M392C NOTES: REPRESENTATION THEORY

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These notes were taken in UT Austin's M392C (Representation Theory) class in Spring 2017, taught by San Gunningham. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Thanks to Jay Hathaway and Surya Raghavendran for correcting a few errors.

Contents

Lecture 1.

Lie groups and smooth actions: 1/18/17

"I've never even seen this many people in a graduate class... I hope it's good."

Today we won't get too far into the math, since it's the first day, but we'll sketch what exactly we'll be talking about this semester.

This class is about representation theory, which is a wide subject: previous incarnations of the subject might not intersect much with what we'll do, which is the representation theory of Lie groups, algebraic groups, and Lie algebras. There are other courses which cover Lie theory, and we're not going to spend much time on the basics of differential geometry or topology. The basics of manifolds, topological spaces, and algebra, as covered in a first-year graduate class, will be assumed.

In fact, the class will focus on the reductive semisimple case (these words will be explained later). There will be some problem sets, maybe 2 or 3 in total. The problem sets won't be graded, but maybe we'll devote a class midsemester to going over solutions. If you're a first-year graduate student, an undergraduate, or a student in another department, you should turn something in, as per usual.

Time for math.

We have to start somewhere, so let's define Lie groups.

Definition 1.1. A *Lie group* G is a group object in the category of smooth manifolds. That is, it's a smooth manifold G that is also a group, with an operation $m : G \times G \to G$, a C^∞ map satisfying the usual group axioms (e.g. a C^∞ inversion map, associativity).

Though in the early stages of group theory we focus on finite or at least discrete groups, such as the dihedral groups, which describe the symmetries of a polygon. These have discrete symmetries. Lie groups are the objects that describe continuous symmetries; if you're interested in these, especially if you come from physics, these are much more fundamental.

Example 1.2. The group of $n \times n$ invertible matrices (those with nonzero determinant) is called the *general linear group* $\mathrm{GL}_n(\mathbb{R})$. Since the determinant is multiplicative, this is a group; since $\det(A) \neq 0$ is an open condition, as the determinant is continuous, $\mathrm{GL}_n(\mathbb{R})$ is a manifold, and you can check that multiplication is continuous. ◀

Example 1.3. The *special linear group* $\mathrm{SL}_n(\mathbb{R})$ is the group of $n \times n$ matrices with determinant 1. This is again a group, and to check that it's a manifold, one has to show that 1 is a regular value of $\det : M_n(\mathbb{R}) \to \mathbb{R}$. But this is true, so $\mathrm{SL}_n(\mathbb{R})$ is a Lie group. ◀

Example 1.4. The *orthogonal group* $\mathrm{O}(n) = \mathrm{O}(n, \mathbb{R})$ is the group of orthogonal matrices, those matrices A for which $A^t = A^{-1}$. Again, there's an argument here to show this is a Lie group. ◀

You'll notice most of these are groups of matrices, and this is a very common way for Lie groups to arise, especially in representation theory.

We can also consider matrices with complex coefficients.

Example 1.5. The *complex general linear group* $\mathrm{GL}_n(\mathbb{C})$ is the group of $n \times n$ invertible complex matrices. This has several structures.

- For the same reason as $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$ is a Lie group.
- $\mathrm{GL}_n(\mathbb{C})$ is also a *complex Lie group*: it's a complex manifold, and multiplication and inversion are not just smooth, but holomorphic.
- It's also a *algebraic group* over $\mathbb{C}$: a group object in the category of algebraic varieties. This perspective will be particularly useful for us.

We can also define the *unitary group* $\mathrm{U}(n)$, the group of $n \times n$ complex matrices such that $A^\dagger = A^{-1}$: their inverses are their transposes. One caveat is that this is *not* a complex Lie group, as this equation isn't holomorphic. For example, $\mathrm{U}(1) = \{z \in \mathbb{C} \text{ such that } |z| = 1\}$ is topologically S^1, and therefore is one-dimensional as a real manifold! This is also $\mathrm{SO}(2)$ (the circle acts by rotating $\mathbb{R}^2$). More generally, a *torus* is a finite product of copies of $\mathrm{U}(1)$. ◀

There are other examples that don't look like this, *exceptional groups* such as G_2, E_6, and F_4 which are matrix groups, yet not obviously so. We'll figure out how to get these when we discuss the classification of simple Lie algebras.

Here's an example of interest to physicists:

Example 1.6. Let q be a quadratic form of signature $(1, 3)$ (corresponding to Minkowski space). Then, $\mathrm{SO}(1, 3)$ denotes the group of matrices fixing q (origin-fixing isometries of Minkowski space), and is called the *Lorentz group.*

Smooth actions. If one wants to add translations, one obtains the *Poincaré group* $\mathrm{SO}(1, 3) \ltimes \mathbb{R}^{1,3}$. ◀

In a first course on group theory, one sees actions of a group G on a set X, usually written $G \curvearrowright X$ and specified by a map $G \times X \to X$, written $(g, x) \mapsto g \cdot x$. Sometimes we impose additional structure; in particular, we can let X be a smooth manifold, and require G to be a Lie group and the action to be smooth, or Riemannian manifolds and isometries, etc.[1]

It's possible to specify this action by a continuous group homomorphism $G \to \mathrm{Diff}(X)$ (or even smooth: $\mathrm{Diff}(X)$ has an infinite-dimensional smooth structure, but being precise about this is technical).

Example 1.7. $\mathrm{SO}(3) \coloneqq \mathrm{SL}_3(\mathbb{R}) \cap \mathrm{O}(3)$ denotes the group of rotations of three-dimensional space. Rotating the unit sphere defines an action of $\mathrm{SO}(3)$ on S^2, and this is an *action by isometries*, i.e. for all $g \in \mathrm{SO}(3)$, the map $S^2 \to S^2$ defined by $x \mapsto g \cdot x$ is an isometry. ◀

Example 1.8. Let $\mathbb{H} \coloneqq \{x + iy \mid y > 0\}$ denote the upper half-plane. Then, $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ by Möbius transformations. ◀

Smooth group actions arise in physics: if S is a physical system, then the symmetries of S often form a Lie group, and this group acts on the space of configurations of the system.

Where do representations come into this? Suppose a Lie group G acts on a space X. Then, G acts on the complex vector space of functions $X \to \mathbb{C}$, and G acts by linear maps, i.e. for each $g \in G$, $f \mapsto f(g \cdot -)$ is a linear map. This is what is meant by a representation, and for many people, choosing certain kinds of functions on X (smooth, continuous, L^2) is a source of important representations in representation theory. Representations on $L^2(X)$ are particularly important, as $L^2(X)$ is a Hilbert space, and shows up as the state space in quantum mechanics, where some of this may seem familiar.

[1] What if X has singular points? It turns out the axioms of a Lie group action place strong constraints on where singularities can appear in interesting situations, though it's not completely ruled out.

Representations.

Definition 1.9. A *(linear) representation* of a group G is a vector space V together with an action of G on V by linear maps, i.e. a map $G \times V \to V$ written $(g, v) \mapsto g \cdot v$ such that for all $g \in G$, the map $v \mapsto g \cdot v$ is linear.

This is equivalent to specifying a group homomorphism $G \to \mathrm{GL}(V)$.[2] Sometimes we will abuse notation and write V to mean V with this extra structure.

If G is in addition a Lie group, one might want the representation to reflect its smooth structure, i.e. requiring that the map $G \to \mathrm{GL}(V)$ be a homomorphism of Lie groups.

The following definition, codifying the idea of a representation that's as small as can be, is key.

Definition 1.10. A representation V is *irreducible* if it has no nontrivial invariant subspaces. That is, if $W \subseteq V$ is a subspace such that for all $w \in W$ and $g \in G$, $g \cdot w \in W$, then either $W = 0$ or $W = V$.

We can now outline some of the goals of this course:

- Classify the irreducible representations of a given group.
- Classify all representations of a given group.
- Express arbitrary representations in terms of irreducibles.

These are not easy questions, especially in applications where the representations may be infinite-dimensional.

Example 1.11 (Spherical harmonics)**.** Here's an example of this philosophy in action.[3]

Let's start with the Laplacian on $\mathbb{R}^3$, a second-order differential operator

$$\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2,$$

which acts on $C^\infty(\mathbb{R}^3)$. After rewriting in spherical coordinates, the Laplacian turns out to be a sum

$$\Delta = \Delta_{\mathrm{sph}} + \Delta_{\mathrm{rad}},$$

of spherical and radial parts independent of each other, so Δ_{sph} acts on functions on the sphere. We're interested in the eigenfunctions for this spherical Laplacian for a few reasons, e.g. they relate to solutions to the *Schrödinger equation*

$$\dot{\psi} = \widehat{H}(\psi),$$

where the Hamiltonian is

$$\widehat{H} = -\Delta_{\mathrm{sph}} + V,$$

where V is a potential.

The action of $\mathrm{SO}(3)$ on the sphere by rotation defines a representation of $\mathrm{SO}(3)$ on $C^\infty(S^2)$, and we'll see that finding the eigenfunctions of the spherical Laplacian boils down to computing the irreducible components inside this representation:

$$V_0 \oplus V_2 \oplus V_4 \oplus \cdots \overset{\text{dense}}{\subseteq} \mathcal{S}(S^2),$$

where $\mathcal{S}(S^2)$ is the space of Schwarz functions. These V_{2k} are the isomorphism classes of the irreducible representations of $\mathrm{SO}(3)$, and in fact are the eigenspaces for the spherical Laplacian, where the eigenvalue for V_{2k} is $\pm k(k+1)$. We'll see more things like this later, once we have more background. ◀

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[2]This general linear group $\mathrm{GL}(V)$ is the group of invertible linear maps $V \to V$.

[3]No pun intended.