M392C NOTES: APPLICATIONS OF QUANTUM FIELD THEORY TO GEOMETRY

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These notes were taken in UT Austin's M392C (Applications of Quantum Field Theory to Geometry) class in Fall 2017, taught by Andy Neitzke. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Thanks to Andy Neitzke for a few corrections.

CONTENTS

Lecture 1.

Donaldson invariants and supersymmetric Yang-Mills theory: 8/31/17

"The wind blowing on it, well, that's not the worst thing that could happen to a pond! Now imagine you have a laser..."

The course website is https://www.ma.utexas.edu/users/neitzke/teaching/392C-qft-geometry/. There are also lecture notes which are hosted at https://github.com/neitzke/qft-geometry, and are currently a work in progress; if you have contributions or improvements, feel free to contribute them, as a pull request or otherwise. (I'm also taking notes, of course, and if you find problems or typos in my notes, feel free to let me know.) There's also a Slack channel for course-related discussions, which may be easier to use than office hours.

There will be exercises in this course, and you should do at least one-fourth of them for the best grade. Of course, you also want to do them in order to gain understanding. Some worked-out computations could be useful for submitting to the professor's lecture notes.

This course will be relatively wide-ranging; today's prerequisites involve some gauge theory, but the next few lectures won't as much.

Suppose you want to study the topology of smooth manifolds X. Surprisingly, it's really effective to introduce a geometrical gadget, e.g. a Riemannian metric g. Using it, we can define the *Laplace operator* on differential forms $\Delta\colon \Omega^k(X)\to\Omega^k(X)$, which has the formula

$$\Delta := \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d},$$

where $\mathrm{d}\colon \Omega^k(X)\to\Omega^{k+1}(X)$ is the de Rham differential, and $\mathrm{d}^*\colon\Omega^{k+1}(X)\to\Omega^k(X)$ is its adjoint in the L^2-inner product on differential forms induced by the metric. Thus d is canonical, but d^* depends on the choice of metric.

Next we consider the equation

$$\Delta\omega = 0. \tag{1.1}$$

This is a linear equation, so its space of solutions $\mathscr{H}_{k,g} := \ker(\Delta\colon\Omega^k\to\Omega^k)$, called the *space of harmonic k-forms*, is a vector space. If X is compact, it's even a finite-dimensional vector space, which is a consequence of the ellipticity of the Laplace operator.[1] Hence we can define a nonnegative integer

$$b_k(X) := \dim \mathscr{H}_{k,g},$$

called the k^{th} *Betti number* of X It's a fact that $b_k(X)$ does not depend on the choice of the metric! Thus they are invariants of the smooth manifold X.

In fact, there's even a categorified version of this. This reflects a recent (last decade or so) trend of replacing numbers with vector spaces, sets with categories, etc.

Theorem 1.2. *If X is compact,[2] there is a canonical isomorphism $\mathscr{H}_{k,g} \cong H^k(X;\mathbb{R})$, where the latter is the singular cohomology of X with coefficients in $\mathbb{R}$.*

This shows $b_k(X)$ doesn't depend on the smooth structure of X, and is even a homotopy invariant. This will not be true for the Donaldson invariants that we'll discuss later.

Exercise 1.3. Work out some of these spaces of harmonic forms for a metric on S^1 and S^2.

You have to choose a metric, and there are more or less convenient ones to pick. But no matter how you change the metric, there will be a canonical way to identify them.[3]

If X is oriented and $4n$-dimensional, there's a small refinement of the middle Betti number b_{2n} and space of harmonic forms $\mathscr{H}_{2n}$. The *Hodge star operator*

$$\star\colon \Omega^p(X) \longrightarrow \Omega^{\dim X - p}(X)$$

is an involution on $\Omega^{2n}(X)$.

Remark. Let's recall the Hodge star operator. This is an operator on differential forms defined using the Riemannian metric satisfying $\star^2 = 1$ in even dimension, and $[\star, \Delta] = 0$. Hence it acts on harmonic forms. On $\mathbb{R}^2$ with the usual metric, $\star(1) = \mathrm{d}x \wedge \mathrm{d}y$, and $\star(f\,\mathrm{d}x) = f\,\mathrm{d}y$. ◀

Hence we can decompose $\Omega^{2n}(X)$ into the (± 1)-eigenspaces of $\star$: let $\Omega^{2n,\pm}(X)$ denote the ± 1-eigenspace for $\star$. Similarly, $\mathscr{H}_{2n}(X)$ splits into $\mathscr{H}_{2n}^{\pm}(X)$. Thus b_{2n} also splits:

$$b_{2n}(X) = b_{2n}^+(X) + b_{2n}^-(X).$$

These spaces and numbers are also topological invariants, and can be understood in that way.

Exercise 1.4. In dimension $4n+2$, the Hodge star squares to -1. You can still extract topological information from this; what do you get?

Linear equations seem to behave more or less the same in all dimensions. But nonlinear equations behave very differently in different dimensions. In the 1980s, Donaldson [6] used nonlinear equations to produce new and interesting invariants of 4-manifolds. Let X be a connected, oriented 4-manifold with a Riemannian metric g.

Fix a compact Lie group G. For Donaldson, $G = \mathrm{SU}(2)$, and it's probably fine to assume that for much of this class. Fix a principal G-bundle $P \to X$. We'll consider connections on P.

Remark. If you don't know what a connection is, that's OK. Locally, a connection on P is represented by a Lie algebra-valued 1-form $A \in \Omega^1_X(\mathfrak{g})$, and has a *curvature 2-form* $F \in \Omega^2_X(\mathfrak{g}_P)$, which locally is written

$$F = \mathrm{d}A + A \wedge A.$$

Because $\mathrm{SU}(2)$ is nonabelian, $A \wedge A$ isn't automatically zero. ◀

Since F is a 2-form and $\dim X = 4$, we can decompose F into its *self-dual part* F^+ and its *anti-self-dual part* F^-, defined by the splitting of Ω^p by the Hodge star.

Exercise 1.5. Show that if you reverse the orientation of X, F^+ and F^- switch.

[1] For a general differential operator on differential forms, nothing like this is true.
[2] Compactness is really necessary for this.
[3] Interesting question: if you change the metric infinitesimally, how does $\mathscr{H}_k$ change?

Donaldson studied the *anti-self-dual Yang-Mills equation* (ASD YM):

$$F^+ = 0. \tag{1.6}$$

By Exercise 1.5, this is not really different than studing the self-dual Yang-Mills equation; the reason one prefers the ASD version is that it occurs more naturally on certain complex manifolds which were test cases for Donaldson theory.

If G is abelian, e.g. U(1), (1.6) is linear. But if G is nonabelian, e.g. SU(2), then (1.6) is nonlinear.

Definition 1.7. The *instanton moduli space* is the space $\mathscr{M}$ of equations on P obeying (1.6), modulo the action of the *gauge group* $\mathscr{G}$, the bundle automorphisms of P.[4]

Exercise 1.8. Show that if $G = \mathrm{U}(1)$, then $\mathscr{M}$ is only governed by linear algebra in that

$$\mathscr{M} \cong H^1(M : \mathbb{R})/H^1(X; \mathbb{Z}).$$

So in this case we don't find anything new, though the way we found it is still interesting.

When G is nonabelian, this is not a vector space. It still has some reasonable structure. We now fix $G = \mathrm{SU}(2)$. In this case, (topological) isomorphism classes of principal SU(2)-bundles are classified by the integers, given by the formula

$$k := \int_X c_2(P) \in \mathbb{Z},$$

where c_2 denotes the second Chern class.

This means the moduli of instantons is a disjoint union over $\mathbb{Z}$ of spaces $\mathscr{M}_k$.

Theorem 1.9. *If $k > 0$ and g is chosen generically, $\mathscr{M}_k$ is a finite-dimensional manifold.*

Hence one could learn topological information about X by studing topological properties of $\mathscr{M}_k$. The first idea would be the Betti numbers, but these turn out not to depend on the smooth structure.

Proposition 1.10. *Assuming $k > 0$ and g is generic,*

$$\dim \mathscr{M}_k = 8k - 3(1 - b_1(X) + b_2^+(X)).$$

But there's more to $\mathscr{M}_k$ than the dimension. Donaldson introduced an orientation on $\mathscr{M}_k$, which is canonically defined (and a lot of hard work!), and one can produce classs $\tau_\alpha \in \Omega^*(\mathscr{M}_k)$ labeled by classes $\alpha \in H_*(X)$. Using these, the *Donaldson invariants* are the real numbers

$$\langle \mathscr{O}_{\alpha_1} \cdots \mathscr{O}_{\alpha_\ell} \rangle := \int_{\mathscr{M}} \tau_{\alpha_1} \wedge \cdots \wedge \tau_{\alpha_\ell} \in \mathbb{R}. \tag{1.11}$$

Theorem 1.12. *If $b_2^+(X) > 1$, the Donaldson invariants are independent of g.*

Moreover, they really depend on smooth information: it's not possible to reconstruct them out of algebraic or differential topology, unlike the Betti numbers. So these are very powerful. Their study is called *Donaldson theory*. One good refernece is Donaldson and Kronheimer's book [7].

Unfortunately, Donaldson theory is technically very hard: the ASD YM equation is hard to study: $\mathscr{M}_k$ is usually noncompact, and (1.11) is an integral over a noncompact space, which is no fun.

What does this have to do with quantum field theory? In 1988, Witten [13], following a suggestion of Atiyah, found an interpretation of the Donaldson invariants in terms of quantum field theory (hence the suggestive notation in (1.11)).

There are many different quantum field theories: the Standard Model describes three of the four fundamental forces of the universe; quantum electrodynamics describes electromagnetism. Witten interpreted the Donaldson invariants in terms of a specific QFT, called "(a topological twist of) $\mathscr{N} = 2$ supersymmetric Yang-Mills theory (SYM) with gauge group SU(2)."

One imagines X to be a "spacetime" or "universe" whose laws of physics are governed by $\mathscr{N} = 2$ supersymmetric Yang-Mills theory, and to compute the Donaldson invariants, one conducts "experimental measurements"

[4] TODO: not sure if I got this right.

(correlation functions). According to the rules of Lagrangian quantum field theory, this means computing an integral over an infinite-dimensional space (which is alarming, but so it goes):

$$\langle \mathscr{O}_\alpha \rangle = \int_{\mathscr{C}} \mathrm{d}\mu\, \Phi_\alpha e^{-S},$$

where

- $\mathscr{C}$ is the *space of fields*, some sort of infinite-dimensional space akin to the space of functions on X or forms on X,
- $S\colon \mathscr{C} \to \mathbb{R}$ is a functional called the *action*,
- $\Phi_\alpha\colon \mathscr{C} \to \mathbb{R}$ is a (set of) *observables*,
- and $\mathrm{d}\mu$ is some measure on $\mathscr{C}$.

In general, computing these correlation functions are very hard,[5] but in $\mathscr{N} = 2$ SYM, Witten found localization, a way to reduce it to Donaldson's integrals over finite-dimensional spaces.

This is undoubtedly cool, and brings geometric topology into quantum field theory, but it does not make it much easier to actually compute Donaldson invariants.

The next step was taken in 1995, by Seiberg and Witten [11, 12], who were interested in a different but related physics problem. They answered a fundamental question about SYM: how it behaves at low energies.

To make an analogy, suppose you have a pond, and you're pond-ering what happens when wind goes across the surface. You're good at physics, so you model the pond as a system of 10^{30} molecules of water and other things, then rent some time on a supercomputer where you model the action on the wind and… somehow this seems wrong. Instead, you model the water and the wind using things like the Navier-Stokes equations. This is not easy, but it's much, much easier.

The idea is there's a "high-energy" description, in terms of 10^{30} particles, but the "low-energy" description[6] involves things like temperature, pressure, liquid, and other things that are hard to define from the high-energy approach. The low-energy picture is very useful for calculations, though if you fire a laser into your pond it wouldn't suffice. Obtaining the description of the low-energy physics from the high-energy physics is typically very hard; in this case, one would have to define temperature and pressure and a lot of things starting from fundamentals. But you just have to do it once, then can apply it to all bodies of water, etc.

Seiberg and Witten applied this to $\mathscr{N} = 2$ SYM with gauge group $\mathrm{SU}(2)$, and showed that its low-energy description is (roughly) $\mathscr{N} = 2$ SYM with gauge group $\mathrm{U}(1)$, coupled to matter (sometimes called monopoles). Since the gauge group is abelian, this is much easier. Now, one can imagine that there's an easier description of the Donaldson invariants in terms of the low-energy theory (though, again, this was not the original intent of Seiberg and Witten), and this is given by the *Seiberg-Witten equations*. They look more complicated but are actually vastly simpler.[7]

In the Seiberg-Witten equations, the fields are

- a connection Θ in a $\mathrm{U}(1)$-bundle $\mathscr{E}$, or equivalently a determinant line of a spin^c-structure, and
- a section ψ of S^+, a spinor bundle associated to a spin^c-structure.

In this case, there's a *Dirac operator* $\not{D}$ and a pairing

$$q\colon S^+ \otimes S^+ \longrightarrow \Lambda^2_+ T^*X.$$

Then, the Seiberg-Witten equations are

$$F^+ = q(\psi, \overline{\psi}) \tag{1.13a}$$

$$\not{D}\psi = 0. \tag{1.13b}$$

Let $\widetilde{\mathscr{M}}$ denote the moduli space of pairs (Θ, ψ) satisfying (1.13) modulo the action of some group. For generic g, this is a compact manifold, so understanding its topology is much easier, and the correlation functions for the low-energy theory can be written as integrals over $\widetilde{\mathscr{M}}$, and there's a simple formula relating these to the correlation functions for the high-energy theory. Once this was realized, there was very rapid progress of its use in applications, though understanding precisely why it's the same came more slowly, beginning from a physical

[5] Unless $\dim X = 0$, where $\mathscr{C}$ is finite-dimensional. We'll talk about this in the next few lectures.

[6] The term "low-energy," despite sounding pejorative, is actually a very useful thing to have.

[7] For a reference, see Morgan [10].

argument by Moore and Witten [9] and proceeding to a very different-looking mathematical proof much more recently.

This is an application of QFT to geometry, as we will study in this course. Somehow the most powerful applications involve taking a low-energy limit, and many of them also involve localization in supersymmetric QFT (from an infinite-dimensional integral to a finite-dimensional one).

We will start more slowly: first considering QFT where $\dim X = 0$, then $\dim X = 1$ (which is quantum mechanics); in these cases, the physics can be made completely rigorous (though it's not necessarily easy). We'll briefly talk about $\dim X = 2$, then jump into $\dim X = 4$.

Lecture 2.

Zero-dimensional QFT and Feynman diagrams: 9/5/17

Last time, we talked about two perspectives on physics, high-energy (or *fundamental*) and low-energy (or *effective*). For example, the high-energy description of a pond is the physics of the 10^{30} or so particles in it, and the low-energy description is the Navier-Stokes equations. We're interested in the relationship between Donaldson theory in the high-energy perspective and Seiberg-Witten theory in the low-energy perspective, which is a story about four-dimensional QFT. But over the next few lectures, we're going to learn about this passage from fundamental to effective in 0-dimensional QFT, one of the few cases where it's known how to make everything rigorous. Nonetheless, it's still an interesting theory, e.g. it has Feynman diagrams.

We also discussed that in the Lagrangian formalism to QFT on a spacetime X, one evaluates integrals over a space $\mathscr{C}(X)$, which is some kind of function space. Hence, it's usually infinite-dimensional, unless $\dim X = 0$. Hence, let's assume $X = \mathrm{pt}$, so $\mathscr{C}(X) = \{X \to \mathbb{R}\} = \mathbb{R}$. There are many choices for $S\colon \mathscr{C} \to \mathbb{R}$,[8] such as

$$S(x) = \frac{m}{2}x^2 + \frac{\lambda}{4!}x^4,$$

where $m, \lambda > 0$. Here m might mean some kind of mass, and λ measures the interaction in the system.

Now we can define something important and fundamental: the *partition function*

$$Z := \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-S(x)}.$$

The *observables* are polynomial functions $f : \mathscr{C} \to \mathbb{R}$, and their *(unnormalized) expectation values* are

$$\langle f \rangle := \int_{-\infty}^{\infty} \mathrm{d}x\, f(x) e^{-S(x)}.$$

We require f to be polynomial so that this integral converges. All of these are functions in m and λ. Also, notice that all of these are completely well-defined; maybe this is a trivial observation, but it won't be true when we ascend to higher dimensions.

Computing these quantities is less trivial. Let's start with Z, or even $Z_0 := Z(m, \lambda = 0)$. This is a Gaussian:

$$Z_0 = \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-mx^2/2} = \sqrt{\frac{2\pi}{m}}.$$

In order for this to be well-defined, we need $m = 0$ of course, but there's a physical reason to throw out this case, as it corresponds to a system with more than one vacuum state and a degenerate critical point of the action.

To compute the partition function for $\lambda > 0$, we're not sure how to directly evaluate the integral, but we can try to expand it out as a Taylor series in λ around 0. This will allow us to understand the system in the presence of weak interactions, which is often exactly what physicists want to know. We'll leave $e^{-mx^2/2}$ alone, since we know how to integrate it exactly. The $\lambda x^4/4!$ term expands to

$$Z(m, \lambda) = \int_{-\infty}^{\infty} \mathrm{d}x \sum_{n=0}^{\infty} \left(-\frac{\lambda}{4!}\right)^n \frac{x^{4n}}{n!} e^{-mx^2/2}.$$

[8] One can also use $\mathbb{C}$-valued actions.

We'd like to switch the sum and integral to obtain

$$= \sum_{n=0}^{\infty} \left(-\frac{\lambda}{4!}\right)^n \underbrace{\int_{-\infty}^{\infty} \frac{x^{4n}}{n!} e^{-mx^2/2}}_{I}, \tag{2.1}$$

but we have to be careful about convergence. If this works, though, the integral I is tractable.

Exercise 2.2. Show that

$$\int_{-\infty}^{\infty} \mathrm{d}x\, x^{2k} e^{-mx^2/2} = \sqrt{\frac{2\pi}{m}} \frac{1}{m^k} \frac{(2k)!}{k!2^k}.$$

Hence, modulo the assumption we made before, if $\widetilde{\lambda} := \lambda/m^2$,

$$\begin{aligned} Z(m,\lambda) &= \sqrt{\frac{2\pi}{m}} \sum_{n=0}^{\infty} \left(-\frac{1}{96}\right)^n \frac{(4n)!}{n!(2n)!} \widetilde{\lambda}^n \\ &= \sqrt{\frac{2\pi}{m}} \left(1 - \frac{1}{8}\widetilde{\lambda} + \frac{35}{384}\widetilde{\lambda}^2 + \cdots + (1390.1)\widetilde{\lambda}^{10} + \cdots\right). \end{aligned} \tag{2.3}$$

This is called the *perturbation series* for this partition function. Though this partition function is a scalar multiple of a Bessel function, often these series are actually divergent for any $\widetilde{\lambda} > 0$. This means the assumption we made in (2.1) was wrong. There's various ways to think about this — if this function did converge to its Taylor series, it would do so in a neighborhood of 0 in $\mathbb{C}$, hence for negative λ. Physically, this doesn't make sense.

Nonetheless, the perturbation series is still useful in those cases.

Definition 2.4. Let $f : \mathbb{R}_+ \to \mathbb{C}$ be a function and $s := \sum_{n=0}^{\infty} c_n t^n$ be a formal series. We say that s is an *asymptotic series* for f as $t \to 0^+$ if for all $N \geq 0$,

$$\lim_{t \to 0^+} t^{-N} \left| f(t) - \left(\sum_{n=0}^{N} c_n t^n \right) \right| = 0.$$

In this case, we write

$$f(t) \underset{t \to 0^+}{\sim} \sum_{n=0}^{\infty} c_n t^n.$$

In particular, this means that

$$\lim_{t \to 0^+} |f(t) - c_0| = 0$$
$$\lim_{t \to 0^+} \frac{1}{t} |f(t) - c_0 t + c_1| = 0,$$

and so on. So even if s doesn't converge, it's still useful, capturing the limits, linear behavior, quadratic behavior, etc., of f. You have encountered other asymptotic series in your life: Stirling's formula for the factorial is an asymptotic series for the gamma function at ∞: it doesn't actually converge in a sensible way, but it captures a lot of useful information.

Proposition 2.5. *The series* (2.3) *is an asymptotic series for the partition function* $Z(m,\lambda)$ *as* $\lambda \to 0^+$.

So it's not equality, but it's a useful and interesting approximation.

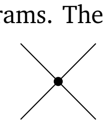
You might wonder whether there's some better series approximating $Z(m,\lambda)$ that actually converges, but this is not true.

Proposition 2.6. *If* f *has a convergent Taylor series at* x_0*, then its Taylor series is an asymptoric series for* f *at* x_0.

Proposition 2.7. *Every smooth function* f *can have at most one perturbation series as* $x \to x_0$.

Sometimes none exists.

We will interpret (2.3) in terms of Feynman diagrams. The basic object is a vertex with four half-edges attached:



A *Feynman diagram* for (2.3) is a placement of some of these vertices and a way of connecting the half-edges. (Feynman diagrams for other systems may look different.)

placeholder

FIGURE 1. Some Feynman diagrams with one or two vertices.

Let D_n denote the set of diagrams with n vertices.

Proposition 2.8. *The number of ways to pair up* $2k$ *objects is* $(2k)!/k!2^k$.

Corollary 2.9.

$$|D_n| = \frac{(4n)!}{(2n)!2^{2n}}.$$

There's also a group action of a group $G_n := (S_4)^n \rtimes S_n$ on D_n, where the i^{th} copy of S_4 permutes the half-edges for the i^{th} vertex, and S_n shuffles the n vertices. In other words, we can restate the asymptotic series for the partition function (2.3) in a more combinatorial manner: since $Z_0 = \sqrt{2\pi/m}$.

$$\frac{Z(m,\lambda)}{Z_0} \sim \sum_{n=0}^{\infty} (-\widetilde{\lambda})^n \frac{|D_n|}{|G_n|}.$$

We want to describe $|D_n|/|G_n|$ as the cardinality of some kind of quotient set, but this is only literally true if the G_n-action on D_n is free. The proper thing to do, as suggested by the orbit-stabilizer theorem, is to sum over orbits, weighted by the order of their stabilizers.[9] Thus

$$\frac{Z(m,\lambda)}{Z_0} \sim \sum_{n=0}^{\infty} (-\widetilde{\lambda})^n \sum_{[\Gamma]\in D_n/G_n} \frac{1}{|\mathrm{Aut}\,\Gamma|}.$$

Since $\widetilde{\lambda} = \lambda/m^2$ and a Feynman diagram in D_n has n^2 edges, we can rewrite (2.3) in a way that is completely a combinatorial sum over Feynman diagrams:

$$\frac{Z(m,\lambda)}{Z_0} \sim \sum_{n\geq 0} \sum_{[\Gamma]\in D_n/G_n} \frac{(-\lambda)^{|V(\Gamma)|}}{m^{|E(\Gamma)|}} \cdot \frac{1}{|\mathrm{Aut}(\Gamma)|}.$$

Here, $V(\Gamma)$ is the set of vertices of Γ, and $E(\Gamma)$ is the set of edges. This leads to the *Feynman rules* for summing over the Feynman diagrams for this theory:

- Draw one representative Γ for each orbit in D_n/G_n.
- Define its weight w_Γ as the product of factors $-\lambda$ for each vertex and $1/m$ for each edge, weighted by $1/|\mathrm{Aut}(\Gamma)|$.

Then,

$$\frac{Z}{Z_0} \sim \sum_{[\Gamma]} w_\Gamma.$$

Example 2.10. Let's calculate some low-order terms.

- The empty Feynman diagram has the weight 1.
- The action of $G_1 \cong S_4$ on D_1 is transitive, so we only need a single representative, such as the "figure-8 diagram." Its stabilizer group has order 8, so there's a contributing factor of $(-\lambda)/8m^2$.
- There are three orbits in D_2/G_2, represented by a graph with zero self-loops, which contributes a term of $\lambda^2/48m^4$, one with one self-loop on each vertex, which contributes $\lambda^2/16m^4$, and one with two self-loops on each vertex, which contributes $\lambda^2/128m^4$.

[9] Another way to think about this is to consider the quotient *groupoid* D_n/G_n, and sum over it in the groupoid measure, which amounts to the same thing.

Thus, the perturbative expansion is

$$\begin{aligned}\frac{Z}{Z_0} &\sim 1 - \frac{\lambda}{8m^2} + \frac{\lambda^2}{48m^4} + \frac{\lambda^2}{16m^4} + \frac{\lambda^2}{128m^4} + O(\lambda^3)\\ &= 1 - \frac{\lambda}{8m^2} + \frac{35}{384}\frac{\lambda^2}{m^4} + O(\lambda^3).\end{aligned}$$

The higher-order terms correspond to diagrams with 3 or more vertices. ◂

If you know the automorphism group of a diagram Γ, then the automorphism group of $\Gamma \amalg \Gamma$ is very similar: a copy of Aut(Γ) for each component, plus the S_2 switching them. If you follow your nose in this line of thought, you can determine the sum in terms of only nonempty, connected diagrams.

Proposition 2.11.

$$\sum_\Gamma w_\Gamma = \exp\left(\sum_{\Gamma \text{ connected, nonempty}} w_\Gamma\right).$$

This suggests that $\log(Z/Z_0)$ is an important physical quantity, and indeed, it's called the *free energy* of the system, as in statistical mechanics. We'd like to say that

$$\log\left(\frac{Z(m,\lambda)}{Z_0}\right) \sim \sum_{\Gamma \text{ connected, nonempty}} w_\Gamma,$$

though there's an analysis argument to check here.

Now we want to compute expectation values. Let's start with

$$\langle x^k \rangle := \int_{-\infty}^{\infty} x^n e^{-S}\,\mathrm{d}x.$$

If k is odd this is 0, but for k even, we can compute an asymptotic series for this function with a similar sum over Feynman diagrams, but with different rules:

- In addition to the 4-valent vertices from before, each diagram must have exactly k univalent vertices.
- We only consider automorphisms which fix these vertices.

You can work this out with a similar argument as for Z/Z_0.

To compute the *normalized expectation values* $\langle x^k \rangle / Z$, use the same diagrams, but with the rule that every connected component of Γ must have at least one univalent vertex. You can then draw out the first few diagrams and conclude things such as

$$\frac{\langle x^2 \rangle}{2} \sim \frac{1}{m} - \frac{\lambda}{2m^3} + O(\lambda^2).$$

More generally, there's no need to constrain ourselves to a quartic interaction: we can isntead consider the action

$$S = \frac{m}{2}x^2 + \sum_{k=3}^{\infty} \frac{\lambda_k x^k}{k!}. \tag{2.12}$$

In this case, we consider Feynman diagrams with vertices of aribitrary valence ≥ 3, and sum with the rules that an edge contributes $-1/m$ and an n-valent vertex contributes $-\lambda_n$. We can actually carry out the analysis even if (2.12) doesn't converge (in which case we don't get an asymptotic series for a function, but that's OK). Anyways, tabulating the Feynman diagrams we get the beginning of the normalized perturbative expansion

$$\frac{Z}{Z_0} \sim 1 - \frac{\lambda_4}{8m^2} + \frac{\lambda_3^2}{12m^3} + \cdots$$

Yet another generalization is to consider actions on $\mathscr{C} = \mathbb{R}^N$, rather than $\mathbb{R}$, corresponding to considering the theory on N points, rather than one point. Now, the quartic term is some 4-tensor, so (using the Einstein summation convention) the most general action is

$$S = \frac{1}{2}x^i M_{ij} x^j + \frac{1}{4!} C_{ijk\ell} x^i x^j x^k x^\ell,$$

and Z_0 is again a Gaussian:

$$Z_0 = \int_{\mathbb{R}^n} e^{-x^i M_{ij} x^j/2} = \frac{(2\pi)^{N/2}}{\sqrt{\det M}}.$$

In this case, one can compute with Feynman diagrams again, but this time labeling the edges with labels $1, \dots, N$.

Lecture 3.

A Little Effective Field Theory: 9/7/17

Today, we're going to illustrate the passage from the fundamental to the effective using zero-dimensional QFT: the fundamental theory will be an action $S(x, y)$ in two variables, and its effective theory S_{eff} will be a simpler theory in a single variable.

Last time, we discussed the fields $\mathscr{C} = \mathbb{R}^N$ in a zero-dimensional QFT with an action

$$S := \frac{1}{2} x^i M_{ij} x^j + \frac{1}{4!} C_{ijk\ell} x^i x^j x^k x^\ell.$$

As $C \to 0$, one wants to compute the asymptotic series, which amounts to a sum over Feynman diagrams. In this context, one can sum over unlabeled diagrams Γ, but with the weight incorporating the labels of the half-edges in $\{1, \dots, N\}$. Explicitly, the weight of an edge i to j should be $(M^{-1})^{ij}$, and that of a vertex with half-edges i, j, k, and ℓ is $C_{ijk\ell}$.

More abstractly, if V is a finite-dimensional vector space with a measure μ, you can choose an $M \in \operatorname{Sym}^2 V^*$ and a $C \in \operatorname{Sym}^4 V^*$, and define the action

$$S(x) := \frac{1}{2} M(x, x) + \frac{1}{4!} C(x, x, x, x).$$

Then, one would compute the partition function

$$\int \mathrm{d}\mu \, e^{-S(x)}.$$

Now let's focus on a specific example. We can start with fields $\mathsf{C} = \mathbb{R}^2$ with coordinates x, y and an action

$$S(x, y) := \frac{m}{2} x^2 + \frac{M}{2} y^2, \tag{3.1}$$

which is two uncoupled systems. So let's turn on coupling in (3.1):

$$S(x, y) := \frac{m}{2} x^2 + \frac{M}{2} y^2 + \frac{\mu}{4} x^2 y^2. \tag{3.2}$$

Say that we're actually interested in x: we want to compute Z and $\langle x^n \rangle$, but *not* $\langle y \rangle$ or $\langle f(x, y) \rangle$ that depends on y. This might happen in a system which naturally comes with both x and y, but y is some extra degrees of freedom. We'll see this is natural when $M \gg m$.

There are only a few kinds of labels in the Feynman diagram, because M and C in (3.2) have a lot of zeroes: we'll use a solid line for $1/m$ (corresponding to x^2) and a dashed line for $1/M$ (for y^2); all vertices must have two solid half-edges and two dashed half-edges, weighted by $-\mu$.

Let's compute $\log(Z/Z_0)$; by Proposition 2.11, this allows us to only sum over connected diagrams. There is only one diagram with a single vertex (order μ), and three with two vertices (order μ^2). Their respective computations are

$$\log\left(\frac{Z}{Z_0}\right) \sim -\frac{\mu}{4mM} + \frac{\mu^2}{16m^2M^2} + \frac{\mu^2}{16m^2M^2} + \frac{\mu^2}{8m^2M^2} + O(\mu^3).$$

For correlation functions, we must add n univalent vertices for x^n. The μ^0-term (the "tree level") calculates exactly the noninteracting theory. When we enumerate the diagrams for $\langle x^2 \rangle$, there's one with zero 4-valent vertices, one for a single 4-valent vertex, and three with two 4-valent vertices, and the sum is

$$\frac{\langle x^2 \rangle}{Z} \sim \frac{1}{m} - \frac{\mu}{2m^2M} + \frac{\mu^2}{4m^3M^2} + \frac{\mu^2}{2m^3M^2} + \frac{\mu^2}{4m^3M^2} + O(\mu^3).$$

This is not the logarithm: since we've normalized this calculation, it's a sum over Feynman diagrams for which every connected component contains a univalent vertex.

This explodes more quickly than other ones we considered: to compute $\langle x^4 \rangle$, there are a lot of diagrams to sum over, even just at the μ^2. The answer will be

$$\frac{\langle x^4 \rangle}{Z} \sim \frac{3}{m^2} - \frac{3\mu}{m^3M} + \frac{33\mu^2}{4m^4M^2} + O(\mu^3).$$

And since we only care about x, there should be some way to simplify this and get all of the dashed lines out of the way first. One idea is: if we only want

$$\langle x^n \rangle = \int_{\mathbb{R}^2} \mathrm{d}x\,\mathrm{d}y\, x^n e^{-S(x,y)},$$

then by Fubini's theorem, we can integrate out the dependence on y, defining S_{eff} such that

$$e^{-S_{\text{eff}}(x)} := \int_{\mathbb{R}} \mathrm{d}y\, e^{-S(x,y)}.$$

Then

$$\langle x^n \rangle = \int_{\mathbb{R}} \mathrm{d}x\, x^n e^{-S_{\text{eff}}(x)}.$$

In this particular example, we can compute S_{eff}, or at least its asymptotic series (which suffices if we want to do the asymptotic series for $\langle x^n \rangle$ in the original theory). The answer for the asymptotic series for $\mu \to 0$ is

$$S_{\text{eff}}(x) \sim \frac{m_{\text{eff}}}{2} x^2 + \sum_{k \geq 3} \lambda_k x^k, \tag{3.3}$$

where m_{eff} is some effective mass. The interacting term is interesting — there are interactions between multiple xs (vertices with four solid edges). These arise because of Feynman diagrams such as the one in Figure 2, where by "ignoring y" we close the gap between these two vertices and obtan an interaction between two copies of x.

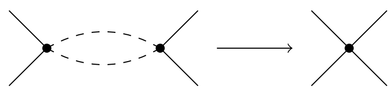


FIGURE 2. Left: a Feynman diagram for the action (3.2). In the effective field theory (3.3), the dashed lines correspond to terms which are integrated out, so this diagram becomes a quartic x-x interaction (on the right).

Specifically, in (3.3), the terms are

$$\begin{aligned} m_{\text{eff}} &= m + \frac{\mu}{2M} \\ \lambda_k &= \begin{cases} 0, & k \text{ odd} \\ -\left(-\dfrac{\mu}{M}\right)^{k/2} \dfrac{1}{2^{k/2+2}k}, & k \text{ even.} \end{cases} \end{aligned}$$

Thus, as $M \to \infty$, $m_{\text{eff}} \to m$: when $M \gg m$, this is a more reasonable approximation.

This is our first baby example of an effective field theory. The fact that we integrated out the degrees of freedom we didn't care about is a useful heuristic to have around.

Symmetries. Let's go back to $\mathscr{C} = \mathbb{R}$ and

$$S = \frac{m}{2}x^2 + \frac{\lambda}{4!}x^4.$$

This is in a sense the simplest nontrivial example: if you had a cubic term instead of a quartic term, $\int e^{-S}$ wouldn't be well-defined (it goes to ∞ as $x \to \pm\infty$).

Proposition 3.4. *$\langle x^n \rangle = 0$ when n is odd.*

Proof.

$$\begin{aligned} \langle x^n \rangle &= \int_{-\infty}^{\infty} \mathrm{d}x\, x^n e^{-S(x)} \\ &= \int_{-\infty}^{\infty} \mathrm{d}(-x)\,(-x)^n e^{-S(-x)} \\ &= (-1)^n \int_{-\infty}^{\infty} \mathrm{d}x\, x^n e^{-S(x)} \\ &= (-1)^n \langle x^n \rangle. \end{aligned}$$

⊠

One takeaway is that this theory is symmetric under the group $\mathbb{Z}/2$ acting on $\mathscr{C}$ as multiplication by $\{\pm 1\}$. This leads to a very general principle.

Proposition 3.5. *Let $S\colon \mathscr{C} \to \mathbb{R}$ and the measure on $\mathscr{C}$ are both G-invariant for a group G, then $\langle \mathscr{O} \rangle = \langle \mathscr{O}^g \rangle$ for any observable $\mathscr{O}\colon \mathscr{C} \to \mathbb{R}$, where $\mathscr{O}^g = g^*\mathscr{O}$.*

If G is a Lie group, we can differentiate this equation: take $g = \exp(tX)$ for some $X \in \mathfrak{g}$: taking

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \bigl(\langle \mathscr{O} \rangle = \langle \mathscr{O}^{tX} \rangle\bigr),$$

we conclude that $\langle X(\mathscr{O}) \rangle = 0$.

In general, symmetries are an extremely important ingredient in QFT.

Fermions and super-vector spaces. You might remember that we wanted to do something topological, but our computations, as functions in the parameters (m, λ), were not deformation-invariant (you could think of them as nonconstant functions on a moduli space of QFTs). To get things that are, we need one more ingredient: fermions.

The way to do this, which will return again and again in this course, is to replace the manifold $\mathscr{C}$ by a supermanifold! Since we've so far only considered vector spaces, we'll get a slightly gentler introduction in the form of super-vector spaces.

For a reference on this material, check out Etingof's course notes for a class on the mathematics of QFT.[10]

Definition 3.6. A *super-vector space* is a $\mathbb{Z}/2$-graded vector space $V = V^0 \oplus V^1$.

For example, if $V^0 = \mathbb{R}^p$ and $V^1 = \mathbb{R}^q$, V is denoted $\mathbb{R}^{p|q}$. This can be done over any field, but we're only going to consider $\mathbb{R}$ or $\mathbb{C}$.

These are not so terrible. But how we do algebra with them is also different: if you are taking tensor products, super-vector spaces are not the same as $\mathbb{Z}/2$-graded vector spaces![11]

Definition 3.7. The symmetric monoidal category of super-vector spaces $(\mathsf{sVec}, \otimes, s_{\text{–},\text{–}})$ is the same as that for ordinary $\mathbb{Z}/2$-graded vector spaces $\mathsf{Vect}^{\mathbb{Z}/2}$, except for the symmetry

$$s_{V,W}\colon V \otimes W \to W \otimes V.$$

For $\mathsf{Vect}^{\mathbb{Z}/2}$, this is the map $v \otimes w \mapsto w \otimes v$, but in sVec, it's defined on homogeneous v, w by

$$v \otimes w \longmapsto (-1)^{|v||w|} w \otimes v,$$

where $v \in V^{|v|}$ and $w \in W^{|w|}$; non-homogeneous elements are sums of homogeneous ones, so this determines $s_{V,W}$.

So the point is if v or w is in V^1, we multiply by -1:

$$s(v \otimes w) = \begin{cases} -w \otimes v & v \text{ or } w \text{ is in } V^1 \\ \ \ w \otimes v, & v, w \in V^0. \end{cases}$$

This category is considerably more useful than it looks. There's a sense in which sVec and $\mathsf{Vect}^{\mathbb{Z}/2}$ are the only two symmetric monoidal structures that can be placed on the monoidal category $(\mathsf{Vect}^{\mathbb{Z}/2}, \otimes)$.

Other algebraic constructions are also different.

Definition 3.8. The *symmetric algebra* on a super-vector space V is the superalgebra ($\mathbb{Z}/2$-graded algebra)

$$\mathrm{Sym}^*(V) := T^*V / \langle v \otimes w - s(w \otimes v) \rangle.$$

Thus, if $V = V^0$, $\mathrm{Sym}^*(V)$ is the usual symmetric algebra, but if $V = V^1$, $\mathrm{Sym}^*(V) = \Lambda^*(V^1)$, the exterior algebra! In general, it'll be a mix of these two things.

We can use this to define polynomial functions: in ordinary algebra, there's a canonical isomorphism between the algebra of polynomials on a vector space V and $\mathrm{Sym}^*(V^*)$.

Definition 3.9. Motivated by this, if $V \in \mathsf{sVec}$, we define its *algebra of polynomial functions* $\mathscr{O}(V)$ to be

$$\mathscr{O}(V) := \mathrm{Sym}^*(V^*).$$

[10] For supermanifolds specifically, see `https://ocw.mit.edu/courses/mathematics/18-238-geometry-and-quantum-field-theory-fall-200` `lecture-notes/sec9.pdf`.

[11] If the base field has characteristic 2, these two notions are actually the same, which quickly follows from Definition 3.7. But this will not be important to us.

Here $V^* := \mathrm{Hom}_{\mathsf{sVec}}(V, \mathbb{R}^{1|0}) = (V^0)^* \oplus (V^1)^*$. $\mathscr{O}(V)$ is itself a super-vector space, in fact a (super)commutative algebra! That is, $p \cdot q = (-1)^{|p||q|} q \cdot p$.

In physics, the even direction corresponds to bosonic stuff, and the odd direction to fermionic stuff. So $\mathscr{C}$ may be a super-vector space, and we can take the action function $S \in \mathscr{O}^0(\mathscr{C})$.

Example 3.10. Let's consider a purely fermionic theory, such as $\mathscr{C} = \mathbb{R}^{0|2}$. Then, $\mathscr{C}$ has coordinate functions $\psi^1, \psi^2 \in \mathscr{O}^1(\mathscr{C})$, which have odd statistics in the sense that

$$\begin{aligned}\psi^1\psi^2 &= -\psi^2\psi^1\\ (\psi^1)^2 &= 0\\ (\psi^2)^2 &= 0.\end{aligned}$$

This, $\mathscr{O}^0(\mathscr{C})$ has basis $\{1, \psi^1\psi^2\}$ and $\mathscr{O}^1(\mathscr{C})$ has basis $\{\psi^1, \psi^2\}$. Thus $\mathrm{Sym}^* \mathscr{C}$ is four-dimensional, which is as expected, since it should be $\Lambda^*\mathbb{R}^2$.

Since there's no quartic terms in ψ^1 and ψ^2, we actually can't introduce interactions, so our action functional is

$$S := \frac{1}{2} M \psi^1 \psi^2. \tag{3.11}$$

This is somewhat like a function, but it behaves very weirdly: $S^2 = 0$!

We'd like to make sense of the partition function in this setting. In order to do this, we need rules for integrating over odd variables. To integrate over $\mathbb{R}^{0|1}$ with odd coordinate ψ, the most general function is $a\psi + b$, so we can stipulate that its integral is

$$\int_{\mathbb{R}^{0|1}} \mathrm{d}\psi\, (a\psi + b) := a.$$

We'll define the exponential via its power series, which means it's much simpler than for bosons!

Now, on $\mathbb{R}^{0|k}$, we have to specify order of integration: to compute

$$\int_{\mathbb{R}^{0|k}} \mathrm{d}\psi^1\, \mathrm{d}\psi^2 \cdots \mathrm{d}\psi^k F = \int_{\mathbb{R}^{0|1}} \mathrm{d}\psi^1 \left(\int_{\mathbb{R}^{0|1}} \mathrm{d}\psi^2 \left(\cdots \int_{\mathbb{R}^{0|1}} F \right) \cdots \right),$$

first evaluate the innermost integral, then the next innermost, and so on, ending at the outermost ($\mathrm{d}\psi^1$ in the above equation).

Hence the partition function is

$$\begin{aligned} Z &= \int_{\mathbb{R}^{0|2}} \mathrm{d}\psi^1\, \mathrm{d}\psi^2 e^{-S(\psi^1, \psi^2} \\ &= \int_{\mathbb{R}^{0|2}} \mathrm{d}\psi^1\, \mathrm{d}\psi^2 \left(1 - \frac{1}{2} M \psi^1 \psi^2\right) \\ &= -\frac{1}{2} M \int_{\mathbb{R}^{0|2}} \mathrm{d}\psi^1\, \mathrm{d}\psi^2\, \psi^1 \psi^2 \\ &= \frac{1}{2} M \int_{\mathbb{R}^{0|2}} \mathrm{d}\psi^1\, \mathrm{d}\psi^2\, \psi^2 \psi^1 \\ &= \frac{1}{2} M. \end{aligned}$$

◀

For bosons (i.e. even fields), we had a Gaussian

$$\int_{-\infty}^{\infty} e^{-Mx^2/2}\, \mathrm{d}x = \frac{\sqrt{2\pi}}{\sqrt{M}}.$$

This is suggestive: if you arrange the masses of bosons and fermions right, things might cancel out to produce a theory whose dependence on the mass cancels out and is deformation-invariant.

Lecture 4.

Supersymmetry in zero dimensions: 9/12/17

We've been doing zero-dimensional quantum field theory, and we will continue to do so today. Last time, we introduced supersymmetry, so $\mathscr{C}$ is a super-vector space. We looked at a particular specific example where $\mathscr{C}$ is odd, e.g. $\mathbb{R}^{0|2}$, which has two odd coordinate functions $\psi^1, \psi^2 \in \mathscr{O}^1(\mathscr{C})$. The total coordinate algebra is $\mathscr{O}(\mathscr{C}) = \Lambda^*(\mathbb{R}^2)$, and the even functions are spanned by $1, \psi^1\psi^2 \in \mathscr{O}^0(\mathscr{C})$. Since ψ^1 and ψ^2 are odd, $\psi^1\psi^2 = -\psi^2\psi^1$.

Let's introduce the action (3.11): since there are only odd terms, there can be no interacting terms, because higher-order powers of ψ^1 and ψ^2 vanish! The partition function is

$$Z = \int_{\mathscr{C}} \mathrm{d}\mu\, e^{-S} = \int_{\mathscr{C}} \mathrm{d}\mu \left(1 - \frac{1}{2} M \psi^1\psi^2\right). \tag{4.1}$$

Last time, we discussed a heuristic way to understand the measure $\mathrm{d}\mu$; today we'll be more explicit.

Definition 4.2. The *parity change operator* $\Pi\colon \mathsf{sVec} \to \mathsf{sVec}$ sends a super-vector space $V = V^0 \oplus V^1$ to the super-vector space with even part V^1 and odd part V^0.

That is, Π just switches the odd and even parts of a super-vector space.

Definition 4.3. A *translation-invariant measure* on an odd super-vector space $V = V^1$ is a $\mathrm{d}\mu \in \Lambda^{\mathrm{top}}(\Pi V)$. For an $f \in \mathscr{O}(V) \cong \Lambda^*((\Pi V)^*)$, let $f^{\mathrm{top}} \in \Lambda^{\mathrm{top}}((\Pi V)^*)$ be its top-degree component; then, the integral of f with respect to $\mathrm{d}\mu$ is

$$\int \mathrm{d}\mu\, f := (\mathrm{d}\mu) \cdot (f^{\mathrm{top}}).$$

There's a one-dimensional space of measures, determined up to a scalar.

Exercise 4.4. For $V = \mathbb{R}^{0|1}$ with odd coordinate ψ, show there's a measure $\mathrm{d}\mu$ on V such that for all $a, b \in \mathbb{R}$,

$$\int \mathrm{d}\mu\, (a\psi + b) = a.$$

This measure is called $\mathrm{d}\psi$. Notice that

$$\int \mathrm{d}\psi\, \psi = 1 \qquad \text{and} \qquad \int \mathrm{d}\psi = 0.$$

The fact that the integral of a constant in an odd direction is 0 is one of the striking features of this "Grassmann integration."

We can also give names to some more measures:

Exercise 4.5. For $V = \mathbb{R}^{0|1}$ and $c \in \mathbb{R}$, show there are measures $c\mathrm{d}\psi$ and $\mathrm{d}(c\psi)$ on V such that

$$\int_V (c\mathrm{d}\psi) f(\psi) = c \int \mathrm{d}\psi\, f(\psi)$$
$$\int_V \mathrm{d}(c\psi) f(c\psi) = \int_V \mathrm{d}\psi\, f(\psi).$$

Then prove the Grassmann change-of-variables formula

$$\mathrm{d}(c\psi) = \frac{1}{c}\mathrm{d}\psi,$$

or equivalently that

$$\int_V \mathrm{d}(c\psi)\, c\psi = 1.$$

Similarly, on $\mathbb{R}^{0|q}$, define $\mathrm{d}\boldsymbol{\psi} = \mathrm{d}\psi^1\, \mathrm{d}\psi^2 \cdots \mathrm{d}\psi^q$ to be the unique measure such that

$$\int_{\mathbb{R}^{0|q}} \mathrm{d}\boldsymbol{\psi}\, \psi^q \psi^{q-1} \cdots \psi^1 = 1.$$

This definitely depends on how the ψ^i are ordered; we'll stick with this convention, which is common in physics.

These behave more like measures than top-degree forms: you need no choice of orientation to integrate. These definitions might be strange, but they're forced on you if you want a good change-of-variables formula.

Now, we know how to calculate the partition function (4.1):

$$Z = \int_{\mathbb{R}^{0|2}} \mathrm{d}\psi \left(1 - \frac{1}{2} M\psi^1\psi^2\right) = \frac{1}{2}M.$$

More fermions. If we add more fermions, we can turn on interactions: if $\mathscr{C}$ is any even-dimensional[12] odd super-vector space with translation-invariant measure $\mathrm{d}\mu$, let

$$M \in \mathrm{Sym}^2(V^*) = \Lambda^2((\Pi V^1)^*)$$
$$C \in \mathrm{Sym}^4(V^*) = \Lambda^4((\Pi V^1)^*).$$

If V is at least four-dimensional, C can be nonzero. In that case we can change the action (3.11) to one with interactions:

$$S := \frac{1}{2}M + \frac{1}{4!}C \in \mathscr{O}(\mathscr{C}).$$

After choosing a basis for ΠV^1, equivalently an isomorphism $V \cong \mathbb{R}^{0|q}$, we can rewrite this in coordinates:

$$S = \frac{1}{2}M_{IJ}\psi^I\psi^J + \frac{1}{4!}C_{IJKL}\psi^I\psi^J\psi^K\psi^L, \tag{4.6}$$

where M_{IJ} is an antisymmetric matrix with real entries, and C_{IJKL} is a totally antisymmetric tensor. Again the partition function is $Z = \int \mathrm{d}\mu\, e^{-S}$, but this time it's possible to evaluate it algebraically.

Exercise 4.7. If $\mathscr{C} = \mathbb{R}^{0|4}$ and

$$S = m\psi^1\psi^2 + m\psi^3\psi^4 + \lambda\psi^1\psi^2\psi^3\psi^4, \tag{4.8}$$

show that

$$Z = m^2 - \lambda.$$

Remark. This is much easier than the bosonic case, where calculations like this flowed through asymptotic series, Feynman diagrams, etc. There is a perturbation-theoretic description of the fermionic case as a Feynman diagram expansion; the rules are quite similar to those for bosons, but with some extra signs. ⋖

In the theory (4.8), $Z_0 = m^2$.

Remark. For the more general theory of the form (4.6), $Z_0 = \mathrm{Pf}(M)$, the *Pfaffian* of the antisymmetric matrix M; this is a number which squares to the determinant. ⋖

To compute Z/Z_0 in (4.8), you can again sum over Feynman diagrams with four-valent vertices, but skew-symmetry introduces a sign rule which forces all Feynman diagrams with more than one vertex to have weight 0.

Bosons and fermions together. Now we consider $\mathscr{C} = V = V^0 \oplus V^1$ with both odd and even parts. We need a theory of integration for such spaces, but that won't be so hard: we'll first integrate over the odd part, then over the even part.

We also want some functions to integrate; polynomials don't have finite integrals on V^0.

Definition 4.9. Let $C^\infty(V) := C^\infty(V^0) \otimes \mathscr{O}(V^1)$.

We also need a measure to integrate with.

Definition 4.10. Let V be a super-vector space.

- The *Berezinian line* of V is
$$\mathrm{Ber}(V) := \Lambda^{\mathrm{top}}V^0 \otimes (\Lambda^{\mathrm{top}}(\Pi V^1))^*.$$
- An *integration measure* is an element of $\mathrm{Ber}(V^*)$.[13]

[12]If $\mathscr{C}$ is odd-dimensional, (4.6) still makes sense, but skew-symmetry forces us to leave out one fermion, so the partition function is 0. However, some correlation functions will be nonzero.

[13]To be completely precise, this would be a measure twisted by the orientation bundle, since measures don't require orientation to integrate.

If V is purely odd, this reduces to the above definition of the space of measures.

Since V now has an even subspace, integration will depend on orientation again.

Definition 4.11. Let V be an oriented super-vector space and $\mathrm{d}\mu = \omega^0 \otimes \omega^1 \in \mathrm{Ber}(V^*)$ be an integration measure. For any $f = f^0 \otimes f^1 \in C^\infty(V)$, its *integral* is

$$\int_V \mathrm{d}\mu\, f := \int_{V^0} \omega^0 f^0 \left(\int_{V^1} \omega^1 f^1 \right).$$

That is: integrate the odd part, then the even part.

On $\mathbb{R}^{p|q}$ there's a canonical measure

$$\mathrm{d}\mu = \mathrm{d}\mathbf{x}\,\mathrm{d}\boldsymbol{\psi} := \left(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^p\right) \otimes \left(\mathrm{d}\psi^1 \cdots \mathrm{d}\psi^q\right).$$

Example 4.12. Take $\mathscr{C} = \mathbb{R}^{1|2}$ with action

$$S(x, \psi^1, \psi^2) := S_1(x) + S_2(x)\psi^1\psi^2. \tag{4.13}$$

Then, the partition function is

$$Z = \int \mathrm{d}x\,\mathrm{d}\boldsymbol{\psi}\, e^{-S} = \int \mathrm{d}x\, S_2(x) e^{-S_1(x)}.$$

In other words, in the purely bosonic theory with action S_1, this is just the correlation function $\langle S_2(x) \rangle$. This can be a helpful perspective, but it also obscures why this is happening. ◀

For general S_1 and S_2, these are not super interesting.[14] But there is a special case that is much better. Fix an $h\colon \mathbb{R} \to \mathbb{R}$ such that as $|x| \to \infty$, $h(x) \to \infty$. Then, set

$$\begin{aligned} S_1(x) &:= \frac{1}{2} h(x)^2 \\ S_2(x) &:= h'(x). \end{aligned}$$

Hence

$$S = \frac{1}{2} h(x)^2 + h'(x)\psi^1\psi^2, \tag{4.14}$$

using the action (4.13). This action is invariant under a certain odd vector field on $\mathscr{C}$; we're going to explain what this means.

Definition 4.15. Let A be a super-commutative super-algebra, a *derivation* D on A with degree $|D|$ is a function $D\colon A \to A$ such that $D(a + a') = D(a) + D(a')$ and

$$D(aa') = (Da)a' + (-1)^{|a||D|} a(Da').$$

The set of all derivations of $\mathscr{O}(V)$ is a super-vector space, which we'll denote $\mathrm{Vect}(V)$.

Exercise 4.16. Show that $\mathrm{Vect}(V)$ is a *super-Lie algebra*, in the same way that vector fields on a vector space are a Lie algebra.[15]

On $\mathbb{R}^{p|q}$, we have the usual derivations/vector fields $\partial_{x_i} \in \mathrm{Vect}^0(V)$, but now also some odd vector fields $\partial_{\psi^I} \in \mathrm{Vect}^1(V)$, defined to satisfy

$$\begin{aligned} \partial_{\psi^I}(x^i) &= 0 \\ \partial_{\psi^I}(\psi^J) &= \delta_I^J. \end{aligned}$$

Hence

$$\partial_{\psi^1}(\psi^1\psi^2) = \psi^2 \qquad \text{and} \qquad \partial_{\psi^1}(\psi^2\psi^1) = -\psi^2.$$

Now we'll discuss the symmetry in the action (4.14). Let

$$\begin{aligned} Q_1 &:= \psi^1 \partial_x + h(x)\partial_{\psi^2} \\ Q_2 &:= \psi^2 \partial_x - h(x)\partial_{\psi_1}. \end{aligned} \tag{4.17}$$

[14] No pun intended.

[15] There's a whole theory of super-manifolds and Lie super-groups and more. But it's possible to go a long way before needing to understand the whole package.

Then,

$$Q_1 S = \psi^1 h'(x)h(x) + h(x)h'(x)\partial_{\psi^2}(\psi^1\psi^2) = 0,$$

and similarly for $Q_2 S$.

Exercise 4.18. This means that if $X = [Q_1, Q_2]$, then X is an even vector field and $XS = 0$. Find X and show this explicitly.

There's also a sense in which Q_1 and Q_2 are divergence-free.

Definition 4.19. Let

$$X := h^i\partial_{x^i} + g^I\partial_{\psi^I}.$$

Then, the *Lie derivative* along X of a section of $\mathrm{Ber}(V^*)$ is

$$\mathscr{L}_X(\mathrm{d}\mathbf{x}\,\mathrm{d}\boldsymbol{\psi}) := \left(\partial_{x^i}h^i + \partial_{\psi^I}g^I\right)\mathrm{d}\mathbf{x}\,\mathrm{d}\boldsymbol{\psi}.$$

If $\mathscr{L}_X(\mathrm{d}\mu) = 0$, we say X is *divergence-free*.

There is a coordinate-free definition of this, which can be found in [15]. Other references on the general theory:

- Deligne-Morgan, "Notes on supersymmetry (following Joseph Bernstein)" [5].
- Witten recently wrote some notes on integration on supermanifolds in [14], which are pretty down-to-Earth.

Lemma 4.20. *Let Q be a divergence-free vector field on an oriented super-vector space V with measure $\mathrm{d}\mu$. For any $f \in C^\infty(V)$,*

$$\int_V \mathrm{d}\mu\, Qf = 0.$$

Proof. We'll compute in coordinates: suppose $Q = h^i\partial_{x^i} + g^I\partial_{\psi^I}$. Then,

$$\begin{aligned}
\int_V \mathrm{d}\mu\, Qf &= \int_{V^0} \mathrm{d}\mathbf{x}\,(Qf)^{\mathrm{top}} \\
&= \int_{V^0} \mathrm{d}\mathbf{x}\left(h^i\partial_{x^i}f + g^I\partial_{\psi^I}f\right)^{\mathrm{top}} \\
&= \int_{V^0} \mathrm{d}\mathbf{x}\left(-(\partial_{x^i}h^i)f + (-1)^{|g^I|}(\partial_{\psi^I}g^I)f\right)^{\mathrm{top}} \\
&= 0,
\end{aligned}$$

because Q is divergence-free. ⊠

Using this, we can show that a certain deformation of these theories is actually constant.

Proposition 4.21. *Let V be an oriented super-vector space with measure $\mathrm{d}\mu$, $S \in C^\infty(V)$, and Q be a divergence-free odd vector field on V with $[Q,Q] = 0$ and $QS = 0$. For any smooth family of odd elements $\{\Psi_t\} \in C_c^\infty(V)$ with $\Psi_0 = 0$, let*

$$S_t := S + Q\Psi_t,$$

which is called a Q-exact deformation *of S. Then, Z_t is independent of t.*

Proof. Let $\Psi_t' := \partial_t\Psi_t$. Since

$$Z_t = \int_{\mathscr{C}} \mathrm{d}\mu\, e^{-(S+Q\Psi_t)},$$

then

$$\begin{aligned}
\partial_t Z_t &= -\int_{\mathscr{C}} \mathrm{d}\mu\,(Q\Psi_t')e^{-(S+Q\Psi_t)} \\
&= -\int_{\mathscr{C}} \mathrm{d}\mu\, Q\left(\Psi_t' e^{-(S+Q\Psi_t)}\right) \\
&= 0
\end{aligned}$$

by Lemma 4.20. ⊠

Lecture 5.

Localization in supersymmetry: 9/14/17

Today, we're going to use the $(0+1)$-dimensional field theory that we've been developing to do something actually topological. Recall that our state space is $\mathscr{C} = \mathbb{R}^{1|2}$, and given a smooth $h\colon \mathbb{R} \to \mathbb{R}$ such that $|h(x)| \to \infty$ as $|x| \to \infty$, we defined the action (4.14), and we'd like to compute its partition function $Z = \int_{\mathscr{C}} e^{-S}$.

Rather than boldly going forward as in previous lectures, we first observed that the partition function is invariant under two symmetries Q_1 and Q_2 (4.17). If

$$Q := Q_1 + Q_2 = (\psi^1 + \psi^2)\partial_x + h(x)(\partial_{\psi^2} - \partial_{\psi^1}),$$

then Q acts on $C^\infty(\mathscr{C})$ and $[Q,Q] = 0$. By Proposition 4.21, for any deformation $\psi_t \in C_c^\infty(\mathscr{C})$. That is, if $S_t := S + Q(\psi_t)$ and $Z_t := \int e^{-S_t}$, then $\partial_t Z_t = 0$. One way to think of this is to take Q as a differential operator and consider "Q-cohomology" — then, Proposition 4.21 tells us that Z_t only depends on the cohomology class of S.

Consider deforming $h(x)$ to a family $h_t(x)$ in a compactly supported manner, which defines a variation S_t of S. Using dots to denote $\frac{\mathrm{d}}{\mathrm{d}t}$,

$$\dot{S}(x) = h(x)\dot{h}(x) + \dot{h}'(x)\psi^1\psi^2.$$

Since $\dot{S}(x) = Q\Psi$ with $\Psi = -\dot{h}(x)\psi^1$, Proposition 4.21 tells us that Z *does not depend on* $h(x)$, as long as you only take compactly supported deformations.

Exercise 5.1. Bootstrap this to show that Z only depends on the behavior at infinity: it's only a function of $\varepsilon_\pm$, where $\lim_{h\to\pm\infty} = \varepsilon_\pm\infty$.

This is in a sense topological; certainly, there's no dependence on the metric.

One way to think of this which will come up again and again is that the action makes the configuration space only care about compact things. If you switch the behavior of h at $\pm\infty$, which requires doing something noncompact, it will change the invariants. Donaldson theory has the same behavior, with chambers in which the invariants do not change (where $b_2^+(X) > 1$), plus "wall-crossing phenomena" on their boundaries (where $b_2(X) = 1$).

Now let's compute Z, using topological invariance and a trick called localization. Since Z doesn't depend on our choice of h, let's do something nice: Z does not depend on λ in the variation $h(x) \to \lambda h(x)$ for $\lambda > 0$, so let's compute the limit as $\lambda \to \infty$. That is, we need to understand the asymptotics of

$$\lim_{\lambda\to\infty} \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-\lambda F(x)}, \tag{5.2}$$

where $F(x) \to \infty$ as $x \to \infty$. The asymptotics are controlled by something called the *method of steepest descent*, which may be surprising at first.

Proposition 5.3. *Assume F has a unique global minimum at x_c.*[16] *Then, as $\lambda \to \infty$,*

$$\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-\lambda F(x)} \sim \sqrt{\frac{2\pi}{\lambda F''(x_c)}}\, e^{-\lambda F(x_c)}. \tag{5.4}$$

That is, neither side of (5.4) has a limit at $\lambda \to \infty$, but their ratio does, and its limit is 1.

The proof is a little bizarre: first you only look at a tiny neighborhood of x_c, and then expand that neighborhood to the whole real line, and each of these contributes an exponentially small amount to the integral. For a full proof, check out [2]; it has a big list of cool tricks for computing asymptotic expansions like this one.

Another way to interpret (5.3) is that it gives us permission to truncate $F(x)$ to quadratic order around F_c. Thus, let's reshape h such that all of its global minima are 0, and make a quadratic approximation of S by summing over all of these local minima. The fermionic part is already quadratic, so we just have to look at the bosonic part. As $\lambda \to \infty$, we get that

$$Z(\lambda) \simeq \sum_{x_c : h(x_c)=0} \int \mathrm{d}x\, \mathrm{d}\psi\, \exp\!\left(-\frac{1}{2}\lambda^2 h'(x_c)^2(x-x_c)^2 - \lambda h'(x_c)\psi^1\psi^2\right).$$

[16] Otherwise, there would be a sum over local minima.

This is a Gaussian in the bosonic and in the fermionic parts:

$$\begin{aligned}
&= \sum_{x_c} \left(\sqrt{\frac{2\pi}{\lambda^2 h'(x_c)^2}} \left(\lambda h'(x_c)\right) \right) \\
&= \sqrt{2\pi} \sum_{h_c} \frac{h'(x_c)}{|h'(x_c)|} \\
&= \sqrt{2\pi} \sum_{h_c} \operatorname{sign}(h'(x_c)).
\end{aligned}$$

This is actually not so hard to evaluate directly, first integrating out the fermionic part then the bosonic part, but it's a useful example nonetheless.

Now we can look at one function in each deformation class.

- If $\lim_{x\to\pm\infty} h(x) = \pm\infty$, then there are an odd number of points x_c with sign $+1$ and an even number with sign -1, so we get $Z/\sqrt{2\pi} = -1$.
- If $\lim_{x\to\pm\infty} h(x) = \mp\infty$, this reverses: there are an even number with sign $+1$ and an odd number with sign -1, so $Z/\sqrt{2\pi} = -1$ again.
- If $\lim_{x\to\pm\infty} h(x) = \infty$ (or if it goes to $-\infty$), the number of critical points with positive and negative signs are the same, so $Z = 0$.

Hence you can express this in terms of $\varepsilon_\pm$:

$$Z = \sqrt{2\pi} \frac{|\varepsilon_+ - \varepsilon_-|}{2}.$$

Localization in a 0-dimensional σ-model. Let (M, ω) be a compact, $2n$-dimensional symplectic manifold: this means ω is a differential 2-form on M with $\mathrm{d}\omega = 0$ and $\omega^n \neq 0$, and assume there is U(1)-action on M generated by the vector field $Y := \omega^{-1}(\mathrm{d}H)$, where $H\colon M \to \mathbb{R}$ is some function.

Let's assume all the fixed points of Y are isolated,[17] and pick an $\alpha \in \mathbb{R}$. We're going to use all this stuff to prove something cool, a formula for

$$\int_M \frac{\omega^n}{n!} e^{i\alpha H}. \tag{5.5}$$

Example 5.6. In examples, this integral is something people actually care about. Let $M = S^2$ with $\omega := \sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\varphi$ and $H := z = \cos\theta$, so $Y = \partial_\varphi$. Then,

$$\int_M \frac{\omega^n}{n!} e^{i\alpha H} = \int_{S^2} e^{i\alpha\cos\theta} \sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\varphi = 2\pi \int_0^\pi e^{i\alpha\cos\theta} \sin\theta\,\mathrm{d}\theta.$$

This is a Bessel function, and it's also funny to notice it is a great example of the kinds of integrals you teach for u-substitution and never expect to see anywhere else. It is hence easy to solve:

$$\begin{aligned}
2\pi \int_0^\pi e^{i\alpha\cos\theta} \sin\theta\,\mathrm{d}\theta &= -2\pi \int_1^{-1} e^{i\alpha z}\,\mathrm{d}z \\
&= \frac{2\pi}{i\alpha}\left(e^{i\alpha} - e^{-i\alpha}\right) \\
&= 4\pi \frac{\sin\alpha}{\alpha}.
\end{aligned}$$

◀

This answer demonstrates a localization phenomenon: it's a sum of contributions only from the north and south poles. In general, the integral (5.5) is a sum of contributions

$$(\pm)\frac{2\pi}{i\alpha} e^{i\alpha H(x_c)},$$

summed over the fixed points x_c of the U(1)-action. This is an instance of the Duistermaat-Heckman theorem [8], and we're going to prove it using localization in supersymmetry.

To do this, we're going to need a supermanifold that's not a super-vector space, but it's not so bad.

[17] It is possible to excise this assumption, but it's helpful for now.

Definition 5.7. Let $E \to M$ be a vector bundle. Its *parity change* ΠE is a supermanifold whose algebra of functions is $C^\infty(\Pi E) := C^\infty(M, \Lambda^*(E))$.

We won't go into the general theory of supermanifolds here. Concretely, for $E = TM$, in local coordinates on M, we have even coordinates x^i, $i = 1, \dots, 2n$, and odd coordinates ψ^i for $i = 1, \dots, 2n$, and we can translate between functions on $\Pi(TM)$ and differential forms on M by exchanging ψ^i and $\mathrm{d}x^i$.

If we want to write down a zero-dimensional quantum field theory, we ought to have an action. Let $\mathscr{C} := \Pi(TM)$ and take

$$S := -i\alpha(H + \omega),$$

or in coordinates,

$$= -i\alpha\big(H + \omega_{ij}\psi^i\psi^j\big).$$

There's a canonical measure (up to scaling) $\mathrm{d}\mathbf{x}\,\mathrm{d}\boldsymbol{\psi}$ on $\Pi(TM)$, which in local coordinates is exactly $\mathrm{d}\mathbf{x}\,\mathrm{d}\boldsymbol{\psi}$ from before, and is invariant under change-of-charts. This might be surprising. A more abstract way to think of this is that the super-tangent bundle $T\mathscr{C}$ to $\mathscr{C}$ factors into a short exact sequence

$$0 \longrightarrow \Pi(\pi^*TM) \longrightarrow T\mathscr{C} \longrightarrow TM \longrightarrow 0,$$

so $\mathrm{Ber}(T\mathscr{C}) = \mathrm{Ber}(TM) \otimes \mathrm{Ber}(\Pi(TM))$, hence must be trivial. Hence the partition function is

$$Z := \int_{\mathscr{C}} \mathrm{d}\mathbf{x}\,\mathrm{d}\boldsymbol{\psi}\, e^{-S}.$$

If we integrate over fermions first, we get

$$Z = (i\alpha)^n \int_M \frac{\omega^n}{n!} e^{i\alpha H}.$$

We want to compute this by localization. This means first writing down a vector field under which S is invariant. We'll take

$$Q := \mathrm{d} + \iota_Y = \psi^i\partial_{x^i} + Y^j\partial_{\psi^j},$$

where ι_Y is contraction;[18] then, Q is odd and S is invariant under it:

$$\begin{aligned} QS &= (\mathrm{d} + \iota_Y)(H + \omega) \\ &= \mathrm{d}H + \iota_Y\omega \\ &= 0, \end{aligned}$$

Moreover,

$$\begin{aligned} \frac{1}{2}[Q,Q] &= [\mathrm{d}, \iota_Y] = \mathscr{L}_Y \\ &= \psi^i(\partial_{x^i}Y^j)\partial_{\psi^j} + Y^i\partial_{x^i}. \end{aligned}$$

We want to use localization to obtain the fixed points of Y through a perturbation $S \to S + \lambda Q\Psi$ as $\lambda \to \infty$ for some Ψ. To do this, we need to choose a U(1)-invariant metric g on M, which we can always do, and the answer will turn out to not depend on it. Then, let

$$\Psi := g(Y) = g_{ij}\psi^i Y^j = \psi^i Y_i.$$

Exercise 5.8. Check that

$$Q\Psi = g(Y,Y) - d(gY) \tag{5.9}$$

$$Q^2\Psi = 0, \tag{5.10}$$

e.g. by computing in coordinates.[19]

[18] This is the differential in the Cartan model for the U(1)-equivariant cohomology for M, but that's not important right now.

[19] Even though the derivations only form a (super)-Lie algebra, so Q^2 doesn't make sense on that level, it's still acting on vector fields, and we can iterate its action. This differs from $[Q,Q]$ by 1/2, so it doesn't make a difference.

Thus, we know the perturbation (5.9). As before, Z is independent of λ, which uses (5.10).

Now, you can take $\lambda \to \infty$ and use the method of steepest descent to conclude that

$$Z \sim \sum_{x_c \in M : Y(x_c)=0} e^{i\alpha H(x_c)}(2\pi)^n \frac{(\mathrm{d}(gY)(x_c))^n/n!}{\sqrt{\det(D^2(g(Y,Y))(x_c)}} \tag{5.11}$$

This looks complicated, but just like before, the fermionic and bosonic pieces almost cancel each other out, leaving behind a topological contribution. Here, $g(Y,Y)$ is a real-valued function on M, so we can take its Hessian $D^2(g(Y,Y))$ and evaluate it at the critical point x_c.

Both the numerator and the denominator of the fraction in (5.11) are naturally valued in $\Lambda^{\mathrm{top}} T^*_{x_c} M$. We'll exploit this to calculate their ratio in a local model by diagonalizing the U(1)-action on $T_{x_c}M$. That is, we choose an isomorphism

$$T_{x_c}M \cong \bigoplus_{i=1}^n \mathbb{R}^2_i,$$

where U(1) acts on $\mathbb{R}^2_i$ by

$$\theta \longmapsto \begin{pmatrix} \cos k_i\theta & \sin k_i\theta \\ -\sin k_i\theta & \cos k_i\theta \end{pmatrix}$$

for some weights $k_1, \dots, k_n \in \mathbb{R}$.

Example 5.12. The 2-dimensional case is simplest: take $T_{x_c}M = \mathbb{R}^2$ with weight k. Let $\omega = r\,\mathrm{d}r \wedge \mathrm{d}\theta$ be the symplectic form and the standard metric $g := \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2$ is U(1)-invariant. Then, $Y -= k\partial_\theta$, $H = (1/2)kr^2$, and $g(Y,Y) = k^2r^2$, then

$$\mathrm{d}(gY) = 2kr\,\mathrm{d}r \wedge \mathrm{d}\theta \sqrt{\det(D^2(g(Y,Y)))} = 2k^2 r\,\mathrm{d}r \wedge \mathrm{d}\theta. \tag{5.13}$$

Again, almost everything cancels out, so we get

$$Z = (2\pi)^n \sum_{x_c} \frac{e^{i\alpha H(x_c)}}{\prod_{i=1}^n k_i(x_c)},$$

i.e.

$$\int_M \frac{\omega^n}{n!} e^{i\alpha H} = \left(\frac{2\pi}{i\alpha}\right)^n \sum_{x_c} \frac{e^{i\alpha H(x_c)}}{\prod_{i=1}^n k_i(x_c)}. \tag{5.14}$$

◀

(5.14) is known as the *Duistermaat-Heckman formula*. We've just given a completely rigorous proof of it, which probably differs greatly from their original proof in [8].

Next time, we'll wrap up this story and begin thinking about higher dimensions.

Lecture 6.

One-dimensional QFT: 9/19/17

Reminder: there are exercises in the professor's notes, and you should try them!

We've been talking about localization in the past week, and we're on the way to thinking about effective field theory. Last time, we discussed the Duistermaat-Heckman formula (5.14) for a compact symplectic manifold (M,ω) (i.e. M is a compact $2n$-dimensional manifold, and $\omega \in \Omega^2(M)$ is a closed form with $\omega^n \neq 0$). We assumed we had a vector field Y which generates a U(1)-action,[20] This Y is generated by a Hamiltonian $H\colon M \to \mathbb{R}$, in the sense that $Y = \omega^{-1}(\mathrm{d}H)$; we assume H has isolated fixed points.[21]

Then, we showed that the integral

$$\int_M \frac{\omega^n}{n!} e^{i\alpha H}$$

depends only on the fixed points of H, and the precise formula is (5.14). The equation uses the infinitesimal U(1)-action on the tangent space of a fixed point.

[20] This is a strong assumption: a general smooth function $H\colon M \to \mathbb{R}$ can be taken for a Hamiltonian, and we can let $Y = \omega^{-1}(\mathrm{d}H)$, which generally does not generate a U(1)-action, as $e^{2\pi Y} \neq \mathrm{id}$. So as an equation, we're assuming $e^{2\pi Y} = \mathrm{id}$.

[21] Unlike the previous assumption, this is generically true. We also assumed H is Morse in the final step; this assumption probably can be removed, but the argument will be nontrivial.

Exercise 6.1. Suppose V is a $2n$-dimensional vector space with an orientation and an inner product, i.e. a reduction of the structure group from $\mathrm{GL}_{2n}(\mathbb{R})$ to $\mathrm{SO}(2n)$. Then, define a natural line $\mathrm{Pf}(V)$ with $\mathrm{Pf}(V)^{\otimes 2} = \mathrm{Det}(V) := \Lambda^{2n}(V)$.

Exercise 6.2. We saw what happens for $M = S^2$ last time, in Example 5.6. Try it with $\mathbb{CP}^2$.[22]

Remark. Another way to interpret (5.14) is that "the stationary phase approximation to

$$\int \frac{\omega^n}{n!} e^{i\alpha H}$$

is exact." This is an asymptotic analysis as $\alpha \to \infty$; we've already done this for things like $\int e^{-tF}$, but in this case there's something weirder going on: as $\alpha \to \infty$, the function is oscillating more and more rapidly. The fact that it only depends on the critical points in the end is a manifestation of the fact that these oscillations cancel each other out.

This stationary phase analysis is much like the method of steepest descent that we've been doing: approximate the integrand by its quadratic Taylor expansion around each critical point. There are some tricky technicalities, and you have to make rigorous the idea that you're integrating something only conditionally convergent

The point is, if you hear someone saying the stationary phase approximation is exact, that's a different statement with a different proof than the approach we used. There's a really great exposition of this approach in [1]. ◀

In our proof of the Duistermaat-Heckman formula, we used localization for

$$\int_{\mathscr{C}} \mathrm{d}\mu\, e^{-S},$$

where $\mathscr{C} = \Pi TM$, the parity change of the tangent bundle, and

$$S = -i\alpha(\omega + H) \in C^\infty(\mathscr{C}) = \Omega^*(M).$$

If $Q := \iota_Y + \mathrm{d}$, then $QS = 0$.

There's an interpretation of this in terms of U(1)-equivariant cohomology which allows for a more general formula than (5.14). Namely, we think of Q as an "equivariant differential," and we can generalize to any $S \in C^\infty(\mathscr{C})$ with $QS = 0$, i.e. any *equivariantly closed* form $\alpha \in \Omega^*(M)$ on any compact manifold M with a U(1)-action.

Theorem 6.3 (Atiyah-Bott-Berline-Vergne [1, 4])**.** *Let M be a compact manifold with a* U(1)*-action with isolated fixed points $\{x_c\}$, and let $\beta \in \Omega^*(M)$ be an equivariantly closed form. Then,*

$$\int_M e^\beta = (-2\pi i)^n \sum_{x_c} \frac{e^{\beta^{\mathrm{bot}}(x_c)}}{\prod_{i=1}^n k_i(x_c)}.$$

Here, β^{bot} denotes the piece of β in Ω^0; equivariantly closed forms are generally non-homogeneous.

The equivariant folks also call this theorem "the localization theorem in equivariant cohomology," and like this formulation of it better.

We can also generalize to non-isolated fixed points, and we will need to use this later. In this case, the steepest descent analysis of

$$\int \mathrm{d}\mu\, e^{-S+\lambda Q\Psi}$$

as $\lambda \to \infty$ is now localized on the fixed *set* P, and the integrand is determined by the local structure around P.[23]

Let NP denote the normal bundle, and recall that in the steepest descent analysis, we introduced a U(1)-invariant metric g on M. Since the volume form ω^n on a symplectic manifold defines an orientation, NP is also oriented, and the orientation and the metric g define an SO($2n$)-structure.

Definition 6.4. Let X be a manifold with a U(1)-equivariant vector bundle $E \to X$ together with a reduction of its structure group to SO($2n$) compatible with the U(1)-action. Let $Y \in \Omega^0(\mathfrak{so}(E))$ denote the action of U(1), and choose a U(1)-invariant metric g on E and let $F \in \Omega^2(\mathfrak{so}(E))$ denote its curvature form. Then, the *equivariant Euler form* of E is

$$\mathrm{Eul}(E) := \mathrm{Pf}\left(\frac{1}{2\pi}(Y + F)\right).$$

[22] TODO: which circle action?

[23] We may need to make a transversality assumption on P, but it's OK.

In general, $\mathrm{Eul}(E)$ is concentrated in even degrees of $\Omega^*(X)$.[24] If $n=1$, the Euler form has a simpler formula:

$$\mathrm{Eul}(E) = \frac{1}{2\pi}(ik+F),$$

with k as in Example 5.12.

More generally, the bottom piece of the Euler form is $\prod ik_i/2\pi \neq 0$, so if all $k_i \neq 0$, there's an inverse to the Euler form $1/\mathrm{Eul}(E) \in \Omega^*(X)$, using the fact that

$$\frac{1}{a+x} = \frac{1}{a}\left(\frac{1}{1+a^{-1}x}\right) = \frac{1}{a}\left(1 - a^{-1}x + a^{-2}x^2 - \cdots\right).$$

This leads to the most general version of the ABBV formula, which is one of the coolest things you can do with 0-dimensional supersymmetric quantum field theory.

Theorem 6.5 (Atiyah-Bott-Berline-Vergne [1, 4])**.** *With M, β, and P as above,*

$$\int_M e^\beta = \int_P \frac{e^\beta}{\mathrm{Eul}(NP)}.$$

Quantum field theory in one dimension. Now we'll move on to the one-dimensional case, which specializes to undergraduate quantum mechanics. Choose a compact Riemannian 1-manifold (X,η): either $X=[0,T]$, or $X \cong S^1$ with circumference T. We'll parametrize X by t, which you can think of as time. Now, the space $\mathscr{C}_X$ of (some kind of generalized) functions on X will be infinite-dimensional.

Let's define a theory. Fix a Riemannian manifold (Y,g), which we'll call the *target*, and $V: Y \to \mathbb{R}$, called the *potential*.

- For $X \cong S^1$, let $\mathscr{C}_{S^1} := \{\phi : S^1 \to Y\}$, the C^∞ maps from S^1 to Y, and
- for $X=[0,T]$, fix $y_0, y_1 \in Y$ and let $\mathscr{C}_{[0,T]_{y_0}^{y_1}} := \{\phi : [0,T] \to Y \mid \phi(0) = t_0, \phi(T) = y_1\}$.

So for S^1 we get loops, and for $[0,T]$ we get paths with chosen endpoints. Let $\mathrm{d}V_X$ denote the volume form on X and R denote the scalar curvature of Y; then, we define the action $S : \mathscr{C}_X \to \mathbb{R}$ to be

$$S(\phi) := \int_X \mathrm{d}V_X \, \frac{1}{2}\left(g(\dot\phi,\dot\phi) + V(\phi) - \frac{1}{3}R(\phi)\right). \tag{6.6}$$

Or in coordinates,

$$= \int_X \mathrm{d}t \sqrt{\eta_{tt}}\left(\frac{1}{2}g_{ij}(\phi(t))\dot\phi^i(t)\dot\phi^j(t)\eta^{tt}(t) + V(\phi(t)) - \frac{1}{3}R(\phi(t))\right). \tag{6.7}$$

If you parametrize X by arc length, $\eta_{tt}=1$ and this simplifies:

$$= \int_X \frac{1}{2}g_{ij}\dot\phi^i\dot\phi^j + V(\phi(t)) - \frac{1}{3}R(\phi(t)). \tag{6.8}$$

We would like to define the partition function

$$Z_X \text{ “=” } \int_{\mathscr{C}_X} \mathrm{d}\phi \, e^{-S(\phi)}, \tag{6.9}$$

but here we run aground: $\mathrm{d}\phi$ is now a measure on an infinite-dimensional Banach space. There's no analogue of the Lesbegue measure here: a unit ball contains infinitely many balls of radius $1/4$, so there's no consistent way to define the volume of anything to be nonzero and finite. Nonetheless, in a sense $\mathrm{d}\phi$ doesn't exist, but the whole expression (6.9) will exist; it's something that statistical mechanics researchers call the Weiner measure.

Physicists make sense of (6.9) by discretization. For concreteness, set $X=[0,T]$ and fix $y_0, y_1 \in Y$. We'll replace X by a *lattice*: for some $N > 0$, let $t_0, t_1, \ldots, t_N \in X$ such that $\Delta_t = t_j - t_{j-1} = T/N$. The discretized field space $\mathscr{C}_{X;N}$ is the space of piecewise geodesic paths $\phi : X \to Y$ that are smooth on (t_{j-1}, t_j) and such that the path from $\phi(t_{j-1})$ to $\phi(t_j)$ is the unique minimal geodesic between them.[25] The map $\phi \mapsto (\phi(t_0), \ldots, \phi(t_N))$ defines

[24] TODO: Does the cohomology class of the Euler form depend on the metric?

[25] One might be surprised to learn this stuff was formalized and written down surprisingly recently, in the mid-2000s.

an embedding $\mathscr{C}_{X;N} \subset Y^{N+1}$, and this is a finite-dimensional manifold, so we can use the *product measure*

$$\mathrm{d}\mu_N := \frac{1}{(4\pi\Delta t)^{N \dim Y/2}} \prod_{n=1}^{N-1} \mathrm{d}u\, \mathrm{d}y(\phi(t_n)).$$

Then, we can define the discretized partition function

$$Z_{X;N} = \int_{\mathscr{C}_{X;N}} e^{-S}\, \mathrm{d}\mu_N,$$

and try to take the limit as $N \to \infty$. This does exist!

Theorem 6.10. *The limit* $\lim_{N\to\infty} Z_{X;N}$ *exists, and is the heat kernel* $k_T(y_0, y_1)$*.*

Interestingly, it only depends on the endpoints y_0, y_1 and the total length.

Definition 6.11. Fix Y and V as above, For $t \in \mathbb{R}_+$, the *heat kernel (deformed by V)* is a smooth function $k_t \colon Y \times Y \to \mathbb{R}$ satisfying the *heat equation*

$$\partial_t k_t(x,y) + (-\Delta_X + V(x))k_t(x,y) = 0, \tag{6.12}$$

and as a distribution,

$$\lim_{t\to 0} k_t(x,y) = \delta(x,y).$$

You can also characterize the heat kernel as the fundamental solution to the heat equation (6.12).

Exercise 6.13. Show that when $Y = \mathbb{R}^n$ and $V = 0$, the heat kernel is

$$k_t(x,y) = \left(\frac{1}{4\pi t}\right)^{n/2} \exp\left(-\frac{1}{4t}\|x-y\|^2\right).$$

The heat kernel is the kernel of an integral operator, the operator U_t of heat evolution for time t. This is the operator evolving solutions to (6.12) forward in time. As an integral kernel, this has the formula

$$(U_t f)(x) = \int_M \mathrm{d}u\, \mathrm{d}y\, k_t(x,y) f(y)\, \mathrm{d}y.$$

U_t is a *smoothing* operator: it maps distributions to smooth functions. It also defines a linear operator on $L^2(M)$ which has the formula

$$U_t = e^{-t(-\Delta+V)}.$$

Heuristic proof of Theorem 6.10 when $V = 0$. Let's discretize the heat operator: let $U_T = (U_{\Delta t})^N$ and

$$K_T(y_N, y_0) := \int_{Y^{N-1}} \prod_{n=1}^{N-1} \mathrm{dvol} \prod_{n=0}^{N-1} k_{\Delta t}(y_{n-1}, y_n). \tag{6.14}$$

When Δt is sufficiently small (N is sufficiently large), we have short-time asymptotics of $k_{\Delta t}$:

$$k_{\Delta t}(x,y) \sim \left(\frac{1}{4\pi\Delta t}\right)^{\dim Y/2} \exp\left(-\frac{1}{4t} d(x,y)^2\right).$$

This is the piece that we're not making precise. If you substitute this into (6.14), you get

$$\begin{aligned} k_T(y_N, y_0) &\sim \int_{Y^{N-1}} \prod_{n=1}^{N-1} \mathrm{dvol} \prod_{n=0}^{N-1} \left(\frac{1}{4\pi\Delta t}\right)^{\dim Y/2} \exp\left(-\frac{1}{4\pi\Delta t} d(y_{n+1}, y_n)^2\right) \\ &= \int \mathrm{d}\mu_N \exp\left(-\frac{\Delta t}{4}\left(\frac{d(y_{n+1},y_n)}{\Delta t}\right)^2\right) \\ &= Z_{X;N}. \end{aligned}$$

⊠

Though this was not a proof, this proof can be made rigorous, and has been done so. Exactly where the scalar curvature goes is somewhat of a mystery, though some more careful analysis of the asymptotics above can be found in [3].

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