M392C NOTES: MATHEMATICAL GAUGE THEORY

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These notes were taken in UT Austin's M392C (Mathematical gauge theory) class in Spring 2019, taught by Dan Freed. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Any mistakes in the notes are my own. Thanks to Yixian Wu for finding and fixing a typo.

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Note: I have handwritten notes for the missing lectures, and will try to type them up at some point.

Lecture 1.

Some useful linear algebra: 1/22/19

"Why did the typing stop?"

Today we'll discuss some basic linear algebra which, in addition to being useful on its own, is helpful for studying the self-duality equations. You should think of this as happening pointwise on the tangent space of a smooth manifold.

Let V be a real n-dimensional vector space. The exterior powers of V define more vector spaces: the scalars $\mathbb{R}$, V, $\Lambda^2 V$, and so on, up to $\Lambda^n V = \operatorname{Det} V$. We can also apply this to the dual space, defining $\mathbb{R}$, V^*, $\Lambda^2 V^*$, etc, up to $\Lambda^n V^* = \operatorname{Det} V^*$.

There is a duality pairing

$$\begin{aligned} \theta\colon \Lambda^k V^* \times \Lambda^k V &\longrightarrow \mathbb{R} \\ (v^1 \wedge \cdots \wedge v^k, v_1 \wedge \cdots \wedge v_k) &\longmapsto \det(v^i(v_j))_{i,j}, \end{aligned} \tag{1.1}$$

where $v^i \in V^*$ and $v_j \in V$.

Now fix a $\mu \in \operatorname{Det} V^* \setminus 0$, which we call a *volume form.* Then we get another duality pairing

$$\begin{aligned} \Lambda^k V \times \Lambda^{n-k} V &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto \theta(\mu, x \wedge y). \end{aligned} \tag{1.2}$$

Thus $\Lambda^k V \cong \Lambda^{n-k} V^*$.

Suppose we have additional structure: an inner product and an orientation. Let $e_1, \dots, e_n$ be an oriented, orthonormal basis of V, and $e^1, \dots, e^n$ be the dual basis. Now we can choose $\mu = e^1 \wedge \cdots \wedge e^n$.

Definition 1.3. The *Hodge star operator* is the linear operator $\star\colon \Lambda^k V^* \to \Lambda^{n-k} V^*$ characterized by

$$\alpha \wedge (\star\beta) = \langle \alpha, \beta \rangle_{\Lambda^k V} \cdot \mu. \tag{1.4}$$

The inner product on $\Lambda^k V^*$ is defined by

$$\langle v^1 \wedge \cdots \wedge v^k, w^1 \wedge \cdots \wedge w^k \rangle \coloneqq \det(\langle v^i, w^j \rangle)_{i,j}. \tag{1.5}$$

The Hodge star was named after W.V.D. Hodge, a British mathematician. Notice how we've used both the metric and the orientation – it's possible to work with unoriented vector spaces (and eventually unoriented Riemannian manifolds), but one must keep track of some additional data.

Example 1.6.

- $\star(e^{i_1} \wedge \cdots \wedge e^{i_k}) = e^{j_1} \wedge \cdots \wedge e^{j_{n-k}}$ if the permutation $1, \dots, n$ to $i_1, \dots, i_k, j_1, \dots, j_{n-k}$ of $[n] \coloneqq \{1, \dots, n\}$ is even. Otherwise there's a factor of -1.
- Suppose $n = 4$. Then $\star(e^1 \wedge e^2) = e^3 \wedge e^4$ and $\star(e^1 \wedge e^3) = -e^2 \wedge e^4$, and so on. ◀

Remark 1.7. The Hodge star is natural. First, you can see that we didn't make any choices when defining it, other than an orientation and a volume form, but there's also a functoriality property. Let $T\colon V \to V$ be an automorphism; this induces $(\Lambda^k T^*)^{-1}\colon \Lambda^k T^* \to \Lambda^k T^*$, and if T is an orientation-preserving isometry,

$$\star \circ (\Lambda^k T^*)^{-1} = (\Lambda^{n-k} T^*)^{-1} \circ \star. \tag{1.8}$$

Hence $\star\star\colon \Lambda^k V^* \to \Lambda^k V^*$ is some nonzero scalar multiple of the identity, and we can determine which multiple it is. Certainly we know

$$\star\star(e^1 \wedge \cdots \wedge e^k) = \star(e^{k+1} \wedge \cdots \wedge e^1) = \lambda e^1 \wedge \cdots \wedge e^k, \tag{1.9}$$

and we just have to compute the parity of these permutations: one uses k transpositions, and the other uses $n - k$. Therefore we conclude that

$$\star\star = (-1)^{k(n-k)}\colon \Lambda^k V^* \to \Lambda^k V^*. \tag{1.10}$$

◀

Now suppose $n = 2m$, so we have a middle dimension m, and $\star\star\colon \Lambda^m \to \Lambda^m$ is $(-1)^m$. This induces additional structure on $\Lambda^m V^*$.

- If m is even (so $n \equiv 0 \bmod 4$), the double Hodge star is an endomorphism squaring to 1. This defines a $\mathbb{Z}/2$-grading on $\Lambda^m V^*$, given by the ± 1-eigenspaces, which we'll denote $\Lambda^m_\pm V^*$. The $+1$-eigenspace is called *self-dual* m-forms, and the -1-eigenspace is called the *anti-self-dual* m-forms.
- If m is odd (so $n \equiv 2 \bmod 4$), the double Hodge star squares to -1, so this defines a complex structure on $\Lambda^m V^*$, where i acts by the double Hodge star.

Exercise 1.11. Especially for those interested in physics, work out this linear algebra in indefinite signature (particularly Lorentz). The signs are different, and in Lorentz signature the two bullet points above switch!

Exercise 1.12. Show that if $4 \mid n$, the direct-sum decomposition $\Lambda^m V^* = \Lambda^m_+ V^* \oplus \Lambda^m_- V^*$ is orthogonal. See if you can find the one-line proof that self-dual and anti-self-dual forms are orthogonal.

Next we introduce conformal structures. This allows the sort of geometry which knows angles, but not lengths.

Definition 1.13. A *conformal structure* on a real vector space V is a set C of inner products on V such that any $g_1, g_2 \in C$ are related by $g_1 = \lambda g_2$ for a $\lambda \in \mathbb{R}_+$.

In this setting, one can obtain g_2 from g_1 by pulling back g_1 along the dilation $T_\lambda\colon v \mapsto \lambda v$. This induces an action of $(T_\lambda^*)^{-1}$ on $\Lambda^k V^*$, which is multiplication by λ^{-k}: if μ_i is the volume form induced from g_i, so that

$$\alpha \wedge \star\beta = g_1(\alpha, \beta)\mu_1, \tag{1.14}$$

then

$$\lambda^{-2k}\alpha \wedge \star\beta = g_2(\alpha, \beta)\lambda^{-n}\mu_2. \tag{1.15}$$

Thus pulling back by dilation carries the Hodge star to $\lambda^{n-2k}\star$. Importantly, if $n = 2m$, then $\star\colon \Lambda^m V^* \to \Lambda^m V^*$ is preserved by this dilation, so it only depends on the orientation and the conformal structure.

Remark 1.16. A conformal structure is independent from an orientation. For example, on a one-dimensional vector space, a conformal structure is no information at all (all inner products are multiples of each other), but an orientation is a choice. ◀

Example 1.17. Suppose $n = 2$ and choose an orientation and a conformal structure on V. As we just saw, this is enough to define the Hodge star $\star\colon V^* \to V^*$, which defines a complex structure on V. Pick a square root i of -1 and let $\star$ act by it (there are two choices, acted on by a Galois group).

We get more structure by complexifying: $V^* \otimes \mathbb{C}$ splits as a the $\pm i$-eigenspaces of the Hodge star; we denote the i-eigenspace by $V^{(1,0)}$ (the $(1,0)$-*forms*) and the $-i$-eigenspace by $V^{(0,1)}$ (the $(0,1)$-*forms*).

Now let's globalize this: everything has been completely natural, so given an oriented, conformal 2-manifold X, it picks up a complex structure, hence is a Riemann surface, and the Hodge star is a map $\star\colon \Omega^1_X \to \Omega^1_X$. Moreover, we can do this on the complex differential forms, which split into $(1,0)$-forms and $(0,1)$-forms.

How do 1-forms most naturally appear? They're differentials of functions, so given an $f\colon X \to \mathbb{C}$, we can ask what it means for $\mathrm{d}f \in \Omega^{1,0}_X$. This is the equation

$$\star\,\mathrm{d}f = i\,\mathrm{d}f. \tag{1.18}$$

This is precisely the Cauchy-Riemann equation; its solutions are precisely the holomorphic functions on X. ◀

Remark 1.19. More generally, one can ask about functions to $\mathbb{C}^n$ or even sections of complex vector bundles; the analogue gives you notions of holomorphic sections. In this case, the equations have the notation

$$\overline{\partial} f = \left(\frac{1 + i\star}{2}\right)\mathrm{d}f. \tag{1.20}$$ ◀

We'll spend some time in this class understanding a four-dimensional analogue of all of this structure.

Symmetry groups. Symmetry is a powerful perspective on geometry. If we think about V together with some structure (orientation, metric, conformal structure, some combination,...), we can ask about the symmetries of V preserving this structure. Of course, to know this, we must know V, but we can instead look at a model space $\mathbb{R}^n$ to define a *symmetry type*, and ask about its symmetry group G: then an isomorphism $\mathbb{R}^n \to V$ preserving all of the data we're interested in defines an isomorphism from G to the symmetry group of V.

Example 1.21. When $\dim V = 2$, the most general symmetry group is $\mathrm{GL}_2(\mathbb{R})$, the invertible matrices acting on $\mathbb{R}^2$. Adding more structure we get more options.

- If we restrict to orientation-preserving symmetries, we get $\mathrm{GL}_2^+(\mathbb{R})$.
- If we restrict to symmetries preserving a conformal structure, the group is called $\mathrm{CO}_2 = \mathrm{O}_2 \times \mathbb{R}^{>0}$.
- If we ask to preserve an orientation and a complex structure, we get $\mathrm{CO}_2^+ = \mathrm{SO}_2 \times \mathbb{R}^{>0}$. This is isomorphic to $\mathbb{C}^\times = \mathrm{GL}_1(\mathbb{C})$: an element of $\mathrm{SO}_2 \times \mathbb{R}^{>0}$ is rotation through some angle θ and a positive number r; this is sent to $re^{i\theta} \in \mathbb{C}^\times$.

This provides another perspective on why an orientation and a conformal structure give us a complex structure. ◀

Example 1.22. Now suppose $n = 4$, and choose a conformal structure C and an orientation on V. Then orthogonal makes sense, though orthonormal doesn't, and the Hodge star induces a $\mathbb{Z}/2$-grading on $\Lambda^2 V^* = \Lambda^2_+ V^* \oplus \Lambda^2_- V^*$, the self-dual and anti-self-dual 2-forms. The total space $\Lambda^2 V^*$ is six-dimensional, and these two subspaces are each three-dimensional.

Suppose e^1, e^2, e^3, e^4 is an orthonormal basis for some inner product in C. We can use these to define bases of $\Lambda^2_\pm V^*$, given by

$$\begin{aligned} \alpha_1^\pm &:= e^1 \wedge e^2 \pm e^3 \wedge e^4 \\ \alpha_2^\pm &:= e^1 \wedge e^3 \mp e^2 \wedge e^4 \\ \alpha_3^\pm &:= e^1 \wedge e^4 \pm e^2 \wedge e^3. \end{aligned} \tag{1.23}$$

Now, what symmetry groups do we have? Inside $\mathrm{GL}_4(\mathbb{R})$, preserving an orientation lands in the subgroup $\mathrm{GL}_4^+(\mathbb{R})$; preserving a conformal structure lands in $\mathrm{O}_4 \times \mathbb{R}^{>0}$; and preserving both lands in $\mathrm{SO}_4 \times \mathbb{R}^{>0}$. The first three of these act irreducibly on $\Lambda^2(\mathbb{R}^4)^*$, but the action of $\mathrm{SO}_4 \times \mathbb{R}^{>0}$ has two irreducible summands, $\Lambda^2_\pm(\mathbb{R}^4)^+$.

To understand this better, we should learn a little more about SO_4. Recall that Sp_1 is the Lie group of unit quaternions. This is isomorphic to SU_2, the group of determinant-1 unitary transformations of $\mathbb{C}^2$. This group has an irreducible 3-dimensional representation ρ in which Sp_1 acts by conjugation on the imaginary quaternions (since $\mathbb{R} \subset \mathbb{H}$ is preserved by this action).

Remark 1.24. Another way of describing ρ is: let ρ' denote the action of SU_2 on $\mathbb{C}^2$ by matrix multiplication. Then $\rho \cong \mathrm{Sym}^2 \rho'$. ◀

Proposition 1.25. *There is a double cover* $\mathrm{Sp}_1 \times \mathrm{Sp}_1 \to \mathrm{SO}_4$*. Under this cover, the* SO_4*-representation* $\Lambda^4_\pm(\mathbb{R}^4)^*$ *pulls back to a real three-dimensional representation in which one copy of* Sp_1 *acts by* ρ *and the other acts trivially.*

Proof. Let W' and W'' be two-dimensional Hermitian vector spaces with compatible quaternionic structures J', resp. J''.[1] Then, $V := W' \otimes_{\mathbb{C}} W''$ has a real structure $J' \otimes J''$: two minuses make a plus, and compatibility of J' and J'' means the real points of V have an inner product. (These kinds of linear-algebraic spaces are things you should prove once in your life.)

By tensoring symmetries we obtain a homomorphism $\mathrm{Sp}(W') \times \mathrm{Sp}(W'') \to \mathrm{O}(V)$. This factors through $\mathrm{SO}(V) \hookrightarrow \mathrm{O}(V)$, which you can see for two reasons:

- $\mathrm{Sp}(W')$ and $\mathrm{Sp}(W'')$ are connected, so this homomorphism must factor through the identity component of $\mathrm{O}(V)$, which is $\mathrm{SO}(V)$; or
- a complex vector space has a canonical orientation, and using this we know these symmetries are orientation-preserving.

Now we want to claim this map is two-to-one. One can quickly check that $(-1, -1)$ is in the kernel; the rest is an exercise. ◀

Since Spin_n is the double cover of SO_n, this is telling us $\mathrm{Spin}_4 = \mathrm{Sp}_1 \times \mathrm{Sp}_1$. This splitting is the genesis of a lot of what we'll do in the next several lectures.

Consider the 16-dimensional space

$$V^* \otimes V^* = (W')^* \otimes (W')^* \otimes (W'')^* \otimes (W'')^*. \tag{1.26}$$

Because the map

$$\begin{aligned} \omega' \colon W' \times W' &\longrightarrow \mathbb{C} \\ \xi', \eta' &\longmapsto h'(J'\xi', \eta') \end{aligned} \tag{1.27}$$

is skew-symmetric, it lives in $\Lambda^2(W')^* \subset (W')^* \otimes (W')^*$. In particular, the embedding

$$\mathrm{Sym}^2(W')^* \oplus \mathrm{Sym}^2(W'')^* \hookrightarrow (W')^* \otimes (W')^* \otimes (W'')^* \otimes (W'')^* \tag{1.28}$$

[1] That is, J' is an antilinear endomorphism of W' squaring to -1, and similarly for J''. Compatible means with the Hermitian metric: h is a map $\overline{W} \times W \to \mathbb{C}$ and J is a map $W \to \overline{W}$, and if $\xi, \eta \in W'$, we want

$$h(J'\xi, \overline{J'\eta}) = \overline{h(\xi, \eta)} \qquad \text{and} \qquad h(J\xi, \eta) = -h(J\eta, \xi).$$

is the map sending

$$\alpha, \beta \longmapsto \alpha \otimes w'' + \omega' \otimes \beta. \tag{1.29}$$

Remark 1.30. This story can be interpreted in terms of representations of $\mathrm{Sp}(W') \times \mathrm{Sp}(W'')$. Let $\mathbf{1}$ denote the trivial representation of Sp_1 and $\mathbf{3}$ be the three-dimensional irreducible representation we discussed above. Then (1.26) enhances to

$$V^* \otimes V^* = \mathbf{1}_{\mathrm{Sp}(W')} \otimes \mathbf{3}_{\mathrm{Sp}(W')} \otimes \mathbf{1}_{\mathrm{Sp}(W'')} \otimes \mathbf{3}_{\mathrm{Sp}(W'')}. \tag{1.31}$$

The skew-symmetric part is $\mathbf{3}_{\mathrm{Sp}(W')} \otimes \mathbf{1}_{\mathrm{Sp}(W'')} \oplus \mathbf{1}_{\mathrm{Sp}(W')} \otimes \mathbf{3}_{\mathrm{Sp}(W'')}$, and the "rest" (complement) is symmetric. ◄

The group $\mathrm{Sp}_1 \times \mathrm{Sp}_1 = \mathrm{Spin}_4$ has complex (quaternionic) two-dimensional representations $S^\pm$, the *spin representations*, and $\Lambda^2_\pm V \cong \mathrm{Sym}^2 S^\pm$.

So two-forms have self-dual and anti-self-dual parts, and curvature is a natural source of 2-forms! ◄

Lecture 2.

Fantastic 2-forms and where to find them: 1/24/19

"I've taught this before, so I know it's true."

Last time, we discussed some linear algebra which is a local model for phenomena we will study in differential geometry. For example, we saw that on an oriented even-dimensional vector space with an inner product, the Hodge star defines a self-map of the middle-dimensional part of the exterior algebra, which induces extra structure, such a splitting into self-dual and anti-self-dual pieces in dimensions divisible by 4. This therefore generalizes to a $4k$-dimensional manifold with a metric and an orientation: the space of $2k$-forms splits as an orthogonal direct sum of self-dual and anti-self-dual forms. (We also discussed other examples, such as how 1-forms on an oriented 2-manifold split into holomorphic and antiholomorphic pieces.)

We're particularly interested in the case $k = 1$, where this splitting depends only on a conformal structure, and applies to 2-forms. To study its consequences we'll discuss where one can find 2-forms in differential geometry.

Definition 2.1. A *fiber bundle* is the data of a smooth map $\pi\colon E \to X$ of smooth manifolds if for all $x \in X$ there's an open neighborhood U of x and a diffeomorphism $\varphi\colon U \times \pi^{-1}(x) \to \pi^{-1}(U)$ such that the diagram

$$\begin{tikzcd} U \times \pi^{-1}(x) \arrow[rr, "\varphi"] \arrow[dr, "\mathrm{proj}_1"'] & & \pi^{-1}(U) \arrow[dl, "\pi"] \\ & U & \end{tikzcd} \tag{2.2}$$

commutes. In this case we call X the *base space* and E the *total space*. If there is a manifold F such that in the above definition we can replace $\pi^{-1}(x)$ with F, we call π a *fiber bundle with fiber* F.[2] The map φ is called the *local trivialization*.

Example 2.3. The *trivial bundle* with fiber F is the projection map $X \times F \to X$. ◄

Remark 2.4. Fiber bundles were first defined by Steenrod [Ste51] in the 1940s, albeit in a different-looking way. His key insight was local triviality. There are variants depending on what kind of space you care about: for example, you can replace manifolds with spaces and smooth maps with continuous maps.

Keep in mind that a fiber bundle is data (π) and a condition. Often people say "E is a fiber bundle" when they really mean "π is a fiber bundle"; specifying E doesn't uniquely specify π. ◄

If F has more structure, such as a Lie group, torsor, vector space, algebra, Lie algebra, etc., we ask that $\varphi|_{\pi^{-1}(x)}\colon F \to \pi^{-1}(x)$ preserve this structure. For example, in a fiber bundle whose fibers are vector spaces, we want φ to be linear; in this case we call it a *vector bundle*.

Definition 2.5. If $\pi\colon E \to X$ is a vector bundle, the space of *k-forms valued in E*, denoted $\Omega^k_X(E)$, is the space of C^∞ sections of $\Lambda^k T^*X \otimes E \to X$.

For ordinary differential forms (so when E is a trivial bundle), we have the de Rham differential $\mathrm{d}\colon \Omega^k_X \to \Omega^{k+1}_X$, but we do not have this in general.

[2] Not all fiber bundles have a fiber in this sense, e.g. a fiber bundle with different fibers over different connected components.

Definition 2.6. Let X be a smooth manifold.

(1) A *distribution* on X is the subbundle $E \subset TX$.
(2) A vector field ξ on X *belongs to* E if $\xi_x \in E_x \subset T_xX$ for all X.
(3) A submanifold $Y \subset X$ is an *integral submanifold* for E if for all $y \in Y$, $T_yY = E_y$ inside T_yX.

Do integral submanifolds exist? This is a local question and a global question (the latter about maximal integral submanifolds). In general, the answer is "no," as in the next example.

Example 2.7. Consider a distribution on $\mathbb{A}^3$ with coordinates (x, y, z) given by

$$E_{(x,y,z)} = \operatorname{span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}\right\}. \tag{2.8}$$

There is no integral surface for this distribution. TODO: I missed the argument, sorry. $\blacktriangleleft$

This is the basic example that illustrates curvature. It turns out that the existence of an integral submanifold is determined completely by the (non)vanishing of a tensor.

Definition 2.9. Let $E \subset TX$ be a distribution. The *Frobenius tensor* $\phi_E \colon E \times E \to TX/E$ given by

$$\xi_1, \xi_2 \longmapsto [\xi_1, \xi_2] \bmod E.$$

Let's think about this: the Lie bracket is defined for vector fields, not vectors. So we have to extend ξ_1 and ξ_2 to vector fields (well, sections of E, since they're in E), which is a choice, and then check that what we obtain is independent of this choice. It suffices to know that this is linear over functions: that

$$[f_1\xi_1, f_2\xi_2] \overset{?}{=} f_1f_2[\xi_1, \xi_2]. \tag{2.10}$$

Of course, this is not what the Lie bracket does: it differentiates in both variables, so we have the extra terms $f_1(\xi \cdot f_2)\xi_2$ and $f_2(\xi \cdot f_1)\xi_1$. But both of these are sections of E, so vanish mod E, and therefore we do get a well-defined, skew-symmetric form, a section of $\Lambda^2E^* \otimes TX/E$ – not quite a differential form.

Frobenius did many important things in mathematics, across group theory and representation theory and this theorem, which is about differential equations!

Theorem 2.11 (Frobenius theorem)**.** *An integral submanifold of E exists locally iff $\phi_E = 0$.*

This is a nonlinear ODE. As such, our proof will rely on some facts from a course on ODEs.

Lemma 2.12. *Let X be a smooth manifold, ξ be a vector field on X, and $x \in X$ be a point where ξ doesn't vanish. Then there are local coordinates $x^1, \dots, x^n$ around x such that $\xi = \partial x^1$ in this neighborhood.*

Proof. Let φ_t be the local flow generated by ξ, and choose coordinates $y^1, \dots, y^n$ near x such that $\xi_x = \left.\frac{\partial}{\partial y^1}\right|_x$. Define a map $U \colon \mathbb{R}^n \to X$ by

$$x^1, \dots, x^n \longmapsto \varphi_{x^1}(0, x^2, \dots, x^n). \tag{2.13}$$

The right-hand side is expressed in y-coordinates. Now we need to check this is a coordinate chart, which follows from the inverse function theorem, because the differential of φ is invertible at 0 (in fact, it's the identity). The theorem then follows because x^1 is the time direction for flow along ξ in this coordinate system. $\boxtimes$

Lemma 2.14. *With notation as above, let $\xi_1, \dots, \xi_k$ be vector fields which are linearly independent at x and suc that $[\xi_i, \xi_j] = 0$ for all $1 \le i, j \le k$. Then there exist local coordinates $x^1, \dots, x^n$ such that for $1 \le i \le k$, $\xi_i = \frac{\partial}{\partial x^i}$.*

In fact, the converse is also true, but trivially so: it's the theorem in multivariable calculus that mixed partials commute.

Proof. Let $\varphi_1, \dots, \varphi_k$ be the local flows for $\xi_1, \dots, \xi_k$. Because the pairwise Lie brackets commute, $\varphi_i\varphi_j = \varphi_j\varphi_i$. Since these vector fields are linearly independent at x, we can choose local coordinates $y^1, \dots, y^n$ around x such that $\xi_i|_x = \left.\frac{\partial}{\partial y^i}\right|_x$. Then, as above, define

$$x^1, \dots, x^n \longmapsto (\varphi_1)_{x_1}(\varphi_2)_{x_2}\cdots(\varphi_k)_{x_k}(0, \dots, 0, x^{k+1}, \dots, x^n). \tag{2.15}$$

You can check that $\mathrm{d}\varphi$ is invertible, so this is a change of coordinates, and then, using the fact that the flows commute, you can see that the lemma follows. $\boxtimes$

These lemmas are important theorems in their own right.

Proof of Theorem 2.11. Since the theorem statement is local, we can work in affine space $\mathbb{A}^n$. Let $\pi\colon \mathbb{A}^n \to \mathbb{A}^k$ be an affine surjection such that $\mathrm{d}\pi_0$ restricts to an isomorphism $E_0 \to \mathbb{R}^k$. Restrict to a neighborhood U of 0 in $\mathbb{A}^n$ such that $\mathrm{d}\pi_p|_{E_p}\colon E_p \to \mathbb{R}^n$ is an isomorphism for all $p \in U$, and choose $\xi_i|_p \in E_p$ such that $\mathrm{d}\pi_p(\xi_p) = \frac{\partial}{\partial y^i}$. Then, $[\xi_i, \xi_j] = 0$: we know it's in E, and

$$\mathrm{d}\pi[\xi_i, \xi_j] = [\mathrm{d}\pi(\xi_i), \mathrm{d}\pi(\xi_j)] = \left[\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right] = 0. \tag{2.16}$$

Now apply Lemma 2.14; then $\{y^{k+1} = \cdots = y^n = 0\}$ gives the desired integral submanifold. ⊠

The idea of the theorem is that it's a local normal form for an involutive distribution (one whose Frobenius tensor vanishes): locally it looks like the splitting of $\mathbb{R}^n$ into the first k coordinates and the last $(n-k)$ coordinates. And in that local model, we know what the integral manifolds are.

Consider a fiber bundle with a discrete fiber (i.e. the inverse image of every point has the discrete topology). This is also known as a *covering space*. On a "nearby fiber," whatever that means (without more data, we don't have a metric on the base space), we have some sort of parallel transport. The precise statement is that there's a neighborhood of any x on the base space such that any path in that neighborhood lifts to a path on the total space, unique if you specify a point in the fiber. More generally, you can lift families of paths, which illustrates a homotopy-theoretic generalization of a fiber bundle called a *fibration*. But globally, given an element of $\pi_1(X)$, it might lift to a nontrivial automorphism of the fiber.

We'd like to do this for more general fiber bundles $\pi\colon E \to X$, in which case we'll need more data. The kernel of $\mathrm{d}\pi$ is a distribution, and consists of the "vertical" vectors (projection down to X kills them). A complement is "horizontal".

Without any choice, we get a short exact sequence at every $e \in E$:

$$0 \longrightarrow \ker(\mathrm{d}\pi_e) \longrightarrow T_eE \xrightarrow{\mathrm{d}\pi_e} T_xX \longrightarrow 0, \tag{2.17}$$

and a splitting is exactly the choice of a complement $H_e\colon T_xX \to T_eE$. We would like to do this over the whole base, which motivates the next definition.

Definition 2.18. Let $\pi\colon E \to X$ be a fiber bundle. A *horizontal distribution* is a subbundle $H \subset TE$ transverse to $\ker(\mathrm{d}\pi)$, or equivalently a section of the (surjective) map $TE \to \pi^*TX$ of vector bundles on E.

We must address existence and uniqueness. At e the space of splittings is an affine space modeled on $\mathrm{Hom}(T_xX, \ker(\mathrm{d}\pi_e))$, because TODOsomething with a short exact sequence.

Therefore existence and uniqueness of a horizontal distribution is a question about existence and uniqueness of a section of an affine bundle over X. Using partitions of unity, we can construct many of these: existence is good, but uniqueness fails.

What about path lifting? Suppose $\gamma\colon [0,1] \to X$ is a path in X beginning at x_0 and terminating at x_1. We can pull back both E and H by γ, to obtain a rank-1 distribution γ^*H in γ^*TE, and the projection map to $T[0,1]$ is a fiberwise isomorphism. Therefore given a vector at $x_0 = \gamma(0)$ we get a unique horizontal lift along $[0,1]$ to a vector field, and therefore get a unique integral curve above γ.

Note that you cannot always lift higher-dimensional submanifolds, and again the obstruction is the Frobenius tensor, because that's the obstruction to the existence of an integral submanifold. In this context the Frobenius tensor is called *curvature* – right now it's on the total space, but in some settings we can descend it to the base.

Lecture 3.

Principal bundles, associated bundles, and the curvature 2-form: 1/29/19

"For whatever reason I'm being a little impressionistic..."

Last time, we discussed a way in which 2-forms appear in geometry: as the obstruction to integrability of a distribution $E \subset TX$. That is, a distribution contains vectors, and we can ask whether integral curves of those vectors have tangent vectors contained within E. Associated to E we defined a Frobenius tensor $\phi_E\colon \Lambda^2 E \to TX/E$ sending

$$\xi_1, \xi_2 \longmapsto [\widetilde{\xi_1}, \widetilde{\xi_2}] \bmod E, \tag{3.1}$$

where $\widetilde{\xi_i}$ is a vector field extending ξ (and we showed this doesn't depend on the choice of extension). In Theorem 2.11, we saw that ϕ_E is exactly the local obstruction to integrability; we can then move to global questions.

More generally, suppose that $\pi\colon E \to X$ is a fiber bundle. Then TE fits into a short exact sequence (2.17), and we can ask for a horizontal lift from TX to TE, which is a section H of (2.17). Then, given a vector $e \in T_xX$ and a path $\gamma\colon [0,1] \to X$ with $\gamma(0) = x$, we can pull back[3] π and H to obtain a distribution in γ^*E. The Frobenius tensor vanishes, because $[0,1]$ is one-dimensional, so we can extend to an integral curve and therefore parallel-transport along γ. However, if we choose different paths in a ball, there's no guarantee that parallel transport along nearby paths agree at all; the Frobenius tensor may still be nonzero on X.

Steenrod's elegant perspective on fiber bundles (see his book [Ste51]) considered in the spirit of Felix Klein symmetry groups associated to fiber bundles. This leads to the definition of a *principal G-bundle* as a fiber bundle of right G-torsors.

Definition 3.2. Let G be a Lie group and recall that a *right G-torsor* is a smooth manifold T and a smooth right G-action on T such that the action map $T \times G \to T \times T$ sending $(t, g) \mapsto (t, t \cdot g)$ is an isomorphism.

Example 3.3. The prime example of a torsor is to let V be a real vector space; then, the manifold $\mathcal{B}(V)$ of bases of V is a $\mathrm{GL}_n(\mathbb{R})$-torsor: $\mathrm{GL}_n(\mathbb{R})$ acts by precomposition. This also works over $\mathbb{C}$ and $\mathbb{H}$. ◂

Example 3.4. Now let X be a smooth manifold. Our first example of a principal bundle spreads Example 3.3 over X: let $\mathcal{B}(X)$ be the smooth manifold[4] of pairs (x, b) where $x \in X$ and b is a basis of T_xX, i.e. an isomorphism $b\colon \mathbb{R}^n \xrightarrow{\cong} T_xX$. There's a natural forgetful map $\pi\colon \mathcal{B}(X) \to X$ sending $(x, b) \mapsto x$.

This fiber bundle is a principal $\mathrm{GL}_n(\mathbb{R})$-bundle: given $g \in \mathrm{GL}_n(\mathbb{R})$ and a basis $b\colon \mathbb{R}^n \to T_xX$, we let $b \cdot g := b \circ g\colon \mathbb{R}^n \to T_xX$, using the standard action of $\mathrm{GL}_n(\mathbb{R})$ on $\mathbb{R}^n$. This is called the *frame bundle* of X. ◂

This principal bundle controls a lot of the geometry of X, via its associated fiber bundles.

Definition 3.5. Let $\pi\colon P \to X$ be a principal G-bundle and F be a smooth (left) G-manifold. The *associated fiber bundle* with fiber F is the quotient $P \times_G F := (P \times F)/G$, which is a fiber bundle over X with fiber F. Here, G acts on $P \times F$ on the right by $(p, f) \cdot g = (p \cdot g, g^{-1} \cdot f)$.

One has to check this is a fiber bundle, and in particular that the total space is a smooth manifold. Since G acts freely on P, it acts freely on $P \times F$, but for G noncompact there's more to say.

Example 3.6. $\mathrm{GL}_n(\mathbb{R})$ acts linearly on $\mathbb{R}^n$. Because $\mathbb{R}^n$ carries the additional structure of a vector space, the associated bundle $\mathcal{B}(X) \times_{\mathrm{GL}_n(\mathbb{R})} \mathbb{R}^n$ has additional structure: it's a vector bundle. In general, additional structure on F manifests in additional structure on $P \times_G F$.

Anyways, what vector bundle do we get? (Can you guess?) An element of the fiber of $\mathcal{B}(X) \times_{\mathrm{GL}_n(\mathbb{R})} \mathbb{R}^n$ is an equivalence class of an element $v \in \mathbb{R}^n$ and a basis $p\colon \mathbb{R}^n \to T_xX$; let $\xi := p(v) \in T_xX$. Another representative of this equivalence class are represented by $g^{-1}v$ and $p \circ g$ for some $g \in \mathrm{GL}_n(\mathbb{R})$, so this pair defines the same tangent vector ξ. Therefore we recover the tangent bundle. ◂

In general, a principal bundle is telling you some internal coordinates. You know these coordinates up to some symmetry G, and the principal bundle tracks that: you have to make a choice to get coordinates, and it tells you how different choices are related.

[3] In general, we can form the pullback of $[0,1] \to X \leftarrow E$ in the category of sets or spaces, but we want to put a smooth manifold structure on it. We can do it when these two maps are transverse – and since $\pi\colon E \to X$ is a submersion, this is always satisfied.

[4] You have to put a smooth manifold structure on this set! The way to do this is the only tool we have right now: work in an atlas $\mathfrak{U}$ of X which trivializes TX, do this locally, and check that the transition maps are smooth. This will also show that the map $\pi\colon \mathcal{B}(X) \to X$ is a fiber bundle.

We want to show local triviality of a principal G-bundle $\pi\colon P \to X$, which will follow from local triviality as a fiber bundle. Consider a local section $s\colon U \to \pi^{-1}(U)$, where $U \subset X$; we would like to exhibit an isomorphism of fiber bundles $U \times G \to \pi^{-1}(U)$ over U. The map is exactly

$$x, g \longmapsto s(x) \cdot g. \tag{3.7}$$

This exhibits $U \times G \cong \pi^{-1}(U)$ as principal G-bundles, so we have local trivialization. Then in every associated bundle to P, we also obtain local triviality, hence local coordinates. For example, if the bundle of frames is trivialized over U, we get local coordinates (i.e. a local trivialization of TU).

Definition 3.8. A *connection* on a principal G-bundle $\pi\colon P \to X$ is a G-invariant horizontal distribution.

Specifically, given $g \in G$, we have the right action map $R_g\colon P \to P$, and can therefore define $H_{p\cdot g} := (R_g)_*(H_p)$ for a distribution H.

In this setting, the Frobenius tensor is going to do something nice: it's a map

$$\phi_H\colon H \wedge H \to TP/H \cong \ker(\pi_*), \tag{3.9}$$

so given two horizontal vectors, we get a vertical vector. Since H is G-invariant, the Frobenius tensor is also G-invariant, so we ought to be able to descend it to the base: there's only one piece of information on each fiber. That is, given vectors ξ_1, ξ_2 on X, we can lift them to P and compute the Frobenius tensor there, and G-invariance means it doesn't matter how we lift.

If $\mathfrak g$ is the Lie algebra of the Lie group G, we have an isomorphism $\underline{\mathfrak g} \xrightarrow{\cong} \ker(\pi_*)$ as vector bundles on P. Specifically, let $\xi \in \mathfrak g$, and consider the exponential map $\exp\colon \mathfrak g \to G$. Given $p \in P$ with $\pi(p) = x$, we get a curve in P given by $t \mapsto p \cdot \exp(t\xi)$ sending $0 \mapsto p$, and this curve is contained entirely within P_x. Therefore its tangent vector at p is in $\ker(\pi_*)$.

So the Frobenius tensor is a map $\phi_H\colon H \wedge H \to \underline{\mathfrak g}$. Now let's descend to the base. We'd like to claim that what we get in $\mathfrak g$ is invariant, but that's just not true: if $g \in G$, the action of g on $p \cdot \exp(t\xi)$ is not the same as $p \cdot g \cdot \exp(t\xi)$: the issue is that $g\exp(t\xi)$ and $\exp(t\xi)g$ may not agree. This will make it slightly more interesting to descend to the base.

First, extend ϕ_H to a map

$$\widetilde\phi_H\colon TP \wedge TP \longrightarrow \underline{\mathfrak g} \tag{3.10}$$

by projecting $p_H\colon TP \twoheadrightarrow H$, which has kernel $\ker(\pi_*)$. That is, $\widetilde\phi_H(\eta_1 \wedge \eta_2) := p_H\eta_1 \wedge p_H\eta_2$. Thus $\widetilde\phi_H \in \Omega^2_P(\mathfrak g)$.

Lemma 3.11. *Let $g \in G$. Then in $\Omega^2_P(\mathfrak g)$, $R_g^*\widetilde\phi_H = \mathrm{Ad}_{g^{-1}}\widetilde\phi_H$.*

So once we choose a basis for $\mathfrak g$, we can think of elements of $\Omega^2_P(\mathfrak g)$ as matrix-valued differential forms.[5] The proof of Lemma 3.11 comes from the observation above that to get from $p \cdot g \cdot \exp(t\xi)$ to

$$p \cdot \exp(t\xi)g = p \cdot g \cdot (g^{-1}\exp(t\xi)g) = p \cdot g \cdot \mathrm{Ad}_{g^{-1}}(\xi). \tag{3.12}$$

So this is exactly an example of an associated bundle to P, where the G-manifold F is $\mathfrak g$ with the adjoint G-bundle. So associated to P is the *adjoint bundle* $\mathfrak g_P \to X$ defined as $P \times_G \mathfrak g$. This is a vector bundle, in fact a bundle of Lie algebras because the adjoint action preserves the Lie bracket.

A section of $\mathfrak g_P$ is a function upstairs valued in $\mathfrak g$, which is exactly what $\widetilde\phi_H$ is.

Corollary 3.13. *$\widetilde\phi_H$ descends to a 2-form $-\Omega_H \in \Omega^2_X(\mathfrak g_P)$.*

In this case Ω_H is called the *curvature* of H. In particular, if X is a 4-manifold with a conformal structure, we can ask for this to be self-dual or anti-self-dual.

In the short exact sequence

$$0 \longrightarrow \underline{\mathfrak g} \longrightarrow TP \xrightarrow{\pi_*} \pi^*TX \longrightarrow 0, \tag{3.14}$$

a section $H\colon \pi^*TX \to TP$ is equivalent to a section $\Theta\colon TP \to \underline{\mathfrak g}$, i.e. a form $\Theta \in \Omega^1_P(\mathfrak g)$. This is called the *connection form*, and $H = \ker(\Theta)$. It has to satisfy some properties.

- Θ must be G-invariant: $R_g^*\Theta = \mathrm{Ad}_{g^{-1}}\Theta$. This is a linear equation inside the infinite-dimensional vector space $\Omega^1_P(\mathfrak g)$.

[5]Here I suppose we need to use a Lie group G that admits a faithful finite-dimensional representation, but all compact Lie groups, and most noncompact Lie groups that you'll encounter, have this property.

- The other constraint is affine: $\Theta|_{\text{vertical}} = \mathrm{id}$.

So the space $\mathcal{A}_P$ of one-forms Θ satisfying these conditions is affine. This is the space of connections, and in particular tells us that there are lots of connections.

We can also interpret the Frobenius tensor in terms of Θ. Let ζ_1 and ζ_2 be horizontal vectors, and extend them to vector fields $\widetilde{\zeta}_1$ and $\widetilde{\zeta}$. Then $\zeta \cdot \Theta(\widetilde{\zeta}_{1-i}) = 0$, so

$$\begin{aligned} \mathrm{d}\Theta(\zeta_1,\zeta_2) &= \zeta_1\Theta(\widetilde{\zeta}_2) - \zeta_2\Theta(\widetilde{\zeta}_1) - \Theta([\widetilde{\zeta}_1,\widetilde{\zeta}_2]) \\ &= -\Theta([\widetilde{\zeta}_1,\widetilde{\zeta}_2]) = -\phi_H(\zeta_1,\zeta_2). \end{aligned} \tag{3.15}$$

Thus we have proved

Proposition 3.16. $\pi^*\Omega_H = -\widetilde{\phi}_H = \mathrm{d}\Theta + (1/2)[\Theta\wedge\Theta]$.

The notation $[\Theta\wedge\Theta]$ means: $\Theta\wedge\Theta \in \Omega^2_P(\mathfrak{g}\otimes\mathfrak{g})$, and this has a Lie bracket map $[\cdot]\colon \Omega^2_P(\mathfrak{g}\otimes\mathfrak{g}) \to \Omega^2_P(\mathfrak{g})$.

Corollary 3.17 (Bianchi identity)**.** $\mathrm{d}\Omega + [\Theta\wedge\Omega] = 0$.

Proof.

$$\begin{aligned} \mathrm{d}\Omega_H &= [\mathrm{d}\Theta\wedge\Theta] \\ &= \left[\Omega - \frac{1}{2}[\Theta\wedge\Theta]\wedge\Theta\right] \\ &= [\Omega\wedge\Theta] \end{aligned} \tag{3.18}$$

by the Jacobi identity. ⊠

This has been more theory than examples of principal bundles, but we will see plenty of examples when we delve into gauge theory.

Now given a principal G-bundle $\pi\colon P\to X$ with a connection, and any associated bundle F_P with fiber F, we get a horizontal distribution. There's a hands-on way to construct this, or you could think of it in terms of path lifting: given an $x\in X$ and a lift $p\in P$, the connection lifts a path $\gamma\colon[0,1]\to X$ based at x to a path $\widetilde{\gamma}\colon[0,1]\to P$ based at p, so given an $f\in F$, we can define the path $\widehat{\gamma}\colon[0,1]\to F_P$ by $t\mapsto(\widetilde{\gamma}(t),f)$.

Suppose V is a G-representation, so its associated vector bundle $V_P\to X$ is a vector bundle. Then the horizontal distribution we obtain on V_P is tangent to the zero section of V_P. Let $\psi\colon X\to V_P$ be a section and $\xi\in T_xX$; we would like to differentiate ψ in the direction ξ. If ψ were valued in a fixed vector space, we could do this as usual: extend ξ to a curve $\gamma\colon(-\varepsilon,\varepsilon)\to M$, and then define

$$\nabla_\xi\psi \coloneqq \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\psi(\gamma(t)). \tag{3.19}$$

This is precisely the directional derivative. In V_P, the fibers are different vector spaces, which seems like a problem except that the connection on P defines parallel transport τ_t along γ for the fibers of V_P, and therefore we can define the directional derivative of ψ as

$$\nabla_\xi\psi \coloneqq \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\tau_{-t}\psi(\gamma(t)). \tag{3.20}$$

This is called the *covariant derivative.*

Exercise 3.21. Show that this satisfies the Leibniz rule: if f is a function on X, then

$$\nabla_\xi(f\cdot\psi) = (\xi\cdot f)\psi + f(x)\nabla_\xi\psi. \tag{3.22}$$

In other words, the existence of the horizontal distribution is somehow telling us about the Leibniz rule, though this is a somewhat mysterious fact.

Lecture 4.

Harmonic forms and (anti)-self-dual connections: 1/31/19

"The key to humor is. . . . . timing!"

Last time, we discussed connections on principal bundles, and what they induce on associated vector bundles. We also briefly saw the covariant derivative associated to a connection. We begin with more on covariant derivatives.

Definition 4.1. Let $E \to X$ be a vector bundle. A *covariant derivative* is a linear map $\nabla\colon \Omega^0_X(E) \to \Omega^1_X(E)$ satisfying the *Leibniz rule*

$$\nabla(fs) = \mathrm{d}f \cdot s + f\nabla s, \tag{4.2}$$

where f is a smooth function on X and s is a smooth section of E.

If E is a trivial bundle with constant fiber V, the usual directional derivative is a covariant derivative, but there can be others.

We can extend ∇ to a sequence of first-order differential operators

$$0 \longrightarrow \Omega^0_X(E) \xrightarrow{\mathrm{d}_\nabla} \Omega^1_X(E) \xrightarrow{\mathrm{d}_\nabla} \Omega^2_X(E) \xrightarrow{\mathrm{d}_\nabla} \cdots \tag{4.3}$$

defined by

$$\mathrm{d}_\nabla(\omega \cdot s) \coloneqq \mathrm{d}\omega \cdot s + (-1)^k \omega \wedge \nabla s, \tag{4.4}$$

where $\omega \in \Omega^k_X$ and $s \in \Omega^0_X(E)$. Thus the first map $\mathrm{d}_\nabla\colon \Omega^0_X(E) \to \Omega^1_X(E)$ is just ∇.

Exercise 4.5. Show that $\mathrm{d}^2_\nabla(fs) = f\mathrm{d}^2_\nabla(s)$.

In other words, this says the *symbol* of d^2_∇ vanishes; this second-order operator is really a first-order operator. Therefore there exists an $F_\nabla \in \Omega^2_X(\operatorname{End} E)$, called the *curvature*, such that $\mathrm{d}^2_\nabla(s) = F_\nabla \cdot s$.

Digression 4.6. We recall what the symbol of an operator is. Let $E, F \to X$ be vector bundles and $D\colon \Omega^0_X(E) \to \Omega^0_X(F)$ be a differential operator. By definition, D is first-order if for every function f and section s,

$$D(fs) = \sigma(\mathrm{d}f)s + fDs \tag{4.7}$$

for some $\sigma\colon T^*X \otimes E \to F$, which is called the *symbol* of D. ◀

Exercise 4.8. Compute d^3_∇. (Answer: it's zero.)

Now we have two notions of curvature: the curvature associated to a covariant derivative as above, and the curvature associated to a principal bundle with connection and an associated vector bundle.

Exercise 4.9. Let G be a Lie group, $\pi\colon P \to X$ be a principal G-bundle with connection $\Theta \in \Omega^1_P(\mathfrak{g})$, and $\rho\colon G \to \mathrm{Aut}(\mathbb{E})$ be a linear representation of G. Let $E \coloneqq \mathbb{E}_P = P \times_G \mathbb{E} \to X$ be the associated bundle, which carries a covariant derivative $\nabla\colon \Omega^0_X(E) \to \Omega^1_X(E)$. Compute d^2_∇ in terms of $\Omega = \mathrm{d}\Theta + (1/2)[\Theta \wedge \Theta]$.

Example 4.10. Let's think about connections on a principal $\mathbb{T}$-bundle.[6] Consider $\mathbb{C}^2$ with coordinates z^0, z^1 and metric

$$\langle (z^0, z^1), (w^0, w^1)\rangle \coloneqq \overline{z^0}w^0 + \overline{z^1}w^1. \tag{4.11}$$

The circle group $\mathbb{T}$ acts on $S^3 \subset \mathbb{C}^2$ on the right by $(z^0, z^1) \cdot \lambda \coloneqq (z^0\lambda, z^1\lambda)$. This is a free action, so its quotient is a smooth manifold, specifically $\mathbb{CP}^1 \cong S^2$, the manifold of complex lines through the origin in $\mathbb{C}^2$. Thus we obtain a principal $\mathbb{T}$-bundle $\pi\colon S^3 \to \mathbb{CP}^1$, called the[7] *Hopf bundle*.

Now let's put a connection on π. We want a horizontal distribution on the total space S^3. Inside $T_{(z^0,z^1)}S^3$, there's a one-dimensional subspace of vectors in the direction of the fiber $\{(z^0, z^1) \cdot \lambda\}$. The standard Riemannian metric on $\mathbb{C}^2 = \mathbb{R}^4$ allows us to choose a complementary line at each point, which is a horizontal distribution. Because $\mathbb{T}$ acts by isometries, this is an invariant distribution, hence a connection.

This is all pretty and geometric, but we need to compute the connection form $\Theta \in \Omega^1_{S^3}(i\mathbb{R})$ (the Lie algebra of $\mathbb{T}$ is a line with trivial bracket, and is more canonically $i\mathbb{R}$). Specifically,

$$\Theta = \operatorname{Im}\Big(\overline{z^0}\,\mathrm{d}z^0 + \overline{z^1}\,\mathrm{d}z^1\Big). \tag{4.12}$$

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[6] Here $\mathbb{T} \subset \mathbb{C}^\times$ is the group of unit-magnitude complex numbers, sometimes also denoted U_1 or S^1.

[7] Well, there's more than one Hopf bundle, and we'll see some others later, but this is the first example.

In the vertical direction, $\Theta = \mathrm{id}$, $(z^0e^{it}, z^1e^{it}) = (iz^0, iz^1)$. Looking inside the complexified tangent bundle (a four-dimensional complex vector bundle), which has basis $\{\partial_{z^0}, \partial_{\overline{z^0}}, \partial_{z^1}, \partial_{\overline{z^1}}\}$, we get

$$iz^0\frac{\partial}{\partial z^0} - i\overline{z^0}\frac{\partial}{\partial\overline{z^0}} + iz^1\frac{\partial}{\partial z^1} - i\overline{z^1}\frac{\partial}{\partial\overline{z^1}}. \tag{4.13}$$

So on vertical vectors, this is the identity. One (you) can check that on a vector normal to S^3, this vanishes – this is just linear algebra over the complex numbers, so nothing too intimidating.

Next we'd like to see

$$\Omega = \mathrm{d}\Theta = \mathrm{Im}\Big(\overline{\mathrm{d}z^0}\wedge \mathrm{d}z^0 + \overline{\mathrm{d}z^1}\wedge\mathrm{d}z^1\Big), \tag{4.14}$$

though this is already imaginary, so we can remove the 'Im' in front. You can check this descends to $\mathbb{CP}^1$. It's a 2-form on $\mathbb{C}^2$, visibly of type $(1,1)$, and we restrict it to S^3; the claim is that there's a form on $\mathbb{CP}^1$ whose pullback by π is $\Omega|_{S^3}$. This involves verifying two things: that Ω is $\mathbb{T}$-invariant, and that it's trivial in the vertical direction. This is a good practice computation.

Let Ω also denote the form on $\mathbb{CP}^1$: $\Omega\in\Omega^2_{\mathbb{CP}^1}(i\mathbb{R})$. We claim

$$\int_{\mathbb{CP}^1}\frac{1}{2\pi}i\Omega = 1. \tag{4.15}$$

To compute this, we need some coordinates on $\mathbb{CP}^1$. We'll construct a section s of π over $\mathbb{CP}^1\setminus\infty\cong\mathbb{C}$. Specifically, given $z\in\mathbb{C}$, which we think of as $[z:1]\in\mathbb{CP}^1$, let

$$s(z) = \frac{(z,1)}{\sqrt{1+|z|^2}}. \tag{4.16}$$

The term in the denominator means that the function decays at infinity in $\mathbb{C}$, so we expect this integral to converge. (But you should still do it!) ◄

Consider a more general principal $\mathbb{T}$-bundle $\pi\colon P\to X$, where X is a smooth manifold. Is it a pullback of the Hopf bundle by a map $X\to\mathbb{CP}^1$? This need not be true, but something weaker is. Consider the generalized Hopf bundle $S^{2N+1}\to\mathbb{CP}^N$, defined in the same way as the Hopf bundle.

Theorem 4.17. *Every principal $\mathbb{T}$-bundle P over a smooth manifold X arises as a pullback of a Hopf bundle $S^{2N+1}\to\mathbb{CP}^N$ for some N.*

We can choose N independent of P, but it will depend on X. So in general you can think of pulling back from $\mathbb{CP}^\infty$.

Proof sketch. A pullback is a $\mathbb{T}$-equivariant map $\varphi\colon P\to S^{2N+1}$; the quotient by $\mathbb{T}$ defines a map $X\to\mathbb{CP}^N$ satisfying the theorem. But this is equivalent data to a section of the associated bundle $S^{2N+1}_P\to X$. This is good: there are tools in topology for constructing sections. First, using an approximation theorem, one shows that it suffices to find a continuous section. Then, one uses obstruction theory: choose a CW structure on X and a q-cell $D\to X$. We'd like to extend a section over this cell; since D is contractible, it's equivalent to ask that the map $S^{q-1}=\partial D\to S^{2N+1}_P$ is trivial (up to homotopy). This is a question about homotopy groups, and for N large enough, the relevant homotopy group vanishes. ⊠

So the next question is: can we construct universal connections Θ^{univ} on these Hopf bundles such that every connection arises as a pullback? This is finickier. Supposing it exists, and $\varphi\colon(P,\Theta)\to(S^{2N+1},\Theta^{\mathrm{univ}})$, then since connections form an affine space, there's an $\alpha\in\Omega^1_X(i\mathbb{R})$ such that

$$\varphi^*\Theta^{\mathrm{univ}} - \Theta = \pi^*\alpha, \tag{4.18}$$

and hence

$$\overline{\varphi}^*\Omega^{\mathrm{univ}} - \Omega = \mathrm{d}\alpha. \tag{4.19}$$

This therefore implies $\mathrm{d}\Omega = 0$, where $\Omega\in\Omega^2_X(i\mathbb{R})$, so it has a de Rham cohomology class $[i\Omega/2\pi]\in H^2_{\mathrm{dR}}(X)$. This is the pullback of a class $(c_1)_{\mathbb{R}}\in H^2_{\mathrm{dR}}(\mathbb{CP}^N)$. We can see this class explicitly; $\mathbb{CP}^N$ has a very simple CW structure with one cell in each even dimension. Therefore the cochain complex for CW cohomology with $\mathbb{Z}$ coefficients looks like $\mathbb{Z}\to 0\to\mathbb{Z}\to 0\to\mathbb{Z}\to\cdots$, and we claim c_1 is the generator[8] of $H^2(\mathbb{CP}^N;\mathbb{Z})$. Then

[8] We need to pick a sign, but this is determined by the canonical orientation of $\mathbb{CP}^N$ coming from the complex structure.

there's an argument for why these two agree, namely just calculate on $\mathbb{CP}^N$, and this is the beginning of Chern-Weil theory, relating curvature and characteristic classes.

Remark 4.20. There's a similar story for higher Chern classes, but it's sufficiently complicated enough that it's generally easier to calculate using the splitting principle to split a vector bundle as a direct sum of line bundles. ◀

Let's come back to 4-manifolds and self-duality: we let X be an oriented 4-manifold with a conformal structure $[g]$. This is enough to define the Hodge star $\star\colon \Omega^2_X \to \Omega^2_X$, which squares to the identity. Tensoring with a vector bundle allows us to define $\star\colon \Omega^2_X(E) \to \Omega^2_X(E)$ for any vector bundle $E \to X$, which also squares to the identity; therefore we can also define self-dual and anti-self-dual forms valued in E in the same way.

Definition 4.21. Let $P \to X$ be a principal G-bundle with connection Θ and $\Omega \in \Omega^2_X(\mathfrak{g}_P)$ be the associated connection form. We say Θ is *self-dual* (resp. *anti-self-dual*) if $\star\Omega = \Omega$ (resp. $\star\Omega = -\Omega$).

As we discussed in the first lecture, this is the four-dimensional analogue of a two-dimensional question on oriented, conformal surfaces: whether a function (form, ...) is holomorphic or antiholomorphic. The sign isn't all that intrinsic: changing the orientation on X changes it.

Anti-self-dual connections are of interest to physicists, since the 1970s, beginning with work of Polyakov and others looking at flat space. Uhlenbeck produced a condition guaranteeing that solutions to $\star\Omega = -\Omega$ extend over S^4, and later Atiyah, Bott, Hitchin, and Singer claimed there are more solutions, and used algebraic geometry to produce them. We will study more of this story in this class, but first some examples.

The simplest case is $G = \mathbb{T}$. Often this is called "the" abelian case, though there are certainly other abelian Lie groups, such as $\mathbb{T}^2$. Anyways, in this case Ω lives in $\Omega^2_X(i\mathbb{R})$, $\mathrm{d}\Omega = 0$, and if $\star\Omega = \pm\Omega$, then $\mathrm{d}\star\Omega = 0$ iff $\mathrm{d}^*\Omega = 0$. Together these imply that Ω is a harmonic form if X is closed.

Digression 4.22. Let M be a Riemannian manifold (though for just dimension 4, we're only going to need the conformal class of the metric.) For example, we could take $M = \mathbb{E}^n$, which denotes $\mathbb{R}^n$ with the standard Riemannian metric. Then the *Laplacian* is

$$\Delta \coloneqq -\left(\frac{\partial^2}{\partial(x^1)^2} + \cdots + \frac{\partial^2}{\partial(x^n)^2}\right). \tag{4.23}$$

Why the minus sign? This has a discrete spectrum, and we'd like it to be nonnegative rather than nonpositive.

The de Rham derivative has the form

$$\mathrm{d} = \varepsilon(\mathrm{d}x^i)\frac{\partial}{\partial x^i}, \tag{4.24}$$

where ε denotes exterior multiplication (which is its symbol). Using the metric, the formal adjoint is

$$\mathrm{d}^* = -\iota(\mathrm{d}x^i)\frac{\partial}{\partial x^i}. \tag{4.25}$$

(whose symbol is $-\iota$; here ι is interior multiplication). Then you can check that $\Delta \coloneqq \mathrm{dd}^* + \mathrm{d}^*\mathrm{d}$.

Now we can bring this to any Riemannian manifold M: we know what d is, and can define d^* by integrating by parts to construct the formal adjoint of d, or construct it locally. But, for the same reason that interior multiplication requires a metric, d^* depends on the metric. And therefore we can define the Laplacian Δ on M to be $\mathrm{dd}^* + \mathrm{d}^*\mathrm{d}$. This means the analogue of (4.23) on M in local coordinates $(x^1, \ldots, x^n)$ is

$$\Delta = -\sum_{1\le i\le j\le n} -g^{ij}\frac{\partial^2}{\partial x^i \partial x^j}. \tag{4.26}$$

Here $g_{ij} \coloneqq \langle \partial_i, \partial_j\rangle$, and g^{ij} is the (components of the) inverse to the matrix $(g_{ij})_{i,j}$. If you haven't seen this before, it's good to work it out.

Now suppose M is closed and $\Delta\omega = 0$. Then

$$\begin{aligned}0 &= \mathrm{dd}^*\omega + \mathrm{d}^*\mathrm{d}\omega\\ &= \langle \mathrm{dd}^*\omega, \omega\rangle + \langle \mathrm{d}^*\mathrm{d}\omega, \omega\rangle\\ &= \int_M (\langle \mathrm{dd}^*\omega, \omega\rangle + \langle \mathrm{d}^*\mathrm{d}\omega, \omega\rangle)\,\mathrm{dvol}\\ &= \int_M (\langle \mathrm{d}^*\omega, \mathrm{d}^*\omega\rangle + \langle \mathrm{d}\omega, \mathrm{d}\omega\rangle)\,\mathrm{dvol}.\end{aligned}$$

In fact, the converse is true.

Theorem 4.27. *On a closed Riemannian manifold,* $\mathrm{d}\omega = 0$ *and* $\mathrm{d}^*\omega = 0$ *iff* $\Delta\omega = 0$.

Such a form ω is called *harmonic*. The space of harmonic k-forms is denoted $\mathcal{H}^k_M(g) \subset \Omega^k_M$. Elliptic theory shows this is finite-dimensional,[9] and in fact more is true.

Theorem 4.28 (Hodge decomposition)**.** *There is a splitting*

$$\Omega^k_M \cong \underbrace{\mathcal{H}^k_M(g) \oplus \mathrm{Im}(\mathrm{d})}_{\text{closed}} \oplus \mathrm{Im}(\mathrm{d}^*).$$

Since harmonic forms are closed, there's a projection $\mathcal{H}^k_M(g) \to H^k_{\mathrm{dR}}(M)$, and in fact this is an isomorphism! So every cohomology class has a unique harmonic representative. ◂

And now back to 4-manifolds. If X is an oriented Riemannian 4-manifold, we have $\mathcal{H}^2_X(g) \cong H^2(X;\mathbb{R})$, and $\mathcal{H}^2_X(g)$ has two distinguished subspaces: the self-dual forms $\mathcal{H}^2_+(g)$ and the anti-self-dual forms $\mathcal{H}^2_-(g)$. These are distinct subspaces, so every harmonic 2-form ω decomposes as a sum $\omega = \omega_+ + \omega_-$, where $\omega_\pm \in \mathcal{H}^2_\pm(g)$. Explicitly,

$$\omega_\pm = \frac{\omega \pm \star\omega}{2}. \tag{4.29}$$

All of this depended on the metric, so we can ask how this changes as the metric moves, which involves some Sard-Smale theory, as we discussed in Morse theory last semester. But Chern-Weil theory tells us that if Ω comes from a connection on a principal $\mathbb{T}$-bundle, then $i\Omega/2\pi$ defines an integer-valued cohomology class. Therefore self-dual or anti-self-dual connections are the intersection of an integer lattice in H^2 with two lines $\mathcal{H}^2_\pm(g)$. Generically, this has no solutions, unless one of $\mathcal{H}^2_\pm(g)$ is zero (so all forms are self-dual, or are anti-self-dual). Perhaps that's a little disappointing.

To study this, we'll look at the intersection form, a symmetric bilinear 2-form on $H^2(X;\mathbb{Z})$ sending $c_1, c_2 \mapsto \langle c_1 \smile c_2, [X]\rangle$. Let b^2_+ (resp. b^2_-) denote the dimension of the largest subspace on which this form is positive (resp. negative). Then $b^2_+ + b^2_- = b^2(M)$, and their difference is the sigature. We'll put conditions on $b^2_\pm$ which make it possible to find (anti)-self-dual connections.

Example 4.30.

1. On S^4, $b^2 = 0$, so $b^2_\pm = 0$. So no self-dual forms here.
2. On $\mathbb{CP}^2$, $b^2_+ = 1$ and $b^2_- = 0$. In this case, self-dual forms exist! Hooray.
3. But on a K3 surface, $b^2_- = 19$ and $b^2_+ = 3$, so no self-dual forms generically. ◂

This is a little annoying. Maybe we should work with a different Lie group.

The next simplest example is $\mathrm{SU}_2 = \mathrm{Sp}_1$. Associated to it is another Hopf bundle: Sp_1 acts on $S^7 \subset \mathbb{H}^2$, as (right) multiplication by unit quaternions, and the quotient is $\mathbb{HP}^1 \cong S^4$. We can use this to follow the same story as above, defining a connection geometrically and so on.

Lecture 5.

The Yang-Mills functional: 2/5/19

Let X be an oriented, conformal 4-manifold, $P \to X$ be a principal G-bundle, and $\Theta \in \Omega^1_P(\mathfrak{g})$ be a connection. We will study gauge theory in this situation; sometimes we will use a Riemannian metric in the conformal class for X.

[9] We'll use some elliptic theory later this semester, and will therefore go over some of the ingredients that you'd use to prove this.

In gauge theory, people typically use slightly different notation.

- The connection Θ is usually denoted A.
- Its curvature is denoted $F = F_A = \mathrm{d}A + (1/2)[A \wedge A] \in \Omega^2_P(\mathfrak{g})$ – but F_A is also used to denote the curvature form on X, $F_A \in \Omega^2_X(\mathfrak{g}_P)$.

Definition 5.1. We say that A is *self-dual*, resp. *anti-self-dual*, if $\star F_A = F_A$, resp. $\star F_A = -F_A$.

These are first-order nonlinear PDEs in A. Let's say something about where they come from.

Hodge theory and minimization. Let (M, g) be a closed, oriented Riemannian n-manifold.[10] Given a cohomology class $c \in H^k(M; \mathbb{R})$, what's the "best" differential form representative for c? That is, what's the "best" $\omega \in \Omega^k_M$ with $[\omega] = c$?

Well, what does "best" mean? Maybe smallest-norm: let's ask for an ω which minimizes

$$f\colon \omega \longmapsto \int_M \|\omega\|^2 \,\mathrm{dvol}_g = \int_M \omega \wedge \star\omega \tag{5.2}$$

such that $[\omega] = c$.

Remark 5.3. If M is not oriented, then we don't have a volume form, and $\omega \wedge \star\omega$ is a density. Asking to minimize this norm still makes sense. ◀

Fix an ω_0 such that $[\omega_0] = c$. On the affine line $\omega_0 + \mathrm{d}\Omega^{k-1}_M$, consider the function

$$f(\omega_0 + \mathrm{d}\eta) = \int_M (\omega_0 + \mathrm{d}\eta) \wedge \star(\omega_0 + \mathrm{d}\eta). \tag{5.4}$$

This is a quadratic function on a real affine line. We know what those look like – parabolas. So we can find the unique minimum where the derivative of f is zero. The derivative is

$$\mathrm{d}f_{\omega_0}(\mathrm{d}\eta) = 2\int_M \mathrm{d}\eta \wedge \star\omega_0 = \pm 2\int_M \eta \wedge \mathrm{d}\star\omega_0, \tag{5.5}$$

using Stokes' theorem, since M is closed. The output equations are

$$\begin{aligned} \mathrm{d}\star\omega_0 &= 0 \\ \mathrm{d}\omega_0 &= 0. \end{aligned} \tag{5.6}$$

These are the Euler-Lagrange equations for this problem.[11] They're satisfied iff $\Delta\omega_0 = 0$, where $\Delta \coloneqq \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$ is the *Laplacian*. Solutions to $\Delta\omega_0 = 0$ are called *harmonic* forms.

Lemma 5.7. *On a closed manifold M, ω is harmonic iff $\mathrm{d}\omega = 0$ and $\mathrm{d}\star\omega = 0$.*

Proof sketch. Suppose ω is harmonic. Then

$$\begin{aligned} 0 &= \int_M \Delta\omega \wedge \star\omega &(5.8)\\ &= \int_M (\mathrm{d}\mathrm{d}^*\omega \wedge \star\omega + \mathrm{d}\mathrm{d}^*\omega \wedge \star\omega) &(5.9)\\ &= \int_M \left(\|\mathrm{d}^*\omega\|^2 + \|\mathrm{d}\omega\|^2\right) \mathrm{dvol}_g. &(5.10) \end{aligned}$$

Since this is the integral of a nonnegative function, that function must be 0 everywhere, so we conclude that $\mathrm{d}^*\omega = 0$ and $\mathrm{d}\omega = 0$. To get from d^* to $\mathrm{d}\star$, use the fact that the formal adjoint of d, namely d^*, is also $\pm\star\mathrm{d}\star$, where the sign depends on the degree of ω and the dimension of M.

The other direction is up to you. ⊠

Now let's apply this to connections on 4-manifolds. Let $\mathcal{A}_P$ denote the affine space of connections on the principal G-bundle $P \to X$.

[10] One can generalize to open manifolds, but then one needs some vanishing or growth conditions at infinity, or a boundary condition. We're not going to worry about this in this motivational section.

[11] Well, this is a little silly in this setting, since all we did is take a derivative. But in general they're more involved.

Definition 5.11. The *Yang-Mills functional* $Y\colon \mathcal{A}_P \to \mathbb{R}$ is

$$Y(A) := \int_X \|F_A\|^2 \,\mathrm{dvol}_g. \tag{5.12}$$

This looks nice and all that, but we haven't yet defined everything: we need to make sense of the norm on $\Omega^2_P(\mathfrak{g})$. We'll come back to this.

Example 5.13. Suppose $G = \mathbb{T}$, so $\mathfrak{g} = i\mathbb{R}$. Then $F_A \in \Omega^2_X(i\mathbb{R})$, and we know how to take the norm of these kinds of differential forms: the Yang-Mills functional is

$$Y(A) = -\int_X F_A \wedge \star F_A. \tag{5.14}$$

We can decompose F_A into its self-dual and anti-self-dual pieces $F_A^\pm$: $F_A = F_A^+ + F_A^-$, and then $\star F_A = F_A^+ - F_A^-$. Thus we can rewrite the Yang-Mills functional as[12]

$$\begin{aligned}
Y(A) &= \int_X \underbrace{F_A^- \wedge F_A^-}_{\ge 0} - \underbrace{F_A^+ \wedge F_A^+}_{\le 0} && (5.15)\\
&\ge \int_X F_A^- \wedge F_A^- + F_A^+ \wedge F_A^+ && (5.16)\\
&= \int_X F_A \wedge F_A && (5.17)\\
&= 4\pi^2 \langle c_1(P)^2, [X]\rangle. && (5.18)
\end{aligned}$$

Here $c_1(P) = (i/2\pi)[F_A] \in H^2_{\mathrm{dR}}(M)$ is the first (well only) Chern class of the principal $\mathbb{T}$-bundle P. So we obtain a lower bound on the Yang-Mills functional in terms of characteristic classes. Equality is achieved exactly when $F_A^+ \wedge F_A^+ = 0$, i.e. A is anti-self-dual.

Next, what are the critical points of Y? The differential is

$$\mathrm{d}Y_A(\dot{A}) = -2\int_X \dot{A} \wedge \mathrm{d}{\star F_A}, \tag{5.19}$$

where $A \in \mathcal{A}_P$ and $\dot{A} \in \Omega^1_X(i\mathbb{R})$ (a variation of A), so the critical points are those such that $\mathrm{d}{\star F_A} = 0$. The Bianchi identity says that, in addition, $\mathrm{d}F_A = 0$, so the critical points have harmonic curvature forms. These two PDEs are second-order (curvature is a derivative, and then we take one more derivative), but linear.

Definition 5.20. The *Yang-Mills equations* are $\mathrm{d}{\star F_A} = 0$ and $\mathrm{d}F_A = 0$.

Suppose $b_2^+(M) > 0$. Then for generic g, the minimum of Y is *not* realized: we're trying to intersect a line and a lattice. But we can get arbitrarily close, producing a sequence of connections approaching the minimum. ◀

What changes for G more general? First we need a G-invariant inner product $\langle -,-\rangle_{\mathfrak{g}}$ on $\mathfrak{g}$; this will induce an inner product $\langle -,-\rangle_{\mathfrak{g}_P}$ on $\Omega^2_X(\mathfrak{g}_P)$: we think of $\omega, \eta \mapsto \omega\wedge\eta \in \Omega^4_X(\mathfrak{g}\otimes\mathfrak{g})$, then use $\langle -,-\rangle_{\mathfrak{g}}$ to get from $\mathfrak{g}\otimes\mathfrak{g}$ to $\mathbb{R}$, and then integrate to get a number.

G acts on $\mathfrak{g}$ by the adjoint action, so we want something invariant under this action. Such a form always exists: you can take for example the Killing form. Then the Yang-Mills functional is

$$Y(A) = \int_X \langle F_A \wedge \star F_A\rangle_{\mathfrak{g}_P}, \tag{5.21}$$

[12] If the signs look weird, keep in mind F_A is imaginary. But there may yet be sign errors.

and the above calculation generalizes to

$$Y(A) = \int_X \underbrace{\langle F_A^- \wedge F_A^- \rangle_{\mathfrak{g}_P}}_{\geq 0} - \underbrace{\langle F_A^+ \wedge F_A^+ \rangle_{\mathfrak{g}_P}}_{\leq 0} \tag{5.22}$$

$$\geq \int_X \langle F_A^- \wedge F_A^- \rangle_{\mathfrak{g}_P} + \langle F_A^+ \wedge F_A^+ \rangle_{\mathfrak{g}_P} \tag{5.23}$$

$$= \int_X \langle F_A \wedge F_A \rangle_{\mathfrak{g}_P} \tag{5.24}$$

$$= 4\pi^2 \langle c(P), [X] \rangle, \tag{5.25}$$

where $c(P)$ is some degree-4 characteristic class for principal G-bundles. In particular, this is constant, so we once again obtain a lower bound.

Exercise 5.26. Show that

$$\mathrm{d}Y_A(\dot{A}) = -2 \int_X \langle \dot{A} \wedge \mathrm{d}_A \wedge \star F_A \rangle_{\mathfrak{g}_P},$$

where $\dot{A} \in \Omega^1_X(\mathfrak{g})$ is a variation of A.

Proof. First, let's differentiate the curvature operator $F\colon \mathcal{A}_P \to \Omega^2_X(\mathfrak{g}_P)$.

$$\mathrm{d}F_A(\dot{A}) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F(A + t\dot{A}) \tag{5.27}$$

$$= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{d}(A + t\dot{A}) + \frac{1}{2}[(A + t\dot{A}) \wedge (A + t\dot{A})] \tag{5.28}$$

$$= \mathrm{d}\dot{A} + [A \wedge \dot{A}] = \mathrm{d}_A A. \tag{5.29}$$

We'll use this to differentiate the Yang-Mills functional, starting with (5.21). Then

$$\mathrm{d}Y_A(\dot{A}) = 2 \int_X \langle \mathrm{d}_A \dot{A} \wedge \star F_A \rangle \tag{5.30}$$

$$= -2 \int_X \langle \dot{A} \wedge \mathrm{d}_A \star F_A \rangle, \tag{5.31}$$

using the Leibniz rule

$$\mathrm{d}\langle \omega \wedge \eta \rangle = \langle \mathrm{d}_A \omega \wedge \eta \rangle \pm \langle \omega \wedge \mathrm{d}_A \eta \rangle \tag{5.32}$$

where $\omega, \eta \in \Omega^*_X(\mathfrak{g}_P)$. $\boxtimes$

Finding the critical points of the Yang-Mills functional is now a nonlinear second-order PDE. This is part of a very general story, including work of many people. One doesn't have to restrict to dimension 4; for example, Atiyah and Bott studied the Yang-Mills functional on Riemann surfaces and the topology of (certain equivalence classes of) the space of connections and its relationship with the algebraic geometry of the Riemann surface.

For the rest of today's lecture, we'll discuss some more classical geometric preliminaries, beginning with some cool facts about the conformal group. Let W be a finite-dimensional real vector space,[13] and suppose $\langle -,- \rangle\colon W \times W \to \mathbb{R}$ is a nondegenerate, symmetric bilinear form. We aren't assuming it's positive definite, e.g. it could be Lorentz. Inside W, we have the *null cone* N_W of vectors $\xi \in W$ with $\langle \xi, \xi \rangle = 0$. This is a cone because if $\xi \in N_W$, then $t\xi \in N_W$ too. If the form is definite, the null cone is just $\{0\}$.

Let Q_W denote the image of $N_W \setminus 0$ under the quotient $W \setminus 0 \twoheadrightarrow \mathbb{P}W$ by $\mathbb{R}^\times$. Then Q_W is a compact real manifold, a real quadric, and carries a natural conformal structure. Suppose $\ell \in Q_W$, meaning it's a line in the null cone. Then $T_\ell Q_W \subset T_\ell \mathbb{P}W = \mathrm{Hom}(\ell, W/\ell)$, and under this identification, $T_\ell Q_W = \mathrm{Hom}(\ell, \ell^\perp/\ell)$: since ℓ is null, $\ell \subset \ell^\perp$, and in fact $\ell^\perp$ is a codimension-one subspace of W: it's cut out by one equation.

The bilinear form $\langle -,- \rangle$ descends to $\ell^\perp/\ell \times \ell^\perp/\ell \to \mathbb{R}$, and therefore defines a conformal structure on $\mathrm{Hom}(\ell, \ell^\perp/\ell) \cong \ell^\perp/\ell \otimes \ell^*$. This obviously varies smoothly in ℓ, hence defines a conformal structure on Q_W.

Suppose, for example, $W = V \oplus H$ is an orthogonal direct sum, where $\langle -,- \rangle_V$ is an inner product and H is a hyperbolic plane, so it has signature $(1,1)$ and exactly two lines. Then $\mathbb{P}H$ is a circle, and the null cone

[13] We could work with complex vector spaces, or infinite-dimensional vector spaces.

is two points inside it. The entire quadric cone splits as $Q_W = V \amalg N_V \amalg Q_V$. How does this work? Inside $\mathbb{P}W = \{[v : s : t]\}$, $V = \{[v : -\|v\|^2 : 1]\}$, $N_V = \{[v : 1 : 0]\}$, and $Q_V = \{[v] : 0 : 0\}$.

If V is n-dimensional, then N_V is $(n-1)$-dimensional and Q_V is $(n-2)$-dimensional. Therefore Q_W is a compactification of V: we've added strata of codimensions one and two to it, and picked up a conformal structure.

Example 5.33. Suppose $\langle -,- \rangle$ is an inner product. Then N_V is a point and Q_V is the empty set, so Q_W is diffeomorphic to S^n with the conformal structure induced from the usual metric.[14] In this case, the symmetry group is $\mathrm{O}(V)$, and with an orientation we can get $\mathrm{SO}(V)$. ◂

Example 5.34. More generally, let $V = \mathbb{R}^{p,q}$, where $p + q = n$, with the indefinite-signature form

$$\langle \xi, \eta \rangle = \sum_{i=1}^{p} \xi^i \eta^i - \sum_{i=p+1}^{q} \xi^i \eta^i. \tag{5.35}$$

For example, $H = \mathbb{R}^{1,1}$. In this case, $W \cong \mathbb{R}^{p+1,q+1} = \mathbb{R}^{p+1} \oplus \mathbb{R}^{q+1}$ (though the inner product on the second has a minus sign put in front). If $\xi \in \mathbb{R}^{p+1}$ and $\eta \in \mathbb{R}^{q+1}$, then $(\xi, \eta) \in N_W$ iff $\|\xi\| = \|\eta\| = 1$, so Q_W is diffeomorphic to $(S^q \times S^q)/\{\pm 1\}$, and $\mathrm{PO}(W)$ acts via conformal transformations on Q_W. Inside this we have $\mathrm{O}(V) \cong \mathrm{O}_{p,q}$. So this says that the conformal symmetries in signature (p, q) are the orthogonal symmetries in signature $(p + 1, q + 1)$. ◂

For example, the group of conformal symmetries of $\mathbb{R}^4$ with positive-definite inner product is $\mathrm{O}_{5,1}$, which acts on S^4, the conformal compactification of $\mathbb{R}^4$. The conformal group of $\mathbb{R}^2$, sitting inside its compactification S^2, is $\mathrm{O}_{3,1}$; with an orientation we get $\mathrm{SO}_{3,1} \cong \mathrm{PSL}_2(\mathbb{C})$, one of the low-dimensional exceptional isomorphisms of Lie groups. These act by Möbius transformations. Analogously, there's a special isomorphism $\mathrm{SO}_{5,1} \cong \mathrm{PSL}_2(\mathbb{H})$, allowing us to get at conformal symmetries in dimension 4 via the quaternions. Next time we'll apply this to the self-duality equations in dimension 4.

Lecture 6.

Spinors in low dimensions and special isomorphisms of Lie groups: 2/7/19

"I'll tell a story for three minutes and then you can go."

Today we'll continue discussing preliminaries to the ADHM construction, sometimes using them to introduce interesting nearby ideas in their own right. Today we'll discuss exceptional isomorphisms of Lie groups in low dimensions, which are applicable in other cases. For example, if you care about fermions in, e.g. supersymmetric quantum field theories in dimensions 6 or below, these ideas appear.

Fix a 4-dimensional complex vector space $\mathbb{S}$, which could be $\mathbb{C}^4$. We can take exterior powers $\Lambda^2\mathbb{S}$, $\Lambda^3\mathbb{S}$, and $\Lambda^4\mathbb{S} = \mathrm{Det}\,\mathbb{S}$, and similarly for $\mathbb{S}^*$. Choose a volume form $\mu \in \mathrm{Det}\,\mathbb{S}^* \setminus 0$; we want to consider the symmetries of $(\mathbb{S}, \mu)$. A linear map $T\colon \mathbb{S} \to \mathbb{S}$ has an associated volume $\det T$, so we're essentially asking for automorphisms with determinant 1. This is literally true for $\mathbb{S} = \mathbb{C}^4$, in which case $\mathrm{Aut}(\mathbb{S}, \mu) = \mathrm{SL}_4(\mathbb{C})$.

Now given such an automorphism T, we obtain an automorphism $\Lambda^2 T\colon \Lambda^2\mathbb{S} \to \Lambda^2\mathbb{S}$. The condition that T preserves the volume form is mapped to the bilinear pairing $B\colon \Lambda^2\mathbb{S} \times \Lambda^2\mathbb{S} \to \mathbb{C}$ sending

$$x, y \longmapsto \langle \mu, x \wedge y \rangle_{\mathrm{Det}\,\mathbb{S}^*, \mathrm{Det}\,\mathbb{S}}. \tag{6.1}$$

This is symmetric and nondegenerate, hence an inner product, so $\Lambda^2 T$ preserves this inner product! Therefore the map $\phi = \Lambda^2\colon\ \mathrm{Aut}(\mathbb{S}, \mu) \to \mathrm{Aut}(\Lambda^2\mathbb{S}, B)$ amounts to a homomorphism $\mathrm{SL}_4(\mathbb{C}) \to \mathrm{O}_6(\mathbb{C})$.

You can directly check that $\{\pm 1\} \subset \ker(\phi)$. Since $\mathrm{SL}_4(\mathbb{C})$ is connected, the image of ϕ is contained in $\mathrm{SO}_6(\mathbb{C})$.

Claim 6.2. $\phi\colon \mathrm{SL}_4(\mathbb{C}) \to \mathrm{SO}_6(\mathbb{C})$ is surjective.

The proof would amount to checking that it's an isomorphism on Lie algebras, so the image is open, and that the image is closed, hence all of $\mathrm{SO}_6(\mathbb{C})$. One corollary is that we've identified the spinor representation as $\mathbb{S}$.

Moreover, since $\mathrm{SL}_4(\mathbb{C})$ is simply connected, the map $\phi\colon \mathrm{SL}_4(\mathbb{C}) \to \mathrm{SO}_6(\mathbb{C})$ is the nontrivial double cover map. Therefore we have produced an isomorphism $\mathrm{SL}_4(\mathbb{C}) \cong \mathrm{Spin}_6(\mathbb{C})$.

[14] TODO: I would like to double-check this.

Remark 6.3. We can define the complex spin groups in the same way as the real ones: for $n \geq 3$, $\pi_1 \mathrm{SO}_n(\mathbb{C}) = \mathbb{Z}/2$, and for $n = 2$, $\pi_1 \mathrm{SO}_2(\mathbb{C}) = \mathbb{Z}$, so we can ask for the connected double cover of $\mathrm{SO}_n(\mathbb{C})$, which has a canonical Lie group structure, and define it to be $\mathrm{Spin}_n(\mathbb{C})$. ◀

There can be two different realizations of this double cover, but you can prove there's a unique Lie group homomorphism between them respecting the covering map, and it's an isomorphism.

There are several other exceptional isomorphisms, and they all follow from this one.

Example 6.4. Let $J\colon \mathbb{S} \to \mathbb{S}$ be a *quaternionic structure*, i.e. an antilinear map such that $J^2 = -\mathrm{id}_{\mathbb{S}}$. Then $\Lambda^k J\colon \Lambda^k\mathbb{S} \to \Lambda^k\mathbb{S}$ is also antilinear, and squares to $(-1)^k \mathrm{id}_{\mathbb{S}}$. So on $\Lambda^2\mathbb{S}$ it defines a real structure, and on $\Lambda^3\mathbb{S}$ it's quaternionic.

We can impose another constraint, that $(\det J)^*\mu = \mu$, i.e. μ is real. That is, it's in the subspace of $\mathrm{Det}\,\mathbb{S}^*$ fixed by $\det J$, which is a one-dimensional real vector space. Therefore we obtain a map

$$\phi\colon \mathrm{Aut}(\mathbb{S}, J, \mu) \longrightarrow \mathrm{Aut}(\Lambda^2\mathbb{S}, B, \Lambda^2 J). \tag{6.5}$$

Now suppose $\mathbb{S} = \mathbb{C}^4$. Then the codomain is O_{p+q} for some $p+q = 6$, but B might not be positive definite. The domain is $\mathrm{SL}_2(\mathbb{H})$ – though you have to be careful with what this means. There's no determinant map $\mathrm{GL}_n(\mathbb{H}) \to \mathbb{H}^\times$, but we can take the determinant to be a complex number (via regarding quaternionic matrices as complex matrices), and ask for it to be 1.

Working out the details, the kernel will once again be $\{\pm 1\}$, and this will be a double cover, so this will identify $\mathrm{SL}_2(\mathbb{H}) \cong \mathrm{Spin}_{p,q}$, an isomorphism of 15-dimensional real Lie groups. Then $\mathbb{S}$ is the spinor representation again – but it's quaternionic, so we know we can't get Spin_6.[15]

To determine the signature, let $\{e_1, Je_1, e_2, Je_2\}$ be a basis for $\mathbb{S}$. Then we can write down a basis for $(\Lambda^{@}\mathbb{S})_{\mathbb{R}}$ as follows:

$$\begin{aligned} &e_1 \wedge Je_1 && e_2 \wedge Je_2 \\ &e_1 \wedge Je_2 + e_2 \wedge Je_1 && i(e_1 \wedge Je_2 - e_2 \wedge Je_1) \\ &e_1 \wedge e_2 + Je_1 \wedge Je_2 && i(e_1 \wedge e_2 - Je_1 \wedge Je_2). \end{aligned} \tag{6.6}$$

You can check these are orthogonal. The first two form a hyperbolic pair (signature $(1,1)$), and the last four all self-pair to -1. Therefore the signature is $(1,5)$, and we conclude $\mathrm{SL}_2(\mathbb{H}) \cong \mathrm{Spin}_{1,5}$.

Taking the quotient, we also get $\mathrm{PSL}_2(\mathbb{H}) \cong \mathrm{SO}^0_{1,5}$[16] – in indefinite signature, special orthogonal groups aren't connected. Now, $\mathrm{SO}^0_{1,5}$ acts as the group of automorphisms of the conformal compactification of $\mathbb{R}^4$ with a definite metric, which gives you S^4: $\mathrm{PSL}_2(\mathbb{H})$ is the conformal group of the 4-sphere. We can also identify $\mathrm{PSL}_2(\mathbb{H}) \cong \mathrm{PGL}_2(\mathbb{H})$ via the diffeomorphism $S^4 \cong \mathbb{HP}^1$.

So in summary, we have $\mathrm{SL}_2(\mathbb{H}) \cong \mathrm{Spin}_{1,5}$ and $\mathrm{PSL}_2(\mathbb{H}) \cong \mathrm{PGL}_2(\mathbb{H}) \cong \mathrm{SO}^0_{1,5}$. ◀

Remark 6.7. If you want to imitate the above story but get Spin_6, you'll want to get the maximal compact in $\mathrm{SL}_4(\mathbb{C})$, which is SU_4. So throw out J and instead ask that your automorphisms fix a Hermitian inner product. ◀

Example 6.8. Now let's introduce a symplectic form $\omega \in \Lambda^2\mathbb{S}^*$, and let's say that $\mu = (1/2)\omega \wedge \omega$. We'll ask about the automorphisms of $\mathbb{S}$ that fix μ; if $\mathbb{S} = \mathbb{C}^4$, this group is called $\mathrm{Sp}_4(\mathbb{C})$.

Passing up to $\Lambda^2\mathbb{S}$, such an automorphism must preserve $\ker(\omega)$, which has codimension 1, and the bilinear form B from before. The automorphisms of $\ker(\omega)$ and B form the group $\mathrm{O}_5(\mathbb{C})$. As in the previous case, the map $\mathrm{Sp}_4(\mathbb{C}) \to \mathrm{O}_5(\mathbb{C})$ has image the identity component $\mathrm{SO}_5(\mathbb{C}) \subset \mathrm{O}_5(\mathbb{C})$, and is a double cover onto it. Therefore we obtain an isomorphism $\mathrm{Spin}_5(\mathbb{C}) \cong \mathrm{Sp}_4(\mathbb{C})$. ◀

Example 6.9. If you ask for automorphisms of $\mathbb{S}$ which preserve both J and ω? In this case we get a map $\mathrm{Aut}(\mathbb{S}, J, \omega) \to \mathrm{Aut}(\ker(\omega), \Lambda^2 J, B)$. The basis of $\ker(\omega)$ contains the last four vectors in (6.6), but instead of the first two we have the vector $e_1 \wedge Je_1 - e_2 \wedge Je_2$, which self-pairs to -1. Therefore we're in definite signature, so this map is identified with the map $\mathrm{Sp}_2 \to \mathrm{O}_5$. Here, Sp_2 is the symmetries of $\mathbb{H}^2$ with its standard inner product.

As usual, this only sees $\mathrm{SO}_5 \subset \mathrm{O}_5$, and is a double cover, providing for us an isomorphism $\mathrm{Spin}_5 \cong \mathrm{Sp}_2$. ◀

[15] You can also check that $Z(\mathrm{SL}_2(\mathbb{H})) \cong \mathbb{Z}/2$ but $Z(\mathrm{Spin}_6) \cong \mathbb{Z}/4$.

[16] $\mathrm{PSL}_2(\mathbb{H})$ is defined to be $\mathrm{SL}_2(\mathbb{H})$ modulo its center $\{\pm 1\}$.

Let's digress from the linear algebra a little bit and talk about quaternions. A general quaternion is of the form $x = x^0 + x^1 i + x^2 j + x^3 k$.

Definition 6.10. The *conjugate* of a quaternion x as above is

$$\overline{x} \coloneqq x^0 - x^1 i - x^2 j - x^3 k.$$

The *imaginary part* of x is $\mathrm{Im}(x) \coloneqq x^1 i + x^2 j + x^3 k$.

Therefore $x\overline{x} = (x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$; this is the norm squared of x. We have $\mathrm{d}x = \mathrm{d}x^0 + \mathrm{d}x^1 i + \mathrm{d}x^2 j + \mathrm{d}x^3 k$, and therefore

$$\mathrm{d}x \wedge \mathrm{d}\overline{x} = -(\mathrm{d}x^0 \wedge \mathrm{d}x^1 + \mathrm{d}x^2 \wedge \mathrm{d}x^3)i - (\mathrm{d}x^0 \wedge \mathrm{d}x^2 - \mathrm{d}x^1 \wedge \mathrm{d}x^3)j - (\mathrm{d}x^0 \wedge \mathrm{d}x^3 + \mathrm{d}x^1 \wedge \mathrm{d}x^2)k. \tag{6.11}$$

You'll notice this is self-dual, and in fact the coeffcients are a basis for the space of self-dual 2-forms. Similarly, $\mathrm{d}\overline{x} \wedge \mathrm{d}x$ is anti-self-dual, and its coefficients are a basis for the space of anti-self-dual forms.

Example 6.12. Continuing Example 6.9, consider $S^7 \subset \mathbb{H}^2$, which is preserved by $\mathrm{Sp}_2 = \mathrm{Spin}_5$. Restricting to $\mathbb{H}^2 \setminus 0$, we have a projection down to $\mathbb{HP}^1 \cong S^4$, the quaternionic projective line.[17] This restricts to a submersion $\pi\colon S^7 \to S^4$. Two points in S^7 have the same image iff the are acted on by a unit-norm quaternion. The group of unit-norm quaternions is $\mathrm{Sp}_1 = \mathrm{SU}_3 = \mathrm{Spin}_2$. Since $S^7 = \mathrm{Sp}_2/\mathrm{Sp}_1$, $S^4 \cong \mathrm{Sp}_2/\mathrm{Sp}_1 \times \mathrm{Sp}_1$. In other words, $\pi\colon S^7 \to S^4$ is a principal Sp_1-bundle, and it is homogeneous for the left Sp_2-action. This is another example of a Hopf fibration.[18]

Now who acts on S^4? Well, SO_5 acts by isometries. The whole situation is invariant under the isometry group, so we can move the total space around by isometries of the base, and therefore SO_5 lifts to the Spin_5-action we already identified on S^7. Inside the isometries, there's a bigger group $\mathrm{SO}_{1,5}^0$ of conformal transformations. We'd like to ask whether these lift to conformal isometries on S^7. The identification $S^4 \cong \mathbb{HP}^1$ carries SO_5 to PSp_2 and $\mathrm{SO}_{1,5}^0 \cong \mathrm{PSL}_2(\mathbb{H})$, and this lifts to the $\mathrm{SL}_2(\mathbb{H})$-action on $\mathbb{H}^2$.

Now we introduce connections, mimicking the basic story we already saw over $\mathbb{C}$ in §4. We want to consider a connection Θ for $S^7 \to S^4$ regarded as an Sp_1-bundle, an Sp_1-invariant horizontal distribution. Specifically, you can check that

$$\Theta \coloneqq -\mathrm{Im}\Big(q^0\,\overline{\mathrm{d}q^0} + q^1\,\overline{\mathrm{d}q^1}\Big) \tag{6.13}$$

is a connection, by checking it's orthogonal to orbits everywhere. Then $\Omega_{0,1}$, restricted to the horizontal distribution, is $-\mathrm{Im}(\mathrm{d}q^0 \wedge \overline{\mathrm{d}q^0})$. But from the calculation in (6.11), we already know this is imaginary, so $\Omega_{0,1}|_{\text{horiz.}} = -\mathrm{d}q^0 \wedge \overline{\mathrm{d}q^0}$; we also know it's self-dual. ◀

So we've found one solution to the self-dual equations; we can discover others by transforming by conformal transformations. Therefore $\mathrm{SL}_2(\mathbb{H}) = \mathrm{Spin}_{1,5}$ acts on Θ to produce self-dual connections. This is an $\mathrm{SL}_2(\mathbb{H})$-orbit inside $\mathcal{A}_P$; we know it's a homogeneous manifold, so if we want to know what it is, we should compute the stabilizer of Θ. This is precisely the isometries, which are Spin_5, so the orbit is $\mathrm{Spin}_{1,5}/\mathrm{Spin}_5 = \mathrm{SO}_{1,5}/\mathrm{SO}_5$, which is a hyperbolic 5-ball M, and Θ is actually the center. As you get closer to the "edge" S^4, which we think of as the base S^4, the curvature concentrates more and more at the boundary point, and we could think of the connections at infinity as having curvature in a δ-function (this doesn't actually work, of course). The only connection which has no priviledged concentration of curvatuee is Θ, at the center.

Theorem 6.14 (Atiyah-Drinfeld-Hitchin-Manin)**.** $\mathrm{SO}_{1,5}/\mathrm{SO}_5$ *is the moduli space of self-dual* Sp_1*-instantons with Chern class* $c_2 = 1$.

This is the only characteristic class data; for an $\mathrm{Sp}_1 = \mathrm{SU}_2$-bundle, $c_1 = 0$, and there are no more Chern classes. On S^4, $c_2 \in H^4(S^4)$, and the orientation identifies this cohomology group with $\mathbb{Z}$, so we can make sense of $c_4 = 1$. This is an instance of fixing discrete data in a moduli problem, which is common.

This is a basic case where we can picture what's going on, and illustrates a good part of the general story. But how do you prove Theorem 6.14? How do we know that every connection is one of these? Stay tuned; we'll prove this later. Then there are other questions involving other Chern classes, other manifolds, and so on.

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[17]Because $\mathbb{H}$ isn't commutative, we have to specify that we're taking the quotient of the right action.

[18]There are also Hopf fibrations over the reals, including the double cover $\mathbb{Z}/2 \to S^1 \to \mathbb{RP}^1$. There's also one over the octonions.

This kind of question was first investigated by physicists, Polyakov and others. Atiyah and Singer then checked the dimension by linearizing the equations and found that the physicists had missed some. When Simon Donaldson was a graduate student, he had the brilliant idea of taking this example, but generalizing to 4-manifolds where the intersection form is positive definite. Once again, the moduli space is five-dimensional, and you can take connections with curvature concentrated near a point, and extend by zero. Taubes show you can wiggle this a little bit and actually get a solution – and therefore you again get a copy of the manifold at infinity. Through this (and of course, plenty more) Donaldson was able to prove his first theorem – this exhibits a bordism from this manifold to something else.

Lecture 7.

Some linear algebra underlying spinors in dimension 4: 2/12/19

"Is it clear that —" "Yes, it's clear."

Recall that we've been discussing connections on S^4 with Chern number 1. Fix the usual orientation and round metric (we only need a conformal structure in dimension 4, but we have a canonical metric so we might as well). The conformal symmetries $\mathrm{SO}_{1,5} \cong \mathrm{PSL}_2(\mathbb{H})$ act on S^4, and hence also on the moduli space $\mathcal{M}$ of self-dual Sp_1-connections. This moduli space is diffeomorphic to a 5-ball.

Inside this space, there's a special point, as we discussed last time: the principal Sp_1-bundle $S^7 \to S^4$ realized as the Hopf fibration $\mathbb{H}^2 \setminus \{0\}/\mathbb{R}^{>0} \to \mathbb{HP}^1$, and $\mathbb{HP}^1 \cong S^4$. Then we can choose a Sp_1-invariant horizontal distribution on S^7, as in (6.13).

Now for some more cool facts about spinors. Previously, we discussed how to obtain several exceptional isomorphisms of Lie groups by starting with a four-dimensional complex vector space $\mathbb{S}$ and considering various wedge powers of $\mathbb{S}$ and $\mathbb{S}^*$. Now, let's suppose $\mathbb{S} = \mathbb{S}' \oplus \mathbb{S}''$, where each summand is two-dimensional. Then

$$\Lambda^2\mathbb{S}^* = \Lambda^2(\mathbb{S}')^* \oplus \Lambda^2(\mathbb{S}'')^* \oplus (\mathbb{S}')^* \otimes (\mathbb{S}'')^*. \tag{7.1}$$

The summands have dimensions 1, 1, and 4, respectively.

Suppose we fix a $\mu \in \operatorname{Det}\mathbb{S}^*$, which splits as $\epsilon' \wedge \epsilon''$, where $\epsilon' \in \Lambda^2(\mathbb{S}')^*$ and $\epsilon'' \in \Lambda^2(\mathbb{S}'')^*$. Then $\mathbb{S}' \oplus \mathbb{S}''$ has a bilinear pairing

$$B(s' \otimes s'', \widetilde{s}' \otimes \widetilde{s}'') := \epsilon'(s', \widetilde{s}')\epsilon''(s'', \widetilde{s}''). \tag{7.2}$$

Given an isometry of $\mathbb{S}'$ (i.e. preserving ϵ') and an isometry of $\mathbb{S}''$, their tensor product preserves B, so we obtain a map

$$\operatorname{Aut}(\mathbb{S}', \epsilon') \times \operatorname{Aut}(\mathbb{S}'', \epsilon'') \longrightarrow \operatorname{Aut}(\mathbb{S}' \otimes \mathbb{S}'', B), \tag{7.3a}$$

which if $\mathbb{S} = \mathbb{C}^4$ with the usual decomposition as $\mathbb{C}^2 \oplus \mathbb{C}^2$, is a map

$$\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{O}_4(\mathbb{C}), \tag{7.3b}$$

and as usual, this has image $\mathrm{SO}_4(\mathbb{C})$, and is a double cover onto it. Hence $\mathrm{Spin}_4(\mathbb{C}) \cong \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$.

We're also interested in the real spin group. Hence, fix a *quaternionic* structure $J = J' \oplus J''$, where $J' \colon \mathbb{S}' \to \mathbb{S}'$ is antilinear and squares to $-\mathrm{id}$, and similarly for J''. Then $\operatorname{Aut}(\mathbb{S}', \epsilon', J') \cong \mathrm{Sp}_1$, and similarly for $\mathbb{S}''$, and $\Lambda^2 J$ restricts to a real structure on $\mathbb{S}' \otimes \mathbb{S}''$, so we obtain a map

$$\mathrm{Sp}_1 \times \mathrm{Sp}_1 \longrightarrow \mathrm{SO}_4, \tag{7.4}$$

and this is a double cover, so $\mathrm{Spin}_4 \cong \mathrm{Sp}_1 \times \mathrm{Sp}_1$.

We can produce a real basis of $\mathbb{S}' \otimes \mathbb{S}''$ with respect to this real structure. Let e', Je' be a basis of $\mathbb{S}'$, where we ask $\epsilon'(e', Je') = 1$, and define e'' and J'' similarly. Then our real basis is

$$\begin{gathered} e' \otimes e'' + Je' \otimes Je'' \\ i(e' \otimes e'' - Je' \otimes Je'') \\ e' \otimes Je'' - Je' \otimes e'' \\ i(e' \otimes Je'' + Je' \otimes e''). \end{gathered} \tag{7.5}$$

Remark 7.6. You can also choose a real structure J on $\mathbb{S} = \mathbb{S}' \oplus \mathbb{S}''$, and stipulate that $J(\mathbb{S}') = \mathbb{S}''$, and can play a similar game as above. ◀

Now consider the exterior product $\Lambda^2\mathbb{S}\otimes\mathbb{S}\to\Lambda^3\mathbb{S}$. Restricted to $V^*:=\mathbb{S}'\otimes\mathbb{S}''\subset\Lambda^2\mathbb{S}$ we obtain a map

$$\begin{aligned}V^*\otimes(\mathbb{S}'\oplus\mathbb{S}'')&\longrightarrow\mathbb{S}'\oplus\mathbb{S}''\\(s'\otimes s''),(\psi',\psi'')&\longmapsto(\epsilon'(s',\psi')s'',\epsilon''(s'',\psi'')s').\end{aligned}\tag{7.7}$$

In particular, if $v\in V^*$, $s'\in\mathbb{S}'$, and $s''\in\mathbb{S}''$, then $v,s'\otimes 1$ lands in $\mathbb{S}''$, and correspondingly $v,1\otimes s''$ lands in $\mathbb{S}'$. Therefore we obtain maps $\gamma\colon V^*\otimes\mathbb{S}'\to\mathbb{S}''$ and $V^*\otimes\mathbb{S}''\to\mathbb{S}'$. These will be the Clifford multiplication maps when we pass to associated bundles on a spin 4-manifold. The notation is suggestive: V^* will be vectors and $\mathbb{S}'$ and $\mathbb{S}''$ will be the spinors.

Proposition 7.8. *If $\theta_1,\theta_2\in V^*$, then*

$$\gamma(\theta_1)\gamma(\theta_2)+\gamma(\theta_2)\gamma(\theta_1)=B(\theta_1,\theta_2).$$

But first we need a quick fact.

Lemma 7.9. *Let W be a two-dimensional vector space and ϵ be an area form for W. For any $w_1,w_2,w_3\in W$,*

$$\epsilon(w_1,w_2)w_3+\epsilon(w_2,w_3)w_1+\epsilon(w_3,w_1)w_2=0.$$

Proof sketch. It suffices to check that the map

$$\begin{aligned}W\times W\times W&\longrightarrow\Lambda^2W\otimes W\\w_1,w_2,w_3&\longmapsto(w_1\wedge w_2)\otimes w_3+(w_2\wedge w_3)\otimes w_1+(w_3\wedge w_1)\otimes w_2\end{aligned}\tag{7.10}$$

factors through Λ^3W. ⊠

Proof of Proposition 7.8. Write $\theta_1=s_1'\otimes s_1''$ and $\theta_2=s_2'\otimes s_2''$. If $\psi'\in\mathbb{S}'$, then

$$\begin{aligned}\gamma(s_1'\otimes s_1'')\gamma(s_2'\otimes s_2'')\psi'&=\gamma(s_1'\otimes s_1'')\epsilon'(s_2',\psi')s_2''\\&=\epsilon'(s_2',\psi')\epsilon''(s_1'',s_2'')s_1'.\end{aligned}\tag{7.11a}$$

Similarly,

$$\gamma(s_2'\otimes s_2'')\gamma(s_1'\otimes s_1'')=-\epsilon'(s_1',\psi')\epsilon''(s_1'',s_2'')s_2'.\tag{7.11b}$$

Adding these together, you get

$$\gamma(s_1'\otimes s_1'')\gamma(s_2'\otimes s_2'')\psi'+\gamma(s_2'\otimes s_2'')\gamma(s_1'\otimes s_1'')=-\epsilon_1'(s_1',s_2')\epsilon''(s_1'',s_2'')\psi'.\tag{7.12}$$

⊠

Proposition 7.13. *If $\theta_1,\theta_2\in V^*$, then the assignment*

$$\theta_1\wedge\theta_2\longmapsto\gamma(\theta_1)\gamma(\theta_2)-\gamma(\theta_2)\gamma(\theta_1)\tag{7.14a}$$

defines a map

$$\Lambda^2V^*\longrightarrow\mathrm{Aut}(\mathbb{S}')\times\mathrm{Aut}(\mathbb{S}''),\tag{7.14b}$$

such that $\Lambda^2V^*_+$ acts by zero on $\mathbb{S}''$ and $\Lambda^2V^*_-$ acts by zero on $\mathbb{S}'$.

Recall first that if W is a vector space, there's a canonical isomorphism

$$W\otimes W\cong\mathrm{Sym}^2W\otimes\Lambda^2W.\tag{7.15}$$

The general theory of decomposing a tensor product involves a tool called Young diagrams. In our setting,

$$\begin{aligned}V^*\otimes V^*&\cong\mathbb{S}'\otimes\mathbb{S}''\otimes\mathbb{S}'\otimes\mathbb{S}''\\&\cong\underbrace{\mathrm{Sym}^2\mathbb{S}'\otimes\Lambda^2\mathbb{S}''}_{\Lambda^2_+V^*}\oplus\underbrace{\Lambda^2\mathbb{S}'\otimes\mathrm{Sym}^2\mathbb{S}''}_{\Lambda^2_-V^*}\oplus\cdots\end{aligned}\tag{7.16}$$

Now choose a real subspace $L'\subset\mathbb{S}'$ such that $\mathbb{S}'=L'\oplus JL'$. Then $V*$ splits as

$$V^*=\mathbb{S}'\otimes\mathbb{S}''=(L'\oplus JL')\otimes\mathbb{S}''=\underbrace{L'\otimes\mathbb{S}''}_{(1,0)}\oplus\underbrace{JL'\otimes\mathbb{S}''}_{(0,1)},\tag{7.17}$$

and J interchanges the two factors, defining a complex structure on $V^*_{\mathbb{R}}$. Similarly,

$$\begin{aligned}\Lambda^2_+V^*&\cong\mathrm{Sym}^2\mathbb{S}'\oplus\Lambda^2\mathbb{S}''\\&\cong\underbrace{(L')^{\otimes2}\otimes\Lambda^2\mathbb{S}''}_{(2,0)}\oplus\underbrace{(JL')^{\otimes2}\otimes\Lambda^2\mathbb{S}''}_{(0,2)}\oplus\underbrace{(L'\otimes JL')\otimes\Lambda^{@}\mathbb{S}''}_{(1,1)}.\end{aligned}\tag{7.18}$$

and

$$\Lambda^2_- V^* \cong \Lambda^2 \mathbb{S}' \otimes \operatorname{Sym}^2 \mathbb{S}'' = \underbrace{(L' \otimes JL') \otimes \operatorname{Sym}^2 \mathbb{S}''}_{(1,1)}. \tag{7.19}$$

Here the annotations mean the *type* of a form in the complexificiation of a real vector space. Specifically, suppose V is a real vector space, so that $V \otimes \mathbb{C} = W \oplus \overline{W}$. Then $V^* \otimes \mathbb{C} = W^* \oplus \overline{W}^*$ too, and therefore there is a splitting

$$\Lambda^k(V^* \otimes \mathbb{C}) = \bigoplus_{p+q=k} \Lambda^p W^* \otimes \Lambda^q \overline{W}^*. \tag{7.20}$$

The forms in this summand are said to have type (p,q). Since $\dim V = 4$, then

$$\Lambda^2(V^* \otimes \mathbb{C}) = \Lambda^{2,0} \oplus \Lambda^{1,1} \oplus \Lambda^{0,2} \tag{7.21}$$

of dimensions 1, 4, and 1 respectively.

Theorem 7.22. *The intersection over all subspaces L' of the $(1,1)$-forms with respect to L' is $\Lambda^2_- V^*$.*

This follows directly from a symmetry argument: this is the only $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$-invariant subspace of $\Lambda^{1,1}$.

This is very useful, as it establishes a link between self-duality in dimension 4 and complex geometry. If you have some connection and want to know whether it's anti-self-dual, it suffices to show that it's type $(1,1)$ in every complex structure.

∽·∼

We're now in a position to discuss the Dirac operator, which we will do next time. In the last ten minutes of this lecture, we'll review some basics of complex geometry.

Let M be a manifold, and suppose $I \in \operatorname{End}(TM)$ squares to $-\mathrm{id}_{TM}$. This is what's called an *almost complex structure* on M, and just as above allows us to decompose

$$\Omega^k_M(\mathbb{C}) = \bigoplus_{p+q=k} \Omega^{p,q}_M. \tag{7.23}$$

In more detail, $TM \otimes \mathbb{C}$ splits as $W \oplus \overline{W}$, where W is the subspace where I acts by i (where we've chosen a square root i of -1) and $\overline{W}$ is where I acts by $-i$. Thus $\Lambda^k T^*M$ splits as a sum of $\Lambda^p W^*$ and $\Lambda^q \overline{W}^*$ over all (p,q) with $p+q=k$, and hence sections do as well, giving (7.23).

In this setting, what happens to the de Rham differential? It looks like it gets complicated, but it turns out that $\mathrm{d}_{\Omega^{1,0}}$ has no $(0,2)$-component, and more generally, $\mathrm{d}|_{\Omega^{p,q}}$ lands only in $(p+1,q)$ and $(p,q+1)$. Therefore we can let ∂ denote the part of d valued in $\Omega^{p+1,q}$ and $\overline{\partial}$ the part of d valued in $\Omega^{p,q+1}$, and we get a diagram of maps

$$\begin{tikzcd}[column sep=tiny] & & \Omega^{p,q} \arrow[dl, "\overline{\partial}"'] \arrow[dr, "\partial"] & & \\ & \Omega^{p,q+1} \arrow[dl, "\overline{\partial}"'] \arrow[dr, "\partial"] & & \Omega^{p+1,q} \arrow[dl, "\overline{\partial}"'] \arrow[dr, "\partial"] & \\ \Omega^{p,q+2} & & \Omega^{p+1,q+1} & & \Omega^{p+2,q} \end{tikzcd} \tag{7.24}$$

We know $\mathrm{d}^2 = 0$ iff $\partial^2 = 0$, $\overline{\partial}^2 = 0$, and $\partial\overline{\partial} + \overline{\partial}\partial = 0$, so we would like this to be true.

Claim 7.25. This condition is exactly the vanishing of the complex Frobenius tensor of the complex distribution $\overline{W} \subset TM \otimes \mathbb{C}$.

This isn't too hard to see. But the crucial equivalent condition is harder:

Theorem 7.26 (Neulander-Nirenberg)**.** *This condition holds iff we can cover M by local coordinates in which the change-of-charts map is* holomorphic.

Lecture 8.

Twistors and Dirac operators: 2/14/19

"Dimension 4 is, as always, the problem child, or the interesting child."

Last time, we discussed that if M is an almost complex manifold, meaning it comes equipped with a map $I\colon TM \to TM$ with $I^2 = -\mathrm{id}$, then the complex differential forms $\Omega_M^k(\mathbb{C})$ split into $\Omega_M^{p,q}$ indexed over p, q with $p + q = k$ based on how I acts on them. We then mentioned Theorem 7.26, which says that if d restricted to a map $\Omega_M^{0,1} \to \Omega_M^{2,0}$ is zero (which is an integrability condition), then there is an atlas for M whose change-of-charts maps are holomorphic, i.e. M is a complex manifold. In particular, in these local coordinates $z_1, \dots, z_n$, $\mathrm{d}z_1, \dots \mathrm{d}z_n$ are of type $(1, 0)$ and pointwise form a basis of $\Omega_M^{1,0}$.

Assume now that M is a complex manifold, and let $E \to M$ be a C^∞ complex vector bundle, i.e. a complex bundle in the usual sense, and not necessarily holomorphic. Suppose we have a linear operator $\overline{\partial}_E\colon \Omega_M^{0,0}(E) \to \Omega_M^{0,1}(E)$ such that

$$\overline{\partial}_E(f \cdot s) = \overline{\partial} f \cdot s + f\overline{\partial}_E s, \tag{8.1}$$

where f is a function and s is a section of E. We can then extend this to a complex

$$0 \longrightarrow \Omega_M^{0,0}(E) \xrightarrow{\overline{\partial}_E} \Omega_M^{0,1}(E) \xrightarrow{\overline{\partial}_E} \Omega_M^{0,2}(E) \longrightarrow \cdots, \tag{8.2}$$

and $\overline{\partial}_E^2\colon \Omega_M^{0,0}(E) \to \Omega_M^{0,2}(E)$ is multiplication by some tensor $\Phi_E \in \Omega_M^{0,2}(\operatorname{End} E)$.

Theorem 8.3. *If $\Phi_E = 0$, then there exists a local basis of sections $s_1, \dots, s_n$ of sections such that $\overline{\partial}_E s_j = 0$.*

This is easier than Theorem 7.26, though we'll defer the proof. In this case we can place the structure of a complex manifold on E, and E is what's called a *holomorphic vector bundle*.

The twistor approach to the anti-self-dual equations. Now we'll briefly take a peek at one approach to the anti-self-dual equations, as part of some great activity in the 1970s by researchers including the Oxford school. This involves some algebro-geometric techniques which show off the uses of the linear algebra we did last time, and is the original approach to this equation on S^4.

Let X be an oriented Riemannian 4-manifold. If $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ is a 4-dimensional complex vector space (which we'll turn into the spinor bundle soon enough) and $\dim \mathbb{S}^\pm = 2$, then $\mathrm{Sp}_1 \times \mathrm{Sp}_1$ acts on it as the spin group. If $V^+ := \mathbb{S}^+ \otimes \mathbb{S}^-$, then it's isomorphic to $(\mathbb{R}^4)^*$, and SO_4 acts on it. Choose a complex line $L \subset \mathbb{S}^+$, which as we discussed last time defines a complex structure on V^*. The space of choices is $\mathbb{P}(\mathbb{S}^+)$, which is also acted on by SO_4.

Bringing in the topology, let $\mathcal{B}_{\mathrm{SO}}(X) \to X$ be the principal SO_4-bundle of oriented orthonormal bases. The Riemannian metric defines the Levi-Civita connection Θ^{LC} on $\mathcal{B}_{\mathrm{SO}}(X)$, which is a beautiful and still somewhat mysterious fact. Because SO_4 acts on $\mathbb{P}(\mathbb{S}^+)$, we obtain an associated bundle $\mathbb{P}(\mathbb{S}^+) \to X$, and the Levi-Civita connection induces a horizontal distribution on it.

Remark 8.4. We don't have an action of SO_4 on $\mathbb{S}^+$ – that's what we'd need the spin structure for. But we do have a projective action, so the action on the projective space is well-defined. ◂

$\mathbb{P}(\mathbb{S}^+)$ is a six-dimensional real manifold.

Exercise 8.5. Choose $S^4 = \mathbb{HP}^1$ with the round metric. Then $\mathbb{P}(\mathbb{S}^+)$ is diffeomorphic to $\mathbb{CP}^3$, and the projection map $\mathbb{CP}^2 \to \mathbb{HP}^1$ is the map sending a complex line in $\mathbb{C}^2 \to \mathbb{H}^1$ to the unique quaternionic line containing it. The fiber is $\mathbb{CP}^1$.

In this case, $\mathbb{S}^+$ is a complex manifold. In general, we will only have an almost complex manifold: given a point $x, L \in \mathbb{P}(\mathbb{S}^+)$, we have the two lines $T_x X$ and the vertical line in $T_{x,L}\mathbb{P}(\mathbb{S}^+)$ (TODO: I might have this wrong), and so we can take the usual almost complex structure where $T_x X$ is real and the vertical line is imaginary.

Definition 8.6. If M is a Riemannian manifold of dimension at least 5, then its Riemann curvature tensor splits as a sum of three pieces: scalar curvature, Ricci curvature, and something called *Weyl curvature*, which is the piece that's invariant under conformal transformations.

In more detail, the Riemann curvature tensor has some symmetries that mean it's a section of $\mathrm{Sym}^2(\Lambda^2(T^*M))$. If V is an n-dimensional vector space with an orthogonal basis, the standard O_n-action induces an O_n-action on $\mathrm{Sym}^2(\Lambda^2 V^*)$, and this decomposes into three irreducible components, giving us the scalar curvature (for the trivial subrepresentation), the Ricci curvature, and the Weyl curvature.

The Weyl curvature requires four antisymmetric indices, so the Weyl curvature vanishes in dimensions 2 and 3: in dimension 2, the Riemannian curvature tensor is just the scalar curvature, and is called the *Gauss curvature*. In dimension 3, we also have Ricci curvature, but that's it, and so Riemannian geometry in 3 dimensions is nice, e.g. when studying Ricci flow. In dimension 4, the Weyl curvature tensor splits into two pieces: its self-dual and anti-self-dual components. This makes life interesting, as always, in dimension 4.

Theorem 8.7 (Atiyah-Hitchin-Singer)**.** *This almost complex structure is integrable iff $W_+ = 0$, where W_+ is the self-dual Weyl curvature tensor on X.*

In particular: we've passed from an integrability question on the fiber to one on the base.

Now suppose $E \to X$ is a complex vector bundle,[19] and let A be a connection on E, with curvature $F \in \Omega^2_X(\operatorname{End} E)$. Now pull back E and A to $\pi^* E \to \mathbb{P}(\mathbb{S}^+)$. Now, we're over an almost complex base, so we can decompose the curvature into its type components:

$$\pi^* F \in \Omega^2_{\mathbb{P}(\mathbb{S}^+)}(\operatorname{End} E) = \bigoplus_{p+q=2} \Omega^{p,q}(\operatorname{End} E). \tag{8.8}$$

A bit of linear algebra leads to the following lemma.

Lemma 8.9. *The connection A is anti-self-dual iff $\pi^* F$ has type $(1,1)$.*

Now let $P \to M$ be a principal G-bundle with a connection $\Theta \in \Omega^1_P(\mathfrak{g})$. This satisfies two equations: if $g \in G$, then $R_g^* \Theta = \operatorname{Ad}_g^{-1} \Theta$, and if $\xi \in \mathfrak{g}$, then $\iota_{\widehat{\xi}} \Theta = \xi$.

Now let $\mathbb{E}$ be a vector space and $\rho\colon G \to \operatorname{Aut}(\mathbb{E})$ be a representation. It differentiates to a Lie algebra representation $\dot\rho\colon \mathfrak{g} \to \operatorname{End}(\mathbb{E})$. Let $E := \mathbb{E}_P$ be the associated bundle to the data of ρ, and let ψ be a section. We can think of ψ as a map $P \to \mathbb{E}$ or as an $\mathbb{E}$-valued function on P; then ψ descends to $\Omega^0_M(E)$ iff $R_g^* \psi = \rho(g^{-1}) \cdot \psi$ for all $g \in G$. More generally, if $\alpha \in \Omega^k_P(\mathbb{E})$, then it descends to $\Omega^k_M(E)$ iff

$$\begin{aligned} R_g^* \alpha &= \rho(g^{-1}) \alpha \\ \iota_{\widehat{\xi}} \alpha &= 0, \end{aligned} \tag{8.10}$$

where as before $g \in G$ and $\xi \in \mathfrak{g}$.

The covariant derivative of ψ is

$$\nabla_\Theta \psi = \mathrm{d}\psi + \dot\rho(\Theta)\psi, \tag{8.11}$$

which *a priori* lives in $\Omega^1_P(\mathbb{E})$, but you can directly check that it descends, which is a useful exercise. More generally, if α is a k-form as above,

$$\mathrm{d}_\Theta \alpha = \mathrm{d}\alpha + \dot\rho(\Theta) \wedge \alpha, \tag{8.12}$$

and this also descends, which you can check.

Exercise 8.13 (Bianchi identity)**.** The curvature $F_\Theta = \mathrm{d}\Theta + (1/2)[\Theta \wedge \Theta]$ satisfies $\mathrm{d}F = 0 + [\mathrm{d}\Theta \wedge \Theta]$, i.e.

$$\mathrm{d}_\Theta F := \mathrm{d}F + [\Theta \wedge F] = 0,$$

because F and $\mathrm{d}\Theta$ differ by $[\Theta \wedge \Theta]$, and the Jacobi identity shows the triple bracket you get by substituting it in is zero.

Now let M be a Riemannian n-manifold with its principal O_n-bundle bundle of frames $\mathcal{B}(M) \to M$ and the Levi-Civita connection Θ^{LC}. This therefore gives us horizontal vector fields $\partial_1, \dots, \partial_n$ on $\mathcal{B}(M)$ defined as follows: given an $x \in X$ and a basis $b\colon \mathbb{R}^n \xrightarrow{\cong} T_x M$, hence a point in $\mathcal{B}(M)$, we can take $b(e_1) \in T_x M \subset T_{(x,b)}\mathcal{B}(M)$, which is a horizontal vector, and this defines ∂_1; then $\partial_2, \dots, \partial_n$ are analogous. Moreover,

$$[\partial_k, \partial_\ell] = \underbrace{-\frac{1}{2} T^i_{k\ell} \partial_i}_{=0} - \frac{1}{2} R^i_{jk\ell} E^j_i, \tag{8.14}$$

[19] As X is not necessarily a complex manifold, we can't ask for E to be holomorphic.

where E_i^j is some matrix (TODO: I did not parse the definition) which is 0 everywhere except for a 1 in column i and a -1 in column j. The first term vanishes because the Levi-Civita connection is torsion-free.

We can flow along these by geodesics by solving the geodesic equation, which frames the horizontal tangent bundle on $\mathcal{B}(M)$.

We can use this to write the covariant derivative: if O_n acts on a vector space $\mathbb{S}$ and $\psi\colon \mathcal{B}(M) \to \mathbb{S}$ is an equivariant (TODO: ?) map, then

$$\nabla\psi = e^k \cdot \partial_k\psi\colon \mathcal{B}(M) \longrightarrow \mathbb{S} \otimes (\mathbb{R}^n)^*. \tag{8.15}$$

Now we can say something about the Dirac operator. Assume now that X is a 4-dimensional spin manifold with a Riemannian metric. All the linear algebra we did last time tells us that if $\widetilde{\mathcal{B}} \to X$ is the principal Spin_4-bundle of frames, we have Spin_4-equivariant maps $\psi^\pm\colon \widetilde{\mathcal{B}} \to \mathbb{S}^\pm$, where $\mathbb{S}^\pm$ are as in the previous lecture, so we obtain Clifford multiplication $\gamma\colon (\mathbb{R}^4)^* \otimes \mathbb{S}^\pm \to \mathbb{S}^\mp$, and we proved a lemma that

$$\gamma(e^i)\gamma(e^j) + \gamma(e^j)\gamma(e^i) = -\delta^{ij}. \tag{8.16}$$

Definition 8.17. Let $S^\pm$ denote the spinor bundles on X. The *Dirac operator* $D\colon \Omega^0_X(S^\pm) \to \Omega^0_X(S^\mp)$ is defined by

$$D\psi \coloneqq \gamma(e^k)\partial_k\psi.$$

It's easy to see this is a first-order differential operator: we differentiated once.

In the last ten minutes,[20] we'll compute the square of the Dirac operator.

$$D^2\psi = \gamma(e^k)\gamma(e^\ell)\partial_k\partial_\ell\psi \tag{8.18}$$

$$= -\sum_{k=1}^n \partial_k^2\psi + \sum_{k<\ell}\gamma(e^k)\gamma(e^\ell)[\partial_k, \partial_\ell]\psi \tag{8.19}$$

$$= -\nabla^*\nabla\psi + \frac{1}{4}R_{\mathrm{scal}}. \tag{8.20}$$

This is called the *Weitzenböck formula*.[21] Here $\nabla^*\nabla$, meaning the composition of ∇ with its formal adjoint, is called the *covariant Laplacian*, and exists for any associated bundle for the bundle of frames on a Riemannian manifold.

Now lets choose a Hermitian vector bundle $E \to X$ of rank N with a connection A and curvature F_A, and let $P \to X$ denote the principal U_N-bundle of unitary bases for E; then A is a connection here, in the sense that $A \in \Omega^1_P(\mathfrak{u}_N)$. Cross with the spin bundle to obtain $\widetilde{\mathcal{B}} \times_X P \to X$, which is a principal $\mathrm{Spin}_4 \times \mathrm{U}_N$-bundle, and this has a connection which heuristically is $\Theta^{\mathrm{LC}} + A$, and in particular we still have the vector fields $\partial_1, \dots, \partial_4$.

If $\mathbb{E} = \mathbb{C}^N$ is the model vector space for E, we can consider $\mathrm{Spin}_4 \times \mathrm{U}_N$-equivariant maps $\psi\colon \widetilde{\mathcal{B}} \times_X P \to \mathbb{S}^\pm \otimes \mathbb{E}$, and hence obtain a Dirac operator

$$D_A\colon \Omega^0_X(S^+ \otimes E) \longrightarrow \Omega^0_X(S^- \otimes E), \tag{8.21}$$

and this squares to TODO: I had to go.

Lecture 9.

The Nahm transform for anti-self-dual connections: 2/17/19

"That's why I wrote it that way, thinking you might try to pull that one on me."

Today, we're going to discuss a kind of Fourier transform for anti-self-dual forms on a Euclidean torus, mostly following Donaldson-Kronheimer [DK97].

Definition 9.1. A *lattice* Λ in a real vector space V is a finitely generated abelian subgroup of V, necessarily free. If its rank is equal to the dimension of V, it's called *full*.

[20] Measured in my local frame, we have -3 minutes.

[21] The beer is spelled differently: weizenbock.

Equivalently, a full lattice is the $\mathbb{Z}$-span of a basis of V.

Now let V be a 4-dimensional oriented inner product space and $\Lambda \subset V$ be a full lattice. The model example is $V = \mathbb{R}^4$ and $\Lambda = \mathbb{Z}^4$. Then $T := V/\Lambda$ is a torus with an orientation and Riemannian metric induced from V. It is also an abelian Lie group.

Let $\Lambda^* := \mathrm{Hom}(\Lambda, \mathbb{Z})$ denote the *dual lattice*, which sits inside $V^* = \mathrm{Hom}(V, \mathbb{R})$. There is an isomorphism $\Lambda^* \xrightarrow{\cong} \mathrm{Hom}(T, \mathbb{T})$ sending $\theta\colon \Lambda \to \mathbb{Z}$ to $v \mapsto e^{2\pi i\theta(v)}$.

Definition 9.2. The *dual torus* to T is $T^* := V^*/\Lambda^*$.

There is a universal cheracter $\chi\colon T \times \Lambda^* \to \mathbb{T}$: given $\theta \in \Lambda^*$ and $x \in T$, $\chi(x, \theta) := \exp(2\pi i\theta(x))$ (essentially, "evaluate θ on x"). There are projection maps

$$\begin{tikzcd} & T \times \Lambda^* \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ T & & \Lambda^*, \end{tikzcd} \tag{9.3}$$

from which we can define a Fourier transform from (a certain class of) functions on T to (a certain class of) functions on Λ^*. Specifically, given a function f on T, we'd like to define

$$\widehat{f} := (p_2)_*\chi^{-1}p_1^*f, \tag{9.4}$$

or more explicitly,

$$\widehat{f}(e^v) = \int_T \mathrm{d}t\, e^{-2\pi i\theta(v)} \cdot f(e^v). \tag{9.5}$$

Remark 9.6. This is an instance of a more general phenomenon for locally compact abelian groups called *Pontrjagin duality*. Another example is a finite analogue: letting $\mu_n \subset \mathbb{C}$ denote the n^{th} roots of unity, we have a pairing $\mathbb{Z}/n \times \mu_n \to \mathbb{T}$ sending $(k, \lambda) \mapsto \lambda^k$. These two groups are noncanonically isomorphic,, but each is canonically isomorphic to the character group of the other. In our setting, T and Λ^* aren't isomorphic, but are still each others' character groups. ◀

We will discuss a categorified version of this, which in the algebro-geometric setting is called the Fourier-Mukai transform. We will use T^* instead of Λ^*, and push-pull along the diagram

$$\begin{tikzcd} & T \times T^* \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ T & & T^*. \end{tikzcd} \tag{9.7}$$

Instead of the universal character we will have a universal line bunde $\mathscr{L} \to T \times T^*$ with a Hermitian connection, allowing us to exchange vector bundles on T and on T^*. This $\mathscr{L}$ will be called the *Poincaré line bundle*. Heuristically, the formula will look like (9.4): if $E \to T$ is a vector bundle, we let

$$\widehat{E} = (p_2)_*(\mathscr{L} \otimes p_1^*E), \tag{9.8}$$

In order to make this precise, we have some details to figure out, namely

(1) constructing $\mathscr{L}$ and its Hermitian connection,
(2) interpreting $(p_2)_*$, and
(3) incorporating connections on E and $\widehat{E}$ into the story.

Once we do this, though, we will be able to prove some nice theorems: generic anti-self-dual connections on E pass to anti-self-dual connections on $\widehat{E}$, and there will be an inversion formula.

Remark 9.9. If we give a complex structure on V, then T and T^* acquire the structure of complex manifolds, and we can do all of this with sheaves. This is what's called the Fourier-Mukai transform. ◀

Instead of constructing the Poincaré line bundle, we'll construct a principal $\mathbb{T}$-bundle $P \to T \times T^*$ with connection; this is equivalent, via passing to the associated bundle $P \times_{\mathbb{T}} \mathbb{C} \to T \times T^*$. Begin with the trivial principal $\mathbb{T}$-bundle $V \times V^* \times \mathbb{T} \to V \times V^*$, with its trivial connection $\Theta \in \Omega^1_{V\times V^*\times\mathbb{T}}(i\mathbb{R})$; specifically,

$$\Theta_{(v,\theta,z)}(\dot{v}, \dot{v}^*, \dot{z}) = 2\pi i\theta(\dot{v}) + z^{-1}z. \tag{9.10}$$

This is a universal family of flat connections; you can check that the holonomy vanishes.

Now $\Lambda\times\Lambda^*$ acts on this trivial principal $\mathbb{T}$-bundle by

$$\lambda\cdot(v,\theta,z) \coloneqq (v+\lambda,\theta,z) \tag{9.11a}$$

$$\lambda^*\cdot(v,\theta,z) \coloneqq (v,\theta+\lambda^*,\exp(-2\pi i\lambda^*(v))z), \tag{9.11b}$$

and this covers the usual $\Lambda\times\Lambda^*$-action on the base, so it descends to a principal $\mathbb{T}$-bundle $P\to T\times T^*$ with a flat connection $\overline{\Theta}$. You can think of this as a family of flat connections on T parameterized by T^*; these bundles are trivializable,[22] but not canonically so. Similarly, you can view this as a family of flat connections on T^* parameterized by T, and these are also trivializable but not canonically trivialized. However, the total bundle is not trivial.

The connection form Ω is a translation-invariant (under the group operation) purely imaginary 2-form satisfying the formula

$$\Omega\Big((\dot v_1,\dot\theta_1),(\dot v_2,\dot\theta_2)\Big) = 2\pi i\Big(\dot\theta_1(\dot v_1)-\dot\theta_2(\dot v_2)\Big), \tag{9.12}$$

or, in other words,

$$\Omega = 2\pi i\langle \mathrm{d}\theta\wedge\mathrm{d}v\rangle. \tag{9.13}$$

Here $\mathrm{d}\theta\in\Omega^1_{T^*}(V^*)$ and $\mathrm{d}v\in\Omega^1_T(V)$ are precisely the Maurer-Cartan forms for these two tori.

Now let $E\to T$ be a Hermitian vector bundle with covariant derivative ∇_A. We can pull both of these back to $T\times T^*$ and tensor with $\mathscr{L}$, but how do we push forward? This will involve the Dirac operator! Let $\mathbb{S}^+$ and $\mathbb{S}^-$ be the spinor spaces, so that $V^*=\mathbb{S}^+\otimes\mathbb{S}^-$. The spinor bundles over V are the trivial bundles $\underline{\mathbb{S}}^\pm\to V$, which then descend to the torus as trivial bundles. Given $\theta\in T^*$, we have Dirac operators

$$D^\pm_{A,\theta}\colon \Gamma_T(\underline{\mathbb{S}}^\pm\otimes E\otimes\mathscr{L}_\theta)\longrightarrow\Gamma_T(\underline{\mathbb{S}}^\mp\otimes E\otimes\mathscr{L}_\theta). \tag{9.14}$$

We will use this to define the pushforward $(p_2)_*$ – crucially, since T is compact, these are Fredholm operators, which play the role of sheaves in differential geometry: we have the kernel and cokernel of a Fredholm operator, but in a family these can jump. We would like to extract an honest vector bundle, so we'll ask for a hypothesis such that $\ker(D^+_{A,\theta})=0$. This is equivalent to saying that $D^-_{A,\theta}=(D^+_{A,\theta})^*$ is surjective.[23] Assuming this, $\ker(D^-_A)\to T^*$ (i.e. the fiber above θ is $\ker(D^-_{A,\theta})$) is a vector bundle.

Example 9.15. Before we discuss that hypothesis, let's look at a toy model. Consider the matrix

$$D_z \coloneqq \begin{pmatrix}1&0&0\\0&z&0\end{pmatrix}\colon\mathbb{C}^3\to\mathbb{C}^2, \tag{9.16}$$

where $z\in\mathbb{C}$; hence this is a family of linear operators parametrized by $\mathbb{C}$. This is surjective for $z\neq0$, with kernel $\mathbb{C}\cdot(0,0,1)$. At 0, the kernel is $\mathbb{C}\cdot(0,0,1)\oplus\mathbb{C}\cdot(0,1,0)$. The dimension of the kernel minus the dimension of the cokernel is always 1, but the kernel isn't a vector bundle; it does define a sheaf, however. ◂

Proposition 9.17. *If A is anti-self-dual, then $\ker(D^+_{A,\theta})=0$, hence we can define the pushforward.*

Proof. This is a vanishing theorem, and many vanishing theorems follow a similar proof. We will use the Weitzenböck formula

$$D^-_AD^+_A = \nabla^*_A\nabla_A - F^+_A, \tag{9.18}$$

and we know $F^+_A=0$. Suppose $\psi\in\Gamma_T(\underline{\mathbb{S}}^+\otimes E\otimes\mathscr{L}_\theta)$ and $D^+_A\psi=0$, but $\psi\neq0$. Then

$$0=\int_T\langle\psi,\nabla^*_A\nabla_A\psi\rangle\,\mathrm{d}x=\int_T\|\nabla_A\psi\|^2. \tag{9.19}$$

Since $\|\nabla_A\psi\|^2$ is nonnegative, this means it's zero, so ψ is covariantly constant. This is a typical application of the Weitzenböck formula. ⊠

[22] This is a flat connection on a torus; Chern-Weil theory implies the Chern class of P vanishes, so it's abstractly trivializable.

[23] Though this is a simple linear-algebra exercise in finite dimensions, in the infinite-dimensional Fredholm setting it's a harder theorem known as the *Fredholm alternative*.

Spelling this out in more detail, let s_1, s_2 be a basis for $\mathbb{S}^+$. We can write

$$\psi(t) = \psi_1(t)s_1 + \psi_2(t)s_2 \tag{9.20}$$

for $t \in T$, where $\psi_i \in \Gamma_T(E \otimes \mathscr{L}_\theta)$ and $\nabla_A \psi_i = 0$. This section ψ defines a subbundle of $E \otimes \mathscr{L}_\theta$ of rank 1, and the fact that it's covariantly constant means that it splits off! Therefore there is some other bundle $\widetilde{E} \subset E \otimes \mathscr{L}_\theta$ with $E \otimes \mathscr{L}_\theta = \widetilde{E} \oplus \underline{\mathbb{C}}$; the connection form is diagonal, and is the standard connection on $\underline{\mathbb{C}}$. Now, tensoring with $\mathscr{L}_\theta^*$,

$$E = \widetilde{E} \otimes \mathscr{L}_\theta^* \oplus \mathscr{L}_\theta^* = E' \oplus \mathscr{L}_\theta^*. \tag{9.21}$$

Definition 9.22. We say E is *without flat factors* (WFF) if there is no decomposition $E = E' \oplus L$ where L is flat.

This is a generic condition.

Anyways, we have our Fourier-transformed bundle $\widehat{E} \coloneqq \ker(D_A^-)$. The next step is to define the covariant derivative on $\widehat{E}$, via a more general definition.

Definition 9.23. of vector bundles $K, L \to M$, where M is a smooth manifold, and a map $R\colon K \to L$. We will assume K and L come with covariant derivatives ∇^K, resp. ∇^L, and that either R is Fredholm, or both K and L have finite rank. Assuming that R is fiberwise surjective, then $\ker R \to M$ is a vector bundle. Choose a projection $\pi\colon K \to \ker R$ such that $\pi \circ i = \mathrm{id}_{\ker R}$, where $i\colon \ker R \hookrightarrow K$ is inclusion. Then the *compressed covariant derivative* on $\ker R$ is

$$\pi \circ \nabla^K \circ i \colon \Omega_M^0(\ker R) \longrightarrow \Omega_M^1(\ker R).$$

This general construction applies to our situation, where $K = \underline{\mathbb{S}}^- \otimes E \otimes \mathscr{L}_\theta$, $L = \underline{\mathbb{S}}^+ \otimes E \otimes \mathscr{L}_\theta$, and $R = D_{A,\theta}^-$. (TODO: how did we get π?) Therefore we pick up a connection $\widehat{A}$ on $\widehat{E} \to T^*$.

Theorem 9.24. *Assuming A is anti-self-dual and (E, ∇_A) is without flat factors, then $\widehat{A}$ is anti-self-dual.*

We want to prove that $F_{\widehat{A}} \in \Omega_{T^*}^{2;-}(\operatorname{End} \widehat{E})$, and will do so by proving that in every complex structure on T^*, $F_{\widehat{A}}$ is of type $(1,1)$, which by Theorem 7.22 suffices. The complex structures on T^* are parameterized by $\mathbb{P}(\mathbb{S}^+)$. We will need some more ingredients to do this, which we will discuss next time:

(1) the Chern connection on a holomorphic, Hermitian vector bundle;
(2) how to express the Dirac operator in terms of $\overline{\partial}$; and
(3) how to fit these vector bundles into a family of Dolbeault complexes.

This will allow us to identify the connection with the Chern connection, which always has type $(1,1)$; we can avoid a calculation in favor of a geometric proof.

Lecture 10.

The Chern connection: 2/19/19

Today, we'll prove Theorem 9.24, that the Fourier transform of an anti-self-dual connection on a 4-torus is anti-self-dual. To do this, we'll fill in some facts about geometry, as promised last time.

Definition 10.1. Let $\pi\colon E \to M$ be a complex vector bundle, and suppose that both E and M are complex manifolds and π is holomorphic. Then $E \to M$ is called a *holomorphic* vector bundle.

This implies in particular that locally on M, there is a basis $s_1, \dots, s_N$ of holomorphic sections of E. Thus any section locally has the form $f^i s_i$ for $\mathbb{C}$-valued f^i, allowing us to define the $\overline{\partial}_E$ operator by $\overline{\partial}_E(f^i s_i) \coloneqq \overline{\partial} f^i s_i$. Of course, one must check that this is independent of coordinates.

Remark 10.2. Conversely, suppose $\pi\colon E \to M$ is a C^∞ complex vector bundle, M is a complex manifold, and we have an operator $\overline{\partial}_E\colon \Omega_M^{0,0}(E) \to \Omega_M^{0,1}(E)$ satisfying a Leibniz rule and $\overline{\partial}_E^2 = 0$. Then $\pi\colon E \to M$ admits the structure of a holomorphic vector bundle for which $\overline{\partial}_E$ is the operator defined above.[24] ◀

Theorem 10.3. *Let $\pi\colon E \to M$ be a holomorphic vector bundle over a complex manifold, and endow E with a Hermitian metric $\langle -, - \rangle\colon \overline{E} \otimes E \to \underline{\mathbb{C}}$. Then there exists a unique connection ∇ on $E \to M$ such that*

[24] This is less difficult to prove that you might expect: you can certainly lean on the Neulander-Nirenberg theorem, but there's a more elementary proof you can find in Donaldson-Kronheimer [DK97].

(1) *if* $\nabla = \nabla' + \nabla''$ *is the decomposition induced by* $\nabla\colon \Omega_M^{0,0}(E) \to \Omega_M^{1,0}(E) \oplus \Omega_M^{0,1}(E)$, *then* $\nabla'' = \overline{\partial}_E$;

(2) ∇ *is compatible with the metric, in the sense that*

$$\mathrm{d}\langle \overline{s}_1, s_2\rangle = \langle \nabla\overline{s}_1, s_2\rangle + \langle \overline{s}_1, \nabla s_2\rangle; \tag{10.4}$$

(3) *and the curvature of* ∇ *is type* $(1,1)$.

In this setting, ∇ is called the *Chern connection*.

Proof. We work locally in a holomorphic chart U, allowing us to choose a basis $s_1, \dots, s_N$ of holomorphic sections of E. Define $h_{ij} \coloneqq \langle \overline{s}_i, s_j\rangle$ and write $\nabla s_j = \alpha_j^i s_i$, so $\alpha_j^i \in \Omega_U^{1,0}$. Using (10.4), we have

$$\mathrm{d}h_{\bar{\imath}j} = \overline{\alpha}_i^k h_{\overline{k}j} + h_{\bar{\imath}k}\alpha_j^k \tag{10.5}$$

$$\partial h_{\bar{\imath}j} = h_{\bar{\imath}k}\alpha_j^k \tag{10.6}$$

$$\alpha_j^i = h^{i\overline{\ell}}\partial h_{\overline{\ell}j}. \tag{10.7}$$

Here $h^{i\overline{\ell}}$ denotes the components of the inverse matrix. Now (10.7) completely characterizes ∇, forcing uniqueness, and we can take it as a definition and check that it's consistent, guaranteeing existence. ⊠

Another proof. Let $t_1, \dots, t_N$ be a local unitary basis of C^∞ sections, which means that $\langle \overline{t}_i, t_j\rangle = \delta_{ij}$, and write $\nabla t_j = \beta_j^i t_i$. Then $\beta = (\beta_j^i)$ is skew-Hermitian, so we can write $\beta = \beta' + \beta''$, where $\beta' \in \Omega_M^{1,0}$ and $\beta'' \in \Omega_M^{0,1}$. Since β is skew-Hermitian, $\beta' = -(\beta'')^*$, but β'' is given by $\overline{\partial}_E$, so we've again completely determined ∇. ⊠

Remark 10.8. The fact that the curvature is type $(1,1)$ is saying something about invariance under scalars: a $\lambda \in \mathbb{C}^\times$ acts on a form of type (p,q) by λ^{p-q}, so it acts by the identity on (p,p)-forms. ◂

A third proof sketch of Theorem 10.3. Let $P \to M$ be the principal $\mathrm{GL}_N(\mathbb{C})$-bundle of frames of E. This is a *holomorphic principal bundle*, meaning the total space and the projection map are both holomorphic. Inside it is the principal U_N-bundle of unitary frames with respect to $\langle -,-\rangle$. We would like to produce a connection on P, namely a $\mathrm{GL}_N(\mathbb{C})$-invariant horizontal distribution $H \subset TP$.

Define $H \coloneqq TQ \cap I(TQ)$, where $I \in \mathrm{End}(TP)$ is the action of i. Then one can check (TODO) that the induced covariant derivative satisfies the conditions in the theorem statement. ⊠

This proof is due to Is Singer.

Remark 10.9. Suppose $E = TM$ and M has a Riemannian metric; then in the last proof, $P = \mathcal{B}_{\mathrm{GL}}(M)$ and $Q = \mathcal{B}_{\mathrm{U}}(M)$. The proof above tells us that $\mathcal{B}_{\mathrm{U}}(M)$ picks up the Chern connection Θ^c. There's another connection on $\mathcal{B}_{\mathrm{U}}(M)$ induced from the Levi-Civita connection $\Theta^{\ell c}$ on $\mathcal{B}_{\mathrm{O}}(M)$, using the inclusion $\mathcal{B}_{\mathrm{U}}(M) \hookrightarrow \mathcal{B}_{\mathrm{O}}(M)$ induced from the inclusion $\mathrm{U}_N \hookrightarrow \mathrm{O}_{2N}$.

Are these connections equal? This is expressing that the complex structure and Riemannian metric are compatible, so of course what we get is that this is equivalent to M being Kähler. This is perhaps the most beautiful definition of Kähler manifolds. ◂

More generally, given any kind of tangential structure (e.g. spin structure, almost symplectic structure) on a manifold, and a Riemannian metric, you can ask whether there's a canonical torsion-free connection on the bundle of frames.

Recall that on a complex manifold M, the *canonical bundle* is the complex line bundle $K_M \coloneqq \mathrm{Det}_{\mathbb{C}}(T^*M)$.

Theorem 10.10 (Atiyah-Bott-Shapiro [ABS64], Hitchin [Hit74])**.** *Let M be a Kähler spin manifold. Then the spin structure induces a holomorphic Hermitian line bundle $L \to M$ together with an isomorphism $L^{\otimes 2} \to K_M$, and the Dirac operator is*

$$\overline{\partial}_L + \overline{\partial}_L^*\colon \Omega_N^{0,\mathrm{even/odd}} \longrightarrow \Omega_M^{0,\mathrm{odd/even}}(L).$$

To elaborate on the "even/odd" and "odd/even" pieces, recall that $\overline{\partial}_L$ maps from (p,q)-forms to $(p,q+1)$-forms, and therefore $\overline{\partial}_L^*$, its adjoint with respect to the Riemannian metric on M, sends $(p,q+1)$-forms to (p,q)-forms.

Remark 10.11. Not every Kähler manifold is spin, e.g. $\mathbb{CP}^2$. And a spin structure is extra data, which in the holomorphic world is given by a square root of the canonical bundle. ◂

We're not going to prove Theorem 10.10 in full generality, but we've developed enough of the theory of spin geometry in dimension 4 to do it, using the linear algebra of the spin representations that we've worked through. The details are worked out in Donaldson-Kronheimer [DK97].

Now let K^0, K^1, K^2 be complex vector spaces, so the trivial bundles $\underline{K}^i \to M$ over a complex manifold M are holomorphic. Suppose we have maps $\alpha\colon \underline{K}^0 \to \underline{K}^1$ and $\beta\colon \underline{K}^1 \to \underline{K}^2$, and suppose $\beta\circ\alpha = 0$, so that we have a complex. The cohomology groups might not be vector bundles: the dimension could jump. But if we assume α is injective and β is surjective, then the zeroth and second cohomology vanish, so as long as α and β are Fredholm (automatic if these spaces are finite-dimensional) the Euler characteristic of the complex is equal to $\dim H^1$, and therefore the dimension can't jump, and H^1 is a vector bundle.[25]

Theorem 10.12. *The cohomology $H^1 \to M$ has a holomorphic structure. Moreover, given Hermitian metrics on $\underline{K}^i \to M$, the compressed connection on $H^1 \to M$ is its Chern connection.*

Proof. For the first part, define a section of $H^1 \to M$ to be holomorphic if locally it has a lift to a holomorphic section $\ker(\beta) \to M$. Then we need to show that there's locally a basis of holomorphic sections. Fix an $m_0 \in M$ and $\overline{s}_0 \in H^1_{m_0}$ which has a holomorphic lift to $e \in K^1_{m_0}$ with $\beta_{m_0} e = 0$. Choose a $P\colon K^2_{m_0} \to K^1_{m_0}$ such that $\beta_{m_0} P = \mathrm{id}$, and trivialize each $\underline{K}^i \to M$ near m_0. Then in a neighborhood U of m_0, we can write $s_m = e + j_m$, where $m_{m_0} = 0$ and $m_j \in \mathrm{Im}(P)$, and we can write $\beta_m = \beta_{m_0} + \eta_m$, where of course $\eta_{m_0} = 0$. So we now have the equation

$$0 = \beta s = P(\beta_{m_0} + \eta_m)(e + j_m) = P\eta_m e + P\beta_{m_0} j_m + P\eta_m j_m, \tag{10.13}$$

i.e. we want to solve

$$(1 + P\eta_m) j_m = -P\eta_m e. \tag{10.14}$$

That is, a solution to this equation allows us to extend $\overline{s}_0$ to a holomorphic section of H^1. We can just write down the solution: let

$$j_m := -\sum_{i=0}^{\infty} (-P\eta_m)^i P\eta_m e. \tag{10.15}$$

This converges (roughly because $1 + \text{small}$ is invertible), and is holomorphic. In particular, we can lift a basis of $K^1_{m_0}$ to some sections that we hope are a local basis. It will be left as an exercise to show that every holomorphic section can be written as a linear combination of these.

For the second part, let $\pi\colon \underline{K}^1 \to \ker\alpha^* = (\mathrm{Im}\,\alpha)^\perp$, where $\alpha^*\colon \underline{K}^1 \to \underline{K}^0$ is the adjoint in the Hermitian metrics. Then π is given by

$$k \longmapsto k - \alpha(\alpha^*\alpha)^{-1}\alpha^* k. \tag{10.16}$$

Since $[\overline{\partial}, \alpha] = 0$, then $[\overline{\partial}, \pi] \subset \mathrm{Im}(\alpha)$, and therefore $\pi[\overline{\partial}, \pi] = 0$. Suppose k is a local holomorphic section of $\ker\beta \to U$; then, $\pi(k)$ is a section of $(\ker\alpha^*) \cap (\ker\beta) \to U$ and

$$\pi\overline{\partial}_{K^1}(\pi(k)) = \pi[\overline{\partial}_{K^1}, \pi]k + \pi^2(\overline{\partial}_{K^1}, k) = 0. \tag{10.17}$$

There's a little more to do, but we're out of time. ⊠

This is an instance of something general, that holomorphy and unitarity are often in tension. For example, even just in complex analysis, a norm-1 holomorphic function is constant.

Lecture 11.

Proof of the Nahm transform, part 1: 2/26/19

"We're not spinning our wheels here. I mean, we are*, but..."*

[25] Implicit in this paragraph is the definition of a holomorphic vector bundle of possibly infinite rank. One can make sense of this by determining the right notion of a holomorphic function on a complex Banach space, and then using the same words as in Definition 10.1.

We're still discussing the Nahm transform (the analogue of the Fourier transform for anti-self-dual connections on tori). This story is nice because tori are quotients of vector spaces; but there are also more quotients of vector spaces we could study.

As usual, let V be a four-dimensional real oriented inner product space, $\Lambda \subset V$ be a full lattice, V^* be the dual, and $\Lambda^* := \mathrm{Hom}(\Lambda, \mathbb{Z})$. We formed the torus $T = V/\Lambda$ and its dual torus $T^* = V^*/\Lambda^*$. Then there is a correspondence $T \leftarrow T \times T^* \rightarrow T^*$, and as we usually do with correspondences, we'd like to do a pullback-and-pushforward weighted by an integral kernel. On functions, this looks like the usual kernel transform,

$$\widehat{f}(x) = \int \mathrm{d}y\, K(x,y) f(y). \tag{11.1}$$

But we're interesting in vector bundles (and connections), not functions, so the kernel is the Poincaré bundle. Well, there are two Poincaré bundles, just as how the Fourier transform uses $e^{-i\pi x\cdot\xi}$ and its inverse uses $e^{i\pi x\cdot \xi}$. The two Poincaré bundles $\mathbb{P}, \widetilde{\mathbb{P}} \to T \times T^*$ are Hermitian line bundles with connection. Recall that a point $v \in T$ defines a flat line bundle over T^*, and vice versa; therefore we say that $\mathbb{P}|_{T\times\{\theta\}} \to T$ is the flat line bundle on T with holonomy θ, and analogously for the bundle over $\{v\} \times T^*$. For $\widetilde{\mathbb{P}}$, one of the holonomies switches to $-v$. However, there is curvature on $T \times T^*$, given by $\pm i\langle \mathrm{d}v \wedge \mathrm{d}\theta\rangle$ (here $v \in T$ and $\theta \in T^*$; $\mathrm{d}v$ is a translation-invariant one-form on V, and hence descends to T, and similarly for $\mathrm{d}\theta$).

On to the Nahm transform: given a Hermitian vector bundle $E \to T$ with connection A, we would like to define a Hermitian vector bundle $\widehat{E} \to T^*$ with connection $\widehat{A}$. Fix spin spaces $\mathbb{S}^\pm$ with $V = \mathbb{S}^+ \otimes \mathbb{S}^-$, so the spinors on T are functions $T \to \mathbb{S}^\pm$ (in other words, on T, the spinor bundles are trivial bundles with fiber $\mathbb{S}^\pm$, so sections of them are functions to $\mathbb{S}$). We have a Dirac operator

$$D_A^-|_\theta \colon \Gamma(\underline{\mathbb{S}}^- \otimes E \otimes \mathbb{P}_\theta) \longrightarrow \Gamma(\underline{\mathbb{S}}^+ \otimes E \otimes \mathbb{P}_\theta). \tag{11.2}$$

If A is anti-self-dual and without flat factors, then $D_A^-|_\theta$ is surjective, and we define $\widehat{E}_\theta := \ker(D_A^-|_\theta)$; because the Dirac operator is Fredholm, this is finite-dimensional. We also want a covariant derivative $\widehat{A}$, which we define to the compressed covariant derivative associated to the Fredholm map $D_A^-|_\theta$. We want to prove Theorem 9.24, that $\widehat{A}$ is anti-self-dual; we will prove this by showing it's type $(1,1)$ in every complex structure on T^*.

Proof of Theorem 9.24. A line $L \in \mathbb{P}(\mathbb{S}^+)$ defines a complex structure (V, I), as we discussed; call this two-dimensional complex vector space U. The dual space $U^* = (V^*, -I^*)$ – the reason we use $-I^*$ is so that the duality pairing works nicely. Specifically, if $v \in V$ and $\theta^* \in V^*$, we want $\langle\theta, v\rangle = \langle I_{V^*}\theta, I_V v\rangle$; since $I^2 = -1$, if we took I^* instead of $-I^*$, there would be an extra minus sign.

Now split $\nabla_A = \nabla'_A + \nabla''_A$, by types, so $\nabla''_A \colon \Omega_T^{0,0}(E) \to \Omega_T^{0,1}(E)$. Since A is anti-self-dual, its curvature has type $(1,1)$, and therefore $(\nabla''_A)^2 = 0$. Therefore $\overline{\partial}_A := \nabla''_A$ defines a holomorphic structure on E; we will call this holomorphic vector bundle $\mathcal{E} \to T$.

Remark 11.3. Since the curvatures of $\mathbb{P}, \widetilde{P} \to T \times T^*$ also have type $(1,1)$, they also pick up holomorphic structures in this way; we denote these holomorphic bundles $\mathcal{P}, \widetilde{\mathcal{P}} \to T \times T^*$. ◀

Lemma 11.4. *Let (M, I) be an almost complex manifold and $\omega \in \Omega_M^2(\mathbb{C})$. Then ω has type $(1,1)$ iff $\omega(I\xi_1, I\xi_2) = \omega(\xi_1, \xi_2)$ for all $\xi_1, \xi_2 \in TM$.*

Proof. Split ω by type:

$$\omega = \underbrace{\omega_{ij}\,\mathrm{d}z^i \wedge \mathrm{d}z^j}_{(2,0)} + \underbrace{\omega_{i\overline{j}}\,\mathrm{d}z^i \wedge \overline{\mathrm{d}z^j}}_{(1,1)} + \underbrace{\omega_{\overline{i}\overline{j}}\,\overline{\mathrm{d}z^i} \wedge \overline{\mathrm{d}z^j}}_{(0,2)}. \tag{11.5}$$

Now $\mathbb{C}^\times$ acts on $\Omega_M^2(\mathbb{C})$ by scalar multiplication, and a $\lambda \in \mathbb{C}^\times$ acts by λ^2 on $\Omega_M^{2,0}(\mathbb{C})$, by 1 on $\Omega_M^{1,1}(\mathbb{C})$, and by λ^{-2} on $\Omega_M^{0,2}(\mathbb{C})$. Since I acts by multiplication by i, ω can only be fixed by I if it's of type $(1,1)$. ⊠

Now that everything is holomorphic, we have the Dolbeault complex

$$\Omega_T^{0,0}(\mathcal{E} \otimes \mathcal{P}_\theta) \underset{\overline{\partial}_A^*}{\overset{\overline{\partial}_A}{\rightleftarrows}} \Omega_T^{0,1}(\mathcal{E} \otimes \mathcal{P}_\theta) \underset{\overline{\partial}_A^*}{\overset{\overline{\partial}_A}{\rightleftarrows}} \Omega_T^{0,2}(\mathcal{E} \otimes \mathcal{P}_\theta), \tag{11.6}$$

and the Dirac operator is

$$D_A = \overline{\partial}_A + \overline{\partial}_A^* \colon \Omega_T^{0,1}(\mathcal{E} \otimes \mathcal{P}_\theta) \longrightarrow \tag{11.7}$$

Hence, by the usual Hodge theory story,

$$\ker(D_A^-|_\theta) \cong H^1(T, \mathcal{E} \otimes \mathcal{P}_\theta) \tag{11.8a}$$

$$\operatorname{coker}(D_A^-|_\theta) \cong H^0(T, \mathcal{E} \otimes \mathcal{P}_\theta) \oplus H^2(T, \mathcal{E} \otimes \mathcal{P}_\theta). \tag{11.8b}$$

Since E is without flat factors, though, $H^0 \oplus H^2$ vanishes. Therefore the fiber of $\widehat{\mathcal{E}}$ at θ is exactly $H^1(T, \mathcal{E} \otimes \mathcal{P}_\theta)$. This is exactly what the *Mukai transform* in algebraic (or complex) geometry does: given the correspondence, pull back from T to $T \times T^*$, then push forward to T^*. Here, though "push forward" means something different: we have to generalize from vector bundles to sheaves. Given a sheaf $\mathscr{F} \to X$ and a map $f \colon X \to Y$, we obtain a pushforward sheaf $f_*\mathscr{F} \to Y$ whose space of sections on an open $U \subset Y$ is defined to be $\mathscr{F}(f^{-1}(U))$. However, the pushforward isn't exact, so we have to take its derived functors, which amounts to cohomology. Here, we only can have H^0 through H^2, and H^0 and H^2 vanish, leaving just H^1. Moreover, the fact that E is without flat factors implies that what we get is a vector bundle, rather than just a sheaf. In particular, as C^∞ vector bundles, $\widehat{\mathcal{E}}$ is identified with what we got from the Nahm transform.

The point is (TODO details) that, just as we proved last time, the compressed connection is the Chern connection, and therefore it has type $(1,1)$. Since the complex structure was arbitrary, this implies $\widehat{A}$ is anti-self-dual. ⊠

Now we want to study the moduli space of such $E \to T$. Generally one fixes discrete invariants to make the problem simpler.

Proposition 11.9. *The discrete invariants of a complex (i.e. just C^∞) vector bundle $E \to T$ are: the rank in $\mathbb{Z}^{\geq 0}$ and its Chern classes $c_1(E) \in H^2(T)$ and $c_2(E) \in H^4(T) \cong \mathbb{Z}$. That is, these determine the C^∞ isomorphism class of E.*

This is a good exercise in algebraic geometry.

Theorem 11.10. *The discrete invariants of $\widehat{E} \to T^*$ can be given in terms of those of E:*

- $\operatorname{rank}(\widehat{E}) = c_2(E) - (1/2)c_1(E)^2$,
- $c_1(\widehat{E}) = \sigma(c_1(E))$, *and*
- $c_2(\widehat{E}) = \operatorname{rank} E + (1/2)c_1(E)^2$.

These follow from the Atiyah-Singer index theorem, though mind that the formulas in the book have a few mistakes. Here $\sigma \colon H^2(T) \to H^2(T^*)$ is a map induced from Poincaré duality. Specifically, there is a canonical class $\kappa \in H^1(T) \otimes H^1(T) \hookrightarrow H^2(T \times T^*)$.[26] Then we define

$$\sigma(x) := (p_2)_*(\kappa^2 p_1^* x). \tag{11.11}$$

The pushforward $(p_2)_*$ is the Gysin map, which comes from Poincaré duality.

Corollary 11.12. *There cannot exist an anti-self-dual connection on a bundle with $c_1 = 0$ and $c_2 = 1$.*

Proof. Suppose such a connection exists, and choose one of minimal rank. Then there exists an anti-self-dual connection on $\widehat{E}$ with rank 1 and $c_2 \neq 0$, but this is a contradiction: the second Chern class of any line bundle vanishes. ⊠

Now we want to prove an analogue of Fourier inversion.

Definition 11.13. Let $\mathcal{F} \to T^*$ be a holomorphic vector bundle without flat factors, and define $\check{\mathcal{F}} \to T$ to be the holomorphic vector bundle whose fiber at $v \in T$ is $H^1(T^*, \mathcal{F} \otimes \widetilde{\mathcal{P}}_v)$.

Given a Hermitian structure on $\mathcal{F}$, we also get one on $\check{\mathcal{F}}$ as before.

Theorem 11.14 (Mukai inversion)**.** *Let $\mathcal{E} \to T$ be a Hermitian holomorphic vector bundle without flat factors. Then $\widehat{\mathcal{E}} \to T^*$ is without flat factors, and there is a natural isomorphism $\omega \colon \check{\widehat{\mathcal{E}}} \to \mathcal{E}$ of Hermitian holomorphic vector bundles on T.*

[26] The fact that we can decompose $H^2(T \times T^*)$ as a sum of $(H^0(T) \otimes H^2(T^*)) \oplus (H^1(T) \otimes H^1(T^*)) \oplus (H^2(T) \otimes H^0(T^*))$ relies on the fact that the homology of T is torsion-free, and is not true in general.

Letting $p_1\colon T\times T^*\to T$ be projection onto the first factor, the complex $\Omega^{0,\bullet}_{T\times T^*}(p_1^*\mathcal{E}\otimes\mathcal{P})$ is actually a double complex $C^{\bullet,\bullet}$. The bigrading (p,q) exists on forms on any Cartesian product: given a k-form, how many of its factors are from T and how many are from T^*? We also have that the differential splits as $\overline{\partial}_1+\overline{\partial}_2$, the former of which sends (p,q)-forms to $(p+1,q)$-forms, and the latter of which sends (p,q)-forms to $(p,q+1)$-forms.

Given a double complex we obtain a spectral sequence: consider the bigraded vector space $E_1^{\bullet,\bullet}:=H^\bullet(C^{\bullet,\bullet};\overline{\partial}_1)$. Since $\overline{\partial}_1$ and $\overline{\partial}_2$ commute, $\overline{\partial}_2$ is a differential on $E_1^{\bullet,\bullet}$, so we can take the cohomology of this complex, and obtain another bigraded vector space $E_2^{\bullet,\bullet}$ often called the *E_2-page* of the spectral sequence. One can in general continue on in this way, finding more differentials using homological algebra, but in this case they all vanish, and this approximates the cohomology of $\Omega^{0,\bullet}_{T\times T^*}(p_1^*\mathcal{E}\otimes\mathcal{P})$, in the sense that it's the associated graded for some filtration of it.

Remark 11.15. We could have also defined $E_1^{\bullet,\bullet}$ and $E_2^{\bullet,\bullet}$ by first taking cohomology with respect to $\overline{\partial}_2$, then using $\overline{\partial}_1$. Then $E_1^{\bullet,\bullet}$ may be different, but $E_2^{\bullet,\bullet}$ will not change. ◀

Theorem 11.16. *The E_2-page of this spectral sequence is as follows.*

$$\begin{array}{c|ccc} 2 & 0 & H^2(T^*,\widehat{\mathcal{E}}) & 0 \\ 1 & 0 & H^1(T^*,\widehat{\mathcal{E}}) & 0 \\ 0 & 0 & H^0(T^*,\widehat{\mathcal{E}}) & 0 \\ \hline & 0 & 1 & 2 \end{array} \tag{11.17}$$

The total cohomology of the double complex vanishes except in (total) degree 2, and $H^2(T\times T^*,p_1^*\mathcal{E}\otimes\mathcal{P})\cong E_0$.

We will delve into the proof next time.

Lecture 12.

Proof of the Nahm transform, part 2: 2/28/19

"I have two things to do and ten minutes to do them, and no time for either."

As before, let V be an oriented real four-dimensional inner product space, and choose a complex structure I on V. Let $U:=(V,I)$ and $U^*:=(V^*,-I^*)$. Let $\Lambda\subset T$ be a full lattice and $T:=V/\Lambda$, which acquires a flat Kähler structure from the inner product and complex structure on V. The same is true for the dual torus $T^*:=V^*/\Lambda^*$. We also defined the Poincaré bundle $\mathcal{P}\to T\times T^*$, which we have made into a holomorphic line bundle.

With this structure in place, suppose $\mathcal{E}\to T$ is a holomorphic vector bundle without flat factors. Then we can define $\widehat{\mathcal{E}}:=p_{2_*}(p_1^*\mathcal{E}\otimes\mathcal{P})$, where $p_1\colon T\times T^*\to T$ and $p_2\colon T\times T^*\to T^*$ are the projections onto the two factors. Here, "pushforward" means sheaf cohomology, but the assumption that $\mathcal{E}$ is without flat factors means that $H^0(T,\mathcal{E})=0$ and $H^2(T,\mathcal{E})=0$, so we can just take H^1. We would like to prove a Fourier inversion formula for this transform.

We then defined a double Dolbeault complex

$$C:=(\Omega^{0,\bullet}_{T\times T^*}(p_1^*\mathcal{E}\otimes\mathcal{P}),\overline{\partial}_1+\overline{\partial}_2). \tag{12.1}$$

The second grading comes from the fact that we're over a product manifold. Associated to this double complex is a spectral sequence whose E_2-page is $H^*_{\overline{\partial}_2}(H^*_{\overline{\partial}_1}(C))$. Our first task today is to prove Theorem 11.16, first verifying that the E_2 page looks like (11.17) and then that the spectral sequence collapses because the total cohomology vanishes except in degree 2, where it's $\mathcal{E}_0$.

Corollary 12.2.

1. $H^0(T^*,\widehat{\mathcal{E}})=0$ *and* $H^2(T^*,\widehat{\mathcal{E}})=0$, *and*
2. *we obtain an isomorphism* $\omega_I\colon H^1(T^*,\widehat{\mathcal{E}})\xrightarrow{\cong}\mathcal{E}_0$.

The second point is a piece of the inversion formula: $\check{\widehat{\mathcal{E}}}_0$ is $H^1(T^*, \widehat{\mathcal{E}})$, and this shows us it's $\mathcal{E}_0$ as it should be. We'll be able to buoy this up into the full inversion theorem.

Proof of Theorem 11.16. Introduce coordinates z_1, z_2 for T and ζ_1, ζ_2 for T^*. Now, we can view C as a bundle of complexes over T^*:

$$\mathcal{V}^0 \xrightarrow{\overline{\partial}_1} \mathcal{V}^1 \xrightarrow{\overline{\partial}_2} \mathcal{V}^2. \tag{12.3}$$

A point in $\mathcal{V}^0$ in the fiber at $\theta \in T^*$ is an element of $\Gamma(\mathcal{E} \otimes \mathcal{P}_\theta)$, hence a section of $\mathcal{V}^0$ looks sort of like a function $\lambda(z, \zeta)$, with the caveat that $\mathcal{E}$ is nontrivial. In the same way, a section of $\mathcal{V}^1$ looks like a 2-form $\lambda_i(z,\zeta)\,\overline{\mathrm{d}z^i}$; and a section of $\mathcal{V}^2$ looks like a 2-form $\lambda_{12}(z,\zeta)\,\overline{\mathrm{d}z^1} \wedge \overline{\mathrm{d}z^2}$.

Since $C^{p,0} = \Gamma_{T^*}(\mathcal{V}^p)$, then the cohomology of the global sections of (12.3) is what we're after. We claim this is the same as the global sections of the cohomology of (12.3). That these two functors commute is a statement about the vanishing of derived functors. In this case, the sheaves associated to these vector bundles are flabby, which follows from elliptic theory, and therefore higher cohomology vanishes. These are infinite-rank vector bundles, so we can't lean on the usual vanishing results.

The upshot is that on the E_1-page, $E_1^{p,q}$ is 0 if $p \neq 1$, and if $p = 1$, is $\Omega^{0,q}_{T^*}(\widehat{\mathcal{E}})$:

$$\begin{array}{c|ccc}
2 & 0 & \Omega^{0,2}_{T^*}(\widehat{\mathcal{E}}) & 0 \\
 & & \uparrow\overline{\partial}_2 & \\
1 & 0 & \Omega^{0,1}_{T^*}(\widehat{\mathcal{E}}) & 0 \\
 & & \uparrow\overline{\partial}_2 & \\
0 & 0 & \Omega^{0,0}_{T^*}(\widehat{\mathcal{E}}) & 0 \\
\hline
 & 0 & 1 & 2
\end{array} \tag{12.4}$$

The differential is $\overline{\partial}_2 \colon E_1^{p,q} \to E_1^{p,q+1}$: it increases vertical degree by 1. The cohomology is clearly $H^*(T^*, \widehat{\mathcal{E}})$, proving the first part of the theorem.

To get at the second part, we're going to compute the E_2-page in a different way. Fixing $v \in T$, we get a bundle $\mathcal{E}_v \otimes \mathcal{P}_v$ over T^*, a slice of $\mathcal{E} \otimes \mathcal{P}_\theta$. We care about this because $\widehat{\mathcal{E}}_\theta = H^1(T, \mathcal{E} \otimes \mathcal{P}_\theta)$.

We need a quick lemma about the cohomology with coefficients in a flat line bundle.

Lemma 12.5. *If $v \neq 0$, then $H^*(T^*, \mathcal{P}_v) = 0$; at $v = 0$, $H^q(T^*, \mathcal{P}_0) \cong \Lambda^q U$.*

Recall that U is the complex vector space (V, I).

Proof. For $v = 0$, consider the Dolbeault complex

$$\Omega^{0,0}_{T^*} \xrightarrow{\overline{\partial}_2} \Omega^{0,1}_{T^*} \xrightarrow{\overline{\partial}_2} \Omega^{0,2}_{T^*}. \tag{12.6}$$

The tangent space to T^* is, as a complex vector space, $\overline{U}^*$, and therefore the $(0,1)$ part of the cotangent bundle to T^* is $\overline{\overline{U}^*}^* = U$.

Now suppose $v \neq 0$. Given a $\lambda \in \Lambda$, we obtain a function $e_\lambda \colon T^* \to \mathbb{C}$ by

$$e_\lambda(\theta) := e^{2\pi i \langle \theta, \lambda \rangle}. \tag{12.7a}$$

Explicitly, in local coordinates ζ_1, ζ_2 coming from V^*, this is

$$e_\lambda(\zeta_1, \zeta_2) := e^{2\pi i \operatorname{Re}(\lambda^1 \zeta_1 + \lambda^2 \zeta_2)}. \tag{12.7b}$$

These give us examples of Dolbeault forms for the complex

$$\Omega^{0,0}_{T^*}(\mathcal{P}_v) \xrightarrow{\overline{\partial}_2} \Omega^{0,1}_{T^*}(\mathcal{P}_v) \xrightarrow{\overline{\partial}_2} \Omega^{0,2}_{T^*}(\mathcal{P}_v): \tag{12.8}$$

e_λ in degree zero, $e_\lambda\,\overline{\mathrm{d}\zeta_1}$ and $e_\lambda\,\overline{\mathrm{d}\zeta_2}$ in degree one, and $e_\lambda\,\overline{\mathrm{d}\zeta_1}\wedge\overline{\mathrm{d}\zeta_2}$ in degree two. Moreover, the theory of Fourier series says that, as long as we consider forms which rapidly decay in (TODO: I think) the fiber, these are a Hilbert space basis for $\Omega^{0,\bullet}_{T^*}(\mathcal{P}_v)$.[27]

Let $v \coloneqq (z^1, z^2)$. We can choose $z^i\,\overline{\mathrm{d}\zeta_i}$ as a connection form of $P_\theta \to T^*$. Then, you can just compute that

$$\overline{\partial}_v(e_\lambda) = (\lambda^i + z^i)e_\lambda\,\overline{\mathrm{d}\zeta_i} \tag{12.9a}$$
$$\overline{\partial}_v(e_\lambda\,\overline{\mathrm{d}\zeta_1}) = -(\lambda^2 + \zeta^2)e_\lambda\,\overline{\mathrm{d}\zeta_1}\wedge\overline{\mathrm{d}\zeta_2} \tag{12.9b}$$
$$\overline{\partial}_v(e_\lambda\,\overline{\mathrm{d}\zeta_2}) = (\lambda^1 + z^1)e_\lambda\,\overline{\mathrm{d}\zeta_1}\wedge\overline{\mathrm{d}\zeta_2}, \tag{12.9c}$$

and this is manifestly acyclic away from $v = 0$. ⊠

So λ defines a subcomplex A_λ generated by e_λ, $e_\lambda\,\overline{\mathrm{d}\zeta_i}$, and $\varepsilon_\lambda\,\overline{\mathrm{d}\zeta_1}\wedge\overline{\mathrm{d}\zeta_2}$, and the total complex is a completed direct sum of these over $\lambda\in\Lambda$, via the Fourier decomposition.

Now let $FD^{\bullet,\bullet}$ denote the germs of forms in a neighborhood of T_0^*, i.e. equivalence classes of differential forms on open neighborhoods of $(0,\theta)\in T\times T^*$, where two forms are equivalent if they agree on some (possibly smaller) open neighborhood. The map sending a form to its germ defines a map of complexes $\phi\colon C^{\bullet,\bullet}\to D^{\bullet,\bullet}$, which induces a map of spectral sequences for these two double complexes.

Lemma 12.5 implies that ϕ induces an isomorphism of all groups on the E_1-page, and a general theorem on spectral sequences implies the induced maps on all later pages are also isomorphisms (though here everything stabilizes at E_2).

Let N denote a formal neighborhood of zero on T, in the sense that $\Omega^{0,\bullet}_N$ denotes germs of $(0,\bullet)$-forms at zero. We again have a Fourier-theoretic decomposition

$$D^{\bullet,\bullet} = \widehat{\bigoplus_{\lambda\in\Lambda}} D^{\bullet,\bullet}_\lambda, \tag{12.10a}$$

where

$$D^{\bullet,\bullet}_\lambda \coloneqq \Omega^{0,\bullet}_N\otimes A_\lambda\otimes\mathcal{E}_0. \tag{12.10b}$$

The double grading here arises from $\overline{\partial}_1$ acting on $\Omega^{0,\bullet}_N$ and from $\overline{\partial}_2$ acting on A_λ.

Now run the spectral sequence for $D^{\bullet,\bullet}_\lambda$: the E_1-page is

$$\begin{array}{c|ccc} 2 & \tilde{A}^2_\lambda & 0 & 0\\ & \uparrow\overline{\partial}_2 & & \\ 1 & \tilde{A}^1_\lambda & 0 & 0\\ & \uparrow\overline{\partial}_2 & & \\ 0 & \tilde{A}^0_\lambda & 0 & 0\\ \hline & 0 & 1 & 2\end{array}\ \otimes\mathcal{E}_0, \tag{12.11}$$

where $\tilde{A}^q_\lambda$ is the space of holomorphic functions from N to A^q_λ (i.e. germs of these functions). Hence the homology of thr E_1-page, which is the E_2-page for $D^{\bullet,\bullet}$ and hence also for $C^{\bullet,\bullet}$, is

$$\begin{array}{c|ccc} 2 & \mathcal{E}_0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \hline & 0 & 1 & 2\end{array} \tag{12.12}$$

⊠

[27] A little more work has to be done to remove the "rapidly decaying" hypothesis.

Remark 12.13. There's a picture of this in terms of something called a *Koszul complex*. Fixing $\lambda = 0$, let $v \in T_0U$, which defines a complex

$$\underline{\Lambda^0 U} \xrightarrow{\epsilon(v)} \underline{\Lambda^1 U} \xrightarrow{\epsilon(v)} \underline{\Lambda^2 U}, \tag{12.14}$$

where ϵ denotes exterior product. The homology of this complex vanishes except in the top degree (2), where we get the determinant line $\operatorname{Det} U$.

Or, more algebraically, consider the complex

$$K^\bullet \coloneqq (\mathrm{Sym}^\bullet(U^*) \otimes \Lambda^\bullet(U), d_K), \tag{12.15}$$

where d_K is multiplication by $u^i \otimes u_i$. This complex can be thought of as $\Lambda^\bullet(U)$-valued polynomials on U. Then one can prove that the cohomology of this complex is 0 except in degree 2, where it's $\operatorname{Det} U$. There's a similar story for $\lambda \neq 0$. ◀

So we have an abstract isomorphism $H^1(T^*, \widehat{\mathcal{E}}) \cong \mathcal{E}_0$ coming from the spectral sequence argument. Can we get our hands on precisely what this isomorphism is? TODO: I didn't follow this part as well, so the logic might be wonky.

In (12.12), $\mathcal{E}_0$ lives in bidgree $(0, 2)$, but in (11.17), $H^1(T^*, \widehat{\mathcal{E}})$ lives in bidegree $(1, 1)$. The way to understand this is: consider $\alpha \in E_1^{1,1}$ and $\beta \in E_1^{0,2}$. If we can get that

$$0 = (\overline{\partial}_1 + \overline{\partial}_2)(-\alpha + \beta) = \overline{\partial}_1\beta - \overline{\partial}_2\alpha, \tag{12.16}$$

then we have an identification. Knowing that $\overline{\partial}_1\alpha = 0$ and $\overline{\partial}_2\alpha = 0 \bmod \mathrm{Im}(\overline{\partial}_1)$, we want to find a β such that $\overline{\partial}_1\beta = \overline{\partial}_2\alpha$.

Let $r_0\colon \Omega^{*,*}_{T\times T^*} \to \Omega^{0,0}_{\{0\}\times T^*}$ be restriction to $\{0\} \times T^* \subset T \times T^*$. Let ε^- denote a constant in $\Omega^{2,0}_{T^*}$. Then we need β to satisfy

$$\omega_I([\alpha]) = \int_{T^*} r_0(\beta) \wedge \varepsilon^-, \tag{12.17}$$

which leads to an explicit answer, and in particular ω_I is the isomorphism we want.

Now we want to finish out the proof of Mukai's theorem, passing back to the C^∞ world and asking about the relationship between $\check{\widehat{A}}$ and A. We know that $(\widehat{E}, \widehat{A})$ is without flat factors: if it weren't, then it would have H^0 and H^2, which we just saw is not true. We want to show that ω_I not only induces an isomorphism of $\check{\widehat{A}}$ and A, but also that it doesn't depend on I. The argument goes by trying to rewrite (12.17) in purely C^∞ terms. After that, we know it suffices to understand this for I and $-I$; I takes care of the $(1, 0)$ part and $-I$ takes care of the $(0, 1)$ part. The details can be found in Donaldson-Kronheimer [DK97].

Lecture 13.

Fredholm theory: 3/7/19

"We don't know that, except that I told you so."

In the next few lectures, we'll do something different: explaining how in a differential-geometric context, one constructs moduli spaces of solutions to nonlinear equations. Today, we'll mostly discuss the underlying linear theory, the theory of Fredholm operators between (usually) infinite-dimensional spaces. A linear operator between two finite-dimensional vector spaces is always Fredholm, and theorems about operators on finite-dimensional vector spaces often generaize to the Fredholm setting.

Lemma 13.1. *Let $0 \to V^0 \to V^1 \to \cdots \to V^n \to 0$ be an exact sequence of finite-dimensional vector spaces (over any field). Then*

$$\sum_{q=0}^n (-1)^q \dim V^q = 0. \tag{13.2}$$

Proof. Let T_q denote the map $V^q \to V^{q+1}$. If $C_q \subset V^q$ is a complement to $\mathrm{Im}(T^q)$, then $T^{q+1}|_{C_q}$ is injective, hence an isomorphism onto its image. But in (13.2), C_q and $\mathrm{Im}(C_q)$ are counted with opposite signs, so the total contribution is zero. ⊠

Corollary 13.3 (Rank-nullity theorem)**.** *Let $T\colon V\to W$ be a linear map. Then the sequence*

$$0 \longrightarrow \ker T \longrightarrow V \xrightarrow{T} W \longrightarrow \operatorname{coker} T \longrightarrow 0 \tag{13.4a}$$

is exact, and in particular

$$\dim V - \dim W = \dim\ker T - \dim\operatorname{coker} T. \tag{13.4b}$$

If we try to consider a families version of Corollary 13.3 over some curve in $\operatorname{Hom}(V,W)$, we might ask whether (13.4a) becomes a short exact sequence of vector bundles. This is not true in general: the kernel of a map of vector bundles need not be a vector bundle, because the dimension can jump. A simple example is

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \end{pmatrix} : \mathbb{R}^3 \to \mathbb{R}^2, \tag{13.5}$$

as $t\to 0$. However, the dimension of the cokernel also jumps, so the expression "$\dim(V-W)$" (as $\dim\ker T - \dim\operatorname{coker} T$) is constant, even if the expression is a little weird.

We would like to generalize this to the infinite-dimensional setting (where "dimension" doesn't play as well). There are different kinds of topology one considers for infinite-dimensional vector spaces; we will mostly work with Banach spaces, which are vector spaces complete with respect to a given norm. Sometimes we'll have to restrict to Hilbert spaces. We will always work over $\mathbb{C}$.

Definition 13.6. Let X and Y be Banach spaces. A continuous linear map $T\colon X\to Y$ is *Fredholm* if

1. $\operatorname{Im}(T)\subset Y$ is closed,
2. $\ker(T)$ is finite-dimensional, and
3. $\operatorname{coker}(T)$ is finite-dimensional.

In this case the *index* of T is defined to be $\operatorname{ind} T \coloneqq \dim\ker T - \dim\operatorname{coker} T$.

The first condition is redundant.

Remark 13.7. There is a sense in which the definition of the index is "the wrong way around": an element of $\operatorname{Hom}(X,Y)$ defines an element of $Y\otimes X^*$, not $X\otimes Y^*$, so we should do cokernel minus kernel. We'll stick with the standard definition, though, and sometimes this sign will come back and surprise us. ◀

Let $\operatorname{Hom}(X,Y)$ denote the space of continuous linear maps, equipped with the operator norm. This makes it a Banach space, and induces the strong topology. Let $\operatorname{Fred}(X,Y)$ denote the subspace of Fredholm operators, which is not a vector space.

Theorem 13.8. *Suppose X and Y are Hilbert spaces. Then the index* $\operatorname{ind}\colon \operatorname{Fred}(X,Y)\to\mathbb{Z}$ *is an isomorphism on π_0.*

This is not in general true for X and Y Banach spaces: consider X infinite-dimensional and Y finite-dimensional. However, it should be true if there's an isomorphism $X\cong Y$.

Remark 13.9. The higher topology of $\operatorname{Fred}(X,Y)$ is very interesting; each connected component has the topology of the classifying space of the infinite unitary group. ◀

Definition 13.10. Let $T\colon X\to Y$ be Fredholm and W be a subspace of Y. We say T is *transverse* to W, denoted $T\pitchfork W$, if $T(X)+W=Y$.

For each finite-dimensional $W\subset Y$, let

$$\mathcal{O}_W \coloneqq \{T\in\operatorname{Fred}(X,Y)\mid T\pitchfork W\}. \tag{13.11}$$

In this case we have (13.4a) again: the sequence

$$0 \longrightarrow \ker T \longrightarrow T^{-1}(W) \xrightarrow{T} W \longrightarrow \operatorname{coker} T \longrightarrow 0 \tag{13.12}$$

is exact, and all of its terms are finite-dimensional. In particular, using Lemma 13.1,

$$\operatorname{ind} T = \dim T^{-1}(W) - \dim W. \tag{13.13}$$

There is a vector bundle $V\to\mathcal{O}_W$ whose fiber at T is $T^{-1}(W)$, and there's a map of vector bundles $V\to\underline{W}$ which at T is just T.

Lemma 13.14. $\mathcal{O}_W\subset\operatorname{Hom}(X,Y)$ *is open.*

Therefore we have a canonical open cover of $\mathrm{Fred}(X,Y)$. It's uncountable, yes, but it's still nice to have.

Proof. Let $T_0 \in \mathcal{O}_W$, and choose a closed complement X' to $T_0^{-1}(W)$.[28] Then $X = T_0^{-1}(W) \oplus X'$.

There is also a splitting $Y = W \oplus T_0(X')$. Why is this?

- First, we have to show that these don't intersect except at zero. If $y \in W \cap T_0(X')$, then $y = T_0 x$ for some $x \in X$, and $x \in T_0^{-1}(W) \cap X'$, so $x = 0$.
- Second, we want to write $y \in Y$ as a sum of a $w \in W$ and $y' \in T_0(X')$. TODO...

Now, consider the composition

$$\mathrm{Hom}(X,Y) \longrightarrow \mathrm{Hom}(X',Y) \longrightarrow \mathrm{Hom}(X',T_0(X')) \tag{13.15}$$

where the first map is restriction and the second is projection. Each map is continuous and linear. Invertibility is an open condition, so the space of isomorphisms $X' \xrightarrow{\cong} T_0(X')$ is open in $\mathrm{Hom}(X',T_0(X'))$. Thus its preimage in $\mathrm{Hom}(X,Y)$, which is $\mathcal{O}_W$, is also open. ⊠

Corollary 13.16. *The index map* $\mathrm{ind}\colon \mathrm{Fred}(X,Y) \to \mathbb{Z}$ *is locally constant.*

This follows from $V \to \mathcal{O}_W$ being a vector bundle.

Proof of Theorem 13.8. Suppose $T_0, T_1 \in \mathrm{Fred}(X,Y)$ have the same index. Choose a finite-dimensional subspace of Y transverse to both T_0 and T_1, and let $X' \subset X$ be a closed complement of both $T_0^{-1}(W)$ and $T_1^{-1}(W)$. Why can you do this? Well, you can take the sum $T_0^{-1}(W) + T_1^{-1}(W)$ and ask for a complement to that, then work within that sum, which is finite-dimensional.

Therefore we can write T_0 and T_1 as maps $T_0^{-1}(W) \oplus X' \to W \oplus T_0(X')$. Then T_0 and T_1 have the block decompositions

$$T_0 = \begin{pmatrix} T_0|_{T_0^{-1}(W)} & 0 \\ 0 & T_0|_{X'} \end{pmatrix} \qquad T_1 = \begin{pmatrix} B & * \\ * & A \end{pmatrix}, \tag{13.17}$$

where A is invertible. By considering the family

$$t \longmapsto \begin{pmatrix} B & t* \\ t* & A \end{pmatrix} \tag{13.18}$$

and letting $t \to 0$, we just have to think about B and A. Since B has the same rank as $T_0|_{T_0^{-1}(W)}$, then it suffices to show that the space of invertible operators $X' \to X'$ is connected.

Lemma 13.19. *Let H be a Hilbert space; then* $\mathrm{GL}(H) \subset \mathrm{Hom}(H,H)$ *is connected.*

Proof. If $A \in \mathrm{GL}(H)$, then A^*A is positive, so it has a square root P, i.e. $P^2 = A^*A$, and P is also invertible. Define U by $A = UP$; then

$$U^*U = (AP^{-1})^*(AP^{-1}) = P^{-1}A^*AP^{-1} = A^*AP^{-2} = \mathrm{id}. \tag{13.20}$$

We can write $P = e^B$ for some unique self-adjoint B, using the spectral theorem for self-adjoint operators, and can write $U = e^{iC}$ for some self-adjoint C. This follows because the eigenvalues of U are contained in the unit complex numbers, and the logarithm map from the circle to $[0, 2\pi)$ is measurable, which suffices for the spectral theorem.

Anyways, now we have the paths $t \mapsto e^{tB}$ and $t \mapsto e^{tC}$. ⊠

In fact, it is a (harder) theorem of Kuiper that $\mathrm{GL}(H)$ is contractible when H is infinite-dimensional.

Now one has to argue that π_0 is surjective; this first involves producing a Fredholm operator ⊠

Exercise 13.21. Using the local models $\mathcal{O}_W$, show that the composition of two Fredholm maps is Fredholm, and the indices add.

This in particular means that if $X = Y$, $\mathrm{Fred}(X,X)$ has a composition structure, meaning π_0 is a monoid (in fact, it's a group), and the map to $\mathbb{Z}$ is in fact a group isomorphism.

We will occasionally consider a variant.

[28] The existence of such a closed complement follows from the Hahn-Banach theorem. If X and Y are Hilbert spaces, then it's easier – you can just take the orthogonal complement.

Definition 13.22. Consider a complex of Banach spaces

$$0 \longrightarrow X^0 \xrightarrow{T_1} X^1 \xrightarrow{T_2} \cdots \xrightarrow{T_n} X^n \longrightarrow 0,$$

so $T_q \circ T_{q-1} = 0$. This is a *Fredholm complex* if each T_q has closed image and the cohomology groups are all finite-dimensional.

If $n = 1$, this is equivalent to $T_1 \colon X^0 \to X^1$ being a Fredholm map. Some of the notions we've considered above generalize to Fredholm complexes.

Remark 13.23. Peering into the proof of Lemma 13.14, $T_0 \colon X' \to T_0(X')$ is an isomorphism, so we can let S be an inverse. Then $T \circ S$ is not quite equal to the identity, but its image is finite-dimensional; one says it has *finite rank*. The same applies to $S \circ T$.[29] Therefore we've shown any Fredholm operator is invertible up to a finite-rank operator, and in fact the converse is true.

The more common version of that result is that T is Fredholm iff it's invertible up to a *compact* operator (i.e. an operator under which the image of the unit ball is compact). The compact operators are precisely the closure of the finite-rank ones in the strong topology, so what we just said is a little bit stronger. ◄

Next we discuss determinants, first in the finite-dimensional setting, then in the Fredholm setting.

Recall that over any field k, the *determinant line* of a finite-dimensional vector space V is $\operatorname{Det} V \coloneqq \Lambda^{\dim V} V$. The determinant line of the zero vector space is therefore canonically k.

If L is a line over $k = \mathbb{R}$ or $\mathbb{C}$ (i.e. a one-dimensional k-vector space), there's a notion of an "inverse" for L: over $\mathbb{R}$, $L^{-1} = L^*$, because given any $v \in L$, there's a unique $\ell \in L^*$ with $\ell(v) = 1$. Over $\mathbb{C}$, $L^{-1} = \overline{L}^*$ for a similar reason.

Definition 13.24. Consider a complex of finite-dimensional vector spaces over $\mathbb{R}$ or $\mathbb{C}$:

$$0 \longrightarrow V^0 \xrightarrow{T_1} V^1 \xrightarrow{T_2} \cdots \xrightarrow{T_n} V^n \longrightarrow 0, \tag{13.25a}$$

so in particular $T_q \circ T_{q-1} = 0$. Define the *determinant* of this compelex to be

$$\operatorname{Det} V^\bullet \coloneqq \bigotimes_{q=0}^{n} (\operatorname{Det} V^q)^{(-1)^q}, \tag{13.25b}$$

and similarly define the determinant of its cohomology to be

$$\operatorname{Det} H^*(V^\bullet) \coloneqq \bigotimes_{q=0}^{n} (\operatorname{Det} H^q(V^\bullet))^{(-1)^q}. \tag{13.25c}$$

Lemma 13.26. *The maps T define an isomorphism* $\det T \colon \operatorname{Det} H^*(V^\bullet) \xrightarrow{\cong} \operatorname{Det} V^\bullet$.

Corollary 13.27. *Letting $n = 1$, given a linear map $T \colon V^0 \to V^1$, there is a canonical element* $\det T \in \operatorname{Det}(V^0 \otimes (V^1)^*)$.

This is called the *determinant* of T, and generalizes the determinant of an endomorphism (the case $V^0 = V^1$).

Proof sketch of Lemma 13.26. Let C^q be a complement to $T_q(V^{q-1})$ in $\ker(T_{q+1})$ and let $r_q \coloneqq \operatorname{rank} T_q = \dim T_q(V^{q-1})$ and $b_q \coloneqq \dim H^q(V^\bullet)$. Now choose:

- $v_q \in \Lambda^r T(V^{q-1}) \setminus 0$,
- $c_q \in \operatorname{Det} C^q$,
- $h_q \in \operatorname{Det} H^q(V^\bullet)$, and
- $\widetilde{h}_q$ a lift of h_q to $\Lambda^{b_q} V^q$,

and define

$$\det\left(\prod_{q=0}^{n} h_q^{(-1)^q}\right) \coloneqq \prod_{q=0}^{n} \left(\widetilde{h}_q \wedge c_q \wedge v_q\right)^{(-1)^q}. \tag{13.28}$$

It therefore suffices to show this doesn't depend on choices. ⊠

[29] TODO: I think.

Exercise 13.29. Check that the construction in (13.28) does not depend on the choices of v_q, c_q, h_q, and $\widetilde{h}_q$.

One might hope for a slicker construction, but this seems to be the most straightforward way to do it.

Given a short exact sequence of complexes $0 \to V_0^\bullet \to V_1^\bullet \to V_2^\bullet \to 0$, the determinants add.

Now let X and Y be Banach spaces as before. We would like to construct a vector bundle $\mathrm{Det} \to \mathrm{Fred}(X, Y)$. Fixing a finite-dimensional subspace $W \subset Y$, let $\mathrm{Det}_W \to \mathcal{O}_W$ be $\mathrm{Det}(T\colon V \to \underline{W})$, i.e. $\mathrm{Det}\, V \otimes \mathrm{Det}\, W^*$, where $V \to \mathcal{O}_W$ is the vector bundle with $V_T := T^{-1}(W)$.

Now we want to patch for subspaces $W, W' \subset Y$ with $\mathcal{O}_W \cap \mathcal{O}_{W'} \neq \varnothing$. It suffices to assume $W' \subset W$, since any two finite-dimensional subspaces of Y are contained within their sum, which is also finite-dimensional. We then have a diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & T^{-1}(W') \arrow[r] \arrow[d, "T'"] & T^{-1}(W) \arrow[r] \arrow[d, "T"] & T^{-1}(W)/T^{-1}(W')) \arrow[r] \arrow[d, "\overline{T}"] & 0 \\ 0 \arrow[r] & W' \arrow[r] & W \arrow[r] & W/W' \arrow[r] & 0. \end{tikzcd} \tag{13.30}$$

Then $\overline{T}$ is an isomorphism (TODO: why?), so its determinant

$$\det \overline{T} \in \mathrm{Det}(T^{-1}(W)/T^{-1}(W')) \otimes (\mathrm{Det}\, W/W')^{-1} \tag{13.31}$$

is nonzero. So we can write this as $T\overline{v} \otimes (\overline{w})^{-1}$ for some $\overline{v} \in T^{-1}(W)/T^{-1}(W')$ and $\overline{w} \in W/W'$. Lift $\overline{v}$ to some $v \in T^{-1}(W)$ and $\overline{w}$ to some $w \in W$. There are $v' \in T^{-1}(W')$ and $w' \in W'$ such that

$$\det T' = T'v' \otimes (w')^{-1} \in \mathrm{Det}\, T;^{-1}(W) \otimes (\mathrm{Det}\, W')^{-1}, \tag{13.32}$$

and therefore

$$\det T = \frac{Tv \wedge Tv'}{w \wedge w'} \in \mathrm{Det}\, T^{-1}(W) \otimes (\mathrm{Det}\, W)^{-1}. \tag{13.33}$$

Lecture 14.

Transversality and the obstruction bundle: 3/12/19

"Did I use the word 'like?'... Then I used it as a millenial."

Let X and Y be smooth manifolds and $f\colon X \to Y$ be smooth. Recall that a $y \in Y$ is a *regular value* if $\mathrm{d}f_x$ is surjective for all $x \in N := f^{-1}(y)$. In this case N is a submanifold of X.

In general, $\mathrm{d}f$ defines a map of vector bundles $TX \to f^*TY$ over X; asking for y to be a regular value means that the cokernel of the restriction

$$\mathrm{d}f\colon TX|_N \longrightarrow f^*TY|_N \tag{14.1}$$

is zero. Suppose this is true; then, thinking of (14.1) as a cochain complex, its cohomology is TN in degree 0 and 0 in degree 1.

Example 14.2. Let $X = \mathbb{R}^3$, $Y = \mathbb{R}$, and $f(x^1, x^2, x^3) = (x^1)^2 + (x^2)^2 + (x^3)^2$. Then $y = 1$ is a regular value, which you can see by computing the derivative of f. The preimage is S^2. Since all vector bundles on $\mathbb{R}^3$ are trivializable but TS^2 isn't (the nonzero Euler characteristic of S^2 provides an obstruction), $TS^2 \to S^2$ does not extend to a vector bundle on $\mathbb{R}^3$. ◀

So $TN \to N$ doesn't always extend to X, but it *does* extend as a *virtual bundle*, meaning a formal difference of two vector bundles. Concretely, this means there are vector bundles $\xi, \eta \to X$ such that $TN \oplus \xi|_N \cong \eta|_N$, so we say heuristically that "TN is the restriction of η minus ξ."

First, using the exact sequence

$$0 \longrightarrow TN \longrightarrow TX|_N \xrightarrow{\mathrm{d}f} f^*TY|_N \longrightarrow 0 \tag{14.3}$$

of vector bundles over N, then taking determinants, $\mathrm{Det}\, TN \to N$ is the restriction to N of

$$(\mathrm{Det}\, TX) \otimes (\mathrm{Det}\, f^*TY)^{-1} \longrightarrow X. \tag{14.4}$$

In Example 14.2, $\mathrm{Det}\, TS^2$ is trivializable, because S^2 is orientable, but this of course isn't always the case.

Example 14.5. We can generalize from functions to sections of vector bundles.

Let V be a four-dimensional vector space and $H \to \mathbb{P}(V) \cong \mathbb{RP}^3$ be the *Hopf line bundle*: a point in $\mathbb{P}(V)$ is a line $L \subset V$ through the origin, and we declare the fiber of H at L to be L^*.

There's a map $\varphi\colon V^* \to \Gamma(\mathbb{P}(V), H)$ sending $\theta \mapsto (L \mapsto \theta|_L)$. Therefore, the assignment $f_\theta(L) := \theta|_L$ defines a section s of $H \to \mathbb{P}(V)$. If Z denotes the zero section and $N := f_\theta^{-1}(Z)$, then Z and s are transverse and so N is a submanifold, diffeomorphic to $\mathbb{RP}^2 \subset \mathbb{RP}^3$. Then $\operatorname{Det} TN \to N$ is the restriction of $\operatorname{Det} T\mathbb{P}(V) \otimes H^{-1} \to \mathbb{P}(V)$, and it's nontrivial: $\mathbb{RP}^2$ is nonorientable, and $\operatorname{Det} TN$ has nonzero first Stiefel-Whitney class. ◀

This approach describes finite-dimensional manifolds as cut out by finitely many equations satisfying transversality. In the infinite-dimensional setting, things will look similar, though we may cut out the solutions to infinitely many equations.

Remark 14.6. Another common generalization is to consider two maps $f\colon X \to Y$ and $f'\colon X' \to Y$. We want to take the fiber product of those two maps, the subset of $(x, x') \in X \times X'$ such that $f(x) = f'(x')$, and we want it to be a submanifold. A sufficient, but not necessary, condition, is for $f \pitchfork f'$, and then the formulas about tangent bundles generalize. When $X' = \mathrm{pt}$ and $f'\colon \mathrm{pt} \mapsto y$, transversality is equivalent to asking for y to be a regular value of f. ◀

Remark 14.7 (The non-transverse setting)*.* If $a := \dim X$ and $b := \dim Y$, then in the transverse setting we expect $\dim N = a - b$. Sometimes Y isn't a regular value, but N is nonetheless still a manifold; in this case $\dim N$ might be larger than expected.

Let's assume N is a submanifold of X but that y isn't necessarily a regular value. We have $\ker \mathrm{d}f_x = T_xN$, and hence $E := \operatorname{coker} \mathrm{d}f \to N$ is also a vector bundle, called the *obstruction bundle*.

For example, let $\phi\colon S^2 \to S^2$ be a diffeomorphism and consider the map $f\colon S^2 \to S^2 \times S^2$ sending $p \mapsto (p, \phi(p))$. The intersection with the diagonal would have expected dimension zero, tracking the fixed points of ϕ, but of course if $\phi = \mathrm{id}$ then we get all of S^2, which has positive dimension. As an exercise, show that in this case, the obstruction bundle is identified with TS^2.

In this case, the way to study the intersection is to also keep track of the obstruction bundle. But this doesn't work for all non-transverse intersections – consider the map $f_a := x^2 - a$ from $\mathbb{R} \to \mathbb{R}$. For $a > 0$, this is fine, and we get two points as expected, but at $a = 0$, we only get a single point. Differential geometry can't really tell that you were supposed to get two points there, but algebraic methods can: consider the graph of f_a in $\mathbb{R} \times \mathbb{R}$, and intersect it with the graph $y = 0$.

In algebraic geometry, we do this by considering rings of functions. The functions on $\mathbb{R} \times \mathbb{R}$ are $R := \mathbb{R}[x, y]$; the algebra of functions on $Z := \{y = 0\}$ is $M_Z := \mathbb{R}[x, y]/(y)$, and the algebra of functions on $P := \{y = x^2\}$ is $M_P := \mathbb{R}[x, y]/(y - x^2)$.

The intersection is a pullback, so its algebra of functions is a pushout, which is given by the tensor product

$$M_P \otimes_R M_Z. \tag{14.8}$$

(In more elaborate settings it can be better to consider derived terms as well, which here are factors of Tor.) To compute this, let's take a resolution of R:

$$0 \longleftarrow \mathbb{R}[x, y] \xleftarrow{\;y\;} \mathbb{R}[x, y] \longleftarrow 0 \tag{14.9}$$

Multiplication by y is injective, so this is fine, and the cohomology of this complex has M_Z in degree 0 and vanishes elsewhere. Now, tensor with M_P:

$$0 \longleftarrow \mathbb{R}[x, y]/(y - x^2) \xleftarrow{\;y\;} \mathbb{R}[x, y]/(y - x^2) \longleftarrow 0 \tag{14.10}$$

Now, the zeroth cohomology is a two-dimensional vector space, and in this sense we've remembered that the intersection is of multiplicity 2. ◀

We can also consider this story in families. A "family of equations" is a smooth manifold S and a map $f\colon S \times X \to Y$; we think of fixing the parameter $s \in S$ to obtain an equation $x \mapsto f_s(x) := f(s, x)$.

Lemma 14.11. *Let $y \in Y$ and $N := f^{-1}(y) \subset S \times X$. Let $\pi\colon N \to S$ be the restriction of projection onto the first factor $S \times X \to S$. Then s is a regular value of π iff y is a regular value of f_s.*

Sard's theorem tells us that regular values in Y form a subset of the second category, meaning in particular that it's dense, and therefore nonempty. So there are regular values. The takeaway is that if we start with a nontransverse map and add enough parameters, we can adjust the parameters a little bit and obtain a transverse map. Though algebraic methods are better at directly dealing with nontransverse situations, they don't allow this kind of argument; each perspective is useful in a different way.

Now we throw in symmetries. Suppose G is a Lie group acting on X and Y and that $f\colon X \to Y$ is G-equivariant. If $y \in Y$ is a fixed point of G, then G acts on TX and f^*TY, and $\mathrm{d}f\colon TX \to f^*TY$ is G-equivariant. There's also a Lie algebra action here: at each $x \in X$, there's a map $\mathfrak{g} \to T_xX$: a $\xi \in \mathfrak{g}$ defines a left-invariant vector field on G, and hence a curve $t \mapsto e^{t\xi}$ in G through the identity. Acting by these elements defines a curve in X, and we can differentiate at $t = 0$ to obtain a tangent vector at x. As x varies, this stitches into a vector bundle map, and so altogether we have a complex

$$\underline{\mathfrak{g}} \xrightarrow{\alpha} TX \xrightarrow{\mathrm{d}f} f^*TY. \tag{14.12}$$

in degrees -1, 0, and 1, and its cohomology is $H^{-1} = \ker\alpha$, $H^0 = TN/\operatorname{Im}\alpha$, and $H^1 = E$, the obstruction bundle. However, $\ker\alpha$ isn't always a vector bundle – it tells us the Lie algebra of the isotropy groups at each point, but we don't yet know that these are always the same dimension. If they are, then N/G is a smooth orbifold, and $TN/\operatorname{Im}(\alpha) = T(N/G)$. One way to force this would be to ask for all isotropy groups to be discrete, in which case $\ker\alpha = 0$. So H^1 is non-transversality, H^{-1} is isotropy groups, and H^0 is the tangent space.

∽ · ∼

We would like to generalize all of this to infinite dimensions. Recall that if V and W are Banach spaces and $T\colon V \to W$ is a continuous linear map, then T is Fredholm if its kernel and cokernel are finite-dimensional.

Definition 14.13. Let A and B be affine spaces modeled on the Banach spaces V and W, respectively, and let $U \subset A$ be open. A C^1 map (not necessarily linear) $f\colon U \to B$ is *Fredholm* if $\mathrm{d}f_p\colon V \to W$ is Fredholm for all $p \in U$.

In this case, $\mathrm{d}f$ defines a map $U \to \mathrm{Fred}(V, W)$: it's a family of linear Fredholm operators parameterized by U, and this is the infinitesimal information contained in f.

Remark 14.14. One can generalize to C^1 maps between Banach manifolds. The basic theory of Banach manifolds is set up similarly to the finite-dimensional setting that may be more familiar, at least once you have the inverse function theorem. ◀

Theorem 14.15. *Let $y \in B$ be a regular value of a Fredholm map $f\colon U \to B$. Then $f^{-1}(y) \subset U$ is a finite-dimensional submanifold, with dimension* $\operatorname{ind}\mathrm{d}f_x$.

Now U might not be connected, so the index of $\mathrm{d}f_x$ need only be locally constant, since it's a surjective map $\mathrm{Fred}(V, W) \to \mathbb{Z}^{\geq 0}$, at least of V and W are infinite-dimensional. So $f^{-1}(y)$ may have components of different dimensions.

Proof. Let $x_0 \in f^{-1}(y)$ and $T_0 = \mathrm{d}f_{x_0}\colon V \to W$. If $V' \subset V$ is a complementary subspace to $\ker(T_0)$, then as we discussed last time, $T_0|_{V'}\colon V' \to T_0(V')$ is invertible. Let W' be a complement to $T_0(V')$.

Let's introduce a local change of coordinates φ around x_0 such that

$$(f \circ \varphi)(x_0 + \xi' + \eta) = y + T_0(\xi') + \phi(\xi', \eta), \tag{14.16}$$

where $\xi' \in V'$, $\eta \in \ker(T_0)$, and ϕ is a map $V \to W'$. (To do this, we need to know the inverse function theorem.) The point is, in the direction corresponding to V', $f \circ \varphi$ is just T_0, and in the complementary direction, it's something else, which might be nonlinear. Define $g\colon \ker(T_0) \to W'$ to send $\eta \mapsto \phi(0, \eta)$; we've just shown a bijection between $f^{-1}(y)$ and $g^{-1}(0)$.

We haven't yet used that y is a regular value, so as a bonus we learn that we can locally model the inverse image of a Fredholm map as a map between finite-dimensional vector spaces. Since y is a regular value of f, then 0 is a regular value of g, and therefore the component of $f^{-1}(y)$ containing x_0 is a manifold of the expected dimension, which is $\operatorname{ind} T_0 = \dim\ker(T_0)$, since the cokernel vanishes. ⊠

This is an echo of the techniques we used last time: we used the inverse function theorem to write a nonlinear Fredholm map locally as a nonlinear finite-dimensional map and a linear invertible map.

There's also an analogue of Sard's theorem in this setting, called the Sard-Smale theorem; next time, we'll see how to use this to construct moduli spaces.

Lecture 15.

Constructing moduli spaces: 3/14/19

We continue on our journey to the construction of moduli spaces of solutions to (certain nice) PDEs. One potential roadblock is that we haven't discussed Sobolev spaces in this class, but we'll carry on nonetheless.

Recall that if A and B are affine spaces modeled on the Banach spaces V and W respectively and $U \subset A$ is open, we defined what it means for a C^1 map $f\colon U \to B$, not necessarily linear, to be Fredholm: that $\mathrm{d}f_p\colon V \to W$ is Fredholm for all $p \in U$. Thus we can also make sense of this definition for Banach manifolds locally modeled on V and W.

The crucial lemma about Fredholm maps is that, after a possibly nonlinear change of coordinates around any $p_0 \in U$, we can rewrite f as "linear plus finite-dimensional." More precisely, let $T_0 = \mathrm{d}f|_{p_0}\colon V \to W$, and choose complements V_0 of $\ker(T_0) \subset V$ and W' of $T_0(V_0) \subset W$, so that f is a map $\ker(T_0) \oplus V_0 \to W' \oplus T_0(V_0)$. The inverse function theorem for Banach spaces then tells us there's a $\delta > 0$ and a nonlinear function $\phi\colon B_\delta(0) \subset V \to W$ such that

$$f(\eta, \xi_0) = (\phi(\eta, \xi_0), T_0(\xi_0)). \tag{15.1}$$

Because f is Fredholm, W' is finite-dimensional. This will greatly simplify some of what we're going to do: instead of worrying about an infinite system of equations, we can reduce to considering only finitely many.

Remark 15.2. Suppose Γ is a Lie group acting on U and B, perhaps nonlinearly, and suppose f is Γ-equivariant and p_0 is a fixed point for Γ. Then, differentiating, we get a linear action on V and W, and the running through the construction above, T_0 and ϕ are equivariant for these linear actions. ◀

Corollary 15.3.

(1) *If $q \in B$ is a regular value, then $f^{-1}(q) \subset U$ is a finite-dimensional submanifold.*
(2) *The Sard-Smale theorem: regular values are a Baire set (second category), which in particular means there are a lot of them.*

Now we will use this theory to construct moduli spaces – first, the moduli space of pseudoholomorphic curves in symplectic topology, and then a moduli space in gauge theory, which has a Γ-action as in Remark 15.2.

Pseudoholomorphic curves. Let Σ be a *Riemann surface*, which is an oriented 2-manifold with a conformal structure, or equivalently a manifold of complex dimension 1. Let $j \in \mathrm{End}(TM)$ denote the almost complex structure induced from the complex structure, meaning j is multiplication by i.

Let (X, J) be an almost complex manifold, meaning that $J \in \mathrm{End}(TM)$ squares to $-\mathrm{id}$.

Definition 15.4. A map $\phi\colon \Sigma \to X$ is *pseudoholomorphic* (or *J-holomorphic*) if $\mathrm{d}\phi \circ j = J \circ \mathrm{d}\phi$.

In a little more detail, at any $\sigma \in \Sigma$, $\mathrm{d}\phi_\sigma$ is a map $T_\sigma\Sigma \to T_{\phi(\sigma)}X$; j acts on the domain and J on the codomain, and we want $\mathrm{d}\phi_\sigma$ to intertwine those actions.

The space $\mathrm{Hom}(T_\sigma\Sigma, T_{\phi(\sigma)}X)$ splits as

$$\mathrm{Hom}(T_\sigma\Sigma, T_{\phi(\sigma)}X) = \mathrm{Hom}^+(T_\sigma\Sigma, T_{\phi(\sigma)}X) \oplus \mathrm{Hom}^-(T_\sigma\Sigma, T_{\phi(\sigma)}X), \tag{15.5}$$

the spaces which commute and anticommute with J and j, respectively. Alternatively, there is an involution on this space sending $L \mapsto J \circ L \circ j^{-1}$, and Hom^+ is the eigenspace for 1, and Hom^- the eigenspace for -1.[30]

Definition 15.6. Let $\pi^-\colon \mathrm{Hom}(T_\sigma\Sigma, T_{\phi(\sigma)}X) \to \mathrm{Hom}^-(T_\sigma\Sigma, T_{\phi(\sigma)}X)$ be the projection. Then we define $\overline{\partial}\phi = \overline{\partial}_J\phi \coloneqq \pi^- \circ \mathrm{d}\phi$.

[30]These eigenspaces have the same dimension, which you can see by replacing J with $-J$.

Where does this live? Well, $\mathrm{d}\phi \in \Omega^1_\Sigma(\phi^* TX)$, and this splits like in (15.5):

$$\Omega^1_\Sigma(\phi^* TX) = \Omega^1_\Sigma(\phi^* TX)^+ \oplus \Omega^1_\Sigma(\phi^* TX)^-, \tag{15.7}$$

and $\overline{\partial}$ is projection onto $\Omega^1_\Sigma(\phi^* TX)^-$. Thus $\overline{\partial}\phi = 0$ iff ϕ is pseudoholomorphic.

We would like to study the space of solutions to the equation $\overline{\partial}\phi = 0$ on the space of such ϕ. First we need to write a function $\phi \mapsto \overline{\partial}_J \phi$. The domain is $\mathrm{Map}(\Sigma, X)$, but the codomain *a priori* depends on ϕ: for ϕ given it lands in $\Omega^1_\Sigma(\phi^* TX)^-$. So f is instead a section of a vector bundle $\mathcal{E} \to \mathrm{Map}(\Sigma, X)$, where $\mathcal{E}_\phi \coloneqq \Omega^1_\Sigma(\phi^* TX)^-$.

Remark 15.8. There is a bit of nuance involved in constructing $\mathcal{E} \to \mathrm{Map}(\Sigma, X)$. As with anything involving infinite-dimensional manifolds, there's the question of what kind of regularity we want; if we just consider smooth functions, we'll obtain manifolds modeled on Fréchet spaces, which are weaker than Banach spaces. Calculus in Fréchet spaces is generally more difficult than in Banach spaces, so it's often better to complete in some other norm which leads to a Hilbert manifold, or something like that.

But if you just want to see why $\mathcal{E} \to \mathrm{Map}(\Sigma, X)$ is a vector bundle, you don't really need to worry about that. One way to produce local trivializations of $\mathcal{E} \to \mathrm{Map}(\Sigma, X)$ is to use a connection on TX to identify different fibers nearby using parallel transport, but all approaches will involve thinking about what nearby points in $\mathrm{Map}(\Sigma, X)$ are. ◀

Given a covariant derivative ∇ on X, we can hit f with it, obtaining a map $\nabla f_\phi \colon T_\phi(\mathrm{Map}(\Sigma, X)) \to \mathcal{E}_\phi$, i.e. a map $\Omega^0_\Sigma(\phi^* TX) \to \Omega^1_\Sigma(\phi^* TX)^-$. We can write this as a map $\xi \mapsto (\nabla\xi)^-$.

Claim 15.9. $\nabla f_\phi(\xi) = \pi^-(\nabla\xi)$, and this operator is Fredholm.

The first claim is not so surprising: if you differentiate the $\overline{\partial}_J$ operator, you get almost the same thing applied to ξ. The second claim is much deeper: it involves not only a significant chunk of elliptic theory, but also replacing $\mathrm{Map}(\Sigma, X)$ and $\mathcal{E}$ into a Hilbert manifold and Hilbert bundle, respectively, by taking completions with respect to some norm – "thickening" them, in a sense. So "this" in the theorem statement isn't entirely right; instead we replace it with its thickened version.

Once we know it's Fredholm, which we'll discuss later in a more general setting, we'd like to compute the index of this operator. One reasonable choice for calculating is the Riemann-Roch theorem, but as ϕ isn't holomorphic, it doesn't apply, and instead one has to use the more general Atiyah-Singer index theorem; indeed, this was one of the first successes of this more general theorem.

Then, however, we have to worry about whether 0 is a regular value for f. In fact, maybe it isn't – but we can get around this using the Sard-Smale theorem: almost everything is a regular value, so if we can embed in a family of moduli problems (concretely, writing the equation with parameters), then we have a better chance of establishing transversality. One option is to vary the almost complex structure J on X, and another is to consider $\overline{\partial}_J\phi = \nu$ for $\nu \neq 0$, where ν is a section of the vector bundle $V \to \Sigma \times X$ whose fiber at (σ, x) is $\mathrm{Hom}^-(T_\sigma\Sigma, T_x X)$. Thus we have a section $f \colon \mathrm{Map}(\Sigma, X) \times \Gamma(V) \to \mathcal{E}$, and it turns out that just varying ν is enough to establish transversality. Because $\Gamma(V)$ is a linear space, the calculations are a little easier. So the moduli space isn't quite pseudoholomorphic curves, so if you want topological invariants, it's important to check that what you get doesn't depend on ν for generic ν. If you wanted to let $\nu = 0$ even when you don't have transversality, you might have to work with the derived geometry, as we discussed last time.

The moduli space of self-dual connections. Unlike the previous case, there's a group acting on connections, and we want solutions up to equivalence.

Specifically, let G be a Lie group and M be a smooth manifold. We want to study principal bundles on M modulo those which are "the same."

Definition 15.10. If $\pi \colon P \to M$ and $\pi' \colon P' \to M$ are principal G-bundles, a *morphism* from π to π' is a smooth, G-equivariant map $\varphi \colon P \to P'$ commuting with the maps down to the base; that is, $\pi' \circ \varphi = \pi$. If $P = P'$, we call φ a *gauge transformation.*

As it turns out, this definition forces every morphism to be an isomorphism. Therefore the category $\mathsf{Bun}_G(M)$ of principal G-bundles on M is a groupoid. This is the thing which is local, not its set of isomorphism classes: given intersecting opens $U, V \subset M$, we can't glue isomorphism classes of bundles on U and V (how do we work with equivalence classes nicely?), but given two principal bundles $P \to U$ and $P' \to V$ and an isomorphism $P|_{U\cap V} \xrightarrow{\cong} P'|_{U\cap V}$, we can glue.

We'd like to study these gauge transformations $\varphi\colon P \to P$. From φ we can extract data of a map $g_\varphi\colon P \to G$: $\varphi(p)$ is in the same fiber as p, so it's equal to $g \cdot p$ for a unique $g \in G$, and we let $g_\varphi(p) \coloneqq g$. This is equivariant for G acting on itself by conjugation:

$$g_\varphi(p \cdot h) = h^{-1} g_\varphi(p) h. \tag{15.11}$$

Moreover, from g_φ we can recover φ, and $g_{(\cdot)}$ sends composition to pointwise multiplication. So the group of gauge transformations is the group of such g_φ, which is clearly infinite-dimensional (if G isn't zero-dimensional). Concretely, this is the space of sections of the associated fiber bundle $G_P \to M$ given by mixing P with the G-manifold G with action $\rho\colon G \to \mathrm{Aut}(G)$ sending $H \mapsto (g \mapsto hgh^{-1})$; then the group of gauge transformations is $\mathcal{G}_P \coloneqq C^\infty(M, G)$.

Since $\mathcal{G}_P$ is (some suitable infinite-dimensional version of a) Lie group, it has a Lie algebra, and we should determine what it is. Given a curve on M, if we differentiate a section over that curve, we obtain a vertical vector field on P, so the Lie algebra of $\mathcal{G}_P$ is contained in the algebra vertical vector fields on $P \to M$. Specifically, we know how this transforms under G, so more precisely it's the algebra of G-invariant vertical vector fields on $P \to M$. If this is confusing, it may be helpful to think of this as an infinitesimal gauge symmetry: vertical because it preserves the fibers, and G-invariant because gauge symmetries are G-equivariant.

We can then describe the Lie algebra of $\mathcal{G}_P$ by differentiating g_φ: we want maps $\zeta\colon P \to \mathfrak{g}$ such that

$$\zeta(p \cdot h) = \mathrm{Ad}_{h^{-1}} \zeta(p). \tag{15.12}$$

In other words, ζ is a section of the associated bundle to the adjoint action $\lambda\colon G \to \mathrm{Aut}(\mathfrak{g})$ sending $h \mapsto \mathrm{Ad}_h$. In other words, $\zeta \in \Omega^0_M(\mathfrak{g}_P)$. This $\mathfrak{g}_P$ is more than a vector bundle – the adjoint action preserves the Lie bracket, so this is a bundle of Lie algebras. This makes the fact that these came from infinitesimal symmetries less clear, but is easier to compute with.

Remark 15.13. So far, we've been doing everything with smooth maps, smooth sections, etc., and therefore obtained Fréchet manifolds. As before, we'll have to thicken to Hilbert spaces to make calculations more tractable. ◀

If $\mathcal{A}_P$ denotes the space of connections on $\pi\colon P \to M$ (i.e. G-invariant horizontal distributions), there is a right $\mathcal{G}_P$-action on $\mathcal{A}_P$, because a gauge transformation pulls back a horizontal distribution to a horizontal distribution.

Exercise 15.14. Recall that a connection is equivalent to its connection form $A \in \Omega^1_P(\mathfrak{g})$ such that $R_h^* A = \mathrm{Ad}_{h^{-1}}(A)$ and the restriction of A to the fiber is the Maurer-Cartan form. Describe the $\mathcal{G}_P$-action in these terms.

Lecture 16.

Gauge transformations: 3/26/19

"Can I trust you to prove it, or should I? Or do you want Arun to prove it in the notes?"

Today we continue discussing gauge transformations, illustrating an example of the construction of a moduli space where one has to divide out by a symmetry group.

Let G be a Lie group and $\pi\colon P \to G$ be a principal G-bundle. Last time, we defined $\mathcal{G}_P$ to be the group of G-equivariant diffeomorphisms $\varphi\colon P \to P$ commuting with the projection map back onto M. If $\varphi \in \mathcal{G}_P$, then φ sends fibers to fibers, so there's a function $g_\varphi\colon P \to G$ defined to satisfy

$$\varphi(p) = p \cdot g_\varphi(p), \tag{16.1a}$$

and G-equivariance of φ implies

$$g_\varphi(p \cdot h) \coloneqq h^{-1} g_\varphi(p) h. \tag{16.1b}$$

Hence we can descend g_φ to a section of the associated bundle of groups[31] $G_P \to M$. That is, a gauge transformation is a section of an associated bundle of groups.

We'd like to have a Lie algebra of "infinitesimal gauge transformations," but quickly we run into a problem: we would need to make sense of $\mathcal{G}_P$ as an infinite-dimensional Lie group. This is easier if you take Sobolev

[31] This is not the same as a principal bundle: it is a locally trivial family of groups. In particular, there is a section given by the identity element.

completions; we'll begin discussing the general theory of this approach next time.[32] Today, we'll use a more ad hoc approach, allowing us to finish the construction of a moduli space of a problem with symmetries.

Our ad hoc approach to the Lie algebra of infinitesimal gauge transformations will be to consider a path of sections of $G_P \to M$. The derivative of such a path at $t = 0$ is a G-invariant vertical vector field on P. We'll eventually see that all G-invariant vector fields on P arise in this way, so this will be our Lie algebra.

Given a G-invariant vertical vector field on P, we obtain a function $\xi\colon P \to \mathfrak{g}$: the G-invariant vertical vector fields on each fiber of P are identified with $\mathfrak{g}$, since the fiber is a G-torsor. But G-invariance imposes additional constraints: specifically, if $g_t \coloneqq e^{t\xi}\colon P \to G$, then

$$g_t(p \cdot h) = h^{-1} g_t(p) h, \tag{16.2a}$$

and differentiating at $t = 0$,

$$\xi(p \cdot h) = \operatorname{Ad}_{h^{-1}} \xi(p). \tag{16.2b}$$

So ξ descends to a section of the associated bundle of Lie algebras $\mathfrak{g}_P \to M$. In other words, the Lie algebra of G-invariant vertical vector fields (hence also of infinitesimal gauge transformations) is $\Omega^0_M(\mathfrak{g}_P)$. The Lie bracket is pointwise.

Now let's talk about connections: how do gauge transformations and infinitesimal gauge transformations act on connections? Let $\theta \in \Omega^1_G(\mathfrak{g})$ be the *Maurer-Cartan form*: its value on a tangent vector v is the unique left-invariant vector field on G extending v. This form satisfies the equation

$$\mathrm{d}\theta + \frac{1}{2}[\theta \wedge \theta] = 0. \tag{16.3}$$

The connection forms $\mathcal{A}_P \subset \Omega^1_P(\mathfrak{g})$ are those A satisfying the equations

$$R_h^* A = \operatorname{Ad}_{h^{-1}} A \tag{16.4a}$$
$$i_m^* A = \theta, \tag{16.4b}$$

for any $m \in M$, where $i_m\colon P_m \hookrightarrow P$ is the inclusion of the fiber. Because (16.4a) is linear and (16.4b) is affine, we immediately see that the space of solutions, i.e. the space of connections, is an affine space.

Exercise 16.5. Show that if $\varphi \in \mathcal{G}_P$ and $A \in \mathcal{A}_P$, then $\varphi^* A \in \mathcal{A}_P$. This amounts to checking that $\varphi^* A$ satisfies (16.4a) and (16.4b). For (16.4a), it suffices to apply φ^* to both sides – because φ is G-equivariant, φ^* commutes with R_g^* and $\operatorname{Ad}_{h^{-1}}$. The proof for (16.4b) is similar, though you also have to check that the Mauer-Cartan form is invariant. This is ultimately because the vector field it gives you is left-invariant.

So $\mathcal{G}_P$ acts on $\mathcal{A}_P$. Our next step will be to write down an explicit formula for this action. Then we'll discuss orbits, stabilizer groups, etc. First the formula.

Proposition 16.6. *Let $\varphi \in \mathcal{G}_P$ and $g \coloneqq g_\varphi\colon P \to G$. Then*

$$\varphi^* A = \operatorname{Ad}_{g^{-1}} A + g^*\theta. \tag{16.7}$$

Moreover, if G is a matrix group, the adjoint action is conjugation, so

$$\varphi^* A = g^{-1} A g + g^{-1}\,\mathrm{d}g. \tag{16.8}$$

If G is a matrix group, the Maurer-Cartan form is $g^{-1}\,\mathrm{d}g$.

Proof. Let p_t be a parameterized curve in P with $p_0 = p$ and $\dot{p}_0 = \eta \in T_pP$. ⊠

To be continued...

[32] In the context of similar moduli-theoretic problems, controlling this kind of analytic obstacle is one of the many things Karen Uhlenbeck got the Abel prize for.

Lecture 17.

: 3/28/19

Lecture 18.

More Sobolev spaces and a spectral theorem: 4/2/19

Lecture 19.

Dirac operators: 4/4/19

"It's my job... chief refresher."

Last time, we stated and mostly proved the spectral theorem for positive compact self-adjoint operators: if V is a separable Hilbert space and $T\colon V \to V$ is positive, compact, and self-adjoint, then T is *diagonalizable*: there's an orthonormal basis (e_n) such that for all n, $Te_n = \lambda_n e_n$, where $\lambda_n \to 0$ as $n \to \infty$.

To fill in the missing step, let $S(V)$ denote the unit sphere in V and $Q\colon S(V) \to \mathbb{R}$ denote the quadratic form $\xi \mapsto \langle \xi, T\xi\rangle$. (That this is a quadratic form at all uses that T is positive and self-adjoint.) Let $\lambda_1 := \sup_{\xi\in S(V)} Q(\xi)$. Thus there is a sequence $(\xi_n) \subset S(V)$ such that $\langle \xi_n, T\xi_n\rangle \to \lambda_1$.

Let $P := \lambda_1 - T$; then $\langle \xi, P\xi_n\rangle \to 0$. The pairing

$$\xi', \xi'' \longmapsto \langle \xi', P\xi''\rangle \tag{19.1}$$

is an inner product on $(\ker P)^\perp$, and applying Cauchy-Schwarz,

$$|\langle \xi', P\xi''\rangle| \le \langle \xi', P\xi'\rangle^{1/2}\langle \xi'', P\xi''\rangle^{1/2}. \tag{19.2}$$

Applying this when $\xi' = \xi$ and $\xi'' = P\xi$, we have

$$\|P\xi\|^2 \le \langle \xi, P\xi\rangle^{1/2}\langle P\xi, P^2\xi\rangle^{1/2} \tag{19.3a}$$

$$\le \langle \xi, P\xi\rangle^{1/2}\|P\|\|P\xi\| \tag{19.3b}$$

$$\|P\xi\| \le \langle \xi, P\xi\rangle^{1/2}\|P\|. \tag{19.3c}$$

Plugging in ξ_n, we get

$$\|P\xi_n\| \le \langle \xi_n, P\xi_n\rangle^{1/2}\|P\| \longrightarrow 0, \tag{19.4}$$

so $P\xi_n \to 0$. We haven't yet used that T is compact, but now let's use compactness to extract a subsequence (ξ_{n_i}) such that $T\xi_{n_i} \to \eta$ for some η. Since $P\xi_{n_i} \to 0$, then $\lambda_1\xi_{n_i} - T\xi_{n_i} \to 0$, and therefore $\lambda_1\xi_{n_i} \to \eta$. That is, $\lambda_1 T\eta_{n_i}$ converges both to $T\eta$ and $\lambda_1\eta$, so η is an eigenvector for T with eigenvalue λ_1; then we can set $e_1 := \eta/\|\eta\|$. This fixes the hole in the proof.

Today we'll discuss how to use this functional analysis to study Dirac operators and elliptic theory, with the eventual goal of studying moduli spaces. Dirac operators are very general, appearing in both physics and mathematics.

Let M be an n-dimensional compact Riemannian manifold; we'd like to discuss some differential operators on M. The metric gives us the Levi-Civita connection on the frame bundle $\pi\colon \mathcal{B}_{\mathrm{O}}(M) \to M$, and a covariant derivative on all associated vector bundles. There is a global horizontal framing of $\mathcal{B}_{\mathrm{O}}(M)$: for $i = 1, \dots, n$, we have a horizontal vector field ∂_i on $\mathcal{B}_{\mathrm{O}}(M)$: a point of $\mathcal{B}_{\mathrm{O}}(M)$ is an isomorphism $b\colon \mathbb{R}^n \to T_{\pi(b)}M$; we let $\partial_i(b) := b(e_i)$, where e_i is the i^{th} basis vector.

Now suppose $\mathbb{V}$ is a vector space and $\rho\colon \mathrm{O}_n \to \mathrm{Aut}(\mathbb{V})$ is a representation. Let $V \to M$ denote the associated vector bundle. A section $\psi\colon M \to V$ is equivalent data to an O_n-equivariant map $\psi\colon \mathcal{B}_{\mathrm{O}}(M) \to \mathbb{V}$, i.e. such that $R_g^*\psi = \rho(g)^{-1}\psi$ for all $g \in \mathrm{O}_n$. The covariant derivative of ψ is the section of $V \otimes T^*M$ associated to the O_n-equivariant map

$$\nabla\psi = \partial_k\psi \otimes e_k\colon \mathcal{B}_{\mathrm{O}}(M) \longrightarrow \mathbb{V} \otimes (\mathbb{R}^n)^*. \tag{19.5}$$

You can generalize a lot of this discussion to arbitrary connections, but the above formula only holds when the connection is torsion-free.

Now suppose $\mathbb{E}$ and $\mathbb{F}$ are vector spaces with O_n-actions with associated vector bundles $E \to M$ and $F \to M$, respectively. Given an equivariant map $\sigma\colon (\mathbb{R}^n)^* \to \mathrm{Hom}(\mathbb{E}, \mathbb{F})$, we get a differential operator $D_\sigma\colon \Gamma(E) \to \Gamma(F)$ by the formula

$$D_\sigma(\psi) \coloneqq \sigma(e^k)\partial_k\psi. \tag{19.6}$$

Example 19.7. Suppose $\mathbb{E} = \mathbb{F} = \Lambda^\bullet(\mathbb{R}^n)^*$ and $\sigma(e^k) \coloneqq \epsilon(e^k)$ (meaning exterior multiplication). Then we recover the de Rham differential: $\mathrm{d} = \epsilon(e^k)\partial_k$. ◀

Now assume M is spin, so that we have a bundle of spin frames $\mathcal{B}_{\mathrm{Spin}}(M) \to M$, which is a principal Spin_n-bundle. Let $\mathbb{S} = \mathbb{S}^0 \oplus \mathbb{S}^1$ be a $\mathbb{Z}/2$-graded $C\ell((\mathbb{R}^n)^*)$-module and S denote the associated $\mathbb{Z}/2$-graded vector bundle (TODO explicate: this construction does use the spin structure). Suppose we have a Clifford module map $c\colon (\mathbb{R}^n)^* \to \mathrm{End}^1(\mathbb{S})$, meaning

$$c(e^k)c(e^\ell) + c(e^\ell)c(e^k) = 2\delta^{k\ell}. \tag{19.8}$$

Then we get an associated *Dirac operator* $D\colon \Gamma(S) \to \Gamma(S)$ given by

$$D\psi \coloneqq c(e^k)\partial_k\psi. \tag{19.9}$$

This is an odd operator.

Remark 19.10. One common extension of this is to consider a Lie group G (often O_r or U_r) acting on a vector space $\mathbb{V}$. Let $P \to M$ be a principal G-bundle with connection Θ. Then $\mathcal{B}_{\mathrm{Spin}}(M) \times_M P \to M$ is a principal $\mathrm{Spin}_n \times G$-bundle, and there is a Dirac operator acting on $\mathrm{Spin}_n \times G$-equivariant maps $\psi\colon \mathcal{B}_{\mathrm{Spin}}(M) \times_M P \to \mathbb{S} \otimes \mathbb{V}$ given by

$$D = (c(e^k) \otimes \mathrm{id}_{\mathbb{V}})\partial_k. \tag{19.11}$$

◀

Now let's compute D^2. We can and will do this in coordinates.

$$\begin{aligned} D^2 &= c(e^k)\partial_k c(e^\ell)\partial_\ell \\ &= c(e^k)c(e^\ell)\partial_k\partial_\ell \\ &= -\sum_{k=1}^n \partial_k^2 + \sum_{k<\ell} c(e^k)c(e^\ell)[\partial_k, \partial_\ell]. \end{aligned} \tag{19.12}$$

Now recall that the first term is the Laplacian:

$$-\sum_{k=1}^n \partial_k^2 = \nabla^*\nabla, \tag{19.13}$$

and the second term is

$$[\partial_k, \partial_\ell] = -\Gamma^i_{k\ell}\partial_i - R^i_{jk\ell}Z^j_i. \tag{19.14}$$

The first term is the torsion, so it vanishes, because the Levi-Civita connection is torsion-free. The second term comes from the curvature, and we can't quite get rid of it. So we have some operator F built out of the curvature such that

$$\sum_{k<\ell} c(e^k)c(e^\ell)[\partial_k, \partial_\ell] = F \cdot \psi. \tag{19.15}$$

In summary, we've proved:

Theorem 19.16 (Weitzenböck formula)**.** *$D^2 = \nabla^*\nabla + F$.*

Determining the precise formula for F is a good exercise in Riemannian geometry, and it is a very useful thing to have around; but for today we won't need it.

Elliptic theory will apply to any operator D for which $D^@ = \nabla^*\nabla + F$ for some F.

Example 19.17. Let d^* denote the adjoint of the de Rham differential and $\Delta \coloneqq (\mathrm{d} + \mathrm{d}^*)^2 = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$ be the Hodge Laplacian. The Laplacian is a Dirac operator: consider the map $\mathrm{d} + \mathrm{d}^*\colon \Omega_M^{\mathrm{odd}} + \Omega_M^{\mathrm{even}}$, which is associated to the vector spaces $\mathbb{S}^0 \coloneqq \Lambda^{\mathrm{odd}}(\mathbb{R}^n)^*$ and $\mathbb{S}^1 \coloneqq \Lambda^{\mathrm{even}}(\mathbb{R}^n)^*$, and the map $\sigma\colon \mathbb{S}^0 \to \mathbb{S}^1$ given by

$$\sigma(e^k) = \epsilon(e^k) - \iota(e^k), \tag{19.18}$$

where ϵ denotes exterior multiplication and ι denotes interior multiplication (contraction). This satisfies the Clifford relations, and we get out the Laplacian. ◂

Now we'll begin using the Sobolev theory we spent the previous few lectures building, though we won't get too far until next time.

Definition 19.19. Let $E \to X$ be a vector bundle. The *H_{-1}-norm* of an $f \in C^\infty(X, E)$ is the infimum of the $C > 0$ such that $\langle f, \psi\rangle_{L^2} \le C\|\psi\|_{H^1}$ over all $\psi \in H_1(X; E)$. We then define $H_{-1}(X; E)$ to be the completion of $C^\infty(X; E)$ in the H_1-norm.

More or less by construction, this is the dual norm to the H_1-norm: there is a nondegenerate pairing $H_{-1}(X; E) \otimes H_1(X; E) \to \mathbb{C}$.

Lemma 19.20. *The inclusion map $L_2(X; E) \to H_1(X; E)$ is continuous, and with D as above, $D^2 \colon H_1(X; E) \to H_{-1}(X; E)$ is continuous.*

Theorem 19.21 (Gårding's inequality)**.** *There is some constant $\kappa > 0$ such that*

$$\langle D^2\psi, \psi\rangle + \kappa\langle\psi, \psi\rangle \ge \|\psi\|^2_{H_1}. \tag{19.22}$$

Proof. By Theorem 19.16,

$$D^2\psi = \nabla^*\nabla\psi + F\psi \tag{19.23}$$

$$\psi + \nabla^*\nabla\psi = D^2\psi + (1 - F)\psi \tag{19.24}$$

$$\|\psi\|^2_{H_1} = \langle\psi, \psi\rangle + \langle\nabla\psi, \nabla\psi\rangle = \langle D^2\psi, \psi\rangle + \langle(1 - F)\psi, \psi\rangle \tag{19.25}$$

$$\le \langle D^2\psi, \psi\rangle + \kappa\langle\psi, \psi\rangle. \tag{19.26}$$

This is because $1 - F$ is some algebraic operator on M, and M is compact, so it's bounded. ⊠

Corollary 19.27. *The inner product*

$$\langle\langle\psi_1, \psi_2\rangle\rangle := \langle D^2\psi_1, \psi_2\rangle + \kappa\langle\psi_1, \psi_2\rangle$$

is equivalent to the H_1 inner product.

This is a very strong statement: we can control all first derivatives in terms of the Dirac operator. This is one manifestation of ellipticity.

Theorem 19.28. *$D^2 + \kappa \colon H_1 \to H_{-1}$ is an isomorphism.*

Proof. By Theorem 19.21, $\|(D^2 + \kappa)\psi\|_{H_{-1}} \ge \|\psi\|_{H_1}$. Therefore $D^2 + \kappa$ is injective with closed range; it suffices to prove it's surjective. If $f \in H_{-1}$, consider the map $H_1 \to \mathbb{C}$ sending $\varphi \mapsto \langle f, \varphi\rangle$. By the Riesz theorem (and substituting in an equivalent inner product), there is some $\psi \in H_1$ such that for all $\varphi \in H_1$,

$$\langle f, \varphi\rangle = \langle\langle\psi, \varphi\rangle\rangle = \langle(D^@ + \kappa)\psi, \varphi\rangle, \tag{19.29}$$

and therefore $(D^2 + \kappa)\psi = f$. ⊠

So using not even all that much analysis, we've inverted the Laplacian (plus κ). Now we use the Rellich theorem: consider the operator

$$T \colon H_0 \longrightarrow H_{-1} \xrightarrow{(D^2+\kappa)^{-1}} H_1 \longrightarrow H_0. \tag{19.30}$$

Each of these is continuous, and the last is compact. It's also self-adjoint, and because $\kappa > 0$, it's positive. Now invoke the spectral theorem, yielding an orthonorormal basis (ψ_n) of H_0 with $D^2\psi_n = \lambda_n\psi_n$, where $\lambda_n \ge 0$,[33] and $\lambda_n \to \infty$. Thus for any $a > 0$, the set $\{n : \lambda_n < a\}$ is finite.

The last basic theorem we need guarantees smoothness.

Theorem 19.31 (Elliptic regularity)**.** *These ψ_n are C^∞.*

We'll do this next time.

[33]Well, really we got that $T\psi_n = \mu_n\psi_n$ for some $\mu_n \to 0$; then you can solve for λ_n in terms of μ_n, getting $\lambda_n = 1/\mu_n - \kappa$.

Lecture 20.

Elliptic regularity: 4/9/19

"You might think it's game over, but you're wrong."

Today our goal will be to prove elliptic regularity. Notation for Dirac operators may differ from previous lectures, and objects in the mirror may be closer than they appear.

So let M be a closed spin Riemannian manifold of dimension n, $\mathcal{B}_{\mathrm{SO}}(M) \to M$ be the bundle of oriented orthonormal frames, and and $\pi\colon \mathcal{B}_{\mathrm{Spin}}(M) \to M$ be its bundle of spin frames. The former is a principal SO_n-bundle and the latter is a principal Spin_n-bundle.

Definition 20.1. The *Clifford algebra* $C\ell_{-n}$ is the unital algebra generated by $e^1, \dots, e^n$ subject to the relation

$$e^i e^j + e^j e^i = -2\delta^{ij}. \tag{20.2}$$

Beware that there are different sign conventions in the literature for the Clifford algebra.

The Clifford algebra is $\mathbb{Z}/2$-graded: $C\ell_{-n} = C\ell^0_{-n} \oplus C\ell^1_{-n}$, where the even piece is spanned by products of the e^i with an even number of terms, and the odd piece is spanned by products with an odd number of terms. This is in fact a $\mathbb{Z}/2$-grading because (20.2) only contains products of two and zero generators; since (20.2) is not homogeneous in the tensor algebra, we don't get a $\mathbb{Z}$-grading.

Now suppose $\mathbb{V} = \mathbb{V}^0 \oplus \mathbb{V}^1$ is a *supermodule* for $C\ell_{-n}$, i.e. it's a $\mathbb{Z}/2$-graded $C\ell_{-n}$-module such that the action of $C\ell^0_{-n}$ preserves the grading and the action of $C\ell^1_{-n}$ reverses the grading. Given this data, let's form some associated bundles.

- SO_n acts on $C\ell_{-n}$ by conjugation, where we interpret $e^1, \dots, e^n$ as an oriented orthonormal basis of $\mathbb{R}^n$. Let $C\ell(M) := \mathcal{B}_{\mathrm{SO}}(M) \times_{\mathrm{SO}_n} C\ell_{-n}$.
- The inclusion $\mathrm{Spin}_n \hookrightarrow C\ell_{-n}$ means the action of $C\ell_{-n}$ on $\mathbb{V}$ induces a Spin_n-action on $\mathbb{V}$, so we can form the associated bundle $V := \mathcal{B}_{\mathrm{Spin}}(M) \times_{\mathrm{Spin}_n} \mathbb{V}$. This vector bundle is again $\mathbb{Z}/2$-graded; let V^0, resp. V^1 denote the even and odd components.

In this setting we can define a Dirac operator. We will let $\partial_1, \dots, \partial_n$ denote the horizontal vector fields on $\mathcal{B}_{\mathrm{SO}}(M)$ or on $\mathcal{B}_{\mathrm{Spin}}(M)$, as we constructed last time; the specific bundle will be unambiguous from context. In particular, as we discussed last time,

$$[\partial_k, \partial_\ell] = -\frac{1}{2} R^i_{jk\ell} E^j_i \tag{20.3}$$

is vertical. Here E^j_i is the matrix with a 1 in position (i, j) and zeros everywhere else. The *Dirac operator* $D\colon C^\infty(M : V) \to C^\infty(M; V)$ is defined as follows: we can identify

$$C^\infty(M; V) = \{\psi\colon \mathcal{B}_{\mathrm{Spin}}(M) \to \mathbb{V} \mid R^*_g \psi = g^{-1}\psi \text{ for all } g \in \mathrm{Spin}_n\}, \tag{20.4}$$

using the fact that V is an associated bundle to $\mathcal{B}_{\mathrm{Spin}}(M)$. Then

$$D := c(e^k)\partial_k, \tag{20.5}$$

where $c(e^k)\colon \mathbb{V} \to \mathbb{V}$ is the Clifford action of e^k on $\mathbb{V}$.

Last time, we proved some important and nontrivial properties of the Dirac operator.

- The Weitzenböck formula (Theorem 19.16), that $D^2 = \nabla^*\nabla + \mathcal{R}$ for some $\mathcal{R}$ related to the curvature.
- Using this, we proved Gårding's inequality (Theorem 19.21), that $\langle D^2\psi, \psi\rangle + \kappa\langle \psi, \psi\rangle \ge \|\psi\|^2_{H_1}$, and hence $D^2 + \kappa$ can be inverted.
- Invoking spectral theory for D^2 to construct a basis $\{\psi_n\}$ of $L^2(M; V)$ of eigenvectors with eigenvalues λ_n, which increase to ∞ as $n \to \infty$. In particular, for any $a > 0$, the set of n with $\lambda_n < a$ is finite.

Today we'll prove Theorem 19.31, that these ψ_n are smooth.

For the rest of today's lecture, $\|\cdot\|_\ell$ means $\|\cdot\|_{H_\ell}$.

Proposition 20.6 (Elliptic estimate)**.** *Suppose $\psi \in H_{\ell+1}$ for some $\ell \in \mathbb{Z}^{\ge 0}$. Then*

$$\|\psi\|_{\ell+1} \le C(\|D\psi\|_\ell + \|\psi\|_\ell), \tag{20.7}$$

where C is some constant independent of ψ.

Proof. The $\ell = 0$ case follows from Theorem 19.21 and the fact that $\sqrt{a^2+b^2} \le a+b$:

$$\|\psi\|_1 \le C\sqrt{\|D\psi\|_0^2 + \|\psi\|_0^2} \le C(\|D\psi\|_0 + \|\psi\|_0). \tag{20.8}$$

By induction, let's assume the estimate is true for $1, \dots, \ell-1$. We claim that $[\nabla, D]$ is a zeroth-order operator (sometimes called an *algebraic operator*).

To prove this, we first show that $[t(e^k), c(e^\ell)] = 0$, because $t(e^k)\colon \xi \mapsto \xi \otimes e^k \in \mathbb{V} \otimes (\mathbb{R}^n)^*$ and $c(e^\ell)\colon \xi \mapsto c(e^\ell)\xi \in \mathbb{V}$. So both $t(e^k)c(e^\ell)$ and $c(e^\ell)t(e^k)$ send $\xi \mapsto c(e^\ell)\xi \otimes e^k$. Hence

$$\begin{aligned}[\nabla, D] &= [t(e^k)\partial_k, c(e^\ell)\partial_\ell] \\ &= t(e^k)c(e^\ell)[\partial_k, \partial_\ell],\end{aligned} \tag{20.9}$$

and, as in (20.3), this is zeroth-order. This means we can bound the norm without losing any derivatives after applying $[\nabla, D]$ (that is, if $\|\psi\|_\ell$ is finite, so is $\|[\nabla, D]\psi\|_\ell$). Hence

$$\|\nabla\psi\|_\ell \le C(\|D\nabla\psi\|_{\ell-1} + \|\nabla\psi\|_{\ell-1}) \tag{20.10a}$$
$$\le C(\|\nabla D\psi\|_{\ell+1} + \|[\nabla, D]\psi\|_{\ell+1} + \|\nabla\psi\|_{\ell-1}) \tag{20.10b}$$
$$\le C(\|D\psi\|_\ell + \|\psi\|_{\ell-1} + \|\psi\|_\ell) \tag{20.10c}$$
$$\le C(\|D\psi\|_\ell + \|\psi\|_\ell). \tag{20.10d}$$

Here C is again a variable constant: each C is some constant independent of ψ, but we don't need to care which constant it is, and it may change between lines. Hence

$$\|\psi\|_\ell + \|\nabla\psi\|_\ell \le C(\|D\psi\|_\ell + \|\psi\|\ell), \tag{20.11}$$

and the left-hand side is (some constant multiple of) the $H_{\ell+1}$ norm of ψ. ⊠

Corollary 20.12. *If $\psi \in H_{\ell+2}$, for $\ell \ge 0$, then $\|\psi\|_{\ell+2} \le C\big(\|D^2\psi\|_\ell + \|\psi\|_\ell\big)$.*

Proof.

$$\|\psi\|_{\ell+2} \le C(\|D\psi\|_{\ell+1} + \|\psi\|_{\ell+1}) \tag{20.13a}$$
$$\le C\big(\|D^2\psi\|_\ell + \|D\psi\|_\ell + \|\psi\|_{\ell+1}\big) \tag{20.13b}$$
$$\le C\big(\|D^2\psi\|_\ell + \|\psi\|_{\ell+1} + \|\psi\|_{\ell+1}\big). \tag{20.13c}$$

Changing the value of C, this simplifies to

$$\le C\big(\|D^2\psi\|_\ell + \|\psi\|_{\ell+1}\big) \tag{20.13d}$$
$$\le C\big(\|D^2\psi\|_\ell + \|D\psi\|_\ell + \|\psi\|_\ell\big) \tag{20.13e}$$
$$\le C\big(\|D^2\psi\|_\ell + \|D^2\psi\|_{\ell-1} + \|\psi\|_{\ell-1} + \|\psi\|_\ell\big) \tag{20.13f}$$
$$\le C\big(\|D^2\psi\|_\ell + \|\psi\|_\ell\big) \tag{20.13g}$$

using the elliptic estimate. ⊠

This suggests a very appealing argument: apply this corollary recursively to conclude ψ is in $H_{\ell+4}$, then $H_{\ell+6}$, … until you see that it's smooth. But we have no base case for this recursion argument and have to do something different.

We want to approximate our *a priori* non-smooth sections by smooth ones. One standard way to do this is with *mollifiers*, smooth functions which closely approximate a delta-function. Hence convolving a possibly not smooth f with a mollifier produces a smooth function close to the original f.

Another thing we can do is take difference quotients, which has its own minimalist elegance. In general, consider an L^2 function $f\colon \mathbb{T}^n \to \mathbb{V}$, which has a Fourier series:

$$f(x) = \sum_{\nu \in \mathbb{Z}^n} \widehat{f}_\nu e^{i\nu\cdot x}, \tag{20.14}$$

where $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{Z}^n$, $x = (x^1, \dots, x^n) \in \mathbb{R}^n/(2\pi\mathbb{Z})^n$, and $\nu \cdot x \coloneqq \nu_i x^i$. Thus

$$\|f\|_\ell^2 = \sum_\nu |\widehat{f}_\nu|^2 \sum_{j=0}^{\ell} |\nu|^{2j}. \tag{20.15}$$

Given $h \in \mathbb{R}^n$, define

$$f^h(x) \coloneqq \frac{f(x+h) - f(x)}{|h|}. \tag{20.16}$$

Its Fourier coefficients are

$$\widehat{f}^h_\nu = \frac{e^{i\nu\cdot h} - 1}{|h|} \cdot \widehat{f}_\nu. \tag{20.17}$$

Proposition 20.18.

(1) *If $f \in H_\ell$, then $f(\cdot + h) \in H_\ell$ and $\|f(\cdot + h)\|_\ell = \|f\|_\ell$.*
(2) *If $f \in H_{\ell+1}$, then $f^h \in H_\ell$ and $\|f^h\|_\ell \le C\|f\|_{\ell+1}$.*
(3) *If $f \in H_\ell$ and $\|f^h\|_\ell \le C$ for all h sufficiently close to 0, with C independent of h, then $f \in H_{\ell+1}$.*

Proof. (1) is obvious. Next (2): using (20.17),

$$|\widehat{f}^h_\nu|^2 = \frac{|e^{i\nu\cdot h} - 1|^2}{|h|^2}|\widehat{f}_\nu|^2 \le \frac{|\nu\cdot h|^2}{|h|^2}|\widehat{f}_\nu|^2 \le |\nu|^2|\widehat{f}^2_\nu|. \tag{20.19}$$

Thus

$$\|f^h\|^2_\ell \overset{(20.15)}{=} \sum_{\nu\in\mathbb{Z}^n} \big(1 + |\nu|^2 + \cdots + |\nu|^{2\ell}\big)|\widehat{f}^h_\nu|^2 \tag{20.20a}$$

$$\overset{(20.19)}{\le} \sum_{\nu\in\mathbb{Z}^n} \Big(|\nu|^2 + \cdots + |\nu|^{2(\ell+1)}\Big)|\widehat{f}_\nu|^2 \tag{20.20b}$$

$$\le \|f\|_{\ell+1}. \tag{20.20c}$$

For (3), let $e_i \coloneqq (0, \dots, 0, 1, 0, \dots, 0)$, with 1 in the i^{th} position. Then

$$\lim_{\varepsilon\to 0} \sum_{i=1}^n |\widehat{f}^{\varepsilon e_i}_\nu|^2 = |\nu|^2|\widehat{f}_\nu|^2. \tag{20.21}$$

Hence for $N \in \mathbb{Z}^{\ge 0}$,

$$\sum_{|\nu|<N} |\widehat{f}_\nu|^2 \sum_{j=1}^{\ell+1} |\nu|^{2j} = \lim_{\varepsilon\to 0} \sum_{i=1}^n \sum_{|\nu|<N} |\widehat{f}^{\varepsilon e_i}_\nu| \sum_{j=0}^{2\ell} |\nu|^{2j} \tag{20.22a}$$

$$\le \lim_{\varepsilon\to 0} \sum_{i=1}^n \|f^{\varepsilon e_i}\|^2_\ell, \tag{20.22b}$$

which is bounded by some constant. ⊠

Theorem 20.23. *If $\ell \ge 1$ and $\psi \in H_\ell$, then $D^2\psi \in H_{\ell-1}$, so $\psi \in H_{\ell+1}$.*

Corollary 20.24. *If $\ell \ge 1$ and $\psi \in H_\ell$, then $D^2\psi \in H_\ell$, so $\psi \in H_{\ell+2}$. Iterating, we conclude that if $D^2\psi = \lambda\psi$, then $\psi \in C^\infty$.*

Here we use the Sobolev embedding theorem.

Proof sketch of Theorem 20.23. We use the standard trick to pass to the torus: choose a finite atlas on M which trivializes all the vector bundles in scope, so it suffices to prove this on each chart, and embed each chart in $\mathbb{T}^n$. Since D^2 is a second-order differential operator, we can write it as

$$D^2 = L_2 \circ \nabla^2 + L_1 \circ \nabla + L_0, \tag{20.25}$$

where L_0, L_1, and L_2 are matrix-valued functions on $\mathbb{T}^n$. In particular, there is some E (for "error term") such that

$$D^2\psi^h(x) = (D^2\psi)^h(x) + E\psi(x+h). \tag{20.26}$$

Therefore

$$\|\psi^h\|_\ell \le C\big(\|D^2\psi^h\|_{\ell-2} + \|\psi^h\|_{\ell-2}\big) \tag{20.27a}$$
$$\le C\big(\|(D^2\psi)^h\|_{\ell-2} + \|E\psi\|_{\ell-2} + \|\psi\|_{\ell-2}\big) \tag{20.27b}$$
$$\le C\big(\|D^2\psi\|_{\ell-1} + \|\psi\|_{\ell}\big) \tag{20.27c}$$
$$\le C. \tag{20.27d}$$

Therefore $\psi \in H_{\ell+1}$, using Proposition 20.18, part (3). ⊠

Lecture 21.

Chern-Weil theory: 4/11/19

Today, we'll discuss Chern-Weil theory, and at some point also the Chern-Simons form. Let G be a Lie group, with no restrictions for now, and $\mathfrak g$ be its Lie algebra.

Definition 21.1. An *invariant polynomial* of degree $q \ge 0$ is a symmetric q-linear function

$$f\colon \underbrace{\mathfrak g \times \cdots \times \mathfrak g}_{q} \longrightarrow \mathbb R \tag{21.2}$$

which is invariant under conjugation, i.e. for all $g \in G$ and $\xi_1, \dots, \xi_q \in \mathfrak g$,

$$f(g\xi_1 g^{-1}, \dots, g\xi_q g^{-1}) = f(\xi_1, \dots, \xi_q). \tag{21.3}$$

Invariant polynomials form a vector space, which we denote $I^k(G)$, and which can be identified with $(\mathrm{Sym}^q\, \mathfrak g^*)^G$: $\mathrm{Sym}^q\, \mathfrak g^*$ is the space of symmetric q-linear functions on $\mathfrak g$, and we take the invariant subspace under G acting by conjugation. Let

$$I^\bullet(G) := \bigoplus_{q=0}^{\infty} I^q(G), \tag{21.4}$$

which we make $\mathbb Z$-graded by specifying $\deg I^q(G) = 2q$. Multiplication of polynomials makes this into a commutative $\mathbb R$-algebra.

Example 21.5.

(1) Suppose G is a countable, discrete group. Then $\mathfrak g = \{0\}$, so $I^\bullet(G)$ is just the constant functions in degree 0, or just $\mathbb R$.

(2) Now suppose G is connected and abelian, e.g. $\mathbb T^m$, $\mathbb R^n$, or a product of such things. Then $\mathfrak g = \mathbb R^n$ and conjugation is trivial. Thus all polynomials are invariant, so $I^\bullet(G) = \mathrm{Sym}^\bullet((\mathbb R^n)^*)$, but with a different grading: a degree-q homogeneous polynomial has grading $2q$.

(3) Let's do a more interesting example: $G = \mathrm{SU}_2$, the space of 2×2 matrices of the form $\left(\begin{smallmatrix}\alpha & -\overline\beta\\ \beta & \overline\alpha\end{smallmatrix}\right)$, where $\alpha, \beta \in \mathbb C$ and $|\alpha|^2 + |\beta|^2 = 1$. The Lie algebra is

$$\mathfrak{su}_2 = \left\{ \begin{pmatrix} ix & -y+iz \\ y+iz & -ix \end{pmatrix} : x,y,z \in \mathbb R \right\}. \tag{21.6}$$

One example of an invariant polynomial is $f_4\colon X \mapsto \mathrm{tr}(X^2)$: since

$$X^2 = \begin{pmatrix} -x^2-y^2-z^2 & * \\ * & -x^2-y^2-z^2 \end{pmatrix}, \tag{21.7}$$

then $f_4(X) = \mathrm{tr}(X^2) = -2(x^2+y^2+z^2)$. It turns out that $I^*(\mathrm{SU}_2)$ is a polynomial algebra generated by f_4, which has degree 4.

(4) What about $\mathrm{SL}_2(\mathbb R)$? Again the Lie algebra is the algebra of traceless matrices, and $I^*(\mathrm{SL}_2(\mathbb R)) = \mathbb R[g_4]$, where $g_4(X) := \mathrm{tr}(X^2)$ for $X \in \mathfrak{sl}_2(\mathbb R)$. Explicitly, an arbitrary $X \in \mathfrak{sl}_2(\mathbb R)$ is given by a trace-zero real matrix $X = \left(\begin{smallmatrix} x & z \\ y & -x\end{smallmatrix}\right)$, so $g_4(X) = 2(x^2+yz)$. So though the algebra looks the same, the functions are different: f_4 is positive definite, and g_4 is indefinite.

(5) If we look at $\mathrm{SL}_2(\mathbb C)$, we again get $I^\bullet(\mathrm{SL}_2(\mathbb C)) = \mathbb R[h_4]$, given by the same formula: now x, y, and z are complex, but nothing else changes.

(6) If we consider $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$, or U_n, the Lie algebra now includes matrices with nonzero trace, and the trace defines a degree-one invariant polynomial. Focusing on $G = \mathrm{U}_n$; its Lie algebra $\mathfrak{u}_n$ is the algebra of skew-Hermitian matrices. Consider the polynomial

$$\det(t - X) = \sum_{i=0}^{n} f_{2i}(X) t^i. \tag{21.8}$$

The determinant is an invariant polynomial, because if X is unitary, $\det(X)\det(X^{-1}) = 1$. Therefore these coefficients f_{2i} are also conjugation-invariant. The theorem (which we won't prove here) is that $I^\bullet(\mathrm{U}_n) = \mathbb{R}[f_2, \dots, f_{2n}]$. The same holds for $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$. In each case, $f_{2i}(X) = \pm\operatorname{tr}(\Lambda^i X)$.

(7) Suppose $G = \mathrm{SO}_{2m}$. We still have $\det(t - X)$, $X \in \mathfrak{o}_{2m}$, but the trace vanishes, and the trace of any odd exterior power vanishes: the eigenvalues of X come in conjugate pairs. So

$$\det(t - X) = \sum_{j=0}^{m} f_{4j}(X) t^{2j}, \tag{21.9}$$

where $f_{4j} = \pm\operatorname{tr}\Lambda^{2i} X$ as before. There's also a new invariant polynomial P_m of degree m, called the *Pfaffian*, defined as follows. Let $V = \mathbb{R}^{2m}$, with the standard inner product. Use this inner product to identify V and V^*, so we can say that a matrix X is in $\mathfrak{so}_{2m}$ iff, as a map $X\colon V \to V^*$, $X^* = -X$. The map $X\colon V \to V^*$ can be identified with $\omega_X \in \Lambda^2 V^*$, and the condition tells us it's skew. Then the Pfaffian is

$$\frac{\omega_X^m}{m!} = P_m(x) \cdot \mathrm{vol}, \tag{21.10}$$

where vol is the volume form induced from the inner product on V and the orientation (preserved because we're looking at SO_{2m} and not O_{2m}). (21.10) takes place in $\operatorname{Det} V$.

Exercise 21.11. Show that $P_m(X^2) = \det(X)$.

This tells us that $I^\bullet(\mathrm{SO}_{2m})$ isn't a polynomial ring: there's a nontrivial ring. ◁

Lemma 21.12. *Let $f \in I^q(G)$ and $\zeta, \zeta_1, \dots, \zeta_q \in \mathfrak{g}$. Then*

$$\sum_{i=1}^{q} f(\zeta_1, \dots, [\zeta, \zeta_i], \dots, \zeta_q) = 0.$$

Proof. Differentiate and use the fact that f is invariant:

$$f\big(\mathrm{Ad}_{\exp(t\zeta)}\zeta_1, \dots, \mathrm{Ad}_{\exp(t\zeta)}\zeta_q\big) = f(\zeta_1, \dots, \zeta_q). \tag{21.13}$$ ⊠

This expresses the infinitesimal invariance of the invariant polynomials. If the group isn't connected, though, that doesn't tell you everything.

Chern-Weil theory is about applying invariant polynomials to study connections on principal G-bundles. Let $P \to M$ be a principal G-bundle, and recall that a connection $\Theta \in \Omega^1_P(\mathfrak{g})$ satisfies a few equations, including (16.4a) and (16.4b), telling us how Θ transforms under the pullback by right multiplication and what it must restrict to on each fiber.

The curvature of Θ, given by $\Omega = \mathrm{d}\Theta + (1/2)[\Theta \wedge \Theta]$. It satisfies slightly different equations: it transforms under R_g^* exactly as in (16.4a), but $i_m^*\Omega = 0$,[34] which follows from the Maurer-Cartan equation. Finally, what happens if you differentiate Ω? The $\mathrm{d}\Theta$ vanishes, and using the Leibniz rule, we get

$$\mathrm{d}\Omega - = [\mathrm{d}\Theta \wedge \Theta] = [\Omega \wedge \Theta] - \frac{1}{2}[[\Theta \wedge \Theta] \wedge \Theta]. \tag{21.14}$$

The Jacobi identity implies the second term vanishes: plug in three vectors to this 3-form and see what happens. It may help to work in coordinates, letting $\Theta = \Theta^\alpha \xi_\alpha$, where $\{\xi_\alpha\}$ is a basis of $\mathfrak{g}$.

We rearrange (21.14) into a more standard form, called the *Bianchi identity*:

$$\mathrm{d}_\Theta \Omega \coloneqq \mathrm{d}\Omega + [\Theta \wedge \Omega] = 0. \tag{21.15}$$

Though the curvature is in some sense the derivative of the connection, it's not literally the covariant derivative.

[34] These two imply that the curvature form descends to the base, valued in the adjoint bundle.

Remark 21.16. Suppose $\zeta \in \mathfrak{g}$ and $\widehat{\zeta}$ is the induced vertical vector field on P; we want to show $\iota_{\widehat{\zeta}}\Omega = 0$. Well, let's compute:

$$\iota_{\widehat{\zeta}}\Omega = \iota\left(\mathrm{d}\Theta + \frac{1}{2}[\Theta \wedge \Theta]\right). \tag{21.17a}$$

Using Cartan's formula,

$$= -\mathrm{d}\iota_{\widehat{\zeta}}\Theta + \mathcal{L}_{\widehat{\zeta}}\Theta + [\zeta, \Theta]. \tag{21.17b}$$

The first term is zero because contraction gives us a constant function $P \to \mathfrak{g}$, and when we differentiate we get zero. For the Lie derivative, we have

$$\mathcal{L}_{\widehat{\zeta}}\Theta = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} R^*_{\exp(t\zeta)}\Theta = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}_{\exp(-t\zeta)}\,\Theta = -[\zeta, \Theta]. \tag{21.18}$$

So that adds to $[\zeta, \Theta]$ and we get zero. ◀

Now, given data of $P \to M$ and a connection, we'll construct a map $I^\bullet(G) \to \Omega^\bullet(M)$, called the Chern-Weil homomorphism, and will show that it's a map of differential graded algebras. First we need to define a differential on $I^\bullet(G)$, but since it's concentrated in even degrees, our hand is forced: $d = 0$. So the condition "the Chern-Weil homomorphism is a map of differential graded algebras" means that the image of any invariant polynomial is closed.

Definition 21.19. Let $P \to M$ be a principal G-bundle with connection Θ and curvature form Ω. The *Chern-Weil homomorphism* $\omega\colon I^\bullet(G) \to \Omega^\bullet(M)$ sends $f \in I^q(G)$ to

$$\omega_f \coloneqq f(\underbrace{\Omega \wedge \cdots \wedge \Omega}_{q \text{ times}}). \tag{21.20}$$

Let's quickly typecheck this. Since $\Omega \in \Omega^2_P(\mathfrak{g})$. Hence $\Omega \wedge \cdots \wedge \Omega \in \Omega^{2q}_P(\mathfrak{g}^{\otimes q})$, and therefore $f(\Omega \wedge \cdots \wedge \Omega) \in \Omega^{2q}_P$. There's still plenty to check here – why does this descend to M? Why is it closed? But first, an example.

Example 21.21. Let $G = \mathrm{U}_n$ and $f(X) = \mathrm{tr}(X^3)$. Then $\omega_f = \mathrm{tr}(\Omega \wedge \Omega \wedge \Omega)$. If you write $\Omega = (\Omega^i_j)$, for $\Omega^i_j \in \Omega^2_P(\mathbb{C})$, then you can explicitly obtain matrix-valued forms, multiply them together, and take their trace. ◀

If $\{\zeta_\alpha\}$ is a basis of $\mathfrak{g}$, we can write $\Omega = \Omega^\alpha \zeta_\alpha$, where $\Omega^\alpha \in \Omega^2_P$. Then

$$f(\Omega \wedge \cdots \wedge \Omega) = f(\zeta_{\alpha_1}, \ldots, \zeta_{\alpha_q})\Omega^{\alpha_1} \wedge \cdots \wedge \Omega^{\alpha_q}. \tag{21.22}$$

Now, as promised:

Lemma 21.23. *For any $f \in I^\bullet(G)$, ω_f is closed, and it descends to M.*

Proof. To show it's closed, we just compute:

$$\mathrm{d}\omega_f = \mathrm{d}f(\Omega \wedge \cdots \wedge \Omega) \tag{21.24a}$$

$$= \sum_{i=1}^q f(\Omega \wedge \cdots \wedge \mathrm{d}\Omega \wedge \cdots \wedge \Omega), \tag{21.24b}$$

where $\mathrm{d}\Omega$ is in the i^{th} place. By (21.15),

$$= -\sum_{i=1}^q f(\Omega \wedge \cdots \wedge [\Theta \wedge \Omega] \wedge \cdots \wedge \Omega) \tag{21.24c}$$

$$= 0 \tag{21.24d}$$

by Lemma 21.12.

To show ω_f descends to M, we'll show that the contraction with any vertical vector field is zero. Let $\widehat{\zeta}$ be a vertical vector field; then

$$\iota_{\widehat{\zeta}}\omega_f = \iota_{\widehat{\zeta}} f(\Omega \wedge \cdots \wedge \Omega) = \sum_i f(\Omega \wedge \cdots \wedge \iota_{\widehat{\zeta}}\Omega \wedge \cdots \wedge \Omega) = 0, \tag{21.25}$$

because we saw that $\iota_{\widehat{\zeta}}\Omega = 0$. ☒

The Chern-Weil homomorphism is natural in the following sense: let $\overline{\varphi}\colon M' \to M$ be a smooth map and $\varphi\colon P' = \overline{\varphi}^*P \to P$ be the induced map for the pullback bundle. Let $\Theta' \coloneqq \varphi^*\Theta$ be the pullback map.

Proposition 21.26. $\omega_f(\Theta') = \overline{\varphi}^*\omega_f(\Theta)$.

This leads us to ask whether there is a universal target: is there a manifold BG and a principal G-bundle $EG \to BG$ such that every principal bundle and connection arises this way? Then we could just do Chern-Weil theory there and carry it over to everywhere else. Unfortunately, this dream doesn't quite work: BG and EG exist in topology, but are only unique up to homotopy. So we could fix a model, but we'd need for connections and forms to make sense on it, which is not guaranteed. Moreover, then pullback maps exist but aren't unique; the space of such maps is contractible, which is great for homotopy theory (where contractible spaces are hardly different from points) but not quite for geometry.

Nonetheless there are some nice things to say. If G is compact, there is a manifold model for $EG \to BG$ – albeit an infinite-dimensional, Hilbert manifold. First, let's do this for U_n. Fix a separable infinite-dimensional complex Hilbert space $\mathcal{H}$.

Definition 21.27. The *Stiefel manifold* $\mathrm{St}_n(\mathcal{H})$ is the space of isometries $b\colon \mathbb{C}^n \to \mathcal{H}$.

This is a Hilbert manifold modeled on $\mathcal{H}$, where the topology is the subspace topology of $\mathcal{H}^n$ (in fact, it's a submanifold of that space). The *Grassmannian* of $\mathcal{H}$, denoted $\mathrm{Gr}_n(\mathcal{H})$, is the Hilbert manifold of n-dimensional subspaces $W \subset \mathcal{H}$. There is a map $\mathrm{St}_n(\mathcal{H}) \to \mathrm{Gr}_n(\mathcal{H})$ sending a map $\mathbb{C}^n \to \mathcal{H}$ to its image; since the map is an isometry, it's injective, so the image is an n-dimensional subspace. Then U_n acts on the fiber by precomposition: precomposing $\mathbb{C}^n \to \mathcal{H}$ with a unitary map doesn't change the image.

To see why this is universal, we have to check one more thing.

Lemma 21.28. $\mathrm{St}_n(\mathcal{H})$ *is contractible.*

We'll discuss this more next time (I think). The proof will induct on n; for $n = 1$, this is a fairly explicit question about the unit sphere in an infinite-dimensional Hilbert space. Then, one can induct by showing that $\mathrm{St}_n(\mathcal{H})$ fibers over $\mathrm{St}_{n-1}(\mathcal{H})$ and things are good. This then implies $\mathrm{Gr}_n(\mathcal{H})$ has the homotopy type of BG.

For general G, we can use the fact that G is compact to know there's a faithful representatin $G \hookrightarrow \mathrm{U}_n$, and use this to produce smooth manifold models for BG and EG.

The next step will be to construct a universal connection Θ^{univ} on $\mathrm{St}_n(\mathcal{H})$ and to apply Chern-Weil theory. The de Rham theorem is trickier in infinite dimensions, but this can be worked around, so we do get a map $I^\bullet(G) \to H^*(BG;\mathbb{R})$, and it will be interesting to see what classes we get.

Remark 21.29. In general, this map need not be an isomorphism: $I^\bullet(\mathbb{Z}) = 0$, but $H^*(B\mathbb{Z};\mathbb{R}) = \mathbb{R}[x]/(x^2)$, with x in degree 1, because $B\mathbb{Z} = S^1$. ◄

Lecture 22.

Chern-Weil theory on classifying spaces: 4/16/19

"This is called transgression, even though it doesn't seem very sinful."

As we continue our discussion of Chern-Weil theory, we've been discussing classifying spaces of compact Lie groups, realized as Hilbert manifolds.

Theorem 22.1. *Let $\mathcal{H}$ be a separable*[35] *Hilbert space and $S(\mathcal{H}) \subset \mathcal{H}$ denote the unit sphere (the vectors of norm* 1*). Then $S(\mathcal{H})$ is contractible.*

There are several different proofs; this one is due to Dick Palais.

Proof. Let $\{e_n\}_{n\in\mathbb{Z}}$ be an orthonormal basis of $\mathcal{H}$, and define an embedding $i\colon \mathbb{R} \hookrightarrow S(\mathcal{H})$ as follows: if $x = n + \theta$, where $n \in \mathbb{Z}$ and $0 \le \theta \le 1$, then

$$i(x) \coloneqq \cos\Big(\frac{\pi}{2}\theta\Big)e_n + \sin\Big(\frac{\pi}{2}\theta\Big)e_{n+1}. \tag{22.2}$$

The Tietze extension theorem tells us that if X and Y are metric spaces (or more generally, if X is normal), $C \subset X$ is a closed subset, and $f\colon C \to Y$ is continuous, there is a map $\widetilde{f}\colon X \to Y$ with $\widetilde{f}|_C = f$.

[35]This theorem likely holds for all Hilbert spaces, but we'd need to find a different proof.

Hence we can use the Tietze extension theorem to extend the map $x \mapsto x+1$ on $i(\mathbb{R})$ to a map $g\colon D(\mathcal{H}) \to i(\mathbb{R})$, where $D(\mathcal{H})$ denotes the closed unit ball. Let $f\colon D(\mathcal{H}) \to D(\mathcal{H})$ be g followed by inclusion. This has no fixed points, so we can argue as in Hirsch's proof of the Brouwer fixed-point theorem (see Figure 1): for any $\xi \in D(\mathcal{H})$, consider the ray based at $f(\xi)$ and in the direction ξ), which hits $S(\mathcal{H})$ at a single point p. Define $h(\xi) := p$; because f is continuous, $h\colon D(\mathcal{H}) \to S(\mathcal{H})$ is continuous. It's the identity on $S(\mathcal{H})$, and therefore is a deformation retraction of $D(\mathcal{H})$ onto $S(\mathcal{H})$. Hence $D(\mathcal{H}) \simeq S(\mathcal{H})$, but $D(\mathcal{H})$ is contractible by the radial deformation retraction onto the origin. ⊠

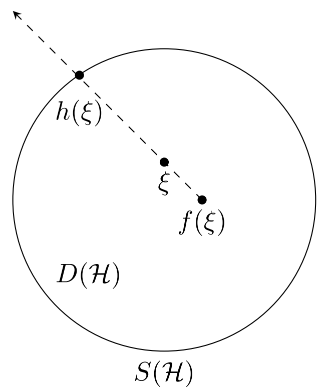

FIGURE 1. As in Hirsch's proof of the Brouwer fixed-point theorem, we construct a deformation retraction of $D(\mathcal{H})$ onto $S(\mathcal{H})$.

Corollary 22.3. *The Stiefel manifold* $\mathrm{St}_n(\mathcal{H})$ *is contractible.*

Proof. Well, let's induct: an isometry $\mathbb{C}^1 \to \mathcal{H}$ determines and is determined by where it sends $1 \in \mathbb{C}$, so $\mathrm{St}_1(\mathcal{H}) = S(\mathcal{H})$.

Now let's induct: there is a fiber bundle (though not a principal bundle) $\mathrm{St}_n(\mathcal{H}) \to S(\mathcal{H})$ given by evaluating at $(1, 0, \dots, 0)$. The fiber is $\mathrm{St}_{n-1}(\mathcal{H})$: an isometric embedding $\mathbb{C} \oplus \mathbb{C}^{n-1} \hookrightarrow \mathcal{H}$ where we've fixed where the first argument goes is equivalent data to an isometric embedding $\mathbb{C}^{n-1}$ into the orthogonal complement of the image of $\mathbb{C}$, but as all separable Hilbert spaces are isomorphic, we get a space homeomorphic to $\mathrm{St}_{n-1}(\mathcal{H})$.

There is a theorem that if $F \to E \to B$ is a fiber bundle where all three spaces are metrizable and F and B are contractible, then E is contractible. In this case, all three spaces are metrizable (as are all Hilbert manifolds), so we're done. ⊠

Palais wrote some elegant and useful papers about using this kind of point-set topology to study the homotopy theory of various spaces useful in geometry.

As we discussed last time, U_n acts freely on $\mathrm{St}_n(\mathcal{H})$ by precomposition: given an embedding $i\colon \mathbb{C}^n \hookrightarrow \mathcal{H}$ and a $g \in \mathrm{U}_n$, we let $i \cdot g := i \circ g\colon \mathbb{C}^n \hookrightarrow \mathcal{H}$. The quotient is the Grassmannian $\mathrm{Gr}_n(\mathcal{H})$ of n-dimensional subspaces of $\mathcal{H}$. Therefore $\pi_n\colon \mathrm{St}_n(\mathcal{H}) \to \mathrm{Gr}_n(\mathcal{H})$ is a principal U_n-bundle and the fiber is contractible, so this is a model for $E\mathrm{U}_n \to B\mathrm{U}_n$.

And since $\mathrm{St}_n(\mathcal{H})$ and $\mathrm{Gr}_n(\mathcal{H})$ are Hilbert manifolds, we'll be able to do geometry, by putting a connection on π_n. First we have to identify the tangent space to $\mathrm{St}_n(\mathcal{H})$. If we were asking about all injective maps $\mathbb{C}^n \hookrightarrow \mathcal{H}$, the tangent space would be $\mathrm{Hom}(\mathbb{C}^n, \mathcal{H})$; but we impose that our maps are isometries. For maps $\mathbb{C}^n \to \mathbb{C}^n$, this imposes the condition of being skew-Hermitian, and similarly if $b\colon \mathbb{C}^n \hookrightarrow \mathcal{H}$ is an isometric embedding and $b^*\colon \mathcal{H} \to \mathbb{C}^n$ is its adjoint, then the condition we need is that

$$(b^* b)^* (b^* b) = \mathrm{id}_{\mathbb{C}^n}. \tag{22.4}$$

Differentiating this condition yields the equations that cut out the tangent space.

Ok, next we need a metric. But we have a natural one. The space $\mathrm{Hom}(\mathbb{C}^n, \mathcal{H})$ has a natural inner product, which on $T_1, T_1 \in \mathrm{Hom}(\mathbb{C}^n, \mathcal{H})$ is given by[36]

$$\langle T_1, T_2\rangle \coloneqq \mathrm{tr}(T_1^* \circ T_2 \colon \mathbb{C}^n \to \mathbb{C}^n). \tag{22.5}$$

This inner product is U_n-equivariant: for $g \in \mathrm{U}_n$,

$$\langle T_1 g, T_2 g\rangle = \mathrm{tr}((T_1 g)^* T_2 g) = \mathrm{tr}(g^* T_1^* T_2 g) = \mathrm{tr}(g g^* T_1^* T_2) = \mathrm{tr}(T_1^* T_2). \tag{22.6}$$

This uses the fact that the trace is cyclic: $\mathrm{tr}(ABCD) = \mathrm{tr}(DABC)$ and so on.

This allows us to define a connection on $\mathrm{St}_n(\mathcal{H}) \to \mathrm{Gr}_n(\mathcal{H})$ as follows: we know the vertical vectors, which are $\ker((\pi_n)_*)$ in $T\mathrm{St}_n(\mathcal{H})$. And using the metric, we can take the orthogonal complement; because the metric is U_n-invariant, this defines a U_n-invariant horizontal distribution.

This immediately generalizes to other compact Lie groups: the Peter-Weyl theorem implies every compact Lie group G has a faithful finite-dimensional unitary representation, which amounts to an embedding $G \hookrightarrow \mathrm{U}_n$ for some n. Therefore we get a principal G-bundle $\mathrm{St}_n(\mathcal{H}) \to \mathrm{St}_n(\mathcal{H})/G$, which is a model for $EG \to BG$, and as before, the metric on the total space is G-invariant and allows us to define a connection in the same way.

Remark 22.7. If G is a noncompact Lie group with $\pi_0 G$ finite, then G deformation retracts onto a compact subgroup, allowing us to extend this construction to such G. ◀

Now let's apply this to Chern-Weil theory. Using the connection on $EG \to BG$ described above, the Chern-Weil map $\omega_G \colon H^k(G) \to \Omega^{2k}(BG)$ lands in closed forms (Lemma 21.23). Hence, using de Rham's theorem, we map into $H^{2k}(BG; \mathbb{R})$.

Proposition 22.8. *Let $P \to M$ be a principal G-bundle. Then there exists a G-equivariant map $\varphi \colon P \to EG$ such that, if $\overline{\varphi} \colon M \to BG$ is the quotient, $P \cong \overline{\varphi}^* EG$.*

Proof. Such a φ is equivalent data to a section of the associated bundle $P \times_G EG \to M$. If EG and M are metrizable, then a section exists, since EG is contractible. ⊠

Things get interesting when you add in connections. Using our model of $EG \to BG$ and its connection, it is true that every connection on a principal G-bundle on a manifold can be written as the pullback of the universal connection by some map to BG. But the map is not unique. If you want to impose uniqueness then there is no such universal connection, unless you broaden the class of spaces you work with to some class of simplicial sheaves. But that's not something we're going to do.

Example 22.9. Let $G = \mathbb{T}$, with Lie algebra $\mathfrak{t} = i\mathbb{R}$. If x denotes the coordinate in $\mathfrak{t}$, then $I^\bullet(\mathbb{T}) = \mathbb{R}[x]$.

Though we know we have a model for $B\mathbb{T}$ above, we can produce some more direct constructions. One is that if $\mathcal{H}$ is a complex separable Hilbert space, its space of lines $\mathbb{P}(\mathcal{H})$ is a model for $B\mathbb{T}$. To see this, notice that $\mathbb{T}$ acts freely on $S(\mathcal{H})$ by scalar multiplication, and the quotient is $P(\mathcal{H})$.

Alternatively, we could take $\mathbb{CP}^\infty$, defined to be the colimit of $\mathbb{CP}^n$ along the inclusions $\mathbb{CP}^n \hookrightarrow \mathbb{CP}^{n+1}$ induced from $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ as the first n coordinates; these two models for $B\mathbb{T}$ are homotopy equivalent, but have very different geometry: $\mathbb{P}(\mathcal{H})$ is a Hilbert manifold, and $\mathbb{CP}^\infty$ isn't.

In either case, we see $H^*(B\mathbb{T}; \mathbb{R}) \cong \mathbb{R}[c_1]$, where c_1 is called the *first Chern class*, and lives in degree 2. The map $I^\bullet(\mathbb{T}) \to H^{2\bullet}(B\mathbb{T}; \mathbb{R})$ sends x to a multiple of c_1. To see this, we'll apply the Serre spectral sequence to the fibration $\mathbb{T} \to S(\mathcal{H}) \to \mathbb{P}(\mathcal{H})$.

This spectral sequence converges to the cohomology of the total space, which is contractible. Therefore all classes on the E_2-page other than $1 \in E_2^{0,0}$ must get killed by differentials. We know $H^*(S^1) = \mathbb{Z}[t]/(t^2)$

[36]In general, if $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces, possibly infinite-dimensional, one can define a similar inner product on a restricted class of operators, namely Hilberrt-Schmidt operators. Because $\mathbb{C}^n$ is finite-dimensional, we don't need to worry about this today.

with t in degree 1, so $t \in E_2^{0,1}$ must get killed by the only differential that it can support:

$$\begin{tikzcd}
1 & t \arrow[drr, "d_2"'] & & \\
0 & 1 & & ? \\
 & 0 & 1 & 2
\end{tikzcd} \tag{22.10}$$

Therefore there's some $c \in H^2(B\mathbb{T}) = E_2^{2,0}$ with $d_2(c) = t$. There can be no other linearly independent classes in $E_2^{2,0}$, since they could not be killed by any differentials. Similarly, there can be nothing in $H^1(B\mathbb{T}) = E_2^{1,0}$, as it would have to survive to the E_∞-page.

This spectral sequence is multiplicative, so $E_2^{2,1}$ is cyclic, generated by ct, and $d_2(ct) = c^2$, since d_2 satisfies the Leibniz rule. This continues along to $d_2(c^n t) = c^{n+1}$, and we see that the spectral sequence collapses on the E_3-page, and $H^*(B\mathbb{T}) = \mathbb{Z}[c]$ as claimed (well, this implies the result over $\mathbb{R}$).

$$\begin{tikzcd}
1 & t \arrow[drr, "d_2"'] & 0 & ct \arrow[drr, "d_2"'] & 0 & c^2 t \\
0 & 1 & 0 & c & 0 & c^2 \\
 & 0 & 1 & 2 & 3 & 4
\end{tikzcd} \tag{22.11}$$

$\blacktriangleleft$

This is an instance of a general phenomenon in spectral sequences called *transgression*, where a differential sends a class on the vertical line to a class on the horizontal line.

TODO: there's a geometric interpretation of this transgression, and it has something to do with canonical differential forms called the Maurer-Cartan and Chern-Simons forms, but I missed it.

Example 22.12. Suppose $G = B\mathrm{SU}_2$. As we discussed in Example 21.5, $I^\bullet(\mathrm{SU}_2) = \mathbb{R}[s]$, with s in degree 4; explicitly, $s(A) = \mathrm{tr}(A^2)$ for $A \in \mathfrak{su}_2$. One can compute the cohomology of $B\mathrm{SU}_2$ in a similar manner as $B\mathbb{T}$ in the previous example: apply the Serre spectral sequence to the fibration $\mathrm{SU}_2 \to E\mathrm{SU}_2 \to B\mathrm{SU}_2$, and use the fact that $E\mathrm{SU}_2$ is contractible and $\mathrm{SU}_2 \cong S^3$. The result is $H^*(B\mathrm{SU}_2; \mathbb{R}) \cong \mathbb{R}[c_2]$, with c_2 in degree 4, and the Chern-Weil map is an isomorphism, sending s to a nonzero multiple of c_2. $\blacktriangleleft$

Example 22.13. The story for $\mathrm{SL}_2(\mathbb{C})$ is very similar to SU_2: $I^\bullet(\mathrm{SL}_2(\mathbb{C})) = \mathbb{R}[s]$, with s in degree 4 again given by $s(A) = \mathrm{tr}(A^2)$. There is a retraction of $B\mathrm{SL}_2(\mathbb{C})$ onto $B\mathrm{SU}_2$, and therefore its cohomology is the same. The Chern-Weil map is once again an isomorphism. $\blacktriangleleft$

Example 22.14. The Chern-Weil map is not always an isomorphism: for $\mathrm{SL}_2(\mathbb{R})$, $I^\bullet(\mathrm{SL}_2(\mathbb{R})) = \mathbb{R}[s]$, with s in degree 4, and since $\mathrm{SL}_2(\mathbb{R}) \simeq \mathrm{SO}_2 = \mathbb{T}$, $H^*(B\mathrm{SL}_2(\mathbb{R}); \mathbb{R}) \cong \mathbb{R}[c]$, with c in degree 2. The Chern-Weil map sends $s \mapsto c^2$. $\blacktriangleleft$

As long as G is compact, though, things are nice.

Theorem 22.15. *If G is a compact Lie group, the Chern-Weil homomorphism $\omega\colon I^{2*}(G) \to H^*(BG; \mathbb{R})$ is an isomorphism.*

Lecture 23.

Chern-Weil and Chern-Simons forms: 4/18/19

"So take a 0-simplex – that's a fancy word for point..."

Today, we'd like to sketch the proof of Theorem 22.15: that for a compact Lie group, the Chern-Weil homomorphism is an isomorphism from the ring of invariant polynomials on G, i.e. $(\mathrm{Sym}^\bullet \mathfrak{g}^*)^G$, onto the real cohomology of BG.

Definition 23.1. Let G be a Lie group. A *torus* $T \subset G$ is a compact abelian Lie subgroup T of G. If T is not strictly contained in another torus, it's called a *maximal torus*.

Compact abelian Lie groups are necessarily isomorphic to $\mathbb{T}^n$, hence the name "torus." In our proof of Theorem 22.15, we'll need a few facts.

Proposition 23.2.

(1) *Every compact connected Lie group G has a maximal torus $T \subset G$; furthermore, all maximal tori are conjugate.*
(2) *G/T is a compact complex manifold, and it has a CW decomposition with only even-dimensional cells called the* Bruhat decomposition.
(3) *Let $N := N(T) \subset G$ be the normalizer of T in G. Then $W := N(T)/G$ is a finite group called the* Weyl group. *The number of cells in the Bruhat decomposition is $|W|$.*
(4) *$H^*(G/N; \mathbb{R}) \cong \mathbb{R}$.*

Example 23.3. If $G = \mathrm{U}_n$, the diagonal matrices are a maximal torus. Specializing to U_2,

$$N = \left\{ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \begin{pmatrix} 0 & \mu_1 \\ \mu_2 & 0 \end{pmatrix} \right\}, \tag{23.4}$$

so the Weyl group is $\mathbb{Z}/2$. Then $\mathrm{U}_2/T \cong \mathbb{CP}^1 = S^2$, with a single 0-cell and a single 2-cell. This is a double cover of U_2/N, which doesn't leave us many options, and indeed $\mathrm{U}_2/N \cong \mathbb{RP}^2$, whose nonzero-degree real cohomology does vanish. ⊲

Why is G/T complex? Consider the restriction of the adjoint action of G on $\mathfrak{g}$ to $T \subset G$. One can prove in Lie theory that this representation splits as

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\text{roots } \rho} \mathfrak{g}_\rho, \tag{23.5}$$

where each $\mathfrak{g}_\rho$ is two-dimensional, and $\mathfrak{t}$ is the Lie algebra of T. The tangent space of G/T at the coset T is $\mathfrak{g}/\mathfrak{t}$, which by (23.5) splits as an $\mathfrak{t}$-representation as a sum of these $\mathfrak{g}_\rho$. So it suffices to choose a complex structure on each $\mathfrak{g}_\rho$, and then check that the Frobenius tensor vanishes. The way to construct the complex structure on $\mathfrak{g}_\rho$ is to complexify: the analogue of (23.5) for $\mathfrak{g}_\mathbb{C} := \mathfrak{g} \otimes \mathbb{C}$ is

$$\mathfrak{g}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\text{roots } \alpha} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}), \tag{23.6}$$

where $\mathfrak{t}_\mathbb{C} := \mathfrak{t} \otimes \mathbb{C}$, and $-\alpha$ is some negation operation on the roots. Thus, we can get a complex structure by choosing which of $\mathfrak{g}_{\pm\alpha}$ is $\mathfrak{g}^{1,0}$ and which is $\mathfrak{g}^{0,1}$, and Lie theory provides nice ways to make this choice. Then one has to argue that this almost complex structure integrates, but it does and things are good.

Remark 23.7. Another approach is to know that G has a complexification $G_\mathbb{C}$, and that G/T is diffeomorphic to $G_\mathbb{C}$ modulo a Borel subgroup, and this is manifestly a complex manifold, because $G_\mathbb{C}$ and the Borel are complex manifolds. ⊲

In general, the principal T-bundle $G \to G/T$ controls the geometry of G/T, in that the tangent bundle is an associated bundle to $G \to G/T$. This is smaller, and therefore makes life easier.

Remark 23.8. There's a quick proof of Proposition 23.2, part (4), assuming the other three statements: we know $\pi\colon G/T \to G/N$ is a degree-$|W|$ cover, so $|W|\chi(G/N) = \chi(G/T)$. Since G/T has cells only in even degrees, and has exactly $|W|$ of them, then $\chi(G/T) = |W|$, hence $\chi(G/N) = 1$.

Now, consider the maps

$$H^*(G/N; \mathbb{R}) \underset{\pi_*}{\overset{\pi^*}{\rightleftarrows}} H^*(G/T; \mathbb{R}), \tag{23.9}$$

where π^* is pullback as usual, and π_* is the Gysin map, which on differential forms is summing over the fiber. Thus $\pi_* \circ \pi^* = |W|$, so π^* is an injection, and therefore $H^*(G/N; \mathbb{R})$ is concentrated in even degrees, so there can only be one nontrivial cohomology group, and it must be $H^0 = \mathbb{R}$. ⊲

Proof sketch of Theorem 22.15. First, we prove the theorem in the case where $G = T$ is a compact connected abelian Lie group. TODO: Serre spectral sequence calculation for the fiber bundle $T \to ET \to BT$ to show $H^*(BT;\mathbb{R}) \cong \mathrm{Sym}^\bullet(\mathfrak{t}^*)$; the adjoint action is trivial, because T is abelian. The idea is: we know $\mathbb{R}$ in deg zero, then $d_2\colon E_2^{0,1} \to E_2^{2,0}$ is an isomorphism $\mathfrak{t}^* \to \mathfrak{t}^*$, because everything has to vanish. On the next line, we get $E_2^{0,3} = \Lambda^2\mathfrak{t}^*$, which maps to $\mathfrak{t}^* \otimes \mathfrak{t}^*$ by d_2, which therefore must map to the thing which kills it, which is $\mathrm{Sym}^2 \mathfrak{t}^*$. And so on.

Now we assume G is connected and compact, but not necessarily abelian. Choose a maximal torus T and let N and W be as above. Since T and N are subgroups of G, they act freely on EG and therefore we get a model for $ET \to BT$ as $EG \to EG/T$ and $EN \to BN$ as $EG \to EG/N$. With these models, $BT \to BN$ is a principal W-bundle. Therefore

$$H^*(BN;\mathbb{R}) \cong H^*(BT;\mathbb{R})^W \cong (\mathrm{Sym}^\bullet(\mathfrak{t}^*))^W \cong \mathrm{Sym}^\bullet(\mathfrak{g}^*)^G. \tag{23.10}$$

Now we descend one step further: the realization of BN above gives us a fiber bundle $BN \to BG$ with fiber G/N. The real cohomology of G/N is trivial (concentrated in degree 0), so if you apply the Serre spectral sequence to this fiber bundle, it collapses, and we conclude the map $H^*(BG;\mathbb{R}) \to H^*(BN;\mathbb{R})$ is an isomorphism.

We have two things left to do: first, remove the assumption that G is connected, and then show that the Chern-Weil homomorphism implements (some scalar multiple of) the above identification $H^*(BG;\mathbb{R}) \cong (\mathrm{Sym}^\bullet \mathfrak{g}^*)^G$.

First, let G be a general compact Lie group, and let $G_0 \subset G$ be the connected component of the identity, which is a Lie subgroup. There is a short exact sequence

$$1 \longrightarrow G_0 \longrightarrow G \longrightarrow \pi_0 G \longrightarrow 1, \tag{23.11}$$

hence a principal π_0G-bundle $BG_0 \to BG$ (constructed in a simialar way to the principal W-bundle $BT \to BN$ above). Therefore

$$H^*(BG;\mathbb{R}) \cong H^*(BG_0;\mathbb{R})^{\pi_0 G} = ((\mathrm{Sym}^\bullet \mathfrak{g}^*)^{G_0})^{\pi_0 G} \cong (\mathrm{Sym}^\bullet \mathfrak{g}^*)^G. \tag{23.12}$$

Now the last part, which will be the sketchiest. As above, we'll start with the torus. The key here is to identify the transgression $d_2\colon E_2^{0,1} \to E_2^{2,0}$ in the Leray-Serre spectral sequence with an invariant polynomial; then, the higher degrees follow, because on both sides of the identification they're polynomials in $\mathfrak{t}^*$. Next, one passes from T to N; because this is a finite cover this isn't too bad, though one will have to also argue why the universal connections are compatible across that cover. The last two steps are similar. ⊠

Exercise 23.13. Fill in the details of the last step of the proof, that the identification $H^*(BG;\mathbb{R}) \cong (\mathrm{Sym}^\bullet \mathfrak{g}^*)^G$ is compatible with the Chern-Weil homomorphism.

Now let's change gears a bit: let G be any Lie group, not necessarily compact, and let $\pi\colon P \to M$ be a principal G-bundle. Let $\mathcal{A}_P \subset \Omega_P^1(\mathfrak{g})$ be the affine space of connections. Then there is a principal G-bundle $\mathcal{A}_P \times P \to \mathcal{A}_P \times M$, where G acts trivially on the first factor, and this carries a universal connection $\Theta_P \in \Omega^1_{\mathbb{A}_P \times P}(\mathfrak{g})$. At the point (Θ, p), where $\Theta \in \mathcal{A}_P$ and $p \in P$, this connection is a map

$$(\Theta_P)_{(\Theta,p)}\colon T_{(\Theta,p)}(\mathcal{A}_P \times P) = T_\Theta\mathcal{A}_P \oplus T_pP \longrightarrow \mathfrak{g}, \tag{23.14}$$

and we have an identification $T_\Theta\mathcal{A}_P = \Omega^1_M(\mathfrak{g}_P) = \Omega^1_P(\mathfrak{g})$. So, thought of as a map $\Omega_P P^1(\mathfrak{g}) \oplus T_pP \to \mathfrak{g}$, this is a map $(\tau,\xi) \mapsto \Theta_p(\xi)$. Let $\Omega_P \in \Omega^2_{\mathcal{A}_P\times M}(\mathcal{A}_P \times M, \mathfrak{g}_{\mathcal{A}_P\times P})$ denote the curvature form.

Proposition 23.15. *The curvature form splits as follows: its $(2,0)$-piece is equal to zero, its $(0,2)$-piece is $\Omega(\Theta)$, and its $(1,1)$-piece is*

$$(\Omega_P)_{(\Theta,m)}(\tau,\eta) = \tau_m(\eta).$$

Here the (p,q) decomposition is in terms of horizontal and vertical, since this is over a direct product. TODO: I didn't really follow this, and so I can't fill in the proof. Sorry about that. :(

Now, given an invariant polynomial f, the Chern-Weil construction produces some $\omega_f(\Theta_P) \in \Omega^{2k}_{\mathcal{A}_P\times M}$ (yes, we are using the calculus of differential forms on an infinite-dimensional space, but will only look on finite-dimensional submanifolds), and it has the property that the restriction to any point $\Theta \in \mathcal{A}_P$ is $\omega_f(\Theta)$, the Chern-Weil form for f and this connection Θ.

Now, given two connections Θ_0 and Θ_1, there is a unique affine line $\phi\colon \Delta^1 \times M \to \mathcal{A}_P \times M$, and we can pull back and integrate to obtain a $(2k-1)$-form

$$\alpha_{\Theta_0,\Theta_1} \coloneqq \int_{\Delta^1} \phi^*\omega_f(\Theta_P). \tag{23.16}$$

on M.

Proposition 23.17. *$\omega_f(\Theta_1) - \omega_f(\Theta_0) = \mathrm{d}\alpha(\Theta_0,\Theta_1)$.*

The proof uses Stokes' theorem and the fact that $\mathrm{d}\omega_f(\Theta_P) = 0$. In particular, not only is the difference between two forms associated to f via different connections exact, but we have a canonical way to see that it's exact. In particular,

Corollary 23.18. *Given a principal G-bundle $\pi\colon P \to M$ and an invariant polynomial f, the cohomology class of the Chern-Weil form for f and P does not depend on the choice of connection on π.*

This α is our first example of a Chern-Simons form. Let's see what it looks like. Explicitly, there is some one-form τ such that $\Theta_1 = \Theta_0 + \tau$, and $\phi(t) = \Theta_0 + t\tau$. Using Proposition 23.15,

$$(\phi^*\Omega_P)_{t,m} = \Omega(\Theta_t)_m + \mathrm{d}t \wedge \tau_m. \tag{23.19}$$

When we apply f, we get

$$\phi^*\omega_f(\Theta_p) = k\,\mathrm{d}t \wedge f(\tau,\Omega,\dots,\Omega) + f(\Omega,\dots,\Omega), \tag{23.20}$$

and there are $k-1$ copies of Ω. Integrating picks out the first factor.

Lecture 24.

Chern-Simons forms, II: 4/23/19

"One day I threw my hands up – not really, it was some strong language – and decided to write down a set of consistent sign conventions."

Let G be a Lie group and $\pi\colon P \to M$ be a principal G-bundle, where M is a manifold. Let $\mathcal{A}_P \subset \Omega^1_P(\mathfrak{g})$ be the affine space of connections. Last time, we considered the "universal connection" on P: the principal G-bundle

$$\mathrm{id} \times \pi\colon \mathcal{A}_P \times P \to \mathcal{A}_P \times M \tag{24.1}$$

carries a tautological 1-form $\Theta_P \in \Omega^{0,1}_{\mathcal{A}_P\times P}(\mathfrak{g}) \subset \Omega^1_{\mathcal{A}_P\times P}(\mathfrak{g})$, where "(0,1)" refers to the bigrading induced by the product. The de Rham differential also splits: let δ is the de Rham differential in the $\mathcal{A}_P$ direction, and d be the de Rham differential in the P direction. (We will also use d to denote the de Rham differential on M.)

Now Θ_P is defined by the formula $(\Theta_P)_{(\Theta,p)} = \Theta_p$: that is, given a point $p \in P$ and a connection $\Theta \in \mathcal{A}_P$, the value of Θ_P at $(\Theta,p) \in \mathcal{A}_P \times P$ is the value of Θ at p. Last time, we proved in Proposition 23.15 that the curvature of Θ_P is

$$\Omega_P = (\delta + \mathrm{d})\Theta_P + \frac{1}{2}[\Theta_P \wedge \Theta_P] \tag{24.2a}$$

$$= \delta\Theta_P + \mathrm{d}\Theta + \frac{1}{2}[\Theta\wedge\Theta] \tag{24.2b}$$

$$= \underbrace{\delta\Theta_P}_{\in\Omega^{1,1}} + \underbrace{\Omega(\Theta)}_{\in\Omega^{0,2}}. \tag{24.2c}$$

At a point (Θ,p), the first term is

$$(\delta\Theta_P)_{(\Theta,p)}((\dot\Theta_1,\dot p_1),(\dot\Theta_2,\dot p_2)) = \dot\Theta_1(\dot p_2) - \dot\Theta_2(\dot p_1). \tag{24.3}$$

Definition 24.4. Given $f \in I^k(G)$, let

$$\omega_f(\Theta_P) \coloneqq f(\Omega_P\wedge\cdots\wedge\Omega_P) \in \Omega^{2k}_{\mathcal{A}_P\times M},$$

which is a closed form.

Example 24.5. If $G = \mathrm{U}_n$ and $f\colon \mathfrak{u}_n \to \mathbb{R}$ is given by $f(A) = c\,\mathrm{tr}(A^3)$ for some $c \in \mathbb{R}$, then $\omega_f(\Theta_P) = c\,\mathrm{tr}(\Omega_P\wedge\Omega_P\wedge\Omega_P)$. Wedging together three matrices of 2-forms gives a single matrix of 6-forms, and we take the trace to obtain a 6-form. ◀

At the end of the previous lecture, we considered a Chern-Simons form associated to two connections Θ_0 and Θ_1. Specifically, because $\mathcal{A}_P$ is affine, there's a unique affine line $\phi\colon [0,1] \to \mathcal{A}_P$ with $\phi(0) = \Theta_0$ and $\phi(1) = \Theta_1$. Then we define

$$\alpha(\Theta_0, \Theta_1) \coloneqq \int_0^1 \phi^* \omega_f(\Theta_P) \in \Omega^{2k-1}(M). \tag{24.6}$$

We will compute this explicitly for $k = 1, 2$.

Example 24.7 ($k = 1$)**.** In this case f is linear. The displacement vector along ϕ is $\dot\Theta = \Theta_1 - \Theta_0$, so $\Theta_t \coloneqq \phi(t) = \Theta_0 + t\dot\Theta$. Since

$$\omega_f(\Theta_P) = f(\Omega_P) = f(\delta\Theta_P + \Omega), \tag{24.8}$$

then

$$\phi^*\Omega_P = \mathrm{d}t \wedge \dot\Theta + \Omega_P \in \Omega^2_{[0,1]\times P}(\mathfrak{g}), \tag{24.9}$$

where t denotes the coordinate on $[0,1]$. Hence

$$\int_0^1 f(\phi^*\Omega_P) = \int_0^1 \mathrm{d}t\, f(\dot\Theta) = f(\dot\Theta). \tag{24.10}$$

◀

Example 24.11 ($k = 2$)**.** This time, f is quadratic, so we need to care about second-order terms. So let's compute:

$$\Omega_t = \mathrm{d}(\Theta_0 + t\dot\Theta) + \frac{1}{2}\Big((\Theta_0 + t\dot\Theta) \wedge (\Theta_0 + t\dot\Theta)\Big) \tag{24.12a}$$

$$= \Omega_0 + t(\mathrm{d}\dot\Theta + [\Theta_0 \wedge \dot\Theta]) + \frac{t^2}{2}[\dot\Theta \wedge \dot\Theta] \tag{24.12b}$$

$$= \Omega_0 + t\mathrm{d}_{\Theta_0}\dot\Theta + \frac{t^2}{2}[\dot\Theta \wedge \dot\Theta]. \tag{24.12c}$$

Thus

$$\omega_f(\Theta_P) = f\Big((\mathrm{d}t \wedge \dot\Theta + \Omega_t) \wedge (\mathrm{d}t \wedge \dot\Theta + \Omega_t)\Big) \tag{24.13a}$$

$$= 2\,\mathrm{d}t \wedge f(\dot\Theta \wedge \Omega_t) + f(\Omega_t \wedge \Omega_t). \tag{24.13b}$$

Since f is a bilinear form, we can compute

$$\int_0^1 \omega_f(\Theta_P) = 2\int_0^1 \mathrm{d}t\, f(\dot\Theta \wedge \Omega_t) \tag{24.14a}$$

$$= 2\int_0^1 \mathrm{d}t\,\left(f(\dot\Theta \wedge \Omega_0) + t f(\dot\Theta \wedge \mathrm{d}_{\Theta_0}\dot\Theta) + \frac{t^2}{2} f(\dot\Theta \wedge [\dot\Theta \wedge \dot\Theta])\right) \tag{24.14b}$$

$$= 2f(\dot\Theta \wedge \Omega_0) + f(\dot\Theta \wedge \mathrm{d}_{\Theta_0}\dot\Theta) + \frac{1}{3} f(\dot\Theta \wedge [\dot\Theta \wedge \dot\Theta]) \tag{24.14c}$$

$$= f(\dot\Theta \wedge \Omega_0) + f(\dot\Theta \wedge \Omega_1) - \frac{1}{6} f(\dot\Theta \wedge [\dot\Theta \wedge \dot\Theta]). \tag{24.14d}$$

◀

We're in the situation of a fiber bundle $\pi\colon M \to S$, where the fiber F is a compact n-manifold with boundary and S is compact, and the relative tangent bundle $T(M/S) \to M$ is oriented. In this setting we can make progress on Proposition 23.17 with Stokes' theorem.

There is a map

$$\int_{M/S} \colon \Omega^q(M) \to \Omega^{q-n}(S) \tag{24.15}$$

called *integration along the fiber*. There is a projection map $\Omega^q(M) \to \Omega^n(F) \otimes \Omega^{q-n}(S)$; then we can integrate the first component over F as usual.

Theorem 24.16 (Stokes)**.** *Let $\omega \in \Omega^q(M)$. Then*

$$\mathrm{d}\int_{M/S} \omega = \int_{\partial M/S} \omega \pm \int_{M/S} \mathrm{d}\omega. \tag{24.17}$$

The sign depends on n, but it wasn't clear in lecture what the exact formula is.

Now let's define the more commonly considered Chern-Simons form, which requires data of only a single connection. Consider the pullback bundle $\pi^*P \to P$. It's endowed with a canonical section Δ: the fiber over any $p \in P$ is $P_{\pi(p)}$, which contains p, so $\Delta(p) := p$. You can think of this as the diagonal map into the product.

The trivialization means we can choose the trivial connection $\Theta_\Delta \in \mathcal{A}_{\pi^*P}$, and it's flat (zero curvature). Thus $\Delta^*\Theta_\Delta = 0$. Given a connection on P, we can pull it back to π^*P, which is an affine embedding $\pi^*\colon \mathcal{A}_P \to \mathcal{A}_{\pi^*P}$.

Definition 24.18. Given a $\Theta \in \mathcal{A}_P$, let $\alpha_f(\Theta) := \alpha_f(\Theta_\Delta, \pi^*\Theta) \in \Omega_P^{2k-1}$, which we call the *Chern-Simons form.*

Here α_f is overloaded: we're defining α_f of a single connection in terms of α_f of two connections.

Corollary 24.19. $\mathrm{d}\alpha_f(\Theta) = \pi^*\omega_f(\Theta)$.

What does this tell us? Well, $\omega_f(\Theta)$ is closed but not exact. If we'd like to write it as d of something, we have to pull back to P, and here it is exact, and we have a nice, explicit choice of a form hitting it.

Corollary 24.20. *Restricted to any fiber of π, $\alpha_f(\Theta)$ is closed.*

This is an example of transgression in the Serre spectral sequence for this fiber bundle. The transgression tells us how to go from cohomology classes on the fiber to cohomology classes on the base, but differential geometry tells us a specific choice, and this choice is also useful elsewhere.

Proposition 24.21. *Explicitly, the Chern-Simons form is $f(\Theta \wedge \Omega) - (1/6)f(\Theta \wedge [\Theta, \Theta])$.*

Proof sketch. (TODOsome of this was erased before I could write it down) We use the formulas we computed in Examples 24.7 and 24.11. In particular, instead of $f(\dot\Theta)$ we get $f(\Theta)$, and $f(\dot\Theta \wedge \Omega_0)$ vanishes (TODO: is this because $\Omega_0 = 0$ and $\Omega_1 = \Omega$?). ⊠

Sometimes one writes $\Omega = \mathrm{d}\Theta$, or leaves off the brackets in $[\Theta \wedge \Theta]$.

Proposition 24.22. *Let $m \in M$ and $i_m\colon P_m \hookrightarrow P$ be the inclusion of the fiber. Then*

$$i_m^*\alpha_f(\Theta) = c_k f(\Theta \wedge \underbrace{[\theta \wedge \theta] \wedge \cdots \wedge [\theta \wedge \theta]}_{k-1}) \in \Omega_G^{2k-1}, \tag{24.23}$$

where $c_1 = 1$ and $c_2 = -1/6$, and θ is the Maurer-Cartan form.

That's the Chern-Simons form; what do people do with it? Well, Chern and Simons did some interesting classical things, and Witten did completely different things more recently. We'll start with the former.

Let $k = 2$ and fix a G-invariant symmetric bilinear form $f\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. Asume G is compact, so the space of these is isomorphic to $H^4(BG; \mathbb{R})$. Let $\pi P \to M$ be a principal G-bundle, where M is a 3-manifold. Then $\alpha(\Theta) \in \Omega_P^3$ is closed, because $\mathrm{d}\alpha = \pi^* f(\Omega \wedge \Omega) = 0$.

TODO: then there was something involving integrating over the triangle in $\mathcal{A}_{\pi^*P}$ defined by $\pi^*\Theta_0$, $\pi^*\Theta_1$, and Θ_Δ, but I could not follow any of it.

Another thing we could do is consider the map $F\colon \mathcal{A}_P \times \Gamma(\pi) \to \mathbb{R}$ defined by

$$F(\Theta, s) := \int_M s^*\alpha(\Theta). \tag{24.24}$$

But are there even sections? $\Gamma(\pi)$ could be empty. This is a topological question: when does a principal G-bundle have a section over a 3-manifold? We can use obstruction theory. If G is simply connected, $\pi_0 G$ and $\pi_1 G$ vanish, and there's a general theorem that $\pi_2 G = 0$. This is enough to imply that there's always a section. This works for SU_n and Spin_n ($n > 2$), but not SO_n, $\mathrm{Pin}_n^\pm$, or U_n, which are all either not connected or not simply connected.

Anyways, suppose $\Gamma(\pi)$ is nonempty. Let δ_1 be the de Rham differential on $\mathcal{A}_P \times \Gamma(\pi)$ in the $\mathcal{A}_P$ direction, and let δ_2 be the differential in the $\Gamma(\pi)$ direction. If ζ is a vertical vector field,

$$\iota_\zeta \delta_2 F = \int_M \zeta \cdot [s^*\alpha(\Theta)] \tag{24.25a}$$

$$= \int_M s^*\iota_\zeta \mathrm{d}\alpha(\Theta) \tag{24.25b}$$

$$= \int_M s^*\iota_\zeta \pi^*\omega(\Theta) = 0, \tag{24.25c}$$

because $\omega(\Theta) = 0$. This suggests that F is constant, which is wrong – it's only locally constant. The space of sections is not always connected (though typically it is). Next time we'll discuss a little more Chern-Simons theory.

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