PHYSICS 262 NOTES: GENERAL RELATIVITY

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Part 1. Review of Special Relativity

1. Introduction, The Big Picture, and Special Relativity: 9/22/14

We know there are four fundamental forces: gravity (the subject of this course), electromagnetism, and the wean and strong nuclear forces. E&M was the first relativistic theory developed (even though it took Einstein to realize that Maxwell's formalization of it was relativistic). The nuclear forces (e.g. in QFT) are only taught relativistically, but gravity isn't always taught relativistically (i.e. Newton's theory of gravity). Though the final equations will be quite simple, they require a lot of context to understand.

How do we make the laws of gravity invariant under relativity?

Let's start with a quick review of special relativity. Recall Newton's law, that a body in motion remains in motion with constant velocity unless acted upon by an outside force. Another way to frame this is that there is a class of *inertial observers*: there is a class of objects that will move with constant velocity, no matter what.

More suggestively, these are also called inertial frames (which refers to the world as seen by such an object). The statment of Newton's laws is that the laws are the same in all inertial frames, or are invariant under changes of inertial frame. This is the principle we will generalize in GR, but we'll remove the word "inertial."

In Newtonian physics, all laws are invariant under *Galilean boosts*, i.e. coordinate transformations such that $t' = t$, $x' = x - vt$, $y' = y$, and $z' = z$ (so-called because they boost from one frame into another). This can be reformulated into a postulate, often called the First Postulate of Special Relativity.

Postulate 1. *All inertial observers are equivalent.*

It gets much deeper: the way modern physicists define a theory is by its group of symmetries: Newton's laws are *defined* as the theory invariant under the group of Galilean transformations. This is beautiful, but perhaps backwards from what we're used to looking at. For another example, one could say that the weak force has an SU_2 symmetry, and so forth, and E&M has a certain gauge symmetry.[1]

We do need a second postulate, which historically came from a empirical understanding of E&M, from which we learned that the speed of light is constant, which was really weird, but led to relativity.

Postulate 2. *The speed of light is constant.*

We'll use $c = 1$, becuse this is a theory class. Thus, on spacetime diagrams, light moves at 45° angles to the axes.

These two relatively simple facts are all you need to completely work out special relativity.

Consider two inertial observers: frame F, with coordinates x and t, and frame F', with coordinates x' and t'. Recall that the t- and t'-axes are the worldlines: all of the places that the observer in that frame passes through. The x-axes is much less obvious: it's the step that is rather different in general relativity; it's very important. For example, we believe in locality: right now, without GR, we see Newton's Law as action at a distance, which isn't good. We can construct

[1] "You guys know gauge symmetry, right?"

x' as a system of rods, but alternatively x' can be defined as the locus of points p where, if light is emitted at time $-t_0$ from the observer and then reflected at point p, it reaches the observer at time t_0.

From the perspective of frame F, the t'-axis is a straight line contained within the lightcone of F (the line given by $x = vt$, where v is the velocity of F' relative to F). Then, we construct x' as per the definition: draw light rays at 45° from x (this is crucial: it's still parallel to the lightcone, since c is constant!).

From this perspective, multiplicity of simultaniety is extremely obvious: pick two points on the x'-axis, and they're on different times relative to F. Well, that breaks all sorts of intuition, but it's easy to see the theory why. Drawing a picture here is extraordinarily useful, and almost completely solves special relativity.

Now, one can go on and construct the whole coordinate system, drawing lines parallel to the already given axes (same construction), giving coordinate grids. The shape of the coordinate system tells a lot about physics, but doesn't say a lot about how lengths look in F and F'. There are various beautiful, geometric derivations of this in the books, and the answer is that *Lorentz transformations* relate coordinate changes between F and F'. These are given as follows:

$$\begin{aligned} t' &= \gamma t - \gamma v x \\ x' &= \gamma x - \gamma v t \\ y' &= y \\ z' &= z, \end{aligned}$$

where $\gamma = 1/\sqrt{1 - v^2}$. These replace the Galilean transformations above.

Now, we can define special relativity as the theory that is invariant under the Lorentz group of transformations (i.e. the compositions of those transformations, the rotations, and so on). And then, special relativity just falls out.

Index Notation. This is extremely useful, but is confusing without an explanation. Thus, here is an explanation.

First, instead of (t, x, y, z), denote the four coordinates as (x^0, x^1, x^2, x^3). The whole object is called a 4-vector, x^μ (where μ is some Greek letter). The full spacetime coordinates will be given by Greek letters, so $\mu = 0, 1, 2, 3$, and Latin letters for the space components, $i = 1, 2, 3$.

Why is this useful? It allows us to rewrite the Lorentz transformations in a prettier way. They just become linear transformations (this is one of the things which is generalized in GR):

$$\left[x^{\mu'}\right] = \begin{bmatrix} x^{0\prime} \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}.$$

This latter matrix is called Λ, so we would write the above $\Lambda^{\mu'}_{\nu} x^\nu$. Note that Λ^μ is zero-indexed, because our coordinates are.

This leads to a very useful piece of notation called the summation convention, which makes life much easier: an upper index and a lower index of the same variable is summed across all variables.

$$A_\mu B^\mu = \sum_{\mu=0}^{3} A_\mu B^\mu = A_0 B^0 + A_1 B^1 + A_2 B^2 + A_3 B^3.$$

So the point is, the Lorentz transformations are just of the form $x^{\mu'} = \Lambda^{\mu'}_{\nu} x^\nu$. Another example of Λ is a rotation matrix:

$$\left[\Lambda^{\mu'}_{\nu}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

You should really use index notation; you could do everything in matrix notation, such as $x' = \Lambda x$, but this will be worse, as we'll see.

One important point is that these aren't technically matrices: $x^\nu \Lambda^{\mu'}_{\mu}$ is totally kosher, because once μ' and ν are specified, they're *numbers*, and multiplying numbers is commutative. This represents all of the coordinate calculations at once. This makes a lot of things that look like cheating totally legit, and provides an advantage over matrix notation, where you can't do that.

The Lorentz transformations form a group, called $SO(3, 1)$. If you don't know what a group is, that just means that composing two of the transformations gives another transformation, or another coordinate system. Though $SO(3)$ is often defined otherwise in quantum mechanics, it can be defined as the set of 3×3 matrices such that $\mathbb{1} = M^T \mathbb{1} M$.

So that's all good, but then SO$(3,1)$ is defined as the matrices which preserve

$$\eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

i.e. those Λ such that $\eta = \Lambda^{\mathrm{T}}\eta\Lambda$. This is known as the Minkowski metric. Then, the $(3,1)$ is because there are 3 $+1$s and 1 -1 in this matrix.

In index notation, this is written

$$\eta_{\alpha\beta} = \Lambda^{\mu'}_{\alpha}\eta_{\mu'\nu'}\Lambda^{\nu'}_{\beta}.$$

This seems like abstract formalism (well, not to me, because I'm a math major), but it's useful for some physical intuition too. In frame F, we have a vector Δx^μ (change in position/time) and a transformation $\Delta x^{\mu'} = \Lambda^{\mu'}_{\mu}\Delta x^\mu$ into F'. Keep in mind that geometric objects exist independently of coordinates; it's important, especially within the differential geometry, to keep them coordinate-free, and coordinates can be assigned to them.

Humor me for a second and consider the quantity $\Delta x^{\mu'}\eta_{\mu'\nu'}\Delta x^{\nu'}$; plug in the above formula to obtain $\Delta x^\mu \Lambda^{\mu'}_{\mu}\Delta_{\mu'\nu'}\Lambda^{\nu'}_{\nu}\Delta x^\nu$ (some terms have been rearranged, but they're just numbers). But the contracted indices go away in the middle, because they're the identity that Lorentz vectors satisfy. Thus, we get $\Delta x^\mu \eta_{\mu\nu}\Delta x^\nu$, so this equation is nicely governed by changes in coordinates.

Notice that we don't write Λ^{T}: these are just numbers, so $(X^{\mathrm{T}})_{ab} = X_{ba}$. Instead of writing the transpose sign, one simply chooses the order of the indices correctly. This is one of those tricky things about index notation.

We defined theories by the symmetries that make them invariant; thus, the first several weeks are discussing laws which will be invariant under the groups of transformations we care about. But here, we now know a law invariant under SO$(3,1)$.

For more stuff about index notation, consider matrices A, B, C. Multiplying them ABC gives another 4×4 matrix, there are two uncontracted indices: $A_{\alpha\beta}B_{\beta\gamma}C_{\gamma\delta}$. This corresponds to 4^2 different free quantities (in the resulting matrix).

Another advantage of this formalism is that this quantity works for other four-vectors, such as velocity or momentum. And it illustrates a much more general truth: that if all of the indices are contracted in a quantity, then it's coordinate-invariant.

2. Lorentz Transformations and More Special Relativity: 9/24/14

Welcome to the lightning-fast review of special relativity; we defined that a theory is relativistic if it's invariant under all boosts and all rotations, or the Lorentz transformations, SO$(3,1)$. Alternatively, these are the transformations Λ that preserve the metric η with signature $(-1,1,1,1)$. There's a sign convention; you could also use $(1,-1,-1,-1)$, and some textbooks use this, with subtly differing formulas. Then, we showed that $\Delta x^\mu\eta_{\mu\nu}\Delta x^\nu$ is invariant, and expanded it out to $-(\Delta x_0)^2 + (\Delta x_1) + (\Delta x_2)^2 + (\Delta x_3)^2$. This is the same in all frames. Alternatively, this could be written $-\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$ (a ``spacetime interval'').

Invariants are almost the only interesting things in GR; they tend to have physical meaning. For example, the above spacetime interval is the the clock time on someome's watch as they wander around the world. In a different frame, the position and frame time may differ, but the numbers on the clock (the clock's frame's time) are invariant, because the clock's frame is such that Δx, Δy, and Δz are zero, so the quantity is the clock time, but since it's invariant, then it's the same number in every frame.

This is the sort of argument that will come back often, and very complicated, in GR: once you can show it's true in one frame, it's true in some other frames, for invariant quantities. This means you can change the frame to something where horribly ugly quantities are much simpler. This is an extremely useful technique in GR.

So we can say that infinitesimally, $-\mathrm{d}\tau^2 = \mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$; this quantity is called the proper time. Thus,

$$\tau = \int \mathrm{d}\tau = \int \mathrm{d}\lambda\sqrt{-\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\eta_{\mu\nu}},$$

where λ is a parameter for the world line.

Now we know why η is called the metric; it tells the distance $\mathrm{d}s^2 = \Delta x^\mu\eta_{\mu\nu}\Delta x^\nu$ of two things. This is a simple metric, but will be generalized to the trickier metrics we see in GR. This is useful for subspaces, as they inherit metrics from their parent spaces; that's exactly what the parameterization says. It is the expression for the induced metric on the one-dimensional subspace.

In the spaceline, rather than the timelike direction, this is called the proper distance, rather than proper time; in null lines, it's zero.

Since Lorentz transformations characterize special relativity, then all of the weird paradoxes of special relativity can be explained by it. For example, time dilation is explained by a frame F and a frame F' moving relative to F; then, $\Delta t' = \gamma \Delta t$ (since $v < 1$, then $\gamma > 1$); thus, time is stretched in the moving frame. Length contraction is the same thing: in the stationary frame F, a ruler travels up the t-axis without incident; however, in a frame F' moving relative to F, the space axis is at an angle relative to that of F, so it intersects the world sheet of the ruler at a nonzero angle, so it sees a shorter ruler. That the ruler changes length is clear from the spacetime diagram; but in order to determine whether it's a contraction or an expansion, one must use a Lorentz transformation.

Specifically, $\gamma t - \gamma v L = 0$, and $t = vL$. Thus, $x' = \gamma L - \gamma v(vL) = \gamma L(1 - v^2)$, so $x' = L\sqrt{1 - v^2} < L$. Thus, length contraction.

Consider a scenario where, in frame F, a clock moves at $v = \frac{\mathrm{d}x}{\mathrm{d}t}$, and flashes every time it strikes the hour. Now, consider the following three quantities:

(1) The time between flashes, according to the clock. This is $\Delta\tau$.
(2) The time between flashes, in frame F. This is $\gamma\Delta\tau$.
(3) The time between flashes according to an observer in frame F.

Some of these are the same; which?

To determine the answer, you should really draw a spacetime diagram. These anchor you, and make it much less likely that you'll make mistakes.

The answer is: they're all different. The clock sees the ticks happen at a specific frequency T, but time dilation means the frequency in frame F is different; finally, the observer has speed-of-light delay that differs from the clock to his worldline, so the frequency is different yet again. This causes a linear correction in the frequency, known as the Doppler shift.

Let's put some numbers to this. The clock emits two pulses with frequency $\Delta\tau$, but in frame F, they seem to be at frequency Δt. Thus, $\Delta t = \gamma\Delta\tau$ (from the transformation equations, with the endpoints plugged in).

The whole point in GR is: what does the observer see? This is *distinct* from what happens in the observer's frame. Sometimes, one writes stuff and gets confused. Related to this is the idea that you can't make nonlocal measurements (which I guess is the connection to differential geometry!), since the speed of light is finite.

Now, we know that in frame F, then the change in the clock's distnance is $\Delta x = v\Delta t$, and this, times the speed of light (which is 1) is the delay added to each pulse of the clock. The sign of this change depends on whether the clock heads towards the observer or away from it. The result is thus $\gamma\Delta\tau(1 + v)$.

Confusing the observer and its frame isn't all that bad in special relativity, but this is completely broken in general relativity. Do not think that the coordinates of two events are meaningful at all! Unlike special relativity, the coordinates don't have much of a physical meaning. This is a common mistake, because physicists in their other classes are trained to treat coordinates as physical objects: length, time, and so on.

Recall that vectors aren't just points with direction; the goal is to ensure that they transform like a vector under the correct change of coordinates. Similarly, a 4-vector is an object which transforms under Lorentz transformations as $V^{\mu'} = \Lambda^{\mu'}_{\nu} V^{\nu}$. Here are some examples:

- The position vector x^μ, which is how this was introduced.
- The "4-velocity" $u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$ (i.e. over the length of the curve). To prove this, well, x is a 4-vector and τ is constant, so this still transforms nicely. $\frac{\mathrm{d}x^\mu}{\mathrm{d}t}$ is *not* a 4-vector, and is not good to work with.
- The "4-momentum" $p^\mu = mu^\mu = m\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$. This can be seen by choosing the rest frame for a particle, in which $p^m = (m, 0, 0, 0)$, and this is nice because the total momentum relates to coordinate-invariant quantities. Sometimes, we also have this written as $(E, \mathbf{p})$.

Now, we can define some useful quantities: the length is $p^2 - \eta_{\mu\nu}p^\mu p^\nu$, and $p^2 = -E^2 + \mathbf{p}^2 = -m^2$; if we have a massive particle with velocity $v = \frac{\mathrm{d}x}{\mathrm{d}t}$, the $p^\mu = (\gamma m, \gamma, mv, 0, 0)$, where $\gamma = 1/\sqrt{1 - v^2}$. This is coordinate-invariant, which is really nice: any time we contract all indices, we get an invariant, which leads to laws of nature (which should be invariant). But you can only contract an upper and a lower index.

To lower an index, and obtain something called a one-form, contract with the metric: $u_\mu = \eta_{\mu\nu}u^\nu$; for example, if $u^\mu = (5, 4, 7, 2)$, then $u_\mu = (-5, 4, 7, 2)$. One-forms can eat vectors, which we'll get into later.

Now, we're basically done reviewing special relativity (!), but we'll write the laws of E& M in nice, simple notation, since they're Lorentz-invariant.

Define

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix}.$$

Thus, $F^{0i} = E^i$ and $F^{ij} = \varepsilon^{ijk} B_k$. This is a 2-tensor, but to change coordinates, we need to handle both indices: $F^{\mu'\nu'} = \Lambda^{\mu'}_{\mu} \Lambda^{\nu'}_{\nu} F^{\mu\nu}$. This ends up resembling the definition of a two-index tensor.

Let's also define the current vector (which is a four-vector, though it's not obvious): $J^\mu = (\rho, \mathbf{J})$.

Now, we can write Maxwell's equations insanely easily: if $\partial_\mu = \frac{\partial}{\partial x^\mu}$, then

$$\partial_\mu F^{\mu\nu} = J^\nu$$
$$\partial_{[\mu]} F_{\nu\lambda} = 0.$$

The brackets denote summing with sign over all permutations of indices (there are six of them). But now, we know these things are invariant!

Part 2. The Principle of Equivalence

3. Building Intuition for General Relativity: 9/29/14

Special relativity aside, we can now try to determine a relativistic theory of gravity.

We currently have a Newtonian law of gravity $\mathbf{F} = Gm_1 m_2 / r^2 \hat{r}$. It doesn't look relativistic (after all, there are no four-vectors, no time-invariance, etc.), though neither did Coulomb's law.

Well, since we have this class curriculum in front of us, it isn't relativistic, so there must be inconsistencies. One simple one is a particle with mass m that falls onto a surface, where it is converted into a photon and emitted. This photon has energy $m + mgh$ (since, remember, $c = 1$), as it gained potential energy. But gravity doesn't affect the photon, so back at the top, it could be reconverted to a massive particle, which is a problem: it's got extra energy. Since the world empirically doesn't appear to violate the laws of conservation of energy, then this is a problem.

Clearly, something happened to the photon on the way up. When we try to rewrite this with relativistic mass (γm), the law isn't coordinate-invariant. However, the final law will look somewhat like this, though it accounts for all energy gravitating as well (even kinetic energy of moving mass). But it's pretty nontrivial to turn this into a relativistic statement. We'll begin developing the math eventually, but today will be very, very useful for building intuition and all of the formalism. It may feel simple, but will generalize to more powerful statements.

Notice that the force of gravity seems to be proportional to m_2: $F \propto m_2$, so if $F = ma$, then its acceleration is independent of its mass. Another way to think of this is that there's no difference between a piece of chalk and several pieces of chalk near each other in terms of acceleration. This follows from the Newtonian law or just as an empirical principle. It's just a truism, because it's so beautiful, in some sense.

The Principle of Equivalence. *All objects fall at the same rate.*

This is just something we have seen empirically (to thirteen digits! --- and yet people at Stanford are working on extending it by another factor of 100).

This may yet one day be shown to be very slightly false, but we're taking it to be true for Newtonian gravity, and also for general relativity. Another way to think of it is that *inertial mass is equal to gravitational mass* (the constant is 1, which isn't so important, but is nice). Notice that this is different than what happens for the electric force: everything is just proportional to how much stuff there is.

Another way to state this is:

Corollary 3.1. *Gravity couples to everything universally.*

Perhaps this seems unrigorous; to formalize it would require a detour into quantum field theory.

Corollary 3.2. *No local experiment can distinguish free fall in a gravitational field from just being at rest in a gravitational field from just being at rest in an inertial frame (in uniform motion) in empty space.*

The scenario to imagine is a man in an elevator in free fall. Before it lands, he has no way to know whether he's in a falling elevator or in the absence of any gravitational field.

Relatedly, the same man should not be able to determine if he is in an elevator on the Earth's surface or an elevator in a linearly accelerating frame relative to an inertial frame (i.e. accelerating upwards at a rate of g meters per second squared). This is a useful way to derive laws of gravity: it should look like motion, which can come from geometry. The only corrections will be tidal forces, i.e. corrections due to the object not being a point mass.[2]

A key part of this statement is the locality (or, equivalently, the size of the tidal corrections). This might make one think (hint, hint) that tidal forces are fundamentally what describe gravity; g can even be cancelled by moving to a different frame, but the tidal forces are really impossible to avoid. They might even be called the Riemann curvature!

Now, we've written this principle many times in different ways, but it's almost the definition of GR. Again, it seems simple, but it leads to something deep: if one has a setup of masses and gravitational forces, as long as I throw a particle

[2] In principle, of course, one can make smaller and smaller experimental apparatuses, and in the limit it should work out.

through in a certain way, all other particles thrown in in the same way follow the same trajectory. Thus, this doesn't have anything to do with the particle: it's not a coupling of the particle to the field, but something more universal.

This looks sort of like electromagnetism with its electric and magnetic fields, but gravity looks really different, since all particles are affected in the same way: it's a property of spacetime, some hidden train tracks that particles tend to follow.

This sounds like geometry, doesn't it? Imagine a landspace with some nice hills and a bowling ball rolled through it: whether this or an 8-ball or a tennis ball or whatever, the track is totally independent of what kind of ball it is; it's just set by the geometry of the background. *Gravity sets some kind of background geometry for spacetime*.

We can extend on eof the principles of special relativity to this notion, which means that we don't have to just restrict to a specific class of frames. Why only inertial frames? One could start with a class of accelerating frames (not accelerating with respect to each other), and begin the theory there, which is ugly!

The Principle of General Relativity. *All observers, not just inertial ones, are equvalenct.*

Now, make any coordinate transformation you want (well, it should be well-defined), and the laws should still work.

Equivalent to this is the principle sometimes called the Principle of General Covariance (which is harder to mathematically formalize): all of the fundamental equations of physics should behave well under coordinate transformations: they should look the same in all coordinae systems. This will end up meaning tensor equations, but that's something we'll get into later.

This is at once reasonable (what do laws mean, after all?), but also really weird: remember how spinning frames have extra fictitious forcs pop up. But this seems ugly: we want to have fundamental, simple, beautiful laws. Clearly, this will not be easy.

Exercise 1. Consider a heavy object, such as a star (or a planet), and another star passes by it, but emits light in all directions. A (general-relativistic) phenomenon called gravitational lensing means that the light from one star is bent by the gravitational force of the other.

Without knowing general relativity, one ought to be able to know how about much the light bends.

Solution. Recall the elevator (i.e. a local experiment, in free fall): we can consider the trajectory the light takes in the elevator, in the accelerating frame. Here, it moves in a straight line (since it has no idea what is outside of it, so the normal laws of physics we know still apply). Thus, we need to do a coordinate transformation into the other star's frame, though to do it exactly we need more physics than we have developed (we'll get the order of magnitude, though).

Thus, the light ray is falling, and by measuring how much the elevator falls, we can see how much the light bends: at each point, consider the next elevator and the moment-to-moment change in the elevator's fall.

What's impressive is that we can derive the notion of gravitational lensing with no GR and no math; the correct answer will use both to provide a more accurate answer.

This locally inertial frame argument (or local tangent space, or whatnot) is a very useful technique: jump to a frame where the laws are particularly simple. This is a trick we will use a lot in this class, and the professor likes to use in research.

Now, let's begin talking about the formalism behind general relativity. That is: the next few lectures will be very mathematical. Well, only if you're a physicists, because we're trying to do a lot of differential geometry in a few weeks, so it won't be super mathematical or rigorous. Tally ho!

We have realized that gravity should be geometry, but we can't just use boring old Euclidean geometry; we want to be able to use any coordinate system we want, requiring a very general way of talking about geometry.

The fundamental object in differential geometry is the manifold. Roughly, this is some space (by which we mean a set of points) that locally (formally, that means within a neighborhood, which we haven't defined yet) around any point of the manifold, looks like the normal $\mathbb{R}^n$ we're all familiar with.

Basically, a space is a manifold if it looks like $\mathbb{R}^n$ as long as it looks like it as long as we get very clear: the ant's view of geometry. Locally, we may get the simple, flat-space laws we saw in a local falling-elevtor experiment, though more interesting things we get globally.

The n named above (for $\mathbb{R}^n$) is constant over the whole manifold, and is called the manifold's dimension.

Now, for a slightly better definition.