M392C NOTES: SYMPLECTIC TOPOLOGY

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These notes were taken in UT Austin's M392C (Symplectic Topology) class in Fall 2016, taught by Robert Gompf. I live-TEXed them using vim, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

Contents

Lecture 1.

Symplectic Vector Spaces: 8/24/16

Here are a few references for this class.

- There's a book by McDuff and Salaman; in fact, there are three considerably different editions, but all are useful.
- The book by ABKLR (Aebischer, Borer, Kalin, Leuenberger, and Reimann).
- Finally, the book by the professor and Stipsicz will be useful for some parts.

As an overview, symplectic topology is the study of symplectic manifolds.

Definition 1.1. A *symplectic manifold* is a manifold together with a symplectic form.

We'll define symplectic forms in a moment, but first explain where this field arose from. One one hand, symplectic forms arise naturally from mathematical physics in the Hamiltonian formulation, and these days also appear in quantum field theory. On the other, algebraic and complex geometers found that Kähler manifolds naturally have a symplectic structure.

Intuitively, a symplectic manifold is akin to a constant-curvature Riemannian manifold, but where the symmetric bilinear form is replaced with a skew-symmetric bilinear form. (If you don't know what a Riemannian manifold is, that's okay; it will not be a prerequisite for this class.) The constant-curvature condition means that any two points have isomorphic local neighborhoods, so all questions are global; similarly, we will impose a condition on symplectic manifolds that ensures that all questions about symplectic manifolds are global.

There's also a field called symplectic geometry; it differs from symplectic topology in, among other things, also looking at local questions. But there's a reason there's no such thing as "Riemannian topology:" a Riemannian structure is very rigid, and so cutting and pasting Riemannian manifolds, especially constant-curvature ones, isn't fruitful. But symplectic manifolds have a flexibility that allows cutting and pasting to work, if you're clever. To understand this, we will have to spend a little time understanding the local structure.

Another analogy, this time with three-manifolds, is Thurston's geometrization conjecture (now a theorem, thanks to Perelman). This states that any three-manifold may be cut along sphere and tori into pieces that have natural geometry, and are almost always have constant negative curvature, hence are *hyperbolic*, so three-manifold topologists have to understand hyperbolic geometry. Symplectic topology is the analogue in the world of four-manifolds. Not all four-manifolds have symplectic structures; in fact, there exist smooth four-manifolds that are homeomorphic, but one admits a symplectic structure and the other doesn't, so they're not diffeomorphic. We've classified topological four-manifolds, but not smooth ones, so symplectic topology is a very useful tool for this. Three-manifold topologists might also care about the three-manifolds that are boundaries of four-manifolds: if the four-manifold is symplectic, its boundary has a natural structure as a *contact manifold*. The professor plans to teach a course on contact manifolds in a year.

There's a basic principle in geometry and analysis that, in order to understand nonlinear things, one first must understand linear things. Before you understand multivariable integration, you will study linear algebra and the determinant. Before understanding Riemannian geometry, you will learn about inner product spaces. In the same way, we begin with symplectic vector spaces.

Definition 1.2. A *symplectic vector space* is a finite-dimensional[1] real vector space V together with a skew-symmetric bilinear form ω that is *nondegenerate*, i.e. if $v \in V$ is such that for all $w \in V$, $\omega(v, w) = 0$, then $v = 0$.

Succinctly, nondegeneracy means every nonzero vector pairs nontrivially with something. This is a very similar condition to the ones imposed for inner product spaces as well as the indefinite forms attached to spaces in relativity theory.

Example 1.3. Our prototypical example is $\mathbb{C}^n = \mathbb{R}^{2n}$ as a real vector space. The standard inner product is the dot product $\langle -, - \rangle$; we'll define $\omega(v, w) = (iv, w)$. This is clearly still real bilinear; let's verify this is a symplectic form.

First, why is it skew-symmetric? $\omega(w, v) = \langle iw, v\rangle = \langle v, iw\rangle$. Since multiplication by i is orthogonal (it's a rotation), then it preserves the inner product, so $\langle v, iw\rangle = \langle iv, i^2 w\rangle = -\langle iv, w\rangle = -\omega v, w$, so ω is skew-symmetric.

Nondegeneracy is simple: any $v \neq 0$ has a $w \neq 0$ such that $\langle v, w\rangle \neq 0$, so $\omega v, iw = \langle iv, iw\rangle = \langle v, w\rangle \neq 0$.

If we take the standard complex basis $e_1, \dots, e_n$ for $\mathbb{C}^n$, let $f_j = ie_j$; then, $(e_1, f_1, \dots, e_n, f_n)$ is a real basis for $\mathbb{C}^n$; we will take this to be the standard basis for $\mathbb{C}^n$ as a symplectic vector space. This is because each (e_i, f_i) is a real basis for a $\mathbb{C}^1$ summand corresponding to the usual basis $(1, i)$ for $\mathbb{C}$, so this basis jives with the decomposition $\mathbb{C}^n = \mathbb{C} \oplus \mathbb{C} \oplus \cdots \oplus \mathbb{C}$.

This is a positively oriented basis, and in fact is consistent with the canonical orientation of a complex vector space, because it arises in an orientation-preserving way from the basis $(1, i)$ for $\mathbb{C}$, which defines the canonical orientation.

This basis defines a dual basis $e_1^*, f_1^*, \dots, e_n^*, f_n^*$ for the dual space $(\mathbb{R}^{2n})^*$.[2] This allows us to calculate ω in coordinates:

$$\omega = \sum_{j=1}^{n} e_j^* \wedge f_j^*. \tag{1.4}$$

Thus, $\omega(e_j, f_j) = 1 = -\omega(f_j, e_j)$, and all other pairs of basis vectors are orthogonal (evaluate to 0). This defines the same form ω because they agree on the standard basis, because $f_j = ie_j$ and $e_1, \dots, e_n$ is an orthonormal basis for the inner product.

The analogue to an orthonormal basis for a symplectic vector space is a symplectic basis, where elements come in pairs.

Definition 1.5. If $(e_1, f_1, \dots, e_n, f_n)$ is a basis for a symplectic vector space (V, ω) such that $\omega(e_j, f_j) = 1 = -\omega(f_j, e_j)$ for all j and all other pairs of basis vectors are orthogonal, then the basis is called a *symplectic basis.*

Recall that we also have a Hermitian inner product on $\mathbb{C}^n$, defined by

$$h(v, w) = \sum_{j=1}^{n} \overline{v}_j w_j.$$

This is bilinear over $\mathbb{R}$, but not over $\mathbb{C}$; it's $\mathbb{C}$-linear in the second coordinate, but conjugate linear in the first. The Hermitian analogue of the symmetry of an inner product (or the skew-symmetry of a symplectic form) is $h(w, v) = \overline{h(v, w)}$. Thus, $\operatorname{Re} h$ is symmetric, and $\operatorname{Im} h$ is skew-symmetric: $\operatorname{Re} h$ is the standard real inner product on $\mathbb{C}^n = \mathbb{R}^{2n}$, and $\operatorname{Im} h$ is the symplectic form ω.[3]

[1] One can talk about infinite-dimensional symplectic vector spaces, and there are useful in some contexts, but all of our symplectic vector spaces will be finite-dimensional.

[2] Recall that if V is a finite-dimensional real vector space, its *dual space* is V^*, the space of linear functions from V to $\mathbb{R}$. A basis $e_1, \dots, e_n$ of V induces a basis $e_1^*, \dots, e_n^*$ of V^*, defined by $e_j^*(e_i) = \delta_{ij}$: 1 if i and j agree, and 0 otherwise.

[3] Some authors reverse the order for h, so that it's $\mathbb{C}$-linear in the first coordinate but not the second; in this case, we'd get $\operatorname{Im} h = -\omega$. There are a lot of minus signs floating around in symplectic topology, and different authors place them in different places.

Example 1.6. As a special case of the previous example, $\mathbb{C}^1 = \mathbb{R}^2$ as a symplectic vector space has ω as the usual (positive) area form: $\omega = e \wedge f = \mathrm{d}x \wedge \mathrm{d}y$.

Suppose (V, ω_V) and (W, ω_W) are symplectic vector spaces; then, their direct product (or equivalently, direct sum) $V \times W$ has a symplectic structure defined by

$$\omega_{V \times W} = \pi_1^* \omega_V + \pi_2^* \omega_W,$$

where $\pi_1 : V \times W \twoheadrightarrow V$ and $\pi_2 : V \times W \twoheadrightarrow W$ are projections onto the first and second coordinates, respectively. This is a linear combination of skew-symmetric forms, hence is skew-symmetric, and if $\omega_{V \times W}(u_1, u_2) = 0$ for all $u_2 \in V \times W$, then $\pi_1 u_1 = 0$ and $\pi_2 u_1 = 0$, so $u_1 = 0$. Thus, $(V \times W, \omega_{V \times W})$ has a symplectic structure, called the *symplectic orthogonal sum* of V and W since V and W are orthogonal in it.

Not only is $\mathbb{C}^n$ the direct sum of n copies of $\mathbb{C}$, but also the standard symplectic structure on $\mathbb{C}^n$ is the symplectic orthogonal sum of n copies of the standard structure on $\mathbb{C}$: (1.4) explicitly realizes ω as a sum of pullbacks of area forms. The complex structures fit together, the orientations fit together, and the symplectic structures fit together, all nicely.

Subspaces.

Definition 1.7. Suppose V is a symplectic vector space and $W \subset V$ is a subspace. Then, its *orthogonal complement* is the subspace $W^\perp = \{v \in V \mid \omega(v, w) = 0 \text{ for all } w \in W\}$.

The definition is familiar from inner product spaces, but there are a few major differences in what happens afterwards. The first theorem is the same, though:

Theorem 1.8. *If W is a subspace of a symplectic vector space V, then* $\dim W + \dim W^\perp = \dim V$.

Proof. There's a linear map $\varphi : V \longrightarrow V^*$ assigning v to the linear transformation $\varphi(v) : V \longrightarrow \mathbb{R}$ that sends $w \mapsto \omega(v, w)$. Since ω is nondegenerate, then φ is injective. Since V and V^* have the same dimension, φ is an isomorphism. The image $\varphi(W^\perp)$ is the space of functions in V^* that vanish on W, which is isomorphic to the space of functions on V/W, i.e. $\varphi : W^\perp \longrightarrow (V/W)^*$ is injective, and in fact an isomorphism: any function on V/W lifts to a function on V vanishing on W, and then can be pulled back by φ into W. Thus, $\dim W^\perp = \dim(V/W)^* = \dim(V/W) = \operatorname{codim} W$. ⊠

The above proof also works for symmetric nondegenerate bilinear forms. What's different is that W and $W^\perp$ do not always sum to V in the symplectic case. In particular, every vector is orthogonal to itself.

Lecture 2.

Symplectic Vector Spaces are Complex Vector Spaces: 8/26/16

Recall that last lecture, we talked about symplectic vector spaces. A symplectic vector space is a finite-dimensional real vector space V together with a skew-symmetric, nondegenerate bilinear form ω. For a subspace $W \subset V$, we defined $W^\perp$, the vectors that pair to 0 with W under ω, and showed that $\dim W + \dim W^\perp = \dim V$.

Corollary 2.1. *If $W \subset V$ is a subspace, then $(W^\perp)^\perp = W$.*

Proof. Clearly, $W \subset (W^\perp)^\perp$, and they have the same dimension. ⊠

This all is also true for inner product spaces, but things soon begin looking different. For any one-dimensional subspace $W = \operatorname{span} v$, $\omega(v, v) = -\omega(v, v) = 0$ by skew-symmetry, so in this case $W \subset W^\perp$. Switching W and $W^\perp$, one sees that every codimension-1 space contains its complement.

Definition 2.2. Let W be a subspace of the symplectic vector space V.

- If $W \subset W^\perp$, then W is an *isotropic subspace.*
- If $W^\perp \subset W$, then W is a *coisotropic subspace.*
- If both of these are true, so $W = W^\perp$, then W is a *Lagrangian subspace.*

If W is isotropic, then $\dim W \le (1/2) \dim V$; if W is coisotropic, then $\dim W \ge (1/2) \dim V$, and if W is Lagrangian, then $\dim W = (1/2) \dim V$. Moreover, W is isotropic iff $W^\perp$ is coisotropic, and vice versa.

Example 2.3. Last time, we discussed the standard symplectic structure on $\mathbb{C}^n$. The subspace $\mathbb{R}^n \subset \mathbb{C}^n$ is Lagrangian: if $v, w \in \mathbb{R}^n$, $\omega(v, w) = \langle iv, w \rangle$, but iv is purely imaginary and w is purely real, so $\omega(v, w) = 0$.

Definition 2.4. If $W \subset V$ is a subspace such that ω restricts to a nondegenerate form on ω, then W is a *symplectic subspace.*

For example, the standard inclusion $\mathbb{C}^k \subset \mathbb{C}^n$, for $1 \le k \le n$, is a symplectic subspace.

Lemma 2.5. *Suppose $W \subset V$ is a subspace. The following are equivalent:*

(1) *W is symplectic.*
(2) *$W \cap W^\perp = \{0\}$.*
(3) *$W + W^\perp = V$.*
(4) *$W^\perp$ is symplectic.*
(5) *$V = W \oplus W^\perp$ is a symplectic orthogonal sum of symplectic subspaces.*

Proof. W is symplectic iff $\omega|_W$ is nondegenerate, meaning there's no nonzero $v \in W$ such that $\omega v, w = 0$ for all $w \in W$. This means exactly that there's no nonzero $v \in W$ that's also in $W^\perp$. Thus, (1) and (2) are equivalent; (3) is equivalent to (2) by usual linear algebra, and since (2) is symmetric in W and $W^\perp$, therse are all equivalent to (4).

In this situation, if $\pi_1 : V \twoheadrightarrow W$ and $\pi_2 : V \twoheadrightarrow W^\perp$ are orthogonal projections onto W and $W^\perp$, respectively, then $\omega = \pi_1^*\omega + \pi_2^*\omega$. This follows because any $v \in V$ may be uniquely written as $v = w + w^\perp$ for $w \in W$ and $w^\perp \in W^\perp$ by (3); then,

$$\begin{aligned}\omega(v_1, v_2) &= \omega(w_1 + w_1^\perp, w_2 + w_2^\perp) \\ &= \omega(w_1, w_2) + \omega(w_1^\perp + w_2^\perp) + \omega(w_1, w_2^\perp) + \omega(w_1^\perp, w_2),\end{aligned}$$

but the cross terms vanish. Thus, (5) follows. ⊠

Theorem 2.6. *Every symplectic vector space (V, ω) is isomorphic to the standard example $(\mathbb{C}^n, \omega)$.*

An isomorphism of symplectic vector spaces is what you would expect: an isomorphism of vector spaces that preserves the symplectic form.

Proof. If $\dim V = 0$, there's nothing to say, so assume $\dim V > 0$. In this case, there's a nonzero $v \in V$, so by nondegeneracy a nonzero $w \in V$ such that $\omega(v, w) \ne 0$. Let $e_1 = v$ and $f_1 = (1/\omega(v, w))w$, so $\omega(e_1, f_1) = 1$. Thus, $\operatorname{span}\{e_1, f_1\} \cong \mathbb{C}$ under the unique map that sends $e_1 \mapsto 1$ and $f_1 \mapsto i$. Hence, $\operatorname{span}\{e_1, f_1\}$ is a symplectic subspace, so by Lemma 2.5, so is its orthogonal complement, which is a symplectic vector space of lower dimension, so by induction it's isomorphic to $\mathbb{C}^{\dim V - 2}$. ⊠

Corollary 2.7. *Every symplectic vector space is even-dimensional.*

Corollary 2.8. *A skew-symmetric bilinear form ω on a $2n$-dimensional vector space V is nondegenerate iff $\underbrace{\omega \wedge \cdots \wedge \omega}_{n} \ne 0$.*

Proof. Since ω has even degree, the wedge product is symmetric, so this isn't automatically zero.

By Theorem 2.6, $(V, \omega) \cong (\mathbb{C}^n, \omega_{\mathrm{std}})$, and $\omega_{\mathrm{std}} = \sum e_j^* \wedge f_j^*$. Thus, taking the n^{th} wedge power of this form, all terms where a dual basis element appears more than once is zero, so the only nonzero terms are of the form $e_1^* \wedge e_2^* \wedge e_2^* \wedge f_2^* \wedge \cdots \wedge e_n^* \wedge f_n^*$. There is at least one of these,[4] so $\omega^{\wedge n}$ is a volume form on V, so must be nonzero.

Conversely, if ω is degenerate, then there's a nonzero $v_1 \in V$ such that $\omega(v_1, -) = 0$, so we can extend to a basis $v_1, \dots, v_{2n}$ for V; then, $\omega \wedge \cdots \wedge \omega(v_1, \dots, v_{2n}) = 0$, because every term either has all $2n$ basis vectors, so it has a v_1 which is paired with something and becomes 0. Thus, $\omega \wedge \cdots \wedge \omega = 0$: since $\dim \Lambda^{2n}(V) = 1$, if it were nonzero, it would be a nonzero multiple of the volume form, which is nonzero on any basis of V. ⊠

This is another common way to express the nondegeneracy condition in the literature.

Corollary 2.9. *Every symplectic vector space has a canonical orientation.*

This orientation is the one determined by the volume form $\omega \wedge \cdots \wedge \omega$, which is consistent with the standard orientation on $\mathbb{C}^n$. Switching the sign of the symplectic form produces a valid symplectic form, but depending on whether n is odd or even, this might not change the orientation.

Remark. Every finite-dimensional complex vector space V has a canonical orientation as a real vector space

[4] If you count carefully, there are actually $n!$ such terms, but we don't need this in the proof.

Proof. We have a standard orientation on $\mathbb{C}^n$ ($\mathbb{C}$ has a standard orientation where i is positively oriented; then, take the direct-sum orientation); choose an isomorphism $\varphi : \mathbb{C}^n \to V$ to orient V.

This could work for $\mathbb{R}^n$, which has an orientation; the key is that every complex linear isomorphism $A : \mathbb{C}^n \to \mathbb{C}^n$ is orientation-preserving. Topologically, this follows because $\mathrm{GL}(n, \mathbb{C})$ has a single connected component (whereas $\mathrm{GL}(n, \mathbb{R})$ has two). More explicitly, we show $\det_{\mathbb{C}} A = \|\det_{\mathbb{R}} A_{\mathbb{R}}\|^2$ (where $A_{\mathbb{R}}$ is the matrix of A as an endomorphism of $\mathbb{R}^{2n}$), which is positive, because $A \in \mathrm{GL}(2n, \mathbb{R})$.

Equality is easy if A is diagonal, and hence also if A is diagonalizable. But the real and complex determinants are continuous, and diagonalizable matrices are dense (since any matrix with distinct eigenvalues is diagonalizable, and if two eigenvalues coincide, one can bump one eigenvalue by an arbitrary small amount). ⊠

If V is a symplectic vector space, the canonical orientations induced as a symplectic vector space and as a complex vector space agree. In particular, if V and W are symplectic, $V \oplus W$ is canonically a symplectic vector space, and its canonical orientation is the direct-sum orientation induced from the orientations of V and W: both are oriented by the same $\omega^{\wedge n}$.

Lecture 3.

Symplectic Manifolds: 8/29/16

Recall that last time, we showed that every symplectic vector space is isomorphic to $\mathbb{C}^n$ with the standard symplectic form, so there's really only one symplectic vector space of every even dimension.

Once you know what an inner product is, you can place an inner product on each tangent space on a manifold in a smoothly varying way, which defines a *Riemannian manifold*. This allows one to talk about lengths and angles of vectors, and to start doing geometry, rather than just topology. We would like to play the same game with symplectic forms.

Recall that a differential form α is *closed* if $\mathrm{d}\alpha = 0$.

Definition 3.1. A *symplectic manifold* is a smooth manifold M with a nondegenerate[5] closed 2-form ω, called the *symplectic form*.

Requiring ω to be closed is precisely what homogenizes the local structure of a symplectic manifold.

Corollary 3.2. *Every symplectic manifold is oriented and even-dimensional.*

This is because the symplectic form makes the tangent spaces into symplectic vector spaces.

Moreover, every symplectic manifold has a canonical volume form $\Omega = \omega^{\wedge n}$, since we required ω to be nondegenerate. This volume form determines the orientation on M.

Example 3.3.

(1) $\mathbb{C}^n = \mathbb{R}^{2n}$ with the standard symplectic form is a symplectic manifold: at each point, the tangent space can be canonically identified with $\mathbb{R}^{2n}$ again, with the usual symplectic structure. This can be globally coordinatized, with $\mathbb{C}^n$ coordinates $(z_1, \dots, z_n)$, or if $z_j = x_j + iy_j$, real coordinates $(x_1, y_1, \dots, x_n, y_n)$. In these coordinates, the symplectic form is

$$\omega_{\mathrm{std}} = \sum_{j=1}^{n} \mathrm{d}x_j \wedge \mathrm{d}y_j.$$

This is because $(\mathrm{d}x_1, \mathrm{d}y_1, \dots, \mathrm{d}x_n, \mathrm{d}y_n)$ is pointwise a basis for every tangent space. The associated volume form is $n!$ times the usual volume form.

The only thing we need to check is that ω_{std} is closed; but every term has a $\mathrm{d}\alpha$ in it, for some α, so $\mathrm{d}\omega_{\mathrm{std}} = 0$. Additionally, ω_{std} is *exact*:

$$\omega_{\mathrm{std}} = \mathrm{d}\left(\sum_{j=1}^{n} x_j \wedge \mathrm{d}y_j\right).$$

A symplectic manifold with an exact symplectic form is called an *exact symplectic manifold*; these are an interesting class of manifolds to study. We'll see that an exact symplectic manifold cannot be closed (i.e. compact and boundaryless).

[5] A form is nondegenerate if it's nondegenerate at each point in M, or equivalently if its top exterior power is a volume form.

(2) In the two-dimensional case, every oriented surface is a symplectic manifold: we saw that in (real) dimension 2, a symplectic form is the same thing as an area form, and all 2-forms are closed (since Ω^3 of any surface is 0). So an area form defines a symplectic structure on any oriented surface.

Definition 3.4. Let M be a symplectic manifold and $N \subset M$ be a submanifold. Then, N is *isotropic*, (resp. *coisotropic*, *Lagrangian*, or *symplectic*) if all of its tangent spaces are isotropic (resp. coisotropic, Lagrangian, or symplectic) subspaces of the tangent spaces to M.

For example, every one-dimensional subspace of a symplectic vector space is isotropic, so every curve in a symplectic manifold is isotropic; similarly, every codimension one manifold is coisotropic. Additionally, $\mathbb{R}^n$ and $i \cdot \mathbb{R}^n$ are Lagrangian submanifolds of $\mathbb{C}^n$, since this is true at the vector space level.

Lemma 3.5. *Suppose $i : N \hookrightarrow M$ is an embedding and ω is a closed form on M. Then, $\omega|_N = i^*\omega$ is a closed form on N.*

Proof. The exterior derivative commutes with pullback, so $\mathrm{d}(i^*\omega) = i^*(\mathrm{d}\omega) = 0$. ⊠

This is useful for the following reassuring corollary.

Corollary 3.6. *A symplectic submanifold is a symplectic manifold.*

Remark. This discussion generalizes to placing a symplectic structure on vector bundles, though there's no analogue of the closed condition. These are occasionally useful.

Example 3.7.

(1) If M and N are symplectic, then $M \times N$ inherits a symplectic structure, since at each $(x, y) \in M \times N$, $T_{(x,y)}(M \times N) = T_xM \oplus T_yN$, so we take the orthogonal symplectic sum of these vector spaces at each point. The symplectic form is a sum:

$$\omega = \pi_M^*\omega_M + \pi_N^*\omega_N, \tag{3.8}$$

where π_M and π_N are the projections onto M and N, respectively. For any $x \in M$, $\{x\} \times N$ is a symplectic submanifold, and similarly in the other coordinate. Moreover, if $L_M \subset M$ is Lagrangian (resp. isotropic) and $L_N \subset N$ is Lagrangian (resp. isotropic), then $L_M \times L_N \subset M \times N$ is Lagrangian (resp. isotropic), by plugging into (3.8). This is a useful way to construct nontrivial Lagrangian submanifolds; for example, $S^1 \subset \mathbb{C}$ is Lagrangian, so $S^1 \times S^1 \subset \mathbb{C}^2$ is a Lagrangian torus, called a *Clifford torus.*

(2) Let M be any smooth manifold, and let T^*M be its cotangent bundle.[6] There is a canonical 1-form on T^*M, called the *Liouville 1-form*: given a $z \in T^*M$ and an $X \in T_z(T^*M)$, $\mathrm{d}\pi(X) \in T_xM$, where $x = \pi(z)$. Since $z \in T_x^*M$, we can let $\theta_z(X) = z(\mathrm{d}\pi(X))$, which defines a 1-form.

We'd like to check this is smooth by writing it in local coordinates: if $(q_1, \dots, q_n)$ is a local system of coordinates for M, then T_x^*M has a basis $(\mathrm{d}q_1)_x, \dots, (\mathrm{d}q_n)_x$, so $z = \sum p_i\,\mathrm{d}q_i$. Thus, if z varies smoothly, this expression will also vary smoothly. These p_i are fiber coordinates, so coordinates for T^*M near x are $(q_1, \dots, q_n, p_1, \dots, p_n)$, and $\pi(q_1, \dots, q_n, p_1, \dots, p_n) = (q_1, \dots, q_n)$: π projects from T^*M to M. We will abuse notation slightly to write $\mathrm{d}q_i = \pi^*\mathrm{d}q_i$. With this notation,

$$\begin{aligned} &\theta_z(X) = z(\mathrm{d}\pi(X)) = \left(\sum p_i\,\mathrm{d}q_i\right)(\mathrm{d}\pi(X)) = \sum p_i\,\mathrm{d}q_i(X) \\ \implies &\theta_z = \sum p_i\,\mathrm{d}q_i. \end{aligned}$$

This is very simple, but maybe it's surprising this is coordinate-invariant. Let

$$\omega = -\mathrm{d}\theta = \sum_{i=1}^n \mathrm{d}q_i \wedge \mathrm{d}p_i.$$

Then, ω is a canonical, globally defined 2-form. It's explicitly exact, hence closed, and it's nondegenerate, so we've canonically defined a symplectic structure on any cotangent bundle.

The fibers are purely p_i coordinates, and therefore are Lagrangian submanifolds, giving T^*M a structure of a *Lagrangian fibration.* The zero section is also Lagrangian.

Exercise 3.9. Show that a section of T^*M is Lagrangian iff the 1-form α that cuts it out is closed.

There are several ways to prove this, all of which are a good way to check your understanding.

Next time, we'll relate symplectic geometry to physics.

[6] The *cotangent bundle* is the bundle that is the cotangent space (dual to the tangent space) T_x^*M at each $x \in M$; this defines a smooth vector bundle. A section of the tangent bundle is a one-form.

Lecture 4.

Hamiltonian Vector Fields: 8/31/16

Last time, we showed that if M is any manifold, its cotangent bundle T^*M is a symplectic manifold: it has a canonical 1-form θ, and the symplectic form is $\omega = -\mathrm{d}\theta$. If $(q_1, \dots, q_n)$ is a local coordinate system for M, then $(q_1, \dots, q_n, p_1, \dots, p_n)$ is a local coordinate system for T^*M, and

$$\omega = \sum_{i=1}^{n} \mathrm{d}q_i \wedge \mathrm{d}p_i.$$

For example, $T^*\mathbb{R}^n$ is isomorphic, as symplectic manifolds, to $\mathbb{C}^n$. We saw that the fibers of the projection $T^*M \longrightarrow M$ are Lagrangian, providing a simple example of a cool notion, a Lagrangian fibration; moreover, a section of this map can be identified with a 1-form α, and the section is Lagrangian iff $\mathrm{d}\alpha = 0$ (e.g. the zero section is Lagrangian), which is a good exercise to work out.

Definition 4.1. Let $L : V \longrightarrow W$ be a linear map between symplectic vector spaces (V, ω_V) and (W, ω_W). Then, L is *symplectic* if $L^*\omega_W = \omega_V$.

The analogous condition on a map of inner product spaces defines an *isometric* linear map.

Proposition 4.2. *If L is a symplectic linear map between symplectic vector spaces (resp. isometry of inner product spaces), then L is injective.*

Proof. Suppose $L(v) = 0$. Then, for all $v' \in V$, $\omega(v, v') = L^*\omega(v, v') = \omega(Lv, Lv') = 0$, since $Lv = 0$, so $v = 0$. For the inner product case, replace $\omega\cdot, \cdot$ with $\langle \cdot, \cdot \rangle$. ⊠

Of course, we use the linear case to apply it to manifolds.

Definition 4.3. Let $f : M \longrightarrow N$ be a smooth map of symplectic manifolds (M, ω_M) and (N, ω_N). Then, f is *symplectic* if $\mathrm{d}f|_x$ is symplectic for all $x \in M$, or equivalently $f^*\omega_N = \omega_M$.

Since $\mathrm{d}f|_x$ is injective everywhere, a symplectic map must be an immersion; similarly, a map of Riemannian manifolds preserving the inner product, called a *isometric immersion* or *Riemannian immersion*, is in particular an immersion.

Definition 4.4. If $f : M \longrightarrow N$ is a diffeomorphism, then f is symplectic iff f^{-1} is; such a map is called a *symplectomorphism.*

There are similar notions for contact manifolds, Lipschitz manifolds, etc. The point of a symplectomorphism is that the two manifolds are the same as symplectic manifolds; there is no way for symplectic geometry to tell them apart.

Proposition 4.5. *Let $f : M \longrightarrow B$ be a diffeomorphism. Then, $f^* : T^*N \longrightarrow T^*M$ is a symplectomorphism.*

This follows because ω is canonical: we constructed it in coordinates, but a diffeomorphism induces an isomorphism of coordinate charts between an atlas of M and an atlas of N, so the definitions of ω_M and $f^*\omega_N$ are the same in all coordinate charts. This invariance under diffeomorphism is one reason it's called canonical.

If $f : M \longrightarrow N$ is a local diffeomorphism of manifolds, then we can globally invert f^*, and the inverse $(f^*)^{-1} : T^*M \longrightarrow T^*N$ is symplectic.

Suppose $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is smooth. Then, we can talk about level sets of its *gradient*

$$\nabla f = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} e_j,$$

which are perpendicular to the gradient vector field using the inner product structure. Thus, we can generalize to Riemannian manifolds: on any smooth manifold,

$$\mathrm{d}f = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \mathrm{d}x_j$$

makes sense (this description is in local coordinates), but we need the metric to define an isomorphism $T_xM \longrightarrow T_x^*M$ that smoothly varies in x; this isomorphism identifies $\mathrm{d}f$ with ∇f on any Riemannian manifold.

For symplectic manifolds, we also have an isomorphism $T_xM \cong T_x^*M$, so we can try to do the same thing.

Definition 4.6. Let (M, ω) be a symplectic manifold and $H : M \to \mathbb{R}$ be smooth. Then, let v_H be the vector field defined such that $\mathrm{d}H = \omega(v_H, -)$ at every point. This v_H is called the *Hamiltonian vector field* with *Hamiltonian* or *energy functional* H.

This is the symplectic analogue of the gradient: just as we can analyze gradients to understand geometry in $\mathbb{R}^2$ and $\mathbb{R}^3$, we will exploit the Hamiltonian vector field to learn about geometry.

Let $\gamma(t)$ be a parameterized curve in M. Then, we can see what H does along γ:

$$\frac{\mathrm{d}}{\mathrm{d}t}(H \circ \gamma(t)) = \mathrm{d}H_{\gamma(t)}(\gamma'(t)) = \omega(v_H, \gamma'(t)).$$

Recall that taking the dot product of the gradient and $\gamma'(t)$ tells us now to calculate $(H \circ \gamma)'$ in multivariable calculus. The trick is that row vectors have to become column vectors, which is why we pair using ω.

Now suppose γ is a trajectory of the *Hamiltonian flow* givenby v_H. Then, $v_H(\gamma(t))$ is by definition the velocity vector $\gamma'(t)$; then, the skew-symmetry of ω means

$$\frac{\mathrm{d}}{\mathrm{d}t}(H(\gamma(t))) = \omega(\gamma'(t), \gamma'(t)) = 0.$$

Thus, $H \circ \gamma$ is constant, so γ is contained in a level set of H. Thus, trajectories of the flow are contained in level sets.

If x is a regular value of H, then the level set $P = H^{-1}(x)$ is a codimension-1 manifold, and therefore is coisotropic. In particular, v_H spans $(TP)^\perp \subset TM$.

Suppose we have local coordinates for M $(q_1, \dots, q_n, p_1, \dots, p_m)$, such that $\omega = \sum \mathrm{dq}_i \wedge \mathrm{d}p_i$.[7] Thus,

$$\begin{aligned} \omega(v_H, \cdot) &= \left(\sum_{i=1}^n \mathrm{d}q_i \wedge \mathrm{d}p_i\right)(v_H, \cdot) \\ &= \sum_{i=1}^n (\mathrm{d}q_i(v_H)\,\mathrm{d}p_i - \mathrm{d}p_i(v_H)\,\mathrm{d}q_i). \end{aligned}$$

In these coordinates, we also know

$$\mathrm{d}H = \sum_{i=1}^n \left(\frac{\partial H}{\partial p_i}\,\mathrm{d}p_i + \frac{\partial H}{\partial q_i}\,\mathrm{d}q_i\right).$$

If γ is a trajectory of the flow for v_H, with coordinates $\gamma(t) = (q_1(t), \dots, q_n(t), p_1(t), \dots, p_n(t))$, then its derivative is $v_H = \dot\gamma(t) = (\dot q_1(t), \dots, \dot q_n(t), \dot p_1(t), \dots, \dot p_n(t))$ (here, the dot means a derivative with respect to time, which is physicists' notation). Putting these together, we obtain *Hamilton's equations*

$$\dot q_i = \frac{\partial H}{\partial p_i} \qquad \dot p_i = -\frac{\partial H}{\partial q_i}. \tag{4.7}$$

These equations are essential in classical mechanics: suppose, for example, we have a spherical pendulum, which may take any position in S^2. But the set of all positions and velocities is TS^2, and the configuration space of position and momentum, the *phase space*, is T^*S^2. Hamiltonian mechanics solves this system by deriving a *potential function* $V : T^*S^2 \to \mathbb{R}$ and the total energy

$$H = V + \frac{\|p\|^2}{2M},$$

where M is the mass of the particle in question. This is the statement that total energy is potential energy plus kinetic energy.

The cotangent bundle always has a symplectic form, and we can study the Hamiltonian vector field and flow associated to this function. The key claim is that the Hamiltonian flow tells you how the system evolves through time.

For example, if a particle freely moves in a force field in $\mathbb{R}^3$, Hamilton's equations imply that $\dot q_i = p_i/m$, which boils down to the classical definition of momentum: $m\vec v = \vec p$. The other Hamilton equation claims that $\dot p_i = -\frac{\partial V}{\partial q_i}$, i.e. $m\vec a = \vec p\,' = -\nabla V$: mass times acceleration (derivative of momentum) is equal to force! These are Newton's laws.

[7] We'll see later that there's always a coordinate neighborhood for which this is true.

Lecture 5.

de Rham Cohomology: 9/2/16

Recall that if M is any manifold, its cotangent bundle T^*M has a canonical symplectic form, and is a symplectic manifold. We also saw that if (X,ω) is a symplectic manifold and $H : X \to \mathbb{R}$ is smooth, we can define a "symplectic gradient" called the Hamiltonian vector field v_H such that $\omega(v_H,\cdot) = \mathrm{d}H$. Unlike the gradient, though, this is tangent to level surfaces of H (since these level sets are codimension 1, hence coisotropic), and spans the normal space to P at x when x is a regular value of H.

We can use this to do classical mechanics. For example, the spherical pendulum has S^2 for its configuration space. Given a potential function V, e.g. the height function $V = mgz$, the total energy is $H = V + \|p\|^2/2m$. If we started with $\mathbb{R}^3$ and some forces, so $T^*\mathbb{R}^3 = \mathbb{C}^3$ with the usual symplectic structure, we ended up deriving Newton's laws. This formalism might seem like a lot of effort to derive that, but it simplifies the analysis, especially in more general physical situations.

Another example: suppose we have an asteroid tumbling in space, so the phase space is $\mathrm{SO}(3) \cong \mathbb{RP}^3$ as manifolds. The cotangent bundle of a Lie group is trivial, so $T^*\mathrm{SO}(3) \cong \mathrm{SO}(3)\times\mathbb{R}^3$ as manifolds, but the symplectic structure is nontrivial. Using the Hamiltonian formalism, we can understand the motion of this asteroid, even when there are other sources of gravity present.

The insight that v_H is tangent to level surfaces of H means that, since the Hamiltonian generally represents the total energy of the system, the total energy of the system is constant: energy is conserved. For the spinning asteroid, the Hamiltonian also measures angular momentum, so we recover conservation of angular momentum; for the system deriving Newton's laws, we recover conservation of linear momentum. Results like this led symplectic geometry to become an integral[8] part of physics.

The symplectic form on a cotangent space is exact, but we can't expect this in general.

Proposition 5.1. *A symplectic form on a closed manifold cannot be exact.*

Proof. Suppose M is a closed manifold and $\omega = \mathrm{d}\theta$ is a symplectic form on M, so that $\Omega = \omega\wedge\cdots\wedge\omega = \mathrm{d}(\theta\wedge\omega\wedge\cdots\wedge\omega)$ is a volume form. Thus, by Stokes' theorem,

$$\int_M \Omega = \int_M \mathrm{d}(\theta\wedge\omega\wedge\cdots\wedge\omega) = \int_{\partial M} \theta\wedge\omega\wedge\cdots\wedge\omega = 0$$

because $\partial M = \emptyset$, but a volume form always has nonzero integral, which is a contradiction. ⊠

To go farther, we need de Rham cohomology. Since this wasn't on the prelim syllabus, we're going to review it briefly. Recall that $\Omega^k(M)$ denotes the space of differental k-forms on a manifold M. The *de Rham complex* is

$$0 \longrightarrow \Omega^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \Omega^2(M) \longrightarrow \cdots \longrightarrow \Omega^n(M) \longrightarrow 0.$$

Since $\mathrm{d}\circ\mathrm{d} = 0$, this feels awfully like a chain complex, except that d increases degree; this is a *cochain complex*, and as such its homology groups are called *cohomology groups.*

Definition 5.2. The k^{th} *de Rham cohomology* of M is the real vector space

$$H^k_{\mathrm{dR}}(M) = \frac{\ker(\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M))}{\operatorname{Im}(\mathrm{d} : \Omega^{k-1}(M) \to \Omega^k(M))}.$$

The kernel of d is called the space of *cocycles* or *closed k-forms*, and the image of (the previous) d is the space of *coboundaries* or *exact k-forms.*

The cohomology measures the failure of closed forms to be exact on a manifold. And we already know the basic invariant, its dimension: $\dim H^k_{\mathrm{dR}}(M) = b_k(M)$, the k^{th} Betti number of M.

Unlike homology, cohomology has a natural product structure: the wedge product $\wedge : \Omega^k(M)\times\Omega^\ell(M) \to \Omega^{k+\ell}(M)$ sends exact forms to exact forms, so it descends to a bilinear map on the quotient

$$\wedge : H^k_{\mathrm{dR}}(M)\times H^\ell_{\mathrm{dR}}(M) \to H^{k+\ell}_{\mathrm{dR}}(M).$$

[8] No pun intended.

Algebraic topologists have a notion of *singular cohomology* $H^k(M;G)$ with coefficients in an arbitrary abelian group G, which has a product structure called the *cup product.* There's a theorem that if $G = \mathbb{R}$, singular cohomology agrees with de Rham cohomology by an identification sending the cup product to the wedge product: de Rham cohomology is an analyst's way of approaching a very algebraic idea.

Corollary 5.3. *The de Rham cohomology of a manifold M depends only on its homotopy type.*

Let's begin using this machinery. The symplectic form ω is closed, so we automatically have a distinguished class $[\omega] \in H^k_{\mathrm{dR}}(M)$. Can we use this to further understand Proposition 5.1?

First, since wedge product descends to de Rham cohomology, this form's n^{th} wedge power is still the volume form:

$$[\Omega] = [\omega \wedge \cdots \wedge \omega] = [\omega] \wedge \cdots \wedge [\omega].$$

The proof of Proposition 5.1 applies to any exact form to show that the integral of an exact form on a closed manifold is 0. Thus, the integral descends to top cohomology.

Proposition 5.4. *Let M be a closed, oriented, m-dimensional manifold. Then, integration $\int : \Omega^m(M) \longrightarrow \mathbb{R}$ descends to a surjective linear map*

$$\int_M : H^m_{\mathrm{dR}}(M) \longrightarrow \mathbb{R}$$

that is an isomorphism if M is connected.

Surjectivity follows from the existence of a volume form, so the map hits a nonzero number; injectivity if M is connected follows from a general fact in algebraic topology, that the top Betti number of a connected manifold is 1 (using, e.g. a triangulation on your manifold and simplicial homology).

In the world of algebraic topology, the integral has a different notation:

$$\int_M \theta = \langle \theta, [M] \rangle.$$

Since we have a canonical symplectic form ω, we have a canonical volume form $\Omega = \omega^{\wedge n}$. Thus, we can make sense of the volume of a closed symplectic manifold, but it would be quite strange for it to be negative.

Theorem 5.5. *Let (M, ω) be a closed, $2n$-dimensional symplectic manifold; then, there exists an $\alpha \in H^2_{\mathrm{dR}}(M)$ such that $\langle \alpha^{\wedge n}, [M] \rangle > 0$.*

In particular, $\alpha^{\wedge n} \in H^{2n}_{\mathrm{dR}}(M)$ cannot be 0.

This is useful as an obstruction to the existence of symplectic structures on closed manifolds.

Example 5.6. Let $M = S^m$. If m is odd, there's obviously no symplectic structure on M, so suppose $m = 2n$ for some $n > 0$. Just like homology is only nonvanishing at 0 and $2n$, the de Rham cohomology of S^{2n} is

$$H^i_{\mathrm{dR}}(S^{2n}) \cong \begin{cases} \mathbb{R}, & i = 0, 2n \\ 0, & \text{otherwise.} \end{cases}$$

Thus, if $n > 1$, there's no symplectic structure. We've already seen that an area form on a surface defines a symplectic structure, so S^2 is the only sphere with a symplectic structure.

Example 5.7. Suppose $M = S^{2k} \times S^{2\ell}$ for $k \neq \ell$ and $k, \ell > 0$. Using a product of cell complexes or a Künneth formula,

$$H^i_{\mathrm{dR}}(M) = \begin{cases} \mathbb{R}, & i = 0, 2k, 2\ell, 2k + 2\ell \\ 0, & \text{otherwise.} \end{cases}$$

Thus, there's no symplectic structure on M unless k or ℓ is 1. Looking at $M = S^2 \times S^{2\ell}$, we know $H^2_{\mathrm{dR}}(M) \cong \mathbb{R}$, so let α be a generator for it. If $\pi_1 : M \longrightarrow S^2$ is projection onto the first factor, then $\alpha = \pi_1^*[\omega]$ for an area form ω of S^2. This is cool and all, but we need $\alpha^{\wedge(1+\ell)} \neq 0$, and this is not true:

$$\alpha \wedge \alpha = \pi_1^* \omega \wedge \pi_1^* \omega = \pi_1^*[\omega \wedge \omega] = 0,$$

because $[\omega \wedge \omega] \in H^4_{\mathrm{dR}}(S^2) = 0$ (there can't be nonzero 4-forms on a 2-manifold). Thus, this has no symplectic structure unless $k = \ell = 1$, in which case we get the standard symplectic structure on the product of two symplectic manifolds.

Lecture 6.

Symplectic Forms on Complex Vector Spaces: 9/6/16

Last time, we talked about a cohomological obstruction to a manifold having a symplectic structure, relating to the existence of a nondegenerate top volume form. Todya, we'll present another obstruction, relating to the existence of a nondegenerate skew-symmetic inner product. To understand this, we must untangle the relations between different kinds of structures on vector spaces.

Definition 6.1. Let V be a real (finite-dimensional) vector space. Then, a *complex structure* on V is a linear map $J : V \longrightarrow V$ such that $J^2 = -I$.

The idea is to turn V into a complex vector space; indeed, if V is already a $\mathbb{C}$-vector space, then we can choose J to be multiplication by i, which is $\mathbb{R}$-linear. Conversely, given a complex structure on a real vector space V, we can define a $\mathbb{C}$-vector space structure by letting $(a + bi)v = (a + bJ)v$; since $J^2 = -I$, this defines an action of $\mathbb{C}$ on V.

Definition 6.2. A *Hermitian inner product* on a complex vector space (V, J) is a real bilinear form $h : V \times V \longrightarrow \mathbb{C}$ such that

(1) for all $v, w \in V$, $h(w, v) = \overline{h(v, w)}$,
(2) $h(v, v) > 0$ unless $v = 0$, and
(3) $h(v, Jw) = ih(v, w)$.

The first and third axioms imply that $h(Jv, w) = -ih(v, w)$: this product is $\mathbb{C}$-linear in the first slot, but *conjugate linear* in the first slot. We can also conclude that $h(v, Jw) = -h(Jv, w)$ (so that J is *skew-adjoint*) and $h(Jv, Jw) = h(v, w)$ (so J preserves h). These two are equivalent for any bilinear form squaring to $-I$.

Let $g = \operatorname{Re} h$; this is an inner product. Similarly, $\omega = \operatorname{Im} h$ is a symplectic form on V. The key is nondegeneracy: $\omega(v, Jv) = \operatorname{Im} h(v, Jv) = \operatorname{Im} ih(v, v) = \operatorname{Re} h(v, v) > 0$ when $v \neq 0$, so every vector pairs nontrivially with something. Also, notice that g and ω are preserved by J. The interplay between g, h, and ω is worth attaching to a word.

Definition 6.3. A skew-symmetric form ω on V *tames* a $\mathbb{C}$-structure J if for all nonzero $v \in V$, $\omega(v, Jv) > 0$.

This implies nondegeneracy: every $v \in V$ pairs nontrivially with Jv. Moreover, ω is nondegenerate on every complex subspace $W \subset V$, because if $v \in W$, then $Jv \in W$.

Definition 6.4. If ω tames J and J preserves ω, we say ω and J are *compatible.*

Proposition 6.5. *If ω tames J, then the orientations induced on V from the symplectic structure and the complex structure agree.*

Proof. This is obvious for $\dim_{\mathbb{C}} V = 1$, because they both determine the same area form, which defines these orientations.

If $\dim_{\mathbb{C}} V > 1$, then let's first assume compatibility. Let W be any complex subspace, and let $W^\perp$ be its *symplectic* orthogonal complement. Since J preserves ω, then $JW^\perp$ is perpendicular to JW. Since W is complex, $JW = W$, so $JW^\perp$ is perpendicular to W, and by a dimension-counting argument, $JW^\perp = W^\perp$. Thus, we can split V into one-(complex-)dimensional subspaces that are pairwise orthogonal (under ω). Using the first part of this proof, these are oriented consistently, and V has the direct sum orientation under both J and ω, so these orientations match up.

Lemma 6.6. *Let $\omega_1, \dots, \omega_m$ be forms taming J on a vector space V. Then, any convex combination $\sum t_i\omega_i$ (where $t_1 + \cdots + t_m = 1$) tames J. The same is true with "tame" replaced with "compatible."*

Proof. For any $v \in V$,
$$\left(\sum_{i=1}^m t_i\omega_i\right)(v, Jv) = \sum_{i=1}^m t_i\omega_i(v, Jv),$$
which is a convex combination of positive numbers, and therefore is convex. Compatibility is a similar argument. ⊠

This can be useful to interpolate between two forms ω and ω', one taming and one compatible. Since this is a continuous family and orientations are a discrete set, then ω and ω' determine the same orientation, completing the proof. ⊠

The takeaway is that every Hermitian structure (h, V) on a real vector space V determines 3 things:

(1) a complex structure J,

(2) an inner product g preserved by J, and
(3) a symplectic structure ω compatible with J.

Theorem 6.7. *Any two of these, suitably compatible, in fact uniquely determine the Hermitian structure and hence the third.*

Proof. Let's start with g and J preserving g; we define $\omega(v, w) = g(Jv, w)$. Since $\omega(v, Jv) = g(Jv, Jv) = g(v, v) > 0$, then ω is nondegenerate and in fact tames J; since $J^2 = -1$, ω is skew-symmetric, hence defines a symplectic structure. Finally, we want to check compatibility with J: $\omega(Jv, Jw) = g(J^2 v, Jw) = g(Jv, w) = \omega(v, w)$.

Given g and ω, we can define $h(v, w) = g(v, w) + i\omega(v, w)$. Then we have to check some conditions, which is a routine calculation. ⊠