Eta Invariants for G-Spaces

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Introduction. Let X^{4k} be a compact oriented Riemannian manifold of dimension $4k$. We suppose that X may have non-empty boundary Y^{4k-1}, of dimension $4k - 1$. Moreover we will assume that the metric of X is a product near its boundary Y. By considering the cup product

$$H^{2k}(X, Y) \times H^{2k}(X, Y) \to H^{4k}(X, Y)$$

and evaluating on the fundamental cycle $[X, Y] \in H_{4k}(X, Y)$ one obtains a symmetric bilinear form on $H^{2k}(X, Y)$. The signature of this bilinear form will be denoted by $\mathrm{Sign}(X)$. Now suppose that $L_k(p_1(\Omega), p_2(\Omega), \cdots, p_k(\Omega))$ is the L_k polynomial of Hirzebruch in the Pontriagin forms $p_i(\Omega)$, $1 \leq i \leq k$, of X. The Hirzebruch signature theorem [13] states that the difference

$$\int_X L_k(p_1(\Omega), p_2(\Omega), \cdots, p_k(\Omega)) - \mathrm{Sign}(X) \tag{I.1}$$

vanishes if Y is empty. If X is not a closed manifold then the difference (I.1) is not necessarily zero. However it follows from the Hirzebruch signature theorem applied to the double of X that the quantity given by (I.1) depends only on Y and its Riemannian metric.

In their paper [3], Atiyah, Patodi, and Singer established the surprising result that the invariants (I.1) are in fact spectral invariants of the boundary Y. If $\Lambda(Y)$ is the exterior algebra of Y, then let $\Lambda^{\mathrm{ev}}(Y)$ be the subspace of the exterior algebra which consists of forms having type $2p$ for some $0 \leq p \leq 2k - 1$. There is a first order self-adjoint elliptic operator $B^{\mathrm{ev}} : \Lambda^{\mathrm{ev}}(Y) \to \Lambda^{\mathrm{ev}}(Y)$; B^{ev} is naturally defined in terms of the Riemannian geometry of Y. Now B^{ev} has pure point spectrum consisting of eigenvalues λ with multiplicity $\dim(\lambda)$. The spectral function

$$\eta(s, Y) = \sum_{\lambda \neq 0} (\mathrm{sign}\,\lambda)(\dim \lambda)\,|\lambda|^{-s}$$

converges for $\mathrm{Re}(s)$ sufficiently large and has a meromorphic continuation to the entire complex s-plane. Moreover, $\eta(0, Y)$ is finite. The central result of [3] is that

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$$\eta(0, Y) = \int_X L_k(p_1(\Omega), p_2(\Omega), \cdots, p_k(\Omega)) - \operatorname{sign}(X). \tag{I.2}$$

The formula (I.2) generalizes easily to the signature complex with coefficients in a bundle.

Now suppose that G is a compact Lie group which acts isometrically on X. Then each $g \in G$ preserves the boundary Y and the given G-action is a product near the boundary. Let $g \in G$, then the fixed point set Ω of g is the disjoint union of compact connected totally geodesic submanifolds N of X. The normal bundle $TN^\perp$ of any component $N \subset X$ decomposes as

$$TN^\perp = TN^\perp(-1) \oplus TN^\perp(\theta_1) \oplus \cdots \oplus TN^\perp(\theta_s)$$

where the differential of g acts on $TN^\perp(-1)$ *via* multiplication by -1 and on $TN^\perp(\theta_i)$ *via* rotation through the angle θ_i, $0 < \theta_i < \pi$. For each $N \in \Omega$, one has $\partial N \subset Y$. Now, corresponding to the formula (I.1), we consider the difference

$$\sum_{N \in \Omega} \int_N 2^{(n-m)/2} *_N \Big[\prod_i \left(\sqrt{-1} \tan(\theta_i/2) \right)^{-c(\theta_i)} \mathscr{L}(N) \mathscr{L}(TN(-1))^{-1} e(TN(-1)) \prod_i \mathscr{M}^{\theta_i}(TN(\theta_i)) \Big](a)da - \operatorname{Sign}(g, X). \tag{I.3}$$

Our notation will be explained in Section 2. The important point is that the integrands on the right are polynomials in the characteristic forms of the bundles $TN^\perp(-1)$, $TN^\perp(\theta_i)$. If X is a closed manifold then the G-signature theorem of Atiyah-Singer states that the difference (I.3) is zero. A new proof of this result using the heat equation was given in [10], [11].

We will apply the methods developed in [3], [10], and [11] to extend the G-signature theorem to manifolds with boundary.

Consider the group G acting isometrically on Y and suppose $g \in G$. Then the map defined by g on sections of $\Lambda^{\mathrm{ev}}(Y)$ commutes with B^{ev}. This induces linear maps $g^\#_\lambda$ on each eigenspace, with eigenvalue λ, of B^{ev}. The spectral function

$$\eta_g(x, Y) = \sum_{\lambda \neq 0} (\operatorname{sign} \lambda) \operatorname{Tr}(g^\#_\lambda) |\lambda|^{-s}$$

converges for $\operatorname{Re}(s)$ sufficiently large and has a meromorphic continuation to the entire complex s-plane. Moreover,

$$\eta_g(0, Y) = \sum_{N \in \Omega} \int_N 2^{(n-m)/2} *_N \Big[\prod_i \left(\sqrt{-1} \tan(\theta_i/2) \right)^{-c(\theta_i)} \mathscr{L}(N) \mathscr{L}(TN^\perp(-1))^{-1} e(TN^\perp(-1)) \prod_i \mathscr{M}^{\theta_i}(TN^\perp(\theta_i)) \Big](a)da - \operatorname{sign}(g, X). \tag{I.4}$$

The formula (I.4) generalizes easily to the signature complex with coefficients in a bundle. Similar results also hold for the other classical elliptic complexes.

Suppose that $\hat{Y} \to Y$ is a regular covering space with finite covering group G of order $|G|$. For each irreducible unitary representation α of G, one has an associated flat vector bundle $E_\alpha \to Y$. The invariants $\eta_\alpha(0, Y)$ are defined using the spectrum of the operator $B_\alpha^{\mathrm{ev}} : \Lambda^{\mathrm{ev}}(Y) \otimes E_\alpha \to \Lambda^{\mathrm{ev}}(Y) \otimes E_\alpha$. These invariants were studied in [4] and are closely related to the G eta invariants of the cover $\hat{Y}$. In particular,

$$\eta_\alpha(0, Y) = \frac{1}{|G|} \sum_{g \in G} \eta_g(0, \hat{Y})\chi_\alpha(g) \tag{I.5}$$

where χ_α is the character of α. This gives an alternative and perhaps more direct approach to the results of [4, pp. 408–413].

For $g \neq 1$, the eta invariants $\eta_g(0, \hat{Y})$ are topological invariants and are quite computable in some cases. Therefore formula (I.5) may sometimes be used to calculate the $\eta_\alpha(0, Y)$ if Y is covered by a manifold $\hat{Y}$ with $\eta(0, \hat{Y}) = 0$. This includes lens spaces and flat manifolds. There are interesting applications to the problem of listening to finite group actions on homotopy spheres.

If we take α to be the trivial one-dimensional representation in (I.5) then we have

$$\eta(0, \hat{Y}) - |G|\eta(0, Y) = -\sum_{g \neq 1} \eta_g(0, \hat{Y}) \tag{I.6}$$

where the sum on the right is over group elements $g \in G$, $g \neq 1$.

It was shown in [8] that

$$(2\mu_k)[\eta(0, \hat{Y}) - |G|\eta(0, Y)]$$

is an integer where Y has dimension $4k - 1$ and μ_k is the denominator of the Hirzebruch L_k-polynomial. Therefore

$$(2\mu_k) \sum_{g \neq 1} \eta_g(0, \hat{Y}) \tag{I.7}$$

is an integer. If $\hat{Y}$ is the unit sphere bundle of some Riemannian vector bundle over a manifold N and g acts fiberwise then the invariants $\eta_g(0, \hat{Y})$ may be computed using (I.3). In that case, formula (I.7) gives an integrality theorem for finite group actions which is closely related to a theorem of Zagier [18] for cyclic groups.

1. A G-index formula for manifolds with boundary. Let Y be a compact Riemannian manifold without boundary and E a vector bundle over Y. Denote by dy the smooth measure induced on Y by its Riemannian metric. Assume that E is endowed with a smooth inner product. We will consider a first order elliptic operator A which is self-adjoint with respect to the induced inner product on sections. Then A has pure point spectrum with real eigenvalues λ. The corresponding eigenfunctions ϕ_λ are normalized by the condition

$$\int_Y \langle \phi_\lambda(y), \phi_\lambda(y) \rangle \, dy = 1.$$

P will denote the projection of $C^\infty(Y, E)$ onto the space spanned by the ϕ_λ for $\lambda \geq 0$.

Suppose that G is a subgroup of the isometry group of Y. Assume that G lifts to an action on the vector bundle E for which each $g \in G$ commutes with the given operator $A : C^\infty(Y, E) \to C^\infty(Y, E)$. Then g induces maps $g_\lambda^\#$ on each eigenspace of A. One may define the eta function of g by the formula

$$\eta_g(s) = \sum_{\lambda \neq 0} (\operatorname{sign} \lambda) \operatorname{Tr}(g_\lambda^\#) |\lambda|^{-s}.$$

Here $\operatorname{sign} \lambda$ is defined by $\operatorname{sign} \lambda = 1$ if $\lambda \geq 0$ and $\operatorname{sign} \lambda = -1$ if $\lambda < 0$. Standard methods show that the series on the right converges for $\operatorname{Re}(s)$ sufficiently large and has a meromorphic continuation to the entire complex s-plane. We will be interested in the behavior of this meromorphic continuation near $s = 0$.

Now suppose that Y is the boundary of a compact Riemannian manifold X whose metric is a product near the boundary Y. This means that one is given a collar neighborhood $Y \times I$ of Y in X for which the metric on X coincides with the product metric on $Y \times I$. The induced measure dx is then a product $dx = dydu$ on the collar neighborhood $Y \times I$. Assume that the vector bundle $E \to Y$ is the restriction of a vector bundle, also denoted E, with smooth inner product over X. Let F be another vector bundle over X with smooth inner product.

One is interested in studying first order elliptic differential operators $D : C^\infty(X, E) \to C^\infty(X, F)$ which decompose near the boundary as

$$D = \sigma\left(\frac{\partial}{\partial u} + A\right)$$

with respect to the splitting $x = (y, u)$ near Y. Here σ is the vector bundle isomorphism given by the symbol of D. $A : C^\infty(Y, E) \to C^\infty(Y, E)$ is the self-adjoint operator considered above. In particular, A is independent of u.

We assume that the action of G on Y extends to an action on X which preserves the metric. Then near the boundary the action of G coincides with the product of the given action on Y with the trivial action on I. Assume that the action of G lifts to the vector bundles E and F over X, preserving the inner product, and that the given liftings commute with $D : C^\infty(X, E) \to C^\infty(X, F)$ and are products near the boundary.

The fixed point set Ω of each $g \in G$ is the disjoint union of compact submanifolds N of dimension n. For each $N \in \Omega$ one has $\partial N \subset \partial X = Y$. Moreover, near the boundary Y there are splittings $N = \partial N \times I$. The collection of boundaries of components of Ω are the components of the fixed point set of g acting on Y.

One is particularly interested in studying the operator $D : C^\infty(X, E; P) \to C^\infty(X, F)$ where $C^\infty(X, E; P)$ is the subspace of $C^\infty(X, E)$ consisting of functions satisfying the boundary conditions $Pf(\cdot, 0) = 0$. It is well-known [3, p. 55] that $\operatorname{Ker} D$ is finite dimensional. Similarly $\operatorname{Ker} D^*$ is finite dimensional for $D^* : C^\infty(X, F; 1 - P) \to C^\infty(X, E)$ satisfying the adjoint boundary condition $(1 - P)f(\cdot, 0) = 0$. Since each $g \in G$ preserves $\operatorname{Ker} D$ and $\operatorname{Ker} D^*$ there is a well-defined G-index

$$\operatorname{index}(D, g) = \operatorname{Tr}(g|\operatorname{Ker} D) - \operatorname{Tr}(g|\operatorname{Ker} D^*).$$

Since the action of G on X is a product near the boundary Y of X, there is an induced action by isometries on the double $M = X \cup X$ of X. The vector bundles E, F extend to bundles over M, by doubling, so one may consider $D : C^\infty(M, E) \to C^\infty(M, F)$ and its adjoint $D^* : C^\infty(M, F) \to C^\infty(M, E)$ with respect to the induced inner products. Denote $\Delta_1 = D^*D$, $\Delta_2 = DD^*$ and $\exp(-t\Delta_1)$, $\exp(-t\Delta_2)$ the fundamental solutions to the associated heat equations. Then $\exp(-t\Delta_1)$, $\exp(-t\Delta_2)$ are given by smoothing kernels $f_1(x, w, t)$, $f_2(x, w, t)$ for $(x, w, t) \in M \times M \times R^+$, where R^+ is the non-negative half of the real line.

Now it is well-known [10] that there exist asymptotic expansions:

$$(1.1)\qquad \begin{aligned} \int_X \operatorname{Tr}(g^*f_1(x, gx, t))dx &\sim \sum_{\in \Omega} (4\pi t)^{-n/2} \sum_{i=0}^{\infty} t^i \int_N b_{1,i}(g, a)da \\ \int_X \operatorname{Tr}(g^*f_2(x, gx, t))dx &\sim \sum_{N \in \Omega} (4\pi t)^{-n/2} \sum_{i=0}^{\infty} t^i \int_N b_{2,i}(g, a)da. \end{aligned}$$

Here da is the volume element induced by the Riemannian metric on the totally geodesic submanifold N, of dimension n.

After these preliminary remarks one may state:

Theorem 1.2. *Let X be a compact Riemannian manifold with boundary Y whose metric is a product near the boundary. Suppose that D ($C^\infty(X, E) \to C^\infty(X, F)$ is a linear first order elliptic differential operator. One assumes that near the boundary Y of X, D takes the special form*

$$D = \sigma\left(\frac{\partial}{\partial u} + A\right)$$

where u is the inward normal coordinate and σ is a bundle isomorphism $E \to F$. Here $A : C^\infty(Y; E) \to C^\infty(Y; E)$ is a first order self-adjoint elliptic differential operator. Let $C^\infty(X, E; P)$ denote the space of sections of E satisfying the boundary condition $Pf(\cdot, 0) = 0$ where P is the spectral projection corresponding to the eigenvalues of A which are greater than or equal to zero.

Suppose that G is a subgroup of the isometry group of X and let $g \in G$. Assume that the action of G lifts to E, F, product near the boundary, and that the induced map on sections commutes with D. Then $D : C^\infty(X, E) \to C^\infty(X, F)$ has finite G-index given by

$$(1.2)\qquad \operatorname{index}(D, g) = \sum_{N \in \Omega} (4\pi)^{-n/2} \int_N [b_{1,n/2}(g, a) - b_{2,n/2}(g, a)]da - \frac{1}{2} (h_g + \eta_g(0))$$

where h_g is the trace of the map induced by g on the kernel of $A : C^\infty(Y, E) \to C^\infty(Y, E)$. In particular $\eta_g(0)$ is finite.

Proof. The proof is an easy generalization of the corresponding proof without any G-action [3]. One simply follows through the proof in [3] checking the appropriate additional details. There are two main steps:

(i) Consider the non-compact manifold $Y \times R^+$ which is the product of Y with the real half line $u \geq 0$. E lifts to a vector bundle, also denoted by E, over $Y \times R^+$. The operator $\bar{D} = \partial/\partial u + A$ acts on sections $f(y, u)$ of E. We impose the boundary condition $Pf(\cdot, 0) = 0$ on $\bar{D}$. Similarly, one may consider the formal adjoint $\bar{D}^* = -\partial/\partial u + A$ of $\bar{D}$. We impose the adjoint boundary condition $(1 - P)f(\cdot, 0) = 0$ on $\bar{D}^*$.

If $\bar{\mathscr{D}}$, $\bar{\mathscr{D}}^*$ denote the closures of $\bar{D}$, $\bar{D}^*$ acting on $L^2(Y \times R^+)$ with the given boundary conditions, then the associated Laplace operators are defined by

$$\Delta_1 = \bar{\mathscr{D}}^*\bar{\mathscr{D}} \qquad \Delta_2 = \bar{\mathscr{D}}\bar{\mathscr{D}}^*.$$

$\bar{\Delta}_1$ is the operator given by $\dfrac{-\partial^2}{\partial u^2} + A^2$ with the boundary conditions

$$(1.3) \qquad Pf(\cdot, 0) = 0 \qquad (1 - P)\left(\frac{\partial f}{\partial u} + Af\right)_{u=0} = 0.$$

Similarly $\bar{\Delta}_2$ is the operator given by $\dfrac{-\partial^2}{\partial u^2} + A^2$ with the boundary conditions

$$(1.4) \qquad (1 - P)f(\cdot, 0) = 0 \qquad P\left(\frac{\partial f}{\partial u} + Af\right)_{u=0} = 0.$$

It is of basic importance to study the fundamental solutions $\exp(-t\Delta_1)$, $\exp(-t\Delta_2)$ of the associated heat equations with boundary conditions given by (1.3), (1.4), respectively.

These fundamental solutions are given by smoothing kernels $K_1((y, u), (z, v), t)$ and $K_2((y, u), (z, v), t)$. One may write

$$K_1((y, u), (z, v)t) = \sum_\lambda a_\lambda(u, v, t)\phi_\lambda(y) \otimes \phi_\lambda(z)$$

$$K_2((y, u), (z, v)t) = \sum_\lambda b_\lambda(u, v, t)\phi_\lambda(y) \otimes \phi_\lambda(z)$$

where ϕ_λ are the eigenfunctions of A corresponding to the eigenvalue λ. The functions a_λ, b_λ may be determined by solving the associated heat equation problem corresponding to each eigenspace of A. Explicit formulas for the a_λ, b_λ are given in [3, p. 52].

We let G act on $Y \times R^+$ by taking the product of the given action on Y with the trivial action on R^+. Recall that g induces linear maps, denoted $g^\#_\lambda$, on the eigenspaces of A. Now

$$\mathrm{Tr}(g^*K_1((y, u), (gy, u), t)) = \sum_\lambda a_\lambda(u, u, t) < \phi_\lambda(y), g^*\phi_\lambda(gy)\rangle$$

$$\mathrm{Tr}(g^*K_2((y, u), (gy, u), t)) = \sum_\lambda b_\lambda(u, u, t) \langle \phi_\lambda(y), g^*\phi_\lambda(gy)\rangle .$$

On the left-hand side of each equation, g^* acts with respect to the second argument (gy, u) of K_1, K_2.

One defines

$$\begin{aligned} K(g, t) &= \int_{Y \times R^+} [\mathrm{Tr}(g^*K_1((y, u), (gy, u), t)) \\ &\qquad - \mathrm{Tr}(g^*K_2((y, u), (gy, u), t))]dydu \\ &= \sum \mathrm{Tr}(g^\#_\lambda)\left[\int_{R^+} (a_\lambda(u, u, t) - b_\lambda(u, u, t))du\right] \end{aligned} \tag{1.5}$$

where $\mathrm{Tr}(g^\#_\lambda)$ denotes the trace of the map induced by g on the λ'th eigenspace of A.

It is shown in [3, p. 53] that one has

$$\int_{R^+} [a_\lambda(u, u, t) - b_\lambda(u, u, t)]du = \frac{-\operatorname{sign}\lambda}{2} \operatorname{erfc}\left(|\lambda| \middle/ \sqrt{t}\right)$$

where $\operatorname{erfc}(\alpha)$ is the complementary error function

$$\operatorname{erfc}(\alpha) = \frac{2}{\sqrt{\pi}} \int_\alpha^\infty \exp(-\beta^2)d\beta.$$

Therefore

$$K(g, t) = -\sum_\lambda \frac{\operatorname{sign}\lambda}{2} \mathrm{Tr}(g^\#_\lambda)\operatorname{erfc}\left(|\lambda| \middle/ \sqrt{t}\right).$$

By definition one has $\operatorname{sign}\lambda = 1$ if $\lambda \geq 0$ and $\operatorname{sign}\lambda = -1$ if $\lambda < 0$. Now $K(g, t) \to -1/2\mathrm{Tr}(g^\#_0)$ as $t \to \infty$. The difference $K(g, t) + 1/2\,\mathrm{Tr}(g^\#_0)$ is in fact exponentially small as $t \to \infty$. Moreover, as shown in [3, p. 53] one has $K(g, t) < Ct^{-i/2}$ as $t \downarrow 0$, where i is the dimension of Y. In fact, it is easy to see that $K(g, t) < Ct^{-j/2}$ as $t \downarrow 0$, where j i the largest dimension for a component of the fixed point set of g.

Hence

$$\int_0^\infty \left[K(g, t) + \frac{1}{2} \mathrm{Tr}(g^\#_0)\right]t^{s-1}dt$$

converges for $\mathrm{Re}(s)$ sufficiently large.

Let h_g denote $\mathrm{Tr}(g^\#_0)$. Then integrating by parts one obtains

$$\int_0^\infty \left[K(g, t) + \frac{1}{2} h_g\right] t^{s-1} dt = \frac{\Gamma(s + 1/2)}{2s\sqrt{\pi}} \sum_{\lambda \neq 0} \frac{\operatorname{sign}(\lambda)}{|\lambda|^{2s}} \operatorname{Tr}(g_\lambda^\#)$$
$$= -\frac{\Gamma(s + 1/2)}{2s\sqrt{\pi}} \eta_g(2s).$$

The last equality is just the definition of the eta function η_g.

Consequently, if $K(g, t)$ has an asymptotic expansion as $t \downarrow 0$:

$$K(g, t) \sim \sum_{k \geq -n} c(g) t^{-k/2} \tag{1.6}$$

then

$$\eta_g(2s) = \frac{-2s\sqrt{\pi}}{\Gamma(s + 1/2)} \left[\frac{h_g}{2s} + \sum_{k=-n}^{N} \frac{c_k(g)}{\frac{1}{2} k + s} + \theta_N(s) \right]$$

where $\theta_N(s)$ is holomorphic for $\operatorname{Re}(s) > -1/2(N + 1)$.

In particular, under the assumption (1.6), η_g is holomorphic at $s = 0$ and its value at $s = 0$ is given by

$$\eta_g(0) = -(2c_0(g) + h_g). \tag{1.7}$$

It is important to note [3, p. 53] that the asymptotic behavior of $K(g, t)$ as $t \downarrow 0$ would be unchanged if we had integrated over $Y \times [0, \delta]$ for some $\delta > 0$, instead of over $Y \times R^+$ in the definition (1.4) of $K(g, t)$.

(ii) Now consider D acting on $L^2(X, E)$ with the boundary conditions $Pf(\cdot, 0) = 0$ and D^* acting on $L^2(X, F)$ with the adjoint boundary conditions $(1 - P)f(\cdot, 0) = 0$. Let $\mathscr{D}$, $\mathscr{D}^*$ be the closures of D, D^* acting on $L^2(X)$. The associated Laplace operators are

$$\Delta_1 = \mathscr{D}^*\mathscr{D} \qquad \Delta_2 = \mathscr{D}\mathscr{D}^*$$

with boundary conditions given by (1.2), (1.3), respectively. Δ_1 and Δ_2 have pure point spectrum.

The fundamental solutions $\exp(-t\Delta_1)$, $\exp(-t\Delta_2)$ of the associated heat equations are given by smoothing kernels $e_1(w, x, t)$, $e_2(w, x, t)$ respectively, where $(w, x, t) \in X \times X \times R^+$. Parametrices for the e_i, $i = 1, 2$ may be obtained by patching together with K_i with interior parametrices f_i given by the fundamental solutions of the corresponding heat equations on the double of X.

If $g \in G$ then g induces linear maps on each of the eigenspaces of Δ_1 and Δ_2. If $\mu \neq 0$, the map $\phi \to D\phi$ defines an isomorphism from the μ'th eigenspace of Δ_1 onto the μ'th eigenspace of Δ_2. This isomorphism commutes with the action of each $g \in G$. Consequently

$$\begin{aligned}\operatorname{index}(D, g) &= \operatorname{Tr}(g|\operatorname{Ker} D) - \operatorname{Tr}(g|\operatorname{Ker} D^*)\\ &= \int_X [\operatorname{Tr}(g^*e_1(x, gx, t)) - \operatorname{Tr}(g^*e_2(x, gx, t))]dx\\ &\sim \int_X [\operatorname{Tr}(g^*f_1(x, gx, t)) - \operatorname{Tr}(g^*f_2(x, gx, t))]dx\\ &\quad + \int_{Y \times [0, \delta]} [\operatorname{Tr}(g^*K_1(x, gx, t)) - \operatorname{Tr}(g^*K_2(x, gx, t))]dx\end{aligned}$$

where $\sim$ means asymptotically equivalent as $t \downarrow 0$.

Furthermore

$$\operatorname{index}(D, g) \sim \sum_{N \in \Omega} (4\pi t)^{-n/2} \sum_{i=0}^{\infty} t^i \int_N [b_{1,i}(g, a) - b_{2,i}(g, a)]da + K(g, t)$$

by (1.1) and (1.4).

Therefore $K(g, t)$ has an asymptotic expansion

$$K(g, t) \sim \operatorname{index}(D, g) - \sum_{N \in \Omega} (4\pi t)^{-n/2} \sum_{i=0}^{\infty} t^i \int_N [b_{1,i}(g, a) - b_{2,i}(g, a)]da$$

and one may apply the arguments following (1.5). This gives the desired formula

$$\operatorname{index}(D, g) = \sum_{N \in \Omega} (4\pi)^{-n/2} \int_N [b_{1,n/2}(g, a) - b_{2,n/2}(g, a)]da - \frac{1}{2} (h_g + \eta_g(0)).$$

2. The signature complex. For each of the various classical elliptic complexes, it is interesting to give an appropriate interpretation of the terms in formula (1.2). The main simplification is due to the fact that the local formulas $b_{1,n/2}(g, a) - b_{2,n/2}(g, a)$ may be identified as polynomial expressions in suitable characteristic forms [11], [12].

We will discuss the signature complex in detail. In this case, Theorem 1.2 implies a generalization of the G-signature theorem of Atiyah-Singer [12], for closed compact manifolds, which applies to the case of manifolds with boundary.

Let X be a compact oriented manifold with boundary Y. Suppose that X has even dimension $d = 2\ell$. The metric on X induces metrics on the bundles of i-forms $\Lambda^i(T^*X) \otimes C$ and hence, by integration over X, inner products on the space of sections of $\Lambda^i(T^*X) \otimes C$.

We denote by ∂ the exterior derivative $\partial : \Lambda^i(T^*X) \otimes C \to \Lambda^{i+1}(T^*X) \otimes C$ and ∂^* the adjoint of ∂ with respect to the induced inner products. The operator $\Delta = \partial\partial^* + \partial^*\partial$ is the Laplace operator of Hodge theory. Now consider the first order operator $D = \partial + \partial^*$. It is formally self-adjoint and we have $\Delta = D^*D = D^2$. One defines an operator τ by $\tau(\alpha) = (\sqrt{-1})^{p(p-1)+\ell *}$ for $\alpha \in \Lambda^p(T^*X) \otimes C$ where $*$ is the Hodge star operator. Then $\tau^2 = 1$ and we may decompose the exterior algebra $\Lambda(T^*X) \otimes C$ of X as $\Lambda(T^*X) \otimes C = \Lambda^+(T^*X) + \Lambda^-(T^*X)$ where $\Lambda^+(T^*X)$, $\Lambda^-(T^*X)$ is the $+1$, -1 eigenspace of τ. The signature complex of X is the elliptic complex

$$0 \to \Lambda^+(T^*X) \xrightarrow{D} \Lambda^-(T^*X) \to 0.$$

This is well-defined since $D\tau = -\tau D$.

Now suppose that the metric of X is a product near its boundary Y. Let $Y \times I$ be a neighborhood of Y in X for which the metric on X coincides with the product metric of $Y \times I$. If $\pi_1 : Y \times I \to Y$ is the projection onto the first factor then $\pi_1^*\Lambda(T^*Y)$ will denote the lift of the vector bundle $\Lambda(T^*Y)$ to $Y \times I$. As shown in [3, p. 63] we may identify $\Lambda^+(T^*X)$ and $\Lambda^-(T^*X)$ with $\pi_1^*\Lambda(T^*Y)$ on the collar neighborhood $Y \times I$. For an appropriately chosen isomorphism $\sigma : \Lambda^+(T^*X) \to \Lambda^-(T^*X)$ on $Y \times I$ the operator D is of the form

$$D = \sigma\left(\frac{\partial}{\partial u} + B\right)$$

where $B : \Lambda(T^*Y) \to \Lambda(T^*Y)$ is a first order self-adjoint elliptic differential operator. B preserves the parity of forms and the map $\phi \to (-1)^p{*}\phi$ for $\phi \in \Lambda^p(T^*Y)$ defines an isomorphism between the action of B on even and odd forms. We write $B = B^{\text{ev}} \oplus B^{\text{od}}$ where B^{ev}, B^{od} denotes the operator B acting on even, odd forms.

Now suppose that the compact Lie group G is a subgroup of the isometry group of X and that each $g \in G$ is orientation preserving. Then each g induces an orientation preserving isometry $g : Y \to Y$. Furthermore, the action of G on X is a product near the boundary Y, relative to the given splitting $Y \times I$. Moreover, the given G-action lifts to the bundles $\Lambda^+(T^*X)$, $\Lambda^-(T^*X)$, *via* the pull-back on forms. The map induced on sections by each $g \in G$ will commute with D.

We will now restrict our attention to a particular $g \in G$. The fixed point set Ω of g is the disjoint union of connected submanifolds N of dimension n. For any N, the eigenvalues of the induced map $C : TN^\perp \to TN^\perp$, given by the differential of g, on the normal bundle $TN^\perp$ of N are constant. We may decompose

$$TN^\perp = TN^\perp(-1) \oplus TN^\perp(\theta_1) \oplus TN^\perp(\theta_2) \oplus \cdots \oplus TN^\perp(\theta_s);$$

$\theta_i \neq \pi$. The endomorphism C acts on $TN^\perp(-1)$ *via* multiplication by -1. The bundles $TN^\perp(\theta_i)$ admit the structure of complex vector bundles where C acts *via* multiplication by $\exp(\sqrt{-1}\,\theta_i)$. It is useful to observe that each component N, of the fixed point set of g, has even dimension n. In fact, the spaces $TN^\perp(\theta_i)$ are even-dimensional and the determinant of C acting on them is positive. This forces $TN^\perp(-1)$ to be even-dimensional. Thus N is even-dimensional.

In the special case of the signature complex, the local formula $b_{1,n/2}(g, a) - b_{2,n/2}(g, a)$ may be identified with the integrand of the Atiyah-Singer g-signature theorem [11]. The Atiyah-Singer integrand is a polynomial in the characteristic forms of the bundles TN, $TN^\perp(-1)$, $TN^\perp(\theta_i)$, $1 \le i \le s$.

We denote

$$\mathcal{L}(N) = \prod_j \frac{x_j/2}{\tanh(x_j/2)}$$

where the Pontriagin classes of N are the elementary symmetric functions in the x_j^2.

Let m be the (even) dimension of $TN^{\perp}(-1)$. Then one may write

$$2^{-m/2}\mathcal{L}(TN^{\perp}(-1)^{-1}e(TN^{\perp}(-1)) = \prod_j \tanh(x_j/2)$$

where the Pontriagin forms of $TN^{\perp}(-1)$ are the elementary symmetric functions in the x_j^2 and the Euler form is the product of the x_j's. Since $TN^{\perp}(-1)$ may not be globally orientable, $e(TN^{\perp}(-1))$ must be interpreted as the Euler form relative to some local choice of orientation.

Finally we set

$$\mathcal{M}^{\theta_i} = \prod_j \frac{\tanh(\sqrt{-1}\ \theta_i/2)}{\tanh\left(\dfrac{x_j + \sqrt{-1}\ \theta_i}{2}\right)}$$

where the elementary symmetric functions of the x_j's are the Chern forms of $TN^{\perp}(\theta_i)$.

One has:

Theorem 2.1. *Let X be a compact oriented Riemannian manifold with boundary Y. Suppose that X has even dimension $d = 2\ell$. Assume that near Y the metric of X is isometric to a product.*

Suppose that the compact Lie group G acts isometrically on X. Fix some $g \in G$, then the fixed point set Ω of g is the disjoint union of compact connected submanifolds N. Some of the N may have boundaries which necessarily lie in Y. For each $N \in \Omega$, the normal bundle of N decomposes as

$$TN^{\perp} = TN^{\perp}(-1) \oplus \sum_i TN^{\perp}(\theta_i). \tag{2.2}$$

Let $\mathcal{L}$, $\mathcal{M}_\theta$, e be the characteristic forms which correspond via the Weil homomorphism to the characteristic classes defined above.

Suppose that sign(g, X) *is the G-signature of the quadratic form on $H^\ell(X, Y)$ which is induced by cup product. We will denote by $\eta_g(0)$ the eta invariant of the operator B^{ev} which was described above. Then*

$$\begin{aligned}\operatorname{sign}(g, X) = \sum_{N\in\Omega}\int_N 2^{(n-m)/2} *_N\Bigg[\Bigg(\prod_i (\sqrt{-1}\ \tan(\theta_i/2))^{-c(\theta_i)}\Bigg) \\ \mathcal{L}(N)\mathcal{L}(TN^{\perp}(-1))^{-1}e(TN^{\perp}(-1)\prod_i \mathcal{M}^{\theta_i}(TN^{\perp}(\theta_i))\Bigg](a)da - \eta_g(0).\end{aligned} \tag{2.3}$$

Here $*_N$ is the Hodge star operator relative to a local choice of orientation for TN. Similarly $e(TN^{\perp}(-1))$ is the Euler form of $TN^{\perp}(-1)$ relative to a local choice of orientation for $TN^{\perp}(-1)$. The orientations of TN, $TN^{\perp}(-1)$ are to be chosen as compatible with the decomposition (2.2).

Proof. By applying Theorem 2.1 to the signature complex, one obtains the formula

$$\operatorname{index}(D, g) = \sum_{N \in \Omega} (4\pi)^{-n/2} \int_N [b_{1,n/2}(g, a) - b_{2,n/2}(g, a)]da - (h_g + \eta_g(0)).$$

Here h_g, $\eta_g(0)$ are computed for the operator B^{ev} of the decomposition $B = B^{\mathrm{ev}} \oplus B^{\mathrm{od}}$ described above. This accounts for the disappearance of a factor of one-half in the second term on the right.

By Theorem 3.4 of [11], we have

$$\begin{aligned}&(4\pi)^{-n/2}[b_{1,n/2}(g, a) - b_{2,n/2}(g, a)]\\ &= 2^{(n-m)/2} *_N\Big[\prod_i \Big(\sqrt{-1}\tan\Big(\frac{\theta_i}{2}\Big)\Big)^{-c(\theta_i)} \mathscr{L}(N)\mathscr{L}(TN^{\perp}(-1))^{-1}\\ &\quad e(TN^{\perp}(-1)) \prod_i \mathscr{M}^{\theta_i}(TN^{\perp}(\theta_i))\Big](a).\end{aligned}$$

Consequently

$$\begin{aligned}&\operatorname{index}(D, g) + h_g\\ &= \sum_{N \in \Omega} \int_N 2^{(n-m)/2} *_N\Big[\prod_i \Big(\sqrt{-1}\tan\Big(\frac{\theta_i}{2}\Big)\Big)^{-c(\theta_i)} \mathscr{L}(N)\mathscr{L}(TN^{\perp}(-1))^{-1}\\ &\quad e(TN^{\perp}(-1)) \prod_i \mathscr{M}^{\theta_i}(TN^{\perp}(\theta_i))\Big](a)da - \eta_g(0).\end{aligned}$$

Thus it suffices to show the equality:

$$\operatorname{sign}(g, X) = \operatorname{index}(D, g) + h_g. \tag{2.4}$$

This is shown in [3, pp. 64–66] for $g = 1$. The main point is to clarify the relationship between harmonic forms satisfying the global boundary conditions (1.2) and the ordinary cohomology groups $H^*(X)$, $H^*(X, Y)$. The more general formula (2.4) follows from the relevant isomorphisms established during that investigation.

Theorem 2.1 easily generalizes to the signature complex wtih coefficients in a Hermitian vector bundle ξ. For a more detailed account of the concepts described below the reader is referred to [2], [12].

Let X be a compact oriented Riemannian manifold with boundary Y. Assume that the metric of X is a product near its boundary Y. If $\xi \to X$ is a Hermitian vector bundle, one defines the signature complex with coefficients in ξ:

$$0 \to \Lambda^+(T^*X) \otimes \xi \xrightarrow{D_\xi} \Lambda^-(T^*X) \otimes \xi \to 0.$$

Near the boundary Y of X one may identify $\Lambda^+(T^*X) \otimes \xi$ and $\Lambda^-(T^*X) \otimes \xi$ with the lift $\pi_1^*\Lambda(T^*Y) \otimes \xi$. Modulo these identifications, we have

$$D_\xi = \frac{\partial}{\partial u} + B_\xi$$

where B_ξ is a self-adjoint first order elliptic differential operator. We write $B_\xi = B_\xi^{\mathrm{ev}} \oplus B_\xi^{\mathrm{od}}$ where B_ξ^{ev}, B_ξ^{od} denotes the operator B_ξ acting on even, odd forms.

Let the compact Lie group G act isometrically on X. We suppose that G lifts to a connection and Hermitian metric preserving morphism of ξ. For each $g \in G$ and component N of the fixed point set of g, we may decompose $\xi|N = \xi(\psi_1) \oplus \xi(\psi_2) \oplus \cdots \oplus \xi(\psi_r)$ where the map induced by g on $\xi(\psi_j)$ is just multiplication by $\exp(\sqrt{-1}\ \psi_j)$.

Denote

$$\mathrm{ch}(\xi|N, g) = \sum_{j=1}^{r} \sum_{i=1}^{e(j)} \exp(\sqrt{-1}\ \psi_j)\exp(x_{ji})$$

where, for each j, the Chern classes of the bundle $\xi(\psi_j)$ with dimension $e(j)$ are the elementary symmetric functions in the x_{ji}, $1 \le i \le e(j)$.

Then one has the following generalization of Theorem 2.1:

Theorem 2.5. *In the notation above,*

$$\mathrm{sign}(g, X, \xi) = \sum_{N \in \Omega} \int_N 2^{(n-m)/2} *_N \Big[\mathrm{ch}(\xi|N, g) \prod_i (\sqrt{-1}\ \tan(\theta_i/2))^{-c(\theta_i)}$$
$$\mathscr{L}(N)\mathscr{L}(TN^\perp(-1))^{-1} e(TN^\perp(-1)) \prod_i \mathscr{M}^{\theta_i}(TN^\perp(\theta_i))\Big](a)da - \eta_g(0, \xi)$$

where $\mathrm{sign}(g, X, \xi)$ *is the G-signature of the quadratic form on* $H^\ell(X, Y, \xi)$ *associated to the wedge product. Moreover,* $\eta_g(0, \xi)$ *is the eta invariant associated to the operator* B_ξ^{ev} *described above.*

Proof. This follows by an argument entirely analogous to that given in the proof of Theorem 2.1, applying the results of [3] and [12].

Remark. This section has been devoted entirely to the signature complex. However, similar arguments also give corresponding results for the other classical elliptic complexes. In fact, the main point is to identify the terms

$$b_{1,n/2}(g, a) - b_{2,n/2}(g, a)$$

of Theorem 1.1 with suitable polynomials in appropriate characteristic forms. This is a strictly local problem which was treated in [11], [12]. One needs to recall from [2] that each of the classical elliptic complexes is locally isomorphic to the signature complex wtih coefficients in an appropriate bundle.

3. Free actions of finite groups. Let Y be a compact oriented odd-dimensional Riemannian manifold of dimension $2\ell - 1$. If $E_\alpha \to Y$ is a flat Hermitian vector bundle over Y then we will define the eta invariant $\eta_\alpha(0)$ associated to the pair (Y, E_α).

In fact, one may endow $Y \times I$ with the product metric and product orientation. The signature complex of $Y \times I$ is well-defined

$$0 \to \Lambda^+(T^*(Y \times I) \otimes E_\alpha) \xrightarrow{D} \Lambda^-(T^*(Y \times I) \otimes E_\alpha) \to 0$$

after lifting the bundle E_α by the projection onto the first factor $\pi_1 : Y \times I \to Y$. Moreover, near $Y \times 0$, one may identify

$$\begin{aligned}\Lambda^+(T^*(Y \times I) \otimes E_\alpha) &\cong \Lambda^-(T^*(Y \times I) \otimes E_\alpha)\\ &\cong \Lambda(T^*(Y) \otimes E_\alpha).\end{aligned}$$

Modulo these identifications, one has $D = \partial/\partial u + B$ where $B : \Lambda(T^*Y) \otimes E_\alpha \to \Lambda(T^*Y) \otimes E_\alpha$ is a first order self-adjoint elliptic differential operator. The action of B decomposes as $B = B^{\text{ev}} \oplus B^{\text{od}}$ where the operators B^{ev}, B^{od} on even, odd forms have the same spectrum. We will denote by $\eta_a(0, Y)$ the eta invariant associated to the operator $B^{\text{ev}} : \Lambda(T^*Y) \otimes E_\alpha \to \Lambda(T^*Y) \otimes E_\alpha$.

Now suppose that $\hat{Y} \to Y$ is a regular covering space with finite covering group G. Then $\hat{Y}$ is a free G-manifold and $Y = \hat{Y}/G$. The operator $\hat{B}^{\text{ev}} : \Lambda^{\text{ev}}(T^*\hat{Y}) \to \Lambda^{\text{ev}}(T^*\hat{Y})$ is defined as above. *Via* the pull-back on forms, each $g \in G$ lifts to the bundle $\Lambda^{\text{ev}}(T^*\hat{Y})$ where it commutes with B^{ev}. Therefore the invariants $\eta_g(0, \hat{Y})$ are well-defined.

To investigate the relationship between the invariants $\eta_\alpha(0, Y)$ and $\eta_g(0, \hat{Y})$ we must recall:

Proposition 3.1. *Let G be a finite group of order $|G|$. If χ_α, χ_β are the characters of two irreducible complex unitary representations α, β of G, then*

$$\frac{1}{|G|} \sum_{g \in G} \chi_\alpha(g)\bar{\chi}_\beta(g) = \delta_{\alpha\beta} \tag{3.2}$$

where $\delta_{\alpha\beta} = 1$ if α is equivalent to β and $\delta_{\alpha\beta} = 0$ if α is not equivalent to β.

Proof. [17], p. 140.

If G is the covering group of a regular covering $\hat{Y} \to Y$, then we will consider flat vector bundles $E_\alpha \to Y$ associated to unitary representations $\alpha : G \to U(k)$, for some k. It suffices to assume that α is irreducible. For any real number λ, let $F_{\lambda,\alpha} \subset \Lambda^{\text{ev}}(T^*Y) \otimes E_\alpha$ be the eigenspace with eigenvalue λ of the operator $B^{\text{ev}} : \Lambda^{\text{ev}}(T^*Y) \otimes E_\alpha \to \Lambda^{\text{ev}}(T^*Y) \otimes E_\alpha$.

If λ is not an eigenvalue of the operator B^{ev}, then by definition $F_{\lambda,\alpha} = 0$.

For each eigenvalue λ of $\hat{B}^{\text{ev}} : \Lambda^{\text{ev}}(T^*\hat{Y}) \to \Lambda^{\text{ev}}(T^*\hat{Y})$ we denote by $F_\lambda \subset \Lambda^{\text{ev}}(T^*\hat{Y})$ the corresponding eigenspace. Then one has the decomposition

$$\hat{F}_\lambda = \sum_\alpha \dim(F_{\lambda,\alpha}) \otimes [\alpha^*]$$

of the G-representation space $\hat{F}_\lambda$. Here α^* is the dual representation of α.

Consequently, for each $g \in G$,

$$\mathrm{Tr}(g|\hat{F}_\lambda) = \sum_\alpha \dim(F_{\lambda,\alpha})\bar{\chi}_\alpha(g). \tag{3.3}$$

This gives:

Theorem 3.4. *Let Y be a compact oriented Riemannian manifold having odd dimension $2\ell - 1$. Suppose that $\hat{Y} \to Y$ is a regular covering space of Y with finite covering group G. For each $g \in G$, one may define the spectral invariant $\eta_g(0, Y)$ for the action of g on the eigenspaces of $\hat{B}^{\mathrm{ev}}$. Similarly, for each irreducible representation α of G, one has the spectral invariants $\eta_\alpha(0, Y)$. These invariants are related by the equation*

$$\eta_g(0, \hat{Y}) = \sum_\alpha \eta_\alpha(0, Y)\bar{\chi}_\alpha(g) \tag{3.5}$$

where the summation is over the irreducible representations α of G. Moreover,

$$\eta_\alpha(0, Y) = \frac{1}{|G|} \sum_g \eta_g(0, \hat{Y})\chi_\alpha(g) \tag{3.6}$$

where the sum is over group elements $g \in G$.

Proof. To prove (3.5) one observes that for $\mathrm{Re}(s)$ sufficiently large

$$\begin{aligned}\eta_g(s, \hat{Y}) &= \sum_{\lambda \neq 0} \mathrm{Tr}(g|\hat{F}_\lambda)\,\mathrm{sign}(\lambda)|\lambda|^{-s} \\ &= \sum_{\lambda \neq 0}\left(\sum_\alpha \dim(F_{\lambda,\alpha})\bar{\chi}_\alpha(g)\right)\mathrm{sign}(\lambda)|\lambda|^{-s}\end{aligned}$$

by relation (3.3).

Consequently

$$\begin{aligned}\eta_g(s, \hat{Y}) &= \sum_\alpha\left(\sum_{\lambda \neq 0} \dim(F_{\lambda,\alpha})\,\mathrm{sign}(\lambda)|\lambda|^{-s}\right)\bar{\chi}_\alpha(g) \\ &= \sum_\alpha \eta_\alpha(s, Y)\bar{\chi}_\alpha(g).\end{aligned}$$

The equation (3.6) now follows by the uniqueness of analytic continuation.

Equation (3.6) follows from equation (3.5) using the orthogonality relations (3.2). In fact

$$\begin{aligned}\frac{1}{|G|}\sum_g \eta_g(0, \hat{Y})\chi_\alpha(g) &= \frac{1}{|G|}\sum_{g,\beta} \eta_\beta(0, Y)\bar{\chi}_\beta(g)\chi_\alpha(g) \\ &= \frac{1}{|G|}\sum_g \eta_\alpha(0, Y) = \eta_\alpha(0, Y).\end{aligned}$$

The last theorem suggests a different approach toward the results of [4, pp. 408–413]. Consider the manifold $\hat{Y}$ as a free G-manifold. By the bordism theory of [7] one knows that a finite number J of copies of $\hat{Y}$ bound a free G-manifold $\hat{X}$; that is $\partial \hat{X} = J\hat{Y}$. Applying Theorem 2.1 to the pair $(\hat{X}, J\hat{Y})$ one obtains:

Proposition 3.7. *Let Y be a compact oriented Riemannian manifold of odd dimension $2\ell - 1$. Suppose that $\hat{Y} \to Y$ is a regular covering space of Y with finite covering group G. Choose a free G-manifold $\hat{X}$ so that $\partial \hat{X} = J\hat{Y}$ for some positive integer J. Then for the eta invariants $\eta_g(0, \hat{Y})$ one has*

$$\eta_g(0, \hat{Y}) = \frac{\operatorname{sign}(g, \hat{X})}{J} \qquad g \neq 1.$$

In particular, $\eta_g(0, Y)$ is a topological invariant for $g \neq 1$.

The topological invariants $\sigma_g(\hat{Y}) = \operatorname{sign}(g, \hat{X})/J$ were studied in [4, pp. 408–413]. For each irreducible representation α of G, one defines

$$\rho_\alpha(Y) = \eta_\alpha(0, Y) - (\dim \alpha)\eta(0, Y)$$

where $\dim \alpha$ is the dimension of α and $\eta(0, Y)$ is the eta invariant corresponding to the trivial one-dimensional representation. The invariants $\rho_\alpha(Y)$ are independent of the metric of Y.

Theorem 3.4 is essentially equivalent to:

Proposition 3.8. *Ley Y be a compact oriented Riemannian manifold having odd dimension $2\ell - 1$. Suppose that $\hat{Y} \to Y$ is a regular covering space of Y with finite covering group G. Then one has*

$$\sigma_g(\hat{Y}) = \sum_\alpha \rho_\alpha(Y)\bar{\chi}_\alpha(g) \qquad g \neq 1 \tag{3.9}$$

where the sum is over irreducible representations α of G. Moreover

$$\rho_\alpha(Y) = \frac{1}{|G|} \sum_{g \neq 1} \eta_g(\hat{Y})(\chi_\alpha(g) - \dim \alpha). \tag{3.10}$$

Proof. Formula (3.9) follows from Formula (3.5). Specifically

$$\begin{aligned}\sum_\alpha \rho_\alpha(Y)\bar{\chi}_\alpha(g) &= \sum_\alpha \eta_\alpha(0, Y)\bar{\chi}_\alpha(g) - \sum_\alpha (\dim \alpha)\bar{\chi}_\alpha(g)\\ &= \sum_\alpha \eta_\alpha(0, Y)\bar{\chi}_\alpha(g) = \eta_g(0, \hat{Y}).\end{aligned}$$

One has $\sum_\alpha (\dim \alpha)\chi_\alpha(g) = 0$ since $\sum_\alpha (\dim \alpha)[\alpha^*]$ is the regular representation of G.

Similarly Formula (3.10) follows from (3.6):

$$\begin{aligned}&\frac{1}{|G|} \sum_{g \neq 1} \sigma_g(\hat{Y})(\chi_\alpha(g) - \dim \alpha)\\ &= \frac{1}{|G|} \sum_{g \neq 1} \sigma_g(Y)\chi_\alpha(g) - \frac{1}{|G|} \sum_{g \neq 1} \sigma_g(\hat{Y})(\dim \alpha)\end{aligned}$$

$$= \frac{1}{|G|} \sum_{g \neq 1} \eta_g(0, \hat{Y})\chi_\alpha(g) - \frac{1}{|G|} \sum_{g \neq 1} \eta_g(0, \hat{Y})(\dim \alpha)$$

$$= \frac{1}{|G|} \sum_{g} \eta_g(0, \hat{Y})\chi_\alpha(g) - \frac{1}{|G|} \sum_{g} \eta_g(0, \hat{Y})(\dim \alpha)$$

$$= \eta_\alpha(0, Y) - (\dim \alpha)\eta(0, Y) = \rho_\alpha(Y).$$

Formula (3.6) has an interesting application to the computation of the usual eta invariant. In fact one obtains:

Proposition 3.11. *Let Y be a compact oriented Reimannian manifold of odd dimension $4k - 1$. Suppose that $\hat{Y} \to Y$ is a regular covering of Y with covering group G of finite order $|G|$. Then*

$$(3.12) \qquad |G|\eta(0, Y) - (\dim \alpha)\eta(0, \hat{Y}) = \sum_{g \neq 1} \sigma_g(\hat{Y})\chi_\alpha(g).$$

If $\hat{Y}$ admits an orientation reversing isometry (in particular if $\hat{Y}$ is globally symmetric) then:

$$(3.13) \qquad \eta_\alpha(0, Y) = \frac{1}{|G|} \sum_{g \neq 1} \sigma_g(\hat{Y})\chi_\alpha(g).$$

Proof. Equation (3.12) follows immediately from Formula (3.6) and the definition of $\sigma_g(\hat{Y})$.

If $f : \hat{Y} \to \hat{Y}$ is an orientation reversing isometry, then $f^*\hat{B}^{ev} = -\hat{B}^{ev}f^*$ where $\hat{B}^{ev}$ is the first order self-adjoint elliptic operator in the definition of $\eta(s, \hat{Y})$. Therefore

$$\eta(s, \hat{Y}) = \sum_{\lambda \neq 0} (\operatorname{sign} \lambda)|\lambda|^{-s} = 0$$

for $\operatorname{Re}(s)$ sufficiently large. Consequently $\eta(0, \hat{Y}) = 0$ by the uniqueness of analytic continuation.

Formula (3.13) provides a useful tool for determining the eta invariants of appropriate algebraic spaces Y. The point is that the invariants $\sigma_g(\hat{Y})$, $g \neq 1$ are computable by purely topological methods. A direct calculation of the $\eta_\alpha(0, Y)$ involves working out the spectrum of B_α^{ev} and finding the value at zero for the analytic continuation of $\eta_\alpha(s, Y)$. This direct approach may be quite difficult.

4. Lens spaces and flat manifolds. The lens spaces provide our first illustration of the utility of formula (3.13). Suppose that I is a positive integer and $1 \leq q_1$, $q_2, \cdots q_{2k} \leq I - 1$ are integers which are relatively prime to I. The cyclic group Z/IZ acts on C^{2k} by sending a generator of Z/IZ to the linear map

$$(z_1, z_2, \cdots, z_{2k}) \to \left(\exp\left(\frac{2\pi\sqrt{-1}\, q_1}{I}\right)z_1, \exp\left(\frac{2\pi\sqrt{-1}\, q_2}{I}\right)z_2, \right.$$
$$\left. \cdots, \exp\left(\frac{2\pi\sqrt{-1}\, q_{2k}}{I}\right)z_{2k}\right)$$

for $(z_1, z_2, \cdots, z_{2k}) \in C^{2k}$. This action leaves invariant the sphere S^{4k-1} defined by $|z_1|^2 + |z_2|^2 + \cdots + |z_{2k}|^2 = 1$. The quotient space $S^{4k-1}/(Z/IZ)$ will be denoted by $L^{4k-1}(I; q_1, q_2, \cdots, q_{2k})$.

One has:

Proposition 4.1. *Consider the lens space $Y = L^{4k-1}(I; q_1, q_2, \cdots q_{2k})$ with fundamental group Z/IZ. For any unitary representation α of Z/IZ, the eta invariant $\eta_\alpha(0, Y)$ is given by the explicit formula*

$$\eta_\alpha(0, Y) = \frac{(-1)^k}{I} \sum_{i=1}^{I-1} \left(\prod_{j=1}^{2k} \cot\left(\frac{i\pi q_j}{I} \right)\right) \chi_\alpha(\bar{g}^i)$$

where $\bar{g}$ is the generator of Z/IZ and χ_α is the character of the representation α.

Proof. Since the universal cover S^{4k-1} of Y is globally symmetric, one may apply the formula (3.13) with $G = Z/IZ$. This gives

$$\eta_\alpha(0, Y) = \frac{1}{I} \sum_{i=1}^{I-1} \sigma_{\bar{g}(i)}(\hat{Y}) \chi_\alpha(\bar{g}^i)$$

where $\bar{g}(i)$ is another notation for $\bar{g}^i$. Now $\sigma_{\bar{g}(i)}(\hat{Y}) = \eta_{\bar{g}(i)}(0, \hat{Y})$ by Proposition 3.7.

The cover $\hat{Y} = S^{4k-1}$ of Y bounds the unit ball B^{4k} in C^{2k}. Moreover, the action of Z/IZ extends to B^{4k} with a single isolated fixed point at the origin. The invariants $\eta_{\bar{g}(i)}(0, \hat{Y})$ may be calculated by applying Theorem 2.1 to the pair (B^{4k}, S^{4k-1}). This gives

$$\sigma_{\bar{g}(i)}(\hat{Y}) = \eta_{g(i)}(0, \hat{Y}) = (-1)^k \prod_{j=1}^{2k} \cot\left(\frac{i\pi q_j}{I} \right).$$

The proposition follows immediately.

Proposition 4.1 was first shown by Millson [14] *via* direct calculation. It was later demonstrated by Atiyah-Patodi-Singer [4, p. 412] using a method essentially equivalent to ours.

Formulas (3.5) and (3.6) have some interesting applications to the problem of listening to Z/IZ actions on homotopy spheres.

Proposition 4.2. *Let Y_1 and Y_2 be two compact oriented Riemannian manifolds which are simply homotopy equivalent to the lens space $L^{4k-1}(I; q_1, q_2, \cdots, q_{2k})$. For each irreducible unitary representation α of Z/IZ one has the associated flat Hermitian vector bundles $E_\alpha^1 \to Y_1$, $E_\alpha^2 \to Y_2$. Suppose that the corresponding operators $B_{\alpha,1}^{\text{ev}}$, $B_{\alpha,2}^{\text{ev}}$ acting on sections of these bundles are isospectral. The Y_1 and Y_2 are* PL *(piecewise-linear) homeomorphic.*

Proof. By the isospectrality hypothesis one has $\eta_\alpha(0, Y_1) = \eta_\alpha(0, Y_2)$ for all representations α of the fundamental group $\pi_1(Y_1) = \pi_1(Y_2) = Z/IZ$.

If Y_1, Y_2 are the universal covers of Y_1, Y_2, formula (3.5) implies that

$$\eta_g(0, \hat{Y}_1) = \eta_g(0, \hat{Y}_2)$$

for all $g \in Z/IZ$. In particular this holds for $g \neq 1$ so that

$$\sigma_g(\hat{Y}_1) = \sigma_g(\hat{Y}_2) \qquad g \neq 1$$

by Proposition 3.7.

However, these Atiyah-Singer invariants σ_g are known to classify up to PL-homeomorphism all manifolds which are simply homotopy equivalent to $L^{4k-1}(I; q_1, q_2, \cdots, q_{2k})$, a given lens space [16, p. 215]. In the same vein, one has:

Proposition 4.3. *Consider the two lens spaces* $Y_1 = L^{4k-1}(I; q_1, q_2, \cdots, q_{2k})$ *and* $Y_2 = L^{4k-1}(I; \bar{q}_1, \bar{q}_2, \cdots, \bar{q}_{2k})$ *with the same fundamental group* Z/IZ. *For each irreducible unitary representation* α *of* Z/IZ *one has the associated flat Hermitian vector bundles* $E^1_\alpha \to Y_1$, $E^2_\alpha \to Y_2$. *Suppose that the corresponding operators* $B^{\mathrm{ev}}_{\alpha,1}$, $B^{\mathrm{ev}}_{\alpha,2}$ *acting on sections of these bundles are isospectral. Then* Y_1 *and* Y_2 *are isometric.*

Proof. As in the proof of Proposition 4.2, the isospectrality hypothesis implies that

$$\eta_g(\hat{Y}_1) = \eta_g(\hat{Y}_2) \qquad g \neq 1 \tag{4.4}$$

where $\hat{Y}_1 = \hat{Y}_2 = S^{4k-1}$ is the universal cover of Y_1 and Y_2. The equality (4.4) implies that the representations $Z/IZ \to SO(4k)$ which define Y_1 and Y_2 are equivalent representations [1, p. 478]. Consequently Y_1 and Y_2 are isometric.

A second application of the results of Section (3) to computing the eta invariant involves manifolds covered by the standard flat torus. Let $Z^{4k-1} \subset R^{4k-1}$ be the lattice of integer points. The group Z^{4k-1} acts freely on R^{4k-1} with quotient the flat torus $T^{4k-1} = R^{4k-1}/Z^{4k-1}$. The isometry group of T^{4k-1} is the semidirect product of T^{4k-1} and the group $0(4k-1, Z)$ of all orthogonal matrices which preserve the lattice Z^{4k-1}. The flat manifolds covered by T^{4k-1} are in one-to-one correspondence with fixed point free representations of finite groups into $T^{4k-1} \circledS 0(4k-1, Z)$. Here $\circledS$ denotes the semidirect product.

We will compute the eta invariants associated to some particular manifolds covered by T^{4k-1}. Observe that the Riemannian manifold T^{4k-1} is isometric to $S^1 \times S^1 \times \cdots \times S^1$ where S^1 is the unit circle and there are $4k-1$ factors. Thus T^{4k-1} is isometric to $S^1 \times T^{4k-2}$. If $x \in S^1$ and $\bar{x} \in T^{4k-2}$ then $(x, \bar{x})$ will denote the corresponding point in T^{4k-1}. Let G be a finite group of isometries of T^{4k-1} so that $g \in G$ acts as

$$g(x, \bar{x}) = (x + a, A\bar{x} + \bar{a}) \tag{4.5}$$

where $a \in R/Z$, $\bar{a} \in R^{4k-2}/Z^{4k-2}$ and $A \in SO(4k-2, Z)$. We assume that $a \neq 0$ for $g \neq 1$ so that the identity element $1 \in G$ is the only group element fixing any points of T^{4k-1}. The quotient space $Y = T^{4k-1}/G$ is then a compact oriented flat Riemannian manifold, which is obtained by suspending the action of an isometry of finite order of the standard flat torus T^{4k-2}.

If $g \in G$, $g \neq 1$ then one may compute the spectral invariants $\eta_g(0, T^{4k-1})$ by using the fact that these invariants coincide with the topological invariants $\sigma_g(T^{4k-1})$. We let D^2 denote the unit disc in R^2. Since D^2 has boundary S^1, it follows that $D^2 \times T^{4k-2}$ has boundary T^{4k-1}. Moreover, the action of g on T^{4k-1} extends to $D^2 \times T^{4k-2}$, by rotation through the angle $2\pi a$ in the first factor. Let $\bar{g} : T^{4k-2} \to T^{4k-2}$ be the transformation given by $\bar{g}(\bar{x}) = A\bar{x} + \bar{a}$ where $\bar{x} \in T^{4k-2}$ and $A \in SO(4k-2, Z)$, $\bar{a} \in R^{4k-2}/Z^{4k-2}$ are the elements appearing in the definition of g. Then the fixed points of $g : D^2 \times T^{4k-2} \to D^2 \times T^{4k-2}$ are of the form $(0, \bar{w})$ where 0 is the center of the disc D^2 and $\bar{w}$ is a fixed point of $g : T^{4k-2} \to T^{4k-2}$.

The value of $\eta_g(0, T^{4k-1})$ depends fundamentally upon the eigenvalues of A.

Proposition 4.6. *Let g: $T^{4k-1} \to T^{4k-1}$ be given by the formula (4.5) with $a \neq 0$. If A has 1 as an eigenvalue, then $\eta_g(0, T^{4k-1}) = 0$.*

Proof. Let v be an eigenvector of A corresponding to the eigenvalue 1, *i.e.* $Av = v$. Consider the action of g extended to $D^2 \times T^{4k-2}$. If $(0, \bar{w})$ is a fixed point of g then so is $(0, w + \alpha v)$ for all real numbers α. Thus $g : D^2 \times T^{4k-2} \to D^2 \times T^{4k-2}$ has no isolated fixed points. Since g is an isometry, the components N of its fixed point set Ω are totally geodesic submanifolds. Moreover ∂N is empty for all N since g acts freely on T^{4k-1}, the boundary of $D^2 \times T^{4k-2}$. Thus each component N of Ω is a flat torus of dimension ≥ 1. Consequently the Pontriagin classes of N all vanish. Furthermore the globally flat connection ω of T^{4k-1} preserves the normal bundle $(TN)^\perp$ of N since N is totally geodesic. The connection ω also preserves each of the vector bundles in the decomposition

$$(TN)^\perp = TN^\perp(-1) \oplus TN^\perp(\theta_1) \oplus \cdots \oplus TN^\perp(\theta_s)$$

using the notation of Section 2. Consequently each of the bundles $TN^\perp(-1)$, $TN^\perp(\theta_i)$, $1 \leq i \leq s$, is trivial.

Since each N is a closed manifold, Formula (2.3) reads

$$\begin{aligned}&\operatorname{sign}(g, D^2 \times T^{4k-2})\\ &= \sum_{N \in \Omega} 2^{(n-m)/2}\Big[\prod_i (\sqrt{-1}\ \tan(\theta_i/2))^{-c(\theta_i)}\mathscr{L}(N)\mathscr{L}(TN^\perp(-1))^{-1}\\ &\quad e(TN^\perp(-1)) \prod_i \mathscr{M}^{\theta_i}(TN^\perp(\theta_i))\Big][N] - \eta_g(0, T^{4k-1}).\end{aligned}$$

However the first term on the right vanishes since each N has no non-vanishing characteristic classes and dim $N \geq 1$. Therefore

$$\eta_g(0, T^{4k-1}) = -\operatorname{sign}(g, D^2 \times T^{4k-2}).$$

We need to show that the right-hand side of this equation vanishes. Recall that the exact sequence of the pair $(D^2 \times T^{4k-2}, T^{4k-1})$ contains the terms

$$\cdots \to H^{2k}(D^2 \times T^{4k-2}, T^{4k-1}) \xrightarrow{j^*} H^{2k}(D^2 \times T^{4k-2}) \xrightarrow{i^*} H^{2k}(T^{4k-1}) \to \cdots.$$

The G-signature Sign $(g, D^2 \times T^{4k-2})$ may be computed relative to the bilinear form induced on $\mathrm{Im}(g^*)$ by cup product for $H^{2k}(D^2 \times T^{4k-2}, T^{4k-1})$. However i^* is injective and therefore $\mathrm{Im}(j^*) = 0$. This yields the desired result

$$\eta_g(0, T^{4k-1}) = -\ \mathrm{sign}(g, D^2 \times T^{4k-2}) = 0.$$

Proposition 4.6 reduces the problem of computing $\eta_g(0, T^{4k-1})$ only for those isometries g satisfying $\det(I - A) \neq 0$. For such g, one has:

Proposition 4.7. *Let $g : T^{4k-1} \to T^{4k-1}$ be an isometry of T^{4k-1} which is given by Formula (4.5). Suppose that $+1$ is not an eigenvalue of A. Then the extension of $g : D^2 \times T^{4k-2} \to D^2 \times T^{4k-2}$ has only isolated fixed points. The invariants $\eta_g(0, T^{4k-1})$ are given by*

$$\eta_g(0, T^{4k-1}) = \nu(g)(-1)^k \cot(\pi a) \prod_{i=1}^{2k-1} \cot(\gamma_i/2) \tag{4.8}$$

where $\nu(g)$ is the number of fixed points of the extension of $g : D^2 \times T^{4k-2} \to D^2 \times T^{4k-2}$ and γ_i, $1 \leq i \leq 2k-1$, are the rotation angles of $A \in SO(4k-2, Z)$. The invariants $\nu(g)$ and $\eta_g(0, T^{4k-1})$ are independent of the translation $\bar{a} \in R^{4k-2}/Z^{4k-2}$ in Formula (4.5).

Proof. As observed above, the fixed points of $g : D^2 \times T^{4k-2} \to D^2 \times T^{4k-2}$ are of the form $(0, \bar{w})$ where $\bar{w}$ is a fixed point of $\bar{g} : T^{4k-2} \to T^{4k-2}$. Consider T^{4k-2} as R^{4k-2}/Z^{4k-2} and let $\bar{x}$ be a point in the fundamental domain given by $0 \leq x_i < 1$, for all $1 \leq i \leq 4k-2$. Then x corresponds to a fixed point of $\bar{g}$ if and only if $Ax + a \in x + Z^{4k-2}$. That is, $\bar{x} \in (I - A)^{-1}a + (I - A)^{-1}Z^{4k-2}$ Consequently the fixed points of g are isolated.

The endomorphism induced on the tangent space of T^{4k-2} at each fixed point $\bar{w}$ of $\bar{g}$ is simply given by A. Moreover, the map $\bar{x} \to \bar{x} + (I - A)^{-1}\bar{a}$ gives a one-to-one correspondence between the fixed points of the map $\bar{x} \to A\bar{x}$ and the fixed points of $\bar{g}$. Consequently $\eta_g(0, T^{4k-1})$ is independent of $\bar{a}$.

The translation $x \to x + a$ of $S^1 = R/Z$ corresponds to a rotation through the angle $2\pi a$ for the disc D^2. Therefore, formula (2.3) gives

$$\eta_g(0, T^{4k-1}) = \nu(g)(-1)^k \cot(\pi a) \prod_{i=1}^{2k-1} \cot(\gamma_i/2) - \mathrm{sign}(g, X)$$

where γ_i are the rotation angles of A. Now $\mathrm{Sign}(g, X) = 0$ by the argument of Proposition 4.6. This yields

$$\eta_g(0, T^{4k-1}) = \nu(g)(-1)^k \cot(\pi a) \prod_{i=1}^{2k-1} \cot(\gamma_i/2)$$

as required.

We would like to make Formula (4.8) explicit by obtaining a description of the number $\nu(g)$ of fixed points of g. Clearly, the number of fixed points $\nu(\bar{g})$ of

$\bar{g} : T^{4k-2} \to T^{4k-2}$ is the same as the number of fixed points of g. Since $\nu(\bar{g})$ is independent of $\bar{a}$, as observed in the proof of Proposition 4.7, it suffices to assume that $\bar{g}\bar{x} = A\bar{x}$ for $A \in SO(4k-2, Z)$.

If A has an eigenvalue equal to $+1$ or -1 then $\eta_g(0, T^{4k-2}) = 0$ by Proposition 4.6 and Proposition 4.7. Thus we may assume that A has no eigenvalue equal to $+1$ or -1. Let $\hat{e}_i = (0, 0, \cdots, 0, 1, 0, \cdots, 0)$, where the only non-zero entry is in the ith place, be an element of the standard basis $\{\hat{e}_i\}$, $1 \leq i \leq 4k-2$, of R^{4k-2}. Since $A \in SO(4k-2, Z)$ one has $A\hat{e}_i = \pm\hat{e}_{j(A,i)}$, $i \neq j(A, i)$ for each i. We denote by $\sigma(A)$ the element of $SO(4k-2, Z)$ defined by $\sigma(A)\hat{e}_i = \hat{e}_{j(A,i)}$. Then $\sigma(A)$ is a permutation matrix and we may decompose $\sigma(A) = \sigma_1(A)\sigma_2(A) \cdots \sigma_\ell(A)$ into disjoint cycles.

One has:

Proposition 4.9. *Let $g : T^{4k-1} \to T^{4k-1}$ be an isometry given by*

$$g(x, \bar{x}) = (x + a, A\bar{x} + \bar{a})$$

where $A \in SO(4k-2, Z)$, $a \in R^{4k-2}/Z^{4k-2}$ and a is a non-zero element of R/Z.

Then

$$\eta_g(0, T^{4k-1}) = 0$$

if A has an eigenvalue equal to $+1$ or -1.

Otherwise

$$\eta_g(0, T^{4k-1}) = 2^\ell(-1)^k\cot(\pi a) \prod_{i=1}^{2k-1} \cot\left(\frac{\gamma_i}{2}\right) \tag{4.10}$$

where

(i) The angles γ_i are the rotation angles of the orthogonal matrix $A \in SO(4k-2, Z)$.

(ii) The integer ℓ is the number of distinct cycles in the decomposition of the permutation matrix $\sigma(A) = \sigma_1(A)\sigma_2(A) \cdots \sigma_\ell(A)$, corresponding to A.

Proof. Most of these assertions were shown above, particularly in the proofs of Propositions 4.6 and 4.7.

It only remains to check that if A has no eigenvalue equal to $+1$ or -1 then $\nu(g) = 2^\ell$, where $\nu(g)$ is the number of fixed points of g. Now $\nu(g) = \nu(\bar{g})$ where we may assume that $\bar{g} : T^{4k-2} \to T^{4k-2}$ is given by $\bar{g}\bar{x} = A\bar{x}$.

Suppose that $\sigma(A) = \sigma_1(A)\sigma_2(A) \cdots \sigma_\ell(A)$ where $\sigma_i(A)$ is a cycle of length s_i. There is a corresponding decomposition of $A = A_1A_2 \cdots A_\ell$. Moreover, $\bar{g} : T^{4k-2} \to T^{4k-2}$ is equal to a product $\bar{g} = \prod_{i=1}^{\ell} \bar{g}_i$ of isometries $\bar{g}_i : T^{s_i} \to T^{s_i}$. The action of $\bar{g}_i$ is given by $\bar{g}_i\bar{x}_i = A_i\bar{x}_i$, for $\bar{x}_i \in T^{s_i}$. Consequently the fixed point set of $\bar{g}$ is the product of the fixed point sets of the $\bar{g}_i$.

It is therefore sufficient to show that $\bar{g}_i : T^{s_i} \to T^{s_i}$ has exactly two fixed points. Let $\bar{x}_i$ be a point in the fundamental domain for T^{s_i} given by

$\bar{x}_i = (\bar{x}_i(1), \bar{x}_i(2), \cdots, \bar{x}_i(s_i))$ with $0 \le \bar{x}_i(j) < 1$ for $1 \le j \le s_i$. Then $\bar{x}_i$ corresponds to a fixed point of $\bar{g}_i$ if and only if $(I - A_i)\bar{x}_i \in Z^{4s_i - 2}$. The condition $(I - A_i)\bar{x}_i \in Z^{4s_i - 2}$ is equivalent to s_i relations of the form

$$
\begin{aligned}
&\bar{x}_i(1) \pm \bar{x}_i(\alpha(1) \in Z \\
&x_i(\alpha(1)) \pm x_i(\alpha^2(1)) \in Z \\
&x_i(\alpha^2(1)) \pm x_i(\alpha^3(1)) \in Z \qquad (4.11)\\
&\quad\vdots \qquad\qquad \vdots \\
&x_i(\alpha^{s_i - 1}(1)) \pm x_i(1) \in Z
\end{aligned}
$$

where $(1, \alpha(1), \alpha^2(1), \cdots, \alpha^{s_i - 1}(1))$ is some permutation of the integers $(1, 2, 3, \cdots, s_i)$. The s_i relations given by (4.10) are linearly independent since $\det(I - A_i) \neq 0$. One obtains a linear relation of the form $\bar{x}_i(\alpha^{s_i - 1}(1)) \pm x_i(1) \in Z$ by adding and subtracting in successive order the first $s_i - 1$ relations of (4.10). Since (4.10) consists of linearly independent relations it must imply both of the two relations

$$
\begin{aligned}
&\bar{x}_i(\alpha^{s_i - 1}(1)) + \bar{x}_i(1) \in Z \\
&\bar{x}_i(\alpha^{s_i - 1}(1)) - \bar{x}_i(1) \in Z.
\end{aligned}
$$

Consequently $2\bar{x}_i(1) \in Z$. So $\bar{x}_i(1) = 0$ or $\bar{x}_i(1) = 1/2$. If $\bar{x}_i(1) = 0$ then $\bar{x}_i = (0, 0, \cdots, 0)$ by the relations (4.10). Similarly if $\bar{x}_i(1) = 1/2$ then $\bar{x}_i = (1/2, 1/2, \cdots, 1/2)$. Therefore $\bar{g}_i$ has exactly two fixed points: $(0, 0, \cdots, 0)$ and $(1/2, 1/2, \cdots, 1/2)$.

This completes the proof of Proposition 4.9.

Furthermore we have:

Proposition 4.12. *Let T^{4k-1} be the standard flat torus, $T^{4k-1} = R^{4k-1}/Z^{4k-1}$. Suppose that G is a finite group which acts on T by isometries of the form*

$$g(x, \bar{x}) = (x + a, A\bar{x} + \bar{a}) \qquad g \in G$$

where $a \in R/Z$, $A \in SO(4k - 2, Z)$, and $a \in R^{4k-1}/Z^{4k-1}$. Assume that $a \neq 0$ for $g \neq 1$. Then the G-action is free and the quotient space $Y^{4k-1} = T^{4k-1}/G$ is a compact flat Riemannian manifold.

Moreover, for each unitary representation α of G, one has a flat Hermitian vector bundle $E_\alpha \to Y$. The corresponding eta invariant is given by

$$\eta_\alpha(0, Y) = \frac{1}{|G|} \sum{}' \left(2^\ell(-1)^k \cot(\pi a) \prod_{i=1}^{2k-1} \cot\left(\frac{\gamma_i}{2}\right)\right)(g)\chi_\alpha(g) \qquad (4.13)$$

where the symbol $\sum'$ means summation over the group elements g whose associated A has no eigenvalue equal to $+1$ or -1. Furthermore,

(i) The angles γ_i are the rotation angles of the orthogonal matrix $A \in SO(4k - 2, Z)$.

(ii) ℓ is the number of distinct cycles in the decomposition $\sigma(A) = \sigma_1(A)\sigma_2(A) \cdots \sigma_\ell(A)$ corresponding to A.

Proof. Formula (3.6) reads

$$\eta_\alpha(0, Y) - \frac{\dim \alpha}{|G|}\eta(0, T^{4k-1}) = \frac{1}{|G|} \sum_{g \neq 1} \eta_g(0, T^{4k-1})\chi_\alpha(g).$$

Now $\eta(0, T^{4k-1}) = 0$ since T^{4k-1} is globally symmetric. Thus

$$\eta_\alpha(0, Y) = \frac{1}{|G|} \sum_{g \neq 1} \eta_g(0, T^{4k-1})\chi_\alpha(g)$$

and the desired conclusion follows from Proposition 4.9.

5. An integrality theorem. Suppose that N is a compact oriented Riemannian manifold of even dimension $n = 2\bar{n}$. An oriented Riemannian vector bundle $E \to N$ is an oriented vector bundle with smooth inner product endowed with a connection which preserves that inner product. Each such E admits a canonical metric induced by the Riemannian metric of N, the connection of E, and the inner product of E.

Let $E \to N$ be an oriented Riemannian vector bundle and $\bar{E} \to N$ its associated unit disc bundle. Suppose that the total space $\bar{E}$ has dimension $4k$. The special orthogonal group $SO(4k - n)$ is contained in the isometry group of $\bar{E}$, with respect to its canonical metric. One simply lets $SO(4k - n)$ act fiberwise. The fixed point set of $g \in SO(4k - n)$, $g \neq 1$, is just the zero section N of E.

The unit sphere bundle $\bar{M} = \partial\bar{E}$ of E has dimension $4k - 1$ and contains $SO(4k - n)$ in its isometry group. The invariants $\eta_g(0, \bar{M})$ are therefore well-defined. These invariants may be calculated for $g \neq 1$ by applying Theorem 2.1 to the pair $(\bar{E}, \bar{M})$. In fact, the normal bundle $TN^\perp$ of N in E may be identified with E. For each $g \in SO(4k - n)$, $g \neq 1$, there is a decomposition

$$E = E(-1) \oplus \sum_i E(\theta_i) \tag{5.1}$$

corresponding to the normal form of g. Here g acts on $E(-1)$ *via* multiplication by -1 and on $E(\theta_i)$ *via* rotation through the angle θ_i, $0 < \theta_i < \pi$. Then each $E(\theta_i)$ admits the structure of a complex vector bundle and is thereby oriented. We assume that $E(-1)$ is orientable and is given an orientation so that the original orientation of E is compatible with the decomposition (5.1).

Then one has:

Theorem 5.2. *Suppose that E is an oriented Riemannian vector bundle over the compact oriented manifold N. If the total space E has dimension $4k$ and N has dimension n, then $SO(4k - n)$ is contained in the isometry group of E. If $g \in SO(4k - n)$, $g \neq 1$, then the eta invariant $\eta_g(0, \bar{M})$, for the unit sphere bundle $\bar{M}$ of E, is given by*

$$\eta_g(0, \bar{M}) = 2^{(n-m)/2}\Big(\prod_i (\sqrt{-1}\ \tan(\theta_i/2))^{-c(\theta_i)}\Big) \tag{5.3}$$
$$\mathscr{L}(N)\mathscr{L}(E(-1))^{-1}e(E(-1))\Big(\prod_i \mathscr{M}^{\theta_i}(E(\theta_i))\Big)(g)[N] - \operatorname{sign}(\bar{E}).$$

The notation is analogous to that of Section 2. In particular, m is the dimension of $E(-1)$ and $c(\theta_i)$ is the dimension of $E(\theta_i)$.

Sign($\bar{E}$) is the signature of the quadratic form

$$H^{2k}(\bar{E}, \bar{M}; R) \times H^{2k}(\bar{E}, \bar{M}; R) \to R$$

which is induced by the cup product.

Proof. The result follows from Theorem 2.1 applied to the pair $(\bar{E}, \bar{M})$. One need only observe that $\operatorname{sign}(g, \bar{E}) = \operatorname{sign}(\bar{E})$ for all $g \in SO(4k - n)$. This follows from the homotopy invariance of maps induced on cohomology and the connectedness of $SO(4k - n)$.

Let G be a finite group and suppose that one is given a fixed point free representation $\rho : G \to SO(4k - n)$. A fixed point free representation [17] is a representation ρ for which $\rho(g)$, $g \neq 1$, has no eigenvalue equal to one. Then G acts freely on $\bar{M}$ and we denote the quotient space by $M = \bar{M}/G$. By Formulas (3.6) and (5.3) one has

$$\eta(0, M) - |G|\eta(0, \bar{M})$$
$$= -\sum_{g \neq 1} 2^{(n-m)/2}\Big[\Big(\prod_i (\sqrt{-1}\ \tan(\theta_i/2))^{-c(\theta_i)}\Big)\mathscr{L}(N)\mathscr{L}(E(-1))^{-1} \tag{5.4}$$
$$e(E(-1)) \prod_i \mathscr{M}^{\theta_i}(E(\theta_i))(g)\Big][N] + (|G| - 1)\operatorname{sign}(\bar{E})$$

where the sum is over group elements $g \in G$, $g \neq 1$. Here $|G|$ denotes the order of G.

We will use the symbol μ_k to denote the denominator of the Hirzebruch L_k polynomial. This means that μ_k is the least integer for which $\mu_k L_k(p_1, p_2, \cdots, p_k)$ is an integral class in the cohomology $H^*(BO, Q)$ of the classifying space BO. It is well-known [13] that

$$\mu_k = \prod_q q^{\left[\frac{2k}{q-1}\right]} \tag{5.5}$$

where the product is over odd primes q, $3 \leq q \leq 2k + 1$.

The invariants $\eta(0, \hat{Y}) - |F|\eta(0, \hat{Y})$ were studied in [8] for arbitrary finite coverings $\hat{Y} \to Y$ of compact Riemannian manifolds Y. Here $|F|$ denotes the order of the fiber F for the covering $\hat{Y} \to Y$. If Y had dimension $4k - 1$ then it was shown in [8] that

$$2\mu_k(\eta(0, \hat{Y}) - |F|\eta(0, Y))$$

is an integer. This fact follows by a simple appeal to the naturality of the associated Chern-Simons invariants [6], [15].

Applying the quoted result in our present situation one obtains the integrality theorem:

Theorem 5.6. *Suppose that E is an oriented Riemannian vector bundle over the compact oriented Riemannian manifold N, of dimension n. If the total space E has dimension $4k$, then $SO(4k - n)$ is contained in the isometry group of E.*

Let $\rho : G \to SO(4k - n)$ be a fixed point free representation of a finite group G. For each $g \in G$, $g \neq 1$, there is a decomposition

$$E = E(-1) \oplus \sum_i E(\theta_i)$$

corresponding to the normal form of g.

Then the real number

$$(5.7)\qquad \begin{aligned}(2\mu_k) \sum_{g \neq 1} 2^{(n-m)/2}\Big[&\Big(\prod (\sqrt{-1}\ \tan(\theta_i/2))^{-c(\theta_i)}\Big)\\ &\mathcal{L}(N)\mathcal{L}(E(-1))^{-1}e(E(-1)) \prod_i \mathcal{M}^{\theta_i}(E(\theta_i))(g)\Big][N]\end{aligned}$$

is an integer.

The notation used here is that of Theorem 5.2.

Now suppose that the group G is cyclic of order $p \neq 2$, $G = Z/pZ$. Any fixed point free representation $\rho : Z/pZ \to SO(4k - n) = SO(4k - 2\bar{n})$ is determined up to conjugacy by the normal form of $\rho(1)$, where 1 is a generator of Z/pZ. We may assume that $\rho(1)$ has eigenvalues $\exp(2\pi\sqrt{-1}\ a_i/p)$ for some integers a_i. The integers a_i are understood to be distinct, although some eigenvalues may have multiplicity greater than one. Since the representation is fixed point free, each a_i must be relatively prime to p. Of course, any collection of integers a_i which are relatively prime to p may occur in the $\exp(2\pi\sqrt{-1}\ a_i/p)$.

We have:

Corollary 5.8. *Suppose that the hypotheses of Theorem 5.6 hold for $G = Z/pZ$, the cyclic group of order $p \neq 2$. If 1 is a generator of Z/pZ, then let $\exp(2\pi\sqrt{-1}\ a_i/p)$ be the eigenvalues of $\rho(1)$, with multiplicity $c(i)$. Then*

$$(2\mu_k) \sum_{s=1}^{p-1} 2^{n/2}\Big(\prod_i \sqrt{-1}\ \tan\Big(\frac{\pi s a_i}{p}\Big)\Big)^{-c(i)}\Big)\mathcal{L}(N) \prod_i \mathcal{M}^{\theta_i}\Big(E\Big(\frac{2\pi s a_i}{p}\Big)\Big)[N]$$

is an integer.

In particular, if $\sum_i c(i) = 2k$, then

$$(2\mu_k)\sum_{2=1}^{p-1}\prod_i\left(\cot\left(\frac{\pi s a_i}{p}\right)\right)^{c(i)}$$

is an integer.

Here a_i are any integers which are relatively prime to p.

The cotangent sums given by

$$\sum_{s=1}^{p-1}\prod_i\left(\cot\left(\frac{\pi s a_i}{p}\right)\right)^{c(i)}$$

were studied by Zagier [18]. Using elementary number theory, he proved the following integrality theorem which sharpens the last part of Corollary 5.7:

Proposition 5.9. *(Zagier) Let p be a positive integer and suppose that a_i are integers which are relatively prime to p. If $c(i)$ are integers with $\sum_i c(i) = 2k$, then*

$$\mu_k\sum_{s=1}^{p-1}\prod_i\left(\cot\left(\frac{\pi s a_i}{p}\right)\right)^{c(i)}$$

is an integer.

It is interesting to remark that our methods are quite different from the methods of [18]. In particular, our approach relies essentially upon the naturality of the geometric characteristic classes of Chern-Simons [15].

Zagier's results raise the question of whether Formula (5.7) is still an integer if $(2\mu_k)$ is replaced by μ_k. This is true at least for the actions of cyclic groups. In fact, one has the following sharpening of Corollary 5.8:

Proposition 5.10. *Suppose that the hypotheses of Theorem 5.6 hold for $G = Z/pZ$, the cyclic group of order $p \neq 2$. If 1 is a generator of Z/pZ, let $\exp\left(\frac{2\pi\sqrt{-1}\,a_i}{p}\right)$ be the eigenvalues of $\rho(1)$, with multiplicity $c(i)$. Then*

$$(\mu_k)\sum_{s=1}^{p-1}2^{n/2}\left(\prod_i\left(\sqrt{-1}\,\tan\left(\frac{\pi s a_i}{p}\right)\right)^{-c(i)}\right)\mathscr{L}(N)\prod_i\mathscr{M}^{\theta_i}\left(E\left(\frac{2\pi s a_i}{p}\right)\right)[N]$$

is an integer.

Proof. The result follows by elementary arguments using the integrality theorem of Zagier, Proposition 5.9.

First recall from Section (2) the definition of the characteristic classes $\mathscr{M}^{\theta_i}$:

$$\mathscr{M}^{\theta_i}\left(E\left(\frac{2\pi s a_i}{p}\right)\right) = \prod_j\frac{\tanh\left(\frac{\sqrt{-1}\,\pi s a_i}{p}\right)}{\tanh\left(\frac{x_{ij}}{2}+\frac{\sqrt{-1}\,\pi s a_i}{p}\right)}$$

where the elementary symmetric functions of the x_{ij} are the Chern classes of

$E\left(\frac{2\pi s a_i}{p}\right)$. The proposition sought is then equivalent to the assertion:

$$Q = \mu_k \sum_{s=1}^{p-1} 2^{n/2}\mathscr{L}(N) \prod_i \left(\prod_j \coth\left(\frac{x_{ij}}{2} + \frac{\sqrt{-1}\ \pi s a_i}{p} \right)\right)[N]$$

is an integer.

We digress briefly to prove a lemma concerning the Taylor series expansion of $\coth(z + \alpha)$, in powers of z.

Lemma 5.11. *Let α be a constant which is not of the form $(2i + 1)\pi/2$ for any integer i. Then the Taylor series expansion of $\coth(z + \alpha)$ is given by:*

$$\coth(z + \alpha) = \sum_{r=0}^{\infty} P_{r+1}(\coth(\alpha))z^r$$

where $P_{r+1}(x)$ is a polynomial of order $(r + 1)$ with integer coefficients.

Proof. Clearly

$$\coth(z + \alpha) = \sum_{r=0}^{\infty} \left[\frac{d^r}{dz^r} (\coth(z))\right]_{z=\alpha} z^r.$$

By explicit computation one obtains the formula

$$\frac{d}{dz} (\coth(z)) = 1 - (\coth(z))^2.$$

It follows by induction that

$$\frac{d^r}{dz^r} (\coth(z)) = P_{r+1}(\coth(z))$$

where $P_{r+1}(x)$ is a polynomial of order $(r + 1)$ with integer coefficients.

We may now return to the proof of Proposition 5.10.

Proof. (Proposition 5.10, Continued). In the notation of Lemma 5.11, one has

$$Q = \mu_k \sum_{s=1}^{p-1} 2^{n/2}\mathscr{L}(N) \prod_{i,j} \left(\sum_{r=0}^{\infty} P_{r+1}\left(\coth\left(\frac{\sqrt{-1}\ \pi s a_i}{p} \right)\right) 2^{-r} x_{ij}^r\right)[N].$$

It follows from the explicit Formula (5.5), giving μ_k, that $\mu_t \mu_{k-t}$ divides μ_k for any integer $0 \le t \le k$. Consequently, there exist integers b_t so that

$$Q = \sum_{s=1}^{p-1} \sum_{t=0}^{k} b_t \sum_{r_{ij}} \mu_{k-t} \prod_{i,j} P_{r_{ij}+1}\left(\coth\left(\frac{\sqrt{-1}\ \pi s a_i}{p} \right)\right)$$

where the summation over r_{ij} is only over indices r_{ij} satisfying $\sum_{i,j} r_{ij} = \bar{n} - 2t$. Therefore

$$Q = \sum_{t=0}^{k} b_t \sum_{r_{ij}} \mu_{k-t} \sum_{s=1}^{p-1} \prod_{i,j} P_{r_{ij}+1}\left(\coth\left(\frac{\sqrt{-1}\ \pi s a_i}{p} \right)\right).$$

Thus it suffices to show that for any fixed choice of indices r_{ij} satisfying $\sum_{i,j} r_{ij} = \bar{n} - 2t$, the quantity

$$R = \mu_{k-t} \sum_{s=1}^{p-1} \prod_{i,j} P_{r_{ij}+1}\left(\coth\left(\frac{\sqrt{-1}\ \pi s a_i}{p}\right)\right)$$

is an integer.

Since the number of values for (i, j) is $2k - \bar{n}$, we may write

$$R = \mu_{k-t} \sum_{s=1}^{p-1} \sum_{w_v} \prod_{v} d(w_v)\left(\coth\left(\frac{\sqrt{-1}\ \pi s a_i}{p}\right)\right)^{w_v}$$

where the $d(w_v)$ are integers and the w_v are integers satisfying $\sum_{v} w_v \leq 2k - 2t$. Therefore

$$R = \sum_{w_v} \left(\prod_{v} d(w_v)\right)\mu_{k-t} \sum_{s=1}^{p-1} \prod_{v} \left(\coth\left(\frac{\sqrt{-1}\ \pi s a_i}{p}\right)\right)^{w_v}$$

However, Proposition 5.9 implies that, for each choice of indices w_v with $\sum_{v} w_v \leq 2k - 2t$, the quantity

$$W = \mu_{k-t} \sum_{s=1}^{p-1} \prod_{v} \left(\coth\left(\frac{\sqrt{-1}\ \pi s a_i}{p}\right)\right)^{w_v}$$

is an integer.

This completes the proof of Proposition 5.10.

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