SUMMER 2016 HOMOTOPY THEORY SEMINAR

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Contents

1. Simplicial Localizations and Homotopy Theory: 5/24/16

"It may be a little dry, but it's been raining recently, so perhaps dryness will be good to have."

Today's lecture was given by Ernie Fontes.

The point of this seminar is to study simplicial localizations. This is a somewhat dry topic; today we're going to frame it, suggesting an outline for talks and some motivation. Thus, today we'll discuss homotopy theory in broad strokes.

A good first question: *what is homotopy theory?* Relatedly, *when can we do it?* In general, homotopy theory happens whenever we have a pair of categories (C, W), where W is a subcategory of C. The idea is that W contains morphisms that we'd like to be isomorphisms. If W contains all of the objects of C, then the pair (C, W) is called a *relative category*.

Example 1.1.

(1) Often, we choose $\mathsf{C} = \mathsf{Top}$, and make W the category of a nice class of morphisms, e.g. π_*-isomorphisms or homotopy equivalences.
(2) Another choice is to let $\mathsf{C} = \mathsf{ch}(R)$, the category of chain complexes of R-modules, where W is the category of *quasi-isomorphisms* (maps which induce an isomorphism on homology).

One nice property that W could have is the *two-out-of-six property*: that for all triples of morphisms $f : X \to Y$, $g : Y \to Z$, and $h : Z \to Z'$ in C, if gf and hg are in W, then so are f, g, h, and hgf. This implies the *two-out-of-three property*, that if any two of f, g, and h are in W, then so is the third.

Definition 1.2. If W satisfies the two-out-of-six property, it is called a *homotopical category*.

In either setting, we can form the *homotopy category* $\mathrm{Ho}(\mathsf{C}) = \mathsf{C}[\mathsf{W}^{-1}]$, localizing C at W. This is the initial category among those categories D and functors $\mathsf{C} \to \mathsf{D}$ sending the arrows in W to isomorphisms.

Most questions in homotopy theory can be framed in terms of the homotopy category: two spaces are homotopic iff they're isomorphic in $\mathrm{Ho}(\mathsf{C})$, and the homotopy classes of maps $X \to Y$ are the hom-set $\mathrm{Hom}_{\mathrm{Ho}(\mathsf{C})}(X, Y)$ in the homotopy category.

One question which does require a little more sophistication is understanding homotopy (co)limits. Since we've inverted a lot of arrows, taking limits or colimits in a homotopy category behaves very poorly. For example, there's no pushout of the degree-2 map $S^1 \to S^1$ along with the map $S^1 \to \mathrm{pt}$, since it "should be" $\mathbb{RP}^2$ but this doesn't satisfy it. $\mathbb{RP}^2$ is the homotopy pushout, however.

Often, one obtains more structure from a homotopy category, e.g. there are some ∞-categorical notions hiding in the background here. More concretely, one often obtains a natural model category structure, where in addition to the relative category (C, W), we have classes of cofibrant and fibrant morphisms satisfying a bunch of axioms. This provides tools for computing homotopy limits and colimits, etc., but it's a lot of data; even the definition is redundant (the cofibrations and fibrations determine each other). In fact, the punchline of the three papers we're reading is that only the structure of the relative category (C, W) is necessary to recover the entire model-categorical structure! For this reason, one makes the analogy that if homotopy theory is to linear algebra, picking a model-categorical structure is akin to picking a basis.

Definition 1.3. A *simplicial set* is a simplicial object in Set. That is, it's a collection of sets $\{X_i\}_{i\geq 0}$ and a bunch of maps $d_{ij} : X_i \longrightarrow X_{i-1}$ for $0 \leq j \leq i$ and $s_{ij} : X_i \longrightarrow X_{i+1}$ for $0 \leq j \leq i$ satisfying some relations that look like the boundary and inclusion relations for an i-simplex inside an $(i+1)$-simplex.

This is a vague definition, and we'll have a better one next lecture. These are akin to a "better" version of topological spaces, in that they model topological spaces very well, and can be described purely combinatorially.

Here's how the three papers of Dwyer and Kan break this information down.

(1) The first paper, "Simplicial Localization of Categories," constructs $\mathsf{C}[\mathsf{W}^{-1}]$, first as "just" a category, and then as a simplicially enriched category $L\mathsf{C}$, meaning that for all $X, Y \in \mathsf{C}$, $L\mathsf{C}(X,Y) \in \mathsf{sSet}$: that is, it's a simplicial set. In particular, we recover $\mathsf{C}[\mathsf{W}^{-1}]$ as the path components of this set: $\mathsf{C}[\mathsf{W}^{-1}](X,Y) = \pi_0 L\mathsf{C}(X,Y)$. There's a lot of comonadic computations here that may be confusing, but are applicable in many parts of algebra.
(2) In "Calculating Simplicial Localizations," Dwyer and Kan define a variant called the *hammock localization* $L^H\mathsf{C}(X,Y)_k$. The hammocks in question are commutative diagrams

$$\begin{tikzcd}
 & C_{0,1} \arrow[d, "\wr"] & C_{0,2} \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\wr"] & \cdots & C_{0,n-1} \arrow[l, "\sim"'] \arrow[d, "\wr"] \arrow[dr] & \\
X \arrow[ur] \arrow[r] \arrow[dr] & \vdots \arrow[d, "\wr"] & \vdots \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\wr"] & \cdots \arrow[r, "\sim"] & \vdots \arrow[r] \arrow[d, "\wr"] & Y \\
 & C_{k,1} \arrow[r, "\sim"] & C_{k,2} \arrow[r, "\sim"] & \cdots & C_{k,n-1} \arrow[l, "\sim"'] \arrow[ur] &
\end{tikzcd}$$

This might not seem like the best construction, but it expresses $L^H\mathsf{C}(X,Y)$ as a colimit of nerves of categories, which are easy to compute, and therefore this is surprisingly easy to work with when it comes to actually computing things. In particular, when certain weak (yet technical) properties hold, $L^h\mathsf{C}(X,Y) \simeq L\mathsf{C}(X,Y)$. The calculations in this paper are much more technical than the first, and it's worth going through more slowly.
(3) The third paper, "Function Complexes in Homotopical Algebra," establishes a relationship between (simplicially enriched) model categories and $L^H\mathsf{C}(X,Y)$. The takeaway is that the weak equivalences are all that you need to define a model categorical structure.

In the unlikely event we have time, there's an interesting relationship between this and algebraic K-theory: in a similar way, the algebraic K-theory of a model category actually only depends on the hammock localization, due to a paper of Blumberg and Mandell; this was a cool and surprising result.

Here's the list of planned talks; we can and should deviate from this in order to make sure we understand everything better.

(1) Simplicial sets, especially nerves and classifying spaces. This should definitely include a definition and some important constructions.
(2) Model categories; there's a lot we could talk about here, but we should talk about the definition, how to construct homotopy limits and colimits, mapping spaces, and fibrant and cofibrant replacement. This is intended to be an overview, rather than discussing complicated examples. This will be helpful to see all the structure we don't need!
(3) We then need to talk about localization in general, including the universal property for localizing rings, and discuss the discrete localization of categories. The hard version of this talk would also talk about Bousfield localization.
(4) Now, the first part of the first paper: localization of (C, W), comonadic resolutions, and bar constructions, which detail how one constructs things. This is mostly all in the paper, and needs to be teased apart.
(5) Perhaps also it will be useful to discuss the rest of the model structure on small simple categories. Here Julie Bergner's thesis is a useful reference, as she treats this more clearly and in greater generality, though we may or may not need to refer to this.
(6) Moving to the second paper, introduce hammock localization. This is important to understand very closely; don't leave anything out of the talks.
(7) Then, we need homotopy calculus of fractions, which is useful for ensuring hammocks are small.
(8) We then need the theory of simplicial model categories; these have more structure and are more excellent than ordinary model categories. The key is understanding the axiom SM7 for a simplicial model category.

(9) Finally, we should treat the main theorem of the last paper, that $L^h\mathsf{C}(X, Y)$ models the simplicial derived mapping space in a model category.

At that point, the summer will be over, and we will be done.