FALL 2017 GOODWILLIE CALCULUS SEMINAR

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These notes were taken in Andrew Blumberg's student seminar in Fall 2017. I live-TeXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

Contents

1. Introduction: 9/13/17

Today, Nicky gave an overview of Goodwillie calculus, following Nick Kuhn's notes.

The setting of Goodwillie calculus is to consider two topologically enriched,[1] based model categories C and D and a functor $F\colon \mathsf{C} \to \mathsf{D}$ between them.

Example 1.1.

(1) Top, the category of topological spaces.
(2) Sp, the category of spectra.
(3) If Y is a topological space, we can also consider ${}_{Y\backslash}\mathsf{Top}_{/Y}$, the category of spaces over and under Y, i.e. the diagrams $Y \to X \to Y$ which compose to the identity. ◀

We want F to satisfy some kind of Mayer-Vietoris property, or excision. Hence, we assume C and D are *proper*, in that the pushout of a weak equivalence along a cofibration is also a weak equivalence. We'll also ask that in D, sequential colimits of homotopy Cartesian cubes are again homotopy Cartesian, and we'll elaborate on what this means.

We also place a condition on F: Goodwillie calls it "continuous," meaning that it's an enriched functor: the induced map

$$\mathrm{Map}_{\mathsf{C}}(X,Y) \longrightarrow \mathrm{Map}_{\mathsf{D}}(F(X),F(Y))$$

is a continuous map between topological spaces (or a morphism of simplicial sets; for the rest of this section, we'll let V denote the choice of Top_* or sSet_* that we made). If $X \in \mathsf{C}$ and $K \in \mathsf{V}$, then we have a tensor-hom adjunction

$$\mathsf{C}(X\otimes K, Y) \cong \mathsf{V}(K, \mathsf{C}(X,Y)).$$

From this, F produces the *assembly map*

$$F(X)\otimes K \longrightarrow F(X\otimes K).$$

We'll also require F to be weakly homotopical in that it sends homotopy equivalences to homotopy equivalences.

The idea of Goodwillie calculus is to approximate F by a tower of functors, akin to Postnikov truncations, $\cdots \to P_2 \to P_1 \to P_1 \to P_0$. The fiber D_i of P_i, akin to the i^{th} Postnikov section, is like the i^{th} term in a

[1] As usual, we can take them to be enriched either over Top or over sSet. This has the important consequence that C and D are tensored and cotensored over Top_*, resp. sSet_*.

Taylor series:

$$\begin{aligned} P_0(X) &\simeq P_0(*) \\ D_1(X) &\simeq D_1(*) \otimes X \\ D_2(X) &\simeq (D_2(X) \otimes X \otimes X)_{h\Sigma_2}, \end{aligned}$$

where Σ_2 acts by switching the two copies of X, and so on. Each P_i will satisfy more and more of the Mayer-Vietoris property. This is akin to the first three terms in a Taylor series for f: $f(a)$, $xf'(a)$, and $x^2 f''(a)/2$.

Weak natural transformations. We'll also need to know what a weak equivalence of functors is. This would allow us to study the homotopy category of $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$.

Definition 1.2. A *weak natural transformation* $F \Rightarrow G\colon \mathsf{C} \to \mathsf{D}$ is one of the two zigzags

$$F \xleftarrow{\sim} H \longrightarrow G \qquad \text{or} \qquad F \longleftarrow H \xrightarrow{\sim} G,$$

where $F \overset{\sim}{\to} G$ means an objectwise weak equivalence.

Commutativity of a diagram of weak natural transformations is computed in $\mathsf{ho}(\mathsf{D})$.[2] You can also form spectra in D in the usual way (inverting suspension, etc).

Diagrams[3]. Let S be a finite set. We'll let $\mathcal{P}(S)$ denote its power set, made into a poset category under inclusion. Similarly, we'll let $\mathcal{P}_0(S) \coloneqq \mathcal{P}(S) \setminus \{\varnothing\}$ and $\mathcal{P}_1(S) \coloneqq \mathcal{P}(S) \setminus \{S\}$, again regarded as poset categories.

Definition 1.3.

(1) A *d-cube* in C is a functor $\mathcal{X}\colon \mathcal{P}(S) \to \mathsf{C}$, where $|S| = d$.
(2) A d-cube $\mathcal{X}$ is *Cartesian* if
$$\mathcal{X}(\varnothing) \xrightarrow{\sim} \operatorname*{holim}_{T \in \mathcal{P}_0(S)} \mathcal{X}(T).$$
(3) A d-cube $\mathcal{X}$ is *co-Cartesian* if
$$\mathcal{X}(S) \xrightarrow{\sim} \operatorname*{hocolim}_{T \in \mathcal{P}_1(S)} \mathcal{X}(T).$$
(4) A d-cube $\mathcal{X}$ is *strongly co-Cartesian* if $\mathcal{X}|_{\mathcal{P}(T)}\colon \mathcal{P}(T) \to \mathsf{C}$ is co-Cartesian for all $T \in \mathcal{P}(S)$ with $|T| \geq 2$.

Example 1.4.

(1) If $d = 0$, a (Cartesian or co-Cartesian) 0-cube is something weakly equivalent to the initial object.
(2) A (Cartesian or co-Cartesian) 1-cube is an equivalence.
(3) A 2-cube is something of the form

$$\begin{tikzcd} \mathrm{fib}_f \arrow[r] \arrow[d] & \mathrm{fib}_g \arrow[d] \\ A \arrow[r] \arrow[d, "f"] & B \arrow[d, "g"] \\ C \arrow[r] & D. \end{tikzcd}$$

We let $\partial\mathcal{X}$ denote the *boundary* of $\mathcal{X}$, the top row; the middle row is $\mathcal{X}_\top$, and the bottom row is $\mathcal{X}_\perp$.[3] In this case, Cartesian and co-Cartesian correspond to (homotopy) pushout and pullback squares, which explains their names in the general case. ◀

There's a way to produce co-Cartesian cubes canonically from a finite set. Let $\phi\colon X^{\amalg T} \to X$ denote the fold map.

[2] There are more worked-out expositions of the homotopy theory of functors, in case this one looks ad hoc, but we don't need the entire background.

[3] These are also written $\mathcal{X}_{\mathrm{top}}$ and $\mathcal{X}_{\mathrm{bottom}}$.

Definition 1.5. Let T be a finite set and $X \in \mathsf{C}$, and let

$$X \star T \coloneqq \operatorname{cofib}\left(\phi\colon \coprod_T X \to X\right).$$

Now, for $T \subset [d]$, the assignment $T \mapsto X \star T$ defines a co-Cartesian $(d+1)$-cube.

For example, when $d = 1$, this is the homotopy pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d] & CX \simeq * \arrow[d] \\ CX \simeq * \arrow[r] & \Sigma X. \end{tikzcd}$$

This formalism allows us to define the analogue of the Mayer-Vietoris principle that we'll need for the Goodwillie tower.

Definition 1.6. An $F\colon \mathsf{C} \to \mathsf{D}$ with F, C, and D as above is *d-excisive* if for all strongly co-Cartesian $(d+1)$-cubes $\mathcal{X}$, $F(\mathcal{X})$ is a Cartesian $(d+1)$-cube in D.

Example 1.7.
(1) 0-excisive functors are homotopy constant.
(2) 1-excisive functors are those that satisfy the Mayer-Vietoris property. In Sp, $\mathrm{Map}_{\mathsf{Sp}}(C, -)$ and L_E are both 1-excisive. $\blacktriangleleft$

There are some nice properties about how d-excisive functors behave with respect to cofibration sequences, etc., and we will return to them in due time. We will now construct Taylor towers.

Fix an $X \in \mathsf{C}$, and let

$$T_d F(X) \coloneqq \operatorname*{holim}_{T \in \mathcal{P}_0([d+1])} F(X \star T).$$

Remark. There is a natural map $t_d F\colon F \to T_d F$, and by definition, this is an equivalence if F is d-excisive. $\blacktriangleleft$

Set $P_d F\colon \mathsf{C} \to \mathsf{D}$ to be the functor sending

$$X \longmapsto \operatorname{hocolim}\left(F(X) \xrightarrow{t_d F} T_d F(X) \xrightarrow{t_d t_d F} T_d T_d F(X) \longrightarrow \cdots \right).$$

For example, if $F(*) \simeq *$, then $T_1 F(X)$ is the homotopy pullback of

$$\begin{tikzcd} & F(CX) \simeq * \arrow[d] \\ * \simeq F(CX) \arrow[r] & F(\Sigma X), \end{tikzcd}$$

and hence is $\Omega F(\Sigma X)$. In this case

$$P_1 F(X) = \operatorname*{hocolim}_{n \to \infty} \Omega^n F \Sigma^n X.$$

For example, if $F = \mathrm{id}$ and $\mathsf{C} = \mathsf{D}$, then $P_1(\mathrm{id}) = \Omega^\infty \Sigma^\infty$, which is cool: the "first derivative" of the identity tells us information of stable homotopy! The calculation of the second derivative will be harder.

2. Interpolating between stable and unstable phenomena: 9/20/17

Today, Adrian gave an overview of what we're going to learn about this semester.

Functors are like functions. We have an analogy between smooth functions and nice functors from Top_* to Top_* or Sp.[4] This analogy sends

- degree-n polynomials to n-excisive functors,
- homogeneous degree-n polynomials to homogeneous n-excisive functors (defined using Cartesian cubes), and
- Taylor series to Taylor towers of functors.

[4]Perhaps more generality is possible, but we'll worry about that later.

In Higher Algebra, Lurie takes the idea that an ∞-category is like a manifold as an anchor for doing a lot of very interesting mathematics, which is one angle for interpreting this analogy.

Let $\mathsf{Homog}_n(\mathsf{C}, \mathsf{D})$ denote the category of homogeneous n-excisive functors $F\colon \mathsf{C} \to \mathsf{D}$, where C and D are categories with the assumptions we placed on them last time.

Theorem 2.1 (Goodwillie, Lurie)**.** *The functor*

$$\Omega^\infty \circ -\colon \mathsf{Homog}_n(\mathsf{Top}_*, \mathsf{Sp}) \longrightarrow \mathsf{Homog}_n(\mathsf{Top}_*, \mathsf{Top})$$

is an equivalence.

Let $\mathsf{Lin}_n(\mathsf{C}, \mathsf{D})$ denote the category of multilinear functors in n variables and FS_{Σ_n} denote the category of *FS-spectra* for Σ_n,[5] the category of spectra together with an action of Σ_n by automorphisms.

Theorem 2.2 (Goodwillie, Lurie)**.** *When $\mathsf{C} = \mathsf{Top}_*$ or Sp, the functors*

$$\mathsf{FS}_{\Sigma_n} \xrightarrow{A} \mathsf{Lin}_n(\mathsf{C}, \mathsf{C}) \xrightarrow{B} \mathsf{Homog}_n(\mathsf{C}, \mathsf{C})$$

are both equivalences, where

- *A sends C_n to the multilinear functor*

$$(X_1, \dots, X_n) \longrightarrow (C_n \wedge X_1 \wedge \cdots \wedge X_n)_{h\Sigma_n},$$

  and
- *$B = - \circ \Delta$, where $\Delta\colon X \mapsto (X, \dots, X)$ is the diagonal.*

So there's not really a difference between these different perspectives.

We'd like to push this analogy further: is it true that n-excisive functors are precisely the things you get by extending $(n-1)$-excisive functors by n-homogeneous excisive functors? Fortunately, this is true, for "nice" n-excisive functors (where "nice" isn't too restrictive).

Another thing about polynomials is that they're uniquely determined by $n+1$ points. There's an analogue for functors. Let $\mathsf{Set}_*^{\leq n+1}$ denote the full subcategory of Set_* consisting of sets with cardinality at most $n+1$ (including the basepoint) and $i\colon \mathsf{Set}_*^{\leq n+1} \hookrightarrow \mathsf{Top}_*$ be the usual inclusion.

Theorem 2.3 (Lurie)**.** *The n-excisive functors $F\colon \mathsf{Top}_* \to \mathsf{Sp}$ are precisely the functors arising as left Kan extension of a functor $\widetilde{F}\colon \mathsf{Set}_*^{\leq n+1} \to \mathsf{Sp}$ along i.*

Interpolating between stable and unstable homotopy theory. Unfortunately, I didn't get everything that happened here, but the idea is to consider the Taylor tower of the identity $\mathsf{Top}_* \to \mathsf{Top}_*$. The first homogeneous piece is $\Omega^\infty\Sigma^\infty$, which somehow says that we see stable information, and after that is $\Omega^\infty(C_2 \wedge X \wedge X)_{\Sigma_2}$ and so on. You can get a spectral sequence out of this.

The Blakers-Massey theorem is another manifestation or maybe explanation of the fact that Goodwillie calculus gets stable phenomena out of unstable ones.

Theorem 2.4 (Blakers-Massey)**.** *Consider a diagram indexed on the unit n-cube (the objects are the vertices, interpreted as a poset category using the dictionary order), and assume the map from the space at $(0, \dots, 0)$ to the space at e_i is k_i-connected. Then, the arrow from the homotopy limit of this diagram to the space at $(0, \dots, 0)$ is $(-1 + n + \sum k_i)$-connected.*

So we don't quite have spectra at any finite level, but if you impose higher and higher excisiveness, you can't have bounded connectivity.

Calculus of embeddings. Let M be a manifold, and consider presheaves of topological spaces on it, i.e. functors $F\colon O(M)^{\mathsf{op}} \to \mathsf{Top}$, where $O(M)$ is the poset category of open sets on M, ordered by inclusion. We restrict to the F such that

- if $U \subset V$ is an isotopy equivalence, then $F(U) \to F(V)$ is a homotopy equivalence, and
- $$F\left(\bigcup_i U_i\right) = \operatorname{holim} F(U_i),$$

  indexed by the inclusion relations among the U_i.

[5] This term is due to C. Wu. You might also hear *doubly naïve Σ_n-spectra* or *spectra with a Σ_n-action.*

Definition 2.5. Such an F is an *n-excisive sheaf* if for any closed subsets $A_1, \dots, A_n \subseteq U$, the homotopy colimit of the "cube" diagram of $U \setminus \mathcal{A}$ for all $\mathcal{A} \subset \{A_1, \dots, A_n\}$ is $F(U)$.

For $n = 1$, this is the same as the usual sheaf condition (which is the strongest condition: the least amount of information is needed to determine it from local information).

3. Two paths to homotopy colimits: 9/27/17

"This was recently alluded to in Derived Memes for Spectral Schemes."

Today, Adrian spoke again, about two ways to think about homotopy colimits.

Recall that a *relative category* is a pair (C, W), where $\mathsf{W} \subseteq \mathsf{C}$ is a subcategory containing all isomorphisms. A *relative functor* between relative categories (C, W) and $(\mathsf{C}', \mathsf{W}')$ is a functor $F\colon \mathsf{C} \to \mathsf{C}'$ such that $F(\mathsf{W}) \subset \mathsf{W}'$. These are the settings for general abstract homotopy theory.

To really talk about homotopy (co)limits, we need ∞-categories. But there are five facts about ∞-categories that might make them easier to digest.

(1) ∞-categories generalize ordinary categories. This is true both as a statement to help with intuition, and as an embedding $\mathsf{Cat} \subset \mathsf{Cat}_\infty$.
(2) Any relative category determines an ∞-category.
(3) Any relative functor determines an ∞-functor.[6]
(4) Let (C, W) be a relative category and $\underline{\mathsf{C}}$ be the ∞-category it determines. Then, there's a canonical functor $L_{\mathsf{C}}\colon \mathsf{C} \to \underline{\mathsf{C}}$.
(5) In nice cases, the set of relative functors from (C, W) to $(\mathsf{C}', \mathsf{W}')$ determines the space of ∞-functors $\underline{\mathsf{C}} \to \underline{\mathsf{C}}'$.

Thus we can also work with relative categories, though with some niceness assumptions present.

Definition 3.1. Let (C, W) be a relative category and J be a small category. The homotopy colimit of a functor $D\colon \mathsf{J} \to \mathsf{C}$ is a presentation of $\varinjlim L_{\mathsf{C}} \circ D$ inside C.

Our running examples will be homotopy pushouts (and dually, homotopy pullbacks as homotopy limits).

Another way to think about this comes from the universal property for colimits: if C^J denotes the functor category, there's an adjunction

$$\mathsf{C}^\mathsf{J} \underset{\Delta}{\overset{\varinjlim}{\rightleftarrows}} \mathsf{C}, \tag{3.2}$$

where $\Delta(X)$ is the constant functor $\mathsf{J} \to \mathsf{C}$ sending all objects to X and all morphisms to id_X. This is true for any category C, but if in addition (C, W) is a relative category, we can formally invert the morphisms in W to define the homotopy category $\mathrm{Ho}(\mathsf{C})$; then, we have a derived version of (3.2):

$$\mathrm{Ho}(\mathsf{C}^\mathsf{J}) \underset{\mathrm{Ho}(\Delta)}{\overset{\mathrm{hocolim}}{\rightleftarrows}} \mathrm{Ho}(\mathsf{C}), \tag{3.3}$$

One simple idea is that it's possible to encode ∞-functors in relative categories, by functors F that *aren't* relative, as long as for every relative equivalence $E\colon \mathsf{D} \simeq \mathsf{C}$, $F \circ E$ is relative.

Definition 3.4. Let (C, W) and $(\mathsf{C}', \mathsf{W}')$ be relative categories, an endofunctor Q of C, and a functor $F\colon \mathsf{C} \to \mathsf{C}'$, a *left deformation* is a natural transformation $Q \Rightarrow \mathrm{id}_{\mathsf{C}}$ such that $F|_{\mathrm{Im}\, Q}$ is relative.

This includes examples such as (co)fibrant replacement, e.g. in the category of complexes of A-modules, let Q be cofibrant replacement (taking a projective resolution), and F tensoring with something which isn't necessarily flat over A. Then, F behaves badly, but not on projectives.

Proposition 3.5. *Given a left deformation Q such that* $\mathrm{Im}(Q) \simeq \mathsf{C}$ *under the natural inclusion, then $F \circ Q$ is automatically relative.*

[6] ∞-functors are the correct notion of functor between ∞-categories; in most situations, these are just called "functors."

It turns out which left deformation you use doesn't really matter, much like for cofibrant replacement: the natural transformation to the identity means that if Q and Q' are left deformations, you have a diagram

$$\begin{tikzcd}
Q'(Q(x)) \arrow[r, "\sim"] \arrow[d, "\wr"] & Q(x) \arrow[d, "\wr"] \\
Q'(x) \arrow[r, "\sim"] & x,
\end{tikzcd}$$

where $\sim$ denotes weak equivalences (i.e. morphisms in W). You can use this to draw a diagram to define the homotopy colimit as a pushout:

$$\begin{tikzcd}
F(\varnothing) \arrow[r] \arrow[d] \arrow[dr, "\sim"] & F(0) \arrow[dr, "\sim"] & \\
F(1) \arrow[dr, "\sim"] & Q(F(\varnothing)) \arrow[r] \arrow[d] & Q(F(0)) \arrow[d] \\
& Q(F(1)) \arrow[r] & \operatorname{hocolim} F. \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

That is, one way to compute the homotopy colimit is to cofibrantly replace, then compute an ordinary limit.

Example 3.6. One concrete model for the (homotopy type of the) homotopy pushout of X_0 and X_1 along maps $f\colon X_\varnothing \to X_0$ and $g\colon\ : X_\varnothing \to X_1$ in topological spaces is a *mapping cylinder* $X_0 \amalg X_\varnothing \times I \amalg X_1/\sim$, where we glue X_0 to $X_\varnothing \times \{0\}$ using f and X_1 to $X_\varnothing \times \{1\}$ using g. $\blacktriangleleft$

Another perspective is that this is the same data as a homotopy coherent data $h_0\colon X_0 \to Z$ and $h_1\colon X_1 \to Z$ (where Z is the mapping cylinder), in that $h_0 \circ f, h_1 \circ g\colon X_\varnothing \rightrightarrows Z$ are homotopic.

One can generalize this to the homotopy colimit over an arbitrary diagram involving a disjoint union indexed over n-simplices for every composition of n morphisms in the diagram, modulo an equivalence relation. The idea is that maps out of this space into Z corresponds exactly to a homotopy coherent diagram indexed by J.

It's possible to reconcile this perspective and the more abstract, categorical one, involving a way to replace homotopy colimits with ordinary colimits.

4. The Blakers-Massey theorem: 10/4/17

Today, Rok spoke on the proof of the Blakers-Massey theorem. All limits (colimits) in today's lecture are homotopy limits (homotopy colimits).

Let's start by recalling some things we already know. Recall that if S is a set, an *S-cube* is a map $\mathcal{X}\colon \mathcal{P}(S) \to S$, where we denote $\mathcal{X}(T) = X_T$. Such a $\mathcal{X}$ is *k-Cartesian* if the natural map

$$X_\varnothing \longrightarrow \underrightarrow{\lim}_{T\neq\varnothing} X_T$$

is k-connected. The dual notion of *k-co-Cartesian* asks for the natural map

$$\underleftarrow{\lim}_{T\subsetneq S} X_T \longrightarrow X_S$$

is k-connected. $\mathcal{X}$ is *strongly (homotopy) co-Cartesian* if all of its faces are co-Cartesian (i.e. k-co-Cartesian for every k).

Lemma 4.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be n-cubes. Then, $f\colon \mathcal{X} \to \mathcal{Y}$ is k-Cartesian as an $(n+1)$-cube iff $\mathcal{F}_y \coloneqq \operatorname{fib}_y(f)$ is a k-Cartesian n-cube for all $y \in Y_\varnothing$.*

By the fiber we mean the homotopy fiber.

Proof. Let $\mathcal{Z}$ be $f\colon \mathcal{X} \to \mathcal{Y}$ interpreted as an $(n+1)$-cube, and $\widetilde{\mathcal{Y}}$ be $\mathrm{id}\colon \mathcal{Y} \to \mathcal{Y}$ interpreted as an $(n+1)$-cube. Therefore we have a diagram

$$\begin{tikzcd} X_\varnothing \arrow[r] \arrow[d] & \varinjlim_{T\neq\varnothing} \mathcal{Z}_T \arrow[d] \\ Y_\varnothing \arrow[r, "\sim"] & \varinjlim_{T\neq\varnothing} \widetilde{\mathcal{Y}}_T. \end{tikzcd}$$

Therefore we obtain a map

$$\mathrm{fib}(X_\varnothing \to Y_\varnothing) \longrightarrow \mathrm{fib}\left(\varinjlim_{T\neq\varnothing} \mathcal{Z}_T \to \varinjlim_{T\neq\varnothing} \widetilde{\mathcal{Y}}_T\right) \simeq \varinjlim_{\substack{T\neq\varnothing \\ T\subseteq[n+1]}} \left(\mathcal{Z}_T - \widetilde{\mathcal{Y}}_T\right). \tag{4.2}$$

But looking at the diagram

$$\begin{tikzcd} \mathrm{fib}(\mathcal{X}\to\mathcal{Y}) \arrow[r] \arrow[d] & \mathcal{X} \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d, "\mathrm{id}"] \\ * \arrow[r] & \mathcal{Y} \arrow[r, "\mathrm{id}"] & \mathcal{Y}, \end{tikzcd}$$

the right-hand side of (4.2) is also weakly equivalent to

$$\varinjlim_{\substack{T\neq\varnothing \\ T\subseteq[n]}} \mathrm{fib}(\mathcal{X}_T - \mathcal{Y}_T),$$

so we're done. ⊠

We can use this to interpret the Blakers-Massey theorem in terms of more familiar results in algebraic topology.

Theorem 4.3 (Blakers-Massey, dimension 2)**.** *Suppose $\mathcal{X}$ is the diagram*

$$\begin{tikzcd} X_\varnothing \arrow[r, "f_2"] \arrow[d, "f_1"] & X_2 \arrow[d] \\ X_1 \arrow[r] & X_{12}, \end{tikzcd} \tag{4.4}$$

and suppose it is co-Cartesian. If each f_i is k_i-connected, then $\mathcal{X}$ is (k_1+k_2-1)-Cartesian.

There's also a dual version. This implies that

$$X_\varnothing \longrightarrow X_1 \times_{X_{12}} X_2$$

is (k_1+k_2-1)-connected.

Corollary 4.5 (Freudenthal suspension theorem)**.** *Suppose X is k-connected. Then, the map $X \to \Omega\Sigma X$ is $(2k-1)$-connected.*

Proof. Consider the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma X. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

The two arrows coming out of X are k-connected, so by Theorem 4.3, the map

$$X \longrightarrow * \times_{\Sigma X} * \simeq \Omega\Sigma X$$

is $(2k-1)$-connected. ⊠

This says that highly connected spaces are close to being stable: taking $\Omega\Sigma$ of a highly connected space doesn't change it within a large range.

Definition 4.6. An *excisive triad* $(X; A, B)$ is three spaces X, A, and B such that $A, B \subset X$, $X = A \cup B$, and $A \cap B$ is a nonempty, connected space.

Corollary 4.7 (Homotopy excision)**.** *Let $(X; A, B)$ be an excisive triad. Suppose that $(A, A \cap B)$ is k-connected and $(B, A \cap B)$ is ℓ-connected. Then, the inclusion map $(A, A \cap B) \to (X, B)$ is $(k + \ell - 1)$-connected.*

Proof. By Lemma 4.1, it suffices to prove that the map $A \cap B \to A \times_X B$ is $(k + \ell - 1)$-connected. Then, by Van Kampen's theorem, the diagram

$$\begin{tikzcd} A \cap B \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & X \end{tikzcd}$$

is co-Cartesian, and the arms are k- and ℓ-connected, so Theorem 4.3 applies and we're done. ⊠

The proof of the general Blakers-Massey theorem is inductive on the dimension, and Theorem 4.3 will be our base case.

Proof of Theorem 4.3. First, let's tackle a special case: we'll show that if e^d denotes a d-dimensional cell, the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X \cup e^{d_2} \arrow[d] \\ X \cup e^{d_1} \arrow[r] & X \cup e^{d_1} \cup e^{d_2} \end{tikzcd}$$

induces a $(d_1 + d_2 - 3)$-connected.

This ultimately depends on a transversality argument, which is where the topology sneakes in. The sketch is that if p is in the interior of e^{d_1} and q is in the interior of e^{d_2}, we want to consider a diagram

$$\begin{tikzcd} Y \setminus \{p, q\} \arrow[r] \arrow[d] & Y \setminus \{p\} \arrow[d] \\ Y \setminus \{q\} \arrow[r] & Y, \end{tikzcd}$$

inducing a map

$$g\colon (D', \partial D') \longrightarrow (Y \setminus p \times_Y Y \setminus q, Y \setminus \{p, q\}).$$

Let

$$G(x, t_1, t_2) := (g(x_1, t_1), g(x_2, t_2)) \in \check{e}^{d_1} \times \check{e}^{d_2}.$$

This is transverse to (p, q) if $i + 2 < d_1 + d_2$, hence $(p, q) \notin \mathrm{Im}(G)$ in this range. (Checking transversality is neither trivial nor terrible.)

Now we'll use this to prove the general theorem (still in dimension 2). By CW approximation, we can replace (4.4) with

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X \cup Y_1 \arrow[d] \\ X \cup Y_2 \arrow[r] & X \cup Y, \end{tikzcd}$$

where Y_1 (resp. Y_2) is the set of cells of dimension greater than k_1 (resp. k_2), and Y is the set of all of the cells. Since we're interested in the attaching map $(D^i, \partial D^i) \to (X \cup Y, X)$, which necessarily only hits finitely many cells, we can assume we're only attaching a finite number of cells.

This means we can induct over the set of cells, attaching them one at a time, and this is the special case we proved above. ⊠

Let's also talk about the general case.

Theorem 4.8 (Blakers-Massey (Goodwillie))**.** *Let $\mathcal{X}$ be a strongly co-Cartesian n-cube, and assume $\mathcal{X}_\varnothing \to \mathcal{X}_{\{i\}}$ is k_i-connected. Then, $\mathcal{X}$ is $(k_1 + \cdots + k_n + 1 - n)$-Cartesian.*

What does this mean geometrically? We have n spaces, and we want to do as many pushouts as we can. There's another, more geometric statement, which is the original one

Theorem 4.9 (Blakers-Whitehead (1953))**.** *Let $\mathfrak{U}$ be a finite open cover of X, and for each $U \in \mathfrak{U}$, let*

$$A_{(U)} \coloneqq \bigcap_{\substack{V \in \mathfrak{U} \\ V \neq U}} U.$$

If the map $A_{(U)} \hookrightarrow A_U$ is k_U-connected, then for $i < 1 - |\mathfrak{U}| + \sum_{U \in \mathfrak{U}} k_U$, $\pi_i(X; A_U \text{ for } U \in \mathfrak{U}) = 0$.

Proof sketch. Let's assume $n = |\mathfrak{U}| = 3$. In this case, we can reduce to a cube of attaching cells as in the proof of Theorem 4.3: we want to prove that

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] \arrow[dd] & & X \cup e_1 \arrow[dr] \arrow[dd] & \\
& X \cup e_2 \arrow[rr] \arrow[dd] & & X \cup e_2 \cup e_1 \arrow[dd] \\
X \cup e_3 \arrow[rr] \arrow[dr] & & X \cup e_1 \cup e_3 \arrow[dr] & \\
& X \cup e_3 \cup e_2 \arrow[rr] & & X \cup e_1 \cup e_2 \cup e_3
\end{tikzcd}$$

is $(d_1 + d_2 + d_3)$-Cartesian (where the attaching map for e_i is d_i-connected). To prove this, one applies Theorem 4.3 to each of the three faces containing the vertex X. This gets you that each face is $(d_1+d_2+d_3-1)$-co-Cartesian, but that's not strong enough — we actually need a stronger version of Theorem 4.3: under the theorem assumptions, if $\mathcal{X}$ j-connected, then it's $\min\{k_1 + k_2 - 1, j - 1\}$-Cartesian. This is not hard to prove, and gets you the $d_1 + d_2 + d_3 - 2$ needed. $\boxtimes$