TOPOLOGICAL QUANTUM FIELD THEORY

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These notes were taken in a class given by Katrin Wehrheim at UC Berkeley in Spring 2020. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

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1. TQFT: definition and Atiyah's examples: 2/19/20

We begin with the definition of a topological quantum field theory due to Atiyah, now over 30 years ago.

Definition 1.1. Fix a base field k. A d-dimensional *topological quantum field theory* (TQFT) consists of data of, for every closed, oriented, smooth d-manifold, a finitely generated k-vector space $Z(\Sigma)$, and for every compact, oriented, smooth $(d+1)$-manifold M, an element $Z(M) \in Z(\partial M)$, satisfying some axioms.

Many interrelated ideas went into this definition: Segal's mathematical formalization of two-dimensional conformal field theory, mathematical perspectives on quantum field theory (fields, Hilbert spaces, etc.).

Later, Atiyah's definition was packaged more concisely into asking for Z to be a symmetric monoidal functor

$$Z\colon \mathcal{C}ob_{n,n-1} \longrightarrow \mathcal{V}ect_k, \tag{1.2}$$

where $\mathcal{V}ect_k$ is the symmetric monoidal category of k-vector spaces with tensor product, and $\mathcal{C}ob_{n,n-1}$ is the cobordism category, whose objects are closed, oriented $(n-1)$-manifolds and whose morphisms are (diffeomorphism classes of) oriented bordisms between them. Cylinders in the cobordism category can be thought of as time evolution, but the inclusion of all other bordisms has something to do with a relativistic perspective.

Remark 1.3. Atiyah used d to denote the dimension of space, i.e. the dimension of manifolds assigned vector spaces. These days, it's more common to refer to the cobordism category using the top dimension (what we just called n), the "spacetime dimension." ◀

There are many different flavors of the cobordism category. Some of these involve technical details that we have to account for: for example, even compact 0-manifolds are too big to form a set, so to more accurately define $\mathcal{C}ob_{1,0}$ (or in any dimension) we should pick a set of representatives of oriented diffeomorphism classes of $(n-1)$-manifolds.

Remark 1.4. There are other ways to work around set-theoretic issues: for example, the topological cobordism category of Galatius, Madsen, Tillmann, and Weiss begins with the space $\mathbb{R}^\infty$ and works with manifolds and bordisms explicitly embedded in $\{t\}\times\mathbb{R}^\infty$, resp. $[t_1, t_2]\times\mathbb{R}^\infty$. Then one must quotient out by diffeomorphisms, just as in the abstract cobordism category, but now we don't just have the "internal diffeomorphisms" of an embedded M, but also "external diffeomorphisms" of the ambient space that carry M to something diffeomorphic, but embedded via a different map. We will not work with embedded bordisms, at least for now. ◀

There are several other generalizations we won't discuss today, but are worth mentioning.

- There's notions of *topological conformal field theory* (TCFT) and *homological conformal field theory* (HCFT), in which $\mathcal{C}ob_{2,1}$ is upgraded to a category where bordisms carry some additional structure (e.g. a conformal structure), and we only identify conformally equivalent bordisms.
- In *fully extended* topological quantum field theory, $\mathcal{C}ob_{n,n-1}$ becomes an (∞, n)-category, by allowing manifolds in all dimensions n and below.

In both cases, we must replace the target category $\mathcal{V}ect_k$ with something related, but different.

In these, and in any, generalizations, the overarching question is: *what kind of algebraic structure do we get from these field theories?* To address this question, we generally must first fix a target category. But there are a few "holy grail" theorems in some of these settings.

Theorem 1.5 (Cobordism hypothesis (Lurie))**.** *A fully extended topological field theory is determined by its value on the* 0*-manifold* pt_+*.*

This is more of a slogan than a theorem, but one can pin it down into a precise theorem, e.g. by making precise what kinds of TQFTs one considers. Here, by "pt_+" we might more generally mean looking at generators and relations of the appropriate bordism category.

Example 1.6. In (spacetime) dimension $n = 1$, TQFTs are vacuously fully extended (with the caveat that 1-categorical and $(\infty, 1)$-categorical TQFT aren't quite the same). Then, the theorem is that for any symmetric monoidal category $\mathcal{C}$, $\mathcal{F}un^{\otimes}(\mathcal{C}ob_{1,0}, \mathcal{C})$ is equivalent to the groupoid of *dualizable* objects in $\mathcal{C}$.[1]

Exercise 1.7. For $\mathcal{C} = \mathcal{V}ect_k$, check that dualizability is equivalent to being finite-dimensional.[2]

So fixing $\mathcal{C} = \mathcal{V}ect_k$ for now, given a one-dimensional *unoriented* (i.e. manifolds and bordisms in $\mathcal{C}ob_{1,0}$ are not oriented) TQFT Z we get a finite-dimensional vector space $V := Z(\mathrm{pt}_+)$, and a biliear pairing $e\colon V \otimes V \to k$. This pairing must be nondegenerate, as one can show via the "Z-diagram" being equivalent to an interval (which is the identity $\mathrm{pt} \to \mathrm{pt}$).

Conversely, given a finite-dimensional vector space V and an inner product $e\colon V \otimes V \to k$, we can build a TQFT $Z_{V,e}\colon \mathcal{C}ob \to \mathcal{V}ect_k$, because there aren't that many diffeomorphism classes of 1-manifolds, so we know generators and relations: the interval, regarded as a bordism from $\mathrm{pt} \to \mathrm{pt}$, is sent to id_V; the interval, regarded as a bordism from $\mathrm{pt} \amalg \mathrm{pt} \to \varnothing$, is sent to e; and the interval, regarded as a bordism $\emptyset \to \mathrm{pt} \amalg \mathrm{pt}$, is sent to the adjoint of e. ◂

Remark 1.8. Some things change in the oriented 1-dimensional case. We don't need the inner product: if you keep careful track of the orientations induced on a boundary, the interval is now a bordism between $\mathrm{pt}_+ \amalg \mathrm{pt}_-$ and $\varnothing$, and one can show that $\mathrm{pt}_- \mapsto V^*$. Then these intervals are sent to the evaluation map $V \otimes V^* \to k$ and the coevaluation map $k \to V \otimes V^*$. ◂

In dimension 1, the cobordism hypothesis feels somewhat silly. But in higher dimensions things can quickly get nontrivial, and difficult. For example, for the oriented 2-dimensional cobordism category (before we extend), this is known by the classification of surfaces: the pair of pants, regarded as a bordism $S^1 \amalg S^1 \to S^1$, and, separately, regarded as a morphism $S^1 \to S^1 \amalg S^1$; the disc, both as a bordism $S^1 \to \varnothing$ and as a bordism $\varnothing \to S^1$; and the cylinder $S^1 \to S^1$. In dimension 1 we just have the circle. But if we try to extend down to points, then discovering generators is more complicated — now we have to determine generators and relations using surfaces with corners. The surface theory isn't that bad, and this will get worse when we care about higher-dimensional manifolds.

And we do care about higher-dimensional manifolds: two key questions in this course will be:

(1) how does this (both the axiomatic structure of TQFT and tools such as the cobordism hypothesis) help build invariants for 3- and 4-manifolds, and
(2) how do geometric/PDE-based invariants of 3- and 4-manifolds yield TQFTs?

With regards to question (2) specifically, Atiyah gave a few examples in his original paper on TQFT.

Example 1.9. This example, built on work of Floer and Gromov, is a 2-dimensional TQFT. Fix a symplectic manifold (X, ω); the quantum field theory here will be about maps $S^1 \to X$. We begin with a "classical phase

[1] *A priori*, the subcategory of dualizable objects in $\mathcal{C}$ is not a groupoid, but we can make it one by throwing out the non-invertible morphisms.

[2] In higher dimensions, "dualizable" generalizes to "fully dualizable," and the fact that "fully dualizable" and "finite-dimensional" have the same initials makes for a good mnemonic.

space" $\mathrm{Map}(S^1, X)$; to a closed, oriented 2-manifold Σ, we should associate the number of pseudoholomorphic maps $u\colon \Sigma \to X$. There's a lot to define here; what is a pseudoholomorphic map? Defining the number of such maps is also nontrivial; in some settings, there are infinitely many, and we must impose point constraints somehow, which makes the theory feel less topological.

The definition of a pseudoholomorphic map involves a PDE, which will be an interesting thing to dig into.

The theory also has a Lagrangian form. In the Lagrangian form, we instead look at paths in X, rather than loops, though we ask that they end on prescribed Lagrangian submanifolds of X. These are a kind of boundary condition. ◂

Atiyah doesn't go into much more detail about this theory, but Schwarz did (assuming $\omega|_{\pi_2(X)}$ vanishes), and we will discuss this example in detail. Ultimately, $Z(S^1)$ will be $H_*(X)$, and the pair-of-pants is sent to a quantum deformation of the cup product which counts pseudoholomorphic curves — Schwarz proves this with Floer theory, but it also makes contact with Gromov-Witten theory.

Example 1.10 (Chern-Simons theory)**.** There are several different flavors of this next example, a 3-dimensional theory. Pick a Lie group G, maybe compact; the classical phase space associated to a closed surface Σ is the moduli space of flat G-bundles on Σ. This isn't infinite-dimensional, because we imposed that our connections are flat, though the space of all connections is infinite-dimensional. If G is nonabelian, this is nonlinear (i.e. not a vector space).

The Lagrangian functional for this theory is the Chern-Simons functional associated to a connection. There's been plenty of work on this example, from different perspectives not just including TQFT, including work by Jones, Witten, Casson, Johnson, and Thurston. ◂

Example 1.11 (Floer theory/Donaldson theory)**.** This is a 4-dimensional example, in which the invariant assigned to a closed 4-manifold X is the Donaldson polynomials on $H_2(M)$ (a tool encoding all of the Donaldson invariants). Atiyah doesn't say what we should do with cobordisms, but fora closed 3-manifold Y, following the Hamiltonian perspective in physics, one should do Floer theory for the Chern-Simons functional on Y (for some Lie group that you have to pick — though only $G = \mathrm{SU}_2$ and $G = \mathrm{U}_2$ have really been worked out, which is Donaldson theory).

Unfortunately, this cannot be an oriented theory — Donaldson polynomials depend on more data. ◂

Awesomely, Atiyah ends with the question *why does the Chern-Simons functional appear in both the three- and four-dimensional cases?* There ought to be an answer in terms of extended TQFT: Chern-Simons theory really seems to be about dimensions 4, 3, and 2.

Example 1.12. After Atiyah's paper came out, Seiberg-Witten theory appeared, as a variant of Example 1.11, and it should fit into a TQFT framework in the same way. This is again a 4-dimensional theory. ◂

We will begin by digging into Example 1.9. Pseudoholomorphic curves are a huge subject; good references include Salamon's lecture notes and the book of Audin-Damian, which is very detailed but doesn't illustrate the analysts' perspective as well as Salamon. The big book of McDuff-Salamon is also good. The professor also has a survey paper, "Lagrangian boundary conditions for anti-self-dual instantons and the Atiyah-Floer conjecture," which is a good way to get an overview of this perspective.

Before we get into pseudoholomorphic curves, here's an important convention: *when we say "symplectic manifold," we always mean closed* (compact and without boundary).

Definition 1.13. A *symplectic manifold* (X, ω) is a manifold X and a 2-form $\omega \in \Omega^2(X)$ which is closed and *nondegenerate*, i.e. $\omega^{\wedge n}$ is a volume form.

This immediately implies $\dim X = 2n$, and is in particular even; and $[\omega] \neq 0$ in $H^2_{\mathrm{dR}}(X)$, which rules out, e.g., S^4.

You can get through a good part of the course thinking of these as even-dimensional manifolds with a particular functional on them. Let $\mathcal{L}X := \mathrm{Map}(S^1, X)$, the unbased loop space of X.

Definition 1.14. The *symplectic action functional* associated to a symplectic manifold (X, ω) is the functional $\mathcal{A}\colon \mathcal{L}X \to \mathbb{R}$ sending a loop $\gamma\colon S^1 \to X$ to the number

$$\int_{[0,1]\times S^1} u^*\omega. \tag{1.15}$$

Here $u\colon [0,1] \times S^1 \to X$ a smooth map with $u(0, –)$ a fixed reference loop u_0 and $u(1, –) = \gamma$.

Often, u_0 is constant, in which case this is choosing a disc whose boundary is γ. There are issues defining this, so the actual target is $\mathbb{R}$ modulo the possible values of ω on tori. If you want to study all of $\mathcal{L}X$, you need to fix a basepoint in each connected component (homotopy class), though often people only study the connected component containing the constant loops, as Floer did.

Given a nice functional, one should want to try gradient flow and Morse theory with it, even though $\mathcal{L}X$ is infinite-dimensional; we will see the definition of a pseudoholomorphic curve pop out naturally from this definition. We will also do Morse theory with the Chern-Simons functional. Doing Morse theory with a function valued in a circle is a bit different, but we'll be able to do it. And in fact, it's the reason we work with the Novikov ring.

2. : 2/21/20

3. 2d TFTs from symplectic manifolds: 2/26/20

We will spend the first part of class carefully setting up a precise statement to the following theorem.

Theorem 3.1 (Schwarz, Floer). *Let (M, ω) be a symplectic manifold such that either $\omega|_{\pi_2(M)} = 0$ or $\omega = \lambda c_1(M)$, with $\lambda > 0$. Then thee is a TQFT $Z\colon \mathcal{C}ob_{\langle 2,1\rangle} \to \mathcal{V}ect_{\mathbb{F}_2}$ with $Z(S^1) \cong H_*(M)$.*

Here (TODO: I think) $c_1(M)$ is measured in any compatible almost complex structure for the symplectic form; the choice doesn't matter.

Remark 3.2.

- The algebra structure on $Z(S^1)$ is not just the usual intersection product; it's deformed by counting pseudoholomorphic curves.
- We can relax the niceness assumptions on the symplectic form, but then our target category is modules over some universal thing called the *Novikov ring*. We're not going to delve into this. ◀

Somehow the existence of this TQFT is not the entire point; instead, it leads us to interesting analytic and geometric questions.

Choose a map $H\colon S^1 \times M \to \mathbb{R}$ such that the time-1 flow

$$\{(p_0, p_1) \mid \text{ there exists a } \gamma\colon [0,1] \to M, \dot\gamma(t) = X_{H_t}(\gamma(t)), \gamma(0) = 0, \gamma(1) = 1\} \subset M \times M \tag{3.3}$$

is transverse to the diagonal $\Delta_M \coloneqq \{(p,p) \mid p \in M\}$. Such a map induces a $\mathbb{Z}/2N$-graded complex $CF_*(H)$ generated by periodic loops of X_H, where N is the minimal positive value of $\langle c_1(TM, J), [S^2]\rangle$ over all embeddings $S^2 \hookrightarrow M$ — pseudoholomorphic or not.[3]

Let $\mathcal{J}(M, \omega)$ denote the space of compatible almost complex structures on M for ω. Given the map H above, there is a comeager subset $S \subset \mathrm{Map}(S^1, \mathcal{J}(M, \omega))$ for which each $J \in S$ induces a differential $\partial\colon CF_*(H) \to CF_{*-1}(H)$; in particular, $\partial^2 = 0$. Fixing these two choices, we can define $Z(S^1)$ to be the homology of this chain complex.

This differential arises by an analogue to Morse theory on an infinite-dimensional Banach manifold, though instead of counting curves, we count solutions to a PDE (only counting them in the case where there's a zero-dimensional moduli space). Specifically, we consider the space of solutions

$$\{u\colon \mathbb{R} \times S^1 \to M \mid u(\pm\infty, -) = \gamma_\pm, \overline{\partial}_J u = X_H\}, \tag{3.4}$$

which $\mathbb{R}$ acts on freely by time translations (just as in Morse theory); the moduli space $\mathcal{M}(\gamma_-, \gamma_+)$ is the quotient of (3.4) by this $\mathbb{R}$-action. Here $\gamma_\pm$ are choices of points in M, so that we are counting pseudoholomorphic strips in M which at infinity converge on $\gamma_\pm$.

Remark 3.5. Why is the Hamiltonian so complicated? You can build TQFTs with simpler Hamiltonians (akin to doing Morse homology with simpler Morse functions). But this example doesn't come from nowhere: Floer used these techniques to solve a piece of the Arnold conjecture. Other confusing-sounding choices sometimes also come from geometric applications. ◀

[3]The magic of symplectic geometry is how much we can do without actually knowing what the pseudoholomorphic curves actually are — or if there are any at all!

The local dimension of $\mathcal{M}(\gamma_-, \gamma_+)$ is $\deg(\gamma_-) - \deg(\gamma_+) - 1$, where the degrees are in the grading of $CF_*(H)$ we discussed above; the factor of -1 arises because we quotiented by the $\mathbb{R}$-action. We will stick to the cases when this dimension is 0.

The proof that the differential squares to zero follows the same line of reasoning as in ordinary Morse theory, though it looks fancier in this setting. In Morse theory, a flow line can break into two. Here, we consider the moduli space of pseudoholomorphic strips from γ_- to γ_+, where $\deg(\gamma_-) - \deg(\gamma_+) = 2$. A great deal of analysis goes into showing this is a smooth 1-manifold with ends, and some of the assumptions we made in the theorem statement eliminate some unsavory possibilities (e.g. bubbling). Anyways, we compactify, to obtain a 1-manifold with boundary, and show that it factors as

$$\coprod_{\gamma : |\gamma| = |\gamma_-| - 1} \mathcal{M}(\gamma_-, \gamma) \times \mathcal{M}(\gamma, \gamma_+). \tag{3.6}$$

The high-level idea for what's happening is that the pseudoholomorphic strip breaks into two. There are many other situations in gauge theory or infinite-dimensional Morse theory where breaking (and sometimes bubbling) can happen.

Compactness is another important ingredient, and it's also fundamentally analytic. Given a sequence $\{u_n\} \subset \mathcal{M}(\gamma_-, \gamma_+)$, there is a subsequence that converges in $C^\infty_{\ell oc}$, unless the energy of the sequence behaves badly. The energy is a functional

$$E(u) = \int_{\mathbb{R}_s \times S^1} |\partial_s u|^2 = \mathcal{A}(\gamma_-) - \mathcal{A}(\gamma_+). \tag{3.7}$$

The Morse-theoretic version of this is

$$\int_{\mathbb{R}} |\nabla(f(\gamma))|^2 = f(p_-) - f(p_+), \tag{3.8}$$

provided $\gamma\colon \mathbb{R} \to X$ is a gradient flow line for the Morse function; in Floer theory, $\partial_s u = -\nabla \mathcal{A}_H u$.

The name "energy" is because $|\partial_s u|^2$ is always locally positive, so we can think of it as an energy density. So before we even ask about convergence of a subsequence, we can ask how the energy behaves. It can concentrate near a point in the cylinder, or it can separate, piling up near the ends of $\mathbb{R}$. Separating is good, in that it leads to breaking, which we thought about in the differential. But concentration is often trickier to deal with: it leads to bubbling, which is a PDE effect, and which we'll largely sweep under the rug.

The next big theorem we need is a gluing theorem. This will rule out pathological broken trajectories in which the ends aren't actually different (TODO: I think?). There is a gluing map

$$g\colon \coprod_{\gamma} \mathcal{M}(\gamma_{-1}, \gamma) \times \mathcal{M}(\gamma, \gamma_+) \times (R_0, \infty) \longrightarrow \mathcal{M}(\gamma_-, \gamma_+). \tag{3.9}$$

where the disjoint union is over the γ for which these moduli spaces are zero-dimensional, and R_0 is some number. The idea is that for every broken trajectory, we embed an interval into the manifold, which chooses which end you converge to.

Theorem 3.10 (Gluing theorem (Schwarz)). *The gluing map g is an homeomorphism onto its image, and $\mathcal{M}(\gamma_-, \gamma_+) \setminus \mathrm{Im}(g)$ is compact.*

So the moduli space without these bad examples is nice. This is a key result from Schwarz's thesis.

Often, papers will only think about one of compactness or gluing, and sketch the proof of the other; this is where the mistakes creep in, so be careful.

Anyways, now we have the Floer complex. Choose $a, b, g \in \mathbb{Z}_{\geq 0}$, and consider any Riemann surface $\Sigma \coloneqq \Sigma_{g,a+b}$ (i.e. genus g, $a+b$ boundary components), regarded as a bordism with a incoming circles and b outgoing circles. This Σ induces a chain map

$$\Phi(\widetilde{X}, \widetilde{J})\colon \bigotimes_{i=1}^{a} CF_*(H_i) \longrightarrow \bigotimes_{j=1}^{b} CF_*(H_j), \tag{3.11}$$

where H_i, H_j, $\widetilde{X}$, and $\widetilde{J}$ are data we haven't discussed yet, but will induces the same map when we take homology. This chain map has degree $(\dim M)/2(2g - 2 + a + b)$, and will be what the TQFT assigns to the bordism Σ.

How do we define this chain map? Again, it counts something, which is the number of points in a zero-dimensional moduli space of pseudoholomorphic maps $\Sigma \to M$, such that the boundary circles of Σ map to specified loops γ_i^-, γ_j^+ in M. There are again choices to make, including disjoint embeddings $\varphi_i\colon (-\infty, 0) \hookrightarrow \Sigma$ and $\varphi_j\colon (0, \infty) \hookrightarrow \Sigma$, which give us cylindrical coordinates near each boundary circle; and a choice of a complex structure on the interior of Σ which is the standard (cylinder) complex structure on the ends $\mathbb{R} \times S^1 \cong \mathbb{C}/\mathbb{Z}$. We also choose (H_i, J_i) on each end, and $\widetilde{X}, \widetilde{J}$ are generic (comeager subset) of a certain 1-form $\widetilde{X}$ with $\varphi_i^* \widetilde{X} = X_{H_i}\, \mathrm{d}t$, and $\widetilde{J}$ a map from the interior of Σ into $\mathcal{J}(M, \omega)$, such that $\varphi_i^* \widetilde{J} = J_i$. All of these are contractible choices, which is reassuring. Some thought has to go into ensuring this is a well-posed PDE.

Again, we want to make a count of a zero-dimensional moduli space, so the same questions come up: is this space zero-dimensional? Is it compact? And so on. Then, to verify that it's a chain map, we need to know that it commutes with the differentials. This again arises by studying the ends of the one-dimensional moduli spaces – this should correspond to breaking of pseudoholomorphic strips, caused by energy running out at the ends. Breaking can happen at each boundary circle of Σ, incoming or outgoing, and these give you the different components of the boundary of the moduli space.

This TFT involves a whole bunch of choices — we will continue with three more theorems that imply the TFT, on homology, is independent of choices. This is a common method of proof in this area, and it's difficult to avoid. Find your way of understanding how these diagrams and pictures work.