M392c NOTES: ALGEBRAIC GEOMETRY

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These notes were taken in UT Austin's M392c (Algebraic geometry) class in Fall 2018, taught by Sam Raskin. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Any mistakes in the notes are my own. Thanks to Alberto San Miguel for correcting a few typos.

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Lecture 1.

Some questions in algebraic geometry: 8/29/18

Office hours are Fridays from 11-1, in room 9.164 (at least for now). Today we'll talk about some questions (and some answers, too!) relating to algebraic geometry and why one might find it interesting. We're going to focus on concreteness.

Broadly speaking, algebraic geometry studies zero sets of polynomials. These could be polynomials over $\mathbb{Q}$, or $\mathbb{R}$, or $\mathbb{C}$, or finite fields, or more. The first question you might ask is, *are there solutions*? This is an *arithmetic question*: in arithmetic situations, there might not be solutions.

Example 1.1 (Taylor-Wiles, 1994)**.** If $n \geq 3$, the polynomial $x^n + y^n = 1$ has no solutions over $\mathbb{Q}$ when $x, y \neq 0$. ◂

You might recognize this as a reformulation of Fermat's last theorem.

Another form of the same question is *can you parameterize solutions of the equation*? For example, let's try it with $x^2 + y^2 = 1$, which we know has solutions. In this case, it is possible to parameterize solutions, via the one-parameter family

$$x = \frac{\lambda^2 - 1}{\lambda^2 + 1}, \qquad y = \frac{2\lambda}{\lambda^2 + 1}. \tag{1.2}$$

These kinds of questions are called *rationality questions*. One can also ask these questions over $\mathbb{C}$ (or over other algebraically closed fields), where they can feel a bit different.

There is a general result that any quadric hypersurface with a rational point is rational. What this means is that if you assume the existence of one solution (x_0, y_0) to a degree-2 polynomial in x and y over, say, $\mathbb{Q}$, then you can use that one solution to parameterize all other solutions. If you plot the solutions in the xy-plane, the parameter of another solution (x_1, y_1) is the slope of the line between (x_0, y_0) and (x_1, y_1). Indeed, in (1.2), the parameter λ is this slope. Because the equation is a quadric, one expects such a line to intersect in exactly two points, the first solution and another one. This is all extremely explicit, to the point that you could explain why you care to a middle schooler.

There are a few other rationality results.

Theorem 1.3 (Segre, 1940s; Manin, 1970s; Kollár, 2000)**.** *Smooth cubics in at least three variables are rational.*

So $x^2 + y^2 = 1$ isn't rational, but $x^2 + y^2 + z^2 = 1$ is. However, this doesn't give you everything.

Theorem 1.4 (Clemens-Griffiths, 1974)**.** *There are cubics in at least four variables which are unirational but not rational, i.e. that one cannot parameterize all solutions in a one-to-one manner.*

This was a hard theorem. How would you prove something like this?

Recent work (2012-15) by many people (Voisin, Colliot-Thèlene, Pirutka, Totaro[1]) generalizes this.

Theorem 1.5. *For cubics in at least five variables, one can also not parameterize solutions in a one-to-one way, even by adding additional "dummy variables."*

For four-variables cubics, this is open.

Schemes. Though this result is stated completely explicitly, it was studied using some very abstract-looking machinery. In this course, we'll also work with this abstract machinery, namely the language of schemes. These are things like solutions to systems of polynomials, but not quite — they encode among other things the equivalence of such systems under changes of coordinates, which doesn't really change the underlying geometry of the solution set. Classification problems with this perspective are a big area of research, and Birkar just won a Fields medal for work in this area from 2006.

Algebraic geometry over $\mathbb{C}$. A third thing you could care about is specific stuff about algebraic geometry over your favorite field (typically $\mathbb{C}$, but not always). In many cases (such as $\mathbb{C}$), you have topology around, and you can ask how it interacts with the algebraic geometry we've been talking about.

For example, if $q \in \mathbb{C}^\times$ isn't a root of unity, then there's a cubic equation $y^2 = x^2 + ax + b$ whose solutions are parameterized by $\mathbb{C}^\times/q\mathbb{Z}$. This may be a bit surprising, and indicates a way in which analytic or topological information can be useful: now we can learn about the universal cover of the solution space, and other topological invariants. Then you might ask whether something like this is true in positive characteristic, which tends to be harder.

More generally, one can study the topology of algebraic varieties over $\mathbb{C}$.

Theorem 1.6. *The odd Betti numbers of smooth proper varieties are even.*

The proof uses the study of the Hodge Laplacian operator on a variety X. This needs a metric, but projective means that X embeds in some $\mathbb{CP}^n$, and we can borrow its metric. There is a purely algebro-geometric proof of this, but first you need to come up with the right notion of Betti numbers (so étale cohomology, which is hard), and then invoke Deligne's proof of the Weil conjectures (also hard). Nonetheless, it's true in characteristic p.

More generally, the cohomology of a complex projective variety has more structure, and is much richer than that of a random manifold.[2]

Conjecture 1.7 (Hodge conjecture, imprecise statement)**.** *The differential topology of a projective algebraic variety over $\mathbb{C}$ knows everything about its algebraic geometry.*

This is a Millennium Prize problem, meaning it comes with a $1 million reward. You can infer that it's hard.

[1] If you like pictures of cats, check out Totaro's math blog: `https://burttotaro.wordpress.com/`.

[2] This doesn't require smoothness *per se*, but it's more difficult to formulate in the singular case.

Algebraic geometry over $\mathbb{Z}$. If you work over $\mathbb{Z}$ instead of over $\mathbb{C}$, meaning your polynomial has integer coefficients, then you can reduce mod p and solve it there. This is the first thing anyone does in number theory, because it often simplifies the problem to a finite question. This naturally leads one to ask, *how do the systems of equations at different primes p relate to each other?*

There's a lot to say about this, beginning with quadratic reciprocity, which is very classical yet a little weird, and continuing all the way to the Langlands program.

Supposing X encodes the system of solutions to your polynomial with $\mathbb{Z}$ coefficients. Then one can define a zeta function, reminiscent of the Riemann zeta function, as follows:

$$\zeta_X(s) := \prod_{p \text{ prime}} \exp\left(\sum \frac{1}{n}(\text{number of solutions in } \mathbb{F}_{p^n})p^{-ns}\right). \tag{1.8}$$

For $X = \operatorname{Spec}\mathbb{Z}$, corresponding to solutions to an empty set of polynomials, this recovers the usual Riemann zeta function.

For any particular X, one conjectures this is meromorphic (and almost entire, in some sense), and that the analogue of the Riemann hypothesis holds; for some X, this is known due to Deligne. There are some other related conjectures related to this known as Sato-Tate conjectures.

Cohomology theories. Over $\mathbb{C}$, you have topology, and therefore can invoke algebraic topology to compute cohomology of algebraic varieties. Over other fields or rings, you might not have these techniques, and there are several other approaches.

- Over an algebraically closed field, one has *étale cohomology*, whose ideas are built from covering space theory, has $\mathbb{Z}_\ell$ coefficients, where ℓ is a prime that's not the characteristic of the field.
- Over any field k, there's *de Rham cohomology*, which uses the idea that $\mathrm{d}z/z$ understands $\mathbb{C}^\times$ isn't simply connected (since $\oint \mathrm{d}z/z \neq 0$). This has coefficients in k.

There are others, too. One wants these to all be the same, or at least closely related; if $k = \mathbb{Q}_p$ and $\ell = p$ ($\mathbb{Q}_p$ has characteristic zero!), then these two are related by p-adic Hodge theory. This is related to deep and recent work by Fontaine, Scholze, and others, and relates to Scholze's Fields medal work. In 2016, Bhatt-Morrow-Scholze showed that one can sometimes interpolate between different cohomology theories. See Scholze's ICM address for more on this. The ultimate question in this corner of algebraic geometry is whether there's some universal cohomology theory interpolating between everything we have, and which is also the source of the ζ-functions mentioned above.

Degenerations. We get additional power by studying solutions in families. For example, we can degenerate $x^2 + y^2 = 1$ to $x^2 + y^2 = 0$, which is much simpler. One asks questions such as, *what invariants are preserved under degenerations?* Therefore one might be able to use a degeneration to reduce a harder problem to an easier problem.

Computations. This subfield of algebraic geometry tries to make these abstract invariants concrete, by writing good algorithms to compute these invariants for explicit systems of polynomials.

Geometric complexity theory. This is another way to relate algebraic geometry and computer science. The goal of this field is to approach another Millennium Prize problem, P vs. NP, using algebraic geometry techniques. This roughly involves studying certain varieties and analyzing whether they're as complicated as they seem. Algebraic geometry has lots of techniques which might help, but on the other hand they haven't yet.

Probably the best way to learn algebraic geometry is to have an application or research focus in mind that you can apply the things you learn to. This method of learning tends to produce algebraic geometers.

Lecture 2.

Defining schemes, I: 8/31/18

The goal of today's lecture is to define a scheme, first heuristically and then rigorously.

"Definition" 2.1. A scheme is a "space" that is a Zariski sheaf which admits an "open cover" by affine schemes.

Of course, in order to do this, we need to know what all of these words — spaces, Zariski sheaves, affine schemes, and open covers — mean in this setting.

Remark 2.2. There's another approach to schemes using the formalism of *locally ringed spaces*, which is followed by Hartshorne, Vakil, and many others. It's more concrete, but it makes it harder to think about what a specific scheme, such as projective space, is supposed to be. ◂

The motivation for "Definition" 2.1 is that a scheme should be something which is locally defined by algebraic equations. For example, let's look at the *Fermat equation* $X_n = \{x^n + y^n = z^n\}$. Fermat was interested in solutions in $\mathbb{Z}$, but the set of solutions makes sense in any commutative ring. This suggests our definition of space, which is not the same as a topological space.

Definition 2.3. A *space* is a functor $X\colon \mathsf{CommRing} \to \mathsf{Set}$.

Concretely, this means that for every ring A, we get a set $X(A)$, and for every map of commutative rings $f\colon A \to B$, we get a map of sets $X(f)\colon X(A) \to X(B)$, and these morphisms should compose well (meaning that $X(f \circ g) = X(f) \circ X(g)$ and $X(\mathrm{id}) = \mathrm{id}$). For example, we could let $X_n(A)$ denote the set of solutions to the Fermat equation in the ring A; then, if we've solved it in A, we can map the solution into B via $f\colon A \to B$, and we'll obtain a solution in B, so this defines a space X_n.

We should also say how spaces interact.

Definition 2.4. A *morphism of spaces* $f\colon X \to Y$ is data of, for all commutative rings A, a map $f_A\colon X(A) \to Y(A)$ such that for all ring homomorphisms $g\colon A \to B$, the diagram

$$\begin{tikzcd} X(A) \arrow[r, "f_A"] \arrow[d, "X(g)"] & Y(A) \arrow[d, "Y(g)"] \\ X(B) \arrow[r, "f_B"] & Y(B) \end{tikzcd}$$

commutes.

Schemes are special examples of spaces, in a way that feels surprisingly down-to-Earth.

Our first example of a space is the solutions to the Fermat equation in A, as discussed above. Here's another example.

Example 2.5. Let A be a commutative ring. We'll define the space $\operatorname{Spec} A$ to be the functor $(\operatorname{Spec} A)(B) = \operatorname{Hom}(A, B)$; given a ring homomorphism $\varphi\colon B \to C$, we use the map $\operatorname{Hom}(A, B) \to \operatorname{Hom}(A, C)$ given by postcomposition with φ. ◂

Definition 2.6. An *affine scheme* is a space of the form $\operatorname{Spec} A$ for some A.

You don't have to be a commutative algebra expert to learn algebraic geometry, but you can see that commutative algebra is built into the definitions of algebraic geometry, so some commutative algebra knowledge is helpful.

Example 2.7. The space X_n sending A to the solutions of the Fermat equation in A is an affine scheme; explicitly,

$$X_n \cong \operatorname{Spec} \mathbb{Z}[x, y, z]/(x^n + y^n - z^n).$$

This is because a ring homomorphism $\mathbb{Z}[x, y, z]/(x^n + y^n - z^n) \to A$ is exactly the data of $x, y, z \in A$ satisfying the relation $x^n + y^n - z^n = 0$. ◂

Lemma 2.8 (Yoneda lemma)**.** *For all spaces X, $\operatorname{Hom}_{\mathsf{Spaces}}(\operatorname{Spec} A, X) \cong X(A)$.*

Proof sketch. First we define a map from $\operatorname{Hom}_{\mathsf{Spaces}}(\operatorname{Spec} A, X)$ to $X(A)$. Specifically, a map $f\colon \operatorname{Spec} A \to X$ is the data of for all commutative rings B, $\operatorname{Spec}(A)(B) \to X(B)$. Take $B = A$; then, $\operatorname{Spec}(A)(A) = \operatorname{Hom}(A)$, so take the image of the identity. It remains to check this is an equivalence. ⊠

Corollary 2.9. $\operatorname{Hom}_{\mathsf{Spaces}}(\operatorname{Spec} A, \operatorname{Spec} B) \cong \operatorname{Hom}_{\mathsf{CommRing}}(B, A)$.

It's interesting that the direction reverses!

Proof. By the Yoneda lemma, $\operatorname{Hom}_{\mathsf{Spaces}}(\operatorname{Spec} A, \operatorname{Spec} B) = \operatorname{Spec}(B)(A) = \operatorname{Hom}(B, A)$. ⊠

This tells you that as long as you make sure to reverse the arrows, anything you can do with commutative rings, you can do with affine schemes, and vice versa.

Fiber products. This is a categorical construction which we're going to use a lot.

Definition 2.10. Let X, Y, and Z be sets and $f\colon X \to Z$ and $g\colon Y \to Z$ be set maps. Then the *fiber product* of X and Y over Z is

$$X \times_Z Y := \{(x, y) \in X \times Y \mid f(x) = g(y)\}. \tag{2.11}$$

If X, Y, and Z are spaces, and f and g are maps of spaces, then the *fiber product* of X and Y over Z is the space defined by

$$(X \times_Z Y)(A) := X(A) \times_{Z(A)} Y(A). \tag{2.12}$$

Technically, the notation should include f and g, but in practice there's usually no ambiguity.

Example 2.13. Suppose we're given commutative rings A, B, and C and maps $\operatorname{Spec} B \to \operatorname{Spec} C$ and $\operatorname{Spec} A \to \operatorname{Spec} C$ (which are equivalent data to maps $\varphi\colon C \to A$ and $\psi\colon C \to B$). Then

$$\operatorname{Spec} A \times_{\operatorname{Spec} C} \operatorname{Spec} B \cong \operatorname{Spec}(A \otimes_C B),$$

where C acts on A, resp. B, through φ, resp. ψ. It's worth working through this one on your own, though it's not extremely hard. ◀

We'll define some properties of affine schemes with geometric names, but the definitions will rest on algebraic properties of rings. One of the real miracles of algebraic geometry is that this really works to define geometry, and even extends geometric intuition to places such as finite fields that are otherwise very hard to reason about.

Definition 2.14. A morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is a *closed embedding* if the induced map $A \to B$ is surjective.

Equivalently, $B = A/I$ for some ideal I of A.

The geometric idea behind defining $\operatorname{Spec} A$ is that geometric objects have a ring of functions on them, e.g. a smooth manifold M has a ring $C^\infty(M)$ of smooth $\mathbb{R}$-valued functions, and a map of manifolds $M \to N$ induces a map in the other direction by pullback: $C^\infty(N) \to C^\infty(M)$. Functional analysis results such as the Gelfand-Naimark theorem tell you what data you need to add to $C^\infty(M)$ to recover M as a topological space, and we're trying to imitate this in a more abstract algebraic setting.

This context allows us to explain why Definition 2.14 deserves to be called a closed embedding: let $I = (f_1, f_2, \dots)$, so

$$\operatorname{Spec} A/I = \{f_i = 0 \text{ for all } i\} = \{f = 0 \text{ for all } f \in I\}. \tag{2.15}$$

So we think of $\operatorname{Spec} B$ as some kind of closed subspace of $\operatorname{Spec} A$, and I as the ideal of functions on $\operatorname{Spec} A$ which vanish on $\operatorname{Spec} B$. This intuition can be turned into something precise.

Using fiber products, we can extend this to all spaces.

Definition 2.16. A map $X \to Y$ of spaces is a *closed embedding* if for all maps $\operatorname{Spec} A \to Y$, the "pullback" φ in the fiber product diagram

$$\begin{tikzcd} \operatorname{Spec} A \times_Y X \arrow[r, "\varphi"] \arrow[d] & \operatorname{Spec} A \arrow[d] \\ X \arrow[r] & Y \end{tikzcd} \tag{2.17}$$

is a closed embedding of affine schemes. In particular, we require $\operatorname{Spec} A \times_Y X$ to be an affine scheme, which is not always satisfied.

For a quick consistency check, we should ask that Definitions 2.14 and 2.16 agree on affine schemes, and indeed, if $I \subset A$ is an ideal, and $\operatorname{Spec} B \to \operatorname{Spec} A$ is a closed embedding in the sense of Definition 2.14, then (2.17) looks like

$$\begin{tikzcd} \operatorname{Spec}(A/I \otimes_A B) \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\ \operatorname{Spec}(A/I) \arrow[r] & \operatorname{Spec} A, \end{tikzcd} \tag{2.18}$$

and since $A/I \otimes_A B \cong B/BI$, this is a closed embedding in the more general sense as well.

We'd also like to know what an open embedding is. We'd like to say that it's something whose complement is a closed embedding. Let's make this precise.

Definition 2.19. Let $Z \hookrightarrow X$ be a closed embedding of spaces. The *complement* $X \setminus Z$ of Z in X is the space with $(X \setminus Z)(A)$ the set of $x \in X(A) = \mathrm{Hom}_{\mathsf{Spaces}}(\operatorname{Spec} A, X)$ such that the diagram

(2.20)
$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & Z \arrow[d] \\ \operatorname{Spec} A \arrow[r, "x"] & X \end{tikzcd}$$

is a fiber product diagram. Here $\varnothing = \operatorname{Spec}(0)$, which sends every ring to the empty set.[3]

Definition 2.21. If $X = \operatorname{Spec} A$ is an affine scheme, an *open embedding* is a map of spaces $j \colon U \to X$ such that $U = X \setminus Z$ for some closed embedding $Z \hookrightarrow X$.

Example 2.22. Letting $X = \operatorname{Spec} A$, if $f \in A$ and $Z = \operatorname{Spec}(A/f)$, the map $A \twoheadrightarrow A/f$ induces a closed embedding $Z \hookrightarrow X$. Its complement is $\operatorname{Spec} A[f^{-1}]$, the *localization* of A at f, so $\operatorname{Spec} A[f^{-1}] \to \operatorname{Spec} A$ is an open embedding. ◀

The intuition is that f generates the ideal of functions that vanish precisely on the closed subset Z. Therefore on the complement of Z, they should be invertible, so we adjoin an inverse to f.

Lemma 2.23. *Let $X = \operatorname{Spec} A$ and $Z = \operatorname{Spec} A/I$. Then maps $\operatorname{Spec} B \to X \setminus Z$ correspond bijectively to maps $A \to B$ such that $B \cdot I = B$.*

Proof. The diagram (2.20) specializes to

(2.24)
$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & \operatorname{Spec}(A/I) \arrow[d] \\ \operatorname{Spec} B \arrow[r] & \operatorname{Spec} A, \end{tikzcd}$$

and this fiber product is $\operatorname{Spec}(B \otimes_A A/I) = \operatorname{Spec}(B/IB) = \varnothing$, which is equivalent to $IB = B$. ⊠

Example 2.25. *Affine n-space* over $\mathbb{Z}$ is the affine scheme $\mathbb{A}^n_{\mathbb{Z}} := \operatorname{Spec} \mathbb{Z}[x_1, \dots, x_n]$, and $0 \hookrightarrow \mathbb{A}^n_{\mathbb{Z}}$ is the closed embedding corresponding to the ideal $(x_1, \dots, x_n)$. The complement $\mathbb{A}^n_{\mathbb{Z}} \setminus 0$ is not affine for $n > 1$! We'll prove that later when we have more tools. ◀

Exercise 2.26. Show that $\mathbb{A}^n_{\mathbb{Z}} \setminus 0$ is the space which maps a ring A to the set of n-tuples $(x_1, \dots, x_n) \in A^n$ such that the equation $\sum x_i y_i = 1$ has a solution.

Lecture 3.

Open covers: 9/5/18

We've been talking about functors as if they were honest geometric objects. And they *are*: the crucial reason is that we're defining open and closed subspaces of affine schemes. You can picture these as akin to open or closed sets in a topological space, and they will allow us to make sense of geometry by giving us notions of locality.

Recall that $Z \hookrightarrow X = \operatorname{Spec} A$ is a closed embedding means that this embedding is of the form $\operatorname{Spec}(A/I) \to \operatorname{Spec} A$ induced by the map $A \twoheadrightarrow A/I$, and that open embeddings are complements of closed ones. You might think of the complement as $(X \setminus Z)(B) = X(B) \setminus Z(B)$, but **this is wrong**: it's not even functorial! Instead, we want to say $(X \setminus Z)(B) = \{\operatorname{Spec} B \to X \setminus Z\}$. What this means is maps $\operatorname{Spec} B \to X$ such that the pullback $\operatorname{Spec} B \times_X Z = \varnothing$. Geometrically, this fiber product is telling you the intersection of the image of $\operatorname{Spec} B$ with X.

Last time, we also talked about $\mathbb{A}^1$ (also $\mathbb{A}^1_{\mathbb{Z}}$ if you want to specify the base), which is by definition $\operatorname{Spec} \mathbb{Z}[t]$. It would be nice to think of this as a line, in the sense you can draw; but it behaves more like a complex line

[3]Caution: this is only true if we work with functors on nonzero rings. However, $\varnothing = \operatorname{Spec} 0$ still counts as affine. There are other ways to correct this issue, but this is among the fastest and cheapest.

(that is, a plane). For example, $\mathbb{A}^1$ minus a point is connected. So thinking of it as a complex line is good, but for drawing pictures you'll run out of dimensions, so the picture of a real line is also helpful.

If B is a commutative ring, $\mathbb{A}^1(B) = \{\operatorname{Spec} B \to \mathbb{A}^1\}$, i.e. $\operatorname{Hom}(\mathbb{Z}[t], B) = B$, because the map is determined by where it sends t. This makes precise the notion that the ring of functions on $\operatorname{Spec} B$ is B. This is another avatar of geometry as we know it: functions on a geometric object (say, a complex manifold) are functions to a complex line, and in this setting we replace the complex line by $\mathbb{A}^1$.

Consider the embedding $0 \hookrightarrow \mathbb{A}^1_{\mathbb{Z}}$, where 0 denotes the locus where $t = 0$, i.e. $\operatorname{Spec} \mathbb{Z}[t]/(t)$. As an affine scheme, this is isomorphic to $\operatorname{Spec} \mathbb{Z}$, because $\mathbb{Z}[t]/(t) \cong \mathbb{Z}$, but this defines a particular closed embedding $0 \hookrightarrow \mathbb{A}^1_{\mathbb{Z}}$. Last time, we discussed $\mathbb{A}^1 \setminus 0$. A map $\operatorname{Spec} B \to \mathbb{A}^1 \setminus 0$ is a function that avoids zero, which means that it's invertible.

Exercise 3.1. Show that $(\mathbb{A}^1 \setminus 0)(B) = B^\times$, and therefore that $\mathbb{A}^1 \setminus 0 \cong \operatorname{Spec} \mathbb{Z}[t, t^{-1}]$.

If we did this with $\mathbb{A}^2 \setminus 0$ instead of $\mathbb{A}^1 \setminus 0$, we'd obtain a nonaffine scheme.

Open coverings are another important geometric notion, and they exist in this setting too.

Definition 3.2. If $X = \operatorname{Spec} A$ is an affine scheme, a *(Zariski) open covering* of X is a collection of open embeddings $\mathfrak{U} = \{(U, i_U : U \hookrightarrow X)\}$ such that for every nonempty $S = \operatorname{Spec} B$ and $f : S \to X$, there's some $(U, i_U) \in \mathfrak{U}$ such that $U \times_X S \neq \varnothing$.

This is the first notion of open covering in algebraic geometry; there are some others around.

The intuition behind open coverings is that points of X are given by maps $\operatorname{Spec} B \to X$, and we want every point in X to intersect some open embedding in the cover.

Proposition 3.3. *Let $X = \operatorname{Spec} A$ and $\mathfrak{U} = \{(U, i_U : U \to X)\}$ be a collection of open embeddings. The following are equivalent:*

(1) *$\mathfrak{U}$ is an open covering.*
(2) *$\mathfrak{U}$ has a finite subset $\mathfrak{V} \subset \mathfrak{U}$ which is also an open covering of X.*
(3) *For all fields k and maps $x : \operatorname{Spec} k \to X$, there's some $(U, i_U) \in \mathfrak{U}$ such that x factors through i_U.*
(4) *Letting $U = X \setminus Z_U$ for each $U \in \mathfrak{U}$, and writing $Z_U = \operatorname{Spec}(A/I_U)$, then*

$$\sum_{U \in \mathfrak{U}} I_U = A.$$

Point (2) is very weird coming from topology, where the open covering $\{(i-1, i+1) \mid i \in \mathbb{Z}\}$ is an open cover of $\mathbb{R}$ with no finite subcover. In other words, affine schemes feel like compact spaces from the perspective of open coverings!

The idea behind (3) is that points are affine schemes of the form $\operatorname{Spec} k$ for k a field. There are different fields, and therefore different kinds of points. The reason for including (4) is that it's very useful for checking in practice. It has a similar feel to partitions of unity in manifold topology, but if you don't know what that is, that's OK.

Proof. We'll first show (1) $\implies$ (4). Suppose $\mathfrak{U}$ is an embedding for which (4) does not hold. Then let

$$B := A / \sum_{U \in \mathfrak{U}} I_U. \tag{3.4}$$

By hypothesis, $B \neq 0$, and we have a closed embedding $\operatorname{Spec} B \hookrightarrow \operatorname{Spec} A$. We'll show that $\operatorname{Spec} B \times_X U = \varnothing$ for all $U \in \mathfrak{U}$.

Lemma 3.5. *Let $Z = \operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A = X$ be a closed embedding and $f : \operatorname{Spec} B \to X$ be a map. Then*

$$(\operatorname{Spec} B) \setminus f^{-1}(Z) = \operatorname{Spec} B \times_X (X \setminus Z).$$

This is more or less a tautology.

Returning to the claim, $\operatorname{Spec} B \times_X U$ is the complement of $Z_U \times_X \operatorname{Spec} B = \operatorname{Spec}(B/BI_U)$. But $B/BI_U = B$, so the complement of $Z_U \times_X \operatorname{Spec} B$ is the empty set.

Next, we'll show (4) $\implies$ (3). Let B be as in (3.4), so $B \neq 0$, k be a field, and $x : \operatorname{Spec} k \to X$ be a map. We want to show this map factors through some U. Since $X = \operatorname{Spec} A$, x corresponds to a map $\varphi : A \to k$. We claim there's a $U \in \mathfrak{U}$ with $\varphi(I_U) \neq 0$; otherwise $\varphi(\sum I_U) = 0$, and therefore $\varphi(A) = 0$. However, $\varphi(1) = 1$, so this is impossible. By Lemma 2.23, since $\varphi(I_U) \neq 0$, $B \cdot f(I_U) = B$, and therefore $x\colon \operatorname{Spec} k \to X$ factors through U.

Next we'll show (3) $\implies$ (1). As usual, let B be as in (3.4) and $f\colon S = \operatorname{Spec} B \to X$ be a map. We want to show that $S \times_X U \neq 0$ for some $U \in \mathfrak{U}$. Since $B \neq 0$, it has a maximal ideal $\mathfrak{m}$, and $B/\mathfrak{m}$ is a field k (TODO: to be continued...)

$\boxtimes$

Lecture 4.

Defining schemes, II: 9/7/18

"I'll let Fun(Y)*, which is such a fun notation, denote..."*

Last time, we talked about open embeddings and open covers for affine schemes; today, we'll generalize this to spaces.

Definition 4.1. Let X be a space.

(1) A map $U \to X$ of spaces is an *open embedding* if for all affine schemes $S = \operatorname{Spec} A$ and maps $f\colon S \to X$ of spaces, the pullback $g\colon U \times_X S \to S$ arising in the diagram

$$\begin{tikzcd} U \times_X S \arrow[r] \arrow[d, "g"] & U \arrow[d] \\ S \arrow[r, "f"] & X \end{tikzcd}$$

is an open embedding of affine schemes.

(2) A *Zariski open covering* of X is the same as in Definition 3.2, but for open embeddings of spaces, rather than affine schemes.

In this case, Proposition 3.3 need not hold: there are open coverings of some spaces X (such as an infinite disjoint union of points) which have no finite subcoverings.

Definition 4.2. A space X is a *Zariski sheaf* if for all $S = \operatorname{Spec} A$ and open coverings $\mathfrak{U}$ of S, the map

$$\operatorname{Hom}(S, X) \longrightarrow \{(f_U : U \to X \text{ for all } U) \in \mathfrak{U} \mid f_U|_{U\cap V} = f_V|_{U\cap V} \text{ for all } U, V \in \mathfrak{U}\}$$

is an isomorphism. (Here $U \cap V = U \times_X V$.)

Not everything is a Zariski sheaf, but the things that aren't are terrible, and you shouldn't worry too much about them.

Now we have all the definitions at hand to define schemes!

Definition 4.3. A *scheme* is a space which is a Zariski sheaf and admits an open cover $\mathfrak{U}$ such that all $U \in \mathfrak{U}$ are affine schemes.

Exercise 4.4. Let X be the space with

$$X(A) = \{t \in A \mid t \in A^\times \text{ or } (1-t) \in A^\times\}.$$

Show that X is *not* a Zariski sheaf. Also, if you know what sheafification is, show that the sheafification of X is $\mathbb{A}^1$.

Proposition 4.5. *If X is an affine scheme, then it's a scheme.*

Obviously X admits an open cover by affines, given by $\mathrm{id}\colon S \to S$; the meat of the proof (or, if you prefer, tofu) is that it's a Zariski sheaf. Unlike EGA, we will start with a special case and use it to bootstrap to the general case.

Let $X = \mathbb{A}^1$.

Definition 4.6. A *function* on a space Y is a map to $\mathbb{A}^1$. We'll let $\operatorname{Fun}(Y) := \operatorname{Hom}(Y, \mathbb{A}^1)$.

We're explicitly trending towards geometric notation and intuition for things: one of the key processes of learning scheme theory is to start thinking geometrically rather than with commutative algebra – except when you need to prove something.

We want to show that for all affine schemes S and open coverings $\mathfrak{U}$ of S, the map

$$\mathrm{Fun}(S) \longrightarrow \{(f_U \in \mathrm{Fun}(U)) \mid f_U|_{U\cap V} = f_V|_{U\cap V}\} \tag{4.7}$$

is an isomorphism.

First we'll prove this for a nice class of open covers.

Lemma 4.8. *Let A be a commutative ring and $f_1, \dots, f_n \in A$. Let $D(f_i) \coloneqq \mathrm{Spec}\, A \setminus \mathrm{Spec}(A/(f_i))$. Then*

1. $D(f_i) = \mathrm{Spec}\, A[f_i^{-1}]$*, and*
2. $\{D(f_i)\}$ *is an open cover iff $\{f_i\}$ generates the unit ideal.*

The proof will be left as an exercise.

In the case (2) holds, the open cover $\{D(f_i)\}$ is called a *basic open cover*. It's really nice because it's an affine open cover; we'll see that there are a lot of these coverings, and enough that we will eventually be able to reduce to this case.

One can alternately characterize $D(f_i)$ as the pullback

$$\begin{tikzcd} D(f_i) \arrow[r] \arrow[d] & \mathrm{Spec}\, A \arrow[d] \\ \mathbb{A}^1 \setminus 0 \arrow[r] & \mathbb{A}^1. \end{tikzcd} \tag{4.9}$$

Lemma 4.10. *Let $\varphi\colon M \to N$ be a map of A-modules. Then f is injective (resp. surjective, resp. bijective) iff for all i, the map $M[f_i^{-1}] \to N[f_i^{-1}]$ is injective (resp. surjective, resp. bijective).*

Recall that $M[f_i^{-1}] \coloneqq M \otimes_A A[f_i^{-1}]$.

Remark 4.11. Let's review some facts about localization. If M is a $\mathbb{Z}[t]$-module, which is equivalent data to an abelian group with an endomorphism $t\colon M \to M$, then we can form $M[t^{-1}] \coloneqq M \otimes_{\mathbb{Z}[t]} \mathbb{Z}[t, t^{-1}]$. Then:

- This construction is *exact*; that is, it preserves kernels and cokernels.
- There's a natural map $M \to M[t^{-1}]$, and its kernel is the submodule of $m \in M$ with $t^n m = 0$ for some n.

The way you prove all of this is to write $\mathbb{Z}[t, t^{-1}]$ as the union of $t^{-n}\mathbb{Z}[t]$ for all n, or as the colimit of the multiplication-by-t map on $\mathbb{Z}[t]$. These are all free modules, hence flat, and one can prove that filtered colimits of flat modules are flat without too much angst. ◀

Now we can get back to the lemma.

Proof of Lemma 4.10. For injectivity, let's compare $\ker\varphi$ with $\ker(\varphi[f_i^{-1}])$. By Remark 4.11, $(\ker\varphi)[f_i^{-1}] = \ker(\varphi[f_i^{-1}])$, so we can reduce to showing that $M = 0$ iff $M[f_i^{-1}] = 0$ for all i. One direction is immediate, of course; conversely, if $M[f_i^{-1}] = 0$ for all i, then for all $m \in M$ and all f_i, then $f_i^{n_i} m = 0$ for some $n_i \gg 0$. There are finitely many f_i, so we can let N be the biggest one, and then $f_i^N m = 0$ for all i.

Lemma 4.12. $(f_1, \dots, f_n) = A$ *iff* $(f_1^N, \dots, f_n^N) = A$.

Proof. There's a naïve argument which isn't too bad, but the geometric reason is that f_i is a function, and f_i vanishes on the same locus where f_i^N vanishes, and therefore $D(f_i^N) = D(f_i)$. Therefore $\{D(f_i^N)\}$ is also an open cover, which by (4.8) means $(f_1^N, \dots, f_n^N) = A$.

If this proof feels sketchy, here's a more careful one (which unfortunately masks the geometry): if $(f_1^N, \dots, f_n^N) \subsetneq A$, then it's contained in some maximal ideal $\mathfrak{m}$. Therefore for all i, $f_i^N = 0$ in $A/\mathfrak{m}$, and therefore $f_i = 0$ in $\mathfrak{m}$, because $A/\mathfrak{m}$ is a field; hence $f_i \in \mathfrak{m}$, which is a contradiction. ⊠

Now, using Lemma 4.12, there are some $g_1, \dots, g_n$ with $\sum_i g_i f_i^N = 1$, and therefore

$$M = \sum_i f_i f_i^N \cdot M = 0. \tag{4.13}$$

The proof of surjectivity is similar, but using cokernels instead of kernels. ⊠

This lemma is a bridge between the geometry of schemes and the linear algebra of modules. You should think of inverting f_i as restricting to $D(f_i)$; we will return to this idea.

Proof of Proposition 4.5, special case. Now we'll prove that an affine scheme is a Zariski sheaf for basic open covers $\{D(f_i)\}$. We want to show that (4.7) is an isomorphism, and by Lemma 4.10 it suffices to show this after inverting f_i.

Let $g_i \in \mathrm{Fun}(D(f_i))$ be a collection of functions that agree on overlaps... TODO: I missed the last part. ⊠

Lecture 5.

$\mathbb{A}^1$ is a Zariski sheaf: 9/10/18

Today, we're going to continue proving Proposition 4.5, that affine schemes are schemes. We're still working on the special case that $\mathbb{A}^1$ is a scheme; the key piece of the proof is showing that it's a Zariski sheaf.

Definition 5.1. Let S be a space and $\mathfrak{U}$ be an open cover of S. A *refinement* of $\mathfrak{U}$ is an open covering $\mathfrak{V}$ of S such that for all $U \in \mathfrak{U}$, $\mathfrak{V}_U := \{V \in \mathfrak{V} \mid V \subset U\}$ is an open covering of U.

The Zariski sheaf condition for maps $X \to S$ is a constraint on compatible functions on all open covers of S. If we only ask about a specific open cover $\mathfrak{U}$, we say "the Zariski sheaf property for X with respect to $\mathfrak{U}$."

Lemma 5.2. *Let X and S be spaces, $\mathfrak{U}$ be an open cover of S, and $\mathfrak{V}$ be a refinement of $\mathfrak{U}$. Suppose the Zariski sheaf property holds for X with respect to $\mathfrak{V}$, and for each $U \in \mathfrak{U}$ with respect to $\mathfrak{V}_U$, then it holds with respect to $\mathfrak{U}$.*

After you unwind all the definitions, this is a definition check which isn't very hard.

Remark 5.3. One corollary of Lemma 5.2 is that in the definition of the sheaf property, we may replace "for all affine schemes S" with "for all spaces S." All of the definitions were built from the beginning to favor affine schemes as important or special, and this is one consequence. ◀

Definition 5.4. A *big basic open covering* of an affine scheme S is an open covering by sets of the form $D(f_i)$ as in Lemma 4.8, but over a possibly infinite indexing set.

This is only a temporary definition. The Zariski sheaf property for X and every basic open covering of an affine scheme S implies the Zariski sheaf for all big basic open coverings.

Proposition 5.5. *Let S be an affine scheme and $\mathfrak{U}$ be an open cover of S. Then there's a big basic open covering of S refining $\mathfrak{U}$.*

Proof. Write $S = \operatorname{Spec} A$ and for each $U \in \mathfrak{U}$, let $Z_U := S \setminus U$; the inclusion $Z_U \hookrightarrow S$ is a closed embedding, so $Z_U = \operatorname{Spec}(A/I_U)$ for some ideal $I_U \subset A$. Recall from Proposition 3.3 that since $\mathfrak{U}$ is an open covering,

$$\sum_{U \in \mathfrak{U}} I_U = A, \tag{5.6}$$

and this is an equivalent condition. Consider the big basic open cover

$$\mathfrak{V} := \{D(f) \mid f \in I_U \setminus 0 \text{ for some } U \in \mathfrak{U}\}. \tag{5.7}$$

That this is a big basic open cover is because an ideal is generated by its elements. It's also a refinement of $\mathfrak{U}$, which follows from a more general lemma.

Lemma 5.8. *Let $U = S \setminus Z$, where $S = \operatorname{Spec} A$ and $Z = \operatorname{Spec}(A/I)$. Then $\{D(f) \mid f \in I \setminus 0\}$ is an open cover of U.*

Proof. We want to show that for all $T = \operatorname{Spec} B$ and maps $g\colon T \to U$, the set $\mathfrak{V}_g := \{g^{-1}(D(f)) \mid f \in I \setminus 0\}$ is an open cover of T.[4] Then... TODO ⊠

Thus we've proven the proposition. ⊠

Corollary 5.9. *Let S be an affine scheme with an open covering $\mathfrak{U}$. Then there's a big basic open covering $\mathfrak{V}$ refining $\mathfrak{U}$ and with the property that for all $U \in \mathfrak{U}$, $\{V \in \mathfrak{V} \mid V \subset U\}$ is a big basic open covering of U.*

This is the technical proposition that lets us reduce to algebra.

[4] The preimage is defined to be $g^{-1}(D(f)) := D(f) \times_U T$.

Remark 5.10. Corollary 5.9 also tells us that a big basic open covering of a space X is an open covering $\mathfrak{U}$ of X such that for all maps of affine schemes to X, the pullback of $\mathfrak{U}$ is also a big basic open covering. ◀

Corollary 5.11. $\mathbb{A}^1$ *is a Zariski sheaf.*

Proof. We showed that $\mathbb{A}^1$ is a Zariski sheaf with respect to all basic open covers of affine schemes, hence for all big basic open covers of affine schemes, hence by Remark 5.10 with respect to all spaces with big basic open covers, hence by Proposition 5.5 any affine scheme and any open cover, and therefore any space and any open cover. ⊠

Corollary 5.12. *Let I be a set and let $\mathbb{A}^I \coloneqq \operatorname{Spec} \mathbb{Z}[\{x_i \mid i \in I\}]$. Then $\mathbb{A}^I$ is a Zariski sheaf.*

Proof. The sheaf property is preserved under arbitrary products. ⊠

If I is an n-element set, then $\mathbb{A}^I$ is also written $\mathbb{A}^n$.

Proof sketch of Proposition 4.5. We can use this to show that if $X = \operatorname{Spec} A$ is an affine scheme, then it's a Zariski sheaf. Let I be a generating set for A and $J \subset \mathbb{Z}[\{x_i \mid i \in I\}]$ be the ideal of relations; then, the quotient map $\mathbb{Z}[\{x_i \mid i \in I\}] \twoheadrightarrow A$ defines a closed embedding $X \subseteq \mathbb{A}^I$ cut out by $X = \{x \mid f(x) = 0 \text{ for all } f \in J\}$.

One then has to check that the sheaf property is preserved under closed embeddings, which is formal. ⊠

We'll spend the next lecture giving examples of schemes, but here are a few to start with.

- As we just showed, affine schemes are schemes.
- A *quasi-affine scheme* is an open subset of an affine scheme, such as $\mathbb{A}^2 \setminus 0$. These are indeed schemes (though not always affine): if U is the complement of $\operatorname{Spec}(A/I) \subset A$, then U admits a covering by $\{D(f) \mid f \in I \setminus 0\}$.

We can use this to prove $\mathbb{A}^2 \setminus 0$ isn't affine.

Lecture 6.

Relative algebraic geometry: 9/12/18

One of the advantages of algebraic geometry is the ability to work relative to a given space, which generalizes choosing a base field (or ring).

Definition 6.1. Let S be a space. A *scheme over S* is a space X with a map $X \to S$, often just written X/S, such that for all affine schemes T and maps $T \to S$, $X \times_S T$ is a scheme. A morphism of schemes over S is a morphism of schemes which commutes with the two maps to S.

In the same way one can define affine schemes over S. If $S = \operatorname{Spec} A$, for A a ring, we might write X/A instead of X/S and say schemes over A; often A will be a field.

We defined spaces as functors $\mathsf{CommRing} \to \mathsf{Set}$, and there's a similar description for schemes over A.

Proposition 6.2. *Let A be a commutative ring. There's an equivalence of categories between spaces over A and functors $\mathsf{CommAlg}_A \to \mathsf{Set}$ (where we ignore the zero algebra).*

Proof sketch. Given $X\colon \mathsf{CommAlg}_A \to \mathsf{Set}$, we can define a functor on all commutative rings by sending B to the set of pairs of (i, x) where $i\colon A \to B$ is an A-algebra structure on B and $x \in X(B)$. Then the forgetful map $(i, x) \mapsto i$ defines the desired map to $\operatorname{Spec} A$.

In the other direction, let $p\colon X \to \operatorname{Spec} A$ be a scheme over A. We'll define a functor on commutative A-algebras by sending $(B, i\colon A \to B)$ to the set of maps $\varphi\colon \operatorname{Spec} A \to X$ for which the diagram

$$\begin{tikzcd} & X \arrow[d, "p"] \\ \operatorname{Spec} B \arrow[ur, "\varphi"] \arrow[r, "i^*"'] & \operatorname{Spec} A \end{tikzcd} \tag{6.3}$$

commutes. ⊠

Example 6.4. Complex conjugation is $\mathbb{Z}$-linear (and even $\mathbb{R}$-linear) but not $\mathbb{C}$-linear, and therefore induces a map of schemes $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{C}$ which is a map of schemes over $\mathbb{R}$, but not of schemes over $\mathbb{C}$. ◀

Proposition 6.5. *Let X, Y, and Z be schemes together with maps $X \to Z$ and $Y \to Z$. Then $X \times_Z Y$ is a scheme.*

Proof. If $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, and $Z = \operatorname{Spec} C$ are affine, this is certainly true: the pullback is $\operatorname{Spec} A \otimes_C B$. Now we'll show more general cases reduce to this one.

If Y and Z are affine but X isn't, then X admits an open cover $\mathfrak{U}$ by affines, and $\{U \times_Z Y \mid U \in \mathfrak{U}\}$ is an affne open cover of $X \times_Z Y$. In the same way, we may assume only that X and Z are affine.

Therefore if you only assume Z is affine, you can pick affine open covers of X and Y called $\mathfrak{U}$ and $\mathfrak{V}$, respectively. Then $\{U \times_Z V \mid U \in \mathfrak{U}, V \in \mathfrak{V}\}$ is an affine open cover of $X \times_Z Y$.

Next, we assume X and Y are affine, but Z might not be.[5] Let $\mathfrak{W}$ be an affine open cover of Z, and $W \in \mathfrak{W}$. By definition, the map

$$X \times_Z W \longrightarrow X \tag{6.6}$$

is an open embedding, and this implies that $X \times_Z W$ is a scheme (we called these quasi-affine): it's the complement of a closed embedding $\operatorname{Spec} A/I \to X = \operatorname{Spec} A$, and is covered by $\{D(f)\}$ where $\{f\}$ generates I. Anyways, then $X \times_Z Y$ is covered by

$$\mathfrak{W}' := \{(X \times_Z W) \times_W (Y \times_Z W) \mid W \in \mathfrak{W}\}. \tag{6.7}$$

Since W is affine, this is a scheme by one of the earlier cases. Therefore $X \times_Z Y$ is covered by schemes, so it must be a scheme (choose an affine cover of each element of $\mathfrak{W}'$, and check this is an affine open cover of $X \times_Z Y$).

Finally, we assume none of them are affine. This is the same as the case where X and Y are affine, but now we can use the previous step to show that if $\mathfrak{U}$ is an open cover of X, $\mathfrak{V}$ is an open cover of Y, $U \in \mathfrak{U}$, and $V \in \mathfrak{V}$, then $U \times_Z V$ is a scheme.

We've ignored the Zariski sheaf property, but it's relatively simple to show that it's preserved by fiber products. ⊠

Corollary 6.8. *If S is a scheme, schemes over S are the same thing as schemes with a map to S.*

Proof. We can check the definition on an affine open cover of S; Proposition 6.5 tells us that pulling back to T preserves scheminess. ⊠

If S is a space that's not a space, Corollary 6.8 isn't necessarily true.

Quasicoherent sheaves and/or linear algebra. In commutative algebra, one often studies a ring by studying its modules; these are linear-algebraic in nature, which can make them easier to reason about. The analogue for schemes is quasicoherent sheaves.

Definition 6.9. Let X be a scheme. A *quasicoherent sheaf* (QC sheaf) $\mathscr{F}$ on X is data of, for all maps $f\colon \operatorname{Spec} A \to X$, an A-module $\mathscr{F}_f$, and for every map $g\colon \operatorname{Spec} B \to \operatorname{Spec} A$, an isomorphism

$$\alpha_{f,g}\colon \mathscr{F}_{g\circ f} \xrightarrow{\cong} \mathscr{F}_f \otimes_A B \tag{6.10}$$

of B-modules, and such that a cocycle condition holds: given a triple

$$\operatorname{Spec} C \xrightarrow{h} \operatorname{Spec} B \xrightarrow{g} \operatorname{Spec} A \xrightarrow{f} X, \tag{6.11}$$

$\alpha_{f,g\circ h} = \alpha_{f\circ g,h}$ as maps $\mathscr{F}_{f\circ g\circ h} \cong (\mathscr{F}_f \otimes_A B) \otimes_B C$, using the natural isomorphism $(\mathscr{F}_f \otimes_A B) \otimes_B C \cong \mathscr{F}_f \otimes_A C$.[6] A morphism of quasicoherent sheaves $\mathscr{F} \to \mathscr{G}$ is data of maps of A-modules $\mathscr{F}_f \to \mathscr{G}_f$ for all $f\colon \operatorname{Spec} A \to X$, such that all induced diagrams commute. The category of QC sheaves on X is denoted $\mathsf{QCoh}(X)$.

Remark 6.12. The word "quasicoherent" isn't really great unless you're playing Scrabble. It grew out of a generalization of coherent sheaves, which originally came from the analytic setting, where the name was more reasonable. You should think of analogues of modules when you hear QC sheaves. ◂

This is a lot of data! So we're going to find a way to express a quasicoherent sheaf with less data.

[5] From here, the proof was finished up in Friday's lecture.

[6] The cocycle condition can be expressed more concisely by asking that $\mathscr{F}$ is a functor from the category of affine schemes to abelian groups.

Proposition 6.13. *If $X = \operatorname{Spec} A$, the functor $\Gamma\colon \mathsf{QCoh}(X) \to \mathsf{Mod}_A$ sending $\mathscr{F} \mapsto \mathscr{F}_{\mathrm{id}}$ is an equivalence of categories, with inverse sending an A-module M to the sheaf $\mathscr{F}_M$ defined by $(\mathscr{F}_M)_f \coloneqq M \otimes_A B$ for all $f\colon \operatorname{Spec} B \to X$.*

Example 6.14. For any scheme X, there's a quasicoherent sheaf $\mathscr{O}_X$, called the *structure sheaf* of X, defined to send $f\colon \operatorname{Spec} A \to X$ to $(\mathscr{O}_X)_f = A$. The maps are what you think they are. ◀

Lecture 7.

Quasicoherent sheaves: 9/14/18

"You know when you're looking for your phone and it was in your hand the whole time? This proof was like that."

Here are two exercises we've been sort of implicitly using, and are good to do to get some comfort with this language.

Exercise 7.1.

1. Let $U \to X$ be a map of schemes and U has an open cover $\mathfrak{V}$ such that for all $V \in \mathfrak{V}$, $V \to X$ is an open embedding. Then $U \to X$ is an open embedding.
2. If $V \to U$ and $U \to X$ are open embeddings, their composition $V \to U$ is an open embedding.

Now back to quasicoherent sheaves. On an affine scheme $X = \operatorname{Spec} A$, these are a lot like A-modules (in fact, exactly like A-modules, according to Proposition 6.13).

Definition 7.2. Let $f\colon X \to Y$ be a map of schemes and $\mathscr{F} \in \mathsf{QCoh}(Y)$. The *pullback* of $\mathscr{F}$, denoted $f^*\mathscr{F} \in \mathsf{QCoh}(X)$, is the quasicoherent sheaf given by the following data: for every map $g\colon \operatorname{Spec} A \to X$, $(f^*\mathscr{F})_g \coloneqq \mathscr{F}_{f\circ g}$.

One must check the compatibility conditions, but these aren't so bad.

If $S = \operatorname{Spec} A$ is affine, then an A-module M defines a quasicoherent sheaf $\mathscr{M}$ by sending $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ to $\mathscr{M}_f \coloneqq M \otimes_A B$. The pullback of $\mathscr{M}$ along f is exactly the quasicoherent sheaf defined by the module $M \otimes_A B$.

Since we understand quasicoherent sheaves on affine schemes, let's next see how they behave on open covers. We'll start with a different-looking definition, then show it's equivalent. This second definition will be useful because it involves substantially less data.

Definition 7.3. Let X be a scheme and $\mathfrak{U}$ be an open cover of X. Let $\mathsf{QCoh}(X;\mathfrak{U})$ denote the category of tuples of $\mathscr{F}_U \in \mathsf{QCoh}(U)$ for all $U \in \mathfrak{U}$ together with, for all intersecting $U, V \in \mathfrak{U}$, isomorphisms

$$\alpha_{UV}\colon \mathscr{F}_U|_{U\cap V} \xrightarrow{\cong} \mathscr{F}_V|_{U\cap V} \tag{7.4}$$

satisfying a cocycle condition on triple intersections.

This is what's sheafy about quasicoherent sheaves: they are determined from compatible local data.

There's a functor $\Phi\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(X;\mathfrak{U})$ which takes a quasicoherent sheaf and produces its pullback on all $U \in \mathfrak{U}$.

Theorem 7.5 (Serre)**.** *The functor Φ is an equivalence of categories.*

Proof sketch. This will look a lot like what we did before. The first step is to reduce to the case where $X = \operatorname{Spec} A$ is affine and $\mathfrak{U}$ is a basic open cover, using a similar argument to the one from two lectures ago. The second step is similar to the proof that $\mathbb{A}^1$ is a Zariski sheaf.

Explicitly, after we've reduced to $X = \operatorname{Spec} A$ and $\mathfrak{U} = \{D(f_i) \mid (f_1, \dots, f_n) = A\}$, then a quasicoherent sheaf on $D(f_i)$ is (equivalent data to) an $A[f_i^{-1}]$-module M_i, together with the natural isomorphisms $\alpha_{ij}\colon M_i[f_j^{-1}] \xrightarrow{\cong} M_j[f_i^{-1}]$ as $A[(f_if_j)^{-1}]$-modules.

Given this data, we want to functorially build an A-module. The answer will be

$$M \coloneqq \{s_i \in M_i, 1 \le i \le n \mid \text{in } M_i[f_j^{-1}] \cong M_j[f_i^{-1}],\ s_i = s_j\}. \tag{7.6}$$

Now the proof is the same as in the $\mathbb{A}^1$-setting, though there we only worried about functions, not sections. The other way is simple once one invokes the flatness of $A[f_i^{-1}]$. ⊠

We might not have defined it yet, but for a field k, $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$. This is slightly nicer to work with for some applications than $\mathbb{A}^2_{\mathbb{Z}}$. Let $X := \mathbb{A}^2_k \setminus 0$, our favorite non-affine scheme, with its open cover $U := \mathbb{A}^1 \times (\mathbb{A}^1 \setminus 0)$ and $V := (\mathbb{A}^1 \setminus 0) \times \mathbb{A}^1$. Then Theorem 7.5 says a quasicoherent sheaf on $\mathbb{A}^2_k \setminus 0$ is the data of

- a $k[x, x^{-1}, y]$-module M,
- a $k[x, y, y^{-1}]$-module N, and
- an isomorphism $\alpha\colon M[y^{-1}] \cong N[x^{-1}]$ of $k[x, x^{-1}, y, y^{-1}]$-modules.

Modules can be big, so it will be useful to have some finiteness hypotheses.

Definition 7.7. Let X be a scheme and $\mathscr{F} \in \mathsf{QCoh}(X)$. Then $\mathscr{F}$ is *locally finitely generated* (l.f.g.) if for all open embeddings $j\colon \operatorname{Spec} A \to X$, $j^*\mathscr{F}$ is a finitely generated A-module.

Theorem 7.8 (Nakayama's lemma)**.** *Let $\mathscr{F}$ be a locally finitely generated scheme X, k be a field, and $x\colon \operatorname{Spec} k \to X$ be such that $x^*\mathscr{F} = 0$. Then there's an open $j\colon U \hookrightarrow X$ containing x (i.e. x factors through j) and such that $h^*\mathscr{F} = 0$.*

Geometrically, this is saying that if an l.f.g. sheaf vanishes at a point, it also vanishes in a neighborhood of that point.

Proof. First we'll reduce to the affine case: we know there's an affine open $V \subseteq X$ such that x factors through V (geometrically, the point x lies in V), so we'll replace X by V (and call it X). Let $X = \operatorname{Spec} A$, so that $\mathscr{F}$ corresponds to a finitely generated A-module M, and x corresponds to a map $\varphi\colon A \to k$. Our hypothesis means that $M \otimes_A k = 0$.

Let's induct on the number of generators of M. If M is generated by zero elements, we're done, so assume we know it for all modules generated by n elements... we'll finish this Monday. ⊠

Lecture 8.

: 9/17/18

We're in the middle of proving Nakayama's lemma, Theorem 7.8. We're proving it by induction on the number of generators of the A-module M, and the base case is trivial. So let's assume it's true for all modules generated by n elements.[7]

Remark 8.1. Let's pause to ask what a finitely generated A-module looks like. If it has one generator, it's isomorphic to A/I for some ideal I. If it has two generators, it's an extension of A/I by A/J for some ideals I and J of A. More generally, a module M with m generators is an extension $0 \to N \to M \to A/I \to 0$, where N has $m-1$ generators.

This means some specific subcases of Nakayama's lemma, such as that for local rings, are close to trivial.[8] You could prove Theorem 7.8 by reducing to the local case, though we're using a different approach.

The fact that finitely generated modules have quotients which look like A/I is the catalyst of the proof: it's untrue for modules which aren't finitely generated, such as $\mathbb{Q}$ as a $\mathbb{Z}$-module, which has no quotients of the form $\mathbb{Z}/n$. ◀

So M is an extension of an A-module N generated by n elements by A/I:

$$0 \longrightarrow N \longrightarrow M \longrightarrow A/I \longrightarrow 0. \tag{8.2}$$

By assumption, $M \otimes_A k = 0$, which means that, since tensor product is right exact,

$$(A/I) \otimes_A k \cong k/Ik = 0. \tag{8.3}$$

Recall that we had data of a map $\varphi\colon A \to k$; since k is a field, this and (8.3) imply there's some $f \in I$ wth $\varphi(f) \neq 0$. Let's localize at f; the map $\varphi\colon A \to k$ passes to a map $\widetilde{\varphi}\colon A[f^{-1}] \to k$, and since localization is exact, (8.2) induces a short exact sequence

$$0 \longrightarrow N[f^{-1}] \longrightarrow M[f^{-1}] \longrightarrow (A/I)[f^{-1}] \longrightarrow 0, \tag{8.4}$$

[7] The theorem is true for non-affine schemes, but we've already reduced to the affine case.

[8] There are many different things called Nakayama's lemma; ours is not the most general one.

but since $A[f^{-1}] = 0$, $N[f^{-1}] \cong M[f^{-1}]$, which (crucially) is generated by n elements as an $A[f^{-1}]$-module. Since $\varphi(f) \neq 0$, then $x \in D(f)$, so there's an open $U \subset D(f)$ containing x such that $(\mathscr{F}|_{D(f)})|_U = 0$ by the inductive hypothesis, and that's exactly what we wanted to prove.

Definition 8.5. Let M be an A-module. Then its *annihilator* $\mathrm{Ann}(M) \coloneqq \{f \in A \mid f \cdot M = 0\}$, which is an ideal of A.

Corollary 8.6. *Let X be a scheme and $\mathscr{F} \in \mathsf{QCoh}(X)$ be locally finitely generated. Then the subset $U_{\mathscr{F}} \coloneqq \{f\colon \operatorname{Spec} B \to X \mid f^*\mathscr{F} = 0\}$ is an open subscheme of X. In particular, if $X = \operatorname{Spec} A$ is affine, then $U_{\mathscr{F}}$ is the complement of the locus of X on which all $f \in \mathrm{Ann}(\mathscr{F})$ vanish.*

That is, the locus where $\mathscr{F}$ vanishes is open. This fits into your intuition: if you're on $\operatorname{Spec} A$ and $\mathscr{F}$ corresponds to $A/(f)$, then $\mathscr{F}$ vanishes wherever f doesn't.

Proof. It suffices to prove the affine statement, and this is a matter of unwinding its definition: let $X = \operatorname{Spec} A$ and $\mathscr{F}$ be an A-module. Given $\varphi\colon A \to B$, it suffices to prove the following are equivalent: $\mathrm{Ann}(\mathscr{F}) \cdot B = B$ and $\mathscr{F} \otimes_A B = 0$.

First, the forward implication: we know there are $f_i \in \mathrm{Ann}(\mathscr{F})$ and $g_i \in B$ such that

$$\sum_{i=1}^{n} \varphi(f_i) g_i = 1. \tag{8.7}$$

Therefore 1 acts by 0 on $\mathscr{F} \otimes_A B$, so that module must be the zero module.

The reverse direction is a bit harder. Suppose for a contradiction that $\mathrm{Ann}(\mathscr{F}) \cdot B \subsetneq B$, so it's contained in some maximal ideal $\mathfrak{m}$; let $k \coloneqq B/\mathfrak{m}$, which is a field. Then

$$\mathscr{F} \otimes_A k = (\mathscr{F} \otimes_A B) \otimes_B k = 0. \tag{8.8}$$

Hence, by Theorem 7.8, there's a $U \subset \operatorname{Spec} A$ containing $\operatorname{Spec} k$ such that $\mathscr{F}|_U = 0$. We can assume $U = D(f)$ for some $f \in A$, so we're assuming $\mathscr{F}[f^{-1}] = 0$. Because $\mathscr{F}$ is finitely generated, this means $f^N \mathscr{F} = 0$ for some $N \gg 0$, or $f^N \in \mathrm{Ann}(f)$. Since $\varphi(\mathrm{Ann}(\mathscr{F})) \subset \mathfrak{m}$, then $\varphi(f^N) = 0 \bmod \mathfrak{m}$, so $\varphi(f) = 0 \bmod \mathfrak{m}$, which contradicts the assumption that $\operatorname{Spec} k \in U$. ⊠

You can draw a picture of this: given a locally finitely generated sheaf $\mathscr{F}$, $\mathrm{Ann}(\mathscr{F})$ has a vanishing locus; if $\mathscr{F}$ corresponds to the module A/I (here we should be on an affine scheme), then this is also the closed subset $\operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A$.

Exercise 8.9. Deduce every other version of Nakayama's lemma that you know (e.g. the one in Matsumara) from these versions.

Definition 8.10. A *vector bundle* on a scheme X is a quasicoherent sheaf $\mathscr{E} \in \mathsf{QCoh}(X)$ which is locally finitely generated and *locally projective*, i.e. for some (equivalently any) affine open cover $\mathfrak{U}$ of X, for every $U = \operatorname{Spec} A \in \mathfrak{U}$, the pullback of $\mathscr{E}$ to U is a projective A-module.

Proposition 8.11. *Let $\mathscr{E} \in \mathsf{QCoh}(X)$. The following are equivalent:*

(1) *$\mathscr{E}$ is a vector bundle.*
(2) *There is an affine open cover $\mathfrak{U}$ of X such that for all $U \in \mathfrak{U}$, $\mathscr{E}|_U$ is a finitely generated free module.*