# M390C NOTES: GEOMETRIC LANGLANDS

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These notes were taken in UT Austin's M390C (Geometric Langlands) class in Spring 2021, taught by David Ben-Zvi. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Any mistakes in the notes are my own.

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Lecture 1.

## Overview and a perspective on modular forms: 1/19/21

This is a class on the geometric Langlands program from a particular perspective, incorporating its relationship to electric-magnetic duality. The class is over Zoom.

The geometric Langlands program lies halfway between number theory and physics. Maybe we are Odysseus and trying to navigate back home, between the two perils of Charybdis (physics, classically the big whirlpool) and Scylla (number theory, classically the monster). You can probably extend the analogy further, e.g. derived algebraic geometry is the Calypso islands. Extending the analogy further is left as an exercise.

There isn't any particular recommended background for this class — in particular, you don't need to know physics or number theory. Mackey [Mac80] wrote a nice overview of a perspective on the relationship between symmetry and harmonic analysis which could be fun to read. In this and the next lecture, we'll talk about modular forms and some relationships to physics; after that, we will begin the course properly: in a sense, the geometric Langlands program is a vast generalization of the Fourier transform, so we will begin with the Fourier transform, in a way that will be helpful when we do generalize.

**Modular and automorphic forms, and physics** We're not going to be super technical about number theory. The idea of modular and automorphic forms is to do a kind of harmonic analysis or quantum mechanics on *arithmetic locally symmetric spaces*. As an example, the upper half-plane $\mathbb{H}$ has a model as $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2$. The *modular group* $\Gamma \coloneqq \mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}$, with a fundamental domain $\Gamma\backslash\mathbb{H}$ (TODO: picture): the fundamental domain is noncompact, and goes off to infinity along the $y$-axis, and there are a couple of orbifold points, where the $\Gamma$-action has stabilizer.

More generally, we might consider a Lie group $G$ with maximal compact $K \subset G$ and a lattice $\Gamma \subset G$. Then we consider the space $\Gamma\backslash G/K$ and study the space of functions on it. You might imagine a particle moving on this locally symmetric space, so we're interested in $L^2(\Gamma\backslash G/K)$, with a Laplacian $\Delta$ acting on this, and we can decompose the space of functions in terms of subspaces of eigenfunctions. This is one way in which modular forms can arise.

There are myriad variants of this. You can yeet $K$ out of the story and study $L^2(\Gamma\backslash G)$ with its $K$-action. (TODO: something about the unit tangent bundle.) Plus, there's no need to restrict ourselves to linearizing with functions: you can use forms or sections of other vector bundles, such as $\Gamma(\Gamma\backslash\mathbb{H}, \omega^{\otimes k/2})$. De Rham says this is related to the cohomology of $\Gamma\backslash\mathbb{H}$, possibly with twisted coefficients. All of these variants are examples of things related to modular forms.

Now maybe you're thinking that if you pass to cohomology, you're no longer doing quantum mechanics, but in fact this is the domain of something called *topological quantum mechanics*; for example, this is discussed by Witten in his paper [Wit82] on supersymmetric quantum mechanics and its relationship to Morse theory.

Automorphic forms follow a similar story, but $G$ is a more general Lie group. For example, pick your favorite reductive algebraic group such as $\mathrm{GL}_n$ or $\mathrm{Sp}_n$, let $G$ be the real points of this group, $\Gamma$ be the integral points of this group, and $K$ be the maximal compact of $G$. There is a long history of studying spaces of functions on $\Gamma\backslash G/K$ via harmonic analysis, and thinking of it as quantum mechanics. For example, if we started with $\mathrm{Sp}_{2n}$, we get $\mathrm{Sp}_{2n}(\mathbb{Z})\backslash\mathrm{Sp}_{2n}(\mathbb{R})/\mathrm{U}_n$.

But there's a *lot* more structure here than in a typical quantum-mechanical setup. You can see this already for modular forms ($G = \mathrm{SL}_2(\mathbb{R})$). Namely, there's an additional variable: we can generalize from $\mathbb{Z}$ to other rings of integers in number fields. That is, given the field $\mathbb{Q}$, we think of $\mathbb{Z}$ as $\mathcal{O}_\mathbb{Q}$, the ring of integers of this number field, and obtain the group $\mathrm{SL}_2(\mathcal{O}_\mathbb{Q})$. Now we can replace $\mathbb{Q}$ with any finite extension $F/\mathbb{Q}$ and let $\mathcal{O}_F$ be the ring of integers of $F$, and consider a new lattice $\mathrm{SL}_2(\mathcal{O}_F)$. To make this completely precise, one has to fiddle with $\mathrm{SL}_2(\mathbb{R})$, since $F$ may have more than one place at infinity, but this is the kind of technical detail we'll avoid for now.

And there is another way to vary the data: fix $F$, say $F = \mathbb{Q}$. Then we can vary the *conductor* or the *ramification data*. That is, the fundamental domain of $\Gamma$ on $\mathbb{H}$ has a lot of covering spaces $\Gamma'\backslash\mathbb{H}$, where $\Gamma' \subset \mathrm{SL}_2(\mathbb{Z})$ is a *congruence subgroup*. One example of a congruence subgroup is, given $N \in \mathbb{Z}$, the subgroup

$$\Gamma(N) \coloneqq \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \equiv \mathrm{id} \bmod N\right\}. \tag{1.1}$$

A variant is $\Gamma_0(N)$, the subgroup of matrices which are upper triangular mod $N$.

As is common in number theory, we can look at different places for primes in $\mathcal{O}_F$. For example, with $F = \mathbb{Q}$, this means looking at the local data at a prime $p$, which involves looking at $\mathrm{SL}_2(\mathbb{Q}_p)$. So the Hilbert space that we wanted to produce in the end depends on all this data: $G$ and $\Gamma$ and $K$, but also possibly $F$ and the congruence subgroup and the prime.

Anyways, we'll get a Hilbert space and can study the spectral theory of the Laplacian. Maybe surprisingly, the eigenspaces are usually not one-dimensional. This "degeneracy" is because of *Hecke operators*, which are a crucial part of this story. At a high level, the Laplacian fits into a large family of commuting operators, and if $p$ is a prime not dividing $N$, this family has a member $T_p$ called the *Hecke operator*, giving an action of $\mathbb{C}[T_p]$ on $L^2(\Gamma\backslash G/K)$. And these all commute, so the tensor product of all of these $\mathbb{C}[T_p]$ over all primes acts on the Hilbert space, preserving the eigenspaces.

From the quantum mechanics perspective, this amount of commuting operators is unusual. You can think of this as an *integrable system*, with lots of conserved quantities. Usually (TODO: if I understood correctly), integrable systems are the opposite of chaos, but these arithmetic systems are studied as good examples of quantum chaos! This is a feature of the arithmetic story, and "arithmetic quantum chaos" behaves a lot more like quantum integrable systems than one might expect.

In this system, there is a special collection of measurements/states for a modular (or automorphic) form, called *periods*. One basic example is, given a modular function $f$ on the fundamental domain $\Gamma/\backslash H$, integrate it:

$$\int_{i\mathbb{R}_+} f. \tag{1.2}$$

We will study modular functions/forms with these invariants. Hecke used $L$-functions to produce examples of these invariants.

**Definition 1.3.** A *Maass form* is an eigenfunction for the Laplacian on $L^2(\Gamma\backslash G/K)$. Specifically, *modular forms* are the holomorphic sections of $\omega^{\otimes k/2}$.

Modular forms can also arise by looking at the (twisted) cohomology of $\Gamma\backslash\mathbb{H}$; this is what's called *Eichler-Shimura theory*.

Our emphasis in this class will be more about topological quantum mechanics, rather than quantum mechanics; we care mostly about ground states. Modular forms are sort of like ground states here.

And there's one more piece of essential structure, to add to our already large pile of structures. There are these mysterious operators that allow you to vary the group! That is, these Hilbert spaces for different groups talk to each other! This is called *Langlands functoriality*. Part of the goal of this class is to explain this structure within physics.

But what the Langlands program itself does is to take these automorphic forms and spectrally decompose them in a prism (TODO: picture of the prism from Dark Side of the Moon, or maybe because this has something to do with physics, Dark Side of the Muon?). Automorphic forms enter on the left, and the prism spectrally decomposes them under the Hecke algebra (the algebra of all these commuting Hecke operators). And the different "colors" (eigenvalues) are given by representations of Galois groups of number fields, which is a surprising and magical statement. Moreover, there is a duality: these Galois representations are not into the complex points of $G$, but instead into $G^\vee_{\mathbb{C}}$, where $G^\vee$ is a dual group under something called Langlands duality.

For example, if $G = \mathrm{PSL}_2(\mathbb{R})$, then $G^\vee = \mathrm{SL}_2(\mathbb{C})$. A relatively explicit way to see how this enters is that if $E$ is an elliptic curve over $\mathbb{Q}$, then $H^1(E)$ is a two-dimensional vector space carrying a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ action, and this is part of how modular forms appear in the story. This is one of the "colors" (eigenvalues) in this theory. The representation morally has image in $\mathrm{SL}_2(\mathbb{R})$, though to see this idea precisely requires setting things up a little more carefully. Elliptic curves appear in two ways in this, as the moduli space of elliptic curves is an arithmetic locally symmetric space. This isn't necessary to see the high-level overview, but it's crucial for actually proving anything! It's useful to know that elliptic curves have covers which are automorphic curves, and this provides a bridge between the two sides of the Langlands program. This is useful, but only applies for $\mathrm{GL}_2$ — in general, you don't have this bridge, and the two sides are very far apart. For example, $\mathrm{GL}_3(\mathbb{Z})\backslash\mathrm{GL}_3(\mathbb{R})/\mathrm{O}_3$ is not a moduli space of anything: it's not a manifold. And this makes your proofs much harder and the duality more mysterious: why should function theory on these spaces have anything to do with Galois representations?[1]

One of the major goals of this class is to show how the (geometric) Langlands program arises in physics, not in quantum mechanics, but in four-dimensional (topological) field theory. Rather than beginning with a quantum mechanics system, we replace it with something much richer and more complicated — and scary. The key adjective "topological" helps mollify this: we throw out dynamics and look at ground states of the Laplacian, like looking only at harmonic forms rather than everything. We will try to match the structure of the Langlands program with the structure of this TFT.

Why 4? Quantum mechanics seems canonical enough, but 4d physics seems less so. We introduce another adjective, *arithmetic quantum field theory*, following the paradigm of arithmetic topology. This is an idea making an analogy between number fields and geometric objects that arise in physics, often manifolds. With a robust enough analogy, you can envision constructions with manifolds as having meaning in the world of number fields. The basic tenets of this theory are outlines in Weil's Rosetta stone (TODO: cite), which establishes a dictionary between number fields, function fields, and Riemann surfaces.

- Given a number field $F/\mathbb{Q}$, we consider $\mathrm{Spec}(\mathcal{O}_F)$, which has points labeled by primes in $\mathcal{O}_F$.
- The analogy between number fields and functional fields is older and better understood. We replace $F$ with a (smooth, projective) curve $C$ over a finite field $\mathbb{F}_q$. The field of rational functions $\mathbb{F}_q(C)$ on $C$ has a lot of structure reminiscent of $F$, and the ring of regular functions $\mathbb{F}_q[C]$ resembles $\mathcal{O}_F$ (e.g. they're both Dedekind domains). The points of $C$ are like the primes in $\mathcal{O}_F$.
- But why stop at $\mathbb{F}_q$? Let $\Sigma$ be a compact Riemann surface. Points of $\Sigma$ are the analogues of primes, in Weil's dictionary, and one can try to make geometric analogues of number-theoretic questions. The field of meromorphic functions on $\Sigma$ is analogous to $F$ and $\mathbb{F}_q(C)$, and the analogue of the ring of integers is a little complicated — $\Sigma$ has no nonconstant entire functions, so we have to remove some points, analogues of points at infinity in the number field setting.

The crucial change in the arithmetic topology analogy is to replace Riemann surfaces with 3-manifolds. The reason behind this surprising change is that Riemann surfaces has strong similarities to curves over algebraically closed fields of positive characteristic. When you study a point $\mathrm{Spec}\,\mathbb{F}_q \hookrightarrow C$, you should remember the internal structure given by the Galois group action. Étale topology tells us to think of $\mathrm{Spec}(\mathbb{F}_q)$ as sort of like a circle, because the étale fundamental group of $\mathrm{Spec}(\mathbb{F}_q)$ is $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \cong \widehat{\mathbb{Z}}$. The Frobenius is a topological generator of this fundamental group. So there's too much structure here to match with a Riemann surface. If you base-change to $\overline{\mathbb{F}}_q$, replacing these "circles" with their "universal covers," we obtain an extra line direction which topology/cohomology doesn't see, because it's contractible, but now we obtain something that feels a little more like a Riemann surface.

[1] Another technical detail to not worry about: when we replace $\mathbb{C}$ with $\overline{\mathbb{Q}_\ell}$, which is necessary for making some of these things precise, one must use étale cohomology instead of singular/Zariski cohomology. But that's not crucial for the point of this lecture.

So from the point of view of Galois theory, function fields of curves over $\mathbb{F}_q$ feel less like Riemann surfaces and more like bundles of Riemann surfaces over $S^1$. This is equivalent data to a Riemann surface $\Sigma$ and a diffeomorphism $\phi\colon \Sigma \to \Sigma$. We then build the bundle as

$$\Sigma \times [0,1]/((x,0) \sim (\phi(x),1)), \tag{1.4}$$

the construction called the *mapping torus*. On the function-field side, we think of the Frobenius as $\phi$. There's a difference here, in that we don't have a canonical choice of $\phi$ on the Riemann surface side (the identity is boring so let's not use that one), so we think of $\phi$ as "generic."

And so we arrive at the arithmetic topology dictionary, built by many workers, including Mumford writing to Mazur, Mazur, Kapranov, Reznakov, Morijita, and Kim. This is also known as the "knots and primes" dictionary: number fields are analogues of 3-manifolds. This is not something totally born out of nowhere; it's a refinement of Weil's dictionary.

Just how not all number fields have unique Frobenii (Frobeniuses?), but rather different ones at different primes, our 3-manifolds $Y$ are not just surface bundles over curves. Primes on the number field side now correspond to embedded circles in $Y$, i.e. knots. Local fields, such as $\mathbb{Q}_p$, look like 2-manifolds. There aren't a lot of 2-manifolds fibered over the circle, but that's okay. And there are many more aspects of the analogy, such as the relationship between Legendre symbols and linking numbers, and more. The nLab page on arithmetic topology has a great list.

This is not an incredibly precise dictionary, and don't make the mistake of trying to associate specific primes to specific knots. For example, if yous said $\mathbb{Q}$ is the sphere, then you'd discover the Poincaré conjecture is false in the number-field setting, which is unfortunate. Rather, let's imagine that number fields are a new class of examples of 3-manifolds, with some commonalities and some other properties, and function fields are another family. So we can then study our new, rich class of examples.

Returning to the question of why four-dimensional topological field theory, well, first we have to discuss exactly what a topological field theory is, but we will see that one of the basic invariants of such a creature is that to ever $(n-1)$-manifold, one obtains a vector space. So the Langlands program assigns vector spaces (or things related to it, such as graded vector spaces, or chain complexes) to function and number fields, which are 3-dimensional in our analogy, and therefore we expect a four-dimensional story in physics.

More generally, an $n$-dimensional quantum field theory has dynamics: you get in addition to your vector space on an $(n-1)$-manifold, a Hilbert space structure and a Hamiltonian. Again, you might have something like a chain complex instead of a vector space. But the Hamiltonian makes this more like a quantum mechanics problem on your codimension-1 manifold. In topological theories, the Hamiltonian is 0.

Now, back to locally symmetric spaces: if $F$ is a number field, we think of the field theory as assigning to it the vector space $L^2(\Gamma_{\mathcal{O}_F}\backslash G/K)$. The space $\Gamma\backslash G/K$ does not directly appear; instead, it is a moduli space of solutions to certain relevant equations on a 3-manifold.

Other parts of the story carry over too. Turning on the conductor/ramification $N$, we have not just a 3-manifold, but also a knot or link inside it, which we think of as the locus along which singularities can appear. In physics, these are called "codimension-2 defects," an important piece of data in general QFT.

To recap: we went very quickly today, and will go quickly on Thursday, but the class will mostly go more slowly, starting next week, where we more carefully keep track of the structures on both sides of this story, trying to stay on the safe, geometric tightrope between these two paradigms.

Thursday we will dig into more of the physics analogues of the variables we can twiddle on the number-theoretic side: what happens if we vary the conductor, if we vary the number field $F$, if we play with functoriality, etc.

Lecture 2.

## A tale of two TFTs: 1/21/21

Last time, we talked about a perspective on modular forms (or automorphic forms): pick your favorite reductive algebraic group or matrix group, such as $\mathrm{GL}_n$ or $\mathrm{PSL}_2$,[2] and let $F/\mathbb{Q}$ be a number field. You can let $F = \mathbb{Q}$ if you want. Let $\mathcal{O}_F$ be the ring of integers of $F$; if $F = \mathbb{Q}$, $\mathcal{O}_F = \mathbb{Z}$.

[2] We're not thinking specifically of these groups as over a specific field, such as $\mathrm{GL}_n(\mathbb{R})$, but a machine for assigning to a field $k$ a group $\mathrm{GL}_n(k)$. This technicality is important because $F$ varies today.

Today we will discuss what happens when we vary $F$, and how this affects a moduli space of principal bundles (TODO: missed this). We obtained a locally symmetric space by taking the real points of the group and taking the double quotient by an "arithmetic" lattice and the maximal compact. For example, we can take $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathrm{PSL}_2(\mathbb{R})/\mathrm{SO}_2$. This is for $S = \mathbb{Q}$.

Now let's consider more general $F$. We have $\mathrm{PSL}_2(\mathcal{O}_F)$ with no issue, but what should replace $\mathrm{PSL}_2(\mathbb{R})$ and its maximal compact? Instead we consider $F \otimes_{\mathbb{Q}} \mathbb{R}$, which is a ring of the form

$$F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}, \tag{2.1}$$

where $F$ has $r_1$ embeddings into $\mathbb{R}$ and $r_2$ pairs of conjugate embeddings into $\mathbb{C}$. Then we replace $\mathrm{PSL}_2(\mathbb{R})$ with $\mathrm{PSL}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$.

**Example 2.2.** Suppose $F = \mathbb{Q}(\sqrt{d})$, where $d$ is squarefree.

- If $d > 0$, so this is a real quadratic extension, $r_1 = 2$ and $r_2 = 0$.
- If $d < 0$, so this is an imaginary quadratic extension, $r_1 = 0$ and $r_2 = 1$. ◀

Then we can take the maximal compact $K$ of $\mathrm{PSL}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$ as normal, and obtain a locally symmetric space. If $F$ is a real quadratic field, this leads us to *Hilbert modular forms*, via $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$, acting on $\mathbb{H} \times \mathbb{H}$. In the imaginary quadratic case, we get $\mathrm{PSL}_2(\mathbb{C})/\mathrm{SO}_3 \cong \mathbb{H}^3$, hyperbolic 3-space, and there are connections to hyperbolic geometry. Usually these double quotients are not algebraic varieties, as this example demonstrates.

For other $F$, we'll get products of $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{C})$; what's most interesting is the arithmetic lattice $\mathrm{PSL}_2(\mathcal{O}_F)$.

Once we have these arithmetic locally symmetric spaces $\mathcal{M}$, we want to produce vector spaces out of them, including $L^2(\mathcal{M})$, $H^*(\mathcal{M})$, and twisted versions thereof. Importantly for the geometric Langlands program, these vector spaces carry an action of a huge commutative algebra, which is a tensor product over all the primes in $F$ of a polynomial ring in $\mathrm{rank}(G)$ generators.

One could also allow ramification, obtaining generalizations $\mathcal{M}_{G,F,N}$, where $N \in \mathrm{O}_F$, and you replace $\mathrm{PSL}_2(\mathcal{O}_F)$ with a congruence subgroup $\Gamma_N$ in which we impose conditions on our matrices mod $N$. These are a few different conditions you might impose (e.g. identity mod $N$, or upper triangular mod $N$). The arithmetic locally symmetric space is $\Gamma_N\backslash G_{\mathbb{R}}/K$, and the large commutative algebra is now "only" a tensor product over the primes $p$ not dividing $N$.

This kind of idea, of a vector space associated to a number field, or maybe a vector space associated to a number field and some primes, is reminiscent under the arithmetic topology analogy to the state spaces in a 4d topological field theory. As we discussed last time, this is a refinement of Weil's Rosetta stone, where $\operatorname{Spec}\mathbb{Z}$ feels like a curve with points $\operatorname{Spec}\mathbb{F}_p$ associated to primes $p$, and $\operatorname{Spec}\mathbb{Z}_p$ as a small disc around this point. Inside that there is the punctured disc $\operatorname{Spec}\mathbb{Q}_p$. This is analogous to having a smooth, reduced algebraic curve over a finite field $\mathbb{F}_q$, which locally looks like $\operatorname{Spec}\mathbb{F}_q[t]$. Here the points are $\operatorname{Spec}\mathbb{F}_q$, and around this is the disc $\operatorname{Spec}\mathbb{F}_q[[t]]$ with the punctured disc $\operatorname{Spec}\mathbb{F}_q((t))$.

Now we look at the étale topology of $\operatorname{Spec}\mathbb{Z}$, which is a fancy way to say we care about the cohomology of Galois groups. From this perspective, the Rosetta stone isn't quite rich enough: $\operatorname{Spec}\mathbb{F}_p \hookrightarrow \operatorname{Spec}\mathcal{O}_F$ is a point, but not étale-topologically: the étale topos tells us that this "point" has a whole bunch of interesting covering spaces, such as $\operatorname{Spec}(\mathbb{F}_{p^n}) \to \operatorname{Spec}(\mathbb{F}_p)$. Its étale fundamental group is $\widehat{\pi}_1^{\text{ét}}(\operatorname{Spec}\mathbb{F}_p) = \mathrm{Gal}(\mathbb{F}_p) \cong \widehat{\mathbb{Z}}$. You can think of this profiniteness as not really there: we can only see finite extensions or, said differently, finite covering spaces, and at this level there's no way to distinguish $\mathbb{Z}$ and $\widehat{\mathbb{Z}}$. This is a common feature of étale fundamental groups.

So the point is that the point $\operatorname{Spec}\mathbb{F}_p$ behaves a lot like a circle if you want to do étale things. (TODO: picture of $\operatorname{Spec}\mathbb{Z}$ with a circle at each prime). Therefore $\operatorname{Spec}\mathbb{Z}$, and its siblings $\operatorname{Spec}\mathcal{O}_F$, feel more like 3-manifolds than Riemann surfaces. And there are other ways to make this fuzzy analogy less fuzzy: there is a version of Poincaré duality, for example, with the correct dimension.

The monodromy around these circles is the Frobenius, but different Frobenii at different primes don't talk to each other. Because the curve $C/\mathbb{F}_q$ maps to $\operatorname{Spec}(\mathbb{F}_q)$, which is sort of like a circle, we think of these 3-manifolds as fibered over a circle.

Given this perspective, what is $\operatorname{Spec}(\overline{\mathbb{F}}_q)$? Étale-topologically, this is actually a point, but if you want a good dictionary between covering spaces and Galois representations, it should be a covering space of the circle, and in fact the universal one, $\mathbb{R}$. This is fine: for the purposes of topology and cohomology, $\mathbb{R}$ is a fine stand-in for a point. Now base-change $C$ to $\overline{C} \coloneqq C_{\overline{\mathbb{F}}_q} \coloneqq C \times_{\operatorname{Spec}\mathbb{F}_q} \operatorname{Spec}\overline{\mathbb{F}}_q$; now we have something which feels like a bundle of Riemann surfaces, i.e. curves over algebraically closed fields, fibered over the real line $\operatorname{Spec}\overline{\mathbb{F}}_q$.

As we discussed last time, the monodromy around the circle is the Frobenius, so we can think of these surface bundle analoues as mapping tori for the Frobenius.

Let's discuss one more piece of evidence for this arithmetic topology dictionary: what happens with $\operatorname{Spec}\mathbb{Z}$? Let $p$ be prime so we get a "circle" $\operatorname{Spec}\mathbb{F}_p$ in $\operatorname{Spec}\mathbb{Z}$. The neighborhood $\operatorname{Spec}\mathbb{Q}_p$ now behaves like a tubular neighborhood of this circle inside $\operatorname{Spec}\mathbb{Z}$. More generally, we can work with $\operatorname{Spec}\mathcal{O}_F$ and a prime $p \in F$ and a place $v$ to complete $\mathcal{O}_F$ at, and obtain a local field $F_v$; then we might expect $\operatorname{Spec}(F_v)$ to be a tubular neighborhood of the circle $\operatorname{Spec} F/p$ — though (TODO) the place has to know something about $p$.

If $F_v$ is a non-Archimedian local field, such as $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$, then $\operatorname{Gal}(\overline{F}_v/F_v)$ surjects onto $\mathbb{Z}_\ell \rtimes \widehat{\mathbb{Z}}$. Fun fact for those interested in group theory: this semidirect product is an example of a *Baumslag-Solitar group*

$$\mathrm{BS}(1,p) \coloneqq \langle \sigma, u \mid \sigma u \sigma = u^p \rangle. \tag{2.3}$$

Here $\sigma$ is the Frobenius and $u$ is a generator of $\mathbb{Z}_\ell$. This interpolates between $\mathbb{Z}\times\mathbb{Z} = \pi_1(T^2)$, for $p = 1$, and $p = -1$, which is $\pi_1$ of the Klein bottle. So this does feel sort of like a torus neighborhood of a knot in a 3-manifold. So primes look like circles, and local fields look like 2-manifolds — not just any 2-manifolds, but 2-manifolds fibered over the circle.

In general, the Galois group $\operatorname{Gal}(\overline{F}_v/F_v)$ can be assembled from three pieces: the Galois group of the residue field $\widehat{\mathbb{Z}}\cdot\sigma$, the *tame part*, which is a product of $\mathbb{Z}_\ell$s for $\ell \neq p$ (here $p$ is the characteristic of the residue field), and the *wild part*, which is a $p$-group.

This analogy is nice and important, tying arithmetic and geometric Langlands together, but we will spend the most time in places where it is the most concrete. Let's summarize the analogy.

- The following objects are thought of as three-dimensional: number fields $\operatorname{Spec}\mathcal{O}_F$ and function fields of curves $C/\mathbb{F}_q$, and mapping tori of diffeomorphisms $\phi\colon \Sigma \to \Sigma$ of Riemann surfaces. The first two of these are related to the *global Langlands program*, and we refer to the "global arithmetic setting."
- Here are some two-dimensional objects: local fields $F_v/\mathbb{Q}_p$ and their spectra, which are lik punctured discs; and $\mathbb{F}_q((t))$, which is also sort of a punctured disc. This is the setting of the *local Langlands program* in the "local arithmetic setting". There are two more kinds of 2-dimensional objects: a curve over an algebraically closed field of positive characteristic $\overline{C}/\overline{\mathbb{F}}_q$, or a closed Riemann surface $\Sigma$. These latter two objects form the "global geometric setting."
- One-dimensional objects: $\operatorname{Spec}\mathbb{F}_q$ and $\operatorname{Spec}\mathbb{C}((t))$ both are analogues of circles. The latter is a punctured disc, so not exactly one-dimensional, but it's close enough to be useful; it is the "local geometric setting."
- And lastly, zero-dimensional objects: $\operatorname{Spec}\overline{\mathbb{F}}_q$ and $\operatorname{Spec}\mathbb{C}$.

One major theme in this class is to apply a 4d topological field theory to these objects. If you complain that there aren't any 4-manifolds, that's a good question, but we will only consider a few 4-manifolds, such as products of 3-manifolds with circles or more generally mapping tori.

The Langlands program is an equivalence of 4d "arithmetic topological field theories." Arithmetic TFTs are not an entirely well-defined object, but we have much of the data that such a definition would need. We pick a group $G$ and get a dual group $G^\vee$; then the two arithmetic TFTs are called $\mathcal{A}_G$ and $\mathcal{B}_{G^\vee}$; $\mathcal{A}_G$ is called the *automorphic* or *magnetic side*, and $\mathcal{B}_{G^\vee}$ is called the *spectral* or *electric side*. There is a sense in which this is a 4-dimensional version of mirror symmetry (which is usually a story about 2d QFT). These two TFTs $\mathcal{A}_G$ and $\mathcal{B} + G^\vee$ are fully extended, in that they assign higher-categorical objects to lower-dimensional objects. That is, we will be able to assign things to two-, one-, and maybe zero-dimensional objects in the above dictionary: a two-manifold gets a category, a 1-manifold gets a 2-category, and if you're very ambitious, a 0-manifold gets a 3-category.

$\mathcal{A}_G$ is a machine for taking a 3-manifold $M$ and attaching a vector space $\mathcal{A}_G(M)$, which we will see is built from functions on arithmetic locally symmetric spaces. For example, $\operatorname{Spec}\mathcal{O}_F$ gets some sort of functions on $\mathcal{M}_{G,F}$. This is a large amount of structure, and one advantage is that it will explain some of the weird properties of modular forms. We will also spend some time on the $\mathcal{B}$-side, which is easier to describe.

There's an interesting tradeoff involving dimension, category number, and difficulty: making sense of what these arithmetic TFTs assign to 4-manifolds is very difficult: there are infinities and difficult renormalizations to deal with, and analyis that is beyond the scope of the course. Vector spaces are nicer and easier to make, but 3-manifolds are difficult. Dimension 2 is the sweet spot: categories aren't that bad, and 2-manifolds are pretty tractable. By the time we get to 1-manifolds, we have to work the category theory harder, and understanding what happens in dimension 0 is almost entirely open. We will not solve this open question in this class.

Now a TFT has additional structure: you can take a manifold with some additional structure, called defects, and assign algebraic data to this too. These bells and whistles line up very nicely on the arithmetic and topological sides: for example, number theorists will tell you the importance of allowing ramification/congruence subgroups in defining your arithmetic locally symmetric spaces. Under the arithmetic topology dictionary, this corresponds to studying what your TFT assigns to a 3-manifold with an embedded link with a label. In physics language, the link is a *codimension 2 defect*. The additional data of the link gets the modified vector space built using the congruence subgroup.

The large commutative algebra acting on the space of functions on $\mathcal{M}$ contains elements called *Hecke operators*, and these correspond to codimension 3 defects, or *line operators*. Physics-wise, you can think of these as "creating magnetic monopoles." And periods of automorphic forms correspond to boundary conditions, which are a codimension 1 phenomenon. This is related to recent work and work in progress of the professor!

Finally, there is Langlands functoriality, which also fits into this picture: more than just boundaries, there are codimension 1 phenomena called *domain walls*, which you can think of as an interface between two regions on a manifold which have two different theories on them.

So to summarize this, all the bells and whistles in the theory of automorphic forms belong to this QFT story.

**The $\mathcal{B}$-side.** Here we've taken the theory of automorphic forms and passed it through a prism to decompose it into "colors" related to Galois representations. Number-theoretically, this side is very very hard, because Galois groups of number fields are complicated, and the $\mathcal{A}$-side is often used to gain information about the $\mathcal{B}$-side. Geometrically, fundamental groups of Riemann surfaces are much easier, so the $\mathcal{B}$-side is used to learn about the $\mathcal{A}$-side.

But at least the $\mathcal{B}$-side is easier to state: we study the algebraic geometry of spaces of Galois representations; the $\mathcal{A}$-side has to do with the topology of the arithmetic locally symmetric space $\mathcal{M}$, by contrast. Geometrically, we might fix a manifold $M$ and consider the space $\mathrm{Loc}_n(M)$ of representations $\pi_1(M) \to \mathrm{GL}_n(\mathbb{C})$. These are nice objects, called *character varieties*, and you can study the algebra of functions on them. You also don't just have to restrict to $\mathrm{GL}_n(\mathbb{C})$: we in particular care about $\mathrm{Loc}_{G^\vee}(M) \coloneqq \{\pi_1(M) \to G^\vee\}$. The vector space that $\mathcal{B}_{G^\vee}$ assigns to a 3-manifold is the vector space of functions on $\mathrm{Loc}_{G^\vee}(M)$, and in fact defining this theory as an extended TFT is considerably easier than for the $\mathcal{A}$-side. Back on the arithmetic side, $\pi_1(M)$ is analogous to a Galois group, so in the arithmetic setting, we are looking at varieties of Galois representations.

The conjectured equivalence of these two (arithmetic) TFTs is something amazing: the huge amount of structure on the $\mathcal{A}$-side is equivalent to the simpler-to-define $\mathcal{B}$-side, and all of the structure passes back and forth. But "conjectured" is a very big word here: in both the arithmetic and geometric settings, there's a lot left to do, and even to define, to make these analogies precise. In the geometric setting, more is known, but there's still plenty of work in progress, including work of the professor, Arinkin, Gaitsgory, Raskin, and many more. The number field story is the work of Lafforgue and many others, but not a lot of this is proven.

The dictionary is not just nice to look at: you can use work done in the geometric setting to learn about what you should be working towards in the arithmetic setting, for example.

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