

# M392C NOTES: A COURSE ON SEIBERG-WITTEN THEORY AND 4-MANIFOLD TOPOLOGY

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These notes were taken in UT Austin's M392C (A course on Seiberg-Witten theory and 4-manifold topology) class in Spring 2016, taught by Tim Perutz. I live-TeXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to [a.debray@math.utexas.edu](mailto:a.debray@math.utexas.edu). Any mistakes in the notes are my own.

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Lecture 1.

## Classification problems in differential topology: 1/18/18

*"This is my opinion, but it's the only reasonable opinion on this topic."*

This course will be on gauge theory; specifically, it will be about Seiberg-Witten theory and its applications to the topology of 4-manifolds. The course website is <https://www.ma.utexas.edu/users/perutz/GaugeTheory.html>; consult it for the syllabus, assignments, etc.

The greatest mystery in geometric topology is: *what is the classification of smooth, compact, simply-connected four-manifolds up to diffeomorphism?* The question is wide open, and the theory behaves very differently than the theory in any other dimension.

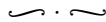
There's a fascinating bit of partial information known, mostly via PDEs coming from gauge theory, e.g. the *instanton equation*  $F_A^+ = 0$  as studied by Donaldson, Uhlenbeck, Taubes, and others. More recently, people have also studied the *Seiberg-Witten equations*

$$(1.1a) \quad D_A \psi = 0$$

$$(1.1b) \quad \rho(F_A^+) = (\psi \otimes \psi^*)_0.$$

Even without defining all of this notation, it's evident that the Seiberg-Witten equations are more complicated than the instanton equation, and indeed they were discovered later, by Seiberg and Witten in 1994. However, they're much easier to work with — after their discovery, the results of Donaldson theory were quickly reproven, and more results were found, within the decade after their discovery. This course will focus on results from Seiberg-Witten theory.

In some sense, this is a closed chapter: the stream of results on 4-manifolds has slowed to a trickle. But Seiberg-Witten theory has in the meantime found new applications to 3-manifolds, contact topology (including the remarkable proof of the Weinstein conjecture by Taubes), knots, high-dimensional topology, Heegaard-Floer homology, and more. Throughout this constellation of applications, there are many results whose only known proofs use the Seiberg-Witten equations.



The central problem in differential topology is to classify manifolds up to diffeomorphism. To make the problem more tractable, let's restrict to smooth, compact, and boundaryless. An ideal solution would solve the following four problems for some class of manifolds (e.g. compact of a particular dimension, and maybe with some topological constraints).

- (1) Write down a set of “standard manifolds”  $\{X_i\}_{i \in I}$  such that each manifold is diffeomorphic to precisely one  $X_i$ . For example, a list of diffeomorphism classes of closed oriented connected surfaces is given by the sphere and the  $n$ -holed torus for all  $n \geq 0$ .
- (2) Given a description of a manifold  $M$ , a way to compute invariants to decide for which  $i \in I$   $M \cong X_i$ . For example, if  $M$  is a closed, connected, oriented surface, we can completely classify it by its Euler characteristic.

A variant of this problem asks for an explicit algorithm to do this when  $M$  is encoded with finite information, e.g. a solution set to polynomial equations in  $\mathbb{R}^N$  with rational coefficients.

- (3) Given  $M$  and  $M'$ , compute invariants to decide whether  $M$  is diffeomorphic to  $M'$ ; once again, there's an algorithmic variant to that problem.
- (4) Understand families (fiber bundles) of manifolds diffeomorphic to  $M$ . In some sense, this means understanding the homotopy type of the topological group  $\text{Diff}(M)$  of self-diffeomorphisms of  $M$ .

This is an ambitious request, but much is known in low dimensions. In dimension 1, the first three questions are trivial, and the last is nontrivial, but solved.

**Example 1.2.** For compact, orientable, connected surfaces, we have a complete solution: a list of diffeomorphism classes is the sphere and  $(T^2)^{\#g}$  for all  $g \geq 0$ , and the Euler characteristic  $\chi := 2 - 2g$  is a complete invariant which is algorithmically computable from any reasonable input data, solving the second and third questions. Here, “reasonable input data” could include a triangulation, a good atlas (meaning nonempty intersections are contractible), or monodromy data for holomorphic a branched covering map  $\Sigma \rightarrow S^2$ , where here we're thinking of surfaces as Riemann surfaces, with chosen complex structures. Here, the Riemann-Hurwitz formula can be used to compute the Euler characteristic.

For the fourth question, let  $\text{Diff}^+(\Sigma)$  denote the topological group of orientation-preserving self-diffeomorphisms of  $\Sigma$ .

**Theorem 1.3** (Earle-Eells).

- The inclusion  $\text{SO}_3 \hookrightarrow \text{Diff}^+(S^2)$  is a homotopy equivalence.
- The identification  $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$  defines a map  $T^2 \hookrightarrow \text{Diff}^+(T^2)$  as translations; this map is a homotopy equivalence into the connected component of the identity in  $\text{Diff}^+(T^2)$ , and  $\pi_0 \text{Diff}^+(T^2) \cong \text{SL}_2(\mathbb{Z})$ .
- If  $g > 1$ , every connected component of  $\text{Diff}^+(\Sigma_g)$  is contractible, and the mapping class group  $\text{MCG}(\Sigma_g) := \pi_0 \text{Diff}^+(\Sigma_g)$  is a finitely presented infinite group which acts with finite stabilizers on a certain contractible manifold called Teichmüller space.<sup>1</sup>

So all four questions have satisfactory answers, though understanding the mapping class groups of surfaces is still an active area of research. ◀

**Example 1.4.** The classification of compact, orientable 3-manifolds looks remarkably similar to the classification of surfaces (albeit much harder!), through a vision of Thurston, realized by Hamilton and Perelman.

<sup>1</sup>Heuristically, but not literally, Teichmüller space is a classifying space for this group.

The solution is almost as complete. The proof uses geometry, and nice representatives are quotients by groups acting on hyperbolic space.

As for invariants, the fundamental group is very nearly a complete invariant.<sup>2</sup>

◀

In higher dimensions, there are a few limitations. Generally, the index set  $I$  will be uncountable. For example, there are an uncountable number of smooth 4-manifolds homeomorphic to  $\mathbb{R}^4$ ! So there will be no nice list, and no nice moduli space either. But restricted to compact manifolds, there are countably many classes, which follows from triangulation arguments or work of Cheeger in Riemannian geometry.

The next obstacle involves the fundamental group. If  $M$  is presented as an  $n$ -handlebody (roughly, a CW complex with cells of dimension at most  $n$ ), there is an induced presentation of  $\pi_1(M)$ , and if  $M$  is compact, this is a finite presentation (finitely many generators, and finitely many relations).

*Fact.* For each  $n \geq 4$ , all finite presentations arise from compact  $n$ -handlebodies (namely, closed  $n$ -manifolds).

◀

This is pretty cool, but throws a wrench in our classification goal.

**Theorem 1.5** (Markov). *There is no algorithm that decides whether a given finite group presentation gives the trivial group.*

The proof shows that an algorithm which could solve this problem could be used to construct an algorithm that solves the halting problem for Turing machines.

**Corollary 1.6.** *There is no algorithm to decide whether a given  $n$ -handlebody,  $n \geq 4$ , is simply connected.*

This means that a general classification algorithm cannot possibly work for  $n \geq 4$ ; thus, we will have to restrict what kinds of manifolds we classify.

A third issue in higher-dimensional topology is that in dimension  $n \leq 3$ , there are existence and uniqueness theorems of “optimal” Riemannian metrics (e.g. constraints on their isometry groups), but for  $n \geq 5$ , this is not true for any sense of optimal; some choices fail existence, and others fail uniqueness. This is discussed further (and more precisely) in Shmuel Weinberger’s “Computers, Rigidity, and Moduli,” which has some very interesting things to say about the utility of Riemannian geometry to classify manifolds (or lack thereof).

So four dimensions is special, but for many reasons, not just one.

Those setbacks notwithstanding, we can still say useful things.

- We will restrict to closed manifolds.
- We will focus on the simply-connected case, eliding Markov’s theorem.<sup>3</sup>

With these restrictions, we have good answers to the first three questions.

**Example 1.7.** There is a countable list of compact, simply-connected 5-manifolds, and invariants (cohomology, characteristic classes) which distinguish any two.

◀

**Example 1.8.** Kervaire-Milnor produced a classification of homotopy spheres in dimensions  $5 \leq n \leq 18$ , and a conceptual answer in higher dimensions, and further work has applied this in higher dimensions.

◀

There is a wider range of conceptual answers to all four questions, more or less explicit, through *surgery theory*, when  $n \geq 5$  (surgery theory fails radically in dimension 4). This gives an answer to the following questions.

- Given a finite,  $n$ -dimensional CW complex  $X$  (where  $n \geq 5$ ), when is it the homotopy type of a compact  $n$ -manifold?
- Given a simply-connected compact manifold  $M$ , what are the diffeomorphism types of manifolds homotopy equivalent to  $M$ ? (Again, we need  $\dim M \geq 5$ .)

Here are necessary and sufficient conditions for the existence question.

- $X$  must be an  $n$ -dimensional *Poincaré duality space*, i.e. there is a fundamental class  $[X] \in H_n(X; \mathbb{Z})$  which implements the Poincaré duality isomorphism. This basic fact about closed manifolds gets you an incredibly long way towards the answer.

<sup>2</sup>The fundamental group cannot distinguish lens spaces, and that’s pretty much the only exception.

<sup>3</sup>More generally, one could pick some fixed group  $G$  and ask for a classification of closed  $n$ -manifolds with  $\pi_1(M) \cong G$ ; people do this, but we won’t worry about it.

- Next,  $X$  must have a tangent bundle — but it's not clear what this means for a general Poincaré duality space. Here we mean a rank- $n$  vector bundle  $T \rightarrow X$  which is associated to the homotopy type in a certain precise sense: the unit sphere bundle of the stabilization of  $T$ , considered as a spherical fibration, has to be manifest in  $X$  in a certain way.
- If  $n \equiv 0 \pmod{4}$ , there's another obstruction — a certain  $\mathbb{Z}$ -valued invariant must vanish, interpreted as asking that  $T \rightarrow X$  satisfies the Hirzebruch signature theorem: the signature of the cup product form on  $H^{n/2}(X)$  must be determined by the Pontrjagin classes of  $T$ .
- If  $n \equiv 2 \pmod{4}$ , the obstruction is a similar  $\mathbb{Z}/2$ -valued invariant related to the Arf invariant of the intersection form.
- If  $n$  is odd, there are no further obstructions.

That's it. Uniqueness is broadly similar — once you specify a tangent bundle, there are only finitely many diffeomorphism types!

Now we turn to dimension 4, the hardest case. We want to classify smooth, closed, simply-connected 4-manifolds. The first basic invariant (even of 4-dimensional Poincaré duality spaces) is the intersection form  $Q_P$ , which we'll begin studying in detail next week. You can realize it as a unimodular matrix modulo integral equivalence. That is, it's a symmetric square matrix over  $\mathbb{Z}$  with determinant  $\pm 1$ , and integral equivalence means up to conjugation by elements of  $\mathrm{GL}_b(\mathbb{Z})$ .

**Theorem 1.9** (Milnor). *The intersection form defines a bijection from the set of homotopy classes of 4-dimensional simply-connected Poincaré spaces to the set of unimodular matrices modulo equivalence.*

So this form captures the entire homotopy type! That's pretty cool.

**Theorem 1.10** (Freedman). *The intersection form defines a bijection from the set of homeomorphism classes of 4-dimensional simply-connected topological manifolds to the set of unimodular matrices modulo equivalence.*

Thus this completely classifies (closed, simply-connected) topological four-manifolds. This theorem won Freedman a Fields medal.

The next obstruction, having a tangent bundle, is a mild constraint told to us by Rokhlin.

**Theorem 1.11** (Rokhlin). *Let  $X$  be a closed 4-manifold. If  $Q_X$  has even diagonal entries, then its signature is divisible by 16.*

The signature is the number of positive eigenvalues minus the number of negative eigenvalues. Algebra tells us this is already divisible by 8, so this is just a factor-of-2 obstruction, which is not too bad.

But the rest of the story of surgery theory is just wrong in dimension 4. This is where analysis of an instanton moduli space comes in.

**Theorem 1.12** (Donaldson's diagonalizability theorem). *Let  $X$  be a compact, simply-connected 4-manifold. If  $Q_X$  is positive definite, i.e.  $xQ_Xx > 0$  for all nonzero  $x \in \mathbb{Z}^b$ , then  $Q_X$  is equivalent to the identity matrix.*

Donaldson proved this theorem as a second-year graduate student!

There's a huge number of unimodular matrices which are positive definite, but not equivalent to the identity; the first example is known as  $E_8$ . So this is a strong constraint on their realizability by 4-manifolds.

In subsequent years, Donaldson devised invariants distinguishing infinitely many diffeomorphism types within a single homotopy class. Then, from 1994 onwards, there came new proofs of these results via Seiberg-Witten theory, which tended to be simpler,<sup>4</sup> and to provide sharper, more general results. We will prove several of these in the second half of the class.

Lecture 2.

## Review of the algebraic topology of manifolds: 1/23/18

Though today might be review for some students, it's important to make sure we're all on the same page, and we'll get to the good stuff soon enough. We won't do too many examples today, but will see many in the future.

<sup>4</sup>That said, Donaldson's original proof of the diagonalizability theorem stands as one of the most beautiful things in gauge theory.

**Cup products.** Cup products make sense in a more general sense than manifolds. Let  $X$  and  $Y$  be CW complexes; then, there is a canonical induced CW structure on  $X \times Y$ : the product of a pair of discs is homeomorphic to a disc, and we take the cells of  $X \times Y$  to be the products of cells of  $X$  and cells of  $Y$ .

Recall that the *cellular chain complex*  $C_*(X)$  is the free abelian group on the set of cells, and the *cellular cochain complex* is the dual:  $C^*(X) := \text{Hom}(C_*(X), \mathbb{Z})$ .

**Proposition 2.1** (Künneth formula). *Let  $X$  and  $Y$  be CW complexes. There is a canonical isomorphism*

$$(2.2) \quad C^*(X \times Y) \cong C^*(X) \otimes C^*(Y).$$

This follows because the cells of  $X \times Y$  are the products of those in  $X$  and those in  $Y$ . There is an analogue of the Künneth formula for pretty much any kind of (ordinary) cohomology theory.

The *diagonal map*  $\Delta: X \rightarrow X \times X$  sending  $x \mapsto (x, x)$  is, annoyingly, *not* a cellular map (i.e. it does not preserve the  $k$ -skeleton). However, it is homotopic to a cellular map  $\delta: X \rightarrow X \times X$ .

**Definition 2.3.** The *cup product of cochains* is the map  $\smile: C^*(X) \otimes C^*(X) \rightarrow C^*(X)$  which is the composition

$$C^*(X) \otimes C^*(X) \xrightarrow[(2.2)]{\cong} C^*(X \times X) \xrightarrow{\delta^*} C^*(X).$$

We haven't said anything about coboundaries, but the cup product plays well with them, and therefore induces a cup product on cellular cohomology,  $\smile: H^*(X) \otimes H^*(X) \rightarrow H^*(X)$ . This is an associative map, and it's *graded*, meaning it sends  $H^i(X) \otimes H^j(X)$  into  $H^{i+j}(X)$ . It's unital and *graded commutative*, meaning

$$(2.4) \quad x \smile y = (-1)^{|x||y|} y \smile x.$$

This turns  $H^*(X)$  into a graded commutative ring.

*Remark.* The cup product is *not* graded commutative on the level of cochains. However, there are coherent homotopies between  $x \smile y$  and  $(-1)^{|x||y|} y \smile x$ .  $\blacktriangleleft$

The fact that we had to choose  $\Delta \simeq \delta$  is annoying, since it's non-explicit and non-canonical. The cup product in singular cohomology does not have this problem, as you can just work with  $\Delta$  itself, but the tradeoff is that the Künneth formula is less explicit.

There are a few other incarnations of the cup product which are more geometrically transparent, and this will be useful for us when studying manifolds. These have other drawbacks, of course.

- (1) Čech cohomology is a somewhat unintuitive way to define cohomology, but has the advantage of providing a completely explicit formula for the cup product.
- (2) de Rham cohomology provides a model for the cup product which is graded-commutative on cochains, but only works with  $\mathbb{R}$  coefficients.
- (3) The intersection theory of submanifolds is a beautiful model for the cup product, but is not always available.

We'll discuss these in turn.

**Čech cohomology.** Let  $M$  be a manifold,<sup>5</sup> and  $\mathfrak{U} = \{U_i\}_{i \in I}$  be an open cover of  $M$ . For  $J \subset I$ , write

$$U_J := \bigcap_{i \in J} U_i.$$

**Definition 2.5.** We say that  $\mathfrak{U}$  is a *good cover* if it is locally finite and all  $U_J$ ,  $J \neq \emptyset$ , are empty or contractible.

In particular, on a compact manifold, a good cover is finite.

**Lemma 2.6.** *Any manifold admits a good cover.*

There are two standard proofs of this — one chooses small geodesic balls around each point for a Riemannian metric on  $M$ , and the other chooses an embedding  $M \hookrightarrow \mathbb{R}^N$  and then uses the intersections of small balls in  $\mathbb{R}^N$  with  $M$ .

There is also a uniqueness (really cofinality) statement.

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<sup>5</sup>Čech cohomology works on a more general class of spaces, but we work with manifolds for simplicity.

**Lemma 2.7.** *Any two good covers of a manifold  $M$  admit a good common refinement.*

For  $k \in \mathbb{Z}_{\geq 0}$ , let  $[k] := \{0, \dots, k\}$ .

**Definition 2.8.** Let  $k \in \mathbb{Z}_{\geq 0}$ . A  $k$ -simplex of  $\mathfrak{U}$  is a way of indexing a  $k$ -fold intersection in  $\mathfrak{U}$ ; specifically, it is an injective map  $\sigma: [k] \hookrightarrow I$  such that  $\mathfrak{U}_{\sigma([k])}$  is nonempty. The set of  $k$ -simplices of  $\mathfrak{U}$  is denoted  $S_k(\mathfrak{U})$ .

There is a *boundary map*  $\partial_i: S_k(\mathfrak{U}) \rightarrow S_{k-1}(\mathfrak{U})$  which deletes  $\sigma(i)$ .

**Definition 2.9.** Let  $A$  be a commutative ring. The *Čech cochain complex valued in  $A$*  is the cochain complex  $\check{C}^*(M, \mathfrak{U}; A)$  defined by

$$C^k(M, \mathfrak{U}; A) := \prod_{S_k} A$$

and with differential  $\delta: \check{C}^k(M, \mathfrak{U}; A) \rightarrow \check{C}^{k+1}(M, \mathfrak{U}; A)$  defined by

$$(\delta\eta)(\sigma) := \sum_{i=0}^{k+1} (-1)^{i+1} \eta(\partial_i \sigma),$$

where  $\eta$  is a cochain and  $\sigma: [k] \hookrightarrow I$ .

One can show that  $\delta^2 = 0$ , hence define the *Čech cohomology groups*  $\check{H}^*(M, \mathfrak{U}; A) := \ker(\delta) / \text{Im}(\delta)$ .

**Proposition 2.10.** *Let  $\mathfrak{U}$  and  $\mathfrak{V}$  be good covers of a manifold  $M$ . Then, there is an isomorphism  $\check{H}^*(M, \mathfrak{U}; A) \cong \check{H}^*(M, \mathfrak{V}; A)$ .*

*Proof idea.* By Lemma 2.7,  $\mathfrak{U}$  and  $\mathfrak{V}$  admit a common refinement  $\mathfrak{W}$ ; then, check that a refinement map of good covers induces an isomorphism in Čech cohomology.  $\square$

Thus the Čech cohomology is often denoted  $\check{H}^*(M; A)$ .

In Čech cohomology, there is a finite, combinatorial model for the cup product: let  $\alpha \in \check{C}^i$ ,  $\beta \in \check{C}^j$ , and  $\sigma: [i+j] \hookrightarrow I$ . Then, we let

$$(2.11) \quad (\alpha \smile \beta)(\sigma) := \alpha(\text{beginning of } \sigma) \cdot \beta(\text{end of } \sigma).$$

To be sure, this works in a more general setting (and indeed is the definition of cup product in singular cohomology), but the finiteness of Čech cochains on a compact manifold makes it a lot nicer in this setting. However, it's not at all transparent that the cup product is graded commutative on cohomology.

**Theorem 2.12.** *There is an isomorphism of graded rings  $\check{H}^*(M; A) \cong H^*(M; A)$  (where the latter means cellular cohomology).*

**de Rham cohomology.** Recall that  $\Omega^k(M)$  denotes the space of differential  $k$ -forms on a manifold  $M$ , and

$$\Omega^*(M) := \bigoplus_{k \geq 0} \Omega^k(M).$$

There is an exterior derivative  $d: \Omega^*(M) \rightarrow \Omega^{*+1}(M)$  with  $d^2 = 0$ , so we can define the *de Rham cohomology*  $H_{\text{dR}}^*(M) := \ker(d) / \text{Im}(d)$  in the usual way.

In this case, the cup product is induced by the wedge product of differential forms

$$\wedge: \Omega^i(M) \otimes \Omega^j(M) \rightarrow \Omega^{i+j}(M).$$

**Proposition 2.13.** *The wedge product is graded commutative on differential forms, hence makes  $\Omega^*(M)$  into a DGA (differential graded algebra).*

This is really nice, but can only occur in characteristic zero; if you tried to do this over a field of positive characteristic, you would run into obstructions called Steenrod squares to defining a functorial graded-commutative cochain model for cohomology.

**Theorem 2.14** (de Rham). *Let  $\mathfrak{U}$  be a good cover of a manifold  $M$ . Then there is a natural isomorphism of graded  $\mathbb{R}$ -algebras  $H_{\text{dR}}^*(M) \cong \check{H}^*(M, \mathfrak{U}; \mathbb{R})$ .*

There are several different ways of proving this. One is to show that they both satisfy the Eilenberg-Steenrod axioms with  $\mathbb{R}$  coefficients, and that up to natural isomorphism there is a single cohomology theory satisfying these isomorphisms. Another is to observe that Čech cohomology is a model for sheaf cohomology, and that both Čech and de Rham cohomology are derived functors of the same functor of sheaves on  $M$  applied to the constant sheaf valued in  $\mathbb{R}$ .

An alternative way to prove it, whose details can be found in Bott-Tu's book, is to form the *Čech-de Rham complex*, a double complex  $\check{C}^*(M, \mathfrak{U}; \Omega^\bullet)$ . Let  $D^*$  denote its *totalization*. Then there are quasi-isomorphisms  $\check{C}^* \hookrightarrow D^*$  and  $\Omega^*(M) \hookrightarrow D^*$  respecting products, hence inducing isomorphisms  $\check{H}^* \cong H^*(D^*) \cong H_{\text{dR}}^*(M)$ .

**Poincaré duality and the fundamental class** Poincaré duality is one of the few (relatively) easy facts about topological manifolds, and one of the only things known until the work of Kirby and Siebenmann in the 1970s. Throughout this section,  $X$  denotes a nonempty, connected topological manifold of dimension  $n$ . For a reference for this section, see May's *A Concise Course in Algebraic Topology*.

**Proposition 2.15.**

- (1) If  $k > n$ ,  $H_k(X) = 0$ .
- (2)  $H_n(X) \cong \mathbb{Z}$  if  $X$  is compact and orientable, and is 0 otherwise.

If  $X$  is compact, a choice of orientation defines a generator  $[X] \in H_n(X)$ , called the *fundamental class* of  $X$ . A homeomorphism  $f: X \rightarrow Y$  sends  $[X] \mapsto [Y]$  if  $f$  preserves orientation and  $[X] \mapsto -[Y]$  if  $f$  reverses orientation. If  $X$  is a CW complex with no cells of dimension  $> n$  and a single cell  $e_n$  in dimension  $n$ , then in cellular homology,  $[X] = \pm[e_n]$ .

There is a *trace map* or *evaluation map*  $H^n(X; A) \rightarrow A$  sending  $c \mapsto \text{eval}(c, [X])$  (that is, evaluate  $c$  on  $[X]$ ); in the de Rham model on a smooth manifold, this is the integration map

$$\eta \mapsto \int_X \eta.$$

The graded abelian group  $H_{-*}(X)$  is a graded module over the graded ring  $H^*(X)$  via a map called the *cap product*

$$\frown: H^k(X) \otimes H_i(X) \longrightarrow H_{i-k}(X).$$

Place a CW structure on  $X$ , and recall that  $\delta: X \rightarrow X \times X$  was our cellular approximation to the diagonal. Then, we can give a cellular model for the cap product:

$$C^*(X) \otimes C_*(X) \xrightarrow{\text{id} \otimes \delta_*} C^*(X) \otimes C_*(X) \otimes C_*(X) \xrightarrow{\text{eval} \otimes \text{id}} C_*(X).$$

Let  $X$  and  $Y$  be smooth  $n$ -manifolds, where  $X$  is closed and oriented. Then,  $-\frown f_*[X]: H_{\text{dR}}^n(Y) \rightarrow H_{\text{dR}}^n(X)$  has the explicit model

$$\eta \mapsto \int_X f^* \eta,$$

showing how the cap product relates to the evaluation map.

**Theorem 2.16** (Poincaré duality). *For  $X$  a closed, oriented manifold, the map*

$$D_X := -\frown [X]: H^*(X) \rightarrow H_{n-*}(X)$$

*is an isomorphism.*

For a proof, see May. In the case of smooth manifolds, there's a slick proof using Morse theory; but Poincaré duality is true for topological manifolds as well.

We will let  $D^X := (D_X)^{-1}$ .

**Intersections of submanifolds.** Intersection theory, though not its relation to the cup product, was discussed in the differential topology prelim. Let  $X$ ,  $Y$ , and  $Z$  be closed, oriented manifolds of dimensions  $n$ ,  $n-p$ , and  $n-q$  respectively, and let  $f: Y \rightarrow X$  and  $g: Z \rightarrow X$  be smooth maps. Let  $c_Y := D^X(f_*[Y]) \in H^p(X)$ , and similarly let  $c_Z := D^X(g_*[Z]) \in H^q(X)$ . We will be able to give a nice interpretation of  $c_Y \smile c_Z$ .

First, let  $f'$  be a smooth map homotopic to  $f$  and transverse to  $g$ ; standard theorems in differential topology show that such a map exists. Transversality means that if  $y \in Y$  and  $z \in Z$  are such that  $f'(y) = g(z)$ , then

$$T_x X = Df'(T_y Y) + Dg(T_z Z).$$



Let  $P := Y_{f'} \times_g X$ , which is exactly the space of pairs  $(y, z)$  such that  $f'(y) = g(z)$ ; transversality guarantees this is a smooth manifold of codimension  $p + q$  in  $X$ . The orientations on  $X$ ,  $Y$ , and  $Z$  induce one on  $P$ , and there is a canonical map  $\phi: P \rightarrow X$  sending  $(y, z) \mapsto f'(y) = g(z)$ .

**Theorem 2.17.** *Let  $c_P := D^X(\phi_*([P]))$ . Then,  $c_P = c_Y \smile c_Z$ .*

If  $Y$  and  $Z$  are transverse submanifolds of  $X$ ,  $P$  is exactly their intersection. We will use this result frequently.

Intersection of submanifolds gives a geometric realization of the cup product, but only for those classes represented by maps from manifolds; not all homology classes are realized in this way.

Classes of codimension at most 2 always have representatives arising from embedded submanifolds. The idea is that in general, there's a natural isomorphism

$$H^n(X) \cong [X, K(\mathbb{Z}, n)],$$

where brackets denote homotopy classes of maps and  $K(\mathbb{Z}, n)$  is an *Eilenberg-Mac Lane space* for  $\mathbb{Z}$  in dimension  $n$ , i.e. a space whose only nontrivial homotopy group is  $\pi_n \cong \mathbb{Z}$ . These spaces always exist, and any two models for  $K(\mathbb{Z}, n)$  are homotopic.

Usually Eilenberg-Mac Lane spaces are not smooth manifolds, but there are a few exceptions, including  $K(\mathbb{Z}, 1) \simeq S^1$ . Hence there is a bijection  $[X, S^1] \rightarrow H^1(X)$ . In the de Rham model, this is the map

$$[f] \mapsto f^*\omega,$$

where  $\omega \in H^1(S^1) \cong \mathbb{Z}$  is the generator. Alternatively, you could think of  $\omega$  as  $D^{S^1}[\text{pt}]$ , for any choice of  $\text{pt} \in S^1$ .

Thus, take  $f: X \rightarrow S^1$  to be a smooth map, where  $X$  is a closed, oriented manifold. Let  $H_t := f^{-1}(t) \subset X$ , where  $t \in S^1$  is a regular value. Then,  $H_t$  comes with a co-orientation, hence an orientation, and  $[H_t] = D_X(f^*\omega)$ . Thus codimension-1 submanifolds are realizable.

In this course, the case of codimension 2 will be more useful.

**Proposition 2.18.** *Let  $\mathbb{CP}^\infty := \text{colim}_n \mathbb{CP}^n$  (the union via the inclusions  $\mathbb{CP}^n \hookrightarrow \mathbb{CP}^{n+1}$ ). Then,  $\mathbb{CP}^\infty$  is a  $K(\mathbb{Z}, 2)$ .*

Hence there is a class  $c \in H^2(\mathbb{CP}^\infty)$  and a natural bijection  $[X, \mathbb{CP}^\infty] \rightarrow H^2(X)$  sending  $[f] \mapsto f^*(c)$ .

$\mathbb{CP}^\infty$  is not a smooth manifold, but its low-dimensional skeleta are, and this leads to codimension-2 realizability. Specifically, the inclusion  $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty$  defines the pullback map  $H^2(\mathbb{CP}^\infty) \rightarrow H^2(\mathbb{CP}^1) \cong H_0(\mathbb{CP}^1) \cong \mathbb{Z}$ . This maps the tautological class  $c$  to  $[\text{pt}]$  for any  $\text{pt} \in \mathbb{CP}^1$ .

Let  $f: \mathbb{CP}^\infty \rightarrow X$  be a map, where  $X$  is a smooth, oriented, closed manifold, which is homotopic to a smooth map  $\mathbb{CP}^N$  followed by the inclusion  $\mathbb{CP}^N \hookrightarrow \mathbb{CP}^\infty$ . Let  $D \subset \mathbb{CP}^N$  be a hyperplane and  $H_D := g^{-1}(D)$ . Assuming  $g \pitchfork D$  (which can always be done for some  $g$  in the homotopy class of  $f$ ), then  $H_D$  is a codimension-2 oriented submanifold of  $X$ , and  $[H_D] = D^X(g^*c)$ . Thus codimension-2 homology classes are representable. We will most commonly use this in dimension 4, for which any class  $a \in H^2(X)$  is represented by an embedding of a closed, oriented surface  $\Sigma \hookrightarrow X$ .

In general, realizability is controlled by oriented cobordism, which in higher codimension is different from cohomology, governed by maps into Thom spaces rather than Eilenberg-Mac Lane spaces. This was studied in the 1950s by René Thom.

Lecture 3.

## Unimodular forms: 1/25/18

Today's lecture will be more algebraic in flavor, though with topology in mind; we'll be discussing the algebra that arises in the middle cohomology of even-dimensional manifolds.

Let  $M$  be a closed, oriented manifold of dimension  $2n$ ; its middle cohomology  $H^n(M)$  carries a bilinear form  $\cdot: H^n(M) \otimes H^n(M) \rightarrow \mathbb{Z}$  sending

$$(3.1) \quad x, y \mapsto x \cdot y := \langle x \smile y, [M] \rangle,$$

where  $\langle -, - \rangle$  denotes evaluation. If  $n$  is even (i.e.  $4 \mid \dim M$ ), this is a symmetric form; if  $n$  is odd, it's skew-symmetric, which follows directly from the graded-commutativity of the cup product.

Poincaré duality means there are three different ways to think of this product.



- As defined, it's a pairing  $H^n \otimes H^n \rightarrow \mathbb{Z}$ , where the pairing is evaluation and the cup product.
- Using Poincaré duality, we could reinterpret it as a map  $H^n \otimes H_n \rightarrow \mathbb{Z}$ . In this case, the pairing is evaluation. This is because the Poincaré duality map is capping with the fundamental class, so

$$(x \smile y) \frown [M] = x \frown (y \frown [M]) = x \frown D_M(y).$$

- Using Poincaré duality again, it's a pairing  $H_n \otimes H_n \rightarrow \mathbb{Z}$ , which is the intersection product.

For an abelian group  $A$ , let  $A_{\text{tors}} \subset A$  denote its torsion subgroup and  $A' := A/A_{\text{tors}}$ . In this case, the form (3.1) descends to a form on  $H^n(M)'$ , and we usually use this version of the form.

*Remark.* If  $M$  is 4-dimensional, the universal coefficients theorem guarantees a short exact sequence

$$0 \longrightarrow (H_1(M))_{\text{tors}} \longrightarrow H^2(M) \longrightarrow H^2(M)' \longrightarrow 0,$$

and that it splits, but non-canonically. In particular, if  $H_1(M) = 0$ ,  $H^2(M)$  is torsion-free.  $\blacktriangleleft$

Let  $\{e_i\}$  be a  $\mathbb{Z}$ -basis for  $H^n(M')$  and  $Q = (Q_{ij})$  be the matrix with entries  $Q_{ij} := e_i \cdot e_j$ . This is a symmetric matrix if  $n$  is even, and is skew-symmetric if  $n$  is odd.

The universal coefficients theorem also implies that  $H^n(M') \cong \text{Hom}(H_n(M), \mathbb{Z})$ , where the map sends a cohomology class  $y$  to the evaluation pairing of  $y$  and a homology class. This, plus the fact that  $\cdot$  is dual to evaluation, implies the following proposition.

**Proposition 3.2.** *The pairing  $\cdot$  is nondegenerate on  $H^n(M)'$ , i.e. the map  $H^n(M)' \rightarrow \text{Hom}(H^n(M)', \mathbb{Z})$  sending  $x \mapsto (y \mapsto x \cdot y)$  is an isomorphism of abelian groups.*

**Corollary 3.3.**  $\det Q \in \{\pm 1\}$ .

Now we focus on the case of manifolds which are boundaries. Suppose there is a compact oriented  $(2n+1)$ -dimensional manifold  $N$  such that  $M = \partial N$ , and let  $i: N \hookrightarrow M$  denote inclusion.

**Proposition 3.4.** *Let  $L = \text{Im}(i^*) \subset H^n(M; \mathbb{R})$ . Then,*

- (1)  $L$  is isotropic, i.e. for all  $x, y \in L$ ,  $x \cdot y = 0$ .
- (2)  $\dim L = (1/2) \dim H^n(M; \mathbb{R})$ .

*Proof.* Perhaps unsurprisingly, this proof uses algebraic topology of manifolds with boundary, namely *Poincaré-Lefschetz duality*, the analogue of Poincaré duality on a compact manifold with boundary.

Part (1) follows from the fact that  $i^*$  is a ring homomorphism:

$$\begin{aligned} i^*u \cdot i^*v &= (i^*u \smile i^*v) \frown [M] \\ &= i^*(u \smile v) \frown [M] \\ &= (u \smile v) \frown i_*[M], \end{aligned}$$

but since  $M = \partial N$ ,  $[M] = \partial[M, N]$ , where  $\partial: H_{2n+1}(M, N; \mathbb{R}) \rightarrow H_{2n}(N; \mathbb{R})$  is the boundary map in the long exact sequence of a pair and  $[M, N] \in H_{2n+1}$  is the relative fundamental class. Hence  $i_*[M] = 0$ , since  $i_* \circ \partial = 0$ .

For part (2), Poincaré-Lefschetz duality implies the following diagram is commutative with exact rows:

$$(3.5) \quad \begin{array}{ccccccccc} \cdots & \longrightarrow & H^n(N; \mathbb{R}) & \xrightarrow{i^*} & H^n(M; \mathbb{R}) & \xrightarrow{\delta} & H^{n+1}(N, M; \mathbb{R}) & \xrightarrow{q} & H^{n+1}(N; \mathbb{R}) & \longrightarrow & \cdots \\ & & \cong \downarrow & & \cong \downarrow D_M & & \cong \downarrow D & & \cong \downarrow D & & \\ \cdots & \longrightarrow & H_{n+1}(N, M; \mathbb{R}) & \xrightarrow{\partial} & H_n(M; \mathbb{R}) & \xrightarrow{i_*} & H_n(M; \mathbb{R}) & \xrightarrow{p} & H_n(N, M; \mathbb{R}) & \longrightarrow & \cdots \end{array}$$

Fix a complement  $K$  to  $L$  in  $H^n(M; \mathbb{R})$ ; it suffices to show that  $\dim K = \dim L$ . Since  $L = \text{Im}(i^*) = \ker(\delta)$ , then  $K \cong H^n(M; \mathbb{R}) / \ker(\delta) \cong \text{Im}(\delta)$ . Since the upper row of (3.5) is exact,  $\text{Im}(\delta) = \ker(q)$ , and by Poincaré-Lefschetz duality this is isomorphic to  $\ker(p)$ . Since the lower row of (3.5) is exact, this is isomorphic to  $\text{Im}(i_*) \subset H_n(N; \mathbb{R})$ . However,  $i_*$  and  $i^*$  are dual (in the sense of  $\text{Hom}(-, \mathbb{R})$ ), and linear algebra tells us that a map and its dual have the same rank.<sup>6</sup>  $\square$

Now let's focus on the case when  $n$  is even, so  $4 \mid \dim(M)$ .

<sup>6</sup>The matrix version of this statement is that a matrix and its transpose have the same rank.

**Definition 3.6.** A *unimodular lattice*  $(\Lambda, \sigma)$  is a finite-rank free abelian group  $\Lambda$  together with a nondegenerate symmetric bilinear form  $\sigma: \Lambda \otimes \Lambda \rightarrow \mathbb{Z}$ .

Therefore a closed, oriented  $4m$ -manifold defines a unimodular lattice  $(H^{2m}, \cdot)$ .

Recall that for  $(V, \sigma)$  a symmetric bilinear form on a real vector space  $V$ , there is an orthogonal decomposition

$$(3.7) \quad V = R \oplus V^+ \oplus V^-,$$

where  $R$  is the subspace orthogonal to everything,  $V^+$  is the subspace on which  $\sigma$  is positive definite, and  $V^-$  is the subspace on which  $\sigma$  is negative definite. Clearly  $(V, \sigma)$  determines  $\dim V^+$  and  $\dim V^-$ .

**Proposition 3.8** (Sylvester's law of inertia). *If  $(V, \sigma)$  is a symmetric bilinear form on a real vector space, then  $\dim R$ ,  $\dim V^+$ , and  $\dim V^-$  determine  $(V, \sigma)$  up to isomorphism.*

**Definition 3.9.** The *signature* of  $(V, \sigma)$  is  $\tau := \dim V^+ - \dim V^-$ . If  $(\Lambda, \sigma)$  is a unimodular lattice, then  $\tau(\Lambda, \sigma) := \tau(\Lambda \otimes \mathbb{R}, \sigma \otimes \text{id}_{\mathbb{R}})$ . If  $M$  is a closed, oriented,  $4m$ -dimensional manifold, its signature is  $\tau(M) := \tau(H^{2m}(M)', \cdot)$ .

*Fact.* If  $M$  is a closed, oriented  $4m$ -manifold, then  $M$  admits an orientation-reversing self-diffeomorphism iff  $\tau(M) = 0$ .  $\blacktriangleleft$

**Theorem 3.10.**

- (1) *Let  $X_1$  and  $X_2$  be closed, oriented 4-manifolds and  $Y$  be an oriented cobordism between them, i.e.  $Y$  is a compact, oriented 5-manifold together with an orientation-preserving diffeomorphism  $\partial Y^5 \cong (-X_1) \amalg X_2$ .<sup>7</sup> Then,  $\tau(X_1) = \tau(X_2)$ .*
- (2) *Conversely, if  $\tau(X_1) = \tau(X_2)$ , then  $X_1$  and  $X_2$  are cobordant.*

Therefore, in particular, the signature is a complete cobordism invariant.

*Partial proof.* For part (1),  $H^2(-X_1 \amalg X_2) \otimes \mathbb{R}$  admits a middle-dimensional isotropic subspace by Proposition 3.4, hence has signature zero. But

$$\begin{aligned} \tau(-X_1 \amalg X_2) &= \tau((-H^2(X_1) \otimes \mathbb{R}) \oplus (H^2(X_2) \otimes \mathbb{R})) \\ &= \tau(-X_1) + \tau(X_2) \\ &= -\tau(X_1) + \tau(X_2) = 0. \end{aligned} \quad \boxtimes$$

We will not give a full proof of part (2). The idea is that cobordism classes of oriented 4-manifolds form an abelian group  $\Omega_4^{\text{SO}}$  under disjoint union, and by part (1), the signature defines a homomorphism

$$\tau: \Omega_4^{\text{SO}} \rightarrow \mathbb{Z}.$$

This map must be surjective, because  $\tau(\mathbb{CP}^2) = 1$  (where the orientation is the standard one coming from its complex structure). To prove it's injective, one uses Thom's cobordism theory, which identifies  $\Omega_4^{\text{SO}}$  with a homotopy group of a space called a *Thom space*, then calculates that group using the Hurewicz theorem and calculation of the homology of the Thom space in question.

Next we discuss unimodular lattices mod 2.

**Definition 3.11.** A *characteristic vector*  $c$  for a unimodular lattice  $(\Lambda, \sigma)$  is a  $c \in \Lambda$  such that  $\sigma(c, x) \equiv \sigma(x, x) \pmod{2}$ .

**Lemma 3.12.** *The characteristic vectors form a coset of  $2\Lambda$  in  $\Lambda$ .*

*Proof.* Let  $\lambda = \Lambda/2\Lambda$ , which is a vector space over  $\mathbb{F}_2$ . The freshman's dream mod 2 implies that the map  $\lambda \rightarrow \mathbb{Z}/2$  sending  $[x] \mapsto \sigma(x, x) \pmod{2}$  is linear! Hence there is a symmetric bilinear form  $\bar{\sigma}$  on  $\lambda$  induced by  $\sigma$  with determinant 1, hence is nondegenerate. Hence there exists a unique  $\bar{c} \in \lambda$  such that  $\bar{\sigma}(x, x) = \bar{\sigma}(\bar{c}, x) \pmod{2}$  for all  $x \in \lambda$ . The characteristic vectors are exactly the lifts of  $\bar{c}$  to  $\Lambda$ , hence are a coset of  $2\Lambda$ .  $\boxtimes$

<sup>7</sup>Here  $-X_1$  denotes  $X_1$  with the opposite orientation.

*Remark.* In the case of a simply connected 4-manifold  $M$ , the element  $\bar{c} \in H^2(M)/2H^2(M) \cong H^2(M; \mathbb{Z}/2)$  is exactly the second Stiefel-Whitney class  $w_2(TM)$ . We'll talk about characteristic classes more next lecture. This follows from the *Wu formula*.

Moreover, the characteristic vectors  $c \in H^2(M; \mathbb{Z})$  are exactly the first Chern classes of  $\text{spin}^c$  structures on  $M$ , and the Seiberg-Witten invariants are functions on the set of characteristic vectors to  $\mathbb{Z}$ . We'll say more about this later.

Most of this is true even in the case of non-simply-connected manifolds, but is harder. It is not true, however, that  $H^2(M)/2H^2(M) \cong H^2(M; \mathbb{Z}/2)$ .  $\blacktriangleleft$

**Lemma 3.13.** *Let  $c$  and  $c'$  be characteristics for  $(\Lambda, \sigma)$ . Then,*

$$\sigma(c, c) \equiv \sigma(c', c') \pmod{8}.$$

*Proof.* Write  $c' - c = 2x$  for some  $x \in \Lambda$ . Then,

$$\sigma(c', c') = \sigma(c + 2x, c + 2x) = \sigma(c, c) + 4 \underbrace{(\sigma(c, x) + \sigma(x, x))}_{(*)},$$

and  $(*)$  is even.  $\square$

**Definition 3.14.** Let  $(\Lambda, \sigma)$  be a unimodular form. Its type  $t \in \mathbb{Z}/2$  is even if  $\sigma(x, x)$  is even for all  $x \in \Lambda$ , and otherwise, it's odd.

**Theorem 3.15** (Hasse-Minkowski theorem on unimodular forms). *An indefinite unimodular form  $(\Lambda, \sigma)$  is classified up to isomorphism by three invariants:*

- its rank  $\dim_{\mathbb{R}}(\Lambda \otimes \mathbb{R})$ ,
- its signature  $\tau \in \mathbb{Z}$ , and
- its type  $t \in \mathbb{Z}/2$ .

For the (quite nontrivial) proof, see Serre's *A Course in Arithmetic*. The proof idea is to solve the quadratic equation  $\sigma(x, x) = 0$  for  $x \in (\Lambda \otimes \mathbb{Q}) \setminus 0$ . This is achieved via a *local-to-global principle* which says it suffices to find solutions  $x_{\infty} \in \Lambda \otimes \mathbb{R}$  and  $x_p \in \Lambda \otimes \mathbb{Q}_p$  (the  $p$ -adic numbers), and this can be done using unimodularity.

This is completely different from the positive definite (equivalently, negative definite) case, for which there are finitely many isomorphism classes below a given rank  $r$ , though this number grows rapidly with  $r$  and is only known in relatively few cases.

**Example 3.16.** Let  $I_+ := (\mathbb{Z}, 1)$  with  $\sigma(x, y) = xy$  and  $I_- := (\mathbb{Z}, -1)$  with  $\sigma(x, y) = -xy$ . Then,  $I_+^{\oplus m} \oplus I_-^{\oplus n}$  has rank  $m + n$ , signature  $m - n$ , and odd type.

There is a characteristic vector  $c := (1, \dots, 1)$ . In particular,  $c^2 = m - n - \tau$ , and therefore for any characteristic vector  $c$ ,  $c^2 \equiv \tau \pmod{8}$ .  $\blacktriangleleft$

**Example 3.17.** Let

$$U := \left( \mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$$

with quadratic form  $(a, b)^2 := 2ab$ . This is an even unimodular lattice with rank 2 and signature 0.  $\blacktriangleleft$

The Hasse-Minkowski principle implies the following.

**Corollary 3.18.** *Let  $\Lambda$  be a unimodular lattice with signature  $\tau$  and  $c$  be a characteristic vector for  $\Lambda$ . Then,  $c \cdot c \equiv \tau \pmod{8}$ . In particular, if  $\Lambda$  is even, then  $\tau \equiv 0 \pmod{8}$ .*

*Proof.* Either  $\Lambda \oplus I_+$  or  $\Lambda \oplus I_-$  is indefinite. It has odd type, and signature  $\tau(\Lambda \oplus I_{\pm}) = \tau(\Lambda) \pm 1$ , and if  $c$  is a characteristic vector for  $\Lambda$ , a characteristic vector for  $\Lambda \oplus I_{\pm}$  is  $c \oplus 1$  with square  $c \cdot c = \pm 1$ .

By Theorem 3.15,  $\Lambda \oplus I_{\pm} \cong mI_+ \oplus nI_-$ , so  $c^2 \pm 1 \equiv \tau(\Lambda) \pm 1 \pmod{8}$ .  $\square$

Hence an even, positive-definite, unimodular lattice has rank  $8k$  for some  $k$ .

**Example 3.19** ( $E_8$  lattice). The basic example is the  $E_8$  lattice, associated to the Dynkin diagram for the exceptional simple Lie group  $E_8$ . As a matrix, it has the form

$$(3.20) \quad \begin{pmatrix} 2 & -1 & & & & & & -1 \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ -1 & & & & & & -1 & 2 \end{pmatrix}.$$

Hasse-Minkowski implies that any even unimodular lattice  $\Lambda$  is isomorphic to  $mU \oplus (n \pm E_8)$ . The intersection forms of interesting 4-manifolds tend to have  $E_8$  terms. ◀

For more on the  $E_8$  lattice, see the professor's class notes.

Lecture 4.

### The intersection form and characteristic classes: 1/30/18

Let's start by writing down the homology and cohomology of a closed, oriented 4-manifold. Poincaré duality narrows the search space considerably.

$$\begin{aligned} H_4(X) &\cong H^0(X) = \mathbb{Z} \cdot 1 \\ H_3(X) &\cong H^1(X) = \text{Hom}(\pi_1(X), \mathbb{Z}) \\ H_2(X) &\cong H^2(X) \cong \text{Hom}(H_2(X), \mathbb{Z}) \oplus H_1(X)_{\text{tors}} \\ \pi_1(X)^{\text{ab}} &= H_1(X) \cong H^3(X) \\ \mathbb{Z} \cdot [\text{pt}] &= H_0(X) \cong H^4(X). \end{aligned}$$

Additively, the homology and cohomology are determined up to isomorphism by  $\pi_1$  and  $H_2$ . Together with the intersection form  $Q_X$  on  $H_2$  (or its torsion-free quotient), we have a lot of information, but we still don't have everything: there could be cup products from  $H^1$  and  $H^2$  to  $H^3$ , and various other questions, e.g. what's the Hurewicz map  $\pi_2(X) \rightarrow H_2(X)$ ? What about mod  $p$  coefficients?

If  $X$  is simply connected, the story is much simpler:

$$\begin{aligned} H_4(X) &\cong H^0(X) = \mathbb{Z} \cdot 1 \\ H_3(X) &\cong H^1(X) = 0 \\ H_2(X) &\cong H^2(X) \cong \text{Hom}(H_2(X), \mathbb{Z}) \\ H_1(X) &\cong H^3(X) = 0 \\ \mathbb{Z} \cdot [\text{pt}] &= H_0(X) \cong H^4(X). \end{aligned}$$

The only real information is  $H_2$ , which determines the cohomology additively. The intersection form determines  $H^*(X)$  as a graded ring and  $H_*(X)$  as a graded  $H^*(X)$ -module. All mod  $p$  cohomology classes are reductions of integral classes, which follows from the universal coefficient theorem. The natural map  $\pi_2(X) \rightarrow H_2(X)$  is an isomorphism, by the Hurewicz theorem, which applies to any simply connected space. You can think about higher homotopy groups, and next time we'll think about the homotopy type of  $X$ , but thus far everything we can see has been determined by  $Q_X$ . (In fact, next time we'll see that it determines the homotopy type of  $X$ .)

So that knocks out the homotopy type, but we want further invariants. The next step is to investigate the tangent bundle, a distinguished rank-4 vector bundle. In particular, it has characteristic classes.

*Remark.* Spoiler alert: all characteristic classes we discuss today are determined by  $Q_X$ , so they don't define any new invariants.<sup>8</sup> Nonetheless, they provide useful tools for computing the intersection form, and therefore will be useful to us. ◀

<sup>8</sup>In fact,  $Q_X$  determines the homotopy type of the classifying map for  $TX$ !

We now review some of the theory of characteristic classes.

**Example 4.1.** Let  $V \rightarrow X$  be a finite-rank real vector bundle over an arbitrary topological space  $X$ . The *Stiefel-Whitney classes* are characteristic classes of  $V$  in mod 2 cohomology with the following properties.

- The  $i^{\text{th}}$  *Stiefel-Whitney class*  $w_i(V) \in H^i(X; \mathbb{Z}/2)$  for  $i \geq 0$ . The *total Stiefel-Whitney class* is

$$w(V) := w_0(V) + w_1(V) + \cdots$$

- $w_0(V) = 1$ .
- If  $i > \text{rank } V$ ,  $w_i(V) = 0$ . Hence the total Stiefel-Whitney class is a finite sum.

It's a theorem that the Stiefel-Whitney classes are characterized by the following properties.

- (1) If  $f: Y \rightarrow X$  is continuous,  $w_i(f^*V) = f^*w_i(V)$  for all  $i$ .
- (2) If  $i > \text{rank } V$ ,  $w_i(V) = 0$ .
- (3) The *Whitney sum formula*: if  $U, V \rightarrow X$  are vector bundles, then  $w(U \oplus V) = w(U)w(V)$ .<sup>9</sup>
- (4) If  $L \rightarrow \mathbb{RP}^1$  denotes the *tautological line bundle* whose fiber over a point  $\ell \in \mathbb{RP}^1$  is the line  $\ell$  itself,  $w_1(L) \neq 0$  in  $H^1(\mathbb{RP}^1; \mathbb{Z}/2) \cong \mathbb{Z}/2$ .

For a proof, see Hatcher's notes on vector bundles and  $K$ -theory, or Milnor-Stasheff, which provides a somewhat baffling construction in terms of Steenrod algebra on the Thom space. There are various differing constructions which make various facts about Stiefel-Whitney classes easier to prove. ◀

If  $X$  is path-connected, the isomorphism  $H^1(X; \mathbb{Z}/2) \cong \text{Hom}(\pi_1(X); \mathbb{Z}/2)$  sends  $w_1(V)$  to the orientation character of  $V$ . In particular,  $w_1(V) = 0$  iff  $V$  is orientable, so if  $M$  is a manifold,  $M$  is orientable iff  $w_1(TM) = 0$ .

Now suppose  $M$  is a closed manifold and  $V \rightarrow M$  is a rank- $r$  vector bundle. Its top Stiefel-Whitney class  $w_r(V) \in H^r(M; \mathbb{Z}/2) \cong H_{n-r}(M; \mathbb{Z}/2)$  maps to some homology class, which admits a representation by some codimension- $r$  cycle. This cycle has an explicit description: let  $s$  be a section of  $V$  transverse to the zero section; then  $s^{-1}(0)$  is codimension  $r$  and represents the homology class which is Poincaré dual to  $w_r(V)$ .

If  $V$  is an orientable bundle on a closed submanifold, its top Stiefel-Whitney class is the mod 2 reduction of its Euler class.

**Proposition 4.2.** Let  $H \in H^1(\mathbb{RP}^2; \mathbb{Z}/2) \cong \mathbb{Z}/2$  denote the generator, which is Poincaré dual to a hyperplane  $\mathbb{RP}^{n-1} \subset \mathbb{RP}^n$ . Then  $w(TR\mathbb{P}^n) = (1 + H)^{n+1}$ .

For a proof, see Milnor-Stasheff or the professor's notes.

Now we specialize to 4-manifolds.

**Theorem 4.3 (Wu).** Let  $X$  be a closed 4-manifold and  $w := w_1^2(TX) + w_2(TX)$ . Then  $w$  is the characteristic element of  $H^2(X; \mathbb{Z}/2)$ , i.e. for all  $u \in H^2(X; \mathbb{Z}/2)$ ,  $w \smile u = u \smile u$ .

There is a more general statement of Wu's theorem for manifolds in other dimensions. See Milnor-Stasheff.

For simply-connected 4-manifolds, the Stiefel-Whitney classes are determined by information we already have.

- If  $X$  is a closed, simply-connected 4-manifold, then  $w_1(TX)$  vanishes, since  $H^1$  does.
- Hence  $w_2(TX) \cup u = u \smile u$  for all  $u \in H^2(X; \mathbb{Z}/2)$ . In this case,  $H^2(X; \mathbb{Z}/2) = H^2(X)/2H^2(X)$ ; therefore  $w_2(TX)$  is the mod 2 reduction of any characteristic vector for  $Q_X$ . Therefore  $w_2$  provides no new information.
- $w_4(TX)$  is Poincaré dual to the zeroes of a vector field, so  $w_4(TX) \smile [X]$  is the number of zeroes mod 2 of a generic vector field, i.e. the Euler characteristic mod 2. This is again not new information, since it can be read off  $H^*(X)$ .
- Finally, since  $H^3$  vanishes, so must  $w_3$ . It's a theorem of Hirzenbuch and Hopf that  $w_3(TX)$  vanishes for any closed, orientable 4-manifolds, and this is quite relevant for our class.

*Remark.* The more general version of Wu's theorem shows that on any closed, oriented 4-manifold  $X$ ,  $w_3(TX) = \text{Sq}^1 w_2(TX)$ . Here  $\text{Sq}^1$  is the first *Steenrod square*, a cohomology operation, which has an explicit identification as the *Bockstein map* which measures whether a mod 2 cohomology class lifts to  $\mathbb{Z}$  coefficients. In particular,  $w_3(TX) = 0$  iff  $w_2(TX)$  is the reduction of an integral class, and such lifts are the first Chern

<sup>9</sup>If  $X$  is paracompact, every short exact sequence of vector bundles over  $X$  splits, so we may replace  $U \oplus V$  with an extension of  $U$  by  $V$ .

classes of  $\text{spin}^c$  structures. Thus the Hirzebruch-Hopf theorem is important for us because it implies that all closed, oriented 4-manifolds admit a  $\text{spin}^c$  structure. Since this theorem is trivial in the simply-connected case, though, we will not prove it.  $\blacktriangleleft$

**Example 4.4.** Chern classes are characteristic classes  $c_{2i}(E) \in H^{2i}(X; \mathbb{Z})$  for complex vector bundles  $E \rightarrow X$ ; again the total Chern class

$$c(E) := c_0(E) + c_1(E) + \cdots$$

and again  $c_0(E) = 1$  and  $c_i(E) = 0$  for  $i > \text{rank } E$ . The Chern classes are uniquely characterized by similar axioms:

- (1)  $c_i(E) = 0$  if  $i > \text{rank } E$ .
- (2)  $c(E \oplus F) = c(E)c(F)$ .
- (3) If  $L \rightarrow \mathbb{CP}^1$  denotes the tautological line bundle  $L \rightarrow \mathbb{CP}^1$ , then the Poincaré dual of  $c_1(L) \in H^2(\mathbb{CP}^1; \mathbb{Z})$  is  $-1 \in H_0(\mathbb{CP}^1)$  (where we take as a generator any positively oriented point).  $\blacktriangleleft$

**Proposition 4.5.** If  $H \in H^2(\mathbb{CP}^n)$  denotes the Poincaré dual to a hyperplane, so  $H = -c_1$  of the tautological line bundle over  $\mathbb{CP}^n$ , then  $c(T\mathbb{CP}^n) = (1 + H)^{n+1}$ . The argument is formally identical to the real case.

**Definition 4.6.** Isomorphism classes of complex line bundles on  $X$  form a group under tensor product; this is called the *topological Picard group* and denoted  $\text{Pic}(X)$ .

If  $M$  is a closed, oriented manifold and  $E \rightarrow M$  is a rank- $r$  complex vector bundle, then the Poincaré dual of  $c_r(E)$  is the zero locus of a generic section of  $E$  (i.e. transverse to the zero section).

**Proposition 4.7.** The first Chern class defines a homomorphism  $c_1: \text{Pic}(X) \rightarrow H^2(X; \mathbb{Z})$ , and this is an isomorphism.

In particular, for line bundles  $L_1$  and  $L_2$ ,  $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$ . One way to use this is to use the above characterization of the Poincaré dual of the top Chern class; another is to show that  $BU_1$ , the classifying space for complex line bundles, is a  $K(\mathbb{Z}, 2)$ , an Eilenberg-Mac Lane space, hence representing cohomology.

For a reference for the following theorem, see Hatcher's notes on vector bundles and  $K$ -theory.

**Theorem 4.8.** Let  $E \rightarrow X$  be a complex vector bundle and  $E_{\mathbb{R}}$  denote the underlying real vector bundle.

- $w_{2i}(E_{\mathbb{R}})$  is the mod 2 reduction of  $c_i(E)$ .
- $w_{2i+1}(E_{\mathbb{R}}) = 0$ .

In particular,  $w_2 = c_1 \bmod 2$ .

In order to apply Chern classes to manifolds, we need some sort of complex structure.

**Definition 4.9.** Let  $M$  be an even-dimensional manifold. An *almost complex structure* on  $M$  is a  $J \in \text{End}(TM)$  such that  $J^2 = -\text{id}$ .

This structure makes  $TM$  into a complex vector bundle, where  $i$  acts as  $J$ . Then we have access to Chern classes, albeit depending on  $J$ .

*Remark.* As the notation suggests, complex manifolds are almost complex.  $\blacktriangleleft$

**Example 4.10.** One example of a complex manifold is a complex hypersurface in  $\mathbb{CP}^n$ : let  $F$  be a degree- $d$  homogeneous polynomial in  $x_0, \dots, x_d$  and  $X := \{F = 0\} \subset \mathbb{CP}^n$ . Homogeneity means this makes sense in projective space; if we additionally assume that whenever  $F = 0$  at least one partial derivative of  $F$  is nonzero, then  $X$  is smooth.

To study complex manifolds, we can import tools from algebraic geometry. A holomorphic hypersurface  $D$  in a complex manifold  $M$  defines an invertible sheaf (i.e. complex line bundle)  $\mathcal{O}_M(D)$  whose sections over  $U \subset M$  are meromorphic functions on  $U$  with only simple poles along  $D \cap U$ . Moreover, if  $N_{D/M} := TM|_D / TD$  denotes the normal bundle to  $D \hookrightarrow M$ , there's an isomorphism  $\mathcal{O}_M(D)|_D \cong N_{D/M}$ .

On  $\mathbb{CP}^n$ , a holomorphic line bundle  $E$  is determined by its degree  $d := c_1(E) \in H^2(\mathbb{CP}^n)$ , because the holomorphic sheaf cohomology  $H^1(\mathbb{CP}^n; \mathcal{O}_{\mathbb{CP}^n}) = 0$ . If  $L \rightarrow \mathbb{CP}^n$  denotes the tautological line bundle, then  $L^*$  is the positive generator of  $\text{Pic } \mathbb{CP}^n$ , so there's an isomorphism  $E \cong (L^*)^{\otimes d}$ .

It follows that  $\mathcal{O}_{\mathbb{CP}^n}(X) \cong (L^*)^{\otimes d}$ , so  $c_1(\mathcal{O}_{\mathbb{CP}^n}(X)) = dH$ , where  $H = c_1(L^*)$  as before. Hence if  $h = i^*H \in H^2(X)$ , where  $i: X \hookrightarrow \mathbb{CP}^n$  is inclusion, then

$$(4.11) \quad c_1(N_{X/\mathbb{CP}^n}) = i^*c_1(\mathcal{O}_{\mathbb{CP}^n}(X)) = dh.$$

Using the short exact sequence

$$(4.12) \quad 0 \longrightarrow TX \longrightarrow T\mathbb{CP}^n|_X \longrightarrow N_{X/\mathbb{CP}^n} \longrightarrow 0,$$

we get that

$$\begin{aligned} i^*c(T\mathbb{CP}^n) &= c(TX)c(N_{X/\mathbb{CP}^n}) \\ (1+h)^{n+1} &= c(TX)(1+dh), \end{aligned}$$

and therefore

$$(4.13) \quad c_j(TX) + dhc_{j-1}(TX) = \binom{n+1}{j}h^j.$$

It's possible to explicitly solve this when you have a specific  $X$ ; the base case is

$$(4.14) \quad c_1(TX) = (n+1-d)h. \quad \blacktriangleleft$$

**Example 4.15.** Let  $n = 3$ , so  $X$  is a degree- $d$  complex surface, hence a closed, oriented 4-manifold. In that case

$$(4.16) \quad \begin{aligned} c_1(TX) &= (4-d)h \\ c_2(TX) &= (d^2 - 4d + 6)h^2. \end{aligned}$$

Since  $[X] \in H_4(\mathbb{CP}^3) \cong H^2(\mathbb{CP}^3)$  is identified with  $dH$  under Poincaré duality, then

$$(4.17) \quad c_2(TX) \frown [X] = d(d^2 - 4d + 6).$$

Since this is again the number of zeros (with orientation) of a generic vector field, this integer is the Euler characteristic, so we have an explicit formula for the Euler characteristic of a degree- $d$  complex surface:

$$(4.18) \quad \chi(X) = d(d^2 - 4d + 6). \quad \blacktriangleleft$$

**Theorem 4.19** (Lefschetz hyperplane theorem). *Suppose  $n \geq 3$  and  $X$  is a hypersurface in  $\mathbb{CP}^n$ . Then  $X$  is simply connected.*

There is a more general version of this theorem.

Therefore if  $X$  is a hypersurface in  $\mathbb{CP}^n$ , its first and third Betti numbers vanish, so

$$(4.20) \quad \chi(X) = 1 + b_2(X) + 1,$$

so

$$(4.21) \quad b_2(X) = d(d^2 - 4d + 6) - 2.$$

So we know the dimension of  $H^2$ . It's possible to write down bases using this information, though we won't get into this.

**Example 4.22.** The *Pontrjagin classes*  $p_i(V) \in H^{4i}(X)$  of a real vector bundle  $V$  are defined by

$$p_i(V) := (-1)^i c_{2i}(V \otimes \mathbb{C}). \quad \blacktriangleleft$$

Pontrjagin classes satisfy very similar axioms to Stiefel-Whitney and Chern classes; however, be aware that which ones vanish might be tricky. For example, the complexification of a rank-3 vector bundle is a rank-3 complex vector bundle, hence only has access to  $c_0$  and  $c_2$  for defining Pontrjagin classes.

If  $X$  is a closed oriented 4-manifold,  $TX$  has only one nonvanishing Pontrjagin class, which is  $p_4(TX) \in H^4(X) \cong \mathbb{Z}$ .

**Lemma 4.23.** *Let  $X$  be a closed, oriented 4-manifold. Then  $\sigma(X) := p_1(TX) \frown [X] \in \mathbb{Z}$  is an oriented cobordism invariant.*

The basic idea is that if  $W$  is a 5-manifold bounding  $X$ , then  $TW = TM \oplus \mathbb{R}$ , which implies  $p_1(W) = i^*p_1(M)$ , where  $i: M \hookrightarrow W$  is inclusion.

**Lemma 4.24.** *If  $V \rightarrow X$  is a complex tangent bundle,  $p_1(V_{\mathbb{R}}) = c_1(V)^2 - 2c_2(V)$ .*



For a proof, see the notes.

Recall that this cobordism group  $\Omega_4^{\text{SO}} \cong \mathbb{Z}$ , and that the signature  $\tau: \Omega_4^{\text{SO}} \rightarrow \mathbb{Z}$  is an isomorphism. Hence  $\sigma(X)$  must be proportional to  $\tau(X)$ .

**Theorem 4.25** (Hirzebruch signature theorem).  $\sigma = 3\tau$ , i.e. on a closed, oriented 4-manifold  $X$ ,  $p_1(X) \frown [X] = 3\tau(X)$ .

*Proof.* The proof is corollary of Thom's work: we just have to check on a generator of  $\Omega_4^{\text{SO}}$ , such as  $\mathbb{CP}^2$ , which has signature 1. Then we use Lemma 4.24:  $c_1(T\mathbb{CP}^2) = 3H$  and  $c_2(T\mathbb{CP}^2) = \chi(\mathbb{CP}^1) = 3$ , so

$$(4.26) \quad p_1(T\mathbb{CP}^2) = c_1^2(\mathbb{CP}^2) - c_2(\mathbb{CP}^2) = 9 - 2 \cdot 3 = 3. \quad \square$$

This also implies that if  $X_d$  is a degree- $d$  hypersurface, its signature is

$$\begin{aligned} \tau(X_d) &= \frac{1}{3}(c_1^2(TX_d) - c_2(X_d)) \frown [X_d] \\ &= -\frac{1}{3}(d-2)d(d+2). \end{aligned}$$

Certainly  $(d-2)d(d+2)$  is divisible by 3, so this produces an integer, even if we didn't already know that. Moreover,

$$\begin{aligned} w_2(TX_d) &= c_1(TX_d) \bmod 2 \\ &= (4-d)h \bmod 2 \\ &= dh \bmod 2. \end{aligned}$$

Hence  $d$  is even iff  $w_2(TX_d) = 0$  iff  $X_d$  has even type (i.e. its intersection form has even type). For  $d \geq 2$ , the intersection form is indefinite, so Theorem 3.15 tells us there's a unique intersection form in this class.

**Example 4.27.** For  $d = 4$ ,  $X_d$  is a *K3 surface*, and one concludes that  $c_1 = 0$ ,  $b_2 = 24$ ,  $\tau = -16$ , and the intersection form has even type. Hence by Theorem 3.15 the intersection form is  $3U \oplus 2(-E_8)$ . ◀

The point is that we can explicitly compute intersection forms in examples of interest, and how characteristic classes made this a bit easier.

Lecture 5.

## Tangent bundles of 4-manifolds: 2/1/18

This lecture has two goals. The first is to show that if  $X$  is a closed, simply-connected manifold, then its tangent bundle is essentially determined by  $w_2$ ,  $\tau$ , and  $\chi$ . The second is to discuss the following theorem.

**Theorem 5.1** (Rokhlin). *Let  $X$  be a closed, oriented 4-manifold. Then  $16 \mid \tau(M)$ .*

There are different proofs of this; today we're going to focus on its equivalence to the following fact about stable homotopy groups of the spheres.

**Proposition 5.2.** *Let  $k \geq 5$ . Then  $\pi_{3+k}(S^k) \cong \mathbb{Z}/24$ .*

**Obstruction theory** For a reference for this part of the lecture, see Hatcher's notes on vector bundles and  $K$ -theory.

Obstruction theory is about the following question: let  $\pi: E \rightarrow X$  be a fiber bundle where  $X$  is a CW complex. When is there a section of  $X$ ?

At first, we will assume  $X$  is simply connected; we will be able to weaken this hypothesis later. Let  $F$  denote the typical fiber of  $\pi$ , i.e. the fiber over a chosen  $x \in X$  (which we'll assume is in the 0-skeleton). We'll construct the section  $s$  inductively as a series of sections  $s^k: X^{(k)} \rightarrow E|_{X^{(k)}}$  (where  $X^{(k)}$  denotes the  $k$ -skeleton). If we're given  $s^k$ , when does it *not* extend to  $s^{k+1}$ ?

Let  $\Phi: (D^{k+1}, \partial D^{k+1}) \rightarrow (X^{(k+1)}, X^{(k)})$  denote the inclusion of a  $(k+1)$ -cell and  $\phi := \Phi|_{\partial D^{k+1}}: S^k \rightarrow X^{(k)}$  be the attaching map. Then  $\phi^*(s^k) := s^k \circ \phi$  is a section of the fiber bundle  $\phi^*E \rightarrow S^k$ . This bundle sits inside of  $\Phi^*E \rightarrow D^{k+1}$ , which is trivially trivial (i.e. canonically trivialized), because  $D^{k+1}$  is contractible. Therefore you can think of  $\phi^*s^k$  as a map to a fiber:  $S^k \rightarrow E_{\Phi(0)}$ , and canonically up to homotopy,  $E_{\Phi(0)} \cong F$  since  $X$  is simply connected.

Therefore we have a map  $\{(k+1)\text{-cells}\} \rightarrow \pi_k F$  sending  $\Phi$  to the map  $\phi^* s^k: S^k \rightarrow F$ . Since  $\pi_k(F)$  is an abelian group (here  $k > 1$ ), this map defines a cellular cochain

$$o^{k+1}(E, S^k) \in C^{k+1}(X; \pi_k(F)).$$

Let  $\mathfrak{o}^{k+1} \in H^{k+1}(X; \pi_k(F))$  denote the cohomology class of  $o^{k+1}$ ; this depends only on the homotopy class of  $s^k$ , and if  $\mathfrak{o}^{k+1}$  vanishes, then  $s^k$  extends to a section  $s^{k+1}$  on the  $(k+1)$ -skeleton.<sup>10</sup>

**Definition 5.3.** Suppose  $\pi_i F = 0$  for  $i < k$ . Then the *primary obstruction* for  $E$  is  $\mathfrak{o}^{k+1}(X; \pi_k(F))$ .

This is an invariant of the fiber bundle, hence is easier to understand than the higher obstructions, which depend on the choices of  $s^k$  that ones makes.

*Remark.* If we relaxed the assumption that  $X$  is simply connected, then we'd have to use local coefficients with the action of  $\pi_1(X)$  on  $\pi_k(F)$ . So if  $\pi_1(X)$  acts trivially on  $\pi_i F$  for  $i \leq k$ , the story continues in almost the same way. ◀

**Stiefel-Whitney classes as primary obstructions.** Though they are usually presented differently, Stiefel-Whitney classes were historically discovered as obstructions to collections of sections of vector bundles.

**Definition 5.4.** Let  $E \rightarrow X$  be a real, rank- $n$  vector bundle with a Euclidean metric. Then  $V_k(E) \rightarrow E$  denotes the *Stiefel bundle*, the fiber bundle whose fiber at  $x$  is  $V_k(E_x)$ , the *Stiefel manifold* of orthonormal  $k$ -frames for  $E_x$ ; standard fiber-bundle methods construct  $V_k(E) \rightarrow X$  in a canonical fashion.

The typical fiber of  $V_k(E)$  is  $V_k(\mathbb{R}^n)$ . A section of  $V_k(E)$  is a  $k$ -tuple of orthonormal, hence linearly independent, sections of  $E \rightarrow X$ .

Stiefel and Whitney applied obstruction theory to  $V_k(E)$  to understand when  $E$  admits linearly independent sections.

**Proposition 5.5.**  $V_k(\mathbb{R}^n)$  is  $(n-k-1)$ -connected, and

$$\pi_{n-k}(V_k(\mathbb{R}^n)) \cong \begin{cases} \mathbb{Z}, & n-k \text{ even or } k=1, \\ \mathbb{Z}/2, & \text{otherwise.} \end{cases}$$

For example, when  $n=1$ , this is telling us that the first homotopy group of  $S^k$  is  $\pi_k S^k = \mathbb{Z}$ . We're not going to prove Proposition 5.5; the idea is to use the long exact sequence of homotopy groups associated to a fiber bundle.

*Remark.* It's worth thinking through what information is needed to identify a homotopy group with  $\mathbb{Z}$ ; there are two choices, so this is something like requiring an orientation. ◀

Hence the primary obstruction to finding a section of  $V_k(E) \rightarrow X$  is an  $\mathfrak{o}^{n-k+1} \in H^{n-k+1}(X; \pi_{n-k}(V_k(\mathbb{R}^n)))$ . Thus we have characteristic classes for  $E$ ,

$$\mathfrak{o}_n^k(E) \in \begin{cases} H^{n-k+1}(X; \mathbb{Z}), & n-k \text{ even or } k=1, \\ H^{n-k+1}(X; \mathbb{Z}/2), & \text{otherwise.} \end{cases}$$

The canonical nature of this construction means these are natural under pullback of vector bundles, hence are indeed characteristic classes.

In either case, there is a mod 2 characteristic class  $\bar{\mathfrak{o}}_k^n \in H^{n-k+1}(X; \mathbb{Z}/2)$ , and this actually makes sense irrespective of  $\pi_1 X$ , because  $\text{Aut}(\mathbb{Z}/2) = 1$ , so all  $\mathbb{Z}/2$  local systems are trivial.

**Theorem 5.6.**  $\bar{\mathfrak{o}}_k^n(E) = w_{n-k+1}(E)$ .

This perspective on Stiefel-Whitney classes shows that some of them come with canonical integral lifts. For  $k=1$ ,  $\mathfrak{o}_1^n(E) \in H^n(X; \mathbb{Z})$  is the Euler class of  $E$ , which requires a choice of orientation on  $E$  (unless you use coefficients twisted by the orientation bundle of  $E$ ). You can take this as the definition of the Euler class if you wish.

<sup>10</sup>One may have to replace  $s^k$  by a section homotopic to it.

**Applications to 4-manifolds.** For a reference, see the classical papers by Dold-Whitney and Hirzebruch-Hopf.

When  $E \rightarrow X$  is a rank-4 real vector bundle, Theorem 5.6 tells us that  $w_2(E)$  is the obstruction to finding 3 linearly independent sections of  $E$  over  $X^{(2)}$ . If  $\ell$  denotes an orthogonal complement to these three sections given a metric on  $E$ , then  $\ell \rightarrow X$  is a line subbundle of  $E$ , which is trivial iff  $w_1(E) = 0$ .

**Corollary 5.7.** *Let  $X$  be a closed, oriented 4-manifold. Then  $w_2(TX)$  is the complete obstruction to trivializing  $TX$  over  $X \setminus \text{pt}$ .*

Such an  $X$  is called *almost parallelizable*.

*Proof.* Choose a metric  $g$  for  $TX$  and a CW structure for  $Y = X \setminus \text{pt}$ . Over  $Y^{(2)}$ , we can find orthonormal vector fields  $v_1, v_2, v_3 \in \Gamma(TX|_Y)$  by Theorem 5.6, and also a unit vector  $v_4 \in (v_1, v_2, v_3)^\perp$ . This is because the line bundle  $\ell \subset TX$  is trivial:  $X$  is orientable, so  $w_1(TX) = 0$ .

Let  $P \rightarrow Y$  denote the principal  $\text{SO}_4$ -bundle of oriented orthonormal frames for  $Y$ ; then,  $(v_1, v_2, v_3, v_4)$  is a section of  $P|_{Y^2} \rightarrow Y^2$ . The obstruction to extending this to  $Y^3$  lies in  $H^3(Y; \pi_2 \text{SO}_4)$ . We'll see later that the universal cover for  $\text{SO}_4$  is  $S^3 \times S^3$ , so

$$(5.8) \quad \pi_2(\text{SO}_4) = \pi_2(S^3 \times S^3) = \pi_2(S^3) \times \pi_2(S^3) = 0.$$

(It's actually true that for any Lie group  $G$ ,  $\pi_2(G) = 0$ . But here we don't need the full, harder result.) The upshot is, the obstruction vanishes, so  $(v_1, \dots, v_4)$  extends to a section on  $Y^{(3)}$ . And the cohomology of a punctured 4-manifold vanishes above dimension 3, so all further obstructions vanish.  $\square$

**Theorem 5.9.** *Let  $X$  be a closed, oriented 4-manifold, and suppose that  $T, T' \rightarrow X$  are two rank-4 oriented vector bundles such that  $w_2(T) = w_2(T') = 0$ .<sup>11</sup> Then*

- (1)  $T \oplus \mathbb{R} \cong T' \oplus \mathbb{R}$  iff  $p_1(T) = p_1(T')$ , and
- (2)  $T \cong T'$  as oriented vector bundles iff  $p_1(T) = p_1(T')$  and  $e(T) = e(T')$ .

Recall that the Euler class is the Poincaré dual to the zero set of a generic section.

To prove this, we'll need a result that will be useful again.

**Lemma 5.10.** *There exists a rank-4 vector bundle  $E \rightarrow S^4$  with  $\langle p_1(E), [S^4] \rangle = -2$  and  $\langle e(E), [S^4] \rangle = 1$ .*

*Proof sketch.* One can produce this bundle as a representative of the Bott element  $\beta \in K^0(S^4)$ . A more explicit construction: if  $\mathbb{H}\mathbb{P}^1$  denotes the *quaternionic projective line*, then  $\mathbb{H}\mathbb{P}^1 \cong S^4$  (for the same reason  $\mathbb{R}\mathbb{P}^1 \cong S^1$  and  $\mathbb{C}\mathbb{P}^1 \cong S^2$ ), and  $\mathbb{H}\mathbb{P}^1$  admits a tautological line bundle  $\Lambda \rightarrow \mathbb{H}\mathbb{P}^1$ , which has the correct Pontrjagin and Euler classes.  $\square$

*Proof of Theorem 5.9.* We'll use *Pontrjagin-Thom collapse* to construct a map  $f: X \rightarrow S^4$  which is smooth (at least near an  $x \in X$ ) of degree 1 with  $D_x f$  an isomorphism such that  $f^{-1}(f(x)) = x$ . The idea is to product a map  $X \rightarrow D^4$  which sends everything outside a disc neighborhood of  $X$  to  $\partial D^4$ , then collapse by the identification  $D^4/\partial D^4 \cong S^4$ .

Since  $w_2(T) = 0$ , there's some ball  $B \subset X$  containing  $x$  such that  $T$  is trivial over  $X \setminus B$ . Therefore  $T \cong f^*U$  for some rank-4 oriented bundle  $U \rightarrow S^4$ . Thus  $p_1(T) = f^*p_1(U)$  and  $e(T) = f^*e(U)$ , and of course the same is true for  $T'$  and a  $U' \rightarrow S^4$ . Therefore it suffices to prove the theorem on  $S^4$ .

Bundles on the 4-sphere aren't so complicated. Let  $D_+$  and  $D_-$  denote the two hemispheres, so  $S^4 = D_+ \cup_{S^3} D_-$ .  $U, U' \rightarrow D_\pm$  are canonically trivialized, but the two trivializations of  $U$  over  $D_+$  and  $D_-$  need not agree; instead, they are related by a map  $S^3 \rightarrow \text{SO}_4$  called the *clutching function*, which defines an element of  $\pi_3 \text{SO}_4$ .

Conversely, given a  $[\gamma] \in \pi_3 \text{SO}_4$ , one can construct a vector bundle  $E_\gamma \rightarrow \text{SO}_4$  by gluing the trivial bundles on  $D_+$  and  $D_-$  by any representative  $\gamma$  for  $[\gamma]$ . In the same way, rank 5 vector bundles on  $S^4$  correspond to elements of  $\pi_3 \text{SO}_5$ .

<sup>11</sup>So we could say that  $T$  and  $T'$  are two spin vector bundles.

Now we prove part (1). By what we've shown so far,  $U \oplus \mathbb{R} \cong U' \oplus \mathbb{R}$  iff they have the same clutching function in  $\pi_3\mathrm{SO}_5 \cong \mathbb{Z}$ .<sup>12</sup> The first Pontrjagin number defines a homomorphism

$$\begin{aligned}\pi_3\mathrm{SO}(5) &\longrightarrow \mathbb{Z} \\ [\gamma] &\longmapsto \langle p_1(E_\gamma), [S^4] \rangle,\end{aligned}$$

and by Lemma 5.10, there exists a vector bundle on  $S^4$  with nonzero Pontrjagin number. Hence this map is injective.

To prove part (2), we look at  $\pi_3\mathrm{SO}_4 \cong \pi_3(S^3 \times S^3) = \pi_3S^3 \times \pi_3S^3 = \mathbb{Z}^2$ . Hence we can define a map

$$\begin{aligned}\pi_3(\mathrm{SO}_4) \cong \mathbb{Z}^2 &\longrightarrow \mathbb{Z}^2 \\ [\gamma] &\longmapsto (\langle p_1(E_\gamma), [S^4] \rangle, \langle e(E_\gamma), [S^4] \rangle).\end{aligned}$$

Again, using Lemma 5.10, there's a bundle  $E$  whose image under the above map is  $(-2, 1)$ . For  $TS^4$ ,  $p_1 = 0$  and  $e = 2$ , so the above map is given by the matrix

$$\begin{pmatrix} -2 & 0 \\ 1 & 2 \end{pmatrix},$$

hence is injective. \(\square\)

*Remark.* The theorem is true more generally assuming that  $w_2(T) = w_2(T') \neq 0$ :  $p_1$  determines their stable isomorphism class and  $(p_1, e)$  determines their unstable isomorphism class. This requires a more sophisticated use of obstruction theory, and the proof is more involved. \(\blacktriangleleft\)

**Corollary 5.11.** *Suppose that  $X$  and  $X'$  are closed, oriented, simply-connected 4-manifolds and  $f: X' \rightarrow X$  is a homotopy equivalence. Then  $f^*TX \cong TX'$  as oriented vector bundles.*

*Proof.* The intersection form  $Q_X$  determines  $p_1 = 3\tau$  and  $e = \chi$ , hence the isomorphism classes of  $f^*TX$  and  $TX'$  coincide. \(\square\)

Thus  $TX$  knows no more than the (oriented) homotopy type of a closed, oriented, simply-connected manifold.

Next time we'll turn to Rokhlin's theorem.

Lecture 6.

### Rokhlin's theorem and the homotopy theorem: 2/6/18

Today we'll relate Rokhlin's theorem to the homotopy-theoretic fact that  $\pi_8(S^5) \cong \mathbb{Z}/24$ . We will then prove a theorem of Milnor that the intersection form of a simply-connected 4-manifold determines its homotopy type (which also knows the tangent bundle, as we saw last time).

*Remark.* Rokhlin was Russian, and there are at least three different ways of spelling his name in English. Keep this in mind when looking for references on this theorem. \(\blacktriangleleft\)

**Theorem 6.1** (Rokhlin). *Let  $X$  be a closed, oriented, smooth manifold with  $w_2(TX) = 0$ . Then  $16 \mid \tau(X)$ .*

We've seen that the signature of an even unimodular form is  $0 \bmod 8$ , so the key is getting another factor of 2. By the Hirzenbruch signature theorem, Rokhlin's theorem is equivalent to  $48 \mid \langle p_1(TX), [X] \rangle$ . This is why the  $\mathbb{Z}/24$  that we mentioned arises.

**Theorem 6.2.** *Rokhlin's theorem is equivalent to the theorem that  $\pi_8(S^5) \cong \mathbb{Z}/24$ .*

We will later directly prove Rokhlin's theorem using index theory.

*Remark.* This is a *stable homotopy group*: for  $k \geq 5$ , the suspension map  $\pi_{3+k}(S^k) \rightarrow \pi_{3+k+1}(S^{k+1})$  is an isomorphism, which is a nontrivial theorem in homotopy theory. For  $k < 5$ , different groups can arise. \(\blacktriangleleft\)

<sup>12</sup>Again, if  $G$  is any simple Lie group,  $\pi_3G = \mathbb{Z}$ .

*Remark.* One can compute  $\pi_8(S^5)$  using methods internal to homotopy theory. Rokhlin was the first to do this, albeit with a more geometric flavor. Before this, Serre invented a localization method for calculating the first  $p$ -torsion of  $\pi_*(S^k)$  for  $p$  prime, which shows that  $\pi_8(S^5)$  has order  $2^k \cdot 3$  for some  $k$ . One can characterize the 2-primary part using, e.g., the Adams spectral sequence.

Alternatively, there's a comprehensive story due to Adams and Quillen on the image of the  $J$ -homomorphism in  $\pi_*^s(S^0)$ , which touches on deep and difficult ideas. This is the first nontrivial example, and in this dimension it's surjective.  $\blacktriangleleft$

The proof of Theorem 6.2 proceeds via the Pontrjagin-Thom construction. This is an awesome thing, and you should definitely fill in the details and/or read the references.

**Definition 6.3.** Consider closed  $k$ -dimensional manifolds  $M$  embedded in  $\mathbb{R}^{k+m}$ . Assume we're in the *stable range*, which means  $m > k$ . A *normal framing* for  $M$  is a trivialization of the normal bundle  $\nu_M \rightarrow M$ , an isomorphism  $\phi: \nu_M \rightarrow \underline{\mathbb{R}}^m$ . Typically only the homotopy class of  $\phi$  is important.

**Definition 6.4.** Let  $(M_0, \phi_0)$  and  $(M_1, \phi_1)$  be normally framed submanifolds of  $\mathbb{R}^{m+k}$ . A *framed cobordism* is data  $(P, \Phi)$ , where  $P$  is a compact submanifold  $P \subset \mathbb{R}^{m+k} \times [0, 1]$  with boundary  $\partial P \subset \mathbb{R}^{m+k} \times \{0, 1\}$  such that

$$(6.5) \quad \partial P = (M_0 \times \{0\}) \amalg (M_1 \times \{1\}).$$

We assume a transversality condition; see the notes for details. The second piece of data is a framing  $\Phi: \nu_P \rightarrow \underline{\mathbb{R}}^m$  which extends  $\phi_0$  and  $\phi_1$ .

The set of equivalence classes of framed submanifolds of  $\mathbb{R}^{m+k}$  under framed cobordism forms an abelian group under disjoint union.<sup>13</sup> This group is called the *framed cobordism group*<sup>14</sup> and denoted  $\Omega_k^{fr}$ .

**Lemma 6.6.** *As the notation suggests, the inclusion  $\mathbb{R}^{m+k} \hookrightarrow \mathbb{R}^{m+k+1}$  induces an isomorphism on framed bordism groups for normally framed  $k$ -manifolds.*

This is not too hard to see; the upshot is that there's not too much homotopical information contained in a normal framing in high codimension.

**Definition 6.7.** There is a group homomorphism  $PT: \pi_{k+m}(S^m) \rightarrow \Omega_k^{fr}$  called the *Pontrjagin-Thom homomorphism* defined as follows: given an  $[f] \in \pi_{k+m}(S^m)$ , represent it by a smooth map  $f: S^{k+m} \rightarrow S^m$ . Then,  $PT(f) := f^{-1}(x)$ , where  $x$  is a regular value. The framing comes from the framing of a neighborhood of  $x$ , which is diffeomorphic to  $\mathbb{R}^n$ , and this framing pulls back to  $f^{-1}(x)$ .

There is a lot to check here, including:

- Any map  $f: S^{k+m} \rightarrow S^m$  is homotopic to a smooth map.
- The framed cobordism class of  $PT(f)$  doesn't depend on the choice of smooth representative of  $[f]$ .
- The framed cobordism class of  $PT(f)$  doesn't depend on the regular value chosen. This is because any two regular values in  $S^n$  are connected by a regular path.

**Theorem 6.8** (Pontrjagin-Thom). *The Pontrjagin-Thom map is an isomorphism.*

The rough idea is to construct an inverse map, called *Pontrjagin-Thom collapse*.

**Example 6.9.** For  $k = 0$ , this is a fact you likely already know: homotopy classes of maps  $S^m \rightarrow S^m$  are classified by their degree, and the degree is defined to be the number of preimages of a regular value.  $\blacktriangleleft$

Another ingredient that we need for Theorem 6.2 is the  $J$ -homomorphism, a map

$$(6.10) \quad J_k^n: \pi_k(\mathrm{SO}_m) \longrightarrow \pi_{k+m}(S^m).$$

Suppose  $[\theta] \in \pi_k(\mathrm{SO}_m)$ , and let  $\theta: (S^k, *) \rightarrow (\mathrm{SO}_m, I)$  be a map; in particular, it's a map of pointed spaces. Thus for each  $x \in S^k$ ,  $\theta(x): \mathbb{R}^m \rightarrow \mathbb{R}^m$  is an isometry, hence restricts to a map

$$(6.11) \quad \theta(x): (\overline{D}^n, \partial \overline{D}^n) \longrightarrow (\overline{D}^n, \partial \overline{D}^n),$$

<sup>13</sup>The fact that cobordism is an equivalence relation comes from the fact that one can glue cobordisms if they share a boundary component.

<sup>14</sup>Sometimes it's instead called a *framed bordism group*.

where  $\overline{D}^n \subset \mathbb{R}^n$  denotes the closed unit disc. Let

$$(6.12) \quad \tilde{J}(\theta): (S^k \times \overline{D}^m, S^k \times \partial \overline{D}^m) \longrightarrow (\overline{D}^m, \partial \overline{D}^m)$$

send

$$(6.13) \quad (x, y) \longmapsto \theta(x)(y).$$

This map descends to a map on the quotients, a map

$$(6.14) \quad J(\theta): (S^k \times \overline{D}^m) / (S^k \times \partial \overline{D}^m) \cong S^{k+m} \longrightarrow \overline{D}^m / \partial \overline{D}^m \cong S^m.$$

We made choices, and one can show that the homotopy class of  $J(\theta)$  is independent of those choices.

*Remark.* In the stable range  $k > m + 1$ , the inclusion  $\mathrm{SO}_m \hookrightarrow \mathrm{SO}_{m+1}$  induces an isomorphism on  $\pi_k$ , and suspension induces an isomorphism  $\pi_{k+m}(S^m) \rightarrow \pi_{k+m+1}(S^{m+1})$ ; the  $J$ -homomorphism is compatible with these isomorphisms.  $\blacktriangleleft$

The Pontrjagin-Thom theorem provides a geometric interpretation of the homotopy groups of the spheres; in view of that, we would like a geometric interpretation of the  $J$ -homomorphism.  $\pi_k(\mathrm{SO}_m)$  acts simply transitively on the set of framings of  $S^k \subset \mathbb{R}^{k+m}$ , and there is a basepoint, namely the framing coming from the usual framing on  $\mathbb{R}^n$ . Therefore the set of framings of  $S^k \subset \mathbb{R}^{k+m}$  is canonically identified with  $\pi_k(\mathrm{SO}_m)$ .

**Proposition 6.15.** *Under the identifications of  $\pi_k(\mathrm{SO}_m)$  with the framings of  $S^k \subset \mathbb{R}^m$  and  $\pi_{m+k}(S^m) \cong \Omega_k^{\mathrm{fr}}$ , the  $J$ -homomorphism is the map sending  $S^k$  with a framing to its framed cobordism class.*

This was left as an exercise. It's not clear who originally came up with this, but it is used heavily by Kervaire and Milnor.

Now we specialize to  $k = 3$  and  $m = 5$ , so we're interested in framings of  $S^3 \subset \mathbb{R}^8$ , and more generally of 3-manifolds in  $\mathbb{R}^8$ .

**Theorem 6.16.**  *$J_3$  is surjective.*

The proof uses spin cobordism, which we haven't discussed; a sketch is in the notes. Therefore we have a sequence

$$\pi_3 \mathrm{SO}_5 \cong \mathbb{Z} \xrightarrow{J_3} \Omega_3^{\mathrm{fr}} \cong \pi_8(S^5) \longrightarrow 0,$$

as  $\pi_3$  of any simple Lie group is  $\mathbb{Z}$ .

*Proof sketch of Theorem 6.2.* First, we'll assume  $\pi_8(S^5) \cong \mathbb{Z}/24$ , so  $\ker(J_3) = 24\mathbb{Z}$ . Let  $X$  be a closed, oriented 4-manifold with  $w_2(TX) = 0$ . By Corollary 5.7,  $TX$  is trivial after any disc  $D$  is removed from  $X$ . By the Whitney theorem, we can embed  $X \hookrightarrow \mathbb{R}^9$ , and over  $X \setminus D$ , we can choose a trivialization  $\Phi: N_{X \setminus D} \rightarrow \mathbb{R}^5$ .

In general, we can't extend  $\Phi$  over  $D$ ; the obstruction  $\mathfrak{o}(\nu, \Phi) \in \pi_3(\mathrm{SO}_5)$  is the element represented by  $\Phi$  on  $\partial(X \setminus D) = S^3$ . In particular,  $\Phi|_{S^3}$  is a framing of  $S^3$  that lies in  $\ker(J_3)$ , because it's a framing of a 3-manifold that extends to a framed 4-manifold  $X \setminus D$ . Thus  $\mathfrak{o}(\nu, \Phi) \in 24\mathbb{Z}$ .

Recall from Lemma 5.10 that there's a rank-4 bundle  $E \rightarrow S^4$  with  $\langle p_1(E), [S^4] \rangle = -2$  and  $\langle e(E), [S^4] \rangle = 1$ . This implies that the clutching function of  $E \oplus \underline{\mathbb{R}}$  is a generator of  $\pi_3(\mathrm{SO}_5)$ . The upshot (there's a step or two to fill in here) is that

$$(6.17) \quad 2\mathfrak{o}(\nu, \Phi) = \pm p_1(X).$$

Since  $\langle \mathfrak{o}(\nu, \Phi), [X] \rangle$  is divisible by 24, then  $\langle \pi_1(X), [X] \rangle$  is divisible by 48.

Conversely, assume Rokhlin's theorem and let  $\phi$  be a framing for  $S^3$ , and suppose it represents a class  $[a] \in \ker(J_3) \subset \mathbb{Z}$ , so there is a framed 4-manifold  $(P, \Phi)$  bounding  $(S^3, \phi)$ . Let  $X := P \cup_{S^3} D^4$ ; since  $w_2(TP) = 0$ ,  $w_2(TX) = 0$ . Rokhlin's theorem implies  $48 \mid \langle p_1(X), [X] \rangle$ , and this is twice the obstruction  $a$ , so  $a$  is divisible by 24. Hence  $\ker(J_3) \subset 24\mathbb{Z}$ .

To get that it's equal, you need an example. Let  $X$  be a K3 surface (a quartic in  $\mathbb{CP}^3$ ); one can show that  $\langle p_1(X), [X] \rangle = -48$ , which suffices.  $\square$

Now we'll have an interlude on relative homotopy groups. Let  $(X, A, x)$  be a based pair of topological spaces, i.e. a topological space  $X$ , a subspace  $A$ , and a basepoint  $x \in A$ .

**Definition 6.18.** The  $n^{\text{th}}$  relative homotopy set of  $(X, A, x)$  is the set  $\pi_n(X, A, x)$  of based maps

$$(D^n, \partial D^n, 0) \longrightarrow X, A, x)$$

This is a pointed set, whose basepoint is the class of any map landing in  $A$ : since  $D^n$  is contractible, these are all homotopic.

For  $n \geq 2$ , this pointed set is a group with identity element  $e$ . The idea is that there is a collapsing map  $c: D^n \rightarrow D^n \vee D^n$  (pinch the sides of the disc inward), which is a based relative map

$$(6.19) \quad (D^n, \partial D^n) \longrightarrow (D^n \vee D^n, \partial D^n \vee \partial D^n).$$

Therefore we can define  $f * g := (f \vee g) \circ c$ , and this defines a group structure on  $\pi_n(X, A, x)$ . If  $n \geq 3$ , this group is abelian, but it might not be for  $n = 2$ , for the same reason that fundamental groups need not be abelian.

We now summarize a few properties of these groups.

**Proposition 6.20** (Long exact sequence of a pair). *If  $(X, A, x)$  is a based pair of spaces, there is a long exact sequence of pointed sets*

$$\cdots \longrightarrow \pi_n(A, x) \longrightarrow \pi_n(X, x) \longrightarrow \pi_n(X, A, x) \xrightarrow{\delta} \pi_{n-1}(A, x) \longrightarrow \cdots$$

For  $n \geq 3$  this is a long exact sequence of abelian groups; for  $n \geq 2$  this is a long exact sequence of groups.

**Definition 6.21.** The Hurewicz map

$$h: \pi_n(X, A, x) \longrightarrow H_n(X, A)$$

is defined to send  $[f] \mapsto f_*[D^n, \partial D^n]$ .

This is a group homomorphism for  $n \geq 2$ .

**Theorem 6.22.** *The Hurewicz maps induce a commutative diagram*

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \pi_n(A, x) & \longrightarrow & \pi_n(X, x) & \longrightarrow & \pi_n(X, A, x) \xrightarrow{\delta} \pi_{n-1}(A, x) \longrightarrow \cdots \\ & & \downarrow h & & \downarrow h & & \downarrow h \\ \cdots & \longrightarrow & H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, A) \xrightarrow{\delta} H_{n-1}(A) \longrightarrow \cdots \end{array}$$

**Definition 6.23.** A pair of spaces  $(X, A)$  is  $n$ -connected if for all  $i \leq n$ , every map  $(D^i, \partial D^i) \rightarrow (X, A)$  is homotopic to a map  $(D^i, \partial D^i) \rightarrow (A, A)$  (as these are all homotopic).

This implies that  $\pi_i(X, A, x) = 0$  for  $1 \leq i \leq n$  and all  $x \in A$ .

The key piece of this that we'll need next time is a relative version of the Hurewicz theorem.

**Theorem 6.24** (Relative Hurewicz). *Let  $(X, A)$  be an  $(n-1)$ -connected pair, where  $n \geq 2$  and  $A$  is nonempty and simply connected. Then,*

- (1)  $H_i(X, A) = 0$  for all  $i < n$ , and
- (2) the Hurewicz map  $h: \pi_n(X, A) \rightarrow H_n(X, A)$  is an isomorphism.

That is, the first nontrivial homotopy and homology groups coincide. This is useful in similar ways to the classical Hurewicz theorem.

Next time, we'll define an isomorphism between  $\pi_3((S^2)^{\vee n})$  and the  $n \times n$  symmetric matrices over  $\mathbb{Z}$ , which is the key step in the homotopy theorem.



Lecture 8.

**The Hodge star: 2/13/18**

Note: I was out of town and missed this lecture; consult the professor's lecture notes (<https://www.math.utexas.edu/users/perutz/Gauge%20Theory/L8.pdf>) or George Torres' lecture notes ([https://www.math.utexas.edu/users/gdavor/notes/gauge\\_notes.pdf](https://www.math.utexas.edu/users/gdavor/notes/gauge_notes.pdf), §2.6) instead.

Lecture 9.

**Hodge theory and self-duality in 4 dimensions: 2/15/18**

Today we're going to talk about self-duality in dimension 4 from the perspective of Hodge theory and global differential geometry. We'll start with the general story, then specialize to dimension 4.

**The Hodge theorem.** Let  $(M, g)$  be a closed, oriented Riemannian manifold of dimension  $n$ . Recall from last time that this structure induces a linear map called the *Hodge star*  $\star: \Lambda^k(T_x^*M) \rightarrow \Lambda^{n-k}(T_x^*M)$  for any  $x \in M$ . Globally this induces a map of vector bundles  $\star: \Lambda^k(T^*M) \rightarrow \Lambda^{n-k}(T^*M)$ .

**Definition 9.1.** The *co-differential*  $d^*: \Omega^{k+1}(M) \rightarrow \Omega^k(M)$  is the map

$$d^* := (-1)^{k+1} \star^{-1} d \star = (-1)^{n-k+1} \star d \star.$$

Since  $d^2 = 0$  and  $\star^2 = \pm 1$ , then  $(d^*)^2 = 0$ .

We don't need compactness to define  $d^*$ , but if  $M$  is compact, something nice happens.

**Proposition 9.2.**  $d^*$  is the formal adjoint to  $d$  with respect to the  $L^2$  inner product on  $\Omega^k(M)$  defined by

$$(9.3) \quad \langle \alpha, \beta \rangle_{L^2} := \int_M g(\alpha, \beta) \, dx.$$

In particular, the definition of  $d^*$  was completely local, but its identification as the adjoint is global.

*Proof.* Applying Stokes' theorem to the identity

$$(9.4) \quad d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d\beta$$

shows that

$$(9.5) \quad \int_M d\alpha \wedge \beta = (-1)^{|\alpha|+1} \int_M \alpha \wedge d\beta,$$

so letting  $\beta = \star\gamma$ , we get

$$(9.6) \quad \int_M g(d\alpha, \gamma) \, dx = (-1)^{k+1} \int_M g(\alpha, \star^{-1} d\star\gamma) \, dx.$$

That is,  $\langle d\alpha, \beta \rangle_{L^2} = \langle \alpha, d^*\beta \rangle_{L^2}$ . □

**Definition 9.7.** Let  $(M, g)$  be a closed, oriented Riemannian manifold. The *Hodge Laplacian* on  $X$  is the map  $\Delta: \Omega^k(M) \rightarrow \Omega^k(M)$  defined by

$$\Delta := dd^* + d^*d = (d + d^*)^2.$$

The kernel of  $\Delta$  is called the space of *harmonic  $k$ -forms* and denoted  $\mathcal{H}_g^k$ .

In general,  $\mathcal{H}_g^k$  is some infinite-dimensional space; clearly, it contains  $\ker(d + d^*)$ .

**Proposition 9.8.** If  $M$  is compact,  $\mathcal{H}_g^k = \ker(d + d^*)$ .

*Proof.* It suffices to prove the forward inclusion. Since

$$\begin{aligned} \langle \alpha, \Delta\alpha \rangle_{L^2} &= \langle \alpha, d^*d\alpha + dd^*\alpha \rangle_{L^2} \\ &= \langle d\alpha, d\alpha \rangle_{L^2} + \langle d^*\alpha, d^*\alpha \rangle_{L^2} \\ &= \|d\alpha\|_{L^2}^2 + \|d^*\alpha\|_{L^2}^2, \end{aligned}$$

then if  $\Delta\alpha = 0$ , then  $(d + d^*)\alpha = 0$ . □

Therefore on a compact manifold, harmonic forms are also solutions to a first-order differential equation.  $d + d^*$  is a kind of Dirac operator, which implies lots of nice things about harmonic forms.

There's also a variational characterization of harmonic forms. Harmonic forms are closed, hence have classes in de Rham cohomology.

**Lemma 9.9.** *Harmonic forms strictly minimize  $\|\cdot\|_{L^2}^2$  of closed  $k$ -forms in their de Rham cohomology class. That is, if  $\eta \in \mathcal{H}_g^k$  and  $\tilde{\eta}$  is cohomologous to  $\eta$ , then  $\|\eta\|_{L^2}^2 \leq \|\tilde{\eta}\|_{L^2}^2$ , and equality occurs iff  $\tilde{\eta} = \eta$ .*

*Proof.* Let  $\gamma \in \Omega^{k-1}(M)$ . Then

$$\begin{aligned} \|\eta + d\gamma\|_{L^2}^2 - \|\eta\|_{L^2}^2 &= 2\langle \eta, d\gamma \rangle_{L^2} + \|d\gamma\|_{L^2}^2 \\ &= 2\langle d^*\eta, \gamma \rangle_{L^2} + \|d\gamma\|_{L^2}^2 \\ &= \|d\gamma\|_{L^2}^2, \end{aligned}$$

so the difference is nonzero if  $\gamma$  is nonzero. □

**Corollary 9.10.** *Every de Rham cohomology class contains at most one harmonic representative.*

Lemma 9.9 admits a converse.

**Lemma 9.11.** *Let  $\eta \in \Omega^k(M)$  be closed. If  $\eta$  minimizes  $\|\cdot\|_{L^2}^2$  in its de Rham cohomology class, then  $\eta$  is harmonic.*

*Proof.* Since  $\eta$  is a minimizer, then for any  $\gamma \in \Omega^{k-1}(M)$ ,

$$(9.12) \quad \left. \frac{d}{dt} \right|_{t=0} \|\eta + t d\gamma\|_{L^2}^2 = 0.$$

Thus  $2\langle \eta, d\gamma \rangle_{L^2} = 0$ . Since  $\gamma$  is arbitrary, choose  $\gamma = d^*\eta$ . Hence

$$(9.13) \quad 0 = \langle \eta, dd^*\eta \rangle_{L^2} = \|d^*\eta\|_{L^2}^2. \quad \square$$

*Remark.* Even if  $\eta$  isn't closed, minimizing the norm still implies something interesting: that it's *co-closed* (i.e.  $d^*\eta = 0$ ). ◀

Inside  $\Omega^k(M)$ ,  $\ker(d) \perp \text{Im}(d^*)$  under the  $L^2$  inner product, and in fact  $\ker(d) = (\text{Im}(d^*))^\perp$ . This is because if  $\langle \eta, d^*\omega \rangle_{L^2} = 0$  for all  $\omega$ , then  $\langle d\eta, \omega \rangle_{L^2} = 0$ , so  $\|d\eta\|_{L^2}^2 = 0$ , so  $d\eta = 0$ .

The next theorem is powerful, and also requires more nontrivial analysis.

**Theorem 9.14** (Hodge theorem). *Let  $(M, g)$  be a compact, oriented Riemannian manifold. Then there exists an  $L^2$ -orthogonal decomposition*

$$\begin{aligned} \Omega^k(M) &= \ker(d) \oplus \text{Im}(d^*) \\ \ker(d) &= \mathcal{H}_g^k \oplus \text{Im}(d). \end{aligned}$$

*The quotient map  $\mathcal{H}_g^k \rightarrow H_{\text{dR}}^k(M)$  is an isomorphism.*

*Proof idea.* The key part of the proof is the existence of a harmonic representative in any cohomology class. This problem breaks down into two steps.

- (1) First, one finds a *weak solution* to  $\Delta\eta = 0$ , meaning a solution lying in some Sobolev space  $L_\ell^2$ .
- (2) Second, one uses *elliptic regularity* to show that  $\eta \in \bigcap_\ell L_\ell^2 = C^\infty$ , and hence the solution is smooth.

Both steps use elliptic estimates to show that  $\Delta$  is a bounded operator between certain Sobolev spaces. □

So instead of working with quotient spaces of infinite-dimensional vector spaces, which can be unwieldy, you can work with finite-dimensional vector spaces, and this is sometimes nicer.

*Remark.* In general, the wedge product of two harmonic forms isn't harmonic, so it's difficult to see the ring structure of cohomology using Hodge theory. ◀

We'll have more to say about this later, but for now we'll assume the Hodge theorem and specialize to dimension 4.

**(Anti)-self-dual harmonic 2-forms.** Let  $(X, g)$  be a closed, oriented, Riemannian 4-manifold. Then  $\star: \Lambda^2(T^*X) \rightarrow \Lambda^2(T^*X)$  squares to 1. Therefore, as in the last lecture, there is a splitting

$$(9.15) \quad \Lambda^2(T^*X) = \Lambda_{[g]}^+ \oplus \Lambda_{[g]}^-,$$

where  $\Lambda_{[g]}^\pm$  is the  $\pm 1$ -eigenspace of  $\star$  on  $\Lambda^2(T^*X)$ . Let

$$(9.16) \quad \Omega_{[g]}^\pm := C^\infty(X, \Lambda_{[g]}^\pm);$$

these are called the *self-dual* (for 1) or *anti-self-dual* (for  $-1$ ) 2-forms on  $X$ .

**Definition 9.17.** Let  $M$  be a manifold. A *conformal structure* on  $M$  is an equivalence class of Riemannian metrics, where  $g_1 \sim g_2$  if  $g_1 = fg_2$  for some function  $f: M \rightarrow \mathbb{R}_{>0}$ .

That is, a conformal structure is a metric up to rescaling.

Last lecture's results apply wholesale: in particular,  $\star$  only depends on the conformal class of the metric. This is special to  $k = 2$ : in general scaling the metric by  $\lambda$  scales  $\star$  by  $\lambda^{k-n/2}$ .

Since  $d^* = -\star d \star$  in this dimension, then if  $\eta \in \Omega^2(X)$ ,  $\eta \in \ker(d + d^*)$  iff  $\star\eta \in \ker(d + d^*)$ . Hence if  $\eta \in \mathcal{H}_g^2$ , then its components

$$(9.18) \quad \eta^\pm := \frac{1}{2}(1 + \star)\eta \in \Omega_{[g]}^\pm$$

are also harmonic. Therefore if  $\mathcal{H}_g^\pm := \mathcal{H}_g^2 \cap \Omega_{[g]}^\pm$ , there's a splitting

$$(9.19) \quad \mathcal{H}_g^2 = \mathcal{H}_g^+ \oplus \mathcal{H}_g^-.$$

Suppose  $\eta \in \mathcal{H}_g^+ \setminus 0$ . Then

$$(9.20) \quad \int_X \eta \wedge \eta = \int_X \eta \wedge \star\eta = \int_X |\eta|^2 dV > 0,$$

and similarly, if  $\omega \in \mathcal{H}_g^- \setminus 0$ ,  $\int_X \omega \wedge \omega < 0$ .

Therefore the conformal class of  $g$  determines maximally positive and negative definite subspaces of the quadratic form  $\eta \mapsto \int_X \eta \wedge \eta$  on  $H_{\text{dR}}^2(X) \cong \mathcal{H}_g^2$ , namely  $\mathcal{H}_g^\pm$ . Let  $b^\pm(X) := \dim \mathcal{H}_g^\pm$ , which is the dimension of the maximally positive (resp. negative) subspace of the intersection pairing on de Rham cohomology.

In general,  $b^+ \neq b^-$ , and in fact their difference is the signature.

*Remark.* Let  $\eta$  be self-dual (resp. anti-self-dual). Then  $\eta$  is harmonic iff it's closed; this is because  $d^* = -\star d \star$ , so  $d^*\eta = -\star d(\star\eta)$  and  $\star\eta = \pm\eta$ .  $\blacktriangleleft$

**Definition 9.21.** Let  $(X, g)$  be as above and

$$d^\pm := \frac{1}{2}(1 \pm \star) \circ d.$$

The *signature complex*<sup>15</sup> is the cochain complex

$$0 \longrightarrow \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d^+} \Omega_g^+ \longrightarrow 0.$$

We will denote the  $i^{\text{th}}$  term of this complex by  $\mathcal{E}^i$  and the differentials by  $D$ , and will denote the whole complex by  $(\mathcal{E}^*, D)$ .

**Theorem 9.22.** *There are isomorphisms*

$$H^i(\mathcal{E}) \cong \begin{cases} H_{\text{dR}}^i(X), & i = 0, 1 \\ \mathcal{H}_g^+(X), & i = 2. \end{cases}$$

<sup>15</sup>This is only half of the signature complex; it's all that we need, but in other sources the signature complex generally refers to something larger.

*Proof.* The case  $i = 0$  is true by definition. For  $i = 1$ , it suffices to prove that if  $\alpha \in \Omega^1(X)$  is in  $\ker(d^+)$ , then it's closed. Since  $d\alpha = d^+\alpha + d^-\alpha$ , then

$$(9.23) \quad \int_X d\alpha \wedge d\alpha = \int_X |d^+\alpha|^2 dV - \int_X |d^-\alpha|^2 dV.$$

Since  $M$  is closed,

$$(9.24) \quad \int_X d\alpha \wedge d\alpha = \int_X d(\alpha \wedge d\alpha) = 0,$$

so for all  $\alpha$ ,  $\|d^+\alpha\|_{L^2}^2 = \|d^-\alpha\|_{L^2}^2$ . Therefore if  $d^+\alpha = 0$ , then  $d^-\alpha = 0$  too, so  $d\alpha = 0$ .

For  $i = 2$ , we'll show that  $\Omega_{[g]}^+ / \text{Im}(d^+) \cong \mathcal{H}_g^+$ , or more precisely, that the composition

$$(9.25) \quad \mathcal{H}_g^+ \hookrightarrow \Omega_{[g]}^+ \longrightarrow \Omega_{[g]}^+ / \text{Im}(d^+)$$

is an isomorphism.

Let  $\omega \in \Omega_{[g]}^+$ ; then, it has a *Hodge decomposition*

$$(9.26) \quad \omega = \omega_h + d\alpha + \star d\beta,$$

where  $\omega_h \in \mathcal{H}_g^2$ ,  $d\alpha \in \text{Im}(d)$ , and  $\star d\beta \in \text{Im}(d^*)$ . The components are unique; hence, if  $\omega$  is self-dual,  $\star\omega_h = \omega_h$  and  $d\beta = d\alpha$ . Thus

$$(9.27) \quad \omega = \underbrace{\omega_h}_{\mathcal{H}_g^+} + \underbrace{d\alpha + \star d\alpha}_{2d^+\alpha \in \text{Im}(d^+)}. \quad \boxtimes$$

This calculation will be relevant for computing the dimensions of Seiberg-Witten moduli spaces, as will another involving Euler characteristics that we will discuss later.

**Definition 9.28.** Fix an  $r \geq 3$  and let  $\text{Conf } X$  denote the space of conformal structures on  $X$ , i.e. the space of  $C^r$ -Riemannian metrics modulo the space of positive  $C^r$  functions.

In the future, we will need to do some analysis for which  $C^r$  regularity, for which things are Banach spaces, are better than  $C^\infty$  regularity, where we merely have a Fréchet space.

Fix a  $[g_0] \in \text{Conf } X$  and let  $\Lambda^\pm := \Lambda_{[g_0]}^\pm$ . This provides an identification of  $\text{Conf } X$  with the space of  $C^r$  bundle maps  $m: \Lambda^+ \rightarrow \Lambda^0 -$  such that **TODO**: given a conformal structure  $[g]$ ,  $\Lambda_{[g]}^- = \Gamma_m$ , the graph of  $m$ , for some such  $m$ , and  $\Lambda_{[g]}^+ = \Gamma_{m^*}$ ; this was discussed last time.

Hence  $\text{Conf } X$  is an open subset of a Banach space, and is particular a Banach manifold. We have an identification of its tangent spaces as Banach spaces:

$$(9.29) \quad T_{[g_0]} \text{Conf } X = C^r(X, \text{Hom}(\Lambda^-, \Lambda^+)).$$

**Definition 9.30.** The *period map* is the map  $P: \text{Conf } X \rightarrow \text{Gr}_{b-(X)} H_{\text{dR}}^2(X)$  sending  $[g] \mapsto \mathcal{H}_g^- \subset \mathcal{H}_g^2 \cong H_{\text{dR}}^2(X)$ .

Studying the period map informs us how the space of anti-self-dual forms changes as the metric changes. The first thing we should do is compute the derivative of  $P$ .

Since  $[g_0] \in \text{Conf } X$  has a neighborhood which is a neighborhood of 0 in  $C^r(X, \text{Hom}(\Lambda^-, \Lambda^+))$ , then a neighborhood of  $\mathcal{H}_{[g_0]}^-$  is identified with a neighborhood of 0 in  $\text{Hom}(\mathcal{H}_{[g_0]}^-, \mathcal{H}_{[g_0]}^+)$ : a subspace is sent to a map which has it as its graph.

Thus, near  $[g_0]$ , we may think of  $P$  as a map  $C^r(X, \text{Hom}(\Lambda^-, \Lambda^+)) \rightarrow \text{Hom}(\mathcal{H}_{g_0}^-, \mathcal{H}_{g_0}^+)$ . Therefore its derivative is a map

$$(9.31) \quad D_{[g_0]} P: C^r(X, \text{Hom}(\Lambda^-, \Lambda^+)) \longrightarrow \text{Hom}(\mathcal{H}_{g_0}^-, \mathcal{H}_{g_0}^+).$$

**Proposition 9.32.** Under this identification, if  $m$  is a bundle map and  $\alpha^- \in \mathcal{H}_{g_0}^-$ ,

$$(D_{[g_0]} P)(m)(\alpha^-) = (m(\alpha^-))_h.$$

**Corollary 9.33.** The period map is a submersion.

Thus you can move around the space of anti-self-dual forms pretty freely. This will be useful for transversality theory.

Lecture 10.

**Covariant derivatives: 2/20/18***“I hate signs.”*

Today we return to local differential geometry, studying connections on vector bundles. Next time we will cover local aspects. Two references on this that may be useful:

- Donaldson-Kronheimer, “The geometry of 4-manifolds,” which is terse but useful.
- Berline-Getzler-Vergne, “Heat kernels and Dirac operators,” chapter 1. This is not aimed at Seiberg-Witten theory but is useful nonetheless.

Let  $E \rightarrow M$  be a complex vector bundle. We will let  $\Gamma(M, E)$  denote the complex vector space of smooth sections of  $E \rightarrow M$ .

**Definition 10.1.** Let  $E \rightarrow M$  be a complex vector bundle and  $\langle -, - \rangle$  be a Hermitian inner product on  $E$ . A *covariant derivative* or a *connection* in  $E$  is a  $\mathbb{C}$ -linear map

$$\nabla: \Gamma(M, E) \longrightarrow \Omega_M^1(E) := \Gamma(M, T^*M \otimes_{\mathbb{R}} E)$$

obeying the *Leibniz rule*

$$(10.2) \quad \nabla(f, s) = df \otimes s + f \nabla s,$$

where  $f \in C^\infty(M)$  and  $s \in \Gamma(M, E)$ . The connection  $\nabla$  is called *unitary* if

$$(10.3) \quad d(s_1, s_2) = (\nabla s_1, s_2) + (s_1, \nabla s_2).$$

We will let  $\mathcal{C}(E)$  denote the space of covariant derivatives on  $E \rightarrow M$ .

**Lemma 10.4.** *Covariant derivatives are local operators, i.e.  $(\nabla s)(x)$  only depends on the germ of  $s$  near  $x$ .*

*Proof.* Let  $U$  be a neighborhood of  $x$  and  $s_1, s_2 \in \Gamma(M, E)$  be such that  $s_1|_U = s_2|_U$ . Let  $\chi$  be a *cutoff function* for  $U$ , i.e. a smooth function supported in  $U$  and equal to 1 in a neighborhood of  $x$ . Hence  $\chi s_1 = \chi s_2$ . Applying (10.2),  $\nabla(\chi s_i)(x) = (\nabla s_i)(x)$ , so  $(\nabla s_0)(x) = (\nabla s_1)(x)$ .  $\square$

This does not generalize to other forms of geometry, e.g. complex analytic geometry, real analytic geometry, or algebraic geometry, as cutoff functions don't exist. There locality is part of the definition.

**Example 10.5.** Consider a trivialized line bundle  $\mathbb{C} \rightarrow M$ . Then the exterior derivative is a covariant derivative, as it satisfies the Leibniz rule. This defines the *trivial connection*, and is still available in higher-rank vector bundles: let  $V$  be a finite-dimensional complex vector space and consider  $\underline{V} := V \times M \rightarrow M$ . Then we define the trivial connection by

$$d := d \otimes \text{id}_V: \Gamma(M, V) \longrightarrow \Gamma(M, T^*M \otimes V).$$

The trivial connection is unitary with respect to a trivialized Hermitian inner product on  $V$ .  $\blacktriangleleft$

Let  $\nabla, \nabla'$  be connections on a vector bundle  $E \rightarrow M$ . Then (10.2) implies  $\nabla - \nabla'$  is  $C^\infty(M)$ -linear, and in fact it can be an arbitrary  $C^\infty(M)$ -linear map. Hence  $\mathcal{C}(E)$  is an affine space over  $\Omega_M^1(\text{End } E) := \Gamma(M, T^*M \otimes_{\mathbb{R}} \text{End}_{\mathbb{C}}(E))$ : if you choose one reference connection  $\nabla$ , every other connection  $\nabla'$  is equal to  $\nabla + \omega$  for some  $\omega \in \Omega_M^1(\text{End } E)$ .

If you think through it a bit, the space of unitary connections is in the same way an affine space modeled on  $\Omega_M^1(\mathfrak{u}(E))$ , where  $\mathfrak{u}(E)_x$  is the bundle of skew-Hermitian endomorphisms of  $E_x$  for  $x \in M$ .

**Example 10.6.** On a trivialized vector bundle  $\underline{V} \rightarrow M$ , we have a canonical trivial connection  $d$ , hence  $\mathcal{C}(\underline{V})$  is canonically isomorphic to  $\Omega_M^1(\text{End } V)$ : any connection  $\nabla$  satisfies  $\nabla = d + A$  for some  $\text{End}_{\mathbb{C}}(V)$ -valued 1-form  $A$ .  $\blacktriangleleft$

**Lemma 10.7.** *Covariant derivatives are first-order operators, in that  $(\nabla s)(x)$  depends only on  $s(x)$  (zeroth-order information) and  $D_x s: T_x M \rightarrow T_{s(x)} E$  (first-order information).*

*Proof.* Since  $(\nabla s)(x)$  is local, we may assume  $E$  is trivialized, so  $\nabla = d + A$ , which is first-order.  $\square$

If  $E \rightarrow M$  is a vector bundle and  $f: M' \rightarrow M$  is smooth, then a connection  $\nabla$  on  $E$  pulls back to the pullback bundle  $f^*E \rightarrow M'$  in the usual way. This satisfies  $(g \circ f)^* \nabla = f^*(g^* \nabla)$ , as it should.

**Lemma 10.8.** *Connections exist on any vector bundle  $E \rightarrow M$ , as do unitary connections.*

*Proof.* Let  $\mathfrak{U}$  be an open cover of  $M$  which trivializes  $E$ , i.e. we have trivializations of  $E|_U \rightarrow U$  for all  $U \in \mathfrak{U}$ . Then we have connections  $\nabla_U$  on  $E|_U$ . Let  $\{\rho_U\}_{U \in \mathfrak{U}}$  be a partition of unity subordinate to  $\mathfrak{U}$ ; then

$$(10.9) \quad \nabla := \sum_{U \in \mathfrak{U}} \rho_U \nabla_U$$

is a connection on  $E$ . \(\square\)

Later, when you have multiple different definitions and perspectives on connections, hold onto this one: it's concrete and straightforward, which can be important when you're confused.

Associated to any connection  $\nabla \in \mathcal{C}(E)$  is a *coupled exterior derivative*

$$(10.10) \quad \begin{aligned} d_\nabla: \Omega_M^k(E) &\longrightarrow \Omega_M^{k+1}(E) \\ d_\nabla(\eta \otimes s) &:= (-1)^k \eta \wedge \nabla s + d\eta \otimes s, \end{aligned}$$

where  $\eta \in \Omega_M^k(E)$  and  $s \in \Gamma(M, E)$ ; then, we extend  $\mathbb{C}$ -linearly. This obeys a Leibniz rule: if  $f \in C^\infty(M)$ ,

$$(10.11) \quad d_\nabla(f\eta \otimes s) = df \wedge \eta \otimes s + f d_\nabla(\eta \otimes s).$$

In the case  $k = 0$ ,  $d_\nabla$  is just  $\nabla$ .

**Lemma 10.12.**  $d_\nabla^2: \Omega_M^*(E) \rightarrow \Omega_M^{*+2}(E)$  is linear over  $C^\infty(M)$  and over  $\Omega_M^*$ .

*Proof.* Suppose  $\eta \in \Omega_M^*$  and  $s \in \Gamma(M, E)$ . Then

$$\begin{aligned} d_\nabla \circ d_\nabla(\eta \otimes s) &= d_\nabla((-1)^k \eta \wedge \underbrace{\nabla s}_{d_\nabla s} + d\eta \otimes s) \\ &= (-1)^{2k} \eta \wedge d_\nabla \circ d_\nabla s + (-1)^k d\eta \wedge \nabla s + d^2 \nabla \otimes s + (-1)^{k+1} d\nabla \wedge \nabla s \\ &= \eta \wedge d_\nabla^2 s. \end{aligned}$$

The same calculation applies when  $s$  is instead an  $E$ -valued differential form. \(\square\)

**Definition 10.13.** The *curvature* of a connection  $\nabla$ , denoted  $F_\nabla \in \Omega_M^2(\text{End } E)$  is defined by

$$d_\nabla \circ d_\nabla s = F_\nabla \wedge s.$$

When you're doing calculations, it's important to know what's going on in a forest of confusing operators; it's important to remember where things live. For example, keep in mind that the curvature is an endomorphism-valued 2-form.

If  $\nabla$  is unitary,  $F_\nabla \in \Omega_M^2(\mathfrak{u}(E))$ . Since the definition of the curvature is completely intrinsic, it follows immediately that it's natural:  $F_{f^*\nabla} = f^*F_\nabla$ .

There are many perspectives on curvature, and we're favoring one of them: that the curvature is the obstruction to  $d_\nabla^2 = 0$ .

**Definition 10.14.** A connection is *flat* if its curvature vanishes.

For example, since  $d^2 = 0$ , the trivial connection is flat.

A connection  $\nabla$  on  $E \rightarrow M$  induces a connection on  $\text{End } E \rightarrow M$ , as well as its coupled exterior derivatives: we define  $d_\nabla: \Omega_M^k(\text{End } E) \rightarrow \Omega_M^{k+1}(\text{End } E)$  by

- $d_\nabla \alpha := [d_\nabla, \alpha]$  on  $\Omega_M^0(\text{End } E)$ , and in general
- $(d_\nabla \alpha)(s) := d_\nabla(\alpha s) - \alpha(d_\nabla s)$ .

Given a connection  $\nabla$  and an endomorphism-valued 1-form  $A$ , we have another connection  $\nabla + A$ .

**Proposition 10.15.**

$$F_{\nabla+A} = F_\nabla + d_\nabla A + A \wedge A,$$

where  $A \wedge A$  denotes a combination of wedge of forms and composition of endomorphisms.

In particular, on a trivial bundle,

$$(10.16) \quad F_{d+A} = dA + A \wedge A.$$

**Example 10.17.** We're going to care the most about connections on line bundles. In this case, the quadratic term  $A \wedge A$  disappears, because it's in  $\Omega^2$  of something of dimension 1. Hence  $F_{\nabla+A} = F_{\nabla} + d_{\nabla}A$ . if  $L \rightarrow M$  is Hermitian, then  $A \in \Omega_M^1(\mathfrak{u}_1) = i\Omega_M^1$ , since  $\mathfrak{u}_1 = i\mathbb{R}$ . ◀

Let  $v$  be a vector field. Then, we will let  $\nabla_v: \Gamma(E) \rightarrow \Gamma(E)$  denote the contraction of  $\nabla$  by  $v$ . Thus if  $f \in C^\infty(M)$ ,  $\nabla_{fv} = f\nabla_v$ .

Let  $(x_1, \dots, x_n)$  be local coordinates on  $M$  and  $\partial_i := \frac{\partial}{\partial x_i}$  be the coordinate vector fields; let  $\nabla_i := \nabla_{\partial_i}$ . We can locally trivialize  $E \rightarrow M$ , hence write a connection  $\nabla = d + A$ . Hence

$$\nabla = d + \sum_k A_k \otimes dx_k,$$

where  $A_k(x) \in \text{End}(\mathbb{C}^r)$ , i.e. they're matrices.

Similarly, there are  $F_{ij}(x) \in \text{End}(\mathbb{C}^r)$  such that the curvature satisfies

$$(10.18) \quad F_{d+A} = \sum_{i,j} F_{ij} dx_i \wedge dx_j.$$

Then (10.16) simplifies to

$$(10.19) \quad F_{ij} = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} + [A_i, A_j].$$

This is what curvature is, concretely. In particular,

$$(10.20) \quad [\nabla_i, \nabla_j] = \left[ \frac{\partial}{\partial x_i} + A_i, \frac{\partial}{\partial x_j} + A_j \right] = F_{ij},$$

so curvature computes the failure of  $\nabla_i$  to commute.

**Lemma 10.21.** *Let  $u$  and  $v$  be vector fields. As elements of  $\Gamma(M, \text{End } E)$ ,*

$$F_{\nabla}(u, v) = [\nabla_u, \nabla_v] - \nabla_{[u, v]}.$$

Thus we have another perspective on curvature: it measures the failure of  $\nabla$  to commute with the Lie bracket of vector fields.

*Proof.* If  $u$  and  $v$  are coordinate vector fields, this is (10.20). Since every vector field is locally a linear combination of coordinate vector fields, it suffices to show that the right-hand side is  $C^\infty(M)$ -bilinear (since we already know the left-hand side is), which is just a computation. ◻

**Proposition 10.22** (Bianchi identity). *The curvature is covariantly closed, i.e.  $d_{\nabla}F_{\nabla} = 0$ .*

The proof is in the professor's notes.

### Gauge transformations.

**Definition 10.23.** A *gauge transformation* of a (Hermitian) vector bundle  $E \rightarrow M$  is a (unitary) bundle automorphism  $u: E \rightarrow E$ , hence a smoothly varying family  $u_x: E_x \xrightarrow{\cong} E_x$ . These form a group which we'll denote  $\mathcal{G}_E$ .

If  $\text{GL}(E)$  denotes the bundle of Lie groups whose fiber at  $x$  is  $\text{GL}(E_x)$ , then  $\mathcal{G}_E = \Gamma(M, \text{GL}(E))$ .<sup>16</sup>

There's an action of  $\mathcal{G}_E$  on  $\mathcal{C}(E)$  by  $u \cdot \nabla := u^* \nabla$ , i.e.

$$(10.24) \quad (u^* \nabla)(s) = u \nabla(u^{-1} s).$$

The curvature transforms by

$$(10.25) \quad F_{u^* \nabla} = u^* F_{\nabla} = u F_{\nabla} u^{-1}.$$

Since  $u^* \nabla$  and  $\nabla$  are both connections on the same bundle, we can compare them and obtain an endomorphism-valued 1-form:

$$(10.26) \quad u^* \nabla - \nabla = -(d_{\nabla} u) u^{-1}.$$

Here to take  $d_{\nabla} u$  we regard  $u$  as a section of  $\text{End } E$ .

<sup>16</sup> $\text{GL}(E)$  is *not* a principal bundle, as it doesn't have a canonical right action of  $\text{GL}_{\text{rank } E}(\mathbb{C})$ ; rather, it's an associated bundle of the principal frame bundle by the adjoint action.



On the trivial bundle we have

$$(10.27) \quad au^*(d + A) - d = -(du)u^{-1} + uAu^{-1}.$$

This has been a lot of formalism; now let's prove something.

**Theorem 10.28.** *Flat bundles are locally trivial; that is, if  $F_\nabla = 0$ , then for any  $x \in M$  there's a local trivialization of  $E$  near  $x$  in which  $\nabla = d$ .*

It suffices to prove the following.

**Proposition 10.29.** *Let  $H := (-1, 1)^n \subset \mathbb{R}^n$  and  $\mathbb{C}^r \rightarrow H$  be a trivial bundle. If  $\nabla$  is a flat (unitary) connection in  $\mathbb{C}^r$  then there's a unitary gauge transformation  $u$  such that  $u^*\nabla$  is trivial.*

*Proof.* Write

$$(10.30) \quad \nabla = d + A = d + \sum_k A_k dx_k,$$

where  $A_k: H \rightarrow \text{End}(\mathbb{C}^r)$  (and is skew-Hermitian in the unitary case). Since  $\nabla$  is flat,

$$(10.31) \quad \partial_i A_j - \partial_j A_i + [A_i, A_j] = 0.$$

Inductively, we assume we've gauge-transformed to  $A_i = 0$  for  $i = 1, \dots, m$ . The base case of  $m = 0$  is vacuous. We want to find a  $u$  such that  $u^*\nabla$  has  $A_i = 0$  for  $i = 1, \dots, m+1$ .

Well

$$(10.32) \quad u^*\nabla = d + \sum_k B_k dx_k,$$

where

$$(10.33) \quad B_k = -(\partial_k u)u^{-1} + uA_k u,$$

and several of the  $A_k$  are already zero. So we want to solve

$$(10.34) \quad \begin{aligned} \partial_i u &= 0, & i &= 1, \dots, m \\ \partial_{m+1} u + uA_{m+1} &= 0. \end{aligned}$$

This is a system of ODEs linear in  $x_{m+1}$ , independent of  $x_1, \dots, x_m$  (because  $\partial_i A_{m+1} = 0$  by flatness), and whose coefficients depend smoothly on  $x_{m+1}, \dots, x_m$ . Therefore it has a unique solution  $u$  which is the identity when  $x_{m+1}$  vanishes, which is smooth in all  $x_i$ , and is independent of  $x_1, \dots, x_m$ .

For unitarity, compute that  $\frac{\partial}{\partial x_{m+1}}(uu^\dagger) = 0$ . □

Lecture 11.

## Instantons in line bundles: 2/22/18

Today, we'll specialize to  $U_1$ -connections, i.e. in complex line bundles. This leads to the story of instantons in line bundles, which can be completely solved, and is very important in Seiberg-Witten theory.

Let  $\nabla$  be a connection on  $E \rightarrow M$ . We can encode it in a collection of local data: let  $\mathfrak{U}$  be an open cover such that  $E$  is trivialized over each  $U_\alpha \in \mathfrak{U}$ . Then we can give transition functions  $\tau_{\alpha\beta}: U_\alpha \cap U_\beta \rightarrow \text{GL}_r(\mathbb{C})$ . These satisfy various *cocycle properties*, e.g.  $\tau_{\beta\alpha} = \tau_{\alpha\beta}^{-1}$ .

On each  $U_\alpha$ ,  $\nabla = d + A_\alpha$  for some  $A_\alpha \in \Omega_{U_\alpha}^1(\text{End } \mathbb{C}^r)$ , so you have explicit matrix-valued 1-forms. They satisfy a *defining condition*

$$(11.1) \quad A_\beta = \tau_{\alpha\beta} A_\alpha \tau_{\beta\alpha} - (d\tau_{\alpha\beta})\tau_{\beta\alpha},$$

which comes from the rule for how covariant derivatives transform under gauge transformations.

If  $\nabla$  is flat, Theorem 10.28 implies it's locally trivial, so we may assume  $A_\alpha = 0$  for all  $U_\alpha \in \mathfrak{U}$ . Then (11.1) simplifies to  $d\tau_{\alpha\beta} = 0$  for all  $\alpha, \beta$ .

**Corollary 11.2.** *A vector bundle with a flat connection determines and is determined by a vector bundle with locally constant transition functions.*

A vector bundle with locally constant transition functions is also called a *local system*. There are other ways to think of them, including locally constant sheaves of vector spaces or as associated to representations of  $\pi_1(M)$ . Since our goal in this course is to gain insight into simply connected 4-manifolds, we're not going to adopt this perspective very much.

Seiberg-Witten theory has  $U_1$ -connections. Let  $L \rightarrow M$  be a Hermitian line bundle and  $\mathcal{A}_L$  denote the space of unitary covariant derivatives on  $L$ . If you pick a connection  $\nabla \in \mathcal{A}_L$ , then  $\mathcal{A}_L = \nabla + \Omega_M^1(i\mathbb{R})$ , because  $\mathfrak{u}(L) \cong i\mathbb{R}$  inside  $\text{End}(L) \cong \mathbb{C}$ .

The gauge group  $\mathcal{G}_L = \Gamma(M, U(L)) = C^\infty(M, U_1)$ , because  $U(L)$  is canonically the trivial principal  $U_1$ -bundle. This is unusual: usual the gauge transformations aren't a trivial bundle, because the endomorphism bundle is usually nontrivial.

$\mathcal{A}_L$  is an affine Fréchet space, with a  $C^\infty$  topology induced from a family of  $C^r$  norms. Similarly,  $\mathcal{G}_L$  has a topology inherited from the Fréchet space  $\Gamma(M, \text{End } L)$ . The action of  $\mathcal{G}_L$  on  $\mathcal{A}_L$  is continuous, and we may therefore form the space  $B_L := \mathcal{A}_L / \mathcal{G}_L$ , the set of gauge orbits of connections.

It's not clear from its definition that  $B_L$  is a reasonable space, e.g. is it Hausdorff? For higher-rank bundles, it's true but difficult to prove; the proof exhibits a metric.

**Theorem 11.3.** *Let  $L$  be a line bundle. Then there is a homeomorphism*

$$(11.4) \quad B_L \cong (S^1)^{b_1(M)} \times F,$$

where  $F$  is a Fréchet space.

First, let's see what Chern-Weil theory has to say about this. If  $\nabla \in \mathcal{A}_L$ , then  $iF_\nabla \in \Omega^2(M)$  is closed, because it's locally exact: locally  $\nabla = d + A$  and  $F_\nabla = dA$ . The class  $c_L := [iF_\nabla] \in H_{\text{dR}}^2(M)$  is independent of  $\nabla$ : any other connection is  $\nabla + iA$  for some  $A \in \Omega^1(M)$ , and its curvature is

$$(11.5) \quad F_{\nabla+iA} = F_\nabla + i dA.$$

*Remark.* In the higher-rank case, there's a quadratic term in this expression, which makes things a bit harder.  $\blacktriangleleft$

**Proposition 11.6.** *There's a universal  $\lambda \in \mathbb{R}$  (i.e. not depending on  $M$ ) such that  $c_L = \lambda c_1(L)$ .*

*Proof.* As we discussed last time, curvature pulls back along a smooth map  $f: N \rightarrow M$ , which implies  $f^*(c_L) = c_{f^*L}$ . Chern classes are also natural under pullback:  $c_1(f^*L) = f^*c_1(L)$ .

All line bundles on manifolds arise as the pullback of the tautological bundle  $\Lambda_N \rightarrow \mathbb{CP}^N$  along a smooth map  $f: M \rightarrow \mathbb{CP}^N$  for some large  $N$ . (This is because  $\mathbb{CP}^\infty$  is a  $BU_1$ .) Hence it suffices to prove this for  $\mathbb{CP}^N$ .

The inclusion  $i: \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^N$  induces an isomorphism on  $H^2$ , and  $i^*\Lambda_M \cong \Lambda_1$ . Hence it suffices to show that  $c_{\Lambda_1} = \lambda c_1(\Lambda_1) \in H^2(\mathbb{CP}^1; \mathbb{R}) \cong \mathbb{R}$ . Thus this boils down to an explicit calculation over  $S^2$ , which is found, e.g. in Bott-Tu.<sup>17</sup>  $\square$

In fact,  $\lambda = 2\pi$ , i.e.

$$(11.7) \quad \frac{1}{2\pi} c_L = \left[ \frac{i}{2\pi} F_\nabla \right] = c_1(L).$$

The action of  $\mathcal{G}_L$  on  $\mathcal{A}_L$  is by

$$(11.8) \quad u^* \nabla = \nabla - (du)u^{-1},$$

where  $\nabla$  is a unitary connection and  $u: M \rightarrow U_1$  is a gauge transformation.

**Definition 11.9.** Let a group  $G$  act on a space  $X$ . The action is *semifree* if there is a normal subgroup  $N \trianglelefteq G$  which acts trivially on  $X$  and such that the induced action of  $G/N$  on  $X$  is free.

**Proposition 11.10.** *The normal subgroup  $U_1 \trianglelefteq \mathcal{G}_L$  of constant gauge transformations acts trivially, and hence the  $\mathcal{G}_L$ -action on  $\mathcal{A}_L$ -action is semifree.*

*Partial proof.* If  $u$  is constant,  $du = 0$ ; so (11.8) simplifies to  $u^* \nabla = \nabla$ .  $\square$

<sup>17</sup>Bott-Tu use the Euler class, but in this setting the two agree.

*Proof of Theorem 11.3.* We have  $\pi_0 \mathcal{G}_L \cong [M, S^1]$ , which is naturally isomorphic to  $H^1(M; \mathbb{Z})$ . The isomorphism is very concrete: send a map  $f: M \rightarrow S^1$  to what it pulls back a generator of  $H^1(S^1)$  to.

The identity component  $\mathcal{G}_L^\circ$  of  $\mathcal{G}_L$  is  $\{u = e^{i\xi} \mid \xi \in C^\infty(M; \mathbb{R})\}$ , the space of gauge transformations with well-defined logarithms. In this case, (11.8) specializes to

$$(11.11) \quad (e^{i\xi})^* \nabla = \nabla - i d\xi.$$

Hence after fixing a connection  $\nabla$ , we get an identification  $\mathcal{A}_L / \mathcal{G}_L^\circ \cong i(\Omega^1(M) / d\Omega^0(M))$ . Let  $S := \{\nabla + ia \mid d^*a = 0\}$ , which is called a *Coulomb gauge slice*. Hodge theory identifies an isomorphism  $\Omega^1(M) / d\Omega^0(M) \cong \ker(d^*)$ , so the inclusion map  $S \rightarrow \mathcal{A}_L / \mathcal{G}_L^\circ$  is a homeomorphism.  $\mathcal{A}_L / \mathcal{G}_L^\circ$  is an infinite-dimensional affine space modeled on the Fréchet space of co-closed 1-forms.

Let's next describe the action of  $\pi_0(\mathcal{G}_L) \cong H^1(M; \mathbb{Z})$  on  $S \cong \mathcal{A}_L / \mathcal{G}_L^\circ$ ; given a  $u \in \mathcal{G}_L$ , (11.8) implies

$$(11.12) \quad u^* \nabla = \nabla - d(\log u).$$

This logarithm may not be uniquely defined, but  $d(\log u)$  is, and is a closed 1-form with periods in  $2\pi i\mathbb{Z}$ . Hence, given such a  $u$ , there's a unique cohomologous 1-form  $d(\log u) + d\xi$  such that

$$(11.13) \quad \nabla + d(\log u) + d\xi \in S.$$

Hence  $d(\log u) + d\xi \in \ker(d) \cap \ker(d^*)$ , so it's a harmonic 1-form.

Since  $\ker(d^*) = \mathcal{H}_g^1 \oplus \text{Im}(d^*)$ , then

$$\begin{aligned} B_L &\cong S / \pi_0 \mathcal{G}_L = \frac{\mathcal{H}_g^1}{2\pi(\text{a } \mathbb{Z}\text{-lattice})} \times \text{Im}(d^*) \\ &\cong \frac{H^1(X; \mathbb{R})}{2\pi H^1(X; \mathbb{Z})} \times \text{Im}(d^*). \end{aligned}$$

Let  $P := H^1(X; \mathbb{R}) / 2\pi H^1(X; \mathbb{Z})$ , which is called the *Picard torus*; it's diffeomorphic to  $(S^1)^{b_1(X)}$ , and  $\text{Im}(d^*)$  is a Fréchet space, as we wanted.  $\square$

*Remark.* Gauge theory tends to be complicated because we're working in a quotient space, but for rank 1, the techniques we just saw allow us to replace the quotient by something nicer, so the proofs are easier.  $\blacktriangleleft$

Next we'll discuss instantons, which are a specifically 4-dimensional story.

**Definition 11.14.** Let  $X$  be a compact 4-manifold with a conformal structure  $[g]$ , and let  $E \rightarrow X$  be a vector bundle with a Hermitian metric. An *instanton* or an *anti-self-dual connection* is a unitary connection  $\nabla$  such that  $(F_\nabla)^+ = 0$ .

Here  $(-)^+ := (1/2)(1 + \star)$ , the projection onto self-dual forms. This doesn't affect the endomorphism part, and is only about 2-forms.

*Remark.* The term “instanton,” or more properly “Yang-Mills instanton,” comes to us from physics: Yang-Mills theory is present in the background of this story, and from that perspective there are particle-physics reasons to call these instantons.  $\blacktriangleleft$

Let  $u \in \mathcal{G}_E$ . Then

$$(11.15) \quad (F_{u^* \nabla})^+ = (u F_\nabla u^{-1})^+ = u F_\nabla^+ u^{-1},$$

so  $\mathcal{G}_E$  sends instantons to instantons. The equation  $(F_{\nabla+A})^+ = 0$  is a first-order PDE in  $A$ .

Donaldson theory, as developed in the 1980s and 1990s, is the study of instantons, chiefly in rank-2 bundles. Today we're going to focus on the rank-1 case ( $U_1$ -instantons).

**Proposition 11.16.** *Suppose  $L \rightarrow X$  is a line bundle with an instanton. Then  $c_1(L) \in \mathcal{H}_{[g]}^-(\mathbb{Z}) := \mathcal{H}_{[g]}^- \cap H^2(X; \mathbb{Z}) \subset H_{\text{dR}}^2(X)$ .*

*Proof.* If  $F_\nabla^+ = 0$ , then  $(i/2\pi)F_\nabla$  is anti-self-dual; since it's also closed, then it's harmonic.  $\square$

If  $c_1(L) \in \mathcal{H}_{[g]}^-(\mathbb{Z})$ , we can represent it by an anti-self-dual form  $\omega$ . Then any connection  $\nabla$  with  $(i/2\pi)F_\nabla = \omega$  is an instanton.

In general,  $H_{\text{dR}}^2(X)$  is a vector space with two complementary subspaces  $\mathcal{H}_{[g]}^+$  and  $\mathcal{H}_{[g]}^-$ , and an integer lattice  $H^2(X; \mathbb{Z})$ . These generically don't satisfy any relations, so that the only element of  $H^2(X; \mathbb{Z}) \cap \mathcal{H}_{[g]}^-$  is the Chern class of the trivial bundle. Sometimes, however, there are others.

Now let's talk about uniqueness.

**Proposition 11.17.** *Suppose  $\nabla \in \mathcal{A}_L$  is an instanton and  $a \in \Omega^1(M)$ . Then  $\nabla + ia$  is an instanton iff  $(F_\nabla + i da)^+ = 0$ .*

$(F_\nabla + i da)^+ = 0$  means that  $d^+a = 0$  (this was defined in Definition 9.21). We saw in Theorem 9.22 that

$$(11.18) \quad \frac{\ker(d^+)}{\text{Im}(d)} = \frac{\ker(d)}{\text{Im}(d)} = H_{\text{dR}}^1(X),$$

So if  $\mathcal{M}_L$  denotes the space of instantons in  $L$  modulo the action of  $\mathcal{G}_L$ ,  $\mathcal{M}_L \subset B_L$ . We'll pick this up next time; uniqueness for instantons is governed by the operator  $d^+ + d^*$ .

Lecture 12.

### Instantons do not exist generically: 2/27/18

Today we'll finish up the discussion of  $U_1$  instantons, and then discuss generalities of differential operators on manifolds.

Like last time, let  $(X, g)$  be a closed, oriented Riemannian manifold and  $L \rightarrow X$  be a Hermitian line bundle with unitary connection  $\nabla$ . Assume  $\nabla$  is an instanton, i.e.  $F_\nabla^+ = 0$ .

Last time, we proved an existence criterion for instantons, Proposition 11.16: there exists an instanton in  $L$  iff  $c_1(L) \in \mathcal{H}_{[g]}^-(\mathbb{Z})$ . This generically does not hold.

If we do have an instanton  $\nabla$ , we can ask about uniqueness; we stated Proposition 11.17 last time, but we're actually more interested in instantons modulo gauge equivalence. Recall that  $\mathcal{A}_L/\mathcal{G}_L^0 \cong \Omega^1/d\Omega^0$  by the map sending  $[\nabla + ia] \mapsto [a]$ . Let  $\mathcal{I}_L$  denote the space of instantons inside  $\mathcal{A}_L$ ; then  $\mathcal{I}_L/\mathcal{G}_L^0 \cong \ker(d^+)/d\Omega^0 = H_{\text{dR}}^1(X)$ . Since  $\pi_0\mathcal{G}_L \cong H^1(X; \mathbb{Z})$ , then

$$(12.1) \quad \mathcal{I}_L/\mathcal{G}_L \cong \frac{H_{\text{dR}}^1(X)}{2\pi H^1(X; \mathbb{Z})} = P.$$

That is, the moduli space of instantons modulo gauge equivalence, if nonempty, is the Picard torus. In particular, it's a finite-dimensional smooth manifold, which is quite nice, and even a little surprising.

Why should this be the case? Let's work in the Coulomb gauge slice  $\mathcal{S} := \nabla + i\ker(d^*)$ , which is diffeomorphic to  $\mathcal{A}_L/\mathcal{G}_L^0$ . Then  $\mathcal{I}_L \cap \mathcal{S} = \nabla + i\ker(d^* \oplus d^+)$ , so we're interested in the operator  $d^* \oplus d^+$ . The ellipticity of this operator is important, but we'll return to that later.

Recall that  $d^*: \Omega^0 \hookrightarrow \Omega^1$  and  $d^+: \Omega^1 \rightarrow \Omega_g^+$ .  $d^+$  respects the splittings

$$(12.2) \quad \begin{aligned} \Omega^1 &= \mathcal{H}_g^1 \oplus d\Omega^- \oplus d^+\omega^2 \\ \Omega_g^+ &= \mathcal{H}_{[g]}^+ \oplus d^+\Omega^1, \end{aligned}$$

in that  $d^+: d^*\Omega^2 \rightarrow d^+\Omega^1$  is an isomorphism. Moreover,  $d^*: d\Omega^0 \rightarrow d^*\Omega_1 = \Omega_0^0$ , the functions whose integral is zero, is an isomorphism.

Putting all these together, the map

$$(12.3) \quad d^* \oplus d^+: \Omega^1 \longrightarrow \Omega_0^0 \oplus \Omega_{[g]}^+$$

has kernel  $\mathcal{H}_g^1$  and cokernel  $\mathcal{H}_{[g]}^+$ . Hence the instanton moduli space  $\mathcal{I}_L/\mathcal{G}_L$  is cut out by a function whose derivative  $d^* \oplus d^+$  is not surjective, but has a cokernel of constant rank  $b^+(X)$ . Hence the moduli space is cut out cleanly, if not transversely, by its defining equations.

*Remark.* We're used to thinking about manifolds arising as preimages of regular values, but must be careful here: we're in the infinite-dimensional setting. The crucial ingredient is the inverse function theorem — which still holds for infinite-dimensional Banach manifolds, but not infinite-dimensional Fréchet manifolds. Hence, to technically set up the moduli space of instantons and make related constructions, one must start with connections of some given Sobolev regularity, though then the story proceeds similarly. ◀

We've mentioned a few times that generically, instantons don't exist on a line bundle, and the idea is that we're intersecting a fixed lattice with a generic subspace, so generically cannot expect any intersection points. Let's make this idea precise.

**Theorem 12.4.** *For  $k < b^+(X)$ ,  $r \geq 3$ , and any  $C^r$  family of conformal structures  $[g_t]_{t \in T}$  parameterized by a  $k$ -dimensional Riemannian manifold  $T$ , there exist perturbations  $\hat{g}_t$  to  $g_t$ , arbitrarily close to  $[g_t]$  in the  $C^r$ -norm induced by the Riemannian metrics on  $X$  and  $T$ , such that  $\mathcal{H}_{[\hat{g}_t]}^-(\mathbb{Z}) = 0$  for all  $t \in T$ .*

*Proof.* Let  $C_X$  denote the space of conformal structures of class  $C^r$ , where  $r \geq 3$  is fixed. We identified this with an open set on  $\Gamma(X, \text{Hom}(\Lambda^-, \Lambda^+))$ , where  $\Lambda^\pm = \Lambda_{[g_0]}^\pm$  for some reference metric  $g_0$ . In particular,  $[g] \mapsto m$  where  $\Lambda_{[g]}^- = \Gamma_m$ . This defines the period map  $P: C_X \rightarrow \text{Gr}^-$ , the Grassmannian of negative-definite subspaces of dimension  $b^-(X)$  in  $H_{\text{dR}}^2(X)$ :  $P[g] = \mathcal{H}_{[g]}^-$ .

Then,  $D_{[g]}P: T_{[g]}C_X = \Gamma(M, \text{Hom}(\Lambda^-, \Lambda^+)) \rightarrow T_{\mathcal{H}_{[g]}^-} \text{Gr}^- = \text{Hom}(\mathcal{H}^-, \mathcal{H}^+)$  is the map

$$(12.5) \quad (DP)(m)(\alpha_-) = m(\alpha_-)_{\text{har}},$$

where  $\alpha_- \in \mathcal{H}^-$ .

We'll need one more tool: let  $c \in H_{\text{dR}}^2(X) \setminus 0$  and

$$(12.6) \quad S_c := \{H \in \text{Gr}^- \mid c \in H\},$$

which is a closed subset of the Grassmannian. Here's why: if  $H \in S_c$ , choose an  $H'$  such that  $H^2(X) = H \oplus H'$ . Then, for any linear map  $\mu: H \rightarrow H'$ ,  $c \in \Gamma_\mu$  iff  $\mu(c) = 0$ , which is a closed condition. Moreover,  $S_c \subset \text{Gr}^-$  is a closed submanifold, and its tangent space is

$$(12.7) \quad T_H S_c = \{\mu \in \text{Hom}(H, H') \mid \mu(c) = 0\}.$$

This identification identifies the normal space  $N_H S_c$  with  $H'$ , sending  $[\mu] \mapsto \mu(c)$ . Therefore, assuming transversality,  $S_c$  has codimension  $b^+(X)$ .

**Lemma 12.8.** *The period map  $P$  is transverse to  $S_c$ .*

*Proof.* We'll show that if  $P[g] \in S_c$ , then  $\text{Im}(D_{[g]}P)$  spans  $N_{P[g]}S_c$ . Let's unravel what this means. We want to show that if  $\alpha^- \in \mathcal{G}_{[g]}^-$  represents  $c$ , then for all  $\alpha^+ \in \mathcal{H}_{[g]}^+$  there's a map  $m: \Lambda^- \rightarrow \Lambda^+$  such that  $m(\alpha^-)_{\text{har}} = \alpha^+$ .

If this were not true, there would exist an  $\alpha^+ \in \mathcal{H}_{[g]}^+$  which is  $L^2$ -orthogonal to  $m(\alpha^-)$  for all  $m$ , i.e. for all  $m$ ,

$$\begin{aligned} 0 &= \langle \alpha^+, m(\alpha^-)_{\text{har}} \rangle_{L^2} \\ &= \langle \alpha^+, m \rangle (\alpha^-)_{L^2} \end{aligned}$$

Choose an  $x \in X$  such that  $\alpha^+(x) \neq 0$  and  $\alpha^-(x) \neq 0$ . Such an  $x$  exists because harmonic forms are nonvanishing on an open dense subset, by analytic continuation. Let  $B$  be a small geodesic ball about  $x$  and choose an  $m_0: \Lambda_x^- \rightarrow \Lambda_x^+$  such that  $m_0(\alpha^-)_x = \alpha^+_x$ . Choose a cutoff function  $\chi$  supported in  $(1/2)B$  and identically 1 in a neighborhood of  $x$ .

In  $B$ , we can extend  $m_0$  to an  $m$  such that  $\langle m(\alpha^-), \alpha^+ \rangle \neq 0$ ; then

$$(12.9) \quad \langle (\chi m)(\alpha^-), \alpha^+ \rangle_{L^2} = \int_B \chi \langle m(\alpha^-), \alpha^+ \rangle \text{dvol} \neq 0. \quad \square$$

Therefore we can assume for any  $c \neq 0$  in  $H_{\text{dR}}^2(X)$   $P^{-1}(S_c) \subset C_X$  is a submanifold of codimension  $b^+(X)$ . Here we have used the inverse function theorem for Banach spaces. All we have to do is avoid these.

Let  $\{[g_t]\}_{t \in T}$  be defined by a smooth map  $G: T \rightarrow C_X$ , and let  $\mathcal{G}$  denote the space of  $C^r$  maps  $T \rightarrow C_X$  lying with a fixed distance of  $G$  with respect to a  $C^r$  norm.<sup>18</sup> Then, there's an open subset  $U_c \subset \mathcal{G}$  of maps  $\hat{G}$  transverse to  $P^{-1}(S_c)$ . By dimensionality reasons,  $\hat{G} \pitchfork P^{-1}(S_c)$  means they do not intersect, because  $\dim T < b^+(X)$ .

Now we eliminate the choice of  $c$ :  $\bigcap_{c \in H^2(X; \mathbb{Z})} U_c$  is a countable intersection of open dense subsets, hence by the Baire category theorem is dense, since we're in a complete metric space. Therefore any  $\hat{G}$  in this intersection does the job.  $\square$

<sup>18</sup>The norm is a choice here, but all choices are equivalent for the goals of this proof.

The technicalities in transversality arguments take some getting used to, but are grounded in concrete differential topology. There are a thousand other transversality arguments in gauge theory, pseudoholomorphic curves, etc., and they follow very similar lines.

In the remaining minutes of this lecture, we'll discuss differential operators. The major point is that we can do this in a coordinate-free way. Let  $M$  be a manifold and  $E, F \rightarrow M$  be real vector bundles. First-order linear differential operators should be certain maps  $D: \Gamma(E) \rightarrow \Gamma(F)$ . In local coordinates  $(x_1, \dots, x_n)$  trivializing both  $E$  and  $F$ , a section  $s$  of  $E$  looks like a column vector  $(s_1, \dots, s_p)$ , and a section  $s'$  of  $F$  is represented by a column vector  $(s'_1, \dots, s'_q)$ . Then we want our operator  $D$  to satisfy a formula like

$$(12.10) \quad (Ds)_\alpha = \sum_{i,\beta} P_{\alpha\beta}^i(x) \frac{\partial}{\partial x_i} s_\beta + \sum_{\beta} Q_{\alpha\beta}(x) s_\beta(x).$$

Here  $P$  and  $Q$  are smooth, and Latin indices refer to coordinates on the manifold and Greek indices refer to coordinates on the vector bundle.

The notation in (12.10) is not pleasing to look at, and is worse to work with. Let's try to do it invariantly.

**Definition 12.11.** A *zeroth-order operator*  $L: \Gamma(M, E) \rightarrow \Gamma(M, F)$  is a  $C^\infty(M)$ -linear function. The vector space of zeroth-order operators is denoted  $\mathcal{D}_0(E, F)$ .

There is an isomorphism

$$(12.12) \quad \mathcal{D}_0(E, F) \xrightarrow{\sigma} \Gamma(M, \text{Hom}(E, F)),$$

called the *symbol*, defined by

$$(12.13) \quad \sigma(L)(x, e) = (Ls)(x),$$

where  $s$  is an arbitrary section with  $s(x) = e \in E_x$ ; one checks this is well-defined, which is not hard.

**Definition 12.14.** A *first-order differential operator* is an  $\mathbb{R}$ -linear map  $D: \Gamma(M, E) \rightarrow \Gamma(M, F)$  such that for all function  $f \in C^\infty(M)$ , the commutator  $[f, D]$  is a zeroth-order operator. The vector space of first-order operators is denoted  $\mathcal{D}_1(E, F)$ .

This is cleaner than (12.10), though it turns out to be equivalent. In particular, it implies zeroth-order operators are also first-order operators.

**Example 12.15.** Any covariant derivative  $\nabla: \Gamma(M, E) \rightarrow \Gamma(M, T^*M \otimes E)$  is a first-order operator, because by definition  $[f, D]s = df \otimes s$ . This is algebraic in  $s$ , hence  $C^\infty(M)$ -linear. ◀

Lecture 13.

## Differential operators: 3/1/18

Last time, we introduced differential operators between vector bundles  $E, F \rightarrow M$ .

- The zeroth-order differential operators  $\mathcal{D}_0(E, F)$  are the  $C^\infty(M)$ -linear functions  $L: \Gamma(E) \rightarrow \Gamma(F)$ , which form an  $\mathbb{R}$ -vector space isomorphic to  $\Gamma(\text{Hom}(E, F))$ .
- The first-order differential operators  $\mathcal{D}_1(E, F)$  are the  $\mathbb{R}$ -linear functions  $D: \Gamma(E) \rightarrow \Gamma(F)$  such that for all  $f \in C^\infty(M)$ ,  $[D, f] \in \mathcal{D}_0(E, F)$ .

The basic example of a first-order operator is the covariant derivative in  $E$ , since  $[\nabla, f] = df \otimes -$ .

We discussed what this would look like in coordinates, namely (12.10), but we can also recast this story in terms of jet bundles.

**Definition 13.1.** The *1-jet bundle*  $J_1(E) \rightarrow M$  is a vector bundle whose fiber at an  $x \in M$  is the space of *1-jets of sections of  $E$*  at  $x$ , i.e. equivalence classes of  $s \in \Gamma(M, E)$  where  $s \sim s'$  if  $s(x) = s'(x)$  and  $s - s'$  is tangent to 0 at  $x$ , i.e.  $D_x(s - s'): T_x M \rightarrow T_{(x,0)} E$  maps to the tangent space of the zero section of  $E$ .

That is, we locally identify sections which agree to first order. In local coordinates which trivialize  $E$ , this implies  $s$  and  $s'$  have the same Taylor expansion up to quadratic terms.

*Remark.* There's a general notion of  $n^{\text{th}}$ -order jets generalizing this directly, where we take equivalence classes of sections which agree to  $n^{\text{th}}$  order. ◀

Given a 1-jet  $[s]$ , its value at any  $x \in M$  does not depend on the choice of representative, so we have an evaluation map  $\text{ev}: J^1 E \rightarrow E$ . This fits into a short exact sequence

$$(13.2) \quad 0 \longrightarrow (T^* M) \otimes E \longrightarrow J^1 E \xrightarrow{\text{ev}} E \longrightarrow 0.$$

There's a tautological map  $j^1: \Gamma(M, E) \rightarrow \Gamma(M, J^1 E)$  sending  $s \mapsto (x \mapsto j_x^1(s) := [s]_x)$ .

**Definition 13.3.** A *first-order jet operator* is a map  $D: \Gamma(E) \rightarrow \Gamma(F)$  of the form  $(Ds)(x) = L(j_x^1(s))$ , where  $L \in \Gamma(\text{Hom}(J^1 E, F))$ . These form a vector space  $\mathcal{D}_1(E, F)_{\text{jet}} \cong \Gamma(\text{Hom}(J^1 E, F))$ .

In local coordinates which trivialize  $E$ , a first-order jet operator  $D$  looks like (12.10), which is not terribly complicated: just spell out what such an operator can do.

**Theorem 13.4** (Grothendieck).  $\mathcal{D}_1(E, F) = \mathcal{D}_1(E, F)_{\text{jet}}$ .

*Remark.* The fact that Grothendieck was interested in this suggests that it secretly applies to algebraic geometry, and indeed this perspective is quite useful there.  $\blacktriangleleft$

*Proof.* First we'll prove that a first-order jet operator  $D$  is a first-order differential operator. We have  $D = L \circ j^1$  for some  $L: \Gamma(J^1 E) \rightarrow \Gamma(F)$ . If  $f \in C^\infty(M)$  and  $s \in \Gamma(M, E)$ ,

$$(13.5) \quad j_x^1(fs) = f(x)j_x^1(s) + \underbrace{df(x) \otimes s(x)}_{(*)}.$$

The part marked  $(*)$  is in  $T_x^* M \otimes E_x$ , so by (13.2) is in  $(J^1 M)_x$ . In particular,

$$(13.6) \quad [f, D](s) = L(df \otimes s).$$

Next, we compute

$$\begin{aligned} [D, f](gs) &= L \circ j^1(fgs) - fL(j^1(gs)) \\ &\stackrel{(13.5)}{=} L \circ (df \otimes gs + fj^1(gs)) - fL(j^1(gs)) \\ &= gL(df \otimes s) \\ &\stackrel{(13.6)}{=} g[D, f](s). \end{aligned}$$

Therefore  $D$  is a differential operator.

Before we go the other direction, (13.2) means there's a restriction map  $\rho$  fitting into a short exact sequence<sup>19</sup>

$$(13.7) \quad 0 \longrightarrow \Gamma(\text{Hom}(E, F)) \longrightarrow \mathcal{D}_1(E, F)_{\text{jet}} = \Gamma(\text{Hom}(J^1 E, F)) \xrightarrow{\rho} \Gamma(\text{Hom}(T^* M \otimes E, F)) \longrightarrow 0.$$

This map  $\rho$  is called the *principal symbol* of  $D$ ; in the coordinate form (12.10), it captures the data of  $\{P_{\alpha\beta}^i\}$ .

It's also possible to define the symbol  $\sigma_D^1$  of a first-order differential operator  $D$  by

$$(13.8) \quad \sigma_D^1(f) := \sigma_{[D, f]}^0 \in \Gamma(\text{Hom}(E, F)).$$

One can check that

$$(13.9) \quad \sigma_D^1(fg) = f\sigma_D^1(g) + g\sigma_D^1(f).$$

**Lemma 13.10.** If  $D \in \mathcal{D}(E, F)_1$ ,  $f(x) = 0$ , and  $df|_x = 0$ , then  $\sigma_D^1(f)_x = 0$ .

*Proof sketch.* Near  $x$ , such an  $f$  is a sum of products  $g_i h_i$  where  $g_i(x) = h_i(x) = 0$ . Then, apply the product rule (13.9) to each  $g_i h_i$ .  $\square$

Moreover, if  $c$  is constant near  $x$ , then  $\sigma_D^1(c)_x = 0$ , since  $[D, c]_x = 0$ . The point of this is that the symbol only depends on  $df_x \in T_x^* M$ .<sup>20</sup> Therefore we can define  $\sigma_D^1(\xi)_x$  for a  $\xi \in T_x^* M$  to be  $\sigma_D^1(f)$  for any  $f$  with  $df_x = \xi$ .

<sup>19</sup>The fact that  $\Gamma$  is exact follows because we're taking smooth sections, rather than holomorphic sections of some complex vector bundle. Therefore local sections can always be extended globally and we don't have to worry about derived functors.

<sup>20</sup>If you prefer to think of the cotangent space as  $\mathfrak{m}_x/\mathfrak{m}_x^2$ , where  $\mathfrak{m}_x$  is the unique maximal ideal of the local ring of germs of functions near  $x$ , which is the more common perspective in algebraic geometry, then  $df_x = [f - f(x)] \in \mathfrak{m}_x/\mathfrak{m}_x^2$ .



The comparison of these two symbol operators allows us to finish the proof: we already know about the inclusions

$$(13.11) \quad \begin{array}{ccc} \mathcal{D}_1(E, F) & \xrightarrow{\sigma_D^1} & \Gamma(\text{Hom}(T^*M \otimes E, F)), \\ \downarrow & \nearrow \rho & \\ \mathcal{D}_1(E, F)_{\text{jet}} & & \end{array}$$

so just from definitions, if  $D \in \mathcal{D}_1(E, F)$ , then  $\sigma_D^1 \circ j^1 \in \mathcal{D}_1(E, F)_{\text{jet}}$ , and  $D - \sigma_D^1 j^1 \in \mathcal{D}(E, F)_0$ . Therefore  $D \in \mathcal{D}_1(E, F)_{\text{jet}}$  also.  $\square$

Once you absorb this formalism, it generalizes straightforwardly to higher-order differential operators.

**Definition 13.12.** We inductively define the  $n^{\text{th}}$ -order linear differential operators  $\mathcal{D}_n(E, F)$  to be the space of  $\mathbb{R}$ -linear functions  $D: \Gamma(E) \rightarrow \Gamma(F)$  such that  $[D, f] \in \mathcal{D}_{n-1}(E, F)$  for all smooth functions  $f$ .

We define the  $n$ -jet bundle  $J^n E \rightarrow M$  to be the space of equivalence classes of sections where  $s \sim s'$  if their Taylor expansions agree to order  $n$ . Then (13.2) generalizes to

$$(13.13) \quad 0 \longrightarrow \text{Sym}^n(T^*X) \otimes E \longrightarrow J^n E \longrightarrow J^{n-1} E \longrightarrow 0,$$

and a tautological map  $j^n$  as before, so we can define the space of  $n^{\text{th}}$ -order jet operators  $\mathcal{D}_n(E, F)_{\text{jet}}$  to be operators of the form  $L \circ j^n$ , where  $L \in \text{Hom}(J^n E, F)$ .

Theorem 13.4 generalizes to  $\mathcal{D}_n(E, F) = \mathcal{D}_n(E, F)_{\text{jet}}$ , and restricting  $n^{\text{th}}$ -order jet operators to  $J^{n-1} E$  defines the symbol

$$(13.14) \quad \sigma^n: \mathcal{D}_n(E, F)/\mathcal{D}_{n-1}(E, F) \xrightarrow{\cong} \Gamma(\text{Hom}(\text{Sym}^n(T^*M) \otimes E, F)).$$

**Proposition 13.15.** Let  $D_1$  be an  $m^{\text{th}}$ -order differential operator and  $D_2$  be an  $n^{\text{th}}$ -order differential operator. Then  $D_1 \circ D_2$  is an  $(m+n)^{\text{th}}$ -order differential operator and

$$\sigma_{D_1 \circ D_2}^{m+n} = \sigma_{D_1}^m \circ \sigma_{D_2}^n.$$

The proof uses the composition in  $\text{Sym}^\bullet(T^*M)$ , which is identified with the polynomial functions on  $T^*M$ .

If  $\mathcal{D}(E, F) := \bigcup_n \mathcal{D}_n(E, F)$ , then  $\mathcal{D}(E, F)$  is filtered by the order of an operator. We can take its associated graded  $\text{gr}\mathcal{D}(E, F) := \bigoplus_n \mathcal{D}_n/\mathcal{D}_{n+1}$ ; if  $E = F$  composition makes it into a graded ring.

**Proposition 13.16.** The symbol defines an isomorphism of graded rings

$$\sigma: \text{gr}\mathcal{D}(E, E) \xrightarrow{\cong} \Gamma(\text{Hom}(\text{Sym}^\bullet(T^*M), \text{End } E)).$$

We'll now discuss some examples of this abstract formalism.

**Example 13.17.** The exterior derivative  $d: \Omega^k(M) \rightarrow \Omega^{k+1}(M)$  is a first-order differential operator, because  $[d, f] = df \wedge -$ . Hence the symbol  $\sigma_d^1: T^*M \rightarrow \text{Hom}(\Lambda^k(T^*M), \Lambda^{k+1}(T^*M))$  is the map  $\sigma_d^1(\xi)(\alpha) = \xi \wedge \alpha$ .  $\blacktriangleleft$

**Example 13.18.** Let's next look at a coupled exterior derivative  $d_\nabla: \Omega_M^k(E) \rightarrow \Omega_M^{k+1}(E)$ , where  $\nabla$  is a covariant derivative on  $E$ . Then  $[d_\nabla, f] = df \wedge -$ , so the symbol is again  $\sigma_{d_\nabla}^1(\xi) = \xi \wedge -$ .  $\blacktriangleleft$

**Example 13.19.** Now suppose  $M$  is compact and  $E, F \rightarrow M$  have metrics. Let  $D \in \mathcal{D}_1(E, F)$  and  $D^* \in \mathcal{D}_1(F, E)$ . We say that  $D$  and  $D^*$  are formal adjoints if

$$(13.20) \quad \langle s', Ds \rangle_{L^2(F)} = \langle D^*s, s' \rangle_{L^2(E)}$$

for all  $s \in \Gamma_{L^2}(M, E)$  and  $s' \in \Gamma_{L^2}(M, F)$ .

One can quickly check (by integration by parts) that if  $D$  and  $D^*$  are formal adjoints, then  $[D^*, f] = -[D, f]^*$ , and therefore  $\sigma_{D^*}^1(\xi) = -\sigma_D^1(\xi)^*$ .

For example, if  $M$  is an oriented Riemannian manifold,  $\sigma_{d^*}^1$  is the adjoint to  $\xi \wedge -$ . You can check this is contraction  $\iota_\xi$ . The story is similar for  $d_\nabla^*$ .  $\blacktriangleleft$

**Example 13.21.** Next let's look at the Hodge Laplacian  $\Delta := dd^* + d^*d = (d + d^*)^2$ . Proposition 13.15 tells us that

$$(13.22) \quad \sigma_\Delta(\xi, \xi) = \sigma_d(\xi) \circ \sigma_{d^*}(\xi) + \sigma_{d^*}(\xi) \circ \sigma_d(\xi),$$

which sends

$$(13.23) \quad \alpha \mapsto -\xi \wedge \iota_\xi \alpha - \iota_\xi(\xi \wedge \alpha) = -|\xi|^2 \alpha.$$

In particular,  $\sigma_\Delta(\xi, \xi) = -|\xi|^2 \text{id}$ .

You can calculate the symbol on two different cotangent vectors using a polarization.  $\blacktriangleleft$

**Definition 13.24.** An *elliptic operator* is a linear differential operator  $D \in \mathcal{D}_n(E, F)$  such that for all  $x \in M$  and  $\xi \in T_x^*M \setminus 0$ , the symbol  $\sigma_D^n(\xi, \dots, \xi)$  is an isomorphism.

In particular, the Hodge Laplacian is elliptic. There are plenty of interesting elliptic operators.

Spelling it out a bit, if  $\xi = df|_x$ , then

$$(13.25) \quad \sigma_D^n(\xi, \dots, \xi) = \frac{1}{n!} [\dots [D, f], f] \dots, f].$$

**Definition 13.26.**

- A *generalized Laplacian* is a second-order differential operator  $\Delta \in \mathcal{D}_2(E, E)$  such that  $\sigma_\Delta^2(\xi, \xi) = -|\xi|^2 \text{id}_E$ . By (13.25), this is equivalent to

$$(13.27) \quad \frac{1}{2} [[\Delta, f], f] = -|df|^2.$$

- A *Dirac operator* is a first-order differential operator  $D \in \mathcal{D}_1(E, E)$  such that  $D^2$  is a generalized Laplacian. Equivalently,  $\sigma_D(\xi)^2 = -|\xi|^2 \text{id}_E$ .

**Example 13.28.** Let  $(X, g)$  be an oriented, 4-dimensional Riemannian manifold. Then  $d^* \oplus d^+ : \Omega^1(X) \rightarrow \Omega^0(X) \oplus \Omega_g^+$  is elliptic: its symbol  $\sigma^1(\xi) : T_x^*X \rightarrow \mathbb{R} \oplus \Lambda^+$  sends

$$(13.29) \quad \sigma^1(\xi)(a) = -\iota_\xi a + (\xi \wedge a)^+.$$

For  $\xi \neq 0$ , this is an isomorphism. We saw last time that  $d^* \oplus d^+$  has finite-dimensional kernel and cokernel. This is a manifestation of its ellipticity.  $\blacktriangleleft$

**Example 13.30.** Consider  $d^* \oplus d : \Omega^\bullet(M) \rightarrow \Omega^\bullet(M)$ . Its square is the Hodge Laplacian, so it's a Dirac operator.  $\blacktriangleleft$

*Remark.* Dirac operators usually come out of spin geometry, but our perspective sheds a different light on it. Nonetheless, it's there in the background: the relation  $\sigma_D^1(\xi) = -|\xi|^2 \text{id}$  for a Dirac operator  $D$ , called the *Clifford relation*, shows that the map  $\sigma_D : T^*X \rightarrow \text{End } E$  extends to a Clifford algebra representation  $\sigma : \text{Cl}(T_x^*X) \rightarrow \text{End}(E_x)$ .

Recall that the *Clifford algebra* on an inner product space  $(V, \langle \cdot, \cdot \rangle)$  is the free associative algebra  $\text{Cl}(V)$  on  $V$  with unit 1 modulo the relation  $\xi \cdot \xi = -\|\xi\|^2 \cdot 1$ . By construction, this satisfies the universal property that any linear map  $\sigma$  into an algebra with  $\sigma(\xi)^2 = -|\xi|^2 \cdot 1$  extends to an algebra homomorphism out of the Clifford algebra.

The upshot is that spin geometry is unavoidable, though this is not a bad thing!  $\blacktriangleleft$

Lecture 14.

## Clifford algebras, spinors, and spin groups: 3/6/18

Seiberg-Witten theory makes heavy use of spin geometry. Since it's focused in dimension 4, and  $\text{Spin}_4 \cong \text{SU}_2 \times \text{SU}_2$ , it's possible to do everything concretely and explicitly — at the expense of losing sight of an interesting general story. We will spend this lecture covering the algebra behind spin geometry, and next lecture on spin geometry.

There are many references for this material; one nice one is Deligne's notes on spinors in the Quantum Fields and Strings book. Like everything Deligne writes, it's terse, yet packed with insights. Our approach will be terse in a different way: we have a lot to get through, so some proofs will be deferred to references.

Throughout today's lecture, fix a commutative ring  $A$  containing  $1/2$ . Usually we'll only use  $A = \mathbb{R}$  or  $A = \mathbb{C}$ , but there are times when the increased generality will be useful.

**Definition 14.1.** A *quadratic  $A$ -module* is an  $A$ -module  $M$  together with a quadratic form  $q: M \rightarrow A$ .

In this case, there is an  $A$ -bilinear form

$$(14.2) \quad \langle m, n \rangle := \frac{1}{2}(q(u+v) - q(u) - q(v)),$$

and  $q(v) = \langle v, v \rangle$ . (Typically we'll take  $A = \mathbb{R}$ ,  $M = \mathbb{R}^n$ , and  $q = |\cdot|^2$ .)

**Definition 14.3.** Given a quadratic  $A$ -module  $(M, q)$ , its *Clifford algebra*  $\text{Cl}(M, q)$  is the free unital associative  $A$ -algebra on  $M$  such that for all  $m \in M$ ,  $m \cdot m = -q(m) \cdot 1$ .

*Remark.* The choice of  $-q(m) \cdot 1$  instead of  $q(m) \cdot 1$  is a convention; some authors use the other convention.  $\blacktriangleleft$

There's an explicit construction using the tensor algebra of  $M$ , as with all algebras defined with generators and relations.

**Proposition 14.4.** *The Clifford algebra  $\text{Cl}(M, q)$  satisfies a universal property: if  $B$  is a unital associative  $A$ -algebra and  $f: M \rightarrow B$  is an  $A$ -linear map with  $f(m)^2 = -q(m)1_B$  for all  $m \in M$ , then there's a unique  $\tilde{f}: \text{Cl}(M, q) \rightarrow B$  such that the following diagram commutes:*

$$\begin{array}{ccc} M & \xrightarrow{f} & B \\ \downarrow & \searrow \tilde{f} & \uparrow \exists! \\ \text{Cl}(M, q) & & \end{array}$$

Since  $m^2 = -q(m) \cdot 1$ , then for all  $m, n \in M$ ,

$$(14.5) \quad \{m, n\} := mn + nm = -2\langle m, n \rangle \cdot 1.$$

Formation of Clifford algebras is compatible with extension of scalars  $A \rightarrow B$ . With  $A$  fixed, it's also functorial in  $(M, q)$ , and in particular, the action of the orthogonal group  $\text{O}(M, q) \rightarrow \text{Aut}(M)$  extends to an action  $\text{O}(M, q) \rightarrow \text{Aut } \text{Cl}(M, q)$ .

Because of the defining relation, the length of a monomial  $m_1 \cdots m_\ell \in \text{Cl}(M, q)$ , with  $m_i \in M$ , is not well-defined, but the length mod 2 is well-defined. Therefore  $\text{Cl}(M, q)$  splits as a direct sum

$$(14.6) \quad \text{Cl}(M, q) = \text{Cl}^0(M, q) \oplus \text{Cl}^1(M, q),$$

where  $\text{Cl}^i(M, q)$  is generated by the monomials with length equal to  $i \bmod 2$ . Alternatively, since  $-I \in \text{O}(M, q)$  squares to 1, its eigenvalues are in  $\{\pm 1\}$ , and  $\text{Cl}^i(M, q)$  is the  $(-1)^i$ -eigenspace of  $-I$ . Hence  $\text{Cl}(M, q)$  is a  $\mathbb{Z}/2$ -graded algebra, or a *superalgebra*.

**Definition 14.7.** Let  $B$  be an  $A$ -algebra. Then the *opposite algebra*  $B^{\text{op}}$  is the  $A$ -algebra with the same underlying  $A$ -module as  $B$ , but with multiplication  $a \cdot^{\text{op}} b := b \cdot a$ .

This is in the ungraded sense.

**Lemma 14.8.** *There is a unique  $A$ -algebra isomorphism  $\beta: \text{Cl}(M, q) \rightarrow \text{Cl}(M, q)^{\text{op}}$  extending  $\text{id}_M$ .*

*Proof.* Check the universal property (Proposition 14.4) for  $\text{id}_M$ .  $\square$

This map  $\beta$ , regarded as an automorphism of  $\text{Cl}(M, q)$ , is called the *prime anti-automorphism* of  $\text{Cl}(M, q)$ , and is characterized as the automorphism such that

$$(14.9) \quad \beta(m_1 m_2 \cdots m_\ell) = m_\ell \cdots m_2 m_1.$$

*Remark.* Many facts about Clifford algebras appear ad hoc from the algebra perspective, but make more sense using superalgebra, which naturally encodes all of the signs that appear.  $\blacktriangleleft$

**Definition 14.10.** Let  $B$  be an  $A$ -superalgebra. Then, its *opposite superalgebra*  $B^{\text{sop}}$  is the  $A$ -superalgebra with the same underlying  $\mathbb{Z}/2$ -graded  $A$ -module as  $B$ , but with multiplication

$$(14.11) \quad a \cdot^{\text{sop}} b := (-1)^{|a||b|} b \cdot a.$$

**Proposition 14.12.** *There's an isomorphism  $\text{Cl}(M, q)^{\text{sop}} \cong \text{Cl}(M, -q)$ .*

Generally  $\text{Cl}(M, q)$  and  $\text{Cl}(M, -q)$  are not isomorphic.

Let  $F^\ell \text{Cl}(M, q)$  denote the submodule spanned by monomials of length at most  $\ell$ . Then  $F^\ell \cdot F^m \subset F^{\ell+m}$ , so the Clifford algebra is filtered by length.

**Proposition 14.13.** *The associated graded of  $\text{Cl}(M, q)$  is isomorphic to  $\Lambda^* M$  as graded  $A$ -algebras.*

*Proof sketch.* If  $m \in M$ , then  $[m] \in F^1/F^0$ , so  $[m]^2 = 0$ , which is the defining relation for an exterior algebra, so we have an algebra map  $i^0: \Lambda^* M \rightarrow \text{gr}^* \text{Cl}(M, q)$  sending  $v \mapsto [v]$ , and one can check this is an isomorphism of graded algebras.  $\square$

**Corollary 14.14.** *If  $M$  is free of rank  $r$ , then  $\text{Cl}(M, q)$  is free of rank  $2^r$ , and  $\text{Cl}^0(M, q)$  is free of rank  $2^{r-1}$ .*

**Definition 14.15.** Let  $B_1$  and  $B_2$  be  $A$ -superalgebras. Then, their *super-tensor product*  $B_1 \otimes B_2$  is their tensor product as  $A$ -modules with the  $A$ -algebra structure satisfying

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{|b'| |a'|} aa' \otimes bb'.$$

Now we have an ambiguity when we write  $\otimes$ , which we resolve as follows: *we only take the super-tensor product of superalgebras unless otherwise specified.*

**Proposition 14.16.** *There's an isomorphism of  $A$ -superalgebras  $\text{Cl}(M_1 \oplus M_2, q_1 \oplus q_2) \cong \text{Cl}(M_1, q_1) \otimes \text{Cl}(M_2, q_2)$ .*

This is one of the advantages of using superalgebra for Clifford algebras, rather than ungraded algebra.

### Spinors.

**Definition 14.17.** Let  $U = U_0 \oplus U_1$  be a *supermodule* over  $A$  (i.e. a  $\mathbb{Z}/2$ -graded module). Its *endomorphism algebra*  $\text{SEnd}(U)$  is the  $A$ -superalgebra of endomorphisms of  $U$ , with the grading  $\text{SEnd}^0(U)$  the parity-preserving endomorphisms and  $\text{SEnd}^1(U)$  the parity-reversing ones.

For example, if we realize endomorphisms as matrices,  $\text{SEnd}^0(U)$  is those which are block diagonal for  $U_0$  and  $U_1$ , and  $\text{SEnd}^1(U)$  is those which are block anti-diagonal.

Now we specialize to a field  $k$  and an extension  $k \hookrightarrow K$ .

**Definition 14.18.** Let  $(V, q)$  be a quadratic vector space over  $k$ , where  $V$  is even-dimensional<sup>21</sup> and  $q$  is nondegenerate. A *spinor module* (defined over  $K$ ) is a  $\text{Cl}(V, q)$ -supermodule  $S = S^+ \oplus S^-$ , i.e. a representation  $\rho: \text{Cl}(V, q) \rightarrow \text{SEnd}_K(S)$ , whose  $K$ -linear extension  $\rho_K: \text{Cl}(V, q) \otimes_k K \rightarrow \text{SEnd}_K(S)$  is an isomorphism of superalgebras.

**Definition 14.19.** Let  $(U, Q)$  be a quadratic  $K$ -vector space, where as before  $U$  is even-dimensional and  $Q$  is nondegenerate. A *polarization* for  $(U, Q)$  is a pair of subspaces  $L, L' \subset U$  such that  $U = L \oplus L'$ ,  $Q|_L = 0$ , and  $Q|_{L'} = 0$ . In this case  $L$  and  $L'$  are called *Lagrangian subspaces*.

The form  $Q$  induces an isomorphism  $L' \cong L^*$ , which induces  $(U, Q) \cong (L \oplus L', \text{ev})$ , where  $\text{ev}$  denotes evaluation.

**Example 14.20.** Over  $\mathbb{C}$ , all quadratic vector spaces of a given dimension are isomorphic, and a polarization exists for the standard one, so polarizations always exist. Over  $\mathbb{R}$ , a polarization exists iff the signature is zero.  $\blacktriangleleft$

The point is that a polarization  $P = (L, L')$  induces a spinor module  $S_P := \Lambda^*(L^*)$ , with the  $\mathbb{Z}/2$ -grading as the even- and odd-graded parts of  $\Lambda^*(L^*)$ . In this case,  $\mu \in L^*$  acts by the *creation operator*

$$(14.21a) \quad c(\mu) := \mu \wedge - \in \text{SEnd}^1(S),$$

and  $\lambda \in L$  acts by the *annihilation operator*

$$(14.21b) \quad a(\lambda) := \iota_\lambda \in \text{SEnd}^1(S).$$

<sup>21</sup>One can work this theory out in odd dimensions, but it's clunkier, and we won't need it.

**Proposition 14.22.** *These satisfy the canonical anticommutation relations*

$$(14.23a) \quad \{c(\mu), c(\mu')\} = 0$$

$$(14.23b) \quad \{a(\lambda), a(\lambda')\} = 0$$

$$(14.23c) \quad \{c(\mu), a(\lambda)\} = \mu(\lambda) \cdot \text{id}.$$

In particular,  $\text{Cl}(V, q)$  acts on  $S$  by  $\rho: \text{Cl}(L \oplus L^*, \text{ev}) \rightarrow \text{SEnd} S$  defined by

$$(14.24) \quad \rho(\lambda, \mu) := c(\mu) - a(\lambda).$$

**Proposition 14.25.**  *$(S, \rho)$  is a spinor module.*

*Proof sketch.* Write  $L = L_1 \oplus \cdots \oplus L_d$  as a sum of lines, so  $L^* = L_1^* \oplus \cdots \oplus L_d^*$  and as quadratic vector spaces,

$$(14.26) \quad (L \oplus L^*, \text{ev}) = \bigoplus_i (L_i \oplus L_i^*, \text{ev}),$$

so if  $S_i = \Lambda^* L_i^*$ , then  $S = \bigotimes_i S_i$ , and

$$(14.27) \quad \text{Cl}(L \oplus L^*, \text{ev}) = \bigotimes_i \text{Cl}(L_i \oplus L_i^*, \text{ev}).$$

(Here we're using the super tensor product.) Now, there are two things to check, both quick: that  $\rho$  is the tensor product of  $\rho_i: \text{Cl}(L_i \oplus L_i^*) \rightarrow \text{SEnd}(S_i)$ , and that the proposition is true for lines.  $\square$

Hence, in particular,  $(\mathbb{R}^{2d}, |\cdot|^2)$  has a spinor module over  $\mathbb{C}$ .

**Definition 14.28.** Let  $A$  be a superalgebra and  $M = M^0 \oplus M^1$  be a supermodule for  $A$ . The *parity change* of  $M$ , denoted  $\Pi M$ , is the  $A$ -supermodule with even part  $M^1$  and odd part  $M^0$ .

**Corollary 14.29.** *Suppose  $(V, q)$  is a quadratic  $k$ -vector space with a polarization  $P$ .*

- (1) *Any finite-dimensional indecomposable  $\text{Cl}(V, q)$ -module is isomorphic to  $S_P$  as ungraded modules.*
- (2) *Any finite-dimensional indecomposable  $\text{Cl}(V, q)$ -supermodule is isomorphic to  $S_P$  or  $\Pi S_P$ .*

This is reminiscent of matrix algebras, whose only finite-dimensional indecomposables are isomorphic to the defining representations given by matrix multiplication. Thus polarized Clifford algebras are some kind of  $\mathbb{Z}/2$ -graded versions of matrix algebras.

**Projective actions.** Let  $S$  be a spinor bundle (over  $K$ ) for  $\text{Cl}(V, q)$  (over  $k$ ). Then, there's a *projective action* of  $\text{O}(V, q)$  on  $S$ , i.e. a map

$$(14.30) \quad \Theta: \text{O}(V) \longrightarrow \text{PGL}_K(S) := \text{Aut}_K(S)/K^\times.$$

We construct this in two steps. First, we have the map  $\text{O}(V, q) \rightarrow \text{Aut}(\text{Cl}(V, q)) \otimes K$ , which we denoted by  $g \mapsto \text{cl}(g)$ . Akin to the fact that all automorphisms of a matrix algebra are inner, given a  $g \in \text{O}(V, q)$ , there's an  $F(g) \in \text{Cl}(V, q)^\times$  such that for all  $a$ ,

$$(14.31) \quad \text{cl}(g)(a) = F(g)aF(g)^{-1},$$

and  $F(g)$  is well-defined modulo scalars. Hence we have a map  $F: \text{Cl}(V, q) \rightarrow \text{Cl}(V, q)^\times/K^\times$  such that  $\text{cl}(g) = \text{Ad } F(g)$ . Then, we define  $\Theta = \rho \circ F: \text{O}(V, q) \rightarrow \text{Aut}_K(S)/K^\times$ .

Correspondingly, there's a projective action of the orthogonal Lie algebra. This time, the intermediary is the Lie algebra  $\text{Der } \text{Cl}(V, q)$  of *derivations*:<sup>22</sup>

$$(14.32) \quad \delta: \mathfrak{o}(V, q) \longrightarrow \text{Der } \text{Cl}(V, q).$$

Just as  $\text{cl}(g) \in \text{Aut } \text{Cl}(V, q)$  is inner for a  $g \in \text{O}(V, q)$ ,  $\delta(\xi)$  for  $\xi \in \mathfrak{o}(V, q)$  is also inner: there's some  $f(\xi) \in \text{Cl}(V, q)$  such that  $\delta(\xi) = [f(\xi), -]$ ,<sup>23</sup> and again  $f(\xi)$  is well-defined mod  $K$ . Normalizing, we have a map

$$(14.33) \quad f: \mathfrak{o}(V) \longrightarrow [\text{Cl}(V, q), \text{Cl}(V, q)] \cong \text{Cl}^0(V, q)/K$$

characterized by  $\xi(v) = [f(\xi), v]$  for  $v \in V$  and  $\xi \in \mathfrak{o}(V)$ , and  $f$  is uniquely defined.

<sup>22</sup>This is elaborated on in the professor's notes.

<sup>23</sup>The Lie bracket on  $\text{Cl}(V, q)$ , as in any associative algebra, is  $[A, B] := AB - BA$ .

**Theorem 14.34.** Let  $(e_1, \dots, e_{2n})$  be a basis for  $V$  and  $\xi \in \mathfrak{o}(V)$ . Then,

$$f(\xi) = \frac{1}{4} \sum_{i,j} \xi_{ij} e_i \cdot e_j \in \text{Cl}^0(V, q).$$

This formula will come back in a crucial way later when we want to show the Lichnerowicz formula which guarantees the compactness of the Seiberg-Witten moduli space, and is the reason the  $1/4K$  appears in it.

**Spin groups.** Let  $(V, q)$  be a quadratic vector space over a field  $k$ , with  $q$  nondegenerate. Then,  $\text{Cl}(V, q)^\times$  acts on  $\text{Cl}(V, q)$  by conjugation:  $u \cdot v = uvu^{-1}$ .

**Definition 14.35.** The *Clifford group*  $G$  is the normalizer of  $V \subset \text{Cl}(V, q)^\times$  in  $\text{Cl}(V, q)$ .

Explicitly,  $G = \{g \in \text{Cl}(V, q)^\times \mid gvg^{-1} \in V \text{ for all } v \in V\}$ . The action map  $G \rightarrow \text{Aut } V$  actually factors through a map  $\alpha: G \rightarrow \text{O}(V, q)$ : if  $u \in V$  and  $q(u) \neq 0$ , then  $-\alpha(u)$  is reflection across the hyperplane  $u^\perp$ . In particular, since  $\text{O}(V, q)$  is generated by reflections,  $\alpha$  is surjective, and fits into a short exact sequence

$$(14.36) \quad 1 \longrightarrow K^\times \longrightarrow G \xrightarrow{\alpha} \text{O}(V, q) \longrightarrow 1.$$

Let  $G^+ := G \cap \text{Cl}^-(V, q)$ , products of even numbers of vectors that are in  $G$ . In this case  $\alpha$  factors through  $\text{SO}(V, q)$  and fits into a short exact sequence

$$(14.37) \quad 1 \longrightarrow K^\times \longrightarrow G^+ \xrightarrow{\alpha} \text{SO}(V, q) \longrightarrow 1.$$

This is nearly the spin group. If  $\beta$  denotes the prime anti-automorphism of  $\text{Cl}(V, q)$ , then

$$(14.38) \quad \beta(g)g = v_r \cdots (v_1 v_1) \cdots v_r \in K^\times.$$

**Definition 14.39.** The *spinor norm*  $\nu: G \rightarrow K^\times$  is the map sending  $g \mapsto \beta(g)g$ .

**Definition 14.40.** The *spin group*  $\text{Spin}(V, q) := G^+ \cap \ker \nu$ .

This is a central extension

$$(14.41) \quad 1 \longrightarrow \{\pm 1\} \longrightarrow \text{Spin}(V, q) \xrightarrow{\alpha} \text{SO}(V, q) \longrightarrow 1.$$

That is, this is a short exact sequence and  $\{\pm 1\} \in Z(\text{Spin}(V, q))$ .

Since we've defined  $\text{Spin}(V, q)$  by algebraic equations, it's an algebraic group. Therefore there's a corresponding Lie algebra  $\mathfrak{spin}(V, q)$ , and  $D(\alpha): \mathfrak{spin}(V, q) \rightarrow \mathfrak{so}(V, q)$  is an isomorphism. In fact, you can realize  $\mathfrak{spin}(V, q)$  inside  $(\text{Cl}(V, q), [-, -])$  as the image of  $f: \mathfrak{o}(V) \rightarrow \text{Cl}^0(V, q)$ , just as  $\mathfrak{o}(V, q)$  sits inside the matrix algebra for  $V$ .

Recall that the projective representation  $\Theta: \text{SO}(V, q) \rightarrow \text{Aut}_K(S)/K^\times$  doesn't lift. But it does lift to a representation  $\rho$  of the spin group: the following diagram commutes.

$$(14.42) \quad \begin{array}{ccc} \text{Spin}(V, q) & \xrightarrow{\rho} & \text{Aut}_K(S) \\ \downarrow & & \downarrow \\ \text{SO}(V, q) & \xrightarrow{\Theta} & \text{Aut}_K(S)/K^\times. \end{array}$$

In some sense, this is the reason to consider the spin group.

Lecture 15.

## Spin groups and spin structures: 3/8/18

Let  $V$  be a real vector space of dimension  $n$  and  $\langle \cdot, \cdot \rangle$  be a positive definite inner product on  $V$ . Last time, we constructed the group  $\text{Spin}(V) \subset \text{Cl}^0(V, \|\cdot\|^2)^\times$ , as the group of even-length products of unit vectors in  $V$ .

Since this is cut out by algebraic equations, it's an algebraic group, but it's also a Lie group:  $\text{Cl}^0(V, \|\cdot\|^2)^\times$  is a Lie group, and  $\text{Spin}(V)$  is a closed Lie subgroup.

Last time, near the end, we constructed its Lie algebra  $\mathfrak{spin}(V, \|\cdot\|^2)$  as a subalgebra of  $\text{Cl}^0(V, \|\cdot\|^2)$  with the commutator and an action  $f: \mathfrak{o}(V) \rightarrow \mathfrak{spin}(V, \|\cdot\|^2)$ , describing the infinitesimal action of a  $\xi \in \mathfrak{o}(V)$  by derivations on the Clifford algebra.

The spin group acts on  $V$  through inner automorphisms of  $\text{Cl}(V, \|\cdot\|^2)$ , and this defines a short exact sequence (14.41). Since the inner product is positive definite,  $\text{SO}(V)$  is compact, and therefore  $\text{Spin}(V)$  is a compact Lie group, since it's a two-fold cover of another compact Lie group.

Inside  $\text{O}(V)$ , we can consider the subgroup of things hit by the exponential map

$$(15.1) \quad \exp \xi := \sum_{n=0}^{\infty} \frac{1}{n!} \xi^n \in \text{End } V.$$

The image of  $\exp$  is precisely  $\text{SO}(V)$ .

**Proposition 15.2.** *In a similar way,  $\exp: \mathfrak{spin}(V) \rightarrow \text{Spin}(V)$  is an isomorphism.*

*Proof sketch.* By (14.41),  $\mathfrak{spin}(V) \cong \mathfrak{so}(V)$ , and the only obstruction is the kernel of  $\alpha$ . One has to show that  $-1 \in \text{Im}(\exp)$ , but if  $e_1$  and  $e_2$  are orthonormal in  $V$ , then  $-1 = \exp((\pi/2)[e_1, e_2])$ , so we're good.  $\square$

**Corollary 15.3.**  *$\text{Spin}(V)$  is connected, and  $\alpha: \text{Spin}(V) \rightarrow \text{SO}(V)$  admits no continuous section. Therefore  $\text{Spin}(V)$  is characterized as the unique connected Lie group  $G$  (up to isomorphism) admitting a double cover Lie group homomorphism  $G \rightarrow \text{SO}(V)$ .*

So you could skip all of the Clifford algebra stuff and use this to *define* the spin group; then you have a handle on the topology of the spin group, but when you want to do differential geometry Clifford algebra is unavoidable. Uniqueness follows because  $\pi_1(\text{SO}_n) = \mathbb{Z}$  when  $n = 2$  and  $\mathbb{Z}/2$  when  $n > 2$ .

We'll let  $\text{Spin}_n := \text{Spin}(\mathbb{R}^n)$ .

**Spinors.** Suppose  $n = 2m$ . Last time, we constructed a representation  $\rho: \text{Cl}(V, \|\cdot\|^2) \otimes \mathbb{C} \rightarrow \text{SEnd } S$ , with  $S = S^+ \oplus S^-$  and  $\dim S^\pm = 2^{m-1}$ . Therefore we get two representations

$$(15.4) \quad \rho^\pm: \text{Spin}_{2m} \longrightarrow \text{Aut}_{\mathbb{C}}(S^\pm).$$

These representations are irreducible.

If  $n = 2m + 1$ , then we have a single, non-super representation  $\rho: \text{Cl}(V, \|\cdot\|^2) \otimes \mathbb{C} \rightarrow \text{End } S$ , and therefore a single irreducible representation of the spin group in this dimension. We can't pick a polarization, but we can split  $V_{\mathbb{C}} = \mathbb{C} \oplus (L \oplus L^*)$ , a polarization plus an extra line. If  $S' := \Lambda^* L^*$ , then

$$(15.5) \quad \text{Cl}(V) \cong \text{Cl}(\mathbb{C}) \otimes \text{SEnd}(S').$$

Explicitly,  $D := \text{Cl}(\mathbb{C}) = \mathbb{C}[\varepsilon]/(\varepsilon^2 + 1)$ , where  $\varepsilon$  is odd. Therefore

$$(15.6) \quad \text{Cl}(V) \cong D \otimes \text{SEnd}(S') \cong \text{SEnd}_D(D \otimes S').$$

Therefore if  $S := D \otimes S'$ , the representation is via endomorphisms of a spinor module, but we're over  $\text{Cl}(\mathbb{C})$  instead of  $\mathbb{C}$ .

*Remark.* The spinor representations are the unique irreducible representations of the Clifford algebra over  $\mathbb{C}$ , but there are additional distinct ones over  $\mathbb{R}$ . This is related to orientations.  $\blacktriangleleft$

**Definition 15.7.** Let  $(S, \rho)$  be a (super)representation of  $\text{Cl}(V, \|\cdot\|^2)$ . A Hermitian inner product  $\langle \cdot, \cdot \rangle$  on  $S$  is called *spin* if for all  $v \in V$ ,  $\rho(v) \in \mathfrak{u}(S)$ , i.e.

$$\langle \rho(v)s_1, s_2 \rangle + \langle s_1, \rho(v)s_2 \rangle = 0.$$

**Proposition 15.8.** *If  $\langle \cdot, \cdot \rangle$  is a spin inner product, then  $\rho: \text{Spin}(V) \rightarrow \text{SEnd}(S)$  is a unitary representation with respect to  $\langle \cdot, \cdot \rangle$ .*

*Proof.* Suppose  $g = v_1 \cdots v_{2r} \in \text{Cl}^0(V, \|\cdot\|^2)$ . Since the inner product is spin,

$$(15.9) \quad \langle \rho(g)s_1, s_2 \rangle = \langle \rho(v_1 \cdots v_{2r})s_1, s_2 \rangle = -\langle \rho(v_2 \cdots v_{2r})s_1, \rho(v_1)s_2 \rangle.$$

Iterating, we get  $\langle \rho(g)s_1, s_2 \rangle = \langle s_1, \rho(\beta g)s_2 \rangle$ . Therefore  $\langle \rho(g)s_1, \rho(g)s_2 \rangle = \langle s_1, \rho(g \cdot \beta g)s_2 \rangle$ . Since  $g \in \text{Spin}(V)$ ,  $g \cdot \beta g = 1$ .  $\square$

**Corollary 15.10.** *Spin inner products always exist.*

*Remark.* We could also allow products of odd numbers of elements in  $\text{Cl}^0(V, \|\cdot\|^2)$ , defining an analogue of the spin group called the *pin group*  $\text{Pin}^-(V)$ . Then the above proof still applies, but with a few minus signs throughout, and one has to average an inner product over this to obtain a spin inner product.  $\blacktriangleleft$



**Example 15.11** ( $n = 2$ ). There's a double cover  $\mathrm{SO}_2 \rightarrow \mathrm{SO}_2$  defined by angle-doubling, and  $\mathrm{SO}_2$  is connected, so by Corollary 15.3,  $\mathrm{Spin}_2 = \mathrm{SO}_2$ . There are standard representations  $S^+, S^-$  of  $\mathrm{SO}_2 = \mathrm{U}_1$  on  $\mathbb{C}$ , where  $e^{i\theta} \in \mathrm{U}_1$  acts by  $e^{\pm i\theta}$ ; then  $S^- \cong (S^+)^*$ . We will show that these are the spinors.

Let  $S := S^+ \oplus S^-$  with the standard Hermitian inner product. The Clifford map is a map  $\rho: \mathbb{R}^2 \rightarrow \mathfrak{u}(S)$  defined by

$$(15.12) \quad \rho(v) = \begin{pmatrix} 0 & -\rho^-(v) \\ \rho^+(v) & 0 \end{pmatrix},$$

where  $\rho^-(v) = \rho^+(v)^\dagger$  and  $\rho^+(v)^\dagger \rho^+(v) = \|v\|^2 \mathrm{id}_{S^+}$ . Showing this boils down to understanding  $\rho^+: \mathbb{C} \rightarrow \mathrm{Hom}_{\mathbb{C}}(S^+, S^-) = (S^-)^{\otimes 2}$ , which is a  $\mathbb{C}$ -linear isometry.

**Definition 15.13.** Define a *spin structure* on a 2-dimensional oriented inner product space  $V$  (i.e. a *Hermitian line*) is a Hermitian line  $L$  and a  $\mathbb{C}$ -linear isometry  $\rho: V \rightarrow L^{\otimes 2}$ . That is, it's a “square root” of  $V$ .

Given a spin structure, set  $S^- := L$ ,  $S^+ = L^*$ , and construct a Clifford action of  $V$  on  $S^+ \oplus S^-$  as above. One can check that  $S$  is still a spinor representation. In particular, there's an identification  $\mathrm{Spin}(V) \cong \mathrm{U}(L)$ , and  $\alpha: \mathrm{U}(L) \rightarrow \mathrm{SO}(V)$  is the map sending  $g \mapsto \rho \circ g^{\otimes 2} \circ \rho^{-1}$ .

This definition is coordinate-free, so applies equally well to a rank-2 oriented vector bundle with a metric  $V \rightarrow M$ . ◀

**Example 15.14** ( $n = 3$ ). Let  $\mathrm{Sp}_1$  denote the group of unit quaternions in  $\mathbb{H}$ , i.e. those quaternions  $q$  with  $q\bar{q} = 1$ . This acts on  $\mathbb{H}$  by left multiplication; there's an identification  $\mathrm{Sp}_1 \cong \mathrm{SU}_2$  carrying this to the defining representation of  $\mathrm{SU}_2$ .

There is a map  $\beta: \mathrm{Sp}_1 \rightarrow \mathrm{SO}(\mathrm{Im}(\mathbb{H}))$  (here  $\mathrm{Im}(\mathbb{H}) := \mathrm{span}_{\mathbb{R}}\{i, j, k\}$ ) sending  $q$  to the map  $x \mapsto qxq^{-1}$ . The kernel is  $\{\pm 1\}$ , and this is a map of three-dimensional Lie groups, so it's a local isomorphism, and in particular a double cover. Since  $\mathrm{Sp}_1$  is connected, this tells us that  $\mathrm{Spin}_3 \cong \mathrm{SU}_2 \cong \mathrm{Sp}_1$ .

If we think of  $\mathrm{Spin}_3$  as  $\mathrm{SU}_2$ , it's the symmetries of  $\mathbb{C}^2$  with a Hermitian inner product (unitary) and a complex volume form  $\Omega \in \Omega^2(\mathbb{C}^2)^* \setminus 0$  (special). In this case, the spinors are  $S = \mathbb{C}^2$ , which  $\mathrm{Spin}_3$  acts on via the defining representation of  $\mathrm{SU}_2$ . If you think of  $\mathrm{Spin}_3$  as  $\mathrm{Sp}_1$ , the spinor representation is the defining action of  $\mathrm{Sp}_1$  on  $\mathbb{H}$ .

To describe the Lie algebra representation  $\rho: \mathbb{R}^3 \rightarrow \mathfrak{u}(S)$ , we'll hypothesize that this is actually valued in  $\mathfrak{su}(S)$ , the trace-free endomorphisms. We also want the Clifford relations to hold, and a little bit of thought tells you that this is equivalent to  $\rho$  being an isometry, where on  $\mathfrak{su}(S)$ ,

$$(15.15) \quad \langle a, b \rangle := -\frac{1}{2} \mathrm{tr}(ab) = \frac{1}{2} \mathrm{tr}(a^\dagger b).$$

So, given an isometry  $\rho: \mathbb{R}^3 \rightarrow \mathfrak{su}(S)$ , we get a Clifford module, and you can check that it's the spinor module.

Let  $(e_1, e_2, e_3)$  be an orthonormal, positively oriented basis for  $\mathbb{R}^3$  and  $\omega := e_1 e_2 e_3 \in \mathrm{Cl}(\mathbb{R}^3)$ . Then  $\omega$  is central and  $\omega^2 = 1$ , so it acts by  $\pm I$  on  $S$ . We will also hypothesize that  $\rho(\omega) = 1$ , which is an orientation condition.<sup>24</sup>

**Definition 15.16.** Let  $V$  be a 3-dimensional oriented real inner product space. A *spin structure* on  $V$  is the data of a 2-dimensional Hermitian vector space  $S$ , a unit-length volume form  $\Omega \in \Lambda^2(S^*)$ , and an isometry  $\rho: V \rightarrow \mathfrak{su}(S)$  such that if  $(e_1, e_2, e_3)$  is a positively oriented orthonormal basis of  $V$ , then  $\rho(e_1)\rho(e_2)\rho(e_3) = I$ .

Again, this applies without change to oriented 3-dimensional real vector bundles with a metric.

We can also compute  $\mathrm{Spin}V$ , the symmetries of the spin structure. It's the set of pairs  $(g, \tilde{g})$  with  $g \in \mathrm{SO}(V)$  and  $\tilde{g} \in \mathrm{SU}(S)$  such that  $\rho$  intertwines the actions of  $g: V \rightarrow V$  and  $\mathrm{Ad}(\tilde{g}): \mathfrak{su}(S) \rightarrow \mathfrak{su}(S)$ . ◀

**Example 15.17** ( $n = 4$ ). There's an isomorphism  $\mathrm{Spin}_4 \cong \mathrm{SU}_2 \times \mathrm{SU}_2 \cong \mathrm{Sp}_1 \times \mathrm{Sp}_1$ . The identification comes as always through Corollary 15.3: there's a map  $\gamma: \mathrm{Sp}_1 \times \mathrm{Sp}_1 \rightarrow \mathrm{SO}(\mathbb{H})$  sending

$$(15.18) \quad (q_1, q_2) \mapsto (x \mapsto q_1 x q_2^{-1}).$$

This exists because  $\mathrm{Sp}_1$  acts on  $\mathbb{H}$  on the left and on the right by multiplication, and these multiplications commute, so it's a representation of the product. Since  $\mathrm{Sp}_1 \times \mathrm{Sp}_1$  is connected, we map to  $\mathrm{SO}(\mathbb{H})$ , even though *a priori* we were only on  $\mathrm{O}(\mathbb{H})$ .

<sup>24</sup>This is an instance of the warning from earlier, that there are distinct irreducible representations of real Clifford algebras.



What's the kernel of  $\gamma$ ? We need  $q_1x = xq_2$  for all  $x \in \mathbb{H}$ ; setting  $x = 1$ , we need  $q_1 = q_2$ , so we're conjugating by a single quaternion, and  $\ker(\gamma) = \{\pm I\}$  as in the three-dimensional case. Since both  $\mathrm{Sp}_1 \times \mathrm{Sp}_1$  and  $\mathrm{SO}(\mathbb{H})$  are six-dimensional, then  $\gamma$  is a two-to-one cover and exhibits  $\mathrm{Spin}_4 \cong \mathrm{Sp}_1 \times \mathrm{Sp}_1$ .

The spinors are  $S^+ = \mathbb{H}$ , viewed as a two-dimensional complex vector space, and  $S^- = \mathbb{H}$ .  $\mathrm{Sp}_1 \times \mathrm{Sp}_1$  acts on  $S^+$ , resp.  $S^-$  through projection onto the first, resp. second coordinate in the group. These actions preserve the quaternionic structure and the Hermitian metrics.

Now we need the Clifford map  $\rho: \mathbb{R}^4 \rightarrow \mathrm{Hom}_{\mathbb{C}}(S^+, S^-)$ . As before, we'll set  $\rho^-(v) := -\rho(v)^\dagger$ .

The space  $\mathrm{Hom}_{\mathbb{H}}(S^+, S^-)$  has dimension 4 over  $\mathbb{R}$ , and carries a norm (the operator norm) and an orientation. As before, we'll make the ansatz that  $\rho^+: \mathbb{R}^4 \rightarrow \mathrm{Hom}_{\mathbb{H}}(S^+, S^-)$  is an isometry, and then check that this gives you spinors.

**Definition 15.19.** A *spin structure* on an oriented, real, 4-dimensional vector space  $V$  is a pair of rank-2 Hermitian vector bundles  $S^+$  and  $S^-$  with quaternionic structure, and an oriented isometry  $\rho^+: V \rightarrow \mathrm{Hom}_{\mathbb{H}}(S^+, S^-)$ .

As before, this applies to oriented real 4-dimensional vector bundles with a metric.

If  $\mathrm{Spin}V$  denotes the symmetries of the spin structure, then it's identified with pairs  $(g, \tilde{g})$  such that  $g \in \mathrm{SO}(V)$ , and  $\tilde{g}$  has the form

$$(15.20) \quad \tilde{g} = \begin{pmatrix} \mathrm{SU}(S^+) & 0 \\ 0 & \mathrm{SU}(S^-) \end{pmatrix} \subset \mathrm{SU}(S^+ \oplus S^-),$$

such that  $\rho$  intertwines the actions of  $g$  and  $\mathrm{Ad}(\tilde{g})$ . ◀

Lecture 16.

### Spin and $\mathrm{spin}^c$ structures: topology: 3/20/18

The professor will be out of town next week, so we will have to make up those two lectures at some point; details will be in an email.

We've thus far studied spin structures from a differential-geometric perspective; today we will address the topological side.

**Definition 16.1.** Let  $V \rightarrow M$  be a rank- $n$  real vector bundle. A *spin structure* on  $V$  is a pair  $\mathfrak{s} = (P_{\mathrm{Spin}}(V), \tau)$ , where  $P_{\mathrm{Spin}}(V) \rightarrow M$  is a principal  $\mathrm{Spin}_n$ -bundle and  $\tau$  is an isomorphism of vector bundles

$$(16.2) \quad \tau: P_{\mathrm{Spin}}(V) \times_{\mathrm{Spin}_n} \mathbb{R}^n \xrightarrow{\cong} V.$$

A *spin structure on  $M$*  is a spin structure on  $T^*M$ .

Recall that if  $G$  is a Lie group, a *principal  $G$ -bundle*  $\pi: P \rightarrow X$  is data of a space  $P$  with a smooth, free, right  $G$ -action and such that  $\pi$  is the quotient map onto the orbit space. The construction in (16.2) is an example of the *mixing construction*:  $P_{\mathrm{Spin}}(V) \times_{\mathrm{Spin}_n} \mathbb{R}^n$  is defined to be the quotient of  $P_{\mathrm{Spin}}(V) \times \mathbb{R}^n$  by the equivalence relation  $(p, v) \sim (p \cdot g, g^{-1}v)$  for all  $g \in \mathrm{Spin}_n$ ; here  $\mathrm{Spin}_n$  acts on  $\mathbb{R}^n$  through the map to  $\mathrm{SO}_n$ .

A spin structure  $\mathfrak{s}$  induces both an orientation and a Euclidean metric on  $V$ : there's a principal  $\mathrm{SO}_n$ -bundle whose fiber at  $x$  is the  $\mathrm{SO}_n$ -torsor of oriented orthonormal bases for  $V_x$ , namely  $P_{\mathrm{Spin}}(V) \times_{\mathrm{Spin}_n} \mathrm{SO}_n$ , where  $\mathrm{Spin}_n$  acts on  $\mathrm{SO}_n$  through the usual map. However, one often thinks of the orientation and metric as given in advance, in which case a spin structure amounts to a choice of a lift of the principal  $\mathrm{SO}_n$ -bundle of oriented orthonormal frames to a principal  $\mathrm{Spin}_n$ -bundle.

**Definition 16.3.** Let  $(V, \mathfrak{s} = (P_{\mathrm{Spin}}(V), \tau))$  be a vector bundle with a spin structure. Then its *spinor bundle* is the complex vector bundle  $\mathbb{S} := P_{\mathrm{Spin}}(V) \times_{\mathrm{Spin}_n} S$ , where  $S$  is the spinor representation for  $\mathrm{Spin}_n$ .

If rank  $V$  is even, there's a Hermitian splitting  $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$  and a Clifford representation  $\rho: V \rightarrow \mathfrak{u}(\mathbb{S})$  which is odd: it splits as

$$(16.4) \quad \rho^\pm := \rho|_{\mathbb{S}^\pm}: \mathbb{S}^\pm \rightarrow \mathbb{S}^\mp,$$

and  $\rho^-(v) = -\rho^+(v)^\dagger$ .

Last time, in Definitions 15.13, 15.16 and 15.19, we gave more geometric definitions of spin structures in low dimensions. These definitions apply to vector bundles more generally, and agree with the more topological definition given today.

**Example 16.5.** Let  $V \rightarrow M$  be an oriented rank-2 bundle, which is equivalent to a Hermitian line bundle. Then a spin structure is equivalent to a square root of  $V$  as a line bundle. This implies in particular that  $\deg V$  must be even, or equivalently,  $w_2(V) = 0$ : the degree is  $\langle c_1(V), [M] \rangle$ , and the mod 2 reduction of  $c_1(V)$  is  $w_2(V)$ .  $\blacktriangleleft$

For Definition 15.16, the rank-2 Hermitian vector bundle we ask for is precisely the spinor bundle  $\mathbb{S} \rightarrow M$ , and the oriented isometry  $\rho: V \rightarrow \mathfrak{su}(\mathbb{S})$  is the spinor representation we mentioned above. For Definition 15.19, the rank-2 Hermitian vector bundles we ask for are  $\mathbb{S}^+$  and  $\mathbb{S}^-$ .

Another perspective on the case  $n = 4$ , which also uses the isomorphism  $\text{Spin}_4 \cong \text{SU}_2 \times \text{SU}_2$ , is that the fiber of  $P_{\text{Spin}}(V)$  at  $x \in M$  is a set  $\{\theta, \theta^+, \theta^-\}$ , where  $\theta: \mathbb{R}^4 \rightarrow V_x$  is an oriented isometry and  $\theta^\pm: \mathbb{H} \rightarrow \mathbb{S}^\pm$  intertwine  $\rho^+$ . These quaternionic structures give you the data of an  $\text{SU}_2 \times \text{SU}_2$ -principal bundle.

**Definition 16.6.** Let  $\mathfrak{s}, \mathfrak{s}'$  be spin structures on  $V$ . An *isomorphism of spin structures*  $\mathfrak{s} \rightarrow \mathfrak{s}'$  is an isomorphism of their principal  $\text{Spin}_n$ -bundles lying over the identity map of the principal  $\text{SO}_n$ -bundle of oriented orthonormal frames.

**Proposition 16.7.** Suppose  $V$  has a spin structure. Then the isomorphism classes of spin structures for  $V$  form an  $H^1(M; \mathbb{Z}/2)$ -torsor, i.e.  $H^1(M; \mathbb{Z}/2)$  acts transitively on the set of spin structures for  $V$ .

*Proof.* Let  $\mathfrak{s}, \mathfrak{s}'$  be spin structures on  $V$ . We want a  $\text{Spin}_n$ -equivariant isomorphism  $\varphi: P_{\text{Spin}}(V) \rightarrow P'_{\text{Spin}}(V)$  fitting into the triangle

$$(16.8) \quad \begin{array}{ccc} P_{\text{Spin}}(V) & \xrightarrow[\cong]{\varphi} & P'_{\text{Spin}}(V) \\ & \searrow & \swarrow \\ & P_{\text{SO}}(V) & \end{array}$$

In the fibers over an  $x \in M$ , (16.8) asks for an isomorphism of double covers over  $(P_{\text{SO}}(V))_x$ , and there are two of these. Varying  $x$ , we get a double cover  $\text{iso}(\mathfrak{s}, \mathfrak{s}')$ , and a global isomorphism is a section of this cover. The obstruction to this is the monodromy of this double cover, which is a map  $\pi_1(M) \rightarrow \mathbb{Z}/2$ , hence an element  $\mathfrak{s}' - \mathfrak{s} \in H^1(M; \mathbb{Z}/2)$ .

Next we define the freely transitive action of  $H^1(M; \mathbb{Z}/2)$  on the set of spin structures. Let  $\mathfrak{s}$  be a spin structure,  $\mathfrak{U}$  be a good cover of  $M$ , and  $\delta \in \check{C}^1(\mathfrak{U}; \{\pm 1\})$  be a Čech cocycle. If we trivialize  $P_{\text{Spin}}(V)$  over each  $U \in \mathfrak{U}$ , we obtain transition functions  $\chi_{UV}: U \cap V \rightarrow \text{Spin}_n$  for every pair of intersecting  $U, V \in \mathfrak{U}$ . Then, the collection  $\delta_{UV}\chi_{UV}$  defines transition functions for a new principal  $\text{Spin}_n$ -bundle, which defines an  $H^1(M; \mathbb{Z}/2)$ -action. The previous argument showed that it's freely transitive.  $\square$

Now we address existence.

**Theorem 16.9.** Let  $V \rightarrow M$  be an oriented vector bundle,  $\mathfrak{U}$  be a good cover of  $M$ , and  $\omega \in \check{C}^2(\mathfrak{U}; \mathbb{Z}/2)$  be a representative for  $w_2(V)$ . Then a spin structure on  $V$  is the same thing as a trivialization of  $w_2$ , meaning that

- (1) a spin structure exists iff  $w_2(V) = 0$ ,
- (2) a 1-cochain  $\eta$  such that  $\delta\eta = \omega$  defines a spin structure  $\mathfrak{s}_\eta$ , and
- (3) if  $\eta, \eta'$  are 1-cochains with  $\delta\eta = \omega = \delta\eta'$ , then  $\mathfrak{s}_\eta - \mathfrak{s}_{\eta'} = [\eta - \eta'] \in H^1(M; \mathbb{Z}/2)$ . Moreover, a 0-cochain  $\zeta$  with  $\delta\zeta = \eta - \eta'$  determines an isomorphism  $\mathfrak{s}_\eta \xrightarrow{\cong} \mathfrak{s}_{\eta'}$ .

We won't give the proof in detail, but we should say something about why the obstruction is  $w_2$ . Locally there's no obstruction, so we can find a spin structure on  $V|_U$  for each  $U \in \mathfrak{U}$ , which is unique up to isomorphism (since  $H^1(U; \mathbb{Z}/2) = 0$ ). We would like to assemble them by choosing isomorphisms  $\theta_{UW}$  on intersections  $U \cap W$ , and set  $\theta_{WU} = \theta_{UW}^{-1}$ . For coherence, we need

$$(16.10) \quad \omega_{U_1 U_2 U_3} := \theta_{U_3 U_1} \theta_{U_2 U_3} \theta_{U_1 U_3}$$

to be a 2-cocycle.

Once we do that, we can check that the cohomology class of  $\omega$  is natural under pullback and is stable under adding a trivial bundle, which implies it's a stable characteristic class. In  $H^2(\text{BSO}; \mathbb{Z}/2)$ , there's only 0 and  $w_2$ , and we saw that there are bundles without spin structures, so it's  $w_2$ .

*Remark.* If you restrict to tangent bundles, this proof is a lot harder — but working with all vector bundles gives us access to the characteristic-class argument outlined above, which is really nice. ◀

Now we'll talk about spin<sup>c</sup> structures, and why they exist on tangent bundles to 4-manifolds.

**Definition 16.11.** Let  $(V, \langle \cdot, \cdot \rangle)$  be a real inner product space (so  $\langle \cdot, \cdot \rangle$  is positive definite). The *spin<sup>c</sup> group*  $\text{Spin}^c(V, \langle \cdot, \cdot \rangle)$  is the subgroup of  $\text{Cl}^0(V \otimes \mathbb{C})^\times$  generated by the scalars  $U_1$  and  $\text{Spin}(V, \langle \cdot, \cdot \rangle)$ .

If  $V = \mathbb{R}^n$  with the usual inner product, we let  $\text{Spin}_n^c := \text{Spin}^c(\mathbb{R}^n, \cdot)$ .

Inside the complexified Clifford algebra,  $\text{Spin}(V, \langle \cdot, \cdot \rangle) \cap U_1 = \{\pm 1\}$ , so

$$(16.12) \quad \text{Spin}^c(V, \langle \cdot, \cdot \rangle) = (\text{Spin}(V, \langle \cdot, \cdot \rangle) \times U_1) / \{\pm 1\}.$$

Thus, for example, we have a concrete model of  $\text{Spin}_4^c$  as  $(\text{SU}_2 \times \text{SU}_2 \times U_1) / \{\pm 1\}$ . There is also a short exact sequence

$$(16.13) \quad 1 \longrightarrow U_1 \longrightarrow \text{Spin}^c(V, \langle \cdot, \cdot \rangle) \longrightarrow \text{SO}(V, \langle \cdot, \cdot \rangle) \longrightarrow 1.$$

**Definition 16.14.** Let  $V \rightarrow M$  be a rank- $n$  real vector bundle, a *spin<sup>c</sup> structure*  $\mathfrak{s} = (P_{\text{Spin}^c}(V), \tau)$  is a principal spin<sup>c</sup>-bundle  $P_{\text{Spin}^c}(V) \rightarrow M$  together with an isomorphism

$$(16.15) \quad \tau: P_{\text{Spin}^c}(V) \times_{\text{Spin}_n^c} \mathbb{R}^n \xrightarrow{\cong} V.$$

Why study spin<sup>c</sup> structures? We'll see that there are more of them, and that the obstruction is weaker. This makes invariants defined using spin<sup>c</sup> structures easier to work with than those requiring spin structures. For example, on any simply-connected 4-manifold, there's at most one spin structure (up to isomorphism), and frequently there are none.

Given a spin<sup>c</sup> structure, there's a *spinor bundle*

$$(16.16) \quad \mathbb{S} := P_{\text{Spin}^c}(V) \times_{\text{Spin}_n^c} \mathbb{S},$$

and if  $n$  is even, it splits as  $\mathbb{S}^+ \oplus \mathbb{S}^-$ ; there is a Clifford representation  $\rho: V \rightarrow \mathfrak{u}(\mathbb{S})$ , which has odd parity when  $n$  is even.

Conversely, if  $n = 2m$ , then we can use this to define a spin<sup>c</sup> structure: given Hermitian vector bundles  $\mathbb{S}^+$  and  $\mathbb{S}^-$  of rank  $2^{m-1}$  and a map

$$(16.17) \quad \rho: V \longrightarrow \text{Hom}_{\mathbb{C}}(\mathbb{S}^+, \mathbb{S}^-)$$

such that for all  $x \in M$  and  $e, f \in V_x$ ,

$$(16.18) \quad \rho(e)^\dagger \rho(f) + \rho(f)^\dagger \rho(e) = 2\langle e, f \rangle \text{id}_{\mathbb{S}^+},$$

then we can recover a spin<sup>c</sup> structure on  $V$ . The reason this works is that if  $V$  is a finite-dimensional oriented inner product space, then  $\text{Spin}^c(V)$  is the group of pairs  $(\theta, \tilde{\theta})$  where  $\theta \in \text{SO}(V)$  and  $\tilde{\theta} \in \mathfrak{u}(\mathbb{S})$ , such that the diagram

$$(16.19) \quad \begin{array}{ccc} V & \xrightarrow{\rho} & \mathfrak{u}(\mathbb{S}) \\ \theta \downarrow & & \downarrow \text{Ad}(\tilde{\theta}) \\ V & \xrightarrow{\rho} & \mathfrak{u}(\mathbb{S}) \end{array}$$

commutes.

**Example 16.20.** Let's look at  $n = 2$ . A spin<sup>c</sup> structure on a rank-2 vector bundle  $V$  is a pair of Hermitian line bundles  $L_+$  and  $L_-$  together with a  $\mathbb{C}$ -linear isometry  $V \xrightarrow{\cong} \text{Hom}_{\mathbb{C}}(L_+, L_-)$ . It's in addition a spin structure if we can take  $L_+ = (L_-)^*$ . ◀

The story of existence and uniqueness of spin<sup>c</sup> structures is similar to that of spin structures, but we can reuse some of what we've already done.

**Proposition 16.21.** *If  $V \rightarrow M$  admits a spin<sup>c</sup> structure, then the set of isomorphism classes of its spin<sup>c</sup> structures is an  $H^2(M; \mathbb{Z})$ -torsor.*

One way to see this is that  $H^2(M; \mathbb{Z})$  is isomorphic to the group of isomorphism classes of complex line bundles on  $M$ . The action of a complex line bundle  $L$  on  $\mathfrak{s}$  is by twisting the spinor bundle:  $\mathbb{S} \mapsto L \otimes \mathbb{S}$ .

Let  $\lambda: \text{Spin}_n^c \rightarrow U_1$  send  $(g, z) \mapsto z^2$ ; since  $z$  is only defined up to  $\pm 1$ , we have to take  $z^2$  here. Then the map  $(\alpha, \lambda): \text{Spin}_n^c \rightarrow \text{SO}_n \times U_1$  is a double cover. Since  $U_1 = \text{SO}_2$ , then we have an inclusion as block diagonals  $\text{SO}_n \times U_1 \hookrightarrow \text{SO}_{n+2}$ .

**Proposition 16.22.**  $\text{Spin}_n^c$  fits into a pullback square

$$\begin{array}{ccc} \text{Spin}_n^c & \xrightarrow{2:1} & \text{SO}_n \times U_1 \\ \downarrow & & \downarrow \\ \text{Spin}_{n+2} & \longrightarrow & \text{SO}_{n+2}. \end{array}$$

Given a  $\text{spin}^c$  structure  $\mathfrak{s}$  on  $V$ , we can define an associated line bundle  $L_{\mathfrak{s}} := P_{\text{Spin}^c}(V) \times_{\text{Spin}_n^c} \mathbb{C}$ , where  $\text{Spin}_n^c$  acts on  $\mathbb{C}$  through  $\lambda$ .

**Corollary 16.23.** Let  $L \rightarrow M$  be an oriented, rank-2 vector bundle. Then, a  $\text{spin}^c$  structure  $\mathfrak{s}$  on  $V$  with  $L_{\mathfrak{s}} \cong L$  is equivalent to a spin structure on  $V \oplus L$ , equivalently to a trivialization of  $w_2(V \oplus L)$ .

**Corollary 16.24.** Hence,  $V$  admits a  $\text{spin}^c$  structure iff there's a line bundle  $L$  such that  $w_2(V) = w_2(L)$ , or  $w_2(V) = c_1(L) \bmod 2$ . Equivalently,  $V$  admits a  $\text{spin}^c$  structure iff there's a  $c \in H^2(M; \mathbb{Z}/2)$  such that  $w_2(V) \equiv c \bmod 2$ .

Here is our obstruction to a  $\text{spin}^c$  structure: an integer lift of  $w_2(V)$ . This is easier than the obstruction for a spin structure (which requires  $w_2(V) = 0$ ).

**Theorem 16.25** (Hirzebruch-Hopf). If  $X$  is a closed, oriented 4-manifold, then  $w_2(T^*X)$  admits an integral lift, and hence  $X$  has a  $\text{spin}^c$  structure.

*Remark.* If  $X$  is simply connected (or more generally,  $H_1(X)$  has no 2-torsion), this is trivial, because  $H^2(X) \rightarrow H^2(X; \mathbb{Z}/2)$ . It's also true for non-closed 4-manifolds (this is due to Teichner and someone else). ◀

We now sketch the proof of Theorem 16.25.

**Lemma 16.26.** Let  $Z$  be a finite CW complex and  $r$  denote reduction mod 2. Then there is a short exact sequence of abelian groups

$$0 \longrightarrow r(H^k(Z)_{\text{tors}}) \longrightarrow H^k(Z; \mathbb{Z}/2) \xrightarrow{ev} \text{Hom}(H_k(Z), \mathbb{Z}/2) \longrightarrow 0.$$

The proof follows, with a little care, from the universal coefficient theorem.

**Proposition 16.27.** Let  $M$  be a compact oriented  $n$ -manifold. Under the cup product pairing  $\smile: H^k(M; \mathbb{Z}/2) \otimes H^{n-k}(M; \mathbb{Z}/2) \rightarrow \mathbb{Z}/2$ ,  $r(H^k(M)_{\text{tors}})$  and  $r(H^{n-k}(M))$  are mutual annihilators.

*Proof sketch.* This follows fairly quickly from Lemma 16.26 and Poincaré duality. ◻

*Proof of Theorem 16.25.* The Wu formula guarantees that  $w_2(T^*X) \cdot x = x^2$  for all  $x \in H^2(X; \mathbb{Z}/2)$ . Hence if  $t$  is a torsion integral class,  $w_2(T^*X) \cdot r(t) = r(t)^2 = r(t^2) = 0$  (because  $H^4(X)$  has no torsion). Therefore  $w_2(T^*X)$  annihilates all torsion classes, and therefore has an integral lift. ◻

Lecture 17.

## Dirac operators and the Lichnerowicz formula: 3/22/18

*“Every good differential geometer has their own spelling of Weitzenböck... you occasionally see it spelled correctly.”*

The Lichnerowicz formula that we discuss today is somewhat involved, but very useful — it's important in establishing compactness of Seiberg-Witten moduli spaces, but also for various other things in differential geometry. Therefore we'll look at it in detail.

Recall that on a Riemannian manifold  $(M, g)$ , there's a distinguished connection  $\nabla$  in  $TM$  called the *Levi-Civita connection*, uniquely characterized by the properties that it's orthogonal (i.e. compatible with the metric), meaning

$$(17.1a) \quad \langle \nabla u, v \rangle + \langle u, \nabla v \rangle = d\langle u, v \rangle;$$

and it's torsion-free, i.e.

$$(17.1b) \quad \nabla_u v = \nabla_v u - [u, v] = 0.$$

The concept of torsion of a connection only makes sense on the tangent bundle.

The Levi-Civita connection has a curvature tensor  $R = \nabla \circ \nabla \in \Omega_M^2(\mathfrak{so}(TM))$ ; it lands in  $\mathfrak{so}(TM)$  because it's orthogonal. It has an explicit formula,

$$(17.2) \quad R_{u,v} := \nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u,v]}.$$

With respect to a local coordinate system  $x_1, \dots, x_n$ , the curvature has components

$$(17.3) \quad R_{ijkl} := g(R_{\partial_i, \partial_j}(\partial_k), \partial_l).$$

The group  $S_4$  acts on these components by permuting the indices; under this action,  $R_{ijkl}$  transforms according to the sign representation  $S_4 \rightarrow \{\pm 1\}$ . There's also a cyclic symmetry

$$(17.4) \quad R_{i(jkl)} := R_{ijkl} + R_{iklj} + R_{iljk} = 0,$$

and there's a precise sense in which these are all of the symmetries of  $R_{ijkl}$  on a general Riemannian manifold.

**Definition 17.5.** With  $(M, g)$  and  $\nabla$  as above, let  $E$  be a complex vector bundle with a Hermitian connection and  $\rho: T^*M \rightarrow \mathfrak{u}(E)$  is a Clifford map, i.e.  $\rho(e)^2 = -|e|^2 \text{id}_E$ . A *Clifford connection* is a unitary connection  $\tilde{\nabla}$  on  $E$  for which  $\rho$  is parallel, i.e. for any local section  $e$  of  $T^*M$ ,

$$[\tilde{\nabla}_v, \rho(e)] = \rho(\nabla_v e).$$

**Proposition 17.6.** *If  $\mathbb{S}$  is the spinor bundle of a spin structure on  $M$ , there is a distinguished Clifford connection.*

*Proof sketch.* We work in a local trivialization  $U_\alpha$  of the principal  $\text{Spin}_n$ -bundle associated to the spin structure on  $M$ . This induces a trivialization of  $T^*M$ , so  $\nabla = d + A_\alpha$  for some  $A_\alpha \in \Omega_{U_\alpha}^1(\mathfrak{so}(T^*M))$ . Since the map  $\mathfrak{spin}(T^*M) \rightarrow \mathfrak{so}(T^*M)$  is an isomorphism of Lie algebras, we can lift  $A_\alpha$  to a connection on  $P_{\text{Spin}}(T^*M)$ , and it satisfies the equations needed for it to be a connection. Then, one checks that it's a Clifford connection.  $\square$

*Remark.* There are several different, but equivalent, perspectives on connections. For example, one can think of them as one-forms in principal bundles. From this perspective the proof of Proposition 17.6 is much easier.  $\blacktriangleleft$

We would like a formula for this connection. We need as an ingredient a formula for  $f: \mathfrak{so}(T_x^*M) \rightarrow \mathfrak{spin}(T_x^*M) \subset \mathcal{Cl}^0(T_x^*M, g)$ . Choosing an oriented orthonormal basis  $\{e_1, \dots, e_n\}$  for  $T_x^*M$ , Theorem 14.34 tells us that for a  $\xi \in \mathfrak{so}(T_x^*M)$ ,

$$(17.7) \quad f(\xi) = \frac{1}{4} \sum_{i,j} \langle \xi e_i, e_j \rangle e_i e_j.$$

In local coordinates  $x_1, \dots, x_n$ , let  $\nabla_{\partial_i} := \partial_i + A_i$ , for  $A_i(x) \in \mathfrak{so}(n)$ . Then

$$(17.8) \quad \begin{aligned} \nabla_{\partial_i}^{\text{Spin}} &= \partial_i + f(A_i) \\ &= \partial_i + \frac{1}{4} \sum_{\alpha, \beta} A_i^{\alpha\beta} \rho_\alpha \rho_\beta, \end{aligned}$$

where  $\rho_\alpha := \rho(\partial_\alpha)$  and  $A_i^{\alpha\beta}$  denotes the matrix coefficients for  $A_i$ .

Though the formula in (17.8) might look mysterious, it's really just the consequence of the explicit isomorphism between  $\mathfrak{so}(T_x^*M)$  and  $\mathfrak{spin}(T_x^*M)$  that we established in Theorem 14.34.

**Proposition 17.9.** *Let  $\mathfrak{s}$  be a  $\text{spin}^c$  structure on  $M$  and  $\mathbb{S}$  be its spinor bundle. Then the Clifford connections form an affine space modeled on  $\Omega_M^1(i\mathbb{R})$ .*

In particular, the distinguished connection of Proposition 17.6 isn't unique.

*Proof.* Let  $\tilde{\nabla}$  be a Clifford connection on  $\mathbb{S}$ . Any other unitary connection is of the form  $\tilde{\nabla} + A$  for an  $A \in \Omega_M^1(\mathfrak{u}(\mathbb{S}))$ . If  $\rho$  is parallel, then  $A_v \in \mathfrak{u}(\mathbb{S})$  is an infinitesimal element of the spinors as a representation of  $\text{Cl}^0(T^*M, g)$ . This is an irreducible representation, and its automorphisms are scalars; hence its infinitesimal automorphisms are the Lie algebra  $i\mathbb{R}$ .

It remains to establish existence, but locally we can choose  $\tilde{\nabla} + a \cdot \text{id}_{\mathbb{S}}$  for  $a$  a purely imaginary 1-form, then use a partition of unity to globalize.  $\square$

Recall that given a  $\text{spin}^c$  structure  $\mathfrak{s}$ , we get an associated line bundle  $L_{\mathfrak{s}}$ . A Clifford connection on  $\mathbb{S}$  is equivalent data to a unitary connection  $\tilde{\nabla}^0$  on  $L_{\mathfrak{s}}$ , but their local descriptions are different: because the map  $\text{Spin}_n^c \rightarrow \text{U}_1$  sends  $(g, z) \mapsto z^2$  instead of to  $z$ , the local description is

$$(17.10) \quad (\tilde{\nabla} + a)^0 = \tilde{\nabla}^0 + 2a.$$

Similarly, the curvature transforms as

$$(17.11) \quad F_{(\tilde{\nabla}+a)^0} = F_{\tilde{\nabla}^0} + 2a.$$

Conversely, we can determine the curvature of  $\tilde{\nabla}$  from that of  $\tilde{\nabla}^0$ :

$$(17.12) \quad F_{\tilde{\nabla}} = f \circ R + \frac{1}{2} F_{\tilde{\nabla}^0} \otimes \text{id}_{\mathbb{S}}.$$

This can be computed locally, where  $\mathfrak{s}$  refines to a spin structure, and we can compare  $\tilde{\nabla}$  to  $\nabla^{\text{Spin}}$ .

**Dirac operators** Let  $\mathfrak{s}$  be a  $\text{spin}^c$  structure on a Riemannian manifold  $(M, g)$  with Levi-Civita connection  $\nabla$  and a Clifford connection  $\tilde{\nabla}$ .

**Definition 17.13.** The *Dirac operator* for  $\tilde{\nabla}$  is  $D := \rho \circ \tilde{\nabla}: \Gamma(\mathbb{S}) \rightarrow \Gamma(\mathbb{S})$ , which is a first-order differential operator.

Explicitly,  $\tilde{\nabla}$  turns a section of  $\mathbb{S}$  into an  $\mathbb{S}$ -valued 1-form (a section of  $T^*M \otimes \mathbb{S}$ ), and  $\rho$  turns cotangent vectors into elements of  $\text{End}(\mathbb{S})$ , so we get back a section of  $\mathbb{S}$  by  $f \otimes \phi \mapsto f(\phi)$ .

If  $e^1, \dots, e^n$  is a local orthonormal frame for  $T^*M$ , then the Dirac operator has the formula

$$(17.14) \quad D\phi = \sum_{i=1}^n \rho(e^i) \circ \tilde{\nabla}_{e_i} \phi.$$

*Remark.*

- (1) If  $M$  is even-dimensional, so that  $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$  is  $\mathbb{Z}/2$ -graded, then the Dirac operator is odd, i.e.  $D: \Gamma(\mathbb{S}^{\pm}) \rightarrow \Gamma(\mathbb{S}^{\mp})$ , because  $\rho(e)$  is also odd.
- (2)  $D$  is *formally self-adjoint*, meaning that if  $\phi$  is compactly supported, then

$$(17.15) \quad \int_M g(D\phi, \chi) \, \text{dvol} = \int_M g(\phi, D\chi) \, \text{dvol}.$$

It therefore follows from a general result in index theory that  $D$  must have index zero, which is a little bit uninteresting. Therefore we will consider “half of  $D$ .”  $D|_{\Gamma(\mathbb{S}^-)}$  is formally self-adjoint to  $D|_{\Gamma(\mathbb{S}^+)}$ , and these may have interesting indices.  $\blacktriangleleft$

The symbol of  $D$  is  $\sigma_D(e) = \rho(e)$ , just Clifford multiplication. This means that  $D$  satisfies the abstract definition of a (generalized) Dirac operator, and deserves the name we gave it.

**The Lichnerowicz formula.** We continue to use the Riemannian manifold  $(M, g)$  with a  $\text{spin}^c$  structure  $\mathfrak{s}$ . In this case the Dirac operator satisfies a nice formula.

**Proposition 17.16.**

$$D^2 = \tilde{\nabla}^* \tilde{\nabla} + \frac{1}{4} K \text{id}_{\mathbb{S}} + \frac{1}{2} \rho(F_{\tilde{\nabla}^0}),$$

where  $K$  denotes the scalar curvature and  $\tilde{\nabla}_v^*: \Gamma(\mathbb{S}) \rightarrow \Gamma(\mathbb{S})$  is the formal adjoint to  $\tilde{\nabla}_v$ .

We also need to define  $\rho(F_{\widetilde{\nabla}^0})$ ; it arises from an action  $\rho: \Lambda^2(M) \rightarrow \text{End } \mathbb{S}$  of 2-forms on spinors by

$$(17.17) \quad \rho(e \wedge f) := \frac{1}{2}[\rho(e), \rho(f)].$$

In particular, when  $\mathfrak{s}$  arises from a spin structure and  $\widetilde{\nabla} = \nabla^{\text{Spin}}$ ,

$$(17.18) \quad D^2 = (\nabla^{\text{Spin}})^* \nabla^{\text{Spin}} + \frac{1}{4}K,$$

since  $F_{(\nabla^{\text{Spin}})^0} = 0$ . Scalar curvature comes up a lot in Seiberg-Witten theory, and this is why.

*Remark.* The formula in Proposition 17.16 is known as a *Weitzenböck formula*:  $D^2$  is *a priori* a second-order operator, but with carefully chosen  $D$  and  $\widetilde{\nabla}$ ,  $D^2 - \widetilde{\nabla}^* \widetilde{\nabla}$  is zeroth-order! Weitzenböck formulas like this one are common in differential geometry.  $\blacktriangleleft$

*Proof of Proposition 17.16.* We will work locally near  $x = 0$  in a local geodesic coordinate system  $(x_1, \dots, x_n)$ , letting  $e_i := dx_i(0)$ . Let  $\widetilde{\nabla}_i := \widetilde{\nabla}_{\partial/\partial x_i}$  and  $\rho_i := \rho(dx_i)$ , so that

$$(17.19) \quad \widetilde{\nabla} = \sum_i \widetilde{\nabla}_i \otimes dx_i.$$

It's a fact that we won't prove that

$$(17.20) \quad \widetilde{\nabla}^* = - \sum_i \widetilde{\nabla}_i \otimes \iota \left( \frac{\partial}{\partial x_i} \right) : \Gamma(T^*M \otimes \mathbb{S}) \rightarrow \Gamma(\mathbb{S}).$$

Since we're in geodesic coordinates,  $\nabla_i(dx_j)(0) = 0$ . We also have the formulas

$$(17.21) \quad D = \sum_i \rho_i \nabla_i$$

$$(17.22) \quad \widetilde{\nabla}^* \widetilde{\nabla} = - \sum_i \widetilde{\nabla}_i \widetilde{\nabla}_i.$$

The equation (17.22) is another instance of a general fact: the Laplacian as it naturally arises for us has a minus sign, as it tends to in geometry; for applications in analysis the Laplacian has no minus sign.

Now we compute. We'll use the notation  $\{A, B\} := AB + BA$  for the anticommutator.<sup>25</sup> At 0,

$$\begin{aligned} -\widetilde{\nabla}^* \nabla + D^2 &= \sum_i \widetilde{\nabla}_i \widetilde{\nabla}_i + \sum_{i,j} \rho_i \widetilde{\nabla}_i \circ \rho_j \widetilde{\nabla}_j \\ &= \sum_i \widetilde{\nabla}_i \widetilde{\nabla}_i + \frac{1}{2} \sum_{i,j} \{\rho_i, \rho_j\} \widetilde{\nabla}_i \widetilde{\nabla}_j + \sum_{i,j} \rho_i [\widetilde{\nabla}_i, \rho_j] \widetilde{\nabla}_j + \frac{1}{2} \sum_{i,j} \rho_i \rho_j [\widetilde{\nabla}_i, \widetilde{\nabla}_j]. \end{aligned}$$

Using the Clifford relations,

$$\begin{aligned} &= \sum_i \widetilde{\nabla}_i \widetilde{\nabla}_i - \sum_{i,j} \delta_{ij} \widetilde{\nabla}_i \widetilde{\nabla}_j + \sum_{i,j} \rho_i \rho(\underbrace{\nabla_i dx_j}_{=0}) \widetilde{\nabla}_j + \sum_{i < j} \rho_i \rho_j (F_{\widetilde{\nabla}})_{ij} \\ &= \sum_{i < j} (F_{\widetilde{\nabla}})_{ij} \rho_i \rho_j. \end{aligned}$$

From (17.12),

$$(17.23) \quad F_{\widetilde{\nabla}} = f(R) + \frac{1}{2} F_{\widetilde{\nabla}^0} \otimes \text{id}_{\mathbb{S}},$$

and therefore

$$(17.24) \quad \frac{1}{2} \sum_{i < j} (F^0)_{ij} \rho_i \rho_j = \frac{1}{2} \rho(F^0),$$

<sup>25</sup>From the perspective of superalgebra, we could subsume all commutators and anticommutators in this calculation with supercommutators: we understand the commutator of two odd elements to pick up a sign, and hence become an anticommutator. Here  $\rho_i$  is odd and  $\widetilde{\nabla}_j$  is even.

as in the formula. It remains to establish

$$(17.25) \quad \sum_{i < j} f(R_{ij}) \rho_i \rho_j = \frac{1}{4} \text{KId}_S.$$

Expanding the left-hand side, we have

$$(17.26) \quad \begin{aligned} \sum_{i < j} f(R_{ij}) \rho_i \rho_j &= \frac{1}{4} \sum_{\substack{i < j \\ k, \ell}} R_{\ell k i j} \rho_i \rho_j \rho_k \rho_\ell \\ &= \frac{1}{8} \sum_{i, j, k, \ell} R_{\ell k i j} \rho_i \rho_j \rho_k \rho_\ell. \end{aligned}$$

Knowing that  $R_{\ell(kij)} = 0$ , let's fix  $\ell$  and permute the other coordinates. We define

$$(17.27) \quad S_{ijk\ell} = \sum_{\sigma \in S_3} R_{\ell\sigma(k)\sigma(i)\sigma(j)} \rho_{\sigma(i)} \rho_{\sigma(j)} \rho_{\sigma(k)},$$

and therefore

$$(17.28) \quad \sum_{i, j, k, \ell} R_{\ell k i j} \rho_i \rho_j \rho_k \rho_\ell = \sum_{\ell} \sum_{i < j < k} S_{ijk\ell} \rho_\ell + \sum_{\ell} \sum_{i < j = k} S_{ijk\ell} \rho_\ell.$$

This would probably be the most opaque step in the proof if you were trying to reproduce it yourself — why is it this symmetry of  $R$  that matters?

To understand (17.28), we'll consider its two pieces as two cases. If  $i < j < k$ ,

$$\begin{aligned} S_{ijk\ell} &= \sum_{\sigma \in S_3} R_{\ell\sigma(k)\sigma(i)\sigma(j)} \rho_{\sigma(i)} \rho_{\sigma(j)} \rho_{\sigma(k)} \\ &= \left( \sum_{\sigma \in S_3} \text{sign}(\sigma) R_{\ell\sigma(k)\sigma(i)\sigma(j)} \right) \rho_i \rho_j \rho_k \\ &= (R_{\ell(kij)} - R_{\ell(ijk)}) \rho_i \rho_j \rho_k = 0. \end{aligned}$$

For  $i < j = k$ ,  $\rho_{\sigma(i)} \rho_{\sigma(j)} \rho_{\sigma(k)} = -\rho_{\sigma(i)}$ , so

$$(17.29) \quad S_{ijk\ell} = - \sum_{\sigma \in S_3} R_{\ell\sigma(k)\sigma(i)\sigma(k)} \rho_{\sigma(i)}.$$

Therefore

$$\sum_{i, j, k, \ell} R_{\ell k i j} \rho_i \rho_j \rho_k \rho_\ell = \sum_{\ell} \sum_{i < k} S_{ikk\ell} \rho_\ell.$$

Skipping a few steps, this comes out as

$$\begin{aligned} &= -2 \sum_{i, k, \ell} R_{\ell i k i} \rho_k \rho_\ell \\ &= -2 \sum_{i, k} R_{i k i k}, \end{aligned}$$

which is exactly the scalar curvature. \(\boxtimes\)