CALCULATING DPIN BORDISM GROUPS

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The purpose of this document is to use the Adams spectral sequence to compute some low-dimensional dpin bordism groups. (We will explain what a dpin structure is below.) In dimensions 2 and 3, these bordism groups are computed a different way and used by Kaidi, Parra-Martinez, and Tachikawa [KPMT19a, KPMT19b] to classify certain invertible field theories which can appear on the worldsheet in type I superstring theory.

Theorem 0.1. *The low-degree dpin bordism groups are:* $\Omega_0^{\mathrm{DPin}} \cong \mathbb{Z}/2$, $\Omega_1^{\mathrm{DPin}} \cong \mathbb{Z}/2$, $\Omega_2^{\mathrm{DPin}} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, $\Omega_3^{\mathrm{DPin}} \cong \mathbb{Z}/8$, $\Omega_4^{\mathrm{DPin}} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, $\Omega_5^{\mathrm{DPin}} \cong 0$, *and* $\Omega_6^{\mathrm{DPin}} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

A dpin structure on a smooth manifold M is the data of a spin structure on the orientation double cover of M.

Lemma 0.2 ([KPMT19b, §6.2])**.** *A dpin structure is equivalent to a choice of two real line bundles $\ell_1, \ell_2 \to M$ and a spin structure on*

$$TM \oplus (\ell_1 \otimes \ell_2) \oplus (\ell_1)^{\oplus 3}. \tag{0.3}$$

One consequence of Lemma 0.2 is that if *MTDPin* denotes the Thom spectrum for dpin-structures, so that $\pi_k(MTDPin) \cong \Omega_k^{\mathrm{DPin}}$, then

$$MTDPin \simeq MTSpin \wedge (B\mathbb{Z}/2 \times B\mathbb{Z}/2)^{\ell_1\ell_2 + 3\ell_2 - 4}. \tag{0.4}$$

Here the second summand, $(B\mathbb{Z}/2 \times B\mathbb{Z}/2)^{\ell_1\ell_2+3\ell_2-4}$ which we denote X to tame the notation, is the Thom spectrum of the virtual vector bundle

$$V := (\ell_1 \otimes \ell_2) \oplus \ell_2^{\oplus 3} - \mathbb{R}^4 \longrightarrow B\mathbb{Z}/2 \times B\mathbb{Z}/2. \tag{0.5}$$

By (0.4), $\Omega_k^{\mathrm{DPin}} \cong \widetilde{\Omega}_k^{\mathrm{Spin}}(X)$. We will compute $\widetilde{\Omega}_k^{\mathrm{Spin}}(X)$ for $0 \le k \le 6$ using the Adams spectral sequence, employing a standard trick to work over $\mathcal{A}(1) := \langle \mathrm{Sq}^1, \mathrm{Sq}^2 \rangle$ rather than the entire Steenrod algebra. For details on how this works and many worked examples, see Beaudry-Campbell [BC18], who carefully explain and summarize how to use the Adams spectral sequence for these kinds of computations. The idea is that we must determine $\widetilde{H}^*(X;\mathbb{F}_2)$ as an $\mathcal{A}(1)$-module. Then, the E_2-page of this Adams spectral sequence is

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}(1)}^{s,t}(\widetilde{H}^*(X;\mathbb{F}_2), \mathbb{F}_2). \tag{0.6}$$

(Definitions and notation are as in [BC18].) The spectral sequence converges to something which in degrees $t - s \le 7$ is isomorphic to $\widetilde{\Omega}_{t-s}^{\mathrm{Spin}}(X)$.[1]

Proof of Theorem 0.1. First we argue $\widetilde{\Omega}_*^{\mathrm{Spin}}(X)$ has no p-torsion for odd primes p, justifying Footnote 1. In fact, we will show that if p is an odd prime, $\widetilde{\Omega}_*^{\mathrm{Spin}}(X) \otimes \mathbb{F}_p = 0$. For any finitely generated abelian group A, the p-torsion subgroup of A includes into the p-torsion subgroup of $A \otimes \mathbb{F}_p$, so this suffices.

By definition, $\widetilde{\Omega}_k^{\mathrm{Spin}}(X) \cong \widetilde{H}_k(MTSpin \wedge X)$. Tensoring with $\mathbb{F}_p$, the map

$$\widetilde{H}_k(MTSpin \wedge X) \otimes \mathbb{F}_p \longrightarrow \widetilde{H}_k(MTSpin \wedge X; \mathbb{F}_p) \tag{0.7}$$

is injective, by the universal coefficient theorem. The Künneth theorem computes $\widetilde{H}_*(MTSpin \wedge X; \mathbb{F}_p)$ as a sum of tensor products of the form $\widetilde{H}_i(MTSpin; \mathbb{F}_p) \otimes \widetilde{H}_j(X; \mathbb{F}_p)$, so it suffices to show $\widetilde{H}_j(X;\mathbb{F}_p)$ vanishes for all j. The twisted-coefficients Thom isomorphism tells us there is a (in this case nontrivial) $\mathbb{Z}[\mathbb{Z}/2 \times \mathbb{Z}/2]$-module structure $\widetilde{\mathbb{F}}_p$ on $\mathbb{F}_p$ such that

$$\widetilde{H}_j(X;\mathbb{F}_p) \cong H_j(\mathbb{Z}/2 \times \mathbb{Z}/2; \widetilde{\mathbb{F}}_p). \tag{0.8}$$

[1] This is not *a priori* true; in general we get $\widetilde{\Omega}_{t-s}^{\mathrm{Spin}}(X) \otimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ denotes the 2-adic integers. In our case, we will see that $\widetilde{\Omega}_*^{\mathrm{Spin}}(X)$ lacks torsion for odd primes, so tensoring it with $\mathbb{Z}_2$ does not lose any information. In general, information can be lost when tensoring with $\mathbb{Z}_2$, but that information can be computed by other means.

Maschke's theorem implies that since $\#(\mathbb{Z}/2\times\mathbb{Z}/2)$ and p are coprime, and since $\widetilde{\mathbb{F}}_p$ is p-torsion, $H_j(\mathbb{Z}/2\times\mathbb{Z}/2;\widetilde{\mathbb{F}}_p)$ vanishes in degrees $j>0$. Using that 0^{th} group homology is the abelian group of coinvariants, one can check directly that $H_0(\mathbb{Z}/2\times\mathbb{Z}/2;\widetilde{\mathbb{F}}_p)=0$ as well. Thus $\widetilde{\Omega}_*^{\mathrm{Spin}}(X)$ has no p-torsion.

On to the Adams spectral sequence. First we determine $\widetilde{H}^*(X;\mathbb{F}_2)$. As a graded abelian group, this is characterized by the Thom isomorphism: if $U\in\widetilde{H}^0(X;\mathbb{F}_2)$ denotes the Thom class, cup product with U is an isomorphism

$$(U\cdot)\colon H^k(B\mathbb{Z}/2\times B\mathbb{Z}/2;\mathbb{F}_2)\xrightarrow{\cong}\widetilde{H}^k(X;\mathbb{F}_2). \tag{0.9}$$

There is no degree shift because the virtual vector bundle $V\to B\mathbb{Z}/2\times B\mathbb{Z}/2$ (from (0.5)) has rank zero. Let $a\coloneqq w_1(\ell_1)$ and $b\coloneqq w_1(\ell_2)$ in $H^1(B\mathbb{Z}/2\times B\mathbb{Z}/2;\mathbb{F}_2)$; then

$$H^*(B\mathbb{Z}/2\times B\mathbb{Z}/2;\mathbb{F}_2)\cong\mathbb{F}_2[a,b]. \tag{0.10}$$

The $\mathcal{A}(1)$-module structure on $\widetilde{H}^*(X;\mathbb{F}_2)$ is determined by the following rules.

(1) $\mathrm{Sq}^i(U)=Uw_i(V)$, where w_i denotes the i^{th} Stiefel-Whitney class. In this case, $w_1(V)=a$ and $w_2(V)=ab$.

(2) The Cartan formula determines the Steenrod squares of a product. We only need Sq^1 and Sq^2, for which the Cartan formula specializes to

$$\mathrm{Sq}^1(xy)=\mathrm{Sq}^1(x)y+x\mathrm{Sq}^1(y) \tag{0.11a}$$

$$\mathrm{Sq}^2(xy)=\mathrm{Sq}^2(x)y+\mathrm{Sq}^1(x)\mathrm{Sq}^1(y)+x\mathrm{Sq}^2(y). \tag{0.11b}$$

(3) From the axiomatic properties of Steenrod squares, $\mathrm{Sq}^1(a)=a^2$, $\mathrm{Sq}^1(b)=b^2$, and $\mathrm{Sq}^2(a)=\mathrm{Sq}^2(b)=0$.

Using these three rules one can determine the action of Sq^1 and Sq^2 on any cohomology class of X, as it is a sum of products of U, a, and b. This is routine, and indeed we used a computer program to make these calculations. The answer is displayed in Figure 1.

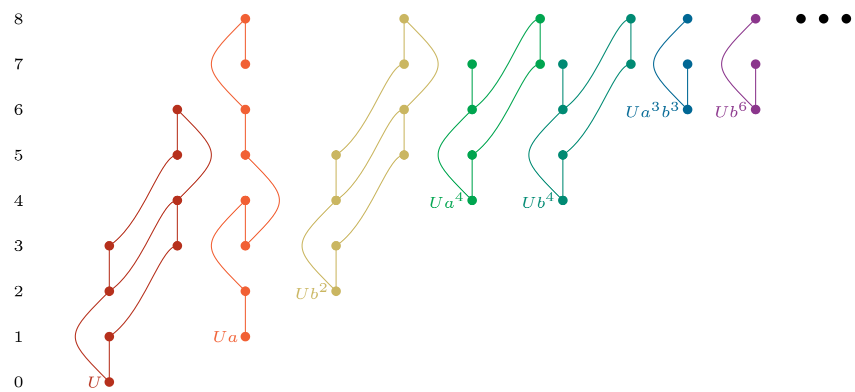


FIGURE 1. $\widetilde{H}^*(X;\mathbb{F}_2)$ as an $\mathcal{A}(1)$-module in degrees 8 and below. Each dot represents an $\mathbb{F}_2$ summand, with its cohomological degree given by its height. The connecting lines, resp. curves, indicate an action by Sq^1, resp. Sq^2, carrying the lower dot to the upper dot. This $\mathcal{A}(1)$-module factors as several different summands; we give each summand a different color.

From this figure, we see that, as an $\mathcal{A}(1)$-module, $\widetilde{H}^*(X;\mathbb{F}_2)$ splits into several summands. All of them except the orange summand look like shifts of $\mathcal{A}(1)$, though because we haven't gone above degree 8, some of them look truncated. The orange summand, which is generated by Ua, is isomorphic to the mod 2 cohomology of the spectrum MO_1, the Thom spectrum of the tautological line bundle $\sigma\to BO_1$ (see [BC18, Figure 4]); therefore we denote that summand by $\widetilde{H}^*(MO_1)$.

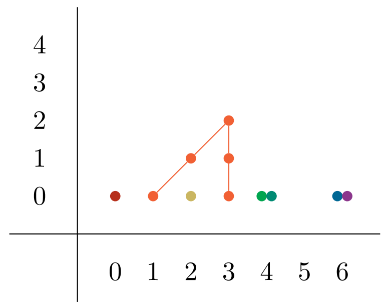
Remark 0.12. We have not fully determined the structure of these summands above degree 8, so it is possible that, for example, the green summands aren't actually isomorphic to $\Sigma^4\mathcal{A}(1)$. Any such discrepancy would only appear above degree 8, so the minimal resolutions for computing the E_2-page would only differ in degrees $t-s\geq 8$, so such a discrepancy would not change the E_2-page we draw in (0.14). ◀

Specifically, up to degree 7,

$$\widetilde{H}^*(X;\mathbb{F}_2) \cong \mathcal{A}(1) \oplus \widetilde{H}^*(MO_1) \oplus \Sigma^2\mathcal{A}(1) \oplus \Sigma^4\mathcal{A}(1) \oplus \Sigma^4\mathcal{A}(1) \oplus \Sigma^6\mathcal{A}(1) \oplus \Sigma^6\mathcal{A}(1). \tag{0.13}$$

The E_2-page of the Adams spectral sequence (0.6) is the direct sum of the E_2-pages of these summands, and these have all been calculated. For $\Sigma^k\mathcal{A}(1)$, there is a single $\mathbb{F}_2$ summand in bidegree $s=0$, $t=k$; for $\widetilde{H}^*(MO_1)$, see [Cam17, Example 6.3]. Putting these together, the E_2-page for this spectral sequence is

(0.14)



In this diagram, the x-axis is $t-s$ and the y-axis is s. Therefore a differential d_r moves one degree to the left and $r-1$ degrees upwards. Each dot represents an $\mathbb{F}_2$ summand of the E_2-page; the different colors indicate which summands of $\widetilde{H}^1(X;\mathbb{F}_2)$ are responsible for which data on the E_2-page.

The E_2-page carries an action by $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)}(\mathbb{F}_2,\mathbb{F}_2)$. The vertical lines indicate action by an element $h_0 \in \mathrm{Ext}^{1,1}_{\mathcal{A}(1)}(\mathbb{F}_2,\mathbb{F}_2)$, and the diagonal lines indicate action by $h_1 \in \mathrm{Ext}^{2,1}_{\mathcal{A}(1)}(\mathbb{F}_2,\mathbb{F}_2)$; see [BC18, Example 4.1.1] for more on h_0 and h_1. Crucially, all differentials are h_0- and h_1-linear, i.e. $d_r(h_ix) = h_id_r(x)$ $(i=0,1)$. In particular, if $h_ix=0$ and $h_iy\neq 0$, then $d_r(x)\neq y$. This forces all differentials in the range shown to vanish.

There is still a question of extension problems: the line $t-s=k$ is the associated graded of a filtration, possibly nontrivial, on $\widetilde{\Omega}_k^{\mathrm{Spin}}(X)$. Let $\overline{x},\overline{y}$ be elements on the E_∞-page, i.e. elements of this associated graded; if $h_0\overline{x}=\overline{y}$, then there are preimages $x,y\in\widetilde{\Omega}_*^{\mathrm{Spin}}(X)$ of $\overline{x}$, resp. $\overline{y}$, such that $2x=y$. Thus, for example, $\widetilde{\Omega}_3^{\mathrm{Spin}}(X)$, which *a priori* could be an arbitrary abelian group of order 8, has two nonzero elements x_1,x_2 with $x_2=4x_1$, and therefore $\widetilde{\Omega}_3^{\mathrm{Spin}}(X)\cong\mathbb{Z}/8$.

Similarly, if $h_1\overline{x}=\overline{y}$, one can choose preimages x and y in $\widetilde{\Omega}_*^{\mathrm{Spin}}(X)$ such that $\eta\cdot x=y$, where η is the generator of $\pi_1\mathbb{S}\cong\mathbb{Z}/2$. (Concretely, if x is the dpin bordism class of some manifold M, then $\eta\cdot x$ is the bordism class of $S^1\times M$, where S^1 has the dpin structure induced from the nonbounding framing.)

However, not all extensions arise in this way: *hidden extensions* are nontrivial extensions that are not detected by the action of $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)}(\mathbb{F}_2,\mathbb{F}_2)$ on the E_2-page. *A priori*, there could be a hidden extension by 2 in bidegree $t-s=2$, in which case $\widetilde{\Omega}_2^{\mathrm{Spin}}(X)$ would be $\mathbb{Z}/4$ rather than $\mathbb{Z}/2\oplus\mathbb{Z}/2$. However, we can rule this out: suppose it were $\mathbb{Z}/4$, and let x be a generator. Then the image of x in the associated graded of $\widetilde{\Omega}_2^{\mathrm{Spin}}(X)$ (i.e. the $t-s=2$ line of the Adams E_∞-page) is the nontrivial element of the yellow $\mathbb{Z}/2$ summand in bidegree $(2,0)$, and the image of $2x$ is the nonzero element of the orange $\mathbb{Z}/2$ summand in bidegree $(2,1)$. The h_1-action carries this to the nonzero element of the orange $\mathbb{Z}/2$ summand in bidegree $(3,2)$, so $\eta\cdot 2x\neq 0$. Since $2\eta=0$, however, this is a contradiction, forcing $\widetilde{\Omega}_2^{\mathrm{Spin}}(X)\cong\mathbb{Z}/2\oplus\mathbb{Z}/2$. ⊠

References

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