Riemannian Geometry

UT Austin, Spring 2017

M392C NOTES: RIEMANNIAN GEOMETRY

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These notes were taken in UT Austin's M392C (Riemannian Geometry) class in Spring 2017, taught by Dan Freed. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Thanks to Martin Bobb for some corrections.

The cover image is the Cosmic Horseshoe (LRG 3-757), a gravitationally lensed system of two galaxies. Einstein's theory of general relativity, written in the language of Riemannian geometry, predicts that matter bends light, so if two galaxies are in the same line of sight from the Earth, the foreground galaxy's gravity should bend the background galaxy's light into a ring, as in the picture. The discovery of this and other gravitational lenses corroborates Einstein's theories. Source: https://apod.nasa.gov/apod/ap111221.html.

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"Do you have all these equations?"

Before we begin with Riemannian manifolds, it'll be useful to do a little geometry in flat space.

Definition 1.1. Let V be a real vector space; then, an *affine space over V* is a set A with a simply transitive right V-action.

That this action is simply transitive means for any $a, b \in A$, there's a unique $\xi \in V$ such that $a \cdot \xi = b$.

Definition 1.2. A set with a simply transitive (right) V-action is called a *(right) V-torsor*.

V-torsors look like copies of V without a distinguished identity.

One of the distinct features of affine space is *global parallelism*: if I have a vector ξ at a point a, I immediately get a vector at every point, which defines a vector field on the entire space.

What is the analogue of a basis for an affine space? This is a collection of points $a_0, \dots, a_n$ such that any $a \in A$ is uniquely written as

$$a = \lambda^0 a_0 + \lambda^1 a_1 + \cdots + \lambda^n a_n \tag{1.3}$$

for some $\lambda^i \in \mathbb{R}$ with $\lambda^0 + \cdots + \lambda^n = 1$.

Equation (1.3) may be written more concisely with *index notation*: any variable written as both a superscript and a subscript is implicitly summed over. That is, we may rewrite (1.3) as

$$a = \lambda^i a_i.$$

Note that in an affine space, we don't know how to add vectors (since we don't have an origin), but we can take weighted averages.

Theorem 1.4 (Giovanni Ceva, 1678)**.** *Let A be an affine plane and $a, b, c \in A$ be a triangle (i.e. three distinct, noncollinear points). Suppose $p \in \overline{bc}$, $q \in \overline{ca}$, and $r \in \overline{ca}$. Then, $\overline{ap}$, $\overline{bq}$, and $\overline{cr}$ are coincident iff*

$$[ar : rb][bp : pc][cq : ca] = 1.$$

Typically, this is thought of as a ratio of lengths, but we don't necessarily have lengths: instead, we can use barycentric coordinates. There is a unique λ such that if $r = (1-\lambda)a + \lambda b$, then $[ar : rb] = \lambda/(1-\lambda)$.

Proof. Let

$$\begin{aligned} r &:= (1-\lambda)a + \lambda b \\ p &:= (1-\mu)b + \mu c \\ q &:= (1-\nu)c + \nu a. \end{aligned}$$

Set

$$x := \alpha a + \beta b + \gamma c, \tag{1.5}$$

where $\alpha + \beta + \gamma = 1$. Since $x \in \overline{ap}$, then

$$x = \alpha a + C((1-\mu)b + \mu c). \tag{1.6}$$

Comparing (1.5) and (1.6), $\mu/(1-\mu) = \gamma/\beta$.

⊠

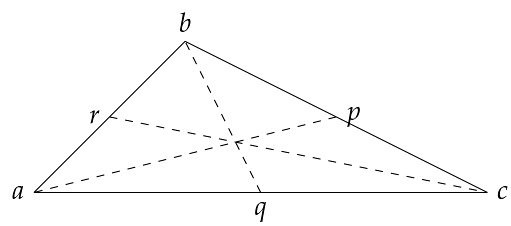

Figure 1. Depiction of Ceva's theorem (Theorem 1.4).

Standard affine space $\mathbb{A}^n := \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^i \in \mathbb{R}\}$. You may complain this is the same as $\mathbb{R}^n$, but $\mathbb{A}^n$ only comes with an affine structure, not a vector-space structure.

Definition 1.7. Let A be an affine space modeled on V and B be an affine space modeled on W. Then, a map $f : A \to B$ is *affine* if there exists a linear map $T : V \to W$ such that $f(a + \xi) = f(a) + T\xi$ for all $a \in A$ and $\xi \in V$.

In other words, an affine map is a linear map plus some constant, which is not uniquely defined.

Definition 1.8. An *affine coordinate system* on A is an affine isomorphism $x = (x^1, \dots, x^n) : A \to \mathbb{A}^n$.

Them, the differentials $\mathrm{d}x_a^1, \dots, \mathrm{d}x_a^N$ are independent of basepoint a and form a basis for V^*, the dual vector space and dual basis to V and $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$, the tangent space to any $a \in A$.

But affine space is not the only flat geometry we could consider: more generally, we consider a structure on a vector space V which can be promoted to a translationally invariant structure on A. This leads to metric geometry, symplectic geometry, etc.

Definition 1.9. An *inner product* on a (finite-dimensional) vector space V is a bilinear map $\langle -, - \rangle : V \times V \to \mathbb{R}$ which is symmetric and positive definite, i.e. for all $\xi, \eta \in V$, $\langle \xi, \eta \rangle = \langle \eta, \xi \rangle$, $\langle \xi, \xi \rangle \geq 0$, and $\langle \xi, \xi \rangle = 0$ iff $\xi = 0$.

Since $\langle -, - \rangle$ is bilinear, then this can be determined in terms of n^2 numbers: let $v_1, \dots, v_n$ be a basis for V and define $g_{ij} := \langle v_i, v_j \rangle$ for $i, j = 1, \dots, n$. Of course, these numbers areen't independent: $g_{ij} = g_{ji}$, so there are really only $n(n+1)$ choices of information.

Definition 1.10. A basis $e_1, \dots, e_n$ for V is *orthonormal* if

$$\langle e_i, e_j \rangle = \delta_{ij} := \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}.$$

Our first major result of flat Euclidean geometry is that these exist.

Theorem 1.11. *There exist orthonormal bases.*

Proof. Let $v_1, \dots, v_n$ be any basis of V. Let

$$e_1 = \frac{v_1}{\langle v_1, v_1 \rangle^{1/2}},$$

and for $i = 2, \dots, n$, let

$$v_i' = v_i - \langle v_i, e_1 \rangle e_1.$$

Then, $\langle e_1, e_1 \rangle = 1$ and $\langle e_1, v_i' \rangle = 0$. Then, repeat with $v_2', \dots, v_n'$. ⊠

This explicit algorithm is called the *Gram-Schmidt process*.

In an inner product space, we get some familiar geometric constructions: the *length* of a vector $\xi \in V$ is $|\xi| = \langle \xi, \xi \rangle^{1/2}$, and the *angle* between $\xi, \eta \in V \setminus 0$ is the θ such that

$$\cos\theta = \frac{\langle \xi, \eta \rangle}{|\xi||\eta|}.$$

Definition 1.12. A *Euclidean space* E is an affine space over an inner product space V.

This has a notion of distance: $d_E : E \times E \to \mathbb{R}^{\geq 0}$, where $a, b \mapsto |\xi|$, where $b = a + \xi$. This generalizes to notions of area, volume, etc.

Theorem 1.13 (Napoleon, 1820)**.** *Let abc be a triangle in a plane and attach an equilateral triangle to each edge. The centers of these three triangles form an equilateral triangle.*

Exercise 1.14. Prove this.

We want to understand curved analogues of this classical material, and will pick up where differential topology left off. We work on smooth manifolds: a *smooth manifold* is a space X together with an atlas of charts $U \subset X$ with homeomorphisms $x : U \to \mathbb{A}^n$ such that every point is contained in the domain of some chart and the transition maps are smooth. We do not require a manifold to have a global dimension: the different connected components may have different dimensions, e.g. $S^1 \amalg S^2$.[1]

A chart map $x : U \to \mathbb{A}^n$ is a set of n continuous maps $(x^1, \dots, x^n)$. If p is in the domain of both x and y, we can consider $x \circ y^{-1} : \mathbb{A}^n \to \mathbb{A}^n$; calculus as usual tells us what it means for this transition map to be smooth.

At any $x \in X$, we have a tangent space $T_x X$ and a cotangent space $T_x X$: a chart defines a basis of the tangent space $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ and a basis of the contangent space $\mathrm{d}x^1_x, \dots, \mathrm{d}x^n_x$. This depends strongly on x: unlike for flat space, we may not be able to parallel-transport these globally, even on something as simple as S^2.

In this course, we will study what happens when we go from a curved analogue of affine space to a curved analogue of Euclidean space, whence the following central definition.

Definition 1.15. A *Riemannian metric* on a smooth manifold X is a choice of inner product $\langle -, - \rangle_x$ on $T_x X$ for all $x \in X$ which varies smoothly in x.

Now, we can compute lengths of tangent vectors and the angle that two smooth curves intersect at (or rather, the angle their tangent vectors intersct at). We also obtain a notion of distance between points, and can develop analogues of Euclidean geometry on manifolds.

[1] This is important for, e.g. a space of solutions of certain PDEs.

What does "varying smoothly" mean, exactly? Suppose $x^1, \dots, x^n$ is a set of local coordinates on $U \subset X$; then, for $i, j = 1, \dots, n$, define

$$g_{ij} := \left\langle \frac{\partial}{\partial x^i}\bigg|_x, \frac{\partial}{\partial x^j}\bigg|_x \right\rangle_{T_x X}.$$

One can check that if the metric is smoothly varying in one chart, then it's smoothly varying in all charts.

We'll write the metric as

$$g = g_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j.$$

This again uses the summation convention, and it's useful to think about where exactly this lives: it identifies the metric as a tensor.

Many manifolds arise as embedded submanifolds of Euclidean space, and the Whitney embedding theorem shows that all may be emnbedded. Many authors say it's best to meet manifolds as embedded submanifolds first, but there are some which arise without a natural embedding, e.g. the Grassmanian $\mathrm{Gr}_2(\mathbb{R}^4)$, the space of two-dimensional subspaces of $\mathbb{R}^4$.

In any case, if $X \subset \mathbb{E}^N$ is embedded, then X inherits a metric, since $T_x X \subset \mathbb{R}^n$ is also a subspace, and we can restrict the inner product. Classical Riemannian geometry is the study of *plane curves* (one-dimensional submanifolds of $\mathbb{R}^2$), *space curves* (one-dimensional submanifolds of $\mathbb{R}^3$), and *surfaces* (two-dimensional submanifolds of $\mathbb{R}^3$).

To study Riemannian manifolds, we should begin with the simplest cases. The zero-dimensional manifolds are disjoint unions of points with zero-dimensional tangent spaces and the trivial Riemannian metric. In the one-dimensional case, there is a little more to tell. A smooth map $X \to Y$ of Riemannian manifolds is an *isometry* if it's a map that preserves the inner product on each tangent space. This automatically implies it's injective.

Theorem 1.16. *Let C be a (complete) Riemannian 1-manifold which is diffeomorphic to $\mathbb{R}$. Then, C is isometric to $\mathbb{E}^1$.*

Before we prove this, we need a change-of-coordinates lemma. (We'll address completeness later, to avoid finite intervals.)

Remark. Let $x^1, \dots, x^n$ and $y^1, \dots, y^n$ be coordinate systems and suppose a metric can be written as

$$g = g_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j = h_{ab}\, \mathrm{d}y^a \otimes \mathrm{d}y^b.$$

Then,

$$g_{ij} = h_{ab} \frac{\partial y^a}{\partial x^i} \frac{\partial y^b}{\partial x^j}. \tag{1.17}$$

This is n^2 equations: there is no implicit summation here. ◀

Proof of Theorem 1.16. Let $x : C \to \mathbb{R}$ be a diffeomorphism, which defines a global coordinate on C. Let $g(x) = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \rangle$. We seek a new coordinate $y : C \to \mathbb{R}$ such that $h(y) = \langle \frac{\partial}{\partial y}, \frac{\partial}{\partial y} \rangle = 1$ everywhere. By (1.17),

$$g = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2, \tag{1.18}$$

so fix an $x_0 \in C$ and define

$$y(x) = \int_{x_0}^x \sqrt{g(t)}\, \mathrm{d}t.$$

This y satisfies (1.18) and therefore is an isometry. ⊠

The analogue to Theorem 1.16 in n dimensions (where $n > 1$) is as follows: if $x^1, \dots, x^n$ is a local coordinate system and g_{ij} is the Riemannian metric in these coordinates, is there a local change of coordinates $y^a(x^1, \dots, x^n)$ such that $h_{ab} = \delta_{ab}$? This is the analogue in Riemannian geometry to finding orthonormal coordinates, guaranteed by Theorem 1.11.

This requires solving an analogue to (1.17), but this time it's a PDE

$$g_{ij} = \sum_a \frac{\partial y^a}{\partial x^i} \frac{\partial y^a}{\partial x^j}.$$

This time, we need to ask whether there are solutions. The only thing we know how to do is differentiate:

$$\frac{\partial g_{ij}}{\partial x^k} = \sum_a \frac{\partial^2 y^a}{\partial x^k \partial x^i}\frac{\partial g^a}{\partial x^j} + \frac{\partial y^a}{\partial x^i}\frac{\partial^2 y^a}{\partial x^k \partial x^j}. \tag{1.19a}$$

By permuting indices, we obtain

$$\frac{\partial g_{ik}}{\partial x^j} = \sum_a \frac{\partial^2 y^a}{\partial x^j \partial x^i}\frac{\partial g^a}{\partial x^k} + \frac{\partial y^a}{\partial x^i}\frac{\partial^2 y^a}{\partial x^j \partial x^k} \tag{1.19b}$$

$$\frac{\partial g_{jk}}{\partial x^i} = \sum_a \frac{\partial^2 y^a}{\partial x^i \partial x^j}\frac{\partial g^a}{\partial x^k} + \frac{\partial y^a}{\partial x^j}\frac{\partial^2 y^a}{\partial x^i \partial x^k}. \tag{1.19c}$$

Taking (1.19a) + (1.19b) − (1.19c), we obtain

$$\frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right) = \sum_a \frac{\partial y^a}{\partial x^i}\frac{\partial^2 y^a}{\partial x^j \partial x^k}.$$

Now we multiply by $\frac{\partial y^b}{\partial x^\ell} g^{\ell i}$, concluding

$$\frac{\partial y^b}{\partial x^\ell}\underbrace{\frac{g^{\ell i}}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right)}_{\Gamma^\ell_{jk}} = \sum_a \frac{\partial y^a}{\partial x^i}\frac{\partial^2 y^a}{\partial x^j \partial x^k} g^{\ell i}\frac{\partial y^b}{\partial x^\ell}.$$

These Γ^ℓ_{jk} symbols therefore satisfy

$$\frac{\partial^2 y^b}{\partial x^j \partial y^k} = \Gamma^i_{jk}\frac{\partial y^b}{\partial x^i}.$$

If we differentiate once again (with respect to x^ℓ), we get

$$\begin{aligned}\frac{\partial^3 y^b}{\partial x^\ell \partial x^j \partial x^k} &= \frac{\partial \Gamma^i_{jk}}{\partial x^\ell}\frac{\partial y^b}{\partial x^i} + \Gamma^i_{jk}\frac{\partial^2 y^b}{\partial x^\ell x^i}\\ &= \left(\frac{\partial \Gamma^i_{jk}}{\partial x^\ell} + \Gamma^m_{jk}\Gamma^i_{m\ell}\right)\frac{\partial y^b}{\partial x^i}.\end{aligned}$$

Since mixed partials commute, then one discovers that if such an isometry exists, the *Riemannian curvature tensor*

$$R^i_{jk\ell} := \frac{\partial \Gamma^i_{j\ell}}{\partial x^\ell} - \frac{\partial \Gamma^i_{jk}}{\partial x^\ell} + \Gamma^m_{jk}\Gamma^i_{m\ell} - \Gamma^m_{j\ell}\Gamma^i_{mk} \tag{1.20}$$

must vanish. In simple cases, one can calculate that it's not always zero, so we don't always have global parallelism.

Riemann derived this in the middle of the 1800s. It's possible to see the glimmer of special relativity in them, though of course this was discovered later.

There's no text, though there is a website: `http://www.ma.utexas.edu/users/dafr/M392C/index.html`. There are problem sets, so undergraduates have to do some problem sets, and graduate students should. Feel free to talk to the professor about the problems, and especially to establish groups to work on the problem sets. Office hours are Wednesdays 2 to 3.

Lecture 2.

Existence of Riemannian metrics: 1/19/17

"There are so many of you... so quiet... I'll be more provocative until I get questions. Or I'll go faster."

Due to the large size of the class, it's being moved to RLM 6.104 starting next week. This means everyone who wants to sign up should be able to.

Some readings are up on the website, including a translation of Riemann's original work on curvature.

Last time, we defined affine space, which leads to the notion of a smooth manifold, and then introduced Euclidean space, an affine space over an inner product space. The curved version of that is a Riemannian manifold.

Recall that a Riemannian metric g on a smooth manifold X is a smoothly varying family of inner products on T_xX, and a Riemannian manifold is a smooth manifold together with a Riemannian metric. We also defined an isometry: if X and Y are Riemannian manifolds, then a diffeomorphism $f : X \to Y$ is an isometry if for all $x \in X$ and $\xi_1, \xi_2 \in T_xX$,

$$\langle f_*\xi_1, f_*\xi_2 \rangle_{T_{f(x)}Y} = \langle \xi_1, \xi_2 \rangle_{T_xX}.$$

Here, $f_* : T_xX \to T_{f(x)}Y$ is the linear pushforward of tangent vectors, also called the *differential*. If f is merely a smooth function, this is called an *isometric immersion* (the inverse function theorem automatically implies it's an immersion). If f is an embedding, this is called an *isometric embedding*.

Existence of Riemannian metrics. Suppose V is a real vector space and $g_0, g_1 : V \times V \to \mathbb{R}$ are inner products. Then for $t \in [0, 1]$, $(1-t)g_0 + tg_1$ is also an inner product (you can check this directly).

The set of bilinear maps $V \times V \to \mathbb{R}$, denoted $\mathrm{Bil}(V \times V, \mathbb{R})$, is a real vector space, naturally isomorphic to $\mathrm{Hom}(V \otimes V, \mathbb{R})$ and to $V^* \otimes V^*$. Here, "natural" means this works for all finite-dimensional vector spaces at once, and commutes with linear maps.

Inner products are elements of this vector space, and our observation above means that if g_0 and g_1 are inner products, the line between them in $\mathrm{Bil}(V \times V, \mathbb{R})$ consists of inner products. In particular, *inner products form a convex set*. This only uses the affine structure on $\mathrm{Bil}(V \times V, \mathbb{R})$, since we can take convex combinations in an affine space.

This is used to generalize to the curved case, showing Riemannian metrics always exist.

Theorem 2.1. *Let X be a smooth manifold. Then, there is a Riemannian metric on X.*

Proof. Let $\mathfrak{U} = \{(U, x)\}$ be a cover of X by coordinate charts $x : U \to \mathbb{A}^n$, and let g_U denote the metric on U such that $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ are orthonormal. That is, take the standard metric on $\mathbb{A}^n$ making it into Euclidean space $\mathbb{E}^n$, and pull it back to U, where it becomes a metric (you can check that metrics pull back along closed immersions).

Now, the bases on two different charts in $\mathfrak{U}$ don't agree, and don't necessarily differ by orthonormal bases. Thus, we use a standard argument in differential geometry to globalize local objects living in a convex set: let $\{\rho_U\}_{U \in \mathfrak{U}}$ be a partition of unity subordinate to $\mathfrak{U}$; then,

$$g = \sum_{U \in \mathfrak{U}} \rho_U g_U$$

is a Riemannian metric. $\boxtimes$

Remark. Global existence is *not* assured for every geometric structure. For example, a *complex structure* on a real vector space V is an endomorphism $J : V \to V$ such that $J^2 = -\mathrm{id}_V$. This is akin to multiplication by i in a complex vector space, which squares to -1 and commutes with addition.

You can place this structure on affine space, and there's an immediate obstruction: $\dim_{\mathbb{R}} V$ must be even. Now we globalize: given an even-dimensional manifold, do we have such a structure? That is, can we place a smoothly varying complex structure on T_xX for all $x \in X$? This is called an *almost complex structure*, and not every even-dimensional manifold admits one.

Exercise 2.2. Show that S^4 has no almost complex structure.

There is an almost complex structure on S^6, and it's a famous open question whether there's a complex structure (i.e. complex coordinates with holomorphic transition functions). The known almost complex structure does not work.

Another local structure that doesn't automatically globalize is a mixed-signature metric (e.g. a Minkowski metric). In such a metric, the *null vectors*, those ξ for which $\langle \xi, \xi \rangle = 0$, form a cone whose interior is the *positive vectors* (for which the metric is positive). Trying to globalize this produces, more or less, a line in each tangent space T_xX. Passing to a double cover, one can choose an orientation, and therefore a nonzero vector field on X, and this can't be done in general. For example, a surface of genus 2 admits no metric of signature $(1,1)$. These kinds of metrics arise in general relativity. ◂

In this class, we care about Riemannian metrics, which do globalize.

Let $x^1, \dots, x^n$ be local coordinates; then, we defined some local quantities in the metric in terms of these coordinates. Namely,

$$g_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle,$$

so that $g = g_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$. We then used this to define symbols Γ^i_{jk} and the Riemann curvature tensor $R^i_{jk\ell}$. We proved Theorem 1.16; here's a better version.

Theorem 2.3. *Let C be a Riemannian 1-manifold diffeomorphic to $\mathbb{R}$. Then, there exists an isometry $C \to I$, where $I \subset \mathbb{E}^1$ is an open interval.*

The argument we gave defining the Riemann curvature tensor generalizes this.

Theorem 2.4. *Suppose (U, g) is a Riemannian manifold and $x : U \to \mathbb{A}^n$ is a global coordinate such that*

$$g = \sum_{i=1}^n (\mathrm{d}x^i)^2.$$

Then, $R^i_{jk\ell} = 0$ on U.

One important thing to check here is that

$$R = R^i_{jk\ell} \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k \otimes \mathrm{d}x^\ell$$

is independent of the coordinate system (which is not clear from its definition). This means that the Riemann curvature tensor is a tensor, i.e. $R \in T_xX \otimes T_x^*X \otimes T_x^*X \otimes T_x^*X$. In the next few weeks, we will add some geometry to this discussion.

Example 2.5. Let $X = \mathbb{E}^2$ be Euclidean space with the standard metric g. Then, we have global coordinates $(x, y) : \mathbb{E}^2 \to \mathbb{A}^2$, so $g = \mathrm{d}x^2 + \mathrm{d}y^2$.

We can also introduce *polar coordinates*, another coordinate system which isn't global. This is a coordinate map $(r, \theta) : \mathbb{E}^2 \setminus \{(x, 0) : x \le 0\} \to \mathbb{A}^2$ (so $r > 0$, $-\pi < \theta < \pi$). In this case, the metric has the form

$$g = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2.$$

This means that the vector field $\frac{\partial}{\partial r}$ has constant length 1, but the vector field $\frac{\partial}{\partial \theta}$ has length r at (r, θ). ◂

Symmetry. We've now seen vector spaces, affine spaces, Euclidean spaces, and Riemannian manifolds. As in any mathematical context, it's important to ask what the proper notion of symmetry is for these objects.

If V is a vector space, its *general linear group* is $\mathrm{GL}(V) = \mathrm{Aut}(V) := \{T : V \to V \text{ invertible}\}$. The standard example is $\mathrm{GL}_n(\mathbb{R}) := \mathrm{GL}(\mathbb{R}^n)$, the group of invertible $n \times n$ matrices, acting on the column vectors of $\mathbb{R}^n$ by scalar multiplication. For example $\mathrm{GL}_1(\mathbb{R}) = \mathbb{R}^\times$, the group of nonzero numbers under multiplication.

What about affine space? Affine space on V is a V-torsor, as V acts by translation. The symmetry group is the group of *affine transformations*

$$\mathrm{Aff}(A) := \{\alpha : A \to A \mid \alpha \text{ is invertible and affine}\}.$$

Recall that an affine map is one that preserves the affine structure: the image of a finite weighed average is the weighted average of the images. The derivative of an affine map is a linear map, so if A is an affine space modeled by V, the derivative defines a group homomorphism $\mathrm{d} : \mathrm{Aff}(A) \to \mathrm{GL}(V)$, whose kernel is the translations, a group isomorphic to V. Thus, we have a *group extension* (short exact sequence of groups)

$$1 \longrightarrow V \longrightarrow \mathrm{Aff}(A) \xrightarrow{\mathrm{d}} \mathrm{GL}(V) \longrightarrow 1. \tag{2.6}$$

The key is that in affine space, there's no canonical origin. However, (2.6) splits, if noncanonically: choose an $a \in A$. Then, any $b \in A$ can be uniquely written as $a + \xi$ for some $\xi \in V$, so for any linear transformation T, $a + \xi \mapsto a + T\xi$ is an affine transformation of A.

(2.6) is a sequence of manifolds with smooth group homomorphisms, making it a short exact sequence of *Lie groups*; we'll discuss Lie groups more later.

If V is an n-dimensional vector space, its bases are the set $\mathscr{B}(V) = \{b : \mathbb{R}^n \xrightarrow{\cong} V\}$. If $V = \mathbb{R}^n$, this is $\mathrm{GL}_n(\mathbb{R})$. In general, this makes $\mathscr{B}(V)$ into a right $\mathrm{GL}_n(\mathbb{R})$-torsor, defined by the simply transitive action $\mathscr{B}(V) \times \mathrm{GL}_n(\mathbb{R}) \to \mathscr{B}(V)$ sending $\beta, g \mapsto \beta \circ g$. (There is a corresponding left action by $\mathrm{GL}(V)$). The action on the right is akin to numbering elements of the basis, and the action on the left is more geometric; this is an instance of a general idea that internal actions tend to be from the right, and geometric ones from the left.

What's the analogue for an affine space A modeled on V? Let $\mathscr{B}(A)$ denote the collection of pairs (a, β) where $a \in A$ and $\beta \in \mathscr{B}(V)$, identified with the set of affine isomorphisms $\alpha : A \xrightarrow{\cong} \mathbb{A}^n$. These are the bases at specific points of A. There is a forgetful map $\pi : \mathscr{B}(A) \to A$ sending $(a, \beta) \to a$, and the fiber is $\mathscr{B}(V)$, the bases at a. In a similar way, there is a left action of $\mathrm{Aff}(A)$ on $\mathscr{B}(A)$, and a right action of $\mathrm{Aff}_n := \mathrm{Aff}(\mathbb{A}^n)$ on $\mathscr{B}(A)$.

We'll use these torsors of bases a lot in this class. In this way, we're enacting Felix Klein's *Erlangen* program, where the kind of geometry we do is reflected by the symmetry group we place on the geometric structures.

Let's see what happens to these ideas in the Euclidean and Riemannian cases. If V is an inner product space, its *orthogonal group* $\mathrm{O}(V) \subset \mathrm{GL}(V)$ is the group of linear isomorphisms preserving the inner product, i.e. $T : V \to V$ such that $\langle T\xi_1, T\xi_2 \rangle = \langle \xi_1, \xi_2 \rangle$ for all $\xi_1, \xi_2 \in V$. For $V = \mathbb{R}^n$, we let $\mathrm{O}_n := \mathrm{O}(\mathbb{R}^n)$.

Example 2.7. If $n = 1$, $\mathrm{O}_1 \subset \mathrm{GL}_1$ is $\{\pm 1\} \subset \mathbb{R}^\times$, so it's isomorphic to the cyclic group of order 2.

If $n = 2$, we can rotate by angles θ or reflect across lines, and playing with an orthonormal basis shows that all elements of O_2 must be rotations or reflections. Since O_2 is a Lie group, we can draw a picture as in Figure 2.

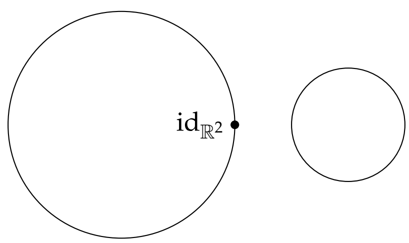

FIGURE 2. A picture of O_2. The left circle is the rotations; the right circle is the reflections, which in a sense form a circle half as long.

As with the affine symmetries, there's an extension

$$1 \longrightarrow \mathrm{SO}_2 \longrightarrow \mathrm{O}_2 \longrightarrow \{\pm 1\} \longrightarrow 1.$$ ◀

Similarly, the isomorphisms of Euclidean space E, denoted $\mathrm{Euc}(E)$, are the affine isomorphisms preserving the inner product at each point. This again fits into an extension sequence

$$1 \longrightarrow V \longrightarrow \mathrm{Euc}(E) \longrightarrow \mathrm{O}(V) \longrightarrow 1.$$

All this is nice, but let's talk about manifolds. If X is a smooth manifold, we no longer have translations, and the linear symmetries talk about the tangent space. We'll see what kind of structures we get in this case.

The analogue of the torsor of bases is $\mathscr{B}(X) := \{(x, \beta) : x \in X, \beta : \mathbb{R}^n \xrightarrow{\sim} T_xX\}$. This admits a right action of $\mathrm{GL}_n(\mathbb{R})$ by precomposition, as on a vector space, and there is again a forgetful map $\pi : \mathscr{B}(X) \to X$ that ignores the basis.

If $x : U \to \mathbb{A}^n$ is a chart, then it defines a local section $U \to \mathscr{B}(X)$ sending

$$(x^1, \ldots, x^n) \longmapsto \left((x^1, \ldots, x^n), \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right) \right).$$

If X is a Riemannian manifold, then we can also speak of orthonormal bases:

$$\mathscr{B}_{\mathrm{O}}(X) := \{(x, \beta) : x \in X, \beta : \mathbb{R}^n \xrightarrow{\cong} T_x X \text{ is an isometry}\}.$$

Again there is a forgetful map to X, but now a coordinate does *not* always determine a section: if the Riemann curvature tensor doesn't vanish, the image of an orthonormal basis of the tangent space at a point might not be orthonormal.

$\mathscr{B}(X)$ and $\mathscr{B}_{\mathrm{O}}(X)$ are not just sets but smooth manifolds, and the forgetful maps back to X are called fiber bundles (even principal bundles). We'll go back and discuss this in more detail.

Curvature. Let's end with something concrete. Let E be a Euclidean plane, an affine space with an underlying 2-dimensional inner product space.

Let $C \subset E$ be a 1-dimensional submanifold. Let's choose a *co-orientation* of C: an orientation of C is an orientation of its tangent bundle, so a co-orientation is an orientation of its normal bundle. In essence, this is choosing a side of the curve.[2] We'll use this to define a function $\kappa : C \to \mathbb{R}$ called the *(signed) curvature*. Intuitively, this should be positive if C is curved towards the side chosen by the co-orientation, and negative if it curves away, and a larger magnitude means a stronger curvature.

The Euclidean structure on E induces an inner product structure on $T_x C$ for all $x \in C$ that varies smoothly, so C becomes a Riemannian manifold. Theorem 1.16 means there's nothing intrinsic about C we can measure, but the way in which it sits inside E is what κ will measure. This is an important dichotomy, between intrinsic geometry and extrinsic geometry. The Riemann curvature tensor is intrinsic, since it doesn't depend on an embedding, but the signed curvature will be extrinsic.

Lecture 3.

The curvature of a curve: 1/24/17

"And if you follow your nose... well, Euler's nose..."

In the next two lectures, we'll march through the theory of extrinsic curvature (which can fill an entire undergraduate course).

Let E be a Euclidean plane modeled on an inner product space of V, which acts on E by translations, and let $i : C \hookrightarrow E$ be an immersed 1-manifold.[3] Suppose C is co-oriented, meaning we've oriented its normal bundle (picking a side of C, so to speak). This determins a unit co-oriented normal vector e_1 at every $x \in C$, meaning the unique unit vector in $(\nu_{C \hookrightarrow E})_x$ with a positive orientation. We can also choose a unit tangent vector e_2 perpendicular to e_1, and there are two choices. Together they define an orthonormal basis at each point: $(e_1, e_2) : C \to \mathscr{B}_{\mathrm{O}}(V)$.

You learned how to do calculus with real-valued differential forms; in exactly the same way, it's possible to do calculus with vector-valued differential forms $\Omega_C^*(V)$, the forms modeled on functions $C \to V$. For $i, j \in \{1, 2\}$, we can define $e_i \in \Omega_C^0(V)$ and $\mathrm{d}e_i \in \Omega_C^1(V)$, such that $\langle e_i, e_j \rangle = \delta_{ij}$ and the Leibniz rule is satisfied:

$$\langle \mathrm{d}e_i, e_j \rangle + \langle e_i, \mathrm{d}e_j \rangle = 0.$$

Thus, there exists an $\alpha \in \Omega_C^1$ such that

$$\mathrm{d}e_1 = -\alpha e_2 \qquad \text{and} \qquad \mathrm{d}e_2 = \alpha e_1.$$

In other words, applying d to the row vector $(e_1\ e_2)$ multiplies it by a skew-symmetric matrix:

$$\mathrm{d}\begin{pmatrix} e_1 & e_2 \end{pmatrix} = \begin{pmatrix} e_1 & e_2 \end{pmatrix} \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}.$$

Let $\theta^1, \theta^2 : C \to V^*$ define the dual basis at each point, i.e. at every $x \in C$, $\theta^i(e_j) = \delta^i_j$ as functions $C \to \mathbb{R}$. Then, $i^*\theta^2 \in \Omega_C^1$ and we can write

$$\alpha = k \cdot i^*\theta^2$$

for some function $k : C \to \mathbb{R}$.

[2]If $N \hookrightarrow M$ is an embedding and M is oriented, an orientation of N and a co-orientation of N determine each other.

[3]Especially if C is immersed but not embedded, it is helpful to remember i: when C self-intersects, remembering i is necessary for computing curvature.

Definition 3.1. The *curvature* of C is the function k.

Example 3.2. Let C denote the circle of radius R in the Euclidean plane $\mathbb{E}^2$. It's parameterized by coordinates $x = R\cos\phi$ and $y = R\sin\phi$, so

$$\begin{aligned} \mathrm{d}x &= -R\sin\phi\,\mathrm{d}\phi \\ \mathrm{d}y &= \quad R\cos\phi\,\mathrm{d}\phi. \end{aligned}$$

Let's choose the co-orientation in which the inward-pointing unit normal is positively oriented. Then,

$$e_1 = -\cos\phi\frac{\partial}{\partial x} - \sin\phi\frac{\partial}{\partial y}.$$

We also have to choose e_2: suppose it points clockwise along the circle. Then,

$$e_2 = \sin\phi\frac{\partial}{\partial x} - \cos\phi\frac{\partial}{\partial y}.$$

Thus, the dual basis is defined by

$$\begin{aligned} \theta^1 &= -\cos\phi\,\mathrm{d}x - \sin\phi\,\mathrm{d}y \\ \theta^2 &= \quad \sin\phi\,\mathrm{d}x - \cos\phi\,\mathrm{d}y, \end{aligned}$$

so $i^*\theta^2 = R\,\mathrm{d}\phi$. Then,

$$\mathrm{d}e_2 = \cos\theta\,\mathrm{d}\theta\frac{\partial}{\partial x} + \sin\theta\,\mathrm{d}\theta\frac{\partial}{\partial y} = -\mathrm{d}\theta e_1.$$

Thus, $\mathrm{d}e_2 = (1/R)i^*\theta^2(e_1)$. In particular, the curvature is $1/R$. It has units of $1/\text{length}$.

If we chose e_2 to point counterclockwise, there would be a sign change in θ^2, and another one in α, so they would cancel out to give the same result. ◂

Since the unit vector always has unit length in V, you can think of e_1 as a map $C \to S(V)$ (called the *Gauss map*), where $S(V)$ is the unit sphere inside V. At a point $p \in C$, we can define the tangent line T_pC at $i(p)$; the tangent line is a subspace of V. We can also consider the tangent line to $e_1(p) \in S(V)$, $T_{e_1(p)}S(V)$; both of these are the same space, the space of vectors in V perpendicular to $e_1(p)$.

This means the differential

$$(\mathrm{d}e_1)_p : T_pC \longrightarrow T_{e_1(p)}S(V) \tag{3.3}$$

is a map from a line to itself.

Theorem 3.4. *The map in (3.3) is multiplication by $-k(p)$.*

Proof.

$$\mathrm{d}e_1(e_2) = \alpha(e_2)\cdot d_2 = -ki^*\theta^2(e_2)e_2 = -k\cdot e_2. \qquad \boxtimes$$

Remark (History). The curvature may have been initially defined by Nicole Oresme in about 1350. It was again discovered by Huygens in c. 1650 and Newton in c. 1664. ◂

Here's a third approach to curvature. Let $i : C \hookrightarrow E$ be a co-oriented curve as usual, and assume C is embedded. For some $p \in C$, we can identify the normal line to $i(p)$ with $\mathbb{R}$, letting the positive numbers point into the positively oriented direction. Call this coordinate y. Given a choice of a unit tangent vector e_2, we can identify the tangent line with $\mathbb{R}$, again pointing the positive numbers in the x-direction. Call this coordinate x.

Lemma 3.5. *There exists an open set $U \subset E$ about p such that $C \cap U$ is the graph of a function $f : \mathbb{R} \to \mathbb{R}$ in the above xy-coordinate system such that*

- *$f(0) = f'(0) = 0$, and*
- *$f''(0) = k(p)$.*

Proof. The x-coordinate map $x|_C : C \to \mathbb{R}$ satisfies $\mathrm{d}x_p = \mathrm{id}_{T_pC}$; in particular, it's invertible. By the inverse function theorem, there's a local inverse $g : I \to C$, where $I \subset \mathbb{R}$ is an open interval. Define f to be $y \circ i \circ g$: since $i : C \hookrightarrow E$ and $y : E \to \mathbb{R}$, this is a map $I \to \mathbb{R}$. Write

$$e_1 = \frac{(-f', 1)}{\sqrt{1 + (f')^2}} \qquad \text{and} \qquad e_2 = \frac{(1, f')}{\sqrt{1 + (f')^2}}.$$

Then,

$$\mathrm{d}e_1 = \left(\frac{(-f'', 0)}{\sqrt{1 + (f')^2}} + \frac{(-f', 1)}{(1 + (f')^2)^{3/2}} f' \right) \mathrm{d}t.$$

At p,

$$\mathrm{d}e_1 = (-f''(0), 0)\,\mathrm{d}t = (-f''(0)\,\mathrm{d}t)e_2. \qquad \boxtimes$$

In calculus, we think of the tangent line as the best linear approximation to a function at a point, which only requires an affine space. Curvature is the process that goes one degree higher: you could ask for the *osculating parabola* to a curve at a point, the parabola that best approximates a curve at a point, or for the *osculating circle*, the circle that best approximates the curve at that point. Then, the curvature can be read off of the constants, e.g. it's 1 over the radius of the osculating circle. But knowing these parameters requires an inner product, hence a Euclidean space.

Prescribing curvature. We aim to solve the following problem: given an abstract curve C and a function $k : C \to \mathbb{R}$, construct an immersion $i : C \to E$ and a co-orientation such that k is the curvature of i.

Curvature requires thinking about a frame at each point if $i(C)$, so we should think about the bundle of orthonormal frame $\pi : \mathscr{B}_\mathrm{O}(E) \to E$. A point in $\mathscr{B}_\mathrm{O}(E)$ is a triple $(p; e_1, e_2)$, where $p \in E$ and (e_1, e_2) is an orthonormal basis of V. In particular, $\mathscr{B}_\mathrm{O}(E)$ is naturally a product $E \times \mathscr{B}_\mathrm{O}(V)$. We want to construct a lift $\widetilde{\imath} : C \to \mathscr{B}_\mathrm{O}(E)$ making the following diagram commute:

$$\begin{tikzcd} & \mathscr{B}_\mathrm{O}(E) \arrow[d, "\pi"] \\ C \arrow[ur, "\widetilde{\imath}"] \arrow[r, "i"] & E. \end{tikzcd}$$

This $\widetilde{\imath}$ is specified as a triple of functions on C, $\widetilde{\imath} = (p, e_1, e_2)$. Prescribing the curvature means we need this to satisfy

$$\mathrm{d}\begin{pmatrix} p & e_1 & e_2 \end{pmatrix} = \begin{pmatrix} p & e_1 & e_2 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & k\,\mathrm{d}t \\ \mathrm{d}t & -k\,\mathrm{d}t & 0 \end{pmatrix}}_{A(t)}. \tag{3.6}$$

We'll interpret A as a time-varying vector field on the manifold $\mathscr{B}_\mathrm{O}(E)$; then, we can evoke the basic theory of ordinary differential equations to prove there's a solution.

Digression. Let's recall what this basic theory of ordinary differential equations says. Let X be a smooth manifold and $(a, b) \subset \mathbb{R}$ be an interval. Projection onto the second factor defines a map $\pi_2 : (a, b) \times X \to X$, and we can pull the tangent bundle back along it:

$$\begin{tikzcd} \pi_2^* TX \arrow[r] \arrow[d, "p"] & TX \arrow[d] \\ (a, b) \times X \arrow[r, "\pi_2"] & X. \end{tikzcd}$$

Definition 3.7.

- A *time-varying vector field* is a section $\xi : (a, b) \times X \to \pi_2^* TX$ of $p : \pi_2^* TX \to (a, b) \times X$.
- An *integral curve* of ξ is an open interval $I \subset (a, b)$ and a function $\gamma : I \to X$ such that

$$\dot{\gamma}(t) = \xi_{(t, \gamma(t))}.$$

Time-varying vector fields correspond to ODEs and integral curves correspond to their solutions.

Theorem 3.8. *Given $(t_0, x_0) \in (a, b) \times X$, there exists an $\varepsilon > 0$ and an integral curve $\gamma : (t_0 - \varepsilon, t_0 + \varepsilon)$ such that $\gamma(t_0) = x_0$, and any two choices for γ agree on their common domain. Moreover, there is a maximal domain $J \subset (a, b)$ on which a solution exists and an integral curve $\gamma : J \to X$*

That is, solutions exist and are unique given an initial condition. However, they may not be globally defined.[4]

Just as $\mathscr{B}_{\mathrm{O}}(V)$ is a torsor for a right action of O_2 (an orthogonal basis composed with an orthogonal transformation is again an orthogonal basis), $\mathscr{B}_{\mathrm{O}}(E)$ is a torsor for the right action of Euc_2, the group of Euclidean transformations of $\mathbb{E}^2$. This torsor structure means the derivative of a curve in any neighborhood of the origin of the group defines a vector field on the torsor.

If $P(t)$ is a curve in O_2 such that $P(0) = \mathrm{id}$, then ${}^tP \cdot P = I$, so differentiating this condition, ${}^t \cdot P + \cdot P = 0$. That is, $T_e\mathrm{O}_2$ is the line of 2×2 skew-symmetric matrices over $\mathbb{R}$. Looking again at (3.6), the lower right entries of $A(t)$ are exactly such a matrix, so $A(t)$ is in fact a time-varying vector field on $\mathscr{B}_{\mathrm{O}}(E)$.

Now, using Theorem 3.8, given an initial $p \in E$ and an initial frame (e_1, e_2) on T_pE, there is a local and in fact a maximal solution to the prescribed curvature problem. This solution is unique up to the choice of (p, e_1, e_2), and this is usually expressed by saying that the group of symmetries of Euclidean space acts transitively on the solutions (so there's only one up to rotations and translations).

This is a somewhat elementary context for this material, but we'll adopt this perspective again and again. Eventually there will also be second-order conditions, e.g. when we define geodesics later.

Now, let's step up a dimension: let E be a Euclidean 3-space modeled on an inner product space V and $i : S \hookrightarrow V$ be an immersion of a 2-manifold together with a co-orientation. We can again define the unit co-oriented normal $\nu : S \to S(V)$. How can we define the curvature of this surface?

Euler solved this problem in 1760 by reducing it to something we've already done: let $L \in \mathbb{P}(T_pS)$ be a 1-dimensional subspace of the tangent space. There's a unique affine plane $\Pi(L)$ passing through p and containing L, and $\Pi(L) \cap S$ is a co-oriented curve in $\Pi(L)$. Let $k_p : \mathbb{P}(T_pS) \to \mathbb{R}$ be the function assigning to L the curvature of the curve $\Pi(L) \cap S$. Euler studied this function.

As before, locally we can write S as the graph of a function $f : T_pS \to \mathbb{R}$ with $f(0) = 0$ and $\mathrm{d}f_0 = 0$. The function k_p encodes the second derivative of f. This is expressed through the Hessian

$$\operatorname{Hess} f_0 : T_pS \times T_pS \longrightarrow \mathbb{R},$$

which is a symmetric bilinear form. In the context of geometry of surfaces, this Hessian is called the *second fundamental form* and denoted II_p.

Corollary 3.9. *For any $L \in \mathbb{P}(T_pS)$, $k_p(L) = II_p(\xi, \xi)$, where $|\xi| = 1$ and $\xi \in L$.*

The *first fundamental form* is the inner product

$$I_p := \langle -, - \rangle : T_pS \times T_pS \to \mathbb{R},$$

The second fundamental form may be nondegenerate (e.g. if S is flat), but we know the first is nondegenerate. This means the second fundamental form may be expressed in terms of the first fundamental form and some other operator S, called the *shape operator*:

$$II_p(\xi, \eta) = I_p(\xi, S(\eta)) = \langle \xi, S(\eta) \rangle.$$

Since II_p is symmetric, then S is self-adjoint. This means it has two real eigenvalues, so we can look at the eigenspaces, which are called the *principal lines* of S at p — unless the curvature is constant at p, in which case p is called an *umbilic point*.

Interestingly, we started with a very extrinsic notion of curvature of surfaces, but from this we've obtained some intrinsic geometry.

Lecture 4.

Curvature for surfaces: 1/26/17

"I didn't go into comedy, because I thought I would be safe here..."

[4] In this class, we assume everything is smooth, but Theorem 3.8 is true in much greater generality, requiring only *Lipschitz continuity*, a condition slightly stronger than continuity. Many other things in this class may be relaxed, e.g. to C^2.

Last time, we talked about the curvature of surfaces in a Euclidean plane; today, we will consider surfaces in a 3-dimensional Euclidean space E modeled on an inner product space $(V, \langle -,- \rangle)$, the vector space of translations of E.

Though E is abstractly isomorphic to $\mathbb{E}^3$, we won't fix an isomorphism by choosing coordinates; later, we'll want to pick special coordinates for E, so this would only complicate things.

Let $\Sigma \subset E$ be an embedded 2-manifold (some of our results will still apply when Σ is immersed), and assume Σ is co-oriented. Let $\nu : \Sigma \to V$ be the co-oriented positive unit normal.

Given a $p \in \Sigma$ and a plane $L \subset V$, $\Pi(L)$ denotes the plane through p containing L and ν. Then, $\Sigma \cap \Pi(L)$ is a curve, which is intuitively the curve "pointing in the L-direction at p."

The map assigning to L the curvature of $\Sigma \cap \Pi(L)$ at p is a function

$$k_p \colon \mathbb{P}(T_p\Sigma) \to \mathbb{R}.$$

Here, $\mathbb{P}(V)$ is the manifold of 1-dimensional subspaces of a vector space V.

We're going to get some information out of k_p. Let's first introduce special coordinates: choose an orthonormal basis in $\mathscr{B}_{\mathrm{O}}(E)$, so we obtain coordinates x^1 and x^2 in $T_p\Sigma$. As in the last lecture, the inverse function theorem provides for us an open set $U \subset T_p\Sigma$ containing 0, a function $f \colon U \to \mathbb{R}$, and an open $J \subset \mathbb{R}$ containing 0 such that $\Sigma \cap ((p+U) \times (p + J\nu))$ is the graph of f.

That is, there's a box inside E with an "xy-plane" $p + U$ and a "z-axis" pointing in the ν-direction, and inside this box, Σ is the graph of a function $f(x, y)$ on $p + U$. Furthermore, $f(p) = 0$ and $\mathrm{d}f_p = 0$, which is easy to check.

Last time, we defined the second fundamental form at p, $II_p = \mathrm{Hess}_p\, f : T_p\Sigma \times T_p\Sigma \to \mathbb{R}$. Based on what we proved last time, using the third incarnation of curvature, we got Corollary 3.9: $k_p(L) = II_p(\xi, \xi)$, where $\xi \in L$ is a unit vector.

This says the Hessian on the diagonal determines the curvature. This is because this is the second derivative of f, and we showed that if $\mathrm{d}f_p = 0$ for an f parameterizing a plane curve, then its second derivative computes the curvature.

On $T_p\Sigma$ we have two fundamental forms: the inner product, also known as the first fundamental form I_p, and the second fundamental form defined above. Since the first fundamental form is nondegenerate, then we can (and did) define the shape operator $S_p \in \mathrm{End}(T_p\Sigma)$ to satisfy the relation

$$\langle \xi, S_p(\eta) \rangle = II_p(\xi, \eta).$$

Since the inner product is nondegenerate, this uniquely defines $S_p(\eta)$. Moreover, since II_p is symmetric, then S_p is self-adjoint, i.e. $\langle \xi, S_p(\eta) \rangle = \langle S_p(\xi), \eta \rangle$ for all ξ and η. In particular, it's diagonalizable, and since $T_p\Sigma$ is two-dimensional, there are two possibilities:

(1) If there's only one eigenvalue $\lambda \in \mathbb{R}$, then $S_p = \lambda \cdot \mathrm{id}_{T_p\Sigma}$. In this case, p is called an umbilic point.

(2) If there are two eigenvalues λ_1 and λ_2 (suppose without loss of generality $\lambda_1 > \lambda_2$), then the two eigenspaces L_1 and L_2 form an orthogonal direct-sum decomposition $T_p\Sigma = L_1 \oplus L_2$. In this case, $S_p|_{L_i}$ is multiplication by λ_i. The L_i are called the *principal directions*, and the λ_i are called the *principal curvatures*. For any plane L,

$$k_p(L) = \frac{II_p(\xi, \xi)}{I_p(\xi, \xi)}.$$

The maximum of k_p is at L_1, and the minimum is at $L_2 II_p(\xi, \xi) I_p(\xi, \xi)$.

If you reverse the co-orientation, then $k \mapsto -k$ and $\lambda_i \mapsto -\lambda_i$. From this we get the *mean curvature* (named after one Mr. Mean)

$$H := \frac{\lambda_1 + \lambda_2}{2} = \frac{1}{2}\,\mathrm{Tr}(S_p),$$

a function $\Sigma \to \mathbb{R}$. Reversing the co-orientation sends $H \mapsto -H$. The *Gauss curvature* (named after Gauss) is

$$K := \lambda_1 \lambda_2 = \det S,$$

also a function $\Sigma \to \mathbb{R}$. This is unchanged when you reverse the co-orientation, which suggests that it comes from an intrinsic invariant! The units of the Gauss curvature has units $1/\mathrm{length}^2$.

We also have the unit normal vector field $\nu \colon \Sigma \to S(V) \subset V$, and it tells us things about the curvature too.

Proposition 4.1. $\mathrm{d}\nu_p : T_p\Sigma \to T_p\Sigma$ *equals* $-S_p$.

Proof. Introduce "Euclidean coordinates" x^1, x^2 on $p + T_p\Sigma$, and let $f = f(x^1, x^2)$ be such that near p, Σ is the graph of f. Then,

$$\nu = \nu(x^1, x^2) = \frac{(-f_1, -f_2, 1)}{\sqrt{1 + f_1^2 + f_2^2}},$$

where $f_i = \frac{\partial f}{\partial x^i}$.

Exercise 4.2. Check that this is in fact a unit normal vector.

You can then calculate

$$\mathrm{d}\nu_p = \begin{pmatrix} -\partial_{11} f & -\partial_{12} f \\ -\partial_{21} f & -\partial_{22} f \end{pmatrix}\Bigg|_p,$$

and this is $-\operatorname{Hess}_p f = -II_p$ as desired. (Here, it may help to remember that p is identified with $(0, 0)$.) ⊠

Many people bemoan computations and coordinates, but certainly computations are useful, and coordinates are useful for computations. The solution is to judiciously choose coordinates to make computations simpler.

Now we can cover two beautiful theorems of Gauss, one global, one local.

Theorem 4.3 (Gauss-Bonnet)**.** *Let* $\Sigma \subset E$ *be a closed, co-oriented surface and* $K : \Sigma \to \mathbb{R}$ *be its Gauss curvature. Let* $|\mathrm{d}A|$ *denote its Riemannian measure. Then,*

$$\int_\Sigma K\,|\mathrm{d}A| = 2\pi\chi(\Sigma). \tag{4.4}$$

Some of these words merit an explanation.

- A *closed manifold* is not the same thing as a closed subset: it means Σ is compact and has no boundary. It turns out all closed surfaces in E are co-orientable, but this is not necessarily true for immersed surfaces (e.g. the standard immersion of the Klein bottle).
- The Riemannian measure is discussed in the homework, but the essential idea is that on a Riemannian manifold, we know the lengths and angles of vectors, and therefore of the volume of the parallelogram that a basis $v_1, \ldots, v_n$ of a tangent space spans, namely $|\det(\langle v_i, v_j\rangle_{ij})|$. Thus, we know how to compute volumes, which defines a measure that we can use to integrate functions.
- $\chi(\Sigma)$ is the Euler characteristic of Σ.

Though the proof we'll see uses the embedding (and implicitly the fact that Σ is orientable), all of the notions in (4.4) turn out to be extrinsic, and the theorem holds for abstract closed surfaces with a Riemannian metric, orientable or not.

Example 4.5. Consider a sphere $S^2(R)$ of radius R inside E. Then, every point is umbilic, and the Gauss curvature is $1/R^2$ everywhere. The surface area of the sphere is $4\pi R^2$, so

$$\int_{S^2} K\,|\mathrm{d}A| = 4\pi = 2\pi \cdot 2,$$

and indeed $\chi(S^2) = 2$. ◀

Theorem 4.3 is the first of many theorems which relate local and global geometry. It can be used to calculate global quantities, and to constrain local ones: for example, the sphere cannot have a metric with negative curvature, because its Euler characteristic is positive. The torus T^2 has Euler characteristic $\chi(T^2) = 0$, so any metric on it is either everywhere flat (no curvature) or has points of both positive and negative curvature. The standard embedding into $\mathbb{E}^3$ has points of both positive and negative curvature, but the flat torus can't be embedded isometrically into $\mathbb{E}^3$. It can be embedded into $\mathbb{E}^4$, as the product of two copies of the unit circle in $\mathbb{E}^2$.

Proof of Theorem 4.3. The proof will use the language of differential topology. Recall that if M and M' are oriented manifolds of the same dimension n, we can define the degree of a smooth map $\nu : M' \to M$, and if $\omega \in \Omega^n_M$, then

$$\int_{M'} \nu^*\omega = (\deg \nu) \int_M \omega.$$

In our case, ν is the unit vector map $\nu : \Sigma \to S(V)$; we computed that $\mathrm{d}\nu = -S$ (where S is the shape operator) in Proposition 4.1. Thus,

$$\det(\mathrm{d}\nu) = \det(-S) = K.$$

Let $\omega \in \Omega^2_{S(V)}$ be the area form; then,

$$v^*\omega = (\det \mathrm{d}\nu) \cdot \mathrm{d}A = K\,\mathrm{d}A.$$

Thus, when we integrate,

$$\int_\Sigma K\,\mathrm{d}A = \int_\Sigma \nu^*\omega = (\deg \nu) \int_{S(V)} \omega = 4\pi \deg \nu,$$

since the area of the unit sphere is 4π. Thus, it suffices to show $\deg \nu = \chi(\Sigma)/2$.

The Euler number emerges from the Poincaré-Hopf theorem, that if $\mathbf{v}$ is a vector field with isolated zeroes on Σ, the sum of the indices of $\mathbf{v}$ at its zeroes produces $\chi(\Sigma)$.

Compose ν with the quotient map $S(V) \twoheadrightarrow \mathbb{P}(V)$, and let q be a regular value of this composition, with two preimages $\pm\eta \in S(V)$. η pulls back to a vector field on Σ (constantly pointing in the direction η with unit length). Let ξ_p denote the vector field produced by projecting η onto $T\Sigma$; this has isolated zeros $x_1, \ldots, x_n$.

You can do the computation without coordinates, but it's not hard in them: if $\eta = (0,0,1)$ (which is true up to a rotation), then at any x_i,

$$\xi = \frac{(f_1, f_2, f_1^2 + f_2^2)}{1 + f_1^2 + f_2^2},$$

and you don't have to worry about the denominator in the derivative, so

$$\mathrm{d}\nu_p = \mathrm{d}\xi_p = \begin{pmatrix} \partial_{11} f & \partial_{12} f \\ \partial_{21} f & \partial_{22} f \end{pmatrix}\Bigg|_p. \qquad \boxtimes$$

This is the first connection between topology and geometry.

You might wonder how this can be generalized. In odd dimensions, the Euler characteristic is zero, but for even dimensions, Chern proved the Gauss-Bonnet-Chern theorem in the 1940s which expresses the Euler characteristic in more complicated terms involving the Riemann curvature tensor.

Lecture 5.

Extrinsic and intrinsic curvature: 1/31/17

On the first day, we derived some equations as to when a Riemannian manifold is locally isometric to Euclidean space. Namely, if

$$A_{ijk} := \frac{\partial g_{\ell j}}{\partial x^k} + \frac{\partial g_{\ell k}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^\ell}$$

and

$$\Gamma^i_{jk} := \frac{1}{2} g^{i\ell} A_{\ell jk},$$

then we derived in (1.20)

$$E^i_{jk\ell} = \frac{\partial \Gamma^i_{j\ell}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^\ell} + \Gamma^m_{j\ell}\Gamma^i_{mk} - \Gamma^m_{jk}\Gamma^i_{m\ell},$$

and the Riemann curvature tensor

$$R = R^i_{ijk\ell} \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k \otimes \mathrm{d}x^\ell$$

is an obstruction to a Riemannian manifold being locally isometric to flat, Euclidean space. There's an exercise in the homework to show this is invariant under change of coordinates, and therefore R is an intrinsic object.

Today, we will tie this to the study of curvature of a surface Σ embedded in Euclidean 3-space E. Suppose Σ is co-oriented; then, at any $p \in \Sigma$, we defined the second fundamental form $II_p : T_p\Sigma \times T_p\Sigma \to \mathbb{R}$ and the shape operator $S_p : T_p\Sigma \to T_p\Sigma$ satisfying $II_p(\xi, \eta) = \langle \xi, S_p(\eta) \rangle$. The Gauss curvature is $k_p = \det S_p$, and the normal curvature is $II_p(\xi,\xi)/I_p(\xi,\xi)$.

Locally, Σ is the graph of a function $f = f(x^1, x^2)$ defined on an open neighborhood U in the x^1x^2-plane; here, x^1 and x^2 are special coordinates determined up to an element of O_2.

Theorem 5.1 (Gauss' *Theorema egregium*, c. 1823)**.** *In any of these special local coordinates at p,*

$$R^1_{212}(p) = k_p.$$

The right-hand side is defined extrinsically, determining how curves contained in orthogonal planes bend when embedded in the surface. But the left-hand side is defined intrinsically, depending only on the metric. Thus, the Gauss curvature is an intrinsic quantity, and does not depend on the co-orientation or embedding.

Corollary 5.2. *If Σ, Σ' are two surfaces embedded in E and $\varphi : \Sigma' \to \Sigma$ is an isometry, then $\varphi^* k = k'$.*

This is because the isometry preserves the metric, and the Gauss curvature can be computed only from the metric. This version is closer to how Gauss stated it.

Looking at Corollary 5.2, we know one embedding of the sphere of radius R into E such that the Gauss curvature is $k = 1/R^2$, and that the flat plane has curvature 0. Thus, map projections must be inaccurate: there's no way to map a plane onto any part of the sphere without distorting some length or angle.

The Riemannian curvature tensor on a Riemannian manifold X has a lot of symmetry. From (1.20), one can show that $R^i_{jk\ell} = -R^i_{j\ell k}$: it's skew-symmetric in these arguments. Thus,

$$R = \frac{1}{2} R^i_{jk\ell} \left(\frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j \right) \otimes \mathrm{d}x^k \wedge \mathrm{d}x^\ell.$$

That is, $R \in \Omega^2_X(\operatorname{End} TX)$: the i and j indices give you an endomorphism of each tangent space. In fact, $R \in \Omega^2(\operatorname{SkewEnd} TX)$: the endomorphism is skew-symmetric.

Applying this to when $\dim X = 2$, if $V := T_pX$, then $R_p \in \operatorname{SkewEnd}(V) \otimes \Lambda^2 V^*$. The second component is the top exterior power, hence the *determinant line* $\operatorname{Det} V^*$. Moreover, $\operatorname{SkewEnd}(V) \xrightarrow{\cong} \Lambda^2 V^*$ through the map sending

$$T \longmapsto (\xi, \eta \longmapsto \langle \xi, T\eta \rangle).$$

This is akin to the way we got the shape operator out of the second fundamental form.

Anyways, this means $R_p \in (\operatorname{Det} V^*)^{\otimes 2} = (\operatorname{Det} V^{\otimes 2})^*$. What is this determinant line? The idea is that for every pair of vectors ξ, η, $\xi \wedge \eta$ can be identified with its area. We don't know what area 1 is *per se*, but we know given ξ', η' how to figure out the ratio of the area of $\xi' \wedge \eta'$ to that of $\xi \wedge \eta$, giving us a one-dimensional subspace.

But we do have an orthonormal basis produced by the metric, so we obtain a distinguished unit vector $e \in \operatorname{Det} V$. Thus, we can express $R^1_{212}(p)$ coordinate-independently, by evaluating $R_p \in ((\operatorname{Det} V)^{\otimes 2})^*$ on $e \otimes e \in (\operatorname{Det} V)^{\otimes 2}$.

Proof of Theorem 5.1. Near p, the surface is the graph of a function $(x^1, x^2) \mapsto (x^1, x^2, f(x^1, x^2))$. Let $f_i := \frac{\partial f}{\partial x^i}$, so

$$\left.\frac{\partial}{\partial x^1}\right|_{(x^1,x^2)} = (1, 0, f_1) \in T_{(x^1, x^2, f(x^1, x^2))}\Sigma \subset V$$

$$\left.\frac{\partial}{\partial x^2}\right|_{(x^1,x^2)} = (0, 1, f_2).$$

Let $\Delta := 1 + f_1^2 + f_2^2$. Then, you can calculate that the metric and its inverse satisfy

$$g_{11} = 1 + f_1^2 \qquad g^{11} = \frac{1 + f_2^2}{\Delta}$$

$$g_{12} = f_1 f_2 \qquad g^{12} = -\frac{f_1 f_2}{\Delta}$$

$$g_{22} = 1 + f_2^2 \qquad g^{22} = \frac{1 + f_1^2}{\Delta}.$$

The right-hand side is obtained from the left by inverting the 2×2 matrix for g_{ij}.

Exercise 5.3. Check that $A_{\ell jk} = 2f_\ell f_{jk}$.

Recall that $f(0,0) = f_\ell(0,0) = 0$, so $A_{\ell ij}(0) = 0$ and $\Gamma^i_{jk}(0) = 0$. Thus,

$$R^1_{212}(0,0) = \left.\frac{\partial\Gamma^1_{22}}{\partial x^1}\right|_{(0,0)} - \left.\frac{\partial\Gamma^1_{21}}{\partial x^2}\right|_{(0,0)}.$$

Another plug-and-chug shows that

$$\begin{aligned}\Gamma^1_{22} &= \frac{1}{2}g^{11}A_{122} + \frac{1}{2}g^{12}A_{222}\\ &= \frac{2}{2\Delta}\Big((1+f_2^2)f_1f_{22} - f_1f_2f_2f_{22}\Big)\\ &= \frac{f_1f_{22}}{\Delta}.\end{aligned}$$

A similar calculation shows

$$\Gamma^1_{21} = \frac{f_1f_{21}}{\Delta}.$$

Therefore

$$\begin{aligned}R^1_{212}(0,0) &= (f_{11}f_{22} - f_{12}f_{21})|_{(0,0)}\\ &= \det \mathrm{Hess}_{(0,0)} f\\ &= k_p.\end{aligned}$$
⊠

You should run through these calculations to make sure you understand them.

This provides us an interpretation of R, measuring curvature in different directions on the manifold. If it's equal to 0, the manifold is flat. We'd also like to interpret the Γ^i_{jk} symbols. This should be easier because they're built from first derivatives, whereas R was built from second derivatives.

Let's think about parallelism. In the Euclidean plane E, we have *global parallelism*, that given a vector field $\eta; E \to V$, we can compute its directional derivatives by considering the function $t \mapsto p + t\xi_p$ along a direction ξ_p (thought of as rooted at p). That is, the directional derivative of η in the direction ξ_p is

$$D_{\xi_p}\eta := \lim_{t\to 0}\frac{\eta(p+t\xi_p) - \eta(p)}{t}.$$

If $\gamma : (-\varepsilon,\varepsilon) \to E$ is a curve with $\gamma(0) = p$ and $\dot\gamma(0) = \xi_p$, then

$$D_{\xi_p}\eta = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\eta(\gamma(t)).$$

This doesn't work quite so well on embedded surfaces $\Sigma \hookrightarrow E$. There's a "poor man's parallelism" that translates a vector using the ambient parallelism on E, but there are lots of issues with this: it does not preserve tangency. So you project down onto $T\Sigma$, you say, but then sometimes you get the zero vector, and it feels like parallelism should preserve lengths and angles, right?

Let's ask a smaller question: given an immersed curve $\gamma\colon (-\varepsilon,\varepsilon) \to \Sigma$ with $\gamma(0) = p$ and $\dot\gamma(0) = \xi_p$, can we parallelize?

Definition 5.4. The *covariant derivative* $\nabla_{\xi_p}\eta$ is the orthogonal projection of $D_{\xi_p}\eta \in V$ onto $T_p\Sigma$.

Here, η is a section of the vector bundle $T\Sigma \to \Sigma$, and $\xi_p \in T_p\Sigma$, so $D_{\xi_p}\eta$ is in $T_pE = V$.

Definition 5.5. We say η is *parallel along* $\gamma : (a,b) \to \Sigma$ if $\nabla_{\dot\gamma}\eta = 0$ for all $t \in (a,b)$. If $\nabla_{\dot\gamma}\dot\gamma = 0$ (i.e. $\dot\gamma$ is parallel along γ), then γ is called a *geodesic*.

Here, η is a *vector field along* γ, meaning a section of the pullback bundle $\gamma^*T\Sigma \to (a,b)$. That is, at each t, $(\gamma^*T\Sigma)_t := T_{\gamma(t)}\Sigma$, and these fit together smoothly. So at each t, η chooses a tangent vector in $T_{\gamma(t)}\Sigma$. Thus, if γ is self-intersecting, we get a different tangent vector each time $\gamma(t)$ reaches the intersection point, so everything is still well-behaved.

Geodesics are the curves which have no acceleration along the curve, so the only acceleration is normal to the surface. For example, if you have a geodesic on a sphere (which is a great circle), it's only accelerating perpendicular to the sphere, the minimal acceleration necessary to stay on the sphere.

One of the first things we prove in multivariable calculus is that the directional derivative is linear in the direction. This is still true here, where we derived it from parallelism, among the oldest notions in geometry.

Lemma 5.6.

(1) $\nabla_{\xi_p}\eta$ *is linear in* ξ_p*, i.e.* $\nabla\eta \in T_p^*\Sigma$.

(2) ∇_{ξ_p} *satisfies a Leibniz rule:*

$$\nabla_{\xi_p}(f\eta) = (\xi_p \cdot f)\eta + f\nabla_{\xi_p}\eta.$$

(3)

$$\nabla_{\xi_p}(\eta + \eta') = \nabla_{\xi_p}\eta + \nabla_{\xi_p}\eta'.$$

(4)

$$\xi_p\langle \eta, \eta'\rangle = \langle \nabla_{\xi_p}\eta, \eta'\rangle + \langle \eta, \nabla_{\xi_p}\eta'\rangle.$$

Though we've defined geodesics extrinsically, they are intrinsic, and we'll be able to describe them using the symbols Γ^i_{jk}.

Theorem 5.7. *Let* η *be a vector field on* Σ*. Then,* $\nabla\eta$ *is intrinsic, i.e. determined solely by the metric.*

In particular, $\nabla\eta \in \Omega^1_\Sigma(T\Sigma)$.

Proof. Use coordinates $(x^1, x^2, f(x^1, x^2))$ as before, so Σ is the graph of f. A basis for the tangent space is $\frac{\partial}{\partial x^1} = (1, 0, f_1)$ and $\frac{\partial}{\partial x^2} = (0, 1, f_2)$ as before.

Write $\eta = \eta^i \frac{\partial}{\partial x^i}$ with $\eta^i = \eta^i(x^1, x^2)$ for $i = 1, 2$. Thus, $\eta = (\eta^1, \eta^2, \eta^i f_i)$, so by a Leibniz rule

$$D\eta = (\mathrm{d}\eta^2 1, \mathrm{d}\eta^2, f_i\,\mathrm{d}\eta^i + \eta^i\,\mathrm{d}f_i).$$

In particular, $D\eta_p = (\mathrm{d}\eta^1_p, \mathrm{d}\eta^p_2, *)$ and $\nabla\eta_p = (\mathrm{d}\eta^1_p, \mathrm{d}\eta^2_p, 0)$, or

$$\nabla\eta = \mathrm{d}\nabla^i \cdot \frac{\partial}{\partial x^i},$$

so $\nabla\frac{\partial}{\partial x^i} = 0$ at p.

We used special coordinates x^1, x^2; let's change to arbitrary coordinates y^1, y^2. Calculus on manifolds (or, for grade students, canceling fractions) shows that

$$\frac{\partial}{\partial y^a} = \frac{\partial x^i}{\partial y^a}\frac{\partial}{\partial x^i},$$

so at p,

$$\begin{aligned}\nabla\frac{\partial}{\partial y^a} &= \frac{\partial^2 x^i}{\partial y^b \partial y^a}\,\mathrm{d}y^b \cdot \frac{\partial}{\partial x^i} \\ &= \underbrace{\frac{\partial^2 x^i}{\partial y^b \partial y^a}\frac{\partial y^c}{\partial x^i}}_{Q^c_{ab}}\,\mathrm{d}y^b \cdot \frac{\partial}{\partial y^c}.\end{aligned}$$

We'll finish the proof by showing $Q^c_{ab} = \Gamma^c_{ab}$ as computed in the (y^1, y^2)-coordinate system. Since Γ^c_{ab} doesn't depend on the metric, neither can $\nabla\eta$.

At p,

$$\begin{aligned}g_{ab} = \left\langle \frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b}\right\rangle &= \frac{\partial x^i}{\partial y^a}\frac{\partial x^j}{\partial y^b} g_{ij} \\ &= \sum_i \frac{\partial x^i}{\partial y^a}\frac{\partial x^i}{\partial y^b},\end{aligned}$$

so (again at p),

$$\frac{\partial g_{ab}}{\partial y^c} = \sum_i \frac{\partial^2 x^i}{\partial y^c \partial y^a}\frac{\partial x^i}{\partial y^b} + \frac{\partial x^i}{\partial y^a}\frac{\partial^2 x^i}{\partial y^c \partial y^b}.$$

Therefore

$$A_{dab} = 2\sum_i \frac{\partial^2 x^i}{\partial y^a \partial y^b}\frac{\partial x^i}{\partial y^d}$$

and

$$g^{cd} = \sum_j \frac{\partial y^c}{\partial x^j}\frac{\partial y^d}{\partial x^j}.$$

Thus,

$$\Gamma^c_{ab} = \frac{1}{2}\gamma^{cd}A_{dab} = \sum_{i,j}\frac{\partial y^c}{\partial x^j}\underbrace{\frac{\partial y^d}{\partial x^j}\frac{\partial x^i}{\partial y^d}}_{\delta^i_j}\frac{\partial^2 x^i}{\partial y^a \partial y^b}.$$

Thus, we can collapse to when $i = j$, which recovers Q^c_{ab}. ⊠

Embedded in this proof is the calculation as to how the Γ^k_{ij} change when the coordinates change.

This allows us to define a differential equation for geodesics: if $\eta = \eta^a \frac{\partial}{\partial y^a}$, so that

$$\nabla\eta = \left(\frac{\partial \eta^c}{\partial y^b} + \Gamma^c_{ab}\eta^a\right)\mathrm{d}y^b\frac{\partial}{\partial y^c},$$

then the *geodesic equation* for $\dot\gamma = \xi = \xi^b\frac{\partial}{\partial y^b}$ is

$$\nabla_\xi \xi = \left(\ddot y^a + \Gamma^c_{ab}\dot y^a \dot y^b\right)\frac{\partial}{\partial y^c} = 0. \tag{5.8}$$

That is, for surfaces, we have intrinsic notions of parallelism and geodesics. This holds in more generality. Next time, we'll say one more thing about surfaces in space (looking at the normal component of the directional derivative), and recover the second fundamental form on it. Then, we'll do some background lectures on differential geometry.

Lecture 6.

: 2/2/17

We've talked about how for surfaces, the sectional curvature at a point p is a map $k_p : \mathbb{P}(V) \to \mathbb{R}$. More generally, the Riemann curvature tensor is $R \in \Omega^2_X(\mathrm{SkewEnd}\,TX)$, so for any $x \in X$, if $V = T_xX$, $R_x : \Lambda^2 V \to \mathrm{SkewEnd}(V) \cong \Lambda^2 V^*$, hence determined by a bilinear map $\Lambda^2 V \times \Lambda^2 V \to \mathbb{R}$. If $\Pi \subset V$ is a two-dimensional subspace, we can evaluate $R_x(\Pi, \Pi) \in \mathbb{R}$, so letting Π vary, we obtain the *sectional curvature* $K_x : \mathrm{Gr}_2(T_xX) \to \mathbb{R}$. Here, $\mathrm{Gr}_2(V)$ is the *Grassmannian*, the manifold of 2-dimensional subspaces of V.

Let's return to the case of a co-oriented surface Σ embedded in a 3-dimensional Euclidean space E, and let η be a vector field on Σ. Then, the directional derivative in the direction ξ_p (a vector ξ rooted at p) is $D^{(E)}_{\xi_p}\eta \in \mathbb{R}^3$. This has tangential and normal components:

$$D_{\xi_p}\eta = \underbrace{\nabla_{\xi_p}\eta}_{\text{tangential}} + \underbrace{B(\xi_p, \eta)\cdot \nu}_{\text{normal}}.$$

Last time, we showed in Theorem 5.7 that the tangential part is intrinsic to Σ. If $\gamma : (-\varepsilon, \varepsilon) \to \Sigma$ is a curve, then $\nabla_{\dot\gamma}\eta$ is the covariant derivative of η along γ. We said η is parallel along γ if $\nabla_{\dot\gamma}\eta = 0$, and γ is a geodesic if $\dot\gamma$ is parallel along γ.

Last time, we saw that geodesics are the solutions to the ODE (5.8); by the general theory of ODEs, solutions exist and are unique. Given a smooth curve $\gamma\colon (a,b) \to \Sigma$, a $t_0 \in (a,b)$, and an $\eta_0 \in T_{\gamma(t_0)}\Sigma$, there exists a unique parallel vector field η along γ such that $\eta_{\gamma(t_0)} = \eta_0$.

But local parallelism doesn't imply global parallelism. Consider a geodesic triangle on a sphere,[5] as in Figure 3. If you start with a vector tangent along the upper left piece and parallel-transport it to the lower

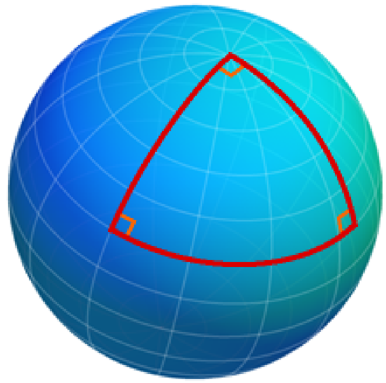

FIGURE 3. A geodesic triangle on the sphere. Each line is a piece of a great circle, and all three angles are right angles. Source: http://world.mathigon.org/Dimensions_and_Distortions.

left corner, then parallel-transport it to the lower-right corner, then parallel-transport it back upm to the pole, you'll end up with a different vector than you started with.

Parallel-transport can be thought of as a way of isometrically identifying tangent spaces (which we can canonically do in Euclidean space, but not always on manifolds).

Proposition 6.1. *Let $\gamma \colon (a, b) \to \Sigma$ be a curve and η_1, η_2 be parallel vector fields along γ. Then, $\gamma\eta_1, \eta_2 \colon \gamma \to \mathbb{R}$ is constant.*

Proof. Let $\xi = \dot{\gamma}$, so

$$\xi\langle \eta_1, \eta_2\rangle = \langle \nabla_\xi \eta_1, \eta_2 + \eta_1, \nabla_\xi \eta_2\rangle = 0. \qquad \boxtimes$$

If γ is closed, traveling along γ from a point p to itself will produce an isometry of $T_p\Sigma$, but not always the identity: in the above example, it was a nontrivial rotation. This is called the *holonomy* around the loop. If the top angle is θ (and the sphere has radius 1), the area of the triangle is θ.

Now let's discuss the geometry of the normal component $B(\xi_p, \eta)$.

Lemma 6.2.

(1) *If $f \in \Omega^0_\Sigma$, $B(\xi_p, f\eta) = fB(\xi_p, \eta)$.*
(2) *B is the second fundamental form.*

Remark. If X is a manifold, $\mathcal{X}(X)$ denotes the space of vector fields on X. We say T is linear if it's $\mathbb{R}$-linear, i.e. for all $\xi, \xi' \in \mathcal{X}(X)$ and $\lambda \in \mathbb{R}$,

$$T(\lambda\eta + \eta') = \lambda T(\eta) + T(\eta').$$

We say T is *linear over functions* if it's Ω^0_X-linear, i.e. for any $f \in \Omega^0_X$ (i.e. $C^\infty(X)$), $T(f\eta) = fT(\eta)$. This means T doesn't differentiate f or anything like that, e.g. $\nabla_{\xi_p}(f\eta) = (\xi_p f)\eta + f\nabla_{\xi_p}\eta$ is not linear over functions. If T is linear over functions, it defines a cotangent vector field. $\blacktriangleleft$

Proof of Lemma 6.2. For the first part, $B(\xi_p, \eta) = \langle D_{\xi_p}\eta, \nu\rangle$, so at p,

$$\begin{aligned} B(\xi_p, f\eta) &= \langle D_{\xi_p} f\eta, \nu\rangle = \langle (\xi_p f)\eta + fD_{\xi_p}\eta, \nu\rangle \\ &= f(p)\langle D_{\xi_p}\eta, \nu\rangle = f(p)B(\xi_p, \eta). \end{aligned}$$

So B determines a bilinear map $B : T_p\Sigma \times T_p\Sigma \to \mathbb{R}$.[6] Let's see that it agrees with II.

[5]There are other surfaces than spheres, of course! Check out the homework for some examples.

[6]Equivalently, $B(\xi_p, \eta)$ only depends on the value of η at p.

Choose coordinates (x^1, x^2) such that Σ is the graph of $f(x^1, x^2)$. Then, $\frac{\partial}{\partial x^1} = (1, 0, f_1)$ and $\frac{\partial}{\partial x^2} = (0, 1, f_2)$ as normal. Write

$$\xi_p = \xi^i \frac{\partial}{\partial x^i}\bigg|_p \qquad \text{and} \qquad \eta = \eta^i \frac{\partial}{\partial x^i}\bigg|_p,$$

for $\xi^i, \eta^i \in \mathbb{R}$, we want to extend η_p to a map on local vector fields. We have liberty in this extension, so let's make our life easier and set $\eta^i(x^1, x^2) = \eta^i$, so it's constant. Thus, $\eta = (\eta^1, \eta^2, \eta^i f_i)$, so at $(x^1, x^2) = (0, 0)$,

$$D_{\xi_p}\eta = (0, 0, \eta^i(\xi_p f_i)) = (0, 0, \eta^i \xi^j f_{ij}).$$

Since $\nu_p = (0, 0, 1)$,

$$B(\xi_p, \eta_p) = f_{ij}\xi^i\eta^j = II(\xi_p, \eta_p). \qquad \boxtimes$$

This provides a coordinate-free interpretation of the second fundamental form, which is nice.

The first chapter of Warner's "Foundations on Differentiable Manifolds and Lie Groups" is a good reference for a lot of this material.

Anyways, this means the directional derivative is

$$D_{\xi_p}\eta = \nabla_{\xi_p}\eta + II_p(\xi_p, \eta_p)\nu_p.$$

We can use this to derive a coordinate-free interpretation of the shape operator:

$$\begin{aligned} II(\xi, \eta) &= \langle D_\xi \eta, \nu \rangle = \xi\langle \eta, \nu\rangle - \langle \eta, D_\xi \nu\rangle \\ &= -\langle \eta, D_\xi \nu\rangle = -\langle \eta, \mathrm{d}\nu(\xi)\rangle, \end{aligned}$$

so $S_p = -\mathrm{d}\nu_p$.

Though we'll begin talking about abstract Riemannian manifolds, these concrete examples, which you can draw, make these ideas clearer, and have fairly direct analogues in the abstract setting.

The choice of a unit tangent vector on a curve is discrete: on each connected component, you can flip between e_1 and $-e_1$. On a surface, though, it's possible to rotate a local frame $\{e_1, e_2, e_3\}$ (where e_1 is normal and e_2 and e_3 are tangential), so there's a continuous choice, and this continues to be true in higher dimensions.

In mathematics, a common approach to studying a situation where one needs to make a choice is to study all choices (sometimes you can make a convenient choice). Thus, we'll have to study this and other structures attached to smooth manifolds, including Lie groups, Lie derivatives, and a little geometry of smooth manifolds. But the payoff is that understanding how the frames change determines a lot of the Riemannian geometry.

Vector fields. Let X be a smooth manifold and ξ be vector field on X.

Definition 6.3. A piecewise-smooth curve $\gamma : (a, b) \to X$ is an *integral curve* of ξ if for all $t \in (a, b)$, $\dot{\gamma}(t) = \xi_{\gamma(t)}$.

That is, ξ is tangent along γ. We'd like to impose some constant-velocity constraint on this, but need a Riemannian metric to do that. It's possible to show that integral curves always exist, by starting with a finite approximation, iterating in a nice manner, and using some soft analysis (the contraction mapping theorem) to show there's a solution.

Theorem 6.4. *Given an $x_0 \in X$ and a vector field ξ on X, there's a unique maximal integral curve $\gamma : (a(x_0), b(x_0)) \to x$ (where $a, b \in [-\infty, \infty]$), such that $\gamma(0) = x_0$ and if $\mu(a, b) \to X$ is an integral curve for ξ with $\mu(0) = x_0$, then $a(x_0) \le a < 0 < b \le b(x_0)$ and $\mu = \gamma|_{(a,b)}$.*

This curve will be called γ_{x_0}.

Definition 6.5. With notation as in the above definition, γ is *complete* if for all $x_0 \in X$, $a(x_0) = -\infty$ and $b(x_0) = \infty$.

Here's a useful sufficient condition:

Theorem 6.6. *If for some Riemannian metric $\langle -, - \rangle$, $\|\xi\| : X \to \mathbb{R}^{\ge 0}$ is bounded, then ξ is complete.*

Corollary 6.7. *If X is a compact manifold, all ξ are complete.*

We would like to travel along a vector field. Let $\varphi(t, x) := \gamma_x(t)$; if ξ is complete, then $\varphi\colon \mathbb{R} \times X \to X$ is well-defined. Otherwise, you may find yourself flowing off the end of the world!

Definition 6.8. A *flow* on a manifold X is a (discrete) group homomorphism $\widehat{\varphi} : \mathbb{R} \to \mathrm{Diff}(X)$ (the latter group is under composition) such that the action map $\varphi\colon \mathbb{R} \times X \to X$ is C^∞.

That is, we ask for $\widehat{\varphi}(t_1 + t_2) = \widehat{\varphi}(t_2) \circ \widehat{\varphi}(t_1)$ and $\varphi(t_1 x) = \widehat{\varphi}(t)(x)$. We don't know how to express smoothness on $\mathrm{Diff}(X)$, so the smoothness criterion is stated in terms of the finite-dimensional manifolds $\mathbb{R}$ and X.

Given a vector field ξ, define for some $t \in \mathbb{R}$ the set

$$\mathcal{D}_t := \{x \in X \mid t \in (a(x), b(x))\}.$$

The following theorem rests on a proof of Theorem 6.4. This is often left unproven in geometry textbooks, but can be found, e.g. in Lang's ODE book or in Coddington-Levinson.

Theorem 6.9.

(1) *$\mathcal{D}_t$ is open.*
(2) *The map $\varphi_t : \mathcal{D}_t \to \mathcal{D}_{-t}$ is a diffeomorphism.*
(3) *The domain of $\varphi_{t_2} \circ \varphi_{t_1}$ is a subset of the domain of $\varphi_{t_1+t_2}$, and on that domain, $\varphi_{t_2} \circ \varphi_{t_1} = \varphi_{t_1+t_2}$.*
(4) *If $x \in X$ and $U \subset X$ is an open set containing x, then there's a $V \subset U$ and an $\varepsilon > 0$ such that $\varphi(-\varepsilon, \varepsilon) \times V$ maps into U.*
(5) *If ξ is complete, then $\mathcal{D}_t = X$ for all t, and φ is a global flow.*

This all relies on ξ being fixed with time (an *autonomous system*). If $\xi = \xi(t)$ varies with time, then a lot of these arguments don't work; in particular $\varphi_{t_2} \circ \varphi_{t_1} \neq \varphi_{t_1+t_2}$. Fortunately, we can use a neat technique to dispatch these.

A vector field ξ is a section of the tangent bundle $p : TX \to X$, and a time-varying vector field $\xi(t)$ is a section of the pullback: if $\pi_2\colon (a, b) \times X \to X$ denotes projection onto the second component, $\widetilde{\xi}$ is a section of the pullback $\pi_2^* TX \to (a, b) \times X$. That is, on each time-slice, you get a section of TX, and these vary smoothly.

Let $\widehat{\widetilde{\xi}} = \frac{\partial}{\partial t} + \widetilde{\xi}$, so $\widehat{\widetilde{\xi}}$ is a vector field on $(a, b) \times X$. Given an initial condition (t_0, x_0), Theorem 6.4 says there's an integral curve $\widehat{\gamma}\colon (\widehat{a}, \widehat{b}) \to (a, b) \times X$. Letting $\widehat{\gamma}(t) = (t, \gamma(t))$, then $\dot{\widehat{\gamma}}(t) = \widehat{\widetilde{\xi}}_{\widehat{\gamma}(t)}$. This is $(1, \dot{\gamma}(t)) = (1, \widetilde{\xi}(t))$, so $\gamma(t)$ is what we were looking for, and the solution exists, at least locally!

Once we have this flow, we're going to look at what happens if you carry various objects along the flow, e.g. vector fields or differential forms.