

GEOMETRY AND STRING THEORY SEMINAR: FALL 2018

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1. MODULI OF FLAT SL_3 -CONNECTIONS AND EXACT WKB: 9/5/18

The first talk this semester was given by Andy Neitzke.

Let C be a thrice-punctured \mathbb{CP}^3 , say punctured at $\{1, \omega, \omega^2\}$, and let \mathcal{M} denote the moduli space of flat $\mathrm{SL}_3(\mathbb{C})$ -connection over C with unipotent holonomy around the punctures; this is an example of a *character variety*. This talk will discuss Andy's work (in progress) with Lotte Hollands on constructing nice coordinate systems on this space, using ideas coming from physics.

Let's start with the simpler case of SL_2 , and consider the *Mathieu equation*, a Schrödinger equation with periodic potential. Let $\hbar > 0$; then, the Mathieu equation is

$$(1.1) \quad \left(-\frac{\hbar^2}{2} \partial_x^2 + \cos x - E \right) \psi(x) = 0.$$

Parallel transport (i.e. evolution of solutions) of this equation defines a flat $\mathrm{SL}_2(\mathbb{R})$ -connection ∇ on \mathbb{R} . You might think it's $\mathrm{GL}_2(\mathbb{R})$, because there are two solutions, but they're related by the Wronskian. Since the potential is periodic, this is a connection on $\mathbb{R}/2\pi\mathbb{Z} = S^1$; now we can ask about its monodromy, or about its eigenvalues (which are easier to write down without making additional choices). In physics, the eigenvalues are known as *quasi-momenta* for a particle moving with respect to this potential.

Let ψ be an eigenfunction with eigenvalue λ . If $E \gg 1$, then $\cos x$ is small, so

$$(1.2) \quad \psi_{\pm}(x) := \exp\left(\pm i \frac{\sqrt{2E}}{\hbar} x\right)$$

is a basis of the solutions. The eigenvalues are

$$(1.3) \quad \lambda_{\pm} = \exp\left(\pm 2\pi i \frac{\sqrt{2E}}{\hbar}\right),$$

and the trace is $2 \cos(2\pi \sqrt{2E}/\hbar)$. Then $|\lambda_{\pm}| = 1$ for all E .

So the trace is periodic in \sqrt{E} . If this is close to ± 2 , we're in a region called the "gap": ΔE is exponentially small, and so solutions are stable. When the absolute value of the trace is smaller, we're in the "band," where the monodromy is complex. This means that solutions exponentially blow up or exponentially decay.

Remark 1.4. In solid-state physics, one example of periodic potentials are crystals. One can show that bands and gaps correspond to conducting and insulating states. ◀

Because of this application, physicists have developed lots of techniques for studying these systems, which we can adapt to geometry to study the monodromy.

First, let's complexify: let $z = e^{ix}$; then we have a complex Schrödinger equation

$$(1.5) \quad (\hbar^2 \partial_z^2 + P(z)) \psi = 0,$$

where

$$(1.6) \quad P(z) = \frac{1}{z^3} - \frac{2E - \hbar^2/4}{z^2} + \frac{1}{z}.$$

The $\hbar^2/4$ correction isn't that important.

Remark 1.7. You can do this on any Riemann surface as long as P is a holomorphic quadratic differential; this requires choosing a complex projective structure. But the ideas can be gotten across in coordinates. ◀

To understand the monodromy, we need to get at the solutions. The exact WKB method constructs solutions of the form

$$(1.8) \quad \psi(z) = \exp\left(\frac{1}{\hbar} \int_{z_*}^z \lambda dz\right).$$

In order to satisfy (1.5), λ must satisfy the *Riccati equation*

$$(1.9) \quad \lambda^2 + P + \hbar \partial_z \lambda = 0.$$

This is easier to solve than the original equation. Namely, to leading order in \hbar , $\lambda^2 + P = 0$. We will then plug this back in to get at higher orders in \hbar . Specifically, we get

$$(1.10) \quad \lambda = \sqrt{-P} - \hbar \frac{P'}{4P} + \hbar^2 \sqrt{-P} \frac{5(P')^2 - 4PP''}{32P^3} + \dots$$

This naturally lives on the *spectral curve* for the equation, i.e. the Riemann surface for $\sqrt{-P}$, $\Sigma := \{y^2 + P(z) = 0\}$, a double cover of the original surface.

This isn't the end of the story, though: solutions will have monodromy around the zeros of P . But we also can't have monodromy (TODO: I missed why). Looking more closely at (1.10), it doesn't actually converge: it's just an asymptotic series. But it's still useful; it admits Borel summation for $\hbar > 0$ and away from a locus called the *Stokes graph* $W(P)$.¹

The Stokes graph cuts the Riemann surface into domains; inside each domain, everything works, and you learn a lot about the solutions. But you can't do anything in a neighborhood of a zero of P , which prevents the paradox we chanced upon earlier. The upshot is that in each domain, there's a canonical basis (up to scaling) of the solution space: the solutions are a line bundle over the spectral curve, together with a connection ∇^{ab} represented by $\hbar \lambda dz$. And there's a canonical way to glue these line bundles over $W(P)$, to obtain a line bundle $L \rightarrow \Sigma$ together with a flat connection. It's almost flat (the monodromy around branch points might be -1).

It's natural to compute the holonomy $X_\gamma \in \mathbb{C}^\times$ around a curve γ , and this has nice properties. As $\hbar \rightarrow 0$, the asymptotic series of this is computable, e.g. $X_\gamma \sim \exp(\hbar^{-1} Z_\gamma)$, where $Z_\gamma = \oint_\gamma \sqrt{-P} dz$. In a given example (choose P , draw the spectral network, fix a loop), this is completely concrete. The trace is almost an eigenvalue of the monodromy, but it has to cross one of the lines in $W(P)$, and the formula shows that. Specifically, one gets a term for the cosine and a term responsible for the gaps (and hence can be studied to learn about the gaps).

So any particular picture/problem comes with its own picture and defines a coordinate system.

What changes for SL_3 ? We need a higher-rank analogue of the Schrödinger equation, which will have two potentials P_2 and P_3 :

$$(1.11) \quad \left(\partial_z^3 + \hbar^{-2} P_2(z) \partial_z + \left(\hbar^{-3} P_3(z) + \frac{1}{2} \hbar^{-2} P_2'(z) \right) \right) \psi(z) = 0.$$

There's a higher WKB method to deal with such equations, but let's look at a specific example, in which

$$(1.12) \quad P_3 = -\frac{u}{(z^3 - 1)^2} \quad \text{and} \quad P_2 = \frac{9\hbar^2}{(z^3 - 1)^2}.$$

The $9\hbar^2$ term in P_2 won't matter for the spectral curve, though we can't completely ignore higher-order terms in \hbar .

Now parallel transport of solutions gives us (I think?) a flat SL_3 -connection on C . We want to study the connections with $u > 0$. The higher WKB machinery gives you a basis $\{\psi_1, \psi_2, \psi_3\}$ inside a chamber (the

¹The same locus appears in $\mathcal{N} = 2$ supersymmetry, where it's called a *spectral network*, but its origin is older.

Stokes graph divides the Riemann surface into two chambers), and the three monodromies around points A , B , and C must satisfy

$$(1.13) \quad C\psi_1, B^{-1}\psi_2 \in \text{span}\{\psi_1, \psi_2\},$$

along with all cyclic permutations of this condition. This is an algebraic geometry question, and has a cool answer: A , B , and C are unipotent, and in this case there's a continuous family of solutions (not as interesting) plus four exceptional ones, and WKB produces one of these.

2. VERTEX ALGEBRAS TOWARDS HIGGS BRANCHES: 9/12/18

Today, David spoke about vertex algebras, providing an introduction and background, albeit an ahistorical one.

You can think of vertex algebras as coming from topological field theory. Consider an oriented 2D TQFT Z , whose space of local operators/observables is $V := Z(S^1)$. The pair-of-pants bordism $S^1 \amalg S^1 \rightarrow S^1$ defines a multiplication map $V \otimes V \rightarrow V$; you can think of this as taking two small circles inside a larger annulus.²

If you favor one of the pant legs, you can think of this bordism as a cylinder together with the insertion of a small circle at some point z in the cylinder, which you can label by any $v \in V$. Once you do this, you get a map $V \rightarrow V$ given by the cylinder, and therefore get a map $V \rightarrow \text{End } V$, which we call $v \mapsto Y(v, z)$.

We're working with a topological field theory, so $Y(v, z)$ must be locally constant in z . Passing to the annulus, we have two inner discs given by the incoming S^1 and a small disc around z . Though $Y(v, z)$ is locally constant in z , interesting things can happen when you move z around v ; the structure given by V and $Y(v, z)$ is called an E_2 -algebra or a 2 -disc algebra.

If our TQFT is valued in $\text{Vect}_{\mathbb{C}}$, an E_2 -algebra is fairly simple to understand: we can move v and z around each other, so it's just a commutative algebra. But there are more operations in what's called the *cohomological setting*, where the TQFT is valued in something like graded complex vector spaces. Local constancy means that we have an action of the homology of the $C_2(\mathbb{R}^2)$, the configuration space of two points in \mathbb{R}^2 on V , and since $C_2(\mathbb{R}^2) \simeq S^1$, we get data of a map $H^*(S^1) \otimes V \otimes V \rightarrow V$, i.e. a map

$$(2.1) \quad V \longrightarrow \text{End } V \otimes H^*(S^1).$$

The cohomology of S^1 is pretty simple; in degree 0 we get back the commutative multiplication, and in degree -1 we get a graded Lie bracket $\{, \}$. This behaves well with respect to the multiplication, and this structure is called a *graded Poisson algebra*, or in this case also a *Gerstenhaber algebra*.³

But we can upgrade this to the *derived* setting, replacing cohomology by cochains, which is what supersymmetry taught us to do. This is the setting people usually refer to when they say E_2 -algebra. We have a diagram

$$(2.2) \quad \begin{array}{ccc} & \text{End } V \otimes C^*(D) & \\ & \downarrow & \\ V & \xrightarrow{\quad} \text{End } V \otimes C^*(S^1) & \\ & \downarrow & \\ & \text{End } V \otimes H^1(S^1)[-1] & \end{array}$$

Here D is the disc. The top map from V is an honestly commutative map for every pair of points, the middle one is the E_2 -algebra structure, and the bottom map is the Lie operad (since it gave us the bracket). So the E_2 -operad is built out of these two operads, and gives a commutative multiplication for pairs of points plus other data.

Example 2.3 (String topology). One simple example of a 2D TQFT is called *string topology* on a manifold M , or the A -model on T^*M . The local operators are $H_*(\text{Map}(S^1, M))$: the homology of the loop space. Setting up the multiplication map takes some thought, and there are papers working this out.

²For these to be the same, we need to be doing oriented TQFT, not framed TQFT.

³For an interesting and recent application to physics, see <https://arxiv.org/abs/1809.00009> by Beem-Ben-Zvi-Bullimore-Dimofte-Neitzke.

There's also a space of *observables*, which comes as part of the description arising from physics, but doesn't fit into the functorial perspective. This is a pity, because they're how vertex algebras enter the story. Specifically, the observables are an algebra $H_*(\text{Map}_c(D, M))$, or $H_*(\Omega^2 M)$. For example, if $M = BG$ is the classifying space of a compact Lie group G with complex form $G_{\mathbb{C}}$, $\Omega^2 BG$ is the affine Grassmannian $LG_{\mathbb{C}}/LG_{\mathbb{C}^\times}$. ◀

We've been thinking of all of this in terms of 2D TQFT, but all of the algebraic structure appears in higher dimensions too: vertex algebras appear in, e.g., 4D gauge theories where there are two topological directions and one holomorphic direction, and the intuition we've been using will carry over to there too.

Vertex algebras are the analogues of E_2 -algebras, but for 2D conformal field theory rather than 2D topological field theory. The analogue of the circle is $\text{Spec } \mathbb{C}[[z]]$, and the vector space of observables on that space is again denoted V . Given a vector v and a point z , we again get a $Y(v, z) \in \text{End } V$, but this time, we want it to depend holomorphically⁴ on z in the neighborhood $\text{Spec } \mathbb{C}((z))$. That is, given $w \in W$, $Y(v, z) \in V((z))$. This may seem weird, but it's typical for how Laurent series are used to study local neighborhoods in formal algebraic geometry, and the upshot is $Y(v, z) \in \text{End } V[[z^{\pm 1}]]$. We'll think of $Y(v, z)$ as an operator-valued distribution supported at $z \in D$.

Why should we think of these as distributions? Our model of functions is Laurent polynomials, and the algebraic dual of $\mathbb{C}[[z]]$ is $\mathbb{C}[[z^{\pm 1}]]$, so they can be called (algebraic) distributions.

Example 2.4. For example, the δ -function at z can be represented as

$$\delta_1(z) = \sum_{n=-\infty}^{\infty} z^n \in \mathbb{C}[z^{\pm 1}].$$

If you calculate its residues, you get just one at 1. ◀

We'll also write $Y(z, v)$ as $v(z)$, thinking of v acting at z .

Definition 2.5. A *vertex algebra* is the data of

- a vector space V ,
- a *unit* or *vacuum element* $|0\rangle \in V$,
- a *translation* $T \in \text{End } V$, and
- a map $Y: V \rightarrow \text{End } V[[z^{\pm 1}]]$,

subject to the following axioms:

- $Y(|0\rangle, z) = \text{id}$,
- $Y(v, z) \cdot |0\rangle = v + z(\text{stuff})$,
- T encodes the $\frac{d}{dz}$ -equivariance, in that $[T, Y(v, z)] = \partial_z Y(v, z)$, and
- a locality axiom, that $[a(z), b(w)]$ ought to be supported at $z = w$, i.e. there's an $N \gg 0$ such that

$$(z - w)^N [a(z), b(w)] = 0.$$

Again, $\mathbb{C}[[z^{\pm 1}]]$ is a space of test functions; you can think of this as a place where you can solve algebraic equations, somewhat like \mathcal{D} -modules.

The last axiom sometimes is written in different ways recalling associativity, Lie brackets, etc. It might seem surprising, because we don't have Laurent series supported at points, but we're working with distributions: letting

$$(2.6) \quad \delta(z - w) := \sum_{n=-\infty}^{\infty} \frac{w^n}{z^{n+1}},$$

and $(z - w)\delta(z - w) = 0$, so we're OK. As usual, $\delta(z - w)$ and its derivatives span all distributions supported at $z = w$. Therefore locality is equivalent to asking that

$$(2.7) \quad [a(z), b(w)] = \sum_{n=0}^N (a_{(n)}b)(w) \frac{1}{n!} \partial_w^n \delta(z_w)$$

⁴This means algebraically if you're thinking algebro-geometrically, which we're mostly doing. You can formulate vertex algebras in either algebraic or analytic language, but all of the structure ends up being completely formal, in the sense of formal algebraic geometry.

for some coefficients $a_{(n)}b$.

This is it: you might want to add associativity or a Jacobi identity, but you don't actually need to. So next we'll talk about how to think about vertex algebras.

First of all, $[a(z), b(w)]$ isn't *quite* supported at $z = w$: it also depends on n^{th} -order derivatives for $n \leq N$, so on the N^{th} -order jets, which requires an N^{th} -order neighborhood of $z = w$. This doesn't change much, though.

Definition 2.8. A vertex algebra is *commutative* if $Y(a, z) \cdot b \in V[[z]]$, rather than just $V((z))$.

This is a very strong assumption: there are no poles. In this case, we can define a new multiplication $\cdot : V \otimes V \rightarrow V$ by

$$(2.9) \quad a \cdot b = \lim_{z \rightarrow 0} a(z) \cdot b.$$

Since $[a(z), b(w)]$ is a Taylor series supported at a single point, it must vanish! And therefore V is just a commutative (associative, unital) algebra with a derivation T (with respect to \cdot), i.e. a differential commutative algebra. And conversely, given a differential commutative algebra, you can just define

$$(2.10) \quad a(z) := \sum_{n \gg 0} \frac{z^n}{n!} (T^n a),$$

and you can check this is a commutative vertex algebra. Of course, any commutative algebra defines a commutative differential algebra with $T = 0$! But there are also nontrivial examples, thankfully.

We can rewrite Y as a map $V \otimes V \rightarrow V((z)) = V \otimes \mathcal{O}(D^\times)$; $\mathcal{O}(D^\times)$ can be thought of as the de Rham complex on S^1 , a souped-up version of locally constant functions. Therefore a Vertex algebra is such a map satisfying some axioms.

Y induces a map $Y^- : V \otimes V \rightarrow V((z))/V[[z]] \cong V \otimes H_{\text{loc}}^1(\{0\}, \mathcal{O})$, and Y^- is a Lie algebra structure (but with a differential; these have different names, such as Lie- $*$ algebras). And if we can restrict to $V[[z]]$, we get a commutative algebra. So much like an E_2 -algebra is something like a commutative algebra plus a Lie algebra, a vertex algebra is something like a differential commutative algebra and a differential Lie algebra.

The fact that Y^- has a Lie bracket is saying something about δ -functions, because $\mathbb{C}((z))/\mathbb{C}[[z]] \cong \langle \partial^n \delta_0 \rangle = \mathbb{C}[\delta] \delta_0$. To get the Lie bracket, though, we start with a general story on a vector bundle $V \rightarrow X$: given a section s , there's a natural pointwise multiplication $s(z) \cdot v$, which is \mathcal{O}_X -linear. But you can also define multiplication depending on the Taylor series at a point, which is local in the physical sense. Since $\mathbb{C}[[z]]^* = \mathbb{C}((z))/\mathbb{C}[[z]]$, then Y^- defines maps $Y_t : (V \otimes V)[[t]] \rightarrow V$, which gives us the Lie bracket structure.

Like for E_2 -algebras, vertex algebras are almost commutative: there's a filtration on either whose associated gradeds are commutative. This means the analogue of a Poisson structure: the description in terms of commutative and Lie algebras splits, and we get both structures.

Therefore vertex algebras are some sort of deformation/quantization of the notion of a differential Poisson algebra.