QUANTUM TOPOLOGY AND CATEGORIFICATION SEMINAR, SPRING 2017

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CONTENTS

Part 1. Quantum topology: Chern-Simons theory and the Jones polynomial

1. The Jones Polynomial: 1/24/17

Today, Hannah talked about the Jones polynomial, including how she sees it and why she cares about it as a topologist.

1.1. Introduction to knot theory.

Definition 1.1. A **knot** is a smooth embedding $S^1 \hookrightarrow S^3$. We can also talk about **links**, which are embeddings of finite disjoint unions of copies of S^1 into S^3.

One of the major goals of 20th-century knot theory was to classify knots up to isotopy.

Typically, a knot is presented as a **knot diagram**, a projection of $K \subset S^3$ onto a plane with "crossing information," indicating whether the knot crosses over or under itself at each crossing. Figure 1 contains an example of a knot diagram.

FIGURE 1. A knot diagram for the left-handed trefoil knot. Source: Wikipedia.

Given a knot in S^3, there's a theorem that a generic projection onto $\mathbb{R}^2$ is a knot diagram (i.e. all intersections are of only two pieces of the knot).

Link diagrams are defined identically to knot diagrams, but for links.

Theorem 1.2. *Any two link diagram for the same link can be related by planar isotopy and a finite sequence of* ***Reidemeister moves****.*

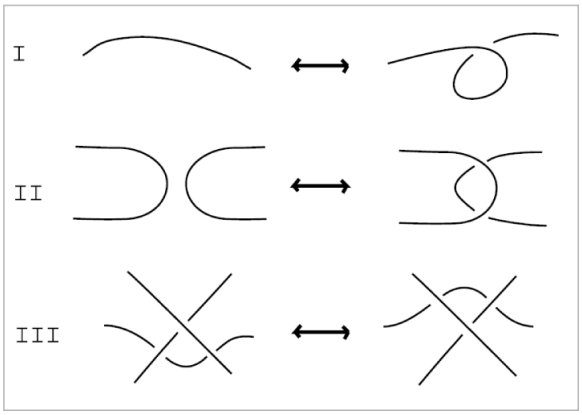

FIGURE 2. The three Reidemeister moves. Source: https://www.computer.org/csdl/trans/tg/2012/12/ttg2012122051.html.

1.2. **Polynomials before Jones.** The first knot polynomial to be defined was the Alexander polynomial $\Delta_K(x)$, a Laurent polynomial with integer coefficients that is a knot invariant, defined in the 1920s.

Here are some properties of the Alexander polynomial:

- It's symmetric, i.e. $\Delta_K(x) = \Delta_K(x^{-1})$.
- It cannot distinguish handedness. That is, if K is a knot, its **mirror** $\overline{K}$ is the knot obtained by switching all crossings in a knot diagram, and $\Delta_K(x) = \Delta_{\overline{K}}(x)$.[1]
- The Alexander polynomial doesn't detect the unknot (which is no fun): there are explicit examples of knots 11_{34} and 11_{42} whose Alexander polynomials agree with that of the unknot.[2]

So maybe it's not so great an invariant, but it's somewhat useful.

1.3. **The Jones polynomial.** The Jones polynomial was defined much later, in the 1980s. The definition we give, in terms of skein relations, was not the original definition. There are three local models of crossings, as in Figure 3.

FIGURE 3. The three local possibilities for a crossing in a knot diagram (technically, L_0 isn't a crossing). Source: https://en.wikipedia.org/wiki/Skein_relation.

The idea is that, given a knot K, you could try to calculate a knot polynomial for K in terms of knot polynomials on links where one of the crossings in K has been changed from L_- to L_+ (or vice versa), or **resolved** by replacing it with an L_0. A relationship between the knot polynomials of these three links is a **skein relation**. This is a sort of inductive calculation, and the base case is the unknot. In particular, you can use the value on the unknot and the skein relations for a knot polynomial to describe the knot polynomial!

Example 1.3. The Alexander polynomial is determined by the following data.

- On the unknot, $\Delta(U) = 1$.
- The skein relation is $\Delta(L_+) - \Delta(L_-) = t\Delta(L_0)$. ◂

Definition 1.4. The **Jones polynomial** is the knot polynomial v determined by the following data.

- For the unknot, $v(U) = 1$.

[1] The mirror of the left-handed trefoil is the right-handed trefoil, for example.

[2] The notation for these knots follows Rolfsen.

- The skein relation is
$$\left(t^{1/2}-t^{-1/2}\right)v(L_0)=t^{-1}v(L_+)-tv(L_-).$$

Example 1.5. Let's calculate the Jones polynomial on a Hopf link H, two circles linked together once. The standard link diagram for it has two crossings, as in Figure 4.

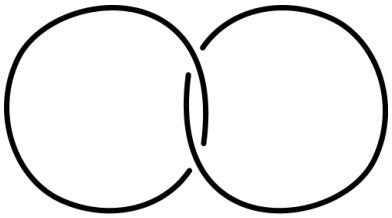

FIGURE 4. A Hopf link. Source: https://en.wikipedia.org/wiki/Link_group.

- Resolving one of the crossings produces an unknot: $v(L_0)=1$.
- Replacing the L_- with an L_- produces two unlinked circles. One more skein relation produces the unknot, so $v(L_+)=-(t^{1/2}-t^{-1/2})$.

Putting these together, one has
$$(t^{1/2}-t^{-1/2})\cdot 1=t^{-1}(-(t^{1/2}+t^{-1/2})-tv(H)),$$
so $v(H)=-t^{-1/2}-t^{-5/2}$. ◁

There are many different definitions of the Jones polynomial; one of the others that we'll meet later in this seminar is via the Kauffman bracket.

Definition 1.6. The **bracket polynomial** of an unoriented link L, denoted $\langle L\rangle$, is a polynomial in a variable A defined by the skein relations

- On the unknot: $\langle O\rangle=1$.
- There are two ways to resolve a crossing C: as two vertical lines V or two horizontal lines H. We impose the skein relation
$$\langle C\rangle=A\langle V\rangle+A^{-1}\langle H\rangle.$$
- Finally, suppose the link L is a union of one unlinked unknot and some other link L' (sometimes called the **distant union**). Then,
$$\langle L\rangle=(-A^2-A^{-2})\langle L'\rangle.$$

Example 1.7. Once again, we'll compute the Kauffman bracket for the Hopf link. (TODO: add picture).
The result is $\langle H\rangle=-A^4-A^{-4}$. ◁

You can show that this bracket polynomial is invariant under type II and III Reidemeister moves, but not type I. We obviously need to fix this.

Definition 1.8. Let D be an *oriented* link, and $|D|$ denote the link without an orientation. The **normalized bracket polynomial** is defined by
$$X(D):=(-A^3)^{-\omega(D)}\langle|D|\rangle).$$

Here, $\omega(D)$ is the writhe of D, an invariant defined based on a diagram. At each crossing, imagine holding your hands out in the shape of the crossing, where (shoulder → finger) is the positively oriented direction along the knot. If you hold your left hand over your right hand, the crossing is a **positive crossing**; if you hold your right hand over your left, it's a **negative crossing**.

Let ω_+ denote the number of positive crossings and ω_- denote the number of negative crossings. Then, the **writhe** of D is $\omega(D):=\omega_+-\omega_-$. For example, the writhe of the Hopf link (with the standard orientation) is 2, and $X(H)=-A^{10}-A^2$.

Thankfully, this is invariant under all types of Reidemeister moves. The proof is somewhat annoying, however.

Theorem 1.9. *By substituting $A=t^{-1/4}$, the normalized bracket polynomial produces the Jones polynomial.*

So these two invariants are actually the same.

Here are some properties of the Jones polynomial.

- $v_{\overline{K}}(t) = v_K(t^{-1})$. Since the Jones polynomial is not symmetric, it can sometimes distinguish handedness, e.g. it can tell apart the left- and right-handed trefoils.
- It fails to distinguish all knots: once again, 11_{34} and 11_{42} have the same Jones polynomial.[3]
- It's unknown whether the Jones polynomial detects the unknot: there are no known nontrivial knots with trivial Jones polynomial.
- Computing the Jones polynomial is **P**-hard: there's no polynomial-time algorithm to compute it. (Conversely, the Alexander polynomial is one of very few knot invariants with a polynomial-time algorithm.)

If a knot does have trivial Jones polynomial, we know:

- it isn't an **alternating knot** (i.e. one where the crossings alternate between positive and negative).
- It has crossing number at least 18 (which is *big*).

One interesting application of what we'll learn in this seminar is that there are knots (9_{42} and 10_{11}) that can't be distinguished by the Jones or Alexander (or HOMFLY, or ...) but *are* distinguished by SU(2)-Chern-Simons invariants.

2. Introduction to quantum field theory: 1/31/17

Today, Ivan talked about quantum field theory (QFT), including what QFT is, why one might want to study it, how it relates to other physical theories, classical field theories, and quantum mechanics, and how to use canonical quantization to produce a QFT.

So, why should we study QFT? One good reason is that its study encompasses a specific example, the **Standard model**, the "theory of almost everything." This is a theory that makes predictions about three of the four fundamental forces of physical reality (electromagnetism, the weak force, and the strong force), leaving out gravity. These predictions have been experimentally verified, e.g. by the Large Hadron Collider.

Unfortunately, the mathematical theory of QFTs is not well formulated; **free theories** are well understood, but if you can rigorously formulate the mathematical theory of **interacting QFTs**, you'll win a million-dollar prize! Perhaps that's a good reason to study QFT.

There's also the notion of a topological quantum field theory (TQFT), which has been rigorously formulated as mathematics, but many of the most important QFTs, including the Standard Model, do not fit into this framework.

QFTs fit into a table with other physical theories: the theory you want to use depends on how fast your particles move and how big they are.

- If your particles are larger than atomic scale and moving considerably slower than the speed of light c, you use *classical mechanics*.
- If your particles are atomic-scale, but moving much slower than c, you use *quantum mechanics*.
- If your particles are larger than atomic-scale, but moving close to the speed of light, you use *special relativity* or *general relativity*: the latter if you need to account for gravity, and the former if you don't.
- If your particles are at atomic-scale and moving close to the speed of light, but you don't need to take gravity into account, you use *quantum field theory*. In this sense, QFT is the marriage of special relativity and quantum mechanics.
- If your particles are small, but moving at about c, *and* you need to consider gravity, you end up in the domain of *string theory*. Here be dragons, of course: string theory hasn't been experimentally verified yet...

With the big picture in place, let's talk a little about classical field theory.

Let $\mathbb{R}^{1,3}$ denote **Minkowski spacetime**, $\mathbb{R}^4$ with the normal **Minkowski metric**

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

Definition 2.1. A **field** is a section of a vector bundle over $\mathbb{R}^{1,3}$, or a connection on a principal G-bundle over $\mathbb{R}^{1,3}$. In the latter case, it's also called a **gauge field**.

[3] This is ultimately for the same reason as for the Alexander polynomial: there's a technical sense in which they're **mutant knots** of each other. It's notoriously hard to write down knot polynomials that detect mutations, and the Jones polynomial cannot detect them.

In this context, we'l care the most about trivial vector bundles and principal bundles!

Definition 2.2. A **classical field theory** is a collection of PDEs that specify the time evolution of a collection of fields.

Example 2.3. Electromagnetism is a famous example of a classical field theory: there are electric and magnetic fields $\vec{E}$ and $\vec{B}$, respectively, and the **Maxwell equations** govern how they evolve in time:

$$\begin{aligned} \nabla \cdot \vec{E} &= \rho \\ \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0 \\ \nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} &= J. \end{aligned}$$

Here, J is the **electric current** and ρ is the **charge density**. There may be some constants missing here. ◁

Usually (always?), you can present the evolution of the classical field theory as the "critical points" of a functional of the form[4]

$$S(\varphi_1, \dots, \varphi_n) = \int_{\mathbb{R}^4} \mathrm{d}^4 x \, \mathscr{L}(\varphi_1, \dots, \varphi_n, \partial_\mu \varphi_1, \dots, \partial_\mu \varphi_n).$$

The functional S is called the **action functional**, and the function $\mathscr{L}$ is called the **Lagrangian**. Using calculus of variations, this notion of critical points is placed on sound footing. Physicists sometimes call these critical points **minimizers**, but sometimes we want to maximize S, not minimize it.

In this context, one can show that the critical points of S are the solutions to the **Euler-Lagrange equations**. In the case of a single field φ, these equations take the form

$$\frac{\partial \mathscr{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial (\partial_\mu \varphi)} \right) = 0.$$

(We are using and will continue to use Einstein notation: any index μ that's both an upper and lower index has been implicitly summed over.) So the Lagrangian contains all the information about the dynamics of the system.

Example 2.4. Let's look at electromagnetism again: if $\rho = 0$ and $J = 0$, then let

$$A := A_\mu \, \mathrm{d}x^\mu$$

be the **electromagnetic potential**. If $F = \mathrm{d}A$, then

$$F = F_{\mu\nu} \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu.$$

Then, the Lagrangian is

$$\mathscr{L}_{\text{Maxwell}} := -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{2.5}$$

where $F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}$ and $g^{\mu\nu}$ denotes the coefficients of the standard Minkowski metric. Then, the Euler-Lagrange equations and the fact that $\mathrm{d}F = 0$ (since F is already exact) directly imply the Maxwell equations, where

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}.$$

◁

Definition 2.6. A **free field theory** is one whose Lagrangian is quadratic in the fields and their partial derivatives. A field theory which is not free is called **interacting**.

In a free field theory, the Euler-Lagrange equations become linear, making them much easier to solve.

[4] To be precise, we should say what space of functions this takes place on. The right way to do this is to consider distributions, but we're not going to delve into detail about this.

Example 2.7. One example of a free field theory uses the **Dirac Lagrangian**

$$\mathscr{L}_{\text{Dirac}} := \overline{\psi}\big(i\gamma^\mu\partial_\mu - m\big)\psi,$$

where $\mu\colon U \subset \mathbb{R}^{1,3} \to \mathbb{C}^4$, γ^μ are **Dirac matrices**, and m is a **mass parameter**, and $\overline{\psi} = \psi^\dagger\gamma^0$. This is used to describe the behavior of a free fermion (e.g. an electron). You can explicitly check this is quadratic in ψ and γ.

The Maxwell Lagrangian (2.5) also defines a free field theory. ◁

Example 2.8. Here's an example of an interacting field theory; its classical solutions don't represent anything physical, but we'll see it again.

$$\mathscr{L}_{\text{QED}} := \mathscr{L}_{\text{Dirac}} + \mathscr{L}_{\text{Maxwell}} + ie\overline{\psi}\gamma^\mu\gamma A_\mu. \tag{2.9}$$

Here e is the charge of an electron, not ≈ 2.78. The first two terms are free, but then it's coupled to an interacting term. ◁

From classical to quantum. To understand how we move from classical field theory to quantum field theory, we'll learn about quantum mechanics, albeit very quickly. This formalism extracts three aspects of a physical system.

- The **states** are the configurations that the system can be in.
- The **observables** are things which we can measure/observe about a system.
- **Time evolution** describes how observables or states evolve with time.

In quantum mechanics:

- The states are unit vectors in some (complex) Hilbert space $\mathscr{H}$.
- The observables are self-adjoint operators $A : \mathscr{H} \to \mathscr{H}$. They are not necessarily bounded. The things you can measure for A are in its spectrum $\operatorname{Spec} A \subset \mathbb{R}$ (since A is self-adjoint). For example, if A represents the position in a coordinate you chose, the spectrum denotes the set of allowed positions in that coordinate.
- Time evolution has two equivalent formulations.
  - The **Schrödinger picture** describes time evolution of the states. There's a distinguished observable, usually representing the energy of the system, called the **Hamiltonian** $H : \mathscr{H} \to \mathscr{H}$. Then, a state $\psi \in \mathscr{H}$ in this system evolves as

$$i\hbar\frac{\partial}{\partial t}\psi(t) = H\psi(t).$$

  - The **Heisenberg picture** describes time evolution of observables as satisfying the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}A(t) - = i\hbar[H, A(t)].$$

    These two perspectives predict the same physics.

Generally, quantum field theories are obtained by taking a classical field theory and quantizing it. This is a process creating a dictionary based on the one between classical mechanics and quantum mechanics:

- The states in classical mechanics are points in T^*M, where M is a smooth manifold; quantum mechanics uses a Hilbert space.
- The observables in classical mechanics are smooth functions $T^*M \to \mathbb{R}$. In coordinates $(q^1, \dots, q^n, p_n, \dots, p_n)$, we have relations

$$\{q^i, q^j\} = 0 \qquad \{p_i, p_j\} = 0 \qquad \{q^i, p_j\} = \delta^i_j,$$

  where $\{-,-\}$ is the Poisson bracket coming from the symplectic structure on T^*M. Quantum mechanics replaces functions with self-adjoint operators. In quantum mechanics, if X_i and P_i are the position and momentum operators in coordinate i, they satisfy the relations

$$[X_i, X_j] = 0 \qquad [P_i, P_j] = 0 \qquad [X_i, P_j] = i\hbar\delta_{ij}.$$

- Time evolution in classical mechanics satisfies $\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) = \{H, \gamma(t)\}$; quantum mechanics assigns the Schrödinger or Heisenberg pictures as above.

So for a classical field theory, we want a way to get a Hilbert space, a Hamiltonian, and position and momentum operators.

Example 2.10 (One-dimensional harmonic oscillator)**.** The harmonic oscillator in one dimension satisfies the equation

$$H(x,p) = \frac{p^2}{2m} + n\omega^2 x^2.$$

Then, the Hilbert space is $\mathscr{H} = L^2(\mathbb{R})$, $Xf = x \cdot f$, and $Pf = -i\hbar \frac{\partial f}{\partial x}$.[5] These automatically satisfy the relations $[X,P] = 1$, $[X,X] = 0$, and $[P,P] = 0$. The Hamiltonian is

$$H(X,P) = \frac{P^2}{2m} + m\omega^2 X^2.$$

◂

It's worth noting that quantization is not a deterministic process, more of an art: choosing the right position and momentum operators and showing why they satisfy the relations doesn't follow automatically from some general theory. But if you can get the commutation relations to work and it describes a physical system, congratulations! You've done quantization.

Next time, we'll discuss quantum field theory, where the fields are replaced with quantum fields.

Part 2. Categorification: Khovanov homology

References

[5] Well, not all L^2 functions are differentiable, but there are ways of working around this, especially since differentiable functions are dense in L^2.