Riemannian Geometry

UT Austin, Spring 2017

M392C NOTES: RIEMANNIAN GEOMETRY

ARUN DEBRAY
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These notes were taken in UT Austin's M392C (Riemannian Geometry) class in Spring 2017, taught by Dan Freed. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

The cover image is the Cosmic Horseshoe (LRG 3-757), a gravitationally lensed system of two galaxies. Einstein's theory of general relativity, written in the language of Riemannian geometry, predicts that matter bends light, so if two galaxies are in the same line of sight from the Earth, the foreground galaxy's gravity should bend the background galaxy's light into a ring, as in the picture. The discovery of this and other gravitational lenses corroborates Einstein's theories. Source: https://apod.nasa.gov/apod/ap111221.html.

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Geometry in flat space: 1/17/17

"Do you have all these equations?"

Before we begin with Riemannian manifolds, it'll be useful to do a little geometry in flat space.

Definition 1.1. Let V be a real vector space; then, an *affine space over* V is a set A with a simply transitive right V-action.

That this action is simply transitive means for any $a, b \in A$, there's a unique $\xi \in V$ such that $a \cdot \xi = b$.

Definition 1.2. A set with a simply transitive (right) V-action is called a *(right) V-torsor*.

V-torsors look like copies of V without a distinguished identity.

One of the distinct features of affine space is *global parallelism*: if I have a vector ξ at a point a, I immediately get a vector at every point, which defines a vector field on the entire space.

What is the analogue of a basis for an affine space? This is a collection of points $a_0, \ldots, a_n$ such that any $a \in A$ is uniquely written as

$$a = \lambda^0 a_0 + \lambda^1 a_1 + \cdots + \lambda^n a_n \tag{1.3}$$

for some $\lambda^i \in \mathbb{R}$ with $\lambda^0 + \cdots + \lambda^n = 1$.

Equation (1.3) may be written more concisely with *index notation*: any variable written as both a superscript and a subscript is implicitly summed over. That is, we may rewrite (1.3) as

$$a = \lambda^i a_i.$$

Note that in an affine space, we don't know how to add vectors (since we don't have an origin), but we can take weighted averages.

Theorem 1.4 (Giovanni Ceva, 1678)**.** *Let A be an affine plane and $a, b, c \in A$ be a triangle (i.e. three distinct, noncollinear points). Suppose $p \in \overline{bc}$, $q \in \overline{ca}$, and $r \in \overline{ca}$. Then, $\overline{ap}$, $\overline{bq}$, and $\overline{cr}$ are coincident iff*

$$[ar : rb][bp : pc][cq : ca] = 1.$$

Typically, this is thought of as a ratio of lengths, but we don't necessarily have lengths: instead, we can use barycentric coordinates. There is a unique λ such that if $r = (1-\lambda)a + \lambda b$, then $[ar : rb] = \lambda/(1-\lambda)$.

Proof. Let

$$\begin{aligned} r &:= (1-\lambda)a + \lambda b \\ p &:= (1-\mu)b + \mu c \\ q &:= (1-\nu)c + \nu a. \end{aligned}$$

Set

$$x := \alpha a + \beta b + \gamma c, \tag{1.5}$$

where $\alpha + \beta + \gamma = 1$. Since $x \in \overline{ap}$, then

$$x = \alpha a + C((1-\mu)b + \mu c). \tag{1.6}$$

Comparing (1.5) and (1.6), $\mu/(1-\mu) = \gamma/\beta$.

⊠

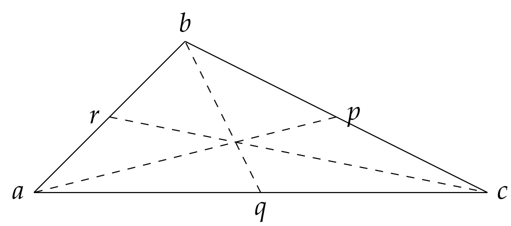

Figure 1. Depiction of Ceva's theorem (Theorem 1.4).

Standard affine space $\mathbb{A}^n := \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^i \in \mathbb{R}\}$. You may complain this is the same as $\mathbb{R}^n$, but $\mathbb{A}^n$ only comes with an affine structure, not a vector-space structure.

Definition 1.7. Let A be an affine space modeled on V and B be an affine space modeled on W. Then, a map $f : A \to B$ is *affine* if there exists a linear map $T : V \to W$ such that $f(a + \xi) = f(a) + T\xi$ for all $a \in A$ and $\xi \in V$.

In other words, an affine map is a linear map plus some constant, which is not uniquely defined.

Definition 1.8. An *affine coordinate system* on A is an affine isomorphism $x = (x^1, \dots, x^n) : A \to \mathbb{A}^n$.

Them, the differentials $\mathrm{d}x_a^1, \dots, \mathrm{d}x_a^N$ are independent of basepoint a and form a basis for V^*, the dual vector space and dual basis to V and $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$, the tangent space to any $a \in A$.

But affine space is not the only flat geometry we could consider: more generally, we consider a structure on a vector space V which can be promoted to a translationally invariant structure on A. This leads to metric geometry, symplectic geometry, etc.

Definition 1.9. An *inner product* on a (finite-dimensional) vector space V is a bilinear map $\langle -, - \rangle : V \times V \to \mathbb{R}$ which is symmetric and positive definite, i.e. for all $\xi, \eta \in V$, $\langle \xi, \eta \rangle = \langle \eta, \xi \rangle$, $\langle \xi, \xi \rangle \geq 0$, and $\langle \xi, \xi \rangle = 0$ iff $\xi = 0$.

Since $\langle -, - \rangle$ is bilinear, then this can be determined in terms of n^2 numbers: let $v_1, \dots, v_n$ be a basis for V and define $g_{ij} := \langle v_i, v_j \rangle$ for $i, j = 1, \dots, n$. Of course, these numbers areen't independent: $g_{ij} = g_{ji}$, so there are really only $n(n+1)$ choices of information.

Definition 1.10. A basis $e_1, \dots, e_n$ for V is *orthonormal* if

$$\langle e_i, e_j \rangle = \delta_{ij} := \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}.$$

Our first major result of flat Euclidean geometry is that these exist.

Theorem 1.11. *There exist orthonormal bases.*

Proof. Let $v_1, \dots, v_n$ be any basis of V. Let

$$e_1 = \frac{v_1}{\langle v_1, v_1 \rangle^{1/2}},$$

and for $i = 2, \dots, n$, let

$$v_i' = v_i - \langle v_i, e_1 \rangle e_1.$$

Then, $\langle e_1, e_1 \rangle = 1$ and $\langle e_1, v_i' \rangle = 0$. Then, repeat with $v_2', \dots, v_n'$. ⊠

This explicit algorithm is called the *Gram-Schmidt process*.

In an inner product space, we get some familiar geometric constructions: the *length* of a vector $\xi \in V$ is $|\xi| = \langle \xi, \xi \rangle^{1/2}$, and the *angle* between $\xi, \eta \in V \setminus 0$ is the θ such that

$$\cos\theta = \frac{\langle \xi, \eta \rangle}{|\xi||\eta|}.$$

Definition 1.12. A *Euclidean space* E is an affine space over an inner product space V.

This has a notion of distance: $d_E : E \times E \to \mathbb{R}^{\geq 0}$, where $a, b \mapsto |\xi|$, where $b = a + \xi$. This generalizes to notions of area, volume, etc.

Theorem 1.13 (Napoleon, 1820)**.** *Let abc be a triangle in a plane and attach an equilateral triangle to each edge. The centers of these three triangles form an equilateral triangle.*

Exercise 1.14. Prove this.

We want to understand curved analogues of this classical material, and will pick up where differential topology left off. We work on smooth manifolds: a *smooth manifold* is a space X together with an atlas of charts $U \subset X$ with homeomorphisms $x : U \to \mathbb{A}^n$ such that every point is contained in the domain of some chart and the transition maps are smooth. We do not require a manifold to have a global dimension: the different connected components may have different dimensions, e.g. $S^1 \amalg S^2$.[1]

A chart map $x : U \to \mathbb{A}^n$ is a set of n continuous maps $(x^1, \dots, x^n)$. If p is in the domain of both x and y, we can consider $x \circ y^{-1} : \mathbb{A}^n \to \mathbb{A}^n$; calculus as usual tells us what it means for this transition map to be smooth.

At any $x \in X$, we have a tangent space $T_x X$ and a cotangent space $T_x X$: a chart defines a basis of the tangent space $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ and a basis of the contangent space $\mathrm{d}x_x^1, \dots, \mathrm{d}x_x^n$. This depends strongly on x: unlike for flat space, we may not be able to parallel-transport these globally, even on something as simple as S^2.

In this course, we will study what happens when we go from a curved analogue of affine space to a curved analogue of Euclidean space, whence the following central definition.

Definition 1.15. A *Riemannian metric* on a smooth manifold X is a choice of inner product $\langle -, - \rangle_x$ on $T_x X$ for all $x \in X$ which varies smoothly in x.

Now, we can compute lengths of tangent vectors and the angle that two smooth curves intersect at (or rather, the angle their tangent vectors intersct at). We also obtain a notion of distance between points, and can develop analogues of Euclidean geometry on manifolds.

What does "varying smoothly" mean, exactly? Suppose $x^1, \dots, x^n$ is a set of local coordinates on $U \subset X$; then, for $i, j = 1, \dots, n$, define

$$g_{ij} := \left\langle \left.\frac{\partial}{\partial x^i}\right|_x, \left.\frac{\partial}{\partial x^j}\right|_x \right\rangle_{T_x X}.$$

One can check that if the metric is smoothly varying in one chart, then it's smoothly varying in all charts.

We'll write the metric as

$$g = g_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j.$$

[1]This is important for, e.g. a space of solutions of certain PDEs.

This again uses the summation convention, and it's useful to think about where exactly this lives: it identifies the metric as a tensor.

Many manifolds arise as embedded submanifolds of Euclidean space, and the Whitney embedding theorem shows that all may be emnbedded. Many authors say it's best to meet manifolds as embedded submanifolds first, but there are some which arise without a natural embedding, e.g. the Grassmanian $\mathrm{Gr}_2(\mathbb{R}^4)$, the space of two-dimensional subspaces of $\mathbb{R}^4$.

In any case, if $X \subset \mathbb{E}^N$ is embedded, then X inherits a metric, since $T_xX \subset \mathbb{R}^n$ is also a subspace, and we can restrict the inner product. Classical Riemannian geometry is the study of *plane curves* (one-dimensional submanifolds of $\mathbb{R}^2$), *space curves* (one-dimensional submanifolds of $\mathbb{R}^3$), and *surfaces* (two-dimensional submanifolds of $\mathbb{R}^3$).

To study Riemannian manifolds, we should begin with the simplest cases. The zero-dimensional manifolds are disjoint unions of points with zero-dimensional tangent spaces and the trivial Riemannian metric. In the one-dimensional case, there is a little more to tell. A smooth map $X \to Y$ of Riemannian manifolds is an *isometry* if it's a map that preserves the inner product on each tangent space. This automatically implies it's injective.

Theorem 1.16. *Let C be a (complete) Riemannian 1-manifold which is diffeomorphic to $\mathbb{R}$. Then, C is isometric to $\mathbb{E}^1$.*

Before we prove this, we need a change-of-coordinates lemma. (We'll address completeness later, to avoid finite intervals.)

Remark. Let $x^1, \dots, x^n$ and $y^1, \dots, y^n$ be coordinate systems and suppose a metric can be written as

$$g = g_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j = h_{ab}\, \mathrm{d}y^a \otimes \mathrm{d}y^b.$$

Then,

$$g_{ij} = h_{ab} \frac{\partial y^a}{\partial x^i} \frac{\partial y^b}{\partial x^j}. \tag{1.17}$$

This is n^2 equations: there is no implicit summation here. ◁

Proof of Theorem 1.16. Let $x : C \to \mathbb{R}$ be a diffeomorphism, which defines a global coordinate on C. Let $g(x) = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \rangle$. We seek a new coordinate $y : C \to \mathbb{R}$ such that $h(y) = \langle \frac{\partial}{\partial y}, \frac{\partial}{\partial y} \rangle = 1$ everywhere. By (1.17),

$$g = \left(\frac{\mathrm{d}y}{\mathrm{d}x} \right)^2, \tag{1.18}$$

so fix an $x_0 \in C$ and define

$$y(x) = \int_{x_0}^{x} \sqrt{g(t)}\, \mathrm{d}t.$$

This y satisfies (1.18) and therefore is an isometry. ⊠

The analogue to Theorem 1.16 in n dimensions (where $n > 1$) is as follows: if $x^1, \dots, x^n$ is a local coordinate system and g_{ij} is the Riemannian metric in these coordinates, is there a local change of coordinates $y^a(x^1, \dots, x^n)$ such that $h_{ab} = \delta_{ab}$? This is the analogue in Riemannian geometry to finding orthonormal coordinates, guaranteed by Theorem 1.11.

This requires solving an analogue to (1.17), but this time it's a PDE

$$g_{ij} = \sum_a \frac{\partial y^a}{\partial x^i} \frac{\partial y^a}{\partial x^j}.$$

This time, we need to ask whether there are solutions. The only thing we know how to do is differentiate:

$$\frac{\partial g_{ij}}{\partial x^k} = \sum_a \frac{\partial^2 y^a}{\partial x^k \partial x^i} \frac{\partial g^a}{\partial x^j} + \frac{\partial y^a}{\partial x^i} \frac{\partial^2 y^a}{\partial x^k \partial x^j}. \tag{1.19a}$$

By permuting indices, we obtain

$$\frac{\partial g_{ik}}{\partial x^j} = \sum_a \frac{\partial^2 y^a}{\partial x^j \partial x^i}\frac{\partial g^a}{\partial x^k} + \frac{\partial y^a}{\partial x^i}\frac{\partial^2 y^a}{\partial x^j \partial x^k} \tag{1.19b}$$

$$\frac{\partial g_{jk}}{\partial x^i} = \sum_a \frac{\partial^2 y^a}{\partial x^i \partial x^j}\frac{\partial g^a}{\partial x^k} + \frac{\partial y^a}{\partial x^j}\frac{\partial^2 y^a}{\partial x^i \partial x^k}. \tag{1.19c}$$

Taking (1.19a) + (1.19b) − (1.19c), we obtain

$$\frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right) = \sum_a \frac{\partial y^a}{\partial x^i}\frac{\partial^2 y^a}{\partial x^j \partial x^k}.$$

Now we multiply by $\frac{\partial y^b}{\partial x^\ell} g^{\ell i}$, concluding

$$\frac{\partial y^b}{\partial x^\ell}\underbrace{\frac{g^{\ell i}}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right)}_{\Gamma^\ell_{jk}} = \sum_a \frac{\partial y^a}{\partial x^i}\frac{\partial^2 y^a}{\partial x^j \partial x^k} g^{\ell i}\frac{\partial y^b}{\partial x^\ell}.$$

These Γ^ℓ_{jk} symbols therefore satisfy

$$\frac{\partial^2 y^b}{\partial x^j \partial y^k} = \Gamma^i_{jk}\frac{\partial y^b}{\partial x^i}.$$

If we differentiate once again (with respect to x^ℓ), we get

$$\begin{aligned}\frac{\partial^3 y^b}{\partial x^\ell \partial x^j \partial x^k} &= \frac{\partial \Gamma^i_{jk}}{\partial x^\ell}\frac{\partial y^b}{\partial x^i} + \Gamma^i_{jk}\frac{\partial^2 y^b}{\partial x^\ell x^i} \\ &= \left(\frac{\partial \Gamma^i_{jk}}{\partial x^\ell} + \Gamma^m_{jk}\Gamma^i_{m\ell}\right)\frac{\partial y^b}{\partial x^i}.\end{aligned}$$

Since mixed partials commute, then one discovers that if such an isometry exists, the *Riemannian curvature tensor*

$$R^i_{jk\ell} := \frac{\partial \Gamma^i_{j\ell}}{\partial x^\ell} - \frac{\partial \Gamma^i_{jk}}{\partial x^\ell} + \Gamma^m_{jk}\Gamma^i_{m\ell} - \Gamma^m_{j\ell}\Gamma^i_{mk}$$

must vanish. In simple cases, one can calculate that it's not always zero, so we don't always have global parallelism.

Riemann derived this in the middle of the 1800s. It's possible to see the glimmer of special relativity in them, though of course this was discovered later.

There's no text, though there is a website: `http://www.ma.utexas.edu/users/dafr/M392C/index.html`. There are problem sets, so undergraduates have to do some problem sets, and graduate students should. Feel free to talk to the professor about the problems, and especially to establish groups to work on the problem sets. Office hours are Wednesdays 2 to 3.

Lecture 2.

Existence of Riemannian metrics: 1/19/17

> *"There are so many of you... so quiet... I'll be more provocative until I get questions. Or I'll go faster."*

Due to the large size of the class, it's being moved to RLM 6.104 starting next week. This means everyone who wants to sign up should be able to.

Some readings are up on the website, including a translation of Riemann's original work on curvature.

Last time, we defined affine space, which leads to the notion of a smooth manifold, and then introduced Euclidean space, an affine space over an inner product space. The curved version of that is a Riemannian manifold.

Recall that a Riemannian metric g on a smooth manifold X is a smoothly varying family of inner products on T_xX, and a Riemannian manifold is a smooth manifold together with a Riemannian metric.

We also defined an isometry: if X and Y are Riemannian manifolds, then a diffeomorphism $f : X \to Y$ is an isometry if for all $x \in X$ and $\xi_1, \xi_2 \in T_x X$,

$$\langle f_* \xi_1, f_* \xi_2 \rangle_{T_{f(x)}Y} = \langle \xi_1, \xi_2 \rangle_{T_x X}.$$

Here, $f_* : T_x X \to T_{f(x)} Y$ is the linear pushforward of tangent vectors, also called the *differential*. If f is merely a smooth function, this is called an *isometric immersion* (the inverse function theorem automatically implies it's an immersion). If f is an embedding, this is called an *isometric embedding*.

Existence of Riemannian metrics. Suppose V is a real vector space and $g_0, g_1 : V \times V \to \mathbb{R}$ are inner products. Then for $t \in [0,1]$, $(1-t)g_0 + tg_1$ is also an inner product (you can check this directly).

The set of bilinear maps $V \times V \to \mathbb{R}$, denoted $\mathrm{Bil}(V \times V, \mathbb{R})$, is a real vector space, naturally isomorphic to $\mathrm{Hom}(V \otimes V, \mathbb{R})$ and to $V^* \otimes V^*$. Here, "natural" means this works for all finite-dimensional vector spaces at once, and commutes with linear maps.

Inner products are elements of this vector space, and our observation above means that if g_0 and g_1 are inner products, the line between them in $\mathrm{Bil}(V \times V, \mathbb{R})$ consists of inner products. In particular, *inner products form a convex set*. This only uses the affine structure on $\mathrm{Bil}(V \times V, \mathbb{R})$, since we can take convex combinations in an affine space.

This is used to generalize to the curved case, showing Riemannian metrics always exist.

Theorem 2.1. *Let X be a smooth manifold. Then, there is a Riemannian metric on X.*

Proof. Let $\mathfrak{U} = \{(U, x)\}$ be a cover of X by coordinate charts $x : U \to \mathbb{A}^n$, and let g_U denote the metric on U such that $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ are orthonormal. That is, take the standard metric on $\mathbb{A}^n$ making it into Euclidean space $\mathbb{E}^n$, and pull it back to U, where it becomes a metric (you can check that metrics pull back along closed immersions).

Now, the bases on two different charts in $\mathfrak{U}$ don't agree, and don't necessarily differ by orthonormal bases. Thus, we use a standard argument in differential geometry to globalize local objects living in a convex set: let $\{\rho_U\}_{U \in \mathfrak{U}}$ be a partition of unity subordinate to $\mathfrak{U}$; then,

$$g = \sum_{U \in \mathfrak{U}} \rho_U g_U$$

is a Riemannian metric. ⊠

Remark. Global existence is *not* assured for every geometric structure. For example, a *complex structure* on a real vector space V is an endomorphism $J : V \to V$ such that $J^2 = -\mathrm{id}_V$. This is akin to multiplication by i in a complex vector space, which squares to -1 and commutes with addition.

You can place this structure on affine space, and there's an immediate obstruction: $\dim_{\mathbb{R}} V$ must be even. Now we globalize: given an even-dimensional manifold, do we have such a structure? That is, can we place a smoothly varying complex structure on $T_x X$ for all $x \in X$? This is called an *almost complex structure*, and not every even-dimensional manifold admits one.

Exercise 2.2. Show that S^4 has no almost complex structure.

There is an almost complex structure on S^6, and it's a famous open question whether there's a complex structure (i.e. complex coordinates with holomorphic transition functions). The known almost complex structure does not work.

Another local structure that doesn't automatically globalize is a mixed-signature metric (e.g. a Minkowski metric). In such a metric, the *null vectors*, those ξ for which $\langle \xi, \xi \rangle = 0$, form a cone whose interior is the *positive vectors* (for which the metric is positive). Trying to globalize this produces, more or less, a line in each tangent space $T_x X$. Passing to a double cover, one can choose an orientation, and therefore a nonzero vector field on X, and this can't be done in general. For example, a surface of genus 2 admits no metric of signature $(1,1)$. These kinds of metrics arise in general relativity. ◂

In this class, we care about Riemannian metrics, which do globalize.

Let $x^1, \dots, x^n$ be local coordinates; then, we defined some local quantities in the metric in terms of these coordinates. Namely,

$$g_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle,$$

so that $g = g_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$. We then used this to define symbols Γ^i_{jk} and the Riemann curvature tensor $R^i_{jk\ell}$. We proved Theorem 1.16; here's a better version.

Theorem 2.3. *Let C be a Riemannian 1-manifold diffeomorphic to $\mathbb{R}$. Then, there exists an isometry $C \to I$, where $I \subset \mathbb{E}^1$ is an open interval.*

The argument we gave defining the Riemann curvature tensor generalizes this.

Theorem 2.4. *Suppose (U, g) is a Riemannian manifold and $x : U \to \mathbb{A}^n$ is a global coordinate such that*

$$g = \sum_{i=1}^{n} (\mathrm{d}x^i)^2.$$

Then, $R^i_{jk\ell} = 0$ on U.

One important thing to check here is that

$$R = R^i_{jk\ell} \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k \otimes \mathrm{d}x^\ell$$

is independent of the coordinate system (which is not clear from its definition). This means that the Riemann curvature tensor is a tensor, i.e. $R \in T_x X \otimes T^*_x X \otimes T^*_x X \otimes T^*_x X$. In the next few weeks, we will add some geometry to this discussion.

Example 2.5. Let $X = \mathbb{E}^2$ be Euclidean space with the standard metric g. Then, we have global coordinates $(x, y) : \mathbb{E}^2 \to \mathbb{A}^2$, so $g = \mathrm{d}x^2 + \mathrm{d}y^2$.

We can also introduce *polar coordinates*, another coordinate system which isn't global. This is a coordinate map $(r, \theta) : \mathbb{E}^2 \setminus \{(x, 0) : x \le 0\} \to \mathbb{A}^2$ (so $r > 0$, $-\pi < \theta < \pi$). In this case, the metric has the form

$$g = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2.$$

This means that the vector field $\frac{\partial}{\partial r}$ has constant length 1, but the vector field $\frac{\partial}{\partial \theta}$ has length r at (r, θ). ◀

Symmetry. We've now seen vector spaces, affine spaces, Euclidean spaces, and Riemannian manifolds. As in any mathematical context, it's important to ask what the proper notion of symmetry is for these objects.

If V is a vector space, its *general linear group* is $\mathrm{GL}(V) = \mathrm{Aut}(V) := \{T : V \to V \text{ invertible}\}$. The standard example is $\mathrm{GL}_n(\mathbb{R}) := \mathrm{GL}(\mathbb{R}^n)$, the group of invertible $n \times n$ matrices, acting on the column vectors of $\mathbb{R}^n$ by scalar multiplication. For example $\mathrm{GL}_1(\mathbb{R}) = \mathbb{R}^\times$, the group of nonzero numbers under multiplication.

What about affine space? Affine space on V is a V-torsor, as V acts by translation. The symmetry group is the group of *affine transformations*

$$\mathrm{Aff}(A) := \{\alpha : A \to A \mid \alpha \text{ is invertible and affine}\}.$$

Recall that an affine map is one that preserves the affine structure: the image of a finite weighed average is the weighted average of the images. The derivative of an affine map is a linear map, so if A is an affine space modeled by V, the derivative defines a group homomorphism $\mathrm{d} : \mathrm{Aff}(A) \to \mathrm{GL}(V)$, whose kernel is the translations, a group isomorphic to V. Thus, we have a *group extension* (short exact sequence of groups)

$$1 \longrightarrow V \longrightarrow \mathrm{Aff}(A) \xrightarrow{\mathrm{d}} \mathrm{GL}(V) \longrightarrow 1. \tag{2.6}$$

The key is that in affine space, there's no canonical origin. However, (2.6) splits, if noncanonically: choose an $a \in A$. Then, any $b \in A$ can be uniquely written as $a + \xi$ for some $\xi \in V$, so for any linear transformation T, $a + \xi \mapsto a + T\xi$ is an affine transformation of A.

(2.6) is a sequence of manifolds with smooth group homomorphisms, making it a short exact sequence of *Lie groups*; we'll discuss Lie groups more later.

If V is an n-dimensional vector space, its bases are the set $\mathscr{B}(V) = \{b : \mathbb{R}^n \xrightarrow{\cong} V\}$. If $V = \mathbb{R}^n$, this is $\mathrm{GL}_n(\mathbb{R})$. In general, this makes $\mathscr{B}(V)$ into a right $\mathrm{GL}_n(\mathbb{R})$-torsor, defined by the simply transitive action $\mathscr{B}(V) \times \mathrm{GL}_n(\mathbb{R}) \to \mathscr{B}(V)$ sending $\beta, g \mapsto \beta \circ g$. (There is a corresponding left action by $\mathrm{GL}(V)$). The action on the right is akin to numbering elements of the basis, and the action on the left is more geometric; this is an instance of a general idea that internal actions tend to be from the right, and geometric ones from the left.

What's the analogue for an affine space A modeled on V? Let $\mathscr{B}(A)$ denote the collection of pairs (a, β) where $a \in A$ and $\beta \in \mathscr{B}(V)$, identified with the set of affine isomorphisms $\alpha : A \xrightarrow{\cong} \mathbb{A}^n$. These are the bases at specific points of A. There is a forgetful map $\pi : \mathscr{B}(A) \to A$ sending $(a, \beta) \to a$, and the fiber is $\mathscr{B}(V)$, the bases at a. In a similar way, there is a left action of $\mathrm{Aff}(A)$ on $\mathscr{B}(A)$, and a right action of $\mathrm{Aff}_n := \mathrm{Aff}(\mathbb{A}^n)$ on $\mathscr{B}(A)$.

We'll use these torsors of bases a lot in this class. In this way, we're enacting Felix Klein's *Erlangen* program, where the kind of geometry we do is reflected by the symmetry group we place on the geometric structures.

Let's see what happens to these ideas in the Euclidean and Riemannian cases. If V is an inner product space, its *orthogonal group* $\mathrm{O}(V) \subset \mathrm{GL}(V)$ is the group of linear isomorphisms preserving the inner product, i.e. $T : V \to V$ such that $\langle T\xi_1, T\xi_2 \rangle = \langle \xi_1, \xi_2 \rangle$ for all $\xi_1, \xi_2 \in V$. For $V = \mathbb{R}^n$, we let $\mathrm{O}_n := \mathrm{O}(\mathbb{R}^n)$.

Example 2.7. If $n = 1$, $\mathrm{O}_1 \subset \mathrm{GL}_1$ is $\{\pm 1\} \subset \mathbb{R}^\times$, so it's isomorphic to the cyclic group of order 2.

If $n = 2$, we can rotate by angles θ or reflect across lines, and playing with an orthonormal basis shows that all elements of O_2 must be rotations or reflections. Since O_2 is a Lie group, we can draw a picture as in Figure 2.

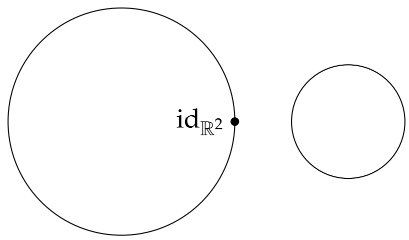

FIGURE 2. A picture of O_2. The left circle is the rotations; the right circle is the reflections, which in a sense form a circle half as long.

As with the affine symmetries, there's an extension

$$1 \longrightarrow \mathrm{SO}_2 \longrightarrow \mathrm{O}_2 \longrightarrow \{\pm 1\} \longrightarrow 1. \qquad \blacktriangleleft$$

Similarly, the isomorphisms of Euclidean space E, denoted $\mathrm{Euc}(E)$, are the affine isomorphisms preserving the inner product at each point. This again fits into an extension sequence

$$1 \longrightarrow V \longrightarrow \mathrm{Euc}(E) \longrightarrow \mathrm{O}(V) \longrightarrow 1.$$

All this is nice, but let's talk about manifolds. If X is a smooth manifold, we no longer have translations, and the linear symmetries talk about the tangent space. We'll see what kind of structures we get in this case.

The analogue of the torsor of bases is $\mathscr{B}(X) := \{(x, \beta) : x \in X, \beta : \mathbb{R}^n \xrightarrow{\sim} T_xX\}$. This admits a right action of $\mathrm{GL}_n(\mathbb{R})$ by precomposition, as on a vector space, and there is again a forgetful map $\pi : \mathscr{B}(X) \to X$ that ignores the basis.

If $x : U \to \mathbb{A}^n$ is a chart, then it defines a local section $U \to \mathscr{B}(X)$ sending

$$(x^1, \dots, x^n) \longmapsto \left((x^1, \dots, x^n), \left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right) \right).$$

If X is a Riemannian manifold, then we can also speak of orthonormal bases:

$$\mathscr{B}_{\mathrm{O}}(X) := \{(x, \beta) : x \in X, \beta : \mathbb{R}^n \xrightarrow{\cong} T_xX \text{ is an isometry}\}.$$

Again there is a forgetful map to X, but now a coordinate does *not* always determine a section: if the Riemann curvature tensor doesn't vanish, the image of an orthonormal basis of the tangent space at a point might not be orthonormal.

$\mathscr{B}(X)$ and $\mathscr{B}_{\mathrm{O}}(X)$ are not just sets but smooth manifolds, and the forgetful maps back to X are called fiber bundles (even principal bundles). We'll go back and discuss this in more detail.

Curvature. Let's end with something concrete. Let E be a Euclidean plane, an affine space with an underlying 2-dimensional inner product space.

Let $C \subset E$ be a 1-dimensional submanifold. Let's choose a *co-orientation* of C: an orientation of C is an orientation of its tangent bundle, so a co-orientation is an orientation of its normal bundle. In essence, this is choosing a side of the curve.[2] We'll use this to define a function $\kappa : C \to \mathbb{R}$ called the *(signed) curvature*. Intuitively, this should be positive if C is curved towards the side chosen by the co-orientation, and negative if it curves away, and a larger magnitude means a stronger curvature.

The Euclidean structure on E induces an inner product structure on T_xC for all $x \in C$ that varies smoothly, so C becomes a Riemannian manifold. Theorem 1.16 means there's nothing intrinsic about C we can measure, but the way in which it sits inside E is what κ will measure. This is an important dichotomy, between intrinsic geometry and extrinsic geometry. The Riemann curvature tensor is intrinsic, since it doesn't depend on an embedding, but the signed curvature will be extrinsic.

[2]If $N \hookrightarrow M$ is an embedding and M is oriented, an orientation of N and a co-orientation of N determine each other.