GROUP ACTIONS ON S^3

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MARCH 28, 2016

ABSTRACT. This week, we'll discuss group actions on S^3 and construct the Poincaré homology sphere as the quotient of S^3 by the binary icosahedral group I^*. Then, we will show that this really is the Poincaré homology sphere by demonstrating its equivalence to the link of the singularity for $p(z_1, z_2, z_3) = z_1^2 + z_2^3 + z_3^5$ at the origin.

1. GROUP ACTIONS ON S^3

Recall that a group action of a group G on a set S is the data of maps $\varphi_g : S \to S$ for each $g \in G$ such that $\varphi_e = \mathrm{id}$ and $\varphi_g \circ \varphi_h = \varphi_{gh}$. That is, we realize the group as some group of symmetries of a set. If S has additional structure, we can require that the φ_g preserve that structure; therefore, when one speaks of a group acting on a smooth manifold M, the maps φ_g must be smooth, and in fact diffeomorphisms. Similarly, if G has additional structure compatible with M, we often want this action to be compatible with that structure; for example, if G is a Lie group, we would like the action map $G \times M \to M$ sending $(g, m) \mapsto \varphi_g(m)$ to be a smooth map; in this case, one says that the action is **smooth**.

Given a smooth action of a Lie group G on a smooth manifold M, we can form the quotient space M/G as the set of G-orbits, and we'd like this to have more structure too. Specifically, is it a manifold? Not always: $\mathbb{R}^\times$ acts on $\mathbb{R}$ by multiplication, and the quotient isn't even Hausdorff: the closure of the orbit of all nonzero points contains 0, the other orbit.

However, for sufficiently well-behaved actions, there is a smooth manifold structure on the quotient.

Theorem 1.1. *Let G be a Lie group acting smoothly on a manifold M, and suppose that*

(1) the action is free, and
(2) the action is ***proper*** *(meaning the map $(g, m) \mapsto (\varphi_g(m), m)$ is a proper map).*

Then, M/G has a unique smooth manifold structure such that the quotient map $M \twoheadrightarrow M/G$ is a submersion, and has dimension $\dim M - \dim G$.

This is Theorem 9.16 in [2], where it is proven.

In particular, suppose Γ is a finite subgroup of a compact Lie group G. Then, Γ acts on G by left multiplication, which is smooth. Since G is compact, this is automatically proper, and if $g \in \Gamma$ and $h \in G$ are such that $g \cdot h = h$, then $g = e$, so this action is free.

Corollary 1.2. *If G is a compact Lie group and $\Gamma \leq G$ is a finite subgroup acting by left multiplication, then the quotient $M = G/\Gamma$ is a smooth manifold with the same dimension as G.*

Since Γ is finite, the projection $G \twoheadrightarrow M$ is a covering map with deck transformation group Γ: Γ is discrete, so each orbit of Γ is discrete, so every $x \in G$ has a neighborhood that doesn't intersect $\Gamma \cdot x$ except at x itself, and so this neighborhood projects down homeomorphically onto the quotient.

This also tells us that $\pi_n(G) = \pi_n(M)$ if $n \geq 2$, and $\pi_1(G) = \pi_1(M)/\Gamma$.

We're going to use this to construct the Poincaré homology sphere as the quotient of a group action on the 3-sphere. To do this, we would like to think of S^3 as the Lie group $\mathrm{SU}(2)$.

- S^3 sits inside $\mathbb{R}^4$ as the unit sphere, and since $\mathbb{R}^4$ can be identified with the quaternions $\mathbb{H}$, S^3 is identified with the group of unit quaternions, making it into a compact Lie group.
- It's also possible to explicitly identify the unit quaternions with $\mathrm{SU}(2)$, the 2×2 complex matrices of the form $\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}$ such that $|a|^2 + |b|^2 = 1$.
- If q is a quaternion, then conjugation $r_q : \mathbb{H} \to \mathbb{H}$ defined by $x \mapsto qxq^{-1}$ is smooth, and if q is a unit quaternion, it's norm-preserving and orientation-preserving. Moreover, it fixes 1, and therefore also

must fix span$(1)^\perp$, which is a copy of $\mathbb{R}^3$; thus, r_q restricts to a rotation of $\mathbb{R}^3$. This defines a map $r : \mathrm{SU}(2) \to \mathrm{SO}(3)$ sending $q \mapsto r_q$, which one can show is a smooth group homomorphism. Moreover, the kernel is $\{\pm 1\}$, since both of those don't affect span$(1)^\perp$; thus, r is a double covering map.

This double cover is the same double cover $S^3 \twoheadrightarrow \mathbb{RP}^3$, and is an example of the more general double cover $\mathrm{Spin}(n) \twoheadrightarrow \mathrm{SO}(n)$ (hence, SU(2) is also Spin(3)).

Putting these two threads together, we see that if $G < \mathrm{SU}(2)$ is a finite subgroup, then the left multiplication action of G on SU(2) can be regarded as an action of G on S^3, and the quotient S^3/G is a smooth 3-manifold. Since S^3 is simply connected, $\pi_1(S^3/G) \cong G$. This is a nice way to generate 3-manifolds; for example, there are **lens spaces** $L(p,1)$, which are the quotients of S^3 by a $\mathbb{Z}/p$-subgroup acting through a rotation by $2\pi/p$.

2. The Binary Icosahedral Group and the Poincaré Homology Sphere

Consider an icosahedron in $\mathbb{R}^3$.

- The symmetry group of the icosahedron is A_5: any symmetry can be generated by the rotations around a face, an edge, and a vertex, because these allow you to send any vertex to any other vertex, and then rotate. Using these rotations, the symmetries of the icosahedron are $\langle a, b \mid (ab)^2 = a^3 = b^5 = 1\rangle$, which is a presentation of A_5. These symmetries are rotations of $\mathbb{R}^3$, and so define an inclusion $A_5 \hookrightarrow \mathrm{SO}(3)$.
- The preimage of A_5 under the double cover $\mathrm{SU}(2) \twoheadrightarrow \mathrm{SO}(3)$ is a subgroup $I^* \leq \mathrm{SU}(2)$;[1] since this was a double cover, $|I^*| = 120$. This group is called the **binary icosahedral group**.
- Left multiplication defines a continuous action of I^* on SU(2), and the diffeomorphism $\mathrm{SU}(2) \cong S^3$ means this action can be regarded as an action of I^* on S^3.
- I^* has a similar-looking presentation, $\langle s, t \mid (st)^2 = s^3 = t^5\rangle$.
- Since this is an action by left multiplication of a finite subgroup of a Lie group, it's a smooth, free, and proper action, so the quotient $P = S^3/I^*$ is a smooth 3-manifold. This will be the Poincaré homology sphere.
- We can also realize P as $\mathrm{SO}(3)/A_5$, since the $\mathbb{Z}/2$-action realizing SO(3) as a quotient of S^3 makes A_5 the quotient of I^*. This is pretty cool and geometric: it means that the Poincaré homology sphere can be thought of as the rotations of 3-space where we identify rotations that preserve the icosahedron (up to symmetry).

3. This Really is the Poincaré Homology Sphere

Define $p : \mathbb{C}^3 \to \mathbb{C}$ by $p(z_1, z_2, z_3) = z_1^2 + z_2^3 + z_3^5$ and $V = Z(p)$ be its zero set in $\mathbb{C}^3$. At a previous talk, Michelle showed that a link of the origin in V is the Poincaré homology sphere.

Proposition 3.1. *P is diffeomorphic to a link of the origin of C, and hence is the Poincaré homology sphere.*

Proof. We follow the proof in [1, pp. 128-32]. Since we've regarded I^* as a subgroup of SU(2), its action on S^3 extends to an action on all of $\mathbb{R}^4 = \mathbb{C}^2$, where an $A \in \mathrm{SU}(2)$ acts by matrix multiplication. We will show there is a biholomorphic map $g : (\mathbb{C}^2/I^*) \setminus 0 \to V \setminus 0$, which will restrict to a map from $P = S^3/I^*$ to a link around the origin in V. There are four steps:

(1) First, we'll define three homogeneous polynomials $g_1, g_2, g_3 : \mathbb{C}^2 \to \mathbb{C}$, and let $\overline{g} = (g_1, g_2, g_3) : \mathbb{C}^2 \to \mathbb{C}^3$; $\overline{g}$ is I^*-invariant and therefore defines a map $g : \mathbb{C}^2/I^* \to \mathbb{C}^3$.
(2) $g_1^2 + g_2^3 + g_3^5 = 0$, so $\mathrm{Im}(g) \subset V$.
(3) Then, we will show that $\mathrm{d}\overline{g}$ has rank 2 on $\mathbb{C}^2 \setminus 0$, so g is a covering map away from the origin.
(4) We'll show that $\overline{g}$ is a 120-fold cover, and therefore g is a bijection. Since $\mathrm{d}\overline{g}$ has full rank away from the origin, this further implies g is biholomorphic, and since g is homogeneous, this restricts to a diffeomorphism from S^3/I^* to a link around the origin in V.

We begin by explicitly writing down the generators s and t of $I^* < \mathrm{SU}(2)$ as 2×2 complex matrices.[2] Let $\phi = (1+\sqrt{5})/2$ be the "golden ratio;" then,

$$s = \frac{1}{2}\begin{pmatrix} 1+i & 1+i \\ -1+i & 1-i \end{pmatrix} \qquad \text{and} \qquad t = \frac{1}{2}\begin{pmatrix} \phi + \phi^{-1} & 1 \\ -1 & \phi - \phi^{-1} \end{pmatrix}.$$

[1] There are many different notations for the binary icosahedral group, including $2I$, $\mathrm{SL}(2,5)$, II, and $(2,3,5)$.

[2] [1] writes down all the elements of I^*, though they also don't define their notation! In any case, to check I^*-invariance of a polynomial, it suffices to check on the generators.

Now, one can check that the polynomials

$$\begin{aligned}
g_1(z_1,z_2) &= (z_1^{30}+z_2^{30}) + 522(z_1^{25}z_2^5 - z_1^5 z_2^{25}) - 10005(z_1^{30}+z_1^{10}z_2^{20})\\
g_2(z_1,z_2) &= -(z_1^{20}+z_2^{20}) + 228(z_1^{15}z_2^5 - z_1^5 z_2^{15}) - 494 z_1^{10} z_2^{10}\\
g_3(z_1,z_3) &= -\sqrt[5]{1728}\, z_1 z_2 (z_1^{10} + 11 z_1^5 z_2^5 - z_2^{10})
\end{aligned}$$

are invariant under s and t, and therefore under all of I^*, satisfying (1).

You can also directly check that $g_1^2 + g_2^3 + g_3^5 = 0$, but Kirby and Scharlemann give a geometric argument. In either case, we have (2).

The next course is to compute the rank of

$$\mathrm{d}\overline{g} = \begin{pmatrix} \dfrac{\partial g_1}{\partial z_1} & \dfrac{\partial g_1}{\partial z_2} & \dfrac{\partial g_1}{\partial z_3} \\ \dfrac{\partial g_2}{\partial z_1} & \dfrac{\partial g_2}{\partial z_2} & \dfrac{\partial g_2}{\partial z_3} \end{pmatrix}.$$

Again, we could do this by hand, but instead, we can do three smaller calculations: let Δ_{ij} be the determinant of columns i and j. Then, each Δ_{ij} is homogeneous and I^*-invariant.

Δ_{13} has degree 40. Since it's I^*-invariant, its zero set is a union of lines through the origin, so it defines a map $\mathbb{CP}^1 \to \mathbb{C}$, and since $\deg(\Delta_{13}) = 40$, we expect the number of roots to divide 40.

$\mathbb{CP}^1 \cong S^2$ and we can project the icosahedron onto this in a way compatible with the induced action of I^* on $\mathbb{CP}^1$. Since Δ_{13} is I^*-invariant, its zero set is a union of orbits. However, these orbits are:

- the set of 12 vertices,
- the set of 30 centers of edges,
- the set of 20 centers of faces, and
- other orbits that have size 60.

Thus, the only possible zeros of Δ_{13} are at the centers of faces. Similarly, Δ_{12} is homogeneous of degree 48, so the only possible zeros are at the vertices; and Δ_{23} is homogeneous of degree 30, so its zeros must be at the centers of edges. In particular, at every point of $\mathbb{C}^2$ save the origin, at least 2 of these are nonzero, so $\mathrm{d}\overline{g}$ has full rank.

Since $\overline{g}$ is a polynomial map, then it's proper, and so $\overline{g}(\mathbb{C}^2 \setminus 0)$ is closed in $V \setminus 0$; since g is a local homeomorphism, this image is also open. Since $V \setminus 0$ is connected, this means $\overline{g}$ is surjective, and therefore is a covering map. Therefore g is also surjective, and hence a covering map too, meaning we're done with (3).

Finally, for (4), there's another geometric argument: since the g-preimage of a point in $\mathbb{C}^3$ is at least one point, the $\overline{g}$-preimage is at least 120 points. Then, Kirby and Scharlemann give another geometric argument for why there are at most 120 points in the preimage: the preimage of a vertex point can be shown to be the intersection of a 30-gon and a 20-gon within 12 different complex lines. Thus, there are at most 10 intersection points inside each line, so we get at most 120 points. $\square$

References

[1] Kirby, R.C. and M.G. Scharleman. "Eight faces of the Poincaré homology 3-sphere." Geometric topology, Proc. Conf., Athens/Ga. (1979), 113–146.

[2] Lee, John M. *Introduction to Smooth Manifolds*. Springer-Verlag New York, 2002.