# FALL 2017 GEOMETRIC SATAKE SEMINAR

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These notes were taken in David Ben-Zvi's student seminar in Fall 2017. I live-TeXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to `a.debray@math.utexas.edu`. Thanks to Tom Gannon and Richard Hughes for some helpful comments.

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## 1. Overview on the geometric Satake theorem: 9/1/17

This overview was given by David Ben-Zvi.

This semester, we're studying the geometric Satake theorem, one of the most important results in geometric representation theory, and even a central result in the geometric Langlands program.

This theorem involves some potentially unfamiliar words; we'll define them in the course of this seminar.

**Theorem 1.1** (Geometric Satake)**.** *Let $G$ be a reductive group over a field $k$.*[1] *Then, there is a category of $G(\mathscr{O})$-equivariant perverse sheaves on the affine Grassmannian of $G$, $G(K)/G(\mathscr{O})$, symmetric monoidal under convolution, together with a fiber functor $H^\bullet(-)$, and this is equivalent to $(\mathsf{Rep}^{\mathrm{fd}}_{G^\vee}, \otimes)$ (where $G^\vee$ is the Langlands dual group) as symmetric monoidal categories, with the fiber functor the forgetful map to $\mathsf{Vect}_k$.*

By $\mathsf{Rep}^{\mathrm{fd}}$ we mean the full subcategory of finite-dimensional representations. $K$ will be some local field, and $\mathscr{O}$ is its ring of integers. For example, if $k = \mathbb{C}$, $K = \mathbb{C}((t))$ and $\mathscr{O} = \mathbb{C}[[t]]$, and over $\mathbb{F}_p$, you have $K = \mathbb{F}_p((t))$ and $\mathscr{O} = \mathbb{F}_p[[t]]$.[2]

Okay, first what's a reductive group? For $k = \mathbb{C}$, these are complexifications of compact groups. For example: $\mathrm{GL}_n$, $\mathrm{SL}_n$, $\mathrm{PGL}_n$, $\mathrm{SU}_n$, $\mathrm{Sp}_n$, and $\mathrm{E}_7$.

Now this theorem is saying that we start with one reductive group and we get another, $G^\vee$. This relationship is such that if $G = T$ is a torus, i.e. $(\mathbb{C}^\times)^k$, its Langlands dual is the dual torus $T^\vee$: if $T$ is the quotient of $\mathbb{C}^n$ by a lattice, $T^\vee$ is the quotient of $(\mathbb{C}^n)^*$ by the dual lattice.

Theorem 1.1 is a kind of Fourier transform, a quite fancy one. For example, if $G = \mathrm{GL}_1$, the affine Grassmannian is a (scheme which behaves more or less like) $\mathbb{Z}$: $\mathrm{GL}_1(\mathbb{C}((t)) = \mathbb{C}((t))^\times$ and $\mathrm{GL}_1(\mathbb{C}[[t]])$ is the group of power series with nonzero constant term. When you mod these out, the leading term of the Laurent series becoms the only important thing, in a sense. The next ingredient is the equivariant perverse sheaves, but ends up being vector bundles over Gr in this case, so we get (modulo some reducedness which doesn't

[1] You can let $k$ be a ring $R$, the coefficients. The algebraic geometry we do will still be over $\mathbb{C}$, though; the representations you get end up also being representations over the ring $R$.

[2] In particular, they will never be $\mathbb{Q}_p$ and $\mathbb{Z}_p$. However, the geometric Satake theorem is a living piece of mathematics, and only in the past year Peter Scholze proved a version of this for the $p$-adics.

come into play here) the category of graded vector spaces. In this case, Theorem 1.1 says the category of graded vector spaces is equivalent to the category of representations of $\mathbb{G}_m$, just like the Fourier transform exchanges functions on $\mathbb{Z}$ with representations of $S^1$.

You can interpret the geometric Satake theorem as the source of the Langlands dual group: it admits a definition in terms of tori and root data, but it feels somewhat ad hoc, and one is left wondering: where did it all come from? Instead, by the Tannakian perspective on representation theory, Theorem 1.1 is telling us that the category of $G(\mathcal{O})$-equivariant sheaves with its fiber functor is canonically the category of representations of a group, and in fact gives us enough information to reconstruct the group! So the geometric Satake theorem is a bridge from $G$ to $G^\vee$, and is one of the only bridges.

**Example 1.2.** Langlands duality is often somewhat surprising: $G$ and $G^\vee$ don't look like each other, and it's not clear how to obtain one from the other. Of course, $(G^\vee)^\vee \cong G$.

$$\begin{aligned} \mathrm{GL}_n &\longleftrightarrow \mathrm{GL}_n \\ \mathrm{SL}_n &\longleftrightarrow \mathrm{PGL}_n \\ \mathrm{SO}_{2n+1} &\longleftrightarrow \mathrm{Sp}_{2n} \\ \mathrm{SO}_{2n} &\longleftrightarrow \mathrm{SO}_{2n}. \end{aligned}$$ ◀

You can also use the geometric Satake theorem to explain some things which at first appear to not be geometric. For example, $H^*(\mathbb{CP}^n) \cong \mathbb{Z}[x]/(x^{n+1})$ with $|x| = 2$ is acted on by $\mathrm{SL}_2$ by raising and lowering operators, which comes out of complex geometry, but this does not arise from an $\mathrm{SL}_2$-action on $\mathbb{CP}^n$ itself. More generally, $\mathrm{SL}_k$ acts on $H^*(\mathrm{Gr}_k(\mathbb{C}^n))$ in a similar way, and is similarly mysterious.

But Theorem 1.1 identifies it with an equivariant perverse sheaf for the action of $\mathrm{PGL}_k$ on a Grassmannian, and the action of $\mathrm{PGL}_k$ on a Grassmannian is more evident. So we've obtained either new interesting representations, or geometric models for representations of your group, and solved the mysteries of this representation.

It also flows the other direction: if you take a reductive group over $R$, you recover information about $\mathsf{Rep}_{G^\vee,R}$, the category of representations over $R$. This is an active topic of research, and people including Geordie Williamson have used it to uncover interesting consequences in modular representation theory.

Of course, the geometric Satake theorem is also entangled with the geometric Langlands conjectures in interesting ways.

We're going to first discuss affine Grassmannians, then perverse sheaves (which generalize cohomology of smooth projective varieties, and could be an entire seminar unto themselves), then their convolution (the fact that it's symmetric monoidal is very deep, and related to commutativity of Hecke algebras). Finally, we'll talk about Tannakian reconstruction, a beautiful abstract story that allows us to extract $G^\vee$ from the theorem.

Here's a very provisional schedule:

- First, a few lectures on the affine Grassmannian: Rustam (next week), and then Richard H. (the week after).
- Then, perverse sheaves (and also intersection cohomology): Arun, Sebastian, and Yan.
- Then, convolution and its commutativity: Richard W. and Vaibhav.
- Then, Tannakian reconstruction: Isabelle and Nicky.
- Finally, putting it together into a proof of the geometric Satake correspondence: Rok and probably also someone else.

There's a lot of further topics and cool applications if anyone is interested after that.

## 2. Bruhat-Tits trees: 9/1/17

In the second part of the first meeting, Tom Gannon spoke about trees (à la Serre).

Throughout this lecture, let $K$ be a complete field (with respect to some norm), together with a discrete valuaton $v\colon K^\times \to \mathbb{Z}$.[3] Let $\mathscr{O}_K$ denote its ring of integers, which is a local ring, and $\mathfrak{m}$ denote its unique

[3]Recall that a *discrete valuation* is a surjective group homomorphism $K^\times \to \mathbb{Z}$. You can think of it as measuring how many times a uniformizer $\pi$ divides a given field element.

maximal ideal. Let $\pi$ be a *uniformizer*, i.e. a generator of $\mathfrak{p}$.[4] We'll let $q := |\mathscr{O}_K/\mathfrak{m}|$, and assume that $q$ is finite, so that the residue field $\mathscr{O}_K/\mathfrak{m} \cong \mathbb{F}_q$.

Though we didn't define $v(0)$, we think of it as $\infty$: the valuation values how many times you can divide an element by $\pi$, and for 0 you can do this infinitely often.

Let $K$ be a field with a discrete valuation $v\colon K^\times \to \mathbb{Z}$ and $c \in (0,1)$ be fixed. Then, the map $\|\cdot\|_c\colon K \to [0,\infty)$ with $|x|_c := c^{v(x)}$ is a norm, and moreover is non-Archimedean, satisfying a stronger form of the triangle inequality:

$$|x+y|_c \le \max\{|x|_c, |y|_c\}.$$

**Proposition 2.1.** *With notation as above, the set $\{x \in K \mid |x| \le 1\}$ is a ring, and in fact a discrete valuation ring; its unique maximal ideal is $\{x \in K \mid |x|_c < 1\}$.*

That it's a discrete valuation ring means the unique maximal ideal is principal. This ring is called the *associated ring of integers* of $K$. Let's pick a specific value of $c$, which is $1/q$.

**Example 2.2** (2-adic rationals)**.** The 2-adic rationals, $\mathbb{Q}_2$, form a complete field with a discrete valuation. One way to think about this is that there's a norm on $\mathbb{Q}$ given by how many 2s you can factor out; completing it with respect to that norm defines $\mathbb{Q}_2$.

There's also a lower-brow way to think of this, as Laurent series in 2: an element of $\mathbb{Q}_2$ is something like

$$2^{-4} + 2^{-3} + 2^{-1} + 2 + 2^3 + 2^5 + \cdots.$$

Equality is termwise, and addition and multiplication are like those of Laurent series. The coefficients are mod 2, so if you consider $p$-adics for $p > 2$, you have more options. In this case, the valuation is the smallest $N$ such that the $N$-coefficient is nonzero.

There's also an algebraic interpretation of $\mathbb{Z}_2$ and $\mathbb{Q}_2$. ◀

**Example 2.3.** Another example if $K = \mathbb{F}_p((t))$ with $\mathscr{O}_K = \mathbb{F}_p[[t]]$. The valuation is the minimal power of $t$ that appears with a nonzero coefficient, like for $\mathbb{Q}_2$. ◀

Now we'll discuss the Bruhat-Tits tree for $\mathrm{SL}_2(K)$. There's not a lot of motivation, except that this stuff is awesome.

The tree will be a set of vertices and edges; its vertices will be a set of lattices in $K^2$.

**Definition 2.4.** A *lattice* in $K^2$ is an $\mathscr{O}_K$-submodule $\Lambda$ of $K^2$ such that $\Lambda \otimes_{\mathscr{O}_K} K = K^2$.

Concretely, these are subsets of $K^2$ of the form $\mathscr{O}_K \cdot v_1 + \mathscr{O}_K \cdot v_2$, where $\{v_1, v_2\}$ is a basis for $K^2$. These correspond to the usual lattices in $\mathbb{R}^2$.

Since $\mathrm{GL}_2(K)$ acts on the set of bases of $K^2$, it acts on the set of lattices. The stabilizer of each lattice is $\mathrm{GL}_2(\mathscr{O}_K)$, and therefore the space of lattices is naturally isomorphic to $\mathrm{GL}_2(K)/\mathrm{GL}_2(\mathscr{O}_K)$. Hey, that space appeared in the statement of the geometric Satake isomorphism!

**Theorem 2.5** (Principal divisor theorem)**.** *Let $L_1$ and $L_2$ be lattices. Then, there's a basis $\{e, f\}$ for $L_1$ and $m, n \in \mathbb{Z}$ such that $\{\pi^m e, \pi^n f\}$ is a basis for $L_2$.*

The proof is linear algebra, spiced up somewhat by the fact that it's over discrete valuation rings. It's also the only place where we assume the residue field is finite.

*Remark.* If you consider $\mathrm{GL}_1$ instead of $\mathrm{GL}_2$, you get the statement we discussed before, that $\mathrm{GL}_1(K)/\mathrm{GL}_1(\mathscr{O}_K)$ is the integers (and therefore representations of $\mathbb{G}_m$ are equivalent to graded vector spaces). ◀

Now, say that two lattices $L_1$ and $L_2$ are equivalent if $L_1 = \pi^\ell L_2$ for some $\ell$. The space of equivalence classes is $\mathrm{PGL}_2(K).\mathrm{PGL}_2(\mathscr{O}_K)$. We define the vertices of the Bruhat-Tits tree to be this set.

Now we should talk edges. Let $v$ and $w$ be two vertices, and $L_1$ and $L_2$ be lattice representatives for $v$ and $w$, respectively. By Theorem 2.5, there are $m$ and $n$ carrying a basis for $L_1$ to a basis for $L_2$, and we add an edge iff $|m-n| = 1$.[5] Equivalently, we add an edge if there's a rescaling of $L_1$ called $L_1'$ such that $L_2 \supsetneq L_1' \supsetneq \pi L_2$.

Call this graph $G$. We'll eventually show it's a tree.

[4] You can think of this as a coordinate on the curve.

[5] There's enough uniqueness in the proof for this to be well-defined, even if $m$ and $n$ aren't unique.

**Proposition 2.6.** *$G$ is a connected graph.*

*Proof.* Let $L_1$ and $L_2$ be lattices. Then, there are $e, f \in L_1$ and $m, n \in \mathbb{Z}$ such that $\{e, f\}$ is a basis for $L_1$ and $\{\pi^m e, \pi^m f\}$ is a basis for $L_2$. Without loss of generality, assume $m \geq n$. Then, there's an edge from $L_2$ to $\mathscr{O}_K \cdot \pi^{m-n} e + \mathscr{O}_K \cdot f$. Continuing in this way, we must eventually reach $L_1$. $\square$

A string of points produced by this method is called an *apartment.* More generally, any path of vertices which is finite or half-infinite is called a *chain.* A *simple chain* is one where you never step forward and then back (or vice versa).[6]

*Remark.* These lattices satisfy a Noetherian-esque property: if you have an infinite chain of vertices $w_0 - w_1 - \cdots$, then there exist representative lattices $L_i$ for $w_i$ such that for all $i$, $L_i \supsetneq L_{i+1} \supsetneq \pi L_i$. $\blacktriangleleft$

**Proposition 2.7.** *For any simple chain $C$, there's a $g \in \mathrm{GL}_2(K)$ such that $g \cdot C$ is the chain*

$$\mathscr{O}_K \cdot e_1 + \mathscr{O}_K \cdot e_2 \supsetneq \mathscr{O}_K \cdot \pi e_1 + \mathscr{O}_K \cdot \pi e_2 \supsetneq \mathscr{O}_K \cdot \pi^2 e_1 + \mathscr{O}_K \cdot \pi^2 e_2 \supsetneq \cdots,$$

*where $\{e_1, e_2\}$ is the standard basis for $K^2$.*

**Corollary 2.8.** *$G$ is actually a tree.*

*Proof.* Assume $C$ is a simple chain that's a cycle in $G$. Then, Proposition 2.7 preserves connectiveness, but replaces it with something which could not possibly be a cycle. $\square$

*Proof sketch of Proposition 2.7.* Let the starting chain be $C = L_0 \supsetneq L_1 \supsetneq L_2 \supsetneq \ldots$. We know there's a $g \in \mathrm{GL}_2(K)$ such that $g_0 L_0$ is the starting vertex $\mathscr{O}_K \cdot e_1 + \mathscr{O}_K \cdot e_2$. So $g_0$ is our candidate. But we don't know whether $g_0 L_1$ is the same as $\mathscr{O}_K \cdot \pi e_1 + \mathscr{O}_K \cdot \pi e_2$, but there's a $g_1$ in the stabilizer of $g_0 L_0$ that moves it to $\mathscr{O}_K \cdot \pi e_1 + \mathscr{O}_K \cdot \pi e_2$. Then, inductively, one can assume there exists an element in the stabilizer of the first $i$ that brings the next element of the chain into position, and so on.[7] This inductive argument is a little delicate, and uses the fact that the resiude field is finite.

You may have to do this infinitely many times, which is actually fine: you can conjugate the $g_i$ such that

$$g_i = \begin{pmatrix} 1 & x_i \\ 0 & 1 \end{pmatrix}$$

for $x_i \in \mathfrak{m}^i$; then, the infinite product is

$$\begin{pmatrix} 1 & \sum x_i \\ 0 & 1 \end{pmatrix},$$

and, using the local topology of $K$, you can show this sum converges. $\square$

If you act by $\mathrm{SL}_2(K)$ (through the inclusion in $\mathrm{GL}_2(K)$), the parity of $m - n$ is preserved, so you can decompose the tree into a bipartite tree. From the geometric Satake perspective, this says that the affine Grassmannian for $\mathrm{PGL}_2(K)$ has two connected components.

Another fun fact is that $\mathrm{PSL}_2(\mathscr{O}_K)$ acts on the tree by graph automorphisms, and the double coset space

$$\mathrm{PSL}_2(\mathscr{O}_K) \backslash \mathrm{PSL}_2(K) / \mathrm{PSL}_2(\mathscr{O}_K)$$

is in bijection with the positive integers — well actually, the highest weights, or the irreducible representations of $\mathrm{SL}_2$. This already looks Langlandsy, and more of the story appears: you can define Hecke operators on the tree: for each $n \in \mathbb{N}$, let

$$T_n f(v) = \frac{1}{n} \sum_{|w-v|=n} f(w).$$

That is, the Hecke operator acts on the space of functions on the tree averages over things that are distance $n$ away.

**Theorem 2.9** ((Classical) Satake theorem)**.** *These Hecke operators $T_n$ commute, and generate an algebra isomorphic to the representation ring of* $\mathrm{SL}_2$.

[6] For example, the basic steps of salsa define chains, but not a simple chain; the basic steps of waltz, which return to the same point but after more than one step, are a simple chain.

[7] One way to think of this is that $\mathrm{GL}_2(K)$ is filtered by the discrete valuation; $g_0$ is the first piece, $g_1 g_0$ is the second piece, and so forth.

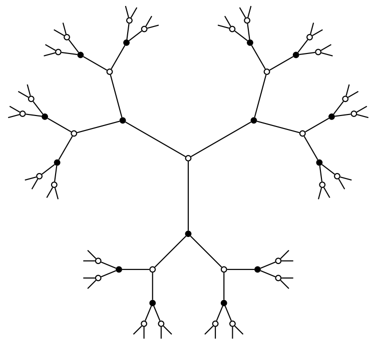

FIGURE 1. The Bruhat-Tits tree for $\mathrm{SL}_2(\mathbb{Q}_3)$. The two parity classes of vertices are in black and white. Source: `https://tex.stackexchange.com/a/135764`.

The geometric Satake theorem is a categorified analogue of this theorem. Indeed, the Bruhat-Tits tree appears in the story of the geometric Satake theorem as well, helping us understand the geometry of the affine Grassmannian in the case $G = \mathrm{PGL}_2$.

## 3. An introduction to the affine Grassmannian: 9/8/17

*"That's an affine Grassmannian you've got there. Shame if something happened to it."*

Today Rustam spoke about the affine Grassmannian, beginning with the uniformization of $G$-bundles as motivation from the topological case, then move to the algebraic world.

**Uniformization of $G$-bundles, topological setting.** Let $\Sigma$ be a compact, connected, oriented (real) surface and $G$ be a connected topological group. Let $x \in \Sigma$ and $\mathbb{D}$ be a small disc around it. Let $\mathbb{D}^0 := \mathbb{D} \setminus x$.

**Definition 3.1.** The *loop groups* of $G$ are:

- the *positive loop group* $L^{\mathrm{top},+}$, the continuous maps $f\colon \mathbb{D} \to G$, under pointwise multiplication;
- the *loop group* $L^{\mathrm{top}}(G)$, the continuous maps $f\colon \mathbb{D}^0 \to G$, and
- $L_X(G) := \{f\colon X \setminus x \to G\}$.

There's a restriction map $L^{\mathrm{top},+}(G) \to L^{\mathrm{top}}(G)$.

**Proposition 3.2.** *There's a bijection of sets*

$$L_X(G)\backslash L^{\mathrm{top}}(G)/L^{\mathrm{top},+}(G) \cong \mathrm{Bun}_G(X).$$

The idea is that $X \setminus x$ is homotopic to a graph, and since isomorphism classes of principal bundles are classified by maps to $BG$, and $BG$ is simply connected because $G$ is connected. Hence all principal $G$-bundles on $X \setminus x$ are trivial, so it matters crucially how we put $x$ back in, and then quotient by redundant data.

**Uniformization of $G$-bundles, algebraic setting.** Now, let $X$ be smooth, connected, projective curve over $\mathbb{C}$ and $G$ be a semisimple algebraic group. Let $x \in X$ be a $\mathbb{C}$-point and $\mathbb{D}_x := \mathrm{Spec}(\widehat{\mathscr{O}}_{X,z}) = \mathrm{Spec}(\mathbb{C}[[t]])$ be the formal disc around $x$, so $x \in \mathbb{D}_x \hookrightarrow X$. Hence, the punctured disc $\mathbb{D}_x \setminus x = \mathrm{Spec}\,\mathbb{C}((t))$.

**Definition 3.3.** The *loop groups* of $G$ are given by the following functors of points.[8]

- The *positive loop group* $L^+(G)(R) := \mathrm{Hom}(\mathbb{D}_R, G)$.
- The *loop group* is $L(G)(R) := \mathrm{Hom}(\mathbb{D}_x \setminus x, G)$.
- $L_X(G) := \mathrm{Hom}(X_R \setminus x \to G)$.

Hence $L(G)(\mathbb{C}) = G(\mathbb{C}((t)))$ and $L^+(G)(\mathbb{C}) = G(\mathbb{C}[[t]])$.

**Theorem 3.4** (Beauville-Laszlo). *There's an equivalence of stacks*

$$L_X G\backslash LG/L^+G \cong \mathrm{Bun}_G(X).$$

[8]Two of them are not representable as schemes; $L^+(G)$ is a scheme, and $L(G)$ is an ind-scheme but not a formal scheme. An *ind-scheme* is a filtered colimit of schemes, where all maps are closed embeddings, e.g. $\mathbb{A}^\infty$ or $\mathbb{P}^\infty$. $\mathbb{A}^\infty = \mathrm{Spec}\,\mathbb{C}[x_1, x_2, \dots]$ is also a scheme.

This is a hard proof: it doesn't follow from the usual descent arguments. Semisimplicity is crucial, and means that you can't apply this to $\mathrm{GL}_1$.

Let $\mathscr{O}_K := \mathbb{C}[[t]]$, which corresponds to the disc, and $K := \mathbb{C}((t))$, which corresponds to the punctured disc. Recall that the *affine Grassmannian* is $\mathrm{Gr}_G := L(G)/L^+(G) = G(K)/G(\mathscr{O}_K)$.[9]

If $G = \mathrm{GL}_n$, this is the same as the set of (full-rank) lattices on $K^n$: $G(K)$ acts transitively on all lattices, and the stabilizer is $G(\mathscr{O}_K)$.[10] For other groups, there's a similar description in terms of lattices.

This definition of the Grassmannian characterizes it as a set. Our first goal will be to give it a topology, at least in the case where $G = \mathrm{GL}_n$.

The first tool we'll use is a valuation: if $\vec{v}_1, \dots, \vec{v}_n$ is a basis for a lattice $\Lambda$, then $\det(\vec{v}_1, \dots, \vec{v}_n) \in K^\times$. This is not an invariant of $\Lambda$, but can be made into one.

**Definition 3.5.** Let $\Lambda$ be a lattice. Its *valuation* $v(\Lambda) \in \mathbb{N}$ is the minimum $n$ such that $t^n$ is the determinant of a basis of $\Lambda$.

For the trivial lattice $\Lambda^0 := (\mathscr{O}_K)^n$, $n = 2$. Also, if you do this for $G = \mathrm{GL}_1$, the affine Grassmannian is literally $\mathbb{Z}$; you can think of the valuation as a kind of determinant map from $\mathrm{GL}_n \to \mathrm{GL}_1$, and hence to $\mathrm{Gr}_{\mathrm{GL}_1} = \mathbb{Z}$.

Our next tool will be to compare lattices with the standard lattice.

**Lemma 3.6.** *For all lattices $\Lambda$, there's an $a \in \mathbb{N}$ such that*

$$t^a\Lambda^0 \subseteq \Lambda \subseteq t^{-a}\Lambda^0. \tag{3.7}$$

This will allow us to get the ind-structure.

**Definition 3.8.** Let $\mathrm{Gr}_{\mathrm{GL}_n}^{\ell,a}$ be the set of lattices $\Lambda \in \mathrm{Gr}_{\mathrm{GL}_n}$ such that $v(\Lambda) = \ell$ and (3.7) is satisfied.

Then $\mathrm{Gr}^{\ell,a} \hookrightarrow \mathrm{Gr}^{\ell,b}$ if $b \geq a$, and

$$\mathrm{Gr}_{\mathrm{GL}_n} = \bigcup_{\ell,a} \mathrm{Gr}_{\mathrm{GL}_n}^{\ell,a}.$$

Let $J_{a,n}$ denote the space of $(na - k)$-dimensional subspaces of $t^{-a}\Lambda^0/t^a\Lambda^0 \cong \mathbb{C}^{2an}$. Then, $\mathrm{Gr}_{\mathrm{GL}_n}^{\ell,a}$ embeds into $J_{a,n}$ by taking the quotient by $t^a\Lambda^0$, and $J_{a,n}$ is a Grassmannian! By the Plücker embedding, it's a projective variety.

**Definition 3.9.** An *ind-projective ind-variety* is an ind-scheme $X = \mathrm{colim}_i\, X_i$ such that each $X_i$ is projective variety.

*Remark.* In schemes, there's a significant difference between limits and colimits. Limits of affine schemes always exist, because colimits of rings do. For example, $\mathbb{A}^\infty = \operatorname{Spec} \mathbb{C}[x_1, x_2, \dots] \cong \varprojlim \mathscr{O}_K/t^n\mathscr{O}_K$. But ind-schemes are not always schemes. For example, $K$, the functor $R \mapsto R((t))$, is the colimit of $K^{\geq -N} \cong \mathbb{A}^\infty$. This is the case where you're only allowed to have finitely many coordinates. Said another way, the positive part of the Laurent series in the affine Grassmannian is fine, albeit infinite-dimensional; the negative tails produce the ind-ness.

For a concrete example, $\mathrm{Gr}_{\mathrm{GL}_1} \cong \mathbb{Z}$, and this is an infinite disjoint union of points, a nice ind-projective ind-variety. You might try to realize it as Spec of an infinite direct product of $\mathbb{C}$s, but these are not isomorphic! The correspondence between coproducts of schemes and products of rings only works fully at the finite level. ◂

**Theorem 3.10.** *The affine Grassmannian $\mathrm{Gr}_{\mathrm{GL}_n}$ is an ind-projective ind-variety.*

The idea is to use the $J_{a,n}$.

There's an important stratification of the affine Grassmannian: using Gauss-Jordan elimination, the orbits of $\mathrm{GL}_n(\mathscr{O}_K)$ on $\mathrm{Gr}_{\mathrm{GL}_n}$ are identified with coweights $\mathbb{C}^\times \to T$ (where $T$ is a maximal torus for our group).

[9]The name comes from the affine Weyl group because it has translations in it; the affine Grassmannian is not affine in any usual sense. For this reason, it's sometimes referred to as the *infinite Grassmannian* or the *loop Grassmannian.*

[10]These are just like lattices in $\mathbb{R}$: a lattice $\Lambda$ is an $\mathscr{O}_K$-submodule of $K^n$ such that $\Lambda \otimes_{\mathscr{O}_K} K = L^n$.

The set of coweights is often denoted $X_\bullet(T)$. Specifically,

$$\mathrm{GL}_n(K) = \coprod_{\substack{\lambda=(a_1,\dots,a_n)\in\mathbb{Z}\\ a_1\ge\cdots\ge a_n}} \mathrm{GL}_n(\mathscr{O}_K)\begin{pmatrix} t^{a_1} & & \\ & \ddots & \\ & & t^{a_n}\end{pmatrix}\mathrm{GL}_n(\mathscr{O}_K). \tag{3.11}$$

The idea: using row reduction, you can get rid of everything except for powers of $t$ (since you're only using Taylor series, not Laurent series). There's a similar perspective for other groups $G$, for which one might write

$$G(K) = \coprod_{\lambda\in X_\bullet(T)_+} G(\mathscr{O}_K)t^\lambda G(\mathscr{O}_K). \tag{3.12}$$

One can identify $X_\bullet(T)_+ \cong X_\bullet(T)/W$, where $W$ is the Weyl group for $G$.[11]

This decomposition is really nice: the orbits are all projective varieties. There's a nice Morse-theoretic approach to all this.

The decomposition (3.12) is sort of a "set-theoretic Satake theorem:" the Langlands dual is a little implicit, but tells us that finite-dimensional irreducible representations of $G^\vee$ are indexing this decomposition of the affine Grassmannian. For $G = \mathrm{GL}_n$, which is Langlands self-dual, (3.11) can also be interpreted as indexed by the representations of $\mathrm{GL}_n$.

**The algebro-geometric approach to the affine Grassmannian.** The affine Grassmannian admits a functor-of-points definition, as the functor $\mathrm{Gr}\colon \mathsf{Alg}_\mathbb{C} \to \mathsf{Set}$ sending a $\mathbb{C}$-algebra $R$ to the set of finitely-generated projective $R[[t]]$-submodules $\Lambda$ of $R((t))^n$ such that

$$\Lambda \otimes_{R[[t]]} R((t)) = R((t))^n.$$

**Theorem 3.13.** *This is represented by an ind-projective ind-scheme.*

This scheme is isomorphic to the one described in Theorem 3.10. You'd prove Theorem 3.13 by finding a cover by subfunctors that are represented by projective varieties and whose colimit is Gr again.

We also get a functor-of-points approach to the orbit decomposition (3.11): $\mathrm{Gr}_{\mathrm{GL}_n}(R)$ is the set of $(\Sigma,\beta)$ where $\Sigma$ is a vector bundle on $\mathbb{D}_R$ and $\beta\colon \Sigma|_{\mathbb{D}_R^0} \to \underline{\mathbb{C}}^n$ is an isomorphism with the trivial bundle. If $Y\subset X$, let $\mathrm{Bun}_G(X,Y)$ denote the set of principal $G$-bundles on $X$ together with data of a trivialization on $Y$, then we can replace vector bundles with principal $G$-bundles to obtain a more general description:

$$\mathrm{Gr}_G(R) = \mathrm{Bun}_G(\mathbb{D}_R, \mathbb{D}_R^0).$$

*Remark.* Unlike in algebraic topology, defining principal $G$-bundles is tricky: if you try things which are Zariski-locally $G$-torsors, you get the wrong thing. Keeping in mind that it should always be possible to form an associated vector bundle from a principal $G$-bundle and a $G$-representation, followed by some messing around with Grothendieck topologies, leads to the right notion. ◀

For general $G$, $\mathrm{Gr}_G$ is an ind-projective ind-scheme; the proof idea is to embed $G\hookrightarrow \mathrm{GL}_n$.

The $R$-points of $L(G)$ are pairs $(\Sigma,\beta)\in\mathrm{Bun}_G(\mathbb{D}_R,\mathbb{D}_R^0)$ together with a trivialization $\varepsilon\colon\Sigma\to\underline{G}$ on all of $\mathbb{D}_R$. Forgetting $\varepsilon$ defines the quotient map to $\mathrm{Gr}_G$; if you think about it, you'll find that this is actually the quotient by $L^+(G)$.

**Back to topology.** There's a topological version of this story — in topology, the Grassmannian arises in a very different way. Let $G$ be a complex Lie group and $K$ be its maximal compact subgroup, so $K$ is homotopic to $G$.

Let $\Omega K$ denote the (based) loop space of $K$, the space of basepoint-preserving, continuous maps $S^1\to K$.[12] Then, there's a model for the affine Grassmannian $\mathrm{Gr}_G$ that's homotopic to $\Omega K$.

Another way to think of this is to let $LK = \mathrm{Map}(S^1,K)$ be the free loop space. Then, $\Omega K = LK/K$, and this is like taking the quotient of $L(G)/L^+(G)$.

Now, we want this to be a moduli space of something. Well, $\Omega K\simeq\Omega^2 BK$, where $BK$ is the homotopy type such that $\mathrm{Map}(X,BK)$ is naturally $\mathrm{Bun}_K(X)$ (as a set). This is representable, which is a theorem.

[11] $X_\bullet(T)_+$ is the set of *dominant weights*: given a chamber $C$, the set of weights that pair positively with the elements of that chamber.

[12] See Pressley-Segal for details.

Two-fold loops in $Y$ are identified with maps from a disc into $Y$ such that (a small neighborhood of) the boundary maps to the basepoint. Hence $\Omega K \simeq \Omega^2 BK$ is the set of $K$-bundles on $\mathbb{C} = \mathbb{R}^2$ that are pointed, in that there's extra data of a trivialization on $\mathbb{C} \setminus \mathbb{D}$, where $\mathbb{D}$ is a small disc. This is precisely what we said the affine Grassmannian was: $G$-bundles trivialized at a point. So the algebraic geometry and the homotopy theory are the same – since $BK \simeq BG$, you could also take the set of $G$-bundles on a disc trivialized at a point.

The fact that $\Omega K$ is a 2-fold loop space means that it's a homotopical kind of abelian group, which is key – it means the affine Grassmannian is in some sense an abelian group. For example, for $G = \mathrm{GL}_1$, $\Omega^2 BS^1 = \mathbb{Z}$, which *is* a group. This groupiness ($E_2$-structure) will be crucial to the geometric Satake theorem.

The description of the affine Grassmannian as based loops on a compact Lie groups is what enabled Bott and others to attack it with Morse theory.

## 4. Geometry of the Affine Grassmannian: 9/15/17

*"Now I have a blank page, which is where I wrote the examples."*

Last time, we spent some time defining the affine Grassmannian and thinking about it from a functor-of-points perspective. Today we want to focus on its geometry, and how we're going to think about it; this will be more useful than being vaguely afraid that it's some terrifying infinite-dimensional space.

**Hecke operators.** We saw in Tom's talk that there are Hecke operators which act on the Bruhat-Tits tree, and that they commute. Hecke operators exist in much greater generality, but are usually a hair more complicated.

Fix a curve $C$ and reductive group $G$; then,

$$\mathscr{H}\!ecke \coloneqq \{(c, P_1 \xrightarrow{G} C, P_2 \xrightarrow{G} C, \beta\colon P_1|_{C\setminus c} \xrightarrow{\cong} P_2|_{C\setminus c})\}.$$

That is, $\mathscr{H}\!ecke$ is the pairs of principal $G$-bundles and isomorphisms between them away from a point.

There is a correspondence

$$\begin{tikzcd} & \mathscr{H}\!ecke \arrow[dl] \arrow[dr] & \\ C \times \mathrm{Bun}_G(C) & & \mathrm{Bun}_G(C), \end{tikzcd}$$

by forgetting $(P_2, \beta)$, resp. $(c, P_1, \beta)$. Let $\underline{G}$ denote the trivial principal $G$-bundle; then, for any $c \in C$, the point $(c, \underline{G}) \in C \times \mathrm{Bun}_G(C)$. Using this, we can define the affine Grassmannian, and $\mathscr{H}\!ecke$ contains $\mathrm{Gr}_G$ in a way compatible with the forgetful map $\mathscr{H}\!ecke \to C \times \mathrm{Bun}_G(C)$.

If you wanted to do this with nontrivial principal $G$-bundles, you would obtain some kind of twisted version of the affine Grassmannian. Twisted means sheaves, so one might imagine that some class of sheaves on $\mathrm{Gr}_G$ act on some class of sheaves on $\mathrm{Bun}_G$. This is true, and important: we'll learn about perverse sheaves this semester, and they will act on $\mathcal{D}$-modules on $\mathrm{Bun}_G$. In this sense, geometric Satake acts on geometric Langlands.

There is a classical perspective on this. Instead of $\mathscr{H}\!ecke$, we consider the space of data $(c, E, \ell)$, where $c \in C$, $E \to C$ is a rank-$n$ vector bundle, and $\ell$ is a line in $E|_c$. This fibers over $C \times \mathrm{Bun}_n(C)$ (where $\mathrm{Bun}_n$ is the moduli space of vector bundles), and the fiber is $\mathbb{P}^{n-1}$, which is contained in the affine Grassmannian. You can think of this $\mathbb{P}^{n-1}$ as all of the modifications you can make to your vector bundle in a neighborhood of a point when you remove that point, and the Hecke-like operators act as averaging, like for the Bruhat-Tits tree. These are called *elementary modifications* in algebraic geometry.

**Some (not too hard) combinatorics.** This is a lot of data, but it will all be useful soon.

Let $G$ be a reductive group, $B$ be a Borel subgroup, and $T$ be a maximal torus, such that $T \subset B \subset G$.

**Example 4.1.** If you chose $G = \mathrm{GL}_n$, then one can let $B$ be the subgroup of upper triangular matrices and $T$ be the subgroup of diagonal matrices. ◀

Let $\Phi$ denote the set of *roots* of $(G, T)$, with respect to the adjoint action of $T$ on $\mathfrak{g}$. That is, $G$ acts on its Lie algebra $\mathfrak{g}$ by conjugation, and the action of $t \in G$ on $x \in \mathfrak{g}$ is denoted $x \mapsto \mathrm{Ad}_t(x)$. Under this action, $\mathfrak{g}$ decomposes as a direct sum

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

Here, $\mathfrak{t}$ is called the *zero-weight space*. Concretely, we realize $\Phi$ as a subset of $\mathfrak{t}^*$.

The choice of torus and Borel subgroup defines for us a decomposition $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is the Lie algebra of the normalizer of the maximal torus.[13]

**Definition 4.2.** We introduce the following notation.

- Let $\Phi^+ \subset \Phi$ be the *positive roots*, i.e. those in $\mathfrak{b} := \mathrm{Lie}(B)$.
- $\Phi_s \subset \Phi^+$ will denote the *simple roots*, those which are not sums of other positive roots in a nontrivial way.
- Let $X^*(T) := \mathrm{Hom}(T, \mathbb{G}_m)$, the *character lattice*, which contains $\Phi$.
- Let $X_*(T) := \mathrm{Hom}(\mathbb{G}_m, T)$, the *cocharacter lattice*, which contains $\Phi^\vee$, the coroots (see below).

For each root $\alpha$, there exists a *coroot* $\alpha^\vee \in X_*(T)$ satisfying

$$\alpha \cdot \alpha^\vee = z^2,$$

where $\alpha \cdot \lambda(z) := z^{\langle \alpha, \lambda \rangle}$.

**Definition 4.3.** More notation:

- Let $Q := \mathbb{Z}\Phi$ be the *root lattice.*
- Let $Q^\vee := \mathbb{Z}\Phi^\vee$, the *coroot lattice.*

There's a partial order on $Q^\vee$, extending to $X_*(T)$, where $0 \leq \lambda$ iff $\lambda$ is a nonnegative combination of positive coroots.

**Definition 4.4.** The *dominant cocharacters* are

$$X_*(T)_+ := \{\lambda \in X_*(T) \mid \langle \lambda, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Phi^+\}.$$

Finally, and most mysteriously, let

$$\rho := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha,$$

which defines a function $X_*(T) \to \frac{1}{2}\mathbb{Z}$. The appearance of $\rho$ is a subtle and mysterious aspect of the representation theory of compact Lie groups or algebraic groups; David has a MathOverflow post with a good explanation.[14]

Okay, now for some examples.

**Example 4.5.** Let $G = \mathrm{SL}_2(\mathbb{C})$. We can then choose

$$\begin{aligned} B &= \left\{ \begin{pmatrix} \lambda & \mu \\ 0 & \lambda^{-1} \end{pmatrix} \right\} \\ T &= \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \right\}. \end{aligned} \tag{4.6}$$

The roots are $\Phi = \{\pm 2\}$ corresponding to the functions

$$\pm\alpha\colon \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix} \longmapsto \lambda^{\pm 2}.$$

The character lattice is hence $Q = 2\mathbb{Z} \subset \mathbb{Z} \cong X^*(T)$, where $n$ is sent to the $n^{\text{th}}$ power map.

The coroots are $\{\pm 1\}$, corresponding with the functions

$$\pm\alpha^\vee\colon \lambda \longmapsto \begin{pmatrix} \lambda^{\pm 1} & \\ & \lambda^{\mp 1} \end{pmatrix}.$$

In this case, $X_*(T) \cong \mathbb{Z}$ ($n$ sends $\lambda \mapsto \left(\begin{smallmatrix} \lambda^n & \\ & \lambda^{-n} \end{smallmatrix}\right)$), and the cocharacter lattice $Q^\vee$ is all of $X_*(T)$, even though the character lattice isn't all of $X^*(T)$!

In this case, the dominant cocharacters are $X_*(T)_+ \cong \mathbb{Z}_{\geq 0}$, with the partial order sent to the usual order. In this case, $\rho = (1/2) \cdot 2 = 1$. ◄

[13] TODO: this might be wrong; double-check.

[14] There are many perspectives on $\rho$: spin structures on the flag manifold, Serre duality, the Weyl character formula, and more.

**Example 4.7.** Now consider $G = \mathrm{PGL}_2(\mathbb{C}) \coloneqq \mathrm{SL}_2(\mathbb{C})/\mu_2$, where $\mu_2 \coloneqq \{\pm 1\}$ is the square roots of unity. We can choose $B$ and $T$ to be basically the same as in (4.6). In this case, we obtain a dual-looking setup (which is not a coincidence).

- The character lattice is $Q \cong \mathbb{Z}$, and this is all of
$$X^*(T) = \left\{ \left[ \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix} \right] \longmapsto \lambda^{2n} \right\}_{n \in \mathbb{Z}}.$$
- The cocharacter lattice is $Q^\vee \cong 2\mathbb{Z}$ inside
$$X_*(T) = \left\{ \lambda \longmapsto \left[ \begin{pmatrix} \lambda^{n/2} & \\ & \lambda^{-n/2} \end{pmatrix} \right] \right\}_{n \in \mathbb{Z}}.$$ $\blacktriangleleft$

**Applications to geometry of the affine Grassmannian.** As before, fix $K \coloneqq \mathbb{C}((t))$ and $\mathscr{O}_K \coloneqq \mathbb{C}[[t]]$.

Let $\lambda \colon \mathbb{G}_m \to T$ be a cocharacter for $G$, which determines a map $K^\times \to T((t)) \subset G(\mathscr{O}_K)$ sending $t \mapsto t^\lambda$.

Let $L_\lambda \coloneqq t^\lambda \cdot G(\mathscr{O}_K) \in \mathrm{Gr}_G$ (since this is $G(K)/G(\mathscr{O}_K)$), and define $\mathrm{Gr}_\lambda \coloneqq G(\mathscr{O}_K) \cdot L_\lambda$. This $\mathrm{Gr}_\lambda$ is called a *Schubert cell.* Last time, in (3.11) and (3.12), we saw a decomposition of $\mathrm{Gr}_G$ into a disjoint union; this is actually as Schubert cells:
$$\mathrm{Gr}_G = \coprod_{\lambda \in X_*(T)_+} \mathrm{Gr}_\lambda.$$
Let
$$\mathrm{Gr}_{\leq \mu} \coloneqq \bigcup_{\substack{\lambda \leq \mu \\ \lambda \in X_*(T)_+}} \mathrm{Gr}_\lambda,$$
which is called a *(spherical) Schubert variety* containing $\mathrm{Gr}_\mu$ as an open subvariety.

**Proposition 4.8.** *$\mathrm{Gr}_\mu$ is a smooth, quasiprojective*[15] *variety of dimension $\langle 2\rho, \mu \rangle$.*

So $\rho$ makes its first surprise appearance (as the sum of all the positive roots).

*Proof.* Let
$$P_\mu^\alpha \coloneqq G(\mathscr{O}_K) \cap (t^\mu G(\mathscr{O}_K) t^{-\mu}),$$
which is the stabilizer for $G(\mathscr{O}_K)$ acting on $\mathrm{Gr}_G$: $\mathrm{Gr}_\mu \cong G(\mathscr{O}_K)/P_\mu^a$. Thus $\mathrm{Gr}_\mu$ is a homogeneous space for a group in characteristic 0, so it's quasiprojective. Smoothness comes from the map
$$G(\mathscr{O}_K)/P_\mu^a \longrightarrow G(K)/G(\mathscr{O}_K)$$
sending $g \mapsto g^{t^\mu}$, so it remains to calculate its dimension.

We have an identification
$$T_{L_\mu} \mathrm{Gr}_\mu \cong \mathfrak{g}(\mathscr{O}_L)/(\mathscr{G}(\mathscr{O}_K) \cap \mathrm{Ad}_{t^\mu}(\mathfrak{g}(\mathscr{O}_K))),$$
and we have the root decomposition
$$\mathfrak{g}(\mathscr{O}_K) =^{\mathrm{fr}} (\mathscr{O}_K) \oplus \bigoplus_{\alpha \in \Phi^+} (\mathfrak{g}_\alpha(\mathscr{O}_K) \oplus \mathfrak{g}_{-\alpha}(\mathscr{O}_K)).$$
For $\mathfrak{t}$, which is abelian, the adjoint action is trivial, and for $x_\alpha \in \mathfrak{g}_\alpha$,
$$\mathrm{Ad}_{t^\mu}(x_\alpha) = t^{\langle \alpha, \mu \rangle} x_2.$$
Therefore we can calculate
$$\mathrm{Ad}_{t^\mu} \mathfrak{g}(\mathscr{O}_K) = \mathfrak{t}(\mathscr{O}_K) \oplus \left( \bigoplus_{\alpha \in \Phi^+} t^{\langle \alpha, \mu \rangle} \mathfrak{g}_\alpha(\mathscr{O}_K) \oplus t^{-\langle \alpha, \mu \rangle} \mathfrak{g}_{-\alpha}(\mathscr{O}_K) \right). \tag{4.9}$$
Hence
$$\mathfrak{g}(\mathscr{O}_K) \cap \mathrm{Ad}_{t^\mu}(\mathfrak{g}(\mathscr{O}_K)) = \mathfrak{t}(0) \oplus \bigoplus_{\alpha \in \Phi^+} (\mathfrak{g}_\alpha(\mathscr{O}_K) \oplus \mathfrak{g}_{-\alpha}(\mathscr{O}_K)). \tag{4.10}$$

[15] This means it's an open subvariety of a projective variety

Therefore

$$\mathfrak{g}(\mathscr{O}_K)/(\mathfrak{g}(\mathscr{O}_K)\cap \mathrm{Ad}_{t^\mu}\,\mathfrak{g}(\mathscr{O}_K)) \cong \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha(\mathscr{O}_K)/t^{\langle\alpha,\mu\rangle}\mathfrak{g}_\alpha(\mathscr{O}_K). \tag{4.11}$$

Since $\dim(\mathfrak{g}_\alpha) = 1$, the dimension of (4.11) is

$$\sum_{\alpha\in\Phi^+} \langle\alpha,\mu\rangle = \langle 2\rho,\mu\rangle. \qquad \square$$

Bam, geometry.

We also know its closure, which is a projective variety, though far from smooth, except in very special and very interesting cases.

**Proposition 4.12.** *The Zariski closure of* $\mathrm{Gr}_\mu$ *is* $\mathrm{Gr}_{\le\mu}$*, which in particular is a projective variety.*

*Proof sketch.* Suppose $\lambda \ge \mu$, so that there exists a positive coroot $\alpha^\vee$ such that $\mu - \alpha^\vee$ is dominant and $\lambda \le \mu - \alpha^\vee \le \mu$. This means it suffices to prove that $t^{\mu-\alpha^\vee} \in \overline{\mathrm{Gr}_\mu}$. To do this, construct a curve $X \subset \mathrm{Gr}_{\le\mu}$ such that $t^{\mu-\alpha^\vee} \in X$ and $X \cdot \{t^{\mu-\alpha^\vee}\} \subset \mathrm{Gr}_\mu$. $\square$

**Example 4.13.** Let $G = \mathrm{SL}_2$. For any $m \in \mathbb{Z}$,

$$t^{\lambda_m} = \left[\begin{pmatrix} t^m & 0 \\ 0 & 1\end{pmatrix}\right] \in \mathrm{PGL}_2(K).$$

Let $K_m \coloneqq \mathrm{Ad}_{t^{\lambda_m}}(\mathrm{SL}_2(\mathscr{O}_K)) \subset \mathrm{SL}_2(K)$. The idea is that we want to use copies of $\mathrm{SL}_2(\mathscr{O}_K)$ to move things around in $\mathrm{SL}_2(K)$.

Let $\{e,h,f\}$ be the standard triple of generators in $\mathfrak{sl}_2$, so if $\mathfrak{k}_m$ is the Lie algebra of $K_m$, $\mathfrak{k}_m = \mathrm{span}_{\mathscr{O}_K}\{t^m e, h, t^{-m} f\}$.

Consider the point

$$\sigma_m \coloneqq \begin{pmatrix} 0 & -t^m \\ t^m & 0\end{pmatrix} \in K_m$$

and

$$L^{>0}\mathrm{SL}_2 \coloneqq \ker(ev\colon \mathrm{SL}_2(\mathscr{O}_K) \to \mathrm{SL}_2(\mathbb{C})).$$

Let $K_m^{(1)} \coloneqq \mathrm{Ad}_{t^{\lambda_m}}(L^{>0}\mathrm{SL}_2)$.

One can show that $K_m/K_m^{(1)} \cong \mathrm{SL}_2(\mathbb{C})$, so if $i_\alpha\colon \mathrm{SL}_2(\mathbb{C}) \to G$ is the canonical homomorphism induced by $\alpha \in \Phi$, the Lie algebra of its image is $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$.

Now, let $m = \langle\mu,\alpha\rangle - 1$. Let

$$C_{\mu,\alpha} \coloneqq (Li_\alpha)(K_m)\cdot L_\mu.$$

This is a homogeneous space for $\mathrm{SL}_2(\mathbb{C})$, hence must be a $\mathbb{P}^1$, and this is the curve we were looking for (and punted on in the proof of Proposition 4.12).

**Exercise 4.14.** Show that if $B$ is the subgroup of upper triangular matrices inside $\mathrm{SL}_2(\mathbb{C})$, $\mathrm{SL}_2(\mathbb{C})/B \cong \mathbb{P}^1$.

Therefore one can make a *balloon decomposition* of the entire space as a sequence of $\mathbb{P}^1$ "balloons," akin to a Morse decomposition of a manifold. There's a great paper of Goresky-MacPherson which shows how incredibly useful this is. $\blacktriangleleft$

**Parity of Schubert cells.** The Schubert cells have a natural even-odd parity directly analogous to the parity in the Bruhat-Tits tree for $\mathrm{SL}_2$, defined by the map $p\colon X_*(T) \to \mathbb{Z}/2$ sending

$$\mu \longmapsto (-1)^{\langle 2\rho,\mu\rangle}.$$

If $\mu$ is a coroot, so that $\langle\rho,\mu\rangle \in \mathbb{Z}$, this defines a map $X_*(T)/Q^\vee \to \mathbb{Z}/2$. But $X_*(T)/Q$ is something we've seen before: $\pi_1(G)$![16] This is because $\pi_1(G) \cong \pi_0(\mathrm{Gr}_G)$, because $\pi_1(G) = \pi_0(LG)$, and $\mathrm{Gr}_G = LGL^+G$ and $L^+G$ is connected.

Therefore we have a map $p\colon \pi_0(\mathrm{Gr}_G) \to \mathbb{Z}/2$, which decomposes $\mathrm{Gr}_G$ into even and odd pieces $\mathrm{Gr}_G^{(+)}$ and $\mathrm{Gr}_G^{(-)}$. We say $\mathrm{Gr}_\mu$ is *even*, resp. *odd*, if it's in $\mathrm{Gr}_G^{(+)}$, resp. $\mathrm{Gr}_G^{(-)}$. By (4.8), $\mathrm{Gr}_\mu$ is even (resp. odd) iff $\dim \mathrm{Gr}_\mu$ is even (resp. odd). This means the parity map doesn't depend on the choice of $B$ or $T$, which is nice.

[16]There are various notions of the fundamental group in algebraic geometry, but we're over $\mathbb{C}$, so they coincide here. One concrete definition is the fundamental group of $G(\mathbb{C})$ in the complex topology.

**Schubert cells and partial flag varieties.** For $\mu \in X_*(T)$, let $P_\mu$ be the *parabolic subgroup* of $G$ corresponding to $\mu$. That is, $P_\mu$ is generated by the root subgroups $U_\alpha \subset G$ (isomorphic to $\mathrm{SL}_2$) for $\alpha \in \Phi$ satisfying $\langle \alpha, \mu \rangle \leq 0$.

Let $ev\colon G(\mathscr{O}_K) \to G$ send $g(t) \mapsto g(0)$ again.

**Proposition 4.15.** *There is a natural projection*

$$p_\mu\colon \mathrm{Gr}_\mu \cong G(\mathscr{O}_K)/\underbrace{(G(\mathscr{O}_K) \cap (t^\mu G(\mathscr{O}_K) t^{-\mu}))}_{P_\mu^a} \longrightarrow G/P_\mu$$

*whose fibers are affine spaces.*

The quotient $G/P_\mu$ is called a *partial flag variety*, and generalizes the flag variety $G/B$, the variety of full flags, filtrations

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = V$$

such that $\dim V_i = i$. But we could just as well take partial flags $0 \subsetneq V_1 \subsetneq V_2 \subsetneq V_n = V$, and this is $\mathrm{GL}_n$ over the subgroup of upper triangular matrices. Hence these can and should be called partial flag varieties.

*Proof of Proposition 4.15.* Recall (4.9) and (4.10), the formulas for $\mathrm{Ad}_{t^\mu} \mathfrak{g}(\mathscr{O}_K)$ and $\mathfrak{g}(\mathscr{O}_K) \cap \mathrm{Ad}_{t^\mu} \mathfrak{g}(\mathscr{O}_K)$, respectively; evaluation defines a map

$$\mathfrak{g}(\mathscr{O}_K) \cap \mathrm{Ad}_{t^\mu} \mathfrak{g}(\mathscr{O}_K) \xrightarrow{ev} \mathfrak{t} \oplus \bigoplus_{\langle \alpha, \mu \rangle \leq 0} \mathfrak{g}_\alpha,$$

and the latter space is the Lie algebra of $P_\mu$. That is, we have a commutative diagram

$$\begin{tikzcd}
G(\mathscr{O}_K) \arrow[r, "ev"] & G \\
P_\mu^a \arrow[u, hook] \arrow[r, "ev_\mu"] & P_\mu. \arrow[u, hook]
\end{tikzcd}$$

$\ker(ev) = L^{>0}G$, which is a pro-unipotent group (a limit of unipotent groups). This implies $\ker(ev)/\ker(ev_\mu)$ is a finite-dimensional unipotent group, and these are all affine spaces. ☐

**Example 4.16.** Let $G = \mathrm{PGL}_2(\mathbb{C})$, so as in Example 4.7, $X_*(T) = \mathbb{Z}$ and $Q^\vee = 2\mathbb{Z}$ inside $X^*(T)$. The partial order is

$$\cdots \leq -2 \leq 0 \leq 2 \leq \cdots$$
$$\cdots \leq -3 \leq -1 \leq 1 \leq 3 \leq \cdots$$

and $X_*(T)_+ = \mathbb{Z}_{\geq 0}$.

Now let's make some Schubert cells. The easiest is $\mathrm{Gr}_0$:

$$\mathrm{Gr}_0 = G(\mathscr{O}_K) \cdot 1 \cdot G(\mathscr{O}_K) = G(\mathscr{O}_K) = \mathrm{pt}.$$

More excitingly,

$$Gr_1 = \mathrm{Gr}_{\leq 1} = G(\mathscr{O}_K) \cdot \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \cdot G(\mathscr{O}_K),$$

which is a $\mathbb{P}^1$. This is all of the modifications one can make in the neighborhood of something, like we saw for the Bruhat-Tits tree.

This demonstrates an interesting phenomenon: the following are equivalence (for any reductive $G$):

- $\mathrm{Gr}_\mu$ is smooth.
- $\mathrm{Gr}_\mu = \overline{\mathrm{Gr}_\mu}$.
- $\mu$ is such that $\langle \mu, \alpha \rangle \leq 1$, where $\alpha \in \Phi_+$ is a *miniscule* root.

Some groups don't even have miniscule roots, so smoothness is generally very uncommon.

Though it's complicated to write out at the board, the answer for $\mathrm{Gr}_2$is that $\mathrm{Gr}_2 \cong \mathscr{O}(-n) \to \mathbb{P}^1$ for some $n$, which is indeed an affine bundle. If you think in terms of lattices, these are lattices containing some given lattice for $\mathrm{Gr}_1$. Similarly, $\mathrm{Gr}_3$ will be a fiber bundle over the total space for $\mathrm{Gr}_2$, which is a variety you may have encountered before. (For general groups, everything is a little more complicated.)

As soon as we understand an orbit, we can calculate its cohomology, and these have natural $\mathrm{SL}_2$-actions. The cohomology of $\mathrm{Gr}_1$ is the irreducible 1-dimensional representation (the trivial one), and the cohomology of $\mathrm{Gr}_2$ is the two-dimensional one. There's plenty of excellent geometry here. ◂

## 5. Perverse sheaves: 9/22/17

*"I thought, I understood sheaves, so all I have to understand is one adjective! But it turns out perverse sheaves aren't really sheaves, and they're also not*

Today, Sebastian spoke about perverse sheaves and intersection cohomology. Today is intended to be a less scary and more heuristic overview of why we want to consider perverse sheaves, with the technical details left to a future talk. Today, all sheaves are valued in $\mathbb{Q}$-vector spaces, and we assume all spaces are locally path-connected and locally simply connected, so that we can take universal covers of path-connected spaces.

To begin, let's recall a definition.

**Definition 5.1.**

- A *locally constant sheaf* $\mathcal{L}$ on a space $X$ is a sheaf such that $X$ such that for every $x \in X$, there's a neighborhood $U$ of $x$ such that $\mathcal{L}(U) \cong \mathcal{L}_x$.
- A *local system* is a locally constant sheaf with finite-dimensional stalks.

**Example 5.2.**

(1) Let $V$ be a vector space. Then, the *constant sheaf* $\underline{V}_X$ is a locally constant sheaf, and if $V$ is finite-dimensional, is a local system.
(2) The *orientation sheaf* $O_X$, defined by $O_X(U) \coloneqq H_n(U;\mathbb{Q})^*$, is a locally constant sheaf, and a local system on nice spaces. ◂

Given a locally constant sheaf $\mathcal{L}$ and an $x_0 \in X$, there's a *monodromy representation* of $\pi_1(X, x_0)$ on $\mathcal{L}_{x_0}$: given a loop $\ell$ from $x_0$ to itself, consider an open cover $\mathfrak{U}$ of a neighborhood of $\ell$, such that we get a chain of transition maps $\mathcal{L}(U_0) \to \mathcal{L}(U_1) \to \cdots \to \mathcal{L}(U_m) \to \mathcal{L}(U_0)$; the composition of these is the automorphism we assign to $\ell$, which is called its *monodromy*, and one can check this defines a group homomorphism $\pi_1(X, x_0) \to \mathrm{GL}(\mathcal{L}_{x_0})$.

*Remark.* Really, this defines a functor from the category of local systems on $X$, as a full subcategory of $\mathsf{Sh}(X, \mathbb{Q})$, to the category of finite-dimensional representations of $\pi_1(X, x_0)$, and this functor is an equivalence of categories! ◂

One can also ask about extensions of local systems: if $U \subset X$ is open, where $X$ is a smooth manifold, and $\mathcal{L}$ is a local system on $U$, can we extend it to $X$? Is such an extension unique?

Using the equivalence of categories mentioned above, we can find a few easy-to-understand criteria.

(1) If $\ell$ is a loop in $U$ that's contractible in $X$, the monodromy around $\ell$ of $\mathcal{L}$ must be trivial.
(2) We need $U$ to be "big enough," in that $\pi_1(U)$ surjects onto $\pi_1(X)$. This is automatic if $X \setminus U$ has codimension at least 2.

This is great, but what if $X$ is singular? For concreteness, take a stratified space $X$, and let $U$ be the smooth locus. If you try to extend a local system $\mathcal{L}$ on $U$ to $X$, you end up with something new — a perverse sheaf. Though these are quite technical to define, you can define an analogue of sheaf cohomology for perverse sheaves, which is good.

A perverse sheaf is, more or less, a collection of local systems on the individual strata, where the behavior on when you pass between strata depends on how severe the singularities are. The slogan is, "perverse sheaves are the singular version of local systems." For a short overview, check out `http://www.ams.org/notices/201005/rtx100500632p.pdf`.

*Remark.* The extension of a constant sheaf in this setting is in general *not* constant! ◂

You also get different perverse sheaves for spaces which are homotopic but not homeomorphic, which is unusual for things we're used to taking cohomology of.

Let $\mathsf{P}(X)$ denote the category of perverse sheaves on $X$. Let's discuss some of its formal properties:

- $\mathsf{P}(X)$ is an abelian category, a full subcategory of $D(X)$, the derived category of $X$ (which itself is a full subcategory of the category of sheaves on $X$).
- $D(X)$ has a $t$-structure, and its heart $D(X)^\heartsuit$ is exactly $\mathsf{P}(X)$.

- There's a duality endomorphism on $D(X)$, and $\mathsf{P}(X)$ is stable under this duality. This is Verdier duality, an extension of Poincaré duality.
- Though $\mathsf{P}(X)$ is not in general semisimple,[17] it's still nice: there are finitely many simple objects, which are intersection cohomology complexes, and any object can be obtained as a sequence of extensions of simple objects. One says that $\mathsf{P}(X)$ has *finite length*.

So one of the puzzles of perverse sheaves is that they have really nice formal properties, but the definition is a mess. Frequently, one takes them as a black box satisfying the formal properties.

**Definition 5.3.** A *stratified pseudomanifold* is a space $X$ together with a filtration $X \supset X_0 \supset X_1 \supset \cdots \supset X_n = \emptyset$, such that each $X_i \subset X$ is closed and
- the *k-stratum* $S_k := X_k \setminus X_{k-1}$ is a topological $(n-k)$-manifold,
- $S_1 = \emptyset$,
- if $\Sigma = X_1 = X_2$ (the *singular locus*), then $S_0 = X \setminus \Sigma$ is dense in $X$,
- and one more technical consideration called local normal triviality.

The Whitney stratification of a complex variety makes it into a stratified pseudomanifold.
We want to compute (co)homology, which means we need a version of chains for these spaces.

**Definition 5.4.** A *geometric chain* in $S_0$ means a singular chain $\xi$ in $X$ whose support $|\xi|$ in $S_0$ is closed in $S_0$.

These are less important than what you call cycles, though.

**Definition 5.5.**
- A *perversity* is a map $p\colon \{2, 3, \ldots, n\} \to \mathbb{Z}$.
- Given an $i \in \mathbb{Z}$ and a perversity $p$, a $Z \subseteq X$ is $(p, i)$*-allowable* if $\dim Z \le i$ and for all strata $S_\alpha \subset X$,

$$\dim(S_\alpha \cap Z) \le i - \operatorname{codim} S_\alpha + p(\alpha).$$

This is a generalization of transversality: how can a 1-chain intersect the singular locus in a reasonably transverse manner?

**Definition 5.6.** A geometric $i$-chain $\xi$ in $S_0$ with coefficients in $\mathcal{L}$ is *p-allowable* if
- $|\xi| \subset X$ is $(p, i)$-allowable, and
- $|\partial\xi|$ is $(p, i-1)$-allowable.

The space of $p$-allowable chains is dneoted $I^pC_i(X, \mathcal{L})$, and these define a complex

$$\cdots \longrightarrow I^pC_{i+1}(X, \mathcal{L}) \longrightarrow I^pC_i(X, \mathcal{L}) \longrightarrow I^pC_{i-1}(X, \mathcal{L}) \longrightarrow \cdots$$

called the *intersection complex*. The homology of this complex is denoted $I^pH_i(X, \mathcal{L})$, and is called the *intersection cohomology*.

**Example 5.7.** Take two spheres and identify their north poles together, then their south poles together. This space $X$, the (unreduced) suspension of $S^1 \amalg S^1$, is a stratified pseudomanifold with singular locus $\{N, S\}$, the (identified) north and south poles. Let $a$ be a meridian in the first sphere and $b$ be a meridian in the second sphere.
Let $\mathcal{L} = \underline{\mathbb{Q}}$ be the constant sheaf. Its (usual) homology is:
- $H_2(X)$ is generated by $[\Sigma a]$ and $[\Sigma b]$.[18]
- $H_1(X)$ is generated by $\Sigma a - \Sigma b]$.
- $H_0(X)$ is generated by $[a] = [b]$.

But for intersection cohomology, we get something different. For $p = -1$,
- $I^{-1}H_2(X)$ is zero, because nothing can intersect the strata with the correct dimension.
- $I^{-1}H_1(X)$ is spanned by the two equators $A$ and $B$.
- $H^{-1}H_0(X)$ is generated by $[a]$ and $[b]$, which are not equal.

For $p = 0$,

[17] It will be for the affine Grassmannian!
[18] In this example, $\Sigma$ denotes unreduced suspension.

- $I^0H_2(X)$ is generated by $[\Sigma A]$ and $[\Sigma B]$.
- $I^0H_1(X)$ is zero.
- $I^0H_0(X)$ is spanned by $[a]$ and $[b]$, which are not equal.

For $p = 1$,

- $I^1H_1(X)$ is spanned by $[\Sigma A]$ and $[\Sigma B]$.
- $I^0H_1(X)$ is spanned by $[\Sigma a]$ and $[\Sigma b]$.
- $I^1H_0(X)$ is zero. $\blacktriangleleft$

We'll define a class of spaces on which intersection cohomology interpolates between (Borel-Moore) homology and ordinary cohomology.

**Definition 5.8.** An $n$-dimensional stratified pseudomanifold is *normal* if for any $x \in \Sigma$ there's a neighborhood $U$ of $x$ such that $U \setminus \Sigma$ is connected.

*Remark.* Every stratified pseudomanifold $X$ has a *normalization* $\pi\colon \widetilde{X} \to X$ such that $\widetilde{X}$ is normal, $\pi|_{\pi^{-1}(S_0)}$ is one-to-one, and $\pi|_{\pi^{-1}(\Sigma)}$ is $n$-to-1, where $n$ is the number of connected components of $U \setminus \Sigma$. $\blacktriangleleft$

The normalization map defines a map of chain complexes $C_*(\widetilde{X}) \to C_*(X)$, such that if $t\colon \alpha \mapsto \alpha - 2$, we get a map on intersection cohomology

$$I^tH_*(\widetilde{X}) \longrightarrow I^tH_*(X).$$

**Proposition 5.9.** *If $X$ is normal, then there are chain maps $I^tC_i(X) \to C_i(X)$ and $C^{n-1}(X) \to I^0C_i(X)$ inducing isomorphisms*

$$I^tH_*(X) \longrightarrow H_*(X)$$
$$H^{n-i}(X) \longrightarrow I^0H_i(X).$$

This is the sense in which for normal spaces, intersection homology generalizes ordinary homology and cohomology.

Recall that Poincaré duality says for a smooth, oriented manifold $X$,

$$H_i(X) \otimes H^c_{i-1}(X) \longrightarrow H^c_0(X) \longrightarrow H^c_0(\mathrm{pt}) \cong \mathbb{Q}$$

is a perfect pairing. This fails for singular spaces, but intersection cohomology fixes this!

**Definition 5.10.**

- Two local systems $L$ and $L'$ are *dual* if there is a map $L \otimes L' \to O_{S_0}$ which is a perfect pairing on each fiber.
- Two perversities $p$ and $q$ are *dual* (written $q = p^*$) if $p + q = t$, i.e. for each $\alpha$, $p(\alpha) + q(\alpha) = \alpha - 2$.

**Theorem 5.11** (Goresky-MacPherson)**.** *Let $k = \dim X$ and $L$ and $L'$ be dual local systems. Then, there is a perfect pairing*

$$I\colon I^pH_i(X, L) \otimes I^{p^*}H_{k-i}(X, L') \longrightarrow \mathbb{Q}.$$

You should think of this as an intersection number.[19]

For $U \hookrightarrow X$ open, we get morphisms $C_i(X) \to C_i(U)$ and $I^pC_i(X) \to I^pC_i(U)$, but for Borel-Moore chains, we get covariant functoriality: $C^c_i(U) \hookrightarrow C^c_i(U)$. This allows us to define two sheaves on $X$,

$$\mathcal{D}^{-i}_X(U) \coloneqq C_i(U)$$
$$I^p\mathcal{C}^{-i}_X(U) \coloneqq I^pC_i(U).$$

[19]It seems like the passage from Poincaré duality to this theorem has followed the usual "fancification functor" of algebraic structures, where something you thought you understood is replaced with something ostensibly geometric, more formal, and considerably more general. . .

These form cochain complexes $\mathcal{D}_X^\bullet, I^p\mathcal{C}_X^\bullet$ of sheaves; one can take (compactly supported) global sections to recover the original vector spaces:

$$\begin{aligned}
H_i^c(X) &= H^{-i}(\Gamma_c(\mathcal{D}_X^\bullet))\\
I^pH_i^c(X) &= H^{-i}(\Gamma_c(I^p\mathcal{C}_X^\bullet))\\
H_i(X) &= H^{-i}(\Gamma(\mathcal{D}_X^\bullet))\\
I^pH_i(X) &= H^{-i}(\Gamma(I^p\mathcal{C}_X^\bullet)).
\end{aligned}$$

If you want something cohomological, change the grading.

**Definition 5.12.** The *intersection cohomology* is the hypercohomology of $I^p\mathcal{C}_X^\bullet$:

$$I^pH^k(X) \coloneqq \mathbb{H}^k(X, I^p\mathcal{C}_X^\bullet).$$

Thus $I^pH^k(X) = I^pH_{-k}(X)$. So these are concentrated in negative degrees, which is a little strange, but ultimately okay. And what's good about this is that you can do it sheafily, since everything is local.

## 6. The derived category theory surrounding perverse sheaves: 9/29/17

These are Arun's prepared notes for his talk.

6.1. **The six-functor formalism and Verdier duality.** We'll begin with the six-functor formalism as an introduction to Verdier duality. There will be no proofs.

Motivated by the fundamental importance of Poincaré duality in algebraic topology, we're going to try to make it work in algebraic geometry. On smooth varieties, everything is fine — but it does not work in general. We'll present two ways to fix it: the first relaxes to the derived category, but it turns out we'll be able to establish a duality on a curious abelian subcategory of that category, which is the category of perverse sheaves.

Recall that if $f\colon X \to Y$ is a morphism of schemes, it defines an adjunction $(f^*, f_*)$ between the direct and inverse image functors:

$$f_*(\mathscr{F})(U) = \mathscr{F}(f^{-1}(U))f^*(\mathscr{G}) = \left(V \longmapsto \varinjlim_{f(U)\subseteq V} \mathscr{G}(V)\right)^{\mathrm{sh}},$$

and a third covariant functor $f_!$ which only keeps the $s \in \mathscr{F}(f^{-1}(U))$ for which $f|_{\mathrm{supp}(s)}$ is proper. We'd like this to also have an adjoint, and in order to do so, we must pass to the derived category. In this case, $Rf_!$ has a right adjoint, $Rf^!\colon D^b(Y) \to D^b(X)$. Thus we have six functors associated to derived categories of sheaves, which come in three adjoint pairs:

- $(- \otimes^L \mathscr{F}, R\mathscr{H}\!om(\mathscr{F}, -))$,
- $(Rf^*, f_*)$, and
- $(Rf_!, Rf^!)$.

Henceforth we will drop the $L$s and $R$s. There's a natural transformation $f_! \to f_*$, an isomorphism when $f$ is proper, and $f^! = f^*$ if $f$ is an open embedding.

Let $p\colon X \to \mathrm{pt}$ denote the crush map and $\omega_X \coloneqq p^!(\underline{\mathbb{C}}) \in D^b(X)$. This is in general a complex of sheaves, and is called the *dualizing complex*.

These functors are in particular generalizations of cohomology. Recall that $H^*$ is the right derived functor of the global sections functor $\Gamma = p_*$. We also have

$$\begin{aligned}
H^*(X) &= H^*(p_*\underline{\mathbb{C}}) & \qquad H_*^!(X) &= H^*(f_*\omega_X)\\
H_!^*(X) &= H^n(p_!\underline{\mathbb{C}}) & \qquad H_*(X) &= H^{-n}(f_!\omega_X).
\end{aligned}$$

Now, define $\mathbb{D} \coloneqq \mathscr{H}\!om(-, \omega_X)$, which is an endofunctor of $D^b(X)$.

**Theorem 6.1** (Verdier duality)**.**

1. *There's a natural isomorphism* $\mathbb{D}^2 \simeq \mathrm{id}$.
2. *A map* $f\colon X \to Y$ *induces a natural isomorphism* $D_Y \circ f_! = f_* \circ \mathbb{D}_X$.
3. *Verdier duality reduces to Poincaré duality in that if* $X$ *is smooth and* $\mathcal{L}$ *is a local system on* $X$, $\mathbb{D}\mathcal{L} \cong \mathcal{L}^\vee[2\dim X]$.

This dimension-shifting result is really nice, but it just doesn't work on singular varieties: you might imagine understanding a complex on each stratum of, say, its Whitney stratification, and these strata will have different dimensions. So if we'd like a nice duality result, we'll have to restrict to a class of sheaves which account for this offset.

6.2. **The perverse $t$-structure.** We're going to define the category of perverse sheaves using an unusual $t$-structure on $D^b(X)$.

$t$-structures are formalizations of two things you've probably already encountered, where shifts of something simpler generate a triangulated category.

- If $\mathsf{A}$ is an abelian category, there's a fully faithful functor $\mathsf{A} \to D(\mathsf{A})$ sending a complex $A$ to $\cdots \to 0 \to A \to 0 \to \cdots$, and this sends short exact sequences to distinguished triangles. Moreover, the entire derived category is built from these and "attaching maps."
- In algebraic topology, one can understand a space in terms of its Postnikov decomposition, which builds it as a tower from attaching maps between spaces $K(G, n)$ with a single homotopy group. A fiber sequence defines a long exact sequence on homotopy groups (better: a distinguished triangle in the homotopy category).

**Definition 6.2.** Let $\mathsf{C}$ be a triangulated category. A *$t$-structure* on $\mathsf{C}$ is a pair $\mathsf{C}^{\geq 0}$ and $\mathsf{C}^{\leq 0}$ of full subcategories such that

- if $\mathsf{C}^{\leq i} = \mathsf{C}^{\leq 0}[i]$ and $\mathsf{C}^{\geq i} = \mathsf{C}^{\geq 0}[i]$, then $\mathrm{Hom}(\mathsf{C}^{\leq 0}, \mathsf{C}^{\geq 1}) = 0$.
- $\mathsf{C}^{-1} \subseteq \mathsf{C}^{\leq 0}$ and $\mathsf{C}^{\geq 1} \subseteq \mathsf{C}^{\geq 0}$.
- any $X \in \mathsf{C}$ belongs to a distinguished triangle
$$A \longrightarrow X \longrightarrow B \longrightarrow A[1],$$
where $A \in \mathsf{C}^{\leq 0}$ and $B \in \mathsf{C}^{\geq 1}$.

In this case, we make the following additional definitions:

- The inclusion $\mathsf{C}^{\leq n}$ has a right adjoint, which we denote $\tau_{\leq n}$.
- Similarly, the inclusion $\mathsf{C}^{\geq n}$ has a left adjoint, which we denote $\tau_{\geq n}$.
- A $t$-structure is *nondegenerate* if the intersection of all of the subcategories $\mathsf{C}^{\geq n}$ is empty, and the intersection of all of the $\mathsf{C}^{\leq n}$ is empty.
- The *heart* of a $t$-structure is $\mathsf{C}^{\heartsuit} := \mathsf{C}^{\geq 0} \cap \mathsf{C}^{\leq 0}$.

**Definition 6.3.** Let $\mathsf{C}$ be a triangulated category and $\mathsf{A}$ be an abelian category. A functor $F\colon \mathsf{C} \to \mathsf{A}$ is *cohomological* if it sends a distinguished triangle $A \to B \to C \to A[1]$ to a long exact sequence
$$\cdots \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow F(A[1]) \longrightarrow \cdots$$
The most important example is the $i^{\text{th}}$ *cohomology sheaf* $\mathcal{H}^i(X) := \tau_{\geq 0}\tau_{\leq 0}(X[i])$.

**Theorem 6.4.** *The heart of a $t$-structure is an abelian category, and the functor $\tau_{\leq 0}\tau_{\geq 0}\colon \mathsf{C} \to \mathsf{C}^{\heartsuit}$ is cohomological.*

**Example 6.5.**

(1) Indeed, if $X$ is a variety, $D^b(X)$ has a $t$-structure in which $D^{\leq 0}$ is the complexes whose cohomology vanishes in all negative gradings, and $D^{\geq 0}$ is the complexes whose cohomology vanishes in all positive gradings. You can check this is a triangulated structure and that its heart is the category of sheaves on $X$.

(2) The stable homotopy category $\mathsf{Spc}$ has a $t$-structure in which $\mathsf{Spc}^{\geq 0}$ is the *connective spectra* (those whose negative homotopy groups are zero) and $\mathsf{Spc}^{\leq 0}$ is the *coconnective spectra* (positive homotopy groups are zero). The heart of this $t$-structure is the subcategory of spectra with only a single homotopy group in degree 0, which is the Eilenberg-Mac Lane spectra, and this is equivalent to $\mathsf{Ab}$. ◀

But we can define another, different $t$-structure on $D^b(X)$, and its heart is the perverse sheaves.

Let $X$ be a space with a stratification
$$X = \coprod_{\lambda \in \Lambda} X_\lambda,$$

such that $D^b(X_\lambda)$ is preserved by Verdier duality (e.g. any Whitney stratification). Let $d_\lambda \coloneqq \dim(X_\lambda)$ and $i_\lambda\colon X_\lambda \hookrightarrow X$ be inclusion. Let[20]

$$\begin{aligned}{}^pD_\lambda^{\le 0} &\coloneqq \{\mathscr{F} \in D^b_{\mathrm{const}}(X_\lambda) \mid \mathcal{H}^i(\mathscr{F}) = 0 \text{ for } i > -d_\lambda\}\\ {}^pD_\lambda^{\ge 0} &\coloneqq \{\mathscr{F} \in D^b_{\mathrm{const}}(X_\lambda) \mid \mathcal{H}^i(\mathscr{F}) = 0 \text{ for } i < -d_\lambda\};\end{aligned}$$

the intersection of these two subcategories is $\mathsf{Loc}(X_\lambda)[d_\lambda]$. In particular, this shift is preserved by Verdier duality, and is the only shift on $X_\lambda$ that is.

**Lemma 6.6.** *$({}^pD_\lambda^{\le 0}, {}^pD_\lambda^{\ge 0})$ is a $t$-structure on $X_\lambda$.*

*Proof.* Any shift of a $t$-structure is a $t$-structure, so we may replace $-d_\lambda$ by 0, and then we have the standard $t$-structure on $D^b(X_\lambda)$. ☐

**Definition 6.7.** The *perverse $t$-structure* on $D^b(X)$ is specified by

$$\begin{aligned}{}^pD^{\le 0} &\coloneqq \{\mathscr{F} \in D^b(X) \mid i_\lambda^*\mathscr{F} \in {}^pD_\lambda^{\le 0} \text{ for all } \lambda \in \Lambda\}\\ {}^pD^{\ge 0} &\coloneqq \{\mathscr{F} \in D^b(X) \mid i_\lambda^*\mathscr{F} \in {}^pD_\lambda^{\ge 0} \text{ for all } \lambda \in \Lambda\}.\end{aligned}$$

Its heart, denoted $\mathsf{P}_\Lambda(X)$, is the category of *perverse sheaves* on $X$ with respect to the stratification $\Lambda$.

**Proposition 6.8.** *This is indeed a $t$-structure, and $\mathsf{P}_\Lambda(X)$ is an abelian category.*

To prove this, we need to use something about geometry.

**Theorem 6.9** (Recollement)**.** *Let $i\colon Z \hookrightarrow X$ be a closed embedding, where $X$ is quasiprojective, $U \coloneqq X \setminus Z$, and $j\colon U \hookrightarrow Z$. Then,*

1. *$j^*i_* = 0$, $i^*j_! = 0$, and $i^!j_* = 0$, so $\mathrm{Hom}(j_!A, i_*B) = 0$ and $\mathrm{Hom}(i_*A, j_*B) = 0$.*
2. *For any $A \in D^b(X)$, the adjunction maps define distinguished triangles*

$$j_!j^!A \longrightarrow A \longrightarrow i_*i^*A \longrightarrow$$
$$i_!i^!A \longrightarrow A \longrightarrow j_*j^*A. \longrightarrow$$

3. *$i_* = i_!$, and $j^! = j^*$ is fully faithful, so there are natural isomorphisms $i^*i_* \Rightarrow \mathrm{id} \Rightarrow i^!i_!$ and $j^*j_* \Rightarrow \mathrm{id} \Rightarrow j^!j_!$.*

**Theorem 6.10** (Beilinson-Bernstein-Drinfeld)**.** *Assume the setup of the previous theorem,[21] and let $(D_Z^{\le 0}, D_Z^{\ge 0})$ and $(D_U^{\le 0}, D_U^{\ge 0})$ be $t$-structures on $D^b(Z)$ and $D^b(U)$ respectively. Then, the full subcategories*

$$\begin{aligned}D^{\le 0} &= \{X \mid i^*X \in D_Z^{\le 0} \text{ and } j^*X \in D_U^{\le 0}\}\\ D^{\ge 0} &= \{X \mid i^*X \in D_Z^{\ge 0} \text{ and } j^*X \in D_U^{\ge 0}\}\end{aligned}$$

*defines a $t$-structure on $D^b(X)$.*

*Proof if Proposition 6.8.* Induct across the stratification $\Lambda$: if $X_0$ denotes the smallest nonempty stratum, then $X_0 \hookrightarrow X$ is closed, so we're in the situation of Theorems 6.9 and (6.10). The perverse $t$-structure on $X$ is a $t$-structure if the $t$-structure $({}^pD_0^{\le 0}, {}^pD_0^{\ge 0})$ on $X_0$ is and if the perverse $t$-structure on $X \setminus X_0$ is, so it suffices by induction to think about $X_0$, and we already proved this is a $t$-structure on $X_0$ in Lemma 6.6. ☐

To eliminate the dependence on the stratification, we do the same thing we did for Čech cohomology: a refinement $\Lambda'$ of $\Lambda$ defines a fully faithful embedding $\mathsf{P}_\Lambda(X) \hookrightarrow \mathsf{P}_{\Lambda'}(X)$, and taking the colimit across all of these embeddings, we obtain the category of *perverse sheaves* $\mathsf{P}(X)$. (So a perverse sheaf is a perverse sheaf with respect to some stratification, and two perverse sheaves are isomorphic if they agree on some common refinement.)

[20]Here $\mathcal{H}$ denotes the $i^{\text{th}}$ cohomology, as a sheaf on $X$, and $D^b_{\mathrm{const}}(X)$ denote the full subcategory of sheaves with locally constant cohomology sheaves.

[21]The proof is categorical, so you could forget about the varieties and assume only that you have triangulated categories satisfying the consequences of that theorem.

6.3. **Nonetheless, perverse sheaves have nice properties.**

**Theorem 6.11.** *$\mathbb{D}$ exchanges ${}^pD^{\le 0}$ and ${}^pD^{\ge 0}$, so $\mathbb{D}$ preserves the heart and therefore is an involution on $\mathsf{P}(X)$.*

**Proposition 6.12.** *A complex $\mathscr{F}$ is a perverse sheaf iff for all $i$,*
1. $\dim\operatorname{supp}\mathcal{H}^i(\mathscr{F}) \le i$, *and*
2. $\dim\operatorname{supp}\mathcal{H}^i(\mathbb{D}(\mathscr{F})) \le -i$.

**Proposition 6.13.** *$\mathsf{P}(X)$ is Noetherian and Artinian, meaning ascending and descending chains of objects in it stabilize.*

This is *not* true for constructible sheaves, which are Noetherian but not Artinian.

**Proposition 6.14.** *Let $Y \subset X$ be a smooth, locally closed subvariety and $\mathcal{L} \in \mathsf{Loc}(Y)$, and let $d_Y := \dim(Y)$. Then, there is a perverse sheaf $IC(Y, \mathcal{L}) \in \mathsf{P}(X)$ such that the following are true, and it is unique up to unique isomorphism.*
1. *If $i < -d_Y$, $\mathcal{H}^i(IC(Y,\mathcal{L})) = 0$.*
2. $\mathcal{H}^{-d}(IC(Y,\mathcal{L}))|_Y \cong \mathcal{L}$.
3. *The inequalities in Proposition 6.12 for $IC(Y,\mathcal{L})$ are strict.*

This perverse sheaf is called the *intersection cohomology complex*. Here are a few more nice properties.

**Proposition 6.15.** *Let $Y$ and $\mathcal{L}$ be as above, and $j\colon Y \hookrightarrow X$ be the embedding.*
1. $\mathcal{H}^i(IC(Y,\mathcal{L})) = 0$ *unless* $i \in [-d_Y, 0)$,
2. $\mathcal{H}^{-d}(IC(Y,\mathcal{L})) \cong \mathcal{H}^0(j_*\mathcal{L})$, *and*
3. $IC(Y,\mathcal{L}^*) \cong IC(Y,\mathcal{L})^\vee$. *(Here $^*$ denotes duality of local systems and $^\vee$ denotes Verdier duality).*

**Proposition 6.16.** *Let $Y$ and $\mathcal{L}$ be as above and $U \subset Y$ be an open subvariety. The intersection complex $IC(Y,\mathcal{L})$ is the minimal perverse sheaf extending its restriction $U$, where "minimal" means that it in $\mathsf{P}(X)$, it has no sub- or quotient object supported on $Y \setminus U$.*

**Theorem 6.17.** *The simple objects in $\mathsf{P}(X)$ are the intersection complexes $IC(Y,\mathcal{L})$, where*
- *$Y \hookrightarrow X$ is a smooth, locally closed subvariety of $X$, and*
- *$\mathcal{L}$ is an irreducible, locally constant sheaf on $Y$.*

**Corollary 6.18.**
1. *For any $k < 0$ and intersection complexes $IC(Y,\mathcal{L})$ and $IC(Y',\mathcal{L}')$,*
$$\operatorname{Ext}^k_{\mathsf{P}(X)}(IC(Y,\mathcal{L}), IC(Y',\mathcal{L}')) = 0.$$
2. *Suppose $\mathcal{L}$ and $\mathcal{L}'$ are irreducible and locally constant. Then,*
$$\operatorname{Hom}_{D^b(X)}(IC(Y,\mathcal{L}), IC(Y',\mathcal{L}')) = \operatorname{Hom}_{\mathsf{P}(X)}(IC(Y,\mathcal{L}), IC(Y',\mathcal{L}')) = 0.$$

**Proposition 6.19** (Perverse continuation principle)**.** *Let $U \subset X$ be a smooth, Zariski-open subset. Then, any map $a\colon \mathcal{L} \to \mathcal{L}'$ of local systems on $U$ uniquely extends to a map $IC(a)\colon IC(X,\mathcal{L}) \to IC(X,\mathcal{L}')$, and the map*
$$IC\colon \operatorname{Hom}_{\mathsf{Loc}(U)}(\mathcal{L},\mathcal{L}') \longrightarrow \operatorname{Hom}_{\mathsf{P}(X)}(IC(X,\mathcal{L}), IC(X,\mathcal{L}'))$$
*is an isomorphism.*

**Theorem 6.20** (Perverse Artin vanishing)**.** *Let $P$ be a perverse sheaf on an affine variety $X$.*
- $\mathcal{H}^i(P) = 0$ *if* $i \notin [-\dim X, 0]$.
- $\mathcal{H}^i_c(P) = 0$ *if* $i \notin [0, \dim X]$.

*Partial proof.* We'll use the Grothendieck composition-of-functors spectral sequence
$$E_2^{p,q} = H^p(X, \mathcal{H}^q(P)) \Longrightarrow H^{p+q}(P).$$
We know $\dim\operatorname{supp}\mathcal{H}^q(P) \le -i$, and $Y_q := \operatorname{supp}\mathcal{H}^q(P)$ is a closed affine subvariety. Using Artin vanishing for constructible sheaves, $H^p(Y_q, \mathcal{H}^q(P)) = 0$ when $p > -q \ge \dim Y_q$, so if $p+q > 0$, $E_2^{p,q}$ vanishes, and therefore the $E_\infty$-page must also vanish there, providing the upper bound.

The rest of the proof uses Verdier duality… somehow. ☐

**Corollary 6.21** (Perverse Lefschetz hyperplane theorem)**.** *Let $X$ be a quasiprojective variety, $P \in \mathsf{P}(X)$, and $X_1 \subseteq X$ be a generic hyperplane section.*[22] *The map $H^i(X;P) \to H^i(X_1; P|_{X_1})$ is*
- *an isomorphism if $i \le -2$, and*
- *injective for $i = -1$.*

There's a similar statement for cohomology with compact supports.

*Proof when $X$ is projective.* Let $U \coloneqq X \setminus X_1$, $i\colon X_1 \hookrightarrow X$ be inclusion, and $j\colon U \hookrightarrow X$ be inclusion. The distinguished triangle $j_*j^!P \to P \to i_*i^*P \to$ (from Theorem 6.9) induces a long exact sequence in cohomology (not cohomology sheaves):

$$\cdots \longrightarrow H^k(X; j_!j^*P) \longrightarrow H^k(X;P) \longrightarrow H^k(X_1;P|_{X_1}) \longrightarrow H^{k+1}(X; j_*j^!P) \longrightarrow \cdots$$

Now, it suffices to show that when $k < 0$,

$$H^k(X, j_!j^*P) = H^k_c(X, j_!j^*P) \cong H^k_c(U; j^*P) = 0,$$

and since $U$ is affine, this is Theorem 6.20. (The equivalence of $H$ and $H_c$ on $X$ is what used the assumption of projectivity.) □

## 7. Neutral Tannakian categories: 10/13/17

*"This is going to be a great talk for bingo."*

Today, most of the time will be devoted to the proof of the following theorem (all terms will be explained). All vector spaces are finite-dimensional, and $\mathsf{Vect}_k$ denotes the category of finite-dimensional $k$-vector spaces.

**Theorem 7.1.** *Let $(\mathsf{C}, \otimes)$ be a rigid, abelian, $\mathbb{C}$-linear tensor category such that $\mathrm{End}(\mathbf{1}) = \mathbb{C}$ together with an exact, faithful, linear tensor functor $\omega\colon \mathsf{C} \to \mathsf{Vect}_\mathbb{C}$. Then, there is a group $G$ such that $\mathsf{C} \cong \mathsf{Rep}_\mathbb{C}(G)$, and $G \cong \mathrm{Aut}^\otimes(\omega)$.*[23]

That was a lot of adjectives! It's a cool-sounding theorem, but the number of hypotheses limits its utility. Fortunately, the category of perverse sheaves on the affine Grassmannian will satisfy the hypotheses.

The idea of the proof is the Barr-Beck theorem. Let $G$ be a finite group and $\omega\colon \mathsf{Rep}_\mathbb{C}(G) \to \mathsf{Vect}_\mathbb{C}$ be the forgetful functor. Then, we have an adjunction $\mathrm{Ind}_e^G \dashv \mathrm{Res}_e^G = \omega$, so

$$\mathrm{Hom}_{\mathsf{Rep}_G}(\mathrm{Ind}_e^G, \mathrm{Ind}_e^G V) \cong \mathrm{Hom}_{\mathsf{Vect}_\mathbb{C}}(V, \mathrm{Res}_e^G \mathrm{Ind}_e^G V),$$

and $\mathrm{Res}_e^G \mathrm{Ind}_e^G V \cong G \times V$. This means $\mathsf{Rep}_\mathbb{C}(G)$ is (equivalent to) the category of algebras in $\mathsf{Vect}_\mathbb{C}$ over the monad $V \mapsto G \times V$.

We don't have induction or restriction in $\mathsf{C}$, but we will attack Theorem 7.1 by realizing it as a category of algebras over a monad.

**Glossary 7.2.** Let's define about all of the words in the statement of Theorem 7.1.
- By a *tensor category* we mean a symmetric monoidal category.[24] This is the data of a functor $\otimes\colon \mathsf{C}\times\mathsf{C} \to \mathsf{C}$ and a unit $\mathbf{1}$ along with data[25] of natural transformations guaranteeing its associativity ($(A\otimes B)\otimes C \overset{\cong}{\to} A \otimes (B \otimes C)$), commutativity ($A \otimes B \overset{\cong}{\to} B \otimes A$), and identity ($\mathbf{1} \otimes A \cong A$), together with some coherence conditions. But the point is, $\otimes$ behaves like a commutative tensor product should.
- *Rigidity* is what will get us a group and not a monoid in the end; it is the condition that all objects are dualizable. This implies that if $V^* \coloneqq \mathrm{Hom}_\mathsf{C}(V, \mathbf{1})$, there's a natural isomorphism $(\text{–})^{**} \to \mathrm{id}_\mathsf{C}$ (so every object is reflexive), along with a natural isomorphism

$$\mathrm{Hom}_\mathsf{C}(A, B) \otimes \mathrm{Hom}_\mathsf{C}(A', B') \cong \mathrm{Hom}_\mathsf{C}(A \otimes A', B \otimes B').$$

  From this it follows that there is an internal Hom.

[22] This means: $X$ is a locally closed subset of some $\mathbb{P}^N$; choose a generic hyperplane $H \subset \mathbb{P}^N$ and let $X_1 \coloneqq H \cap X$.
[23] This theorem is true for any field $k$ in place of $\mathbb{C}$, but nice applications require a characteristic 0 field.
[24] Some people use "tensor category" to mean just a monoidal category, so be careful!
[25] The fact that this is data is important: there are multiple ways to put a symmetric structure on the tensor product of graded vector spaces, depending on whether one wants a trivial sign rule or the Koszul sign rule.

- *Tensorial* is data of a natural isomorphism $\omega(X \otimes Y) \cong \omega(X) \otimes \omega(Y)$, just like how the underlying vector space of the tensor product of two representations is the tensor product of their underlying vector spaces (up to natural isomorphism). We assume all tensor functors are $\mathbb{C}$-linear.

One can prove that for abelian symmetric monoidal categories, the tensor functor is multilinear (i.e. distributes over finite direct sums), which is a nice fact to have.

The requirement that $\omega$ is faithful is also a huge assumption: it means maps of objects in $\mathsf{C}$ are particular maps of underlying vector spaces (as is true for representations).

**Definition 7.3.** Let $F, G\colon (\mathsf{C}, \otimes) \rightrightarrows (\mathsf{C}', \otimes)$ be tensorial functors of tensor categories. A natural transformation $(\lambda, c)\colon F \Rightarrow G$ is *tensorial* is $\lambda\colon F \Rightarrow G$ is a natural transformation and $c\colon \lambda_{A\otimes B} \to \lambda_A \otimes \lambda_B$ is a natural isomorphism.

We'll write $\mathrm{Hom}^{\otimes}(F, G)$ for the set of tensorial natural transformations.

The end goal is to get an algebraic group $G$ out of $\mathsf{C}$, so how will that work? Well, let $\mathsf{C}$ be a tensor category and $F, G\colon \mathsf{C} \rightrightarrows \mathsf{Vect}_{\mathbb{C}}$ be tensor functors. Then, for any $\mathbb{C}$-algebra $R$, we can define

$$\underline{\mathrm{Hom}}^{\otimes}(F, G)(R) \coloneqq \mathrm{Hom}^{\otimes}(R \otimes F(-), R \otimes G(-)).$$

This takes commutative algebras and gives you sets. Thus, $\mathrm{Aut}(\omega)$ can be applied to any $\mathbb{C}$-algebra $R$, so if we can show it's representable, we get a group scheme, not just a group.

*Remark.* One general perspective on this is that if $\mathsf{C}$ is a $k$-linear category and $R$ is a $k$-algebra, there's an $R$-linear category $\mathsf{C} \otimes_k R$, e.g. if $\mathsf{C} = \mathsf{Rep}_k(G)$, then $\mathsf{C} \otimes_k R = \mathsf{Rep}_R(G)$.

The way to do this is to let the objects in $\mathsf{C} \otimes_k R$ be the same as those in $\mathsf{C}$ and

$$\mathrm{Hom}(M, N) \coloneqq R \otimes \mathrm{Hom}_{\mathsf{C}}(M, N).$$

Unfortunately, this is not an abelian category, as it doesn't have kernels and cokernels. Thus, the actual tensor product $\mathsf{C} \otimes_k R$ is the *saturation* of this naïve tensor product (add in the missing kernels and cokernels). This is called the *Deligne tensor product.* ◀

There's a version of Theorem 7.1 for finite groups, rather than group schemes; in this case, you don't assume that $\mathsf{C}$ is $k$-linear (hence can't make this representability). The proof is also much easier, because restriction in $\mathsf{Rep}_G$ (where $G$ is finite) is both a left and a right adjoint. This means the monad we discussed above is also a comonad, so everything is an algebra over it!

*Remark.* Let $A$ be a finite-dimensional algebra and $V$ be a vector space. Then, there's a canonical bijection between the set of $A$-module structures on $V$ and $A^\vee$-comodule structures on $V$: $A^\vee$ has a canonical coalgebra structure (since multiplication on $A$ is turned around). ◀

All right, here comes the hard part.

Let $\mathsf{Vect}_{\mathbb{C}}^s$ be the full subcategory of $\mathsf{Vect}_{\mathbb{C}}$ on the objects $\mathbb{C}^n$ for each $n \geq 0$. The inclusion $\mathsf{Vect}_{\mathbb{C}}^s \hookrightarrow \mathsf{Vect}_{\mathbb{C}}$ is an equivalence, so it has an inverse $c\colon \mathsf{Vect}_{\mathbb{C}} \to \mathsf{Vect}_{\mathbb{C}}^s$ (up to natural isomorphism).

**Definition 7.4.** Define the functor $\otimes\colon \mathsf{Vect}_{\mathbb{C}} \times \mathsf{C} \to \mathsf{C}$ as follows: if $V = \mathbb{C}^n$, then

$$V \otimes X \coloneqq \coprod_n X = \prod_n X = X^n.$$

For a more general $V$, let $V \otimes X \coloneqq \gamma(V) \otimes X$.

This is expressing that any $\mathbb{C}$-linear category is tensored over $\mathsf{Vect}_{\mathbb{C}}$; the definition chooses a basis, then checks it's independent of basis.

**Definition 7.5.** Using this, we can define $\mathrm{Hom}(V, X) \coloneqq V^\vee \otimes X$, *co*tensoring $\mathsf{C}$ over $\mathsf{Vect}_{\mathbb{C}}$.

**Lemma 7.6.** *Let $F\colon \mathsf{C} \to \mathsf{Vect}_{\mathbb{C}}$ be a tensor functor and $X, V$ be as above. Then, there's a natural isomorphism $F(\mathrm{Hom}(V, X)) \cong \mathrm{Hom}_{\mathsf{Vect}_{\mathbb{C}}}(V, FX)$.*

**Definition 7.7.** Let $V, X$ be as above, $W \subset V$, and $Y \subset X$. The *transporter* of $W$ to $Y$ is

$$(Y : W) \coloneqq \ker(\underline{\mathrm{Hom}}(V, X) \longrightarrow \underline{\mathrm{Hom}}(W, X/Y)).$$

**Lemma 7.8.** *The following two subobjects are equal.*

(1) *The largest subobject $P \subset \underline{\mathrm{Hom}}(\omega X, X)$ whose image in $\underline{\mathrm{Hom}}((\omega X)^n, X^n)$ is contained in $(Y : \omega Y)$ for all $Y \subset X^n$.*
(2) *The smallest subobject $P' \subset \underline{\mathrm{Hom}}(\omega X, X)$ such that $\omega(P') \subset \underline{\mathrm{Hom}}(\omega X, \omega X)$ contains $\mathrm{id}_{\omega X}$.*

The idea is that $\underline{\mathrm{Hom}}(\omega X, X)$ is playing the role of $\mathrm{Hom}(\mathrm{Ind}\,\mathrm{Res}\, X, X) = \mathrm{Hom}(\mathrm{Res}\, X, \mathrm{Res}\, X)$, and this is $G \times X$, so we're constructing something akin to the projection map $G \times X \to X$. $P = P'$ will be identified with the regular representation in the Tannakian theorem.

*Proof.* First, why do $P$ and $P'$ exist? If $\omega(X) = 0$, then $\mathrm{End}(X) = 0$ and hence $X = 0$. Thus for all $Y \subset X$, if $\omega(Y) = \omega(X)$, then $Y = X$. Thus all objects are both Noetherian and Artinian, which implies $P$ and $P'$ exist.

In fact, we can explicitly identify

$$P = \bigcap_{n, Y \subset X^n} (\mathrm{Hom}(\omega X, X) \cap (Y : \omega Y)),$$

which under $\omega$ maps to

$$\omega P = \bigcap_{n, Y} (\mathrm{Hom}(\omega X, \omega X) \cap (\omega Y : \omega Y)).$$

This means $\omega(P)$ is the largest subring of $\mathrm{End}(\omega X)$ stabilizing $\omega Y$ for all $Y \subset X^n$. In particular, $\mathrm{id}_{\omega X}$ is a stabilizer, so $P \supseteq P'$.

Now, let $V \in \mathsf{Vect}_{\mathbb{C}}$; there's a natural map $\mathrm{Hom}(\omega X, X) \to \mathrm{Hom}(\omega(V \otimes X), V \otimes X)$, which under $\omega$ is sent to the map $f \mapsto \mathrm{id}_V \otimes f$, a map $\mathrm{Hom}(\omega X, \omega X) \to \mathrm{Hom}(\omega(V \otimes X), \omega(V \otimes X))$.

By definition, $\omega P \subset \mathrm{End}(\omega X)$ stabilizes $\omega Y$ for all $y \subset V \otimes X$, so

$$P' \subset \underline{\mathrm{Hom}}(\omega X, X) = (\omega X)^\vee \otimes V.$$

Since $\omega P$ preserves $\omega P'$ and $\mathrm{id}_{\omega X} \in \omega P'$, then we get $\omega P \subseteq \omega P'$, hence $P \subseteq P'$. ☐

All right. Let $A := \omega P$ and $\langle X \rangle$ denote the full subcategory of $\mathsf{C}$ on the objects $Y \subset X^n$. Then, $\langle X \rangle \cong \mathsf{Mod}_A$: this is because $A \subset \mathrm{Hom}(\omega X, \omega X)$ preserves all $\omega Y$, so we get a map $A \to \mathrm{End}(\omega Y)$ for each $Y \in \langle X \rangle$. Another way to think of this is that the forgetful functor factors through $\mathsf{Mod}_A$.

**Definition 7.9.** Let $M \in \mathsf{Mod}_A$, and define

$$P \otimes_A M := \mathrm{coker}(P \otimes A \otimes M \rightrightarrows P \otimes M).$$

Since $\omega$ is exact,

$$\omega(P \otimes_A M) = (\omega P) \otimes_A M = A \otimes_A M \cong M.$$

One can then show this is full, and it's essentially surjective and by assumption faithful, so it's an equivalence of categories.[26]

Suppose $X' = X \oplus Z$ inside $\mathsf{C}$, $A$ be what we defined above for $X$, and $A'$ be that for $X'$. Then, there's a restriction map $A' \to A$. This allows us to make sense of the fact that $\lim \mathrm{End}(\omega|_{\langle X \rangle}) = \mathrm{End}(\omega)$, and if $\mathcal{A} = \lim_X \mathrm{End}(\omega|_{\langle X \rangle})$, $\mathsf{C} \cong \mathsf{Mod}_{\mathcal{A}}$.

## 8. The perverse decomposition theorem: 10/20/17

Today, Yan spoke about the decomposition theorem: first the classical case for a smooth map of smooth projective varieties, then the version for perverse sheaves, where neither the varieties nor the morphism need to be smooth. Then, we'll specialize to the case of semismall maps.

Let $f\colon X \to Y$ be a smooth map of smooth projective manifolds in the sense of algebraic geometry — in complex geometry, this means $f$ is a smooth submersion. Let $\underline{\mathbb{Q}}_X$ denote the constant sheaf valued in $\mathbb{Q}$ on $X$. We're going to consider the direct image complex $Rf^i_* \underline{\mathbb{Q}}_X$; since $f$ is a locally trivial fibration (in the sense of differential geometry), it's proper, and hence this is locally the derived direct image along the map $f^{-1}(t) \to t$, i.e. cohomology! That is, for a $t \in Y$,

$$(R^i_* \underline{\mathbb{Q}}_X)_t \cong \mathcal{H}^i(f^{-1}(t); \underline{\mathbb{Q}}).$$

As $i$ varies, we'd like this to behave well in the derived category.

[26]TODO: I'm not sure what happened here.

**Theorem 8.1** (Decomposition theorem (Deligne))**.** *There is a quasi-isomorphism*

$$Rf_*\underline{\mathbb{Q}}_X \xrightarrow{\simeq} \bigoplus_{i\geq 0} Rf^i\underline{\mathbb{Q}}_X[-i].$$

From this, one can conclude

$$\begin{aligned}\mathcal{H}^i(X;\underline{\mathbb{Q}}_X) &= \mathbb{H}^i(Y;Rf_*\underline{\mathbb{Q}}_X)\\ &\cong \mathbb{H}^i\Big(Y;\bigoplus R^q f_*\underline{\mathbb{Q}}_X[-q]\Big)\\ &\cong \bigoplus_{q\geq 0}\mathcal{H}^{i-q}(Y,R^q f_*\underline{\mathbb{Q}}_X).\end{aligned}$$

**Example 8.2.** This is something special to algebraic geometry of smooth, projective varieties (it's a case of the Hodge theorem): consider the Hopf fibration $\pi\colon S^3 \to S^2$. Then, $R\pi_*\underline{\mathbb{Q}}_{S^3}$ lives in the derived category of local systems. The heart of this category is the category of local systems on $S^2$ (which is just $\mathsf{Vect}$, because $S^2$ is simply connected). The cohomology of the fiber $S^1$ is $\underline{\mathbb{Q}} \oplus \underline{\mathbb{Q}}[-1]$, and the hypercohomology is $\mathbb{H}(-) = \mathrm{Ext}(\underline{\mathbb{Q}},-)$, so $\mathrm{Ext}^1(\underline{\mathbb{Q}}[-1],\underline{\mathbb{Q}}) \cong \mathrm{Ext}^2(\underline{\mathbb{Q}},\underline{\mathbb{Q}}) = H^2(S^2;\mathbb{Q})$, so the $i^{\text{th}}$ cohomology of the fiber is not the same as $R^i_*\underline{\mathbb{Q}}_X$. ◀

Deligne proved Theorem 8.1 by calculating that the Leray-Serre spectral sequence

$$E_2^{p,q} = \mathcal{H}^p(Y;R^q f_*\underline{\mathbb{Q}}_X) \Longrightarrow \mathcal{H}^{p+q}(X;\underline{\mathbb{Q}}_X)$$

collapses at the $E_2$-page. This implies

$$\mathcal{H}^k(X;\underline{\mathbb{Q}}_X) \twoheadrightarrow E_\infty^{0,k} = E_2^{0,k} = \Gamma(Y,R^k f_*\underline{\mathbb{Q}}_X).$$

There is a monodromy action of $\pi_1(Y)$ on the right-hand group, and one can look at its invariants.

**The non-smooth case.** Now, we lose the smoothness assumptions on $X$, $Y$, and $f$, though we still need to assume it's proper. Now we need to use perverse cohomology ${}^p\mathcal{H}^i$ and the direct image of the intersection complex $Rf_*IC_X$.

**Theorem 8.3** (Perverse decomposition theorem)**.** *There is a quasi-isomorphism*

$$Rf_*IC_X \cong \bigoplus_{i\geq 0} {}^p\mathcal{H}^i(Rf_*IC_X)[-i].$$

Here, the *perverse cohomology* of $Rf_*IC_X$ is

$${}^p\mathcal{H}^i(Rf_*IC_X) = \bigoplus_\beta IC_{\overline{S}_\beta}(L_\beta),$$

where $Y = \coprod S_\beta$ is a stratification into closed, smooth, irreducible subvarieties and $L_\beta$ is a local system on $S_\beta$.

*Remark.* If $X$ is any algebraic space and $X_0 \subset X$ is a smooth subset, then $IC_X|_{X_0} \cong \underline{\mathbb{Q}}_{X_0}[\dim X_0]$. ◀

Recall that the intersection cohomology $IC_X$ was defined in Propositions 6.12 and 6.14. We can also provide a generalization of Corollary 6.21 to the relative setting.

**Theorem 8.4** ("Harder Lefschetz theorem")**.** *Let $f\colon X \to Y$ be a projective morphism, meaning there's an embedding $X \hookrightarrow Y \times \mathbb{P}^n$; let $\pi\colon Y \times \mathbb{P}^n \twoheadrightarrow \mathbb{P}^n$ be projection and $\eta := c_1(\pi^*\mathcal{L}|_X)$. Then, the map*

$$\smile \eta^i\colon {}^pH^{-i}(Rf_*IC_X) \longrightarrow {}^pH^i(Rf_*IC_X)$$

*is an isomorphism.*

This is an extremely deep theorem even in the case $Y = \mathrm{pt}$ and $X$ is smooth, where it's one of the stronger consequences of the Hodge theorem for algebraic varieties. In the complex analytic setting, $\eta$ is the Kähler class, and induces an $\mathrm{SL}_2$-action on cohomology.

**Example 8.5.** Let $f\colon X \to \mathbb{P}^2$ be the blowup of $\mathbb{P}^2$ at a point $x$.[27] Then, $IC_X \cong \underline{\mathbb{Q}}_X[2]$, so

$$R^i f_* \underline{\mathbb{Q}}_X[2] = \begin{cases} \underline{\mathbb{Q}}_Y, & i = -2 \\ \mathbb{Q}_{\{x\}}, & i = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Since $f$ is proper, $f_* = f_!$, so

$$Rf_* \underline{\mathbb{Q}}_X[2] = Rf_! \underline{\mathbb{Q}}_X[2] = Rf_* \underline{\mathbb{Q}}_x[2].$$

The theorem calculates this decomposition to be something geometrically meaningful; this is an example of a semismall map, and semismall maps will display similar behavior in general. ◀

**Semismall maps.** The definition of a semismall map is a little strange, but lots of examples in representation theory are semismall, and the decomposition theorem has a very nice form in this case, so the definition is worth considering.

**Definition 8.6.** A map $f\colon X \to Y$ is *semismall* if $\dim X \times_Y X = \dim X$.

In general, $X \times YX$ is reducible, and its irreducible components can have dimensions at least as large as $\dim X$. So this is saying that all components of $X \times_Y X$ have the same dimension, which is the minimum possible.

Another equivalence goes through Chevalley's theorem.

**Theorem 8.7** (Chevalley)**.** *For any $k$, $\{y \in Y \mid \dim f^{-1}(y) \geq k\}$ is a closed subvariety of $X$.*

**Proposition 8.8.** *Let $f\colon X \to Y$ be a map and $S_k := \{y \in Y \mid \dim f^{-1}(y) = k\}$, which defines a stratification of $Y$. Then, $f$ is semismall iff for all $k$,*

$$\dim S_k + 2k \leq \dim X.$$

The intuition is that the fibers of $f$ can't be too fat: if you have something of very high codimension, you're not doing much to it.

**Example 8.9.** Let $f\colon X \to \mathbb{C}^n$ be the blowup of $\mathbb{C}^m \subset \mathbb{C}^n$. Then, $f$ is semismall iff $n - m \leq 2$. ◀

**Example 8.10.** The *Springer resolution* for $\mathrm{SL}_2$ is the map $T^*\mathbb{P}^1 \cong \mathscr{O}(-2) \to \mathbb{P}^1$, which is another example of a semismall map. Here, $X \times_Y X$ has two irreducible components, and is called the *Steinberg variety* $\mathrm{St}_{\mathrm{SL}_2}$ of $\mathrm{SL}_2$. ◀

The reason semismall maps become relevant to the decomposition theorem is that if $X$ is smooth and $f$ is semismall, $Rf_* \underline{\mathbb{Q}}_X[\dim X]$ is perverse.

Fix a stratification $Y = \coprod_{\alpha \in \mathcal{A}} S_\alpha$.

**Definition 8.11.** A stratum $S_\alpha$ is *relevant* if there's a $k_\alpha$ such that $\dim S_\alpha + 2k_\alpha = \dim X$. The set of relevant strata is denoted $\mathcal{A}_{\mathrm{rel}}$.

In this case, the result from the decomposition theorem simplifies: we only care about relevant strata, and understand their local systems well.

$$Rf_* IC_X \cong \bigoplus_{\alpha \in \mathcal{A}_{\mathrm{rel}}} IC_{\overline{S}_\alpha}(L_\alpha),$$

where $L_\alpha|_{S_\alpha}$ is TODO (I didn't write it down in time – possibly $Rf_* \underline{\mathbb{Q}}_{S_\alpha}[\dim S_\alpha]$?).

This has a nice consequence that behaves like Schur's lemma and Wedderburn's theorem: when $f$ is semismall, $\mathrm{End}_{\mathsf{P}(Y)}(Rf_* \underline{\mathbb{Q}}_X[\dim X])$ is a semisimple algebra, and there's a decomposition of $L_\alpha$ indexed by the irreducible representations of $\pi_1(S_\alpha)$.

The decomposition theorem is one of the deepest theorems in geometry — for a long time, the only proof known required characteristic $p$ methods. Then, there was a characteristic 0 proof, and it was crazy. But recently, only 40 years after the first proof, there's a less intense characteristic 0 proof.

One perspective on this is that interesting sheaves arise as summands of pushforwards of constant sheaves by proper maps; such a sheaf arising in this way is said to be *of geometric origin*. Certainly, this is how we think of local systems.

[27] As a manifold, $X \cong \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$.

We'd like all sheaves to arise in this way, but the decomposition theorem is a no-go theorem in this regard: it says that any sheaf of geometric origin is semisimple. Certainly, there are lots of non-semisimple sheaves, so we're far from seeing everything.

A more optimistic way of looking at the theorem is that for a pushforward along a proper map, there are very few options for what the pushforward can be, so you can often just check on some stalks.

**Example 8.12** (Springer resolution)**.** Let $N \coloneqq \{x \in \mathfrak{sl}_2 \mid \det x = 0\}$ (the *nilpotent cone*). Let $B$ be a Borel subgroup of $\mathrm{SL}_2$, e.g. the upper triangular matrices. Then, $\mathrm{SL}_2/B \cong \mathbb{P}^1$. The *Springer resolution* is $f\colon T^*\mathbb{P}^1 \to \mathbb{P}^1$ (understood in this representation-theoretic sense).

Say you want to understand the *Springer sheaf* $\mathcal{S} \coloneqq f_*\mathbb{Q}[-2]$. The decomposition theorem makes this much easier: you check on a few stalks and can show that $f_*\mathbb{Q}[-2] \cong IC_N \oplus IC_{\{0\}}$.

The decomposition theorem also says the endomorphism algebra is semisimple; in this case, we get $\mathrm{End}(\mathcal{S}) \cong \mathbb{Q}[\mathbb{Z}/2]$, which certainly is semisimple.

More generally, if $G$ is a reductive group and $B$ is a Borel subgroup, let $N \subset \mathfrak{g}^*$ be the cone of nilpotent matrices (the preimage of 0 under some characteristic polynomial map). The *Springer resolution* is a map $T^*G/B \to N$: $T^*G/B$ can be identified with the pairs $(B', x)$ where $B'$ is a Borel subgroup and $x \in N$, and the resolution just forgets the Borel. In symplectic geometry, this is a moment map.

This is the setting for one of the earliest applications of the decomposition theorem: the *Springer sheaf* is $\mathcal{S} \coloneqq \pi_*\underline{\mathbb{Q}}[\dim T^*G/B]$, and $\mathrm{End}(\mathcal{S}) \cong \mathbb{Q}[W]$, where $W$ is the Weyl group of $G$.

This implies in particular that

$$\mathcal{S} \cong \bigoplus_{\mathrm{Irr} W} IC_{S_\rho}(\mathbb{C}) \otimes V_\rho,$$

for some vector spaces $V_\rho$ (providing the right dimension) and strata $S_\rho$. One cool consequence of this is that the irreducible representations of $W$ are realized in the cohomology of fibers of the Springer resolution.

For $\mathrm{SL}_n$, $N$ is a disjoint union over the set of partitions $P$ of $n$ elements of a simply connected orbit $O_P$. The Weyl group is $S_n$, and the irreducible representations $V_P$ of $S_n$ are also indexed over the partitions $P$ of $n$ elements. This is not a coincidence: $\mathrm{Hom}(IC(O_P), \mathcal{S}) \cong V_P$.

This really depends on the fact that we're in characteristic 0, ultimately because we need representations of $W$ to be semisimple.[28] $\blacktriangleleft$

## 9. Tannakian categories: friend or foe?

Today, Yuri spoke about Tannakian categories through examples.

Geometric Satake says that if $G$ is a reductive group, the category of $G(\mathcal{O}_K)$-equivariant perverse sheaves on the affine Grassmannian $G(K)/G(\mathscr{O}_K)$ is equivalent to the category of representations of $G^\vee$, the Langlands dual of $G$. This dual is generally defined in a different way, but this is the "why" of the definition. We'll start by showing that category is a Tannakian category, hence equivalent to the category of representations of *some* group, and then we'll pin down which group it is.

**Example 9.1.** Let $G = \mathbb{G}_m$ (also known as $\mathrm{GL}_1$). Then, $\mathsf{Rep}(\mathbb{G}_m) \cong \mathsf{GrVect}$, the category of graded finite-dimensional vector spaces, and the Langlands dual of $\mathbb{G}_m$ is $\mathbb{G}_m$.

The fiber functor $\omega\colon \mathsf{GrVect} \to \mathsf{Vect}$[29] sends

$$V_\bullet \longmapsto \bigoplus_n V_n.$$

We want to understand $\mathrm{Aut}(\omega)$, which is determined by what happens to $R[n]$ for all $\mathbb{C}$-algebras $R$, where $R[n] = (R[1])^{\otimes n}$, so $\mathrm{Aut}_R(\omega) = R^\times$. Hence $\mathrm{Aut}(\omega) = \mathbb{G}_m$: $\lambda \in \mathbb{G}_m$ acts on $v \in V_n$ by $\lambda^n$. $\blacktriangleleft$

More generally, a map $G \to G'$ defines an induction functor $\mathsf{Rep}(G') \to \mathsf{Rep}(G)$. Thus, any cocharacter $\chi\colon \mathbb{G}_m \to G$ defines a functor $\mathsf{Rep}(G) \to \mathsf{Rep}(\mathbb{G}_m) \cong \mathsf{GrVect}$, hence a way of assigning gradings to $G$-representations.

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[28] We can work in positive, but non-modular characteristic for $W$, which is fine. In modular characteristic, we replace $IC$ sheaves with something called *parity sheaves*.

[29] Here and for all of today's talk, $\mathsf{Vect}$ denotes the category of *finite-dimensional* vector spaces.

**Example 9.2** (Hodge structures)**.** Recall that a *Hodge structure* is a vector space over $\mathbb{R}$,[30] together with a decomposition

$$V \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus V^{p,q},$$

where $V^{p,q}$ are $\mathbb{C}$-subspaces, such that $\overline{V^{p,q}} = V^{q,p}$. Hodge structures will form a Tannakian category.

To make this happen, we need to describe the fiber functor $\omega$, which is the forgetful functor $\mathsf{HS}_{\mathbb{R}} \to \mathsf{Vect}_{\mathbb{R}}$ sending $(V, (V^{p,q})) \mapsto V$. To understand its automorphisms, it suffices to look at position $(p,q)$, and the condition that $\overline{V^{p,q}} = V^{q,p}$ forces (somehow?) the action of a $\lambda \in \mathbb{C}^\times$ on $v \in V^{p,q}$ to be

$$v \longmapsto \lambda^{-p}\overline{\lambda}^{-q}v.$$

The automorphism group is the *restriction* $\operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$, which is sometimes denoted $\mathbb{S}$. Its functor of points sends an $\mathbb{R}$-algebra $A$ to $(A \otimes_{\mathbb{R}} \mathbb{C})^\times$, e.g. $\mathbb{S}(\mathbb{R}) \cong \mathbb{C}^\times$ and $\mathbb{S}(\mathbb{C}) \cong \mathbb{C}^\times \times \mathbb{C}^\times$. This is a real reductive group.

The functor $\mathsf{HS} \to \mathsf{GrVect}$ sending

$$(V^{p,q})_{p,q} \longmapsto \left(V_n := \bigoplus_{p+q=n} V^{p,q}\right)$$

is induction across the map $\mathbb{G}_m(\mathbb{R}) \to \mathbb{S}(\mathbb{R})$ sending $t \mapsto t^{-1}$. ◂

Our next example is a little weirder.

**Example 9.3.** Let $\Gamma$ be a topological group and $k$ a field. Then, the category of continuous representations $\mathsf{Rep}_k(G)$ is a neutral Tannakian category, hence isomorphic to the category of $k$-representations of an algebraic group $\Gamma^{\mathrm{alg}}$, called the *algebraic envelope* or *algebraic hull* of $\Gamma$.

This is usually large enough to be unwieldy, but sometimes will be nice (e.g. when $\Gamma$ is the real form of a reductive complex Lie group). ◂

If $\mathsf{C}$ is a netural Tannakian category and $X \in \mathsf{C}$, let $\langle X\rangle_\otimes$ denote the smallest neutral Tannakian subcategory of $\mathsf{C}$ containing $X$, which is the category of subquotients of direct sums of terms of the form $X^{\otimes r} \otimes (X^\vee)^{\otimes s}$.

**Example 9.4.** Let $\rho\colon \Gamma\mathrm{GL}(V)$ be a representation of an algebraic group, so $V \in \mathsf{Rep}_G$, and the group of automorphisms of the fiber functor for $\langle V\rangle_\otimes$ is isomorphic to the Zariski closure of $\operatorname{Im}(\rho)$. ◂

If $\omega\colon \mathsf{C} \to \mathsf{Vect}$ is the fiber functor for a neutral Tannakian category $\mathsf{C}$, the algebraic group of automorphisms $\underline{\mathrm{Aut}}_{\mathsf{C}}(\omega)$ is also denoted $\pi_1^{\mathrm{Tan}}(\mathsf{C})$.

**Example 9.5.** Let $X$ be a connected topological space.[31] The category of local systems on $X$ is equivalent to the category of representations $\pi_1(X) \to \mathsf{Vect}$. Choose a basepoint $x \in X$; then, $\mathsf{Loc}(X)$ is a neutral Tannakian category whose fiber functor is $\mathcal{L} \mapsto \mathcal{L}_x$.

Hence $\mathsf{Loc}_X \cong \mathsf{Rep}(\pi_1(X,x))$, and this is also equivalent to the category of representations of the group scheme $\pi_1(X,x)^{\mathrm{alg}}$. For any local system $\mathcal{L}$, $\pi_1^{\mathrm{Tan}}(\langle\mathcal{L}\rangle_\otimes)$ is isomorphic to the Zariski closure of the monodromy representation $\pi_1(X,x) \to \mathrm{GL}(\mathcal{L}_x)$. ◂

The next examples might not literally be the Tannakian reconstruction theorem, but embody its spirit, and will be interesting and useful in that way.

If you like stacks, you might think of $\mathsf{Rep}(G)$ as the category of quasicoherent sheaves on the stack $\bullet/G$. Jacob Lurie ran with this approach to define Tannaka duality for geometric stacks. A stack is *affine* if the diagonal $\Delta\colon X \to X \times X$ is representable and affine.[32] Lurie uses this to produce a GAGA-like theorem, and the proof uses Tannakian methods.

Given a tensor category $\mathsf{C}$, define $\operatorname{Spec}\mathsf{C}\colon \mathsf{Ring} \to \mathsf{Grpd}$ to send $R \mapsto \operatorname{Hom}^\otimes(\mathsf{C}, \mathsf{Mod}_R)$.

**Theorem 9.6** (Tannaka duality for geometric stacks (Lurie))**.** *For reasonable $X$ (quasicompact with affine diagonal), $X = \operatorname{Spec}\mathsf{QCoh}(X)$.*

[30] Or over $\mathbb{Q}$, or a $\mathbb{Z}$-module, etc.

[31] If $X$ is not connected, $\mathsf{Loc}(X) \cong \mathsf{Rep}(\pi_{\le 1}(X))$, the representations of the fundamental groupoid, but one has to choose a basepoint anyways to define a fiber functor.

[32] There are a few other, minor conditions.

So knowing how quasicoherent sheaves map to conventional module categories determines $X$ completely.

This ties to noncommutative geometry! There are many approaches, but one says that you care about a scheme (or stack) through its stable $\infty$-category of complexes in quasicoherent sheaves. This motivates the following, somewhat curious, definition.

**Definition 9.7.** A *noncommutative scheme* is a stable $\infty$-category.

Then, one can consider "noncommutative motives," a universal category that factors through maps to abelian categories (or various variants thereof).

Anyways, on to the crucial theorem. We'll need it to see that the category of equivariant perverse sheaves on the affine Grassmannian produces a reductive group.

**Theorem 9.8.** *Let $G$ be a connected affine group scheme in characteristic $0$. Then, $\mathsf{Rep}_k(G)$ is semisimple iff $G$ is reductive.*

The proof will be through a series of lemmas.

**Lemma 9.9.** *We may assume $k$ is algebraically closed, because $\mathsf{Rep}_k(G)$ is semisimple iff $\mathsf{Rep}_{\overline{k}}(G)$ is.*

*Proof.* Let $X \in \mathsf{Rep}_k(G)$. Then, $X$ is semisimple as a $G$-representation iff it's semisimple as a $\mathfrak{g}$-representation iff it's semisimple as a representation of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. This is equivalent to $X \otimes \overline{k}$ being semisimple over $\mathcal{U}(G) \otimes \overline{k} = \mathcal{U}(G_{\overline{k}})$.

Thus, if $\mathsf{Rep}_{\overline{k}}(G_{\overline{k}})$ is semisimple, so is $\mathsf{Rep}_k(G)$. Conversely, consider an $\overline{X} \in \mathsf{Rep}_{\overline{k}}(G_{\overline{k}})$. If $k'/k$ is finite, then $\overline{X} = \mathrm{Ind}_{k'}^{k} X'$, and $X = \mathrm{Res}_{k}^{k'} X'$, and facts about induction and restriction suffice. $\square$

The following lemma sounds trivial, but is true because semisimple *a priori* means different things for Lie algebras and representations.

**Lemma 9.10.** *If $\mathfrak{g}$ is a semisimple Lie algebra, then every finite-dimensional representation of $\mathfrak{g}$ is semisimple.*

**Lemma 9.11.** *Let $N \lhd G$ be a closed normal subgroup. If $\rho\colon G \to \mathrm{GL}(V)$ is semisimple, then $\rho|_N\colon N \to \mathrm{GL}(V)$ is also semisimple.*

*Proof.* It suffices to assume $V$ is simple, and choose a nonzero simple $N$-submodule $Y$. For all $g \in G(k)$, $gY$ is still simple over $N$, so $\sum_{g\in G(k)} gY$ is a nonzero, $G$-stable subspace of $X$, hence must be $X$. Thus, $X$ is a sum of simple $N$-modules. $\square$

*Proof of Theorem 9.8.* First, we'll assume $G$ is reductive. Then, there's a torus $Z$ such that $G = Z \cdot G'$ where $G'$ is the derived subgroup, which is semisimple. Hence for any $X \in \mathsf{Rep}_k(G)$, as a $Z$-representation, $X$ is a direct sum of $G'$-stable $Z$-subrepresentations $X_i$. Since $G'$ is semisimple, $X$ is semisimple too.

The other direction will be more useful to us. Assume $G$ is finite type over an algebraically closed field (using the lemmas, we can do this), and assume $\mathsf{Rep}(G)$ is semisimple; we will prove $G$ is reductive.

Let $V$ be a faithful representation of $G$, and $N$ be the unipotent radical. Then, $V$ is semisimple as an $N$-module, by Lemma 9.11, so it's a sum of simple $N$-modules $V_i$. The Lie-Kolchin theorem says that solvable implies each $V_i$ is one-dimensional.

Since $N$ is unipotent, it has a fixed vector, and therefore $V$ is a trivial $N$-representation. Since $V$ is faithful, this implies $N$ is trivial, so $G$ is semisimple. $\square$

The Lie-Kolchin theorem is not a crazy proof-killer: it's the general formulation of the fact that if you have a solvable group, you can write it as a group of upper triangular matrices. Hence a semisimple representation of such a group is necessarily one-dimensional.

**Example 9.12** (Gabriel-Kuhn-Popesco)**.** Let $\mathsf{A}$ be a cocomplete abelian category with a projective generator $P$, so that $\mathsf{A} \cong \mathsf{Mod}_{\mathrm{End}_{\mathsf{A}}(P)}$, and this equivalence comes as a pair of adjoint functors from $\mathsf{Mod}_{\mathrm{End}_{\mathsf{A}}(P)}$ and $\mathsf{A}$. If $\{C_\alpha\}$ is a generating set of $\mathsf{A}$ and $\mathsf{R}$ is the full subcategory generated by this set, then $\mathsf{Mod}_{\mathsf{R}} = \mathsf{Fun}(\mathsf{R}, \mathsf{Ab})$, and you can get adjunctions between $\mathsf{Mod}_{\mathsf{R}}$ and $\mathsf{A}$. This says that every (cocomplete, with a projective generator) abelian category $\mathsf{A}$ is a localization of a category of modules, which is nice. $\blacktriangleleft$