is a corollary of the Artin-Rees lemma, which is in §12.9.

Proposition 7.11 (Vakil 12.5.2)**.** *If $(A, \mathfrak{m})$ is a Noetherian local ring, then*

$$\bigcap_{i=1}^{\infty} \mathfrak{m}^i = 0.$$

This is actually a geometric statement: for example, it tells us that a holomorphic function that vanishes to all orders must be zero. Moreover, if one has a smooth, non-analytic function on a scheme, then its stalks must not be Noetherian.

Thus, we can extend Theorem 7.8. There are lots of nice algebraic properties that are equivalent.

Theorem 7.12. *The conditions in Theorem 7.8 are also equivalent to:*

(3) *All ideals of* $(A, \mathfrak{m})$ *are of the form* $\mathfrak{m}^i$*, or* (0).
(4) *A is a PID.*
(5) *A is a discrete valuation ring.*
(6) *A is a UFD.*
(7) *A is an integrally closed integral domain.*

These imply lots of things, e.g. characterizing the ideals of a DVR, and that a valuation on an integral domain is unique. One geometric application is to define zeros and poles of integral orders using the valuation $v : k^\times \to \mathbb{Z}$. That is, for a locally Noetherian scheme, we know how to define zeros and poles of various orders at any codimension 1 point!

8. Quasicoherent Sheaves: 6/6/16

Today Yuri will ramble quasicoherently about different types of sheaves, albeit without real proofs. However, we'll still hear which proofs are worth doing, and which aren't.

The setup for today is that over a scheme X, one can develop a theory of sheaves of modules, and one might want this to correspond to ordinary modules over a ring. We'll figure out which part of the theory carries over.

A starting insight is if X is a topological space, the category $\mathsf{Sh}(X)$ of sheaves of sets on X behaves a lot like the category of sets (which is a special case: $\mathsf{Set} = \mathsf{Sh}(\mathrm{pt})$), and so statements such as "module over a ring" can be directly translated into sheaf-theoretic logic. However, there are some bizarre-looking restrictions.

- The axiom of choice does not translate, because there exist surjective morphisms of sheaves $\mathscr{F} \twoheadrightarrow \mathscr{G}$ that do not admit sections (which correspond to choice functions in Set).
- The law of excluded middle does not hold: a statement about sheaves need not just be true or false: it can be true on some open sets and false on others. As such, any statement about sets that requires a proof by contradiction does not necessarily translate to sheaves.

However, many familiar constructions do not need these: the proof of existence of sheaf hom, tensor product, and direct sum, for example, goes through word-for-word, so these proofs aren't "interesting," as they're translations of proofs for modules that you already know.

Recall that a **presentation** of an R-module M is an exact sequence

$$R^{\oplus r} \longrightarrow R^{\oplus g} \longrightarrow M \longrightarrow 0,$$

where r denotes the relations and g denotes the generators. These always exist, since there is a "least efficient" presentation generated by all elements of M, with relations given by all of their relations.

Can we translate this to sheaves? That is, if X is a locally ringed space, does every sheaf $\mathscr{F}$ of R-modules on X admit a presentation

$$\mathscr{O}_X^{\oplus I} \longrightarrow \mathscr{O}_X^{\oplus J} \longrightarrow \mathscr{F} \longrightarrow 0,$$

at least locally?

Disappointingly, the answer is no.

Example 8.1. Consider a sheaf $\mathscr{F}$ of $\mathbb{Z}$-modules over $\mathbb{R}$ where $\mathscr{F}(U) = \mathbb{Z}$ if U doesn't contain the origin; if U contains the origin, we take $\mathscr{F}(U) = 0$. There is no local presentation of $\mathscr{F}$: it would have to at least surject onto the constant sheaf $\underline{\mathbb{Z}}$, but there is no way to surject both onto 0 (for a neighborhood of the origin) and onto $\mathbb{Z}$ over anything else. However, we can provide presentations for all stalks.

Having presentations, at least locally, is a good thing.

Definition 8.2. Let $(X, \mathscr{O}_X)$ be a locally ringed space and $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$-modules on X.

- $\mathscr{F}$ is **quasicoherent** if it's locally presentable.
- $\mathscr{F}$ is **finite type** if it's locally finitely generated (i.e. there is a presentation where the generators form a finite-dimensional free $\mathscr{O}_X$-module).
- $\mathscr{F}$ is **finitely presented** if its presentation is (i.e. both the generators and relations are finite-dimensional).
- $\mathscr{F}$ is **coherent** if it's of finite type, and for any n, the kernel of the map $\mathscr{O}_X^n \to \mathscr{F}$ is finite type.

- $\mathscr{F}$ is **locally free** if, locally, there is an isomorphism $\mathscr{F} \cong \mathscr{O}_X^n$ of $\mathscr{O}_X$-modules.[7]

There are other, equivalent definitions over schemes or over Noetherian schemes. Quasicoherent sheaves will form an abelian category, as will coherent sheaves.

Locally free sheaves are analogous to vector bundles: if $\pi : E \to B$ is a vector bundle, we can take its sheaf $\mathscr{F}$ of sections. Since the vector bundle is locally trivializable, there's an open cover $\mathfrak{U}$ of B such that for each $U \in \mathfrak{U}$, $\pi|_U$ is isomorphic to the projection $U \times \mathbb{R}^n \to U$ (n may vary, but is constant on connected components). Hence, translating to sheaves, we recover the local freeness condition. In the other direction, we can take relative Spec: if $\mathscr{F}$ is locally free, $\operatorname{Spec}(\operatorname{Sym} \mathscr{F})$ will recover the vector bundle (there's a bit to check here).

Hence, as objects, locally free sheaves are the same as vector bundles. **Warning**: this is *not* an equivalence of categories! The morphisms are not the same: there are linear maps of locally free $\mathscr{O}_X$-modules that are not morphisms of vector bundles. Vector bundles are not an abelian category; there are issues with cokernels.

In any case, we know what vector bundles are for manifolds, which suggests that for schemes, we can define vector bundles to be locally free sheaves.

Proposition 8.3. *Let $\mathscr{E}$ be a locally free sheaf on X and $\mathscr{F}$ be an $\mathscr{O}_X$-module. Then, there is an isomorphim of $\mathscr{O}_X$-modules $\mathscr{E}^\vee \otimes_{\mathscr{O}_X} \mathscr{F} \cong \operatorname{Hom}_{\mathscr{O}_X}(\mathscr{E}, \mathscr{F})$.*

Here, $\operatorname{Hom}_{\mathscr{O}_X}$ denotes sheaf hom.

Proof. We have an evaluation map $\mathrm{ev} : \mathscr{E} \otimes \mathscr{E}^\vee \to \mathscr{O}_X$, so composing $\mathrm{ev} \otimes 1 : \mathscr{E} \otimes \mathscr{E}^\vee \otimes \mathscr{F} \to \mathscr{O}_X \otimes \mathscr{F}$ with the isomorphism $\mathscr{O}_X \otimes \mathscr{F} \to \mathscr{F}$, we obtain a map $\mathscr{E} \otimes \mathscr{E}^\vee \otimes \mathscr{F} \to \mathscr{F}$; by the tensor-hom adjunction, this is the same data as a map $\mathscr{E}^\vee \otimes \mathscr{F} \to \operatorname{Hom}_{\mathscr{O}_X}(\mathscr{E}, \mathscr{F})$.

To check this map is an isomorphims, it suffices to check locally, and therefore assume $\mathscr{E} \cong \mathscr{O}_X^{\oplus n}$. The map becomes a map $\mathscr{O}_X^n \otimes \mathscr{F} \to \operatorname{Hom}_{\mathscr{O}_X}(\mathscr{O}_X^n, \mathscr{F})$, which becomes an isomorphism $\mathscr{F}^n \to \mathscr{F}^n$. ☐

Proposition 8.4 (Frobenius reciprocity)**.** *Suppose $f : X \to Y$ is a morphism of locally ringed spaces, $\mathscr{F}$ is an $\mathscr{O}_X$-module, and $\mathscr{E}$ is a locally free $\mathscr{O}_Y$-module. Then, there is an isomorphism $f_*\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{E} \to f_*(\mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{E})$.*

Now, we'll actually talk about schemes, and give a nice characterization of quasicoherent sheaves over schemes: they're constructed from modules over a ring of functions.

Proposition 8.5. *Let X be a scheme and $\mathscr{F}$ be an $\mathscr{O}_X$-module. Then, $\mathscr{F}$ is quasicoherent iff for all affine opens $U = \operatorname{Spec} A \hookrightarrow X$, $\mathscr{F}|_U \cong \widetilde{M}$ for some $\Gamma(U, \mathscr{O}_X)$-module M.*

Here, $\widetilde{M}$ is the sheaf of $\mathscr{O}_X$-modules constructed by localization, in the same way that $\mathscr{O}_X$ was constructed: for a distinguished open $D(f) \subset \operatorname{Spec} A \hookrightarrow X$, we can choose $\widetilde{M}(D(f)) = M_f$.

Proving Proposition 8.5 takes some work, but is essentially a follow-your-nose argument. One important ingredient is that distinguished affine inclusions, i.e. inclusions of the form $\operatorname{Spec}(A_f) \hookrightarrow \operatorname{Spec} A \hookrightarrow X$, are cofinal in $\mathsf{Top}(X)$, the category of open subsets of X (meaning every open subset contains a distinguished affine open). This allows one to work only with these inclusions, and therefore reduce the proof to the affine case.

Lemma 8.6. *If $X = \operatorname{Spec} A$ is an affine scheme, then quasicoherent sheaves on X are the same as A-modules.*

In this case, given a quasicoherent sheaf $\mathscr{F}$, let $M = \Gamma(\operatorname{Spec} A, \mathscr{F})$. Quasicoherence implies $\Gamma(\operatorname{Spec} A_f, \mathscr{F}) \cong \Gamma(\operatorname{Spec} A, \mathscr{F})_f$, which means $\widetilde{M}(U) \cong \mathscr{F}(U)$ for all distinguished affine opens $U \subset \operatorname{Spec} A$, which implies it for all open subsets. We can show this locally, using the presentation of $\mathscr{F}$. Most of the proofs interspersed throughout the text are fairly formal, e.g. verifying definitions. Many of these hold for all locally ringed spaces and aren't so interesting. But the exercises near the end, which are specifically about schemes (or specific kinds, e.g. over number fields), are definitely worth your time.

Remark. The fact that $\Gamma(\operatorname{Spec} A_f, \mathscr{F}) \cong \Gamma(\operatorname{Spec} A, \mathscr{F})_f$ is true for more than affine schemes: if X is QCQS (quasicoherent and quasiseparated) and $f \in \Gamma(X, \mathscr{O}_X)$, we can define $X_f = \{p \in X \mid f(p) \notin \mathfrak{m}_{X,p}\}$ as a subscheme; then, there is a natural isomorphism $\Gamma(X_f, \mathscr{O}_X) \cong \Gamma(X, \mathscr{O}_X)_f$.

Quasicoherent sheaves are locally determined by modules; what about coherent sheaves?

[7] Sometimes people consider infinite-rank locally free sheaves, but today all of our locally free sheaves will be finite rank, akin to finite-dimensional vector spaces.

Proposition 8.7. *Let X be a Noetherian scheme. Then, an $\mathscr{O}_X$-module is a coherent sheaf iff it's locally isomorphic to $\widetilde{M}$ for $\Gamma(U, \mathscr{O}_X)$-modules M that are finite type.*

As locally free sheaves are akin to vector bundles, we can define line bundles.

Definition 8.8. A **line bundle** (or **invertible sheaf**[8]) is a locally free sheaf of rank 1.

By multiplicativity of dimension, if $\mathscr{E}$ and $\mathscr{F}$ are line bundles, so are $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{F}$ and $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{E}, \mathscr{F})$ are line bundles too.

Focally free sheaves form a monoid under tensor product, and so line bundles are exactly the invertible elements. That is, they form a group $\mathrm{Pic}(X)$, called the **Picard group** of X.

Remark. Later, when we know what cohomology is (there are many definitions: derived functors are the fancy version, but in many cases it relates to how one assembles cocycles), we will see an isomorphism $H^1(X, \mathscr{O}_X) = \mathrm{Pic}(X)$. This is because a line bundle on X is defined by trivial data on an open cover of X, along with the data of how to glue them together, and this is the same cocycle condition that defines a cohomology class in H^1. Thinking through this for vector bundles (e.g. take $X = S^1$) may be helpful.

One can define $\mathscr{O}_X^*$ to be the subsheaf of $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_X, \mathscr{O}_X)$ of isomorphisms (or the automorphism sheaf of $\mathscr{O}_X$), which corresponds to $\mathrm{GL}(1, \mathbb{R})$ in the differentiable category. The cocycle condition is that the transition functions have to be valued in $\mathscr{O}_X^*$, which is what Čech cohomology gives you.

Since $\mathrm{GL}_1(\mathbb{R}) = \mathbb{R}^\times$, then it's homotopy equivalent to $\mathbb{F}_2 \cong \mathrm{O}_1$. Hence, if X is a compact CW complex, $H^1(X; \mathbb{F}_2)$ recovers isomorphism classes of line bundles on X, and since $\mathbb{C}^\times$ deformation retracts onto its unit circle $S^1 \cong \mathrm{U}_1$, then $H^1(X; S^1) \cong H^2(X; \mathbb{Z})$ classifies isomorphism classes of line bundles on X.[9]

9. Coherent Sheaves: 6/9/16

Today's talk was given by Yan Zhou, on the second part of Chapter 13.

Recall that if X is a scheme, an $\mathscr{O}_X$-module $\mathscr{F}$ is quasicoherent if for all affine opens $\operatorname{Spec} A \subset X$, $\mathscr{F}|_{\operatorname{Spec} A} \cong \widetilde{M}$, where $\widetilde{M}$ is the $\mathscr{O}_X$-module associated to a $\Gamma(\operatorname{Spec} A, \mathscr{O}_X)$-module M.

There are a bunch of finiteness conditions one can put on quasicoherent sheaves (or the modules locally defining them).

- One could ask that all such M are finitely generated, i.e. there's a surjection $A^{\oplus n} \to M \to 0$.
- One could require M to be finitely presented, meaning there's an exact sequence

$$A^{\oplus n} \longrightarrow A^{\oplus m} \longrightarrow M \longrightarrow 0.$$

- A module M is **coherent** if it's finitely generated and if for every map $A^{\oplus n} \to M$, the kernel is finitely generated.

In particular, if A is Noetherian, the kernel K of such a surjection is a submodule of A^n, which is finitely generated, and therefore K must also be finitely generated. Hence, *if A is Noetherian, all three of these finiteness conditions are the same.*

Recall that a closed embedding defines a sheaf of ideals: if $i : Y \hookrightarrow X$ is a closed embedding (closed subscheme), it defines the sheaf of ideals $\mathscr{I}_{X/Y}$ that fits into the short exact sequence

$$0 \longrightarrow \mathscr{I}_{X/Y} \longrightarrow \mathscr{O}_X \longrightarrow i_*\mathscr{O}_Y \longrightarrow 0,$$

so the kernel of $\mathscr{O}_X \to i_*\mathscr{O}_Y$. The converse isn't necessarily true: a sheaf of ideals might not define a closed embedding. In this case, quasicoherence rescues us.

Proposition 9.1. *There is a bijection between the isomorphism classes of quasicoherent sheaves of ideals on a scheme X and the closed subschemes of X.*

That is, if $\mathscr{I}$ is a quasicoherent sheaf of ideals, then it determines a closed subscheme $Y = \operatorname{supp} \mathscr{I}$, with structure sheaf $\mathscr{O}_X/\mathscr{I}$. Affine-locally, we can explicate this: if $\operatorname{Spec} A \hookrightarrow X$ is an affine open, then $Y = \operatorname{Spec}(A/\mathscr{I}(\operatorname{Spec} A))$; quasicoherence is what guarantees that these glue together to define a scheme.

Now, we'll talk about a bunch of exercises. These all use Nakayama's lemma in the case of local rings, which is actually a different statement than Lemmas 6.13, 6.14, and 6.15 that we already discussed.

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[8]The name "invertible sheaf" comes from the fact that these are exactly the sheaves that are invertible under $\otimes_{\mathscr{O}_X}$. The idea is that $\mathscr{E}^\vee \otimes \mathscr{E} \cong \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E})$, and the latter sheaf has a global section given by the identity map, hence is trivial.

[9]Since $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$, then $H^2(X; \mathbb{Z}) = [X, \mathbb{CP}^\infty]$, and $\Omega\mathbb{CP}^\infty \simeq S^1$.

Lemma 9.2 (Nakayama's lemma (local rings))**.** *Let $(R, \mathfrak{m})$ be a local ring and M be a finitely generated R-module. Then a basis of $M/\mathfrak{m}M$ lifts to a minimal set of generators for M.*

Before we discuss a geometric consequence of Nakayama's lemma, recall that if $\mathscr{F}$ is a sheaf on a scheme X, its fiber at a $p \in X$ is $\mathscr{F}|_p = \mathscr{F}_p \otimes_{\mathscr{O}_{X,p}} k(p)$, where $k(p)$ denotes the residue field at p.

Finite-rank vector bundles are analogous to locally free coherent sheaves, and so we should expect that for locally free coherent sheaves, the rank of the fiber, as a $k(p)$-vector space, should be locally constant. On coherent sheaves more generally, the rank may jump, but it will still be relatively well-behaved.

Lemma 9.3 (Geometric Nakayama's lemma, Vakil ex. 13.7.E)**.** *Let X be a scheme, $U \subset X$ be open, $p \in U$, and $\mathscr{F}$ be a finite-type quasicoherent sheaf on X. If $a_1, \dots, a_n \in \mathscr{F}(U)$ are such that their images $\overline{a}_1, \dots, \overline{a}_n \in \mathscr{F}_p$ form a basis for $\mathscr{F}_p/\mathfrak{m}_p\mathscr{F}_p$, then there exists an affine open $\operatorname{Spec} A \subseteq U$ containing p such that*

(1) *$a_1|_{\operatorname{Spec} A}, \dots, a_n|_{\operatorname{Spec} A}$ generate $\mathscr{F}(\operatorname{Spec} A)$, and*
(2) *for any $q \in \operatorname{Spec} A$, $a_1|_q, \dots, a_n|_q$ generate $\mathscr{F}_q$.*

The idea is that if the fiber is finitely generated, then it generates the sheaf nearby.

Proof. This proposition's name suggests that we should use Nakayama's lemma, version 9.2. This tells us that $\overline{a}_1, \dots, \overline{a}_n$ lift to a minimal set of generators for the stalk $\mathscr{F}_p$. Hence, since $\mathscr{F}$ is finite-type quasicoherent, there is some affine open $\operatorname{Spec} A' \subset U$ containing p such that $\mathscr{F}|_{\operatorname{Spec} A'} \cong \widetilde{M}$, where M is an A'-module.

Let $b_1, \dots, b_k$ be a set of generators for M. At p, each $b_i|_p \in (a_1|_p, \dots, a_n|_p)$, so for each i, there's an open neighborhood on which this is true as functions, not just as germs. Since there are finitely many, their intersection is still an open neighborhood of p, and hence contains a distinguished affine open $\operatorname{Spec} A = \operatorname{Spec} A'_f \subset \operatorname{Spec} A'$ containing p, and therefore $b_i|_{\operatorname{Spec} A} \in (a_1|_{\operatorname{Spec} A}, \dots, a_n|_{\operatorname{Spec} A})$ for each i; since $\mathscr{F}$ is quasicoherent, $b_1|_{\operatorname{Spec} A}, \dots, b_k|_{\operatorname{Spec} A}$ generate $\mathscr{F}(\operatorname{Spec} A)$. $\square$

Definition 9.4. If $\mathscr{F}$ is a finite-type quasicoherent sheaf on a scheme X, its **rank** at a $p \in X$ is the dimension of its fiber: $\varphi(p) = \dim_{k(p)}(\mathscr{F}_p \otimes_{\mathscr{O}_{X,p}} k(p))$.

The rank φ is upper semicontinuous, meaning at the set $\{p \mid \varphi(p) > n\}$ is closed, ultimately following from Lemma 9.3. In particular, if X is irreducible, we can look at the generic point η: if $\varphi(\eta) = n$, then the rank at any point in X is at least n, and there is a dense open set where the rank is exactly n.

Proposition 9.5 (Vakil ex. 13.7.F)**.** *Let $\mathscr{F}$ be a coherent sheaf on a scheme X, and suppose that for some $p \in X$, $\mathscr{F}_p$ is a free $\mathscr{O}_{X,p}$-module. Then, $\mathscr{F}$ is locally free on an open neighborhood of p.*

The takeaway is that being locally free is a stalk-local property: $\mathscr{F}$ is locally free iff for all $p \in X$, $\mathscr{F}_p$ is a free $\mathscr{O}_{X,p}$-module.

Proof. We once again use Lemma 9.3. We may assume $\mathscr{F}|_p \neq 0$, because if it is zero, then geometric Nakayama's lemma implies it's zero in a neighborhood, which is locally free, if silly.

If it's nonzero, there's an open neighborhood U of p and a finite set of sections $a_1, \dots, a_n \in \mathscr{F}(U)$ such that $a_1|_p, \dots, a_n|_p$ are a basis for $\mathscr{F}|_p$. Hence, there is an open neighborhood $Y \subset U$ such that for all $q \in Y$, $a_1|_q, \dots, a_n|_q$ generate $\mathscr{F}|_q$. Since $\mathscr{F}$ is coherent, there is a surjection $\phi : (\mathscr{O}_X|_Y)^{\oplus n} \to \mathscr{F}|_Y \to 0$ that is an isomorphism at p (since $\mathscr{F}|_p$ is free), and $\ker(\phi)$ is coherent.

One can show that the support of a coherent sheaf is closed, and $p \notin \operatorname{supp}(\ker \varphi)$, as $\phi|_p$ is an isomorphism. Thus, $V = Y \setminus \operatorname{supp}(\ker \phi)$ is an open neighborhood of p on which ϕ is an isomorphism, so $\mathscr{F}|_V$ is a free $\mathscr{O}_X$-module. $\square$

We won't prove the next proposition, but Vakil pretty much walks you through it. It's also an exercise in Hartshorne, albeit with no hint.

Proposition 9.6. *Let X be a reduced scheme and $\mathscr{F}$ be a finite-type quasicoherent sheaf on X. If the rank of $\mathscr{F}$ is constant, then $\mathscr{F}$ is locally free.*

When proving this, you will once again use Lemma 9.3 to produce a proof that looks similar to the one for Proposition 9.5: you will concoct a surjection $(\mathscr{O}_X|_{\operatorname{Spec} A})^{\oplus n} \to \mathscr{F}|_{\operatorname{Spec} A} \to 0$ for some $\operatorname{Spec} A \subset X$, and then show that it's an isomorphism.

These are Arun's lecture notes on line bundles and divisors, corresponding to sections 14.1 and 14.2 in Vakil's notes. I'm planning on talking about the following topics:

- A few nice examples of line bundles on $\mathbb{P}^n$.
- Weil divisors and their relation to invertible sheaves.
- Using the class group to compute the Picard group, and if time, some actual examples.

Throughout this lecture, X will be a normal, reduced, Noetherian scheme that's regular in codimension 1. `#sorrynotsorry`

Line bundles on $\mathbb{P}^n$. The first part of this lecture will be an extended example, of nice classes of line bundles on projective spaces. Throughout this section, let A be a ring.

Example 10.1. First, we'll define a line bundle $\mathscr{O}(1) = \mathscr{O}_{\mathbb{P}^1_A}(1)$ on $\mathbb{P}^1_A = \operatorname{Proj} A[x_0, x_1]$. Recall that $\mathbb{P}^1_A$ is covered by two affine subsets $U_0 = D(x_0) = \operatorname{Spec} A[x_{1/0}]$ and $U_1 = D(x_1) = \operatorname{Spec} A[x_{0/1}]$; $\mathscr{O}(1)$ is trivial on those subsets, so it's completely specified by the two transition functions. Over U_0, a section of $\mathscr{O}(1)$ is an element of $A[x_{1/0}]$, and similarly for U_1.

We define $\mathscr{O}(1)$ to be the line bundle whose transition functions are: from U_0 to U_1, multiply by $x_{0/1} = x_{1/0}^{-1}$, and from U_1 to U_0, multiply by $x_{1/0} = x_{0/1}^{-1}$. These satisfy the cocycle condition, so we obtain a line bundle $\mathscr{O}(1)$.

Suppose $s \in \Gamma(\mathbb{P}^1_A, \mathscr{O}(1))$; then, s is the data of polynomials $f \in A[x_{1/0}]$ and $g \in A[x_{0/1}]$ such that $f(1/x_{0/1})x_{0/1} = g(x_{0/1})$. This forces f to be linear: $f(x_{1/0}) = ax_{1/0} + b$, and therefore $g(x_{0/1}) = a + bx_{0/1}$. Thus, $\dim \Gamma(\mathbb{P}^1_A, \mathscr{O}(1)) = 2$. Since $\dim \Gamma(\mathbb{P}^1_A, \mathscr{O}_{\mathbb{P}^1_A}) = 1$, then $\mathscr{O}(1)$ is a nontrivial line bundle: it's not isomorphic to the structure sheaf. Notice also that if we homogenize, $x_{1/0} = x_1/x_0$ and so $ax_{1/0} + b$ is naturally identified with $ax_1 + bx_0$. Thus, the global sections of $\mathscr{O}(1)$ are naturally identified with the degree-1 homogeneous polynomials in $A[x_0, x_1]$.

Example 10.2. In the same way, we can define $\mathscr{O}(n)$ on $\mathbb{P}^1_A$, where the transition functions are instead multiplication by $x_{0/1}^n = x_{1/0}^{-n}$ and vice versa. If $n \geq 0$, a section $s \in \Gamma(\mathbb{P}^1_A, \mathscr{O}(n))$ is identified with a degree-n polynomial in $x_{1/0}$ on U_0, or a homogeneous degree-n polynomial in $A[x_0, x_1]$. Thus, $\Gamma(\mathbb{P}^1_A, \mathscr{O}(n)) = n + 1$. However, if $n < 0$, a global section would determine polynomials $f \in A[x_{1/0}]$ and $g \in A[x_{0/1}]$ such that $f(1/x_{0/1})x_{0/1}^n = g(x_{0/1})$, so we're forced to conclude $f, g = 0$. Thus, if $n < 0$, $\dim \Gamma(\mathbb{P}^1_A, \mathscr{O}(n)) = 0$.

Under this identification, the tensor product of line bundles turns into polynomial multiplication, so $\mathscr{O}(m) \otimes \mathscr{O}(n) = \mathscr{O}(m + n)$. Additionally, $\mathscr{O}(0)$ is the structure sheaf. This implies $\mathscr{O}(-n) = \mathscr{O}(n)^\vee$, since $\mathscr{O}(-n) \otimes \mathscr{O}(n) = \mathscr{O}(0) = \mathscr{O}_{\mathbb{P}^1_A}$. Hence, if $m \neq n$, $\mathscr{O}(m) \not\cong \mathscr{O}(n)$: if at least one of m or n is nonnegative, this is clear because their global sections have different dimensions, and if otherwise, then the global sections of $\mathscr{O}(m)^\vee$ and $\mathscr{O}(n)^\vee$ have different dimensions. That is, the map $n \mapsto \mathscr{O}(n)$ defines an injection $\mathbb{Z} \hookrightarrow \operatorname{Pic}(\mathbb{P}^1_A)$.

Example 10.3. In the same way, we can define $\mathscr{O}(n) = \mathscr{O}_{\mathbb{P}^m_A}(n)$ on $\mathbb{P}^m_A$. Here, we have $n + 1$ affine opens $U_i = \operatorname{Spec} A[x_{0/i}, \ldots, x_{m/i}]/(x_{i/i} - 1)$. We let $\mathscr{O}(n)$ be trivial on these affines, with the transition function from U_i to U_j being multiplication by $x_{i/j}^n = x_{j/i}^{-n}$. Thus, these also satisfy the cocycle condition, so define a line bundle over $\mathbb{P}^n_A$.

If $n \geq 0$, a global section restricts on an affine to a polynomial of degree at most n, and therefore after homogenizing, a global section is defined by a homogeneous, degree-n polynomial in $A[x_0, \ldots, x_m]$, and vice versa. Hence, $\dim \Gamma(\mathbb{P}^m_A, \mathscr{O}(n)) = \binom{m+n}{m}$.

Once again, $\mathscr{O}(\ell) \otimes \mathscr{O}(n) = \mathscr{O}(\ell + n)$, so by the same line of reasoning as before, $n \mapsto \mathscr{O}(n)$ defines an injection $\mathbb{Z} \hookrightarrow \operatorname{Pic}(\mathbb{P}^m_A)$.

It turns out that over a field k, these are the only line bundles over $\mathbb{P}^n_k$. In order to prove this, we introduce the formalism of Weil divisors and their imperfect dictionary to line bundles.

Weil divisors.

Definition 10.4. The **group of Weil divisors** of a scheme X, denoted $\operatorname{Weil} X$, is the free abelian group on the set of codimension-1 irreducible closed subsets of X. Thus, a Weil divisor D is a formal linear combination

$$D = \sum_{Y \subset X \text{ codim. } 1} n_Y [Y], \tag{10.5}$$

where $n_Y \in \mathbb{Z}$ and all but finitely many n_Y are zero.

- If $Y \subset X$ is an irreducible closed subset, $[Y]$ is called an **irreducible** divisor.
- If D is as in (10.5) and $n_Y \geq 0$ for all Y, then D is called **effective**. We define a partial ordering on $\operatorname{Weil} X$ in which $D_1 \leq D_2$ iff $D_2 - D_1$ is effective.
- The **support** of a Weil divisor (10.5) is the set $\bigcup_{n_Y \neq 0} Y$.
- If $U \subset X$ is an open subset, we have a **restriction map** $\operatorname{Weil} X \to \operatorname{Weil} U$ by defining $[Y] \mapsto [Y \cap U]$ and extending $\mathbb{Z}$-linearly.

For example, if X is a curve, the Weil divisors are linear combinations of closed points.

Definition 10.6. Let $\mathscr{F}$ be a sheaf on X; then, a **rational section** s of $\mathscr{F}$ is a section of $\mathscr{F}|_U$, where U is an open, dense subset of X. Two rational sections are equal if they agree on a dense open subset. I'll write the space of rational sections of $\mathscr{F}$ over an open set V as $K(V, \mathscr{F})$.

In particular, on a variety over a field k, a rational function (i.e. to $\mathbb{A}^1_k$) is the same as a rational section of $\mathscr{O}_X$. This is analogous to the generalization from meromorphic functions to meromorphic 1-forms in the theory of Riemann surfaces.

Let $\mathscr{L}$ be a line bundle on X and s be a rational section of $\mathscr{L}$ that does not vanish on any irreducible component of X. If $Y \subset X$ is a codimension-1 irreducible component of X and η_Y is its generic point, then $\mathscr{O}_{X,\eta_Y}$ is a discrete valuation ring, and a trivialization determines an isomorphism $\mathscr{O}_{X,\eta_Y} \cong \mathscr{L}_{\eta_Y}$. Thus, $s|_Y$ has a valuation $\operatorname{val}_Y(s)$, which is independent of the choice of trivialization because any two trivializations will differ by an invertible germ. As such, s determines a Weil divisor

$$\operatorname{div}(s) = \sum_Y \operatorname{val}_Y(s)[Y],$$

called its **divisor of zeros and poles**. If $Q = \{(\mathscr{L}, s)\}/\cong$ denotes the set of isomorphism classes of line bundles and rational sections, Q is an abelian group under tensor product, and div is a group homomorphism $\operatorname{div} : Q \to \operatorname{Weil} X$. We're going to use this homomorphism to calculate the Picard group.

Lemma 10.7 (Vakil ex. 13.1.K)**.** *A rational section with no poles is regular.*

Proposition 10.8 (Vakil prop. 14.2.1)**.** div *is injective.*

Proof. Suppose $\operatorname{div}(\mathscr{L}, s) = 0$, so s has no poles by Lemma 10.7. Since $\mathscr{L}$ is an $\mathscr{O}_X$-module, acting on s defines a morphism $\times s : \mathscr{O}_X \to \mathscr{L}$. We'll show this is an isomorphism, so $(\mathscr{O}_X, 1) \cong (\mathscr{L}, s)$. In fact, it suffices to show $\times s$ is an isomorphism on an open cover $\mathfrak{U}$ of X that trivializes $\mathscr{L}$.

Let $U \in \mathfrak{U}$, so that there is an isomorphism $i : \mathscr{L}|_U \to \mathscr{O}_X|_U$, and let $s' = i(s)$. Then, multiplication by s' defines a map $\times s' = i \circ \times s : \mathscr{O}_X|_U \to \mathscr{O}_X|_U$. Since s' has neither zeros nor poles, it's a regular section and $1/s'$ is a regular section, so $\times s'$ is invertible, and hence an isomorphism; thus, s is also an isomorphism on U. ∎

The next construction will be a kind of inverse.

Definition 10.9. If D is a Weil divisor, define a sheaf $\mathscr{O}_X(D)$ whose sections on an open $U \subset X$ are the rational functions t on U whose zeros and poles are constrained by D, i.e. $\operatorname{div}|_U t + D|_U \geq 0$, along with the zero section. If U is contained in an irreducible component of X, then

$$\Gamma(U, \mathscr{O}_X(D)) = \{t \in K(X)^\times : \operatorname{div}|_U t + D|_U \geq 0\} \cup \{0\}.$$

If $\mathscr{L}$ is a line bundle, define $\mathscr{L}(D) = \mathscr{L} \otimes \mathscr{O}_X(D)$.

$\mathscr{L}(D)$ can be interpreted as rational sections of $\mathscr{L}$ with zeros and poles constrained by D; by algebraic Hartogs' lemma, it's isomorphic to $\mathscr{L}$ away from $\operatorname{supp} D$.

Lemma 10.10 (Vakil ex. 14.2.C)**.** *$\mathscr{O}_X(D)$ and $\mathscr{L}(D)$ are quasicoherent sheaves.*

Using the distinguished affine criterion for quasicoherence, this follows because $\mathscr{O}_X$ and $\mathscr{L}$ are quasicoherent. In fact, in pleasant circumstances, we can do better than quasicoherence.

Proposition 10.11 (Vakil ex. 14.2.E.)**.** *Let $\mathscr{L}$ be an invertible sheaf and $s \in K(X, \mathscr{L})^\times$. Then, there is an isomorphism $\mathscr{O}_X(\operatorname{div} s) \cong \mathscr{L}$ such that if $\sigma : K(X) \to K(X, \mathscr{L})$ is the induced map on rational sections, $\sigma(1) = s$.*

Example 10.12 (Vakil ex. 14.2.F)**.** For example, when $X = \mathbb{P}^m_A$ and $\mathscr{L} = \mathscr{O}(n)$, then a degree-$n$ homogeneous polynomial in $A[x_0, \dots, x_m]$ defines a rational section s of $\mathscr{L} = \mathscr{O}(n)$. Therefore $\mathscr{O}_{\mathbb{P}^n_A}(\operatorname{div} s) \cong \mathscr{O}(m)$ by Proposition 10.11.

Definition 10.13.

- If D is a divisor such that $D = \operatorname{div} f$ for a rational *function* f, D is called a **principal divisor**. The principal divisors form a subgroup $\operatorname{Prin} X \subseteq \operatorname{Weil} X$.
- If D is a divisor that restricts to principal divisors on an open cover of X, then D is called **locally principal**. Locally principal divisors form a subgroup $\operatorname{LocPrin} X \subseteq \operatorname{Weil} X$.
- The **class group** $\operatorname{Cl} X = \operatorname{Weil} X / \operatorname{Prin} X$.

Notice that if $D = \operatorname{div} f$ is principal, Proposition 10.11 tells us $\mathscr{O}_X(D) \cong \mathscr{O}_X$, since f is a rational function, i.e. a rational section of $\mathscr{O}_X$. Thus, if D is locally principal, $\mathscr{O}_X(D)$ is locally isomorphic to $\mathscr{O}_X$, and therefore is a line bundle.

Proposition 10.14 (Vakil ex. 14.2.G)**.** *The converse is true: if $\mathscr{O}_X(D)$ is invertible, then $\operatorname{div}(\sigma(1)) = D$, and D is locally principal.*

In particular, $\operatorname{LocPrin} X$ is the image of div, so we have a commutative diagram

$$\begin{tikzcd}
Q \arrow[r, bend right, "\mathrm{div}", "\sim"'] \arrow[d] & \operatorname{LocPrin} X \arrow[l, bend right, "D\mapsto(\mathscr{O}_X(D),\sigma(1))"'] \arrow[r, hook] \arrow[d, "/\operatorname{Prin} X"] & \operatorname{Weil} X \arrow[d, "/\operatorname{Prin} X"] \\
\operatorname{Pic} X & \operatorname{LocPrin} X/\operatorname{Prin} X \arrow[l, "D\mapsto\mathscr{O}_X(D)"'] \arrow[r, hook] & \operatorname{Cl} X.
\end{tikzcd} \tag{10.15}$$

In particular, the map $D \mapsto \mathscr{O}_X(D)$ along the bottom left is surjective.

Proposition 10.16. *This map is an isomorphism.*

Computing Picard groups. We saw that in a UFD A, all codimension-1 prime ideals are principal, so all Weil divisors on $\operatorname{Spec} A$ are principal. Thus, $\operatorname{Cl} \operatorname{Spec} A = 0$, and therefore $\operatorname{Pic} \operatorname{Spec} A = 0$. For example, $k[x_1, \dots, x_n]$ is a UFD, so $\operatorname{Pic}(\mathbb{A}^n_k) = 0$.

Geometrically, this makes sense: $\mathbb{A}^n$ is akin to the complex manifold $\mathbb{C}^n$, which is contractible, and so "shouldn't have nontrivial line bundles." It's also true that $\mathbb{A}^n_k$ has no nontrivial vector bundles, but this is the much harder Quillen-Suslin theorem.

Another tool for computing Picard groups is excising subsets of schemes. Removing a subset of codimension greater than 1 doesn't affect the class group (though it may affect the Picard group), and removing a subset of codimension 1 affects it in a controlled way.

If Z is an irreducible, codimension-1 subset of X, then the following is a short exact sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{1 \mapsto [Z]} \operatorname{Weil} X \longrightarrow \operatorname{Weil}(X \setminus Z) \longrightarrow 0,$$

and when we quotient by $\operatorname{Prin} X$, we obtain the **excision exact sequence for class groups**, which is merely right exact:

$$\mathbb{Z} \longrightarrow \operatorname{Cl} X \longrightarrow \operatorname{Cl} X \setminus Z \longrightarrow 0. \tag{10.17}$$

For example, open subschemes of $\mathbb{A}^n$ have trivial class groups and therefore trivial Picard groups.

Example 10.18. Suppose $X = \mathbb{P}^n_k$; then, the hyperplane $Z = V(x_0)$ is a codimension-1 closed, irreducible subset, and therefore (10.17) specializes to

$$\mathbb{Z} \longrightarrow \operatorname{Cl} \mathbb{P}^n_k \longrightarrow \operatorname{Cl} \mathbb{A}^n_k = 0 \longrightarrow 0,$$

so $\mathbb{Z} \twoheadrightarrow \operatorname{Cl} \mathbb{P}^n_k$. However, the line bundles we saw at the beginning defined an injection $\mathbb{Z} \hookrightarrow \operatorname{Pic} \mathbb{P}^n_k \hookrightarrow \operatorname{Cl} \mathbb{P}^n_k$, so we're forced to conclude $\operatorname{Pic} \mathbb{P}^n_k \cong \mathbb{Z}$, generated by $\mathscr{O}(1)$. Using this, we can define the **degree** of a line bundle on $\mathbb{P}^n_k$ to be $\deg \mathscr{O}(d) = d$.

We can generalize this to understand factorial schemes more generally.

Proposition 10.19 (Vakil ex. 14.2.I)**.** *If X is factorial and $D \in \operatorname{Weil} X$, then $\mathscr{O}_X(D)$ is an invertible sheaf.*

Corollary 10.20 (Vakil prop. 14.2.10)**.** *Suppose X is factorial. Then, the map $\operatorname{Pic} X \hookrightarrow \operatorname{Cl} X$ is an isomorphism.*

We know for every Weil divisor D, $\mathscr{O}_X(D)$ is invertible, so the map $\operatorname{LocPrin} X \hookrightarrow \operatorname{Weil} X$ is an isomorphism, and this remains true when we quotient by $\operatorname{Prin} X$.

Example 10.21 (Vakil ex. 14.2.K)**.** Let $Y \subset \mathbb{P}_k^n$ be a hypersurface cut out by an irreducible degree-d polynomial $f \in k[x_0, \ldots, x_n]$, and $X = \mathbb{P}_k^n \setminus Y$. Thus, the (irreducible) divisor $[Y] = \operatorname{div} f$, so by Example 10.12, $\mathscr{O}_{\mathbb{P}_k^n}([Y]) = \mathscr{O}(\deg f) = \mathscr{O}(d)$, which in the isomorphism $\mathbb{Z} \cong \operatorname{Cl} \mathbb{P}_k^n$ is identified with $d \in \mathbb{Z}$. Hence, (10.17) for X specializes to

$$\mathbb{Z} \xrightarrow{\cdot d} \mathbb{Z} \longrightarrow \operatorname{Cl} X \longrightarrow 0,$$

so $\operatorname{Cl}(X) \cong \mathbb{Z}/d$. Since X is factorial, then by Corollary 10.20, $\operatorname{Pic}(X) \cong \mathbb{Z}/d$.