GEOMETRY AND STRING THEORY SEMINAR: FALL 2018

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Contents

1. Moduli of flat SL_3-connections and exact WKB: 9/5/18

The first talk this semester was given by Andy Neitzke.

Let C be a thrice-punctured $\mathbb{CP}^3$, say punctured at $\{1, \omega, \omega^2\}$, and let $\mathcal{M}$ denote the moduli space of flat $\mathrm{SL}_3(\mathbb{C})$-connection over C with unipotent holonomy around the punctures; this is an example of a *character variety*. This talk will discuss Andy's work (in progress) with Lotte Hollands on constructing nice coordinate systems on this space, using ideas coming from physics.

Let's start with the simpler case of SL_2, and consider the *Mathieu equation*, a Schrödinger equation with periodic potential. Let $\hbar > 0$; then, the Mathieu equation is

$$\left(-\frac{\hbar^2}{2}\partial_x^2 + \cos x - E\right)\psi(x) = 0. \tag{1.1}$$

Parallel transport (i.e. evolution of solutions) of this equation defines a flat $\mathrm{SL}_2(\mathbb{R})$-connection ∇ on $\mathbb{R}$. You might think it's $\mathrm{GL}_2(\mathbb{R})$, because there are two solutions, but they're related by the Wronskian. Since the potential is periodic, this is a connection on $\mathbb{R}/2\pi\mathbb{Z} = S^1$; now we can ask about its monodromy, or about its eigenvalues (which are easier to write down without making additional choices). In physics, the eigenvalues are known as *quasi-momenta* for a particle moving with respect to this potential.

Let ψ be an eigenfunction with eigenvalue λ. If $E \gg 1$, then $\cos x$ is small, so

$$\psi_\pm(x) \coloneqq \exp\left(\pm i\frac{\sqrt{2E}}{\hbar}x\right) \tag{1.2}$$

is a basis of the solutions. The eigenvalues are

$$\lambda_\pm = \exp\left(\pm 2\pi\frac{\sqrt{2E}}{\hbar}\right), \tag{1.3}$$

and the trace is $2\cos(2\pi\sqrt{2E}/\hbar)$. Then $|\lambda_\pm| = 1$ for all E.

So the trace is periodic in $\sqrt{E}$. If this is close to ± 2, we're in a region called the "gap": ΔE is exponentially small, and so solutions are stable. When the absolute value of the trace is smaller, we're in the "band," where the monodromy is complex. This means that solutions exponentially blow up or exponentially decay.

Remark 1.4. In solid-state physics, one example of periodic potentials are crystals. One can show that bands and gaps correspond to conducting and insulating states. ◀

Because of this application, physicists have developed lots of techniques for studying these systems, which we can adapt to geometry to study the monodromy.

First, let's complexify: let $z = e^{ix}$; then we have a complex Schrödinger equation

$$\big(\hbar^2\partial_z^2 + P(z)\big)\psi = 0, \tag{1.5}$$

where

$$P(z) = \frac{1}{z^3} - \frac{2E - \hbar^2/4}{z^2} + \frac{1}{z}. \tag{1.6}$$

The $\hbar^2/4$ correction isn't that important.

Remark 1.7. You can do this on any Riemann surface as long as P is a holomorphic quadratic differental; this requires choosing a complex projective structure. But the ideas can be gotten across in coordinates. ◀

To understand the monodromy, we need to get at the solutions. The exact WKB method constructs solutions of the form

$$\psi(z) = \exp\left(\frac{1}{\hbar}\int_{z_*}^{z} \lambda\, \mathrm{d}z\right). \tag{1.8}$$

In order to satisfy (1.5), λ must satisfy the *Riccati equation*

$$\lambda^2 + P + \hbar\partial_z\lambda = 0. \tag{1.9}$$

This is easier to solve than the original equation. Namely, to leading order in $\hbar$, $\lambda^2 + P = 0$. We will then plug this back in to get at higher orders in $\hbar$. Specifically, we get

$$\lambda = \sqrt{-P} - \hbar\frac{P'}{4P} + \hbar^2\sqrt{-P}\frac{5(P')^2 - 4PP''}{32P^3} + \cdots \tag{1.10}$$

This naturally lives on the *spectral curve* for the equation, i.e. the Riemann surface for $\sqrt{-P}$, $\Sigma \coloneqq \{y^2 + P(z) = 0\}$, a double cover of the original surface.

This isn't the end of the story, though: solutions will have monodromy around the zeros of P. But we also can't have monodromy (TODO: I missed why). Looking more closely at (1.10), it doesn't actually converge: it's just an asymptotic series. But it's still useful; it admits Borel summation for $\hbar > 0$ and away from a locus called the *Stokes graph* $W(P)$.[1]

The Stokes graph cuts the Riemann surface into domains; inside each domain, everything works, and you learn a lot about the solutions. But you can't do anything in a neighborhood of a zero of P, which prevents the paradox we chanced upon earlier. The upshot is that in each domain, there's a canonical basis (up to scaling) of the solution space: the solutions are a line bundle over the spectral curve, together with a connection ∇^{ab} represented by $\hbar\lambda\,\mathrm{d}z$. And there'sa canonical way to glue these line bundles over $W(P)$, to obtain a line bundle $L \to \Sigma$ together with a flat connection. It's almost flat (the monodromy around branch points might be -1).

It's natural to compute the holonomy $X_\gamma \in \mathbb{C}^\times$ around a curve γ, and this has nice properties. As $\hbar \to 0$, the asymptotic series of this is computable, e.g. $X_\gamma \sim \exp(\hbar^{-1}Z_\gamma)$, where $Z_\gamma = \oint_\gamma \sqrt{-P}\,\mathrm{d}z$. In a given example (choose P, draw the spectral network, fix a loop), this is completely concrete. The trace is almost an eigenvalue of the monodromy, but it has to cross one of the lines in $W(P)$, and the formula shows that. Specifically, one gets a term for the cosine and a term responsible for the gaps (and hence can be studied to learn about the gaps).

So any particular picture/problem comes with its own picture and defines a coordinate system.

What changes for SL_3? We need a higher-rank analogue of the Schrödinger equation, which will have two potentials P_2 and P_3:

$$\left(\partial_z^3 + \hbar^{-2}P_2(z)\partial_z + \left(\hbar^{-3}P_3(z) + \frac{1}{2}\hbar^{-2}P_2'(z)\right)\right)\psi(z) = 0. \tag{1.11}$$

There's a higher WKB method to deal with such equations, but let's look at a specific example, in which

$$P_3 = -\frac{u}{(z^3-1)^2} \qquad \text{and} \qquad P_2 = \frac{9\hbar^2}{(z^3-1)^2}. \tag{1.12}$$

[1] The same locus appears in $\mathcal{N} = 2$ supersymmetry, where it's called a *spectral network*, but its origin is older.

The $9\hbar^2$ term in P_2 won't matter for the spectral curve, though we can't completely ignore higher-order terms in $\hbar$.

Now parallel transport of solutions gives us (I think?) a flar SL_3-connection on C. We want to study the connections with $u > 0$. The higher WKB machinery gives you a basis $\{\psi_1, \psi_2, \psi_3\}$ inside a chamber (the Stokes graph divides the Riemann surface into two chambers), and ther three monodromies around points A, B, and C must satisfy

$$C\psi_1, B^{-1}\psi_2 \in \mathrm{span}\{\psi_1, \psi_2\}, \tag{1.13}$$

along with all cyclic permutations of this condition. This is an algebraic geometry question, and has a cool answer: A, B, and C are unipotent, and in this case there's a continuous family of solutions (not as interesting) plus four exceptional ones, and WKB produces one of these.

2. Vertex algebras towards Higgs branches, I: 9/12/18

Today, David spoke about vertex algebras, providing an introduction and background, albeit an ahistorical one.

You can think of vertex algebras as coming from topological field theory. Consider an oriented 2D TQFT Z, whose space of local operators/observables is $V := Z(S^1)$. The pair-of-pants bordism $S^1 \amalg S^1 \to S^1$ defines a multiplication map $V \otimes V \to V$; you can think of this as taking two small circles inside a larger annulus.[2]

If you favor one of the pant legs, you can think of this bordism as a cylinder together with the insertion of a small circle at some point z in the cylinder, which you can label by any $v \in V$. Once you do this, you get a map $V \to V$ given by the cylinder, and therefore get a map $V \to \operatorname{End} V$, which we call $v \mapsto Y(v, z)$.

We're working with a topological field theory, so $Y(v, z)$ must be locally constant in z. Passing to the annulus, we have two inner discs given by the incoming S^1 and a small disc around z. Though $Y(v, z)$ is locally constant in z, interesting things can happen when you move z around v; the structure given by V and $Y(v, z)$ is called an *E_2-algebra* or a *2-disc algebra*.

If our TQFT is valued in $\mathsf{Vect}_\mathbb{C}$, an E_2-algebra is fairly simple to understand: we can move v and z around each other, so it's just a commutative algebra. But there are more operations in what's called the *cohomological setting*, where the TQFT is valued in something like graded complex vector spaces. Local constancy means that we have an action of the homology of the $C_2(\mathbb{R}^2)$, the configuration space of two points in $\mathbb{R}^2$ on V, and since $C_2(\mathbb{R}^2) \simeq S^1$, we get data of a map $H^*(S^1) \otimes V \otimes V \to V$, i.e. a map

$$V \longrightarrow \operatorname{End} V \otimes H^*(S^1). \tag{2.1}$$

The cohomology of S^1 is pretty simple; in degree 0 we get back the commutative multiplication, and in degree -1 we get a graded Lie bracket $\{,\}$. This behaves well with respect to the multiplication, and this structure is called a *graded Poisson algebra*, or in this case also a *Gerstenhaber algebra*.[3]

But we can upgrade this to the *derived* setting, replacing cohomology by cochains, which is what supersymmetry taught us to do. This is the setting people usually refer to when they say E_2-algebra. We have a diagram

$$\begin{tikzcd} & \operatorname{End} V \otimes C^*(D) \arrow[d] \\ V \arrow[ur] \arrow[r] \arrow[dr] & \operatorname{End} V \otimes C^*(S^1) \arrow[d] \\ & \operatorname{End} V \otimes H^1(S^1)[-1]. \end{tikzcd} \tag{2.2}$$

Here D is the disc. The top map from V is an honestly commutative map for every pair of points, the middle one is the E_2-algebra structure, and the bottom map is the Lie operad (since it gave us the bracket). So the E_2-operad is built out of these two operads, and gives a commutative multiplication for pairs of points plus other data.

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[2]For these to be the same, we need to be doing oriented TQFT, not framed TQFT.

[3]For an interesting and recent application to physics, see https://arxiv.org/abs/1809.00009 by Beem-Ben-Zvi-Bullimore-Dimofte-Neitzke.

Example 2.3 (String topology)**.** One simple example of a 2D TQFT is called *string topology* on a manifold M, or the A-model on T^*M. The local operators are $H_*(\mathrm{Map}(S^1, M))$: the homology of the loopspace. Setting up the multiplication map takes some thought, and there are papers working this out.

There's also a space of *observables*, which comes as part of the description arising from physics, but doesn't fit into the functorial perspective. This is a pity, because they're how vertex algebras enter the story. Specifically, the observables are an algebra $H_*(\mathrm{Map}_c(D, M))$, or $H_*(\Omega^2 M)$. For example, if $M = BG$ is the classifying space of a compact Lie group G with complex form $G_\mathbb{C}$, $\Omega^2 BG$ is the affine Grassmannian $LG_\mathbb{C}/LG_{\mathbb{C}^\times}$. ◂

We've been thinking of all of this in terms of 2D TQFT, but all of the algebraic structure appears in higher dimensions too: vertex algebras appear in, e.g., 4D gauge theories where there are two topological directions and one holomorphic direction, and the intuition we've been using will carry over to there too.

Vertex algebras are the analogues of E_2-algebras, but for 2D conformal field theory rather than 2D topological field theory. The analogue of the circle is $\operatorname{Spec}\mathbb{C}[[z]]$, and the vector space of observables on that space is again denoted V. Given a vector v and a point z, we again get a $Y(v, z) \in \operatorname{End} V$, but this time, we want it to depend holomorphically[4] on z in the neighborhood $\operatorname{Spec}\mathbb{C}((z))$. That is, given $w \in W$, $Y(v, z) \in V((z))$. This may seem weird, but it's typical for how Laurent series are used to study local neighborhoods in formal algebraic geometry, and the upshot is $Y(v, z) \in \operatorname{End} V[[z^{\pm 1}]]$. We'll think of $Y(v, z)$ as an operator-valued distribution supported at $z \in D$.

Why should we think of these as distributions? Our model of functions is Laurent polynomials, and the algebraic dual of $\mathbb{C}[[z]]$ is $\mathbb{C}[[z^{\pm 1}]]$, so they can be called (algebraic) distributions.

Example 2.4. For example, the δ-function at z can be represented as

$$\delta_1(z) = \sum_{n=-\infty}^{\infty} z^n \in \mathbb{C}[z^{\pm 1}].$$

If you calculate its residues, you get just one at 1. ◂

We'll also write $Y(z, v)$ as $v(z)$, thinking of v acting at z.

Definition 2.5. A *vertex algebra* is the data of

- a vector space V,
- a *unit* or *vacuum element* $|0\rangle \in V$,
- a *translation* $T \in \operatorname{End} V$, and
- a map $Y\colon V \to \operatorname{End} V[[z^{\pm 1}]]$,

subject to the following axioms:

- $Y(|0\rangle, z) = \mathrm{id}$,
- $Y(v, z) \cdot |0\rangle = v + z(\text{stuff})$,
- T encodes $\frac{\mathrm{d}}{\mathrm{d}z}$-equivariance, in that $[T, Y(v, z)] = \partial_z Y(v, z)$, and
- a locality axiom, that $[a(z), b(w)]$ ought to be supported at $z = w$, i.e. there's an $N \gg 0$ such that

$$(z - w)^N [a(z), b(w)] = 0.$$

Again, $\mathbb{C}[[z^{\pm 1}]]$ is a space of test functions; you can think of this as a place where you can solve algebraic equations, somewhat like $\mathcal{D}$-modules.

The last axiom sometimes is written in different ways recalling associativity, Lie brackets, etc. It might seem surprisng, because we don't have Laurent series supported at points, but we're working with distributions: letting

$$\delta(z - w) \coloneqq \sum_{n=-\infty}^{\infty} \frac{w^n}{z^{n+1}}, \tag{2.6}$$

[4]This means algebraically if you're thinking algebro-geometrically, which we're mostly doing. You can formulate vertex algebras in either algebraic or analytic language, but all of the structure ends up being completely formal, in the sense of formal algebraic geometry.

and $(z-w)\delta(z-w)=0$, so we're OK. As usual, $\delta(z-w)$ and its derivatives span all distributions supported at $z=w$. Therefore locality is equivalent to asking that

$$[a(z),b(w)] = \sum_{n=0}^{N} (a_{(n)}b)(w)\frac{1}{n!}\partial_w^n\delta(z_w) \tag{2.7}$$

for some coefficients $a_{(n)}b$.

This is it: you might want to add associativity or a Jacobi identity, but you don't actually need to. So next we'll talk about how to think about vertex algebras.

First of all, $[a(z),b(w)]$ isn't *quite* supported at $z=w$: it also depends on n^{th}-order derivatives for $n\le N$, so on the N^{th}-order jets, which requires an N^{th}-order neighborhood of $z=w$. This doesn't change much, though.

Definition 2.8. A vertex algebra is *commutative* if $Y(a,z)\cdot b\in V[[z]]$, rather than just $V((z))$.

This is a very strong assumption: there are no poles. In this case, we can define a new multiplication $\cdot\colon V\otimes V\to V$ by

$$a\cdot b = \lim_{z\to 0} a(z)\cdot b. \tag{2.9}$$

Since $[a(z),b(w)]$ is a Taylor series supported at a single point, it must vanish! And therefore V is just a commutative (associative, unital) algebra with a derivation T (with respect to $\cdot$), i.e. a differential commutative algebra. And conversely, given a differential commutative algebra, you can just define

$$a(z) \coloneqq \sum_{n\gg 0} \frac{z^n}{n!}(T^n a), \tag{2.10}$$

and you can check this is a commutative vertex algebra. Of course, any commutative algebra defines a commutative differential algebra with $T=0$! But there are also nontrivial examples, thankfully.

We can rewrite Y as a map $V\otimes V\to V((z)) = V\otimes\mathscr{O}(D^\times)$; $\mathscr{O}(D^\times)$ can be thought of as the de Rham complex on S^1, a souped-up version of locally constant functions. Therefore a Vertex algebra is such a map satisfying some axioms.

Y induces a map $Y^-\colon V\otimes V\to V((z))/V[[z]]\cong V\otimes H^1_{\mathrm{loc}}(\{0\},\mathscr{O})$, and Y^- is a Lie algebra structure (but with a differential; these have different names, such as Lie-$*$ algebras). And if we can restrict to $V[[z]]$, we get a commutative algebra. So much like an E_2-algebra is something like a commutative algebra plus a Lie algebra, a vertex algebra is something like a differential commutative algebra and a differential Lie algebra.

The fact that Y^- has a Lie bracket is saying something about δ-functions, because $\mathbb{C}((z))/\mathbb{C}[[z]]\cong\langle\partial^n\delta_0\rangle = \mathbb{C}[\delta]\delta_0$. To get the Lie bracket, though, we start with a general story on a vector bundle $V\to X$: given a section s, there's a natural pointwise multiplication $s(z)\cdot v$, which is $\mathscr{O}_X$-linear. But you can also define multiplication depending on the Taylor series at a point, which is local in the physical sense. Since $\mathbb{C}[[z]]^* = \mathbb{C}((z))/\mathbb{C}[[z]]$, then Y^- defines maps $Y_t\colon (V\otimes V)[[t]]\to V$, which gives us the Lie bracket structure.

Like for E_2-algebras, vertex algebras are almost commutative: there's a filtration on either whose associated gradeds are commutative. This means the analogue of a Poisson structure: the description in terms of commutative and Lie algebras splits, and we get both structures.

Therefore vertex algebras are some sort of deformation/quantization of the notion of a differential Poisson algebra.

3. Vertex algebras towards Higgs branches, II: 9/26/18

Today David spoke again, continuing from his previous talk.

Vertex algebras are an algebraic structure capturing the observables in a 2D holomorphic field theory on a Riemann surface Σ, such as $\mathbb{C}$. Given an open $U\subset\Sigma$, we get a vector space $\mathscr{F}(U)$ of observables on U, and this should vary holomorphically in U. If $U = U_1\amalg U_2$, we want $\mathscr{F}$ to satisfy

$$\mathscr{F}(U) = \mathscr{F}(U_1)\otimes\mathscr{F}(U_2). \tag{3.1}$$

Beilinson-Drinfeld realized how to start from this ansatz and write down the definition of a vertex algebra. Specifically, we only consider "opens" which are formal completions of finite subsets of $\mathbb{C}$: they introduce a *Ran space* of Σ, a space of finite subset built as a colimit from ordered finite subsets in a certain way. Then they give data of a certain quasicoherent sheaf $\mathscr{F}$ on these subsets which satisfies (3.1).

This isn't quite a vertex algebra — it's a related structure called a *factorization algebra*. In a vertex algebra, we say that for all singletons $x \in \Sigma$, $\mathscr{F}(x) = V$, and we need to specify what happens when two points collide, which is the map $Y \colon V \otimes V \to V((z))$ that we described last time. Beilinson-Drinfeld showed this algebraic operation, which depends meromorphically on z, defines gluing data for this geometric perspective on vertex algebras.

We saw that this is an amalgam of two related algebraic structures: the quotient $Y_- \colon V \otimes V \to V((z))/[[z]]$ and the sub $Y_+ \colon V \otimes V \to V[[z]]$. If $(V, Y, T, |0\rangle)$ is *holomorphic*, meaning $Y = Y_+$, then V is a commutative ring with derivation by

$$Y_+(a, z) = a(z) = \sum_{n \geq 0} \frac{z^n}{n!} T^n a, \tag{3.2}$$

and conversely, this data defines a vertex algebra. Now, in this lecture, we'll study some examples.

Example 3.3. These are somewhat silly examples, but let R be any commutative ring with the derivation $T = 0$. ◀

Example 3.4. More interestingly, choose a commutative ring R and let $V := R\langle\partial\rangle$, freely adjoining a derivation. This is an algebraic construction, but has a geometric meaning: suppose $R = \mathbb{C}[X]$, the algebra of functions on a variety X, and $X = \operatorname{Spec} R$. Then we can take the space of jets on X, JX, and $R\langle\partial\rangle = \mathbb{C}[JX]$. Specifically, let $J_n X := \operatorname{Map}(\operatorname{Spec} \mathbb{C}[z]/(z^{n+1}), X)$; then $JX := \varprojlim_n J_n X$. That is, we're looking at n^{th}-order information near a point in X, for some n. This is a scheme, but isn't of finite type. This is what ∂, ∂^2, etc. are tracking. JX is a scheme, but not a variety, as it's not finite type. ◀

Not all vertex algebras are spaces of jets, since some vertex algebras are noncommutative. But these are really good examples, so you could take as your intuition the idea that vertex algebra is a quantization of the space of jets, replacing commutative vertex algebras with Poisson ones.

Since $(\mathbb{C}[[z]])^* \cong \mathbb{C}((z))/[[z]]$, then $\mathbb{C}[\partial_z] \cdot \delta_0$.

Example 3.5. Let $X = \mathfrak{a}^*$ be a vector space. Then $J\mathfrak{a}^* = \mathfrak{a}^*[[z]]$, because

$$\begin{aligned} J\mathfrak{a}^* &= \operatorname{Spec} \operatorname{Sym}((\mathfrak{a}^* \otimes \mathbb{C}[[z]])^*) \\ &= \operatorname{Spec} \operatorname{Sym}(\mathfrak{a} \otimes \mathbb{C}[\partial] \cdot \delta) \\ &= \operatorname{Spec}(\operatorname{Sym} \mathfrak{a}((z))/(\operatorname{Sym} \mathfrak{a}((z)))(\mathfrak{a}[[z]])). \end{aligned}$$

This computation is telling us something about the appearance of power series in the definition of the vertex operator. For example, suppose we choose a basis for $\mathfrak{a}^*$, writing $\mathfrak{a}^* = \operatorname{Spec} \mathbb{C}[x_1, \dots, x_N]$. Then

$$JX = \operatorname{Spec} \mathbb{C}[x_{1,n} \dots, x_{N,n}]_{n \leq 0}, \tag{3.6}$$

and the derivation is

$$T(x_{i,n}) = -(n-1)x_{i,n-1}. \tag{3.7}$$

If $\{J^a\}$ is a basis for $\mathfrak{a}$, then $\{J_n^a := J_a z^n \mid n < 0\}$ is a basis for $\mathfrak{a}((z))/\mathfrak{a}[[z]]$. Inside $\operatorname{End} V[[z]]$,

$$J^a(z) = Y_+(J^a, z) = \sum_{n<0} J_n^a z^{-n-1} \tag{3.8}$$

and

$$Y(J^a, z) = \sum_{n \in \mathbb{Z}} J_n^a z^{-n-1}. \tag{3.9}$$

This is spelled out in greater detail in the book by Edward Frenkel and David Ben-Zvi. ◀

Another class of examples are *vertex Poisson (Coisson) algebras*. In this case, we start with $X = \operatorname{Spec} R$ a Poisson variety; then $V := \mathbb{C}[JX]$ is a vertex Poisson algebra. For example, suppose $\mathfrak{g}$ is a Lie algebra for the group G and $X = \mathfrak{g}^*$. Then $\mathbb{C}[X] = \operatorname{Sym} \mathfrak{g}$ and the *arc group* $JG := G(\mathbb{C}[[z]])$ acts on $J\mathfrak{g}^*$, which is in fact the Lie algebra of JG. There's also an action of $\mathfrak{g}[[z]] := \mathfrak{g} \otimes \mathbb{C}[[z]]$ on $V = \operatorname{Sym}(\mathfrak{g}((z))/\mathfrak{g}[[z]])$. Now we get similar formulas as in Example 3.5: if J^a is a basis of $\mathfrak{g}$ and $J_n^a = J^a t^n \in \mathfrak{g}[[t]]$, (3.8) is the same, but we also have a Y_-, whose formula is

$$Y_-(J^a, z) - \sum_{n \geq 0} J_n^a z^{-n-1}. \tag{3.10}$$

These describes maps $\mathfrak{g} \to \operatorname{End} V((z))/[[z]]$ or $\mathfrak{g}[[z]] \to \operatorname{End} V$, so a vertex Poisson algebra is data of $(V, |0\rangle, T)$ together with a map $Y_+ : V \to \operatorname{End} V[[z]]$, which gives us a commutative vertex algebra, and a map $Y_- : V \to \operatorname{End} V((z))/[[z]]$, which has the structure of a differential Lie algebra and acts on Y_+.

Examples of vertex Poisson algebras are things like jets on a Poisson variety. The Y operator degenerates into two parts: $Y \mapsto Y_+$, and taking $\hbar$-linear terms, you get Y_-. This is like ordinary quantization, where associative algebras can become Poisson algebras.

As an example of this deformation (TODO: I think?), consider $\operatorname{Sym} \mathfrak{g} \to \mathcal{U}\mathfrak{g}$; then $\mathbb{C}[\mathfrak{g}^*]$ passes to the distributions at the identity on G. Then the affine Grassmannian makes an appearance:

$$\mathbb{C}[J\mathfrak{g}^*] = \operatorname{Sym} \mathfrak{g}(K)/(\operatorname{Sym} \mathfrak{g}(K)\mathfrak{g}(G)) = \mathbb{C}[T_e^* G(K)/G(\mathscr{O}). \tag{3.11}$$

Here $K = \mathbb{C}((t))$ is Laurent series and $\mathscr{O} = \mathbb{C}[[t]]$ is Taylor series.

This is the "arc version"; now we'll see the "loop version." If we start with $\mathcal{U}\mathfrak{g}$ instead of $\operatorname{Sym} \mathfrak{g}$, then we get distribtuions on the identity of $G(K)/G(\mathscr{O})$, i.e.

$$U\mathfrak{g}(K)/U\mathfrak{g}(K) \cdot \mathfrak{g}(\mathscr{O}) = \operatorname{Ind}_{\mathfrak{g}(\mathscr{O})}^{\mathfrak{g}(K)} \mathbb{C} = \mathcal{U}\mathfrak{g}(K) \otimes_{\mathcal{U}\mathfrak{g}(\mathscr{O})} \mathbb{C}. \tag{3.12}$$

This is denoted $V_{\mathfrak{g},0}$, and is called the (TODO vacuum?) of the affine Kac-Moody algebra at level 0.

For example, letting $|0\rangle$ be a nonzero vector in $\mathbb{C}$ (before we induce to $\mathfrak{g}(K)$), in the representatation of $\widehat{\mathfrak{g}}_0 \coloneqq \mathfrak{g}(K)$,

$$J^a(z) = Y(J_{-1}^a|0\rangle, z) = J_n^a z^{-n-1}. \tag{3.13}$$

Topologically, the affine Grassmannian $G(K)/G(\mathscr{O}) \simeq \Omega^2 BG$. So we're looking at maps from $\mathbb{C}$ rel $\mathbb{C} \setminus \mathbb{D}$ (where $\mathbb{D}$ denotes a disc) to BG, i.e. G-bundles on $\mathbb{C}$ together with trivializations away from a disc (thought of as not much more than a point).

The affine Grassmannian has the structure of a factorization algebra: given a collection of discs, consider the G-bundles trivialized away from these discs. This satisfies a product axiom, so we get a factorization algebra in ind-schemes. Moreover, we always have the trivial bundle, so this is pointed. Therefore any time you linearize this (so take a $\otimes$-functor to Vect, such as homology or distributions), you get a factorization algebra in vector spaces, and in particular a vertex algebra. This is closely related to the observables in string topology.

Next time, we'll talk about dimensional reduction, from vertex to associative (or Poisson) algebras, and the physics thereof.

4. Vertex algebras towards Higgs branches, III: 10/3/18

> *"I don't want to spend a lot of time on Zhu algebras, but... it looks like I don't have a lot of time, so it works out."*

Today David spoke again, continuing his previous talk, discussing the Higgs branch conjecture studied by Beem, Rastelli, and others.

Let's start with a 4D $\mathcal{N} = 2$ superconformal field theory; whatever this is, we can attach to it its moduli of vacua, which contains a subset called the Higgs branch. It's also possible to associated a vertex algebra to this theory, which by some version of dimensional reduction and/or chiralization, is closely related to the Higgs branch.

Higgs branches. For a 4D $\mathcal{N} = 2$ gauge theory, there are eight supercharges. Let's give a Lagrangian description. The gauge group is a compact Lie group G. The theory also includes data of a hyperKähler manifold, together with an action of G on M by hyperKähler isometries, so we have a Hamiltonian moment map $\mu_{\mathbb{H}} : M \to \mathfrak{g}_{\mathbb{H}}$. For example, we could take $M = V \oplus V^*$ for a complex G-representation V.

Heuristically, we'll think of the theory as a σ-model with target the stack M/G. One obvious invariant to extract is $M /\!/\!/ G$, a holomorphic symplectic variety (not necessarily smooth) which is defined as an affine GIT quotient, defined as Spec of the ring of $G_{\mathbb{C}}$-invariant functions on $\mu_{\mathbb{C}}^{-1}(0)$. This is the *Higgs branch* of the theory, and is denoted $\mathcal{M}_{\mathrm{Higgs}}$.

Not all 4D $\mathcal{N} = 2$ theories are Lagrangian, e.g. class S theories, so we'd like to know how to access $\mathcal{M}_{\mathrm{Higgs}}$ intrinsically, without the Lagrangian description. The idea is to look at the observables: observables on a theory define functions on its moduli space, so we're going to try to do that.

In general, the observables in a 4D QFT form a factorization algebra on $\mathbb{R}^4$. Approximately this means that we associate to every open subset of $\mathbb{R}^4$, we get a vector space of observables, and disjoint unions give you tensor products. There should also be a cosheaf property: observables on U are also observables on $V \supset U$.

In general, this is a horrible mess, containing the full structure of operator product expansion, etc. So let's try to simplify it, by using twisting: if our theory is supersymmetric, with an odd symmetry Q squaring to zero, and then look at Q-cohomology.

There's a map $[Q,-] \to T_{\mathbb{C}}\mathbb{R}^n$, and we can ask for this to be surjective: every direction is in the image of Q. In this case we say the twist is *topological*. If $[Q,-]$ surjects onto $T^{0,1}\mathbb{C}^2$, so you can impose Cauchy-Riemann equations, we say the twist is *holomorphic*. You can also ask for a twist to be holomorphic in some directions and topological in others, e.g. splitting $\mathbb{R}^4$ and $\mathbb{R}^2 \oplus \mathbb{C}$, getting something topological in the $\mathbb{R}^2$ directions and holomorphic in the $\mathbb{C}$ directions.

It's possible to classify these using superalgebra; a recent paper of Elliot-Safronov gives a comprehensive list. Topological, holomorphic, and holomorphic-topological twists can all appear for 4D $\mathcal{N} = 2$ theories.

The topological twist in this setting is called the *Donaldson-Witten twist*. The observables are the Q-cohomology of the untwisted observables, which become roughly a topological factorization algebra, or an E_4-algebra.[5] That is, it's a commutative algebra, together with a Poisson bracket of degree -3.[6]

Unfortunately, this is the wrong commutative algebra: it's the algebra of holomorphic functions on the Coulomb branch, not the Higgs branch! The Higgs branch came from studying matter (studying M), but the Coulomb branch comes from the gauge group, and looks roughly like the G-equivariant cohomology of a point. The overall moduli space contains the Higgs and Coulomb branches, but also possibly some other stuff.

The holomorphic-topological twist in this setting is called the *Kapustin twist*. In this case, the observables are a factorization algebra on $\mathbb{R}^2 \times \mathbb{C}$ (so, topological and holomorphic). That is, fixing a point in $\mathbb{R}^2$, you get a holomorphic factorization algebra on $\mathbb{C}$, which is what we said a vertex algebra is. If you fix a $z \in \mathbb{C}$, you'll get an E_2-algebra in the topological directions. Therefore the vertex algebra is locally constant in the topological direction, and in fact the whole thing is commutative (i.e. boring): secretly it's coming from an E_4-algebra. If you do run through the construction, you'll get a vertex Poisson algebra.

This is a hint that there's something noncommutative around, and in fact you can quantize this algebra. This is *different* from what Beem and Rastelli do; it's believed to be equivalent, but there's no proof. Let S^1 act on $\mathbb{R}^2$ by rotation, which we think of as rotating around $\mathbb{C} \subset \mathbb{R}^2 \times \mathbb{C}$. Then we look at the S^1-invariant observables, which are a module over $H^*_{S^1}(\mathrm{pt}) = H^*(BS^1) = \mathbb{C}[\varepsilon]$ with $|\varepsilon| = 2$. So this is a family depending on a parameter ε, and if $\varepsilon = 0$, we get back what we started with. So this is a deformation.

Now observables on $\mathbb{C}$ are suck on $\mathbb{C}$, so the argument above that we got a commutative vertex algebra no longer applies, and the ε-deformation of the original algebra is really a deformation quantization into a noncommutative vertex algebra.

Remark 4.1. For the mathematics of the Higgs branch conjecture, we only need the original vertex algebra and its Poisson algebra, but it is nice to know that this isn't completely bland, and that a noncommutative algebra does appear. ◂

It's not quite clear how to make the Higgs branch conjecture into a conjecture: one side is defined mathematically and the other isn't. In physics, it's just known. Anyways, the conjecture says that *this vertex Poisson algebra V is a "chiral version" of the Higgs branch*, i.e. it is (maybe something closely related to) taking the ring of functions on the jets on $\mathcal{M}_{\mathrm{Higgs}}$, akin to what we discussed last time.

More precisely, we'll construct from V another Poisson algebra $V/C_2(V)$ (also called R_V), which was introduced by Yongchang Zhu, and then $\mathbb{C}[\mathcal{M}_{\mathrm{Higgs}}] \cong C_2(V)$. Zhu defined two different constructions in his thesis given a vertex algebra: one produces an associative algebra, and another (R) produces a Poisson algebra. There's a related construction giving a Poisson algebra from a vertex Poisson algebra, also called R.

Definition 4.2. Given a vertex algebra V, the Poisson algebra R_V is defined as $R_V \coloneqq V/\operatorname{span}\{a_{-2}b\}$, with the product $\overline{a} \cdot \overline{b} = \overline{a_{-1}b}$ (where a is a representative for $\overline{a}$ in the quotient, and so on), and $\{\overline{a}, \overline{b}\} = \overline{a_0 b}$.

A vertex algebra has infinitely many products $a \cdot_n b = a_n b$, but they generally don't have nice structure. In this case it's solved by modding out by the (-2)-product.

Associated to V we therefore get a Poisson variety $X_V \coloneqq \operatorname{Spec} R_V$. Is $X_V = \mathcal{M}_{\mathrm{Higgs}}$? We do have a tautological map $JR_V \twoheadrightarrow V$, and there's research about how close this is to an isomorphism for various kinds

[5]There's a subtlety that we're not going to worry about, but the details can be found in Elliot-Safronov.

[6]Curiously, in all known examples, this Poisson bracket is zero!

of vertex algebras. The upshot is that for the vertex algebras appearing in the Higgs branch conjecture, they're very close (only differing by nilpotents), which is good: X_V is almost the same thing as $\mathcal{M}_{\mathrm{Higgs}}$.

Zhu's constructions have a physical explanation: an associative algebra is exactly the algebra of observables of a 1D topological field theory, and his process of getting an associative algebra from an associative algebra is a dimensional reduction from $\mathbb{C} \times \mathbb{R}^2 \to \mathbb{R} \times \mathbb{R}^2$. A topological factorization algebra on $\mathbb{R} \times \mathbb{R}^2$ is an E_3-algebra, which is precisely a Poisson algebra with even grading. Physically, the Kapustin twist passes to a Rozansky-Witten theory.

In topology, so using E_2-algebras instead of vertex algebras, there's a natural way to do dimensional reduction. There's an S^1-action on the E_2-algebra $Z(S^1)$. Looking at the target M, we can consider $H^*(LM)^{S^1}$, and do equivariant localization (this is a Tate construction), and recover

$$H^*((LM)^{S^1}) = H^*(M) \otimes \mathbb{C}[\varepsilon, \varepsilon^{-1}], \tag{4.3}$$

where ε is the equivariant parameter. So dimensional reduction means looking at loops in your manifold but you can get back to M by rotation.

TODO: not sure what happened after that, with $H_*(\Omega^2 M)$ on a cylinder to get $H_*(L\Omega M)$ on a cylinder, which we squeeze into a line. This is something like Hochschild homology, leading to the associative algebra of observables on a 1D σ-model into M: $H_*(\Omega M) \otimes \mathbb{C}[\varepsilon, \varepsilon^{-1}]$. The Zhu construction is the holomorphic analogue of this, reducing a holomorphic dimension.

5. Infinite chiral symmetry in four dimensions: 10/10/18

Today Behzat spoke about the paper by Beem-Lemos-Liendo-Peelaers-Rastelli-van Rees, "Infinite Chiral Symmetry in Four Dimensions." (https://arxiv.org/abs/1312.5344).

The idea is to start with 4D $\mathcal{N} = 2$ superconformal field theories and produce 2D chiral algebras. Here "chiral algebras" are closely related to the vertex algebras discussed in the last few lectures. Supersymmetry is important for this to work.

In a 4D $\mathcal{N} = 2$ SCFT, there are bosonic and fermionic symmetries. The bosonic symmetries include Poincaré terms $p_{\alpha\dot\alpha}$ $M_\alpha{}^\beta$, $\overline{M}^{\overline{\beta}}{}_{\overline{\alpha}}$, and $K^{\dot\alpha\alpha}$. There's also a bosonic R-symmetry $R^\pm$, R, which generate an $\mathfrak{sl}(2)_R$, and another generator r of a $\mathfrak{u}(1)_r$. The fermionic symmetries include Poincaré supercharges Q^I_α and $\widetilde{Q}_{I\dot\alpha}$ and superconformal terms S_I^α and $\widetilde{S}^{I\dot\alpha}$, where I is an $\mathfrak{sl}(2)_R$ index ($I = 1, 2$), and $\alpha, \dot\alpha = \pm$. The Lie algebra generated by all of these things is denoted $\mathfrak{sl}(4|2)$; the relations are in the appendix of that paper.

We want the fermionic terms too so that the chiral algebra we end up with has the right symmetries (I think). This involves producing an $\mathfrak{sl}(2) \times \mathfrak{sl}(2) \subseteq \mathfrak{sl}(4|2)$, which will correspond to the charges in $\mathbf{Q}$-cohomology, and will be meromorphic ($\mathbf{Q}$ will be defined soon).

This means we're looking for a supercharge $\mathbf{Q}$ which is a nilpotent inside $\mathfrak{sl}(2) \times \overline{\mathfrak{sl}(2|2)}$,[7] We'd like $\mathfrak{sl}(2)$ to generate the holomorphic transformations on the plane and $\overline{\mathfrak{sl}(2)}$ to the generate the antiholomorphic transformations. These two should commute up to $\mathbf{Q}$-exact terms.

Then we write

$$M^+ = M_+{}^+ - \overline{M}^{\dot+}{}_{\dot+} \tag{5.1a}$$

$$M = M_+{}^+ + \overline{M}^{\dot+}{}_{\dot+}, \tag{5.1b}$$

and some more relations: $L_{-1} = P_{+\dot+}$, $L_{+1} = K^{\dot++}$, and $L_0 = (1/2)(D + M)$. Correspondingly, $\overline{L_{-1}} = P_{-\dot-}$, $\overline{L_{+1}} = K^{\dot--}$, and $L_0 = (1/2)(D - M)$. You can verify these generate an $\mathfrak{sl}(2)$, resp. $\overline{\mathfrak{sl}(2)}$.

Then we let $Q^I = Q^I{}_-$, $\widetilde{Q}_I = \widetilde{Q}_{I\dot-}$, $S_i = S_I^-$, and $\widetilde{S}^I = \widetilde{S}^{I\dot-}$. Let $Z = M^\perp$ tr.

Letting $\mathbf{Q}^1 = Q^1 + \widetilde{S}^2$, $\mathbf{Q}_1^+ = S_1 + \widetilde{Q}_2$, $\mathbf{Q}^2 = S_1 - \widetilde{Q}_2$, and $\mathbf{Q}_2^+ = Q^1 - \widetilde{S}^2$, we have

$$\widehat{L}_{-1} = \{\mathbf{Q}, \widetilde{Q}_1\} = \{\mathbf{Q}^2, Q^2_-\} = \overline{L_{-1}} + R^-, \tag{5.2}$$

and in a similar way,

$$\widehat{L}_+ = \overline{L_{+1}} - R^+ \tag{5.3}$$

$$\widehat{L}_0 = \{\mathbf{Q}_1, \mathbf{Q}_1^+\} = \{\mathbf{Q}_2, \mathbf{Q}_2^+\} = 2(\overline{L_0} - R), \tag{5.4}$$

[7] The second part is sometimes denoted $\mathcal{N} = (0, 4)$ in 2D notation, (TODO: I think).

and again this gives you an $\mathfrak{sl}(2)$. The operator $\mathcal{O}(0)$ is $\mathbf{Q}$-exact.

Now consider an operator

$$\mathcal{O}(z,\overline{z}) = e^{zL_{-1}+\overline{z}L_{-1}}\mathcal{O}(0)e^{-zL_{-1}-\overline{z}\widehat{L}_{-1}}. \tag{5.5}$$

Then $\partial_{\overline{z}}$ is $\mathbf{Q}$-closed, so its $\mathbf{Q}$-cohomology class is an $\mathcal{O}(z)$.

If you do an OPE of these operators, you get

$$\mathcal{O}_1(z,\overline{z})\mathcal{O}_2(0) = \sum_k \lambda_{12k}\frac{\overline{z}^{R_1+R_2-R_k}}{z^{h_1+h_2-h_k}\overline{z}^{-\overline{h}_1+\overline{h}_2-\overline{h}_k}}\mathcal{O}_k(0). \tag{5.6}$$

TODO: some of these might be Schur indices? Not Schur? I'm not sure. Here h and $\overline{h}$ are apparently quantum numbers, and the OPE simplifies to

$$\mathcal{O}_1(z,\overline{z})\mathcal{O}_2(0) = \sum_{k\text{ Schur}} \lambda_{12k}\frac{1}{z^{h_1+h_2-h_k}}\mathcal{O}_k(0) + \sum_{\text{non-Schur}}\cdots, \tag{5.7}$$

but the latter is $\mathbf{Q}$-exact, so it's not important in $\mathbf{Q}$-cohomology.

Then we can write down some more quantities: $\widehat{B}_{1/2}$, which is a free hypermultiplet (TODO: maybe?), $\widehat{B}_1$, which is the flavor currents, $\widehat{C}_{0(00)}$, the stress tensor, which has $\mathfrak{su}(2)_R$ and $\mathfrak{u}(1)_R$ currents. We then do have a Schur operator

$$J_R(z,\overline{z}) = u_I u_J J^{IJ}_{+\dot{+}} \tag{5.8}$$

and

$$T^(z) \coloneqq k[J_R(z,\overline{z})]_{\mathbf{Q}}. \tag{5.9}$$

Using this and

$$J^{IJ}_\mu(x)J^{KL}_\nu(0) \sim \frac{3}{4\pi^4}c_{4d}\epsilon^{???}\epsilon^{???}\left(\frac{x^2g_{\mu\nu}-x_\mu x_\nu}{x^8}\right) + \frac{2i}{2\pi^2}x_\mu x_\nu x\cdot J^?\epsilon^{???}, \tag{5.10}$$

it's possible to calculate the OPE of the Schur operators.

6. From 4D $\mathcal{N}=2$ SCFTs to VOAs: 10/24/18

Jacques spoke today. We'll start with a review of some of what Behzat said a few weeks ago. I (Arun) didn't really understand today's lecture, so beware: there may be more typos than normal.

We want to, given a 4D $\mathcal{N}=2$ superconformal field theory, build a vertex operator algebra. We work in $\mathbb{R}^4$ and restrict to the plane $x_1 = x_2 = 0$, and let $z = x_3 + ix_4$. We'll build the VOA out of correlation functions of the field theory, but only some of these, namely the Schur operators

$$\Delta - d_1 - d_2 - 2R = r + d_1 - d_2 = 0. \tag{6.1}$$

Given such an operator a, $\chi[a]$ will denote its image in the VOA.

Here are four classes of these Schur operators.

- $\widehat{B}_R$, which is the primary, when $\Delta = 2R$ and $r = d_1 = d_2 = 0$. The rest are descendants (TODO: I think).
- $\widehat{C}_{R(d_1,d_2)}$, where $\Delta = 2R + d_1 + d_2 + 2$ and $r = d_2 - d_1$.
- $D_{R(d_1,d_2)}$, where $\Delta = 2R + d_2 + 1$ and $r = d_2 + 1/2$.
- $\overline{D}_{R(d_1,d_2)}$, where $\Delta = 2R + d_1 + 1$ and $r = -d_1 - 1/2$.

We move these away from the orign by a form of twisted translation, letting

$$\mathcal{O}(z,\overline{z}) = e^{zL_{-1}+\overline{Z}\widehat{L}_{-1}}\mathscr{O}(0)e^{-zL_{-1}-\overline{z}\widehat{L}_{-1}} = u_{I_1}(\overline{z})\cdots u_{I_{2R}}\underbrace{\mathscr{O}^{I_1\ldots I_{2R}}(z,\overline{z})}_{(*)}, \tag{6.2}$$

where $(*)$ denotes the usual translation. Here

$$\widehat{L}_{-1} = \overline{L}_{-1} + R^- \tag{6.3}$$
$$\widehat{L}_0 = 2(\overline{L}_0 - R) = D - M - 2R \tag{6.4}$$
$$\widehat{L}_1 = \overline{L}_1 - R^+. \tag{6.5}$$

and M is a rotation in the plane.

Remark 6.6. This way of going from 4D to 2D very much depends on the choice of the origin, moreso than the usual mild dependence of a VOA on the choice of origin. ◀

There's a supercharge $\mathbf{Q}$ squaring to zero and such that $\partial_{\overline{z}}\mathcal{O}(z,\overline{z}) = [\mathbf{Q},\dots]$. In particular, for an OPE

$$\mathcal{O}_1(z,\overline{z})\mathcal{O}_2(0) = \sum_{k\ \text{Schur}} \frac{\lambda_{12k}}{z^{h_1+h_2-h_{12}}}\mathcal{O}_k(0) + \sum_{\text{non-Schur}} [\mathbf{Q},\dots] \tag{6.7}$$

This means [more equations on the board that I don't understand]. I guess in the VOA, $\widehat{C}_{0(0,0)}$ goes to $T(z)$, and $\widehat{B}_1$ goes to $\chi(m^a) = J^a(z)$. This data forms an affine Kac-Moody algebra with $k_{2d} = -(1/2)k_{4d}$. $\widehat{B}_{1/2}$ is a free half-hypermultiplet. We also have that

$$q_a(z)q_a(w) \simeq \frac{\epsilon_{ab}}{z-w} + \cdots \tag{6.8}$$

and

$$T(z) = q_1\partial_z q_2 - \frac{1}{2}\partial_z(q_1,q_2) = \frac{1}{2}\epsilon^{ab}q_a\partial q_b. \tag{6.9}$$

Exercise 6.10. Show that

$$T(z)q_a(w) = \frac{(1/2)q_a}{(z-w)^2} + \frac{\partial q_a}{z-w} + \cdots \tag{6.11a}$$

$$T(z)T(w) = \frac{-1/2}{(z-w)^4} + \frac{2T}{(z-w)^2} + \frac{\partial T}{z-w} + \cdots \tag{6.11b}$$

This vertex operator algebra is called the *spin*-1/2 $\beta\gamma$ *system*. For brevity, we'll call it H.

If we start with the free vector multiplet in 4D, it produces a VOA called the *spin*-1 *bc system*, for which $D_{0(0,0)} + \overline{D_{0(0,0)}}$. Here the vertex operators are gauginos: we have $b = \chi[\lambda]$, $\partial c = \chi[\widetilde{\lambda}]$, and $T(z) = b\partial c$. We have $b(z)c(w) = c(z)b(w)$ by some computation analogous to Exercise 6.10. To obtain a local description, we employ a standard trick: we have to expand from our Hilbert space $\mathcal{H}_{\text{small}}$ to a bigger Hilbert space $\mathcal{H}_{\text{big}}$, such that $\mathcal{H}_{\text{small}} = \mathcal{H}_{\text{big}}/\ker(c_0)$.

Next let's consider the 4D theory starting with a pure SU_2 theory and adding four hypermultiplets. In other words, there are eight half-hypermultiplets Q_a^I for $i = 1,\dots,8$. For $A = 1,2,3$, we let $q_a^i = \chi[Q_a^i]$, $b_A = \chi[\lambda_A]$, and $\partial c^A = \chi[\widetilde{\lambda}^A]$. Then

$$J^A(z) = -\frac{1}{4}\delta_{ij}q_a^i(\mathcal{T}^a)^{ab}q_b^d, \tag{6.12}$$

where $(\mathcal{T}^A)^{ab} = \epsilon^{ac}(\sigma^A)_c^b$, where σ^A denotes the Pauli matrices.

Exercise 6.13. These J^A generate a Kac-Moody algebra, and specifically $\widehat{\mathfrak{su}}(2)_{-2}$.

Now let's do something more interesting: define a BRST current

$$J_{\text{BRST}}(z) = c_AJ^A - \frac{1}{2}f^{AB}{}_Cb^Cc_Ac_B \tag{6.14a}$$

$$Q_{\text{BRST}} = \oint \frac{\mathrm{d}z}{2\pi i}\,J_{\text{BRST}}(z). \tag{6.14b}$$

Assuming Exercise 6.13, the OPE we obtain is

$$J^A(z)J^B(w) = \frac{-(n/4)\delta^{AB}}{(z-w)^2} + \frac{f^{AB}{}_CJ^C}{z-w} + \cdots \tag{6.15a}$$

$$J_{\text{gh}}^A(z)J_{\text{gh}}^B(w) = \frac{2\delta^{AB}}{(z-w)^2} + \frac{f^{AB}{}_CJ_{\text{gh}}^C}{z-w} + \cdots \tag{6.15b}$$

We get the affine Kac-Moody algebra $\widehat{\mathfrak{so}}(n)_{-2}$ if

$$J^{ij}(z) = -i\epsilon^{ab}q_a^iq_b^d, \tag{6.16}$$

so comparing with (6.15), we get $Q_{\text{BRST}}^2 = 0$ iff $n = 8$, so we do get the Kac-Moody algebra corresponding to the global symmetry of the original theory.

TODO: then I missed some stuff about the Sugawara algebra.

There are many more examples you could study. The next we'll look at is SU(N) with n hypermultiplets. In this case,

$$J^A(z) = \delta_0^1 \widetilde{q}_i^a (\mathcal{T}^A)_a^b q_b^d, \tag{6.17}$$

where $\mathcal{T}$ is in the fundamental representation of SU(N). The affine Kac-Moody algebra we end up with is $\widetilde{\mathfrak{su}}(N)_{-n/2}$, and $J_1^0(z) = \widetilde{q}_i^a q_a^b$.

TODO: not sure what happened after that.

7. From 4D $\mathcal{N} = 2$ SCFTs to VOAs, II: 10/31/18

Jacques spoke again today, as a continuation of what he spoke on last week.

We were in the middle of considering a 4D $\mathcal{N} = 2$ SU(N) gauge theory with $2N$ hypermultiplets. This means taking $2N$ copies of the VOA corresponding to a free hypermultiplet, which is also called the spin-1/2 β-γ system:

$$q_a^i(z)\widetilde{q}_d^n(w) = -\widetilde{q}_d^b(z) q_a^i(w) = \frac{\delta_d^1 \delta_a^b}{z - w}. \tag{7.1}$$

Then we have a BRST charge

$$Q_{\mathrm{BRST}} = \oint \frac{\mathrm{d}z}{2\pi i}\left(c_A J^A - \frac{1}{2} f^{AB}{}_C b^C c_A c_B\right). \tag{7.2}$$

The vertex algebra is the BRST cohomology of $H^{\otimes 2N^2} \otimes V^{\otimes(N^2-1)}$. If

$$b_0^A = \oint \frac{\mathrm{d}z}{2\pi i} b^A(z), \tag{7.3}$$

then

$$\{Q, b_0^A\} = (J_{\mathrm{tot}}^A)_0 = \oint \frac{\mathrm{d}z}{2\pi i}\big(J^A(z) + f_C^{AB} b^C c_B\big). \tag{7.4}$$

Then $\mathcal{H}_{\mathrm{small}} = \ker(b_0^A)$, and the vertex operator algebra is the Q-cohomology of this space.

This vertex operator algebra contains a copy of the affine Kac-Moody algebra $\widehat{\mathfrak{su}}(2N)_{-N} \times \widehat{\mathfrak{u}}(1)_{-2N^2}$, and

$$J_a^d(z) = \widetilde{q}_i^a q_a^{-d} - \frac{1}{2N}\delta_A^d \widetilde{q}_k^a q_a^k \tag{7.5a}$$

$$J(z) = -\widetilde{q}_i^a q_a^i. \tag{7.5b}$$

So we have particles B you could call *baryons*, which live in $(\Lambda^N)_N$ of the fundamental representation of $\mathfrak{su}(2N)$, and which have $\mathfrak{u}(1)$ charge

$$\epsilon^{a_1 \dots a_N} q_{a_1}^{i_1} \cdots q_{a_N}^{i_N}. \tag{7.6}$$

Correspondingly, in $(\Lambda^N)_{-N}$ there are particles denoted $\widetilde{B}$, and $h(B) = h(\widetilde{B}) = N/2$.

Remark 7.7. The global symmetry of this theory is not SU($2N$), but U($2N$) — except when $N = 2$, in which case you get SO(8) instead of U(4). This happens because one could write down two different stress tensors, and usually they don't agree, but when $N = 2$ they do. ◀

Example 7.8. Let's consider an interesting class of non-Lagrangian 4D $\mathcal{N} = 2$ SCFTs, the rank-1 E_6, E_7, and E_8 Minahan-Nemeschansky theories. Here the global symmetry is one of the exceptional Lie groups E_6, E_7, or E_8, and k_{4D} is 6, 8, and 12, respectively. The VOAs are $\widehat{\mathfrak{e}}(6)_{-3}$, $\widehat{\mathfrak{e}}(7)_{-4}$, and $\widehat{\mathfrak{e}}(8)_{-6}$, respectively. The Sugawara central charges are -12 times the corresponding 4D central charge, hence are -26, -38, and -62, respectively.

Another thing to know about these theories is that the Higgs branch is the minimal nilpotent orbit in the corresponding Lie algebra.

If you pick an $\mathfrak{su}(2) \subset \mathfrak{e}(6)$, and introduce two free hypermultiplets and an $\mathfrak{su}(2)$ vector multiplet, then

$$H_Q^*(\widehat{\mathfrak{e}}(6)_{-3} \otimes H^2 \otimes V^3) \cong H_{Q'_{\mathrm{BRST}}}^*(H^{18} \otimes V^8), \tag{7.9}$$

and the latter is the VOA for th SU(3) theory with six hypermultiplets. This generalizes work of Neitzke and collaborators, which showed the corresponding Higgs branches are isomorphic: for the E_6 theory this is (some kind of special) quotient of an E_6 nilpotent orbit times $\mathbb{H}^2$ by SU_2, and for the SU(3) theory it's $\mathbb{H}^{18} /\!\!/\!\!/ \mathrm{SU}(3)$. ◀

Example 7.10. For higher N, these theories are known as $R_{0,N}$ theories, with an $\mathrm{SU}(2N)_{2N} \times \mathrm{SU}(2)$ global symmetry , a 4D central charge $c_{4D} = (2N^2 - 5)/6$, and a VOA $\widehat{\mathfrak{su}}(2N)_{-N} \times \widehat{\mathfrak{su}}(2)_{-3}$ (not a Kac-Moody algebra). The Sugawara central charge, which is the same as the 2D central charge, is $-2(2N^2 + 5)$. In this case the VOA arises as

$$H_Q^*(\chi(R_{0,N}) \times H^2 \times V^3) \cong H_{Q'}^*(H^{2N^2} \otimes V^{N^2-1}). \tag{7.11}$$

◀

All of these theories have class S constructions. In some cases, these are the only known constructions.

Example 7.12. For each ADE type Lie algebra $\mathfrak{g}$, there's a distinguished isolated SCFT of class S whose VOA contains $(\widehat{G}_{-h^\vee(\mathfrak{g})})^3$. Conjecturally, the case $\mathfrak{g} = A_{n-1}$ is generated by $(\widehat{\mathfrak{su}}(N)_{-N})^3$, T, and W_ℓ for $\ell = 1, \dots, N-1$.

There's also a gluing conjecture on the BRST charge: if $T = \chi(T_{\mathfrak{g}})$, $E = \chi(G) = V^{\dim G}$, and $\Gamma_{g,n}$ is a trivalent graph with n external edges and g loops (corresponding to a pants decomposition of $\Sigma_{g,n}$), let

$$W := T^{\otimes(2g-2+n)} \otimes E^{\otimes(3g-3+n)}. \tag{7.13}$$

Then, for each internal edge γ,

$$Q^\gamma_{\mathrm{BRST}} = \oint \frac{\mathrm{d}z}{2\pi i}\left(c^\gamma_A(J^A_L + J^A_R) - \frac{1}{2} f^{AB}{}_C b^C_\gamma c^\gamma_A c^\gamma_B\right), \tag{7.14}$$

the conjecture is

$$Q^{\mathrm{tot}}_{\mathrm{BRST}} = \sum_{n=1}^{3g-3+n} Q^{\gamma_i}_{\mathrm{BRST}}. \tag{7.15}$$

That is, we've computed the total BRST charge by chopping up the Riemann surface into a pants decomposition. The result should not depend on the decomposition we chose, which is what this conjecture asserts. This conjecture is still open, but progress has been made on it.

Work of Lemos-Peelaers (https://arxiv.org/abs/1411.3252) works this story out for $N = 4$. ◀

8. VOAs and Seiberg-Witten invariants: 11/17/18

Today, Shehper spoke about a paper of Gukov and TODO, part of a general program: given a 4-manifold M, we associate to it a vertex operator algebra VOA$[M]$, very much like what we've been doing: take the 6D theory $\mathfrak{X}$ and compactify it along M. The result, which we'll denote $T[M]$, should be a 2D $\mathcal{N} = (0, 2)$ supersymmetric theory, and we will extract a VOA from this data. This VOA can be considered as an invariant of the 4-manifold M, and related to other invariants; today we'll relate the chiral correlation functions of the VOA to the Seiberg-Witten invariants of M.

Of course, there's a choice of an ADE-type Lie algebra $\mathfrak{g}$ in defining Theory $\mathfrak{X}$; the corresponding vertex operator algebras haven't been defined for all choices of $\mathfrak{g}$. Today we'll talk about the case where there's just one fivebrane, so when $\mathfrak{g}$ is abelian!

The three papers are (TODOformat) 1306.4320, 1705.01645, and 1806.02740; this talk focuses on the first two.

We will assume M is connected, which simplifies the story slightly.

8.1. Field content of $T[M, \mathfrak{u}(1)]$.

We consider Theory $\mathfrak{X}$ on $X = M \times \Sigma$, where M is a 4-manifold and Σ is a Riemann surface. In this case the $\mathfrak{so}(6)$ symmetry decomposes as $\mathfrak{so}(4)_M \oplus \mathfrak{so}(2)_\Sigma$, and then further into left- and right-movers:

$$\mathfrak{su}(2)_\ell \oplus \mathfrak{su}(2)_r \oplus \mathfrak{u}(1)_\Sigma. \tag{8.1}$$

There's also an R-symmetry $\mathfrak{so}(5)_R$, which in this case breaks down to

$$\mathfrak{so}(3)_R \oplus \mathfrak{u}(1)_t \cong \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_t. \tag{8.2}$$

The supercharges become

$$(4_+, 4) \longrightarrow (2, 1, 2)^{(1,\pm1)} \oplus (1, 2, 2)^{(-1,\pm1)}. \tag{8.3}$$

Here these denote the charges under $\mathfrak{u}(1)_\sigma$ and $\mathfrak{u}(1)_t$.

We're going to topologically twist this theory, so as to make it depend on M in a topological manner. Replace $\mathfrak{su}(2)_r$ with the diagonal subalgebra $\mathfrak{su}(2)'_r \subset \mathfrak{su}(2)_r \oplus \mathfrak{su}(2)_R$. In this case the supercharges are

$$(4_+, 4) \longrightarrow (2,2)^{(-1,\pm 1)} \oplus (1,3)^{(-1,\pm 1)} \oplus \underbrace{(1,1)^{(-1,\pm 1)}}_{Q_+, \overline{Q}_+}. \tag{8.4}$$

The scalars go from $(1,5)$ to $(1,3)^{(0,0)} \oplus (1,1)^{(0,\pm 2)}$; here these denote the charges under $\mathfrak{su}(2)_\ell \oplus \mathfrak{su}(2)'_r$. The self-dual 2-form passes to

$$(15,1) \longrightarrow (2,1)^{(2,0)} \oplus (1,1)^{(0,0)} \oplus (2,2)^{(-2,0)} \oplus \underbrace{(3,1)^{(0,0)}}_{X_L^i} \oplus \underbrace{(1,3)^{(0,0)}}_{X_R^i}. \tag{8.5}$$

Now, let's twist on Σ, to obtain something which depends holomorphically on Σ: let $\mathfrak{u}(1)'_\Sigma$ be the diagonal subalgebra of $\mathfrak{u}(1)_\Sigma \oplus \mathfrak{u}(1)_t$. In this case, on Σ,

$$(1,1)^{(-1,\pm 1)} \longrightarrow \underbrace{(1,1)^{(0,1)}}_{\overline{Q}_+} \oplus (1,1)^{(-2,1)}. \tag{8.6}$$

What is the field content of the compactified theory $T[M;\mathfrak{u}(1)]$?

- For $i = 1, \dots, b_2^+$, there are right-movers X_R^i, valued in $\mathrm{U}(1)$ ("compact").
- For $i = 1, \dots, b_2^+$, there are non-chiral fields σ^i, valued in $\mathbb{R}$ ("non-compact").
- For $i = 1, \dots, b_2^+$, there are right-movers ψ_+^i which are spinors.
- A non-chiral field ϕ_0 valued in $\mathbb{C}$ ("non-compact complex").
- A right-mover χ_+, which is a spinor.
- Left-movers X_L^j for $j = 1, \dots, b_2^-$.

These organize into chiral multiplets $(X_R^i, \sigma^i, \psi_+^i)$, (ϕ_0, χ_+), and a trivial multiplet (X_L^j).

More explicitly, we can obtain the left- and right-movers by decomposing the B-field into its zero modes for the Laplacian.

$$B = \sum_{i=1}^{b_2^+} a_i \omega_i^+ + \sum_{j=1}^{b_2^-} b_j \omega_j^-, \tag{8.7}$$

where $\omega^\pm$ are harmonic forms. Since we want $\mathrm{d}B = \star \mathrm{d}B$, then the a_i are holomorphic, and give us X_L, and the b_j are antiholomorphic, giving us X_R.

8.2. **Vertex operators in $T[M, \mathfrak{u}(1)]$.** For now, we assume $b_2^+ = b_2^- = 1$. Consider the vertex operator

$$e^{ik_L x_L + ik_R x_R + p\sigma}. \tag{8.8}$$

In $\overline{Q}_+$-cohomology,

$$\begin{aligned}
[\overline{Q}_+, \sigma + iX_R] &= 0 \\
[\overline{Q}_+, \phi_0] &= 0 \\
[\overline{Q}_+, X_L^j] &= 0.
\end{aligned}$$

Two important points:

(1) (TODO: something about Virasoro?) we see that $H = (1/2)(K_L^2 - p^2)$ and $\overline{h} = (1/2)(K_R^2 - p^2)$, and
(2) K_L and K_R are quantized.

So in $\overline{Q}_+$-cohomology, $p = K_R$, $\overline{h} = 0$, and $h = (1/2)(K_L^2 - K_R^2)$. So this is the free theory on these fields, hence only knows $b_2^\pm(M)$ as an invariant of M! But we will be able to see more structure later by adding more operators.

For more general values of $b_2^\pm$, we get

$$e^{ik_L^j X_L^j + K_R^k \sigma^k}, \tag{8.9}$$

where $j = 1, \dots, b_2^+$ and $k = 1, \dots, b_2^-$. In this case, $\lambda := (K_L, K_R) \in \Gamma = H^2(M; \mathbb{Z})$. In the end, the VOA we'll get is the one associated to this lattice, Letting

$$\lambda_-^2 = \frac{1}{2}\Sigma k_L^2 \tag{8.10}$$

$$\lambda_+^2 = \frac{1}{2}\Sigma k_R^2, \tag{8.11}$$

for

$$V_\lambda = \exp\Big(ik_L^j X_L^j + K_R^k \sigma^k\Big), \tag{8.12}$$

$h = \lambda_-^2 - \lambda_+^2$.

8.3. **Seiberg-Witten invariants.** Roughly speaking, these are correlation functions in topologically twisted $\mathcal{N}=2$ QED with N_f flavors. We have a bosonic scalar field ϕ, fermionic fields ρ, ψ_M, and $\chi_{\mu\nu}$ (the latter of which is self-dual), and the gauge field A_μ. Here μ, ν are $\mathfrak{so}(4)$ indices. These assemble into a topological vector multiplet; there's also a topological hypermultiplet, with bosonic fields $\psi_\alpha, h_{\dot\alpha}$ and fermionic fields $\mu_\alpha, \mu^{-\alpha}, \nu_{\dot\alpha}, \nu^{-\dot\alpha}$.

The action has a lot of terms; some of the important ones are

$$S = \int_M \mathrm{d}V \left(\frac{1}{2}\Big(F_{\alpha\beta}^+ - i\overline{\psi}^i_{(\alpha}\psi_{\beta)i} + \eta_{\alpha\beta}\Big)^2 + D^{\dot\alpha}{}_\alpha \overline{\psi}^{\dot\alpha i} D^i_{\beta\dot\alpha}\psi^\beta + \overline{\psi}^{\dot\alpha i}(\phi+\overline{z}_i)(\phi+z_i)\psi_{\alpha i} + \cdots\right) \tag{8.13}$$

Here z_i is the mass of ψ_i. This is slightly more complicated than the usual formulation, but the extra stuff will make the calculations easier. Here $i = 1, \dots, N_f$.

In the limit $\kappa \to \infty$, the path integral localizes to the *Seiberg-Witten equations*:

$$F_{\alpha\beta}^+ + \eta_{\alpha\beta} = i\overline{\psi}^i_{(\alpha}\psi_{\beta)i} D_{\beta\dot\alpha}\psi_i^\beta \qquad = 0 \tag{8.14a}$$

$$(\phi + z_i)\psi_i = 0, \tag{8.14b}$$

where $i = 1, \dots, N_f$ in the latter two equations.

When $z_i = 0$, this implies $\phi\psi_i = 0$. There are two possibilities: on the *Higgs branch*, $\phi = 0$, and on the *Coulomb branch*, $\psi_i = 0$. For a generic metric on M, there are no solutions to the Seiberg-Witten equations on the Coulomb branch, so the path integral localizes to the Higgs branch, meaning the ψ_i are governed by the Seiberg-Witten equations.

We want to compute some correlation functions in topologically twisted sectors of this theory. Generically, they vanish because of an anomaly in the R-symmetry $\mathfrak{u}(1)_t$; we need to insert just the right number of Q-closed operators to cancel this anomaly.[8]

Shapere-Tachikawa offer the densities

$$\partial_\mu R^\mu = \frac{c-2a}{16\pi^2} R_{\mu\nu\rho\sigma}\widehat{\widehat{R}}_{\mu\nu\rho\sigma} - \frac{c}{16\pi^2} R_{\mu\nu\rho\sigma}\widehat{R}_{\mu\nu\rho\sigma} + \frac{\kappa_G}{32\pi^2} F^A_{\mu\nu}\widetilde{F}^{A\mu\nu}, \tag{8.15}$$

where $c = (2n_V + n_n)/16$ and $a = (5n_v + n_n)/24$. The integrals of these densities can be expressed just in terms of the Euler characteristic and signature of M (though some pieces have more direct interpretations, such as the second Chern character).

Specifically, if $n_v = 1$ and $n_n = 0$, $\Delta R = -(1/2)(\chi + \sigma)$; if $n_v = 0$ and $n_n = 1$, $\Delta R = \lambda^2/4 - \sigma/4$, where $\lambda = c_1(L) \in H^2(M)$.

Since ϕ has charge 2 under $\mathfrak{u}(1)_t$, the simplest correlation function you can consider is $\langle \phi^{k/2} \rangle$, which is the Seiberg-Witten invariant for N_f flavors and our chosen λ.

This was all for $z_i = 0$. If we turn on masses, the story is similar; there are again no contributions to the path integral from the Coulomb branch. In this case, the solutions have $\phi = -z_i$ for a single i, and $\psi_j = 0$ for $j \neq i$. In this case, the Seiberg-Witten invariant factorizes as a product of the Seiberg-Witten invariant for $N_f = 1$ and the determinant of $N_f - 1$ hypermultiplets:

$$\mathrm{SW}(N_f, \lambda) = \mathrm{SW}(1, \lambda) \prod_{i=j}^{N_f} \left(\prod_{j \neq i}^{N_f} (z_j - z_i)^{(1/8)(\lambda^2 - \sigma)} \right)^{-1}. \tag{8.16}$$

[8] Well really, they vanish because of the selection rule; the anomaly is what makes the selection rule not what you'd initially imagine.

The term inside the (outer) product looks a lot like an OPE of some operators, and this is correct. There's more to say here: the $\mathfrak{u}(1)$ R-symmetry here is the same as on the 2D side, and this tells us, when we want to insert operators, what their charges should be.

9. 3D $\mathcal{N}=4$ boundary VOAs and 4D $\mathcal{N}=2$ in the B-type Ω-background: 11/14/18

This talk was given by Dylan Butson.

Let X be a smooth complex algebraic variety or a smooth complex manifold, which we will think of as our spacetime. We'll study classical field theories over X in the BV formalism, which will amount to studying (derived) algebraic symplectic geometry in $D(X)$, the category of $\mathcal{D}$-modules on X. The forgetful functor from D-modules to quasicoherent sheaves has a left adjoint $(-)_D = - \otimes_{\mathscr{O}_X} \mathcal{D}_X$, and

$$\operatorname{Hom}_D(M_D, N_D) = \operatorname{Hom}_{\mathscr{O}_X}(M, N_D) = \Gamma(X, N \otimes_{\mathscr{O}_X} D \otimes_{\mathscr{O}_X} M), \tag{9.1}$$

and $N \otimes_{\mathscr{O}_X} D \otimes_{\mathscr{O}_X} M$ is the sheaf of differential operators from M to N. So studying D-modules is closely related to studying linear differential operators on X.

For example, if $P \in \operatorname{Diff}(M, N)$, let L_P be the complex $M_D \to N_D$, with the map induced by P via (9.1). The zeroth cohomology is the de Rham cohomology, defining a functor to $\mathsf{Vect}_{\mathbb{C}}$.

Often in classical field theory, however, we need nonlinear differential equations. For this reason we consider commutative algebras in $D(X)$, using the monoidal structure $\otimes_{\mathscr{O}_X}$. Taking the spectrum, we obtain the category AffSch_{D_X} of affine schemes over D_X. Crucially, these are examples of factorization algebras; we think of factorization algebras as describing quantum field theories, and our affine D_X-schemes as describing classical field theories.

The de Rham functor $D(X) \to \mathsf{Vect}_{\mathbb{C}}$ extends to where we take commutative algebras, defining a functor from affine D_X-schemes to affine $\mathbb{C}$-schemes, which we denote $\langle \cdot \rangle$. Concretely,

$$\langle A \rangle(F) = \operatorname{Hom}_{\mathsf{CAlg}(D(X))}(A, \mathscr{O}_X \otimes_{\mathbb{C}} F). \tag{9.2}$$

If A is the *Koszul resolution* $\operatorname{Sym}^\bullet_{\mathscr{O}_X}(L_P^0)$, then

$$\langle A \rangle(\mathbb{C}) = \operatorname{Hom}_{D(X)}(L_P^0, \mathscr{O}_X) = \operatorname{Hom}_{D(X)}(\mathscr{O}_X, L_P). \tag{9.3}$$

We're going to bring this to study some specific, very nonlinear, field theories. They will be expressible in terms of simpler data: Lie_∞-algebras.

Definition 9.4. A *Lie_∞-algebra* is a graded vector space L together with operations $\ell_n \colon \Lambda^n L \to L[2-n]$ such that the Lie_∞ relations hold. We encode these compactly using the *Chevalley-Eilenberg cochains* functor $C^\bullet \colon \mathsf{Lie}_\infty \to \mathsf{CDGA}$: $C^\bullet(L) \coloneqq \operatorname{Sym}^\bullet(L^\vee[1])$; the differential is induced from the map

$$d_n \colon L^\vee[-1] \longrightarrow \bigoplus_n \Lambda^n L^\vee[-n]. \tag{9.5}$$

We ask that $d^2 = 0$.

For example, ℓ_1 is d, ℓ_2 is a bracket, and ℓ_3 is the homotopy witnessing the Jacobi identity.

Example 9.6. Let V be a vector space, which we regard as a graded vector space in degree 0, and let $L \coloneqq V[-1]$. Then $C^\bullet(L) = \operatorname{Sym}^\bullet(V^\vee) = \mathscr{O}(V)$. There can be no ℓ_n for degree reasons. ◀

Example 9.7. For a slightly more involved example, suppose $L = V[-1] \oplus W[-2]$. Then the only possible differential is $d \colon V \to W$; this induces a $\sigma \in \operatorname{Sym}^\bullet(V^\vee) \otimes W$, such that the Chevalley-Eilenberg complex is

$$\operatorname{Sym}^\bullet(W^\vee[1]) \otimes \operatorname{Sym}^\bullet(V^\vee), \tag{9.8}$$

with σ encoding the differential. ◀

In general, degree -1 stuff is geometric, and is related to the space of fields. The degree -2 stuff is antifields: they impose the equations of motions. Degrees 0 and -3 are ghosts and antighosts, completing the BV formalism.

Example 9.9. Let $\mathfrak{g}$ be a Lie algebra acting on a vector space V; then we can take $L \coloneqq \mathfrak{g} \ltimes V[-1]$. In this case $C^\bullet(L) = C^\bullet(\mathfrak{g}; C^\bullet(V[-1]) = \mathscr{O}(V)^{\mathfrak{g}}$. In this case, there's something a litte stacky going on, as what we get is a formal completion $\widehat{\mathscr{O}}_0([V/G])$, which relates to the fact that the category of Lie_∞-algebras is homotopical, so we should equip our CDGAs with an augmentation, etc. ◀

Remark 9.10. Corresponding to the fact that CDGAs over $D(X)$ use the monoidal structure denoted $D(X)^!$, the associated data on the Lie_∞ side is called a $*$-product. ◂

Example 9.11 (Chern-Simons theory)**.** Let M be a smooth 3-manifold and G be an algebraic group. Let $L \coloneqq \Omega_M^\bullet \otimes \mathfrak{g}$, with $\ell_1 = \mathrm{d}_{\mathrm{dR}} \otimes 1$ and $\ell_2 = \Lambda \otimes [\cdot,\cdot]$. One can show that the functor of points that this Lie_∞-algebra defines is the Maurer-Cartan functor, and in this setting that encodes the flatness of a connection, hence the same as the extremizers of the Chern-Simons functional.

You can write L as an iterated semidirect product

$$(\mathfrak{g} \ltimes \mathfrak{v}) \ltimes (\mathfrak{v}^! \ltimes \mathfrak{g}^!), \tag{9.12}$$

where, reading from left to right, the three semidirect products are encoding gauge invariance, the equations of motion, and the moment map. If we take factorization homology $\int_M C^\bullet(L)$, we get the formal completion of $\mathscr{O}_0(\mathrm{Maps}(M^3_{\mathrm{dR}}, BG))$. ◂

Now we consider a more central example for this talk. Let X be an algebraic (or complex) surface, N be a 1-manifold, C be an algebraic (or complex) curve,[9] G be an algebraic group, and R be a symplectic representation. The theorem/result is

Theorem 9.13. *The minimal holomorphic-topological twists of 3D $\mathcal{N}=4$ and 4D $\mathcal{N}=2$ theories with gauge group G and matter R is*

$$L^{4d}_{\mathcal{N}=2} = \Omega_X^{0,\bullet} \otimes_{\mathbb{C}} \mathbb{C}[\varepsilon] \otimes_{\mathbb{C}} \mathfrak{g}_R, \tag{9.14}$$

where $|\varepsilon| = \pm 1$ and $\mathfrak{g}_R$ is the stacky symplectic reduction

$$\mathfrak{g}_R = \pi_0[-1][R/\!\!/G] = \mathfrak{g} \ltimes R[-1] \ltimes \mathfrak{g}^\vee[-2], \tag{9.15}$$

and

$$L^{3D}_{\mathcal{N}=4} = \Omega_N^\bullet \boxtimes \Omega_C^{0,\bullet}\mathbb{C}[\varepsilon] \otimes \mathfrak{g}_R. \tag{9.16}$$

You can think of the Lie_∞-algebra in (9.14) as $T^*[1]\mathrm{Maps}(X, [R/\!\!/G])$. The surface X is spacetime. Speaking concretely mathematically, one has a family of differential equations, which has been studied before, and its symmetry defines a canonical deformation to these easier-to-understand Lie_∞-algebras, leading to explicit calculations.

The idea of the proof is to exploit supersymmetry to deform Sch_{D_X}. For example, for the algebra in (9.16), the B-twist deforms to

$$\Omega_C^\bullet = \Omega_C^{0,\bullet} \xrightarrow{\partial} \Omega_C^{1,\bullet}. \tag{9.17}$$

The A-twist looks similar, but is simpler:

$$\Omega_C^\bullet = \Omega_C^{0,\bullet} \overset{\partial}{\mathrm{id}} \Omega_C^{0,\bullet}. \tag{9.18}$$

So the space of fields is contractible, but still not quite trivial. The upshot is that, like many other famous twists, calculations involve intersection theory in moduli spaces.

Example 9.19. Suppose that X is a product of curves; then the Kapustin twist is a twist along one of the two factors:

$$L^{4d}_{\mathrm{Kap}} = \Omega_\Sigma^\bullet \boxtimes \Omega_C^{0,\bullet} \otimes \mathfrak{g}_R. \tag{9.20}$$

Under the replacement of $\mathfrak{g}_R$ by $\mathfrak{g}$, this is Costello's Yangian theory. So the space of fields is $\mathrm{Maps}(\Sigma_{\mathrm{dR}} \times C, [R/\!\!/G])$, where $X = \Sigma \times C$. ◂

This framework also allows for the study of boundary theories. Suppose $\mathcal{X}$ is a (-1)-shifted Coisson algebra that's actually symplectic. There's a way to associate to it a 0-shifted Coisson algebra $\mathcal{X}^\partial$, which behaves like a phase space for the theory. Given a 3-manifold M with boundary Σ for the system, we should pick a boundary B, a D_Σ-scheme, which should be Lagrangian, in a suitable sense, for $\mathcal{X}^\partial$. There's one more technical hypothesis (regular embedding).

Theorem 9.21 (Butson-Yoo)**.** *There is a unique (-1)-shifted Coisson structure on B extending the coisotropic structure.*

[9] For simplicity, we assume X and C are Calabi-Yau. This is not necessary, but makes formulas simpler.

The proof idea has existed for a while, but in a different setting. There are relevant papers by Pavel Safronov and collaborators, as well as one by Theo Johnson-Freyd.

Rather than a proof, it may be more helpful to see it in an example, namely 3D Chern-Simons theory. In this case, $L^\partial = \Omega^\bullet_\Sigma \otimes \mathfrak{g} = \Omega^{0,\bullet}_\Sigma \otimes \mathfrak{g} \ltimes \Omega^{1,\bullet}_\Sigma \otimes \mathfrak{g}[-1]$, with a differential from $0, \bullet$ to $1, \bullet$. Here we had to choose a complex structure on Σ. Choose $B = \Omega^{1,\bullet}_\Sigma \otimes \mathfrak{g}[-1]$, $\partial \in \mathrm{Hom}(B^!, B) \cong B^{\otimes 2}$, which is exactly what we need for a constant-coefficient Poisson bivector. Moreover, $\Lambda_{01} \otimes [\cdot, \cdot] \in \mathrm{Hom}(B^! \otimes B, B) \cong B^! \otimes B^{\otimes 2}$, so we get a linear term. This is the same structure as a Lie algebra structure plus a central extension on the original vector space B. So the Poisson structure on B is coming from the Lie algebra structure on $B^!$ — this is a semiclassical analogue of the Poisson envelope construction. The answer in the end is the semiclassical chiral enveloping algebra, or in VOA terminology, the vacuum module for the corresponding affine Kac-Moody.

This is not quite what you'd expect (which is something WZW), which is telling us that perturbation theory doesn't quite have the right thing, but it's certainly pleasing that the boundary story for Chern-Simons has a linear-algebraic encoding.