M392C NOTES: SYMPLECTIC TOPOLOGY

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These notes were taken in UT Austin's M392C (Symplectic Topology) class in Fall 2016, taught by Robert Gompf. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

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Lecture 1.

Symplectic Vector Spaces: 8/24/16

Here are a few references for this class.

- There's a book by McDuff and Salaman; in fact, there are three considerably different editions, but all are useful.
- The book by ABKLR (Aebischer, Borer, Kalin, Leuenberger, and Reimann).
- Finally, the book by the professor and Stipsicz will be useful for some parts.

As an overview, symplectic topology is the study of symplectic manifolds.

Definition 1.1. A *symplectic manifold* is a manifold together with a symplectic form.

We'll define symplectic forms in a moment, but first explain where this field arose from. One one hand, symplectic forms arise naturally from mathematical physics in the Hamiltonian formulation, and these days also appear in quantum field theory. On the other, algebraic and complex geometers found that Kähler manifolds naturally have a symplectic structure.

Intuitively, a symplectic manifold is akin to a constant-curvature Riemannian manifold, but where the symmetric bilinear form is replaced with a skew-symmetric bilinear form. (If you don't know what a Riemannian manifold is, that's okay; it will not be a prerequisite for this class.) The constant-curvature condition means that any two points have isomorphic local neighborhoods, so all questions are global; similarly, we will impose a condition on symplectic manifolds that ensures that all questions about symplectic manifolds are global.

There's also a field called symplectic geometry; it differs from symplectic topology in, among other things, also looking at local questions. But there's a reason there's no such thing as "Riemannian topology:" a Riemannian structure is very rigid, and so cutting and pasting Riemannian manifolds, especially constant-curvature ones, isn't fruitful. But symplectic manifolds have a flexibility that allows cutting and pasting to work, if you're clever. To understand this, we will have to spend a little time understanding the local structure.

Another analogy, this time with three-manifolds, is Thurston's geometrization conjecture (now a theorem, thanks to Perelman). This states that any three-manifold may be cut along sphere and tori into pieces that have natural geometry, and are almost always have constant negative curvature, hence are *hyperbolic*, so three-manifold topologists have to understand hyperbolic geometry. Symplectic topology is the analogue in the world of four-manifolds. Not all four-manifolds have symplectic structures; in fact, there exist smooth four-manifolds that are homeomorphic, but one admits a symplectic structure and the other doesn't, so they're not diffeomorphic. We've classified topological four-manifolds, but not smooth ones, so symplectic topology is a very useful tool for this. Three-manifold topologists might also care about the three-manifolds that are boundaries of four-manifolds: if the four-manifold is symplectic, its boundary has a natural structure as a *contact manifold*. The professor plans to teach a course on contact manifolds in a year.

There's a basic principle in geometry and analysis that, in order to understand nonlinear things, one first must understand linear things. Before you understand multivariable integration, you will study linear algebra and the determinant. Before understanding Riemannian geometry, you will learn about inner product spaces. In the same way, we begin with symplectic vector spaces.

Definition 1.2. A *symplectic vector space* is a finite-dimensional[1] real vector space V together with a skew-symmetric bilinear form ω that is *nondegenerate*, i.e. if $v \in V$ is such that for all $w \in V$, $\omega(v, w) = 0$, then $v = 0$.

Succinctly, nondegeneracy means every nonzero vector pairs nontrivially with something. This is a very similar condition to the ones imposed for inner product spaces as well as the indefinite forms attached to spaces in relativity theory.

Example 1.3. Our prototypical example is $\mathbb{C}^n = \mathbb{R}^{2n}$ as a real vector space. The standard inner product is the dot product $\langle -, - \rangle$; we'll define $\omega(v, w) = (iv, w)$. This is clearly still real bilinear; let's verify this is a symplectic form.

First, why is it skew-symmetric? $\omega(w, v) = \langle iw, v\rangle = \langle v, iw\rangle$. Since multiplication by i is orthogonal (it's a rotation), then it preserves the inner product, so $\langle v, iw\rangle = \langle iv, i^2 w\rangle = -\langle iv, w\rangle = -\omega v, w$, so ω is skew-symmetric.

Nondegeneracy is simple: any $v \neq 0$ has a $w \neq 0$ such that $\langle v, w\rangle \neq 0$, so $\omega v, iw = \langle iv, iw\rangle = \langle v, w\rangle \neq 0$.

If we take the standard complex basis $e_1, \dots, e_n$ for $\mathbb{C}^n$, let $f_j = ie_j$; then, $(e_1, f_1, \dots, e_n, f_n)$ is a real basis for $\mathbb{C}^n$; we will take this to be the standard basis for $\mathbb{C}^n$ as a symplectic vector space. This is because each (e_i, f_i) is a real basis for a $\mathbb{C}^1$ summand corresponding to the usual basis $(1, i)$ for $\mathbb{C}$, so this basis jives with the decomposition $\mathbb{C}^n = \mathbb{C} \oplus \mathbb{C} \oplus \cdots \oplus \mathbb{C}$.

This is a positively oriented basis, and in fact is consistent with the canonical orientation of a complex vector space, because it arises in an orientation-preserving way from the basis $(1, i)$ for $\mathbb{C}$, which defines the canonical orientation.

This basis defines a dual basis $e_1^*, f_1^*, \dots, e_n^*, f_n^*$ for the dual space $(\mathbb{R}^{2n})^*$.[2] This allows us to calculate ω in coordinates:

$$\omega = \sum_{j=1}^{n} e_j^* \wedge f_j^*. \tag{1.4}$$

Thus, $\omega(e_j, f_j) = 1 = -\omega(f_j, e_j)$, and all other pairs of basis vectors are orthogonal (evaluate to 0). This defines the same form ω because they agree on the standard basis, because $f_j = ie_j$ and $e_1, \dots, e_n$ is an orthonormal basis for the inner product.

The analogue to an orthonormal basis for a symplectic vector space is a symplectic basis, where elements come in pairs.

Definition 1.5. If $(e_1, f_1, \dots, e_n, f_n)$ is a basis for a symplectic vector space (V, ω) such that $\omega(e_j, f_j) = 1 = -\omega(f_j, e_j)$ for all j and all other pairs of basis vectors are orthogonal, then the basis is called a *symplectic basis*.

Recall that we also have a Hermitian inner product on $\mathbb{C}^n$, defined by

$$h(v, w) = \sum_{j=1}^{n} \overline{v}_j w_j.$$

This is bilinear over $\mathbb{R}$, but not over $\mathbb{C}$; it's $\mathbb{C}$-linear in the second coordinate, but conjugate linear in the first. The Hermitian analogue of the symmetry of an inner product (or the skew-symmetry of a symplectic form) is $h(w, v) = \overline{h(v, w)}$. Thus, $\operatorname{Re} h$ is symmetric, and $\operatorname{Im} h$ is skew-symmetric: $\operatorname{Re} h$ is the standard real inner product on $\mathbb{C}^n = \mathbb{R}^{2n}$, and $\operatorname{Im} h$ is the symplectic form ω.[3]

Example 1.6. As a special case of the previous example, $\mathbb{C}^1 = \mathbb{R}^2$ as a symplectic vector space has ω as the usual (positive) area form: $\omega = e \wedge f = \mathrm{d}x \wedge \mathrm{d}y$.

[1] One can talk about infinite-dimensional symplectic vector spaces, and there are useful in some contexts, but all of our symplectic vector spaces will be finite-dimensional.

[2] Recall that if V is a finite-dimensional real vector space, its *dual space* is V^*, the space of linear functions from V to $\mathbb{R}$. A basis $e_1, \dots, e_n$ of V induces a basis $e_1^*, \dots, e_n^*$ of V^*, defined by $e_j^*(e_i) = \delta_{ij}$: 1 if i and j agree, and 0 otherwise.

[3] Some authors reverse the order for h, so that it's $\mathbb{C}$-linear in the first coordinate but not the second; in this case, we'd get $\operatorname{Im} h = -\omega$. There are a lot of minus signs floating around in symplectic topology, and different authors place them in different places.

Suppose (V, ω_V) and (W, ω_W) are symplectic vector spaces; then, their direct product (or equivalently, direct sum) $V \times W$ has a symplectic structure defined by

$$\omega_{V\times W} = \pi_1^*\omega_V + \pi_2^*\omega_W,$$

where $\pi_1 : V \times W \twoheadrightarrow V$ and $\pi_2 : V \times W \twoheadrightarrow W$ are projections onto the first and second coordinates, respectively. This is a linear combination of skew-symmetric forms, hence is skew-symmetric, and if $\omega_{V\times W}(u_1, u_2) = 0$ for all $u_2 \in V \times W$, then $\pi_1 u_1 = 0$ and $\pi_2 u_1 = 0$, so $u_1 = 0$. Thus, $(V \times W, \omega_{V\times W})$ has a symplectic structure, called the *symplectic orthogonal sum* of V and W since V and W are orthogonal in it.

Not only is $\mathbb{C}^n$ the direct sum of n copies of $\mathbb{C}$, but also the standard symplectic structure on $\mathbb{C}^n$ is the symplectic orthogonal sum of n copies of the standard structure on $\mathbb{C}$: (1.4) explicitly realizes ω as a sum of pullbacks of area forms. The complex structures fit together, the orientations fit together, and the symplectic structures fit together, all nicely.

Subspaces.

Definition 1.7. Suppose V is a symplectic vector space and $W \subset V$ is a subspace. Then, its *orthogonal complement* is the subspace $W^\perp = \{v \in V \mid \omega(v, w) = 0 \text{ for all } w \in W\}$.

The definition is familiar from inner product spaces, but there are a few major differences in what happens afterwards. The first theorem is the same, though:

Theorem 1.8. *If W is a subspace of a symplectic vector space V, then* $\dim W + \dim W^\perp = \dim V$.

Proof. There's a linear map $\varphi : V \longrightarrow V^*$ assigning v to the linear transformation $\varphi(v) : V \longrightarrow \mathbb{R}$ that sends $w \mapsto \omega(v, w)$. Since ω is nondegenerate, then φ is injective. Since V and V^* have the same dimension, φ is an isomorphism. The image $\varphi(W^\perp)$ is the space of functions in V^* that vanish on W, which is isomorphic to the space of functions on V/W, i.e. $\varphi : W^\perp \longrightarrow (V/W)^*$ is injective, and in fact an isomorphism: any function on V/W lifts to a function on V vanishing on W, and then can be pulled back by φ into W. Thus, $\dim W^\perp = \dim(V/W)^* = \dim(V/W) = \operatorname{codim} W$. ⊠

The above proof also works for symmetric nondegenerate bilinear forms. What's different is that W and $W^\perp$ do not always sum to V in the symplectic case. In particular, every vector is orthogonal to itself.

Lecture 2.

Symplectic Vector Spaces are Complex Vector Spaces: 8/26/16

Recall that last lecture, we talked about symplectic vector spaces. A symplectic vector space is a finite-dimensional real vector space V together with a skew-symmetric, nondegenerate bilinear form ω. For a subspace $W \subset V$, we defined $W^\perp$, the vectors that pair to 0 with W under ω, and showed that $\dim W + \dim W^\perp = \dim V$.

Corollary 2.1. *If $W \subset V$ is a subspace, then* $(W^\perp)^\perp = W$.

Proof. Clearly, $W \subset (W^\perp)^\perp$, and they have the same dimension. ⊠

This all is also true for inner product spaces, but things soon begin looking different. For any one-dimensional subspace $W = \operatorname{span} v$, $\omega(v, v) = -\omega(v, v) = 0$ by skew-symmetry, so in this case $W \subset W^\perp$. Switching W and $W^\perp$, one sees that every codimension-1 space contains its complement.

Definition 2.2. Let W be a subspace of the symplectic vector space V.

- If $W \subset W^\perp$, then W is an *isotropic subspace*.
- If $W^\perp \subset W$, then W is a *coisotropic subspace*.
- If both of these are true, so $W = W^\perp$, then W is a *Lagrangian subspace*.

If W is isotropic, then $\dim W \le (1/2)\dim V$; if W is coisotropic, then $\dim W \ge (1/2)\dim V$, and if W is Lagrangian, then $\dim W = (1/2)\dim V$. Moreover, W is isotropic iff $W^\perp$ is coisotropic, and vice versa.

Example 2.3. Last time, we discussed the standard symplectic structure on $\mathbb{C}^n$. The subspace $\mathbb{R}^n \subset \mathbb{C}^n$ is Lagrangian: if $v, w \in \mathbb{R}^n$, $\omega(v, w) = \langle iv, w\rangle$, but iv is purely imaginary and w is purely real, so $\omega(v, w) = 0$.

Definition 2.4. If $W \subset V$ is a subspace such that ω restricts to a nondegenerate form on ω, then W is a *symplectic subspace.*

For example, the standard inclusion $\mathbb{C}^k \subset \mathbb{C}^n$, for $1 \le k \le n$, is a symplectic subspace.

Lemma 2.5. *Suppose $W \subset V$ is a subspace. The following are equivalent:*

(1) *W is symplectic.*
(2) $W \cap W^\perp = \{0\}$.
(3) $W + W^\perp = V$.
(4) *$W^\perp$ is symplectic.*
(5) *$V = W \oplus W^\perp$ is a symplectic orthogonal sum of symplectic subspaces.*

Proof. W is symplectic iff $\omega|_W$ is nondegenerate, meaning there's no nonzero $v \in W$ such that $\omega v, w = 0$ for all $w \in W$. This means exactly that there's no nonzero $v \in W$ that's also in $W^\perp$. Thus, (1) and (2) are equivalent; (3) is equivalent to (2) by usual linear algebra, and since (2) is symmetric in W and $W^\perp$, therse are all equivalent to (4).

In this situation, if $\pi_1 : V \twoheadrightarrow W$ and $\pi_2 : V \twoheadrightarrow W^\perp$ are orthogonal projections onto W and $W^\perp$, respectively, then $\omega = \pi_1^*\omega + \pi_2^*\omega$. This follows because any $v \in V$ may be uniquely written as $v = w + w^\perp$ for $w \in W$ and $w^\perp \in W^\perp$ by (3); then,

$$\begin{aligned}\omega(v_1, v_2) &= \omega(w_1 + w_1^\perp, w_2 + w_2^\perp)\\ &= \omega(w_1, w_2) + \omega(w_1^\perp + w_2^\perp) + \omega(w_1, w_2^\perp) + \omega(w_1^\perp, w_2),\end{aligned}$$

but the cross terms vanish. Thus, (5) follows. ⊠

Theorem 2.6. *Every symplectic vector space (V, ω) is isomorphic to the standard example* $(\mathbb{C}^n, \omega)$.

An isomorphism of symplectic vector spaces is what you would expect: an isomorphism of vector spaces that preserves the symplectic form.

Proof. If $\dim V = 0$, there's nothing to say, so assume $\dim V > 0$. In this case, there's a nonzero $v \in V$, so by nondegeneracy a nonzero $w \in V$ such that $\omega(v, w) \neq 0$. Let $e_1 = v$ and $f_1 = (1/\omega(v, w))w$, so $\omega(e_1, f_1) = 1$. Thus, $\operatorname{span}\{e_1, f_1\} \cong \mathbb{C}$ under the unique map that sends $e_1 \mapsto 1$ and $f_1 \mapsto i$. Hence, $\operatorname{span}\{e_1, f_1\}$ is a symplectic subspace, so by Lemma 2.5, so is its orthogonal complement, which is a symplectic vector space of lower dimension, so by induction it's isomorphic to $\mathbb{C}^{\dim V - 2}$. ⊠

Corollary 2.7. *Every symplectic vector space is even-dimensional.*

Corollary 2.8. *A skew-symmetric bilinear form ω on a $2n$-dimensional vector space V is nondegenerate iff* $\underbrace{\omega \wedge \cdots \wedge \omega}_{n} \neq 0$.

Proof. Since ω has even degree, the wedge product is symmetric, so this isn't automatically zero.

By Theorem 2.6, $(V, \omega) \cong (\mathbb{C}^n, \omega_{\mathrm{std}})$, and $\omega_{\mathrm{std}} = \sum e_j^* \wedge f_j^*$. Thus, taking the n^{th} wedge power of this form, all terms where a dual basis element appears more than once is zero, so the only nonzero terms are of the form $e_1^* \wedge e_2^* \wedge e_2^* \wedge f_2^* \wedge \cdots \wedge e_n^* \wedge f_n^*$. There is at least one of these,[4] so $\omega^{\wedge n}$ is a volume form on V, so must be nonzero.

Conversely, if ω is degenerate, then there's a nonzero $v_1 \in V$ such that $\omega(v_1, -) = 0$, so we can extend to a basis $v_1, \dots, v_{2n}$ for V; then, $\omega \wedge \cdots \wedge \omega(v_1, \dots, v_{2n}) = 0$, because every term either has all $2n$ basis vectors, so it has a v_1 which is paired with something and becomes 0. Thus, $\omega \wedge \cdots \wedge \omega = 0$: since $\dim \Lambda^{2n}(V) = 1$, if it were nonzero, it would be a nonzero multiple of the volume form, which is nonzero on any basis of V. ⊠

This is another common way to express the nondegeneracy condition in the literature.

Corollary 2.9. *Every symplectic vector space has a canonical orientation.*

This orientation is the one determined by the volume form $\omega \wedge \cdots \wedge \omega$, which is consistent with the standard orientation on $\mathbb{C}^n$. Switching the sign of the symplectic form produces a valid symplectic form, but depending on whether n is odd or even, this might not change the orientation.

Remark. Every finite-dimensional complex vector space V has a canonical orientation as a real vector space

Proof. We have a standard orientation on $\mathbb{C}^n$ ($\mathbb{C}$ has a standard orientation where i is positively oriented; then, take the direct-sum orientation); choose an isomorphism $\varphi : \mathbb{C}^n \longrightarrow V$ to orient V.

This could work for $\mathbb{R}^n$, which has an orientation; the key is that every complex linear isomorphism $A : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is orientation-preserving. Topologically, this follows because $\mathrm{GL}(n, \mathbb{C})$ has a single connected component (whereas $\mathrm{GL}(n, \mathbb{R})$ has two). More explicitly, we show $\det_{\mathbb{C}} A = \|\det_{\mathbb{R}} A_{\mathbb{R}}\|^2$ (where $A_{\mathbb{R}}$ is the matrix of A as an endomorphism of $\mathbb{R}^{2n}$), which is positive, because $A \in \mathrm{GL}(2n, \mathbb{R})$.

[4] If you count carefully, there are actually $n!$ such terms, but we don't need this in the proof.

Equality is easy if A is diagonal, and hence also if A is diagonalizable. But the real and complex determinants are continuous, and diagonalizable matrices are dense (since any matrix with distinct eigenvalues is diagonalizable, and if two eigenvalues coincide, one can bump one eigenvalue by an arbitrary small amount). ⊠

If V is a symplectic vector space, the canonical orientations induced as a symplectic vector space and as a complex vector space agree. In particular, if V and W are symplectic, $V \oplus W$ is canonically a symplectic vector space, and its canonical orientation is the direct-sum orientation induced from the orientations of V and W: both are oriented by the same $\omega^{\wedge n}$.