

GEOMETRY AND STRING THEORY SEMINAR: SPRING 2018

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1. COMMENTS ON GLOBAL SYMMETRY, ANOMALIES, AND DUALITY IN $(2+1)d$: 1/24/18

Today's talk was given by Val Zakharevich, on the paper [BHS17].

Definition 1.1. Let A and B be UV theories which, under renormalization group flow, flow to the same IR theory C . Then we'll say that theories A and B are dual.

Example 1.2. Let $N_f \leq N$. Then there is a conjectured duality between $SU(2)_k$ -Chern-Simons theory with N_f scalars, also known as *Wilson-Fischer theory*, and $U(k)_{-N+N_f/2}$ -Chern-Simons theory with N_f fermions. \blacktriangleleft

The paper [BHS17] computes the higher symmetries and anomalies of both sides of this duality and of several others; 't Hooft anomaly matching tells us that these should be the same.

This is related to our overarching goal of understanding QCD_4 with a single fermion ψ , which has a Lagrangian

$$(1.3) \quad \mathcal{L} = \text{tr}(F \wedge \star F) + \bar{\psi} \not{D} \psi + m \bar{\psi} \psi,$$

where $m \in \mathbb{C}$ is a parameter whose phase diagram we're interested in. Let \bar{m} denote the mass of the domain wall theory, which is a 3D QCD theory, so \bar{m} is real. If m is real and negative, there's a phase transition: for $\bar{m} \ll 0$, the low energy theory is believed to be trivial, and for $\bar{m} \gg 0$ (m negative of larger magnitude), the low-energy theory is believed to be $SU(N)$ -Chern-Simons theory at level 1. The transition point, at $\bar{m} = 0$, should be described by $SU(N)_{1/2}$ with a single fermion.

1.1. Level-rank duality. *Level-rank duality* is the conjecture that $SU(N)_k$ -Chern-Simons theory and $U(k)_{-N}$ -Chern-Simons theory are isomorphic. A natural generalization is to consider $SU(N)_k$ together with N_f scalar fields of mass m , where $m \in \mathbb{R}$ and $N_f < N$.

- If $m \ll 0$, the Higgs mechanism implies this should be the $SU(N - N_f)_k$ theory.
- if $m \gg 0$, we should expect the $SU(N)_k$ theory again.

On the dual side, let's consider $U(k)_{-N-N_f/2}$ with N_f fermions of mass $m \in \mathbb{R}$.

- If $m \gg 0$, we expect to get $U(k)_{-N+N_f/2}$.
- If $m \ll 0$, we expect to get $U(k)_{-N}$, with Lagrangian shifted by N_f :

$$(1.4) \quad \mathcal{L} = \frac{-N + N_f}{4\pi} \text{tr} \left(b \, db - \frac{2i}{3} b^3 \right) + \bar{\psi} \not{D} \psi + m \bar{\psi} \psi + \text{c.c.}$$

Level-rank duality switches positive-mass scalars and negative-mass fermions, promising dualities between $SU(N - N_f)_k \longleftrightarrow U(k)_{-N+N_f/2}$ and $SU(N)_k \longleftrightarrow U(k)_{-N}$.

1.2. Symmetries. We now see what symmetries these theories have. First, $SU(N)_k$ with N_f scalars. On a 3-manifold M , the fields are triples (P, A, Φ) , where

- $P \rightarrow M$ is a principal $SU(N)$ -bundle with connection,
- A is a connection on P , and
- $\varphi \in \Gamma(P \times_{SU(N)} (\mathbb{C}^N \otimes \mathbb{C}^{N_f}))$ is the N scalar fields.

The Lagrangian is

$$(1.5) \quad \mathcal{L}(A, \Phi) = \frac{k}{4\pi} \text{tr} \left(A \wedge dA + \frac{2}{3} A^3 \right) + |D_A \varphi|^2 + m|\varphi|^2 + \lambda|\varphi|^4.$$

As usual, we have an $SU(N)$ -gauge symmetry, and there's also a $U(N_f)$ -symmetry acting on \mathbb{C}^{N_f} , in which $e^{2\pi i/N} \mathbf{1}$ acts by a gauge symmetry. Hence the global symmetry group (for these symmetries) is $U(N_f)/(\mathbb{Z}/N)$.

Ansatz 1.6. Let G be a compact Lie group and k be a level for G , and let \mathcal{L}_{G_k} denote the Lagrangian for Chern-Simons theory with group G and level k . Let

$$(1.7) \quad 1 \longrightarrow G \xrightarrow{\rho} H \xrightarrow{\sigma} L \longrightarrow 1$$

be a short exact sequence of Lie groups. Then, we take as an ansatz that coupling the G_k theory to a principal L -bundle (i.e. given a principal L -bundle $P \rightarrow M$, we sum over the groupoid of all principal H -bundles which quotient to L) produces a classical gauge theory for H with Lagrangian $\mathcal{L}_{\tilde{k}}$ such that

$$(1.8) \quad \mathcal{L}_{G_k}(P_G, A_G) = \mathcal{L}_{\tilde{k}}((P_G, A_G) \times_G H).$$

When G is finite (so we're in the setting of Dijkgraaf-Witten theory) this is studied in [KT14].

In our setting, (1.7) specializes to $G = SU(N)$, $H = (SU(N) \times U(N_f))/(\mathbb{Z}/N)$, and $L = U(N_f)/(\mathbb{Z}/N)$. Chern-Simons theories for G are labeled by $H^4(BG; \mathbb{Z})$, and the map $\rho: G \rightarrow H$ defines a pullback

$$\rho^*: H^4(BH; \mathbb{Z}) \longrightarrow H^4(BG; \mathbb{Z}).$$

Given a $k \in H^4(BG; \mathbb{Z})$, we want to know whether we can implement the theory with a global L -symmetry; hence we want to know whether $k \in \text{Im}(\rho^*)$; the theory is anomalous iff this is not true.

If the theory is anomalous, we'd like to compute the anomaly. Suppose that we have a $\widehat{k}_{\mathbb{R}} \in H^4(BH; \mathbb{R})$ such that $\rho^*(\widehat{k}_{\mathbb{R}}) = k_{\mathbb{R}}$ (i.e. the image of k in real cohomology). Then, we can't eliminate the anomaly, but we can couple to a bulk theory: suppose that we can extend $(P, A) \rightarrow M$ to $(P_H, A_H) \rightarrow X$, where X is a compact 4-manifold with $\partial X = M$. Then, we have an action

$$(1.9) \quad "S_{\widehat{k}_{\mathbb{R}}}(A_H)": ((P_H, A_H) \rightarrow X) \longmapsto \int_X \widehat{k}_{\mathbb{R}}(F_H),$$

where F_H is the curvature of A_H .

This depends on the choice of X and $P_H \rightarrow X$ extending P , but we can hope that the dependence goes away after exponentiating the action. Let X' be another compact 4-manifold bounding M , and let $(P'_H, A') \rightarrow X'$ be another extension of (P, A) . Let $\widehat{X} := X \cup_M X'$; then, (P_H, A_H) and (P'_H, A'_H) glue to a principal G -bundle $\tilde{P}_H \rightarrow \tilde{X}$ with connection \tilde{A} . Then we have that

$$(1.10) \quad e^{2\pi i S_{\widehat{k}_{\mathbb{R}}}(\tilde{P}_H, \tilde{A}_H)} = S_{\widehat{k}}(\tilde{P}_H \times_H L) \in \mathbb{R}/2\pi i\mathbb{Z}$$

for some $\widehat{k} \in H^4(BL; \mathbb{R}/\mathbb{Z})$. This \widehat{k} tells us the anomaly, so we're interested in computing it. Ultimately, this comes from a question purely in algebraic topology: we have a big commutative diagram

$$(1.11) \quad \begin{array}{ccccc} H^4(BL; \mathbb{R}/\mathbb{Z}) & \longrightarrow & H^4(BH; \mathbb{R}/\mathbb{Z}) & \longrightarrow & H^4(BG; \mathbb{R}/\mathbb{Z}) \\ & & \uparrow & & \uparrow \\ & & H^4(BH; \mathbb{R}) & \longrightarrow & H^4(BG; \mathbb{R}) \\ & & \uparrow & & \uparrow \\ & & H^4(BH; \mathbb{Z}) & \longrightarrow & H^4(BG; \mathbb{Z}) \end{array}$$

Then we have \hat{k} in the upper left, $\hat{k}_{\mathbb{R}}$ in the middle, and k in the lower right. In this case the anomaly theory is purely topological. The computation for the dual theory follows a similar story, but is harder.

To actually calculate this, you can use the Leray-Serre spectral sequence; $k \in H^4(BG; \mathbb{Z})$ transgresses to something in $H^5(BL; \mathbb{Z})$, which tells you which component \hat{k} is in.

2. ON GAUGING FINITE SUBGROUPS: 1/31/18

Today's talk was given by Dan Freed, on Tachikawa's paper [Tac17].

Let's start with classical electromagnetism on n -dimensional Minkowski spacetime \mathbb{M} . Choose an $A \in \Omega_{\mathbb{M}}^1/d\Omega_{\mathbb{M}}^0$, and let $F_A = dA$. Maxwell's laws tell us that

$$\begin{aligned} dF_A &= 0 \\ d\star F_A &= 0, \end{aligned}$$

but more generally we could let $dF_A = j_B \in \Omega^3$, a *magnetic current*, and $d\star F_A = j_E \in \Omega^{n-1}$, an *electric current*. If both j_E and j_B are nonzer, the theory has an anomaly.

We next consider the quantum theory, by doing some things such as Wick rotation, charge quantization, and downshifting the degree of A .¹ The Wick-rotated quantum theory is formulated on an oriented² Riemannian manifold X , and A is a map $X \rightarrow \mathbb{R}\mathbb{Z}$, or its exponentiated version $\lambda: X \rightarrow \mathbb{T}$.

If we introduce point charges $p_1, \dots, p_m \in X$ with charges $k_1, \dots, k_m \in \mathbb{Z}$,³ then the electric current, inserted in the exponentiated action, is

$$(2.1) \quad \prod_{j=1}^m \lambda(p_j)^{k_j} = \exp \left(2\pi \sum_{j=1}^m i k_j A(p_j) \right),$$

which has degree n with \mathbb{Z} coefficients.

The magnetic current is defined using a circle bundle $P \rightarrow X$ with connection Θ ; one can think of λ as a section of P , and this data is used to define the kinetic term. This is a degree-2 term with \mathbb{Z} coefficients.

If $L := P \times_{\mathbb{T}} \mathbb{C}$ is the complex line bundle associated to P , then the *electric coupling* is

$$(2.2) \quad \prod_{i=1}^m \lambda(p_i)^{k_i} \in \bigotimes_{i=1}^m L_{p_i}^{\otimes k_i}.$$

From this perspective, the anomaly is “ $\int_X j_B \cdot j_E$.”

Remark. The term λ only exists if $P \rightarrow X$ is topologically trivializable.

This is akin to something that happens in topological field theory. Let Z denote the 4D oriented TQFT defined by summing the trivial theory over spin structures. Then $Z(\mathbb{CP}^2) = 0$, since \mathbb{CP}^2 admits no global spin structure. But if one varies the manifold in a family, interesting things may nonetheless happen. ◀

Remark. One could also replace \mathbb{T} by any finite abelian group A . In this case a lot of things are still the same, though we don't choose a connection for P . In this case, the magnetic current lives in $H^1(X; A)$ rather than $H^2(X; \mathbb{Z})$ (and we could have thought of it as $H^1(X; \mathbb{T})$ for the \mathbb{T} -theory). However, the electric current lives in $H^n(X; A^\vee)$, where $A^\vee := \text{Hom}(A, \mathbb{T})$ denotes the Pontrjagin dual of A .

We could think of the magnetic current as a map $X \rightarrow BA$; in this case the electric current is a map $X \rightarrow B^n A^\vee := K(A^\vee, n)$. In the rest of this talk, we will adopt this more abstract approach, but you should keep the rigid, geometric approach that we started with in mind for intuition or an example. ◀

From this perspective, the anomaly is a map

$$BA \times B^n A^\vee \longrightarrow B^{n+1} \mathbb{T},$$

which is induced from the pairing $A \otimes A^\vee \rightarrow \mathbb{T}$. On a closed, oriented manifold X with an electric and magnetic current we can pull this back to X and integrate it; this is the partition function for the anomaly theory.

¹TODO: I have no idea what just happened.

²One could impose time-reversal symmetry and study the theory more generally on unoriented manifolds, but for our purposes this will not be necessary.

³TODO: I may have gotten this wrong.

Gauging. Suppose T is some kind of theory (here we probably mean a Wick-rotated field theory on Riemannian manifolds, perhaps with extra structure and background fields), and suppose it has a (global) Γ -symmetry,⁴ where Γ is a finite group. This means that we can couple the theory to Γ -bundles, formulating it on manifolds with the above data and a background principal Γ -bundle.

Sometimes this symmetry gets tangled up with other symmetries, e.g. if T is a σ -model to a space X and Γ is a symmetry of X . Then, depending on how we implement the symmetry, we might end up with sections of some associated bundle.

But if this is not the case (in a σ -model sense, if $B\Gamma$ splits off from the target), then we can sum over the maps to $B\Gamma$. This process is called *gauging*.

Now suppose $A := \Gamma$ is abelian. Then the gauged theory has a higher symmetry akin to electromagnetism, a $B^{n-2}A^\vee$ symmetry, and we can couple the theory to a background $B^{n-2}A^\vee$ -field, which is exactly putting in the electric current. If you try to gauge this symmetry, you'll end up back where you started with, which is a kind of Fourier transform.

On a compact oriented manifold X , the electric coupling lives in $H^1(X; A) \times H^{n-1}(X; A^\vee)$; there's a product map

$$H^1(X; A) \times H^{n-1}(X; A^\vee) \longrightarrow H^n(X; \mathbb{T});$$

then we can evaluate on the fundamental class to obtain an element of \mathbb{T} , which is what one inserts into the action. This exhibits the two cohomology groups as Pontrjagin duals of each other, so the Fourier transform is an isomorphism between spaces of functions on them.

Turning to the material in the paper, let

$$1 \longrightarrow A \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

be a short exact sequence of groups, where A is abelian. In particular, A is normal in Γ ; we do not assume it is central. An example (in which A is not central) is

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow S_3 \longrightarrow \mathbb{Z}/2 \longrightarrow 1.$$

We consider the situation of a theory T with a Γ -symmetry, hence an A -symmetry, and we assume we can gauge A . What happens when we do this?

The new theory should have a G -symmetry, which arises as follows: given a principal G -bundle $Q \rightarrow X$, we can sum over pairs $(P \rightarrow X, \varphi)$, where $P \rightarrow X$ is a principal Γ -bundle and

$$\begin{array}{ccc} Q & \xrightarrow[\cong]{\varphi} & P/A \\ & \searrow G & \swarrow G \\ & X & \end{array}$$

is an isomorphism of principal G -bundles. In this case the magnetic current arises from the map $BG \rightarrow B^2A$ coming from extending the fiber sequence $BA \rightarrow B\Gamma \rightarrow BG$.

There's another symmetry associated to $B^{n-2}A^\vee$, which arises in a similar way as before. So, following what we did with electromagnetism, if you try to couple the theory to $BG \times B^{n-1}A^\vee$, two things might happen (and are not mutually exclusive).

- (1) The two symmetries might interact nontrivially, producing an extension

$$B^{n-1}A^\vee \longrightarrow \mathcal{X} \longrightarrow BG.$$

- (2) There may be an anomaly $\mathcal{X} \rightarrow B^{n+1}\mathbb{T}$.

For example, \mathcal{X} might be $B\mathcal{G}$, where \mathcal{G} is an extension of G by $B^{n-2}A^\vee$, so an extension of a group by a higher group.

First suppose Γ is anomaly-free, so that we can gauge it. Then, the anomaly for the theory coupled to G -symmetry is the compositions of the maps

$$BG \times B^{n-1}A^\vee \longrightarrow B^2A \times B^{n-1}A^\vee \longrightarrow B^{n+1}\mathbb{T},$$

where the first map comes from the connecting map $BG \rightarrow B^2A$ and the second is the Pontrjagin dual pairing.

⁴“Global symmetry” is redundant, because there is no other kind of symmetry.

Now suppose the Γ theory has an anomaly. There are different ways to produce anomalies, such as beginning with a map $MSO \wedge (B\Gamma)_+ \rightarrow B^{n+1}\mathbb{T}$, which produces a *gauge-gravity anomaly*, but let's begin with just $B\Gamma \rightarrow B^{n+1}\mathbb{T}$, or a *pure gauge anomaly*. This data is equivalent to a cohomology class $[\alpha] \in H^{n+1}(B\Gamma; \mathbb{T})$.

The presence or absence of the anomaly arises from a filtration on $H^{n+1}(B\Gamma; \mathbb{T})$ induced from the fiber sequence $BA \rightarrow B\Gamma \rightarrow BG$: BG has a cell structure, and we can ask whether a cohomology class over the basepoint extends over the n -skeleton. We will discuss what happens in the case when the cohomology class is in the last two pieces of the filtration, which are simpler.

To compute this, one uses the *Leray-Serre spectral sequence*

$$E_2^{p,q} := H^p(G; H^q(A; \mathbb{T})) \implies \text{gr}_F H^{p+q}(\Gamma; \mathbb{T}).$$

Here $H^q(A; \mathbb{T})$ is a nontrivial G -module from the residual G -action induced by Γ . Now let's look at the two cases.

- (1) Suppose $[\alpha]$ lives in the highest filtered part; then, there's an $\bar{\alpha}: BG \rightarrow B^{n+1}\mathbb{T}$ lifting α across the map $B\Gamma \rightarrow BG$. In this case, $\bar{\alpha}$ is the anomaly: if you gauge the A -symmetry, the theory couples to $BG \times B^{n-1}A^\vee$, and the anomaly is a sum of the electromagnetic anomaly and $\bar{\alpha}$. This corresponds to a transgression in the spectral sequence.
- (2) If $[\alpha]$ lives in the next highest filtered part, we get something in $H^n(G; \underline{A}^\vee)$ (the underline representing a nontrivial G -action). The paper [Tac17] considers only the special case, where A is central and we get a bundle

$$B\Gamma \times_{BG} \mathcal{X},$$

where $B\Gamma \rightarrow BG$ is a BA -bundle and $\mathcal{X} \rightarrow BG$ is a $B^{n-1}A^\vee$ bundle. Then we have maps

$$BG \rightarrow B^2A \times B^nA^\vee \rightarrow B^{n+2}\mathbb{T},$$

and we consider the case where this composite is null. This means it lifts to a map

$$\beta: B\Gamma \times_{BG} \mathcal{X} \rightarrow B^{n+1}\mathbb{T}.$$

Then, we take $\beta|_{B\Gamma}$ is the anomaly in Γ . If this was the original anomaly in Γ , then the theory couples to the extension $B^{n-1}A^\vee \rightarrow \mathcal{X} \rightarrow BG$, corresponding to some kind of extension theory, and the anomaly for this theory is $\beta|_{\mathcal{X}}$. Then you could gauge the subgroup $B^{n-1}A^\vee$ and go back to the original theory.

- (3) Suppose $[\alpha]$ lives in the n^{th} piece of the filtration, and assume A is central. Then the map $B\Gamma \rightarrow B^{n+1}\mathbb{T}$ doesn't descend to BG , but does descend to a map $BG \rightarrow E$, where E fits into a sequence $B^{n+1}\mathbb{T} \rightarrow E \rightarrow B^nA^\vee$. This corresponds to an anomaly in a generalized cohomology theory, albeit a relatively simple one. We could try to use this to build \mathcal{X} . But you also have to add the electromagnetic anomaly, and this is a little bit unclear.

3. THETA, TIME-REVERSAL, AND TEMPERATURE I: 2/7/18

Today's talk was given by Andy Neitzke, on the paper [GKKS17].

The goal of this paper is to say something new about the phases of 4D Yang-Mills theory with group $G = \text{SU}_N$. There's no supersymmetry here.

Before we tackle this, however, let's look at a simpler example, a 2D U_1 gauge theory with N charged scalar fields z^i , $i = 1, \dots, N$, subject to the constraint that $\sum |z_i|^2 = 1$. This is equivalent data to a map to \mathbb{CP}^{N-1} , and hence this model is sometimes thought of as a σ model with target \mathbb{CP}^{N-1} . The U_1 gauge field is denoted a , and its curvature is F_a . The action on a surface Σ is

$$(3.1) \quad S = \int_{\Sigma} \sum_i |D_a z^i|^2 + \frac{i\theta}{2\pi} \int_{\Sigma} F_a + \frac{1}{g^2} \int_{\Sigma} F_a \wedge \star F_a.$$

Here θ is a real parameter, but we typically think of the θ theory as equivalent to the $\theta + 2\pi$ theory, because on a closed surface Σ , $\int_{\Sigma} F_a = 2\pi n \in 2\pi\mathbb{Z}$, so after exponentiating, $e^{in\theta} = e^{in(\theta+2\pi)}$.

The first question you might ask is: what's the (global) symmetry of this theory? The answer is PSU_N , which might be a surprise: there's a U_N symmetry on the fields, but we gauged the U_1 subgroup, so only $\text{PSU}_N := \text{U}_N/\text{U}_1$ acts faithfully on the (gauge-invariant) local operators.

Example 3.2. The operator z^1 transforms nontrivially under the U_1 symmetry, hence is not gauge invariant. The operator $z^1 \bar{z}^1$ is gauge invariant. \blacktriangleleft

The more modern way of saying that this theory has a PSU_N symmetry is to say that it couples to background PSU_N gauge fields. Let A_{bkgd} be a PSU_N -connection; we want to include A_{bkgd} in the theory such that if it's the trivial connection, we get back (3.1).

Somewhat like what we did last time, instead of U_1 -connections, we'll consider lifts A of A_{bkgd} to a U_N -connection. Locally $A = a \cdot I_N + A_{\text{bkgd}}$. When writing the action, the first term doesn't change much, but the θ term is interesting, since we no longer have a U_1 -connection.

If $A_{\text{bkgd}} = 0$, then $A = a \cdot I_N$, and $F_a = (1/N) \text{tr}(F_A)$. Hence we'll replace F_a with $(1/N) \text{tr}(F_A)$, even when $A_{\text{bkgd}} \neq 0$. That is, the new θ term is

$$(3.3) \quad \frac{i\theta}{2\pi N} \int_{\Sigma} \text{tr}(F_A).$$

We do something analogous for the kinetic term. But we pay a price — since we divided by N , this term is not invariant under $\theta \mapsto \theta + 2\pi$. Instead, it's invariant under $\theta \mapsto \theta + 2\pi N$. If we shift $\theta \mapsto \theta + 2\pi$, the exponentiated action changes by

$$(3.4) \quad \exp\left(\frac{2\pi i}{N} \int_{\Sigma} w_2(A_{\text{bkgd}})\right).$$

In particular, the change only depends on the background field. Therefore this is also true for the partition functions:

$$(3.5) \quad Z(\theta + 2\pi, A_{\text{bkgd}}) = Z(\theta, A_{\text{bkgd}}) \cdot \exp\left(\frac{2\pi i}{N} \int_{\Sigma} w_2(A_{\text{bkgd}})\right).$$

This exhibits a mixed anomaly between the shift symmetry $\theta \mapsto \theta + 2\pi$ of \mathbb{Z} and the PSU_N symmetry. You can't gauge both of them at once.

Remark. From the perspective of symmetries as coupling to background bundles, we want to express this shift symmetry with a background \mathbb{Z} -bundle. The mixed anomaly is saying that θ becomes a section of an associated bundle, such that $e^{i\theta}$ is still a function to U_1 , but not a constant function. ◀

Another option is to add some function of the background fields to the action; this is called a *counterterm*. One natural choice is, for some $p \in \mathbb{Z}$,

$$(3.6) \quad p \frac{2\pi i}{N} \int_{\Sigma} w_2(A_{\text{bkgd}}) \in (2\pi i \mathbb{Z}) / (2\pi i N \mathbb{Z}).$$

If we exponentiate this, it's an N^{th} root of unity, and therefore the theory with parameters θ and p should be equivalent to the theory with parameters $\theta + 2\pi$ and $p + 1$.

One thing we can do with this is implement time reversal, thought of as reversing the orientation of Σ . This acts on S by $\theta \mapsto -\theta$. In the absence of a background PSU_N field, this is only a symmetry when $\theta = 0$ or $\theta = 2\pi$, in which case the time-reversed theory is equivalent.

With a background field and the counterterm (3.6), time-reversal acts as $(\theta, p) \mapsto (-\theta, -p)$. Therefore we can get time-reversal symmetry for $\theta = 0$ and $p = 0$. And what about $\theta = \pi$? In this case, $(-\pi, -p) \simeq (\pi, -p + 1)$, and to get this equivalent to (π, p) , we need $p \equiv -p + 1 \pmod{N}$, i.e. $2p \equiv 1 \pmod{N}$. Therefore if N is even, there's no way to preserve time-reversal at $\theta = \pi$.

This means the theory has a mixed anomaly between time-reversal symmetry and a PSU_N symmetry. Anomaly matching means this tells us something about the infrared theory. There are two possibilities.

- The IR theory “has the same anomaly,” in that it can be coupled to PSU_N background fields in such a way that, if we shift the background coupling by p , then $p \mapsto -1 + p$ under time reversal.
- At least one of the PSU_N or time-reversal symmetries is broken in the infrared. This is a little weird, and is an example of spontaneous symmetry breaking.

What actually happens is the second option, and specifically time-reversal symmetry is broken.⁵ The theory is gapped, and therefore the IR theory should be an explicit topological field theory we could look at.

⁵More generally, any continuous symmetry cannot be spontaneously broken in 2D.

Back to Yang-Mills theory. Remember that we're really interested in Yang-Mills theory with $G = \mathrm{SU}_N$, whose action is

$$(3.7) \quad aS = \frac{1}{g^2} \int \mathrm{tr}(F \wedge \star F) + \frac{i\theta}{8\pi^2} \int \mathrm{tr}(F \wedge F).$$

Again, after exponentiating the action, we have an equivalence between the theory with parameter θ and the theory with parameter $\theta + 2\pi$. Now there's a \mathbb{Z}/N one-form symmetry, or a $B\mathbb{Z}/N$ symmetry, so we should be able to couple the theory to background $B\mathbb{Z}/N$ fields. Whatever these are, they should have a characteristic class $[B] \in H^2(X; \mathbb{Z}/N)$.

As before, this can be done, and the price is that the theory is not periodic in θ . And as before, there's a specific term expressing that failure: assume N is even. When we shift $\theta \mapsto \theta + 2\pi$, e^S shifts by

$$\exp\left(\frac{2\pi i(N_1)}{2N} \int_X \mathfrak{P}_2(B)\right),$$

where $\mathfrak{P}_2: H^2(X; \mathbb{Z}/N) \rightarrow H^4(X; \mathbb{Z}/2N)$ is a cohomology operation called the *Pontrjagin square*. Therefore we can regard this as an element of \mathbb{Z}/N , and introduce a counter-term as above: $(\theta, p) \sim (\theta + 2\pi, p + N - 1)$. Therefore one can calculate that time-reversal symmetry cannot hold when N is even at $\theta = \pi$.

4. THETA, TIME-REVERSAL, AND TEMPERATURE II: 2/7/18

Today, Andy Neitzke continued to speak on [GKKKS17].

Today, instead of focusing on a toy model, we will look at the case of interest, 4D Yang-Mills theory with gauge group SU_N , whose action on X has a term of the form

$$\frac{i\theta}{8\pi^2} \int \mathrm{tr}(F \wedge F).$$

Before coupling to background fields, the theory with parameter θ is (believed to be) equivalent to the theory with parameter $\theta + 2\pi$. We will write this as $\theta \simeq \theta + 2\pi$.

This theory has a $B\mathbb{Z}/N$ symmetry, i.e. a \mathbb{Z}/N one-form symmetry; here \mathbb{Z}/N arises as the center of SU_N . At $\theta = 0, \pi$, there's also time-reversal symmetry T : $T(\theta) = -\theta \simeq \theta$ iff $\theta = 0, \pi \bmod 2\pi$. When we try to implement both of them simultaneously, we'll discover a mixed anomaly.

This theory couples to a background field, a \mathbb{Z}/N -gerbe B , which is classified by its characteristic class $[B] \in H^2(X; \mathbb{Z}/N)$.⁶

Remark. In [GKKKS17], this coupling is described explicitly, and they provide a nice model for the background $B\mathbb{Z}/N$ -field, a pair (B, C) where $B \in \Omega^2(X)$ and C is the connection of a line bundle, such that $NB = F_C$, where F_C denotes the curvature of C . This is something kind of like \mathbb{Z}/N de Rham differential forms, but instead of something being exact, we want it to be N -torsion up to exact things.

In this case, if $\lambda \in \Omega^1(X)$, the symmetry sends $B \mapsto B + d\lambda$ and $C \mapsto C + N\lambda$. ◀

You can also add counterterms depending purely on the background field B (or its characteristic class), such as

$$(4.1) \quad \frac{2\pi ip}{2N} \int \mathfrak{P}_2([B]),$$

where $\mathfrak{P}_2: H^2(X; \mathbb{Z}/N) \rightarrow H^4(X; \mathbb{Z}/2N)$ is the Pontrjagin square, a cohomology operation, and $p \in \mathbb{Z}/2N$, so that this makes sense after exponentiation.

The Pontrjagin square has an abstract definition, but in the explicit model described above,

$$(4.2) \quad \int \mathfrak{P}_2([B]) = \int dC \wedge dC.$$

Now there is an equivalence of theories

$$(4.3) \quad (\theta, p) \simeq (\theta + 2\pi, p + N - 1)$$

and time-reversal symmetry acts by $T(\theta, p) = (-\theta, -p)$.

⁶If you don't really know what a gerbe is, that's OK; we're mostly just going to use its characteristic class.

So at $\theta = \pi$, we need $p = -p + N - 1$ to implement both time-reversal and the $B\mathbb{Z}/N$ symmetry. If this is not the case (e.g. N is even), time-reversal symmetry is broken by coupling to $B\mathbb{Z}/N$ fields, exhibiting a mixed anomaly between these two symmetries.

This story has been an UV story so far; 't Hooft anomaly matching posits that there are IR consequences, which could include

- (1) the vacuum supporting a TQFT which has the same anomaly (i.e. coupling the background fields in the same way),
- (2) the theory is gapless (which is regarded as unlikely, and would imply a low-energy conformal field theory rather than a topological field theory),
- (3) the theory is gapped, and the $B\mathbb{Z}/N$ -symmetry is broken, or
- (4) the theory is gapped, and the time-reversal symmetry is broken.

So we expect there's a TQFT (case (2) is considered unlikely), but its nature is unclear — is it invertible? Case (1) is also considered unlikely, especially for large N . Case (3) is also unlikely, which we'll say more about later, which implies that time-reversal symmetry is probably broken in IR. This implies at $\theta = \pi$, time-reversal symmetry is broken and there are two vacua, exchanged by T .

Remark. In a different theory, $\mathcal{N} = 1$ supersymmetric Yang-Mills theory, it's known (at a physical level of rigor) that there are N vacua. By adding a mass term $m \in \mathbb{R}_+$, one can perturb the theory slightly, breaking supersymmetry. For $\theta \neq 0, \pi$, you get two vacua, but at $\theta = 0, \pi$, there are two degenerate vacua and time-reversal symmetry is broken.

This parallel story is analogous to ours, and suggests that we're on the right track. ◀

You can draw a phase diagram for these theories in θ : at $\theta = \pi$, it looks like a quartic with two global minima, and elsewhere it looks like a quartic with one global minimum and two local minima. At $\theta = 0, 2\pi$, the quartic has a unique local minimum. So there must be a phase transition at π .

We promised to discuss why the $B\mathbb{Z}/N$ symmetry is unbroken, which has something to do with confinement. The theory has line defects with charges $(a, b) \in (\mathbb{Z}/N)^\vee \times \mathbb{Z}/N$ ($(\mathbb{Z}/N)^\vee \cong \mathbb{Z}/N$ abstractly, of course, but this illustrates how it arises). The first component is an electric charge, and the second is magnetic. But these aren't usual line defects: for $b = 0$ they are, but for $b \neq 0$, we need additional data to formulate the defect on a closed 1-manifold ℓ , which is the topological data of a surface which bounds ℓ .

Remark. If we had started with PSU_N , the roles of a and b would be switched, with $a \neq 0$ defects arising as line defects on boundaries of surfaces. ◀

If W is a Wilson line given by a representation V of SU_N , then its charge (a, b) is $a = V|_{\mathbb{Z}/N}$ and $b = 0$, though of as charges of “probe particles” we can insert in the theory.

The basic question here is: which line bundles are confined? Here, Q is confined iff a defect L of charge Q has

$$(4.4) \quad \langle L(\text{loop}) \rangle \sim \exp(-\alpha \Delta X \Delta T)$$

asymptotically, i.e. the energy cost of creating the loop is linear in ΔX . Heuristically, this means that correlation functions with an insertion of L on a line vanish. The higher-symmetry way to say this is that the charge Q transforms nontrivially under the unbroken 1-form symmetry. The idea is that invariant plus transforming nontrivially forces it to be zero.

So the question of who's confined is related to the question of which symmetries remain unbroken in a given vacuum. There have been many numerical simulations of SU_N -Yang-Mills suggesting that at any θ , all Wilson lines are confined. In the language of higher symmetries, this implies the $B\mathbb{Z}/N$ symmetry is unbroken.

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