Riemannian Geometry

UT Austin, Spring 2017

M392C NOTES: RIEMANNIAN GEOMETRY

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These notes were taken in UT Austin's M392C (Riemannian Geometry) class in Spring 2017, taught by Dan Freed. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Any mistakes in the notes are my own. Thanks to Martin Bobb, Gill Grindstaff, Jonathan Johnson, and Sebastian Schulz for some corrections.

The cover image is the Cosmic Horseshoe (LRG 3-757), a gravitationally lensed system of two galaxies. Einstein's theory of general relativity, written in the language of Riemannian geometry, predicts that matter bends light, so if two galaxies are in the same line of sight from the Earth, the foreground galaxy's gravity should bend the background galaxy's light into a ring, as in the picture. The discovery of this and other gravitational lenses corroborates Einstein's theories. Source: https://apod.nasa.gov/apod/ap111221.html.

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"Do you have all these equations?"

Before we begin with Riemannian manifolds, it'll be useful to do a little geometry in flat space.

Definition 1.1. Let V be a real vector space; then, an *affine space over* V is a set A with a simply transitive right V-action.

That this action is simply transitive means for any $a, b \in A$, there's a unique $\xi \in V$ such that $a \cdot \xi = b$.

Definition 1.2. A set with a simply transitive (right) V-action is called a *(right) V-torsor*.

V-torsors look like copies of V without a distinguished identity.

One of the distinct features of affine space is *global parallelism*: if I have a vector ξ at a point a, I immediately get a vector at every point, which defines a vector field on the entire space.

What is the analogue of a basis for an affine space? This is a collection of points $a_0, \dots, a_n$ such that any $a \in A$ is uniquely written as

$$a = \lambda^0 a_0 + \lambda^1 a_1 + \cdots + \lambda^n a_n \tag{1.3}$$

for some $\lambda^i \in \mathbb{R}$ with $\lambda^0 + \cdots + \lambda^n = 1$.

Equation (1.3) may be written more concisely with *index notation*: any variable written as both a superscript and a subscript is implicitly summed over. That is, we may rewrite (1.3) as

$$a = \lambda^i a_i.$$

Note that in an affine space, we don't know how to add vectors (since we don't have an origin), but we can take weighted averages.

Theorem 1.4 (Giovanni Ceva, 1678)**.** *Let A be an affine plane and $a, b, c \in A$ be a triangle (i.e. three distinct, noncollinear points). Suppose $p \in \overline{bc}$, $q \in \overline{ca}$, and $r \in \overline{ca}$. Then, $\overline{ap}$, $\overline{bq}$, and $\overline{cr}$ are coincident iff*

$$[ar : rb][bp : pc][cq : ca] = 1.$$

Typically, this is thought of as a ratio of lengths, but we don't necessarily have lengths: instead, we can use barycentric coordinates. There is a unique λ such that if $r = (1-\lambda)a + \lambda b$, then $[ar : rb] = \lambda/(1-\lambda)$.

Proof. Let

$$\begin{aligned} r &:= (1-\lambda)a + \lambda b \\ p &:= (1-\mu)b + \mu c \\ q &:= (1-\nu)c + \nu a. \end{aligned}$$

Set

$$x := \alpha a + \beta b + \gamma c, \tag{1.5}$$

where $\alpha + \beta + \gamma = 1$. Since $x \in \overline{ap}$, then

$$x = \alpha a + C((1-\mu)b + \mu c). \tag{1.6}$$

Comparing (1.5) and (1.6), $\mu/(1-\mu) = \gamma/\beta$.

⊠

Standard affine space $\mathbb{A}^n := \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^i \in \mathbb{R}\}$. You may complain this is the same as $\mathbb{R}^n$, but $\mathbb{A}^n$ only comes with an affine structure, not a vector-space structure.

Definition 1.7. Let A be an affine space modeled on V and B be an affine space modeled on W. Then, a map $f : A \to B$ is *affine* if there exists a linear map $T : V \to W$ such that $f(a + \xi) = f(a) + T\xi$ for all $a \in A$ and $\xi \in V$.

In other words, an affine map is a linear map plus some constant, which is not uniquely defined.

Definition 1.8. An *affine coordinate system* on A is an affine isomorphism $x = (x^1, \dots, x^n) : A \to \mathbb{A}^n$.

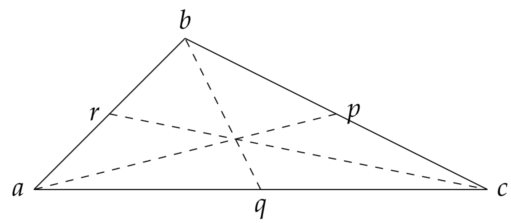

FIGURE 1. Depiction of Ceva's theorem (Theorem 1.4).

Them, the differentials $\mathrm{d}x_a^1, \dots, \mathrm{d}x_a^N$ are independent of basepoint a and form a basis for V^*, the dual vector space and dual basis to V and $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$, the tangent space to any $a \in A$.

But affine space is not the only flat geometry we could consider: more generally, we consider a structure on a vector space V which can be promoted to a translationally invariant structure on A. This leads to metric geometry, symplectic geometry, etc.

Definition 1.9. An *inner product* on a (finite-dimensional) vector space V is a bilinear map $\langle -,- \rangle : V \times V \to \mathbb{R}$ which is symmetric and positive definite, i.e. for all $\xi, \eta \in V$, $\langle \xi, \eta \rangle = \langle \eta, \xi \rangle$, $\langle \xi, \xi \rangle \geq 0$, and $\langle \xi, \xi \rangle = 0$ iff $\xi = 0$.

Since $\langle -,- \rangle$ is bilinear, then this can be determined in terms of n^2 numbers: let $v_1, \dots, v_n$ be a basis for V and define $g_{ij} := \langle v_i, v_j \rangle$ for $i, j = 1, \dots, n$. Of course, these numbers areen't independent: $g_{ij} = g_{ji}$, so there are really only $n(n+1)$ choices of information.

Definition 1.10. A basis $e_1, \dots, e_n$ for V is *orthonormal* if

$$\langle e_i, e_j \rangle = \delta_{ij} := \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}.$$

Our first major result of flat Euclidean geometry is that these exist.

Theorem 1.11. *There exist orthonormal bases.*

Proof. Let $v_1, \dots, v_n$ be any basis of V. Let

$$e_1 = \frac{v_1}{\langle v_1, v_1 \rangle^{1/2}},$$

and for $i = 2, \dots, n$, let

$$v_i' = v_i - \langle v_i, e_1 \rangle e_1.$$

Then, $\langle e_1, e_1 \rangle = 1$ and $\langle e_1, v_i' \rangle = 0$. Then, repeat with $v_2', \dots, v_n'$. ⊠

This explicit algorithm is called the *Gram-Schmidt process.*

In an inner product space, we get some familiar geometric constructions: the *length* of a vector $\xi \in V$ is $|\xi| = \langle \xi, \xi \rangle^{1/2}$, and the *angle* between $\xi, \eta \in V \setminus 0$ is the θ such that

$$\cos\theta = \frac{\langle \xi, \eta \rangle}{|\xi||\eta|}.$$

Definition 1.12. A *Euclidean space* E is an affine space over an inner product space V.

This has a notion of distance: $d_E : E \times E \to \mathbb{R}^{\geq 0}$, where $a, b \mapsto |\xi|$, where $b = a + \xi$. This generalizes to notions of area, volume, etc.

Theorem 1.13 (Napoleon, 1820)**.** *Let abc be a triangle in a plane and attach an equilateral triangle to each edge. The centers of these three triangles form an equilateral triangle.*

Exercise 1.14. Prove this.

We want to understand curved analogues of this classical material, and will pick up where differential topology left off. We work on smooth manifolds: a *smooth manifold* is a space X together with an atlas of charts $U \subset X$ with homeomorphisms $x : U \to \mathbb{A}^n$ such that every point is contained in the domain of some chart and the transition maps are smooth. We do not require a manifold to have a global dimension: the different connected components may have different dimensions, e.g. $S^1 \amalg S^2$.[1]

A chart map $x : U \to \mathbb{A}^n$ is a set of n continuous maps $(x^1, \dots, x^n)$. If p is in the domain of both x and y, we can consider $x \circ y^{-1} : \mathbb{A}^n \to \mathbb{A}^n$; calculus as usual tells us what it means for this transition map to be smooth.

At any $x \in X$, we have a tangent space $T_x X$ and a cotangent space $T_x X$: a chart defines a basis of the tangent space $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ and a basis of the contangent space $\mathrm{d}x^1_x, \dots, \mathrm{d}x^n_x$. This depends strongly on x: unlike for flat space, we may not be able to parallel-transport these globally, even on something as simple as S^2.

In this course, we will study what happens when we go from a curved analogue of affine space to a curved analogue of Euclidean space, whence the following central definition.

Definition 1.15. A *Riemannian metric* on a smooth manifold X is a choice of inner product $\langle -, - \rangle_x$ on $T_x X$ for all $x \in X$ which varies smoothly in x.

Now, we can compute lengths of tangent vectors and the angle that two smooth curves intersect at (or rather, the angle their tangent vectors intersct at). We also obtain a notion of distance between points, and can develop analogues of Euclidean geometry on manifolds.

What does "varying smoothly" mean, exactly? Suppose $x^1, \dots, x^n$ is a set of local coordinates on $U \subset X$; then, for $i, j = 1, \dots, n$, define

$$g_{ij} \coloneqq \left\langle \left.\frac{\partial}{\partial x^i}\right|_x, \left.\frac{\partial}{\partial x^j}\right|_x \right\rangle_{T_x X}.$$

One can check that if the metric is smoothly varying in one chart, then it's smoothly varying in all charts.

We'll write the metric as

$$g = g_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j.$$

This again uses the summation convention, and it's useful to think about where exactly this lives: it identifies the metric as a tensor.

Many manifolds arise as embedded submanifolds of Euclidean space, and the Whitney embedding theorem shows that all may be emnbedded. Many authors say it's best to meet manifolds as embedded submanifolds first, but there are some which arise without a natural embedding, e.g. the Grassmanian $\mathrm{Gr}_2(\mathbb{R}^4)$, the space of two-dimensional subspaces of $\mathbb{R}^4$.

In any case, if $X \subset \mathbb{E}^N$ is embedded, then X inherits a metric, since $T_x X \subset \mathbb{R}^n$ is also a subspace, and we can restrict the inner product. Classical Riemannian geometry is the study of *plane curves* (one-dimensional submanifolds of $\mathbb{R}^2$), *space curves* (one-dimensional submanifolds of $\mathbb{R}^3$), and *surfaces* (two-dimensional submanifolds of $\mathbb{R}^3$).

To study Riemannian manifolds, we should begin with the simplest cases. The zero-dimensional manifolds are disjoint unions of points with zero-dimensional tangent spaces and the trivial Riemannian metric. In the one-dimensional case, there is a little more to tell. A smooth map $X \to Y$ of Riemannian manifolds is an *isometry* if it's a map that preserves the inner product on each tangent space. This automatically implies it's injective.

Theorem 1.16. *Let C be a (complete) Riemannian 1-manifold which is diffeomorphic to $\mathbb{R}$. Then, C is isometric to $\mathbb{E}^1$.*

Before we prove this, we need a change-of-coordinates lemma. (We'll address completeness later, to avoid finite intervals.)

Remark 1.17. Let $x^1, \dots, x^n$ and $y^1, \dots, y^n$ be coordinate systems and suppose a metric can be written as

$$g = g_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j = h_{ab}\, \mathrm{d}y^a \otimes \mathrm{d}y^b.$$

[1] This is important for, e.g. a space of solutions of certain PDEs.

Then,

$$g_{ij} = h_{ab} \frac{\partial y^a}{\partial x^i} \frac{\partial y^b}{\partial x^j}. \tag{1.18}$$

This is n^2 equations: there is no implicit summation here. ◂

Proof of Theorem 1.16. Let $x : C \to \mathbb{R}$ be a diffeomorphism, which defines a global coordinate on C. Let $g(x) = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \rangle$. We seek a new coordinate $y : C \to \mathbb{R}$ such that $h(y) = \langle \frac{\partial}{\partial y}, \frac{\partial}{\partial y} \rangle = 1$ everywhere. By (1.18),

$$g = \left(\frac{\mathrm{d}y}{\mathrm{d}x} \right)^2, \tag{1.19}$$

so fix an $x_0 \in C$ and define

$$y(x) = \int_{x_0}^{x} \sqrt{g(t)}\, \mathrm{d}t.$$

This y satisfies (1.19) and therefore is an isometry. ⊠

The analogue to Theorem 1.16 in n dimensions (where $n > 1$) is as follows: if $x^1, \dots, x^n$ is a local coordinate system and g_{ij} is the Riemannian metric in these coordinates, is there a local change of coordinates $y^a(x^1, \dots, x^n)$ such that $h_{ab} = \delta_{ab}$? This is the analogue in Riemannian geometry to finding orthonormal coordinates, guaranteed by Theorem 1.11.

This requires solving an analogue to (1.18), but this time it's a PDE

$$g_{ij} = \sum_a \frac{\partial y^a}{\partial x^i} \frac{\partial y^a}{\partial x^j}.$$

This time, we need to ask whether there are solutions. The only thing we know how to do is differentiate:

$$\frac{\partial g_{ij}}{\partial x^k} = \sum_a \frac{\partial^2 y^a}{\partial x^k \partial x^i} \frac{\partial g^a}{\partial x^j} + \frac{\partial y^a}{\partial x^i} \frac{\partial^2 y^a}{\partial x^k \partial x^j}. \tag{1.20a}$$

By permuting indices, we obtain

$$\frac{\partial g_{ik}}{\partial x^j} = \sum_a \frac{\partial^2 y^a}{\partial x^j \partial x^i} \frac{\partial g^a}{\partial x^k} + \frac{\partial y^a}{\partial x^i} \frac{\partial^2 y^a}{\partial x^j \partial x^k} \tag{1.20b}$$

$$\frac{\partial g_{jk}}{\partial x^i} = \sum_a \frac{\partial^2 y^a}{\partial x^i \partial x^j} \frac{\partial g^a}{\partial x^k} + \frac{\partial y^a}{\partial x^j} \frac{\partial^2 y^a}{\partial x^i \partial x^k}. \tag{1.20c}$$

Taking (1.20a) + (1.20b) − (1.20c), we obtain

$$\frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right) = \sum_a \frac{\partial y^a}{\partial x^i} \frac{\partial^2 y^a}{\partial x^j \partial x^k}.$$

Now we multiply by $\frac{\partial y^b}{\partial x^\ell} g^{\ell i}$, concluding

$$\frac{\partial y^b}{\partial x^\ell} \underbrace{\frac{g^{\ell i}}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right)}_{\Gamma^\ell_{jk}} = \sum_a \frac{\partial y^a}{\partial x^i} \frac{\partial^2 y^a}{\partial x^j \partial x^k} g^{\ell i} \frac{\partial y^b}{\partial x^\ell}.$$

These Γ^ℓ_{jk} symbols therefore satisfy

$$\frac{\partial^2 y^b}{\partial x^j \partial y^k} = \Gamma^i_{jk} \frac{\partial y^b}{\partial x^i}.$$

If we differentiate once again (with respect to x^ℓ), we get

$$\begin{aligned}
\frac{\partial^3 y^b}{\partial x^\ell \partial x^j \partial x^k} &= \frac{\partial \Gamma^i_{jk}}{\partial x^\ell} \frac{\partial y^b}{\partial x^i} + \Gamma^i_{jk} \frac{\partial^2 y^b}{\partial x^\ell x^i} \\
&= \left(\frac{\partial \Gamma^i_{jk}}{\partial x^\ell} + \Gamma^m_{jk} \Gamma^i_{m\ell} \right) \frac{\partial y^b}{\partial x^i}.
\end{aligned}$$

Since mixed partials commute, then one discovers that if such an isometry exists, the *Riemannian curvature tensor*

$$R^i_{jk\ell} := \frac{\partial \Gamma^i_{j\ell}}{\partial x^\ell} - \frac{\partial \Gamma^i_{jk}}{\partial x^\ell} + \Gamma^m_{jk}\Gamma^i_{m\ell} - \Gamma^m_{j\ell}\Gamma^i_{mk} \tag{1.21}$$

must vanish. In simple cases, one can calculate that it's not always zero, so we don't always have global parallelism.

Riemann derived this in the middle of the 1800s. It's possible to see the glimmer of special relativity in them, though of course this was discovered later.

There's no text, though there is a website: http://www.ma.utexas.edu/users/dafr/M392C/index.html. There are problem sets, so undergraduates have to do some problem sets, and graduate students should. Feel free to talk to the professor about the problems, and especially to establish groups to work on the problem sets. Office hours are Wednesdays 2 to 3.

Lecture 2.

Existence of Riemannian metrics: 1/19/17

"There are so many of you... so quiet... I'll be more provocative until I get questions. Or I'll go faster."

Due to the large size of the class, it's being moved to RLM 6.104 starting next week. This means everyone who wants to sign up should be able to.

Some readings are up on the website, including a translation of Riemann's original work on curvature.

Last time, we defined affine space, which leads to the notion of a smooth manifold, and then introduced Euclidean space, an affine space over an inner product space. The curved version of that is a Riemannian manifold.

Recall that a Riemannian metric g on a smooth manifold X is a smoothly varying family of inner products on T_xX, and a Riemannian manifold is a smooth manifold together with a Riemannian metric. We also defined an isometry: if X and Y are Riemannian manifolds, then a diffeomorphism $f : X \to Y$ is an isometry if for all $x \in X$ and $\xi_1, \xi_2 \in T_xX$,

$$\langle f_*\xi_1, f_*\xi_2\rangle_{T_{f(x)}Y} = \langle \xi_1, \xi_2\rangle_{T_xX}.$$

Here, $f_* : T_xX \to T_{f(x)}Y$ is the linear pushforward of tangent vectors, also called the *differential*. If f is merely a smooth function, this is called an *isometric immersion* (the inverse function theorem automatically implies it's an immersion). If f is an embedding, this is called an *isometric embedding*.

Existence of Riemannian metrics. Suppose V is a real vector space and $g_0, g_1 : V \times V \to \mathbb{R}$ are inner products. Then for $t \in [0,1]$, $(1-t)g_0 + tg_1$ is also an inner product (you can check this directly).

The set of bilinear maps $V \times V \to \mathbb{R}$, denoted $\mathrm{Bil}(V \times V, \mathbb{R})$, is a real vector space, naturally isomorphic to $\mathrm{Hom}(V \otimes V, \mathbb{R})$ and to $V^* \otimes V^*$. Here, "natural" means this works for all finite-dimensional vector spaces at once, and commutes with linear maps.

Inner products are elements of this vector space, and our observation above means that if g_0 and g_1 are inner products, the line between them in $\mathrm{Bil}(V \times V, \mathbb{R})$ consists of inner products. In particular, *inner products form a convex set*. This only uses the affine structure on $\mathrm{Bil}(V \times V, \mathbb{R})$, since we can take convex combinations in an affine space.

This is used to generalize to the curved case, showing Riemannian metrics always exist.

Theorem 2.1. *Let X be a smooth manifold. Then, there is a Riemannian metric on X.*

Proof. Let $\mathfrak{U} = \{(U, x)\}$ be a cover of X by coordinate charts $x : U \to \mathbb{A}^n$, and let g_U denote the metric on U such that $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ are orthonormal. That is, take the standard metric on $\mathbb{A}^n$ making it into Euclidean space $\mathbb{E}^n$, and pull it back to U, where it becomes a metric (you can check that metrics pull back along closed immersions).

Now, the bases on two different charts in $\mathfrak{U}$ don't agree, and don't necessarily differ by orthonormal bases. Thus, we use a standard argument in differential geometry to globalize local objects living in a convex set: let $\{\rho_U\}_{U\in\mathfrak{U}}$ be a partition of unity subordinate to $\mathfrak{U}$; then,

$$g = \sum_{U\in\mathfrak{U}} \rho_U g_U$$

is a Riemannian metric. ⊠

Remark 2.2. Global existence is *not* assured for every geometric structure. For example, a *complex structure* on a real vector space V is an endomorphism $J : V \to V$ such that $J^2 = -\mathrm{id}_V$. This is akin to multiplication by i in a complex vector space, which squares to -1 and commutes with addition.

You can place this structure on affine space, and there's an immediate obstruction: $\dim_{\mathbb{R}} V$ must be even. Now we globalize: given an even-dimensional manifold, do we have such a structure? That is, can we place a smoothly varying complex structure on T_xX for all $x \in X$? This is called an *almost complex structure*, and not every even-dimensional manifold admits one.

Exercise 2.3. Show that S^4 has no almost complex structure.

There is an almost complex structure on S^6, and it's a famous open question whether there's a complex structure (i.e. complex coordinates with holomorphic transition functions). The known almost complex structure does not work.

Another local structure that doesn't automatically globalize is a mixed-signature metric (e.g. a Minkowski metric). In such a metric, the *null vectors*, those ξ for which $\langle \xi, \xi\rangle = 0$, form a cone whose interior is the *positive vectors* (for which the metric is positive). Trying to globalize this produces, more or less, a line in each tangent space T_xX. Passing to a double cover, one can choose an orientation, and therefore a nonzero vector field on X, and this can't be done in general. For example, a surface of genus 2 admits no metric of signature $(1, 1)$. These kinds of metrics arise in general relativity. ◀

In this class, we care about Riemannian metrics, which do globalize.

Let $x^1, \dots, x^n$ be local coordinates; then, we defined some local quantities in the metric in terms of these coordinates. Namely,

$$g_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle,$$

so that $g = g_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$. We then used this to define symbols Γ^i_{jk} and the Riemann curvature tensor $R^i_{jk\ell}$. We proved Theorem 1.16; here's a better version.

Theorem 2.4. *Let C be a Riemannian 1-manifold diffeomorphic to $\mathbb{R}$. Then, there exists an isometry $C \to I$, where $I \subset \mathbb{E}^1$ is an open interval.*

The argument we gave defining the Riemann curvature tensor generalizes this.

Theorem 2.5. *Suppose (U, g) is a Riemannian manifold and $x : U \to \mathbb{A}^n$ is a global coordinate such that*

$$g = \sum_{i=1}^n (\mathrm{d}x^i)^2.$$

Then, $R^i_{jk\ell} = 0$ on U.

One important thing to check here is that

$$R = R^i_{jk\ell} \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k \otimes \mathrm{d}x^\ell$$

is independent of the coordinate system (which is not clear from its definition). This means that the Riemann curvature tensor is a tensor, i.e. $R \in T_xX \otimes T_x^*X \otimes T_x^*X \otimes T_x^*X$. In the next few weeks, we will add some geometry to this discussion.

Example 2.6. Let $X = \mathbb{E}^2$ be Euclidean space with the standard metric g. Then, we have global coordinates $(x, y) : \mathbb{E}^2 \to \mathbb{A}^2$, so $g = \mathrm{d}x^2 + \mathrm{d}y^2$.

We can also introduce *polar coordinates*, another coordinate system which isn't global. This is a coordinate map $(r, \theta) : \mathbb{E}^2 \setminus \{(x, 0) : x \le 0\} \to \mathbb{A}^2$ (so $r > 0$, $-\pi < \theta < \pi$). In this case, the metric has the form

$$g = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2.$$

This means that the vector field $\frac{\partial}{\partial r}$ has constant length 1, but the vector field $\frac{\partial}{\partial \theta}$ has length r at (r, θ). ◀

Symmetry. We've now seen vector spaces, affine spaces, Euclidean spaces, and Riemannian manifolds. As in any mathematical context, it's important to ask what the proper notion of symmetry is for these objects.

If V is a vector space, its *general linear group* is $\mathrm{GL}(V) = \mathrm{Aut}(V) := \{T : V \to V \text{ invertible}\}$. The standard example is $\mathrm{GL}_n(\mathbb{R}) := \mathrm{GL}(\mathbb{R}^n)$, the group of invertible $n \times n$ matrices, acting on the column vectors of $\mathbb{R}^n$ by scalar multiplication. For example $\mathrm{GL}_1(\mathbb{R}) = \mathbb{R}^\times$, the group of nonzero numbers under multiplication.

What about affine space? Affine space on V is a V-torsor, as V acts by translation. The symmetry group is the group of *affine transformations*

$$\mathrm{Aff}(A) := \{\alpha : A \to A \mid \alpha \text{ is invertible and affine}\}.$$

Recall that an affine map is one that preserves the affine structure: the image of a finite weighed average is the weighted average of the images. The derivative of an affine map is a linear map, so if A is an affine space modeled by V, the derivative defines a group homomorphism $\mathrm{d} : \mathrm{Aff}(A) \to \mathrm{GL}(V)$, whose kernel is the translations, a group isomorphic to V. Thus, we have a *group extension* (short exact sequence of groups)

$$1 \longrightarrow V \longrightarrow \mathrm{Aff}(A) \xrightarrow{\mathrm{d}} \mathrm{GL}(V) \longrightarrow 1. \tag{2.7}$$

The key is that in affine space, there's no canonical origin. However, (2.7) splits, if noncanonically: choose an $a \in A$. Then, any $b \in A$ can be uniquely written as $a + \xi$ for some $\xi \in V$, so for any linear transformation T, $a + \xi \mapsto a + T\xi$ is an affine transformation of A.

(2.7) is a sequence of manifolds with smooth group homomorphisms, making it a short exact sequence of *Lie groups*; we'll discuss Lie groups more later.

If V is an n-dimensional vector space, its bases are the set $\mathscr{B}(V) = \{b : \mathbb{R}^n \xrightarrow{\cong} V\}$. If $V = \mathbb{R}^n$, this is $\mathrm{GL}_n(\mathbb{R})$. In general, this makes $\mathscr{B}(V)$ into a right $\mathrm{GL}_n(\mathbb{R})$-torsor, defined by the simply transitive action $\mathscr{B}(V) \times \mathrm{GL}_n(\mathbb{R}) \to \mathscr{B}(V)$ sending $\beta, g \mapsto \beta \circ g$. (There is a corresponding left action by $\mathrm{GL}(V)$). The action on the right is akin to numbering elements of the basis, and the action on the left is more geometric; this is an instance of a general idea that internal actions tend to be from the right, and geometric ones from the left.

What's the analogue for an affine space A modeled on V? Let $\mathscr{B}(A)$ denote the collection of pairs (a, β) where $a \in A$ and $\beta \in \mathscr{B}(V)$, identified with the set of affine isomorphisms $\alpha : A \xrightarrow{\cong} \mathbb{A}^n$. These are the bases at specific points of A. There is a forgetful map $\pi : \mathscr{B}(A) \to A$ sending $(a, \beta) \to a$, and the fiber is $\mathscr{B}(V)$, the bases at a. In a similar way, there is a left action of $\mathrm{Aff}(A)$ on $\mathscr{B}(A)$, and a right action of $\mathrm{Aff}_n := \mathrm{Aff}(\mathbb{A}^n)$ on $\mathscr{B}(A)$.

We'll use these torsors of bases a lot in this class. In this way, we're enacting Felix Klein's *Erlangen* program, where the kind of geometry we do is reflected by the symmetry group we place on the geometric structures.

Let's see what happens to these ideas in the Euclidean and Riemannian cases. If V is an inner product space, its *orthogonal group* $\mathrm{O}(V) \subset \mathrm{GL}(V)$ is the group of linear isomorphisms preserving the inner product, i.e. $T : V \to V$ such that $\langle T\xi_1, T\xi_2 \rangle = \langle \xi_1, \xi_2 \rangle$ for all $\xi_1, \xi_2 \in V$. For $V = \mathbb{R}^n$, we let $\mathrm{O}_n := \mathrm{O}(\mathbb{R}^n)$.

Example 2.8. If $n = 1$, $\mathrm{O}_1 \subset \mathrm{GL}_1$ is $\{\pm 1\} \subset \mathbb{R}^\times$, so it's isomorphic to the cyclic group of order 2.

If $n = 2$, we can rotate by angles θ or reflect across lines, and playing with an orthonormal basis shows that all elements of O_2 must be rotations or reflections. Since O_2 is a Lie group, we can draw a picture as in Figure 2.

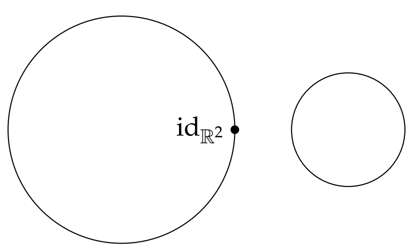

FIGURE 2. A picture of O_2. The left circle is the rotations; the right circle is the reflections, which in a sense form a circle half as long.

As with the affine symmetries, there's an extension

$$1 \longrightarrow SO_2 \longrightarrow O_2 \longrightarrow \{\pm 1\} \longrightarrow 1. \qquad \blacktriangleleft$$

Similarly, the isomorphisms of Euclidean space E, denoted $\mathrm{Euc}(E)$, are the affine isomorphisms preserving the inner product at each point. This again fits into an extension sequence

$$1 \longrightarrow V \longrightarrow \mathrm{Euc}(E) \longrightarrow \mathrm{O}(V) \longrightarrow 1.$$

All this is nice, but let's talk about manifolds. If X is a smooth manifold, we no longer have translations, and the linear symmetries talk about the tangent space. We'll see what kind of structures we get in this case.

The analogue of the torsor of bases is $\mathscr{B}(X) := \{(x,\beta) : x \in X, \beta : \mathbb{R}^n \xrightarrow{\sim} T_xX\}$. This admits a right action of $\mathrm{GL}_n(\mathbb{R})$ by precomposition, as on a vector space, and there is again a forgetful map $\pi : \mathscr{B}(X) \to X$ that ignores the basis.

If $x : U \to \mathbb{A}^n$ is a chart, then it defines a local section $U \to \mathscr{B}(X)$ sending

$$(x^1, \dots, x^n) \longmapsto \left((x^1, \dots, x^n), \left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right)\right).$$

If X is a Riemannian manifold, then we can also speak of orthonormal bases:

$$\mathscr{B}_{\mathrm{O}}(X) := \{(x,\beta) : x \in X, \beta : \mathbb{R}^n \xrightarrow{\cong} T_xX \text{ is an isometry}\}.$$

Again there is a forgetful map to X, but now a coordinate does *not* always determine a section: if the Riemann curvature tensor doesn't vanish, the image of an orthonormal basis of the tangent space at a point might not be orthonormal.

$\mathscr{B}(X)$ and $\mathscr{B}_{\mathrm{O}}(X)$ are not just sets but smooth manifolds, and the forgetful maps back to X are called fiber bundles (even principal bundles). We'll go back and discuss this in more detail.

Curvature. Let's end with something concrete. Let E be a Euclidean plane, an affine space with an underlying 2-dimensional inner product space.

Let $C \subset E$ be a 1-dimensional submanifold. Let's choose a *co-orientation* of C: an orientation of C is an orientation of its tangent bundle, so a co-orientation is an orientation of its normal bundle. In essence, this is choosing a side of the curve.[2] We'll use this to define a function $\kappa : C \to \mathbb{R}$ called the *(signed) curvature*. Intuitively, this should be positive if C is curved towards the side chosen by the co-orientation, and negative if it curves away, and a larger magnitude means a stronger curvature.

The Euclidean structure on E induces an inner product structure on T_xC for all $x \in C$ that varies smoothly, so C becomes a Riemannian manifold. Theorem 1.16 means there's nothing intrinsic about C we can measure, but the way in which it sits inside E is what κ will measure. This is an important dichotomy, between intrinsic geometry and extrinsic geometry. The Riemann curvature tensor is intrinsic, since it doesn't depend on an embedding, but the signed curvature will be extrinsic.

Lecture 3.

The curvature of a curve: 1/24/17

"And if you follow your nose... well, Euler's nose..."

In the next two lectures, we'll march through the theory of extrinsic curvature (which can fill an entire undergraduate course).

Let E be a Euclidean plane modeled on an inner product space of V, which acts on E by translations, and let $i : C \hookrightarrow E$ be an immersed 1-manifold.[3] Suppose C is co-oriented, meaning we've oriented its normal bundle (picking a side of C, so to speak). This determins a unit co-oriented normal vector e_1 at every $x \in C$, meaning the unique unit vector in $(\nu_{C\hookrightarrow E})_x$ with a positive orientation. We can also choose a unit tangent vector e_2 perpendicular to e_1, and there are two choices. Together they define an orthonormal basis at each point: $(e_1, e_2) : C \to \mathscr{B}_{\mathrm{O}}(V)$.

[2] If $N \hookrightarrow M$ is an embedding and M is oriented, an orientation of N and a co-orientation of N determine each other.

[3] Especially if C is immersed but not embedded, it is helpful to remember i: when C self-intersects, remembering i is necessary for computing curvature.

You learned how to do calculus with real-valued differential forms; in exactly the same way, it's possible to do calculus with vector-valued differential forms $\Omega^*_C(V)$, the forms modeled on functions $C \to V$. For $i, j \in \{1, 2\}$, we can define $e_i \in \Omega^0_C(V)$ and $\mathrm{d}e_i \in \Omega^1_C(V)$, such that $\langle e_i, e_j \rangle = \delta_{ij}$ and the Leibniz rule is satisfied:

$$\langle \mathrm{d}e_i, e_j \rangle + \langle e_i, \mathrm{d}e_j \rangle = 0.$$

Thus, there exists an $\alpha \in \Omega^1_C$ such that

$$\mathrm{d}e_1 = -\alpha e_2 \qquad \text{and} \qquad \mathrm{d}e_2 = \alpha e_1.$$

In other words, applying d to the row vector $(e_1\ e_2)$ multiplies it by a skew-symmetric matrix:

$$\mathrm{d}\begin{pmatrix} e_1 & e_2 \end{pmatrix} = \begin{pmatrix} e_1 & e_2 \end{pmatrix} \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}.$$

Let $\theta^1, \theta^2 : C \to V^*$ define the dual basis at each point, i.e. at every $x \in C$, $\theta^i(e_j) = \delta^i_j$ as functions $C \to \mathbb{R}$. Then, $i^*\theta^2 \in \Omega^1_C$ and we can write

$$\alpha = k \cdot i^*\theta^2$$

for some function $k : C \to \mathbb{R}$.

Definition 3.1. The *curvature* of C is the function k.

Example 3.2. Let C denote the circle of radius R in the Euclidean plane $\mathbb{E}^2$. It's parameterized by coordinates $x = R\cos\phi$ and $y = R\sin\phi$, so

$$\begin{aligned} \mathrm{d}x &= -R\sin\phi\,\mathrm{d}\phi \\ \mathrm{d}y &= R\cos\phi\,\mathrm{d}\phi. \end{aligned}$$

Let's choose the co-orientation in which the inward-pointing unit normal is positively oriented. Then,

$$e_1 = -\cos\phi\frac{\partial}{\partial x} - \sin\phi\frac{\partial}{\partial y}.$$

We also have to choose e_2: suppose it points clockwise along the circle. Then,

$$e_2 = \sin\phi\frac{\partial}{\partial x} - \cos\phi\frac{\partial}{\partial y}.$$

Thus, the dual basis is defined by

$$\begin{aligned} \theta^1 &= -\cos\phi\,\mathrm{d}x - \sin\phi\,\mathrm{d}y \\ \theta^2 &= \sin\phi\,\mathrm{d}x - \cos\phi\,\mathrm{d}y, \end{aligned}$$

so $i^*\theta^2 = R\,\mathrm{d}\phi$. Then,

$$\mathrm{d}e_2 = \cos\theta\,\mathrm{d}\theta\frac{\partial}{\partial x} + \sin\theta\,\mathrm{d}\theta\frac{\partial}{\partial y} = -\mathrm{d}\theta e_1.$$

Thus, $\mathrm{d}e_2 = (1/R)i^*\theta^2(e_1)$. In particular, the curvature is $1/R$. It has units of $1/\text{length}$.

If we chose e_2 to point counterclockwise, there would be a sign change in θ^2, and another one in α, so they would cancel out to give the same result. ◀

Since the unit vector always has unit length in V, you can think of e_1 as a map $C \to S(V)$ (called the *Gauss map*), where $S(V)$ is the unit sphere inside V. At a point $p \in C$, we can define the tangent line T_pC at $i(p)$; the tangent line is a subspace of V. We can also consider the tangent line to $e_1(p) \in S(V)$, $T_{e_1(p)}S(V)$; both of these are the same space, the space of vectors in V perpendicular to $e_1(p)$.

This means the differential

$$(\mathrm{d}e_1)_p : T_pC \longrightarrow T_{e_1(p)}S(V) \tag{3.3}$$

is a map from a line to itself.

Theorem 3.4. *The map in (3.3) is multiplication by $-k(p)$.*

Proof.

$$\mathrm{d}e_1(e_2) = \alpha(e_2)\cdot d_2 = -ki^*\theta^2(e_2)e_2 = -k\cdot e_2.$$ ⊠

Remark 3.5 (History). The curvature may have been initially defined by Nicole Oresme in about 1350. It was again discovered by Huygens in c. 1650 and Newton in c. 1664. ◂

Here's a third approach to curvature. Let $i : C \hookrightarrow E$ be a co-oriented curve as usual, and assume C is embedded. For some $p \in C$, we can identify the normal line to $i(p)$ with $\mathbb{R}$, letting the positive numbers point into the positively oriented direction. Call this coordinate y. Given a choice of a unit tangent vector e_2, we can identify the tangent line with $\mathbb{R}$, again pointing the positive numbers in the x-direction. Call this coordinate x.

Lemma 3.6. *There exists an open set $U \subset E$ about p such that $C \cap U$ is the graph of a function $f : \mathbb{R} \to \mathbb{R}$ in the above xy-coordinate system such that*

- $f(0) = f'(0) = 0$, *and*
- $f''(0) = k(p)$.

Proof. The x-coordinate map $x|_C : C \to \mathbb{R}$ satisfies $\mathrm{d}x_p = \mathrm{id}_{T_pC}$; in particular, it's invertible. By the inverse function theorem, there's a local inverse $g : I \to C$, where $I \subset \mathbb{R}$ is an open interval. Define f to be $y \circ i \circ g$: since $i : C \hookrightarrow E$ and $y : E \to \mathbb{R}$, this is a map $I \to \mathbb{R}$. Write

$$e_1 = \frac{(-f', 1)}{\sqrt{1 + (f')^2}} \qquad \text{and} \qquad e_2 = \frac{(1, f')}{\sqrt{1 + (f')^2}}.$$

Then,

$$\mathrm{d}e_1 = \left(\frac{(-f'', 0)}{\sqrt{1 + (f')^2}} + \frac{(-f', 1)}{(1 + (f')^2)^{3/2}} f' \right) \mathrm{d}t.$$

At p,

$$\mathrm{d}e_1 = (-f''(0), 0)\,\mathrm{d}t = (-f''(0)\,\mathrm{d}t)e_2.$$ ⊠

In calculus, we think of the tangent line as the best linear approximation to a function at a point, which only requires an affine space. Curvature is the process that goes one degree higher: you could ask for the *osculating parabola* to a curve at a point, the parabola that best approximates a curve at a point, or for the *osculating circle*, the circle that best approximates the curve at that point. Then, the curvature can be read off of the constants, e.g. it's 1 over the radius of the osculating circle. But knowing these parameters requires an inner product, hence a Euclidean space.

Prescribing curvature. We aim to solve the following problem: given an abstract curve C and a function $k : C \to \mathbb{R}$, construct an immersion $i : C \to E$ and a co-orientation such that k is the curvature of i.

Curvature requires thinking about a frame at each point if $i(C)$, so we should think about the bundle of orthonormal frame $\pi : \mathscr{B}_O(E) \to E$. A point in $\mathscr{B}_O(E)$ is a triple $(p; e_1, e_2)$, where $p \in E$ and (e_1, e_2) is an orthonormal basis of V. In particular, $\mathscr{B}_O(E)$ is naturally a product $E \times \mathscr{B}_O(V)$. We want to construct a lift $\widetilde{\imath} : C \to \mathscr{B}_O(E)$ making the following diagram commute:

$$\begin{tikzcd} & \mathscr{B}_O(E) \arrow[d, "\pi"] \\ C \arrow[ur, "\widetilde{\imath}"] \arrow[r, "i"] & E. \end{tikzcd}$$

This $\widetilde{\imath}$ is specified as a triple of functions on C, $\widetilde{\imath} = (p, e_1, e_2)$. Prescribing the curvature means we need this to satisfy

$$\mathrm{d}\begin{pmatrix} p & e_1 & e_2 \end{pmatrix} = \begin{pmatrix} p & e_1 & e_2 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & k\,\mathrm{d}t \\ \mathrm{d}t & -k\,\mathrm{d}t & 0 \end{pmatrix}}_{A(t)}. \tag{3.7}$$

We'll interpret A as a time-varying vector field on the manifold $\mathscr{B}_O(E)$; then, we can evoke the basic theory of ordinary differential equations to prove there's a solution.

Digression. Let's recall what this basic theory of ordinary differential equations says. Let X be a smooth manifold and $(a, b) \subset \mathbb{R}$ be an interval. Projection onto the second factor defines a map $\pi_2 : (a, b) \times X \to X$, and we can pull the tangent bundle back along it:

$$\begin{tikzcd} \pi_2^* TX \arrow[r] \arrow[d, "p"] & TX \arrow[d] \\ (a, b) \times X \arrow[r, "\pi_2"] & X. \end{tikzcd}$$

Definition 3.8.

- A *time-varying vector field* is a section $\xi : (a, b) \times X \to \pi_2^* TX$ of $p : \pi_2^* TX \to (a, b) \times X$.
- An *integral curve* of ξ is an open interval $I \subset (a, b)$ and a function $\gamma : I \to X$ such that

$$\dot{\gamma}(t) = \xi_{(t, \gamma(t))}.$$

Time-varying vector fields correspond to ODEs and integral curves correspond to their solutions.

Theorem 3.9. *Given $(t_0, x_0) \in (a, b) \times X$, there exists an $\varepsilon > 0$ and an integral curve $\gamma : (t_0 - \varepsilon, t_0 + \varepsilon)$ such that $\gamma(t_0) = x_0$, and any two choices for γ agree on their common domain. Moreover, there is a maximal domain $J \subset (a, b)$ on which a solution exists and an integral curve $\gamma : J \to X$*

That is, solutions exist and are unique given an initial condition. However, they may not be globally defined.[4]

Just as $\mathscr{B}_{\mathrm{O}}(V)$ is a torsor for a right action of O_2 (an orthogonal basis composed with an orthogonal transformation is again an orthogonal basis), $\mathscr{B}_{\mathrm{O}}(E)$ is a torsor for the right action of Euc_2, the group of Euclidean transformations of $\mathbb{E}^2$. This torsor structure means the derivative of a curve in any neighborhood of the origin of the group defines a vector field on the torsor.

If $P(t)$ is a curve in O_2 such that $P(0) = \mathrm{id}$, then ${}^t P \cdot P = I$, so differentiating this condition, ${}^t \cdot P + \cdot P = 0$. That is, $T_e \mathrm{O}_2$ is the line of 2×2 skew-symmetric matrices over $\mathbb{R}$. Looking again at (3.7), the lower right entries of $A(t)$ are exactly such a matrix, so $A(t)$ is in fact a time-varying vector field on $\mathscr{B}_{\mathrm{O}}(E)$.

Corollary 3.10. *Using Theorem 3.9, given an initial $p \in E$ and an initial frame (e_1, e_2) on $T_p E$, there is a local and in fact a maximal solution to the prescribed curvature problem. This solution is unique up to the choice of (p, e_1, e_2).*

Uniqueness is usually expressed by saying that the group of symmetries of Euclidean space acts transitively on the solutions (so there's only one up to rotations and translations).

This is a somewhat elementary context for this material, but we'll adopt this perspective again and again. Eventually there will also be second-order conditions, e.g. when we define geodesics later.

Now, let's step up a dimension: let E be a Euclidean 3-space modeled on an inner product space V and $i : S \hookrightarrow V$ be an immersion of a 2-manifold together with a co-orientation. We can again define the unit co-oriented normal $\nu : S \to S(V)$. How can we define the curvature of this surface?

Euler solved this problem in 1760 by reducing it to something we've already done: let $L \in \mathbb{P}(T_p S)$ be a 1-dimensional subspace of the tangent space. There's a unique affine plane $\Pi(L)$ passing through p and containing L, and $\Pi(L) \cap S$ is a co-oriented curve in $\Pi(L)$. Let $k_p : \mathbb{P}(T_p S) \to \mathbb{R}$ be the function assigning to L the curvature of the curve $\Pi(L) \cap S$. Euler studied this function.

As before, locally we can write S as the graph of a function $f : T_p S \to \mathbb{R}$ with $f(0) = 0$ and $\mathrm{d}f_0 = 0$. The function k_p encodes the second derivative of f. This is expressed through the Hessian

$$\mathrm{Hess}\, f_0 : T_p S \times T_p S \longrightarrow \mathbb{R},$$

which is a symmetric bilinear form. In the context of geometry of surfaces, this Hessian is called the *second fundamental form* and denoted II_p.

Corollary 3.11. *For any $L \in \mathbb{P}(T_p S)$, $k_p(L) = II_p(\xi, \xi)$, where $|\xi| = 1$ and $\xi \in L$.*

---

[4]In this class, we assume everything is smooth, but Theorem 3.9 is true in much greater generality, requiring only *Lipschitz continuity*, a condition slightly stronger than continuity. Many other things in this class may be relaxed, e.g. to C^2.

The *first fundamental form* is the inner product

$$I_p := \langle -,- \rangle : T_pS \times T_pS \to \mathbb{R},$$

The second fundamental form may be nondegenerate (e.g. if S is flat), but we know the first is nondegenerate. This means the second fundamental form may be expressed in terms of the first fundamental form and some other operator S, called the *shape operator*:

$$II_p(\xi,\eta) = I_p(\xi, S(\eta)) = \langle \xi, S(\eta)\rangle.$$

Since II_p is symmetric, then S is self-adjoint. This means it has two real eigenvalues, so we can look at the eigenspaces, which are called the *principal lines* of S at p — unless the curvature is constant at p, in which case p is called an *umbilic point*.

Interestingly, we started with a very extrinsic notion of curvature of surfaces, but from this we've obtained some intrinsic geometry.

Lecture 4.

Curvature for surfaces: 1/26/17

"I didn't go into comedy, because I thought I would be safe here..."

Last time, we talked about the curvature of surfaces in a Euclidean plane; today, we will consider surfaces in a 3-dimensional Euclidean space E modeled on an inner product space $(V, \langle -,- \rangle)$, the vector space of translations of E.

Though E is abstractly isomorphic to $\mathbb{E}^3$, we won't fix an isomorphism by choosing coordinates; later, we'll want to pick special coordinates for E, so this would only complicate things.

Let $\Sigma \subset E$ be an embedded 2-manifold (some of our results will still apply when Σ is immersed), and assume Σ is co-oriented. Let $\nu : \Sigma \to V$ be the co-oriented positive unit normal.

Given a $p \in \Sigma$ and a plane $L \subset V$, $\Pi(L)$ denotes the plane through p containing L and ν. Then, $\Sigma \cap \Pi(L)$ is a curve, which is intuitively the curve "pointing in the L-direction at p."

The map assigning to L the curvature of $\Sigma \cap \Pi(L)$ at p is a function

$$k_p \colon \mathbb{P}(T_p\Sigma) \to \mathbb{R}.$$

Here, $\mathbb{P}(V)$ is the manifold of 1-dimensional subspaces of a vector space V.

We're going to get some information out of k_p. Let's first introduce special coordinates: choose an orthonormal basis in $\mathscr{B}_{\mathrm{O}}(E)$, so we obtain coordinats x^1 and x^2 in $T_p\Sigma$. As in the last lecture, the inverse function theorem provides for us an open set $U \subset T_p\Sigma$ containing 0, a function $f \colon U \to \mathbb{R}$, and an open $J \subset \mathbb{R}$ containing 0 such that $\Sigma \cap ((p+U) \times (p+J\nu))$ is the graph of f.

That is, there's a box inside E with an "xy-plane" $p+U$ and a "z-axis" pointing in the ν-direction, and inside this box, Σ is the graph of a function $f(x,y)$ on $p+U$. Furthermore, $f(p) = 0$ and $\mathrm{d}f_p = 0$, which is easy to check.

Last time, we defined the second fundamental form at p, $II_p = \mathrm{Hess}_p\, f : T_p\Sigma \times T_p\Sigma \to \mathbb{R}$. Based on what we proved last time, using the third incarnation of curvature, we got Corollary 3.11: $k_p(L) = II_p(\xi,\xi)$, where $\xi \in L$ is a unit vector.

This says the Hessian on the diagonal determines the curvature. This is because this is the second derivative of f, and we showed that if $\mathrm{d}f_p = 0$ for an f parameterizing a plane curve, then its second derivative computes the curvature.

On $T_p\Sigma$ we have two fundamental forms: the inner product, also known as the first fundamental form I_p, and the second fundamental form defined above. Since the first fundamental form is nondegenerate, then we can (and did) define the shape operator $S_p \in \mathrm{End}(T_p\Sigma)$ to satisfy the relation

$$\langle \xi, S_p(\eta)\rangle = II_p(\xi,\eta).$$

Since the inner product is nondegenerate, this uniquely defines $S_p(\eta)$. Moreover, since II_p is symmetric, then S_p is self-adjoint, i.e. $\langle \xi, S_p(\eta)\rangle = \langle S_p(\xi), \eta\rangle$ for all ξ and η. In particular, it's diagonalizable, and since $T_p\Sigma$ is two-dimensional, there are two possibilities:

(1) If there's only one eigenvalue $\lambda \in \mathbb{R}$, then $S_p = \lambda \cdot \mathrm{id}_{T_p\Sigma}$. In this case, p is called an umbilic point.

(2) If there are two eigenvalues λ_1 and λ_2 (suppose without loss of generality $\lambda_1 > \lambda_2$), then the two eigenspaces L_1 and L_2 form an orthogonal direct-sum decomposition $T_p\Sigma = L_1 \oplus L_2$. In this case, $S_p|_{L_i}$ is multiplication by λ_i. The L_i are called the *principal directions*, and the λ_i are called the *principal curvatures*. For any plane L,

$$k_p(L) = \frac{II_p(\xi,\xi)}{I_p(\xi,\xi)}.$$

The maximum of k_p is at L_1, and the minimum is at $L_2 II_p(\xi,\xi) I_p(\xi,\xi)$.

If you reverse the co-orientation, then $k \mapsto -k$ and $\lambda_i \mapsto -\lambda_i$. From this we get the *mean curvature* (named after one Mr. Mean)

$$H := \frac{\lambda_1 + \lambda_2}{2} = \frac{1}{2}\operatorname{Tr}(S_p),$$

a function $\Sigma \to \mathbb{R}$. Reversing the co-orientation sends $H \mapsto -H$. The *Gauss curvature* (named after Gauss) is

$$K := \lambda_1\lambda_2 = \det S,$$

also a function $\Sigma \to \mathbb{R}$. This is unchanged when you reverse the co-orientation, which suggests that it comes from an intrinsic invariant! The units of the Gauss curvature has units $1/\text{length}^2$.

We also have the unit normal vector field $\nu\colon \Sigma \to S(V) \subset V$, and it tells us things about the curvature too.

Proposition 4.1. $\mathrm{d}\nu_p\colon T_p\Sigma \to T_p\Sigma$ *equals* $-S_p$.

Proof. Introduce "Euclidean coordinates" x^1, x^2 on $p + T_p\Sigma$, and let $f = f(x^1, x^2)$ be such that near p, Σ is the graph of f. Then,

$$\nu = \nu(x^1, x^2) = \frac{(-f_1, -f_2, 1)}{\sqrt{1 + f_1^2 + f_2^2}},$$

where $f_i = \frac{\partial f}{\partial x^i}$.

Exercise 4.2. Check that this is in fact a unit normal vector.

You can then calculate

$$\mathrm{d}\nu_p = \left.\begin{pmatrix} -\partial_{11} f & -\partial_{12} f \\ -\partial_{21} f & -\partial_{22} f \end{pmatrix}\right|_p,$$

and this is $-\operatorname{Hess}_p f = -II_p$ as desired. (Here, it may help to remember that p is identified with $(0,0)$.) ⊠

Many people bemoan computations and coordinates, but certainly computations are useful, and coordinates are useful for computations. The solution is to judiciously choose coordinates to make computations simpler.

Now we can cover two beautiful theorems of Gauss, one global, one local.

Theorem 4.3 (Gauss-Bonnet). *Let $\Sigma \subset E$ be a closed, co-oriented surface and $K : \Sigma \to \mathbb{R}$ be its Gauss curvature. Let $|\mathrm{d}A|$ denote its Riemannian measure. Then,*

$$\int_\Sigma K\,|\mathrm{d}A| = 2\pi\chi(\Sigma). \tag{4.4}$$

Some of these words merit an explanation.

- A *closed manifold* is not the same thing as a closed subset: it means Σ is compact and has no boundary. It turns out all closed surfaces in E are co-orientable, but this is not necessarily true for immersed surfaces (e.g. the standard immersion of the Klein bottle).
- The Riemannian measure is discussed in the homework, but the essential idea is that on a Riemannian manifold, we know the lengths and angles of vectors, and therefore of the volume of the parallelogram that a basis $v_1, \ldots, v_n$ of a tangent space spans, namely $|\det(\langle v_i, v_j\rangle_{ij})|$. Thus, we know how to compute volumes, which defines a measure that we can use to integrate functions.
- $\chi(\Sigma)$ is the Euler characteristic of Σ.

Though the proof we'll see uses the embedding (and implicitly the fact that Σ is orientable), all of the notions in (4.4) turn out to be extrinsic, and the theorem holds for abstract closed surfaces with a Riemannian metric, orientable or not.

Example 4.5. Consider a sphere $S^2(R)$ of radius R inside E. Then, every point is umbilic, and the Gauss curvature is $1/R^2$ everywhere. The surface area of the sphere is $4\pi R^2$, so

$$\int_{S^2} K\,|\mathrm{d}A| = 4\pi = 2\pi \cdot 2,$$

and indeed $\chi(S^2) = 2$. ◀

Theorem 4.3 is the first of many theorems which relate local and global geometry. It can be used to calculate global quantities, and to constrain local ones: for example, the sphere cannot have a metric with negative curvature, because its Euler characteristic is positive. The torus T^2 has Euler characteristic $\chi(T^2) = 0$, so any metric on it is either everywhere flat (no curvature) or has points of both positive and negative curvature. The standard embedding into $\mathbb{E}^3$ has points of both positive and negative curvature, but the flat torus can't be embedded isometrically into $\mathbb{E}^3$. It can be embedded into $\mathbb{E}^4$, as the product of two copies of the unit circle in $\mathbb{E}^2$.

Proof of Theorem 4.3. The proof will use the language of differential topology. Recall that if M and M' are oriented manifolds of the same dimension n, we can define the degree of a smooth map $\nu : M' \to M$, and if $\omega \in \Omega^n_M$, then

$$\int_{M'} \nu^*\omega = (\deg \nu) \int_M \omega.$$

In our case, ν is the unit vector map $\nu : \Sigma \to S(V)$; we computed that $\mathrm{d}\nu = -S$ (where S is the shape operator) in Proposition 4.1. Thus,

$$\det(\mathrm{d}\nu) = \det(-S) = K.$$

Let $\omega \in \Omega^2_{S(V)}$ be the area form; then,

$$v^*\omega = (\det \mathrm{d}\nu) \cdot \mathrm{d}A = K\,\mathrm{d}A.$$

Thus, when we integrate,

$$\int_\Sigma K\,\mathrm{d}A = \int_\Sigma \nu^*\omega = (\deg \nu) \int_{S(V)} \omega = 4\pi \deg \nu,$$

since the area of the unit sphere is 4π. Thus, it suffices to show $\deg \nu = \chi(\Sigma)/2$.

The Euler number emerges from the Poincaré-Hopf theorem, that if $\mathbf{v}$ is a vector field with isolated zeroes on Σ, the sum of the indices of $\mathbf{v}$ at its zeroes produces $\chi(\Sigma)$.

Compose ν with the quotient map $S(V) \twoheadrightarrow \mathbb{P}(V)$, and let q be a regular value of this composition, with two preimages $\pm\eta \in S(V)$. η pulls back to a vector field on Σ (constantly pointing in the direction η with unit length). Let ξ_p denote the vector field produced by projecting η onto $T\Sigma$; this has isolated zeros $x_1, \dots, x_n$.

You can do the computation without coordinates, but it's not hard in them: if $\eta = (0,0,1)$ (which is true up to a rotation), then at any x_i,

$$\xi = \frac{(f_1, f_2, f_1^2 + f_2^2)}{1 + f_1^2 + f_2^2},$$

and you don't have to worry about the denominator in the derivative, so

$$\mathrm{d}\nu_p = \mathrm{d}\xi_p = \begin{pmatrix} \partial_{11} f & \partial_{12} f \\ \partial_{21} f & \partial_{22} f \end{pmatrix}\Bigg|_p. \qquad \boxtimes$$

This is the first connection between topology and geometry.

You might wonder how this can be generalized. In odd dimensions, the Euler characteristic is zero, but for even dimensions, Chern proved the Gauss-Bonnet-Chern theorem in the 1940s which expresses the Euler characteristic in more complicated terms involving the Riemann curvature tensor.

Lecture 5.

Extrinsic and intrinsic curvature: 1/31/17

On the first day, we derived some equations as to when a Riemannian manifold is locally isometric to Euclidean space. Namely, if

$$A_{ijk} := \frac{\partial g_{\ell j}}{\partial x^k} + \frac{\partial g_{\ell k}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\ell}$$

and

$$\Gamma^i_{jk} := \frac{1}{2} g^{i\ell} A_{\ell jk},$$

then we derived in (1.21)

$$E^i_{jk\ell} = \frac{\partial \Gamma^i_{j\ell}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^\ell} + \Gamma^m_{j\ell}\Gamma^i_{mk} - \Gamma^m_{jk}\Gamma^i_{m\ell},$$

and the Riemann curvature tensor

$$R = R^i_{ijk\ell} \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k \otimes \mathrm{d}x^\ell$$

is an obstruction to a Riemannian manifold being locally isometric to flat, Euclidean space. There's an exercise in the homework to show this is invariant under change of coordinates, and therefore R is an intrinsic object.

Today, we will tie this to the study of curvature of a surface Σ embedded in Euclidean 3-space E. Suppose Σ is co-oriented; then, at any $p \in \Sigma$, we defined the second fundamental form $II_p : T_p\Sigma \times T_p\Sigma \to \mathbb{R}$ and the shape operator $S_p : T_p\Sigma \to T_p\Sigma$ satisfying $II_p(\xi, \eta) = \langle \xi, S_p(\eta)\rangle$. The Gauss curvature is $k_p = \det S_p$, and the normal curvature is $II_p(\xi,\xi)/I_p(\xi,\xi)$.

Locally, Σ is the graph of a function $f = f(x^1, x^2)$ defined on an open neighborhood U in the x^1x^2-plane; here, x^1 and x^2 are special coordinates determined up to an element of O_2.

Theorem 5.1 (Gauss' *Theorema egregium*, c. 1823)**.** *In any of these special local coordinates at p,*

$$R^1_{212}(p) = k_p.$$

The right-hand side is defined extrinsically, determining how curves contained in orthogonal planes bend when embedded in the surface. But the left-hand side is defined intrinsically, depending only on the metric. Thus, the Gauss curvature is an intrinsic quantity, and does not depend on the co-orientation or embedding.

Corollary 5.2. *If Σ, Σ' are two surfaces embedded in E and $\varphi : \Sigma' \to \Sigma$ is an isometry, then $\varphi^* k = k'$.*

This is because the isometry preserves the metric, and the Gauss curvature can be computed only from the metric. This version is closer to how Gauss stated it.

Looking at Corollary 5.2, we know one embedding of the sphere of radius R into E such that the Gauss curvature is $k = 1/R^2$, and that the flat plane has curvature 0. Thus, map projections must be inaccurate: there's no way to map a plane onto any part of the sphere without distorting some length or angle.

The Riemannian curvature tensor on a Riemannian manifold X has a lot of symmetry. From (1.21), one can show that $R^i_{jk\ell} = -R^i_{j\ell k}$: it's skew-symmetric in these arguments. Thus,

$$R = \frac{1}{2} R^i_{jk\ell} \left(\frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j \right) \otimes \mathrm{d}x^k \wedge \mathrm{d}x^\ell.$$

That is, $R \in \Omega^2_X(\operatorname{End} TX)$: the i and j indices give you an endomorphism of each tangent space. In fact, $R \in \Omega^2(\operatorname{SkewEnd} TX)$: the endomorphism is skew-symmetric.

Applying this to when $\dim X = 2$, if $V := T_pX$, then $R_p \in \operatorname{SkewEnd}(V) \otimes \Lambda^2 V^*$. The second component is the top exterior power, hence the *determinant line* $\operatorname{Det} V^*$. Moreover, $\operatorname{SkewEnd}(V) \xrightarrow{\cong} \Lambda^2 V^*$ through the map sending

$$T \longmapsto (\xi, \eta \longmapsto \langle \xi, T\eta \rangle).$$

This is akin to the way we got the shape operator out of the second fundamental form.

Anyways, this means $R_p \in (\operatorname{Det} V^*)^{\otimes 2} = (\operatorname{Det} V^{\otimes 2})^*$. What is this determinant line? The idea is that for every pair of vectors ξ, η, $\xi \wedge \eta$ can be identified with its area. We don't know what area 1 is *per se*, but we know given ξ', η' how to figure out the ratio of the area of $\xi' \wedge \eta'$ to that of $\xi \wedge \eta$, giving us a one-dimensional subspace.

But we do have an orthonormal basis produced by the metric, so we obtain a distinguished unit vector $e \in \operatorname{Det} V$. Thus, we can express $R^1_{212}(p)$ coordinate-independently, by evaluating $R_p \in ((\operatorname{Det} V)^{\otimes 2})^*$ on $e \otimes e \in (\operatorname{Det} V)^{\otimes 2}$.

Proof of Theorem 5.1. Near p, the surface is the graph of a function $(x^1, x^2) \mapsto (x^1, x^2, f(x^1, x^2))$. Let $f_i := \frac{\partial f}{\partial x^i}$, so

$$\begin{aligned} \left.\frac{\partial}{\partial x^1}\right|_{(x^1,x^2)} &= (1, 0, f_1) \in T_{(x^1,x^2,f(x^1,x^2))}\Sigma \subset V \\ \left.\frac{\partial}{\partial x^2}\right|_{(x^1,x^2)} &= (0, 1, f_2). \end{aligned}$$

Let $\Delta := 1 + f_1^2 + f_2^2$. Then, you can calculate that the metric and its inverse satisfy

$$\begin{aligned} g_{11} &= 1 + f_1^2 & g^{11} &= \frac{1 + f_2^2}{\Delta} \\ g_{12} &= f_1 f_2 & g^{12} &= -\frac{f_1 f_2}{\Delta} \\ g_{22} &= 1 + f_2^2 & g^{22} &= \frac{1 + f_1^2}{\Delta}. \end{aligned}$$

The right-hand side is obtained from the left by inverting the 2×2 matrix for g_{ij}.

Exercise 5.3. Check that $A_{\ell jk} = 2 f_\ell f_{jk}$.

Recall that $f(0,0) = f_\ell(0,0) = 0$, so $A_{\ell ij}(0) = 0$ and $\Gamma^i_{jk}(0) = 0$. Thus,

$$R^1_{212}(0,0) = \left.\frac{\partial \Gamma^1_{22}}{\partial x^1}\right|_{(0,0)} - \left.\frac{\partial \Gamma^1_{21}}{\partial x^2}\right|_{(0,0)}.$$

Another plug-and-chug shows that

$$\begin{aligned} \Gamma^1_{22} &= \frac{1}{2} g^{11} A_{122} + \frac{1}{2} g^{12} A_{222} \\ &= \frac{2}{2\Delta}\Big((1 + f_2^2) f_1 f_{22} - f_1 f_2 f_2 f_{22}\Big) \\ &= \frac{f_1 f_{22}}{\Delta}. \end{aligned}$$

A similar calculation shows

$$\Gamma^1_{21} = \frac{f_1 f_{21}}{\Delta}.$$

Therefore

$$\begin{aligned} R^1_{212}(0,0) &= (f_{11} f_{22} - f_{12} f_{21})|_{(0,0)} \\ &= \det \operatorname{Hess}_{(0,0)} f \\ &= k_p. \end{aligned}$$ ⊠

You should run through these calculations to make sure you understand them.

This provides us an interpretation of R, measuring curvature in different directions on the manifold. If it's equal to 0, the manifold is flat. We'd also like to interpret the Γ^i_{jk} symbols. This should be easier because they're built from first derivatives, whereas R was built from second derivatives.

Let's think about parallelism. In the Euclidean plane E, we have *global parallelism*, that given a vector field $\eta; E \to V$, we can compute its directional derivatives by considering the function $t \mapsto p + t\xi_p$ along a direction ξ_p (thought of as rooted at p). That is, the directional derivative of η in the direction ξ_p is

$$D_{\xi_p}\eta := \lim_{t\to 0} \frac{\eta(p+t\xi_p) - \eta(p)}{t}.$$

If $\gamma : (-\varepsilon, \varepsilon) \to E$ is a curve with $\gamma(0) = p$ and $\dot\gamma(0) = \xi_p$, then

$$D_{\xi_p}\eta = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \eta(\gamma(t)).$$

This doesn't work quite so well on embedded surfaces $\Sigma \hookrightarrow E$. There's a "poor man's parallelism" that translates a vector using the ambient parallelism on E, but there are lots of issues with this: it does not preserve tangency. So you project down onto $T\Sigma$, you say, but then sometimes you get the zero vector, and it feels like parallelism should preserve lengths and angles, right?

Let's ask a smaller question: given an immersed curve $\gamma\colon (-\varepsilon, \varepsilon) \to \Sigma$ with $\gamma(0) = p$ and $\dot\gamma(0) = \xi_p$, can we parallelize?

Definition 5.4. The *covariant derivative* $\nabla_{\xi_p}\eta$ is the orthogonal projection of $D_{\xi_p}\eta \in V$ onto $T_p\Sigma$.

Here, η is a section of the vector bundle $T\Sigma \to \Sigma$, and $\xi_p \in T_p\Sigma$, so $D_{\xi_p}\eta$ is in $T_pE = V$.

Definition 5.5. We say η is *parallel along* $\gamma : (a, b) \to \Sigma$ if $\nabla_{\dot\gamma}\eta = 0$ for all $t \in (a, b)$. If $\nabla_{\dot\gamma}\dot\gamma = 0$ (i.e. $\dot\gamma$ is parallel along γ), then γ is called a *geodesic*.

Here, η is a *vector field along* γ, meaning a section of the pullback bundle $\gamma^*T\Sigma \to (a, b)$. That is, at each t, $(\gamma^*T\Sigma)_t := T_{\gamma(t)}\Sigma$, and these fit together smoothly. So at each t, η chooses a tangent vector in $T_{\gamma(t)}\Sigma$. Thus, if γ is self-intersecting, we get a different tangent vector each time $\gamma(t)$ reaches the intersection point, so everything is still well-behaved.

Geodesics are the curves which have no acceleration along the curve, so the only acceleration is normal to the surface. For example, if you have a geodesic on a sphere (which is a great circle), it's only accelerating perpendicular to the sphere, the minimal acceleration necessary to stay on the sphere.

One of the first things we prove in multivariable calculus is that the directional derivative is linear in the direction. This is still true here, where we derived it from parallelism, among the oldest notions in geometry.

Lemma 5.6.

(1) *$\nabla_{\xi_p}\eta$ is linear in ξ_p, i.e. $\nabla\eta \in T_p^*\Sigma$.*

(2) *∇_{ξ_p} satisfies a Leibniz rule:*

$$\nabla_{\xi_p}(f\eta) = (\xi_p \cdot f)\eta + f\nabla_{\xi_p}\eta.$$

(3)

$$\nabla_{\xi_p}(\eta + \eta') = \nabla_{\xi_p}\eta + \nabla_{\xi_p}\eta'.$$

(4)

$$\xi_p\langle\eta, \eta'\rangle = \langle\nabla_{\xi_p}\eta, \eta'\rangle + \langle\eta, \nabla_{\xi_p}\eta'\rangle.$$

Though we've defined geodesics extrinsically, they are intrinsic, and we'll be able to describe them using the symbols Γ^i_{jk}.

Theorem 5.7. *Let η be a vector field on Σ. Then, $\nabla\eta$ is intrinsic, i.e. determined solely by the metric.*

In particular, $\nabla\eta \in \Omega^1_\Sigma(T\Sigma)$.

Proof. Use coordinates $(x^1, x^2, f(x^1, x^2))$ as before, so Σ is the graph of f. A basis for the tangent space is $\frac{\partial}{\partial x^1} = (1, 0, f_1)$ and $\frac{\partial}{\partial x^2} = (0, 1, f_2)$ as before.

Write $\eta = \eta^i\frac{\partial}{\partial x^i}$ with $\eta^i = \eta^i(x^1, x^2)$ for $i = 1, 2$. Thus, $\eta = (\eta^1, \eta^2, \eta^i f_i)$, so by a Leibniz rule

$$D\eta = (\mathrm{d}\eta^2 1, \mathrm{d}\eta^2, f_i\,\mathrm{d}\eta^i + \eta^i\,\mathrm{d}f_i).$$

In particular, $D\eta_p = (\mathrm{d}\eta_p^1, \mathrm{d}\eta_2^p, *)$ and $\nabla\eta_p = (\mathrm{d}\eta_p^1, \mathrm{d}\eta_p^2, 0)$, or

$$\nabla\eta = \mathrm{d}\nabla^i \cdot \frac{\partial}{\partial x^i},$$

so $\nabla\frac{\partial}{\partial x^i} = 0$ at p.

We used special coordinates x^1, x^2; let's change to arbitrary coordinates y^1, y^2. Calculus on manifolds (or, for grade students, canceling fractions) shows that

$$\frac{\partial}{\partial y^a} = \frac{\partial x^i}{\partial y^a}\frac{\partial}{\partial x^i},$$

so at p,

$$\begin{aligned}\nabla\frac{\partial}{\partial y^a} &= \frac{\partial^2 x^i}{\partial y^b \partial y^a}\,\mathrm{d}y^b \cdot \frac{\partial}{\partial x^i}\\ &= \underbrace{\frac{\partial^2 x^i}{\partial y^b \partial y^a}\frac{\partial y^c}{\partial x^i}}_{Q^c_{ab}}\,\mathrm{d}y^b \cdot \frac{\partial}{\partial y^c}.\end{aligned}$$

We'll finish the proof by showing $Q^c_{ab} = \Gamma^c_{ab}$ as computed in the (y^1, y^2)-coordinate system. Since Γ^c_{ab} doesn't depend on the metric, neither can $\nabla\eta$.

At p,

$$\begin{aligned}g_{ab} = \left\langle \frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b}\right\rangle &= \frac{\partial x^i}{\partial y^a}\frac{\partial x^j}{\partial y^b}g_{ij}\\ &= \sum_i \frac{\partial x^i}{\partial y^a}\frac{\partial x^i}{\partial y^b},\end{aligned}$$

so (again at p),

$$\frac{\partial g_{ab}}{\partial y^c} = \sum_i \frac{\partial^2 x^i}{\partial y^c \partial y^a}\frac{\partial x^i}{\partial y^b} + \frac{\partial x^i}{\partial y^a}\frac{\partial^2 x^i}{\partial y^c \partial y^b}.$$

Therefore

$$A_{dab} = 2\sum_i \frac{\partial^2 x^i}{\partial y^a \partial y^b}\frac{\partial x^i}{\partial y^d}$$

and

$$g^{cd} = \sum_j \frac{\partial y^c}{\partial x^j}\frac{\partial y^d}{\partial x^j}.$$

Thus,

$$\Gamma^c_{ab} = \frac{1}{2}\gamma^{cd}A_{dab} = \sum_{i,j}\frac{\partial y^c}{\partial x^j}\underbrace{\frac{\partial y^d}{\partial x^j}\frac{\partial x^i}{\partial y^d}}_{\delta^i_j}\frac{\partial^2 x^i}{\partial y^a \partial y^b}.$$

Thus, we can collapse to when $i = j$, which recovers Q^c_{ab}. ⊠

Embedded in this proof is the calculation as to how the Γ^k_{ij} change when the coordinates change.

This allows us to define a differential equation for geodesics: if $\eta = \eta^a\frac{\partial}{\partial y^a}$, so that

$$\nabla\eta = \left(\frac{\partial \eta^c}{\partial y^b} + \Gamma^c_{ab}\eta^a\right)\mathrm{d}y^b\frac{\partial}{\partial y^c},$$

then the *geodesic equation* for $\dot\gamma = \xi = \xi^b\frac{\partial}{\partial y^b}$ is

$$\nabla_\xi \xi = \left(\ddot y^a + \Gamma^c_{ab}\dot y^a \dot y^b\right)\frac{\partial}{\partial y^c} = 0. \tag{5.8}$$

That is, for surfaces, we have intrinsic notions of parallelism and geodesics. This holds in more generality. Next time, we'll say one more thing about surfaces in space (looking at the normal component of the directional derivative), and recover the second fundamental form on it. Then, we'll do some background lectures on differential geometry.

Lecture 6.

Vector fields and integral curves: 2/2/17

We've talked about how for surfaces, the sectional curvature at a point p is a map $k_p : \mathbb{P}(V) \to \mathbb{R}$. More generally, the Riemann curvature tensor is $R \in \Omega^2_X(\mathrm{SkewEnd}\, TX)$, so for any $x \in X$, if $V = T_x X$, $R_x : \Lambda^2 V \to \mathrm{SkewEnd}(V) \cong \Lambda^2 V^*$, hence determined by a bilinear map $\Lambda^2 V \times \Lambda^2 V \to \mathbb{R}$. If $\Pi \subset V$ is a two-dimensional subspace, we can evaluate $R_x(\Pi, \Pi) \in \mathbb{R}$, so letting Π vary, we obtain the *sectional curvature* $K_x : \mathrm{Gr}_2(T_x X) \to \mathbb{R}$. Here, $\mathrm{Gr}_2(V)$ is the *Grassmannian*, the manifold of 2-dimensional subspaces of V.

Let's return to the case of a co-oriented surface Σ embedded in a 3-dimensional Euclidean space E, and let η be a vector field on Σ. Then, the directional derivative in the direction ξ_p (a vector ξ rooted at p) is $D^{(E)}_{\xi_p}\eta \in \mathbb{R}^3$. This has tangential and normal components:

$$D_{\xi_p}\eta = \underbrace{\nabla_{\xi_p}\eta}_{\text{tangential}} + \underbrace{B(\xi_p, \eta) \cdot \nu}_{\text{normal}}.$$

Last time, we showed in Theorem 5.7 that the tangential part is intrinsic to Σ. If $\gamma : (-\varepsilon, \varepsilon) \to \Sigma$ is a curve, then $\nabla_{\dot\gamma}\eta$ is the covariant derivative of η along γ. We said η is parallel along γ if $\nabla_{\dot\gamma}\eta = 0$, and γ is a geodesic if $\dot\gamma$ is parallel along γ.

Last time, we saw that geodesics are the solutions to the ODE (5.8); by the general theory of ODEs, solutions exist and are unique. Given a smooth curve $\gamma\colon (a, b) \to \Sigma$, a $t_0 \in (a, b)$, and an $\eta_0 \in T_{\gamma(t_0)}\Sigma$, there exists a unique parallel vector field η along γ such that $\eta_{\gamma(t_0)} = \eta_0$.

But local parallelism doesn't imply global parallelism. Consider a geodesic triangle on a sphere,[5] as in Figure 3. If you start with a vector tangent along the upper left piece and parallel-transport it to the lower

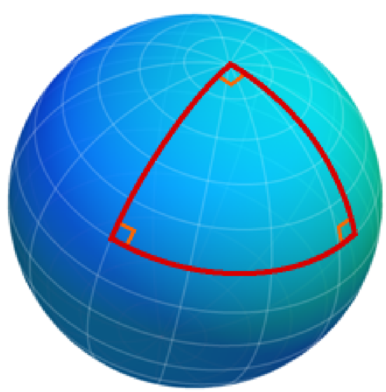

FIGURE 3. A geodesic triangle on the sphere. Each line is a piece of a great circle, and all three angles are right angles. Source: `http://world.mathigon.org/Dimensions_and_Distortions`.

left corner, then parallel-transport it to the lower-right corner, then parallel-transport it back upm to the pole, you'll end up with a different vector than you started with.

Parallel-transport can be thought of as a way of isometrically identifying tangent spaces (which we can canonically do in Euclidean space, but not always on manifolds).

Proposition 6.1. *Let $\gamma\colon (a, b) \to \Sigma$ be a curve and η_1, η_2 be parallel vector fields along γ. Then, $\gamma\eta_1, \eta_2\colon \gamma \to \mathbb{R}$ is constant.*

[5] There are other surfaces than spheres, of course! Check out the homework for some examples.

Proof. Let $\xi = \dot\gamma$, so
$$\xi\langle\eta_1, \eta_2\rangle = \langle\nabla_\xi\eta_1, \eta_2 + \eta_1, \nabla_\xi\eta_2\rangle = 0. \qquad \boxtimes$$

If γ is closed, traveling along γ from a point p to itself will produce an isometry of $T_p\Sigma$, but not always the identity: in the above example, it was a nontrivial rotation. This is called the *holonomy* around the loop. If the top angle is θ (and the sphere has radius 1), the area of the triangle is θ.

Now let's discuss the geometry of the normal component $B(\xi_p, \eta)$.

Lemma 6.2.

(1) *If $f \in \Omega^0_\Sigma$, $B(\xi_p, f\eta) = fB(\xi_p, \eta)$.*
(2) *B is the second fundamental form.*

Remark 6.3. If X is a manifold, $\mathcal{X}(X)$ denotes the space of vector fields on X. We say T is linear if it's $\mathbb{R}$-linear, i.e. for all $\xi, \xi' \in \mathcal{X}(X)$ and $\lambda \in \mathbb{R}$,
$$T(\lambda\eta + \eta') = \lambda T(\eta) + T(\eta').$$
We say T is *linear over functions* if it's Ω^0_X-linear, i.e. for any $f \in \Omega^0_X$ (i.e. $C^\infty(X)$), $T(f\eta) = fT(\eta)$. This means T doesn't differentiate f or anything like that, e.g. $\nabla_{\xi_p}(f\eta) = (\xi_p f)\eta + f\nabla_{\xi_p}\eta$ is not linear over functions. If T is linear over functions, it defines a cotangent vector field. ◀

Proof of Lemma 6.2. For the first part, $B(\xi_p, \eta) = \langle D_{\xi_p}\eta, \nu\rangle$, so at p,
$$\begin{aligned} B(\xi_p, f\eta) &= \langle D_{\xi_p} f\eta, \nu\rangle = \langle(\xi_p f)\eta + fD_{\xi_p}\eta, \nu\rangle \\ &= f(p)\langle D_{\xi_p}\eta, \nu\rangle = f(p)B(\xi_p, \eta). \end{aligned}$$

So B determines a bilinear map $B : T_p\Sigma \times T_p\Sigma \to \mathbb{R}$.[6] Let's see that it agrees with II.

Choose coordinates (x^1, x^2) such that Σ is the graph of $f(x^1, x^2)$. Then, $\frac{\partial}{\partial x^1} = (1, 0, f_1)$ and $\frac{\partial}{\partial x^2} = (0, 1, f_2)$ as normal. Write
$$\xi_p = \xi^i \frac{\partial}{\partial x^i}\bigg|_p \qquad \text{and} \qquad \eta = \eta^i \frac{\partial}{\partial x^i}\bigg|_p,$$
for $\xi^i, \eta^i \in \mathbb{R}$, we want to extend η_p to a map on local vector fields. We have liberty in this extension, so let's make our life easier and set $\eta^i(x^1, x^2) = \eta^i$, so it's constant. Thus, $\eta = (\eta^1, \eta^2, \eta^i f_i)$, so at $(x^1, x^2) = (0, 0)$,
$$D_{\xi_p}\eta = (0, 0, \eta^i(\xi_p f_i)) = (0, 0, \eta^i\xi^j f_{ij}).$$
Since $\nu_p = (0, 0, 1)$,
$$B(\xi_p, \eta_p) = f_{ij}\xi^i\eta^j = II(\xi_p, \eta_p). \qquad \boxtimes$$

This provides a coordinate-free interpretation of the second fundamental form, which is nice.

The first chapter of Warner's "Foundations on Differentiable Manifolds and Lie Groups" is a good reference for a lot of this material.

Anyways, this means the directional derivative is
$$D_{\xi_p}\eta = \nabla_{\xi_p}\eta + II_p(\xi_p, \eta_p)\nu_p.$$
We can use this to derive a coordinate-free interpretation of the shape operator:
$$\begin{aligned} II(\xi, \eta) &= \langle D_\xi\eta, \nu\rangle = \xi\langle\eta, \nu\rangle - \langle\eta, D_\xi\nu\rangle \\ &= -\langle\eta, D_\xi\nu\rangle = -\langle\eta, \mathrm{d}\nu(\xi)\rangle, \end{aligned}$$
so $S_p = -\mathrm{d}\nu_p$.

[6] Equivalently, $B(\xi_p, \eta)$ only depends on the value of η at p.

Though we'll begin talking about abstract Riemannian manifolds, these concrete examples, which you can draw, make these ideas clearer, and have fairly direct analogues in the abstract setting.

The choice of a unit tangent vector on a curve is discrete: on each connected component, you can flip between e_1 and $-e_1$. On a surface, though, it's possible to rotate a local frame $\{e_1, e_2, e_3\}$ (where e_1 is normal and e_2 and e_3 are tangential), so there's a continuous choice, and this continues to be true in higher dimensions.

In mathematics, a common approach to studying a situation where one needs to make a choice is to study all choices (sometimes you can make a convenient choice). Thus, we'll have to study this and other structures attached to smooth manifolds, including Lie groups, Lie derivatives, and a little geometry of smooth manifolds. But the payoff is that understanding how the frames change determines a lot of the Riemannian geometry.

Vector fields. Let X be a smooth manifold and ξ be vector field on X.

Definition 6.4. A piecewise-smooth curve $\gamma : (a, b) \to X$ is an *integral curve* of ξ if for all $t \in (a, b)$, $\dot{\gamma}(t) = \xi_{\gamma(t)}$.

That is, ξ is tangent along γ. We'd like to impose some constant-velocity constraint on this, but need a Riemannian metric to do that. It's possible to show that integral curves always exist, by starting with a finite approximation, iterating in a nice manner, and using some soft analysis (the contraction mapping theorem) to show there's a solution.

Theorem 6.5. *Given an $x_0 \in X$ and a vector field ξ on X, there's a unique maximal integral curve $\gamma : (a(x_0), b(x_0)) \to x$ (where $a, b \in [-\infty, \infty]$), such that $\gamma(0) = x_0$ and if $\mu(a, b) \to X$ is an integral curve for ξ with $\mu(0) = x_0$, then $a(x_0) \le a < 0 < b \le b(x_0)$ and $\mu = \gamma|_{(a,b)}$.*

This curve will be called γ_{x_0}.

Definition 6.6. With notation as in the above definition, γ is *complete* if for all $x_0 \in X$, $a(x_0) = -\infty$ and $b(x_0) = \infty$.

Here's a useful sufficient condition:

Theorem 6.7. *If for some Riemannian metric $\langle -, - \rangle$, $\|\xi\| : X \to \mathbb{R}^{\ge 0}$ is bounded, then ξ is complete.*

Corollary 6.8. *If X is a compact manifold, all ξ are complete.*

We would like to travel along a vector field. Let $\varphi(t, x) := \gamma_x(t)$; if ξ is complete, then $\varphi\colon \mathbb{R} \times X \to X$ is well-defined. Otherwise, you may find yourself flowing off the end of the world!

Definition 6.9. A *flow* on a manifold X is a (discrete) group homomorphism $\widehat{\varphi} : \mathbb{R} \to \mathrm{Diff}(X)$ (the latter group is under composition) such that the action map $\varphi\colon \mathbb{R} \times X \to X$ is C^∞.

That is, we ask for $\widehat{\varphi}(t_1 + t_2) = \widehat{\varphi}(t_2) \circ \widehat{\varphi}(t_1)$ and $\varphi(t_1 x) = \widehat{\varphi}(t)(x)$. We don't know how to express smoothness on $\mathrm{Diff}(X)$, so the smoothness criterion is stated in terms of the finite-dimensional manifolds $\mathbb{R}$ and X.

Given a vector field ξ, define for some $t \in \mathbb{R}$ the set

$$\mathcal{D}_t := \{x \in X \mid t \in (a(x), b(x))\}.$$

The following theorem rests on a proof of Theorem 6.5. This is often left unproven in geometry textbooks, but can be found, e.g. in Lang's ODE book or in Coddington-Levinson.

Theorem 6.10.

(1) *$\mathcal{D}_t$ is open.*
(2) *The map $\varphi_t : \mathcal{D}_t \to \mathcal{D}_{-t}$ is a diffeomorphism.*
(3) *The domain of $\varphi_{t_2} \circ \varphi_{t_1}$ is a subset of the domain of $\varphi_{t_1+t_2}$, and on that domain, $\varphi_{t_2} \circ \varphi_{t_1} = \varphi_{t_1+t_2}$.*
(4) *If $x \in X$ and $U \subset X$ is an open set containing x, then there's a $V \subset U$ and an $\varepsilon > 0$ such that $\varphi(-\varepsilon, \varepsilon) \times V$ maps into U.*
(5) *If ξ is complete, then $\mathcal{D}_t = X$ for all t, and φ is a global flow.*

This all relies on ξ being fixed with time (an *autonomous system*). If $\xi = \xi(t)$ varies with time, then a lot of these arguments don't work; in particular $\varphi_{t_2} \circ \varphi_{t_1} \neq \varphi_{t_1+t_2}$. Fortunately, we can use a neat technique to dispatch these.

A vector field ξ is a section of the tangent bundle $p : TX \to X$, and a time-varying vector field $\xi(t)$ is a section of the pullback: if $\pi_2 \colon (a,b) \times X \to X$ denotes projection onto the second component, $\widetilde{\xi}$ is a section of the pullback $\pi_2^* TX \to (a,b) \times X$. That is, on each time-slice, you get a section of TX, and these vary smoothly.

Let $\widehat{\xi} = \frac{\partial}{\partial t} + \widetilde{\xi}$, so $\widehat{\xi}$ is a vector field on $(a,b) \times X$. Given an initial condition (t_0, x_0), Theorem 6.5 says there's an integral curve $\widehat{\gamma} \colon (\widehat{a}, \widehat{b}) \to (a,b) \times X$. Letting $\widehat{\gamma}(t) = (t, \gamma(t))$, then $\dot{\widehat{\gamma}}(t) = \widehat{\xi}_{\widehat{\gamma}(t)}$. This is $(1, \dot{\gamma}(t)) = (1, \widetilde{\xi}(t))$, so $\gamma(t)$ is what we were looking for, and the solution exists, at least locally!

Once we have this flow, we're going to look at what happens if you carry various objects along the flow, e.g. vector fields or differential forms.

Lecture 7.

Tangential structures of manifolds: 2/7/17

Today, we'll talk about tangential structures: vector fields, subspaces of the tangent bundle, etc. We'll later dualize and look at functions and differential forms, and still later use this to understand Lie groups.

Let X be a smooth manifold and $\xi \in \mathcal{X}(X)$ be a vector field on it. Let φ_t be the (local) flow generated by ξ: given a vector field, traveling along its integral curves moves points along the flow, and we can therefore flow all sorts of other objects: functions, vectors, differential forms, connections. . .

The Lie derivative is the instantaneous change in a quantity as you flow it. By a *covariant* object we mean something which pushes forward along maps, e.g. vectors. Similarly, *contravariant* things are those which pull back under maps.

Definition 7.1. Let T be a covariant object. Then, the *Lie derivative* of T is

$$\mathcal{L}_\xi T := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\varphi_{-t})_* T.$$

If T is a contravariant object, then the Lie derivative of T is

$$\mathcal{L}_\xi T := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\varphi_t)^* T.$$

Example 7.2. Let $f \in \Omega^0_X$ be a function. Functions pull back, so this is contravariant: $\varphi_t^* f(p) = f(\varphi_t(p))$. Thus, the Lie derivative of f is

$$\mathcal{L}_\xi f(p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varphi_t^* f(p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\varphi_t(p)).$$

Thus, this is the directional derivative along a curve $\gamma : t \mapsto \varphi_t(p)$, since $\gamma(0) = p$ and $\dot{\gamma}(p) = \xi_p$. In symbols, $\mathcal{L}_\xi f(p) = \mathrm{d}f|_p(\xi_p)$. ◀

Example 7.3. Let $\eta \in \mathcal{X}(X)$ be a vector field. To compute its Lie derivative, let's introduce local coordinates $x^1, \dots, x^n$, so $\xi = \xi^i \frac{\partial}{\partial x_i}$, $\eta = \eta^j \frac{\partial}{\partial x^j}$, and $\varphi(t,x) = (\varphi^1(t,x), \dots, \varphi^n(t,x))$. Since φ_t is the flow, it satisfies

$$\dot{\varphi}(t,x) = \xi_{\varphi(t,x)}, \qquad \left.\frac{\partial \varphi^i}{\partial x^j}\right|_{t=0} = \delta^i_j, \quad \text{and} \quad \left.\frac{\partial^2 \varphi^i}{\partial x^j \partial x^k}\right|_{t=0} = 0.$$

Therefore we compute

$$(\varphi_{-t})\eta = (\varphi_{-t})_* \left(\eta^i \frac{\partial}{\partial x^i} \right) = \left(\varphi_t^* \eta^i \right) (\varphi_{-t})_* \frac{\partial}{\partial x^i}.$$

This pullback and pushforward are

$$\begin{aligned}
\varphi_t^*\eta^i &= \eta^i\Big(\varphi^1(t,x),\ldots,\varphi^n(t,x)\Big)\\
(\varphi_{-t})_*\frac{\partial}{\partial x^i} &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{t=0}\varphi_{-t}\Big(x^1,\ldots,x^i+s,\ldots,x^n\Big)\\
&= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}\Big(\varphi^j\Big(-t,x^1,\ldots,x^i+s,\ldots,x^n\Big)\Big)_j\\
&= \frac{\partial\varphi^j}{\partial x^i}(\varphi(-t,x))\cdot\frac{\partial}{\partial x^j}.
\end{aligned}$$

Putting these together,

$$(\varphi_{-t})\eta = \eta^i(\varphi(t,x))\cdot\frac{\partial\varphi^j}{\partial x^i}(\varphi(-t,x))\frac{\partial}{\partial x^j}$$

and

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(\varphi_{-t})_*\eta &= \left.\left(\frac{\partial\xi^i}{\partial\eta^j}\dot\varphi^k\frac{\partial\varphi^j}{\partial x^i}\frac{\partial}{\partial x^j} - \eta^i\frac{\partial^2\varphi^j}{\partial x^k\partial x^i}\dot\varphi^k\frac{\partial}{\partial x^j} - \eta^i\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\varphi^j}{\partial x^i}\right)\frac{\partial}{\partial x^j}\right)\right|_{t=0}\\
&= \frac{\partial\eta^i}{\partial x^k}\xi^k\delta_i^j\frac{\partial}{\partial x^j} - \eta^i\frac{\partial\xi^j}{\partial x^i}\frac{\partial}{\partial x^j}\\
&- \left(\xi^i\frac{\partial\eta^j}{\partial x^i} - \eta^i\frac{\partial\xi^j}{\partial x^i}\right)\frac{\partial}{\partial x^j}.
\end{aligned}$$

You might think this formula in coordinates is boring; if so, do it yourself. ◂

Therefore we have proven:

Proposition 7.4 (Lie derivative of vector fields in local coordinates)**.**

$$\mathcal{L}_{\xi^i\partial/\partial x^i}\eta^j\frac{\partial}{\partial x^j} = \xi^i\frac{\partial\eta^j}{\partial x^i}\frac{\partial}{\partial x^j} - \eta^j\frac{\partial\xi^i}{\partial\eta^j}\frac{\partial}{\partial x^i}.$$

Notice the almost symmetry between ξ and η in the above result.

Corollary 7.5. $\mathcal{L}_\xi\eta = -\mathcal{L}_\eta\xi$.

Let's take a different point of view. Recall that the vector fields on X are alternatively the *derivations* $\xi : \Omega^0_X \to \Omega^0_X$, i.e. linear maps satisfying the Leibniz rule

$$\xi(fg) = (\xi f)g + f(\xi g).$$

Hopefully you proved this in a differential topology class; if not, it's an exercise! You could also use the more general criterion that anything tensorial over functions is a tensor.

Definition 7.6. If $\xi,\eta\in\mathcal{X}(X)$, their *Lie bracket* $[\xi,\eta]$ is the derivation

$$f \longmapsto \xi(\eta f) - \eta(\xi f). \tag{7.7}$$

Lemma 7.8. (7.7) *satisfies the Leibniz rule and hence is actually a derivation.*

Proof. Applying it to fg, we get

$$\begin{aligned}
\xi(\eta(fg)) - \eta(\xi(fg)) &= \xi(\eta(f)\cdot g - f\eta(g)) - \eta(\xi(f)\cdot g - f\xi(g))\\
&= \xi\eta f\cdot g + \eta f\cdot\xi g + \xi f\cdot\eta g + f\xi\eta g - \eta\xi f\cdot g - \xi f\cdot\eta g - \eta f\cdot\xi g - f\eta\xi g.
\end{aligned}$$

The cross terms cancel, so this is

$$= ([\xi,\eta]f)\cdot g + f\cdot[\xi,\eta]g.$$ ☒

This is in fact the same thing as the Lie derivative!

Proposition 7.9. *Let* $\xi,\eta\in\mathcal{X}(X)$.

(1) *The Lie bracket is the Lie derivative:* $[\xi,\eta] = \mathcal{L}_\xi\eta$.
(2) *The Lie bracket is antisymmetric:* $[\eta,\xi] = -[\xi,\eta]$.

(3) *(Jacobi identity)*
$$[[\xi,\eta],\zeta]+[[\eta,\zeta],\xi]+[[\zeta,\xi],\eta]=0.$$
(4)
$$[f\xi,g\eta]=fg[\xi,\eta]+f(\xi g)\eta-g(\eta f)\xi.$$

Parts (2) and (3) follow formally from properties of the commutator in an associative algebra, and parts (1) and (4) can be checked in local coordinates. This uses the fact that mixed partials commute, or for all i and j,
$$\left[\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right]=0.$$
So the Lie bracket vanishes for coordinate vector fields, but may not vanish in general, reflecting yet another obstruction to global parallelism; the Lie bracket measures the failure of vector fields to commute as derivations.

There are a couple of other forms of the Jacobi identity. Some of them say that $\mathcal{L}_\xi=[\xi,-]$ is a derivation, meaning it satisfies the Leibniz rule:
$$\mathcal{L}_\xi[\eta,\zeta]=[\mathcal{L}_\xi\eta,\zeta]+[\eta,\mathcal{L}_\xi\zeta].$$
Let $\xi,\eta\in\mathcal{X}(X)$, and let φ_t, ψ_s be their local flows. Consider flowing along a rectangle: t and s are small, and we flow by $\psi_{-s}\varphi_{-t}\psi_s\varphi_t$. That the Lie bracket isn't always zero means this flow might not get back to where it started.

Specifically, choose local coordinate $x^1,\dots,x^n$, and let $(t,s)\mapsto x^i(t,s)$ be the map sending $(t,s)\mapsto\psi_{-s}\varphi_{-t}\psi_s\varphi_t(p)$. This maps the rectangle into the manifold. The two axes (where $s=0$ or $t=0$) are collapsed onto p.

Proposition 7.10. *In this case,*
$$\mathcal{L}_\xi\eta(p)=\left.\frac{\partial^2x^i}{\partial t\partial s}\right|_{\substack{s=0\\t=0}}\frac{\partial}{\partial x^i}.$$

Proof. The proof is, again, a calculation.
$$\left.\frac{\partial x^i}{\partial s}\frac{\partial}{\partial x^i}\right|_{s=0}=-\eta=(\varphi_{-t})_*\eta.$$
The first term is constant with respect to t, so disappears when we differentiate with respect to r, and the second term becomes $\mathcal{L}_\xi\eta(p)$ by the definition of the Lie derivative. ☒

These computations aren't just a nuisance: there's lots of ways to think about them or to choose notation, and these choices of notation are particularly nice for not making mistakes, etc. They also demonstrate how to think about local coordinates.

Now, suppose ξ and η are complete (so the flow exists for all time), so we can make a global statement. Pushing forward a vector field along a map $X'\to X$ doesn't in general define a vector field: if the map isn't surjective, there's not a vector at every point, and if it's not injective, there may be multiple choices for the vector at a given point. If $\psi\colon X\to X$ is a diffeomorphism, however, you can push vector fields forward, and $\psi_*\xi$ generates the flow $\psi\circ\varphi_t\circ\psi^{-1}$. (This is a nice exercise using the existence and uniqueness of ODEs.) Thus, $\psi_*\xi=\xi$ iff $\psi\varphi_t\psi^{-1}=\varphi_t$ for all t, and therefore $(\varphi_s)_*\xi=\xi$ for all s iff $\psi_s\varphi_t\psi_{-s}\varphi_{-t}=0$ for all s and t. This is one direction of the following.

Proposition 7.11. $[\xi,\eta]=0$ *iff for all s and t, $\psi_s\varphi_t\psi_{-s}\varphi_{-t}=\mathrm{id}$, where ψ_s and φ_t are the flows associated with η and φ, respectively.*

The direction we didn't prove follows from observing that $[\eta,\xi]=\frac{\mathrm{d}}{\mathrm{d}s}(\psi_{-s})_*\xi=0$. The situation in this proposition might hold more generally, however.

Definition 7.12. Let $\psi\colon X'\to X$ be a smooth map, $\xi'\in\mathcal{X}(X')$, and $\xi\in\mathcal{X}(X)$. Then, ξ' and ξ are *ψ-related* if $(\psi_*)_{p'}\xi'_{p'}=\xi_{\psi(p')}$ for all $p'\in X'$.

The consequence is that this preserves Lie bracket data.

Proposition 7.13. *Suppose ξ' and ξ are ψ-related and η' and η are ψ-related. Then, $[\xi',\eta']$ and $[\xi,\eta]$ are ψ-related.*

You can (and should) prove this yourself, using the formula for the Lie bracket as the commutator of the derivatives of the flows. This is part of a general principle, that to prove something about vector fields, it's useful to think of them as infinitesimally small curves. Alternatively, you could translate this into the language of derivations and prove it that way.

Another useful thing you can do with vector fields is use them to define local coordinate systems.

Theorem 7.14. *Let $\xi \in \mathcal{X}(X)$ and $p \in X$ be such that $\xi_p \neq 0$. Then, there exists a local chart U containing p and local coordinates $(x^1, \dots, x^n)\colon U \to X$ such that $\frac{\partial}{\partial x^1} = \xi$ on all of U.*

Proof sketch. Let $(y^1, \dots, y^n)$ be local coordinates about p such that p maps to the origin, so $y^i(p) = 0$, and $\left.\frac{\partial}{\partial y^1}\right|_p = \xi_p$. Let φ_t be the flow generated by ξ. Then, define

$$(x^1, \dots, x^n) \longmapsto \varphi_{x^1}(0, x^2, \dots, x^n), \tag{7.15}$$

where on the right-hand side, the argument of φ_{x^1} is written in y^i-coordinates. Then, using y^i-coordinates, you can check that the differential at 0 is invertible. This means (7.15) defines a local coordinate system, and the computation will show that $\frac{\partial}{\partial x^1} = \xi$. ⊠

Now let's do this with multiple vector fields.

Theorem 7.16. *Let $k \leq \dim X$ and $\xi_1, \dots, \xi_k \in \mathcal{X}(X)$. If $\{\xi_1|_p, \dots, \xi_k|_p\}$ are linearly independent at a $p \in X$, then there exists a local chart U containing p and local coordinates $(x^1, \dots, x^n) : U \to X$ with $\frac{\partial}{\partial x^i} = \xi_i$ for $i = 1, \dots, k$ iff $[\xi_i, \xi_j] = 0$ for all $1 \leq i, j \leq k$.*

Proof sketch. Let $y^1, \dots, y^n$ be local coordinates about p such that $y^i(p) = 0$ for all i and $\left.\frac{\partial}{\partial y^i}\right|_p = \xi_p$ for $i = 1, \dots, k$. Let $\varphi_t^{(i)}$ be the flow generated by ξ_i, and let

$$(x^1, \dots, x^n) \longmapsto \varphi_{x^k}^{(k)} \cdots \varphi_{x^2}^{(2)} \varphi_{x^1}^{(1)} (0, \dots, 0, x^{k+1}, \dots, x^n).$$

Again, the argument of φ is in y^i-coordinates. Now, check that the differential is the identity (in y^i-coordinates) and $\frac{\partial}{\partial x^i} = \xi_i$ in the same way. However, this will use that $\varphi^{(i)}$ and $\varphi^{(j)}$ commute, which is true iff the Lie brackets vanish. ⊠

So defining a coordinate system through vector fields only works if they commute. Think back to Riemannian geometry: we studies surfaces by introducing special coordinates at each point, but we may not be able to promote this to an orthonormal frame. Being a coordinate system and being orthonormal are in tension, and what measures this tension is the Riemann curvature tensor.

You might want to generalize this to plane fields or hyperplane fields at a point instead of just vector fields. Then, you'll get integral manifolds, and this perspective can be useful to define maps between manifolds. Again, there will be a condition about commutators, encoded in a theorem due to Klebsch (and attributed to Frobenius).

Lecture 8.

Distributions and Foliations: 2/9/17

Today, we'll work through distributions, the local Frobenius theorem, foliations, and some other things preparing us for Lie groups. After that, we'll be able to return to Riemannian geometry. Throughout today's lecture, X is a smooth manifold.

Definition 8.1.

(1) A *distribution* is a vector subbundle $E \subset TX \to X$.
(2) If E is a distribution, a vector field $\xi \in \mathcal{X}(X)$ *belongs to* E if ξ is a section of $E \to X$, i.e. $\xi_p \in E_p$ for all $p \in X$.
(3) E is *involutive* (or *integrable*) if whenever $\xi, \eta \in E$, $[\xi, \eta] \in E$ as well.

(4) A submanifold $Y \subset X$ is an *integral manifold* of E if for all $p \in Y$, $T_pY = E_p$.

You can think of a distribution as a hyperplane field at every point, distinguishing the directions in T_pX that are contained in E_p. If ξ belongs to E, then at every point it's contained in the hyperplane defined at that point. So a distribution is first-order information for constructing a manifold: we've specified the tangent space, and want to find a curved manifold which satisfies it.

Proposition 8.2. *If* $\operatorname{rank} E = 1$, *$E$ is involutive.*

These E are also called *line fields*.

Proof. We can work locally: choose an open $U \subset X$ and a nonzero section e of $E|_U \to U$, so that any $\xi, \eta \in \Gamma(E|_U)$ can be written as $\xi = fe$ and $\eta = ge$ for $f, g \in \Omega^0_U$. Then, compute:

$$\begin{aligned}[\xi, \eta] &= [fe, ge] = fg[e,e] + f(e \cdot g)e - g(e \cdot f)e \\ &= [f(e \cdot g) - g(e \cdot f)]e,\end{aligned}$$

which is also a section of $E|_U$. ☒

Integral manifolds to line fields also exist: locally choose a nonvanishing ξ belonging to E and choose its integral curve.

Remark 8.3. However, not every line field admits a global nonvanishing section. Let X be the Möbius band, the total space of a line bundle $\pi \colon X \to S^1$ defined by gluing $\mathbb{R} \times [0,1] \to [0,1]$ with degree -1. We'll let E be the copy of X inside the tangent bundle. That is, π is a submersion, so we have a short exact sequence of vector bundles

$$0 \longrightarrow T(X/S^1) \longrightarrow TX \longrightarrow \pi^* TS^1 \longrightarrow 0.$$

This has no natural splitting (splittings will be called covariant derivatives later). Since the fiber X/S^1 is a vector space V, then its tangent space at each $v \in V$ is identified with V again. We choose E to be this subspace of TX, and ultimately because the Möbius strip isn't orientable, E admits no global nonvanishing section. ◀

Example 8.4. Let $X = \mathbb{A}^3_{x,y,z}$ and

$$E := \operatorname{span}\left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} + x\frac{\partial}{\partial z} \right\}.$$

Then, for any $(x_0, y_0, z_0) \in \mathbb{A}^3$, there's a piecewise smooth map γ from $(0,0,0)$ to (x_0, y_0, z_0) with $\dot{\gamma} \in E$. The idea is to zigzag from varying x to varying y and z.

However, E is not involutive:

$$\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y} + x\frac{\partial}{\partial z} \right] = \frac{\partial}{\partial z} \notin E.$$ ◀

Checking involutivity seems kind of hard, but there's a nice criterion. We know $\operatorname{span}\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$ is involutive, and this turns out to be the key.

Theorem 8.5 (Local Frobenius)**.** *Let $E \subset TX$ be a distribution of rank k.*

(1) *E is involutive iff about every $p \in X$, there exist local coordinates $x^1, \dots, x^n$ on some $U \subset X$ containing p such that*

$$E|_U = \operatorname{span}\left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^k} \right\}. \tag{8.6}$$

(2) *If this is true, then any connected integral manifold $Y \subset U$ has the form $x^m = c^m$, for $m = k+1, \dots, n$, for some $c^{k+1}, \dots, c^n \in \mathbb{R}$.*

One can imagine a structure on a manifold which is an atlas only of charts satisfying (8.6). This is a geometric structure in the same way that a Riemannian metric is a geometric structure. Part (2) says that in this case, the integral manifolds are parallel to the $x^1 \cdots x^k$-plane on U.

Proof. It suffices to prove the forward direction; the reverse direction is a computation. Choose local coordinates $y^1, \dots, y^n$ about p such that

(1) $y^i(p) = 0$ for each i,
(2) $y(U)$ has the form $-\varepsilon < y^i < \varepsilon$ for some $\varepsilon > 0$ (i.e. it's a cube),[7] and
(3) $E_p = \operatorname{span}\left\{\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^k}\right\}$.

Let $\pi : \mathbb{A}^n \to \mathbb{A}^k$ be projection onto the first k coordinates; then, shrinking ε if necessary, $\pi_* : \mathbb{R}^n \to \mathbb{R}^k$ sends E_y isomorphically onto $\mathbb{R}^k$ for all $y \in y(U)$. Let $\xi_1, \dots, \xi_k$ be defined to satisfy $\pi_* \xi_i = \frac{\partial}{\partial y^i}$.

Now we have a basis $\xi^1, \dots, \xi^k$ in E that is π_*-related to $\frac{\partial}{\partial y^i}$. In particular,

$$\pi_*[\xi_i, \xi_j] = \left[\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right] = 0.$$

If E is involutive, then we conclude that $[\xi_i, \xi_j] = 0$, so by Theorem 7.16, we can find local coordinates $x^1, \dots, x^n$ (shrinking ε if necessary) such tht $\frac{\partial}{\partial x^i} = \xi_i$ for $i = 1, \dots, k$. ⊠

This is a very geometric approach to a question originally motivated by systems of differential equations; involutivity is the analogue of mixed partials commuting.

The coordinate system guaranteed by Theorem 8.5 for an involutive E is called an *E-coordinate system* or a *slice*.

Example 8.7. Consider $\mathbb{A}^2$ and E be a one-dimensional constant distribution. Then, the integral submanifolds of $\mathbb{A}^2$ are the lines parallel to E. Quotient out by $\mathbb{Z}^2 \subset \mathbb{R}^2$ and let $X = \mathbb{A}^2/\mathbb{Z}^2$ be the torus. The integral manifolds project to integral submanifolds of the torus, but their structure depends on the embedding $\mathbb{Z}^2 \subset \mathbb{R}^2$.

If the images of $(1,0)$ and $(0,1)$ are related by rational numbers, then each integral manifold has finite length and closes up to a circle. For example, it could be a circle that traverses each direction of the torus once, or more than once. If the integral manifolds traverse one direction p times and the other q times, this characterizes the homology class $(p,q) \in H_1(\mathbb{A}^2/\mathbb{Z}^2) \cong \mathbb{Z}^2$. This is a *foliation*, and the integral submanifolds are called *leaves*. In this case, the quotient space parameterizing the leaves is a manifold, specifically S^1.

If instead the images of $(1,0)$ and $(0,1)$ aren't related by rational numbers, then each integral manifold has infinite length, and is in fact dense in the torus! So we get an immersed submanifold, not an embedded one. This is a more pathological case, and is one of the reasons we require manifolds to be second countable. The quotient space is still a topological space, but not Hausdorff. ◂

Example 8.8. The *Hopf fibration* is a map $\pi\colon S^3 \to S^2$. Identify S^3 as the unit sphere in $\mathbb{C}^2$, the set $\{(\xi^1, \xi^2) \in \mathbb{C}^2 \mid |\xi^1|^2 + |\xi^2|^2 = 1\}$. Then, identify $S^2 = \mathbb{CP}^2$, the space of 1-dimensional subspaces in $\mathbb{C}^2$. The Hopf fibration sends an $x \in S^2$ to the line containing it.

Each fiber of π is a circle $S^1 \subset S^3$, which is actually unknotted. Thinking of $S^3 = \mathbb{A}^3 \cup \{*\}$, there's one fiber which is a straight line, and the rest are circles winding around this line. Any two circlular fibers are linked with linking number 1. There's a distribution $E = \ker \pi_* \subset TS^3$. The fibers of the Hopf fibration are integral manifolds for E. ◂

The leaf space of a foliation is sometimes a manifold, but not always; in fact, it's an example of a *noncommutative space*, a significant example in Alain Connes' noncommutative geometry.

Definition 8.9. A *k-dimensional foliation* on X is a decomposition of X as

$$X = \coprod_{\alpha \in A} \mathscr{F}_\alpha,$$

where each $\mathscr{F}_\alpha$ is a k-dimensional immersed submanifold, such that for every $p \in X$, there exist local coordinates $x^1, \dots, x^n$ on a $U \subset X$ such that $\mathscr{F}_\alpha \cap U$ is a union of slices $x^m = c^m$ (for $m = k+1, \dots, n$). These $\mathscr{F}_\alpha$ are called the *leaves* of the foliation $\mathscr{F}$.

The *tangent bundle* to a foliaton $\mathscr{F}_\alpha$ is

$$T\mathscr{F} := \coprod_{p \in X} T_p \mathscr{F}_{\alpha(p)},$$

where $p \in \mathscr{F}_{\alpha(p)}$.

[7]This is easy to accomplish: given any local coordinates, we can pick the open cube of side length ε around 0 and restrict y to it.

$T\mathscr{F} \to X$ is locally trivial; in particular, it's a vector bundle. Foliations are a richly studied subject.

Theorem 8.10. *Let $E \subset TX$ be an involutive distribution. Then, there exists a foliation $\mathscr{F}$ such that $T\mathscr{F} = E$. In particular, the leaves are maximal integral manifolds for E, and there is a unique such maximal integral submanifold though every point.*

So this is the solution to the differential equation that E defines; rather than solve it at each point, we solve it at all points at once.

We will often only need one leaf when we apply this, but the full generality of the theorem is too beautiful to pass up.

Proof sketch of Theorem 8.10. Second-countability of X is important in this theorem: we cover X by a *sequence* of charts $U_1, U_2, \ldots$ of E-coordinate systems. If $S \subset U_i$ is a slice, then $\pi_0(S \cap U_j)$ is either empty, finite, or countably infinite.

Introduce an equivalence relation on slices where $S \subset U_i$ and $T \subset U_j$ are equivalent if there's a fintie sequence $i_0 = i, i_1, i_2, \ldots, i_N = j$ and slices $S_{i_j} \subset U_{i_j}$ such that $S_{i_j} \cap S_{i_{j+1}} \neq \varnothing$, $S_0 = S$, and $S_N = T$. Let $\mathscr{F}_\alpha$ be the union of the slices S in an equivalence class; these will be the leaves in the foliation.

Clearly $\{\mathscr{F}_\alpha\}$ is a decomposition of X into a disjoint union of pieces, but we need each inclusion map $\mathscr{F}_\alpha \to X$ to be an immersion, which is data. The construction we've defined already does this: it's covered by these U_{i_j}. Then, since there's only countably many charts, $\mathscr{F}_\alpha$ must be second countable. ⊠

This implies the distribution in Example 8.4 isn't involutive: we saw how to get from the origin to any point, but if it were involutive, there would have to be a foliation, and there would be no way to travel between different leaves in a foliation.

We'll now pass to something more formal, which involves functions on X. This involves drawing fewer pictures, which is maybe a little unfortunate, but the calculations we can make are useful.

Let $\Omega_X^\bullet$ denote the algebra of differential forms on X, which is a $\mathbb{Z}$-graded algebra, graded by

$$\Omega_X^\bullet = \bigoplus_{k\in\mathbb{Z}} \Omega_X^k.$$

The algebra structure is wedge product: $\alpha, \beta \mapsto \alpha \wedge \beta$. It's commutative, in the sense of a $\mathbb{Z}$-graded algebra: if α and β are homogeneous elements of degrees $|\alpha|$, resp. $|\beta|$, then

$$\alpha \wedge \beta - (-1)^{|\alpha||\beta|}\beta \wedge \alpha. \tag{8.11}$$

This is called the *Koszul sign rule*.

Definition 8.12. Let $A^\bullet$ be a commutative (meaning (8.11) holds) $\mathbb{Z}$-graded algebra and $T\colon A^\bullet \to A^\bullet$. T is a *derivation* if it's a linear map of degree s[8] and T satisfies the Liebniz rule

$$T(\alpha \wedge \beta) = T\alpha \wedge \beta + (-1)^{s|\alpha|}\alpha \wedge T\beta.$$

Example 8.13.

(1) The *Cartain-de Rham differential* $\mathrm{d} : \Omega_X^\bullet \to \Omega_X^{\bullet+1}$ is a derivation of degree 1.
(2) Let $\xi \in \mathscr{X}(X)$. If $\alpha \in \Omega_X^\bullet$ and ξ generates the flow φ_t, then the Lie derivative is a derivation of degree 0. This boils down to it satisfying the Leibniz rule: to verify this, use the fact that the pullback commutes with wedge product:

$$\varphi_t^*(\alpha \wedge \beta) = \varphi_t^*\alpha \wedge \varphi_t^*\beta.$$

Now, you can differentiate this with the usual Leibniz rule. There's no sign here, but that's okay, because the degree is 0. ◂

[8] A linear map of $\mathbb{Z}$-graded algebras has *degree s* if it sends homogenenous elements of degree n to homogeneous elements of degree $n + s$ for all $n \in \mathbb{Z}$.

Given a vector field ξ, there's a derivation of degree -1 called contraction ι_ξ, which can be defined somewhat algebraically. We'll talk about this next time; so we now have three operators on differential forms of degrees 1, 0, and -1; under Lie bracket, we get some interesting relations between them, namely

$$\mathscr{L}_\xi = [\mathrm{d}, \iota_\xi].$$

There's one more operation on differential forms, integration over the fiber in the context of a fiber bundle. If you know all the brackets between these four operators, you can recover the calculus of differential forms and therefore calculus in the usual sense on X.

Lecture 9.

Lie algebras and Lie groups: 2/14/17

Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a $\mathbb{Z}$-graded algebra. (That is, multiplication sends $A_m \cdot A_n$ into A_{m+n}.) For example, if V is a real vector space, $\Lambda^\bullet V$ is a commutative $\mathbb{Z}$-graded algebra, because we specified commutativity by the Koszul sign rule

$$z \wedge w = (-1)^{|z||w|} w \wedge z.$$

Last time, we defined a derivation of degree d to be a linear map $T : A \to A$ of degree d (so $A_n \to A_{n+d}$ on homogeneous elements) such that

$$T(\alpha\wedge\beta) = T\alpha\wedge\beta + (-1)^{|T||\alpha|}\alpha\wedge T\beta.$$

Definition 9.1. Let V be a vector space and $\xi \in V$. Define $\iota_\xi\colon \Lambda^k V^* \to \Lambda^{k-1}V^*$ by

$$(\iota_\xi\alpha)(\xi_2,\dots,\xi_k) := \alpha(\xi,\xi_2,\xi_3\dots,\xi_k).$$

This is called *contracting with the first index*.

There is a natural identification $\Lambda^k V^*$ with $(\Lambda^k V)^*$ defined through a pairing $\Lambda^k V^* \otimes \Lambda^k V \to \mathbb{R}$ sending

$$\alpha_1\wedge\dots\wedge\alpha_k \otimes \xi,\dots,\xi \longmapsto \det(\alpha_i(\xi_j)_{i,j}),$$

for $\alpha_i \in V^*$ and $\xi_j \in V$.

Proposition 9.2.

(1) *ι_ξ is adjoint to left exterior multiplication $\varepsilon_\xi\colon \Lambda^{k-1}V \to \Lambda^k V$, the map sending $z \mapsto \xi\wedge z$.*
(2) *$\iota_\xi : \Lambda^\bullet V^* \to \Lambda^\bullet V^*$ is a derivation of degree -1.*
(3) *If $\xi,\eta \in V$, then $\iota_\xi\iota_\eta + \iota_\eta\iota_\xi = 0$.*

If we define the *commutator* of $a, b \in A$ to be

$$[a,b] := ab - (-1)^{|a||b|}ba,$$

then (3) can be restated as $[\iota_\xi,\iota_\eta] = 0$. For any odd-degree element a, $[a,a] = 2a^2$.

In a basis, $\xi = \xi^i e_i$ and

$$\alpha = \frac{1}{k!}\alpha_{i_1\dots i_k} e^{i_1}\wedge\dots\wedge e^{i_k},$$

so that contraction is

$$\iota_\xi\alpha = \frac{1}{(k-1)!}\xi^i\alpha_{i_1 i_2\dots i_k}e^{i_2}\wedge\dots\wedge e^{i_k}.$$

Proof sketch of Proposition 9.2. Part (1) is immediate from the definition.

For part (2), let $\alpha_1,\dots,\alpha_{k+\ell} \in V^*$, let $\alpha = \alpha_1\wedge\dots\wedge\alpha_k$ and $\beta = \alpha_{k+1}\wedge\dots\wedge\alpha_{k+\ell}$. Then,

$$\iota_{\xi_1}(\alpha\wedge\beta)(\xi_2\wedge\dots\wedge\xi_{k+\ell}) = \det(\alpha_i(\xi_j)),$$

and you can finish the proof by expanding along $j = 1$.

For part (3),

$$\begin{aligned}(\iota_\xi\iota_\eta\alpha)(z) &= \iota_\eta\alpha(\xi\wedge z) = \alpha(\eta\wedge\xi\wedge z).\\ (\iota_\eta\iota_\xi\alpha)(z) &= \iota_\xi\alpha(\eta\wedge z) = \alpha(\xi\wedge\eta\wedge z) = -\alpha(\eta\wedge\xi\wedge z).\end{aligned}$$

⊠

Now suppose X is a smooth manifold and $\xi \in \mathcal{X}(X)$. Then, $\iota_\xi \colon \Omega_X^\bullet \to \Omega_X^{\bullet-1}$ defined pointwise is a derivation of degree -1. Now we have three kinds of derivations on differential forms: d has degree 1, ι_ξ has degree -1, and $\mathcal{L}_\xi$ has degree 0.

The vector space of derivations on $\Omega_X^\bullet$ is closed under commutators, so it's worth asking what derivations we obtain from commutators of our three amigos. By symmetry, there are six ones to consider.

Theorem 9.3.

- $[\iota_\xi, \iota_\eta] = 0$.
- $[\mathcal{L}_\xi, \iota_\eta] = \iota_{[\xi,\eta]}$.
- $[\mathrm{d}, \iota_\eta] = \mathcal{L}_\eta$ *(the Cartan formula).*
- $[\mathcal{L}_\xi, \mathcal{L}_\eta] = \mathcal{L}_{[\xi,\eta]}$.
- $[\mathcal{L}_\xi, \mathrm{d}] = 0$.
- $[\mathrm{d}, \mathrm{d}] = 0$.

Some of these we've proven before; others are new, but can be computed from the definitions. You can quickly check that the degree of a commutator is the sum of the degrees of its two arguments.

The calculations above are in a sense generators and relations for a $\mathbb{Z}$-graded Lie algebra (with the Koszul sign rule).

Definition 9.4. A *Lie algebra* is a (real) vector space $\mathfrak{g}$ together with a bilinear form $[–,–] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that $[\xi, \eta] = -[\eta, \xi]$ and the *Jacobi identity* holds:

$$[[\xi, \eta], \zeta] + [[\eta, \zeta], \xi] + [[\zeta, \xi], \eta] = 0$$

for all $\xi, \zeta, \eta \in \mathfrak{g}$.

To define a graded Lie algebra, one must insert signs corresponding to the Koszul sign rule, e.g. the bracket satisfies

$$[\xi, \eta] = (-1)^{|\xi||\eta|}[\eta, \xi],$$

and there are additional signs in the graded Jacobi identity.

The commutation relations either follow directly or have been already proven, except for the second and third ones. Let's prove those. To check that two derivations on $\Omega_X^\bullet$ agree, it suffices to check on Ω_X^0 and $\mathrm{d}\Omega_X^0$, since every element of $\Omega_X^\bullet$ is (locally) a sum of wedges of these forms, and the Leibniz rule behaves the same for both derivations.

Let's prove that $[\mathcal{L}_\xi, \iota_\eta] = \iota_{[\xi,\eta]}$. If $f \in \Omega_X^0$, the right-hand side is 0 and the left-hand side is

$$[\mathcal{L}_\xi, \iota_\eta]f = -\iota_\eta \mathcal{L}_\xi f = 0.$$

Now let's check on $\mathrm{d}f$:

$$\begin{aligned}
[\mathcal{L}_\xi, \iota_\eta]\mathrm{d}f &= \mathcal{L}_\xi \iota_\eta \mathrm{d}f - \iota_\eta \mathcal{L}_\xi \mathrm{d}f \\
&= \mathcal{L}_\xi(\eta \cdot f) - \iota_\eta \, \mathrm{d}\mathcal{L}_\xi f \\
&= \xi\eta \cdot f - \eta\xi \cdot f \\
&= \iota_{[\xi,\eta]}\mathrm{d}f.
\end{aligned}$$

The Cartan formula is proved in a similar manner.

Corollary 9.5. *Let $\alpha \in \Omega_X^1$ and $\xi, \eta \in \mathcal{X}(X)$. Then,*

$$\mathrm{d}\alpha(\xi, \eta) = \xi \cdot \alpha(\eta) - \eta \cdot \alpha(\xi) - \alpha([\xi, \eta]).$$

This is a very useful formula, and sometimes is used to define d!

Proof.

$$\begin{aligned}
\mathrm{d}\alpha(\xi, \eta) &= \iota_\eta \iota_\xi \mathrm{d}\alpha \\
&= -\iota_\eta \, \mathrm{d}\iota_\xi \alpha + \iota_\eta \mathcal{L}_\xi \alpha \\
&= -\eta \cdot \alpha(\xi) + \mathcal{L}_\xi \iota_\eta \alpha - \iota_{[\xi,\eta]}\alpha \\
&= -\eta \cdot \alpha(\xi) + \xi \cdot \alpha(\eta) - \alpha([\xi, \eta]).
\end{aligned}$$

⊠

Let X be a smooth manifold and $E \subset TX$ be a distribution as last time. Let

$$\mathscr{I}(E) := \{\alpha \in \Omega_X^\bullet : \alpha|_E = 0\}.$$

That is, $\mathscr{I}(E)$ is the annihilator of E. For example, on $\mathbb{A}^3$, if

$$E = \operatorname{span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}\right\},$$

$\mathscr{I}(E) = \operatorname{span}\{x\,\mathrm{d}y - \mathrm{d}z\}$.

Lemma 9.6. *$\mathscr{I}(E)$ is closed under* d *iff E is involutive.*

We defined integrability in terms of vector fields, and the dual condition on differential forms is a little simpler. The key step in the proof is showing that if ξ and η belong to E and $\alpha \in \Omega_X^1 \cap \mathscr{I}(E)$, Corollary 9.5 shows that

$$\mathrm{d}\alpha(\xi,\eta) = \xi\cdot\alpha(\eta) - \eta\cdot\alpha(\xi) - \alpha([\xi,\eta]) = -\alpha([\xi,\eta]).$$

In the example, it's easy to check that E isn't involutive: $\mathrm{d}(x\,\mathrm{d}y - \mathrm{d}z) = \mathrm{d}x\wedge\mathrm{d}y \notin \mathscr{I}(E)$.

Let's talk about Lie groups.

Definition 9.7. A *Lie group* G is a smooth manifold equipped with a group structure such that multiplication $G\times G\to G$ and inversion $G\to G$ are smooth.

Like many definitions in mathematics, this is a marriage of two structures, with compatibility conditions (the group structure is smooth). There should be a compatibility condition for the identity, that inclusion of the identity is smooth, but this follows automatically for manifolds. It is needed for defining topological groups more generally, however.

Example 9.8. We have already seen many examples of Lie groups.

- If V is a finite-dimensional, real vector space, $(V,+)$ is an abelian Lie group.
- $\mathrm{GL}(V) = \mathrm{Aut}(V)$ is a Lie group, in general nonabelian. If $V = \mathbb{R}^n$, $\mathrm{GL}(\mathbb{R}^n)$ is called $\mathrm{GL}_n(\mathbb{R})$, and is the group of invertible $n\times n$ matrices. As a manifold, it's an open subset of the space of all $n\times n$ matrices (inversion is the preimage of an open subset of $\mathbb{R}$ under a polynomial, hence smooth, map).
- If V has an inner product, we can look at its orthogonal group $\mathrm{O}(V)$. If $V = \mathbb{R}^n$ and the inner product is standard, this is called O_n.
- If A is an affine space modeled on V, then the group of affine transformations $\mathrm{Aff}(A)$ is a Lie group, and fits into a short exact sequence of Lie groups

$$1 \longrightarrow V \longrightarrow \mathrm{Aff}(A) \longrightarrow \mathrm{GL}(V) \longrightarrow 0.$$

  Similarly, if E is a Euclidean space modeled on V, the Euclidean transformations $\mathrm{Euc}(E)$ form a Lie group. If $A = \mathbb{A}^n$, $\mathrm{Aff}(\mathbb{A}^n)$ is denoted Aff_n, and similarly $\mathrm{Euc}(\mathbb{E}^n)$ is called Euc_n. ◀

Observe that on any manifold X, $\mathcal{X}(X)$ is a Lie algebra under Lie bracket, but it's infinite-dimensional, which makes life hairy.

Lie groups have extra geometric structure: the identity is a distiguished point, and there are distinguished symmetries given by left and right multiplication and conjugation: for each $g\in G$, let

- $L_g\colon G\to G$ send $x\mapsto gx$,
- $R_g\colon G\to G$ send $x\mapsto xg$, and
- $A_g\colon G\to G$ send $x\mapsto gxg^{-1}$.

Definition 9.9.

- Let $\xi\in\mathcal{X}(G)$ be a vector field. Then, ξ is *left-invariant* if $(L_g)_*\xi = \xi$ for all $g\in G$.
- Let $\alpha\in\Omega_G^\bullet$ be a differential form. Then, α is *right-invariant* if $(L_g)^*\alpha = \alpha$ for all $g\in G$.

In the same way, one can define left-invariance of Riemannian metrics, etc. Replacing L_g with R_g defines *right-invariance* of vector fields, forms, etc. And requiring invariance under any two of L_g, R_g, and A_g forces invariance under the third, and this is called *bi-invarinace*.

Since the differential of L_g is an isomorphism (since $L_{g^{-1}}$ is an inverse), then G acts on G transitively by left multiplication. This means in particular the dimension of G is constant (on arbitrary manifolds, such as $S^1 \amalg S^2$, the dimension is only locally constant).

Definition 9.10. Let G be a Lie group. Then, its Lie algebra $\mathfrak{g} \subset \mathcal{X}(G)$ is the subspace of left-invariant vector fields on G.

Evaluation at the identity defines a linear map $\mathrm{ev}_e\colon \mathfrak{g} \to T_eG$, sending $\xi \mapsto \xi|_e$. This map is an isomorphism, so $\dim \mathfrak{g} = \dim G$.

Proposition 9.11. $\mathfrak{g}$ *is closed under* $[-,-]$.

Proof. Let $\xi, \eta \in \mathfrak{g}$ and $g \in G$. Since ξ is left-invariant, it's L_g-related to itself, and similarly for η. We showed in Proposition 7.13 that if ξ and ξ' are related under ψ and η and η' are related under ψ, then so are $[\xi, \eta]$ and $[\xi', \eta']$. Thus, for our ξ and η, $[\xi, \eta]$ is L_g-related to itself for all g, hence is L_g-invariant. ☒

In a very weak sense, the bracket is dual to d, so let's see what the corresponding statement is for differential forms.

Proposition 9.12. *The subspace* $(\Omega_G^\bullet)^G$ *of left-invariant forms is closed under* d.

Proof. Let $\omega \in (\Omega_G^\bullet)^G$, i.e. $L_g^*\omega = \omega$ for all $g \in G$. Then, by the commutation relations,

$$L_g^* \mathrm{d}\omega = \mathrm{d}L_g^*\omega = \mathrm{d}\omega.$$

☒

Observe that evaluation at the identity defines a map

$$\mathrm{ev}_e\colon (\Omega_G^\bullet)^G \longrightarrow \Lambda^\bullet T_e^*G = \Lambda^\bullet \mathfrak{g}^*.$$

Thus one obtains a complex

$$0 \longrightarrow \mathfrak{g} \xrightarrow{\mathrm{d}} \Lambda^2\mathfrak{g}^* \xrightarrow{\mathrm{d}} \Lambda^3\mathfrak{g}^* \longrightarrow \cdots \tag{9.13}$$

The map $\mathrm{d}\colon \mathfrak{g} \to \Lambda^2\mathfrak{g}^*$ can be computed with Corollary 9.5 to be the map

$$\alpha \longmapsto (\xi, \eta \longmapsto -\alpha([\xi, \eta])). \tag{9.14}$$

But this makes sense for any abstract Lie algebra, whether or not it came from G, and so one could define the complex (9.13) through the map (9.14). In this way one can talk about Lie algebra cohomology.

Suppose that $\xi_1, \dots, \xi_n$ is a basis for $\mathfrak{g}$, so that there exist $C_{jk}^i \in \mathbb{R}$ such that

$$[\xi_j, \xi_k] = C_{jk}^i \xi_i.$$

These are called the *structure constants* for $\mathfrak{g}$ in this basis. Skew-symmetry implies that $C_{jk}^i + C_{kj}^i = 0$, and there's a similar formula that comes from the Jacobi identity.

Exercise 9.15. Suppose that $\theta^1, \dots, \theta^n$ is the dual basis to $\xi_1, \dots, \xi_n$ for $\mathfrak{g}^* \subset \Omega_G^1$. Show that

$$\mathrm{d}\theta^i + \frac{1}{2} C_{jk}^i \theta^j \wedge \theta^k = 0.$$

If $\theta := \theta^i \xi_i \in \Omega_G^1(\mathfrak{g})$, then θ does not depend on the choice of basis, coming from $\mathfrak{g}^* \otimes \mathfrak{g} \cong \mathrm{End}(\mathfrak{g})$ (as vector spaces): θ maps to $\mathrm{id}_{\mathfrak{g}}$. This θ is a canonical 1-form that restricts to the form corresponding to the identity at each point; θ is called the *Maurer-Cartan* 1*-form*.

Proposition 9.16. θ *satisfies the* Maurer-Cartan equation

$$\mathrm{d}\theta + \frac{1}{2}\theta \wedge \theta = 0. \tag{9.17}$$

Lecture 10.

The Maurer-Cartan form: 2/16/17

Let's continue where we left off, letting G be a Lie group. There are a lot of nice facts about Lie groups, some of which are homework problems (e.g. using the inverse function theorem to prove that if multiplication is smooth, inversion is automatically smooth). The Lie algebra of G, denoted $\mathfrak{g}$, is the Lie algebra of left-invariant vector fields on G. We also defined a $\mathfrak{g}$-valued 1-form $\theta \in \Omega^1_G(\mathfrak{g})$, called the Mauer-Cartan 1-form.

If G is a Lie group, the path component of the identity G_e is a subgroup: for any $g, h \in G_e$, there's a path through the identity to g and one from the identity to h, so since multiplication is continuous, we get a path from g to gh. Moreover, all components are diffeomorphic: left translation $L_g : G_e \xrightarrow{\cong} g \cdot G_e$ defines a diffeomorphism from G_e to the component of G containing g. Many Lie groups that we care about aren't connected, e.g. the orthogonal groups O_n and the general linear groups $\mathrm{GL}_n(\mathbb{R})$, as well as any nontrivial finite group.

The Maurer-Cartan equation (9.17) encodes a lot of structure about G. For example, if $\xi_1, \dots, \xi_n$ is a basis of $\mathfrak{g}$ and $\theta = \theta^i \xi_i$, so $\theta^i \in \Omega^1_G$ is an ordinary 1-form, then

$$[\theta \wedge \theta] = [\theta^i \xi_i \wedge \theta^j \xi_j] = \theta^i \wedge \theta^j [\xi_i, \xi_j].$$

Proof of Proposition 9.16. We'll compute at $x \in G$, evaluating on $\xi_x, \eta_x \in T_x G$. We can extend them to left-invariant vector fields ξ, η on G; since the Maurer-Cartan equation is tensorial, its truth or falsity doesn't depend on the choice of extension. Then,

$$\begin{aligned} \mathrm{d}\theta_x[\xi_x, \eta_x] &= \xi_x \cdot \theta(\eta) - \eta_x \cdot \theta(\xi) - \theta_x([\xi, \eta]_x) \\ &= -\theta_x([\xi, \eta]_x) = -[\xi, \eta]_x. \\ \frac{1}{2}[\theta \wedge \theta]_x(\xi_x, \eta_x) &= \frac{1}{2}([\theta_x(\xi_x), \theta_x(\eta_x)] - [\theta_x(\eta_x), \theta_x(\xi_x)]) \\ &= [\theta_x(\xi_x), \theta_x(\eta_x)] \\ &= [\xi, \eta_x]. \end{aligned}$$ ⊠

It's locally possible to get from the Lie algebra to the Lie group.

Definition 10.1. If $\xi \in \mathfrak{g}$, then there's a unique integral curve $\gamma_\xi(t)$ such that $\gamma_\xi(0) = e$ and $\cdot\gamma_\xi(0) = \xi$. Define $\exp\colon \mathfrak{g} \to G$ by

$$\exp \xi := \gamma_\xi(1).$$

That is, flow along the curve in the direction of ξ for time 1. In particular, $\gamma_\xi(t) = \exp(t\xi)$, which will also be written $e^{t\xi}$. You can check that the integral curve with initial position $x \in G$ is $t \mapsto xe^{t\xi}$, so the flow φ_t generated by ξ is $\varphi_t = R_{e^{t\xi}}$, i.e. right translation by $\exp(t\xi)$.

A lot of Lie groups arise as matrix groups, in which the exponential map is just the matrix exponential.

Example 10.2. The *general linear group* $\mathrm{GL}_n(\mathbb{R})$ is the group of $n \times n$ invertible matrices. This is an open condition in all $n \times n$ matrices, since it's asking the determinant to be nonzero, so this is a Lie group. Inclusion defines a canonical function $g\colon G \hookrightarrow M_n(\mathbb{R})$. ◀

Since $\mathrm{GL}_n(\mathbb{R})$ is an open subset of the vector space $M_n(\mathbb{R})$, then its Lie algebra $\mathfrak{gl}_n(\mathbb{R})$ can be canonically identified with $M_n(\mathbb{R})$.

Theorem 10.3. *For $G = \mathrm{GL}_n(\mathbb{R})$,*

(1) *the Lie bracket on $\mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$ is $[A, B] = AB - BA$,*
(2) *the exponential is the matrix exponential $\exp\colon M_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ defined by*

$$e^A = I + A + \frac{A^2}{2!} + \cdots,$$

(3) *and the Maurer-Cartan form is*

$$\theta = g^{-1}\,\mathrm{d}g.$$

As most Lie groups we think about are matrix groups, hence subgroups of $\mathrm{GL}_n(\mathbb{R})$ for some n, this makes all of these notions nicely concrete, and Theorem 10.3 holds for them too.

Proof sketch of Theorem 10.3. For (2), first prove that for an $A \in M_n(\mathbb{R})$, the function

$$f(t) := I + tA + \frac{(tA)^2}{2!} + \cdots + \frac{(tA)^N}{N!} + \cdots$$

converges, and then differentiate term-by-term to show that $\dot{f}(t) = f(t) \cdot A$, which is what we needed.

For (1), the flow generated by ξ_A is $\varphi_t := R_{e^{tA}}$ and for ξ_B it's $\psi_s := R_{e^{sB}}$. Thus,

$$\begin{aligned}
[\xi_A, \xi_B]_I &= \left.\frac{\partial^2}{\partial s \partial t}\right|_{s,t=0} \psi_{-s}\varphi_{-t}\psi_s\varphi_t(I) \\
&= \left.\frac{\partial^2}{\partial s \partial t}\right|_{s,t=0} e^{tA}e^{sB}e^{-tA}e^{-sB} \\
&= \left.\frac{\partial^2}{\partial s \partial t}\right|_{s,t=0} (I + tA)(I + sB)(I - tA)(I - sB) \\
&= AB - BA.
\end{aligned}$$

The calculation of the Maurer-Cartan form comes directly from its definition. ☒

The Maurer-Cartan equation $\mathrm{d}\theta + \theta \wedge \theta = 0$ turns into matrix multiplication for G a matrix group; in particular, if θ^i_j denotes the coefficients of the matrix θ in some basis, then matrix multiplication in coordinates implies that

$$\mathrm{d}\theta^i_j + \theta^i_k \wedge \theta^k_j = 0.$$

This will be important to us.

Example 10.4. Let's compute the Maurer-Cartan form for Aff_n. The trick is to embed $\mathrm{Aff}_n \hookrightarrow \mathrm{GL}_{n+1}(\mathbb{R})$ as follows: any affine transformation is of the form $x \mapsto Mx + \xi$ for some $M \in \mathrm{GL}_n(\mathbb{R})$ and $\xi \in \mathbb{R}^n$, so it can be represented in $\mathrm{GL}_{n+1}(\mathbb{R})$ as the block matrix

$$\begin{pmatrix} M & \xi \\ 0 & 1 \end{pmatrix},$$

which acts on $x = (x^1, \dots, x^n, 1)$. In other words, looking at the line $\{x^{n+1} = 1\}$ in $\mathbb{R}^{n+1}$ recovers $\mathbb{A}^n$ and Aff_n.

Therefore the Lie algebra also embeds: $\mathfrak{aff}_n \hookrightarrow \mathfrak{gl}_{n+1}(\mathbb{R}) = M_{n+1}(\mathbb{R})$: the affine Lie algebra consists of transformations $x \mapsto Mx + \xi$ where M need not be invertible, and this is sent to the matrix

$$\begin{pmatrix} M & \xi \\ 0 & 0 \end{pmatrix}.$$

With this embedding, the Maurer-Cartan form is

$$\begin{pmatrix} \theta^i_j & \theta^i \\ 0 & 0 \end{pmatrix},$$

and the equations of matrix multiplication inform us that

$$\begin{aligned}
\mathrm{d}\theta^i + \theta^i_j \wedge \theta^j &= 0 \\
\mathrm{d}\theta^i_j + \theta^i_k \wedge \theta^k_j &= 0.
\end{aligned}$$

◀

Example 10.5. For curved space, we care about the orthogonal group $\mathrm{O}_n \hookrightarrow M_n(\mathbb{R})$ of orthogonal $n \times n$ matrices, those matrices M such that $M^{\mathrm{T}}M = I$. The Lie algebra $\mathfrak{o}_n$ is the vector space of skew-symmetric matrices, i.e. $M^{\mathrm{T}} + M = 0$. In this case, the Maurer-Cartan form satisfies

$$\theta^i_j + \theta^j_i = 0.$$

◀

Similarly, we can embed $\mathrm{Euc}_n \hookrightarrow M_{n+1}(\mathbb{R})$ through the embedding $\mathrm{Euc}_n \hookrightarrow \mathrm{Aff}_n \hookrightarrow \mathrm{GL}_{n+1}(\mathbb{R}) \hookrightarrow M_{n+1}(\mathbb{R})$, and obtain its Maurer-Cartan formula in that way. The same is true for the *conformal group* $\mathrm{CO}_n \subset \mathrm{GL}_n(\mathbb{R})$ of matrices M such that $\langle M\xi, M\eta\rangle = c\langle \xi, \eta\rangle$ for some $c > 0$ and all $\xi, \eta \in \mathbb{R}^n$. You can play the same game for any kind of geometric structure: symplectic geometry with the symplectic group, spin geometry with the spin group, and so forth.

The adjoint action. Let $g \in G$. Then, the conjugation action $A_g = L_g \circ R_{g^{-1}} = R_{g^{-1}} \circ L_g$ sends $x \mapsto gxg^{-1}$. Therefore A defines a map $G \to \mathrm{Aut}(G)$.

Definition 10.6. Identifying $\mathfrak{g} \cong T_eG$, the *adjoint action* $\mathrm{Ad}_g \in \mathrm{End}(\mathfrak{g})$ is defined by $\mathrm{Ad}_g = \mathrm{d}(A_g)_e$.

Lemma 10.7.

(1) *The adjoint action preserves the Lie algebra, hence is a Lie algebra endomorphism.*
(2) *The differential of* $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$ *(sending* $g \mapsto A_g$*) at the identity is*

$$d\mathrm{Ad}_g|_e(\xi) = [\xi, -].$$

The Maurer-Cartan form behaves nicely under the left and right actions.

Proposition 10.8. *In* $\Omega^1_G(\mathfrak{g})$*,* $L_g^*\theta = \theta$*, but* $R_g^*\theta = \mathrm{Ad}_{g^{-1}}\theta$*.*

These equations are simple, but will come up again.

Proof. Let $x \in G$, so that if $\xi_x \in T_eG$,

$$(R_g^*\theta)_x(\xi_x) = \theta_{xg}(R_{g*}\xi_x).$$

Let ξ be a left-invariant extension of $\xi_x = \theta_x(\xi_x)$. Then, at the identity,

$$\begin{aligned}(R_g^*\theta)_x(\xi_x) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \theta_{xg}\left(xe^{t\xi}g\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (xg)^{-1}\left(xe^{t\xi}g\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} g^{-1}e^{t\xi}g\\ &= \mathrm{Ad}_{g^{-1}}\xi.\end{aligned}$$

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We've met the Maurer-Cartan form on groups, but it will also arise on G-torsors. This is an instance of the notion that left-invariant objects on G extend to right G-torsors, and right-invariant objects extend to left torsors.

Recall that a *right* G*-torsor* G is a manifold with a simply transitive right action $T \times G \to T$, sending $t, g \mapsto t \cdot g$, so the map $T \times G \to T \times T$ sending $t, g \mapsto t, t \cdot g$ is a diffeomorphism.

Torsors are like groups with multiplication but no preferred origin. Examples include affine space $\mathbb{A}^n$, a torsor over $\mathbb{R}^n$; and $\mathscr{B}(V)$, the space of bases for a vector space V, which is a $\mathrm{GL}(V)$-torsor.

If $t \in T$, $\varphi_t\colon GT \to T$ sending $g \mapsto t \cdot g$ is a diffeomorphism, in fact an isomorphism of right G-torsors. If I have two of these, φ_{t_0} and φ_{t_1}, then the map $\varphi_{t_1}^{-1} \circ \varphi_{t_0}\colon G \to G$ is a right-invariant diffeomorphism, and in fact is left multiplication by some $h \in G$, such that $t_1(hg) = t_0g$ for all $g \in G$; in particular, $t_0 = t_1h$. This means that the Maurer-Cartan form, which is left-invariant, can be pulled back along some (well, any) φ_t^{-1} and therefore makes sense on a right G-torsor.

This concludes our crash course on Lie groups; we'll see this stuff again later.

Plane curves. Let E be a Euclidean plane modeled on an inner product space V.[9] Let $\mathscr{B}_{\mathrm{O}}(E)$ denote the space of affine isometries $\alpha\colon \mathbb{E}^2 \to E$. These are ways of choosing an origin and orthonormal coordinates for E. That is, α can be identified with pairs (p, b) where $p \in E$ and b is a basis of V. Forgetting the basis defines a map $\pi\colon \mathscr{B}_{\mathrm{O}}(E) \to E$, called the *frame bundle*; it is a right Euc_2-torsor, where Euc_2 acts by changing the coordinates of b.

[9] Some of what we do here will work for more general Euclidean spaces.

Therefore all left-invariant information on Euc_2 comes over to $\mathscr{B}_{\mathrm{O}}(E)$. In particular, there are Maurer-Cartan forms θ^1 and θ^2 with $\theta^1_2 = -\theta^2_1$, coming from the same equation on Euc_2. This will be useful: differential forms make computations particularly easy compared to coordinate systems and vector fields (though those are not bad). Figuring out what this form actually calculates on a pair (p, b) is the first step towards understanding the Riemann curvature tensor: θ^i is the component of translation in the direction e^i.

Let $\gamma\colon (a, b) \to E$ be an embedded curve, which lifts to a map $\widetilde{\gamma}$ of frames, so e_1 is normal and e_2 is tangent. What is the pullback $\gamma^*\theta^i$ and $\gamma^*\theta^i_j$? Since there's no translation in direction e^1, $\gamma^*\theta^1 = 0$, and consequently $\gamma^*\theta^1 = \mathrm{d}t$. The pullback $\gamma^*\theta^1_2 = -k\,\mathrm{d}t$. So the Maurer-Cartan forms encode curvature on the torsor of frames, even in the simple situation of a plane curve.

Lecture 11.

Characterizing the Maurer-Cartan form: 2/21/17

"General confusion reigns in the land here..."

Recall that the Maurer-Cartan form for a Lie group G is the canonical 1-form $\theta \in \Omega^1_G(\mathfrak{g})$ that at f pulls back a vector field ξ to $L_{g^{-1}}(\xi_g)$, i.e. produces the unique left-invariant vector field extending ξ_g. This uses the fact that evaluation defines an isomorphsim $\mathfrak{g} \to T_xG$ for each $x \in G$.

Since $\mathrm{GL}_n(\mathbb{R})$ is an open submanifold of $M_n(\mathbb{R})$, then $T_A\mathrm{GL}_n(\mathbb{R})$ is canonically identified with $M_n(\mathbb{R})$ again. In particular, this means a vector field is a function $\xi\colon \mathrm{GL}_n(\mathbb{R}) \to M_n(\mathbb{R})$, and ξ is left-invariant means $\xi_{AB} = A \cdot \xi_B$.

Let $g\colon \mathrm{GL}_n(\mathbb{R}) \to M_n(\mathbb{R})$ be the embedding, so $\mathrm{d}g \in \Omega^1_{\mathrm{GL}_n(\mathbb{R})}(M_n\mathbb{R})$, and $g^{-1}\,\mathrm{d}g$ is also a $\mathfrak{gl}_n(\mathbb{R})$-valued 1-form. It operates on a matrix $\dot{A}$ as $g^{-1}\,\mathrm{d}g_A(\dot{A}) = A^{-1}\dot{A}$, which is indeed pulling back to the identity, then plugging in the vector $(\dot{A})$ in question.

This applies just as well to any matrix group, i.e. a group with an embedding $G \hookrightarrow M_n(\mathbb{R})$. This is equivalent to the data of a faithful (real) representation of G.

Remark 11.1. Not every Lie group is a matrix group. For example, the double cover of $\mathrm{SL}_2(\mathbb{R})$ isn't. This is most easily shown using representation theory. $\blacktriangleleft$

Some of the following proposition was in Proposition 10.8, but not all of it. In any case, checking these tells you how the Maurer-Cartan form behaves under various pullbacks.

Proposition 11.2. *Let $g \in G$.*

1. $L_g^*\theta = \theta$ *and* $R_g^*\theta = \mathrm{Ad}_{g^{-1}}\theta$.
2. *If $i\colon G \to G$ is the inversion map $g \mapsto g^{-1}$, $(i^*\theta)_g = -\mathrm{Ad}_g\theta_g$.*
3. *If $m\colon G \times G \to G$ is the multiplication map $g, h \mapsto gh$, then $m^*\theta_{(g,h)} = \mathrm{Ad}_{h^{-1}}\pi_1^*\theta + \pi_2^*\theta$, where π_i is the projection onto the i^{th} component.*

Proof of parts (2) *and* (3) *for matrix groups.* When G is a matrix group, $\theta = g^{-1}\,\mathrm{d}g$, so we can calculate:

$$i^*\theta = (g^{-1})^{-1}\,\mathrm{d}(g^{-1}) = gg^{-1}\,\mathrm{d}gg^{-1} = -\mathrm{Ad}_g(g^{-1}\,\mathrm{d}g).$$

For part (3), it comes from the calculation

$$(g_1g_2)^{-1}\,\mathrm{d}(g_1g_2) = g_2^{-1}g_1^{-1}(\,\mathrm{d}g_1g_2 + g_1\,\mathrm{d}g_2) = g_2^{-1}(g_1^{-1}\,\mathrm{d}g_1)g_2 + g_2^{-1}\,\mathrm{d}g_2. \qquad \boxtimes$$

We're going to return to curves and surfaces, and next time generalize to higher dimensions, but before we do that, we need to discuss a theorem that we'll return to several times.

Theorem 11.3. *Let Y be a manifold, G be a Lie group, and $\theta_Y \in \Omega^1_Y(\mathfrak{g})$.*

1. *If Y is connected and $F, F'\colon Y \to G$ are such that $F^*\theta = (F')^*\theta = \theta_Y$, then there exists a $g \in G$ such that $F' = L_g \circ F$.*
2. *If*

$$\mathrm{d}\theta_Y + \frac{1}{2}[\theta_Y \wedge \theta_Y] = 0, \tag{11.4}$$

then for any $y_0 \in Y$ and $g_0 \in G$, there's a neighborhood $U \subset Y$ containing y_0 and an $F\colon U \to G$ such that $F(y_0) = g_0$ and $F^*\theta = \theta_Y|_U$, and if U is connected F is unique.

(3) *If Y is simply connected and (11.4) holds, then there's a unique $F\colon Y \to G$ such that $F(y_0) = g_0$ and $F^*\theta = \theta_Y$.*

Example 11.5.

(1) Let $Y = \mathbb{R}$ with a t-coordinate and $\xi \in \mathfrak{g}$, and let $\theta_Y = \xi\,\mathrm{d}t \in \Omega^1_{\mathbb{R}}(\mathfrak{g})$. Then, Theorem 11.3 shows there's a unique $F\colon \mathbb{R} \to G$ such that $F(0) = e$ and $F^*\theta = \xi\,\mathrm{d}t$. This is the exponential map $F(t) = e^{t\xi}$ (which requires a check).

(2) Theorem 11.3 can be used to lift some morphisms of Lie algebras to Lie groups. Let G' be a simply connected Lie group and $\dot\phi\colon \mathfrak{g}' \to \mathfrak{g}$ be a Lie algebra homomorphism. Let $\theta' \in \Omega^1_{G'}(\mathfrak{g}')$ be the Maurer-Cartan form and $\theta_{G'} := \dot\phi\theta'$. Then, (11.4) holds:

$$\begin{aligned} \mathrm{d}\theta_{G'} + \frac{1}{2}[\theta_{G'} \wedge \theta_{G'}] &= \mathrm{d}\dot\phi\theta' + \frac{1}{2}[\dot\phi\theta' \wedge \dot\phi\theta'] \\ &= \dot\phi\,\mathrm{d}\theta' + \frac{1}{2}\dot\phi[\theta' \wedge \theta'] \\ &= \dot\phi\left(\mathrm{d}\theta' + \frac{1}{2}[\theta' \wedge \theta']\right) \\ &= 0. \end{aligned}$$

By Theorem 11.3 we conclude that there's a unique $\phi\colon G \to G$ such that $\phi(e') = e$ and $\phi^*\theta_{G'} = \theta$. That G' is simply connected is essential. ◂

Proof of Theorem 11.3. Part (1) is a calculation. If G is a matrix group,

$$\begin{aligned} (F'F^{-1})^*\theta = (F'F^{-1})^{-1}\,\mathrm{d}(F'F^{-1}) &= (F(F')^{-1})\left(\mathrm{d}F'F^{-1} - F'F^{-1}\,\mathrm{d}FF^{-1}\right) \\ &= F((F')^{-1}\,\mathrm{d}F')F^{-1} - F(F^{-1}\,\mathrm{d}F)F^{-1} \\ &= F\left((F')^{-1}\,\mathrm{d}F' - F^{-1}\,\mathrm{d}F\right)F^{-1} \\ &= F\left((F')^*\theta - F^*\theta\right)F^{-1} = 0. \end{aligned}$$

For a general group, one must decompose $F'F^{-1}$ as the composition

$$Y \xrightarrow{(F',F)} G \times G \xrightarrow{(\mathrm{id},i)} G \times G \xrightarrow{m} G.$$

For part (2) we use an old tactic: using the graph of a function to pass between sets and functions. Let π_i be the projection onto the i^{th} factor out of $Y \times G$. Let $\xi_1, \dots, \xi_n$ be a basis of $\mathfrak{g}$, $\theta_Y = \theta^i_Y\xi_i$, and $\theta = \theta^i\xi_i$. Then, let $\mathscr{I} \subset \Omega^\bullet_{Y\times G}$ be the ideal generated by $\pi_1^*\theta^i_Y - \pi_2^*\theta^i_Y$. This is the vanishing ideal for a distribution $E \subset T(Y \times G) \to Y \times G$ of codimension n, so $\dim E = \dim Y$.

We'll check that $\mathrm{d}(\pi_1^*\theta_Y - \pi_2^*\theta)$ is an ideal. For ease of writing, this will be written $\mathrm{d}(\theta_Y - \theta)$, with the projections implicit. Since

$$\mathrm{d}\theta_Y - \mathrm{d}\theta = \mathrm{d}\theta_Y + \frac{1}{2}[\theta \wedge \theta]$$

and

$$\frac{1}{2}[(\theta - \theta_Y) \wedge (\theta + \theta_Y)] = \frac{1}{2}[\theta \wedge \theta] - \frac{1}{2}[\theta_Y \wedge \theta_Y],$$

then $\mathrm{d}\theta_Y - \mathrm{d}\theta \in \mathscr{I}$ iff (11.4) holds. Then, Theorem 8.5 guarantees the existence of a graph locally, proving (2).

For part (3), first observe that E is left-invariant under G:[10] if $g \in G$, then $L_{g*}E_{(y,h)} = E_{(y,gh)}$. Equivalently, $L_g^*\mathscr{I} = \mathscr{I}$, which is true because $L_g^*\theta = \theta$. The global Frobenius theorem (Theorem 8.10) says that around each $(y,h) \in Y \times G$ there's a neighborhood $U \times V \subset Y \times G$ containing (y,h) and local coordinates such that $E|_{U\times V}$ is a product of a distribution that's constant on U and one that's constant on V.

By left-invariance, we can assume $V = G$. Let Γ be the leaf of the foliation through (y_0, g_0); then, $\Gamma \cap (U \times G)$ has at most countably many components which are horizontal. It follows that π_1 is both open and closed, so $\mathrm{Im}(\pi_1)$ is both open and closed, hence connected; since Y is connected, then $\mathrm{Im}(\pi_1) = Y$.

[10] It's also right-invariant, but we don't need that.

Furthermore, the local form (that U is evenly covered) shows that $\pi_1|_\Gamma$ is a covering map. Since Y is simply connected and Γ is connected, then $\pi_1|_\Gamma$ has to be a diffeomorphism. Therefore we have the graph, and recover the function as $F = \pi_2 \circ (\pi_1|_\Gamma)^{-1}$. ⊠

The key here (aside from the local Frobenius theorem) is the left-invariance, which is what guarantees the leaves of the foliation can't do anything funny.

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This fancy technology of differential forms and Lie groups takes us a long way quickly even just in the case of curves and surfaces.

Let E be a Euclidean plane, modeled on a two-dimensional inner product space V, and let $\mathscr{B}_{\mathrm{O}}(E)$ denote the space of isometries $\mathbb{E}^2 \to E$, i.e. pairs (p, b) with $p \in E$ and $b\colon \mathbb{R}^2 \to V$ a basis. As we've discussed before, $\mathscr{B})_{\mathrm{O}}(E)$ is a right Euc_2-torsor.

In this context and in a basis, the Maurer-Cartan form is a matrix

$$\begin{pmatrix} 0 & \theta_2^1 & \theta^1 \\ \theta_1^2 & 0 & \theta^2 \\ 0 & 0 & 0 \end{pmatrix}$$

Here $\theta_2^1 = -\theta_1^2$ and $\theta^i, \theta_j^i \in \Omega^1_{\mathscr{B}_{\mathrm{O}}(E)}$. The Maurer-Cartan equation imposes some important relations between these forms:

$$\mathrm{d}\theta^1 + \theta_2^1 \wedge \theta^2 = 0 \tag{11.6a}$$

$$\mathrm{d}\theta^2 + \theta_1^2 \wedge \theta^1 = 0 \tag{11.6b}$$

$$\mathrm{d}\theta_1^2 = 0. \tag{11.6c}$$

These are also called the *Maurer-Cartan equations*, and we're about to get a lot more familiar with them. In curved space, the matrix form of θ will be different, and things will be different.

Let $i\colon (a, b) \hookrightarrow E$ be a co-oriented curve, and lift it across $\pi\colon \mathscr{B}_{\mathrm{O}}(E) \to E$ to a map $\widetilde{\imath}$ such that e_2 is the oriented unit normal. This is a choice of what happens to e_1.

In general, a short exact sequence of vector bundles $0 \to V' \to V \to V'' \to 0$ splits, so one of V' is a quotient and the other is a subspace. In the absence of orientations, which is which doesn't really matter, but if V, V', and V'' are oriented vector spaces, then the induced orientations on $V \cong V' \oplus V''$ force V' to be the quotient and V'' to be the subspace. Thus, the tangent-normal sequence of oriented vector bundles on C is

$$0 \longrightarrow TC \longrightarrow TE|_C \longrightarrow \nu \longrightarrow 0.$$

Then, we can calculate components of the Maurer-Cartan form: $\widetilde{\imath}^*\theta^1 = \mathrm{d}t$ and $\widetilde{\imath}^*\theta^2 = 0$, based on how e^1 and e^2 change with time. Then, $\widetilde{\imath}^*\theta_1^2$ measures the rate of turning of e_1 in the direction of e_2, which is precisely $k\,\mathrm{d}t$.

We can also revisit the problem of prescribing curvature: given $Y = (a, b)$ and a function $k\colon Y \to \mathbb{R}$, is there an immersed curve with curvature k? We saw in Corollary 3.10 that this is (locally) possible and the curve is unique up to a Euclidean motion. Here's another proof.

Another proof of Corollary 3.10. Define $\theta_Y \in \Omega_Y^1(\mathfrak{euc}_2)$ to satisfy the identities we just calculated:

$$\theta_Y := \begin{pmatrix} 0 & k\,\mathrm{d}t & \mathrm{d}t \\ -k\,\mathrm{d}t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By Theorem 11.3 we get a map $\widetilde{\imath}\colon Y \to \mathscr{B}_{\mathrm{O}}(E)$ such that $\widetilde{\imath}\theta = \theta_Y$ and if we fix an initial condition, $\widetilde{\imath}$ is unique. ⊠

For a co-oriented surface Σ embedded in Euclidean 3-space E, things work differently: instead of a discrete set of choices of lift of the embedding to a map $\Sigma \to \mathscr{B}_{\mathrm{O}}(E)$, there's an O_2 worth of them, where O_2 acts as the automorphisms of a framing in which e_3 is fixed.

Now the Maurer-Cartan form contain six pieces of information: $\theta^1, \theta^2, \theta^3, \theta^1_2, \theta^1_3, \theta^2_3$. Here $\theta^3 = 0$ and (θ^1, θ^2) is a local orthonormal (co)framing of E, which is called a *moving frame* (or *repère mobile* in French). Elie Cartan discovered moving frames and used them to make calculations.

The Maurer-Cartan equations take the form

$$\begin{aligned} \mathrm{d}\theta^i + \theta^i_j \wedge \theta^j &= 0 \\ \mathrm{d}\theta^i_j + \theta^i_k \wedge \theta^k_j &= 0. \end{aligned}$$

Next time, we'll write these out explicitly, and they will cause results like Gauss' theorema egregium to fall out in the blink of an eye! We'll also see the second fundamental form and the Gauss curvature and find some relations between them. This leads to a version of the prescribed curvature problem for surfaces, which involves solving a PDE instead of an ODE.

Lecture 12.

Applications to immersed surfaces: 2/23/17

Let E be Euclidean 3-space modeled on a 3-dimensional inner product space V. Then, $\mathscr{B}_{\mathrm{O}}(E)$, the space of isometries $\mathbb{E}^3 \to E$, is a right Euc_3-torsor, and the map $\pi\colon \mathscr{B}_{\mathrm{O}}(E) \to E$ realizes it as a principal O_3-bundle: the fibers are acted on by the isometries of $\mathbb{R}^3$. This describes a section to the short exact sequence

$$1 \longrightarrow \mathbb{R}^3 \longrightarrow \mathrm{Euc}_3 \longrightarrow \mathrm{O}_3 \longrightarrow 1.$$

The Maurer-Cartan form on $\mathscr{B}_{\mathrm{O}}(E)$ is determined by the 1-forms θ^1, θ^2, θ^3, θ^2_1, θ^3_1, and θ^3_2 such that $\theta^i_j = -\theta^j_i$ and the Maurer-Cartan equations are satisfied:

$$\mathrm{d}\theta^1 + \theta^1_2 \wedge \theta^2 + \theta^1_3 \wedge \theta^3 = 0 \tag{12.1a}$$

$$\mathrm{d}\theta^2 + \theta^2_1 \wedge \theta^1 + \theta^2_3 \wedge \theta^3 = 0 \tag{12.1b}$$

$$\mathrm{d}\theta^3 + \theta^3_1 \wedge \theta^1 + \theta^3_2 \wedge \theta^2 = 0 \tag{12.1c}$$

$$\mathrm{d}\theta^2_1 + \theta^3_2 \wedge \theta^3_1 = 0 \tag{12.1d}$$

$$\mathrm{d}\theta^3_1 + \theta^3_2 \wedge \theta^2_1 = 0 \tag{12.1e}$$

$$\mathrm{d}\theta^3_2 + \theta^3_1 \wedge \theta^1_2 = 0. \tag{12.1f}$$

Let $i\colon \Sigma \hookrightarrow E$ be an immersed surface, and choose a lift $\widetilde{\imath}\colon \Sigma \to \mathscr{B}_{\mathrm{O}}(E)$, an orthonormal frame on Σ. Let e_3 be the unit normal to Σ. We'll restrict the pieces of the Maurer-Cartan form to Σ via $\widetilde{\imath}$, though we'll leave the $\widetilde{\imath}^*$ out of the equation.

Suppose $\widetilde{\gamma}\colon (-\varepsilon, \varepsilon) \to \mathscr{B}_{\mathrm{O}}(E)$ is the lift of a curve $\gamma\colon (-\varepsilon, \varepsilon) \to E$ such that $\dot\gamma_0 = \xi$ and $\dot{\widetilde{\gamma}} = \widetilde{\xi}$. Then, $\theta^i(\widetilde{\xi})$ is the e_i-component of $\xi = \pi_*\widetilde{\xi}$, and $\theta^i_j(\widetilde{\xi}) = \langle \dot e_j(0), e_i\rangle$.

Let $U \subset \Sigma$ be a neighborhood. Then, on U, $\theta^3 = 0$ and $\{\theta^1, \theta^2\}$ is a basis for Ω^1_U. For $\mu, \nu \in \{1, 2\}$, write $\theta^3_\mu = h_{\mu\nu}\theta^\nu$ for some $h_{\mu\nu}\colon U \to \mathbb{R}$, which defines a 2×2 matrix $h := (h_{\mu\nu})$.

Lemma 12.2. *$h_{12} = h_{21}$ and h is the second fundamental form in the basis $\{e_1, e_2\}$, i.e. $II = h_{\mu\nu}\theta^\mu \otimes \theta^\nu$.*

Proof. By (12.1c),

$$0 = h_{12}\theta^2 \wedge \theta^1 + h_{21}\theta^1 \wedge \theta^2 = (h_{12} - h_{21})\theta^2 \wedge \theta^1,$$

so $h_{21} - h_{12} = 0$.

Recall that in these coordinates, we have $e_3\colon U \to V$, and the shape operator is $-\mathrm{d}e_3\colon TU \to V$. Since $\theta^i_j(\widetilde{\xi}) = \langle \dot e_j(0), e_i\rangle$, then $-\mathrm{d}e_3 = -\theta^\mu_3 e_\mu = h_{\mu\nu}\theta^\nu e_\mu$, and $II(\xi_1, \xi_2) = \langle \xi_1, S(\xi_2)\rangle$, so h describes II. ⊠

This is part of a theme: once you write down what the Maurer-Cartan form actually is, everything falls out, and the objective is to recognize it before it falls past you.

Proposition 12.3. $\mathrm{d}\theta^2_1 = -K\theta^1 \wedge \theta^2$, *where K is the Gauss curvature.*

Proof. From (12.1d),

$$\begin{aligned} 0 &= \mathrm{d}\theta_1^2 - (h_{21}\theta^1 + h_{22}\theta^2) \wedge (h_{11}\theta^1 + h_{12}\theta^2) \\ &= \mathrm{d}\theta_1^2 + \underbrace{(h_{11}h_{22} - h_{12}h_{21})}_{\det(h)} \theta^1 \wedge \theta^2 \\ &= \mathrm{d}\theta_1^2 + K\theta^1 \wedge \theta^2. \end{aligned}$$

⊠

Proposition 12.4 (Gauss' *Theorema egregium*)**.** *θ_1^2 is determined by θ^1 and θ^2.*

Proof. Suppose $\theta_1^2 = a\theta^1 + b\theta^2$ for some $a, b \in \Omega_U^0$. By (12.1a) and (12.1b),

$$\begin{aligned} \mathrm{d}\theta^1 + a\theta^1 \wedge \theta^2 &= 0 \\ \mathrm{d}\theta^2 - b\theta^1 \wedge \theta^2 &= 0. \end{aligned}$$

This means that a and b are determined by computing $\mathrm{d}\theta^1$ and $\mathrm{d}\theta^2$. ⊠

The relation with the more conventional statement of Theorem 5.1 is that θ_1^2 is intrinsic, and therefore so is $\mathrm{d}\theta_1^2$, hence also K.

The last two equations, (12.1e) and (12.1f), called the *Codazzi-Mainardi equations*, haven't been used yet, but they are constraints on the first and second fundamental forms of an immersed surface. You can ask, given an abstract surface and choices of the first and second fundamental form, is there an immersion such that the induced metric produces the chosen first and second fundamental forms? This is the surface-level analogue of the prescription of curvature problem for plane curves. The fact that the Gauss curvature matches the second fundamental form forces a relation between the first and second fundamental form, and the derivative of the second fundamental form is constricted by the Codazzi-Mainardi equations.

Older proofs of this boil everything down to solutions of systems of partial differential equations, and the solutions exist because mixed partials commute. However, we've managed to take a more geometric viewpoint which encodes everything into symmetries of the Maurer-Cartan form.

Example 12.5. Suppose r, θ are local coordinates on a two-dimensional Riemannian manifold akin to polar coordinates, in that the metric is

$$\mathrm{d}s^2 = \mathrm{d}r^2 + G(r)\mathrm{d}\theta^2,$$

where G is some positive function. One can then show that $\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \rangle = 1$ and $\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \rangle = 0$, so these are always perpendicular. One can show that such a coordinate system exists locally around any point in any Riemannian surface, and an analogous theorem is true in higher dimension.

For example, on $\mathbb{E}^2$ (i.e. $\mathbb{R}^2$ with the standard Euclidean metric), one can choose (x, y) (so $G = 1$) or polar coordinates (r, θ) where the metric is $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2$. On the sphere (with the induced metric as the unit sphere in $\mathbb{E}^3$), we have spherical coordinates (ϕ, θ) and the metric is $\mathrm{d}s^2 = \mathrm{d}\phi^2 + \sin^2\phi\,\mathrm{d}\theta$.

We can compute the Gauss curvature K in terms of G. Namely, if $g(r)$ is such that $g^2 = G$, then $\theta^1 = \mathrm{d}r$ and $\theta^2 = g\,\mathrm{d}\theta$. Therefore $\mathrm{d}\theta^1 = 0$ and $\mathrm{d}\theta^2 = g'\,\mathrm{d}r \wedge \mathrm{d}\theta = (g'/g)\theta^1 \wedge \theta^2$. Thus, $\theta_1^2 = -(g'/g)\theta^2 = -g'\,\mathrm{d}\theta$ and $\mathrm{d}\theta_1^2 = -g''\,\mathrm{d}r \wedge \mathrm{d}\theta = -(g''/g)\theta^1 \wedge \theta^2$, so we conclude

$$K = -g''/g.$$

If you plug this into (x, y) or (r, θ) on $\mathbb{E}^2$, g'' vanishes, so the Gauss curvature is 0; for the sphere, the second derivative of $\sin\phi$ is $-\sin\phi$, so $K = 1$. Thus, we have a surface of constant flat curvature and one of constant positive curvature; negative curvature is missing from this list, but one can realize it using hyperbolic space, replacing $\sin^2\phi$ with $\sinh^2\phi$. ◀

In the next few lectures, we'll continue on to higher dimensions. Suppose X is an n-dimensional Riemannian manifold, and let $\pi\colon \mathscr{B}_{\mathrm{O}}(X) \to X$ be the bundle of pairs (x, b) with $x \in X$ and $b\colon \mathbb{R}^n \xrightarrow{\cong} T_xX$ an isometry. This means we've switched to an abstract, intrinsic story: one can set up the extrinsic story again, and there are a few differences, e.g. in higher dimensions there are extra normal directions.

Anyways, $\mathscr{B}_{\mathrm{O}}(X)$ is called the *bundle of orthonormal frames* of X, and has a free right O_n-action, and π is the quotient map. Therefore it's possible to construct the pieces of the Maurer-Cartan forms $\theta^1, \ldots, \theta^n$ on

$\mathscr{B}_{\mathrm{O}}(X)$ and θ^i_j from the structure equations. Then, the equations $\theta^i_j = -\theta^j_i$ and $\mathrm{d}\theta + \theta \wedge \theta = 0$. This will define the Levi-Civita equations for us.

The orthogonal group is not the only choice here: you could ask for bases for $T_x X$ that preserve a $\mathrm{GL}_n(\mathbb{R})$ action, which is weaker; in this case, you get θ^i but not unique θ^j_i. If you ask for a complex structure on the tangent space, this leads to the notion of a complex structure and local holomorphic coordinates. It's possible to develop a general theory for these θ^i_j for general structure groups.

In any case, the existence *and* uniqueness in the case of Riemannian manifolds, which is the fundamental theorem of Riemannian geometry, is completely mysterious: except in a few cases, such as Kähler manifolds, where there are beautiful formulas, it's completely unclear *why* the unique connection compatible with the metric should exist.

In the general setting, we'll need a definition.

Definition 12.6. Let X be a smooth manifold and G be a Lie group. Then, a *principal G-bundle over X* is a manifold P together with a free right G-action and quotient map $\pi\colon P \to X$ such that π admits local smooth sections.

That π is a quotient means that for every $x \in X$, the fiber $P_x := \pi^{-1}(x)$ is an orbit of G, so for any $p_1, p_2 \in P_x$, there's a unique $g \in G$ such that $g \cdot p_1 = p_2$. The condition of local smooth sections means that for every $x \in X$, there's a neighborhood $U \subset X$ of x and a section $s\colon U \to P$ such that $\pi \circ s = \mathrm{id}_U$.

Intuitively, the local smooth sections criterion says that the fibers are "locally constant," and don't move too much if x doesn't.

Keep in mind that P is not the principal bundle: we need the data of the base X and the quotient π.

Example 12.7. Let $P = X \times G$, with the action on G by right multiplication and π projection onto the first component. This principal G-bundle is called the *trivial bundle*. ◀

Lemma 12.8. *If $\pi\colon P \to X$ is a principal G-bundle, then π admits* local trivializations*. That is, for any $x \in X$, there's a neighborhood U of x and an isomorphism $\pi^{-1}(U) \xrightarrow{\cong} U \times G$ that commutes with the projection back to U.*

Proof. Given x, choose a local section $s\colon U \to P$, and define $\varphi\colon U \times G \to P$ to send $y, g \mapsto s(y) \cdot g$. You can check φ is a diffeomorphism $U \times G \to P|_U := \pi^{-1}(U)$, but better is to show it commutes with the right G-actions, and therefore is an *isomorphism of principal G-bundles*. ⊠

We'll let $R_g\colon P \to P$ denote the action of a $g \in G$ on the principal G-bundle $P \to X$.

Proposition 12.9. *$\mathscr{B}_{\mathrm{O}}(X) \to X$ is a principal O_n-bundle.*

Proof. First, let X be any manifold and let $\pi\colon \mathscr{B}(X) \to X$ be the bundle of all frames, the pairs (x, b) such that $b\colon \mathbb{R}^n \to T_x X$ is a linear isomorphism. This is a principal $\mathrm{GL}_n(\mathbb{R})$-bundle — you should check that it's a manifold, e.g. by producing a chart $U \times \mathrm{GL}_n(\mathbb{R})$ for $\mathscr{B}(X)$ for every chart U, and use gluing on X and local sections to glue (there's more to show here). To obtain the local section near x, choose local coordinates $x^1, \dots, x^n$ neat x; then, the local section is given by $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$.

Great, so how about orthonormal frames? If X has a Riemannian metric, then the orthonormal frames $\mathscr{B}_{\mathrm{O}}(X)$ form a submanifold of $\mathscr{B}(X)$, and the quotient by O_n is X, but we need to check that there's a local section. Given a local section of $\mathscr{B}(X)$, one can use the Gram-Schmidt process to smoothly deform it into a section of $\mathscr{B}_{\mathrm{O}}(X)$. ⊠

Next time we'll talk about connections in this context.

Lecture 13.

Principal G-bundles: 2/28/17

Recall that if X is a smooth manifold and G is a Lie group, then a principal G-bundle over X is a map $P \to X$ such that P is a smooth manifold equipped with a free right G-action, such that π is the quotient map, and π admits local sections.[11]

[11] More on principal bundles can be found at http://www.ma.utexas.edu/users/dafr/M392C/Notes/lecture13.pdf.

Local sections are maps $s\colon U \to P$ for a chart $U \subset X$ such that $\pi \circ s = \mathrm{id}$. This is equivalent to a local trivialization, a commutative diagram

$$\begin{tikzcd} U \times G \arrow[rr, "\varphi"] \arrow[dr, "\pi_1"'] & & P|_U \arrow[dl, "\pi|_U"] \\ & U. & \end{tikzcd}$$

Here, $\varphi(x, g) := s(x) \cdot g$.

If G is compact, then it suffices to specify its free action on P, as the quotient of a manifold by a free action of a compact Lie group is again a manifold. However, this is not true for G noncompact: let $\mathbb{R}$ act on the torus $\mathbb{R}^2/\mathbb{Z}^2$ by translation by $(1/2, a)$ where a is an irrational number. Then, the orbits are dense, and in fact the quotient space isn't even Hausdorff!

Example 13.1. The *trivial G-bundle* is $P = X \times G$, with G acting by right multiplication on G and trivially on X. Here π is projection onto the first factor. ◀

Definition 13.2. A *morphism of principal G-bundles* is a G-equivariant map $\varphi\colon P \to P'$ that commutes with projection to the base:

$$\begin{tikzcd} P \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & P' \arrow[dl, "\pi'"] \\ & X. & \end{tikzcd}$$

Local triviality means that principal G-bundles are examples of *fiber bundles* as defined by Norman Steenrod.

Example 13.3. In this class, we care the most about frame bundles, but there are lots of other examples.

(1) Let $G = \mathrm{GL}_n(\mathbb{R})$, so, as we discussed last time, $\mathscr{B}(X) \to X$ is a principal $\mathrm{GL}_n(\mathbb{R})$-bundle. The fiber over an $x \in X$ is the space of bases $b\colon \mathbb{R}^n \to T_xX$.
(2) Similarly, if X is a Riemannian manifold, we can restrict to orthonormal frames, which defines a principal O_n-bundle $\mathscr{B}_{\mathrm{O}}(X) \to X$.
(3) In specific cases, you can say more. For example, if $X = \mathbb{E}^n$ (so Euclidean space with the standard metric), $\mathscr{B}_{\mathrm{O}}(\mathbb{E}^n)$ is a right Euc_n-torsor: any (x, b) and (x', b') are related by a unique Euclidean transformation.
(4) If S^n carries the usual metric, $\mathscr{B}_{\mathrm{O}}(S^n)$ is an O_{n+1}-torsor, as it's determined by $n+1$ unit vectors: the first determines the point $e_0 \in S^n$, and the rest determine the frame $e_1, \dots, e_n$.
(5) Let $X = \mathbb{H}^n$ be hyperbolic space, e.g. a hyperboloid of two sheets in $\mathbb{R}^{n,1}$ inheriting a Riemannian metric (even though the metric on $\mathbb{R}^{n,1}$ has signature $(n, 1)$, and in particular is not Riemannian). Then, $\mathscr{B}_{\mathrm{O}}(\mathbb{H}^n)$ is an $\mathrm{O}^+_{n,1}$-torsor. By $\mathrm{O}_{n,1}$ we mean the group of matrices preserving the (Lorentzian) metric on $\mathbb{R}^{n,1}$, and then we choose the connected component containing the identity.
(6) There's a $\mathbb{T}$-bundle $\pi\colon S^3 \to S^2$ which is the restriction of the projection $\mathbb{C}^2 \twoheadrightarrow \mathbb{CP}^1$ to $S^3 \subset \mathbb{C}^2$, and using the identification $S^2 \cong \mathbb{CP}^1$. This is called the *Hopf fibration*. The same construction more generally defines a principal $\mathbb{T}$-bundle $\pi\colon S^{2n+1} \twoheadrightarrow \mathbb{CP}^n$.
(7) Let G be a Lie group and $H \subset G$ be a closed subgroup. Then, G/H is a manifold, and the quotient map $\pi\colon G \to G/H$ is a principal H-bundle. Verifying this is on the homework. ◀

Though we've just seen some examples where the group of isometries is transitive, this is not true for every Riemannian manifold. For example, the curved torus (with the standard embedding in $\mathbb{R}^3$) doesn't have a transitive group of isometries.

Definition 13.4. Let F be a smooth manifold with a left G-action and $P \to X$ be a principal G-bundle. Then, the *mixing construction* or *associated bundle* is the fiber bundle $F_P = P \times_G F \to X$, where

$$P \times_G F := P \times F/((pg, f) \sim (p, gf)).$$

The right G-action is $(p, f) \cdot g = (p \cdot g, g^{-1}f)$, which you can check is well-defined in the quotient.

The idea is that the quotient tells you how far the G-action is from being a product action: this is always true locally, so the mixing construction is locally trivial, and therefore a fiber bundle. The principal bundle controls everything: the fibers look like F, but they're twisted in a way dictated by P.

This is how Steenrod originally defined fiber bundles, and in fact every fiber bundle arises in this way. This perspective means that if P or F has extra structure, so does any fiber bundle obtained by mixing them.

Example 13.5. Consider the frame $\mathrm{GL}_n(\mathbb{R})$-bundle $\mathscr{B}(X) \to X$ over an n-dimensional manifold X. The tangent bundle $TX \to X$ is the result of the mixing construction applied to $F = \mathbb{R}^n$ with the usual left $\mathrm{GL}_n(\mathbb{R})$-action. If one takes $F = (\mathbb{R}^n)^*$ instead, the result is the cotangent bundle. The extra structure on F in both cases carries over to the mixing construction, which is a vector bundle. Similarly, one can look at $\mathrm{End}(\mathbb{R}^n)$ or the space of inner products in $\mathrm{Sym}^2 \mathbb{R}^n$; the latter mixes to the bundle of Riemannian metrics (which is not a vector bundle).

$\mathrm{GL}_n(\mathbb{R})$ also acts on the set of two points $\{\pm 1\}$ by $g \mapsto \operatorname{sign}\det g$. The associated bundle is a principal $\mathbb{Z}/2$-bundle, also known as a double cover, and specifically is the orientation double cover of X. ◂

Example 13.6. Let G be a Lie group and $H \subset G$ be a closed subgroup, so there is a principal H-bundle $G \to G/H$. There is an adjoint representation of H on $\mathfrak{g}/\mathfrak{h}$, and the mixing construction can be canonically identified with $T(G/H) \to G/H$ — but with structure group H, rather than $\mathrm{GL}_n(\mathbb{R})$. If H is small, this is a lot of extra information.

For example, let $X = S^6$ with the round metric. It can be written as the homogeneous space $\mathrm{O}_7/\mathrm{O}_6$, or more exotically $\mathrm{G}_2/\mathrm{SU}_3$ (which is smaller: $\dim \mathrm{O}_7 = 21$, but $\dim \mathrm{G}_2 = 14$). Thus we obtain an O_6- or SU_3-structure on TS^6. ◂

This can be thought of as a differential-geometric realization of Felix Klein's *Erlangen* program, which says that geometric properties of an object should be understood in terms of the symmetries of that object.

Let $\pi\colon P \to X$ be a principal G-bundle and $p \in P$. Then, one can push forward along $\pi_{p*}\colon T_pP \to T_{\pi(p)}X$, which defines a short exact sequence of vector spaces

$$0 \longrightarrow \ker(\pi_{*p}) \longrightarrow T_pP \longrightarrow T_{\pi(p)}X \longrightarrow 0,$$

or, doing this for all $p \in P$ simultaneously,

$$0 \longrightarrow \ker(\pi_*) \longrightarrow TP \longrightarrow \pi^*TX \longrightarrow 0, \tag{13.7}$$

a short exact sequence of vector bundles over P. The kernel of π_* is the bundle of vectors tangent to the G-orbits, and is called the *vertical vector bundle*, denoted $T(P/X)$ or $T(\pi)$.

Lemma 13.8. *There's a canonical identification $T(P/X) \cong P \times \mathfrak{g}$ as vector bundles, with the projection $P \times \mathfrak{g} \to P$ onto the first factor.*

Proof. Given $(p, \xi) \in P \times \mathfrak{g}$, the isomorphism sends it to $t \mapsto pe^{t\xi}$. That is, we use the G-action to define a map $\mathfrak{g} \to \mathcal{X}(P)$; with a right G-action this preserves Lie bracket, but for a left G-action there would have to be a sign. Then, evaluation at p defines a map $\mathcal{X}(P) \to T_pP$. ⊠

Anyways, the point is that the vertical tangent bundle is trivializable, and is trivialized by $\mathfrak{g}$.

The frame bundle $\pi\colon \mathscr{B}(X) \to X$ has extra structure, a canonical form $\theta \in \Omega^1_{\mathscr{B}(X)}(\mathbb{R}^n)$ called the *soldering form*. Let's fix some notation: let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^n$ and $e^1, \dots, e^n$ be the dual basis for $(\mathbb{R}^n)^*$. Let $e_i^j \coloneqq e_i \otimes e^j$ in $\mathrm{End}(\mathbb{R}^n) \cong (\mathbb{R}^n)^* \otimes \mathbb{R}^n$, i.e. $e_i^j(e_k) = \delta_k^j e_i$: that is, $e_j \mapsto e_i$ and $e_k \mapsto 0$ for $k \neq j$.

The soldering form is defined by the formula $\theta_p(\eta) = b(p)^{-1}\pi_*\eta$, where $\eta \in T_p\mathscr{B}(X)$ and $b(p)\colon \mathbb{R}^n \to T_{\pi(p)}X$ is the basis associated to $p = (x, b) \in \mathscr{B}(X)$. Another way to say this is that $\pi_*\eta = b(\theta^i(\eta)e_i)$.

Vertical vector fields $\widehat{\xi} \in \mathcal{X}_{\mathscr{B}(X)}(\mathfrak{g}^*)$ are killed by θ, so θ is "horizontal" in a sense. You might imagine that there's a horizontal vector field $\widehat{\zeta} \in \mathcal{X}_{\mathscr{B}(X)}((\mathbb{R}^n)^*)$ and a $\Theta \in \Omega^1_{\mathscr{B}(X)}(\mathfrak{g})$ that kills the horizontal vector field. You could get that information if you had a distribution that's complimentary to the vertical bundle, equivalent to a section for (13.7), which would express TP as a direct sum of $T(P/X)$ and π^*TX. This structure is called a connection.

Definition 13.9. A *connection* on a principal G-bundle $\pi\colon P \to X$ is a G-invariant distribution $H \subset TP$ complementary to the vertical $\ker(\pi_*)$.

A vector in $H_p \subset T_pP$ is called *horizontal*, and a vector in $\ker(\pi_*)$ is called *vertical*.

What this means is that if $\xi \in T_{\pi(p)}X$ (a vector downstairs), the connection determines a horizontal lift of it, a $\widehat{\xi} \in H_p$. We hope to integrate that to convert paths on X to paths on P, and if P is the frame bundle, we get something beautiful: one gets a horizontal lift of basis elements and obtains a vector field for each basis element, at least locally. So on the frame bundle, these horizontal spaces are identified with $\mathbb{R}^n$, and the vertical is already identified with $\mathfrak{o}_n$. The existence of a connection (and the integrability condition we'll get back to) parallelizes the neighborhood of a point!

The integral curves in the frame bundle project down to particular curves on X. What's special about these curves? Tune in next time to find out.

Anyways, we also have the form Θ, which can be thought of as splitting (13.7) as a map $TP \to \ker(\pi_*)$. The data of the connection is determined by Θ, but since $\ker(\pi_*) \cong P \times \mathfrak{g}$, this means the connection is determined by a Lie-algebra-valued 1-form $\Theta \in \Omega^1_{\mathscr{B}(X)}(\mathfrak{g})$.

When we discussed distributions, we asked whether they were integrable. We know they're always locally integrable, but what about globally? We'll introduce curvature on a general Riemannian manifold as the obstruction to global integrability of the distribution.

Meanwhile, let's discuss the geometry that a connection buys. Recall that covering spaces $\pi\colon \widetilde{X} \to X$ have *path lifting* (so are examples of fibrations in homotopy theory): if $\gamma\colon (a,b) \to X$ is a path sending $0 \mapsto x_0$ and $\widetilde{x}_0 \in \pi^{-1}(x_0)$, then there's a unique path $\widetilde{\gamma}\colon (a,b) \to \widetilde{X}$ sending $0 \mapsto \widetilde{x}_0$ projecting down to γ, i.e. such that $\pi \circ \widetilde{\gamma} = \gamma$.[12]

If G is a discrete group, principal G-bundles are Galois covering spaces with covering group G. But more generally, we need a connection H to do path lifting on a principal G-bundle $P \to X$.

Definition 13.10. A curve $\widetilde{\gamma}\colon (a,b) \to P$ is *horizontal* if $\dot{\widetilde{\gamma}} \in H_{\gamma(t)}$ for all t.

Theorem 13.11. *Given a connection H on a principal bundle $\pi\colon P \to X$, a path $\gamma\colon (a,b) \to X$ with $\gamma(0) = x_0$, and a $\widetilde{x}_0 \in \pi^{-1}(x_0)$, there is a unique horizontal lift $\widetilde{\gamma}\colon (a,b) \to P$ such that $\widetilde{\gamma}(0) = \widetilde{x}_0$.*

If one specifies that the curve must begin and end at the same points, so the curve closes up, its lift need not close up; its *holonomy* measures the difference (in the fiber, as a G-torsor) between its starting and ending points.

Proof. You can check that the pullback of a principal G-bundle $P \to X$ by a map $f\colon Y \to X$ is again a principal G-bundle $f^*P \to Y$. So let's pull back $P \to X$ by γ, producing a principal G-bundle $\gamma^*P \to (a,b)$, and a map $\widehat{\gamma}\colon \gamma^*P \to P$. Concretely, $\gamma^*P = \{(t,p) \in (a,b) \times P \mid \gamma(t) = \pi(p)\}$.

The connection also pulls back, just as one-forms pull back: $\gamma^*H_p := \{\eta \in T_p\gamma^*P \mid \widehat{\gamma}_*\eta \in H_{\widehat{\gamma}(p)}\}$. This is a rank-1 distribution, hence integrable (or involutive), so let Γ be the maximal leaf of the foliation through $\widetilde{x}_0$. Then, one can show that $\pi|_\Gamma\colon \Gamma \to (a,b)$ is a diffeomorphism, and we can define $\widetilde{\gamma} = (\pi|_\Gamma)^{-1}$, which is unique by the general theory of integrating distributions.

The argument that $\pi|_\Gamma$ is a diffeomorphism is the same as above: it's a covering map where the cover is connected, but the base (a,b) is simply connected. The G-invariance is what keeps it from going to infinity. ⊠

The connection defines an isomorphism of fibers $P_{x_0} \to P_{x_1}$ given a path $x_0 \to x_1$, which is called *parallel transport*. This comes along for all associated bundles, and in particular it's possible to parallel-transport vectors, covectors, etc. Unfortunately, we can only do this along curves, not globally.

Next time, we'll return to the Riemannian situation, and see that in Riemannian geometry, there is a distinguished connection[13] that satisfies the first Maurer-Cartan equation.

[12] For fibrations, such a path lift always exists, but need not be unique.

[13] A typical partition-of-unity argument shows that connections always exist on a manifold.

Lecture 14.

Connections on frame bundles: 3/2/17

Let G be a Lie group and X be a smooth manifold. Last time, we talked about the mixing construction: if F is a smooth manifold with a left G-action, then $F_P = P \times_G F$, which forms a principal G-bundle over X. A section of $F_P \to X$ is a G-equivariant map $\psi\colon P \to F$, i.e. $\psi(pg) = g^{-1}\psi(p)$.

We also defined a connection on a principal G-bundle $\pi\colon P \to X$ to be a G-invariant distribution $H \subset TP$ such that $H \oplus T(P/X) \cong TP$. By G-invariance, we mean under the right G-actioin: $(R_g)_* H_p = H_{p\cdot g}$. We proved that a connection induces a unique lift of horizontal paths, which in particular induces parallel transport in every associated fiber bundle $F_P \to X$. Specifically, if γ is a path from x_0 to x_1, the connection induces a path γ' from p_0 to p_1 in P, and we want to lift this to F_P. We do this by making the F-component constant: on F_P, the path lift starting at (p_0, f) is $(\gamma'(t), f)$. Since there's a quotient by an equivalence relation here, one should check that this behaves well under the G-action, which it does.

Remark 14.1. Another way to think about this is that H defines a distribution on $P \times F$ which is G-invariant, and therefore descends to a distribution on F_P. The parallel transport on F_P is horizontal with respect to this distribution. This is an instance of the idea that additional geometric structure on a principal bundle carries over to all of its associated fiber bundles, where this geometric structure is the connection. ◀

The main case for us is where $P \to X$ is a frame bundle, orthonormal or not. The parallel transport we recover resembles the parallelism that exists in an affine space — but here, we can only transport along curves, and there may be holonomy.

You could take subspaces of tangent spaces, symmetric bilinear forms, and any object that defines a fiber bundle can be parallel-transported using the associated bundle construction. This will also enable us to define a derivative: derivatives require subtraction of values obtained from nearby points, and this requires parallel transport. This doesn't require the bundle of frames, as it can be done more generally.

If $H \subset TP$ is a connection, where $\pi\colon P \to X$ is a principal G-bundle, we get a short exact sequence of vector bundles

$$0 \longrightarrow T(P/X) \longrightarrow TP \xrightarrow{\pi^*} \pi^* TX \longrightarrow 0,$$

and $T(P/X) \cong P \times \mathfrak{g} = \underline{\mathfrak{g}}$ (i.e. the constant vector bundle with fiber $\mathfrak{g}$). Given an $\eta \in T_pP$, let $\Theta_p(\eta) \in T_p(P/X)\colon \mathfrak{g}$ be the vertical projection of η along H_p, so Θ_p is the identity when restricted to $T_p(P/X)$ and is 0 on H_p. This defines a map $\Theta_p\colon T_pP \to \mathfrak{g}$, hence a $\mathfrak{g}$-valued 1-form $\Theta \in \Omega^1_P(\mathfrak{g})$.

This notation looks familiar, and that's no coincidence.

Lemma 14.2. *For any $x \in X$, $\Theta|_{P_x} = \theta_G$ is the Maurer-Cartan form for G. Moreover, for any $g \in G$, $R_g^*\Theta = \mathrm{Ad}_{g^{-1}}\Theta$. Conversely, any $\Theta \in \Omega^1_P(\mathfrak{g})$ satisfying these two properties determines a connection.*

Proof. The first part comes from unwinding the definition: the tangent space of any G-torsor can be identified with $\mathfrak{g}$, which is how we wrote down the Maurer-Cartan form on a G-torsor. Thus, $\Theta|_{P_x} = \theta$.

For the second part, $(R_g^*\Theta)_p(\eta) = \Theta_{pg}(R_{g*}\eta)$ when $\eta \in T_pP$. Choose a curve $p_t\colon (-\varepsilon, \varepsilon) \to P$ with $p(0) = p$ and $\eta = \cdot p(0)$, and write $\eta = \eta_H + \eta_V$, with η_H and η_V denoting the horizontal and vertical components of η, respectively. Then, $\eta_V = \widehat{\xi}_p$ for some $\xi \in \mathfrak{g}$ such that $\xi = \Theta_p(\eta)$, and in particular

$$\eta_V = \widehat{\xi}_p = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} pe^{t\xi}.$$

Then, $(R_g)_*\eta = (R_g)_*\eta_H + (R_g)_*\eta_V$. The first part is in $H_{p\cdot g}$, and

$$\begin{aligned}
(R_g)_*\eta_V &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} R_g\left(pe^{t\xi}\right)\\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} pe^{t\xi}g\\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} pg\left(g^{-1}e^{t\xi}g\right)\\
&= \left.\left(\mathrm{Ad}_{g^{-1}\xi}\right)\right|_{pg}.
\end{aligned}$$

Then, compare this with

$$(\mathrm{Ad}_{g^{-1}}\Theta)_p(\eta) = \mathrm{Ad}_{g^{-1}}\Theta_p\eta = \mathrm{Ad}_{g^{-1}}\eta. \qquad \boxtimes.$$

The two equations in (14.2) are an affine equation: a constant value through P_x and a linear equation. This implies that the space of solutions, namely connections on X, is an affine space: the difference of any two connections is a vector space.

Recall that $\mathscr{B}(X) \to X$ is the $\mathrm{GL}_n(\mathbb{R})$-bundle of frames on a smooth n-manifold X, and there's a soldering form $\theta = \theta^- e_i \in \Omega^1_{\mathscr{B}(X)}(\mathbb{R}^n)$, which transforms under the equation $R_g^*\theta = g^{-1}\cdot\theta$, where the action is matrix multiplication. Moreover, if ζ is vertical, then $\iota_\zeta\theta = 0$, so you might want to push θ down to the base, but the action of G on $\mathbb{R}^n$ is nontrivial.

However, you can bring it over to the associated fiber bundle modeled on $\mathbb{R}^n$, i.e. the tangent bundle, so we obtain a form $\theta \in \Omega^1_X(TX)$. This construction is canonical, and so the only choice we have is for it to be id_{TX}. It's a good exercise to check that what you actually get is the identity.

Now suppose $\Theta \in \Omega^1_{\mathscr{B}(X)}(\mathfrak{gl}_n(\mathbb{R}))$ is a connection (since the frame bundle is an example of a principal bundle). Then, we can write $\Theta = \Theta^i_j e^j_i$, where $\{e^j_i\}$ is the basis for the Lie algebra consisting of matrices with a 1 in entry (i,j) and 0s elsewhere. Then, the forms θ^i and Θ^i_j give n^2+n linearly independent forms which give a global trivialization of $T^*\mathscr{B}(X) \to \mathscr{B}(X)$. This gloabl parallelism makes this a very nice place to do calculus.

You could also take the dual trivialization: dual to θ^i is ∂_i, the horizontal component, which we've seen before; and dual to Θ^i_j is the vertical component $\tilde{e}^j_i$, which is something new. More explicitly, $\partial_i|_p$ is the horizontal lift of the i^{th} basis element of the basis $b(p)$ of $T_{\pi(p)}X$.

Definition 14.3. A curve $\gamma\colon (a,b) \to X$ is a *geodesic* (relative to Θ) if $\dot\gamma$ is parallel.

So geodesics are those which aren't turning: there's no acceleration. To know whether the velocity is changing, you have to compute instantaneous change through parallel transport, which requires the *affine connection* on $\mathscr{B}(X)$.

Proposition 14.4. *Integral curves of ∂_1 project under π to geodesics on X.*

The idea is that the acceleration of an integral curve for ∂_1 is only in the vertical direction.

Proof. Let γ be such a curve, so that the horizontal lift of $\dot\gamma$ is $\dot{\tilde\gamma} = \partial_1$. Writing $\dot\gamma$ as a function $\gamma^*\mathscr{B}(X) \to \mathbb{R}^n$, it's the constant function with value e_1; since it's constant along a horizontal curve, then $\dot\gamma$ is parallel, and hence γ is a geodesic. $\boxtimes$

This perspective gives you all of the usual theorems on geodesics: for example, given a point and an initial velocity, one finds a unique parallel solution starting at a given point in the frame bundle, hence a unique geodesic with that initial position and velocity data on X.

Torsion. There are lots of possible connections on a manifold. But we're going to impose a condition – that the torsion vanishes – which singles out a unique connection in the Riemannian case.

Recall that when $X = \mathbb{A}^n$, $\mathscr{B}(X) = \mathrm{Aff}_n$, and if the Maurer-Cartan forms define the soldering form and connection, then we had equations

$$\begin{aligned} d\theta + \Theta\wedge\theta &= 0\\ \mathrm{d}\Theta + \Theta\wedge\Theta &= 0, \end{aligned}$$

or in indices,

$$\begin{aligned} \mathrm{d}\theta^i + \Theta^i_j\wedge\theta^j &= 0\\ \mathrm{d}\Theta^i_j + \Theta^i_k\wedge\Theta^k_j &= 0. \end{aligned} \tag{14.5}$$

However, this is *not* true for general X! Instead, we give them names.

Definition 14.6. Let Θ be a connection on the frame bundle. Then, the *curvature* is

$$\Omega := \mathrm{d}\Theta + \Theta \wedge \Theta \in \Omega^2_{\mathscr{B}(X)}(\mathfrak{gl}_n(\mathbb{R}))$$

and the *torsion* is

$$\tau := \mathrm{d}\theta + \Theta \wedge \theta \in \Omega^2_{\mathscr{B}(X)}(\mathbb{R}^n).$$

Remark 14.7. The curvature can be defined more generally, in the context of a principal G-bundle $P \to X$, in which case we would say

$$\Omega := \mathrm{d}\Theta + \frac{1}{2}[\Theta \wedge \Theta] \in \Omega^2_P(\mathfrak{g}),$$

which agrees with our definition when we pass to $\mathscr{B}(X)$. ◂

To interpret the torsion, let's compute it on basis vectors.

$$\begin{aligned}
\tau(\partial_k, \partial_\ell) &= \mathrm{d}\theta(\partial_k, \partial_\ell) + (\Theta \wedge \theta)(\partial_k, \partial_\ell) \\
&= \partial_k\theta(\partial_\ell) - \partial_\ell\theta(\partial_k) - \theta([\partial_k, \partial_\ell]) + \Theta(\partial_k)\theta(\partial_\ell) - \Theta(\partial_\ell)\theta(\partial_k) \\
&= -\theta([\partial_k, \partial_\ell]).
\end{aligned}$$

To figure out what this is, let ∂_k generate the flow φ_t and ∂_ℓ generate the flow ψ_s. Then,

$$[\partial_k, \partial_\ell] = \left.\frac{\partial^2}{\partial s \partial t}\right|_{s,t=0} \psi_{-s}\varphi_{-t}\psi_s\varphi_t.$$

The idea is that, as $s, t \to 0$, flowing in the x^k-direction, then the x^j-direction, then back along the $-x^k$-direction, then back along the $-x^j$-direction. You don't always end up back where you started, though you do in affine space. If the connection is *torsion-free*, meaning infinitesimally the connection looks a bit like affine space, the geometry is very nice, and in general the torsion provides a way to quantify how differently X and its connection behave from affine space.

We'll restrict to the Riemannian case soon, but the existence of a torsion-free connection compatible with a geometric structure – complex structure, symplectic structure, etc. – is an integrability condition, and such connections may or may not exist. One of the beautiful aspects of Riemannian geometry is that there always exists a unique connection that's compatible with the Riemannian metric and that is torsion-free.

Lecture 15.

The Levi-Civita connection and curvature: 3/7/17

"Depending on your vision, all of these indices may be a blur."

Lemma 15.1. *Let $\Delta^k_{ij} \in \mathbb{R}$ for $i, j, k = 1, \dots, n$. If $\Delta^i_{jk} = \Delta^i_{kj}$ and $\Delta^k_{ij} = -\Delta^j_{ik}$, then $\Delta^i_{jk} = 0$ for all i, j, k.*

Proof. $\Delta^i_{jk} = \Delta^i_{kj} = -\Delta^k_{ji} = \Delta^j_{ki} = \Delta^j_{ik} = -\Delta^i_{jk}$. ☒

We'll use Lemma 15.1 a few times in today's lecture; what comes out of it is the Levi-Civita connection, a canonical connection defined on a Riemannian manifold.

Theorem 15.2 (Fundamental theorem of Riemannian geometry)**.** *Let X be a Riemannian manifold. Then, there is a unique torsion-free connection on $\mathscr{B}_{\mathrm{O}}(X) \to X$.*

This connection is called the *Levi-Civita connection*. In some cases, e.g. when X is Kähler, there's a beautiful geometric construction of this connection, but in general we must calculate.

Recall that we wrote a connection as a 1-form $\Theta = \Theta^i_j e^j_i \in \Omega^1_{\mathscr{B}_{\mathrm{O}}(X)}(\mathfrak{o}_n)$, such that $\Theta^i_j = -\Theta^j_i \in \Omega^1_{\mathscr{B}_{\mathrm{O}}(X)}$. Here, $e^j_i \colon \mathbb{R}^n \to \mathbb{R}^n$ is defined to send $e_k \mapsto \delta^j_k e_j$. If $\theta = \theta^i e_i \in \Omega^1_{\mathscr{B}_{\mathrm{O}}(X)}(\mathbb{R}^n)$ is the soldering form, then the connection defined by Θ is torsion-free if $\mathrm{d}\theta + \Theta \wedge \theta = 0$. In coordinates, $\mathrm{d}\theta^i + \Theta^i_j \wedge \theta^j = 0$.

Lemma 15.3. *The Levi-Civita connection is unique if it exists.*

Proof. Let $\Delta = \Delta^i_j e^j_i$ be the difference of two such Θ and Θ' satisfying the equations defining a torsion-free connection. Since $\Theta^i_j = -\Theta^j_i$ is a linear equation,

$$\Delta^i_j = -\Delta^j_i, \tag{15.4a}$$

and since the torsion-free condition is affine,

$$\Delta^i_j \wedge \theta^j = 0. \tag{15.4b}$$

If $x \in X$, then restricting a connection Θ to TP_x produces the Maurer-Cartan form for O_n, and therefore $\Delta|_{TP_x} = 0$.

At a $p \in \mathscr{B}_{\mathrm{O}}(X)$, there's a short exact sequence

$$0 \longrightarrow V' \longrightarrow T_p\mathscr{B}_{\mathrm{O}}(X) \longrightarrow V'' \longrightarrow 0,$$

where V is the vertical bundle and $V'' = T_{\pi(p)}X$. Since we have local coordinates, we get an identification $T_{\pi(p)}X \cong \mathbb{R}^n$, and Δ_p and θ_p are pulled back from forms $\overline{\Delta}$ and $\overline{\theta}$ on V''. Write

$$\begin{aligned}\overline{\Delta} &= \overline{\Delta}^i_{jk} e^j_i \otimes e^k\\ \overline{\theta} &= \overline{\theta}^i_k e_i \otimes e^k,\end{aligned}$$

where $\overline{\Delta}^i_{jk}, \overline{\theta}^i_k \in \mathbb{R}$. But the soldering form is defined to satisfy $\overline{\theta}^i_k = \delta^i_k$, so (15.4a) implies $\overline{\Delta}^i_{jk} = -\overline{\Delta}^j_{ik}$, and (15.4b) implies

$$0 = \overline{\Delta}^i_{jk} e^j_i e_\ell \otimes e^k \wedge e^\ell = \overline{\Delta}^i_{jk} e_i \otimes e^k \wedge e^j.$$

But this is skew-symmetric in j and k, so we also have to add in $\overline{\Delta}^i_{kj}$, and in particular $\overline{\Delta}^i_{jk} = \overline{\Delta}^i_{kj}$, so by Lemma 15.1, $\overline{\Delta} = 0$. ⊠

This is not the fastest proof, but all proofs involve some sort of computation.

With uniqueness in hand, it suffices to check existence locally.

Proof sketch of Theorem 15.2. Since Lemma 15.3 takes care of uniqueness, we sketch existence locally: let $U \subset X$ be a chart with a moving frame, i.e. a section $s\colon U \to \mathscr{B}_{\mathrm{O}}(X)|_U$. Then, we can solve for $\overline{\Theta} = s^*\Theta$.

Let $\overline{\theta} = s^*\theta \in \Omega^1_U(\mathbb{R}^n)$, and write

$$\mathrm{d}\overline{\theta}^i = \frac{1}{2} A^i_{jk} \overline{\theta}^j \wedge \overline{\theta}^k.$$

By skew-symmetry, $A^i_{jk} = -A^i_{kj}$. Similarly, write $\overline{\Theta}^i_j = B^i_{jk}\overline{\theta}^k$; we'd like $B^i_{jk} = -B^j_{ik}$. Then,

$$\begin{aligned}0 &= \mathrm{d}\overline{\theta}^j + \overline{\Theta}^i_j \wedge \overline{\theta}^j\\ &= \left(\frac{1}{2}A^i_{jk} - B^i_{jk}\right)\overline{\theta}^j \wedge \overline{\theta}^k\end{aligned}$$

Again using skew-symmetry in j and k,

$$= \frac{1}{2}\left(A^i_{jk} - \frac{1}{2}B^i_{jk} + \frac{1}{2}B^i_{kj}\right)\overline{\theta}^j \wedge \overline{\theta}^k.$$

Thus,

$$\begin{aligned}B^i_{jk} &= A^i_{jk} + B^i_{kj}\\ &= A^i_{jk} - B^k_{ij}\\ &= A^i_{jk} - A^k_{ij} - B^k_{ji}\\ &= A^i_{jk} - A^k_{ij} + B^j_{ki}\\ &= A^i_{jk} - A^k_{ij} + A^j_{ki} + B^j_{ik}\\ &= A^i_{jk} - A^k_{ij} + A^j_{ki} - B^i_{jk}.\end{aligned}$$

So we've gone around three times and ended up with something useful:

$$B^i_{jk} = \frac{1}{2}\Big(A^i_{jk} - A^k_{ij} + A^j_{ki}\Big).$$

Now we can construct Θ, using s to identify $U \times \mathrm{O}_n \xrightarrow{\cong} \mathscr{B}_{\mathrm{O}}(X)|_U$ through the map $x, g \mapsto s(x)\cdot g$. Then, set

$$\Theta_{(x,g)} := \mathrm{Ad}_{g^{-1}}\overline{\Theta} + \theta_{\mathrm{O}_n}.$$

Then, you can check this defines a connection on $\mathscr{B}_{\mathrm{O}}(X)|_U \to U$. ⊠

This is unsatisfying, perhaps, but that's how the Levi-Civita connection works.

We've done a lot of differential-geometric prerequisites in the first half of this semester, and we'll wrap up discussing curvature. In the second act, we can go on to some more fun topics, possibly including Hodge theory, the Gauss-Bonnet-Chern theorem, or other topics, depending on student interest.

∽ · ∼

Anyways, recall that the Frobenius theorem (Theorem 8.5) concerned itself with the integrability of a distribution. Let $H \subset TM$ be a distribution on a manifold M; we define a skew-symmetric bilinear map of vector bundles on M $\Phi_H\colon H \times H \to TM/H$ as follows: for $\xi_1|_m, \xi_2|_m \in H_m$, extend them to local sections ξ_1, ξ_2 of $H \to M$, and let

$$\Phi_H(\xi_1, \xi_2) := [\xi_1, \xi_2] \pmod{H}.$$

Lemma 15.5. *Φ_H is linear over functions, i.e. is a tensor.*

Proof. Let $f_1, f_2 \in C^\infty(M)$. With all equalities taken $\operatorname{mod} H$,

$$\begin{aligned}\Phi_H(f_1\xi_1, f_2\xi_2) &= [f_1\xi_1, f_2\xi_2]\\ &= f_1 f_2[\xi_1,\xi_2] + f_1(\xi_1 f_2)\xi_2 - f_2(\xi_2 f_1)\xi_1\\ &= f_1 f_2[\xi_1,\xi_2].\end{aligned}$$ ⊠

Moreover, H is involutive iff $\Phi_H = 0$. This Φ_H, called the *Frobenius tensor*, is a useful thing to have around: if you have a distribution associated to additional structure, e.g. an almost complex structure or a metric, the Frobenius tensor provides information about its behavior. In our case, where the distribution comes from the Levi-Civita connection, it quantifies curvature.

Recall that for Euc_n or $\mathscr{B}_{\mathrm{O}}(\mathbb{E}^n)$, the Maurer-Cartan form satisfies the structure equations

$$\Theta^i_j + \Theta^j_i = 0 \tag{15.6a}$$

$$\mathrm{d}\theta^i + \Theta^i_j \wedge \theta^j = 0 \tag{15.6b}$$

$$\mathrm{d}\Theta^i_j + \Theta^i_k \wedge \Theta^k_j = 0. \tag{15.6c}$$

Equations (15.6a) and (15.6b) are satisfied by the Levi-Civita connection on $\mathscr{B}_{\mathrm{O}}(X)$, since it's torsion-free. However, the Levi-Civita connection doesn't satisfy (15.6c), and the obstruction is called the *curvature*

$$\Omega^i_j := \mathrm{d}\Theta^i_j + \Theta^i_k \wedge \Theta^k_j. \tag{15.7}$$

This obstruction makes sense more generally, since (15.7) doesn't depend on the Maruer-Cartan form, only the connection. Thus, we can generalize to connections on principal G-bundles. Let G be a Lie group and $\pi\colon P \to X$ be a principal G-bundle. Let $\Theta \in \Omega^1_p(\mathfrak{g})$ be a connection, so $H := \ker(\Theta)$ is a distribution in TP which satisfies $R_{g*}H = H$ for all $g \in G$. Thus, Θ satisfies $\Theta|_{P_x} = \theta_G$, where θ_G is the Maurer-Cartan form on G and $x \in X$, and $R_g^*\Theta = \mathrm{Ad}_{g^{-1}}\Theta$. For any $\zeta \in \mathfrak{g}$, let $\widehat{\zeta} \in \mathcal{X}(P)$ be induced by the G-action. Then the first condition (that Θ restricts to the Maurer-Cartan form) is equivalent to $\iota_{\widehat{\zeta}}\Theta = \zeta$.

Definition 15.8. With G, P, and Θ as above, the *curvature* of Θ is

$$\Omega := \mathrm{d}\Theta + \frac{1}{2}[\Theta \wedge \Theta] \in \Omega^2_P(\mathfrak{g}).$$

The curvature is not a connection: $R_g^*\Omega = \mathrm{Ad}_{g^{-1}}\Omega$, but $\iota_{\widehat{\zeta}}\Omega = 0$. Thus, it descends to a twisted 2-form $\Omega \in \Omega_X^2(\mathfrak{g}_P)$. Checking these properties is once again a computation, e.g.

$$\begin{aligned}\iota_{\widehat{\zeta}}\Omega &= \iota_{\widehat{\zeta}}\left(\mathrm{d}\theta + \frac{1}{2}[\Theta\wedge\Theta]\right)\\ &= -\mathrm{d}\iota_{\widehat{\zeta}}\Theta + \mathcal{L}_{\widehat{\zeta}}\Theta + \frac{1}{2}[\iota_{\widehat{\zeta}}\Theta\wedge\Theta] - \frac{1}{2}[\Theta\wedge\iota_{\widehat{\zeta}}\Theta].\end{aligned}$$

Contracting with a constant function produces 0, so this simplifies to

$$\begin{aligned}&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} R_{e^{t\zeta}}^*\Theta + [\zeta,\Theta]\\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}_{e^{-t\zeta}}\Theta + [\zeta,\Theta]\\ &= -[\zeta,\Theta] + [\zeta,\Theta] = 0.\end{aligned}$$

This allows us to understand the curvature formally; now let's see what it actually is. We've just computed that the curvature of two vectors vanishes when either one is vertical, so the only information it carries is about horizontal vectors.

Let $\xi|_p, \xi_2|_p \in H_p$, so that they're horizontal. We'd like to extend them to vector fields to calculate, and we may as well extend them horizontally, producing (local) sections ξ_1, ξ_2 of H. Then,

$$\begin{aligned}\Omega(\xi_1,\xi_2) &= \mathrm{d}\Theta(\xi_1,\xi_2) + \frac{1}{2}[\Omega\wedge\Omega](\xi_1,\xi_2)\\ &= \xi\cdot\Theta(\xi_2) - \xi_2\Theta(\xi_1) - \Theta([\xi_1,\xi_2] + [\Theta(\xi_1),\Theta(\xi_2)]).\end{aligned}$$

Since we extended horizontally, most things vanish:

$$= -\Theta([\xi_1,\xi_2]).$$

What does that tell us? The connection defines a splitting $TP \cong H \oplus \underline{\mathfrak{g}}$, so $TP/H \cong \underline{\mathfrak{g}}$. In particular, the connection is -1 times the Frobenius form: $\Omega(\xi_1,\xi_2) = -\Phi_H(\xi_1,\xi_2)$. This is one way of understanding the curvature: it's an obstruction to the distribution being integrable.

So the curvature is an obstruction to locally finding an integral manifold whose tangent space is the distribution. A small curve can be lifted, and a loop lifts to a path, but that path might not be closed. Another way to think of this is that small loops bound discs, and we ask whether the disc lifts. Another way to view this is that the curvature defines an obstruction to the horizontal map $\mathcal{X}(X) \to \mathcal{X}(P)$ being a Lie algebra homomorphism.

In coordinates, the Levi-Civita connection is $\Theta = \Theta_j^i e_i^j = \Gamma_{jk}^i e_i^j \theta^k$, where $\Gamma_{jk}^i\colon \mathscr{B}_{\mathrm{O}}(X) \to \mathbb{R}$ are smooth functions, and the coordinate form of the curvature uses the Riemann curvature tensor:

$$\Omega = \Omega_j^i e_i^j = \frac{1}{2}R_{jk\ell}^i e_i^j \theta^k \wedge \theta^\ell.$$

These are equations in the orthonormal frame bundle, at least for now. In the first lecture, we considered the bundle $\mathscr{B}(X) \to X$ of all frames, whose fiber is n^2-dimensional instead of $\binom{n}{2}$-dimensional. Sections that lad in $\mathscr{B}_{\mathrm{O}}(X)$ are the local orthonormal frames, but local coordinates $\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^n}$ typically aren't in $\mathscr{B}_{\mathrm{O}}(X)$. The Levi-Civita connection exists on $\mathscr{B}_{\mathrm{O}}(X)$, and we'd like to extend it to $\mathscr{B}(X)$. Since the connection is O_n-invariant, insisting its extension to be $\mathrm{GL}_n(\mathbb{R})$-invariant is the right thing to do — concretely, we'll right-multiply by elements of $\mathrm{GL}_n(\mathbb{R})$.

Next time, we'll talk about covariant derivatives, which allow us to reinterpret this material and derive the Levi-Civta connection more quickly.

Lecture 16.

Covariant derivatives: 3/9/17

"So you need to reconcile this concrete picture of driving to Minneapolis with these abstract symbols... they're part of the same intellectual soup."

In the first lecture, we derived the Riemann curvature tensor as an obstruction to choosing local coordinates whose derivatives behaved like those for $\mathbb{E}^n$. Then, we worked out a lot of differential geometry, eventually concluding by finding the unique torsion-free connection on a Riemannian manifold, the Levi-Civita connection. Today, we'll interpret the Riemann curvature tensor in this context (which was, historically, the original context): as an obstruction to parallelism, as specified by the connection.

Let $\pi\colon P \to X$ be a principal G-bundle with connection. We've talked about how a connection can be defined as a distribution $H \subset TP$ or a $\Theta \in \Omega^1_P(\mathfrak{g})$, setting $H = \ker(\Theta)$: H contains the horizontal directions with respect to the connection. Then, given a curve $\gamma\colon (a,b) \to X$ with $t_0 \in (a,b)$ and a $p \in \pi^{-1}(\gamma(t_0))$, there's a unique way to lift the path γ along π, producing a new curve $\widetilde{\gamma}\colon (a,b) \to P$ with $\widetilde{\gamma}(t_0) = p$ and $\pi \circ \widetilde{\gamma} = \gamma$. This lift doesn't depend on the parameterization of γ.

More generally, suppose F is a space with a left G-action, so we can form the associated bundle $F_P \to X$, where $F_P = P \times_G F = \{(p,f) \in P \times F\}/(pg,f) = (p,gf)$.[14] A section ψ of F_P lifts to a function $\widetilde{\psi}\colon P \to F$, which satisfies a transformation law: $R_g^*\widetilde{\psi} = g^{-1}\widetilde{\psi}$.

Conversely, suppose $\widetilde{\psi}\colon P \to F$ is a function such that $R_g^*\widetilde{\psi} = g^{-1}\widetilde{\psi}$. Then, $\widetilde{\psi}$ descends to a section ψ of $F_P \to X$. If G acts trivially, this is just a function.

An element of the fiber $(F_P)_x$ defines an element $f \in F$ for every $p \in P_x$. Then, parallel transport along γ is the constant function with value f along $\widetilde{\gamma}$, because of how we quotiented by the G-action. But this can be thought of as parallel transport of the whole fiber, i.e. parallel transport defines a map

$$\tau_t^\gamma\colon (F_P)_{\gamma(0)} \longrightarrow (F_P)_{\gamma(t)},$$

where $\gamma\colon [0,1] \to X$ is a smooth curve.

Example 16.1. Let S^2 be the unit sphere in $\mathbb{E}^3$ and γ be a path from x to y on the sphere. If γ is a geodesic, parallel-transport sends tangent vectors to tangent vectors, since there's no need to turn. For example, if I-35 were a geodesic, you could drive on it all the way from Austin to Minneapolis without turning. However, if the path from x to y isn't a geodesic, you do need to turn (there's nonzero acceleration), and so parallel transport does not preserve tangent vectors. ◀

If F has extra structure that the action of G preserves, then parallel transport also preserves that structure. For example, if F is a Riemannian manifold and G acts by isometries, then parallel-transport also acts by isometries.

We'll be more interested in the case where F is a vector space and G acts linearly through a representation $\rho\colon G \to \mathrm{Aut}(F)$. Then, $F_P \to X$ is a vector bundle and τ_t^γ is linear. We can therefore use it to define differentiation.

Definition 16.2. With F as above, for an $x \in X$ and a $\xi \in T_xX$, let $\gamma\colon (-\varepsilon,\varepsilon) \to X$ satisfy $\gamma(0) = x$ and $\dot{\gamma}(0) = \xi$. Let γ be a section of $F_P \to X$; the *covariant derivative* of ψ at x in the direction ξ is

$$\nabla_\xi \psi = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\tau_t^\gamma\right)^{-1} \psi(\gamma(t)).$$

ψ is *parallel* if its covariant derivative vanishes. Letting x, ξ vary, we obtain an operator $\nabla\colon \Omega^0_X(F_P) \to \Omega^1_X(F_P)$.

The idea is that we want to take $\psi(\gamma(x+t)) - \psi(\gamma(x))$ as $h \to 0$, like in ordinary calculus, but they live in different fibers. Thus we have to parallel-transport them back to $(F_P)_x$.

The covariant derivative satisfies a Leibniz rule, whose proof is the same as for ordinary calculus.

Definition 16.3. Let $\pi\colon E \to X$ be a vector bundle. A *covariant derivative* is a linear map

$$\nabla\colon \Omega^0_X(E) \longrightarrow \Omega^1_X(E)$$

satisfying the *Leibniz rule*

$$\nabla(f\psi) = \mathrm{d}f \cdot \psi + f\nabla\psi.$$

We say a section is *parallel* if its covariant derivative vanishes.

[14] This is a kind of nonlinear analogue of the tensor product: $M \otimes_R L$ takes tuples $m \otimes \ell$ modulo the relation $mr \otimes \ell = m \otimes r\ell$, and we're doing something similar here.

This perspective of covariant derivatives on vector bundles is more concrete and more common than our abstract approach with principal G-bundles.

Let's specialize to $P = \mathscr{B}(X)$, the principal $\mathrm{GL}_n(\mathbb{R})$-bundle of frames on a manifold (no Riemannian metric, yet), and choose a connection on P. If F is a vector space with a $\mathrm{GL}_n(\mathbb{R})$-action and $\psi \in \Gamma(F_P)$, then ψ lifts to a function $\widetilde{\psi}\colon P \to F$, such that $R_g^*\widetilde{\psi} = \rho(g)^{-1}\widetilde{\psi}$.

Lemma 16.4. *$\nabla\psi$ lifts to the function $\partial\psi := \partial_k\psi \otimes e^k\colon P \to F \otimes (\mathbb{R}^n)^*$.*

The upshot is that the covariant derivative of ψ lifts to the directional derivative of the vector field $\partial\psi$.

Proof. Let $x \in X$ and $\xi \in T_xX$, and fix a $p \in \mathscr{B}(X)_x$. This defines a basis of T_xX under which ξ corresponds to $(\xi^1, \dots, \xi^n) \in \mathbb{R}^n$, so ξ lifts to the horizontal vector $\xi^k\partial_k|_p$. If $\widetilde{\gamma}\colon (-\varepsilon, \varepsilon) \to P$ is an integral curve of $\xi^k\partial_k$ with $\widetilde{\gamma}(0) = p$, then $\gamma := \pi \circ \widetilde{\gamma}$ satisfies $\gamma(0) = x$ and $\dot{\gamma}(0) = \xi$, and therefore

$$\nabla_\xi\psi = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \widetilde{\psi}(\widetilde{\gamma}(t)) = \xi^k\partial_k\widetilde{\psi}. \qquad \boxtimes$$

Lemma 16.5. *With notation as above, if $\zeta \in \mathfrak{g}$ and $\widehat{\zeta} \in \mathcal{X}(P)$ is a vertical vector field, then*

$$\widehat{\zeta} \cdot \widetilde{\psi} = -\dot{\rho}(\zeta)\widetilde{\psi},$$

where $\dot{\rho}\colon \mathfrak{g} \to \mathrm{End}(F)$ is the Lie algebra representation corresponding to $\rho\colon G \to \mathrm{Aut}(F)$.

Proof. Differentiate the equation

$$\widetilde{\psi}\left(p \cdot e^{t\zeta}\right) = \rho\left(e^{-t\zeta}\right)\widetilde{\psi}(p)$$

at $t = 0$. $\boxtimes$

Lemma 16.6. *Let $\xi_1, \dots, \xi_n$ be a local framing of X on $U \subset X$, i.e. a section s of $\mathscr{B}(X)|_U \to U$. Then,*

$$\nabla_{\xi_k}\xi_j = (s^*\Theta^i_j)(\xi_k)\xi_i.$$

Recall that $\Theta = \Theta^i_j e^j_i \in \Omega^1_{\mathscr{B}(X)}(\mathfrak{gl}_n(\mathbb{R}))$. We're differentiating in the associated bundle, which is the tangent bundle.

Proof. We want to take the section ξ_k and see what it becomes upstairs under $s_*\xi_k$: we know the horizontal part is ∂_k, and the vertical part is $\Theta^i_j(s_*\xi_k)\widehat{e}^j_i$ ($\widehat{e}^j_i$ is the vector field obtained from the Lie algebra element e^j_i), because the connection Θ is exactly connection onto the vertical part. Thus,

$$s_*\xi = \partial_k + \Theta^i_j(s_*\xi_k)\widehat{e}^j_i. \tag{16.7}$$

Now, ξ_j lifts to some function $\widetilde{\xi}_j\colon \mathscr{B}(X)|_U \to \mathbb{R}^n$, and on the image of s, $\widetilde{\xi}_j = e_j$ is constant. Apply (16.7) to it: since $\widetilde{\xi}_k$ is constant, its covariant derivative is 0, so we get

$$0 = \partial_k\widetilde{\xi}_j - \Theta^i_j(s_*\xi_k) \cdot e_i,$$

so

$$\nabla_{\xi_k}\xi_k = (s^*\Theta^i_j)(\xi_k) \cdot \xi_i. \qquad \boxtimes$$

We now have time to translate these facts into formulas. There will be indices. Recall that the torsion is defined to be $\tau = \mathrm{d}\theta + \Theta \wedge \theta \in \Omega^2_{\mathscr{B}(X)}(\mathbb{R}^n)$, and it descends to $\Omega^2_X(TX)$. If ξ and η are vector fields on X, $\tau(\xi, \eta) = -\theta([\widetilde{\xi}, \widetilde{\eta}])$. Let's reinterpret this in terms of the covariant derivative.

Proposition 16.8.

$$\tau(\xi, \eta) = \nabla_\xi\eta - \nabla_\eta\xi - [\xi, \eta]. \tag{16.9}$$

So to evaluate this at a $p \in X$, we need information on the vector fields in a neighborhood: $\nabla_\eta\xi$ depends on η at x but ξ in a neighborhood, and $[\xi, \eta]$ depends on both in a neighborhood. But τ is a tensor, so it can only depend on their values at x, so the right-hand side of Proposition 16.8 also only depends on ξ and η at x. It's not too difficult to check this directly.

Proof. Again, lift to the frame bundle: $\nabla_\xi\eta$ lifts to the $\mathbb{R}^n$-valued function $\widetilde{\xi}_k(\partial_k\widetilde{\eta}^\ell)e_\ell$, and similarly, $\nabla_\eta\xi$ lifts to $\widetilde{\eta}^\ell(\partial_\ell\widetilde{\xi}^k)e_k$. The bracket $[\widetilde{\xi}^k\partial_k, \widetilde{\eta}^\ell\partial_\ell]$ projects to $[\xi,\eta]$, and expanding,

$$[\widetilde{\xi}^k\partial_k, \widetilde{\eta}^\ell\partial_\ell] = \widetilde{\xi}^k\widetilde{\eta}^k[\partial_k,\partial_\ell] + \widetilde{\xi}^k(\partial_k\widetilde{\eta}^\ell)\partial_\ell - \widetilde{\eta}^\ell(\partial_\ell\widetilde{\xi}^k)\partial_k.$$

The first term of the right-hand side projects down to $-\tau(\xi,\eta)$, the second term projects to $\nabla_\xi\eta$, and the third projects to $-\nabla_\eta\xi$. Thus

$$[\xi,\eta] = -\tau(\xi,\eta) + \nabla_\xi\eta - \nabla_\eta\xi. \qquad \boxtimes$$

Proposition 16.10. *Choose any connection on $\mathscr{B}_{\mathrm{O}}(X)\to X$, where X is a Riemannian manifold. Then, the metric is parallel with respect to the covariant derivative, i.e. if $\xi,\eta\in\mathcal{X}(X)$,*

$$\mathrm{d}\langle\xi,\eta\rangle = \langle\nabla\xi,\eta\rangle + \langle\xi,\nabla\eta\rangle \in \Omega^1_X. \tag{16.11}$$

The idea is that d differentiates three things, so the Leibniz rule should have three pieces, but the term corresponding to differentiating the metric vanishes.

Proof. Lift ξ and η to functions $\widetilde{\xi},\widetilde{\eta}\colon\mathscr{B}_{\mathrm{O}}(X)\to\mathbb{R}^n$. Since the connection was on $\mathscr{B}_{\mathrm{O}}(X)$, ∂_k is tangent to $\mathscr{B}_{\mathrm{O}}(X)$, and therefore $\mathrm{d}\langle\xi,\eta\rangle$ lifts to

$$e^k\partial_k\langle\widetilde{\xi},\widetilde{\eta}\rangle = e^k\Big(\langle\partial_k\widetilde{\xi},\widetilde{\eta}\rangle + \langle\widetilde{\xi},\partial_k\widetilde{\eta}\rangle\Big).$$

The first part of the sum projects to $\langle\nabla\xi,\eta\rangle$, and the second part projects to $\langle\xi,\nabla\eta\rangle$. $\boxtimes$

Recall that a connection is torsion-free if (16.9) vanishes and is orthogonal (compatible with the metric) if (16.11) vanishes. The fundamental theorem of Riemannian geometry, Theorem 15.2, says that on a Riemannian manifold X, there's a unique connection that's torsion-free and orthogonal. In particular, there should be a unique covariant derivative satisfying (16.9) and (16.11), which can be checked more easily.

Second proof of Theorem 15.2 (uniqueness). Let ∇ be a torsion-free, orthogonal connection. Then,

$$\begin{aligned}\langle\nabla_\zeta\xi,\eta\rangle &= \zeta\langle\xi,\eta\rangle = \langle\xi,\nabla_\zeta\eta\rangle\\ &= \zeta\langle\xi,\eta\rangle - \langle\xi,[\zeta,\eta]\rangle - \langle\xi,\nabla_\eta\zeta\rangle - \eta\langle\xi,\zeta\rangle.\end{aligned}$$

Since ∇ is torsion-free,

$$= \langle[\eta,\xi],\zeta\rangle + \langle\nabla_\xi\eta,\zeta\rangle + \xi\langle\eta,\zeta\rangle - \langle\eta,\nabla_\xi\zeta\rangle - \langle\zeta,[\xi,\zeta]\rangle - \langle\eta,\nabla_\xi\zeta\rangle,$$

and since ∇ is orthogonal,

$$= \langle[\eta,\xi],\zeta\rangle + \xi\langle\eta,\zeta\rangle - \langle\eta,[\xi,\zeta]\rangle.$$

Thus, the covariant derivative's value is uniquely defined. TODO: the above calculation is wrong. $\boxtimes$

Corollary 16.12. *If ∇ denotes the Levi-Civita connection, then*

$$2\langle\nabla_\zeta\xi,\eta\rangle = \zeta\langle\xi,\eta\rangle - \eta\langle\xi,\zeta\rangle + \xi\langle\eta,\zeta\rangle - \langle\xi,[\zeta,\eta]\rangle + \langle\zeta,[\eta,\xi]\rangle - \langle\eta,[\xi,\zeta]\rangle.$$

We can use this to compute the connection in local coordinates. As we did in the first lecture, let $x^1,\dots,x^n$ be local coordinates, and define

$$\nabla_{\partial/\partial x^k}\frac{\partial}{\partial x^j} = \Gamma^i_{jk}\frac{\partial}{\partial x^i}.$$

Set $\zeta = \frac{\partial}{\partial x^k}$, $\xi = \frac{\partial}{\partial x^j}$, and $\eta = \frac{\partial}{\partial x^m}$, and apply Corollary 16.12: $g_{ij} = \langle\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\rangle$, and the Lie brackets of coordinate vector fields vanish. Thus,

$$\begin{aligned}2\Gamma^i_{jk}g_{im} &= \frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{km}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m}\\ \Gamma^i_{jk} &= \frac{1}{2}g^{im}\left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{km}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m}\right).\end{aligned}$$

Lecture 17.

Symmetries of the Riemann curvature tensor: 3/21/17

Recall that if X is a Riemannian manifold, then there's a unique orthogonal, torsion-free connection on the principal O_n-bundle $\mathscr{B}_{\mathrm{O}}(X) \to X$ called $\Theta \in \Omega_{\mathscr{B}_{\mathrm{O}}(X)}(\mathfrak{o}_n)$. This induces a covariant derivative $\nabla\colon \Omega^0_X(TX) \to \Omega^1_X(TX)$ which satisfies

$$\begin{aligned} \mathrm{d}\langle \xi, \eta\rangle &= \langle \nabla\xi, \eta\rangle + \langle \xi, \nabla\eta\rangle \\ [\xi, \eta] &= \nabla_\xi \eta - \nabla_\eta \xi. \end{aligned}$$

If $\xi_1, \dots, \xi_n$ is a local framing, i.e. a section $s\colon U \to \mathscr{B}(X)|_U$ (where $U \subset X$), then $\nabla \xi_j = s^*\Theta^i_j \cdot \xi_i$ and $\nabla_{\xi_k}\xi_k = (s^*\Theta^i_j)(\xi_k)\cdot \xi_i$.

Recall that the curvature of the connection Θ is $\Omega := \mathrm{d}\Theta + \Theta\wedge\Theta \in \Omega^2_{\mathscr{B}(X)}(\mathfrak{gl}_n(\mathbb{R}))$, as in (15.7). It satisfies $\iota_{\widehat{\zeta}} = 0$, where $\widehat{\zeta}$ is the vertical vector field generated by a $\zeta \in \mathfrak{gl}_n(\mathbb{R})$, and if $g \in \mathrm{GL}_n(\mathbb{R})$, $R_g^*\Omega = \mathrm{Ad}_{g^{-1}}\Omega$. Thus, Ω descends to a form on X, also called $\Omega \in \Omega^2_X(\operatorname{End} TX)$.

The curvature is measuring the failure of the map from a vector field to its covariant derivative to be a Lie algebra homomorphism. We can say this precisely.

Proposition 17.1. *If $\xi, \eta \in \mathcal{X}(X)$, then*

$$\Omega(\xi,\eta) = \nabla_\xi\nabla_\eta - \nabla_\eta\nabla_\xi - \nabla_{[\xi,\eta]} = [\nabla_\xi, \nabla_\eta] - \nabla_{[\xi,\eta]}. \tag{17.2}$$

In particular, the right-hand side, *a priori* a vector field, is actually a tensor: the non-local information cancels out.

Proof. Let ζ be a vector field near a point $x \in X$, which lifts to a function $Z\colon \mathscr{B}(X) \to \mathbb{R}^n$. Then, the covariant derivative is computed as $\nabla_\xi\zeta = \widetilde{\xi}\cdot Z$, where $\widetilde{\xi}$ is the horizontal lift of ξ.

This allows us to compute: let $\widetilde{\xi}$ and $\widetilde{\eta}$ be the lifts of ξ and η, respectively. Then, the left-hand side of (17.2) is

$$\begin{aligned} \Omega(\xi,\eta) &= (\mathrm{d}\Theta + \Theta\wedge\Theta)(\widetilde{\xi},\widetilde{\eta}) \\ &= \widetilde{\xi}\cdot\Theta(\widetilde{\eta}) - \widetilde{\eta}\cdot\Theta(\widetilde{\xi}) - \Theta([\widetilde{\xi},\widetilde{\eta}]) + \Theta(\widetilde{\xi})\Theta(\widetilde{\eta}) - \Theta(\widetilde{\eta})\Theta(\widetilde{\xi}) \\ &= -\Theta([\widetilde{\xi},\widetilde{\eta}]). \end{aligned}$$

This is a function $\mathscr{B}(X) \to \mathfrak{gl}_n(\mathbb{R})$. The right-hand side of (17.2) is

$$\begin{aligned} \nabla_\xi\nabla_\eta - \nabla_\eta\nabla_\xi - \nabla_{[\xi,\eta]} &= \widetilde{\xi}\widetilde{\eta} - \widetilde{\eta}\widetilde{\xi} - \sim([\xi,\eta]) \\ &= [\widetilde{\xi},\widetilde{\eta}] - \sim([\xi,\eta]), \end{aligned}$$

which is the vertical part of $[\widetilde{\xi},\widetilde{\eta}]$. This is a vertical vector field on $\mathscr{B}(X)$, and when this acts on Z, which transforms by $R_g^*Z = g^{-1}Z$, then differentiation accounts for the $-$ sign, by Lemma 16.5. ⊠

We can also compute the curvature in local coordinates $x^1, \dots, x^n$, which is how we get the explicit formula for the Riemann curvature tensor (and compute). This is what tells you whether there exist local coordinates in which the metric looks like the standard metric, as we computed in the first lecture: the curvature tensor is the obstruction to the integrability of the connection.

Let $x^1, \dots, x^n$ be local coordinates and $\xi_j := \frac{\partial}{\partial x^j}$ be a local framing. In particular, $[\xi_k, \xi_\ell] = 0$. In particular,

$$\nabla_{\partial/\partial x^k}\frac{\partial}{\partial x^j} = (s^*\Theta^i_j)\left(\frac{\partial}{\partial x^k}\right)\cdot\frac{\partial}{\partial x^i} = \Gamma^i_{jk}\frac{\partial}{\partial x^i}$$

for some functions Γ^i_{jk}. Write

$$\Omega\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell}\right)\frac{\partial}{\partial x^j} = R^i_{jk\ell}\frac{\partial}{\partial x^i}.$$

Then,

$$\Omega = \frac{1}{2} R^i_{jk\ell} \left(\frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j \right) \otimes \left(\mathrm{d}x^k \wedge \mathrm{d}x^\ell \right).$$

In particular,

$$\begin{aligned}
\Omega\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell} \right) \frac{\partial}{\partial x^j} &= \nabla_{\partial/\partial x^k} \left(\Gamma^m_{\ell j} \frac{\partial}{\partial x^m} \right) - \nabla_{\partial/\partial x^\ell} \left(\Gamma^m_{\ell k} \frac{\partial}{\partial x^m} \right) \\
&= -\left(\frac{\partial \Gamma^i_{\ell j}}{\partial x^k} - \frac{\partial \Gamma^i_{kj}}{\partial x^\ell} + \Gamma^m_{\ell j} \Gamma^j_{km} - \Gamma^m_{kj} \Gamma^j_{\ell m} \right) \frac{\partial}{\partial x^i}.
\end{aligned}$$

This agrees with what we naïvely derived in the first lecture, which is always reassuring.

The votes are in, and people are interested in learning about symmetric spaces and special holonomy; in our discussion of these topics, we'll also touch on some more traditional topics, such as geodesics. Before we embark on this journey, however, we should discuss some properties of the Riemann curvature tensor while it's still on the board.

Let $V = \mathbb{R}^n \cong T_x X$, and $e_1, \dots, e_n$ be an orthonormal basis of V. We can use it to lower an index of the Riemann curvature tensor, defining

$$R_{ijk\ell} := \langle R(e_k, e_\ell) e_j, e_i \rangle.$$

We can view R as a multilinear function $V \times V \times V \times V \to \mathbb{R}$, i.e. an element of $(V^*)^{\otimes 4}$, but it has symmetries: it's not just any function.

The orthogonal group O_n acts on $(V^*)^{\otimes 4}$, and the curvature tensor lives in a subrepresentation, but not an irreducible one. The subrepresentation it lives in splits into m_n irreducible components, where m_n depends on the dimension n for small n. We'll return to this when we discuss special holonomy.

There are also more down-to-Earth symmetries of $R_{ijk\ell}$.

Proposition 17.3.

(1) *R is alternating in its first two indices:* $R_{ijk\ell} = -R_{ij\ell k}$.
(2) *R is alternating in its last two indices:* $R_{ijk\ell} = -R_{jik\ell}$.
(3) *R sums to* 0 *under cyclic permutation:* $R_{ijk\ell} + R_{ik\ell j} + R_{i\ell jk} = 0$.
(4) *R is symmetric under switching the first two and last two indices:* $R_{k\ell ij} = R_{ijk\ell}$.

Proof. (1) follows because Ω is a 2-form, so $\Omega(e_\ell, e_k) = -\Omega(e_k, e_\ell)$. (2) follows because$\Omega(e_k, e_\ell)$ is a skew-symmetric endomorphism of V.

(3) is a *Bianchi identity*; there's another which is about covariant derivatives of R. Anyways, we can use Proposition 17.1 to prove it by computing the cyclic sum $\Omega(\xi_1, \xi_2)\xi_3$. Extend $\xi_i|_x$ to local vector fields ξ_i such that $[\xi_i, \xi_j] = 0$. Then,

$$\begin{aligned}
\Omega(\xi_1, \xi_2)\xi_3 + \Omega(\xi_2, \xi_3)\xi_1 + \Omega(\xi_3, \xi_1)\xi_2 = \nabla_{\xi_1} \nabla_{\xi_2} \xi_3 - \nabla_{\xi_2} \nabla_{\xi_1} \xi_3 \\
+ \nabla_{\xi_2} \nabla_{\xi_3} \xi_1 - \nabla_{\xi_3} \nabla_{\xi_2} \xi_1 \\
+ \nabla_{\xi_3} \nabla_{\xi_1} \xi_2 - \nabla_{\xi_1} \nabla_{x_3} \xi_2.
\end{aligned}$$

Since the connection is torsion-free, $\nabla_{\xi_1} \nabla_{\xi_2} \xi_3 - \nabla_{\xi_1} \nabla_{\xi_3} \xi_2$ is a Lie bracket $\nabla_{\xi_3}[\xi_1, \xi_2]$, which vanishes. Similarly, $\nabla_{\xi_2} \nabla_{\xi_3} \xi_1$ and $\nabla_{\xi_2} \nabla_{\xi_1} \xi_3$ cancel, as do $\nabla_{\xi_3} \nabla_{\xi_1} \xi_2$ and $\nabla_{\xi_3} \nabla_{\xi_2} \xi_1$.

The fourth claim is a formal consequence of the previous three. Milnor's Morse theory book has a nice depiction of these symmetries, labeling them by an octahedron and coloring in the edges that satisfy a Bianchi identity. From this, one can deduce (4). ⊠

Now we can interpret R more precisely. The identities (1) and (2) show that $R \in \mathrm{SkewEnd}\, V \otimes \Lambda^2 V^* = \Lambda^2 V^* \otimes \Lambda^2 V^*$. If you raise two indices using the metric, to obtain $\widetilde{R} := R^{ij}_{k\ell}$, you get $\widetilde{R} \in \Lambda^2 V \otimes \Lambda^2 V^* \cong \mathrm{End}(\Lambda^2 V)$. As an endomorphism $\widetilde{R} \colon \Lambda^2 V \to \Lambda^2 V$, $\widetilde{R}$ is called the *curvature operator*.

$\Lambda^2 V$ has an inner product

$$\langle \xi_1 \wedge \xi_2, \eta_1 \wedge \eta_2 \rangle := \det\big(\langle \xi_i, \eta_j \rangle \big)_{ij},$$

and identity (4) says that $\widetilde{R}$ is a symmetric operator. The Bianchi identity is still independent, so we haven't completely characterized the Riemann curvature tensor, but this is still pretty good.

We can use this to construct a numerical invariant: given a $\pi \in \Lambda^2 V$, consider the quadratic function $\langle \widetilde{R}(\pi), \pi \rangle$, and construct the homogeneous function

$$K(\pi) := \frac{\langle \widetilde{R}(\pi), \pi \rangle}{\langle \pi, \pi \rangle}.$$

Its homogeniety means it passes to a function $K\colon \mathbb{P}(\Lambda^2 V) \to \mathbb{R}$. Riemann actually found this invariant before the curvature tensor.

To characterize this, consider the *Grassmannian* $\mathrm{Gr}_2 V$, the manifold of two-dimensional subspaces of V.

Lemma 17.4. *Let (ξ_1, ξ_2) and (η_1, η_2) be ordered bases of $\pi \in \mathrm{Gr}_2 V$. Then, inside $\mathbb{P}(\Lambda^2 V)$,*

$$\mathrm{span}\{\xi_1 \wedge \xi_2\} = \mathrm{span}\{\eta_1 \wedge \eta_2\}.$$

The reason is that they're related by a determinant, which is nonzero because they're both bases. Lemma 17.4 means that $\xi_1, \xi_2 \mapsto \xi_1 \wedge \xi_2$ induces an embedding $P\colon \mathrm{Gr}_2 V \hookrightarrow \mathbb{P}(\Lambda^2 V)$, which is called the *Plücker embedding*.

Anyways, we have our invariant $K\colon \mathbb{P}(\Lambda^2 V) \to \mathbb{R}$, and can restrict it through the Plücker embedding to $\mathrm{Gr}_2 V$, defining a function on 2-planes $K\colon \mathrm{Gr}_2 V \to \mathbb{R}$. This is called the *sectional curvature* (explicitly, for an $x \in X$, $V = T_x X$, and this is the sectional curvature at x).

Remark 17.5. The relationship between two symmetric bilinear forms as a way to obtain a function is formally the same as what we saw before for surfaces immersed in Euclidean 3-space, where we compated the second and first fundamental forms. ◂

Letting x vary, the sectional curvature is a map $K_X\colon \mathrm{Gr}_2(TX) \to \mathbb{R}$.

Remark 17.6.

(1) It's possible to recover the Riemann curvature tensor from K_X. This is an exercise in linear algebra, since everything's a tensor, so think about how to do it on a single vector space.
(2) If ξ and η are orthonormal, then there's a nice formula for K in terms of R: $K(\xi \wedge \eta) = \langle R(\xi, \eta)\eta, \xi \rangle$.
(3) Less trivially (and possibly a future homework exercise), given a two-dimensional subspace $\pi \subset T_x X$, let $\Sigma_\pi \subset X$ be the surface which is the union of geodesics starting at x with tangents in π (where we flow for some time ε). This surface inherits the Riemannian metric and has a curvature, which is a scalar. In particular, $K_X(\pi)$ is the Gauss curvature of Σ_π at x. ◂

We saw that integrating the curvature of a surface produced (a fixed multiple of) the Euler characteristic. Thus, constraining the curvature to be positive places strong constraints on the topology of a surface, and there are more general ways in which restricting the sign of the sectional curvature constrains the geometry and topology.[15] Here are some examples.

Definition 17.7. A Riemannian manifold X is a metric space in which

$$d(x, y) = \inf_{\gamma\colon x \to y} \mathrm{length}(\gamma).$$

We say that X is *complete* if it's complete as a metric space.

Proposition 17.8. *Assume X is a Riemannian manifold whose curvature is positive everywhere.*

(1) *(Bonnet-Meyers) If X is complete and $K_X > \varepsilon > 0$ for some fixed ε, then X is compact and has finite fundamental group.*
(2) *(Brendle-Schoen) If X is compact and* quarter-pinched, *i.e. $1/4 < K_X \le 1$, then in addition the universal cover of X is diffeomorphic to S^n.*[16]
(3) *If $1/4 \le K_X \le 1$, then X is diffeomorphic to one of S^n, $\mathbb{CP}^n$, $\mathbb{HP}^n$, or $\mathbb{OP}^2$.*

Proposition 17.9. *Assume X is a Riemannian manifold whose curvature is nonpositve everywhere.*

[15] In general, when someone says the curvature of a Riemannian manifold with no additional context, they mean sectional curvature, and otherwise will clarify.

[16] It's been known for a while that a quarter-pinched compact manifold with positive sectional curvature is homeomorphic to a sphere, but in dimensions at least 7, there are manifolds homeomorphic to spheres but not diffeomorphic (exotic differentiable structures). The result of Simon Brendel and Rick Schoen is recent, coming in the past 15 years.

(1) *If X is complete, then the universal cover of X is diffeomorphic to affine space $\mathbb{A}^n$.*
(2) *The torus $(S^1)^n$ does not admit a metric with strictly negative curvature.*

The *Hopf conjecture* (well, one of several Hopf conjectures) is that there's no metric on $S^2 \times S^2$ with entireky positive curvature.

Next time, we'll talk about flat curvature, and begin discussing holonomy.

Lecture 18.

Constant curvature and holonomy: 3/23/17

Last time, we talked about the sectional curvature, and stated some theorems (Propositions 17.8 and 17.9) that show how fixing the sign of sectional curvature is a strong constraint on a manifold. Today, we'll discuss what constant sectional curvature implies.

Let X be a Riemannian manifold and suppose we have a global framing of $\pi\colon \mathscr{B}_{\mathrm{O}}(X) \to X$, namely vertical vector fields $\widehat{\zeta}_i^j$ and horizontal vector fields ∂_k. Each ζ_i^j is an element of the orthogonal Lie algebra $\mathfrak{o}_n$ with a 1 in entry ij, a -1 in entry ji, and 0 everywhere else. This defines a morphism of Lie algebras $\mathfrak{o}_n \to \mathsf{Vect}(\mathscr{B}_{\mathrm{O}}(X))$, i.e. $[\widehat{\zeta}_i^j, \widehat{\zeta}_k^\ell]$ recovers the structure equations of the orthogonal group: there are four terms that look like $\delta_k^j \widehat{\zeta}_i^\ell$.

If you take the commutator of a horizontal piece and a vertical piece, you recover the structure equation of $\mathfrak{euc}_n$:

$$[\widehat{\zeta}_i^j, \partial_k] = \delta_k^j \partial_i - \delta_k^i \partial_j. \tag{18.1}$$

If you bracket two horizontal pieces, the result in general has torsion, but we're using the Levi-Civita equation, which is torsion-free, and we get

$$[\partial_k, \partial_\ell] = -\frac{1}{2} R^i_{jk\ell} \widehat{\zeta}_i^j. \tag{18.2}$$

The $1/2$ appears because

$$\Omega(\partial_k, \partial_\ell) = -\Theta([\partial_k, \partial_\ell]) = \frac{1}{2} R^i_{jk\ell} \zeta_i^j.$$

Equations (18.1) and (18.2) are a statement about curvature. In particular, if R has constant sectional curvature K, then the coefficients $R^i_{jk\ell}$ are constant functions, and (18.1), (18.2), and the structure equations for $[\widehat{\zeta}_i^j, \widehat{\zeta}_k^\ell]$ are the structure equations for an $n(n+1)/2$-dimensional Lie algebra.

Exercise 18.3. Calculate which Lie algebra you get. The answer depends on sign:

- When $K = 0$, we can consider Euclidean space $\mathbb{E}^n$, whose orthonormal frame bundle is acted on simply transitively by Euc_n, and therefore the brakcets define $\mathfrak{euc}_n$.
- When $K > 0$, we can consider the sphere S^n with the round metric, whose orthonormal frame bundle is acted on simply transitively by O_{n+1}, and therefore we obtain the Lie algebra $\mathfrak{o}_{n+1}$.
- If $K < 0$, we can consider hyperbolic space $\mathbb{H}^n$, whose frame bundle is acted on simply transitively by $\mathrm{O}_{n,1}^+$, the group of isometries of $\mathbb{R}^{n,1}$ (i.e. preserving a Minkowski metric, one with signature $(n, 1)$) that are orientation-preserving in the time-like (negative definite) direction. The Lie algebra in queation is $\mathfrak{o}_{n,1}$.

When $K \neq 0$ and $n \geq 1$, the Lie group we obtain has two connected components.

Asking for constant sectional curvature is a very strong constraint — in fact, the examples we saw above are virtually the only possible examples.

Theorem 18.4. *Let X be a simply connected Riemannian manifold of constant curvature K. Then, X is isometric to one of S^n, $\mathbb{E}^n$, or $\mathbb{H}^n$, depending on whether $K > 0$, $K = 0$, or $K < 0$ respectively.*

Definition 18.5. A manifold X is *homogeneous* if it admits a transitive action of a Lie group G. If X is Riemannian, we ask for G to act by isometries.

When X has constant curvature, it doesn't just look the same in every point, but also in every direction: not just is X homogeneous, but its frame bundle is too. The groups O_{n+1}, Euc_{n+1}, and $\mathrm{O}^+_{n,1}$ are the largest groups that can act by isometries on a Riemannian n-manifold. Their point stabilizer groups are the same, O_n, acting on the orthonormal bases of the tangent space at that point.

Let G act on X by isometries; then, for any $x \in X$, let $f_x\colon G \to X$ send $g \mapsto g\cdot x$. If the *stabilizer group* $H_x := f_x^{-1}(x)$ is a closed subgroup of G, then it's a Lie subgroup, and in particular f_x induces a diffeomorphism $G/H_x \cong X$. In the cases of constant curvature, this exhibits $S^n \cong \mathrm{O}_{n+1}/\mathrm{O}_n$, $\mathbb{E}^n \cong \mathrm{Euc}_n/\mathrm{O}_n$, and $\mathbb{H}^n \cong \mathrm{O}^+_{n,1}/\mathrm{O}_n$.

We'll now discuss holonomy, in order to understand Berger's classification of holonomy of Riemannian manifolds. Throughout this class, we've switched between the specific perspective of Riemannian geometry and the general perspective of principal bundles (the Riemannian case is the frame bundle). Holonomy also makes sense in the general case.

Let $\pi\colon P \to X$ be a principal G-bundle with connection Θ. Then, as we've discussed, given a piecewise C^1 curve γ from x to y and a, horizontal lifting induces a map $\rho_\gamma\colon P_x \to P_y$: $\rho_\gamma(\widetilde{x})$ starts at $\widetilde{x}$, takes the unique horizontal lift, and sees where it ends up in P_y. All of this follows from the existence and uniqueness theorem for ODEs, and does not depend on a parameterizaton of γ.

The G-invariance of Θ means that $\rho_\gamma(pg) = \rho_\gamma(p)\cdot g$, i.e. ρ_γ is a map of G-torsors.

Now we specialize to loops. Fix an $x \in X$ and consider loops γ based at x. Then, $\rho_\gamma \in \mathrm{Aut}(P_x)$, i.e. it's a diffeomorphism of the fiber that commutes with the G-action. Furthermore, $\rho_{\gamma_1\gamma_2} = \rho_{\gamma_1}\circ\rho_{\gamma_2}$. That is, if you trace through one loop, then the other, the parallel transports compose.

Definition 18.6.

- The *holonomy group* $\mathrm{Hol}_X(\Theta)$ is the subgroup of $\mathrm{Aut}(P_x)$ consisting of ρ_γ for all loops γ based at x.
- The *restricted holonomy group* $\mathrm{Hol}^0_X(\Theta)$ is the subgroup of $\mathrm{Hol}_X(\Theta)$ of ρ_γ for which γ is null-homotopic.

If we fix a $p \in P_x$, then we can define $h_\gamma(p) \in G$ by $\rho_\gamma(p) = p\cdot h_\gamma(p)$. For a different $pg \in P_x$, $h_\gamma(pg) = g^{-1}h_\gamma(p)g$.

Remark 18.7. A choice of $p \in P_x$ gives an identification $G \to P_x$ of right G-torsors taking $g \mapsto p\cdot g$, and this identification takes ρ_γ to left multiplication by $h_\gamma(p)$. Thus, we can identify $\mathrm{Hol}^0_p(\Theta) \subseteq \mathrm{Hol}_p(\Theta) \subseteq G$. ◂

If $p, q \in P$ are joined by a (piecewise C^1) horizontal line, then $\mathrm{Hol}_p(\Theta) = \mathrm{Hol}_q(\Theta)$ as subgroups of G, and the same is true of reduced holonomy groups. The idea is that the curve c from p to q, plus a loop γ based at p, defines a loop $c\gamma c^{-1}$ based at q, and the holonomy is the same by uniqueness of parallel transport.

Thus, P is partitioned by the equivalence relation where $p \sim q$ if p and q are joined by a horizontal curve (since you can glue a curve from p to q to a curve from q to r). Let P_p denote the equivalence class of a $p \in P$; we'll prove shortly that P_p is a manifold and that it surjects smoothly onto a connected component of X. It's called the *holonomy bundle*, and we'll return to it later.

Theorem 18.8.

(1) $\mathrm{Hol}^0_p(\Theta)$ *is a connected Lie subgroup of G.*[17]

(2) $\mathrm{Hol}_p(\Theta)$ *is a Lie group whose identity component is* $\mathrm{Hol}^0_p(\Theta)$.

(3) *There is a surjective homomorphism*

$$\pi_1(X, x) \longrightarrow \mathrm{Hol}_p(\Theta)/\mathrm{Hol}^0_p(\Theta) = \pi_0(\mathrm{Hol}_p(\Theta)).$$

For a reference on this, see Kobayashi-Nomizu, "Foundations of differential geometry," volumes 1 and 2. In particular, they discuss a theorem (probably due to a few people).

[17] It's not always a closed subgroup. For an example of a Lie subgroup of a Lie group that's not closed, consider the torus $\mathbb{R}^2/\mathbb{Z}^2$ and a line emerging from the identity with irrational slope. It's dense but not the whole space, and since the line is a Lie subgroup of $\mathbb{R}^2$, it's s a Lie subgroup of the torus.

Theorem 18.9. *Let G be a Lie group and $H \subset G$ be a sunbgroup such that, for all $h \in H$, there exists a piecewise C^1 path from e to h. Then, H is a connected Lie subgroup of G.*

The proof idea is to construct the Lie algebra for H, then exponentiate it and show that the connected Lie subgroup you necessarily obtain is H. The theorem is true, but harder, with weaker regularity hypotheses.

We'll use this to show that $\mathrm{Hol}_p^0(\Theta)$ is C^1-path-connected: given any loop $h_\gamma(p) \in \mathrm{Hol}_p^0(\Theta)$, the null-homotopy from it to the constant loop can be lifted to a path from $h_\gamma(p)$ to the constant loop. The lift solves ODEs with a parameter; this again uses the fundamental theorem of ODE, specifically the part where solutions depend smoothly on initial data.

Riemannian holonomy. Let (X, g) be a Riemannian manifold. For a $p \in \mathscr{B}_{\mathrm{O}}(X)$, the holonomy groups specialize to $\mathrm{Hol}_p^0(g) \subset \mathrm{Hol}_p(g) \subset \mathrm{O}_n$, the holonomy groups for the frame bundle with the Levi-Civita connection, called the *Riemannian holonomy groups*.

Theorem 18.10 (Borel-Lichnerowicz)**.** $\mathrm{Hol}_p^0(g)$ *is a closed (and therefore compact) Lie subgroup of* O_n.

We won't prove this, but next time we will prove some results relating holonomy and curvature.

Examples are like cherries on a sundae, so let's see what the Riemannian holonomy is in a few concrete cases.

Example 18.11. Let's start with S^2 in the round metric. Take a loop based at the north pole that's a geodesic triangle with angles θ_1, θ_2, and θ_3. Since S^2 is simply connected, the holonomy and special holonomy groups are the same. In particular, we know that $\mathrm{Hol}_p(g)$ is a subgroup of the connected component of the identity of O_2, which is SO_2. $\mathrm{SO}_2 \cong \mathbb{T}$ is the circle group, so the holonomy is an angle.

Let ∂_1 be the unit tangent vector at the north pole in the direction of the first leg of the triangle. An integral curve for∂_1 projects to the geodesic that is the first leg of the triangle. At the next leg, we want to take the integral curve of ∂_1, but it's pointing in the wrong direction: we have to rotate it again by the *exterior* angle at the juncture of the first and second legs, which is $\pi - \theta_2$. At the next leg, we also have to rotate, this time by $\pi - \theta_3$. Then, when we get back, we have to rotate by θ_1. Thus, the total angle is rotation by $\pi - (\theta_1 + \theta_2 + \theta_3)$. The trick was to use that horizontal lifts are invariant under the action of O_n: not all computations use indices. ◂

Example 18.12. Consider $\mathbb{R} \times \mathbb{E}^2$ with the flat metric, which has no holonomy at all. But let's quotient by the equivalence relation $(x, e) \sim (x + 1, R_\alpha(e))$, where R_α is rotation through an angle α about the origin.

The quotient X, diffeomorphic to $S^1 \times \mathbb{E}^2$, is still a flat manifold, and so $\mathrm{Hol}^0 = \{1\}$ is trivial. What about nonrestricted holonomy? If you go around the circle, the holonomy is the rotation R_α, so $\mathrm{Hol} \cong \langle R_\alpha \rangle \subset \mathrm{SO}_2 \subset \mathrm{SO}_3 \subset \mathrm{O}_3$. The nature of this group depends on whether α is a rational multiple of π: if so, it's a finite cyclic group, and if not, it's isomorphic to $\mathbb{Z}$, and is a 0-dimensional Lie subgroup that's not a closed subgroup. Its closure is all rotations, $\mathrm{SO}_2 \subset \mathrm{O}_3$. In particular, the holonomy group is not always closed, but the restricted holonomy group is always closed. ◂

Example 18.13. The *Klein bottle* is the surface you obtain by gluing two ends of a cylinder by a reflection. If you give the cylinder the flat metric, the restricted holonomy group is trivial, and the holonomy group contains a reflection, which detects the fact that the Klein bottle is not orientable. ◂

Berget's theorem classifies restricted holonomy, and is a local theorem, meaning we can attack it with curvature.

Theorem 18.14 (Berger)**.** *Let X be a simply connected Riemannian n-manifold, and suppose X is irreducible. Then, ether X is* symmetric *(i.e. a symmetric space, which we'll discuss later), or exactly one of the following holds:*

(1) $\mathrm{Hol} = \mathrm{SO}_n$, *which is the generic case.*
(2) $\mathrm{Hol} = \mathrm{U}_m$ *where $n = 2m$, in which case X is a* Kähler manifold.
(3) $\mathrm{Hol} = \mathrm{SU}_m$, *where $n = 2m$, in which case X is a* Calabi-Yau manifold.[18]
(4) $\mathrm{Hol} = \mathrm{Sp}_k \times \mathrm{Sp}_1$, *where $n = 4k$ and $k \geq 2$, in which case X is called* quaternionic Kähler.
(5) $\mathrm{Hol} = \mathrm{Sp}_k$, *where $n = 4k$ and $k \geq 2$, in which case X is called* hyperKähler.
(6) $\mathrm{Hol} = \mathrm{G}_2 \subset \mathrm{SO}_7$, *where $n = 7$, in which case X is called a* G_2-manifold.

[18] In older works, Calabi-Yau manifolds were called *special Kähler manifolds*, though that term now means something different.

(7) $\mathrm{Hol} = \mathrm{Spin}_7 \subset \mathrm{SO}_8$, *where* $n = 8$, *in which case* X *is called a* Spin_7-manifold.

When we say X is irreducible, we mean roughly that it's not a product. We'll discuss this in due time.

These different kinds of manifolds arise in many different places: Kähler manifolds arise in complex algebraic geometry, e.g. $\mathbb{CP}^n$ is Kähler, as are its algebraic submanifolds. The Calabi-Yau condition is equivalent to a Kähler manifold with vanishing Ricci curvature. Yau famously proved the Calabi conjecture relating this to the vanishing of the first Chern class. Various other examples force conditions such as the Ricci curvature vanishing, Einstein manifolds (the Ricci tensor is a multiple of the metric), etc.: the holonomy controls a lot of the geometry.

Quaternionic projective space $\mathbb{HP}^n$ is an example of a quaternionic Kähler manifold, but is a symmetric space, so doesn't count. Compact hyperKähler manifolds were first understood as a corollary of Yau's theorem, which was proven after Berger. The exceptional examples were left as open possibilities by Berger, with examples found in Dominic Joyce's thesis. There were other ones left open by Berger, which were later found to only happen for symmetric spaces.

The classification resembles the classification of division algebras: G_2 acts as symmetries of the imaginary octonions, for example.

Lecture 19.

Holonomy and Kähler manifolds: 3/28/17

Today, we'll discuss some more general facts about holonomy and discuss one of Berger's cases, namely Kähler manifolds.

First, though, let's return to Example 18.11: let S^2 have the round metric of radius 1, and consider a geodesic triangle with angles θ_1, θ_2, and $\theta)3$ based at the north pole. Let ξ_1 and ξ_2 be the unit tangent vectors in the directions of the legs of the triangle leaving the north pole (where to calculate holonomy, we're going to travel in the ξ_1, direction and return in the ξ_2-direction).

We can lift ξ_1 to a vector ∂_1 in $\mathscr{B}_{\mathrm{O}}(S^2)$, and choose an integral curve which brings us to the second vertex of the triangle (the one whose angle is θ_2). To rotate it to the direction of the leg leaving this point, we need to subtract the exterior angle $\pi - \theta_2$. Then, the same thing happens when we get to the third vertex, and then back at the first vertex, so the holonomy is $R_{-(\pi-\theta_1)} \circ R_{-(\pi-\theta_2)} \circ R_{-(\pi-\theta_3)}$, which is rotation by $\theta_1 + \theta_2 + \theta_3 - \pi$. This is a general geometric computation: it's true in more contexts than the sphere.

Anyways, let's talk some more about the general setting. Let G be a Lie group and $\pi\colon P \to X$ be a principal G-bundle.

Definition 19.1. Let $H \subset G$ be a subgroup. Then, a *reduction* of $P \to X$ to H is a submanifold $Q \subset P$ such that $Q \cdot H = Q$ and $\pi|_Q\colon Q \to X$ is a principal H-bundle.

Example 19.2. The bundle of orthonormal frames $\mathscr{B}_{\mathrm{O}}(X)$ is a reduction of the frame bundle $\mathscr{B}(X) \to X$ to the subgroup $\mathrm{O}_n \subset \mathrm{GL}_n(\mathbb{R})$. ◀

Definition 19.3. If Θ is a connection on $P \to X$, then it *reduces* to a connection on $Q \to X$ if the horizontal distribution $H \subset TP$ is contained in TQ.

The following theorem is left as a homework exercise.

Theorem 19.4. *If $H \subset G$ is a closed subgroup, then reductions (with connection) of $P \to X$ to H correspond bijectively to flat sections of $P/H \to X$.*

$P/H \to X$ is not always a principal bundle; rather, it's a fiber bundle with fiber G/H.

Lemma 19.5. *Let X be a principal G-bundle and $Q \subset P$ be a sub*set *satisfying the following conditions.*

(1) *$\pi|_Q\colon Q \to X$ is surjective.*
(2) *$Q \cdot H = Q$.*
(3) *If $\pi(q) = \pi(q')$ for $q, q' \in Q$, then there's an $h \in H$ such that $q' = q \cdot h$.*
(4) *For any $x \in X$, there's a local smooth section σ of π whose image lies in Q.*

Then, $\pi|_Q\colon Q \to X$ is a reduction of $\pi\colon P \to X$ to H.

Proof sketch. We have to construct a manifold structure on Q, which we'll do with the local section σ. It defines a diffeomorphism $\varphi\colon U \times G \to \pi^{-1}(U)$ commuting with projection to U, and such that $\varphi(U \times H) = Q \cap \pi^{-1}(U)$. We can use these maps $\varphi|_{U\times H}$ to patch together the smooth structures into a manifold. ⊠

Let $P \to X$ be a principal G-bundle with connection Θ, where X is connectwed. We'll introduce an equivalence relation on P in which $p \sim p'$ iff p and p' can be joined by a piecewise C^1-curve that's horizontal (i.e. its tangent is horizontal everywhere). We'll let $P(p)$ denote the equivalence class of p.

Theorem 19.6. *$\pi|_{P(p)}\colon P(p) \to X$ is a reduction of $\pi\colon P \to X$ to the subgroup $\mathrm{Hol}_p(\Theta)$, and Θ reduces to a connection Θ' on $P(p) \to X$.*

This reduction is called the *holonomy bundle*, and in a sense, it's the most efficient principal bundle you can produce from the connection. For example, if the connection is trivial, then the equivalence classes are foliations, reducing the structure group to the trivial group. If you take the bundle of frames of a Riemannian manifold and the Levi-Civita connection, $\mathrm{Hol}_p(\Theta) \subset \mathrm{O}_n$, so the holonomy bundle will be contained in the orthonormal frame bundle (it might be a subgroup).

Proof of Theorem 19.6. The proof will go through Lemma 19.5. Suppose $\pi(p) = x$.

- Since X is connected, there's a piecewise C^1 path from x to x' for any $x' \in X$, and therefore this path lifts to a path $p \to p' \in \pi^{-1}(x')$, so $\pi|_{P(p)}$ is surjective.
- We want to show that if p' is connected to p, then so is $p' \cdot h$ for any h in the holonomy group. In particular, there's some loop over $\pi(p')$ with holonomy h, which therefore connects p' and $p' \cdot h$, so p is connected to $p' \cdot h$.
- The third point is basically the same: if p'_0 and p'_1 are both in the same fiber and in $P(p)$, then there's a path from p'_0 to p, and one from p to p'_1, which projects down to a loop in X, hence defines an element of the holonomy group sending p'_0 to p'_1.
- To get a local section at x', take radial curves in every direction out of x' and lift them; there's parallel transport locally, which defines a local section. ⊠

Now suppose $\pi\colon P \to X$ is a principal G-bundle with connection Θ and F is a manifold with a left G-action. Then, the associated fiber bundle $F_P = P \times_G F \to X$ has an induced connection. An $f_x \in (F_P)_x$ lifts to a function $\widetilde{f}\colon P_x \to F$ such that

$$\widetilde{f}_x(p \cdot g) = g; \widetilde{f}_x(p), \qquad p \in P_x, g \in G.$$

If $\gamma\colon [0,1] \to X$ is a curve, then the connection lifts it to a horizontal curve starting at f_x called $\widetilde{\gamma}$. Then, $\widetilde{f}_t\colon P_{\gamma(t)} \to F$ is such that $\widetilde{f}_t \circ \widetilde{\gamma}$ is constant. Thus, we've carried over the horizontal lift from P to F_P.

If f is a global section of $F_P \to X$ which is parallel, then it lifts to a constant function $\widetilde{f}\colon P(p) \to F$, and its value is a $\mathrm{Hol}_p(\Theta)$-invariant point of F. That is, to understand the global parallel sections of $F_P \to X$, look at the fixed points of the holonomy.

Proposition 19.7. *The set of parallel sections of $F_P \to X$ is in bijective correspondence with the set of fixed points of the $\mathrm{Hol}_P(\Theta)$-action on F.*

This is often applied in the case where F is a vector space and G acts linearly, producing a covariant derivative on the associated vector bundle, and we can understand the covariantly constant sections. If we instead look at the frame bundle, we get covariantly constant tensors.

Example 19.8. Suppose X is a Riemannian manifold and consider the bundle of orthonormal frames $\mathscr{B}_{\mathrm{O}}(X) \to X$ with the Levi-Civita connection. The metric defines a parallel section of $\mathrm{Sym}^2(T^*X) \to X$, which is a fixed point of the O_n-action induced from the O_n-action on $\mathrm{Sym}^2(\mathbb{R}^{n*})$. Something similar happens for the volume form. ◂

So we have manifolds and nice notions of parallelism, and this produces very general notions of parallelism, including lengths, angles, area, volumes, etc. In Riemannian geometry, the Levi-Civita connection produces a distinguished notion of parallelism: asking for the metric to be torsion-free pins it down uniquely.

Last time, we stated Theorem 18.14, which (subject to a few reductions) computes the possible Riemannian holonomies. The fundamental group adds complications, so we look at the restricted holonomy group. Products also add complications, so we restrict to the case where the metric is not a product. Then, there are two classes: symmetric spaces, which have lots of parallel structures, and a finite list of possibly holonomies. The list is important because in each type of geometry, the parallel invariants tell you a lot about the nature of that geometry.

The generic case is SO_n, where we know we have metric, length, and angle, and all concepts derived from them (e.g. area and top-dimensional volume). The next case is where the restricted holonomy is $\mathrm{U}_m \subset \mathrm{SO}_{2m}$; this has additional, extra parallel structures.

Definition 19.9. A Riemannian manifold X is *Kähler* if its holonomy group (with respect to the Levi-Civita connection) is a subgroup of the unitary group $\mathrm{U}_m \subset \mathrm{O}_{2m}$.[19]

In particular, we mean the entire holonomy group, not the restricted holonomy group.

The first thing U_m acts on is $\mathbb{R}^{2m}$, and the associated vector bundle on a Kähler manifold is the tangent bundle. There are no parallel vector fields in general.

Example 19.10. Let $K = \mathbb{R} \times S^1/\mathbb{Z}$ where $(t, \theta) \sim (t+1, -\theta)$: we've divided out by this equivalence relation, which is a circle bundle over S^1. We've glued the ends of a cylinder by a reflection, so obtain a Klein bottle. It inherits a quotient metric, and the holonomy group is trivial: $R \times S^1$ is a quotient of the Euclidean plane, hence has global parallelism, which you can picture explicitly. Thus, the restricted holonomy group is trivial.[20]

To compute the unrestricted holonomy, we have to trace around the two generating loops. The loop around the fiber has no holonomy, because the metric on the fiber is flat, but the other loop was produced by a reflection, hence is the reflection $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ (differentiate the equivalence relation we glued by). In general, we get a cyclic subgroup of order 2.

Is K Kähler? We'd like the holonomy group to be contained in $\mathrm{U}_1 = \mathrm{SO}_2 \subset \mathrm{O}_2$, but $-1 \notin \mathrm{SO}_2$, so this metric is not Kähler. In fact, there's no Kähler metric on the Klein bottle.

This reflection fixes a line, and therefore we get a line of parallel vector fields: these are the tangents to the fiber. ◀

Though it isn't on Berger's list (because it's not irreducible), the trivial holonomy group is an example of a holonomy group: there is global parallelism, which implies the other structures. For example, the torus is parallel, and therefore is in particular Kähler.

The basic invariant of U_m is the matrix

$$I = \begin{pmatrix} \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} & & \\ & \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} & \\ & & \ddots \end{pmatrix}. \tag{19.11}$$

Then, $I^2 = -\mathrm{id}$, and $\mathrm{U}_m = \mathrm{O}_{2m} \cap \mathrm{Stab}(I) \subset \mathrm{GL}_{2m}(\mathbb{R})$. Since O_{2m} is the stabilizer of the metric, then a Kähler manifold comes with a metric and a parallel almost complex structure. Conversely:

Theorem 19.12. *Let X be a Riemannian manifold with a parallel almost complex structure J. Then, X is Kähler.*

Definition 19.13. Let V be a real vector space of dimension $2m$, $g\colon V \times V \to \mathbb{R}$ be an inner product, and $I\colon V \to V$ be a complex structure, i.e. $I^2 = -\mathrm{id}_V$, that's an isometry under g (equivalently, I is skew-symmetric).

- The associated *symplectic form* is a form $\omega(\xi, \eta) = -g(\xi, I\eta)$.
- The associated *hermitian form* is $h(\xi, \eta) = g(\xi, \eta) + i\omega(\xi, \eta)$.

There's some algebra here, e.g. to verify that ω is skew-symmetric. Since both structures are fixed under the U_m-action, then these forms extend to parallel forms g, ω, and I on any Kähler manifold. Moreover the

[19] You also have to choose a basepoint, but changing the basepoint affects the holonomy subgroup up to conjugation, but this is acceptable.

[20] Next time, we'll see that this is implied because the curvature is zero.

induced covariant derivative $\nabla\omega = 0$, and since d can be expressed in terms of ∇ when the connection is torsion-free, $d\omega = 0$, anoher argument in favor of torsion-free connections.

You can talk about Kähler manifolds by imposing some subset of these structures, or you might get one of these structures in whatever application you're using this for, and you may want to know what to say about these structures.

First, let's discuss the case of complex manifolds, where we just have the complex structure I.

Definition 19.14. Let X be a smooth manifold of dimension $2m$.

- An *almost complex structure* on X is a global section $I \in \Gamma(\mathrm{End}(TX) \to X)$ such that $I^2 = -\mathrm{id}$.

There is no obstruction to finding a Riemannian metric, since $\mathrm{GL}_n(\mathbb{R})/\mathrm{O}_n$ is contractible and even convex, but there are obstructions to finding a Kähler metric, because the space of endomorphisms I with $I^2 = -\mathrm{id}$ is not convex, and is empty on some manifolds. For example, the 4-sphere does not admit an almost complex structure. The 2-sphere, in the guise of $\mathbb{CP}^1$, does, and S^6 does.

Theorem 19.15. *S^4 does not admit an almost complex structure.*

Proof. Suppose S^4 has an almost complex structure. Then, $T_{\mathbb{C}}S^4 \cong E \oplus \overline{E}$, where E is a rank-2 complex subbundle, where E is the i-eigenspace of I and $\overline{E}$ is the $-i$-eigenspace.[21] Therefore the underlying real bundle $E_{\mathbb{R}}$ of E must be TS^4.

The second Chern class of E, $c_2(E) \in H^4(S^4;\mathbb{Z})$ is by definition $p_1(E_{\mathbb{R}}) = 0$, because $E_{\mathbb{R}} = TS^4$ is stable trivial: embedding $S^4 \hookrightarrow \mathbb{R}^5$ defines a splitting $TS^4 \oplus \nu \cong \underline{\mathbb{R}}^5$. However, $c_2(E)$ is also the Euler class $e(E_{\mathbb{R}})$, which is nonzero, because S^4 has nonzero Euler characteristic. ⊠

The almost complex structure I defines a complex distribution $E \subset T_{\mathbb{C}}X$, and the Frobenius is $\Phi_E\colon E \times E \to T_{\mathbb{C}}X/E$, where E is the i-eigenspace of I, as in the proof above. Passing to real tangent spaces, we get a map $\Phi'_E\colon TX \times TX \to TX$, called the *Nijenhuis tensor*. The identification $TX \cong E$ sends $\xi \mapsto \xi - iI\xi$, and the map $TX \to T_{\mathbb{C}}X/E$ sends $\xi \mapsto \xi + iI\xi \bmod E$.

Next time, we'll show what it means for this to be integrable. That is:

Theorem 19.16 (Neulander-Nirenberg)**.** *The following are equivalent:*

(1) $\Phi_E = 0$.
(2) *$\mathscr{B}_I(X) \to X$ admits a torsion-free connection.*
(3) *X admits an atlas of complex coordinate systems $z^1, \dots, z^m$ such that $E = \mathrm{span}\{\frac{\partial}{\partial z^\mu}\}$.*

Here, $\mathscr{B}_I(X) \subset \mathscr{B}(X)$ is the reduction of $\mathscr{B}(X)$ to $\mathrm{GL}_m(\mathbb{C}) \subset \mathrm{GL}_{2m}(\mathbb{R})$; concretely, $\mathscr{B}_I(X) = \{(\mathbb{R}^{2m}, I) \to (T_xX, I)\}$, where the standard almost complex structure on $\mathbb{R}^{2m}$ from (19.11).

So like in every case, the geometric structure defines a distribution, and the question is always: is there a torsion-free connection? This is an instance of a general statement about integrability of a distribution in terms of a bundle of frames.

Lecture 20.

Complex manifolds: 3/30/17

Last time, we were talking about complex manifolds and Kähler manifolds in the context of Berger's classification of special holonomy.

To understand complex manifolds, we should do a little linear algebra. Let V be a finite-dimensional real vector space and $I \in \mathrm{End}(V)$ be such that $I^2 = -\mathrm{id}$. Then, $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ splits as $V_{(1,0)} \oplus V_{(0,1)}$, where $V_{(1,0)}$ is the $+i$-eigenspace of I and $V_{(0,1)}$ is the $-i$-eigenspace of I.

Exercise 20.1. Check the following.

- $\overline{V_{(1,0)}} = V_{(0,1)}$.
- The map $V \to (V_{(1,0)})_{\mathbb{R}}$ sending $\xi \mapsto \xi \otimes 1 - I\xi \otimes i$ is an isomorphism.
- The map $V \to (V_{(0,1)})_{\mathbb{R}}$ sending $\xi \mapsto \xi \otimes 1 + I\xi \otimes i$ is an isomorphism.
- $V_{\mathbb{C}}^* = V^* \otimes_{\mathbb{R}} \mathbb{C} \cong (V_{(1,0)})^* \oplus (V_{(0,1)})^*$.

[21] This splitting into $\pm i$-eigenspaces is equivalent to an almost complex structure in general.

- $$\Lambda^k V_{\mathbb{C}}^* \cong \bigoplus_{p+q=k} \Lambda^p(V_{(1,0)})^* \otimes \Lambda^q(V_{(0,1)})^*.$$

  The summand for (p, q) is called the forms of *type* (p, q).

Hence, if X is a smooth manifold and I is an almost complex structure on X, we get a splitting

$$\Omega_X^k \otimes \mathbb{C} \cong \bigoplus_{p+q=k} \Omega_X^{p,q}.$$

The almost complex structure defines a complex distribution $T_{1,0} \subset T_{\mathbb{C}}X := TX \otimes \underline{\mathbb{C}}$. It has a Frobenius tensor $\Phi\colon T_{1,0} \times T_{1,0} \to T_{0,1}$ (here, $T_{1,0}$ is the piece of $T_{\mathbb{C}}X$ that is the i-eigenspace on each fiber, and $T_{0,1}$ is the $-i-$-eigenspace on each fiber).

Recall that if $\dim X = 2m$, there is a reduction of the $\mathrm{GL}_{2m}(\mathbb{R})$-bundle of frames $\mathscr{B}(X) \to X$ to $\mathrm{GL}_m(\mathbb{C})$, and we called this $\mathrm{GL}_m(\mathbb{C})$-bundle $\mathscr{B}_I(X) \to X$. A point $p \in \mathscr{B}_I(X)$ is a pair $(x \in X, (\xi_1, \dots, \xi_{2m}))$, where $\xi_{2k} = I\xi_{2k-1}$ for $1 \le k \le m$. Last time, we stated Theorem 19.16, that $\mathscr{B}_I(X)$ admits a torsion-free connection iff $\Phi = 0$, and in this case, we get a collection of complex charts for the manifold that are related by holomorphic maps.

Remark 20.2. This is a complexified account of the local Frobenius theorem (Theorem 8.5), which said that if $\Phi_E = 0$ for a real distribution $E \subset TX$, then there are local coordinates $x^1, \dots, x^n$ on X such that $E|_U = \mathrm{span}\{\frac{\partial}{\partial x^i}\}_{i=1}^k$, where U is the coordinate neighborhood for $x^1, \dots, x^n$. ◂

Proof of Theorem 19.16. Suppose $\mathscr{B}_I(X) \to X$ admits a torsion-free connection, and we'll prove that $\Phi = 0$. Then, the vector bundle $T_{1,0}X \to X$ is associated to $\mathscr{B}_I(X) \to X$ via the defining representation of $\mathrm{GL}_m(\mathbb{C})$ on $\mathbb{C}$. Thus, $T_{1,0}X \to X$ inherits a covariant derivative. If $\xi, \eta \in \Gamma(T_{1,0}X)$, then they're also sections of $T_{\mathbb{C}}X$, and $[\xi, \eta] = \nabla_\xi \eta - \nabla_\eta \xi$, because ∇ is torsion-free. Since $\nabla_\xi \eta$ and $\nabla_\eta \xi$ are sections of $T_{1,0}X$, then $[\xi, \eta]$ also has type $(1, 0)$, and hence $\Phi = 0$.

The other direction isn't as easy. First, oberve that the torsion-free condition $\mathrm{d}\theta + \Theta \wedge \theta$ is an affine equation for θ, and therefore the torsion-free connections form an affine subspace of all connections. In particular, this means they can be constructed globally from local torsion-free connections using a partition of unity. Hence, we may assume that we're working in an open $U \subset X$.

Let $\theta^1, \dots, \theta^m$ be a *complex coframing*, i.e. a basis of $(1, 0)$-frames. If $\Phi = 0$, then the ideal generated by $\{\theta^\mu\}$ is closed under d, and therefore

$$\mathrm{d}\theta^\mu = \frac{1}{2} A^\mu_{\nu\lambda} \theta^\nu \wedge \theta^\lambda + A^\mu_{\nu\overline{\lambda}} \theta^\nu \wedge \overline{\theta^\lambda},$$

for some $\mathbb{C}$-valued tensor $A^{\bullet}_{\bullet\bullet}$ such that $A^\mu_{\lambda\nu} = -A^\mu_{\nu\lambda}$. Thus, we can just write down the connection 1-form: define

$$\Theta^\mu_\nu := \frac{1}{2} A^\mu_{\nu\lambda} \theta^\lambda + A^\mu_{\nu\overline{\lambda}} \overline{\theta^\lambda}.$$

Then

$$\mathrm{d}\theta^\mu + \Theta^\mu_\nu \wedge \theta^\nu = 0 \mathrm{d}\overline{\theta}^\mu + \overline{\Theta}^\mu_\nu \wedge \overline{\theta}^\nu \qquad = 0,$$

so

$$\Theta = \begin{pmatrix} \Theta^\mu_\nu & 0 \\ 0 & \overline{\Theta}^\mu_\nu \end{pmatrix}$$

defines a torsion-free connection which preserves I, i.e. passes to $\mathscr{B}_I(X) \to X$. ☒

Remark 20.3. The connection defined in the second part of the proof is not unique: for any tensor $B^{\bullet}_{\bullet\bullet}$ with $B^\mu_{\nu\lambda} = B^\mu_{\lambda\nu}$, $\Theta^\mu_\nu + B_{\nu\lambda^\mu}\theta^\lambda$ is also a torsion-free connection on $\mathscr{B}_I(X) \to X$. ◂

Let's talk a little bit about complex manifolds.

Definition 20.4. A smooth map $f\colon X \to Y$ of complex manifolds is *holomorphic* if $f_*(T_{1,0}X) \subset T_{1,0}Y$. This is equivalent to $f_* \circ I_X = I_Y \circ f_*$.

In particular, a map $f\colon X \to \mathbb{C}$ is holomorphic iff $\mathrm{d}f \in \Omega_X^{1,0}$. Thus this generalizes the usual notion of holomorphic functions from $\mathbb{C}$ to $\mathbb{C}$.

There's a decomposition of the de Rham differential $\mathrm{d} = \partial + \overline{\partial}$ on

$$\Omega_X^\bullet \otimes \mathbb{C} = \bigoplus_{p,q} \Omega_X^{p,q}.$$

Specifically, $\partial\colon \Omega_X^{p,q} \to \Omega_X^{p+1,q}$ raises the p-degree by 1 and $\overline{\partial}\colon \Omega_X^{p,q} \to \Omega_X^{p,q+1}$ raises the q-degree by 1. We have $\partial^2 = 0$, $\overline{\partial}^2 = 0$, and $\partial\overline{\partial} + \overline{\partial}\partial = 0$.

Example 20.5.

(1) Let W be a complex vector space. Then, *projective W-space* $\mathbb{P}(W)$ is the set of complex lines through the origin (i.e. one-dimensional complex subspaces) in W. We'll put a set of complex charts on $\mathbb{P}(W)$ as follows: given a line $\ell \in \mathbb{P}(W)$, let U be a complimentary subspace, i.e. $\ell \oplus U = W$. Then, $\mathrm{Hom}_{\mathbb{C}}(\ell, U) \hookrightarrow \mathbb{P}(W)$ is a chart containing ℓ, where the inclusion sends T to the graph of T in $\ell \oplus U = W$. We need to check that the change-of-charts maps are holomorphic, but this follows from the fact that complex linear matrices are holomorphic. The space $\mathbb{P}(\mathbb{C}^{n+1})$ is also denoted $\mathbb{CP}^n$, and is called *complex projective space*.

(2) Let P be a homogeneous polynomial in $n+1$ variables. Then, its zero set is a subset of $\mathbb{CP}^n$, which in nice cases is a complex manifold (though in general it may have singularities). These zero sets are called *complex varieties*.

(3) Let X be a real two-dimensional manifold with an orientation and a metric. Then, letting I be rotation through the angle $\pi/2$ on each tangent space defines a complex structure on X. ◁

Now we'll turn to a beautiful piece of geometry for Kähler manifolds. Let X be a manifold with a metric g and a complex structure I. We can then define some unique connections on X. We've already seen the unique torsion-free connection with respect to the metric, the Levi-Civita connection. But there's another connection, called the Chern connection, that we can define. Let $\mathrm{U}_m \subset \mathrm{GL}_m(\mathbb{C})$ denote the unitary group, whose Lie algebra $\mathfrak{u}_n \subset \mathfrak{gl}_n(\mathbb{C})$ is the algebra of *skew-Hermitian matrices*. The inclusion splits: $\mathfrak{gl}_m(\mathbb{C}) = \mathfrak{u}_m \oplus i\mathfrak{u}_m$: $i\mathfrak{u}_m$ is the algebra of Hermitian matrices.

Theorem 20.6 (Chern)**.** *Let X be a complex manifold, $\pi\colon P \to X$ be a holomorphic principal $\mathrm{GL}_m(\mathbb{C})$-bundle, and $Q \to X$ be a reduction of P to U_m, Then, there exists a unique connection on $Q \to X$ whose horizontal distribution, viewed as a subbundle of TP, is complex.*

By complex we mean that this distribution $H \subset TQ$ is preserved by the complex structure: if $I\colon TP \to TP$ is the complex structure, then $I(H) = H$. By a *holomorphic principal bundle* we mean a principal bundle whose transition maps are not just smooth, but holomorphic.

So in other words, $H \oplus T(Q/X) = TQ$, H is U_m-invariant, and $I(H) = H$. This connection is called the *Chern connection* Ξ.

Proof. This proof is due to Singer: let $H = TQ \cap I(TQ)$. This is evidently I-invariant. Since the action of U_m is holomorphic, this is also U_m-invariant. So all we have to check is that it's horizontal.[22] ⊠

Remark 20.7. The same theorem and proof apply when $\mathrm{GL}_m(\mathbb{C})$ is replaced with any complex Lie group and U_m is replaced with its maximal compact subgroup. ◁

There's a version of this applied to vector bundles: let $E \to X$ be a vector bundle with structure group $\mathrm{U}_n \subset \mathrm{GL}_m(\mathbb{C})$. This is the data of a holomorphic vector bundle with a Hermitian metric. Let $e_1, \dots, e_m$ be a local basis of holomorphic sections. We'd like a unique covariant derivative ∇ such that

$$\begin{aligned} \mathrm{d}\langle e_\mu, \overline{e}_\nu\rangle &= \langle \nabla e_\mu, \overline{e}_\nu\rangle + \langle e_\mu, \nabla\overline{e}_\nu\rangle \\ \nabla e_\nu &= \Xi^\mu_\nu e_\mu, \end{aligned}$$

for some $\Xi^\mu_\nu \in \Omega^{1,0}_{\mathbb{C}}$. Writing $h_{\mu\overline{\nu}} \coloneqq \langle e_\mu, \overline{e}_\nu\rangle$, the solution has to be

$$\Xi^\mu_\nu = h^{\mu\overline{\lambda}}\partial h_{\nu\overline{\lambda}},$$

[22] TODO: I zoned out and missed this. What happens here?

where $(h^{\mu\overline{\lambda}})$ is the inverse matrix to $(h_{\mu\overline{\lambda}})$. Recall that ∂ is the projection of the de Rham differential onto the $(1,0)$ part, so this is a $(1,0)$-operator.

Theorem 20.8. *Let X be a complex manifold with complex structure I and a Hermitian metric defined by a 2-form $\omega \in \Omega^2_X$. Then, the following are equivalent:*

(1) *X is Kähler, i.e. the holonomy of the Levi-Civita connection is contained in U_m.*
(2) *I is parallel.*
(3) *ω is parallel.*
(4) *$\mathrm{d}\omega = 0$.*
(5) *The Levi-Civita and Chern connections are equal.*

One way to think of this is that Kähler manifolds are those whose geometry is controlled by the unitary group. Inside $\mathrm{GL}_{2m}(\mathbb{R})$, $\mathrm{U}_m = \mathrm{O}_{2m} \cap \mathrm{GL}_m(\mathbb{C})$, so we're asking the same question in two ways: when does the Levi-Civita connection Θ on $\mathscr{B}_{\mathrm{O}}(X)$ descend to $\mathscr{B}_{\mathrm{U}}(X)$? When does the connection defined by the complex structure descend to $\mathscr{B}_{\mathrm{U}}(X)$? The Levi-Civita connection is characterized by being torsion-free, and the Chern connection by its compatibility with the complex structure, so this says on Kähler manifolds, the Levi-Civita connection is compatible with the complex structure, or the Chern connection is torsion-free.

Last time, we talked about how restricting the holonomy group makes more things parallel; this is a realization of this idea, as I and ω are now both parallel.

Proof of Theorem 20.8. (1) $\implies$ (2): since X is Kähler, then Θ passes to a connection on $\mathscr{B}_{\mathrm{U}}(X) \to X$, and I lifts to a constant function $\widetilde{I}\colon \mathscr{B}_{\mathrm{U}}(X) \to \mathrm{End}(\mathbb{C}^n)$ with value I from (19.11), and therefore I is Θ-parallel.

(2) $\implies$ (3): since $\omega(\xi,\eta) = -g(\xi, I(\eta))$, then we're done.[23]

(3) $\implies$ (4): since Θ is torsion-free, then $\mathrm{d}\omega = \varepsilon \circ \nabla\omega = 0$, where $\nabla\colon \Omega^2_X \to \Omega^2_X \otimes \Omega^1_X$ and ε maps to Ω^3_X.

(4) $\implies$ (5) is harder: let $z^1, \dots, z^m$ be local coordinates on X, and $\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^m}$ be the corresponding local framing for $T_{1,0}X$. Let

$$h_{\mu\overline{\nu}} := \left\langle \frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu} \right\rangle$$

and $\theta^\mu := \mathrm{d}z^\mu$ be the dual coframing.

Now we can explicitly write down the torsion of the Chern connection:

$$\begin{aligned}\tau^\mu &= \mathrm{d}\theta^\mu + \Xi^\mu_\nu \wedge \theta^\nu \\ &= 0 + h^{\mu\overline{\lambda}} \frac{\partial h_{\nu\overline{\lambda}}}{\partial z^\rho}\,\mathrm{d}z^\rho \wedge \mathrm{d}z^\nu.\end{aligned}$$

In these coordinates,

$$\omega = \frac{i}{2} h_{\mu\overline{\nu}}\,\mathrm{d}z^\mu \wedge \overline{\mathrm{d}z^\nu},$$

and since $\mathrm{d}\omega = 0$, then

$$\frac{\partial h_{\mu\overline{\nu}}}{\partial z^\rho} = \frac{\partial h_{\rho\overline{\nu}}}{\partial z^\mu},$$

and therefore $\tau = 0$. Thus, the Chern connection is the unique torsion-free connection for the metric, so must be equal to the Levi-Civita connection.

For (5) $\implies$ (1), Θ reduces to $\mathscr{B}_{\mathrm{U}}(X)$, so the holonomy of Θ is contained in U_m. ⊠

[23] There's also a Hermitian form defined by $h(\xi,\eta) = g(\xi,\eta) + i\omega(\xi,\eta)$.

Lecture 21.

: 4/4/17

Lecture 22.

Irreducibility : 4/6/17

Lecture 23.

Homogeneous spaces, I: 4/11/17

"It would be iconoclastic to take the basepoint of a Lie group to be anything but the identity."

Recall that we're in the business of Berger's program of classifying Riemannian manifolds through their holonomy groups, at least for manifolds that are irreducible (as we defined last time) and not symmetric spaces.

Definition 23.1. Let X be a connected Riemannian manifold and $H := \mathrm{Hol}_X(\Theta)$, where Θ is the Levi-Civita connection. The holonomy acts on the tangent space as a representation, and we say X is *irreducible* if this representation is.

Last time, we discussed a theorem on irreducibility, which connects this representation-theoretic notion to a geometric one.

Theorem 23.2. *Let X be a connected Riemannian manifold, Θ be its Levi-Civita connection, and $H := \mathrm{Hol}_X(\Theta)$. Then, there exists an isomorphism*

$$H \cong \prod_{i=0}^{k} H_i$$

and an orthogonal direct sum

$$T_x X \cong \bigoplus_{i=0}^{k} T_X^{(i)}$$

such that as an H-representation, $T_x X$ decomposes as a direct sum of the H_i-actions on $T_X^{(i)}$ and H_i is the restricted holonomy group of a Riemannian manifold.

Today, we'll talk about the other class of examples we have to throw out, symmetric spaces.

Definition 23.3. A *homogeneous space* is a smooth manifold with a transitive action of a Lie group G.

For each x in a homogeneous space x, we define $\alpha_x \colon G \to X$ by $g \mapsto g \cdot x$. Then, $\alpha_x^{-1}(x)$ is a closed Lie subgroup of G, which we'll call H, and α_x induces a diffeomorphism $G/H \cong X$. Thus homogeneous spaces all arise as quotients of Lie groups by closed Lie subgroups. Differentiating at H/H, this defines an isomorphism $\mathfrak{g}/\mathfrak{h} \cong T_x X$.

The Lie group G is parallelizable, and we'd like this to descend to X, but this doesn't quite work: X is parallelizable up to the adjoint action of H on $\mathfrak{g}/\mathfrak{h}$: we have parallel transport, but it's not unique, and there's an H worth of choices.

Example 23.4.

(1) There are many ways to write S^{n-1} as a homogeneous space: it's $\mathrm{O}_n/\mathrm{O}_{n-1}$, $\mathrm{SO}_n/\mathrm{SO}_{n-1}$, $\mathrm{Spin}_n/\mathrm{Spin}_{n-1}$, and so forth (e.g. there are versions for pin groups). These choices differ only by finite groups: the first two by the group of connected components, the second two by a double cover. Some of these are *ineffective*, in that there is a nonidentity element acting by the identity, but desipte the name, these ineffective actions are still useful models for S^{n-1}.

Theorem 23.5 (Montgomery-Samelson, Borel). *The classification of compact, connected Lie groups acting effectively on a sphere is:*

(a) $S^{n-1} \cong \mathrm{SO}_n/\mathrm{SO}_{n-1}$,
(b) $S^{2n-1} \cong \mathrm{SU}_n/\mathrm{SU}_{n-1} \cong \mathrm{U}_n/\mathrm{U}_{n-1}$, *and*
(c) $S^{4n-1} \cong \mathrm{Sp}_n/\mathrm{Sp}_{n-1} \cong \mathrm{Sp}_n\mathrm{U}_1/\mathrm{Sp}_{n-1}\mathrm{U}_1 \cong \mathrm{Sp}_n\mathrm{Sp}_1/\mathrm{Sp}_{n-1}\mathrm{Sp}_1$.

(d) $S^6 \cong \mathrm{G}_2/\mathrm{SU}_3$.
(e) $S^7 \cong \mathrm{Spin}_7/\mathrm{G}_2$.
(f) $S^{15} \cong \mathrm{Spin}_9/\mathrm{Spin}_7$.

The first case comes from SO_n acting on a real vector space; the second case comes from U_n and SU_n acting on complex vector spaces; and the third case comes from Sp_n and its friends acting on quaternionic vector spaces. We can either require symplectic symmetry in all directions, or allow complex (unitary) or quaternionic (synplectic) symmetry in one direction, and we'll still get a transitive action. The last three cases are exceptional.

Another way to think of Theorem 23.5 is as a list of Lie groups which can act transitively on a sphere: SO_n, SU_n, U_n, Sp_n, $\mathrm{Sp}_n\mathrm{U}_1$, $\mathrm{Sp}_n\mathrm{Sp}_1$, G_2, Spin_7, and Spin_9. You can also see most of Berger's classification: the generic case is SO_n, and we also see the common cases and exceptional cases, but $\mathrm{Sp}_n\mathrm{U}_1$ and Spin_9 don't appear in that classification. This is because there are theorems that any spaces with such holonomy are symmetric spaces.

(2) If G is a Lie group, we can realize it as a homogeneous space $G/\{e\}$ or $G \times G/G$, where $(g,h) \cdot k := gkh^{-1}$. The stabilizer at the identity is the diagonal subgroup of G. In the first example, we have actual parallelism, but in the second, uniqueness fails, and two tangent spaces are identified up to the adjoint action of G on $\mathfrak{g}$.
(3) Projective spaces $\mathbb{RP}^n$, $\mathbb{CP}^n$, $\mathbb{HP}^n$, and the *Cayley plane* $\mathbb{OP}^2$.
(4) The Grassmanians $\mathrm{Gr}_k(\mathbb{R}^n)$, $\mathrm{Gr}_k(\mathbb{C}^n)$, and $\mathrm{Gr}_k(\mathbb{H}^n)$. ⊲

One key feature of a homogeneous space is that projection $\pi\colon G \to G/H$ is a principal H-bundle, and at the basepoint H/H (whose fiber contains the identity of G), the fiber of this projection is H, and the vertical tangent space is $\mathfrak{g}$.

This allows us to set up a passageway between G-invariant objects on G/H and H-invariant objects at a point; the forward direction is called *restriction* and the reverse direction is called *induction*.

Example 23.6.

(1) There is an equivalence of categories from the category of G-equivariant vector bundles on G/H and the representations of H. A *G-equivariant vector bundle $P \to X$* on a space acted on by G is a G-space P such that the projection to X is G-equivariant and the fibers are vector spaces. The idea is that G-equivariance allows one to recover the whole vector bundle from the fiber at H/H, which is an H-representation (by restricting the action). Conversely, if V is an H-representation, we obtain a G-equivariant vector bundle from the mixing construction $G \times_H V \to G/H$, which inherits a G-action from the first component, which commutes with the right H-action on V. This example generalizes some good theorems in the world of finite groups.
(2) There's a natural bijection between the G-invariant connections on $G \to G/H$ and the H-invariant connections over a point, i.e. H-invariant decompositions $\mathfrak{g} \cong \mathfrak{h} \oplus \mathfrak{m}$, where the Lie bracket is trivial on $\mathfrak{m}$: the idea is that $\mathfrak{h}$ is the vertical part and $\mathfrak{m}$ is the horizontal part. We want the left- and right-translates of a vector by some $h \in H$ to produce identical copies of $\mathfrak{m}$, and this is why we ask for the decomposition to be H-invariant.

The restriction map is easier: a connection is a horizontal distribution, so just take the subspace at the identity.
(3) There's a natural bijection between the space of G-invariant Riemannian metrics on G/H and the H-invariant Riemannian metrics on $\mathfrak{g}/\mathfrak{h}$. ⊲

So this is really nice: G-invariant questions on homogeneous spaces are determined by their answers at the basepoint.

Example 23.7. Since $S^{n-1} \cong \mathrm{O}_n/\mathrm{O}_{n-1}$, we should be able to write $\mathfrak{o}_n \cong \mathfrak{o}_{n-1} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is a vector space (i.e. has trivial Lie bracket) that's O_{n-1}-invariant. We can let $\mathfrak{o}_{n-1}$ be the skew-symmetric matrices with zeros filling the first row and column; then, the direct sum adds a vector ξ in the first column and $-\xi^{\mathrm{T}}$ in the first row (to preserve skew-symmetry). Thus, $\mathfrak{m}$ is the vector space of matrices in block form

$$\begin{pmatrix} 0 & -\xi^{\mathrm{T}} \\ \xi & 0 \end{pmatrix}.$$

This should be O_{n-1}-invariant, and indeed it is: the action of O_{n-1} is by conjugation, and you can check that

$$\begin{pmatrix} 1 & \\ & A \end{pmatrix} \begin{pmatrix} & -\boldsymbol{\xi}^{\mathrm{T}} \\ \boldsymbol{\xi} & \end{pmatrix} \begin{pmatrix} 1 & \\ & A^{-1} \end{pmatrix} = \begin{pmatrix} & -(A\boldsymbol{\xi})^{\mathrm{T}} \\ A\boldsymbol{\xi} & \end{pmatrix}.$$

The identification of $\mathfrak{m}$ with $\mathbb{R}^{n-1}$ that picks out $\boldsymbol{\xi}$ induces an O_{n-1}-action on $\mathbb{R}^{n-1}$, which is the standard representation. The associated vector bundle is the tangent bundle to S^{n-1}. ◂

Example 23.8. Note that G-invariant metrics don't always exist. Consider $X = \mathbb{RP}^1 = \mathrm{U}(1)/2\mathrm{U}(1)$, the connected double cover of the circle, or $\mathbb{A}^1 \cup \{\infty\}$. The group $G = \mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{RP}^1$ by Möbius transformations (the same as for $\mathrm{SL}_2(\mathbb{C})$ acting on $\mathbb{CP}^1 \cong S^2$): if $ad - bc = 1$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{ax+b}{cx+d},$$

and if the denominator is 0, we say that it sends $x \mapsto \infty$. There are several kinds of transformations: dilations that fix a single point or two points, translations at ∞, and more.

But there's no metric invariant under this action: in particular, there are dilations on $\mathbb{A}^1$ that fix ∞, and therefore they do not preserve length, the key invariant of a metric on a 1-manifold. Correspondingly, there's no invariant line in the decomposition.

The stabilizer H is the group of projective transformations that fix a point, the group generated by dilations and translations at ∞, i.e. on $\mathbb{A}^1$. These are the Möbius transformations with $b = 0$. ◂

Many other examples exist; you should play with some of them.

The absence of G-invariant metrics should frighten you. But fortunately, they exist in a large class of cases.

Theorem 23.9 (Maschke)**.** *Let H be a compact Lie group acting on a real vector space V, and let $V' \subset V$ be an H-invariant subspace. Then,*

(1) *there exists an H-invariant inner product on V, and*
(2) *there exists an H-invariant splitting $V \cong V' \oplus V''$ for some other H-invariant subspace V''.*

Proof. Let $\langle -,- \rangle'$ be any inner product on V, and define

$$\langle \xi, \eta \rangle := \int_H \mathrm{d}h \, \langle h \cdot \xi, h \cdot \eta \rangle',$$

where $\mathrm{d}h$ is the Haar measure on H. If $h' \in H$, then

$$\langle h'\xi, h'\eta \rangle = \int_H \mathrm{d}h \, \langle h' \cdot h\xi, h' \cdot h\eta \rangle' = \int_H \mathrm{d}(h'h) \, \langle (h'h)\xi, (h'h)\eta \rangle' = \langle \xi, \eta \rangle,$$

so $\langle -,- \rangle$ is H-invariant. To prove the second piece, we set $V'' = (V')^\perp$ with respect to the inner product $\langle -,- \rangle$, so that the decomposition respects the H-action, because the inner product is H-invariant. ⊠

This trick, averaging something in the Haar measure, is a common and good trick in representation theory. However, it requires you to know what Haar measure is.

A *measure* on a vector space W is a function $\mu \colon \mathscr{B}(W) \to \mathbb{R}$ such that

$$\mu(b \cdot g) = |\det g| \cdot \mu(b)$$

whenever $b \in \mathscr{B}(W)$ and $g \in \mathrm{GL}(W)$. That is, under a change of basis, the measure must scale by the volume of the parallelepiped it defines. This is a one-dimensional vector space called $|\mathrm{Det}\, W^*|$, and if you specify a basis b for W, the *positive measures* (i.e. those sending b to a positive number) are a ray in it, making $|\mathrm{Det}\, W^*|$ oriented.

A *smooth (positive) measure* on a smooth manifold X is a positive section of $|\mathrm{Det}\, T^*X| \to X$. A Riemannian metric induces a smooth measure by taking the norm of a vector in the metric.

Theorem 23.10. *Let H be a compact Lie group. Then, there exist bi-invariant (i.e. both left and right invariant) positive measures on H, called* Haar measures*.*

Proof sketch. The idea is that, since $H \cong (H \times H)/H$, a Haar measure on H is equivalent data to an H-invariant measure on $\mathfrak{h}$, which is equivalent to an H-fixed point of $|\mathrm{Det}\, \mathfrak{h}^*|$. The action is a Lie group homomorphism $H \to \mathrm{Aut}(|\mathrm{Det}\, \mathfrak{h}^*|) \cong \mathbb{R}^{>0}$, and since H is compact, the image has to be $\{1\}$. ⊠

Haar measure exists on more general groups (locally compact Hausdorff topological groups), but it's a much deeper theorem for those groups.

The principal H-bundle $\pi\colon G \to G/H$ is a reduction of the bundle of frames $\mathscr{B}(G/H) \to G/H$, which is a $\mathrm{GL}_n(\mathbb{R})$-bundle. To verify this, we need a homomorphism $H \to \mathrm{GL}_n(\mathbb{R}) = \mathrm{Aut}(\mathbb{R}^n)$ and an identification of $\mathfrak{g}/\mathfrak{h} \cong \mathbb{R}^n$. Fix a basis $b\colon \mathbb{R}^n \to \mathfrak{g}/\mathfrak{h}$. Then, we can induce $\rho\colon H \to \mathrm{GL}_n(\mathbb{R})$ by $h \mapsto b^{-1} \circ \mathrm{Ad}_G(h) \circ b$, and we can define the map $G \times_H \mathrm{GL}_n(\mathbb{R}) \to \mathscr{B}(G/H)$ to send $(g, A) \mapsto (gH, \alpha(g)_*(Ab))$, where α is the action of G. You can use this to understand the geometry of the homogeneous manifold G/H in terms of this bundle, and we will do this next time.

Lecture 24.

Homogeneous spaces, II: 4/13/17

Recall that if G is a Lie group and $H \subset G$ is a closed subgroup, we were considering the homogeneous manifolds G/H. The projection $\pi\colon G \to G/H$ is a principal H-bundle. Let $n := \dim(G/H)$.

Suppose we have an H-invariant splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, so the left translates of $\mathfrak{m}$ form an H-invariant distribution on G, and $\pi_*\colon \mathfrak{m} \overset{\cong}{\to} T_{H/H}G/H$ is an isomorphism. Fix a basis $b\colon \mathbb{R}^n \overset{\cong}{\to} \mathfrak{m}$, so $\pi_* \circ b$ is a basis for the tangent space to G/H at H/H.

Define a map $\varphi\colon G \times_H \mathrm{Aut}(\mathfrak{m}) \to \mathscr{B}(G/H)$ to send $(g, \alpha) \mapsto (gH, (L_g)_* \circ \pi_*\alpha \circ b)$, which is a basis $\mathbb{R}^n \overset{\cong}{\to} T_{gH/H}G/H$. Then, φ is the witness that reduces $\mathscr{B}(G/H) \to G/H$, with structure group $\mathrm{GL}_n(\mathbb{R})$, to $G \to G/H$, with structure group H.

We can pull the soldering form back from $\Omega^1_{\mathscr{B}(X)}(\mathbb{R}^n)$ to a soldering form in $\Omega^1_G(\mathbb{R}^n) \cong \Omega^1_G(\mathfrak{m})$ (this identification depends on b). Let $\Theta \in \Omega^1_G(\mathfrak{h})$ be the connection form (since we've reduced the structure group to H).

Lemma 24.1. *Let $\theta_G \in \Omega^1_G(\mathfrak{g})$ be the Maurer-Cartan form. Then, the soldering form is $\pi_{\mathfrak{m}}(\theta_G)$ and the connection from is $\pi_{\mathfrak{h}}(\theta_G)$, where $\pi_{\mathfrak{h}}\colon \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \twoheadrightarrow \mathfrak{h}$ and $\pi_{\mathfrak{m}}$ are the projections onto the first and second factors, respectively.*

Proof. The soldering form, connection, and $\pi_{\mathfrak{h}}\theta_G$ are all left-invariant, so it suffices to check the lemma at the identity of G.

At the identity, the soldering form is the projection onto the horizontal, which is exactly projection onto $\mathfrak{h}$, and the connection is projection onto the vertical, which is projection onto $\mathfrak{h}$. ⊠

Recall the Maurer-Cartan equation (9.17). We can use it to quickly calculate the torsion and curvature in a homogeneous space.

Proposition 24.2.

1. *The torsion of Θ evaluated on $\xi_1, \xi_2 \in \mathfrak{m}$ is $-\pi_{\mathfrak{m}}[\xi_1, \xi_2]$.*
2. *The curvature of Θ evaluated on $\xi_1, \xi_2 \in \mathfrak{m}$ is $-\pi_{\mathfrak{h}}[\xi_1, \xi_2]$.*
3. *Assume Θ is torsion-free. Let $\mathfrak{h}' \subset \mathfrak{h}$ be the subspace generated by $\{[\xi_1, \xi_2] \mid \xi_1, \xi_2 \in \mathfrak{m}\}$. Then, $\mathfrak{h}'$ is a Lie subalgebra and even a Lie ideal, and is the Lie algebra of $\mathrm{Hol}_e(\Theta)$.*
4. *Assume Θ is torsion-free. The total space of the holonomy bundle $P(e)$ based at e is the connected Lie subgroup of G with Lie algebra $\mathfrak{h}' \oplus \mathfrak{m}$.*
5. *If $\xi \in \mathfrak{m}$, the projection of a 1-parameter group $t \mapsto e^{t\xi}$ is a geodesic.*
6. *A G-invariant tensor field on G/H is parallel.*
7. *The torsion of Θ is parallel.*
8. *The curvature of Θ is parallel.*

This is pretty neat: the Lie subalgebra exponentiates to a Lie subgroup, and a useful one. The many pieces of Proposition 24.2 share a common idea: to understand anything invariant on a homogeneous space, you only need to know what it does at the identity.

Proof sketch of Proposition 24.2. For (1) and (2), the torsion of (ξ_1, ξ_2) is the horizontal component of $-[\widetilde{\xi}_1, \widetilde{\xi}_2]$, where $\widetilde{\xi}_1, \widetilde{\xi}_2$ are lifts of ξ_1 and ξ_2 to the total space. If we extend ξ_1 and ξ_2 to vector fields in a neighborhood that commute, they lift to horizontal vector fields, and so their bracket makes sense. In the same way, the curvature is the vertical part. This comes from the equations $\tau = \mathrm{d}\theta + \theta \wedge \theta$ and $\Omega = \mathrm{d}\Theta + \Theta \wedge \Theta$: if you throw out the horizontal (resp. vertical) parts, this is what you get. At the basepoint, we know how to take the horizontal and vertical components: they're $\mathfrak{m}$ and $\mathfrak{h}$ respectively.

Alternatively, we could've used the Maurer-Cartan equation:

$$\tau = \mathrm{d}(\pi_{\mathfrak{m}}\theta_G) + \pi_{\mathfrak{h}}\theta_G \wedge \pi_{\mathfrak{m}}\theta_G = \mathrm{d}(\pi_{\mathfrak{m}}\theta_G) + \frac{1}{2}[\pi_{\mathfrak{h}}\theta_G \wedge \pi_{\mathfrak{m}}\theta_G],$$

and the Maurer-Cartan equation says

$$= \pi_{\mathfrak{m}}\left(-\frac{1}{2}[\theta_G \wedge \theta_G]\right) + \frac{1}{2}[\pi_{\mathfrak{h}}\theta_G \wedge \pi_{\mathfrak{m}}\theta_G]. \tag{24.3}$$

This also uses the fact that d and $\pi_{\mathfrak{m}}$ commute, since the latter is a projection. Now, evaluate (24.3) on $\xi_1, \xi_2 \in \mathfrak{m}$. The proof for curvature is similar.

For (3), let $\xi_1, \dots, \xi_4 \in \mathfrak{m}$ and $\eta = [\xi_1, \xi_2]$. Since Θ is torsion-free, $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$, and in particular $\eta \in \mathfrak{h}$, so

$$[\eta, [\xi_3, \xi_4]] = [[\eta, \xi_3], \xi_4] + [\xi_3, [\eta, \xi_4]],$$

and since $[\eta, \xi_i] \in \mathfrak{m}$, then this is in $\mathfrak{h}'$.

The Ambrose-Singer theorem tells us that the Lie algebra of $\mathrm{Hol}_e(\Theta)$ is spanned by $\Omega_g(\xi_1, \xi_2)_{\in \mathfrak{h}}$ where $g \in P(e)$ and $\xi_1, \xi_2 \in (L_g)_*\mathfrak{h}$. By G-invariance it suffices to consider $g = e$, so we get exactly $\mathfrak{h}'$.

For (4), first observe that by what we've already shown, $\mathfrak{h}' \oplus \mathfrak{m} \subset \mathfrak{g}$ is a Lie subalgebra. Let $G' \subset G$ be the connected Lie subgroup of G with this Lie algebra. We'd like to draw paths on G that complete the proof, but got confused and will see the rest of the proof next time.

For (5), we want to show its tangent vector is parallel. Up on the frame bundle, a tangent vector is represented as a function along a curve from G to $\mathfrak{m}$. Using that $T(G/H) = G \times_H \mathfrak{m}$< then along the curve $t \mapsto e^{t\xi}$ (where $\xi \in \mathfrak{m}$), the tangent vector is $(L_{\exp(t\xi)})_*\xi$, which translates to the constant function with value ξ, and therefore the tangent vector is parallel.

For (6), consider more generally a manifold F with a left H-action, so we have the associated fiber bundle $F_P := P \times_H F \to G/H$, and G acts on F_P: $g \cdot [g', f] = [gg', f]$. A section of $F_P \to G/H$ is equivalent data to an equivariant function $s\colon G \to F$. In particular, s is parallel iff $s(g) = f_0$ for some $f_0 \in F$ fixed by $\mathrm{Hol}_e(\Theta)$. Since the holonomy group is a subgroup of H, it's sufficient for f_0 to be fixed by H, and this is equivalent to s being G-invariant.

For (7) and (8), this follows from the observation that the torsion and curvature are G-invariant. ⊠

So torsion-free connections correspond bijectively to splittings $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ that are invariant under the adjoint action of H and such that $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$.

Remark 24.4. Why the focus on torsion-free connections? Why is this a good condition? We want to model our geometry on affine space $\mathbb{A}^n$, which has global parallelism, so the displacements at any two points are identified (i.e. you can uniquely extend a vector into a constant vector field).

Forgetting the origin on tangent spaces makes the tangent bundle into a bundle of affine spaces A_xX. This is interesting because parallel transport is an affine map $A_xX \to A_xY$. Were x equal to y, this would be an element of the affine group. We've been focusing on the holonomy part, which lies in the quotient $\mathrm{Aff}_n / \mathbb{R}^n \cong \mathrm{GL}_n(\mathbb{R})$ (the $\mathbb{R}^n$ is the grouo of translations). The torsion integrates to tell you exactly the translational component, so the connection is torsion-free if for all curves, the basepoint maps the the basepoint. This is a good condition, because it means that the zero displacement always maps to the zero displacement, or equivalently that the zero displacement is parallel, and this is something that we want. ◂

Given such a splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, let $\dot\sigma$ be the involution of $\mathfrak{g}$ that acts as the identity on $\mathfrak{h}$ and -1 on $\mathfrak{m}$. Therefore $\dot\sigma^2 = \mathrm{id}_{\mathfrak{g}}$ and $\mathfrak{g}^{\dot\sigma} = \mathfrak{h}$.

Definition 24.5. A triple $(\mathfrak{g}, \mathfrak{h}, \dot\sigma)$ such that $\mathfrak{g}$ is a Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra, and $\dot\sigma\colon \mathfrak{g} \to \mathfrak{g}$ is an involution and a Lie algebra homomorphism such that $\mathfrak{g}^{\dot\sigma} = \mathfrak{h}$ is called a *symmetric Lie algebra*.

This is the infinitesimal analogue of a symmetric space.

Suppose $\mathfrak{m}$ has an H-invariant inner product; by Theorem 23.9, this can always be done if H is compact. Then, by left multiplication we obtain a G-invariant Riemannian metric on G/H

Lemma 24.6. *In this metric, Θ is the Levi-Civita connection.*

Proof. Since the metric is G-invariant, it must be Θ-parallel, i.e. Θ is torsion-free, hence the Levi-Civita connection. ⊠

Elie Cartan classified symmetric spaces in the 1920s. There are tables of them, and here are some examples that are homogeneous manifolds.

Example 24.7. The space $\mathrm{SO}_{p+q}/(\mathrm{SO}_p \times \mathrm{SO}_q)$ is determined by $(p+q)\times(p+q)$-matrices with determinant 1, stabilized by matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where $A \in \mathrm{SO}_p$ and $B \in \mathrm{SO}_q$. SO_{p+q} acts on $\mathbb{R}^{p+q}$ and therefore any linear-algebraic concept modeled on this (after fixing an orientation and a metric). The subspace of matrices in $\mathrm{SO}_p \times \mathrm{SO}_q$ fixes the subspace of vectors of the form $(\mathbf{x}, 0)$, and therefore also fixes its orthogonal complement $(0, \mathbf{y})$. The stabilizer is still slightly bigger (a transformation that reverses orientation on both subspaces preserves it on the total space). So $\mathrm{SO}_p \times \mathrm{SO}_q$ fixes oriented subspaces of dimension p. Thus, this space is the *oriented Grassmannian* $\mathrm{Gr}_p^{\mathrm{SO}}(\mathbb{R}^{p+q})$, the manifold of oriented p-dimensional subspaces of $\mathbb{R}^{p+q}$. Then, you can check that $\mathfrak{m}$ is matrices of the form

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & -A^{\mathrm{T}} \\ A & 0 \end{pmatrix} \right\},$$

and that $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$, so this is a symmetric space of dimension pq.

If V is a fixed vector space, let $\underline{V} \to \mathrm{Gr}_p(V)$ be the constant vector bundle with fiber V.[24] There's a canonical subbundle $S \to \mathrm{Gr}_p(V)$ called the *tautological bundle*: a point $W \in \mathrm{Gr}_p(V)$ is a p-dimensional subspace of V, and the fiber of S at W is W itself. This fits into an exact sequence

$$0 \longrightarrow S \longrightarrow \underline{V} \longrightarrow Q \longrightarrow 0,$$

where the fiber of Q at a $W \in \mathrm{Gr}_p(V)$ is V/W, which is also a canonical bundle. You can identify $T\mathrm{Gr}_p(V) \cong \mathrm{Hom}(S, Q)$.

For example, we saw that $\mathfrak{m} = \mathrm{Hom}(\mathbb{R}^p, \mathbb{R}^q) \cong \mathbb{R}^q \otimes (\mathbb{R}^p)^*$, and SO_q acts on the first component and SO_p acts on the second. This is the H-representation that we wanted: $\mathrm{SO}_p \times \mathrm{SO}_q$ acts on $\mathbb{R}^{pq}$.

The holonomy group is the entire group $\mathrm{SO}_p \times \mathrm{SO}_q$, a product of two lower-dimensional groups, but the representation is not a direct-sum representation. This is the kind of phenomenon that doesn't occur in Berger's list (it's not the same as the criterion for reducibility). Symmetric spaces will show us things not on Berger's list, the kinds of things we want to rule out. ◂

Exercise 24.8. Check that when $p, q > 1$, $\mathrm{SO}_p \times \mathrm{SO}_q$ does not act transitively on the unit sphere in $\mathbb{R}^{pq}$.

Example 24.9.

(1) $\mathrm{E}_6/\mathrm{Sp}_4$ is a 42-dimensional symmetric space, but arises from an unusual Sp_4-representation on a 16-dimensional space rather than the quaternions, so it's not accounted for on Berger's list.
(2) $\mathrm{E}_6/\mathrm{F}_4$ is a 24-dimensional symmetric space. F_4 isn't one of the listed holonomy groups on Berger's list. ◂

Next time, we'll talk about locally symmetric spaces, such as lens spaces $\Gamma\backslash\mathrm{SO}_4/\mathrm{SO}_3 = (\mathbb{Z}/n)\backslash S^3$.

Lecture 25.

Affine local diffeomorphisms: 4/18/17

Today, we'll begin with a couple of lemmas we need to begin the study of symmetric spaces.

Lemma 25.1. *Let X be a smooth manifold and η be a nonzero vector field on it. Let $\alpha \in \Omega^1_X$ and $\gamma\colon [0, \varepsilon) \to X$ be an integral curve of η. If $\mathcal{L}_\eta \alpha = 0$ along γ and $\alpha|_{\gamma(0)} = 0$, then $\alpha = 0$ along γ.*

[24] You can run this story for different variants of the Grassmanian, e.g. oriented, unoriented, complex, and so forth.

Proof. Choose local coordinates $x^1, \dots, x^n$ near $\gamma(0)$ such that $\eta = \frac{\partial}{\partial x^1}$, and write

$$\alpha = \alpha_i(x^1, \dots, x^n)\, \mathrm{d}x^i.$$

Then, we can compute:

$$\begin{aligned}
\mathcal{L}_\eta \alpha &= \iota_{\partial/\partial x^1}(\mathrm{d}\alpha) + \mathrm{d}\iota_{\partial/\partial x^1}\alpha \\
&= \iota_{\partial/\partial x^1}\left(\frac{\partial \alpha_j}{\partial x^k}\, \mathrm{d}x^k \wedge \mathrm{d}x^j\right) + \frac{\partial \alpha_1}{\partial x^j}\, \mathrm{d}x^j \\
&= \sum_{j\neq 1}\left(\frac{\partial \alpha_j}{\partial x^1}\, \mathrm{d}x^j - \sum_{k\neq 1}\frac{\partial \alpha_1}{\partial x^k} + \frac{\partial \alpha_1}{\partial x^j}\, \mathrm{d}x^j\right) \\
&= \frac{\partial \alpha_j}{\partial x^1}(x^1, \dots, x^n)\, \mathrm{d}x^j.
\end{aligned}$$

Now, γ is the curve $x^1 = t$ and $x^j = 0$ for $j > 1$, so if $\gamma(0) = (0, \dots, 0)$, the hypotheses tell us

$$\begin{aligned}
\frac{\mathrm{d}\alpha_j}{\mathrm{d}t}(t, 0, \dots, 0) &= 0 \\
\alpha_j(0, \dots, 0) &= 0,
\end{aligned}$$

so by the uniqueness of solutions to an ODE, $\alpha_j = 0$. ⊠

Now we'll discuss a recognition theorem for when a map on the frame bundles is the differential of a smooth map on the underlying manifolds.

Lemma 25.2. *Suppose X and X' are smooth manifolds of dimension n and $\varphi\colon X \to X'$ and $\widetilde{\varphi}\colon \mathscr{B}(X) \to \mathscr{B}(X')$ are smooth maps. The following are equivalent:*

(1) *$\widetilde{\varphi}$ is the differential of φ.*
(2) *If θ' denotes the soldering form on X' and θ denotes the soldering form on X, then $\widetilde{\varphi}^*(\theta') = \theta$.*

In the presence of a connection, one can dualize the second statement and obtain horizontal vector fields, and the analogous criterion for $\widetilde{\varphi}$ is equivalent to the two statements in the lemma.

The idea is that if $\varphi\colon X \to X'$ is a local diffeomorphism, it induces a pushforward $\varphi_*\colon TX \to TX'$ and hence a map $\varphi_*\colon \mathscr{B}(X) \to \mathscr{B}(X')$. Then, tracing through the definitions, it's almost a tautology.

Proof. Let $p \in \mathscr{B}(X)$ be in the fiber of an $x \in X$, and let $\pi\colon \mathscr{B}(X) \to X$ be projection (and similarly for π'). Let $\widetilde{\xi} \in T_p\mathscr{B}(X)$ be in the fiber of $\xi \in T_xX$, and let $\pi_*\colon T\mathscr{B}(X) \to \mathscr{B}(X)$ be projection (and similarly for π'). Thus

$$\pi'_*\widetilde{\varphi}_*(\widetilde{\xi}) = \varphi_*(\xi)$$

amd

$$(\widetilde{\varphi}^*\theta')_p(\widetilde{\xi}) = \theta'_{\widetilde{\varphi}(p)}(\widetilde{\varphi}_*\widetilde{\xi}),$$

which is the components of $\pi'_*\widetilde{\varphi}_*\widetilde{\xi} = \varphi_*(\xi)$ in the basis $\widetilde{\varphi}(p)$. By contrast, $\theta_p(\widetilde{\xi})$ is the components of ξ in the basis p. The vectors in $\mathbb{R}^n$ are equal iff $\widetilde{\varphi}(p) = \varphi_*(p)$, as desired. ⊠

Let X and X' be n-dimensional manifolds, Θ be a connection on $\mathscr{B}(X) \to X$, and Θ' be a connection on $\mathscr{B}(X') \to X'$. Let T and R (resp. T' and R') be the torsion and curvature of Θ (resp. Θ').

Definition 25.3. A local diffeomorphism $\varphi\colon X \to X'$ is *affine* if $\widetilde{\varphi}^*\Theta' = \Theta$ for the differential $\widetilde{\varphi}\colon \mathscr{B}(X) \to \mathscr{B}(X')$.

Affine maps preserve the parallel structure, e.g. sending geodesics to geodesics.

Fix a $p \in \mathscr{B}(X)_x$ and a $p' \in \mathscr{B}(X')_{x'}$. Then, we obtain an isomorphism $\psi\colon T_xX \to T_{x'}X'$: $b\colon T_xX \to \mathbb{R}^n$ is an isomorphism, and similarly with p', so $\psi := (p')^{-1} \circ p$.

Let ∂_k denote a horizontal vector field on $\mathscr{B}(X)$. Then, its integral curves project to geodesics on X: the geodesic equation is that the tangent vector field to a geodesic curve is parallel, which is a second-order ODE. But on the frame bundle, we keep track of the derivative, so this is a system of first-order ODEs. Thus, the map $\widetilde{\psi}$ sends the integral curve $\exp(t\xi^k\partial_k)$ based at p with initial velocity (tangent vector) $\xi^k\partial_k$ to

$\exp(t\xi^k\partial'_k)$, i.e. the same integral curve relative to the basis, but measured in the basis p' instead of p. This extends to a map ψ in a neighborhood of x by flowing along a geodesic.

Theorem 25.4. *Assume $\dot\psi^* R'_{x'} = R_x$, $\dot\psi^* T_{x'} = T_x$, and that T, R, T', and R' are parallel. Then, ψ is an affine diffeomorphism of the neighborhoods U (resp. U') of x (resp. x').*

In symbols, we're assuming $\nabla T = 0$, $\nabla R = 0$, $\nabla T' = 0$, and $\nabla R' = 0$. This map can't be a global diffeomorphism in general, e.g. the covering map $\mathbb{R} \twoheadrightarrow S^1$.

Proof. We must show that $\widetilde{\psi}^*\theta' = \theta$ and $\widetilde{\Theta}' = \Theta$. Write

$$\begin{aligned} T &= \frac{1}{2} T^i_{k\ell}\theta^k \wedge \theta^\ell\, e_i \\ R &= \frac{1}{2} R^i_{jk\ell}\theta^k \wedge \theta^\ell\, e^j_i, \end{aligned}$$

where $T^i_{k\ell} = -T^i_{\ell k}$. The assumption that both torsions and curvatures are parallel means that $T^i_{k\ell}$, $R^i_{jk\ell}$, ${T'}^{\,i}_{k\ell}$, and ${R'}^{\,i}_{jk\ell}$ are all constant functions on U and U'. Furthermore, since the curvature and torsion pull back, the constants for T and T' agree, as do those for R and R'.

Recall the structure equations for torsion and curvature:

$$\frac{1}{2} T^i_{k\ell}\theta^k \wedge \theta^\ell = \mathrm{d}\theta^i + \Theta^i_j \wedge \theta^j \tag{25.5a}$$

$$\frac{1}{2} R^i_{jk\ell}\theta^k \wedge \theta^\ell = \mathrm{d}\Theta^i_j + \Theta^i_k \wedge \Theta^j_k. \tag{25.5b}$$

Set $\eta := \xi^k\partial_k$ for $(\xi^k) \in S(\mathbb{R}^n)$ (the unit sphere inside $\mathbb{R}^n$), and apply ι_η to (25.5a) and (25.5b) to yield

$$\begin{aligned} \mathcal{L}_\eta\theta^i &= T^i_{k\ell}\xi^k\theta^\ell + \xi^j\Theta^i_j \\ \mathcal{L}_\eta\Theta^i_j &= R^i_{jk\ell}\xi^k\theta^\ell. \end{aligned}$$

These equations are also satisfied by $\widetilde{\psi}^*{\theta'}^i$ and $\widetilde{\psi}^*{\Theta'}^i_j$, so the differences

$$\begin{aligned} \delta^i &:= \widetilde{\psi}^*{\theta'}^i - \theta^i \\ \Delta^i_j &:= \widetilde{\psi}^*{\Theta'}^i_j - \Theta^i_j \end{aligned}$$

satisfy

$$\begin{aligned} \mathcal{L}_\eta\delta^i &= 0 \\ \mathcal{L}_\eta\Delta^i_j &= 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \delta^i|_p &= 0 \\ \Delta^i_j|_p &= 0. \end{aligned}$$

Thus, these neighborhoods U and U' are the same in the affine sense. ⊠

This is a common style of argument in differential geometry.

Corollary 25.6. *Let X and X' be Riemannian manifolds with curvatures R, resp. R' parallel near x, resp. x'. Let p and p' be orthonormal bases. Then, ψ is a local isometry.*

Proof. The proof of Theorem 25.4 applies to the orthonormal frame bundle without much change, and some steps simplify. In particular, $\widetilde{\psi}\colon \mathscr{B}_{\mathrm{O}}(X)|_U \to \mathscr{B}_{\mathrm{O}}(X')|_{U'}$ because it's also the differential by Lemma 25.2. Thus, it maps orthonormal bases to orthonormal bases, hence is an isometry. ⊠

Now let's change gears. We want to study symmetric spaces, hence homogeneous manifolds G/H with principal H-bundles $P \to G/H$.

We had the general principle that G-invariant information on G/H corresponds to H-invariant information on $\mathfrak{g}/\mathfrak{h}$. Here are three examples.

- A G-invariant metric on G/H is equivalent to an H-invariant inner product on $\mathfrak{g}/\mathfrak{h}$.
- A G-invariant connection on $\pi\colon P \to G/H$ is equivalent to an H-invariant spliting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.
- A G-invariant torsion-free connection on π is equivalent to an H-invariant splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ such that $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. This defines a Lie algebra involution $\dot\sigma\colon \mathfrak{g} \to \mathfrak{g}$ such that $\dot\sigma|_{\mathfrak{h}} = \mathrm{id}_{\mathfrak{h}}$ and $\dot\sigma|_{\mathfrak{m}} = -\mathrm{id}_{\mathfrak{m}}$.

These ways to study the geometry of G/H view it as a framed manifold, with the framing coming from the framing on G.

Example 25.7.

(1) Consider $G = G/\{e\}$, which has a G-action by left translation, and $\pi\colon G \to G$ is the identity. The splitting is $\mathfrak{g} = 0 \oplus \mathfrak{g}$, and $[\mathfrak{m}, \mathfrak{m}] \not\subset 0$ unless $\mathfrak{g}$ is abelian.

(2) We can also consider $G = (G \times G)/G$, with the left and right G-actions. Then, the principal G-bundle is $G \times G \to G$. We have a bigger symmetry group, hence more room to play: $\mathfrak{h} \subset \mathfrak{g} \oplus \mathfrak{g}$ is the diagonal, and

$$\mathfrak{m} = \{(\xi, -\xi) \mid \xi \in \mathfrak{g}\}.$$

Hence $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. If G is compact, then there are $(G \times G)$-invariant metrics, and you can show they have nonnegative sectional curvature. ◀

These kinds of spaces, like homogeneous Riemannian manifolds in general, have nice formulas for their curvature, and hence are an excellent playground for testing conjectures in Riemannian geometry.

Given a (finite-dimensional) Lie algebra $\mathfrak{g}$, there's a unique connected, simply-connected Lie group G (up to isomorphism) whose Lie algebra is $\mathfrak{g}$. If $H \subset G$ is a Lie subgroup whose Lie algebra is H, then $\dot{\sigma}$ exponentiates to a $\sigma\colon G \to G$ which is an involution of Lie groups whose fixed point set is H. In nice circumstances.[25] Therefore σ induces an involution $\overline{\sigma}\colon G/H \to G/H$ with $\overline{\sigma}(H/H) = H/H$, and $\mathrm{d}\overline{\sigma}_{H/H} = -\mathrm{id}$. If G/H is Riemannian, with a metric arising from a G-invariant metric on G, then $\overline{\sigma}$ is an isometry.

This provides another approach to understanding symmetric spaces: if X is any Riemannian manifold and $x \in X$, we can define a local diffeomorphism $s)_x\colon U_x \to U_x$ (where U_x is a neighborhood containing x) such that $s_x(x) = x$ and $\mathrm{d}s_x = -\mathrm{id}_{T_xX}$ by following the geodesic in the other direction: for $y \in U_x$, $y = \exp(\xi)$ for a unique $\xi \in T_xX$ if U_x is sufficiently small. Then, let $s_x(y) = \exp(-\xi)$. This s_x is a local isometry anywhere, but in general will not glue to a global isometry.

Definition 25.8. A Riemannian manifold X is *locally symmetric* if s_x is a local isometry for all $x \in X$.

Theorem 25.9. *X is locally symmetric iff its curvature is parallel.*

Thus we have a local criterion for a local property, as expected.

Proof. In the forward direction, if $s(x)$ is an isometry, it preserves the Riemann curvature tensor, which has skew-symmetry:

$$s_x^*\langle(\nabla_{\xi_1}R)(\xi_2,\xi_3)\xi_4,\xi_5\rangle = \langle\nabla_{-\xi_1}R(-\xi_2,-\xi_3)(-\xi_4),-\xi_5\rangle = -\langle(\nabla_{\xi_1}R)(\xi_2,\xi_3)\xi_4,\xi_5\rangle,$$

so $\nabla_\xi R$ is identically 0 for all ξ.

Conversely, if $s_x^*R = R$, then from a theorem at the beginning of class, s_x is an isometry. ⊠

Next Thursday (the 27th) there's no class; the last two classes will be self-contained, and taught by Lewis Bowen.

Lecture 26.

Locally symmetric spaces: 4/20/17

Last time, we showed that if X is a Riemannian manifold and $x \in X$, there's a local involution $s_x\colon B_r(x) \to B_r(x)$ for some radius r defined by $\exp(\xi) \mapsto \exp(-\xi)$ (Theorem 25.4). Here, exp is the *exponential map* $\exp_x\colon T_xX \to X$ sending $\xi \mapsto \gamma_\xi(1)$, where $\gamma_\xi(t)$ is the unique geodesic such that $\gamma_\xi(0) = x$ and $\dot{\gamma}_\xi(0) = \xi$. The exponential $\exp_x$ is a local diffeomorphism of a neighborhood of the identity to a neighborhood of x.

We said that X is locally symmetric if s_x is an isometry for all x. If X is connected, we say X is *(globally) symmetric*, or is a *Riemannian symmetric space*, if s_x extends to a global isometry for all x.

This notion of symmetry makes sense any time there's a notion of parallelism on a manifold; in particular, one can define local symmetry in the context of a bundle with a connection.

Theorem 26.1.

[25] This is a nice situation: in more general circumstances, $\dot{\sigma}$ doesn't exponentiate to such a map.

(1) *X is locally symmetric iff $\nabla R = 0$.*
(2) *If X is symmetric, then X is homogeneous, i.e. there exists a transitive Lie group action on X by isometries.*

We proved part (1) last time, as Theorem 25.9. Today we'll tackle part (2).

Remark 26.2.

(1) If we're in case (2) and $X \cong G/H$, where G is the isometry group, then there exists a Lie group involution $\sigma\colon G \to G$ such that $G_0^\sigma \subseteq H \subseteq G^\sigma$.
(2) Furthermore, if H acts irreducibly on $\mathfrak{g}/\mathfrak{h}$ and X is simply connected, then the holonomy group is H. Thus there are lots of holonomy groups for symmetric spaces, in contrast with Berger's classification for non-symmetric spaces. ◀

Really, though, the proof of Theorem 26.1, part (2) will be an excuse to work with geodesics.

Proof sketch of Theorem 26.1, part (2). The first step will be to show that geodesics extend infinitely far, i.e. if $x \in X$, then $\exp_x$ is defined on the entirety of T_xX. This entails showing that $\exp(t \cdot \lambda\xi) = \gamma_{\lambda\xi}(t) = \gamma_\xi(\lambda t)$, so we can scale back to something where we know the exponential map is defined.

Now suppose the geodesic stops existing after some finite time T at some point g. Then, we can apply the local symmetry at $2/3^{\text{rds}}$ of the way along (in the time parameter), apply the symmetry. This makes the geodesic extend to $4/3^{\text{rds}}$ of what the maximum was supposed to be, so the maximum time can't exist.

The second step is to apply the Hopf-Rinow theorem to conclude that X is *geodesically complete*, i.e. for any $x, y \in X$, there exists a geodesic[26] γ from x to y.

Step 3 will be to show that if m is the midpoint of the geodesic γ joining x and y, then $s_m(x) = y$.

We'll spend the time to make this rigorous, as the geodesic techniques in the details are useful in many other places.

For the rest of this lcture, let X be a connected Riemannian manifold. Let $\gamma\colon [0,1] \to X$ be a piecewise C^1 curve (i.e. it's continuous, and at all but finitely many points, it's C^1); then, define

$$L(\gamma) := \int_0^1 \mathrm{d}t\, \|\dot\gamma(t)\|.$$

Given $x, y \in X$, let

$$\rho(x,y) := \inf_\gamma L(\gamma),$$

as γ ranges over all piecewise C^1 curves $\gamma\colon [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Since the length is nonnegative and X is connected, this infimum exists.

Theorem 26.3. *With ρ as the distance function, X is a metric space. Moreover, the metric space topology induced by ρ equals the manifold topology on X.*

Proof. Elementary properties of the infinum show the triangle inequality, and symmetry of ρ is evident, along with nonnegativity of $\rho(x,y)$ for all x and y. What we do have to show is that if $x \neq y$, then $\rho(x,y) > 0$.

As x and y are distinct, there's a neigborhood U containing x but not y. If $\phi\colon U \to \mathbb{R}^n \setminus 0$ is a chart for X, then $K := \phi^{-1}(\overline{B_r(0)})$ is a compact subset of U. Choose an $r > 0$ such that $y \notin K$. Let $\pi\colon S(TX) \to X$ be the unit sphere bundle, so $\pi^{-1}(K)$ is compact. Now, the image of $\phi_*\colon \pi^{-1}(K) \to \mathbb{R}^n$ does not contain 0, so there exist constants $c, C > 0$ such that

$$c \le \|\phi_*(\xi)\| \le C,$$

where $\xi \in \pi^{-1}(K)$.

Hence, if $\gamma_0\colon [0, L] \to K$ is parameterized by arc length,

$$cL \le L(\phi \circ \gamma_0) = \int_0^: \mathrm{d}t\, \|\phi_*\dot\gamma_0(t)\| \le CL.$$

Thus if $\gamma\colon [0, L'] \to X$ has $\gamma(0) = x$ and $\gamma(L') = y$, let t_0 be the minimal time such that $d(0, \phi \circ \gamma_0(t_0)) = r$ and set $\gamma_0 = \gamma|_{[0,t_0]}$. Then

$$L(\gamma) \ge L(\gamma_0) \ge \frac{L(\phi \circ \gamma_0)}{C} \ge \frac{r}{C},$$

[26] Said geodesic need not be unique.

and taking the infimum, $\rho(x,y) \geq r/C > 0$.

The second part, that the topologies are the same, is a local statement, so it suffices to check on K. Since U is a chart, K is homeomorphic to $\overline{B_r(0)}$ with the Euclidean topology, which is a topology induced by a metric. Hence it suffices to show that the Euclidean metric d_E determines the same topology as ρ on K (and hence are equivalent as metrics). To show this, it suffices to bound one on both sides by the other. In particular, we'll show that

$$cL(\gamma) \leq d_E(x,y) \leq CL(\gamma).$$

The right-hand inequality follows because the minimal-distance curve in $\mathbb{R}^n$ is a straight line, and taking the infimum, we see that $d_E(x,y) \leq C\rho(x,y)$. In the other direction, consider the straight-line curve, whose length we bounded above in the first part of the proof. ⊠

Now we want to show that these length-minimizing curves are actually geodesics. (Recall that γ is a geodesic if its tangent vector is parallel. In this case the theory of ODEs guarantees that γ is smooth.) The key tool is the *first variation formula*, which tells us how to differentiate the length.

Let $x, y \in X$ and $\gamma\colon [0,1] \to X$ be a path from x to y with constant speed, so $L := \langle \dot\gamma, \dot\gamma \rangle$ is constant. Now, extend γ in another direction, to a *variation*, amap $\Gamma\colon (-\varepsilon, \varepsilon) \times [0,1] \to X$. Call the first coordinate s and the second coordinate t, and let $\tau := \Gamma_* \frac{\partial}{\partial t}$ and $\xi := \Gamma_* \frac{\partial}{\partial s}$. Since $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial s}$ are coordinate vector fields, $[\xi, \tau] = 0$ too.

The length of $\Gamma_s := \Gamma(s, -)$ is a function in s, and we can differentiate it:

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} L(\Gamma_s) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \int_0^1 \mathrm{d}t\, \langle \tau, \tau \rangle^{1/2}.$$

As this is a continuous function on a compact interval, this is

$$\begin{aligned}
&= \int_0^1 \mathrm{d}t\, \xi \cdot \langle \tau, \tau \rangle^{1/2} \Big|_{s=0} \\
&= \int_0^1 \mathrm{d}t\, \langle \tau, \tau \rangle^{1/2} \langle \nabla_\xi \tau, \tau \rangle. \Big|_{s=0}
\end{aligned}$$

Since the Levi-Civita connection is torsion-free,

$$\begin{aligned}
&= \int_0^1 \mathrm{d}t\, \langle \tau, \tau \rangle^{1/2} \langle \nabla_\tau \xi, \tau \rangle \Big|_{s=0} \\
&= \int_0^1 \mathrm{d}t\, \langle \tau, \tau \rangle^{1/2} (\tau \cdot \langle \xi, \tau \rangle - \langle \xi, \nabla_\tau \tau \rangle) \Big|_{s=0} \\
&= \frac{1}{L} \left(\langle \xi, \tau \rangle |_x^y - \int_0^1 \mathrm{d}t\, \langle \xi, \nabla_\tau \tau \rangle \right).
\end{aligned}$$

We want to minimize this, which is solving an elliptic PDE.

Corollary 26.4. *If γ is length-minimizing, then γ is a smooth geodesic.*

Proof. If γ is length-minimizing, then $\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} L(\Gamma_s) = 0$ for all variations, where we take $\Gamma(0,s) = x$ and $\Gamma(1,s) = y$. Then, $\xi_x = 0$ and $\xi_y = 0$, so $\nabla_\tau \tau = 0$. ⊠

Remark 26.5. This is an argument by the standard lemma in the calculus of variations, and shows up in other contexts. In particular, if $f\colon [0,1] \to \mathbb{R}$ is continuous and

$$\int_0^1 \mathrm{d}t\, f(t) g(t) = 0$$

for all continuous $g\colon [0,1] \to \mathbb{R}$ such that $g(0) = g(1) = 0$, then $f = 0$ (the idea is to approximate it by g whose endpoints are fixed). ◀

Corollary 26.6 (Gauss lemma)**.** *Suppose $\eta, \zeta \in T_x X$ are orthogonal. Then, $d\exp_\eta \eta$ and $d\exp_\eta \zeta$ are also orthogonal.*

The idea is that the exponential map preserves orthogonality.

Proof. Let $c\colon (-\varepsilon, \varepsilon) \to T_xX$ be a curve with $c(0) = \eta$, $\dot{c}(0) = \zeta$, and $\|c(t)\| = \|\eta\|$. Then, consider the variation

$$\Gamma(t,s) := \exp(t \cdot c(s)).$$

Since radial lines in T_xX map to geodesics, then $\nabla_\tau \tau = 0$ for all t, s, where $\tau = \Gamma_* \frac{\partial}{\partial t}$ as before. Moreover, $\xi_x = 0$ and $L(\Gamma_s)$ is constant, so the variation formula tells us that

$$0 = \frac{1}{L} \langle \xi, \tau \rangle|_{\exp \eta}.$$

But $\xi|_{\exp \eta} = d\exp_\eta(\zeta)$ and $\tau|_{\exp\eta} = d\exp_\eta \eta$. ⊠

On some open $U \subset T_xX$ containing the origin, $\exp_x$ embeds U into X, and its inverse is a coordinate system on X. After choosing a basis on T_xX, we obtain standard affine coordinates. If you omit the origin, you can choose *polar coordinates* $(r, \theta^1, \dots, \theta^{n-1})$, where r is distance from the origin and $(\theta^1, \dots, \theta^{n-1})$ is a coordinate system on S^{n-1}. The Gauss lemma then tells us that

$$\mathrm{d}s^2 = \mathrm{d}r^2 + G_{ij}(r, \theta^1, \dots, \theta^{n-1})\, \mathrm{d}\theta^i + \mathrm{d}\theta^j.$$

Here, $\mathrm{d}s$ is the area form (in rectangular coordinates), and the point is that there are no $\mathrm{d}r\,\mathrm{d}\theta^i$ terms: they started perpendicular and remain perpendicular. The G_{ij} are constant in two dimensions, but not in general. This is called *geodesic polar coordinates*, an example of a *normal coordinate system*.

Exercise 26.7. In geodesic normal coordinates, show that $\Gamma^i_{jk}(0) = 0$.

Also, $\frac{\mathrm{d}}{\mathrm{d}r} = \operatorname{grad}(r)$. This is the gradient on a Riemannian manifold:

Definition 26.8. If X is Riemannian and $f\colon X \to \mathbb{R}$, its *(Riemannian) gradient* is the vector field $\operatorname{grad}(f) \in \mathcal{X}(X)$ satisfying

$$\mathrm{d}f_x(\eta_x) = \langle \operatorname{grad}_x f, \eta_x \rangle$$

for all $x \in X$ and $\eta_x \in T_xX$.

This has the nice properties you want it to, e.g. pointing in the direction of steepest increase.

Lecture 27.

Geodesics: 4/25/17

First, remember there's no lecture Thursday, and next week Lewis Bowen will give two lectures on Hodge theory. Today, X will always denote a connected Riemannian manifold.

Recall the first variation formula: if $\Gamma\colon (-\varepsilon, \varepsilon) \times [0,1] \to X$ is a family of curves (the family is in the s-direction), such that $\Gamma_0 := \Gamma(0, -)$ has constant speed, then

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} L(\Gamma_s) = \langle \tau, \xi \rangle|_{t=0}^{t=1} - \int_0^1 \mathrm{d}t\, \langle \xi, \nabla_\tau \tau \rangle.$$

Last time, we discussed one consequence: that if $\gamma\colon [0,1] \to M$ is a piecewise-C^1 length-minimizing curve, it's actually a smooth geodesic.

For another consequence, fix an $x \in X$ and consider a ball $B_{r_0}(x)$ such that the exponential map $\exp\colon B_{r_0}(0) \to B_{r_0}(x)$ is a diffeomorphism. Such a ball is called a *normal coordinate ball*. Then, the radius is a function $r\colon B_{r_0}(x) \to \mathbb{R}$ sending $y \mapsto \rho(x,y)$ (length measured by the shortest geodesic from x to y). This is continuous, but not smooth at x, and its gradient (away from x) is $\operatorname{grad} r = \frac{\partial}{\partial r}$. This follows from the Gauss lemma (Corollary 26.6): if η is a radial vector in T_xX with length less than r_0 and $\zeta \perp \eta$, then $\langle \mathrm{d}\exp_\eta(\zeta), \mathrm{d}\exp_\eta(\eta) \rangle = 0$, and the Gauss lemma shows that they remain perpendicular when exponentiated, so the gradient is the radial vector field $\frac{\partial}{\partial r}$.

We'd like to understand length minimizing curves, but so far we don't even know when they exist!

Theorem 27.1. *Let $x \in X$ and $B_{r_0}(x)$ be a normal exponential ball. Then,*

1. *For an $\eta \in B_{r_0} \subset T_xX$, the geodesic $t \mapsto \exp(t\eta)$ is the unique (up to reparameterization) length-minimizing curve from x to $\exp(\eta)$.*
2. *If $y \notin B_{r_0}(x)$, then there exists a $z \in \partial B_{r_0}(x)$ such that $\rho(x,y) = r_0 + \rho(z,y)$.*

The second result is useful so that we can attack the global result using the local result.

Proof. For (1), let $\gamma\colon [0,1] \to X$ be any piecewise-C^1 curve from x to $\exp(\eta)$, and assume $0 \le r(\gamma(t)) < r_0$ for $0 \le t \le t_0$ (since the geodesic might leave the normal exponential ball — but by altering the curve, we can obtain one with smaller length that's contained within the ball). Then,

$$\begin{aligned} L(\gamma) &= \int_0^{t_0} \mathrm{d}t\, \|\gamma\| + \int_{t_0}^1 \mathrm{d}t\, \|\dot\gamma\| \\ &\ge \int_0^{t_0} \mathrm{d}t \left\langle \dot\gamma, \frac{\partial}{\partial r} \right\rangle + \int_{t_0}^1 \mathrm{d}t\, \|\dot\gamma\| \\ &= \int_0^{t_0} \mathrm{d}t\, \frac{\mathrm{d}}{\mathrm{d}t} r(\gamma(t)) + \int_{t_0}^1 \mathrm{d}t\, \|\dot\gamma\| \\ &= r(\gamma(t_0)) + \int_{t_0}^1 \mathrm{d}t\, \|\dot\gamma\|. \end{aligned}$$

Choose the minimal t_0 such that $r(\gamma(t_0)) = \|\eta\|$; then,

$$L(\gamma) \ge \|\eta\| + \int_{t_0}^1 \mathrm{d}t\, \|\dot\gamma\|.$$

Therefore $L(\gamma) = \|\eta\|$ iff both of the following are true:

(1) $\int_{t_0}^1 \mathrm{d}t\, \|\dot\gamma\| = 0$ and hence $\dot\gamma(t) = 0$ for $0 \le t \le t_0$; and
(2) $\dot\gamma(t)$ is a multiple of $\frac{\partial}{\partial r}$ for $0 \le t \le t_0$.

Thus γ is a radial geodesic.

For (2), let γ be a curve from x to y, and let t_0 be minimal such that $\gamma(t_0) \in \partial B_{r_0}(x)$, so that $r(\gamma(t_0)) = r_0$. Then

$$\begin{aligned} L(\gamma) &\ge r_0 + \rho(\gamma(t_0), y) \\ &\ge r_0 + \rho(\partial B_{r_0}(x), y). \end{aligned}$$

The right-hand side is independent of γ, so taking the infimum over all γ,

$$\rho(x,y) \ge r_0 + \rho(\partial B_{r_0}(x), y),$$

and the triangle inequality gives the opposite bound, so

$$\rho(x,y) = r_0 + \rho(\partial B_{r_0}(x), y).$$

Since $\partial B_{r_0}(x)$ is compact, we can find a $z \in \partial B_{r_0}(x)$ such that $\rho(z,y) = \rho(\partial B_{r_0}(x), y)$. ⊠

With this out of the way, we can tackle the big theorem, characterizing when a manifold is complete.

Theorem 27.2 (Hopf-Rinow)**.** *Let X be a connected Riemannian manifold. Then, the following are equivalent:*

(1) *(X, ρ) is a complete metric space.*
(2) *For some $x \in X$, $\exp_x\colon T_xX \to X$ is everywhere defined.*
(3) *$\exp\colon TX \to X$ is everywhere defined.*

If these hold, then

(4) *given any $x, y \in X$, there exists a geodesic γ from x to y such that $L(\gamma) = \rho(x,y)$.*

The length-minimizing curve in (4) need not be unique: consider two antipodal points on a sphere, which is geodesically complete because it's compact. In general, completeness means there's no "holes," which geodesics can't travel through. If the manifold isn't complete, $\rho(x,y)$ is the infimum, not the minimum, and a curve realizing $\rho(x,y)$ might not exist.

Proof. Clearly (3) implies (2) for any x. We'll show that for any x, (2) holds for x implies (4) for x, and together these imply (1). Then, we'll show (1) implies (3).

First, (2) to (4) for x. Let $B_{r_0}(x)$ be a normal ball about x. If $y \in B_{r_0}(x)$, we're done, so assume otherwise. By Theorem 27.1, we can find a $z \in \partial B_{r_0}(x)$ such that $\rho(x,y) = r_0 + \rho(z,y)$.

Let $\gamma\colon [0,\infty) \to X$ be the *regular geodesic* (i.e. $\|\dot\gamma\| = 1$) such that $\gamma|_{[0,t_0]}$ is the minimal radial geodesic from x_0 to z, and suppose $\gamma|_{[0,\rho(x,y)]}$ isn't a length-minimizing geodesic from x to y.

Choose the maximal $t_0 < \rho(x,y)$ such that

$$\rho(x,\gamma(t_0)) + \rho(\gamma(t_0),y) = \rho(x,y).$$

Let $B_{r_1}(\gamma(t_0))$ be a normal ball. Choose a $w \in \partial B_{r_1}(\gamma(t_0))$ such that $\rho(\gamma(t_0),y) = r_1 + \rho(w,y)$. Then,

$$\begin{aligned}\rho(x,y) &= \rho(x,\gamma(t_0)) + r_1 + \rho(w,y)\\ &\geq \rho(x,w) + \rho(w,y)\\ &\geq \rho(x,y),\end{aligned}$$

so $\rho(x,w) = \rho(x,\gamma(t_0)) + r_1$.

Let σ be a radial length-minimizing geodesic from $\gamma(t_0)$ to w. Then, $\gamma|_{[0,t_0]} \cup \sigma$ is a length-minimizer from x to w, hence a smooth geodesic. Since geodesics are solutions to an ODE and hence are unique, this must be $\gamma|_{[0,\rho(x,w)]}$, but $\rho(x,w) > t_0$ is a contradiction to the definition of t_0.

Now, we'll show (2) plus (4) at x implies (1). Let $\{y_i\} \subset X$ be a Cauchy sequence and $\gamma_i \colon [0, L_i] \to X$ be a normal geodesic from x to y_i. Then, $\{L_i\}$ is a Cauchy subset of $\mathbb{R}$, hence has a limit L which is a nonnegative number. Similarly, $\{\dot\gamma_i\} \subset S(T_xX)$ has a convergent subsequence $\dot\gamma_{i_j}(0) \to \eta$. Let $\gamma \colon [0, L] \to X$ be the normal geodesic, with $\gamma(t) = \exp_x(t\eta)$, which exists for all time by (2). Now, by the theory of ODEs, $\gamma_{i_j}(L_{i_j})$ converges to $\gamma(L)$ because the solutions of ODEs depend smoothly on the parameters.

Lastly, we want to show that (1) implies (3). Let $\gamma \colon [0, t_0] \to X$ be a geodesic whose initial velocity is η, and such that t_0 is maximal. Now, if $\{t_i\}$ is a Cauchy sequence approaching t_0, $\gamma(t_i)$ is Cauchy in X, so by completeness extends to a limit point y. Let $\gamma(t_0) = y$; then, we can extend a little farther than t_0 by considering a normal ball around $\gamma(t_0)$ and extending the geodesic on that normal ball; thus, such a maximal t_0 cannot exist. ⊠

See Cheeger-Even's book for references for all of these arguments.

For the past several weeks, we've been thinking about things related to Berger's theorem classifying the restricted holonomy groups of Riemannian manifolds that are irreducible and are not symmetric spaces. In particular, either Hol^0 acts transitively on the unit sphere $S(T_xX)$ or X is locally symmetric at x.

Berger's proof reduces this to an algebraic problem, studying systems of an inner product space V, a curvature tensor R (i.e. a tensor of the correct type on V), and a group $G \subset \mathrm{O}(V)$ that acts irreducibly on V. Thus we get a whole orbit of curvature tensors on V. If that orbit is a single point, it's an algebraic version of (local) parallelism, which is the locally symmetric space. Thus we can establish a dichotomy between G acting transitively on the unit sphere and R being fixed in the induced representation on tensors.

We also have a relationship between $\mathfrak{g}$ and curvature in the form of the Ambrose-Singer theorem, which we haven't yet introduced to the algebraic framework. This provides additional buying power, though the arguments to prove the algebraic version of Berger's theorem are quite complicated. Then, there's another step bringing it back to the manifold.

We'll learn about Hodge theory next week, a use of differential equations in the study of manifolds. In this case, the differential equation is linear, at least. There are lots of interesting examples, e.g. the geodesic equations, harmonic maps of Riemannian surfaces (two-dimensional versions of the geodesic equation), or for 4-dimensional equations, things such as the Yang-Mills equations or instanton equations.

The simplest thing we can write down is the Laplacian $\Delta\omega = 0$. This is linear, so the space of solutions is a vector space; nonlinear equations have manifolds for solution spaces, and global features of the manifold tell you information about X. For a linear equation and a vector space, the only thing we can ask is the dimension of the space of solutions; we'll show the dimensions associated to the Laplacian are independent of the metric, hence are topological invariants. In particular, this recovers the Betti numbers, which is complicated but beautiful.

Lecture 28.

Spectral geometry: 5/2/17

Lewis Bowen gave today's lecture, and will be giving the next lecture. We'll talk about spectral geometry.

The first part of the lecture will constitute an introduction to spectral geometry. On $\mathbb{R}^n$, there's a second-order differential operator called the *Laplacian*

$$\Delta := -\sum_{i=1}^{n} \frac{\partial^2}{\partial {x_i}^2}.$$

This is a positive definite operator: $\langle \Delta f, f\rangle \geq 0$ for all twice-differentiable f.

Fix a closed Riemannian manifold M. Then, the Laplacian $\Delta_M\colon H^2(M) \to L^2(M)$ (here H^2 is the Sobolev space) is an unbounded operator, but has a discrete set of nonnegative real eigenvalues,[27] and there's an orthonormal basis of $L^2(M)$ of eigenfunctions of the Laplacian.

The basic question of spectral geometry is: what can the eigenvalues of Δ_M tell us about the geometry of M? This was raised in an article of V. Kac called "Can one hear the shape of a drum?" What he meant by the title is the following: let $\Omega \subset \mathbb{R}^2$ be a compact domain with smooth boundary. We can consider the Laplacian on this domain, along with some boundary conditions to ensure well-posedness. We think of Ω as a drumskin, and functions $f\colon \Omega \to \mathbb{R}$ as height functions. We'd like f to satisfy the *wave equation*:

$$f_{tt} + \Delta f = 0 \tag{28.1}$$

and such that $f_{\partial\Omega} = 0$.

Now suppose $\Delta\varphi = \lambda\varphi$, and define the *standing wave*

$$f(t,x) := \Big(a\cos(\sqrt{\lambda}t) + b\sin(\sqrt{\lambda}t)\Big)\varphi(x),$$

which you can check satisfies (28.1).

Physically, $f(t,x)$ specifies how the drum head vibrates, and the eigenvalues we get dictate the frequency of the sound it makes. So, recovering Ω up to isometry from its spectrum would be determining it from "the sounds it makes."

Counterexamples were found by Milnor (and easier ones by Sunada and Carolyn Gordon): there are drums with the same spectra. It's known that the space of Riemannian manifolds with a given spectrum is compact in the Gromov-Hausdorff metric, but it's open whether it's finite in general.

Though Kac's question was answered, this is still an active area of research. For example, suppose $\lambda_0 \leq \lambda_1 \leq \cdots$ is the set of eigenvalues, with multiplicity, of the Laplacian, and let $\{\varphi_i\}$ be an eigenbasis for $L^2(M)$. If appropriately normalized, $|\varphi_i|^2\,\mathrm{dvol}$ is a probability measure on M. If we think of quantum mechanics for a particle on M, φ_i are states. People would like to understand these measures better. It's known that if the geodesic flow on (the unit tangent bundle of) M is ergodic,[28] there's a sense in which these probability measures converge to the volume form on M, so $|\varphi_i|^2 \to 1$ weakly (i.e. when integrated against an arbitrary test function).

This is not technically true as stated. Here's the correct formulation.

Theorem 28.2 (Quantum ergodicity)**.** *If the geodesic flow on (the unit tangent bundle of) M is ergodic, then there's a subsequence $\{i_k\} \subset \mathbb{N}$ of density 1 such that $|\varphi_{i_k}|$ converges weakly to 1.*

Here, *density* 1 means that

$$\lim_{k\to\infty} \frac{|\{i_1,\dots,i_k\} \cap \{1,\dots,i_k\}|}{i_k} = 1.$$

If $\{i_k\} = \mathbb{N}$ (so we don't have to pass to a subsequence), then M is said to satisfy *quantum unique ergodicity* (QUE).

Conjecture 28.3. *A closed surface of constant negative curvature satisfies quantum unique ergodicity.*

[27] Compactness is crucial here; if M is noncompact, the spectrum of its Laplacian may have a continuous part. For example, on $M = \mathbb{R}^n$ with the standard metric, choose any $\mathbf{v}$ and let $f(\mathbf{x}) := \exp(i\mathbf{v}\cdot\mathbf{x})$. Then, $\Delta f = \|\mathbf{v}\|^2 f$, so $\|\mathbf{v}\|$ is an eigenvalue and $\mathrm{Spec}(\Delta) = [0,\infty)$. However, $(\Delta - \|\mathbf{v}\|^2 I)^{-1}$ is an unbounded operator.

[28] This means it's not possible to decompose the space into measurable subspaces, both of positive measure, such that both are fixed under geodesic flow. This is a dynamical kind of irreducibility.

We'll spend the next two days proving some of the ingredients in spectral geometry: defining the Laplacian on a manifold and proving that $L^2(M)$ admits an orthonormal basis of eigenfunctions. These results are called Hodge theory. We'll also discuss Cheeger's inequality, an isoperimetric inequality

$$\lambda_1 \geq \frac{h(M)^2}{4},$$

where λ_1 is the smallest eigenvalue of Δ_M, and we'll explain $h(M)$ in due time. Proofs will usually go through the heat kernel, which we'll discuss, and then turn to Weyl's theorem that

$$|\{\lambda_i \mid i \leq N\}| \sim \frac{|B_n| \operatorname{vol}(M) N^{n/2}}{(2\pi)^n}.$$

Here, $n := \dim(M)$ and $|B_n|$ is the volume of the unit n-ball. Finally, we'll discuss a canonical isomorphism from $\ker(\Delta^k)$ to the k^{th} cohomology (de Rham or singular with $\mathbb{R}$-coefficients). Thus the cohomology is determined by harmonic forms.[29] A readable reference for this is "The Laplacian on a Riemannian manifold" by Rosenberg.

Example 28.4. Consider the circle $S^1 = [-\pi, \pi)$ so $\Delta = -\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}$. Thus

$$\Delta e^{in\theta} = n^2 e^{in\theta},$$

so $\operatorname{Spec} \Delta = \{1, 4, 9, 16, \dots\}$, and $\{e^{in\theta}\}$ forms an orthonormal basis for $L^2(S^1)$, which we already knew by Fourier theory. ◀

Definition 28.5. On a manifold M, the *Laplacian* is the divergence of the gradient: $\Delta := \operatorname{div} \nabla$.

Of course, this means we've reduced to another thing to define. Suppose $\{\partial_i\}$ is a basis for T_pM and X is a vector field on M. Then, the *divergence* of X is

$$\operatorname{div} X := \sum_i \nabla_{\partial_i} X.$$

To check invariance under choices, let $F_X \colon T_p(M) \to T_p(M)$ be the operator such that $F_X(\xi) = \nabla_\xi X$. Then, $\operatorname{div}(X) = \operatorname{tr}(F_X)$, so the divergence is invariant under change of coordinates.

Lemma 28.6. *Let $f \in C_c^\infty(M)$ and X be a vector field on M. Then, the divergence is the formal adjoint of the gradient, i.e.*

$$\langle \nabla f, X \rangle = \langle f, -\operatorname{div} X \rangle.$$

We can use this to derive a formula for the Laplacian:

$$\begin{aligned}
\langle \nabla f, X \rangle &= \int_M g^{ij} (\partial_j f) X^k g_{ik} \, \mathrm{d}V \\
&= \int_M (\partial_j f) X^j \, \mathrm{d}V \\
&= \int_M (\partial_j f) X^j \sqrt{g} \, \mathrm{d}x_1 \cdots \mathrm{d}x_n \\
&= -\int f \partial_j (X^j \sqrt{g}) \, \mathrm{d}x_1 \cdots \mathrm{d}x_n \\
&= \langle f, -\partial_j (X^j \sqrt{g}) g^{-1/2} \rangle,
\end{aligned}$$

so the Laplacian is

$$\Delta f = -\partial_j \Big(\sqrt{g} (g^{ij} \partial_j f) \Big) g^{-1/2}.$$

There's a sense in which this is $-g^{ij} \partial_j \partial_i f$ plus lower-order terms.

Lemma 28.7. *The Laplacian is self-adjoint.*

[29] Δ^k is not a k-fold iteration of the Laplacian; rather, it's an extension of the Laplacian to differential k-forms. This is not hard, but also not formal.

Proof. This is formal:

$$\begin{aligned}\langle \Delta f, g\rangle &= \langle -\operatorname{div}\nabla f, g\rangle \\ &= \langle \nabla f, \nabla g\rangle \\ &= \langle f, -\operatorname{div}\nabla g\rangle = \langle f, \nabla g\rangle.\end{aligned}$$ ⊠

To go further we'll need to discuss Sobolev spaces.

Definition 28.8. Let $s \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^n$ be a domain, and $f \in C_c^\infty(\Omega)$. Then, the *Sobolev s-norm* is

$$\|f\|_s := \left(\sum_{|\mathbf{ff}|\le s} \|D^{\mathbf{ff}} f\|_2^2\right)^{1/2}.$$

Here, $\mathbf{ff} = (\alpha_1, \dots, \alpha_n)$ is a multi-index, and

$$D^{\mathbf{ff}} := \frac{\partial^{\alpha_1}}{\partial {x_1}^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial {x_n}^{\alpha_n}}.$$

The Hilbert-space completion of $C^\infty(\Omega)$ under the s-norm, called the *Sobolev space* of s derivatives, is denoted $H_s(\Omega)$.

Notice that $H_0(\Omega) = L^2(\Omega)$ as Hilbert spaces.
The Fourier transform

$$\widehat{f}(\xi) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} f(x)\,\mathrm{d}x$$

plays well with the Sobolev norms:

$$\|f\|_s = \left(\int_\Omega |\widehat{f}(\xi)|^2 (1+|\xi|^2)^s\,\mathrm{d}\xi\right)^{1/2}. \tag{28.9}$$

Using this, you can define $\|\cdot\|_s$ for all $s \in \mathbb{R}$ to satisfy (28.9).

Proposition 28.10.

(1) *If $s > r$, there's a comtinuous embedding $H^s(\Omega) \subseteq H^r(\Omega)$.*
(2) *If $s < k - n/2$, then $H^k(\Omega) \subset C^s(\overline{\Omega})$.*
(3) *For $t > s$, if $\overline{\Omega}$ is compact, the inclusion $H^t(\Omega) \subseteq H^s(\Omega)$ is compact.*

Now, the first major black box.

Theorem 28.11. *There exists a smooth map $e\colon R^{\ge 0} \times M \times M \to \mathbb{R}$ called the* heat kernel *satisfying*

(1) *$(\partial_t + \Delta_x) e(t, x, y) = 0$.*
(2) *For any $f \in C^\infty(M)$,*

$$\lim_{t\to 0}\int_M e(t,x,y) f(y)\,\mathrm{d}V_y = f(x)$$

The *heat propagation* $e^{-t\Delta}\colon C^\infty(M) \to C^\infty(M)$ is defined to be

$$(e^{-t\Delta} f)(x) := \int_M e(t,x,y) f(y)\,\mathrm{d}V_y.$$

This is a self-adjoint operator satisfying a *semigroup property*, i.e. $e^{-t\Delta} \circ e^{-s\Delta} = e^{-(t+s)}\Delta$. Moreover, $e(t,x,y) = e(t,y,x)$.

Now we can state and work towards the main theorem:

Theorem 28.12. *Let M be a closed Riemannian manifold. Then, there exists an orthonormal basis for $L^2(M)$ consisting of eigenfunctions of Δ.*

Lemma 28.13. *$e^{-t\Delta}\colon L^2(M) \to L^2(M)$ is compact.*

Proof sketch. For any $f \in L^2(M)$, $e^{-t\Delta} f \in C^\infty(M)$, and hence it's contained in $H^s(M)$ for all s. As a map $L^1(M) \to H^1(M)$, you can verify that it's bounded, and then forgetting back to $L^2(M)$ is compact by Proposition 28.10. ⊠

Now we use a spectral theorem.

Theorem 28.14 (Spectral theorem for compact self-adjoint operators)**.** *Let $\Phi\colon \mathscr{H} \to \mathscr{H}$ be a compact, self-adjoint operator on a Hilbert space $\mathscr{H}$. Then,*

(1) *$\mathscr{H}$ admits an orthonormal basis of eigenvectors for Φ,*
(2) *the eigenspaces of Φ are finite-dimensional,*
(3) *and the only possible accumulation point for the eigenvalues of Φ is at 0.*

The semigroup property means that $e^{-t\Delta}$ and $e^{-s\Delta}$ commute, and hence are simultaneously diagonalizable. If $\gamma_i(t)$ is the i^{th} eigenvalue of $e^{-t\Delta}$, then $\gamma_i(t)\gamma_i(s) = \gamma_i(t+s)$, and therefore $\gamma_i(t) = e^{-\lambda_i t}$, where λ_i is the i^{th} eigenvalue of Δ.

Lecture 29.

: 5/4/17

Lewis Bowen spoke again today.

Once again, let M be a closed Riemannian manifold. We defined the Laplacian $\Delta f = -\operatorname{div} \nabla f$ last time; it's also describable as $\mathrm{d}^*\mathrm{d}f$. We then stated Theorem 28.11, that there's an operator $e\colon \mathbb{R}^{\geq 0} \times M \times M \to |R$ which has nice properties. We used this to define the heat propagation operator

$$e^{-t\Delta} f(x) := \int_M e(t,x,y) f(y)\, \mathrm{d}y,$$

for $f \in L^2(M)$. As $t \to 0$, this converges to f pointwise almost everywhere. We also saw that this is a one-parameter family, as $e^{-(t+s)\Delta} = e^{-t\Delta} \circ e^{-s\Delta}$. In particular, they commute.

Let ω_i denote the i^{th} eigenvector of $e^{-t\Delta}$, with eigenvalue $\gamma_i(t)$. Then, $\gamma_i(t+s) = \gamma_i(t)\gamma_i(s)$, so therefore there's a $\lambda_i > 0$ such that $\gamma_i(t) = e^{-\lambda_i t}$. In particular, as

$$(\partial_t + \Delta)^{-\lambda_i t} \omega_i(x) = (\partial_t + \Delta) e^{-\Delta t} = 0,$$

then

$$-\lambda_i e^{-\lambda_i t} \omega_i(x) + e^{-\lambda_i t}(\Delta \omega_i(x)) = 0.$$

Hence $\Delta \omega_i = \lambda_i \omega_i$, so we've found the eigenvalues (and some eigenvectors) of the Laplacian. This suggests that you can write

$$e^{-t\Delta} = \sum_{n=0}^{\infty} \frac{t^n \Delta^n}{n!},$$

but there are nontrivial convergence issues.

Last time, we also discussed how Δ is a compact, self-adjoint operator, so $L^2(M)$ has an orthonormal basis of eigenfunctions for Δ. Moreover, $\operatorname{Spec}(\Delta)$, the set of eigenvalues with multiplicity, has only ∞ for an accumulation point, and each eigenvalue has finite multiplicity.

On a Riemannian manifold, we can define volumes of manifolds and submanifolds. We'd like to know how the "shape" of the manifold, and one way is through the Cheeger constant.

Definition 29.1. The *Cheeger constant* of M is

$$h(M) := \inf\left\{ \frac{\operatorname{Area}(\partial M')}{\operatorname{Vol}(M')} \mid M' \subseteq M, \operatorname{Vol}(M') \leq \frac{\operatorname{Vol}(M)}{2} \right\}.$$

A low Cheeger constant means that the manifold has a bottleneck. This has consequences in ergodic theory: it means that the manifold is in two parts that don't interact much, so a system will converge more slowly than one on a manifold with a higher Cheeger constant.

Theorem 29.2 (Cheeger's inequality)**.** *$h(M)^2/4 \leq \lambda_1$.*

There's also an upper bound, so that $\lambda_1 = \Theta(h(M)^2)$, due to Buser. So the smallest nonzero eigenvalue of the Laplacian controls the Cheeger constant.[30]

[30] In theoretical computer science, people also consider the first eigenvalue of the *graph Laplacian* on a finite graph, which controls several graph-theoretic properties in a similar way.

Lemma 29.3 (Co-area formula)**.** *Let $f \in C^\infty(M)$. Then,*

$$\int_M \|\nabla f\| \, \mathrm{d}V = \int_{-\infty}^{\infty} \mathrm{Area}(f^{-1}(t)) \, \mathrm{d}t.$$

By Sard's theorem, a generic $t \in \mathbb{R}$ is a regular value, so $f^{-1}(t)$ is a submanifold for almost all t, and therefore the integral on the right-hand side makes sense.

Proof. Let $R_f \subseteq \mathbb{R}$ be the set of regular values of f, which is an open dense subset of $\mathrm{Im}(f)$. Suppose $a < b < c$ and $(a, c) \subseteq R_f$. Then, there's a diffeomorphism $\Phi \colon (a, c) \times f^{-1}(b) \to f^{-1}(a, c)$.

Let $X = \nabla f / \|\nabla f\|^2$ and $\Psi(x, t)$ denote the image of X under the flow generated by x for time $t - b$. Then

$$\left| \frac{\partial \Psi}{\partial t}(t, x) \right| = \|\nabla f\|^{-1}.$$

The idea is that this flows from one fiber of f to another, and the amount of change in volume infinitesimally is

$$\mathrm{d}V_M(x, t) = \|\nabla f\|^{-1} \mathrm{d}A.$$

Thus, when we integrate, we get

$$\int_M \|\nabla f\| \, \mathrm{d}V = \int_{-\infty}^{\infty} \mathrm{Area}(f^{-1}(t)) \, \mathrm{d}t. \qquad \boxtimes$$

With this in hand, we can prove Cheeger's inequality.

Proof of Theorem 29.2. Let φ be an eigenfunction for Δ with eigenvalue λ_1. Since constant functions are in the kernel of Δ, then φ is orthogonal to constant functions.

Assume that 0 is a regular value for φ, so $\varphi^{-1}(0)$ splits M into two manifolds-with-boundary, $M_- := \varphi^{-1}(-\infty, 0)$ and $M_+ := \varphi^{-1}(0, \infty)$.

First, where $M_\pm$ is either of M_+ and M_-,

$$\begin{aligned} \lambda \|\varphi\|^2_{L^2(M_\pm)} &= \int_{M_\pm} \Delta\varphi \cdot \varphi \, \mathrm{d}V \\ &= \int_{M_\pm} (\nabla\varphi)^2 \, \mathrm{d}V \\ 0 &= \|\nabla\varphi\|^2_{L^2(M_\pm)} \end{aligned}$$

on each half of the manifold, using the fact that $-\operatorname{div}$ is the adjoint of ∇. The boundary term goes away because $\varphi(M_\pm) = 0$.

Now, we use the Cauchy-Schwarz inequality: $\nabla(\varphi^2) = 2\varphi\nabla\varphi$, and therefore

$$\|\nabla(\varphi)^2\|_{L^1(M_\pm)} \le 2^2 \|\nabla\varphi\|^2_{L^2(M_\pm)} \|\varphi\|^2_{L^2(M_i)},$$

and therefore

$$\lambda \|\varphi\|_{L^2(M_\pm)} 62 \ge \frac{1}{4} \|\nabla(\varphi)^2\|^2_{L^1(M_\pm)} \|\varphi\|^{-2}_{L^2(M_\pm)}. \tag{29.4}$$

Now, we'll use Lemma 29.3. Let $A_\pm(t)$ denote the area of $M_\pm \cap \varphi^{-1}(t)$, which makes sense almost everywhere. Then

$$\|\nabla(\varphi^2)\|_{L^1(M_\pm)} = \int_0^\infty A_\pm(t) \, \mathrm{d}t.$$

Without loss of generality, assume $\mathrm{Vol}(M_+) \le \mathrm{Vol}(M_-)$. Let

$$V_+(t) := \mathrm{Vol}\{x \in M_+ \mid \varphi^2(x) \ge t\}.$$

Then,

$$\int_0^\infty A_+(t) \, \mathrm{d}t \ge \int_0^\infty h(M) V_+(t) \, \mathrm{d}t,$$

and

$$\int_0^\infty V_1(t) = \int_0^\infty \int \mathbf{1}_{x \in M_+ \mid \varphi^2(x) \ge t} \, \mathrm{d}V_x \, \mathrm{d}t.$$

By Fubini's theorem,

$$\begin{aligned} &= \int_{M_+} \varphi^2(x)\,\mathrm{d}V \\ &= \|\varphi\|^2_{L^2(M_+)}. \end{aligned}$$

Substituting this into (29.4), we get what we were looking for. ☒

Remark 29.5. Nowhere did we use the fact that λ_1 is the smallest nonzero eigenvalue. In fact, the result holds for all nonzero eigenvalues of Δ, but the bound is strongest for λ_1. ◂

Theorem 29.6 (Weyl's law)**.** *Let $N(\lambda)$ denote the number of eigenvalues less than x. Then,*

$$N(\lambda) \sim \frac{|B_n|\operatorname{Vol}(M)\lambda^{n/2}}{(2\pi)^n},$$

where $|B_n|$ is the volume of the unit ball in $\mathbb{R}^n$.

The proof rests on some technical facts about the heat kernel which would be too much of a digression to prove. For example, the heat kernel is some analogue of a matrix; the next lemma is about its trace.

Lemma 29.7. *Let $\{\gamma_j\}$ be an orthonormal basis of $L^2(M)$ consisting of eigenfunctions for the Laplacian. Then*

$$e(t,x,y) = \sum_{n=0}^{\infty} e^{-\lambda_n t}\gamma_n(x)\gamma_n(y),$$

and

$$\sum_{n=0}^{\infty} e^{-\lambda_n t} = \int_M e(t,x,x)\,\mathrm{d}V.$$

Proof. Since these γ_i form an orthonormal basis, then there are functions in t $a_{n,m}(t)$ such that

$$e(t,x,y) = \sum_{m,n} a_{n,m}(t)\gamma_n(x)\gamma_m(y).$$

Thus

$$e^{-\lambda_m t}\gamma_m(x) = \int_M e(t,x,y)\gamma_m(y)\,\mathrm{d}V = \sum_n a_{n,m}(t)\gamma_n(x).$$ ☒

The next lemma is harder; we will not prove it.

Lemma 29.8 (Short-time asymptotics)**.** *On $\mathbb{R}^n$, the heat kernel is*

$$e(t,x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-|x-y|^2/4t}.$$

If $x, y \in M$ are close and $t \approx 0$, then

$$e(t,x,y) \approx \frac{1}{(4\pi t)^{n/2}} e^{-d(x-y)^2/4t}.$$

More precisely, there exists a $\delta > 0$ and smooth functions u_j, $j \in \mathbb{N}$ (with $u_0(x,x) = 1$ for all x) such that for all $k > 0$ and all $x, y \in M$ with $d(x,y) < \delta$,

$$e(t,x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-d(x,y)^2/4t}\left(\sum_{j=0}^{k} t^j u_j(x,y) + O(t^{k+1})\right).$$

Corollary 29.9.

$$\sum_{j=0}^{\infty} e^{-\lambda_j t} = \frac{\operatorname{Vol}(M)}{(4\pi t)^{n/2}} + O(t^{1-n/2}).$$

To prove Weyl's law, we'll need one more theorem.

Theorem 29.10 (Karamata)**.** *Let μ be any positive measure on $[0, \infty)$ and $\alpha > 0$. Then,*

$$\lim_{t \to 0} t^\alpha \int_0^\infty e^{-tx}\, \mathrm{d}\mu = \Gamma(\alpha + 1) \lim_{\lambda \to \infty} \int_0^\lambda \mathrm{d}\mu,$$

assuming the limit on the left-hand side exists.

Proof of Theorem 29.6. We'll apply Theorem 29.10 where $\mu(E)$ is the number of eigenvalues of Δ in E and $\alpha = n/2$ (here $n = \dim M$). Thus

$$\lim_{t \to 0} t^{n/2} \int_0^\infty e^{-tx} \mathrm{d}\mu = \Gamma\left(\frac{n}{2} + 1\right) \lim_{\lambda \to \infty} \lambda^{-n/2} N(\lambda).$$

Rearranging, this means that

$$N(\lambda) \sim \frac{\lambda^{n/2}}{\Gamma(n/2+1)} \lim_{t \to 0} t^{n/2} \int_0^\infty e^{-tx}\, \mathrm{d}\mu.$$

Since $|B_n| = \pi^{n/2}/\Gamma(n/2+1)$, we recover the original formula. ⊠

The last thing we'll do is upgrade the Laplacian from functions to differential k-forms. One important ingredient is the *Hodge star*, an isomorphism $\star\colon \Lambda^k T^*M \to \Lambda^{n-k} T^*M$ which requires the Riemannian metric to define. This allows us to define a pairing

$$\langle \omega, \eta \rangle := \star(\omega \wedge \star\eta),$$

which produces a 0-form, i.e. a function. We can then take the formal adjoint of the de Rham differential d under this pairing; call it δ.

Definition 29.11. The *Laplacian* on k-forms is

$$\Delta(\omega) := \delta \mathrm{d}\omega + \mathrm{d}\delta\omega.$$

Theorem 29.12. $\Gamma(\Lambda^k T^*M) = \ker(\Delta^k) \oplus \mathrm{Im}(\mathrm{d}) \oplus \mathrm{Im}(\delta)$.

To prove this, we'll need a third black box related to the heat equation.

Theorem 29.13. *There is a smooth section $e(t, x, y) \in \Lambda^k T_x^* M \otimes T_y^* M$ for $t > 0$ such that*

1. $(\partial_t + \Delta_x) e(t, x, y) = 0$.
2. *For all k-forms ω,*

$$\lim_{t \to 0^+} \int_M \langle e(t, x, y), \omega(y) \rangle\, \mathrm{d}V = \omega(x).$$

A lot of similar theorems go through, e.g. the kernel of Δ^k consists of smooth k-forms, and there's an orthonormal absis for the L^2 sections of $\Lambda^k T^*M$ consisting of eigenfunctions of something.

Lemma 29.14. $\ker(\Delta^k) = \ker(\delta) \cap \ker(\mathrm{d})$.

The proof is a computation.