M392C NOTES: INDEX THEORY

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These notes were taken in UT Austin's M392C (Index theory) class in Spring 2018, taught by Dan Freed. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Any mistakes in the notes are my own. Thanks to Rok Gregoric for fixing a few errors.

CONTENTS

Lecture 1.

Overview, History, and some Linear Algebra: 1/17/18

"This formula should look fake if you haven't seen it before."

We'll start with and overview and some history of index theory. The overview will use a little bit of complex geometry, but if you don't know it that's okay; the rest of the class will not depend on it.

One of the earliest manifestations of index theory was in the theory of algebraic curves. Let M be a compact smooth connected complex curve, i.e. a Riemann surface, and let D be a divisor on M, a finite formal sum of points of M with integer coefficients. For example, if $p_1, p_2, p_3 \in M$, one divisor is $4p_1 - 2p_2 + 7p_3$.

Definition 1.1. Let f be a meromorphic function on M; then, its divisor $\mathrm{div}(f)$ is the zeros of f minus the poles of f, where both are counted with multiplicity. For $f = 0$, we let $\mathrm{div}(0) = 0$.

For example, if $M = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, then a meromorphic function on M is a rational function. If we took $f(z) = (z-1)^2/(z+2)$, then $\mathrm{div}(f) = 2\cdot 1 - 1\cdot(-2) - 1\cdot\infty$: f has a double zero at 1 and a single pole at -2, and at ∞ there is a simple pole.[1]

A divisor has a *degree* which is the sum of its terms.

Theorem 1.2. *The degree of the divisor of a meromorphic function is zero.*

This is a consequence of the Cauchy integral formula.

A divisor specifies the zeros and poles of a meromorphic function, and it's a classical problem to, given a degree-zero divisor D on a Riemann surface, construct a function whose divisor is D. More generally, let $\mathscr{L}(D)$ denote the set of meromorphic f such that $\mathrm{div}(f) + D \geq 0$.[2] $\mathscr{L}(D)$ is a vector space, and if $\deg(D) < 0$, $\mathscr{L}(D) = 0$; we also have $\mathscr{L}(0) = \mathbb{C}$, given by constant functions.

Another classical question is to compute $\dim \mathscr{L}(D)$. Riemann provided an estimate:

$$\dim \mathscr{L}(D) \geq 1 - g + \deg(D), \tag{1.3}$$

where g is the *genus* of M, defined to be

$$g := \frac{1}{2}\operatorname{rank} H_1(X). \tag{1.4}$$

The next natural question is to identify the discrepancy, and Riemann's student Roch found the answer.

Theorem 1.5 (Riemann-Roch)**.** *here is a canonical divisor K_M such that*

$$\dim \mathscr{L}(D) - \dim \mathscr{L}(K_X - D) = 1 - g + \deg D. \tag{1.6}$$

[1] To see this, use the change-of-variables $z = 1/w$ and evaluate f at $w = 0$.

[2] This is missing a zero element, so one needs to adjoint 0 for everything to work.

We won't say much about K_M, though $\deg(K_M) = 2g - 2$.

Corollary 1.7. *The genus is an integer.*

A more modern interpretation of this story is that D determines a holomorphic line bundle $L \to M$, and $\mathscr{L}(D)$ is the vector space of holomorphic sections of L, i.e. $\mathscr{L}(D) \cong H^0(M; L)$. If s is any smooth section of L, s is holomorphic iff $\overline{\partial} s = 0$. That is, in local coordinates $z = x + iy$, and

$$\overline{\partial} = \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right). \tag{1.8}$$

Thus, $\overline{\partial} x = 0$ is a first-order differential equation, and computing $\dim \mathscr{L}(D)$ is asking for the dimension of the space of solutions to the equation. Thus one way you might prove Theorem 1.5 is to analyze the differential operator $\overline{\partial}$, which is a linear operator

$$\overline{\partial}\colon \Omega^{0,0}(M; L) \longrightarrow \Omega^{0,1}(M; L).$$

Then, $\mathscr{L}(D) = \ker(\overline{\partial})$ and $\mathscr{L}(K_M - D) \cong \operatorname{coker}(\overline{\partial})$.

Definition 1.9. The *index* of $\overline{\partial}$ is $\operatorname{ind}(\overline{\partial}) := \dim\ker(\overline{\partial}) - \dim\operatorname{coker}(\overline{\partial})$.

Broadly speaking, this course will be about indices of this sort, and their applications: for example, the Riemann-Roch theorem from this perspective is about computing the index of $\overline{\partial}$.

For a simpler case, let V and W be finite-dimensional vector space and $T\colon V \to W$ be a linear map. Then, $\ker(T) \subset V$ and $\operatorname{coker} T := W/T(V)$. Computing the index is a fundamental theorem in linear algebra.

Theorem 1.10.

$$\operatorname{ind}(T) := \dim(\ker T) - \dim(\operatorname{coker} T) = \dim V - \dim W.$$

In particular, it's independent of T! One way you might prove this is to observe that it's true when $T = 0$ and then try to prove that it's locally constant.

In this class, we're interested in operators between infinite-dimensional vector spaces, such as $\Omega^{p,q}(M; L)$, whose kernels and cokernels are finite-dimensional (such that the definition of an index makes sense). There will be no nice formula like Theorem 1.10, but some aspects stay the same: though the dimension of the kernel or cokernel may jump along a continuous path, their difference is constant.

Another classical subject that relates to index theory is that of the Euler number of a compact smooth n-manifold M. Betti defined *Betti numbers* $b_0, \dots, b_n$ associated to M, and Noether realized they can be identified with ranks of abelian groups (or dimensions of certain real vector spaces).[3]

Definition 1.11. The *Euler characteristic* of M is

$$\chi(M) := \sum_{i=0}^{n} (-1)^i b_i.$$

The Betti numbers are defined via simplices, and how M is built out of cells. Since M is a smooth manifold, one might want to compute them in another way, using the smooth structure of the manifold. To do this, one introduces the *de Rham complex*

$$0 \longrightarrow \Omega^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \Omega^2(M) \longrightarrow \cdots \longrightarrow \Omega^n(M) \longrightarrow 0, \tag{1.12}$$

with linear maps d such that $\mathrm{d}^2 = 0$. Unlike in the previous example, this is built out of real functions and real differential forms.

Definition 1.13. The *de Rham cohomology* of M is the sequence of real vector spaces

$$H^i_{\mathrm{dR}}(M) := \frac{\ker(\mathrm{d}\colon \Omega^i(M) \to \Omega^{i+1}(M))}{\operatorname{Im}(\mathrm{d}\colon \Omega^{i-1}(M) \to \Omega^i(M))}.$$

[3] These days, this would be called *categorification*: it can often be useful to identify a number as the dimension of some vector space attached to your object.

Theorem 1.14 (de Rham)**.** *There is an isomorphism* $H^i_{\mathrm{dR}}(M) \cong H^i(M;\mathbb{R})$*, and therefore* $\dim H^i_{\mathrm{dR}}(M) = b_i$.

From this perspective, the Euler characteristic looks more like an index, where we stack together the pieces of the de Rham complex:

$$\bigoplus_{i \text{ even}} \Omega^i(M) \longrightarrow \bigoplus_{i \text{ odd}} \Omega^i(M). \tag{1.15}$$

However, the index of this is *not* the Euler characteristic! The issue is that the de Rham cohomology groups are a subquotient, not just a subspace or just a quotient. To compute the Euler characteristic as an index, we'll need some way of turning them into pure subspaces or quotients. One way to do this is to use an inner product and take orthogonal complements.

Let M be a Riemannian manifold. Then, there is a Laplace operator $\Delta\colon \Omega^i(M) \to \Omega^i(M)$, which is a linear second-order elliptic differential operator.

Remark. There are three basic kinds of differential operators studied in a typical differential equations course: elliptic, parabolic, and hyperbolic. The Laplacian is the basic example of an elliptic operator; the heat operator is the basic example of a parabolic operator; and the Schrödinger operator is the basic example of a hyperbolic operator. We will focus on elliptic operators in this course, but both the heat equation and the Schrödinger equation will appear. ◄

Example 1.16. Let $\mathbb{E}^n$ denote n-dimensional Euclidean space with coordinates $x^1, \dots, x^n$. Then, the Laplacian on $\mathbb{E}^n$ is

$$\Delta = \left(\frac{\partial}{\partial x^1}\right)^2 + \cdots + \left(\frac{\partial}{\partial x^n}\right)^2.$$

◄

For more general Riemannian manifolds, the definition of the Laplacian is more complicated, but not much more so.

Definition 1.17. If M is a Riemannian manifold, there is an L^2 inner product on $\Omega^i(M)$ defined by

$$\langle \alpha, \beta \rangle_{L^2} := \int_M \langle \alpha(M), \beta(M) \rangle \, \mathrm{dvol}_m.$$

Using these inner products, we can let $\mathrm{d}^*\colon \Omega^{i+1}(M) \to \Omega^i(M)$ be the formal adjoint to d.

Fact. d^* exists and is a first-order differential operator. ◄

Definition 1.18. The *Laplace operator* on M is $\Delta := \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$.

A form in the kernel of Δ is called *harmonic*, and the space of harmonic forms is denoted $\mathscr{H}^i(M) \subset \Delta^i(M)$.

Theorem 1.19 (Hodge theorem)**.** *The natural map* $\mathscr{H}^i(M) \to H^i_{\mathrm{dR}}(M)$ *is an isomorphism. In particular,* $\dim \mathscr{H}^i(M) = b_i$.

This is how index theory enters the picture: if we can access the space of harmonic forms as kernels and cokenels of operators, we could compute the Euler characteristic as an index. And indeed, we can fix (1.15) as folows:

$$\bigoplus_{i \text{ even}} \Omega^i(M) \xrightarrow{\mathrm{d}+\mathrm{d}^*} \bigoplus_{i \text{ odd}} \Omega^i(M). \tag{1.20}$$

The index of this operator is the Euler characteristic.

A third example of index theory is the higher-dimensional Riemann-Roch theorem. Let M be a compact complex manifold; then, the $\overline{\partial}$ operator defines a *Dolbeault complex* analogous to the de Rham complex. If M is 2-(complex-)dimensional, the Euler characteristic satisfies a formula

$$\chi(M) = \frac{1}{12}\big(c_1^2(M) + c_2(M)\big)[M]. \tag{1.21}$$

Here c_1 and c_2 are examples of *characteristic classes*, which we'll start on in the next few lectures. In particular, the right-hand side is an integer. In higher dimensions, there are similar expressions with larger denominators and more characteristic classes.

These were studied by Todd and his student Egger, by Weyl, and others. But the general forms remained conjectures until 1954, when Hirzebruch proved these generalizations of the Riemann-Roch theorem, and an additional, similar result called the signature theorem. He wove together two very new pieces of mathematics: the cobordism theory of René Thom (published only earlier that year!) and the theory of sheaves.

Hirzebruch and others in this field introduced a rational combination of different characteristic numbers, called *Pontrjagin numbers*, called the *$\widehat{A}$-genus* (said "A-hat genus"). This is defined on closed oriented manifolds, and on a spin manifold is an integer.

That $\widehat{A}(M)$ is an integer is a suggestion that it's a dimension of something, and when Singer went to visit Oxford in 1963, Atiyah asked him what object has the $\widehat{A}$-genus as its dimension, and this is the problem that they solved: they constructed a differential operator called the Dirac operator on a spin manifold, and showed that its index is the $\widehat{A}$-genus.

The Dirac operator

$$D := \gamma^\mu \frac{\partial}{\partial x^\mu}$$

for some γ^μ (this notation means the index μ is implicitly summed over) is a first-order linear differential operator. We'd like this to be a square root of the Laplacian operator.

Exercise 1.22. Show that $D^2 = \Delta$ iff

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2\delta^{\mu\nu}. \tag{1.23}$$

Here, $\delta^{\mu\nu}$ means 1 if $\mu = \nu$ and 0 otherwise.

So the operator has to satisfy n^2 equations. If you try to solve this for functions on $\mathbb{E}^n$, you can show that no such γ^μ exist, but one could instead ask for vector-valued functions which satisfy (1.23), and indeed we will spend some time studying the abstract theory of matrices which satisfy this condition, rephrased as the algebraic theory of Clifford modules. In particular, we will be able to show that a spin structure is precisely what one needs to be able to construct the Dirac operator on a Riemannian manifold.

Before Atiyah and Singer told this story, Grothendieck took the Hirzebruch-Riemann-Roch theorem and generalized it still further, and Atiyah and Hirzenbruch saw how to translate his ideas from algebraic geometry to topology, and replace sheaves with vector bundles. They then defined K-theory and rapidly developed it from 1958 to 1962. When Atiyah asked Singer his question, it was in this context.

At the same time, parallel work was undertaken in the Soviet Union under Gelfand and his students. He observed that the index sometimes can be computed topologically, and asked whether this is true in general, and Atiyah-Singer's answer also incorporates this question.

Subsequently, in the 1970s, Gilkey, Patodi, and others were able to provide more rigid, simpler proofs with analytic methods, and in the 1980s Getzler made another important simplifying step to what's now called the heat equation proof of the index theorem, which we'll follow.

We'll use John Roe's book in this course. It's analytic in flavor, but also treats many other nice results, and if we go quickly enough, we'll get to see some of them, including Witten's physical treatment of Morse theory, the Lefschetz theorem, the Hodge theorem, and more.

In this class, the students will give lectures, two each week, and we hope to go through two chapters a week. You don't have to use all three hours!

On the course website (`https://www.ma.utexas.edu/users/dafr/M392C/`), there will be some useful information, including some old course notes, some historical background, and more to come. These will be there so that you do not forget the beauty of the material amongst all the details in the lectures.

Not everybody may know all of the prerequisites for this course, since it draws in lots of different parts of mathematics. One can ask the professor for references or talk to other students in the course.

The second half of the first day is on the first chapter of the book, reviewing some of the basics of Riemannian geometry.

Let's first start with some linear algebra and differential forms. Let V be an n-dimensional real vector space. Eventually, V will be a tangent space at a point to a manifold, and if the manifold has a Riemannian metric, V picks up an inner product.

Associated to V are several canonical vector spaces built from it: its wedge powers $\Lambda^2 V, \dots, \Lambda^n V$, and $\Lambda^0 V$, which is canonically $\mathbb{R}$. The top exterior power is also called the *determinant line*, $\operatorname{Det} V := \Lambda^n V$. Dually, there are the exterior powers of the dual space V^*: $\mathbb{R}$, V^*, $\Lambda^2 V^*, \dots, \operatorname{Det} V^*$.

An inner product on V canonically induces inner products on all of these exterior powers. One way to see this is to let $e_1, \dots, e_n$ be an orthonormal basis of V; then, there is a dual basis $e^1, \dots, e^n$ of V^*, defined by the relation

$$e^\mu(e_\nu) = \delta^\mu_\nu, \tag{1.24}$$

i.e. 1 if $\mu = \nu$ and 0 otherwise.

We specify the inner product on V^* by declaring this dual basis orthonormal, which suffices, though you have to check that if you change the orthonormal basis of V you started with, you'll end up with the same inner product nontheless.

We also obtain bases for the exterior powers of V and V^*: for $\Lambda^q V$, the basis is

$$\{e_{i_1} \wedge \cdots \wedge e_{i_q} : 1 \le i_1 < \cdots < i_q \le n\}, \tag{1.25}$$

and for $\Lambda^q V^*$, it's

$$\{e^{i_1} \wedge \cdots \wedge e^{i_q} : 1 \le i_1 < \cdots < i_q \le n\}. \tag{1.26}$$

Again we define the inner products on $\Lambda^q V$ and $\Lambda^q V^*$ by asking for these bases to be orthonormal, and again the inner product in question does not depend on the specific choice of orthonormal basis of V.

Definition 1.27. An *orientation* of V is an orientation of its determinant line. That is, $\operatorname{Det} V \setminus 0$ has two components, and an orientation is a choice of one of them.

Given n vectors $e_1, \dots, e_n \in V$, we can wedge them together to an $e_1 \wedge \cdots \wedge e_n \in \operatorname{Det} V$; $\{e_1, \dots, e_n\}$ is a basis iff $e_1 \wedge \cdots \wedge e_n \ne 0$. Thus a basis singles out one of the two rays in $\operatorname{Det} V \setminus 0$, hence defines an orientation. Since $(\operatorname{Det} V)^* = \operatorname{Det}(V^*)$ canonically, then this also defines an orientation on $(\operatorname{Det} V)^*$: the duality pairing implies there's a single $\theta \in \operatorname{Det} V^*$ which sends $e_1 \wedge \cdots \wedge e_n \mapsto 1$; we call it the *volume form* and denote it vol.

On an oriented Riemannian n-manifold, this is a differential n-form, hence can integrate it to determine the volume of the manifold. If it's not oriented, there are two at each point, which may twist globally into something called a density. Nonetheless, this can be integrated, and the volume of, e.g. $\mathbb{RP}^2$ still makes sense.

The pairing $\Lambda^q V^* \otimes \Lambda^{n-q} V^* \to \operatorname{Det} V^*$ defined by

$$\alpha, \beta \longmapsto \alpha \wedge \beta \tag{1.28}$$

is nondegenerate. An orientation of V defines a trivialization of $\operatorname{Det} V^*$ (where $\mathrm{vol} = 1$), so this pairing is $\mathbb{R}$-valued. Therefore we obtain an isomorphism $\Lambda^q V^* \cong \Lambda^{n-q}(V)$, though it depends on the inner product and the orientation.

Example 1.29. In three dimensions, we use this frequently, to shift from the perspective of vector fields and scalars and div, ∇, and curl to differential forms. ◀

There's also an isomorphism $\star\colon \Lambda^q V^* \to \Lambda^{n-q} V^*$ which only uses the inner product; this is called the *Hodge star*. Putting everything together, the Hodge star is defined uniquely by the stipulation that

$$\alpha_1 \wedge \star\alpha_2 = \langle \alpha_1, \alpha_2 \rangle \mathrm{vol} \tag{1.30}$$

for any $\alpha_1, \alpha_2 \in \Lambda^q V^*$.

Exercise 1.31. For example, check that $\star(e_{i_1} \wedge \cdots \wedge e_{i_q})$ is the wedge of all of the e_j not in $(i_1, \dots, i_q)$, possibly multiplied by -1.

Exercise 1.32. Show that $\star^2 = (-1)^{q(n-q)}$.

Here, "inner product" means nondegenerate inner product; much of this story still goes through for a Lorentz-signature metric, but not all of it.

Exercise 1.33. Show that on a closed, oriented Riemannian manifold M, $\mathrm{d}^* = \pm \star \mathrm{d} \star$, and determine the sign (which depends on n and q).

You can type-check that the right-hand side is a first-order differential operator which lowers the degree by 1. Solving the exercise boils down to checking that

$$\int_M \langle \mathrm{d}\alpha, \beta\rangle \,\mathrm{vol} = \pm \int_M \langle \alpha, \star\mathrm{d}\star\beta\rangle \,\mathrm{vol}.$$

You'll end up using Stokes' theorem.

Now let's think about parallelism. Let $\mathbb{A}^n$ be n-dimensional affine space (no distinguished origin), where we learn calculus. This has parallel transport: if $\xi \in \mathbb{R}^n$ is a tangent vector at some point, we can translate it everywhere to a vector field. This allows us to define differentiation: if $f: U \to \mathbb{R}$, where $U \subset \mathbb{A}^n$ is open, then we define the derivative of f at p in the direction of ξ to be

$$\xi_p f := \lim_{h\to 0} \frac{f(p+h\xi) - f(p)}{h}. \tag{1.34}$$

This uses parallelism in the expression $p + h\xi$.

More generally, if M is a smooth manifold, we don't always have a canonical parallel transport between tangent spaces for different points of the manifold, so we can't compare tangent vectors in different places and differentiate.

For example, if $\gamma: [a,b] \to M$ is a curve, its tangent vectors at two different points can't be compared (without extra structure), so there's no way to make the subtraction in (1.34). We'll introduce the structure that allows us to do this.

Definition 1.35. Let $V \to M$ be a vector bundle and $C^\infty(M;V)$ denote its space of smooth sections, which is a real vector space. A *covariant derivative* is a bilinear operator

$$\nabla: C^\infty(M;TM) \times C^\infty(M;V) \longrightarrow C^\infty(M;V),$$

denoted

$$X, s \longmapsto \nabla_X s,$$

such that

(1) $\nabla_{fX} s = f\nabla_X s$, and
(2) $\nabla_X(fs) = (X\cdot f)s + f\nabla_X s$,

where $(X \cdot f)$ is the usual directional derivative associated to a vector field.

For $V = TM$, we have the usual Lie bracket

$$[-,-]: C^\infty(M;TM) \times C^\infty(M;TM) \longrightarrow C^\infty(M;TM)$$

sending $X, Y \mapsto [X,Y]$; if $f, g: M \to \mathbb{R}$ are functions, then

$$[fX, gY] = fg[X,Y] + f(X\cdot g)Y - g(Y\cdot f)X.$$

This operator is the commutator of an infinitesimal flow of X and an infinitesimal flow of Y.

Definition 1.36. Let ∇ be a covariant derivative for the tangent bundle. Its *torsion* is

$$\tau(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y].$$

Exercise 1.37. Show that $\tau(fX, gY) = fg\tau(X,Y)$ and $\tau(X,Y) = -\tau(Y,X)$.

Let's write this out in local coordinates. There are two things we could mean – coordinates on M or on V. Since V is a vector bundle, we can use for its coordinates the coordinates of M and a (local) basis of sections $s_1, \ldots, s_r$. (Global nonvanishing sections might not exist at all, e.g. $TS^2 \to S^2$). In this case, you can differentiate s_j, obtaining some linear combination of the sections depending on x in a neighborhood U:

$$\nabla_X s_j = \Gamma^i_j(x) s_i.$$

This is just parameterized linear algebra. These Γ^i_j are 1-forms on U. We can also obtain coordinates for these 1-forms: if we let

$$\nabla_{\partial/\partial x^\mu} s_j = \Gamma^i_{j\mu} s_i,$$

then $\Gamma^i_j = \Gamma^i_{j\mu}\,\mathrm{d}x^\mu$.

If $V \to M$ has an inner product (metric), a positive definite pairing $C^\infty(M;V)\times C^\infty(M;V) \to C^\infty(M)$ sending $s_1, s_2, \mapsto \langle s_1, s_2\rangle$, we can ask how a covariant derivative interacts with it.

Definition 1.38. A covariant derivative is *compatible with the metric* if for all $X \in C^\infty(M;TM)$ and $s_1, s_2 \in C^\infty(M;V)$,

$$X \cdot \langle s_1, s_2 \rangle = \langle \nabla_X s_1, s_2 \rangle + \langle s_1, \nabla_X s_2 \rangle.$$

Definition 1.39. A section $s \in C^\infty(M;V)$ is *parallel* if $\nabla_X s = 0$ for all X.

Parallel sections exist in $\mathbb{A}^n$ but not in general; the obstruction is called the curvature.

Definition 1.40. The *curvature* of a covariant derivative ∇ is

$$K(X,Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]},$$

i.e.

$$K(X,Y)(s) := \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s.$$

If M is Riemannian, there's a beautiful theorem about how all of these structures interact.

Theorem 1.41 (Levi-Civita)**.** *Let M be a Riemannian manifold. Then, there is a unique connection on $TM \to M$ which is torsion-free and compatible with the metric.*

Exercise 1.42. Prove this theorem. The way you do this is to compute $\langle \nabla_X Y, Z \rangle$, because if you know this for all Z, you know $\nabla_X Y$. Using the torsion-free and metric compatibility conditions, you can expand it out, and after some number of steps, you'll get the answer.

This local but non-global parallelism is an important property of Riemannian manifolds.

Next we will write a local formula for this connection. Suppose we have local coordinates $x^1, \dots, x^n$ on an open set $U \subset M$; then, we obtain the symbols $\Gamma^i_{jk} \colon U \to \mathbb{R}$. If we define the inner product and the Lie bracket, we can write down formulas for them. Namely, if we let

$$g_{ij} := \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle,$$

and since

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^i} \right] = 0,$$

then we can determine equations that the Γ^i_{jk} must satisfy. These can be encoded in the Riemann curvature tensor $R(X,Y)Z$, and in coordinates, on elets

$$R^i_{jk\ell} \frac{\partial}{\partial x^i} = R\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell} \right) \frac{\partial}{\partial x^j}.$$

This tensor has a bunch of important symmetries. The curvature is a 2-form on the manifold, but valued in $\mathrm{End}(TM)$: X and Y are the two directions you're testing, and are the 2 components of the 2-form.

The symmetry $R^i_{jk\ell} = -R^i_{j\ell k}$ means that $R(X,Y)$– is a skew-symmetric endomorphism of TM.

You can also lower an index by defining

$$R_{ijk\ell} = \langle R(\partial_k, \partial_\ell)\partial_j, \partial_i \rangle, \tag{1.43}$$

and skew-symmetry means

$$R_{ijk\ell} = -R_{jik\ell}.$$

These are the two "easier" symmetries, in that they don't use much specifically about R. A more interesting one is

$$R_{ijk\ell} + R_{ik\ell j} + R_{i\ell jk} = 0,$$

and the fourth identity, which follows from the other three, is

$$R_{ijk\ell} = R_{k\ell ij}.$$

Exercise 1.44. Compute the dimension of the vector space of tensors which satisfy these identities, as a subspace of $(V^*)^{\otimes 4}$.