M392c NOTES: TOPICS IN ALGEBRAIC GEOMETRY

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These notes were taken in UT Austin's M392c (Topics in algebraic geometry) class in Fall 2019, taught by Bernd Seibert. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Any mistakes in the notes are my own. Thanks to Tom Gannon for several corrections.

Contents

Lecture 1.

Historical overview of mirror symmetry, I: 8/29/19

"I saw this happening, which makes me realize how old I am."

The first two lectures will contain an overview of mirror symmetry, the broad-scope context of this class; the specific details, e.g. how fast-paced we go, will be determined by who the audience is.

There are about as many perspectives on mirror symmetry as there are researchers in mirror symmetry, but a consensus of sorts has emerged.

Recall that the *canonical bundle* of a complex manifold X is $K_X := \operatorname{Det} T^*X$. A *Calabi-Yau manifold* is a complex manifold with a trivialization of its canonical bundle, i.e. $K_X \cong \mathcal{O}_X$. Though the definition doesn't imply it, we also often assume $b_1(X) = 0$ and that X is irreducible.

Let X be a Calabi-Yau threefold (i.e. it's a Calabi-Yau manifold of complex dimension 3).

Example 1.1. A *quintic threefold* $X \subset \mathbb{P}^4$ is the zero locus in $\mathbb{P}^4$ of a homogeneous, degree-5 polynomial f in the 5 variables $x_0, \ldots, x_4$. For a generically chosen f, X is smooth. We'll prove X is Calabi-Yau.

Let $\mathcal{I}$ denote the vanishing sheaf of ideals of X, i.e. $(f) \subset \mathcal{O}_{\mathbb{P}^4}$. We therefore have a short exact sequence

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathbb{P}^4}|_X \longrightarrow \Omega_X \longrightarrow 0, \tag{1.2}$$

and since $\mathcal{I}/\mathcal{I}^2 \cong \mathcal{I} \otimes_{\mathcal{O}_{\mathbb{P}^4}} \mathcal{O}_X$, it's an invertible sheaf. Using (1.2),

$$K_{\mathbb{P}^4}|_X = \operatorname{Det} \Omega_{\mathbb{P}^4}|_X \cong \mathcal{I}/\mathcal{I}^2 \otimes K_X. \tag{1.3}$$

By standard methods, one can compute that $K_{\mathbb{P}^4} = \mathcal{O}_{\mathbb{P}^4}(-5)$, hence $K_{\mathbb{P}^4}|_X \cong \mathcal{O}_X(-5)$. Since $\mathcal{I} \hookrightarrow \mathcal{O}_{\mathbb{P}^4}$, this means $\mathcal{I} \simeq \mathcal{O}_{\mathbb{P}^4}(-5)$, and therefore $\mathcal{I}/\mathcal{I}^2 \cong \mathcal{O}_X(-5)$, and as a corollary $K_X \cong \mathcal{O}_X$. ◀

Remark 1.4. Mirror symmetry is related to string theory! If you ask physicists, even theoretical ones, they'll tell you there's plenty to do still in setting up string theory, but there are two related classes of string theories called IIA and IIB, which are supersymmetric σ-models with a target $\mathbb{R}^{1,3} \times X$, where X is some Calabi-Yau threefold. Phenomenologists are interested in the $\mathbb{R}^{1,3}$ piece, which hopes to describe our world, and X tells us some information about particle dynamics in the $\mathbb{R}^{1,3}$ factor via the Kaluza-Klein mechanism.

Now, supersymmetric σ-models are better understood in physics than string theories in general, and in fact these give you two superconformal field theories (SCFTs), one corresponding to IIA, and to IIB, with target X. Using physics arguments, you can calculate the Hodge numbers of X; since X is a Calabi-Yau threefold, you can (and we will) show that its only nonzero Hodge numbers are $h^{1,1}$ and $h^{2,1}$.

But if you do this for both the A- and B-type SCFTs, you get flipped answers: $h^{1,1}$ computed via the A-type SCFT is $h^{2,1}$ computed via the B-type SCFT. We think there's only one string theory, which is puzzling. Dixon and Lerihe-Vafa-Warner noticed that sometimes, we can find another Calabi-Yau threefold Y such that the A-type SCFT of X is equivalent to the B-type SCFT of Y, and the A-type SCFT of Y is equivalent to the B-type SCFT for X, hence in particular $h^{1,1}(X) = h^{2,1}(Y)$ and $h^{2,1}(X) = h^{1,1}(Y)$. In fact, we'd expect the IIA string theory for X should be equivalent to the IIB string theory for Y, and likewise the IIB string theory for X should be equivalent to the IIA string theory for Y.

Greene and Plesser postulated such a duality, constructing the dual theory via an orbifolding construction. These were all in the late 1980s or early 1990s, but it was another decade before Hori-Vafa proved (at a physics level of rigor) this duality for complete intersections in toric varieties. ◀

This is good if you like physics, but what if you don't? It turns out that mirror symmetry is still useful – it helps us calculate things in pure mathematics that we didn't have access to before.

Remark 1.5. Let's address a possible source of confusion in the literature.

In 1988, Witten introduced the notion of a *topological twist* of a supersymmetric σ-model. These are topological field theories in the physical sense, not the mathematical ones: we only mean that the variation in the metric vanishes. We can obtain from this data two topologically twisted σ-models called the *A-model* $A(X)$ and the *B-model* $B(X)$, which are *a priori* unrelated to the A- and B-type SCFTs — but it turns out $A(X)$ and $B(X)$ compute certain limits, called *Yukawa couplings*, for these SCFTs. In particular, an equivalence of the A-type SCFT for X and the B-type SCFT for Y (and vice versa) implies an equivalence of $A(X)$ and $B(Y)$.

Caution: the A-model tells you about type IIB string theory, and the B-model tells you about type IIA string theory.

Some mathematicians zoom in on this, and say that mirror symmetry is just the equivalence of the $A(X)$ and $B(Y)$, and of $A(Y)$ and $B(X)$. ◀

Interestingly, the A-model only depends on the symplectic structure on X, and the B-model depends only on the complex structure.

In 1991, Candelas, de la Ossa, Greene, and Parkes [CDGP91] studied the quintic threefold and its mirror Y_t (here t is a parameter, which we'll say more about later), and computed the Yukawa couplings F_A and F_B. Geometrically, the A-model has to do with counts of rational (i.e. genus-zero) holomorphic curves;[1] some of

[1] If we don't have a complex structure, but only a symplectic structure, this seems nonsensical, but these curve counts can nonetheless be defined.

these were known classically. The B-model has to do with period integrals

$$F_B(t) = \int_\alpha \Omega_{Y_t}, \tag{1.6}$$

where $\alpha \in H_3(Y_t)$ and Ω_{Y_t} is a (suitably normalized) holomorphic volume form. These are generally much easier to compute. This was an astounding computation, and they made a further prediction which turned out to be true, and led to astonishing divisibility properties.

A reasonable next question is: can we do this on other Calabi-Yau threefolds? Morrison, building on ideas of Deligne, computed $F_B(Y)$ in terms of Hodge theory, giving more parameters for the Calabi-Yau moduli space. On the A-side, this led to the creation of *Gromov-Witten theory* around 1993, which makes $F_A(X)$ precise. On the symplectic side, this was the work of many people, including Y. Ruan, Tian, Fukaya-Ono, and Siebert; on the algebro-geometric side, this included work of Jun Li and Behrend-Fantechi.

Kontesvich's 1994 ICM address (and subsequent lecture notes) proposed a conjecture called *homological mirror symmetry*. In symplectic geometry, one can extract a triangulated category called the *Fukaya category* from a symplectic manifold X; if Y denotes its mirror, homological mirror symmetry postulates that this is equivalent to the bounded derived category of Y.

This was a charismatic, visionary conjecture, and people have spent a lot of time and thought on it. It's influenced many fields, to the point that people have focued less on the other contexts (e.g. the enumerative formulation). But this is a formulation, not an explanation. We don't quite have a mathematical explanation yet, though ingredients are in place to construct mirrors and make a systematic proof possible.

In 1996, Givental provided a proof of the equivalence of the counts established by Candelas, de la Ossa, Greene, and Parkes; Givental's proof was for hypersurfaces, and Lian, Liu, and Yau provided the general proof. The proof wasn't explanatory: it didn't express these equalities as being true for a reason. These proofs proceeded via localization methods: find a $\mathbb{C}^\times$-action and use methods akin to those of Atiyah-Bott and Berline-Vergne.

Progress on homological mirror symmetry came a little later, first established for quartic twofolds (in $\mathbb{P}^3$), i.e. for K3 surfaces. So the statement has to be modified somehow, but this can be done. This was done by Seidel in 2003, then to more general Calabi-Yau hypersurfaces by his student Nick Sheridan in 2011. This was very hard work, but was strong evidence that mirror symmetry in its various avatars is real. (One of these avatars is the geometric Langlands program.)

In the course of proving homological mirror symmetry for various cases, such as SZY-fibered symplectic manifolds on the A-side and rigid spaces on the Y-side (see Abouazid, Fukaya-Oh-Ohta-Ono), we needed a way to produce mirrors. This led to research into intrinsic construction of mirrors, and this has gone on to have applications outside of mirror symmetry: this allows for some computations to be simplified by passing to the mirror and working there. This includes work of Gross-Siebert, Gross-Hacking-Keel, and more.

This is all the genus-zero part of the story, which physicists call the *tree-level* part of the theory. People also study higher-genus (or second quantized mirror symmetry), such as Costello and Si Li, or look at the method of topolgical recursion, e.g. Eynard and Orantin.

The plan for this class is, roughly:

- Sketch the computation of Candelas, de la Ossa, Greene, and Parkes in [CDGP91].
- Gromov-Witten theory, and its construction via virtual fundamental classes and moduli stacks.
- Potentially an introduction to toric geometry.
- Toric degenerations and mirror constructions. This has undergone several refinements, and we'll take a pretty modern perspective.
- Using this, you can compute homogeneous coordinate rings (which is a lot of information: it knows the variety, hence also the derived category). On the A-side, a result of Polischuk forces that there's only one possible Fukaya category (as an A_∞-category), which leads to a proposal for a plan to prove homological mirror symmetry in great generality. The mirror statement (using the Fukaya category and its A_∞-structure to determine the derived category of the mirror) is considered a hard open problem in symplectic geometry.
- Next, we could discuss higher-genus information. In Gromov-Witten theory, the genus is part of the input data, but we could also compute *Donaldson-Thomas invariants*, where we count ideal sheaves rather than holomorphic curves. This organizes the count differently, because curves of different genera may be part of the same count. The role of Donaldson-Thomas theory in mirror symmetry

is somewhat unclear, and there's an interesting statistical-mechanics model called *crystal melting*, which ports this down to genus zero. This is work of Okounkov and others.

This can be adjusted depending on class interest.

In the last few minutes, let's begin talking about the quintic threefold, its mirror, and the work of Candelas, de la Ossa, Greene, and Parkes.

The quintic threefold comes in a big family: we're looking at degree-5 homogeneous polynomials in five variables, so to enumerate monomials, we need to know the number of ways to draw lines between five points in a line. For example, $x_0^2x_2$ corresponds to 12|345 and $x_0x_1^2x_2$ corresponds to 1|2‖345. The answer is

$$\binom{n+d-1}{n-1} = \binom{n+d-1}{d}, \tag{1.7}$$

which here is $\binom{9}{5} = 126$. Hence the dimension of the moduli space of quintic polynomials in $\mathbb{P}^4$ is $126-1 = 125$. However, to get the space of quintics, we need to divide out by the symmetries of the problem, which is PGL_5. This has dimension $5^2 - 1 = 24$, so the moduli space of quintic threefolds is 101-dimensional.

This is *huge* — you may think it's a long way down the road to the chemist, but that's just peanuts compared to the dimension of this moduli space. It's way too big for us to get a good grasp on.

Indeed, for a projective Calabi-Yau manifold X, the moduli space of Calabi-Yau manifolds deformation-equivalent to X is a smooth orbifold[2] of complex dimension $h^1(\Theta_X)$, where Θ_X is the holomorphic tangent bundle, and we can show that this is 101 for the quintic threefold.

Lecture 2.

Hodge diamonds of Calabi-Yau threefolds: 9/3/19

Last time, we studied the quintic threefold in $\mathbb{P}^4$, which is Calabi-Yau, and whose moduli space is terribly high-dimensional, but remarkably is a smooth orbifold! (That is, the stabilizer groups are finite.) This is unusual, and related to the Calabi-Yau property — for general varieties there's a "Murphy's law" property guaranteeing all sorts of terrible singularities in the moduli space. For a general projective Calabi-Yau manifold X, the moduli of Calabi-Yau deformations of X is a smooth orbifold of dimension $h^1(\Theta_X)$; for the quintic threefold this is 101. Here Θ_X is the holomorphic tangent bundle.

We'll begin with a brief description of how to compute this number, then look at the Hodge theory of the quintic threefold and its mirror. The *Euler sequence* is the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \longrightarrow \Theta_{\mathbb{P}^n} \longrightarrow 0. \tag{2.1}$$

To describe the maps, write $\mathbb{P}^n = \operatorname{Proj} \mathbb{C}[x_0, \dots, x_n]$; then $x_i\partial_{x_i}$ is a well-defined logarithmic vector field on $\mathbb{P}^n$. Then the two maps in (2.1) are $1 \mapsto \sum e_i$ and $e_i \mapsto x_i \partial x_i$, respectively, where e_i is the i^{th} standard basis vector in $\mathcal{O}(1)^{\oplus n}$.

Remark 2.2. TODO: I (Arun) think this looks like a short exact sequence I'd recognize in differential topology relating $T\mathbb{CP}^n$ and its tautological bundle; I'd like to think this through. ◀

We also have the *conormal sequence* for any variety $X \subset \mathbb{P}^4$. Let $\mathcal{I}_X$ denote the sheaf of ideals cutting out X; then the following sequence is short exact:

$$0 \longrightarrow \mathcal{I}_X/\mathcal{I}_X^2 \xrightarrow{g \mapsto \mathrm{d}g} \Omega^1_{\mathbb{P}^4}|_X \xrightarrow{\mathrm{restr}_x} \Omega^1_X \longrightarrow 0. \tag{2.3}$$

Since $\mathcal{I}_X/\mathcal{I}_X^2$ is the conormal bundle of X, this resembles the conormal sequence in differential geometry. Dualizing, we get the *normal sequence*, which is more likely to look familiar:

$$0 \longrightarrow \Theta_X \longrightarrow \Theta_{\mathbb{P}^4}|_X \longrightarrow N_{X\subset\mathbb{P}^4} \longrightarrow 0, \tag{2.4}$$

and since X has degree 5, $N_{X\subset\mathbb{P}^4} \cong \mathcal{O}_{\mathbb{P}^4}(5)|_X$.

[2] We'll say more about this later, but an orbifold is locally modeled on a manifold quotient by a nice group action, and you can think of it as that, as a singular topological space.

Finally, we have two *restriction sequences*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-5) \longrightarrow \mathcal{O}_{\mathbb{P}^4} \longrightarrow \mathcal{O}_X \longrightarrow 0 \tag{2.5a}$$

$$0 \longrightarrow \Theta_{\mathbb{P}^4}(-5) \longrightarrow \Theta_{\mathbb{P}^4} \longrightarrow \Theta_{\mathbb{P}^4}|_X \longrightarrow 0. \tag{2.5b}$$

Now take the long exact sequence in cohomology associated to (2.4):

$$H^0(\Theta_X) \longrightarrow H^0(\Theta|_{\mathbb{P}^4}|_X) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^4}(5)|_X) \longrightarrow H^1(\Theta_X) \longrightarrow H^1(\Theta_{\mathbb{P}^4}|_X) \longrightarrow \cdots \tag{2.6}$$

We will show that

(1) $H^0(\Theta_X) = 0$,
(2) $H^0(\Theta_{\mathbb{P}^4}|_X) \cong \mathbb{C}^{24}$,
(3) $H^0(\mathcal{O}_{\mathbb{P}^4}(5)|_X) \cong \mathbb{C}^{125}$, and
(4) $H^1(\Theta_{\mathbb{P}^4}|_X) = 0$,

which collectively imply that $H^1(\Theta_X) \cong \mathbb{C}^{101}$ (since $101 = 125 - 24$).

First, (4). Take the long exact sequence in cohomology associated to (2.1):

$$\underbrace{H^1(\mathcal{O}_{\mathbb{P}^4}(1))^{\oplus 5}}_{=0} \longrightarrow H^1(\Theta_{\mathbb{P}^4}) \longrightarrow \underbrace{H^2(\mathcal{O}_{\mathbb{P}^4})}_{=0}, \tag{2.7}$$

so $H^1(\Theta_{\mathbb{P}^4}) = 0$. TODO: then restrict to X.

Now take (2.1), tensor with $\mathcal{O}(-5)$, and take the long exact sequence in cohomolgy.[3]:

$$\underbrace{H^i(\mathcal{O}_{\mathbb{P}^4}(-4))}_{=0} \longrightarrow H^i(\Theta_{\mathbb{P}^4}(-5)) \longrightarrow \underbrace{H^i(\mathcal{O}_{\mathbb{P}^4}(-5))}_{=0}, \tag{2.8}$$

and therefore $H^i(\Theta_{\mathbb{P}^4}(-5)) = 0$.

TODO: several more arguments like this, which I couldn't follow in realtime and couldn't reconstruct from the board. Sorry about that. For example, we used the first restriction sequence to use info on $H^1(\Theta_{\mathbb{P}^4})$ and $H^2(\Theta_{\mathbb{P}^4(-5)}$ to conclude $H^1(\Theta_{\mathbb{P}^4}|_X)$ vanishes...

∽·∼

OK, now let's discuss the Hodge diamond of the quintic threefold. On a compact Kähler manifold of complex dimension n, we have some nice facts about the Dolbeault cohomology $H^{i,j}_{\overline{\partial}} := H^j(\mathcal{A}^{i,\bullet}, \overline{\partial})$, where $\mathcal{A}^{\bullet,\bullet}$ is the sheaf of holomorphic differential forms, bigraded via ∂ and $\overline{\partial}$ as usual. Let $\Omega^i_X := (\Omega_X)^{\otimes i}$ and $K_X := \Omega^n_X$. Then,

(1) There are canonical isomorphisms $H^{i,j}_{\overline{\partial}} \cong H^j(X, \Omega^i_X) = \overline{H^{j,i}_{\overline{\partial}}}$ (i.e. the conjugate complex vector space). Hence $h^{ij} = h^{ji}$.
(2) Serre duality tells us $H^{n-j}(X, \Omega^{n-i}_X) \cong H^j(X, K_X \otimes (\Omega^{n-i}_X)^*)^* = H^j(X, \Omega^i_X)^*$, so we have a canonical isomorphism $H^{n-i,n-j}_{\overline{\partial}} \cong H^{i,j}_{\overline{\partial}}$ and $h^{i,j} = h^{n-i,n-j}$.
(3) Let $b^k := \dim_{\mathbb{C}} H^k(X;\mathbb{C}) = H^k_{\mathrm{dR}}(X) \otimes \mathbb{C}$. This group is the direct sum of $H^{i,j}_{\overline{\partial}}$ over $i + j = k$.

These facts are proven using some difficult analysis.

Now if in addition X is Calabi-Yau, $b_1 = 0$, and therefore $h^{1,0} = h^{0,1} = 0$. Moreover, $H^{n,0} \cong H^0(X, K_X) \cong H^0(X, \mathcal{O}_X) \cong \mathbb{C}$, so $h^{n,0} = h^{0,n} = 1$. We further assume X is *irreducible*: neither X nor its universal cover are a product of Calabi-Yau manifolds in a nontrivial way.[4] Beauville showed this is equivalent to $H^{k,0} = 0$, $k = 1, \ldots, n-1$.

[3] Why is $\mathcal{O}(-5)$ flat?

[4] If you like Riemannian geometry and metrics of special holonomy, irreducible Calabi-Yau corresponds exactly to having holonomy landing in SU_n.

It is traditional to arrange the Hodge numbers $h^{i,j}$ in a diamond, known as (surprise!) the *Hodge diamond*. For a 3-fold, we have

$$\begin{array}{ccccccc} & & & h^{3,3} & & & \\ & & h^{2,3} & & h^{3,2} & & \\ & h^{1,3} & & h^{2,2} & & h^{3,1} & \\ h^{0,3} & & h^{1,2} & & h^{2,1} & & h^{3,0} \\ & h^{0,2} & & h^{1,1} & & h^{2,0} & \\ & & h^{0,1} & & h^{1,0} & & \\ & & & h^{0,0}. & & & \end{array} \tag{2.9}$$

But the Calabi-Yau condition tells us this collapses to very few parameters:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{2,2} & & 0 & \\ 1 & & h^{1,2} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1, & & & \end{array} \tag{2.10}$$

and the two red values are equal, as are the two blue values. The red values are both 101 for the quintic threefold.

To get at the last piece of information in the Hodge diamond, we'll relate $h^{1,1}$ to the Picard group.

Definition 2.11. The *Néron-Severi group* $NS(X)$ is the preimage of $H^{1,1}(X) \subset H^2(X;\mathbb{C})$ under the map $H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{C})$.

In complex analytic geometry, we have the *exponential exact sequence* of sheaves of abelian groups

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{\cdot 2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \longrightarrow 1. \tag{2.12}$$

The fact that we began with 0 and ended with 1 isn't significant; it only represents that the first two sheaves of abelian groups are written additively, and the last is written multiplicatively.

Anyways, (2.12) induces a long exact sequence in cohomology.

$$H^1(X,\mathbb{Z}) \longrightarrow H^1(X,\mathcal{O}_X) \longrightarrow H^1(X,\mathcal{O}_X^\times) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathcal{O}_X). \tag{2.13}$$

We have identifications $H^1(X,\mathcal{O}_X) = H^{0,1}$, and $H^1(X,\mathcal{O}_X^\times)$ with the *Picard group* $\mathrm{Pic}(X)$, the isomorphism classes of holomorphic line bundles under tensor product.

Theorem 2.14 (Lefschetz theorem on $(1,1)$ classes)**.** *The image of* $c_1\colon \mathrm{Pic}(X) \to H^2(X,\mathbb{Z})$ *is exactly the Néron-Severi group.*

Thus, for a projective Calabi-Yau threefold, $h^{1,1}(X) = \operatorname{rank} NS(X)$ and $\mathrm{Pic}(X) \cong NS(X)$. This is telling you that a projective Calabi-Yau threefold has no non-projective deformations! This is not true in general, e.g. for K3 surfaces.

Remark 2.15. Serre's GAGA theorem explans why we can so cavalierly pass between the algebro-geometric and complex-analytic world: as long as we restrict to projective varieties and projective manifolds, there are appropriate equivalences of categories between the two perspectives. ◀

To actually compute $h^{1,1}$, though, we need another general theorem from Kähler geometry.

Theorem 2.16 (Lefschetz hyperplane theorem). *Let $X \subset \mathbb{P}^{n+1}$ be a hypersurface, so* $\dim X = n$. *The map* $H^k(\mathbb{P}^{n+1};\mathbb{Z}) \to H^k(X;\mathbb{Z})$ *is an isomorphism for* $k < n-1$ *and is surjective for* $k = n-1$.

In the case of a Calabi-Yau threefold, $H^1(X;\mathbb{Z}) = H^1(\mathbb{P}^4;\mathbb{Z}) = 0$, and doing this for H^2 shows $NS(X) \cong \mathbb{Z}$, and $\mathrm{Pic}(X) = \mathbb{Z} \cdot c_1(\mathcal{O}_X(1))$. So $h^{1,1} = h^{2,2} = 1$.

For the mirror quintic, these should be swapped: we should get $h^{1,1} = h^{2,2} = 1$ and $h^{1,2} = h^{2,1} = 101$. This is a bit weird: it has a huge Picard group and a very small moduli space (it will be an orbifold $\mathbb{P}^1$).

Lecture 3.

The mirror quintic: 9/5/19

As part of mirror symmetry, we want to find a Calabi-Yau threefold Y whose Hodge diamond is the mirror of that of the quintic threefold. In particular, it should have $h^{1,1} = h^{2,2} = 1$ (small space of deformations) and $h^{1,2} = h^{2,1} = 101$ (very large Picard group).

Remark 3.1. The construction we discuss today is physically motivated by *minimal conformal field theories* and their tensor products, and by a procedure to orbifold them. ◂

Specifically, begin with the *Fermat quintic* $X := \{x_0^5 + \cdots + x_4^5 = 0\} \subset \mathbb{P}^4$. Now $(\mathbb{Z}/5)^5$ acts on $\mathbb{P}^4$ through its action on $\mathbb{C}^5$, and the diagonal $\mathbb{Z}/5$ subgroup fixes X, so we have a $(\mathbb{Z}/5)^5/(\mathbb{Z}/5) \cong \mathbb{Z}/4$-action on X. Let $\overline{Y} := X/(\mathbb{Z}/5)^4$.

However, we have a problem: X is smooth, by the Jacobian criterion, but $\overline{Y}$ is not: if, for example, $x_i = x_j = 0$, then the stabilizer of $\mathbf{x}$ contains a copy (TODO: possibly more?) of $\mathbb{Z}/5$. There's a curve $\widetilde{C}_{ij} = Z(x_i, x_j) \subset X$ where the local action is $\zeta \cdot (z_1, z_2, z_3) = (\zeta z_1, \overline{\zeta} z_2, z_3)$, so the singularity looks like that of $uv = w^4$ in $\mathbb{C}^3$, which is an A_4 singularity.[5]

We can do worse, however: when $x_i = x_j = x_k = 0$ for disjoint i, j, k, we get $(\mathbb{Z}/5)^2$ in the stabilizer, and this locally looks like $\mathbb{C}^3/(\mathbb{Z}/5)^2$, with the action

$$(\zeta, \xi) \cdot (z_1, z_2, z_3) = (\zeta\xi z_1, \zeta^{-1} z_2, \xi^{-1} z_3). \tag{3.2}$$

We want to resolve these singularities by blowing them up. Since we're not just blowing up points, this takes a little care. Note that $C_{01} = Z(x_0, x_1, x_2^5 + x_3^5 + x_4^5)/(\mathbb{Z}/5)^3 \simeq Z(u + v + w) \subset \mathbb{P}^2_{u,v,w}$; here $u = x_2^5$, $v = x_3^5$, and $w = x_4^5$, and this is a $\mathbb{P}^1$ inside $\overline{Y}$.

Proposition 3.3. *There exists a projective resolution* $Y \to \overline{Y}$.

One can do this by hand, or in a more general way using methods from toric geometry.

We want to count the number of independent exceptional divisors in Y. Resolving an A_4 gives four $\mathbb{P}^1$s over each C_{ij}, and similarly we'll get six over each P_{ijk}, and each $\mathbb{P}^1$ produces 10, so we have 100 linearly independent elements of $H^2(Y)$. The hyperplane class is also independent, which is how (albeit with some more work) one obtains rank 101. This is shown by hand.

Proposition 3.4. Y *is Calabi-Yau,* $h^{1,1}(Y) = 101$, *and* $h^{2,1}(Y) = 1$.

There's a direct proof due to S.S. Roan, and a more general approach with toric methods due to Batyrev.

Now Y fits into a one-dimensional family, and this is small enough that we might hope to write it down. In fact, this works — it's an example of a general construction called the *Dwork family*. In this case we deform with a parameter ψ and consider

$$f_\psi := x_0^5 + \cdots + x_4^5 - 5\psi x_0 x_1 x_2 x_3 x_4 = 0. \tag{3.5}$$

This again is fixed by a diagonal $\mathbb{Z}/5$-action, giving us a $(\mathbb{Z}/5)^3$-action. Let $X_\psi = Z(f_\psi)$, which is a family over $\mathbb{P}^1$ in ψ. If we let $z := (5\psi)^{-5}$, then $\mathbb{P}^1_\psi \to P^1_z$ is a quotient by a $\mathbb{Z}/5$-action, and Y fits into a family $\mathcal{Y} \to \mathbb{P}^1_z$ of Calabi-Yau threefolds which is smooth away from 0 and ∞. This family has some special fibers.

- At $z = 0$, $f_\psi = x_0 \cdots x_4 = 0$, so the zero locus is a union of five copies of $\mathbb{P}^3$ in $\mathbb{P}^4$, specifically the coordinate hyperplanes. This is a bad-looking degeneration! But it will be important in the computations, in that we will often consider z near zero.

[5]More generally, the singularity of type A_{n-1} can be found in $\{uv = w^n\}$.

- At $z = 5^{-5}$, i.e. $\psi = 1$, there's a three-dimensional A_1 singularity. To see this, let's first pass to the cover X_1, which has 125 three-dimensional A_1 singularities, which locally look like $\{x^2+y^2+z^2+w^2 = 0\}$. These all live in the same $(\mathbb{Z}/5)^3$-orbit, hence all get identified in the quotient. This is called a *conifold*, and isn't a great singularity to have — it behaves like letting your complex structure go to infinity.
- The *Fermat point* $z = \infty$, which is what we started with, the Fermat quintic. This has an additional $\mathbb{Z}/5$-symmetry.

So the moduli space of mirror quintics, namely $\mathbb{P}^1_z$, is really an orbifold $\mathbb{P}^1$, with these two singularities. The singularity at $z = 0$ is called the *large complex structure (LCS)* limit point, $z = 5^{-5}$ is called the *conifold point*, and $z = \infty$ is called the *orbifold point*. All of these points have some meaning in mirror symmetry.

Physicists are interested in computing Yukawa couplings, certain numbers extracted from an effective field theory. We can compute them in two ways, either using X or using Y, and they should agree. These take the form

$$\langle h, h, h\rangle_A = \sum_{d\in\mathbb{N}} N_d d^3 q^d, \tag{3.6}$$

where h is the hyperplane class in $H^2(X)$ (or more precisely, its Poincaré dual). When (3.6) was first written down, people did not know what these N_d were completely mathematically, but now we know they're Gromov-Witten invariants, a count of genus-0, degree-d curves C in your Calabi-Yau threefold. The d^3 comes form fixing the points of intersection with three copies of h.

There's a subtlety in N_d: it's not a naïve integer-valued count, because there could be maps which aren't embeddings, so this *a priori* gives rational numbers. You end up with rational power series in q, expressed in terms of *primitive counts*, which aren't exactly Gromov-Witten invariants, and haven't yet been made mathematical in general. But the Gromov-Witten invariants exist, and the numbers we get out at the end agree, which was one of the first manifestations of mirror symmetry historically. Physics suggested that these are symplectic invariants (in this setting you use pseudoholomorphic curves, following Gromov, Floer, and Fukaya), and in particular should be invariant under deformations of the complex structure.

But before we knew how to define and compute Gromov-Witten invariants, the computations that people did used the B-model on the mirror quintic, which sees the complex structure but not the symplectic structure. In this setting the Yukawa coupling on the family Y_z (with $z = (5\psi)^{-5}$) is

$$\langle \partial_z, \partial_z, \partial_z\rangle_B := \int_{Y_Z} \Omega^\nu(z) \wedge \partial_z^3 \Omega^\nu(z), \tag{3.7}$$

where Ω^ν is a (suitably normalized) holomorphic volume form: we fix $\int_{\beta_0} \Omega^\nu$ to be some constant, given $\beta_0 \in H_3(Y;\mathbb{Z})$.

Now, why is ∂_z a mirror to h? The idea is that h is equivalent data to a vector field on the moduli space of symplectic structures on X (well, really $\exp(2\pi i h)$ is that vector field). The mirror is a vector field ∂_w, a vector field on the moduli space of complex structures on Y, and it turns out

$$w = \int_{\beta_1} \Omega^\nu(z) \tag{3.8}$$

for a family of 3-cycles $\beta_1 \in H_2(Y;\mathbb{Z})$. The mirror symmetry statement is that

$$\langle h, h, h\rangle_A = \langle \partial_z, \partial_z, \partial_z\rangle_B \tag{3.9}$$

where $q = \exp(2\pi i w(z))$.

Now we want to compute these periods. We'll omit some details; there's a good account in Gross' lecture notes from the Nordfjordeid summer school.

$H_3(Y_\psi, \mathbb{Z}) \cong \mathbb{Z}^4$. Near $\psi = \infty$ (the large complex structure limit), we have a vanishing cycle. The idea of what's going on is to consider a singularity of the form $zw = t$ for t small. When $t \neq 0$, this is a one-sheeted hyperboloid, so we have a cycle diffeomorphic to S^1. When $t = 0$, there are two paraboloids, so the cycle has gone away, in a sense. We're in a higher-dimensional setting, but the basic idea is the same. We can write down an explicit choice for β_0, which will be diffeomorphic to a T^3, and next time we'll continue the computation.

Lecture 4.

Period integrals and the Picard-Fuchs equation: 9/10/19

Today we continue our discussion of the mirror quintic $\overline{Y}_\psi$, which fits into a one-dimensional family: ψ is a coordinate on an orbifold $\mathbb{P}^1$. Last time we discussed the vanishing cycle β_0, which is diffeomorphic to a T^3, and today we'll begin discussing the holomorphic 3-form Ω.

We can relatively easily write down this form by working inside $\mathbb{P}^4$, by taking the residue of a meromorphic (in fact rational) 4-form on $\mathbb{P}^4$ with simple poles along $X_\psi = Z(f_\psi)$. There are not so many choices to do this, and we might be able to guess the right answer.

$$\Omega(\psi) \coloneqq 5\psi \cdot \mathrm{res}_{X_\psi} \frac{\widetilde{\Omega}}{f_\psi} \in \Gamma(X_\psi, \Omega^3_{X_\psi}), \tag{4.1a}$$

where

$$\widetilde{\Omega} \coloneqq \sum_{i=0}^4 x_i \, \mathrm{d}x_0 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_4. \tag{4.1b}$$

(4.1a) doesn't quite make literal sense, but in homogeneous coordinates it's perfectly fine. Choose local holomorphic coordinates on X_ψ with $x_4 = 1$ and $\partial_{x_3} f \neq 0$; then

$$\Omega(\psi) = 5\psi \left. \frac{\mathrm{d}x_0 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2}{\partial_{x_3} f_\psi} \right|_{X_\psi}. \tag{4.2}$$

Now we'd like to normalize. If $\phi_0 \coloneqq \int_{\beta_0} \Omega(\psi)$, then $\widetilde{\Omega} \coloneqq \phi_0^{-1}\Omega(\psi)$ is normalized to have total integral 1. We can explicitly determine ϕ_0 with the (higher-dimensional) residue theorem:

$$\int_{\beta_0} \Omega(\psi) = \int_{T^4} 5\psi \frac{\mathrm{d}x_0 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3}{f_\psi} \tag{4.3}$$

$$= \int_{T^4} \frac{\mathrm{d}x_0 \wedge \cdots \wedge \mathrm{d}x_3}{x_0x_1x_2x_3} \left(\frac{1 + x_0^5 + \cdots + x_3^5}{5\psi x_0x_1x_2x_3} - 1 \right)^{-1}. \tag{4.4}$$

We can expand the second term as a geometric series:

$$= -\sum_{n\geq 0} \int_{T^4} \frac{\mathrm{d}x_0 \wedge \cdots \wedge \mathrm{d}x_3}{x_0x_1x_2x_3} \frac{(1 + x_0^5 + \cdots + x_3^5)^n}{(5\psi)^n (x_0x_1x_2x_3)^n}. \tag{4.5}$$

All summands in the numerator are fifth powers, so the summands in the denominator must be as well in order to contribute:

$$= -\sum_{n\geq 0} \int_{T^4} \frac{\mathrm{d}x_0 \wedge \cdots \wedge \mathrm{d}x_3}{x_0x_1x_2x_3} \frac{(1 + x_0^5 + \cdots + x_3^5)^{5n}}{(5\psi)^{5n} (x_0x_1x_2x_3)^{5n}} \tag{4.6}$$

$$= -(2\pi i)^4 \sum_{n\geq 0} \frac{(5n)!}{(n!)^5 (5\psi)^{5n}}, \tag{4.7}$$

telling us $\phi_0(z)$. This is enumerating the number of possibilities for these contributions. To completely pin it down, we'd need two more, similar integrals.

Griffiths' theory of period integrals, developed in the 1980s, allows one to compute more period integrals via the Picard-Fuchs equation. The argument goes as follows. Since cohomology is topological in nature, $H^3(Y_\psi; \mathbb{C})$ is locally constant, and is four-dimensional, so we can realize it as a trivial holomorphic vector bundle (at least over some subspace of the moduli space for the mirror quintic). The holomorphic volume form $\Omega(\psi)$ gives a section of this bundle — it trivializes $H^{3,0}$, but the Hodge structure on H^3 varies in ψ, which is part of the general story of *variation of Hodge structure*. The flat connection on H^3 is called the *Gauß-Manin connection* and denoted ∇^{GM}.

In particular, the derivatives $\partial_z^i \Omega(z)$ need not be holomorphic, since the trivialization of H^3 doesn't trivialize $H^{3,0}$. But the five elements of $\{\Omega(z), \partial\Omega(z), \partial^2\Omega(z), \partial^3\Omega(z), \partial^4\Omega(z)\}$ are sections of a four-dimensional vector bundle, hence must satisfy a relation, called the *Picard-Fuchs equation*,[6] a 4^{th}-order ODE with holomorphic coefficients.

To derive the equation, we'll produce more 3-forms from forms with higher-order poles; this part of the story is called *Griffiths' reduction of pole order.* As usual, let $X = Z(f_\psi) \subset \mathbb{P}^4$; then, associated to the pair of spaces $(\mathbb{P}^4, \mathbb{P}^4 \setminus X)$, we have the long exact sequence in cohomology

$$H^4(\mathbb{P}^4;\mathbb{C}) \longrightarrow H^4(\mathbb{P}^4 \setminus X;\mathbb{C}) \longrightarrow \underbrace{H^5(\mathbb{P}^4, \mathbb{P}^4 \setminus X;\mathbb{C})}_{(*)} \longrightarrow \underbrace{H^5(\mathbb{P}^4;\mathbb{C})}_{=0}. \tag{4.8}$$

If U is a tubular neighborhood of X, then $\dim U = 8$ and excision implies

$$(*) \cong H^5(U, U \setminus X;\mathbb{C}) \cong H^5(U, \partial U;\mathbb{C}). \tag{4.9}$$

Lefschetz duality, a version of Poincaré duality with boundary, establishes an isomorphism $H^q(M, \partial M) \cong H_{n-q}(M)$ for any compact oriented manifold M. Using this, and the fact that U retracts onto X,

$$(4.9) \cong H_3(U;\mathbb{C}) \cong H_3(X;\mathbb{C}) \cong H^3(X;\mathbb{C}), \tag{4.10}$$

where the last map is Poincaré duality.

Returning to (4.8), we've exhibited a surjective map $H^4(\mathbb{P}^4 \setminus X;\mathbb{C}) \twoheadrightarrow H^3(X;\mathbb{C})$; moreover, the use of differential forms to represent cohomology classes (TODO: I think that's what happened) tells us $H^0(\mathbb{P}^4 \setminus X, \Omega^4_{\mathbb{P}^4\setminus X}) \twoheadrightarrow H^4(\mathbb{P}^4 \setminus X;\mathbb{C})$, so we can represent any degree-3 cohomology class on X by differential 4-forms on $\mathbb{P}^4 \setminus X$.

Specifically, consider something of the form $g\widetilde{\Omega}/f^\ell \in H^0(\mathbb{P}^4, \Omega^4_{\mathbb{P}^4\setminus X})$, where $\deg g = 5\ell - 5$ (and $f = f\psi$). The exact forms are those of the form

$$\mathrm{d}\left(\frac{\sum_{i<j}(-1)^{i+j}(x_i g_j - x_j g_i')\,\mathrm{d}x_0 \wedge\cdots\wedge \widehat{\mathrm{d}x_i} \wedge\cdots\wedge \widehat{\mathrm{d}x_j} \wedge\cdots\wedge \mathrm{d}x_4 f^\ell}{=}\left(\ell \sum g_j \partial_{x_j} f - f \sum \partial_[x_j\right) g_j\right)\frac{\widetilde{\Omega}}{f^{\ell+1}}. \tag{4.11}$$

The upshot is that the numerator is in the ideal generated by $\partial_{x_i} f$, and we can therefore reduce ℓ. Taking four derivatives seems onerous but is perfectly tractable with the help of a physicist friend or a computer, and we obtain a relation, expressing g as a linear combination of $\partial_z^i \Omega(z)$, $i = 0, \dots, 4$. The answer is actually pretty simple.

Proposition 4.12. *Any period* $\phi = \int_\alpha \Omega(\psi)$ *fulfills the* Picard-Fuchs equation, *the ODE*

$$\theta^4 - 5z(5\theta+1)(5\theta+2)(5\theta+3)(5\theta+4)\phi(z) = 0, \tag{PF}$$

where $\theta = z\partial_z$.

It's not too hard to verify that ϕ_0 from (4.7) fulfills the equation — you might imagine there's a simpler one, and to prove that's not true requires more work.

Remark 4.13. Generalizing this to other hypersurfaces, or in general complete intersections in toric varieties, is less of a mess than the general story. Sometimes one has to delve into the more general theory of hypergeometric functions. ◀

(PF) is an ODE with a regular single pole

$$\Theta \cdot \phi(z) = A(z) \cdot \phi(z), \tag{RS}$$

where $\phi(z) \in \mathbb{C}^5$. This fits into a beautiful theorem that's sadly absent from the modern American ODE curriculum.

Theorem 4.14. (RS) *has a fundamental system of solutions of the form* $\Phi(z) = S(z)\cdot z^R$, *where* $S(z) \in M_s(\mathcal{O})$, $R \in M_s(\mathbb{C})$, *and*

$$z^R = I + (\log z)R + (\log z)^2 R^2 + \cdots. \tag{4.15}$$

If the eigenvalues do not differ by integers, we can take $R = A(0)$.

[6]The version of the Picard-Fuchs equation that you might find on, say, Wikipedia is in the setting of elliptic curves, which is the simplest setting for variations of Hodge structures. It fits into a more general story, though today we're only going to look at 3-folds.

Throw some linear algebra at (PF) and you can calculate that $A(0)$ has Jordan normal form with a single Jordan block, and $S = (\psi_0, \psi_1, \psi_2, \psi_3)$, where each ψ_i is a germ of a holomorphic function. This yields a fundamental system of solutions $\phi_0(z) = \psi_0$ — up to scaling, there's a unique single-valued (i.e. no $\log z$ terms) solution. Including logarithmic terms, we have additional solutions:

$$\phi_1(z) = \psi_0(z)\log z + \psi_1(z) \tag{4.16a}$$

$$\phi_2(z) = \psi_0(z)(\log z)^2 + \psi_1(z)\log z + \psi_2(z) \tag{4.16b}$$

$$\phi_3(z) = \psi_0(z)(\log z)^4 + \cdots + \psi_4(z). \tag{4.16c}$$

These solutions are multivalued, which means there's monodromy. This seems like a mystery, and one concludes the cycles must have monodromy. Though β_0 doesn't, everthing else has monodromy. Specifically, the monodromy of $z^{A(0)}$ refelcts the monodromy of $H^3(Y_z;\mathbb{C})$ around $z = 0$ (equivalently, $\psi = \infty$). More specifically, one can show that there's a symplectic basis $\beta_0, \beta_1, \alpha_1, \alpha_0$ of $H_3(Y_z;\mathbb{Q})$ such that the monodromy sends

$$\alpha_0 \longmapsto \alpha_1 \longmapsto \beta_1 \longmapsto \beta_0 \longmapsto 0. \tag{4.17}$$

Therefore $\phi_0 = \int_{\beta_0}\Omega(z)$, $\phi_1 = \int_{\beta_1}\Omega(z)$, $\phi_2 = \int_{\alpha_1}\Omega(z)$, and $\phi_3 = \int_{\alpha_0}\Omega(z)$. We're not far from the final computation of the Yukawa couplings!

Now let's write down the canonical coordinates. Let $q = e^{2\pi i w}$, where

$$w = \frac{\int_{\beta_1}\Omega(z)}{\int_{\beta_0}\Omega(z)} = \int_{\beta_1(z)}\widetilde{\Omega}(z). \tag{4.18}$$

Then $\phi_1(z) = \phi_0(z)\log z + \psi_1(z)$ is easy to obtain as a series solution to (PF); specifically, up to some constant,

$$\psi_1(z) = 5\sum_{n\geq 1}\frac{(5n)!}{(n!)^5}\left(\sum_{j=n+1}^{5n}\frac{1}{j}\right)^n. \tag{4.19}$$

Lecture 5.

Yukawa coupling: 9/12/19

"Typically physicists are right when it comes to numbers."

So far we've run through the computations of period integrals, and we're almost done with everything; the last step is to compute the Yukawa coupling. These computations are sometimes done by a different research community. They have to do with the (projective) special Kähler structure on the moduli space, special coordinates related to that, etc.

So we want to compute

$$\langle\partial_z, \partial_z, \partial_z\rangle_B = \int_{Y_z}\widetilde{\Omega}(z)\wedge\partial_z^3\widetilde{\Omega}(z), \tag{5.1}$$

where $\widetilde{\Omega}(z)$ is the normalized holomorphic volume form we discussed last time.

First, because the normalization is a bit messy, let's simplify the calculation. Introduce the auxiliary quantities

$$W_k \coloneqq \int_{Y_z}\Omega(z)\wedge\partial_z^k\Omega(z), \tag{5.2}$$

where $k = 0, \ldots, 4$, so (5.1) is W_3. Using the Picard-Fuchs equation

$$\left(\frac{\mathrm{d}^4}{\mathrm{d}z^4} + \sum_{k=0}^{3}c_k\frac{\mathrm{d}^k}{\mathrm{d}z^k}\right)\Omega(z) = 0, \tag{5.3}$$

we have a relation between the W_k with different k:

$$W_4 + \sum_{k=0}^{3}c_kW_k = 0. \tag{5.4}$$

The next ingredient we need to make progress is *Griffiths transversality.* Let U be an open subspace of the moduli space for Y and $\mathcal{F} := H^3(Y_z;\mathbb{C}) \otimes_{\mathbb{Z}} \mathcal{O}_U$. Then $\mathcal{F}$ has the *Hodge filtration* $\mathcal{F} = \mathcal{F}^0 \supseteq \mathcal{F}^1 \supseteq \mathcal{F}^2 \supseteq \mathcal{F}^3$, where

$$\mathcal{F}^k := \bigoplus_{q\geq k} R^q\pi_*\Omega^{3-q}. \tag{5.5}$$

Recall from last time we had something called the Gauß-Manin connection ∇^{GM}; the image of $\mathcal{F}^k$ under this connection is contained in $\mathcal{F}^{k-1} \otimes \Omega^1_U$, which one can check by looking at the definition of ∇^{GM} and some Hodge theory. This is part of a general story about Kähler manifolds.

Recall that under the wedge product, $H^{p,q}$ and $H^{p',q'}$ are orthogonal unless $p' = 3-p$ and $q' = 3-q$. This immediately forces $W_0 = W_1 = W_2 = 0$. Hence $W_2''(z) = 0$ as well and so forth, and in particular $2W_3' - W_4 = 0$. Plugging this into the Picard-Fuchs equation, we conclude

$$W_3 + \frac{1}{2}c_3W_3 = 0. \tag{5.6}$$

That this only depends on c_3 is nice, but c_3 is not necessarily easy to compute. In this form, the answer is

$$c_3(z) = \frac{6}{z} - \frac{25^5}{1-5^5z}, \tag{5.7}$$

which is simple enough that we can solve for W_3 in closed form:

$$W_3 = \frac{C}{(2\pi i)^3z^3(5^2z-1)}, \tag{5.8}$$

where C is a constant of integration we have no control over; we can't resolve this ambiguity.

The final step is to rewrite this in terms of $q = e^{2\pi i w}$, where $w = \phi_1(z)/\phi_0(z)$ is the canonical coordinate, and then expand. The answer is

$$\langle \partial_w, \partial_w, \partial_w\rangle_B = -c_1 - 575\frac{c_1}{c_2}q - \frac{1950750}{2}\frac{c_1}{c_2^2}q^2 - \frac{10277490000}{6}\frac{c_1}{c_2^3}q^3 + \dots \tag{5.9}$$

We can rewrite this as

$$5 + \sum_{d\geq 1} d^3 n_d\frac{q^d}{1-q^d} = 5 + n_1q + (8n_2+n_1)q^2 + (27n_3+n_1)q^3 + \cdots \tag{5.10}$$

Here n_d is a Gromov-Witten invariant, counting curves of degree d in the quintic. This makes the following predictions about curve counts.

- $n_1 = 2875$ is classically known.
- $n_2 = 609250$, computed by S. Katz just a few years before the initial mirror symmetry paper.
- $n_3 = 317206375$, which disagreed with a preexisting computation – but the authors of that computation took a more careful look at their computations, and it turned out that this agrees with the correct answer, as determined by Ellingswood and Strømme in 1990.

At this point, it was clear something was going on. The conjecture for all d was proven by Givental in 1996 and Lian, Lin, and Yau in 1997.

Remark 5.11. Two takeaways: first, computations are hard, of course, but another takeaway is that the prediction depends on the large complex structure limit (so the Kähler cone), though the Yukawa couplings exist in the entire moduli space, and in particular depends on the monodromy in $H^3(Y)$. These two facts seem to stand in opposition to each other. ◀

One also asks, in what generality does mirror symmetry apply? The orbifolding construction of the mirror is special to the quintic threefold; Batyrev and Borisov constructed mirrors for complete intersections in toric varieties.

We will next discuss Gromov-Witten theory, but from a somewhat broader perspective beginning with moduli spaces and stacks. This will streamline the discussions of log Gromov-Witten theory and Donaldson-Thomas theory later in the course.

Gromov-Witten theory lives within the general framework of curve counting, e.g. how many genus-zero holomorphic curves in the quintic? This is a very classical question, and also attracted the interest of the Italian school of algebraic geometry.

The rigidity of algebraic geometry sometimes works against you: for example, if you want to consider the conics through three points, there's an issue for non-generic point arrangements. Typical computations assumed the existence of enough deformations to work around this; classically, this involved "general position arguments" which were often simple — but proving that general-position objects exist at all can be quite difficult, especially in higher-dimensional enumerative questions. These are essentially transversality arguments.

From a more modern point of view, one works with topology. One considers a moduli space of objects, and each condition imposed (e.g. curve must go through a given point) is a cohomology class, and then one computes in the cohomology ring. For example, intersection theory in the Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$ governs the classical theory called *Schubert calculus*.

You can get a fair ways with this, e.g. you can compute the number of lines on a quintic using Schubert calculus, and some higher-genus counts proceed along essentially similar methods. But there is a conjecture[7] that for every curve, there's some quintic without a discrete moduli space of such curves. This is still open, and if it's true these methods will fail in general.

Generally, moduli spaces of curves on a quintic don't have the expected dimension. For example, on the Dwork family $\{f_\psi = 0\}$, the moduli space of lines contains 375 isolated lines (each isomorphic to $\mathbb{P}^1$), e.g.

$$(u, v, -\zeta^k u, -\zeta^\ell v, 0), \tag{5.12}$$

where $[u : v] \in \mathbb{P}^1$, ζ is a primitive fifth root of unity, and $0 \le k, \ell \le 4$. But there are also two irreducible families. How do we count in the absence of general deformations?

Moreover, in higher degree, there are multiple covers, e.g. the degree-d cover $\mathbb{P}^1 \to \mathbb{P}^1$. How do we count these?

The answer is to use virtual counts, which produces invariants which are constant in families of targets.

Moduli spaces. The basic idea of a moduli space is to consider the set of isomorphism classes of some object under interest, and give it a topology, and, ideally, the structure of an algebraic variety, ideally in some systematic way. We want the set of closed points of this space $\mathcal{M}$ to be the set of isomorphism classes of the geometric objects in question (e.g. varieties).

To upgrade this into a variety or scheme, you need to know what the structure sheaf is, or equivalently what all the holomorphic functions are. We can test this by pulling back along maps $T \to \mathcal{M}$. This tells us an object for each point of T, and we want this map to be holomorphic if these objects vary algebraically in T.

Example 5.13. The *Hilbert scheme*[8] is the moduli space of closed subschemes $Z \subset \mathbb{P}^N$, where N is fixed. Often one also fixes the Hilbert polynomial of Z. This forms a moduli space $\mathrm{Hilb}(\mathbb{P}^N)$, with a canonical subscheme $\mathcal{Z} \subset \mathrm{Hilb}(\mathbb{P}^N) \times \mathbb{P}^N$, and together these satisfy a universal property. Given any flat proper map $Z \to T$, where $Z \subset T \times \mathbb{P}^N$ is closed, there is a unique map $\varphi\colon T \to \mathrm{Hilb}(\mathbb{P}^N)$ such that the pullback of $\mathcal{Z} \subset \mathrm{Hilb}(\mathbb{P}^N) \times \mathbb{P}^N$ along φ is identified with the inclusion $Z \subset T \times \mathbb{P}^N$. ◀

This is just about the nicest moduli problem out there; things are usually more difficult (or, depending on your perspective, more interesting).

To set this in more general terms, let $\mathcal{S}ch$ denote the category of schemes.[9] There is a functor $F\colon \mathcal{S}ch \to \mathcal{S}et$ sending a scheme T to the set of isomorphism classes of flat proper maps $Z \to T$ where $Z \subset T \times \mathbb{P}^N$ is closed. That is, this is the set of families of closed subschemes of Z parametrized by T. That $\mathrm{Hilb}(\mathbb{P}^N)$ is the moduli space for such data is encoded in the theorem that it *represents* the functor F, i.e. F is naturally isomorphic to $\mathrm{Hom}_{\mathcal{S}ch}(-, \mathrm{Hilb}(\mathbb{P}^N))$. Plugging in $\mathrm{Hilb}(\mathbb{P}^N)$ and its identity map, we obtain an element of $F(\mathrm{Hilb}(\mathbb{P}^N))$, which is exactly the universal family $\mathcal{Z}$, which you can check.

This is a nice picture — too nice, in fact, for some very nice moduli spaces, such as families of curves! The reason for this failure is that curves have automorphisms. This wasn't an issue for the Hilbert scheme, but in general you might be interested in moduli problems for objects which have automorphisms. For example,

[7]This conjecture is often attributed to Klemens, except that Klemens claims to not have conjectured it!

[8]There are more general examples also called Hilbert schemes.

[9]It actually suffices to use affine schemes, if you'd prefer to do that.

there are families of curves whose fibers are all isomorphic (so the map to the moduli space is constant) but the fiber bundle itself is nontrivial (so the map should be nonconstant). We will see an example next time.

Lecture 6.

Stacks: 9/17/19

> *"French [is] almost English, right? It's a little hard if you insist on English pronunciation, though."*

In addition to these lectures on stacks/moduli spaces, it may be helpful to keep some references in hand:

(1) Dan Edidin's notes [Edi98] on the construction of the moduli space of curves.
(2) The original article by Deligne and Mumford [DM69] on the irreducibility of the moduli space of curves.

Last time, in Example 5.13, we discussed a moduli space of subvarieties of $\mathbb{P}^n$. We would like to repeat this story to construct a moduli space of complete[10] curves of a fixed genus g. To do so, we need a notion of families of curves.

Definition 6.1. Let S be a scheme. A *curve of genus g* over S is a morphism of schemes $\pi\colon C \to S$ such that

(1) π is proper and flat.
(2) Each *geometric fiber* $C_S := \operatorname{Spec} k \times_S C$, where k is an algebraically closed field and $\operatorname{Spec} k \to S$ is a k-point, is reduced,[11] connected, one-dimensional, and has *arithmetic genus* equal to g, i.e. $h^1(C_S, \mathcal{O}_{C_S}) = g$.

You can think of this as follows: S is a space of parameters, and we're thinking about a family of curves parametrized by S. Some of the conditions tell us this is a suitably nice family. One can also impose additional restrictions, e.g. imposing that C is nonsingular — though it was a major insight of Deligne and Mumford that if you want a compact space, you have to allow certain singularities, such as nodes.

However, there is a fundamental and serious problem obstructing the construction of a moduli space: the functor $F_g\colon \mathcal{S}ch \to \mathcal{S}et$ sending S to the set of isomorphism classes of (nonsingular) genus-g curves over S is not representable. Ultimately this is because there are *isotrivial* families of curves (i.e. trivial after a finite base change) that are nontrivial: if M_g is the hypothetical representing object and $\varphi\colon S \to M_g$ determines an isotrivial family, then we have a finite map $T \to S$ such that the composition with φ is constant. Hence φ is also constant – but there are examples of nonconstant φ, so M_g can't exist.

Example 6.2. Let's explicitly construct one of these families in any genus $g > 0$. Let C_0 be a curve with a nontrivial automorphism, such as a hyperelliptic curve, which is a branched double cover of $\mathbb{P}^1$. One explicit example is the projective closure of

$$y^2 = (x-g-1)(x-g)\cdots(x-1)(x+1)\cdots(x+g+1) = 0. \tag{6.3}$$

This has genus g. The nontrivial automorphism ϕ flips the sign of y. See Figure 1 for a picture.

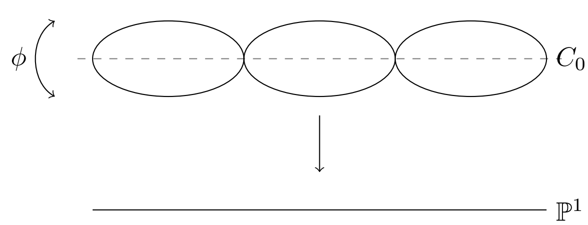

FIGURE 1. A hyperelliptic curve is a branched double cover of $\mathbb{P}^1$.

[10] This property is the analogue of compactness in differential geometry.

[11] This rules out a problem which can only happen in algebraic geometry, i.e. nilpotent elements of the ring of functions. Thus C_S is a variety.

Recall that $\mathbb{G}_m := \operatorname{Spec}\mathbb{C}[t, t^{-1}]$, which as a space is $\mathbb{C}^\times$. Define a $\mathbb{Z}/2$-action on $C_0 \times \mathbb{G}_m$ by asking the nontrivial element of $\mathbb{Z}/2$ to act by $(\phi, -1)$. Then $(C_0 \times \mathbb{G}_m)/(\mathbb{Z}/2)$ is a family of genus-g curves over $\mathbb{G}_m$, and is nontrivial, but is trivialized by pulling back by the multiplication-by-2 map $\mathbb{G}_m \to \mathbb{G}_m$. ◀

So there must be some kind of internal structure of the moduli ... thingy which keeps track of this data, and indeed we'll see how that happens.

Remark 6.4. Well it seems like the problem is automorphsims, so why don't we just remove curves which have automorphisms? This actually works; it just isn't useful. The resulting space isn't compact: in a sense you've just poked a bunch of holes in it, and moreover, some of the holes you've poked through were things you wanted to study. ◀

The upshot is that we will solve the moduli problem by constructing an *algebraic stack*. Algebraic stacks are a generalization of the notion of schemes with automorphisms baked into the story from the very beginning. As we go along, the book *Champs algébriques* [LMB00] by Laumon and Moret-Bailly is a great reference. It's in French, but is a great enough reference that it's still worth recommending.[12]

We want to first formalize the notion of a family of objects parameterized by a scheme, along with fiberwise automorphisms. Fix a base scheme S (you can think $S = \mathbb{C}$ for concreteness), and recall that the category $\mathcal{S}ch_S$ of *schemes over* S is the category whose objects are pairs of a scheme T and a map of schemes $f_T : T \to S$, and whose morphisms $(T_1, f_{T_1}) \to (T_2, f_{T_2})$ are maps $T_1 \to T_2$ of schemes intertwining f_{T_1} and f_{T_2}.

Let $\mathcal{S} := \mathcal{S}ch_S$; thus the following definitions can all be extended to categories, groupoids, etc. over arbitrary categories, though we won't need that level of generality.

Definition 6.5.

(1) A *category over* $\mathcal{S}$ is a category $\mathcal{C}$ together with a functor $p_{\mathcal{C}} : \mathcal{C} \to \mathcal{S}$. If $T \in \mathcal{S}$, its *fiber category* $\mathcal{C}(T)$ is the subcategory of $\mathcal{C}$ whose objects are those $x \in \mathcal{C}$ with $p_{\mathcal{C}}(x) = T$ and whose morphisms are those morphisms in $\mathcal{C}$ whose image under $p_{\mathcal{C}}$ is the identity on T.

(2) A category $\mathcal{C}$ over $\mathcal{S}$ is a *groupoid over* $\mathcal{S}$, or a *fibered groupoid*, if it satisfies the following conditions.
  (a) For all $f : B' \to B$ in $\mathcal{S}$ and objects $X \in \mathcal{C}$, there is a map $\phi : X' \to X$ in $\mathcal{C}$ such that $p_{\mathcal{C}}(\phi) = f$; in particular, $p_{\mathcal{C}}(X) = B$ and $p_{\mathcal{C}}(X') = B'$.
  (b) Suppose we have $X, X', X'' \in \mathcal{C}$, maps $\phi' : X' \to X$ and $\phi'' : X'' \to X$, and a map $h : p_{\mathcal{C}}(X') \to p_{\mathcal{C}}(X'')$ such that $p_{\mathcal{C}}(\phi') = p_{\mathcal{C}}(\phi'') \circ h$. Then, there is a *unique* map $\chi : X' \to X''$ such that $\phi' = \phi'' \circ \chi$.

Remark 6.6. If $\mathcal{C}$ is a groupoid over $\mathcal{S}$, all of its fiber categories are groupoids, but in general this is a stronger notion. We also see that:

- $\phi : X' \to X$ is an isomorphism if and only if $p_{\mathcal{C}}(\phi)$ is an isomorphism.
- Given a map $f : B' \to B$ in $\mathcal{S}$ and an $X \in \mathcal{C}$ with $p_{\mathcal{C}}(X) = B$, there is a unique $X' \in \mathcal{C}$ (up to unique isomorphism) X' with a map $\phi : X' \to X$ such that $p_{\mathcal{C}}(X') = B'$ and $p_{\mathcal{C}}(\phi) = f$. We call X' the *pullback* of X, and denote it f^*X. Pullback is natural enough to define a functor $f^* : \mathcal{C}(B) \to \mathcal{C}(B')$. ◀

Example 6.7. Let $F : \mathcal{S}^{op} \to \mathcal{S}et$ be a contravariant functor. This yields a groupoid $\mathcal{F}$ over $\mathcal{S}$:

Objects: the pairs (B, β) such that $B \in \mathcal{S}$ and $\beta \in F(B)$.

Morphisms: the maps between (B, β) and (B', β') are the maps $f : B \to B'$ such that $F(f)(\beta') = \beta$.

The fiber categories don't have any nontrivial morphisms, but this is still useful: any S-scheme X defines such a functor via its *functor of points*: the functor $\mathcal{S}^{op} \to \mathcal{S}et$ sending $B \mapsto \operatorname{Hom}_{\mathcal{S}}(B, X)$. This seems a little silly, but will later tell us that schemes embed into stacks, and therefore stacks extend things we already think about. ◀

The next example is "more stacky," i.e. the fiber categories are more interesting.

Example 6.8. Let X be a scheme over S with an action of a (flat) group scheme G over S. For example, you could let $S = \operatorname{Spec}\mathbb{C}$, $G = \mathrm{GL}_n(\mathbb{C})$, and $X = \mathbb{A}^n_{\mathbb{C}}$ via the standard representation. The *quotient groupoid* $[X /\!\!/ G]$ is the groupoid over $\mathcal{S}$ given by the following data.

[12] One thing which might be confusing: in French, *champ* means both "stack" and "field", the latter both in the algebraic sense and the physics sense.

- The objects are pairs $(\pi\colon E \to B, f\colon E \to X)$, where $\pi\colon E \to B$ is a principal G-bundle and f is G-equivariant.
- The morphisms (π, E, B, f, X) to (π', E', B', f', X') are pairs $\phi\colon E' \to E$ and $\psi\colon B' \to B$ such that $f \circ \phi = f'$ and

$$\begin{tikzcd} E' \arrow[r, "\phi"] \arrow[d, "\pi'"] & E \arrow[d, "\pi"] \\ B' \arrow[r, "\psi"] & B \end{tikzcd} \tag{6.9}$$

  is Cartesian.

It turns out that if G acts freely on X and X/G exists as a scheme, then $[X/\!\!/G] \simeq X/G$ (where we regard X/G as a groupoid over $\mathcal{S}$ as in Example 6.7). ◀

Remark 6.10. If $X = \mathrm{pt}$, this tells us that $[\mathrm{pt}/\!\!/G]$ is the moduli "space" of principal G-bundles. It's worth comparing to the analogue in algebraic topology, which is called BG; this is at least an actual space, but doesn't have nearly as nice of a canonical construction. ◀

Lecture 7.

The 2-category of groupoids: 9/19/19

Recall from last time that $\mathcal{S}$ denotes the category of schemes over a base S. An S-scheme X defines a groupoid fibered over $\mathcal{S}$ by $\underline{X}(T) := \mathrm{Hom}_{\mathcal{S}}(T, X)$. We also saw tht a group acting on X induces a quotient groupoid $[X/\!\!/G]$. There is a functor $F_g\colon \mathcal{S} \to \mathcal{S}et$ sending an S-scheme T to the groupoid of (families of) curves over T.

Now we define the *moduli groupoid of curves* of genus g, $\mathcal{M}_g$. Its objects are curves $X \to B$ of genus g, where B is any scheme and for all geometric points s of B, X_s is nonsingular. The morphisms from $X' \to B'$ to $X \to B$ are pairs of maps $X' \to X$ and $B' \to B$ such that the diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ B' \arrow[r] & B \end{tikzcd} \tag{7.1}$$

commutes and is Cartesian, i.e. $X' \cong B' \times_B X$. This is *not* the groupoid associated to F_g.

Similarly, one can define the *universal curve* $\mathcal{C}_g$ over $\mathcal{M}_g$, whose objecrs are pairs of a smooth curve $X \to B$ of genus g (with the same conditions as before) and a section $\sigma\colon B \to X$, and whose morphisms include compatibility of the sections under pullback.

We'd like to say that forgetting the section defines a map $\mathcal{C}_g \to \mathcal{M}_g$, and to do this we need to know what maps of groupoids are.

Definition 7.2. Let $p_1\colon \mathcal{C}_1 \to \mathcal{S}$ and $p_2\colon \mathcal{C}_2 \to \mathcal{S}$ be groupoids over $\mathcal{S}$. A functor $\varphi\colon \mathcal{C}_1 \to \mathcal{C}_2$ is a *morphism of groupoids over* $\mathcal{S}$ if $p_1 = p_2 \circ \varphi$. Here we ask for equality of functors, not just up to natural isomorphism.

Thus forgetting the section indeed specifies a map $\mathcal{C}_g \to \mathcal{M}_g$.

We would also like to say that a morphism $f\colon X \to Y$ of schemes is equivalent to a morphism $\underline{f}\colon \underline{X} \to \underline{Y}$ of groupoids over $\mathcal{S}$; this would tell us that $\mathcal{S}$ embeds into the category of groupoids over $\mathcal{S}$, and allow us to think of X and $\underline{X}$ as the same.

- In the forward direction, given f, we can define $\underline{f}$ on objects by

$$\underline{f}\colon \left(B \stackrel{u}{\to} X\right) \longmapsto \left(B \stackrel{f\circ u}{\to} Y\right), \tag{7.3}$$

  and on morphisms by

$$\underline{f}\colon \left(\begin{tikzcd} B \arrow[d, "s"'] \arrow[dr, "u"] & \\ B' \arrow[r, "u'"'] & X \end{tikzcd}\right) \longmapsto \left(\begin{tikzcd} B \arrow[d, "s"'] \arrow[dr, "f\circ u"] & \\ B' \arrow[r, "f\circ u'"'] & X \end{tikzcd}\right). \tag{7.4}$$

- Conversely, given $\underline{f}$ the map of groupoids over $\mathcal{S}$, $\underline{f}(\mathrm{id}\colon X \to X)$ is some map $X \to Y$, which we call f.

You can unwind the diagrams to check that these are mutually inverse operations, which is exactly what is happening in Yoneda's lemma; $\underline{f}$ can be viewed as a natural transformation between the two functors $\mathrm{Hom}_{\mathcal{S}}(-, X)$ and $\mathrm{Hom}_{\mathcal{S}}(-, Y)$ from $\mathcal{S}$ to $\mathcal{S}et$.

Remark 7.5. Similarly, if $B \in \mathcal{S}$ and $\mathcal{C}$ is a groupoid over $\mathcal{S}$, there's a natural isomorphism $\mathrm{Hom}(\underline{B}, \mathcal{C}) \xrightarrow{\cong} \mathcal{C}(B)$ by $p \mapsto p(\mathrm{id}_B)$. ◀

Example 7.6. This is a somewhat silly example, but for any groupoid $\mathcal{C}$ over $\mathcal{S}$, the structure map $p_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{S}$ is a morphism of groupoids $\mathcal{C} \to \underline{S}$. ◀

Example 7.7. Let X be a scheme over S with an action of a group S-scheme G; this yields a quotient map $q\colon \underline{X} \to [X/\!\!/G]$. Recall that an object of $[X/\!\!/G]$ is a scheme E with a G-action and an identification of the quotient with B, and a G-equivariant map $E \to X$.

The map q sends the object $s\colon B \to X$ to $G \times B$, which carries the left G-action, and whose quotient is identified with B in the standard way. The map $G \times B \to X$ sends $g, b \mapsto g \cdot s(b)$, and this is G-equivariant.

On morphisms, q is the map

$$\left(\begin{tikzcd} B' \arrow[rr, "f"] \arrow[dr, "s'"'] & & B \arrow[dl, "s"] \\ & X & \end{tikzcd}\right) \longmapsto \left(\begin{tikzcd} G \times B' \arrow[r, "-\mathrm{id}\times f"'] \arrow[d] \arrow[rr, bend left] & G \times B \arrow[r] \arrow[d] & X \\ B' \arrow[r, "f"'] & B & \end{tikzcd}\right). \tag{7.8}$$ ◀

We now have groupoids over $\mathcal{S}$ and morphisms thereof, so it seems reasomable to ask whether two groupoids over $\mathcal{S}$ are isomorphic. But this is a subtle issue, as with all questions involving isomorphisms of categories, because in addition to objects (categories, or in this case groupoids over $\mathcal{S}$) and morphisms (suitable functors), we have natural transformations, and in order to even formulate commutativity of diagrams in a useful way, we need to take natural transformations into account.

Specifically, $\mathcal{S}$-groupoids form what's called a *2-category*: roughly speaking, instead of just a set of morphisms between objects X and Y, we have a category; the objects in this category are called 1-*morphisms* and the maps are called 2-*morphisms*. For $\mathcal{S}$-groupoids:

- The objects are groupoids over $\mathcal{S}$.
- The 1-morphisms are the functors we considered in Definition 7.2, so functors commuting with the functors to $\mathcal{S}$.
- The 2-morphisms are the natural isomorphisms between these functors.

Proposition 7.9. *Let X and Y be schemes. Then X and Y are isomorphic if and only if $\underline{X}$ and $\underline{Y}$ are isomorphic in this* 2*-category $\mathcal{G}pd_{\mathcal{S}}$.*

Here, "isomorphic in $\mathcal{G}pd_{\mathcal{S}}$" means that there are 1-morphisms $\underline{f}\colon \underline{X} \to \underline{Y}$ and $g\colon \underline{Y} \to \underline{X}$ and 2-morphisms (natural isomorphisms) $h_1\colon \underline{f} \circ \underline{g} \to \mathrm{id}$ and $h_2\colon \underline{g} \circ \underline{f} \to \mathrm{id}$.[13] TODO: I didn't write down the proof in time. The key point, though, is that we can't upgrade this into "an isomorphism determines an isomorphism" because $\underline{f} \circ \underline{g}$ and $\underline{g} \circ \underline{f}$ might not be *equal* to id, just *naturally isomorphic* to it.

Anyways, the upshot is that schemes over S embeds as a full subcategory of $\mathcal{G}pd_{\mathcal{S}}$, even in this 2-categorical setting. Therefore we will no longer distinguish X and $\underline{X}$, etc., nor S and $\mathcal{S}$.

With our enriched understanding of the structure of $\mathcal{G}pd_S$, we can define fiber products and Cartesian diagrams of groupoids over S.

Definition 7.10. Let $F, G, H \in \mathcal{G}pd_S$ and $f\colon F \to G$ and $h\colon G$ be morphisms of S-groupoids. The *fiber product* $F \times_G H$ is the S-groupoid with the following data.

- The objects over $B \in \mathcal{S}$ are triples (x, y, ψ) with $x \in F(B)$, $y \in H(B)$, and $\psi\colon f(X) \xrightarrow{\cong} h(y)$ is an isomorphism in $G(B)$.
- The morphisms $(x, y, \psi) \to (x', y', \psi')$ over a map $B \to B'$ are the pairs $(\alpha\colon x' \to x, \beta\colon y' \to y)$ such that $\psi \circ f(\alpha) = h(\beta) \circ \psi'$.

[13]Compare to the definition of a homotopy equivalence of topological spaces.

You're probably thinking of this as sitting in a square

$$\begin{tikzcd} F \times_G H \arrow[r, "q"] \arrow[d, "p"] & H \arrow[d, "h"] \\ F \arrow[r, "f"] & G, \end{tikzcd} \tag{7.11}$$

where p is remembers x and forgets y and ψ, and q remembers y and forgets x and ψ, but *this diagram is not commutative*, as

$$fp(x, y, \psi) = f(x) \neq h(y) = gq(x, y, \psi). \tag{7.12}$$

However, there is a natural isomorphism $fp \simeq hq$, meaning (7.11) is 2-*commutative* (i.e. commutative up to 2-morphisms, which in this 2-category are isomorphisms). Explicitly, the natural isomorphism is $\eta_{(x,y,\psi)} := \psi\colon f(x) \overset{\cong}{\to} h(y)$.

This allows us to formulate the universal property of the pullback: if

$$\begin{tikzcd} T \arrow[r, "\alpha"] \arrow[d, "\beta"'] & H \arrow[d, "h"] \\ F \arrow[r, "f"] & G \end{tikzcd} \tag{7.13}$$

is a 2-commutative diagram in $\mathcal{G}pd_S$, then there is a 1-morphism (i.e. functor over S) $\phi\colon T \to F \times_G H$, unique up to 2-morphsims (i.e. natural isomorphisms) such that the following diagram is 2-commutative:

$$\begin{tikzcd} T \arrow[dr, "\phi", "\exists!"'] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "\beta"'] & & \\ & F \times_G H \arrow[r, "q"] \arrow[d, "p"] & H \arrow[d, "h"] \\ & F \arrow[r, "f"] & G. \end{tikzcd} \tag{7.14}$$

Example 7.15. Because $\mathcal{S}ch_S$ embeds in $\mathcal{G}pd_S$ as a full subcategory, if X, Y, and Z are schemes and $f\colon X \to Z$ and $g\colon Y \to Z$ are maps of schemes over S, then $\underline{X} \times_{\underline{Z}} \underline{Y} \simeq \underline{X \times_Z Y}$. ◂

Example 7.16. We can use pullback to define base change. Let F be a groupoid over S and $T \to S$ be a map of schemes. The $T \times_S F$ is a groupoid over T. In fact, for all schemes B over T, $F(B)$ and $(T \times_S F)(B)$ are naturally isomorphic as functors $\mathcal{S}ch_T^{op} \to \mathcal{S}et$. ◂

This last example allows us to (finally!) define stacks, though we'll need some more material to define algebraic stacks.

Definition 7.17. Let (F, p_F) be a groupoid over S, B be a scheme over S, and $X, Y \in F(B)$. The *iso-functor* determined by this data is the functor $\underline{\mathrm{Iso}}_B\colon \mathcal{S}ch_B^{op} \to \mathcal{S}et$ which

- sends an object $f\colon B' \to B$ to the set of isomorphisms $\{\phi\colon f^*X \overset{\cong}{\to} f^*Y\}$, and
- sends the morphism $h\colon B'' \to B'$ to the morphism

$$(\phi\colon f^*X \to f^*Y) \longmapsto (h^*\phi\colon h^*f^*X \to h^*f^*Y). \tag{7.18}$$

Theorem 7.19 (Deligne-Mumford)**.** *If X and Y are curves over B of the same genus $g \geq 2$, then $\underline{\mathrm{Iso}}_B(X, Y)$ is represented by a scheme $\mathrm{Iso}_B(X, Y)$.*

Proof sketch. Since $g \geq 2$, then $\omega_{X/B}$ and $\omega_{Y/B}$ are ample, and therefore for any map $f\colon B' \to B$, any isomorphism $f^*X \to f^*Y$ preserves the polarization. Then the representing object is the relative Hilbert scheme for the graph of $f^*X \to f^*Y$. ⊠

Remark 7.20. $\mathrm{Iso}_B(X, Y)$ is finite and unramified over B, which relates to the fact that automorphism groups of genus-≥ 2 curves are finite. However, $\mathrm{Iso}_B(X, Y) \to B$ is not in general flat; the cardinalities of the fibers can jump. ◂

We're almost at the definition of a stack: we just need $\underline{\mathrm{Iso}}_B$ to satisfy a sheaf condition and a gluing condition. We'll do this next time.

Lecture 8.

The étale topology and stacks: 9/24/19

We now have all the ingredients in place!

Definition 8.1. A groupoid (F, p_F) over S is a *stack* if

(1) for all S-schemes B and $X, X' \in F(B)$, $\underline{\mathrm{Iso}}(X, X')$ is an étale sheaf,[14]
(2) For any étale covering $\mathfrak{U}$ of B, given objects $\{X_U \in F(U) : U \in \mathfrak{U}\}$ and isomorphisms

$$\phi_{UV} : X_V|_{U\times_B V} \longrightarrow X_U|_{U\times_B V}, \qquad U, V \in \mathfrak{U} \tag{8.2}$$

satisfying the cocycle condition $\phi_{UV}\phi_{VW}\phi_{WU} = \mathrm{id}$, we can glue: there is an $X \in F(B)$ and isomorphisms $X|_U \cong X_U$ inducing the ϕ_{UV}.

The first condition can be thought of as "descent (i.e. gluing) for isomorphisms," and the second as "descent for objects."

So we need to talk about the étale topology. We will be replacing Zariski-open subsets with maps $U \to X$ called étale morphisms; these should be thought of as finite-to-one covering maps which are unramified. These are a notion of open set, so the intersection of two opens ought to be open. The intersection of $f\colon U \to X$ and $g\colon V \to X$ is $U \times_X V \to V$, and if f and g are étale, the intersection will be to. This is an example of a notion called a *Grothendieck topology.*

Let's talk about étale morphisms, and their siblings, (formally) smooth and unramified morphisms. The perspective you see here didn't quite make it into Hartshorne; for a reference see EGA IV or Milne's notes on étale cohomology.

Definition 8.3. A finite-type map of affine schemes $\operatorname{Spec} A \hookrightarrow \operatorname{Spec} A'$ is an *infinitesimal extension* if it identifies $A \cong A'/I$, where $I^n = 0$ for some n. If $n = 2$, we also call it a *square-zero extension.*

For example, $\operatorname{Spec} k \hookrightarrow \operatorname{Spec} k[\varepsilon]/(\varepsilon^n)$ is an infinitesimal extension, and is the algebro-geometric analogue of n-jets at a point in differential geometry.

Given a map $f\colon X \to Y$, we will consider commutative diagrams of the form

$$\begin{tikzcd} T \arrow[r, "\varphi"] \arrow[d, hook] & X \arrow[d, "f"] \\ T' \arrow[r, "\psi"] & Y, \end{tikzcd} \tag{8.4}$$

whjere $T \hookrightarrow T'$ is an infinitesimal extension, and ask whether a lift $\widetilde{\psi}\colon T' \to X$ of ψ exists or is unique.

Definition 8.5. A map $f\colon X \to Y$ is

- *(formally) smooth* if in every diagram of the form (8.4), a lift $\widetilde{\psi}$ exists,
- *unramified* if in every such diagram, a lift is unique if it exists, and
- *étale* if in every such diagram, a lift exists and is unique.

"Smooth" isn't quite like the notion of a smooth map in differential geometry; it's the version of a submersion. We're saying that any tangent vector of Y (well, in fact any n-jet, for any n) lifts, possibly nonuniquely, to a tangent vector on X.

Étaleness can be thought of as the algebraic geometry analogue of being a local isomorphism: tangent vectors pull back uniquely.

To understand unramified morphisms, let's look at a nonexample, the map $\mathbb{A}^1 \to \mathbb{A}^1$ induced from $z \mapsto z^2$. The induced map on tangent spaces at the origin is the zero map, so the zero tangent vector has a nonunique lift. This map is a double cover branched at the origin.

Now let's look at some examples of stacks.

[14]Precisely, one defines a certain kind of topology on a scheme, called the *étale topology*, and asks for $\underline{\mathrm{Iso}}(X, X')$ to sheafify in that topology. We'll say more about this. One could make this definition for the Zariski topology, hence sheaves in the usual sense, but that produces a considerably less useful notion of stacks. People also study stacks for other interesting topologies, such as fppf.

Example 8.6. Let $\mathcal{F}$ be the groupoid assciated to a functor $F\colon \mathcal{S}ch_S \to \mathcal{S}et$. Then $\mathcal{F}$ is a stack iff F is a sheaf (of sets) in the étale topology. For example, the functor of points $\mathrm{Hom}(-, X)$, where X is an S-scheme, is a stack, and this needs étale descent rather than Zariski descent for morphisms. However, the moduli functor F_g from last lecture is not a stack, because étale descent for objects doesn't hold. However, we'll later show that the moduli groupoid $\mathcal{M}_g$ is. ◀

Proposition 8.7. *Let G be a flat (affine and separated) group scheme over S acting on an S-scheme X. Then $[X/\!\!/G]$ is a stack.*

Proof sketch. To prove this, we need to establish étale descent for principal G-bundles (i.e. G-torsors). This is true because principal G-bundles are étale-locally trivial. This establishes descent for objects.

For descent for morphisms, we claim that given data of $B \leftarrow E \xrightarrow{f} X$ and $B \leftarrow E' \xrightarrow{f'} X$, $\underline{\mathrm{Iso}}_B(f, f')$ is represented by a scheme. It suffices to check (étale-)locally on B, and hence may assume E and E' are both trivialized principal G-bundles over B. Choose sections $\sigma\colon B \to E$ and $\sigma'\colon B \to E'$; then we have two maps $f \circ \sigma, f' \circ \sigma'\colon B \rightrightarrows X$, so $\underline{\mathrm{Iso}}_B(f, f')$ is represented by the scheme $B \times_X B$ (with the pullback along $f \circ \sigma$ and $f' \circ \sigma'$). ⊠

Remark 8.8. We have étale descent for *morphisms* of schemes, where the domain and target are fixed. Do not confuse this with étale descent for schemes, which is not true; that produces something different called an *algebraic spaces*. Because of this, sometimes it's convenient to work with algebraic spaces when studying stacks, following Artin and Knutson; fortunately we don't need to do this today, because étale descent is baked into the definition of a stack, so we don't have to think about stacks over an algebraic space. ◀

Example 8.9. If F, G, and H are stacks, and we have maps $F \to G$ and $H \to G$, then $F \times_G H$ is a stack. ◀

There are some morphisms of stacks which behave like schemes in an interesting way.

Definition 8.10. A morphism $f\colon \mathcal{F} \to \mathcal{G}$ of stacks is *representable* if for any scheme B and map $B \to G$, the fiber product $\mathcal{F} \times_{\mathcal{G}} B$, which is *a priori* a stack, is representable by a scheme.[15]

For example, $X \to [X/\!\!/G]$ is representable: if you pull back to a scheme, you get a principal G-bundle. Another more elaborate example is that the universal curve $\mathcal{C}_g \to \mathcal{M}_g$ is a representable morphism: when we pull it back to a scheme B, we get a genus-g curve over B.

Given a representable morphism $\mathcal{F} \to \mathcal{G}$, you can attach additional properties (finite, flat, etc.) by asking that for every scheme B and map $B \to \mathcal{G}$, the map of schemes $\mathcal{F} \times_{\mathcal{G}} B \to B$ satisfies that property. This is well-defined as long as this property is stable under pullback, such as finite type, separated, affine, flat, proper, and more.

Example 8.11. Let G be a smooth group scheme acting on X. Then the quotient morphism $X \to [X/\!\!/G]$, which is a representable morphism, is smooth. ◀

Now we turn to algebraic and Deligne-Mumford stacks.

Definition 8.12. A stack $\mathcal{F}$ is *Deligne-Mumford* if

(1) the diagonal map $\Delta_{\mathcal{F}}\colon \mathcal{F} \to \mathcal{F} \times_S \mathcal{F}$ is representable,[16] quasicompact and separated.[17] If in addition $\Delta_{\mathcal{F}}$ is proper, we ask for $\mathcal{F}$ to be separated.[18]
(2) There is an étale surjective map (an *étale atlas*) $U \to \mathcal{F}$, where U is a scheme.

Lecture 9.

Deligne-Mumford and algebraic stacks: 9/26/19

Recall from last time the definition of a Deligne-Mumford stack X (over a base scheme S). We require that the diagonal $\Delta_X\colon X \to X \times_S X$ is representable, quasicompact, and separated; that there is a representable map $\phi\colon U \to X$, where U is a scheme; and that ϕ is étale and surjective.

[15] Sometimes it's useful to check with algebraic spaces rather than schemes.

[16] This is telling you something strong about automorphisms, which is part of the general zen of the diagonal morphism: $\underline{\mathrm{Iso}}_B(f, f')$ is not just a sheaf, but is represented by a scheme.

[17] The last two conditions are sometimes relaxed, in view of important examples which for which they don't hold. But we always require the diagonal to be representable.

[18] Recall that separatedness of a scheme is a property solely of its diagonal morphism, so given a stack with representable diagonal, we can define separatedness in the same way as above.

Proposition 9.1. *Let X be a stack. Then Δ_X is representable iff for any morphism $B \to X$, where B is a scheme, is representable.*

Proof. In the forward direction, it suffices to show that for all schemes B and B' with maps $f\colon B \to X$ and $g\colon B' \to X$, the pullback $B' \times_X B$ is a scheme. But the natural maps $B' \times_X B \to X$ also fits into the diagram

$$\begin{tikzcd} B' \times_X B \arrow[r] \arrow[d] & X \arrow[d, "\Delta_X"] \\ B' \times_S B \arrow[r, "f\times g"] & X \times_S X. \end{tikzcd} \tag{9.2}$$

Now $B' \times_S B$ is a scheme, and since Δ_X is representable, $B' \times_X B$ must also be a scheme.

To go in the other direction, we can basically rewind this proof. ⊠

Proposition 9.3. *Let F be a stack with representable diagonal and $X, Y \in F(B)$. Then $\underline{\mathrm{Iso}}_B(X, Y)$ is represented by a scheme.*

Proof. Explicitly, given any maps $f, g\colon B \to X$, where B is a scheme, we can take the pullback

$$\begin{tikzcd} B \times_{X \times_S X} X \arrow[r] \arrow[d] & X \arrow[d, "\Delta_X"] \\ B \arrow[r, "(f{,}g)"] & X \times_S X. \end{tikzcd} \tag{9.4}$$

Then $B \times_{X \times_S X} X$ represents $\underline{\mathrm{Iso}}_B(X, Y)$. ⊠

So if X is a Deligne-Mumford stack, we've imposed several conditions on Δ_X.

(1) We want Δ_X to be representable, which buys us Propositions 9.1 and 9.3.
(2) We also will ask for Δ_X to be quasicompact; this means that $\underline{\mathrm{Iso}}_B(X, Y)$ isn't too wild.
(3) We ask that Δ_X is separated, which means that an isomorphism is the identity if it's generically the identity.

We might also want Δ_X to be proper. This is equivalent to X being separated, which for schemes is equivalent to Δ_X being a closed embedding.

The étale map $\phi\colon U \to X$ should be thought of as an atlas for X, and exhibits it as locally the quotient of a scheme by an étale equivalence relation. The atlas provides *versal deformation spaces* for deformations over Artinian rings. To elaborate, deformation theory, as founded by Schlessinger, asks whether a map $\operatorname{Spec} \overline{A} \to X$ can extend to a "thickening" $\operatorname{Spec} A \to X$, where $\overline{A} = A/I$, for I a nilpotent ideal. By étale lifting, we can study this question over U. For this we only need ϕ to be smooth, in fact; this leads us to vastly more general kinds of stacks, called *Artin stacks* or *algebraic stacks*.

But the fact that the atlas is étale, rather than smooth, is equivalent to the fact that Δ_X is unramified.[19] This means that automorphism groups are discrete: concretely, there are no infinitesimal deformations of geometric points.

Example 9.5. Let G be a group scheme over S which is flat, but not étale. Then $BG \coloneqq [\mathrm{pt}/\!\!/G]$ is an Artin stack that's not a Deligne-Mumford stack. ◀

We can also define separatedness of morphisms of stacks.

Lemma 9.6. *Let $\mathcal{F}$ and $\mathcal{G}$ be stacks with representable diagonals. For any map $f\colon \mathcal{F} \to \mathcal{G}$ of stacks, the* relative diagonal $\Delta_{\mathcal{F}/\mathcal{G}}\colon \mathcal{F} \to \mathcal{F} \times_{\mathcal{G}} \mathcal{F}$ *is representable.*

For a proof, see, e.g. [LMB00].

Definition 9.7. With $\mathcal{F}$, $\mathcal{G}$, and f as above, we say f is *separated* if $\Delta_{\mathcal{F}/\mathcal{G}}$ is proper.

Theorem 9.8 ([LMB00, Theorem 8.1], [Edi98, Theorem 2.1])**.** *Let $\mathcal{F}$ be an algebraic stack over a Noetherian base scheme S, and let $\phi\colon U \to \mathcal{F}$ be a smooth atlas. Assume that ϕ is finite type; then $\mathcal{F}$ is Deligne-Mumford iff $\Delta_{\mathcal{F}}$ is unramified.*

[19] In fact, within a larger class of stacks, being Deligne-Mumford is equivalent to having unramified diagonal.

One takeaway is that finite-type, which sounds like an innocent niceness hypothesis, is actually quite strong.

Corollary 9.9. *Assume S is Noetherian, and let G be a smooth affine*[20] *group scheme over S acting on a Noetherian S-scheme X, and assume both G and X are of finite type. Assume the geometric points have finite and reduced stabilizers. Then,*

(1) *$[X/\!\!/G]$ is a Deligne-Mumford stack, and*
(2) *$[X/\!\!/G]$ is separated iff the G-action on X is proper.*

Proof. First, let's unwrap the condition of finite, reduced stabilizers. Let B be a scheme, so a map $B \to [X/\!\!/G]$ is equivalent data to a principal G-bundle $B \to X$ and a G-equivariant map $E \to X$. Finite, reduced stabilizers then means that $\underline{\mathrm{Iso}}_B(E, E)$ is unramified over B. ⊠

Now let's apply this to moduli spaces of curves. The moduli space of smooth curves has no chance to be compact, which is frustrating — but we can obtain a complete (read: compact) space of curves by admitting certain singularities.

You can think of this as: you have a family of curves, or just a curve evolving in time. It might degenerate into something singular.

It was a fundamental insight of Deligne and Mumford that nodal singularities suffice, i.e. the moduli space of smooth and nodal curves is nice — well, if you allow all nodal curves, it's nice but not separated. We can even do better by removing superfluous components (those which "bubble," adding an additional $\mathbb{P}^1$), and then obtain something Deligne-Mumford (at least when $g \geq 2$).

Definition 9.10. A *nodal singularity* on a curve, or *node* for short, is a singular point which étale-locally looks like $\{zw = 0\} \subset \mathbb{A}^2$.

Now we discuss the kinds of nodal curves we include. As always in moduli questions, we work in families.

Definition 9.11 (Deligne-Mumford)**.** A *stable curve* over a base scheme S is a proper flat morphism $\pi\colon C \to S$ such that

(1) the geometric fibers C_s, for $s \in S$ a geometric point, are reduced, connected, and one-dimensional of genus $g \geq 2$[21] with at worst nodal singularities, and
(2) if $E \subset C_s$ is a non-singular rational curve with normalization $\nu\colon \widetilde{E} \to E$, then the preimage under ν of the singular locus consists of at least 3 points.

In this setting, the genus is the *arithmetic genus* $g \coloneqq h^1(C_s, \mathcal{O}_{C_s})$. Since π is flat, this is locally constant.

Remark 9.12. Condition (2) is equivalent to asking for $\mathrm{Aut}(C_s)$ to be finite: when $g \geq 2$, a genus-g curve has finite automorphism group, so we just have to control the genus-zero components. ◀

The groupoid of stable curves of genus g and their isomorphisms is fairly clearly a stack, since we can glue curves. We call this stack $\overline{\mathcal{M}}_g$; thus far, it's a stack over $\operatorname{Spec}\mathbb{Z}$.

Theorem 9.13. *$\overline{\mathcal{M}}_g$ is a Deligne-Mumford stack.*

Recall (hopefully) from curve theory that a stable curve $\pi\colon C \to S$ is an *(étale-)locally complete intersection morphism* (l.c.i. morphism). That is, C has an open cover $\mathfrak{U}$ such that for each $U \in \mathfrak{U}$, U is a hypersurface in $S \times \mathbb{A}^2$ (in general, one asks to specify one equation for each codimension); hence, π has a *relative dualizing sheaf* $\omega_{C/S}$ — in the smooth case, we can just use the conormal bundle, but if there are nodes, we must do something different: the sheaf of holomorphic differentials has simple poles at the nodal points. So we impose the condition that the residues sum to zero.[22]

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[20] It might be possible to remove this hypothesis; in class we weren't sure.

[21] We will be interested in genus-0 curves as well, and will later be able to fix this definition to allow them, and to allow genus-1 curves with enough marked points.

[22] Explicitly, if $\nu\colon C_s' \to C_s$ is the normalization, then

$$\nu^*\omega_{C_s'} = \omega_{C_s'}(x_1 + \cdots + x_r + y_1 + \cdots + y_r), \tag{9.14}$$

where $x_1, \ldots, x_r, y_1, \ldots, y_r \in C_s'$. Then we ask that

$$\mathrm{res}_{x_i}(\nu^*\alpha) + \mathrm{res}_{y_i}(\mu^*\alpha) = 0, \tag{9.15}$$

where α is TODO.

We will have to finish the proof next time. The key point is that for $g \geq 2$, $\omega_{C/S}^{\otimes n}$ is relatively very ample, and its pushforward under π is locally free of rank $(2n-1)(g-1)$. This will give us a nice embedding into projective space over S.

Lecture 10.

$\overline{\mathcal{M}}_g$ is a Deligne-Mumford stack, $g \geq 2$: 10/1/19

Last time, we described a compactification of $\mathcal{M}_g$, the moduli groupoid of curves, to include certain singular curves with singularities that aren't too bad, and make the automorphism groups finite, leading to a moduli functor $\overline{\mathcal{M}}_g$. Today, we'll show this is a Deligne-Mumford stack.

Theorem 10.1. *Let $g \geq 2$ and $\pi\colon C \to S$ be a stable curve over the base scheme S. Then the relative dualizing sheaf $\omega_{C/S}^{\otimes n}$ is (relatively) very ample for $n \geq 3$, and $\pi_*\omega_{C/S}^{\otimes n}$ is locally free of rank $(2n-1)(g-1)$. If C is smooth, we can take $n \geq 2$.*

Proof. To show that something is very ample, we need to show that sections separate points and separate tangent vectors, so the sheaf defines an embedding into projective space.

Without loss of generality, say $S = \operatorname{Spec} k$ for a field k. Then by the Riemann-Roch theorem,

$$h^0(C, \omega_C^{\otimes n}) = h^1(C, \omega_C^{\otimes n}) + \deg(\omega_C^{\otimes n}) + 1 - g \tag{10.2}$$

By Serre duality,

$$= h^0(C, \omega_C^{\otimes(-n)} \otimes \omega_C) + n(2g-2) + 1 - g. \tag{10.3}$$

If C is smooth, $\omega^{\otimes(1-n)}$ has degree $(1-n)(2g-2) < 0$, so

$$= 0 + (2n-1)(g-1). \tag{10.4}$$

This tells us we have enough sections to separate tangents (TODO: I think); to separate points p_1 and p_2, we want to compute $h^0(C, \omega_C^{\otimes n}(-p_1 - p_2))$; this is positive, so $\omega_C^{\otimes n}$ is very ample.

If C is nodal, let $\nu\colon C' \to C$ be its normalization. Let $C_{\mathrm{sing}} = \{z_1, \dots, z_n\}$ be the singular locus of C, and write $\nu^{-1}(z_i) = \{x_i, y_i\}$. If $\alpha \in H^0(U, \omega_C)$, then $\nu^*\alpha$ is a rational[23] section of $\omega_{C'} = \Omega^1_{C'}$ with simple poles at $\{x_i, y_i\}$, and such that

$$\operatorname{res}_x(\nu^*\alpha) + \operatorname{res}_y(\nu^*\alpha) = 0. \tag{10.5}$$

Thus,

$$\nu^*\omega_C = \Omega^1_{C'}\Big(\sum x_i + \sum y_i\Big). \tag{10.6}$$

Then, repeating the above argument on C' shows that $\omega_C^{\otimes n}$ is very ample when $n \geq 3$. In the worst case, a genus-zero component with three special points could give you $\nu^*\omega_C|_D \cong \mathcal{O}_{\mathbb{P}^1}(1)$, so $n \geq 3$ is sharp.

The second fact, that $\pi_*(\omega_{C/S}^{\otimes n})$ is locally free, follows from a cohomology and base change theorem. ⊠

Why is this helpful? Well, $\omega_{C/S}$ is canonically associated to any variety with at worst Gorenstein singularities, which we certainly have, and this theorem gives us an embedding into an intrinsically defined projective space (over S). This will allow us to embed $\overline{\mathcal{M}}_g$ into a quotient space.

Corollary 10.7. *Every stable curve $C \to S$ of genus $g \geq 2$ can be embedded into $\mathbb{P}^N$, where $N = (2n-1)(g-1) - 1$ and $n \geq 3$, with Hilbert polynomial $P_{g,n}(t) = (2nt-1)(g-1)$.*

This embedding is called the *n-canonical embedding.*

Definition 10.8. Let $\overline{H}_{g,n}$ denote the subspace of $\mathrm{Hilb}_{\mathbb{P}^N}^{P_{g,n}}$ consisting of the n-canonical embeddings.

Ultimately because nodes can at most smooth out to other nodes, $\overline{H}_{g,n} \subset \mathrm{Hilb}_{\mathbb{P}^N}^{P_{g,n}}$ is an open subscheme.

Remark 10.9. A map $S \to \overline{H}_{g,n}$ is equivalent to data of a stable curve $\pi\colon C \to S$ of genus g and an isomorphism $\mathbb{P}(\pi_*(\omega_{C/S}^{\otimes n})) \xrightarrow{\cong} S \times \mathbb{P}^N$, i.e. a trivialization of the projectivization of $\pi_*(\omega_{C/S}^{\otimes n})$. ◀

The action of PGL_{N+1} on $\mathbb{P}^N$ restricts to a PGL_{N+1}-action on $\overline{H}_{g,n}$.

[23]In complex geometry, you'd say "meromorphic," and this is the algebro-geometric analogue.

Theorem 10.10. *There is an equivalence of stacks* $\overline{\mathcal{M}}_g \simeq [\overline{H}_{g,n} /\!\!/ \mathrm{PGL}_{N+1}]$.

Proof. First, let's define a functor $p\colon \overline{\mathcal{M}}_g \to [\overline{H}_{g,n} /\!\!/ \mathrm{PGL}_{N+1}]$. To a stable curve $\pi\colon C \to B$, assign the principal PGL_{N+1}-bundle $E \to B$ which is the projective bundle of frames of $\pi_*\omega_{C/S}^{\otimes n}$ — that is, since this is a locally free sheaf on B of rank $N+1$, its bundle of frames $\mathcal{B} \to B$ is a principal GL_{N+1}-bundle, and we can take the associated bundle $E \coloneqq \mathcal{B} \times_{\mathrm{GL}_{N+1}} \mathrm{PGL}_{N+1}$, which is a principal PGL_{N+1}-bundle.

We also need a PGL_{N+1}-equivariant map to $\overline{\mathcal{H}}_{g,n}$, which in view of Remark 10.9 is a stable curve of genus g over E and a trivialization of the projectivized pushforward of the relative dualizing sheaf. To provide this, consider the pullback

$$\begin{tikzcd} C \times_B E \arrow[r] \arrow[d, "\widetilde{\pi}"] & C \arrow[d, "\pi"] \\ E \arrow[r] & B, \end{tikzcd} \tag{10.11}$$

which is a stable curve over E, and $\mathbb{P}(\widetilde{\pi}_*(\omega_{C\times_B E/E}^{\otimes n}))$ has a tautological section, hence a trivialization as desired. Moreover, this is equivariant with respect to the PGL_{N+1}-action on E, as you can check by pulling back (10.11) by the action of any $g \in \mathrm{PGL}_{N+1}$.

A morphism in $\overline{\mathcal{M}}_g$ is a pullback diagram

$$\begin{tikzcd} C' \arrow[r] \arrow[d, "\pi'"] & C \arrow[d, "\pi"] \\ B' \arrow[r, "\phi"] & B, \end{tikzcd} \tag{10.12}$$

where $C \to B$, and hence also $C' \to B'$, is a stable curve of genus g. This implies a natural isomorphism

$$\pi'_*(\omega_{C'/B'}^{\otimes n}) \cong \phi^*\pi_*(\omega_{C/B}^{\otimes n}), \tag{10.13}$$

so we obtain pullback diagrams from all of the data p assigns to B (E, etc.) to the data assigned to B'.

Now that we've defined p, we must check that it's fully faithful and essentially surjective. First, that p is *faithful*, meaning that the map

$$\mathrm{Hom}_{\overline{\mathcal{M}}_g}(C/B, C/B) \xrightarrow{p} \mathrm{Hom}_{[\overline{H}_{g,n}/\!\!/\mathrm{PGL}_{N+1}]}(p(C/B), p(C/B)) \tag{10.14}$$

is injective. This follows because a non-identity automorphism of C induces a nontrivial automorphism of $\mathbb{P}(H^0(\omega_C^{\otimes n}))$

Next, we must check that p is *full*, meaning that (10.14) is surjective. If $\Phi \in \mathrm{PGL}_{N+1}$ and $\Phi(C) = C$, then Φ is induced from an automorphism of C, which follows because the n-canonical embedding is not contained within a linear subspace of $\mathbb{P}^N$. This implies that p is full.

Finally, we want to show that p is *essentially surjective*, meaning every isomorphism class of objects of $[\overline{H}_{g,n}/\!\!/\mathrm{PGL}_{N+1}]$ is in the image of p. So given such an object $B \leftarrow E \to \overline{H}_{g,n}$, we have from the map to $\overline{H}_{g,n}$ data of a stable curve $\pi_E\colon C_E \to E$ and an isomorphism

$$\mathbb{P}(\pi_{E*}(\omega_{C_E/E}^{\otimes n})) \cong E \times \mathbb{P}^N. \tag{10.15}$$

Then, we can descend to $B \coloneqq E/\mathrm{PGL}_{N+1}$, $C \coloneqq C_E/\mathrm{PGL}_{N+1}$, and so on, and thus obtain something in the preimage of $B \leftarrow E \to \overline{H}_{g,n}$. ⊠

Corollary 10.16. *$\overline{\mathcal{M}}_g$ is a separated, Deligne-Mumford stack of fintie type over $S = \operatorname{Spec}\mathbb{Z}$ or $\operatorname{Spec} k$.*

For separatedness, we can use a criterion we used earlier, checking the finiteness of $\underline{\mathrm{Iso}}_B(C', C) \to B$. For the rest, see the original paper of Deligne-Mumford [DM69].

Proposition 10.17. *$\overline{\mathcal{M}}_g$ is proper, over $\operatorname{Spec}\mathbb{Z}$ or over $\operatorname{Spec} k$.*

Before we can do this, we need to digress a bit — we know what properties of representable morphisms of stacks are, but what does it mean for $\overline{\mathcal{M}}_g$, which is not a representable stack, to be proper? Of course there are several other properties we care about that we should define too.

For those properties which are local in the smooth topology (i.e. given by open smooth morphisms), or in the étale topology when we care about Deligne-Mumford stacks, the story is easier. “Local in the *blah* topology" means that you can check on an atlas in the *blah* topology. For the smooth topology, this includes some properties that you probably like thinking about — being flat, smooth, finite type, or unramified — and also some other more technical ones: being normal, locally Noetherian, etc. For these kinds of morphisms, we can produce a smooth atlas of affine schemes for our stacks and check there.

Properness, however, is not local. But we can rephrase it for schemes in a way that goes through for stacks.

Definition 10.18. A morphism $f\colon X \to Y$ is *proper* if it is separated, of finite type, and universally closed.

Asking what "universally closed" means is a little subtle. Here's what happens: we know what open maps of schemes are, and they can be checked on a smooth atlas, so we can define them for stacks. This defines a topology on stacks, called the *Zariski topology for stacks*; as for schemes, the Zariski topology is very coarse and generally we'll prefer the étale topology, but we can use it to define a universally closed morphism of stacks just like for schemes.

However, if you want to check that a stack is proper, you don't use the definition — just like for schemes, you use the valuative criterion.

Theorem 10.19 (Valuative criterion for schemes)**.** *Let Y be a Noetherian scheme and $f\colon X \to Y$ be a quasiseparated map of schemes. Let R be a discrete valuation ring and k be its fraction field, and consider the diagram*

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} R \arrow[r] & Y \end{tikzcd} \tag{10.20}$$

If a lift $\operatorname{Spec} R \to X$ *exists, then f is separated; if it's unique, f is proper.*

Noetherianness and quasiseparatedness are very mild conditions on schemes. And since R is a discrete valuation ring, it only has two points, making this easier to work with.

For stacks, we can't quite use this definition; instead, we need to pull back by a finite cover. Specifically, we replace $\operatorname{Spec} k$ by $\operatorname{Spec} k'$, where $k \hookrightarrow k'$ is a finite extension of fields, and replace $\operatorname{Spec} R$ by $\operatorname{Spec} R'$, where R' is the normalization of R in k'.

Lecture 11.

Stable maps: 10/3/19

We want to show that $\overline{\mathcal{M}}_g$ is proper (with $g \geq 2$ as always). We just discussed the valuative criterion for schemes; today, we'll first describe how it generalizes to stacks.

Theorem 11.1 (Valuative criterion for stacks)**.** *Let $f\colon X \to Y$ be a quasi-separated map of stacks, where Y is Noetherian (in particular, a scheme). Consider diagrams of the form* (10.20)*, where R is a discrete valuation ring and k is its field of fractions. Suppose that for all such diagrams, there exists a finite field extension $k \hookrightarrow k'$ such that the diagram*

$$\begin{tikzcd} \operatorname{Spec} k' \arrow[r] \arrow[d] & \operatorname{Spec} k \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} R' \arrow[r] \arrow[urr, dashed, "\psi?"] & \operatorname{Spec} R \arrow[r] & Y, \end{tikzcd} \tag{11.2}$$

where R' is the normalization of R in k',

(1) admits *a lift ψ as indicated; then f is proper; or*
(2) *has a* unique *lift ψ; then f is separated.*

In moduli theory (so X is some sort of moduli stack), this is related to the notion of *stable reduction*: you have a family of thingies on some domain minus a point. Can you fill it in? If yes, then the moduli stack is proper; if the filling is unique, it's separated.

Now let's look at $\overline{\mathcal{M}}_g$ as a stack over $\operatorname{Spec} \mathbb{Z}$, and take $g \geq 2$.

Theorem 11.3. *Let R, k, and R' be as in Theorem* 11.1 *and $f\colon X \to \operatorname{Spec} k$ be a stable curve. Then there is a finite extension $k \hookrightarrow k'$ and a* unique *stable curve $X' \to \operatorname{Spec} R'$ with $X_{k'} \cong X \times_{\operatorname{Spec} R} \operatorname{Spec} k'$.*

You can think of having a family over $\operatorname{Spec} R$, and trying to fill in a point. The theorem says we can do this after a finite extension.

Proof assuming $\operatorname{char}(k) = 0$. First, extend arbitrarily by projectivity as a reduced scheme. The upshot is that $X \to \operatorname{Spec} k$ sits inside $\overline{X} \subset \mathbb{P}^N_R$, which sits over $\operatorname{Spec} R$, so X is a smooth surface. We can desingularize, e.g. explicitly by blowing up a bunch of times.

TODO: the proof was pretty opaque to me, but the key points are (1) that we *need to* base change by a finite extension, and (2) that we *can*. The first point is true because the stable curve might have nodal singularities of some multiplicativity, so we need to perform the base change $t \mapsto t^\ell$, where ℓ is the least common multiple of these multiplicaitivities, and this works. ⊠

What goes wrong in positive characteristic? If $p \mid \ell$, then $t \mapsto t^\ell$ is not a good base change.

Corollary 11.4. *$\overline{\mathcal{M}}_g$ is proper.*

Let's discuss a few more nice properties of $\overline{\mathcal{M}}_g$.

Theorem 11.5. *$\overline{\mathcal{M}}_g$ is smooth and connected. The locus of singular curves $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ is a normal crossings divisor with $s := \lfloor (g+1)/2 \rfloor$ irreducible components $D_1, \ldots, D_s$.*

The interior of D_h for $h > 0$ consists of curves of the form $\Sigma_h \vee \Sigma_{g-h}$. For $h = 0$, we instead have genus-g curves where one of the donut holes is pinched, like a croissant.

Proof sketch. For smoothness, it suffices to check on a smooth atlas, and we already have one: $\overline{H}_{g,n} \twoheadrightarrow \overline{\mathcal{M}}_g$. Smoothness of $\overline{H}_{g,n}$ amounts to doing some deformation theory, as is often true when checking smoothness in algebraic geometry.

Connectedness leads to classical considerations in algebraic geometry. Consider the *Hurwitz space* $\mathrm{Hur}_{k,b}$, the moduli space of simply branched covering maps $\pi\colon C \to \mathbb{P}^1$ where C is connected, π is degree k, and π is branched at b points. The Riemann-Hurwitz formula shows $g(C) = b/2 - k + 1$.

Hurwitz [Hur91] showed over a century ago that $\mathrm{Hur}_{k,g}$ is connected, and that if $k > g+1$, $\mathrm{Hur}_{k,b} \twoheadrightarrow \mathcal{M}_g$. The proof amounts to an algorithm transforming a curve to its normal form, which is a path in Hurwitz space; then it's much easier to see that the subspace of curves in normal form is connected.

Now, since $\mathcal{M}_g$ admits a surjective map from a connected space, it's also connected, and since $\mathcal{M}_g$ is dense in $\overline{\mathcal{M}}_g$, then $\overline{\mathcal{M}}_g$ is also connected. ⊠

There are several variations on $\overline{\mathcal{M}}_g$.

(1) The first thing we can do is add marked points, leading to the moduli space $\overline{\mathcal{M}}_{g,k}$ of genus-g stable curves with k marked points. A stable curve with k marked points over a base B is a stable curve $\pi\colon C \to B$ together with k sections of π.

This changes the notion of stability: we want the group of automorphisms fixing the marked points to be finite, which amounts to $2g - 2 + k > 0$. Now we can consider moduli spaces of genus-0 and genus-1 curves: a $\mathbb{P}^1$ with three marked points is rigid, so we allow $\overline{\mathcal{M}}_{0,k}$ for $k \geq 3$. Similarly, a genus-1 curve with any marked point has a finite automorphism group, so we allow $\overline{\mathcal{M}}_{1,k}$ for $k \geq 1$.

This allows us to understand the boundary divisors of $\overline{\mathcal{M}}_g$ better: for $h > 0$, $D_h \cong \overline{\mathcal{M}}_{h,1} \times \overline{\mathcal{M}}_{g-h,1}$, and $D_0 \cong \overline{\mathcal{M}}_{g-1,2}$.

(2) We can consider the moduli space of *prestable curves*, where we simply drop stability. This wasn't really in the literature before the Stacks project carefully wrote it down; the argument is roughly the same, but done in much greater generality, e.g. the $\underline{\mathrm{Iso}}$ functor can be represented as an algebraic space in stunning generality, though not a scheme in general. The moduli space of prestable curves is an Artin stack, and is often used in bootstrapping-type arguments, where one uses a previously constructed stack to build a new one.

(3) Gromov-Witten theory is interested in moduli spaces of *stable maps*, i.e. maps of curves into a given space satisfying a stability condition.

We're interested in Gromov-Witten theory, so let's dig into the last example.

Definition 11.6. A *stable map* of genus g, with k marked points, with target X (a scheme), and over a base B, is a genus-g, k-punctured curve $\pi\colon C \to B, \mathbf{x} = (x_1, \ldots, x_k)$ over B, together with a map $f\colon C \to X$.

Stability means that for all geometric points $s \to B$, the automorphism group of the data $(C_s, \mathbf{x}_s, f|_{C_s})$ is finite.

The stack of genus-g, k-punctured, nodal stable maps to X is denoted $\overline{\mathcal{M}}_{g,k}(X)$.[24] Often we also fix the image of C to be a particular homology class β (in $H_2(X;\mathbb{Z})$ or $H_2(X;\mathbb{Q})$ or $A_1(X)$); the space of such smooth stable maps is denoted $\overline{\mathcal{M}}_{g,k}(X,\beta)$, and is an open substack of $\overline{\mathcal{M}}_{g,k}(X)$.

Unlike $\overline{\mathcal{M}}_g$, the moduli stack of stable maps in general has several connected components.

Let's unpack stability a bit more. Over $\operatorname{Spec}\mathbb{C}$, let's consider a map $f\colon (C,\mathbf{x}) \to X$. Stability impies that for all irreducible components $C' \subset C$, $f|_{C'}\colon C' \to X$ is either constant or finite (in particular, has finite automorphism group). So the stability condition is equivalent to asking for every rational irreducible component which is contracted by f should have at least 3 marked points, as for stable curves, and that a node gives two special points on the normalization, which need not be in the same irreducible component. (We also exclude genus-1 constant maps, which are a special case.)

Stability is important to have a proper moduli stack; in somewhat more geometric terms, you can write down families of smooth maps which limit to a map from a nodal curve. One example is: consider the resolution map $\mathbb{CP}^2 \to \{y^2 - x^3 = 0\}$ (TODO: I think), which leads to a family of smooth genus-1 curves over the normalization of $\{y^2 - x^3 = 0\}$. If we pass to the singular curve $\{y^2 - x^3 = 0\}$, though, what we get is nodal: a genus-0 curve attached to a genus-1 curve at a node. At the cusp of $\{y^2 - x^3 = 0\}$, the genus-1 component is contracted to a cusp.

The pictures of these examples can be a source of great insight about these families of curves. It's fun if you like geometry with pictures, but it won't be essential for this class.

Remark 11.7. For $n \geq 3$, the space of embedded curves in $\mathbb{P}^n$ of fixed arithmetic genus is highly singular. Understanding just the case $n = 3$ is the subject of a great deal of Robin Hartshorne's research. But by imposing the nodal condition, $\overline{\mathcal{M}}_{0,k}(\mathbb{P}^n)$ is smooth! This is an insight of Kontsevich, and again involves deformation theory: the stack $\widetilde{\mathcal{M}}_{0,k}$ of nodal prestable curves is smooth, by roughly the same argument as for $\overline{\mathcal{M}}_g$; there's a forgetful morphism $\phi\colon \overline{\mathcal{M}}_{0,k}(\mathbb{P}^n) \to \widetilde{\mathcal{M}}_{0,k}$ forgetting the map,[25] and it suffices to show this is a smooth map of stacks.

This amounts to checking that if A is a ring, $I \subset A$ is an ideal with $I^2 = 0$, and $\overline{A} \coloneqq A/I$, and we're given maps

$$\begin{tikzcd} \operatorname{Spec}\overline{A} \arrow[r] \arrow[d, hook] & \overline{\mathcal{M}}_{0,k}(\mathbb{P}^n) \arrow[d, "\phi"] \\ \operatorname{Spec} A \arrow[r] & \widetilde{\mathcal{M}}_{0,k}, \end{tikzcd} \tag{11.8}$$

then we have a lift $\operatorname{Spec} A \to \overline{\mathcal{M}}_{0,k}(\mathbb{P}^n)$. In more concrete terms, if we have a prestable curve $C \to \operatorname{Spec} A$ such that the pullback along $\operatorname{Spec}\overline{A} \to \operatorname{Spec} A$ has a map $f_{\overline{A}}\colon C_{\overline{A}} \to X$, does $f_{\overline{A}}$ extend to a map $f\colon C \to X$? In general, the obstruction to this question lives in $H^1(C_{\overline{A}}, f_{\overline{A}}^* T_X)$.

Stability implies that $f_{\overline{A}}^* T_X$ is globally generated on each irreducible component $C'_{\overline{A}} \subset C_{\overline{A}}$. Therefore for some ℓ we have a short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}_{C'_{\overline{A}}}^{\oplus \ell} \longrightarrow f_{\overline{A}}^* T_{\mathbb{P}^n}|_{C'_{\overline{A}}} \longrightarrow 0, \tag{11.9}$$

which, upon taking cohomology, shows the obstruction class comes from something in H^2, which is zero because we're on a curve. ◀

Lecture 12.

The moduli space of stable maps and Gromov-Witten invariants: 10/8/19

TODO: I was 20 minutes late and probably missed some stuff. Also, no class Thursday.

[24] The professor denotes this $\mathcal{M}_{g,k}(X)$; the notation I used is also common, and I favor it in order to keep myself from getting confused.

[25] Since the defintiions of stability for stable curves and stable maps are slightly different, so *a priori* when we forget the map, we only get a prestable curve.

Today, we again study the stack $\overline{\mathcal{M}}_{g,n}(X,\beta)$, the stack of stable maps $f\colon C \to X$, where C is a curve with at worst nodal singularities, with genus g and n marked points, and $\beta \in H_2(X)$ is the image of the fundamental class of C. (More generally, we consider a stable curve over some base T).

Theorem 12.1. *$\overline{\mathcal{M}}_{g,n}(X,\beta)$ is Deligne-Mumford, proper, and separated.*

Recall that we imposed some constraints on n for $g = 0, 1$: for $g = 0$, we need $n \geq 3$, and for $g = 1$, we need $n \geq 1$.

Theorem 12.2. *$\mathcal{M}_{0,k}(\mathbb{P}^n)$ is smooth.*

Proof sketch of Theorem 12.1. Let $\overline{\mathcal{M}}_{g,k}$ be the Artin stack of nodal genus-g curves with k marked points.

Let $f\colon C \to X$ be a stable map, where $\pi\colon C \to T$ is a stable curve of genus g with k marked points, given by a section $\mathbf{x}$. Then we have a diagram

$$\begin{tikzcd} C \arrow[r, "(f,\pi)"] \arrow[d, "\pi"] & X \times T \arrow[r] \arrow[d, "pr_2"] & X \arrow[d] \\ T \arrow[r, equal] & T \arrow[r] & \mathrm{pt}. \end{tikzcd} \tag{12.3}$$

This is natural in X, hence defines an embedding $\mathcal{M}_g(X) \hookrightarrow \mathrm{Hom}_{\mathcal{M}_g}(\mathcal{C}_g, \mathcal{M}_g \times X)$, where $\mathcal{C}_g \to \mathcal{M}_g$ is the universal curve. $\mathrm{Hom}_{\mathcal{M}_g}(\mathcal{C}_g, \mathcal{M}_g \times X)$ is an Artin stack locally of finite type.[26]

The next thing to prove is that this is in fact an open embedding; this comes from the fact that stability is an open condition. Great.

Once we fix $\beta \in H_2(X)$, there are only finitely many *combinatorial types* of stable maps. The combinatorial type of a stable map is the data of the intersection pattern of its irreducible components, the genera of the irreducible components, and the images β_i of their fundamental classes. The upshot is that $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is quasicompact.

Showing that this stack is separated and proper, e.g. by stable reduction is harder. We follow Fulton and Pandharipande [FP96]. First, assume that X is projective. This means for some d and r, there's a closed embedding $\overline{\mathcal{M}}_{g,n}(X,\beta) \hookrightarrow \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$. Therefore we may without loss of generality assume $X = \mathbb{P}^r$.

The point is that $\mathcal{L} := \omega_C \otimes f^*\mathcal{O}_{\mathbb{P}^r}(1)$ is relatively ample for $C \to T$, meaning that there is some ℓ (depending on g, k, and d) such that $\mathcal{L}^{\otimes \ell}$ is very ample. Let $H_0, \dots, H_r$ be hyperplanes in $\mathbb{P}^r$ transverse to C_s; then the intersection of each H_j with C_s is zero-dimensional, hence defines additional marked points on C. Without loss of generality, let $H_i = \{t_i = 0\}$, where $t_i \in \Gamma(\mathbb{P}^r, \mathcal{O}(1))$. Then, locally, $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ is isomorphic to the rigidified space R with objects stable curves $\pi\colon C \to T$ and sections $\mathbf{x}, q = (q_{ij})\colon T \to C$ of π, together with data of, for all i, isomorphisms

$$\pi^*\pi_*(\mathcal{H}_i^{-1} \otimes \mathcal{H}_0) \longrightarrow \mathcal{H}_i^{-1} \times \mathcal{H}_0, \tag{12.4}$$

where $\mathcal{H}_i := \mathcal{O}_C(q_{i1} + \cdots + q_{id})$, and also data of maps $\lambda_0, \dots, \lambda_r\colon T \to \mathbb{G}_m$. Here, the λ_i scale the canonical sections s_i of $\mathcal{H}_i$, and $\mathbb{G}_m$ acts diagonally. Anyways, now we can do stable reduction here, at this rigidified level. ⊠

Now let's define some Gromov-Witten invariants for hypersurfaces in $\mathbb{P}^{n+1}$. Let $X \subset \mathbb{P}^{n+1}$ be a degree-ℓ hypersurface; explicitly, write $X = Z(F)$. Then $\overline{\mathcal{M}}_0(X,d) \subset \overline{\mathcal{M}}_0(\mathbb{P}^{n+1}, d)$, where the latter is smooth, and this embedding can be defined by the zero locus of a section of a vector bundle.

To see this, consider the universal curve $\mathcal{C}_0(\mathbb{P}^{n+1}) \to \mathcal{M}_0(\mathbb{P}^{n+1})$ with its universal stable map $f\colon \mathcal{C}_0(\mathbb{P}^{n+1}) \to \mathbb{P}^{n+1}$. On geometric fibers s, $f_s^*\mathcal{O}(\ell)$ has nonnegative degree on each irreducible component, so since $g = 0$, $H^1(f_s^*\mathcal{O}(\ell)) = 0$.

This then implies that $\mathbf{R}^1\pi_* f^*\mathcal{O}(\ell) = 0$, so $\mathcal{E} := \pi_* f^*\mathcal{O}(\ell)$ is locally free. The Riemann-Roch theorem computes its rank to be

$$\operatorname{rank} \mathcal{E} = h^0(C_s, f_s^*\mathcal{O}(\ell)) = \deg f_s^*\mathcal{O}(\ell) + 1 - g = d\ell + 1, \tag{12.5}$$

Now F defines $\sigma \in \Gamma(\overline{\mathcal{M}}_0(\mathbb{P}^{n+1}), \mathcal{E})$ with $\sigma((C_s, f_s)) = 0$ iff $f_s(s) \subset X$.

Definition 12.6 (Kontsevich)**.** Let $[\overline{\mathcal{M}}_0(X,d)]_{\mathrm{virt}} := c_{d\ell+1}(\mathcal{E}) \cap [\mathcal{M}_n(\mathbb{P}^{n+1}, d)] \in A_*(\mathcal{M}_0(\mathbb{P}^{n+1}, d)) \to H_{2*}(\mathcal{M}_0(\mathbb{P}^{n+1}, d))$. This is called the *virtual fundamental class* of $\overline{\mathcal{M}}_0(X,d)$.

[26] TODO: replace $\mathcal{M}_g$ with $\overline{\mathcal{M}}_{g,n}$, which is OK

For example, on the quintic threefold, $n = 3$ and $\ell = 5$, so $\operatorname{rank} \mathcal{E} = 5d+1$, and hence $\dim \overline{\mathcal{M}}_0(\mathbb{P}^4, d) = 5d+1$. In general, using Riemann-Roch again,

$$\dim \overline{\mathcal{M}}_0(\mathbb{P}^{n+1}, d) = \deg f_s^* \Theta_{\mathbb{P}^{n+1}} + (n+1)(1-g) - 3 = \mathrm{d}(n+2) + n - 2. \tag{12.7}$$

Here, the 3 is $\dim \operatorname{Aut}(\mathbb{P}^1)$.

∽ · ∼

In order to understand this a little better, we'll need to digress into intersection theory; working with Gromov-Witten invariants will involve some nontrivial facts about Chow groups of stacks. We'll follow Fulton for this part.

Let X be a scheme over a field. A *subvariety* of X is a reduced, closed subscheme of finite type. An *algebraic cycle* is a finite formal sum of irreducible subvarieties of X; the set of all algebraic cycles of X is a graded abelian group denoted $Z_*(X)$. The subgroup of algebraic cycles where all subvarieties have degree d is denoted $Z_d(X)$; these define a $\mathbb{Z}$-grading on $Z_*(X)$.

This group is way too big to do anything with, so let's quotient out by an equivalence relation called *rational equivalence*...

TODO: he defined rational equivalence, the fundamental class, and Chow groups, and discussed $A_*(\mathbb{A}^n)$ and $A_i(\mathbb{P}^n)$.

Lecture 13.

Chern classes: 10/9/19

Let X be a scheme and $X_1, \dots, X_n$ be its irreducible components. Then, in the Chow group, $[X] = \sum m_i[X_i]$, where m_i is the length of $\mathcal{O}_{X,X_i}$ as an $\mathcal{O}_{X,X_i}$-module. So, if X is reduced, its length is 1. In general, the length is the length of a chain of modules $M_i \subset M_{i+1} \subset \cdots \subset A$ with $M_{i+1}/M_i \cong A/\mathfrak{p}_i$, where $\mathfrak{p}_i$ is prime.

Example 13.1. Let $X = \operatorname{Spec} A$ with A a finite-dimensional algebra over a field k, and let $m \subset A$ be a maximal ideal.

For example, we could take $A = k[\varepsilon]/(\varepsilon^{k+1})$. Then A is filtered by $(\varepsilon^j) \subset (\varepsilon^{j+1})$, a filtration which has total length $k+1$, and therefore the length of A is $k+1$. ◀

The length of an Artinian local ring agrees with its dimension.

Last time we also discussed rational equivalence and cycles as ingredients in intersection theory. Next, we discuss pushforwards under proper maps; since cycles aren't necessarily compact, we must impose the properness hypothesis. So let $f\colon X \to Y$ be a proper map and $V \subset X$ be a subvariety. Let W denote the reduction of $f(V)$ in Y, so W is also a subvariety of Y. Then f induces a field extension $f^\sharp\colon K(W) \hookrightarrow K(V)$. We then define[27]

$$f_*[V] := \begin{cases} 0, & \dim V < \dim W \\ k[f_*V], & \dim V - \dim W = k \geq 0. \end{cases} \tag{13.2}$$

Theorem 13.3 (Excision)**.** *Let $i\colon Y \hookrightarrow X$ be a closed embedding, so that $j\colon X \setminus Y \to X$ is open. Then the sequence*

$$A_*(Y) \longrightarrow A_*(X) \xrightarrow{j^*} A_*(X \setminus Y) \longrightarrow 0 \tag{13.4}$$

is right exact.

TODO: I probably missed some other stuff after that.

Definition 13.5. Suppose X has pure dimension n. Then its *Weil divisors* are the formal sums of codimension 1 subvarieties in $Z_{n-1}(X)$.

There's a different, but related, notion of divisor, coming from the Čech approach to sheaves.

[27]Not sure if I got this inequality wrong, sorry. TODO

Definition 13.6. Let X be a scheme. A *Cartier divisor* is data of a finite open cover $\mathfrak{U}$ of X and, for each $U \in \mathfrak{U}$, $f_U \in R(U)$, the total ring of quotients (so an equivalence class of g/h, where h is nonzero), such that for all $U, V \in \mathfrak{U}$, $f_U/f_V \in \mathcal{O}^*_{U\cap V}$. We take these under a notion of equivalence that I missed (TODO).

If $D = (\mathfrak{U}, \{f_U\})$ is a Cartier divisor, this data determines transition functions for an invertible sheaf, which we denote $\mathcal{O}(D)$.

If X is purely n-dimensional, there's a map from the Cartier divisors to $A_{n-1}(X)$ sending $(\mathfrak{U}, \{f_U\})$ to the sum of $\operatorname{div}(f_U)$ over all $U \in \mathfrak{U}$.

We can define an intersection product: if $j\colon V \hookrightarrow X$ is a k-dimensional subvariety of X, then we let $D \cdot [V]$ be $[j^{-1}(D)]$ if $V \not\subset D$; otherwise, we define it to be the class of C, where $C \subset V$ is the Cartier divisor with $\mathcal{O}_V(C) \cong \mathcal{O}_X(D)|_V$. This defines a map $D \cdot -\colon A_k(X) \to A_{k-1}(X)$.

Definition 13.7. Let L be an invertible sheaf. Then its (dual) *first Chern class* is the map $c_1(L)\colon A_*(X) \to A_{*-1}(X)$ sending $Z \mapsto D \cdot Z$, where D is a Cartier divisor with $L = \mathcal{O}_X(D)$.

In a similar way, we get at the other Chern classes, though they'll appear through their duals, which are called Segre classes. Let $E \to X$ be a vector bundle of rank k and $\mathcal{E}$ denote its sheaf of sections. Then $E = \mathcal{S}pec\operatorname{Sym}^\bullet(\mathcal{E}^\vee)$ (here $\mathcal{E}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$). The *projective bundle* of E is $\mathbb{P}(E) := \mathcal{P}roj\operatorname{Sym}^\bullet(\mathcal{E}^\vee)$, which comes with a line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ and a proper, flat map $p\colon \mathbb{P}(E) \to X$.

Definition 13.8. For $i \geq 0$, the i^{th} *Segre class* is the map $s_i(E) := A_k(X) \to A_{k-i}(X)$ sending

$$\alpha \longmapsto p_*(c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) \frown p^*\alpha). \tag{13.9}$$

Sometimes we denote this $s_i(E) \frown \alpha$.

The projection formula for Cartier divisors induces one for Segre classes.

Proposition 13.10 (Projection formula for Segre classes)**.** *Let $F\colon X \to Y$ be a proper map and $E \to Y$ be a vector bundle of rank r. Then*

$$f_*(s_i(f^*E) \frown \alpha) = s_i(E) \frown f_*(\alpha). \tag{13.11}$$

Corollary 13.12. $s_0(E) = \mathrm{id}$.

Proof of Proposition 13.10. Using the projection formula for $i\colon V \hookrightarrow X$, where V is a subvariety, we can reduce[28] to the case where $X = V$. In this case there is some $m \in \mathbb{Z}$ such that

$$f_*(s_i(f^*\alpha) \frown p^*\alpha) = m[X], \tag{13.13}$$

because $A_k(X) = \mathbb{Z} \cdot [X]$, and we can compute m locally. Locally, $\mathbb{P}(E) \cong X \times \mathbb{P}^{r-1}$ and $\mathcal{O}_{\mathbb{P}(E)}(1)$ is just pulled back from $\mathbb{P}^{r-1}$, so

$$c_1(\mathcal{O}_{\mathbb{P}^{r-1}}(1))^{r-1} \frown [\mathbb{P}^{r-1}] = [\mathrm{pt}] \in A_0(\mathbb{P}^{r-1}), \tag{13.14}$$

from which we conclude that $m = 1$. ⊠

To obtain Chern classes, we invert: let

$$s_t(E) := \sum_i s_i(E)t^i \in \operatorname{End}(A_*(X))[[t]], \tag{13.15}$$

and let $c_t(E) := s_t(E)^{-1}$. To make sense of this, we need $s_is_j = s_js_i$, but this is true. Then we see that $c_0(E) = \mathrm{id}$, $c_1(E) = -s_1(E)$, and so forth. The following formula is helpful for computations.

Theorem 13.16 (Whitney sum formula)**.** *If $0 \to E' \to E \to E'' \to 0$ is a short exact sequence, $c_t(E) = c_t(E')c_t(E'')$.*

We can also pull back classes by regular embeddings $i\colon X \hookrightarrow Y$. That this is a regular embedding means that it's locally given by the ideal $(a_1, \dots, a_d)$, such that no a_i is a zero divisor in $A/(a_1, \dots, a_{i-1})$. Given this, let $f\colon V \to Y$ be a map from a subvariety; we can then take its pullback over u and obtain a map $W \to X$.

If $\mathcal{I}$ denotes the sheaf of ideals of a codimension-d regular embedding (meaning the ideal was generated by d elements, as above), then $\mathcal{I}/\mathcal{I}^2$ is locally free, and the normal bundle for $X \hookrightarrow Y$ is $\mathcal{S}pec(\operatorname{Sym}^\bullet \mathcal{I}/\mathcal{I}^2)$.

[28]No pun intended.

Definition 13.17. The *normal cone* $C_{W/V} := \mathcal{S}pec_W \bigoplus_{d\geq 0} \mathcal{I}^d/\mathcal{I}^{d+1}$, which is a subspace of $N_{W/V} := \mathrm{Sym}^d \mathcal{I}/\mathcal{I}^2$.

Proposition 13.18. *If V is pure dimensional, then $C_{W/V}$ is purely k-dimensional.*

Now we can define the Gysin map. Let $s_0\colon W \to C_{W/V}$ be the zero section; then we define $\iota^![V] := s_0^*[C_{W/V}]$. Since $s_0 = (\pi^*)^{-1}$, where $\pi^*\colon A_{k-d}(X) \to A_k(E)$ is the pullback map, then we conclude π^* is an isomorphism. Surjectivity follows from excision, but injectivity is harder.

Morally, what you should take away is that this is the intersection of the normal cone with the zero section.

Lecture 14.

Deformation to the normal cone and the cotangent complex: 10/15/19

Last time, we defined the Gysin pullback on the normal cone. This allows us to realize the intersection product on the chow group as

$$\alpha \cdot \beta = \iota^![\alpha \times \beta], \tag{14.1}$$

where $\iota\colon X \to X \times X$ is the diagonal map and $\alpha, \beta \in A_*(X)$.

But a more exciting reason to care about the normal cone is something called deformation (degeneration) to the normal cone. Let $X \hookrightarrow Y$ be a closed subscheme, and let $M := B\ell_{X\times\{0\}} Y \times \mathbb{A}^1$, where $B\ell$ denotes the blowup. Over $t \in \mathbb{A}^1 \setminus 0$, we have

$$X \hookrightarrow C_{X/Y} \hookrightarrow \mathbb{P}(C_{X/Y} \oplus \mathbb{A}^1) \cup B\ell_X Y. \tag{14.2}$$

The crucial upshot is that for problems that vary nicely enough in flat families and which are local enough in X, $X \hookrightarrow Y$ is as good as $X \hookrightarrow C_{X/Y}$.

The application we're interested in is the construction of the virtual fundamental class for $\overline{\mathcal{M}}_0(X)$, where X is the zero locus of a homogeneous, degree-ℓ polynomial in $\mathbb{P}^{n+1}$. Let

$$E := \pi_*\varphi^*\mathcal{O}(\ell) \to \overline{\mathcal{M}}_0(\mathbb{P}^{n+1}), \tag{14.3}$$

which is the "deformation bundle," a rank-r vector bundle. Note that $\mathcal{M}_0(\mathbb{P}^{n+1})$ is smooth.

The polynomial f defines a section s of E, such that the zero locus of f is $\mathcal{M}_0(X) \subset \mathcal{M}_0(\mathbb{P}^{n+1})$. Now, intersect $\mathrm{Im}(s)$ with the zero section, defining $s_0^![\mathcal{M}_0(X)] \in A_{\dim \mathcal{M}_0(X) - r}(\mathcal{M}_0(X))$; this is the virtual fundamental class.

We would like to think of this virtual fundamental class as an actual fundamental class of some sort of smooth object standing in for $\mathcal{M}_0(X)$, but this isn't literally true right now. If you do derived algebraic geometry, there is a sense in which this is true, though. Today we will not do that, instead following Li and Tian's original approach to obstruction theories and the virtual fundamental class. Locally, say on $U \subset \mathcal{M}_0(X)$, embed U in a smooth space V, and let $\mathcal{I}$ be the corresponding sheaf of ideals. Then we have an embedding

$$\mathrm{Spec}_U\left(\bigoplus_d \mathcal{I}^d/\mathcal{I}^{d+1}\right) = C_{U/V} \hookrightarrow N_{U/V} = \mathrm{Spec}_U(\mathrm{Sym}^\bullet \mathcal{I}/\mathcal{I}^2), \tag{14.4}$$

and $T_V|_U = \mathrm{Spec}_U(\mathrm{Sym}^\bullet \Omega_V|_U)$ also surjects onto $N_{U/V}$, and the additive action of $T_V|_U$ on $N_{U/V}$ leaves $C_{U/V}$ invariant.

Now embed $N_{U/V}$ inside some vector bundle E_1 and globalize. To that, we need patching data, i.e. a commutative diagram

$$\begin{tikzcd} T_V|_U \arrow[r] \arrow[d] & E_0 \arrow[d] \\ N_{U/V} \arrow[r, hook] & E_1, \end{tikzcd} \tag{14.5}$$

where E_0 and E_1 are vector bundles, together with maps $T_V|_U \to E_0 \oplus N_{U/V} \to E_1 \to 0$, where the second map restricts to a map $E_0 \oplus C_{U/V} \to C = E_0 \oplus C_{U/V}/T_V|_U$.

How do we obtain compatibility? There are a few options.

- Li and Tian do an Artin-style obstruction-theoretic argument.
- Behrend and Fantechi use the cotangent complex and derived categories.

The cotangent complex sounds scary! It's really not all that bad; it's just that existing references tend to be better for reference than for learning. (The Stacks project is all right, but as always is more of a reference — the ideal, say 20 pages or so, reference is still yet to be written.) We'll mostly take the cotangent complex as a black box.

Let $f\colon X\to Y$ be a map of schemes. Then we obtain a sequence of quasicoherent sheaves

$$f^*\Omega_Y \longrightarrow \Omega_X \longrightarrow \Omega_{X/Y} \longrightarrow 0, \tag{14.6}$$

which is right exact, but not exact in general. (It is exact if X is smooth.) In general, we need to replace $\Omega_{X/Y}$, Ω_X, and Ω_Y by complexes of quasicoherent sheaves $L^\bullet_{X/Y}$, $L^\bullet_X$, and $L^\bullet_Y$. Then, in the derived category of X, which we denote $D_{\mathcal{QC}oh}(X)$, there is a distinguished triangle

$$\mathbf{L}f^*L^\bullet_Y \longrightarrow L^\bullet_X \longrightarrow L^\bullet_{X/Y} \longrightarrow \mathbf{L}f^*L^\bullet_Y[1]. \tag{14.7}$$

Hence in particular there is a long exact sequence of cohomology sheaves:

$$\mathcal{H}^{-1}(L^\bullet_{X/Y}) \longrightarrow \underbrace{\mathcal{H}^0(\mathbf{L}f^*L^\bullet_Y)}_{=\Omega_Y} \longrightarrow \underbrace{\mathcal{H}^0(L^\bullet_X)}_{=\Omega_X} \longrightarrow \underbrace{\mathcal{H}^0(L^\bullet_{X/Y})}_{=\Omega_{X/Y}} \longrightarrow \underbrace{\mathcal{H}^1(\mathbf{L}f^*L^\bullet_Y)}_{=0}. \tag{14.8}$$

That is, these things are complexes of sheaves, and we take the cohomology of this complex (kernels mod images), not sheaf cohomology.

Remark 14.9 (Reviewing the derived category). This is all happening in the derived category $D_{\mathcal{QC}oh}(X)$ of $\mathcal{QC}oh(X)$, the abelian category of quasicoherent sheaves on X.

We begin with the category of complexes $\mathcal{F}^\bullet$ of quasicoherent sheaves, which are $\mathbb{Z}$-graded sheaves together with a differential $d_{\mathcal{F}}\colon \mathcal{F}^\bullet\to\mathcal{F}^\bullet[1]$ — the "[1]" denotes raising the degree of all elements by 1 — and we ask $d_{\mathcal{F}}^2=0$. The morphisms of complexes $\varphi\colon\mathcal{F}^\bullet\to\mathcal{G}^\bullet$ are data of maps $\varphi^i\colon\mathcal{F}^i\to\mathcal{G}^i$ on each graded component commuting with the differentials.

A *homotopy equivalence* of two morphisms $\varphi,\psi\colon\mathcal{F}^\bullet\rightrightarrows\mathcal{G}^\bullet$ is a map $h\colon\mathcal{F}^\bullet\to\mathcal{G}^\bullet[-1]$ such that $\varphi-\psi = h\circ d_{\mathcal{F}} - d_{\mathcal{G}}\circ h$. A *qausi-isomorphism* is a morphism which induces an isomorphism on all cohomology sheaves; all homotopy equivalences are quasi-isomorphisms, but not vice versa.

The derived category is obtained by localizing at the quasi-isomorphisms (i.e. formally making them invertible by including inverses into the category). Thus a morphism in the derived category $\varphi\colon\mathcal{F}^\bullet\to\mathcal{G}^\bullet$ is data of a quasi-isomorphism $\widetilde{\mathcal{F}}^\bullet\to\mathcal{F}^\bullet$, and any map of complexes $\widetilde{\varphi}\colon\widetilde{\mathcal{F}}^\bullet\to\mathcal{G}$.

Injective resolutions correspond to something in the derived category, namely replacing a complex by a quasi-isomorphic complex of injective sheaves. (The analogous statement for projectives is also true.) One can do this functorially to define right derived functors of left exact functors of sheaves. For example, if $f\colon X\to Y$ is a map of schemes, f_* is left exact, and we can obtain a right derived functor $\mathbf{R}f_*\colon D_{\mathcal{QC}oh}(X)\to D_{\mathcal{QC}oh}(Y)$: first functorially replace with an injective complex, then push forward.

The left exact case, with projectives, allows one to derive the tensor product $-\otimes^{\mathbf{L}}-$ and pullback $\mathbf{L}f^*$, though this case is a little more subtle. ◂

We return to the cotangent complex. First, the affine case: let $\varphi\colon A\to B$ be a homomorphism of commutative rings, and resolve B freely as an A-algebra: $P_\bullet\to B\to 0$, e.g.

$$P_\bullet = \big(\cdots\longrightarrow A[A[B]]\longrightarrow A[B]\big). \tag{14.10}$$

Then the *cotangent complex* of φ is

$$L^\bullet_{B/A} = \Omega^\bullet_{P_\bullet/A}\otimes_{P_\bullet} B, \tag{14.11a}$$

i.e.

$$L^{-n}_{B/A} = \Omega_{P_n/A}\otimes_{P_n} B. \tag{14.11b}$$

For $f\colon X\to Y$ a morphism of schemes or algebraic stacks, there is a similar *simplicial resolution* for $f^{-1}\mathcal{O}_Y\to\mathcal{O}_X$; for details see Illusie or the Stacks project. But the important fact is that if f factors first as an inclusion $i\colon X \hookrightarrow Z$ and then a smooth map $Z \to Y$, then

$$\tau_{\geq -1} L^\bullet_{X/Y} = \left(\mathcal{I}_{X/Z}/\mathcal{I}^2_{X/Z} \longrightarrow \Omega_{Z/Y}|_X \right) \tag{14.12}$$

in degrees -1 and 0. Here $\tau_{\geq k}$ denotes the *truncation* of a complex above degree k, meaning we replace everything below degree k with 0 and the degree-k piece with $\ker(d^k)$; the rest is the same.

The upshot is that at the level of linear fiber spaces, we have $T_{Z/Y}|X \to N_{X/Z}$ in degrees 0 and 1, as needed for the globalization of $C_{X/Z} \hookrightarrow B_{X/Z}$.

Definition 14.13 (Behrend-Fantechi)**.** Let $f\colon X \to Y$ be a map of algebraic stacks, where X is Deligne-Mumford. A *perfect obstruction theory* on X/Y is a two-term complex $\mathcal{F}^\bullet = (\mathcal{F}^{-1} \to \mathcal{F}^0)$ of locally free coherent sheaves, together with a map in $D_{\mathcal{QC}oh}(\mathcal{O}_X)$, denoted $\phi\colon \mathcal{F}^\bullet \to L^\bullet_{X/Y}$, such that $\mathcal{H}^0(\phi)$ is an isomorphism and $\mathcal{H}^{-1}(\phi)$ is an epimorphism.

As for why this has anything to do with obstruction theory, tune in next time.

Lecture 15.

A more perfect obstruction theory: 10/17/19

Recall the definition of a perfect obstruction theory due to Behrend-Fantechi in Definition 14.13. Let's tease apart what this means.

First assume ϕ is an honest morphism of complexes. Let Z contain X and $\tilde{f}\colon Z \to Y$ be an extension of f which is smooth. Then

$$\tau_{\geq 1} L^\bullet_{X/Y} = \left(\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{Z/Y}|_X \right), \tag{15.1}$$

then we have a commutative, but not Cartesian, diagram

$$\begin{CD} \mathcal{F}^{-1} @>>> \mathcal{I}/\mathcal{I}^2 \\ @VVV @VVV \\ \mathcal{F}^0 @>>> \Omega_{Z/Y}|_X. \end{CD} \tag{15.2}$$

This induces a sequence

$$\mathcal{F}^{-1} \longrightarrow \mathcal{F}^0 \oplus \mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{Z/Y}|_X, \tag{15.3}$$

which is not exact in general. Instead, exactness tells us key properties of ϕ.

- (15.3) is exact at $\mathcal{F}^{-1}$ iff $\mathcal{H}^{-1}(\phi)$ is injective.
- (15.3) is exact at $\mathcal{F}^0 \oplus \mathcal{I}/\mathcal{I}^2$ iff $\mathcal{H}^{-1}(\phi)$ is surjective and $\mathcal{H}^0(\phi)$ is injective.
- (15.3) is exact at $\Omega_{Z/Y}|_X$ iff $\mathcal{H}^0(\phi)$ is surjective.

But the definition of a perfect obstruction theory means $\mathcal{H}^0(\phi)$ is an isomorphism and $\mathcal{H}^{-1}(\phi)$ is surjective, so dualizing, we have a left exact sequence

$$0 \longrightarrow T_{Z/Y} \longrightarrow \mathcal{F}_0 \oplus N_{Z/X} \longrightarrow \mathcal{F}_1, \tag{15.4}$$

where $\mathcal{F}_i = L(\mathcal{F}^{-i})$. Here, L denotes the linear space associated to a sheaf, i.e. $L(\mathcal{F}) = \operatorname{Spec}_X \operatorname{Sym}^\bullet \mathcal{F}$.

Lemma 15.5.

(1) *There is a unique cone $C_\phi \subset \mathcal{F}_1$ with*

$$0 \longrightarrow T_{Z/Y}|_X \longrightarrow \mathcal{F}_0 \oplus C_{Z/X} \longrightarrow C_\phi \longrightarrow 0 \tag{15.6}$$

"exact" in the sense that there are exact sequences of vector bundles

$$\begin{CD} 0 @>>> E' @>>> E @>{q}>> E'' @>>> 0 \\ @. @| @AAA @AAA @. \\ 0 @>>> E' @>>> q^{-1}(C) @>{q}>> C @>>> 0. \end{CD} \tag{15.7}$$

(2) *C_ϕ is independent of the choice of Z and invarinat under homotopy equivalences of complexes.*

Proof sketch. For (1), $\mathcal{F}_1$ is locally free, so we can split (15.4) locally, and then it's trivial. For (2), TODO: I didn't follow what was written on the board. For homotopies, I think we get a map from $\mathcal{F}_0 \oplus C_{Z/X}$ to itself, and that means C_ϕ doesn't change. ⊠

Behrend-Fantechi take a more intrinsic approach. Let $\mathcal{N}_{X/Y} := [N_{X/Z} /\!\!/ T_{Z/Y}|_X]$, which we call the *intrinsic normal sheaf* or a *Picard stack*. This is an algebraic stack, and not a particularly simple one. The *intrinsic normal cone* is $\mathcal{C}_{X/Y} := [C_{X/Z} /\!\!/ T_{Z/Y}|_X] \subset \mathcal{N}_{X/Y}$, which is purely zero-dimensional. Then, ϕ defines a perfect obstructioon theory iff the induced map $\mathcal{N}_{X/Y} \to \mathcal{H}^0(\mathcal{F}^\bullet)/\mathcal{H}^1(\mathcal{F}^\bullet)$ is a monomorphism. In this case, we can define C_ϕ to be the pullback in

$$\begin{tikzcd}
 C_\phi \arrow[r, hook] \arrow[d] & & \mathcal{F}_1 \arrow[d] \\
 \mathcal{C}_{X/Y} \arrow[r, hook] & \mathcal{N}_{X/Y} \arrow[r, hook] & \mathcal{H}^0(\mathcal{F}^\bullet)/\mathcal{H}^1(\mathcal{F}^\bullet).
\end{tikzcd} \tag{15.8}$$

Now, as promised: what does this have to do with obstruction theory? Consider the following lifting problem: let $T \hookrightarrow \overline{T}$ be a square-zero extension, so the associated sheaf $\mathcal{I}$ of ideals squares to zero. We want to know whether, given a diagram

$$\begin{tikzcd}
 T \arrow[r, "h"] \arrow[d, hook] & X \arrow[d, "f"] \\
 \overline{T} \arrow[r, "g"'] \arrow[ur, dashed, "\tilde{g}", "\exists ?"'] & Y,
\end{tikzcd} \tag{15.9}$$

there's a lift $\tilde{g}$ as indicated. Functoriality of $L^\bullet_{X/Y}$ gives us an obstruction class

$$ob(h) := \left(\mathbf{L}h^* L^\bullet_{X/Y} \longrightarrow L^\bullet_{T/\overline{T}} \longrightarrow \tau_{\geq -1} L^\bullet_{T/\overline{T}}\right) \in \mathrm{Ext}^1(\mathbf{L}h^* L^\bullet_{X/Y}, \mathcal{I}), \tag{15.10}$$

and $\tau_{\geq -1} L^\bullet_{T/\overline{T}} \simeq \mathcal{I}[1]$ (i.e. the complex with $\mathcal{I}$ in position -1 and 0 everywhere else).

This is a complete obstruction, in that such a lift $\tilde{g}$ exists iff $ob(h) = 0$. If this is the case, the set of such lifts is an $\mathrm{Ext}^0(\mathbf{L}h^* L^\bullet_{X/Y}, \mathcal{I})$-torsor. What Behrend and Fantechi prove is an analogue of this fact.

Proposition 15.11 (Behrend-Fantechi)**.** *The following are equivalent.*

(1) $\phi\colon \mathcal{F}^\bullet \to L^\bullet_{X/Y}$ *is a perfect obstruction theory.*
(2) $\phi^* ob(h) \in \mathrm{Ext}^1(\mathbf{L}h^* \mathcal{F}^\bullet, \mathcal{I})$ *is a complete obstruction, and if it vanishes, the set of lifts is an* $\mathrm{Ext}^0(\mathbf{L}h^* \mathcal{F}^\bullet, \mathcal{I})$*-torsor.*
(3) *The map* $\mathcal{N}_{X/Y} \to \mathcal{H}^0(\mathcal{F}^\bullet)/\mathcal{H}^1(\mathcal{F}^\bullet)$ *is an embedding.*

The takeaway is that $\mathcal{N}_{X/Y}$ is a universal (and minimal!) "obstruction theory," but not a vector bundle stack unless $X \to Y$ is a local complete intersection. $\mathcal{F}^\bullet \to L^\bullet_{X/Y}$ proves an embedding into the vector bundle stack $[\mathcal{F}_1/\mathcal{F}_0] = \mathcal{H}^0(\mathcal{F}^\bullet)/\mathcal{H}^1(\mathcal{F}^\bullet)$.

To apply this to Gromov-Witten theory, it remains to construct $\phi\colon \mathcal{F}^\bullet \to L^\bullet_{\overline{\mathcal{M}}_{g,k}(X)/\overline{\mathcal{M}}_{g,k}}$. The map $s\colon \overline{\mathcal{M}}_{g,k}(X) \to \overline{\mathcal{M}}_{g,k}$ isn't just forgetting the map to X — the result might not be a stable curve. But there's a stabilization process, so we can apply it to map into $\overline{\mathcal{M}}_{g,k}$.

Once we do this, we can define the virtual fundamental class as

$$[\overline{\mathcal{M}}_{g,k}(X)]_{virt,\phi} := s_0^![C_\phi] \in A_*(\overline{\mathcal{M}}_{g,k}(X)), \tag{15.12}$$

where $s_0\colon \overline{\mathcal{M}}_{g,k}(X) \to \mathcal{F}_1$ is the zero section.

Remark 15.13. If you only want the virtual fundamental class in the rational Chow group or rational cohomology, it actually suffices to work with the coarse moduli space $\overline{M}_{g,k}(X)$, which is a scheme with a locally finite map to $\overline{\mathcal{M}}_{g.k}(X)$. ◂

Now we actually have to construct this obstruction theory. Fix $\beta \in H_2(X)$; we'll work with $\overline{\mathcal{M}}_{g,k}(X,\beta)$. The universal diagram is

(15.14)
$$\begin{tikzcd} \mathcal{C}_{g,k}(X) \arrow[r, "f"] \arrow[d, "\pi"] \arrow[dr] & X \\ \overline{\mathcal{M}}_{g,k}(X) \arrow[dr, "s"'] & \mathcal{C}_{g,k} \arrow[d, "\overline{\pi}"] \\ & \overline{\mathcal{M}}_{g,k}. \end{tikzcd}$$

In the sequence

(15.15)
$$\mathbf{L}f^*L^\bullet_X \longrightarrow L^\bullet_{\mathcal{C}_{g,k}(X)} \longrightarrow L^\bullet_{\mathcal{C}_{g,k}(X)/\mathcal{C}_{g,k}},$$

we have isomorphisms $f^*\Omega_X \simeq \mathbf{L}f^*L^\bullet_X$ and, since $\overline{\pi}$ is flat,

(15.16)
$$\pi^* L_{\overline{\mathcal{M}}_{g,k}(X)/\overline{\mathcal{M}}_{g,k}} \cong L^\bullet_{\mathcal{C}_{g,k}(X)/\mathcal{C}_{g,k}}.$$

Now let's apply relative Serre duality $\mathbf{R}\pi_*(- \otimes \omega_\pi)$, yielding

(15.17a)
$$\phi\colon \mathbf{R}\pi_*(f^*\Omega_X \otimes \omega_\pi) \longrightarrow \mathbf{R}\pi_*(\pi^* L^\bullet_{\overline{\mathcal{M}}_{g,k}(X)/\overline{\mathcal{M}}_{g,k}} \otimes \omega_\pi) \simeq L^\bullet_{\overline{\mathcal{M}}_{g,k}(X)/\overline{\mathcal{M}}_{g,k}} \otimes \mathbf{R}.$$

Since the map $\mathrm{tr}\colon \mathbf{R}\pi_*(\omega_\pi) \to \mathcal{O}_{\overline{\mathcal{M}}_{g,k}(X)}$ is an equivalence,

(15.17b)
$$\simeq L^\bullet_{\overline{\mathcal{M}}_{g,n}(X)/\overline{\mathcal{M}}_{g,k}}.$$

Let $E \coloneqq f^*\Omega_X \otimes \omega_\pi$.

Lemma 15.18. *There exist locally free $\mathcal{F}^i$ such that $\mathbf{R}\pi_*E \simeq [\mathcal{F}^{-1} \to \mathcal{F}^0]$.*

Proof. The line bundle $\mathcal{L} \coloneqq \omega_{\mathcal{C}_{g,k}(X)/\overline{\mathcal{M}}_{g,k}(X)}(\underline{X}) \otimes (\mathcal{O}_X(1))^{\otimes 3}$ is relatively ample for $\mathcal{C}_{g,k}(X) \to \overline{\mathcal{M}}_{g,k}(X)$. This has a few useful consequences: there's some $N \gg 0$ such that

(1) $\pi^*\pi_*(E \otimes \mathcal{L}^{\otimes N}) \to E \otimes \mathcal{L}^{\otimes N}$ is surjective (as $E \otimes \mathcal{L}^{\otimes N}$ is fiberwise globally generated),
(2) $\mathbf{R}^1\pi_*(E \otimes \mathcal{L}^{\otimes N}) = 0$, and
(3) for all geometric points $s \hookrightarrow \overline{\mathcal{M}}_{g,k}(X)$, $H^0(C_s, \mathcal{L}^{\otimes(-N)}) = 0$.

Now let

(15.19)
$$F \coloneqq \pi^*\pi_*(E \otimes \mathcal{L}^{\otimes N}) \otimes \mathcal{L}^{\otimes(-N)},$$

which by the above observations and cohomology and base change, is a vector bundle. Likewise, $H \coloneqq \ker(F \twoheadrightarrow E)$ is a vector bundle, and the sequence

(15.20)
$$0 \longrightarrow H \longrightarrow F \longrightarrow E \longrightarrow 0$$

is exact.

Next, since

(15.21a)
$$H^0(C_s, F) = H^0(C_s, \pi_*(E \otimes \mathcal{L}^{\otimes N})_s \otimes \mathcal{L}_s^{\otimes(-N)})$$

(15.21b)
$$= \underbrace{H^0(C_s, \mathcal{L}_s^{\otimes(-N)})}_{=0 \text{ by } (3)} \otimes \underbrace{\pi_*(E \otimes \mathcal{L}^{\otimes N})_s}_{=0 \text{ by } (2)} = 0,$$

then $\pi_*F = 0$ and $\pi_*H = 0$. Hence if π is a family of curves, $\mathbf{R}^1\pi_*F$ and $\mathbf{R}^1\pi_*H$ are locally free, so we've written

(15.22)
$$\mathbf{R}\pi_*E = [\mathbf{R}^1\pi_*H \longrightarrow \mathbf{R}^1\pi_*F]$$

as desired. ⊠

Great! Now we can define Gromov-Witten invariants. Fixing g, k, and β, we have our virtual fundamental class $[\overline{\mathcal{M}}_{g,k}]_{virt,\phi} \in A_d(\overline{\mathcal{M}}_{g,k}(X,\beta))$, where

$$d = 3g - 3 + k + c_1(X)\cdot\beta + n(1-g). \tag{15.23}$$

The first term is $\dim \overline{\mathcal{M}}_{g,k}$, and the second comes from Riemann-Roch. This d is called the *expected dimension*.

Lecture 16.

Gromov-Witten invariants, monoids, and cones: 10/22/19

Let X be a smooth projective variety over $\mathbb{C}$, $g,k \geq 0$, and $\beta \in H_2(X)$. We have defined the moduli space $\overline{\mathcal{M}}_{g,k}(X,\beta)$ of stable maps from k-pointed, genus-g curves representing the homology class β, together with its virtual fundamental class $[\overline{\mathcal{M}}_{g,k}(X,\beta)]_{virt} \in A_d(\overline{\mathcal{M}}_{g,k}(X,\beta))$, which is invariant under deformations of X (as we'll discuss later today).

Here, d is the expected dimension of $\overline{\mathcal{M}}_{g,k}(X,\beta)$ — it might actually have components of a larger dimension, but this is the minimum. If $n = \dim X$, then

$$d = \langle c_1(X), \beta\rangle + (n-3)(1-g) + k. \tag{16.1}$$

Hence, on a Calabi-Yau 3-fold, if $k = 0$, then $d = 0$ for any genus. This will be useful for us: we can count stable maps.

In general, though, rather than elements of some Chow ring of some stack, we want numbers. To do this, we use the *evaluation map*

$$\begin{aligned} \overline{\mathcal{M}}_{g,k}(X,\beta) &\longrightarrow \underbrace{X\times\cdots\times X}_{k} \\ (C,(x_1,\dots,x_k),f) &\longmapsto (f(x_1),\dots,f(x_k)) \end{aligned} \tag{16.2}$$

and the *stabilization map*

$$\begin{aligned} \overline{\mathcal{M}}_{g,k}(X,\beta) &\longrightarrow \overline{\mathcal{M}}_{g,k} \\ (C,(x_1,\dots,x_k),f) &\longmapsto (C,(x_1,\dots,x_k))^{st}, \end{aligned} \tag{16.3}$$

i.e. we forget the map, but then might have an unstable curve, so we stabilize. Now we can define numerical invariants: for $\alpha_1,\dots,\alpha_k \in H^*(X)$ and $\gamma \in H^*(\overline{\mathcal{M}}_{g,k})$, the *Gromov-Witten invariant* associated to this data is

$$\langle \alpha_1,\dots,\alpha_k;\gamma\rangle_{\mathrm{GW}}^{g,k,\beta} \coloneqq \langle \mathrm{ev}^*(\alpha_1\times\cdots\times\alpha_k)\smile p^*\gamma, [\overline{\mathcal{M}}_{g,k}(X,\beta)]_{virt}\rangle. \tag{16.4}$$

Here, "$\times$" denotes the cross product in cohomology, the map $\times\colon H^m(X)\otimes H^n(X)\to H^{m+n}(X\times X)$. Pulling back by the diagonal defines a map $\Delta^*\colon H^*(X\times X)\to H^*(X)$, and $\Delta^*\circ\times$ is the cup product. The notation $\langle\dots,[\overline{\mathcal{M}}_{g,k}(X,\beta)]_{virt}\rangle$ means to evaluate on this homology class — one might suggestively write this as

$$\int_{[\overline{\mathcal{M}}_{g,k}(X,\beta)]_{virt}} \mathrm{ev}^*(\alpha_1\times\cdots\times\alpha_k)\smile p^*\gamma. \tag{16.5}$$

We mentioned above that Gromov-Witten invariants are deformation-invariant; let's see why. Let $q\colon X\to S$ be a projective, smooth map, where the base S is connected and smooth. (We are specifically interested in the case where S is a locally complete intersection.) We would like to show that, given the same input data $\alpha_1,\dots,\alpha_n$, g, k, β, and γ, the Gromov-Witten invariants of X_s and $X_{s'}$, where $s,s'\in S$, are equal.

I wrote that β is "the same," but we have to worry about monodromy if S isn't simply connected. So we choose $\beta\in H_2(X_s)$ and a path from s to s'; the answer will not depend on this data, which has the nice implication that Gromov-Witten invariants do not have monodromy.

For any $s\in S$, let $\iota_s\colon X_s\hookrightarrow X$ be the inclusion of the fiber, and let $\mathcal{M}\coloneqq\overline{\mathcal{M}}_{g,k}(X,(\iota_s)_*\beta)$, where $(\iota_s)_*$ is the Gysin map. Then we have a forgetful map $\rho\colon\mathcal{M}\to S$, and ... TODO: the rest of this was erased before I could write it down :(— but the idea was, for S a locally complete intersection, one can prove that the virtual fundamental classes of the fibers are identified, and therefore that the Gromov-Witten invariants are equal.

Now let's talk about toric geometry. We work over an algebraically closed field k of characteristic zero (well, with a little more work, we could work over $\mathbb{Z}$). Toric geometry will be useful for actually computing Gromov-Witten invariants. It's roughly about equivariant compactifications of tori $(\mathbb{G}_m)^n$. That is, we want to produce a space X with a $\mathbb{G}_m$-action and a $\mathbb{G}_m$-equivariant inclusion $(\mathbb{G}_m)^n \hookrightarrow X$ such that the torus is Zariski-dense. Here $\mathbb{G}_m \coloneqq \operatorname{Spec} k[z, z^{-1}] = \mathbb{A}^1 \setminus \{0\}$; over $\mathbb{C}$, $\mathbb{G}_m = \mathbb{C}^\times$, justifying the name "torus." Toric geometry has a very combinatorial flavor.

Literature: Fulton [Ful93] has a book on toric geometry, as does Oda [Oda88]; the latter has a few useful things hard to find elsewhere but unusual notation. There's a more recent book by Cox, Little, and Schenk [CLS11].

We begin with monoids and cones. We could spend a lot of time proving all of the details, but that would make your head spin and is probably not worth it; instead, we'll take a few plausible-sounding statements on faith, and leave their (difficult) proofs to the references. We use the following notation.

- M will be a free abelian group of rank d (so abstractly isomorphic to $\mathbb{Z}^d$).
- $N \coloneqq \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, the dual of M.
- We let $M_{\mathbb{Q}} \coloneqq M \otimes_{\mathbb{Z}} \mathbb{Q}$, which is naturally isomorphic to $\operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Q})$.
- We let $M_{\mathbb{R}} \coloneqq M \otimes_{\mathbb{Z}} \mathbb{R} = M_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$.
- We define $N_{\mathbb{Q}}$ and $N_{\mathbb{R}}$ analogously.

Definition 16.6. Let V be a vector space over $\mathbb{Q}$ or $\mathbb{R}$. A subset $C \subset V$ is called a *cone* if $0 \in C$ and for all $\lambda > 0$, $\lambda C \subset C$. If in addition $C + C \subset C$, C is called *convex.*

C is convex iff the closure of C in $V_{\mathbb{R}}$ is a convex set. Here $V_{\mathbb{R}} \coloneqq V$ is what you expect: V is over $\mathbb{R}$ and is $V \otimes_{\mathbb{Q}} \mathbb{R}$ if V is over $\mathbb{Q}$.

Remark 16.7. In this section, by a "monoid" we mean a commutative monoid: it has an associative, commutative binary operation with an identity, but not necessarily inverses. A monoid is *cancellative* if whenever $a + c = b + c$, then $a = b$. ◀

A commutative monoid M has a *group completion* $K^0(M)$, which is defined by formally adding inverses for all elements of M; for example, $K^0(\mathbb{N}, +) \cong \mathbb{Z}$. If M is cancellative, the map $M \to K^0(M)$ is injective, and we identify M with its image in $K^0(M)$.

Definition 16.8. A cancellative monoid M is *saturated* if whenever $m \in K^0(M)$ and $\lambda \in \mathbb{N} \setminus \{0\}$ are such that $\lambda m \in M$, then $m \in M$.

For example, consider the submonoid of $\mathbb{Z}$ generated by 2 and 3; its group completion is $\mathbb{Z}$. This is not saturated, because it doesn't contain 1, but it does contain 2×1.

Lemma 16.9. *If $C \subset M_{\mathbb{R}}$ is a convex cone, then $C \cap M$ is a saturated monoid.*

This follows directly from the definition.

Example 16.10. Suppose $C \subset \mathbb{R}^2$ is the convex hull of 0, $(0,1)$, and $(k+1, -1)$, which is a convex cone which is more explicitly

$$\mathbb{R}_{\geq 0} \cdot (0,1) + \mathbb{R}_{\geq 0} \cdot (k+1, -1). \tag{16.11}$$

Then $C \cap M$[29] is the submonoid generated by $(0,1)$, $(1,0)$, and $(k+1, -1)$; we will write this as

$$C \cap M = \langle (0,1), (1,0), (k+1, -1) \rangle. \tag{16.12}$$

One can show this by computing that $\det\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = 1$ and $\det\left(\begin{smallmatrix} k+1 & 1 \\ -1 & 0 \end{smallmatrix}\right) = 1$.

We can therefore write $C \cap M$ as the monoid quotient

$$C \cap M \cong \mathbb{N}^3 / \left\langle \left(\begin{smallmatrix} 0 \\ k+1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}\right) \right\rangle, \tag{16.13}$$

i.e. if u, v, and w are the three generators, the sole relation is $u + v = (k+1)w$. ◀

Toric geometry produces fairly complicated examples if you unravel all the monoidal details, but also provides tools for dealing with them.

[29] TODO: what is M?

Definition 16.14. If R is a commutative ring and S is a monoid, the *monoid ring* $R[S]$ is, as a set, the set of functions $a\colon S \to R$, which we write as formal sums $\sum_S a_s\chi^s$, meaning $a(s) = a_s$. The multiplication rule is defined by imposing $\chi^s \cdot \chi^{s'} \coloneqq \chi^{s+s'}$ and extending R-linearly, which defines it uniquely.

This is a generalization of the group ring from representation theory.

If S is infinite, one might ask, when is $k[S]$ finitely generated as a k-algebra? It's obviously finitely generated if S is, but what if it isn't? For example, suppose $C \coloneqq 0 \cup (\mathbb{R}_{\geq 0} \times \mathbb{R}_{>0})$; then $S \coloneqq C \cap \mathbb{Z}^2$ is infinitely generated: we need to include $(a, 1)$ as a generator for all $a \in \mathbb{N}$.

But the good news is, there's an easy criterion.

Definition 16.15. A cone $C \subset M_\mathbb{R}$ is *rational polyhedral* if there exist $m_1, \dots, m_r \in M_\mathbb{Q}$ such that $C = \mathbb{R}_{\geq 0} \cdot m_1 + \cdots + \mathbb{R}_{\geq 0} \cdot m_r$.

If C is rational polyhedral, it's automatically closed and convex.

Proposition 16.16 (Gordon's lemma)**.** *If $C \subset M_\mathbb{R}$ is rational polyhedral, then $C \cap M$ is finitely generated, and hence $k[C \cap M]$ is also finitely generated.*

Proof. Choose $m_1, \dots, m_r$ as in Definition 16.15, except we impose $m_i \in M$. We can do this without loss of generality by clearing denominators. Then, the set

$$K \coloneqq \left\{ \sum_{i=1}^r t_i m_i \mid 0 \leq t_i \leq 1 \right\} \tag{16.17}$$

is compact, and hence $K \cap M$ is finite, and contains all of the m_i. The claim is that $K \cap M$ generates $C \cap M$: if $u = \sum r_i m_i \in C \cap M$, with each $r_i \geq 0$, then write $r_i = s_i + t_i$, where $s_i \in \mathbb{N}$ and $t_i \in [0, 1]$; then

$$u = \sum s_i m_i + \sum t_i m_i, \tag{16.18}$$

and $m_i, t_i m_i \in K \cap M$. ⊠

Lecture 17.

Attack of the Cones: 10/24/19

Recall from last time that we defined (rational) polyhedral cones. Choose a finitely generated free abelian group M, and let $N \coloneqq \operatorname{Hom}(M, \mathbb{Z})$. Let C be a finitely generated cone in $M_\mathbb{R}$.

We begin with some facts from convex geometry. We could prove them, but this would drag us two weeks off course, so we're not going to.

Definition 17.1. Let $\sigma \subset N_\mathbb{R}$ be a polyhedral cone. Its *dual cone* is $\sigma^\vee \coloneqq \{u \in M_\mathbb{R} \mid \langle u, v\rangle \geq 0 \text{ for all } v \in \sigma\} \subset M_\mathbb{R}$.

More generally, given a vector $u \in M_\mathbb{R}$, we let $u^\perp = \{v \in N_\mathbb{R} \mid \langle u, v\rangle = 0\}$.

Theorem 17.2.

1. *$\sigma^\vee$ is polyhedral. If σ is rational polyhedral, so is $\sigma^\vee$.*
2. *The canonical identification $N_\mathbb{R} \cong N_\mathbb{R}^{**}$ identifies $\sigma^{\vee\vee} \cong \sigma$.*
3. *If σ_1 and σ_2 are polyhedral cones, $(\sigma_1 \cap \sigma_2)^\vee = \sigma_1^\vee + \sigma_2^\vee$.*
4. *$\sigma + (-\sigma)$ is equal to the linear subspace spanned by σ in $N_\mathbb{R}$.*

In view of part (4), we define the *dimension* of a polyhedral cone σ to be the dimension of $\sigma + (-\sigma)$.

Definition 17.3. A polyhedral cone σ is *sharp* if $\sigma \cap (-\sigma) = 0$.

In other words, a cone is sharp iff the only linear subspace it contains is the zero subspace. See Figure 2.

Proposition 17.4. *The following are equivalent:*

1. *σ is sharp.*
2. $\dim \sigma^\vee = \operatorname{rank} M$.
3. *There exists a $u \in \sigma^\vee$ with $\sigma \cap u^\perp = \{0\}$.*

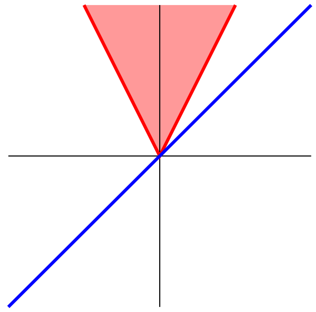

FIGURE 2. The red cone is sharp, but the blue cone is not sharp.

In toric geometry, *cones in $N_\mathbb{R}$ are always sharp, and cones in $M_\mathbb{R}$ are always d-dimensional.*

If σ is not sharp, we can reduce it. Let $C \coloneqq \sigma^\vee \subset M_\mathbb{R}$; then $L \coloneqq C \cap (-C) = \sigma^\perp = (\mathbb{R} \cdot \sigma)^\perp$ is the largest linear subspace contained in C. Then, there is a sharp cone $\overline{C} \subset M_\mathbb{R}/L$ such that if $q\colon M_\mathbb{R} \twoheadrightarrow M_\mathbb{R}/L$ is the projection map, then $C = q^{-1}(\overline{C})$. A section of q defines a splitting $C \cong \overline{C} \times L$.

Proposition 17.5. *A cone $\sigma \subset N_\mathbb{R}$ is polyhedral iff it is the intersection of finitely many half-spaces.*

Proof sketch. In the forward direction, $\sigma^\vee$ is polyhedral, so $\sigma^\vee = \sum \mathbb{R}_{\ge 0} u_i$. Let $H_i \coloneqq (\mathbb{R}_{\ge 0} u_i)^\vee$; then $\sigma = \bigcap_i H_i$. The reverse direction is similar: since H_i is a half-space, its dual is spanned by a single vector $u_i \in N_\mathbb{R}$. ⊠

Definition 17.6. A *supporting hyperplane* for σ is a hyperplane $u^\perp \subset N_\mathbb{R}$ such that $u \ne 0$ and $\sigma \subset (\mathbb{R}_{\ge 0} u)^\perp$. A *face* is the intersection of σ with $u^\perp$ for some $u \in \sigma^\vee$.

Most faces are just the origin, but not all of them.

Proposition 17.7. *Faces are rational polyhedral cones, and*

$$\partial\sigma = \bigcup_{\textit{faces } \tau \subsetneq \sigma} \tau \tag{17.8}$$

inside $\mathbb{R} \cdot \sigma \subset N_\mathbb{R}$.

Proposition 17.9.

1. *If τ_1 and τ_2 are faces of σ, then $\tau_1 \cap \tau_2$ is a face of σ.*
2. *If τ_1, τ_2 are faces of σ with $\tau_1 \subseteq \tau_2$, then $\tau_1 = \tau_2$ iff $\dim \tau_1 = \dim \tau_2$.*
3. *The minimum of $\dim \tau$ as τ ranges over all positive-dimensional faces of σ is equal to $\dim_\mathbb{R} \sigma \cap (-\sigma)$.*

Definition 17.10. A *facet* of σ is a face of codimension 1.

Proposition 17.11.

1. *Any face $\tau \subsetneq \sigma$ is an intersection of facets.*
2. *A face of codimension 2 is contained in exactly two facets.*

Definition 17.12. A one-dimensional face is also called an *extremal ray.*

Theorem 17.13. *The map $\Phi\colon \{\text{faces of } \sigma\} \to \{\text{faces of } \sigma^\vee\}$ sending $\tau \mapsto \tau^\perp \cap \sigma^\vee$ is an inclusion-reversing bijection; moreover, $\dim \tau + \dim \Phi(\tau) = \operatorname{rank} M$.*

With all this out of the way, we can begin defining toric varieties. We work over a field k, which does not need to be characteristic zero.

Definition 17.14. An *affine toric variety* V is a scheme of the form $\operatorname{Spec} k[\sigma^\vee \cap M]$, where $\sigma \subset N_\mathbb{R}$ is a rational polyhedral cone.

The *dimension* of V, as an affine toric variety, is $\operatorname{rank} M$.

The inclusion $k[\sigma^\vee \cap M] \hookrightarrow k[M]$ induces an open embedding of a torus into V:

$$\mathbb{G}_m^{\operatorname{rank} M} \cong \operatorname{Spec} k[M] \hookrightarrow \operatorname{Spec} k[\sigma^\vee \cap M]. \tag{17.15}$$

Let $\mathbb{T}_N \coloneqq \operatorname{Spec} k[M]$, as a subscheme of V.

Remark 17.16. By Proposition 16.16, affine toric varieties are of finite type over k (and, once we define toric varieties, they will also be finite type). ◀

Proposition 17.17. *Suppose $m_1, \dots, m_s \in M$ generate $\sigma^\vee \cap M$ as a monoid, and let I denote the kernel of the map*

$$\begin{aligned} k[x_1, \dots, x_s] &\xrightarrow{q} k[\sigma^\vee \cap M] \\ x_1^{a_1} \cdots x_s^{a_s} &\longmapsto \chi^{a_1 m_1 + \dots + a_s m_s}. \end{aligned} \tag{17.18}$$

Then I is generated by

$$G := \left\{ x_a^{a_1} \cdots x_s^{a_s} - x_1^{b_1} \cdots x_s^{b_s} \mid \sum_{i=1}^s a_i m_i = \sum_{i=1}^s b_i m_i, a_i, b_i \geq 0 \right\}. \tag{17.19}$$

Remark 17.20. The map q in (17.18) identifies $k[\sigma^\vee \cap M] \cong k[x_1, \dots, x_s]/I$. ◀

Proof of Proposition 17.17. Choose an identification $M \cong \mathbb{Z}^d$, which induces a $\mathbb{Z}^d$-grading on $k[\sigma^\vee M]$ in which $\deg \chi^{m_i} := m_i$. If we grade $k[x_1, \dots, x_s]$ by $\deg x_i = m_i$, then q is homogeneous, and therefore I is a homogeneous ideal, so is generated by homogeneous expressions of the sort

$$f := \sum_{\text{multi-indices } A} \alpha_A x^A. \tag{17.21}$$

For any such homogeneous element f, let $m := \sum a_i m_i$, so that $q(f) = (\sum \alpha_A)\chi^m$, and therefore $f \in I$ iff $\sum \alpha_A = 0$. The proof then follows by descending induction on the number of nonzero a_i; we can replace x^A by $x^{A'}$ if $x^A - x^{A'} \in G \subseteq I$. ⊠

Definition 17.22. An ideal generated by generators of the form of those in G is called a *toric ideal*.

Recall that if X is a scheme and K is a field, its *K-valued points* $X(K)$ are those scheme-theoretic points $x \in X$ such that the residue field $\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x \cong K$. If K is algebraically closed and X is finite type over K, these are precisely the closed points of X: since X is finite type, any closed point's residue field is a finite field extension of K.

Proposition 17.23. *If $X := \operatorname{Spec} k[\sigma^\vee \cap M]$, there is a natural identification $X(k) \cong \operatorname{Hom}_{\mathcal{M}on}(\sigma^\vee \cap M, (k, \cdot))$.*

This is a little strange: we're considering k with its multiplicative structure, which is non-cancellative because of 0.

Proof. Well, $X(k) = \operatorname{Hom}_{\mathcal{A}lg_k}(k[\sigma^\vee \cap M], k) = \operatorname{Hom}_{\mathcal{M}on}(\sigma^\vee \cap M, (k, \cdot))$. ⊠

Example 17.24.

(1) If $\sigma^\vee = M_\mathbb{R}$, then

$$\operatorname{Hom}_{\mathcal{M}on}(M, (k, \cdot)) \cong \operatorname{Hom}_{\mathcal{M}on}(M, k^\times) \cong N \otimes_\mathbb{Z} k^\vee \cong (k^\vee)^d. \tag{17.25}$$

(2) If $\sigma^\vee = \mathbb{R}^2_{\geq 0}$, then $k[\sigma^\vee \cap M] \cong k[x, y]$ under the identification $x := \chi^{(1,0)}$ and $y := \chi^{(0,1)}$. Then $\operatorname{Hom}(\sigma^\vee \cap M, (k, \cdot)) \cong k^2$. ◀

Suppose that $k = \mathbb{C}$; then, there is a *momentum map* $X(\mathbb{C}) \to \sigma^\vee$. If $p \in \sigma^\vee$, the fiber over p is contained in the interiors of faces of σ, and inside an ℓ-dimensional space, the piece of the fiber is $(S^1)^\ell$. One example is the map $\mathbb{C}^2 \to \mathbb{R}^2_{\geq 0}$ sending $(\lambda, \mu) \mapsto (|\lambda|, |\mu|)$.

Proposition 17.26. *Let $\sigma \subset N_\mathbb{R}$ be a rational polyhedral cone and $\tau \subseteq \sigma$ be a face. The inclusion $\sigma^\vee \hookrightarrow \tau^\vee$ defines a homomorphism $k[\sigma^\vee \cap M] \hookrightarrow k[\tau^\vee \cap M]$, which is a localization at a single element. In particular, $\operatorname{Spec} k[\tau^\vee \cap M] \hookrightarrow \operatorname{Spec} k[\sigma^\vee \cap M]$ is a distiguished open subscheme.*

Recall that a *distinguished open* or a *fundamental open* is a subscheme of the form $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A$, where $f \in A$.

Proof. We can write $\tau = \sigma \cap m^\perp$, where $m \in \sigma^\vee \cap M$. Then $m(\sigma \setminus \tau) \subset \mathbb{N} \setminus 0$, so $\tau^\vee \cap M = \sigma^\vee M + \mathbb{N} \cdot (-m)$. (TODO: one inclusion is clear; I missed the argument for the other.)

This means that for all $u \in \tau^\vee \cap M$, $u + \lambda m \in \sigma^\vee$ for $\lambda \gg 0$, and therefore $k[\tau^\vee \cap M] = k[\sigma^\vee \cap m]_{\chi^m} \subset k[M]$, as promised. ⊠

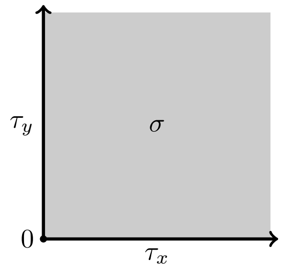

FIGURE 3. The faces of σ in Example 18.2.

Lecture 18.

Fans and toric varieties: 10/29/19

Today, we continue discussing (affine) toric geometry. Let M be a finite-rank free abelian group, $N \coloneqq \mathrm{Hom}(M, \mathbb{Z})$, and k be a field (of any characteristic).

Let $\sigma \subset N_\mathbb{R}$ be a sharp, strictly convex cone and $\tau \subset \sigma$ be a face. Then, inside $M_\mathbb{R}$, $\sigma^\vee \subset \tau^\vee$, which induces a map $k[\sigma^\vee \cap M] \hookrightarrow k[\tau^\vee \cap M]$. This is a localization, so after taking Spec, it's an open embedding of $\operatorname{Spec} k[\tau^\vee \cap M]$ into the affine toric variety $X \coloneqq \operatorname{Spec} k[\sigma^\vee \cap M]$ defined by σ.

Definition 18.1. We call $U_\tau \coloneqq \operatorname{Spec} k[\tau^\vee \cap M] \subset X$ the *fundamental open set* associated to τ.

Example 18.2. Suppose $\sigma \coloneqq \mathbb{R}^2_{\geq 0}$ inside $N_\mathbb{R} \coloneqq \mathbb{R}^2$. Then $X = \mathbb{A}^2$. Let τ_x and τ_y be the nonnegative x- and y-axes, respectively, and 0 denote the origin; these are the faces of σ, as depicted in Figure 3.

Inside $M_\mathbb{R}$, $\sigma^\vee$ is the (closure of the) first quadrant; $\tau_x^\vee$ is the right half of the plane; $\tau_y^\vee$ is the upper half-plane; and $0^\vee$ is the entire plane. Therefore

$$\begin{aligned} U_{\tau_x} &= \operatorname{Spec} k[x,y]_y = \mathbb{A}^1 \times \mathbb{G}_m \\ U_{\tau_y} &= \operatorname{Spec} k[x,y]_x = \mathbb{G}_m \times \mathbb{A}^1 \\ U_0 &= \operatorname{Spec} k[x,y]_{(x,y)} = \mathbb{G}_m^2. \end{aligned} \tag{18.3}$$

◀

Next, we will discuss rational functions, dimension, and normality.

Proposition 18.4.

(1) *Let $C \subset M_\mathbb{R}$ be a rational polyhedral cone and $L \coloneqq \mathbb{R} \cdot C = C + (-C)$. Then the field of fractions of $k[C \cap M]$ is equal to the field of fractions of $k[L \cap M]$, and these are $k(L \cap M)$.*
(2) *Any affine toric variety is normal.*
(3) $\dim(\operatorname{Spec} k[C \cap M]) = \dim_\mathbb{R} L$.

Proof. Part (1) follows directly from the definitions. For (2), write C as an intersection of half-spaces: $C = v_1^\vee \cap \cdots \cap v_r^\vee$; then, after choosing an identification $M \cong \mathbb{Z}^d$,

$$C \cap M = \bigcap_{i=1}^r (M \cap v_i^\vee), \tag{18.5}$$

and therefore inside $k[M]$,

$$k[C \cap M] = \bigcap_{i=1}^r k[M \cap v_i^\vee]. \tag{18.6}$$

The intersection of normal rings is normal, so it suffices to prove this for $k[M \cap v_i^\vee]$. But since

$$k[M \cap v_i^\vee] \cong k[x_1, \dots, x_d]_{x_d}, \tag{18.7}$$

then it's in particular a localization of a polynomial ring, so it is normal.

Finally, for (3), observe that $\operatorname{Spec} k[C \cap M]$ contains $\operatorname{Spec} k[L \cap M] \cong \mathbb{G}_m^{\dim L}$ as an open dense subset. ☒

Next, we will discuss a nice stratification of toric varieties, The strata are closures of $\mathbb{G}_m^d$-orbits.

Lemma 18.8. *Let $\sigma \subset N_\mathbb{R}$ be a sharp rational polyhedral cone, and let $d := \dim_\mathbb{R} N_\mathbb{R}$. Let $\tau \subset \sigma$ be a face and $L := \mathbb{R} \cdot \tau = \tau + (-\tau)$. Then $\tau^\perp \cap \sigma^\vee = L^\perp \cap \sigma^\vee$ is the dual cone of σ/L in $N_\mathbb{R}/L$.*

This follows fairly straightforwardly when one unwinds the definitions.

Dualizing yields an inclusion

$$k[(\tau^\perp \cap \sigma^\vee) \cap M] \hookrightarrow k[\sigma^\vee \cap M] \tag{18.9}$$

induced from the map $\sigma/L \twoheadleftarrow \sigma$, and using, from Lemma 18.8, that $(\sigma/L)^* = \tau^\perp \cap \sigma^\vee$. Equivalently, the map

$$\operatorname{Spec} l[\sigma^\vee \cap M] \twoheadrightarrow \operatorname{Spec} k[(\tau^\perp \cap \sigma^\vee) \cap M] \tag{18.10}$$

is dominant, inducing a projection map $\pi\colon X = U_\sigma \twoheadrightarrow U_{\sigma/L}$. We'd like to understand this map.

Since $L \subset N_\mathbb{R}$ is rational (as a cone), it induces a subtorus $\mathbb{T}_L := \operatorname{Spec} k[M/(L^\perp \cap M)] \subset \mathbb{T} = \mathbb{G}_m^d$. Then, it turns out that π is the (categorical) quotient of X by $\mathbb{T}_L$! You can think of this as sort of a fiber bundle with $\mathbb{T}_L$ fibers. There is a canonical section ι, constructed as follows: $\tau^\perp \cap \sigma^\vee \subset \sigma^\vee$ is a face, and

$$I_\tau := (\chi^m : m \in (\sigma^\vee \cap M) \setminus \tau^\perp) \subset k[\sigma^\vee \cap M] \tag{18.11}$$

is an ideal. Therefore we can let ι^* denote the quotient map $k[\sigma^\vee \cap M] \twoheadrightarrow k[\sigma^\vee \cap M]/I_\tau$ and ι be the induced map on schemes. The composition

$$k[(\tau^\perp \cap \sigma^\vee) \cap M] \xrightarrow{\pi^*} k[\sigma^\vee \cap M] \xrightarrow{\iota^*} k[\sigma^\vee \cap M]/I_\tau \tag{18.12}$$

induces more or less by definition a bijection between the bases of the relevant k-vector spaces given by monomials in $(\tau^\perp \cap \sigma^\vee) \cap M$. Therefore $\pi \circ \iota = \mathrm{id}$, so ι is really a section.

Remark 18.13. There is an identification

$$k[(\tau^\perp \cap \sigma^\vee) \cap M] = k[\sigma^\vee \cap M]^{\mathbb{T}_L}. \tag{18.14}$$

If $q\colon M \twoheadrightarrow M/L$ is the quotient map and $M \in \sigma^\vee \cap M$, then $\deg_{\mathbb{T}_L} \chi^m = q(m)$. ◀

Definition 18.15. A *toric substratum* of $X = U_\sigma$ is a closed subscheme of the form $U_{\sigma/L}$.

In particular, these are toric varieties themselves for $\mathbb{T}_L$.

Toric geometry is generally not conceptually difficult, but there are lots of duals and embeddings and things floating around, and keeping them all straight can be tricky.

Example 18.16. With σ, $\sigma^\vee$, and τ_x as in Example 18.2, $k[(\tau^\perp \cap \sigma^\vee) \cap M] = k[y] \hookrightarrow k[x, y]$, which induces on schemes the map $\mathbb{A}^2 \to \mathbb{A}^1$ sending $(x, y) \mapsto y$. The section sends $\mathbb{A}^1 = U_{\sigma/L}$ to the y-axis, so the y-axis is a toric substratum. ◀

If you're ever confused in toric geometry, definitely work through examples: things are very explicit.

Example 18.17. Let $\sigma^\vee := \mathbb{R}_{\ge 0}(1,0,0) + \mathbb{R}_{\ge 0}(1,1,0) + \mathbb{R}_{\ge 0}(1,1,1) + \mathbb{R}_{\ge 0}(1,0,1)$. Then

$$k[\sigma^\vee \cap M] = k[x, y, z, w]/(xy - zw), \tag{18.18}$$

because both xy and zw are $\chi^{(2,1,1)}$. Let $\tau := \mathbb{R}_{\ge 0} \cdot (1, 0, 0)$; then,

$$\tau^\perp \cap \sigma^\vee = \mathbb{R}_{\ge 0}(1,0,0) + \mathbb{R}_{\ge 0}(1,1,0). \tag{18.19}$$

Then $k[(\sigma^\vee \cap \tau^\perp) \cap M] \cong k[x, z]$ and $I_\tau = (w, y)$, so we know π^* and ι^*, namely the quotient maps

$$\begin{aligned} \pi^*&\colon k[x, z] \longrightarrow k[x, y, z, w]/(xy - zw) \\ \iota^*&\colon k[x, y, z, w]/(xy - zw) \longrightarrow (k[x, y, z, w]/(xy - zw))/(w, y) \cong k[x, z]. \end{aligned} \tag{18.20}$$ ◀

These constructions globalize. That is, we can patch together affine toric varieties, and all this structure is compatible.

Definition 18.21. A *fan* in $N_\mathbb{R}$ is a nonempty collection Σ of rational polyhedral cones in $N_\mathbb{R}$ such that

(1) if $\sigma \in \Sigma$ and τ is a face of σ, then $\tau \in \Sigma$, and
(2) if $\sigma, \sigma' \in \Sigma$, then $\sigma \cap \sigma'$ is a face of both σ and σ'.

The *support* of a fan Σ is $|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma$; if this is all of $N_\mathbb{R}$, Σ is called *complete*.

Here are a few examples of fans.

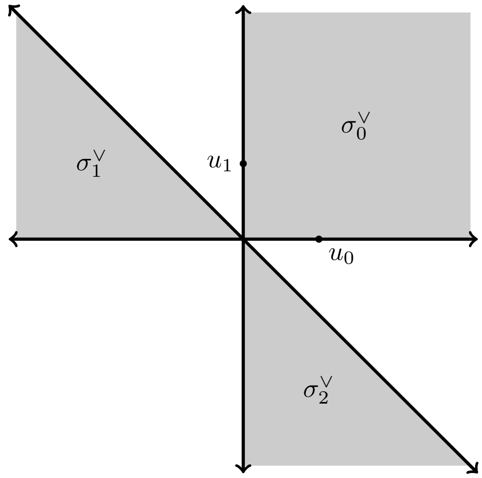

FIGURE 4. The dual cones to the fan in Example 18.23.

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- Inside $\mathbb{R}^3$, consider the fan containing (1) $\{0\}$; (2) the nonnegative x-, y-, and z-axes; and (3) the first quadrants inside the xy-, xz-, and yz-planes.
- We could also throw out the xz- and yz-planes from the previous example and keep everything else.
- Inside $\mathbb{R}$, the options for fans are just 0, 0 and a ray, or 0 and both rays.

Given a fan Σ, we can produce a recipe for gluing the affine toric varieties $U_\sigma \supset \mathbb{G}_m^d$ for $\sigma \in \Sigma$. Specifically, if $\sigma, \sigma' \in \Sigma$, then $\sigma \cap \sigma'$ is contained in both σ and σ', so $U_{\sigma\cap\sigma'} \hookrightarrow U_\sigma$ and $U_{\sigma\cap\sigma'} \hookrightarrow U_{\sigma'}$ are both open embeddings. This defines a directed system, and so

$$X(\Sigma) \coloneqq \varinjlim_{\sigma\in\Sigma} U_\sigma \tag{18.22}$$

is a scheme. (We'll discuss this in more detail later.)

Example 18.23. We will build $\mathbb{P}^2$ as a toric variety. Consider the fan Σ in $\mathbb{R}^3$ which contains 0, the positive axes, and the positive planes as above. The dual picture is unusually nice, as in Figure 4.

Let $u_0 \coloneqq \chi^{(1,0)}$, $i_1 \coloneqq \chi^{(0,1)}$, $v_0 \coloneqq \chi^{(-1,1)}$, $v_1 \coloneqq \chi^{(-1,0)}$, $w_0 \coloneqq \chi^{(0,-1)}$, and $w_1 \coloneqq \chi^{(1,-1)}$. From Figure 4, one can see that u_0 and u_1 generate $\sigma_0^\vee$, and therefore

$$U_{\sigma_0} = \operatorname{Spec} k[u_0, u_1] \cong \mathbb{A}^2, \tag{18.24}$$

and similarly, $U_{\sigma_1} = \operatorname{Spec} k[v_0, v_1]$ and $U_{\sigma_2} = \operatorname{Spec} k[w_0, w_1]$.

TODO: I did not finish writing down the rest (gluing, and generalizing to $\mathbb{P}^d$). Will add later, hopefully. $\blacktriangleleft$

Lecture 19.

Maps between fans and maps between toric varieties: 10/31/19

"I love dumb questions! They're typically not dumb at all!"

Recall that last time, we discussed how to use a fan to glue together affine toric varieties into a toric variety, by gluing them together along the cones in the fan. We used this to produce $\mathbb{P}^2$, for example, and that construction generalizes to $\mathbb{P}^n$.

We next discuss how the toric strata behave under such a gluing. Recall that in the affine toric variety $\operatorname{Spec}(k[\sigma^\vee \cap M])$, given a face τ of σ, the toric stratum given by τ is

$$\operatorname{Spec}(k[\sigma^\vee \cap M]^{\mathbb{T}_\tau}) = \operatorname{Spec}(k[(\tau^\perp \cap \sigma^\vee) \cap M). \tag{19.1}$$

Globalizing, let Σ be a fan in N and $\tau \in \Sigma$. Define the *quotient fan* Σ_τ, a fan in $N/\mathbb{R}\cdot\tau$, by

$$\Sigma_\tau \coloneqq \{\sigma/\mathbb{R}\cdot\tau \subset N/\mathbb{R}\cdot\tau \mid \sigma\in\Sigma, \sigma \supseteq \tau\}. \tag{19.2}$$

Then there is a closed embedding $X(\Sigma_\tau) \hookrightarrow X(\Sigma)$ of codimension equal to $\dim \tau$. We can see this on affine charts, where given $\sigma \supseteq \tau$, that map comes from the map on rings $k[\sigma^\vee] \twoheadrightarrow k[\sigma^\vee \cap \tau^\perp]$, which is a closed embedding.

Example 19.3. Briefly, consider the fan for $\mathbb{P}^3$ inside $\mathbb{R}^4$; if τ denotes the positive x-axis, then quotienting, Σ_τ is the usual fan for $\mathbb{P}^2$ inside $\mathbb{R}^3$, and the closed embedding is the usual embedding $\mathbb{P}^3 \hookrightarrow \mathbb{P}^4$ as the first three coordinates. ◀

Toric varieties are not automatically quasicompact.

Example 19.4. Let $N_\mathbb{R} = \mathbb{R}^2$ and let Σ be the fan containing the cones spanned by $(i, 1)$ and $(i + 1, 1)$ for all $i \in \mathbb{Z}$, together with all of their faces, as in Figure 5. Then

$$|\Sigma| = \mathbb{R} \times \mathbb{R}_{>0} \cup \{(0, 0)\}, \tag{19.5}$$

which isn't closed, and $X(\Sigma)$ is an infinite union of copies of $\mathbb{A}^2$, which is not quasicompact!

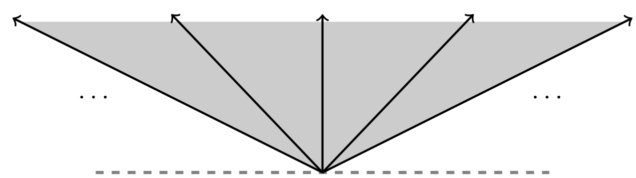

FIGURE 5. The fan Σ in Example 19.4.

However, you can describe $X(\Sigma)$ as an infinite blowup of $\mathbb{A}^1 \times \mathbb{G}_m$, so $X(\Sigma)$ has an infinite chain of $\mathbb{P}^1$s, one for each ray. The $\mathbb{G}_m$ corresponds to these $\mathbb{P}^1$s, and this projects down to $\mathbb{A}^1$ — the fiber over 0 is an infinite union of $\mathbb{P}^1$s, so again we see this isn't quasicompact. The fiber over the generic point is a $\mathbb{G}_m$ over the function field $k(t)$. ◀

Proposition 19.6. *$X(\Sigma)$ is quasicompact iff Σ is finite.*

Proof. The proof in the $\Longleftarrow$ direction is clear, so let's go in the forward direction. Since $X(\Sigma)$ is quasicompact, then $\bigcup_{\tau\in\Sigma} X(\Sigma_\tau)$ can only have finitely many irreducible components, and therefore $\{\tau \in \Sigma \mid \dim \tau = 1\}$ must be a finite set. This forces Σ to also be a finite set. ⊠

The assignment from fans to toric varieties extends to a covariant functor from a suitable category of fans to schemes.

Definition 19.7. Let Σ be a fan in N and Σ' be a fan in N'. A *morphism of fans* $\varphi\colon \Sigma \to \Sigma'$ is a linear map $N_\mathbb{R} \to N'_\mathbb{R}$ such that

(1) $\varphi(N) \subset N'$, and
(2) for all $\sigma \in \Sigma$, there's some $\sigma' \in \Sigma'$ with $\varphi(\sigma) \subseteq \sigma'$.

In condition (2), σ' need not be unique, but there is a unique minimal such σ'.

Now φ induces a *toric morphism* $\Phi\colon X(\Sigma) \to X(\Sigma')$, i.e. a morphism of schemes equivariant for the $\mathbb{G}_m(N)$-action. Explicitly, $\mathbb{G}_m(N)$ acts on $X(\Sigma)$ as usual, and acts on $X(\Sigma')$ through the group homomorphism $\mathbb{G}_m(N) \to \mathbb{G}_m(N')$ induced by φ.

We define Φ locally; it will be clear that the constructin is compatible with gluing. Given $\sigma \in \Sigma$ and $\sigma' \supseteq \varphi(\sigma)$, define $\Phi\colon U_\sigma \to U_{\sigma'}$ to be Spec of the map of rings

$$\begin{aligned} \varphi^*\colon k[(\sigma')^\vee \cap M'] &\longrightarrow k[\sigma^\vee \cap M] \\ \chi^m &\longmapsto \chi^{\varphi^*(m)}. \end{aligned} \tag{19.8}$$

Here, $\varphi^*(m)$ is defined as follows: identifying $M' = (N')^*$, it produces a map $N' to \mathbb{R}_{\geq 0}$, and we precompose with φ to obtain an element $N^* = M$.

We finish today's lecture with several different examples displaying what toric morphisms can look like.

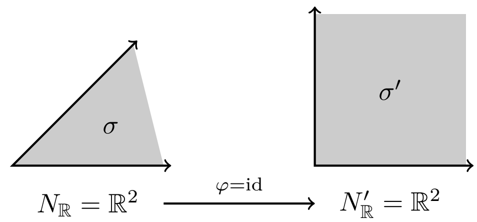

FIGURE 6. The map of fans considered in Example 19.10.

Example 19.9. The first example will just be affine: let Σ be the fan in $\mathbb{R}$ containing $[0, \infty)$ and its faces, and let Σ' be the fan in $\mathbb{R}^2$ containing the first quadrant and its faces. Then $X(\Sigma) = \mathbb{A}^1$ and $X(\Sigma) = \mathbb{A}^2$.

Let $\varphi\colon \mathbb{R} \to \mathbb{R}^2$ send $\lambda \mapsto (p\lambda, q\lambda)$. Then $\varphi^*\colon M' \to M$ is identified with the map $\mathbb{N}^2 \to \mathbb{N}$ sending $(m_1, m_2) \mapsto pm_1 + qm_2$, and therefore defines the map $k[x, y] \to k[u]$ sending $x \mapsto u^p$ and $y \mapsto u^q$. Assuming $\gcd(p, q) = 1$, $\ker(\varphi^*) = (x^q - y^p)$, and therefore Φ embeds $\mathbb{A}^1$ as the nodal curve $\{x^p - y^q = 0\}$. ◀

Example 19.10. Our next example is also affine, and is the toric model of an affine blowup. Let Σ be the fan in $\mathbb{R}^2$ consisting of the cone σ generated by $(1, 1)$ and $(1, 0)$ and its faces; similarly let Σ' be the fan in $\mathbb{R}^2$ consisting of the cone σ' generated by $(1, 0)$ and $(0, 1)$ and its faces. Let φ be the identity on $\mathbb{R}^2$. This is depicted in Figure 6.

For the dual map, let $x := \chi^{(1,0)}$ and $y := \chi^{(0,1)}$ in M', and $v := \chi^{(0,1)}$ and $u := \chi^{(1,-1)}$ in M. Then the dual map sends $x \mapsto uv$ and $y \mapsto v$, which is the affine chart for $B\ell\mathbb{A}^2 \to \mathbb{A}^2$.

In a little more detail, if $\mathcal{I} \subset k[x, y]$ is an ideal, then

$$B\ell_{\mathcal{I}}\mathbb{A}^2 := \mathrm{Proj}_{\mathcal{O}_X} \bigoplus_{d \geq 0} \mathcal{I}^d. \tag{19.11}$$

To blow up at the origin, $\mathcal{I} = (x, y)$. The right-hand side of (19.11) has the general formula

$$\mathrm{Proj}_{k[x,y]}\, A[u_1, \dots, u_r]/(f_1, \dots, f_s), \tag{19.12}$$

where the u_i are degree 1 and the f_j are homogeneous. Here, $u \mapsto x$ and $v \mapsto y$, so the sole relation is $f_1 = xV - yU$, i.e.

$$B\ell_{(x,y)}\mathbb{A}^2 = \mathrm{Proj}\, k[x, y][u, v]/(xv - yu). \tag{19.13}$$

Here $|x| = |y| = 0$ and $|u| = |v| = 1$.

Then, the blowup itself is the union of $\{u \neq 0\}$ and $\{v \neq 0\}$ — when $u \neq 0$, we can divide by it and establish an isomorphism $k[x, y] \to k[x, v]$ sending $x \mapsto x$ and $y \mapsto xv$, and so here this looks like a local isomorphism of schemes. Similarly, when $v \neq 0$, we get an isomorphism $k[x, y] \to k[u, y]$ sending $x \mapsto yu$ and $y \mapsto y$, and this looks nice. But the fiber at the origin is a $\mathbb{P}^1$. ◀

This example fits into a more general story. A *subdivision* is a morphism of fans $\varphi\colon \Sigma \to \Sigma'$ in which $N_{\mathbb{R}} = N'_{\mathbb{R}}$, φ is the identity, and for all $\varphi' \in \Sigma'$, φ' is a union of $\varphi(\sigma)$ for $\sigma \in \Sigma$. A map of toric varieties induced by a subdivision is always a *modification*, i.e. it's birational, proper, and integral.

Example 19.14. Now let's consider a projection. Let Σ be the fan in $\mathbb{R}^2$ containing the cone σ generated by $(0, 1)$ and $(k, 1)$ and its faces, and let Σ' be the fan generated by the cone $(0, \infty)$ in $\mathbb{R}$. Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ send $(x, y) \mapsto y$.

On the dual side, let $x := \chi^{(1,0)}$, $y := \chi^{(-1,k)}$, and $t := \chi^{(0,1)}$ in M, and $w := \chi^1$ in M'. Then the induced map on rings $k[w] \to k[x, y, t]/(xy - t^k)$ sends $w \mapsto xy$. This is the standard example of an A_{k-1} surface singularity, a fibration of the A_{k-1}-surface $\mathrm{Spec}\, k[x, y, t]/(xy - t^k) \to \mathbb{A}^1_w$. Since this sends $w \mapsto t$, we identify w and t. For $t \neq 0$, the fibers are smooth and isomorphic to $\mathbb{G}_m$, but at $t = 0$ we get a nodal curve, isomorphic to $\mathbb{A}^1 \cup \mathbb{A}^1$. ◀

Example 19.15. Next, we consider what can happen when the lattice changes. Let Σ and Σ' be the fans generated in $N_{\mathbb{R}} = \mathbb{R}^2$ by the first quadrant, but where $\varphi = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$, where $a, b \in \mathbb{N}$. This induces the map of rings $k[x, y] \to k[x, y]$ sending $x \mapsto x^a$ and $y \mapsto y^b$. This is a toric branched cover of $X(\Sigma')$ — and indeed, in general, changing the lattice corresponds to a branched covering map. ◀

TODO: there was another example that I missed, which was the inclusion of a face into a fan as a toric morphism.

Lecture 20.

Polytopes: 11/5/19

Last time, we promoted the construction of a toric variety from a fan into a functor; we begin by translating several properties of (morphisms of) fans into properties of the induced (morphisms of) toric varieties. Then, we'll discuss polytopes and their use in a different important perspective on toric varieties.

To fix notation, let $\varphi\colon \Sigma \to \Sigma'$ be a morphism of fans, where Σ is a fan in $N_\mathbb{R}$ and Σ' is a fan in $N'_\mathbb{R}$. Let $\varphi\colon N_\mathbb{R} \to N'_\mathbb{R}$ also denote the underlying map of vector spaces. That φ is a morphism of fans means that for all $\sigma \in \Sigma$, there is some $\sigma' \in \Sigma'$ with $\varphi(\sigma) \subseteq \sigma'$. Let $\Phi\colon X(\Sigma) \to X(\Sigma')$ denote the induced map of toric varieties.

The *preimage* of the fan Σ' under a linear map $f\colon N_\mathbb{R} \to N'_\mathbb{R}$ is $f^{-1}(\Sigma') \coloneqq \{f^{-1}(\sigma') \mid \sigma' \in \Sigma'\}$. This is not automatically a fan (TODO: I think).

Proposition 20.1. *Φ is a closed embedding iff $\Sigma = f^{-1}(\Sigma')$ and for all $\sigma' \in \Sigma'$ and $\sigma \in \varphi^{-1}(\sigma')$, the map $(\sigma')^\vee \cap M' \to \sigma^\vee \cap M'$ is surjective.*

In particular, $\varphi\colon N_\mathbb{R} \to N'_\mathbb{R}$ must be injective. The proof is a tautology: the description of Σ' as $\varphi^{-1}(\Sigma)$ gives you a nice cover by affines.

The next two results are also fairly easy to check.

Proposition 20.2. *The following are equivalent.*

1. *Φ is dominant.*
2. *$N'/\varphi(N)$ is torsion.*
3. *$[N' : \varphi(N)]$ is finite.*

Proposition 20.3. *Φ is always separated.*

The next one, though, is harder: it relies on the valuative criterion.

Proposition 20.4. *Φ is proper iff $\varphi^{-1}(|\Sigma'|) = |\Sigma|$.*

These are the morphisms in the category of toric varieties. We'd like to understand this category better, which means we'll need a more intrinsic definition of a toric variety, one which depends on fewer choices.

Definition 20.5. Let k be a field. A *toric variety* over k is a connected, normal, and separated scheme X of finite type over k, together with an action of an algebraic torus $\mathbb{T}_N \coloneqq \operatorname{Spec} k[N^\times]$ and a $\mathbb{T}_N$-equivariant open embedding $\mathbb{T}_N \hookrightarrow X$.

Given data of a map $N \to N'$, we get an algebraic group homomorphism $\mathbb{T}_N \to \mathbb{T}_{N'}$; then, a *morphism of toric varieties* $X \to X'$ is a map $s\colon N \to N'$ and a map $X \to X'$ which is $\mathbb{T}_N$-equivariant; here, $\mathbb{T}_N$ acts on X' through s.

The adjectives on a toric variety imply that it's reduced and irreducible.

Theorem 20.6. *The functor X from the category of fans to the category of toric varieties is an equivalence of categories.*

We will now discuss an alternate perspective on toric varieties, using polytopes. The equivalence of these two pictures is a baby form of mirror symmetry, albeit baked into the theory of toric varieties.

Definition 20.7. A *polyhedron* is a finite intersection of half-planes inside a finite-dimensional $\mathbb{R}$-vector space V. A *polytope* in V is the convex hull of finitely many points in V.

A priori, polyehdra need not be convex, but we will always take "polyhedron" to mean "convex polyhedron." Polyhedra can be unbounded; polytopes cannot.

Definition 20.8. Suppose that $V = M_\mathbb{R}$, where M is a finite-rank free abelian group. Then a polytope is *rational* if it's the convex hull of a finite set of points in $M_\mathbb{Q} \subset M_\mathbb{R}$, and is *integral* if those points are in $M \subset M_\mathbb{R}$.

A polyhedron is *rational* if the half-spaces in its definition can be defined over $M_\mathbb{Q}$.

Defining integral polyhedra is a little trickier.

Theorem 20.9. *A bounded polyhedron is a polytope, and vice versa.*

Polytopes and polyhedra have faces and *facets* (i.e. codimension-one faces). Polytopes also have *vertices* (zero-dimensional faces); not all polyhedra have vertice s (e.g. $\mathbb{R}^{n-1}$ inside $\mathbb{R}^n$).

Definition 20.10. If $\Xi \subset N_\mathbb{R}$ is a polyhedron, the *cone over* Ξ is

$$C(\Xi) := \overline{\{(m,h) \in M_\mathbb{R} \times \mathbb{R} \mid m \in h \cdot \Xi\}}. \tag{20.11}$$

The *recession cone* or *asymptotic cone* of Ξ is $C(\Xi) \cap (M_\mathbb{R} \times \{0\}) \subset M_\mathbb{R}$.

Remark 20.12. The naïve definition of a cone over a polyhedron would leave out the closure, but this is wrong! It's never closed if Ξ is unbounded. ◀

Example 20.13. Let $\Xi \subset \mathbb{R}^2$ be the intersection of $\{x \geq 0\}$, $\{y \geq 0\}$, and $\{y \geq x\}$. To compute $C(\Xi)$, we rescale Ξ by h; this amounts to updating the last half-plane to $\{y \geq x + h\}$. ◀

Definition 20.14. Let $\Xi \subset M_\mathbb{R}$ be a polyhedron, and assume $0 \in \mathrm{Int}(\Xi)$. The *dual polyhedron* to Ξ iS

$$\Xi^\circ := \{v \in N_\mathbb{R} \mid \langle u, v\rangle \leq 1 \text{ for all } u \in \Xi\}. \tag{20.15}$$

Of course, if Ξ is a polytope, we call Ξ° the *dual polytope.*

Remark 20.16. There are two sign conventions for (20.15); the other one asks to include vectors v with $\langle u, v\rangle \geq -1$ for all $u \in \Xi$. This sign convention has the potential to make things more confusing, though. ◀

Lemma 20.17. $C(\Xi)^\vee = C(\Xi^\circ)$.

Since $C(\Xi)$ knows Ξ, then the fact that duality of cones is involutive implies $\Xi^{\circ\circ} = \Xi$. There is a bijective correspondence between the proper faces of $C(\Xi)$ not contained in $M_\mathbb{R} \times \{0\}$ and the proper faces of Ξ.

We will use polyhedra to build examples of Calabi-Yau varieties, and then to approach mirror symmetry.

Example 20.18. Let Ξ be the (convex hull of the) triangle in $\mathbb{R}^2$ with vertices $(-1,-1)$, $(2,-1)$, and $(-1,2)$; then the dual is the convex hull of $(-1,0)$, $(1,1)$, and $(0,-1)$.

In $\mathbb{R}^3$, the dual of the unit cube (defined as the convex hull of $\{(a_1,a_2,a_3) \mid a_1,a_2,a_3 \in \{\pm 1\}\}$) is the unit octahedron (defined as the span of $\{\pm e_1, \pm e_2, \pm e_3\}$). ◀

Now we use polyhedra to build toric varieties. Let $\Xi \subset M_\mathbb{R}$ be a rational polyhedron. Given a face $\Xi' \subset \Xi$, we get a cone $\sigma_{\Xi'} \subset N_\mathbb{R}$ defined by

$$\sigma_{\Xi'} := \{v \in N_\mathbb{R} \mid \langle u - u', v\rangle \geq 0 \text{ whenever } u \in \Xi, u' \in \Xi'\}. \tag{20.19}$$

This is typically not sharp. We can also write $\sigma_{\Xi'}$ as

$$\sigma_{\Xi'} = \left(\overline{\{u - u' \in M_\mathbb{R} \mid u \in \Xi, u' \in \Xi'\}}\right)^\vee. \tag{20.20}$$

This isn't very interesting in dimension 2, so let Ξ be the first octant in $\mathbb{R}^3$ and Ξ' be the (nonnegative) x-axis in Ξ. Then... TODO not sure what happened.

Sending $\Xi' \mapsto \sigma_{\Xi'}$ exchanges dimension and codimension, so if Ξ' is a facet, we'll just get a line, and if Ξ' is a vertex, we'll get a full cone.

The *normal fan* of Ξ is $\Sigma_\Xi := \{-\sigma_{\Xi'} \mid \Xi' \text{ is a face of } \Xi\}$.

Proposition 20.21. Σ_Ξ *is a fan in* $N_\mathbb{R}$*, complete iff* Ξ *is bounded. If* $0 \in \Xi$,[30] *then*

$$\Sigma_\Xi = \{0\} \cup \{\overline{\mathbb{R}_{\geq 0} \cdot \omega} \mid \omega \text{ a face of } \Xi^0\}. \tag{20.22}$$

If we take Ξ to be the triangle in (20.18), the fan we get is the one generated by the positive pieces of the xy-, xz-, and yz-planes.

Rescaling Ξ doesn't change Σ_Ξ, and in general you should think of small deformations of Ξ as not affecting the normal fan very much.

Definition 20.23. The *toric variety associated to* Ξ is $X(\Sigma_\Xi)$.

For example, with Ξ the triangle as above, we get $\mathbb{P}^2$.

[30] For this part, we might need the additional hypothesis that Ξ is bounded.

Example 20.24. For a more elaborate example, a *Hirzebruch surface* is a surface one of the from $X_a := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a))$, $a \geq 0$. Then $X_a = X(\Sigma_\Xi)$, where Ξ is the convex hull of the points $(0,0)$, $(0,1)$, $(1,1)$, and $(a+1, 0)$. ◀

Lecture 21.

Divisors on toric varieties: 11/7/19

Notation is the same as the previous lecture. In particular, Ξ is a polyedron and Σ_Ξ is its associated fan.

Proposition 21.1. *There is an isomorphism* $\mathrm{Proj}(k[C(\Xi) \cap (M \oplus \mathbb{Z})]) \cong X(\Sigma_\Xi)$.

TODO: what's the grading? Presumably points in $M \oplus \{n\} \subset M \oplus \mathbb{Z}$ have grading n, but I didn't figure it out.

We next ask: do all toric varieties appear in this way? Surprisingly, the answer is no.

Remark 21.2. The map $\Xi \mapsto \Sigma_\Xi$ defines a bijection from the set of rational polyhedra in $M_\mathbb{Q}$ to the set of $(\Sigma, \varphi\colon |\Sigma| \to \mathbb{R})$, where Σ is a fan in $N_\mathbb{R}$, and such that φ is strictly convex and piecewise linear. The inverse map takes a fan and produces its *Newton polyhedron*; this is studied in other settings, such as singularity theory. ◀

If $m \in M$, we can translate Ξ by m, and $\Xi + m \mapsto (\Sigma_\Xi, \varphi + m)$.

Not every fan arises in this way! Figure 7 provides a counterexample: its fan cannot admit a strictly convex, piecewise linear function. This ultimately happens because this triangulation is not *regular*.

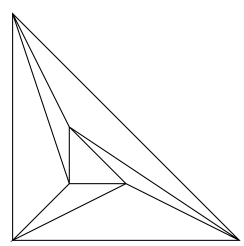

FIGURE 7. The fan over this polyhedron has seven maximal cones, and therefore does not support a convex, piecewise linear function.

TODO: I missed about 20 minutes of class because I wasn't feeling well, including another example.

Next, let's talk about Weil divisors on a toric variety X given by a fan Σ with rays $\mathbb{R}_{\geq 0} v_1, \dots, \mathbb{R}_{\geq 0} v_r$, with $v_i \in N$; these v_i are called *primitives*, and each one determines a codimension-one toric stratum $D_i \subset X$.

Definition 21.3. The *toric prime divisors* are those Weil divisors which are invariant under the torus action. The space of toric prime divisors is denoted $\mathrm{Div}^T(X) \subset \mathrm{Div}(X)$.

The class group $\mathrm{Cl}(X) = A_{d-1}(X)$ is the group of Weil divisors modulo rational equivalence. Let $\varphi\colon \mathrm{Div}^T(X) \to \mathrm{Cl}(X)$ denote restriction of the quotient map to $\mathrm{Div}^T(X) \subset \mathrm{Div}(X)$.

Proposition 21.4. *If* $\psi\colon M \to \mathrm{Div}^T(X)$ *denotes the map sending*

$$m \longmapsto (\chi^m) := \sum_{i=1}^r \mathrm{ord}_{D_i} \chi^m \cdot D_i, \tag{21.5}$$

then the sequence

$$0 \longrightarrow M \xrightarrow{\psi} \mathrm{Div}^T(X) \xrightarrow{\varphi} \mathrm{Cl}(X) \longrightarrow 0 \tag{21.6}$$

is short exact.

Proof. Since $k[M]$ is a unique factorization domain and $\mathbb{T} = \mathrm{Spec}\, k[M]$, $\mathrm{Cl}(X \setminus D) = \mathrm{Cl}(\mathbb{T}) = 0$. Therefore $\mathrm{Div}^T(X)$ surjects onto $\mathrm{Cl}(X)$, and $A_{d-1}(D) = \mathrm{Div}^T(X)$.

Next, suppose $\sum n_i D_i \in \ker(\varphi)$; then, there's an $f \in K(X)^\times$ with $(f) = \sum n_i D_i$, so $f|_{X \setminus D} \in k[M]^\times = k^\times \times M$. Since $k(M) = \Gamma(X \setminus D)$, then $f = \lambda \cdot \chi^m$, where $\lambda \in k^\times$ and $m \in M$ — and without loss of generality, we may assume $\lambda = 1$. ⊠

Note that $\operatorname{ord}_{D_i}\chi^m = \langle v_i, m\rangle$; we use the basis for N with $e_i = v_i$, and obtain a corresponding dual basis $\{e_i^*\}$ for M. Then

$$\mathbb{T}_{D_i} = \operatorname{Spec} k[\chi^{\pm e_1^*}, \dots, \chi^{\pm e_d^*}] \tag{21.7}$$

(TODO: this can't be correct) sits inside the distinguished open

$$U_i = \operatorname{Spec} k[\chi^{\pm e_1^*}, \dots, \chi^{\pm e_d^*}] \tag{21.8}$$

of X.

Corollary 21.9. $\mathrm{Cl}(\mathbb{P}^n) = A_{n-1}(\mathbb{P}^n) \cong \mathbb{Z}\cdot H$*, where* H *is a hyperplane class. The degree map to* $\mathbb{Z}$ *is an isomorphism.*

Proof sketch. Placing the standard toric structure on $\mathbb{P}^n$, the short exact sequence from Proposition 21.4 has the form

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z}^{n+1} \longrightarrow \mathbb{Z} \longrightarrow 0, \tag{21.10}$$

where the first map sends $m \mapsto (\langle v_i, m\rangle)$ and the second map sends $(a_i) \mapsto \sum a_i$.

Recall that this standard toric structure is generated by the rays $\mathbb{R}_{\ge 0} e_i$, $i = 1, \dots, n$, and $\mathbb{R}_{\ge 0}\cdot(-e_1, \dots, -e_n)$. ⊠

Next, we discuss Cartier divisors. Recall that if $\mathcal{K}$ denotes the sheaf of total quotient rings on X, then the group of Cartier divisors is $\mathrm{CaDiv}(X) := \Gamma(X, \mathcal{K}^*/\mathcal{O}_X^*)$. If X is integral, $\mathcal{K}^*/\mathcal{O}_X^*$ is locally constant, and its stalks are all isomorphic to $K(X)$. Inside $\mathrm{CaDiv}(X)$, we have divisors of the form

$$(U_i, f_i) : f_i \in K(X)^\times, \text{ and for all } i \text{ and } j,\ \frac{f_i}{f_j} \in \mathcal{O}^\times(U_i \cap U_j). \tag{21.11}$$

We also have an embedding $\mathrm{CaDiv}(X) \hookrightarrow \mathrm{Div}(X)$, though this is generally not an isomorphism.

Lemma 21.12. *Let* $X = \operatorname{Spec} k[\sigma^\vee \cap M]$ *be an affine toric variety. Then* $D \in \mathrm{Div}^T(X)$ *is Cartier iff there is an* $m \in M$ *with* $D = (\chi^m)$.

The reverse direction follows by definition, but TODO: we didn't get to the forward direction in class today.

Proposition 21.13. *If* Σ *is a fan in* $N_\mathbb{R}$*, then* $\mathrm{CaDiv}(X_\Sigma)$ *is isomorphic to the space of PL functions* $\varphi\colon |\Sigma| \to \mathbb{R}$ *linear with respect to* Σ.

Specifically, the map in the reverse direction sends

$$\varphi \longmapsto \sum_i \varphi(v_i)\cdot D_i. \tag{21.14}$$

Proposition 21.15. *The map* $\mathrm{PL}(\Sigma)/M \to \mathrm{Pic}(X_\Sigma)$ *sending* $m \mapsto \langle m, \cdot\rangle$ *is an isomorphism. This also identifies* $\mathrm{Pic}(X_\Sigma)$ *with the group of Cartier divisors modulo rational equivalence.*

This should follow from Propositions 21.4 and 21.13.

Corollary 21.16. $\mathrm{Pic}(X_\Sigma)$ *is discrete and torsion-free.*

Lecture 22.

Toric degenerations and mirror symmetry: 11/12/19

Our first goal today is to relate ampleness and convexity.

Proposition 22.1. *Let* Σ *be a fan with* $|\Sigma|$ *convex,* $\varphi \in \mathrm{PL}(\Sigma)$*, and* $D := \sum \varphi(v_i) D_i$ *be the corresponding Cartier divisor. Then* $\mathcal{O}_X(D)$ *is ample iff* φ *is strictly convex.*

Proof. In the $\Leftarrow$ direction, let $\Xi \subset M_\mathbb{R}$ be the Newton polyhedron for Σ; then $X := X(\Sigma) = \operatorname{Proj}(k[C(\Xi) \cap M \oplus \mathbb{Z})])$ and $\mathcal{O}_X(D) = \mathcal{O}_X(1)$, which is very ample.

For the $\Rightarrow$ direction, we will use an equivalent characterization of strict convexity, by measuring the "kink" of φ alng a codimension-one cone $\rho \in \Sigma^{[d-1]}$ that is not in $\partial|\Sigma|$. Specifically, $\rho = \sigma_1 \cap \sigma_2$ for σ_1, σ_2 maximal in Σ. Since φ is piecewise linear, there are $m_1, m_2 \in M$ with $\varphi|_{\sigma_i} = m_i$ for $i = 1, 2$.

There is a unique primitive generator $\check{d}_\rho \in M$ of $s^\perp \cong \mathbb{Z}$ such that $\check{d}_\rho|_{\sigma_2} > 0$; then, we deifne the *kink* $\kappa_\rho(\varphi) \in \mathbb{Z}$ by

$$m_2 - m_1 = \kappa_\rho(\varphi) \cdot \check{d}_\rho. \tag{22.2}$$

Now observe that φ is strictly convex iff for all ρ, $\kappa_\rho(\varphi) > 0$. And, to conclude, we determine what $\kappa_\rho(\varphi)$ means within toric geometry: $\kappa_\rho(\varphi) = \deg \mathcal{O}_X(D)|_{X(\Sigma_\rho)}$; D is ample iff this is positive, since $X(\Sigma_\rho) \cong \mathbb{P}^1$. You can check the relation of $\kappa_\rho(\varphi)$ with the degree by explicitly computing what the difference in trivializations between the two charts on $\mathbb{P}^1$ are. ⊠

Next, we discuss toric degenerations and mirror symmetry. The Batyrev mirror construction, built using toric degenerations, will immediately give us about $400,000,000$ mirror pairs.

Definition 22.3. A lattice polytope $\Xi \subset M_\mathbb{R}$ is *reflexive* if

(1) $\mathrm{Int}(\Xi) \cap M = \{0\}$, and
(2) each facet of Ξ has $\mathbb{Z}$-affine distance 1 from 0.

Condition (2) is equivalent to $\mathrm{Int}(\Xi^\circ) \cap M = \{0\}$; the definition is equivalent to both Ξ and Ξ° being lattice polytopes.

Clearly, Ξ is reflexive iff Ξ° is reflexive.

Example 22.4.

(1) If Ξ is the triangle with vertices $(-1, -1)$, $(2, -1)$, and $(-1, 2)$, then its dual is the triangle with vertices $(0, -1)$, $(-1, 0)$, and $(1, 1)$. Thus Ξ and Ξ° are reflexive.
(2) The dual of the square with vertices $(\pm 1, \pm 1)$ is the square with vertices $\pm e_1, \pm e_2$. Thus both are reflexive.
(3) An interesting nonexample is the quadrilateral with vertices $(0, 2)$, $(1, 0)$, $(-1, 0)$, and $(-1, -1)$. Its dual intersects $M = \mathbb{Z}^2$ in a line. ◀

In dimension 2, there are exactly 16 reflexive polygons. Curiously, they don't come in dual pairs — one is self-dual up to a rotation. The point is that there are only finitely many reflexive polytopes in each dimension. The precise number isn't known in general, though we saw it's 16 in dimension 2; higher dimensions use a computer search. In dimension 3, there are 319, and in dimension 4, there are $473, 800, 776$. This is the dimension that will give us Calabi-Yau threefolds and mirror pairs.[31] There's a big database of the Hodge numbers of these Calabi-Yaus.

Definition 22.5. A *Fano variety* is a variety X with at worst *Gorenstein singularities* (i.e. K_X is Cartier, or equivalently, X has a dualizing line bundle) and such that $-K_X$ is ample.

The second condition guarantees Fano varieties are projective.

Proposition 22.6. *Let X be a complete toric variety. Then X is Fano iff X is torically equivalent to $X(\Sigma_\Xi)$, where Ξ is reflexive.*

Proof. Let $\sigma \subset N_\mathbb{R}$. Then $U := \operatorname{Spec} k[\sigma^\vee \cap M]$ has canonical divisor $K_U = -\sum D_i$, where $D_i \subset U$ are the toric prime divisors.

Remark 22.7. It is both remarkable and useful that the holomorphic volume form on $\mathbb{G}_m^d \subset U$ given by

$$\frac{1}{z_1}\,\mathrm{d}z_1 \wedge \cdots \wedge \frac{1}{z_d}\,\mathrm{d}z_d \tag{22.8}$$

has simple poles along *any* toric divisor. ◀

Returning to the proof, K_U is Cartier iff there is some $m_\sigma \in M$ such that for all vertices v_i of σ, $\langle m_\sigma, v_i \rangle = 1$. Now translate into the language of polytopes: if Ξ specifies X and $-K_X$, then Ξ is the convex hull of $\{m_\sigma \mid \sigma \text{ maximal}\}$. ⊠

[31] Ultimately, this is where the claim that there are hundreds of millions of different string theories arises from.

And once we have a smooth Fano variety X, we can easily produce Calabi-Yau hypersurfaces. Assume $\Gamma(X, K_X) \neq 0$ (which is not always true). Then, by Bertini's theorem, for a general section s, the zero locus $Z(s) \subset X$ is smooth. Then the adjunction formula tells you $Z(s)$ is Calabi-Yau:

$$K_{Z(s)} = (-K_X + K_X)|_{Z(s)} = 0. \tag{22.9}$$

This recovers something from the beginning of class: $\mathbb{P}^4$ is a smooth Fano variety, and $-K_{\mathbb{P}^4}$ has nonzero sections. Therefore the zero locus of a general quintic (quintic because $K_{\mathbb{P}^4} = \mathcal{O}(-5)$) is Calabi-Yau. This is nice, except that very few of the Fano varieties we produced from toric geometry are smooth.

To solve this problem, we need to desingularize somehow. Toric geometry makes this (as always) nice and explicit, albeit so long as you're okay with partial desingularizations. Specifically, we will use *maximal projective crepant partial resolutions*, and to save on syllables we'll call these *MPCP resolutions*.

Recall that Σ_Ξ is the fan over the faces of Ξ°.

Definition 22.10. An *MPCP subdivision* Σ of Σ_Ξ is a subdivision of Σ_Ξ such that Σ is simplicial and projective, and such that the set $\Sigma(1)$ of primitive generators of rays of Σ is

$$\Sigma(1) = \Xi^0 \cap (N \setminus 0) = \partial\Xi \cap N. \tag{22.11}$$

The upshot is that Σ is a subdivision of Σ_Ξ such that the induced dual triangulations of the faces of Ξ^0 are both regular and maximal.

Remark 22.12. In dimension 3 and above, MPCP subdivisions need not be unique. And starting in dimension 4, there are lattice simplices containing no other integral points than their vertices, and which are not the standard simplex. The easiest example is the cone over the three-dimensional tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,0,1)$, and $(0,p,q)$, when p and q are coprime. ◀

Proposition 22.13. *Let Σ be an MPCP subdivision of Σ_Ξ. Then*

(1) *X_Σ is a Gorenstein orbifold,*[32] *in particular smooth in codimension* 3*, and*
(2) *Ξ is still the polytope associated to $-K_{X_\Sigma}$, and*
(3) *$-K_{X_\Sigma}$ is semi-ample, i.e. generated by global sections and* big*, meaning $-K_{X_\Sigma}^d > 0$.*

Moreover, $f\colon K_{X_\Sigma} \to K_{X(\Xi)}$ is crepant*, i.e. $K_{X_\Sigma} \cong f^*K_{X(\Xi)}$.*

"Crepant" is a neologism, coined to indicate that there is no discrepancy.

A preview of the construction of mirror pairs: given a reflexive pair Ξ, Ξ°, choose MPCP subdivisions Σ of Σ_Ξ and $\check{\Sigma}$ of Σ_{Ξ°. The mirror pairs will be families, as we expect: $Z(s) \subset X_\Sigma$ parameterized by $s \in \Gamma(X_\Sigma, -K_{X_\Sigma})$ general, and on the other side $Z(\check{s})$ parameterized by $\check{s} \in \Gamma(X_{\check{\Sigma}}, -K_{X_{\check{\Sigma}}})$.

In fact, $\Gamma(X, -K_{X_\Sigma})$ is identified with the space of maps $(a_m)\colon \Xi^0 \cap M \to k$, where

$$(a_m) \longmapsto s := \sum a_m \chi^m \, \mathrm{d}\log z_1 \wedge \cdots \wedge \mathrm{d}\log z_m. \tag{22.14}$$

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[32] Any toric variety is an orbifold, so the interesting fact is that this is Gorenstein.

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