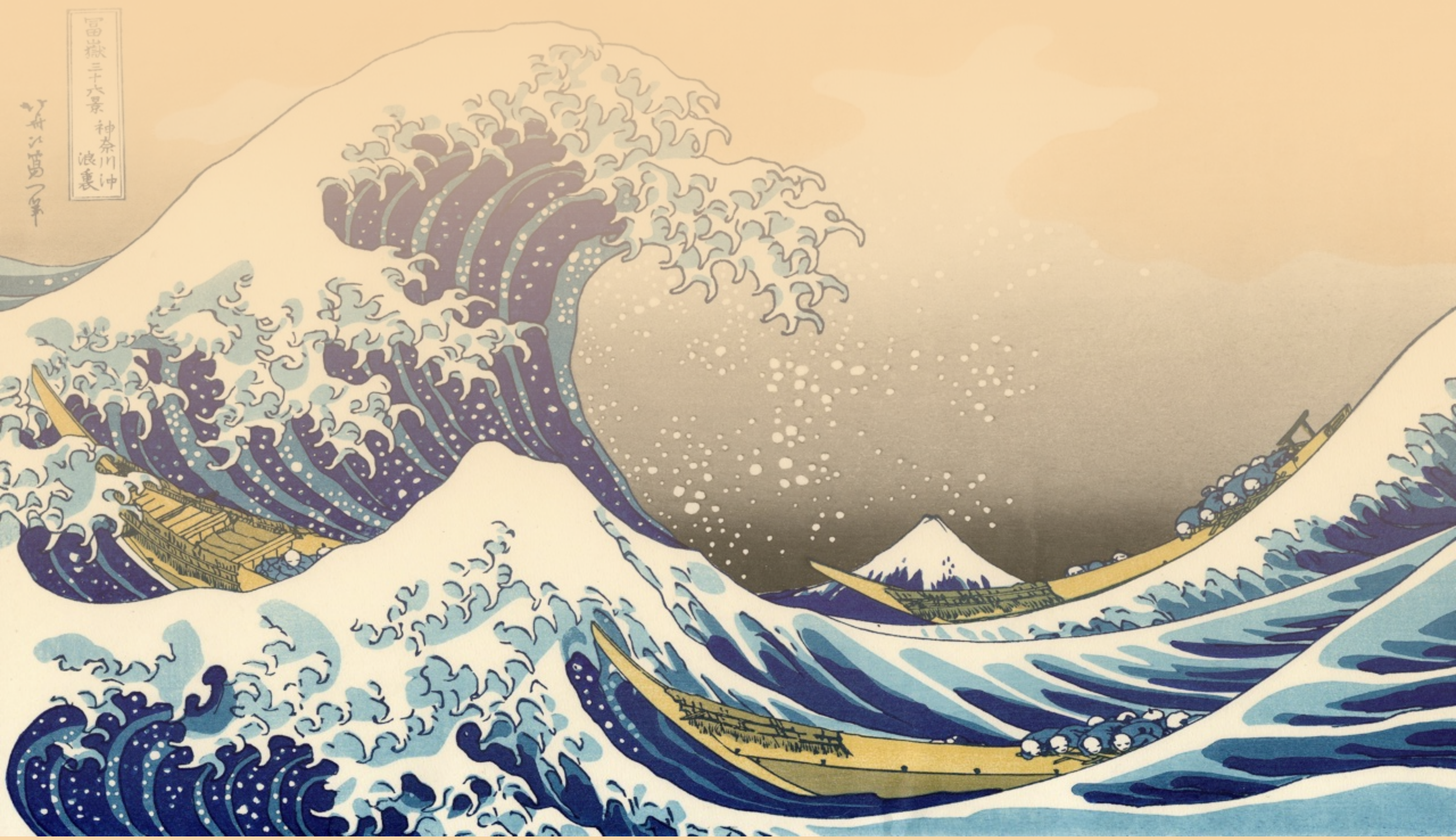
Algebraic Geometry

UT Austin, Spring 2016



M390C NOTES: ALGEBRAIC GEOMETRY

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These notes were taken in UT Austin's Math 390c (Algebraic Geometry) class in Spring 2016, taught by David Ben-Zvi. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

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Lecture 1.

The Course Awakens: 1/19/16

> *"There was a mistranslation in Grothendieck's quote, 'the rising sea:' he was actually talking about raising an X-wing fighter out of a swamp using the Force."*

There are a lot of things that go under the scheme of algebraic geometry, but in this class we're going to use the slogan "algebra = geometry;" we'll try to understand algebraic objects in terms of geometry and vice versa.

There are two main bridges between algebra and geometry: to a geometric object we can associate algebra via functions, and the reverse construction might be less familiar, the notion of a spectrum. This is very similar to the notion of the spectrum of an operator.

We will follow the textbook of Ravi Vakil, *The Rising Sea*. There's also a course website.[1] The prerequisites will include some commutative algebra, but not too much category theory; some people in the class might be bored. Though we're not going to assume much about algebraic sets, basic algebraic geometry, etc., it will be helpful to have seen it.

Let's start. Suppose X is a space; then, there's generally a notion of $\mathbb{C}$-valued functions on it, and this space might be $F(X)$. For example, if X is a smooth manifold, we have $C^\infty(X)$, and if X is a complex manifold, we have the holomorphic functions $\mathrm{Hol}(X)$.[2] Another category of good examples is *algebraic sets*, $X \subset \mathbb{C}^n$ that is given by the common zero set of a bunch of polynomials: $X = \{f_1(x) = \cdots = f_k(x) = 0\}$ for some $f_1, \ldots, f_k \in \mathbb{C}[x_1, \ldots, x_n]$. These have a natural notion of function, *polynomial functions*, which are polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ restricted to X, If $I(X)$ is the functions vanishing on X, then these functions are given by $\mathbb{C}[x_1, \ldots, x_n]/I$.

The point is, on all of our spaces, the functions have a natural ring structure.[3] In fact, there's more: the constant functions are a map $\mathbb{C} \to F(X)$, and since $\mathbb{C}$ is a field, this map is injective. This means $F(X)$ is a $\mathbb{C}$-algebra, i.e. it is a $\mathbb{C}$-vector space with a commutative, $\mathbb{C}$-linear multiplication.

One of the things Grothendieck emphasized is that one should never look at a space (or an anything) on its own, but consider it along with maps between spaces. For example, given a map $\pi : X \to Y$ of spaces, we always have a *pullback* homomorphism $\pi^* : F(Y) \to F(X)$: if $f : Y \to \mathbb{C}$, then its pullback is $\pi^* y(x) = y(\pi(x))$. This tells us that we have a *functor* from spaces to commutative rings.

[1] `https://www.ma.utexas.edu/users/benzvi/teaching/alggeom/syllabus.html`.

[2] The best examples here are Riemann surfaces; when the professor imagines a "typical" or example algebraic variety, he sees a Riemann surface.

[3] In this class, all rings will be commutative and have a 1. Ring homomorphisms will send 1 to 1.

Categories and Functors. This is all done in Vakil's book, but in case you haven't encountered any categories in the streets, let's revisit them.

Definition. A *category* C consists of a set[4] of *objects* $\operatorname{Ob}\mathsf{C}$; if $X \in \operatorname{Ob}\mathsf{C}$, we just say $X \in \mathsf{C}$. We also have for every $X, Y \in \mathsf{C}$ the set $\operatorname{Hom}_{\mathsf{C}}(X, Y)$ of *morphisms*. For every $X, Y, Z \in \mathsf{C}$, there's a *composition map* $\operatorname{Hom}_{\mathsf{C}}(X, Y) \times \operatorname{Hom}_{\mathsf{C}}(Y, Z) \to \operatorname{Hom}_{\mathsf{C}}(Y, Z)$ and a unit $1_X \in \operatorname{Hom}_{\mathsf{C}}(X, X) = \operatorname{End}_{\mathsf{C}}(X)$ satisfying a bunch of axioms that make this behave like associative function composition.

To be precise, we want categories to behave like monoids, for which the product is associative and unital. In fact, a category with one object is a monoid. Thus, we want morphisms of categories to act like morphisms of monoids: they should send composition to composition.

Definition. A *functor* $F : \mathsf{C} \to \mathsf{D}$ is a function $F : \operatorname{Ob}\mathsf{C} \to \operatorname{Ob}\mathsf{D}$ with an induced map on the morphisms:

- If the map acts as $\operatorname{Hom}_{\mathsf{C}}(X, Y) \to \operatorname{Hom}_{\mathsf{D}}(F(X), F(Y))$, F is called a *covariant* functor.
- If it sends $\operatorname{Hom}_{\mathsf{C}}(X, Y) \to \operatorname{Hom}_{\mathsf{D}}(F(Y), F(X))$, then F is *contravariant*.

When we say "functor," we always mean a covariant functor, and here's the reason. Recall that for any monoid A there's the *opposite monoid* A^{op} which has the same set, but reversed multiplication: $f \cdot_{\mathrm{op}} g = g \cdot f$. Similarly, given a category C, there's an *opposite category* C^{op} with the same objects, but $\operatorname{Hom}_{\mathsf{C}^{\mathsf{op}}}(X, Y) = \operatorname{Hom}_{\mathsf{C}}(Y, X)$. Then, a contravariant functor $\mathsf{C} \to \mathsf{D}$ is really a covariant functor $\mathsf{C}^{\mathsf{op}} \to \mathsf{D}$. Hence, in this class, we'll just refer to functors, with opposite categories where needed.

Exercise. Show that a functor $\mathsf{C}^{\mathsf{op}} \to \mathsf{D}$ induces a functor $\mathsf{C} \to \mathsf{D}^{\mathsf{op}}$.

When presented a category, you should always ask what the morphisms are; on the other hand, if someone tells you "the category of smooth manifolds," they probably mean that the morphisms are smooth functions.

Now, we see that pullback is a functor $F : \mathsf{Spaces} \to \mathsf{Ring}^{\mathsf{op}}$. One of the major goals of this class is to define a category of spaces on which this functor is an equivalence. This might not make sense, *yet*. This is the seed of "algebra = geometry."

Definition. Let $F, G : \mathsf{C} \rightrightarrows \mathsf{D}$ be functors. A *natural transformation* $\eta : F \Rightarrow G$ is a collection of maps: for every $X \in \mathsf{C}$, there's a map $\eta_X : F(X) \to G(X)$ satisfying a consistency condition: for every $f : X \to Y$ in C, there's a commutative diagram

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"] & G(Y) \end{tikzcd}$$

That is, a natural transformation relates the objects and the morphisms, and reflects the structure of the category.

Definition. A natural transformation η is a *natural isomorphism* if for every $X \in \mathsf{C}$, the induced $\eta_X \in \operatorname{Hom}_{\mathsf{D}}(F(X), G(X))$ is an isomorphism.

This is equivalent to having a natural inverse to η.

So one might ask, what is the notion for which two categories are "the same?" One might naïvely suggest two functors whose composition is the identity functor, but this is bad. The set of objects isn't very useful: it doesn't capture the structure of the category. In general, asking for equality of objects is worse than asking for isomorphism of objects. Here's the right notion of sameness.

Definition. Let C and D be categories. Then, a functor $F : \mathsf{C} \to \mathsf{D}$ is an *equivalence of categories* if there's a functor $G : \mathsf{D} \to \mathsf{C}$ such that there are natural isomorphisms $FG \to \mathrm{Id}_{\mathsf{D}}$ and $GF \to \mathrm{Id}_{\mathsf{C}}$.

This is a very useful notion, and as such it will be useful to see an equivalence that is not an isomorphism.

[4]This is wrong. But if you already know that, you know that worrying about set-theoretic difficulties is a major distraction here, and not necessary for what we're doing, so we're not going to worry about it.

Exercise. Let k be a field, and let $\mathsf{D} = \mathsf{fdVect}_k$, the category of finite-dimensional vector spaces and linear maps, and let C be the category whose objects are $\mathbb{Z}_{\geq 0}$, the natural numbers, with an object denoted $\langle n \rangle$, and with $\mathrm{Hom}(\langle n \rangle, \langle m \rangle) = \mathrm{Mat}_{m \times n}$. This is a category with composition given by matrix multiplication.

Let $F : \mathsf{C} \to \mathsf{D}$ send $\langle n \rangle \mapsto k^n$, and with the standard realization of matrices as linear maps. Show that F is an equivalence of categories.

This category C has only some vector spaces, but for those spaces, it has all of the morphisms.

Definition. Let $F : \mathsf{C} \to \mathsf{D}$ be a functor.

- F is *faithful* if all of the maps $\mathrm{Hom}_{\mathsf{C}}(X, Y) \hookrightarrow \mathrm{Hom}_{\mathsf{D}}(F(X), F(Y))$ are injective.
- F is *fully faithful* if all of these maps are isomorphsism.
- F is *essentially surjective* if every $X \in \mathsf{D}$ is isomorphic to $F(Z)$ for some $Z \in \mathsf{C}$.

The following theorem will also be a useful tool.

Theorem 1.1. *A functor $F : \mathsf{C} \to \mathsf{D}$ is an equivalence iff it is fully faithful and essentially surjective.*

So, to restate, we want a category of spaces that is the opposite category to the category of rings; this is what Grothendieck had in mind. In fact, let's peek a few weeks ahead and make a curious definition:

Definition. The *category of affine schemes* is $\mathsf{Rings}^{\mathsf{op}}$.

Of course, we'll make these into actual geometric objects, but categorically, this is all that we need.

Recall that if $f : M \to N$ is a set-theoretic map of manifolds, then f is smooth iff its pullback sends C^∞ functions on N to C^∞ functions on M. The first step in this direction is the following theorem, sometimes called *Gelfand duality*.

Theorem 1.2 (Gelfand-Naimark)**.** *The functor $X \mapsto C^0(X)$ (the ring of continuous functions) defines an equivalence between the category of compact Hausdorff spaces and the (opposite) category of commutative C^*-algebras.*

This is an algebro-geometric result: it identifies a category of spaces with the opposite category of a category of algebraic objects.

However, we need to think harder than Gelfand duality in terms of compact, complex manifolds or in terms of algebraic spaces: for example, for $X = \mathbb{CP}^1$, $\mathrm{Hol}(X) = \mathbb{C}$: the only holomorphic functions are constant. The issue is that there are no partitions of unity in the holomorphic or algebraic world. This means we'll need to keep track of local data too, which will lead into the next few lectures' discussions on *sheaf theory*.

Returning to the example of algebraic sets, suppose X and Y are algebraic sets. What is the set of their morphisms? We decided the ring of functions was the polynomial functions $Y \to \mathbb{C}$, so we want maps $X \to Y$ to be those whose pullbacks send polynomial functions to polynomial functions. To be precise, the *ideal of* X is $I(X) = \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid f|_X = 0\}$, defining a map I from algebraic subsets of $\mathbb{C}^n$) to ideals in $\mathbb{C}[x_1, \ldots, x_n]$. There's also a reverse map V,[5] sending an ideal I to $V(I) = \{x \in \mathbb{C}^n \mid f(x) = 0 \text{ for all } f \in I\}$. From classical commutative algebra, it's a fact that this is finitely generated, so it's the vanishing locus of a finite number of polynomials, and therefore in fact an algebraic set.

The dictionary between algebraic sets and ideals of $\mathbb{C}[x_1, \ldots, x_n]$ is one of many versions of the Nullstellensatz (more or less German for the "zero locus theorem"): if J is an ideal, $I(V(J)) = \sqrt{J}$, its radical.

Definition. Let R be a ring and $J \subset R$ be an ideal. Then, the *radical* of J is $\sqrt{J} = \{r \in R \mid r^n \in J \text{ for some } n > 0\}$. One says that J is *radical* if $J = \sqrt{J}$.

What this says is that J is radical iff R/J has no nonzero nilpotents.[6] Why are these kinds of ideals relevant? If $X \subset \mathbb{C}^n$ and f vanishes on X, then so does f^n for all n. That is, radicals encode the geometric property of vanishing, which is why $I(X)$ is a radical ideal.

This is an outline of what classical algebraic geometry studies: it starts by defining algebraic subsets, and establishing a bijection between algebraic subsets of $\mathbb{C}^n$ and radical ideals of $\mathbb{C}[x_1, \ldots, x_n]$. This isn't yet an equivalence of categories. Radical ideals correspond to finitely generated $\mathbb{C}$-algebras with no (nonzero) nilpotents: an ideal I corresponds to the $\mathbb{C}$-algebra $\mathbb{C}[x_1, \ldots, x_n]/I$.

[5] V stands for "vanishing," "variety," or maybe "vendetta."

[6] Recall that if R is a ring, an $r \in R$ is *nilpotent* if $r^n = 0$ for some n.

This is all what the course is *not* about; we're going to replace the category of finitely generated, nilpotent-free $\mathbb{C}$-algebras with the category of *all* rings, but we want to keep some of the same intuition. This involves generalizing in a few directions at once, but we'll try to write down a dictionary; the defining principle is to identify spaces X with rings $R = F(X)$, their ring of functions.

A point $x \in X$ is a map $i_x : x \to X$, so we get a pullback $i_x^* : F(X) \to \mathbb{C}$ given by evaluation at x. Let $\mathfrak{m}_x = \ker(i_x^*)$; since $\mathbb{C}$ is a field, this is a maximal ideal.[7] If k is a field and R is a k-algebra, then R/I is also a k-algebra, so in particular if I is maximal, then $k \hookrightarrow R/I$ is a map of fields, and therefore a field extension. Thus, if k is algebraically closed (e.g. we're studying $\mathbb{C}$) and R is a finitely generated k-algebra, then maximal ideals of R are in bijection with homomorphisms $R \to k$.

Thus, given a ring R, we'll associate a set $\mathrm{MSpec}(R)$, the set of maximal ideals of R, such that R should be its ring of functions. To do this, we'll say that an $r \in R$ is a "function" on $\mathrm{MSpec}(R)$ by acting on an $\mathfrak{m}_x \subset R$ as $r \bmod \mathfrak{m}_x$. This is a "number," since it's in a field, but the notion may be different at every point in $\mathrm{MSpec}(R)$! For example, if $R = \mathbb{Z}$, then $\mathrm{MSpec}(\mathbb{Z})$ is the set of primes, and $n \in \mathbb{Z}$ is a function which at 2 is $n \bmod 2$, at 3 is $n \bmod 3$, and so on.

A perhaps nicer example is when $R = \mathbb{R}[x]$, which has maximal ideals $(x - t)$ for all $t \in \mathbb{R}$. Here, evaluation sends $f(x) \mapsto f(x) \bmod (x - t) = f(t)$. That is, this is really evaluation, and here the quotient field is $\mathbb{R}$. So these look like good old real-valued functions, but these aren't all the maximal ideals: $(x^2 + 1)$ is also a maximal ideal, and $\mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C}$. Then, we do get a kind of evaluation again, but we have to identify points and their complex conjugates.

So we'll have to find a good notion of geometry which generalizes from $\mathbb{C}$-algebras to k-algebras for any field k, to any commutative rings. We'll also have to think about nilpotents: we threw them away by thinking about zero sets, but they play a huge role in ring theory.

Lecture 2.

Some Category Theory: 1/21/16

> *"To this end, we're going to give a crash course in category theory over the next few lectures; the door is over there."*

Remember that our general agenda is to match algebra and geometry; one way to express this idea is to take the category of rings and identify it with some category of geometric objects. However, we're going to reverse the arrows, and we'll get the category of affine schemes. These are some geometric spaces, with a contravariant functor from affine schemes to rings given by taking the ring of functions and a functor in the opposite direction called Spec.

One potential issue is that spaces may not have enough functions, e.g. $\mathbb{CP}^1$ as a complex manifold only has constant functions; as such, we'll enlarge our category to a whole category of schemes, which will also have an algebraic interpretation. Another weird aspect is that functions may take values in varying fields.

Schemes generalize geometry in three different directions: gluing spaces together to ensure we have enough functions is topology, like making manifolds; functions having varying codomains is useful for arithmetic and number theory; and allowing for rings with nilpotents feels a little like analysis.

Last time, we defined $\mathrm{MSpec}(R)$ for a ring R, the set of maximal ideals. It turns out that topology is not sufficient to understand these spaces; for example, the class of *local rings* are those with only one maximal ideal. There are many such rings, e.g. $\mathbb{C}[x]/(x^n)$, whose maximal ideal is (x). In short, MSpec doesn't see nilpotents.

To any ring R, one can attach the category Mod_R, whose objects are R-modules and morphisms are R-linear maps (those commuting with the action of R). This category is one of the more important things one studies in algebra, and we also want to express them in terms of geometric objects that are related somehow to $\mathrm{Spec}\, R$. This should also help us understand the algebraic properties of R-modules too.

Crash Course in Categories. There's a lot of categorical notions in algebraic geometry; it does strike one as a painful way to start a course, but hopefully we can get it out of our systems and move on to geometry knowing what we need. This corresponds to chapters 1 and 2 in the book.

[7] Recall that an ideal $I \subset R$ is *maximal* iff R/I is a field. This is about the level of commutative algebra that we'll be assuming.

We've seen several examples of categories: sets, groups, rings, etc. The next example is a useful class of categories.

Definition. A *poset* is a set S and a relation $\leq$ on S that is
- *reflexive*, so $x \leq x$ for all $x \in \mathsf{S}$;
- *transitive*, so if $x \leq y$ and $y \leq z$, then $x \leq z$; and
- *antisymmetric*, so if $x \leq y$ and $y \leq x$, then $x = y$.

S has the structure of a category: the objects are the elements of S, and $\mathrm{Hom}(x, y)$ is $\{\mathrm{pt}\}$ if $x \leq y$ and is empty otherwise.

Transitivity means that we have composition, and reflexivity gives us identity maps.

This is an unusual example compared to things like "the category of all (somethings)," but is quite useful: a functor from the poset $\bullet \to \bullet$ to another category C is a choice of $A, B \in \mathsf{C}$ and a map $A \to B$; a functor from the poset $\mathbb{N}$ is the same as an infinite sequence in C, and a commutative diagram is the same as a functor out of the category

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

into C.

Example 2.1. A particularly important example of this: if X is a topological space, then its open subsets form a poset under inclusion. Hence, they form a category, called $\mathsf{Top}(X)$. This category is important for sheaf theory, which we will say more about later. For example, if A is an abelian group and $U \subset X$ is open, then let $\mathcal{O}_A(U)$ denote the abelian group of A-valued functions on U (for example, A might be $\mathbb{C}$, so $\mathcal{O}_A(U) = C^\infty(U)$). If $V \subset U$, then restriction of functions defines a map $\mathrm{res}_U^V : \mathcal{O}_A(U) \to \mathcal{O}_A(V)$. Since restriction obeys composition, then we've defined a functor $\mathcal{O}_A : \mathsf{Top}(X)^{\mathsf{op}} \to \mathsf{Ab}$ (or perhaps to $\mathbb{C}$-algebras, or another category); this is a *presheaf of abelian groups* (or $\mathbb{C}$-algebras, etc.).

To be precise, a *presheaf* on X is a functor out of $\mathsf{Top}(X)^{\mathsf{op}}$. This is a way of organizing functions in a way that captures restriction; it will be very useful throughout this class.

Returning to category theory, one of its greatest uses is to capture structure through universal properties, rather than using explicit details of a given category. We'll give a few universal properties here.

Definition. Let C be a category.
- A *final* (or *terminal*) object in C is a $* \in \mathsf{C}$ such that for all $X \in \mathsf{C}$, there's a unique map $X \to *$.
- An *initial* object is a $* \in \mathsf{C}$ such that for all $X \in \mathsf{C}$, there's a unique map $* \to X$.

This is not the last time we'll have dual constructions produced by reversing the arrows.

Example 2.2. If C is a poset, then a terminal object is exactly a maximum element, and an initial object is a minimum element. Thus, in particular, they do not necessarily exist.

Nonetheless, if a final (or initial) object exists, it's necessarily unique.

Proposition 2.3. *Let $*$ and $*'$ be terminal objects in C; then, there's a unique isomorphism $*$ to $*'$.*

Proof. There's a unique map $* \to *$, which therefore must be the identity, and there are unique maps $* \to *'$ and $*' \to *$, so composing these, we must get the identity, so such an isomorphism exists, and it must be unique, since there's only one map $* \to *'$. ⊠

By reversing the arrows, the same thing is true for initial objects. Thus, if such an object exists, it's unique, so one often hears "the" initial or final object. These will be useful for constructing other universal properties.

Example 2.4.

(1) In the category of sets, or in the category of topological spaces, the final object is a single point: everything maps to the point. The initial object is the empty set, since there's a unique (empty) map to any set or space.

(2) In Ab or Vect_k (abelian groups and vector spaces, respectively), 0 is both initial and terminal: the unique map is the zero map. An object that is initial and final is called a *zero object*; as in the case of sets, it may not exist.
(3) In the category of rings, 0 is terminal, but not initial (since a map out of 0 must send $0 = 1$ to 0 and 1). $\mathbb{Z}$ is initial, with the unique map determined by $1 \mapsto 1$.[8]
(4) Even though we don't really understand what an affine scheme is yet, we know that $\operatorname{Spec} \mathbb{Z}$ has to be a terminal object, and $\operatorname{Spec} 0$ has to be the initial object. Since we want this to be geometric, then $\operatorname{Spec} \mathbb{Z}$ will play the role of a point. It might not look like a point, but categorically it behaves like one.
(5) The category of fields is also interesting: setting $1 = 0$ isn't allowed, so there are neither initial nor terminal objects! If we specialize to fields of a given characteristic, then we get a unique map out of $\mathbb{Q}$ or $\mathbb{F}_p$, so the category of fields of a given characteristic is initial.
(6) The poset $\mathsf{Top}(X)$ has $\varnothing$ initial and X terminal: it has top and bottom objects.

The fact that initial and terminal objects are unique means that if you characterize an object in terms of initial or terminal objects, then you know they're unique as soon as they exist.

Definition. If R is a ring, we have the category Alg_R of R-algebras (rings T with the extra structure of a map $R \to T$; morphisms must commute with this map). This is an example of something more general, called an *undercategory*: if C is a category and $X \in \mathsf{C}$, then the undercategory $X \downarrow \mathsf{C}$ is the category whose objects are data of $Y \in \mathsf{C}$ with C-morphisms $a_Y : X \to Y$ and whose morphisms are commutative diagrams

$$\begin{tikzcd} Y_1 \arrow[rr, "\varphi"] & & Y_2 \\ & X. \arrow[ul, "a_{Y_1}"] \arrow[ur, "a_{Y_2}"'] & \end{tikzcd}$$

In the same way, the *overcategory* $X \uparrow \mathsf{C}$ is the same idea, but with maps to X rather than from X (e.g. spaces over a given space X).

Thus, it's possible to concisely define $\mathsf{Alg}_R = R \downarrow \mathsf{Ring}$. We will see other examples of this.

Example 2.5 (Localization)**.** Let R be a ring and $S \subset R$ be a multiplicative subset. Then, the *localization at S* is $S^{-1}R = \{r/s \mid r \in R, s \in S : r/s = r/s' \text{ when } s''(rs' - r's) = 0 \text{ for some } s'' \in S\}$. This is a construction we'll use a lot, so it will be useful to have a canonical characterization of them.

Now, let C be the category of R-algebras T with maps ($\varphi_T : R \to T$ such that (and this is a property, not structure) $\varphi_T(s)$ is invertible in T for all $s \in S$.

Exercise. Show that $S^{-1}R$ is the initial object in C.

Note that the naïve idea that localization is "fractions in S" is true if R is an integral domain, but if we have zero divisors, the R-algebra structure map $R \to S^{-1}R$ need not be injective. But the point is that if T is an R-algebra where the elements of S become invertible, the map φ_T factors through $S^{-1}R$; this means that $S^{-1}R$ is the element of C that's "closest to R." However, you still have to concretely build it to show that it exists; however, we know already that it's determined up to unique isomorphism, so we say "the" localization.

Another very fundamental language for making constructions is that of limits and colimits. It may seem a little strange, but it's quite important.

Definition. Let I be a *small category* (so its objects form a set); in the context of limits, we will refer to it as an *index category*. Then, a functor $A : I \to \mathsf{C}$ is called a *I-shaped* (or *I-indexed*) *diagram in* C.

That is, if $m : i \to j$ is a morphism in I, then this diagram contains an arrow $A(m0 : A_i \to A_j$.

Definition. Let A be an I-shaped diagram in C. Then, a *cone* on A is the data of an object $B \in \mathsf{C}$ and maps $A_i \to B$ for every $i \in I$ commuting with the morphisms in I. The cones on A form a category Cones_A, where the morphisms are maps $B \to B'$ commuting with all the maps in the cone.

[8] That rings and ring homomorphisms are unital is important for this to be true.

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d] \arrow[drr, bend right] & A_3 \arrow[dr] & \\
A_2 \arrow[ur] \arrow[rr] & & B
\end{tikzcd}$$

FIGURE 1. A cone on a diagram A.

We can also take the category of "co-cones," which are data of maps from B *into* the diagram. This is not quite the opposite category (since we want maps $B \to B'$ commuting with the maps into the diagram).[9]

Definition.

- The *colimit* $\varinjlim_I A$ is the initial object in the category of cones of A.
- The *limit* $\varprojlim_I A$ is the terminal object in the category of co-cones of A.

As before, colimits and limits may or may not exist, but if they do, they're unique up to unique isomorphism.

Colimits act like a quotient, and it's easier to map out of them. Correspondingly, limits behave like a subobject, and it's easier to map into them.

Example 2.6 (Products and Coproducts)**.** Let $I = \bullet\ \bullet$ be a two-element discrete set (no non-identity arrows). Thus, an I-shaped diagram is just a choice of two spaces A_1 and A_2, so a colimit C_1 is the data of a unique map φ_B for each $B \in \mathsf{C}$ fitting into the following diagram.

$$\begin{tikzcd}
A_1 \arrow[dr] \arrow[drr, bend left] & & \\
& C_1 \arrow[r, "\varphi_B"] & B \\
A_2 \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}$$

This is called the *coproduct* of A_1 and A_2, denoted $A_1 \sqcup A_2$ or $A_1 \coprod A_2$.

Similarly, the limit of A is called the *product* of A_1 and A_2, is denoted $A_1 \times A_2$, and fits into the diagram

$$\begin{tikzcd}
& & A_1 \\
B \arrow[r] \arrow[urr] \arrow[drr] & A_1 \times A_2 \arrow[ur] \arrow[dr] & \\
& & A_2
\end{tikzcd}$$

In the same way, if I is a larger discrete set, we get coproducts and products of objects in C indexed by I, denoted $\coprod_I A_i$ and $\prod_I A_i$, respectively.

In the category of sets, the product is Cartesian product, and the coproduct is disjoint union. The same is true in topological spaces.

In the category of groups, the product is once again Cartesian product, but the coproduct is the free product (mapping out of it is the same as mapping out of the individual components, which is not true of the direct product). Note that this is distinct as underlying sets from the coproduct of sets.

In linear categories, e.g. Ab, Mod_R, or Vect_k, $V \oplus W$ is the product and coproduct, and the same is true over all finite I. However, this is *not* true when I is infinite: the coproduct is the direct sum, which takes finite sums of elements, and the product is the Cartesian product, which takes arbitrary sums of elements. It's worth working out why this is, and how it works.

Many of these categories are "sets with structure," e.g. groups, vector spaces, topological spaces, and so on. In these cases, there is a *forgetful functor* which forgets this structure: indeed, a group homomorphisms (continuous map, linear map) is a map of sets too.[10]

[9]Some people switch the definitions of cones and co-cones, but since we're not going to use these words very much, it doesn't matter all that much.

[10]If this seems vague, that's all right; it's possible to define and find forgetful functors more formally.

There's a useful principle here: *forgetful functors preserve limits*: if F is a forgetful functor, then there is a canonical isomorphism $F(\varprojlim A) \cong \varprojlim F(A)$. This is something that can be defined more rigorously and proven. But one important corollary is that if you know what the limit looks like for sets, it's the same in groups, rings, vector spaces, topological spaces, and so on. However, this is very false for coproducts, e.g. the coproduct on groups is not the same as the one on sets.

This becomes a little cooler once we see limits that aren't just products.

Example 2.7. Consider the diagram of rings

$$\cdots \longrightarrow \mathbb{Z}/p^n \longrightarrow \cdots \longrightarrow \mathbb{Z}/p^3 \longrightarrow \mathbb{Z}.p^2 \longrightarrow \mathbb{Z}/p,$$

where each map is given by modding out by p. One can show that the limit exists, and it'll be the same as the limit of the underlying sets, a sequence of compatible elements; this limit is called the *p-adic integers*, denoted $\mathbb{Z}_p$. More generally, the same thing works for $\varprojlim R/I^n$ for an ideal $I \subset R$, and defines the *I-adic completion* $\widehat{R}_I$, which we'll revisit, since it has useful geometric meaning.