2019 PCMI PREPARATORY LECTURES

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These notes were taken at UT Austin as part of a learning seminar in preparation for PCMI's 2019 graduate summer school. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

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1. BPS states: 5/21/19

Today, Shehper talked about BPS states in 4D $\mathcal{N} = 2$ supersymmetric theories. This is not the only place you can have BPS states, but this is probably the one most relevant to our interests. For a reference, check out Moore's PITP lectures on BPS states.[1]

First, 4D means the dimension of the theory: we have three space coordinates and one time coordinate. There's an underlying symmetry group called the *Poincaré group* of $\mathbb{R}^{1,3}$, whose Lie algebra is

$$\mathfrak{iso}_{1,3} \cong \mathfrak{so}_{1,3}^+ \rtimes \mathbb{R}^{1,3}. \tag{1.1}$$

The "+" means that we want transformations to preserve the arrow of time. That is, these transformations correspond to changes between different reference frames. In one, we have local coordinates (t, x, y, z), and from one reference frame to another, the time coordinate t is scaled by something; we want this to be a nonnegative number. The transformations coming from $\mathfrak{so}_{1,3}^+$ are called *(orthochronous special) Lorentz transformations*, but we'll call them Lorentz transformations.

Another way to describe the Poincaré group is as the group of isometries of $\mathbb{R}^{1,3}$.

Now we should clarify what "underlying symmetry" means. This is a statement about QFT, which means we have to indicate how to actually discuss or work with QFT. There are a few different formalisms, e.g. the *Hamiltonian formalism* or *canonical quantization formalism*; or the *path integral formalism*, which comes with the following data:

- A space of *field configurations* $\mathcal{F}$.
- An *action*, a function $S\colon \mathcal{F} \to \mathbb{R}$.
- A set of *local operators*.

From this data one can compute correlation functions associated to local operators $\Phi_1, \dots, \Phi_n$ at points $x_1, \dots, x_n$ in spacetime via the path integral

$$\langle \Phi_1(x_1) \cdots \Phi_n(x_n) \rangle = \int_{\mathcal{F}} \mathcal{D}\varphi \, e^{-S(\varphi)} \Phi_1 \cdots \Phi_n. \tag{1.2}$$

Of course, this is not mathematically well-defined in general, but physicists have ways of working with it which agree extremely well with experimental data.

The Hamiltonian formalism in a d-dimensional quantum field theory associates to a $(d-1)$-manifold a Hilbert space $\mathcal{H}$. Elements of $\mathcal{H}$ are called *states*, because they represent states of the physical system. Inside $\mathfrak{iso}_{1,3}$, there's an element P_τ which is time translation by τ: explicitly, under the isomorphism (1.1), these are the elements in $\mathbb{R} \cdot t \subset \mathbb{R}^{1,3}$. This element acts on $\mathcal{H}$ by the Hamiltonian, and this is how the system evolves under time. An eigenvector for the Hamiltonian with eigenvalue λ is said to have *energy* λ.

[1] The lecture notes can be found at http://www.sns.ias.edu/pitp2/2010files/Moore_LectureNotes.rev3.pdf.

Assumption 1.3. There is a unique vector $|v\rangle \in \mathcal{H}$, called the *vacuum*, with minimum energy.

There's a sense in which the vacuum generates all of the states: one can act by local operators to obtain the other states. And in this formalism, the correlation functions are given by

$$\langle \Phi_1 \cdots \Phi_n \rangle \coloneqq \langle v \mid \Phi_1 \cdots \Phi_n \mid v \rangle. \tag{1.4}$$

Explicitly, assume that $\phi(x)$ is a *Lorentz scalar*, which means it's a field transforming in the trivial representation of $\mathfrak{so}_{1,3}^+$. Here x is position, i.e. the coordinate in the spacetime manifold.

Remark 1.5. A field is not an operator, but it does determine a local operator, e.g. ϕ, as a scalar field (function), has a value at a point x. We will think of ϕ as a local operator sometimes in what follows. ◀

How do we use this to create states in $\mathcal{H}$? The first step is to Fourier transform ϕ, leading to $\widetilde{\phi}(p)$. Now this depends on the momentum p. We can act on $|v\rangle$ by $\widetilde{\phi}$ to obtain other states in $\mathcal{H}$.[2] There are things which have positive momenta and with negative momenta; these should be thought of as particle creation $\widetilde{\phi}^\dagger$, resp. particle annihilation operators $\widetilde{\phi}$ on the space of states. This is analogous to the raising and lowering operators on $\mathfrak{su}_2$-representations.

The physical interpretation is that the vacuum has no particles and no momentum. Acting by one creation operator creates a single particle with a prescribed momentum. Acting by another means two particles, and so on.

Remark 1.6. All of this is in a *free theory*, meaning the action is quadratic in the fields. In general, the story is a little more complicated. ◀

Anyways, back to "underlying symmetry." This means the following.

- The fields are all in representations of $\mathfrak{iso}_{1,3}$ (i.e. governing how it transforms under a change of coordinates).
- The Hilbert space is a unitary representation of $\mathfrak{iso}_{1,3}$. Additionally, we want every operator to be unitary, i.e. $U^\dagger U = \mathbf{1}$.

This means that Poincaré symmetries do not change the norm of states, which is important.

Example 1.7. Here are some irreducible representations of $\mathfrak{so}_{1,3}^+$.

- The *trivial* or *scalar representation* $\mathbb{C}$.
- The *vector representation*, which is the defining representation of $\mathfrak{so}_{1,3}^+$ on $\mathbb{R}^{1,3} \otimes \mathbb{C}$.
- The *tensor representations*, which are obtained from the vector representation by symmetric or exterior powers.
- The *spinor representations*, two 2-dimensional representations which are complex conjugates of each other, but are not isomorphic. In physics these are also called *Weyl spinors*; there's a different thing called a *Dirac spinor*, which transforms in the direct sum of the two spinor representations. ◀

So we've discussed what 4D QFT is. What does $\mathcal{N} = 2$ mean? This is specifying "how much supersymmetry" is present in the theory. Supersymmetry means that we extend the Poincaré algebra to a $\mathbb{Z}/2$-graded Lie algebra (sometimes called a *Lie superalgebra*) $\underline{\mathfrak{g}} = \underline{\mathfrak{g}}^0 \oplus \underline{\mathfrak{g}}^1$. In our situation ($\mathcal{N} = 2$), we'd like

$$\underline{\mathfrak{g}}^0 = \mathfrak{iso}_{1,3} \oplus \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_R \oplus \mathbb{C}, \tag{1.8}$$

where $\mathfrak{su}(2)_R$ denotes $\mathfrak{su}(2)$, but we write "R" to denote that this tracks something called *R-symmetry*, and likewise for $\mathfrak{u}(1)_R$ – $\mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_R$ is the *R-symmetry algebra*. Then, $\mathbb{C}$ is generated by an element Z called the *central charge* of the theory.

Then, we want $\underline{\mathfrak{g}}^1$ to be a spinor representation of $\underline{\mathfrak{g}}^0$; specifically, for $\mathcal{N} = 2$,

$$\underline{\mathfrak{g}}^1 = (2, 1; 2)_{+1} \oplus (1, 2; 2)_{-1}. \tag{1.9}$$

The notation $(a, b; c)_d$ means the irreducible $\mathfrak{so}^+(1,3)$-representation given by (a, b), the irreducible $\mathfrak{su}(2)_R$-representation of dimension c, and the irreducible $\mathfrak{u}(1)_R$-representation of weight d (i.e. the corresponding to the Lie group representation $\mathrm{U}_1 \to \mathrm{U}_1$ sending $z \mapsto z^d$). $\mathbb{C} \cdot Z$ and $\mathbb{R}^{1,3}$ act trivially.

[2] Contextualizing this, and why we can think of this as associated to position and momentum, is really related to how quantum field theory arises via quantization from classical field theory.

Since $\underline{\mathfrak{g}}^1$ is odd, the Lie bracket restricted to $\underline{\mathfrak{g}}^1 \times \underline{\mathfrak{g}}^1$ is actually an anticommutator (or Poisson bracket), so it lives in $\mathrm{Sym}^2(\underline{\mathfrak{g}}^1)$.

Let $\{Q_\alpha^A\}$ be a basis of $(2,1;2)_{+1}$, where $\alpha \in \{1,2\}$ and $A \in \{1,2\}$; similarly, let $\{\overline{Q}_{\dot\alpha A}\}$ be a basis for $(1,2;2)_{-1}$. So we have eight basis elements in total; they're called *supercharges*.

Remark 1.10. The two-dimensional irreducible representation of $\mathfrak{su}(2)_R$ is pseudoreal. There's a notion of a complex representation being *real*, which means that it's self-conjugate – or at least, the representation and its conjugate are related through a symmetric matrix. A representation is *pseudoreal* if instead we have an antisymmetric matrix: $(M^a)^\dagger = \epsilon^{ab} M_b$ (here ϵ is the Levi-Civita tensor). ◂

The point is that complex conjugation identifies some of these basis vectors, so we have to impose the relation

$$(Q_\alpha^A)^\dagger = \overline{Q}_{\dot\alpha A}. \tag{1.11}$$

Once we've imposed this, we have a real 8-dimensional representation.

We can specify the commutation relations between the supercharges:

$$\{Q_\alpha^A, \overline{Q}_{\dot\beta B}\} = 2\sigma^m_{\alpha\dot\beta} P_m \delta^A{}_B \tag{1.12a}$$

$$\{Q_\alpha^A, Q_\beta^B\} = 2\epsilon_{\alpha\beta}\epsilon^{AB}\overline{Z} \tag{1.12b}$$

$$\{\overline{Q}_{\dot\alpha A}, \overline{Q}_{\dot\beta B}\} = -2\epsilon_{\dot\alpha\dot\beta}\epsilon_{AB} Z. \tag{1.12c}$$

Once we understand $\mathrm{Sym}^2\, \underline{\mathfrak{g}}^1$ as a representation, we can analyze this and learn, e.g. that $\sigma^m_{\alpha\dot\beta} P_m$ transforms in the $(2,2)$ representation of $\mathfrak{so}^+_{1,3}$.

Definition 1.13. A *4D $\mathcal{N} = 2$ supersymmetric quantum field theory* is a QFT with an underlying symmetry algebra $\underline{\mathfrak{g}}$.

To construct BPS states, we need some representations of $\underline{\mathfrak{g}}$. We'll do this by finding an analogue of the Casimir operator inside $\mathfrak{iso}_{1,3}$ – an operator which commutes with all other operators. Explicitly, it's

$$P^2 := -P_0^2 + P_1^2 + P_2^2 + P_3^2. \tag{1.14}$$

This mimics the $\mathfrak{su}_2$ story, where the Casimir is the sum of the squares of the three Pauli matrices. In physics, P^2 is also thought of as the mass squared. For example, if the momentum is zero, this relates to the familiar equation $E^2 = M^2c^2$ – in general momentum changes this.

Now one can choose a particular basis in which $P = (M, 0, 0, 0)$, called the *rest frame*. One place you might want this is if you want a state with particular momenta $M^\mu = (P^0, P^1, P^2, P^3)$, and can obtain it from the rest frame by a Lorentz transformation.

Anyways, once you have $(M, 0, 0, 0)$, you can act on it by $\mathfrak{so}_3$ in the last three coordinates, which produces more things of the same mass. So to create "massive" irreducible representations of $\mathfrak{iso}_{1,3}$ with a fixed mass $M > 0$, we need to look for representations of $\mathfrak{so}_3 \oplus \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_R \oplus \mathbb{C}$ as follows: we want eight generators R_α^A and T_α^A such that $\{R, R\} \neq 0$, $\{T, T\} \neq 0$, $\{R, T\} = 0$, such as

$$\{R_\alpha^A, R_\beta^B\} = 4(M - |Z|)\epsilon_{\alpha\beta}\epsilon^{AB} \tag{1.15a}$$

$$\{T_\alpha^A, T_\beta^B\} = -4(M + |Z|)\epsilon_{\alpha\beta}\epsilon^{AB} \tag{1.15b}$$

$$\{R_\alpha^A, T_\beta^B\} = 0. \tag{1.15c}$$

So we have two copies of a Clifford algebra. Explicitly, if $\zeta \in \mathfrak{u}(1) \setminus 0$,

$$R_\alpha^A := \zeta^{-1} Q_\alpha^A + \zeta \sigma^0_{\alpha\dot\beta} \overline{Q}^{\dot\beta A} \tag{1.16a}$$

$$T_\alpha^A := \zeta^{-1} Q_\alpha^A - \zeta \sigma^0_{\alpha\dot\beta} \overline{Q}^{\dot\beta A}. \tag{1.16b}$$

These have reality constraints coming from those of the supercharges, e.g. $(R_1^1)^\dagger = -R_2^2$ and $(R_1^2)^\dagger = R_2^1$. This means

$$(R_1^1 = (R_1^1)^\dagger)^2 = (R_1^2 + (R_1^2)^\dagger)^2 = 4\left(M + \mathrm{Re}\left(\frac{Z}{\zeta^{-2}}\right)\right). \tag{1.17}$$

This is important for unitarity: we want $A^\dagger = A$: we want $\|A|\psi\rangle\|^2 > 0$ if $\psi \neq 0$, so we want $A^2 \geq 0$.

Suppose we choose $\zeta^{-2} = -Z/|Z|$; then the right-hand side of (1.17) simplifies to $4(M - |Z|)$. Therefore we want $M \geq |Z|$, which is called the *BPS bound*. (Other choices of ζ give you weaker constraints.) That is, in any state in a 4D $\mathcal{N} = 2$ supersymmetric theory, the mass of any state is at least $|Z|$.

There are two cases: $M = |Z|$, which is called a *BPS state*, and $M > |Z|$, which is called a *non-BPS state*. If $M = |Z|$, $\{R^A_\alpha, R^B_\beta\} = \pm 4(M - |Z|) = 0$, which acts trivially, so for BPS states, we only get one copy of the Clifford algebra (called a *short representation* rather than the usual *long representation* with two copies). In particular, the T^A_α split into creation and annihilation operators, and we get four states: $|v\rangle$, $T^\beta_\alpha|v\rangle$, $T^\delta_\gamma|v\rangle$, and $T^\beta_\alpha T^\delta_\gamma|v\rangle$. In a non-BPS state, then we'd be able to create eight states instead of four.

Great, and why do we care about BPS states? In QFT, a lot of things can happen – QFTs usually come in families, meaning there are various parameters in a quantum field theory that one can adjust. In general these parameters vary over a moduli space. If you try to move in this moduli space, short representations do not usually combine into long representations, and usually stay as they are. So BPS states are relatively rigid – or said in other words, the Hilbert space of states can change, but the spectrum of BPS states is generally invariant. Moreover, we can compute it in important situations (which is not true for the general Hilbert space), thanks to work of Gaiotto-Moore-Neitzke. In mathematics, the ways of computing BPS states have to do with things called spectral networs, which are tied to the geometry of Riemann surfaces.

The BPS representations are generally of the form $\rho \otimes s$, where ρ is the representation of $\mathfrak{so}_3 \oplus \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_R$ that we began with, and s is a short representation; similarly, the non-BPS states are in ρ tensored with a long representation. So giving representations of $\mathfrak{su}(2)_R$ and $\mathfrak{u}(1)_R$ gives you new BPS representations.

2. 3-manifold topology: 5/22/19

"You just glue noodles and pancakes to basketballs, and that's it."

Today, Charlie spoke about low-dimensional topology. Here "low-dimensional" means in dimensions 2 through 4. Today's goal is to cover surgery presentations of 3-manifolds and when two of them are equivalent. Often, interesting 3-manifold invariants are defined or calculated via surgery diagrams (such as the Witten-Reshetikhin-Turaev invariants, in particular); showing that you get an invariant is a matter of checking that it stays the same under those equivalences.

But first, dimension 2.

Definition 2.1. Let Σ be a closed, oriented surface. Let $\mathrm{Diff}^+(\Sigma)$ denote the topological group of orientation-preserving diffeomorphisms and $\mathrm{Diff}^+_0(\Sigma)$ denote the connected component of the identity, which is a normal subgroup. The *mapping class group* of Σ, denoted $\mathrm{MCG}(\Sigma)$, is $\mathrm{Diff}^+(\Sigma)/\mathrm{Diff}^+_0(\Sigma)$.

We could also have defined this using homeomorphisms instead of diffeomorphisms, and we get the same mapping class group.[3]

Theorem 2.2 (Dehn, Lickorish)**.** $\mathrm{MCG}(\Sigma)$ *is finitely generated, and is generated by Dehn twists.*

So let's talk about Dehn twists.

Definition 2.3. Let $a\colon S^1 \to \Sigma$ be an embedding and $N \cong S^1 \times I$ be a tubular neighborhood of I. The *Dehn twist* associated to a, denoted $T_a\colon \Sigma \to \Sigma$, is the (equivalence class in $\mathrm{MCG}(\Sigma)$ of) a diffeomorphism which is the identity on $\Sigma \setminus N$, and which on $N \cong S^1 \times I$ is the map

$$T_a\colon (\theta, t) \longmapsto (\theta + 2\pi t, t). \tag{2.4}$$

See Figure 1 for a picture. There's a refinement of Theorem 2.2 producing explicit generators for $\mathrm{MCG}(\Sigma)$: if Σ is connected and g denotes the genus of Σ, then we can take $3g - 1$ generators. Writing Σ as a connected sum of g tori, we can take the Dehn twists associated to two curves generating the homology of each torus, together with one other family. This is shown in the mapping class group book (*A Primer on Mapping Class Groups*).

Definition 2.5. Assume Σ is connected. A curve $a\colon S^1 \hookrightarrow \Sigma$ is *separating* if $\Sigma \setminus a(S^1)$ has two components. A curve is *simple* if it doesn't intersect itself.

[3] The mapping class group generalizes to other manifolds, but this fact presumably doesn't.

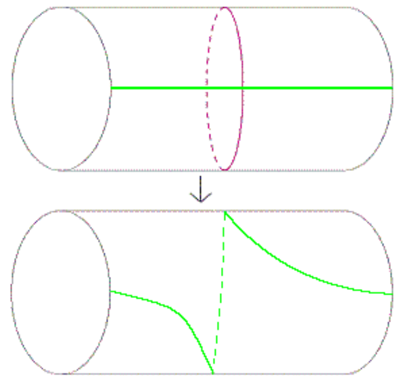

FIGURE 1. A Dehn twist. Source: https://en.wikipedia.org/wiki/Dehn_twist.

Question 2.6. Let Σ_g denote the closed, connected, oriented surface of genus g and $a_1, \dots, a_g, b_1, \dots, b_g \subset \Sigma_g$ be nonseparating simple closed curves such that no a_i and a_j intersect and no b_i and b_j intersect. Is there a diffeomorphism f such that $f \circ a_g = b_g$?

It seems reasonable, and in fact is true. One common trick for thinking about curves like this is – consider $\Sigma_g \setminus (a_1 \cup \cdots \cup a_g)$ and $\Sigma_g \setminus (b_1 \cup \cdots \cup b_g)$. These are compact surfaces, so we know their classification (each connected component is classified by its genus and number of boundary components), and you can check that these invariants match on each connected component, so there must be a diffeomorphism – and then you can extend that across the curves you removed.

Now we'll talk a little bit about handle decompositions.

Definition 2.7. A *d-dimensional k-handle* is a disc $D^d \cong D^k \times D^{n-k}$ glued to a d-manifold along the boundary $S^{k-1} \times D^{d-k} \subset \partial(D^k \times D^{d-k})$.

For example, you can build the torus out of handles: begin with a (two-dimensional) zero-handle, then attach two (two-dimensional) one-handles, then a (two-dimensional) two-handle. This is actually an instance of a general fact.

Theorem 2.8. *Any closed d-manifold can be built by attaching a series of d-dimensional handles to a d-dimensional 0-handle.*

Such a description is called a *handle decomposition* of the manifold.

Example 2.9. Let's write down a handle decomposition of $\mathbb{CP}^2$. Using homogeneous coordinates, we can write $\mathbb{CP}^2 = X \cup Y \cup Z$, where

$$X = \{[1 : y : z] : |y| \le 1, |z| \le 1\} \tag{2.10a}$$

$$Y = \{[x : 1 : z] : |x| \le 1, |z| \le 1\} \tag{2.10b}$$

$$Z = \{[x : y : 1] : |x| \le 1, |y| \le 1\}. \tag{2.10c}$$

These are the handles in a handle decomposition of $\mathbb{CP}^2$. Topologically, each is a $D^2 \times D^2 \cong D^4$. But which piece is which handle depends on the order you glue them in.

If we start with X, it's the zero-handle. Let's next glue in Y: $X \cap Y \cong \{[1 : y : z] : |y| = 1, |z| \le 1\}$. Thus $y \in S^1$ and $z \in D^2$, so $X \cap Y \cong S^1 \times B^2$, and therefore gluing Y to X along their intersection is attaching a 4-dimensional 2-handle. Finally let's glue in Z. Since $\partial Z = S^3$, this is attaching a 4-handle. ◀

Theorem 2.8 provides a way of classifying manifolds, at least in principle – above $d = 2$ it's intractable. But in dimension 2, it allows one to show that closed, connected surfaces are classified by whether the surface is orientable and how many handles are attached.

When $d = 3$, this is still useful, though: on a closed, connected 3-manifold, we begin with a 0-handle and some 1-handles, which maybe look like noodles that you attach to the disc. The 2-handles now look like pancakes (note: $d = 3$, so these pancakes are thick). The pancakes (i.e. 2-handles) are determined by circles on the boundary of the 1-*handlebody* (i.e. just the 0- and 1-handles glued in), and then where the 3-handle is uniquely determined. That is: given a 1-handlebody with a bunch of circles on the boundary, we know how to get a 3-manifold M out of it. Such a description is called a *Heegaard diagram* of M.

In fact, we can get more out of this. On any handlebody, you can glue the handles in reverse order, in which case k-handles become $(d-k)$-handles; this is called the *reverse handle decomposition.* So the Heegard diagram of M defines two 1-handlebodies: the one we made from the 0- and 1-handles, and the one made from the 2- and 3-handles, but for the reverse handle decomposition. These two handlebodies H_1 and H_2 have the same number of 1-handles (this number is called the *genus* of the 1-handlebody), and M is H_1 glued to H_2 across their common boundary.

Definition 2.11. A *Heegaard splitting* of a 3-manifold M is a decomposition $M = H_1 \cup H_2$, where H_1 and H_2 are 1-handlebodies and $H_1 \cap H_2$ is an embedded connected surface in M.

So we've just argued that Heegaard splittings always exist. If $\Sigma \coloneqq H_1 \cap H_2$ has genus g, you can represent M by two lists of g embedded nonseparating, nonintersecting circles in Σ, which tells us how to glue H_1 to H_2.

A reasonable next question is – can any such diagram occur? In the answer to Question 2.6, we saw that the answer is yes: there is a diffeomorphism that allows us to glue them, and we obtain a 3-manifold.

Next question: when do two Heegaard diagrams determine the same 3-manifold? An isotopy of any of the embedded discs doesn't change the diffeomorphism type of M, but the converse is false: there are nonisotopic Heegaard diagrams which define the same 3-manifold.

This motivates the notion of a *handle slide.* Looking first at $d = 2$, you could take two handles which look like $\cap\cap$, and move one "inside" the other to obtain something that looks like $\Cap$. This does not change the diffeomorphism type of the surface we obtain. The same thing works when $d = 3$ – and if you trace through what happens on the Heegaard diagram, you get nonisotopic curves, but the same 3-manifold! Handle slides define a useful equivalence relation on Heegaard diagrams – but there are additional diffeomorphisms between 3-manifolds presented as Heegaard diagrams that don't come from isotopies or handle slides.

Definition 2.12. A *framed circle* in a 3-manifold is an embedded circle $\gamma\colon S^1 \hookrightarrow M$ together with a trivialization of its normal bundle.[4]

It's useful to think of a framing as a choice of normal vector field. Given a framed circle $\gamma\colon S^1 \hookrightarrow M$, let N be a tubular neighborhood of γ, which is a solid torus; let b be a longitude curve on ∂N (i.e. winding along γ) and a be a curve on ∂N winding once around γ. If you glue an $S^1 \times B^2$ to $M \setminus N$ along their boundary, where we can attach $S^1 \times D^2$ along some diffeomorphism, the resulting manifold only depends on the image of a under this diffeomorphism. Moreover, isotopic diffeomorphisms produce diffeomorphic 3-manifolds, so we really only need to ask about the image of the diffeomorphism in the mapping class group.

The mapping class group of T^2 is $\mathrm{SL}_2(\mathbb{Z})$, so $a \mapsto pa + qb$, where $p, q \in \mathbb{Z}$. One can show that the resulting 3-manifold, called *(Dehn) surgery in M along γ*, only depends on p/q. In particular, given a framed link in M, together with a rational number x_C on each component C, we get a new 3-manifold obtained by doing surgery in M along each component, with $p/q = x_C$.

Theorem 2.13. *Any closed, oriented* 3*-manifold is diffeomorphic to one arising as surgery on a link in* S^3*, and in fact we can let* $p/q = \pm 1$*.*

Proof sketch. Let $M = H_1 \cup H_2$ be a Heegaard splitting, where $\Sigma_g \coloneqq H_1 \cap H_2$ is a closed, connected, oriented surface of genus g. Using Theorem 2.2, we can describe the gluing along Σ_g, *a priori* an element of $\mathrm{MCG}(\Sigma_g)$ as a sequence of Dehn twists. You can think of this as a presentation of M as a sequence of bordisms: first H_1, then several copies of $\Sigma_g \times I$ glued via Dehn twists, then H_2. This kind of looks like a hamburger.

The key is to see that Dehn surgery and Dehn twists are related, which maybe isn't such a surprise given their names. Suppose in the i^{th} bordism we're gluing by a Dehn twist along the curve $a \in \Sigma_g$. A neighborhood of a looks like a thickened washer, and Dehn surgery by either 1 or -1 accomplishes the Dehn twist: a meridian curve goes once around a, in some direction, and once around the other generator. As you do successive Dehn surgeries, these curves can become linked. ⊠

You can think of the numbers on each component as specifying what combinations of loops you want to make contractible. The presentation in Theorem 2.13, called a *surgery diagram*, is generally not unique.

[4]This definition goes through more generally for a framed submanifold in an ambient manifold.

Later, we'll define a 3-manifold invariant called the *Reshetikhin-Turaev invariant*, associated to a surgery diagram. One must check that it's invariant under the equivalences, called *Kirby moves*, between surgery diagrams that generate equivalent 3-manifolds, and this is kind of a pain, but it works.

Remark 2.14. A surgery diagram whose coefficients p/q are integers also tells you how to make a compact, oriented 4-manifold W whose boundary is M: start with a D^4, so $S^3 = \partial D^4$, and attach 4-dimensional 2-handles along the components of the link in S^3; the framings needed to glue the handles are specified by the coefficients on each component of the link. ◀

This is important in the definition of the Reshetikhin-Turaev invariants: we'll have to use this 4-manifold in the definition, and the invariant in general depends on the signature of the 4-manifold we choose.