FALL 2018 HOMOTOPY THEORY SEMINAR

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1. Overview: 9/5/18

This short overview was given by Richard.

In the beginning, there were homotopy groups $\pi_n(X) \coloneqq [S^n, X]$. Homotopy theory begins with the study of these groups, which are hard to calculate. Even the homotopy groups of the spheres, $\pi_k(S^n)$, are complicated. However, there are patterns.

Theorem 1.1 (Freudenthal suspension theorem)**.** *For $n \geq k+2$, $\pi_{k+n}(S^n) \cong \pi_{k+n+1}(S^{n+1})$.*

The first few of these stable homotopy groups are $\pi_n(S^n) = \mathbb{Z}$, $\pi_{n+1}(S^n) = \mathbb{Z}/2$, $\pi_{n+2}(S^n) = \mathbb{Z}?2$, $\pi_{n+3}(S^n) = \mathbb{Z}/24$, $\pi_{n+4}(S^n) = \pi_{n+5}(S^n) = 0$, $\pi_{n+6}(S^n) = \mathbb{Z}/2$, and $\pi_{n+7}(S^n) = \mathbb{Z}/120$.

You can encode all of this stability data in one place using spectra. There's an object $\mathbb{S}$ called the *sphere spectrum* built in a precise way from spheres, and the homotopy groups of $\mathbb{S}$ are the stable homotopy groups of the spheres.

These stable homotopy groups are very hard to calculate. However, we can work locally (at primes), which simplifies the problem a little bit.

Theorem 1.2 (Fracture square)**.** *Let X be a space, $X_\mathbb{Q}$ be its rationalization, and for p a prime let X_p denote the p-completion of X. Then the following square is a homotopy pullback:*

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & X_\mathbb{Q} \arrow[d] \\
\displaystyle\prod_{p \text{ prime}} X_p \arrow[r] & \left(\displaystyle\prod_{p \text{ prime}} X_p\right)_\mathbb{Q}.
\end{tikzcd}$$

Here $\pi_*(X_p) = \pi_*(X) \otimes \mathbb{Z}_p$ and $\pi_*(X_\mathbb{Q}) \cong \pi_*(X) \otimes \mathbb{Q}$. The upshot of Theorem 1.2 is that these groups determine the original homotopy groups of X.

The rational homotopy groups of spheres are known, due to an old theorem of Serre. Over p, there are other techniques, such as the Adams and Adams-Novikov spectral sequences. The Adams-Novikov spectral sequences uses a filtration on X_p to produce a spectral sequence with E_2-term

$$E_2^{*,*} = \mathrm{Ext}_{BP_*BP}(BP_*, BP_*(X)), \tag{1.3}$$

and converging to $\pi_*(X)_{(p)}$ (p-local, not p-complete!). Here BP is a spectrum, but you don't actually need to know much about it (yet): BP_* is some algebra, and BP_*BP is a Hopf algebra, and they can be described explicitly. We'll learn more about this spectral sequence in time.

If you look at a picture of the E_∞-page of the Adams-Novikov spectral sequence for any p (maybe just p odd for now), there are strong patterns: a pattern along the bottom, which is the α-family (said to be v_1-periodic), and some periodic things along the diagonal (said to be v_2-periodic), containing the β-family. Both of these are families in the homotopy groups of spheres, providing structue in the complicated story —

we don't know the stable homotopy groups of spheres past about 60, so producing families is very helpful for our understanding! In a similar way, one can find v_3-periodic elements, including something called the γ-family, and so forth.

Of course, there's a lot of work to do even from here: how to we get here from the E_2-page? Do the extension problems go away, giving us actual elements of the stable stem? For the α-, β-, and γ-families, these are known, and there are even geometric interpretations for small n (up to 3 or 4) and large p (usually something like $p > 5$ or $p > 7$). Specifically, if $V(0)$ denotes cofiber of the multiplication-by-p map $\mathbb{S} \to \mathbb{S}$, the α-family comes from self-maps $\Sigma^k V(0) \to V(0)$, together with the maps to and from $\Sigma^k \mathbb{S}$ coming from the cofiber sequence. There are less explicit complexes $V(1)$ and $V(2)$ which give you the β- and γ-families, and there is a similar story.