TENSOR, SYMMETRIC, AND EXTERIOR ALGEBRAS

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"Where is everyone? The time change is on Sunday, right?"

Last time, it was stated without proof that $S^{-1}M \cong S^{-1}A \otimes_A M$. This can be proven in a more general context: suppose $A \xrightarrow{f} B$, so that B can be viewed as an A-algebra; in particular, B is an A-module. Then, if M is another A-module, then $B \otimes_A M$ is *a priori* an A-module, but also has a B-module structure, with action of a $b \in B$ given by $\operatorname{act}(b) : (b_1, m) \mapsto bb_1 \otimes m$. By the definition of tensor product, this is A-linear, but the action must give the structure of a B-module (basically, ensuring that it's compatible with multiplication within B), so it's necessary to check this.

The functor $B \otimes_A - : A\text{-Mod} \to B\text{-Mod}$ is called extension of scalars (from A to B; since A and B act on the modules, then they are called scalars, and often, B is larger than A). There's also the forgetful functor $B\text{-Mod} \to A\text{-Mod}$, which is called restriction of scalars.

Proposition 0.1. *$B \otimes_A -$ is left adjoint to restriction of scalars.*

Proof sketch. The goal is to show that $\operatorname{Hom}_B(B \otimes_A M, N) \cong \operatorname{Hom}_A(M, N)$. The correspondence is described as:

$$\varphi : B \otimes_A M \to N \longmapsto \psi : M \to N \text{ given by } m \mapsto \varphi(1 \otimes m).$$

On the right, $\psi : M \to N \mapsto \widetilde{\psi} : B \times M \to N$ is A-bilinear, so it extends to an A-linear $B \otimes_A M \to N$. It remains to check these are inverses of each other, etc. ⊠

Proposition 0.2. *$S^{-1}M \cong S^{-1}A \otimes_A M$ as A-modules.*

Proof. For any $S^{-1}A$-module N, $\operatorname{Hom}_{S^{-1}A}(S^{-1}M, N) = \operatorname{Hom}_A(M, N)$. Recall that $S^{-1}(-)$ is lft adjoint to the forgetful functor $S^{-1}A\text{-Mod} \to A\text{-Mod}$, but $S^{-1}A \otimes_A -$ is also left adjoint to the same forgetful functor. Thus, $\operatorname{Hom}_{S^{-1}A}(S^{-1}A \otimes_A M, N) = \operatorname{Hom}_A(M, N)$. Thus, $S^{-1}M$ and $S^{-1}A \otimes_A M$ represent the same functor $S^{-1}A\,\text{Mod} \to \text{Set}$ given by $N \mapsto \operatorname{Hom}_A(M, N)$. Since they give the same Hom space, they must be canonically isomorphic as A-modules. ⊠

This is a useful test: to show that two objects are isomorphic, one can show that they represent the same functor in some category.

Tensor Product of Algebras. Suppose A is a commutative ring, and B and C are commutative A-algebras.

Proposition 0.3. *$B \otimes_A C$, which is a priori an A-module, also has a natural ring structure, giving a ring homomorphism $A \to B \otimes_A C$ that makes it an A-algebra.*

In effect, this just says that the tensor product of two rings is still a ring. The proof of this proposition will be deferred while multi-tensors are mentioned. This is a straightforward generalization of the tensor product outlined in the exercises; let $M_1, \dots, M_r, N$ be A-modules and $L_A(M_1 \times \cdots \times M_r; N)$ be the set of A-multilinear functions $M_1 \times \cdots \times M_r \to N$.

Claim. $L_A(M_1 \times \cdots \times M_r, -)$ is representable, and the object that represents it is denoted $M_1 \otimes_A \cdots \otimes_A M_r$.

In other words, for any A-module N, $\operatorname{Hom}_A(M_1 \otimes_A \cdots \otimes_A M_r) = L_A(M_1 \times \cdots \times M_r, N)$.

From this definition, we can deduce the associativity of the tensor product: $M_1 \otimes_A (M_2 \otimes_A M_3) \cong (M_1 \otimes_A M_2) \otimes_A M_3 = M_1 \otimes_A M_2 \otimes_A M_3$. This is true because all 3 represent the same functor, which sends an A-module L to the set of trilinear maps $M_1 \times M_2 \times M_3 \to L$. This generalizes in the reasonable way to greater numbers of factors, and means that parentheses in the construction of the multi-tensor product are unimportant. Since each of the three constructions is slightly different, it is not completely tautological to show that the functors are the same; but it is not particularly difficult.

Proof of Proposition 0.3. Define a map $B \times C \times B \times C \to B \otimes_A C$ by $(b_1, c_1, b_2, c_2) \mapsto b_1 b_2 \otimes c_1 c_2$. It's clear that this is A-linear in each argument, so this is an A-quadrilinear function. Thus, this induces an A-linear map $B \otimes_A C \otimes_A B \otimes_A C \xrightarrow{\mu} B \otimes_A C$. By the associativity of the tensor product, this is equal to the map

$(B \otimes_A C) \otimes_A (B \otimes_A C) \xrightarrow{\mu} B \otimes_A C$. Thus, again using the universal property, there is a unique bilinear map ψ such that the following diagram commutes:

$$\begin{tikzcd}
(B \otimes_A C) \times (B \otimes_A C) \arrow[r, "\psi"] \arrow[d, "(b,c)\mapsto b\otimes c"] & B \otimes_A C \arrow[d, "\wr"] \\
(B \otimes_A C) \otimes_A (B \otimes_A C) \arrow[r] & B \otimes_A C
\end{tikzcd}$$

This induces the required ring structure, though it's necessary to check that this actually is in fact associative and commutative, and that $1 \otimes 1$ is the unit. Finally, one will have to show that the ring homomorphism making $B \otimes_A C$ into an A-algebra is $a \mapsto a \otimes 1 = 1 \otimes a$. (Since this is a tensor product over A, scalars can be pushed around like this.) ⊠

Though it's possible to write down a ring homomorphism and check it, this fuller argument cleanly ensures that it works on sums of simple tensors.

Example 0.1. If A is an abelian group, then $A \otimes_{\mathbb{Z}} \mathbb{Z}[x_1, \dots, x_n] \cong A[x_1, \dots, x_n]$ (in the exercises, this was checked as groups, but it also holds true for rings). It's also true that $A[x] \otimes_A A[y] \cong A[x, y]$.

Be careful, though: $A \otimes_{\mathbb{Z}} \mathbb{Z}[[x]] \to A[[x]]$ is in general not surjective. In essence, this is because finite linear combinations don't map to infinite power series very well. This is because $\sum a_i \otimes f_i$ maps to something whose coefficients are all linear combinations of the $a_1, \dots, a_n$. Thus, to show that surjectivity fails, pick a power series whose set of coefficients isn't finitely generated, e.g. if $A = \mathbb{Q}$, $\sum_{n\geq 1} x^n/n \notin \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}[[x]]$.

Tensor Algebras.

Definition. Given an A-module M, the tensor algebra of M is

$$T(M) = A \oplus M \oplus (M \otimes_A M) \oplus (M \otimes_A M \otimes_A M) \oplus \cdots = A \oplus \bigoplus_{n=1}^{\infty} M^{\otimes n}.$$

Here, the notation $M^{\otimes n}$ refers to $M \otimes_A \cdots \otimes_A M$, where there are n terms in the tensor product.

Claim. $T(M)$ is naturally an associative A-algebra.

Proof sketch. Define multiplication as follows: for an $x_1 \otimes \cdots \otimes x_m \in M^{\otimes m}$ and a $y_1 \otimes \cdots \otimes y_n \in M^{\otimes n}$, their product is

$$(x_1 \otimes \cdots \otimes x_m)(y_1 \otimes \cdots \otimes y_n) = x_1 \otimes \cdots \otimes x_m \otimes y_1 \otimes \cdots \otimes y_n \in M^{\otimes(m+n)},$$

giving an A-bilinear map $M^{\otimes m} \otimes M^{\otimes n} \to M^{\otimes(m+n)}$, though it's necessary to check that this multiplication behaves well on sums of tensors, and in particular that it is well-defined.

Since every element in $T(M)$ is a finite linear combination of these, one obtains a map $T(M) \times T(M) \to T(M)$ with the required properties. ⊠

This algebra is associative (as are all algebras in this class), but it is *not* commutative: in particular, looking just at $M \times M \to M^{\otimes 2}$, $(x, y) \mapsto x \otimes y \neq y \otimes x$ in general.

Proposition 0.4. *Let R be an associative A-algebra. Then,* $\mathrm{Hom}_{A\text{-Alg}}(T(M), R) \cong \mathrm{Hom}_A(M, R)$*, and the isomorphism is canonical. That is, the forgetful functor* A-Alg $\to A$-Mod *has T as its left adjoint.*

Proof sketch. Again, build maps in both directions.

Suppose $\varphi : T(M) \xrightarrow{A\text{-Alg}} R$; then, take $\psi = \varphi|_M : M \to R$, which is clearly A-linear.

Conversely, if $\psi : M \to R$ is A-linear, first define $\varphi_n : M \times \cdots \times M \to R$ by $(x_1, \dots, x_n) \mapsto \psi(x_1)\psi(x_2)\cdots\psi(x_n)$, which is multilinear, so it extends to $\widetilde{\psi}_n : M^{\otimes n} \to R$. Then, let $\varphi = \bigoplus \widetilde{\psi}_n$ (i.e. $\varphi|_{M^{\otimes n}} = \tilde{\psi}$). This is *a priori* A-linear, but it remains to be checked that it's also a ring homomorphism, and then that these two maps are inverses of each other. ⊠

Symmetric Algebras. There's a similar universal property that gives a commutative A-algebra called the symmetric algebra. Let $\mathcal{C}$ denote the category of commutative A-algebras.

Proposition 0.5. *The forgetful functor* $\mathcal{C} \to A$-Mod *admits a left adjoint, denoted S, i.e. for every A-module M, there exists a commutative A-algebra $S(A)$ such that* $\mathrm{Hom}_{\mathcal{C}}(S(M), B) = \mathrm{Hom}_A(M, B)$ *(where the forgetfulness of B is assumed in the right side).*

Proof sketch. The construction is given by quotienting $T(M)$ by the universal relations that force the quotient to be commutative. This is a general procedure: if R is a ring, then $[R, R] = \mathrm{Span}\{xy - yx \mid x, y \in R\}$, so that $[R, R]$ is a two-sided ideal (which remains to be checked). If you're lucky, then $1 \notin [R, R]$, so one can quotient to obtain $R^{\mathrm{ab}} = R/[R, R]$, which is a commutative ring (if $1 \in [R, R]$, then the quotient is zero, which is sad). This is the smallest set of relations that force commutativity, so apply this procedure to $T(M)$: take the ideal generated by $x \otimes y - y \otimes x$ for all $x, y \in M$, and quotient by it.[1] ⊠

From the construction, it's easy to show that $\mathrm{Hom}_{\mathcal{C}}(S(M), B) = \mathrm{Hom}_A(M, B)$, because Proposition 0.4 applies to B as a (not necessarily commutative) A-algebra: $\mathrm{Hom}_{A\text{-Alg}}(T(M), B) = \mathrm{Hom}_A(M, B)$. So it only remains to check that $\mathrm{Hom}_{A\text{-Alg}}(T(M), B) = \mathrm{Hom}_{\mathcal{C}}(S(M), B)$, i.e., that such a map factors through the quotient, which is not that bad.

The most important examples of a symmetric algebra is the polynomial ring.

Claim. Suppose M is a free A-module of rank n. Then, $S(M) = A[x_1, \ldots, x_n]$.

Proof sketch. Pick a basis of M, as $M = Ae_1 \oplus \cdots \oplus Ae_n$. Given some A-linear map $M \to A[x_1, \ldots, x_n]$, that sends $e_i \mapsto x_i$ (which extends uniquely to an A-linear map because M is free), this is by Proposition 0.5 equivalent to $\varphi : S(M) \to A[x_1, \ldots, x_n]$.

In the other direction, consider the map $A[x_1, \ldots, x_n] \to T(M)$ given by

$$\prod_{i=1}^{n} x_i^{\ell_i} \longmapsto \bigotimes_{i=1}^{n} \bigotimes_{j=1}^{\ell_i} x_i \in M^{\otimes(\ell_1 + \cdots + \ell_n)}.$$

Taking the quotient, this yields a map $\psi : A[x_1, \ldots, x_n] \to S(M)$, and it can be shown this is an inverse to φ. ⊠

This can be generalized; if M is a free A-module with basis S, then $S(M)$ is the set of polynomials over A in $|S|$ variables, even if M isn't finite-dimensional. Similarly, $T(A^n) = A\langle x_1, \ldots, x_n \rangle$, and a similar result holds for the infinite-dimensional case.

Exterior Algebras. Given an A-module M, one wants a universal, graded, anti-commutative A-algebra $\Lambda(M)$. This will end up being nearly commutative, and is also given by a quotient of the tensor algebra:

$$\Lambda(M) = T(M)/(x \otimes y + y \otimes x \mid x, y \in M)$$

(i.e. taking linear combinations of these generators), which is *not* commutative. The image of an $x \otimes y \in T(M)$ in $\Lambda(M)$ is denoted $x \wedge y$, and the ideal forces the relation $x \wedge y + y \wedge x = 0$ (there was in general no relation in $T(M)$), or, equivalently, $x \wedge y = -y \wedge x$. This is what is meant by anti-symmetric or anti-commutative.

More generally, to switch the order of pure wedges, the sign doesn't always change: given an $x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_\ell \in \Lambda(M)$, $x_1 \wedge \cdots \wedge x_k$ is the image of $x_1 \otimes \cdots \otimes x_k \in T(M)$ (and similarly for the other term). Multiplication in $\Lambda(M)$ is also denoted $\wedge$, but this isn't actually ambiguous, because $(x_1 \wedge \cdots \wedge x_k) \wedge (y_1 \wedge \cdots \wedge y_\ell) = x_1 \wedge \cdots \wedge x_k \wedge y_1 \wedge \cdots \wedge y_\ell$ (since this is an associative algebra), then this is OK. However, to switch the order the sign might have to change:

$$(x_1 \wedge \cdots \wedge x_k) \wedge (y_1 \wedge \cdots \wedge y_\ell) = (-1)^{k\ell}(y_1 \wedge \cdots \wedge y_\ell) \wedge (x_1 \wedge \cdots \wedge x_k),$$

because $k\ell$ pairs need to be exchanged and each flips the sign. Thus, even numbers of wedges commute with everything.

The construction of $\Lambda(M)$ is universal in the category of graded, anti-commutative A-algebras, and satisfies the same adjoint property, which is further developed in the exercises.

This has applications to the real world: every topological space X has an associated ring $H^*(X)$ called the cohomology ring, which is a graded anti-commutative algebra. Thus, this category is useful. In fact, sometimes the cohomology ring is an exterior power, e.g. the n-dimensional torus T^n has $H^*(T^n) = \Lambda(\mathbb{Z}^{\oplus n})$.

[1] These lie in $M^{\otimes 2}$, but by left- or right-multiplying by other things, one obtains all of the other necessary elements.