CHARACTERISTIC CLASSES

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CONTENTS

These are lecture notes for a series of five lectures I gave to other graduate students about characteristic classes through UT Austin's summer minicourse program (see https://www.ma.utexas.edu/users/richard.wong/Minicourses.html for more details). Beware of potential typos. In these notes I cover the basic theory of Stiefel-Whitney, Wu, Chern, Pontrjagin, and Euler classes, introducing some interesting topics in algebraic topology along the way. In the last section the Hirzebruch signature theorem is introduced as an application. Many proofs are left out to save time. There are many exercises, which emphasize getting experience with characteristic class computations. Don't do all of them; you should do enough to make you feel comfortable with the computations, focusing on the ones interesting or useful to you.

Prerequisites. Formally, I will assume familiarity with homology and cohomology at the level of Hatcher, chapters 2 and 3, and not much more. There will be some differential topology, which is covered by UT's prelim course. Some familiarity with vector bundles will be helpful, but not strictly necessary.

The exercises may ask for more; in particular, you will probably want to know the standard CW structures on $\mathbb{RP}^n$ and $\mathbb{CP}^n$, as well as their cohomology rings.

References. Most of this material has been synthesized from the following sources.

- Milnor-Stasheff, "Characteristic classes," which fleshes out all the details we neglect.
- Freed, "Bordism: old and new." https://www.ma.utexas.edu/users/dafr/bordism.pdf. The material in §§6–8 is a good fast-paced introduction to classifying spaces, Pontrjagin, and Chern classes.
- Hatcher, "Vector bundles and K-theory," chapter 3. https://www.math.cornell.edu/~hatcher/VBKT/VB.pdf.
- Bott-Tu, "Differential forms in algebraic topology," chapter 4.

1. Five approaches to characteristic classes

Today, we're going to discuss what characteristic classes are. The definition is not hard, but there are at least four ways to think about them, and each perspective is important. This will also be an excuse to introduce some useful notions in geometry and topology — though this will be true every day.

1.1. **Characteristic classes: what and why.**

Characteristic classes are natural cohomology classes of vector bundles. Let's exposit this a bit.

Definition 1.1. Recall that a *(real) vector bundle* over a space M is a continuous map $\pi\colon E \to M$ such that

(1) each fiber $\pi^{-1}(m)$ is a finite-dimensional real vector space, and
(2) there's an open cover $\mathfrak{U}$ of M such that for each $U \in \mathfrak{U}$, $\pi^{-1}(U) \cong U \times \mathbb{R}^n$, and this isomorphism is linear on each fiber.

That is, it's a continuous family of vector spaces over some topological space. We allow $\mathbb{C}^n$ and *complex vector bundles*. Often our spaces will be manifolds, and our vector bundles will usually be smooth. We will often assume the dimension of a vector bundle on a disconnected space is constant.

Example 1.2.

(1) The *tangent bundle* $TM \to M$ to a manifold M is the vector bundle whose fiber above $x \in M$ is T_xM.
(2) A *trivial bundle* $\underline{\mathbb{R}}^n := \mathbb{R}^n \times M \twoheadrightarrow M$.
(3) The *tautological bundle* $S \to \mathbb{RP}^n$ is a line bundle defined as follows: each point $\ell \in \mathbb{RP}^n$ is a line in $\mathbb{R}^{n+1}$; we let the fiber above ℓ be that line. The same construction works over $\mathbb{CP}^n$, and Grassmannians. ◀

It's also possible to make new vector bundles out of old: the usual operations on vector spaces (direct sum, tensor product, dual, Hom, symmetric power, and so on) generalize to vector bundles without much fuss. Vector bundles also pull back.

Definition 1.3. Let $\pi\colon E \to M$ be a vector bundle and $f : N \to M$ be continuous. Then, the *pullback* of E to N, denoted $f^*E \to N$, is the vector bundle whose fiber above an $x \in N$ is $\pi^{-1}(f(x))$.

One should check this is actually a vector bundle.

With these words freshly in our minds, we can define characteristic classes.

Definition 1.4. A *characteristic class* c of vector bundles is an assignment to each vector bundle $E \to M$ a cohomology class $c(E) \in H^*(M)$ that is *natural*, in that if $f : N \to M$ is a map, $c(f^*E) = f^*(c(E)) \in H^*(N)$.

Characteristic classes can be for real or complex vector bundles, but usually not both at once. The coefficient group for $H^*(M)$ will vary.

You probably have motivations in mind for learning characteristic classes, but here are some more just in case.

- Vector bundles interpolate between geometric and algebraic information on manifolds — often they arise in a geometric context, but they're classified with algebra. Characteristic classes provide useful algebraic invariants of geometric information.
- More specifically, the obstructions to certain structures on a manifold (orientation, spin, etc) are captured by characteristic classes, so computations with characteristic classes determine which manifolds are orientable, spin, etc.
- Pairing a product of characteristic classes against the fundamental class defines a *characteristic number*. These are cobordism invariants, and in many situations the set of characteristic numbers is a complete cobordism invariant, and a computable one. Fancier characteristic numbers have geometric meaning and are useful for proving geometric results, e.g. in the Atiyah-Singer index theorem.

We'll now discuss four approaches to characteristic classes. These are not the only approaches; however, they are the most used and most useful ones. All approaches work in the setting of Chern classes, characteristic classes of complex vector bundles living in integral cohomology; most generalize to other characteristic classes, but not all of them.

1.2. **Axiomatic approach.**

The axiomatic definition of Chern classes is due to Grothendieck.

Definition 1.5. The *Chern classes* are characteristic classes for a complex vector bundle $E \to M$: for each $i \ge 0$, the i^{th} Chern class of E is $c_i(E) \in H^{2i}(M;\mathbb{Z})$. The *total Chern class* $c(E) = c_0(E) + c_1(E) + \cdots$. One writes $c_i(M)$ for $c_i(TM)$, and $c(M)$ for $c(TM)$.

These classes are defined to be the unique classes satisfying naturality and the following axioms.

(1) $c_0(E) = 1$.
(2) The *Whitney sum formula* $c(E \oplus F) = c(E)c(F)$, and hence

$$c_k(E \oplus F) = \sum_{i+j=k} c_i(E)c_j(F).$$

(3) Let x be the generator of $H^2(\mathbb{CP}^n) \cong \mathbb{Z}$; then, $c(S \to \mathbb{CP}^n) = 1 - x$.[1]

[1]There are two choices of such x; we define it to be Poincaré dual to a hyperplane $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$ with the orientation induced from the complex structure.

Of course, it's a theorem that these exist and are unique! Thus, all characteristic-class calculations can theoretically be recovered from these, though other methods are usually employed. However, some computations follow pretty directly, including one in the exercises.

So what are these telling us?

Example 1.6. Let $\underline{\mathbb{C}}^n \to M$ be a trivial bundle. Then, $c(\underline{\mathbb{C}}^n) = 1$. This is because $\underline{\mathbb{C}}^n$ is a pullback of the trivial bundle over a point. ⊲

Thus the Chern classes (and characteristic classes more generally) give us a necessary condition for a vector bundle to be trivial.

Definition 1.7. A complex vector bundle $E \to M$ is *stably trivial* if $E \oplus \underline{\mathbb{C}}^n$ is a trivial vector bundle.

We'll also use the analogous definition for real vector bundles.

Lemma 1.8. *$c(E \oplus \underline{\mathbb{C}}) = c(E)$, and hence if E is stably trivial, then $c(E) = 1$.*

Proof. Whitney sum formula. ⊠

This approach is kind of rigid, and also provides no geometric intuition.

1.3. **Linear dependency of generic sections.** This approach is geometric and slick, but one must show it's independent of choices.

To discuss it, we need one important fact, Poincaré duality.

Theorem 1.9 (Poincaré duality)**.** *Let M be a closed manifold.*

(1) *Let A be an abelian group. An orientation of M determines an isomorphism $\mathrm{PD}\colon H^k(M;A) \to H_{n-k}(M;A)$ given by cap product with the fundamental class.*
(2) *There is isomorphism $\mathrm{PD}\colon H^k(M;\mathbb{Z}/2) \to H_{n-k}(M;\mathbb{Z}/2)$ given by cap product with the mod 2 fundamental class.*

This theorem is pretty much the best.

Definition 1.10. Let M and N be oriented manifolds and $i\colon N \hookrightarrow M$ be an embedding. Hence it defines a pushforward $i_*[N] \in H^*(M)$; we will refer to this as the *homology class represented by N*, and N as a *representative* for this homology class.

We'll do the same thing in homology with coefficients in any abelian group A; when $A = \mathbb{Z}/2$, no orientation is necessary.

Definition 1.11. Let $y \in H^k(M)$. A *Poincaré dual submanifold* to y is an embedded, oriented submanifold $N \subset M$ which represents $\mathrm{PD}(y) \in H_{n-k}(M)$. Correspondingly, the *Poincaré dual* to an embedded oriented submanifold $i\colon N \hookrightarrow M$ is $\mathrm{PD}(i_*[N]) \in H^{\operatorname{codim} N}(M)$.

Again, the above applies, *mutatis mutandis*, to cohomology with $\mathbb{Z}/2$-coefficients, but without orientations.

Definition 1.12. Let $\pi\colon E \to M$ be a complex vector bundle over a manifold M. Then, choose k sections $s_1, \dots, s_k \in \Gamma(E)$ that are transverse to each other and to the zero section. (It's a theorem in differential topology that this is always possible.)

Let Y_k be the *locus of dependency* of $s_1, \dots, s_k$, i.e. the subset of $x \in M$ on which $\{s_1(x), \dots, s_k(x)\} \in \pi^{-1}(x)$ is linearly dependent. Then, Y_k is a smooth k-dimensional submanifold of M. The k^{th} *Chern class* of E, denoted $c_k(E)$, is the Poincaré dual of Y_k.

This definition provides a perspective: a Chern class is an obstruction to finding everywhere linearly independent sections of your vector bundle.

1.4. **Chern-Weil theory.** Any concept that appears in the real cohomology of a manifold can be expressed with de Rham theory, and Chern-Weil theory does this for Chern classes.

Definition 1.13. Let $E \to M$ be a vector bundle. A *connection* on E is an $\mathbb{R}$-linear map $\nabla\colon \Gamma(TM) \otimes_{\mathbb{R}} \Gamma(E) \to \Gamma(E)$ that is $C^\infty(M)$-linear in its first argument and satisfies the Leibniz rule

$$\nabla_v(f\psi) = (v \cdot f)\psi = f\nabla_v\psi.$$

where v is a vector field, $\psi \in \Gamma(E)$, and $f \in C^\infty(M)$.

This is a way of differentiating vector fields. Locally (i.e. in coordinates U), a connection is like the de Rham differential, but plus some matrix-valued one-form $A \in \Gamma(T^*U \otimes \mathrm{End}(E|_U))$: $\nabla|_U = \mathrm{d} + A$. So if you have coordinates, you can define a connection through a matrix.

Definition 1.14. Let ∇ be a connection. Its *curvature* is $F_\nabla \in \Omega^2_M(\mathrm{End}\, E) := \Gamma(\Lambda^2 T^*M \otimes \mathrm{End}\, E)$ defined by

$$F_\nabla := \nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X - \nabla_{[X,Y]}.$$

That is, it's a 2-form, but instead of being valued in T^*M, it's valued in $\mathrm{End}\, E$. If E is a line bundle, this is canonically trivial, so the curvature of a connection on a line bundle is just a differential 2-form, and in fact it's closed, so it represents a class on $H^2_{\mathrm{dR}}(M)$. This is $2\pi i$ times the first Chern class of that line bundle.

The *trace* $\mathrm{tr}\colon \Omega^k_M(\mathrm{End}\, E) \to \Omega^k_M$ is the map induced from the map $\Gamma(\mathrm{End}\, E) \to C^\infty(M)$ which takes the trace at each point. As before, one can show that $\mathrm{tr}((F_\nabla)^k) \in \Omega^{2k}_M$ is closed, hence defines a de Rham cohomology class.

Definition 1.15. The k^{th} *Chern class* of E is $(1/2\pi i)[\mathrm{tr}((F_\nabla)^k)] \in H^{2k}_{\mathrm{dR}}(M)$.

Though this is *a priori* only in $H^{2k}_{\mathrm{dR}}(M) \otimes \mathbb{C}$, it's an integral class (as the other definitions we've given were for $\mathbb{Z}$-cohomology), and it doesn't depend on the choice of connection. The proof idea is that the space of connections is convex, so you can interpolate between two connections.

So from this perspective, a Chern class measures curvature.

Corollary 1.16. *If E admits a flat connection, its (rational) Chern classes are* 0*, and its integral Chern classes are torsion.*

1.5. **The search for the universal bundle.** The final approach for today is moduli-theoretic. It's possible to construct a maximally twisted vector bundle: all vector bundles (of a given kind) are pullbacks of a universal vector bundle over a universal space.

Let G be a Lie group. Recall that a *principal G-bundle* is kind of like a vector bundle, but for G-torsors instead of $\mathbb{R}^n$: it's a map $\pi\colon E \to M$ such that E carries a G-action and locally, $\pi^{-1}(U) \cong G \times U$ as G-spaces.

By EG we will mean any contractible space with a free G-action, and $BG := EG/G$. Hence $EG \to BG$ is a principal G-bundle.

Proposition 1.17. *Any two choices for BG are homotopy equivalent.*

Example 1.18. Let $\mathscr{H}$ be a separable Hilbert space and S^∞ denote the unit sphere in $\mathscr{H}$, which is contractible. The antipodal map defines a free $\mathbb{Z}/2$-action on S^∞, and its quotient, denoted $\mathbb{RP}^\infty$, is a model for $B\mathbb{Z}/2$. ◂

This model for $B\mathbb{Z}/2$ realizes it as a Hilbert manifold, and in fact for any compact Lie group G, BG has a model as a Hilbert manifold. There are other constructions, e.g. defining $\mathbb{RP}^\infty$ as a colimit of finite-dimensional spaces (which is not homeomorphic to the Hilbert manifold description) or using the bar construction, which works in great generality.

Let $\mathrm{Bun}_G\, M$ denote the set of isomorphism classes of principal G-bundles over M.

Theorem 1.19. *Let M be a space. Then, the assignment $[M, BG] \to \mathrm{Bun}_G\, M$ sending $f : M \to BG$ to the pullback $f^*(EG) \to M$ is a bijection.*

That is, every principal G-bundle arises from $EG \to BG$ in an essentially unique way.

Definition 1.20. Let $E \to M$ be a complex vector bundle with a metric. Its *bundle of unitary frames* is the principal U_n-bundle that over a point $x \in M$, is the set of unitary bases for E_x, i.e. isometries $b\colon E_x \to \underline{\mathbb{C}}^n$.

The analogous definition may be made for a real bundle with a metric. The isomorphism class of $\mathscr{B}(E)$ does not depend on the choice of metric.

Proposition 1.21. *There's a natural bijection between the isomorphism classes of complex vector bundles of rank n and $\mathrm{Bun}_{\mathrm{U}_n}(M)$ defined by sending $E \mapsto \mathscr{B}(E)$. The same is true for real vector bundles and $\mathrm{Bun}_{\mathrm{O}_n}$.*

"Natural" here means this bijection is compatible with pullback.

So in other words, given a complex vector bundle $E \to M$ of rank n, we get a principal U_n-bundle, hence a homotopy class of maps $f_E\colon M \to B\mathrm{U}_n$. If $c \in H^*(B\mathrm{U}_n)$, then let $c(E) := f_E^* c$. This satisfies naturality, hence is a characteristic class, and all characteristic classes for rank-n vector bundles arise this way, because all principal U_n-bundles are pullbacks of $E\mathrm{U}_n \to B\mathrm{U}_n$!

In other words, *a characteristic class is a cohomology class of the classifying space*.

Of course, we'd like to treat characteristic classes for all vector bundles at once, not just those of rank n. This is where stability jumps in: a rank-n vector bundle E defines a rank-$(n+1)$-vector bundle $E\oplus\underline{\mathbb{C}}$ which should have the same Chern classes. In the classifying-space framework, there's a map $\mathrm{U}_n \hookrightarrow \mathrm{U}_{n+1}$ sending

$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix},$$

which induces a map $B\mathrm{U}_n \to B\mathrm{U}_{n+1}$.[2] If $f_E\colon M \to B\mathrm{U}_n$ is the classifying map for E, then the classifying map for $E\oplus\underline{\mathbb{C}}$ is the map $M \xrightarrow{f_E} B\mathrm{U}_n \to B\mathrm{U}_{n+1}$.

So now we have a directed system $B\mathrm{U}_1 \hookrightarrow B\mathrm{U}_2 \hookrightarrow \cdots$, and any vector bundle defines compatible maps to objects in this system. Hence, the classifying space for vector bundles of any (finite) rank is

$$B\mathrm{U} := \operatorname*{colim}_{n\to\infty} B\mathrm{U}_n.$$

That is, a homotopy class of maps $M \to B\mathrm{U}$ defines a stable isomorphism class of vector bundles $E \to M$, and characteristic classes are exactly elements of the cohomology of $B\mathrm{U}$! Exactly the same story goes forth to define $B\mathrm{O}$ and characteristic classes for real vector bundles.[3]

Theorem 1.22. *$H^*(B\mathrm{U}) \cong \mathbb{Z}[c_1, c_2, \dots]$, with $|c_k| = 2k$.*

Thus we can define the k^{th} *Chern class* to be c_k. Naturality and stability follow almost immediately.

Remark. This approach tells us that cohomology classes of BG define characteristic classes for principal G-bundles, not just vector bundles, and this approach is sometimes useful. ◂

1.6. **Exercises.** Most important:

(1) In this exercise, we'll compute $c(\mathbb{CP}^n) = (1+x)^{n+1}$, where $x \in H^2(\mathbb{CP}^n) \cong \mathbb{Z}$ is a generator, Poincaré dual to $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$.
   (a) Let $Q = \underline{\mathbb{C}}^{n+1}/S$, the universal quotient bundle: its fiber over an $\ell \in \mathbb{CP}^n$ is $\mathbb{C}^{n+1}/\ell$. Show that $\mathrm{Hom}(S,Q) \cong T\mathbb{CP}^n$. (Hint: let ℓ be a complex line in $\mathbb{C}^{n+1}$ and $\ell^\perp$ be a complimentary subspace, i.e. $\ell \oplus \ell^\perp \cong \mathbb{C}^{n+1}$. Then, $\mathrm{Hom}(\ell,\ell^\perp)$ can be identified with the neighborhood of $\ell \in \mathbb{CP}^n$ of lines which are graphs of functions $\ell \to \ell^\perp$.)
   (b) Using this, show that $T\mathbb{CP}^n \oplus \mathrm{Hom}(S,S) \cong (S^*)^{\oplus(n+1)}$.
   (c) If E is any line bundle, show that $\mathrm{Hom}(E,E)$ is trivial.
   (d) If $E \to \mathbb{CP}^n$ is a line bundle, show that $c_1(E^*) = -c_1(E)$. (Hint: use the fact that $E^* \cong \overline{E}$ and naturality of Chern classes.)
   (e) Applying (1c) and (1d) to (1b), conclude $c(\mathbb{CP}^n) = (1+x)^{n+1}$.

(2) If $E \to M$ is a vector bundle, its determinant bundle $\mathrm{Det}\, E \to M$ is its top exterior power, which is a line bundle. Use the locus-of-dependency definition of Chern classes to show that $c_1(E) = c_1(\mathrm{Det}\, E)$.

(3) Use Chern-Weil theory to compute the Chern classes of $\mathbb{CP}^1$ and $\mathbb{CP}^2$.

Also important, especially if you're interested:

(1) Show that TS^2 is stably trivial, but not trivial. What's an example of a manifold whose tangent bundle isn't stably trivial?

(2) Show that if G is discrete, any Eilenberg-Mac Lane space $K(G,1)$ is a model for BG, and vice versa. Hence $S^1 = B\mathbb{Z}$ and $\mathbb{RP}^\infty = B\mathbb{Z}/2 = B\mathrm{O}_1$.

(3) In this exercise, we construct $B\mathrm{O}_n$ as an infinite-dimensional manifold. Fix a separable Hilbert space, such as ℓ^2. The Stiefel manifold $\mathrm{St}_n(\ell^2)$ is the set of linear isometric embeddings $\mathbb{R}^n \hookrightarrow \ell^2$ (i.e. injective linear maps preserving the inner product), topologized as a subspace of $\mathrm{Hom}(\mathbb{R}^n, \ell^2)$. O_n acts on $\mathrm{St}_k(\ell^2)$ by precomposition.

   The infinite-dimensional Grassmannian $\mathrm{Gr}_n(\ell^2)$ is the space of n-dimensional subspaces of ℓ^2, topologized in a similar way to finite-dimensional Grassmannians. There's a projection $\pi\colon \mathrm{St}_n(\ell^2) \twoheadrightarrow \mathrm{Gr}_n(\ell^2)$ sending a map $b\colon \mathbb{R}^n \to \ell^2$ to its image.

[2]Technically, it induces a homotopy class of maps. But there are models for BG which make B a functor on the nose.

[3]The notation is suggestive, and in fact $B\mathrm{U}$ is the classifying space for the infinite unitary group U, the colimit of U_n over all n.

(a) Show that $\mathrm{St}_n(\ell^2)$ is contractible. (Hint: if e_i denotes the sequence with a 1 in position i and 0 everywhere else, define two homotopies, one which pushes any embedding to one orthogonal to the standard embedding $s\colon \mathbb{R}^n \to \ell^2$ as the first n coordinates, and the other which contracts the subspace of embeddings orthogonal to s onto s).
(b) Show that the O_n-action on $\mathrm{St}_n(\ell^2)$ is free, so $\mathrm{St}_n(\ell^2)$ is an $E\mathrm{O}_n$.
(c) Show that $\pi\colon \mathrm{St}_n(\ell^2) \to \mathrm{Gr}_n(\ell^2)$ is the quotient by the O_n-action, so $\mathrm{Gr}_n(\ell^2)$ is a $B\mathrm{O}_n$.

(4) Show that the definition of Chern classes as cohomology classes on $B\mathrm{U}$ satisfies the axiomatic characterization of Chern classes. Hint: $\mathbb{CP}^\infty = \mathrm{colim}_n\, \mathbb{CP}^n$ is a $B\mathrm{U}_1$ with a standard CW structure, and the inclusion $\mathbb{CP}^n \hookrightarrow \mathbb{CP}^\infty$ is cellular (for the standard CW structure on $\mathbb{CP}^n$). Conversely, show that the axiomatic definition of Chern classes implies they pull back from characteristic classes on $B\mathrm{U}_n$, and agree under the map $B\mathrm{U}_n \to B\mathrm{U}_{n+1}$, and hence are unique.

Additional exercises:

(1) Why is the category of vector bundles not abelian?
(2) Make sense of the idea that "vector bundles can untwist under pullback, but can't become more twisted."
(3) Verify that S^∞ is contractible.

2. Stiefel-Whitney classes

The first characteristic classes we'll discuss are Stiefel-Whitney classes, which are characteristic classes for real vector bundles in $\mathbb{Z}/2$ cohomology. This will make things slightly easier, so when the same ideas appear again for Chern and Pontrjagin classes on Thursday, they will already be familiar.

2.1. A Definition of Stiefel-Whitney classes.

Last time we emphasized that there are many ways to define and think about characteristic classes. To get off the ground, we're going to use one approach, and then state some properties. Other definitions are possible.

Theorem 2.1. *As graded rings, $H^*(B\mathrm{O};\mathbb{F}_2) \cong \mathbb{F}_2[w_1, w_2, w_3, \dots]$, with $|w_i| = i$.*

Hence any characteristic class for real vector bundles in mod 2 cohomology is a polynomial in these classes.

Definition 2.2. The characteristic class defined by $w_i \in H^i(B\mathrm{O};\mathbb{F}_2)$ is called the i^{th} *Stiefel-Whitney class*. We also let $w_0 = 1$. The *total Stiefel-Whitney class* is $w(E) := 1 + w_1(E) + w_2(E) + \cdots$. If M is a manifold, $w(M) := w(TM)$ and $w_i(M) := w_i(TM)$.

Proposition 2.3. *Some basic properties of Stiefel-Whitney classes:*

(1) *The Stiefel-Whitney classes are natural, i.e. $f^*(w_i(E)) = w_i(f^*(E))$.*
(2) *The Whitney sum formula: $w(E \oplus F) = w(E)w(F)$, and hence*

$$w_k(E \oplus F) = \sum_{i+j=k} w_i(E) w_j(F).$$

(3) *If x denotes the generator of $H^1(\mathbb{RP}^n;\mathbb{F}_2) \cong \mathbb{F}_2$, then $w(S \to \mathbb{RP}^n) = 1 + x$.*
(4) *The Stiefel-Whitney classes are stable, i.e. $w(E \oplus \underline{\mathbb{R}}) = w(E)$.*
(5) *If $k > \operatorname{rank} E$, then $w_k(E) = 0$.*
(6) *If E has a set of ℓ everywhere linearly independent sections, then $w_j(E) = 0$ for any $j \geq \ell$.*

2.2. Tangential structures.

Our first application of characteristic classes will be to obstructing certain structures on manifolds. The idea is that some structures, such as an orientation, can be expressed as a condition on the characteristic classes of the tangent bundle. These structures tend to be more "topological;" geometric structures (complex structure, Kähler structure, etc.) can't be captured by this formalism.

Recall that one way to define a real vector bundle E on a manifold M is through transition functions: if $\mathfrak{U}$ is an open cover trivializing E, then for every pair of intersecting opens $U, V \in \mathfrak{U}$, E defines a smooth function $g_{UV}\colon U \cap V \to \mathrm{GL}_n(\mathbb{R})$.

Definition 2.4. Let G be a group with a specified map to $\rho\colon G \to \mathrm{GL}_n(\mathbb{R})$. A *G-structure* on E (sometimes *reduction of the structure group* to G) is a choice of transition functions $h_{UV}\colon U \cap V \to G$ such that for all intersecting $U, V \in \mathfrak{U}$, the following diagram commutes.

$$\begin{tikzcd} & G \arrow[d, "\rho"] \\ U \cap V \arrow[ur, "h_{UV}"] \arrow[r, "g_{UV}"'] & \mathrm{GL}_n(\mathbb{R}). \end{tikzcd}$$

A G-structure on M is by definition a G-structure on TM. We define two such G-structures to be equal if they're homotopic (possibly after taking a common refinement of open covers).

This is a formalization of the idea that, for example, an orientation is the structure such that all change-of-charts maps preserve the orientation of tangent vectors.

Remark. There's an equivalent classifying-space perspective: a vector bundle determines a map $f : M \to B\mathrm{GL}_n(\mathbb{R})$, and a G-structure is a lift of that map to BG:

$$\begin{tikzcd} & BG \arrow[d, "B\rho"] \\ M \arrow[ur, dashed, "\widetilde{f}"] \arrow[r, "f"'] & B\mathrm{GL}_n(\mathbb{R}). \end{tikzcd}$$

We declare two such lifts to be equivalent if they're homotopy equivalent. ◂

Proposition 2.5. *Every vector bundle has a unique* O_n*-structure.*

Proof sketch. Introduce a Riemannian metric on M (the space of metrics is contractible, so the resulting structures will be equivalent). Thus, there is an inner product on each fiber of E, so we can apply the Gram-Schmidt process to homotope each function so it maps into O_n into $\mathrm{GL}_n(\mathbb{R})$. ⊠

Proposition 2.6. *An* SO_n*-structure on* E *is equivalent data to an orientation of* E*.*

Instead of Spin_n-structure, we'll say spin structure, and so forth.

Remark. The same story goes through, of course, for complex vector bundles, with transition functions valued in $\mathrm{GL}_n(\mathbb{C})$. Again the Gram-Schmidt process allows a canonical reduction of structure group to U_n in the presence of a Riemannian metric. ◂

The point is: these structures are obstructed by characteristic classes.

Theorem 2.7. *Let* M *be a manifold.*

- M *is orientable iff* $w_1(M) = 0$.
- M *is spinnable iff* $w_1(M) + w_2(M) = 0$.

We'd like to say more, and say something like "an orientation is a choice of trivialization of w_1," but we can't make sense of that on the nose, since $H^1(M;\mathbb{Z}/2)$ is a set. To make sense of this, we should replace it with a space: because $H^n(M;A) = [M, K(A,n)]$, we'll look at $\mathrm{Map}(M, K(\mathbb{Z}/2,1))$.

That is, if M is orientable, an orientation is a path from w_1 to 0 in this mapping space, so we want to know $\pi_1(\mathrm{Map}(M, K(\mathbb{Z}/2,1)), 0)$. This we can compute with abstract nonsense:

$$\begin{aligned} \pi_1(\mathrm{Map}(M, K(\mathbb{Z}/2,1),0)) &= \pi_0(\Omega\,\mathrm{Map}(M, K(\mathbb{Z}/2,1))) \\ &= \pi_0(\mathrm{Map}_*(S^1, \mathrm{Map}(M, K(\mathbb{Z}/2,1)))) \\ &= \pi_0(S^1 \wedge M_+, K(\mathbb{Z}/2,1)) \\ &= [\Sigma M_+, K(\mathbb{Z}/2,1)] \\ &= H^1(\Sigma M_+;\mathbb{Z}/2) \cong H^0(M;\mathbb{Z}/2). \end{aligned}$$

Hence, the number of orientations of M is $|H^0(M;\mathbb{Z}/2)|$. Of course, this makes sense, because an element of $H^0(M;\mathbb{Z}/2)$ is a $\mathbb{Z}/2$-valued function on $\pi_0(M)$, corresponding to positively or negatively orienting each connected component. This isomorphism is not canonical, however; the set of orientations is a torsor for $H^0(M;\mathbb{Z}/2)$.

Look, you knew how many orientations a manifold has. The point is, this formalism applies to lots of other structures. The same argument as above proves the following.

Proposition 2.8. *If a structure S is equivalent to the trivialization of a characteristic class $c \in H^k(M;A)$, the set of S-structures on M is either empty or an $H^{k-1}(M;A)$-torsor.*

Corollary 2.9. *Let M be an oriented manifold. Then, the set of spin structures on M is an $H^1(M;\mathbb{Z}/2)$-torsor.*

2.3. **Stiefel-Whitney numbers and unoriented cobordism.** Fix a dimension $n \geq 0$; we'll allow the empty set to be an n-manifold. Recall that two n-manifolds M and N are *(unoriented) cobordant* if there's an $(n+1)$-manifold X such that $\partial X = M \amalg N$; one says X is a *cobordism* from M to N.

By gluing cobordisms, cobordism is an equivalence relation; the set of equivalence classes is denoted Ω_n^{O}. This is an abelian group under disjoint union, and

$$\Omega_*^{\mathrm{O}} := \bigoplus_{n \geq 0} \Omega_n^{\mathrm{O}}$$

is a graded ring under Cartesian product. This is called the *(unoriented) cobordism ring*.

Remark. Fix a tangential structure G. The above goes through when restricted to manifolds and cobordisms with G-structure, and therefore defines G-cobordism groups and rings, denoted Ω_n^G and Ω_*^G. Frequently considered are oriented cobordism, spin cobordism, and framed cobordism. ◂

It's a classical question in algebraic topology, and a hard one, to compute cobordism rings. Somewhat easier is the construction of *cobordism invariants*, maps out of Ω_*^{O} to some other ring that are easier to compute. For example, one can show that the mod 2 Euler characteristic is a cobordism invariant: if M is cobordant to N, then $\chi(M) \equiv \chi(N) \bmod 2$. (This admits a direct cellular argument, but we'll prove it later with characteristic classes.) We're going to construct some more.

Definition 2.10. Let M be a closed n-manifold, so that it admits a unique fundamental class in $\mathbb{F}_2$ cohomology, and let $n = i_1 + \cdots + i_k$ be a partition of n. Then, the *Stiefel-Whitney number*

$$w_{i_1 i_2 \cdots i_k} := \langle w_{i_1}(M) w_{i_2}(M) \cdots w_{i_k}(M), [M] \rangle.$$

That is, multiply all of the specified Stiefel-Whitney classes together, then cap with the fundamental class.

In the exercises you'll prove this is a cobordism invariant. Great! But it turns out the Stiefel-Whitney numbers are a *complete* invariant.

Theorem 2.11 (Thom)**.** *As graded rings,*

$$\Omega_*^{\mathrm{O}} \cong \mathbb{F}_2[x_i \mid i \neq 2^j - 1] \cong \mathbb{F}_2[x_2, x_4, x_5, x_6, x_8, \dots],$$

where if i is even, $x_i = [\mathbb{RP}^i]$. Moreover, two n-manifolds M and N are cobordant iff their Stiefel-Whitney numbers all agree.

The significance of this theorem is difficult to overstate: Thom more or less invented differential topology in order to prove it.

Remark. The odd-dimensional generators are certain *Dold manifolds* $P(m,n) := (S^m \times \mathbb{CP}^n)/\mathbb{Z}/2$, where $\mathbb{Z}/2$ acts by the antipodal map on S^m and complex conjugation on $\mathbb{CP}^n$. ◂

The lesson today is: we know how to compute Stiefel-Whitney numbers, so we can tell whether two manifolds are cobordant. Later we'll give analogous results for other kinds of cobordism.

2.4. **Some example calculations.**

Proposition 2.12. *There is no immersion $\mathbb{RP}^9 \hookrightarrow \mathbb{R}^{14}$.*

Proof. Suppose $f : \mathbb{RP}^9 \hookrightarrow \mathbb{R}^{14}$ is such an immersion. Then, there is a short exact sequence of vector bundles on $\mathbb{RP}^9$

$$0 \longrightarrow T\mathbb{RP}^9 \longrightarrow f^*(T\mathbb{R}^{14}) \longrightarrow \nu \longrightarrow 0,$$

where ν is the normal bundle. Hence by the Whitney sum formula,

$$w(\mathbb{RP}^9) w(\nu) = w(f^* T\mathbb{R}^{14}) = 1,$$

because $T\mathbb{R}^3$ is trivial. Expanding,

$$w(\mathbb{RP}^9) = (1+x)^{10} = 1 + x^2 + x^8,$$

so if you solve for $w(\nu)$, it has to be

$$w(\nu) = 1 + x^2 + x^4 + x^6.$$

However, ν is 5-dimensional, so $w_6(\nu) = 0$. $\boxtimes$

Some more useful facts about Stiefel-Whitney classes follow. Recall that the *determinant* of a vector bundle E is its top exterior power $\operatorname{Det} E := \Lambda^{\operatorname{rank} E} E$.

Proposition 2.13. *If $E \to M$ is a real vector bundle, $w_1(E) = w_1(\operatorname{Det} E)$.*

The analogous result for Chern classes was an exercise yesterday, and this is true for the same reasons.

Proposition 2.14. *Let $E, E' \to M$ be real line bundles, where M is a closed manifold. Then, the following are equivalent:*

(1) $E \cong E'$.
(2) $w(E) = w(E')$.
(3) $w_1(E) = w_1(E')$.

Corollary 2.15. *Let M be a closed n-manifold. The following three maps are group isomorphisms:*

$$\operatorname{Bun}_{\mathbb{Z}/2}(M) \xrightarrow{-\times_{\mathbb{Z}/2}\mathbb{R}} \operatorname{Line}(M) \xrightarrow{w_1} H^1(M; \mathbb{Z}/2) \xrightarrow{\mathrm{PD}} H_{n-1}(M; \mathbb{Z}/2).$$

The first map is the associated bundle construction, the second is the first Stiefel-Whitney class, and the third is Poincaré duality.

It is possible, and enlightening, to describe compositions or maps going the other way. For example, given an embedded $(n-1)$-manifold $N \subset M$, one can construct a principal $\mathbb{Z}/2$-bundle on M by declaring it to be trivial on $M \setminus N$, and on N, glue by switching the two fibers.

Proposition 2.16. *The top Stiefel-Whitney number $\langle w_n, [M]\rangle$ of a closed manifold is its Euler characteristic modulo 2.*

Later we'll see that if M is orientable, w_n is the reduction of another characteristic class which encodes the Euler characteristic in $\mathbb{Z}$.

2.5. **Exercises.** Most important:

(1) Analogous to yesterday's calculation of $c(\mathbb{CP}^n)$, show that $w(\mathbb{RP}^n) = (1+x)^{n+1}$, where x is the nonzero element of $H^1(\mathbb{RP}^n; \mathbb{Z}/2) \cong \mathbb{Z}/2$.
(2) For which n is $\mathbb{RP}^n$ orientable? Spin?
(3) We provided a definition of the k^{th} Chern class as the Poincaré dual of the dependency locus of k generic sections. Can you provide the analogous definition for the k^{th} Stiefel-Whitney class and prove it's equivalent to the one given in lecture?
(4) Show that the top Stiefel-Whitney class of an odd-dimensional manifold vanishes.
(5) Show that when $n \neq 2^k - 1$, $\mathbb{RP}^n$ does not embed in $\mathbb{R}^{n+1}$.

Also important, especially if you're interested:

(1) There are two groups Pin_n^+ and Pin_n^- which are double covers of O_n; for each one, the connected component of the identity is Spin_n. Thus, one may speak of Pin^+- and Pin^--structures on manifolds; the former is a trivialization of w_2, and the latter is a trivialization of $w_2 + w_1^2$. For which n is $\mathbb{RP}^n$ Pin^+? Pin^-?
(2) Show that an orientation and either a Pin^+ or a Pin^- structure determines a Spin structure. (This is not the same as: an orientable and $\mathrm{Pin}^\pm$ manifold is spin: we're choosing structures.)
(3) If M admits a Pin^+ structure, how many does it have? What about Pin^--structures?
(4) Find a manifold M which is not parallelizable, but with $w(M) = 1$.
(5) Express $w(M \times N)$ in terms of $w(M)$ and $w(N)$.
(6) Show that if E is any vector bundle, $E \oplus E$ is orientable. Can you make sense of this geometrically?
(7) Show that a Stiefel-Whitney number defines a group homomorphism $\Omega_n^{\mathrm{O}} \to \mathbb{F}_2$.
(8) Show that if an n-manifold M embeds in $\mathbb{R}^{n+1}$, then $w_j(M) = w_1(M)^j$.
(9) Consider the fiber bundle $S^2 \to E \to S^1$ where we quotient $S^2 \times [0,1]$ by $(x,0) \sim (f(x),1)$, where f has degree -1. What are its Stiefel-Whitney classes? Is it orientable? If instead you use a degree-1 map, what's the total space?

(10) Show there's no immersion $\mathbb{RP}^{2^k} \hookrightarrow \mathbb{R}^{2^{k+1}-2}$ (hence Whitney's theorem is optimal).
(11) Show a real vector bundle E is orientable iff $\operatorname{Det} E$ is trivial.

Additional exercises:

(1) If E_1 and E_2 are vector bundles such that two of E_1, E_2, and $E_1 \oplus E_2$ are spin, show that the third is also spin.
(2) Find two Pin^+ manifolds M and N such that $M \times N$ is not Pin^+. Repeat for Pin^-. (This is ultimately the reason why the cobordism groups $\Omega_*^{\mathrm{Pin}^+}$ and $\Omega_*^{\mathrm{Pin}^-}$ aren't rings. As a spin structure determines a $\mathrm{Pin}^\pm$ structure, at least they're still modules over Ω_*^{Spin}. Said another way, $M\mathrm{Pin}^+$ and $M\mathrm{Pin}^-$ aren't ring spectra, but they are module spectra over $M\mathrm{Spin}$.)
(3) Show that all Stiefel-Whitney numbers of M vanish iff the Stiefel-Whitney numbers of its stable normal bundle vanish.
(4) Show that if M is a closed, oriented 4-manifold that embeds in $\mathbb{R}^6$, it's spin.
(5) Let $y \in H^1(M;\mathbb{Z}/2)$ and $N \hookrightarrow M$ be a Poincaré dual to y. Obtain a formula for the mod 2 Euler characteristic of N as $\langle c, [M]\rangle$ for some $c \in H^n(M;\mathbb{Z}/2)$. Hint: feel free to assume that if $L \to M$ is a line bundle and $N \subset M$ is Poincaré dual to $w_1(L)$, then $\nu_{N\hookrightarrow M} \cong L|_N$.
(6) Show that if n is an odd number and M is a closed, n-dimensional manifold then for $0 \le k \le (d-1)/2$ and any $y \in H^1(M;\mathbb{Z}/2)$, $w_{n-2k}(M)y^{2k} = 0$.
(7) Show there is no vector bundle $E \to \mathbb{RP}^\infty$ whose direct sum with the tautological bundle S is trivial.

3. Stable cohomology operations and the Wu formula

Today, we're going to discuss Wu classes, which are also characteristic classes for real vector bundles in $\mathbb{Z}/2$ cohomology. This means they're polynomials over the Stiefel-Whitney classes, but they way in whcih they arise is interesting and useful.

3.1. Stable cohomology operations.

Wu classes arise through stable cohomology operations, which are a worthwhile digression.

Definition 3.1. A *cohomology operation* is a natural transformation of functors $\theta\colon H^p(-;A) \to H^q(-;B)$, meaning it commutes with pullback. If in addition it commutes with suspension, θ is said to be *stable*.

Example 3.2.

- One simple example is the squaring map $x \mapsto x^2$ in any degree and any coefficients. This is not stable.
- The *Pontrjagin square* $\mathscr{P}\colon H^2(X;\mathbb{Z}/2) \to H^4(X;\mathbb{Z}/4)$ is a more interesting example, which is the squaring map, but using the fact that if $x \in \mathbb{Z}$, knowing x mod 2 suffices to determine x^2 mod 4.
- Here's an explicit example of a stable operation. The short exact sequence

  $$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

  induces a short exact sequence of cochain complexes

  $$0 \longrightarrow C^*(M;\mathbb{Z}) \xrightarrow{\cdot 2} C^*(M;\mathbb{Z}) \longrightarrow C^*(M;\mathbb{Z}/2) \longrightarrow 0,$$

  and hence a long exact sequence in cohomology:

  $$\cdots \longrightarrow H^n(M;\mathbb{Z}) \longrightarrow H^n(M;\mathbb{Z}) \longrightarrow H^n(M;\mathbb{Z}/2) \xrightarrow{\beta_0} H^{n+1}(M;\mathbb{Z}) \longrightarrow \cdots.$$

  The connecting morphism β_0 is called the *Bockstein homomorphism*. Both of these are stable.

  If we instead started with the short exact sequence

  $$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2 \longrightarrow 0,$$

  we'd obtain a different Bockstein homomorphism $\beta_4\colon H^i(M;\mathbb{Z}/2) \to H^{i+1}(M;\mathbb{Z}/2)$. ◀

Since Eilenberg-Mac Lane spaces represent cohomology, a cohomology operation of type $H^p(-;A) \to H^q(-;B)$ is determined by a homotopy class of maps $K(A,p) \to K(B,q)$. That is, the abelian group of cohomology operations from $H^p(-;A) \to H^q(-;B)$ is $[K(A,p), K(B,q)] = H^q(K(A,p);B)$. This is a complicated problem.

Stable cohomology operations admit an axiomatic description. It turns out that over $\mathbb{Z}$, all stable cohomology operations are either multiples of the identity, or come from stable cohomology operations over $\mathbb{F}_p$. We'll only need the case $p = 2$.

Definition 3.3. The stable cohomology operations $H^*(-;\mathbb{F}_2) \to H^*(-;\mathbb{F}_2)$ form a graded $\mathbb{F}_2$-algebra called the *Steenrod algebra* $\mathscr{A}$, which is generated by classes $\mathrm{Sq}^n \in \mathscr{A}_n$ for $n \geq 0$, called *Steenrod squares*, such that:

- $\mathrm{Sq}^n \colon H^k(-;\mathbb{F}_2) \to H^{k+n}(-;\mathbb{F}_2)$ commutes with pullback and is additive.
- $\mathrm{Sq}^0 = \mathrm{id}$.
- $\mathrm{Sq}^1 = \beta_4$.
- Restricted to classes of degree n, Sq^n is the map $x \mapsto x^2$.
- If $n > |x|$, then $\mathrm{Sq}^n x = 0$.
- The *Cartan formula*
$$\mathrm{Sq}^n(xy) = \sum_{i+j=n} \mathrm{Sq}^i(x)\mathrm{Sq}^j(y).$$
Equivalently, the total Steenrod square $\mathrm{Sq} := 1 + \mathrm{Sq}^1 + \mathrm{Sq}^2 + \cdots$ is a ring homomorphism.

It's a theorem that these axioms uniquely determine $\mathscr{A}$, and actually constructing the Steenrod squares is involved.

As a consequence, the Steenrod squares satisfy the *Ádem relations*

$$\mathrm{Sq}^i\mathrm{Sq}^j = \sum_{k=0}^{\lfloor i/2 \rfloor} \binom{j-k-1}{i-2k} \mathrm{Sq}^{i+j-k}\mathrm{Sq}^k.$$

Since we can apply any element of $\mathscr{A}$ to any cohomology class, $H^*(M;\mathbb{F}_2)$ is a module over $\mathscr{A}$ for any M. Pullback maps are $\mathscr{A}$-module homomorphisms, as is the connecting morphism in a long exact sequence.

Example 3.4. Let's determine the $\mathscr{A}$-module structure on $H^*(\mathbb{RP}^4;\mathbb{Z}/2) \cong \mathbb{Z}/2[a]/(a^5)$ with $|a| = 1$. We know $\mathrm{Sq}^0 a = a$ and $\mathrm{Sq}^1 a = a^2$, and all higher Steenrod squares vanish. Now we can use the Cartan formula:

- $\mathrm{Sq}(a^2) = \mathrm{Sq}(a)\mathrm{Sq}(a) = (a+a^2)^2 = a^2 + a^4$. Hence $\mathrm{Sq}^1 a^2 = 0$, $\mathrm{Sq}^2 a^2 = a^4$, and all others vanish.
- $\mathrm{Sq}(a^3) = \mathrm{Sq}(a)\mathrm{Sq}(a^2) = (a+a^2)(a^2+a^4) = a^3 + a^4$, so $\mathrm{Sq}^1 a^3 = a^4$ and all others vanish. ◂

3.2. **The Wu class and Wu formula.** We're going to use Poincaré duality to turn the Steenrod squares into characteristic classes. One formulation of Poincaré duality is that for any closed n-manifold M,

$$H^k(M;\mathbb{Z}/2) \otimes H^{n-k}(M;\mathbb{Z}/2) \xrightarrow{\smile} H^n(M;\mathbb{Z}/2) \xrightarrow[\sim]{\frown[M]} \mathbb{Z}/2$$

is a nondegenerate pairing. This is the adjoint to the usual Poincaré duality statement (an isomorphism between H^k and H_{n-k}).

In particular, $H^k(M;\mathbb{Z}/2) \cong (H^{n-k}(M;\mathbb{Z}/2))^*$, so if we can produce linear functionals on $H^{n-k}(M;\mathbb{Z}/2)$, they will define cohomology classes for us. And $\mathrm{Sq}^k \colon H^{n-k}(M;\mathbb{Z}/2) \to H^n(M;\mathbb{Z}/2)$ is such a linear functional, so it's represented by some class $v_k \in H^k(M;\mathbb{Z}/2)$: $v_k \smile x = \mathrm{Sq}^k(x)$. This class is called the k^{th} *Wu class* of M. Similarly, the *total Wu class* is $v = 1 + v_1 + v_2 + \cdots$. The total Wu class satisfies

$$\langle v \smile x, [M] \rangle = \langle \mathrm{Sq}x, [M] \rangle$$

for all $x \in H^*(M;\mathbb{Z}/2)$.

Lemma 3.5. *The Wu classes are natural, and hence are $\mathbb{Z}/2$ characteristic classes of real vector bundles.*

By natural we mean the pullback of the total Wu class on M by $f : N \to M$ is the total Wu class on N.

Proof sketch. The Steenrod squares are natural with respect to pullback. ⊠

The Wu classes are something we haven't seen before: there's no vector bundle, just the manifold. So the theorem that every $\mathbb{Z}/2$ characteristic class for real vector bundles is a polynomial in Stiefel-Whitney classes doesn't literally apply. But the Wu classes are still closely related to Stiefel-Whitney classes.

Theorem 3.6 (Wu). $\mathrm{Sq}(v) = w$.

Proof. ⊠

Corollary 3.7. *The Stiefel-Whitney classes of a manifold are homotopy invariants.*

Corollary 3.8. *Homotopy equivalent manifolds of the same dimension are unoriented cobordant.*

Here's another application of Theorem 3.6:

Proposition 3.9 (Wu formula)**.**

$$\mathrm{Sq}^i w_k = \sum_{j=0}^{i} \binom{k+j-i-1}{j} w_{i-j} w_{k+j}.$$

3.3. **Some example applications.** The point of all this formalism is to be useful, so let's see some applications.

Proposition 3.10. *If M is a closed* 2*- or* 3*-manifold,* $w_1(M)^2 = w_2(M)$.

Proof. Here we use the fact that $w = \mathrm{Sq}(v)$. Looking at the homogeneous terms,

$$\begin{aligned} w_1 &= \mathrm{Sq}^1 v_0 + \mathrm{Sq}^0 v_1 = v_1 \\ w_2 &= \mathrm{Sq}^2 v_0 + \mathrm{Sq}^1 v_1 + \mathrm{Sq}^0 v_2 = v_1^2 + v_2 = w_1^2, \end{aligned}$$

because $v_2 = 0$ on a 3-manifold. ⊠

Corollary 3.11. *Every orientable manifold of dimension at most* 3 *is spin.*

So the Wu classes force certain Stiefel-Whitney numbers to vanish. It's a theorem of Brown and Peterson that all such relationships between Stiefel-Whitney classes arise in this way.

Proposition 3.12. *Let M be an orientable* 4*-manifold. Then, M is spin iff all embedded surfaces have even intersection number.*

Proof. Since the intersection product is Poincaré dual to cup product, it suffices to show $\langle a^2, [M]\rangle = 0$ for all $a \in H^2(M; \mathbb{Z}/2)$ iff $w_2(M) = 0$.

Now we use the Wu formula. w_1 is the degree-1 piece of Sq^v, so

$$w_1 = \mathrm{Sq}^1 v_0 + \mathrm{Sq}^0 v_1 = v_1,$$

and hence $v_1 = 0$. Next,

$$w_2 = \mathrm{Sq}^2 v_0 + \mathrm{Sq}^1 v_1 + \mathrm{Sq}^0 v_2,$$

so $w_2 = v_2$. For any $a \in H^2(M; \mathbb{Z}/2)$,

$$\langle a^2, [M]\rangle = \langle \mathrm{Sq}^2 a, [M]\rangle = \langle v_2 a, [M]\rangle = \langle w_2 a, [M]\rangle.$$

Poincaré duality tells us the cup product pairing $H^2(M; \mathbb{Z}/2) \otimes H^2(M; \mathbb{Z}/2) \to \mathbb{Z}/2$ is nondegenerate, so $w_2 = 0$ iff $\langle a^2, [M]\rangle = 0$ for all a, as desired. ⊠

The Wu classes tell you that you can get the Stiefel-Whitney classes directly out of the $\mathscr{A}$-module structure on $H^*(M; \mathbb{Z}/2)$, which can be useful if you don't have a good geometric description of your space.

Example 3.13. Just as one has real and complex projective spaces, one can define *quaternionic projective space* $\mathbb{HP}^n := \mathbb{H}^{n+1}/\mathbb{H}^\times$, a $4n$-dimensional manifold which behaves quite a bit like $\mathbb{RP}^n$ and $\mathbb{CP}^n$. For example, $H^*(\mathbb{HP}^n) \cong \mathbb{Z}[a]/(a^{n+1})$, where $|a| = 4$. This fact completely determines the Stiefel-Whitney classes of $\mathbb{HP}^n$.

For example, let $n = 4$. By degree reasons, $\mathrm{Sq}^4 a = a^2$ and no other Steenrod squares are nonzero, so $\mathrm{Sq}(a) = a + a^2$. By the Cartan formula, $\mathrm{Sq}(a^k) = (\mathrm{Sq} a)^k$ and so

$$\begin{aligned} \mathrm{Sq}(a^2) &= (a + a^2)^2 = a^2 + a^4 \\ \mathrm{Sq}(a^3) &= (a + a^2)(a^2 + a^4) = a^3 + a^4 + a^5 + a^6 = a^3 = a^4 \\ \mathrm{Sq}(a^4) &= a^4. \end{aligned}$$

Often this is encoded in a diagram such as Figure 1.

The only possible nonzero Wu classes are v_0, v_4, and v_8, and looking at the $\mathscr{A}$-action, $v_4 = a$ and $v_8 = a^2$. Thus

$$\begin{aligned} w(\mathbb{HP}^4) &= \mathrm{Sq}(v) = \mathrm{Sq}(1 + a + a^2) \\ &= 1 + (a + a^2) + (a^2 + a^4) \\ &= 1 + a + a^4, \end{aligned}$$

so $w_4(\mathbb{HP}^4) = a$, $w_{16}(\mathbb{HP}^4) = a^4$, and all other Stiefel-Whitney classes are zero. ⊲

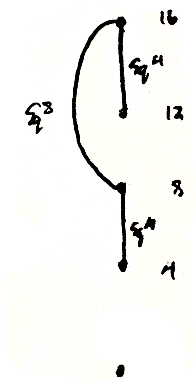

FIGURE 1. The $\mathbb{Z}/2$-cohomology of $\mathbb{HP}^4$, as an $\mathscr{A}$-module.

Example 3.14. The *Wu manifold* $W := \mathrm{SU}_3/\mathrm{SO}_3$ is a five-dimensional manifold. One can show that its mod 2 cohomology is $H^*(W;\mathbb{Z}/2) \cong \mathbb{Z}/2[z_2, z_3]/(z_2^2, z_3^2)$, and the $\mathscr{A}$-action is $\mathrm{Sq}^1 z_2 = z_3$ and $\mathrm{Sq}^2 z_3 = z^5$. Hence $v(W) = 1 + v_2$, which determines the Stiefel-Whitney classes. Only w_2 and w_3 can be nonzero, by looking at cohomology. And indeed,

$$w_2(W) = \mathrm{Sq}^2 v_0 + \mathrm{Sq}^1 v_1 + \mathrm{Sq}^0 v_2 = v_2 = z_2$$
$$w_3(W) = \mathrm{Sq}^3 v_0 + \mathrm{Sq}^2 v_1 + \mathrm{Sq}^1 v_2 + v_3 = \mathrm{Sq}^1 z_2 = z_3,$$

so $w(W) = 1 + z_2 + z_3$.

This is noteworthy because it means the Stiefel-Whitney number $w_{2,3} = \langle w_2(W) w_3(W), [W] \rangle = 1$, and you'll show in the exercises that in dimension 5, all Stiefel-Whitney numbers are either 0 or equal to $w_{2,3}$. Thus, $\Omega_5^{\mathrm{O}} \cong \mathbb{Z}/2$ with W as a generator, and you can check you don't get a generator from any 5-dimensional product of projective spaces. ◀

3.4. The Bockstein and integral Stiefel-Whitney classes.

Definition 3.15. Let $E \to M$ be a real vector bundle. The k^{th} *integral Stiefel-Whitney class* of E, denoted $W_n(E)$, is $\beta_0 w_{n-1}(E) \in H^n(M;\mathbb{Z})$.

For every n, there's a Lie group Spin_c^n which can be defined in a few ways: it's the quotient

$$\mathrm{Spin}_c^n := (\mathrm{Spin}_n \times \mathrm{U}_1)/\mathbb{Z}/2,$$

where $\mathbb{Z}/2$ acts as -1 on both components.

Proposition 3.16. *A* Spin^c*-structure on an oriented manifold is obstructed by the third integral Stiefel-Whitney class.*

Using the Bockstein long exact sequence, this is the same thing as w_2 being in the image of the reduction map $H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{Z}/2)$.

Remark. $W_3(M)$ is also an obstruction to the existence of a contact structure on M, though it's not the only obstruction. ◀

3.5. Exercises.
Most important:

(1) Which Wu classes vanish on a 5-manifold? What about an orientable 5-manifold?
(2) Show that all Stiefel-Whitney numbers on a closed 5-manifold either vanish or are equal to $w_{2,3}$.
(3) Show that any orientable 4-manifold is spin^c.

Also important, especially if you're interested:

(1) Let M be a $2n$-dimensional manifold. Show that there exists an n-dimensional embedded submanifold Y such that for any other n-dimensional embedded submanifold $N \subset M$, $I_2(N,N) = I_2(Y,N)$. (Here I_2 denotes the mod 2 intersection number.)
(2) Show that if M is a closed, orientable manifold of dimension 6 or 10, $\chi(M)$ is even.
(3) Determine the action of the Steenrod algebra on $H^*(\mathbb{RP}^n;\mathbb{Z}/2)$.

(4) Show that for any vector bundle $E \to M$, the smallest $k \geq 1$ such that $w_k(E) \neq 0$, if one exists, is a power of 2.
(5) Show that $\beta_4 w_{2k+1}(E) = w_1(E)w_{2k+1}(E)$ and $\beta_4 w_{2k}(E) = w_{2k+1}(E) + w_1(E)w_{2k}(E)$. Hint: check this on the universal bundle $E\mathrm{O}_n \to B\mathrm{O}_n$.

Additional exercises:

(1) Show that if M is an oriented manifold and $H^*(M)$ contains no torsion, then M is spin^c. Conclude that $\mathbb{CP}^n$ is spin^c for all n.
(2) There's a group $\mathrm{Pin}_n^c = (\mathrm{Pin}_n^+ \times \mathrm{U}_1)/\mathbb{Z}/2$ analogous to the definition of Spin^c. The obstruction to a Pin^c-structure on $E \to M$ is exactly $W_3(E)$. Show that $\mathbb{RP}^n$ is pin^c iff it's Pin$^+$ iff it's Pin$^-$ (and hence, $\mathbb{RP}^n$ is spin^c iff it's spin).
(3) Show that if M is a spin 5-manifold, $w(M) = 1$. If M is a Pin$^-$ 5-manifold, show that $w(M) = 1 + w_1(M)$.
(4) Show that $w_3(M) = 0$ for a closed 4-manifold M.
(5) Generalize Proposition 3.12 to the unoriented setting.
(6) How many spin^c structures are there on an oriented manifold M?

4. Chern, Pontrjagin, and Euler classes

4.1. Chern classes. We've been here before. Let's quickly recall a definition, and then discuss some properties. Many are directly analogous to properties of Stiefel-Whitney classes, in a way that's strongly reminiscent of the passage from mod 2 intersection theory of unoriented submanifolds to integral intersection theory with orientations. This analogy is not a coincidence.

We've provided several definitions of Chern classes already. From a universal perspective, $H^*(B\mathrm{U}) \cong \mathbb{Z}[c_1, c_2, \dots]$, with $|c_k| = 2k$, thus defining characteristic classes for complex vector bundles. Things like naturality, stability, and the Whitney sum formula follow.

If M is an almost complex manifold, its tangent bundle has the structure of a complex vector bundle. In this case we may define Chern numbers of M as usual. We can also do this if M is a *stably almost complex* manifold, meaning we've placed a complex structure on $TM \oplus \underline{\mathbb{R}}^k$; this uses the fact that Chern classes are stable.

Here are some more properties of Chern classes. Some of these will be reminiscent of analogous properties for Stiefel-Whitney classes.

Proposition 4.1. *Let $E \to M$ be a complex vector bundle.*

(1) $c_1(E) = c_1(\operatorname{Det} E)$.
(2) *If $\overline{E}$ denotes the complex conjugate bundle, then $\overline{E} \cong E^*$ and $c_k(\overline{E}) = (-1)^k c_k(E)$.*
(3) *If M is a stably almost complex manifold, its top Chern number is equal to $\chi(M)$.*
(4) *Under the reduction homomorphism $H^*(M) \to H^*(M;\mathbb{Z}/2)$, $c_n(E) \mapsto w_{2n}(E)$, and $w_{2n+1}(E) = 0$.*

Just as w_1 classifies real line bundles, c_1 classifies complex line bundles.

Proposition 4.2. *Let $E, E' \to M$ be complex line bundles, where M is closed. Then, the following are equivalent:*

(1) $E \cong E'$.
(2) $c(E) = c(E')$.
(3) $c_1(E) = c_1(E')$.

Corollary 4.3. *Let M be a closed manifold. Then, the following maps are group isomorphisms.*

$$\mathrm{Bun}_{\mathrm{U}_1}(M) \xrightarrow{-\otimes_{\mathrm{U}_1}\mathbb{C}} \mathrm{Line}_{\mathbb{C}}(M) \xrightarrow{c_1} H^2(M).$$

If M is orientable, we also have the Poincaré duality isomorphism $H^2(M) \to H_{n-2}(M)$.

Though Chern classes are obstructions to tangential structures, these structures aren't considered as often as orientations or spin structures.

Proposition 4.4. *A complex vector bundle E has an* SU*-structure iff $c_1(E) = 0$.*

Though we can't define a cobordism ring of complex manifolds (what's a complex structure on an odd-dimensional manifold?), stably almost complex structures work fine. The stably almost complex cobordism ring is denoted Ω_*^{U}.[4]

[4] There's a similar issue with defining a cobordism ring of symplectic manifolds, and what one obtains is stably almost symplectic cobordism.

Theorem 4.5 (Milnor, Novikov). *As graded rings,*

$$\Omega_*^{\mathrm{U}} \cong \mathbb{Z}[x_1, x_2, \dots],$$

where $|x_k| = 2k$*. Moreover, two stably almost complex manifolds are cobordant iff all of their Chern numbers agree.*

In $\Omega_*^{\mathrm{U}} \otimes \mathbb{Q}$, we can take $\mathbb{CP}^k$ as a generator of the degree-$2k$ piece, but over $\mathbb{Z}$, things are more complicated.

Remark. The identification of Ω_*^{U} with the ring of formal group laws is a major organizing principle in stable homotopy theory, allowing one to define generalized cohomology theories that see a lot of the structure of stable homotopy theory. This is an active area of research known as the *chromatic program*. ◁

There isn't a single characteristic class which obstructs a stably almost complex structure. However, a stably almost complex structure is exactly what it means to have Chern classes, so we obtain a necessary condition.

Proposition 4.6. *If* M *is a stably almost complex manifold,* $w_{2k+1}(M) = 0$ *and* $W_{2k+1}(M) = 0$ *for all* k*.*

That is, the odd-degree Stiefel-Whitney classes are zero and the even-degree ones are reductions of integral classes (namely, Chern classes of the tangent bundle).

4.2. **The splitting principle.** When it comes to computing Chern classes, if a vector bundle is a direct sum of line bundles, life is good. But there are vector bundles E which aren't direct sums of line bundles, and this is sad. But if we knew of a space Y and a map $p\colon Y \to M$ such that

- $p^*\colon H^*(M) \to H^*(Y)$ is injective, and
- after pullback to Y, E is a direct sum of line bundles,

then things would be OK: one could compute the Chern class of $E \to M$ by pulling it back to Y, decomposing it as a direct sum of line bundles, and computing their Chern classes. Since p^* is injective, we've lost no information, so this determines the Chern class of E. This is known as the *splitting principle*.

Such a Y exists, and we will construct it.

Definition 4.7. Let V be a finite-dimensional complex Hilbert space. The *flag manifold* $\mathrm{F}\ell(V)$ is the manifold whose points are orthogonal decompositions of V as a direct sum of one-dimensional subspaces.

The diffeomorphism class of the flag manifold does not depend on the choice of Hermitian metric.

Example 4.8. $\mathrm{F}\ell(\mathbb{C}^2) = \mathbb{CP}^1$: in an orthogonal decomposition of $\mathbb{C}^2$, one line determines the other, so $\mathrm{F}\ell(\mathbb{C}^2)$ is the space of lines through the origin in $\mathbb{C}^2$. ◁

For the rest of this subsection, we fix a complex vector bundle $E \to M$ of rank k.

Definition 4.9. The *flag bundle* $p\colon \mathrm{F}\ell(E) \to M$ is the fiber bundle whose fiber at an $x \in M$ is $\mathrm{F}\ell(E_x)$. The total space is called the *flag manifold*.

This requires a Hermitian metric to construct, as $\mathrm{F}\ell(V)$ did, but is independent of the choice of metric.

There are tautological line bundles $L_1, \dots, L_k \to \mathrm{F}\ell(E)$: the fiber of L_i over $(x, W_1 \oplus \dots \oplus W_k)$ is W_i.

Proposition 4.10. *The pullback* $p^*\colon H^*(M) \to H^*(\mathrm{F}\ell(E))$ *is injective.*

This is an application of the Leray-Hirsch theorem.

Proposition 4.11. *After being pulled back to* $\mathrm{F}\ell(E)$*,* $p^*E \cong L_1 \oplus \dots \oplus L_k$*.*

Why is this? At every point $(x, W_1 \oplus \dots \oplus W_k)$, we have an isomorphism

$$E_x \cong W_1 \oplus \dots \oplus W_k \cong (L_1 \oplus \dots \oplus L_k)_x,$$

and these vary smoothly.

Let $x_i := c_1(L_i)$; these are called the *Chern roots* of E. By the Whitney sum formula, $c_j(E)$ is the j^{th} symmetric polynomial in the Chern roots. In particular,

$$c(E) = \prod_{i=1}^{k} (1 + x_i).$$

You can actually use the splitting principle to provide yet another definition of Chern classes: once c_1 is defined, then all the others follow from the Whitney sum formula on the flag bundle.

4.3. **Pontrjagin classes.** We'll leverage the Chern classes to define integral cohomology classes for real vector bundles. At this point you broadly know how the story goes.

Definition 4.12. Let $E \to M$ be a real vector bundle. Then, $E_{\mathbb{C}} := E \otimes \underline{\mathbb{C}}$ is a complex vector bundle, which we call the *complexification* of E.

Note that complexification doubles the rank.

Definition 4.13. Let $E \to M$ be a real vector bundle. Then, its k^{th} *Pontrjagin class* is $p_k(E) := (-1)^k c_{2k}(E_{\mathbb{C}}) \in H^{4k}(M)$. The *total Pontrjagin class* is $p(E) := 1 + p_1(E) + \cdots$. As usual, $p_i(M) := p_i(TM)$, and $p(M) := p(TM)$.

Remark. Not everyone uses the same sign convention when defining Pontrjagin classes. ◄

The Pontrjagin classes satisfy most of the usual axioms; in particular, they are stable. However, they do *not* follow the Whitney sum formula! Thankfully, the difference $p(E \oplus F) - p(E)p(F)$ is 2-torsion, so if you work over $\mathbb{Q}$ (or even $\mathbb{Z}[1/2]$) Pontrjagin classes satisfy the Whitney sum formula.

Proposition 4.14 (Splitting principle for the Pontrjagin class)**.** *The Chern roots of $E_{\mathbb{C}}$ come in pairs $x_1, -x_1, \dots, x_k, -x_k$, and so*

$$p(E) = \prod_{i=1}^{k} (1 + x_i^2).$$

Pontrjagin numbers are used to classify oriented cobordism. The answer is not as clean as for unoriented cobordism

Theorem 4.15 (Thom, Wall)**.**

(1) *All torsion in Ω_*^{SO} is 2-torsion.*
(2) *As graded rings,*

$$\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q} \cong \mathbb{Q}[x_1, x_2, \dots,]$$

where $|x_k| = 4k$, and $x_k = [\mathbb{CP}^{2k}]$.
(3) *Two oriented n-manifolds are oriented cobordant iff their Pontrjagin and Stiefel-Whitney numbers agree.*

Remark. Ultimately because $\mathrm{Spin}_n \to \mathrm{SO}_n$ is a double cover, the forgetful map $\Omega_*^{\mathrm{Spin}} \to \Omega_*^{\mathrm{SO}}$ is an isomorphism after tensoring with $\mathbb{Z}[1/2]$. In particular, $\Omega_*^{\mathrm{Spin}} \otimes \mathbb{Q} \cong \mathbb{Q}[\widetilde{x}_1, \widetilde{x}_2, \dots]$ with $|\widetilde{x}_k| = 4k$. However, we can't take $\mathbb{CP}^{2k}$ to be generators anymore.

To get characteristic numbers that characterize spin cobordism, one has to define characteristic classes for real K-theory, a generalized cohomology theory. ◄

4.4. **The Euler class.** The Euler class is an unstable characteristic class for oriented vector bundles, arising because the map $H^*(B\mathrm{O}_n) \to H^*(B\mathrm{SO}_n)$ induced by the inclusion $\mathrm{SO}_n \hookrightarrow \mathrm{O}_n$ is not surjective. Throughout this section, $E \to M$ is an oriented real vector bundle of rank k.

Definition 4.16. The *Euler class* of E, $e(E) \in H^k(M)$, is the Poincaré dual to the zero locus of a generic section of E.[5]

That is, choose a section $s \in \Gamma(E)$ that's transverse to the zero section, and let $N = s^{-1}(0)$, which is a codimension-k submanifold of M. Then, $e(E)$ is Poincaré dual to the class N represents in $H_{n-k}(M)$.

Proposition 4.17.

(1) *The Euler class is natural.*
(2) *The Euler class satisfies the Whitney sum formula: $e(E_1 \oplus E_2) = e(E_1)e(E_2)$.*
(3) *If E possesses a nonvanishing section, $e(E) = 0$.*
(4) *If E^{op} denotes E with the opposite orientation, then $e(E^{\mathrm{op}}) = -e(E)$.*
(5) *If k is odd, $e(E)$ is 2-torsion.*

Most of these follow directly from the definition.

Proposition 4.18 (Relationship with other characteristic classes)**.**

(1) *Reduction mod 2 $H^k(M) \to H^k(M; \mathbb{Z}/2)$ carries $e(E) \to w_k(E)$.*

[5] If $k > \dim M$, this does not make sense, but then $H^k(M) = 0$ anyways, so we let $e(E) = 0$.

(2) *If $F \to M$ is a complex vector bundle of rank $2k$, $e(F) = c_k(F)$.*
(3) *$e(E)^2 = c(E_{\mathbb{C}})$. Hence if k is even, $e(E)^2 = p_{k/2}(E)$.*

The characteristic number associated to the Euler class is familiar.

Proposition 4.19. *For any oriented manifold M, $\langle e(M), [M]\rangle = \chi(M)$, its Euler characteristic.*

Sometimes, people define the Euler class for *sphere bundles*, i.e. fiber bundles whose fibers are spheres. This definition is equivalent to ours: given a sphere bundle $S^k \to E \to M$, we can create a vector bundle $V(E) \to M$ whose unit sphere bundle is E. The Euler class of E is defined to be that of $V(E)$.

Sphere bundles are good examples to play with: you can build them out of manifolds you already understand, but they may twist in interesting ways. Moreover, there are tools for computing with them.

Definition 4.20. Let A be an abelian group and $\pi\colon E \to M$ be a fiber bundle, where M is n-dimensional and the fiber is k-dimensional, and (if $A \neq \mathbb{Z}/2$) assume that both E and M are oriented. For each j, there's a sequence of maps

$$H^{k+j}(E;A) \xrightarrow{\mathrm{PD}} H_{n-j}(E;A) \xrightarrow{\pi_*} H_{n-j}(M;A) \xrightarrow{\mathrm{PD}} H^j(M;A),$$

where the first and third arrows are Poincaré duality. The composition of these maps is called the *Gysin map* $\pi_!\colon H^{k+j}(E;A) \to H^j(M;A)$.

The Gysin map goes by a variety of colorful names, including the *wrong-way map*, the *umkehr map*, the *shriek map*, the *pushforward map*, and the *surprise map*. Indeed, it's surprising: we have a covariant map in cohomology!

Remark. For intuition, you can look to de Rham cohomology, where the Gysin map is integration on the fiber. That is, since E is locally $S^k \times U$, we can integrate a differential $(j+k)$-form over S^k to obtain a j-form on U. This is precisely the Gysin map. ◀

Theorem 4.21 (Gysin long exact sequence)**.** *Let A be an abelian group and $\pi\colon E \to M$ be a sphere bundle with fiber S^k. Assume (unless $A = \mathbb{Z}/2$) that the fibers of $E \to M$ are consistently oriented. Then, there is a long exact sequence*

$$\cdots \longrightarrow H^m(E;A) \xrightarrow{\pi_!} H^{m-k}(M;A) \xrightarrow{\cdot e(E)} H^{m+1}(M;A) \xrightarrow{\pi^*} H^{m+1}(E;A) \longrightarrow \cdots$$

That is, Gysin map, cup with the Euler class, pullback.

Remark. The Gysin long exact sequence is a special case of the Serre spectral sequence, and may be proven in that way. ◀

4.5. **Exercises.** Most important:

(1) Show that $T\mathbb{CP}^n$ is not isomorphic to its complex conjugate.
(2) Show that $\mathbb{CP}^4$ cannot be embedded in $\mathbb{R}^{11}$.
(3) Let M be a manifold with an orientation-reversing diffeomorphism. Show that $[M] \in \Omega_*^{\mathrm{SO}}$ is torsion. (Hint: this diffeomorphism sends $[M] \mapsto -[M]$. How does it affect the Pontrjagin classes? Alternatively, by a direct argument, you could find a manifold bounding $M \amalg M$, showing $[M]$ is 2-torsion.)
(4) Show that if $E \subset TS^{2n}$, E is either trivial or all of TS^{2n}.
(5) The Euler class of a complex vector bundle is equal to its top Chern class, but the Euler class is unstable and Chern classes are stable. How can this be?
(6) Prove Proposition 4.19. Hint: use the definition of the Euler characteristic as the sum of local indices of a vector field.

Also important, especially if you're interested:

(1) Why is $p(S^n) = 1$?
(2) In contrast to Chern, Pontrjagin, and Stiefel-Whitney numbers, there are manifolds with nonzero Euler characteristic that bound. What's an example?
(3) Exhibit two manifolds cobordant as unoriented manifolds, but not oriented manifolds.
(4) Show that $\Omega_5^{\mathrm{SO}} \cong \mathbb{Z}/2$, and the Wu manifold is a generator. This is the lowest-degree torsion in Ω_*^{SO}.
(5) Show that the mod 2 reduction of $p_k(E)$ is $w_{2k}(E)^2$.
(6) Show that odd Chern classes are 2-torsion.
(7) Let $N \subset M$ be an embedded submanifold with normal bundle ν. Show that $\langle [N], e(\nu)\rangle = I_2(N,N)$ (i.e. the mod 2 intersection number).

(8) Complexification of line bundles commutes with tensor product, hence defines a group homomorphism $H^1(X;\mathbb{Z}/2) \to H^2(X)$ for any space X.
   (a) Show this is a cohomology operation.
   (b) Show this is the Bockstein homomorphism β_0. Hence, if $E \to M$ is a real line bundle, $c_1(E \otimes \mathbb{C}) = \beta_0 w_1(E)$.
   (c) Using the splitting principle, show that if $E \to M$ is a real vector bundle, $c_1(E \otimes \mathbb{C}) = \beta_0 w_1(E)$.

Additional exercises:

(1) For which n is $\mathbb{CP}^n$ spin?
(2) Let $u \in H^4(\mathbb{HP}^n)$ be the generator. Show that $p(\mathbb{HP}^n) = (1+u)^{2n+2}/(1+4u)$.
(3) Complexification turns a real vector bundle into a complex vector bundle. Hence it turns a principal O_n-bundle into a principal U_n-bundle. Describe this process.
(4) Let $E \to M$ be an oriented $(2k+1)$-dimensional vector bundle. Show that $e(E) = \beta_0 w_{2k}(E)$.
(5) Prove part (3) of Proposition 4.18.
(6) Give an example of
   (a) an even-dimensional stably almost complex manifold which is not almost complex, and
   (b) an odd-dimensional stably almost complex manifold.

5. Genera and the Hirzebruch signature theorem

Today, we're going to talk about genera and a few of their applications. Genera touch on homotopy theory, topology, and even physics; our tour will be more modest.

Definition 5.1. A *genus* is a homomorphism out of a cobordism ring.

Example 5.2. The mod 2 Euler characteristic is a signature $\chi_2 \colon \Omega_*^{\mathrm{O}} \to \mathbb{F}_2$ and $\chi_2 \colon \Omega_*^{\mathrm{SO}} \to \mathbb{F}_2$. The Euler characteristic is a signature $\chi \colon \Omega_*^{\mathrm{U}} \to \mathbb{Z}$. ◂

Example 5.3. Let M be a closed, oriented $4k$-manifold. Then, the cup product $H^{2k}(M;\mathbb{R}) \otimes H^{2k}(M;\mathbb{R}) \to H^{4k}(M;\mathbb{R})$ is commutative, and Poincaré dual to the *intersection pairing* on half-dimensional homology. As a quadratic form on a real vector space, it's conjugate to one of the form

$$x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2.$$

The difference $\sigma \coloneqq p - q$ is called the *signature*.

If M is a manifold whose dimension isn't a multiple of 4, define $\sigma(M) \coloneqq 0$.

The signature is our first nontrivial example of a genus: since it's defined using intersection numbers, you can check that it vanishes on a manifold that bounds. It's fairly clearly additive under disjoint union, and by the Künneth formula, multiplicative under Cartesian product. Hence we obtain a map $\Omega_*^{\mathrm{SO}} \to \mathbb{Z}$. ◂

This is a pretty broad definition, especially since relatively few, specific genera are considered. But some genera are better than others. What makes a particular genus worth considering?

- Some genera arise from geometric considerations, as the signature did.
- Some genera arise as the indices of Dirac operators. In this way they apply to geometry and to physics.
- A genus $\Omega_*^G \to R$ can be thought of as a ring homomorphism $\Omega_*^G(\mathrm{pt}) \to H^*(\mathrm{pt};R)$, so it's reasonable to ask whether it lifts to a natural transformation of generalized cohomology theories: $\Omega_*^G(\text{–}) \to H^*(\text{–};R)$.
- In homotopy theory, you might ask to refine further to ring morphisms of the objects representing these theories (i.e. ring spectra) $MG \to HR$.

What's interesting about the theory is that mostly the same genera arise from all these considerations.

We're going to build some genera. We already have a bunch of awesome cobordism invariants lying around, namely characteristic numbers, let's use them. We discussed yesterday how any symmetric function in the Chern roots defines a polynomial in Chern classes, so given a power series $a(x) \in \mathbb{Q}[\![x]\!]$, consider the class

$$G_a(M) \coloneqq \prod_{i=1}^{n} x_i,$$

where $x_1, \ldots, x_n$ are the Chern roots of the n-manifold M. Hence we get a number $\langle G_a(M), [M]\rangle$. Since the Chern numbers are cobordism invariants, this is a cobordism-invariant function, and similarly one can show it's additive and multiplicative. You can do the same thing with Pontrjagin classes.

Using this, we'll quickly define some genera.

Example 5.4.

(1) The *L-genus* or *L-polynomial* is associated to the power series

$$\frac{x}{\tanh x},$$

which ends up being a power series in x^2, hence polynomials in Pontrjagin classes: $L_4 = (1/3)p_1$, $L_8 = (1/45)(7p_2 - p_1^2)$, and so forth.

(2) The *Todd genus* is associated to the power series for

$$\frac{z}{1-e^{-z}}.$$

(3) The *$\hat{A}$-genus* is associated to the power series

$$\frac{z/2}{\sinh(z/2)},$$

which ends up being a power series Pontrjagin classes, e.g. $\hat{A}_4 = -(1/24)p_1$, $\hat{A}_8 = (1/5760)(-4p_2 + 7p_1^2)$, and so forth.

(4) The *Chern character*

(5) The *Pontrjagin character* $ph(E) := \mathrm{ch}(E_{\mathbb{C}})$.

◂

We're going to focus on the L-polynomial, for the following reason.

Theorem 5.5 (Hirzebruch signature theorem)**.** *For any closed manifold M, $\sigma(M) = \langle L(M), [M]\rangle$.*

Since these are both $\mathbb{Q}$-valued genera for oriented cobordism, it'll suffice to check on a generating set for $\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q}$.

Remark. From the perspective of Dirac operators, this theorem is a quick corollary of the Atiyah-Singer index theorem (which is lurking in the background of this whole lecture). The genera we mentioned as examples also appear in this way, and the Atiyah-Singer index theorem also applies to them, proving useful theorems in topology and geometry. For example, applied to the Euler characteristic, one finds a generalization of the Gauss-Bonnet theorem.

We're not going to go into this in detail, partly because it would take a lot of time, but also because we don't need its full power to prove the Hirzebruch signature theorem and recover its applications. ◂

Lemma 5.6.

$$\sigma(\mathbb{CP}^n) = \begin{cases} 1, & n \text{ even} \\ 0, & n \text{ odd.} \end{cases}$$

You'll prove this in an exercise.

Lemma 5.7.

$$\langle L(\mathbb{CP}^n), [\mathbb{CP}^n]\rangle = \begin{cases} 1, & n \text{ even} \\ 0, & n \text{ odd.} \end{cases}$$

Proof. Earlier this week, we saw that $T\mathbb{CP}^n \oplus \underline{\mathbb{C}} \cong (S^*)^{\oplus(n+1)}$. This is a direct sum of line bundles, so its Chern roots are all $c_1(S^*) = a$, where a is the positive generator of $H^2(\mathbb{CP}^n)$ as usual. Hence we need to isolate the degree-n coefficient in $L(\mathbb{CP}^n) = (a/\tanh a)^{n+1}$. The Cauchy integral formula tells us this is

$$L(\mathbb{CP}^n)[n] = \frac{1}{2\pi i}\oint_{B_\varepsilon(0)} \frac{1}{z^{n+1}}\left(\frac{z}{\tanh z}\right)^{n+1}.$$

Let $y=\tanh z$, so $\mathrm{d}y=(1-z^2)\,\mathrm{d}z$.

$$= \frac{1}{2\pi i}\oint_{B_\varepsilon(0)} \frac{\mathrm{d}y}{(1-y^2)y^{n+1}}$$
$$= \frac{1}{2\pi i}\oint_{B_\varepsilon(0)} \frac{1+y^2+y^4+\cdots}{y^{n+1}}\,\mathrm{d}y$$
$$= \begin{cases} 1, & n \text{ even} \\ 0, & n \text{ odd}. \end{cases} \qquad \boxtimes$$

Proof of Theorem 5.5. Because $\mathbb{Z}\hookrightarrow\mathbb{Z}\otimes\mathbb{Q}=\mathbb{Q}$ is injective, the signature and L-genus are determined by their values on $\Omega_*^{\mathrm{SO}}\otimes\mathbb{Q}$. Using Theorem 4.15, this ring is generated by $[\mathbb{CP}^{2k}]$ for $k\ge 1$.

By Lemmas 5.6 and 5.7, the signature and L-genus agree on $\mathbb{CP}^{2k}$. Hence they agree on all of $\Omega_*^{\mathrm{SO}}\otimes\mathbb{Q}$. $\boxtimes$

Corollary 5.8. *If M is an oriented 4-manifold, $\langle p_1(M),[M]\rangle = 3\sigma(M)$, and in particular $p_1/3$ is an integer.*

This is one of many integrality theorems. Here's another, which is harder.

Theorem 5.9 (Borel-Hirzebruch)**.** *Let M be a closed spin manifold. Then, $\langle \hat{A}(M),[M]\rangle$ is an integer, and even if* $\dim(M)\equiv 4 \bmod 8$.

One way to prove this in dimension 4 is to check on the generators of $\Omega_4^{\mathrm{Spin}}\cong\mathbb{Z}^2$, which are a K3 surface and $(S^1)^4$.

Corollary 5.10 (Rokhlin)**.** *If M is a closed spin 4-manifold, $16\mid\sigma(M)$.*

Proof. We know $\langle\hat{A}(M),[M]\rangle = \langle -1/24p_1(M),[M]\rangle$ is even, so $48\mid\langle p_1(M),[M]\rangle$, and hence $16\mid\langle p_1(M)/3,[M]\rangle = \sigma(M)$. $\boxtimes$

Definition 5.11. The *Rokhlin invariant* $\mu(M)\in\mathbb{Z}/16$ of a spin 3-manifold M is the signature of any spin 4-manifold bounding M; this is well-defined by Corollary 5.10.

Every orientable 3-manifold is spinnable (do you remember why?), but the Rokhlin invariant may depend on the spin structure. However, every homology 3-sphere admits a unique spin structure (exercise (6), below), so we can envision the Rokhlin invariant as an invariant of homology 3-spheres.

It's a theorem that if a homology 3-sphere embeds in S^4, it splits S^4 into two homology 3-balls, which necessarily have signature 1. Hence a nonzero Rokhlin invariant is an obstruction to embedding a homology 3-sphere into $\mathbb{R}^4$! For example, the *Poincaré homology sphere* (the quotient of SO_3 by the subgroup of symmetries of an icosahedron) has Rokhlin invariant 1, and therefore cannot be smoothly embedded into $\mathbb{R}^4$. This is interesting because it's parallelizable, so all of its characteristic classes vanish, but we were able to find an embedding result anyways.

Remark. The genera we discussed today extend to natural transformations of cohomology theories and maps of spectra:

- The mod 2 Euler characterstic extends to a natural transformation $\Omega_{\mathrm{O}}^*\to H^*(-;\mathbb{F}_2)$ and to a morphism of ring spectra $M\mathrm{O}\to H\mathbb{F}_2$.
- The $\widehat{A}$-genus extends to a natural transformation of cohomology theories $\Omega_{\mathrm{Spin}}^*\to KO^*$; that is, it goes from spin cobordism to real K-theory. This extends to a ring map of ring spectra $M\mathrm{Spin}\to KO$, which is exactly the Spin orientation of KO of Atiyah-Bott-Shapiro.
- The Todd genus similarly extends to a natural transformation $\Omega_{\mathrm{Spin}^c}^*\to K^*$, i.e. from Spin^c-cobordism to complex K-theory. This extends to the complex Atiyah-Bott-Shapiro orientation $M\mathrm{Spin}^c\to KU$.
- The Chern character extends to a natural homomorphism of cohomology theories $K^*\to H^*(-;\mathbb{Q})$.

◀

5.1. Exercises.

(1) Why does the mod 2 Euler characteristic define a ring homomorphism $\Omega_*^{\mathrm{O}}\to\mathbb{Z}/2$?
(2) Why does the Euler characteristic define a ring homomorphism $\Omega_*^{\mathrm{U}}\to\mathbb{Z}$?
(3) Show that the signature is cobordism-invariant.
(4) Show that the signature is a ring homomorphism $\Omega_*^{\mathrm{SO}}\to\mathbb{Z}$.
(5) Prove Lemma 5.6.

(6) Prove that every homology 3-sphere admits a unique spin structure.
(7) Why is the Poincaré homology sphere parallelizable?