SUMMER 2016 HOMOTOPY THEORY SEMINAR

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Contents

1. Simplicial Localizations and Homotopy Theory: 5/24/16

"It may be a little dry, but it's been raining recently, so perhaps dryness will be good to have."

Today's lecture was given by Ernie Fontes.

The point of this seminar is to study simplicial localizations. This is a somewhat dry topic; today we're going to frame it, suggesting an outline for talks and some motivation. Thus, today we'll discuss homotopy theory in broad strokes.

A good first question: *what is homotopy theory?* Relatedly, *when can we do it?* In general, homotopy theory happens whenever we have a pair of categories $(\mathbf{C}, \mathbf{W})$, where $\mathbf{W}$ is a subcategory of $\mathbf{C}$. The idea is that $\mathbf{W}$ contains morphisms that we'd like to be isomorphisms. If $\mathbf{W}$ contains all of the objects of $\mathbf{C}$, then the pair $(\mathbf{C}, \mathbf{W})$ is called a *relative category*.

Example 1.1.

(1) Often, we choose $\mathbf{C} = \mathbf{Top}$, and make $\mathbf{W}$ the category of a nice class of morphisms, e.g. π_*-isomorphisms or homotopy equivalences.
(2) Another choice is to let $\mathbf{C} = \mathbf{ch}(R)$, the category of chain complexes of R-modules, where $\mathbf{W}$ is the category of *quasi-isomorphisms* (maps which induce an isomorphism on homology).

One nice property that $\mathbf{W}$ could have is the *two-out-of-six property*: that for all triples of morphisms $f : X \to Y$, $g : Y \to Z$, and $h : Z \to Z'$ in $\mathbf{C}$, if gf and hg are in $\mathbf{W}$, then so are f, g, h, and hgf. This implies the *two-out-of-three property*, that if any two of f, g, and h are in $\mathbf{W}$, then so is the third.

Definition 1.2. If $\mathbf{W}$ satisfies the two-out-of-six property, it is called a *homotopical category*.

In either setting, we can form the *homotopy category* $\mathrm{Ho}(\mathbf{C}) = \mathbf{C}[\mathbf{W}^{-1}]$, localizing $\mathbf{C}$ at $\mathbf{W}$. This is the initial category among those categories $\mathbf{D}$ and functors $\mathbf{C} \to \mathbf{D}$ sending the arrows in $\mathbf{W}$ to isomorphisms.

Most questions in homotopy theory can be framed in terms of the homotopy category: two spaces are homotopic iff they're isomorphic in $\mathrm{Ho}(\mathbf{C})$, and the homotopy classes of maps $X \to Y$ are the hom-set $\mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}(X, Y)$ in the homotopy category.

One question which does require a little more sophistication is understanding homotopy (co)limits. Since we've inverted a lot of arrows, taking limits or colimits in a homotopy category behaves very poorly. For example, there's no pushout of the degree-2 map $S^1 \to S^1$ along with the map $S^1 \to \mathrm{pt}$, since it "should be" $\mathbb{RP}^2$ but this doesn't satisfy it. $\mathbb{RP}^2$ is the homotopy pushout, however.

Often, one obtains more structure from a homotopy category, e.g. there are some ∞-categorical notions hiding in the background here. More concretely, one often obtains a natural model category structure, where in addition to the relative category $(\mathbf{C}, \mathbf{W})$, we have classes of cofibrant and fibrant morphisms satisfying a bunch of axioms. This provides tools for computing homotopy limits and colimits, etc., but it's a lot of data; even the definition is redundant (the cofibrations and fibrations determine each other). In fact, the punchline of the three papers we're reading is that only the structure of the relative category $(\mathbf{C}, \mathbf{W})$ is necessary to recover the entire model-categorical structure! For this reason, one makes the analogy that if homotopy theory is to linear algebra, picking a model-categorical structure is akin to picking a basis.

Definition 1.3. A *simplicial set* is a simplicial object in **Set**. That is, it's a collection of sets $\{X_i\}_{i\geq 0}$ and a bunch of maps $d_{ij} : X_i \to X_{i-1}$ for $0 \leq j \leq i$ and $s_{ij} : X_i \to X_{i+1}$ for $0 \leq j \leq i$ satisfying some relations that look like the boundary and inclusion relations for an i-simplex inside an $(i+1)$-simplex.

This is a vague definition, and we'll have a better one next lecture. These are akin to a "better" version of topological spaces, in that they model topological spaces very well, and can be described purely combinatorially.

Here's how the three papers of Dwyer and Kan break this information down.

(1) The first paper [DK1] constructs $\mathbf{C}[\mathbf{W}^{-1}]$, first as "just" a category, and then as a simplicially enriched category $L\mathbf{C}$, meaning that for all $X, Y \in \mathbf{C}$, $L\mathbf{C}(X, Y) \in \mathbf{sSet}$: that is, it's a simplicial set. In particular, we recover $\mathbf{C}[\mathbf{W}^{-1}]$ as the path components of this set: $\mathbf{C}[\mathbf{W}^{-1}](X, Y) = \pi_0 L\mathbf{C}(X, Y)$. There's a lot of comonadic computations here that may be confusing, but are applicable in many parts of algebra.

(2) In [DK2], Dwyer and Kan define a variant called the *hammock localization* $L^H\mathbf{C}(X, Y)_k$. The hammocks in question are commutative diagrams

$$\begin{tikzcd}
 & C_{0,1} \arrow[d, "\wr"] & C_{0,2} \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\wr"] & \cdots & C_{0,n-1} \arrow[l, "\sim"'] \arrow[d, "\wr"] \arrow[dr] & \\
X \arrow[ur] \arrow[r] \arrow[dr] & \vdots \arrow[d, "\wr"] & \vdots \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\wr"] & \cdots \arrow[r, "\sim"] & \vdots \arrow[r] \arrow[d, "\wr"] & Y \\
 & C_{k,1} \arrow[r, "\sim"] & C_{k,2} \arrow[r, "\sim"] & \cdots & C_{k,n-1} \arrow[l, "\sim"'] \arrow[ur] &
\end{tikzcd}$$

This might not seem like the best construction, but it expresses $L^H\mathbf{C}(X, Y)$ as a colimit of nerves of categories, which are easy to compute, and therefore this is surprisingly easy to work with when it comes to actually computing things. In particular, when certain weak (yet technical) properties hold, $L^H\mathbf{C}(X, Y) \simeq L\mathbf{C}(X, Y)$. The calculations in this paper are much more technical than the first, and it's worth going through more slowly.

(3) The third paper [DK3] establishes a relationship between (simplicially enriched) model categories and $L^H\mathbf{C}(X, Y)$. The takeaway is that the weak equivalences are all that you need to define a model categorical structure.

In the unlikely event we have time, there's an interesting relationship between this and algebraic K-theory: in a similar way, the algebraic K-theory of a model category actually only depends on the hammock localization, due to a paper [BM] of Blumberg and Mandell; this was a cool and surprising result.

Here's the list of planned talks; we can and should deviate from this in order to make sure we understand everything better.

(1) Simplicial sets, especially nerves and classifying spaces. This should definitely include a definition and some important constructions.

(2) Model categories; there's a lot we could talk about here, but we should talk about the definition, how to construct homotopy limits and colimits, mapping spaces, and fibrant and cofibrant replacement. This is intended to be an overview, rather than discussing complicated examples. This will be helpful to see all the structure we don't need!

(3) We then need to talk about localization in general, including the universal property for localizing rings, and discuss the discrete localization of categories. The hard version of this talk would also talk about Bousfield localization.
(4) Now, the first part of [DK1]: localization of $(\mathbf{C}, \mathbf{W})$, comonadic resolutions, and bar constructions, which detail how one constructs things. This is mostly all in the paper, and needs to be teased apart.
(5) Perhaps also it will be useful to discuss the rest of the model structure on small simple categories. Here Julie Bergner's thesis is a useful reference, as she treats this more clearly and in greater generality, though we may or may not need to refer to this.
(6) Moving to [DK2], introduce hammock localization. This is important to understand very closely; don't leave anything out of the talks.
(7) Then, we need homotopy calculus of fractions, which is useful for ensuring hammocks are small.
(8) We then need the theory of simplicial model categories; these have more structure and are more excellent than ordinary model categories. The key is understanding the axiom SM7 for a simplicial model category.
(9) Finally, we should treat the main theorem in [DK3], that $L^H\mathbf{C}(X, Y)$ models the simplicial derived mapping space in a model category.

At that point, the summer will be over, and we will be done.

2. Simplicial Sets: 5/31/16

These are Arun's prepared notes for this talk.

Simplicial sets are a combinatorial analogue of topological spaces that are often simpler to work with, yet in a sense contain the same information from the perspective of homotopy theory. At the same time, they also behave like a nonlinear analogue of chain complexes.

2.1. Two Definitions of Simplicial Sets.

One definition is formal, and easier to write down; the other is more geometric, but requires more words.

Definition 2.1. The *simplex category* $\mathbf{\Delta}$ is the category whose objects are the ordered sets $[n] = \{0, 1, \dots, n\}$ for $n \geq 0$, and whose morphisms are order-preserving functions.

Definition 2.2. A *simplicial set* is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$. With natural transformations as morphisms, these form the category $\mathbf{sSet}$. More generally, for any category $\mathbf{C}$, a *simplicial object* in $\mathbf{C}$ is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{C}$.

That is, a simplicial set X is a set X_n for each $[n]$ (called the set of *n-simplices*) with compatible actions by the morphisms in $\mathbf{\Delta}$. A morphism of simplicial sets $X \to Y$ is a collection of maps $X_n \to Y_n$ for each n that commutes with those actions.

This doesn't seem very topological or geometric; here's another definition.

Definition 2.3. A *simplicial set* X is a collection of sets X_n for each $n \geq 0$, along with functions $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ for $0 \leq i \leq n$, called the *face maps* and *degeneracy maps*, respectively, satisfying the relations

$$\begin{aligned} d_i \circ d_j &= d_{j-1} \circ d_i, \quad i < j \\ s_i \circ s_j &= s_{j+1} \circ s_i, \quad i \leq j \\ d_i \circ s_j &= \begin{cases} 1, & i = j \text{ or } i = j+1 \\ s_{j-1} \circ d_i, & i < j \\ s_j \circ d_{i-1} & i > j+1. \end{cases} \end{aligned} \tag{2.4}$$

A morphism of simplicial sets $f : X \to Y$ is a collection of maps $f_n : X_n \to Y_n$ that commute with the face and degeneracy maps.

We can think of the object $[n] \in \mathbf{\Delta}$ as the standard n-simplex (triangle, tetrahedron, …); in this case, the face map d_i is induced from the inclusion of the i^{th} face (which is a copy of $[n-1]$) and the degeneracy map s_i is induced from the projection onto the i^{th} face. If you play with this picture, you end up writing down the definitions in (2.4).

Example 2.5.

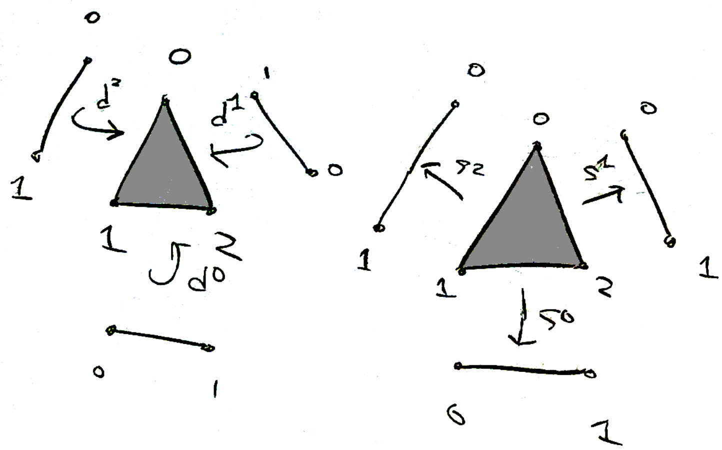

FIGURE 1. Examples of generating maps in Δ that induce the face and degeneracy maps of a simplicial set.

(1) The *standard n-simplex* is the $\Delta^n = \mathrm{Hom}_\Delta(-,[n])$. Thus, by the Yoneda lemma, for any simplicial set X, $\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, X) = X_n$. Geometrically, think of standard n-simplex as, well, the n-dimensional simplex: the i^{th} face map is the assignment to the i^{th} face of this simplex, and the i^{th} degeneracy map realizes Δ^n as a degenerate $(n+1)$-simplex where vertices i and $i+1$ coincide. See Figure 2 for a depiction of the standard 3-simplex.

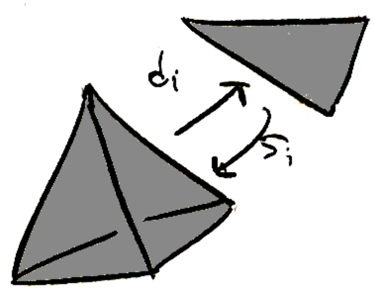

FIGURE 2. The standard 3-simplex, with example face and degeneracy maps.

By the Yoneda lemma, Δ^n corepresents the functor $X \mapsto X_n$. That is, there is a natural isomorphism (of sets)

$$\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, X) \cong X_n. \tag{2.6}$$

(2) Given a simplicial set X, we can form its *k-skeleton* in much the same way as for CW complexes, by preserving $X_0, \dots, X_k$ and the maps between them, but making all higher simplices degenerate.
(3) The *simplicial n-sphere*, denoted $\partial\Delta^n$, is the $(n-1)$-skeleton of Δ^n. Geometrically, this is the n-simplex minus its interior, which is a reasonable thing to call a sphere (to homotopy theorists, at least). Another equivalent formulation is to take $\Delta^n \setminus \{\mathrm{id}\}$ (regarding it as a functor), or the union (or colimit) of all of the faces of Δ^n across the morphisms gluing *their* faces (which are $(n-2)$-simplices).
(4) The *simplicial horn* Λ^n_k is the union (or colimit) of all faces of Δ^n except for the k^{th} face. The notation Λ is suggestive of the geometry. If X is a simplicial set, a *horn in X* is a map of simplicial sets $\Lambda^n_k \to X$.

Definition 2.7. A simplicial set X is a *Kan complex* if every horn $\Lambda^n_k \to X$ can be extended to a map $\Delta^n \to X$, i.e. it factors through the inclusion $\Lambda^n_k \hookrightarrow \Delta^n$. If this is only true for *inner horns*, i.e. $\Lambda^n_k \hookrightarrow X$ where $0 < k < n$, then X is called a *weak Kan complex*.

One says that "every horn has a filler."

Like **Top**, the category **sSet** is complete and cocomplete: all limits and colimits exist, and in fact can be constructed levelwise. In particular, products exist.

Another nice property of this category is that we can build a simplicial set of the morphisms between two simplicial sets, rather than just a set.

Definition 2.8. Given $X, Y \in \mathbf{sSet}$, their *function complex* is the simplicial set $\mathbf{sSet}(X, Y)$ whose n-simplices are the set $\mathbf{sSet}(X, Y)_n = \mathrm{Hom}_{\mathbf{sSet}}(X \times \Delta^n, Y)$, with face and degeneracy maps induced from those on Δ^n.

For any simplicial set Y, there is an adjunction $(- \times Y, \mathbf{sSet}(Y, -))$; one says that **sSet** is *Cartesian closed*. Other Cartesian closed categories include **Set** and the category of compactly generated spaces.

Definition 2.9. A *simplicially enriched category* **C** is defined in exactly the same way as a category, but for every $X, Y \in \mathbf{C}$, there is a simplicial set (sometimes called the *function complex*) $\mathbf{C}(X, Y)$ of morphisms between them, instead of a set.

We require an associative composition law as usual; the identity is a distinguished 0-simplex in $\mathbf{C}(X, X)$ satisfying the same properties as usual.

Sometimes these are called "simplicial categories," but that term is also used to refer to simplicial objects in **Cat**; here **Cat** is the category of small categories with functors as morphisms. However, we can identify simplicially enriched categories with the simplicial categories whose objects are the same in every dimension.

If X and Y are simplicial sets, we defined their function complex $\mathbf{sSet}(X, Y)$, so **sSet** is a simplicially enriched category. The categories **Set** and **Top** can also be simplicially enriched, e.g. $\mathbf{Top}(X, Y)_n = \mathrm{Hom}_{\mathbf{Top}}(X \times |\Delta^n|, Y)$.

2.2. **Geometric Realization and the Total Singular Complex.** Simplicial sets are closely related to topological spaces: they're built out of n-simplices, which are manifestly topological objects. As such, there is an adjunction

$$|-| : \mathbf{sSet} \rightleftarrows \mathbf{Top} : S \tag{2.10}$$

relating simplicial sets and topological spaces.

The left adjoint is called *geometric realization*, and does in fact geometrically realize a simplicial set as a topological space: start with a concrete n-simplex in **Top** for every (abstract) n-simplex in X. Then, the face and degeneracy maps identify some of the faces of these n-simplices, so glue the corresponding concrete simplices together along those edges. Rigorously, "gluing" means a colimit. A simplicial set is essentially the data of n-simplices glued together in a specific way, and in particular

$$X \cong \varinjlim_{\Delta^n \to X} \Delta^n,$$

where the colimit is taken across all maps $\Delta^n \to X$ directed under arrows $\theta : \Delta^n \to \Delta^m$ that commute with the maps to X. We already know how to realize Δ^n as the standard n-simplex $|\Delta^n|$, so the geometric realization can be defined in parallel:

$$|X| = \varinjlim_{\Delta^n \to X} |\Delta^n|.$$

The geometric realization of a simplicial set is a CW complex.

The right adjoint is called the *total singular complex*, and belongs to the analogy

simplicial sets : chain complexes :: total singular complex : singular chain complex.

If Y is a topological space, we've already defined a chain complex of maps from the standard n-simplices into Y; this is a refinement. The total singular complex SY is defined by setting $SY_n = \mathrm{Hom}_{\mathbf{Top}}(\Delta^n, Y)$, the set of all continuous maps of the standard n-simplex (as a topological space) into Y. Given a map $f : \Delta^n \to Y$, we can restrict it to the i^{th} face; this is exactly what the i^{th} face map does. Applying the degeneracy map is given by collapsing Δ^{n+1} onto Δ^n at the i^{th} vertex, then composing with f, giving a map $\Delta^{n+1} \to Y$ as desired.

From this definition, the adjunction isn't too hard to see.

Proposition 2.11. *The functors* $|-|$ *and* S *defined above are adjoint, as in* (2.10).

Proof. We want to show for all spaces Y and simplicial sets X, there's a natural isomorphism $\mathrm{Hom}_{\mathbf{Top}}(|X|, Y) \cong \mathrm{Hom}_{\mathbf{sSet}}(X, SY)$. First, it's true when $X = \Delta^n$: $SY_n = \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, Y)$ by definition, and $SY_n = \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, SY)$

by (2.6). Since $\mathrm{Hom}_{\mathbf{C}}(A, -)$ sends colimits to limits,

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Top}}(|X|, Y) &\cong \varprojlim_{\Delta^n \to X} \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, Y)\\ &\cong \varprojlim_{\Delta^n \to X} \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, SY)\\ &\cong \mathrm{Hom}_{\mathbf{sSet}}(X, SY).\end{aligned}$$

⊠

One can recover the singular chain complex $C_\bullet(X)$ from the total singular complex by setting $C_n(X)$ to be the free abelian group on X_n with the boundary map

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i. \tag{2.12}$$

Fact. The total singular complex SY is a Kan complex. There's a sense in which this adjunction defines an equivalence of the homotopy theories of **sSet** and **Top**.

2.3. **The Nerve of a Category.** For any small category $\mathbf{C}$, we can build a simplicial set $N\mathbf{C}$, called the *nerve* of $\mathbf{C}$; this is functorial in $\mathbf{C}$, defining a functor $\mathbf{Cat} \to \mathbf{sSet}$ (here **Cat** is the category of small categories, with functors as morphisms).

The construction is as follows:

- $N\mathbf{C}_0$ be the set of objects in $\mathbf{C}$.
- $N\mathbf{C}_1$ is the morphisms of $\mathbf{C}$.
- $N\mathbf{C}_2$ is the set of pairs of composable morphisms $X \to Y \to Z$.
- If $n \geq 2$, $N\mathbf{C}_n$ is the set of n-tuples of composable morphisms $X_0 \to X_1 \to \cdots \to X_n$.

In other words, if $[n]$ is regarded as a poset category (so there's a unique map $i \to j$ iff $i \leq j$), $N\mathbf{C}_n$ is the set of functors $[n] \to \mathbf{C}$.

The degeneracy map $s_i : N\mathbf{C}_n \to N\mathbf{C}_{n+1}$ takes a string of arrows and inserts the identity at the i^{th} position. The face map $d_i : N\mathbf{C}_n \to N\mathbf{C}_{n-1}$ replaces the i^{th} and $(i+1)^{\text{th}}$ arrows with their composition, unless $i = 0$ or $i = n$, in which case it just drops the first or last arrow, respectively.

Fact. The nerve of a category is a weak Kan complex.

Example 2.13 (Classifying spaces)**.** If one interprets a group G as a category with a single object, its nerve will correspond to the classifying space BG.

More precisely, a discrete group G defines a category $\mathbf{G}$ with a single object $\bullet$ and $\mathrm{Hom}_{\mathbf{G}}(\bullet, \bullet) = G$, with group multiplication as composition. Its nerve $N\mathbf{G}$ is the simplicial set whose set of n-simplices is G^n: the i^{th} degeneracy map includes e at index i, and the i^{th} face map $d_i : G^n \to G^{n-1}$ multiplies indices i and $i+1$ together (unless $i = 0$ or n, in which case that index is dropped).

Define another simplicial set X whose n-simplices are $X_n = G^{n+1}$ with the same degeneracy maps and face maps, except for d_n, which sends $(g_1, \dots, g_n, g_{n+1}) \mapsto (g_1, \dots, g_n g_{n+1})$ instead of dropping the last index. Then, projection onto the first n coordinates defines maps $p_n : X_n \to N\mathbf{G}_n$ commuting with the face and degeneracy maps, so we obtain a map of simplicial sets $p : X \to N\mathbf{G}$.

Multiplication on the last coordinate defines a right action of G on X: if $h \in G$, $(g_1, \dots, g_{n+1}) \cdot h = (g_1, \dots, g_n, g_{n+1}h)$. This commutes with the face and degeneracy maps of X, making it a simplicial G-set, and the fibers of p are G-torsors.

Now, we geometrically realize, suggestively defining $EG = |X|$ and $BG = |N\mathbf{G}|$. Projection $\pi = |p| : EG \to BG$ is a fiber bundle whose fibers are G-torsors, so $\pi : EG \to BG$ is a principal G-bundle. It's true, albeit harder to show, that EG is contractible, and therefore BG is a model for the classifying space of G.[1] Since G is discrete, BG is also a concrete model for $K(G, 1)$.

Example 2.14 (Bar construction)**.** We can generalize Example 2.13 and obtain a surprisingly useful class of simplicial objects.

Let $\mathbf{C}$ be a monoidal category, M be a monoid in $\mathbf{C}$, and $X, Y \in \mathbf{C}$ be acted on by M from the right and left, respectively.

[1]You might be wondering what happens if G isn't discrete, the case where classifying spaces are more interesting. Nearly the same story applies: we regard G as a single-object category enriched over **Top**, so NG is a *simplicial space* (i.e. simplicial object in **Top**). Geometric realization of simplicial spaces goes through to define the principal G-bundle $EG \to BG$ in the same way.

- If $\mathbf{C} = \mathbf{Top}$, this is the notion of a continuous monoid action (from the right or the left), akin to that of a continuous group action.
- If $\mathbf{C} = \mathbf{Mod}_R$ for a commutative ring R a monoid S in $\mathbf{C}$ is an R-algebra, X is a right S-module and Y is a left S-module.

We'll build a simplicial object in $\mathbf{C}$ called the *bar construction* $B(X, M, Y)$, reminiscent of the nerve:

- The n-simplices $B_n(X, M, Y) = X \otimes M^{\otimes n} \otimes Y$ (here, $\otimes$ denotes the monoidal product; for $\mathbf{C} = \mathbf{Top}$ or $\mathbf{C} = \mathbf{Set}$, this is just Cartesian product).
- If $0 < i < n$, the i^{th} face map multiplies together the i^{th} and $(i+1)^{\text{th}}$ indices:

$$d_i : (x, m_1, \dots, m_n, y) \mapsto (x, m_1, \dots, m_i m_{i+1}, \dots, m_n, y).$$

- The 0^{th} face map sends $(x, m_1, \dots, m_n, y) \mapsto (x \cdot m_1, m_2, \dots, m_n, y)$, and correspondingly the n^{th} face map sends $(x, m_1, \dots, m_n, y) \mapsto (x, m_1, \dots, m_{n-1}, m_n \cdot y)$.
- The i^{th} degeneracy map s_i inserts the identity $e \in M$ at the i^{th} index.

If we know how to geometrically realize simplicial $\mathbf{C}$-objects, then this produces a genuine object of $\mathbf{C}$.

- Suppose $\mathbf{C} = \mathbf{Top}$ and $M = G$ is a group. Then, $|B(*, G, *)| = BG$ and $|B(*, G, G)| = EG$ are exactly the constructions we gave in Example 2.13.
- Suppose $\mathbf{C} = \mathbf{Mod}_R$, so the monoid $M = S$ is an R-algebra. Then, $B(X, S, Y)$ is a simplicial R-module, so we can define a chain complex $K_\bullet(X, S, Y)$ of R-modules by letting the boundary map be as in (2.12). This chain complex is the usual resolution for computing $\mathrm{Tor}_S(X, Y)$!

2.4. **Simplicial Homotopies.** Homotopies of topological spaces are defined via unit interval $[0, 1]$; for simplicial sets, Δ^1 plays the analogous role. Everything in the next two sections comes from [GJ].

Definition 2.15. Let $f, g : X \rightrightarrows Y$ be two morphisms of simplicial sets. A *homotopy* $\eta : f \Rightarrow g$ is a morphism $\eta : X \times \Delta^1 \to Y$ such that the diagram

$$\begin{tikzcd} X \cong X \times \Delta^0 \arrow[r, "(\mathrm{id}, d^1)"] \arrow[dr, "f"'] & X \times \Delta^1 \arrow[d, "\eta"] & X \times \Delta^0 \cong X \arrow[l, "(\mathrm{id}, d^0)"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

commutes. Here, $d^0, d^1 : \Delta^0 \rightrightarrows \Delta^1$ are the maps realizing Δ^0 as the zeroth and first vertices of Δ^1, respectively.

This is probably more or less what you were expecting. However, what's more surprising is that homotopy is not an equivalence relation! The 1-simplex Δ^1 defines a homotopy from d^0 to d^1 as maps $\Delta^0 \to \Delta^1$, but since $1 > 0$, there's no 1-simplex which can produce a homotopy from d^1 to d^0. This is awkward, but we do have the following result.

Proposition 2.16. *Homotopy is an equivalence relation on maps $X \to Y$ iff Y is a Kan complex.*

As such, we define simplicial homotopy groups for Kan complexes. Instead of picking an arbitrary basepoint, as we did in $\mathbf{Top}$, we choose a vertex $x : \Delta^0 \to X$.

Definition 2.17. If X is a Kan complex, $x : \Delta^0 \to X$ is a vertex, and $n > 0$, we define the n^{th} *simplicial homotopy group* based at x to be the set of homotopy classes of maps $\Delta^n \to X$ that fix the boundary $\partial\Delta^n$, such that $\partial\Delta^n$ maps to x. That is, we consider maps $f : \Delta^n \to X$ fitting into a commutative diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \Delta^0 \arrow[d, "x"] \\ \Delta^n \arrow[r, "f"] & X, \end{tikzcd}$$

where two maps are equivalent if there is a homotopy between them fixing $\partial\Delta^n$. To define the group structure, let $f, g : \Delta^n \rightrightarrows X$ represent two elements of $\pi_n(X, x)$. Let

$$v_i = \begin{cases} x, & 0 \le i \le n-2 \\ f, & i = n-1 \\ g, & i = n+1. \end{cases}$$

Then, the assignment $i \mapsto v_i$ defines a map $\tilde{h} : \Lambda_n^{n+1} \to X$, so since X is a Kan extension, this extends to a map $h : \Delta^{n+1} \to X$. Then, we define $[a] \cdot [b] = [d_n h]$,[2] which one can show makes $\pi_n(X, x)$ into a group (the constant map to v is the identity), and an abelian group if $n \ge 2$.

We define $\pi_0(X)$, the set of *path components* of X, to be the set of homotopy classes of vertices of X.

Definition 2.18. Let X and Y be Kan complexes. A map $f : X \to Y$ is a *weak equivalence* if for all vertices x of X and $n \ge 1$, the induced map $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism, and $f_* : \pi_0(X) \to \pi_0(Y)$ is a bijection.

2.5. **Bisimplicial Sets.** Dwyer and Kan mention in [DK1] that they "will often use, explicitly or implicitly," a result about bisimplicial sets (Proposition 2.21, below). As such, it's probably a good idea to at least explain what they're saying.

Bisimplicial sets fit into the analogy

simplicial sets : chain complexes :: bisimplicial sets : double complexes.

As double complexes are important in the genesis of spectral sequences, you might guess bisimplicial objects are too, and you'd be right.

Definition 2.19. A *bisimplicial set* is a simplicial object in **sSet**; equivalently, it is a functor $\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} = (\mathbf{\Delta} \times \mathbf{\Delta})^{\mathrm{op}} \to$ **Set**. Replacing sets with another category **C** defines the notion of a *bisimplicial object* in **C**.

Given a bisimplicial set viewed as a functor $K : (\mathbf{\Delta} \times \mathbf{\Delta})^{\mathrm{op}} \to \mathbf{Set}$, $K([m], [n])$ is written $K_{m,n}$ and called the *degree-(m, n) bisimplices* of K. The face and degeneracy maps are bigraded, denoted d_{ij} and s_{ij}.

Definition 2.20. If K is a bisimplicial set, its *diagonal* $\operatorname{diag} K$ is the simplicial set with n-simplices $(\operatorname{diag} K)_n = K_{n,n}$ and whose face and degeneracy maps are the diagonal maps d_{ii} and s_{ii}.

That is, if K is the functor $\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ and $\mathrm{Diag} : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}$ is the diagonal functor, then $\operatorname{diag} K = K \circ \mathrm{Diag}$.

Alternatively, thinking of K as a simplicial object in **sSet**, its n-simplices are a simplicial set $K_{n,\bullet}$. These are called the *vertical simplicial sets* associated to K.

Proposition 2.21 [GJ, Prop. IV.1.9]. *If $K \to L$ is a map of bisimplicial sets such that, for every integer $n \ge 0$, the restriction $K_{n,\bullet} \to L_{n,\bullet}$ is a homotopy equivalence, then its diagonal* $\operatorname{diag} K \to \operatorname{diag} L$ *is also a weak homotopy equivalence.*

3. Model Categories: 6/7/16

"This isn't German, so '∞-category' isn't one word."[3]

Today's talk, also titled "I wish I had known this when I started learning about model categories," was given by Adrian. He will demystify the formalism of model categories, which you can read about in any book, by explaining that they correct two serious deficiencies of relative categories, that

- localization in relative categories is not well-behaved, and
- relative functors between relative categories do not pass well to derived functors after localization.

Today, the word "simplicial category" means a simplicially enriched category, as in the papers of Dwyer and Kan.

[2] There's a lot to check here: why is $d_n h$ constant on $\partial\Delta^n$? Why is this independent of choice of representative for $[a]$ and $[b]$?

[3] In German, the word is *Unendlichkategorie*.

3.1. Introduction: Relative Categories.

Definition 3.1. A *relative category* $(\mathbf{C}, \mathbf{W})$ is a category $\mathbf{C}$ and a subcategory $\mathbf{W} \subseteq \mathbf{C}$ containing all objects and all isomorphisms.

For example, if R is a commutative ring, the category $\mathbf{Ch}_{\geq 0,R}$ of chain complexes of R-modules in nonnegative degree is a relative category, with $\mathbf{W}$ the (full) subcategory of weak equivalences.

Somehow related to this is the notion of an ∞*-category*, which is, vaguely speaking, a category enriched in homotopy types.

Definition 3.2. Let $\mathbf{C}$ be a simplicial category. Then, its *homotopy category* $\mathbf{ho}(\mathbf{C})$ or $\pi_0\mathbf{C}$ is the category with the same objects as $\mathbf{C}$ and whose morphisms are $(\pi_0\mathbf{C})(X, Y) = \pi_0\mathbf{C}(X, Y)$.

That is, we've identified a morphism with all others in its connected component. Given a functor $F : \mathbf{C} \to \mathbf{C}'$, one obtains a functor $\pi_0 F : \pi_0\mathbf{C} \to \pi_0\mathbf{C}'$.

Definition 3.3. Let $\mathbf{C}$ and $\mathbf{C}'$ be simplicial categories. A functor $F : \mathbf{C} \to \mathbf{C}'$ is a *Dwyer-Kan equivalence*, or *DK-equivalence*, if $\pi_0 F : \pi_0\mathbf{C} \to \pi_0\mathbf{C}'$ is essentially surjective (so for all $X, Y \in \mathbf{C}$, $F : \mathbf{C}(X, Y) \to \mathbf{C}'(FX, FY)$ is an isomorphism).

That is, a Dwyer-Kan equivalence is not an equivalence of categories, but is an equivalence of homotopy types.

The hammock localization we're going to study will be an assignment from relative categories to simplicial categories. If in addition we work with small categories, there's a theorem of Barwick and Kan that shows that relative categories under DK equivalence are the same as simplicial categories under DK equivalence, with sameness in a precise sense.

Since simplicial categories present ∞-categories, this also means that relative categories present ∞-categories. One has obtained quite a lot of structure from very little information. The analogy is that $\mathbf{C}$ is the generators and $\mathbf{W}$ is the relations, akin to generators and relations of most other algebraic structures.

Classically, the usual notion of localization produces an ordinary category from a relative category, very akin to the localization of a ring or module at a multiplicative subset. The zigzags in this equivalence relation are akin to the zigzags in the hammock localization, and in fact, after taking π_0 of a hammock localization, one obtains the ordinary localization. This story continues in [GZ].

All this is great, but relative categories have problems, which is why we'll introduce model categories.

(1) A big (meaning objects do not form a set) relative category need not be localizable. Consider a category with a proper class of objects, and two distinguished objects $\bullet_1$ and $\bullet_2$, and an isomorphism from $\bullet_1$ and $\bullet_2$ to every other object. Then, if we localize at everything except $\bullet_1$ and $\bullet_2$, the morphisms between them do not form a set, which is bad.
(2) Even when we can localize, it's quite hard to detect.
(3) Even when we know we can localize, it's often hard to describe the localization.[4]
(4) Another drawback is that relative functors are often not sufficient for discussing ∞-functors in practice. This comes up in practice when dealing with homotopy (co)limits: in nice situations one can recover them from regular (co)limits and the relative-categorical structure, but other times one cannot. Model categories increase the range of situations where we may do this.

3.2. Derived Functors.

Definition 3.4. Let $(\mathbf{C}, \mathbf{W})$ be a relative category, $\mathbf{D}$ be a category, and $F : \mathbf{C} \to \mathbf{C}$ be a functor. Suppose that $(\mathbf{C}, \mathbf{W})$ is localizable. Then, the *left derived functor* of F, denoted $LF : L\mathbf{C} \to \mathbf{D}$, is the right Kan extension of the diagram

$$\begin{tikzcd} \mathbf{C} \arrow[r, "F"] \arrow[d] \arrow[dr, phantom, "\Nwarrow"] & \mathbf{D} \\ L\mathbf{C}. \arrow[ur, "\mathrm{RKan}\,F"'] & \end{tikzcd}$$

The right Kan extension is the indicated functor and natural transformation, such that any other functor (with natural transformation) mapping to the diagram has a unique natural transformation to RKan F. There is an analogous notion of a left Kan extension.

[4] If you only want to solve this problem, you can work with simpler structures than model categories: it suffices to have a relative category with left (or right) calculus of fractions. See [GZ] again.

Definition 3.5. If $(\mathbf{C}, \mathbf{W})$ and $(\mathbf{C}', \mathbf{W}')$ are (localizable) relative categories, $F : \mathbf{C} \to \mathbf{C}'$ is a functor, and $\varphi : \mathbf{C}' \to L\mathbf{C}'$ is the localization functor, then the *total left derived functor* of F is the left derived functor of $\varphi \circ F$, and maps $L\mathbf{C} \to L\mathbf{C}'$.

This also exists in a right-hand variant.

Definition 3.6. If $(\mathbf{C}, \mathbf{W})$ is a relative category, then a *left deformation* (Q, ε) of $(\mathbf{C}, \mathbf{W})$ is a functor $Q : \mathbf{C} \to \mathbf{C}$ and a natural transformation $\varepsilon : Q \Rightarrow \mathrm{id}$.

Theorem 3.7 [DHKS]. *Let $(\mathbf{C}, \mathbf{W})$ and $(\mathbf{C}', \mathbf{W}')$ be relative categories, (Q, ε) be a deformation on $\mathbf{C}$, and $F : \mathbf{C} \to \mathbf{C}'$ be a functor. If $F|_{Q\mathbf{C}}$ is relative, then the following diagram commutes.*

$$\begin{tikzcd}
& \mathbf{C} \arrow[r, "F"] \arrow[dd] & \mathbf{C}' \arrow[dd] \\
Q\mathbf{C} \arrow[ur, hook, "Q"] \arrow[dd] & & \\
& L\mathbf{C} \arrow[r, "\mathrm{t.l.}\,F"] & L\mathbf{C}' \\
LQ\mathbf{C}. \arrow[ur, hook, "\simeq"] \arrow[urr, "LF|_{Q\mathbf{C}}"'] & &
\end{tikzcd}$$

As an example, if we take the relative category of chain complexes, we could take Q to be projective replacement, and this can actually tell us something concrete. I guess the point is that deformations can allow us to enlarge the amount of functors that present ∞-functors. For more on derived functors, check out Kahn-Maltsineotis, who provide a considerably more general theory of derived functors.

There are plenty of theorems that tell us how to localize categories, and there are plenty that inform us how to take derived functors. The advantage of model categories is that they allow us to do both.

3.3. **Model Categories.**

Definition 3.8. Consider a commutative square of morphisms in a category

$$\begin{tikzcd}
\bullet \arrow[r, "a"] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\
\bullet \arrow[r, "b"'] & \bullet,
\end{tikzcd}$$

then f has the *left lifting property* (LLP) with respect to g, and g has the *right lifting property* (RLP) with respect to f if for all a, b making the diagram commute, there exists a φ making the following diagram commute:

$$\begin{tikzcd}
\bullet \arrow[r, "a"] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\
\bullet \arrow[r, "b"'] \arrow[ur, "\varphi"] & \bullet,
\end{tikzcd}$$

In this case, one writes that $f \boxslash g$.

If M is a subcategory of $\mathbf{C}$ that contains all objects, denoted $M \subseteq \operatorname{mor} \mathbf{C}$, we define

$$^{\boxslash}M = \{f \in \operatorname{mor} \mathbf{C} \mid \text{ for all } g \in M, f \boxslash g\}$$
$$M^{\boxslash} = \{g \in \operatorname{mor} \mathbf{C} \mid \text{ for all } f \in M, f \boxslash g\}.$$

Proposition 3.9. *$M^{\boxslash}$ is a category (i.e. it's closed under composition), contains all isomorphisms, and is closed under retracts and base change.*

Proposition 3.10. *Suppose $F : \mathbf{C} \rightleftarrows \mathbf{D} : G$ is an adjunction, $L \subseteq \operatorname{mor} \mathbf{C}$, and $R \subseteq \operatorname{mor} \mathbf{D}$. Then, $F(L) \boxslash R$ iff $L \boxslash G(R)$.*

These are both not difficult to prove, and highlight the essentially algebraic nature of factorization systems.

Definition 3.11. A *weak factorization system* is a pair $A, B \subseteq \operatorname{mor} \mathbf{C}$ such that

- for all $\mathbf{C}$-morphisms $f : X \to Y$, there exist morphisms $u \in A$ and $p \in B$ such that $f = p \circ u$, and
- $A = B^{\boxslash}$ and $A^{\boxslash} = B$.

Definition 3.12. Let $(\mathbf{M}, \mathbf{W})$ be a relative category. Then, a *model structure* on $(\mathbf{M}, \mathbf{W})$ is a pair of classes of morphisms $C, F \subseteq \operatorname{mor} \mathbf{M}$ such that:

- $\mathbf{M}$ contains all finite limits and colimits.
- $\mathbf{W}$ has the two-out-of-three property: if $f = g \circ h$ and any two of f, g, and h are in $\mathbf{W}$, so is the third.
- $(C \cap \mathbf{W}, F)$ and $(C, F \cap \mathbf{W})$ are weak factorization systems.

In this case, we have a bunch of words.

- The morphisms in C are called *cofibrations* and denoted $\twoheadrightarrow$, and those in F are called *fibrations*, denoted $\hookrightarrow$.[5]
- A morphism in $C \cap \mathbf{W}$ is called a *acyclic cofibration*, and one in $F \cap \mathbf{W}$ is called an *acyclic fibration*.
- If $*$ denotes the final object in $\mathbf{M}$ and $\varnothing$ denotes the initial object, then an $X \in \mathbf{M}$ is *fibrant* if $X \to *$ is a fibration, and similarly X is *cofibrant* if $\varnothing \to X$ is a cofibration. The fibrant objects are denoted $\mathbf{M}_f$, the cofibrant ones are denoted $\mathbf{M}_c$, and the *fibrant-cofibrant* (i.e. both fibrant and cofibrant) objects are denoted $\mathbf{M}_{fc}$.
- The weak factorization systems imply that for any $X \in \mathbf{M}$, the map $X \to *$ factors as $X \overset{\sim}{\hookrightarrow} X_f \twoheadrightarrow *$ (i.e. the first arrow is an acyclic fibration). This X_f is fibrant, and is called the *fibrant replacement* of X.
- In the same way, the map $\varnothing \to X$ factors as $\varnothing \hookrightarrow X_c \overset{\sim}{\twoheadrightarrow} X$; X_c is cofibrant and is called the *cofibrant replacement* of X.

Example 3.13. Consider again $\mathbf{Ch}_{\geq 0,R}$, the bounded-below chain complexes of R-modules; we already know the weak equivalences to be the quasi-isomorphisms. We can place a model structure on this category, where:

- the cofibrations are the monomorphisms $X \hookrightarrow Y$ such that Y/X is levelwise projective, and
- the fibrations are the epimorphisms.

Thus, all objects are fibrant, and levelwise projective objects are fibrant. Cofibrant replacement is an epimorphism onto a levelwise projective module, which is projective replacement.

Proposition 3.14. *Let $F : \mathbf{M} \rightleftarrows \mathbf{M}' : G$ be an adjunction between the model categories $\mathbf{M}$ and $\mathbf{M}'$. Then, the following are equivalent:*

(1) F preserves cofibrations and acyclic cofibrations.
(2) G preserves fibrations and acyclic fibrations.
(3) F preserves cofibrations and G preserves fibrations.

In this case, (F, G) are called a *Quillen adjunction.* We'll see that Quillen adjunctions induce adjunctions between the localized categories (one must show that the relevant localizations exist, but this is fortunately true).

Proposition 3.15. *With notation as in Proposition 3.14, if (F, G) form a Quillen adjunction, the following are equivalent.*

(1) (F, G) induce an equivalence between $L\mathbf{M}$ and $L\mathbf{M}'$.
(2) For all fibrant $A \in \mathbf{M}$ and cofibrant $X \in \mathbf{M}'$, $f : FA \to X$ is a weak equivalence iff its adjoint $f^* : A \to GX$ is.

In this case, (F, G) are called a *Quillen equivalence.*

One should justify why these localizations exist. This is ultimately due to a particular corollary of Ken Brown's lemma.

Lemma 3.16 (Ken Brown)**.** *With the notation of Proposition 3.14, if (F, G) is a Quillen adjunction and if fibrant and cofibrant replacement are functorial on $\mathbf{M}$ and $\mathbf{M}'$, then F and G are localizable.*

The takeaway is: functorial fibrant and cofibrant replacement allow us to compute left and right derived functors by taking cofibrant and fibrant replacement, respectively, exactly how we computed Ext and Tor in homological algebra (by taking projective and injective replacements).

3.4. **Model Categories are Localizable.**

Definition 3.17. Let $\mathbf{M}$ be a model category and $X \in M$.

- A *cylinder object* for X in $\mathbf{M}$, denoted $\operatorname{Cyl} X$, is an object such that the co-diagonal map $X \amalg X \to X$ factors through $X \amalg X \hookrightarrow \operatorname{Cyl} X \xrightarrow{\sim} X$.

[5]It's nice to know that fibrations and cofibrations often behave like the ones in topology, though not always: one can take the model structure on an opposite category, and in that case the intuition is reversed.

- A *path space object* for X, denoted PX, is an object such that the diagonal map $X \to X \times X$ factors as $X \xrightarrow{\sim} PX \twoheadrightarrow X \times X$.

The names are suggestive: in the model structure on **Top**, $\mathrm{Cyl}\, X = I \times X$ and $PX = \mathbf{Top}(I, X)$, exactly what we would call cylinders and path spaces. And just like for topological spaces, we can use these to do homotopy theory.

Definition 3.18. Let $f, g : A \rightrightarrows X$ be two morphisms in a model category $\mathbf{M}$.

- A *left homotopy* from f to g is a map $H : \mathrm{Cyl}\, A \to X$ such that the diagram

$$\begin{tikzcd} & \mathrm{Cyl}\, A \arrow[dr, "H"] & \\ A \amalg A \arrow[ur] \arrow[rr, "f \amalg g"] & & X \end{tikzcd}$$

  commutes.
- A *right homotopy* from f to g is a map $H : A \to PX$ such that the diagram

$$\begin{tikzcd} & PX \arrow[dr] & \\ A \arrow[ur, "H"] \arrow[rr, "f \times g"] & & X \times X \end{tikzcd}$$

  commutes.

Two different notions might be surprising; it's true in **Top** that left and right homotopy are the same, but this is not true in general. More worryingly, these are not always equivalence relations! However, they are well-behaved on nice objects.

Theorem 3.19. *Left and right homotopy agree on $\mathbf{M}_{fc}$, and are equivalence relations there.*

This can be stated in greater generality. In any case, when homotopy is an equivalence relation, it will be denoted $\sim$.

Definition 3.20. A morphism $f : A \to X$ in $\mathbf{M}_{fc}$ is a *homotopy equivalence* if there's a $g : X \to A$ in $\mathbf{M}_{fc}$ such that $g \circ f \sim \mathrm{id}_A$ and $f \circ g \sim \mathrm{id}_X$.

These can be defined more generally, but here is where they are nice.

Theorem 3.21. *The homotopy equivalences and weak equivalences of $\mathbf{M}_{fc}$ coincide.*

Moreover, weak equivalence is the same as homotopy equivalence (for nice objects)!

Theorem 3.22. *Let $A, X \in \mathbf{M}_{fc}$. Then, $\mathbf{M}(A, X)/\sim\, \cong L\mathbf{M}(A, X)$.*

This tells us, for example, that a zigzag of weak equivalences in $\mathbf{M}$ is represented by a single morphism $A \to X$, which is pretty great. Moreover, even for non-fibrant-cofibrant objects, using (co)fibrant replacement allows one to do the same thing. Thus, model categories are good at both dealing with derived functors and dealing with localizations.

One example of this is that there's a model structure on $\mathbf{Cat}(\bullet \leftarrow \bullet \to \bullet, \mathbf{M})$ and a Quillen adjunction of this to $\mathbf{M}$, where $\mathbf{M}$ is any model category. This means that one can compute homotopy pushouts by deriving the usual notion of pushout.

Again, if this seems like way too much structure, there are weaker notions, such as categories of fibrant objects, but in these categories one can't take both left and right derived functors. This is something one often would like to do, so we'd want the notion of a model category.

4. Localization: Classical and Bousfield: 6/17/16

"So once I had a Russian professor... yes, this is funny already."

Today Nicky spoke about localizations of categories.

4.1. **Classical localizations.** You're probably familiar with the classical localization: if $\mathbf{C}$ is a category and S is a subset of the morphisms of $\mathbf{C}$, then we can form the naïve localization $\mathbf{C}[S^{-1}]$, whose objects are the same objects as $\mathbf{C}$ and whose morphisms are arbitrary zigzags of morphisms in $\mathbf{C}$, morphisms in S, and formal inverses of morphisms in S. This has set-theoretic issues, which was mentioned last time: in a large category, the morphisms between two objects might not be a set, so restrictions have to be placed on S.

We'd like localization to satisfy a universal property: that for any functor $\mathbf{C} \to \mathbf{D}$ taking all arrows in S to isomorphisms factors uniquely through the map $\mathbf{C} \hookrightarrow \mathbf{C}[S^{-1}]$.

One solution, in the form of an assumption on S, is that it has a *left calculus of fractions*.[6] The following definition is from [Sta, Tag 04VB].

Definition 4.1. If the following axioms hold, S forms a *left multiplicative system*.

LMS1. S is a subcategory, i.e. it's closed under composition and contains all identity arrows.

LMS2. All *solid diagrams* can be completed, i.e. for any $f : X \to Y$ in $\mathbf{C}$ and $t : X \to Z$ in S, there's a $W \in \mathbf{C}$ and arrows $h : Z \to W$ in $\mathbf{C}$ and $s : Y \to W$ in S such that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "t"] & Y \arrow[d, "s"] \\ Z \arrow[r, "g"] & W \end{tikzcd}$$

commutes.

LMS3. For all pairs $f, g : X \rightrightarrows Y$ of arrows in $\mathbf{C}$ equalized by an arrow $t : A \to X$ in S, i.e. diagrams of the form

$$\begin{tikzcd} A \arrow[r, "t \in S"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y, \end{tikzcd}$$

f and g can be coequailized, i.e. there's an $s : Y \to B$ in S such that $sf = sg$.

Localization of categories generalizes localization of rings; the last axiom generalizes the notion that we don't like to localize by zero divisors.

Dual to the notion of a left multiplicative system is that of a *right multiplicative system*, whose axioms **RMS1**, **RMS2**, and **RMS3** are the mirror images of those for a left multiplicative system. If **LMS***i* and **RMS***i* both hold, one says axiom **MS***i* holds.

Definition 4.2. Suppose S is a left multiplicative system. Then, the *left calculus of fractions* is the category $S^{-1}\mathbf{C}$ given by the following data.

Objects: The objects of $S^{-1}\mathbf{C}$ are the same as the objects of $\mathbf{C}$.

Morphisms: The morphisms $\mathrm{Hom}_{S^{-1}\mathbf{C}}(X, Y)$ are the equivalence classes of cospans, i.e. pairs $(f : X \to Y', s : Y \to Y')$ where $s \in S$, under the equivalence relation where $(f_1 : X \to Y_1, s_1 : Y \to Y_1)$ is equivalent to $(f_2 : X \to Y_2, s_2 : Y \to Y_2)$ if there's a zigzag $(f_3 : X \to Y_3, s_3 : Y \to Y_3)$ that they both map to, i.e. there exist $u : Y_1 \to Y_3$ and $v : Y_2 \to Y_3$ making the following diagram commute:

$$\begin{tikzcd} & Y_1 \arrow[d, "u"] & \\ X \arrow[ur, "f_1"] \arrow[r, "f_3"] \arrow[dr, "f_2"'] & Y_3 & Y \arrow[ul, "s_1"'] \arrow[l, "s_3"'] \arrow[dl, "s_2"] \\ & Y_2. \arrow[u, "v"'] & \end{tikzcd}$$

Composition: Given $(f : X \to Y', s : Y \to Y')$ and $(g : Y \to Z', t : Z \to Z')$, axiom **LMS2** tells us the diagram

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "s"] & Z' \arrow[d, dashed, "u"] \\ Y' \arrow[r, dashed, "h"] & Z' \end{tikzcd}$$

[6] The right-handed analogue of this story, with spans instead of cospans, is very similar. In many cases it satisfies the same universal property, hence is equivalent.

can be filled in for some $h \in \mathbf{C}$ and $u \in S$. Then, we let $g \circ f$ be the class of $(h \circ f : X \to Z'', u \circ t : Z \to Z'')$.

Lemma 4.3. *This data makes $S^{-1}\mathbf{C}$ into a category.*

This is a chore to prove, so we won't do this. There's some set-theoretic stuff hiding in the background, but not crucially.

The cospan $X \xrightarrow{f} Y' \xleftarrow{s} Y$ will be denoted by $s^{-1}f$. There are a bunch of reasons that this is good notation, which we won't delve into.

A lot of proofs hinge on the following important fact, which is analogous to the fact that any finite number of fractions have a common denominator when one localizes a ring.

Proposition 4.4. *If $\{g_1, \dots, g_n\}$ is a finite set of arrows in $S^{-1}\mathbf{C}$, there exist $s : Y \to Y'$ in S and $f_i : X_i \to Y'$ such that $g_i = (f_i : X_i \to Y', s : Y \to Y')$.*

Remark. Given an object $Y \in \mathbf{C}$, one can form the category Y/S, the category of morphisms $sY \to Y'$ such that $s \in S$ (so the morphisms are commutative diagrams). This is a filtered category, which is also quite useful.[7] Filtered categories are nice for taking colimits, and in fact one can show that

$$\mathrm{Hom}_{S^{-1}\mathbf{C}}(X, Y) = \operatorname*{colim}_{s : Y \to Y' \in Y/S} \mathrm{Hom}_{\mathbf{C}}(X, Y').$$

An important part of the universal property for this localization was the data of a map from $\mathbf{C}$ into its localization.

Lemma 4.5. *Let S be a left multiplicative system.*

(1) The assignment $X \mapsto X$ and $(f : X \to Y) \mapsto (f, \mathrm{id}_Y)$ defines a functor $Q : \mathbf{C} \to S^{-1}\mathbf{C}$.
(2) For all $s \in S$, $Q(s)$ is an isomorphism.
(3) If $G : \mathbf{C} \to \mathbf{D}$ maps all $s \in S$ to isomorphisms, then it factors through Q: there's a unique $G' : S^{-1}\mathbf{C} \to \mathbf{D}$ such that $G = G' \circ Q$.
(4) Q preserves finite colimits.

There's also a right calculus of fractions and right localization; everything can be dualized: given a right multiplicative system, one can take the overcategory S/X rather than the undercategory, which is cofiltered rather than filtered, and so forth. In this case, the notation is also dualized, to give us fs^{-1} rather than $s^{-1}f$.

Since each satisfies the same universal property, given a multiplicative system, there's a natural isomorphism between the left and right localizations: $S^{-1}\mathbf{C} = \mathbf{C}S^{-1}$.

Definition 4.6. A multiplicative set S is a *saturated multiplicative set* if whenever $fg, gh \in S$, then $g \in S$.

One particular consequence is that a saturated multiplicative set contains all isomorphisms.

Proposition 4.7. *If S is a multiplicative system, then $Q(f)$ is an isomorphism iff $f \in S$.*

4.2. **Bousfield localizations.** There's a ton of things that could be said about Bousfield localizations. In a model category, the idea is to create a new model structure on a model category with the same cofibrations, but more weak equivalences (so the fibrant structure changes). For example, one might want to study the model category of rational, or p-local, homotopy types. Fibrant replacement becomes localization: in the rational homotopy category, fibrant replacement is $X \mapsto X_{\mathbb{Q}}$, and so forth.

In triangulated categories, it's possible to take quotients: if the cone of a morphism lies in S, we localize by that morphism. Even in a more general sense, it can be useful to think of Bousfield localization as a quotient.

Let $\mathbf{C}$ be a category, $L : \mathbf{C} \to \mathbf{C}$ be an endofunctor, and $\iota : \mathbb{1}_{\mathbf{C}} \to L$ be a natural transformation.

Definition 4.8. The pair (L, ι) is a *Bousfield localization* if "$L\iota = \iota_L$ with the common value an isomorphism:" concretely, for all objects $X \in \mathbf{C}$, $L(\iota_X : X \to LX)$ is the same map as $\iota_{LX} : LX \to LLX$, and this common map is an isomorphism.

Definition 4.9.

- If σ is a $\mathbf{C}$-morphism such that $L(\sigma)$ is an isomorphism, f is called an *L-equivalence.* The set of L-equivalences is denoted $\mathbf{W} \subset \mathbf{C}$.

[7] Recall that a category is *filtered* if for all pairs of objects, there's a common object they map to, and for any pair of arrows between two objects, there's an arrow coequalizing them.

- For any subclass $\mathbf{W}$ of the morphisms of $\mathbf{C}$, an object $Z \in \mathbf{C}$ is $\mathbf{W}$-*local* if for all $f : X \to Y$ in $\mathbf{W}$, the induced map $f^* : \mathrm{Hom}_{\mathbf{C}}(Y, Z) \to \mathrm{Hom}_{\mathbf{C}}(X, Y)$ is an isomorphism.[8] The full subcategory of $\mathbf{W}$-local objects will be denoted $\mathbf{C}_{\mathbf{W}}$.

In the model-categorical case, we're trying to produce more weak equivalences, rather than more isomorphisms, so in this case, we'll replace the condition for W-locality to be weak equivalence (e.g. of simplicial sets for a simplicial model category).

Proposition 4.10. *If (L, ι) is a Bousfield localization, then Z is $\mathbf{W}$-local iff ι_Z is an isomorphism.*

Proof. By the definition of a Bousfield localization, if $Z \in \mathbf{C}_{\mathbf{W}}$, then $\iota_Z : Z \to LZ$ is in $\mathbf{W}$, and therefore $\iota_Z^* : \mathrm{Hom}(LZ, Z) \to \mathrm{Hom}(Z, Z)$ is an isomorphism, more or less by the Yoneda lemma: $\iota_Z^{-1}(\mathrm{id}_Z)$ is an inverse of ι_Z.

Conversely, suppose ι_Z is an isomorphism. Proving things are $\mathbf{W}$-local tends to involve diagram chases; so be it. Let $f : X \to Y$ be in $\mathbf{W}$ and $g : X \to Z$ be arbitrary. We need to show that g comes from a unique $h \in \mathrm{Hom}(X, Z)$, i.e. $f^*(h) = h \circ f = g$. The answer is to set $h = \iota_Z^{-1}(Lg)(Lf^{-1})\iota_Y$; recall Lf is invertible because $f \in \mathbf{W}$.

Why does this work? By naturality, $\iota_Z^{-1}(Lg)(Lf)^{-1}\iota_Y f = \iota_Z^{-1} Lg(Lf^{-1})Lf\iota_X = \iota_Z^{-1}\iota_Z g = g$. Magic! Uniqueness is a similar diagram chase. ⊠

Proposition 4.11. *If (L, ι) is a Bousfield localization and $\mathbf{W}$ is its L-equivalences, then $\mathbf{C}_{\mathbf{W}} = \mathbf{C}[\mathbf{W}^{-1}]$.*

Proof. We'd like to find a morphism $\mathbf{C} \to \mathbf{C}_{\mathbf{W}}$ factoring L; by Proposition 4.10, L maps $\mathbf{C}$ to $\mathbf{C}_{\mathbf{W}}$. By definition, it takes maps in $\mathbf{W}$ to isomorphisms, so we know L should factor through the localization. We have no assumptions on $\mathbf{W}$, so we have to use the naïve localization $\mathbf{C}[\mathbf{W}^{-1}]$; in any case, this means there's a map $G : \mathbf{C}[\mathbf{W}^{-1}] \to \mathbf{C}_{\mathbf{W}}$ that L factors through.

The final claim is that G is an equivalence of categories, which we prove by constructing an inverse functor $F : \mathbf{C}_{\mathbf{W}} \hookrightarrow \mathbf{C} \to \mathbf{C}[\mathbf{W}^{-1}]$, where the second arrow is the localization functor. The idea is that $LX = FGX$, and $\iota_X : X \to LX$ is an isomorphism. ⊠

Now, we can use this to construct Bousfield localizations: if we're not handed (L, ι), but instead were handed $\mathbf{W}$, we can check whether localizing at $\mathbf{W}$ produces a Bousfield localization, and determine conditions for when this is true.

There's also a precise sense in which the Bousfield localization is *not* terrible.

Proposition 4.12. *If $\mathbf{W}$ comes from a Bousfield localization, then it is a left multiplicative system.*

Proof. **LMS1** follows from functoriality of L. To get **LMS2**, we can complete $Y \xleftarrow{\beta} X \xrightarrow{\gamma} Z$ with LZ, which is $\mathbf{W}$-local. Hence, $\beta^* : \mathrm{Hom}(Y, LZ) \to \mathrm{Hom}(X, LZ)$ must be an isomorphism, and the preimage of $\iota_Z \circ \gamma : X \to LZ$ is the map δ that makes the requisite diagram commute:

$$\begin{tikzcd} X \arrow[r, "\gamma"] \arrow[d, "\beta"] & Z \arrow[d, "\iota_Z"] \\ Y \arrow[r, "\exists!", "\delta"'] & LZ. \end{tikzcd}$$

Finally, for **LMS3**, $\iota_Z : Z \to Z$ coequalizes anything in $\mathbf{W}$ that equalizes two arrows. ⊠

So now, given any $\mathbf{W}$ contained in the arrows of $\mathbf{C}$, we still have definitions of $\mathbf{W}$-local objects and $\mathbf{C}_{\mathbf{W}}$. If we have a nice way to make $\mathbf{W}$-localizations, we can extract a Bousfield localization.

Definition 4.13. A $\sigma : A \to B$ is called a $\mathbf{W}$-*localization* of A if $\sigma \in \mathbf{W}$ and $B \in \mathbf{C}_{\mathbf{W}}$. The category of $\mathbf{W}$-localizable objects in $\mathbf{C}$ is written $\mathbf{C}^{\mathbf{W}}$.

Choosing a localization for each object in $\mathbf{C}^{\mathbf{W}}$ defines a functor $L : \mathbf{C}^{\mathbf{W}} \to \mathbf{C}_{\mathbf{W}}$, because any two localizations of an $A \in \mathbf{C}^{\mathbf{W}}$ are uniquely isomorphic. There's a little work to here, but not too much. But once you've done this, the natural transformation $\mathbb{1}_{\mathbf{C}} \to L$ (when $\mathbf{C}^{\mathbf{W}} = \mathbf{C}$) just drops out.

[8]Using pushforward instead of pullback produces the dual notion of $\mathbf{W}$-*colocal*.

Proposition 4.14. *If all objects in* $\mathbf{C}$ *are localizable and* $\mathbf{W}$ *is weakly closed,*[9] *then* (L, ι) *is a Bousfield localization and* $\mathbf{W}$ *is exactly the class of* L*-equivalences.*

Proof. We defined (L, ι) by making choices; it suffices to show that $L\iota = \iota_L$ and these are isomorphisms. Since ι is a natural transformation, then for any $A \in \mathbf{C}$, the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "L_A"] \arrow[d, "\iota_A"] & LA \arrow[d, "\iota_{LA}"] \\ LA \arrow[r, "L\iota_A"] & LLA. \end{tikzcd}$$

Since $LLA \in \mathbf{C}_{\mathbf{W}}$, then applying this to ι_A shows $L\iota_A = \iota_{LA}$. This map is an isomorphism because the diagonal $A \to LLA$ is localization, but the localization is unique, so $L\iota_A$ must be an isomorphism. Showing that this recovers $\mathbf{W}$ is similar. ⊠

If $\mathbf{C}$ is a model category, then its homotopy category has a triangulated structure; taking the homotopy category of a Bousfield localization of $\mathbf{C}$ descends to a *Verdier quotient* on the homotopy category. Thus, for practical purposes, one very frequently thinks of Bousfield localization as a quotient. In this case, if $\mathbf{C}$ is pointed, $\mathbf{C}^{\mathbf{W}}$ sometimes denotes the collection of $X \in \mathbf{C}$ such that $* \to X$ is a $\mathbf{W}$-equivalence. In algebraic K-theory, one can obtain a fiber sequence $K(\mathbf{C}^{\mathbf{W}}) \to K(\mathbf{C}) \to K(L\mathbf{C})$, another sense in which this behaves like a kernel.

So in a model category, we're not thinking about inverting morphisms, but rather inverting them up to homotopy, or after passing to left and right derived functors. In particular, for a model category, we should replace isomorphism with weak equivalence, so being $\mathbf{W}$-local will require $f^* : \mathbf{C}(Y, Z) \to \mathbf{C}(X, Z)$ to be a weak equivalence of simplicial sets. In this case, the Bousfield localization is a new model structure on the same category where the weak equivalences are $\mathbf{W}$-local equivalences and the cofibrations are the same; in this case, the identity functor $\mathbf{C} \to \mathbf{C}[\mathbf{W}^{-1}]$ is part of a left Quillen adjunction. There's also right Bousfield localizations and conditions on when they exist…

One cool fact is that taking $\mathbf{W} = \{f\}$ for any morphism f of topological spaces, it's possible to take the Bousfield localization on $\mathbf{W}$!

5. Simplicial Localizations of Categories: 6/21/16

Today's talk, an exposition of [DK1], was given by Danny and Arun.

5.1. Free Categories and the Simplicial Localization.

The first half was given by Danny.

The crux of this paper is that the standard localization is the shadow of a much richer structure, the simplicial localization. One reason that you might care about this is that if one starts with a closed simplicial model category, the simplicial structure gives rise to function complexes $\mathbf{C}(X, Y)$ between two objects, and the simplicial localization produces weakly equivalent function complexes $L\mathbf{C}(X, Y)$. Thus, we can use the simplicial localization, which does not require any model structure or even simplicial structure, to define function complexes in a much broader setting.

Definition 5.1. Fix a set O of objects. The category O-**Cat** is the category of small categories $\mathbf{C}$ whose set of objects are O; the morphisms are functors that are the identity on objects.

The coproduct in O-**Cat**, called *free product*, is denoted $*$. If $\mathbf{C}, \mathbf{D} \in O$-**Cat**, there is a one-to-one correspondence between the non-identity arrows in $\mathbf{C} * \mathbf{D}$ and finite compositions of non-identity arrows in $\mathbf{C}$ and $\mathbf{D}$ such that no two adjacent arrows are both from $\mathbf{C}$ or from $\mathbf{D}$.

Definition 5.2. The category sO-**Cat** is the category of simplicial objects over O-**Cat**, i.e. the simplicially enriched categories on a set O of objects. A morphism $f : \mathbf{C} \to \mathbf{D}$ in sO-**Cat** is a *weak equivalence* if for all objects $X, Y \in O$, it induces a weak equivalence $\mathbf{C}(X, Y) \xrightarrow{\simeq} \mathbf{D}(X, Y)$.

We will think of O-**Cat** $\hookrightarrow sO$-**Cat** as discrete simplicial objects.

Fact. If $f : \mathbf{C} \to \mathbf{D}$ is a weak equivalence, then the induced map $f_* : N\mathbf{C} \to N\mathbf{D}$ is a weak homotopy equivalence.

Definition 5.3. A *free category* is a category $\mathbf{F} \in O$-**Cat** along with a collection S of morphisms in $\mathbf{F}$, called *generators*, such that every non-identity map in $\mathbf{F}$ is a unique composition of morphisms in S.

[9] $\mathbf{W}$ is *weakly closed* if it satisfies **LMS1** and the two-out-of-three property. This is similar to, but not the same as, being saturated.

One can show, using a different but equivalent definition, that there is a uniqueness condition on S for a given free category.

Proposition 5.4 [DK1, 2.2,2.3]. *The free product of free categories is free; conversely, every free category is a free product of free categories each on a single generator.*

Given a $\mathbf{C} \in O\text{-}\mathbf{Cat}$, one can take the category $F\mathbf{C}$ "free on $\mathbf{C}$" generated by the set of morphisms of $\mathbf{C}$, where we've forgotten the composition relations. The functor F is a composition of adjoints, as will be explained later.

You can totally do this as many times as you'd like: the assignment $\psi : F\mathbf{C} \to F^2\mathbf{C}$ defines a natural transformation $F \to F^2$, and $\varphi : Fc \mapsto c$ defines a natural transformation $F \to \mathbb{1}$. The triple (F, ψ, φ) forms a comonad on $O\text{-}\mathbf{Cat}$; φ is called the *augmentation map*. That these form a comonad means they satisfy a few comonadic relations, which allow us to define a free resolution $F_*\mathbf{C} \in sO\text{-}\mathbf{Cat}$ for a given $\mathbf{C} \in O\text{-}\mathbf{Cat}$.

Definition 5.5. Let $\mathbf{C} \in O\text{-}\mathbf{Cat}$. Then, the *standard resolution* of $\mathbf{C}$, denoted $F_*\mathbf{C} \in sO\text{-}\mathbf{Cat}$, is the simplicially enriched category defined as follows:

- The k-simplices are $F_k\mathbf{C} = F^{k+1}\mathbf{C}$.
- The i^{th} face map $d_i^k : F_k\mathbf{C} \hookrightarrow F_{k-1}\mathbf{C}$ is $d_i^k = F^i\varphi F^{k-i}$.
- The i^{th} degeneracy map $s_i^k : F_k\mathbf{C} \to F_{k+1}\mathbf{C}$ is $s_i^k = F^i\psi F^{k-1}$.

When we put a model structure on $sO\text{-}\mathbf{Cat}$, this will be a cofibrant resolution; it feels very similar to projective resolutions in homological algebra.

Proposition 5.6 [DK1, 2.6]. *If* $\mathbf{C} \in O\text{-}\mathbf{Cat}$ *is regarded as a discrete object in* $sO\text{-}\mathbf{Cat}$*, then the map* $\varphi : F_*\mathbf{C} \to \mathbf{C}$ *is a weak equivalence.*

This is a good exercise.

One big reason we care about free categories is that localization preserves the homotopy type of the nerve for free categories, but not in general; using a free resolution allows us to get around this just enough to prove things.

Proposition 5.7 [DK1, 3.7]. *Suppose* $\mathbf{C} = \mathbf{D} * \mathbf{W}$*, where* $\mathbf{W}$ *is free. Then,* $N\mathbf{C} \to N(\mathbf{D} * \mathbf{W}[\mathbf{W}^{-1}])$ *is a weak homotopy equivalence.*

This is the main result of §3; to prove it, one starts by proving it when $\mathbf{W}$ has a single generator, and then use the more general fact that $N\mathbf{D} \cup N\mathbf{W} \hookrightarrow N\mathbf{C}$ is a weak homotopy equivalence to carry this to the general case.

Definition 5.8. Let $(\mathbf{C}, \mathbf{W})$ be a relative category on O (i.e. $\mathbf{C}, \mathbf{W} \in O\text{-}\mathbf{Cat}$ and $\mathbf{W} \subset \mathbf{C}$). Then, the *(standard) simplicial localization* $L\mathbf{C} = L(\mathbf{C}, \mathbf{W}) = F_*\mathbf{C}[F_*\mathbf{W}^{-1}]$.

In particular, this means $\pi_0(L\mathbf{C})(X, Y) = \mathbf{C}[\mathbf{W}^{-1}](X, Y)$, so it recovers the naïve localization, and in this sense is at least as good.

This also has the nice fact that is preserves the homotopy type of the nerve, thanks to a zigzag of weak homotopy equivalences

$$N\mathbf{C} \xleftarrow{N\varphi} NF_*\mathbf{C} \xrightarrow{N\mathrm{loc}} NL\mathbf{C}.$$

5.2. **Comonadic Resolutions and the Model Structure on sO-Cat.** These are Arun's prepared notes for the talk, on approximately the second half of [DK1], which introduces the homotopical algebra necessary to prove two important theorems about the simplicial localization. I will call particular attention to the use of the bar construction in these proofs.

Recall that $O\text{-}\mathbf{Cat}$ denotes the category of small categories with a given set O of objects and $sO\text{-}\mathbf{Cat}$ denotes the category of simplicially enriched small categories with the set O of objects. For a relative category $(\mathbf{C}, \mathbf{W})$, we will let $L\mathbf{C} = L(\mathbf{C}, \mathbf{W})$ denote its simplicial localization.

There's a nice abstract approach to defining the simplicial localization using comonads, which I'll briefly discuss.

Definition 5.9. A *comonad* over a category $\mathbf{C}$ is a comonoidal object in the category of endofunctors of $\mathbf{C}$; that is, there is a *counit* natural transformation $\varepsilon : F \to \mathbb{1}$ and a *comultiplication* $\Delta : F \to F^2$ satisfying the analogues of associativity and identity, e.g. the diagram

$$\begin{tikzcd} F \arrow[r, "\Delta"] & F^2 \arrow[r, shift left, "(\varepsilon,\mathrm{id})"] \arrow[r, shift right, "(\mathrm{id},\varepsilon)"'] & F \end{tikzcd}$$

should commute.

Example 5.10. Comonads arise from adjunctions. Suppose $F : \mathbf{C} \leftrightarrows \mathbf{D} : G$ is an adjunction. Then, FG defines a comonad: the counit $\varepsilon : FG \to \mathbb{1}$ valued in $X \in \mathbf{C}$ is mapped to id_{GX} under the identification $\mathrm{Hom}_{\mathbf{C}}(FGX, X) = \mathrm{Hom}_{\mathbf{D}}(GX, GX)$, and the comultiplication $\Delta : FG \to FGFG$ is defined by $F \circ \eta \circ G$, where $\eta : \mathbb{1} \to FG$ is dual to ε: it's the image of id_{FY} under the identification $\mathrm{Hom}_{\mathbf{C}}(FY, FY) = \mathrm{Hom}_{\mathbf{D}}(Y, GFY)$.

We care today because a comonad gives us a way to produce simplicial objects, by repeatedly iterating it.

Example 5.11. If (F, ε, Δ) is a comonad over $\mathbf{C}$, it defines a functor $G : \mathbf{C} \to \mathbf{C}^{\Delta^{\mathrm{op}}}$ called the *standard (simplicial) resolution* as follows. Let $X \in \mathbf{C}$.

- The n-simplices of GX are $GX_n = F^{n+1}X$.
- The i^{th} face map $d_i : F^{k+1}X \to F^kX$ inserts ε in the i^{th} position of $F \circ \cdots \circ F$.
- The i^{th} degeneracy map $s_i : F^{k+1}X \to F^{k+2}X$ inserts Δ in the i^{th} position of $F \circ \cdots \circ F$.

This already looks very familiar, so choosing the correct adjunction will give us the right definition of simplicial localization.

Definition 5.12. Let O be a set.

- The category of *reflexive O-graphs*, O-**Gr**, is the category of graphs on O such that every vertex has a self-loop, and whose morphisms send edges to edges.
- The category sO-**Gr** of *simplicial O-graphs* is the category of simplicial objects in O-**Gr**.

That is, O-**Gr** is everything about O-**Cat** except composition. This means there are forgetful functors $\alpha : O\text{-}\mathbf{Cat} \to O\text{-}\mathbf{Gr}$ and $s\alpha : sO\text{-}\mathbf{Cat} \to sO\text{-}\mathbf{Gr}$ defined by ignoring composition of morphisms. These have right adjoints $\beta : O\text{-}\mathbf{Gr} \to O\text{-}\mathbf{Cat}$ and $s\beta : sO\text{-}\mathbf{Gr} \to sO\text{-}\mathbf{Cat}$ respectively.

Let $F_* : O\text{-}\mathbf{Cat} \to sO\text{-}\mathbf{Cat}$ denote the standard resolution functor associated to (α, β).

Definition 5.13. The *simplicial localization* of a relative category $(\mathbf{C}, \mathbf{W})$ is $L(\mathbf{C}, \mathbf{W}) = F_*\mathbf{C}[F_*\mathbf{W}^{-1}]$.

We can also do this with simplicially enriched categories. This time, the standard resolution functor $sF_* : sO\text{-}\mathbf{Cat} \to sO\text{-}\mathbf{Cat}^{\Delta^{\mathrm{op}}}$ lands in the category of *bi*simplicial categories on O, so we take the diagonal to obtain $F_* = \mathrm{diag} \circ sF_* : sO\text{-}\mathbf{Cat} \to sO\text{-}\mathbf{Cat}$, and define the *simplicial localization* of $(\mathbf{C}, \mathbf{W})$, where $\mathbf{C} \in sO\text{-}\mathbf{Cat}$, to be $F_*\mathbf{C}[F_*\mathbf{W}^{-1}]$.

The aim is to prove the following results.

Lemma 5.14 (Homotopy lemma, [DK1, Lem. 6.2])**.** *Let $\mathbf{A}, \mathbf{B} \in sO\text{-}\mathbf{Cat}$ be free and $\mathbf{U} \subset \mathbf{A}$ and $\mathbf{V} \subset \mathbf{B}$ be free factors in each dimension. If $S : \mathbf{A} \to \mathbf{B}$ maps $\mathbf{U}$ into $\mathbf{V}$, and both S and $S|_{\mathbf{U}}$ are weak equivalences, then so is the induced map $\mathbf{A}[\mathbf{U}^{-1}] \to \mathbf{B}[\mathbf{V}^{-1}]$.*

Corollary 5.15 [DK1, Cor. 6.3]. *If $\mathbf{A}, \mathbf{B} \in sO\text{-}\mathbf{Cat}$ are arbitrary, $\mathbf{U} \subset \mathbf{A}$ and $\mathbf{V} \subset \mathbf{B}$, and $S : \mathbf{A} \to \mathbf{B}$ maps $\mathbf{U}$ into $\mathbf{V}$, then if both S and $S|_U$ are weak equivalences, then so is the induced map $L(\mathbf{A}, \mathbf{U}) \to L(\mathbf{B}, \mathbf{V})$.*

Lemma 5.16 (Closure lemma, [DK1, Lem. 6.4])**.** *Let $\mathbf{C} \in sO\text{-}\mathbf{Cat}$, $\mathbf{W} \subset \mathbf{C}$, and $\overline{\mathbf{W}}$ denote the "π_0-closure," i.e. the preimage of $\overline{\pi_0\mathbf{W}} \subset \pi_0\mathbf{C}$.*[10] *Then, the inclusion $L(\mathbf{C}, \mathbf{W}) \hookrightarrow L(\mathbf{C}, \overline{\mathbf{W}})$ is a weak equivalence.*

Corollary 5.17 [DK1, Cor. 6.5]. *The weak equivalence classes of simplicial localizations of a simplicial category $\mathbf{C} \in sO\text{-}\mathbf{Cat}$ are in one-to-one correspondence with the closed subcategories of $\pi_0\mathbf{C}$.*

These proofs require homotopical machinery, in the form of a model structure on sO-**Cat**.

Definition 5.18. A morphism $f : \mathbf{C} \to \mathbf{D}$ in sO-**Cat** is *free* if

- f is injective,
- for each $k \geq 0$, there's a free category $\mathbf{F}_k$ such that $\mathbf{D}_k = f(\mathbf{C}_k) * \mathbf{F}_k$, and the degeneracies of the generators of $\mathbf{F}_k$ are the generators of $\mathbf{F}_{k+1}$.

This in particular implies that $\mathbf{C} \in sO\text{-}\mathbf{Cat}$ is a free category iff the map $* \to \mathbf{C}$ from the initial object is a free map.

Proposition 5.19 [DK1, 7.2, 7.6]. *With the following data, sO-**Cat** is a closed, simplicial model category.*

[10] Recall that the closure for ordinary categories is throwing in the minimal collection of morphisms such that $\mathbf{W}$-equivalence becomes an equivalence relation.

- *The weak equivalences are those maps $f : \mathbf{C} \to \mathbf{D}$ that induce weak equivalences (of simplicial sets) $\mathbf{C}(X, Y) \to \mathbf{D}(X, Y)$ for all $X, Y \in O$.*
- *The fibrations are, in the same way, the maps that induce fibrations of simplicial sets on all hom-spaces.*
- *The cofibrations are retracts of free maps.*

Thus, $\mathbf{C}$ is fibrant iff all of its hom-sets are fibrant, and is cofibrant iff it's a retract of a free category. We will call a cofibration between two cofibrant objects a *strong cofibration.*

Proposition 5.20 [DK1, 8.2]. *Let*

$$\begin{tikzcd} \mathbf{A} \arrow[r] \arrow[d, "f"] & \mathbf{C} \arrow[d] \\ \mathbf{B} \arrow[r] & \mathbf{D} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

be a pushout square in sO-$\mathbf{Cat}$ (so $\mathbf{D} = \mathbf{B} \amalg_{\mathbf{A}} \mathbf{C}$) such that f is a cofibration. If $(\mathbf{B}, \mathbf{A}, \mathbf{C}) \in sO\text{-}\mathbf{Cat}^{\Delta^{\mathrm{op}}}$ denotes the bar construction, then the natural map $\operatorname{diag}(\mathbf{B}, \mathbf{A}, \mathbf{C}) \to \mathbf{D}$ is a weak equivalence.

Recall that the n-simplices of $(\mathbf{B}, \mathbf{A}, \mathbf{C})$ are $(\mathbf{B}, \mathbf{A}, \mathbf{C})_n = \mathbf{B} * \mathbf{A} * \overset{n}{\cdots} * \mathbf{A} * \mathbf{C}$.

Proof. Without loss of generality, we may assume f is a free map.

First, assume $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are discrete (in O-$\mathbf{Cat}$). Since f is free, there's a free category $\mathbf{F}$ such that $\mathbf{B} = \mathbf{F} * \mathbf{A}$; then, by checking the universal property, $\mathbf{D} = \mathbf{F} * \mathbf{C}$. Thus, $(\mathbf{B}, \mathbf{A}, \mathbf{C})_n = \mathbf{F} * \mathbf{A}^{*(n+1)} * \mathbf{C}$, so it suffices to describe a weak equivalence

$$\mathbf{F} * \mathbf{A}^{*(n+1)} * \mathbf{C} \xrightarrow{\simeq} \mathbf{F} * \mathbf{C}.$$

For the more general case, recall that for any category $\mathscr{C}$, the pushout in the category of simplicial $\mathscr{C}$-objects may be taken levelwise [Sta, Tag 016V], so for every $n \geq 0$, $\mathbf{D}_n = \mathbf{B}_n \amalg_{\mathbf{A}_n} \mathbf{C}_n$, and $f_n : \mathbf{A}_n \to \mathbf{B}_n$ is free. In this case,

$$\operatorname{diag}(\mathbf{B}, \mathbf{A}, \mathbf{C})_n = \mathbf{B}_n * \mathbf{A}_n * \overset{n}{\cdots} * \mathbf{A}_n * \mathbf{C}_n.$$

By the discrete case, the natural map $\operatorname{diag}(\mathbf{B}, \mathbf{A}, \mathbf{C})_n \to \mathbf{D}_n$ is a weak equivalence, and we can stitch these together for all n. ⊠

Corollary 5.21 [DK1, 8.1]. *Consider a map of pushout squares*

$$\begin{tikzcd} \mathbf{A} \arrow[r] \arrow[d, "f"] & \mathbf{C} \arrow[d] & & \mathbf{A}' \arrow[r] \arrow[d, "f'"] & \mathbf{C}' \arrow[d] \\ \mathbf{B} \arrow[r] & \mathbf{D} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, phantom, "\longrightarrow"] & & \mathbf{B}' \arrow[r] & \mathbf{D}' \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

inducing weak equivalences $\mathbf{A} \simeq A'$, $\mathbf{B} \simeq \mathbf{B}'$, and $\mathbf{C} \simeq \mathbf{C}'$. If f and f' are cofibrations, the induced map $\mathbf{D} \to \mathbf{D}'$ is also a weak equivalence.

This follows because the free product of weak equivalences is a weak equivalence.

Definition 5.22. The *groupoid completion* of a simplicially enriched category $\mathbf{C} \in sO$-$\mathbf{Cat}$ is the simplicial groupoid $\mathbf{C}[\mathbf{C}^{-1}]$, i.e. obtained by adjoining inverses for all maps in each dimension.

In particular, $\pi_0(\mathbf{C}[\mathbf{C}^{-1}]) = (\pi_0\mathbf{C})[\pi_0\mathbf{C}^{-1}]$.

Proposition 5.23 [DK1, 9.3]. *If $\mathbf{C}$ and $\mathbf{D}$ are cofibrant objects in sO-$\mathbf{Cat}$, then a map $\mathbf{C} \to \mathbf{D}$ induces a weak equivalence $\mathbf{C}[\mathbf{C}^{-1}] \simeq \mathbf{D}[\mathbf{D}^{-1}]$ iff it induces a weak homotopy equivalence $N\mathbf{C} \simeq N\mathbf{D}$.*

Definition 5.24. Let $f : \mathbf{C} \to \mathbf{D}$ be a morphism in sO-$\mathbf{Cat}$. Then, the $\mathbf{C}$-*localization of* $\mathbf{D}$ is the simplicially enriched category $\mathbf{D}[\mathbf{C}^{-1}]$ fitting into the pushout square

$$\begin{tikzcd} \mathbf{C} \arrow[r] \arrow[d, "f"] & \mathbf{C}[\mathbf{C}^{-1}] \arrow[d] \\ \mathbf{D} \arrow[r] & \mathbf{D}[\mathbf{C}^{-1}]. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{5.25}$$

Remark. This generalizes the localization of algebras, because the pushout in the category of A-algebras is $- \otimes_A -$. The analogue of (5.25) is

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & S^{-1}A \arrow[d] \\ B \arrow[r] & S^{-1}A \otimes_A B, \end{tikzcd}$$

which indeed defines $S^{-1}B$.

Proposition 5.26 [DK1, 10.3]. *Let*

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\simeq"] \arrow[d, "S"] & \mathbf{B} \arrow[d, "S"] \\ \mathbf{C} \arrow[r, "\simeq"] & \mathbf{D} \end{tikzcd}$$

*be a commutative diagram in sO-***Cat***, where the horizontal arrows are weak equivalences and the horizontal maps are strong cofibrations. Then, the induced map* $\mathbf{C}[\mathbf{A}^{-1}] \to \mathbf{D}[\mathbf{B}^{-1}]$ *is also a weak equivalence.*

Since free categories are cofibrant, this implies Lemma 5.14.

Proof. Well, let's draw the cube:

$$\begin{tikzcd} \mathbf{A} \arrow[rr, "\simeq"] \arrow[dr] \arrow[dd, "S"] & & \mathbf{B} \arrow[dr] \arrow[dd, "S"] & \\ & \mathbf{A}[\mathbf{A}^{-1}] \arrow[rr] \arrow[dd] & & \mathbf{B}[\mathbf{B}^{-1}] \arrow[dd] \\ \mathbf{C} \arrow[rr, "\simeq"] \arrow[dr] & & \mathbf{D} \arrow[dr] & \\ & \mathbf{C}[\mathbf{A}^{-1}] \arrow[rr] & & \mathbf{D}[\mathbf{B}^{-1}]. \end{tikzcd}$$

The weak equivalence $\mathbf{A} \to \mathbf{B}$ induces a weak homotopy equivalence of their nerves, so by Proposition 5.23 makes the top front arrow a weak equivalence. Since the left and right faces are pushout squares, this is a map of pushout squares inducing weak equivalences on the first three objects, so by Corollary 5.21, the lower front map is a weak equivalence. ⊠

Proving Lemma 5.16 follows a similar path. Most of the hard work is in the proof of the following proposition.

Proposition 5.27 [DK1, 9.5]. *If* $\mathbf{C} \in sO$-**Cat** *is cofibrant, then the localization map* $\mathbf{C} \to \mathbf{C}[\mathbf{C}^{-1}]$ *is a weak equivalence iff* $\pi_0\mathbf{C}$ *is a groupoid.*

Corollary 5.28 [DK1, 9.6]. *If* $\mathbf{C}$ *and* $\mathbf{D}$ *are cofibrant objects in sO-***Cat** *such that* $\pi_0\mathbf{C}$ *and* $\pi_0\mathbf{D}$ *are groupoids, then a map* $\mathbf{C} \to \mathbf{D}$ *is a weak equivalence iff the induced map* $N\mathbf{C} \to N\mathbf{D}$ *is a weak homotopy equivalence.*

The key ingredient in the proof is a different bar construction. Let $N(\mathbf{C}, \mathbf{C})$ be the simplicial set whose k-simplices are the sequences of $k + 1$ morphisms in $\mathbf{C}$, with the usual face and degeneracy maps; forgetting the last map defines a morphism $p : N(\mathbf{C}, \mathbf{C}) \to N\mathbf{C}$. This is analogous to the construction of $EG \to BG$ through a bar resolution, in the sense that it adds one entry of extra information unaffected by the face or degeneracy maps. For the same reasons that EG is contractible, $N(\mathbf{C}, \mathbf{C})$ retracts onto its set of components, which is in bijection with O.

6. The Hammock Localization: 6/28/16

Today, Richard spoke about the first part of [DK2], discussing the hammock localization. (Before that, we worked through a little bit of the stuff in my lecture notes, in the previous lecture notes, understanding the things that I didn't understand.)

The simplicial localization has nice properties, but it's irreversibly defined with free categories: $L(\mathbf{C}, \mathbf{W}) = F_*\mathbf{C}[F_*\mathbf{W}^{-1}]$. As such, it is hard to compute with. The hammock localization is a little harder to define, but has almost as nice properties, and is much easier to compute with.

Definition 6.1. If $\mathbf{C} \in s\mathcal{O}$-**Cat** and $\mathbf{W}$ is a subset of $\mathbf{C}$, the *hammock localization* of $\mathbf{C}$, denoted $L^H\mathbf{C}$, is the simplicially enriched category whose k-simplices $L^H\mathbf{C}(X,Y)_k$ is the set of *hammocks* of height k, i.e. commutative diagrams of the form

$$\begin{tikzcd}
 & C_{0,1} \arrow[r, dash] \arrow[d] & C_{0,2} \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] \arrow[d] & C_{0,n-1} \arrow[d] \arrow[dr] & \\
X \arrow[ur] \arrow[r] \arrow[ddr] & C_{1,1} \arrow[r, dash] \arrow[d] & C_{1,2} \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] \arrow[d] & C_{1,n-1} \arrow[r] \arrow[d] & Y, \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \ddots \arrow[d] & \vdots \arrow[d] & \\
 & C_{k,1} \arrow[r, dash] & C_{k,2} \arrow[r, dash] & \cdots \arrow[r, dash] & C_{k,n-1} \arrow[uur] &
\end{tikzcd}$$

such that:

- The horizontal arrows may go either left or right, but all horizontal arrows in a given column go the same direction.
- The vertical arrows, and any horizontal arrows going left, are in $\mathbf{W}$.
- The hammocks are *reduced*: there are no columns consisting of entirely identity maps.

For this hammock, n is called the *width*, and k is called the *height*.

$L^H\mathbf{C}(X,Y)$ is made into a simplicial set as follows: the i^{th} face map composes rows i and $i+1$ together, and the i^{th} degeneracy map inserts a row of identity morphisms at position i.

To make $\mathbf{W}$ explicit in the notation, this is sometimes written $L^H\mathbf{C}(\mathbf{W})$.

Here are some of the nice properties of this construction.

Proposition 6.2. *There is a chain of weak equivalences* $L^H\mathbf{C} \xleftarrow{\sim} \operatorname{diag} L^H(F_*\mathbf{C}) \xrightarrow{\sim} L\mathbf{C}$.

This will follow from the following two lemmas.

Lemma 6.3 (Comparison lemma)**.** *Suppose* $\mathbf{D}$ *and* $\mathbf{W}$ *are free categories and* $\mathbf{C} = \mathbf{D} * \mathbf{W}$*. Then,* $L^H\mathbf{C} \to \mathbf{C}[\mathbf{W}^{-1}]$ *is a weak equivalence.*

Lemma 6.4 (Homotopy lemma)**.** *Let* $\mathbf{A}, \mathbf{B} \in s\mathcal{O}$-**Cat** *and* $\mathbf{U} \subset \mathbf{A}$ *and* $\mathbf{V} \subset \mathbf{B}$ *be subcategories. Let* $S : \mathbf{A} \to \mathbf{B}$ *be a functor such that* $S(\mathbf{U}) \subseteq \mathbf{V}$*, and suppose that* S *and* $S|_{\mathbf{U}}$ *are both weak equivalences. Then, the induced map* $\operatorname{diag} L^H\mathbf{A}(\mathbf{U}) \to \operatorname{diag} L^H\mathbf{B}(\mathbf{V})$ *is a weak equivalence.*

Moreover, just as for the usual simplicial localization, $\pi_0 L^H\mathbf{C} = \mathbf{C}[\mathbf{W}^{-1}]$: 0-simplices of $L^H\mathbf{C}$ are chains of maps in $\mathbf{C}$ and formal inverses of maps in $\mathbf{W}$.

There's also a natural map $\mathbf{C} \to L^H\mathbf{C}$ in $s\mathcal{O}$-**Cat**, which is a big advantage over $L\mathbf{C}$: we map $f : X \to Y$ to the hammock with one column, all maps in the grid equal to the identity, and the remaining maps equal to f.

Suppose $S : \mathbf{C} \to \mathbf{C}'$ is a morphism in $s\mathcal{O}$-**Cat**, $\mathbf{W} \subset \mathbf{C}$ and $\mathbf{W}' \subset \mathbf{C}'$ are subcategories, and $S(\mathbf{W}) \subseteq \mathbf{W}'$. Then, we can define $L^H S : L^H\mathbf{C}(\mathbf{W}) \to L^H\mathbf{C}'(\mathbf{W}')$ by applying S levelwise, and then reducing (which is important).

Stay tuned for next week, when we prove some of these statements, discuss how to realize hammocks as limits over an indexing category, and how there are nice weak equivalences with the hammock localization when $\mathbf{C}$ and $\mathbf{W}$ admit calculi of fractions.

7. The Indexing Category II: 7/8/16

"This notation is getting very **CW***-complex." – Yuri Sulyma*

Today, Richard spoke about the homotopy lemma from [DK2] and all of the scary formalism that goes into proving it.

The homotopy lemma is one of the two important ingredients in the proof that the simplicial localization and hammock localization are weakly equivalent.

Lemma 7.1 (Homotopy lemma)**.** *Suppose* $\mathbf{A}, \mathbf{B} \in s\mathcal{O}$-**Cat** *and* $\mathbf{U} \subset \mathbf{A}$ *and* $\mathbf{V} \subset \mathbf{B}$ *are subcategories. If* $S : \mathbf{A} \to \mathbf{B}$ *is a functor such that* $S(\mathbf{U}) \subset \mathbf{V}$*, then the map* $\operatorname{diag} L^H\mathbf{A} \to \operatorname{diag} L^H\mathbf{B}$ *that it induces is a weak homotopy equivalence.*

7.1. **The Indexing Category.** Let $\mathbb{N}$ denote the set of positive integers,[11] and for an $n \in \mathbb{N}$, let $[n]$ denote $\{1, \dots, n\}$.

Definition 7.2. We define a category $\mathbf{II}$ as follows:

Objects: The sets (S, T), where S and T are finite, disjoint subsets of $\mathbb{N}$ and $S \cup T = [|S \cup T|]$.

Morphisms: A morphism $(S, T) \to (S', T')$ is a weakly order-preserving[12] function $f : S \cup T \to S' \cup T'$ such that $f(S) \subset S'$ and $f(T) \subset T'$.

We also define $\mathbf{II}_n$ to be the full subcategory of pairs (S, T) such that $|S \cup T| \le n$.

These subcategories are useful for filtering colimits.

Proposition 7.3. *Suppose that* $\sigma : \mathbf{II} \to \mathbf{sSet}$ *is a functor. Then,* $\operatorname{colim}_{\mathbf{II}} \sigma = \operatorname{colim}_{n\in\mathbb{N}} \operatorname{colim}_{\mathbf{II}_n} \sigma_n$*, where* $\sigma_n = \sigma|_{\mathbf{II}_n}$.

Some example morphisms:

- There are trivial arrows $(\varnothing, \varnothing) \to (\{1\}, \varnothing)$ and $(\varnothing, \varnothing) \to (\varnothing, \{1\})$.
- There are two arrows $(\{1\}, \varnothing) \rightrightarrows (\{1, 2\}, \varnothing)$: one sends $1 \mapsto 1$ and the other sends $1 \mapsto 2$.
- Correspondingly, there's an arrow $(\{1, 2\}, \varnothing) \to (\{1\}, \varnothing)$ sending $1, 2 \mapsto 1$.
- There's a single arrow $(\{1\}, \varnothing) \to (\{1\}, \{2\})$ sending $1 \mapsto 1$.

These objects should be thought of as strings of arrows pointing left or right: the n^{th} arrow points right if in S and left if in T. For example,

$$(\{1, 2, 4, 5\}, \{3, 6, 7, 8\}) \Rightarrow (\to, \to, \leftarrow, \to, \to, \leftarrow, \leftarrow, \leftarrow). \tag{7.4}$$

Now, the morphisms can be described by two rules:

- we're allowed to add an arrow anywhere in the diagram, and
- we may compose two adjacent arrows pointing in the same direction.

For example, the two morphisms $(\{1\}, \varnothing) \rightrightarrows (\{1, 2\}, \varnothing)$ are the maps $(\to) \rightrightarrows (\to, \to)$ that are inclusion as the first and as the second components.

Let I_n^0 denote the set of even integers in $[n]$, and I_n^1 denote the odd ones.

Definition 7.5. Given a functor $\sigma : \mathbf{II} \to \mathbf{sSet}$ and an $\varepsilon \in \{0, 1\}$, the *boundary* is the set

$$\operatorname{bd} \sigma(I_n^\varepsilon, I_n^{1-\varepsilon}) = \bigcup_{f \in \mathrm{PI}(\varepsilon, n)} \operatorname{Im}(\sigma_n f),$$

where $\mathrm{PI}(\varepsilon, n)$ is the set of proper injections[13] $f : (S, T) \to (I_n^\varepsilon, I_n^{1-\varepsilon})$.

Proposition 7.6. *For any* $n \ge 0$ *and* $\sigma : \mathbf{II} \to \mathbf{sSet}$*, the diagram*

$$\begin{tikzcd}
\operatorname{bd} \sigma(I_n^0, I_n^1) \cup \operatorname{bd}(I_n^1, I_n^0) \arrow[r] \arrow[d] & \operatorname{colim}_{\mathbf{II}_{n-1}} \sigma_{n-1} \arrow[d] \\
\sigma(I_n^0, I_n^1) \cup \sigma(I_n^1, I_n^0) \arrow[r] & \operatorname{colim}_{\mathbf{II}_n} \sigma_n
\end{tikzcd}$$

is a pushout square.

Dwyer and Kan claim this "is not difficult to verify," which is contentious. Nonetheless, it's reasonable to get a hold on the cases with smaller n, and then to try to induct. In any case, its utility in allowing us to inductively get a handle on colimits is a good thing.

Definition 7.7. First, observe that an injection in $\mathbf{II}$ is completely determined by its image and its target

- If f and g are injections in $\mathbf{II}$ into a target (S, T), we let $f \cap g$ denote the injection whose target is (S, T) and whose image is $\operatorname{Im} f \cap \operatorname{Im} g$.
- A functor $\sigma : \mathbf{II} \to \mathbf{sSet}$ is *proper* if
  (1) it carries injections to injections, and
  (2) if f, g are injections in $\mathbf{II}$ with the same target, then $\operatorname{Im}(\sigma(f \cap g)) = \operatorname{Im}(\sigma(f)) \cap \operatorname{Im}(\sigma(g))$.

[11] In [DK2], this set is denoted J. This notation (or $\mathbb{J}$) appears in more than one old paper; does anyone know why?

[12] This is what many people call order-preserving: if $a \le b$, then $f(a) \le f(b)$; contrast with strictly order-preserving or nondecreasing.

[13] Recall that an injection is proper if it's not a bijection.

Proper functors are nice because we can say more about them.

Proposition 7.8. *Suppose $\sigma, \sigma' : \mathbf{II} \rightrightarrows \mathbf{sSet}$ are proper functors and $t : \sigma \to \sigma'$ is a natural transformation such that for all $(S, T) \in \mathbf{II}$, $t(S, T) : \sigma(S, T) \to \sigma'(S, T)$ is a weak equivalence. Then, the induced map $\operatorname{colim}_{\mathbf{II}} \sigma \to \operatorname{colim}_{\mathbf{II}} \sigma'$ is a weak equivalence.*

Proof. By Proposition 7.3, $\operatorname{colim}_{\mathbf{II}} \sigma = \operatorname{colim}_n \operatorname{colim}_{\mathbf{II}_n} \sigma$, and similarly for σ', so it suffices to prove this when indexed over $\mathbf{II}_n$, as a colimit of weak equivalences is a weak equivalence.

Now, we draw a cube.[14]

$$\begin{tikzcd}
& \operatorname{bd} \sigma' \arrow[rr] \arrow[dd, hook] & & \operatorname{colim}^{\mathbf{II}_0} \sigma' \arrow[dd, hook] \\
\operatorname{bd} \sigma \arrow[rr] \arrow[ur, "s"] \arrow[dd, hook] & & \operatorname{colim}^{\mathbf{II}_0} \sigma \arrow[ur, "s"] \arrow[dd, hook] \arrow[dr, phantom, "\ulcorner"] & \\
& \sigma' \arrow[rr] & & \operatorname{colim}^{\mathbf{II}_1} \sigma' \\
\sigma \arrow[rr] \arrow[ur, "s"] & & \operatorname{colim}^{\mathbf{II}_1} \sigma \arrow[ur, "s"] \arrow[from=2-1, phantom, "\ulcorner"] &
\end{tikzcd}$$

⊠

Finally, all of the above extends to functors $\sigma : \mathbf{II} \to sO\text{-}\mathbf{Gr}$ without great change.

7.2. **Hammock Graphs.** Now, we apply this to hammock graphs. Let $\mathbf{C} \in O\text{-}\mathbf{Cat}$ and $\mathbf{W} \subset \mathbf{C}$ be a subcategory. Let $n \in \mathbb{N}$ and $\mathbf{m}$ be a word in $\{\mathbf{C}, \mathbf{W}^{-1}\}$ of length n.[15] We will also use $\mathbf{m}$ to denote the simplicial graph (object in $sO\text{-}\mathbf{Gr}$) such that for all $X, Y \in O$, the k-simplices of $\mathbf{m}(X, Y)$ are the hammocks of height k, length n, and type $\mathbf{m}$: if the i^{th} letter of $\mathbf{m}$ is $\mathbf{C}$, then the arrows in row $n - i + 1$ point to the right (and are in $\mathbf{C}$), and if it's $\mathbf{W}^{-1}$, the arrows in that row point to the left (and are in $\mathbf{W}$). All vertical arrows are in $\mathbf{W}$.

The simplicial set $\mathbf{m}(X, Y)$ is the nerve of the category whose objects are the hammocks of length n, type $\mathbf{m}$, and width 0, and whose morphisms are the hammocks of length n, type $\mathbf{m}$, and length 1. This category is denoted $N^{-1}\mathbf{m}(X, Y)$.[16]

Definition 7.9. Given $(\mathbf{C}, \mathbf{W})$ as above, we define a functor $\lambda\mathbf{C} : \mathbf{II} \to sO\text{-}\mathbf{Gr}$ sending (S, T) to the simplicial graph $\mathbf{m}$, where the i^{th} letter of $\mathbf{m}$ is $\mathbf{C}$ if $i \in S$, and is $\mathbf{W}^{-1}$ otherwise.

This functor formalizes the analogy we made in (7.4): given (S, T), it returns the hammocks whose columns have the arrow patterns we wrote down.

Proposition 7.10. *$\lambda\mathbf{C}$ is proper.*

This isn't too hard to check; the hardest part is figuring out what intersections are.

The hammocks that $\lambda\mathbf{C}$ produces may or may not be reduced; however, reducing them defines the *reduction map* $r : \lambda\mathbf{C}_{(S,T)}(X, Y) \to L^H\mathbf{C}(X, Y)$, which is a morphism in $sO\text{-}\mathbf{Gr}$.

Proposition 7.11. *The reduction maps induce an isomorphism $r : \operatorname{colim}_{\mathbf{II}} \lambda\mathbf{C} \xrightarrow{\sim} L^H\mathbf{C}$.*

The way to prove this is to tease apart the pushout square in Proposition 7.6.

Now, someday you might want to compute the space of morphisms in a localization. Using $L\mathbf{C}$ is awful for this; $L^H\mathbf{C}$ doesn't break universes as badly,[17] and this filtration produces a filtered way to compute this from finite pushout diagrams.

Anyways, we'll prove the homotopy lemma 7.1 using another homotopy lemma:

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[14]This was live-TEXed by Yuri, saving me some pain and confusion.

[15]That is, we use the symbols $\mathbf{C}$ and $\mathbf{W}^{-1}$, rather than drawing from composable morphisms in $\mathbf{C}$ and $\mathbf{W}^{-1}$; this point sometimes can be confusing.

[16]This notation is cute, which is not necessarily a good thing.

[17]This is also true of the LHC under Switzerland.

Lemma 7.12 (Another homotopy lemma)**.** *In the situation of Lemma 7.1, the induced map* $\operatorname{diag}\lambda\mathbf{A}(S,T) \to \operatorname{diag}\lambda\mathbf{B}(S,T)$ *is a weak equivalence.*

Assuming this lemma, we can prove Lemma 7.1, starting with Proposition 7.11.

8. Homotopy Calculi of Fractions and You!: 7/19/16

Today, Ernie talked about homotopy calculi of fractions.

Definition 8.1. Recall that a pair of categories $(\mathbf{C},\mathbf{W})$ admits a *left calculus of fractions* if

(1) given a diagram in $\mathbf{C}$
$$\begin{tikzcd} X' \arrow[r, "\sim", "u"'] & X \arrow[d, "f"] \\ & Y, \end{tikzcd}$$
with $u \in \mathbf{W}$, it can be completed to a diagram
$$\begin{tikzcd} X' \arrow[r, "\sim", "u"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "\sim", "v"'] & Y, \end{tikzcd}$$
where $v \in \mathbf{W}$.

(2) Given a commutative diagram
$$\begin{tikzcd} X' \arrow[r, "\sim", "u"'] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y, \end{tikzcd}$$
i.e. such that $u \in \mathbf{W}$ and $fu = gu$, then there exists a $v : Y \to Y'$ in $\mathbf{W}$ coequalizing this diagram, i.e. $vf = vg$.

This is "left" in the sense that maps in $\mathbf{W}$ may be moved to the left. The mirror concept is called a *right calculus of fractions*, and if $(\mathbf{C},\mathbf{W})$ has both a left and a right calculus of fractions, it is said to have a *calculus of fractions*.

Definition 8.2. Given $(\mathbf{C},\mathbf{W})$ as before and $i.j \geq 0$, define two maps $\alpha_{ij} : \mathbf{W}^{-1}\mathbf{C}^{i+j} \to \mathbf{W}^{-1}\mathbf{C}^{i}\mathbf{W}^{-1}\mathbf{C}^{j}$ and $\beta_{ij} : \mathbf{W}^{-1}\mathbf{W}^{i+j} \to \mathbf{W}^{-1}\mathbf{W}^{i}\mathbf{W}^{-1}\mathbf{W}^{j}$ in sO-$\mathbf{Gr}$ by adding an identity arrow. $(\mathbf{C},\mathbf{W})$ admits a *left homotopy calculus of fractions* if all α_{ij} and β_{ij} are weak equivalences in sO-$\mathbf{Gr}$.

The idea is that we can move weak equivalences to the left (coalescing two $\mathbf{W}^{-1}$ terms), at least up to homotopy.

Remark. Most categories admitting a homotopy calculus of fractions satisfy the two-out-of-three property: for any pair of composable maps f and g, if any two out of $\{f, g, g \circ f\}$ are weak equivalences, so is the third.

The last key point we want to extract from [DK2] is that, in these cases, we can use the hammock localization as a model for the naïve localization.

Theorem 8.3. *If* $(\mathbf{C},\mathbf{W})$ *admits a left homotopy calculus of fractions, then the natural maps* $\mathbf{W}^{-1}\mathbf{C} \to L^H\mathbf{C}$ *and* $\mathbf{W}^{-1}\mathbf{W} \to L^H\mathbf{W}$ *are equivalences.*

We'll come back to proving this. First, let's look at some examples where this holds.

Proposition 8.4. *If* $(\mathbf{C},\mathbf{W})$ *admits a left calculus of fractions, then it admits a left homotopy calculus of fractions.*

Proof. We want to prove that for all $X, Y \in \mathbf{C}$, the map $A : \mathbf{W}^{-1}\mathbf{C}^{i+j}(X,Y) \to \mathbf{W}^{-1}\mathbf{C}^{i}\mathbf{W}^{-1}\mathbf{C}^{j}(X,Y)$ in sO-$\mathbf{Gr}$ is a weak equivalence. These objects are nerves of the categories $N^{-1}\mathbf{m}$ and $N^{-1}\mathbf{m}'$, respectively, where $\mathbf{m}$ is the word $\mathbf{W}^{-1}\mathbf{C}^{i+j}$ and $\mathbf{m}'$ is the word $\mathbf{W}^{-1}\mathbf{C}^{i}\mathbf{W}^{-1}\mathbf{C}^{j}$. (Recall that $N^{-1}\mathbf{m}(X,Y)$ was the category whose objects are height-0, type-$\mathbf{m}$ hammocks from X to Y, and whose morphisms are height-1, type-$\mathbf{m}$ morphisms.) Inclusion of an identity arrow means that A is the nerve of a functor $N^{-1}A : N^{-1}\mathbf{m}(X,Y) \to N^{-1}\mathbf{m}'(X,Y)$, and we would like its nerve to be an equivalence. Thankfully, there's a theorem for this.

Theorem 8.5 (Quillen's theorem A)**.** *Let* $f : \mathbf{C} \to \mathbf{D}$ *be a functor such that for all objects* $d \in \mathbf{D}$, $N(d \downarrow f)$ *is contractible. Then,* $Nf : N\mathbf{C} \to N\mathbf{D}$ *is a weak homotopy equivalence of simplicial sets.*

Here, $d \downarrow f$ is a *comma category*, whose objects are (c, φ) with $c \in \mathbf{C}$ and $\varphi : d \to f(c)$ a morphism in $\mathbf{D}$ and whose morphisms are those $\mathbf{C}$-morphisms commuting with the maps from d.

Let's check that Quillen's theorem A applies to this situation.

Claim. For any $d \in N^{-1}\mathbf{W}^{-1}\mathbf{C}^i\mathbf{W}^{-1}\mathbf{C}^j(X, Y)$, $d \downarrow (N^{-1}A)$ is filtered, and therefore its nerve is contractible.

Recall that a category $\mathbf{C}$ is *filtered* if for all $a, b \in \mathbf{C}$, there's a $c \in \mathbf{C}$ admitting maps from both a and b. This implies that its nerve is contractible: for any two simplices, there's a higher-dimensional simplex joining them.

The claim is true, though a complete proof would be laborious; the idea is that given two objects (which are chains of arrows in $\mathbf{C}$ and $\mathbf{W}^{-1}$), one can make a long argument in calculus of fractions (and keep track of the maps to d), and end up with maps to $\leftarrow\leftarrow\rightarrow \cdots \rightarrow$, and then compose the first two arrows. It's instructive to work this out for $i = j = 1$, where it's more tractable.

Assuming this argument, Theorem 8.5 shows $N(N^{-1}A) = A$ is a weak equivalence. ⊠

The upshot is that one can think of zigzags of arrows as having just one arrow, on the left, in $\mathbf{W}^{-1}$.

There's a similar result for right fractions, or for calculi of fractions (both sides).

Proposition 8.6. *Suppose $(\mathbf{C}, \mathbf{W})$ has the two-out-of-three property and that for any $u : X \to X'$ in $\mathbf{W}$ and $f : X \to Y$ in $\mathbf{C}$, the square*

$$\begin{tikzcd} X' \arrow[d, dashed, "g"'] & X \arrow[l, "\sim", "u"'] \arrow[d, "f"] \\ Y' & Y \arrow[l, dashed, "v", "\sim"'] \end{tikzcd}$$

can be filled in functorially (here $v \in \mathbf{W}$), then $(\mathbf{C}, \mathbf{W})$ admits a homotopy calculus of fractions.

Remark. If $\mathbf{C}$ admits pushouts and $\mathbf{W}$ is closed under pushouts, then they hypotheses of Proposition 8.6 are satisfied. In particular, given a model category $\mathbf{M}$ with functorial factorizations, then $\mathbf{M}^f$, $\mathbf{M}^c$, $\mathbf{M}^{cf}$, and $\mathbf{M}$ (each $\mathbf{M}^*$ paired with $\mathbf{W} \cap \mathbf{M}^*$) each admit two-sided homotopy calculi of fractions. This in particular implies that the functor diagram induced from inclusions

$$\begin{tikzcd} L^H(\mathbf{M}^{cf}) \arrow[r] \arrow[d] & L^H(\mathbf{M}^f) \arrow[d] \\ L^H(\mathbf{M}^c) \arrow[r] & L^H(\mathbf{M}) \end{tikzcd}$$

induces weak equivalences on every pair of objects.

Now, we can prove Theorem 8.3. This will tie in several results from [DK2] that we've been discussing, including that $L^H\mathbf{C} = \operatorname{colim}^{\mathbf{II}}(\lambda\mathbf{C})$ over the diagram category $\mathbf{II}$. Recall that we had functors $A, B : \mathbf{II} \to \mathbf{II}$ sending

$$(S, T) \overset{A}{\longmapsto} (\{1, 2, \dots, |S|\}, \varnothing)$$
$$(S, T) \overset{B}{\longmapsto} (S, T \cup \{|S \cup T| + 1\}).$$

Inside $\lambda\mathbf{C}$, i.e. thinking of objects in $\mathbf{II}$ as arrows, A forgets all of the leftwards arrows and B appends a single leftwards arrow. Notice that $\mathbf{W}^{-1}\mathbf{C} = (\lambda\mathbf{C})BA(\{1\}, \varnothing)$.

We prove Theorem 8.3 in a series of lemmas.

Lemma 8.7. *The inclusion $i : \mathbf{W}^{-1}\mathbf{C} = (\lambda\mathbf{C})BA(\{1\}, \varnothing) \to \operatorname{colim}^{\mathbf{II}}(\lambda\mathbf{C})BA$ is an isomorphism.*

Proof. Something in $\operatorname{colim}^{\mathbf{II}}(\lambda\mathbf{C})BA$ is a single leftwards arrow followed by a bunch of rightwards arrows; composing all the rightwards arrows together defines a map back to $\mathbf{W}^{-1}\mathbf{C}$. Thus, $\mathbf{W}^{-1}\mathbf{C}$ is terminal, so it must be the colimit. ⊠

We'll expand the scope of this idea.

Lemma 8.8. *There's a single injection $A(S, T) \to (S, T)$ by including identities where the leftwards arrows forgotten by A were; the map $j : \operatorname{colim}^{\mathbf{II}}(\lambda\mathbf{C})BA \to \operatorname{colim}^{\mathbf{II}}(\lambda\mathbf{C})B$ is a weak equivalence.*

Proof. Now, we can have arbitrary many $\mathbf{W}^{-1}$ terms on the right. We'll induct on the number of these terms, i.e. on $|T|$. Notice that if $|T| = 0$, this is trivially true, as using A doesn't do anything.

Recall that $\lambda\mathrm{C}$ is proper, in that it takes injections to (categorical) injections; hence, $(\lambda\mathrm{C})BA$ and $(\lambda\mathrm{C})B$ are also proper. This means it suffices to check weak equivalences pointwise, i.e. on each choice of (S, T). In the appendix of [DK2] (which we'll discuss next week), we'll learn how to induct via concatenation of words: using Grothendieck constructions, we can view these as nerves of categories.

Once we learn how to do this, we can take a word $\mathbf{m}$ corresponding to (S, T), and split it into $\mathbf{m}'\mathbf{W}^{-1}\mathbf{C}^i$, so that $\mathbf{m}'$ has one fewer $\mathbf{W}^{-1}$ term. By induction, the result is true for $\mathbf{m}'$, and on $\mathbf{W}^{-1}\mathbf{C}^i$, A doesn't do anything, so j is an equivalence. ⊠

Lemma 8.9. *For all $X, Y \in \mathbf{C}$, the following diagram commutes up to homotopy.*

$$\begin{tikzcd} & \operatorname{colim}^{\mathbf{II}}(\lambda\mathrm{C})BA(X,Y) \arrow[dr, "j"] & \\ \mathbf{W}^{-1}\mathbf{C}(X,Y) \arrow[ur, "i"] \arrow[dr, "\text{incl.}"'] & & \operatorname{colim}^{\mathbf{II}}(\lambda\mathrm{C})B(X,Y) \\ & \operatorname{colim}^{\mathbf{II}}(\lambda\mathrm{C}) \arrow[ur, "k"'] & \end{tikzcd}$$

Here, k is the map induced by the inclusion $(S, T) \mapsto B(S, T)$.

Proof. We're starting with arrows of the form $\leftarrow\rightarrow$, and we want to find a homotopy identifying $\leftarrow\rightarrow$ with $\leftarrow\leftarrow\rightarrow$, which is the homotopy of composition $\mathbf{W}^{-1}\mathbf{C} \to \mathbf{W}^{-1}\mathbf{C}\mathbf{W}^{-1}\mathbf{C}$ guaranteed as part of the data of a left homotopy calculus of fractions ⊠

Lemma 8.10. *The map k defined in Lemma 8.9 has a left inverse induced by inclusion of* $\operatorname{Im} B \subset \mathbf{II}$.

Proof. This left inverse should take a word $\mathbf{m}$ and prepend a leftwards arrow (a $\mathbf{W}^{-1}$); the other possibility given by the diagram is to start with $\mathbf{m}$ beginning with a $\mathbf{W}^{-1}$, add a $\mathbf{W}^{-1}$, and then compose. Again, composing is a homotopy equivalence because we have the data of a left homotopy calculus of fractions. ⊠

In particular, this inverse is a weak homotopy equivalence, so we have weak homotopy equivalences

$$\mathbf{W}^{-1}\mathbf{C}(X,Y) \xrightarrow[\sim]{i} \operatorname{colim}^{\mathbf{II}}(\lambda\mathrm{C})BA(X,Y) \xrightarrow[\sim]{j} \operatorname{colim}^{\mathbf{II}}(\lambda\mathrm{C})B(X,Y) \xleftarrow[k]{\sim} \operatorname{colim}^{\mathbf{II}}(\lambda\mathrm{C})(X,Y) \xrightarrow{=} L^H\mathbf{C},$$

and therefore have proved Theorem 8.3.

9. The Grothendieck Construction: 7/26/16

"At som epoint, Dwyer probably said to Kan, 'you think anyone will read this?' and Kan said 'Nah.' "

Today, Nicky talked about the Grothendieck construction, which is covered in the appendix of [DK2].

Definition 9.1. Suppose that $\mathbf{W}$ is a category and $F : \mathbf{W}^{\mathrm{op}} \to \mathbf{Cat}$ and $G : \mathbf{W} \to \mathbf{Cat}$ are functors. Then, the *Grothendieck construction* is the category $F \otimes_{\mathbf{W}} G$ defined by the following data.

Objects: The objects are triples (A, W, B), where W is an object in $\mathbf{W}$, A is an object in $F(W)$, and G is an object in $G(W)$.

Morphisms: A morphism $(A, W, B) \to (A', W', B')$ is the data of a $\mathbf{W}$-morphism $\varphi : W \to W'$, a $F(W)$-morphism $A \to F\varphi A'$, and a FW'-morphism $G\varphi B \to B'$.

Recall that if $\mathbf{m}$ is a word in the letters $\{\mathbf{W}^{-1}, \mathbf{C}\}$, then we defined a functor $N^{-1}\mathbf{m}(-,-) : \mathbf{W}^{\mathrm{op}} \times \mathbf{W} \to \mathbf{Cat}$ that sends X, Y to the category $N^{-1}\mathbf{m}(X, Y)$ of hammocks of lengths 0 and 1. Given two different words $\mathbf{m}$ and $\mathbf{m}'$, if $\mathbf{m} \,||\, \mathbf{m}'$ denotes the concatenation of $\mathbf{m}$ and $\mathbf{m}'$, we'd like to understand $N^{-1}(\mathbf{m} \,||\, \mathbf{m}')$ in terms of $N^{-1}\mathbf{m}$ and $N^{-1}\mathbf{m}'$.

Given a fixed $X \in O$, we'd like to understand the functor $N^{-1}\mathbf{m}(X, -) : \mathbf{W} \to \mathbf{Cat}$, which might be covariant or contravariant (the paper is somewhat murky about this, so we're going to be explicit). Let $\tau : Y \to Y'$ be a morphism in O.

- Suppose $\mathbf{m}$ ends with $\mathbf{W}^{-1}$. Then, hammocks end in reverse arrows $Z \leftarrow Y$ and $Z' \leftarrow Y'$, so composing with the latter arrow defines a map τ^* going in the other direction, so if $\mathbf{m}$ ends with $\mathbf{W}^{-1}$, then $N^{-1}\mathbf{m}(X, -)$ is *contravariant.*
- If $\mathbf{m}$ ends with $\mathbf{C}$, then hammocks end with forward arrows, so we precompose with the last map $Z \to Y$, so $N^{-1}\mathbf{m}(X, -)$ is *covariant.*

There are several possibilities when we add $\mathbf{m}'$ into the mix; we'll just discuss one example. Suppose $\mathbf{m}'$ starts with (meaning furthest to the right) $\mathbf{W}^{-1}$ and $\mathbf{m}$ ends with $\mathbf{W}^{-1}$. In this case, $N^{-1}\mathbf{m}'(X, -)$ is contravariant and $N^{-1}\mathbf{m}(-, Y)$ is covariant. In this case, we can apply the Grothendieck construction to conclude

$$N^{-1}(\mathbf{m} \,||\, \mathbf{m}') \cong N^{-1}\mathbf{m}'(X, -) \otimes_{\mathbf{W}} N^{-1}\mathbf{m}(-, Y).$$

This is exactly because objects of this Grothendieck construction are triples $(X \to Z \leftarrow W, W, W \leftarrow Z' \to Y)$, which is the same data as a zigzag

$$X \longrightarrow Z \longleftarrow W \longleftarrow Z' \longrightarrow Y.$$

There are several other variance cases, depending on what $\mathbf{m}$ and $\mathbf{m}'$ start and end with, but the idea is quite similar.

Our next goal is to show that the Grothendieck construction can be computed using simplicial techniques. We will show that the nerve of $F \otimes_W G$ is weakly equivalent to the *triagonal* of a certain *tri*simplicial set.[18]

Let $\mathbf{1}$ denote the trivial category with a single object and a single (identity) morphism, and $* : \mathbf{W} \to \mathbf{1}$ be the trivial functor. Then, there is a forgetful functor $P : F \otimes_{\mathbf{W}} * \to \mathbf{W}$ sending $(A, W, B) \mapsto W$.

Given any $W \in \mathbf{W}$, the category $W \downarrow P$ has objects $\mathcal{A} = (A, U, *) \in F \otimes_{\mathbf{W}} *$ along with data of a map $W \to P(\mathcal{A})$, and the morphisms are those morphisms in $F \otimes_{\mathbf{W}} *$ that, after applying P, commute with the maps from W. This is functorial in $\mathbf{W}$, defining a functor $- \downarrow P : .\mathbf{W}^{\mathrm{op}} \to \mathbf{Cat}$.

The last ingredient we need for Lemma 9.2 is the trisimplicial set. Let F, $\mathbf{W}$, and G be as in Definition 9.1. We'll construct a trisimplicial set $N_{*,*,*}(F, \mathbf{W}, G)$, where an (i, j, k)-simplex is a triple

- a j-simplex of $N(\mathbf{W})$, so a chain of j composable arrows $W_j \to W_{j-1} \to \cdots \to W_1 \to W_0$ in $\mathbf{W}$;
- an i-simplex of $N(F(W_0))$; and
- a k-simplex of $N(G(W_j))$.

We let $N(F, \mathbf{W}, G)$ denote the diagonal of $N_{*,*,*}(F, \mathbf{W}, G)$.

Lemma 9.2. *There is a chain of weak equivalences*

$$N(F, \mathbf{W}, G) \xleftarrow{\sim} N(- \downarrow P, \mathbf{W}, G) \xrightarrow{\sim} N(*, F \otimes_{\mathbf{W}} *, G(P)).$$

Proof. Let's start with the first equivalence. We'll fix the second and third variables, and produce a weak equivalence by showing that there's a weak equivalence $T : W \downarrow P \xrightarrow{\sim} F(W)$. Given an object $\mathcal{A} = (A, U, *)$, so P maps it to U and a map $\alpha : W \to U$, we will assign it to $T(\mathcal{A}) = (F\alpha)(A)$.

Using Quillen's theorem A 8.5, it suffices to show that for all $C \in F(W)$, $C \downarrow T$ is contractible, and therefore suffices to show that it has an initial object. The objects of $C \downarrow T$ are $(F\alpha)(A) = X$ with a map $\gamma : C \to X$, where $\mathcal{A} = (A, U, *)$ is in $W \downarrow P$. Our candidate initial object is $\mathrm{id} : C \to C$, so let's build something that maps to it. In $W \downarrow P$, we'd like to have $(C, W, *)$, which P maps to W, and the identity map $\mathrm{id} : W \to W$. Then, T maps this to $\mathrm{id} : C \to C$. Thus, we have defined an object: it's unpleasant but possible to show this is initial, and you can see why it could come before every arrow: why does $(C, W, *)$ map to $(A, U, *)$ for all $(A, U, *) \in W \downarrow P$? The map $W \to U$ is the map α that we already had, and hitting it with F gives us $\gamma : C \to (F\alpha)A$. Everything unwinds from the fact that this is all happening below C.

Thus, by Quillen's theorem A, there's a weak equivalence $N(F, \mathbf{W}, G) \xrightarrow{\sim} N(- \downarrow P, \mathbf{W}, G)$.

For the second step, things get trickier: look at $N(- \downarrow P, \mathbf{W}, G)$. The simplicial set $N_{i,j,*}(- \downarrow P, \mathbf{W}, G)$ is determined by

- a j-simplex $W_j \to \cdots \to W_0$ of $N\mathbf{W}$, and
- an i-simplex of $N(W_0 \downarrow P)$, which in turn is given by a collection of composable arrows $A_i \to \cdots \to A_0$ in the domain of P, i.e. an i-simplex of $N(F \otimes_{\mathbf{W}} *)$, where $A_l = (S_l, U_l, *)$, along with maps $W_0 \to U_l$ commuting with the maps $U_i \to U_{i-1}$. But this is uniquely determined by the map $b : W_0 \to U_i$.[19]

[18] The triagonal of a trisimplicial set is the simplicial set constructed in the analogous manner as the diagonal of a bisimplicial set. The word "triagonal" sounds made-up, but is really used for this purpose, though [DK2] just uses "diagonal."

[19] In [DK2], they say this should map to U_0, but this is a typo.

Claim. $N_{i,j,j}(- \downarrow P, \mathbf{W}, G)$ consists of the j-simplices of the disjoint union of the homotopy colimits

$$\mathrm{holim}^{\mathbf{W}\downarrow U_i}((W \mapsto U_i) \longmapsto N(G(W))).$$

This is confusing to unwind, but if you ask what all the data is, things fall into place, relating the j-simplices of these simplicial sets with the j-simplices of $N_{i,j,*}(- \downarrow P, \mathbf{W}, G)$. There should be a formula, but there doesn't appear to be a slick way to do this. This is probably the hardest part of everything.

Once we have this claim, j-simplices of $N(*, F \otimes_{\mathbf{W}} *, G(P))$ come from $N(G(U_i))$; since we have a terminal object, namely U_i, we get a natural isomorphism from this homotopy limit to $N(G(U_i))$. ⊠

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