KÄHLER GEOMETRY

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Note: I missed the first two lectures.

3. Holomorphic line bundles: 7/12/17

Today's going to be about holomorphic vector bundles, with a focus on holomorphic line bundles.

Definition 3.1. Let X be a complex manifold. A **holomorphic vector bundle** of rank k over E is a complex manifold E and a holomorphic map $\pi\colon E \to X$ such that

- π makes $E \to X$ into a complex vector bundle of rank k, and
- E admits **holomorphic trivializations**, i.e. there's an open cover $\mathfrak{U}$ of X trivializing E such that for each $U \in \mathfrak{U}$, there's a biholomorphic map $\varphi\colon E|_U \to U \times \mathbb{C}^k$ commuting with projection to U that is complex linear on each fiber.

A rank-1 holomorphic vector bundle is called a **holomorphic line bundle**.

Equivalently, $E \to X$ is holomorphic iff admits local holomorphic sections.

Definition 3.2. A **homomorphism** of holomorphic vector bundles $f\colon E \to F$ over X is a homomorphism of complex vector bundles that is holomorphic as a map between complex manifolds.

In particular, it must commute with the projection down to X and be complex linear on each fiber. If in addition it's invertible on each fiber, f is called an **isomorphism**.

Exercise 3.3. Show that if $f\colon E \to F$ is an isomorphism of holomorphic vector bundles, it's a biholomorphism on their total spaces.

Remark. Some authors, such as Huybrechts, add an extra condition, that the dimension of the rank of a homomorphism of vector bundles is constant, thus ensuring the (fiberwise) kernel and cokernel of a morphism are again holomorphic vector bundles. Other authors, such as Griffiths-Harris, do not require this, and we'll follow that convention. ◀

In the kyperkähler geometry minicourse, we saw a different definition of holomorphic vector bundles in terms of the $\overline{\partial}_E$ operator $\overline{\partial}_E\colon C^\infty(X,E) \to C^\infty(X, T^{0,1}X \otimes E)$. This is equivalent, and one way to understand this is to use a local trivialization: given a holomorphic identification $E|_U \cong U \times \mathbb{C}^k$ (for an open $U \subset X$) and a section $\psi\colon U \to \mathbb{C}^k$, define

$$\overline{\partial}_E(\psi) := \overline{\partial}\psi = \frac{\partial \psi}{\partial \overline{z}^\alpha}\,\mathrm{d}\overline{z}^\alpha.$$

Then, check that this glues on overlaps, producing a well-defined operator on smooth sections of E.

Another way to understand holomorphic vector bundles is through transition functions.

Proposition 3.4. *There is a bijective correspondenc between the set of isomorphism classes of rank-k holomorphic vector bundles on X and the set of open covers $\mathfrak{U}$ on X and holomorphic functions $\varphi_{ab}\colon U_a \cap U_b \to \mathrm{GL}_k(\mathbb{C})$ for all $U_a, U_b \in \mathfrak{U}$ such that $\varphi_{ab}\varphi_{bc} = \varphi_{ac}$ and $\varphi_{aa} = \mathrm{id}$, modulo equivalence on a common refinement of the open cover.*

Proof sketch. Given a vector bundle E, let $\mathfrak{U}$ be an open cover for which E has holomorphic local trivializations. For $U_a, U_b \in \mathfrak{U}$ that intersect, $\varphi_{ab} \colon U_a \cap U_b \to \mathrm{GL}_n(\mathbb{C})$ is the transition function

$$\begin{tikzcd}
& (U_a \cap U_b) \times \mathbb{C}^k \arrow[dd, "\varphi_{ab}"] \\
E|_{U_a \cap U_b} \arrow[ur, "f"] \arrow[dr, "g"'] & \\
& (U - a \cap U_b) \times \mathbb{C}^k.
\end{tikzcd}$$

Here f is the transition function for U_a and g is the transition function for U_b.

Conversely, given the data $\mathfrak{U}$ and $\{\varphi_{ab}\}$, one can define

$$E := \coprod_{U_a \in \mathfrak{U}} U_a \times \mathbb{C}^k/(x, v) \simeq (x, \varphi_{ab} v),$$

where $x \in U_a \cap U_b$ and $v \in \mathbb{C}^k$, over all pairs $U_a, U_b \in \mathfrak{U}$. Then one must check that equivalent data defines isomophic line bundles. ⊠

For $k = 1$, this proposition identifies the set of isomorphism classes of line bundles with the first Čech cohomology $\check{H}^1(X; \mathscr{O}_X^*)$, i.e. valued in the sheaf $\mathscr{O}_X^*$ of holomorphic functions into $\mathbb{C}^\times$. This is because $\mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$.

Pretty much every natural operation you can do to vector spaces extends to holomorphic vector bundles $E, F \to X$, including

- the dual $E^* \to X$,
- the direct sum $E \oplus F \to X$,
- the tensor product $E \otimes F \to X$,
- the wedge product $\Lambda^r E \to X$,
- the pullback $f^* E \to Y$ given a holomorphic map $f \colon Y \to X$,
- and so on.

One way to prove this is to write down their transition functions: suppose $\mathfrak{U}$ is an open cover of X which holomorphically trivializes both E and F (by taking common refinements, such a cover always exists), and suppose φ_{ab} are the transition functions for $\mathfrak{U}$ for E, and ψ_{ab} are those for F. Then,

- E^* has transition functions $(\varphi_{ab}^{\mathrm{T}})^{-1}$,
- $E \oplus F$ has transition functions
$$\begin{pmatrix} \varphi_{ab} & 0 \\ 0 & \psi_{ab} \end{pmatrix},$$
- $E \otimes F$ has transition functions $\varphi_{ab} \otimes \psi_{ab}$, and
- $\Lambda^r E$ has transisiton functions $\Lambda^r \varphi_{ab}$. In particular, if $r = k = \mathrm{rank}(E)$, then $\Lambda^k \varphi_{ab} = \mathrm{det}(\varphi_{ab})$.
- Given a holomorphic map $f \colon Y \to X$, $f^* E$ has transition functions $\varphi_{ab} \circ f$. Hence holomorphicity of f is necessary. This uses the trivializing open cover $f^{-1}(\mathfrak{U})$.

Remark. The set of isomorphism classes of holomorphic line bundles is a group under $\otimes$, called the **Picard group** $\mathrm{Pic}(X)$. The identity is the **trivial bundle** $\underline{\mathbb{C}} := X \times \mathbb{C}$, and the inverse of a line bundle $\mathcal{L}$ is $\mathcal{L}^*$, because $\mathcal{L} \otimes \mathcal{L}^* = \mathrm{End}(\mathcal{L})$, which has a global nonvanishing section that's the identity on each fiber. Hence $\mathcal{L} \otimes \mathcal{L}^* \cong \underline{\mathbb{C}}$. ◀

Example 3.5. Let X be a complex manifold. Then, the holomorphic tangent bundle $T^{1,0}X$ and the holomorphic cotangent bundle $T^{*1,0}X$ are holomorphic vector bundles. Hence, since the wedge product of holomorphic vector bundles is holomorphic, the canonical bundle $K_X := \Lambda^{n,0} T^* X = \Lambda^n (T^{*1,0} X)$ is a holomorphic line bundle. ◀

Proof. Let (z^α) be holomorphic coordinates on (an open neighborhood of a given point in) X. This defines a local trivialization of $T^{1,0}X$, namely

$$\left(z^1, \dots, z^n, \frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n} \right).$$

If (w^β) is another set of holomorphic coordinates, the transition functions are

$$\frac{\partial}{\partial z^\alpha} = \sum_\beta \frac{\partial w^\beta}{\partial z^\alpha}\frac{\partial}{\partial w^\beta}.$$

This is the Jacobi matrix $\left(\frac{\partial w^\beta}{\partial z^\alpha}\right)$, which is holomorphic. $T^{*1,0}X$ is similar. ⊠

However, $\Lambda^{p,q}T^*X$ is *not* a holomorphic vector bundle in general! For example, the transition functions on $T^{*0,1}X$ are antiholomorphic rather than holomorphic.

Example 3.6. The **tautological bundle** on $\mathbb{CP}^n$ is

$$\mathscr{O}_{\mathbb{P}^n}(-1) := \{(\ell, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} \mid v \in \ell\},$$

i.e., a point $\ell \in \mathbb{CP}^n$ is a line in $\mathbb{C}^{n+1}$, hence we can say the fiber over ℓ is ℓ regarded as a line. This is a holomorphic line bundle. The total space looks like $\mathbb{C}^{n+1}$ with a $\mathbb{CP}^n$ "glued in" at the origin; this is the local model of a blowup.

We can describe the local trivializations explicitly. Let $U_0 = \{z_0 \neq 0\} \subset \mathbb{CP}^n$. Then, the map $U_0 \times \mathbb{C} \to \mathscr{O}_{\mathbb{P}^n}(-1)|_{U_0}$ sends

$$([z_0 : \ldots : z_n], \lambda) \longmapsto \left([z_0 : \ldots : z_n], \lambda \cdot \left(1, \frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0}\right)\right),$$

and you can check that the transition functions for $U_0 \cap U_1 \to \mathbb{C}^\times$ (where U_1 is the locus where $z_1 \neq 0$) is the map $[z_0 : \ldots : z_n] \mapsto z_1/z_0$, which is biholomorphic (and hence this actually is a holomorphic line bundle). ◀

Definition 3.7. Using the tautological bundle, we can define a bunch of other line bundles on $\mathbb{CP}^n$:

- Let $\mathscr{O}_{\mathbb{P}^n}(0) := \underline{\mathbb{C}}$, the trivial bundle.
- Let $\mathscr{O}_{\mathbb{P}^n}(1) := \mathscr{O}_{\mathbb{P}^n}(-1)$.
- If $k > 0$, let $\mathscr{O}_{\mathbb{P}^n}(k) := \mathscr{O}_{\mathbb{P}^n}(1)^{\otimes k}$ and $\mathscr{O}_{\mathbb{P}^n}(-k) = \mathscr{O}_{\mathbb{P}^n}(-1)^{\otimes k}$.

Hence $k \mapsto \mathscr{O}_{\mathbb{P}^n}(k)$ defines a group homomorphism $\Phi\colon \mathbb{Z} \to \mathrm{Pic}(\mathbb{CP}^n)$.

Theorem 3.8. *In fact, $\Phi\colon \mathbb{Z} \to \mathrm{Pic}(\mathbb{CP}^n)$ is an isomorphism.*

We won't prove this. It's nontrivial: for complex line bundles, you can use $H^2(\mathbb{CP}^n) \cong \mathbb{Z}$, but then you have to prove that each has a unique holomorphic structure.

Proposition 3.9. *For $k > 0$, the space of holomorphic sections of $\mathscr{O}_{\mathbb{P}^n}(k)$ is isomorphic to the space of degree-k k-homogeneous polynomials in $n+1$ variables.*

Proof sketch. Suppose we're given such a homogeneous polynomial $P(z_1, \ldots, z_n)$. On the trivialization $U_0 \times \mathbb{C}$, define a section by

$$[z_0 : \ldots : z_n] \longmapsto P(1, z_1/z_0, \ldots, z_n/z_0) \in \mathbb{C},$$

which is holomorphic. It hence suffices to check that these local sections transform correctly according to the transition functions. On, for example, $U_0 \cap U_1$, we have that

$$P\left(1, \frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0}\right) = \left(\frac{z_1}{z_0}\right)^k P\left(\frac{z_0}{z_1}, 1, \frac{z_2}{z_1}, \ldots, \frac{z_n}{z_1}\right).$$

One can then show that these sections span $\Gamma(\mathbb{CP}^n, \mathscr{O}_{\mathbb{P}^n}(k))$. ⊠

Proposition 3.10. *The canonical bundle on $\mathbb{CP}^n$ is isomorphic to $\mathscr{O}_{\mathbb{P}^n}(-n-1)$.*

The proof idea is again to use the local trivialization U_i to define the local section

$$[1 : z_1 : \cdots : z_n] \longmapsto \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n$$

and compute transition functions.