HYPERKÄHLER GEOMETRY AND THE MODULI SPACE OF HIGGS BUNDLES

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Contents

1. Introduction to Kähler and hyperkähler geometry

Today will be mostly preliminaries, including some complex and symplectic geometry, such as the symplectic quotient, and an introduction to Kähler and hyperkähler geometry. Over the rest of the week, we'll discuss some examples (which are usually left implicit) such as quiver varieties, introduce the moduli space of Higgs bundles, and more. A good reference for this is Andy Neitzke's lecture notes on the moduli space of Higgs bundles: https://www.ma.utexas.edu/users/neitzke/teaching/392C-higgs-bundles/higgs-bundles.pdf.

Kähler manifolds are great because they involve a whole bunch of angles towards differential geometry: Riemannian, complex, and symplectic manifolds. Riemannian manifolds are likely the most familiar, so we won't discuss them in too much detail.

1.1. Almost complex and complex geometry.

Definition 1.1. Let X be a smooth manifold. A **almost complex structure** on X is a choice of $I \in \mathrm{End}(TX)$ such that $I^2 = -1$. The pair (X, I) is called an **almost complex manifold**.

This implies that $\dim_{\mathbb{R}}(X)$ is even.

You can define a **complex manifold** as the same thing as a real manifold, but with charts valued in $\mathbb{C}^n$, and such that the change-of-charts maps are holomorphic. This implies an almost complex structure, as the tangent space is a complex vector bundle, but (as the terminology "almost" suggests) the two are not the same.

Definition 1.2. An almost complex structure (M, I) is **integrable** if there exists an open cover $\mathfrak{U}$ of M and holomorphic diffeomorphisms $\phi_U : U \to \phi_U(U) \subset \mathbb{C}^n$ for each $U \in \mathfrak{U}$.

Proposition 1.3. *An integrable almost complex structure on X is equivalent to a complex structure on X.*

If (X, I) is an almost complex manifold of dimension n, then the action of I on TX makes TX into a complex vector bundle. You can also complexify it as a real vector bundle, producing a vector bundle $T_{\mathbb{C}}X$ of rank $2n$.

Definition 1.4. Let $T^{1,0}X$ denote the eigenspace for i acting on $T_{\mathbb{C}}X$ and $T^{0,1}X$ denote the $-i$-eigenspace. Hence $T_{\mathbb{C}}X \cong T^{1,0}X \oplus T^{0,1}X$.

These are both complex vector bundles of rank n, and in fact $TX \cong T^{1,0}X$.

You can play the same game with the complexified cotangent bundle: $T^*_{\mathbb{C}}X \cong (T^*)^{1,0}X \oplus (T^*)^{0,1}X$, and $(T^*)^{1,0}X = \mathrm{Ann}(T^{0,1}X)$. More generally, one gets a **type decomposition** or **Hodge decomposition** of complexified exterior powers and complexified differential forms:

$$\Lambda^* T^*_{\mathbb{C}} X \cong \bigoplus_{p+q=n} \Lambda^{p,q} T^* X$$

$$\Omega^*_{\mathbb{C}} X \cong \bigoplus_{p,q=0}^{n} \Omega^{p,q}(X).$$

The piece of degree n is the sum of $\Lambda^{p,q}$ (resp. $\Omega^{p,q}$) for which $p+q=n$.

The intuition is that the holomorphic structure on $\mathbb{C}$ allows one to write

$$\mathrm{d}x \wedge \mathrm{d}y = -\frac{1}{2}\,\mathrm{d}z \wedge \mathrm{d}\overline{z},$$

where $\mathrm{d}z \coloneqq \mathrm{d}x + i\,\mathrm{d}y$ and $\mathrm{d}\overline{z} = \mathrm{d}x - i\,\mathrm{d}y$. The graded part $\Lambda^{p,q}T^*X$ corresponds to the pieces with $\mathrm{d}z$ in p directions and $\mathrm{d}\overline{z}$ in q directions, and similarly for $\Omega^{p,q}X$.[1]

Theorem 1.5. *The following are equivalentfor an almost complex manifold (X, I):*
1. *I is integrable.*
2. *There's a decomposition $\mathrm{d} = \partial + \overline{\partial}$, where $\partial\colon \Omega^{p,q} \to \Omega^{p+1,q}$ and $\overline{\partial}\colon \Omega^{p,q} \to \Omega^{p,q+1}$.*
3. *$T^{0,1}X$ is an integrable distribution, i.e. for sections $s, t \in \Gamma(T^{0,1}X)$, $[s,t]$ is also a section.*

We'll use the first two more often than the third.

Definition 1.6. The **Dolbeault cohomology** is the homology of the complexified differential forms: $H^{p,q}_{\mathrm{Dol}}(X) \coloneqq H^q(\Omega^{p,\bullet}, \overline{\partial})$.

Definition 1.7. Let X be a complex manifold. A **holomorphic vector bundle** is a complex vector bundle $E \to X$ together with a differential $\overline{\partial}_E\colon \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$ such that
1. $\overline{\partial}_E$ satisfies the **Leibniz rule**: if $\alpha \in \Omega^*(X)$ and $\psi \in \Omega^0(E)$,
$$\overline{\partial}_E(\alpha\psi) = (\overline{\partial}\alpha)\psi + (-1)^{|\alpha|}\alpha \wedge \overline{\partial}_E\psi.$$
2. $\overline{\partial}_E^2 = 0$.

Exercise 1.8.
1. If X is a complex manifold, show that $TX \to X$ is holomorphic.
2. The **canonical line bundle** over a complex manifold is $K_X \coloneqq \Lambda^{n,0}T^*X$. Show that K_X is holomorphic.

Elements of K_N are things of the form $f(z)\,\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n$, where $f\colon X \to \mathbb{C}$ is holomorphic.

Definition 1.9. Let $E \to X$ be a holomorphic vector bundle and h be a Hermitian metric on E (i.e. a smoothly varying Hermitian metric on each fiber). The **Chern connection** is the unique connection D on E that is
- unitary with respect to h, and
- compatible with the holomorphic structue, in that $(D\psi)^{0,1} = \overline{\partial}_E\psi$ for any $\psi \in \Omega^0(E)$.

Like the Levi-Civita connection, it's a theorem that the Chern connection exists. That h is unitary means $h(Ix, y) = ih(x, y) + h(x, -Iy)$.

1.2. Kähler geometry.

Let (X, I) be an almost complex manifold with Hermitian metric g, and let ∇ denote the Levi-Civita connection for g.

Definition 1.10. The **fundamental form** is the $\omega \in \Omega^{1,1}_{\mathbb{R}}(X)$ such that $\omega(v, w) = g(Iv, w)$.

Here, $\Omega^{p,p}_{\mathbb{R}}(X)$ is the fixed points of complex conjugation acting on $\Omega^{p,p}(X)$.

There are many different ways to define Kähler manifolds.

Definition 1.11. The triple (X, g, I) is called a **Kähler manifold** if one of the following equivalent conditions is satisfied.
1. $\nabla I = 0$, i.e. I is **covariantly constant**.
2. $\nabla\omega = 0$.
3. I is integrable and the Levi-Civita and Chern connections coincide.
4. I is integrable and $\mathrm{d}\omega = 0$.

In this case ω is also called the **Kähler form**.

Corollary 1.12. *Let X be a Kähler manifold and $Y \subset X$ be a complex submanifold. Then, restricting g and I to Y shows that Y is also a Kähler manifold.*

[1] You can run this same story with tensor products, but the resulting vector bundles don't appear as often in the theory.

Example 1.13. Everyone's first example of a Kähler manifold is $\mathbb{CP}^n$ with the **Fubini-Study metric**. ◂

You can show by a dimensional argument that any Riemann surface is automatically Kähler. Finding counterexamples is kind of tricky: combining Corollary 1.12 and Example 1.13, any smooth projective complex variety is Kähler. There are homological obstructions to being Kähler, and we'll learn more about this in the Kähler geometry minicourse later this summer.

Proposition 1.14. *Let $(X.h)$ be a Kähler manifold of complex dimension n. Then, the holonomy of the Levi-Civita connection around any point p is a subgroup of U_n.*

1.3. **Symplectic geometry.** Hyperkähler manifolds often arise as moduli spaces, which themselves often arise as symplectic quotients of things. To understand symplectic quotients we should first go over some symplectic geometry.

Definition 1.15.

- Let V be a vector space over either $\mathbb{R}$ or $\mathbb{C}$. By a **nondegenerate** 2-form we mean an $\omega \in \Lambda^2(V)$ such that $v \mapsto \omega(v, -)$ defines an isomorphism $V \to V^*$.
- A **symplectic manifold** is a pair (X, ω), where $\omega \in \Omega^2(X)$ is a closed, nondegenerate 2-form (i.e. it's nondegenerate at every $x \in X$). In this case, ω is called a **symplectic form**.
- A symplectic manifold (X, ω) is **exact** if ω is exact, i.e. there's a one-form λ such that $\mathrm{d}\lambda = \omega$.

Example 1.16. The canonical example is the cotangent bundle T^*X to a smooth manifold X, which is an exact symplectic manifold. The **Liouville form**, **Liouville differential**, or (in some contexts in physics) the **Seiberg-Witten differential** $\lambda \in \Omega^1(T^*X)$ is

$$\lambda(x, p) \cdot v = p \cdot \pi_* v,$$

where $\pi \colon T^*X \to X$ is projection. The symplectic form is $\omega = \mathrm{d}\lambda$.

In local coordinates $(p_1, q_1, \ldots, p_n, q_n)$, so $p_1, \ldots, p_n$ are coordinates on X (in physics, position) and $q_1, \ldots, q_n$ are coordinates on the fiber (in physics, momentum), we have

$$\lambda = \sum_{i=1}^n p_i \,\mathrm{d}q_i \qquad \text{and hence} \qquad \omega = \sum_{i=1}^n \mathrm{d}p_i \wedge \mathrm{d}q_i. \qquad ◂$$

For a general nondegenerate $\omega \in \Lambda^2(V)$, there's a **canonical basis** $\{e_1, e_2, \ldots, e_n, f_1, \ldots, f_n\}$ such that $\omega(e_i, e_j) = \delta_{ij}$ and $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$ for all i and j. This means that, in the canonical basis, ω is represented by the block matrix

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Moduli spaces arise as quotients. Let G be a (real) Lie group acting on a symplectic manifold (X, ω),[2] and let $\mathfrak{g}$ be the Lie algebra of G. Differentiating the action defines a Lie algebra homomorphism $\rho \colon \mathfrak{g} \to \mathcal{X}(X)$, where $\mathcal{X}(X)$ denotes the Lie algebra of vector fields on X.

We'd like to dualize this to define a map $\mu \colon X \to \mathfrak{g}^*$ that's G-equivariant (where G acts on $\mathfrak{g}^*$ through the dual of the adjoint action).

Definition 1.17. A **moment map** for the G-action on X is a map $\mu \colon X \to \mathfrak{g}^*$ such that $\iota_{\rho(Z)}\omega = \mathrm{d}(\mu \cdot Z)$ for $Z \in \mathfrak{g}$. (Here, ι is the contraction operator as in Riemannian geometry.)

This is a very important definition — it characterizes to what degree the action commutes with the Hamiltonian.

Remark.

- The moment map does not always exist. One local obstruction comes from the **Cartan formula**:

  $$\mathcal{L}_v \omega = \mathrm{d}(\iota_v \omega) + \iota_v(\mathrm{d}\omega),$$

  and therefore

  $$\mathcal{L}_{\rho(Z)}\omega = \mathrm{d}(\iota_{\rho(Z)}\omega) + \iota_{\rho(Z)}\mathrm{d}\omega = 0,$$

  so the infinitesimal action must preserve ω. This does not suffice; there are other, global obstructions.

[2] Generally, G acts through **symplectomorphisms**, i.e. diffeomorphisms preserving the symplectic form, and defining the moment map will require this.

- When a moment map exists, it need not be unique. For example, consider any $c \in [\mathfrak{g}, \mathfrak{g}]^\perp \subset \mathfrak{g}^*$;[3] then, if μ is a moment map, so is $\mu + c$. ◂

Example 1.18. Consider $X = \mathbb{C}^n$ with the standard Kähler structure

$$\omega \coloneqq \frac{i}{2}\sum_{i=1}^n \mathrm{d}z_i \wedge \mathrm{d}\overline{z}_i.$$

The easiest action you can think of is a U_1-action by rotation: $z_i \mapsto e^{i\alpha} z_i$. Thus, if we rotate z_i by $e^{i\alpha}$, we rotate $\overline{z}$ by $e^{-i\alpha}$, so this action is through local symplectomorphisms.

One moment map is

$$\mu = -\frac{1}{2}\sum \lvert \mathrm{d}z_i\rvert^2 + c$$

for any $c \in \mathbb{R}$, because the commutator subgroup of U_1 is trivial and $\mathfrak{u}_1 \cong \mathbb{R}$. ◂

Definition 1.19. Let G act on a symplectic manifold (X, ω) in such a way that a moment map μ exists. Then, the **symplectic quotient** of X by G is $X /\!\!/_\mu G \coloneqq \mu^{-1}(0)/G$.

The dependence on μ is a little fearsome, and the G-action on $\mu^{-1}(0)/G$ is not always free.

Proposition 1.20. *If G acts freely on $\mu^{-1}(0)$ freely, then $X /\!\!/_\mu G$ is a symplectic manifold. Moreover,*

1. $\dim(X /\!\!/_\mu G) = \dim X - 2\dim G$, *and*
2. *if ω_X denotes the symplectic form on X and $\omega_{X/\!\!/_\mu G}$ denotes the symplectic form on $X /\!\!/_\mu G$, then $\pi^*\omega_{X/\!\!/_\mu G} = \iota^*\omega_X$, where $\iota\colon \mu^{-1}(0) \hookrightarrow X$ is inclusion and $\pi\colon \mu^{-1}(0) \twoheadrightarrow X /\!\!/_\mu G$ is projection.*

Remark. For some more motivation about why this is an okay idea, let X be a compact manifold and G be a compact Lie group acting freely on X. Hence G acts on (T^*X, ω) through symplectomorphisms, and in fact a moment map always exists, and one is given by

$$\mu_Z(x, p) = -p \cdot (\rho(Z)(x)),$$

where $x \in X$ and $p \in T_x^*X$. Moreover, the action of G on $\mu^{-1}(0)$ is free, and $T^*X /\!\!/_\mu G \cong T^*(X/G)$ as symplectic manifolds. ◂

The point of the moment map is to generalize this nice fact.

Proposition 1.21. *Let X be a Kähler manifold and G act on X preserving g, I, and ω, and suppose that there's a moment map μ such that G acts freely on $\mu^{-1}(0)$. Then, $X /\!\!/_\mu G$ has a natural Kähler structure.*

Explicitly, the symplectic structure is as in Proposition 1.20, the metric is the quotient metric, and I is determined by g and ω. This fact is the reason symplectic quotients arise in the study of Kähler manifolds.

Example 1.22. Let U_1 act on $\mathbb{C}^n$ as in Example 1.18.

- If $c < 0$, then $\mu^{-1}(0) = \varnothing$. G acts freely on this, but for silly reasons.
- If $c = 0$, $\mu^{-1}(0) = \{0\}$, so the U_1-action is not free.
- If $c > 0$,

$$\mu^{-1}(0) = \left\{\sum_{i=1}^n \lvert z_i\rvert^2 = 2c\right\} \cong S^{2n-1},$$

and the quotient is $\mathbb{C}^n /\!\!/_\mu \mathrm{U}_1 = \mu^{-1}(0)/\mathrm{U}_1 = \mathbb{CP}^{n-1}$. The induced metric is the Fubini-Study metric.

The last case is interesting: instead of taking the quotient X/G, we took a subset of X and quotiented out by a complex analogue of G: $X \setminus \{0\}/\mathbb{C}^\times$. This applies in more general situations. ◂

Definition 1.23. Let (X, I) be a complex manifold. A **holomorphic symplectic structure** on X is an $\Omega \in \Omega^{2,0}(X)$ such that

- $\mathrm{d}\Omega = 0$ and
- Ω is **holomorphically nondegenerate**, i.e. it induces an isomorphism $T^{1,0}X \to (T^{1,0}X)^*$.

The triple (X, I, Ω) is called a **holomorphic symplectic manifold**.

Proposition 1.24. *Let (X, I, Ω) be a holomorphic symplectic manifold. Then, $\dim_{\mathbb{R}}(X)$ is divisible by 4.*

[3] By this notation, we mean $[\mathfrak{g}, \mathfrak{g}]^\perp = \{f \in \mathfrak{g}^* \mid f(x) = 0 \text{ for all } x \in [\mathfrak{g}, \mathfrak{g}]\}$.

On a holomorphic symplectic manifold, there are particularly nice holomorphic coordinates, called **Darboux coordinates** $(p_1, \ldots, p_n, q_1, \ldots, q_n)$, for which

$$\Omega = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i.$$

This is the analogue of a symplectic structure, but on a complex manifold.

Now we know enough to define a hyperkähler manifold.

Definition 1.25. Let (X, g) be a Riemannian manifold and I, J, and K be three complex structures on X. Then, (X, g, I, J, K) is a **hyperkähler manifold** if

- $IJ = K$ in $\mathrm{End}(TM)$, and
- (X, g, I), (X, g, J), and (X, g, K) are all Kähler manifolds.

The corresponding Kähler forms are denoted ω_I, ω_J, and ω_K. Sometimes, I_1 is used for I, I_2 for J, and I_3 for K, and ω_i for ω_{I_i}.

The relation $IJ = K$ is equivalent to imposing the relations of the quaternions onto I, J, and K, i.e. $IJ = K$, $JK = I$, and $KI = J$; and $JI = -K$, $KJ = -I$, and $IK = -J$.

If you let $\Omega_1 \coloneqq \omega_2 + i\omega_3$, then (X, I, Ω_1) is a holomorphic symplectic manifold, and hence $\dim_{\mathbb{R}} X$ must be divisible by 4.