FALL 2016 ALGEBRAIC GEOMETRY SEMINAR

ARUN DEBRAY
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Contents

1. What is a scheme?: 8/31/16

Today, Tom provided an introduction and overview, with the goal of understanding what a *scheme*, the central object of study in algebraic geometry, is. We'll start with sheaves, a way of understanding locality of things in geometry, then discuss locally ringed spaces, the spectrum of a ring, and finally schemes, their properties, and a little bit about morphisms of schemes. Finally, we'll learn a little about varieties.

1.1. **Sheaves.**

Definition 1.1. A *presheaf of rings*[1] $\mathscr{F}$ on a topological space X consists of the data

(1) for every open $U \subset X$, a ring $\mathscr{F}(U)$, and
(2) for every inclusion of open sets $V \subset U$, a ring homomorphism $\rho_V^U : \mathscr{F}(U) \longrightarrow \mathscr{F}(V)$ called the *restriction map*,

such that for every nested inclusion of opens $W \subset V \subset U$, the restriction maps compose: $\rho_W^U = \rho_W^V \circ \rho_V^U$.

Elements of $\mathscr{F}(U)$ are called *sections*.

The idea is that $\mathscr{F}(U)$ is some collection of data on U, such as the continuous real-valued functions on U, which define a ring. Given such a function, we can restrict it to a $V \subset U$, and this is exactly what the restriction map does. If I want to further restrict to another subset, it doesn't matter whether I restrict to V first.

Presheaves have some problems, and we define sheaves to fix these problems.

Definition 1.2. A *sheaf* $\mathscr{F}$ on a space X is a presheaf such that

(1) sections can be computed locally: if $U \subset X$ is open, $\mathfrak{U}$ is an open cover of U, and $s \in \mathscr{F}(U)$, then if $\rho_{U_i}^U(s) = 0$ for all $U_i \in \mathfrak{U}$, then $s = 0$.
(2) compatible sections can be glued: with U and $\mathfrak{U}$ as above, suppose we have data of $s_i \in \mathscr{F}(U_i)$ for each $U_i \in \mathfrak{U}$ such that for all $U_i, U_j \in \mathfrak{U}$, $\rho_{U_i \cap U_j}^{U_i}(s_i) = \rho_{U_i \cap U_j}^{U_j}(s_j)$, then there is a section $s \in \mathscr{F}(U)$ such that $\rho_{U_i}^U(s) = s_i$ for all $U_i \in \mathfrak{U}$.

To understand this intuitively, think about continuous real-valued functions, which can be uniquely determined from local data, and can be glued together from compatible functions on an open cover.

Example 1.3. Let X be a space. We've already been referring to the sheaf C_X of continuous $\mathbb{R}$-valued functions: $C_X(U) = \{f : U \longrightarrow \mathbb{R} \text{ continuous}\}$. Restriction of functions defines a restriction map, functions are determined by local data, and compatible functions may be glued together.

[1] One can talk about presheaves of sets, groups, or of any other category, by replacing "rings" in this definition by "sets," "groups," or whatever you're using.

This is a good example of sheaves for your intuition: sheaves in general behave a lot like a sheaf of functions, and it's convenient to think of the restriction map as actual restriction of functions.

Example 1.4. Let X be a manifold; then, we can define the sheaf C_X^∞ of smooth functions: $C_X^\infty(U)$ is the ring of smooth functions $U \to \mathbb{R}$. This is very similar, but it's interesting that this sheaf uniquely determines the smooth structure on the manifold X.

That is, smooth structure is determined by what you call smooth functions. This is a rule that applies more generally in geometry: a geometric structure is determined by the sheaf of functions to some base that we allow.

Remark. The empty set is an open subset of a space X. You can prove or define (depending on your taste for empty arguments) that for any sheaf $\mathscr{F}$ on X, $\mathscr{F}(\emptyset) = 0$.

Definition 1.5. Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves on a space X. Then, a morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$ is the data of for all open $U \subset X$, a ring homomorphism $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ that commutes with restriction in the following sense: for all inclusions of open sets $V \subset U$, the following diagram commutes:

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\rho_V^U"] & \mathscr{G}(U) \arrow[d, "\rho_V^U"] \\ \mathscr{F}(V) \arrow[r, "\varphi(V)"] & \mathscr{G}(V). \end{tikzcd}$$

That is, we want to map in a way that doesn't affect how we restrict. A sheaf is data parametrized by a topological space, and we want a morphism of sheaves to respect this parametrization.

Definition 1.6. Let $\mathscr{F}$ be a sheaf on X and $p \in X$. Then, the *stalk* of $\mathscr{F}$ at p is

$$\mathscr{F}_p = \{(s, U) \mid U \subset X \text{ is open}, s \in \mathscr{F}(U)\}/\sim,$$

where $(s, U) \sim (t, V)$ if there's an open $W \subset U \cap V$ containing p such that $\rho_W^U(s) = \rho_W^V(t)$.

That is, we define two functions to be equivalent if they agree on any neighborhood of the point. These are sort of infinitesimal data of functions near the point p.

1.2. **Locally ringed spaces.**

Definition 1.7. A *local ring* is a ring A with a unique maximal ideal $\mathfrak{m} \subset A$, often denoted $(A, \mathfrak{m})$.

This is the same as saying $A^\times = A \setminus \mathfrak{m}$: everything outside the maximal ideal is invertible.

Definition 1.8. A *locally ringed space* is a pair $(X, \mathscr{O}_X)$, where X is a topological space and $\mathscr{O}_X$ is a sheaf of rings, such that all stalks $\mathscr{O}_{X,p}$ are local rings.

Example 1.9. Manifolds are examples of locally ringed spaces: if X is a manifold, let $\mathscr{O}_X = C_X^\infty$, the smooth, real-valued functions. Let $p \in X$ and $\mathfrak{m}_{X,p}$ be the functions vanishing at p inside $\mathscr{O}_{X,p}$, which is an ideal. Then, any $f \in \mathscr{O}_{X,p} \setminus \mathfrak{m}_{X,p}$ is a unit: since it doesn't vanish at p, there's an open neighborhood U of p on which f doesn't vanish, so $1/f$ is smooth on U, and therefore defines an inverse to f in $\mathscr{O}_{X,p}$.

This locally ringed formalism is surprisingly useful: the maximal ideal of a stalk will always be functions vanishing at a point, even in weirder situations.

Of course, we want to understand morphisms of locally ringed spaces.

Definition 1.10. A *morphism of locally ringed spaces* $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is the data $(\varphi, \varphi^\#)$ of a continuous map $\varphi : X \to Y$ and a morphism of sheaves $\varphi^\# : \varphi_* \mathscr{O}_Y \to \mathscr{O}_X$ such that the induced map on stalks preserves the notion of vanishing at a point, i.e. for every $p \in X$, the preimage of the maximal ideal $\mathfrak{m}_{X,p}$ is contained in $\mathfrak{m}_{Y,\varphi(p)}$.

Here, $\varphi_* \mathscr{O}_X$ is the *pushforward* of $\mathscr{O}_X$, which attaches to every open $U \subset Y$ the ring $\varphi_* \mathscr{O}_X(U) = \mathscr{O}_X(\varphi^{-1}(U))$: since φ is continuous, this is again an open set.

The pushforward is an important definition in its own right. It's necessary to check that it actually defines a sheaf, but this isn't too complicated.

As an example, a smooth map $\varphi : X \to Y$ of manifolds defines a morphism of locally ringed spaces: φ is continuous, and a continuous map $f : V \to \mathbb{R}$ is sent to the map $f \circ \varphi : \varphi^{-1}(V) \to \mathbb{R}$. This is called the *pullback* of f. This is curious: we could have started with a merely continuous function that sends smooth functions to smooth functions, and it's forced to be smooth. Thus, the geometry of smooth manifolds is determined entirely by their structure as locally ringed spaces! Similarly, we'll define schemes to be certain kinds of locally ringed spaces.

1.3. **The spectrum of a ring.** A scheme is a particular kind of locally ringed space, locally isomorphic to $\operatorname{Spec} A$ for rings A, in the same way that a manifold is locally $\mathbb{R}^n$. Let's discuss the local model better.

Definition 1.11. The *spectrum* of a (commutative) ring A is $\operatorname{Spec} A = \{\mathfrak{p} \subset A \mid \mathfrak{p} \text{ is prime}\}$.

Let's briefly recall localization of rings.

Definition 1.12. If A is a ring and $S \subset A$ is a sub*set* such that $1 \in S$ and whenever $x, y \in S$, then $xy \in S$, we call S a *multiplicative subset*. Then, we can define the *localization* $S^{-1}A$ to be the ring of fractions $\{a/s \mid a \in A, s \in S\}$, where $a/s = b/s'$ iff there exists a $t \in S$ such that $t(s'b - sa) = 0$.

This is strongly reminiscent of the field of fractions of an integral domain, for which $S = A \setminus 0$; the equivalence relation is what allows us to know that $1/2 = 2/4$. For example, if $A = \mathbb{Z}$ and $S = \mathbb{Z} \setminus 0$, then $S^{-1}A = \mathbb{Q}$. In the same sense, a more general localization is akin to formally adding inverses of S.

Example 1.13. Let $\mathfrak{p} \subset A$ be a prime ideal. Then, $S = A \setminus \mathfrak{p}$ is multiplicative, since if $x \notin \mathfrak{p}$ and $y \notin \mathfrak{p}$, then $xy \notin \mathfrak{p}$. The localization $S^{-1}A$ is denoted $A_\mathfrak{p}$, the set of fractions a/s where $a \in A$ and $s \notin \mathfrak{p}$, with some equivalence relation. This makes everything except $\mathfrak{p}$ for units, so the image of $\mathfrak{p}$ is maximal in $A_\mathfrak{p}$.

Similarly, if $f \in A$, we can define $S = (f)$. The localization $S^{-1}A$ is denoted A_f, fractions of the form a/f^n; this makes f into a unit.

We need to define $\operatorname{Spec} A$ as a topological space, and then place a sheaf structure on it. With this structure, $\operatorname{Spec} A$ will be an *affine scheme*.

Definition 1.14. Let $I \subset A$ be an ideal. Then, let $D(I) \subset \operatorname{Spec} A$ be the set of prime ideals not containing I; if $I = \{f\}$, $D(I) = D(f) = \{\mathfrak{p} \mid f \notin \mathfrak{p}\}$. We define the topology on $\operatorname{Spec} A$ to have as its open sets $D(I)$ for all ideals I.

One has to check that these are closed under finite intersection and arbitrary union, but this is true, so $\operatorname{Spec} A$ is indeed a topological space.

Example 1.15. For example, $\operatorname{Spec} \mathbb{Z}$ as a set is the set of prime numbers and 0, since these account for all the ideals. The topology is curious: $(0) \subset \mathfrak{p}$ for all prime ideals $\mathfrak{p} \subset \mathbb{Z}$, so the zero ideal "lives everywhere."

The open sets are $D(a)$, the set of primes not dividing a, unless $a = 0$, in which case we get $\emptyset$.

The open set $D(f)$ is actually isomorphic as a topological space to $\operatorname{Spec}(A_f)$; for this reason, it's called a *distinguished affine open*.

Now, we just need to define the structure sheaf $\mathscr{O}_A$: what are the functions on $\operatorname{Spec} A$? We define $\mathscr{O}_A(U)$ to be the ring of functions $f : U \longrightarrow \coprod_{\mathfrak{p}\in U} A_\mathfrak{p}$ such that $f(\mathfrak{p}) \in A_\mathfrak{p}$ and for all $\mathfrak{p} \in U$, there's an $a/s \in A_\mathfrak{p}$ and an open $V \subset U$ such that for all $\mathfrak{q} \in V$, $f(\mathfrak{q}) = a/s$.

There are a bunch of equivalent definitions, but this is one of the most concrete: a section is a function to a weird space, but other definitions don't explicitly make the structure sheaf a sheaf of functions, and so it's harder to prove that the structure sheaf is, in fact, a sheaf.

Distinguished opens are particularly nice, in that $\mathscr{O}_X(D(f)) \cong A_f$. Moreover, for any $\mathfrak{p} \in \operatorname{Spec} A$, one can show $\mathscr{O}_{A,\mathfrak{p}} \cong A_\mathfrak{p}$. $A_\mathfrak{p}$ is a local ring, with (the image of) $\mathfrak{p}$ as its unique maximal ideal.

1.4. **Examples.** First, let's understand $\operatorname{Spec} \mathbb{Z}$ as a scheme, not just a topological space. $D(6)$ is the set of all primes except 2 and 3, plus the zero ideal. The acceptable functions on it are isomorphic to $\mathbb{Z}_6 = \{a/6^n \mid n \geq 0, a \in \mathbb{Z}\} = \mathbb{Z}[1/6]$. Thinking of these as functions, the function $21/6$ has value $21/6$ — but in different rings. Over (5), $21/6$ takes the value $21/6 \in \mathbb{Z}_{(5)}$; at (7), $21/6$ takes the value $21/6 \in \mathbb{Z}_{(7)}$. Here, $\mathbb{Z}_{(5)}$ is the ring of fractions whose denominators aren't divisible by 5. We can make sense of this for all primes except 2 or 3, and the function $21/6$ can't exist there (since dividing by zero is bad). At (0), the value is $21/6 \in \mathbb{Z}_{(0)} = \mathbb{Q}$.

Next, we'll do a more geometric example.

Example 1.16. Let k be a field (if you like, $k = \mathbb{C}$ makes for good geometric intuition). We define *affine n-space* $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \dots, x_n]$. All prime ideals of $\mathbb{A}^1_k$ look like (f) for some $f \in k[x]$; this prime ideal is prime iff f is irreducible. If k is algebraically closed, e.g. $k = \mathbb{C}$, this is only the case when $f(x) = x - a$ or $f(x) = 0$.

We associate the point $(x - a)$ to the point $a \in \mathbb{C}$, so we have a complex line of points plus the zero ideal, which is weird: it somehow lives everywhere.

$\mathbb{A}^2_\mathbb{C}$ is a little stranger: not only do we have a $\mathbb{C}^2$ worth of points (a, b) corresponding to $(x - a, y - b)$, and (0) which is once again everywhere, there are additional prime ideals: $(y - x^2)$ is a prime ideal, and it somehow lives at the entire curve $\{y = x^2\} \subset \mathbb{C}^2$. This is disorienting, but sometimes is useful.

2. But really, what is a scheme?: 9/7/16

These are Arun's lecture notes on the functor of points, another way to understand schemes and algebraic geometry that's particularly useful in the world of algebraic groups.

There doesn't seem to be a canonical reference for functor-of-points-style algebraic geometry. The discussion in the comments of [Kam09] provides interesting perspective, though no math. [Mil15, §1.a] has a brief introduction. [Vak15] and [Sta16] have the information, but it's scattered.

2.1. Functors and Yoneda's lemma.

The functor of points describes maps to schemes as "universal" or "natural" families of objects; to understand these cleanly, we need a dash of category theory. This section more or less follows [Sta16, Tag 001L].

Definition 2.1. A *category* C is a collection of *objects* and for every pair of objects $X.Y \in \mathsf{C}$ a set of *morphisms* $\mathrm{Hom}_{\mathsf{C}}(X, Y)$, such that there is always an identity $\mathrm{id}_X \in \mathrm{Hom}_{\mathsf{C}}(X, X)$ and we can compose morphisms $f : X \to Y$ and $g : Y \to Z$ into a morphism $g \circ f : X \to Z$.

For example: Set of sets and functions, Grp of groups and group homomorphisms, $\mathsf{Vect}_{\mathbb{R}}$ of real vector spaces and linear maps, Top of topological spaces and continuous functions.

Categories are useful for defining *universal properties*. We know what the product of two sets is, of rings, of spaces, of manifolds, … but if we can unify these definitions, we know how to define the product in unfamiliar situations: in particular, this is how we'll define the product of schemes.

Definition 2.2. Let $X, Y \in \mathsf{C}$. The *product* of X and Y, denoted $X \times Y$, is the *terminal object* with a pair of morphisms $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$. That is, for any object $Z \in \mathsf{C}$ with maps $p_1 : Z \to X$ and $p_2 : Z \to Y$, there exists a unique map $h : Z \to X \times Y$ such that the following diagram commutes.

$$\begin{tikzcd}
Z \arrow[dr, dashed, "h", "\exists!"'] \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\
& X \times Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"'] & Y \\
& X &
\end{tikzcd}$$

From this definition, one can show that the product doesn't always exist, but when it does, it's determined up to unique isomorphism respecting π_X and π_Y. The product in any category you've seen is an instance of this universal definition, and π_X and π_Y are the projection maps.

Here's a more general example, which we'll need later.

Definition 2.3. Let $\varphi : X \to Z$ and $\psi : Y \to Z$ be morphisms in a category C. Then, the *fiber product* (also *pullback* and *base change*) $X \times_Z Y$ is the terminal object with a pair of morphisms $\pi_X : X \times_Z Y \to X$ and $\pi_Y : X \times_Z \to Y$ such that the following diagram commutes.

$$\begin{tikzcd}
X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "\psi"] \\
X \arrow[r, "\varphi"] & Z
\end{tikzcd}$$

Once again, this might not exist, but is uniquely determined if it does.

Exercise 2.4. If you haven't seen this before, show that in Set, the fiber product of $\varphi : X \to Z$ and $\psi : Y \to Z$ is $X \times_Z Y = \{(x, y) : \varphi(x) = \psi(y)\}$.

Given any category C, we can define the *opposite category* C^{op} by reversing the arrows: $\mathrm{Hom}_{\mathsf{C}^{\mathsf{op}}}(X, Y) = \mathrm{Hom}_{\mathsf{C}}(Y, X)$.

Definition 2.5. A *(covariant) functor* $F : \mathsf{C} \to \mathsf{D}$ is a structure-preserving map of categories: to every object $X \in \mathsf{C}$, we associate $F(X) \in \mathsf{D}$, and to every morphism $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ in C, we associate $F(f) \in \mathrm{Hom}_{\mathsf{D}}(F(X), F(Y))$.

A functor $\mathsf{C}^{\mathsf{op}} \to \mathsf{D}$ is sometimes called a *contravariant functor* from C to D: it sends every $f : X \to Y$ in C to $F(f) : F(Y) \to F(X)$.

Functors abound in mathematics. For example: fundamental group, homotopy, homology, and cohomology groups; free groups, abelian groups, or algebras; pullback of functions; the forgetful functor $\mathsf{Grp} \longrightarrow \mathsf{Set}$ sending a group to its underlying set.

Example 2.6 (Functor of points)**.** Given an object $X \in \mathsf{C}$ the contravariant functor $h_X = \mathrm{Hom}_{\mathsf{C}}(-, X) : \mathsf{C} \longrightarrow \mathsf{Set}$ (sending Y to the set of morphisms $Y \longrightarrow X$) is called the *functor of points* of X. This is contravariant because of precomposition: a map $\varphi : Y \longrightarrow Z$ induces a map $\varphi^* : h_X(Z) \longrightarrow h_X(Y)$, called *pullback*: it sends $f : Z \longrightarrow X$ to $f \circ \varphi : Y \longrightarrow X$.

The functor-of-points approach to algebraic geometry is to understand a geometric object X through its functor h_X, which typically has a cleaner description. This is like understanding a function $f : S^1 \longrightarrow \mathbb{R}$ through its Fourier coefficients, with Hom playing the role of an inner product.

The Yoneda lemma is the statement that this "inner product" is nondegenerate.

Lemma 2.7 (Yoneda)**.** *Let $X, Y \in \mathsf{C}$. An isomorphism $h_X \xrightarrow{\sim} h_Y$ uniquely determines an isomorphism $X \xrightarrow{\sim} Y$.*

In particular, X is completely and canonically determined by h_X.

Finally, we mention a few categorical facts about scheme theory.

Definition 2.8. A *duality of categories* is a pair of adjoint, contravariant functors inducing an equivalence of opposite categories. That is, a duality of categories C and D is the data of two functors $F : \mathsf{C}^{\mathsf{op}} \longrightarrow \mathsf{D}$ and $G : \mathsf{D} \longrightarrow \mathsf{C}^{\mathsf{op}}$ such that $F \circ G$ and $G \circ F$ are naturally isomorphic to the identity functor.

This term is nonstandard.

Theorem 2.9 ("Fundamental theorem", [Sta16, Tag 01HX])**.** *The following pairs of functors define dualities of categories.*

1. (Γ, Spec) *between* AffSch *and* Ring*: the ring of global sections* $\Gamma(X)$ *of an affine scheme* X *and the spectrum* $\mathrm{Spec}\, A$ *of a commutative ring, respectively.*
2. *Fixing a ring A,* (Γ, Spec) *between* AffSch_A *and* Alg_A*: the same functors, but between the category of affine schemes over A*[2] *and the category of A-algebras.*
3. *Fixing a field k,* (I, V) *between* AffVar_k *and* RedAlg_k*: analogous functors between the category of affine varieties over a field k and finitely generated reduced k-algebras.*

Moreover, because every scheme (resp. scheme over A, variety over k) can be constructed by gluing together affine schemes (resp. affine A-schemes, affine varieties over k), a functor of points $h_X : \mathsf{Sch}^{\mathsf{op}} \longrightarrow \mathsf{Set}$ is determined by its restriction to $h_X : \mathsf{AffSch}^{\mathsf{op}} \longrightarrow \mathsf{Set}$ (we'll explain this more in a bit), which by the above theorem is equivalent to a covariant functor $h_X^{\mathsf{op}} : \mathsf{Ring} \longrightarrow \mathsf{Set}$. That is, functors of points on this geometric category can be understood in terms of purely algebraic data. In the same way, a functor of points on schemes over A is determined by a contravariant functor $\mathsf{Alg}_A \longrightarrow \mathsf{Set}$, and one from k-varieties is determined by a covariant functor $\mathsf{RedAlg}_k \longrightarrow \mathsf{Set}$.

2.2. **Representability.** Our game plan is to write down examples of these functors and try to understand the geometry of the objects associated to them. The first obstacle is that not every functor is h_X for some X.

Definition 2.10. Let $F : \mathsf{C}^{\mathsf{op}} \longrightarrow \mathsf{Set}$ be a functor. If $X \in \mathsf{C}$ is such that $F \cong h_X$, then F is called *representable.*

Representable functors are the ones we can do geometry with.

Example 2.11. Algebraic geometry is built out of solution sets to systems of polynomial equations, and the functor of points is a package for solving a system of equations (the geometric constraint) over all coefficient rings simultaneously, as this example illustrates.

Let $f_1, \dots, f_n \in \mathbb{Z}[x_1, \dots, x_m]$. For any ring A, the unique map $\mathbb{Z} \longrightarrow A$ induces a map $\mathbb{Z}[x_1, \dots, x_m] \longrightarrow A[x_1, \dots, x_m]$. Let $F : \mathsf{Ring} \longrightarrow \mathsf{Set}$ send

$$A \longmapsto \{(a_1, \dots, a_m) \in A^n \mid f_1(a_1, \dots, a_m) = \cdots = f_n(a_1, \dots, a_m) = 0\},$$

the set of solutions to $f_1, \dots, f_n$ over A. Using (2.9), F is representable, and is represented by the scheme $X = \mathrm{Spec}\, \mathbb{Z}[x_1, \dots, x_m]/(f_1, \dots, f_n)$.

In this sense, a solution to $(f_1, \dots, f_n)$ over A can be thought of as an "A-valued point" of the representing scheme X; the A-valued points are in natural bijection with the maps $\mathrm{Hom}_{\mathsf{Sch}}(\mathrm{Spec}\, A, X)$. More generally, if X is any scheme, an

[2] A *scheme over a base A* is the data of a scheme X and a map to $\mathrm{Spec}\, A$, called the *structure map.* Morphisms of schemes over A are required to commute with the structure maps.

A-valued point is (the image of) a map $\operatorname{Spec} A \longrightarrow X$. This is the etymology of the functor of points: to a scheme/functor X we have its $\mathbb{C}$-valued points $X(\mathbb{C})$, its $\mathbb{F}_p$-valued points $X(\mathbb{F}_p)$, etc.

The following discussion is adapted from [Vak15, §9.1.6].

Example 2.12 (Fiber products)**.** Let X, Y, and Z be schemes, and $\varphi : X \longrightarrow Z$ and $\psi : Y \longrightarrow Z$ be specified. Let $F : \mathsf{Sch}^{\mathsf{op}} \longrightarrow \mathsf{Set}$ be the functor sending a scheme $W \mapsto h_X(W) \times_{h_Z(W)} h_Y(W)$. Then, F is representable, and is represented by the fiber product $X \times_Z Y$.

In particular, fiber products of schemes always exist. They include restricting to the preimage of an open subset or changing the base ring (e.g. base change with $\operatorname{Spec} \mathbb{C}$ turns schemes over $\mathbb{R}$ to schemes over $\mathbb{C}$).

When we refer to the fiber product of functors $\mathsf{Sch}^{\mathsf{op}} \longrightarrow \mathsf{Set}$, we take them pointwise, i.e. $(F \times_H G)(X) = F(X) \times_{H(X)} G(X)$.[3]

This allows us to discuss a criterion for representability. The key is that a morphism of schemes can be glued together from compatible local data. They define a sheaf $U \mapsto \operatorname{Hom}_{\mathsf{Sch}}(U, X)$, with restriction given by actual restriction of functions.

Definition 2.13. Let $F : \mathsf{Sch}^{\mathsf{op}} \longrightarrow \mathsf{Set}$ be a functor such that for every scheme X, the map $U \mapsto F(U)$ for $U \subset X$ open defines a sheaf of sets on X, and for every morphism $\varphi : X \longrightarrow Y$, the induced map $F(\varphi)$ is a morphism of these sheaves; then, F is called a *Zariski sheaf.*

In other words, morphisms should sheafify in a universal way. This is necessary, but not sufficient.

Definition 2.14. Let $h : \mathsf{Sch}^{\mathsf{op}} \longrightarrow \mathsf{Set}$ be a functor.

- A functor $h' : \mathsf{Sch}^{\mathsf{op}} \longrightarrow \mathsf{Set}$ is called an *open subfunctor* of h if for all representable functors h_X and maps $h_X \longrightarrow h$,[4] the pullback $h' \times_h h_X$ is representable and represents an open subscheme of X.
- A collection $\mathfrak{U}$ of open subfunctors of h is said to *cover* h if for all representable functors h_X and maps $h_X \longrightarrow h$, the schemes representing $h_X \times_h F_i$ for all $F_i \in \mathfrak{U}$ are an open cover of X.

Once again, these are the ordinary notions universalized.

Theorem 2.15. *Let $F : \mathsf{Sch}^{\mathsf{op}} \longrightarrow \mathsf{Set}$ be a Zariski sheaf that has an open cover by representable functors. Then, F is representable.*

Exercise 2.16. Let A be a ring and $F : \mathsf{Sch}_A^{\mathsf{op}} \longrightarrow \mathsf{Set}$ be the functor sending an A-scheme X to the set of data $\{(\mathscr{L}, s_0, s_1)\}$ up to isomorphism, where $\mathscr{L}$ is a line bundle (invertible sheaf) on X and s_0 and s_1 are sections with no common zero. Use Theorem 2.15 to show F is representable; the representing scheme is called *projective 1-space* over A, denoted $\mathbb{P}^1_A$.

Remark. You may be more used to the definition of $\mathbb{P}^1_A$ as two copies of $\mathbb{A}^1_A$ glued together so one's 0 is the other's point at infinity, as for the construction of the Riemann sphere $S^2 \cong \mathbb{P}^1_{\mathbb{C}}$.

Example 2.17. Another use for the functor of points is in *moduli problems*: we want to classify geometric objects as elements of some space. In general, the "moduli space of X-stuff" is the scheme representing the functor sending Y to the set of flat families of X-stuff over Y. These functors aren't always representable, however.

2.3. **Group Schemes.** This section follows [Vak15, §6.6].

Informally, just as a topological group is a topological space with continuous multiplication and inversion, and a Lie group is a manifold with smooth multiplication and inversion, a group scheme is a scheme with multiplication and inversion maps that are morphisms of schemes. We will make this precise in two different ways.

Definition 2.18. Let C be a category in which finite products exist. A *group object* in C is an object $G \in \mathsf{C}$ together with a *multiplication map* $\mu : G \times G \longrightarrow G$, an *identity map* $e : 1 \longrightarrow G$,[5] and an *inversion map* $i : G \longrightarrow G$, all of which are morphisms in C, such that μ, e, and i satisfy the usual axioms of a group. For example, associativity of μ means that the following diagram commutes:

$$G \times G \times G \underset{(\mathrm{id},\mu)}{\overset{(\mu,\mathrm{id})}{\rightrightarrows}} G \times G \xrightarrow{\mu} G.$$

[3] This is in fact the fiber product of these functors in the *functor category* $\mathsf{Fun}(\mathsf{Sch}^{\mathsf{op}}, \mathsf{Set})$.

[4] A morphism of schemes is an *open embedding* if it factors as an isomorphism of schemes followed by an inclusion $(U, \mathscr{O}_X|_U) \hookrightarrow (X, \mathscr{O}_X)$ of an open subset. See [Vak15, §7.1].

[5] Here, 1 is the *terminal object*, which is also the empty product.

You know what the axioms are; the trick is writing them as commutative diagrams rather than element-wise.

Example 2.19. Group objects encode the usual notion of "groups with additional structure:"

- A group object in Set is just a group.
- A group object in Top is a topological group.
- A group object in Man is a Lie group.

Exercise 2.20. Why are the group objects in Grp the abelian groups?

Group objects in C form a category whose morphisms are group homomorphisms that are also C-morphisms.

Definition 2.21. A *group scheme* is a group object in Sch. A *group variety* is a group in Var.

Algebraic groups are well-behaved group varieties.

Let's functor-of-pointsify Definition 2.18. If $G \in \mathsf{C}$ is a group object and $Y \in \mathsf{C}$, we can define multiplication pointwise on $\mathrm{Hom}_{\mathsf{C}}(Y, G)$: $f \cdot g$ is the composition $\mu \circ (f, g)$ (so, if we have elements, $(f \cdot g)(x) = f(x)g(x)$), and can define the identity and inverse maps similarly. The upshot is that $h_G(Y)$ is a group, and h_G sends morphisms to group homomorphisms, so we may regard it as a functor $\mathsf{C}^{\mathsf{op}} \longrightarrow \mathsf{Grp}$. The converse is also true: if a representable functor factors through Grp, it's represented by a group object. Maybe the following definition is actually a theorem.

Definition 2.22. A *group object* in a category C is an object $X \in \mathsf{C}$ whose functor of points $h_X : \mathsf{C}^{\mathsf{op}} \longrightarrow \mathsf{Set}$ factors through the structure-forgetting inclusion $\mathsf{Grp} \longrightarrow \mathsf{Set}$, and hence may be regarded as group-valued.

This generalizes to abelian group objects, ring objects, etc.

There's no shortage of group-valued functors around, making it easy to define group schemes.

Example 2.23.

(1) Fix a ring A. The forgetful functor $\mathrm{For} : \mathsf{Alg}_A \longrightarrow \mathsf{Ab}$ sends an A-algebra to its underlying abelian group, and is covariant. It is representable, and is represented by the *additive group* $\mathbb{G}_a = \mathbb{A}^1_A$, which is hence an abelian group scheme.

(2) Similarly, the group of units is a covariant functor $\mathsf{Alg}_A \longrightarrow \mathsf{Ab}$ sending $B \mapsto B^\times$; its representing A-scheme is called the *multiplicative group* $\mathbb{G}_m = \mathrm{Spec}\, A[x, x^{-1}]$, which is an abelian group scheme.

(3) The functor $\mathrm{GL}_n : \mathsf{Ring} \longrightarrow \mathsf{Grp}$ sending $A \mapsto \mathrm{GL}_n(A)$, the group of $n \times n$ matrices with coefficients in A, is represented by the *general linear group* GL_n, an algebraic group. In the same way, one may define the *special linear group* $\mathrm{SL}_n : A \mapsto \mathrm{SL}_n(A)$, the *orthogonal group* $\mathrm{O}_n : A \mapsto O_n(A)$, and the *special orthogonal group* $\mathrm{SO}_n : A \mapsto \mathrm{SO}_n(A)$. For $n > 2$, these are all nonabelian group schemes.

(4) There is a covariant functor $\mu_n : \mathsf{Alg}_A \longrightarrow \mathsf{Ab}$ sending B to its group of n^{th} roots of unity, i.e. solutions $x \in B$ to $x^n = 1$. This is an instance of Example 2.11; this functor is represented by $\mathrm{Spec}\, A[x]/(x^n - 1)$.

Exercise 2.24. Let A be a ring and G be a group scheme. Show that the identity, multiplication, and inversion maps define a group structure on the set of A-valued points of G. (It will be easier to use Definition 2.22 than Definition 2.18.)

For example, the usual notion of the group $\mathrm{GL}_n(k)$ agrees with that induced on the k-valued points of GL_n.

3. Algebraic Groups, a Definition: 9/14/16

Throughout today's lecture, k will denote a field, and $* = \mathrm{Spec}\, k$ denotes the one-point variety.

The definition of an algebraic group might be a little surprising at first, but it turns out that it's just a restatement of the usual definition of a group, trying to replace the reference to elements with commutative diagrams.

Definition 3.1. A (blah) group (we'll be nonspecific about what (blah) means for now) consists of a (something) G and maps $m : G \times G \longrightarrow G$, $e : * \longrightarrow G$, and $i : G \longrightarrow G$ such that the following diagrams commute.

Associativity: We have two possible ways to apply m to three copies of G (starting with the first factors, or starting with the last factors), and we want them to be the same:

$$\begin{tikzcd} G \times G \times G \arrow[r, "(\mathrm{id}, m)"] \arrow[d, "(m, \mathrm{id})"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"] & G. \end{tikzcd}$$

Identity: We want to encode that multiplication with e as one of the factors doesn't change anything. Since groups in general are noncommutative, we need this to be true both on the left and the right:

$$\begin{tikzcd} * \times G \arrow[r, "(e,\mathrm{id})"] \arrow[dr, "\pi_2"'] & G \times G \arrow[d, "m"] & G \times * \arrow[l, "(\mathrm{id},e)"'] \arrow[dl, "\pi_1"] \\ & G. & \end{tikzcd}$$

Here π_i is projection onto the i^{th} component ($i \in \{1, 2\}$).

Inverse: In the same way, multiplying with the inverse should give you the identity, both on the left and on the right.

$$\begin{tikzcd} G \arrow[r, "(\mathrm{id},i)"] \arrow[d] & G \times G \arrow[d, "m"] & G \arrow[l, "(i,\mathrm{id})"'] \arrow[d] \\ * \arrow[r] & G & * \arrow[l] \end{tikzcd}$$

A (blah) should really be some sort of category, but we don't need the full generality.

If (something) is just a set, then this definition is equivalent to that of an ordinary group.

Definition 3.2. If (something) is

- a scheme, then this group is called a *group scheme*.
- an affine scheme, then this group is caled an *affine group scheme*.
- a variety, this group is called an *algebraic group* or *group variety*.

We also like commutativity, but again need to specify this without using elements.

Definition 3.3. An algebraic group G is *commutative* or *abelian* if $m = m \circ \tau$, where $\tau : G \times G \longrightarrow G \times G$ is the transposition map switching the two copies of G in $G \times G$.

It's also important to know what a homomorphism of algebraic groups is. Recall that for usual groups (defined over sets), a map $\varphi : (G, \cdot_G) \longrightarrow (H, \cdot_H)$ is a homomorphism if for all $g_1, g_2 \in G$, $\varphi(g_1 \cdot_G g_2) = \varphi(g_1) \cdot_H \varphi(g_2)$.

Definition 3.4. Let (G, m_G) and (H, m_H) be algebraic groups, and let $\varphi : G \longrightarrow H$ be a morphism of schemes. Then, φ is a *homomorphism of algebraic groups* if $\varphi \circ m_g = m_h \circ (\varphi, \varphi)$.

Once again, this just means it sends multiplication to multiplication.

Definition 3.5. Let (G, m_G) and (H, m_H) be algebraic groups. Then, H is a *algebraic subgroup* of G if it's a subscheme of G and the inclusion map $i : H \hookrightarrow G$ satisfies $i \circ m_H = m_G$.

Example 3.6. The *special linear group* is

$$\mathrm{SL}_n = \mathrm{Spec}(k[x_{11}, x_{12}, \dots, x_{nn}]/(\det(X) - 1)),$$

which represents the $n \times n$ matrices over k with determinant 1.

Schemes have a lot of structure, e.g. they are topological spaces. Let's see what the structure of an algebraic group buys us in this context. Given a scheme X, we'll let $|X|$ denote the topological space of closed points of X.

Definition 3.7. At any closed point $p \in |X|$, the stalk $\mathcal{O}_{X,x}$ has a unique maximal ideal $\mathfrak{m}_x$. The quotient $\mathcal{O}_{X,x}/\mathfrak{m}_x$ is called the *residue field*; Milne denotes this κ, and we will denote it $\mathring{k}(x)$.

The multiplication map on an algebraic group is regular. This more or less means it looks like a polynomial function locally. This means that if we fix a field κ and let $T = \{x : \mathring{k}(x) = \kappa\}$ be the space of points with that residue field, then the multiplication map restricted to $T \times T$ maps into T.

If $\kappa = \overline{k}$, then $T = |G|$, and m induces a map on the underlying topological space. However, it does *not* define the structure of a topological group on $|G|$ in general! However, in this case, the left multiplication map $\ell_a : G \longrightarrow G$ defined by $x \longmapsto ax$ (for an $a \in G$), is a homeomorphism of underlying topological spaces, and therefore $|G|$ is a homogeneous topological space: its automorphism group acts transitively on it (there's only one orbit).

We can also develop a notion of density. The definition is a little scary, but means intuitively that, just like a dense subset of a topological space, a function on a schematically dense set extends uniquely to one on the whole scheme.

Definition 3.8. Let X be a scheme over an algebraically closed field k. Then, a subset $S \subseteq |X|$ is *schematically dense* if the restriction map $f \mapsto \{(s, f(s)) \mid s \in S\}$ is injective.[6]

If X is reduced, which is often but not always true, this is equivalent to the usual notion of density.

Another view. Yoneda's lemma tells us that an algebraic group G determines a functor $h_G : \mathsf{Alg}_k \longrightarrow \mathsf{Grp}$ (which is the functor of points):[7] specifying what this group does to all algebras allows us to determine a lot about the group in question. The functor $G \mapsto h_G$ is fully faithful.

This functor is group-valued because we can precompose pairs of morphisms by $m : G \times G \longrightarrow G$: if $\varphi, \psi \in h_G(R)$, then their product is $(\varphi, \psi) \circ m$, and this obeys the usual axioms for a group (of sets).

This allows us to provide a better definition for SL_n: it's the algebraic group that represents the functor $R \mapsto \mathrm{SL}_n(R)$, where R is a k-algebra. And we can also use it to make clean definitions about algebraic groups.

Definition 3.9. Let H be a subscheme of the algebraic group G. Then,

- H is an *algebraic subgroup* of G if for all k-algebras R, $h_H(R)$ is a subgroup of $h_G(R)$.
- H is a *normal algebraic subgroup* if for all k-algebras R, $h_H(R) \triangleleft h_G(R)$ (it's a normal subgroup).

The general pattern isn't too different: a notion on algebraic groups often comes from that notion applied to all groups that its functor of points defines.

Proposition 3.10. *The identity and inverse maps are uniquely determined for an algebraic group G. Moreover, if $\varphi : G \longrightarrow H$ is a homomorphism, then $\varphi \circ e_G = e_H$ and $i_H \circ \varphi = \varphi \circ i_G$.*

This is something we already know for groups; then, we invoke Yoneda's lemma.

Proposition 3.11. *The identity subscheme is a subgroup of any algebraic group.*

The proof is the same: we know this for groups (of sets), and using Yoneda's lemma, we can recover it for algebraic groups.

References

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[6] Question we weren't able to resolve during lecture: where should f live?

[7] Milne in [Mil15] writes this as a functor $h_G : \mathsf{Alg}_k^0 \longrightarrow \mathsf{Grp}$. Here, Alg_k^0 means the "small" (i.e. finitely generated) k-algebras, which suffices because he only considers schemes of finite type over k. This notation is confusing (since it's reminiscent of the opposite category).