MSRI: QUANTUM SYMMETRIES INTRODUCTORY WORKSHOP

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These notes were taken at MSRI's introductory workshop on quantum symmetries in Spring 2020. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to `a.debray@math.utexas.edu`.

1. Sarah Witherspoon: Hopf algebras, I

Our perspective on Hopf algebras, their actions on rings and modules, and the structures on their categories of rings and modules, will be to think of them as generalizations of group actions and representations; groups actions are symmetries in the usual sense, and Hopf algebra actions are often related to "quantum symmetries."

We're not going to give the full definition of a Hopf algebra, because it would require drawing a lot of commutative diagrams, but we'll say enough to give the picture.

Throughout this talk we work over a field k; all tensor products are of k-vector spaces.

Definition 1.1. A *Hopf algebra* is an algebra A together with k-linear maps $\Delta\colon A \to A \otimes A$, called *comultiplication*; $\varepsilon\colon A \to k$, called the *counit*; and $S\colon A \to A$, called the *coinverse*. These maps must satisfy some properties, including that ε is an algebra homomorphism and that S is an *anti-automorphism*, i.e. that $S(xy) = S(y)S(x)$.

The definition is best understood through examples.

Example 1.2.

(1) Let G be a group. Then the group algebra $k[G]$ is a Hopf algebra, where for all $g \in G$, $\Delta(g) \coloneqq g \otimes g$, $\varepsilon(g) \coloneqq 1$, and $S(g) \coloneqq g^{-1}$. This is a key example that allows us to generalize ideas from group actions to Hopf algebra actions: whenever we define a notion for Hopf algebras, when we implement it for $k[G]$ it should recover that notion for groups.

(2) Let $\mathfrak{g}$ be a Lie algebra over k. Then its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is a Hopf algebra, where for all $x \in \mathfrak{g}$, $\Delta(x) \coloneqq x \otimes 1 + 1 \otimes x$, $\varepsilon(x) \coloneqq 0$, and $S(x) \coloneqq -x$. Since ε is an algebra homomorphism, $\varepsilon(1_{\mathcal{U}(\mathfrak{g})}) = 1$.

For example,

$$\mathcal{U}(\mathfrak{sl}_2) = k\langle e, f, h \mid ef - fe = h, he - eh = 2e, hf - fh = -2f\rangle, \tag{1.3}$$

given explicitly by the basis of $\mathfrak{sl}_2$

$$e \coloneqq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f \coloneqq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h \coloneqq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.4}$$ ◀

Both of these examples are classical, in that they've been known for a long time. But more recently, in the 1980s, people discovered new examples, coming from quantum groups.

Example 1.5 (Quantum $\mathfrak{sl}_2$)**.** Let $q \in k^\times \setminus \{\pm 1\}$. Then, given a simple Lie algebra $\mathfrak{g}$, we can define a "quantum group," $\mathcal{U}_q(\mathfrak{g})$, which is a Hopf algebra. For example, for $\mathfrak{sl}_2$,

$$\mathcal{U}_q(\mathfrak{sl}_2) = k\left\langle E, F, K^{\pm 1} \mid EF - FE = \frac{K - K^{-1}}{q - q^{-1}}, KE = q^2 EK, KF = q^{-2} EK \right\rangle, \tag{1.6}$$

with comultiplication

$$\Delta(E) \coloneqq E \otimes 1 + K \otimes E \tag{1.7a}$$

$$\Delta(F) \coloneqq F \otimes K^{-1} + 1 \otimes F \tag{1.7b}$$

$$\Delta(K^{\pm 1}) \coloneqq K^{\pm 1} \otimes K^{\pm 1} \tag{1.7c}$$

and counit $\varepsilon(E) = \varepsilon(F) = 0$ and $\varepsilon(K) = 1$. This generalizes to other simple $\mathfrak{g}$, albeit with more elaborate data. ◀

Example 1.8 (Small quantum $\mathfrak{sl}_2$)**.** Let q be an n^{th} root of unity. Then, as before, given a simple Lie algebra $\mathfrak{g}$, we can define a Hopf algebra $u_q(\mathfrak{g})$, called the *small quantum group* for $\mathfrak{g}$ and q, which is a finite-dimensional vector space over k; for $\mathfrak{sl}_2$, this is

$$u_q(\mathfrak{sl}_2) = \mathcal{U}_q(\mathfrak{sl}_2)/(E^n, F^n, K^n - 1). \tag{1.9}$$ ◀

Before we continue, we need some useful notation for comultiplication, called *Sweedler notation.* Let A be a Hopf algebra and $a \in A$; then we can symbolically write

$$\Delta(a) = \sum_{(a)} a_1 \otimes a_2. \tag{1.10}$$

Comultiplication in a Hopf algebra is *coassociative*, in that as maps $A \to A \otimes A \otimes A$,

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta. \tag{1.11}$$

Therefore when we iterate comultiplication, we can symbolically write

$$(\mathrm{id} \otimes \Delta) \circ \Delta(a) = \sum_{(a)} a_1 \otimes a_2 \otimes a_3 \tag{1.12}$$

without worrying about parentheses.

Actions on rings. Hopf algebra actions on rings generalize group actions on rings by automorphisms and actions of Lie algebras on rings by derivations. If a group G acts on a ring R, then for all $g \in G$ and $r, r' \in R$,

$$g(rr') = (gr)(gr') \tag{1.13a}$$

$$g(1_R) = 1_R. \tag{1.13b}$$

In $k[G]$, our Hopf algebra avatar of G, $\Delta(g) = g \otimes g$, and $\varepsilon(g) = 1$.

If a Lie algebra $\mathfrak{g}$ acts on a ring R by derivations, then for all $x \in \mathfrak{g}$ and $r, r' \in R$,

$$x \cdot (rr') = (x \cdot r)r' + r(x \cdot r') \tag{1.14a}$$

$$x \cdot (1_R) = 0. \tag{1.14b}$$

In $\mathcal{U}(\mathfrak{g})$, our Hopf algebra avatar of $\mathfrak{g}$, $\Delta(x) = x \otimes 1 + 1 \otimes x$, and $\varepsilon(x) = 0$. These two examples suggest how we should implement a general Hopf algebra action on a ring: comultiplication tells us how to act on the product of two elements, and the counit tells us how to act on 1.

Definition 1.15. Let A be a Hopf algebra and R be a k-algebra. An *A-module algebra structure* on R is data of an A-module structure on R such that for all $a \in A$ and $r, r' \in R$,

$$a \cdot (rr') = \sum_{(a)} (a_1 \cdot r)(a_2 \cdot r) \tag{1.16a}$$

$$a \cdot (1_R) = \varepsilon(a) 1_R. \tag{1.16b}$$

Thus a group action as in (1.13) defines an action of the Hopf algebra $k[G]$, and a Lie algebra action as in (1.14) defines an action of the Hopf algebra $\mathcal{U}(\mathfrak{g})$.

Example 1.17. The quantum analogue of the $\mathfrak{sl}_2$-action on $k[x, y]$, thought of as (functions on the) plane, there is an action of $\mathcal{U}_q(\mathfrak{sl}_2)$ on the *quantum plane*

$$R \coloneqq k\langle x, y \mid xy = qyx\rangle. \tag{1.18}$$

This is a deformation of $k[x, y]$, which is the case $q = 1$. The explicit data of the action is

$$E \cdot x = 0 \qquad F \cdot x = y \qquad K^{\pm 1} \cdot x = q^{\pm 1} x \tag{1.19}$$

$$E \cdot y = x \qquad F \cdot y = 0 \qquad K^{\pm 1} y = q^{\mp 1} y. \tag{1.20}$$

One has to check that this extends to an action satisfying Definition 1.15, but it does, and R is an A-module algebra. Here E and F act as *skew-derivations*, e.g.

$$E \cdot (rr') = (E \cdot r)r' + (K \cdot r)(E \cdot r') \tag{1.21}$$

for all $r, r' \in R$. ◂

Given a Hopf algebra action of A on R in this sense, we can construct two useful rings: the *invariant subring*

$$R^A := \{r \in R \mid a \cdot r = \varepsilon(a) \cdot r \text{ for all } a \in A\}, \tag{1.22}$$

and the *smash product ring* $R \# A$, which as a vector space is $R \otimes A$, with multiplication given by

$$(r \otimes a)(r' \otimes a') := \sum_{(a)} r(a_1 \cdot r') \otimes a_2 a'. \tag{1.23}$$

The smash product ring knows the A-module algebra structure on R. Often, rings we're interested in for other reasons are smash product rings of interesting Hopf algebra actions, and identifying this structure is useful.

Example 1.24. The *Borel subalgebra* of $\mathcal{U}_q(\mathfrak{sl}_2)$ is $k\langle E, K^{\pm 1} \mid KE = q^{-2}K\rangle$. This is isomorphic to the smash product $k[E] \# k\langle K\rangle$, where $k\langle K\rangle$ is the group algebra of the free group on the single generator K.

In fact, there's a sense in which $\mathcal{U}_q(\mathfrak{sl}_2)$ is a deformation of $k[E,F] \# k\langle K\rangle$: in this smash product ring, E and F commute, and we deform this to $\mathcal{U}_q(\mathfrak{sl}_2)$, in which they don't commute. ◂

Modules. Given a Hopf algebra A, what is the structure of its category of modules? The first thing we can do is take the tensor product of A-modules U and V using comultiplication: for $a \in A$, $u \in U$, and $v \in V$,

$$a \cdot (u \otimes v) = \sum_{(a)} a_1 \cdot u \otimes a_2 \cdot v. \tag{1.25}$$

Moreover, k has a canonical A-module structure via the counit: $a \cdot x := \varepsilon(a)x$ for $a \in A$ and $x \in k$. Finally, if U is an A-module, its vector space dual $U^* := \mathrm{Hom}_k(U, k)$ has an A-module structure via S: for all $a \in A$, $u \in U$, and $f \in U^*$, $(a \cdot f)(u) := f(S(a)u)$.

The existence of tensor products, duals, and the ground field in the world of Hopf algebra modules is a nice feature: these aren't always present for a general associative algebra. Moreover, these constructions interact well with each other.

(1) Coassociativity of Δ implies the tensor product is associative: for A-modules U, V, and W, we have a natural isomorphism $U \otimes (V \otimes W) \overset{\cong}{\to} (U \otimes V) \otimes W$.

(2) In any Hopf algebra A, we have the condition

$$\sum_{(a)} \varepsilon(a_1)a_2 = \sum_{(a)} a_1\varepsilon(a_2) \tag{1.26}$$

for any $a_1, a_2 \in A$. This implies k, as an A-module, is the unit for the tensor product: we have natural isomorphisms $k \otimes U \cong U \cong U \otimes k$ for an A-module U.

(3) Suppose U is an A-module which is a finite-dimensional k-vector space. Then it comes with data of a *coevaluation map* $c\colon k \to U \otimes U^*$ sending

$$1 \longmapsto \sum_i u_i \otimes u_i^*, \tag{1.27}$$

where $\{u_i\}$ is a basis for U over k and $\{u_i^*\}$ is its dual basis; this map turns out to be independent of basis. We also have an *evaluation map* $e\colon U^* \otimes U \to k$ sending $f \otimes u \mapsto f(u)$. Now, not only are these A-module homomorphisms, but the composition

$$U \xrightarrow{c \otimes \mathrm{id}_U} U \otimes U^* \otimes U \xrightarrow{\mathrm{id}_U \otimes e} U \tag{1.28}$$

is the identity map.

Definition 1.29. A *tensor category*, or *monoidal category* is a category $\mathcal{C}$ together with a functor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an object $\mathbf{1} \in \mathcal{C}$ called the *unit*, and natural isomorphisms $U \otimes (V \otimes W) \cong (U \otimes V) \otimes W$ and $\mathbf{1} \otimes U \cong U \cong U \otimes \mathbf{1}$ for all objects U, V, and W in $\mathcal{C}$, subject to some coherence conditions.

Our key examples of tensor categories are the category of modules over a Hopf algebra A, as well as the subcategory of finite-dimensional modules.

If the coinverse of A is invertible, which is always the case when A is finite-dimensional over k, then $\mathcal{C} = \mathcal{M}od_A$ is a *rigid* tensor category, meaning that every object U has a *right dual* ${}^*U := \mathrm{Hom}_k(U, k)$, which means the composition (1.28) is the identity.

Remark 1.30. Notations for left and right duals differ. We're following [EGNO15], but Bakalov-Kirillov [BK01] use a different convention; be careful! ◀

Some Hopf algebras' categories of modules have additional structure or properties: they might be semisimple, or braided, or even symmetric. This amounts to additional information on the Hopf algebra itself.

2. Victor Ostrik, Introduction to fusion categories, I

In the world of classical symmetries, i.e. those given by group actions, there is a particularly nice subclass: finite groups. If you know your symmetry group is finite, you can take advantage of many simplifying assumptions. Likewise, in the setting of quantum symmetries, given by, say, $\mathbb{C}$-linear tensor categories, fusion subcategories form a very nice subclass for which many simplifying assumptions hold. And indeed, if G is a finite group, its category of finite-dimensional representations is a fusion category.

Recall that a *monoidal category* is a category $\mathcal{C}$ together with a functor $\otimes\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ and a distinguished object $\mathbf{1}\in\mathcal{C}$ called the *unit*, together with natural isomorphisms implementing associativity of $\otimes$, via $(X\otimes Y)\otimes Z \overset{\cong}{\to} X\otimes(Y\otimes Z)$; and unitality of $\mathbf{1}$, via $\mathbf{1}\otimes X \overset{\cong}{\to} X \overset{\cong}{\to} X\otimes\mathbf{1}$. These must satisfy some axioms which we won't discuss in detail here; the most important one is the *pentagon axiom* on the associator.

Today, we work over an algebraically closed field k, not necessarily closed. Recall that a *k-linear category* $\mathcal{C}$ is one for which for all objects $x, y\in\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is a k-vector space, such that composition is bilinear. A *k-linear monoidal category* is a monoidal category that is also a k-linear category — and we also impose the consistency condition that the tensor product is a k-linear functor. we will impose a few more niceness conditions before arriving at the definition of a fusion category — in fact, as many as we can such that we still have examples!

In particular, we will only consider k-linear monoidal categories $\mathcal{C}$ such that

- all Hom-spaces are finite-dimensional over k,
- $\mathcal{C}$ is semisimple,[1]
- $\mathcal{C}$ has only finitely many isomorphism classes of simple objects,
- $\mathbf{1}$ is indecomposable, and
- $\mathcal{C}$ is *rigid*, a condition on duals of objects.

A category satisfying all of these axioms is a *fusion category*. (TODO: double-check)

There are three ways we can come to an understanding of these categories: through the definition, through realizations and examples, and through diagrammatics. We will also heavily use semisimplicity, through the principle that *k-linear functors out of $\mathcal{C}$ are determined by their values on simple objects, and all choices are allowed.*

Example 2.1. Our running example is $\mathcal{V}ec^{\omega}_{\mathbb{Z}/n}$, where n is a natural number and ω is a degree-3 cocycle for $\mathbb{Z}/n$, valued in $k^\times$.

The objects of $\mathcal{V}ec^{\omega}_{\mathbb{Z}/n}$ are the elements of $\mathbb{Z}/n$, with the tensor product $i\otimes j := i+j$. If $\omega = 1$, then we use the obvious associator, i.e. the isomorphism

$$(i\otimes j)\otimes k \xrightarrow{\cong} i\otimes(j\otimes k) \tag{2.2}$$

which corresponds to the identity under the identifications with $i+j+k$.[2] But in general, we can do something different: choose the map (2.2) which is $\omega(i, j, k)$ times the standard one.

A priori you can use any function $\mathbb{Z}/n\times\mathbb{Z}/n\times\mathbb{Z}/n\to k^\times$, but the pentagon axiom on associativity imposes the condition that ω is a cocycle.

[1] A k-linear category is *semisimple* if it's equivalent to the category of modules over $k\oplus\cdots\oplus k$, where there is a finite number of summands.

[2] These multiplication rules are really special, in that we were able to just write down an associator. This is generally not true; for general multiplication rules you're interested in, you'll have to work a little harder.

Exercise 2.3. If you have not seen this before, verify that the pentagon axiom forces $\partial\omega = 1$.

The simplest nontrivial example[3] is for $n = 2$ and

$$\omega(i, j, k) := \begin{cases} 1, & \text{if } i = 0,\ j = 0, \text{ or } k = 0 \\ -1, & \text{otherwise.} \end{cases} \tag{2.4}$$

$\mathbb{Z}/n$ was not special here — given any finite group G and a cocycle $\omega \in Z^3(G; k^\times)$, we obtain a fusion category $\mathcal{V}ec_G^\omega$ in the same way.

With ω as in (2.4), $\mathcal{V}ec_{\mathbb{Z}/2}^\omega$ looks like a new example, not equivalent to $\mathcal{V}ec_G^0$ for any G — but in order to understand that precisely, we need to discuss when two tensor categories are equivalent.

Definition 2.5. A *tensor equivalence* of tensor categories $\mathcal{C}$ and $\mathcal{D}$ is a monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$, i.e. a functor together with data of natural isomorphisms $F(X \otimes Y) \xrightarrow{\cong} F(X) \otimes F(Y)$ satisfying some axioms.

Choose cocycles ω and ω' for $\mathbb{Z}/n$, and let's consider tensor functors $F\colon \mathcal{V}ec_{\mathbb{Z}/n}^\omega \to \mathcal{V}ec_{\mathbb{Z}/n}^{\omega'}$. Furthermore, let's assume F is the identity on objects, so the data of F is the natural isomorphism $F(X \otimes Y) \cong F(X) \otimes F(Y)$. This is a choice of an element of $k^\times$ for every pair of objects, subject to some additional conditions:

Proposition 2.6. *F is a tensor functor iff $\omega = \omega' \cdot \partial\psi$.*

Corollary 2.7. *$\mathcal{V}ec_{\mathbb{Z}/n}^\omega \simeq \mathcal{V}ec_{\mathbb{Z}/n}^{\omega'}$ if ω and ω' are cohomologous.*

Recall that $H^3(\mathbb{Z}/n; k^\times) \cong \mathbb{Z}/n$, so we have n possibilities, some of which might coincide. If F isn't the identity on objects, it's fairly easy to see that as a function on objects, identified with a function $\mathbb{Z}/n \to \mathbb{Z}/n$, we must get a group homomorphism; if F is to be an equivalence, this homomorphism must be an isomorphism. One can run a similar argument as above and obtain a nice classification result.

Proposition 2.8. *The tensor equivalence classes of tensor categories $\mathcal{V}ec_{\mathbb{Z}/n}^\omega$ are in bijection with the orbits $H^3(\mathbb{Z}/n; k^\times)/\operatorname{Aut}(\mathbb{Z}/n)$, via the map sending ω to its class in cohomology.*

The action of $\operatorname{Aut}(\mathbb{Z}/n) = (\mathbb{Z}/n)^\times$ on $H^3(\mathbb{Z}/n; k^\times) \cong \mathbb{Z}/n$ is not the first action you might write down! Given $a \in (\mathbb{Z}/n)^\times$ and $s \in H^3(\mathbb{Z}/n; k^\times)$, the action is

$$a \cdot s = a^2 s. \tag{2.9}$$

This is a standard fact from group cohomology.

Now let's discuss some realizations of fusion categories. If H is a semisimple Hopf algebra, then $\mathcal{C} := \mathcal{R}ep_H^{fd}$ is a fusion category. Let $F\colon \mathcal{C} \to \mathcal{V}ec$ denote the forgetful functor to finite-dimensional vector spaces. It turns out that one can reconstruct $\mathcal{C}$ as a fusion category from F, and in fact any fusion category $\mathcal{C}$ with a tensor functor to $\mathcal{V}ec$ is equivalent to $\mathcal{R}ep_H^{fd}$ for some Hopf algebra H. The data of the tensor functor to $\mathcal{V}ec$ is crucial!

Example 2.10. For example, $\mathcal{V}ec_{\mathbb{Z}/n} \simeq \mathcal{R}ep_{\mathbb{Z}/n}^{fd}$; we saw in the previous lecture that representations of $\mathbb{Z}/n$ are equivalent to modules over the Hopf algebra $k[\mathbb{Z}/n] := k[x]/(x^n - 1)$, with comultiplication $\Delta(x) := x \otimes x$.

However, if ω is nontrivial, $\mathcal{V}ec_{\mathbb{Z}/n}^\omega$ admits no tensor functor to $\mathcal{V}ec$, and therefore cannot be seen using Hopf algebras. One can try to generalize the reconstruction program, using quasi-Hopf algebras, weak Hopf algebras, etc. ◀

Bimodules provide another approach to realizations: we look for a ring R and a tensor functor $F\colon \mathcal{C} \to \mathcal{B}imod_R$. Applying this to $\mathcal{V}ec_{\mathbb{Z}/n}^\omega$, we get (R, R)-bimodules $F(i)$ for each $i \in \mathbb{Z}/n$ and isomorphisms $F(i) \otimes_R F(j) \xrightarrow{\cong} F(i + j)$. In particular, each $F(i)$ is (tensor-)invertible.

Example 2.11. An *inner automorphism* of a ring R is conjugation by some $r \in R^\times$. Inner automorphisms form a normal subgroup of $\operatorname{Aut}(R)$, and the quotient is called the *outer automorphism group* of R and denoted $\operatorname{Out}(R)$. An *outer action* of a group G on a ring R is a group homomorphism $\varphi\colon G \to \operatorname{Out}(R)$.

Given an outer automorphism θ of R, one obtains an (R, R)-bimodule R_θ, whose left action is the R-action on R by left multiplication, and whose right action is $r \cdot x = r\theta(x)$. We need to choose an element in $\operatorname{Aut}(R)$ mapping to θ to make this definition, but different choices lead to isomorphic bimodules.

[3] This is nontrivial provided $\operatorname{char}(k) \neq 2$.

Anyways, given an outer action of $\mathbb{Z}/n$ on R, we obtain (R,R)-bimodules $R_{\varphi(i)}$ indexed by the objects $i \in \mathcal{V}ec_{\mathbb{Z}/n}$ and isomorphisms between $R_{\varphi(i)} \otimes R_{\varphi(j)} \xrightarrow{\cong} R_{\varphi(i+j)}$. This data stitches together into a tensor functor $\mathcal{V}ec_{\mathbb{Z}/n} \to \mathcal{B}imod_R$. ◂

Diagrammatics represents the objects of a fusion category $\mathcal{C}$ as points, and morphisms as lines. One can then impose relations on certain morphisms, and therefore diagrammatics provide a generators-and-relations approach to the structure of a given fusion category. Next time, we'll see how to do this for $\mathcal{V}ec^{\omega}_{\mathbb{Z}/n}$, and see more examples.

3. Eric Rowell, An introduction to modular tensor categories I

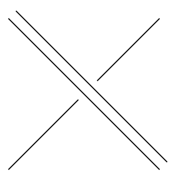
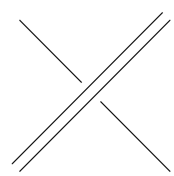
In this lecture, we'll begin with definitions and basic examples of modular tensor categories, and then use them in the next lecture. But first, let's discuss the whys of modular tensor categories.

We're often interested in knot and link invariants which are pictorial in nature, e.g. computed using a diagram. Another seemingly unrelated application is to study statistical-mechanical systems. Witten introduced TQFT into this story, extending the Jones polynomial to 3-manifold invariants using physics. Lately, there are interesting condensed-matter pheomena in topological phases. All of these are goverened by modular tensor categories in different ways, and in related ones.

(TODO: list of references, via handout)

Definition 3.1. Let $\mathcal{C}$ be a fusion category. A *braiding* on $\mathcal{C}$ (after which it's called a *braided fusion category*) is data of a natural transformation $c_{X,Y} \colon X \otimes Y \xrightarrow{\cong} Y \otimes X$ satisfying some relations called the *hexagon identities*.

You can think of $c_{X,Y}$ as taking strands labeled by the objects X and Y, and laying the X strand over the Y strand. The hexagon identities arise by comparing the two strands

(3.2)  and  .

Because the braiding is implemented via a natural transformation, it is functorial: we can braid morphisms as well as objects.

Example 3.3. Given a finite group G, $\mathcal{R}ep_G$ is a braided fusion category. Let V and W be representations; then the braiding $c_{V,W}(v \otimes w) := w \otimes v$. ◂

Definition 3.4. Let $\mathcal{C}$ be a braided fusion category. The *symmetric center* or *Müger center* of $\mathcal{C}$ is the subcategory $\mathcal{C}'$ of $x \in \mathcal{C}$ such that $c_{X,Y}c_{Y,X} = \mathrm{id}_X$ for all $Y \in \mathcal{C}$.

For example, the symmetric center of $\mathcal{R}ep_G$ is once again $\mathcal{R}ep_G$.

Exercise 3.5. Why is the symmetric center of $\mathcal{C}$ a braided fusion category? In particular, why is it closed under tensor products?

Definition 3.6. If the symmetric center of $\mathcal{C}$ is itself, we call $\mathcal{C}$ *symmetric*.[4] If the symmetric center of $\mathcal{C}$ is generated by the unit object (equivalently, $\mathcal{C}' \simeq \mathcal{V}ect$), we call $\mathcal{C}$ *nondegenerate*.

Here, "generated by the unit object" means every object is isomorphic to a direct sum of copies of the unit.

Now let's put some more adjectives in front of these structures. These will make the structure nicer, as usual, but are interesting enough to have examples.

Definition 3.7. Let $\mathcal{C}$ be a braided fusion category. A *twist* on $\mathcal{C}$ is a choice of $\theta \in \mathrm{Aut}(\mathrm{id}_{\mathcal{C}})$.

[4] Notice that being symmetric is a property of braided fusion categories.

Diagrammatically, we think of the twist as acting by the diagram in the first Reidemeister move, except we place right over left, not left over right. By looking at a picture of the twist on $X \otimes Y$, and untangling the picture, you can prove the *balancing equation*

$$\theta_{X\otimes Y} = c_{X,Y} \circ \theta_X \otimes \theta_Y. \tag{3.8}$$

Diagrams make it easier to picture these relations, but aren't strictly necessary. For example, the evaluation map $d_X : X^* \otimes X \to \mathbf{1}$ is represented by a diagram $\frown$ labeled by X, and coevaluation $b_X : \mathbf{1} \to X^* \otimes X$ is represented by a diagram $\smile$ labeled by X. Since braided categories aren't necessarily symmetric, one must be careful with left versus right duals.

Definition 3.9. A *ribbon structure* on a braided fusion category $\mathcal{C}$ is a twist such that $(\theta_X)^* = \theta_{X^*}$.

TODO: picture goes here. Here's where it's useful to use ribbon diagrams rather than string diagrams: really we want to keep track of the normal framings of the strings in our diagrams (thought of as embedded in $\mathbb{R}^3$), and ribbons provide a clean way to understand that.

Let $\mathrm{Irr}(\mathcal{C})$ denote the set of isomorphism classes of irreducible objects in $\mathcal{C}$. This is always a finite set; the *rank* of $\mathcal{C}$ is $\#\mathrm{Irr}(\mathcal{C})$. Choose representatives $x_1, \dots, x_r$ of the isomorphism classes of simple objects; then, by Schur's lemma, $\mathrm{Aut}(X_i) \cong \mathbb{C}^\times$. Let $\theta_i \in \mathbb{C}^\times$ denote the twist of X_i.

Now we have all the words we need to define modular tensor categories.

Definition 3.10. A *modular tensor category* is a nondegenerate ribbon fusion category.

There are other, equivalent definitions.

Definition 3.11. A *pivotal structure* on a fusion category $\mathcal{C}$ is a natural isomorphism $j : X \xrightarrow{\cong} X^{**}$.

If a pivotal structure satisfies a certain niceness condition, it's called *spherical*. Then:

- A braided fusion category with a pivotal structure automatically has a twist.
- If that pivotal structure is spherical, the twist defines a ribbon structure.
- A nondegenerate braided fusion category with a spherical structure is a modular tensor category.

This still hasn't quite made contact with the usual definition.

If $\mathcal{C}$ is a ribbon fusion category, it has a canonical trace on $\mathrm{End}(X)$, valued in $\mathrm{End}(\mathbf{1}) \cong \mathbb{C}$. The *dimension* of an object $X \in \mathcal{C}$ is $\mathrm{tr}(\mathrm{id}_X)$.

Definition 3.12. The *S-matrix* of a ribbon fusion category is the matrix with entries $S_{ij} := \mathrm{tr}(c_{X_i,X_j} \circ c_{X_j,X_i})$ for $X_i, X_j \in \mathrm{Irr}(\mathcal{C})$.

Theorem 3.13 (Brugières-Müger)**.** *A ribbon tensor category $\mathcal{C}$ is modular if and only if the S-matrix is invertible.*

Now let's turn to examples.

Example 3.14. Let G be a finite abelian group and $\mathcal{V}ec_G$ be the category of G-graded vector spaces. These were discussed previously in Example 2.1, albeit in a slightly different way.

Let $c : G \times G \to \mathbb{C}^\times$ be a *bicharacter* of G, i.e. for all $g, h, k \in G$,

$$c(gh, k) = c(g, k)c(h, k). \tag{3.15}$$

Then we obtain a braiding on $\mathcal{V}ec_G$ by $c : g \otimes h \to h \otimes g$ by

$$\theta_g(v \otimes w) = c(g, h) w \otimes v. \tag{3.16}$$

For the twist, use $\theta_g := c(g, g)$. This defines a ribbon tensor category, and it is modular iff $\det((c(g, h)c(h, g))_{g,h}) \neq 0$.

Exercise 3.17. In particular, let $G := \mathbb{Z}/3$ and w be a generator. Show that $c(w, w) = \exp(2\pi i/3)$ extends to a bicharacter that defines a modular tensor structure on $\mathcal{C} := \mathcal{V}ec_G$. Show that we cannot obtain a modular structure on $\mathcal{V}ec_{\mathbb{Z}/2}$ in this way, however.

We can produce a modular structure on $\mathcal{V}ec_{\mathbb{Z}/2}$ in a different way: let z be a generator, and define $c(z, z) := i$ and $c(1, z) = c(z, 1) = c(1, 1) = 1$. This defines a modular tensor category structure on $\mathcal{V}ec^{\omega}_{\mathbb{Z}/2}$ whenever ω is cohomologically nontrivial; this category is of considerable interest in physics, where it's known as the *semion category*. ◀

If you tried to generalize this to G nonabelian, you would not be able to write down a braiding, because $g \otimes h \ncong h \otimes g$.

If all simple objects in $\mathcal{C}$ are invertible, $\mathcal{C}$ is called a *pointed fusion category.* It turns out these have been classified, and the underlying monoidal tensor category id $\mathcal{V}ec_G^\omega$ for some finite group G and some cocycle ω. If in addition $\mathcal{C}$ is braided, then G is abelian, and we can ask about the converse.

Theorem 3.18. *If $|G|$ is odd, $\mathcal{V}ec_G^\omega$ admits a braiding iff ω is cohomologically trivial.*

When $|G|$ is even, things are more complicated, as we saw above, but the answers are known. For $\mathbb{Z}/2$, we can get $\mathcal{R}ep_{\mathbb{Z}/2}$, and for $c(z, z) = -1$, we obtain $s\mathcal{V}ec$. Both of these are symmetric. One can generalize: Deligne [Del02] classified symmetric fusion categories, showing they're all equivalent to $\mathcal{R}ep_G$ or $\mathcal{R}ep_G(z)$, where $z \in G$ is central and order 2 (giving a super-vector space structure on G-representations). Symmetric fusion categories equivalent to $\mathcal{R}ep_G$ are called *Tannakian*; those equivalent to $\mathcal{R}ep_G(z)$ are called *super-Tannakian.*

References

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