M392C NOTES: K-THEORY

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These notes were taken in UT Austin's Math 392c (K-theory) class in Fall 2015, taught by Dan Freed. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

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Part 1. Vector Bundles and Bott Periodicity

Lecture 1.

Families of Vector Spaces and Vector Bundles: 8/27/15

"Is that clear enough? I didn't hear a ding *this time."*

Let's suppose X is a topological space. Usually, when we do cohomology theory, we send in probes, n-simplicies, into the space, and then build a chain complex with a boundary map. This chain complex can be built in many ways; for general spaces we use continuous maps, but if X has the structure of a CW complex we can use a smaller complex. If we have a singular simplicial complex, a triangulation, we get other models, but they really compute the same thing.

Given a chain complex $C_\bullet$, we get a cochain complex by computing $\mathrm{Hom}(-,\mathbb{Z})$, giving us a cochain complex $C^0 \xrightarrow{\mathrm{d}} C^1 \xrightarrow{\mathrm{d}} \cdots$, giving us the cohomology groups $H^0 = H^0(X,\mathbb{Z})$.

If M is a smooth manifold, we have a cochain complex $\Omega^0_M \xrightarrow{\mathrm{d}} \Omega^1_M \xrightarrow{\mathrm{d}} \cdots$, and therefore get the de Rham cohomology $H^\bullet_{\mathrm{dR}}(M)$. de Rham's theorem states this is isomorphic to $H^\bullet(M;\mathbb{R})$, obtained by tensoring with $\mathbb{R}$.

In K-theory, we extract topological information in a very different way, using linear algebra. This in some sense gives us more powerful invariants. Consider $\mathbb{C}^n = \{(\xi^1,\dots,\xi^n) : \xi^i \in \mathbb{C}\}$. This has the canonical basis $(1,0,\dots,0)$, $(0,1,0,\dots,0)$, and so on. This is a rigid structure, in that the automorphism group of this space *with this basis* is rigid (no maps save the identity preserve the linear structure and the basis).

In general, we can consider an abstract complex vector space $(\mathbb{E},+,\cdot,0)$, and assume it's finite-dimensional. Then, $\mathrm{Aut}\,\mathbb{E}$ is an interesting group: every basis gives us an automorphism $b : \mathbb{C}^n \xrightarrow{\cong} \mathbb{E}$, and therefore gives us an isomorphism $b : \mathrm{GL}_n\,\mathbb{C} \xrightarrow{\cong} \mathrm{Aut}\,\mathbb{E}$.

We can also consider automorphisms that have some more structure; for example, $\mathbb{E}$ may have a hermitian inner product $\langle -,-\rangle : \mathbb{E}\times\mathbb{E} \to \mathbb{C}$. Then, $\mathrm{Aut}(\mathbb{E},\langle -,-\rangle) = \mathrm{U}(\mathbb{E})$, which by a basis is isomorphic to U_n, the set of $n\times n$ matrices A such that $A^*A = \mathrm{id}$ (where A^* is the conjugate transpose). U_n is a Lie group, and a subgroup of $\mathrm{GL}_n\,\mathbb{C}$.

For example, when $n = 1$, $\mathrm{U}_1 \hookrightarrow \mathrm{GL}_1\,\mathbb{C}$. U_1 is the set of $\lambda \in \mathbb{C}$ such that $\overline{\lambda}\lambda = 1$, so U_1 is just the unit circle. Then, $\mathrm{GL}_1\,\mathbb{C}$ is the set of invertible complex numbers, i.e. $\mathbb{C}\setminus 0$. In fact, this means the inclusion $\mathrm{U}_1 \hookrightarrow \mathrm{GL}_1\,\mathbb{C}$ is a homotopy equivalence, and we can take the quotient to get $\mathrm{U}_1 \hookrightarrow \mathrm{GL}_1\,\mathbb{C} \twoheadrightarrow \mathbb{R}^{>0}$.

In some sense, the quotient determines the inner product structure on $\mathbb{C}$, since in this case an inner product only depends on scale. But the same behavior happens in the general case: $\mathrm{U}_n \hookrightarrow \mathrm{GL}_n\,\mathbb{C} \twoheadrightarrow \mathrm{GL}_n\,\mathbb{C}/\,\mathrm{U}_n$, and the quotient classifies hermitian inner products on $\mathbb{C}^n$.

Exercise. Identify the homogeneous space $\mathrm{GL}_n/\,\mathrm{U}_n$, and show that it's contractible. (Hint: show that it's convex.)

Now, we return to the manifold. Embedding things into the manifold is covariant: composing with $f : X \to Y$ of manifolds with something embedded into X produces something embedded into Y. K-theory will be contravariant, like cohomology: functions and differential forms on a manifold pull back contravariantly. What we'll look at is families of vector spaces parameterized by a manifold X.

Definition. A *family of vector spaces* $\pi : E \to X$ parameterized by X is a surjective, continuous map together with a continuously varying vector space structure on the fiber.

This sounds nice, but is a little vague. Any definition has data and conditions, so what are they? We have two topological spaces E and X; X is called the *base* and E is called the *total space*, as well as a continuous, surjective map $\pi : E \to X$. The condition is that the fiber $E_x = \pi^{-1}(x)$ is a vector space for each $x \in X$. Specifically, sending x to the zero element of E_x is a zero $z : X \to E$, which is a section or right inverse to π. We also have scalar multiplication $m : C \times E \to E$, which has to stay in the same fiber; thus, m commutes with π. Vector addition $+ : E \times_X E \to E$ is only defined for vectors in the same fiber, so we take the fiberwise product $E \times_X E$. Again, $+$ and π commute. Finally, what does continuously varying mean? This means that z, m, and $+$ are continuous.

Intuitively, if we let $\mathcal{V}$ be the collection of vector spaces, we might think of such a family as a function $X \to \mathcal{V}$. To each point of X, we associate a vector space, instead of, say, a number.

Example 1.1.

(1) The constant function: let $\mathbb{E}$ be a vector space. Then, $\underline{\mathbb{E}} = X \times \mathbb{E} \to X$ given by $\pi = \mathrm{pr}_1$ sends $(x, e) \mapsto x$. This is called the *constant vector bundle* or *trivial vector bundle* with fiber $\mathbb{E}$.

(2) A nonconstant bundle is the *tangent bundle* $TS^2 \to S^2$. For now, let's think of this as a family of real vector spaces; then, at each point $x \in S^2$, we have this 2-dimensional space T_xS^2, and different tangent spaces aren't canonically identified. Embedding $S^2 \hookrightarrow \mathbb{R}^3$ as the unit sphere, each tangent space embeds as a subspace of $\mathbb{R}^3$, and we have something called the Grassmanian. Note that $TS^2 \ncong \underline{\mathbb{R}}^2$, which we proved in algebraic topology as the hairy ball theorem.

Implicit in the second example was the definition of a map; the idea should be reasonably intuitive, but let's spell it out: if we have $\pi : E \to X$ and $\pi' : E' \to X$, a morphism is the data of a continuous $f : E \to E'$ such that the following diagram commutes.

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & X & \end{tikzcd}$$

Then, you can make all of the usual linear-algebraic constructions you like: inverses, direct sums and products, and so on.

Example 1.2. Here's an example of a rather different sort. Let $\mathbb{E}$ be a finite-dimensional complex vector space, and suppose $T : \mathbb{E} \to \mathbb{E}$ is linear. Define for any $z \in \mathbb{C}$ the map $K_z = \ker(z \cdot \mathrm{id} - T) \subset \mathbb{E}$, and let $K = \bigcup_{z \in \mathbb{C}} K_z$.

For a generic z, $z \cdot \mathrm{id} - T$ is invertible, and so $K_z = 0$. But for eigenvalues, we get something more interesting, the eigenspace. But sending $K_z \mapsto z$, we get a map $\pi : K \to \mathbb{C}$. This is interesting because the vector space is 0-dimensional except at a finite number of points, and in fact if we take

$$\varphi : \bigoplus_{z : K_z \neq 0} K_z \to \mathbb{E},$$

induced by the inclusion maps $K_z \to \mathbb{E}$, then φ is an isomorphism. This is the geometric statement of the Jorden block decomposition (or generalized eigenspace decomposition) of a vector space.

Definition. Given a family of vector spaces $\pi : E \to X$, the rank $x \mapsto \dim E_x = \pi^{-1}(X)$ is a function $\mathrm{rank} : X \to \mathbb{Z}^{\geq 0}$.

Example 1.2 seems less nice than the others, and the property that makes this explicit, developed by Norman Steenrod in the 1950s, is called local triviality.

Definition. A family of vector spaces $\pi : E \to X$ is a *vector bundle* if it has *local triviality*, i.e. for every $x \in X$, there exists an open neighborhood $U \subset X$ and isomorphism $E|_U \cong \underline{\mathbb{E}}$ for some vector space $\mathbb{E}$.

This property is sometimes also called being *locally constant*. So the fibers aren't literally equal to $\mathbb{E}$ (they're different sets), but they're isomorphic as vector spaces.

One good question is, what happens if I have two local trivializations? Suppose E_x lies above x, and we have $\varphi_x : \mathbb{E} \to E_x$ and $\varphi'_x : \mathbb{E}' \to E_x$, each defined on open neighborhoods of x in X. The function $\varphi_x^{-1} \circ \varphi'_x : \mathbb{E}' \to \mathbb{E}$ is called a *transition function*, and we can see that it must be linear, and furthermore, isomorphic.

The Clutching Construction. This leads to a way of constructing vector bundles, known as the *clutching construction*. First, consider $X = S^2$, decomposed into $B^2_+ = S^2 \setminus \{-\}$ and $B^2_- = S^2 \setminus \{+\}$ (i.e. minus the south and north poles, respectively). Each of these is diffeomorphic to the real plane, and in particular is contractible. Taking the trivial bundle $\underline{\mathbb{C}}$ over each of these, we have something like

$$\begin{tikzcd}
\underline{\mathbb{C}} \arrow[d] & & \underline{\mathbb{C}} \arrow[d] \\
B^2_+ \arrow[dr, leftrightarrow] & & B^2_- \arrow[dl, leftrightarrow] \\
& B^2_+ \cap B^2_- &
\end{tikzcd}$$

The intersection $B^2_+ \cap B^2_-$ is diffeomorphic to $\mathbb{A}^2 \setminus \{0\}$. Thus, the two structures of $\underline{\mathbb{C}}$ on this intersection are related by a map $\underline{\mathbb{C}} \to \underline{\mathbb{C}}$, which induces a map $\tau : B^2_+ \cap B^2_- \to \operatorname{Aut}(\mathbb{C}) = \operatorname{GL}_1 \mathbb{C} = \mathbb{C}^\times$. This τ has an invariant called its *winding number*, so we can construct a line bundle $L \xrightarrow{\pi} S^2$ by gluing: let L be the quotient of $(B^2_+ \times \mathbb{C}) \amalg (B^2_- \times \mathbb{C})$ with the identification $\{x\} \times \mathbb{C} \sim \{\tau(x)\} \times \mathbb{C}$ (the former from B^2_+ and the latter from B^2_-).

More generally, if $\{U_\alpha\}_{\alpha \in A}$ is an open cover of X, then we get a map

$$\coprod_{\alpha \in A} U_\alpha \xrightarrow{p} X,$$

and so we can construct a gluing: whenever two points in the disjoint union map to the same point, we want to glue them together. The arrows linking two points to be identified have identities and compositions.

The clutching construction gives us a vector bundle over this space: given a vector bundle E_α over each U_α, we glue basepoints using those arrows, and get an associated isomorphism of vector spaces. Then, you can prove that you get a vector bundle.

Notice that maps $f : X \to Y$ of manifolds can be pulled back, and in this regard a vector bundle is a contravariant construction.

Topology and Vector Bundles. We were going to add some topology to this discussion, yes?

Theorem 1.3. *If $E \to [0,1] \times X$ is a vector bundle, then $E|_{\{0\}\times X} \cong E|_{\{1\}\times X}$.*

We'll prove this next lecture. The idea is that the isomorphism classes are homotopy-invariant, and therefore rigid or in some sense discrete. This will allow us to do topology with vector bundles.

Now, we can extract $\operatorname{Vect}^{\cong}(X)$, the set of vector bundles on X up to isomorphism. This has a 0 (the trivial bundle) and a $+$, given by direct sum of vector bundles. This gives a commutative monoid structure from X which is homotopy invariant.

Commutative monoids are a little tricky to work with; we'd rather have abelian groups. So we can complete the monoid, taking the Grothendieck group, obtaining an abelian group $K(X)$.

Using real or complex vector bundles gives $K_{\mathbb{R}}(X)$ and $K_{\mathbb{C}}(X)$, respectively (the latter is usually called $K(X)$). On S^n, one can compute that $K(S^n) = \pi_{n-1} \operatorname{GL}_N$ for some large N. These groups were computed to be periodic in both the real and complex cases, a result which is known as *Bott periodicity.*[1] This periodicity was proven in the mid-1950s.[2] This was worked into a topological theory by players such as Grothendieck and Atiyah, among others.

One of the first things we'll do in this class is provide a few different proofs of Bott periodicity.

Another interesting fact is that K-theory satisfies all of the axioms of a cohomology theory except for the values on S^n, making it a *generalized* (or *extraordinary*) *cohomology theory.* This is nice, since it means most of the computational tools of cohomology are available to help us. And since it's geometric, we can use it to attack problems in geometry, e.g. when is a manifold parallelizable?

For example, for S^n, S^0, S^1, and S^3 are parallelizable (the first two are trivial, and S^3 has a Lie group structure as the unit quaternions). It turns out there's only one more parallelizable sphere, S^7, and the rest are not; this proof by Adams in 1967 used K-theory, and is related to the question of how many division algebras there are.

[1] The sequence of groups you get almost sounds musical. Maybe sing the Bott song!

[2] The professor says, "I wasn't around then, just so you know."

Relatedly, and finer than just parallelizability, how many linearly independent vector fields are there on S^n? Even if S^n isn't parallelizable, we may have nontrivial l.i. vector fields. There are other related ideas, e.g. the Atiyah-Singer index theorem.

K-theory can proceed in different directions: we can extract modules of the ring of functions on X, and therefore using Spec, start with any ring and do algebraic K-theory. One can also intertwine K-theory and operator algebras, which is also useful in geometry. We'll focus on topological K-theory, however. There are also twistings in K-theory, which relate to representations of loop groups.

K-theory has also come into physics, both in high-energy theory and condensed matter, but we probably won't say much about it.

Nuts and bolts: this is a lecture course, so take notes. There might be notes posted on the course webpage[3], but don't count on it. There will also be plenty of readings; four are posted already: [2, 16, 24, 25].

Lecture 2.

Homotopies of Vector Bundles: 9/1/15

> *"You need a bit of Bourbaki imagination to determine the vector bundles over the empty set."*

Recall that all topological spaces in this class will be taken to be Hausdorff and paracompact.

We stated this as Theorem 1.3 last time; now, we're going to prove it.

Theorem 2.1. *Let X be a space and $E \to [0,1] \times X$ be a vector bundle. Let $j_t : X \hookrightarrow [0,1] \times X$ send $x \mapsto (t, x)$. Then, there exists a natural isomorphism $j_0^* E \xrightarrow{\cong} j_1^* E$ of vector bundles over X.*

To define the pullback more precisely, we can characterize it as fitting into the following diagram.

$$\begin{tikzcd} j^*E \arrow[r] \arrow[d] & E \arrow[d] \\ Y \arrow[r, "j"] & Z \end{tikzcd}$$

Then, j^*E is the subset of $Y \times E$ for which the diagram commutes.

We'll want to make an isomorphism of fibers and check that it is locally trivial; in the smooth case, one can use an ordinary differential equation, but in the more general continuous case, we'll do something which is in the end more elementary.

To pass between the local properties of vector bundles and a global isomorphism, we'll use partitions of unity.

Definition. Let X be a space and $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover (which can be finite, countable, or uncountable). Then, a *partition of unity* $\{\rho_\alpha\}_{\alpha \in A}$ indexed by a set A is a set of continuous functions $X \to [0,1]$ with locally finite supports such that $\sum \rho_\alpha = 1$. This partition of unity is said to be *subordinate* to the cover $\mathcal{U}$ if there exists $i : A \to I$ such that $\operatorname{supp} \rho_\alpha \subset U_{i(\alpha)}$.

Theorem 2.2. *Let X be a Hausdorff paracompact space and $\{U_i\}_{i \in I}$ be an open cover.*

(1) There exists a partition of unity $\{\rho_i\}_{i \in I}$ subordinate to $\{U_i\}_{i \in I}$ such that at most countably many ρ_i are not identically zero.

(2) There exists a partition of unity $\{\rho_\alpha\}_{\alpha \in A}$ subordinate to $\{U_i\}_{i \in I}$ such that each ρ_α is compactly supported.

(3) If X is a smooth manifold, we can choose ρ_α to be smooth.

We'll only use part (1) of this theorem.

A nontrivial example is $X = \mathbb{R}$ and $U_x = (x-1, x+1)$ for $x \in \mathbb{R}$ (so an uncountable cover). In this case, we don't need every function to be nonzero; we only need a countable number.

Returning to the setup of Theorem 2.1, if X is a smooth manifold, we will set up a covariant derivative, which will allow us to define a notion of parallel. Then, parallel transport will produce the desired isomorphism. In this case, we'll call $X = M$.

[3] https://www.ma.utexas.edu/users/dafr/M392C/index.html.

Suppose first that $\mathbb{E}$ is a vector space, either real or complex. $\Omega_M^0(\mathbb{E})$ denotes the set of smooth functions $M \to \mathbb{E}$ (written as 0-forms), and we have a basic derivative operator $\mathrm{d} : \Omega_M^0(\mathbb{E}) \to \Omega_M^1(\mathbb{E})$ satisfying the Leibniz rule

$$\mathrm{d}(f \cdot e) = \mathrm{d}f \cdot e + f \,\mathrm{d}e,$$

where $f \in \Omega_M^0$ and $e \in \Omega_M^0(\mathbb{E})$ (that is, e is vector-valued and f is scalar-valued). Moreover, any other first-order differential operator (an operator $\Omega_M^0(\mathbb{E}) \to \Omega_M^1(\mathbb{E})$ that is linear and satisfies the Leibniz rule) has the form $\mathrm{d} + A$, where $A \in \Omega_M^1(\operatorname{End}\mathbb{E})$. This means that if $\mathbb{E} = \mathbb{C}^r$, then e is a column vector of $e^1, \dots, e^r$ with $e^i \in \Omega^0(\mathbb{E})$, and $A = (A_j^i)$ is a matrix of one-forms: $A_j^i \in \Omega_M^1(\mathbb{C})$. Ultimately, this is because the difference between any two differential operators can be shown to be a tensor.

Now, let's suppose $E \to M$ is a vector bundle.

Definition. A *covariant derivative* is a linear map $\nabla : \Omega_M^0(E) \to \Omega_M^1(E)$ satisfying

$$\nabla(f \cdot e) = \mathrm{d}f \cdot e + f \cdot \nabla e$$

when $f \in \Omega_M^0$ and $e \in \Omega_M^0(\mathbb{E})$.

Here, $\Omega_M^0(E)$ is the space of sections of E. In some sense, this is a choice for functions with values in a varying vector space.

Theorem 2.3. *In this case, covariant derivatives exist, and the space of covariant derivatives is affine over $\Omega_M^1(\operatorname{End}\mathbb{E})$.*

Proof. Choose $\{U_i\}_{i\in I}$ and local trivializations $\underline{\mathbb{E}_i} \xrightarrow{\cong} E|_{U_i}$ on U_i. We have a canonical differentiation d of $\mathbb{E}_i$-valued functions on U_i to define ∇_i on the bundle $E|_{U_i} \to U_i$.

To stitch them together, choose a partition of unity $\{\rho_i\}_{i\in I}$ and define

$$\nabla e = \sum_i \rho_i \nabla(j_i^* e),$$

where $j_i : U_i \hookrightarrow M$ is inclusion. ⊠

All right, so what's parallel transport? Let $\mathcal{E} \to [0,1]$ be a vector bundle with a covariant derivative ∇. Parallel transport will be an isomorphism $\mathcal{E}_0 \xrightarrow{\sim} \mathcal{E}_1$.

Definition. A section e is *parallel* if $\nabla e = 0$.

Lemma 2.4. *The set $P \subset \Omega_{[0,1]}^0(\mathcal{E})$ of parallel sections is a subspace. Then, for any $t \in [0,1]$, the evaluation map $\mathrm{ev}_t : P \to \mathcal{E}_t$ sending $e \mapsto e(t)$ is an isomorphism.*

The first statement is just because $\nabla e = 0$ is a linear condition. The second has the interesting implication that for any $(x,t) \in \mathcal{E}$, there's a unique parallel section that extends it.

Proof. Suppose $\mathcal{E} \to [0,1]$ is trivializable, and choose a basis $e_1, \dots, e_r$ of sections. Then, we can write

$$\nabla e_j = A_j^i e_i,$$

where we're summing over repeated indices and $A_j^i \in \Omega_{[0,1]}^1(\mathbb{C})$. Then, any section has the form $e = f^j e_j$ and the parallel transport equation is

$$\begin{aligned} 0 = \nabla e &= \nabla(()f^j e_j) \\ &= \mathrm{d}f^j e_j + f^j \nabla e_j \\ &= (\mathrm{d}f^i + A_j^i f^j) e_j. \end{aligned}$$

If we write $A_j^i = \alpha_j^i \,\mathrm{d}t$ for $\alpha_j^i \in \Omega_{[0,1]}^0(\mathbb{C})$, then the parallel transport equation is

$$\frac{\mathrm{d}f^i}{\mathrm{d}\tau} + \alpha_j^i f^j = 0. \tag{2.1}$$

This is a linear ODE on $[0,1]$, so by the fundamental theorem of ODEs, there's a unique solution to (2.1) given an initial condition.

More generally, if $\mathcal{E}$ isn't trivializable, partition it into $[0,t_1]$, $[t_1,t_2]$, and so on, so that $\mathcal{E} \to [t_i, t_{i+1}]$ is trivialiable, and compose the parallel transports on each interval. ⊠

Now, we can prove Theorem 2.1 in the smooth manifolds case.

Proof of Theorem 2.1, smooth case. Choose a covariant derivative ∇, and use parallel transport along $[0,1]\times\{x\}$ to construct an isomorphism $E_{(0,x)}\to E_{(1,x)}$. The fundamental theorem on ODEs also states that the solution smoothly depends on the initial data, so these isomorphisms vary smoothly in x. ⊠

Note that this fundamental theorem only gives local solutions, but (2.1) is linear, so a global solution exists.

In the continuous case, we can't do quite the same thing, but the same idea of parallel transport is in effect.

Proof of Theorem 2.1, continuous case. By local triviality, we can cover $[0,1]\times X$ by open sets of the form $(t_0,t)\times U$ on which $E\to[0,1]\times X$ restricts to be trivializable.

By the compactness of $[0,1]$, we can cover X by sets $\{U_i\}_{i\in I}$ such that $E|_{[0,1]\times U_i}$ is trivializable: we can get trivializations on a finite number of patches. Thus, at the finite number of boundaries, we can patch the trivialization, choosing a continuous isomorphism of vector spaces.

Choose a partition of unity $\{\rho_i\}_{i\in I}$ subordinate to $\{U_i\}_{i\in I}$ and pare down I to the countable subset of $i\in I$ such that ρ_i isn't identically zero. Let $\varphi_n=\rho_1+\cdots+\rho_n$ for $n=1,2,\ldots$, and let Γ_n be the graph of φ_n, which is a subset of $[0,1]\times X$.

So now we have a countable cover, and Γ_n is only supported on $U_1\cup\cdots\cup U_n$, and only changes from Γ_{n-1} on U_n. But since the sum of the ρ_i is 1, then the graph Γ_n must go across the whole of $[0,1]\times X$ as $n\to\infty$. But over each open set, since we've pared down I, there are only finitely many steps.[4]

Going from Γ_0 (identically 0) to Γ_1 makes a trivialization on U_1, and from Γ_1 to Γ_2 extends the trivialization further, and so on. ⊠

Corollary 2.5. *If $f:[0,1]\times X\to Y$ is continuous and $E\to Y$ is a vector bundle, then $f_0^*E\cong f_1^*E$.*

This is because $f_t(x)=f(t,x)$ is a homotopy.

Corollary 2.6. *A continuous map $f:X\to Y$ induces a pullback map $f^*:\mathrm{Vect}(Y)^{\cong}\to\mathrm{Vect}(X)^{\cong}$, and this map depends only on the homotopy type of f.*

This is a hint that we can make algebraic topology out of the sets of vector bundles of spaces. There are many homotopy-invariant sets that we attach to topological spaces, e.g. π_0, π_1, π_2, H_1, H_2, and so on; these tend to be groups and even abelian groups, and thus tend to be easier to work with.

$\mathrm{Vect}^{\cong}(X)$ is a *commutative monoid*, so there's an associative, commutative + and an identity. The identity is the isomorphism class of the bundle $\underline{\mathbb{O}}$, the zero vector space. Then, we define addition by $[E]+[E']=[E\oplus E']$. Moreover, it is a *semiring*, i.e. there's a $\times$ and a multiplicative identity 1 given by the isomorphism class of $\underline{\mathbb{C}}$. Multiplication is given by (the isomorphism class of) the tensor product.

Commutative monoids are pretty nice; a typical example is the nonnegative integers.

Example 2.7.

(1) The simplest possible space is $\emptyset$. There's a unique vector bundle over it, the zero bundle, so $\mathrm{Vect}^{\cong}(\emptyset)=0$, the trivial monoid.
(2) Over a point, vector bundles are just finite-dimensional vector spaces, which are determined up to isomorphism by dimension, so $\mathrm{Vect}^{\cong}(\mathrm{pt})\xrightarrow{\sim}\mathbb{Z}^{\geq 0}$.

Definition. If X is a compact space, $K(X)$ is the abelian group completion of the commutative monoid $\mathrm{Vect}^{\cong}_{\mathbb{C}}(X)$; the completion of $\mathrm{Vect}^{\cong}_{\mathbb{R}}(X)$ is denoted $KO(X)$.

This definition makes sense when X is noncompact, but doesn't give a sensible answer. We'll see other definitions in the noncompact case eventually.

We'll talk more about the abelian group completion next lecture; the idea is that for any abelian group A and homomorphism $\alpha:\mathrm{Vect}^{\cong}(X)\to A$ of commutative monoids, there should be a unique $\widetilde{\alpha}$ such that the

[4]This argument is likely confusing; it was mostly given as a picture in lecture, and can be found more clearly in Hatcher's notes [16] on vector bundles and K-theory.

following diagram commutes.

$$\begin{tikzcd} \mathrm{Vect}^{\cong}(X) \arrow[r] \arrow[dr, "\alpha"'] & K(X) \arrow[d, "\widetilde{\alpha}"] \\ & A \end{tikzcd}$$

Another corollary of Theorem 2.1:

Corollary 2.8. *If X is contractible and $\pi : E \to X$ is a vector bundle, then π is trivializable.*

Corollary 2.9. *Let $X = U_0 \cup U_1$ for open sets U_0, U_1 and $E_i \to U_i$ be two vector bundles, and let $\alpha : [0,1] \times U_0 \cap U_1 \to \mathrm{Iso}(E_0|_{U_0\cap U_1}, E_1|_{U_0\cap U_1})$: that is, α is a homotopy of isomorphisms $E_0 \to E_1$ on the intersection. Then, clutching with α_t gives a vector bundle $E_t \to X$, and $E_0 \cong E_1$.*

In the last five minutes, we'll discuss a few more partition of unity arguments.

(1) Let X be a topological space, and

$$0 \longrightarrow E' \xrightarrow{i} E \xrightarrow{j} E'' \longrightarrow 0$$

be a short exact sequence of vector bundles over X. Recall that a *splitting* of this sequence is an $s : E'' \to E$ such that $j \circ s = \mathrm{id}_{E''}$. Then, splittings form a bundle of affine spaces over $\mathrm{Hom}(E'', E)$, which happens because linear maps act simply transitively on splittings (adding a linear map to a splitting is still a splitting, and any two splittings differ by a linear map).

Theorem 2.10. *Global splittings exist, i.e. the affine bundle of splittings has a global section.*

Proof. At each point, there's a section, which is a linear algebra statement, and locally on X, there's a splitting, which follows from local trivializations. Then, patch them together with a partition of unity, which works because we're in an affine space, so our partition of unity in each affine space is a weighted average (because the ρ_i are nonnegative) and therefore lies in the convex hull of the splittings. ⊠

(2) We also have Hermitian inner products. The same argument goes through, as inner products are convex (the weighted average of two inner products is convex), so one can honestly use a partition of unity in the same way as above.

Lecture 3.

Abelian Group Completions and $K(X)$: 9/3/15

"First I want to remind you about fiber bundles. . . (pause) *. . . Consider yourself reminded."*

Last time, we said that if $\mathbb{E}$ is a (real or complex) vector space, the space of its inner products is contractible. This is because we have a vector space of sesquilinear (or bilinear in the real case) maps $\mathbb{E} \times \mathbb{E} \to \mathbb{C}$ (or $\mathbb{R}$), and the inner products form a convex cone in this space.

Inner products relate to symmetry groups: the symmetry group of $\mathbb{C}^n$ is $\mathrm{GL}_n \mathbb{C}$, the set of $n \times n$ complex invertible matrices, but the symmetry group of $\mathbb{C}^n$ with an inner product $\langle -, - \rangle$ is the unitary group $\mathrm{U}_n \subset \mathrm{GL}_n \mathbb{C}$, the set of matrices A such that $A^*A = I$. In the real case, the symmetries of $\mathbb{R}^n$ are $\mathrm{GL}_n \mathbb{R}$, and the group of symmetries of $\mathbb{R}^n$ with an inner product is $\mathrm{O}_n \subset \mathrm{GL}_n \mathbb{R}$.

As a consequence, we have the following result.

Proposition 3.1. *There are deformation retractions $\mathrm{GL}_n \mathbb{C} \to \mathrm{U}_n$ and $\mathrm{GL}_n \mathbb{R} \to \mathrm{O}_n$.*

For example, when $n = 1$, $\mathrm{GL}_1 \mathbb{C} = \mathbb{C}^\times$, which deformation retracts onto the unit circle, which is U_1. Then, $\mathrm{GL}_1 \mathbb{R} = \mathbb{R}^\times$ and $\mathrm{O}_1 = \{\pm 1\}$, so there's a deformation retraction in the same way.

Proof. We'll give the proof in the complex case; the real case is pretty much identical.

Since the columns of an invertible matrix determine a basis of $\mathbb{C}^n$ and vice versa, identify $\mathrm{GL}_n \mathbb{C}$ with the space of bases of $\mathbb{C}^n$; then, U_n is the space of orthonormal bases of $\mathbb{C}^n$.

A general basis $e_1, \dots, e_n$ may be turned into an orthonormal basis by the Gram-Schmidt process, which is a composition of homotopies. First, we scale e_1 to have norm 1, given by the homotopy $e_1 \mapsto ((1-t) + t/|e_1|)e_1$. Then, we make $e_2 \perp e_1$, which is given by the homotopy $e_2 \mapsto e_2 - t\langle e_2, e_1 \rangle e_1$. The rest of the steps are given by scaling basis vectors and making them perpendicular to the ones we have so far, so they're also homotopies. $\boxtimes$

Group Completion. Recall that a commutative monoid is the data $(M, +, 0)$, such that $+$ is associative and commutative, and 0 is the identity for $+$.

Definition. (A, i) is a *group completion* of M if A is an abelian group, $i : M \to A$ is a homomorphism of commutative monoids, and for every abelian group B and homomorphism $f : M \to B$ of commutative monoids, there exists a unique abelian group homomorphism $\widetilde{f} : A \to B$ of abelian groups such that $\widetilde{f} \circ i = f$.

That is, we require that there exists a unique $\widetilde{f}$ such that the following diagram commutes.

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[dr, "f"'] & A \arrow[d, "\widetilde{f}"] \\ & B \end{tikzcd}$$

Note that i was never specified to be injective, and in fact it often isn't.

Example 3.2.

- If $M = (\mathbb{Z}^{\geq 0}, +)$, the group completion is $A = \mathbb{Z}$.
- If $M = (\mathbb{Z}^{>0}, \times)$, we get $A = \mathbb{Q}^{>0}$.
- However, if $M = (\mathbb{Z}^{\geq 0}, \times)$, we get $A = 0$. This is because if $i : \mathbb{Z}^{\geq 0} \to A$, then there must be an $a \in A$ such that $i(0) \cdot a = 1$; thus, for any $n \geq 0$,
$$i(n) = i(n)i(1) = i(n)i(0)a = i(n \cdot 0)a = i(0)a = 1.$$

Since the group completion was defined by a universal property, we can argue for its existence and uniqueness; universal properties tend to have very strong uniqueness conditions.

We saw that the vector bundles up to isomorphism are a commutative monoid (even semiring under tensor product), and so taking the group completion can cause a loss of information, as in the last part of the above example. Though abelian groups are nicer to compute with, there are examples where information about vector bundles is lost by passing to abelian groups.

The uniqueness of the group completion is quite nice: given two group completions (A, i) and (A', i') of a commutative monoid M, there exists a unique isomorphism ϕ that commutes with the universal property. That is, in the following diagram, $\phi \circ \widetilde{f}' = \widetilde{f}$.

$$\begin{tikzcd} A' \arrow[dr, "\widetilde{f}'"'] & M \arrow[l, "i'"'] \arrow[r, "i"] \arrow[d, "f"] & A \arrow[dl, "\widetilde{f}"] \arrow[ll, bend right, "\phi"'] \\ & B & \end{tikzcd}$$

To prove this, we'll apply the universal property four times. To see why ϕ is an isomorphism, putting A' in place of B and i' in place of f, we get a ϕ, and switching (A, i) with (A', i') gives us $\psi : A' \to A$. Then, in the following diagram, $i' = \phi i = (\phi\psi)i'$, which satisfies a universal property (which one?) and therefore proves ϕ and ψ are inverses.

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[dr, "i'"'] & A \arrow[d, shift left, "\phi"] \\ & A' \arrow[u, shift left, "\psi"] \end{tikzcd}$$

For existence, define $A = M \times M / \sim$, where $(m_1, m_2) \sim (m_1 + n, m_2 + n)$ for all $m_1, m_2, n \in M$. Then, $0_A = (0_M, 0_M)$ and $-[m_1, m_2] = [m_2, m_1]$. This makes sense: it's how we get $\mathbb{Z}$ from $\mathbb{N}$, and $\mathbb{Q}$ from $\mathbb{Z}$ multiplicatively.

Often, the abelian group completion is called the *Grothendieck group* of M, called $K(M)$.

Back to K-Theory. If X is compact hausdorff, then $\mathrm{Vect}^{\cong}(X)$, the set of isomorphism classes of vector bundles over X, is a commutative monoid, with addition given by $[E'] + [E''] = [E' \oplus E'']$, and a semiring given by $[E'] \times [E''] = [E' \otimes E'']$. There's some stuff to check here.

The group completion of $\mathrm{Vect}^{\cong}_{\mathbb{C}}(X)$ is denoted $K(X)$ (sometimes $KU(X)$, with the U standing for "unitary"), and the group completion of $\mathrm{Vect}^{\cong}_{\mathbb{R}}(X)$ is denoted $KO(X)$, with the O for "orthogonal."

The map $X \mapsto K(X)$ (or $KO(X)$) is a homotopy-invariant functor; that is, if $f : X \to Y$ is continuous, then $f^* : K(Y) \to K(X)$ is a homomorphism of abelian groups. The homotopy invariance says that if $f_0 \simeq f_1$, then $f_0^* = f_1^*$. We could write $K : \mathsf{CptSpace}^{op} \to \mathsf{AbGrp}$, and mod out the homotopy.

There are plenty of other functors that look like this; for example, the n^{th} cohomology group is a contravariant functor from topological spaces (more generally than compact Hausdorff spaces) to abelian groups, and is homotopy-invariant. But this gives us a sequence of groups, indexed by $\mathbb{Z}$ (where the negative cohomology groups are zero by definition). Similarly, we'll promote the K-theory of a space to a sequence of abelian groups indexed by the integers, with $K(X)$ becoming $K^0(X)$; we'll also see that in the typical case, $K^n(X)$ is nonzero for infinitely many n.

For example, if E and E' are vector bundles, $\mathrm{Hom}(E, E') \cong E' \otimes E^*$, by the map sending $e' \otimes \theta \mapsto (e \mapsto \theta(e)e')$. There's some stuff to check; in particular, once you know it for vector spaces, it's true fiber-by-fiber. Moreover, E and E^* are isomorphic as vector bundles, because any metric $E \otimes E \to \underline{\mathbb{R}}$ induces an isomorphism $E \to E^*$; thus, in $KO(X)$, $[E] = [E']$, so $[\mathrm{Hom}(E, E')] = [E] \times [E']$.

In the complex case, the metric is a map $\overline{E} \otimes E \to \underline{\mathbb{C}}$: the conjugate bundle is defined fiber-by-fiber by the conjugate vector space $\overline{\mathbb{E}}$, identical to $\mathbb{E}$ except that scalar multiplication is composed with conjugation. Thus, there's an isomorphism $\overline{E} \xrightarrow{\sim} E^*$. This is sometimes, but not always, an isomorphism: if X is a point, then it's always an isomorphism, but the bundle $\mathbb{C}P^1 \to S^2$ isn't fixed: complex conjugation flips the winding number, and therefore produces a nonisomorphic bundle.

We said that we might lose information taking the group completion, so we want to know what kind of information we've lost. The key is the following proposition.

Proposition 3.3. *Let X be a compact Hausdorff space and $\pi : E \to X$ be a vector bundle. Then, there exists a vector bundle $\pi' : E' \to X$ such that $E \oplus E' \to X$ is trivializable.*

If $X \neq \emptyset$, then there's a map $p : X \to \mathrm{pt}$, and its pullback $p^* : K(\mathrm{pt}) \to K(X)$ is injective. That is, we have an injective map $\mathbb{Z} \hookrightarrow K(X)$, consisting of the trivial bundles (i.e. those pulled back by a point). Proposition 3.3 implies that given a $k \in K(X)$, there's a k' such that $k + k' = n$ for $n \in \mathbb{Z}$. Thus, the inverse is $-k = k' - N$.

Proof of Proposition 3.3. Since X is compact, we can cover it with a finite collection of opens $U_1, \dots, U_N$ such that $E|_{U_i}$ is trivializable for each i.

Choose a basis of sections $e_1^{(i)}, \dots, e_n^{(i)}$ on U_i, and let $\rho_1, \dots, \rho_N$ be a partition of unity subordinate to the cover $\{U_i\}$. Then, let

$$S = \left\{\rho_1 e_1^{(1)*}, \dots, \rho_1 e_n^{(1)*}, \rho_2 e_1^{(2)*}, \dots\right\} \subset C^0(X; E^*),$$

where $e_1^{(i)*}, \dots, e_r^{(i)*}$ is the dual basis of sections of $E^*|_{U_i} \to U_i$.

Then, set $V = \mathbb{C}S^*$, the set of functions $S \to \mathbb{C}$. Then, evaluation defines an injection $E \hookrightarrow \underline{V}$: evaluating at E_x determines a value on each basis element on each ρ_i that doesn't vanish there, so we get values on basis elements. Moreover, since at least one such ρ_i exists for each point, this map is injective.

Let $E' = \underline{V}/E$, so we have a short exact sequence

$$0 \longrightarrow E \longrightarrow \underline{V} \longrightarrow E' \longrightarrow 0.$$

Last time, we proved in Theorem 2.10 that all short exact sequences of vector bundles exist, so there's an isomorphism $E' \oplus E \xrightarrow{\sim} V$. ⊠

Now, we can do some stuff that will look familiar from cohomology.

Definition. The *reduced K-theory* of X is the quotient $\widetilde{K}(X) = K(X)/p^*K(\mathrm{pt})$, where $p : X \to \mathrm{pt}$.

Example 3.4. If $X = \mathrm{pt} \sqcup \mathrm{pt}$, then $K(X) \xrightarrow{\sim} \mathbb{Z} \oplus \mathbb{Z}$ sending bundles to their ranks. Then, $p^* : K(\mathrm{pt}) = \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is the diagonal map Δ, so $\widetilde{K}(X) = \mathbb{Z} \oplus \mathbb{Z}/\Delta \xrightarrow{\sim} \mathbb{Z}$.

Corollary 3.5. *Let $E, E' \to X$ be vector bundles. Then, $[E] = [E']$ in $\widetilde{K}(X)$ iff there exist $r, r' \in \mathbb{Z}^{\geq 0}$ such that $E \oplus \underline{\mathbb{C}}^r \cong E' \oplus \underline{\mathbb{C}}^{r'}$.*

In this case, we say that E and E' are *stably equivalent*. In other words, *K-theory remembers the stable equivalences of vector bundles.* This is the first inkling we have of what K-theory is about, and what the geometric meaning of group completion is.

Example 3.6. Let's look at $\widetilde{KO}(S^2)$. We have a nontrivial bundle of rank 2 over S^2, $TS^2 \to S^2$. However, $TS^2 \oplus \underline{\mathbb{R}} \to S^2$ is trivializaible!

To see this, embed $S^2 \hookrightarrow \mathbb{A}^3$; such an embedding always gives us a short exact sequence of vector bundles

$$0 \longrightarrow TS^2 \longrightarrow T\mathbb{A}^3|_{S^2} \longrightarrow \nu \longrightarrow 0.$$

The quotient ν, by definition, is the *normal bundle* of the submanifold (in this case, S^2). We know that $T\mathbb{A}^3 = \underline{\mathbb{R}}^3$ everywhere, which is almost by definition, and therefore $\nu \cong \underline{\mathbb{R}}$. This means that in $\widetilde{KO}(S^2)$, $[TS^2] = 0$.

So right now, we can calculate the K-theory of a point, and therefore of any contractible space. We want to be able to do more; a nice first step is to compute the K-theory of S^n. Just as in cohomology, this will allow us to bootstrap our calculations on CW complexes.

Definition. Recall that a *fiber bundle* is the data $\pi : E \to X$ over a topological space X such that π is surjective and local trivializations exist. E is called the *total space*.

Thus, a vector bundle is a fiber bundle where the fibers are vector spaces, *and* we require the local trivializations to respect this structure. We can do this more generally, e.g. with affine spaces and affine maps.

Example 3.7. If $V \to X$ is a vector bundle, we get some associated fiber bundles over X. For example, $\mathbb{P}V \to X$, with fiber of lines in the vector space that's the fiber of V. We can generalize to the Grassmanian $\operatorname{Gr}_k V$, which uses k-dimensional subspaces instead of lines. There are plenty more constructions.

Definition. A topological space F is *k-connected* if $Y \to F$ is null-homotopic for every CW complex Y of dimension at most k.

It actually suffices to take only the spheres for Y.

Lemma 3.8. *Let n be a positive integer and $\pi : \mathcal{E} \to X$ be a fiber bundle, where X is a CW complex with finitely many cells and of dimension at most n, and the fibers of π are $(n-1)$-connected. Then, π admits a continuous section.*

Proof. We'll do cell-by-cell induction on the skeleton $X_0 \subset X_1 \subset \cdots \subset X_n = X$. On points, π trivially has a continuous section.

Suppose we have constructed s on X_{k-1}. Then, all the k-cells are attached via maps

$$\begin{tikzcd} D^k \arrow[r, "\Phi"] & X \\ S^{k-1} \arrow[u, hook] \arrow[r, "\partial\Phi"] & X_{k-1}. \arrow[u, hook] \end{tikzcd}$$

Since $D^k \simeq \mathrm{pt}$, then $\Phi^*\mathcal{E} \to D^k$ is trivializable, so we have a map $\theta : \Phi^*\mathcal{E} \to \underline{F}$. The section on X_{k-1} pulls back and composes with θ to create a map $S^{k-1} = \partial D^k \to F$, but by hypothesis, this is null-homotopic, and therefore extends to D^k. ⊠

A different kind of induction is required when X has infinitely many cells; however, what we've proven is sufficient for the K-theory of the spheres.

Theorem 3.9. *Let $n \in \mathbb{Z}^{\geq 0}$ and $N \geq n/2$. Then, there is an isomorphism $\pi_{n-1} \mathrm{U}_N \to \widetilde{K}(S^n)$.*

Corollary 3.10. *The inclusion $\mathrm{U}_N \hookrightarrow \mathrm{U}_{N+1}$ induces an isomorphism $\pi_{n-1} \mathrm{U}_N \to \pi_{n-1} \mathrm{U}_{N+1}$ if $N \geq n/2$.*

Note that the theorem statement doesn't give enough information to say which map induces the isomorphism, but the proof will show that the usual inclusion does it. Specifically, thinking of U_N as a matrix group, U_N embeds in U_{n+1} on the upper left, i.e.

$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

We can take the union (direct limit) of the inclusions $\mathrm{U}_1 \subset \mathrm{U}_2 \subset \mathrm{U}_3 \subset \dots$, and call it U_∞ (sometimes U). These sequences of homotopy groups must stabilize.

Theorem 3.11 (Bott)**.**

$$\pi_{n-1}\,\mathrm{U}_\infty \cong \widetilde{K}(S^n) = \begin{cases} \mathbb{Z}, & n \text{ even} \\ 0, & n \text{ odd.} \end{cases}$$

We have a real analogue to this theorem as well: the analogous inclusion $\mathrm{O}_1 \hookrightarrow \mathrm{O}_2 \hookrightarrow \cdots$ define a limit O_∞.

Theorem 3.12.

$$\pi_{n-1}\,\mathrm{O}_\infty \cong \widetilde{KO}(S^n) = \begin{cases} \mathbb{Z}, & n \equiv 0, 4 \bmod 8 \\ \mathbb{Z}/2\mathbb{Z}, & n \equiv 1, 2 \bmod 8 \\ 0, & n \equiv 3, 5, 6, 7 \bmod 8. \end{cases}$$

These results, known as the *Bott periodicity theorems*, are the foundations of Bott periodicity. We'll give three proofs: Bott's original proof using Morse theory, a more elementary one, and one that uses functional analysis and Fredholm operators.

Lecture 4.

Bott's Theorem: 9/8/15

"Any questions?"
"How was your weekend?"
"I was afraid of that."

We know that vector bundles always have sections (e.g. the zero section), but fiber bundles don't. For example, the following fiber bundles don't have sections.

- The orientation cover of a nonorientable manifold (e.g. the Möbius strip) is a double cover that doesn't have a section.
- The Hopf fibration $S^1 \to S^3 \to S^2$.
- Any nontrival covering map $S^1 \to S^1$.

However, sometimes sections do exist.

Theorem 4.1. *If X is a CW complex of dimension n and $\pi : \mathcal{E} \to X$ is a fiber bundle, then if the fibers of π are $(n-1)$-connected, then π admits a section.*

Definition. A *fibration* is a map $\pi : \mathcal{E} \to B$ satisfying the *homotopy lifting property*: that is, if $h : [0,1] \times S \to X$ is a homotopy and $f : \{0\} \times S \to \mathcal{E}$, then f can be lifted across the whole homotopy, i.e. there exists an $\widetilde{f} : [0,1] \times S \to \mathcal{E}$ that makes the following diagram commute.

$$\begin{tikzcd} \{0\} \times S \arrow[r, "f"] \arrow[d, hook] & \mathcal{E} \arrow[d, "\pi"] \\ {[0,1] \times S} \arrow[r, "h"] \arrow[ur, dashed, "\widetilde{f}"] & X \end{tikzcd}$$

Theorem 4.2. *A fiber bundle is a fibration.*

We won't prove this, but we also won't use it extremely extensively.

Theorem 4.3. *Let $N, n \in \mathbb{Z}^{\geq 0}$ and $N \geq n/2$. Then, there is an isomorphism $\varphi : \pi_{n-1}\,\mathrm{U}_N \to \widetilde{K}(S^n)$ defined by clutching.*

This is part of Theorem 3.11 from last time. Recall that in the reduced K-theory, two bundles are equivalent iff they are stably isomorphic: for example, over S^2, the tangent bundle is stably isomorphic to any trivial bundle, so it's equal to zero.

Proof of Theorem 4.3. We'll show that φ is a composition of three isomorphisms

$$\pi_{n-1}\,\mathrm{U}_N \xrightarrow{i} [S^{n-1}, \mathrm{U}_N] \xrightarrow{j} \mathrm{Vect}_N^{\cong}(S^n) \xrightarrow{k} \widetilde{K}(S^n).$$

To define i, we'll pick a basepoint $* \in S^{n-1}$; then, $\pi_{n-1}\,\mathrm{U}_N$ is equal to $\{f : S^{n-1} \to \mathrm{U}_N : f(*) = e\}$ up to based homotopy (U_N is naturally a pointed space, using its identity element). We want this to be isomorphic to $[S^{n-1}, \mathrm{U}_N]$, the set of maps without basepoint condition up to homotopy, so let $\phi : [S^{n-1}, \mathrm{U}_N] \to \pi_{n-1}\,\mathrm{U}_N$ be defined by $\phi(f) = f(*)^{-1} \cdot f$, where $f : S^{n-1} \to \mathrm{U}_N$. Then, one can check that ϕ is well-defined on homotopy classes and inverts i, so i is an isomorphism.[5]

j is defined by the clutching construction. We can write $S^n = D^n_+ \cup_{S^{n-1}} D^n_-$, and then glue $\underline{\mathbb{C}}^N \to D^n_+$ and $\underline{\mathbb{C}}^N \to D^n_-$ using $f : S^{n-1} \to \mathrm{U}_N$, because U_N is the group of isometries of $\mathbb{C}^N$. So this defines a map j, but why is it an isomorphism? We have to show that j is surjective.

Last time, we showed that the group of isomorphisms deformation retracts onto the group of isometries, so that's fine. To show that j is surjective, we could use that every vector bundle admits a Hermitian metric, or that every vector bundle over D^n is trivializable by orthogonal bases, both of which are true. That j is well-defined follows from an argument that homotopic clutching functions lead to isomorphic vector bundles. Finally, to show that j is injective, all trivializations over D^n are homotopic, since D^n is contractible and U_N is connected.

Then, k just sends a vector bundle to its stable equivalence class. For its surjectivity, we need to show that if $E \to S^n$ has rank $N \geq n/2 + 1$, then there exists an E' of rank $N-1$ and an isomorphism of the $\underline{\mathbb{C}} \oplus E' \cong E$. In words, for large enough N, we can split off a trivial bundle from E. Equivalently, we can show that $E \to S^n$ admits a nonzero section, whose span is a line bundle $L \to X$ which is trivialized; then, we can let $E' = E/L$.

A nonzero section, normalized, is a section of the fiber bundle $S(E) \to S^{n-1}$ with fiber S^{2N-1} (the unit sphere sitting in $\mathbb{C}^N$).[6] This sphere is $(2n-2)$-connected, so by Theorem 4.1, such a section exists.

Why is k injective? We need to show that if a rank-N bundle is stably trivial in $\widetilde{K}(S^n)$, then it is actually trivial. But since it's not clear that $\mathrm{Vect}_N^{\cong}(S^n)$ is an abelian group (yet), then we'll show injectivity of sets. Let $E_0, E_1 \to S^n$ be rank-N vector bundles with an isometry $E_0 \oplus \underline{\mathbb{C}}^r \to E_1 \oplus \underline{\mathbb{C}}^r$; we'll want to produce a homotopic isometry which preserves the last vector $(0, \ldots, 0, 1) \in \mathbb{C}^r$ at each point in X. The evaluation map $\mathrm{ev}_{(0,\ldots,0,1)}$ at the last basis vector is a map of fiber bundles over X; that is, the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Isom}(E_0 \oplus \underline{\mathbb{C}}^r, E_1 \oplus \underline{\mathbb{C}}^r) \arrow[rr, "\mathrm{ev}_{(0,\ldots,0,1)}"] \arrow[dr] & & S(E_1 \oplus \underline{\mathbb{C}}^r) \arrow[dl] \\
& S^n &
\end{tikzcd}$$

An isometry is a section $\varphi : S^n \to \mathrm{Isom}(E_0 \oplus \underline{\mathbb{C}}^r, E_1 \oplus \underline{\mathbb{C}}^r)$, so applying the evaluation map, we get a section $p\varphi : S^n \to S(E_1 \oplus \underline{\mathbb{C}}^r$. We get an additional section $\xi = (0, 0, \ldots, 0, 1)$. Thus, all that's left is to construct a homotopy from $p\varphi$ to ξ, which by the homotopy lifting property defines a section of the pullback $[0,1] \times S(E_1 \oplus \underline{\mathbb{C}}^r) \to [0,1] \times S^n$ over $\{0,1\} \times S^n$. ⊠

Note that, while the K-theory is a ring given by tensor product, the reduced K-theory isn't a ring in most cases.

These arguments are important to demonstrate that when N is high enough, in the stable range, we have this stability.

Corollary 4.4. *If N is in the stable range, i.e. $N \geq n/2$, then the inclusion $\mathrm{U}_N \hookrightarrow \mathrm{U}_{N+1}$ induces an isomorphism $\pi_{n-1}\,\mathrm{U}_N \to \pi_{n-1}\,\mathrm{U}_{N+1}$.*

[5] $[S^{n-1}, \mathrm{U}_N]$ inherits another group structure from that of U_N (i.e. pointwise multiplication of loops); one can reason about it using something called the Eckmann-Hilton argument.

[6] The *sphere bundle* $S(E)$ of a vector bundle E is the fiber bundle whose fiber over each point x is the unit sphere in the E_x.

This means that eventually $\pi_{n-1}\,\mathrm{U}_N$ is identical for large enough N; this group, the *stable isomorphism group* of the unitary groups, is written $\pi_{n-1}(\mathrm{U})$ (and there is a group U that makes this work, the limit of these U_N with the appropriate topology). Then, Bott's theorem, Theorem 3.11, calculates these groups: $\pi_{n-1}\,\mathrm{U}$ is $\mathbb{Z}$ when n is even and 0 when n is odd.

For example, a generator of $\pi_1\,\mathrm{U}_3$ is given by stabilizing a loop $e^{i\theta}$; that is, it's given by the map

$$e^{i\theta} \longmapsto \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with $\theta \in S^1$.

Outlining a Proof of Bott's Theorem. We'll move to providing different proofs of Theorem 3.11; these are explained in our readings, and so the professor won't post lecture notes for a little while.

Let's re-examine $S^2 \cong \mathbb{CP}^1 = \mathbb{P}(\mathbb{C}^2)$ (that is, the space of lines in $\mathbb{C}^2$). More generally, if V is a vector space, $\mathbb{P}V$ will denote its *projectivization*, the space of lines in V. Then, there is a tautological line bundle $H^* \to \mathbb{P}V$, whose fiber at a line $K \subset V$ (which is a point of $\mathbb{P}V$) is the line K.

The dual of H^* is called the *hyperplane bundle*, and denoted $H \to \mathbb{P}V$; a nonzero element of H can be identified with a hyperplane in V, and there is a canonical map $V^* \to \Gamma(\mathbb{P}V, H)$ (where $\Gamma(X, E)$ denotes the sections of $E \to X$): a linear functional on V becomes a linear functional on a line by restriction. Interestingly, if V is a complex vector bundle, then this is an isomorphism onto the holomorphic sections. In particular, the space of holomorphic sections is finite-dimensional.

In fact, if you take $\mathrm{Sym}^k V^*$, the k^{th} symmetric power of V^*, then there's a canonical map $\mathrm{Sym}^k V^* \to \Gamma(\mathbb{P}V, H^{\otimes k})$, which is again an isomorphism in the complex case.

If $V = \mathbb{C}^2$, then write $V = L \oplus \mathbb{C}$; then, L and $\mathbb{C}$ are distinguished points in our projective space. This will enable us to make a clutching-like construction in a projective space.

Let $P_\infty = \mathbb{PC}^2 \setminus \{\mathbb{C}\}$ and $P_0 = \mathbb{PC}^2 \setminus \{L\}$; then, $P_0 \cap P_\infty \cong \mathbb{PC}^2 \setminus \{\mathbb{C}, L\} = L^* \setminus \{0\}$. Our clutching construction will start with a vector bundle $\underline{L} \to P_0$, a vector bundle $\underline{\mathbb{C}} \to P_\infty$, and an isomorphism $\alpha : \underline{L} \to \underline{\mathbb{C}}$ over the intersection $P_0 \cap P_\infty = L^* \setminus \{0\}$. Thus, we'll need to specify an isomorphism $P_0 \cap P_\infty \to L^* \setminus \{0\}$ to determine how to glue $\underline{L}$ and $\underline{\mathbb{C}}$ together.

It's natural to call the identity map z^{-1}, thinking of $z \in L$, and the bundle we get is $H \to \mathbb{PC}^2$. Here again we have a punctured plane and so the winding number classifies things.

Lemma 4.5. *$H \oplus H \cong H^{\otimes 2} \oplus \underline{\mathbb{C}}$ as vector bundles over $\mathbb{CP}^1 \cong S^2$.*

Proof. The two clutching maps are, respectively, $\begin{pmatrix} z^{-1} & \\ & z^{-1} \end{pmatrix}$ and $\begin{pmatrix} z^{-2} & \\ & 1 \end{pmatrix}$. Each has determinant 1, so they're both in $\mathrm{SL}_2\,\mathbb{C}$, which deformation-retracts onto S^2, which is simply connected. Thus, the clutching maps are homotopic. ⊠

Corollary 4.6. *If $t = [H] - 1$ in $K(S^2)$, then $t^2 = 0$.*

This is the first insight we have into the ring structure of a K-theory.

Corollary 4.7. *The map $\mathbb{Z}[t]/(t^2) \to K(S^2)$ sending $t \mapsto [H] - 1$ is an isomorphism of rings.*

Definition. Let X_1 and X_2 be topological spaces; then, there are projection maps

$$\begin{array}{ccc} X_1 \times X_2 & \xrightarrow{p_1} & X_1 \\ \downarrow{\scriptstyle p_2} & & \\ X_2. & & \end{array}$$

Then, the *external product* is a map $K(X_1) \otimes K(X_2) \to K(X_1 \times X_2)$ defined as follows: if $u \in K(X_1)$ and $v \in K(X_2)$, then $u \otimes v \mapsto p_1^* u \cdot p_2^* v$.

Theorem 4.8. *If X is compact Hausdorff, then the external product $K(S^2) \otimes K(X) \to K(S^2 \times X)$ is an isomorphism of rings.*

We'll talk about this more next lecture; the idea is that in general distinguished basepoints of X and S^2 lift to subspaces of $S^2 \times X$.

The reason it doesn't work for S^1 is that if $X = S^1$, we get a torus $S^1 \times S^1$. Then, basepoints in S^1 give us $S^1 \vee S^1$ (the *wedge product*), and the quotient is $S^1 \wedge S^1 \simeq S^2$ (the *smash product*).

In fact, we'll bootstrap Theorem 4.8, using the smash product and reduced K-theory; then, results about smash products of spheres do a bunch of the work of periodicity for us. The proof will be elementary, in a sense, but with a lot of details about clutching functions, which is pretty explicit.

The version you'll read about in the Atiyah-Bott paper [3], or in Atiyah's book [2], is slightly more general. We want a family of S^2 parameterized by X, instead of just one, which is a fiber bundle; but we want two distinguished points, which will allow the clutching construction, and a linear structure.

Thus, more generally, if $L \to X$ is a complex line bundle, then $\mathbb{P}(L \oplus \underline{\mathbb{C}}) \to X$ is a fiber bundle with fiber S^2. We can once again form the hyperplane bundle $H \to \mathbb{P}(L \otimes \underline{\mathbb{C}})$.

Theorem 4.9 ([3])**.** *The map $K(X)[t]/(t[L]-1)(t-1) \to K(\mathbb{P}(L \oplus \underline{\mathbb{C}}))$ defined by sending $t \mapsto [H]$ is an isomorphism of rings.*

Then, if $X = \mathrm{pt}$, we recover Theorem 4.8, which we'll prove next time.

Lecture 5.

The K-theory of $X \times S^2$: 9/10/15

Our immediate goal is to prove the following theorem.

Theorem 5.1. *Let X be compact Hausdorff. Then, the map $\mu : K(X)[t]/(1-t)^2 \to K(X) \otimes K(S^2) \to K(X \times S^2)$, defined by sending $[E] \cdot t \mapsto [E] \otimes [H]$ followed by $[E_1] \otimes [E_2] \mapsto [\pi_1^* E_1 \otimes \pi_2^* E_2]$, is an isomorphism.*

Next time, we'll introduce basepoints and use this to prove Bott periodicity, calculating the K-theory of the spheres in arbitrary dimension; we saw last time that this computes the stable homotopy groups of the unitary group.

The proof we give is due to Atiyah and Bott in [3], and actually proves a stronger result, Theorem 4.9. Hatcher's notes [16] provide a proof of the less general theorem.

The heuristic idea is that a bundle on S^2 is given by clutching data: two closed discs D_∞ and D_0 along with a circle $S^1 = \mathbb{T}$ (i.e. we identify it with the circle group $\mathbb{T} = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$, which is a Lie group under multiplication). Then, the final piece of clutching data is given by a group homomorphism $f : \mathbb{T} \to \mathrm{GL}_r \mathbb{C}$

Suppose f is given by a Laurent series

$$f(\lambda) = \sum_{k=-N}^{N} a_k \cdot \lambda^k,$$

with $a_k \in \operatorname{End} \mathbb{C}^r$ (i.e. they might not be invertible, but their sum is). Then, $f = \lambda^{-N} p$ for a $p \in \mathbb{C}[\lambda] \otimes \operatorname{End} \mathbb{C}^r$. Then, the K-theory class of this bundle is determined by the rank r and the winding number of $\lambda^{-N} p$, which we'll denote $\omega(\lambda^{-N} p) = -Nr + \omega(p)$. That is, it's basically determined by the winding number of a polynomial.

What is the winding number of a polynomial? For simplicity, tke $r = 1$; then, $\omega(p)$ is the number of roots of p interior to $\mathbb{T} \subset \mathbb{C}$.

In some sense, we're taking the winding number as information about S^2, but we're not getting a lot of information about X. We *categorify*: we want to find a vector space whose dimension is $\omega(p)$. Set $R = \mathbb{C}[\lambda]$, which is a commutative ring, and $M = \mathbb{C}[\lambda]$ as an R-module. (If $r > 1$, we need to tensor with $\operatorname{End} \mathbb{C}^r$ again). Then, $p : M \to M$ given by multiplication by p, has a cokernel $\operatorname{coker} p = V$, a $\deg(p)$-dimensional vector space. Thus, we can canonically decompose $V = V_+ \oplus V_-$, where V_+ is the set of roots inside the unit disc. Then, we can soup this up further when $r > 1$ and X comes back into the story.

This is essentially the way that we'll prove the theorem: the proof will construct an inverse map ν to μ. The main steps are:

- approximate an arbitrary clutching by a Laurent series, leading to a polynomial clutching
- convert a polynomial clutching to a linear clutching, and
- convert a linear clutching to a vector bundle V over X.

Proof of Theorem 5.1. The first step, approximating by Laurent series, requires some undergraduate analysis. Suppose $f : X \times \mathbb{T} \to \mathbb{C}$ is continuous. The Fourier coefficients of a function on $\mathbb{T}$ become functions parameterized by X: set

$$a_n(x) = \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} f\big(x, e^{i\theta}\big) e^{-in\theta}, \quad n \in \mathbb{Z},$$

and let $u : X \times [0,1) \times \mathbb{T} \to \mathbb{C}$ be

$$u(x, r, \lambda) = \sum_{n\in\mathbb{Z}} a_n(x) r^{|n|} \lambda^n.$$

Then, u is continuous, because

$$\|a_n\|_{C^0(X)} \le \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \|f\|_{C^0(X\times\mathbb{T})} |e^{-in}| = \|f\|_{C^0(X\times\mathbb{T})}.$$

Proposition 5.2. *$u(x, r, \lambda) \to f(x, \lambda)$ as $r \to 1$ uniformly in x and λ.*

Proof. Introduce the *Poisson kernel* $P : [0,1) \times \mathbb{T} \to \mathbb{C}$, given by

$$P(r, e^{is}) = \sum_{n\in\mathbb{Z}} r^{|n|} e^{ins} = \frac{1 - r^2}{1 - 2r\cos s + r^2}, \tag{5.1}$$

which can be proven by treating the positive and negative parts as two geometric series. Then, since it converges absolutely, we can integrate term-by-term to show that

$$\int_{\mathbb{T}} \frac{\mathrm{d}s}{2\pi} P\big(r, e^{-is}\big) = 1.$$

Additionally, if $\lambda \ne 1$, (5.1) tells us that $\lim_{r\to 1} P(r, \lambda) = 0$. Thus, $\lim_{r\to 1} = \delta_1$ in $C^0(\mathbb{T})^*$ (i.e. $\delta_1(f) = f(1)$ for $f \in C^0(\mathbb{T})$, as a distribution). Now, we can write u as a convolution on $\mathbb{T}$:

$$\begin{aligned} u\big(x, r, e^{i\theta}\big) &= \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} P\Big(r, e^{i(\theta-\phi)}\Big) f\big(x, e^{i\phi}\big) \\ &= P_\theta(r, -) *_{\mathbb{T}} f(x, -) \\ &= \langle \widetilde{P}_\theta(r, -), f(x, -)\rangle, \end{aligned}$$

where our pairing is a map $C^0(\mathbb{T})^* \times C^0(\mathbb{T}) \to \mathbb{C}$. ⊠

This will allow us to approximate a clutching function with a finite step in the Fourier series, producing a Laurent series as intended.

Corollary 5.3. *The space of Laurent functions*

$$\sum_{|k|\le N} a_k(x) \lambda^k$$

is dense in $C^0(X \times \mathbb{T})$.

Proof. If $f \in C^0(X\times\mathbb{T})$, define a_k and u as before. Given an $\varepsilon > 0$, there's an r_0 such that $\|f - u(r)\|_{C^0(\times\mathbb{T})} < \varepsilon/2$ if $r > r_0$, and an N such that

$$\sum_{|n|>N} r_0^N < \frac{\varepsilon}{2\|f\|_{C^0(X\times\mathbb{T})}}.$$

Then, one can show that the norm of the difference is less than ε. ⊠

Thus, we have our approximations of clutching bundles. Note that Hatcher's proof in [16] involves a little less "undergraduate" analysis.

Thinking about S^2 as $\mathbb{P}(\mathbb{C}_0 \oplus \mathbb{C}_\infty) = \mathbb{CP}^1$, we can look at the tautological bundle. If $\lambda \in \mathbb{C}$, then the line $y = \lambda x$ in $\mathbb{C}_0 \times \mathbb{C}_\infty$ projects down, e.g. $(1, \lambda)$ to 1 and λ. In particular, the tautological bundle $H^* \to \mathbb{CP}^1 = S^2$ has clutching function λ, and therefore the hyperplane bundle $H \to \mathbb{CP}^1 = S^2$ has clutching function λ^{-1}.

For a more general $\mathcal{E} \to X \times S^2$, we want to clutch $X \times D_0$ and $X \times D_\infty$ at $X \times \mathbb{T}$. Define $E \to X$ as the restriction of $\mathcal{E} \to X \times S^2$ to $X \times \{1\}$; then, E pulls back to bundles $\pi_0^* E \to X \times D_0$ and $\pi_\infty^* E \to X \times D_\infty$. Since D_0 and D_∞ are contractible, we can choose isomorphisms $\theta_0 : \pi_0^* E \xrightarrow{\cong} \mathcal{E}|_{X\times D_0}$ and $\theta_\infty : \pi_\infty^* E \to \mathcal{E}|_{X\times D_\infty}$.

Then, $f = \theta_\infty^{-1} \circ \theta_0$ is a section of the bundle $\operatorname{Aut}(\pi_{\mathbb{T}}^* E) \to X \times \mathbb{T}$. In other words, $X \times \mathbb{T}$ embeds into $X \times D_0$ and $X \times D_\infty$, and f is the clutching data from $\pi_0^* E \to \pi_\infty^* E$.

Also, we can and will choose θ_0, θ_∞ to be the identity on $X \times \{1\}$, so that f is the identity there too.

Notationally, we'll write $[\mathcal{E}] = [E, f] \in K(X \times S^2)$; we can start with an $E \to X$ and such an f, an automorphism of $E \times \mathbb{T} \to X \times \mathbb{T}$, to get a vector bundle on $X \times S^2$. For example, $[\underline{\mathbb{C}}, \lambda] = [H^*]$, $[\underline{\mathbb{C}}, \lambda^n] = [H^{\otimes(-n)}]$, and $[E, f \cdot \lambda^n] = [E, f] \cdot [H^{\otimes(-n)}]$ in $K(X \times S^2)$ (which one can check).

What this argument shows is the following.

Proposition 5.4. *Any vector bundle on $X \times S^2$ is isomorphic to one of the form (E, f), and any two choices of f are homotopic through normalized clutching functions.*

Here, a normalized clutching function is one homotopic through the basepoint.

Now we have our clutching function, which is continuous, and replace it with a Laurent function.

Proposition 5.5.

(1) In $K(X, S^2)$, $[E, f] = [E, \lambda^{-N} p]$ for some polynomial clutching function

$$p(x, \lambda) = \sum_{k=0}^{2n} a_k(x) \lambda^k,$$

with $a_k(x) \in \operatorname{End} E_x$.

(2) Any two such choices are homotopic via a Laurent clutching function.

Proof. The proof will show that the Laurent endomorphisms of $E \times \mathbb{T} \to X \times \mathbb{T}$. If $E = \underline{\mathbb{C}}$, the proof is the same proof with Poisson kernels at the start of the class; more generally, we'll use a partition of unity $\{\rho_i\}$ subordinate to a cover $\{U_i\}$ such that $E|_{U_i}$ is trivial. Then, $f|_{U_i}$ can be approximated by a Laurent ℓ_i, and one can check that $\sum \rho_i \ell_i$ is Laurent.

For (1), since the invertible matrices are an open set, then choose an $\varepsilon > 0$ such that $B_\varepsilon(f)$ contains only invertible functions, and choose an ℓ Laurent such that $\|f - \ell\|_{C^0(X \times \mathbb{T})} < \varepsilon$, so that ℓ is invertible and $f \simeq \ell$ by a straight-line homotopy. And we know clutching with homotopic functions doesn't change the isomorphism class of the vector bundle, hence nor the K-theory class. ⊠

Thus, we've gone from continuous to Laurent; now, we will go from Laurent to linear. Observe that $[E, f] = [E, -\lambda^N p] = [H^{\otimes N}] - [E, p]$.

Let p be a polynomial clutching function of degree at most n. Then, write

$$p(x, \lambda) = \sum_{k=0}^{n} p_k(x) \lambda^k,$$

and set

$$\mathcal{L}_p^m = \begin{pmatrix} 1 & -\lambda & & & \\ & 1 & -\lambda & & \\ & & \ddots & \ddots & \\ & & & 1 & -\lambda \\ p_n & p_{n-1} & \cdots & p_1 & p_0 \end{pmatrix}.$$

This matrix of polynomials acts linearly on $E^{\oplus(n+1)} \times \mathbb{T} \to X \times \mathbb{T}$.

Proposition 5.6. $[E^{\oplus(n+1)}, \mathcal{L}_p^n] = [E, p] + [E^{\oplus n}, 1]$.

Proof. The clutching function for the right-hand side is

$$\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & p \end{pmatrix},$$

and this is exactly the matrix you get if you diagonalize $\mathcal{L}_p^n$ by elementary row and column operations. Thus, they're homotopic, and so have the same class in K-theory. ⊠

We'll then make a basic spectral construction. Suppose $T \in \operatorname{End}\mathbb{E}$ has no eigenvalues on the unit circle $\mathbb{T} \subset \mathbb{C}$. Then, take the contour integral

$$Q = \frac{1}{2\pi i} \int\limits_{|\omega|=1} (\omega - T)^{-1}\, \mathrm{d}\omega,$$

which is in $\operatorname{End}\mathbb{E}$. One can check that $Q^2 = Q$, so it's a projection, and $QT = TQ$. Thus, we can decompose $\mathbb{E} = Q\mathbb{E} \oplus (1-Q)\mathbb{E}$, which we'll denote $\mathbb{E}_+$ and $\mathbb{E}_-$, respectively. Since T commutes with Q, $T = (T_+, T_-)$, with T_+ acting on $\mathbb{E}_+$, and similarly for T_- on $\mathbb{E}_-$. This is analogous to the spectral theorem's decomposition of an operator into its generalized eigenspaces.

Proposition 5.7. *Let $[E, q]$ be a K-theory class with $q(x, \lambda) = a(x)\lambda + b(x)$. Then, there is a splitting $E = E_+ \oplus E_-$ such that $[E, q] = [E_+, \lambda] + [E_-, 1]$.*

Proof. Define

$$Q = \frac{1}{2\pi i} \int\limits_{|\lambda|=1} q^{-1}\, \mathrm{d}q = \frac{1}{2\pi i} \int\limits_{|\lambda|=1} q^{-1} \frac{\partial q}{\partial \lambda}\, \mathrm{d}\lambda.$$

Choose an $\alpha \in \mathbb{R}^{>1}$ such that $q(x, \alpha)$ is an isomorphism for all x, which works because isomorphism is an open condition. Then, compose with $q(x, \alpha)^{-1}$, so we can assume $q(x, \alpha) = \mathrm{id}$. Then, $w = (1 - \alpha\lambda)/(\lambda - \alpha)$ preserves $\mathbb{T}$ and D_0 as $\alpha \to \infty$. Define $q(\lambda) = (w - T)/(w + \alpha)$ with $T \in C^0(X; \operatorname{End} E)$, and $qT = Tq$. Then,

$$Q = \frac{1}{2\pi i} \int\limits_{|w|=1} (w - T)^{-1}\, \mathrm{d}w - (w + \alpha)^{-1}\, \mathrm{d}w,$$

but the last term goes away. Thus, this is the desired projection: q fails to be invertible exactly where T has an eigenvalue. Denote $q_\pm(\lambda) = a_\pm\lambda + b_\pm$, and $q_+(\lambda)$ is invertible if $\lambda \in D_\infty$ and $q_-(\lambda)$ is if $\lambda \in D_0$.

Thus, $q_+^t = a_+\lambda + tb_+$ and $q_-^t = ta_-^\lambda + b_-$ are homotopies of clutching functions, so

$$\begin{aligned} [E, q] &= [E_+, q_+] + [E_-, q_-] \\ &= [E_+, a_+\lambda] + [E_-, b_-] = [E_+, \lambda] + [E_-, 1]. \end{aligned}$$ ⊠

So if we have $[E, p]$ with $\deg(p) \le n$, then

$$[E, p] = [E^{\oplus(n+1)}], \mathcal{L}_p^n] - [E^{\oplus n}, q],$$

and we just proved that a linear clutching function splits as

$$\begin{aligned} &= [V_n(E, p), \lambda] + [E^{\oplus(n+1)}, 1] - [V_n(E, p), 1] - [E^{\oplus n}, 1] \\ &= [V_n(E, p)] \otimes ([H^*] - 1) + [E] \otimes 1, \end{aligned}$$

where $V_n(E, p)$ is the $+$ part of the decomposition of $E^{\oplus(n+1)}$ by $q = \mathcal{L}_p^n$. So we've gone from polynomial to linear and then split it; this will allow us to define the inverse, check it's well-defined and in fact the inverse, and so on. But this is enough of a proof sketch to follow the references and work out the details. ⊠

Even though the proof is confusing, all of the ideas are relatively elementary.

Lecture 6.

The *K*-theory of the Spheres: 9/15/15

Recall that last time, we mostly proved Theorem 5.1, but didn't pin down our inverse. The details are mostly in [3], as well as in the expositions in [2, 16]. We'll then use it to prove Bott periodicity.[7]

Recall that the idea was to take a bundle $\mathcal{E} \to X \times S^2$ and decompose. Here's the proof at the executive summary level.

(1) Write it as (E, f) for $E \to X$ and f a clutching function, an automorphism of $(E \times \mathbb{T} \to X \times \mathbb{T})$.

[7]There are many proofs of Bott periodicity; there's one in the coda of [23], which is probably well exposited.

(2) Homotope f to a Laurent clutching function, which is canonical: for $n \in \mathbb{Z}$, we get

$$a_n(x) = \int_{\mathbb{T}} \frac{\mathrm{d}\theta}{2\pi} f\big(x, e^{i\theta}\big) e^{-in\theta}.$$

Notice $f(x, e^{i\theta})$ is in $\operatorname{Aut} E_x$, but we're not averaging in this group, just in $\operatorname{End} E_x$, and therefore there's no guarantee that a_n is invertible. We can form

$$u_N(x, \lambda) = \sum_{|x| \le N} a_n(x) \lambda^n,$$

with $N \in \mathbb{Z}^{>0}$. This isn't *a priori* invertible, but there's some N_0 (depending on f) such that if $N \ge N_0$, then u_N is invertible and homotopic to f through invertible clutching functions.

(3) If p is a polynomial clutching function of degree at most d on E, then we constructed a polynomial clutching function $\mathcal{L}^d p$ on $E^{\oplus(d+1)}$, and from this linear clutching function we extracted a bundle $V_d(E, P) \to X$ such that

$$[E, p] = V_d(E, p) \otimes ([H^*] - 1) + [E, 1]. \tag{6.1}$$

in $K(X \times S^2)$; note that $[E, 1] = [E] \otimes 1$.

This is all great, if we have a polynomial clutching function magically at the beginning. But from the construction we also know the following.

(i) If $p_0 \simeq p_1$ through polynomial clutching functions, then $V_d(E, p_0) \simeq V_d(E, p_1)$. This is our basic homotopy invariance.
(ii) $V_{d+1}(E, p) \cong V_d(E, p)$; this depends more explicitly on the construction we gave last time. Notice how this is consistent with (6.1).
(iii) $V_{d+1}(E, \lambda p) \cong V_d(E, p) \oplus E$.

That was all from last time; now, we'll construct an inverse, check that it's well-defined, and show that it's the inverse. The details of the proof from last time need some filling-in, but this is something we'll be able to do.

Construction of the inverse. We're going to cook up a $\nu : K(X \times S^2) \to K(X)[t]/(1-t)^2$. Given an $\mathcal{E} \to X \times S^2$, choose an f such that $\mathcal{E} \cong (E, f)$, where $E = \mathcal{E}|_{X \times \{1\}} \to X$. For N sufficiently large (greater than an N_0 depending on f), we have $f \simeq u_N = \lambda^{-N} p_N$, where p_N is a polynomial of degree at most $2N$. Then, define

$$\nu_N(E, f) = [V_{2N}(E, p_N)](t^{-1} - 1)t^N + [E]t^N.$$

First, we must check that

(1) it's independent of N given f, and then
(2) that it's independent of f,

so that we get a function in $\mathcal{E}$.

For (1), we'll use that $p_{N+1} \simeq \lambda p_N$ via polynomial clutching functions of degree at most $2(N+1)$ if N is sufficiently large: multiplying by λ shifts all of the coefficients, so all that changes is the top-order term λ^{2N+2} and the constant term a_{-1}. Since these are invertible when N is sufficiently large, then we can go from one to the other with a straight-line homotopy, which is polynomial. Then,

$$\begin{aligned}\nu_{N+1}(E, f)[V_{2N+2}(E, p_{N+1})](1-t)t^N + [E]t^{N+1} \\ = [V_{2N+2}(E, \lambda p^N)](1-t)t^N + [E]t^{N+1},\end{aligned}$$

so using property (ii),

$$= [V_{2N+1}(E, \lambda p^N)](1-t)t^N + [E]t^{N+1}.$$

Then, using property (iii),

$$\begin{aligned}&= [V_{2N}(E, p)](1-t)tt^{N-1} + [E]t^N \\ &= [V_{2N}(E, p)](1-t)t^{N-1} + [E]t^N = \nu_N(E, f).\end{aligned}$$

Here, we used the fact that $(1-t)t = 1-t$ in this ring, and so ν is independent of N for sufficiently large N.

To show the independence of ν from f, we'll make a truncation argument: if f_0 and f_1 are sufficiently C^0-close, then their truncations at N are also homotopy equivalent, because they'll both be invertible at the same time. Thus, this is locally constant on homotopy classes, and therefore constant on homotopy classes: $\nu(E, f_0) = \nu(E, f_1)$. In particular, ν factors through the homotopy class of f and therefore depends only on $\mathcal{E}$.

Now, we need to show that $\mu \circ \nu = \mathrm{id}_{K(X \times S^2)}$. Well, it was rigged to be the identity: look at (6.1) and the definition of ν; you get back what you started with. In the opposite direction, to check that $\nu \circ \mu = \mathrm{id}_{K(X)[t]/(1-t)^2}$, use the fact that ν is a $K(X)$-module homomorphism, and therefore some information about tensor products factors through. But then we just have to check on the generators $\nu \circ \mu(t^N)$ for $N \geq 0$. This requires one more fact, that $V_N(\underline{\mathbb{C}}, 1) = 0$, which also follows from what we did last time. ⊠

Once again, this is a little more of a summary; it would be hard to give all of the details in lecture, and you can work them out using these ideas and techniques.

Computing $K(S^n)$. The rest of the lecture will deal with some elementary homotopy theory, useful not just in K-theory but also in plenty of other parts of topology. We'll use it to inductively calculate $K(S^n)$ using Theorem 5.1; note that $S^n \times S^2 \neq S^{n+2}$, but we'll be able to use smash products to do our bidding instead.

More specifically, $S^1 \times S^1$ isn't a sphere; it's a torus. But if you collapse the fundamental rectangle of the torus by two boundaries, you take $(S^1 \times S^1)/(S_1 \vee S^1)$, which gives us S^2, in a sense we'll clarify. We'll want to generalize this to inductively construct n-spheres.

Definition.

(1) A *pointed space* (X, x) is a topological space X along with some point $x \in X$.
(2) A *map of pointed spaces* $f : (X, x) \to (Y.y)$ is a continuous map $f : X \to Y$ such that $f(x) = y$. We also require homotopies $f : [0,1] \times X \to Y$ to preserve the basepoint: $f(x,t) = y$ for all $t \in [0,1]$.

Pointed spaces and their maps form a category, as well as those with additional properties, such as pointed Hausdorff spaces, pointed CW complexes, and so on.

Recall that we have defined the reduced K-theory for any space X, given by $\widetilde{K}(X) = K(X)/\operatorname{Im}(K(\mathrm{pt}) \to K(X))$, where $K(\mathrm{pt}) \to K(X)$ is induced by the unique map $\pi : X \to \mathrm{pt}$. But if (X, x) is a pointed space, then we have a splitting $\mathrm{pt} \mapsto x$ (in particular, X is nonempty). So we have two pullbacks: $x^{\cdot} K(X) \to K(\mathrm{pt})$, and $\pi^* : K(\mathrm{pt}) \to K(X)$. Thus, we can split off a summand: $K(X) \cong k(X) \oplus \pi^* K(\mathrm{pt})$, where $k(X) = \ker x^*$, and the projection $K(X) \to \widetilde{K}(X)$ restricts to an isomorphism $k(X) \xrightarrow{\cong} \widetilde{K}(X)$.

In summary, for pointed spaces, we can take the reduced K-theory to be a subspace rather than a quotient space, and specifically, the subspace that reduces to 0 at the basepoint.

This is a pretty important idea: when we're making topology out of contravariant objects, we can more generally consider the subgroup that restricts to zero at the basepoint. In K-theory specifically, vector bundles can't exactly restrict to 0, then we have that if $E^0, E^1 \to X$, then $[E^0] - [E^1] \in K(X)$ restricts to 0 at an x iff $\operatorname{rank}_x E^0 = \operatorname{rank}_x E^1$.

We can generalize this to subspaces $A \subset X$ rather than basepoints.[8] Then, we can look at things that restrict to zero on A, and use these to define a more general reduced K-theory. This is powerful: for example, if X is a CW complex, we get a filtration from its skeleton, $X_0 \subset X_1 \subset \cdots \subset X$, and this induces a filtration on K-theory.

Given two pointed spaces (X, x) and (Y, y), we can make a few useful constructions in the category of pointed spaces out of them.

Definition.

(1) The *wedge* $X \vee Y = X \coprod Y/(x \sim y)$, which is a pointed space with the identified x and y as its basepoint.
(2) The *smash*[9] is $X \wedge Y = (X \times Y)/(X \vee Y)$; once again, there is a unique image of (x, b) and (a, y), and this becomes our basepoint.
(3) The *suspension* $\Sigma X = S^1 \wedge X$. You can think of this as two cones on X collapsed by the unit interval. The unique image of the old basepoint becomes the new basepoint. Sometimes, this is called the *reduced suspension* of X.
(4) The *cone* $CX = [0,1] \wedge X$, turned into a pointed space by taking 0 to be the basepoint of $[0,1]$. Basically, we collapse to a point at 1; if $X = S^2$, this is the familiar cone in $\mathbb{R}^3$.

[8] Well, the basepoint actually sits inside A, but that won't matter so much.

[9] This can be confusing: in LaTeX, $\wedge$ is called "wedge," and $\vee$ is called "vee."

Notice that a map extending over the cone is a null homotopy; the large number of ideas that can be stated in similar terms illustrate that these can be very useful constructions.

Proposition 6.1. *Let (X, a) be a compact Hausdorff pointed space and $A \subset X$ be a subspace containing A. Then, the sequence*

$$A \xhookrightarrow{i} X \xrightarrow{q} X/A, \tag{6.2}$$

with i given by inclusion and q given by quotient, induces an exact sequence of abelian groups

$$\widetilde{K}(X/A) \xrightarrow{q^*} \widetilde{K}(X) \xrightarrow{i^*} \widetilde{K}(A).$$

Notice that the image of a in A, X, and X/A is our basepoint.

Proof. We'll prove it in the case of CW complexes; for a more general proof, see [16].

The composition $q \circ i$ sends A to a point, so $(q \circ i)^* = i^* \circ q^*$ has image constant vector bundles, which vanish in $\widetilde{K}(A)$. Thus, $i^* \circ q^* = 0$.

If $E \to X$ restricts to be stably trivial on A, then after adding a constant bundle, we can assume $E|_A \to A$ is trivial. So choose a trivialization; then, clutching with it produces a bundle on X/A whose image under q^* is isomorphic to E. In some sense, pt is attached to every point of A, and so we get the same fiber over every point in A, and then can clutch in that way. Certainly, we get a family of vector spaces, but we actually get a vector bundle $E \to X/A$; local triviality is only nontrivial at the basepoint (which is in a sense all of A), which follows because it's true in a deformation retract neighborhood; this exists for CW complexes. ⊠

Now, we can employ a standard construction called the *Puppe sequence*: we'll extend (6.2) to the sequence

$$A \xrightarrow{i} X \xrightarrow{q} X/A \longrightarrow \Sigma A.$$

This is because $X/A \simeq X \cup_A CA$ (since replacing A with CA makes A within X null-homotopic, so we're taking the quotient by it), and $\Sigma A \simeq X \cup_A CA/X$ by definition, and we can do this by attaching a cone on X (this may be confusing; it helps to draw a picture). Thus, we can extend further to

$$A \xrightarrow{i} X \xrightarrow{q} X/A \longrightarrow \Sigma A \xrightarrow{\Sigma i} \Sigma X \xrightarrow{\Sigma q} \Sigma(X/A) \longrightarrow \Sigma^2 A \longrightarrow \cdots$$

This sequence can be made from any contravariant functor of geometric objects.

Corollary 6.2. *There exists a long exact sequence*

$$\cdots \longrightarrow \widetilde{K}(\Sigma(X/A)) \longrightarrow \widetilde{K}(\Sigma X) \longrightarrow \widetilde{K}(\Sigma A) \longrightarrow \widetilde{K}(X/A) \longrightarrow \widetilde{K}(X) \longrightarrow \widetilde{K}(A). \tag{6.3}$$

This can be quite computationally useful, as in the following example.

Lemma 6.3. *Restriction induces an isomorphism $\widetilde{K}(X \vee Y) \to \widetilde{K}(X) \oplus \widetilde{K}(Y)$ for pointed spaces X and Y.*

Proof. We'll apply (6.3) to $x = y \in X \vee Y \subset X \times Y$. We have projections $\pi_1, \pi_2 : X \vee Y \rightrightarrows X, Y$, and then $\pi_1^* \oplus \pi_2^*$ is a section for the map $\widetilde{K}(X \times Y) \to \widetilde{K}(X \vee Y)$ (and similarly, there's a section $\Sigma^n s$ at every level in the long exact sequence). Thus, the long exact sequence breaks into a list of short exact sequences. TODO: what happened here? ⊠

Corollary 6.4. $\widetilde{K}(X \times Y) \cong \widetilde{K}(X) \oplus \widetilde{K}(Y) \oplus \widetilde{K}(X \wedge Y)$.

We'll be able to use this to construct a product map $\widetilde{K}(X) \otimes \widetilde{K}(Y) \to \widetilde{K}(X \wedge Y)$: if $u \in X$ and $v \in Y$, then $\pi_1^* u \cdot \pi_2^* v$ vanishes on $X \vee Y \subset X \wedge Y$, and so $\pi_1^* u \cdot \pi_2^* v \in \widetilde{K}(X \wedge Y)$ in the decomposition in Corollary 6.4.

There's a pointed version of the μ we constructed when proving Theorem 5.1.

Theorem 6.5. *The map $\beta : \widetilde{K}(X) \to \widetilde{K}(\Sigma^2 X) = \widetilde{K}(S^2 \wedge X)$ sending $u \mapsto ([H] - 1) \cdot u$ is an isomorphism.*

All of this is written up more carefully in [16]; next time, we'll turn K-theory into a cohomology theory and use it to prove a result about division algebras.

Lecture 7.

Division Algebras Over $\mathbb{R}$: 9/17/15

"I've been posting problems... they're not just for my health."

First, we'll discuss some points that were rushed through last time. if (X, x) is a pointed space, then $K(X) \cong \widetilde{K}(X) \oplus K(\{x\})$, and the latter summand is infinite cyclic. We will want to think of vector bundles that vanish at the basepoint, so we associate to a class $[E]$ the class $([E] - [\underline{E_x}]) \oplus [E_x]$ (i.e. subtract the constant bundle formed from the fiber at x).

Then, Proposition 6.2 tells us that if $A \hookrightarrow X \twoheadrightarrow X/A$ and X is compact, then we get an exact sequence $\widetilde{K}(A) \leftarrow \widetilde{K}(X) \leftarrow \widetilde{K}(X/A)$, assuming there exists a deformation retraction of a neighborhood of A in X back to A, for example when X is a CW complex and A is a subcomplex. Then, we converted this into a longer sequence called the Puppe sequence, using suspensions of A, X, and A/X.

Proposition 7.1. *If X and Y are compact Hausdorff, the sequence $X \vee Y \to X \times Y \to X \wedge Y$, which induces a split exact sequence*

$$0 \longrightarrow \widetilde{K}(X \wedge Y) \longrightarrow \widetilde{K}(X \times Y) \longrightarrow \widetilde{K}(X \vee Y) \longrightarrow 0.$$

Since $\widetilde{K}(X \vee Y) \cong \widetilde{K}(X) \oplus \widetilde{K}(Y)$, this gives us an isomorphism $\widetilde{K}(X \times Y) \cong \widetilde{K}(X \wedge Y) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y)$.

Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the canonical projections; then, we get an external product $\pi_1^* u \cdot \pi_2^* v$ for $u \in \widetilde{K}(X)$ and $v \in \widetilde{K}(Y)$. This product restricts to 0 in $X \vee Y$, and hence by the above proposition is pulled back from a unique $u * v \in \widetilde{K}(X \wedge Y)$; thus, we have a product $* : \widetilde{K}(X) \otimes \widetilde{K}(Y) \to \widetilde{K}(X \wedge Y)$.

By FOILing, the following diagram commutes.

$$\begin{tikzcd} K(X) \otimes K(Y) \arrow[r, "\cong"] \arrow[d, "*"] & (\widetilde{K}(X) \otimes \widetilde{K}(Y)) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y) \oplus \mathbb{Z} \arrow[d, "{(*,\mathrm{id},\mathrm{id},\mathrm{id})}"] \\ K(X \times Y) \arrow[r, "\cong"] & \widetilde{K}(X \wedge Y) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y) \oplus \mathbb{Z} \end{tikzcd}$$

And we proved that if X is compact Hausdorff, then $\times : K(S^2) \otimes K(X) \to K(S^2 \times X)$ is an isomorphism, and therefore $\beta : \widetilde{K}(X) \to \widetilde{K}(\Sigma^2 X)$ sending $u \mapsto ([H] - 1) * u$ is an isomorphism. Then, by induction, we get our nice result.

Corollary 7.2.

$$\widetilde{K}(S^n) = \begin{cases} 0, & n \text{ odd} \\ \mathbb{Z}, & n \text{ even.} \end{cases}$$

There are many things called Bott periodicity; this one is equivalent to Bott's original one, which used Morse theory and calculated $\pi_{n-1}\,\mathrm{U}$. Things are slightly different in the real case, which we will be able to prove as well.

We'll spend the rest of this lecture and part of next lecture proving the following statements.

Proposition 7.3. *For an $n \in \mathbb{Z}^{>0}$, (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) in the following.*

(1) $\mathbb{R}^n$ admits the structure of a division algebra.
(2) S^{n-1} is parallelizable.
(3) S^{n-1} is an H-space.
(4) There exists an $f : S^{2n-1} \to S^n$ of Hopf invariant 1.

We'll define H-spaces, division algebras, and the Hopf invariant shortly. Then, we can use this to get a nice result.

Theorem 7.4. *If there exists an $f : S^{2n-1} \to S^n$ of Hopf invariant 1, then $n = 1$, 2, 4, or 8.*

Corollary 7.5 (Milnor [21], Kervaire [19])**.** *$\mathbb{R}^n$ admits a divison algebra structure iff $n = 1$, 2, 4, or 8.*

Now, what do all of these words mean?

Definition. A *unital division algebra* is a vector space A and a linear map $m : A \times A \to A$ and an $e \in A$ such that

(1) $m(e,-) = m(-,e) = \mathrm{id}_A$.
(2) $m(x,-)$ and $m(-,y)$ are bijective maps $A \to A$ when $x, y \neq 0$.

Notice that this is required to be neither associative nor commutative.

It's quite striking that, yet again, we're proving a theorem from pure algebra using topology! But for existence, we will have to do a little algebra.

When $n = 1$, we have $\mathbb{R}$, and when $n = 2$, we have $\mathbb{C}$, both familiar fields. When $n = 4$, we have the *quaternions* $\mathbb{H} = \mathbb{R}\{1, i, j, k\}$ with multiplication relations $i^2 = j^2 = k^2 = 1$, $ij = k$, and $ji = -k$. This multiplication is associative, but not commutative, so $\mathbb{H}$ isn't a field. Finally, when $n = 8$, we have the *octonions* or *Cayley numbers* $\mathbb{O}$, an eight-dimensional vector space over $\mathbb{R}$ with basis $\{1, e_1, e_2, \dots, e_7\}$, with a kind of complicated multiplication table given in Figure 1. This is in some sense projective geometry over $\mathbb{F}_2$: there's a lot of interesting math to be said about this structure, and a good article to begin reading is [7].

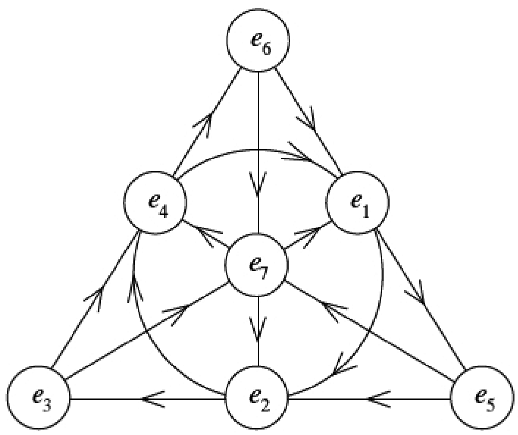


FIGURE 1. The *Fano plane*, a way to remember the rules of octonion multiplication. The rule is, $e_i^2 = -1$, and to determine $e_i \cdot e_j$, choose the third point on the line containing them, and add a minus sign if you went against the direction of the arrows. For example, $e_5 \cdot e_2 = e_3$ and $e_7 \cdot e_3 = -e_1$. Source: [7].

Definition. An *H-space* is a pointed topological space (X, e) together with an unpointed map $g : X \times X \to X$ such that $g(e,-) = g(-,e) = \mathrm{id}_X$.

This is sort of a very lax version of a topological group, with no associativity.

Finally, we'll get to the Hopf invariant later.

Partial proof of Proposition 7.3. Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^n$. Then, for any $x \in S^{n-1}$, $(x \cdot e_1)e_1, \dots, (x \cdot e_n)e_n$ is a basis of $\mathbb{R}^n$, so use the Gram-Schmidt process to convert this into an orthonormal basis $\xi_1(x), \dots, \xi_n(x)$ of $\mathbb{R}^n$. (For example, $\xi_1(x) = x \cdot e_1/\|x \cdot e_1\|$). Then, observe that $S^{n-1} \to S^{n-1}$ sending $x \mapsto \xi_i(x)$ is a diffeomorphism, so (1) $\implies$ (2).

For (2) $\implies$ (3), suppose $\eta_2(x), \dots, \eta_n(x)$ is a basis of $T_x S^{n-1}$; then, use Gram-Schmidt again to get an orthonormal basis $\xi_2(x), \dots, \xi_n(x)$ of $T_x S^{n-1}$, and therefore $x, \xi_2(x), \dots, \xi_n(x)$ is an orthonormal basis of $\mathbb{R}$.

Then, compose with a fixed orthogonal transformation so that $(e_1, \xi_2(e_1), \dots, \xi_n(e_1)) = (e_1, e_2, \dots, e_n)$. Define $\alpha : S^{n-1} \to \mathrm{SO}(n)$ by $\alpha_x(e_1, \dots, e_n) = (x, \xi_2(x), \dots, \xi_n(x))$, and $g : S^{n-1} \times S^{n-1} \to S^{n-1}$ by $x, y \mapsto \alpha_x(y)$. Since $(e_1, y) \mapsto y$ and $(x, e_1) \mapsto x$, then this gives us an H-space structure. ⊠

The following theorem from the 1940s was originally proven with cohomology, but our K-theoretic proof of Theorem 7.4 will be a little cleaner.

Theorem 7.6 (Hopf)**.** *If $\mathbb{R}^n$ is a division algebra, then n is a power of 2.*

Proof. Multiplication $m : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ induces a map $g : S^{n-1} \times S^{n-1} \to S^{n-1}$ given by sending $x, y \mapsto (x \cdot y)/\|x \cdot y\|$. This sends antipodal points to antipodal points, so we get a quotient $\overline{g} : \mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1} \to \mathbb{RP}^{n-1}$.

We'll use $H^\bullet(\mathbb{RP}^{n-1};\mathbb{F}_2)$; specifically, we'll use the cup product. This is a very powerful tool, but it's considerably more obscure than the K-theoretic product. Specifically, we have that $H^\bullet(\mathbb{RP}^{n-1};\mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^n)$, with $\deg x = 1$.

For our map $\overline{g}$, let x, y, and z be the respective generators of our three copies $\mathbb{RP}^{n-1}$ (x nd y for the domain, and z for the range). Cohomology gives us a pullback $\overline{g}^*$, and in fact $\overline{g}^*(z) = x + y$: z must be sent to another 1-dimensional class, which is therefore generated by some projective lines. Looking at exactly what $\overline{g}$ is doing, we can conjugate the second to the identity, and so we get $x+y$ in cohomology.[10] Thus, by the binomial theorem,

$$\begin{aligned} 0 = \overline{g}^*(z^n) &= (x+y)^n \\ &= x^n + \sum_{k=1}^{n-1} \binom{n}{k} x^{n-k} y^k + y^n \\ &= \sum_{k=1}^{n-1} \binom{n}{k} x^{n-k} y^k, \end{aligned}$$

which lies in $H^\bullet(\mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1};\mathbb{F}_2)$, which by the Künneth formula, is isomorphic to $\mathbb{F}_2[x,y]/(x^n, y^n)$. In particular, the monomials x^k and y^k are all independent, and since their sum is zero mod 2, $\binom{n}{k} = 0 \pmod 2$ for each k, and Pascal's triangle tells us that this only happens when n is a power of 2. ⊠

Let's talk about the Hopf invariant now.

Definition. Given a map $f : S^{2n-1} \to S^n$, we can take the *cone* of f, $C_f = S^n \cup_f D^{2n}$. (More generally, if $f : X \to Y$ is a continuous map, $C_f = Y \cup_f CX$; this is defined without basepoints.) For this map between spheres, C_f has the structure of a CW complex with cells e^0, e^n, and e^{2n}, and only depends on the homotopy type of f.

Using the Puppe sequence (collapsing Y gives us ΣX, which in this case is S^{2n}), we get a sequence

$$S^{2n-1} \longrightarrow S^n \longrightarrow C_f \longrightarrow S^{2n}. \tag{7.1}$$

Focusing on the latter three terms, if $n > 1$, we can deduce that $H^\bullet(C_f;\mathbb{Z}) \cong \mathbb{Z}\cdot b_m \oplus \mathbb{Z} \cdot a_{2n}$ (i.e. the generators have degrees n and $2n$, respectively). The ring structure means that $b^2 = ha$ for some $h \in \mathbb{Z}$. This h is called the *Hopf invariant*, and is determined up to sign.

By fixing orientations, we can pin down a sign for h, but we won't need to.

We can give an alternative definition of the Hopf invariant using K-theory. Appling $\widetilde{K}$ to (7.1) when $n = 2m$ is even produces a split short exact sequence (because $\widetilde{K}(S^{2m+1}) = 0$)

$$0 \longrightarrow \widetilde{K}(S^{4m}) \longrightarrow \widetilde{K}(C_f) \longrightarrow \widetilde{K}(S^{2m}) \longrightarrow 0,$$

where the first map sends $([H]-1)^{2m} \mapsto \alpha$ and the second map sends β to a generator of $\widetilde{K}(S^{2m})$, $([H]-1)^m$. By exactness, this means $\beta^2 = h\alpha$ for some $h \in \mathbb{Z}$. However, h isn't well-defined; if $\beta \mapsto \beta + k\alpha$, then $\beta^2 \mapsto \beta^2 + 2k\alpha\beta = (h + 2k\ell)\alpha$, where $\alpha\beta = \ell\alpha$. We can see that $h \bmod 2$ is well-defined, though, and that's all we needed.

If n is odd, then by degree considerations, $b^2 = 0$ in $H^\bullet(C_f;\mathbb{Z})$, and so the Hopf invariant is necessarily zero.

The story behind these proofs is kind of tangled; Milnor and Kervaire. in [21] and [19], respectively, figured out the proof of Theorem 7.4 and therefore the corollary about division algebras. Milnor wrote to Bott about it in [9], and Bott was nicely surprised, so these letters were published. Then, some of the later results were published by Adams and Atiyah in [1]; one of the proofs nicely fit on a postcard. Some of these proofs depended on operations on mod 2 cohomology called Steenrod squares.

For (3) $\implies$ (4), suppose $g : S^{2m-1} \times S^{2m-1} \to S^{2m-1}$ gives S^{2m-1} an H-space structure. Then, we can view

$$S^{4m-1} = \partial(D^{4m}) = \partial(D^{2m} \times D^{2m}) = (\partial D^{2m} \times D^{2m}) \cup_{\partial D^{2m} \times \partial D^{2m}} (D^{2m} \times \partial D^{2m}).$$

Thus, we can define an $f : S^{4m-1} \to S^{2m}$ by extending from S^{2m-1} on each cone, and we'll determine its Hopf invariant next time.

[10] In homology, the induced map sends generators to generators; this is just the dual statement.

Of course, this can all be found in [16].

Lecture 8.

The Splitting Principle: 9/22/15

"If this were a teaching class, I would tell you to not do what I just did."

Recall that we were in the middle of proving Proposition 7.3, which is instrumental in the K-theoretic proof that the only division algebras over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$; the key is linking in Theorem 7.4.

Soon, we'll start talking about Fredholm operators, which lead to another proof of Bott periodicity, and then move into equivariant topics, including Lie groups.

Definition. If (X, A) is a pair with $A \subset X$, the *relative K-theory* $\widetilde{K}(X, A) = \widetilde{K}(X/A)$, assuming A is nonempty.

Proof of (3) $\implies$ (4) *in Proposition 7.3.* Let $n = 2m$ and $g : S^{2n-1} \times S^{2n-1} \to S^{2m-1}$; it's easy to see that if n is odd, the Hopf invariant has to be 0, so we're assuming n is even.

This argument is straight out of [16]; read the details there (or check out the giant diagram).

We want to construct a map $f : S^{4m} \to S^{2m}$ by writing $S^{4m} = \partial(D^{4m}) = \partial(D^{2m} \times D^{2m})$, and since ∂ obeys the Leibniz rule, this is homeomorphic to $D^{2m} \times \partial D^{2m} \cup_{\partial D^{2m} \times \partial D^{2m}} \partial D^{2m} \times D^{2m}$.

We can write S^{2m} as the suspension of S^{2m-1}; thus, we can draw this as a cone with cone parameter t; to construct f, take a point on ΣS^{2m-1} with parameter t, figure out where its projection down to S^{2m-1} goes, and then send to the point above that in $S^{2m} = \Sigma S^{2m-1}$, but with the same parameter. Thus, we can use the decomposition from the previous paragraph to realize this as a map $f : S^{4m-1} \to S^{2m}$.

Let C_f denote the cone of f, which entails attaching a $4m$-cell. Thus, we get $S^{2m} \to C_f \to S^{4m}$, which as we proved gives us a short exact sequence $\widetilde{K}(S^{2m}) \leftarrow \widetilde{K}(C_f) \leftarrow \widetilde{K}(S^{4m})$; since the even-dimensional K-theory of spheres is infinite cyclic, then we've shown that $\widetilde{K}(C_f)$ is also infinite cyclic, by looking at the diagram, so if β generates it, then $\beta^2 \mapsto h\alpha$, where α generates $\widetilde{K}(S^{2m})$. It turns out (though we didn't prove it), this is independent of the lift we chose, and in this specific case, $h = 1$, courtesy of the following diagram.

$$\begin{tikzcd}
\widetilde{K}(C_f) \otimes \widetilde{K}(C_f) \arrow[r, "*"] & \widetilde{K}(C_f) \\
\widetilde{K}(C_f, D^{2m}_-) \otimes \widetilde{K}(C_f, D^{2m}_+) \arrow[u, "\cong"', blue] \arrow[r, "*"] \arrow[d, "\Phi^* \otimes \Phi^*"] & \widetilde{K}(C_f, S^{2m}) \arrow[u] \arrow[d, "\cong"', "\Phi^*"] \\
\widetilde{K}(D^{2m} \times D^{2m}, \partial D^{2m} \times D^{2m}) \otimes \widetilde{K}(D^{2m} \times D^{2m}, D^{2m} \times \partial D^{2m}) \arrow[r, "*"] \arrow[d, "\cong"'] & \widetilde{K}(D^{2m} \times D^{2m}, \partial(D^{2m} \times D^{2m})) \\
\widetilde{K}(D^{2m} \times \{e\}, \partial D^{2m} \times \{e\}) \otimes \widetilde{K}(\{e\} \times D^{2m}, \{e\} \times \partial D^{2m}) \arrow[ur, "*", "\cong"'] &
\end{tikzcd}$$

Here, the blue arrow is given by excision; in each argument, we've excised out a contractible set, so nothing changes.

What we have to prove is that β^2 is a generator, so that $\beta^2 = \alpha$. This diagram commutes, which is a fun exercise, and follows because the product of vector bundles is natural; then, this commutativity, and the isomorphisms in the diagram, allow us to show that in the uppermost map, $\beta \otimes \beta \mapsto \alpha$. ⊠

The splitting principle. Now, we'll switch topics. The question we want to answer is: given a complex vector bundle $E \to X$, can we write E as a direct sum of line bundles? Sometimes, the answer is yet, e.g. when $X = S^2$. However, when $X = S^4$, isomorphism classes of rank-2 vector bundles over S^4 is isomorphic to $[S^3, \mathrm{U}_2]$, but the isomorphism classes of line bundles $L \to S^4$ form $[S^3, \mathrm{U}_1] = 0$.

Returning to rank-2 bundles, $\mathrm{SU}_2 \hookrightarrow \mathrm{U}_2$ creates a map $f : [S^3, \mathrm{SU}_2] \hookrightarrow [S^3, \mathrm{U}_2]$, and $\mathrm{SU}_2 \cong \mathrm{Sp}_1 \cong S^3$, as unit quaternions of length 1, and we know there are homotopically nontrivial maps $S^3 \to S^3$. That f is injective comes from the fact that $\mathrm{U}_2 \to \mathrm{SU}_2$ is a 2 : 1 covering space.

We can actually produce a specific example: there's a *Hopf fibration* $S^3 \to S^7 \to S^42$ by choosing the vectors with unit norm $\{(q_1, q_2) : |q_1|^2 + |q_2|^2 = 1\}$. Thus, we get $S^7 \subset \mathbb{H}^2$, with fibers S^3, and projecting down to $S^4 = \mathbb{HP}^1$ produces the desired fibration.[11] This fiber bundle satisfies Steenrod local triviality; and when you pull it back by a continuous map, it can only untwist; it can't twist more. Writing as a sum of line bundles would be a kind of untwisting.

So we want to construct a map $p : \mathbb{F}(E) \to X$ (where $\mathbb{F}(E)$ is the *flag manifold*) such that

(1) the vector bundle $p^*E \to \mathbb{F}(E)$ is isomorphic to a direct sum of line bundles in the diagram

$$\begin{tikzcd} p^*E \arrow[r] \arrow[d] & E \arrow[d] \\ \mathbb{F}(E) \arrow[r, "p"] & X, \end{tikzcd}$$

(2) and that the map $p^* : K(X) \to K(\mathbb{F}(E))$ is injective. This injectivity will allow us to push our isomorphism into the vector bundle.

This is known as the *splitting principle*; it's a very important argument in the theory of characteristic classes, and we're going to be doing something quite similar, though using K-theory in place of cohomology as the residence of the classes. This is a very common manuever in mathematics.

First, let's simplify the problem. We first want a map $q : \mathbb{P}(E) \to X$ such that $q^*E \supset L$ is a line bundle. This helps us because then $E \cong L \oplus E/L$, as the sequence splits; then, we have reduced the problem.

To do this, we need to make a choice of a line in each E_x. The mathematician's maneuver is to make all choices. Let $q : \mathbb{P}(E) \to X$ be defined by sending $q^{-1}(x)$ to the space of lines in E_x, which works because $\mathbb{P}(E_x) = \mathbb{P}(E)_x$.

When we do this for all x, we describe q as a fiber bundle. Then, the pullback gives the data of a line and a point in the bundle, and working with this, we get the desired line bundle L. Thus, the pullback splits as $0 \to L \to q^*E \to \mathbb{P}(E) \to 0$.

We'd like to make it a complement, rather than just a quotient; if we have a Hermitian metric, this is easy, as we just take the orthogonal complement. We might not have this given, in which case we need to make a choice. Or, again, all choices.

Given a one-dimensional subspace L of a vector space $\mathbb{E}$, what can we say about the space of possible complements to L? If W is one complement, we can think about graphs: we can identify W with $\mathbb{E}/L$, and so given a map in $\mathrm{Hom}(\mathbb{E}/L, L)$, it's also a map $W \to L$, and this has a graph, which is a complement to L. Moreover, all such complements can be realized in this way. These complements are splittings of $0 \to L \to \mathbb{E} \to \mathbb{E}/L \to 0$, so they form an affine space, and one can work this way. Of course, it's usually simpler to choose a metric on E so that everything works.

Now we can take complements, so we can split off bundles until we run out: first, we get

$$\begin{tikzcd} L_1 \oplus E_1 \arrow[r] \arrow[d] & E \arrow[d] \\ \mathbb{P}(E) \arrow[r] & X \end{tikzcd}$$

and then repeating, we get another line bundle L_2, and so on until we run out, and so E has been written as a direct sum of line bundles.

K-theory as a cohomology theory. To get the second criterion, that p^* is injective, we need to discuss K-theory as a cohomology theory. We'll work in the category of pairs of pointed compact Hausdorff spaces (X, A) with $A \subset X$.

Definition. For $n \in \mathbb{Z}^{\geq 0}$, define $\widetilde{K}^{-n}(X, A) = \widetilde{K}(\Sigma^n(X/A))$.

If $n = 0$, $\widetilde{K}^0(X, A) = \widetilde{K}(X, A)$. We can also take $A = \emptyset$; $X/\emptyset = X_+$, defined to be $X \sqcup \mathrm{pt}$, and we can write $K^{-n}(X) = \widetilde{K}^{-n}(X_+) = \widetilde{K}(\Sigma^n(X_+))$, and $K^{-n}(X, A) = \widetilde{K}(X, A)$ if $A \neq \emptyset$.

[11] If you replace $\mathbb{H}^2$ with $\mathbb{C}^2$, you get the more familiar Hopf fibration $S^1 \to S^3 \to S^2$.

Thus, our short exact sequence

$$\widetilde{K}^{-n}(X,A) \longrightarrow \widetilde{K}^{-n}(X) \longrightarrow \widetilde{K}^{-n}(A)$$

becomes by the Puppe sequence a long exact sequence

$$\cdots \longrightarrow \widetilde{K}^{-n}(X,A) \longrightarrow \widetilde{K}^{-n} \longrightarrow \widetilde{K}^{-n}(A) \longrightarrow \widetilde{K}^{-n+1}(X,A) \longrightarrow \widetilde{K}^{-n+1}(X) \longrightarrow \cdots \tag{8.1}$$

Since we haven't defined $\widetilde{K}^n(X)$ for $n > 0$, this sequence terminates at $\widetilde{K}^0(A)$. However, Bott periodicity creates a map $\beta : \widetilde{K}^{-n}(X,A) \to \widetilde{K}^{-n+2}(X,A) = \widetilde{K}^{-n}(S^2 \wedge X/A)$ by $[E] \mapsto ([H]-1) * E$. Thus (8.1) becomes a hexagon.

$$\begin{tikzcd}
\widetilde{K}^{-n}(X,A) \arrow[r] & \widetilde{K}^{-n} \arrow[r] & \widetilde{K}^{-n}(A) \arrow[dd] \\
\widetilde{K}^{-n+2}(X,A) \arrow[u, "\beta"] & & \\
\widetilde{K}^{-n+1}(A) \arrow[u] & \widetilde{K}^{-n+1}(X) \arrow[l] & \widetilde{K}^{-n+1}(X,A) \arrow[l]
\end{tikzcd}$$

Now, we can define $\widetilde{K}^n(X,A) = \widetilde{K}^{n-2}(X,A)$ for any $n \in \mathbb{Z}$. For general cohomological reasons, it makes sense to think of this as graded in $\mathbb{Z}$, rather than $\mathbb{Z}/2$. Then, $K^\bullet(\mathrm{pt})$ is a $\mathbb{Z}$-graded ring, and in some sense is the "ground ring" of this cohomology theory. In fact, $K^\bullet(\mathrm{pt}) = \mathbb{Z}[u, u^{-1}]$, with $\deg(u) = 2$, as $u^{-1} = [H] \in K^{-2}(\mathrm{pt}) \cong \widetilde{K}(S^2)$.

A useful fact is that every map in the long exact sequence is compatible with the $K(X)$-module structures on $K(A)$ and $K(X,A)$.

The second part of the splitting principle (whose proof can be found in [16]), is to prove that for $q : \mathbb{P}(E) \to X$, the pullback $q^* : K(X) \to K(\mathbb{P}(E))$ is injective. We'll give part of the proof next time; it's a more sophisticated example of familiar arguments from algebraic topology. Ultimately, by the Leray-Hirsch theorem, $K(\mathbb{P}(E))$ is a free $K(X)$-module.

The Adams operations. Analogous to the Steenrod operations in cohomology, we have Adams operations in K-theory.

Theorem 8.1. *For $k \in \mathbb{Z}^{\geq 0}$ and X a compact Hausdorff space, there exists a unique a ring homomorphism $\psi^k : K^0(X) \to K^0(X)$ natural in X and satisfying $\psi^([L]) = [L^{\otimes k}] = [L]^K$ for all line bundles $L \to X$. Moreover, ψ^k satisfies the following properties.*

(1) $\psi^k \psi^\ell = \psi^{k+\ell}$.
(2) If p is prime, $\psi^p(x) \equiv x^p \bmod p$ for $x \in K(X)$.
(3) ψ^k is multiplication by k^m on $\widetilde{K}(S^{2m})$.

Proof. By the splitting principle, we can reduce to direct sums of line bundles, by passing back to the flag manifold $\mathbb{F}(E)$. If $E = \bigoplus_{i=1}^r L_k$, then $\psi^k([E]) = [L_1]^k + \cdots + [L_r]^k \in K(\mathbb{F}(E))$, which certainly exists and is unique, and one can check that it descends to X.

Now we need to check all these properties. (1) is trivial: taking the sum of a bunch of k^{th} powers followed by ℓ^{th} powers gives $(k+\ell)^{\text{th}}$ powers. For (2), set $x_i = [L_i]$, so that

$$\begin{aligned}
\psi^p(x_1 + \cdots + x_r) &= x_1^p + \cdots + x_r^p \\
&\equiv (x_1 + \cdots + x_p)^p \pmod p.
\end{aligned}$$

For (3), when $m = 1$,

$$\begin{aligned}
\psi^k([H]-1) &= [H]^k - 1 \\
&= (1+x)^k - 1 = (1+kx) - 1 = kx,
\end{aligned}$$

since in $K(S^2)$, the basic relation is $x = [H]-1$, so $x^2 = ([H]-1)^2 = 0$. ⊠

The proof that this map descends from $\mathbb{F}(E)$ to E will be given next time; we'll also talk more about the splitting principle and characteristic classes.

But now, we can give the postcard proof of Theorem 7.4 by Adams and Atiyah in [1].

Proof of Theorem 7.4. Suppose $f : S^{4m-1} \to S^{2m}$ has Hopf invariant one, and take $C_f = S^{2m} \cup_f D^{4m}$. Then, we have $\widetilde{K}(S^{4m}) \to \widetilde{K}(C_f) \to \widetilde{K}(S^{2m})$, given respectively by maps $([H]-1)^{2m} \mapsto \alpha$ and $\beta \mapsto ([H]-1)^m$.

We know that $\psi^k(\alpha) = k^{2m}\alpha$ and $\psi^k(\beta) = k^m\beta + \mu_k\alpha$, with $\mu_k \in \mathbb{Z}$, so $\psi^2(\beta) = 2^m\beta + \mu_2\alpha \equiv \mu_2\alpha \pmod 2$, but this is also $\beta^2 \pmod 2$, and this is $h\alpha$. Thus, μ_2 is the Hopf invariant.

Since $\psi^2\psi^3(\alpha) = \psi^3\psi^2(\alpha)$, then $2^m(2^{m-1})\mu_3 = 3^m(3^m-1)\mu_2$; the right-hand side is odd because we wanted the Hopf invariant to be odd, and 2^m has to divide it, so $2^m \mid 3^m - 1$, which (one can check) implies m is one of 1, 2, 4, or 8. ⊠

Lecture 9.

Flag Manifolds and Fredholm Operators: 9/24/15

"I see confused faces... speak now."

Next week, the professor will be gone, and Tim Perutz will deliver two lectures about Morse theory and its use in a proof of Bott periodicity. But today, we'll finish talking about flag manifolds and then introduce Fredholm operators, which we'll talk about for a few weeks.

Last time, we promoted K-theory to a cohomology theory; the following result illustrates how one might use that.

Proposition 9.1. *If $n \in \mathbb{Z}^{>0}$, then $K(\mathbb{CP}^n) = K^0(\mathbb{CP}^n)$ is a free abelian group of rank $n+1$, and as a ring $K(\mathbb{CP}^n) \cong \mathbb{Z}[x]/(x^{n+1})$ under the identification $x \mapsto [L]-1$, where $[L]$ is the K-theory class of the tautological bundle $L \to \mathbb{CP}^n$.*

Proof. We'll provide a proof for the group structure; then, check out [16] for the ring structure. The proof will proceed on induction on n, and also show that $K^{\mathrm{odd}}(\mathbb{CP}^n) = 0$.

We have $\mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^n$ by attaching a single $2n$-cell (realizing it as a subcomplex), so we have a sequence $\mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^n \twoheadrightarrow S^{2n}$, and therefore the following long exact sequence.

$$\widetilde{K}^{-1}(\mathbb{CP}^{n-1}) \longrightarrow \widetilde{K}^0(S^{2n}) \longrightarrow \widetilde{K}^0(\mathbb{CP}^n) \longrightarrow \widetilde{K}^0(\mathbb{CP}^{n-1}) \longrightarrow \widetilde{K}^1(S^{2n}) \longrightarrow \cdots$$

But $\widetilde{K}^{-1}(\mathbb{CP}^{n-1}) = 0$ by hypothesis, and $\widetilde{K}^1(S^{2n}) = 0$ by our previous computations, so this is a short exact sequence. We also know that $\widetilde{K}^0(S^{2n}) = \mathbb{Z}$, and by the inductive hypothesis, $\widetilde{K}^0(\mathbb{CP}^{n-1})$ is free of rank $n-1$, so this sequence simplifies to a short exact sequence of abelian groups.

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{K}^0(\mathbb{CP}^n) \longrightarrow \mathbb{Z}^{n-1} \longrightarrow 0.$$

Thus, $\widetilde{K}^0(\mathbb{CP}^n)$ is free of rank n.

For the second half of our inductive assumption, take the following part of the long exact sequence.

$$\widetilde{K}^1(S^{2n}) \longrightarrow \widetilde{K}^1(\mathbb{CP}^n) \longrightarrow \widetilde{K}^1(\mathbb{CP}^{n-1}),$$

but we already know that $\widetilde{K}^1(S^{2n}) = 0$ and $\widetilde{K}^1(\mathbb{CP}^{n-1}) = 0$, so $\widetilde{K}^1(\mathbb{CP}^n) = 0$. ⊠

The result for rings involves figuring out where generators go, and isn't too much more involved.

Theorem 9.2 (Leray-Hirsch)**.** *Let $p : \mathcal{E} \to X$ be a fiber bundle with fiber F, where $\mathcal{E}$ is compact Hausdorff and X is a finite CW complex. Suppose $K^\bullet(F)$ is a free abelian group with basis $f_1, \dots, f_N$ and we have $c_1, \dots, c_N \in K^\bullet(\mathcal{E})$ with $c_i|_{\mathcal{E}_x} = f_i$ for all $x \in X$. Then, $K^\bullet(\mathcal{E}) \cong K^\bullet(X)[c_1, \dots, c_n]$ as a $K^\bullet(X)$-module.*

Proof. Let $X' \subset X$ be a subcomplex, and let $[C^\bullet]^q$ denote the q^{th} degree of the complex $C^\bullet$. Then, we have the following commutative diagram, where $\mathcal{E}' = p^{-1}(X')$.

$$\begin{tikzcd} \cdots \arrow[r] & {[K(X,X') \otimes K(F)]^q} \arrow[r] \arrow[d, "\Psi"] & {[K(X) \otimes K(F)]^q} \arrow[r] \arrow[d, "\Psi"] & {[K(X') \otimes K(F)]^q} \arrow[r] \arrow[d, "\Psi"] & \cdots \\ \cdots \arrow[r] & K^q(\mathcal{E}, \mathcal{E}') \arrow[r] & K^q(\mathcal{E}) \arrow[r] & K^q(\mathcal{E}') \arrow[r] & \cdots \end{tikzcd} \tag{9.1}$$

Here,

$$\Psi\Big(\sum x_i \otimes f_i\Big) = \sum p^*(x_i)c_i$$

for $x_i \in K^\bullet(X)$.

The rows in (9.1) are exact: the top sequence is obtained from the long exact sequence for $X' \subset X$ by tensoring with a free abelian group, and the bottom sequence is the long exact sequence for $\mathcal{E}' \subset \mathcal{E}$. Moreover, the diagram commutes, which you can check from the description of Φ, and is written out more explicitly in [16].

We'll use a typical proof technique: since there are finitely many cells floating around, we can induct on $\dim X$ plus the number of cells in each dimension in order to show that Ψ is an isomorphism.

The inductive step is $X = X' \cup_f D^n$, where $f : S^{n-1} \to X'$. We'll want to apply the five lemma to (9.1); on the right, we have Ψ acting on degree $q-1$, so we win by the inductive assumption, and on the left, the attaching map f gives us $K(X, X') = K(D^n, S^{n-1})$, and therefore a description

$$\begin{tikzcd}
{[K(X,X') \otimes K(F)]^q} \arrow[r, "\cong"] \arrow[d, "\Psi"] & {[K(D^n, S^{n-1})]^q} \arrow[d, "\Psi"] \\
K^q(\mathcal{E}, \mathcal{E}') \arrow[r, "\cong"] & K(D^n \times F, S^{n-1} \times F).
\end{tikzcd} \tag{9.2}$$

In other words, we've reduced to the following box:

$$\begin{tikzcd}
\cdots \arrow[r] & {[K(D^n, S^{n-1}) \otimes K(F)]^q} \arrow[r] \arrow[d, "\Psi"] & {[K(D^n) \otimes K(F)]^q} \arrow[r] \arrow[d, "\Psi"] & {[K(S^{n-1}) \otimes K(F)]^q} \arrow[r] \arrow[d, "\Psi"] & \cdots \\
\cdots \arrow[r] & K^q(D^n \times F, S^{n-1} \times F) \arrow[r] & K^q(D^n \times F) \arrow[r] & K^q(S^{n-1} \times F) \arrow[r] & \cdots
\end{tikzcd}$$

Once again, the rows are exact and the diagram commutes by (9.1), but this time, D^n is contractible, so the blue arrow is an isomoprhism; then, the inductive assumptions give us the other isomorphisms we need for the five lemma, and therefore we get that the right-hand arrow in (9.2) is an isomorphism. Thus, we can apply the five lemma to (9.1), proving the theorem. ⊠

Remark. The same proof works for $H^*(-, R)$ for coefficients in any ring R, and its use in the following discussion on splitting sequences generalizes. We can also remove the assumption that X is a CW complex, though this requires more highbrow techniques such as spectral sequences.

We'll use this to understand how complex subbundles decompose into line bundles. If $E \to X$ is a complex bundle, and we split off a line bundle L_1, so $E \cong L_1 \oplus E_1 \to \mathbb{P}(E)$. The fibers of $\mathbb{P}(E) \to X$ are $\mathbb{CP}^n$, which has free K-theory as we saw above, so we can apply the Leray-Hirsch theorem to the splitting principle.

We also talked about the Adams operations last time. Suppose we have a situation $E \to X$ and $p : \mathbb{F}(E) \to X$, where $p^*E \cong L_1 \oplus \cdots \oplus L_n$, and we have the diagram

$$\begin{tikzcd}
\mathbb{F}(E) \arrow[d, "p"'] & p^*E \arrow[l] \arrow[d] \\
X & E. \arrow[l]
\end{tikzcd}$$

We want to show that for all $k \in \mathbb{Z}^{>0}$, $L_1^{\otimes k} \oplus \cdots \oplus L_n^{\oplus k} = L_1^k + \cdots + L_n^k$ has a K-theory class which descends to X.

When $n = k = 1$, this is silly, so let's consider $n = k = 2$. Here,

$$[L_1^2 + L_2^2] = \underbrace{[(L_1 + L_2)^2]}_{[E]^2 = [E \otimes E]} - \underbrace{2[L_1 \otimes L_2].}_{2[\Lambda^2 E]}$$

Both of these factors descend to E, so we're good. This relies on a useful fact from linear algebra: there's a canonical isomorphism $\Lambda^2(L_1 \oplus L_2) \cong L_1 \otimes L_2$.

To see how beautiful K-theory is as opposed to singular cohomology, consider replacing L_i by its Chern class $c_1 L_i \in H^2(\mathbb{F}(E); \mathbb{Z})$. This involves a nontrivial descent argument, but the exterior powers in K-theory make the argument more smooth (heh) and more geometric.

For the general argument, recall that $\mathbb{Z}[x_1,\dots,x_n]^{\mathrm{Sym}_n} \cong \mathbb{Z}[\sigma_1,\dots,\sigma_n]$, where σ_i is the i^{th} *symmetric polynomial*:

$$\sigma_j = \sum_{i_1<\cdots<i_j} x_{i_1}\cdots x_{i_j}.$$

For example, when $n=3$,

$$\begin{aligned}\sigma_1 &= x_1+x_2+x_3\\ \sigma_2 &= x_1x_2+x_1x_3+x_2x_3\\ \sigma_3 &= x_1x_2x_3.\end{aligned}$$

Crucially, $\sigma_j(L_1,\dots,L_n) = p^*(\Lambda^j E)$, for which the descent argument goes as in the $n=k=2$ case. But we wanted it for $s_k = x_1^k+\cdots+x_n^k$. Thankfully, this is a classical problem, and the solution is the *Newton polynomials*: $s_1=\sigma_1$, $s_2=\sigma_1^2-2\sigma_2$, and in general,

$$s_k-\sigma_1 s_{k-1}+\sigma_2 s_{k-2}-\cdots+(-1)^{k-1}\sigma_{k-1}s_1+(-1)^k k\sigma_k = 0.$$

These ideas are very similar to the theory of characteristic classes for integral cohomology, and similar descent arguments happen.

Another Approach. So far, we've represented an $x\in K(X)$ as the difference between two classes corresponding to complex vector bundles (or real vector bundles for $KO(X)$). But we'd like a more flexible way to use this in geometry, since not everything is a difference of two vector bundles. This is a very important principle for applying algebraic topology to geometry: the greater number of ways you have to realize your objects geometrically, the more powerful your theory is: for example, cohomology shows up whenever you have a CW structure on a topological space, but if you know that de Rham cohomology agrees, then you can use the same ideas in different places to simplify your proofs. Similarly, we want to make K-theory more flexible.

Let H^0 and H^1 be complex vector spaces. Then, a $T: H^0\to H^1$ can be extended to the exact sequence

$$0\longrightarrow \operatorname{Ker} T\longrightarrow H^0\xrightarrow{T} H^1\longrightarrow \operatorname{coker} T\longrightarrow 0.$$

The *cokernel* is $\operatorname{coker} T = H^1/T(H^0)$.

If H^0 and H^1 are finite-dimensional, we want to take an alternating sum, and have it equal to zero for an exact sequence. More generally, for an exact sequence

$$0\longrightarrow E^0\longrightarrow E^1\longrightarrow E^2\longrightarrow\cdots\longrightarrow E^N\longrightarrow 0,$$

we have two alternating-line results:

$$\begin{gathered}\sum_{i=0}^N (-1)^i\dim E^i = 0\\ \bigotimes_{i=0}^N (\det E^i)^{\otimes(-1)^i}\cong\mathbb{C}.\end{gathered}$$

The latter is canonical.

If the E^i are vector bundles over a compact Hausdorff space X, this implies that $\sum_{i=0}^N(-1)^i[E^i]=0$. For example, if $X=\mathbb{R}$ and $H^0=H^1=\underline{\mathbb{C}}$, we can set $x\in X$, $T_x:\mathbb{C}\to\mathbb{C}$ as multiplication by x. Then, the exact sequence degenerates except when $x=0$, where it jumps. There, the K-theory isn't given by a difference of vector bundles… because $\mathbb{R}$ isn't compact.

This is a good motivation to generalize: we can allow H^0 and H^1 to be infinite-dimensional and approach this from the perspective we've outline above. However, we'll still require that the kernel and cokernel are finite-dimensional. A T with that stipulation is called a *Fredholm operator*, and we'll hope to build a K-theory from these operators.

There are a couple wrinkles we'll have to address, though.

- First, for infinite-dimensional vector spaces, we have topology and not just algebra: we want to talk about continuous functionals, not just linear one.
- We need to show that the Fredholms define K-theory classes when X is compact and Hausdorff.
- Then, we'll extend $K(X)$ using Fredholm operators to noncompact X.

- Finally, we'll show these make sense, by using them to prove Bott periodicity. This will bring Clifford algebras into the story, which are quite important.

We'll spend the next four lectures (not counting the two next week, where the professor is absent) on these topics. Two useful references for this section are [2, 22].

So what kind of infinite-dimensional spaces are we going to consider? Norms give us topology, and inner products give us angles (and therefore geometry). So we'll use infinite-dimensional inner product spaces; specifically *Hilbert spaces*: a vector space equipped with a bilinear (or sesquilinear in the complex case), nondegenerate pairing and that is complete.

Definition. If H^0 and H^1 are Hilbert spaces, a linear map $T : H^0 \to H^1$ is *bounded* if there exists a $C > 0$ such that for all $\xi \in H^0$, $|T\xi|_{H^1} \le C|\xi|_{H^0}$.

The following fact and the previous definition are considerably more general than just Hilbert spaces.

Fact. Let H^0 and H^1 be Hilbert spaces. Then, a linear $T : H^0 \to H^1$ is bounded iff it is continuous.

In this case, we may define the *operator norm* $\|T\|$ to be the infimum of C that work to make T bounded. This makes $\mathrm{Hom}(H^0, H^1)$, the set of continuous linear maps, into a *Banach space* (a complete normed space); in general we don't have an inner product, and we can show that if $T_1, T_2 \in \mathrm{Hom}(H^0, H^1)$, then $\|T_2 \circ T_1\| \le \|T_1\|\|T_2\|$. This makes $\mathrm{Hom}(H^0, H^1)$ into a structure called a *Banach algebra*.

We'll define $\mathrm{Hom}(H^0, H^1)^\times \subset \mathrm{Hom}(H^0, H^1)$ to be the subspace of invertible elements, i.e. homeomorphisms, but it turns out we don't need to distinguish between the two.

Theorem 9.3 (Open mapping theorem)**.**
(1) If $T : H^0 \to H^1$ is bounded and bijective, then T^{-1} is bounded.
(2) $\mathrm{Hom}(H^0, H^1)^\times \subset \mathrm{Hom}(H^0, H^1)$ is open (i.e. invertibility is an open condition).

The first part is a standard theorem in functional analysis, and (2) is a fairly easy standard argument.

We'll also use the following theorem.

Definition. A vector space is *separable* if there exists a countable set of vectors such that every $x \in X$ is an *infinite* linear combination of those vectors.

Theorem 9.4 (Kuiper)**.** *If H^0 and H^1 are separable, infinite-dimensional vector spaces, then $\mathrm{Hom}(H^0, H^1)^\times$ is contractible.*

Notice that this isn't true in the finite-dimensional case.

Remark. Let H be a Hilbert space.

- If $V \subset H$ is finite-dimensional, then V is closed.
- If $V \subset H$ is closed, then since we're in a Hilbert space, we can form $V^\perp$, and therefore get a sequence $V^\perp \hookrightarrow H \twoheadrightarrow H/V$, which gives us a Hilbert space structure on $V^\perp$.

Now, we can state the main definition.

Definition. Let H^0 and H^1 be Hilbert spaces and $T : H^0 \to H^1$ be a continuous linear map. Then, T is *Fredholm* if

(1) $T(H^0) \subset H^1$ is closed,
(2) $\ker T \subset H^0$ is finite-dimensional, and
(3) $\operatorname{coker} T$ is finite-dimensional.

It turns out that the first requirement is superfluous.

The idea is that $\mathrm{Hom}(H^0, H^1)$ is a vector space, and therefore contractible; its topology isn't very interesting. But the space of Fredholm operators $\mathrm{Fred}(H^0, H^1)$ has a more interesting topology, and ends up being open. The space of invertible operators sits inside (since then the kernel and cokernel are trivial), and is contractible. But the space of Fredholm is not connected, and the components are indexed by the difference in dimensions of the kernel and cokernel (called the *index*) of the operators in the component. And each component is interesting, having $\pi_{2n} = \mathbb{Z}$ for all n.

We'll study this with open sets: if $W \subset H^1$ is finite-dimensional, then T is nearly surjective on operators, and we can therefore find a W such that T is transverse to it. Then, we'll reverse it, and choose a $W \subset H^1$ and consider the set of Fredholm operators that are transverse to W. This will eventually lead to constructions of K-theory classes.

Lecture 10.

Bott Periodicity and Morse-Bott Theory: 9/29/15

Today's lecture was given by Tim Perutz.

We'll talk about Bott periodicity as proved by Bott, as distinct from how it was proven by later authors.

U will denote the infinite unitary group $\mathrm{U} = \mathrm{U}(\infty) = \bigcup_n \mathrm{U}(n)$. These are infinite matrices which have block form

$$\begin{bmatrix} A & 0 \\ 0 & I_\infty \end{bmatrix},$$

where $A \in \mathrm{U}(n)$ for some n and I_∞ denotes the infinite identity matrix. Note that this is *not* the group of unitary transformations of an infinite-dimensional Hilbert space.

Bott periodicity, in a nutshell, is a homotopy equivalence $\Omega^2 \mathrm{U} \simeq \mathrm{U}$, and therefore isomorphisms $\pi_{k+2}(\mathrm{U}) \cong \pi_k(\mathrm{U})$. In particular, since U is path-connected, then $\pi_{2k}(\mathrm{U}) = \pi_0(\mathrm{U}) = 0$. The odd homotopy groups are $\pi_{2k+1}(\mathrm{U}) = \pi_1(\mathrm{U}) = \mathbb{Z}$ (because $\pi_1(\mathrm{U}(n)) = \mathbb{Z}$ for each n).

Bott talked about this in [8]. Note that this is distinct from stable homotopy theory! This is a very geometric, very down-to-Earth proof, a vindication for actual geometric methods in homotopy theory.

In the 1920s, Morse theory was developed, originally involving geodesics on Riemannian manifolds via calculus of variations. Bott used Morse theory to make detailed calculations of geodesics on Lie groups to prove Bott periodicity. He also obtained similar results for other groups: if $\mathrm{O} = \mathrm{O}(\infty)$ (infinite matrices of the same block form, but with an orthogonal matrix instead of a unitary one) and $\mathrm{Sp} = \mathrm{Sp}(\infty)$ (analogous), then $\Omega^4 \mathrm{O} = \mathrm{Sp}$ and $\Omega^4 \mathrm{Sp} = \mathrm{O}$, and therefore $\Omega^8 \mathrm{O} = \mathrm{O}$. In particular, for all k, $\pi_k(\mathrm{O}) = \pi_{k+8}(\mathrm{O})$ for all k. For example, $\pi_0(\mathrm{O}) = \mathbb{Z}/2$ and $\pi_1(\mathrm{O}) = \mathbb{Z}/2$. $\pi_2(\mathrm{O}) = 0$ (since π_2 of a Lie group is always zero, and we can use $\mathrm{O}(n)$ for our calculations). $\pi_3(\mathrm{O}) = \mathbb{Z}$ (this is true for any simple Lie group). Then, $\pi_4(\mathrm{O}) = \pi_0(\mathrm{Sp}) = 0$, $\pi_5(\mathrm{O}) = \pi_1(\mathrm{Sp}) = 0$, $\pi_6(\mathrm{O}) = \pi_2(\mathrm{Sp}) = 0$ (same deal, it's a Lie group), and $\pi_7(\mathrm{O}) = \pi_3(\mathrm{Sp}) = \mathbb{Z}$, as $\mathrm{Sp}(n)$ is simple. Then, we're back to where we started.

This periodicity was absolutely surprising, and very serendipitous.

This week, these two lectures will cover the unitary case. We'll more or less follow Milnor in [20], but we'll treat loop spaces as actual, infinite-dimensional manifolds.

Definition. A map $f : X \to Y$ of path-connected spaces is called *n-connected* if the induced maps $\pi_k X \to \pi_k Y$ are isomorphisms for $k < n$ and surjective for $k = n$.

Equivalently, for the algebraic topologists in the audience, the homotopy fiber of f is an n-connected space.

Lemma 10.1. *The inclusion* $\mathrm{U}(m) \hookrightarrow \mathrm{U}(m+n)$ *sending*

$$A \longmapsto \begin{bmatrix} A & 0 \\ 0 & I_n \end{bmatrix}$$

is $2m$-connected.

Proof sketch. First, we may without loss of generality assume $n = 1$; then, iterating that result proves it for larger n. In this case, we have a fibration sequence in which $\mathrm{U}(m+1)$ acts on S^{2m+1} inside $\mathbb{C}^{m+1}$, so we get a sequence $\mathrm{U}(m) \to \mathrm{U}(m+1) \to S^{2m+1}$. Then, since $\pi_k(S^{2m+1}) = 0$ for $k \le 2m$, we can invoke the long exact sequence of homotopy groups of this fibration. ⊠

In particular, $\pi_k \mathrm{U}(m) \cong \pi_k \mathrm{U}$ when $k < 2m$ (which is called the *stable range*).

Our next step is to construct maps $j_m : \mathrm{Gr}_m(\mathbb{C}^{2m}) \to \Omega\,\mathrm{SU}(2m)$. let $P_m = \Omega_{I,-I}\,\mathrm{SU}(2m)$, i.e. the space of paths $\gamma : [0,1] \to \mathrm{SU}(2m)$ where $\gamma(0) = I$ and $\gamma(1) = -I$. That is, P_m is the space of paths from I to $-I$. Then, we'll think of the Grassmanian as follows. There is a canonical homeomorphism between $\mathrm{Gr}_m(\mathbb{C}^{2m})$ and the space of Hermitian matrices $A \in \mathrm{Mat}_{\mathbb{C}}(2m, 2m)$ whose eigenvalues are 1 and -1, each with multiplicity m.

In the reverse direction, send $A \mapsto \ker(A - I)$, and in the forward direction, we want to write an m-dimensional subspace as a matrix that acts as I on that subspace and $-I$ on its orthogonal complement, which will be Hermitian.

Now, we'll define a map $i_m : \mathrm{Gr}_m(\mathbb{C}^{2m}) \to P_m$ sending $A \mapsto (t \mapsto \exp(i\pi t A))$: $i\pi t A$ defines a one-parameter subgroup of A, and the conditions on the eigenvalues of A mean that this path starts at I and goes to $-I$.

Then, we can take some reference path β in $\mathrm{SU}(2m)$ from $-I$ to I, so $a_m : P_m \to \Omega\,\mathrm{SU}(2m)$ sending $\gamma \mapsto \beta \circ \gamma$ is a homotopy equivalence. Finally, we'll let $j_m = a_m \circ i_m$.

Theorem 10.2. *j_m is $(2m+1)$-connected.*

That is, the low-degree homotopy groups of the Grassmanian agree with those of the special unitary groups.

Before proving the theorem, we'll digress to talk about how this proves Bott periodicity. Theorem 10.2 provides a relationship between homotopy groups of the Grassmanian and those of unitary groups, but more classical homotopy theory provides other relationships between these groups. We can construct a map $\eta_m : \Omega\,\mathrm{Gr}_m(\mathbb{C}^{2m}) \to \mathrm{U}(m)$ as follows: take the tautological vector bundle $\mathbb{C}^m \to V \to \mathrm{Gr}_m(\mathbb{C}^{2m})$ (the fiber over a subspace L in the Grassmanian is just that subspace). Then, choose a Hermitian metric in V and therefore a Hermitian connection ∇.

If we're given a $\gamma \in \Omega\,\mathrm{Gr}_m(\mathbb{C}^{2m})$, so that $\gamma : S^1 \to \mathrm{Gr}_m(\mathbb{C}^{2m})$ is a based map, then we have a pullback vector bundle $\gamma^* V \to S^1$ and a pullback connection $\gamma^*\nabla$. Then, we'll write that $\eta_m(\gamma)$ is the holonomy of $\gamma^*\nabla$, and this is in $\mathrm{U}(m)$. Specifically, we'll try to trivialize this vector bundle over $[0,1)$; the holonomy is the discrepancy at the basepoint (i.e. at 0 and 1), which is a unitary matrix.

Proposition 10.3. *η_m induces isomorphisms on π_k for $k \le 2m+1$.*

The proof will be omitted, but isn't too difficult: you'll write down a homotopy long exact sequence again. It's an instance of the following general fact.

Fact. Let G be a Lie group and BG be its classifying space. Then, $\Omega BG \xrightarrow{\sim} G$ (the classifying space has a canonical principal G-bundle, and the identification is obtained by pulling back to the circle and taking holonomy).

Finally, we use this to obtain Bott periodicity:

Theorem 10.4 (Bott periodicity)**.** *There exists a map $\mathrm{U}(m) \to \Omega^2\,\mathrm{U}(2m)$ inducing isomorphisms on π_k for $k < 2m+2$, and hence $\pi_k\,\mathrm{U} \cong \pi_{k+2}\,\mathrm{U}$.*

Proof sketch. We have $\eta_m : \Omega\,\mathrm{Gr}_m(\mathbb{C}^{2m}) \to \mathrm{U}(m)$. The first thing we'll do is choose a map $\kappa_m : \mathrm{U}(m) \to \Omega\,\mathrm{Gr}_m(\mathbb{C}^{2m})$ which is approximately a homotopy inverse to η_m: specifically, that κ_m and η_m are inverses on π_k for $k < 2m+2$. This is possible thanks to a version of the Whitehead theorem. Moreover, we have a map $\Omega j_m : \Omega\,\mathrm{Gr}_m(\mathbb{C}^{2m}) \to \Omega^2\,\mathrm{SU}(2m)$, and an inclusion $\iota : \mathrm{SU}(2m) \hookrightarrow \mathrm{U}(2m)$, which is an isomorphism on π_k for all $k > 1$, thanks to the fibration

$$\mathrm{SU}(2m) \hookrightarrow \mathrm{U}(2m) \xrightarrow{\det} \mathrm{U}(1).$$

Thus, we have the following system of maps.

$$\begin{tikzcd} \mathrm{U}(m) \arrow[r, bend right, "\kappa_m"'] & \Omega\,\mathrm{Gr}_m(\mathbb{C}^{2m}) \arrow[l, bend right, "\eta_m"'] \arrow[r, "\Omega j_m"] & \Omega^2\,\mathrm{SU}(2m) \arrow[r, "\Omega^2 \iota"] & \Omega^2\,\mathrm{U}(2m) \end{tikzcd}$$

Theorem 10.2 and Proposition 10.3 then prove that the composition of these maps induces the identity on the homotopy groups we need. ⊠

One unfortunate consequence of this proof is that we don't know how to use this to get generators of the maps. It would be an interesting exercise, but this is one of the advantages of the other proofs of Bott periodicity.

Today, we won't prove Theorem 10.2, but we'll talk about the mechanism of the proof of this theorem, which involves Morse-Bott theory. Though we want to talk about $P_m = \Omega_{I,-I}\,\mathrm{SU}(2m)$, which is an infinite-dimensional manifold, let's start with the finite-dimensional case.

Let M be an n-dimensional manifold and $f \in C^\infty(M)$. Let $\mathrm{crit}(f) = \{c \in M : D_c f = 0\}$, the set of critical points. For all $c \in \mathrm{crit}(f)$, we have a *Hessian* $D_c^2 f : T_cM \times T_cM \to \mathbb{R}$, which is a symmetric bilinear form defined as the second derivative in any coordinate chart centered at c (which is sort of a cheap definition, but suffices, and is indeed independent of the chart).

Definition. The *index* of a critical point $c \in \operatorname{crit}(f)$, denoted $\operatorname{ind}(f; c)$, is the index (or signature) of $D_c^2 f$, i.e. the dimension of the maximal negative-definite subspace of $T_c M$ with respect to $D_c f$ (i.e. its induced inner product).

If $\operatorname{crit}(f)$ is a submanifold of M, then f is locally constant on it, and hence the Hessian descends to the *normal spaces* $N_c = T_c M / T_c(\operatorname{crit} f)$, so we have a pairing $D_c^2 f : N_c \times N_c \to \mathbb{R}$. This doesn't change the index (we just quotiented out by a space where the form was zero).

If C is a connected component of $\operatorname{crit}(f)$, we'll write $\operatorname{ind}(f; C) = \operatorname{ind}(f, c)$ for any $c \in C$ (since it's locally constant on $\operatorname{crit}(f)$), particularly in the Morse-Bott case below.

Definition. f is said to be *Morse-Bott* if

(1) $\operatorname{crit}(f)$ is a submanifold of M, and
(2) for all $c \in \operatorname{crit}(f)$, the Hessian $D_c^2 f : N_c \times N_c \to \mathbb{R}$ is a non-degenerate bilinear form.

Note that for a function to be Morse, the Hessian must not be degenerate on the tangent space, and being Morse-Bott means that we can have some degeneracy, but it must vanish outside of the critical points.

Theorem 10.5 (Morse-Bott). *Let M be an n-dimensional manifold, and assume the following.*

- *We have an $f \in C^\infty(M)$ that is not only Morse-Bott, but also proper*[12] *and bounded below.*
- *If $C_{\min}$ denotes the* manifold of local minima *of f, which is part of* $\operatorname{crit}(f)$*; we'll want to assume $C_{\min}$ is connected.*
- *There's an ℓ such that for all conneced components C of* $\operatorname{crit}(f)$ *other than $C_{\min}$,* $\operatorname{ind}(f; C) > \ell$.

Then, the inclusion $C_{\min} \hookrightarrow M$ is ℓ-connected.

The idea is that all of the π_ℓ of M should come from that of $C_{\min}$. Examples won't be terribly useful right now.

We'd love to apply this to the case $M = P_m$, f is the Riemannian energy functional, and $C_{\min}$ is a path space that will be identified with the Grassmanian, but of course P_m isn't finite-dimensional. The statement is still true, of course, but just requires more work.

First, let's set up the proof. Choose a Riemannian metric g on M, so that we have a gradient vector field $\operatorname{grad} f$. Then, $g(\operatorname{grad} f, v) = df(v)$, so if $\gamma : [0, 1] \to M$, we get a nice ODE

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = -(\operatorname{grad} f) \circ \gamma. \tag{10.1}$$

The intuition is that if M is embedded in $\mathbb{R}^N$ so that f is a height function,[13] then the gradient indicates the direction of greatest increase of the function.

Then, $\operatorname{grad} f$ defines a flow $\phi_t : M \to M$, and $t \mapsto \phi_t(x)$ is a solution to (10.1), and exists at least locally, by general nonsense about differential equations. But since f is proper and bounded below, then ϕ_t exists for all $t \geq 0$! This is because the negative gradient flow points into $f^{-1}(-\infty, c]$, which is compact, so by standard long-time existince theorems on ODEs, the flow exists for all positive times.[14]

Moreover, for all starting points $x \in M$, the limit $x_\infty = \lim_{t\to\infty} \phi_t(x)$ exists, again basically due to compactness (though it does use the Morse-Bott hypothesis).

Definition. For a connected component $C \subset \operatorname{crit}(f)$, define the *stable manifold* $S_C = \{x \in M : x_\infty \in C\}$.

The stable manifold is the set of points that flows into C eventually (e.g. rolling downhill in the height function).

Lemma 10.6. *S_C is a submanifold of M, and has codimension* $\operatorname{ind}(f; C)$.

This is hard to prove.

So we want to prove that in the conditions assumed in Theorem 10.5, the manifold of minima contains all of the information about the low-dimensional homotopy groups.

[12] A smooth function f is *proper* if $f^{-1}(-\infty, c]$ is compact for all c.

[13] Though this picture is primarily for intuition, the Whitney embedding theorem means that for sufficiently large N, this is possible.

[14] There's no guarantee that it'll exist for all negative time, though.

Proof of Theorem 10.5. Recall that if $C \neq C_{\min}$ is a connected component of $\operatorname{crit}(f)$, then $\operatorname{ind}(f; C) > \ell$. Then, take a based map $f : S^k \to M$ where $k \leq \ell$ and the basepoint of M is taken to be in $C_{\min}$; we want to show this is homotopic to a map into $C_{\min}$.

Transversality theory tells us that h is based homotopic to a map transverse to S_C for all connected components of $\operatorname{crit}(f)$. But $\operatorname{Im}(h) \cap S_C = \emptyset$ for $C \neq C_{\min}$, as $\operatorname{Im}(h)$ has dimension at most ℓ and S_C has dimension at least ℓ, so their intersection in the general case has to be empty (i.e. we can adjust h a little bit to get an empty intersection).

Now, let $h_t : S^k \to M$ be a based map defined by $h_t = \phi_t \circ h$: we take our sphere, and flow it downwards. Notice that for all $x \in S^k$, $h(x)_\infty \in C_{\min}$, and so for $t \gg 0$, $\operatorname{Im}(h_t)$ lies in a tubular neighborhood of $C_{\min}$, which deformation retracts to $C_{\min}$. Hence, h is homotopic to some map $S^k \to C_{\min}$.

The next step is to show that $C_{\min} \hookrightarrow M$ induces injections on π_k for $k < \ell$; take an $h : S^k \to C_{\min}$ that extends to an $H : B^{k+1} \to M$, so we need to find a homotopy relative to the boundary that maps it to $C_{\min}$. As before, we may assume that h is transverse to the S_C (thanks to the relative transversality theorem), and then run the same argument; we've chosen the dimensions so that once again, it can't hit the stable manifolds except for $S_{C_{\min}}$, and so flowing once again gives us a homotopy. ⊠

Our task for next time is to run a version of this argument in the infinite-dimensional loop space.

Lecture 11.

Bott Periodicity and Morse-Bott Theory II: 10/1/15

Recall that last time, we deduced periodicity of $\pi_k \mathrm{U}$ from Theorem 10.2, which defined a map $j_m : \mathrm{Gr}_m(\mathbb{C}^{2m}) \to \Omega_{I,-I}\,\mathrm{SU}(2m)$ and showed that it is $(2m+1)$-connected (and therefore an isomorphism on π_k for $k < 2m+1$, and surjective for $k = 2m+1$). But we still haven't proven Theorem 10.2.

We also talked about Morse-Bott theory: we assumed M is a connected manifold, $f : M \to \mathbb{R}$ is a Morse-Bott function (a condition relating to the nondegeneracy of the critical manifolds) that is bounded below, and the indices of the critical manifolds are 0 for $C_{\min}$ and otherwise greater than some ℓ. Then, Theorem 10.5 proved that the inclusion $C_{\min} \hookrightarrow M$ is ℓ-connected.[15]

We also assumed that M was finite-dimensional and f was proper. These are the tricky assumptions: we want to apply this theorem to the Riemannian energy functional E on $\Omega_{I,-I}\,\mathrm{SU}(2m)$, with the goal of identifying $C_{\min}$ with the Grassmanian, identifying j_m with inclusion. Specifically, $C_{\min}$ for the energy functional is the space of *minimal geodesics*, the critical points are more general geodesics, and the nonzero indices will turn out to be at least $2m+2$. If we can do that, then we get the main theorem, Theorem 10.2.

However, this isn't a finite-dimensional manifold, and the energy functional isn't proper, so applying these assumptions would be a little preposterous.

Definition. A *Hilbert manifold* M is a structure akin to a smooth manifold, but in which every point has a neighborhood diffeomorphic to some separable Hilbert space H, which may be infinite-dimensional. One hears that M is *modeled on* H.

Similarly, one defines *Banach manifolds* as modeled on a Banach space and *Fréchet manifolds* as modeled on Fréchet spaces.

Theorem 11.1. *The conclusion of Theorem 10.5 still holds under the following, more general conditions:*
- *M is a Hilbert manifold,*
- *the indices of the critical points of f are finite,*
- *there is a Riemannian metric on M for which the downward gradient flow ϕ_t (satisfying (10.1)) exists for all $t \geq 0$, and*
- *$x_\infty = \lim_{t\to\infty} \phi_t(x)$ always exists.*

With this theorem, we diverge slightly from Milnor's treatment in [20]. The theorem is probably also true for Banach manifolds.

Proof. The proof is roughly as before; we'll homotope maps $S^k \to M$ into $C_{\min}$ using ϕ_t, as long as $k \leq \ell$. Formally speaking, the proof is identical, but what assumptions did we lean on?

[15] Note that, though we assumed in Theorem 10.5 that $C_{\min}$ was connected, this hypothesis isn't really necessary; showing that the map is 1-connected implies an isomorphism on π_0, and therefore $C_{\min}$ is connected because M is.

First, we needed that the stable manifolds S_C of the connected components C of $\operatorname{crit}(f)$ were submanifolds of M, with codimention $\operatorname{ind}(f;C)$. This remains true: Jost proves in [18] that S_C is injectively immersed in M and the result on indices is locally true, from which the global result follows.

The second thing we need is that $h : S^K \to M$ is transverse to S_C, and for $H : B^{k_1} \to M$, we want $H \pitchfork S_C$. For H, though, we want to leave it untouched on the boundary if it's already transverse there. This is proven in [14, Ch. 4].

The rest of the proof is exactly the same, thanks to the assumptions we made. ⊠

Path Spaces. Now, we need to show that our energy functional satisfies these requirements, so let's talk about path spaces.

Definition. Let (M, g) be an n-dimensional Riemannian manifold and $p, q \in M$. Then the *path space* is defined as

$$\Omega_{p,q} = \Omega_{p,q}(M) = \{\gamma : [0,1] \to M \mid \gamma(0) = p, \gamma(1) = q\}.$$

We can take γ to be C^0, giving $\Omega_{p,q}$ the compact-open topology, but we'll want more regularity. One could take γ to be C^∞ (or piecewise C^∞, which [20] does), or to be C^k for some k (which is nice because these functions form a Banach space, whereas the space of C^∞ paths is merely a Fréchet space).

Often, one chooses paths in the *Sobolev space* L^2_k, for $k \geq 1$. This is defined to be the space of paths which have k derivatives in L^2. This is a common approach in modern analysis, and will create Hilbert spaces.

All of these spaces have a natural topology, and since continuous functions can approximate C^k or smooth functions, all of these topologies have the same homotopy type, so in some sense, it doesn't matter; it's just where you want to do the work. In each case, we get some kind of infinite-dimensional manifold.

Let's take the C^∞ case; we'll start by defining our tangent spaces.

Definition. For a $\gamma \in \Omega_{p,q}$, define the *future tangent space* to $\Omega_{p,q}$ at γ to be the set T_γ of vector fields along γ that vanish at the endpoints.

That is, these are sections ξ of $\gamma^* TM \to [0,1]$, where $\xi(0) = \xi(1) = 0$.

Next, we'll define charts. Let $U \subset TM$ be any open neighborhood of the zero-section $M \subset TM$ such that the exponential map $\exp_g : TM \to M$ is an embedding on $U \cap T_x M$ for all $x \in M$, and let $U_\gamma = \{\xi \in T_\gamma \mid \xi(t) \in U \text{ for all } t \in [0,1]\}$. Then, we have a chart $U_\gamma \to \Omega_{p,q}$ sending $\xi \mapsto \exp_g \circ \xi$.

Fact. These are charts for a C^∞ Fréchet manifold structure.

We're not going to get bogged down into transition maps.

In the Sobolev case, we take $\gamma \in C^\infty$ and define $T_\gamma = \left\{\xi \in L^2_1(\gamma^* TM) \mid \gamma(0) = \gamma(1) = 0\right\}$ (L^2_1 is a subset of the continuous sections of $\gamma^* TM$). Then, T_γ is the completion of the space of C^∞ vector fields with respect to the Sobolev norm

$$\|\xi\|^2_{L^2_1} = \int_0^1 (g(\xi,\xi) + g(\nabla_t \xi, \nabla_t \xi))\, \mathrm{d}t,$$

where ∇_t is the covariant derivative for g.

This gives us a Hilbert space (relating to the Sobolev embedding $L^2_1 \subset C^0$).

Remark. The analytic tools that we use here can be worked around for Bott periodicity, but they're often very useful in topology and geometry, especially when dealing with infinite-dimensional spaces, and are unavoidable in other important proofs. For example, there's a theorem that if the fundamental group of a manifold has an unsolvable word problem, then there are powerful results on the number of certain kinds of geodesics on that manifold.

Then, the Sobolev path space $\Omega^{L^2_1}_{p,q}$ is contained in the C^0 path space $\Omega^{C^0}_{p,q}$, and so we can think of these as somewhat smooth paths, with the Hilbert space structure around when we need it. Thus, we get a C^∞ Hilbert manifold modeled on $L^2_1(\mathbb{R}^n)$.

Next, we need to address the energy functional. We can put a Riemannian metric on $\Omega_{p,q}$ by defining on each tangent space T_γ the inner product

$$\langle \delta_1, \delta_2 \rangle = \int_0^1 g(\delta_1(t), \delta_2(t))\, \mathrm{d}t,$$

and then $\|\delta\|^2 = \langle\delta,\delta\rangle$. Note that this is *not* the same as the norm induced from the Sobolev structure. Then, the energy function is $E(\gamma) = (1/2)\|\dot\gamma\|^2$ (akin to kinetic energy), producing a function $E : \Omega_{p,q} \to \mathbb{R}_+$.[16]

Morse Theory of the Energy Functional. The next step is to address the Morse theory of E, which is basically calculus of variations from another perspective. There are a lot of calculations which we don't have time for, so we'll state their results.

From the metric we get a gradient, which should increase the energy functional as much as possible. Specifically, we'll define $\operatorname{grad}(E) = -\nabla_t(\dot\gamma)$: first differentiate γ, and then take its covariant derivative (specifically, with respect to the pullback by γ of the Levi-Civita connection on M).

This means that $\operatorname{crit} E = \{\gamma \mid \nabla_t\dot\gamma = 0\}$, and by definition these are geodesics. We get a downward gradient flow $\Gamma : I \to \Omega_{p,q}$ (where I is an interval) defined by $\Gamma : I \times [0,1] \to M$: $\Gamma(s,0) = p$ and $\Gamma(s,1) = q$. Thus, the equation

$$\partial_s\Gamma + \nabla_t(\partial_t\Gamma) = 0$$

is a PDE on $I \times [0,1]$. In fact, it's parabolic: it looks like the heat equation, so solutions exist for positive time (though perhaps not negative time). Thus, the flow ϕ_t exists for all $t \geq 0$ and $\lim_{t\to\infty}\phi_t(x)$ exists (both are proven in [18]); this relates to a property called the *Palais-Smale condition.* So the point is: the gradient flow is great, so long as you don't try to run it backwards.

The next thing we need is the Hessian. We can apply the Hessian $D^2_\gamma E$ to a pair of tangent vectors, but it's convenient to recast that in terms of a self-adjoint linear operator $H_\gamma : T_\gamma \to T_\gamma$, i.e.

$$\langle H_\gamma(\delta_1), \delta_2\rangle = (D^2_\gamma E)(\delta_1,\delta_2).$$

This ends up meaning that

$$H_\gamma(\delta) = -(\nabla_t\nabla_t\delta + R(\dot\gamma,\delta)\dot\gamma), \tag{11.1}$$

where $R(X,Y)Z = (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})Z$ is the *Riemann curvature tensor.* (11.1) is called the *second variational equation*, and is a second-order linear ODE.

Elements of the kernel of H_γ, which are zeros of (11.1), are called *Jacobi fields* (vanishing at the endpoints); these are standard in Riemannian geometry. But (11.1) is also the equation that linearizes the geodesic equation! So formally, $\ker H_\gamma = T_\gamma(\operatorname{crit} E)$, and this actually makes geometric sense if $\operatorname{crit} E$ is a submanifold of $\Omega_{p,q}$: the only degeneracies are tangent to the critical submanifold.

That is, if $\operatorname{crit} E$ is a submanifold, then E is Morse-Bott.

I know this is a long story, but we have yet one more ingredient.

Definition. For $\gamma \in \operatorname{crit} E$ and $t \in [0,1]$, define the *multiplicity* $\operatorname{mult}(\gamma;t)$ to be the dimension of the space of solutions to (11.1) on $[0,t]$ that vanish at the endpoints.

That is, we just restrict to $[0,t]$ instead of $[0,1]$.

Theorem 11.2 (Morse index theorem)**.** $\operatorname{ind}(E;\gamma)$ *is finite, and moreover*

$$\operatorname{ind}(E;\gamma) = \sum_{t\in(0,1)} \operatorname{mult}(\gamma;t).$$

Thus, the multiplicity is 0 for all but finitely many γ. When it's nonzero, the $\gamma(t)$ are called *conjugate points* for γ. This is proven in the piecewise-smooth case in [20], and there are many other proofs in the literature, some more analytic than others.

In the case of a compact Lie group G, let g denote the left-invariant metric, and $p = e$ be the identity, so that $T_pG = \mathfrak{g}$ is the Lie algebra. In this case, all of this analysis reduces to linear algebra on $\mathfrak{g}$.

In fact, it turns out that the geodesics $\gamma : \mathbb{R} \to G$ with $\gamma(0) = e$ are the one-parameter subgroups! In other words, the geodesic requirement means that γ must be a homomorphism. Then, these one-parameter subgroups are in bijection with $\mathfrak{g}$. Then, E is Morse-Bott, and it's fairly easy to check that $\operatorname{crit} E$ is a submanifold.

One can get a reasonable concrete understanding of the Jacobi field equation in this case: if X, Y, and Z are left-invariant vector fields (so elements of $\mathfrak{g}$), then the curvature tensor simplifies to

$$R(X,Y)Z = \frac{1}{4}[[X,Y],Z],$$

[16] Here, $\dot\gamma = \frac{\mathrm{d}\gamma}{\mathrm{d}t}$, just the ordinary derivative, giving us a vector field along γ, though it probably doesn't vanish at the endpoints.

and so the Jacobi field equation boils down to something happening in the Lie algebra. Specifically, define $K_\xi \in \operatorname{End}\mathfrak{g}$ for a $\xi \in \mathfrak{g}$ as follows: if $\eta \in \mathfrak{g}$, then $K_\xi(\eta) = R(\xi, \eta)\xi = (1/4)[\xi, \eta], \eta]$. Then, the conjugate points along $\gamma(t) = \exp(t\xi)$ are the points $\exp(t\xi)$ where $t - \pi k/\sqrt{\lambda}$, where k is a nonzero integer and λ is a positive eigenvalue of K_ξ; then, the multiplicity of γ at t is the multiplicity of λ! Thus, these computations are just linear algebra in the end.

Now, let's specialize a little further, to $\Omega_{I,-I}\operatorname{SU}(2m)$. Finally. What are our geodesics? They take the form $t \mapsto \exp(t\xi)$ for some $\xi \in \mathfrak{g} = \mathfrak{su}(2m)$, and when $t = 1$, we need to have $\exp(\xi) = -I$. Thus, $\operatorname{crit} E \cong \{\xi \in \mathfrak{g} \mid (1/i\pi)\xi \text{ has odd integer eigenvalues}\}$.

Let's say that $\xi/i\pi$ is conjugate to a diagonal matrix with entries $k_1, \ldots, k_{2m}$ that are odd integers. Then, using Theorem 11.2,

$$\operatorname{ind}(E; \xi) = \sum_{k_i > k_j} k_i - k_j - 2,$$

and therefore the index is only zero if $\xi/i\pi$ is conjugate to something where if $k_i > k_j$, their difference must be equal to 2, which is only true if m of them are 1 and the rest are -1 (these are the only options, since $\mathfrak{su}(2m)$ is the set of trace-free Hermitian matrices). In particular, this (m, m) block structure means that $C_{\min} \cong \operatorname{Gr}_m(\mathbb{C}^{2m})$, and if the index is positive, then after playing around with it for a few minutes, then it has to be at least $2m + 2$.

At this point, we can apply Theorem 11.1 to get Theorem 10.2.

Part 2. *K*-theory Represented by Fredholm Operators

Lecture 12.

Fredholm Operators and *K*-theory: 10/6/15

"That's one of the things Jean-Pierre Serre mocks."

Professor Freed is back, and we're going to talk about Fredholm operators again.

We'll talk about separable, complex Hilbert spaces H^0 and H^1 in this class, but everything should also work in the real case. Recall that a $T : H^0 \to H^1$ is Fredholm if

(1) T has closed range, and
(2) $\ker(T)$ and $\operatorname{coker}(T)$ are finite-dimensional.

It turns out that the first property follows from the second, but that's okay. If T is Fredholm, we define its index to be $\operatorname{ind} T = \dim\ker(T) - \dim\operatorname{coker}(T)$. Where does the minus sign go? It can be confusing. If V and W are finite-dimensional, $\operatorname{Hom}(V, W) \cong W \otimes V^*$, so maybe remember that V^* is where the cokernel lives, and the star is a reminder to take the minus sign.

Example 12.1. We talked earlier about how K-theory is about making algebraic topology out of linear algebra; one can step back from vector spaces to modules over a ring, and one can do K-theory there, too.

(1) Let H have an orthonormal basis $e_1, e_2, \ldots$, and for $k \in \mathbb{Z}$, define
$$T_k(e_i) = \begin{cases} e_{i-k}, & i - k \geq 1 \\ 0, & i - k \leq 0. \end{cases}$$
This operator, called the *shift operator*, shifts every basis element to the left k places, and zeroes out the ones that go past e_0. Then, $\operatorname{ind} T_k = k$, since it is surjective, and its kernel has rank k. Recall that every $\xi \in H$ has the form $\xi = \sum_{i=1}^\infty a^i e_i$, where the sum of the $|a^i|^2$ is finite.
(2) If $H^1 = L^2(S^1, \mathrm{d}x)$, then let $T = i\frac{\mathrm{d}}{\mathrm{d}x}$. (The i makes it formally self-adjoint.) This is an unbounded (so not continuous) differential operator. However, we can take the Sobolev space $H^0 = L^2_1(S^1, \mathrm{d}x)$, which is the space of L^2 functions whose first derivatives are also in L^2. Then, $T : H^0 \to H^1$ is bounded, and also Fredholm, with index 0. This is true more generally: an elliptic differential operator on a manifold is Fredholm on some Sobolev space.
(3) We can also define families of Fredholms by maps $X \to \operatorname{Fred}(H^0, H^1)$, which occur naturally in geometry. Let Σ be a compact Riemann surface and Y be a complex manifold, and we'll consider the space $C^\infty(\Sigma, Y)$ of smooth maps $f : \Sigma \to Y$. Such an f determines a Fredholm operator $\overline{\partial}_f : \Omega^{0,0}_\Sigma(f^*TY) \to \Omega^{0,1}_\Sigma(f^*TY)$ (i.e. from functions on Σ of the pullback of the tangent bundle to 1-forms). Again, we need to take the Sobolev completions L^2_1, but then each of these is Fredholm, so

we have a family of Fredholm operators. Interestingly, the Hilbert space itself depends on f here: the Hilbert spaces are also moving in a locally trivial way.

(4) There is a nonlinear Fredholm operator, as outlined in [?] (TODO: cite Smale), related to the previous example: given a vector bundle $\mathcal{E}$ over $C^\infty(\Sigma, Y)$, we get a section $\overline{\partial} f$ for an $f \in C^\infty(\Sigma, Y)$. One defines this to be Fredholm if all of its differentials are, which does hold in this case. We'll see another example akin to this later, with loop groups.

Since all (infinite-dimensional) separable complex Hilbert spaces are isomorphic, we can talk generally about the index function $\operatorname{ind} : \operatorname{Fred}(H^0, H^1) \to \mathbb{Z}$; in fact, $\operatorname{ind} : \pi_0 \operatorname{Fred}(H^0, H^1) \to \mathbb{Z}$ is an isomorphism.

Recall that $T \pitchfork W$ for a $W \subset H^1$ if $T(H^0) + W = H^1$ (said T is *transverse* to W). Then, we can define $\mathcal{O}_W = \left\{T \in \operatorname{Fred}(H^0, H^1) : T \pitchfork W\right\}$.

Lemma 12.2.

(1) $\mathcal{O}_W$ *is open.*
(2) $\left\{\mathcal{O}_W : W \subset H^1 \text{ is finite dimensional}\right\}$ *is an open cover of* $\operatorname{Fred}(H^0, H^1)$.
(3) If $T : X \to \operatorname{Fred}(X^0, X^1)$ *for a compact Hausdorff* X*, then* $T(X) \subset \mathcal{O}_W$ *for some* W.

In other words, our set of possible $\mathcal{O}_W$ is a canonical (albeit uncountable) open cover of $\operatorname{Fred}(H^0, H^1)$. The last part of the lemma provides some nice conditions on families of Fredholm operators coming from compact spaces.

Proof sketch. For (1), $\mathcal{O}_W$ is open iff the composition $H^0 \xrightarrow{T} H^1 \to H^1/W$ is surjective. Suppose $T_0 \in \mathcal{O}_W$; then, if T is Fredholm, then

$$(T_0^{-1}(W))^\perp \hookrightarrow H^0 \xrightarrow{T} H^1 \longrightarrow H^1/W$$

is an isomorphism, because $\operatorname{Im}(T)$ necessarily contains $T(T_0^{-1}(W))$, and $\mathcal{O}_W$ has the transverseness condition we need.[17] Since $\operatorname{Fred}(H^0, H^1) \to \operatorname{Hom}(T_0^{-1}(W)^\perp, H^1/W)$ is continuous, and the preimage of an open set is open.

For (2), this isn't saying much: any Fredholm operator comes with finite-dimensional subspaces attached to it. Then, (3) follows by taking a finite subcover (see the course notes for a full proof). ⊠

Corollary 12.3. *If* $T \in \mathcal{O}_W$*, then the following sequence is exact.*

$$0 \longrightarrow \ker(T) \longrightarrow T^{-1}(W) \xrightarrow{T} W \longrightarrow \operatorname{coker}(T) \longrightarrow 0 \tag{12.1}$$

Thus, $\ker(T) \oplus W \cong \operatorname{coker}(T) \oplus T^{-1}(W)$.

The last conclusion follows because the alternating sum of a bounded exact sequence is trivial (followed by a diagram chase). That is, in an intuitive sense, $\ker(T) - \operatorname{coker}(T)$ is the same as $T^{-1}W - W$. So the index can be given in terms of W, which is constant on an open neighborhood $\mathcal{O}_W$. We want to think of this as a difference of vector bundles.

Lemma 12.4. *The vector bundle* $K_W \to \mathcal{O}_W$ *defined by* $(K_W)_T = T^{-1}(W)$ *is locally trivial.*

Proof. Fix a $T_0 \in \mathcal{O}_W$ and let $p : H^0 \to T_0^{-1}W$ be orthogonal projection. Then, there's an open neighborhood on which (12.1) is an isomorphism, so p restricts to an isomorphism $T^{-1}W \to T_0^{-1}W$. Thus, $K_W \to \mathcal{O}_W$ is locally constant. ⊠

Corollary 12.5. $\operatorname{ind} : \operatorname{Fred}(H^0, H^1) \to \mathbb{Z}$ *is locally constant.*

The idea is that a Fredholm operator adds some finiteness: on an open set, we have a finite model for a Fredholm operator. The infinite-dimensional pieces are isomorphic, and therefore we care about the finite-dimensional parts Kuiper's theorem also gives us a nice handle on the topology. We can't consider only a single Fredholm operator, since the dimensions of the kernels and cokernels may grow, but we at least have that it's locally constant.

Lemma 12.6. *If* H *is a Hilbert space and* $T_1, T_2 \in \operatorname{Fred}(H, H)$*, then* $T_2 \circ T_1 \in \operatorname{Fred}(H, H)$ *and* $\operatorname{ind} T_2 \circ T_1 = \operatorname{ind} T_2 + \operatorname{ind} T_1$.

[17] This is not an if and only if; the converse is not true.

Proof. If $T_2 \circ T_1 \pitchfork W$, then $T_2 \pitchfork W$ and $T_1 \pitchfork T_2^{-1}W$, so

$$\begin{aligned}\operatorname{ind} T_2 \circ T_1 &= (\dim((T_2 \circ T_1)^{-1}) - \dim(T_2^{-1}W)) + (\dim(T_2^{-1}W) - \dim W)\\ &= \operatorname{ind} T_1 + \operatorname{ind} T_2.\end{aligned}$$ ⊠

Since the identity is obviously Fredholm, then this turns $\operatorname{Fred}(H, H)$ into a noncommutative monoid.

Now, we can return to K-theory, with the following important result: Fredholm operators give us K-theory on compact, Hausdorff spaces.

Theorem 12.7 (Atiyah-Jänich)**.** *Let X be a compact, Hausdorff space; then, the map* $\operatorname{ind} : [X, \operatorname{Fred}(H, H)] \to K(X)$ *sending* $T \mapsto [T^*K_W] - [\underline{W}]$ *is a well-defined isomorphism of abelian groups, where H is an infinite-dimensional separable complex Hilbert space and $W \subset H$ is finite-dimensional and chosen such that $T_x \pitchfork W$ for all $x \in X$. In particular, $[X, \operatorname{Fred}(H)]$ is an abelian group under composition.*

The picture for Fredholm operators is that the kernels jump discontinuously (though, since invertibility is an open condition, it can only jump in one direction, and is lower semicontinuous), as do the cokernels, but their difference is locally constant!

Proof sketch. We have a bunch of things to show; let's unpack them.

(1) First, ind is well-defined, meaning it's independent of W and invariant under homotopy.
(2) Then, that ind is a homomorphism of monoids, preserving composition.
(3) Then, that ind is surjective.
(4) Finally, that ind is injective. This means it's a bijective monoid homomorphism, and since one is an abelian group, the other has to be, since the multiplicative structure is the same.

To see why ind is independent of W, first see that the finite-dimensional subspaces W are partially ordered under inclusion, so it suffices to show that if $W \subset W'$, then if it holds for W, then it holds for W'. This is some linear algebra with exact sequences.

Recall our differential operator $i\frac{\mathrm{d}}{\mathrm{d}x}$. We want to talk about its eigenvalues and eigenvectors; it's an unbounded operator on L^2, but we can compute that its spectrum is discrete, and in fact is $\mathbb{Z}$. Then, one of these subspaces W is a finite piece, and W' is a larger piece, and so when we take the quotient, things are well-behaved. A general Fredholm operator's spectrum may have continuous or discrete parts; the Fredholm condition only implies that 0 is an isolated point.

A homotopy gives us a cylinder $[0, 1] \times X \to \operatorname{Fred}(H, H)$, but this is compact, so we can find a single W that works.

The monoid homomorphism is tricky, relying crucially on compactness. For surjectivity, you just have to cook up a Fredholm, by mapping between two different, but isomorphic (by Kuiper's theorem) spaces with the right kernel, and this isn't too hard. Injectivity comes from producing a homotopy from the difference of two things mapping to zero into the invertible component, which is contractible. ⊠

The full details of the proof are in the lecture notes. It can get complicated, so try it out with some examples. For example, the shift operator isn't invertible, and if we're mapping to $K(S^1) = \mathbb{Z}$, then the inverse of 1 is -1, so the inverse was formally added to the K-theory, but maybe it's less apparent what the inverse should be in $[X, \operatorname{Fred}(H, H)]$. It turns out your inverse is the adjoint! It probably helps to think about this for a while.

So now we have two ways to think about K-theory: isomorphism classes of vector bundles if X is compact Hausdorff, or mapping into the space of Fredholm operators. But the latter is still defined for more general X, which leads us to make the following definition.

Definition. If X is a paracompact, Hausdorff space, then define $K(X) = [X, \operatorname{Fred}(H, H)]$.

Theorem 12.7 shows us that this is an abelian group, and extends our previous definition.

Now, we can play the same game again, defining $\widetilde{K}(X)$ and therefore $\widetilde{K}^{-n}(X)$ for X pointed and $n \geq 0$, by mapping suspensions of X into $\operatorname{Fred}(H, H)$ (or, equivalently, into loopspaces of $\operatorname{Fred}(H)$). We can do this more generally, e.g. $H^0(X; \mathbb{Z}) = [X, \mathbb{Z}]$, and with suspensions this gives us negative cohomology groups, too (which are, unsurprisingly, zero). But it's less clear how to do this with positive indices: we need to de-loop, or we're stuck with half a cohomology theory.

Last time, we defined the whole thing with Bott periodicity, proven using a very geometric construction; for Fredholm operators, we will prove a version of Bott periodicity in this context.

Theorem 12.8.

(1) $\Omega^2\operatorname{Fred}(H_{\mathbb{C}}) \simeq \operatorname{Fred}(H_{\mathbb{C}})$*, where* $H_{\mathbb{C}}$ *is a separable complex Hilbert space.*
(2) $\Omega^8\operatorname{Fred}(H_{\mathbb{R}}) \simeq \operatorname{Fred}(H_{\mathbb{R}})$*, where* $H_{\mathbb{R}}$ *is a separable real Hilbert space.*

This is our last statement of Bott periodicity. We'll prove it by providing spaces of operators that explicitly de-loop; it requires an important new ingredient, the notion of Clifford algebras. Then, we'll be able to move from vector spaces to modules over these Clifford algebras. This all takes place in the worlds of $\mathbb{Z}/2$-graded vector spaces and $\mathbb{Z}/2$-graded algebras (sometimes, thanks to physics, called *super-vector spaces* and *superalgebras*). We'll make this work over the next few lectures.

Lecture 13.

Clifford Algebras: 10/8/15

Recall that we showed that the path components of $\operatorname{Fred}(H)$ are parameterized by the index: if $\operatorname{Fred}_n(H)$ denotes the space of Fredholm operators with index n, then

$$\operatorname{Fred}(H) = \coprod_{n\in\mathbb{Z}} \operatorname{Fred}_n(H).$$

Moreover, $\operatorname{Fred}_0(H) \simeq BU$, the classifying space of $\mathrm{U} = \mathrm{U}_\infty$, the colimit of the unitary groups U_n.

Today, we're going to talk about Clifford algebras, and so also about the orthogonal group. Recall that the orthogonal group O_n, a Lie group, sits inside the associative algebra $M_n(\mathbb{R})$ of $n\times n$ matrices. This is often very useful, e.g. for computing things or realizing the tangent space to O_n, a Lie algebra.

The situation with Clifford algebras will be analogous. A Clifford algebra $\mathrm{Cliff}_{\pm n}(\mathbb{R})$ doesn't exactly contain the orthogonal group, but contains a group called $\mathrm{Pin}_{\pm n}$, which is a double cover of O_n.

Recall that $\pi_0\,\mathrm{O}_n \cong \{\pm 1\}$, and that SO_n is the identity component. SO_1 is trivial and $\mathrm{SO}_2 \cong \mathbb{T}$ (sending a rotation by θ to $e^{i\theta}$, and vice versa), but for $n \geq 3$, $\pi_i\,\mathrm{SO}_n \cong \mathbb{Z}/2\mathbb{Z}$, which we argued earlier in this class.

Suppose G is a Lie group and $\widetilde{G} \to G$ is a connected covering space. Then, we can give $\widetilde{G}$ a unique group structure: the identity is one of the preimages of the identity, and, since multiplication can be uniquely determined if it exists in a neighborhood of the identity, we can pick a neighborhood of $e \in G$ that is covered by a disjoint union of copies of itself, and define multiplication in a neighborhood of the new identity in the same way. Choosing different preimages of e gives us an automorphism.

If $\widetilde{G}$ is not connected, we may not get a unique group structure: for example, there's a double cover of $\mathbb{Z}/2$ that consists of four points, and depending on what the preimages of 1 do, we may get either $\mathbb{Z}/2\times\mathbb{Z}/2$ or $\mathbb{Z}/4$ as our covering groups.

Since SO_n is connected and, for $n \geq 3$, $\pi_1\,\mathrm{SO}_n = \mathbb{Z}/2$, then its connected double cover has a unique Lie group structure. This is called the *Spin group* Spin_n, and is a nice way to construct it abstractly. But this same strategy doesn't work for O_n, which isn't connected.

Definition. Let $\xi \in \mathbb{R}^n$ be such that $|\xi| = 1$. Then, we define the *hyperplane reflection* $\rho_\xi(\eta) = \eta - 2\langle\xi,\eta\rangle\xi$.

This is reflection across ξ in the usual geometric sense, particularly when $n = 2$.

Theorem 13.1 (Sylvester)**.** *Any element of* O_n *is the product of at most* n *hyperplane reflections.*

In its simplest form, this theorem was known circa 200 B.C.!

Proof. We'll induct on n. If $g \in \mathrm{O}_n$ fixes a $\xi \in S(\mathbb{R}^n)$, then $g \in \mathrm{O}(\mathbb{R}\cdot\xi^\perp)$, and therefore g is a product of at most $n-1$ reflections. Then, for a general g, we can find some $\zeta \in \mathbb{R}^n$ such that $g(\zeta) \perp \zeta$; in this case, set $\xi = (g(\zeta)-\zeta)/|g(\zeta)-\zeta|$, and $\rho_\xi \circ g(\zeta) = \zeta$, so we get at most one more reflection. ⊠

Let's try to build an algebra out of this theorem. As a heuristic, if $\xi \in S(\mathbb{R}^n)$, we'll let "ξ" stand in for ρ_ξ, so that $\xi^2 = \pm 1$. Since $\rho_\xi = \rho_{-\xi}$, then there is an ambiguity of ± 1.

Suppose $\langle\xi,\eta\rangle = 0$. Then, $|(\xi+\eta)/\sqrt{2}| = 1$, so

$$\begin{aligned}\pm 1 = \left(\frac{\xi+\eta}{\sqrt{2}}\right)^2 &= \frac{1}{2}\big(\xi^2+\eta^2+\xi\eta+\eta\xi\big)\\ &= \frac{1}{2}(\pm 2+\xi\eta+\eta\xi),\end{aligned}$$

so in particular $\xi\eta + \eta\xi = 0$. Geometrically, we already knew that reflections across perpendicular lines commute.

More generally, for any unit vectors ξ, η, $\rho_\xi(\eta) = -\xi\eta\xi^{-1}$ (since ξ defines a reflection, its inverse exists). Thus, we can define an algebra, the *Clifford algebra* using the two relations $\xi^2 = \pm|\xi|^2$ and $\xi_1\xi_2 + \xi_2\xi_1 = 0$ if $\langle \xi_1, \xi_2 \rangle = 0$. Since $\mathbb{R}^n$ comes with the standard basis $e_1, \dots, e_n$, we can rewrite these relations as

$$\begin{cases} e_i^2 = \pm 1 \\ e_ie_j + e_je_i = 0, \quad i \neq j, \end{cases}$$

or, equivalently, $e_ie_j + e_je_i = \pm 2\delta_{ij}$.

Example 13.2.

- Cliff_1 is generated by $\{1, e_1\}$ with $e_1^2 = 1$. Thus, $\mathrm{Pin}_1 = \{\pm 1, \pm e_1\} \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, the Klein-four group. And as an algebra, $\mathrm{Cliff}_1 \cong \mathbb{R} \times \mathbb{R}$.
- Cliff_{-1} is the same, but with $e_1^2 = -1$. This, as an algebra, $\mathrm{Cliff}_{-1} \cong \mathbb{C}$, and in this case, $\mathrm{Pin}_1^- = \{\pm 1, \pm e_1\} \cong \mathbb{Z}/4$.
- $\mathrm{Cliff}_2 \hookrightarrow M_2\mathbb{C}$: we have the relations $e_1e_2 + e_2e_1 = 0$ and $e_1^2 = e_2^2 = 1$, so we can choose
$$e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad e_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$
- Similarly, $\mathrm{Cliff}_{-2} \hookrightarrow M_2\mathbb{C}$. This time, $e_1^2 = e_2^2 = -1$, so we choose
$$e_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$
That these generators for $\mathrm{Cliff}_{\pm 2}$ are off-diagonal is not a coincidence.

Remark. Dirac considered whether there was a "square root" of the Laplace operator, a differential operator D (called the *Dirac operator*) on $\mathbb{E}^n$ such that

$$D^2 = \Delta = -\sum_{i=1}^n \frac{\partial^2}{(\partial x^i)^2}.$$

(We'll use the implicit summation convention in this remark.)

If $D = \gamma^i \frac{\partial}{\partial x^i}$ operates on a function $\psi : \mathbb{E}^n \to \mathbb{R}^N$ (so that $\gamma^i \in M_N\mathbb{R}^N$), then

$$D^2 = \left(\gamma^i\gamma^j + \gamma^j\gamma^i\right) \frac{\partial^2}{\partial x^i \partial x^j}.$$

Therefore $\gamma^i\gamma^j + \gamma^j\gamma^i = -2\delta^{ij}$, so we have the same generators and relations! This is an important motivation of Clifford algebras, and some useful intuition.

In general, the generators of the Clifford algebra within the matrix algebra are off-diagonal or off-block-diagonal. This means that the product of any two is diagonal, which is a nice way of realizing a $\mathbb{Z}/2$-grading on the Clifford algebra.

Definition.

(1) A *super vector space* is a space $\mathbb{S} = \mathbb{S}^0 \oplus \mathbb{S}^1$. Equivalently, it is a pair $(\mathbb{S}, \epsilon)$ where $\mathbb{S}$ is a vector space and $\epsilon \in \mathrm{End}(\mathbb{S})$ is such that $\epsilon^2 = \mathrm{id}_{\mathbb{S}}$.
(2) If $(\mathbb{S}', \epsilon')$ and $(\mathbb{S}'', \epsilon'')$ are super vector spaces, then their *tensor product* is $(\mathbb{S}' \otimes \mathbb{S}'', \epsilon' \otimes \epsilon'')$.
(3) The *Koszul sign rule* is the symmetry $\mathbb{S}' \otimes \mathbb{S}'' \to \mathbb{S}'' \otimes \mathbb{S}'$: the sign convention $s' \otimes s'' \mapsto (-1)^{|s'||s''|} s'' \otimes s$, where $s' \in \mathbb{S}'^{|s'|}$ and similarly for s'' (this tells us which part of $\mathbb{S}'$ or $\mathbb{S}''$ it's in). A general element of a super vector space isn't homoegenous, but it's a sum of homogeneous elements, so this map is well-defined.
(4) A *superalgebra* $A = A^0 \oplus A^1$ is an algebra for which the multiplication map $A \otimes A \to A$ is an *even* map, i.e. it respects the grading.
(5) $z \in A$ is *central* if $za(-1)^{|a||z|}az$ for all homogeneous $a \in A$ (so that z is necessarily homogeneous). The set of central elements, denoted $Z(A)$, is called the *center.*

There are also notions of *opposite algebras* A^{op} where multiplication is more or less turned around, *ideals* (which must be the sum of its even part and its odd part), and *simple* algebras, which we can read about in TODO: cite.

The idea is that these familiar constructions from algebra still hold, as long as you're careful with the sign convention and the grading.

Example 13.3. If $\mathbb{S} = \mathbb{S}^0 \oplus \mathbb{S}^1$, then $\mathrm{End}(\mathbb{S}) = \mathrm{End}(\mathbb{S})^0 \oplus \mathrm{End}(\mathbb{S})^1$ is a superalgebra. Specifically, block diagonal (with respect to $\mathbb{S}^0$ and $\mathbb{S}^1$) matrices are in $\mathrm{End}(\mathbb{S})^0$, and block off-diagonal matrices are in $\mathrm{End}(\mathbb{S})^1$.

Definition. Let k be a field of characteristic not equal to 2,[18] and V be a vector space over k.

(1) $Q : V \times V \to k$ is *quadratic* if $\xi_1, \xi_2 \mapsto Q(\xi_1 + \xi_2) - Q(\xi_1) - Q(\xi_2)$ is bilinear and $Q(n\xi) = n^2 Q(\xi)$ for $n \in k$.
(2) A pair (C, i) of a unital, associative algebra C and a linear map $i : V \to C$ is a *Clifford algebra* of (V, Q) if $i(\xi)^2 = -Q(\xi)1_C$ and for every unital, associative algebra A and linear $\psi : V \to A$ such that $\psi(\xi)^2 = Q(\xi) \cdot 1_A$ for all $\xi \in V$, then there exists a unique k-algebra homomorphism $\widetilde{\psi} : C \to A$ such that the following diagram commutes.

$$\begin{tikzcd} V \arrow[rr, "i"] \arrow[dr, "\psi"'] & & C \arrow[dl, "\widetilde{\psi}"] \\ & A & \end{tikzcd}$$

In this case, (C, i) is denoted $\mathrm{Cliff}(V, Q)$ or $\mathrm{C}\ell(V, Q)$.

The universal property quickly implies a few things.

(1) First, that such a Clifford algebra exists and is unique given k, V, and Q.
(2) Then, there is a canonical, surjective map $\otimes V \to \mathrm{Cliff}(V, Q)$.[19]
(3) If (C, i) is a Clifford algebra, then i must be injective.
(4) Since $\otimes V$ has an increasing filtration $\otimes^0 V \subset \otimes^{\le 1} V \subset \otimes^{\le 2} V \subset \cdots$, then there is an induced filtration on $\mathrm{Cliff}(V, Q)$, and the associated graded is $\Lambda^\bullet V$.[20]
(5) This means that $\mathrm{Cliff}(V, Q)$ is $\mathbb{Z}/2\mathbb{Z}$-graded (following ultimately from how the quadratic form acts on the filtration).

Notice that the Clifford algebra is not commutative, however, even though its associated graded is commutative. It's in some sense a deformation of the exterior algebra (e.g. when Q is degenerate). These abstract properties will be shored up by concrete things we have to prove in the homework.

Both of these are algebraic pictures of a process called *quantization* in physics, deforming a commutative operator into a noncommutative one.

By applying the universal property, one can show that for any pair (V', Q') and (V'', Q'') of k-vector spaces and quadratic forms on them, there is a canonical isomorphism

$$\mathrm{Cliff}(V' \oplus V'', Q' \oplus Q'') \xrightarrow{\cong} \mathrm{Cliff}(V', Q') \otimes \mathrm{Cliff}(V'', Q''). \tag{13.1}$$

Here, $(x' \otimes x'')(y' \otimes y'') = (-1)^{|x'||y'|} x'y' \otimes x''y''$ is how multiplication works in the tensor product of superalgebras.

Definition. If L is a k-vector space, $\xi \in L$, and $\theta \in L^*$, then *interior multiplication by* ξ is the map $i_\xi \in \mathrm{End}(\Lambda^\bullet L^*)$ defined by $i_\xi(\phi) = \phi(\xi)$ for $\phi \in \Lambda^1 L^*$ and extended as a derivation:

$$i_\xi(\omega_1 \wedge \omega_2) = i_\xi \omega_1 \wedge \omega_2 + (-1)^{|\omega_i|} \omega_1 \wedge i_\xi \omega_2.$$

Then, *exterior multiplication by* θ is $\varepsilon_\theta(\omega) = \theta \wedge \omega$.

Proposition 13.4. *Let L be a k-vector space and $V = L \oplus L^*$ with $Q(\xi, \theta) = \theta(\xi)$ for $\xi \in L$ and $\theta \in L^*$. Then, $i : L \oplus L^* \to \mathrm{End}(\Lambda^\bullet L^*)$ sending $\xi \mapsto i_\xi$ and $\theta \mapsto \varepsilon_\theta$ is a Clifford algebra of (V, Q).*

[18] We'll only use $k = \mathbb{R}$ or $\mathbb{C}$ in this class, though.
[19] Here, $\otimes V$ denotes the tensor algebra of V.
[20] The *associated graded* is the graded algebra of quotients of this filtration.

The idea is to prove by induction: the base case is essentially the same as Example 13.2, and in general we can reduce to a lower dimension using (13.1).

Example 13.5.

(1) If $k = \mathbb{C}$, then any nondegenerate Q on V with $\dim V = 2\mathbb{Z}$ can be written as the form δ_{ij} in a suitable basis (akin to diagonalizing a symmetric matrix), and by rearranging we can make it off-diagonal: there's a basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ of V such that $B(e_i, e_j) = B(f_i, f_j) = 0$, and $B(e_i, f_j) = \delta_{ij}$, and so any nondegenerate Q gives us a Clifford algebra.

(2) If $k = \mathbb{R}$, when we diagonalize, we can't get rid of the signature: there are some 1s and some -1s, and their difference, the *signature*, is an invariant. If we have a split form, we can take the standard basis $e_1, \ldots, e_n$ and the dual basis $e^1, \ldots, e^n$; then, $Q(e_i, e_j) = Q(e^i, e^j) = 0$ and $B(e_i, e^j) = \delta_i^j$, so we get a Clifford algebra (the matrices are block off-diagonal, with the off-diagonal components equal to the identity). However, other signatures don't work here.

Incredibly, Bott periodicity comes up *again* in this guise. Let $\mathrm{C}\ell_{\pm n} = \mathrm{Cliff}(\mathbb{R}^n, \pm Q)$, where Q is the standard quadratic form, and let $\mathrm{C}\ell_n^{\mathbb{C}} = \mathrm{C}\ell_n \otimes \mathbb{C} \cong \mathrm{C}\ell_{-n} \otimes \mathbb{C}$.

Theorem 13.6.

(1) $\mathrm{C}\ell_{-2}^{\mathbb{C}} \cong \mathrm{End}(\mathbb{C}^{1|1})$.

(2) $\mathrm{C}\ell_{-8} \cong \mathrm{End}(\mathbb{R}^{8|8})$.

In particular, $\mathrm{C}\ell_1, \ldots, \mathrm{C}\ell_7$ aren't $\mathbb{Z}/2$-graded matrix algebras, and similarly for $\mathrm{C}\ell_1^{\mathbb{C}}$.

Proof. For (1), we can write $\mathbb{C}^2 = L \oplus L^*$ with the canonical quadratic form; then, the previous example did the work for us.

For (2), $\mathrm{C}\ell_{-2}$ acts on $W = \mathbb{C}^{1|1}$ via

$$e_1 \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad e_2 \longmapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

One can check that this action graded-commutes with the odd antilinear $J : W \to W$ defined by $J(z^0, z^1) = (\overline{z}^1, \overline{z}^0)$ (so that $J^2 = -\mathrm{id}_W$).

We have an odd map that squares to $-\mathrm{id}$, but we wanted an even map squaring to the identity. So taking $-^{\otimes 4}$, we get that $\mathrm{C}\ell_{-8} = \mathrm{C}\ell_{-2}^{\otimes 4}$ acts on $W^{\otimes 4}$ and commutes with $J^{\otimes 4}$, which is even antilinear and squares to $\mathrm{id}_{W^{\otimes 4}}$. In particular, $J^{\otimes 4}$ is a real structure (our space is $\mathbb{R}^{8|8}$). ⊠

Lecture 14.

Kupier's Theorem and Principal *G*-Bundles: 10/13/15

"It's nice to make statements, but this isn't politics. It's mathematics, so we have to carry it out."

Last time, we talked about Clifford algebras, and the time before about Fredholm operators; today, we'll combine the two, and state a theorem whose proof will occupy us for the next few lectures.

Theorem 14.1 (Kuiper)**.** *Let H be an infinite-dimensional real or complex Hilbert space. Then, the group* $\mathrm{Aut}(H)$ *of invertible bounded maps $H \to H$ is contractible in the norm topology.*

$\mathrm{Aut}(H)$ is a subset of the space of bounded maps (endomorphisms) $H \to H$, and thus inherits the topology from its norm. This is one of several topologies you could put on $\mathrm{Aut}(H)$, and it's contractible in some other important ones, which we'll see later on in the course.

Recall that if $P : H \to H$ is a bounded algebra and P^* denotes its adjoint, then P^*P is a nonnegative, self-adjoint operator, and so has a square root, denoted $|P| = \sqrt{P^*P}$, which is also self-adjoint and nonnegative. Forming that square root uses the spectral theorem: in finite dimensions, a self-adjoint operator is represented by a symmetric matrix, which can be made diagonal with real eigenvalues. Then, one can take the nonnegative square root of each eigenvalue. In infinite dimensions, the von Neumann spectral theorem allows us to do the same thing.

We'll apply this to invertible operators to get a deformation from $\mathrm{Aut}(H)$ to the subgroup of unitary automorphisms $\mathrm{U}(H)$ (or $\mathrm{O}(H)$ in the real case):

$$P_t = P\Big((1-t)\,\mathrm{id}_H + t|P|^{-1}\Big).$$

Since all eigenvalues are nonzero, $|P|$ is invertible, so we can do this.

Corollary 14.2. $\mathrm{U}(H)$ *(or* $\mathrm{O}(H)$ *in the real case) is contractible in the norm topology.*

This is definitely not true in finite dimensions: for example, $\mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$, and $\mathrm{U}(1) = S^1$, neither of which is contractible. But the deformation retraction still exists. Contractibility is strange: if you embed $S^n \hookrightarrow S^{n+1}$ as the equator, S^{n+1} is "more contractible" than S^n, since another homotopy group vanishes. But the analytic version of that statement is that the infinite-dimensional unit sphere, the limit of this process, is contractible! That's a little counterintuitive.

Rather than proving directly that $\mathrm{Aut}(H)$ is contractible, we'll establish a weak homotopy equivalence with a point, and by a theorem of Whitehead, this is sufficient.

Definition. A continuous map $f : X \to Y$ of topological spaces is a *weak homotopy equivalence* if

1. $f_* : \pi_0 X \to \pi_0 Y$ is an isomorphism, and
2. for all $x \in X$ and $n > 0$, the induced map $f_* : \pi_n(X, e) \to \pi_n(Y, f(e))$ is an isomorphism.

By a theorem of Whitehead, if X and Y have the homotopy type of CW complexes, then this implies f is a homotopy equivalence.

Proof of Theorem 14.1. We'll sketch the proof that $\pi_n(\mathrm{Aut}(H), \mathrm{id}_H)$ vanishes for all n, which as noted above is sufficient. The full details are in the lecture notes.

The first step will be to reduce to thinking about finite-dimensional operators.

Lemma 14.3. *Let X be a compact simplicial complex and $f_0 : X \to \mathrm{Aut}(H)$ be continuous. Then, there exists a homotopy $f_0 \simeq f_1$ and a finite-dimensional $V \subset \mathrm{End}(H)$ such that $f_1(x) \in V$ for all $x \in H$.*

Proof. Cover $\mathrm{Aut}(H)$ in balls in $\mathrm{Aut}(H)$. Then, the inverses images under f_0 cover X, and we can choose a finite subcover. Then, subdivide these open sets so that for each simplex Δ of X, $f_0(\Delta)$ is contained in some open sets. Since X is compact, there are finitely many such simplicies. The n vertices of $f_0(\Delta)$ are operators, and we can take an affine combination of them. In the end, we get finitely many such affine operators, and passing to each one is a homotopy through the ball (and therefore through invertible operators). Since there are finitely many of them, the space they span is finite-dimensional. ⊠

The second step deals with V but not f_1. We will construct an orthogonal decomposition $H = H_1 \oplus H_2 \oplus H_3$ such that

1. $\alpha(H_1) \perp H_3$ for all $\alpha \in V$,
2. $\dim H_1$ is infinite,
3. there exists an isomorphism $T : H_1 \to H_3$.

Let's do this. Let P_1 be a line in H, so we can choose a finite-dimensional $P_2 \perp P_1$ such that $\alpha(P_1) \subset P_1 \oplus P_2$ for all $\alpha \in V$. Then, we may choose P_3 to be a line perpendicular to $P_1 \oplus P_2$ and fix an isomorphism $T : P_1 \to P_3$.

Let Q_1 be a line perpedicular to $P_1 \oplus P_2 \oplus P_3$, so that we can choose a finite-dimensional Q_2 such that $\alpha(P_1 \oplus Q_1) \subset P_1 \oplus Q_1 \oplus P_2 \oplus Q_2$ for all $\alpha \in V$, and $P_2 \perp Q_2$. Then (surprise) we choose a line Q_3 perpendicular to $P_1 \oplus Q_1 \oplus P_2 \oplus Q_2$ and fix an isomorphism $T : Q_1 \to Q_3$. We set $H_1^{(1)} = P_1 \oplus Q_1$, $H_2^{(1)} = P_2 \oplus Q_2$, and $H_3^{(1)} = P_3 \oplus Q_3$.

At this point, we say "induction" and get $H_1^{(n)}$, $H_2^{(n)}$, and $H_3^{(n)}$, all finite-dimensional, such that $\alpha(H_1^{(n)}) \subset H_1^{(n)} \oplus H_2^{(n)}$ and $T : H_1^{(n)} \to H_3^{(n)}$ is an isomorphism, and all three are orthogonal. Since $H_i^{(n)} \subset H_i^{(n+1)}$, then we can define

$$H_i = \overline{\bigcup_{n=1}^{\infty} H_i^{(n)}}, \qquad i = 1, 3,$$

and then define $H_2 = (H_1 \oplus H_3)^\perp$. Clearly, $\dim(H_1^{(n)}) = \dim(H_3^{(n)}) = n$ (since each time, we add a line), so H_1 is infinite-dimensional, and the actions of V and T extend to have the right properties.

On to the third step. We want to construct homotopies $f_1 \simeq f_2 \simeq f_3$ such that $f_3(x)|_{H_1} = \mathrm{id}_{H_1}$ for all $x \in X$. (Note that $f_1(x)(H_1) \perp H_3$ for all x). This is a trick with rotations, and can be done in two steps.

First, let $H_x = (f_1(x)H_1 \oplus H_3)^\perp$, so there's a map $H_1 \oplus H_x \oplus H_1 \to H_1 \oplus H_x \oplus H_1$ sending $\xi \oplus \eta \oplus \zeta \mapsto -\zeta \oplus \eta \oplus \xi$. This is a rotation by 90°, and therefore is homotopic to the identity. Conjugating by $f_1(x) \oplus \mathrm{id}_{H_x} \oplus T : H_1 \oplus H_x \oplus H_1 \to H_1 \oplus H_x \oplus H_3$, we get a path from id_H to $\varphi_x : f_1(x)H_1 \oplus H_x \oplus H_3 = H \to H$ sending $f_1(x)\xi \oplus \eta + T\zeta \mapsto -f_1(x)\zeta \oplus \eta \oplus T\xi$; further rotation takes us to f_3 (which is easier to read about then listen to).

The fourth step, called the *Eilenberg swindle*, proceeds as follows. If $H = H_1^\perp \oplus H_1$, each component is infinite-dimensional, and $f_3(x)$ is the identity on H_1, so in block form looks like

$$f_3(x) = \begin{pmatrix} u(x) & 0 \\ * & 1 \end{pmatrix},$$

where $u(x)$ is some invertible piece. By replacing $*$ with $t*$, we get a homotopy through invertibles to

$$f_4(x) = \begin{pmatrix} u(x) & 0 \\ 0 & 1 \end{pmatrix}.$$

Next, we want to get rid of $u(x)$. We can write $H_1^\perp = K_1 \oplus K_2 \oplus K_2 \oplus \cdots$, where each K_i is infinite-dimensional and the sum is of closed, orthogonal subspaces — and therefore we fix isomorphisms $K_i \cong H_1^\perp$! Then, the path

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} u & \\ & 1 \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} u^{-1} & \\ & 1 \end{pmatrix}$$

on $H_1^\perp \oplus H_1^\perp$. When $t = 0$, this is the identity, and when $t = 1$, it is the matrix with diagonal (u^{-1}, u). Therefore (and this is the swindle part),

$$f_4 \simeq \begin{pmatrix} u & & & \\ & \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} & & \\ & & \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} & \\ & & & \ddots \end{pmatrix} \simeq \begin{pmatrix} \begin{pmatrix} u & \\ & u^{-1} \end{pmatrix} & & & \\ & \begin{pmatrix} u & \\ & u^{-1} \end{pmatrix} & & \\ & & \begin{pmatrix} u & \\ & u^{-1} \end{pmatrix} & \\ & & & \ddots \end{pmatrix} \simeq \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix}.$$

⊠

Remark. The last step in the Eilenberg swindle looks a lot like the "proof"

$$\begin{aligned} 0 &= (1 + -1) + (1 + -1) + (1 + -1) + \cdots \\ &= 1 + (-1 + 1) + (-1 + 1) + \cdots \\ &= 1. \end{aligned}$$

Principal *G*-bundles.

Definition. Let G be a topological group (often a Lie group); then, a fiber bundle $\pi : P \to X$ is a *principal G-bundle* if G acts freely on P on the right and π is a quotient map for the G-action.

In other words, a fiber bundle is a collection of spaces, but a principal G-bundle is a collection of (right) *G-torsors*, spaces on which G acts simply transitively. Importantly, if $y \in p^{-1}(x)$ and $g \in G$, then $gy \in p^{-1}(x)$.

Local sections give us local trivializations, and vice versa: $s : U \to P|_U$ is equivalent to a map $U \times G \to P|_U$ sending $x, g \mapsto s(x) \cdot g$. This has the useful corollary that a principal G-bundle has global sections iff it's trivial.

Example 14.4. Let $E \to X$ be a rank-r complex vector bundle, and let P be its *bundle of frames*: $P_x = \mathrm{Iso}(\mathbb{C}^r, E_x)$, and $G = \mathrm{Iso}(\mathbb{C}^r, \mathbb{C}^r) = \mathrm{GL}_r \mathbb{C}$. In other words, every point of P_x is a basis for E_x. G acts on the right by precomposition, and so if we go from a $p \in P_x$ to a $pg \in P_x$, then we can think of it as an invertible linear map $\mathbb{C}^r \to \mathbb{C}^r$, given by g^{-1}.

This example was an instance of the *associated fiber bundle*: if F is any space with a left G-action, then the associated fiber bundle is $P \times F/G \to X$ with fiber F. This is Steenrod's picture of principal G-bundles (which you can read more about in the lecture notes); there are lots of G-bundles, and in some sense their behavior is controlled by the principal ones.

Proposition 14.5. *Let $\pi : \mathcal{E} \to M$ be a fiber bundle with fiber F such that F is a contractible, metrizable, topological manifold (albeit perhaps infinite-dimensional)*[21] *and M is metrizable. Then, π admits a section, and if $\mathcal{E}$, M, and F have the homotopy type of CW complexes, then π is a homotopy equivalence.*

In general, topological spaces can get — well, not *bad*; there's nothing morally wrong about them. But they can be pretty vile. That's why we want metrizable ones, though we don't commit to a particular metric.

We won't prove this; a proof is given in TODO: cite. It's a bunch of point-set topology we don't need to get into, but it's important that such theorems are provable. In any case, the slogan to take away is that in these nice cases and with contractible fibers, sections are homotopy equivalences.

Theorem 14.6. *Let G be a Lie group, and suppose $\pi^{\mathrm{univ}} : P^{\mathrm{univ}} \to B$ is a principal G-bundle and P^{univ} is a contractible, metrizable, topological manifold. Then, for any principal G-bundle $\pi : P \to M$ with M metrizable, there exists a G-equivariant pullback φ fitting into the following diagram.*

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d, "\pi"] & P^{\mathrm{univ}} \arrow[d] \\ M \arrow[r, "\overline{\varphi}"] & B. \end{tikzcd}$$

The proof is pretty simple: form the associated bundle over M with fiber P^{univ}, and check that it satisfies the right properties.

Example 14.7. Fix a $k \in \mathbb{Z}^{>0}$ and let H be a separable, complex Hilbert space. Then, define the *Stiefel*[22] *manifold* $\mathrm{St}_k(H)$ to be the set of "partial isometries" $\mathbb{C}^k \hookrightarrow H$, i.e. injections that preserve the norm. Since U_k is the group of isometries of $\mathbb{C}^k$, then it freely acts on $\mathrm{St}_k(H)$ on the right, so we get a bundle $\pi : \mathrm{St}_k(H) \to \mathrm{Gr}_k(H)$: the quotient is the Grassmanian.

It turns out that $\mathrm{St}_k(H)$ is contractible: $\mathrm{U}(H)$ acts transitively, and the stabilizer of $e_1, \dots, e_k$ is $\mathrm{U}(\mathbb{C}\{e_1, \dots, e_k\}^\perp)$. In other words, when we pick a basepoint, $\mathrm{St}_k(H) \cong \mathrm{U}(H)/\mathrm{U}(H_0)$ (the latter being basepoint-preserving unitary maps), and by Theorem 9.4, the unitary groups are contractible, and $\mathrm{U}(H) \to \mathrm{St}_k(H)$ is a principal $\mathrm{U}(H_0)$-bundle, and by Proposition 14.5 is a homotopy equivalence.

Note that $\mathrm{St}_1(H) = S(H)$, the unit sphere.

The Peter-Weyl theorem tells us that any compact Lie group can be embedded in a unitary group, and so allows us to obtain nice manifold models for more general classifying spaces.

Putting Things Together. Let $H = H^0 \oplus H^1$ be a super-Hilbert space. An odd skew-adjoint operator A has block form

$$A = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}.$$

This is not technically skew-adjoint, since there are a few factors of i unaccounted for, but that's OK for the purposes of this discussion.

Definition.

(1) $\mathrm{Fred}_0(H)$ is the space of odd skew-adjoint Fredholm operators on H, which is also $\mathrm{Fred}(H^0, H^1)$ (since skew-adjointness forces the whole operator once you know T).

(2) For $n > -1$, define $\mathrm{Fred}_{-n}(H) \subset \mathrm{Fred}_0(\mathrm{C}\ell^{\mathbb{C}}_{-n} \otimes H)$. This has a left action of $\mathrm{C}\ell^{\mathbb{C}}_{-n}$ induced by the left multiplication of $\mathrm{C}\ell^{\mathbb{C}}_{-n}$ on itself.

In (2), $Ae_i = -e_i A$ for $i = 1, \dots, n$.

If $\mathbb{S} = \mathbb{C}^{1|1}$, which is a complex super-vector space (i.e. $\mathbb{S} = \mathbb{S}^0 \oplus \mathbb{S}^1$, where each $\mathbb{S}^i \cong \mathbb{C}$), then $\mathrm{C}\ell^{\mathbb{C}}_{-2} \cong \mathrm{End}(\mathbb{S})$, so we can talk about algebraic periodicity: there is a map

$$\mathrm{Fred}_0(\mathbb{S}^* \otimes H) \longrightarrow \mathrm{Fred}_{-2} \subset \mathrm{Fred}_3(\mathbb{S} \otimes \mathbb{S}^* \otimes H)$$

[21] To be precise, we want F to be a topological manifold modeled on a locally convex topological vector space.

[22] Pronounced "shteefel."

given by $A \mapsto \mathrm{id}_8 \otimes A$, and it's a homeomorphism. In other words, $\mathrm{Fred}_0 \cong \mathrm{Fred}_{-2} \cong \mathrm{Fred}_{-4} \cong \cdots$, and similarly $\mathrm{Fred}_{-1} \cong \mathrm{Fred}_{-3} \cong \cdots$. So we have two homeomorphism types, and therefore two homotopy types.

So far, this is just the periodicity of the Clifford algebras; there's nothing analytic about it. We can extend to positive n by using more Clifford algebras, though. Analytically, what's going on is the *Atiyah-Singer loop map* $\alpha : \mathrm{Fred}_{-n}(H) \to \Omega\,\mathrm{Fred}_{-(n-1)}(\mathrm{C}\ell_{-1}^{\mathbb{C}} \otimes H)$ sending $A \mapsto (t \mapsto e_n \cos \pi t + A \sin \pi t)$, where $0 \le t \le 1$. Our goal is to prove the following theorem.

Theorem 14.8 (Atiyah-Singer)**.** *The Atiyah-Singer loop map α is a homotopy equivalence.*

Corollary 14.9. $\Omega^2\,\mathrm{Fred}_0(H) \cong \mathrm{Fred}_0(H)$.

There may have been a shift in our separable Hilbert space, but by Kuiper's theorem, that doesn't actually matter.

This is our final version of Bott periodicity: it will allow us to define K-theory on noncompact spaces.

Lecture 15.

Compact Operators: 10/15/15

Though we'll soon move into studying groupoids, equivariant vector bundles, and loop groups, this and the next lecture will address the proof of Theorem 14.8. David Ben-Zvi will give next Thursday's lecture.

Suppose X and Y are pointed spaces; then, a map $f : \Sigma X \to Y$ is equivalent to a map $g : X \to \Omega Y$. In other words, for any point $x \in X$, we get a based loop, because the ends of the suspension coordinate $(t = 0, 1)$ map to the basepoint, so tracing over t for a given x is a loop in g that starts and ends at the basepoint. Conversely, given a map $X \to \Omega Y$, write it as $x \mapsto f(t, x)$, and then $(t, x) \to f(t, x)$ is our map $\Sigma X \to Y$. That is, these maps are adjoints.

Definition.

(1) A *prespectrum* is a sequence $\{T_n\}_{n\in\mathbb{Z}}$ is a sequence of pointed spaces and maps $s_n : \Sigma T_n \to T_{n+1}$.
(2) A prespectrum is an Ω-*prespectrum* if the adjoint maps $t_n : T_n \to \Omega T_{n-1}$ are weak homotopy equivalences.
(3) An Ω-prespectrum is a *spectrum* if the t_n are homeomorphisms.

Notice that it's enough to specify T_n for $n \ge n_0$, given some $n_0 \in \mathbb{Z}$ (a lower bound) by defining $T_n = \Omega^{n_0 - n} T_{n_0}$ when $n < n_0$.

So in a spectrum, we have some sequence where decreasing the degree means taking loops $\Omega(-)$, and increasing the degree is delooping (which is in general harder): it's not just taking suspensions. For example, $\Sigma S^1 \simeq S^2$, but ΩS^2 is an infinite-dimensional manifold, not homeomorphic to S^1.

Example 15.1. If X is any pointed space, set $T_n = \Sigma^n X$ and $s_n : \Sigma\Sigma^n X \to \Sigma^{n+1} X$ to be the identity. This is called the *suspension spectrum* of X.

These spectra are the domain of stable homotopy theory, studying the stable properties of topolgical spaces under these sequences.

So why do we care as K-theorists? If $\{T_n\}$ is a spectrum, then it defines a (reduced) cohomology theory on a category of reasonable topological spaces defined by $k^n(X) = [X, T_n]$. This means it satisfies a few properties. For example, if we have a map $f : X \to Y$, we can extend to the mapping cone: $X \xrightarrow{f} Y \to C_f$. This is required to induce an exact sequence

$$k^n(X) \longleftarrow k^n(Y) \longleftarrow k^n(C_f)$$

This is the most crucial one. We used the Puppe sequence to extend this to a long exact sequence, and since we're taking suspensions again, we can do the same thing. This is useful, because we're defining a sequence of Fredholm operators that is an Ω-prespectrum. There's a way to obtain a spectrum from a prespectrum, which is intuitively a kind of completion, though we might lose the niceness of the properties in the sequence.

Once we pass from spaces to spectra, we may want to do algebraic topology with them, defining homotopy or homology theory. This is done in more detail in the lecture notes.

So we were mired in Fredholm operators, and defined $K^0(X) = [X, \mathrm{Fred}_0(H)]$, where if $H = H^0 \oplus H^1$ is a $\mathbb{Z}/2$-graded Hilbert space,

$$\mathrm{Fred}_0(H) = \left\{ \begin{pmatrix} & -T^* \\ T & \end{pmatrix} : T : H^0 \to H^1 \text{ Fredholm} \right\},$$

which is the same as $\mathrm{Fred}(H^0, H^1)$. Geometrically, $x \in K^0(X)$ is represented by a family of Fredholm operators parameterized by x.

Remark. If $E = E^0 \oplus E^1 \to X$ is a super-vector bundle and $H = H^0 \oplus H^1$ is a fixed Hilbert space, then $E^i \oplus \underline{H}^i$, for each $i = 1, 2$, is a trivializable vector bundle over X. Thus, we can construct a family of Fredholms $T_x = 0_{E_x} \oplus \mathrm{id}_H$. If X is compact Hausdorff, the K-theory class of T is the same as the K-theory class of E, independent of the choices we made.

What about other degrees? We use Clifford algebras to make loops, and define $\mathrm{Fred}_n(H) \subset \mathrm{Fred}_0(\mathrm{C}\ell_n^{\mathbb{C}} \otimes H)$ to be $\{T : e_i T = -Te_i, i = 1, \dots, n\}$.

Remark. Suppose $E = E^0 \oplus E^1 \to X$ is a finite-rank bundleof $\mathrm{C}\ell_1^{\mathbb{C}}$-modules (i.e. we have a left action of the Clifford algebra). We'd think of this as giving us a class in K^1. This is true, but the class is always zero: if e_1 is the Clifford generator and ε is the grading, then let $e_2 = ie_1\varepsilon$, which is odd (since i and ε are even, but e_1 is odd).

Then, $e_2e_1 + e_1e_2 = 0$ and $e_2^2 = e_1^2$, so E is the restriction of a $\mathrm{C}\ell_2^{\mathbb{C}}$-module, so 0_E is homotopic to an invertible through the homotopy $t \mapsto te_2$ of odd endomorphisms of $\mathrm{C}\ell_1^{\mathbb{C}}$-modules. And by Kuiper's theorem, invertibles are trivial in K-theory.

So in the end, we'll define $K^n(X) = [X, \mathrm{Fred}_n(H)]$; the invertibles in $\mathrm{Fred}_n(H)$ are contractible by Kuiper's theorem, so if your family ends up in the invertibles, it's homotopic to the trivial class in K-theory. Sadly, this means we don't have nice finite-dimensional vector bundle representatives of these classes, as we did in the case of compact X.

Compact Operators. We're going back to functional analysis now, so as usual let H^0 and H^1 be complex, separable, ungraded, infinite-dimensional Hilbert spaces.

Definition. If $T : H^0 \to H^1$ is bounded, then

(1) T has *finite rank* if $T(H^0) \subset H^1$ is finite-dimensional, and
(2) T is *compact* (sometimes *completely continuous*) if T of the unit ball is *precompact* (i.e. has compact closure).

The space of compact operators is denoted $\mathrm{cpt}(H^0, H^1)$.

There are many equivalent characterizations of compactness: for example, defining this with the unit ball is equivalent to defining it for any bounded neighborhood of the origin.

Fact. $\mathrm{cpt}(H^0, H^1)$ is a closed, two-sided ideal in $\mathrm{Hom}(H^0, H^1)$ (i.e. a compact operator composed with a bounded operator, on either side, is compact). The closure of the finite-rank operators is the compact operators. And finally, the identity is compact iff H is finite-dimensional.

Thinking back to the definition of Fredholm operators, we said that one of our axioms in the definition was redundant. Let's prove this.

Lemma 15.2. *Let $T : H^0 \to H^1$ be such that* $\ker T$ *and* $\operatorname{coker} T$ *are finite-dimensional. Then, $T(H^0) \subset H^1$ is closed.*

Proof. $\ker T$ is closed, since it's finite-dimensional, and $T : (\ker T)^\perp \to H^1$ is clearly injective with image $T(H^0)$ and a finite-dimensional cokernel, so it suffices to prove it when T is injective.

Choose $V \subset H^1$ to be a finite-dimensional space such that $H^1 = T(H^0) \oplus V$, which means also that $H^1 - V^\perp \oplus V$. $V^\perp$ is closed, because V is (the condition of being an orthogonal complement is a closed condition), so $\pi T : H^1 \to V^\perp$ given by orthogonal projection is a continuous bijection, which means it has a continuous inverse F.

If $\{\xi_n\} \subset H^0$ and $T\xi_n = \eta_n$ converges to an $\eta_\infty \in H^1$, set $\xi_\infty = F\pi\eta_\infty \in H^0$, and then it's easy to check that $T\xi_\infty = \eta_\infty$. ⊠

This lemma is useful for proving the following criterion.

Proposition 15.3. *A continuous operator $T : H^0 \to H^1$ is Fredholm iff there exist $S, S' : H^1 \to H^0$ such that $\mathrm{id}_{H^0} - ST$ and $\mathrm{id}_{H^1} - TS'$ are compact; moreover, we can take $S = S'$ and such that $\mathrm{id} - ST$ and $\mathrm{id} - TS$ are finite rank.*

S and S' are called *parametrices*, which can be thought of as "almost-inverses." We'll end up modding out by the "almost." The idea is that Fredholms are invertible up to small operators, so almost invertible.

Corollary 15.4. *If $k \in \mathrm{cpt}(H^0)$, then the operator $\mathrm{id}_{H^0} + k$ is Fredholm of index 0.*

In Proposition 15.3, we can just take S and S' to be the identity. This is what Erik Fredholm, a Swedish mathematician, was concerned with; it's not clear whether he studied Fredholm operators more generally.

Proof of Proposition 15.3. If T is Fredholm, decompose it as the map $(\ker T) \oplus (\ker T)^{\oplus} \to T(H^0) \perp \oplus T(H^0)$. If π is orthogonal projection onto $T(H^0)$, then $\pi T : (\ker T)^\perp \to T(H^0)$ is bijective (again, this is invertible up to a small space). Then, define $S = S'$ to be its inverse (which is bounded by the open mapping theorem) on $T(H^0)$ and 0 on $T(H^0)^\perp$.

Conversely, if $\mathrm{id}_{H^0} - ST$ is compact, then it's compact restricted to $\ker T$, and therefore $\mathrm{id}_{H_0} \ker T$ must be finite-dimensional, and the same argument holds for $\mathrm{id}_{H^1} - TS'$ and the cokernel. ⊠

From now on, we'll call $\mathrm{Aut}(H) = \mathrm{GL}(H)$: the invertible linear, bounded operators. Then, analogous to the Lie algebra is the space of all bounded operators, denoted $\mathfrak{gl}$ or $\mathfrak{gl}(H)$, and we'll write $\mathfrak{cpt}$ for $\mathrm{cpt}(H)$. These all act on the ungraded vector space H^0.

Definition. $\mathrm{GL}^{\mathrm{cpt}} = \{P \in \mathrm{GL} : P - \mathrm{id} \in \mathrm{cpt}(H)\}$, things that are compact minus the identity.

Then, $\mathrm{GL}^{\mathrm{cpt}}$ is a Banach Lie group (i.e. an infinite-dimensional Banach manifold with a group structure), and its Lie algebra is $\mathfrak{cpt}$. So $\mathrm{GL} \leftrightarrow \mathfrak{gl}$ and $\mathrm{GL}^{\mathrm{cpt}} \leftrightarrow \mathfrak{cpt}$. GL is also a Banach Lie group, which is less of a surprise.

We can also consider the Banach Lie group U of unitary operators on H, and its Lie algebra $\mathfrak{u}$, the space of skew-adjoint operators. In the same way we can take $\mathrm{U}^{\mathrm{cpt}}$ (also a Banach Lie group) and its algebra $\mathfrak{u} \cap \mathfrak{cpt}$.

Now, we can take a filtration $0 \subset H - 1 \subset H_2 \subset \cdots \subset H$ such that $\dim H_n = N$ and

$$\overline{\bigcup_{n=1}^\infty} H_n = H.$$

This induces maps $\mathrm{GL}(H_1) \subset \mathrm{GL}(H_2) \subset \cdots$.

Theorem 15.5 (Palais [22])**.** *The induced map*

$$\bigcup_{n=1}^\infty \mathrm{GL}(H_n) \hookrightarrow \mathrm{GL}^{\mathrm{cpt}}$$

is a homotopy equivalence.

But we know the homotopy type to be $\mathrm{GL}_\infty \simeq \mathrm{U}_\infty$, and by Bott periodicity, we know

$$\pi_q(\mathrm{GL}_\infty) \cong \begin{cases} \mathbb{Z}, & q \text{ odd} \\ 0, & q \text{ even}. \end{cases}$$

So we'll actually prove that the space of operators we get from K-theory sometimes has this homotopy type, which is an ingredient we need for Bott periodicity.

Definition. The *Calkin algebra* is the quotient $\mathfrak{gl}/\mathfrak{cpt}$.

This has lots of structure; it's a Banach space in the usual way,[23] and so it's a Banach algebra and even a C^* algebra.

[23] If X is a Banach space and $Y \subseteq X$, then X/Y has a norm $\|[x]\|_{X/Y} = \inf_{y \in [x]} \|y\|_X$.

It's also a Lie algebra, whose Banach Lie group is $\mathrm{GL}/\mathrm{GL}^{\mathrm{cpt}}$, and there is a principal bundle

$$\begin{tikzcd} \mathrm{GL}^{\mathrm{cpt}} \arrow[r] & \mathrm{GL} \arrow[d] \\ & \mathrm{GL}/\mathrm{GL}^{\mathrm{cpt}}. \end{tikzcd}$$

This is, again, a theorem of Palais. Since Kuiper's theorem implies that $\mathrm{GL} \simeq *$, then $\mathrm{GL}/\mathrm{GL}^{\mathrm{cpt}} \simeq B\,\mathrm{GL}^{\mathrm{cpt}} \simeq B\,\mathrm{GL}_\infty$.

So now we have the two homotopy types $\mathrm{GL}^{\mathrm{cpt}} \simeq \mathrm{GL}_\infty$ and its classifying space. In this context, the Bott periodicity theorem is that the loop spaces repeat: each is the other's loop space, and we'll prove this by using the fact that

$$\mathrm{Fred}_n \simeq \begin{cases} \mathrm{U}_\infty, & n \text{ odd}, \\ \mathbb{Z} \times B\mathrm{U}_\infty, & n \text{ even}. \end{cases}$$

So we have the following diagram, where G will denote $\mathrm{U}/\mathrm{U}^{\mathrm{cpt}}$.

$$\begin{tikzcd} \mathrm{U} \arrow[r, hook, "\text{d.r.}"] \arrow[d] & \mathrm{GL} \arrow[r, hook] \arrow[d] & \mathfrak{gl} \arrow[d, "\pi"] \\ G \arrow[r, hook, "\text{d.r.}"] & \mathrm{GL}/\mathrm{GL}^{\mathrm{cpt}} \arrow[r, hook] & \mathfrak{gl}/\,\mathrm{cpt}. \end{tikzcd}$$

Here, "d.r." means a deformation retraction, and the vertical arrows are the quotient maps. We can take the invertible elements $(\mathfrak{gl}/\mathfrak{cpt})^\times$ within the Calkin algebra, which is a group.

Proposition 15.6 (Freed[24])**.**

(1) $\mathrm{GL}/\mathrm{GL}^{\mathrm{cpt}}$ *is the identity component of* $(\mathfrak{gl}/\mathfrak{cpt})^\times$.
(2) $\pi^{-1}((\mathfrak{gl}/\mathfrak{cpt})^\times) = \mathrm{Fred} \subset \mathfrak{gl}$.

Moreover, $\pi : \mathrm{Fred} \to (\mathfrak{gl}/\mathfrak{cpt})^\times$ is a fibration with contractible fibers, and therefore a homotopy equivalence!

Corollary 15.7. $\mathrm{Fred}^{(0)} \simeq B\,\mathrm{GL}_\infty$ *and* $\mathrm{Fred} \simeq \mathbb{Z} \times B\,\mathrm{GL}_\infty$.

Let $\mathcal{F}$ denote Fred, and $\widehat{\mathcal{F}}$ denote the space of skew-adjoint Fredholm operators, which is an ungraded space. Then, we'll prove the following.

Theorem 15.8. $\widehat{\mathcal{F}}$ *is the disjoint union of three components* $\widehat{\mathcal{F}}_+ \sqcup \widehat{\mathcal{F}}_- \sqcup \widehat{\mathcal{F}}_*$, *where* $\widehat{\mathcal{F}}_\pm$ *are contractible and* $\alpha : \widehat{\mathcal{F}}_* \to \Omega\mathcal{F}$ *sending* $T \mapsto \cos \pi t + T \sin \pi t$ *for* $0 \le t \le 1$ *is a homotopy equivalence.*

We haven't explained how this is related to Clifford algebras in the graded situation, but it'll be easy to go from this to $\widehat{\mathcal{F}}_* = \mathrm{Fred}_1$. This is the crucial theorem that allows us to get Bott periodicity once we get the layout of the structure groups.

Lecture 16.

Quasifibrations and Fredholm Operators: 10/20/15

Today is the last lecture about Fredholm operators and the theorem of Atiyah and Singer connecting K-theory to the space of skew-adjoint operators. Today will be about making deformations, in a way that can be considerably more general than the setting we use today. If $p : E \to B$ is a fiber bundle with contractible fibers, we want p to be a homotopy equivalence; of course, this isn't true in general, so we need some sort of structure.

For example, $p : \mathbb{R}_{\mathrm{discrete}} \to \mathbb{R}$ (the latter with the usual topology) given by the identity set map is a fiber bundle with contractible fibers (since each fiber is a point), but cannot be a homotopy equivalence: $\mathbb{R}$ is connected, and $\mathbb{R}_{\mathrm{discrete}}$ has uncountably many components. As such, we will assume that B is path-connected, and E and B are metrizable.

[24]Yes, this was part of the professor's thesis!

We'll talk about three classes of maps: fiber bundles, fibrations, and quasifibrations. These all have the important property that the preimages of each point are, respectively, homeomorphic, homotopy equivalent, and wealky homotopy equivalent. Thus, to establish a weak equivalence any of these will suffice.

We've talked about fiber bundles before: they locally look like products. Specifically, if B is path-connected, for any $b \in B$, there's a neighborhood $U \subset B$ of b such that the following diagram commutes, where F is the fiber.

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\cong"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

The crucial property of fibrations is that they have the homotopy lifting property: if $p : E \to B$ is a fibration and $f : [0,1] \times X \to B$ is a homotopy, then we can lift f to $\widetilde{f}$ in the following diagram.

$$\begin{tikzcd} \{0\} \times X \arrow[r, "\widetilde{f}_0"] \arrow[d, hook] & E \arrow[d, "p"] \\ {[0,1]} \times X \arrow[r, "f"] \arrow[ur, dashed, "\widetilde{f}"] & B \end{tikzcd}$$

Sometimes these are taken in the category of pointed topological spaces, so that the basepoints are preserved by these commutative diagrams.

Theorem 16.1. *Suppose $p : E \to B$ is a fibration.*

(1) For $n \geq 0$, $p_* : \pi_n(E, p^{-1}(b); b) \to \pi_n(B, b)$ is an isomorphism.
(2) There is a long exact sequence of homotopy groups as follows.

$$\cdots \longrightarrow \pi_n(F, e) \longrightarrow \pi_n(E, e) \longrightarrow \pi_n(B, e) \longrightarrow \pi_{n-1}(F, e) \longrightarrow \cdots$$

For part (1), the idea is that we can lift a map $S^n \to B$ into the fiber, and this plays well with basepoints, but you have to consider relative homotopy. Then, the long exact sequence is ultimately the long exact sequence of the pair (E, F). These are standard in homotopy theory; see Hatcher's book for some of the proofs.

Proposition 16.2. *Let $p : (E, e) \to (B, b)$ be a fibration and $b' \in B$. Then, if*

$$P_e(E, p^{-1}(b')) = \{\gamma : [0,1] \to E \mid \gamma(0) = e, \gamma(1) \in \pi^{-1}(b')\},$$

then p induces a fibration $P_e(E, p^{-1}(b')) \to P_b(B, b')$ with contractible fibers.

If you specify an initial point and take the space of paths that can have any final point, this path space is contractible (just reel in the paths). This proposition is a fibered generalization of that.

Now, what's a quasifibration? We're going to encounter these a few times in this lecture.

Definition. If $p : E \to B$, the *homotopy fiber* over a $b' \in B$ is the space of pairs (x, γ), where $x \in E$ and γ is a path in B from b' to $p(x)$.

If $H_{b'}$ is the homotopy fiber over b', then there's a map $\psi : p^{-1}(b') \to H_{b'}$ sending $x \mapsto (x, \gamma_{\text{constant}})$.

Definition. With p and ψ as above, p is a *quasifibration* if ψ is a homotopy equivalence for all $b' \in B$.

Our map $\mathbb{R}_{\text{discrete}} \to \mathbb{R}$ is not a quasifibration: the homotopy fiber over a point is $\mathbb{R}_{\text{discrete}}$ times the path space, and this is not contractible. See Figure 2 for an example of a quasifibration that isn't a fibration.

Proposition 16.3. *p is a quasifibration iff $p_* : \pi_n(E, p^{-1}(b); e) \to \pi_n(B, b)$ is an isomorphism for all $b \in B$, $e \in p^{-1}(B)$, and $n \geq 0$.*

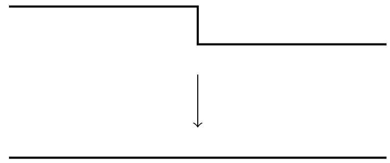


FIGURE 2. A map which is a quasifibration, but not a fibration. The preimage of a point is usually a point, but over one point it's an interval. Nonetheless, the homotopy fiber over every point is contractible, and this is induced by ψ.

Returning to the Fredholm story, we fixed a Hilbert space H and considered the following diagram of Lie algebras and/or groups.

$$\begin{tikzcd}
\mathrm{U} \arrow[r, hook, "\text{d.r.}", "\simeq"'] \arrow[d, "\mathrm{U}^{\mathrm{cpt}}"] & \mathrm{GL} \arrow[r, hook] \arrow[d, "\mathrm{GL}^{\mathrm{cpt}}"] & \mathfrak{gl} \arrow[d, "\pi"'] & \mathrm{Fred} \arrow[l, hook'] \arrow[d, "\pi"'] & \mathrm{Fred}_0 \arrow[l, hook'] \arrow[d, "\pi"'] \\
\mathrm{U}/\mathrm{U}^{\mathrm{cpt}} \arrow[r, hook, "\text{d.r.}", "\simeq"'] & \mathrm{GL}/\mathrm{GL}^{\mathrm{cpt}} \arrow[r, hook] & \mathfrak{gl}/\mathfrak{cpt} & (\mathfrak{gl}/\mathfrak{cpt})^\times \arrow[l, hook'] & \mathrm{GL}/\mathrm{GL}^{\mathrm{cpt}} \arrow[l, hook']
\end{tikzcd}$$

Oh boy. So what do we know here? By Kuiper's theorem, GL is contractible, and since the deformation retraction onto U is a homotopy equivalence, U is contractible as well. Then, $\mathrm{GL}^{\mathrm{cpt}} \simeq \mathrm{GL}_\infty$ annd $\mathrm{GL}/\mathrm{GL}^{\mathrm{cpt}} \simeq B\,\mathrm{GL}_\infty$, so $\mathrm{U}/\mathrm{U}^{\mathrm{cpt}} \simeq \mathrm{GL}_\infty$ too. We also know that $\mathrm{Fred} \simeq \mathbb{Z} \times B\,\mathrm{GL}_\infty$, as does $(\mathfrak{gl}/\mathfrak{cpt})^\times$, and $\mathrm{Fred}_0 \simeq B\,\mathrm{GL}_\infty$.

As in the last lecture, $\widehat{\mathcal{F}}$ will denote the skew-adjoint Fredholm operators.A skew-adjoint Fredholm operator must have index 0 (the kernel and cokernel must be isomorphic), so $\widehat{\mathcal{F}}$ sits inside Fred_0, and therefore π maps it into $\mathrm{GL}/\mathrm{GL}^{\mathrm{cpt}}$. Let $\widehat{G}$ denote the group of unitary, self-adjoint operators, i.e. if $x \in \widehat{G}$, then $xx^* = 1$ and $x = -x^*$, so $x^2 = -1$. Note that there is a deformation retraction of the inclusion $\widehat{G} \hookrightarrow \mathrm{GL}/\mathrm{GL}^{\mathrm{cpt}}$, inducing a homotopy equivalence.

In particular, $\mathrm{Spec}(x) \subset \{\pm i\}$.[25] This gives us three possibilities.

(1) $\widehat{G}_+$, the set where the spectrum is $\{i\}$. The only operator that satisfies this is i, and a single point is contractible.
(2) $\widehat{G}_-$, the set where the spectrum is $\{-i\}$. Again, only $-i$ satisfies this, so this is contractible.
(3) $\widehat{G}_*$ is everything else, which has both i and $-i$ in the spectrum.

We have a decomposition $\widehat{G} = \widehat{G}_+ \sqcup \widehat{G}_- \sqcup \widehat{G}_*$, and we can lift this to a decomposition of $\mathcal{F}$; thus, what we need to prove is that the map $\alpha : \widehat{\mathcal{F}}_* \to \Omega\mathcal{F}$ sending $T \mapsto \cos \pi t + T \sin \pi t$, with $0 \le t \le 1$, is a homotopy equivalence.[26] This map specifically will allow us to build a spectrum of Fredholm operators, once we put Clifford algebras back into the story.

To do this, we need to prove the following theorem.

Theorem 16.4. *The exponential map $\epsilon : \widehat{G}_* \to \Omega G$ sending $x \mapsto \exp \pi t x$, for $0 \le t \le 1$, is a homotopy equivalence.*

Then, we can lift ϵ up to α. We need to define one more space of operators; though, let

$$\widehat{F}_* = \left\{ T \in \pi^{-1}(\widehat{G}_*) \mid \|T\| = 1 \right\}.$$

That is, if $T \in \widehat{F}_*$, then T is Fredholm, $T^* = -T$, and $\|T\| = 1$. Thus, the essential spectrum of T is $\{\pm i\}$.

Lemma 16.5. *$\widehat{F}_*$ is a deformation retraction of $\widehat{\mathcal{F}}_*$.*

(If things are getting confusing at this point, consider checking out the lecture notes, or better yet, the original paper!)

Proof of Lemma 16.5. First, we have a deformation retraction $((1-t) + t\|\pi(T)^{-1}\|)T$ onto the subspace of $S \in \widehat{\mathcal{F}}_*$ with $\|\pi(S)^{-1}\| = 1$. We know that the essential spectrum of S is contained in the imaginary axis and

[25] Here, we're thinking of spectrum in a somewhat abstract set, the $\lambda \in \mathbb{C}$ such that $x - \lambda \cdot \mathrm{id}$ has a nontrivial kernel.

[26] These aren't loops *per se*; $\Omega\mathcal{F}$ consists of paths of Fredholm operators from id to $-$id.

has magnitude at least 1 (since the norm of the inverse is 1, so the largest part of the spectrum of the inverse is at most 1).

Now, we want to deformation retract onto $\widehat{F}_*$, which has only $\pm i$ in its spectrum. This is perfectly possible, since $i\mathbb{R}$ deformation retracts onto $[-i, i]$. That this induces one upstairs in operator-land follows from the spectral theorem (analogously to allowing us to diagonalize matrices in linear algebra, after which everything is pretty nice). ☒

In particular, $\pi : \widehat{F}_* \to \widehat{G}_*$ is a homotopy equivalence. So we're getting closer...

Let $\delta : x \mapsto \exp(\pi t x)$, for $0 \le t \le 1$. Then, we have the following diagram; we know the red arrow is a homotopy equivalence, and we want to prove that ϵ is one (which will imply Theorem 16.4).

$$\begin{tikzcd} \widehat{F}_* \arrow[r, "\delta"] \arrow[d, "\widehat{\pi}"', "\simeq"] & P_1(\mathrm{U}, -\mathrm{U}^{\mathrm{cpt}}) \arrow[d, "\rho"] \\ \widehat{G}_* \arrow[r, "\epsilon"] & P_1(G, -1) \end{tikzcd} \tag{16.1}$$

We'll prove this by showing δ and ρ are homotopy equivalences; this is where the discussion from the beginning of lecture comes in.

Proposition 16.6. *Evaluation at the endpoint is a homotopy equivalence $P_1(\mathrm{U}, -\mathrm{U}^{\mathrm{cpt}})$.*

Recall that $P_1(\mathrm{U}, -\mathrm{U}^{\mathrm{cpt}})$ is the space of paths in U that end in the subspace $-\mathrm{U}^{\mathrm{cpt}}$.

Proof. This is a fibration (even a principal bundle) with fiber $\Omega\,\mathrm{U}$, which is contractible by Kuiper's theorem. Thus, we get a weak homotopy equivalence, but since these spaces can be modeled on CW complexes, Whitehead's theorem means this is also a homotopy equivalence. ☒

Thus, in (16.1), ρ is a homotopy equivalence, because $\mathrm{U} \to G$ is a principal fiber bundle, with fiber the unitary operators that are 1 plus a compact operator. Thus, as we talked about earlier, the relevant map between path spaces is a homotopy equivalence.

That ϵ is a homotopy equivalence comes from the following theorem.

Theorem 16.7. $q : \widehat{F}_* \to -\mathrm{U}^{\mathrm{cpt}}$ *sending* $T \mapsto \exp \pi T$ *is a homotopy equivalence.*

It would suffice to prove that it's a fibration, or even a quasifibration... but it's neither. It's *almost* a quasifibration, though, which will be useful. For example, if $P \in -\mathrm{U}^{\mathrm{cpt}}$, it can be written as $P = -\mathrm{id}_H + \ell$, where $\ell \in \mathrm{cpt}$.

(1) If ℓ has finite rank and $K = \ker(\ell)$. Then, $H = K \oplus K^\perp$, and $K^\perp$ is finite-dimensional. Suppose $T \in q^{-1}(P)$, i.e. $\exp \pi T = P$. Then, $T|_{K^\perp}$ is determined by P, because $\exp(\pi, -) : [-i, i] \to \mathbb{T}$ sends the two endpoints to -1 and wraps around; in particular, it's one-to-one except at -1.

Asking about the fibers of the map is equivalent to asking for a logarithm, and the logarithm exists except at -1. Thus, we're okay except on a finite-dimensional subspace. In particular, there is a decomposition $K = K_+ \oplus K_-$, where each of $K_\pm$ is infinite-dimensional, and $T|_{K_+} = I$ and $T_{K_-} = -i$. Thus, $q^{-1}(P)$ is the Grassmanian of such splittings of K, which is a homogeneous space (U acts transitively on it), so $q^{-1}(P) \cong \mathrm{U}(K)/(\mathrm{U}(K_+) \times \mathrm{U}(K_-))$. By Kuiper's theorem, this is contractible.

Thus, over the subspace where ℓ has fixed rank n, q is a fiber bundle with contractible fibers.

(2) But it's not a quasi-fibration over the whole space. Let $e_1, e_2, \ldots$ be an orthonormal basis of H and define $P_1, P_2 \in -\mathrm{U}^{\mathrm{cpt}}$ by

$$P_1(e_n) = \exp\left(\pi i \left(1 - \frac{1}{n}\right)\right) e_n$$
$$P_2(e_n) = \exp\left(\pi i \left(1 + \frac{(-1)^n}{n}\right)\right) e_n.$$

That is, P_1 has eigenvalues clustering near -1 from one side, and P_2 is similar, but alternating around both sides (on the circle). But neither has -1 as an eigenvalue, so we can take the logarithm. The inverse image of P_1 has eigenvalues converging to i, so we get a skew-adjoint Fredholm operator with essential spectrum i, and therefore it's in $\widehat{\mathcal{F}}_+$: so $q^{-1}(P_1)$ is empty!

However, P_2 pulls back to something approaching both i and $-i$, so we do get a preimage of P_2, which is a point. This is not homotopy equivalent to $q^{-1}(P_1)$, so q isn't a quasi-fibration. Generically, -1 won't be in the spectrum, so inverse images will be unique; if -1 is in the spectrum, then we have extra stuff in the preimage. Ultimately, since a dense subspace of this has nice behavior, we can deformation retract both the domain and the codomain to make the fibers actually contractible, and get a quasifibration.

The point of this part is that you can chase around abstract things all day, but at some point you have to actually delve into the space of operators and work with that.

Unfortunately, we don't have time to put Clifford algebras back in, but this is the key: the bottom line is, we have a model for K-theory involving spectra and Fredholm operators. We'll use this in the second half of the class applied to geometry. In the next few weeks, we'll start with groupoids and the representation theory of compact Lie groups, and moving on to loop groups.

Part 3. Representations of Compact Lie Groups

Lecture 17.

Groupoids: 10/22/15

"I'm not going to tell you about index theorems, because I have no idea what they are."

Today's lecture was given by David Ben-Zvi.

To talk about groupoids, let's first think about equivalence relations. Specifically, an *equivalence relation* on a set X is a relation $E \subset X \times X$ (where one says that $x \sim y$ if $x, y \in E$), subject to some conditions. It's *reflexive*, so that $x \sim x$, meaning E contains the diagonal $\Delta \subset X \times X$; it's *symmetric*, meaning that $x \sim y$ iff $y \sim x$ (so that it's invariant under the transposition $\sigma : X \times X \to X \times X$). Finally, we need $\sim$ to be *transitive*, so if $x \sim y$ and $y \sim z$, then $x \sim z$. This can be thought of in terms of fiber products! Specifically, if we take the product across the two projections $p_1, p_2 : E \to X$, transitivity means that $E \times_X E = E$.

Equivalence relations are really ways of thinking about quotients: if $E \subset X \times X$, we have a quotient $Z = X/E$. This allows one to define an isomorphism of equivalence relations: if E is an equivalence relation on X and F is one on Y, a map $f : X \to Y$ is an isomorphism of E and F if the following diagram commutes.

$$\begin{tikzcd}
E \arrow[r, "f"] \arrow[d] & F \arrow[d] \\
X \times X \arrow[r, "f \times f"] & Y \times Y
\end{tikzcd}$$

This is much more useful than two equivalence relations being the same; an isomorphism of equivalence relations induces an isomorphism $X/E \xrightarrow{\sim} Y/F$. Really, all that we care about is the quotient; you can test everything on the quotient. We'll generalize this into the notion of groupoids.

Suppose a group G acts on a set X, so we say $x, y \in X$ are (orbit) equivalent if there's a $g \in G$ such that $gx = y$, and we can form the quotient X/G. There is a map $A : G \times X \to X \times X$ sending $(g, x) \mapsto (x, g \cdot x)$, and its image is exactly the equivalence relation. We'll change our way of thinking from E to $G \times X$ in order to approach groupoids. This is nicer in one part because E completely forgets about stabilizers.[27] For example, when G and X are topological, $\mathrm{Im}(A)$ might not behave well, e.g. it may not be closed, so the quotient isn't Hausdorff. The image isn't a great way to think about this.

So let's say $\mathcal{G} = G \times X$, and axiomatize what properties it has, which is what the theory of groupoids does.

Definition. A *groupoid* $\mathcal{G}$ acting on a set X is a set $\mathcal{G}$ along with maps $s, t : \mathcal{G} \rightrightarrows X$[28], $i : X \to \mathcal{G}$, and $c : \mathcal{G} \times_X \mathcal{G} \to \mathcal{G}$ (akin to composition) which satisfy the follwing three properties.

[27]The *stabilizer* $\mathrm{Stab}_G\, x \subset G$ of an $x \in X$ is the set of $g \in G$ for which $gx = x$.

[28]This is equivalent to specifying a map $\mathcal{G} \to X \times X$.

(1) The action is *reflexive*, i.e. the following diagram commutes.

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "\Delta"'] & & G \arrow[dl, "(s,t)"] \\ & X \times X & \end{tikzcd}$$

(2) The action is *transitive*, meaning the following diagram commutes.

$$\begin{tikzcd} \mathcal{G} \times_X \mathcal{G} \arrow[rr, "c"] \arrow[dr, "(s,t)"'] & & \mathcal{G} \arrow[dl, "(s,t)"] \\ & X \times X & \end{tikzcd}$$

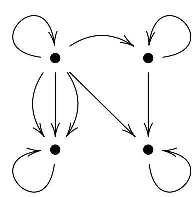
Moreover, c must be associative.

(3) There must be inverses, so with σ as above, the following diagram commutes.

$$\begin{tikzcd} \mathcal{G} \arrow[r] \arrow[d] & \mathcal{G} \arrow[d] \\ X \times X \arrow[r, "\sigma"] & X \times X \end{tikzcd}$$

Rather than think about all of the axioms, keep a good example in your head, for which you can write down all the axioms you want. Specifically, a group G acting on a point is a groupoid acting on $X = \bullet$, and the axioms mean that G has a unit, is associative, and has inverses. In fact, a groupoid acting on a single point is the same notion as a group. Another example: if $\mathcal{G} \hookrightarrow X \times X$, then what we have is exactly the notion of an equivalence relation. So you might think of groupoids as noninjective equivalence relations.

Alternatively, a groupoid acting on X is a bunch of arrows on points of X, but we require that every identity arrow exists and all compositions and inverses exist. (The inverses have been omitted from the following diagram to reduce clutter.)



Another way to think of this is as a "partially defined group," so we may not be able to compose all arrows, but we can invert them all.

Example 17.1. If X is a topological space, the *fundamental groupoid* or *Poincaré groupoid* $\mathcal{G} = \pi_{\le 1}(X)$ is defined as follows: for any $x, y \in X$, $\mathcal{G}_{x,y}$ is the set of paths $x \to y$ up to homotopy. Thus, $\mathcal{G}_{x,x} = \pi_1(X, x)$, and $\operatorname{Im}(\mathcal{G}) \subset X \times X$ is the equivalence relation of path components of X, i.e. $\pi_0(X)$.

There's yet another characterization of groupoids, which depends on categorical notions. It's almost better to have not seen it before: first examples of categories tend to be the category of all sets, of all groups, etc. These aren't necessarily how people actually use categories on a day-to-day basis.

Definition. A *category* $\mathcal{C}$ is a collection $\operatorname{Ob}(\mathcal{C})$ of *objects* and sets $\operatorname{Mor} \mathcal{C} = \mathcal{G}$ of *morphisms* (one writes $\operatorname{Hom}(X, Y) = \mathcal{G}_{x,y}$) such that:

- there is an *identity morphism* $1_X \in \operatorname{Hom}(X, X)$ for all $X \in \operatorname{Ob}(\mathcal{C})$, and
- there is an associative *composition* map $\operatorname{Hom}(X, Y) \times \operatorname{Hom}(Y, Z) \to \operatorname{Hom}(X, Z)$ for all $X, Y, Z \in \operatorname{Ob}(\mathcal{C})$.

You can think of a category as a bunch of arrows on $\operatorname{Ob}(\mathcal{C})$, such that the identity arrow and compositions all exist. This is suspiciously similar to the axioms for a groupoid!

Lemma 17.2. *Indeed, a groupoid is a category in which all morphisms are invertible.*

A category with one object satisfies precisely the same axioms as a *monoid* (intuitively, a group without inverses), so a category can be thought of as a partially defined monoid, which is actually a useful way to think about it. In other words,

$$\text{monoids} : \text{categories} :: \text{groups} : \text{groupoids}\,.$$

Would that we see that on the SAT!

Another mistake people make when thinking of categories is having the wrong pictue for when two categories are equivalent. One can formulate and write down a notion of isomorphism of categories, but this is considerably less useful than the more flexible notion of *equivalence of categories.* This is akin to the idea of a homotopy equivalence, rather than a homeomorphism.

Definition. A *functor* between groupoids (essentially just a map of groupoids) $\mathcal{G} \to X \times X$ and $\mathcal{H} \to Y \times Y$ is the data $f_0 : X \to Y$ along with a map of arrows $f_1 : \mathcal{G} \to \mathcal{H}$ (specifically, $\mathcal{G}_{x,y} \to \mathcal{H}_{f_0(x),f_0(y)}$) which commutes with associativity.

This can be summarized in the diagram

$$\begin{tikzcd} \mathcal{G} \arrow[r, "f_1"] \arrow[d] & \mathcal{H} \arrow[d] \\ X \times X \arrow[r, "(f_0,f_0)"] & Y \times Y. \end{tikzcd}$$

If you use $\mathrm{Hom}(X, Y)$ instead of $\mathcal{G}_{X,Y}$, then we get the familiar categorical notion of a functor. And if your groupoids are actually groups, you just get a homomorphism of groups.

A map of groupoids can be recast in the notion of equivalence relations: it provides a map $f : X/\mathcal{G} \to Y/\mathcal{H}$ on the quotients. We want to define the notion of isomorphism of groupoids to be isomorphism on quotients, not on the sets $\mathcal{G}$ and $\mathcal{H}$ *per se.*

Definition. Let $\mathcal{G} \rightrightarrows X$ and $\mathcal{H} \rightrightarrows Y$ be two groupoids and $f, g : \mathcal{G} \to \mathcal{H}$ be two functors of groupoids. Then, a *natural transformation* $\eta : f \to g$ is a way of connecting f to g by defining maps $\eta : f(x) \to g(x)$. For all $x, x' \in X$ and $\gamma : x \to x'$, we have maps $f(\gamma) : f(x) \to f(x')$ and similarly for g; for η to be a natural transformation we require that the following diagram commutes for all $x, x' \in X$ and $\gamma : x \to x'$.

$$\begin{tikzcd} f(x) \arrow[r, "f(\gamma)"] \arrow[d, "\eta"] & f(x') \arrow[d, "\eta"] \\ g(x) \arrow[r, "g(\gamma)"] & g(x') \end{tikzcd}$$

The same definition works for categories.

Notice that specifying a natural transformation $\mathcal{G} \to \mathcal{H}$ is equivalent to specifying an isomorphism on the quotients $X/\mathcal{G} \to Y/\mathcal{H}$. This allows us to define our analogue of homotopy.

Definition. An *equivalence of groupoids* $\mathcal{G} \sim \mathcal{H}$ is a pair of functors $f : \mathcal{G} \to \mathcal{H}$ and $g : \mathcal{H} \to \mathcal{G}$ such that there are natural transformation $fg \iff \mathrm{id}$ and $gf \iff \mathrm{id}$.

Again, this is exactly the same as specifying an isomorphism on the quotients.

Example 17.3. To make things a little more concrete, let X and Y be topological spaces and $f : X \to Y$ be continuous; thus, it induces a map $\pi_{\leq 1}(X) \to \pi_{\leq 1}(Y)$ given by composing paths with f.

If X is contractible, a map $\bullet \to X$ induces an equivalence of groupoids $\pi_{\leq 1}(\bullet) \to \pi_{\leq 1}(X)$! Though X and consequently $\pi_{\leq 1}(X)$ may be huge, the idea is that these things are "the same." Our map f induces $f : \pi_{\leq 1})(\bullet) \to \pi_{\leq 1}(X)$, and there is a unique map $g : \pi_{\leq 1}(X) \to \pi_{\leq 1}(\bullet)$. gf must be the identity, because there's no other map $\pi_{\leq 1}(\bullet) \to \pi_{\leq 1}(\bullet)$, but fg might not be; it sends a point x to a specific point x_0. Since X is contractible, there's a unique path $x_0 \to x$ up to homotopy, giving us a unique map $fg \to \mathrm{id}_X$.

So equivalence of groupoids is coarse, but remembers something "essential." $\pi_{\leq 1}(X)$ knows $\pi_0(X)$ and $\pi_1(X, x)$ for each $x \in X$ (so really for each connected component), and it turns out that equivalence of groupoids tracks these groups (i.e. an equivalence of groupoids induces an isomorphism on them) and nothing else.

To be precise, there is an equivalence of groupoids between $\pi_{\le 1}(X)$ and the groupoids

$$\pi_{\le 1}\left(\prod_{\alpha\in\pi_0(X)} K(\pi_1(X_\alpha, x_\alpha), 1)\right).$$

This space is sometimes called the 1-*truncation* of X, which has the same π_0 and π_1 as X, but no other homotopy.

It turns out this is a rather general example: if $\mathcal{G} \rightrightarrows X$, then we can actually build a topological space on which $\mathcal{G}$ is $\pi_{\le 1}$; for example, we take $\pi_0(\mathcal{G}) = X/\operatorname{Im}(\mathcal{G})$. Then, equivalence of groupoids is the same as homotopies of 1-truncated spaces, so you can relate homotopy theory and groupoids! And, again, this equivalence is also the same as specifying isomorphisms on the quotients.[29]

The point is, this equivalence relation is pretty floppy; if someone hands you a groupoid, you shouldn't get too attached to it (only up to equivalence).

There are yet more way to think about groupoids: a stack is a groupoid, and equivalence of groupoids is an isomorphism of the quotient stacks $X/\mathcal{G} \simeq Y/\mathcal{H}$.[30]

When we talked about groupoids at first, we used the language of sets. But you can throw any adjective in front of it: for example, a *topolgical groupoid* is the same as a groupoid where $\mathcal{G}$ and X are spaces and the specified maps are continuous; a *differentiable groupoid* requires $\mathcal{G}$ and X to be manifolds and the maps to be smooth; an *algebraic groupoid* uses varieties and algebraic morphisms, and so on, in your favorite category. There's another sense in which a topological groupoid is a functor from topological spaces to groupoids of sets (with some extra conditions; we're relating groupoids up to equivalence, so be careful). This relates to a common presentation of stacks: a sheaf is a functor from spaces (or varieties) to sets, and a stack is a functor to groupoids instead: replacing sets with groupoids is precisely what the generalization does.

Lecture 18.

Compact Lie Groups: 10/27/15

> *"I felt like I didn't have enough problems in my life."*
> *"I'll fix that."*

Professor Freed is back today. He may not post notes for these topics, so take notes.

The next two or three lectures will be a lightning review (or not review, in some cases) of compact Lie groups, their structure, and their representation theory; then, K-theory will come back into the picture (specifically equivariant K-theory) relating to vector bundles over groupoids. Then, we'll eventually talk about loop groups.

Let G be a compact Lie group.[31] A *Lie group* is the marriage of a manifold and a group: a group that is a manifold and such that multiplication and inversion are continuous. To require G to be compact means requiring its underlying manifold to be compact.

There are three basic examples of Lie groups.

(1) Finite groups (as 0-dimensional manifolds).
(2) Tori are particularly useful in structure theorems.
(3) Connected and simply-connected Lie groups are products of *simple* Lie groups, which can be classified: the *classical* groups SU_n, Spin_n, and Sp_n (which are roughly matrix groups); and the *exceptional* groups G_2, F_4, E_6, E_7, and E_8.

Note that U_n and O_n aren't in this list; U_n is a combination of SU_n and a torus, and SO_n is covered by Spin_n, and so we can build O_n and SO_n out of them.

A general Lie group G is off from these by a sort of twisting: it fits into a diagram

$$1 \longrightarrow G_1 \longrightarrow G \longrightarrow \pi_0 G \longrightarrow 1,$$

[29]Think about what this means on groups: we have an equivalence of groups G acting on X and H acting on Y when $X/G \simeq Y/H$, though we have to be careful about stabilizers. For example, $\mathbb{R}$ and $\mathbb{R}^{24}$ acting on $\mathbb{R}^{25}$ are equivalent as groupoids, even though they're quite different!

[30]"If you don't yet live in the world of stacks you should join."

[31]A compact manifold is necessarily finite-dimensional, so all of our compact Lie groups are finite-dimensional. One can think about infinite-dimensional Lie groups, but the results depending on compactness don't necessarily hold.

where $\pi_0(G)$ is finite and G_1 is connected (so, technically speaking, it's an *extension* of a connected Lie group by a finite group). Then, G_1 fits into the following diagram.

$$1 \longrightarrow F \longrightarrow \widetilde{G} \longrightarrow G_1 \longrightarrow 1$$

Here, F is finite and $\widetilde{G}$ is a product of tori and connected, simply-connected groups (whose components we understand, as stated above). So up to twisting, Lie groups have a reasonable product decomposition.

Example 18.1. Here's what this classification looks like for O_n.

$$1 \longrightarrow \mathrm{SO}_n \longrightarrow \mathrm{O}_n \xrightarrow{\det} \{\pm 1\} \longrightarrow 1$$

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}_n \longrightarrow \mathrm{SO}_n \longrightarrow 1$$

Since U_n is connected, we only need the second part of the decomposition: let μ_n denote the n^{th} roots of unity and $\mathbb{T}$ denote the circle group. Then,

$$1 \longrightarrow \mu_n \longrightarrow \mathrm{SU}_n \times \mathbb{T} \longrightarrow \mathrm{U}_n \longrightarrow 1.$$

Any Lie group G has a *Lie algebra* $\mathfrak{g} \subset \mathcal{X}(G)$ (where $\mathcal{X}(G)$ denotes the set of vector fields on G), the set of left-invariant vector fields. This inherits the Lie bracket $[-,-]$ from $\mathcal{X}(G)$, and thus has a Lie algebra structure.

To be precise, "left-invariant" means that the left-multiplication map by a $g \in G$, $L_x : G \to G$ sending $x \mapsto gx$, must commute with the vector field (which can be thought of as a function).

We also have the *exponential map* $\exp : \mathfrak{g} \to G$: given a $\xi \in \mathfrak{g}$, there's a flow $\Phi_t^\xi(x)$; we define $\exp(\xi) = \Phi_1^\xi(e)$. Since G is compact, this flow exists for all $t \in \mathbb{R}$, and gives us a one-parameter subgroup of G. From this definition, we deduce that $d\exp_e : T_0\mathfrak{g} \cong \mathfrak{g} \to T_eG = \mathfrak{g}$ (since $\exp(0) = e$) is just the identity, implying that the exponential map is a local diffeomorphism near $0 \in \mathfrak{g}$. If your Lie group is a matrix group, exponentiation is in fact the matrix exponential, which in general is not a homomorphism (there's a Taylor series formula).

Be careful: for $G = \mathrm{SO}_3$, exp maps a sphere of radius π to e: it stops being locally one-to-one. This is just like the Riemannian exponential map, and that's no coincidence.

In addition to left multiplication, there's also right multiplication $R_g : x \mapsto xg$, and we can talk about things invariant under both left and right multiplication.

Definition. If V is an n-dimensional real vector space, a *density* on V is a functional assigning a volume to every parallelepiped in V. That is, if $\mathcal{B}(F)$ denotes the space of bases on V, then it is a function $\mu : \mathcal{B}(V) \to \mathbb{R}$ such that $\mu(b \cdot g) = |\det g|\mu(b)$ when $g \in \mathrm{GL}_n(\mathbb{R})$.

These densities form an oriented line $\mathrm{Dens}(V)$ (there's a notion of a positive density: does it assign a positive density to your favorite basis?); this is awfully like a top-degree differential form, but with an absolute value for the determinant, so that we get an orientated space.

In other words, this is behaved in the way it should under change of basis.

Theorem 18.2.

(1) *There exists a bi-invariant*[32] *smooth density on G.*
(2) *There exists a bi-invariant Riemannian metric on G.*

An argument from Riemannian geometry can show that the two notions are equivalent.

Proof. For (1), a left-invariant density is equivalent to an element of $\mathrm{Dens}(T_eG)$. For any $g \in G$, pullback by $x \mapsto gxg^{-1}$ is an automorphism of $\mathrm{Dens}(T_eG)$. But automorphisms of an oriented line are only given by positive scalars (a group under multiplication).

So we have a map $G \to \mathbb{R}^{>0}$, but G is compact, so its image is a compact subgroup of $\mathbb{R}^{>0}$, and is therefore trivial. Thus, any left-invariant density is already right-invariant.

For the second part, average an arbitrary metric across this density (or, equivalently, the Haar measure) to get a bi-invariant one. ⊠

The existence of a smooth density is a measure, called the *Haar measure*; we can use it in various different ways to average things in a way that makes them invariant.

[32]That is, it's both left- and right-invariant.

Corollary 18.3. *Let $\mathbb{E}$ be a finite-dimensional $\mathbb{C}$-vector space and $\rho : G \to \mathrm{Aut}(\mathbb{E})$ be a representation. Then, there exists an invariant inner product on $\mathbb{E}$.*

In other words, we can choose a basis such that $\rho(g)$ is a unitary matrix for all $g \in G$, which is quite nice.

Proof. Choose an arbitrary inner product h on $\mathbb{E}$ and integrate $\int_G \rho(g)^* h \, \mathrm{d}\nu$, where $\mathrm{d}\nu$ is the Haar measure, normalized so that $\int_G \mathrm{d}\nu = 1$. The resulting inner product is G-invariant, because $\alpha^* h(\xi, \eta) = h(\alpha(\xi), \alpha(\eta))$. And we need to prove that it's an inner product, but an integral over a space with total measure 1 is the continuous version of a convex combination, which means it remains an inner product (the space of inner products is convex).[33] ⊠

Next, let's talk about tori. The *standard, one-dimensional torus* is $\mathbb{T} \subset \mathbb{C}$, the set of complex numbers with magnitude 1. In higher dimensions, we have $\mathbb{T}^n = \mathbb{T} \times \cdots \times \mathbb{T}$. These are fundamental for classifying abelian groups.

Theorem 18.4. *If G is a compact, connected, n-dimensional abelian Lie group, then $G \cong \mathbb{T}^n$.*

Proof. First, in the abelian case, $\exp : \mathfrak{g} \to G$ is a homomorphism: if $\xi_1, \xi_2 \in \mathfrak{g}$, then

$$\begin{aligned}(\exp \xi_1)(\exp \xi_2) &= \left(\exp \frac{\xi_1}{N}\right)^N \left(\exp \frac{\xi_2}{N}\right)^N \\ &= \left(\exp \frac{\xi_1}{N} \exp \frac{\xi_2}{N}\right)^N.\end{aligned}$$

Using the Taylor series formula,

$$\begin{aligned}&= \left(\exp\left(\frac{\xi_1}{N} + \frac{\xi_2}{N} + o\left(\frac{1}{N}\right)\right)\right)^N \\ &= \exp(\xi_1 + \xi_2 + o(1))\end{aligned}$$

Thus, taking $N \to \infty$, we approach $\exp(\xi_1 + \xi_2)$.

If $\Pi = \exp^{-1}(e) \subset \mathfrak{g}$, then Π is a discrete subgroup of a vector space, and therefore $\Pi = \mathbb{Z}^r$ for some $r \le n$; it has a basis of r linearly independent vectors, so $\mathfrak{g}/\Pi \cong \mathbb{T}^r \times \mathbb{R}^{n-r}$, and $\exp : \mathfrak{g}/\Pi \to G$ is a diffeomorphism.

Since exp is open, it has open image, but if G is a compact, connected Lie group, then it's generated by a neighborhood of e, but since G is compact, this forces $r = n$. ⊠

These are the Lie groups we call *tori*.

If G is a connected abelian Lie group that might not be compact, the general form is $V \times \mathbb{T}^n$, where V is a vector space. An easy example is $\mathbb{R}$ or $\mathbb{R}^n$.

Theorem 18.5. *If T is a torus Lie group, $\mathbb{E}$ is a complex vector space, and $\rho : T \to \mathrm{Aut}(\mathbb{E})$ is irreducible, then $\dim \mathbb{E} = 1$.*

This is exactly like the analogous statement for finite groups: an irreducible complex representation of a finite abelian group is one-dimensional.

Proof. If not, then there exists a $t \in T$ such that $\rho(t)$ isn't a multiple of $\mathrm{id}_{\mathbb{E}}$ (if not, then it's pretty clearly one-dimensional, or reducible). Since $\mathbb{E}$ is a $\mathbb{C}$-vector space, we can choose an eigenvalue λ for $\rho(t)$, so $\ker(\rho(t) - \lambda \,\mathrm{id}_{\mathbb{E}}) \subset \mathbb{E}$ is a proper (since λ is an eigenvalue and $\rho(t) \neq 0$) T-invariant subspace. This latter bit is because every other $\rho(t')$ commutes with $\rho(t)$, and you can write down that this forces it to be invariant. ⊠

Next, we have an analogue to Maschke's theorem.

Theorem 18.6. *If G is a compact Lie group and $\rho : G \to \mathrm{Aut}(\mathbb{E})$ is a representation, then $\mathbb{E}$ decomposes as a direct sum of irreducible subspaces.*

There's something to say here; for example, if $\varphi = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, then φ acts by shearing, and the x-axis is the only invariant subspace. However, the group generated by φ within $\mathrm{GL}_2(\mathbb{R})$ is isomorphic to $\mathbb{Z}$, which is noncompact. This is a useful example to keep in mind for when things go wrong.

[33] This is a continuous version of the trick of constructing inner products on tangent spaces of Riemannian manifolds by stitching local ones together using a partition of unity.

Proof. By Weyl's unitary trick, we have an invariant inner product, so if $\mathbb{E}_1 \subset \mathbb{E}$ is G-invariant, then so is $\mathbb{E}_1^\perp$, and $\mathbb{E} = \mathbb{E}_1 \oplus \mathbb{E}_1^\perp$; then, we repeat the process with $\mathbb{E}_1$ and $\mathbb{E}_1^\perp$. ☒

Corollary 18.7. *Any finite-dimensional representation of a torus is a direct sum of one-dimensional representations.*

If T is a torus, we can define two lattices $\Pi = \mathrm{Hom}(\mathbb{T}, T)$ and $\Lambda = \mathrm{Hom}(T, \mathbb{T})$. Each element of Π defines a one-parameter subgroup of T, so if $\mathfrak{t}$ is the Lie algebra of T, then we have a map $\Pi \to \mathfrak{t}$ sending $\gamma \to \mathrm{d}\gamma_e(i)$, where $\mathrm{d}\gamma_e : T_e\mathbb{T} \to T_eT = \mathfrak{t}$, and $T_e\mathbb{T} = i\mathbb{R}$. Its image is (possibly up to a term of 2π) the kernel of the exponential map. So Π is the group of one-parameter subgroups.

Λ is the group of *characters*, and maps into the dual space: $\Lambda \to \mathfrak{t}^*$ by $\lambda \mapsto \mathrm{d}\lambda_e$.

Example 18.8. If $T = \mathbb{R}^2/\mathbb{Z}^2$, and $\mathbb{R}^2$ has basis $\{x^1, x^2\}$, then $\Pi \cong \mathbb{Z}^2$, and $(n_1, n_2) \in \Pi$ acts as $e^{i\theta} \mapsto (e^{in_1\theta}, e^{in_2\theta})$. $\Lambda \cong \mathbb{Z}^2$ as well, and (m_1, m_2) acts as $(z_1, z_2) \mapsto (z^1)^{m_1}(z^2)^{m_2}$.

These characters are exactly the one-dimensional representations, because they're scalar multiplication, and a one-dimensional representation has to preserve the invariant metric, and thus must be scalar multiplication.

Thus, any representation $\mathbb{E}$ gives us a sum of one-dimensional representations, which are characters in $\Lambda \subset \mathfrak{t}^*$. The isomorphism type is defined by how many times each lattice element appears in this sum, so we get a function $\chi_\mathbb{E} : \Lambda \to \mathbb{Z}^{\geq 0}$ with finite support (indicating how many times a point appears).

Connected compact Lie groups. But we want to talk more generally about connected, compact Lie groups that might not be abelian. In this case, conjugation gives us a map $G \to \mathrm{Aut}(G)$ which is trivial if G is abelian. Thus, the conjugation action represents the failure of G to be abelian. Just as for discrete groups, the orbit of a $g \in G$ under this action is called its *conjugacy class*, and its stabilizer is the *centralizer* of g.

Example 18.9. Let's look at this for SO_3. Any rotation has a fixed axis, and composing rotations moves these axes. But there's a rotation flipping your favorite axis, so if we fix that axis, $\theta \in \mathrm{SO}_3$ is conjugate to $-\theta$. Thus, we get conjugacy classes 0, π, and $\{\pm\theta\}$ for other θ. In other words, we take the unit circle and mod out by reflection across the x-axis.

In general, the conjugacy class of 0 is itself, as usual, and for θ whose angle isn't π, you can consider points in different directions, and so we get a conjugacy class homeomorphic to S^2. At π, we only have half as many, and we get $\mathbb{RP}^2$.

The centralizer of 0 is SO_3, of course; the centralizer of rotation through some axis (except by π) is other rotations along that axis, i.e. SO_2, and of a rotation by π is centralized by these rotations and by reflections perpendicular to the axis, giving us O_2 in total.

Notice that we have a stabilizer that's not connected and a conjugacy class that's not simply connected. In a connected, simply-connected Lie group, this doesn't happen.

Exercise. Play with some other examples where there might be less intuition, specifically SU_2 and SU_3. You may want the theorem that every unitary transformation can be diagonalized.

Definition. A torus $T \subseteq G$ is *maximal* if for any other torus $T' \subseteq G$ with $T \subseteq T' \subseteq G$, then $T = T'$.

In other words, these are just maximal under inclusion.

Fact. Maximal tori exist (since there exist tori). No Zorn's lemma needed, since everything's finite-dimensional.

Here's the main theorem about maximal tori, which we won't completely prove.

Theorem 18.10. *Let G be a compact, connected Lie group and $T \subset G$ be a maximal torus. If $g \in G$, then there's an $x \in G$ such that $x^{-1}gx \in T$.*

That is, every element can be conjugated into a maximal torus. This will eventually imply that all maximal tori are conjugate.

Example 18.11. If $G = \mathrm{SO}_3$, a maximal torus is $T = \mathrm{SO}_2$, rotations about a fixed axis (which is one-dimensional). If we had a two-dimensional torus, it would include a rotation that commutes with all of these, but we found already that this SO_2 is a maximal centralizer for a point.

If $G = \mathrm{U}_n$, we can choose T (maximal tori need not be unique!) to be the diagonal matrices, and these turn out to be maximal. Theorem 18.10 tells us that every unitary matrix can be diagonalized.

You should also know that for $G = \mathrm{SO}_n$, the maximal torus is

$$T = \left\{ \begin{pmatrix} R_{\theta_1} & & & \\ & R_{\theta_2} & & \\ & & R_{\theta_3} & \\ & & & \ddots \end{pmatrix} \mid R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right\}.$$

Here, if n is odd, the last block diagonal entry just has to be a 1.

We'll also see that all maximal tori of a compact, connected Lie group have the same dimension, called the *rank* of the group.

Definition. If G is a Lie group and $g \in G$, then g *generates* G if $\{g^n\}_{n\in\mathbb{Z}}$ is dense in G. A Lie group generated by a single element is called *monogenic*.

Theorem 18.12. *Tori are monogenic.*

For $\mathbb{T}$, we have the generator $e^{2\pi i\theta}$, where $\theta \in \mathbb{R} \setminus \mathbb{Q}$. This is because if we have a rational value, it'll eventually hit the identity and repeat, and is otherwise dense.

On $\mathbb{T}^2$, we now want to avoid the 2-dimensional lattice of integer-valued points, and powers of some element are all going to lie on the same line (wrapping around, since we're on a torus). This is equivalent to choosing two transcendentally independent irrational points, which we can certainly do. This idea, generalizing to n-dimensional tori, becomes a proof of Theorem 18.12.

Lecture 19.

Maximal Tori of Compact Lie Groups: 10/29/15

"We proved it is monogenic, which means it takes good pictures."

Recall that if G is a compact, connected Lie group, we saw that it has a maximal torus $T \subset G$. Let's give some examples.

Example 19.1.

(1) If $G = \mathrm{U}_n$, then a maximal torus is the diagonal matrices,

$$T = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} : \lambda_i \in \mathbb{C}, |\lambda_i| = 1 \right\}.$$

This is maximal because if something commutes with all of these, it must be diagonal, and we already have all the diagonal matrices. This torus is clearly a direct product of circles.

(2) If $G = \mathrm{SO}_n$, one maximal torus is the block diagonal rotations, matrices of the form

$$\begin{pmatrix} R_{\theta_1} & & \\ & R_{\theta_2} & \\ & & \ddots \end{pmatrix}, \qquad R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

for $0 \le \theta \le 2\pi$.

(3) For $G = \mathrm{Spin}_n \subset \mathrm{C}\ell_n^\pm$, we have a maximal torus given by

$$T = \left\{ \prod_{i=1}^{\lfloor n/2 \rfloor} (\cos\theta_i + \sin\theta_i) e_{2i-1} e_{2i} \right\}.$$

It's instructive to check that, under the double covering map $\mathrm{Spin}_n \to \mathrm{SO}_n$, this torus is sent to the maximal torus we wrote down for SO_n (here $n > 1$).

One can also write down maximal tori for Sp_n and the exceptional groups.

We also have a theorem from the general theory of Lie groups.

Theorem 19.2. *If G is a Lie group and $H \subset G$ is a closed subgroup, then H is a Lie group.*

As such, one can check that if T is a torus, then $N(T) = \{n \in G : nTn^{-1} = T\}$ is a closed subgroup (it follows from the monogenicity of T), and therefore $N(T)$ is a Lie group itself.

Proposition 19.3. *If T is a maximal torus, the identity component of $N(T)$ is equal to T.*

Proof. Conjugation is a map $N(T) \to \mathrm{Aut}(T)$. So what's $\mathrm{Aut}(T)$? When $T = S^1$, we just get $\mathbb{Z}/2$ (generated by reflection across the y-axis), and in general $\mathrm{Aut}(T) \subset \mathrm{GL}_n(\mathbb{Z})$: we know $\mathrm{Aut}(T)$ is discrete, because an automorphism of T can be lifted to an automoprhism of the Lie algebra that must preserve our lattice Π from last time. Thus, $\mathrm{Aut}(T) \subset \mathrm{Aut}(\Pi) = \mathrm{GL}_n(\mathbb{Z})$.

In particular, the identity component $N(T)_e$ acts trivially, so it commutes with everything in T, and since T is maximal, then $N(T) \subset T$. ⊠

Let's see what that looks like, just to be sure.

Example 19.4. Let $G = \mathrm{SU}_2$, so we have a torus

$$T = \left\{ \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} : |\lambda_i| = 1 \right\}.$$

$N(T)$ has two components, T and

$$\left\{ \begin{pmatrix} & \mu \\ -\mu^{-1} & \end{pmatrix} : |\mu| = 1 \right\}.$$

Definition. The *Weyl group* of a maximal torus T is $W = N(T)/T = \pi_0 N(T)$.

This fits into a short exact sequence

$$1 \longrightarrow T \longrightarrow N(T) \longrightarrow W \longrightarrow 1.$$

This normalizer $N(T)$ is an interesting compact Lie group in its own right, but it may not be connected.

It's a theorem from the general theory of Lie groups that a quotient of a Lie group by a closed subgroup is a manifold. This means we can make the following definition.

Definition. The quotient homogeneous manifold G/T is called the *flag manifold*.[34]

It's homogeneous in the sense of the G-action, and has a lot of nice structure: one can put invariant Riemannian or even Kähler metrics on it.

Example 19.5. If $G = \mathrm{SU}_2$ and T is as in Example 19.4, then G/T is diffeomorphic to S^2, and $G \to G/T$ is a principal T-bundle. Notice (we'll see this idea again) if $\lambda : T \to \mathbb{T}$ is a character of T, then λ acts on by left multiplication, so there is an associated Hermitian line bundle $\mathcal{L}_\lambda \to G/T$, which is homoegeneous for the left G-action.

Remember Theorem 18.10? It said that if $T \subset G$ is a maximal torus, then every $x \in G$ is conjugate to an element of T. It has the following corollary.

Corollary 19.6. *If T and T' are maximal tori of G, then there's an $x \in G$ such that $x^{-1}T'x = T$.*

Proof. Apply Theorem 18.10 to a generator t' of T'. ⊠

Another corollary is that every element is contained in a maximal torus.

Corollary 19.7. *If G is a compact, connected Lie group, then the exponential map* $\exp : \mathfrak{g} \to G$ *is surjective.*

So if $\mathcal{T}$ is the manifold of maximal tori, then G acts transitively on $\mathcal{T}$. Thus, if we fix a maximal torus T, then there's a map $G \to \mathcal{T}$ sending $x \mapsto xTx^{-1}$, which identifies $\mathcal{T} = G/N(T)$. For example, if $G = \mathrm{SO}_3$, with the torus from before, one defines a torus as all rotations around a given axis, so $\mathcal{T}$ is our space of axes, $\mathbb{R}P^2$. This isn't orientable, and in fact that's usually the case.[35]

We'll give a proof of Theorem 18.10 due to Cartan that uses Lefschetz duality. Let's recall what that says.

[34] More generally, one can do something very similar with a not necessarily maximal torus T and $G/C(T)$, quotienting by the centralizer.

[35] Of course, if G itself is a torus, then it is its own unique maximal torus, and $\mathcal{T}$ is a point. But this is not the typical example.

Definition. Let M be a compact manifold and $f : M \to M$ be smooth. Then, the *Lefschetz number* of f is

$$L(f) = \sum_{q=0}^{\dim M} (-1)^q \operatorname{tr}(f^* : H^q(M;\mathbb{R}) \to H^q(M;\mathbb{R})),$$

which is in $\mathbb{Z}$.

This is useful for its following properties.

- $L(f)$ is a homotopy invariant (which is fairly easy to see from the definition).
- If f has isolated fixed points that form a set $\operatorname{Fix}(f)$, then
$$L(f) = \sum_{x\in\operatorname{Fix}(f)} L_x(f),$$
where $L_x(f) = \operatorname{sign}\det(1 - df_x)$ is the *local Lefschetz number* of f at x.

In particular, if $L(f) \neq 0$, then f must have a fixed point. This beautiful result from topology was probably in a class you had before this one, and we'll use it in our proof.

Proof sketch of Theorem 18.10. Let $g \in G$ and consider $f : G/T \to G/T$ sending $xT \to gxT$. Then, xT is a fixed point iff $x^{-1}gx \in T$. We can homotope this to $\widetilde{f}(xT) = t_0xT$, where $t_0 \in T$ is a generator. Then, $\operatorname{Fix}(\widetilde{f}) = N(T)/N = W$, using the definition of $N(T)$.

Now we just have to calculate the local Lefschetz numbers and show that their sum is nonzero, so $\widetilde{f}$ (and thus also f) has a fixed point.

First, $L_{nT}(\widetilde{f}) = L_{eT}(\widetilde{f})$, so the local Lefschetz numbers are all the same: replace t_0 wuth $n^{-1}t_0n$. This means we only need to compute at the identity; the way we do that is to lift $\widetilde{f}$ to a map $\widetilde{F} : G \to G$ sending $x \mapsto t_0xt_0^{-1}$. Thus, $\mathrm{d}\widetilde{F}_e : T_eG \to T_eG$, is just the adjoint Ad_{t_0}, so (we'll explain this in a second)

$$T_eG \cong \mathfrak{t} \oplus V_1 \oplus V_2 \oplus \cdots,$$

where the action is trivial on $\mathfrak{t}$ and $\dim V_i = 2$. This is because these are irreducible representations over $\mathbb{R}$, and therefore they are either trivial (which is only $\mathfrak{t}$) or two-dimensional.[36] In particular, t_0 acts on V_i by

$$R_{\theta_i} = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix},$$

where $0 < \theta < 2\pi$. This means $\det(1 - R_\theta) = 2(1 - \cos\theta) > 0$, so

$$L_{eT}(\widetilde{f}) = \prod \operatorname{sign}(2(1 - \cos\theta_i)) = 1,$$

so $L(\widetilde{f}) = |W| > 0$. ⊠

There is another proof is due to Bott TODO cite!

Corollary 19.8. *The Euler number $\chi(G/T) = |W|$, and since $G/T \to G/N(T)$ is an order-$|W|$ covering space, then $\chi(G/N(T)) = 1$.*

For example, $\mathbb{R}P^2$ was $G/N(T)$ for some G, and we know its Euler characteristic is 1.

So we've seen that the orbit of a $g \in G$, $\mathcal{O}_g$ (under the conjugation action), intersects the maximal torus. By thinking about $N(T)$, we saw that a finite group, W, acts on T. But we'll also be able to prove that the intersection of $\mathcal{O}_g$ and T is acted on by W, which will be useful for enumerating the conjugacy classes of G.

Corollary 19.9. *If $t_0, t_1 \in T$ are conjugate in G, there is a $w \in W$ such that $w \cdot t_0 = t_1$.*

The converse is also true, though trivial, from the definition of W.

Proof. Suppose $gt_0g^{-1} = t_1$, and let $H = Z(t_1)_e$ (the identity component of our centralizer). H is in general nonabelian, but it is a closed subgroup, hence a Lie subgroup. Since G is compact, this makes H a compact connected Lie group. Then, T and $gTg^{-1} \subset H$ are maximal tori of H. This means there's an $h \in H$ such that $T = hgTg^{-1}h^{-1}$, so $T = (hg)t_0(hg)^{-1}$ and $hg \in N(T)$. ⊠

This is more or less how the arguments with maximal tori tend to go.

[36] We didn't prove this, but such an irreducible representation must factor into two irreducible complex representations; SO_2 acting by rotation is an example of a two-dimensional irreduicble real representation.

Going Back to Our Roots. Suppose $T \subset G$ is a maximal torus, and recall that we have two lattices $\Pi = \mathrm{Hom}(\mathbb{T}, T) \subset \mathfrak{t}$ and $\Lambda = \mathrm{Hom}(T, \mathbb{T}) \subset \mathfrak{t}^*$. We'll be careful not to identify $\mathfrak{t}$ and $\mathfrak{t}^*$; if the group is simple, there's a metric induced by a Killing form, allowing an identification, but often this is not the case. Λ tends to be called the *weight lattice* or *character lattice*, and Π, which was the kernel of the exponential map, is called the *coweight lattice*.

We want to construct some other lattices, which will be the roots. T acts on $\mathfrak{g}$ by the adjoint action (the derivative of conjugation), and we can decompose this action into irreducibles, just as in the proof of Theorem 18.10:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_\alpha V_\alpha,$$

where $\dim V_\alpha = 2$. This is, again, ultimately because irreducible real representations are one- or two-dimensional. Thus, if we complexify, defining $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$, then the decomposition instead looks like

$$\mathfrak{g}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \bigoplus_\alpha (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}),$$

for $\alpha \in \Lambda$.

Definition. The set Δ of these α, where α is identified with $-\alpha$, is called the set of *roots* of G.

Example 19.10. Suppose $G = \mathrm{U}_2$, so that $\mathfrak{g} = \mathfrak{u}_2$ is the space of skew-Hermitian matrices. If you complexify, multiplying a skew-Hermitian matrix by i produces a Hermitian matrix, and every complex matrix is the sum of is Hermitian part and its skew-Hermitian part, so $\mathfrak{g}_\mathbb{C} = \mathfrak{gl}_2\mathbb{C}$.

Our maximal torus is, as usual for U_n, the group of 2×2 diagonal unitary matrices. Its Lie algebra is $\mathfrak{t}$, the space of all diagonal complex matrices, so if

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \overline{e} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \tag{19.1}$$

then we get a decomposition

$$\mathfrak{g}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \mathbb{C} \cdot e \oplus \mathbb{C} \cdot \overline{e}.$$

The *Cartan involution* $\beta : \mathfrak{g}_\mathbb{C} \to \mathfrak{g}_\mathbb{C}$ sending an $X \in \mathfrak{gl}_2\mathbb{C}$ to $-X^*$, has $\mathfrak{g}$ as its fixed points and sends $e \mapsto \overline{e}$.

Thus, if $\alpha = \lambda_1\lambda_2^{-1}$ (for a diagonal unitary matrix with entries λ_1 and λ_2), then $-\alpha = \lambda_1^{-1}\lambda_2$.

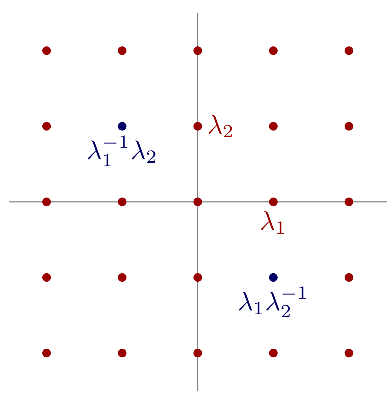


FIGURE 3. The roots in $\mathfrak{t}$; the red elements are the lattice Λ, and the blue ones are the roots, which generate a lattice $R \subset \Lambda$.

So if R is the lattice generated by these roots, it's a rank 1 sublattice of Λ, which is a rank 2 lattice in $\mathfrak{t}$.

Since we have a root lattice, we should also talk about the coroot lattice. Given a root $\pm\alpha$, there are (up to a phase in $\mathbb{T}$) distinguished basis elements $e_\alpha \in \mathfrak{g}_\alpha$ and $\overline{e}_\alpha = e_{-\alpha} \in \mathfrak{g}_\alpha$. We'll define $H_\alpha = -i[e_\alpha, e_{-\alpha}]$, subject to the commutation relations of $\mathfrak{su}_2$: if e_α and $e_{-\alpha}$ are as in (19.1) (with e_α in place of e, and $e_{-\alpha}$ in place of $\overline{e}$), then $[H_\alpha, e_\alpha] = 2ie_\alpha$, $[H_\alpha, e_{-\alpha}] = -2ie_{-\alpha}$, and $[e_\alpha, e_{-\alpha}] = iH_\alpha$.

This is the data of a homomorphism $\mathfrak{su}_2 \to \mathfrak{g}$, and therefore of $\mathrm{SU}_2 \to G$. One can check that $\alpha(H_\alpha) = 2$ and $\lambda(H_\alpha) \in \mathbb{Z}$ for all $\lambda \in \Lambda$; thus, $H_\alpha \in \Pi$, so it spans a sublattice $R^* \subset \Pi$, called the *coroot lattice*.

Example 19.11. If $G = \mathrm{SO}_3$, then our embedding of SU_2 comes from

$$H = \begin{pmatrix} 0 & -2 & \\ 2 & 0 & \\ & & 0 \end{pmatrix}, \quad e_+ = \begin{pmatrix} & & 1 \\ & & -i \\ -1 & i & \end{pmatrix}, \quad \text{and} \quad e_- = \begin{pmatrix} & & 1 \\ & & i \\ -1 & i & \end{pmatrix}.$$

Then, $R^* \subset \Pi$ has index 2, but $R = \Lambda$. (Draw some pictures!)

For $G = \mathrm{SU}_2$, one can show that $R^* = \Pi$, but $R \subset \Lambda$ has index 2. Again, it's definitely worth drawing these pictures.

Lecture 20.

Weyl Chambers, Roots, and Representations: 11/3/15

As usual, let G be a compact, connected Lie group and $T \subset G$ be a maximal torus. Then, we saw that if we complexify the action of G on its Lie algebra, it breaks into a sum of irreducible representations:

$$\mathfrak{g}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\alpha \in \Delta/\{\pm 1\}} \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}.$$

The action on $\mathfrak{t}_\mathbb{C}$ is trivial. These $\alpha : T \to \mathbb{T}$ are characters, and the $\mathfrak{g}_{\pm\alpha}$ are called *root spaces*.

Last time, we saw that for each root α, there is a unique $H_\alpha \in i[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ such that $\mathrm{ad}(H_\alpha)|_{\mathfrak{g}_\alpha} = 2i$; then, we can choose $e_\alpha \in \mathfrak{g}_\alpha$ and $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that H_α, e_α, and $e_{-\alpha}$ obey the relations of $\mathfrak{su}_2$: $[H_\alpha, e_\alpha] = 2ie_\alpha$, $[H_\alpha, e_{-\alpha}] = -2ie_\alpha$, and $[e_\alpha, e_{-\alpha}] = iH_\alpha$. From this we can see that $H_{-\alpha} = -H_\alpha$; H_α is called the *coroot* of α; the set of coroots is denoted Δ^*.

Example 20.1. For $G = \mathrm{SU}_2$, we have:

$$H = \begin{pmatrix} i & \\ & -i \end{pmatrix} \qquad e_\alpha = \begin{pmatrix} & 1 \\ & \end{pmatrix} \qquad e_{-\alpha} = \begin{pmatrix} & \\ -1 & \end{pmatrix}. \tag{20.1}$$

We also have a bunch of lattices: in $\mathfrak{t}$ we have $\Pi = \mathrm{Hom}(\mathbb{T}, T)$ (the closed one-parameter subgroups), but this is also $(1/2\pi)\exp^{-1}(e)$: notice that in the above example, if you multiply by $1/2\pi$ and exponentiate, we get back to the identity matrix. Π is called the *coweight lattice* or *integral lattice*.

Inside Π, the *coroot lattice* R^* is the span of all the coroots H_α. Inside $\mathfrak{t}^*$, we have the *weight lattice* $\Lambda = \mathrm{Hom}(T, \mathbb{T})$ and the *root lattice* $R = \mathbb{Z}\Delta$ (the span of the roots).

Example 20.2. Returning to SU_2 and SO_3, the most basic nontrivial examples, they have the same real, three-dimensional Lie algebra, so the adjoint map induces a two-to-one covering $\mathrm{SU}_2 \twoheadrightarrow \mathrm{SO}_3$. One can compute this action in a basis; start with (20.1), but this is a complex basis, so we need to take linear combinations of these. In any case, the maximal torus maps like this:

$$\begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} \longmapsto \begin{pmatrix} R_{2\theta} & \\ & 1 \end{pmatrix}.$$

This is an excellent example to play with; in the end, you should find that for SU_2, the root lattice has index 2 in Λ, but for SO_3, they're equal. Moreover, the adjoint representation of SU_2 drops to a representation of SO_3. The coroot lattice is equal to the coweight lattice in SU_2, but in SO_3 it has index 2.

If you compute the Weyl group of SU_2 or SO_3, you get $\mathbb{Z}/2$; in general, if one Lie group is a cover of the other, then they have the same Weyl group.

For a simpler example, if G is a torus T, $R = 0 \subset \Lambda$, and $R^* = 0 \subset \Pi$.

Calculating the indices of R in Λ and R^* in Π can be generalized neatly.

Definition. If A is an abelian group, its *Pontrjagin dual* $A^\vee = \mathrm{Hom}(A, \mathbb{T})$, the unitary characters of A.

Theorem 20.3. *If G is a Lie group, there are isomorphisms $\Pi/R^* \xrightarrow{\cong} \pi_1(G)$ and $\Lambda/R \xrightarrow{\cong} Z(G)^\vee$.*[37]

Proof. We can start with the maps $\Pi = \mathrm{Hom}(\mathbb{T}, T) \to \mathrm{Hom}(\mathbb{T}, G) \to \pi_1(G)$. A coroot $H_\alpha \in \Delta^*$ gives a map $\mathfrak{su}_2 \to \mathfrak{g}$, and exponentiation of a map of Lie algebras produces a map of Lie groups, as long as we take the domain to be the connected, simply connected Lie group with that Lie algebra. Thus, R^* is mapped to zero.

[37] That $Z(G)^\vee$ is "dual" to the fundamental group can be thought of in the context of Langlands duality!

For the second part, we want a map $\Lambda \to \mathrm{Hom}(Z(G), \mathbb{T})$. First, $Z(G) \subset T$ for all maximal tori T, and so a map $T \to \mathbb{T}$ restricts to a map $Z(G) \to \mathbb{T}$. Then, where do the roots go? The same idea shows they map to zero. ⊠

If $\pm\alpha \in \Delta$, then by differentiation, $\alpha : \mathfrak{t} \to \mathbb{R}$ is a linear functional, and in particular $\ker \alpha \subset \mathfrak{t}$ is a hyperplane. Thus, $\ker \Delta \subset \mathfrak{t}$ is a union of hyperplanes.

Example 20.4.

(1) For $G = \mathrm{U}_2$, T is the group of diagonal unitary matrices, so if $t = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$, then a root sends $t \mapsto \lambda_1\lambda_2^{-1}$ or $t \mapsto \lambda_1^{-1}\lambda_2$. Thus, $\mathfrak{t}$ is the algebra of matrices of the form $\begin{pmatrix} ix^1 & \\ & ix^2 \end{pmatrix}$, and the roots maps this to $\pm(x^1 - x^2)$. Thus, the kernel is the union of the lines $x^1 = x^2$ and $x^1 = -x^2$. Here, the Weyl group is $\mathbb{Z}/2$.

(2) When $G = \mathrm{SU}_3$, the torus is again the diagonal unitary matrices (so the sum of the diagonal entries is 1), and
$$\mathfrak{t} = \left\{ \begin{pmatrix} ix^1 & & \\ & ix^2 & \\ & & ix^3 \end{pmatrix} : x^1 + x^2 + x^3 = 0 \right\}.$$
This is a two-dimensional Lie algebra, but SU_3 is eight-dimensional, so we should expect six roots. We can let $\theta^j : \mathfrak{t} \to \mathbb{R}$ return the j^{th} diagonal element, so our roots are $\pm(\theta^i - \theta^j)$ when $i \neq j$. In this case, the kernels are three lines spaced 60° from each other, as in Figure 4. Here, the Weyl group is S_3.

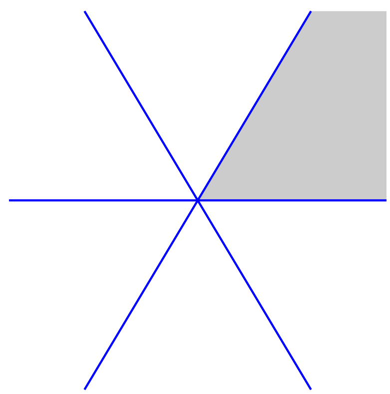


FIGURE 4. A picture of $\mathfrak{t}$, with $\ker(\Delta)$ in blue and a chamber in gray, for $G = \mathrm{SU}_3$. This is the chamber we'll use in Example 20.7.

Definition. A *Weyl chamber* is an element of $\pi_0(\mathfrak{t} \setminus \ker \Delta)$.

Thus, for example, the Weyl chambers of U_2 are the parts of $\mathfrak{t}$ above (resp. below) the line $x^1 = x^2$. The Weyl group $W = N(T)/T$ permutes the Weyl chambers.

Theorem 20.5. *The Weyl group acts simply transitively on the set of Weyl chambers.*

We won't prove this, but notice how it applies to Example 20.4. A lot of this is reminiscent of theorems in linear algebra about diagonalizing various kinds of matrices.

Corollary 20.6. *G acts transitively on pairs (T, C), where $T \subset G$ is a maximal torus and C is a Weyl chamber.*

If $\mathcal{P}$ denotes the set of such pairs, this means the action of G induces a map $G \twoheadrightarrow \mathcal{P}$; the stabilizer of this action is the torus itself (since $N(T)/T$ is permuting these chambers, but simply, so there's no stabilizer). In other words, the flag manifold can be thought of as $\mathcal{P}$, the space of pairs of maximal tori and Weyl chambers.

For the next definition, fix a Weyl chamber C.

Definition.

(1) A root α is *positive* if $\alpha(\xi) > 0$ for all $\xi \in C$. The set of positive roots is denoted Δ^+.
(2) If α is a positive root, H_α is said to be a *positive coroot.*
(3) The *dual Weyl chamber* $C^* \subset \mathfrak{t}^*$ is $C^* = \{\theta \in \mathfrak{t}^* : \theta(H_\alpha) > 0 \text{ for all } \alpha \in \Delta^+\}$.
(4) A weight λ is *dominant* if $\lambda(H_\alpha) \geq 0$ for all $\alpha \in \Delta^+$, i.e. $\lambda \in \Lambda \cap \overline{C^*}$.
(5) A weight λ is *regular* if $\lambda(H_\alpha) \neq 0$ for all $\alpha \in \Delta$.

Notice that either α is positive or $-\alpha$ is, because the Weyl chambers are split by the hyperplane where any roots vanish. Also, dominant weights can live on the walls of the Weyl chambers, but regular ones cannot.

We'll also let

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha,$$

which is in $\mathfrak{t}$, but not necessarily Λ. If G is simply connected, ρ is always a weight.

Remark. There is a complex structure on G/T. First, we get a G-invariant almost complex structure, as

$$T_{[T]}(G/T)_\mathbb{C} \cong \left(\bigoplus_{\alpha>0} \mathfrak{g}_\alpha\right) \oplus \overline{\left(\bigoplus_{\alpha>0} \mathfrak{g}_\alpha\right)}.$$

Then, the Neulander-Nirenberg theorem means we just have to check integrability. But since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ and $\alpha + \beta > 0$ if both α and β are positive, so this ends up working out.

Example 20.7.

(1) If $G = \mathrm{SO}_3$ with the torus we've used before, $\mathfrak{t}$ is given by the infinitesimal rotation matrices:

$$\mathfrak{t} = \left\{ x \begin{pmatrix} R_\theta & \\ & 0 \end{pmatrix} : x \in \mathbb{R} \right\}.$$

Thus, $\mathfrak{t}$ is one-dimensional, and the kernel of the roots is 0, so we have two chambers, $C = \{x > 0\}$ and $-C$; this is illustrated in Figure 5.

FIGURE 5. Depiction of $\mathfrak{t}$ when $G = \mathrm{SO}_3$; the two chambers are the positive and the negative numbers, respectively in red and in blue. The Weyl group is $W = \mathbb{Z}/2$.

If we choose the positive (red) chamber, the positive root is the one that sends

$$\alpha : x \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longmapsto x.$$

(2) When $G = \mathrm{SU}_3$, we've already seen some of this in previous examples. We can choose the chamber $C = \{x^1 > x^2 > x^3\}$. Then, the positive roots are $\Delta^+ = \{\theta^1 - \theta^2, \theta^2 - \theta^3, \theta^1 - \theta^3\}$ and $\rho = \theta^1 - \theta^3$, which is actually regular! Finally, the dual chamber is

$$C^* = \left\{ \sum \alpha_i \theta^i : \alpha_1 + \alpha_2 > 0, \alpha_1 + \alpha_3 > 0, \alpha_2 + \alpha_3 > 0 \right\}.$$

Lemma 20.8. *$\lambda \in \Lambda$ is dominant iff $\lambda + \rho \in C^*$.*

As usual, one direction is quite easy, and the reverse direction is harder.

Representation Theory. Let $\mathbb{E}$ be a complex vector space and $\rho : G \to \mathrm{Aut}(\mathbb{E})$ be a representation.

Definition. The *character* $\chi_\rho : G \to \mathbb{C}$ is $\chi_\rho(g) = \mathrm{Tr}(\rho(g))$.

In particular, the character is invariant within a conjugacy class of G.

This is very general, and even if we restrict to compact Lie groups, these include all finite groups, so we've encompassed the representation theory of finite groups as well.

Example 20.9. SU_2 acts naturally on $\mathbb{C}^2$, so if

$$g = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix},$$

with $\alpha, \beta \in \mathbb{C}$ and $|\alpha| + |\beta| = 1$, then $\chi(g) = 2\operatorname{Re}\alpha = \alpha + \overline{\alpha}$.

Lemma 20.10. *Let G be a compact group and $\mathbb{E}_1$ and $\mathbb{E}_2$ be representations of G.*

(1) $\chi_{\mathbb{E}_1 \oplus \mathbb{E}_2} = \chi_{\mathbb{E}_1} \oplus \chi_{\mathbb{E}_2}$.
(2) $\chi_{\mathbb{E}_1 \otimes \mathbb{E}_2} = \chi_{\mathbb{E}_1} \cdot \chi_{\mathbb{E}_2}$.
(3) $\chi_{\mathbb{E}_1^*} = \overline{\chi_{\mathbb{E}_1}}$.
(4) $\chi_{\mathbb{E}_1}(e) = \dim \mathbb{E}_1$.
(5) $\chi_{\mathbb{E}_1}(g^{-1}) = \overline{\chi_{\mathbb{E}_1}(g)}$.

If G is compact and connected, a character $\chi_{\mathbb{E}}$ is determined by its restriction to a maximal torus T, since everything is conjugate to something in T, and since T is abelian, the action of T on $\mathbb{E}$ splits into a direct sum of one-dimensional representations. Thus, the characters we get (for G) are in $\mathbb{Z}[\Lambda]$.

Since characters are invariant under conjugacy, then they're also invariant under the action of the Weyl group, so if K_G is the Grothendieck group of virtual characters,[38] $K_G \to (K_T)^W$ is an isomorphism.

But what about the irreducible representations of G? We also want to know their characters. The Weyl chambers and roots allow us to make sense of this, which we'll talk about next time.

Lecture 21.

Representation Theory: 11/5/15

"This isn't a baseball game, this is a math lecture."

Recall that, as in the last few lectures, we have a compact, connected Lie group G, a maximal torus $T \subset G$, and our four lattices: the coweight lattice Π, the coroot lattice $R^* \subset \Pi$, the weight lattice Λ, and the root lattice $R \subset \Lambda$.

Proof of Theorem 20.3. Since G is connected, then it has a connected universal cover $\widetilde{G}$. Let A denote the lift of T in $\widetilde{G}$, so we have the diagram

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "\pi_1 G"] & \widetilde{G} \arrow[d, "\pi_1 G"] \\ T \arrow[r, hook] & G. \end{tikzcd}$$

The first thing we can see is that $\pi_1 T = \Pi$, which is pretty much by definition; then, $\pi_1 A \cong R^*$, because by covering space theory, $\pi_1 G = \pi_1 T / \pi_1 A \cong \Pi / R^*$. This can be seen by splitting into two cases: where G is its maximal torus, or where it has other elements. In the former case, A is a vector space, and thus simply connected, for example.

For the second part, we have the *adjoint representation* $\rho : G \to \mathrm{Aut}(\mathfrak{g})$ acting by conjugation, so $\ker(\rho) = Z(G)$; denote $\mathrm{Ad}(G) = \rho(G)$. Thus, we can mod out by conjugation and get the following diagram.

$$\begin{tikzcd} T \arrow[r, hook] \arrow[d] & G \arrow[d] \\ \rho(T) \arrow[r, hook] & \mathrm{Ad}(G) \end{tikzcd}$$

Thus, we get an extention of abelian Lie groups:

$$0 \longrightarrow Z(G) \longrightarrow T \longrightarrow \rho(T) \longrightarrow 0.$$

Then, apply $\mathrm{Hom}(-, \mathbb{T})$.

$$0 \longleftarrow Z(G)^\vee \longleftarrow \Lambda \longleftarrow R \longleftarrow 0 \qquad \boxtimes$$

[38]We're doing the same thing we did in K-theory: start with the monoid of characters under direct sum, and formally take the completion of this monoid.

Hopefully this provides a more geometric way of thinking about the root and coroot lattices.

Now, let's talk about representations. All representation will be complex unless otherwise specified, since algebraic closure is so nice.

Definition. If $(\mathbb{E}, \rho)$ and $(\mathbb{E}', \rho')$ are representations of G, an *intertwiner* is a map $T : \mathbb{E} \to \mathbb{E}'$ such that $T(\rho(g)\xi) = \rho'(g)T\xi$ for all $g \in G$ and $\xi \in \mathbb{E}$, i.e. T commutes with the action of every $g \in G$. The vector space of intertwiners is denoted $\operatorname{Hom}_G(\mathbb{E}, \mathbb{E}')$.

Lemma 21.1 (Schur)**.** *Let $\mathbb{E}$ and $\mathbb{E}'$ be irreducible finite-dimensional representations of G. Then,*

$$\dim \operatorname{Hom}_G(\mathbb{E}, \mathbb{E}') = \begin{cases} 1, & \mathbb{E} \cong \mathbb{E}' \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let $T : \mathbb{E} \to \mathbb{E}'$ be an intertwiner; thus, $\ker(T) \subset \mathbb{E}$ is an invariant subspace, so since $\mathbb{E}$ is irreducible, then it must be either 0 or $\mathbb{E}$; a similar argument shows $\operatorname{Im}(T)$ must be either all of $\mathbb{E}'$ or 0. In particular, either $T = 0$ or T is an isomorphism.

Now, we need to show the space of isomorphisms is one-dimensional if $\mathbb{E} \cong \mathbb{E}'$. If $T_1, T_2 : \mathbb{E} \to \mathbb{E}'$ are isomorphisms, then $S = T_2^{-1} \circ T_1$ is an automorphism of $\mathbb{E}$. Since $\mathbb{E}$ is a complex vector space, S has an eigenvalue λ, so let $\mathbb{E}_\lambda$ be the associated eigenspace. In particular, $\mathbb{E}_\lambda \neq 0$ and is G-invariant, so $\mathbb{E}_\lambda = \mathbb{E}$, so $S = \lambda \operatorname{id}_{\mathbb{E}}$. ⊠

Recall that for a G-representation $\rho : G \to \operatorname{Aut}(\mathbb{E})$, we defined its character $\chi_{\mathbb{E}} : G \to \mathbb{C}$ sending $g \mapsto \operatorname{Tr}(\rho(g))$. This is a *central function*, i.e. it is constant on conjugacy classes, becayse $\chi_{\mathbb{E}}(hgh^{-1}) = \chi_{\mathbb{E}}(g)$. And since every conjugacy class of a compact connected Lie group hits a given maximal torus, $\chi_{\mathbb{E}}$ is determined by $\chi_{\mathbb{E}}|_T$, which is invariant under the Weyl group.

On a Lie group G, let $\mathrm{d}g$ denote the Haar measure (with total measure 1). If $C^0(G) = C^0(G; \mathbb{C})$ denotes the space of continuous functions $G \to \mathbb{C}$, then we can introduct an inner product on it:

$$\langle f_1, f_2 \rangle = \int_G \mathrm{d}g\, \overline{f_1(g)} f_2(g).$$

This space isn't complete under this inner product, but its completion is the Hilbert space $L^2(G)$. Inside of $L^2(G)$ is the closed subspace $L^2(G)^G$.[39]

In general, one wants to use the Haar measure to average things, but this doesn't make a whole lot of sense if we don't have scalar multiplication and addition in the integrand, so we usually average elements of a vector space parameterized by the group G.

Lemma 21.2. *Let $\rho : G \to \operatorname{Aut}(\mathbb{E})$ be a representation. Then,*

$$\pi = \int_G \mathrm{d}g\, \rho(g)$$

is in $\operatorname{End}(\mathbb{E})$ and is projection onto $\mathbb{E}^G \subset \mathbb{E}$.

Proof. Let $h \in G$; then,

$$\rho(h)\pi = \rho(h) \int_G \mathrm{d}g\, \rho(g) = \int_G \mathrm{d}g\, \rho(h)\rho(g) = \int_G \mathrm{d}(hg)\, \rho(hg) = \pi.$$

Conversely, if $\xi \in \mathbb{E}^G$, then

$$\pi(\xi) = \int_G \mathrm{d}g\, \rho(g)\xi = \int_G \mathrm{d}g\, \xi = \xi,$$

since the Haar measure has total measure 1. ⊠

Corollary 21.3 (Schur orthogonality relations)**.** *If $\mathbb{E}$ and $\mathbb{E}'$ are irreducible representations, then*

$$\langle \chi_{\mathbb{E}}, \chi_{\mathbb{E}'} \rangle = \begin{cases} 1, & \mathbb{E} \cong \mathbb{E}' \\ 0, & \text{otherwise.} \end{cases}.$$

[39]This notation may seem a little weird, but the idea is that G acts on itself by conjugation, which pulls back to an action on functions. So we're really taking the fixed subspace under this action.

Proof. The key is that, since $\mathbb{E}$ and $\mathbb{E}'$ are finite-dimensional vector spaces, then $\mathbb{E}^* \otimes \mathbb{E}' = \operatorname{Hom}(\mathbb{E}, \mathbb{E}')$. Thus, using Lemma 20.10,

$$\begin{aligned}
\langle \chi_{\mathbb{E}}, \chi_{\mathbb{E}'} \rangle &= \int_G \mathrm{d}g\, \overline{\chi_{\mathbb{E}}(g)} \chi_{\mathbb{E}'}(g) \\
&= \int_G \mathrm{d}g\, \chi_{\mathbb{E}^* \otimes \mathbb{E}'} \\
&= \int_G \mathrm{d}g\, \operatorname{Tr}(\rho_{\mathbb{E}^* \otimes \mathbb{E}'}(g)) \\
&= \operatorname{Tr} \int_G \mathrm{d}g\, \rho_{\operatorname{Hom}(\mathbb{E}, \mathbb{E}')}(g) \\
&= \dim \operatorname{Hom}_G(\mathbb{E}, \mathbb{E}'). \qquad \boxtimes
\end{aligned}$$

Example 21.4.

(1) First, take $G = \mathbb{T}$. Let $\mathbb{E}_n = \mathbb{C}$ with $\lambda \in \mathbb{T}$ acting as multiplication by λ^n. Thus, $\chi_n(\lambda) = \lambda^n$, or $\chi_n(e^{i\theta}) = e^{in\theta}$. In fact, $\{\chi_n\}$ is an orthonormal basis of $L^2(\mathbb{T})$: if $f \in L^2(\mathbb{T})$, we can write

$$f = \sum_{n \in \mathbb{Z}} \langle \chi_n, f \rangle \chi_n,$$

which is just the Fourier series of f.

(2) How about $G = \mathrm{SU}_2$? Let $V_0 = \mathbb{C}$ be the trivial representation and $V_1 \cong \mathbb{C}^2$ be the defining representation: that is, we're given SU_2 as a group of 2×2 matrices, and we just take that. In general, we'll let $V_n = \operatorname{Sym}^n(V_1)$, the n^{th} *symmetric power* of V_1; this space is obtained by forcing commutativity relations amongst the elements of the tensor algebra. Specifically,

$$\operatorname{Sym}^n V = V^{\otimes n} / (v_1 \otimes v_2 \otimes \cdots \otimes v_n - v_2 \otimes v_1 \otimes \cdots \otimes v_n, \text{ etc.}).$$

Thus, $\dim V_2 = 3$, and in general $\dim V_n = n + 1$, which is a nice combinatorial exercise.[40]

Now, let's look at their characters, so we'll restrict to the torus

$$\left\{ \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix} \mid \lambda \in \mathbb{T} \right\}.$$

The trivial representation acts just as the identity, so its character is $\psi_0(\lambda) = 1$. The defining representation acts by matrix multiplication, so $\psi_1(\lambda) = \lambda + \lambda^{-1} = \chi_1 + \chi_{-1}$. Notice that this is explicitly invariant under the Weyl group $W = \mathbb{Z}/2$.

For ψ_2, a basis of $\operatorname{Sym}^2 V_1$ is $\{e_1 \otimes e_1, e_2 \otimes e_2, (1/2)(e_1 \otimes e_2 + e_2 \otimes e_1)\}$, and calculating what $e_1 \mapsto \lambda e_1$ and $e_2 \mapsto \lambda^{-1} e_2$ does on each element, we get $\psi_2(\lambda) = \lambda^2 + 1 + \lambda^{-2}$. In the same way, $\psi_3(\lambda) = \lambda^3 + \lambda + \lambda^{-1} + \lambda^{-3}$.

In general,

$$\psi_n(\lambda) = \lambda^n + \lambda^{n-2} + \cdots + \lambda^{-(n-2)} + \lambda^{-n} = \frac{\lambda^{n+1} - \lambda^{-(n+1)}}{\lambda - \lambda^{-1}}.$$

(3) If we take the cyclic group of order n and think of it as $\mu_n \subset \mathbb{T}$, the group of n^{th} roots of unity, any representation of μ_n extends to a representation of $\mathbb{T}$, and any representation of $\mathbb{T}$ restricts to one on μ_n, but some representations retrict trivially. In fact, the distinct representations we get are $\chi_0, \ldots, \chi_{n-1}$. So we get an analogue of Fourier series, but in a finite sense, and that can be applied to any finite Lie group; here, $L^2(\mu_n)$ is just functions on these n points, and the Haar measure assigns $1/n$ to each point, as on any finite group.

(4) Let's look at S_3, the symmetric group on 3 items. There are three conjugacy classes, (1), (1 2), and (1 2 3); the first one has Haar measure $1/6$, the second has measure $1/2$, and the third has measure $1/3$. This is interesting, relating to sizes of centralizers of an element in each class.

[40]Sometimes, it's also useful to think of the symmetric power as a subspace of $V^{\otimes n}$; specifically it can be realized as $(V^{\otimes} n)^{S_n}$, where S_n is the symmetric group on n items. This doesn't work so well in finite characteristic, however.

We have a trivial represntation ρ_0, with character χ_0. There's also the *permutation representation* ρ_p, a three-dimensional representation where $\sigma \in S_3$ permutes the basis vectors. However, $(1,1,1)$ generates an invariant subspace, on which ρ_p is trivial, so there's a short exact sequence

$$0 \longrightarrow \rho_1 \longrightarrow \rho_p \longrightarrow \rho_3 \longrightarrow 0,$$

for a two-dimensional ρ_2, which you can check is irreducible (no other subspaces are fixed by the permutation representation). Finally, there's the *sign representation* ρ_1, a one-dimensional representation sending a permutation to its sign. See Table 1 for the character table; you can compute the characters directly for ρ_0 and ρ_1, and $\rho_2((1)) = 2$, since it's two-dimensional. Then, you can use orthogonality to fill in the rest or compute ρ_2 directly and check orthogonality.

	(1)	$(1\ 2)$	$(1\ 2\ 3)$
χ_0	1	1	1
χ_1	1	-1	1
χ_2	2	0	-1

TABLE 1. The character table for the symmetric group of order 3.

The next theorem is one of the major theorems for the representations of compact Lie groups. It relies on some other results we haven't proven, such as the fact that irreducible representations of a compact Lie group must be finite-dimensional.

Theorem 21.5 (Peter-Weyl)**.**
(1) As representations of $G \times G$,

$$L^2(G) \cong \bigoplus_{\mathbb{E} \text{ irreducible}} (\mathbb{E} \otimes \mathbb{E}^*).$$

Here, we take the Hilbert direct sum, i.e. the closure of the algebraic direct sum, and we choose one representative from each isomorphism class.
(2) $\{\chi_\mathbb{E}\}_{\mathbb{E} \text{ irreducible}}$ is a basis of $L^2(G)^G$.

Thus, there's a countable, discrete set of isomorphism classes of irreducible representations.

In general, we would want to enumerate the irreducible representations, hopefully actually construct them, and then compute their characters.

Isomorphism classes of finite-dimensional G-representations form a commutative monoid under direct sum, and tensor product makes this into a semiring, just as with vector bundles. Thus, we can define K_G to be the Grothendieck abelian group completion of this monoid, which has a ring structure. Applying this to Example 21.4, $K_\mathbb{T} \cong \mathbb{Z}[\lambda, \lambda^{-1}]$ and $K_{\mathrm{SU}_2} = \mathbb{Z}[\mu]$ (where $\mu = \chi_1$). There's a map $K_{\mathrm{SU}_2} \to K_\mathbb{T}$ defined by $\mu \to \lambda + \lambda^{-1}$, which is an instance of Theorem 21.6.

In some sense, we have a discrete ring generated by the irreducible characters hiding inside $L^2(G)^G$.

We can also check that $K_{\mu_n} \cong \mathbb{Z}[\mu_n]$; in general for a finite abelian group we get the group ring of its Pontryagin dual.

For K_{S_3}, we start with $\mathbb{Z}[\epsilon, \rho]$, where ϵ denotes the sign representation; then, you can look back at the character table to determine what multiplication looks like. The end result is $K_{S_3} \cong \mathbb{Z}[\epsilon, \rho]/(\epsilon^2 - 1, \epsilon\rho - \rho, \rho^2 - (1 + \epsilon + \rho))$.

These rings can get strange, but they're still interesting. We might want to think just about the monoid of actual characters, though.

Theorem 21.6. *Let G be a compact, connected Lie group and $T \subset G$ be a maximal torus.*
(1) Restriction induces an isomorphism $K_G \to (K_T)^W$.
(2) If G is simply connected, then K_G is a polynomial ring.

This theorem is not immediate, and we won't prove it.

Last time, we set up quite a lot of structure: the roots Δ and coroots Δ^*, a Weyl chamber C and the positive roots Δ^+ relative to C. We can also define the *dual Weyl chamber* $C^* \subset \mathfrak{t}^*$ by $C^* = \{\theta \in \mathfrak{t}^* \mid \theta(H_\alpha) > 0 \text{ for all } \alpha \in \Delta^+\}$. Thus, the ρ we defined yesterday is in C^*. This all comes together with the representation theory from today in the following theorem.

Theorem 21.7. *There are bijections between*

- *the isomorphism classes of irreducible representations of G,*
- *W-orbits in Λ,*
- *$\Lambda \cap \overline{C^*}$,*
- *$(\Lambda + \rho) \cap C^*$ (sending $\lambda \mapsto \lambda + \rho$),*
- *the set of integral coadjoint orbits, and*
- *the regular ρ-shifted integral coadjoint orbits.*

We'll talk about the last two sets in more detail next lecture.

Example 21.8. If $G = \mathrm{SU}_2$ and T is the subgroup of diagonal matrices, then the isomorphism classes of irreducible representation of G are represented by $\mathrm{Sym}^n V_1$ for $n \in \mathbb{N}$, as in Example 21.4.

Lecture 22.

The Geometry of Coadjoint Orbits: 11/10/15

We'll have one more lecture about representation theory today; on Thursday, Andrew Blumberg will talk about algebraic K-theory.

We have a bunch of structure: a compact, connected Lie group G, a maximal torus $T \subset G$, a Weyl chamber $C \subset \mathfrak{t}$ and its dual $C^* \subset \mathfrak{t}^*$, and the weight lattice $\Lambda \subset \mathfrak{t}^*$. We also have $\rho = (1/2)\sum_{\alpha\in\Delta^+} \alpha$.

Theorem 21.7 establishes a bunch of sets that are in bijection to the set of isomorphism classes of irreducible complex G-representations: the W-orbits in Λ, $\lambda \cap \overline{C^*}$, $(\Lambda+\rho)\cap C^*$, the set of integral coadjoint orbits, and the set of regular ρ-shifted coadjoint orbits.

Example 22.1. The canonical example is SU_3, which might be a little more complicated than some of our previous examples. We have $\mathrm{SU}_3 \hookrightarrow \mathrm{U}_3$, whose maximal torus is $\widetilde{T}$, the group of 3×3 diaognal unitary matrices. Its intersection with SU_3, denoted T, is a maximal torus for SU_3.

The weight lattice for U_3, $\widetilde{\Lambda} = \mathbb{Z}[\theta^1, \theta^2, \theta^3]$, surjects onto the weight lattice Λ for SU_3, and the kernel is $\mathbb{Z}[\theta^1+\theta^2+\theta^3]$, where the weights are

$$\theta^j \begin{pmatrix} ix^1 & & \\ & ix^2 & \\ & & ix^3 \end{pmatrix} = x^j.$$

This looks more like a triangular lattice, as depicted in Figure 6.

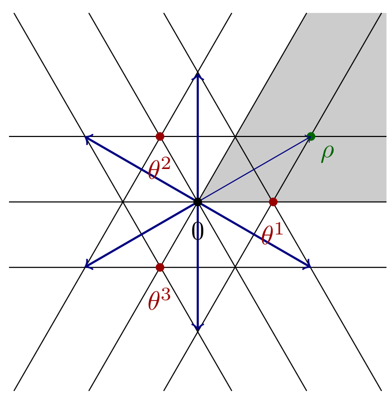


Figure 6. A diagram of $\mathfrak{t}^*$ for $G = \mathrm{SU}_3$. Irreducible representations correspond to lattice elements in the dual Weyl chamber (shaded in gray): 0 to the trivial representation, $\rho = \theta^1 - \theta^3$ to the adjoint representation, and multiplication by 2 corresponds to the symmetric square. The blue arrows denote the roots $\theta^i - \theta^j$ for $i \neq j$.

Theorem 22.2 (Weyl)**.** *The character of the representation with the highest weight $\lambda \in \Lambda \cap \overline{C^*}$ is*

$$\chi_\lambda = \frac{\sum_{w\in W}(-1)^w e^{i(\lambda+\rho)^w}}{\sum_{w\in W}(-1)^w e^{i\rho^w}}.$$

This is a nice, perhaps magical, formula.

We also mentioned the coadjoint action without really defining it, but it's not so bad. For any Lie group, the adjoint representation is how G acts on its Lie algebra $\mathfrak{g}$ by conjugation, and we automatically get a representation on its dual: the *coadjoint representation* is the dual representation to the adjoint, an action of G on $\mathfrak{g}^*$. If $\mathscr{O}$ denotes an orbit of the coadjoint representation, then it's a homogeneous space for G, and in fact a nice manifold; for example, if $G = \mathrm{SO}_3$, the orbit spaces are the spheres of different radii, and in fact

$$g^* = \coprod_{r\geq 0} S(r),$$

where $S(r)$ is the sphere of radius r.

More generally, let $\mu \in \mathscr{O}$, so we get a diffeomorphism $G/H_\mu \to \mathscr{O}$ given by $g \mapsto g\cdot\mu = \mathrm{Ad}_g^*(\mu)$. Thus, the kernel is $H_\mu = \{g \in G : g\cdot\mu = \mu\}$, which is exactly the stabilizer of μ. Passing to Lie algebras, we have a sequence of maps $\mathfrak{g} \to \mathscr{X}(\mathscr{O}) \to T_\mu\mathscr{O}$, first sending a left-invariant vector field to, well, itself, and then evaluating at μ to get a tangent vector. In particular, this defines an isomorphism $\mathfrak{g}/\mathfrak{h}_\mu \xrightarrow{\cong} T_\mu\mathscr{O}$; in our case, $\mathfrak{h}_\mu = \{\xi \in \mathfrak{g} : \mathrm{ad}_\xi^*\mu = 0\}$. But if $\eta,\xi \in \mathfrak{g}$ and $\mu \in \mathscr{O} \subset \mathfrak{g}^*$, then we can calculate that $\langle \mathrm{ad}_\xi^*\mu, \eta\rangle = \langle \mu, \mathrm{ad}_\xi \eta\rangle = \langle \mu, [\xi,\eta]\rangle$, so $\mathfrak{h}_\mu$ is also the set of ξ such that $\langle\mu,[\xi,\eta]\rangle = 0$ for all $\eta \in \mathfrak{g}$. This motivates the following definition.

Definition. The *Kostant-Kirilov-Soriau symplectic form* is an $\omega \in \Omega^2(\mathscr{O})$ given by $\omega_\mu(\xi,\eta) = \langle\mu,[\xi,\eta]\rangle$.

To show that this is well-defined, we need it to vanish if $\xi \in \mathfrak{h}_\mu$ or $\eta \in \mathfrak{h}_\mu$, but this is exactly what we just showed $\mathfrak{h}_\mu$ is. We also see that it's nondegenerate for the same reason (the degeneracy has been quotiented out, so to speak).

Next, we should check that ω is closed; suppose $\xi_1,\xi_2,\xi_3 \in \mathfrak{g}$, so we have three vector fields $\hat\xi_1,\hat\xi_2,\hat\xi_3 \in \mathscr{X}(\mathscr{O})$. Then, there's a nice formula for the exterior differential of a 2-form:

$$\mathrm{d}\omega(\hat\xi_1,\hat\xi_2,\hat\xi_3) = \sum_{\text{cyclic permutations}} \hat\xi_i\omega(\hat\xi_j,\hat\xi_k) \pm \omega([\hat\xi_i,\hat\xi_j],\hat\xi_k).$$

By applying the Jacobi identity twice, this vanishes (though you have to evaluate at μ and check what this means).

So we have a G-invariant, closed, nondegenerate symmetric form. Why is it G-invariant? We haven't made a single choice in this construction, so it has to be; but you can verify it if you want.

$(\mathscr{O},\omega)$ is a symplectic manifold, and these are all of the homogeneous symplectic manifolds. If you study group actions on symplectic manifolds from the perspective of symplectic geometry, you'll soon run into this kind of Lie theory.

Definition. An orbit $\mathscr{O} \subset \mathfrak{g}^*$ is *integral* if $[\omega/2\pi] \in H^2_{\mathrm{dR}}(\mathscr{O}) \cong H^2(\mathscr{O},\mathbb{R})$ lies in the image of $H^2(\mathscr{O},\mathbb{Z}) \to H^2(\mathscr{O},\mathbb{R})$.

What's going on here? The de Rham cohomology, which is isomorphic to the cohomology with real coefficients, is a vector space, and has plenty of lattices in it. One particularly nice choice for a lattice is the image of the integral-valued cohomology. In the de Rham world, this means that certain integrals of these forms end up in $\mathbb{Z}$.

The lattice $H^2(\mathscr{O},\mathbb{Z})$ has full rank inside $H^2(\mathscr{O},\mathbb{R})$, even if there's torsion in the integral cohomology groups, which ultimately follows because each degree in the cochain complex has $C^0 \hookrightarrow C^0 \otimes \mathbb{R}$ as a full-rank lattice (there's still something to prove here).

Theorem 22.3. *ω is an integral 2-form iff there exists a principal circle bundle $T \to \mathscr{O}$ with connection whose curvature is $i\omega/2\pi$.*

This is a general theorem from differential geometry, and provides a nice geometric condition for integrality. It's often true in symplectic geometry that the geometry or connection of the circle bundle says a lot about the form, e.g. if you lift it to this circle bundle, it becomes exact.

Definition. $\mathscr{O}$ is *regular* if the stabilizer $H_\mu \subset G$ is of minimal dimension for every $\mu \in \mathscr{O}$.

This is a more general definition about group actions. Also, since $\mathscr{O}$ is an orbit, then if one μ satisfies this minimality, then all of them do.

Proposition 22.4. *If G is a compact, connected Lie group, then $\mathscr{O}$ is regular iff $(H_\mu)_1$ is a maximal torus.*

That is, once we make this choice-free assumption that $\mathscr{O}$ is regular, we get a maximal torus T_μ for every point in H_μ, and $G/T_\mu \cong \mathscr{O}$ — and we get the whole story of roots for each one. This means we're looking at the flag manifold again.

In particular, we can look at the decomposition of $\mathfrak{g}$ and $\mathfrak{g}^*$ into $\mathfrak{t}$ and $\mathfrak{t}^*$ and some two-dimensional real invariant vector spaces.

Lemma 22.5. $\mu \in \mathfrak{g}^*$ *is a regular element of* $\mathfrak{t}^*$.

Corollary 22.6. μ *determines a choice of* C^* *(and therefore everything else).*

So, in summary, at each point, we have a maximal torus, a Weyl chamber, and so on, and therefore the enumeration of representations.

Proof of Lemma 26.5. We have to show that μ vanishes on everything except $\mathfrak{t}^*$. A basis for the other vector spaces is given by the roots e_α. Then,

$$\langle \mu, e_\alpha\rangle = \frac{1}{2i}\langle \mu, [H_\alpha, e_\alpha]\rangle = \frac{1}{2i}\langle \operatorname{ad}^*_{H_\alpha}\mu, e_\alpha\rangle = 0,$$

so $\mu \in \mathfrak{t}^*$. To show that it's regular, we need to show that $\langle \mu, H_\alpha\rangle \neq 0$ for all $\alpha \in \Delta$. The idea is to show that if it vanishes for some α, then $\operatorname{ad}^*_{e_\alpha}\mu = 0$, and therefore the stabilizer for μ is larger than T_μ, which we know to not be true. ⊠

This is our picture: a generic orbit is regular, and produces a flag manifold with a G-invariant symplectic structure, where every point gives us a maximal torus and Weyl chamber. Thus, we also get a ρ, which we'll call $\rho_\mu \in \mathfrak{g}^*$.

Definition. A regular orbit $\mathscr{O}$ is *ρ-shifted integral* if for all $\mu \in \mathscr{O}$, $\lambda = \mu - \rho_\mu$ is integral (i.e. lies in $\Lambda_\mu \subset \mathfrak{t}^*_\mu$).

We've heard that the integral orbits correspond to representations, but ρ-shifting makes the geometry nicer, and the bijection is still there. For example, for SO_3, our integral orbits are the spheres with integral radii, but $\rho = 1/2$, so when we ρ-shift, we get the spheres with half-integer orbits $1/2$, $3/2$, and so on.

If $\mathscr{O}$ is regular, then it also has a canonical complex structure, compatible with ω in the sense that the two together are the data of a Kähler manifold. What is this complex structure? Well, we can write $T_\mu\mathscr{O} \cong \mathfrak{g}/\mathfrak{t}_\mu$; after complexifying, it's isomorphic to the sum of the root spaces. Once we have a Weyl chamber, we can split into positive and negative roots, giving an almost complex structure. But we saw that these can be integrated, so we do have a complex structure. However, we're out of time, so we'll have to check that these two combine to a Kähler structure on our own. In any case, the coadjoint orbits have a beautiful geometry.

If $\mathscr{O}$ is a regular, ρ-shifted integral orbit, then we can use the character $e^{i\mu} : T_\mu \to \mathbb{T}$ to make an associated circle bundle to the principal T_μ-bundle $G \to \mathscr{O}$. One must check, however, that this is canonically independent of μ, that the action of G lifts to this bundle, and that if $\mathcal{L} \to \mathscr{O}$ is the associated Hermitian line bundle, then it inherits a complex structure. These are quite nice exercises in differential geometry.

The construction is that if one takes the space of holomorphic sections in $\mathcal{L}$, called $H^0(\mathscr{O}, \mathcal{L})$, then that space is an irreducible representation of G with highest weight $\lambda_\mu \in \Lambda_\mu \cap \overline{C^*}$; this is called the *Borel-Weil construction.*

Lecture 23.

Algebraic *K*-theory: 11/12/15

> *"This, however, is modern homotopy theory, the subject which you all really should be studying."*

Today Andrew Blumberg gave the lecture.

The story is that we want to do an analogue of topological K-theory in the algebraic setting, where there's not really a natural notion of a vector bundle.[41] This leads to a surprisingly productive question: *what is a vector bundle?*

[41] Well, there is, but we'll get there, and it's certainly not as obvious.

Rather than a space, we want a ring, or maybe a scheme, as input. This isn't as obviously geometric, especially in the ring case, so how do you define vector bundles? Once we define that, the rest of K-theory follows in a similar way.

The key to the answer is following result, due in this form to Swan, and an analogue of a theorem of Serre.

Theorem 23.1 (Swan)**.** *Let X be a compact, Hausdorff space*[42] *and $C(X)$ denote the algebra of continuous functions $X \to \mathbb{R}$.*[43] *Then, there is an equivalence of categories between the category of finitely generated, projective $C(X)$-modules and the category of vector bundles on X.*

The point is, this suggests that we should replace vector bundles with finitely generated projective modules.

We probably won't prove this today, but at minimum let's construct a functor from vector bundles to finitely generated projective modules. Let $p : E \to X$ be a vector bundle, and let $\Gamma(p)$ be the space of sections, $\{s : X \to E \mid ps = \mathrm{id}_X\}$. This is a $C(X)$-module, which is evident fiberwise, and it's also fairly easy to show this is functorial: a map of vector bundles gives us a map of sections.

Harder to show is that $\Gamma(p)$ is finitely generated, and that it's projective. It's finitely generated because X is compact, so you can choose a partition of unity and use finiteness to pick your generators.

Recall that a finitely generated R-module P is *projective* if there exists another R-module Q such that $P \oplus Q \cong R^n$. There are other characterizations. But for vector bundles, remember that any vector bundle is a summand of a trivial bundle, and trivial bundles are the analogue of free R-modules, so we're set.

We're probably at the level where proving that this functor is an equivalence could be an exercise, albeit a hard one.

Now, we will throw aside our geometric intuition, and work with a ring R and the category Proj_R of finitely generated, projective R-modules.[44] We're going to define the 0^{th} K-group to be $K_0(R) = K_0(\mathsf{Proj}_R)$, the Grothendieck group.

To be clear, we've taken the Grothendieck group of a monoid; today, we're going to take the Grothendieck group of a *category*.

Let C be a small category with a distinguished class of short exact sequences in C, taken as input data, and that is closed under isomorphism. Often, as for Proj_R, this class comes for free: we'll take all short exact sequences. But having the generality can be nice too.

We'll also assume two auxiliary hypothesis.

- C is symmetric monoidal, and
- exact sequences are closed under sum.

Both of these are satisfied in Proj_R, since we're considering all exact sequences.

Definition. Under the above assumptions, $K_0(\mathsf{C})$ is the free abelian group on the isomorphism classes of objects $[X]$ for $X \in \mathrm{ob}(\mathsf{C})$, with relations $[B] = [A] + [C]$ for every short exact sequence $0 \to A \to B \to C \to 0$ in our distinguished class.

Why would you do this?

(1) This is a universal target for the Euler characteristics: if you have a function which behaves like the Euler characteristic does under extension, then it factors through $K_0(\mathsf{C})$. The intuition should be the land of CW complexes: the Euler characteristic behaves well under sums. In fact, algebraic K-theory was originally motivated in this context, involving the Riemann-Roch theorem.

(2) This also "splits exact sequences," in a sense. In Proj_R, we always have the exact sequence

$$0 \longrightarrow A \longrightarrow A \oplus B \longrightarrow B \longrightarrow 0, \tag{23.1}$$

and so we've set $[A] + [B] = [A \oplus B]$. So this is exactly what we did on a monoid. But through the eyes of K_0, the sequence (23.1) is exactly the same as the sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

so $[C] = [A \oplus B]$. Thus, we're thinking of all extensions of A by B as the same. In Proj_R, where all short exact sequences split, this isn't such a big deal, but it's definitely a useful property in general.

[42] It may be possible to do this in greater generality, but some sort of separation hypothesis is necessary.

[43] You can do this with, for example, complex vector bundles instead, and the story is similar.

[44] Some set-theoretic issues arise when we do this, especially when we pass to isomorphism classes. We won't worry about it, though if you care, the issue can be resolved with some cardinal counting.

Now, let's make some observations.

First, K_0 is functorial with respect to functors that preserve the distinguished class of exact sequences. In this case, though, it's covariantly functorial: if $F : \mathsf{C}_1 \to \mathsf{C}_2$ plays well with exactness, then we have a group homomorphism $K_0(\mathsf{C}_1) \to K_0(\mathsf{C}_2)$.

When R and S are rings, and $f : R \to S$ is a ring map, then we have two adjoint functors: extension of scalars $-\otimes_R S : \mathsf{Mod}_R \to \mathsf{Mod}_S$, and restriction of scalars $f^* : \mathsf{Mod}_S \to \mathsf{Mod}_R$. However, restriction of scalars doesn't preserve projectiveness, in some situations, whereas extension always does. Thus, algebraic K-theory is always covariant in rings, and sometimes contravariant.

One of the beautiful things about this is how easily it generalizes: we can just as easily do this with schemes, and now also chain complexes of vector bundles or even many other things.

Now, suppose C is *bipermutative*, so there's a second product $\otimes$ along with $\oplus$. There's some coherence conditions that you might worry about, but broadly speaking, if C is bipermutative, then $K_0(\mathsf{C})$ is a ring. So in general, it captures a lot of structure about C.

There are two classes of particularly salient examples: Proj_R, where R is the ring of integers of a number field; and Proj_R where $R = \mathbb{Z}[\pi_1(M)]$ is the group ring of the fundamental group of a compact manifold. In the former case, this purely algebraic invariant recovers a lot of arithmetic structure; in the latter case, it's the natural home for a lot of constructions in (high-dimensional) geometric topology!

Example 23.2. Suppose F is a field and C is the category of finite-dimensional F-vector spaces, so all exact sequences in C split. In this case, $K_0(F) = \mathbb{Z}$; it captures the dimension. After all, there's a map in where an $n \in \mathbb{Z}$ is sent to the class $[F^n]$, and there's an obvious map out that's the dimension, and it's a nice exercise (which maybe you should do, and isn't very hard) is to check this is an isomorphism.

So what about K_i for $i > 0$? Or, for that matter, $i < 0$?

We built topological K-theory as a spectrum, or at the very least, as a sequence of groups. We'd like to do something similar. A good starting point would be to describe $K_1(\mathsf{C})$; there are multiple equivalent answers, one more algebraic and one in the style of K_0.

If $\mathsf{C} = \mathsf{Proj}_R$, then $K_1(R) = K_1(\mathsf{C}) = \mathrm{GL}(R)/E(R)$. What this says is that K_1 is about linear algebra and limiting processes in linear algebra. Here,

$$\mathrm{GL}(R) = \operatorname{colim}_n \mathrm{GL}_n(R),$$

where $\mathrm{GL}_n(R) \to \mathrm{GL}_{n+1}(R)$ is what you might guess: inclusion in the upper left corner (with the last diagonal entry as 1). $E(R)$ is the set of *elementary matrices* from Gaussian elimination: the matrices E_{ij}^λ that have a λ in the $(i,j)^{\text{th}}$ position, 1s on the diagonal, and 0s elsewhere.

Proposition 23.3. *$E(R)$ is normal in $\mathrm{GL}(R)$.*

This involves some mucking around with matrices, which we won't do, but it involves showing that $E(R) = [\mathrm{GL}(R), \mathrm{GL}(R)]$, which is another thing one could try as an exercise.

One takeaway is that this suggests a picture of K_1 as related to determinants. One cosmic takeaway is that basic constructions in linear algebra give rise to surprisingly sophisticated phenomena in algebraic K-theory: in the last 20 years, one of the most successful developments has been the trace, which is a significant generalization of the trace from linear algebra.

On the other hand, we can describe it as K_0, more or less, of a category of automorphisms. Once again, we'll assume C is a category with a class of exact sequences, and has a symmetric monoidal product.[45] Now, let $\mathrm{Aut}(\mathsf{C})$ be the category whose objects are (M, α), where $M \in \mathrm{ob}(\mathsf{C})$ and $\alpha : M \to M$ is an automorphism, and the morphisms are maps $f : M \to N$ that commute with the specified automorphisms: $f : (M,\alpha) \to (N,\beta)$ is required to satisfy

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\alpha"'] & N \arrow[d, "\beta"] \\ M \arrow[r, "f"] & N. \end{tikzcd}$$

We have a forgetful functor $(M, \alpha) \mapsto M$, and we say a sequence in $\mathrm{Aut}(\mathsf{C})$ is exact if its image under the forgetful functor is exact in C.

[45] In this context, the symmetric monoidal assumption is a pretty weak requirement; one generally considers categories that have finite coproducts, and you could just take this to be the symmetric monoidal structure, as we do in Proj_R.

Definition. $K_1(\mathsf{C})$ is the free abelian group on $[(M,\alpha)]$ subject to the relations $[(M',\beta)] = [(M,\alpha)]+[(M'',\gamma)]$ whenever we have a short exact sequence $(M,\alpha) \to (M',\beta) \to (M'',\gamma)$. We also have to specify that $[(M,\alpha\beta)] = [(M,\alpha)] + [(M,\beta)]$, so this plays nicely with composition.

K_1 is functorial, because $\mathsf{C} \mapsto \mathrm{Aut}(\mathsf{C})$ is functorial. You might wonder about the 2-functoriality of this, i.e. how it behaves under natural transformations, which is an interesting story.

Though K_0 was invented by Grothendieck, K_1 was invented by Hyman Bass, who stared at topological K_1, and cooked this up as an analogue of the suspension and clutching functions from topology.

This is justified in part by Mayer-Vietoris; we want algebraic K-theory to be a cohomology theory. It's a very different kind of cohomology than topological K-theory; for one, there's no analogue to Bott periodicity.[46] if we have a nice diagram of rings

$$\begin{tikzcd} R_1 \arrow[r] \arrow[d] & R_2 \arrow[d] \\ R_3 \arrow[r] & R_4, \end{tikzcd}$$

such that the categories of modules pull back, then we get an induced sequence

$$K_1(R_1) \longrightarrow K_1(R_2) \oplus K_1(R_3) \longrightarrow K_1(R_4) \longrightarrow K_0(R_1) \longrightarrow \cdots$$

which was Bass' greatest insight: it's a short, but tricky proof that the blue connecting morphism exists.

We also want to describe the relationship between $K_1(\mathsf{Proj}_R)$ and $\mathrm{GL}(R)/E(R)$. It's what you would write down: an invertible matrix is an automorphism of R^n: they reason this all works is because $\mathrm{GL}_n(R) = \mathrm{Aut}(R^n)$, so we send an $[M] \in \mathrm{GL}(R)/E(R)$ to (R^n, M), and this turns out to be an isomorphism. To go the other way, suppose $(P,\alpha) \in \mathrm{Aut}(\mathsf{Proj}_R)$, so if we complete P to a free module R^m (since it's projective) and extend α to the identity, then we get an automorphism of R^m. Working out why $E(R)$ is the kernel is a good exercise as well.

Here's another reason this is a good thing. If M is a compact manifold, the *Wall finiteness obstruction* lives in $K_0(\mathbb{Z}[\pi_1(M)])$. This is an Euler characteristic which determines an obstruction to whether M is isomorphic to a finite CW-complex. The $\mathbb{Z}[\pi_1(M)]$ occurs because you pass to a covering space; in general, if you see $\mathbb{Z}[\pi_1(M)]$, you should expect that reasoning.

Now, $K_1(\mathbb{Z}[\pi_1(M)])$ is home to *Whitehead torsion*, which is related to the S-cobordism theorem. Recall that Smale got the Fields medal for proving the h-cobordism theorem.

Definition. An *h-cobordism* is a cobordism $M \hookrightarrow W \hookleftarrow N$ where both arrows are homotopy equivalences.

Theorem 23.4 (h-cobordism (Smale))**.** *If M is at least 5-dimensional and $\pi_1(M) = 0$, then all h-cobordisms are trival (homeomorphic to $M \times I$).*

This implies, after a little work, the smooth Poincaré conjecture in dimensions 5 and above. But it's a handlebody argument, so it's really hard to reduce it to lower dimensions. So maybe we can attack the other assumption, that M is simply connected.

Theorem 23.5 (S-cobordism (Barden, Stallings, Mazur))**.** *If $\dim(M) \geq 5$, then isomorphism classes of h-cobordisms are in bijection with $K_1(\mathbb{Z}[\pi_1(M)])/\{\pm[M]\}$.*

Forty years later, we want to reframe this as follows: we see a *set* of isomorphism classes, and want to think of this as π_0 of a *space*. So we would like to find a space where h-cobordisms live.

As a historical note, after Bass invented K_1, the subject stalled: it took a long time for Milnor to invent K_2, and it seemed ad hoc. But eventually, Quillen managed to define all of them at once, by saying that K_0 and K_1 are π_0 and π_1 of an *algebraic K-theory spectrum* $K(\mathsf{Proj}_R)$, which is much easier to generalize.

The idea is to take the classifying space $B\,\mathrm{GL}(R)$, whose π_1 is violently nonabelian, and abelianize it in a certain way to obtain the spectrum. The story of how exactly this works is long and complicated, but beautiful.

So we know $K_0(\mathbb{Z}[\pi_1(M)])$ and $K_1(\mathbb{Z}[\pi_1(M)])$ for M a compact manifold. But if R is a Dedekind domain, we get different results: one can calculate that $K_0(R)$ is its class group, and $K_1(R) \cong R^\times$. It's much harder

[46] You can force Bott periodicity, and you end up with *étale K-theory*, which is a different and interesting subject.

to show that there's a relationship between $K_2(R)$ and the Brauer group. So this natural invariant links these two very different notions!

Back to h-cobordisms: this is very related to geometric topology, using coordinate or stable isotopy spaces, and is in fact telling a story about the classifying space of the diffeomorphism group. This is interesting because $K_n(\mathbb{Z}[\pi_1(M)])$ is obviously a homotopy invariant, and $B\operatorname{Diff}(M)$ clearly isn't, so somewhere along the way there was a stabilization process.

So someone might also think about applying K-theory to modern homotopy theory, where it allows for constructions analogous to rings and modules involving spectra and loop spaces. This relates to, for example, the fact that $\mathbb{Z}$ is the initial commutative ring, but the sphere spectrum is the initial commutative ring spectrum! There are lots of constructions outside of homotopy theory within geometry and topology, though. See Waldhausen's theorem for the full story. It's kind of a frustrating story: there were gaps in some proofs, which got filled in wrong, and then some people were angry at each other. But the mathematics is great, though hard.

Definition. Two rings R and S are said to be *Morita equivalent* if $\mathsf{Proj}_R \cong \mathsf{Proj}_S$, and there is an $(R.S)$-bimodule P and an (S, R)-bimodule Q such that $Q \otimes_R P \cong S$ and $P \otimes_S Q \cong R$.

Famously, $M_n(R)$ is equivalent to R.

Here's another fun facet of this, though maybe of dubious value. K-theory is not unlike $B\operatorname{GL}(R)$ (which isn't obvious from what we saw today, but is at least plausible). You can resolve the Hochschild homology $HH(R)$ with a *cyclic bar complex*; the takeaway is that you can map $B\operatorname{GL}(R) \to B^{\mathrm{cyc}}\operatorname{GL}(R)$, by adding a 1 at the end, and then embed this into the cyclic homology of $M_n(R)$, which by Morita equivalence maps into the cyclic bar complex of R, which is the Hochschild homology. This composition of maps turns out to be nonzero.

Algebraic K-theory is very hard to compute, but Hochschild homology is much easier to compute, so this is actually really nice. This is called the *trace map*, because the classical trace $M_n(R) \to R$ realizes the Morita equivalence.

Part 4. Equivariant *K*-theory

Lecture 24.

Equivariant Vector Bundles and the Thom Isomorphism: 11/17/15

"Someone's supposed to have a comment at this point... I don't know who."

Recall that a *topological groupoid* is a pair X_0 and X_1 of topological spaces (X_0 akin to the objects, and X_1 to the morphisms) and continuous maps $s, t : X_1 \to X_0$ (source and target, respectively), an identity map $i : X_0 \to X_1$, a composition map $c : X_1\, {}_t{\times}_s X_1 \to X_1$, and an inverse map $\iota : X_1 \to X_1$, which satisfy some axioms, e.g. composition of an arrow $x_1 \in X_1$ with the identity just gives us x_1 again, or the source of the inverse is the target of the original, and so forth. The group of arrows $x_0 \to x_0$ for an $x_0 \in X_0$ is sometimes called the *isotropy group* at x_0.

The stabilizer at a point in X_0 can be thought of encoding a set of symmetries at that point.

Example 24.1.

(1) Any topological space X_0 gives us a groupoid with $X_0 = X_1$, so the only arrows are the identities.
(2) The other extreme is a topological group X_1, giving us $X_0 = \mathrm{pt}$.
(3) More generally, if a group G acts on a space S, then we let $X_0 = S$, and the arrows are points of S with group elements (that is, an arrow is the association $(g, s) \mapsto g \cdot s$), so $X_1 = S \times G$. This *action groupoid* can denoted $X = S /\!\!/ G$; this notation is controversial, however. A special case of this is a principal G-bundle $P \to Y$.
(4) Suppose Y is a space and $\{U_\alpha\}_{\alpha \in A}$ is an open cover of Y. Then, we get a groupoid out of it: let

$$X_0 = \coprod_{\alpha \in A} U_\alpha.$$

Thus, we have a surjection $\pi : X_0 \twoheadrightarrow Y$. Let $X_1 = X_0 \times_Y X_0$ across π. In other words, if I have a $y \in Y$ such that $y \in U_\alpha$ and $y \in U_\beta$, then we add a unique arrow $(y, U_\alpha) \to (y, U_\beta)$; if $\alpha = \beta$ this produces an identity arrow, and the uniqueness forces a unique composition law.[47]

Given a groupoid (X_0, X_1), one can build a simplicial space, where X_0 is the set of 0-simplices and X_1 is the set of 1-simplices. Then, $s, t : X_1 \rightrightarrows X_0$ are the boundary maps. The identity map provides a map $X_0 \to X_1$. For X_2, we take pairs of composable maps: $x_0 \to x_1 \to x_2$, and we can take source, target, or composition to get back to X_1, or compose with the identity to go from $X_1 \to X_2$. Then, triples of composable maps define X_3, and so on; these are also iterated fiber products.

The notion of a principal G-bundle is like that of an open cover: between any two points in the fiber, there's a unique map between them. The difference is that the fiber might not be discrete. So there's a sense in which these two can be equivalent; in the abstract sense of equivalence of categories, these are equivalent, but we also have to remember the topological structure.

We can also talk about vector bundles over groupoids, or even fiber bundles, though we'll only need the former.

Definition. A *vector bundle* over a groupoid (X_0, X_1) is the data of a (classical) vector bundle $E \to X_0$ together with an isomorphism $\psi : s^*E \to t^*E$, subject to the condition that if $(f_0, f_1) : x_0 \to x_1 \to x_2 \in X_2$, then $\psi_{f_2 \circ f_1} = \psi_{f_2} \circ \psi_{f_1}$.

The idea is that ψ needs to be a fiberwise isomorphism on every arrow of X_1, which is how we pull back sources and targets. We have data on X_0, data on X_1, and a condition on X_2... but then it stops.

Just for comparison, what's a function on a groupoid? Instead of a vector space above each point, we have a number, such that every arrow induces an isomorphism of numbers, which, well, has to be an equality. If instead we have internal structure, non-identity isomorphisms may exist.

We should also be aware of the notion of a map of topological groupoids $(X_0, X_1) \to (Y_0, Y_1)$; the idea is to have continuous maps $X_0 \to X_1$ and $Y_0 \to Y_1$ that commute with composition, identity, etc. This means a real-valued function on a groupoid is a homomorphism of groupoids to $\mathbb{R}$ with all arrows equal to the identity. As such, we end up with a notion of maps of vector bundles, which is important: a map between vector bundles over a groupoid X is a map which is a linear map of vector spaces on each fiber, and each arrow in X_1 induces a commutative square with our map. So this is (data, condition) rather than (data, data, condition).

Let's look back at Example 24.1.

- If we turn a topological space into a groupoid, with arrows only the identity, a vector bundle over this groupoid is the same as a vector bundle over our original space.
- If we start with a topological group, which is a groupoid over a point, then the data we have is a vector space $\mathbb{E}$ and a continuous group of automorphisms of $\mathbb{E}$, so we get a continuous representation of our topological group!
- A vector bundle over a group action is what's called an equivariant vector bundle. We'll have more to say about this.

Definition. A map $f : X \to Y$ of topological groupoids is a *local equivalence* if

(1) f is an equivalence of underlying discrete groupoids,[48]

[47] If you don't like fiber products, you can also think of this as

$$X_1 = \coprod_{(\alpha,\beta) \in A \times A} U_\alpha \cap U_\beta.$$

[48] This means an equivalence of categories, an inverse g which need not be continuous that induces natural transformations to the identity on $f \circ g$ and $g \circ f$. In the case of the groupoid of a principal G-bundle, this is weird, because we're used to thinking that they have no sections — but the key is that we don't need the section to be continuous.

(2) f has *local sections*: for all $y \in Y_0$, there exists an open neighborhood U of y_0 and a σ that satisfies the following diagram, where the square is a pullback square, so $\widetilde{X}_0 = Y_1 \times_{Y_0} X_0$.

$$\begin{tikzcd}
 & \widetilde{X}_0 \arrow[r] \arrow[d] & X_0 \arrow[d, "f"] \\
 & Y_1 \arrow[r, "s"] \arrow[d, "t"] & Y_0 \\
U \arrow[r, hook] \arrow[uur, dashed, "\sigma"] & Y_0 &
\end{tikzcd}$$

This second condition may be opaque, but it's worth puzzling over; it ultimately comes from the familiar topological notion of local sections.

Condition (1) is equivalent to requiring that:

- f is *essentially surjective*, i.e. for all $y \in Y_0$ there's an $x \in X_0$ and $\beta \in Y_1$ such that $\beta : f(x) \to y$; and
- f is *fully faithful*, i.e. for all $x_0, x_1 \in X_0$, $f : X(x_0, x_1) \to Y(f(x_0), f(x_1))$ is a bijection.

A good example is that if $P \to Y$ is a principal G-bundle, $P /\!\!/ G \to Y$ is a local equivalence.

Theorem 24.2. *Suppose $f : X \to Y$ is a local equivalence of topological groupoids. Then, f^* induces an equivalence between the categories of vector bundles on X and vector bundles on Y.*

Proof idea. The idea is to construct a *descent* $f_* : \mathsf{Vect}_{X_0} \to \mathsf{Vect}_{Y_0}$. Given a vector bundle $E \to X_0$ and a $y \in Y_0$, we know $f : X \to Y$ is essentially surjective, so there's an $x \in X$ and $\beta \in Y_1$ such that $\beta : f(x) \to y$, and we can push E_x to y using this.

However, this choice of (x, β) is not unique. The way to resolve this is to make all choices at once: we end up getting a groupoid of choices $\mathcal{G}_y$ defined by $(\mathcal{G}_y)_0 = \{(x, \beta) \in X_0 \times Y_1 \mid \beta : f(x) \to y\}$ and $(\mathcal{G}_y)_1 = \{\alpha : (x, \beta) \to (x', \beta') \mid f(\alpha) = (\beta')^{-1} \circ \beta\}$. Since f is fully faithful, this means that, given (x, β) and (x', β'), the choice of α is unique, meaning $\mathcal{G}_y$ is *contractible*, i.e. equivalent to a point (ultimately, this is true of any groupoid where there's a unique arrow between any two points). So now we can take the vector space of parallel sections across α (or taking the limit of a diagram (TODO which diagram), which produces a canonically isomorphic vector space).

This is the construction of descent; now you have to figure out how lifting works, not to mention figure out how these fit together into vector bundles and continuous maps, which comes from the notion of local sections.

This general argument of getting a specific vector space from a groupoid of them is definitely worth investigating if you want to work with stacks or with groupoids.

Example 24.3. Let $X = \mathrm{pt} /\!\!/ (\mathbb{R}, +)$, so that a vector bundle $\mathbb{E} \to X$ is a representation of $(\mathbb{R}, +)$. We can form a vector bundle over $\mathbb{R} \times X$ whose fiber is $\mathbb{C}$ over every point ξ, but each $t \in \mathbb{R}$ acts by $e^{it\xi}$. This does not have homotopy invariance; the thing that goes wrong is that the stabilizer groups aren't compact.

This is a good counterexample to have in mind for the following definition.

Definition. A groupoid X is a *local quotient groupoid* if over a neighborhood of each point, it's locally equivalent to $S /\!\!/ G$, where S is a paracompact, Hausdorff space and G is a compact Lie group.

Theorem 24.4. *In this case, we have homotopy invariance: if X is a local quotient groupoid, $j_t : X \to [0,1] \times X$ sends $x \mapsto (t, x)$, and $E \to [0,1] \times X$ is a vector bundle, the $j_0^* E \cong j_1^* E$.*

We won't prove this, but it has a very similar proof to Theorem 2.1 from the non-equivariant case. However, one has to make the isomorphism compatible with the group action, which can be done by averaging over the Haar measure.[49]

So now, if S is compact, we can make K-theory out of isomorphism classes of equivariant vector bundles, in the same way: taking the monoid of equivalence classes of vector bundles, taking its group completion, and getting an equivariant cohomology theory. In the same way, we could talk about families of Fredholm

[49]This means we can probably generalize to compact Hausdorff groups.

operators parameterized over a groupoid, and generalize to the noncompact case; you can prove that you get a cohomology theory.

The first of these was developed by Segal and Atiyah, and the second by the professor and a few others. We won't develop this in detail, but we will use some of it.

The Thom Isomorphism. First, let's talk about the Thom class in de Rham theory. Suppose $\mathbb{V}$ is a real, n-dimensional vector space with an inner product. Then, the Thom class is a cohomology class represented by an $\eta \in \Omega_c^n(\mathbb{V})$ (i.e. η is compactly supported) such that $\int_{\mathbb{V}} \eta = 1$. Notice there are many of these, even an infinite-dimensional space of them, but the cohomology class should be unique. That is, adding anything exact (and compactly supported) to η should also produce a valid η, and the difference of any two choices should be exact. Here, though, "exact" means it's the derivative of something that was compactly supported. Thus, we get a unique $Y = [\eta]_{\mathrm{dR}} \in H_c^n(\mathbb{V})$.

Theorem 24.5. *The map $H^0(\mathrm{pt}) \to H_c^n(\mathbb{V})$ sending $c \mapsto c \cdot U$ is an isomorphism.*

Think of the point as the origin of $\mathbb{V}$, so that we have not just an inclusion $\{0\} \to \mathbb{V}$, but a projection in the other direction. Thus, the inverse map from $H_c^n(\mathbb{V}) \to H^0(\mathrm{pt})$ is given by $\int_{\mathbb{V}} -$ (since $H^0(\mathrm{pt}) \cong \mathbb{R}$). The full details of the proof can be found in Bott and Tu's book (TODO cite). Keep in mind, however, that this is only well-defined up to orientation, so we need to have an orientation of $\mathbb{V}$ to make this unique.

One interesting consequence of this is that such η can be thought of as approximating the δ distribution, which is supported at the origin and is 0 elsewhere. We would think of this as a $\delta_0 \in \Omega_c^n(\mathbb{V})^*$, i.e. a function: $\delta_0 : \Omega^0(\mathbb{V}) \to \mathbb{R}$ sends $f \mapsto f(0)$. More generally, we would get distributions of the sort $f \mapsto \int_{\mathbb{V}} f\eta$. In equivariant K-theory, we will actually get a reasonable notion of a delta distribution.

Now, if $\pi : V \to X$ is a rank-n vector bundle with an orientation, then the analogous construction produces a *Thom class* in the compactly supported vertical cohomology, $U \in H_{\mathrm{cv}}^n(V)$, and we have a theorem similar to Theorem 24.5 (again proved in Bott and Tu, using spectral sequences!).

Theorem 24.6. *The map $H^\bullet(X) \to H_{\mathrm{cv}}^{\bullet+n}(V)$ sending $[\omega] \to [\pi^*\omega] \cdot U$ is an isomorphism.*

In K-theory, we'll need a little extra structure: a spin structure in the real case and something slightly weaker in the complex case. But we get the δ-distribution for freer.

We'll start with a construction of Atiyah, Bott, and Shapiro. Let $\mathbb{V}$ be a real vector space and Q be a negative definite quadratic form. Let $A = \mathrm{Cliff}(\mathbb{V}, Q)$, so if $\xi \in \mathbb{V} \subset A$, then $\xi^2 = -|\xi|^2$ (we have this $\mathbb{Z}/2$-grading structure). Thus, if $\mathbb{E}$ is a (real or complex) left A-module, compatible with the inner product structure, then Riesz representation defines a map T from $\mathbb{V}$ to the odd-graded, skew-adjoint operators on $\mathbb{E}$, and if $\xi \in \mathbb{V} \subset A$, then $T(\xi)$ is invertible.

This defines an element of $K^0(\mathbb{V}), \mathbb{V} \setminus \{0\})$ (since it's invertible except at the origin), and by excision, this group is the same as $K_c^0(\mathbb{V})$ (K-theory with compact supports).

Another way to look at this is to take the one-point compactification $S^{\mathbb{V}} = \mathbb{V} \cup \{\infty\}$, so we get a class in $\widetilde{K}^0(S^{\mathbb{V}})$. We do this by defining a decomposition of $S^{\mathbb{V}}$ into upper and lower hemispheres, e.g. everything inside and outside the unit sphere in $\mathbb{V}$. Thus, we can clutch $\mathbb{E}^0$ and $\mathbb{E}^1$ by $T(\xi)$. It turns out that both of these viewpoints are isomorphic.

Example 24.7. Suppose $\mathbb{V} = \mathbb{R}$ and $A = \mathbb{R}\{1, e\}$, with $r^2 = -1$. Let $\mathbb{E} = \mathbb{R}^{1|1} = \mathbb{R} \oplus \mathbb{R}$, sending

$$e \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus, we'll get a bundle over the sphere, which acts by $+1$ when $x > 0$ and -1 when $x < 0$, so it actually produces the Möbius bundle.

Lecture 25.

The Thom Isomorphism and *K*-theory: 11/19/15

Before we return to the discussion of Thom classes and their role in K-theory, we'll talk a little bit about orientation and spin structure.

Let $\mathbb{V}$ be an n-dimensional vector space with an inner product. Then, we can define $\mathcal{B}_0(\mathbb{V})$ to be the set of bases of $\mathbb{V}$, i.e. the set of isometries $b : \mathbb{R}^n \to \mathbb{V}$ (here, $\mathbb{R}^n$ has its standard basis and inner product). Then,

O_n acts on $\mathcal{B}_0(\mathbb{V})$ freely and transitively on the right (i.e. there's a unique $h \in O_n$ for any $x, y \in \mathcal{B}_0(\mathbb{V})$ such that $h \cdot x = y$) by precomposition: if $h : \mathbb{R}^n \to \mathbb{R}^n$ is in O_n, then we get another basis $bh : \mathbb{R}^n \to \mathbb{V}$. That this action is free and transitive means that $\mathcal{B}_0(\mathbb{V})$ is a *torsor* for O_n.

O_n acts on $\mathbb{R}^n$ on the left, by the usual action of a function on a set, and in fact there's a canonical isomorphism $\mathcal{B}_0(\mathbb{V}) \times_{O_n} \mathbb{R}^n \to \mathbb{V}$ sending $(b, \xi) \mapsto b(\xi)$; the $\times_{O_n}$ means that we're identifying $(b, \xi) \sim (bh, h^{-1}\xi)$.

Definition. Let P be an H-torsor and $\rho : H' \to H$. Then, a *reduction* or *lift* of P to H' is a pair (Q, θ), where Q is an H'-torsor and $\theta : P \to \rho(Q)$ is an isomorphism, where $\rho(Q)$ is the H-torsor $Q \times_{H'} H$.

This is a linear version of a symmetry group in geometry, and as such is quite useful for keeping track of symmetry data. In different language, this is the same philosophy as Felix Klein's program.

Definition.

1. An *orientation* of $\mathbb{V}$ is a reduction of $\mathcal{B}_0(\mathbb{V})$ to $SO_n \hookrightarrow O_n$.
2. A *spin structure* of $\mathbb{V}$ is a reduction of $\mathcal{B}_0(\mathbb{V})$ to $\mathrm{Spin}_n \to O_n$.
3. A *spin*$^{\mathbb{C}}$ *structure* is a reduction of $\mathcal{B}_0(\mathbb{V})$ to $\mathrm{Spin}_n^{\mathbb{C}} \to O_n$.

Let's unpack these.

Example 25.1.

- If $\mathbb{V} = \mathbb{R}$, then $\mathcal{B}_0(\mathbb{V})$ is just two points, and $O_1 = \{\pm 1\}$, so to reduce to $SO_1 = \{1\}$, we just pick one of the two points, which is what orienting a line usually does.
- If $\mathbb{V} = \mathbb{R}^2$, O_2 is SO_2 union reflections, so this reduction of the torsor of bases is saying that we need to pick a subspace of bases that are a torsor for SO_2. $\mathcal{B}_0(\mathbb{R}^2)$ is two disjoint circles, so we just pick one of them, which is exactly picking an orientation: it tells us what bases are positively oriented. Moreover, any two bases of positive orientation are related by a unique element of determinant 1, as usual.

Spin and spin$^{\mathbb{C}}$ structures are analogous, but neither of these maps are inclusions. $\mathrm{Spin}_n \to O_n$ has image SO_n, so choosing a spin structure automatically chooses an orientation; in fact, it's a further lift across $\mathrm{Spin}_n \to SO_n \hookrightarrow O_n$. The kernel is $\{\pm 1\}$. For a complex spin structure, we do essentially the same thing, but this time the kernel is $U(1) \subset \mathbb{C}$.

The same story applies for vector bundles, thanks to local triviality. In fact, $\mathcal{B}_0(X) \to X$ is a principal O_n-bundle (i.e. a fiber bundle of torsors). Asking for a reduction of a structure group to SO_n is the same thing as a reduction of the torsor, and may be a more familiar definition of an orientation. The two spin structures are analogous.

Now, though, we see that there's an existence and uniqueness question. The space of orientations of a vector space $\mathbb{V}$ is two points (so we have a $\mathbb{Z}/2$-torsor of orientations), so there are two possible orientations of every fiber of your vector bundle $E \to X$. This gives a double cover of X, and since we want continuity, we need a section, reframing this problem as an existence and uniqueness problem of sections over a vector bundle.

What about spin structure? In the case $n = 1$, O_1 and Spin_1 are both cyclic of order 2, but the map must factor through SO_1, so we're looking at the constant map $\mathrm{Spin}_1 \to O_1$. To think of these more generally, we need to know what it means to have a map of reductions of a torsor.

Definition. Let P be an H-torsor, $\rho : H' \to H$, and (Q, θ) and (Q', θ') be reductions of P to H'. Then, a map of these torsors is an H'-equivariant map $\varphi : Q \to Q'$ such that the following diagram commutes.

$$\begin{tikzcd}
\rho(Q) \arrow[rr] & & \rho(Q') \\
& P \arrow[ul, "\theta"] \arrow[ur, "\theta'"'] &
\end{tikzcd}$$

Now, specializing, we have a double cover $\mathrm{Spin}_n \to SO_n$, so for every orientation we have two choices of the spin structure, and now we're looking at a more general class of double covers. But the maps between them are all invertible, so the category of spin structures is actually a groupoid!

There are topological invariants that provide some answers to whether such a structure exists, called *Steifel-Whitney classes*.

Now, suppose $\mathbb{V}$ has a spin structure; then, there is a canonical Clifford module $\mathbb{S} = \mathcal{B}_{\mathrm{spin}}(\mathbb{V}) \times_{\mathrm{Spin}_n} \mathrm{C}\ell_{-n}$. Since the spin group sits inside the Clifford algebra (in the same way $\mathrm{O}_n \subset M_n(\mathbb{R})$), we're identifying $(b, a) \sim (b \cdot \widetilde{h}, \widetilde{h}^{-1} a)$ for all $\widetilde{h} \in \mathrm{Spin}_n$. The picture is that we have a space of spin bases (spin structures with basis), and there's a constant copy of $\mathrm{C}\ell_{-n}$ over each. However, we can act by Spin_n to move between different spin bases, and this construction takes vertical sections. In some sense, we're integrating out the choice of basis.[50]

$\mathbb{S}$ is a vector space, but not an algebra; on $\mathrm{C}\ell_{-n}$, we're acting by right multiplication where we would expect left multiplication, so we don't get the induced algebra structure you might expect. Nonetheless, this is a $\mathrm{Cliff}_-(\mathbb{V}) = \mathrm{Cliff}(\mathbb{S}, -Q)$-module, and a right $\mathrm{C}\ell_{-n}$-module.[51]

One could take this to be the definition of a spin structure, though the definition we presented is considerably more general for geometric constructions.

Now, we can refer to the Thom class: consider the constant bundle $\underline{\mathbb{S}} \to \mathbb{V}$. If $\mu \in \mathbb{V}$ and T_μ denotes left Clifford multiplication by $\mu \in \mathrm{Cliff}_-(\mathbb{V})$ (which acts on $\mathbb{S}$), then T maps V into the space of odd, skew-adjoint endomorphisms of $\mathbb{S}$ and $T_\mu^2 = -|\mu|^2$.

In other words, we've defined a representative $U = [\mathbb{S}, T]$ in some K-theory. It's real K-theory, and the right $\mathrm{C}\ell_{-n}$ structure means we've shifted up to the n^{th} K-theory. Since it's 0 at the origin, this means $U \in KO^n(\mathbb{V}, \mathbb{V} \setminus \{0\})$. This is isomorphic to $KO^n_c(\mathbb{V})$, and under this isomorphism this U is sent to something supported only at 0, so we have a vector-bundle analogue of a δ-function! This U is the *Thom class.*[52]

Theorem 25.2. *The map $KO^0(\mathrm{pt}) \to KO^n_c(\mathbb{V})$ sending $[\mathbb{E}] \mapsto [\pi^*\mathbb{E}] \cdot U$ is an isomorphism.*

These are both isomorphic to $\mathbb{Z}$, and so maybe this theorem isn't so hard; here's the real theorem.

Theorem 25.3. *Let $\pi : V \to X$ be a real vector bundle of rank n with a spin structure. Then, the map $KO^\bullet(X) \to KO^{\bullet+n}_{\mathrm{cv}}(X)$ sending $[\mathbb{E}] \mapsto [\pi^*\mathbb{E}] \cdot U$ is an isomorphism.*

We'd like to generalize a little bit: let G be a compact Lie group acting on $\mathbb{V}$, a real vector space, from the left. Then, this induces an action of G on $\mathcal{B}_0(\mathbb{V})$ which commutes with the action of O_n from the right. It's still an O_n-torsor, but now we have a group of symmetries, so we can talk once again about the orientation, spin structure, or spin$^{\mathbb{C}}$ structure associated to this object, which says that this is a G-equivariant structure.

Last time, we mentioned that this gives us a groupoid of these structures which is acted on by G. Then, the definition would look like lifting a torsor again, but in a G-equivariant way.

Example 25.4. Consider $\mathbb{V} = \mathbb{R}^2$ with an orientation, on which SO_2 acts by rotations. Now, we get $\mathcal{B}_{\mathrm{SO}}(\mathbb{R}^2)$, so we have an "internal" right action of SO_2 and a "geometric" action of SO_2 on the left. This $\mathcal{B}_{\mathrm{SO}}(\mathbb{R}^2)$ is a circle, so to talk about spin structures, we want to find a double cover of the circle whose covering map is equivariant? Well, no; one full rotation of the circle on bottom isn't the identity on the cover, so there's no SO_2-equivariant spin structure.

This example might be simpler than the general case, because SO_2 is abelian.

In any case, if we have an equivariant spin structure, we can run the Thom story over again, but with G-equivariance in the background (and sometimes foreground). In the end, we get an analogue of Theorems 25.2 and 25.3 in equivariant K-theory: there will be an isomorphism $KO^\bullet_G(X) \to KO^{\bullet+n}_G(V)_{\mathrm{cv}}$.[53]

Let G be a compact Lie group and consider the coadjoint action of $\mathfrak{g}^*$ on itself. Kirillov described a one-to-one map from regular ρ-shifted, integral coadjoint orbits to the irreducible complex representations of G. Then, the Freed-Hopkins-Teleman paper (TODO cite) gives us a map in the other direction. We can reframe this in light of things we've talked about today.

[50]There's a $\mathbb{Z}/2$-grading floating around, as usual, and the fact that we're using the spin group rather than the pin group means that these maps respect this grading.

[51]In fact, it's even an invertible module, in the sense of Morita, making it a *Morita isomorphism* between these two algebras.

[52]You may be wondering how this relates to the Thom class in cohomology. It turns out there are several ways to relate K-theory and cohomology; the simplest one would be with cohomology in rational coefficients, using something called a *Chern character.* It's an interesting story involving characteristic classes, which we aren't assuming in this class.

[53]Last time, we talked about vector bundles over a groupoid, and defined equivariant vector bundles over local quotient groupoids. If we have a global quotient of a compact space by a compact Lie group, we define equivariant K-theory by taking the classes of finite-rank equivariant vector bundles, as in the non-equivariant case. Alternatively, one can develop Fredholm operators and Clifford algebras for all local quotient groupoids, again as in the non-equivariant case.

Let $\mathbb{E}$ be a finite-dimensional, complex representation of G (which is, say, n-dimensional); using the Haar measure, we can fix a G-invariant inner product on $\mathbb{E}$. Then, we can consider $\mathbb{E}$ as a constant vector bundle over $\mathfrak{g}^*$; this is equivariant under the action of G on $\mathfrak{g}^*$. To take the Thom construction, we should tensor with U; the result is a family of odd, skew-adjoint operators commuting with the left $\mathrm{C}\ell_{+n}$-action $T : \mathfrak{g}^* \to \mathrm{End}(\mathbb{E} \otimes \mathbb{S})$ sending $\mu \mapsto \mathrm{id}_{\mathbb{E}} \otimes C(\mu)$. Now, we want to know whether this is an equivariant spin structure, i.e. whether the adjoint map lifts.

$$\begin{tikzcd} & \mathrm{Spin}(\mathfrak{g}^*) \arrow[d] \\ G \arrow[ur, dashed, "?"] \arrow[r, "\mathrm{Ad}"] & \mathrm{O}(\mathfrak{g}^*) \end{tikzcd}$$

If this doesn't lift, then G is neither abelian nor simply connected. The simplest connected Lie group satisfying those conditions is SO_3. Thus, the adjoint map is the identity, and Spin_3 is a double cover, so is there a homomorphism $\mathrm{SO}_3 \to \mathrm{Spin}_3$? In this case, no, so it's not equivariantly spin.

In general, though, we can pull the double cover back and get a double cover G^σ of G on which the action lifts to a spin structure. So we're always OK if you're willing to take a cover.

Let's modify this to a family D (for "Dirac") $\mathfrak{g}^* \to \mathrm{End}(\mathbb{E} \otimes \mathbb{S})$ with $D_\mu = c(\mu) + D_0$ for some fixed D_0 independent of μ. This has the same topology, in that it's got the same K-theory class in some equivariant K-theory with compact supports. However, it's a different geometric representative, which is nice.

Next time, we'll prove that:

- D_0 is invertible;
- if $\mathbb{E}$ is irreducible, D_μ is invertible except when $\mu \in \mathscr{O}_{\mathbb{E}}$ (that is, the coadjoint orbit), and
- we have a vector bundle $\ker D_\mu \to \mathscr{O}_{\mathbb{E}}$, which is naturally isomorphic to $\mathbb{S}_\mu \otimes \mathcal{L}_{\mathbb{E}}$, where $\mathcal{L}_{\mathbb{E}}$ is a line bundle, and in fact, precisely the one given by the Borel-Weil construction.

In the last week of class, we'll apply this to loop groups, which requires stepping this up to the infinite-dimensional case.

The D_0 that we'll consider is

$$D_0 = \sqrt{-1}\left(\gamma_a R_a + \frac{1}{12} f_{abc}\gamma^a\gamma^b\gamma^c\right),$$

where $\{e_a\}$ is a basis for $\mathfrak{g}$ and $\{e^a\}$ is a basis for $\mathfrak{g}^*$, $\mathrm{m}\gamma^a = c(e^a)$, R_a the action of e_a on $\mathbb{E}$, and $[e_a, e_b] = f_{ab}^c e_c$, so $\langle [e_a, e_b], e_c\rangle = f_{abc}$. This is a piece of a Dirac operator for a certain left-invariant connection, which appears in more places than just this one.

Lecture 26.

Dirac Operators: 11/24/15

"What does your shirt say? 'Seven days without a pun makes one week?' That's it, class is cancelled, happy Thanksgiving."

If M is an n-dimensional Riemannian manifold, we can consider its *bundle of orthonormal frames* $\mathcal{B}_{\mathrm{O}}(M) \to M$, which is a principal O_n-bundle. Each point in $\mathcal{B}_0(M)$ is a point of M along with an orthonormal basis of its tangent space, and projection forgets the tangent space.

We're going to talk about *the* Dirac operator; there are other Dirac operators, but this is "the" one. M comes with a canonical connection called the *Levi-Civita connection*;[54] we'll start with this one, though when we specialize to Lie groups we'll use a different connection.

A spin structure on M gives us maps of bundles

$$\begin{tikzcd} \mathcal{B}_{\mathrm{spin}}(M) \arrow[r, two heads, "2:1"] \arrow[dr, "\mathrm{Spin}_n"'] & \mathcal{B}_{\mathrm{SO}}(M) \arrow[r, hook, "1:2"] \arrow[d, "\mathrm{SO}_n"] & \mathcal{B}_{\mathrm{O}}(M) \arrow[dl, "\mathrm{O}_n"] \\ & M. & \end{tikzcd}$$

[54]Note that Tullio Levi-Civita was one person.

So we get a vector bundle $\mathbb{S} = \mathcal{B}_{\mathrm{spin}|}(M) \times_{\mathrm{Spin}_n} \mathrm{C}\ell_{+n}$ over M. This is a bundle of right $\mathrm{C}\ell_{+n}$-modules, and therefore is a left $\mathrm{C}\ell_{-n}$-module. At a point $m \in M$, we have a Clifford multiplication $T^*M \otimes \mathbb{S} \to \mathbb{S}$, since $T^*M \cong \mathbb{R}^n \subset \mathrm{C}\ell_{+n}$, so we can send $\xi \otimes a \mapsto \xi a$. If $g \in \mathrm{Spin}_n$, then $g^{-1}\xi g \otimes g^{-1} a \mapsto g^{-1}\xi a$.

So on the manifold, we have geometry, but in the frame bundle, we have linear algebra, and invariants pass to invariants. So far, we haven't made any choices; everything is canonical.

Now, covariant differentiation gives us C^∞ sections of our vector bundle: $\nabla : C^\infty_M(\mathbb{S}) \to C^\infty_M(T^*M \otimes \mathbb{S})$, and Clifford multiplication is a map $c : C^\infty_M(T^*M \otimes \mathbb{S}) \to C^\infty_M(\mathbb{S})$. Composing these gives us the *Dirac operator* $D = c \circ \nabla : C^\infty_M(\mathbb{S}) \to C^\infty_M(\mathbb{S})$.

Exercise. Compute the Dirac operator on $\mathbb{E}^n$, flat Euclidean space.[55] You should get the answer $D = \gamma^\mu \frac{\partial}{\partial x^\mu}$.

Note that Clifford multiplication is odd and self-adjoint, but ∇ is a skew-adjoint operator, since it's a first derivative (interchange the two using integration by parts; there are no boundary terms, since M is closed). Thus, D is odd skew-adjoint, and commutes (in the graded sense) with the left $\mathrm{C}\ell_{-n}$ action, since it ultimately comes from a right $\mathrm{C}\ell_{+n}$-action.

Remember, our model for K-theory (well, in this case, KO-theory) was operators like this: odd, skew-adjoint Fredholm operators commuting with the $\mathrm{C}\ell_{-n}$-action. So imagine a family of such operators parameterized by a smooth manifold Y.

What is this exactly? Well, a family of manifolds parameterized by a smooth manifold is sometimes called a "manifold bundle," but more often we call it a fiber bundle of smooth manifolds.

Definition. A *fiber bundle of smooth manifolds* is a smooth map $\pi : X \to Y$ of smooth manifolds such that the fiber over every point has a smooth manifold structure and π is locally trivial (which in particular implies it's a submersion).

If X and Y are Riemannian manifolds, we also get a map $T(X/Y) = \ker \pi_* \to X$. One might define a Riemannian metric to be a metric on the vector bundle $T(X/Y) \to X$, but this isn't strong enough to recover the Levi-Civita connection, so we also need a criterion called *horizontal distribution*.

Now we have to say what a spin structure is, but once we have a metric, we have a bundle of orthonormal frames over X, and so we can specify that the induced spin structure restricts to a continuously varying spin structure on each fiber. Thus, we get a family of Dirac operators parameterized by Y!

Then, the elliptic theory of PDEs tells us that if M is compact, or if π is proper, then this Dirac operator is Fredholm.[56] So our model of K-theory gives an *analytic index* $\mathrm{ind}_{X/Y} \in KO^{-n}(Y)$, and complexifying gives a class in $K^{-n}(Y)$.

Relevant to this notion of analytic index is the following celebrated theorem.

Theorem 26.1 (Atiyah-Singer index theorem (1963))**.** *The analytic index* $\mathrm{ind}_{X/Y} = \pi_!(1)$,[57] *where* $\pi_! : KO^0(X) \to KO^{-n}(Y)$ *is the topological pushforward defined using the spin structure.*

There are other forms of this theorem, but this is the strongest; the others follow from it. In fact, Atiyah and Singer's proof of this form is the only known proof (though there are different proofs of some generalizations).

Let's talk about $\pi_!$ a little more. For simplicity, assume X is compact; not much changes if you don't assume this. In particular, Y must be compact. Thus, we can embed $i : X \hookrightarrow S^N$ for large N, thanks to the Whitney embedding theorem, and we can choose i so that X misses the basepoint $* \in S^N$.[58]

Any smooth map can be factored into an inclusion and a projection, so we just have to define $\pi_!$ on inclusions and projections, and check that it's well-defined. That is, we have

$$\begin{tikzcd} X \arrow[rr, hook, "\pi \times i"] \arrow[dr, "\pi"'] & & Y \times S^N \arrow[dl, "\mathrm{pr}_1"] \\ & Y. & \end{tikzcd}$$

[55]In general, for any construction in Riemannian geometry, you should try computing it in flat space first.

[56]This is in the sense of manifolds, not Hilbert spaces, but the idea is very similar.

[57]$\pi_!$ is pronounced "pi lower-shriek."

[58]This is because the theorem actually gives us an embedding $X \hookrightarrow \mathbb{R}^N$, and we add the point $*$ at infinity, which is not hit by X, to get S^N.

If ν is the normal bundle of the inclusion i, then the Thom isomorphism theorem gives an isomorphism $KO^0(X) \to KO^{N-n}(\nu, \nu \setminus i(X))$. Then, excision means this is isomorphic to $KO^{N-n}(Y \times S^N, Y \times S^N \setminus i(X))$, which maps to $KO^{N-n}(Y \times S^N, Y \times *) = KO^{-n}(Y)$. The composition of all of these maps is our pushforward: use the Thom isomorphism theorem and extend by zero; pushforward by projection is just desuspension.

This is the topologist's version of integration; if we were doing de Rham theory, there's a similar Thom isomorphism theorem, and the pushforward is defined by integration. Thus, this pushforward is actually a quite general concept.

Thus ends the half-hour discussion of Dirac operators, which can really be fleshed out into a semester course.

Back to Lie Groups. Let's suppose we have a compact Lie group G and a G-invariant inner product $\langle -, - \rangle$ on $\mathfrak{g}$, which gives us a bi-invariant Riemannian metric on G.

We'll need a spin structure on G, which will give us $\mathbb{S}$ as a $(\mathrm{Cliff}(\mathfrak{g}^*), \mathrm{C}\ell_{+n})$-bimodule structure, as we discussed previously.

Recall that the Peter-Weyl theorem states that the direct sum of $\mathbb{E}^* \otimes \mathbb{E}$ over all isomorphism classes of $[\mathbb{E}]$ of irreducible representations is dense in C^∞_G, and therefore we also have a dense inclusion

$$C^\infty_G(\mathbb{S}) \supseteq \bigoplus_{[\mathbb{E}]} \mathbb{E}^* \otimes \mathbb{E} \otimes \mathbb{S}.$$

These decompositions are $(G \times G)$-equivariant (this action given by left, resp. right multiplication). In particular, the Dirac operator is equivariant, so acts separately on each component.

Last time, we wrote down a family of operators, and looking at how the Dirac operator splits shows that it really is a Dirac operator.

However, we aren't using the Levi-Civita connection, so we need to write down which covariant derivative we're using. A covariant derivative is a way of defining parallel transport along paths; in Euclidean space, we have global parallel transport: given a vector, we immediately get a vector field. We don't generally have this on a smooth manifold, but on a Lie group, we do have global parallelism: left translation produces a vector field everywhere when you start with a vector at the identity. Right translation gives us another one.

There are lots of orthogonal[59] covariant derivatives on G, and they form some infinite-dimensional affine space over the space of G-invariant one-forms over the skew-adjoint TG-endomorphisms (to see this, what is the difference between two connections)?

If G isn't abelian, left and right translation induce two connections ∇_L and ∇_R, which determine an affine line. It's a fun exercise to show that the Levi-Civita connection lies at the midpoint of these two.

We're not going to use any of these three; instead, we'll use a fourth connection, which is 1/3 of the way from ∇_L to ∇_R. As for why, that is a great question; it's hard to describe even the Levi-Civita connection synthetically (unless your manifold is Kähler).

Since G is finite-dimensional (well, for this week), then we can fix an irreducible complex representation $\mathbb{E}$, so if $\mu \in \mathfrak{g}^*$, then the Dirac operator $D_\mu \in \mathrm{End}(\mathbb{E} \otimes \mathbb{S})$. Let $\{e_\alpha\}$ be a basis for $\mathfrak{g}$ and $\{e^\alpha\}$ be the corresponding dual basis for $\mathfrak{g}^*$. Then, we write $\langle e_a, e_b \rangle = g_{ab}$ and $\langle [e_a, e_b], e_c \rangle = f_{abc}$; similarly, $\langle e^a, e^b \rangle = g^{ab}$ and $[e_a, e_b] = f^c_{ab} e_c$.

Let $\gamma^a \in \mathrm{End}(\mathbb{S})$ be Clifford multiplication by e^a and $R_a \in \mathrm{End}(\mathbb{E})$ be the action of e_a (every representation of a group gives an endomorphism representation of its Lie algebra, infinitesimally). Finally, let $\sigma \in \mathrm{End}(\mathbb{S})$ be $\sigma_a = (1/4) f_{abc} \gamma^b \gamma^c$, which is induced from the action of the Lie algebra $\mathfrak{spin}_n$ induced from the action of the spin group.

Then, our Dirac operator $D_0 \in \mathrm{End}(\mathbb{E} \otimes \mathbb{S})$ has the formula

$$\begin{aligned} D_0 &= i\gamma^a \left(R_a + \frac{1}{3} \sigma_a \right) \\ &= i\gamma^a \left(R_a + \frac{1}{12} f_{abc} \gamma^a \gamma^b \gamma^c \right), \end{aligned}$$

and this is where the 1/3 in the curious choice of connection ultimately comes from. If $\mu \in \mathfrak{g}^*$ more generally, then $D_\mu = c(\mu) + D_0$, where if $\mu_a e^a = \mu \in \mathfrak{g}^*$, then $c(\mu) = \mu_a \gamma^a$.

[59] Here, "orthogonal" is with respect to the bi-invariant metric on G.

Proposition 26.2.
$$D_0^2 = (1/2)[D_0, D_0] = g^{ab}\left(R_aR_b + \frac{1}{3}\sigma_a\sigma_b\right).$$

The proof is a calculation, which we will skip; in particular, this means it's a first-order operator. On Euclidean space, the Dirac operator is the square root of the Laplace operator, but on a general Riemannian manifold this is only approximately true. Here, however, there is no first-order term, since there is no term $g^{ab}\sigma_aR_b$.

TODO: cite FHT paper, section 2.

As a consequence, we also have the following.

Theorem 26.3. *$D_\mu^2 = D_0^2 - 2i(R(\widehat{\mu}) + \sigma(\widehat{\mu})) - |\mu|^2$, where $\widehat{\mu} = \mu_a g^{ab} e_b$.*

That is, $\widehat{\mu}$ is given by raising an index: in the identification of $\mathfrak{g}$ and $\mathfrak{g}^*$ determined by our G-bi-invariant inner product, μ is sent to $\widehat{\mu}$. In particular, $R(\widehat{\mu}) = g^{ab}\mu_aR_nb$ and $\sigma(\widehat{\mu}) - g^{ab}\mu_a\sigma_b$.

Example 26.4. If $G = \mathbb{T}$, then it's abelian, so lots of things nicely coincide. Our irreducible representations are $\mathbb{E}_n = \mathbb{C}$, with $\lambda \in \mathbb{T} \subset \mathbb{C}$ acting as λ^n. Then, $\mathfrak{g} = i\mathbb{R}$ and $\langle ia, ia'\rangle = aa'$, $\mathbb{S} = \mathbb{C}\oplus\mathbb{C} = \mathbb{C}^{1|1}$. Hence, $\gamma = \left(\begin{smallmatrix} 0 & i \\ i & 0\end{smallmatrix}\right)$, so
$$D_{i\alpha} = \begin{pmatrix} 0 & i(a-n) \\ i(a-n) & 0 \end{pmatrix},$$
which is invertible everywhere except $\mu = i\alpha$. There, $\ker D_\mu = \mathbb{C}^{1|1}$.

Working this example out in detail is helpful; other Lie groups are harder.

Definition. A $\mu \in \mathfrak{g}^*$ is *regular* if its stabilizer $\mathrm{Stab}_\mu \subset G$ is some maximal torus T_μ.

Theorem 26.5. *Let G be a compact, connected Lie group and $\mathbb{E}$ be an irreducible G-representation. Then, D_μ is invertible unless μ is regular and $\mu = \lambda_\mu + \rho_\mu$, where λ_μ is the highest weight of $\mathbb{E}$. In this case, $\ker D_\mu = K_\lambda \otimes \mathbb{S}_\rho$, where K_λ is the λ-weight space of $\mathbb{E}$ under T_μ, and $\mathbb{S}_\rho$ is the same, but for ρ on $\mathbb{S}$.*

If μ is regular and G is compact and connected, then $\mathfrak{g}$ splits as $\mathfrak{t}$ and a sum of root spaces, and $\mathfrak{g}^*$ splits dually, as $\mathfrak{t}^*$ plus some coroot spaces. Part of the statement of Theorem 26.5 is that $\mu \subset \mathfrak{t}^*$ inside $\mathfrak{g}^*$. Moreover, regularity means that, in $\mathfrak{t}^*$, μ is contained in a Weyl chamber C^*, so we have splitting into positive and negative roots and $\rho_\mu = (1/2)\sum_{\alpha\in\Delta^+}\alpha$. This is the ρ_μ that we referred to earlier. Now, if we define $\lambda = \mu - \rho$, our condition is that λ is the highest weight.

Looking more closely at this, the action of $T = T_\mu$ decomposes $\mathbb{E}$ into one-dimensional subspaces
$$\mathbb{E} = \bigoplus_{\lambda\in\Lambda} \mathbb{E}_\lambda \otimes \mathbb{C}_\lambda,$$
where T acts on $\mathbb{C}_\lambda$ with the character λ, and $\mathbb{E}_\lambda$ represents its multiplicity. That λ_0 is the highest weight means that $\mathbb{E}_\lambda = 0$ for λ a sum of λ_0 and any nontrivial sum of positive roots.

How should we interpret Theorem 26.5? We want to align everything up geometrically: topologically, these things give us the same element of K-theory, but the magic here is that doing this allows everything to work geometrically too.

We also have to account for a twisting or central extension from the spin structure, but don't stress about that the first time around.

The representation ring of G is $K_G^0(\mathrm{pt})$, and the Dirac family lives in $K_G^n(\mathfrak{g}^*)_c$ (thinking of these as classes of Fredholm operators). But we also have $\ker(D(\mathbb{E}))$, which is a nice vector bundle over our orbit $\mathscr{O}$; the kernel is actually $\mathcal{L}\otimes\mathbb{S}_{\nu_\mathscr{O}}$, where ν is a normal bundle to $\mathscr{O}$ and $\mathcal{L}$ is a line bundle. This is also a class in a K-group, specifically $K_G^{n-r}(\mathscr{O})$, where r is the dimension of a maximal torus.

These are all related by maps: let i be inclusion $\{0\} \hookrightarrow \mathfrak{g}^*$ and j be the inclusion $\mathscr{O} \hookrightarrow \mathfrak{g}^*$. We also have projections $p : \mathfrak{g}^* \to \{0\}$ and $q : \mathscr{O} \to \{0\}$, so we have a diagram

$$\begin{tikzcd}
K_0^G(\{0\}) \arrow[dr, shift left, "i_*"] & & K_G^{n-r}(\mathscr{O}) \arrow[ll, "q_!"'] \arrow[dl, "j_*"] \\
& K_G^n(\mathfrak{g}^*)_c \arrow[ul, shift left, "p_!"] &
\end{tikzcd}$$

That i_* and p_* are inverses comes ultimately from the Thom isomorphism theorem, and so one can show that this diagram commutes. What it means is that one can lift representations past projection, though in general it will have to be virtual.

In the last two lectures, we will pass this story to loop groups, which are infinite-dimensional but some of the story still holds.

Lecture 27.

Twisted Loop Groups: 12/1/15

We don't have much time left, but we'll be able to finish this course with a discussion on loop groups and how Dirac operators and K-theory work their way into the story.

Definition. If G is a compact Lie group, the set $LG = \mathrm{Map}(S^1, G)$ is a group under pointwise multiplication, and is called the *loop group* of G. To be precise, if $\gamma_1, \gamma_2 : S^1 \to G$, then $(\gamma_1 \cdot \gamma_2)(s) = \gamma_1(s)\gamma_2(s)$.

What kinds of maps are these? We want them to be smooth maps, so that we get a smooth manifold, but making a space of maps into a manifold (well, an infinite-dimensional manifold) is tricky; we want to complete to a Hilbert space, and there's a bunch of surprisingly analytic fussing about that needs to be done. But we won't worry too much about this.

There's an alternative, more geometric interpretation: let $P \to S^1$ be a principal G-bundle, i.e. G acts simply transitively on each fiber. Then, let $L_P G$ denote the group of automorphisms of P that cover id_{S^1}.[60] If $P = S^1 \times G$, then there's a very natural identification with "the" loop group: such an automorphism φ is a loop through the action of G.

If G is disconnected, we may get different groups, but if G is connected, then every principal G-bundle is trivializable, and so every one of the loop groups is, after an isomorphism, the loop group. So we can think of these as twisted a little bit.

As an intuition if G is a finite group, let's look at all $L_P G$ simultaneously: let $\mathcal{G}$ be the groupoid whose objects are principal G-bundles $P \to S^1$ and whose morphisms are automorphisms $\varphi : P \to P'$ such that the following diagram commutes.

$$\begin{tikzcd} P \arrow[rr, "\varphi"] \arrow[dr] & & P' \arrow[dl] \\ & S^1 & \end{tikzcd}$$

$\mathcal{G}$ contains all of the loop groups at once: the maps in it from a P to itself are the group $L_P G$. So it is a pretty huge groupoid; let's make it smaller.

Proposition 27.1. *$\mathcal{G}$ is equivalent to $G /\!\!/ G$, where G acts on itself by conjugation.*

Proof sketch. We'll construct a groupoid $\mathcal{G}_*$ which is more evidently equivalent to both of these: fix a basepoint $s_0 \in S^1$, and let the objects of a groupoid $\widetilde{\mathcal{G}}_*$ be based principal G-bundles: $P \to S^1$ such that $p_0 \in P_{s_0}$. Then, we require its morphisms to send basepoints to basepoints. However, all automorphisms of a given object are the identity (since each bundle is a covering space of the circle, and therefore maps lift and determine everything). That is, $\widetilde{\mathcal{G}}_*$ is *rigid*.

Regarding G as a groupoid, there's an equivalence $\widetilde{\mathcal{G}}_* \to G$, sending (P, p_0) to its holonomy h: with an orientation of S^1, the loop on S^1 starting at s_0 lifts to some perhaps nontrivial path in the fiber above s_0. Since G acts simply transitively, the *holonomy* is defined to be the $h \in G$ whose action sends the first endpoint of this path to the second.

Now, $g \in G$ acts on $\widetilde{\mathcal{G}}_*$ by $p_0 \mapsto p_0 \cdot g$, so let $\mathcal{G}_* = \widetilde{\mathcal{G}}_* /\!\!/ G$ (again, G acts by conjugation). Then, you can prove that $\mathcal{G}_*$ is an equivalence. ⊠

In our groupoid G, the maps of the trivial bundle are just the elements of G, and over an $x \in G$, the self-maps are the things which conjugate x back to itself, i.e. the centralizer $Z_x \subset G$.

[60] An *automorphism of a principal G-bundle* is a map $\varphi : P \to P$ commuting with the right action of G: $\varphi(p \cdot g) = \varphi(p) \cdot g$ for $p \in P$ and $g \in G$. Hence, φ preserves fibers, so it restricts to an automorphism of the base, and here we want this to be the identity.

We want to study representations of these twisted loop groups $L_P G$, and relate this to the K-theory of $G/\!\!/G \simeq K_G(G)$. There will be a twist here, though it's a little tautological when G is finite: here, $L_P G \cong Z_x$ for an $x \in G$, so if $\mathbb{E}$ is a Z_x-representation.

We might have other elements in our conjugacy class; if x' is in the same conjugacy class as x, then we get a noncanonical $Z_{x'}$-representation: stick a copy of $\mathbb{E}$ over x', and then choose an arrow $x \to x'$ that will act as the identity on $\mathbb{E}$. Then, we know what any other arrow does: map back by the inverse of our special arrow and you get an automorphism, and that tells us what the other arrow does.

This means that we get a vector bundle over this groupoid $G/\!\!/G$, supported on the conjugacy class $[x]$ of x. Let me repeat: a representation of the centralizer is, almost visibly, a vector bundle, and therefore an element of K-theory!

This is much clearer when G is finite, but we'll develop it when G is infinite as well; we have to introduce a notion of twisting the things we want to work with: Fredholm operators, representations, vector bundles, etc.

Let G act linearly on a vector space $\mathbb{E}$. One way to twist this action is to take a *central extension* of a projective representation, or a central extension of the action of G. This is a short exact sequence

$$1 \longrightarrow \mathbb{T} \longrightarrow G^\tau \longrightarrow G \longrightarrow 1$$

in which $\operatorname{Im}(\mathbb{T}) \subseteq Z(G^\tau)$. Specifically, we will study representation of G^τ on $\mathbb{E}$ such that the action of $\mathbb{T} \subset G^\tau$ is scalar multiplication (which more or less just restricts our choice of central extension). If $\mathbb{T}$ acts trivially, the representation factors through G, and we get the original representation again.

Since we're going to be looking at unitary complex representations, we chose $\mathbb{T}$ as the kernel; if we were looking at real representations, we would choose $\{\pm 1\}$. For example, if G is the Klein group, this implies there's a nonabelian central extension G^τ that fits into the following sequence.

$$1 \longrightarrow \{\pm 1\} \longrightarrow G^\tau \longrightarrow \mathbb{Z}/2 \times \mathbb{Z}/2 \longrightarrow 1.$$

To be precise, if $\overline{x}$ and $\overline{y}$ are my generators of $\mathbb{Z}/2 \times \mathbb{Z}/2$, then if x and y are their preimages in G^τ, then $x^2 = y^2 = 1$ and $xyx^{-1}y^{-1} = 1$.

Let's write down the representations of this group; since G is finite and therefore compact Lie, these representations are unitary. We know that if $\{\pm 1\}$ acts trivially, then we have four 1-dimensional representations given by choosing each of x and y to act trivially or as -1. Then, by the Peter-Weyl theorem, we need one more two-dimensional representation, which will be the one where $\{\pm 1\}$ acts by, well, ± 1.

This is the simplest example of a *Heisenberg group*, and is an interesting way to see how the representations of $\mathbb{Z}/2 \times \mathbb{Z}/2$ got twisted.

There's a geometric reason this is called a projective representation: we can think of $\mathbb{E}$ as a vector bundle over G as a groupoid (with only one object, so only the one fiber), but this factors through the projective space $P\mathbb{E} \to G$.

So more generally, we want to replace each arrow in a groupoid $\mathcal{G}$ with a whole "tube" of arrows, permuted simply transitively by the circle group, and with a group law. In other words, we're making a central extension of a groupoid by $\mathrm{pt}/\!\!/\mathbb{T} \to \widetilde{\mathcal{G}} \to \mathcal{G}$. Looking only at the automorphism group over a point, this creates a central extension in the sense from before, and therefore we've produced a whole bunch of central extensions simultaneously. In this way, we have twisted a vector bundle over a groupoid![61]

Example 27.2. Let $X = S^3$. This only has identity arrows, but the key is that we can work with an equivalent groupoid: let $U_+ = S^3 \setminus \{-\}$ and $U_- = S^3 \setminus \{+\}$ (so each misses one of the north or south poles). So we get a groupoid $\mathcal{G} = U_+ \sqcup U_- / \sim$, which looks like the same objects, but now we have an arrow from a point on U_+ to its corresponding point on U_-, and vice versa. Hence, a central extension is a principal $\mathbb{T}$-bundle over $U_+ \cap U_- \simeq S^2$. And we know that these are classified by $\mathbb{Z}$.

A nice problem to work on is that it's not possible to find twisted vector bundles in this case: a vector bundle on U_+ and one on U_- with an isomorphism at every point of the intersection. You can work out why this isn't possible. There are nontrivial elements of K-theory, but you have to use an infinite-dimensional model, e.g. the Fredholm model.

[61]There are formal definitions to be found here, but for reasons of time we're not going to be entirely precise, in order to get down more of the story.

There are other kinds of twistings you can do, some of which are easier; for example, if $\mathbb{E}$ has a $\mathbb{Z}/2$-grading, then we usually require G to preserve this grading, but we can instead allow some elements to reverse the grading.

Returning to our loop group LG, there are some uninteresting representations. For example, fix $s_1, \ldots, s_r \in S^1$ and above each one fix G-representations $\mathbb{E}_1, \ldots, \mathbb{E}_r$. Then, we can define a representation $LG \to \mathrm{Aut}(\mathbb{E}_1 \otimes \cdots \mathbb{E}_r)$ by $\gamma \mapsto \rho_1(\gamma(s_1)) \otimes \cdots \otimes \rho_r(\gamma(s_r))$. The picture looks like a cactus: there are finitely many "spines" $\mathbb{E}_i$ over their points s_i, and for an action of a loop, we just look at what it does at each spine.

The representations we want to keep will have a bigger symmetry. The diffeomorphism group of S^1 acts on LG by automorphisms: if $\varphi \in \mathrm{Diff}(S^1)$ and $\gamma \in LG$, then $(\varphi \cdot \gamma)(s) = \gamma(\varphi^{-1}(s))$. Furthermore, the map $\mathrm{Diff}(S^1) \to \mathrm{Aut}(LG)$ is a homomophism, so we can form the semidirect product $LG \rtimes \mathrm{Diff}(S^1)$. Inside this is a group $\widehat{L}G = LG \rtimes \mathbb{T}$, where $\mathbb{T}$ acts as rotation. Since $\mathrm{Diff}(S^1)$ deformation retracts onto $\mathbb{T}$ (the rotations), then topologically this is the same story. $\widehat{L}G$ is the subgroup of automorphisms that cover a rigid rotation of the base. They also fit into a short exact sequence

$$1 \longrightarrow LG \longrightarrow \widehat{L}G = LG \rtimes \mathbb{T} \longrightarrow \mathbb{T} \longrightarrow 1.$$

This is sometimes called a *central coextension* by $\mathbb{T}$ (since it's in the cokernel, rather than the kernel). We want central extensions, but having this symmetry will be nice as well. To be precise, we will study unitary representations of $\widehat{L}G$. But since this is a semidirect product, $\mathbb{T}$ is a subgroup (though not a normal one), and really sit inside $\widehat{L}G$. Thus, if $\mathbb{E}$ is an $\widehat{L}G$-representation, it restricts to a representation of $\mathbb{T}$ (acting as rotations), which may not be irreducible. Nonetheless, we can split it up into irreducible representations, which are labelled by the characters of $\mathbb{T}$, which are the integers, so restricted to $\mathbb{T}$,

$$\mathbb{E} = \bigoplus_{n \in \mathbb{Z}} \mathbb{E}_n,$$

and the algebraic direct sum maps back into $\mathbb{E}$. In general, its image is dense, but may not be everything. For example, if $\mathbb{E} = L^2(S^1)$, with S^1 acting by rotation, then we get

$$\mathbb{E} = L^2(S^1) \longleftarrow \bigoplus_{n \in \mathbb{Z}} \mathbb{C} \cdot e^{ins}.$$

You can't hit every element of $L^2(S^1)$ by a finite sum, so the image isn't everything, but you can with an infinite sum (from Fourier series).

The space $\mathbb{E}_n$ is sometimes called the *energy n subspace*. We want to study representations with positive energy.

Definition. A $\widehat{L}G$-representation $\mathbb{E}$ has *positive energy* if there's an $n_0 \in \mathbb{Z}$ such that $\mathbb{E}_n = 0$ when $n < n_0$.

Really, it's "bounded below," rather than "positive," but that's okay.

There are different motivations for why you might want this definition: these arise in different places in physics, such as quantum mechanics.

Example 27.3. For the rest of the lecture, we'll study an example in detail: let $G = T$ be a torus. In a sense, we only need to understand finite groups, tori, and the simple groups, and we've looked at finite groups, so on to the tori.

The first observation is that LT is abelian (since multiplication was defined pointwise), so any irreducible complex representation is one-dimensional.

Proposition 27.4. *Any irreducible representation of LT which extends to a representation of $\widehat{L}T = LT \rtimes \mathbb{T}$ is trivial.*

Proof. Let $\mathfrak{t}$ denote the Lie algebra of T and $z = e^{is}$. If $\xi \in \mathfrak{t}_{\mathbb{C}}$, then $z^n \xi$ is the loop $(s \mapsto e^{ins}\xi) \in L\mathfrak{t}_{\mathbb{C}}$. (If this is strange, try it with $T = S^1$). Then, the direct sum of all $z^n \mathfrak{t}_{\mathbb{C}}$ is dense in $L\mathfrak{t}_{\mathbb{C}}$, no matter which loops we pick: continuous, smooth, L^2, some kind of Sobolev, etc.

If d generates $\mathbb{T}$, then we have a short exact sequence

$$0 \longrightarrow L\mathfrak{t} \longrightarrow \widehat{L}\mathfrak{t} \underset{\varphi}{\overset{\longrightarrow}{\longleftarrow}} i\mathbb{R} \longrightarrow 0,$$

where $\mathbb{R}$ also acts as rottion and φ sends $i \mapsto d$. $\widehat{L\mathfrak{t}}$ isn't abelian, unlike $L\mathfrak{t}$, and its derivation is $[d, z^n\xi] = inz^n\xi$. Thus, if $e \in \mathbb{E}_n$ and $d \cdot e = ike$. Then,

$$\begin{aligned} d \cdot (z^n\xi \cdot e) &= z^n\xi(de) + [d, z^n\xi]e \\ &= z^n\xi(ike) + inz^n\xi \cdot e \\ &= i(k+n)(z^n\xi + e), \end{aligned}$$

so it takes loops of energy k to energy $n+k$: $z^n\xi : \mathbb{E}_k \to \mathbb{E}_{n+k}$. Thus nothing really fixes the energy spaces, so we don't get anything terribly interesting (well, they would be one-dimensional anyways). ⊠

Next idea: let's try to construct a positive energy representation of LT. We'll do this through the Lie algebra $L\mathfrak{t}_{\mathbb{C}}$, and through the dense algebraic direct sum of the $z^n\mathfrak{t}_{\mathbb{C}}$ for $n \in \mathbb{Z}$.

Since we want the representation to be positive, suppose $e_0 \in \mathbb{E}_0$ and $\mathbb{E}_{-n} = 0$ for $n > 0$. Thus, $z^{-n}\xi \cdot e_0 = 0$ for all $n > 0$, but what about $z\xi \cdot e_0$? We get

$$z^{-1}\eta \cdot (z\xi \cdot e_0) = z\xi \cdot (z^{-1}\eta \cdot e_0) + [z^{-1}\eta, z\xi] \cdot e_0 = 0, \tag{27.1}$$

because $L\mathfrak{t}$ is an abelian Lie algebra and terms in negative energy vanish.

If $\mathbb{E}$ is unitary, then $L\mathfrak{t}$ maps by our representation into the skew-adjoint endomorphsims of $\mathbb{E}$; the fixed points of conjugation map to the skew-adjoint operators. In particular, if $\xi \in \mathfrak{t}_{\mathbb{C}}$, then if $*$ denotes adjoint, then $(z^n\xi)^* = -z^{-n}\overline{\xi}$ (where we're looking at the automorphism given by the representation).

This spells doom: $\langle z\xi \cdot e_0, z\xi \cdot e_0\rangle = -\langle z^{-1}\overline{\xi} \cdot z\xi \cdot e_0, e_0\rangle = e_0$ by (27.1). So we started with a *vacuum vector* e_0 and even at the first energy level we're at zero; you can show this for any energy. So we'll need to take a central extension of LT. But if we have a central extension

$$1 \longrightarrow \mathbb{T} \longrightarrow LT^\tau \longrightarrow LT \longrightarrow 1$$

as Lie groups, then we also get a central extension of Lie algebras.

$$0 \longrightarrow i\mathbb{R} \longrightarrow L\mathfrak{t}^\tau \longrightarrow L\mathfrak{t} \longrightarrow 0.$$

We can also complexify, and write down a section $s : L\mathfrak{t}_{\mathbb{C}} \to L\mathfrak{t}_{\mathbb{C}}^\tau$ as vector spaces by defining $[s(z^n\xi), s(z^m\eta)]_{L\mathfrak{t}_{\mathbb{C}}^\tau}$, but this must pass to 0 in $L\mathfrak{t}_{\mathbb{C}}$ (because $L\mathfrak{t}_{\mathbb{C}}$ is an abelian Lie algebra), so it lives in the kernel $i\mathbb{C}$. Thus, this defines a cocycle, and it turns out there's a unique one satisfying skew-symmetry and the Jacobi identity, which is

$$s(z^n\xi), s(z^m\eta)]_{L\mathfrak{t}_{\mathbb{C}}^\tau} = in\delta^{n+m=0}\langle \xi, \eta\rangle,$$

where the pairing on the right-hand side is symmetric.

In general, if $A = LT$, so that A is abelian, then an extension

$$1 \longrightarrow \mathbb{T} \longrightarrow A^\tau \longrightarrow A \longrightarrow 1$$

defines a map $s : A \times A \to \mathbb{T}$ by lifting $a_1, a_2 \mapsto \widetilde{a}_1\widetilde{a}_2\widetilde{a}_1^{-1}\widetilde{a}_2^{-1}$, their commutator lifted into A^τ; but this passes to 0 in A, so it comes from the kernel $\mathbb{T}$. Thus, s is a map $A \mapsto \operatorname{Hom}(A, \mathbb{T}) = A^\vee$, the *Pontrjagin dual* group; if this map is an isomorphism, A^τ might be called a *(generalized) Heisenberg group*, or a *Heisenberg extension*.

We saw already what this does if $A = \mathbb{Z}/2 \times \mathbb{Z}/2$; in general, we get a *bihomomorphism* (i.e. it's a homomorphism separately in each argument).

The lesson is: if you go to loop groups, you necessarily have to take central extensions to get anywhere interesting.

Lecture 28.

: 12/3/15

> *"[The Heisenberg group] is named after the physicist Werner Heisenberg, not the Heisenberg of Breaking Bad."*

Recall that we were talking about loop groups and central extensions: if G is a compact Lie group, we consider its free loop group LG, the maps $S^1 \to G$. If $G = T$ is a torus, then $A = LT$ is abelian.

We mentioned last time that it's nice to have projective representations, which led us to the question of central extensions.

Definition. Let A be an abelian Lie group such that $\pi_0(A)$ and $\pi_1(A)$ are finitely generated. Then, a central extension

$$1 \longrightarrow \mathbb{T} \longrightarrow A^\tau \longrightarrow A \longrightarrow 1 \tag{28.1}$$

is called a *generalized Heisenberg group*.

For a generalized Heisenberg group, there is a *commutation map* $s : A \times A \to \mathbb{T}$: for any $a, b \in A$, the commutator of their preimages $\widetilde{a}\widetilde{b}\widetilde{a}^{-1}\widetilde{b}^{-1}$ passes to 0 in A, and therefore comes from some element $s(a, b) \in \mathbb{T}$, by exactness. s is a bihomomorphism which is alternating, and $s(a, a) = 1$, which is quickly verifiable.

Let $Z = \{a \in A : s(a, b) = 1 \text{ for all } b \in A\}$, which is a kernel for s; then, the center $Z^\tau \subset A^\tau$ is an extension

$$1 \longrightarrow \mathbb{T} \longrightarrow Z^\tau \longrightarrow Z \longrightarrow 1. \tag{28.2}$$

A generalized Heisenberg group is said to be *nondegenerate* if $Z = 0$, so that $Z^\tau = \mathbb{T}$. In this case, s is a self-Pontryagin duality on A; in general, $a \mapsto s(a, -)$ is a map $A \to A^\vee$, so to have nondegeneracy we need A to be self-dual.

Remark (TODO cite). It turns out that s determines A^τ up to isomorphism, and any s can occur. The construction is to choose any $\psi : A \times A \to \mathbb{T}$ that's *bimultiplicative* (i.e. bilinear), and set $s(a.b) = \psi(a, b)/\psi(b, a)$; then, set $A^\tau = A \times \mathbb{T}$ as a manifold, with multiplication

$$(a, \lambda)(b, \mu) = (a + b, \lambda\mu\psi(a, b)).$$

You can check that this defines a group, and that this group fits into (28.1).[62]

Example 28.1.

(1) Last time, we saw what this was for $A = \mathbb{Z}/2 \times \mathbb{Z}/2$; you could replace 2 with your favorite n if you want.

In this case, $s((a_1, b_1), (a_2, b_2)) = (-1)^{a_1 b_2 - a_2 b_1}$.[63] In this case, $\psi(a_1, b_1), (a_2, b_2)) = (-1)^{a_1 b_2}$. This cocycle tells is how two elements commute (or don't) in the central extension.

You should think of this as a kind of symplectic pairing, and even over $\mathbb{Z}/2 \times \mathbb{Z}/2$, it could be the symplectic plane over $\mathbb{F}_2$.

(2) Making the symplecticity more explicit, let $A = \mathbb{C}$ and $s(\xi, \eta) = e^{2\pi i\omega(\xi,\eta)}$, where $\omega(\xi, \eta) = c(x_1 y_2 - -x_2 y_1)$ is the symplectic form. Thus, $\psi(\xi, \eta) = e^{2\pi i c x_1 y_2}$, and we get the standard Heisenberg group.

When we look at representations, we're going to require that $\mathbb{T} \subset A^\tau$ acts by scalar multiplication, in particular in the following theorem.

Theorem 28.2 (Stone-von Neumann-Shale)**.**

(1) If A is finite-dimensional, then:
   (a) if A^τ is nondegenerate, then there is a unique such irreducible unitary representation, and
   (b) in general, the irreducible unitary representations are classified by splittings of (28.2)*.*

(2) The same is true for an infinite-dimensional A if we restrict to positive-energy representations with respect to a polarization.

TODO cite Segal, "Unitary Representations of Infinite Groups"

To actually construct such a representation, we wander into quantum mechanics. We have to choose a *polarization* of A, so we need a maximal subgroup on which the commutator vanishes (in the symplectic sense, a maximal isotropic subgroup).

Looking back at $A = \mathbb{Z}/2 \times \mathbb{Z}/2$, first, we let $\mathbb{E}$ be the functions $\mathbb{Z}/2 \times \{0\} \to \mathbb{C}$; then, the first component acts through the pullback by translation, and $\{0\} \times \mathbb{Z}/2$ acts by multiplication using s. In particular,

$$(1, 0) \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad (0, 1) \longmapsto \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}.$$

You can check that the commutator maps to -1.

[62] A good question to ask is when two different bimultiplicative functions give the same s (and therefore the same extension).

[63] In general for $\mathbb{Z}/n \times \mathbb{Z}/n$, replace -1 with an n^{th} root of unity

The case for $A = \mathbb{C} = \mathbb{R}^2$ (as groups) is similar, but different. Let $\mathbb{E} = L^2(\mathbb{R})$, and consider $\mathbb{C} = \mathbb{R}_x \times \mathbb{R}_y$. Then, an $x_0 \in \mathbb{R}_x$ sends $f(-) \mapsto f(- - x_0)$: translation is a nice, unitary operator.[64] Then, a $y_0 \in \mathbb{R}_y$ acts by sending f to $x \mapsto e^{2\pi i x y_0} f(x)$, and you can check that this satisfies the commutator identity.

This is the geometric picture; for an algebraic picture, choose a dense subset of algebraic functions in L^2, e.g. polynomials, and then complete. This is another common picture of this representation.

If we have loops on a connected Lie group, then we can look at its maximal torus, and therefore these Heisenberg groups and their representations are an important way to understand loop groups.

The first thing to notice is that if T is a torus, there's a quite natural decomposition $LT \simeq T \times \Pi \times U$, where

- $\Pi = \mathrm{Hom}(\mathbb{T}, T) \subset \mathfrak{t}$, so that $T = \mathfrak{t}/\Pi$. We also had $\Lambda = \mathrm{Hom}(T, \mathbb{T}) \subset \mathfrak{t}^*$, which we'll use again.
- $U = \exp(V)$, where
$$V = \left\{\beta : S^1 \to \mathfrak{t} : \int_{S^1} \beta(s) |\,\mathrm{d}s| = 0\right\}.$$

Let's unwind this: if $\gamma : S^1 \to T$, then in decomposition $LT \simeq T \times \Pi \times U$, T represents the point loops (that don't move anywhere), and Π contains the one-parameter subgroups, since a vector $\pi \in \Pi$ determines a line segment from 0 to π, and exponentiating this gives us $\exp(\pi) = e$, so we get a loop centered at the identity. Hence, if we choose a bi-invariant metric on T (or, since it's abelian, this is the same as a left-invariant metric), $T \times \Pi$ is the group of closed geodesics inside LT. These account for everything in $\pi_1(T)$, meaning that all loops in U are homotopically trivial. There's some normalization involved in this, which was the condition on the inverse.

The idea is that a loop γ has a homotopy class, which determines its class in Π, and after subtracting, it's in the image of the exponential. The preimage might not have integral zero, which determines the point loop part in T, and then the remainder has integral zero.

$V \subset L\mathfrak{t}$, and inside this is the dense subset of polynomial loops: $\bigoplus_{n\in\mathbb{Z}} z^n \mathfrak{t}_\mathbb{C}$, where $z^n : s \mapsto e^{ins}$. Sitting inside V and inside this is the dense subset of all of these except $n = 0$, so it splits into $n > 0$ (positive energy) and $n < 0$ (negative energy). This is our polarization.

We want to look at a central extensions; in this case, they have the form $(T \times \Pi)^\tau \times_\mathbb{T} U^\tau$ (that is, we're taking the fiber product). For general G, it happens, a nice class of central extensions of LG comes from a *level* in $H^4(BG;\mathbb{Z})$. This is the beginning of a beautiful story in low-dimensional geometry, though you have to start with a representative element, rather than an equivalence class.

In our case, $H^4(BT;\mathbb{Z}) \cong \mathrm{Sym}^2 \Lambda$, where Λ is our character lattice and $\mathrm{Sym}^2 \Lambda = \Lambda \otimes \Lambda/(\lambda_1 \otimes \lambda_2 - \lambda_2 \otimes \lambda_1)$.[65] This is actually quite easy to prove with the Serre spectral sequence for $T \to * \to BT$ (the total space is contractible): $E_2^{0,0} = \mathbb{Z}$, and $E_2^{0,1} = \Lambda$, but since the total space is contractible, then there can't be anything in the E_∞ page, so d_2 must be an isomorphism, so $E_2^{2,0} = \Lambda$, and therefore $E_2^{2,1} = \Lambda \otimes \Lambda$, which will hit something in degree $(4, 0)$ by d_2, and is hit by something in the $(0, 2)$-page, which is Λ^2. Then, d_2 acts by $d_2(\lambda_1 \smile \lambda_2) = d_2\lambda_1 \smile \lambda_2 - \lambda_1 \smile d_2\lambda_2$, so its image in $E_2^{2,1} = \Lambda \otimes \Lambda$ is the ideal $I = (\lambda_2 \otimes \lambda_1 - \lambda_1 \otimes \lambda_2)$, and therefore, since it must die by the E_∞-page, exactness of d_2 forces the ($E_2^{4,0}$ term, which is $H^4(BG;\mathbb{Z})$, to be $\Lambda \otimes \Lambda/I = \mathrm{Sym}^2 \Lambda$.

An element of $\mathrm{Sym}^2 \Lambda$ gives a quadratic function $q : \Pi \to \mathbb{Z}$ by evaluation: if $\pi \in \Pi$, $q(\pi)$ is evaluation on $\pi \otimes \pi$. Equivalently, this can be defined by an even symmetric form $k : \Pi \times \Pi \to \mathbb{Z}$, and this gives us a homomorphism $\kappa : \Pi \to \Lambda$, which we will assume is injective.

Returning to central extensions, this gives us a $\kappa_\mathbb{R} : \mathfrak{t} \times \mathfrak{t} \to \mathbb{R}$ which is nondegenerate. We can build a symplectic form $\omega : V \times V \to \mathbb{R}$ by defiuning $\omega(z^n\xi, z^m\eta) = \delta^{n+m=0} m \kappa_\mathbb{R}(\xi, \eta)$, or
$$\omega(\beta_1, \beta_2) = \int_{S^1} \kappa_\mathbb{R}(\beta_1(s), \beta_2'(s))\,\mathrm{d}s,$$
the dual way of writing it. Thus, we have a symplectic form, albeit on this infinite-dimensional space. This allows us to build U^τ, which is nnondegenerate, so there is a unique positive-energy irreducible unitary representation, so everything boils down to the geometric story of this representation.

[64]Sometimes, quantum mechanics people prefer to write everything in terms of Lie algebras; the resulting operators aren't necessary unitary, or even bounded! The formulas are nicer for the groups.

[65]The whole cohomology ring is $H^*(BT;\mathbb{Z}) = \mathrm{Sym}^\bullet \Lambda$, the symmetric algebra.

Now, we construct $(T \times \Pi)^\tau$. We'll use $\psi : (T \times \Pi) \times (T \times \Pi) \to \mathbb{T}$ defined by $(t_1, \pi_1), (t_2, \pi_2) \mapsto \kappa(\pi_1)t_2$, which tells you how to build the central extension.

For example, if $T = \mathbb{T}$ and $\kappa : \mathbb{Z} \to \mathbb{Z}$ is multiplication by 6, then $\mathbb{T} \times \Pi = \mathbb{T} \times \mathbb{Z}$, a circle at each integer. Unfortunately, figuring out which part is trivialized and what the commutator does is a little tricky, so maybe this wasn't the best picture.

Now, suppose $\mathbb{E}$ is an irreducible, unitary representation of $(T \times \Pi)^\tau$. Let's restrict to

$$1 \longrightarrow \mathbb{T} \longrightarrow T^\tau \longrightarrow T_\bullet \longrightarrow 1,$$

with $T \times \Pi \subset T_\bullet$, and there's a section, so we can regard this as a subgroup of T^τ too. Then, we can decompose $\mathbb{E}$ into the character spaces $\mathbb{E}_\lambda$ for each $\lambda \in \Lambda$, with a $\pi \in \Pi$ acting by $\mathbb{E}_\lambda \to \mathbb{E}_{\lambda + k(\pi)}$. Thus, we have a vector bundle over Λ; an irreducible representation will be generated be an $\mathbb{E}_\lambda$ over one $\lambda \in \Lambda$, and shifted by all $k(\pi)$, and also must have $\mathbb{E}_\lambda = \mathbb{C}$. (For example, if κ is multiplication by 6, then one irreducible representation is $\mathbb{C}$ over 0 mod 6 and 0 elsewhere).

Thus, in general, the irreducible representations are classified by $\Lambda/\kappa(\Pi)$. How does that fit with what we've done before? We have $\kappa : \Pi \hookrightarrow \Lambda$ inducing an isomorphism $\kappa_\mathbb{R} : \mathfrak{t} \to \mathfrak{t}^*$, and therefore $\kappa_{\mathbb{R}/\mathbb{Z}} : \mathfrak{t}/\Pi = T \to T^* = \mathfrak{t}^*/\Lambda$. This is in fact a covering map (e.g. for multiplication by 6, we get the six-fold cover $S^1 \to S^1$), so $\kappa_{\mathbb{R}/\mathbb{Z}}^{-1}(1) = F$ is a finite subgroup of T, and in fact the center from (28.2) is $Z = F \times \{0\} \subset T \times \Pi$.

This restricts from a Pontryagin duality pairing $T \times \Lambda \to \mathbb{T}$ to a Pontryagin duality pairing $F \times \Lambda/\kappa(\Pi) \to \mathbb{T}$, so if $C = \mathfrak{t} \times_\Pi \Lambda$, then we have projections

$$\begin{tikzcd} & C \arrow[dl, "p_1"'] \arrow[dr, bend left, "p_2"] & \\ T & & \Lambda/\kappa(\Pi) \arrow[ul, bend left, "\sigma"] \end{tikzcd}$$

and the section $\sigma : \lambda \mapsto (\kappa_\mathbb{R}^{-1}(\lambda), \lambda)$, and the image $p_1 \circ \sigma$ is just F again, a finite group, and C turns out to be a finite (same cardinality) collection of affine spaces. The points of F classify these representations of our loop group.

This is beautiful, yes, but what does it have to do with K-theory? I'm glad you asked.

Theorem 28.3. *There is a Dirac family isomorphism $\Phi : \mathcal{R}^{\tau-\sigma}(LG) \to K_G^{\tau + \dim G}(G)$, where $\mathcal{R}^{\tau-\sigma}(LG)$ is the free abelian group generated by irreducible unitary representations of positive energy of the central extension $LG^{\tau-\sigma}$ and τ is a twisting of the groupoid $G /\!\!/ G$.*

We haven't talked about what happens in the nonabelian case, but here we have a Weyl group, and it acts on anything you can think of, which tells us more of this story, and where σ, the *adjoint shift* comes from.

Since T is abelian, then if $G = T$, $T /\!\!/ T$ acts trivially on T, so in the groupoid, the only arrows are automorphsims. A vector bundle over this is a vector space over each $x \in T$, but we need to twist this action, giving a central extension $\widehat{T}_x$ for the fiber T_x for every $x \in T$. Thus, we get a family of extensions parameterized by T; each one is trivial (split), but there's no continuous splitting. τ tells us the nontriviality of this family of extensions.

This actually leads to some computations with K-theory of T: you can shift the story up to C, where it's the K-theory with compact supports on each of its affine planes. Thus, we get one generator for each element of F by the Thom isomorphism.

If A is the space of connections on the trivial G-bundle $P = S^1 \times G$, then LG acts on A by *gauge transformations*, and in fact there's an equivalence of groupoids $A /\!\!/ LG \to G /\!\!/ G$, and this is a way to get the infinite-dimensional Dirac operators, analogous to the matrix-based way it happens in the finite-dimensional case.

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