GROMOV-WITTEN THEORY LEARNING SEMINAR

ARUN DEBRAY
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Contents

1. An Overview of Gromov-Witten Theory: 1/29/18

Today, Jonathan spoke, delivering an overview of Gromov-Witten theory and how associativity of quantum cohomology leads to applications in enumerative geometry. Today we always work over $\mathbb{C}$, and follow Fulton-Pandharipande's notes [FP96].

Classically, if X is a nonsingular projective variety and $\beta \in H_2(X;\mathbb{Z})$, we want to know how many algebraic curves in X represent the class β. This relates to very classical questions, such as: if you have $3d-1$ points in $\mathbb{P}^n$, how many degree-d curves pass through them?

Definition 1.1. To simplify notation, let $A_d(X) \coloneqq H_{2d}(X;\mathbb{Z})$, and similarly $A^d(X) \coloneqq H^{2d}(X;\mathbb{Z})$.

The moduli space of stable maps. Another important ingredient, whose construction we will punt on, is the *moduli space of stable maps*. Here we summarize its definition. Let X be a smooth projective variety and $\beta \in A_1(X)$. The moduli space of stable maps, denoted $\mathcal{M}_{g,n}(X,\beta)$ is the moduli space of isomorphism classes of pointed maps

$$u\colon (C, p_1, \dots, p_n) \longrightarrow X \tag{1.2}$$

where C is a projective nonsingular curve of genus g, $p_1, \dots, p_n$ are distinct marked points in C, and $u_*([c]) = \beta$. We must impose a stability condition which ensures these maps have finitely many automorphisms, where an automorphism $(C, p_1, \dots, p_n) \to (C', p_1], \dots, p'_n)$ must send $p_i \mapsto p'_i$ and commute with the maps to X.

This is all right, but we really want something compact, and therefore will have to consider stable maps which are slightly worse. The compactification $\overline{\mathcal{M}}_{g,n}(X,\beta) \supset \mathcal{M}_{g,n}(X,\beta)$ is the space of stable maps as in (1.2), subject to the following conditions.

- C is a projective, connected, reduced, genus-g curve with at worst nodal singularities, and the p_j are distinct smooth points.
- Stability: for every irreducible compact $E \subset C$ such that if $E \simeq \mathbb{P}^1$ and $u(E) = \{\mathrm{pt}\}$, then E contains at least 3 of the points p_i.
- If E is genus 1 and $u(E) = \{\mathrm{pt}\}$, then E contains at least one of the points p_i.

Why is this a compactification? The idea is that if $u\colon (C, p_1, \dots, p_n) \to X$ is a smooth curve, we can let two points collide. In the compactified moduli space, the collision is avoided by adding another $\mathbb{P}^1$ to C intersecting near the collision point; then, the two points can live in distinct irreducible components.

The next question is: what's the dimension of $\overline{\mathcal{M}}_{g,n}(X,\beta)$ be? Naïvely, the expected dimension is

$$n + \int_\beta c_1(X) + (\dim X - 3)(1-g) = n + 3g - 3 + \chi(TX|_C). \tag{1.3}$$

Here c_1 is the first Chern class, and $\int_\beta c_1(X)$ represents the cap product pairing $A_1(X) \otimes A^1(X) \to \mathbb{Z}$, and $\chi(TX|_C)$ denotes its Euler characteristic:

$$\chi(TX|_C) := h^0(C; TX|_C) - h^1(C; TX_|C). \tag{1.4}$$

This does not depend on the choice of C representing β, which is a fun fact about characteristic classes.

Why is (1.3) a reasonable guess? Here's what's going on.

- The $3g-3$ represents the dimension of the moduli space of the curve C, hence representing how C can change on its own.
- The $\chi(TX|_C)$ represents how C can deform in X.
- The n is the extra data corresponding to the marked points.

We said "naïve," and indeed (1.3) is not the dimension of $\overline{\mathcal{M}}_{g,n}(X,\beta)$ in all cases. But it is true in nice cases, and then you can do some cool stuff.

Gromov-Witten invariants. There are natural *evaluation maps* $p_i\colon \overline{\mathcal{M}}_{g,n}(X,\beta) \to X$ sending

$$(u\colon (C, p_1, \dots, p_n) \to X) \longmapsto u(p_i). \tag{1.5}$$

We can pull back cohomology classes along these maps: suppose $\gamma_1, \dots, \gamma_n \in A^*(X)$. Then, let

$$I_\beta(\gamma_1, \dots, \gamma_n) := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]} p_1^*(\gamma_1) \smile \cdots \smile p_n^*(\gamma_n). \tag{1.6}$$

This is called a *Gromov-Witten invariant* for X. The thing that we're integrating over requires some very technical work to define in general, but for spaces which are "nice" (convex and homogeneous, which we'll discuss later), it's not so bad. $\mathbb{P}^n$ is an example of such a space.

Suppose $\gamma_1, \dots, \gamma_n$ have the correct dimensions such that (1.6) is a number. Then there's an enumerative interpretation of (1.6) (in the convex case): the number of pointed maps $u\colon \Sigma_g \to X$ such that $u_*([\Sigma_g]) = \beta$ and if Γ_i is a subvariety representing the Poincaré dual to γ_i,[1] then $u(p_i) \in \Gamma_i$. Here Σ_g is a curve of genus g.

Important properties.

Proposition 1.7. *If $\beta = 0$, the only nonzero Gromov-Witten invariants occur when $n = 3$.*

Proof sketch. If $\beta = 0$, there's an identification $\overline{\mathcal{M}}_{0,n}(X,\beta) \cong \overline{\mathcal{M}}_{0,n} \times X$, which carries all of the evaluation maps to projection onto X, where $\overline{\mathcal{M}}_{0,n}$ is the (compactified) moduli space of genus-0 curves with n marked points. Call this map π. Then,

$$\begin{aligned} I_\beta(\gamma_1, \dots, \gamma_n) &= \int_{\overline{\mathcal{M}}_{0,n}(X,0)} p_1^*(\gamma_1) \smile \cdots \smile p_n^*(\gamma_n) \\ &= \int_{\overline{\mathcal{M}}_{0,n} \times X} \pi^*(\gamma_1 \smile \cdots \smile \gamma_n) \\ &= \int_{\pi_*([\overline{\mathcal{M}}_{0,n} \times X])} \gamma_1 \smile \cdots \smile \gamma_n. \end{aligned}$$

If $n < 3$, $\mathcal{M}_{0,n}$ is empty, because any choice of n points in $\mathbb{P}^1$ doesn't have a finite automorphism group. For $n > 3$, π has positive-dimensional fibers. ⊠

If $n = 3$, then

$$I_0(\gamma_1, \gamma_2, \gamma_3) = \int_X \gamma_1 \smile \gamma_2 \smile \gamma_3, \tag{1.8}$$

so this Gromov-Witten invariant isn't too hard to calculate.

Proposition 1.9. *Suppose $\gamma_1 = 1 \in A^0(X)$. Then, $I_\beta(1, \gamma_2, \dots, \gamma_n)$ is nonzero only when $\beta = 0$ and $n = 3$.*

[1] We can dodge the Steenrod realizability problem because every even-degree homology class of $\mathbb{P}^n$ is represented by a complex subvariety.

Proof sketch. If $\beta \neq 0$, $p_1^*(1) \smile \cdots \smile p_n^*(\gamma_n)$ is the pullback of a class in $\overline{\mathcal{M}}_{0,n-1}(X,\beta)$ along the map

$$\overline{\mathcal{M}}_{0,n}(X,\beta) \longrightarrow \overline{\mathcal{M}}_{0,n-1}(X,\beta) \tag{1.10}$$

which forgets the first point. There's a projection formula which then finishes the proof in a similar way to Proposition 1.7. ⊠

In the case $\beta = 0$ and $n = 3$, there's a similar formula to (1.8):

$$I_0(1,\gamma_2,\gamma_3) = \int_X \gamma_2 \smile \gamma_3. \tag{1.11}$$

Proposition 1.12. *If $\gamma_1 \in A^1(X)$, then*

$$I_\beta(\gamma_1,\dots,\gamma_n) = \left(\int_\beta \gamma_1\right) I_\beta(\gamma_2,\dots,\gamma_n).$$

Since $\int_\beta \gamma_1$ is the number of choices for $p_i \in C$ to map to Γ_1, where Γ_1 is a Poincaré dual to γ_1. The proof idea has something to do with the pushforward map (1.10) again.

Next time we'll talk about the quantum cohomology ring, and show that its associativity provides recursive formulas for enumerative invariants.

2. Quantum Cohomology: 2/5/18

Today, Jonathan spoke again, discussing quantum cohomology and an explicit example of how its associativity produces enumerative data on convex varieties.

Recall that last time, we discussed the moduli spaces of stable maps $\overline{\mathcal{M}}_{0,n}(X,\beta)$ given a variety X, a $\beta \in A_1(X)$, and an $n \geq 0$. We can use this moduli space, and the evaluation maps $p_i \colon \overline{\mathcal{M}}_{0,n}(X,\beta) \to X$, to define Gromov-Witten invariants as in (1.6). We then discussed three important properties of Gromov-Witten invariants, namely Propositions 1.7, 1.9 and 1.12; they will be useful when we do calculations.[2]

Now we'll define quantum cohomology in a restricted setting. Some of our notation will be redundant today, but will be useful when we discuss the general case. Fix $X = \mathbb{P}^r$ and $T_0 = 1 \in A^0(X)$. Let $T_1,\dots,T_p$ be a basis for $A^1(X)$ and $T_{p+1},\dots,T_m$ be a basis for the rest of $A^*(X)$. For $\beta \in A_1(X)$ and $n_{p+1},\dots,n_m \in \mathbb{N}$, let

$$N(n_{p+1},\dots,n_m;\beta) \coloneqq I_\beta(T_{p+1}^{n_{p+1}},\dots,T_m^{n_m}). \tag{2.1}$$

For $0 \leq i,j \leq m$, define

$$g_{ij} \coloneqq \int_X T_i \smile T_j \tag{2.2}$$

and g^{ij} be the entries of the matrix inverse to (g_{ij}). By (1.8),

$$T_i \smile T_j = \sum_{e,f} I_0(T_i,T_j,T_e) g^{ef} T_f. \tag{2.3}$$

Definition 2.4. The *quantum potential* of a $\gamma \in A^*(X)$ is

$$\Phi(\gamma) \coloneqq \sum_{n\geq 3} \sum_{\beta \in H_2(X;\mathbb{Z})} \frac{1}{n!} I_\beta(\gamma^n).$$

The summand is nonzero for only finitely many β for a given n, so this converges. Moreover, if $\gamma = \sum y_i T_i$,

$$\Phi(y_0,\dots,y_n) \coloneqq \Phi(\gamma) = \sum_{n_0+\cdots+n_m \geq 3} \sum_\beta I_\beta(T_0^{n_0},\dots,T_m^{n_m}) \frac{y_0^{n_0}}{n_0!}\cdots\frac{y_m^{n_m}}{n_m!}. \tag{2.5}$$

This is a formal power series in $y_0,\dots,y_n$, and hence one may define

$$\Phi_{ijk} \coloneqq \frac{\partial^3 \Phi}{\partial y_i \partial y_j \partial y_k} = \sum_{n\geq 0} \sum_\beta \frac{1}{n!} I_\beta(\gamma^n, T_i, T_j, T_k). \tag{2.6}$$

[2] Kontsevich and Manin [KM94] take these properties as *axioms* for Gromov-Witten theory.

Definition 2.7. The *quantum cup product* is

$$T_i * T_j := \sum_{e,f} \Phi_{ije} g^{ef} T_f.$$

Remark. This definition is kind of unenlightening — it's not clear what it's doing. Hopefully through examples we can figure out why it's defined in this way. ◂

Theorem 2.8. *$A^*(X)$ with the quantum cup product is associative, commutative, and has T_0 as a unit.*

The hardest part is associativity, requiring a full page of calculations. We're not going to do that today, but we'll talk about what it implies. Writing everything out,

$$\begin{aligned}(T_i * T_j) * T_k &= \sum_{e,f} \Phi_{i,e} g^{ef} T_f * T_k \\ &= \sum_{e,f}\sum_{c,d} \Phi_{ije} g^{ef} \Phi_{fkc} g^{cd} T_d.\end{aligned}$$

Similarly,

$$T_i * (T_j * T_k) = \sum_{e,f}\sum_{c,d} \Phi_{jke} g^{ef} \Phi_{ifc} g^{cd} T_d.$$

Therefore associativity is equivalent to

$$\Phi_{ije} g^{ef} \Phi_{fkc} = \Phi_{jke} g^{ef} \Phi_{ifc}, \tag{2.9}$$

so if we define

$$F(i,j \mid k,\ell) := \sum_{e,f} \Phi_{ije} g^{ef} \Phi_{fk\ell}, \tag{2.10}$$

then associativity is equivalent to $F(i,j \mid k,\ell) = F(j,k \mid i,\ell)$ for all i,j,k,ℓ.

We can split the quantum potential into two pieces: the "classical" part $\Phi_{\text{classical}}$, given by $\beta = 0$, and the "quantum" part Φ_{quantum}, for which $\beta \neq 0$. Then $\Phi = \Phi_{\text{classical}} + \Phi_{\text{quantum}}$, and using Proposition 1.7,

$$\Phi_{\text{classical}} = \sum_{n_1+\cdots+n_m=3} \int_X T_0^{n_0} \smile \cdots \smile T_m^{n_m} \prod_{i=1}^m \frac{y_i^{n_i}}{n_i!}. \tag{2.11}$$

TODO: then there was a big formula for $\Gamma(y)$ whose relation to the story was unclear to me.

Example 2.12. Let's actually do this on $X = \mathbb{P}^2$. For $i = 0,1,2$, let $T_i \in H^{2i}(\mathbb{P}^2)$ be the generators corresponding to the orientation coming from the complex structure. That is, T_0 is Poincaré dual to $\mathbb{P}^2$, T_1 to a embedded $\mathbb{P}^1$, and T_2 to a point. Recall that $g_{ij} = \int_{\mathbb{P}^2} T_i \smile T_j$, so

$$g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{2.13}$$

and $g^{-1} = g$.

Associativity of the quantum cup product implies that $F(1,1 \mid 2,2) = F(1,2 \mid 1,2)$, i.e.

$$\sum_{e,f} \Phi_{11e} g^{ef} \Phi_{f22} = \sum_{e,f} \Phi_{12e} g^{ef} \Phi_{f12}. \tag{2.14}$$

Since $g^{ef} \neq 0$ only when $e + f = 2$, this sum simplifies to

$$\Phi_{110}\Phi_{222} + \Phi_{111}\Phi_{122} + \Phi_{112} + \Phi_{022} = \Phi_{120}\Phi_{212} + \Phi_{121}\Phi_{112} + \Phi_{122}\Phi_{012}. \tag{2.15}$$

Now we have to actually compute some of these things.

$$\Phi_{110} = \sum_{n\geq 0}\sum_{\beta} I_\beta(\gamma^n \cdot T_1 \cdot T_0). \tag{2.16}$$

Most of these are zero for degree reasons, and the only nonzero contribution is from $\int_X T_1^2$.

TODO: then there was another thing I didn't follow. . . ◂

3. The moduli space of stable maps: 2/12/18

4. The little quantum product: 2/19/18

Today, Yixian spoke about associativity, the little quantum product, and more, finishing up the talks from [FP96].

We will continue to use notation from previous sections, in particular for Gromov-Witten invariants and the ingredients in the quantum product.

Let's suppose our target X is really nice: it's a projective, nonsingular, convex variety.

Definition 4.1. Let $\beta \in H_2(X)$. We call β an *effective class* if there is a stable map realizing β, i.e. $\overline{\mathcal{M}}_{0,n}(X,\beta)$ is nonempty.[3]

The idea of a boundary divisor is to split a reducible stable map into two components.

Definition 4.2. Let $\mu\colon (C, p_1, \dots, p_n) \to X \in \overline{\mathcal{M}}_{0,n}(X,\beta)$ be a stable map such that the domain curve C is reducible. The *boundary divisor* $D(A, B; \beta_1, \beta_2) \subset \overline{\mathcal{M}}_{0,n}(X,\beta)$ is the locus of stable maps which admit the following data:

- a partition $[n] = A \cup B$,[4] and
- effective classes β_1, β_2 such that $\beta_1 + \beta_2 = \beta$,

such that:

1. If $\beta_1 = 0$, $|A| \geq 2$, and if $\beta_2 = 0$, $|B| \geq 2$.
2. There are curves C_A, C_B such that $C_A \cup C_B = C$ and $C_A \cap C_B = \{\mathrm{pt}\}$.
3. The markings in A lie in C_A and the markings in B lie in C_B.
4. $\mu([C_A]) = \beta_1$ and $\mu([C_B]) = \beta_2$.

Theorem 4.3. *Let $e_1\colon \overline{\mathcal{M}}_{0,A\cup\{\mathrm{pt}\}}(X,\beta) \to X$ be the evaluation map at the extra point, and define $e_2\colon \overline{\mathcal{M}}_{0,B\cup\{\mathrm{pt}\}}(X,\beta) \to X$ analogously. Let $D(A, B; \beta_1, \beta_2)$ be a boundary divisor and define*

$$\widetilde{K} := \overline{\mathcal{M}}_{0,A\cup\{\mathrm{pt}\}}(X,\beta_1) \times_X \overline{\mathcal{M}}_{0,B\cup\{\mathrm{pt}\}}(X,\beta_2) \tag{4.4}$$

along e_1 and e_2. If A and B are nonempty, then $\widetilde{K} \cong D(A, B; \beta_1, \beta_2)$.

Let $i, j, k, \ell \in [n]$. We define a divisor

$$D(i, j \mid k, \ell) := \sum_{\substack{i,j\in A\\ k,\ell\in B}} D(A, B; \beta_1, \beta_2). \tag{4.5}$$

Then $D(i, j \mid k, \ell) = D(i, k \mid j, \ell)$.[5]

Recall that associativity of the quantum product, as defined in a previous lecture, is equivalent to (2.9): we have to prove some equalities about quantum potentials.

Lemma 4.6. *Let*

$$\iota\colon D(A, B, \beta_1, \beta_2) \longrightarrow \overline{\mathcal{M}}_{0,A\cup\{\mathrm{pt}\}}(X,\beta_1) \times \overline{\mathcal{M}}_{0,B\cup\{\mathrm{pt}\}}(X,\beta_2)$$

and

$$\alpha\colon D(A, B; \beta_1, \beta_2) \longrightarrow \overline{\mathcal{M}}_{0,n}(X,\beta)$$

denote inclusion. For $\gamma_1, \dots, \gamma_m \in A^*(X)$,

$$\iota_* \circ \alpha^*(\rho_1^*(\gamma_1) \smile \cdots \smile \rho_n^*(\gamma_n)) = \sum_{e,f} g^{ef} \left(\prod_{a\in A} \rho_a^*(\gamma_a)\rho_{\mathrm{pt}}^*(T_e) \right) \left(\prod_{b\in B} \rho_b^*(\gamma_b)\rho_{\mathrm{pt}}^*(T_f) \right).$$

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[3] TODO: just genus zero?

[4] Here, $[n] := \{1, \dots, n\}$.

[5] TODO: why?

Define

$$\begin{aligned}
G(q,r \mid s,t) &:= \sum_{\substack{q,r\in A\\ s,t\in B}} g^{ef} I_{\beta_1}\left(\prod_{a\in A}\gamma_a T_e\right) I_{\beta_2}\left(\prod_{\beta\in B}\gamma_b T_f\right)\\
&= \sum_{\substack{A\cup B=[n]\\ \beta_1+\beta_2=\beta}} \int_{D(A,B,\beta_1,\beta_2)} \rho_1^*(\gamma_1)\smile\cdots\smile\rho_n^*(\gamma_n)\\
&= \int_{D(q,r|s,t)} \rho_1^*(\gamma_1)\smile\cdots\smile\rho_n^*(\gamma_n).
\end{aligned}$$

Then associativity of the quantum product is asking whether

$$G(q,r \mid s,t) \stackrel{?}{=} G(q,s \mid r,t). \tag{4.7}$$

Next we define

$$F(i,j \mid k,\ell) := \sum_{e,f} \Phi_{ije} g^{ef} \Phi_{fk\ell} = \sum_{\substack{\beta_1+\beta_2=\beta\\ e,f,n_1,n_2}} I_{\beta_1}(\gamma^{n_1}\cdot T_i T_j T_\ell) g^{ef} I_{\beta_2}(\gamma^{n_2}\cdot T_k T_e T_f), \tag{4.8}$$

and associativity would imply this is equal to $F(i,k \mid j,\ell)$.

Remark. We're not going to attach intrinsic geometric meaning to F and G; they are tools in the proof of associativity. However, the notes suggestively use Feynman-diagram-like notation for them, which suggests that these things have an interpretation in physics. ◀

Recall that the quantum product is defined on a basis by

$$T_i * T_j := \sum_{e,f} \Phi_{ije} g^{ef} T_f. \tag{4.9}$$

We can use this to define the *quantum cohomology ring* $QH^*(X)$ as the algebra generated by $A^*(X)$ under this product. This is naturally a $\mathbb{Q}[[y]]$-algebra (where y acts by the extra factor of γ that has appeared in everything), and if $V := A^*(X)\setminus 0$, it's also a $\mathbb{Q}[[V]]$-algebra, which is a more coordinate-free way to say it. That is, $\mathbb{Q}[[V]]$ is the completion of

$$\bigoplus_{i=0}^{\infty} \mathrm{Sym}^i(V)\otimes\mathbb{Q}$$

at its unique maximal ideal.

The embedding map $A^*(X)\to QH^*(X)$ is a group homomorphism, but not a ring homomorphism.

If X is a homogeneous variety, there's an isomorphism

$$QH^*(X)\cong A^*_{\mathbb{Q}}(X)\otimes\mathbb{Q}[[y]].$$

In general this is not true.

TODO: more stuff happened, including a calculation on $\mathbb{P}^2$, but I didn't get it down.

References

[FP96] W. Fulton and R. Pandharipande. Notes on stable maps and quantum cohomology. 1996. https://arxiv.org/pdf/alg-geom/9608011.pdf. 1, 5

[KM94] M. Kontsevich and Yu. Manin. Gromov-witten classes, quantum cohomology, and enumerative geometry. *Comm. Math. Phys.*, 164(3):525–562, 1994. https://projecteuclid.org/download/pdf_1/euclid.cmp/1104270948. 3