APPLICATIONS OF THE THEORY OF MORSE TO SYMMETRIC SPACES.*

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To Marston Morse on his 65th birthday.

Introduction. In this sequel to [6], we bring the proof of results announced in [7] and [8]. Although [6] might be a good introduction, the present paper is relatively self contained.

Let Ω be the loop space on the connected compact Lie group K, and let S be the set of geodesics on K leading from a general point P to the identity e. (A point of K is general if it is contained in only one maximal torus of K.) To each $s \in S$ we assign an index (dimension), δ_s, equal to the number of (properly counted) singular points of K in the interior of s. The main result of [6] then states that $H_*(\Omega;\mathbf{Z})$ is *abstractly* isomorphic to S_*, the free module generated by the elements of S, and graded by the index δ. (Thus as a generator in S_*, the element $s \in S$ has dimension δ_s.)

In the present paper this result is extended in two ways: a) We define an explicit isomorphism $\gamma: S_* \to H_*(\Omega;\mathbf{Z})$; in other words, we describe a base of singular cycles for $H_*(\Omega;\mathbf{Z})$. b) We show that (at least with $\mathbf{Z}_2$ as coefficients) a completely analogous description of $H_*(\Omega)$ holds in several instances, notably when Ω is the loopspace of a symmetric space.

This program is carried out in chapters I and II. We start by constructing a homomorphism $\gamma: S_* \to H_*(\Omega)$ (at least mod 2) under rather general conditions: Ω is the loop space of a manifold M on which a compact Lie group K acts with at least one fixed point. The set S is now defined as above, with the fixed point playing the role of e and P being any point on a K-orbit of maximal dimension. The index $\delta_K(s)$ is computed again as before, the K-singular points of M being the points on lower-dimensional orbits. Using the K-singular points in s, we construct a manifold Γ_s (with $\dim \Gamma_s = \delta_K(s)$) and a homeomorphism $f_s: \Gamma_s \to \Omega$, for each $s \in S$. These sub-

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manifolds of Ω can be thought of as generalizations of the cycles in the loop space of a sphere which Morse constructed (cf. [16]). By assigning to each $s \in S$ the f_s-image of the fundamental cycle on Γ_s, we obtain the homomorphism $\gamma: S_* \to H_*(\Omega)$, in general mod 2, and over the integers if all the Γ_s are orientable. The details of this construction are given in Chapter I, nos. 1-5.

In general this homomorphism has no great virtue. However, and this is our main observation, if the action of K on M satisfies a certain infinitesimal condition, already introduced in [6] and there called variational completeness, then $\gamma: S_* \to H_*(\Omega)$ is an isomorphism onto. This is the content of Theorem I.

Intuitively, we like to think of variational completeness as absence of conjugate points on the decomposition space M/K. This is literally correct when K reduces to the identity, or $M \to M/K$ is a fibering. In the case of K acting on itself by inner automorphisms, this interpretation fits in with E. Cartan's description of this decomposition space as a Euclidean cell (with certain identifications on the boundary); the proof that variational completeness holds for this example was already given in [6].

The proof of Theorem I takes up the rest of Chapter I. The Morse theory enters vitally, and more delicately than in [6], where one merely applied the Morse inequalities.

In Chapter II, we prove that variational completeness holds in various instances (Theorem II). All of these cases are associated with symmetric spaces.

In Chapter III, we describe the ring $H^*(\Gamma_s; \boldsymbol{Z})$ for a generic s on the group K, in terms of the Cartan integers of K (Proposition 4.2). The rest of that chapter then is devoted to applications, of Theorem I and this description of $H^*(\Gamma_s)$, to the topology of Lie groups. The main results are outlined in no. 2 of that chapter. We have reproduced our computations with possibly greater detail than is currently the custom.

Chapter IV contains immediate applications of Theorems I and II to symmetric spaces in general. The results here are less complete than in Chapter III. The manifolds Γ_s need not be orientable so that we have to restrict ourselves to the coefficient domain $\boldsymbol{Z}_2$. We do not know whether the loopspace of a general symmetric space has ony two-torsion.

Chapter I. Action of a Group and the Theory of Morse.

1. Action of a group on a manifold. All manifolds considered in this paper are of class C^∞. They are not necessarily connected, but all components of a manifold are of the same dimension. If x is a point of the manifold M, then M_x denotes the tangent space of M at x. If $f: M \to M'$ is a map (always C^∞) of the manifold M into the manifold M', then $\dot{f}_x: M_x \to M'_{x'}$ denotes the differential of f at x, with $x' = f(x)$. If f is a curve in M, i. e. a map of the real line $E^1 = \boldsymbol{R}$ into M, we write $\dot{f}(t)$ for the tangent vector to f at t, that is, for the image under $\dot{f}_t$ of the standard tangent vector D_t (or $\frac{d}{dt}$) of E^1 at t.

Let K be a compact Lie group, and M a (paracompact) manifold of dimension r. *Let K act on M from the left,* i. e. *let there be given a map*

$$\pi: K \times M \to M \text{ of class } C^\infty \tag{1.1}$$

such that (with $\pi(k,x)$ abbreviated to $k \cdot x$) *the following conditions hold:*

$$e \cdot x = x, \text{ where } e \text{ is the identity of } K, \tag{1.2}$$

$$k \cdot (k' \cdot x) = (k \cdot k') \cdot x \text{ for } k, k' \in K, x \in M. \tag{1.3}$$

The K-orbit of a subset A is denoted by $K \cdot A$. Thus,

$$K \cdot A = \{k \cdot a \colon k \in K, a \in A\}. \tag{1.4}$$

The stabilizer (stability group) of A is denoted by K_A; this is the subgroup of K which keeps A pointwise fixed,

$$K_A = \{k \in K \colon k \cdot a = a, a \in A\}. \tag{1.5}$$

For each $x \in M$, the orbit $K \cdot x$ is a regular submanifold of M, C^∞-homeomorphic to the quotient space K/K_x.

It is also convenient to denote, for any $k \in K$, by $\pi_k: M \to M$ the map defined by $\pi_k(x) = k \cdot x$, and to abbreviate $\pi_k \circ f$ to $k \cdot f$, for any map f with values in M.

It is well known that M can be given a Riemannian metric that is invariant under the action of K, so that all the maps $\pi_k: M \to M$ become isometries. We assume that such a metric is assigned, and use the notation (X, Y) for the corresponding inner product of two vectors in M_x; the norm of X is written $|X|$.

The action of K on M induces a representation of the Lie algebra $\mathfrak{k}$ of

K by vector fields over M. These vector fields—the infinitesimal K-motions—can be defined in the following manner:

Let X be an element of $\mathfrak{k}$, and let $h: \mathbf{R} \to K$ be the corresponding 1-parameter subgroup, with $\dot{h}(0) = X$.

DEFINITION 1.6. *For $x \in M$, let $h_x: R \to M$ be the curve defined by $h_x(\alpha) = h(\alpha) \cdot x$ for any $\alpha \in \mathbf{R}$. The assignment $x \to \dot{h}_x(0)$ defines a vector field $\tilde{X}$ on M, which is called the infinitesimal K-motion corresponding to X.*

It is clear that the tangent space to the K-orbit $K \cdot x$ at x is given by the values of all possible infinitesimal K-motions at x:

$$(1.7) \qquad (K \cdot x)_x = \{\tilde{X}(x) : X \in \mathfrak{k}\}.$$

The Riemannian metric in M defines in standard fashion the concept of distance (greatest lower bounds of length of curves connecting the two points), and the concept of geodesic. To be able to apply the theory of Morse, we shall assume that M is complete under its metric, so that geodesics can be continued indefinitely in both directions. A geodesic will then always be a non-degenerate C^∞-map of $\mathbf{R}$ into M, satisfying the usual differential equations for geodesics, parametrized proportionally to arc length. A geodesic segment s is the restriction of a geodesic g to an interval $[a, b]$ with $a < b$. The point $s(a)$ is the initial point, $s(b)$ is the terminal point of s. We write K_s for the stabilizer of $s([a, b])$; similarly for the stabilizer of a geodesic. The elements of K map geodesics into geodesics (the maps π_k are isometries).

2. K-transversality; the space $\Omega(M; \mathbf{R}, N)$. We introduce some basic definitions.

DEFINITION 2.1. *A geodesic g of M is called K-transversal (briefly: transversal) if for each $t \in \mathbf{R}$ the tangent vector $\dot{g}(t)$ is orthogonal to the K-orbit of the point $g(t)$ (i.e., orthogonal to the tangent space of the submanifold $K \cdot g(t)$ of M at $g(t)$); similarly for geodesic segments.*

There exist many transversal geodesics (unless K is transitive on M, in which case there are none at all!), as the next proposition shows.

PROPOSITION 2.2. *The geodesic g is transversal if there exists a $t_0 \in \mathbf{R}$ such that $\dot{g}(t_0)$ is orthogonal to $K \cdot g(t_0)$.*

In other words, transversality holds, provided it holds at a single point. The proof of Proposition 2.2 is an immediate consequence of Gauss' theorem on orthogonal trajectories of families of geodesics; we will give it in no. 6.

DEFINITION 2.3 (cf. [6], Def. 5.1). *If A, B are subsets of M, then $\Omega(M;A,B)$ denotes the metric space of all piecewise regular curves (maps u of the unit interval $[0,1]$) in M from A to B, i.e. with $u(0) \in A$, $u(1) \in B$, parametrized proportionally to arc length, with the distance between two curves u and u' defined as*

$$\sup_{t \in [0,1]} d(u(t),u'(t)) + |(L(u) - L(u')|;$$

here d is the distance in M, and L denotes the length of the curves.

Note that L is a continuous function on $\Omega(M;A,B)$.

Remark. The set $\Omega(M;A,B)$ is contained in the set $\Omega'(M;A,B)$ of all continuous curves (maps of $[0,1]$ into M) from A to B; if the latter, as customary, is topologized by the compact-open topology, then the inclusion map is continuous, and induces an isomorphism of the singular homology groups and also of the homotopy groups of the two spaces [17].

Let R be a point of M, and let N be the K-orbit of a point of M; to avoid trivial complications, assume $R \notin N$. The main object of study in this paper will be the space $\Omega = \Omega(M;R,N)$, to which we shall apply the theory of Morse. The next definition introduces a subset of Ω that is basic for this application.

DEFINITION 2.4. *$S = S(M;R,N)$ is the set of transversal geodesic segments, parametrized (proportionally to arc length) on $[0,1]$, from R to N, i.e. with initial point R and with terminal point on N.*

3. Fiber bundle concepts. We interpolate a section recalling some concepts from the theory of fiber bundles, specialized slightly to fit our purposes. By a principal bundle E for a compact Lie group G (the structure group) we mean a compact manifold on which G acts, from the right, without fixed points ($p \cdot g = p \Rightarrow g = e$). The decomposition space of E, derived from the decomposition of E into the orbits under G, is the base space B, a manifold. The natural map of E onto B is called the projection; it is differentiable; only the vectors tangent to the fibers are mapped into 0. One often writes E/G for B, particularly in the case where E is a group, G a subgroup and $B = E/G$ the space of left cosets of G.

We describe the standard construction of fiber spaces, associated to the principal bundle E for G: Suppose G operates, from the left, on another space F. The product of E and F over G, written $E \times_G F$, is obtained by identifying $(p \cdot g, x)$ with $(p, g \cdot x)$ in $E \times F$, or equivalently, by letting G

act on $E \times F$ via $(p,x) \cdot g = (p \cdot g, g^{-1} \cdot x)$ (which makes $E \times F$ into a principal G-bundle) and taking the base space. There exists a map of $E \times_G F$ onto B which makes the following diagram commutative:

$$\begin{array}{ccc} E \times F & \longrightarrow & E \\ \downarrow & & \downarrow \\ E \times_G F & \longrightarrow & B, \end{array} \tag{3.1}$$

where the other maps are the natural ones; this map is the induced fiber map with F as fiber.

Let another group G' act principally on F from the right, such that the actions of G and G' on F associate, and let G' act from the left on a third space F'. Then $E \times F \times F'$ is a principal $G \times G'$-bundle by the definition $(p,x,x') \cdot (g,g') = (p \cdot g, g^{-1} \cdot x \cdot g', g'^{-1} \cdot x')$. The base space, denoted by $E \times_G F \times_{G'} F'$, can in an obvious way be regarded also as $E \times_G (F \times_{G'} F')$ or $(E \times_G F) \times_{G'} F'$. Extension to more factors is immediate. Similarly, the two spaces $E \times_G F/G'$ and $(E \times_G F)/G'$ can be identified.

Let E, E' be two principal bundles with structure groups G, G' and base spaces B, B'. A map $\tilde{f}\colon E \to E'$ is a bundle map or equivariant map relative to a homomorphism $f\colon G \to G'$, if $\tilde{f}(p \cdot g) = \tilde{f}(p) \cdot f(g)$ for $p \in E$, $g \in G$. Such a map induces a map $\bar{f}$ of B into B'. The spectral sequence of E (in homology) is then mapped into that of E'; the map has the obvious naturality properties with respect to E_2, and the transgression (cf. [18] for these concepts).

4. The K-cycle of a geodesic segment. Let $s\colon [0,1] \to M$ be an element of the set S (cf. (2.4)), i.e. a transversal geodesic segment with $s(0) = R$ and $s(1) \in N$. We shall attach to s a manifold Γ_s, the "K-cycle of s," and a homeomorphism f_s of Γ_s into Ω. We first describe Γ_s abstractly, and present its geometrical meaning in the next section.

DEFINITION 4.1. *The parameter value $t \in [0,1)$ (note that 1 is excluded) and the point $s(t)$ are called exceptional for s, if for the dimensions of the stabilizers $K_{s(t)}$ and K_s* (cf. no. 1), *the relation*

$$\dim K_{s(t)} > \dim K_s$$

holds.

Geometrically, the exceptional points are those where the segment encounters orbits of lower dimension. We shall show in (9.2) that each s has only a finite number of exceptional values, and that the exceptional values are related to the focal values.

Let $t_1 < t_2 < \cdots < t_n$ be the exceptional values of s; the stabilizers $K_{s(t_i)}$ are then defined, and we put, for brevity [as long as we consider a fixed s]:

$$K_i = K_{s(t_i)}, \qquad i = 1, \cdots, n. \tag{4.2}$$

The stabilizer K_s of the whole segment s is a subgroup of each K_i; it acts on K_i principally from the left and the right, by multiplication, and acts also from the left on the coset space K_n/K_s. Because of the importance of coset spaces with K_s as subgroup, we introduce the following notation:

(4.3) *If G is a group containing K_s, then G^s denotes the left coset space G/K_s.*

In the following fundamental definition, we make use of the notation introduced in no. 3.

DEFINITION 4.4. *The K-cycle of s, denoted by Γ_s, is the manifold*

$$K_1 \times_{K_s} K_2 \times_{K_s} \cdots \times_{K_s} K_{n-1} \times_{K_s} K_n{}^s;$$

note that the last factor, according to (4.3), *is K_n/K_s.*

The dimension of Γ_s is given by

$$\dim \Gamma_s = \sum_i (\dim K_i - \dim K_s). \tag{4.5}$$

If s has no exceptional values at all, we let Γ_s be a single point.

The manifold Γ_s can also be considered as the base space of $K_1 \times_{K_s} K_2 \times_{K_s} \cdots \times_{K_s} K_n$, under the right operation of K_s on K_n. A third description will be used below. Put

$$W_s = K_1 \times \cdots \times K_n, \tag{4.6}$$

and let the n-th power $(K_s)^n = K_s \times \cdots \times K_s$ operate on W_s by the rule

$$p \cdot q = (c_1 a_1, a_1{}^{-1} c_2 a_2, a_2{}^{-1} c_3 a_3, \cdots, a_{n-1}{}^{-1} c_n a_n) \tag{4.7}$$

for $p = (c_1, \cdots, c_n) \in W_s$, $q = (a_1, \cdots, a_n) \in (K_s)^n$.

One verifies that this is a principal bundle operation, and that the base space is exactly Γ_s. We denote by ψ_s the projection of W_s onto Γ_s. It should be noted that although $(K_s)^n$ is a subgroup of W_s, and although the orbit of the point $(e, \cdots, e)$ of W_s under the operation (4.7) is just $(K_s)^n$, the operation (4.7) is not identical with left (or right) group multiplication in W_s, and Γ_s is not homeomorphic with the coset space $W_s/(K_s)^n$; it is easily seen, however, that the two spaces have isomorphic homotopy groups π_i for $i > 1$, and that they agree in π_1 up to an extension.

For an example, let M be the 2-sphere $\{x: x_1^2 + x_2^2 + x_3^2 = 1\}$ in 3-space, and let K, isomorphic to the circle group S_1, be the group of rotations around the x_3-axis. Let $R = (1,0,0)$; $N = \{(0,0,1)\}$. Let s be the geodesic segment from $(1,0,0)$ to the southpole $(0,0,-1)$ to $(-1,0,0)$ to the northpole $(0,0,1)$. There is one exceptional value, $t = \frac{1}{3}$. The stabilizer K_1 is equal to K, the stabilizer K_s reduces to the identity. We have $\Gamma_s = W_s = K$, a circle.

5. Geometric form of the *K*-cycles. We come to the concrete representation of Γ_s as a submanifold of the path space Ω (cf. no. 2). Let again $t_1 < t_2 < \cdots < t_n$ be the exceptional t-values for s. Put $t_0 = 0$, $t_{n+1} = 1$, and denote by s_i the restriction of s to $[t_i, t_{i+1}]$, for $i = 0, \cdots, n$; the first one of these is possibly a single point, the others are non-degenerate geodesic segments. We write

$$s = s_0 + s_1 + \cdots + s_n \tag{5.1}$$

for this decomposition of s. With its help, we shall define a map $\tilde{f}_s$ of the manifold W_s (cf. (4.6)) into Ω. This map is then shown to be constant along the orbits of W_s under the action of $(K_s)^n$ (cf. (4.7)), and so defines the promised map f_s of the base space Γ_s into Ω.

For $(c_1, \cdots, c_n) = p \in W_s$, we put $\pi^k(p) = c_1 \cdot c_2 \cdot \cdots \cdot c_k$, for $k = 1, \cdots, n$, and $\pi^0(p) = e$.

DEFINITION 5.2. *$\tilde{f}_s(p)$ is the map of $[0,1]$ into M, whose restriction to $[t_i, t_{i+1}]$ equals $\pi^i(p) \cdot s_i$; i. e., for $t \in [t_i, t_{i+1}]$, one has $\tilde{f}_s(p)(t) = \pi^i(p) \cdot s_i(t)$.*

We write symbolically

$$\tilde{f}_s(p) = \pi^0(p) \cdot s_0 + \pi^1(p) \cdot s_1 + \cdots + \pi^n(p) \cdot s_n.$$

The map $\tilde{f}_s(p)$ is well defined: The value $\tilde{f}_s(p)(t)$ has a double definition for $t = t_{i+1}$, $0 \leqq i \leqq n-1$, namely $\pi^i(p) \cdot s_i(t_{i+1})$ and $\pi^{i+1}(p) \cdot s_{i+1}(t_{i+1})$. The two values are identical because firstly $s_i(t_{i+1})$ and $s_{i+1}(t_{i+1})$ are identical and secondly $\pi^{i+1}(p)$ differs from $\pi^i(p)$ by the factor c_{i+1} in the stabilizer of $s_{i+1}(t_{i+1})$.

Since K acts by isometries, each $\pi^i(p) \cdot s_i$ is isometric with s_i; it follows easily that the curve $\tilde{f}_s(p)$ is parametrized proportionally to arc length. The initial point of $\tilde{f}_s(p)$ is R (since $\pi^0(p) \cdot s_0(0) = e \cdot R = R$); the terminal point is $\pi^n(p) \cdot s_n(1) = c_1 \cdot \cdots \cdot c_n \cdot s(1)$ and so lies on N because $s(1)$ does. All this implies that the curve $\tilde{f}_s(p)$ is a point of $\Omega(M; R, N)$. It can be described as a polygon, consisting of first s_0, then s_1, moved by c_1, then s_2, moved by $c_1 \cdot c_2$, etc.

Let $\tilde{f}_s\colon W_s \to \Omega$ be the map which assigns to $p \in W_s$ the element $\tilde{f}_s(p)$ of Ω. The continuity of $\tilde{f}_s$ is clear. However, $\tilde{f}_s$ is in general not 1-1; in fact, each orbit under the action (4.7) of $(K_s)^n$ maps into a single point.

Let $q = (a_1, \cdots, a_n) \in (K_s)^n$. From the definition of the action of $(K_s)^n$ on W_s, one concludes that

$$\pi^i(p \cdot q) = \pi^i(p) \cdot a_i. \tag{5.3}$$

Since $a_i \cdot s = s$, and so $a_i \cdot s_i = s_i$, it follows at once that

$$\tilde{f}_s(p \cdot q) = \tilde{f}_s(p).$$

This means that the map $\tilde{f}_s\colon W_s \to \Omega$ can be factored through the base space Γ_s under the projection ψ_s, i.e. that there exists a (unique) map $f_s\colon \Gamma_s \to \Omega$, such that $\tilde{f}_s = f_s \circ \psi_s$:

$$\begin{tikzcd} W_s \arrow[d, "\psi_s"'] \arrow[dr, "\tilde{f}_s"] & \\ \Gamma_s \arrow[r, dashed, "f_s"'] & \Omega. \end{tikzcd} \tag{5.4}$$

PROPOSITION 5.5. *f_s is a homeomorphism of Γ_s into Ω.*

Proof. Suppose $p = (c_1, \cdots, c_n)$ and $p' = (c'_1, \cdots, c'_n)$, two points of W_s, have the same $\tilde{f}_s$-image, i.e. the relations

$$\pi^i(p) \cdot s_i = \pi^i(p') \cdot s_i \text{ hold.}$$

Since none of the s_i for $1 \leqq i \leqq n$ is degenerate (so that $K_s = K_{s_i}$), it follows that $\pi^i(p)^{-1} \cdot \pi^i(p') = a_i$ belongs to K_s, for $1 \leqq i \leqq n$. This implies easily that $p' = p \cdot q$ with $q = (a_1, \cdots, a_n)$, and Proposition 5.5 is an immediate consequence.

Let $\omega_s \in \Gamma_s$ be the point $\psi_s(e, \cdots, e)$, i.e. the point whose inverse image under ψ_s is the fiber $(K_s)^n$; the f_s-image of ω_s is the geodesic segment s itself. Proposition 5.5 implies as a special case

PROPOSITION 5.5'. *The inverse image under f_s of the point s of Ω is the point ω_s of Γ_s.*

6. Jacobi fields; variational completeness. Our purpose is to formulate sufficient conditions for the cycles Γ_s to generate the homology of Ω. Before we can state our main result in this direction (cf. no. 7), we have to recall some facts concerning Jacobi fields. These vector fields can be defined in the following manner.

A geodesic variation $\{V_\alpha\}$ of a geodesic g in M is a C^∞-map $V\colon \boldsymbol{R}\times I\to M$, where I is an open interval containing 0, such that

(6.1) for each $\alpha\in I$ the map $V_\alpha\colon \boldsymbol{R}\to M$, defined by $V_\alpha(t)=V(t,\alpha)$, is a geodesic,

(6.2) $$V_0=g.$$

DEFINITION 6.3. *The vector field η along g, defined by $\eta(t)=\partial V(t,0)/\partial\alpha$, is called the Jacobi field, or J-field, determined by the geodesic variation V.*

It is well known that the J-fields along g form a vector space, which we shall call J_g, of dimension $2r$ (where $r=\dim M$), and that they can also be characterized as the solutions of the Jacobi differential equations (cf. [16]).

If h is a 1-parameter group in K, then the family $h(\alpha)\cdot g$, $\alpha\in\boldsymbol{R}$ is a geodesic variation of g; this implies that the restriction of any infinitesimal K-motion (cf. Definition 1.6) to g is a J-field along g. From the standard formula for the first variation of arc length, one derives easily that for any infinitesimal K-motion X the inner product $(\dot g(t),\tilde X(g(t)))$ is independent of t.

We indicate now the proof of Proposition 2.2 as promised there. By (1.7), transversality of g at t means the vanishing of $(\dot g(t),\tilde X(g(t)))$ for all $X\in\mathfrak{k}$. By the remark just made, this inner product will vanish for all t, provided it does so for any one particular value t_0.

We shall have to consider all those J-fields along g that vanish at some given parameter value t_0.

DEFINITION 6.4. *For any $t_0\in\boldsymbol{R}$, $\Lambda_g(t_0)$ is the subspace of J_g consisting of those J-fields that vanish at t_0;*

$$\Lambda_g(t_0)=\{\eta\in J_g\colon \eta(t_0)=0\}.$$

It can be shown that any J-field in $\Lambda_g(t_0)$ can be derived from a variation with fixed point at t_0, i.e., a variation $\{V_\alpha\}$ such that $V(t_0,\alpha)=g(t_0)$ for all $\alpha\in I$. Since a geodesic is determined by its tangent vector at any point, a variation with fixed point at t_0 is equivalent to a curve $Y(\alpha)$ in the tangent space $M_{g(t_0)}$, with $Y(0)=\dot g(t_0)$; in terms of V, we have $Y(\alpha)=\partial V(t_0,\alpha)/\partial t$. The corresponding J-field, derived by differentiation with respect to α, depends only on the vector $\dot Y(0)$. This amounts to a map θ from $M_{g(t_0)}$ (using the customary identification of a vector space with its tangent space at any of its points) into $\Lambda_g(t_0)$; actually, the map is an isomorphism of the two spaces involved. Suppose the variations are further restricted by the requirement that the tangent vectors of the V_α at t_0 lie in a given subspace W of $M_{g(t_0)}$;

then the associated J-fields will form a subspace W' of $\Lambda_{g(t_0)}$, and θ will map W onto W'. We omit the proofs of the various assertions made; they are classical, although usually stated in a different language.

DEFINITION 6.5. *Suppose g is a K-transversal geodesic (cf. no. 2); a J-field is called transversal if it is derived from a geodesic variation $\{V_\alpha\}$ of g in which all V_α are transversal geodesics.*

PROPOSITION 6.6. *If g is a transversal geodesic, then the restriction of any infinitesimal K-motion to g is a transversal J-field.*

This follows at once from the fact that the maps π_k (cf. no. 1) preserve transversality and that therefore each $h(\alpha)\cdot g$ is a transversal geodesic.

DEFINITION 6.7. *Let g be a transversal geodesic, and let t_0 be a real number. We denote by $J_g^\pi(t_0)$ the subspace of J_g of those transversal J-fields which at t_0 are tangent to the K-orbit of $g(t_0)$.*

This equals the J_g^N of [6], with N the orbit of $g(t_0)$. Remarks similar to those made after 6.4 apply. The dimension of $J_g^\pi(t_0)$ is $r=\dim M$ (to obtain $\eta\in J_g^\pi(t_0)$, one varies the vector $\dot g(t_0)$ in the r-dimensional manifold of vectors ($\neq 0$) transversal to the orbit of $g(t_0)$).

Let $\tilde X$ be an infinitesimal K-motion, which therefore restricts to a transversal J-field η along the transversal geodesic g. It is clear from (1.7) that η is tangent to the orbit at every point of g, i.e. $\eta\in J_g^\pi(t)$ for all t.

The basic property which we require of our group K is laid down in the following definition (cf. [6], p. 261).

DEFINITON 6.8. *The action of K on M is variationally complete if every transversal J-field η, which is tangent to the K-orbits for two different points (parameter values) of the underlying transversal geodesic g (i.e., for which there exist $t_0, t_1\in\mathbf{R}$, $t_0\neq t_1$, such that $\eta\in J_g^\pi(t_0)\cap J_g^\pi(t_1)$) is induced by K, i.e. is the restriction to g of an infinitesimal K-motion.*

We get an equivalent definition, (6.8′), if instead of requiring tangency at two points, we require tangency at one point and vanishing at another point, i.e. $\eta\in J_g^\pi(t_0)\cap\Lambda_g(t_1)$. Clearly, if (6.8) holds, then also (6.8′) holds. We consider the reverse implication: Let (6.8′) hold, and let η be a transversal J-field in $J_g^\pi(t_0)\cap J_g^\pi(t_1)$. Since $\eta(t_0)$ is tangent to the orbit, there exists an infinitesimal K-motion $\tilde X$, such that $\tilde X(g(t_0))=\eta(t_0)$. Denote by ζ the restriction of $\tilde X$ to g. Then $\eta-\zeta$ is a transversal J-field which vanishes at t_0 and is tangent to the orbit at t_1. By (6.8′), it is therefore induced by K; but then $\eta=(\eta-\zeta)+\zeta$ is also induced by K.

7. The main theorem. For any space X and Abelian group $\boldsymbol{G}$, we denote by $H_r(X;\boldsymbol{G})$ the r-th singular homology group of X with coefficients in $\boldsymbol{G}$; similarly, $H_r(X,A;\boldsymbol{G})$ denotes the relative homology group of X modulo the subset A. We put $H_*(X;\boldsymbol{G})=\sum_r H_r(X;\boldsymbol{G})$. We denote the integers by $\boldsymbol{Z}$ and the integers modulo 2 by $\boldsymbol{Z}_2$. In the case $\boldsymbol{G}=\boldsymbol{Z}$, we write simply $H_*(X)$, etc. If $f\colon X\to Y$ is a map, then f_* is the induced homology map.

Cohomology is indicated by upper indices, $H^r(X;\boldsymbol{G})$, etc. If $\boldsymbol{G}$ is a commutative ring with unit, then $H^*(X;\boldsymbol{G})$ is a graded $\boldsymbol{G}$-algebra (cup product). We write $x(y)$ for the value of a cocycle x on a cycle y.

A fundamental cycle on a manifold, with coefficients in $\boldsymbol{Z}$ or $\boldsymbol{Z}_2$, is the sum of fundamental cycles (i.e. generators of the homology groups in the highest dimension) for the various components of the manifold.

DEFINITION 7.1 (cf. [6], Proposition 6.1). *The defect $\delta(Q)$ of a point Q of M is the difference between the maximum of the dimensions of K-orbits in M and the dimension of the K-orbit of Q.*

Let R be a point with maximal orbit dimension, i.e. with $\delta(R)=0$; it is clear that $\delta(Q)$ can be expressed in terms of stabilizers by

$$\delta(Q)=\dim K_Q-\dim K_R. \tag{7.2}$$

For a geodesic segment $s\colon[a,b]\to M$ we define the defect δ_s by

$$\delta_s=\sum_{a\leqq t<b}\delta(s(t)); \tag{7.3}$$

we shall show later (after Proposition 9.2) that δ_s is finite if $\delta(s(a))=0$.

For each s in the set $S=S(M;R,N)$ (cf. no. 2), we have constructed in no. 4, 5 the manifold Γ_s and the map f_s of Γ_s into Ω. Note that Γ_s is not necessarily connected. For each such s, let $\bar{\gamma}_s$ denote the (unique) fundamental cycle over $\boldsymbol{Z}_2$ of Γ_s. We call S orientable (or better, K-orientable) if all Γ_s are orientable manifolds [i.e., all components of each Γ_s orientable]. In this case, an orientation of S is a function γ which to each $s\in S$ assigns a fundamental cycle γ_s over $\boldsymbol{Z}$ of Γ_s. Only the orientation of the principal component, containing the point $\psi_s(e,\cdots,e)$, will actually matter later. Suppose R lies on an orbit of maximal dimension. It is well known that the stabilizer of any point Q on s, sufficiently close to R, is contained in K_R since s is orthogonal to the orbit of R (existence of a slice [15]). The maximality condition on R implies that the e-components of K_Q and K_R, and therefore also those of K_s and K_R, coincide. We conclude now from (4.5), (7.2), (7.3):

If R has defect 0, *then for any $s \in S(M;R,N)$, the relation*

$$\dim \mathbf{\Gamma}_s = \delta_s \text{ holds.} \tag{7.4}$$

Let $\bar{S}_*$, resp. S_*, denote the $\mathbf{Z}_2$-vector space, resp. the free abelian group, with the elements of S as basis, graded by assigning to any s the value δ_s as dimension.

We can now state our first main theorem.

THEOREM I. *Let the action of K on M be variationally complete and let R lie on an orbit of maximal dimension. Then the elements $f_{s*}(\bar{\gamma}_s)$, for $s \in S$, form a basis of (the $\mathbf{Z}_2$-vector space) $H_*(\Omega;\mathbf{Z}_2)$. If S is K-orientable, then $H_*(\Omega;\mathbf{Z})$ is free abelian, with the elements $f_{s*}(\gamma_s)$, for $s \in S$ and γ any orientation of S, as a basis.*

In other words: The assignment $s \to f_{s*}(\bar{\gamma}_s)$, resp. $s \to f_{s*}(\gamma_s)$ in the K-orientable case, induces a gradation preserving isomorphism $H_*(\Omega;\mathbf{Z}_2) \approx \bar{S}_*$, resp. $H_*(\Omega;\mathbf{Z}) \approx S_*$ in the K-orientable case.

The two versions of the theorem are equivalent because of (7.4).

The proof of Theorem I, based on the theory of Morse, will be given in the following sections.

We note that Theorem I constitutes an improvement over the results of [6] since it contains a construction of the homology of Ω, whereas Proposition 6.1 of [6], together with the Morse inequalities, essentially gives a majorisation of the Betti numbers of Ω.

Remarks. (a) Consider the case where K reduces to the identity. Each $\mathbf{\Gamma}_s$ is then a point. Variational completeness becomes simply absence of conjugate points. Theorem I says that Ω has no homology in positive dimensions, and that the cardinality of S is equal to the cardinality of the set of arc-components of Ω, i.e. equal to the order of the fundamental group $\pi_1(M)$ of M. If M is simply connected, this means that any two points are connected by exactly one geodesic.

(b) The action of S_1 on S_2, described in no. 2, is easily seen to be variationally complete. The element $f_{s*}(\gamma_s)$, for the s described there, with $R = (1,0,0)$, $N = \{(0,0,1)\}$, is the first Morse cycle, of dimension 1, in $\Omega(S_2)$; an alternative description is as follows: to each point x on the equator of S_2, associate the path consisting of the great circle arc from $(1,0,0)$ to the southpole plus the meridian through x (from $(0,0,-1)$ to $(0,0,1)$); the image of the equator under this map into Ω is the Morse cycle in question. The constructions of Theorem I were motivated by this example.

8. Review of the theory of Morse. We recall some facts and definitions concerning the theory of Morse (cf. [16], [17]).

The values of the length function L on the subset S (made up of the transversal geodesics) of the function space Ω are the stationary values of L. Assume that, for any real l, the collection of elements of S, whose length is $\leqq l$, is finite. The stationary values, in natural order, form then a strictly increasing finite or infinite sequence $\{c_i\}$, $i = 1, 2, \cdots$; if the sequence is infinite, it diverges. If the sequence is finite, one extends it arbitrarily to a strictly increasing, diverging one. To each c_i corresponds then only a finite number of elements of S. Let Ω_i be the subset of Ω determined by $\Omega_i = \{u \in \Omega : L(u) \leqq c_i\}$, the set *up to* level c_i, and let $\Omega_i^- = \{u \in \Omega : L(u) < c_i\}$, the set *strictly below* level c_i; let Ω_0 and Ω_0^- be the empty set.

We define a group E_1 as the direct sum over i of the relative homology groups $H_*(\Omega_i, \Omega_{i-1}; \boldsymbol{G})$, and similarly E_1^- as the direct sum of the groups $H_*(\Omega_i, \Omega_i^-; \boldsymbol{G})$; the notation E_1, E_1^- is used because of the connection with spectral sequences. The following two propositions express Morse's evaluation of the group E_1.

PROPOSITION 8.1. *The inclusions $(\Omega_i, \Omega_{i-1}) \subset (\Omega_i, \Omega_i^-)$ induce isomorphisms of $H_*(\Omega_i, \Omega_{i-1}; \boldsymbol{G})$ and $H_*(\Omega_i, \Omega_i^-; \boldsymbol{G})$, and induce therefore an isomorphism of E_1 and E_1^-.*

We sketch briefly how this proposition follows from Morse's work. Put $w_i = \{s \in S : L(s) = c_i\}$, and $\Omega(c) = \{u \in \Omega : L(u) \leqq c\}$ for $c \geqq 0$. The main construction (cf. [17, Satz, p. 66]) consists in finding, for any c with $c_i \leqq c < c_{i+1}$, a deformation D_t, $0 \leqq t \leqq 1$, of the pair $(\Omega(c), \Omega_i^- \cup w_i)$ such that always $D_t(\Omega(c)) \subset \Omega(c)$, $D_t(\Omega_i^- \cup w_i) \subset \Omega_i^- \cup w_i$, and moreover $D_1(\Omega(c)) \subset \Omega_i^- \cup w_i$. It follows that the relative homology groups $H_*(\Omega(c), \Omega_i^- \cup w_i)$ vanish. This implies (by the exact sequence of a triple) that $H_*(\Omega(c), \Omega_i)$ vanishes; one concludes easily that also $H_*(\Omega_{i+1}^-, \Omega_i)$ vanishes, and 8.1 follows. (We note that part of the Theorem on p. 275 in [6] is stated incorrectly; the map mentioned there should be replaced by the two inclusions

$$(\Omega_i^- \cup w_i, \Omega_i^-) \subset (\Omega(l_i), \Omega_i^-) \text{ and } (\Omega(l_i), \Omega(l_{i-1})) \subset (\Omega(l_i), \Omega_i^-).)$$

For non-compact, complete manifolds the argument has to be modified slightly.

To each geodesic segment s in S is assigned an index λ_s (reative to the manifold N): Recall that s is parametrized on $[0, 1]$; call g the underlying geodesic. A value $t_0 \in [0, 1)$ and the point $s(t_0)$ are called focal (for N) if there exists a non-trivial transversal Jacobi field along g that is tangent to N for $t=1$ and vanishes for $t=t_0$, i.e., if the space $J_g{}^{\pi}(1) \cap \Lambda_g(t_0)$ is of positive dimension. The index $\lambda_s(t_0)$ of t_0 is the number of linearly independent such fields: $\lambda_s(t_0)=\dim J_g{}^{\pi}(1) \cap \Lambda_g(t_0)$. It is shown in [16] that each s has only a finite number of focal values.

DEFINITION 8.2. *The index λ_s (relative to N) of the geodesic segment s is given by*

$$\lambda_s = \sum_{0 \leqq t < 1} \lambda_s(t) = \sum_{0 \leqq t < 1} \dim J_g{}^{\pi}(1) \cap \Lambda_g(t).$$

The segment s is called non-degenerate (in the sense of Morse) if $\lambda_s(0)=0$, i.e. if 0 is not a focal value. If all $s \in S$ are non-degenerate, then the finiteness assumption, made in the third sentence of this section, is satified. This is an immediate consequence of the implicit function theorem; for another approach cf. [16], Ch. VII, Theorem 11.1.

Assuming s non-degenerate, then there exists, according to Morse, a Euclidean simplex σ_s, of dimension λ_s, and a map $\phi_s\colon \sigma_s \to \Omega$, satisfying the following conditions: (a) the barycenter of σ_s goes into the point s of Ω, (b) the function $L \circ \phi_s$, induced by the length function on Ω, takes its maximum at the barycenter of σ_s, and only there, (c) $L \circ \phi_s$ is of class C^{∞} near the barycenter, and the maximum is non-degenerate (i.e. the quadratic terms of the Taylor expansion constitute a negative definite form). Any singular simplex satisfying (a), (b), (c) will be said to be associated to s or "hanging over s" (using an intuitive term motivated by thinking of L, the length, as indicating the "height" of the points of Ω).

Let $\dot{\sigma}_s$ denote the boundary of σ_s. Then ϕ_s maps the pair $(\sigma_s, \dot{\sigma}_s)$ into the pair (Ω_i, Ω_i^-), with i determined by $c_i = L(s)$, by condition (b). Recall that the relative homology group $H_r(\sigma_s, \dot{\sigma}_s; \boldsymbol{G})$ is non-trivial only for $r = \dim \sigma_s = \lambda_s$.

PROPOSITION 8.3. *Assume that all $s \in S$ are nondegenerate; choose for each s an associated singular simplex (σ_s, ϕ_s). Let ξ_s, resp. $\bar{\xi}_s$, denote a generator of $H_{\lambda_s}(\sigma_s, \dot{\sigma}_s; \boldsymbol{G})$ for $\boldsymbol{G}=\boldsymbol{Z}$, resp. $\boldsymbol{Z}_2$. Then, for each i, the group $H_*(\Omega_i, \Omega_i^-)$ is free abelian, with a basis consisting of the ϕ_{s*}-images of the elements ξ_s, for all s with $L(s)=c_i$; similarly, the ϕ_{s*}-images of the elements $\bar{\xi}_s$, for $L(s)=c_i$, form a basis of the $\boldsymbol{Z}_2$-vector space $H_*(\Omega_i, \Omega_i^-; \boldsymbol{Z}_2)$.*

We note that this implies $H_*(\Omega_i, \Omega_i^-; \boldsymbol{G})=0$ if there is no s with $L(s)=c_i$; this will happen, under our convention, for large i, if the number of stationary values is finite.

The simplest situation in which one can deduce the homology of Ω from E_1 is that in which the condition of "completability" holds:

CONDITION 8.4. *For each i and r the map $j_i\colon H_r(\Omega_i;\mathbf{G})\to H_r(\Omega_i,\Omega_i^-;\boldsymbol{G})$, induced by the inclusion of (Ω_i,Ω_0) in (Ω_i,Ω_i^-) is onto.*

Geometrically this means that every relative cycle of $\Omega_i \bmod \Omega_i^-$, represented by an associated simplex "hanging over" a point s of $\Omega_i - \Omega_i^-$, can be completed "below level $c_i = L(s)$" to an absolute cycle of Ω.

PROPOSITION 8.5. (A) *Suppose all $s \in S$ are non-degenerate and the completability condition (8.4) holds for $\boldsymbol{G}=\boldsymbol{Z}$. For each $s\in S$, let x_s be an element of $H_*(\Omega_i)$, with i determined by $c_i = L(s)$, such that $j_i(x_s) = \phi_{s*}(\xi_s)$. Let z_s be the image of x_s in $H_*(\Omega)$ under the inclusion $\Omega_i \subset \Omega$. Then $H_*(\Omega)$ is a free abelian group, and the set $\{z_s\}$, $s \in S$, is a basis.*

(B) *A similar statement holds for the case $\boldsymbol{G}=\boldsymbol{Z}_2$, with elements $\bar{x}_s \in H_*(\Omega_i)$, and with the set $\{\bar{z}_s\}$, $s\in S$, forming a base of the $\boldsymbol{Z}_2$-vector space $H_*(\Omega;\boldsymbol{Z}_2)$.*

Note that the z_s and $\bar{z}_s$ are completions of the relative cycles $\phi_{s*}(\xi_s)$ and $\phi_{s*}(\bar{\xi}_s)$.

We indicate briefly how Proposition 8.5 follows from (8.1) and (8.3) under condition 8.4, for the case of $\boldsymbol{G}=\boldsymbol{Z}$. From (8.1) and (8.3), one concludes that each $H_*(\Omega_i,\Omega_{i-1})$ is free. From the exactness of the homology sequence of (Ω_i,Ω_{i-1}) and (8.4), one concludes that the sequence

$$0 \to H_*(\Omega_{i-1}) \to H_*(\Omega_i) \to H_*(\Omega_i,\Omega_{i-1}) \to 0$$

is exact. One proves now inductively that each $H_*(\Omega_i)$ is free, and imbedded as direct summand in $H_*(\Omega_{i+1})$ under the inclusion $\Omega_i \subset \Omega_{i+1}$. Since $H_*(\Omega)$ is easily seen to be the direct limit of the $H_*(\Omega_i)$, it follows that it is free abelian. The statement concerning the basis $\{z_s\}$ for $H_*(\Omega)$ is verified by tracing the ξ_s through the construction.

9. Proof of Theorem I. Beginning. We shall show that the hypotheses of Proposition 8.5 are satisfied, if those of Theorem I are.

PROPOSITION 9.1. *Let the point R lie on a K-orbit of maximal dimension and suppose the action of K is variationally complete; then all geodesic segments making up the set $S = S(M;R,N)$ are non-degenate in the sense of Morse.*

Proof. Let s be an element of S, carried by the geodesic g. We have to show that the index $\lambda_s(0)$, which equals the dimension of $J_g{}^\pi(1)\cap\Lambda_g(0)$, vanishes. By the variational completeness assumption 6.8, any J-field in

$J_g^\pi(1) \cap \Lambda_g(0)$ is induced by an infinitesimal K-motion (cf. (1.5)) which in turn is derived from a variation of the form $h(\alpha)\cdot g$, where h is a 1-parameter group in K with the property: $h(\alpha)\cdot R = R$ for all α (the variation must leave R fixed, since the J-field vanishes at R). In other words, the elements $h(\alpha)$ are contained in the stabilizer K_R. As in the proof of (7.4), one sees that the e-components of K_g and K_R coincide. This means that the 1-parameter group h above leaves every point of g fixed, and the associated J-field vanishes identically, q. e. d.—The next step is to prove that the completability condition 8.4 is satisfied, and that in fact the K-cycles Γ_s under the maps f_s can be used as completions. The proof is more complicated than that of Proposition 9.1. Let again s be an element of S.

PROPOSITION 9.2. *For any $t \in [0,1)$, the inequality $\dim K_{s(t)} - \dim K_s \leqq \lambda_s(t)$ holds; if the action of K is variationally complete, then equality obtains.*

Proof. Every X in the Lie algebra $\mathfrak{k}$ produces, through the corresponding infinitesimal K-motion $\tilde{X}$, a transversal Jacobi field; because of (1.7), this field belongs automatically to $J_g^\pi(1)$. The field $\tilde{X}$ vanishes at the point $s(t)$ exactly if X belongs to the Lie algebra $\mathfrak{k}_{s(t)}$ of $K_{s(t)}$. This means that there is a map, obviously linear, of $\mathfrak{k}_{s(t)}$ into $J_g^\pi(1) \cap \Lambda_g(t)$. The kernel of this map consists of those X whose associated $\tilde{X}$ vanishes identically along s; it is clear that this is just the Lie algebra $\mathfrak{k}_s$ of K_s. The inequality

$$\dim \mathfrak{k}_{s(t)} - \dim \mathfrak{k}_s \leqq \dim J_g^\pi(1) \cap \Lambda_g(t)$$

of (9.2) is an immediate consequence. Assume now variational completeness. This means that every J-field in $J_g^\pi(1) \cap \Lambda_g(t)$ is induced by an infinitesimal K-motion. The map of $\mathfrak{k}_{s(t)}$ into $J_g^\pi(1) \cap \Lambda_g(0)$ introduced above is onto, and the equality of (9.2) follows.

We note that Proposition 9.2 implies the statements about exceptional values made after Definition 4.1 and after (7.3).

PROPOSITION 9.3. *If the action of K is variationally complete, then for every $s \in S$, the dimension of the K-cycle Γ_s equals the index λ_s.*

Proof. By 4.5, we have $\dim \Gamma_s = \sum_i (\dim K_{s(t_i)} - \dim K_s)$, where the sum runs over the exceptional values of s. Proposition 9.3 follows immediately from the Definition 8.2 of λ_s and (9.2). We note that (9.3) and (7.4) imply

(9.4) *If the action of K is variationally complete and $\delta(R) = 0$, then $\dim \Gamma_s = \lambda_s = \delta_s$ for every $s \in S$.*

10. A deformation of the K-cycles. Let s be a geodesic of S, and let i be the number of its stationary level: $L(s)=c_i$. In (5.4), we constructed a map $f_s\colon \Gamma_s\to\Omega$, and it is clear from the construction that all polygons making up the image set of f_s have the same length. Hence f_s maps Γ_s into Ω_i.

PROPOSITION 10.1. *Under the hypotheses of Theorem* I, *there is only one geodesic segment in the set* $f_s(\Gamma_s)$, *namely* s *itself*; *all other elements are geodesic polygons with at least one corner.*

Proof. Since 0 now is not an exceptional value by (9.1), it follows that the interval s_0 on s, introduced in (5.1), is non-degenerate. All elements of $f_s(\Gamma_s)$ are geodesic polygons beginning with s_0. If two geodesic segments have a non-degenerate segment in common, they are identical, q. e. d.

It is possible, and basic for the application of the theory of Morse, to deform f_s in such a way that all $f_s(x)$, except s itself, are replaced by shorter curves, by "cutting across the corners" of the polygons. Actually, a sharper statement has to be made.

PROPOSITION 10.2. *If the segment* s *is non-degenerate in the sense of Morse, then there exists a homotopy* $f_s^u\colon \Gamma_s\to\Omega$, $0\leqq u\leqq 1$, *such that*

(a) $f_s^0=f_s$,

(b) $f_s^u(\omega_s)=s$ *for all* $u\in[0,1]$,

(c) f_s^u *maps* $\Gamma_s-\omega_s$ *into* Ω_i^-, *whenever* $0<u\leqq 1$,

(d) *for any* $u\in(0,1]$, *the induced length function* $L_s^u=L\circ f_s^u$ *on* Γ_s, *which by* (b), (c) *takes it maximum,* $L(s)$, *exactly at* ω_s, *has a non-degenerate critical point at* ω_s.

Assertion (d) means that L_s^u is differentiable near ω_s, and that, in terms of a local coordinate system $\{y_i\}$ at ω_s, the Hessian of L_s^u, i.e. the matrix $(\partial^2L_s^u/\partial Y_i\partial Y_j\,|_{\omega_s})$, is non-singular; it will then automatically be negative definite, since L_s^u has a maximum at ω_s.

We first define a homotopy $\tilde f_s^u$ of the map $\tilde f_s$ of W_s into Ω (cf. no. 5).

Let U be an open neighborhood of s in M, invariant under K and with compact closure; such sets exist. It is known from Riemannian geometry that there is a positive $a=a(U)$ such that any closed sphere V of radius a with center in $\bar U$ is geodesically convex (so that any two points of V are joined within V by a unique geodesic segment that varies smoothly with the points and whose length equals the distance of the points). Let ϵ be a number such that

$$(10.3)\quad 0<\epsilon<a(U),$$

(10.3′) $2\epsilon < L(s_i)$, $i = 0, \cdots, n$, where the s_i are the intervals on s, defined by the exceptional points (cf. (5.1)).

For any $u \in [0,1]$, we let s_i^u be the following restrictions of s, with $u' = u \cdot \epsilon / L(s)$:

$$
\begin{aligned}
&s_0^u = s \mid [0, t_1 - u'], \\
(10.4) \qquad &s_i^u = s \mid [t_i + u', t_{i+1} - u'], \text{ for } i = 1, \cdots, n-1, \\
&s_n^u = s \mid [t_n + u', 1].
\end{aligned}
$$

Because of (10.3′), the indicated intervals of $[0,1]$ are all non-degenerate:

$$2u' \leqq 2\epsilon / L(s) < L(s_i)/L(s) = t_{i+1} - t_i.$$

We denote by $\alpha_i(u)$, resp. $\beta_i(u)$, the initial, resp. terminal point of the segment s_i^u.

Let $p = (c_1, \cdots, c_n)$ be any point of W_s. Using the notation introduced in no. 5, we let $r_i^u(p)$ be the unique shortest geodesic segment from $\pi^i(p) \cdot \beta_i(u)$ to $\pi^{i+1}(p) \cdot \alpha_{i+1}(u)$, parametrized from $t_{i+1} - u'$ to $t_{i+1} + u'$, for $i = 0, \cdots, n-1$.

The existence of $r_i^u(p)$ comes from the fact that the two points which it is supposed to connect lie with the ϵ-sphere around the point $\pi^i(p) \cdot s_i(t_{i+1}) \in U$, and (10.3) applies.

DEFINITION 10.5. *$\tilde{f}_s^u(p)$ is the geodesic polygon, parametrized on $[0,1]$ proportionally to arc length, made up of the segments*

$$\pi^0(p) \cdot s_0^u, \quad r_0^u(p), \quad \pi^1(p) \cdot s_1^u, \quad r_1^u(p), \cdots, \quad r_{n-1}^u(p), \quad \pi^n(p) \cdot s_n^u.$$

As indicated before, each $\tilde{f}_s^u(p)$ is a geodesic polygon in Ω (i.e. beginning at R, and ending transversally on N), obtained from $\tilde{f}_s(p)$ by "cutting across the corners." The construction is pictured in fig. 1. It follows from local minimum properties of geodesics in convex spheres that $\tilde{f}_s^u(p)$, for $u > 0$, is actually shorter than $\tilde{f}_s(p)$, provided there is at least one corner on $\tilde{f}_s(p)$.

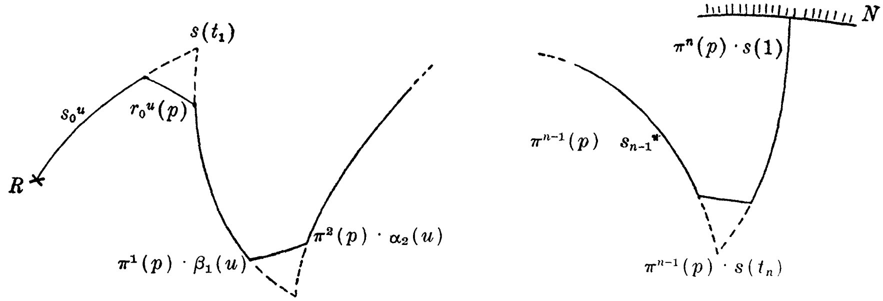


Figure 1.

11. The critical point on Γ_s. It is clear that with the choice of ϵ in (10.3), (10.3′), the point $\tilde{f}_s{}^u(p)$ of Ω depends continuously on the pair (p,u). Since $\tilde{f}_s{}^u(p)$ is derived from the polygon $\tilde{f}_s(p)$ by a geometrical construction, it follows at once that two points p, p' of W_s which are equivalent under the action 4.6 of $(K_s)^n$ on W_s, will give rise to the same modified polygon; i.e. $p' = p \cdot q$ implies $\tilde{f}_s{}^u(p') = \tilde{f}_s{}^u(p)$. This means that the map $\tilde{f}_s{}^u$ of W_s into Ω can be factored through Γ_s, with an induced map $f_s{}^u\colon \Gamma_s \rightarrow \Omega$. Clearly $f_s{}^u$ is a homotopy with properties (a), (b) of Proposition 10.2. The induced length function $L_s{}^u$ is clearly constant for $u = 0$. Since 0 is not exceptional for s, Proposition 10.1 applies. By the last remark of no. 10, the function $L_s{}^u$, for $u > 0$, will take a value less than $L(s)$ at every point x of Γ_s, different from ω_s; this proves (10.2 (c)).

The proof of (10.2 (d)) is more complicated. We note first that each $L_s{}^u$ is of class C^∞, since all the points involved in the construction of the geodesic polygons are C^∞-functions of the parameters occurring. We prove (d) in the following equivalent form.

PROPOSITION 11.1. *Let s be non-degenerate in the sense of Morse [and suppose* $\dim \Gamma_s > 0$ *to rule out a trivial case]; let x be a C^∞-curve in Γ_s, defined on some interval $(-a,a)$ with $x(0) = \omega_s$, and $\dot{x}(0) \neq 0$; put $F = L_s{}^u \circ x$ with any u in $(0,1]$; then the second derivative of F at the origin 0 is negative: $F''(0) < 0$.*

It is clear from what has been proved already that $F(0) = L(s)$ and $F'(0) = 0$, since F has a maximum at 0. For the proof of (11.1), we take a vector Y at $(e,\cdots,e)$ in W_s that projects into $\dot{x}(0)$ under the map $\psi_s\colon W_s \rightarrow \Gamma_s$, and let y be a curve in W_s, through $(e,\cdots,e)$ with $\dot{y}(0) = Y$, that projects into x, so that $\psi_s(y(v)) = x(v)$. Both Y and y exist since ψ_s is a fiber map. We have then $F(v) = L \circ \tilde{f}_s{}^u(y(v))$, since $f_s{}^u$ is induced by $\tilde{f}_s{}^u$.

Suppose first that y is of the form $y(v) = (e,\cdots,e,y_{i_0}(v),e,\cdots,e)$, i.e. that it has only one non-constant component. This means that the geodesic polygon $\tilde{f}_s(y(v))$, for any v, has exactly one corner at parameter value t_{i_0}.

Let $\phi(v)$ stand for the length of the geodesic segment from $s(t_{i_0} - u') = \beta_{i_0-1}(u)$ to $y_{i_0}(v)\cdot s(t_{i_0} + u') = y_{i_0}(v)\cdot\alpha_{i_0}(u)$, which cuts across the corner of $\tilde{f}_s(y(v))$ to form $\tilde{f}_s{}^u(y(v))$. We clearly have $F(v) = L(s) - 2u\cdot\epsilon + \phi(v)$. The vector Y is not in the kernel of ψ_s, and is therefore not tangent to the submanifold $\psi_s{}^{-1}(\omega_s) = (K_s)^n$ of W_s. This means that $\dot{y}_{i_0}(0)$ is not tangent to K_s. Since $t_{i_0}+u'$ is not exceptional for s, this implies that the curve defined by $z(v)=y_{i_0}(v)\cdot s(t_{i_0}+u')$ has a non-zero tangent vector at $v=0$. The assertion $\phi''(0)<0$ reduces therefore to the following: Let a, b be the end points of a geodesic segment of length 2ϵ, whose midpoint c lies in $\bar{U}$; let $b(v)$ be a C^∞-curve such that (1) $b(0)=b$, (2) $b(v)$ lies on the ϵ-sphere around c, (3) $\dot{b}(0)\neq 0$; then for $d(v)$ defined as the distance from a to $b(v)$, the relation $d''(0)<0$ holds. This property of geodesic spheres (elementary for Euclidean space) is well known to hold, provided $a(U)$ is chosen sufficinetly small.

We consider now an arbitrary curve y, without the restriction made above. The vector $Y=(\dot{y}_1(0),\cdots,\dot{y}_n(0))$ is again not tangent to $(K_s)^n$ at $(e,\cdots,e)$, so that at least one of the $\dot{y}_i(0)$, say $\dot{y}_{i_0}$, *is* not tangent to the submanifold K_s of K_{i_0} at e. We put $y_0(v)=(e,\cdots,e,y_{i_0}(v),e,\cdots,e)$, and compare $\bar{f}_s{}^u(y(v))$ with $\bar{f}_s{}^u(y_0(v))$. The shortening process at the corner at t_{i_0} reduces both curves by the same amount from the original value $L(s)$. The latter curve has only this one corner, the former may have several corners; it follows that the former is not longer than the latter. Putting $F_0(v) = L\circ\bar{f}_s{}^u(y_0(v))$, we have therefore $F(v)\leqq F_0(v)$. The reasoning above applies to F_0, since y_0 has only one non-constant component. We have therefore $F_0''(0)<0$. In addition, we have $F(0)=F_0(0)=L(s)$, and $F'(0) = F_0'(0)=0$, as noted before. It is elementary that then $F''(0)\leqq F_0''(0)$, and so finally $F''(0)<0$. Propositions 11.1 and 10.2 are proved.

12. Proof of Theorem I completed. For each $s\in S$, choose a Euclidean simplex σ_s, with $\dim\sigma_s=\dim\Gamma_s$, and a regular C^∞-homeomorphism ρ_s of σ_s into Γ_s such that ρ_s sends the barycenter of σ_s into ω_s. Define $\phi_s\colon\sigma_s\to\Omega$ by $\phi_s=f_s{}^u\circ\rho_s$ for some positive u. By (9.3) and (10.2), the singular simplex (σ_s,ϕ_s) is associated to s in the sense of no. 8. Put $g_s=f_s{}^u$, considered as a map into Ω_i.

$$\begin{array}{ccccc} & & \Gamma_s & \xrightarrow{g_s} & \Omega_i \\ & & \downarrow{\scriptstyle j_s} & & \downarrow{\scriptstyle j_i} \\ (\sigma_s,\dot{\sigma}_s) & \xrightarrow{\rho_s} & (\Gamma_s,\Gamma_s-\{\omega_s\}) & \xrightarrow{g_s} & (\Omega_i,\Omega_i^-) \end{array}$$

We consider the commutative diagram above, where the vertical maps are simply inclusion maps. Let first $\mathbf{Z}_2$ be taken as coefficient group. It is clear by excision that the j_s-image of the fundamental cycle of Γ_s equals the ρ_s-image of the fundamental cycle of $(\sigma_s,\dot{\sigma}_s)$; with the notations of Theorem I and

(8.3) we have $j_{s*}(\bar{\gamma}_s) = \rho_{s*}(\bar{\xi}_s)$. Applying g_{s*} (with $g_{s*} \circ \rho_{s*} = \phi_{s*}$) and using the commutativity of the diagram, one concludes that $\phi_{s*}(\bar{\xi}_s) = j_{i*}(g_{s*}(\bar{\gamma}_s))$, so that $\phi_{s*}(\bar{\xi}_s)$ is contained in $j_{i*}(H_{\lambda_s}(\Omega_i; \boldsymbol{Z}_2))$. Since this holds for every s, it follows now from (8.3) that the completability condition 8.4 holds for $G = \boldsymbol{Z}_2$.

For the case of $\boldsymbol{G} = \boldsymbol{Z}$, assume that S is K-orientable, and let γ be an orientation of S, as defined prior to Theorem I. Again it is clear, by excision, that for a suitably chosen generator ξ_s of $H_{\lambda_s}(\sigma_s, \dot{\sigma}_s)$ the relation $j_{s*}(\gamma_s) = \rho_{s*}(\xi_s)$ holds. The rest of the argument goes as before, and shows that 8.4 holds.

Non-degeneracy of all s has been proved in (9.1) already. With non-degeneracy and completability following from the hypotheses of Theorem I, we can now apply Proposition 8.5 with $\bar{x}_s = g_{s*}(\bar{\gamma}_s)$, or, in the K-orientable case, with $x_s = g_s(\gamma_s)$. Theorem I follows directly if we remark that the image of $g_{s*}(\bar{\gamma}_s)$ under the inclusion $\Omega_i \subset \Omega$ is $f_{s*}{}^u(\bar{\gamma}_s)$, and that $f_{s*}{}^u$ is identical with f_{s*}, by the homotopy axiom; similarly for γ_s.

Chapter II. Variational Completeness in Symmetric Spaces.

1. Action of $\boldsymbol{K} \times \boldsymbol{K}$ and of $\boldsymbol{K}$. Let G be a compact connected Lie group, and let K be a closed connected subgroup, which then itself is also a Lie group. The direct product $K \times K$ acts from the left on G under the definition

$$(1.1) \qquad (a, b) \cdot c = a \cdot c \cdot b^{-1}$$

with $a, b \in K$, $c \in G$; this is an action in the sense of Chapter I, no. 1. We introduce in G a Riemannian metric invariant under left and right translations. It is well known that such metrics exist, and that the geodesics in such a metric are the 1-parameter subgroups and their cosets. In general, we shall follow the notation of [6]. But we write $a \cdot X$ or aX instead $l_a(X)$ for the left translate of the vector X by the element a of G. The adjoint action of G on its Lie algebra $\mathfrak{g}$ is then given by $Ada \cdot x = a \cdot X \cdot a^{-1}$. For the Lie product [,] and the invariant inner product or Killing form (,) in $\mathfrak{g}$ we have then the usual relations:

$$(1.2) \qquad \begin{aligned} &(a) \quad \mathrm{Ad}a \cdot [X, Y] = [\mathrm{Ad}a \cdot X, \mathrm{Ad}a \cdot X], \\ &(b) \quad (X, Y) = (\mathrm{Ad}a \cdot X, \mathrm{Ad}a \cdot X), \\ &(c) \quad ([X, Y], Z) + (Y, [X, Z]) = 0, \\ &(d) \quad d\,\mathrm{Ad}e^{tY} \cdot X/dt\,|_{t=0} = [Y, X] = \mathrm{ad}Y \cdot X. \end{aligned}$$

As usual, G/K denotes the homogeneous space formed by the left cosets xK, $x \in G$, under the decomposition topology, with the induced C^∞-manifold structure; the group G acts on the left on G/K by $y \cdot (xK) = (yx)K$. Let p denote the natural map of G on G/K. The Riemannian metric of G induces a Riemannian metric on G/K, invariant under the action of G: If X' is a vector at the point $x' \in G/K$, let X be a vector at any point x in $p^{-1}(x')$, orthogonal to the manifold xK and with $\dot{p}(X) = X'$; define the norm $|X'|$ of X' to be equal to the norm $|X|$ of X; the definition is independent of the choice of x involved. In particular, let $\mathfrak{p}$ denote the subspace of the Lie algebra $\mathfrak{g}$ orthogonal to the Lie algebra $\mathfrak{k}$; then $\dot{p}$ gives an isometry of $\mathfrak{p}$ with the tangent space of G/K at $p(e) = e'$. The group K acts on G/K, as subgroup of G. Since K is the stability group of e', it acts on the tangent space $(G/K)_{e'}$. It also acts on $\mathfrak{g}$ by the adjoint transformations $\mathrm{Ad}k$. The subspaces $\mathfrak{k}$ and $\mathfrak{p}$ of $\mathfrak{g}$ are invariant under $\mathrm{Ad}K$, and the action of K on $\mathfrak{p}$ so defined is equivalent, via the map $\dot{p}$, to the action of K on $(G/K)_{e'}$.

2. Symmetric spaces and variational completeness. We recall E. Cartan's concept of symmetric space [9].

DEFINITION 2.1. *The pair (G, K) is called a symmetric pair, if there exists an involution* $*$ *(i.e. automorphism of order two), of G, written $x \to x^*$, such that K is the e-component of the fixed group of the involution (i.e. of the group $\{x : x^* = x\}$).*

If (G, K) is a symmetric pair, then G/K is a symmetric space; actually, one can divide G by any closed subgroup of G between K and the fixed group of the involution.

In Chapter I, (8.4), we have defined variational completeness which forms the hypothesis of Theorem I. The following theorem provides cases were this hypothesis holds.

THEOREM II. *Let (G, K) be a symmetric pair; then the action of $K \times K$ on G, the action of K on G/K and the action of K on $\mathfrak{p}$, as defined in no.* 1, *are variationally complete.*

The proof of this theorem is a generalization of that of Proposition 7.1 in [6].

We begin with the action of $K \times K$ on G. Theorem II, for this case, takes the form of Proposition 2.2 below; note that the role of K in Chapter I is now taken by $K \times K$. (For the definitions of transversal geodesic, transversal Jacobi field and infinitesimal $K \times K$-motion see Chapter I, (2.1), (8.2) and no. 1.)

PROPOSITION 2.2. *Let g be a $K \times K$-transversal geodesic of G; let η be a transversal Jacobi field along g that vanishes for $t = 0$ and is tangent to the $K \times K$-orbit of $g(1)$ for $t = 1$; then η is induced by an infinitesimal $K \times K$-motion.*

For the proof, we shall describe, in Lemma 4.1, the space of all transversal J-fields vanishing for $t = 0$; in Lemma 5.1, we describe the space of all infinitesimal $K \times K$-motions; Proposition 2.2 will then follow from a comparison of the two spaces.

3. Tangent and transversal space of an orbit. Basic for the proof are the following relations in the Lie algebra $\mathfrak{g}$ of G, which are well known to be equivalent to the existence (locally) of the involution which defines our symmetric pair (G, K):

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}; \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}; \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}. \tag{3.1}$$

We also recall

$$\mathfrak{p} = \mathfrak{k}^{\perp} \text{ (orthogonal complement in } \mathfrak{g}). \tag{3.2}$$

Next we describe the tangent space of an orbit and its orthogonal complement.

DEFINITION 3.3. *For any $a \in G$, we put*

$$\mathfrak{k}^a = \mathfrak{k} + \mathrm{Ad}a^{-1} \cdot \mathfrak{k}; \quad \mathfrak{k}_a = \mathfrak{k} \cap \mathrm{Ad}a^{-1} \cdot \mathfrak{k}$$
$$\mathfrak{p}_a = \mathfrak{p} \cap \mathrm{Ad}a^{-1} \cdot \mathfrak{p}.$$

The relations (3.1) imply similar relations for the spaces defined in (3.3); we give the ones that we shall need later.

$$[\mathfrak{k}_a, \mathfrak{p}_a] \subset \mathfrak{p}_a; \quad [\mathfrak{p}_a, \mathfrak{p}_a] \subset \mathfrak{k}_a; \tag{3.4}$$

$$\mathfrak{p}_a = (\mathfrak{k}^a)^{\perp}. \tag{3.5}$$

(3.4) follows easily from (3.1) and the fact that $\mathrm{Ad}a^{-1}$ is an automorphism of $\mathfrak{g}$. Similarly, (3.5) follows from (3.2) and the fact that $\mathrm{Ad}a^{-1}$ is an isometry of $\mathfrak{g}$.

PROPOSITION 3.6. *The spaces $\mathfrak{k}^a$ and $\mathfrak{p}_a$ are the left translates, under a^{-1}, of the tangent space and the transversal space to the orbit $K \cdot a \cdot K$ at the point a.*

Proof. The tangent space to $K \cdot a \cdot K$ at a is clearly $\mathfrak{k} \cdot a + a \cdot \mathfrak{k}$. Applying

a^{-1} on the left, one gets $\mathfrak{k}^a$. The assertion for $\mathfrak{p}_a$ follows from (3.5) and the isometric character of left translation.

The space $\mathfrak{p}_a$ admits a decomposition relative to any of its elements, which we derive now.

DEFINITION 3.7. *For any $X \in \mathfrak{g}$, the centralizer $\mathfrak{g}_X$ (a subalgebra of $\mathfrak{g}$) consists of the elements $Z \in \mathfrak{g}$ with $[X, Z] = 0$.*

LEMMA 3.8. *For any $X \in \mathfrak{p}_a$, the space $\mathfrak{p}_a$ admits the orthogonal direct sum decomposition*

$$\mathfrak{p}_a = \mathrm{ad}X(\mathfrak{k}_a) \oplus (\mathfrak{p}_a \cap \mathfrak{g}_X).$$

Proof. By (3.4), $\mathrm{ad}X(\mathfrak{k}_a)$ is a subspace of $\mathfrak{p}_a$. Take Z in $\mathfrak{p}_a$, orthogonal to this subspace. Then, for any $Y \in \mathfrak{k}_a$, we have $0 = ([X, Y], Z) = -(Y, [X, Z])$. Since $[X, Z]$ belongs to $\mathfrak{k}_a$ by (3.4) and since Y is arbitrary in $\mathfrak{k}_a$, we have $[X, Z] = 0$, q. e. d.

4. Transversal Jacobi fields.

Let now g be a transversal geodesic. We put $g(0) = a$; then g is of the form $a \cdot e^{tX}$, where e^{tX} is the 1-parameter group with initial vector X. By (3.6), the vector X belongs to $\mathfrak{p}_a$.

LEMMA 4.1. *Any transversal Jacobi field η along g with $\eta(0) = 0$ can be written as*

$$\eta(t) = a \cdot e^{tX} \cdot (\mathrm{Ad}e^{-tX} \cdot Y - Y + tZ) \quad \textit{with } Y \in \mathfrak{k}_a, Z \in \mathfrak{p}_a \cap \mathfrak{g}_X.$$

Proof. There are two obvious ways of varying g, with $g(0)$ fixed: conjugation of e^{tX} by elements y of G and multiplication of e^{tX} by e^{tZ} with any Z that commutes with X. To keep the geodesics transversal, we restrict y to $K \cap a^{-1} \cdot K \cdot a$ and Z to $\mathfrak{p}_a$. We shall show that with these two processes, one can produce all transversal variations with $g(0)$ fixed, and the associated J-fields are of the form given in 4.1.

Let then Y be any element of $\mathfrak{k}_a$, and Z any element of $\mathfrak{p}_a \cap \mathfrak{g}_X$. We consider the geodesic $g_{Y,Z}$, given by $a \cdot e^Y \cdot e^{t(X+Z)} \cdot e^{-Y}$, with $g_{Y,Z}(0) = a$, and with initial tangent vector $a \cdot \mathrm{Ad}e^Y \cdot (X + Z)$. Since $X \in \mathfrak{p}_a$, and $e^Y \in K \cap a \cdot K \cdot a^{-1}$ (because of $Y \in \mathfrak{k} \cap \mathrm{Ad}a^{-1} \cdot \mathfrak{k}$) this vector belongs to $a \cdot \mathfrak{p}_a$, and $g_{Y,Z}$ is transversal. Assigning $a \cdot \mathrm{Ad}e^Y \cdot (X + Z)$ to the pair (Y, Z) constitutes a map ϕ of $\mathfrak{k}_a \oplus (\mathfrak{p}_a \cap \mathfrak{g}_X)$ into $a \cdot \mathfrak{p}_a$, with $\phi(0,0) = a \cdot X$. Using (1.2(d)), one derives that the differential $\dot{\phi}_0$ at $(0,0)$ maps the pair (Y, Z) into $a \cdot [Y, X] + a \cdot Z$ (we have made use of the customary identification for tangent spaces to vector spaces).

Lemma 3.8 implies that $\dot{\phi}_0$ is a map onto $a \cdot \mathfrak{p}_a$. From the remarks on

J-fields in Chapter I, no. 6, taking $a\cdot\mathfrak{p}_a$ as the space W there, one concludes now that the map which to the vector (Y,Z) assigns the J-field of the variation $a\cdot e^{\alpha Y}\cdot e^{t(X+\alpha Z)}e^{-\alpha Y}$ is a map of $\mathfrak{k}_a\oplus(\mathfrak{p}_a\cap\mathfrak{g}_X)$ onto the space of transversal J-fields vanishing for $t=0$. One verifies by straight forward differentiating (using $[X,Z]=0$) that this J-field is of the type described in (4.1), and so (4.1) is proved.

5. The infinitesimal motions. We first prove

LEMMA 5.1. *The infinitesimal $K\times K$-motions, which vanish at a, restrict along g to the vector fields of the form* $a\cdot e^{tX}\cdot(\mathrm{Ad}e^{-tX}\cdot V-V)$ *with* $V\in\mathfrak{k}_a$.

Proof. All 1-parameter $K\times K$-motions are clearly of the form $y\to e^{\alpha U}\cdot y\cdot e^{-\alpha V}$, with $U,V\in\mathfrak{k}$. The corresponding infinitesimal motion assigns to y the vector $U\cdot y-y\cdot V$. Such a field will vanish at a, if $U\cdot a-a\cdot V=0$, i.e., if $U=\mathrm{Ad}a\cdot V$. In other words, the infinitesimal motions that vanish at a are of the form $(\mathrm{Ad}a\cdot V)y-y\cdot V$, with $V\in\mathfrak{k}$ and $\mathrm{Ad}a\cdot V\in\mathfrak{k}$, i.e with $V\in\mathfrak{k}_a$. For $y=a\cdot e^{tX}$, along g, this becomes $a\cdot e^{tX}\cdot(\mathrm{Ad}e^{-tX}\cdot V-V)$, q.e.d.

We come now to the proof of Proposition 2.2. Let η be as described there, and let Y, Z be the vectors associated with η by (4.1). The assumption that $\eta(1)$ is tangent to the orbit of $b=a\cdot e^X$ at b means, by (3.6),

$$b^{-1}\cdot\eta(1)=\mathrm{Ad}e^{-X}\cdot Y-Y+Z\in\mathfrak{k}^b. \tag{5.2}$$

Any vector B in $\mathfrak{k}^b$ can be written as B_1+B_2 with $B_1\in\mathfrak{k}$ and $B_2\in\mathrm{Ad}b^{-1}\cdot\mathfrak{k}$; we can write B_2 as $\mathrm{Ad}e^{-X}\cdot B_2'$ with $B_2'\in\mathrm{Ad}a^{-1}\cdot\mathfrak{k}$, since $b=a\cdot e^X$. By (3.5), Z is orthogonal to $\mathfrak{k}^a$, so that the inner products (Z,B_1) and (Z,B_2') vanish. We have $\mathrm{Ad}e^X\cdot Z=Z$, since $[X,Z]=0$. It follows that

$$(Z,B_2)=(\mathrm{Ad}e^X\cdot Z,\mathrm{Ad}e^X\cdot B_2)=(Z,B_2')=0.$$

This means that $(Z,B)=0$ and Z is orthogonal to $\mathfrak{k}^b$. From (5.2), we have then $0=(\mathrm{Ad}e^{-X}\cdot Y-Y+Z,Z)=(\mathrm{Ad}e^{-X}\cdot Y,Z)-(Y,Z)+(Z,Z)$. Since $(\mathrm{Ad}e^{-X}\cdot Y,Z)=(Y,Z)$, we conclude $(Z,Z)=0$, and so $Z=0$. But then η is given by $a\cdot e^{tX}\cdot(\mathrm{Ad}e^{-tX}\cdot Y-Y)$, and this is, by (5.1), an infinitesimal motion, since $Y\in\mathfrak{k}_a$.

6. *K* operating on *G/K*. We consider now Theorem II for the action of K on G/K. We will reduce this case, with the help of the projection $p\colon G\to G/K$, to the previous case of $K\times K$ acting on G. It is well known and easily proved that for the Riemannian metric in G/K. as defined in no. 1, the geodesics in G/K are exactly the projections of those geodesics in G that are orthogonal to the left cosets of K (i. e. to the geodesics which are transversal with respect to the action of K on G, defined by multiplying on the right). For any $a \in G$, the projection of the $K \times K$-orbit $K \cdot a \cdot K$ is the K-orbit $K \cdot a'$, with $a' = p(a)$. The space $a \cdot \mathfrak{p}_a$, the orthogonal complement of the tangent space to $K \cdot a \cdot K$ at a in the tangent space to G at a, is mapped by $\dot{p}$ isometrically onto the orthogonal complement of the tangent space at a' of the orbit $K \cdot a'$. It follows that the $K \times K$-transversal geodesics at a map onto the K-transversal geodesics at a' (in a 1-1 way), and that for any such geodesic at a, the space of transversal Jacobi fields that vanish at a maps onto the space of transversal J-fields along the projected geodesic that vanish at a'; moreover, any such J-field is tangent to the $K \times K$-orbit at a point b, if and only if the projected J-field is tangent to the K-orbit at $b' = p(b)$. In other words, p establishes an isomorphism between $\Lambda_g(0) \cap J_g{}^{\pi}(1)$ and $\Lambda_{g'}(0) \cap J_{g'}{}^{\pi}(1)$ for any transversal geodesic g in G and its projection g' in G/K. Finally, the projection is equivariant with respect to the natural map of $K \times K$ onto its first factor; i. e. $p((k_1, k_2) \cdot a) = k_1 \cdot p(a)$. It follows easily that any infinitesimal $K \times K$-motion along g that vanishes at a projects into an infinitesimal K-motion along g' that vanishes at a'. Since all elements in $\Lambda_g(0) \cap J_g{}^{\pi}(1)$ are induced by $K \times K$, it follows now that all elements in $\Lambda_{g'}(0) \cap J_{g'}{}^{\pi}(1)$ are induced by K, q. e. d.

7. The infinitesimal case. We come to the adjoint action of K on $\mathfrak{p}$, the infinitesimal analogue of the case just treated. We give a proof analogous to that of Proposition 2. 2; another possible approach consists in comparing $\mathfrak{p}$ with a small K-invariant neighborhood of e' in G/K. Let A be any point in $\mathfrak{p}$.

PROPOSITION 7. 1. *The tangent space to the orbit of A under the adjoint action of K is* $\mathrm{ad}A(\mathfrak{k})$.

This follows from the relation (1. 2(d)).

DEFINITION 7. 2. $\mathfrak{k}_A = \mathfrak{k} \cap \mathfrak{g}_A$; $\mathfrak{p}_A = \mathfrak{p} \cap \mathfrak{g}_A$ (*here* $\mathfrak{g}_A$ *is the centralizer of* A).

We have the relations:

$$(7.3) \qquad [\mathfrak{k}_A, \mathfrak{p}_A] \subset \mathfrak{p}_A; \quad [\mathfrak{p}_A, \mathfrak{p}_A] \subset \mathfrak{k}_A;$$

$$(7.4) \qquad \mathfrak{p}_A = \mathrm{ad}A(\mathfrak{k})^{\perp} \text{ (orthogonal complement in } \mathfrak{p}).$$

For the proof of (7. 3), we use (3. 1) and note that by the Jacobi identity, the left sides are contained in $\mathfrak{g}_A$. To prove (7.3), take $P \in \mathfrak{p}$ and $Q \in \mathfrak{k}$. The relation $(P, [A, Q]) = -([A, P], Q)$ together with $[A, P] \in \mathfrak{k}$ (by (3.1)) implies the stated result. Note that $A \in \mathfrak{p}_A$.

Recall that the geodesics in $\mathfrak{p}$ are simply the straight lines. Let X by any element of $\mathfrak{p}_A$ and define the geodesic g by $g(t) = A + tX$; with X varying through $\mathfrak{p}_A$, this represents all the K-transversal geodesics starting at A.

LEMMA 7.5. $\mathfrak{p}_A = \mathrm{ad}X(\mathfrak{k}_A) \oplus (\mathfrak{p}_A \cap \mathfrak{g}_X)$ *(orthogonal decomposition).* The proof is analogous to that of (3.8) with (7.3), (7.4) replacing (3.4), (3.5).

LEMMA 7.6. *Any transversal J-field η along g with $\eta(0) = 0$ can be written as $\eta(t) = t(E_1 + E_2)$ with $E_1 \in \mathrm{ad}X(\mathfrak{k}_A)$, $E_2 \in \mathfrak{p}_A \cap \mathfrak{g}_X$.*

Proof. The transversal geodesics are the straight lines of the form $A + tC$, with $C \in \mathfrak{p}_A$, by 7.4. For the study of transversal Jacobi fields vanishing at A, it is clearly sufficient to consider linear variations, i.e. families $\{V_\alpha\}$ with $V_\alpha(t) = A + t \cdot (X + \alpha C)$ for $C \in \mathfrak{p}_A$. The Jacobi field of such a variation is given by tC. Splitting C according to Lemma 7.5, we get Lemma 7.6.

LEMMA 7.7. *The infinitesimal K-motions along g, vanishing at A, are exactly the fields of the form $t[Y, X]$ with $Y \in \mathfrak{k}_A$.*

Proof. The infinitesimal K-motion in $\mathfrak{p}$, determined by the 1-parameter group $\mathrm{Ad}e^{\alpha Y}$, assigns to the point S of $\mathfrak{p}$ the vector $[Y, S]$. Such a field vanishes at A exactly if $Y \in \mathfrak{k}_A$, and the field along g, i.e. for $S = A + tX$, is then $t[Y, X]$.

We prove now that variational completeness holds.

PROPOSITION 7.8. *Let η be a transversal Jacobi field along g with $\eta(0) = 0$ that is tangent to the K-orbit at $B = A + X$ (i.e., for $t = 1$); then η is induced by an infinitesimal K-motion.*

Decompose $\eta(t)$ into $t[X, Y] + tZ$ with $Y \in \mathfrak{k}_A$, $Z \in \mathfrak{p}_A \cap \mathfrak{g}_X$ by Lemma 7.6. To say that $\eta(1)$ is tangent to the orbit is to say that $[X, Y] + Z \in \mathrm{ad}(A + X)(\mathfrak{k})$, by (7.1). We have $([X, Y], Z) = -(Y, [X, Z]) = 0$ because of $[X, Z] = 0$. Take any $D \in \mathfrak{k}$ and form $([A + X, D], Z)$. This equals $-(D, [A, Z]) - (D, [X, Z])$. Since $[A, Z] = [X, Z] = 0$, we conclude that Z is orthogonal to $\mathrm{ad}(A + X)(\mathfrak{k})$. It follows now that $(Z, Z) = 0$, and so $Z = 0$. The proof of Theorem II is complete.

8. G as symmetric space. A well known case of symmetric pairs is obtained by starting with any compact connected Lie group G, forming $G \times G$, and defining an involution in $G \times G$ by $(x,y)^* = (y,x)$. The fixed group is then the diagonal Δ, isomorphic to G under $x \to (x,x)$. The quotient space $(G \times G)/\Delta$ can be identified with G through the map $(x,y) \to x \cdot y^{-1}$. The action of Δ on $(G \times G)/\Delta$ goes over into the action of G on itself by inner automorphisms. Similarly, the adjoint action of Δ on the tangent space of $(G \times G)/\Delta$ at the image of (e,e) (or on the subspace $\{(X, -X) : X \in \mathfrak{g}\}$ of $\mathfrak{g} \oplus \mathfrak{g}$) goes over into the adjoint action of G on $\mathfrak{g}$. Theorem II shows that both these actions are variationally complete; these two cases constitute theorems proved in [6], of which our Theorem II is then a generalization.

Chapter III. Applications to Certain Homogeneous Spaces and Loop Spaces.

1. The diagram of K, review of notation. Before stating, in no. 2, our main results, we collect here some facts about Lie groups (cf. [6], [20]).

K is again a compact connected Lie group, with Lie algebra $\mathfrak{k}$; in this chapter, we assume K semi-simple and simply connected. T is a maximal torus in K, with Lie algebra $\mathfrak{t}$; $\dim T = \operatorname{rank} K = l$. On $\mathfrak{k}$ we have the Killing form $(\ ,\ )$, invariant under the adjoint maps $\mathrm{Ad}x$, for all x in K. The orthogonal complement of $\mathfrak{t}$ in $\mathfrak{k}$ is denoted by $\mathfrak{m}$; it splits into m planes of dimension 2, invariant and irreducible under $\mathrm{Ad}T$. If $\mathfrak{e}$ is such a plane with a definite orientation, then there exists a unique linear function $\theta \colon \mathfrak{t} \to \boldsymbol{R}$, the root belonging to $\mathfrak{e}$, such that $\mathrm{Ad}e^X$, for any $X \in \mathfrak{t}$, operates on $\mathfrak{e}$ as rotation through the angle $2\pi\theta(X)$. It is possible to make a simultaneous choice of orientations for the m planes in such a way that among the corresponding roots θ there are l "fundamental" roots with the property that every θ is a non-negative integral linear combination of the fundamental ones. We assume such a choice made, and call $\mathcal{R} = \{\theta\}$ the corresponding set of m roots. The dominant root, with all coefficients maximal, we denote by μ. We introduce the set $\mathfrak{P} = \{p\}$ of oriented singular planes in $\mathfrak{t}$; such a plane is by definition a pair (θ, n) of a root θ in $\mathcal{R}$ and an integer n; for any such pair p, we write p also for the set $\{X \in \mathfrak{t} \colon \theta(X) = n\}$ and $\bar{p}$ for the image $\exp(p)$ under the exponential map into T or K. Each θ in $\mathcal{R}$ defines a basic translation τ_θ, the element of $\mathfrak{t}$ that is orthogonal to the plane $(\theta, 0)$ and satisfies $\theta(\tau_\theta) = 2$. We write $\mathcal{T}$ for the lattice in $\mathfrak{t}$ that the τ_θ generate; $\mathcal{T}$ is identical with the kernel of the exponential map of $\mathfrak{t}$ into T. For any $p = (\theta, n) \in \mathfrak{P}$, we write $\theta_p = \theta$, $n_p = n$ and $\tau_p = \tau_\theta$. The Cartan integers are defined as the values

$$\theta_p(\tau_q), \text{ for } p, q \in \mathfrak{P}; \tag{1.1}$$

only $0, \pm 1, \pm 2, \pm 3$ occur.

The diagram $D(K)$ is the union of all p in $\mathfrak{P}$; the infinitesimal diagram $D'(K)$ is the union of all $p \in \mathfrak{P}$ with $n_p = 0$. The components of $\mathfrak{t} - D(K)$ are the cells, Δ; the components of $\mathfrak{t} - D'(K)$ are the chambers. We write $\mathfrak{F}$ for the fundamental chamber in which all $\theta \in \mathfrak{R}$ take positive values, and $\Delta_{\mathfrak{F}}$ for the cell in $\mathfrak{F}$ that has the origin 0 in its closure.

Let N_T be the normalizer, in K, of T: The Weyl group $\mathfrak{W}(K)$ or $\mathfrak{W}$ is the (finite) factor group N_T/T. It operates on $\mathfrak{t}$, via the adjoint map, as a group of orthogonal transformations, generated by the reflections across the planes of $D'(K)$; it also operates on T and on K/T (cf. [1]).

2. The main results. The proofs of the results stated here appear in nos. 3-15.

Let s be a line segment in $\mathfrak{t}$, from a point R interior to the fundamental chamber $\mathfrak{F}$ to a point R' interior to the chamber $-\mathfrak{F}$, such that no point of s lies on two or more of the planes of $D'(K)$. Let the roots in $\mathfrak{R}$ be numbered θ_i, $1 \leqq i \leqq m$, in such a way that s meets the planes $(\theta_i, 0)$ in order of increasing index from R to R'. Write τ_i for τ_{θ_i}. Define a graded algebra A_K by

$$A_K = \mathbf{Z}[x_1, \cdots, x_m]/I_m, \tag{2.1}$$

where $\mathbf{Z}[x_1, \cdots, x_m]$ is the polynomial ring over $\mathbf{Z}$ in the variables $x_1, \cdots, x_m$, each x_i of dimension 2, and where I_m is the ideal generated by the elements

$$\rho_k = x_k \cdot \Big(x_k + \sum_1^{k-1} \theta_k(\tau_i) x_i\Big), \qquad 1 \leqq k \leqq m; \tag{2.2}$$

we denote the image of x_i in A_K again by x_i.

Denote by $\mathfrak{I}$ the module of integral forms on $\mathfrak{t}$, i.e. the set of those linear functions on $\mathfrak{t}$ that take integal values at all points of the lattice $\mathfrak{J}$. Finally, let B be the submodule of the linear (two-dimensional) part of A_K formed by the elements of the form $\sum \phi(\tau_i) x_i$ with ϕ ranging over $\mathfrak{I}$.

THEOREM III. *The integral cohomology ring $H^*(K/T)$ of the coset space K/T can be identified with the smallest subring of A_K that* (1) *contains the unit and the module B, and* (2) *is additively a direct summand. In particular, $H^2(K/T)$ is identified with B.*

It is easily seen that B can also be described as follows: Let $\{\zeta_1, \cdots, \zeta_l\}$ be a basis of the lattice $\mathfrak{J}$, and write $\tau_i = \sum a_{ij}\zeta_j$, $1 \leqq i \leqq m$. Then B has as a basis the elements $z_j = \sum a_{ij} x_i$, $1 \leqq j \leqq l$.

We note that this theorem describes the integral cohomology of K/T in terms of infinitesimal invariants of K, namely the Cartan integers and what amounts to the integral relations between the basic translations τ_i (for $\phi \in \mathfrak{J}$, the values $\phi(\tau_i)$ satisfy $\sum a_i\phi(\tau_i)=0$ whenever $\sum a_i\tau_i=0$ with real, necessarily rational, a_i).

As an example, we compute G_2/T for the exceptional group G_2. The results III and III′ were announced in [7].

THEOREM III′. *The cohomology ring $H^*(G_2/T)$ contains two elements α, β of dimension 2 such that $H^*(G_2/T)$ has an (additive) basis consisting of 1, α, β, $\alpha\cdot\beta$, β^2, $\alpha\cdot\beta^2$, $\beta^3/2$, $\alpha\cdot\beta^3/2$, $\beta^4/2$, $\alpha\cdot\beta^4/2$, $\beta^5/2$, $\alpha\cdot\beta^5/2$; moreover, the relations $\alpha^2+3\alpha\cdot\beta+3\beta^2=0$, $\alpha^6=\beta^6=0$ hold.*

Here $\beta^3/2$ means of course an element x such that $2x=\beta^3$.

For any group K, one has the natural action of the Weyl group $\mathfrak{W}$ on K/T and on $H^*(K/T)$. In the description of $H^*(K/T)$, given by Theorem III, this action takes the following form: The operations of $\mathfrak{W}$ on $\mathfrak{t}$ map the lattice $\mathcal{J}$ into itself. Consequently, there is an induced operation on the module $\mathfrak{J}$ of integral linear forms introduced before Theorem III. Finally, $\mathfrak{W}$ acts on the submodule B of A_K, defined there, since B is naturally isomorphic to $\mathfrak{J}$.

PROPOSITION 2.1. *Under the identification of B and $H^2(K/T)$ of Theorem III, the action of $\mathfrak{W}$ on B becomes identical with the action of $\mathfrak{W}$ on $H^2(K/T)$ derived from the action of $\mathfrak{W}$ on K/T. The action on B extends uniquely to the action on $H^*(K/T)$, the latter considered as a subalgebra of A_K.*

In terms of the bases $\{\zeta_i\}$, $\{z_i\}$ above, we express the image $w\cdot\zeta_i$ of ζ_i under the action of $w\in\mathfrak{W}$ as $\sum b_{ij}\zeta_j$; then the relation $w\cdot z_j=\sum b_{ij}z_i$ defines the operation of w on $H^2(K/T)$.

Let now T' be any non-maximal torus of K, and let $K_{T'}$ (denoted by $C(T')$ in [6]) be the centralizer of T' (a connected group by ([10]). We may assume $T'\subset T$; the latter is then a maximal torus of $K_{T'}$, and the Weyl group $\mathfrak{W}'=\mathfrak{W}(K_{T'})$ is a subgroup of $\mathfrak{W}(K)$.

THEOREM III″. *The cohomology ring $H^*(K/K_{T'})$ is isomorphic to the subalgebra of $H^*(K/T)$, consisting of the invariants of $\mathfrak{W}'$ in $H^*(K/T)$.* (The action of $\mathfrak{W}$, and therefore also of $\mathfrak{W}'$, on $H^*(K/T)$ has been described in Proposition 2.1.).

We note that Theorem III, Proposition 2.1 and Theorem III″ together yield a description of $H^*(K/K_{T'})$ in terms of the Cartan integers, the relations between the basic translations, and the action of the Weyl group of $K_{T'}$ on the basic translations.

The following theorems concern the homotopy groups π_i of Lie groups. It is a classical fact that $\pi_2(K)=0$ for semi-simple K; and it is known [6] that $\pi_3(K)=\mathbf{Z}$ for simple K. We use the standard notation A_n, B_n, C_n, D_n for the classical groups, and G_2, F_4, E_6, E_7, E_8 for the exceptional groups.

THEOREM IV. *Let K be simple and let μ be its dominant root.*

(a) $\pi_4(K)$ *is either* 0 *or* $\mathbf{Z}_2$; *it is* 0 *exactly if the plane* $(\mu,1)$ *with equation* $\mu(X)=1$ *in* $\dot{\mathfrak{t}}$ *contains a point of the basic lattice* $\mathfrak{I}$.

(b) *The groups* A_1, B_1, *and* C_n, $n\geqq 1$, *have* $\pi_4=\mathbf{Z}_2$, *all other simple groups have* $\pi_4=0$.

THEOREM V.

(a) $\pi_i(E_6)=0$ for $4\leqq i\leqq 8$; $\pi_9(E_6)=\mathbf{Z}$;

(b) $\pi_i(E_7)=0$ for $4\leqq i\leqq 10$; $\pi_{11}(E_7)=\mathbf{Z}$;

(c) $\pi_i(E_8)=0$ for $4\leqq i\leqq 14$; $\pi_{15}(E_8)=\mathbf{Z}$;

(d) $\pi_{10}(G_2)\otimes\mathbf{Z}_3=0$.

Part (d) is a correction of the value given in Toda's table in [4]; it is in agreement with recent calculations of Toda, and also with recent results of A. Borel-F. Hirzebruch (not yet published).

3. The groups K_p. The proofs of the theorems of no. 2 depend on the construction of certain iterated 2-sphere bundles that appear as K-cycles for suitable actions of K.

For each root $\theta\in\mathfrak{R}$, let K_θ be the stabilizer of the singular plane $(\theta,0)$ of $\mathfrak{t}$, under the adjoint action of K on $\mathfrak{k}$; denote by $\mathcal{K}$ the set of K_θ's. For any oriented singular plane $p\in\mathfrak{P}$, we write K_p for the centralizer of $\bar{p}$ (stabilizer under the action of K on itself by inner automorphisms). It is familiar that for $p=(\theta,n)$, we have $K_p=K_\theta$, independent of n, and that K_p can be described as follows:

Let S_3 denote the special unitary group $SU(2)$ (or equivalently, the group of quaternions of norm 1), topologically a 3-sphere; let S_1 denote the circle group contained in S_3, consisting of the ordinary complex numbers $\cos\phi+i\sin\phi$ in the quaternion form of S_3. For each $p\in\mathfrak{P}$, there exists a monomorphism

(3.1) $$\beta_p\colon S_3\to K_p$$

depending only on θ_p and not on n_p, such that T and $\beta_p(S_3)$ generate K_p and such that $\beta_p(S_3) \cap T = \beta_p(S_1)$, cf. [11]. (That β_p is a monomorphism depends on the fact that the element $\frac{1}{2} \cdot \tau_p$ does not belong to the lattice $\mathfrak{J}$, and that therefore $\beta_p(-1) \neq e$.) S_3 is a principal bundle for its subgroup S_1, with base space S_2, the 2-sphere (Hopf fibering). Similarly, K_p is a principal T-bundle; β_p is a bundle map, and induces a homeomorphism β_p of the base spaces; we note explicitly that every K_p/T is a 2-sphere.

Let j denote the quaternion unit usually denoted by that letter, and put $j_p = \beta_p(j) \in K_p$. The inner automorphism of K by j_p maps T into itself; its restriction to T is an involution which we denote by R_p. Similarly, $\mathrm{Ad} j_p$ maps $\mathfrak{t}$ into itself; its restriction to $\mathfrak{t}$ is also an involution, again denoted by R_p, and given by the formula

$$(3.2) \qquad R_p(X) = X - \theta_p(X)\tau_p, \text{ for } X \in \mathfrak{t}$$

(reflection across the plane $(\theta_p, 0)$). The R_p's generate the Weyl group.

Using the standard isomorphism of $\pi_1(T)$ and $H_1(T)$, and the identification of $\pi_1(T)$ with the group $\mathfrak{J}$ of covering translations of the universal covering space $\mathfrak{t}$ of T, we associate to each basic translation τ_p or τ_θ a homology class $\boldsymbol{\tau}_p$ or $\boldsymbol{\tau}_\theta$ in $H_1(T)$. For the homology map, induced by R_p, we have then from (3.2),

$$(3.3) \qquad R_{p*}(\boldsymbol{\tau}_q) = \boldsymbol{\tau}_q - \theta_p(\tau_q)\boldsymbol{\tau}_p.$$

In particular, on S_1 we have a standard generator $\boldsymbol{\tau}$ of $H_1(S_1)$, such that for the $\beta_p\colon S_1 \to T$, associated with any $p \in P$,

$$(3.4) \qquad \beta_{p*}(\boldsymbol{\tau}) = \boldsymbol{\tau}_p \in H_1(T).$$

4. The manifolds Γ_P. Let $P = \{p_1, \cdots, p_r\}$ be an ordered finite sequence of oriented singular planes in $\mathfrak{P}$. Using definition 4.2 of Chapter I, we put

$$(4.1) \qquad \Gamma_P = K_{p_1} \times_T K_{p_2} \times_T \cdots \times_T K_{p_{r-1}} \times_T K_{p_r}/T.$$

Our aim is to compute the cohomology of Γ_P. We abbreviate K_{p_i} to K_i, and similarly, θ_{p_i} to θ_i, etc. The polynomial ring $\mathbf{Z}[x_1, \cdots, x_r]$ will be considered as a graded ring, the variables having dimension 2.

PROPOSITION 4.2. *The cohomology ring $H^*(\Gamma_p)$ is isomorphic to $\mathbf{Z}[x_1, \cdots, x_r]/I_P$, where I_P is the ideal generated by the elements*

$$\rho_k = x_k \cdot (x_k + \sum_1^{k-1} \theta_k(\tau_i) x_i), \qquad 1 \leq k \leq r.$$

In particular, $H^*(\Gamma_P)$ *has no torsion and is generated by 2-dimensional elements.*

The proof will occupy nos. 4 and 5. The proposition is obvious for $r=1$, since then Γ_P is a 2-sphere (cf. no. 3). We proceed by an inductive construction, and take $r>1$. As in Chapter I, (4.6), we put

$$W_P = K_1 \times \cdots \times K_r. \tag{4.3}$$

The space Γ_P is then also the base space of the principal action of $T^r = T \times \cdots \times T$ (r factors) on W_P, given by

$$(k_1,\cdots,k_r)\cdot(t_1,\cdots,t_r) = (k_1t_1, t_1^{-1}k_2t_2,\cdots,t_{r-1}^{-1}k_rt_r),$$

where $k_i \in K_i$, and $t_i \in T$; let ψ be the projection of W_P onto Γ_P.

Put $P' = \{p_1,\cdots,p_{r-1}\}$, defining $\Gamma_{P'}$, $W_{P'}$ and ψ'. Let $\tilde{\phi}\colon W_P \to W_{P'}$ be defined by suppressing the last coordinate. This is a bundle map, relative to the homomorphism $\phi\colon T^r \to T^{r-1}$, defined similarly, and induces a map $\bar{\phi}\colon \Gamma_P \to \Gamma_{P'}$. Writing T^r as $T^{r-1}\times T$, we can factor the projection ψ by first operating on W_P with the factor T, which results in $W_{P'} \times K_r/T$, and then operating with T^{r-1}. One reads off from the basic diagram of Chapter I, (3.1) that Γ_P is the K_r/T-bundle over $\Gamma_{P'}$, associated to the action of T^{r-1} on K/T, given by $(t_1,\cdots,t_{r-1})\cdot k_rT = t_{r-1}k_rT$. Let $\tilde{s}$ be the natural injection of $W_{P'}$ into $W_P = W_{P'} \times K_r$. One verifies that this is a bundle map, relative to the natural injection of T^{r-1} into $T^r = T^{r-1}\times T$; denote by $\bar{s}$ the induced map of $\Gamma_{P'}$ into Γ_P. Since $\tilde{\phi}\circ\tilde{s} =$ identity, the map $\bar{s}$ is a cross section of the bundle map $\bar{\phi}$.

We now set up suitable bases for homology and cohomology. Let $\tilde{\chi}_i$, for $1 \leqq i \leqq r$, denote the map of S_3 into W_P, defined by the β_i (cf. (3.1)) of S_3 into the factor K_i of W_P. This is a bundle map relative to the homomorphism $\chi_i\colon S_1 \to T$, given by

$$\chi_i(t) = (e,\cdots,e,\beta_i(t),\cdots,\beta_i(t)), \text{ with } i-1\ e\text{'s}. \tag{4.4}$$

Let $\bar{\chi}_i$ be the induced map of S_2 into Γ_P.

We shall apply the transgression map $\partial\colon H_2(\text{base}) \to H_1$ (fiber), for the various bundles under consideration; in all cases occurring here, ∂ can be interpreted, via the Hurewicz isomorphism, as the boundary map (dimension 2) of the homotopy sequence of the bundle. We define a generator y of $H_2(S_2)$ by $\partial y = \tau$, where τ is the standard generator of $H_1(T)$ introduced above (cf. (3.4)), and define elements y_i of $H_2(\Gamma_P)$ by

$$y_i = \bar{\chi}_{i*}(y), \qquad 1 \leqq i \leqq r. \tag{4.5}$$

One verifies immediately that the elements y_i, $1 \leqq i \leqq r-1$, map under $\bar{\phi}_*$ into the similarly defined elements y_i', $1 \leqq i \leqq r-1$, of $H_2(\Gamma_{P'})$, that y_r represents a generator of the second homology group of the fiber K_r/T of $\bar{\phi}$, so that $\bar{\phi}_*(y_r)=0$, and that the y_i', $1 \leqq i \leqq r-1$, map into y_i, $1 \leqq i \leqq r-1$, under the cross section $\bar{s}$.

One concludes now inductively by a simple application of the Gysin sequence (for 2-sphere bundles with cross section) (cf. [18]) that the following proposition holds.

PROPOSITION 4.6. *$H^*(\Gamma_P)$ has no torsion (and Γ_P is simply connected). The elements y_i, $1 \leqq i \leqq r$, defined in (4.5), form a basis for $H_2(\Gamma_P)$. If ξ is any element of $H^2(\Gamma_P)$ whose restriction to the fiber K_r/T of $\bar{\phi}$ is a generator of $H^2(K_r/T)$, then the map $\bar{\phi}^*$, and the map consisting of $\bar{\phi}^*$ followed by cup-product with ξ are isomorphisms into, and define a direct sum decomposition*

$$H^*(\Gamma_P)=\bar{\phi}^*H^*(\Gamma_{P'}) \oplus \xi \cdot \bar{\phi}^*H^*(\Gamma_{P'}).$$

Specifically, the ring structure of $H^*(\Gamma_P)$ is determined by $H^*(\Gamma_{P'})$ and the expression for ξ^2 in this above decomposition.

Let $\{x_i\}_1{}^r$ be the basis of $H^2(\Gamma_P)$ dual to the basis $\{y_i\}$ of $H_2(\Gamma_P)$ introduced above, so that

$$x_i(y_j)=\delta_{ij}, \qquad 1 \leqq i, j \leqq r. \tag{4.7}$$

The element x_r can then serve as ξ in 4.6. From the properties of the y_i's, it follows at once that the elements x_i, $1 \leqq i \leqq r-1$, are the images under $\bar{\phi}^*$ of the similarly defined elements x_i', $1 \leqq i \leqq r-1$, of $\Gamma_{P'}$, and therefore generate $\bar{\phi}^*H^*(\Gamma_{P'})$, and that

$$\bar{s}^*(x_r)=0. \tag{4.8}$$

5. The involution J. To find the expression for $x_r{}^2$, we make use of an involution on Γ_P, which reverses the orientation of the fiber. (This method goes back to [12], cf. also [14].) Let $\tilde{J}\colon W_P \to W_P$ be the map sending

$$(k_1, \cdots, k_r) \text{ into } (k_1, \cdots, k_{r-1}, k_r \cdot j_r{}^{-1}),$$

where $j_r=\beta_r(j)$ as defined in no. 3. This is a bundle map of the bundle ψ, relative to the map $J\colon T^r \to T^r$, defined by $J(t_1, \cdots, t_r)=(t_1, \cdots, t_{r-1}, R_r(t_r))$, with R_r the involution of T determined by conjugation by j_r (cf. no. 3). Let $\bar{J}$ be the induced map of Γ_P into itself. Since $(j_r)^2$ belongs to T, $(\tilde{J})^2$ is a bundle map over the identity and $(\bar{J})^2$ is the identity of Γ_P. Further, since $\tilde{J}$ operates only on the last coordinate of W_P, the map $\bar{J}$ is an involution of the bundle Γ_P over the identity of the base $\Gamma_{P'}$:

$$\bar{\phi} \circ \bar{J} = \bar{\phi}. \tag{5.1}$$

To determine the effect of $\bar{J}$ on $H_2(\Gamma_P)$, we consider, for $1 \leqq i \leqq r$, the map $\tilde{\chi}_i \colon S_3 \to W_P$. We write again $T^r = T^{r-1} \times T$; using e_0 as generic symbol for the positive generator of the 0-th homology group (a point with coefficient 1), we have $H_1(T^r) = H_1(T^{r-1}) \otimes e_0 + e_0 \otimes H_1(T)$ [Künneth]. From (4.5), (4.4) and (3.4), we conclude that

$$\partial y_i = \tilde{\chi}_{i*}(\boldsymbol{\tau}) = \boldsymbol{\tau}'_i \otimes e_0 + e_0 \otimes \boldsymbol{\tau}_i, \tag{5.2}$$

where the $\boldsymbol{\tau}'_i$ are certain elements of $H_1(T^{r-1})$, with $\boldsymbol{\tau}'_r = 0$. (We have used the fact that the composition of χ_i with the projection of $T^{r-1} \times T$ onto T is just β_i.)

Equivariance with respect to J and $\bar{J}$ shows that

$$\partial \circ \bar{J}_*(y_i) = \boldsymbol{\tau}'_i \otimes e_0 + e_0 \otimes R_{r*}(\boldsymbol{\tau}_i),$$

since J operates only on the last factor of T^r, as R_r. By (3.3), we have

$$\partial(\bar{J}_* y_i) = \boldsymbol{\tau}'_i \otimes e_0 + e_0 \otimes \boldsymbol{\tau}_i - \theta_r(\tau_i) e_0 \otimes \boldsymbol{\tau}_r = \partial(y_i - \theta_r(\tau_i) y_r).$$

The well known fact "$\pi_2(W_P) = 0$" implies that ∂ is an isomorphism into, and we conclude that

$$\bar{J}_*(y_i) = y_i - \theta_r(\tau_i) y_r, \qquad 1 \leqq i \leqq r. \tag{5.3}$$

We note that for $i = r$, this means, because of $\theta_r(\tau_r) = 2$,

$$\bar{J}_*(y_r) = -y_r,$$

so that $\bar{J}$ reverses the orientation of the fiber K_r/T of Γ_P. Going to the dual basis $\{x_i\}$ for $H^2(\Gamma_P)$, we find

$$\bar{J}^*(x_r) = -x_r - \sum_1^{r-1} \theta_r(\tau_i) x_i. \tag{5.4}$$

It follows from the remark at the end of no. 4 that $\bar{J}^*(x_r)$ is congruent to $-x_r \bmod \bar{\phi}^* H^*(\Gamma_{P'})$. Using x_r as the ξ in (4.6), one concludes easily (using also (5.1)) that the elements of $H^*(\Gamma_P)$, invariant under $\bar{J}^*$, are exactly the elements in $\bar{\phi}^* H^*(\Gamma_P)$. We consider now $v = x_r \cdot \bar{J}^*(x_r)$. Since $\bar{J}$ is an involution and $\dim x_r = 2$, we have $\bar{J}^*(v) = v$, and by what was just said, v is of the form $\bar{\phi}^*(u)$. By (4.8), we have $\bar{s}^*(v) = 0$, and therefore $u = \bar{s}^* \circ \bar{\phi}^*(u) = \bar{s}^*(v) = 0$ (since $\bar{\phi} \circ \bar{s} =$ identity), and finally $v = 0$. Writing out $x_r \cdot \bar{J}(x_r)$ from (5.4), we obtain

$$x_r\cdot(x_r+\sum_1^{r-1}\theta_r(\tau_i)x_i)=0 \quad \text{or}$$
(5.5)
$$x_r^2=-\sum_1^{r-1}\theta_r(\tau_i)x_ix_r.$$

An obvious induction argument, based on (4.6) and the description of $\bar{\phi}^*H^*(\Gamma_P)$, contained in the last sentence of no. 4, completes the proof of (4.2).

6. Description of K/T. We turn to the proof of Theorem III. For this purpose, we consider, as in [6], the adjoint action of K on $\mathfrak{k}$, and apply the considerations of Chapter I. The defect function δ of this action (Chapter I, Definition 7.1)), for points of $\mathfrak{t}$, is determined by the infinitesimal diagram $D'(K)$; namely, $\delta(Q)$ is equal to twice the number of planes in $D'(K)$ that contain Q. This follows readily from the definition of $D'(K)$ and Chapter I, (7.2). In particular, any point of $\mathfrak{t}-D'(K)$ lies on a maximal K-orbit. As the centralizer of such a point is precisely T, its orbit N can be identified with K/T. Let $\alpha\colon K/T\to N$ be such an identification, and consider the space $\Omega_{\mathfrak{k}}=\Omega(\mathfrak{k};R,N)$, where R is a definite point of $\mathfrak{t}-D'(K)$ not on N. Let $\kappa\colon\Omega_{\mathfrak{k}}\to N$ be the natural projection of paths onto their end points. Because $\mathfrak{k}$ is a Euclidean space, κ induces a homotopy equivalence. Hence $\alpha^{-1}\circ\kappa$ defines a homotopy equivalence: $\Omega_{\mathfrak{k}}\approx K/T$. We will now apply Theorem I to $\Omega_{\mathfrak{k}}$. This is legitimate because of Theorem II.

We recall from [6] that N meets $\mathfrak{t}$ in a finite number of points $Q_1,\cdots,Q_w$ (one in each fundamental chamber). The set $S=S(\mathfrak{k};R,N)$ consists of the straight line segments s_i from R to the Q_i, and δ_{s_i} is equal to twice the number of planes in $D'(K)$ crossed by s_i. We may assume that the s_i intersect the planes of $D'(K)$ one at a time.

The construction of Chapter I, no. 5, determines K-cycles Γ_i, and maps $f_i\colon\Gamma_i\to\Omega_{\mathfrak{k}}$, for each $s_i\in S$. The stabilizer of a generic point on the plane $(\theta_i,0)$ is the group $K_{\theta_i}=K_i$ in the system $\mathcal{K}$ defined in no. 3. The Γ_i are therefore of the type considered in no. 4; in particular, each Γ_i is orientable. Hence, by Theorem I, images of the fundamental classes of the Γ_i under $\alpha^{-1}\circ\kappa\circ f_i$ form a basis for the integral homology of K/T.

Since K/T is an orientable manifold it follows that the manifold Γ_{i_0} of highest dimension, $2m$, maps onto K/T with degree ±1. This K-cycle, to be denoted by Γ_K, is characterized by the fact that R and Q_{i_0} lie in opposite fundamental chambers $\mathcal{F}$ and $-\mathcal{F}$. Adopting the numbering of the roots $\{\theta\}$, described at the beginning of no. 2, with $Q_{i_0}=R'$, we see from Definition 4.1 that

$$(6.1) \qquad \Gamma_K = K_1 \times_T K_2 \times_T \cdots \times_T K_{m-1} \times_T K_m/T,$$

The next proposition expresses a general property of maps of degree one; put $f_K = \alpha^{-1} \circ \kappa \circ f_{i_0}\colon \Gamma_K \to K/T$.

PROPOSITION 6.2. *The map $f_k{}^*$ maps $H^*(K/T)$ isomorphically onto an additively-direct summand of $H^*(\Gamma_K)$.*

Proof. If γ' and γ are suitable fundamental cycles of Γ_K and K/T, we have $f_{K*}(\gamma') = \gamma$, expressing the fact that f_{i_0} is of degree one. We denote by $\mathscr{d}$ the Poincaré duality operator, i.e. the cap-product with the fundamental cycle. The permanence relation $f_{K*}(f_K{}^*(a) \cap \gamma') = a \cap f_{K*}(\gamma')$ implies then $f_{K*} \circ \mathscr{d} \circ f_K{}^* = \mathscr{d}$. Defining $\bar{f}\colon H^*(\Gamma_K) \to H^*(K/T)$ by $\bar{f} = \mathscr{d}^{-1} \circ f_{K*} \circ \mathscr{d}$, we get $\bar{f} \circ f_K{}^* = \mathscr{d}^{-1} \circ \mathscr{d} =$ identity, and the proposition follows by standard group theory.

Since $H^*(\Gamma_K)$ has no torsion, Proposition 6.2 gives another proof of the well known fact that K/T has no torsion. [See [6] for references.]

To give the promised description of $H^*(K/T)$, we now make use of an important proposition due to J. Leray and A. Borel [1].

PROPOSITION 6.3. *The cohomology ring $H^*(K/T;\boldsymbol{Q})$ over the rationals $\boldsymbol{Q}$ is generated by (the unit and) $H^2(K/T;\boldsymbol{Q})$.*

The two Propositions 6.2 and 6.3 and the well-known relations between rational and integral cohomology clearly imply the following proposition.

PROPOSITION 6.4. *$H^*(K/T)$ is isomorphic, under $f_K{}^*$, to the smallest subalgebra of $H^*(\Gamma_K)$ that contains the unit and $f_K{}^*(H^2(K/T))$ and is additively a direct summand.*

In Proposition 4.2, we have shown that $H^*(\Gamma_K)$ is the algebra A_K of Theorem III. The theorem will therefore follow from Proposition 6.4, once we have identified the $f_K{}^*$—image of $H^2(K/T)$ in $H^2(\Gamma_K)$. We begin by noting that the transgression ∂ in the bundle $K \to K/T$ maps $H_2(K/T)$ isomorphically onto $H_1(T)$, because of $\pi_1(K) = \pi_2(K) = 0$ (cf. no. 4). Instead of $f_K{}^*$, we consider $\partial \circ f_{K*}$.

PROPOSITION 6.5. *Let $\{y_i\}_1{}^m$ be the basis of $H_2(\Gamma_K)$, constructed in 4.5; the image of y_i in $H_1(T)$ under $\partial \circ f_{K*}$ is τ_i.*

Proof. Let ψ be the bundle projection of the T^m-bundle $W_K = K_1 \times \cdots \times K_m$ onto Γ_K, as in no. 4, and let g be the map of W_K into K, defined by $g(k_1, \cdots, k_m) = k_1 \cdot k_2 \cdot \cdots \cdot k_m$. One verifies that g is a bundle map relative to the projection of T^m onto its last factor T, with K as T-bundle in the usual fashion, and that the induced map of the base spaces is just f_K. For $1 \leqq i \leqq m$, consider the map $\bar{\chi}_i\colon S_3 \to W_K$. The composition with g is a bundle map, relative to the map $\beta_i\colon S_1 \to T$, over the map $\beta_i\colon S_2 \to K/T$. It follows that

$$\partial \circ f_{K*}(y_i) = \partial \circ f_{K*} \circ \bar{\chi}_i(y) = \partial \circ \bar{\beta}_{i*}(y) = \beta_{i*}(\partial y) = \beta_{i*}(\boldsymbol{\tau}) = \boldsymbol{\tau}_i$$

(by (3.4)).

The map dual to $\partial \circ f_{K*}$ sends the cohomology class $\phi \in H^1(T)$ into the element $\sum \phi(\boldsymbol{\tau}_i) x_i \in H^2(\Gamma_K)$. Each such ϕ can be identified with an integral linear form ϕ on $\mathfrak{t}$ such that $\phi(\boldsymbol{\tau}_i) = \phi(\tau_i)$ (cf. the definition of $\mathfrak{J}$ before Theorem III and the remark before (3.3)). It is now clear that $f_K{}^*H^2(K/T)$ is the module B of Theorem III.

7. Computation of $H^*(G_2/T)$. As an example, we compute the cohomology of G_2/T, where G_2 is the exceptional group of rank 2. As is customary, we represent the Cartan algebra $\mathfrak{t}$ of G_2 as the plane with equation $t_1 + t_2 + t_3 = 0$ in Euclidean $t_1t_2t_3$-space. The 6-roots θ_i, ordered by the prescription of no. 2 and with a suitable choice of signs, are then given by the expression $t_1 - t_2, t_2, t_2 - t_3, -t_3, t_1 - t_3, t_1$. The corresponding vectors τ_i are given by $(1,-1,0)$, $(-1,2,-1)$, $(0,1,-1)$, $(1,1,-2)$, $(1,0,-1)$, $(2,-1,-1)$. The Cartan integers $\theta_r(\tau_s)$ are given by the matrix

$$(7.1)\qquad \begin{bmatrix} 2 & -3 & -1 & 0 & 1 & 3 \\ -1 & 2 & 1 & 1 & 0 & -1 \\ -1 & 3 & 2 & 3 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 3 & 2 & 3 \\ 1 & -1 & 0 & 1 & 1 & 2 \end{bmatrix}.$$

The relations between the variables $x_1, \cdots, x_6$ in the algebra A_{G_2} are then the following:

$$(7.2)\qquad \begin{aligned} x_1{}^2 &= 0 \\ x_2{}^2 &= x_1x_2 \\ x_3{}^2 &= x_1x_3 - 3x_2x_3 \\ x_4{}^2 &= -x_2x_4 - x_3x_4 \\ x_5{}^2 &= -x_1x_5 - x_3x_5 - 3x_4x_5 \\ x_6{}^2 &= -x_1x_6 + x_2x_6 - x_4x_6 - x_5x_6. \end{aligned}$$

As basis for the lattice $\mathcal{J}$ spanned by the τ_i, we can take τ_1 and τ_2. The matrix (a_{ij}) is then the matrix of the relations

$$
(7.3) \qquad
\begin{aligned}
\tau_1 &= \tau_1 \\
\tau_2 &= \tau_2 \\
\tau_3 &= \tau_1 + \tau_2 \\
\tau_4 &= 3\tau_1 + 2\tau_2 \\
\tau_5 &= 2\tau_1 + \tau_2 \\
\tau_6 &= 3\tau_1 + \tau_2.
\end{aligned}
$$

The generators for the image of $H^2(G_2/T)$ in A_{G_2} are therefore

$$
\begin{aligned}
a &= x_1 + x_3 + 3x_4 + 2x_5 + 3x_6 \\
b &= x_2 + x_3 + 2x_4 + x_5 + x_6.
\end{aligned}
$$

We change to other generators defined by

$$
(7.4) \qquad
\begin{aligned}
\alpha &= 3b - a = -x_1 + 3x_2 + 2x_3 + 3x_4 + x_5 \\
\beta &= a - 2b = x_1 - 2x_2 - x_3 - x_4 + x_6.
\end{aligned}
$$

One verifies the relations

$$
(7, 5) \qquad
\begin{aligned}
&\text{a)} \quad \alpha^2 + 3\alpha\beta + 3\beta^2 = 0, \\
&\text{b)} \quad \beta^3 \text{ divisible by } 2, \\
&\text{c)} \quad \alpha \cdot \beta^5 = 2x_1 \cdot x_2 \cdot \cdots \cdot x_6, \\
&\text{d)} \quad \alpha^6 = \beta^6 = 0.
\end{aligned}
$$

Relation (7.5a)) shows that the products of the form β^t, $\alpha \cdot \beta^t$, $0 \leqq t \leqq 5$, span $f^*(H^*(G_2/T))$ rationally, i.e. up to division by integers. The product $x_1 \cdot \cdots \cdot x_6$ represents the fundamental cocycle of Γ_{G_2}. It follows from (7.5c)) that all divisibility relations in $H^*(G_2/T)$ are consequences of (7.5b)), and that a basis for $H^*(G_2/T)$ is given by

$$
\{1, \alpha, \beta, \alpha \cdot \beta, \beta^2, \alpha \cdot \beta^2, \beta^3/2, \alpha \cdot \beta^3/2, \beta^4/2, \alpha \cdot \beta^4/2, \beta^5/2, \alpha \cdot \beta^5/2\}.
$$

This completely determines the cohomology ring $H^*(G_2/T)$, and proves Theorem III′. The computations are facilitated by making the change of variables:

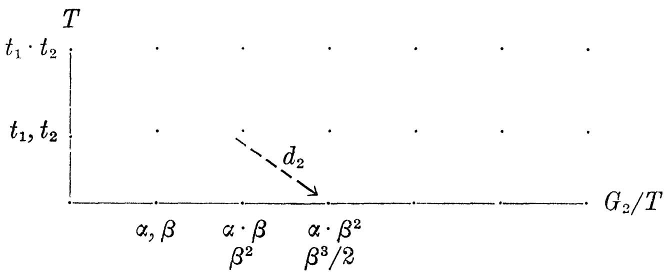
$$
(7.6) \qquad
\begin{aligned}
y_1 &= x_1 \\
y_2 &= x_2 \\
y_3 &= -2x_1 + 3x_2 + x_3 \\
y_4 &= -x_1 + 2x_2 + x_3 + x_4 \\
\alpha &= -x_1 + 3x_2 + 2x_3 + 3x_4 + x_5 \\
\beta &= x_1 - 2x_2 - x_3 - x_4 + x_6.
\end{aligned}
$$

The relations (7.2) go over into

$$
\begin{aligned}
y_1{}^2 &= 0 \\
y_2{}^2 &= y_1 y_2 \\
y_3{}^2 &= 3y_1y_2 - 3y_1y_3 + 3y_2y_3 \\
y_4{}^2 &= -\tfrac{1}{3}y_3{}^2 + y_3y_4 \\
\alpha^2 &= -3y_4{}^2 + 3y_4\cdot\alpha \\
\beta^2 &= \tfrac{1}{3}\alpha^2 + \alpha\beta.
\end{aligned} \tag{7.7}
$$

It is clear that $x_1 \cdot \cdots \cdot x_6 = y_1y_2y_3y_4\alpha\beta$.

As an application, one verifies, by a standard computation with the spectral sequence of $G_2 \to G_2/T$, that $H^*(G_2)$ has $\boldsymbol{Z}_2$ as torsion group in dimensions 6 and 9, as found by Borel in [3].



Note that the differential indicated above has $\boldsymbol{Z}_2$ as cokernel: $\beta^3/2$ is not in the image, although, with $d_2t_1 = \alpha$, $d_2t_2 = \beta$, one has $d_2(\beta^2 \cdot t_2) = 2 \cdot \beta^3/2$.

8. Cohomology of $K/K_{T'}$. We continue to use the notation introduced in no. 6. The proof of Proposition 2.1, to which we turn now, depends on the fact that the transgression $\partial\colon H_2(K/T) \to H_1(T)$, is equivariant with respect to the operation of the Weyl group, and that the same holds for the dual map ∂^* of the cohomology groups. This proves the statement about the action of $\mathfrak{W}$ on B, as a consequence of the remarks at the end of no. 6. The uniqueness of the extension to all of $H^*(K/T)$ follows from Theorem III. Note that $\mathfrak{W}$ does not act as a group of automorphisms on A_K.

The proof of Theorem III″ goes along quite different lines. We consider the spectral sequence of the fiber map $K/T \to K/K_{T'}$, with fiber $K_{T'}/T$, induced by the inclusions $K \supset K_{T'} \supset T$. All dimensions in $H^*(K/K_{T'})$ and in $H^*(K_{T'}/T)$ are even, and the spectral sequence is trivial (all differentials vanish). The Weyl group $\mathfrak{W}'$ of $K_{T'}$ operates on the spectral sequence. The operation is trivial on the base space $K/K_{T'}$, since $K_{T'}$ is connected. The operation on $H^*(K_{T'}/T)$ is known to be equivalent to the regular representation of $\mathfrak{W}'$ (cf. [1]); the elements left fixed by all $w \in \mathfrak{W}'$ are exactly the elements of $H^0(K_{T'}/T)$. It follows now by comparing the E_2-term $H^*(K/K_{T'}) \otimes H^*(K_{T'}/T)$ with $H^*(K/T)$, that $H^*(K/K_{T'})$ [considered as a subalgebra of $H^*(K/T)$] consists exactly of the elements of $H^*(K/T)$ left fixed by all $w \in \mathfrak{W}'$, and Theorem III″ is proved.

9. The holomogy of $\boldsymbol{\Omega}(K)$; applications to homotopy. Let K operate on itself by inner automorphisms. Let R be a point of K, and let $\Omega_R = \Omega(K; R, e)$ be the space of paths, as in Chapter I, no. 2, from R to the orbit of e, i. e. to e. Ω_R can be identified, in standard fashion, with the space $\Omega(K) = \Omega(K; e, e)$ of loops in K at e; since K is simply connected, this identification is unique up to homotopy.

Suppose now that R is a regular point of T, i. e. supposes $R \notin \bar{p}$ for any $p \in \mathfrak{P}$; the orbit of R is then of maximal dimension, and because of Theorem II, we can apply Theorem I to Ω_R. As shown in [6], the set $S = S(K; R, e)$ of Chapter I, no. 2, can be identified with the set S' of line segments in $\mathfrak{t}$, obtained by choosing in each cell Δ of the fundamental chamber $\mathfrak{F}$ a suitable point, and connecting it to the origin 0 by a straight segment s. To Δ or to s, we associate the K-cycle of the geodesic segment $\exp \circ s$ (in the sense of Chapter I, no. 4), denoted by Γ_Δ, and the map of Γ_Δ into $\Omega(K)$ (using the identification of the various Ω_R above), denoted by f_Δ. The exceptional points of $\exp \circ s$ correspond to the points where s crosses singular planes in $\mathfrak{t}$ (we assume that s has no point of higher singularity). The exceptional stabilizers are then groups of the set $\mathfrak{K}$ of no. 3; the stabilizer of $\exp \circ s$ is T itself. It follows that all K-cycles are of the type discussed in no. 4. In more detail, we get from Theorem I the following proposition, describing the homology of $\Omega(K)$.

PROPOSITION 9.1. *For each cell Δ in $\mathfrak{F}$, let $P = \{p_1, \cdots, p_r\}$ be the ordered set of oriented planes in $\mathfrak{P}$ that the segment s from Δ to 0 crosses (in that order). The K-cycle Γ_Δ is then Γ_P, in the sense of* 4.1, *so that in particular* $\dim \Gamma_\Delta = 2r$. *The f_Δ-images of fundamental cycles γ_Δ of the Γ_Δ, with Δ ranging over the cells of $\mathfrak{F}$, form a basis for the (free) group $H_*(\Omega(K))$.*

It should be noted that strictly speaking the geodesic segment $\exp \circ s$ is not in the set S, but that a suitable inner automorphism of K will bring it into S. The abstract K-cycle will not change under this operation; and, K being connected, the image homology class in the loop space will also be the same (using again the identification of all the Ω_R).

For our application to homotopy, we need a partial determination of the cohomology ring of $\Omega(K)$. We quote some well known facts. Assume from now on K to be simple. The cohomology $H^*(K;\boldsymbol{Q})$ over the rationals $\boldsymbol{Q}$ is an exterior algebra with l generators of dimensions $2m_i-1$, with $2=m_1<m_2\leqq\cdots\leqq m_l$ (actually $m_2=m_3$ only for the type D_4). The cohomology $H^*(\Omega(K);\boldsymbol{Q})$ is a polynomial ring with l generators of dimensions $2m_i-2$. Since $H^*(\Omega(K))$ has no torsion, this implies that all $H^{2i+1}(\Omega(K))$ vanish, that $H^{2i}(\Omega(K))$ is infinite cyclic for $0\leqq i\leqq m_2-2$, and that $H^{2m_2-2}(\Omega(K))$ is free abelian of rank 2 or 3. Let η denote a generator of $H^2(\Omega(K))$; no power η^i vanishes. The second non-vanishing homotopy group of K (or $\Omega(K)$) depends on the divisibility properties (by integers) of the powers η^i, $2\leqq i\leqq m_2-1$, as described in the next proposition.

PROPOSITION 9.2. (A) *Suppose no power η^i, for $1\leqq i\leqq m_2-1$, is divisible by an integer >1; then $\pi_r(K)=0$ for $3<r<2m_2-1$, and $\pi_{2m_2-1}(K)$ is free abelian of rank one less than the number of m_i's equal to m_2;*

(B) *In the remaining case, let i_0 be the smallest exponent $i\leqq m_2-1$ for which η^i is divisible, and let η^{i_0} be divisible exactly by the positive integer q (>1); then $\pi_r(K)=0$ for $3<r<2i_0$, and $\pi_{2i_0}(K)=\boldsymbol{Z}_q$.*

Proof. The integral cohomology ring of the Eilenberg-MacLane space $K(\boldsymbol{Z},2)$, the infinite complex projective space, is a polynomial ring $\boldsymbol{Z}[x]$ with $\dim x=2$. By standard principles, there exists a map $\phi_\eta\colon \Omega(K)\to(\boldsymbol{Z},2)$ such that $\phi_\eta{}^*(x)=\eta$, and consequently $\phi_\eta{}^*(x^i)=\eta^i$. One can now easily determine the nature of the first non-vanishing relative homology group of ϕ_η (i.e. of the mapping cylinder of ϕ_η, with $\Omega(K)$ as the subset) in terms of the assumptions in (9.2); the Hurewicz isomorphism theorem and the known nature of the $\pi_r(K(\boldsymbol{Z},2))$ yield then (9.2). We see from (9.1) that η^i is divisible by q exactly if for every Δ with $\dim\Gamma_\Delta=2i$, the cocycle $f_\Delta{}^*(\eta^i)$ (a multiple of the fundamental cocycle of Γ_Δ) is divisible by q. We shall study this question using the description of Γ_Δ in (4.2), and begin by finding the expression for $f_\Delta{}^*(\eta)$.

10. The 3-spheres in K. We identify the Lie algebra of the subgroup S_1 of the group S_3 (cf. no. 3) with the real numbers $\boldsymbol{R}$ such that the exponential map is given by $\phi\to e^{i\phi}$, for $\phi\in\boldsymbol{R}$. There is then one root θ with $\theta(\phi)=\frac{1}{\pi}\phi$; the basic translation τ_θ is the number 2π; the system $\mathfrak{P}$ of oriented singular planes can be identified with the integral multiples of π; the fundamental chamber is given by $\phi > 0$. Let n be any integer; let $s\colon [0,1] \to \boldsymbol{R}$ be a non-degenerate, linear map with $s(1) = 0$ such that for some $\bar{t} \in [0,1)$, we have $s(\bar{t}) = n \cdot \pi$, so that $\bar{t}$ is an exceptional value of the geodesic segment $\exp \circ s$ in S_3, according to Chapter I, no. 4. Put $R = \exp \circ s(0)$. We write $s = s' + s''$ with $s' = [0, \bar{t}]$ (possibly degenerate) and $s'' = [\bar{t}, 1]$. Generalizing slightly the construction of Chapter I, no. 5, we define a map $\tilde{f}_n\colon S_3 \to \Omega(S_3; R, 1)$ by $\tilde{f}_n(x) = \exp \circ s' + x \cdot \exp \circ s'' \cdot x^{-1}$. This is well defined at the junction point since the centralizer of $\exp \circ s(\bar{t})$ is S_3. As in Chapter I, no. 5, the map $\tilde{f}_n$ is constant along the cosets of S_1, and induces a map $f_n\colon S_2 = S_3/S_1 \to \Omega(S_3; R, 1)$.

PROPOSITION 10.1. *Let y be the generator of $H_2(S_2)$ defined in* 3.3. *Then $\bar{y} = f_{1*}(y)$ is a generator of the infinite cyclic group $H_2(\Omega(S_3))$; and $f_{n*}(y) = n \cdot \bar{y}$.*

We identify here $\Omega(S_3; R, 1)$ with $\Omega(S_3) = \Omega(S_3; 1, 1)$; that $\bar{y}$ is a generator of $H_2(\Omega(S_3))$ is a special case of Proposition 9.1 for $K = S_3$, since the interval $(\pi, 2\pi)$ is the only cell Δ whose Γ_Δ has dimension 2, and since S_2 and f_1 are just the associated Γ_Δ and f_Δ.

To treat the factor n, we note first that we may choose s in such a way that $s(0) = 2n\pi$. Let ρ be the natural homomorphism of S_3 onto $S_3/\{1, -1\} = P_3$ (projective 3-space $= SO(3)$). Each $\rho \circ f_n(x)$, for $x \in S_2$, is then a loop in P_3 that goes through e at the parameter values $i/2n$, $0 \leqq i \leqq 2n$; in other words, it is a $2n$-fold Pontryagin product in $\Omega(P_3)$. Using the fact that the Pontryagin product in the loop space of a group is homotopy-commutative one verifies that $\rho \circ f_n$ is homotopic to the n-th Pontryagin power of $\rho \circ f_1$ (for $n \geqq 0$), and to the inverse of $\rho \circ f_1$ for $n = -1$. Since ρ induces a homeomorphism of $\Omega(S_3)$ with the e-component of $\Omega(P_3)$, and since y, being spherical, is primitive, one concludes now easily that $f_{n*}(y) = n \cdot \bar{y}$.

For any root $\theta \in \mathfrak{R}$, let $g_\theta\colon S_3 \to K$ be the composition of the map β_θ of no. 3 with the inclusion of K_θ in K, and let $\bar{g}_\theta$ be the induced map of the loop spaces.

PROPOSITION 10.2. (A) *The element $\bar{g}_{\mu*}(\bar{y}) = y_K$ (where μ is the dominant root and $\bar{y}$ is defined in* 10.1) *is a generator of $H_2(\Omega(K))$.*

(B) *For any root $\theta \in \mathfrak{R}$, the following relation holds:*

$$\bar{g}_{\theta*}(\bar{y}) = ((\tau_\theta, \tau_\theta)/(\tau_\mu \cdot \tau_\mu)) \cdot y_K.$$

(Note that the factor of y_K is always an integer.)

For part (A), we use (9.1). There is exactly one cell Δ in $\mathfrak{F}$ with

$\dim \Gamma_\Delta = 2$, namely the cell obtained by reflecting the fundamental cell $\Delta_{\mathfrak{F}}$ across the plane $(\mu, 1)$; we have $\Gamma_\Delta = K_\mu/T$. One verifies, by considering the definition of f_Δ, Chapter I, no. 5, the relation

$$(10.3) \qquad f_\Delta \circ \bar{\chi}_1 = \bar{g}_\mu \circ f_1,$$

with f_1 as defined above, and $\bar{\chi}_1 \colon S_2 \to \Gamma_\Delta$ defined in no. 4. [Actually, this equation has to be understood as a homotopy. The map f_Δ is defined by a certain segment s in $\mathfrak{t}$. As far as $f_\Delta \circ \bar{\chi}_1$ goes, one can replace s by any segment in $\mathfrak{t}$ that has 0 as end point and for whose initial point the inequality $\mu(s(0)) \geqq 1$ holds, provided that one disregards all exceptional points of s except the one where s meets the plane $(\mu, 1)$. In terms of Chapter I, no. 5, this means that instead of W_n we act on s only with $T \times \cdot \cdot \cdot \times T \times K_n$. It is now possible to take for s the image, under the differential $\dot{g}_\mu$ of g_μ, of the segment used for the construction of f_1 in the Lie algebra $\boldsymbol{R}$ of S_1; this produces actual equality in the above relation.] It follows that $f_{\Delta *}(y_1) = \bar{g}_{\mu *}(\bar{y})$; since y_1 is the fundamental cycle of K_μ/T, part (A) follows.

The standard transgression argument shows that, with z denoting a generator of $H_3(S_3)$, the element $g_{\mu *}(z)$ is a generator of $H_3(K)$. We shall prove (10.2 (B)) by showing that $g_{\theta *}(z) = ((\tau_\theta, \tau_\theta)/(\tau_\mu, \tau_\mu)) \cdot g_{\mu *}(z)$.

Let ψ_K be the Cartan-form of K, i.e. the closed invariant differential 3-form defined, at e, by $\psi_K(X, Y, Z) = (X, [Y, Z])$. Let τ, τ', τ'' be a basis for the Lie algebra of S_3, with τ the basic translation in the Lie algebra of S_1, and $[\tau', \tau''] = \tau$. By (3.4), we have $\beta_\theta(\tau) = \tau_\theta$, and therefore $g_\theta{}^*\psi_K(\tau, \tau', \tau'') = (\tau_\theta, \tau_\theta)$. This implies $g_\theta{}^*\psi_K = ((\tau_\theta, \tau_\theta)/(\tau_\mu, \tau_\mu)) \cdot g_\mu{}^*\psi_K$, and (10.2 (B)) follows.

11. Computation of $f_\Delta{}^*(\eta)$. Let the generator η of $H^2(\Omega(K))$ be so chosen that $\eta(y_K) = 1$, with y_K as in (10.2(A)). Let Δ be any cell in $\mathfrak{F}$, and let $P = \{p_1, \cdot \cdot \cdot, p_r\}$, Γ_Δ and f_Δ be as in (9.1). By no. 4, to each $p_i \in P$, there corresponds a basis element y_i of $H_2(\Gamma_\Delta)$; the determination of $f_\Delta{}^*(\eta) \in H^2(\Gamma_\Delta)$ is equivalent to that of $f_{\Delta *}(y_i)$, $1 \leqq i \leqq r$.

PROPOSITION 11.1.

$$f_{\Delta *}(y_i) = n_i((\tau_i, \tau_i)/(\tau_\mu, \tau_\mu)) \cdot y_K,$$

with $\tau_i = \tau_{p_i}$ and $n_i = n_{p_i}$, the translation and the multiplicity associated with the singular plane p_i.

For the proof, one verifies, as in the proof of (10.2), that the commutativity relation $f_\Delta \circ \bar{\chi}_i = \bar{g}_i \circ f_{n_i}$ holds, where f_{n_i} is as in (10.1), and where $\bar{g}_i$

is the map of the loop spaces induced by the map $g_i\colon S_3 \to K_i \subset K$. (11.1) follows then from (10.1) and (10.2(B)). By simple duality, we have

PROPOSITION 11.2.

$$f_\Delta{}^*(\eta) = (\tau_\mu, \tau_\mu)^{-1} \sum_1^r n_i(\tau_i, \tau_i) x_i;$$

here τ_i, n_i are the basic translation and the multiplicity associated with the i-th singular plane p_i along the segment s from Δ to 0.

12. Computation of $\pi_4(K)$. To determine $\pi_4(K)$, we have to consider the element η^2, according to (9.2); according to (9.1), this amounts to considering all cells Δ with $\dim \Gamma_\Delta = 4$. Assume that the rank l of K is > 1. Let Δ_1 be the cell obtained from the fundamental cell $\Delta_{\mathfrak{F}}$ by reflection across the plane $(\mu, 1)$. It is clear that the cells that correspond to dimension 4 are the cells in $\mathfrak{F}$ adjoining to Δ_1 (different from $\Delta_{\mathfrak{F}}$). Let Δ be such a cell (actually there are one or two such, with two occurring for the type A_n, $n \geqq 2$, only). The segments from Δ to 0 has 2 exceptional points; the first belongs to a certain root, say θ; the second belongs to the dominant root. By (4.2), the ring $H^*(\Gamma_\Delta)$ is then generated by two variables x_1, x_2 subject to $x_1{}^2 = 0$ and $x_2{}^2 = -a x_1 x_2$, where $a = \mu(\tau_\theta)$. The multiplicities n_1, n_2 of (11.2) are both 1, and we have $f_\Delta{}^*(\eta) = ((\tau_\theta, \tau_\theta)/(\tau_\mu, \tau_\mu)) \cdot x_1 + x_2$. Squaring, we obtain

$$f_\Delta{}^*(\eta^2) = (2(\tau_\theta, \tau_\theta)/(\tau_\mu, \tau_\mu) - a) x_1 x_2.$$

Since τ_μ belongs to the closure of $\mathfrak{F}$, we have $\theta(\tau_\mu) > 0$ ($\neq 0$, since τ_μ and τ_θ are clearly not orthogonal). Moreover, since $(\tau_\mu, \tau_\mu) \leqq (\tau_\theta, \tau_\theta)$ (τ_μ is a "shortest" translation), we have $\theta(\tau_\mu) < \theta(\tau_\theta) = 2$, and so $\theta(\tau_\mu) = 1$. It follows that $(\tau_\theta, \tau_\theta)/(\tau_\mu, \tau_\mu) = \mu(\tau_\theta)/\theta(\tau_\mu) = \mu(\tau_\theta)$, and so

$$f_\Delta{}^*(\eta^2) = \mu(\tau_\theta) x_1 x_2 \tag{12.1}$$

It follows from (12.1) that η^2 is divisible by the integer $q \geqq 1$ if and only if the Cartan integers $\mu(\tau_\theta)$, for the θ's involved, are so divisible. Recalling that these roots θ are obtained by reflecting across the plane $(\mu, 1)$ those fundamental roots that are not orthogonal to μ, and recalling that the translations belonging to the fundamental roots generate the lattice $\mathfrak{T}$, one sees easily that η^2 is divisible by q exactly if all values $\mu(X)$, for $X \in \mathfrak{T}$, are so divisible. Since $\mu(\tau_\mu) = 2$, this means $q = 1$ or 2. Moreover, q will be 1 exactly if there exists an $X \in \mathfrak{T}$ with $\mu(X) = 1$. This proves Theorem IV(a). The statements of IV(b) are obtained by a simple inspection of the diagrams of the simple groups. The case $l=1$, i.e. the group S_3 itself, can be treated with the same method; there is only one root, the multiplicity n_1 is 2, as is the Cartan integer a; n_2 is 1; $f_\Delta^*(\eta)=2x_1+x_2$; $f_\Delta^*(\eta^2)=2x_1x_2$. The result $\pi_4(S_3)=\boldsymbol{Z}_2$ is of course well known

13. Homotopy groups of $\boldsymbol{E_6}$, $\boldsymbol{E_7}$, $\boldsymbol{E_8}$. To have a way of picturing the arrangement of the cells in the fundamental chamber, we associate to the Lie group K a graph $\Phi=\Phi(K)$, the 1-skeleton of the dual of the triangulation of $\mathfrak{F}$ by the cells, with labelled edges. To each cell Δ in $\mathfrak{F}$, there is associated a vertex v_Δ, and to any $(l-1)$-face between two cells is associated an edge connecting the two vertices; the edge is labelled with the plane $p\in\mathfrak{P}$ in which the $(l-1)$-face lies. There is a distinguished vertex $v_{\mathfrak{F}}$, corresponding to the lowest cell $\Delta_{\mathfrak{F}}$. We define a function d of the vertices by setting $d(v)=$ smallest number of edges in a path from v to $v_{\mathfrak{F}}$. One verifies that for any cell Δ with segment s from Δ to 0 and K-cycle Γ_Δ, as in (9.1), one has $\dim\Gamma_\Delta=2d(v_\Delta)$, and that the singular planes occur along s in the same order as along a *suitable* minimal path from v_Δ to $v_{\mathfrak{F}}$ in Φ. We write Φ_i for the subgraph of Φ containing all vertices with d-values $\leqq i$.

Then, for any K $(\neq D_4)$, the graph Φ_{m_2-1} (here m_2 is the second primitive exponent of K, cf. no. 9) has the form

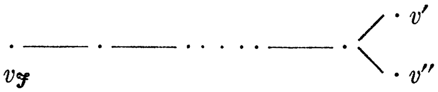


Figure 2.

as follows at once from the behavior of the Betti numbers. Since for all the vertices shown, the minimal path to $v_{\mathfrak{F}}$ is unique, one can read off the nature of the K-cycles Γ_Δ, provided one knows which planes of $\mathfrak{P}$ are attached to the various edges. The first edge (i.e., the edge at $v_{\mathfrak{F}}$) is of course labelled with the dominant root μ, or better, with the plane $(\mu,1)$. We now describe the labelling of Φ_{m_2-1} for the exceptional groups E_6, E_7, E_8, in terms of the usual fundamental roots $\{\phi_i\}_1{}^l$ (cf. the Coxeter (Schläfli) diagrams below). To have a short description, we state first that all the planes p occurring are of the form $(\theta,1)$, i.e. they have $n_p=1$. Further, it turns out that for each edge after the first one, the root attached is obtained by subtracting a suitable fundamental root from the root attached to the edge immediately preceding in Φ. We give below the system consisting of the dominant root and the fundamental roots to be subtracted successively; the last two entries correspond to the two edges leading to v' and v''.

$$(13.1)\qquad \begin{aligned} &E_6: \ \{\mu, \phi_6, \phi_3; \phi_2, \phi_4\} \\ &E_7: \ \{\mu, \phi_6, \phi_5, \phi_4; \phi_3, \phi_7\} \\ &E_8: \ \{\mu, \phi_1, \phi_2, \phi_3, \phi_4, \phi_5; \phi_6, \phi_8\}. \end{aligned}$$

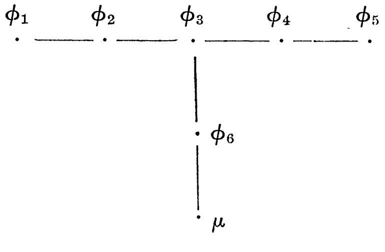
The singular plane attached to the 3rd edge of $\Phi_4(E_6)$, e.g., is then $(\mu - \phi_6 - \phi_3, 1)$; the plane attached to the edge leading to v' is $(\mu - \phi_6 - \phi_3 - \phi_2, 1)$. To prove that these are the correct values, we proceed as follows:

(A) The fundamental roots indicated in (13.1), are the only ones at each stage in the processe that can be subtracted from the root obtained up to the point in question and yield again a root as difference. To verify this we start from the remark that for any two roots θ, θ' of a group K, the difference $\theta - \theta'$ is again a root if $\theta'(\tau_0)$ is positive (cf. [5]); moreover, for E_6, E_7, E_8 the reverse implication is also true, since all basic translations are of the same length, say 1. Further, we have for the fundamental roots ϕ_i and the corresponding basic translations λ_i the usual relations

$$(13.2)\qquad \begin{aligned} &\phi_i(\lambda_i) = 2, \\ &\phi_i(\lambda_j) = 2(\lambda_i, \lambda_j) = -1, \text{ if } \phi_i \text{ and } \phi_j \text{ are adjacent in the Coxeter diagram,} \\ &\phi_i(\lambda_j) = 0 \text{ otherwise.} \end{aligned}$$

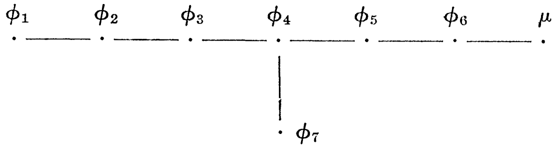
One verifies that in our three cases the dominant root μ has $\mu(\lambda_i)$ positive for exactly one of the λ_i, and 0 for all others, so that exactly one $\mu - \phi_i$ is a root (cf. also [5]). We reproduce the three Coxeter diagrams, with in each case a new vertex, corresponding to the dominant root, joined by an edge to the unique vertex whose corresponding λ_i is not orthogonal to τ_μ.

E_6:



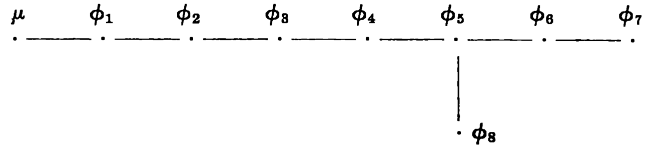
$$\mu = \phi_1 + 2\phi_2 + 3\phi_3 + 2\phi_4 + \phi_5 + 2\phi_6,$$

E_7:



$$\mu = \phi_1 + 2\phi_2 + 3\phi_3 + 4\phi_4 + 3\phi_5 + 2\phi_6 + 2\phi_7,$$

E_8:



$$\mu = 2\phi_1 + 3\phi_2 + 4\phi_3 + 5\phi_4 + 6\phi_5 + 4\phi_6 + 2\phi_7 + 3\phi_8.$$

As an example, we compute, for E_6, $\mu(\lambda_1) = \phi_1(\lambda_1) + 2\phi_2(\lambda_1) = 2 - (2 \cdot -1) = 0$ and $\mu(\lambda_6) = 3\phi_3(\lambda_6) + 2\phi_6(\lambda_6) = 3 \cdot -1 + 2.2 = 1$. One verifies the statement made in (A) about the roots in 13.1 directly from the diagrams. As an example, for E_6, to find a ϕ_i for which $\mu - \phi_6 - \phi_i$ is a root, we have to find a ϕ_i for which $\phi_i(\tau_\mu - \lambda_6) = \phi_i(\tau_\mu) - \phi_i(\lambda_6) > 0$. Clearly i must be different from 6; but then $\phi_i(\tau_\mu) = 0$, and the only root with $\phi_i(\lambda_6) \neq 0$ (and actually $= -1$ by (13.2)) is the root ϕ_3, adjacent to ϕ_6.

(B) If for two planes $p = (\theta, n)$, $p' = (\theta', n')$ of $\mathfrak{P}$, the linear function $\theta/n - \theta'/n'$ is positive on $\mathfrak{F}$, then clearly the first (i.e. nearest to $v_{\mathfrak{F}}$) occurrence of p at an edge of Φ comes before any occurrence of p'. One verifies now that the roots θ constructed by the prescription in (13.1), are successively smaller in this sense, since each time a fundamental root is subtracted (except that the last two roots are not comparable in this ordering), and that they are all greater than $\frac{1}{2} \cdot \mu$ (by explicit computation; e.g., for E_6, $\mu - \phi_6 - \phi_3 - \phi_2 = \phi_1 + \phi_2 + \phi_3 + 2\phi_4 + \phi_5 + \phi_6 > \frac{1}{2} \cdot \mu$ on $\mathfrak{F}$).

It is clear from (A) and (B) that the labelling of Φ_{m_2-1}, described by (13.1), is the correct one.

For each of E_6, E_7, E_8, we shall now compute the cohomology ring of the manifold Γ', attached by (9.1) to the vertex v' of the graph Φ_{m_2-1}, and also the image, under the associated map f', of η (by (11.2)) and of the relevant power of η; using (9.2), this will yield Theorem V(a), (b), (c). If s is a segment from the cell Δ', corresponding to v' in figure 2, to 0, then the singular planes crossed by s are identical with the singular planes attached to the edges of Φ_{m_2-1}, moving from v' to $v_{\mathfrak{F}}$. All the multiplicities are 1, as noted before; all the "weights" $(\tau_i, \tau_i)/(\tau_\mu, \tau_\mu)$ occurring in (11.2) are also 1. The roots along s are given by the description in (13.1) (but in reverse order!). For E_6, for instance, the singular planes p_1, p_2, p_3, p_4 are given by $(\mu - \phi_6 - \phi_3 - \phi_2, 1)$, $(\mu - \phi_6 - \phi_3, 1)$ $(\mu - \phi_6, 1)$ $(\mu, 1)$.

It turns out that all the Cartan integers, appearing in the description (4.2) of the cohomology of our Γ's are 1. As an example, for E_6, the integer $\mu(\tau_\mu - \lambda_6) = \mu(\tau_\mu) - \mu(\lambda_6) = 2 - 1 = 1$. We omit the details.

For the Γ' of E_6, the cohomology is then generated by variables x_1, x_2, x_3, x_4, with the relations

$$(13.1)\qquad \begin{aligned} &x_1{}^2 = 0,\\ &x_2(x_1 + x_2) = 0,\\ &x_3(x_1 + x_2 + x_3) = 0,\\ &x_4(x_1 + x_2 + x_3 + x_4) = 0, \end{aligned}$$

and $f'^*(\eta)$ is given by $x_1 + x_2 + x_3 + x_4$. The fundamental cocycle of Γ' is given by $x_1x_2x_3x_4$; we claim

$$(13.4)\qquad f'^*(\eta^4) = x_1x_2x_3x_4.$$

For the (elementary) computation, we introduce a new basis by setting $z_1 = x_1$, $z_1 = x_1 + x_2$, $z_3 = x_1 + x_2 + x_3$, $z_4 = x_1 + x_2 + x_3 + x_4$. The relations turn into

$$(13.5)\qquad \begin{aligned} &z_1{}^2 = 0,\ z_2{}^2 = z_1z_2,\ z_3{}^2 = z_2z_3,\ z_4{}^2 = z_3z_4,\\ &\text{and } f'^*(\eta) \text{ is given by } z_4. \end{aligned}$$

Now $(z_4)^4 = (z_3z_4)^2 = z_2z_3{}^2\cdot z_4 = z_2{}^2\cdot z_3z_4 = z_1z_2z_3z_4 = x_1x_2x_3x_4$, q. e. d.

It follows that the powers η^i, $1 \leqq i \leqq 4$, are not divisible by any integers $q > 1$, and Theorem V(a) follows from (9.2).

The computations for E_7 and E_8, with 5, resp. 7 variables x_i are entirely analogous.

14. $\pi_{10}(G_2) \otimes \boldsymbol{Z}_3$, preparation. Following a suggestion of J. C. Moore and A. S. Shapiro, we shall compute $\pi_{10}(G_2) \otimes \boldsymbol{Z}_3$ by considering, in addition to the ring structure of $H^*(\Omega(G_2);\boldsymbol{Z}_3)$ up to dimension 10, a certain Steenrod power (cf. [19] for this concept).

We begin by listing the necessary facts about G_2 (in a notation somewhat different from no. 7).

(1) Fundamental roots ϕ_1, ϕ_2; corresponding translations λ_1, λ_2;

(2) Coxeter diagram $\overset{\phi_1}{\circ}\!\equiv\!\equiv\!\overset{\phi_2}{\circ}$; $(\lambda_1, \lambda_1) = 3$, $(\lambda_2, \lambda_2) = 1$, $(\lambda_1, \lambda_2) = -\frac{3}{2}$;

(3) Fundamental Cartan integers: $\phi_1(\lambda_1) = \phi_2(\lambda_2) = 2$, $\phi_1(\lambda_2) = -1$, $\phi_2(\lambda_1) = -3$;

(4) Positive roots (in lexicographic order): $3\phi_1 + 2\phi_2$ (dominant),

$$3\phi_1 + \phi_2,\ 2\phi_1 + \phi_2,\ \phi_1 + \phi_2,\ \phi_1,\ \phi_2;$$

corresponding translations:

$$\lambda_1 + 2\lambda_2,\ \lambda_1 + \lambda_2,\ 2\lambda_1 + 3\lambda_2,\ \lambda_1 + 3\lambda_2,\ \lambda_1,\ \lambda_2;$$

norm-squares of these translations: $1, 1, 3, 3, 3, 1$;

(5) $$m_2 = 5;$$

graph Φ_5:

singular planes p attached to the edges of Φ_5 (in order): $(3\phi_1 + 2\phi_2, 1)$,

$$(3\phi_1 + \phi_2, 1),\ (2\phi_1 + \phi_2, 1),\ (3\phi_1 + 2\phi_2, 2);\ (3\phi_1 + \phi_2, 2)\ (\text{to } v'),$$
$$(\phi_1 + \phi_2, 1)\ (\text{to } v'');$$

beginning of the fundamental chamber:

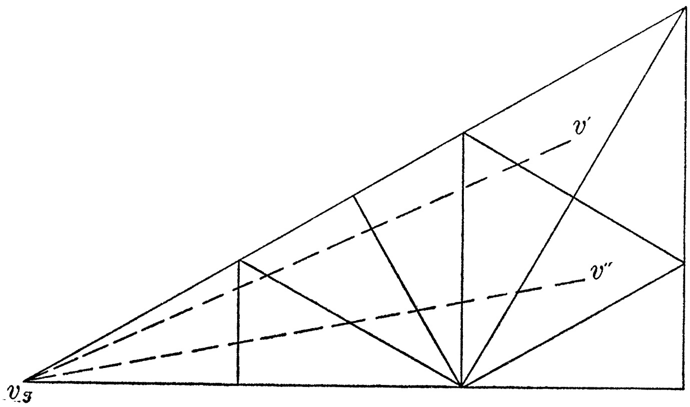


Figure 3.

As before in nos. 9 and 11, we denote by η the generator of $H^2(\Omega(G_2))$, defined by $\eta(y_{G_2}) = 1$. Our first aim is the following partial description of $H^*(\Omega(G_2); \mathbf{Z}_3)$.

PROPOSITION 14.1. *The element η^3 of $H^6(\Omega(G_2))$ is divisible exactly by 3. Put $u = \eta^3/3$. Then a basis for $\sum_0^{10} H^i(\Omega(G_2); \mathbf{Z}_3)$ is given by $\{1; \eta; \eta^2; u; u \cdot \eta; u \cdot \eta^2, P^1 u\}$; here the image of η etc. under the coefficient reduction $\mathbf{Z} \to \mathbf{Z}_3$ is again denoted by η etc.; and P^1 is the Steenrod power that send $H^i(X; \mathbf{Z}_3)$ into $H^{i+4}(X; \mathbf{Z}_3)$.*

Remark. The fact that η^3 is divisible exactly by 3, is responsible for the complexity of the problem of finding $\pi_{10}(G_2) \otimes \mathbf{Z}_3$. From now on we write Ω for $\Omega(G_2)$. We shall prove Proposition 14.1 by considering the two G_2-cycles Γ', Γ'' attached, by (9.1), to the cells Δ', Δ''. [A part of the result, in fact all except the determination of P^1u, follows also, for instance, from the fact that, G_2 having only two-torsion, $H^*(\Omega;\mathbf{Z}_3)$ is a divided polynomial ring.] We denote by f', f'' the maps of Γ', Γ'' into Ω, defined in Chapter I, (5.4).

Γ': The variables $x_1,\cdots,x_5$, generating $H^*(\Gamma')$, correspond, in order, to the roots $3\phi_1+\phi_2$, $3\phi_1+2\phi_2$, $2\phi_1+\phi_2$, $3\phi_1+\phi_2$, $3\phi_1+2\phi_2$. The relations are given by

$$\begin{aligned}&x_1{}^2=0,\\&x_2(x_2+x_1)=0,\\&x_3(x_3+x_2+x_1)=0,\\&x_4(x_4+3x_3+x_2+2x_1)=0,\\&x_5(x_5+x_4+3x_3+2x_2+x_1)=0.\end{aligned}\tag{14.2}$$

As an example, $a_{43}=(3\phi_1+\phi_2)(2\lambda_1+3\lambda_2)=6\phi_1(\lambda_1)+9\phi_1(\lambda_2)+2\phi_1(\lambda_1)+3\phi_2(\lambda_2)=12-9-6+6=3$. The multiplicities n_i of 11.2 are given by 2, 2, 1, 1, 1, and the weights $(\tau_i,\tau_i)/(\tau_\mu,\tau_\mu)$ by 1, 1, 3, 1, 1, so that

$$a'=f'^*(\eta)=2x_1+2x_2+3x_3+x_4+x_5.$$

The following computations are facilitated by making the change of basis

$$\begin{aligned}a'&=2x_1+2x_2+3x_3+x_4+x_5,\\b'&=2x_1+2x_2+3x_3+x_4,\\c'&=x_3,\\d'&=x_1+x_2,\\e'&=x_1.\end{aligned}$$

This transforms the relations into

$$\begin{aligned}a'^2&=a'b'+a'e'-b'e',\\b'^2&=3(b'c'+b'd'-c'd'+c'e')-b'e',\\c'^2&=-c'd',\\d'^2&=d'e',\\e'^2&=0.\end{aligned}\tag{14.3}$$

One clearly has $a'b'c'd'e'=x_1x_2x_3x_4x_5$. One computes

$$(a')^3=3(a'-e')(b'c'+b'd'-c'd'+c'e')\tag{14.4}$$

and, with $u'=(a')^3/3$,

$$u'(a')^2=4a'b'c'd'e'.\tag{14.5}$$

Γ'': The variables $x_1, \cdots, x_5$ in $H^2(\Gamma'')$ correspond to the roots $\phi_1 + \phi_2$, $3\phi_1 + 2\phi_2$, $2\phi_1 + \phi_2$, $3\phi_1 + \phi_2$, $3\phi_1 + 2\phi_2$. The relations are computed to be

$$\begin{aligned} &x_1^2 = 0, \\ &x_2(x_2 + 3x_1) = 0, \\ &x_3(x_3 + x_2 + x_1) = 0, \\ &x_4(x_4 + 3x_3 + 2x_2) = 0, \\ &x_5(x_5 + x_4 + 3x_3 + 2x_2 + 3x_1) = 0. \end{aligned} \tag{14.6}$$

The multiplicities n_i of are $1, 2, 1, 1, 1$; the weights $(\tau_i, \tau_i)/(\tau_\mu, \tau_\mu)$ are $3, 1, 3, 1, 1$, so that

$$a'' = f''^*(\eta) = 3x_1 + 2x_2 + 3x_3 + x_4 + x_5.$$

We change variables according to

$$\begin{aligned} a'' &= 3x_1 + 2x_2 + 3x_3 + x_4 + x_5, \\ b'' &= 3x_1 + 2x_2 + 3x_3 + x_4, \\ c'' &= x_1 + x_3, \\ d'' &= x_2, \\ e'' &= x_1. \end{aligned}$$

The relations go over into

$$\begin{aligned} a''^2 &= a''b'', \\ b''^2 &= 3(b''c'' + b''e'' + b''d'' - c''d'' - 3c''e''), \\ c''^2 &= -c''d'' + c''e'' + d''e'', \\ d''^2 &= -3d''e'', \\ e''^2 &= 0. \end{aligned} \tag{14.7}$$

One computes that

$$a''^3 = 3a''(b''c'' + b''e'' + b''d'' - c''d'' - 3c''e''). \tag{14.8}$$

From (14.4) and (14.8), we conclude, by (9.2), that η^3 is divisible exactly by 3; let $\eta^3/3 = u$, and $a''^3/3 = u''$. We reduce now the coefficients in $H^*(\Gamma'')$ from $\mathbf{Z}$ to $\mathbf{Z}_3$. We have then

$$u''a''^2 = (a'')^3(b''c'' + \cdots) = 0 \text{ in } H^*(\Gamma''; \mathbf{Z}_3), \tag{14.9}$$

since $(a'')^3$ is zero mod 3.

Utilizing the fact that the Steenrod power P^1 is a derivation (Cartan formula), and that P^1 applied to a two-dimensional element yields the third power of the element, we obtain after a short computation

$$P^1u'' = -a''b''c''d''e''. \tag{14.10}$$

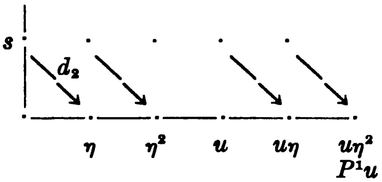
From (14.5) and (14.10), we conclude that $u \cdot \eta^2$ and P^1u are non-zero elements of $H^{10}(\Omega; \boldsymbol{Z}_3)$, since they map into non-zero elements under f', resp. f''. From (14.9) and (14.10), we conclude, applying f'', that $u \cdot \eta^2$ and P^1u are linearly independent. Since $H^{10}(\Omega; \boldsymbol{Z}_3) \approx \boldsymbol{Z}_3 + \boldsymbol{Z}_3$, this proves (14.1).

15. The 3-component of $\pi_{10}(G_2)$. We compute $\pi_i(\Omega) \otimes \boldsymbol{Z}_3$, $i \leqq 9$, by a straightforward application of the method of killing homotopy groups, the principles of which we assume known. Let β denote the Bokstein operator $\beta\colon H^i(X; \boldsymbol{Z}_3) \to H^{i+1}(X; \boldsymbol{Z}_3)$, derived from the coefficient sequence

$$0 \to \boldsymbol{Z}_3 \to \boldsymbol{Z}_9 \to \boldsymbol{Z}_3 \to 0.$$

Throughout this section, $H^i(X)$ will mean $H^i(X; \boldsymbol{Z}_3)$. We utilize known facts about the cohomology of the Eilenberg-MacLane spaces $K(\boldsymbol{Z}_3, n)$, including the behavior of the Bokstein operator.

We kill $\pi_2(\Omega)$ by a suitable bundle X_1, over Ω, with $K(\boldsymbol{Z}, 1)$ or S_1 (the circle) as a fiber. The E_2-term of the corresponding spectral sequence is $H^*(\Omega) \otimes H^*(S_1)$. Let s denote the generator of $H^1(S_1)$ that maps into η under d_2. We have then $d_2(\eta \otimes s) = \eta^2$, $d_2(\eta^2 \otimes s) = 0$, $d_2(u \otimes s) = u \cdot \eta$, $d_2(u \cdot \eta \otimes s) = u \cdot \eta^2$.



It follows that $H^*(X_1)$, up to dimension 10, has the set $\{1, x, v, P^1v\}$ as basis, where $\dim x = 5$, and where v is the image of the element u of $H^6(\Omega)$ under the projection $X_1 \to \Omega$.

Furthermore the relation

$$\beta(x) = \pm v$$

holds. This follows from the fact that, using the integers as coefficients in the above computation, we have $d_2(\eta^2 \cdot s) = 3u$, so that the integral element v', whose reduction mod 3 yields v, satisfies $3v' = 0$.

We now kill x by constructing a suitable bundle X_2 over X_1, with $K(\boldsymbol{Z}_3, 4)$ as fiber. Let y be the generating class of $K(\boldsymbol{Z}_3, 4)$; the cohomology of $K(\boldsymbol{Z}_3, 4)$, up to dimension 9, is given by the basis $\{1, y, \beta y, y^2, P^1y, \beta P^1 y, P^1\beta y, y\cdot\beta y\}$. In the spectral sequence of $X_2 \to X_1$, we have $E_2 = H^*(X_1)\otimes H^*(K(\mathbf{Z}_3, 4)) = E_5$, and $d_5(y) = x$. It follows that

$$d_6(\beta y) = v, d_5(y^2) = 2x\otimes y, d_5(P^1y) = \cdots = d_9(P^1y) = 0,$$
$$d_5(y\cdot\beta y) = x\otimes\beta y, d_{10}(P^1\beta y) = P^1 v, d_{10}(\beta P^1 y) = 0.$$

(Note that βy and P^1y are transgressive, since y is.)

One computes thus that in E_∞ the only elements of dimension $\leqq 9$ are the linear combinations of the elements $1, \bar{w}, \bar{z}$, which come from $1, P^1y, \beta P^1 y$ in E_2. It follows that $H^*(X_2)$, up to dimension 9, has a basis $\{1, w, z\}$, with $\dim w = 8$, $\dim z = 9$. Moreover, $\beta w = z$, since w and z map into P^1y, $\beta P^1 y$ under the inclusion of the fiber $K(\mathbf{Z}_3, 4)$ in X_2.

We now kill w in $H^*(X_2)$ by constructing a suitable bundle X_3 over X_2, with fiber $K(\mathbf{Z}_3, 7)$. If t is the generating class of $H^7(\mathbf{Z}_3, 7)$, then βt generates $H^8(\mathbf{Z}, 7)$. In the spectral sequence we conclude from $d_8 t = w$ and $d_9(\beta t) = \beta w = z$ that $H^i(X_3) = 0$ for $0 < i \leqq 9$.

From the values of the first non-trivial cohomology groups (in positive dimensions) of Ω and the X_i, $i = 1, 2, 3,$, we read off:

$$\pi_i(\Omega)\otimes\mathbf{Z}_3 = \mathbf{Z}_3 \text{ for } i = 2, 5, 8,$$
$$= 0 \text{ for the other } i\text{-values} \leqq 9.$$

Since $\pi_i(\Omega) \approx \pi_{i+1}(G_2)$, this implies in particular $\pi_{10}(G_2)\otimes\mathbf{Z}_3 = 0$, and Theorem V(d) is proved.

Chapter IV. Applications to Symmetric Spaces.

1. Introduction. In Chapter II, three actions were shown to be variationally complete for a symmetric pair (G, K): a) $K\times K$ on G, b) K on G/K, c) K on $\mathfrak{p}$ (by the adjoint action). In this chapter, we study some immediate consequences of Theorem I for these situations. Actually the cases a) and b) are equivalent in the following sense (this has appeared already in Chapter II, no. 6): Let $p\colon G\to G/K$ be the natural projection. If N' is a $K\times K$-orbit in G, then $N = p(N')$ is a K-orbit in G/K. Let R' be a point in G, and put $R = p(R')$. The map p induces a map p' of the path space $\Omega(G; R', N')$ (cf. Chapter I, no. 2) into $\Omega(G/K; R, N)$. It follows easily from the covering homotopy theorem for the fibering $G\to G/K$, that p' induces an isomorphism of the homotopy and therefore also of the homology of the two spaces.

By Theorem I, the mod 2 homology of $\Omega(G;R',N')$ is completely determined by the set $S=S(G;R',N')$ of transversal geodesics from R' to N' in G, as graded by the index δ_s. Our aim is to describe S and δ in each case in terms of invariants of (G,K). As we will show, the diagram of the symmetric pair (G,K) in the sense of E. Cartan [9], a variant of the diagram of a Lie group, contains all the pertinent information. Since there is no description of this diagram in the current literature, we reprove certain of Cartan's results. In particular, we have included Hunt's version of the proof of the conjugacy of the maximal tori of a symmetric space.

Properly understood, all the notions associated with the diagram of a Lie group generalize to the diagram of a symmetric space.

We will start with the adjoint action of K on $\mathfrak{p}$. Here the space $\Omega(\mathfrak{p};R,N)$ is homotopic to the orbit N. Our formulas therefore describe the homology of the orbits of K in $\mathfrak{p}$.

2. The adjoint action of K on $\mathfrak{p}$. We recall the pertinent definitions from Chapter II. The $(+1)$-eigen space of $*$ is the Lie algebra $\mathfrak{k}$ of K; its orthogonal complement, the (-1)-space of $*$, is $\mathfrak{p}$. Thus $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$. For any $A\in\mathfrak{g}$, we have $\mathfrak{g}_A=\{X:[A,X]=0\}$; $\mathfrak{p}_A=\mathfrak{g}_A\cap\mathfrak{p}$; $\mathfrak{k}_A=\mathfrak{g}_A\cap\mathfrak{k}$.

PROPOSITION 2.1. *For any $A\in\mathfrak{p}$, the equations*

(a) $$\mathfrak{g}_A=\mathfrak{k}_A\oplus\mathfrak{p}_A$$

(b) $$\mathfrak{p}=[\mathfrak{k},A]\oplus\mathfrak{p}_A$$

represent orthogonal decompositions.

Relation (2.1(a)) follows from Chapter V, 3.1; for (2.1(b)), we use Chapter II, (7.5), with A replaced by zero and X replaced by A.

Let $\mathfrak{g}_A{}^\perp$ denote the orthogonal complement of $\mathfrak{g}_A$ in $\mathfrak{g}$. We see from (2.1(a)) and (2.1(b)) that $[\mathfrak{k},A]=\mathfrak{g}_A{}^\perp\cap\mathfrak{p}$, and that $\mathrm{ad}A$ maps $\mathfrak{g}_A{}^\perp\cap k$ isomorphically onto $\mathfrak{g}_A{}^\perp\cap\mathfrak{p}$, so that these two spaces have the same dimension. We conclude that

(2.2) $$\dim\mathfrak{k}_A-\dim\mathfrak{p}_A=\dim\mathfrak{t}-\dim\mathfrak{p},$$

independent of A.

This can also be expressed by saying that the trace of $*$ on $\mathfrak{g}_A$ equals the trace of $*$ on $\mathfrak{g}$, for any A.

DEFINITION 2.3. *A maximal abelian subalgebra of $\mathfrak{p}$ is called a Cartan algebra of (G,K). If $\mathfrak{t}$ is such an algebra, $A\in\mathfrak{t}$ is called a general point of $\mathfrak{t}$ if $\mathfrak{p}_A=\mathfrak{t}$. The dimension of $\mathfrak{t}$ is called the rank of (G,K).*

It is clear from general theory that Cartan algebras as well as general points on them exist. In the following, we *denote by* $\mathfrak{t}$ *a fixed Cartan algebra of* (G,K), *and let* A *be a fixed general point of* $\mathfrak{t}$.

Let Q be a point of $\mathfrak{p}$, and let N be the K-orbit of Q. As in Chapter I, no. 2, we denote by $S(\mathfrak{p};A,N)$ the set of geodesics (i. e. straight line segments in $\mathfrak{p}$) which are transversal to the action of K on $\mathfrak{p}$ and join A to points of N. Let $S(\mathfrak{t};A,\mathfrak{t}\cap N)$ be the set of straight lines in $\mathfrak{t}$ joining A to points of $\mathfrak{t}\cap N$.

PROPOSITION 2.4. $S(\mathfrak{p};A,N)=S(\mathfrak{t};A,\mathfrak{t}\cap N)$.

Proof. A transversal line from A to N must be perpendicular to the orbit of A at A (cf. Chapter I, (2.2)). By Chapter II, (7.1) and (7.4), the orthogonal complement to the orbit of A, at A, is precisely $\mathfrak{p}_A$, which is equal to $\mathfrak{t}$. Hence all the elements of $S(\mathfrak{p};A,N)$ lie in $\mathfrak{t}$. Conversely, every line in $\mathfrak{t}$ starting at A is transversal. This proves the proposition.

COROLLARY 2.5. *Every orbit* N *of* K *on* $\mathfrak{p}$ *intersects* $\mathfrak{t}$.

Because every such orbit is compact, there must be a point at which N is closest to A. The line from A to that point will then have to be transversal. Hence $S(\mathfrak{p};A,N)$ is not empty. Now apply Proposition 2.4.

COROLLARY 2.6. *Any two Cartan algebras of* (G,K) *are conjugate under the adjoint action of* K.

Proof. Let $\mathfrak{t}'$ be any Cartan algebra of (G,K), let B be a general point of $\mathfrak{t}'$ and let N be the orbit of B. Since $N\cap\mathfrak{t}\neq 0$ there exists a $k\in K$, such that $\mathrm{Ad}k\cdot B\subset\mathfrak{t}$. But then clearly $\mathrm{Ad}k\cdot\mathfrak{t}'=\mathfrak{t}$, so that $\mathfrak{t}'$ is conjugate to $\mathfrak{t}$.

Note. This corollary goes back to E. Cartan [9]. The proof given here is an immediate extension of the proof due to Hunt for the conjugacy of Cartan algebras in the group case [13].

PROPOSITION 2.7. (a) *Every orbit of* K *in* $\mathfrak{p}$ *intersects* $\mathfrak{t}$ *orthogonally.* (b) *The orbit through* A *has maximal dimension.*

Proof. Clearly $\mathfrak{p}_B\supset\mathfrak{p}_A=\mathfrak{t}$ for any $B\in\mathfrak{t}$. As the transversal space to the orbit through B is $\mathfrak{p}_B$, this inclusion together with Corollary 2.5 proves both statements.

We write δ_K' for the defect function of the adjoint action of K on $\mathfrak{p}$, as defined in Chapter I, (7.4). Because of (2.7(b)) and Chapter II, (7.1), we have

$$\delta_K'(B)=\dim[\mathfrak{k},A]-\dim[\mathfrak{k},B]. \tag{2.8}$$

Applying (2.1(b)), (2.2), we obtain

$$\delta_K'(B) = \dim \mathfrak{p}_B - \dim \mathfrak{p}_A = \dim \mathfrak{k}_B - \dim \mathfrak{k}_A.$$

This proves

PROPOSITION 2.9.

$$\delta_K'(B) = \tfrac{1}{2}(\dim \mathfrak{g}_B - \dim \mathfrak{g}_A)$$

for any $B \in \mathfrak{p}$.

According to Chapter I, (7.3), the defect δ_s of any segment

$$s \in S(\mathfrak{t}; A, N \cap \mathfrak{t}),$$

with end points A and Q, is given by

$$\delta_s = \sum_{0 \leq \alpha < 1} \delta_K'((1-\alpha)A + \alpha Q). \tag{2.10}$$

We extend $\mathfrak{t}$ to a Cartan algebra $\tilde{\mathfrak{t}}$ of the Lie algebra $\mathfrak{g}$, and denote by $D'(G)$ the infinitesimal diagram of G on $\mathfrak{t}$ (cf. Chapter III, no. 1).

As a consequence of (2.9), δ_K' can be described entirely in terms of $D'(G)$. In fact, it is easily seen that

(2. 11) $\delta_K'(B) =$ *number of planes of* $D'(G)$ *containing* B, *but not containing all of* $\mathfrak{t}$.

(For each root $\theta \in \mathfrak{R}$ with $\theta(B) = 0$, the dimension of $\mathfrak{g}_B$ goes up by two.)

The set of points B in $\mathfrak{t}$ with $\delta_K'(B) > 0$ is called the infinitesimal diagram of (G, K), and is denoted by $D'(G, K)$; it consists of a finite number of planes through the origin. The defect δ_s of a segment s is the number of planes, properly weighted, in $D'(G, K)$ crossed by s; note that planes of $D'(G, K)$ containing the end point Q of s do not count in δ_s.

The connected components of the complement of $D'(G, K)$ in $\mathfrak{t}$ are called the *fundamental chambers* of $D'(G, K)$. In complete analogy to the group case, every chamber $\mathfrak{F}$ plays the role of a fundamental domain for the Weyl group of (G, K), as will be seen later. This group, denoted by $\mathfrak{W}(G, K)$, is defined as the quotient of the group consisting of the elements in K which keep $\mathfrak{t}$ setwise fixed (the normalizer of $\mathfrak{t}$) modulo the group of the elements which keep $\mathfrak{t}$ pointwise fixed (i.e. $K_{\mathfrak{t}}$). From Proposition 2.7(a), applied to A (and the fact that all orbits are compact), it follows that $\mathfrak{W}(G, K)$ is a finite group. It is also clear that if B is any general point of $\mathfrak{t}$ and N is the orbit of B, then

$$N \cap \mathfrak{t} = \textit{orbit of } B \textit{ under } \mathfrak{W}(G, K). \tag{2.12}$$

(Actually, a more general statement is true: if A, B are two subsets of $\mathfrak{t}$, conjugate under $k \in K$, then there exists an element of the Weyl group, whose restriction to A equals that of $\mathrm{Ad}k$. One proves this by considering the centralizer, in G, of $\mathfrak{t} \cap \mathrm{Ad}k(\mathfrak{t})$. A similar statement holds for the global case, i.e., for the operation of the Weyl group on the maximal torus T, cf. Definition 3.3.)

For any K-orbit N on $\mathfrak{p}$, we write $\bar{S}_*(N)$ for the graded $\mathbf{Z}_2$-vector space generated by the elements of $S(\mathfrak{t}; A, \mathfrak{t} \cap N)$, the dimension of a segment being the defect δ_s (cf. (2.10)).

We can now state the consequences of Theorems I and II in the following form:

THEOREM VI. *Let N be any orbit of K on $\mathfrak{p}$. Then (as graded modules)*

$$H_*(N; \mathbf{Z}_2) \approx \bar{S}_*(N).$$

Proof. We have $\Omega(\mathfrak{p}; A, N) \approx N$ because $\mathfrak{p}$ is a Euclidean space (cf. Chapter III, no. 6). By Theorem II, the action of K on $\mathfrak{p}$ is variationally complete, and the point A has defect 0. Hence Theorem I is applicable, and describes $H_*(N; \mathbf{Z}_2)$ in terms of $S(\mathfrak{p}; A, N)$, or, because of (2.4), in terms of $S(\mathfrak{t}; A, \mathfrak{t} \cap N)$, graded by δ_s.

Remarks. This theorem is the complete analogue of Theorem III in [6], except that the coefficients have been reduced mod 2. One can of course give the data for $\bar{S}_*(N)$ entirely in terms of the roots of G, and the group $\mathfrak{W}(G, K)$. Such a formulation was given in [6]. We omit these details here, and give only one example in the next corollary.

COROLLARY 2.13. *Let $\mathfrak{W}_*$ be the graded $\mathbf{Z}_2$-vector space generated by the elements of $\mathfrak{W}(G, K)$, the dimension of $w \in \mathfrak{W}(G, K)$ being*

$$\sum_{0 < \alpha \leq 1} \delta_K' \{ (1 - \alpha) A + \alpha (w \cdot A) \}.$$

Let $K_{\mathfrak{t}}$ be the centralizer of $\mathfrak{t}$ in K. Then as graded modules,

$$H_*(K/K_{\mathfrak{t}}; \mathbf{Z}_2) = \mathfrak{W}_*$$

Proof. The elements of $\mathfrak{W}(G, K)$ clearly permute the fundamental chambers of $D'(G, K)$, since the function δ_K' is invariant under $\mathfrak{W}(G, K)$. We need the following proposition:

PROPOSITION 2.14. *The Weyl group $\mathfrak{W}(G, K)$ is transitively and faithfully represented by the permutations it induces in the fundamental chambers of $D'(G, K)$.*

We prove this proposition by means of Theorem VI. Let $\mathfrak{F}$ be any fundamental chamber of $D'(G,K)$, and let B be a general point in it with K-orbit N. Let $\mathfrak{F}_A$ be the chamber that contains A. By (2.10), (2.11), (2.12), the number of points in the $\mathfrak{W}(G,K)$-orbit of B that lie in $\mathfrak{F}_A$ equals the number of segments in $S(\mathfrak{t};A,\mathfrak{t}\cap N)$ of index 0. By Theorem VI, this is also the number of components of N. But K being connected, N is connected, and (2.14) is proved.

The Corollary 2.13 is now immediate. Let B be taken in $\mathfrak{F}_A$. With any $w \in \mathfrak{W}(G,K)$, associate the segment from A to $w\cdot B$. This induces a gradation preserving isomorphism of $\bar{S}_*(N)$ and $\mathfrak{W}_*$. Since B is general in $\mathfrak{t}$, its K-orbit N is homeomorphic to $K/K_{\mathfrak{t}}$. The corollary follows now from Theorem VI.

Let $\mathfrak{t}'$ be any subspace of $\mathfrak{t}$. A result analogous to (2.13) can then be stated for the homology of $K/K_{\mathfrak{t}'}$ in terms of the quotient set of $\mathfrak{W}(G,K)$ by the subgroup that leaves $\mathfrak{t}'$ point-wise fixed.

3. The action of $K \times K$ on G. This section is the global analogue of the previous one. As in Chapter II, we write $\mathfrak{k}_a = \mathfrak{k} \cap \mathrm{Ad}a^{-1}\mathfrak{k}$, $\mathfrak{p}_a = \mathfrak{p} \cap \mathrm{Ad}a^{-1}\mathfrak{p}$ for any $a \in G$. We also write $\mathfrak{g}_a$ for $\{X \in g : \mathrm{Ad}aX = X\}$; note that the definition of $\mathfrak{g}_a$ differs from that of $\mathfrak{k}_a$ and $\mathfrak{p}_a$.

PROPOSITION 3.1. *If $a^* = a^{-1}$, then*

(a) $$\mathfrak{g}_{a^2} = \mathfrak{k}_a \oplus \mathfrak{p}_a$$

(b) $$\mathfrak{p} = \{\mathrm{Ad}a - \mathrm{Ad}a^{-1}\}\, \mathfrak{k} \oplus \mathfrak{p}_a$$

are orthogonal decompositions.

Proof. Clearly, $\mathfrak{g}_{a^2}$ is stable under $*$. Hence $\mathfrak{g}_{a^2} = \mathfrak{g}_{a^2} \cap \mathfrak{k} + \mathfrak{g}_{a^2} \cap \mathfrak{p}$. If $Z \in \mathfrak{p}_a$, then $\mathrm{Ad}aZ \in \mathfrak{p}$. Hence, applying $*$, $\mathrm{Ad}a^{-1}Z = \mathrm{Ad}aZ$, whence $Z \in \mathfrak{g}_{a^2} \cap \mathfrak{p}$. Hence $\mathfrak{p}_a \subset \mathfrak{g}_{a^2} \cap \mathfrak{p}$. Conversely, if $Z \in \mathfrak{g}_{a^2} \cap \mathfrak{p}$, then $(\mathrm{Ad}a^{-1}Z)^* = -\mathrm{Ad}aZ = -\mathrm{Ad}a^{-1}Z$, so that $Z \in \mathfrak{p}_a$. Hence $\mathfrak{p} \supset \mathfrak{g}_{a^2} \cap \mathfrak{p}$. Quite similarly, one finds that $\mathfrak{g}_{a^2} \cap \mathfrak{k} = \mathfrak{k}_a$. This proves 3.1(a). Next, $\{(\mathrm{Ad}a - \mathrm{Ad}a^{-1})Z\}^* = -(\mathrm{Ad}a - \mathrm{Ad}a^{-1})Z^*$. Hence, $(\mathrm{Ad}a - \mathrm{Ad}a^{-1})\mathfrak{k} \subset \mathfrak{p}$, and $(\mathrm{Ad}a - \mathrm{Ad}a^{-1})\mathfrak{p} \subset \mathfrak{k}$. Suppose that $Z \in \mathfrak{p}$ is perpendicular to $(\mathrm{Ad}a - \mathrm{Ad}a^{-1})\mathfrak{k}$, so that

$$(\{\mathrm{Ad}a - \mathrm{Ad}a^{-1}\}\mathfrak{k}, Z) = 0.$$

This implies $(\mathrm{Ad}a^{-1} - \mathrm{Ad}a)Z \in \mathfrak{p}$, whence $\mathrm{Ad}a^{-1}Z = \mathrm{Ad}aZ$, so that $Z \in \mathfrak{g}_{a^2} \cap \mathfrak{p} = \mathfrak{p}_a$. The argument can be reversed, and (3.1(b)) is proved.

COROLLARY 3.2. *If $a^* = a^{-1}$,* then

$$\dim \mathfrak{g}_{a^2} \cap \mathfrak{p} = \dim \mathfrak{g}_{a^2} \cap \mathfrak{k};$$

$$\dim \mathfrak{k}_a - \dim \mathfrak{p}_a = \dim \mathfrak{k} - \dim \mathfrak{p}.$$

This follows from (3.1(a), (b)) as (2.2) followed from (2.1(a), (b)) in no. 2; first one has to verify, using *, that $\mathfrak{k}_a$ is the kernel of the restriction of $\mathrm{Ad}a - \mathrm{Ad}a^{-1}$ to $\mathfrak{k}$.

DEFINITION 3.3. *The image in G, under the exponential map, of a Cartan algebra $\mathfrak{t}$ of (G,K) is called a maximal torus of (G,K), and is denoted by T. The point $a \in T$ is called general if $\mathfrak{p}_a = \mathfrak{t}$.*

The maximality of $\mathfrak{t}$ implies that $\exp \mathfrak{t}$ is closed in G. By general theory, we can find an $a \in T$ such that $\mathfrak{g}_{a^2} \cap \mathfrak{p} = \mathfrak{t}$. Hence by (3.1), T contains general points.

In the sequel, T denotes a fixed maximal torus of (G,K) and a is a fixed general point of T. The tangent space to T at the identity is identified with the Cartan algebra $\mathfrak{t}$.

Let $b \in T$, and let $N = K \cdot b \cdot K$ be the orbit of b under $K \times K$. By Chapter II, (7.1), (7.4) the transversal space to N at b is precisely $b \cdot \mathfrak{p}_b$. The arguments of no. 2 therefore easily yield the following analogue of (2.4), (2.5), and (2.7).

PROPOSITION 3.4. *Let N be any orbit of $K \times K$ on G. Then $N \cap T$ is not vacuous. N intersects T orthogonally (and therefore at a finite number of places). The sets*

$$S(G; a, N) \text{ and } S(T; a, T \cap N)$$

coincide. The orbit of a has maximal dimension.

Here $S(G; a, N)$ of course denotes the set of transversal geodesics from a to N, while $S(T; a, T \cap N)$ stands for the, automatically transversal, geodesics on T from a to points of $N \cap T$.

We denote by δ_K the defect function of $K \times K$ on G, defined as in Chapter I, (7.1). By Chapter II, no. 3 we have $\delta_K(b) = \dim \mathfrak{k}^a - \dim \mathfrak{k}^b$. As in no. 2, one proves that

$$(3.5) \qquad \delta_K(b) = \tfrac{1}{2}(\dim \mathfrak{g}_{b^2} - \dim \mathfrak{g}_{a^2}) \text{ for any } b \in T.$$

(Note that for such b's the relation $b^* = b^{-1}$ holds.)

As before, $\tilde{\mathfrak{t}}$ is a Cartan algebra of g, containing $\mathfrak{t}$; let $D(G)$ be the diagram of G on $\tilde{\mathfrak{t}}$ (cf. Chapter III, no. 1). We define the *defect function* δ of (G,K) by

(3.6) $\delta(B) =$ *number of planes of* $D(G)$ *that contain* B, *but do not contain all of* $\mathfrak{t}$, *for any* $B \in \mathfrak{t}$.

The diagram $D(G,K)$ consists now by definition of all $B \in \mathfrak{t}$ with $\delta(B) > 0$; it can also be described as the union of the intersections $\mathfrak{p} \cap \mathfrak{t}$ for all those singular planes $p = (\theta, n) \in \mathfrak{P}$, whose translation vector τ_θ is not orthogonal to $\mathfrak{t}$.

The defects δ and δ_K are related by the following proposition.

PROPOSITION 3.7. *Let* $\rho\colon \mathfrak{t} \to T$ *be defined by* $\rho(B) = \exp(\frac{1}{2}B)$. *Then* $\delta_K \circ \rho = \delta$, *where* δ_K *is the defect function of* $K \times K$ *on* G, *and* δ *is the defect function of* $D(G,K)$.

This formula is an immediate consequence of (3.5), and the meaning of $D(G)$ for the stabilizers of points in T. Let A be an element of $\mathfrak{t}$ that maps into a under ρ; clearly A is general in $\mathfrak{t}$. For any $K \times K$-orbit N, the elements of $S(T; a, T \cap N)$ then lift under ρ uniquely into straight line segments in $\mathfrak{t}$, with initial point A. Let $S'(N) = S'(\mathfrak{t}; A, \rho^{-1}(T \cap N))$ denote this set of line segments. Let s', with end points A and B, be an element of $S'(N)$, corresponding to $s \in S(T; a, T \cap N)$. Then the δ_K-defect of s is equal to the δ-defect of s', given by

$$(3.8) \qquad \delta_{s'} = \sum_{0 \leq \alpha < 1} \delta((1-\alpha)A + \alpha B),$$

the number of planes, properly weighted, of $D(G,K)$ crossed by s'. Finally, let $\bar{S}'_*(N)$ be the $\mathbf{Z}_2$-vector space generated by $S'(N)$ according to the defect δ. The global analogue of Theorem VI now clearly takes the following form:

THEOREM VII. *Let* N *be any orbit of* $K \times K$ *on* G. *Then, as graded modules,*

$$H_*\{\Omega(G; a, N); \mathbf{Z}_2\} \approx \bar{S}'_*(N).$$

The Corollary 2.13 to Theorem VI also has a global analogue. We will formulate it in terms of the cells of $D(G,K)$. By definition, these are the connected components of $\mathfrak{t} - D(G,K)$. We let $\mathfrak{F}$ be a fundamental chamber of $D'(G,K)$ and denote by $\{\Delta\}_{\mathfrak{F}}$ the set of cells of $D(G,K)$ contained in the closure of $\mathfrak{F}$. We assign an index $\delta(\Delta)$ to $\Delta \in \{\Delta\}_{\mathfrak{F}}$ by the formula

$$(3.9) \qquad \delta(\Delta) = \sum_{0 < t \leq 1} \delta(tB),$$

where δ is the defect function of (G,K) and B is any point of Δ. (In words, $\delta(\Delta) =$ number of properly counted planes in $D(G,K)$ crossed by a line from 0 to a point of Δ.) Finally, we let $\mathcal{J}_*$ be the graded $\mathbf{Z}_2$-vector space generated by $\{\Delta\}_{\mathcal{J}}$ and graded by δ.

COROLLARY 3.10. *Let* $\pi_1(G)=0$. *Then,*

$$H_*(\Omega(G/K);\mathbf{Z}_2)\approx\mathcal{J}_*$$

as graded modules.

Proof. Because $\pi_1(G)=0$ and $\pi_0(K)=0$, G/K is simply connected. Hence $\Omega(G/K)$ has precisely one component. Let $N=K$ be the orbit of $e\in G$ under $K\times K$. Then, as explained in no. 1, $\Omega(G/K)$ can be identified with $\Omega(G;a,K)$. Let $\mathcal{J}_0=\rho^{-1}(K\cap T)$, so that $S'(K)$ consist of the segments from A to the points of $\mathcal{J}_0$. Then Theorem VII implies that $\mathcal{J}_0$ intersects the closure of any cell Δ in $D(G,K)$ precisely once. To show this, we may assume $A\in\Delta$. Since $\Omega(G;a,K)$ is connected, by Theorem VII there is only one segment in $S'(K)$ of index 0; but this is equivalent to the assertion about $\mathcal{J}_0$.

Now to any $s\in S'(K)$, with second end point $B\in\mathcal{J}_0$, there exists a unique $w\in\mathcal{W}(G,K)$ such that $w\cdot(A-B)$ lies in the interior of a cell in $\mathcal{J}$. As in [6], one shows that this defines a one-to-one map of $S'(K)$ onto $\{\Delta\}_{\mathcal{J}}$, and that the induced isomorphism of $\bar{S}'_*(K)$ onto $\mathcal{J}_*$ is gradation preserving. The Corollary now follows from Theorem VII.

As a simple application, we will prove the following proposition, which, as the referee pointed out, is contained in [9], no. 101.

PROPOSITION 3.11. *Let * be an involution of the compact, simply connected Lie group G. Then the set of fixed points of * is connected.*

Proof. Let (G,K) be the symmetric pair determined by *, and let $\hat{K}$ be the fixed point set of *. Thus K is the e-component of $\hat{K}$. We start with the following lemma.

LEMMA 3.12. *If* $K\cap T=\hat{K}\cap T$, *then* $K=\hat{K}$.

Proof. Suppose K_1 is a component of $\hat{K}$ different from K. The $K\times K$-orbit of any point e_1 of K_1 is clearly K_1 itself. By (3.4), this orbit intersects T, so that $K_1\cap T$ is not vacuous. But $K\cap T$ and $K_1\cap T$ are disjoint, and (3.12) follows. Next, we remark:

(3.13) *The set* $\hat{K}\cap T$ *consists of the points* $x\in T$, *with* $x^2=e$.

This follows from the fact that on T the involution * is just inversion.

LEMMA 3.14. *If* $\pi_1(G)=0$, *then* $K\cap T=\hat{K}\cap T$.

Proof. Let $\mathcal{J}_0 = \rho^{-1}(K \cap T)$ as before, and let $\mathcal{J} = \rho^{-1}(\hat{K} \cap T)$. It follows from (3.13) and the definition $\rho(B) = \exp(\frac{1}{2}B)$ that $\mathcal{J}$ consists of the lattice of points in $\mathfrak{t}$ which under the exponential map go into e. We shall show that $\mathcal{J}_0 = \mathcal{J}$. Suppose then that $\mathcal{J}$ is a proper refinement of $\mathcal{J}_0$. Since the closed cells of $D(G,K)$ cover $\mathfrak{t}$, and since $\mathcal{J}_0$ intersects each closed cell of $D(G,K)$ in precisely one point (as noted in the proof of 3.10), the lattice $\mathcal{J}$ will have to intersect the closure of some cell, say Δ, in at least two points. From the construction of $D(G,K)$, it is clear that Δ is contained in the closure of some cell Δ_1 of $D(G)$ in $\tilde{\mathfrak{t}}$. (Here $\tilde{\mathfrak{t}}$ is a Cartan algebra of G containing $\mathfrak{t}$.) Hence the closure of Δ_1 in $D(G)$ intersects $\mathcal{J}$ in at least two points. But it is well known that if $\pi_1(G) = 0$, then each closed cell of $D(G)$ contains exactly one point whose exponential image is e (cf. [20]). This contradiction proves (3.14). The two Lemmas 3.12, 3.14 prove Proposition 3.11.

4. An example. Symmetric spaces of maximal rank. According to (3.10), the mod 2 Poincaré series of $\Omega(G/K)$ is given by $\sum_{\mathcal{F}} t^{\delta(\Delta)}$, where Δ runs over the cells of a fundamental chamber $\mathcal{F}$ in $D(G,K)$. In Theorem B, Part 1, of [6], the Poincaré series of $\Omega(G)$ was given as $\sum_{\mathcal{F}} t^{2\lambda(\Delta)}$, where one now sums over the cells of a fundamental chamber in $D(G)$. Suppose now that the maximal torus of (G,K) is also a maximal torus of G. In this case, $D(G\ )$coincides with $D(G,K)$, and $\delta(\Delta)$ as defined here, agrees with the $\lambda(\Delta)$ of [6]. We therefore get the following somewhat mysterious application of 3.10.

PROPOSITION 4.1. *Suppose that $\pi_1(G) = 0$, and that the rank of the symmetric pair (G,K) equals the rank of G. Then*

$$\dim H_q(\Omega(G/K); \mathbf{Z}_2) = \dim H_{2q}(\Omega(G); \mathbf{Z}_2).$$

Every 1-connected Lie group has an essentially unique involution *, with rank $(G,K) = \operatorname{rank} G$. This involution correspond to "the" real form of the complexified Lie group. We will call a G/K obtained in this way a symmetric space of maximal rank. The mod 2 Betti numbers of $\Omega(G/K)$ are then determined by the Betti numbers of $\Omega(G)$ by (4.1); on the other hand, the Poincaré series of $\Omega(G)$ is determined by the Poincaré polynomial of G (cf. [6]). Indeed, if

$$P(G;t) = (1 + t^{2m_1-1}) \cdots (1 + t^{2m_l-1}),$$

then

$$P(\Omega(G);t) = (1 - t^{2(m_1-1)})^{-1} \cdots (1 - t^{2(m_l-1)})^{-1}.$$

Combining this observation with (4.1), we obtain the following proposition.

PROPOSITION 4.2. *Let G/K be a symmetric space of maximal rank, where G is a 1-connected compact Lie group. If $m_1, \cdots, m_l$ are the primitive exponents of G, then*

$$P(\Omega(G/K); \mathbf{Z}_2; t) = (1-t^{m_1-1})^{-1}(1-t^{m_2-1})^{-1} \cdots (1-t^{m_l-1})^{-1}$$

A similar comparison of (2.15) with Theorem B, Part 2 of [6], leads to the following proposition.

PROPOSITION 4.3. *Let (G, K) be a symmetric pair of maximal rank. Let T_1 be a torus of G contained in a maximal torus of (G, K). Let K_{T_1} and G_{T_1} denote the centralizers of T_1 in K and G respectively. Then*

$$\dim H_q(K/K_{T_1}; \mathbf{Z}_2) = \dim H_{2q}(G/G_{T_1}; \mathbf{Z}_2).$$

As a concrete instance of the halving in dimensions, we mention the real projective space as opposed to the complex projective space.

Applying the Hirsch formula [1] to the right hand side of (4.3), with $T_1 = T$, one obtains the following analogue of (4.2).

PROPOSITION 4.4. *Let (G, K) be a symmetric pair of maximal rank; let T be a maximal torus of (G, K); let $m_1, \cdots, m_l$ be the primitive exponents of G. Then*

$$P(K/K_T; \mathbf{Z}_2; t) = (1-t^{m_1}) \cdots (1-t^{m_l})/(1-t)^l.$$

In certain cases, for instance when $G = SU(n)$ and $K = SO(n)$, the formulas (4.3) and (4.4) also follow from Borel's work with the maximal tori mod 2 of G [2].

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REFERENCES.

[1] A. Borel, "Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts," *Annals of Mathematics*, vol. 57 (1953), pp. 115-207.

[2] ———, "La cohomologie mod 2 de certains espaces homogènes," *Commentarii Mathematici Helvetici*, vol. 27 (1953), pp. 165-197.

[3] ———, "Sur l'homologie et la cohomologie des groupes de Lie compacts connexes," *American Journal of Mathematics*, vol. 76 (1954), pp. 273-302.

[4] ———, "Topology of Lie groups and characteristic classes," *Bulletin of the American Mathematical Society*, vol. 61 (1955), pp. 397-432.

[5] A. Borel and J. de-Siebenthal, "Les sous-groupes fermés de rang maximum des groupes de Lie clos," *Commentarii Mathematici Helvetici*, vol. 23 (1949), pp. 200-221.

[6] R. Bott, "An application of the Morse theory to the topology of Lie groups," *Bulletin de la Société Mathématique de France*, vol. 84 (1956), pp. 251-282.

[7] R. Bott and H. Samelson, "The cohomology ring of G/T," *Proceedings of the National Academy of Sciences*, vol. 41 (1955), pp. 490-493.

[8] ———, "Applications of Morse theory to symmetric spaces," *Proceedings of the Symposium international de Topologia Algebraica*, Mexico City, 1956, (to appear).

[9] E. Cartan, "Sur certaines formes Riemanniennes remarquables des géométries a groups fondamental simple," *Annales Scientifiques de l'Ecole Normale Supérieure*, vol. 44 (1927), pp. 345-467.

[10] H. Hopf, "Über den Rang geschlossener Lie'scher Gruppen," *Commentarii Mathematici Helvetici*, vol. 13 (1940/51), pp. 119-143.

[11] ———, "Maximale Toroide und singuläre Elemente in geschlossenen Lie'schen Gruppen," *ibid.*, vol. 15 (1942/43), pp. 59-70.

[12] ———, "Sur une formule de la theorie des espaces fibrés," *Collogue de Topologie Bruxelles* (1950), pp. 117-121.

[13] G. A. Hunt, "A theorem of Elie Cartan," *Proceedings of the American Mathematical Society*, vol. 7 (1956), pp. 307-308.

[14] S. D. Liao, "On the theory of obstructions of fiber bundles," *Annals of Mathematics*, vol. 60 (1954), pp. 146-191.

[15] D. Montgomery, H. Samelson and C. T. Yang, "Exceptional orbits of highest dimension," *Annals of Mathematics*, vol. 64 (1956), pp. 131-141.

[16] M. Morse, *The Calculus of Variations in the Large*, American Mathematical Society Colloquium Publications, vol. 18, New York, 1934.

[17] H. Seifert and W. Threlfall, *Variationsrechnung im Grossen* (*Theorie von Marston Morse*), Teubner, Leipzig, 1938.

[18] J. P. Serre, "Homologie singulière des espaces fibrés. Applications," *Annals of Mathematics*, vol. 54 (1951), pp. 425-505.

[19] N. E. Steenrod, "Cohomology operations derived from the symmetric group," *Commentarii Mathematici Helvetici*, vol. 31 (1957), pp. 195-218.

[20] E. Stiefel, "Über eine Beziehung zwischen geschlossenen Lie'schen Gruppen und diskontinuierlichen Bewegungsgruppen euklidischer Räume und ihre Anwendung auf die Aufzählung der einfachen Lie'schen Gruppen," *ibid.*, vol. 14 (1941/42), pp. 350-380.