M392C NOTES: RATIONAL HOMOTOPY THEORY

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These notes were taken in UT Austin's Math 392c (rational homotopy theory) class in Fall 2015, taught by Jonathan Campbell. I live-TEXed them using vim, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

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Lecture 1.

Postnikov Towers and Principal Fibrations: 8/27/15

First, we'll outline some aspects of the course.

$X \to Y$ is a rational equivalence if $\pi_*(X) \otimes \mathbb{Q} \xrightarrow{\cong} \pi_*(Y) \otimes \mathbb{Q}$. The goal is to define a category $\mathrm{Ho}(\mathsf{Top}_{\mathbb{Q}})$ where, more or less, the isomorphisms are rational equivalences. The point is that this is a purely algebraic category, equivalent to a category of differential graded algebras, $\mathrm{Ho}(\mathsf{CDGA}_{\mathbb{Q}})$.

The first half of the course will deal with something called Sullivan's method: we'll get our hands on rational equivalence, and produce the rationalization functor $X \mapsto X_{\mathbb{Q}}$. We're developing it as it "could have been done," with some computations to show that things get a lot easier over $\mathbb{Q}$ (e.g. homology of Eilenberg-MacLane spaces is the same as for spheres).

Then, we'll have to talk about model categories, which is a good way of producing homotopy categories or homotopy theories for more than just topological categories. Intuitively, a model category is a category in which one can do homotopy theory. Using this, we'll talk about the homotopy theory of commutative, differential algebras over $\mathbb{Q}$.

This isn't how it was originally done by Sullivan et al., and so we'll also discuss the classical construction. We'll also produce functors from simplicial sets to differential graded $\mathbb{Q}$-algebras and topological spaces, with adjoints and so on. One of these, turning a simplicial set into a differential graded $\mathbb{Q}$-algebra, will resemble the functor Ω^* of differential forms, but is more combinatorial.

This will enable us to prove equivalence, with all sorts of cool consequences: Whitehead products appear in the differential graded algebras category; automorphisms of CDGAs correspond to automorphisms of $\mathsf{Top}_{\mathbb{Q}}$, which relate to automorphisms of topological spaces nicely, and so on.

The rest of the course will discuss Quillen's model, which relates differential graded Lie algebras to rational spaces. That might not mean anything right now, and we'll have to learn a little more machinery for it. Thus, this course will cover some classical and some modern algebraic topology, making the useful notion of model categories nice and concrete.

Here are some good references for this subject.

- Griffiths-Morgan, *Rational Homotopy Theory and Differential Forms*; it's all right, and geometric (they use simplicial complexes, rather than simplicial sets, and therefore don't get as nice of a result). The second edition came out a year ago, but is similar to the first edition. The beginning has a beautiful exposition of algebraic topology in general.
- There's a GTM by Felix, Halperin, and Thomas, called *Rational Homotopy Theory*. It's pretty beefy, and the one gripe the professor has is that it doesn't use model categories at all, making things opaque. But there's definitely a bootlegged copy...
- Katherine Hess, who is a great writer, has a survey paper, about 20 pages, called *Rational Homotopy Theory*.

Those are the only expository works, but there are also some papers.

- Sullivan, "Infinitesimal Computations in Algebraic Topology." Sullivan is crazy, and the paper is very hard to read. Hopefully after the course everything is easier to read.

- Quillen, "Rational Homotopy Theory." This paper also isn't that easy to read.

There are a few other sources; things will be well cited in this class.

Now for some math.

Definition. Let X be a connected topological space. A *Postnikov tower* is a sequence

$$\begin{tikzcd} & X_n \arrow[d] \\ & \vdots \arrow[d] \\ & X_2 \arrow[d] \\ X \arrow[r] & X_1 \end{tikzcd}$$

such that

(1) there are maps $X \to X_i$,
(2) $\pi_i(X) \cong \pi_i(X_n)$ for $i \leq n$, and
(3) $\pi_i(X_n) = 0$ for $i > n$.

As a consequence of the three properties, the homotopy fiber $X_n \to X_{n-1}$ is a $K(\pi_n(X), n)$, i.e. an Eilenberg-MacLane space. In some sense, this is a "co-cellular" way of building a space out of Eilenberg-MacLane spaces.

Theorem 1.1. *Postnikov towers exist.*

The proof is easy: just attach cells to X to kill homotopy above a given degree. But that's not so useful of a characterization. We want to know: what information in stage n determines stuff in stage $(n+1)$?

To produce spaces with certain fibers, classifying maps are useful. Suppose X_{n+1} arises as a (homotopy) pullback: if $\star$ denotes a contractible space, this would look like

$$\begin{tikzcd} X_{n+1} \arrow[r] \arrow[d] & \star \arrow[d] \\ X_n \arrow[r] & K(\pi_{n+1}(X), n+2). \end{tikzcd}$$

It would be nice if fibrations with fiber $K(G, n)$ were classified by maps $X \to K(G, n+1)$, because then we could work with the cohomology group $H^{n+2}(X, \pi_{n+1}(X))$. It's not very easy to compute stuff in this cohomology group, however.

In any case, not every fibration is even classified in this manner!

Definition. A fibration $K(\pi, n) \to E \to B$ is *principal* if it arises as a pullback of a path fibration as follows.

$$\begin{tikzcd} K(\pi, n) \arrow[r, equal] \arrow[d] & K(\pi, n) \arrow[d] \\ E \arrow[r] \arrow[d] & P \arrow[d] \\ B \arrow[r] & K(\pi, n+1) \end{tikzcd}$$

There's an equivalent, less useful, formulation in the lecture notes. The reason we like our formulation is the following theorem.

Theorem 1.2. *A connected CW complex X with $\pi_1(X)$ acting trivially on $\pi_n(X)$ has a Postnikov tower composed of principal fibrations.*

As a consequence, X_{n+1} is determined from X_n by a map $k_n : X_n \to K(\pi_{n+1}(X), n+2)$; this determines a class $[k_n] \in H^{n+2}(X_n, \pi_{n+1}(X))$, called a *k-invariant*. This is why we care about Postnikov towers: they are built up nicely in stages, using cohomology classes that, in nice cases, we can compute. And so in rational homotopy theory, where the homotopy groups are nicer, the k-invariants are nicer.

We'll use spectral sequences in this class; an introduction to them can be found in the professor's lecture notes.

Another takeaway from these results is that Eilenberg-MacLane spaces are pretty fundamental building blocks. Though they have nice homotopy, their cohomology groups are generally pretty nasty, leading to computations called Steenrod operations. But rationally, there's a nice result.

Theorem 1.3.
$$H^*(K(\mathbb{Z},n);\mathbb{Q}) = \begin{cases} \mathbb{Q}[x], & n \text{ even} \\ \Lambda_{\mathbb{Q}}(x), & n \text{ odd}, \end{cases},$$
where the generators x have degree n.

These are the simplest differential graded $\mathbb{Q}$-algebras, and suggest that all spaces' rational homotopy will be built out of them (which is true).

Proof. As a base case, $H^*(K(\mathbb{Z},1);\mathbb{Q}) = H^*(S^1;\mathbb{Q}) = \Lambda_{\mathbb{Q}}(x)$, which is fine. More generally, we'll use the fibration
$$K(\mathbb{Z}, n-1) \longrightarrow \star \longrightarrow K(\mathbb{Z},n).$$
By induction, if n is odd, then $H^*(K(\mathbb{Z},n-1);\mathbb{Q}) = \mathbb{Q}[x]$, with $\deg x = n-1$, since $n-1$ is even. Let's use the Serre spectral sequence, for which
$$E_2^{p,q} = H^p(K(\mathbb{Z},n-1);\mathbb{Q}) \otimes H^q(K(\mathbb{Z},n-1);\mathbb{Q}).$$
For example, when $n=3$, we have
$$\begin{array}{cccccc}
\mathbb{Q}y^3 \\
0 \\
\mathbb{Q}y^2 \\
0 \\
\mathbb{Q}y & 0 & 0 & \mathbb{Q}x \otimes \mathbb{Q}y \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbb{Q}x & H^4 & H^5 & \cdots
\end{array}$$
Here, degree increases from 0 to the right and going upwards. This also uses the Hurevitch theorem. Then, we remark that d_2, from $(0,2)$ to $(3,0)$, has to be an isomorphism, because the E_∞-page is 0.[1] But since the Serre spectral sequence is linear, we also have isomorphisms from $(4,0)$ to $(2,2)$ to $(0,4)$; specifically, $d_3 : \mathbb{Q}y^2 \mapsto \mathbb{Q}x \otimes \mathbb{Q}y$, so $H^6(K(\mathbb{Z},3);\mathbb{Q}) = 0$. And this means that $x^2 = 0$. Then, we continue by induction to show that all higher $H^q(K(\mathbb{Z},3);\mathbb{Q})$ are zero. Thus, $H^*(K(\mathbb{Z},3);\mathbb{Q}) = \Lambda_{\mathbb{Q}}(x)$, and the case for general n is similar. ⊠

Exercise 1. Handle the case where n is even, which is somewhat similar.

Again, this is suggestive: Eilenberg-MacLane spaces build topological spaces up, and they have differential graded algebras for their rational cohomology groups.

Next lecture, we'll discuss Serre theory, the tricks that Serre used to compute the rational homotopy groups of the spheres. These are strong clues that, rationally, things are much nicer.

After that, we'll discuss rational equivalence, and then CDGAs and their homotopy theory, necessitating a discussion of model categories. The course will get less computational at this point.

Lecture 2.

Serre Theory: 9/1/15

"Whistle guy has really got me off my game!"

Last lecture may have gone a little fast, so we'll reboot and say some things that we didn't mean to assume. Then, we'll start Serre theory.

Definition. If G is a group, an *Eilenberg-MacLane space* $K(G,n)$ is a space whose homotopy groups are
$$\pi_i(K(G,n)) = \begin{cases} G, & i = n \\ 0, & i \neq n. \end{cases}$$

For example, S^1 is a $K(\mathbb{Z},1)$, $\mathbb{C}P^\infty$ is a $K(\mathbb{Z},2)$, and $\mathbb{R}P^\infty$ is a $K(\mathbb{Z}/2,1)$. Also, $K(G,n-1) = \Omega K(G,n)$ (i.e. the space of loops). One can think of building spaces out of these, because they're the simplest spaces from the perspective of homotopy theory.

Another of the reasons they're important is the following theorem, which is hard to prove.

[1] Though the first few lectures will use spectral sequences, they won't be very important after that, so don't drop the course if this is the only thing making you uncomfortable.

Theorem 2.1. $[X, K(G, n)] = \widetilde{H}^n(X; G)$.

In particular, the cohomolgy functor is representable.

This is why, last time, a map $k : X \to K(\pi_{n+1}(X), n+2)$ corresponded to $H^{n+2}(X; \pi_{n+1}(X))$: k-invariants arise from the representability of cohomology.

Postnikov towers are a way of using Eilenberg-MacLane spaces to build a space up, one homotopy group at a time.

Theorem 2.2. *Eilenberg-MacLane spaces exist for all n and all G, where G is abelian if $n > 1$.*

Recall by the Eckmann-Hilton argument that $\pi_n(X)$ is always abelian when $n > 1$.

Proof idea. When $n = 1$, one can produce a fiber bundle where the total bundle is trivial and the fiber is G, with the discrete topology, thus producing a sequence $G \to EG \to BG$, and one can show that $\pi_1(BG) = G$ and $\pi_i(BG) = 0$ for $i \geq 2$. This is done in chapter 1 of Hatcher.

If $n > 1$, take a resolution for G:

$$\mathbb{Z}[r_\beta] \longrightarrow \mathbb{Z}[g_\alpha] \longrightarrow G,$$

where the g_α are generators and r_β are relations. Then, consider a bouquet of spheres $\bigvee_\alpha S^n_\alpha$; each relation gives a map $S^n \to \bigvee_\alpha S^n_\alpha$ using the degree of the relation, so glue cells onto this bouquet via the relations, forming pushouts of the form

$$\begin{tikzcd} \bigvee_\beta S^n \arrow[r] \arrow[d] & \bigvee_\alpha S^n_\alpha \\ \bigvee_\beta D^n \arrow[r] & X \arrow[u, hook] \end{tikzcd}$$

Then, the n-skeleton $X^{(n)}$ is given by the generators, and so we have an exact sequence

$$\pi_{n+1}(X, X^{(n)}) \longrightarrow \pi_n(X^{(n)}) \longrightarrow \pi_n(X) \longrightarrow \pi_n(X, X^{(n)}).$$

Then, $\pi_{n+1}(X, X^{(n)}) = \mathbb{Z}[r_\beta]$, $\pi_n(X^{(n)}) = \mathbb{Z}[g_\alpha]$, and $\pi_n(X, X^{(n)}) = 0$, so $\pi_n(X) = G$. There are only n-cells, so there's no lower homotopy, and we can kill all higher homotopy in a standardized manner. ⊠

Anyways, last time we talked about how $H^*(K(\mathbb{Z}, n), \mathbb{Q})$ is $\mathbb{Q}[x]$ when n is even and $\Lambda_\mathbb{Q}(x)$ when n is odd; this suggests that since the constituents of spaces have simple rational homotopy, then maybe rational homotopy provides some insights.

We proved this with the Serre spectral sequence, and we'll use it a few more times to get nice short exact sequences, so let's carefully state what's going on.

Theorem 2.3 (Homological Serre sequence)**.** *Let $F \to E \to B$ be a fibration with $\pi_1(B)$ acting trivially on $H^*(F, G)$. Then, there is a spectral sequence $\{E^r_{p,q}, \mathrm{d}_r\}$ such that:*

(1) $\mathrm{d}_r : E^r_{p,q} \to E^r_{p-r,q-r+1}$, i.e. d_r is of degree $(-r, r-1)$.
(2) $E^\infty_{n,n-p} = F^p_n / F^{p-1}_n$, where $F^\bullet_n$ is some filtration of $H_n(E; G)$, i.e. $H_n(E; G) = F^n_n \supset F^{n-1}_n \supset \cdots \supset F^0_n = 0$.
(3) $E^2_{p,q} = H_p(B; H_q(F; G))$.

Remark. If G is a field, then Künneth gives us the nicer result that $E^2_{p,q} = H_p(B) \otimes H_q(E)$.

Theorem 2.4 (Cohomological Serre sequence)**.** *With the same setup as Theorem 2.3, there exists a spectral sequence $(E^{p,q}_r, \mathrm{d}_r)$ such that:*

(1) $\mathrm{d}_r : E^{p,q}_r \to E^{p+q,q-r+1}_r$.
(2) $E^{p,n-p}_\infty = F^n_p / F^n_{p-1}$, where $F^n_\bullet$ is some filtration of $H^n(E; G)$.
(3) $E^{p,q}_2 \cong H^p(B : H^q(F; G))$.

The cohomological Serre sequence is multiplicative: there's a product that ultimately comes from the cup product. This lends some rigidity to the cohomological theory that is often very useful.

Example 2.5. Suppose we're in a case of the homological Serre spectral sequence such that $E^2_{p,q}$ for $0 < q < n$. It turns out that in situations like this, you can leverage your knowledge of the sequence to obtain useful exact sequences.

In this case, the first differential that does anything interesting is $\mathrm{d}_{n+1} : E^2_{n+1,0} \to E^2_{0,n}$ (previous differentials all map to zero). This means that $E^3_{n+1,0} = E^2_{n+1,0} / \operatorname{Im}(\mathrm{d}_{n+1})$, and nothing else hits this, so this is also $E^\infty_{n+1,0}$. Thus, we have a sequence

$$E^2_{n+1,0} \longrightarrow E^2_{0,n} \longrightarrow E^\infty_{0,n} \longrightarrow 0. \tag{9.1.1}$$

Furthermore, there is a filtration

$$H_n(E) = F_n^n \supset F_n^{n-1} \supset \cdots \supset F_n^0,$$

with $E^\infty_{p,n-p} = F_n^p/F_n^{p-1}$. In particular, $E^\infty_{0,n} = F_n^0$, $F^\infty_{1,n} = 0$, and $E^\infty_{n,0} = F_n^n/F_n^{n-1} = H_n(X)/E^\infty_{0,n}$. That is, we have a sequence

$$0 \longrightarrow E^\infty_{0,n} \longrightarrow H_n(X) \longrightarrow E^\infty_{n,0} \longrightarrow 0,$$

which we can join to (9.1.1) to produce

$$E^2_{n+1,0} \longrightarrow E^2_{0,n} \longrightarrow E^\infty_{0,n} \longrightarrow H_n(E) \longrightarrow E^\infty_{n,0} \longrightarrow 0.$$

This may seem a little contrived, but it happens, for example, when a fiber in a fibration is n-connected.

Serre Theory. This was sometimes called Serre's thesis. It will use some very abstract computations to determine the rational homotopy groups of the spheres.

Definition. Let $\mathcal{C}$ be one of the three classes (the *Serre classes*): *FG* of finitely generated abelian groups, $\mathcal{T}_P$, the torsion abelian groups with orders drawn from a set P, and $\mathcal{F}_P$, the finite groups in $\mathcal{T}_P$.

Lemma 2.6. *The classes $\mathcal{C}$ are closed under extension: that is, if $A, C \in \mathcal{C}$ and $0 \to A \to B \to C \to 0$ is short exact, then $B \in \mathcal{C}$. Moreover, for any $A, B \in \mathcal{C}$, $A \otimes B \in \mathcal{C}$ and $\mathrm{Tor}(A, B) \in \mathcal{C}$.*

The point is that the disgusting machinery of the Serre spectral sequence mostly leaves a Serre class intact, which will be useful for proving the Hurewicz theorem mod $\mathcal{C}$. First, though, we'll need more lemmas.

Lemma 2.7. *Let $F \to E \to B$ be a filtration satisfying the hypotheses of the homological Serre spectral sequence, and assume F, E, and B are all path-connected. If any two of $H_*(B)$, $H_*(E)$, and $H_*(F)$ are in $\mathcal{C}$, then so is the third.*

Proof. We'll show that if $H_*(B)$ and $H_*(F)$ are in $\mathcal{C}$, then $H_*(E)$ is. Recall that

$$\begin{aligned} E^2_{p,q} &= H_p(B; H_q(F)) \\ &= H_p(B;\mathbb{Z}) \otimes H_q(F;\mathbb{Z}) \oplus \mathrm{Tor}(H_{p-1}(B), H_q(F)), \end{aligned}$$

by the universal coefficient theorem. The first two terms are in $\mathcal{C}$ by assumption, and Lemma 2.6 implies the last one is. Thus, $E^2_{p,q} \in \mathcal{C}$, so all subsequent pages must be too (since homology is just kernels and images, which don't pop us out of $\mathcal{C}$). Since $H_n(X)$ has successive filtrations whose quotients are $E^\infty_{p,q}$, which are all in $\mathcal{C}$, then $H_n(E) \in \mathcal{C}$. ⊠

Lemma 2.8. *If $\pi \in \mathcal{C}$, then $H_k(K(\pi, n), \mathbb{Z}) \in \mathcal{C}$ for all k.*

At this point, there aren't many guesses for tools we can use: the Serre sequence is basically the only tool we have for homotopy groups.

Proof. Recall that we have a fibration $K(\pi, n-1) \to * \to K(\pi, n)$; we'll apply the Serre sequence. By induction, it's sufficient to consider the case $n = 1$.

For right now, we'll consider $\mathcal{C} = FG$. By Künneth, it's sufficient to show for $K(\mathbb{Z}, 1)$ and $K(\mathbb{Z}/m, 1)$, which are S^1 and lens spaces, for which this is true. ⊠

Lemma 2.9. *Let X be simply connected. Then, $\pi_n(X) \in \mathcal{C}$ implies $H_n(X) \in \mathcal{C}$.*

Remark. The converse also holds (i.e. if we know this for all n, then X must be simply connected), but we won't show that yet.

Proof of Lemma 2.9. We'll use a Postnikov tower.

$$\begin{tikzcd} K(\pi_n(X), n) \arrow[r] & X_n \arrow[d] \\ & X_{n-1} \end{tikzcd} \tag{9.1.2}$$

From Lemma 2.8, we know that $H_k(K(\pi_n(x), n), \mathbb{Z}) \in \mathcal{C}$, so once again using the homological Serre sequence, $H_*(X_n)$ is computed using $E^2_{p,q} = H_p(X_{n-1}, H_q(K(\pi_n(x), n)))$, and therefore $H_n(X_n; \mathbb{Z}) \in \mathcal{C}$. But this is $H_n(X)$. ⊠

There seems to be a duality, where things with complicated homology seem to have uncomplicated homotopy, and vice versa.

Now we can get to the reason we're doing this abstract nonsense.

In the following theorems, *isomorphic mod $\mathcal{C}$* means that the kernel and cokernel of the map are both in $\mathcal{C}$.

Theorem 2.10 (Mod $\mathcal{C}$ Hurewicz)**.** *If X has $\pi_i(X) \in \mathcal{C}$ for $i < n$, then $h : \pi_n(X) \to H_n(X)$ is an isomorphism mod $\mathcal{C}$.*

Corollary 2.11. *If $\pi_i(X) \in \mathcal{C}$ for all i, then $\pi_n(X) \to H_n(X)$ is isomorphic mod $\mathcal{C}$ for all i.*

Corollary 2.12. *If $\pi_i(X) \in FG$, then $\pi_i(X) \otimes \mathbb{Q} \to H_i(X) \otimes \mathbb{Q}$ is an isomorphism.*

Proof of Theorem 2.10. $\pi_n(X) \to H_n(X)$ is the same as $\pi_n(X_n) \to H_n(X_n)$, where $\{X_i\}$ is the Postnikov tower. Look at the fibration (9.1.2), and use the five-term exact sequence

$$0 \longrightarrow H_{n+1}(X_{n-1}) \longrightarrow H_n(K(\pi_n(X), n)) \longrightarrow E^{\infty}_{0,n} \longrightarrow H_n(X_n) \longrightarrow H_n(X_{n-1}).$$

The composition of the middle maps is the inclusion of the fiber, so we get a four-term exact sequence

$$0 \longrightarrow H_{n+1}(X_{n-1}) \longrightarrow H_n(K(\pi_n(X), n)) \longrightarrow H_n(X_n) H_n(X_{n-1}) \longrightarrow 0.$$

By induction, the first and last terms are in $\mathcal{C}$, which is exactly what we need for the middle arrow to be an isomorphism mod $\mathcal{C}$.

Now, we can put that into the following diagram.

$$\begin{tikzcd} H_n(K(\pi_n(X), n)) \arrow[r, "\cong"] \arrow[d, "h"', "\cong"] & \pi_n(X_n) \arrow[d] \\ H_n(K(\pi_n(X), n)) \arrow[r, "\cong \bmod \mathcal{C}"'] & H_n(X_n) \end{tikzcd}$$

The isomorphism on the left follows from the usual Hurewicz theorem, and the one on the top is from the construction of the Postnikov tower. Then, we just showed the result for the bottom arrow, so the result is that the arrow on the right is an isomorphism mod $\mathcal{C}$. ⊠

You can see the game: Postnikov tower, fibration, Serre sequence, and then hope and pray that there's an equivalence (e.g. one coarse enough such as $\mathcal{C}$).

One corollary here is that homology and homotopy don't differ much over $\mathbb{Q}$, which is pretty nice.