M390C NOTES: GEOMETRIC LANGLANDS

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These notes were taken in UT Austin's M390C (Geometric Langlands) class in Spring 2021, taught by David Ben-Zvi. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Any mistakes in the notes are my own. Thanks to Tom Gannon, Charlie Reid, Saad Slaoui, and Jackson Van Dyke for finding and fixing a few errors.

Contents

Lecture 1.

Overview and a perspective on modular forms: 1/19/21

This is a class on the geometric Langlands program from a particular perspective, incorporating its relationship to electric-magnetic duality. The class is over Zoom.

The geometric Langlands program lies halfway between number theory and physics. Maybe we are Odysseus and trying to navigate back home, between the two perils of Charybdis (physics, classically the big whirlpool) and Scylla (number theory, classically the monster). You can probably extend the analogy further, e.g. derived algebraic geometry is the Calypso islands. Extending the analogy further is left as an exercise.

There isn't any particular recommended background for this class — in particular, you don't need to know physics or number theory. Mackey [Mac80] wrote a nice overview of a perspective on the relationship between symmetry and harmonic analysis which could be fun to read. In this and the next lecture, we'll talk about modular forms and some relationships to physics; after that, we will begin the course properly: in a sense, the geometric Langlands program is a vast generalization of the Fourier transform, so we will begin with the Fourier transform, in a way that will be helpful when we do generalize.

Modular and automorphic forms, and physics We're not going to be super technical about number theory. The idea of modular and automorphic forms is to do a kind of harmonic analysis or quantum mechanics on *arithmetic locally symmetric spaces.* As an example, the upper half-plane $\mathbb{H}$ has a model as $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2$. The *modular group* $\Gamma \coloneqq \mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}$, with a fundamental domain $\Gamma\backslash\mathbb{H}$ (TODO: picture): the fundamental domain is noncompact, and goes off to infinity along the y-axis, and there are a couple of orbifold points, where the Γ-action has stabilizer.

More generally, we might consider a Lie group G with maximal compact $K \subset G$ and a lattice $\Gamma \subset G$. Then we consider the space $\Gamma\backslash G/K$ and study the space of functions on it. You might imagine a particle moving on this locally symmetric space, so we're interested in $L^2(\Gamma\backslash G/K)$, with a Laplacian Δ acting on this, and we can decompose the space of functions in terms of subspaces of eigenfunctions. This is one way in which modular forms can arise.

There are myriad variants of this. You can yeet K out of the story and study $L^2(\Gamma\backslash G)$ with its K-action. (TODO: something about the unit tangent bundle.) Plus, there's no need to restrict ourselves to linearizing with functions: you can use forms or sections of other vector bundles, such as $\Gamma(\Gamma\backslash\mathbb{H}, \omega^{\otimes k/2})$. De Rham says this is related to the cohomology of $\Gamma\backslash\mathbb{H}$, possibly with twisted coefficients. All of these variants are examples of things related to modular forms.

Now maybe you're thinking that if you pass to cohomology, you're no longer doing quantum mechanics, but in fact this is the domain of something called *topological quantum mechanics*; for example, this is discussed by Witten in his paper [Wit82] on supersymmetric quantum mechanics and its relationship to Morse theory.

Automorphic forms follow a similar story, but G is a more general Lie group. For example, pick your favorite reductive algebraic group such as GL_n or Sp_n, let G be the real points of this group, Γ be the integral points of this group, and K be the maximal compact of G. There is a long history of studying spaces of functions on $\Gamma\backslash G/K$ via harmonic analysis, and thinking of it as quantum mechanics. For example, if we started with Sp_{2n}, we get $\mathrm{Sp}_{2n}(\mathbb{Z})\backslash\mathrm{Sp}_{2n}(\mathbb{R})/\mathrm{U}_n$.

But there's a *lot* more structure here than in a typical quantum-mechanical setup. You can see this already for modular forms ($G = \mathrm{SL}_2(\mathbb{R})$). Namely, there's an additional variable: we can generalize from $\mathbb{Z}$ to other rings of integers in number fields. That is, given the field $\mathbb{Q}$, we think of $\mathbb{Z}$ as $\mathcal{O}_\mathbb{Q}$, the ring of integers of this number field, and obtain the group $\mathrm{SL}_2(\mathcal{O}_\mathbb{Q})$. Now we can replace $\mathbb{Q}$ with any finite extension $F/\mathbb{Q}$ and let $\mathcal{O}_F$ be the ring of integers of F, and consider a new lattice $\mathrm{SL}_2(\mathcal{O}_F)$. To make this completely precise, one has to fiddle with $\mathrm{SL}_2(\mathbb{R})$, since F may have more than one place at infinity, but this is the kind of technical detail we'll avoid for now.

And there is another way to vary the data: fix F, say $F = \mathbb{Q}$. Then we can vary the *conductor* or the *ramification data*. That is, the fundamental domain of Γ on $\mathbb{H}$ has a lot of covering spaces $\Gamma'\backslash\mathbb{H}$, where $\Gamma' \subset \mathrm{SL}_2(\mathbb{Z})$ is a *congruence subgroup*. One example of a congruence subgroup is, given $N \in \mathbb{Z}$, the subgroup

$$\Gamma(N) \coloneqq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \mathrm{id} \bmod N \right\}. \tag{1.1}$$

A variant is $\Gamma_0(N)$, the subgroup of matrices which are upper triangular mod N.

As is common in number theory, we can look at different places for primes in $\mathcal{O}_F$. For example, with $F = \mathbb{Q}$, this means looking at the local data at a prime p, which involves looking at $\mathrm{SL}_2(\mathbb{Q}_p)$. So the Hilbert space that we wanted to produce in the end depends on all this data: G and Γ and K, but also possibly F and the congruence subgroup and the prime.

Anyways, we'll get a Hilbert space and can study the spectral theory of the Laplacian. Maybe surprisingly, the eigenspaces are usually not one-dimensional. This "degeneracy" is because of *Hecke operators*, which are a crucial part of this story. At a high level, the Laplacian fits into a large family of commuting operators, and if p is a prime not dividing N, this family has a member T_p called the *Hecke operator*, giving an action of $\mathbb{C}[T_p]$ on $L^2(\Gamma\backslash G/K)$. And these all commute, so the tensor product of all of these $\mathbb{C}[T_p]$ over all primes acts on the Hilbert space, preserving the eigenspaces.

From the quantum mechanics perspective, this amount of commuting operators is unusual. You can think of this as an *integrable system*, with lots of conserved quantities. Usually (TODO: if I understood correctly), integrable systems are the opposite of chaos, but these arithmetic systems are studied as good examples of quantum chaos! This is a feature of the arithmetic story, and "arithmetic quantum chaos" behaves a lot more like quantum integrable systems than one might expect.

In this system, there is a special collection of measurements/states for a modular (or automorphic) form, called *periods*. One basic example is, given a modular function f on the fundamental domain $\Gamma/\backslash H$, integrate it:

$$\int_{i\mathbb{R}_+} f. \tag{1.2}$$

We will study modular functions/forms with these invariants. Hecke used L-functions to produce examples of these invariants.

Definition 1.3. A *Maass form* is an eigenfunction for the Laplacian on $L^2(\Gamma\backslash G/K)$. Specifically, *modular forms* are the holomorphic sections of $\omega^{\otimes k/2}$.

Modular forms can also arise by looking at the (twisted) cohomology of $\Gamma\backslash\mathbb{H}$; this is what's called *Eichler-Shimura theory.*

Our emphasis in this class will be more about topological quantum mechanics, rather than quantum mechanics; we care mostly about ground states. Modular forms are sort of like ground states here.

And there's one more piece of essential structure, to add to our already large pile of structures. There are these mysterious operators that allow you to vary the group! That is, these Hilbert spaces for different groups talk to each other! This is called *Langlands functoriality.* Part of the goal of this class is to explain this structure within physics.

But what the Langlands program itself does is to take these automorphic forms and spectrally decompose them in a prism (TODO: picture of the prism from Dark Side of the Moon, or maybe because this has something to do with physics, Dark Side of the Muon?). Automorphic forms enter on the left, and the prism spectrally decomposes them under the Hecke algebra (the algebra of all these commuting Hecke operators). And the different "colors" (eigenvalues) are given by representations of Galois groups of number fields, which is a surprising and magical statement. Moreover, there is a duality: these Galois representations are not into the complex points of G, but instead into $G^\vee_{\mathbb{C}}$, where $G^\vee$ is a dual group under something called Langlands duality.

For example, if $G = \mathrm{PSL}_2(\mathbb{R})$, then $G^\vee = \mathrm{SL}_2(\mathbb{C})$. A relatively explicit way to see how this enters is that if E is an elliptic curve over $\mathbb{Q}$, then $H^1(E)$ is a two-dimensional vector space carrying a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ action, and this is part of how modular forms appear in the story. This is one of the "colors" (eigenvalues) in this theory. The representation morally has image in $\mathrm{SL}_2(\mathbb{R})$, though to see this idea precisely requires setting things up a little more carefully. Elliptic curves appear in two ways in this, as the moduli space of elliptic curves is an arithmetic locally symmetric space. This isn't necessary to see the high-level overview, but it's crucial for actually proving anything! It's useful to know that elliptic curves have covers which are automorphic curves, and this provides a bridge between the two sides of the Langlands program. This is useful, but only applies for GL_2 — in general, you don't have this bridge, and the two sides are very far apart. For example, $\mathrm{GL}_3(\mathbb{Z})\backslash\mathrm{GL}_3(\mathbb{R})/\mathrm{O}_3$ is not a moduli space of anything: it's not a manifold. And this makes your proofs much harder and the duality more mysterious: why should function theory on these spaces have anything to do with Galois representations?[1]

One of the major goals of this class is to show how the (geometric) Langlands program arises in physics, not in quantum mechanics, but in four-dimensional (topological) field theory. Rather than beginning with a quantum mechanics system, we replace it with something much richer and more complicated — and scary. The key adjective "topological" helps mollify this: we throw out dynamics and look at ground states of the Laplacian, like looking only at harmonic forms rather than everything. We will try to match the structure of the Langlands program with the structure of this TFT.

Why 4? Quantum mechanics seems canonical enough, but 4d physics seems less so. We introduce another adjective, *arithmetic quantum field theory*, following the paradigm of arithmetic topology. This is an idea making an analogy between number fields and geometric objects that arise in physics, often manifolds. With a robust enough analogy, you can envision constructions with manifolds as having meaning in the world of number fields. The basic tenets of this theory are outlines in Weil's Rosetta stone (TODO: cite), which establishes a dictionary between number fields, function fields, and Riemann surfaces.

- Given a number field $F/\mathbb{Q}$, we consider $\operatorname{Spec}(\mathcal{O}_F)$, which has points labeled by primes in $\mathcal{O}_F$.
- The analogy between number fields and functional fields is older and better understood. We replace F with a (smooth, projective) curve C over a finite field $\mathbb{F}_q$. The field of rational functions $\mathbb{F}_q(C)$ on C has a lot of structure reminiscent of F, and the ring of regular functions $\mathbb{F}_q[C]$ resembles $\mathcal{O}_F$ (e.g. they're both Dedekind domains). The points of C are like the primes in $\mathcal{O}_F$.
- But why stop at $\mathbb{F}_q$? Let Σ be a compact Riemann surface. Points of Σ are the analogues of primes, in Weil's dictionary, and one can try to make geometric analogues of number-theoretic questions. The field of meromorphic functions on Σ is analogous to F and $\mathbb{F}_q(C)$, and the analogue of the ring of integers is a little complicated — Σ has no nonconstant entire functions, so we have to remove some points, analogues of points at infinity in the number field setting.

The crucial change in the arithmetic topology analogy is to replace Riemann surfaces with 3-manifolds. The reason behind this surprising change is that Riemann surfaces has strong similarities to curves over algebraically closed fields of positive characteristic. When you study a point $\operatorname{Spec}\mathbb{F}_q \hookrightarrow C$, you should remember the internal structure given by the Galois group action. Étale topology tells us to think of $\operatorname{Spec}(\mathbb{F}_q)$ as sort of like a circle, because the étale fundamental group of $\operatorname{Spec}(\mathbb{F}_q)$ is $\operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \cong \widehat{\mathbb{Z}}$. The Frobenius is a topological generator of this fundamental group. So there's too much structure here to match with a Riemann surface. If you base-change to $\overline{\mathbb{F}}_q$, replacing these "circles" with their "universal covers," we obtain an extra line direction which topology/cohomology doesn't see, because it's contractible, but now we obtain something that feels a little more like a Riemann surface.

So from the point of view of Galois theory, function fields of curves over $\mathbb{F}_q$ feel less like Riemann surfaces and more like bundles of Riemann surfaces over S^1. This is equivalent data to a Riemann surface Σ and a diffeomorphism $\phi\colon \Sigma \to \Sigma$. We then build the bundle as

$$\Sigma \times [0,1]/((x,0) \sim (\phi(x),1)), \tag{1.4}$$

the construction called the *mapping torus*. On the function-field side, we think of the Frobenius as ϕ. There's a difference here, in that we don't have a canonical choice of ϕ on the Riemann surface side (the identity is boring so let's not use that one), so we think of ϕ as "generic."

And so we arrive at the arithmetic topology dictionary, built by many workers, including Mumford writing to Mazur, Mazur, Kapranov, Reznakov, Morijita, and Kim. This is also known as the "knots and primes" dictionary: number fields are analogues of 3-manifolds. This is not something totally born out of nowhere; it's a refinement of Weil's dictionary.

Just how not all number fields have unique Frobenii (Frobeniuses?), but rather different ones at different primes, our 3-manifolds Y are not just surface bundles over curves. Primes on the number field side now correspond to embedded circles in Y, i.e. knots. Local fields, such as $\mathbb{Q}_p$, look like 2-manifolds. There aren't a lot of 2-manifolds fibered over the circle, but that's okay. And there are many more aspects of the analogy,

[1]Another technical detail to not worry about: when we replace $\mathbb{C}$ with $\overline{\mathbb{Q}_\ell}$, which is necessary for making some of these things precise, one must use étale cohomology instead of singular/Zariski cohomology. But that's not crucial for the point of this lecture.

such as the relationship between Legendre symbols and linking numbers, and more. The nLab page on arithmetic topology has a great list.

This is not an incredibly precise dictionary, and don't make the mistake of trying to associate specific primes to specific knots. For example, if yous said $\mathbb{Q}$ is the sphere, then you'd discover the Poincaré conjecture is false in the number-field setting, which is unfortunate. Rather, let's imagine that number fields are a new class of examples of 3-manifolds, with some commonalities and some other properties, and function fields are another family. So we can then study our new, rich class of examples.

Returning to the question of why four-dimensional topological field theory, well, first we have to discuss exactly what a topological field theory is, but we will see that one of the basic invariants of such a creature is that to ever $(n-1)$-manifold, one obtains a vector space. So the Langlands program assigns vector spaces (or things related to it, such as graded vector spaces, or chain complexes) to function and number fields, which are 3-dimensional in our analogy, and therefore we expect a four-dimensional story in physics.

More generally, an n-dimensional quantum field theory has dynamics: you get in addition to your vector space on an $(n-1)$-manifold, a Hilbert space structure and a Hamiltonian. Again, you might have something like a chain complex instead of a vector space. But the Hamiltonian makes this more like a quantum mechanics problem on your codimension-1 manifold. In topological theories, the Hamiltonian is 0.

Now, back to locally symmetric spaces: if F is a number field, we think of the field theory as assigning to it the vector space $L^2(\Gamma_{\mathcal{O}_F}\backslash G/K)$. The space $\Gamma\backslash G/K$ does not directly appear; instead, it is a moduli space of solutions to certain relevant equations on a 3-manifold.

Other parts of the story carry over too. Turning on the conductor/ramification N, we have not just a 3-manifold, but also a knot or link inside it, which we think of as the locus along which singularities can appear. In physics, these are called "codimension-2 defects," an important piece of data in general QFT.

To recap: we went very quickly today, and will go quickly on Thursday, but the class will mostly go more slowly, starting next week, where we more carefully keep track of the structures on both sides of this story, trying to stay on the safe, geometric tightrope between these two paradigms.

Thursday we will dig into more of the physics analogues of the variables we can twiddle on the number-theoretic side: what happens if we vary the conductor, if we vary the number field F, if we play with functoriality, etc.

Lecture 2.

A tale of two TFTs: 1/21/21

Last time, we talked about a perspective on modular forms (or automorphic forms): pick your favorite reductive algebraic group or matrix group, such as GL_n or PSL_2,[2] and let $F/\mathbb{Q}$ be a number field. You can let $F = \mathbb{Q}$ if you want. Let $\mathcal{O}_F$ be the ring of integers of F; if $F = \mathbb{Q}$, $\mathcal{O}_F = \mathbb{Z}$.

Today we will discuss what happens when we vary F, and how this affects a moduli space of principal bundles (TODO: missed this). We obtained a locally symmetric space by taking the real points of the group and taking the double quotient by an "arithmetic" lattice and the maximal compact. For example, we can take $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathrm{PSL}_2(\mathbb{R})/\mathrm{SO}_2$. This is for $S = \mathbb{Q}$.

Now let's consider more general F. We have $\mathrm{PSL}_2(\mathcal{O}_F)$ with no issue, but what should replace $\mathrm{PSL}_2(\mathbb{R})$ and its maximal compact? Instead we consider $F \otimes_{\mathbb{Q}} \mathbb{R}$, which is a ring of the form

$$F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}, \tag{2.1}$$

where F has r_1 embeddings into $\mathbb{R}$ and r_2 pairs of conjugate embeddings into $\mathbb{C}$. Then we replace $\mathrm{PSL}_2(\mathbb{R})$ with $\mathrm{PSL}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$.

Example 2.2. Suppose $F = \mathbb{Q}(\sqrt{d})$, where d is squarefree.

- If $d > 0$, so this is a real quadratic extension, $r_1 = 2$ and $r_2 = 0$.
- If $d < 0$, so this is an imaginary quadratic extension, $r_1 = 0$ and $r_2 = 1$. ◀

Then we can take the maximal compact K of $\mathrm{PSL}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$ as normal, and obtain a locally symmetric space. If F is a real quadratic field, this leads us to *Hilbert modular forms*, via $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$, acting on $\mathbb{H} \times \mathbb{H}$. In the imaginary quadratic case, we get $\mathrm{PSL}_2(\mathbb{C})/\mathrm{SO}_3 \cong \mathbb{H}^3$, hyperbolic 3-space, and there are

[2] We're not thinking specifically of these groups as over a specific field, such as $\mathrm{GL}_n(\mathbb{R})$, but a machine for assigning to a field k a group $\mathrm{GL}_n(k)$. This technicality is important because F varies today.

connections to hyperbolic geometry. Usually these double quotients are not algebraic varieties, as this example demonstrates.

For other F, we'll get products of $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{C})$; what's most interesting is the arithmetic lattice $\mathrm{PSL}_2(\mathcal{O}_F)$.

Once we have these arithmetic locally symmetric spaces $\mathcal{M}$, we want to produce vector spaces out of them, including $L^2(\mathcal{M})$, $H^*(\mathcal{M})$, and twisted versions thereof. Importantly for the geometric Langlands program, these vector spaces carry an action of a huge commutative algebra, which is a tensor product over all the primes in F of a polynomial ring in $\mathrm{rank}(G)$ generators.

One could also allow ramification, obtaining generalizations $\mathcal{M}_{G,F,N}$, where $N \in \mathrm{O}_F$, and you replace $\mathrm{PSL}_2(\mathcal{O}_F)$ with a congruence subgroup Γ_N in which we impose conditions on our matrices mod N. These are a few different conditions you might impose (e.g. identity mod N, or upper triangular mod N). The arithmetic locally symmetric space is $\Gamma_N \backslash G_{\mathbb{R}}/K$, and the large commutative algebra is now "only" a tensor product over the primes p not dividing N.

This kind of idea, of a vector space associated to a number field, or maybe a vector space associated to a number field and some primes, is reminiscent under the arithmetic topology analogy to the state spaces in a 4d topological field theory. As we discussed last time, this is a refinement of Weil's Rosetta stone, where $\operatorname{Spec}\mathbb{Z}$ feels like a curve with points $\operatorname{Spec}\mathbb{F}_p$ associated to primes p, and $\operatorname{Spec}\mathbb{Z}_p$ as a small disc around this point. Inside that there is the punctured disc $\operatorname{Spec}\mathbb{Q}_p$. This is analogous to having a smooth, reduced algebraic curve over a finite field $\mathbb{F}_q$, which locally looks like $\operatorname{Spec}\mathbb{F}_q[t]$. Here the points are $\operatorname{Spec}\mathbb{F}_q$, and around this is the disc $\operatorname{Spec}\mathbb{F}_q[[t]]$ with the punctured disc $\operatorname{Spec}\mathbb{F}_q((t))$.

Now we look at the étale topology of $\operatorname{Spec}\mathbb{Z}$, which is a fancy way to say we care about the cohomology of Galois groups. From this perspective, the Rosetta stone isn't quite rich enough: $\operatorname{Spec}\mathbb{F}_p \hookrightarrow \operatorname{Spec}\mathcal{O}_F$ is a point, but not étale-topologically: the étale topos tells us that this "point" has a whole bunch of interesting covering spaces, such as $\operatorname{Spec}(\mathbb{F}_{p^n}) \to \operatorname{Spec}(\mathbb{F}_p)$. Its étale fundamental group is $\widehat{\pi}_1^{\text{ét}}(\operatorname{Spec}\mathbb{F}_p) = \operatorname{Gal}(\mathbb{F}_p) \cong \widehat{\mathbb{Z}}$. You can think of this profiniteness as not really there: we can only see finite extensions or, said differently, finite covering spaces, and at this level there's no way to distinguish $\mathbb{Z}$ and $\widehat{\mathbb{Z}}$. This is a common feature of étale fundamental groups.

So the point is that the point $\operatorname{Spec}\mathbb{F}_p$ behaves a lot like a circle if you want to do étale things. (TODO: picture of $\operatorname{Spec}\mathbb{Z}$ with a circle at each prime). Therefore $\operatorname{Spec}\mathbb{Z}$, and its siblings $\operatorname{Spec}\mathcal{O}_F$, feel more like 3-manifolds than Riemann surfaces. And there are other ways to make this fuzzy analogy less fuzzy: there is a version of Poincaré duality, for example, with the correct dimension.

The monodromy around these circles is the Frobenius, but different Frobenii at different primes don't talk to each other. Because the curve $C/\mathbb{F}_q$ maps to $\operatorname{Spec}(\mathbb{F}_q)$, which is sort of like a circle, we think of these 3-manifolds as fibered over a circle.

Given this perspective, what is $\operatorname{Spec}(\overline{\mathbb{F}}_q)$? Étale-topologically, this is actually a point, but if you want a good dictionary between covering spaces and Galois representations, it should be a covering space of the circle, and in fact the universal one, $\mathbb{R}$. This is fine: for the purposes of topology and cohomology, $\mathbb{R}$ is a fine stand-in for a point. Now base-change C to $\overline{C} \coloneqq C_{\overline{\mathbb{F}}_q} \coloneqq C \times_{\operatorname{Spec}\mathbb{F}_q} \operatorname{Spec}\overline{\mathbb{F}}_q$; now we have something which feels like a bundle of Riemann surfaces, i.e. curves over algebraically closed fields, fibered over the real line $\operatorname{Spec}\overline{\mathbb{F}}_q$.

As we discussed last time, the monodromy around the circle is the Frobenius, so we can think of these surface bundle analoues as mapping tori for the Frobenius.

Let's discuss one more piece of evidence for this arithmetic topology dictionary: what happens with $\operatorname{Spec}\mathbb{Z}$? Let p be prime so we get a "circle" $\operatorname{Spec}\mathbb{F}_p$ in $\operatorname{Spec}\mathbb{Z}$. The neighborhood $\operatorname{Spec}\mathbb{Q}_p$ now behaves like a tubular neighborhood of this circle inside $\operatorname{Spec}\mathbb{Z}$. More generally, we can work with $\operatorname{Spec}\mathcal{O}_F$ and a prime $p \in F$ and a place v to complete $\mathcal{O}_F$ at, and obtain a local field F_v; then we might expect $\operatorname{Spec}(F_v)$ to be a tubular neighborhood of the circle $\operatorname{Spec} F/p$ — though (TODO) the place has to know something about p.

If F_v is a non-Archimedian local field, such as $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$, then $\operatorname{Gal}(\overline{F}_v/F_v)$ surjects onto $\mathbb{Z}_\ell \rtimes \widehat{\mathbb{Z}}$. Fun fact for those interested in group theory: this semidirect product is an example of a *Baumslag-Solitar group*

$$\mathrm{BS}(1,p) \coloneqq \langle \sigma, u \mid \sigma u \sigma = u^p \rangle. \tag{2.3}$$

Here σ is the Frobenius and u is a generator of $\mathbb{Z}_\ell$. This interpolates between $\mathbb{Z} \times \mathbb{Z} = \pi_1(T^2)$, for $p = 1$, and $p = -1$, which is π_1 of the Klein bottle. So this does feel sort of like a torus neighborhood of a knot in a

3-manifold. So primes look like circles, and local fields look like 2-manifolds — not just any 2-manifolds, but 2-manifolds fibered over the circle.

In general, the Galois group $\mathrm{Gal}(\overline{F}_v/F_v)$ can be assembled from three pieces: the Galois group of the residue field $\widehat{\mathbb{Z}}\cdot\sigma$, the *tame part*, which is a product of $\mathbb{Z}_\ell$s for $\ell \neq p$ (here p is the characteristic of the residue field), and the *wild part*, which is a p-group.

This analogy is nice and important, tying arithmetic and geometric Langlands together, but we will spend the most time in places where it is the most concrete. Let's summarize the analogy.

- The following objects are thought of as three-dimensional: number fields $\operatorname{Spec}\mathcal{O}_F$ and function fields of curves $C/\mathbb{F}_q$, and mapping tori of diffeomorphisms $\phi\colon\Sigma\to\Sigma$ of Riemann surfaces. The first two of these are related to the *global Langlands program*, and we refer to the "global arithmetic setting."
- Here are some two-dimensional objects: local fields $F_v/\mathbb{Q}_p$ and their spectra, which are lik punctured discs; and $\mathbb{F}_q((t))$, which is also sort of a punctured disc. This is the setting of the *local Langlands program* in the "local arithmetic setting". There are two more kinds of 2-dimensional objects: a curve over an algebraically closed field of positive characteristic $\overline{C}/\overline{\mathbb{F}}_q$, or a closed Riemann surface Σ. These latter two objects form the "global geometric setting."
- One-dimensional objects: $\operatorname{Spec}\mathbb{F}_q$ and $\operatorname{Spec}\mathbb{C}((t))$ both are analogues of circles. The latter is a punctured disc, so not exactly one-dimensional, but it's close enough to be useful; it is the "local geometric setting."
- And lastly, zero-dimensional objects: $\operatorname{Spec}\overline{\mathbb{F}}_q$ and $\operatorname{Spec}\mathbb{C}$.

One major theme in this class is to apply a 4d topological field theory to these objects. If you complain that there aren't any 4-manifolds, that's a good question, but we will only consider a few 4-manifolds, such as products of 3-manifolds with circles or more generally mapping tori.

The Langlands program is an equivalence of 4d "arithmetic topological field theories." Arithmetic TFTs are not an entirely well-defined object, but we have much of the data that such a definition would need. We pick a group G and get a dual group $G^\vee$; then the two arithmetic TFTs are called $\mathcal{A}_G$ and $\mathcal{B}_{G^\vee}$; $\mathcal{A}_G$ is called the *automorphic* or *magnetic side*, and $\mathcal{B}_{G^\vee}$ is called the *spectral* or *electric side*. There is a sense in which this is a 4-dimensional version of mirror symmetry (which is usually a story about 2d QFT). These two TFTs $\mathcal{A}_G$ and $\mathcal{B}+G^\vee$ are fully extended, in that they assign higher-categorical objects to lower-dimensional objects. That is, we will be able to assign things to two-, one-, and maybe zero-dimensional objects in the above dictionary: a two-manifold gets a category, a 1-manifold gets a 2-category, and if you're very ambitious, a 0-manifold gets a 3-category.

$\mathcal{A}_G$ is a machine for taking a 3-manifold M and attaching a vector space $\mathcal{A}_G(M)$, which we will see is built from functions on arithmetic locally symmetric spaces. For example, $\operatorname{Spec}\mathcal{O}_F$ gets some sort of functions on $\mathcal{M}_{G,F}$. This is a large amount of structure, and one advantage is that it will explain some of the weird properties of modular forms. We will also spend some time on the $\mathcal{B}$-side, which is easier to describe.

There's an interesting tradeoff involving dimension, category number, and difficulty: making sense of what these arithmetic TFTs assign to 4-manifolds is very difficult: there are infinities and difficult renormalizations to deal with, and analyis that is beyond the scope of the course. Vector spaces are nicer and easier to make, but 3-manifolds are difficult. Dimension 2 is the sweet spot: categories aren't that bad, and 2-manifolds are pretty tractable. By the time we get to 1-manifolds, we have to work the category theory harder, and understanding what happens in dimension 0 is almost entirely open. We will not solve this open question in this class.

Now a TFT has additional structure: you can take a manifold with some additional structure, called defects, and assign algebraic data to this too. These bells and whistles line up very nicely on the arithmetic and topological sides: for example, number theorists will tell you the importance of allowing ramification/congruence subgroups in defining your arithmetic locally symmetric spaces. Under the arithmetic topology dictionary, this corresponds to studying what your TFT assigns to a 3-manifold with an embedded link with a label. In physics language, the link is a *codimension 2 defect*. The additional data of the link gets the modified vector space built using the congruence subgroup.

The large commutative algebra acting on the space of functions on $\mathcal{M}$ contains elements called *Hecke operators*, and these correspond to codimension 3 defects, or *line operators*. Physics-wise, you can think of these as "creating magnetic monopoles." And periods of automorphic forms correspond to boundary conditions, which are a codimension 1 phenomenon. This is related to recent work and work in progress of the professor!

Finally, there is Langlands functoriality, which also fits into this picture: more than just boundaries, there are codimension 1 phenomena called *domain walls*, which you can think of as an interface between two regions on a manifold which have two different theories on them.

So to summarize this, all the bells and whistles in the theory of automorphic forms belong to this QFT story.

The $\mathcal{B}$-side. Here we've taken the theory of automorphic forms and passed it through a prism to decompose it into "colors" related to Galois representations. Number-theoretically, this side is very very hard, because Galois groups of number fields are complicated, and the $\mathcal{A}$-side is often used to gain information about the $\mathcal{B}$-side. Geometrically, fundamental groups of Riemann surfaces are much easier, so the $\mathcal{B}$-side is used to learn about the $\mathcal{A}$-side.

But at least the $\mathcal{B}$-side is easier to state: we study the algebraic geometry of spaces of Galois representations; the $\mathcal{A}$-side has to do with the topology of the arithmetic locally symmetric space $\mathcal{M}$, by contrast. Geometrically, we might fix a manifold M and consider the space $\mathrm{Loc}_n(M)$ of representations $\pi_1(M) \to \mathrm{GL}_n(\mathbb{C})$. These are nice objects, called *character varieties*, and you can study the algebra of functions on them. You also don't just have to restrict to $\mathrm{GL}_n(\mathbb{C})$: we in particular care about $\mathrm{Loc}_{G^\vee}(M) := \{\pi_1(M) \to G^\vee\}$. The vector space that $\mathcal{B}_{G^\vee}$ assigns to a 3-manifold is the vector space of functions on $\mathrm{Loc}_{G^\vee}(M)$, and in fact defining this theory as an extended TFT is considerably easier than for the $\mathcal{A}$-side. Back on the arithmetic side, $\pi_1(M)$ is analogous to a Galois group, so in the arithmetic setting, we are looking at varieties of Galois representations.

The conjectured equivalence of these two (arithmetic) TFTs is something amazing: the huge amount of structure on the $\mathcal{A}$-side is equivalent to the simpler-to-define $\mathcal{B}$-side, and all of the structure passes back and forth. But "conjectured" is a very big word here: in both the arithmetic and geometric settings, there's a lot left to do, and even to define, to make these analogies precise. In the geometric setting, more is known, but there's still plenty of work in progress, including work of the professor, Arinkin, Gaitsgory, Raskin, and many more. The number field story is the work of Lafforgue and many others, but not a lot of this is proven.

The dictionary is not just nice to look at: you can use work done in the geometric setting to learn about what you should be working towards in the arithmetic setting, for example.

Lecture 3.

Spectral theory and sheaf theory: 1/26/21

Today we begin going more slowly and deeply; in the next several lectures, we'll focus on the picture of the prism, in which automorphic forms enter in on the left and are spectrally decomposed into Galois representations. Before we get into spectral decomposition, though, what's a spectrum?

We start with geometry, which might mean different things precisely to different people, but in geometry there is some notion of spaces, and given a space there is a commutative ring of functions, with multiplication taken pointwise. This is the basic starting point for algebraic geometry: to study spaces, study their algebra of functions. In fact, we can think of taking the ring of functions as a contravariant functor $\mathcal{O}$ from spaces to commutative rings: given a map $X \to Y$ of spaces, we want a pullback map of functions which preserves pointwise addition and multiplication.

The fundamental idea of a spectrum is to go backwards: begin with commutative algebra and build a space out of it. Category theory clarifies precisely what we're trying to do: there should be a spectrum functor Spec from commutative algebra to spaces, which satisfies a universal property concisely summarized by asking for it to be a right adjoint of $\mathcal{O}$. Maybe to be completely precise, we need to say what classes of spaces and functions we care about, and there are different options, but in those different situations you have this basic question.

Explicitly, saying that Spec is a right adjoint to $\mathcal{O}$ says that for every commutative algebra R, the set of maps of spaces $X \to \operatorname{Spec} R$ is naturally isomorphic to $\mathrm{Hom}_{\mathcal{R}ing}(R, \mathcal{O}(X))$. You can produce a "weak solution" in the functor category $\mathcal{F}un(\mathcal{S}paces, \mathcal{S}et)$, where $\operatorname{Spec} R$ sends $X \mapsto \mathrm{Hom}_{\mathcal{R}ing}(R, \mathcal{O}(X))$, and we can ask whether this is a true solution, in that it's represented by an actual space. We also want $\mathcal{O}(\operatorname{Spec} R) = R$, where "=" means "naturally isomoprhic".

There are several different settings in which this works pretty well.

Example 3.1. Suppose "space" means "finite set," and "function" means k-valued functions, for your favorite field k. (My favorite field is $\mathbb{C}$. What's yours?) Now, $\mathcal{O}(X)$ is the ring of functions $\{X \to k\}$, i.e. $\prod_{x \in X} k$, or the algebra of diagonal $|X| \times |X|$ matrices. So in this setting, there are plenty of algebras, such as $k[x]/(x^2)$, which are not the k-algebras of functions on spaces. ◀

Example 3.2 (Gelfand)**.** By "spaces" we mean locally compact Hausdorff spaces and by "functions" we mean $\mathbb{C}$-valued continuous functions vanishing at infinity: $\mathcal{O}(X) := C_0(X)$. This has the structure of a *commutative C^*-algebra*, with the $*$-operation given by complex conjugation.[3]

In this setting, there is a nice Spec functor from commutative C^*-algebras to l.c. Hausdorff spaces, the *Gelfand spectrum*. Given such an algebra A, $\mathrm{mSpec}A$ is defined to be the space of maximal ideals of A. These are identified with the set $\mathrm{Hom}_{C^*}(A, \mathbb{C})$, and this can be profitably thought of as the "points" of A: $\mathbb{C}$ is the functions on a point, so this is heuristically the maps $\mathrm{pt} \to \mathrm{mSpec}A$. Alternatively, these are the unitary one-dimensional representations of A.

Theorem 3.3 (Gelfand-Naimark)**.** $(\mathcal{O}, \mathrm{mSpec})$ *are contravariant equivalences of categories from locally compact Hausdorff spaces to commutative C^*-algebras.*

So the world of continuous topology is completely known by algebra.

Under this equivalence, compact Hausdorff spaces (i.e. the nice ones) are exchanged with *unital* commutative C^*-algebras (i.e. the nice ones). This is because the constant function with value 1 is bounded in the C^* norm iff the domain is compact. ◀

Example 3.4. As an even coarser example, we can let "spaces" mean measure spaces and "functions" mean $L^\infty(X)$, which lands in the world of commutative von Neumann algebras. There should be a few more words here to make everything precisely. There is again a contravariant equivalence of categories, and this time there's not very many isomorphism classes of objects: finite unions of points, countable unions of points, intervals, and unions of intervals and some points. ◀

Example 3.5. Algebraic geometry is the best-studied example, but it doesn't work quite as well as some of these other examples. In this case, spaces means *locally ringed spaces*, i.e. topological spaces X together with a sheaf of rings $\mathcal{O}_X$ with a property that we're not going to go into here, and rings means commutative rings. Taking global sections of $\mathcal{O}_X$ defines a contravariant functor to commutative rings, and there is a right adjoint Spec, the spectrum of a ring. It is not essentially surjective: things in the image are called *affine schemes*, and on the full subcategory of affine schemes, $(\mathcal{O}, \mathrm{Spec})$ are contravariant equivalences of categories.

Unfortunately, this doesn't capture lots of important examples: locally ringed spaces which locally look like affine schemes. Lots of important objects in algebraic geometry, such as projective lines, are built out of affine schemes this way, but are not themselves affine. So you expand your notion of geometry a little bit, but from this perspective there are useful things which are weak but not strong solutions to representing a functor to sets, and at that point why not just do geometry with said weak solutions? ◀

Example 3.6 (Quillen-Sullivan rational homotopy theory)**.** In homotopy theory, you might want to study the *homotopy category*, a category built out of (locally compact weakly Hausdorff) topological spaces by inverting homotopy equivalences.[4] Here "rings" means graded commutative rings, and the functor is $H^*(-; \mathbb{Z})$. Cohomology is nice but this isn't quite flexible enough to set up a good spectral theory.

You can get a better correspondence by remembering the entire category of (locally compact weakly Hausdorff) topological spaces and letting the functor be rational cochains, which is valued in the category of commutative differential graded $\mathbb{Q}$-algebras, or CDGAs. Quillen-Sullivan showed that this defines a nice spectral theory: restricting to simply connected spaces, this defines an equivalence of categories from spaces modulo rational homotopy equivalences to simply connected $\mathbb{Q}$-CDGAs.[5]

There are analogous statements by Mandell p-adically, and by Allen Yuan [Yua19] very recently integrally, albeit using cochains with a little more structure. ◀

Remark 3.7. Most of the uses of "spectrum" in math – algebro-geometric, operator-theoretic, even mathematical-physicsy — are all related. The one exception is the homotopy theorists' spectrum, which means something different. Beware this common source of confusion. In this class, "spectrum" will mostly mean integrally. ◀

[3] See [aHRW09] for some discussion on this duality.

[4] You have to do this carefully, to avoid set-theoretic issues, but it can be done.

[5] You can generalize slightly to *nilpotent* spaces, but you must have some sort of condition on π_1. This is sort of like the non-affineness in algebraic geometry, though in practice it behaves a little differently.

Let's get back to spectral decomposition. Let R be a commutative ring (often a k-algebra; for us, always a k-algebra, where $k = \mathbb{C}$) acting on a module (for us, a complex vector space) V. This is data of an algebra homomoprhism $R \to \mathrm{End}(V)$. Our perspective on spectral decomposition is that we want to sheafify, or localize, or spread out, or spectrally decompose this module, over $\operatorname{Spec} R$. To do this, we will use that $\mathcal{M}od_R$ is *symmetric monoidal*: we have a tensor product $\otimes_R\colon \mathcal{M}od_R \times \mathcal{M}od_R \to \mathcal{M}od_R$. Thus we can define the sheaf $\underline{V}$ associated to the module V to be

$$\underline{V}(U) := V \otimes_R \mathcal{O}(U). \tag{3.8}$$

One immediate consequence is that you can talk about the *support* of an element $v \in V$, which is a subset $\operatorname{supp}(v) \subset X$.

For example, if X is a finite set, so $R = \prod_{x\in X} \mathbb{C}$, then

$$V = \bigoplus_{x\in X} V_x, \tag{3.9}$$

It will be useful for the sheaves produced by this construction to have a name.

Definition 3.10. A *quasicoherent sheaf* on $\operatorname{Spec} R$ is one obtained from an R-module in this way. If the R-module is finitely generated, the sheaf is called a *coherent sheaf.*

Not all schemes are affine, so we say that (quasi)coherent sheaves are those which are locally of this form.

TODO: example I missed, where the spectral decomposition is the usual one of a vector space into eigenspaces. Is this $\mathbb{C}[x]$ with x acting by the matrix in question? Or more of the finite set example? This example is a little basic: points are open and closed, and so the eigenspaces are both a hom and a tensor. This is not true in general.

Now, how does the spectral theorem appear in this context? Let V be a vector space (not necessarily finite-dimensional, though in general you need some nice topology here) and $M \in \mathrm{End}(V)$. We think of this matrix as a map of algebras: $\mathrm{Hom}_{\mathcal{S}et}(\mathrm{pt}, \mathrm{End}(V))$. Now $\mathrm{End}(V)$ has a lot of additional structure — it's an associative $\mathbb{C}$-algebra, though not commutative. So there should be an adjunction

$$\mathrm{Hom}_{\mathcal{S}et}(\mathrm{pt}, \mathrm{End}(V)) = \mathrm{Hom}_{\mathcal{A}lg_{\mathbb{C}}}(\mathrm{Free}, \mathrm{End}(V)), \tag{3.11}$$

where Free is the free associative algebra on one generator, which is $\mathbb{C}[x]$. So rather than the matrix M, we will think about the data of V being a $\mathbb{C}[x]$-module. In some contexts, this is called "functional calculus" — once you have a matrix, you can act by polynomials in this matrix. Our approach here is to think of all of these together, rather than just M. In fact, if you have nice enough topology, you can complete, and make sense of things like functions on $\mathbb{R}$, not just polynomials.

Anyways, the idea is that $M \in \mathrm{End}(V)$ is equivalent to V being the global sections of some quasicoherent sheaf on the *affine line* $\mathbb{A}^1 := \operatorname{Spec} \mathbb{C}[x]$. This will be our basic case of spectral decomposition: the simplest case of a family of commuting operators is a single operator.

We want to study how V spreads out over $\mathbb{A}^1$. There is a short exact sequence

$$0 \longrightarrow V_{\mathrm{tors}} \longrightarrow V \longrightarrow V_{\mathrm{free}} \longrightarrow 0, \tag{3.12}$$

using the structure theory of modules over PIDs, and in fact this splits. So $V_{\mathrm{free}} \cong \mathbb{C}[x]^{\oplus r}$, and the torsion part is a direct sum

$$V_{\mathrm{tors}} \cong \bigoplus_{\lambda\in\mathrm{Spec}(V)} V_{\widehat{\lambda}}. \tag{3.13}$$

The free part we will call the *continuous spectrum*, and the torsion part the *discrete spectrum*. Specifically, if V is supported at $\lambda \in \mathbb{A}^1$, this is saying λ is a generalized eigenvalue, and $V_{\widehat{\lambda}}$ is (data equivalent to) the Jordan block for M.

Eigenvectors $Mv = xv = \lambda v$ are equivalent to elements of $\mathrm{Hom}_{\mathbb{C}[x]}(\mathbb{C}_\lambda, V)$, and this has to do with a quotient, rather than a sub (TODO: missed something here). And if you look in the continuous spectrum, there are no eigenvectors, which is to say that over $\mathbb{A}^1$, the free part looks like sections of the trivial bundle, and the discrete spectrum is skyscrapers at points. In both cases we can take fibers (quotients). (TODO: I should draw a picture of something like this.)

Remark 3.14. There are many versions of the spectral dictionary. If we talk about von Neumann algebras and measurable spaces, the corresponding spectral theorem is von Neumann's spectral theorem: $M = A$ is a self-adjoint operator on a Hilbert space V. This theorem, reinterpreted sheafily, says there's a "sheaf," i.e. a projection-valued measure on $\mathbb{R}$ ($\mathbb{R}$ is the spectrum), and the operator A can be reconstructed as a direct integral

$$A = \int_{\mathbb{R}} x \,\mathrm{d}\pi_A, \tag{3.15}$$

with respect to the projection-valued measure π_A.

To make sense of this, let's say what a projection-valued measure is. This is an assignment from every measurable subset $U \subset \mathbb{R}$ to a projection operator $\pi_A(U)$ on V. Different projections should commute. So it's sort of a sheaf of Hilbert spaces, in a particularly weak sense. You can think of the space of sections on U to be $\mathrm{Im}(\pi_A(U))$, and there is a countable additivity property that

$$U \longmapsto \langle w, \pi_A(U)(v)\rangle \tag{3.16}$$

must be a $\mathbb{C}$-valued measure on $\mathbb{R}$. The spectrum of A is the support of π_A. If V is finite-dimensional, the direct integral (3.15) is a direct sum, and is a decomposition of V into A-eigenspaces. In general, the direct integral takes the continuous spectrum into account. ◀

And there are versions with similar pictures in the other spectral settings we discussed, such as Hilbert C^*-modules for C^*-algebras, etc. There's even a homotopical version of this: if $R = C^*(X)$ for a space X, then $\mathcal{M}od_R$ injects into $\mathcal{L}oc(X)$, the category of locally constant (complexes of) sheaves on X, and a module over R, finitely generated in the right sense, has a corresponding (complex of) sheaves on X, and it is finitely generated in the same way. This has to do with the fact that $C^*(X) = \mathrm{End}(k_X)$, where k_X is the constant local system, and $\underline{M}(U) = M \otimes_{C^*(X)} C^*(X)$. There are some homotopical details to fill in here, but everything can be made precise; the point is that there is an analogue of the story here too: you have algebras R as associated to spaces $\operatorname{Spec} R$, and modules over R spread out over $\operatorname{Spec} R$.

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Spectral decomposition shines where there are lots of examples of algebras, spaces, and modules for us to work with. For this reason, we turn to Fourier theory.

In this setting, we want distributions to be the linear dual $\mathrm{Hom}(V, \mathbb{C})$ to V. There are lots of weak eigenvectors, such as δ-functions, which might not be actual eigenvectors (for example, the problem with the continuous spectrum we saw above). For example, if $V = L^2(\mathbb{R})$, with the operator $M = x$, you can't make sense of "a function supported at x" in L^2-land. You can do this for distributions, though. Dually, if $M^\vee = \frac{\mathrm{d}}{\mathrm{d}x}$, we have natural eigenvectors $e^{i\lambda x}$ for λ, but these are not L^2: they live in something bigger, controlled by a different norm. This is what the continous spectrum often looks like: you have a direct integral, which is different than direct sum. The things you're integrating aren't actually subsets. For example, functions on $\mathbb{A}^1$ are functions on a point, directly assembled together, but functions at a point in $\mathbb{A}^1$ aren't a subset of functions on $\mathbb{A}^1$, but instead a quotient. This behaves a little better when there's no torsion, but that obscures the general story. And this general story isn't a weird analysis fact, because it appears for polynomials in algebra too.

Next time, we'll talk about Fourier theory, or abelian duality, from this perspective: as a natural source for commuting operators. Abelian groups G acting on vector spaces V are a good place to look for large algebras of commuting operators. We will spectrally decompose V using these operators. The aim of the class, and in some sense the broader aim of the Langlands program, is a nonabelian generalization of this, and we will spend a few weeks on the abelian story.

Lecture 4.

Some Fourier theory: 1/28/21

Today, we'll say a bit more about spectral decomposition before diving into Fourier theory.

We begin with a commutative algebra A, and build a geometric object $\operatorname{Spec} A$. What precisely these things are depends on your specific formalism, e.g. if you care about C^*-algebras and topological spaces, or commutative rings and affine schemes, or other possibilities.

Last time we saw how an A-module M "spreads out" in a spectral decomposition over $\operatorname{Spec} A$, defining a sheaf on it. You can think of this with physics: there is a physical system with an algebra A of observables and a space M of states. $\operatorname{Spec} A$ has the defining universal property that maps $A \to \mathcal{O}(X)$ are in natural bijetion with maps $X \to \operatorname{Spec} A$. In physics, you might think of making observations as a way of understanding the geometry of X, and observations might be something like functions to a line (so $\operatorname{Spec} A$ is the line here). So maybe X fibers over the line. Observables on the line now tell us something about X.

Modules and states linearize this story: we took M and sheafified it into a sheaf $\mathcal{M} \to \operatorname{Spec} A$. A single function on X is a map to $\mathbb{A}^1 = \operatorname{Spec} k[x]$, and likewise a single matrix (endomorphism) of a vector space gave us a sheaf on $\mathbb{A}^1$ via the Jordan form.

This is exactly how observations happen in quantum mechanics: we don't have a classical phase space like in classial mechanics, only its linearization, the Hilbert space $\mathcal{H}$ of states of the system. Observables are self-adjoint operators on $\mathcal{H}$; it is also useful to talk about the (noncommutative) algebra $\operatorname{End}(\mathcal{H})$ of all operators on $\mathcal{H}$. Let $\mathcal{O}$ be an observable; then, just as for a vector space and an endomorphism, we can sheafify $\mathcal{O}$ into a projection-valued measure (the analogue of a sheaf) on $\mathbb{R}$, where $\mathbb{R}$ is the spectrum of the algebra generated by a single operator.

From this perspective, a state $|\varphi\rangle \in \mathcal{H}$ is a section of the sheaf, i.e. projections of each vector onto the "fibers," which are the images of the projection-valued measures. Given a section, you can ask where it is supported, i.e. you made a measurement, where does it live? That's the support. We can also do something more precise: use the norm. Now $\|\varphi\|^2$ is a complex-valued measure on $\mathbb{R}$: take φ, project onto a fiber, and then take the norm. Suitably normalized, this is a probability measure on $\mathbb{R}$, which tells you where you expect this measurement to live, and ask questions such as what its expectation value is, as

$$\int_{-\infty}^{\infty} x \|\mathrm{proj}_{\mathcal{H}_\lambda} |\varphi\rangle\|^2 \,\mathrm{d}\lambda. \tag{4.1}$$

This is a continuous version of the fact that the expected value over a finite probability space S is a sum:

$$\frac{1}{\langle \varphi \mid \varphi\rangle} \sum_{\lambda \in S} \lambda \langle \psi_i \mid \varphi\rangle |\varphi\rangle, \tag{4.2}$$

where $\{\psi_i\}$ is an S-indexed basis of eigenvectors.

The main observable that is part of the data of a quantum mechanics system is the *Hamiltonian*, a particular self-adjoint operator H. It is the energy functional: its eigenstates are the steady states of the system, fixed by time-evolution, and their eigenvalues measure the energy for each state.

The perspective of algebras as observables, modules as spaces of states, and sheaves as spectral decomposition will come up again and again in the class, even though we will mostly see commutative algebras (which glue a lot better: you can try C^*-algebras, but Gelfand-Naimark tells us that all spaces are affine, so why glue?). The context and language will be fancier, but this philosophy will still shine through.

Let's talk about Fourier theory and abelian duality, which will carry us through the next few weeks. We will think of this as a special case of spectral decomposition, where the source of commuting operators is an abelian group G.

So, fix an abelian group G and a representation $T\colon G \to \operatorname{Aut}(V) \subset \operatorname{End} V$. We'll write that the action of $g \in G$ on V is called T_g. Often, we can cook up good examples of these representations by considering a G-action on a space X, and then taking functions on X. Then G acts by pullback. As we've said before a few times in this class, there are different levels of regularity you can do this in, and the specific one you pick isn't that important right now. If this worries you, sprinkle in the word "finite" where it helps.

The most canonical space that G acts on is itself, giving us the *regular representation*. We want to decompose, so we need to figure out what the spectrum is. If G acts on a one-dimensional vector space, we get a character $\chi\colon G \to \mathbb{C}^\times \subset \mathbb{C}$, and these are the eigenvalues that can arise.

Let $\widehat{G} \coloneqq \operatorname{Hom}_{\mathcal{G}rp}(G, \mathbb{C}^\times)$ be the set of all of the characters; if you make G topological or algebraic or something, also impose that condition on the characters (e.g. continuous or smooth or polynomial). In fact, for a moment imagine G is a finite abelian group — and let's try not to think about the classification of finite abelian groups, because the story will go through in greater generality.

Because not all matrices are invertible, $\operatorname{End}(V)$ is only a monoid, and $G \to \operatorname{Aut}(V) \subset \operatorname{End}(V)$ is a monoid map. This is not a lot of structure; we want more. Specifically, $\operatorname{End}(V)$ is an algebra that we've forgotten down to a monoid. So this looks like one half of an adjunction:

$$\mathrm{Hom}_{\mathcal{M}onoid}(G, \mathrm{For}(\mathrm{End}(V))) = \mathrm{Hom}_{\mathcal{A}lg_{\mathbb{C}}}(\textbf{?}, \mathrm{End}\, V). \tag{4.3}$$

Filling in the **?** is the free algebra on a monoid, i.e. the *group algebra* $\mathbb{C}[G]$, which is the algebra generated by symbols δ_g for $g \in G$ with multiplication $\delta_g \cdot \delta_h = \delta_{gh}$. So you can think of this as formal linear combinations of elements of G — or you can say this is the algebra of functions $G \to \mathbb{C}$, whence the notation δ_g. And this is a suggestion that $\mathbb{C}[G]$ should really be thought of as dual to functions, as measures. The canonical counting measure on a finite set identifies functions and measures for us, but this won't generalize.

For general functions $f_1, f_2\colon G \to \mathbb{C}$, the product is convolution:

$$(f_1 * f_2)(k) = \sum_{g\in G}\left(\sum_{gh=k} f_1(g) f_2(h)\right) = \sum_{g\in G} f_1(g) f_2(kg^{-1}). \tag{4.4}$$

If you've studied convolution in an analysis class, this ought to look familiar. Also, so far we have not needed G to be abelian! And you can recast this in terms of pushforward and pullback.

There are three maps $G \times G \to G$: project onto the first and second factors π_1, resp. π_2, but also multiplication μ. So we can define an *external product*

$$f_1 \boxtimes f_2 \coloneqq \pi_1^* f_1 \pi_1^* f_2 \colon G \times G \longrightarrow \mathbb{C}, \tag{4.5}$$

and because G is finite, we can push this forward along μ, summing over the fibers, and obtain the usual convolution; indeed, this is what (4.4) is telling us. If you care about infinite groups with some sort of regularity, that regularity can buy you the pushforward map in that setting.

The group algebra $\mathbb{C}[G]$ is commutative iff G is abelian, and G acting on V induces a $\mathbb{C}[G]$-action on G, more or less by that adjunction. The key fact is that

$$\mathrm{Spec}(\mathbb{C}[G], *) = \widehat{G}, \tag{4.6}$$

i.e. functions on $\widehat{G}$ are the group algebra. Why is this true? A point of the spectrum is a map $\mathrm{Spec}\,\mathbb{C} \to \mathrm{Spec}\,\mathbb{C}[G]$, so unwinds to a map $\mathbb{C}[G] \to \mathbb{C}$, under multiplication, which gives us a character.

So we have a dictionary

$$(\mathbb{C}[G], *) \overset{\sim}{\longleftrightarrow} (\mathcal{O}(\widehat{G}), \cdot), \tag{4.7}$$

and characters on the left are exchanged with δ-functions on the right. Moreover, $g \in G$ acts by translation on the left, and on the right it defines a function $\widehat{g}$, and translation is exchanged with multiplication by $\widehat{g}$.

This is a form of the Fourier transform. If f is a function on G, then there is an inversion formula

$$f = \sum_{t\in\widehat{G}} \widehat{f}(t) \cdot \chi_t, \tag{4.8}$$

where $\widehat{f}(t)$ is the coefficient of f in an orthonormal basis. And $\widehat{(f)}$ corresponds to $(\widehat{f}\cdot)$. All these statements are restatements of each other, and of the key statement that the group algebra is functions on the dual with pointwise multiplication, as well as functions on G under convolution.

There is an additional symmetry: $\widehat{G}$ is not just a set, but is itself an abelian group.[6] This is because you can pointwise multiply functions on G: the product of two characters is still a character. So $\widehat{G}$ is really the dual group. For this to deserve the name "dual," we need $\widehat{\widehat{G}}$ to be naturally isomorphic to G: we want an assignment for every $g \in G$ a function from characters to $\mathbb{C}^\times$, which of course is $\chi \mapsto \chi(g)$.

There is a more symmetric way to say this: there are two projections

$$\begin{tikzcd} G \times \widehat{G} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \widehat{G} \\ G, \end{tikzcd} \tag{4.9}$$

and on $G \times \widehat{G}$, there is a canonical function

$$K(g, t) \coloneqq \chi_t(g) = \chi_g(t), \tag{4.10}$$

[6]If G is finite, the classification of finite abelian groups shows $\widehat{G}$ is noncanonically isomorphic to G. This is not true in general, so don't use it for your intuition any more than you need to.

which you can think of as a multiplicative kernel. Here χ_g is the character on $\widehat{G}$ given by g. So this setup is more symmetric (and may also remind you of some kernels from a functional analysis class). Now we can describe the Fourier transform as a pullback-pushforward:

$$\widehat{f}(t) = \pi_{2*}(\pi_1^* f \cdot \chi)(t) = \sum_g f(g)\overline{K(g,t)}. \tag{4.11}$$

We get a complex conjugate because there was one in the inner product. And it means the inverse Fourier transform looks slightly different:

$$f(g) = \sum_t \widehat{f}(t) K(g,t). \tag{4.12}$$

Great, we've spectrally decomposed functions on G — said differently, we've simultaneously diagonalized the action of G on the space of functions on G. "Simultaneous diagonalization" is a reminder how important being abelian is to the whole story.

Now this is a lot of work for just one representation, but because all G-representations are $\mathbb{C}[G]$-representations, then for any representation $\mathbb{C}[G] \to \operatorname{End} V$, we get a spectral decomposition of V over the spectrum $\widehat{G}$, i.e.

$$V = \bigoplus_{t \in \widehat{G}} V_{\chi_t}. \tag{4.13}$$

That is, the fiber at t is the χ_t-isotypic component of V. This is sort of overkill for finite abelian groups, but generalizes.

Let G be a locally compact topological abelian group (LCA). There are lots of good examples that are not finite: $\mathbb{Z}$, U_1, $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{Z}_p$, $\mathbb{Q}_p^\times$, and many more. The dual is

$$\widehat{G} = \operatorname{Hom}_{\mathcal{T}op\mathcal{G}rp}(G, \mathrm{U}_1), \tag{4.14}$$

i.e. the continuous, unitary characters of G. If G is finite, this is exactly what we had already: all characters are valued in roots of unity.[7]

It was crucial for us that the dual $\widehat{G}$ was the spectrum of the group algebra. There are different ways to implement that in the LCA setting; the way we'll do it is to endow G with a measure. Specifically, G carries a *Haar measure*, which is a left-invariant measure with respect to G acting on itself by multiplication, and is unique up to scaling. If G is compact, this measure is bi-invariant and can be normalized to total measure 1, but we will often care about noncompact groups.

Anyways, our stand-in for $\mathbb{C}[G]$ is the C^*-algebra $L^1(G)$ (with respect to a chosen Haar measure), with convolution

$$(f_1 * f_2)(h) = \int_G f_1(g) f_2(g^{-1}h)\, \mathrm{d}g. \tag{4.15}$$

You can extract this from a kernel transform just as in the finite-group case. With this definition, $\mathrm{mSpec}(L^1(G), *) \cong \widehat{G}$, not just as topological spaces, but also group structures, and this is a version of the Fourier transform. This is again a duality, often called *Pontrjagin duality*, where the natural map $G \to \widehat{\widehat{G}}$ is an isomorphism of topological abelian groups.

The Fourier transform again has the formula

$$f \longmapsto \pi_{2*}(\pi_1^* f \cdot K), \tag{4.16}$$

where K is the kernel, with the same formula as before. The thing that's new here is that you have to do analysis to see what regularity appears on the other side. For L^1, you get an isomorphism $(L^1(G), *) \cong (C_v(\widehat{G}), \cdot)$, where C_v denotes the space of continuous functions vanishing at infinity. The Plancherel theorem gives you $(L^2(G), *) \cong (L^2(\widehat{G}), \cdot)$. And for both of those, translations by group elements are exchanged with multiplication. (To say this completely precisely, you may need to work with distributions, so that you have δ-functions.) Characters are exchanged with points; this is symmetric, but in the geometric Langlands program and the related topological field theory, the two sides are not symmetric.

[7] If you care about algebraic geometry, you might be used to saying $\mathbb{G}_m$ instead of U_1. There is a version of this story for $\mathbb{G}_m$, too, but this particular kind of regularity requires U_1.

Example 4.17 (Fourier series)**.** Say $G = \mathrm{U}_1$. Then $\widehat{G} = \mathrm{Hom}(\mathrm{U}_1, \mathrm{U}_1)$, which can be identified with $\mathbb{Z}$ via the map

$$n \longmapsto (x \longmapsto \exp(2\pi i n x)). \tag{4.18}$$

That is, $\mathbb{Z} \cong \widehat{\mathrm{U}_1}$.

The theory of Fourier series identifies $L^2(\mathrm{U}_1) \cong L^2(\mathbb{Z})$; the latter is often called ℓ^2. One decomposes a periodic function (a function on U_1) into its Fourier modes. There are versions of this for other kinds of regularity.

To think of this as a kernel transform, there is a function on $\mathrm{U}_1 \times \mathbb{Z}$ sending

$$x, n \longmapsto \exp(2\pi i n x), \tag{4.19}$$

and characters $\mathbb{Z} \to \mathrm{U}_1$ are identified by where 1 goes, which is anywhere, and therefore $\widehat{\widehat{\mathbb{Z}}} = \mathrm{U}_1$. ◀

This illuminates a nice fact about Pontrjagin duality: G is compact iff $\widehat{G}$ is discrete.

One nice reference for all this is Ramakrishnan-Valenza [RV99].

Lecture 5.

Pontrjagin and Cartier duality: 2/2/21

We spent some time the other day discussing Pontrjagin duality. To review, choose a locally compact abelian group G; its dual is

$$\widehat{G} \coloneqq \mathrm{Hom}_{\mathcal{A}b}(G, \mathrm{U}_1), \tag{5.1}$$

which has canonically the structure of a locally compact abelian groups. This has many properties that resemble a Fourier transform, including that L^1 functions on one side are identifed with continuous functions on the other side that vanish at infinity, as well as L^2 functions on one side passing to L^2 functions on the other side. Characters on one side exchange with points on the other, and translation and convolution exchange with multiplication. G is compact iff $\widehat{G}$ is discrete (and, of course, vice versa).

Examples:

- The dual of U_1 is $\mathbb{Z}$: the characters $\mathrm{U}_1 \to \mathrm{U}_1$ are the maps $z \mapsto z^n$, indexed by $n \in \mathbb{Z}$. Correspondingly, the dual of $\mathbb{Z}$ is U_1.
- Let T be a compact torus, which is defined as a quotient $\mathbb{R}^d/\Lambda = \Lambda \otimes_{\mathbb{Z}} \mathrm{U}_1$ for a full-rank lattice $\Lambda \subset \mathbb{Z}^d$. This has an associated *dual lattice* $\Lambda^\vee \subset \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$, and the dual of T is $\Lambda^\vee$.
- $\mathbb{R}^d$ is self-dual — or, more precisely, the dual of $\mathbb{R}^d$ is $(\mathbb{R}^d)^*$. More generally, for a finite-dimensional real vector space V, the dual is V^*, the usual linear dual.

This last identification recovers the theory of the Fourier transform. Say x is the coordinate on the primal $\mathbb{R}$ and t is the coordinate on the dual $\mathbb{R}$; then the canonical character on $\mathbb{R}_x \times \mathbb{R}_t$ is $\exp(2\pi i x t)$, and when one writes down the transform as a pullback-pushforward, one recovers the usual Fourier transform.

More generally, if we began with V and obtained the Pontrjagin dual V^* (i.e. also the usual linear dual), the canonical pairing is $\exp(2\pi i \langle x, t\rangle)$, and the Fourier transform is

$$f(x) = \int_V \widehat{f}(t) e^{2\pi i \langle x, t\rangle}\, \mathrm{d}t. \tag{5.2}$$

In the Fourier transform, we know that differentiation by x (on the primal side) is exchanged with multiplication by t (on the dual side), just like translation by group elements. This makes sense: differentiation is an infinitesimal transfomation. It is a general example of how group theory on one side is exchanged with geometry on the other side.

And in general, if G is an abelian Lie group, then its Lie algebra $\mathfrak{g}$ maps to the Lie algebra of vector fields on G, which sits inside the algebra of differential operators on G, and there is an adjunction between forgetting the structure of an associative algebra to a Lie algebra and a free functor building the *universal enveloping algebra* $\mathcal{U}(\mathfrak{g})$ out of a Lie algebra $\mathfrak{g}$. Since $\mathfrak{g}$ is abelian, $\mathcal{U}(\mathfrak{g})$ is the usual symmetric algebra $\mathrm{Sym}^* \mathfrak{g}$.

By the adjunction, $\mathcal{U}(\mathfrak{g})$ acts on $C^\infty(G)$ as differential operators, and so we can sheafify over $\mathrm{Spec}(\mathcal{U}(\mathfrak{g})) = \mathfrak{g}^*$. So for $G = \mathbb{R}_x$, for example, $\mathfrak{g}^* = \mathbb{R}_t$ can be identified with $\mathrm{Spec}\,\mathbb{R}[t]$, where t *is* $\frac{\mathrm{d}}{\mathrm{d}x}$, which explains why differentiation becomes multiplication by t. So every aspect of the group theory on the primal side is simultaneously diagonalized, because everything is commutative.

Speaking of commutative things, let's talk about quantum mechanics, which is famously commutative.[8] This is a place the Fourier transform will happen. For the classical mechanics of a particle moving on a manifold M, the phase space is T^*M, the cotangent space, with local coordinates q (position) and p (momentum), with the momenta pointing in the bundle direction. These generate the algebra of observables.

In quantum mechanics, we replace T^*M with L^2 functions on half of the variables: just the positions. That is, the Hilbert space is $\mathcal{H} := L^2(M)$, and the observables include differential operators on M, including both functions on M and tangent vectors, with $p_j = i\frac{\mathrm{d}}{\mathrm{d}q_j}$. If $\xi \in TM$ and $f\colon M \to \mathbb{R}$ is a function, there is a commutator

$$\xi f = f\xi + \hbar f'. \tag{5.3}$$

Here $\hbar$ is Planck's constant.

We want to say that "states look like the square root of the observables," which has to do with the fact that we threw out half of the observables classically in order to quantize. The Fourier transform reenters the story because we want to diagonalize the momentum operators and their derivatives. This isn't really meaningful in general, but if M is an abelian Lie group — often $\mathbb{R}^n$ — then we have a natural basis of commuting vector fields, and a Fourier transform exchanging $L^2(\mathbb{R}^n_q) \cong L^2(\mathbb{R}^n_p)$. This ultimately leads to an interesting nontrivial isomorphism between quantum mechanics on G and on $\widehat{G}$, and crucially, this isomorphism cannot be seen at the classical level. This is sort of a very simple version of abelian duality or mirror symmetry in 1d, and we will see echoes of this story in higher-dimensional, less trivial settings.

In the next few weeks, we will discuss Cartier duality, which is the algebraic version of this story, and then pass to its main manifestations in physics and in number theory: electromagnetic duality and class field theory. One takeaway will be that both of these can be thought of as kinds of Fourier transforms. These are abelian cases of the Langlands program.

Cartier duality is about algebraic groups. You can think of these in a few ways: things like in Lie groups but in algebraic geometry, such as group objects in the category of varieties, but it will also be helpful to think of algebraic groups through their functors of points. A variety X defines a functor $\mathcal{A}lg_k \to \mathcal{S}et$ by $X(R) := \mathrm{Hom}(\operatorname{Spec} R, X)$: the set of "$R$-points" of X. Here $\mathcal{A}lg_k$ denotes commutative k-algebras.

If X is a group object in the category of varieties, that is equivalent structure to factoring the functor of points through $\mathcal{G}rp \to \mathcal{S}et$. That is, $\{\operatorname{Spec} R \to X\}$, the R-valued points of X, is a group, and these are compatible as R varies. And through the Yoneda lemma, X is determined by its functor of points.

Example 5.4. Consider the functor $R \mapsto \mathrm{GL}_n(R)$. This is representable, defining an algebraic group GL_n. You can play this game with other familiar groups such as SL_n. ◀

As with Pontrjagin duality, we care more about abelian algebraic groups. By the magic of the Yoneda lemma, this is the same thing as asking for the functor of points to factor through $\mathcal{A}b \hookrightarrow \mathcal{G}rp \to \mathcal{S}et$.

Example 5.5.

(1) The *additive group* $\mathbb{G}_a := \mathbb{A}^1$, which sends an algebra R to the abelian group of functions on $\operatorname{Spec} R$.
(2) The *multiplicative group* $\mathbb{G}_m := \mathbb{A}^1 \setminus 0$. This is $\mathrm{Aut}(k)$. Its functor of points is $\mathbb{G}_m(R) = R^\times$. This is our analogue of U_1: characters are functions to $\mathbb{G}_m$.
(3) Another slightly weirder example: the constant functor valued in $\mathbb{Z}$. ◀

Now given an abelian algebraic group G, we can let $\widehat{G} := \mathrm{Hom}(G, \mathbb{G}_m)$, where this is homomorphisms of algebraic groups (if you like, natural transformations of $\mathcal{G}rp$-valued functors of points).

Useful examples.

- $\widehat{\mathbb{Z}} = \mathbb{G}_m$, just as in the topological case: a map out of $\mathbb{Z}$ is determined by where 1 goes, and 1 can go anywhere.
- Conversely, $\widehat{\mathbb{G}}_m = \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$.
- If we start with $\mathbb{Z}/n$, we get μ_n, the group scheme of n^{th} roots of unity.

Say Λ is a lattice in a complex vector space V. Then the dual of Λ is the dual torus $T^\vee = \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{G}_m$, where $\Lambda^* \subset V^*$ is the dual lattice. And the torus V/Λ has Cartier dual $\Lambda^\vee$. This is the analogue of Fourier series in the algebraic geometry setting.

[8] This is a joke.

To avoid regularity issues, we assume G is finite for now (finite flat group scheme, or spectrum of an Artinian algebra. . .).[9] Then $\widehat{G} = \mathrm{Spec}(\mathcal{O}(G)^*)$. The algebra $\mathcal{O}(G)$ has additional structure coming from G, namely a *coalgebra structure* given by the pullback of the multiplication map, called *comultiplication*:

$$\Delta \coloneqq \mu^*\colon \mathcal{O}(G) \longrightarrow \mathcal{O}(G) \otimes_k \mathcal{O}(G). \tag{5.6}$$

This plays the role of convolution here. There is similarly a *counit map* $\mathcal{O}(G) \to k$ arising as pullback of the inclusion of the identity $\mathrm{id}\colon \mathrm{Spec}\, k \to G$.

So $\mathcal{O}(G)$ has an algebra and a coalgebra structure. Do they play nice together? Yes they do! There are a few commutation relations that are satisfied and this data together gives a *commutative, cocommutative Hopf algebra*. (The abelian assumption on G is required for cocommutativity). And in fact, the theory of finite abelian group schemes over k is contravariantly equivalent to the theory of commutative, cocommutative, finite-dimensional Hopf algebras.

Duality defines an involution on the category of commutative, cocommutative, finite-dimensional Hopf algebras, and when you pass this through Spec, you get Cartier duality for finite abelian group schemes. If you want to study infinite algebraic groups, you need to worry about regularity, because duals of infinite-dimensional vector spaces are a little trickier.

Remark 5.7. Why should comultiplication be convolution? Say G is a finite abelian group, without any algebraic geometry aroud. Then $(\mathbb{C}[G], *)$ can be identified with the algebra of measures on G, and convolution is identified with the pushforward of measures under multiplication. ◀

Now let's see what happens with the Fourier transform here. As in the topological case, we expect $\mathbb{G}_a$ to pass to $\mathbb{G}_a$, but the Cartier dual of $\mathbb{G}_a$ is actually something different, the *formal completion* of $\mathbb{G}_a$, which, confusingly, is also written $\widehat{\mathbb{G}}_a$. For now, I will use $\widehat{\mathbb{G}}_a$ to denote the Cartier dual and $\mathbb{G}_a^\wedge$ to denote the formal completion. What this actually is is the union of all infinitesimal neighborhoods of the origin:

$$\mathbb{G}_a^\wedge = \bigcup_n \mathrm{Spec}\, k[t]/(t^n). \tag{5.8}$$

Let's assume k has characteristic zero, so we don't have to worry about defining the character of $\mathbb{G}_a = \mathrm{Spec}\, k[x]$ to be

$$e^{xt} = \sum_n \frac{(xt)^n}{n!}. \tag{5.9}$$

To allow this in algebraic geometry, this sum is required to be finite, which is why we passed to formal completions: t, the dual coordinate, is nilpotent, so this is fine.

More generally, if V is a finite-dimensional k-vector space, which defines an abelian algebraic group, its Cartier dual is $(V^*)^\wedge$, the formal completion of the dual near 0, and the duality is again given by $\exp(\langle x, t\rangle)$, which makes sense formally (i.e. in this formal neighborhood of the origin).

Spectral theory is asking about G-representations, which are identified with $\mathcal{O}(G)$-comodules, i.e. $\mathcal{O}(\widehat{G})$-modules, or quasicoherent sheaves on $\widehat{G}$. For this to be literally true G has to be finite, but there is a version of this story in general. This is the way in which Cartier duality gives us decompositions of representations.

For example, $\mathcal{R}ep(\mathbb{Z})$ is identified with $k[z, z^{-1}]$-modules: $\mathcal{O}(\mathbb{G}_m) = k[z, z^{-1}]$. So a single matrix (vector space and endomorphism) is identified with a quasicoherent sheaf on the line, i.e. on $\mathbb{A}^1 = \mathrm{Spec}\, k[z]$, and requiring the matrix to be invertible is saying, well, 0 cannot be an eigenvalue, so you are a sheaf on $\mathbb{A}^1 \setminus 0 = \mathbb{G}_m$, the Cartier dual of $\mathbb{Z}$ as expected.

In the other direction, a representation of $\mathbb{G}_m$ is the same thing as a $\mathbb{Z}$-graded vector space, or a sheaf on the discrete abelian group $\mathbb{Z}$. That is, if V is a $\mathbb{G}_m$-representation, we can write

$$V = \bigoplus_{n\in\mathbb{Z}} V_n, \tag{5.10}$$

where V_n is the eigenspace in which $z \in \mathbb{G}_m$ acts by z^n, and this gives us the $\mathbb{Z}$-grading. This is how the Fourier series takes a representation of the multiplicative group and spits out a sheaf on a discrete set.

[9]These technicalities arise only in positive characteristic, where you can have nonregular finite group schemes, such as the kernel of the Frobenius $\varphi\colon \mathbb{A}^1 \to \mathbb{A}^1$, which has no closed points yet is nonreduced of length p. Another (counter)example is the group scheme of the n^{th} roots of unity, which over $\mathbb{C}$ is just $\mathbb{Z}/n$ as you might expect, but in positive-characteristic can look different.

Example 5.11. Here's another algebraic example of Fourier series (lattices and tori exchanged) which takes a little effort to set up, but is nice. This example is in topology. Let M be a 3-manifold and G be the Picard group of M, the group of isomorphism classes of complex line bundles on M up to isomorphism. The group structure is tensor product. We can equivalently use principal U_1-bundles, or $\pi_0(\mathrm{Map}(M, B\mathrm{U}_1))$. One can use as a model for $B\mathrm{U}_1$ the space $\mathbb{CP}^\infty$ or an Eilenberg-Mac Lane space $K(\mathbb{Z}, 2)$.

Because $B\mathrm{U}_1 = K(\mathbb{Z}, 2)$, homotopy classes of maps to $B\mathrm{U}_1$ are naturally identified with $H^2(M;\mathbb{Z})$ as abelian groups, with the map given by sending a line bundle to its first Chern class. For now, assume $H^2(M;\mathbb{Z})$ is free, though we can tell a version of this story in the presence of torsion.

Now $G \coloneqq H^2(M;\mathbb{Z})$ is abelian; let's calculate its Cartier dual. This is a dual torus

$$\begin{aligned}\Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{G}_m &= \mathrm{Hom}(H_1(M), \mathbb{G}_m)\\ &\cong \mathrm{Hom}(\pi_1(M), \mathbb{G}_m).\end{aligned}$$

This can be identified with the group of isomorphism classes of flat $\mathbb{C}^\times$-bundles on M, where given a bundle we use the flat connection to define a monodromy map $\pi_1(M) \to \mathbb{C}^\times$. So we obtain a duality between $\mathrm{Pic}(M)$ and $\mathrm{Loc}_{\mathbb{C}^\times}(M)$ — this isn't super deep, since it's just exchanging $\mathbb{Z}^n$ and $(\mathbb{C}^\times)^n$. You can make this independent of basis by replacing line bundles with T-bundles, where T is a torus. Let $T^\vee$ denote the dual torus.

Now the same story allows us to identify $\mathcal{B}un_T(M)$ and $\mathcal{L}oc_{T^\vee}(M)$, where the latter is the group of isomorphism classes of flat $T^\vee$-bundles. Choosing a basis these are $(\mathbb{C}^\times)^n$-bundles. Again a flat connection is determined by its monodromy. Both of these can be thought of as categories, but we're not using that, just thinking of the sets of isomorphism classes.

So again this is just lattice-torus Cartier duality, but it looks a little more suggestive: bundles and local systems are more geometric, and indeed this examples appears when one studied electromagnetic duality on a 3-manifold. And if you apply this idea to arithmetic topology, thinking of a number field as a 3-manifold, you get an instance of Cartier duality and you get the statement of class field theory!

This abelian duality is a four-dimensional story, and has avatars in all lower dimensions. This is part of the 3-manifold story. And in this story, you have more than just the set of isomorphism classes, but considerably more structure. ◀

Lecture 6.

Electric-magnetic duality: 2/4/21

Last time we discussed Cartier duality, an algebro-geometric version of Pontrjagin duality, e.g. with $\mathbb{G}_m$ (or $\mathbb{C}^\times$ for those of you who always work over $\mathbb{C}$) instead of U_1. But the algebraic and analytic stories do have useful things to say to each other.

There are different versions of representation theory of G, which correspond to different notions of group algebra in the dual world. For example, if we consider the group algebra as a von Neumann algebra, the dual $\widehat{G}$ has the structure of a measure space. If we consider the group C^* algebra, $\widehat{G}$ is a topological space. And if we just consider a discrete algebra (so G is very small), then $\widehat{G}$ is an algebraic variety. You should think of this as one dual object $\widehat{G}$, even if the precise mathematical objects are not the same.

And now for some examples, though they're really the same example as Fourier series, between $\mathbb{Z}$ and the circle, said with a little more flair.

Example 6.1 (Pontrjagin-Poincaré duality)**.** Let M be a compact n-dimensional manifold and G be a locally compact abelian group, so we can use M as a source of abelian groups, specifically $H^i(M;G)$. Pontrjagin-Poincaré duality is the existence of a perfect pairing

$$H^i(M;G) \otimes H^{n-i}(M;\widehat{G}) \longrightarrow H^n(M;G\otimes\widehat{G}) \longrightarrow H^n(M;\mathrm{U}_1) \longrightarrow \mathrm{U}_1, \tag{6.2}$$

hence a natural identification of $H^{n-i}(M;\widehat{G})$ as the Pontrjagin dual of $H^i(M;G)$. In homotopy theory, this is related to a fairly general abstract version of Pontrjagin duality called *Brown-Comenetz duality*.

(TODO: if G has a topology, what happens? I want to make sure I understand this.)

Specializing to dimension 1, we get duality from $\mathbb{Z}$ (or maps $\mathrm{pt} \to \mathbb{Z}$) to $\mathbb{C}^\times$ (or maps $\mathrm{pt} \to \mathbb{C}^\times$). Not terribly exciting. But this can be thought of as an equivalence between quantum mechanics on $\mathbb{Z}$ and quantum mechanics on $\mathbb{C}^\times$.

What about dimension 2? Now we can consider $H^0(S^1;\mathbb{Z})$, which we learned has dual $H^1(S^1;\mathbb{C}^\times)$. In particular, we have a Fourier transform identifying functions on these ableian groups. We can think of $H^0(S^1;\mathbb{Z})$ as $[S^1,\mathbb{Z}]$, the homotopy classes of maps to $\mathbb{Z}$. For $H^1(S^1;\mathbb{C}^\times)$, we want to call it $\mathcal{L}oc_{\mathbb{C}^\times}(S^1)$, but this should really mean the stacky version, so let's just let $\mathrm{Loc}_{\mathbb{C}^\times}(S^1)$ denote the underlying space. ◀

Let's talk about local systems. A G-local system is a sheaf that's locally isomorphic to the constant sheaf $\underline{G}$, and in which the gluing maps are identified with G acting on itself by multiplication. In some sense, this doesn't use the topology or geometry of G, just the underlying set. For example, a local system has a monodromy map $\pi_1(X)\to G$, and it is determined up to isomorphism by this map. Said a little differently, isomorphism classes of local systems are given by $H^1(X;G)$ — even if G is nonabelian! Usually we're used to only defining cohomology for abelian groups, but for nonabelian groups you can define singular or Čech cohomology in degrees 0 and 1 only, and H^1 recovers isomorphism classes of principal bundles.

Local systems have automorphisms, and it is often important to remember that, but right now we just see isomorphism classes. When we consider just the underlying space $\mathrm{Loc}_{\mathbb{C}^\times}(S^1)$, this is the space $\mathbb{C}^\times$, so we once again get the Cartier duality between $\mathbb{Z}$ and $\mathbb{C}^\times$. This seems boring but now that we're one dimension higher and seeing principal bundles, this duality has something to do with gauge theory for U_1.

Example 6.3 (Another 2d example)**.** Now let's consider $H^1(S^1;\mathbb{Z})$, which we call the *A side*. We call the dual $H^0(S^1;\mathbb{C}^\times)$ the *B side*. This is not about albums!

On the B side, we are looking at maps to $\mathbb{C}^\times$, where $\mathbb{C}^\times$ has the discrete topology. In physics this is an exampe of a *scalar field*. On the A side, we're looking at $[S^1,B\mathbb{Z}]$; since $B\mathbb{Z}=K(\mathbb{Z},1)=S^1$, we're looking at maps to S^1. This is another scalar field, and you could think of this as a very very special case of T-duality or mirror symmetry. In T-duality, one exchanges radii $R\leftrightarrow 1/R$ for these two circles.

If you want to soup this up a little bit, you can replace $\mathbb{Z}$ with a lattice Λ, so that you don't have a canonical basis and the duality is a little more interesting. Then the dual is the dual complex torus $T_{\mathbb{C}}^\vee$, but $H^1(S^1;\Lambda)=[S^1,T]$, where T is the compact torus $T\coloneqq\Lambda_{\mathbb{Z}}S^1$. Again this is an example of T-duality. ◀

Example 6.4. Now we step up to 3d. Let Σ be a compact, oriented 2-manifold. Then Pontrjagin-Poincaré duality exchanges $H^1(\Sigma;\mathbb{Z})$ and $H^1(\Sigma;\mathbb{C}^\times)$. On the left we have a scalar field $[\Sigma,B\mathbb{Z}]$, or $[\Sigma,S^1]$, and on the right we get $\mathrm{Loc}_{\mathbb{C}^\times}(\Sigma)$, which is about 3d gauge theory.

Another approach is to exchange $H^2(\Sigma;\mathbb{Z})$ and $H^0(\Sigma;\mathbb{C}^\times)$. Now $H^2(\Sigma;\mathbb{Z})\cong H^1(\Sigma;\mathrm{U}_1)=[\Sigma,B\mathrm{U}_1]$, since $B\mathrm{U}_1=K(\mathbb{Z},2)$. This is about gauge theory for principal U_1-bundles. On the right side, we get $[\Sigma,\mathbb{C}^\times]$, a theory of a scalar field. Again if you want this to be more interesting, you can upgrade to tori and lattices, and you get isomorphisms of vector spaces. ◀

Example 6.5. Now a four-dimensional example: $H^2(M;\mathbb{Z})$ and $H^2(M;\mathbb{C}^\times)$. You can make this work with torsion but for ease of exposition let's assume $H^2(M;\mathbb{Z})$ is torsion-free. The cool thing about dimension 4 is that both sides of this correspondence can be identified with gauge theory: on the A side, $H^2(M;\mathbb{Z})=[M,K(\mathbb{Z},2)]=[M,B\mathrm{U}_1]$ so we get principal U_1-bundles. Said differently, $[M,B\mathrm{U}_1]=\pi_0(\mathrm{Map}(M,\mathbb{CP}^\infty))$.

There is an equivalence between principal U_1-bundles and complex line bundles, given in one direction by taking the associated line bundle to a principal U_1-bundle and in the other direction by choosing a metric and taking the unit circle bunde. So we can think of $H^2(M;\mathbb{Z})$ as the Picard group of M, though this might be a strange version of it if you're used to the Picard group with more structure (e.g. a variety).

On the B side we get $\mathbb{C}^\times$-valued local systems, at least without stackiness.

This is an instance of electric-magnetic duality. See Witten's lectures "Abelian duality I" and "Abelian duality II" in the quantum fields and strings books, as well as Freed [Fre00] and Freed-Moore-Segal [FMS07] for more mathematical approaches.

(TODO: something I missed) and we want to study solutions to equations $\mathrm{d}u=0$ and $\mathrm{d}{\star}u=0$, where $u=\mathrm{d}\varphi$. The first condition is automatic, because u is exact. The second is equivalent to ${\star}\mathrm{d}{\star}u=0$, or equivalently φ is harmonic.

But this reformulation of the harmonic condition shows that we have a pretty symmetry: $\{\mathrm{d}u=0,\mathrm{d}{\star}u=0\}$ is symmetric under the Hodge star. Since ${\star}u$ is closed, it is at least locally $\mathrm{d}\varphi^\vee$. So the theory of a U_1-valued scalar is dual to a theory with another U_1-valued scalar. Or, if you consider maps to one torus, the dual side is about maps to a dual torus. When you introduce metrics so that you can talk about harmonicity better, you will see that if R is the radius of the A side circle, the radius of the B side cicle is $1/R$.

Now let's see what's happening in dimension 3. Let $\varphi\colon\Sigma\to S^1$ be a smooth map and $u=\mathrm{d}\varphi$. Imposing $\mathrm{d}u=0$ and $\mathrm{d}{\star}u=0$ is dual to asking for $F={\star}u=dA$.

But the symmetry in 4d is more interesting. Quick crash course in electromagnetism: one has an *electric field* $E \in \Omega^1(\mathbb{R}^3)$ and a *magnetic field* $B \in \Omega^2(\mathbb{R}^3)$. If you want to work relativistically on Minkowski space, we instead collate these together into the *field strenth*

$$F \coloneqq B - \mathrm{d}t \wedge E \in \Omega^2(\mathbb{M}^4), \tag{6.6}$$

where $\mathbb{M}^4$ denotes Minkowksi space.

Maxwell's equations in a vacuum tell us that $\mathrm{d}F = 0$ and $\mathrm{d}\star F = 0$. This is of course symmetric under $F \leftrightarrow \star F$. If we break relativistic symmetry and write

$$\star F = B^\vee - \mathrm{d}t \wedge E^\vee, \tag{6.7}$$

then $B^\vee = -\star_3 E$ and $E^\vee = \star_3 B$, where $\star_3$ means to take the Hodge star on a time slice, which is a 3-manifold. (That is, space is 3-dimensional, and spacetime is 4-dimensional.)

If you throw in electromagnetic charges with currents, these equations modify, and we now have electric and magnetic currents j_E and j_B, respectively, which are 3-forms, and we ask that $\mathrm{d}F = j_B$ and $\mathrm{d}\star F = j_E$. On a time-slice M, $\int_M j_B = Q_B$ and $\int_M j_E = Q_E$, the magnetic, resp. electric charges. And Stokes' theorem tells you Gauss' law: if $\Sigma \subset M$ is closed, the *magnetic flux*

$$b_\Sigma = \int_\Sigma F = -Q_B. \tag{6.8}$$

This has an interpretation as the average number of field lines leaving Σ. Similarly, the total *electric flux* is

$$e_\Sigma = \int_\Sigma \star F = Q_E. \tag{6.9}$$

(TODO: don't assume Arun's minus signs are correct!)

In nature, there are no magnetic monopoles, so $Q_B = 0$, and Gauss' law says that if you integrate over a closed surface, you get 0. This allows us to introduce a new object, the *electromagnetic potential* A, and writing $F = \mathrm{d}A$. This breaks the symmetry between electricity and magnetism. Given this data, $\nabla \coloneqq \mathrm{d} + A$ is a connection on a principal U_1-bundle on M, and the gauge symmetries

$$\begin{aligned} \nabla &\longmapsto g^{-1}\nabla g \\ A &\longmapsto A + g^{-1}\,\mathrm{d}g, \end{aligned} \tag{6.10}$$

where $g\colon M \to \mathrm{U}_1$ is smooth, does not affect F.

This is a very convenient trick to play but is it meaningful in physics? Yes! There is an experimentally measured principle called the *Aharonov-Bohm effect*, which gives meaning in physics to A. Even when $F = 0$ (the connection is flat), charge particles acquire phases along loops, which measures the holonomy of the flat connection. So A really exists (albeit only up to gauge transformations.)

Now that we've replaced F with $\mathrm{d}A$ (and introduced A) on 3-manifolds, let's also see what happens when we do this in dimension 4. This choice now implements Dirac charge quantization. For $M = N \times \mathbb{R}$, where N is 3-dimensional and M is 4-dimensional, say that ∇ is a connection on the principal U_1-bundle $P \to M$. We still have fluxes even without charged particles, as we can integrate

$$b_\Sigma = \frac{1}{2\pi i}\int_\Sigma F. \tag{6.11}$$

This is no longer zero, and in fact using Chern-Weil theory this is $\langle c_1(P), [\Sigma]\rangle$. This is an integer, rather than an arbitrary complex number, which is Dirac charge quantization. There is an analogue of this for the electirc flux

$$e_\Sigma = \frac{1}{2\pi i}\int_\Sigma \star F. \tag{6.12}$$

This is all of the classical story for now, and we turn to the quantum story. We will discuss it in the Hamiltonian formalism. Quantum *field* theory means that instead of doing quantum mechanics with point particles, we do it with fields. These are things which have a locality property and sheafify (this is not a precise definition!): maps to a space, sections of vector bundles, principal bundles or vector bundles and connections, and so on.

As a heuristic, QFT on $M^{d-1}\times\mathbb{R}$ should feel lie quantum mechanics on a space of fields on M^{d-1}. This is often infinite-dimensional. This is very much like a standard procedure in analysis where you regard a PDE as an infinite-dimensional ODE.

We will have a Hilbert space of states $\mathcal{H}$, which is heuristically L^2 of the space of fields on M. Part of what quantum field theory does is to attach Hilbert spaces to time-slices. For now we're not going to worry about regularity issues: L^2 means "functions that are a Hilbert space," but we note that good solutions exist. So we want to consider the space $\mathcal{C}(M)$ of isomorphism classes of principal line bundles $L\to M$ with unitary connection. Connections always have isomorphisms and we are going to ignore this for now.

So we will try doing quantum mechanics with the Hilbert space $L^2(\mathcal{C}(M))$. $\mathcal{C}(M)$ has a nice structure: it is an infinite-dimensional analogue of a Lie group under tensor product of line bundles and connections, and there is a sense in which it splits as the product of a lattice, a torus, and an infinite-dimensional vector space. So we have both sides of the abelian duality we've discussed today, as well as the vector space where we want to study Fourier series.

The lattice comes from a discrete invariant, the first Chern class $c_1\colon\mathcal{C}(M)\to H^2(M;\mathbb{Z})$; in physics this measures the magnetic flux. Geometrically, one can also recover this by taking the curvature of the connection $F\colon\mathcal{C}(M)\to\Omega^2(M)$. But it's not just any 2-form: it is closed and has integral periods (which means that it talks to $H^2(M;\mathbb{Z})$). The kernel of the curvature map is the space $\mathcal{C}_\flat(M)$ of flat connections, which are the same thing as U_1-valued local systems on M (i.e. principal U_1-bundles, where U_1 has the discrete topology, and we take isomorphism classes of these bundles). We discussed how you can think of local systems up to isomorphism as $H^1(M;\mathbb{R}/\mathbb{Z})$, identifying $\mathbb{R}/\mathbb{Z}\cong\mathrm{U}_1$, and $H^1(M;\mathbb{R}/\mathbb{Z})$ is a torus $\mathrm{U}_1\otimes\mathbb{Z}H^1(M;\mathbb{Z})$, assuming there's no torsion in $H^1(M;\mathbb{Z})$.

The lattice $\Lambda\coloneqq H^2(M;\mathbb{Z})$ is called the *lattice of magnetic fluxes*, and inside $\mathcal{C}(M)$ we have the torus $T=\mathrm{Loc}_{\mathrm{U}_1}(M)$ of local systems, and there's a vector space part too. So we know what L^2 looks like: we have Fourier series for T and Λ, and for L^2 of the vector space, we have a Fourier transform, even though it's infinite-dimensional.

Electric-magnetic duality from this perspective will apply Pontrjagin duality on $\mathcal{C}(M)$, as we will discuss next time. We will be able to write the same Hilbert space in two ways, and magnetic measurements on one side are electric measurements on the other side. ◀

Lecture 7.

Lagrangian quantum field theory: 2/9/21

Today we'll discuss Euclidean quantum field theory (QFT) from the Lagrangian perspective.

Fix a d-dimensional manifold M; here d is the dimension of spacetime. To M we attach a space of fields $\mathcal{F}(M)$. Fields are things which can sheafify, i.e. they can be assembled from local data which glues: for example, principal bundles and connections, functions, sections of vector bundles, differential forms, and so on. Associated to a field $\varphi\in\mathcal{F}(M)$, there is an *action* $S(\varphi)$, which is a function $S\colon\mathcal{F}(M)\to\mathbb{R}$.

The "quantum" in quantum field theory means that rather than computing explicit measurements we get a probability distribution, which we obtain by integrating over the space of fields. This is the notorious path integral, which is not always mathematically well-defined, but we're not going to worry about that for now. Physicists have ways of calculating path integrals.

If M is closed, the most fundamental measurement we can make is the volume of the space of fields, the *partition function*

$$Z(M)\coloneqq\int_{\mathcal{F}(M)}e^{-S(\varphi)/\hbar}\,\mathcal{D}\varphi. \tag{7.1}$$

We also want to calculate expectation values of operators. If $\mathcal{O}_x$ is an operator at a point $x\in M$, it defines a functional on $\mathcal{F}(M)$ which physically one can interpret as making a measurement at x. The measurement is calculated by a modified path integral

$$Z(M)\coloneqq\int_{\mathcal{F}(M)}\mathcal{O}_x(\varphi)e^{-S(\varphi)/\hbar}\,\mathcal{D}\varphi. \tag{7.2}$$

Among these are the *disorder operators*, for which inserting $\mathcal{O}_x$ means to look at fields that might possibly be singular at x, e.g. by looking at nice fields within $\mathcal{F}(M\setminus x)$.

We're not going to make this rigorous, but it will be helpful for us understanding time evolution and the Hamiltonian perspective from last time. Let M be a bordism from ∂M_{in} to $\partial M_{\mathrm{out}}$. Because fields sheafify, we can restrict them to the incoming and outgoing boundaries, defining maps $\mathcal{F}(\partial M_{\mathrm{in}}) \leftarrow \mathcal{F}(M) \to \mathcal{F}(\partial M_{\mathrm{out}})$, and we have (more or less) the path integral measure $\mathcal{D}\varphi$ on $\mathcal{F}(M)$, so we can perform an integral transform, at least heuristically, from functionals on $\mathcal{F}(\partial M_{\mathrm{in}})$ to functionals on $\mathcal{F}(\partial M_{\mathrm{out}})$. Let $\mathcal{H}_{\mathrm{in}}$ and $\mathcal{H}_{\mathrm{out}}$ be these spaces of functionals. There's lots of details to worry about here, including regularity and defining the path integral measure.

ANyways, from all this data we get a linear map $\mathcal{H}_{\mathrm{in}} \to \mathcal{H}_{\mathrm{out}}$. Its value on a functional $f \in \mathcal{H}_{\mathrm{in}}$ and a field $\varphi \in \mathcal{F}(\partial M_{\mathrm{out}})$ can be described by considering the subspace T of $\mathcal{F}(M)$ of the fields together with data of restriction to φ on $\partial M_{\mathrm{out}}$. Then integrate, using the path integral measure for ∂M_{in}, the value of f on the restrictions of the fields in T to ∂M_{in}.

This is the relationship between the Lagrangian and Hamiltonian perspectives. If M is a cylinder interpreted as the identity bordism, we want to think of this map as time evolution by $e^{itH/\hbar}$, where t is the length of the cylinder. The Lagrangian perspective uncovers a lot of additional rich structure which we will find useful.

Example 7.3. Say $d = 1$ and we want to study quantum mechanics on a Riemannian manifold X. Then the fields on a 1-manfold C are $\mathcal{F}(C) := \mathrm{Map}(C, X)$; the critical points of the action are geodesics, and $\mathcal{H} = L^2(\mathrm{Map}(\mathrm{pt}, X)) = L^2(X)$, and the Hamiltonian is the Laplacian. ◀

However, if we try to do this in higher dimensions, the problems that occur because of our infinite-dimensional spaces of fields get worse. But let's continue forwards anyways, and see what we can learn even though it's not mathematically precise.

Example 7.4 (Quantum Maxwell theory)**.** This is a four-dimensional field theory. The fields on M are the (gauge equivalence classes of) principal U_1-bundles on M with a connection $\mathrm{d} + A$. Let F denote the curvature of the connection. The classical equations (i.e. the equations whose solutions are critical points of the action) are $\mathrm{d}F = 0$ and $\mathrm{d}{\star}F = 0$. The action is

$$S = \frac{g}{2\pi i}\int_M F \wedge \star F + \theta \int F \wedge F \tag{7.5}$$

where g and θ are parameters, and there might be some constants to fix. Note that by Chern-Weil theory, $F \wedge F$ is a fixed multiple of $c_1(P)^2$, where $P \to M$ is the principal U_1-bundle.

The Hilbert space $\mathcal{H}$ on a closed 3-manifold N is the functions on the space of fields $\mathcal{C}(N)$, specifically L^2 of the space of connections on principal U_1-bundles, mod gauge.

Because U_1 is abelian, we'd like to think of $\mathcal{C}(N)$ as some sort of infinite-dimensional Lie group. Taking the first Chern class surjects $\mathcal{C}(N) \twoheadrightarrow \Lambda := H^2(N : \mathbb{Z})$, and the kernel of this map is the space of flat connections $T := H^1(N; \mathrm{U}_1) \cong BH^1(N; \mathbb{Z})$. The space of fields splits as $\Lambda \times T \times V$ for some infinite-dimensional vector space V.

The lattice Λ acts on $\mathcal{H}$, related to Dirac monopoles, though in the nonabelian case this is related to *'t Hooft operators*. Specifically, given $\lambda \in \Lambda$, there is a line bundle L with $c_1(P) = \lambda$. Then the action of Λ on $\mathcal{C}(N)$ is to tensor with L, and this induces an action on $\mathcal{H}$.

This creates magnetic monopoles in M! Let γ be a smooth closed curve whose homology class is Poincaré dual to λ. (TODO: physics goes here — this has something to do with a magnetic monopole on γ.)

Now if N is the cylinder bordism for N with an embedded knot γ at time $t = 1/2$, our equations ruled out magneti monpoles, so we need to consider the space of fields on $(M \times I) \setminus \gamma$. The link of this knot is $S^2 \times S^1$. Pick a point on γ where our would-be monopole would be, and let S be the linking 2-sphere. Then the magnetic charge inside S is nonzero.

Let's look at the space $\mathcal{C}_{c=1}$ of connections on $(M \times I) \setminus \gamma$ such that $(1/2\pi i)\int_{S^2} F = 1$, i.e. inside the 2-sphere there is 1 unit of magnetic charge. This space of connections is mathematically well-defined, and we get a correspondence

$$\mathcal{C}(N_{\mathrm{in}}) \longleftarrow \mathcal{C}_{c=1} \longrightarrow \mathcal{C}(N_{\mathrm{out}}). \tag{7.6}$$

Therefore by the usual integral transform, we get a map $\mathcal{H} \to \mathcal{H}$ whose effect is to shift by $\lambda = c_1(P)$. This is an example of a *disorder operator*: we're not measuring our fields, but instead looking at a different space of fields. Our measurement makes the fields singular along γ. The field-theoretic description is complicated, even though the action on $\mathcal{H}$ is simple.

Wilson operators are correspondingly simple to describe field-theoretically. Let $\gamma \subset N$ be a simple closed curve. Even classically, there's a function $W_\gamma\colon \mathcal{C}(N) \to \mathbb{C}$ sending (L, ∇) to the holonomy of ∇ around γ. This is valued in $\mathrm{U}_1 \subset \mathbb{C}$. Once again we can put γ inside $[0,1] \times M$ at $t = 1/2$ to define an integral transform $\mathcal{H} \to \mathcal{H}$. We're no longer measuring at a point, but instead along a loop. And the action is to multiply functionals in $\mathcal{H}$ by W_γ.

These W_γ are eigenfunctions for the action of T on $L^2(\mathcal{C}(M))$, multiplying by the holonomy. This is something one should check, but it's Fourier theory. This provides a decomposition

$$\mathcal{H} = \bigoplus_{e \in H^(M;\mathbb{Z})} \mathcal{H}_e, \tag{7.7}$$

the *electric fluxes*.

This is the first part of electric-magnetic duality, which you can think of as doing Fourier analysis on $\mathcal{C}(N)$, which we decomposed into a lattice, a torus, and a vector space. The Fourier transform identifies $L^2(\mathcal{C}_{\mathrm{U}_1}(M))$ and $L^2(\mathcal{C}_{\mathrm{U}_1^\vee}(M))$, switching the lattice Λ_B with the torus $T_B^\vee$, and conversely T_E to $\Lambda_E^\vee$, and the vector space goes to the vector space.

On one side, the Wilson operators are diagonal; on the other side, the 't Hooft operators are diagonal. This is the hallmark of electric-magnetic duality.

Electric-magnetic duality is inherently quantum: you cannot see it from the classical Lagrangian perspective. You have to quantize to see one half of the operators. The magnetic operator from the Lagrangian perspective destroys a connection by making it singular along the loop. Since electric-magnetic duality switches this with operators that make sense in the classical perspective, we need a different perspective to see it.

Later, when we talk about the geometric Satake isomorphism, we will interpret it as telling us what the 't Hooft operators are in nonabelian gauge theories. ◀

Example 7.8 (Abelian duality in $d = 2$)**.** This will be a simple instance of T-duality. Let M be a 2-manifold; the fields are $\mathrm{Map}(M, S^1)$, and the Hilbert space on S^1 is $L^2(\mathrm{Map}(S^1, S^1))$. This is graded by $H^1(S^1;\mathbb{Z})$, the winding number of the map.

What T-duality does here is that we can take a dual circle $(S^1)^\vee$; if the primal circle has radius R, the dual has radius $1/R$. Then there's a correspondence between L^2 of these two spaces of maps. Concretely, you can think of $L^2(H^0(M;\mathrm{U}_1))$ in the dual case.

On the primal side, we can shift the winding number, which is an analogue of the 't Hooft operators in 4d Maxwell theory. On the dual side, we can evaluate a function at a point, which is a Wilson analogue. Duality exchanges these operators, and is a very simple case of mirror symmetry. ◀

The quantum field theories we care the most about in this class are the topological ones. Our goal is to cut down from these gigantic Hilbert spaces of functions on infinite-dimensional manifolds to finite-dimensional Hilbert spaces, in a way that is still helpful.

For example, let's look at quantum mechanics. The point is assigned a Hilbert space $\mathcal{H}$, and a time interval $[0, T]$ is assigned a time evolution operator $r^{iTH/\hbar}$. The Hamiltonian is the Laplacian, and the entire spectral theory of the Laplacian appears in the story, and there are subtleties.

In topological quantum mechanics, we kill time. Not in the sense of, let's hang around for a bit, but in the sense of, let's force time-evolution to be the identity map. There are a few ways to do this: you could restrict to the zero eigenspace of the Laplacian, but these are harmonic functions, which are constant on closed Riemannian manifolds by the maximum principle. So whatever theory we get is not super interesting.

Alternatively, following Witten [Wit82], you can kill time in a derived sense by introducing supersymmetry.

(1) First, enlarge $L^2(X)$ to the space of L^2 differential forms on the target X. So we're adding new fields.
(2) Using this, we can identify the (super) Lie group of symmetries.

For $\Omega_X^\bullet$, we have in addition to $\mathfrak{u}_1$ acting by grading, we have two operators $Q := \mathrm{d}$ and $\mathbb{Q}^* := d^*$. The commutation relations between Q and Q^* and $\mathfrak{u}_1$ (the *R-symmetry*) tell us that we should think of Q and Q^* in degrees 1 and -1, respectively. Then these jointly define a graded Lie algebra called the *one-dimensional* $\mathcal{N} = 1$ *supersymmetry algebra*, and they all *graded* commute with each other, except $[Q, Q^*] = H$: $\Delta = \mathrm{dd}^* + \mathrm{d}^*\mathrm{d}$. The other commutation relations use $\mathrm{d}^2 = 0$, etc.

That is, Q^* is a (chain) homotopy from H to 0. Therefore on Q-cohomology (which is ordinary de Rham cohomology), H acts by zero: there's some Hodge theory here, saying harmonic forms define representatives in de Rham cohomology. And yet we have more stuff left: maybe not harmonic functions, but now we have cohomology!

That is, topological quantum mechanics assigns to X the "Hilbert space" $(\Omega_X^\bullet, \mathrm{d})$, or $H^*_{\mathrm{dR}}(X)$, with $H = 0$. And this is a topological invariant in X, indicating that the dependence on Riemannian geometry has gone away. This is very drastic!

Lecture 8.

Topological quantum mechanics, topological Maxwell theory, and class field theory: 2/11/21

As we discussed a little bit last time, in topological quantum mechanics we kill the time-evolution operator in a derived sense: first we added a differential Q, making the Hilbert space a chain complex, and then showing that H is exact, or homotopy to 0. You could then work with cohomology as your Hilbert space, but these days people have realized it's more convenient to work in a derived sense with the entire chain complex and H exact.

The grading (by $\mathbb{Z}$ or $\mathbb{Z}/2$) is part of the structure of the action by the supersymmetry algebra, but in some sense, which came first? Maybe you are studying representations of the algebra through its R-symmetry (here this is $\mathfrak{u}_1$, which is where the grading comes from), and the grading comes afterward; alternatively, you begin with the grading and then produce the algebra action.

There are two different ways we can implement this.

(1) In "A-type topological quantum mechanics," we let $\mathcal{H} \coloneqq \Omega_X^\bullet$, $Q \coloneqq \mathrm{d}$, $Q^* \coloneqq \mathrm{d}^*$, and $H \coloneqq \Delta$, the Hodge Laplacian. This is the example we discused last time.

(2) In "B-type topological quantum mechanics," instead of a Riemannian manifold, we let X be a complex manifold. Then there's a similar complex around (you could call it a complex complex): $\mathcal{H} \coloneqq \Omega_X^{0,\bullet}$, $Q \coloneqq \overline{\partial}$, $Q^* \coloneqq \overline{\partial}^*$, and $H \coloneqq \Delta_{\overline{\partial}}$. The cohomology with respect to Q is $H^{0,*}(X)$, which is quasi-isomorphic to $\mathbf{R}\Gamma(\mathcal{O}_X)$, the derived global sections of the sheaf of functions on X.

Both of these are representations of the one-dimensional $\mathcal{N} = 1$ supersymmetry algebra. Here $\mathcal{N}$ measures "how much supersymmetry there is;" there's a precise way to define this, but right now it's best to think of this as telling you this is the minimal amount of supersymmetry possible in dimension 1.

There is also an $\mathcal{N} = 2$ supersymmetry algebra, and an $\mathcal{N} = 2$ version of topological quantum mechanics. Here we combine the two types above by letting X be a compact Kähler manifold, with bigraded chain complex $\Omega_X^{\bullet,\bullet}$, and we need more operators: $\overline{\partial}$, $\overline{\partial}^*$, ∂, and ∂^*. There is a deep mathematical fact (the hard Lefschetz theorem) that $\mathrm{SL}_2(\mathbb{C})$ acts on the cohomology; the diagonal part gives the grading, and there are also raising and lowering operators. The R-symmetry group is $\mathrm{SU}_2 \subset \mathrm{SL}_2(\mathbb{C})$.

From a mathematical point of view this is a difficult, deep theorem, but in physics it emerges in a more elementary way: you ask what the supersymmetry algebra is for quantum mechanics on a Kähler manifold, and a fact about representation theory (which is not necessarily simpler) tells you that we get $\mathcal{N} = 2$, and therefore an $\mathrm{SL}_2(\mathbb{C})$-action, because that's present in the $\mathcal{N} = 2$ supersymmetry algebra in $d = 1$.

One can push this a little further: if X is compact *hyper*Kähler, the supersymmetry algebra enlarges to $\mathcal{N} = 4$, and instead of $\mathrm{SL}_2(\mathbb{C})$, the cohomology has a Spin_5-action, as part of the $\mathcal{N} = 4$ super Lie algebra. And you can even go all the way to $\mathcal{N} = 8$. These algebras are distinguished by the number of linearly independent odd operators (the *supercharges*): two in $\mathcal{N} = 1$, four in $\mathcal{N} = 2$, eight in $\mathcal{N} = 4$, and sixteen in $\mathcal{N} = 8$. The dimension of the space of odd-order operators is a mathematically better way to think about $\mathcal{N}$, though as you can see this is not literally what $\mathcal{N}$ is.

By adding some more fields to these theories, we'll find some additional supersymmetry, i.e. enlarging the super Lie group that acts on the theory. Our desiderata are that the Hamiltonian $H = [Q, –]$, i.e. it's exact. And for a topological theory, we want to do more: there is in quantum mechanics generally an operator T called the *stress tensor* measuring the dependence on the metric, and we want this to also be Q-exact, to make the theory more topological: nothing depends on the metric on the level of cohomology. This is explained in more detail in Costello-Gwilliam's book.

Topological Maxwell theory Now let's soup this up to dimension 4, taking Maxwell theory, adding fields, and finding supersymmetry to give us a topological field theory in this Q-exact sense. This is an $\mathcal{N} = 4$ theory, which has sixteen supercharges and a whole bunch of data, including a complex line bundle with connection, a 1-form which is called the *Higgs field*, a complex scalar u, and four fermions (giving us odd operators).

There are again two options, indexed by A and B. For the A-twist, the Hilbert space on our 3-manifold M is the cohomology of the space of connections, with some nuances. First of all, connections are a stack, because some connections have stabilizer group actions, and second, the gauge group acts and we take equivariant cohomology. That is, $\mathcal{H} = H^\bullet_{\mathrm{U}_1}(\mathcal{C}(M))$. So this is like doing the A-version of topological quantum mechanics (de Rham theory) on the space of U_1-connections.

Alternatively, in the B-twist, we let ∇ be a connection on a principal $\mathbb{C}^\times$-bundle, rather than a U_1-bundle. Then we do B-type topological quantum mechanics (the Dolbeault story) on the space of $\mathbb{C}^\times$-connections.

Back to the A side for a moment. We will abuse notation slightly to say $\mathrm{Pic}(M)$ is the topological space $\mathcal{C}(M)$. Then $H^0(\mathrm{Pic}(M))$ is the locally constant functions on $\mathcal{C}(M)$. The connected components of $\mathcal{C}(M)$ are the topological types of principal U_1-bundles, which is $H^2(M;\mathbb{Z})$. So we get back what we thought about a few lectures ago, but in a fancier context.

For higher cohomology, we decompose $\mathrm{Pic}(M)$ as a lattice times a torus times a vector space times $B\mathrm{U}_1$, and we need to consider. We see $H^2(M;\mathbb{Z})$, the torus $H^1(M;\mathrm{U}_1)$, and $\mathbb{C}[u]$ coming from the cohomology of $B\mathrm{U}_1$. Under the topological twist, things are much simpler.

On the B-side, we care about $\mathrm{Loc}_{\mathbb{C}^\times}(M)$, which looks like a complex torus $T^\vee_{\mathbb{C}}$, the torus of maps $\pi_1(M) \to \mathbb{C}^\times$. These factor through $H_1(M)$, so this is the torus dual to the lattice $H_1(M)$. So our Hilbert spaces look like $\mathbb{C}[H^2(M;\mathbb{Z})]$ or $\mathbb{C}[T^\vee_{\mathbb{C}}]$, and there is Poincaré-Pontrjagin duality that exchanges them.

One takeaway is that what we saw before about Fourier series on the cohomology of a manifold appears as part of the story in topological Maxwell theory, and there are additional new things going on. The A-twisted super Maxwell theory is studying the topology of the space of connections, and the B-twisted version studies the algebraic geometry of the space of flat connections. Electric-magnetic duality switches these two. These theories are not the same theory with two descriptions, but a duality between two genuinely different theories.

Topology of Pic on the A-side being dual to the algebraic geometry of Loc is a perspective we're going to see again and again.

Behind the curtain, these two topological field theories are topological twists of a supersymmetric QFT called $\mathcal{N}=4$ super Maxwell theory. This is a supersymmetric version of the theory of light in our world. $\mathcal{N}=4$ here tells us that we're looking at complex geometry (much like how in dimension 1, $\mathcal{N}=2$ is a signal that X is Kähler). There's a lot of literature about $\mathcal{N}=4$ super-Yang-Mills; this is a special case with gauge group U_1, so if you're looking for references, try super-Yang-Mills.

The duality again exchanges Wilson and 't Hooft operators. On the A side, we have 't Hooft operators, which shift on the lattice; on the B side, we have Wilson operators, which multiply by the monodromy along a curve $\gamma \in H_1(M;\mathbb{Z})$. 't Hooft operators create magnetic monopoles. The physics theory has both electricity and magnetism, but by choosing either of these twists, we've broken that symmetry: on the A side, we only have electric measurements, and on the B side we only have magnetic ones.

Now let's introduce defects. These will change the space of fields slightly.

- First, there are "timelike" 't Hooft operators. Choose a point $x \in M$ and consider electromagnetism in the presence of a monopole sitting at x. That is, in $M \times [0,1]$, the line $\{x\} \times [0,1]$ is the worldline of the monopole: it doesn't move. We will then consider electromagnetism in the presence of this monopole, by consider connections on $M \setminus x$. This space splits as a disjoint union

$$\mathcal{C}(M \setminus x) = \amalg_{c_1(P|_{S_\varepsilon(x)})=n} \mathcal{C}(M, nx) \tag{8.1}$$

  i.e. indexed over the Chern class on a linking sphere of x. We think of n as the charge of the monopole. Then we obtain a Hilbert space of L^2 functions on this space of connections.
- On the B side, we want "timelike" Wilson operators on $\mathrm{Loc}_{\mathbb{C}^\times}(M,x)$, defined to be flat $\mathbb{C}^\times$-connections together with a trivialization of the fiber at x. This is a principal $\mathbb{C}^\times$-bundle over $\mathrm{Loc}_{\mathbb{C}^\times}(M)$. This "extra $\mathbb{C}^\times$" is dual to the "extra $\mathbb{Z}$" of charges on the A side, and Fourier series identify the spaces of functions on these extra groups. This is interpreted in physics as creating an electrically charged particle without adding a new field for the particle.

So, electric-magnetic duality tells us that creating a magnetic monopole is dual to creating an electrically charged particle that's heavy enough to break this symmetry (we're not adding a new field corresponding to this particle).

There are yet more defects to consider. The next one, called *ramification* or adding a *surface defect* or *solenoid*, matches in physics what happens when we consider a current running through a wire wrapped tightly around a cylinder to form a tube. Our fields are going to do something funny here; specifically, we remove a curve β from M and consider $\mathrm{Loc}_{\mathbb{C}^\times}(M \setminus \beta)$. Now there's π_1, which means something in the world in the Aharonov-Bohm effect: if you go around a solenoid, the electric field is affected. (TODO: maybe I, Arun, misinterpreted this.)

So we can look at holomorphic functions on this space of local systems (this is the B, or electric, side). On the A, or magnetic side, we look at cohomology of the space of connections on M with a trivialization along β. This is a map $\beta \to U_1 = S^1$, which has a winding number, adding a $\mathbb{Z}$ where dually we had a $\mathbb{C}^\times$ (holonomy around the extra loop). Again, the functions on $\mathbb{Z}$ and $\mathbb{C}^\times$ are identified via a Fourier transform. The mathematics here is all interpretations of Poincaré duality on a 3-manifold, and it's neat how all of these Fourier-theoretic statements admit different physical meaning.

Our physics story here is about geometric Langlands; what's the corresponding arithmetic story? This is a subject in number theory called class field theory. Once again we're just going to get Fourier theory exchanging $\mathbb{Z}$ and $\mathbb{C}^\times$ (or lattices and tori, more generally), but we will find these groups in interesting places and ways. Again the topology of something called Pic is exchanged with the algebraic geometry of something called Loc. At first, "topology" is just going to mean π_0, and "algebraic geometry" means holomorphic functions, but one can extract more structure.

Now the setup. Let F be a finite extension of $\mathbb{Q}$. Number theorists attach an important object to F, the *(ideal) class group* $C\ell(F) := \mathrm{Pic}(\mathrm{Spec}\,\mathcal{O}_F)$. That is, this is telling us about line bundles on $\mathrm{Spec}\,\mathcal{O}_F$, i.e. rank-1 projective $\mathcal{O}_F$-modules, up to isomorphism. In this setting, though, it has another, more elementary interpretation: fractional ideals of $\mathcal{O}_F$ (i.e. sitting inside F) modulo principal ideals. The class group is always finite, which is pretty neat, and doesn't have a clear analogue in other settings. One of the most important invariants of a number field is the *class number* $n_F := |C\ell(F)|$.

(Unramified) class field theory relates the class group to Galois theory. It identifies $\mathbb{C}[C\ell(F)]$ with $\mathbb{C}[\mathrm{Loc}_{\mathbb{G}_m}(\mathcal{O}_F)]$. If you look at a textbook on class field theory, it will use very different notation, but this is used to match with our more geometric/physical story from earlier today. So what is Loc exactly? These are π_1-representations, and our analogue of π_1 is the Galois group. For a manifold, π_1 is the automorphisms of a universal cover over the base, which under our arithmetic topology dictionary would be the automorphisms of a field extension F^{un}/F, where F^{un} is the *maximal unramified abelian extension* of F.

Why this, and not $\overline{F}$? We're thinking of F not as a point $\mathrm{Spec}\,F$, but as rational functions on a scheme, $\mathrm{Spec}(\mathcal{O}_F)$. The algebraic closure corresponds to allowing fields of functions on curves with punctures, but let's not worry about that for the moment, instead thinking about closed curves.

Definition 8.2. A field extension $F \hookrightarrow E$ is *unramified* at the prime $p \in \mathcal{O}_F$ if $\mathcal{O}_E \otimes_{\mathcal{O}_F} \mathcal{O}_F/p$ has no nilpotents.

These nilpotent operators correspond to branch points in a branched cover: some interesting geometry is happening here. Anyways, the poset of unramified, abelian extensions has a maximal element, and that's what F^{un} is. And this is the étale fundamental group of $\mathrm{Spec}\,\mathcal{O}_F$.

There's a technical condition here where we need things to also be unramified at infinite places: what happens when we take $F \otimes_{\mathbb{Q}} \mathbb{R}$? This is isomorphic to $\mathbb{R}^r \times \mathbb{C}^s$, where r and s count embeddings of F into things. We say that E/F is *unramified at infinity* if there are no extensions $\mathbb{R} \hookrightarrow \mathbb{C}$ here. For example, totally imaginary fields (for which $r = 0$) are unramified at infinity.

Anyways, the main theorem of unramified class field theory is that $C\ell(F) \cong \mathrm{Gal}(F^{un}/F)$. This is the group of automorphisms of the *Hilbert class field* F^{un}. We want to interpret this in terms of Pic and Loc. The class group is a Picard group, so that's fine; on the other side, we interpret $\mathrm{Hom}(\mathrm{Gal}(F^{un}/F), \mathbb{C}^\times)$ as $H_1(\mathrm{Spec}(\mathcal{O}_F))$ (TODO: how?). The character dual of the Galois group is also $\mathrm{Hom}(C\ell(F), \mathbb{C}^\times) = C\ell(F)^\vee$.

There is an analogue of Poincaré duality in this setting known as *Artin-Verdier duality*, which applies to spectra of rings of integers. This is part of the arithmetic topology dictionary: another way in which number fields behave like 3-manifolds. The proof of Artin-Verdier duality uses class field theory, so this is sort of upside down from how we argued in the topological setting.

Also, $\mathrm{Spec}(\mathcal{O}_F)$ is more like an unoriented 3-manifold: there is a thing which gives duality, but it's not the constant sheaf. Instead we get something using cyclotomic characters. This has something to do with the difference between $\mathbb{Z}/n$ and the n^{th} roots of unity μ_n, or $\mathbb{Q}/\mathbb{Z}$ and all roots of unity, or $\mathbb{C}^\times$ and $\mathbb{G}_m$. The dualizing object is rather than $\mathbb{Z}$, the *Tate twist* $\mathbb{Z}(1)$, which can be interpreted back in physics as corresponding to accounting for the factor of $2\pi i$ that appears in the Fourier transform. These details make the precise statement of Artin-Verdier duality more confusing, but the key idea is an analogue of Poincaré duality.

Next time, we'll discuss among other things the function-field version of this story.

Lecture 9.

Class field theory and function fields: 2/23/21

It's been a little while since we had class, thanks to the Texas winter storm, so let's review what's happened. We were in the midst of discussing abelian duality, and how it related the topology of principal bundles with the algebraic geometry of local systems. Among the examples of abelian duality we considered, we discussed a (relatively weak) form of class field theory. Let's go over that again.

Let F be a number field (i.e. a finite extension of $\mathbb{Q}$). Class field theory identifies the Galois group of the maximal unramified abelian Galois extension of F with the class group of $\mathcal{O}_F$. Therefore we can identify $\mathbb{C}[C\ell_F] \cong \mathbb{C}[\mathrm{Loc}_1(\mathrm{Spec}\,\mathcal{O}_F)]$. The right-hand side of this isomorphism can be interpreted as the characters of the Galois group.

This is not always an interesting statement. For example, when $F = \mathbb{Q}$, both sides vanish, even though the Galois theory of $\mathbb{Q}$ is interesting. To see this data, we will have to allow ramifications, which we will discuss later; we'll have to figure out in particular what to replace the class group with. Geometrically, if we pretend that $\mathrm{Spec}\,F$ is something like a Riemann surface, then class field theory is talking about covers; ramifications allow the cover to branch. If there were actual Riemann surfaces, allowing branching makes the problem much more interesting.

The bulk of today's lecture, though, will focus on the more geometric setting of function fields. Let k be a field (no further conditions for now) and C be a smooth projective curve over k. When you draw pictures, $k = \mathbb{C}$ and C is a compact Riemann surface. To such a curve C, we attach a field $F := k(C)$ of rational functions on C. The abelian group $\mathrm{Pic}(C)$ of divisors fits into a short exact sequence

$$0 \longrightarrow \mathrm{Jac}(C) \longrightarrow \mathrm{Pic}(C) \longrightarrow \mathbb{Z} \longrightarrow 0, \tag{9.1}$$

where the surjection onto $\mathbb{Z}$ is given by the degree, $\sum_{x\in C} n_x x \mapsto \sum n_x$, for $x \in C(k)$. This short exact sequence splits: choose $x \in C(k)$; then the skyscraper $\mathbb{Z}\cdot x$ defines a map $\mathbb{Z} \to \mathrm{Pic}(C)$ which is a section. This is already different from the class field theory case: $\mathrm{Pic}(C)$ is our analogue of the class group, but it always is infinite, and the class group is finite.

Now assume $k = \mathbb{F}_q$ is finite. Then there is a map $\pi_1^{\text{ét}}(C) \twoheadrightarrow \widehat{Z} := \varprojlim_n \mathbb{Z}/n$. This $\widehat{\mathbb{Z}}$ is $\pi_1^{\text{ét}}(\mathrm{Spec}\,k)$, i.e. $\mathrm{Gal}^{un}(\overline{k}/k)$, topologically generated by the Frobenius.

We will replace $\pi_1^{\text{ét}}(C)$ with the (unramified) Weil group $W^{\text{un}}(C)$, which is the pullback

$$\begin{tikzcd} W^{\text{un}}(C) \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d, hook] \\ \pi_1^{\text{ét}}(C) \arrow[r] & \widehat{\mathbb{Z}}. \end{tikzcd} \tag{9.2}$$

Another thing we can do is take the maximal abelian quotient $W^{ur,ab}(C)$, which surjects onto $\mathbb{Z}$ and still has a section for every k-point of the curve. This looks more like the class group. Thinking of C as like a Riemann surface, the section at a point $x \in C(k)$ is sort of like an embedding of $\mathrm{Spec}(k)$ as a circle in the 3-manifold (the surface bundle).

At this point, it's your choice whether you use the Weil group or the étale fundamental group. Now we can also change Pic. For k finite, the Jacobian $\mathrm{Jac}(C)$ is also a finite abelian group, so all of the "infinite-ness" comes from the degree map onto $\mathbb{Z}$. So we can replace Pic by its profinite completion (the limit over all finite subgroups), which now surjects onto $\widehat{\mathbb{Z}}$.

Theorem 9.3 (Unramified class field theory)**.** *There is a map* $\mathrm{Pic}(C) \to \pi_1^{\text{ét},ab}(C)$ *which is an isomorphism on profinite completions. Equivalently, there is an isomorphism* $\mathrm{Pic}(C) \cong W^{un,ab}(C)$ *respecting the degree homomorphism and its sections.*

So this isomorphism respects quite a lot of structure. This statement is not interesting in all cases: for example, if $C = \mathbb{P}^1$, both sides are $\mathbb{Z}$.

Class field theory is telling us that characters if $\mathrm{Pic}(C)$ are isomorphic to characters of the (unramified, abelian) Weil group. But just like in class field theory, we don't need to abelianize when we take characters: this is the same thing as characters of $W^{un}(C)$, or the characters of $\pi_1^{\text{ét}}(C)$ — here, though, we have to restrict to finite-order characters. So maps into $\mathbb{Q}/\mathbb{Z}$ instead of $\mathbb{C}^\times$. And characters of π_1 deserve to be called rank-one local systems on C (again with some question about coefficients, and certainly true for systems with finite coefficents). So $\mathrm{Pic}^\vee(C) \cong \mathrm{Loc}(C)$.

Inside the functions on $\mathrm{Pic}(C)$ we have the characters: functions which know and respect the multiplication law; and these are identified with functions on the set of local systems. But this is not the only way to think about these.

Recall that we can think of characters via the push-pull diagram, as we discussed a while ago, and this allows you to see that they are precisely the eigenfunctions for the translation action of $\mathrm{Pic}(C)$ on itself. And since $\mathrm{Pic}(C)$ is generated by $\mathbb{Z}x$ for $x \in C(k)$, characters can be thought of as functions on $\mathrm{Pic}(C)$ which are eigenfunctions for the action of $\mathbb{Z}x$ for all $x \in C(k)$. What is this action explicilty? Given $\mathcal{L} \in \mathrm{Pic}(C)$ and $x \in C$, the action sends $\mathcal{L} \mapsto \mathcal{L}(x)$: add $1 \cdot x$ to the divisor of $\mathcal{L}$.

Being an eigenfunction tells you that a function f behaves nicely under this process:

$$f(\mathcal{L}(x)) = \gamma_x \cdot f(\mathcal{L}). \tag{9.4}$$

Now functions on $\mathrm{Pic}(C)$ are identified with functions on $\mathrm{Loc}(C)$, via a Fourier transform. Let ρ be a local system, thought of as a map $\rho\colon \pi_1^{\text{ét}}(C) \to F$, where F is some field of coefficients. This gives us a bunch of eigendata:

$$f(\mathcal{L}(x)) = \rho(\mathrm{Fr}_x) f(\mathcal{L}). \tag{9.5}$$

The operator $\mathcal{L} \mapsto \mathcal{L}(x)$ is the first example of a *Hecke operator*, analogous to the 't Hooft or Dirac monopole operators that we saw in physics, where we identified H^2 of a 3-manifold M with a lattice, and acted by translating on that lattice. And we thought of$H^2(M)$ as $\pi_0(\mathrm{Pic}(M))$.

The analogue of Wilson operators here is $\rho \in \mathrm{Loc}(C) \mapsto \rho(\mathrm{Fr}_x)$: on the 3-manifold M, we sent a local system to multiplication by the monodromy around each loop.

In algebraic geometry, we can't really talkm about local systems as maps from π_1, so we'll avoid infinite covers. Finite covers correspond to maps $\pi_1 \to \mu_n \subset \mathbb{Q}/\mathbb{Z}$. And in characteristic p, the theory only behaves well when $p \nmid n$. So we will work with a prime ℓ different from p; then there is a nice theory of ℓ-adic local systems. Specifically, we consider maps $\pi_1 \to \mathbb{Z}/\ell^n$, which sits inside $\mathbb{Z}_\ell$, or we could even go all the way to $\mathbb{Q}_\ell$ or $\overline{\mathbb{Q}}_\ell$. There is a good theory of ℓ-adic representations of π_1 into, e.g. $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$.

The nice thing about $\overline{\mathbb{Q}}_\ell$ is that there is a (highly noncanonical and difficult-to-describe) isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$, basically because they're both algebraically closed fields of characteristic zero. And the isomorphism certainly doesn't respect the topology!

So what we'll mean by local systems is something like continuous (in the profinite topology) representations from the unramified Galois group of C to $\overline{\mathbb{Q}}_\ell^\times$. And in this situation, everywhere we would've said "complex-valued functions," we now mean $\mathbb{Q}_\ell$-valued functions. The shape of the story is the same, even if the details are different.

Key to our generalizations will be how we identify $\mathrm{Pic}(C)$. We've been thinking of it as the group of divisors mod principal divisors, a quotient

$$\left(\bigoplus_{x\in C} \mathbb{Z}\right) / k(C)^\times. \tag{9.6}$$

This is related to line bundles as follows: given a line bundle $\mathcal{L}$, we can trivialize it generically, meaning there is a smooth section of $\mathcal{L}$ on all but finitely many points $x_1, \dots, x_n$, defining an isomorphism $\mathcal{O}_C|_{C\setminus\{x_i\}} \xrightarrow{\cong} \mathcal{L}|_{C\setminus\{x_i\}}$.

On the other hand, for all $x \in C$, we can trivialize $\mathcal{L}$ very close to x. By this we mean take a disc around x, $D_x \coloneqq \mathrm{Spec}(\mathcal{O}_x)$ ($\mathcal{O}_x$ means the *completed* local ring at x, which a choice of coordinate identifies with $k[[t]]$). The pullback of $\mathcal{L}$ to D_x is trivialized.

The local degree at x of a section s, $\deg_x s$, is nonzero, but by changing the trivialization of $\mathcal{L}$ on D_x, we get "the same" local degree. So we realy should be looking at $K_x^\times/\mathcal{O}_x^\times$. Choosing a coordinate defines an isomorphism $k((t))^\times/k[[t]]^\times$. Concretely, these are equivalence classes of Laurent series with nonzero leading coefficient, where series are equivalent if they agree after multiplying by an invertible Taylor series. This means that we can zero out all but the leading term t^{-N}, so we end up identifying this quotient with $\mathbb{Z}$, which is where degrees ought to live. Great!

Now we want to vary the line bundle. This means we have to vary the locus $U := C \setminus \{x_i\}$ where the bundle is trivialized. We will address this by using all points, including huge amounts of data: a line bundle $\mathcal{L}$, a rational section, and a trivialization of $\mathcal{L}|_{D_x}$ for *all* $x \in C$. This is some huge product of $K_x^\times$ for all $x \in C$, but we only see some of the elements of this product. This is the *restricted product*

$$\prod_{x\in C}{}' K_x^\times \subset \prod_{x\in C} K_x^\times, \tag{9.7}$$

meaning that for all but finitely many $x \in C$, we obtain an element of $\mathcal{O}_x^\times \subset K_x^\times$. This is still a lot of data! So let's now reduce the data.

Now if you take line bundles and rational sections, you end up with divisors, which is reassuring: we get

$$\prod{}' K_x^\times / \prod \mathcal{O}_x^\times = \prod{}'(K_x^\times/\mathcal{O}_x^\times) = \prod{}' \mathbb{Z}, \tag{9.8}$$

i.e. finite formal sums of points $x \in C$, which is what divisors are. (TODO: I need to fix these restricted products.)

So we can identify the Picard group with a double quotient

$$F^\times \backslash \prod{}' K_x^\times / \prod \mathcal{O}_x^\times, \tag{9.9}$$

where on the left, $F^\times$ accounts for changes in section, and on the right, the product accounts for changes of trivialization on D_x. This is the group of divisors modulo principal divisors ($F^\times$). In the number field setting this corresponds to the *(unramified) idele class group* of F.

Lecture 10.

: 2/25/21

Lecture 11.

Boundary conditions and extended TFT: 3/2/21

> *"There's a pun here, which has been left as an exercise for Arun"* (Note: I have not yet solved this exercise)

Note: I missed Thursday's lecture. I'll watch the video and update notes soon. In the meantime, check out Jackson's notes.

Today we're going to consider ramification, which corresponds in physics to boundary theories, or considering operators on singularities. (TODO: some stuff I missed about physics examples. In particular, solenoids, and Cartier duality)

So under the equivalence $\mathrm{Pic}(C) \simeq \mathrm{Loc}_1(C)$, we can generalize in two ways: we can put *level structures* on line bundles on the left, adding trivializations along positive-codimension regions. On the right, we can add singularities to our local systems. And the full story of class field theory, which takes into account ramifications, shows that the correspondence upgrades to including these generalizations too.

In field theory, we have a space of fields $\mathcal{F}(M)$ on the manifold M, and there is a restriction map $\pi_{\partial M} \colon \mathcal{F}(M) \to \mathcal{F}(\partial M)$. Boundary data modifies how we attach a vector space to $N := \partial M$: specifically, we choose a sheaf $\mathcal{E}_N \to N$, and we use this to write down a different vector space. For example, we could attach to M the vector space $\Gamma_M(\pi_{\partial M}^*\mathcal{E}_N)$.

Example 11.1. In topological Maxwell theory (the A-side), the vector space we attached to M is $H^*(\mathcal{C}(M))$, the cohomology of the space of U_1-connections on M. Recall that at least at the level of components, $\pi_0(\mathcal{C}(M)) \cong H^2(M;\mathbb{Z})$. If we consider a submanifold $N \subset M$ and consider boundary data, we would like to replace this with something that looks like relative cohomology: $H^2(M, N; \mathbb{Z})$.

First, let's remember how we got $H^2(M;\mathbb{Z})$. This arose by showing that $\mathcal{C}(M) \simeq \mathrm{Map}(M, K(\mathbb{Z},2)) = \mathrm{Map}(M, B\mathrm{U}_1)$: the space of connections on a principal U_1-bundle is contractible, and the topological type is classified by $B\mathrm{U}_1 \simeq K(\mathbb{Z},2)$. Now, given a bundle and connection, we can restrict to N, defining a map

$$\mathrm{Map}(M, K(\mathbb{Z},2)) \longrightarrow \mathrm{Map}(N, K(\mathbb{Z},2)) = \mathrm{Map}(N, B\mathrm{U}_1). \tag{11.2}$$

One sheaf we can consider is $\mathcal{E} := i_*\underline{\mathbb{C}}$ (where $i\colon \mathrm{pt} \hookrightarrow N$ is an inclusion map); then $H^*(\mathcal{C}(M); \pi_N^*\mathcal{E})$ is the cohomology of the space of line bundles on M with data of a trivialization on N. So this is an example of a *Dirichlet boundary condition.*

On connected components, this is preciely $H^2(M, N;\mathbb{Z})$, namely the group of connected components of the space of maps $f\colon M \to K(\mathbb{Z},2)$ together with null-homotopies of the restriction $f|_N$. This space is the pullback of the diagram

$$\begin{tikzcd} {} \arrow[r] \arrow[d] & \mathrm{Map}(M, K(\mathbb{Z},2)) \arrow[d] \\ \mathrm{pt} \arrow[r] & \mathrm{Map}(N, K(\mathbb{Z},2)). \end{tikzcd} \tag{11.3}$$

Dually, we can consider the *Neumann boundary condition*: let M_0 denote the complement of a tubular neighborhood of N in M, so M_0 is a manifold with boundary. Again we take the constant sheaf on $\mathcal{F}(\partial M_0) \leftarrow \mathcal{F}(M_0)$, and pull back (to the constant sheaf again), considering $H^*(\mathcal{F}(M_0), \underline{\mathbb{C}})$. This is the cohomology of the space of fields which can have arbitrary singularities on N. But we could also take the pullback of a skyscraper sheaf, which is equivalent to considering the cohomology of the space of connections on M_0 with trivializations on the boundary. There are plenty of more general conditions you can put here; each sheaf on the boundary is a machine that gives you a new vector space on the bulk manifold. ◀

We'll come back to the number-theoretic aspects of this in a bit, but first, extended topoloical field theory!

Recall that topological field theory attaches to a closed n-manifold some number (the partition function), which can be heuristically thought of as the volume of the space of fields. And to a closed $(n-1)$-manifold N, we attached a complex vector space (often a Hilbert space), which we think of as the space of functionals on the space of fields on M. We arrived at this by breaking relativistic symmetry and thinking of N as being at a single instant of time in its cylinder.

Now we will go further: given a closed $(n-2)$-manifold P, we will assign a category, often something like the category of sheaves on $\mathcal{F}(P)$. In general, this category will be "linear," e.g. Hom-spaces are complex vector spaces; thus this is another kind of linearization of the fields, albeit more categorified.

Traditionally, this arose from the Atiyah-Segal (TODO: citations here) definition of topological field theory, defined as a symmetric monoidal functor

$$Z\colon \mathcal{B}ord_{n-1,n} \longrightarrow \mathcal{V}ect. \tag{11.4}$$

Here $\mathcal{B}ord_{n-1,n}$ is the category whose objects are closed, oriented $(n-1)$-manifolds and whose morphisms $N_0 \to N_1$ are (diffeomorphism classes of)[10] bordisms from N_0 to N_1. "Symmetric monoidal" means that disjoint unions of manifolds are sent to tensor products in $\mathcal{V}ect$. Composition is gluing bordisms.

The idea is that a bordism from N_0 to N_1 gives you a linear map $Z(N_0) \to Z(N_1)$ between the vector spaces assigned to its boundary. And a closed n-manifold M is a bordism $\varnothing \to \varnothing$, hence defines a linear map $\mathbb{C} \to \mathbb{C}$, which we identify with a complex number, which is the partition function of M.

The bordism-theoretic perspective tells us that it's possible to compute the partition function by chopping M up into pieces, specifically bordisms, and then computing what Z assigns to those bordisms, then composing to obtain the partition function. Physically speaking, this expresses the locality of the partition function in spacetime.

But state spaces also ought to be local! Extended TFT is telling us how to repeat this story one dimension lower and one category number higher. If N is a closed $(n-1)$-manifold that factors as a composition of bordisms $N = N_1 \cup_P N_2$, we would like to express

$$Z(N) = \langle Z(N_1), Z(N_2)\rangle_{Z(P)}, \tag{11.5}$$

in some categorified sense of the pairing, just as we computed the partition function by gluing. This perspective was articulated by Freed, Lawrence, and Dolan among others.

[10]This parenthetical is important for composition to be associative on the nose.

To make this precise, we build a bordism 2-category $\mathcal{B}ord_{n-2,n}$, supplying the following data.

Objects: Closed, oriented $(n-2)$-manifolds.

Morphisms: Compact, oriented bordisms, as before.

2-morphisms: In a 2-category, there are now "2-morphisms" between morphisms. Here we allow bordisms between bordisms N_0 and N_1, thought of as manifolds with corners.

For 2-morphisms, $\partial N_0 = \partial N_1$, so rather than thinking of $[0,1]\times[0,1]$ as your prototypical manifold-with-corners, you should think of the eye-shaped quotient $([0,1]\times[0,1])/((0,a)\sim(0,b),(1,a)\sim(1,b))$.

The bordism 2-category is again symmetric monoidal with respect to disjoint union, though the definition of a symmetric monoidal 2-category is a bit more involved.

Definition 11.6. A *two-tier TFT* or *once-extended TFT* is a symmetric monoidal functor of 2-categories

$$Z\colon \mathcal{B}ord_{n-2,n} \longrightarrow \mathcal{C}, \tag{11.7}$$

where $\mathcal{C}$ is some symmetric monoidal 2-category.

In this case, it's not immediately clear what $\mathcal{C}$ should be, so we just leave that as a choice. Generally, though, we want this to generalize the Atiyah-Segal definition of TFT, so we stipulate that $\mathcal{C}$ is a *delooping* of $\mathcal{V}ect$. That is, $\Omega\mathcal{C}$, defined to be $\mathrm{End}(1_{\mathcal{C}})$, is equivalent to $\mathcal{V}ect$ as symmetric monoidal categories.

Now, if N is an $(n-1)$-manifold factored as $N_1\cup_P N_2$ as before, we can write N as a composition of morphisms in $\mathcal{B}ord_{n-2,n}$, and therefore exploit locality to compute $Z(N)$, at least in principle.

To compute this in practice we ought to choose $\mathcal{C}$. There are multiple choices that work well, but we'll let $\mathcal{C} = \mathcal{C}at_{\mathbb{C}}$, the symmetric monoidal 2-category of $\mathbb{C}$*-linear categories*. Specifically:

Objects: (Small) categories enriched over $\mathcal{V}ect_{\mathbb{C}}$.

Morphisms: Functors which respect this enrichment.

2-morphisms: Natural transformations which respect this enrichment.

And sure enough, $\Omega\mathcal{C}at_{\mathbb{C}} \simeq \mathcal{V}ect$.

Though we didn't define (topologial) Maxwell theory formally as an Atiyah-Segal TFT, it is still useful to think from this perspective: given a closed, oriented 2-manifold Σ, Maxwell theory ought to assign a $\mathbb{C}$-linear category, and if we factor a closed 3-manifold N as $N \cong N_1 \amalg_\Sigma N_2$, we can compute the vector space assigned to N using this decomposition.

What does abelian duality look like in codimension 2? We have several examples to look into (this is vacuous in dimensions 1 and below, so we start at dimension 2).

Example 11.8. In dimension 2, we had two examples of abelian duality: in one, $H^0(S^1;\mathbb{Z}) \leftrightarrow H^1(S^1;\mathbb{C}^\times)$, and in the other, $H^1(S^1;\mathbb{Z}) \leftrightarrow H^0(S^1;\mathbb{C}^\times)$. These are interpreted as different field theories; let's extend them to codimension 2. First the first example:

- On the A-side ($H^0(S^1;\mathbb{Z})$), the fields are maps to $\mathbb{Z}$. Maybe not the most interesting, but we can extend: $\mathcal{F}(\mathrm{pt}) = \mathbb{Z}$, and we linearize by taking sheaves. There're not many options for what kind of sheaves we take here, and in the end $Z(\mathrm{pt}) = \mathcal{S}hv(\mathbb{Z})$, or the category of $\mathbb{Z}$-graded vector spaces. This is $\mathbb{C}$-linear, which is good.
- Now on the B-side, corresponding to $H^1(S^1;\mathbb{C}^\times)$. The fields are $\mathbb{C}^\times$-local systems, which is pretty boring on pt: the set of isomorphism classes of $\mathbb{C}^\times$-local systems is a singleton, and sheaves on this is not just boring, but doesn't match what we got on the A-side. To remedy this we have to introduce... stacks.[11] The space of local systems enhances to a stack $\mathcal{L}oc_{\mathbb{C}^\times}(\mathrm{pt})$ encoding the fact that local systems have symmetries. In this specific case, we have a single isomorphism class of local systems, but it has lots of isomorphisms, a whole $\mathbb{C}^\times$ of them. Therefore this stack is equivalent to $B\mathbb{C}^\times = \mathrm{pt}/\mathbb{C}^\times$. We want to take sheaves on this, which here means sheaves on a point together with a $\mathbb{C}^\times$-action on the (unique) stalk. That is, what we assign to a point in the B-side field theory is the category of representations of $\mathbb{C}^\times$. Again this is $\mathbb{C}$-linear.

Duality means we better get an equivalence of categories between $\mathcal{V}ect_{\mathbb{Z}}$ and $\mathcal{R}ep_{\mathbb{C}^\times}$, and this is given by the isotypic decomposition: if V is a $\mathbb{C}^\times$-representation, it splits as

$$V = \bigoplus_{n\in\mathbb{Z}} V_n, \tag{11.9}$$

[11] If you don't know anything about stacks, that's OK, and we'll discuss some important things about them next time.

where $V_n \subset V$ is the subspace on which $z \in \mathbb{C}^\times$ acts by z^n.

This can be thought of Fourier-theoretically, by thinking of these two categories as the categories of $\mathbb{C}[\mathbb{Z}]$-modules (A-side) or $\mathbb{C}[\mathbb{C}^\times]$-modules (B-side). The Fourier transform requires thinking a little bit about comodules, so we won't get into the details right now.

Now the other example, $H^1(S^1;\mathbb{Z}) \leftrightarrow H^0(S^1;\mathbb{C}^\times)$. Again, the A-side will involve topology, and the B-side will involve complex geometry.

- The A-side used (functions on) $H^1(S^1;\mathbb{Z})$. We can think of this as isomorphism classes of $\mathbb{Z}$-valued local systems, but it is more convenient to map to $S^1 = B\mathbb{Z}$, so periodic functions. Now we try to extend to a point: fields are maps to a point, so we get $\mathcal{F}(\mathrm{pt}) = S^1 = B\mathbb{Z}$. When we take sheaves, that could mean more than one thing, but there's only one notion from the topological point of view: locally constant sheaves, i.e. $\mathcal{L}oc_{\mathbb{Z}}(S^1)$. These are representations of $\mathbb{Z}$, or sheaves on $B\mathbb{Z}$, where here $\mathbb{Z}$ comes from $\pi_1(S^1)$. So we end up with $\mathbb{C}[\mathbb{Z}]$-modules, with the algebra structure on $\mathbb{C}[\mathbb{Z}]$ coming from convolution.

  This has a fancy name: the *wrapped Fukaya category* of T^*S^1. In general this is a difficult object to study for a given symplectic manifold X, but when X is a cotangent bundle it admits a description as local systems.
- On the B-side, we had $H^0(S^1;\mathbb{C}^\times)$. The fields are maps to $\mathbb{C}^\times$ with the discrete topology (we're doing algebraic geometry on this side, not topology). The fields on a point are therefore $\mathbb{C}^\times$, and when we take sheaves, here we use algebraic geometry and mean quasicoherent sheaves on $\mathbb{C}^\times$. This is the category of $\mathbb{C}[\mathbb{C}^\times]$-modules, where the algebra structure is pointwise multiplication. Pontrjagin duality identifies $\mathbb{C}[\mathbb{C}^\times]$ with pointwise multiplication with $\mathbb{C}[\mathbb{Z}]$ with convolution, and therefore we see that $\mathcal{L}oc(S^1) \simeq \mathcal{QC}(\mathbb{C}^\times)$.

  This is a very special case of homological mirror symmetry, which in general relates Fukaya categories on the A-side (in symplectic topology, which is why things were more topological on the A-side) with categories of quasicoherent sheaves on the B-side (hence we care about algebro-geometric notions of sheaves). ◂

You can run a roughly similar game in 3d, but let's jump to the key example: 4d Maxwell theory. On the A-side, we care about the topology of line bundles, and on the B-side, we care about the algebraic geometry of *flat* line bundles. We understood how this works in codimension 1, where we had a Pontrjagin-Poincaré duality between $H^2(M;\mathbb{Z})$ and $H^1(M;\mathbb{C}^\times)$, where M is a closed, oriented 3-manifold; now let's see what this attaches to a closed, oriented topological surface Σ.

On the A-side, we only have topology, and our fields are line bundles with connection. We got H^2 by taking π_0 of this space. So to Σ, we attach the space of fields

$$\mathrm{Pic}(\Sigma) := \mathrm{Map}(\Sigma, B\mathrm{U}_1). \tag{11.10}$$

Choose a complex structure on Σ; then there is the more familiar $\mathrm{Pic}(\Sigma)$, the space of holomorphic or algebraic line bundles on Σ, an object in algebraic geometry. This space depends on the complex structure, but its topological type does not, and is in fact homotopy equivalent to $\mathrm{Map}(\Sigma, B\mathrm{U}_1)$. Again this factors as

$$0 \longrightarrow \mathrm{Jac}(\Sigma) \longrightarrow \mathrm{Pic}(\Sigma) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0, \tag{11.11}$$

where $\mathrm{Jac}(\Sigma)$ is the space of degree-0 line bundles. Suppose Σ is connected and has genus g; then, the homotopy type of Jac is a torus $\mathbb{C}^g/\mathbb{Z}^{2g}$ of real dimension $2g$.

Remark 11.12. Line bundles have automorphisms, and therefore $\mathrm{Pic}(\Sigma)$ and $\mathrm{Jac}(\Sigma)$ are really stacks, but we don't need to worry about this at present. ◂

The short exact sequence (11.11) is reminiscent of the *exponential exact sequence* of complex geometry:

$$0 \longrightarrow H^1(\Sigma;\mathcal{O}_\Sigma)/H^1(\Sigma;\mathbb{Z}) \longrightarrow H^1(\Sigma;\mathcal{O}_\Sigma^\times) \longrightarrow H^2(\Sigma;\mathbb{Z}). \tag{11.13}$$

The identifications $H^1(\Sigma;\mathcal{O}_\Sigma^\times) \cong \mathrm{Pic}(\Sigma)$ (again, ignoring stackiness) and $H^2(\Sigma;\mathbb{Z}) \cong \mathbb{Z}$, together with the connecting morphism with the degree map, means that

$$\mathrm{Jac}(\Sigma) \cong H^1(\Sigma;\mathcal{O}_\Sigma)/H^1(\Sigma;\mathbb{Z}). \tag{11.14}$$

This identifies the Jacobian with $K(H^1(\Sigma;\mathbb{Z}),1)$, a torus with fundamental group $H^1(\Sigma;\mathbb{Z})$. This torus can also be described as $S^1 \otimes_{\mathbb{Z}} H^1(\Sigma;\mathbb{Z})$.

What we attach to Σ is the $\mathbb{C}$-linear category $\mathrm{Loc}(\mathrm{Pic}(\Sigma))$.[12] We will see on Thursday how this is related to the category that the B-side theory attaches to Σ.

Lecture 12.

Extended TFT and topological Maxwell theory: 3/4/21

As we began discussing last time, extended TFT tells us to not just assign complex numbers to 4-manifolds[13] and vector spaces to 3-manifolds, but also $\mathbb{C}$-linear categories to closed surfaces. And given a 3-manifold Y with boundary Σ, we get a linear functor $Z(\Sigma) \to Z(\varnothing) = \mathcal{V}ect$. This functor assigns a vector space to Y together with data of an object $\mathcal{E} \in Z(\Sigma)$. Dually, we could switch incoming and outgoing, so we get a linear functor $Z(Y)\colon \mathcal{V}ect \to Z(\Sigma)$, giving us a distinguished object $Z(Y)(\mathbb{C})$.

(TODO: an example with sheaves, I think?)

We also discussed some examples last time, most notably topological Maxwell theory. On the A-side, which is about the topology of spaces of U_1-bundles, which we called "Pic," we considered an oriented surface Σ, chose a complex structure, and then considered the space $\mathrm{Pic}(\Sigma)$ of line bundles on Σ. The homotopy type of this space does not depend on the complex structure, and noncanonically it splits as

$$\mathrm{Pic}(\Sigma) \cong \mathbb{Z} \times \mathrm{Jac}(\Sigma) \times B\mathrm{U}_1, \tag{12.1}$$

where the last factor, which we ignore, is about constant maps into $B\mathrm{U}_1$ (giving trivial line bundles). You can think of this as stackiness: $B\mathrm{U}_1$ is trying to be pt/U_1, giving us additional automorphisms of line bundles. But we don't have to worry about it right now.

The space $\mathrm{Jac}(\Sigma) \cong \mathbb{C}^g/\mathbb{Z}^{2g}$, assuming Σ is connected and has genus g. This is because it arises as $H^1(\Sigma; \mathcal{O}_\Sigma)/H^1(\Sigma; \mathbb{Z})$. We care about topological stuff only, and so we end up with a torus, which is also an Eilenberg-Mac Lane space $K(H^1(\Sigma; \mathbb{Z}), 1)$.

We have a space and now should linearize to produce a category, so we will take a category of sheaves. Just as there were different specific kinds of functions you can take when someone says "take functions on a space," which is a functional analysis question, we can consider different kinds of sheaves. We will take locally constant ones, so the category we assign to Σ is $\mathrm{Loc}(\mathrm{Pic}(\Sigma))$. We're not diving into the derived world yet (which means we ignore the $B\mathrm{U}_1$ in (12.1) for now, as $B\mathrm{U}_1$ is simply connected), so $\mathrm{Loc}(\mathrm{Pic}\Sigma)$ decomposes as

$$\mathrm{Loc}(\mathrm{Pic}\Sigma) \simeq \mathcal{V}ect_{\mathbb{Z}} \otimes \mathrm{Loc}(\mathrm{Jac}(\Sigma)), \tag{12.2}$$

where $\mathcal{V}ect_{\mathbb{Z}}$ means $\mathbb{Z}$-graded vector spaces. But because the Jacobian is a torus, local systems on it are canonically identified with representations of $\Lambda \coloneqq H^1(\Sigma; \mathbb{Z})$, and therefore $\mathrm{Loc}(\mathrm{Pic}(\Sigma))$ is identified with the category of $\mathbb{Z}$-graded Λ-representations. This has a monoidal structure under convolution, which is not told to us by the TFT.

Now the B-side, which is about the algebraic geometry of $\mathbb{C}^\times$-local systems. We have our surface Σ again, but we don't even need to choose a complex structure to consider $\mathrm{Loc}_{\mathbb{C}^\times}(\Sigma) = \mathrm{Hom}(\pi_1(\Sigma), \mathbb{C}^\times)/\mathbb{C}^\times$ — but since $\mathbb{C}^\times$ is abelian, it acts trivially by conjugation. And since $H_1(\Sigma) = \pi_1(\Sigma)^{\mathrm{ab}}$, this space of local systems is identified with

$$\mathrm{Hom}(H_1(\Sigma), \mathbb{C}^\times)/\mathbb{C}^\times \cong (H^1(\Sigma; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}^\times)/\mathbb{C}^\times, \tag{12.3}$$

where again $\mathbb{C}^\times$ acts trivially. We should attach a category to this, and something algebraic geometry flavored, such as $\mathcal{QC}$. What do we mean precisely by this? $\mathrm{Loc}_{\mathbb{C}^\times}(\Sigma)$ looks like a torus again, but this extra $\mathbb{C}^\times$-action can't be gotten rid of, and so the category of sheaves is the category of modules for $\mathbb{C}[\mathrm{Loc}_{\mathbb{C}^\times}(\Sigma)]$ under the "multiplication" monoidal structure, tensored with $\mathcal{R}ep_{\mathbb{C}^\times}$.

We want this category to be equivalent to the category of locally constant sheaves on $\mathrm{Pic}(\Sigma)$ that we constructed on the A-side. TODO: I missed this, but the key ingredient is an identification of algebras $\mathbb{C}[\Lambda]$ with convolution and $\mathbb{C}[T_{\mathbb{C}}^\vee]$ under pointwise multiplication, which therefore induces an equivalence of sheaves. This is a first example of *Fourier-Mukai duality*, abelian duality identifying categories of sheaves. We threw in this additional $\mathbb{Z}$-grading on the A-side and the $\mathbb{C}^\times$-action on the B-side, though.

This is the first version of the geometric Langands correspondence, albeit a very simple one. It is specifically the Betti geometric Langlands equivalence for $G = \mathrm{GL}_1$.

[12] Stackiness does not affect this abelian category, though it will affect the derived category.

[13] This particular aspect of TFT will generally not come up in this class.

Next, we'll connect this to the 3-manifold story, and then explain the number-theoretic analogue. Let Y be a compact, oriented 3-manifold with $\partial Y \cong \Sigma$. We can restrict line bundles from Y to Σ, which defines a map $\pi\colon \mathrm{Pic}(Y) \to \mathrm{Pic}(\Sigma)$; we can then try to push forward or pull back sheaves along π. For example, choosing $\mathcal{E} \in \mathrm{Loc}(\mathrm{Pic}\Sigma)$, we want to obtain a vector space, which we did by pulling back to $\mathrm{Pic}(Y)$ and taking sections (rather than just taking functions, as we did when Y is closed).

On the B-side, we have $\pi^\vee\colon \mathrm{Loc}_{\mathbb{C}^\times} Y \to \mathrm{Loc}_{\mathbb{C}^\times}(\Sigma)$, so we can again pull back $\mathcal{E}^\vee \in \mathcal{QC}(\mathrm{Loc}_{\mathbb{C}^\times}(\Sigma))$ and take sections. The Fourier-Mukai transform includes a correspondence $\mathcal{E} \leftrightarrow \mathcal{E}^\vee$ inducing an isomorphism

$$\Gamma(\mathrm{Pic}(Y), \pi^*\mathcal{E}) \cong \Gamma(\mathrm{Loc}_{\mathbb{C}^\times}(Y), (\pi^\vee)^*\mathcal{E}^\vee). \tag{12.4}$$

This all looks fancy, but remember that these spaces are just tori, and everything here is really coming from a combination of Poincaré and character duality.

We can now ramify. Let's consider the subcategory of local systems which when we restrict to Σ land in a particular subset. One of the motiviations for doing this was number-theoretic: ramifications are where everything interesting happens in number theory.

So let's let C be a curve over a finite field $\mathbb{F}_q$. Let x be an $\mathbb{F}_q$-point of C; we will talk about allowing ramification at x. Let K_x be the completion of the field of rational functions at x, so $K_x \cong \mathbb{F}_{q'}((t))$, where $\mathbb{F}_q \hookrightarrow \mathbb{F}_{q'}$ is a finite extension. Then $\operatorname{Spec} K_x$ is a formal disc around $C \setminus x$. Recall that points corresponded to circles in topology, so the boundary of a tubular neighborhood of a knot coresponds to what we're looking at here.

On the B-side (Galois side), we restrict

$$\mathrm{Loc}_1(C \setminus x) \to \mathrm{Loc}_1(D) = \mathrm{Hom}_{\mathrm{cts}}(\mathrm{Gal}(\overline{K}_x/K_x), \overline{\mathbb{Q}}_\ell^\times) = \mathrm{Hom}_{\mathrm{cts}}(\mathrm{Gal}^{\mathrm{ab}} K_x, \overline{\mathbb{Q}}_\ell^\times) \tag{12.5}$$

And what about $\mathrm{Pic}(C)$? We described this as

$$\mathrm{Pic}(C) = F^\times \backslash {\prod_{y\in C}}' K_y^\times \Big/ \prod_{y\in C} \mathcal{O}_Y^\times. \tag{12.6}$$

Now we have to remove the point x. We take the restricted products over C and now work over $C \setminus x$, so we obtain

$$\mathrm{Pic}(C)/(\mathbb{Z}_x = K_x^\times/\mathcal{O}_x^\times). \tag{12.7}$$

This is not a fun space, just like if you killed the $\mathbb{Z}$ in $\mathrm{Pic}(\Sigma)$ happening at a point on a Riemann surface Σ. But it will be useful for us, so let's call it $\mathrm{Pic}(C, \widehat{x})/K_x^\times$; this notation is justified by the third isomorphism theorem. $\mathrm{Pic}(C, \widehat{x})$ is line bundles on C with a trivialization on D_x, or equivalently, line bundles on $C \setminus x$ and a trivialization on $D_x^\times$. Taking the quotient by $K_x^\times$ gets rid of that trivialization on the formal disc.

So we don't need to puncture the curve in other to study the punctured cuve: you just have to know how $K_x^\times$ acts there. There is a single isomorphism class of line bundles on the punctured disc $D_x^\times$, but there are lots of functions on $D_x^\times$, so this line bundle has a lot of automorphisms. So as a stack,

$$\mathrm{Pic}(D_x)^\times \cong \mathrm{pt}/K_x^\times, \tag{12.8}$$

and studying this picture is equivalent to studying the action of $K_x^\times$ on $\mathrm{Pic}(C, \widehat{x})$, which feels much less like abstract nonsense and much more like classical number theory. However, number theorists say to not take infinite level structure. Specifically, $K_x^\times$ contains $\mathcal{O}^\times = \mathbb{F}_{q'}[[t]]$, and this contains subrings corresponding to congruence subgroups $\mathcal{O}^{\times(n)}$, functions of the form $1 + t^n(-)$, thought of as a trivialization to level n. So the action of $K_x^\times$ on $\mathrm{Pic}(C, nx)$ (sort of the n^{th} order neighborhood) factors through $K_x^\times/\mathcal{O}^{\times(n)}$.

Until very recently, there weren't great techniques to deal with this, so let's look at smooth representations of $K^\times$, meaning not what you might usually expect over $\mathbb{C}$, but rather something more p-adic: that every vector is fixed by $\mathcal{O}^{\times N}$ for some $N \gg 0$. For example, if we give the object acted on the discrete topology, this imposes continuity, because it is expressing the topology on $K_x^\times$.

TODO: something I missed, comparing this to a surface bounding a 3-manifold. But analogously to moving sheaves between the bulk and the boundary there, we push sheaves around on $C \setminus x$ and $D_x^\times$.

How do these disparate pieces of data talk to each other? On the A-side we had representations of $K_x^\times$ coming from $\mathrm{Pic}(C, \widehat{x})$, and on the B-side we have sheaves on $\mathrm{Loc}_1(\mathcal{O}^\times)$, a module for functions on $\mathcal{O}^\times$. We need an analogue of the Fourier-Mukai theorem to get between these two things that are like categories of sheaves. Local class field theory is all about this — though it does not usually get expressed in this equivalence-of-categories manner. Just like global class field theory is usually posited as an isomorphism between the maximal unramified abelian Galois group with the completed Picard group, as we discussed, and then reinterpreted in an abelian duality manner using Pontrjagin duality, we will do something similar with local class field theory, an isomorphism

$$\widehat{K}_x^\times \xrightarrow{\cong} \mathrm{Gal}^{\mathrm{ab}}(\overline{K}_x/K_x). \tag{12.9}$$

Here we profinitely complete the left-hand side. Anyways this mere isomorphism of abelian groups tells us that the group ring of $K_x^\times$ is Pontrjagin dual to $\mathrm{Hom}(\mathrm{Gal}^{\mathrm{ab}} K, \overline{\mathbb{Q}}_\ell^\times)$, which we're calling "Loc." Again on one side we have a group algebra with convolution, and on the other we have a group algebra with multiplication. Instead of $\mathbb{C}$, our group algebras are over $\overline{\mathbb{Q}}_\ell$. One has to be careful with topology here.

Now, pass to modules, which tells us that representations of $K_x^\times$ are equivalent to quasicoherent sheaves on Loc_1. Making this precise takes a lot of work, to be clear. But the point is that representations of $K_n^\times$ on one side is identified with the algebraic geometry of a space of local systems, just as in the topological story.

We want to think of (12.9) as identifying representations of $K_x^\times$ with modules for functions on GL_1 Galois representations. We don't have a moduli-theoretic interpretation of $\mathrm{Pic}(C, \widehat{x})$ here, which is why we take adeles:

$$F^\times \backslash \prod_v{}' K_v^\times / \prod_{v\notin S} \mathcal{O}_v^\times, \tag{12.10}$$

which is the ramified idele class group, and which is acted on by a large abelian group, namely $\prod_{v\in S} K_v^\times$.

Next time, we'll finish up abelian groups, understanding the duality we discussed today on Riemann surfaces as a spectral decomposition or Fourier transform. This will pave the way to the nonabelian generalization that is the goal of the course.

Lecture 13.

Geometric class field theory: 3/9/21

Today we'll discuss geometric class field theory, and begin (possibly into the next lecture) discussing how this is a spectral decomposition in a categorical sense.

Let Σ be a Riemann surface. Then $\mathrm{Pic}(\Sigma)$ fits into the short exact sequence

$$0 \longrightarrow \mathrm{Jac}(\Sigma) \longrightarrow \mathrm{Pic}(\Sigma) \longrightarrow \mathbb{Z} \longrightarrow 0, \tag{13.1}$$

and $\mathrm{Jac}(\Sigma)$ is a torus, a quotient of some vector space by $H^1(\Sigma;\mathbb{Z})$, and therefore by Poincaré duality, $\pi_1(\mathrm{Jac}(\Sigma)) \cong H_1(\Sigma;\mathbb{Z})$.[14] That is, the fundamental group of the Jacobian is closely related to that of Σ itself: $\pi_1(\Sigma)^{\mathrm{ab}} \cong \pi_1(\mathrm{Jac}(\Sigma))$.

For A an abelian group, $\mathrm{Loc}_A(\Sigma) = \{\pi_1(\Sigma) \to A\}/\cong$, and maps $\pi_1(\Sigma) \to A$ factor through the abelianization, which is $\pi_1(\mathrm{Jac}(\Sigma))$. So for these local systems, or abelian covers of Σ, we might as well use $\mathrm{Jac}(\Sigma)$ – the isomorphism classes of local covers are in bijection. We are particularly interestd in $A = \mathbb{C}^\times$, so $\mathrm{Loc}_A(\Sigma) = \mathrm{Hom}(\widehat{A}, \mathrm{Loc}_1(\Sigma))$, where $\mathrm{Loc}_1(\Sigma) \cong \mathrm{Loc}_1(\mathrm{Jac}(\Sigma))$.

If we pick a basepoint $e \in \Sigma$, we get a map $\Sigma \to \mathrm{Jac}(\Sigma)$ sending $x \mapsto x - e$, identifying Pic^1 and $\mathrm{Pic}^0 = \mathrm{Jac}$. This gives you the *Abel-Jacobi map* $AJ^*\colon \mathrm{Loc}_1(\mathrm{Pic}^1) \xrightarrow{\cong} \mathrm{Loc}_1(\Sigma)$. This is a way of saying that A-covers of Pic^1 are identified with A-covers of Σ.

TODO: this has something to do with the *Albanese property* (TODO: I missed a little here): if A is any abelian group scheme, maps $\Sigma \to A$ factor through the Abel-Jacobi map $\Sigma \to \mathrm{Pic}^0(\Sigma)$. This is classically due to Lang-Rosenlicht; there are more general modern formulations, such as a nice generalization to stacks due to Justin Campbell [Cam17]. That is, AJ^* is an isomorphism

$$\mathrm{Map}(\Sigma, A) \xleftarrow{\cong} \mathrm{Hom}_{\mathcal{G}rp}(\mathrm{Pic}(\Sigma), A). \tag{13.2}$$

You can conceive of this as the statement that "$\mathrm{Pic}(\Sigma)$ is the abelian group scheme freely generated by Σ."

Something like this works in more generality: if X is a smooth projective variety, there is an *Albanese variety* $\mathrm{Alb}(X)$, an abelian group scheme, which can be constructed from 0-cycles on X modulo an equivalence relation. (That is, 0-cycles are $\mathbb{Z}$-linear combinations of points, and we identify them if they fit into a

[14] In a little more detail, $\pi_1(\mathrm{Jac}(\Sigma))$ is canonically $H^1(\Sigma;\mathbb{Z})$, and Poincaré duality gets us $H_1(\Sigma;\mathbb{Z})$.

$\mathbb{P}^1$-family.) Alb(X) has the universal property that any map from X to an abelian group scheme factors through Alb(X). In higher dimensions, Pic(X) is defined differently (it's about line bundes).

Remark 13.3. This is related to an important theorem in homotopy theory called the *Dold-Thom theorem*, asserting that the free commutative monoid built from a pointed space X, $\mathrm{Sym}^\infty(X,*)$, has homotopy groups isomorphic to the homology groups of X. ◀

We can do something similar for our Riemann suefaces. For example, $\Sigma \hookrightarrow \mathrm{Pic}^1$ as we saw, and $\mathrm{Sym}^2\Sigma$ maps to $\mathrm{Pic}(\Sigma)$, where $\mathrm{Sym}^k(\Sigma)$ denotes the space of k-tuples of distinct points in Σ. The map to Pic^2 sends $(x,y)\mapsto x+y$. We can do the same thing on $\mathrm{Sym}^d(\Sigma)$, and lands in $\mathrm{Pic}^d(\Sigma)$: this map preserves degree. This is another way of writing the Abel-Jacobi map, as

$$AJ\colon \mathrm{Sym}^*(\Sigma) \longrightarrow \mathrm{Pic}(\Sigma). \tag{13.4}$$

The finite combinations of points $\mathrm{Sym}^*(\Sigma)$ are the same thing as the *effective divisors* in $\mathrm{Div}(\Sigma)$, which can be identified with pairs of a holomorphic line bundle $\mathcal{L}$ and a section σ, modulo scalar multiplication on the section; the Abel-Jacobi map simply forgets the section. Therefore the fiber of a line bundle $\mathcal{L}\in\mathrm{Pic}(\Sigma)$ is $\mathbb{P}(H^0(\Sigma,\mathcal{L}))$.

This map is not surjective, e.g. most line bundles don't have holomorphic sections. But when the degree d is high enough, they do, by the Riemann-Roch theorem. Effectively, we need $d > 2g(\Sigma)+2$, and we obtain $\mathbb{P}^{d-g}$ as the fiber.

This is good: projective spaces are simply connected, so the Abel-Jacobi map is an isomorphism on π_1 in degree bigger than $2g(\Sigma)-2$. This implies something about line bundles: you can think of it as a group-completion map from the free commutative monoid generated by Σ to the free abelian group generated by the curve, à la Dold-Thom.

We want to understand the relationship between $\mathcal{L}oc(\mathrm{Pic})$ and $\mathrm{Loc}_1(\Sigma)$ beter. Specifically, we hope to get $\mathcal{L}oc(\mathrm{Pic})\cong \mathcal{QC}(\mathrm{Loc}_1(\Sigma))$. One issue is that Σ only sees one component of $\mathrm{Pic}(\Sigma)$, so we'll have to modify the story above somehow, which geometric class field theory will do.

Rememeber unramified class field theory for a curve C over $\mathbb{F}_q$: we identified rank-1 local systems with characters of $\mathrm{Pic}(C)$, attaching to a local system $E\in\mathrm{Loc}_1(C)$ a character χ_E. "Character" means

$$\chi_E(\mathcal{L}\otimes\mathcal{M}) \simeq \chi_E(\mathcal{L})\cdot\chi_E(\mathcal{M}). \tag{13.5}$$

And since Pic is generated by $\mathbb{Z}_x$ for all $x\in C$, we also know

$$\chi_E(\mathcal{L}(x)) = \chi_E(\mathcal{L}\otimes\mathcal{O}(x)) = \chi_E(\mathcal{O}(x))\chi_E(\mathcal{L}). \tag{13.6}$$

That is, $\chi_E(\mathcal{L})$ is an eigenfunction for the operation of twisting by x, with eigenvalue $\chi_E(\mathcal{O}(x))$. And this represents the value of the local system E on the Frobenius. And $\chi_E(\mathcal{O}(x)) = AJ(x)$.

The geometric version of this is to consider *character local systems* on Pic, analogues of characters on the group Pic. Our context is a curve C over any field k, but really you can think about Riemann surfaces. A character local system on Pic is something that ought to satisfy the multiplicative properties (13.5) and (13.6).

Definition 13.7. A *character local system* $\chi\in\mathcal{L}oc(\mathrm{Pic})$ is a local system together with data of an identification $\mu^*\chi \overset{\cong}{\Rightarrow} \chi\boxtimes\chi$, where μ is the multiplication map on Pic. (TODO: was there another requirement?)

Pointwise, pinning down what this does under pullback by μ means that

$$\chi|_{\mathcal{L}\otimes\mathcal{M}} \cong \chi|_{\mathcal{L}}\otimes\chi|_{\mathcal{M}}, \tag{13.8}$$

but more naturally, or in families. Another way to think of this is that χ is a map $\mathrm{Pic}\to B\mathbb{G}_m$.

You can use the multiplicativity property to show that any local system over an abelian group with this property must be rank-one: $\chi(0+a)=\chi(a)=\chi(0)\otimes\chi(a)$, so $\chi(0)\cong k$.

We can consider the total space $A_\chi\to A$; the condition that χ is a character is equivalent to the condition that A_χ is an abelian group; in particular, in this case it is an extension

$$0 \longrightarrow k^\times \longrightarrow A_\chi \longrightarrow A \longrightarrow 0. \tag{13.9}$$

This is always abelian; you can ask about abelian group objects in schemes (from the functor-of-points perspective, these are representable functors to $\mathcal{A}b$), and this extension is in the category of chain complexes of abelian groups (TODO: I think).

Character local systems are categorified characters that you can define on any abelian group. We will specifically care about $\mathrm{Pic}(C)$, where C is a curve over any field k. Just as we did with ordinary characters, we can ask how $\chi(\mathcal{L}(x))$ is related to $\chi(\mathcal{L})$, analogously to (13.6). We get that

$$\chi(\mathcal{L}(x)) \cong \chi(\mathcal{O}(x)) \otimes \chi(\mathcal{L}), \tag{13.10}$$

and $\mathcal{O}(x) = AJ(x)$. Let $L_\chi \coloneqq AJ^*\chi \in \mathrm{Loc}_1(C)$. Then this multiplicativity property is saying that TODO(missed this; sorry!). You can say this in families: we have a diagram

$$\begin{tikzcd} \mathrm{Pic}\times\mathrm{Pic} \arrow[r, "\mu"] & \mathrm{Pic}; \\ \Sigma\times\mathrm{Pic} \arrow[u, "AJ\times\mathrm{id}"] \arrow[ur, "\mu_\Sigma"'] & \end{tikzcd} \tag{13.11}$$

the statement is that $\mu_\Sigma^*(\chi) \simeq L \boxtimes \chi$, which is a *Hecke eigenproperty.* With this in hand we can really formulate geometric class field theory, which says that *the map* $\chi \mapsto L_\chi = AJ^*\chi$ *gives an equivalence between character local systems on* Pic *and rank-one local systems on* Σ*.* This is an equivalence of groupoids, and is thought of as an equivalence of stacks. The proof is abstract enough to work over any field, and there are analogues over rings.

Among other things, this tells us that abelian covers of Σ are identified not with arbitrary abelian covers of Pic, but with the ones that arise as extensions of Pic. So we really need to know the group structure of Pic. Said differently, any rank-1 local system on Σ, extended canonically to $\mathrm{Sym}^*\Sigma$, descends to Pic.

Deligne's proof of geometric class field theory. Deligne's proof first extends L to $\mathrm{Sym}^*(\Sigma)$.

An element of $\mathrm{Sym}^d(\Sigma)$ is a finite sum of points, so we'll let

$$L^{(d)}\left(\sum_{i=1}^d x_i\right) \coloneqq \bigotimes_{i=1}^d L(x_i), \tag{13.12}$$

which indeed defines a rank-one local system. More formally, you can say this as assigning to L the line bundle $L^{\boxtimes d} \to \Sigma^d$, and quotienting by the symmetric group (after cutting out the diagonal, I think) we can take $\pi\colon \Sigma^d \to \mathrm{Sym}^d(\Sigma)$ and obtain $(\pi_* L^{\boxtimes d})^{S_d}$.

The next step is to use that the map $\mathrm{Sym}^d \to \mathrm{Pic}^d$ is 1-connected for d large enough: the fibers are projective spaces, which are simply connected. This implies that $L^{(d)}$ descends to Pic^d, i.e. $L^{(d)} \cong AJ^*(\chi_L)$ for some local system χ_L on Pic^d. This step, using simple connectivity of projective spaces, requires geometry.

Anyways, we've constructed $\chi_L^\bullet$ for sufficiently high degree ($d > 2g-2$), and χ_L satisfies multiplicativity (albeit in this restricted way, only for high enough degrees). The claim is then that there's a unique extension to all of Pic satisfying multiplicativity: any divisor can be written as a difference of large-degree effective divisors $D = D_1 - D_2$, with $D_1, D_2 \in \mathrm{Im}(AJ)$ and of sufficiently high degree. Multiplicativity tells us to extend χ_L and define

$$\chi_L(D) \coloneqq \chi_L(D) \otimes \chi_L(D_2)^{-1}. \tag{13.13}$$

⊠

Bhatt has notes (an Oberwolfach report) on geometric class field theory which are a good reference for this proof; Bhatt also notes the same proof also works for ramified geometric class field theory: choose $S \subset \Sigma$, and consider the Abel-Jacobi map $\Sigma \setminus S \to \mathrm{Pic}(\Sigma, S)$, the group of line bundles trivialized near S. This uses something we won't have access to in the nonabelian setting: that Pic is very close to the curve Σ. The general story will have more duality that is trivial here, and will be more complicated.

Next time, we'll give another take on class field theory, relating it to a categorified version of spectral decomposition. We want to understand the relationship between local systems on Pic and sheaves on Loc to be some sort of Fourier transform. We will spectrally decompose all local systems on Pic (not just character local systems — like in the ordinary Fourier transform, we can try to write an arbitrary local system as a combination of character local systems). On the B-side, looking at $\mathcal{QC}(\mathrm{Loc}_1(\Sigma))$, our "basis" is points of $\mathrm{Loc}_1(\Sigma)$.

But our notion of "spectrum" is categorified. In the beginning of the class, we discussed how the spectrum is some sort of adjoint or inverse to the functor assigning to a geometric thing (e.g. spaces) its ring of functions. One category number up, we send a space X to the symmetric monoidal category of sheaves on X; therefore the spectrum functor should go in the other direction, sending a symmetric monoidal category $\mathcal{C}$ to a space in the best possible way, i.e. there will be an adjunction

$$\mathrm{Map}(-, \mathrm{Spec}(\mathcal{C})) = \mathrm{Hom}_{\otimes\text{-}\mathcal{C}at}(\mathcal{C}, \mathcal{S}h(-)). \tag{13.14}$$

In this setting, this is called *Tannakian reconstruction.* The punchline will be that

$$\mathrm{Spec}(\mathrm{Loc}(\mathrm{Pic})) = \mathrm{Loc}_1(\Sigma), \tag{13.15}$$

which is a particularly strong version of geometric class field theory.

If we want to do this in positive characteristic, there's the question of what kinds of sheaves we want – sheaves of vector spaces over what field? For functions, we used $\overline{\mathbb{Q}}_\ell$-valued functions (noncanonically the same thing as $\mathbb{C}$-valued functions), and (TODO: I think) there will be something similar here, where we take sheaves with $\overline{\mathbb{Q}}_\ell$ coefficients if C is a curve over a finite field.

The Tannakian point of view will generalize most readily to the nonabelian setting out of all of these cases.

Lecture 14.

Tannakian reconstruction: 3/11/21

The stacks lecture will be recorded and posted in the near future, to be watched asynchronously if you want to learn a crash course in stacks or refresh the material for its use in this course.

Today is about Tannakian reconstruction. The slogan here is that "commutativity indicates geometry:" a commutative ring is an algebra of functions on a geometric object; we will categorify this. Specifically, if R is a commutative ring, $\mathcal{M}od_R$ has a tensor product $M, N \mapsto M \otimes_R N$, which is also an R-module. This gives $\mathcal{M}od_R$ the structure of a symmetric monoidal category.

The definition of a symmetric monoidal category is a bit technical. What we care about (which is not equivalent to the standard definition) is abelian categories with a few niceness properties, including arbitrary direct sums. *Grothendieck abelian categories* satisfy these operations. Such a category $\mathcal{C}$ has a *symmetric monoidal* structure when we define a functor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ caled the *tensor product* and an object $\mathbf{1} \in \mathcal{C}$ called the *unit*, and some further data and conditions. These include encoding the fact that tensoring with the unit is trivial, which is implemented as natural transformations

$$\mathbf{1} \otimes - \simeq - \otimes \mathbf{1} \simeq \mathrm{id}, \tag{14.1}$$

as well as natural transformations encoding associativity and commutativity of $\otimes$ (up to natural isomorphism), and some conditions on that data. There is a corresponding notion of *symmetric monoidal functor* which preserves the data of the tensor product and unit.

The point is, for R a commutative ring, $\mathcal{M}od_R$ is a symmetric monoidal category with $\otimes_R$ and unit R. Before, we just used R-modules for spectral decomposition, but now we will think of $\mathcal{M}od_R$ as a categorified commutative ring. This will be helpful in TFT, where as the codimension increases, so too does the category number.

The geometric space is still $X = \mathrm{Spec}\, R$, but we will categorify the "ring of functions" on X, replacing R with $\mathcal{M}od_R = \mathcal{QC}(X)$. In algebraic geometry, most schemes aren't affine, which is somewhat annoying, and is a major difference from the manifolds case. One advantage of categorifying is that many more things are "affine." Specifically, suppose X is a general scheme, which admits an affine open cover $\mathfrak{U} = \{\mathrm{Spec}(R_i)\}$; then the category of quasicoherent sheaves on X is

$$\mathcal{QC}(X) = \varprojlim_{\mathfrak{U}} \mathcal{M}od_{R_i}, \tag{14.2}$$

which can be a definition or a theorem depending on your approach to $\mathcal{QC}$. Quasicoherent sheaves are symmetric monoidal with tensor product $\otimes_{\mathcal{O}_X}$ and unit $\mathcal{O}_X$. Moreover, if $f\colon X \to Y$ is a map of schemes, there is a *pullback functor* $f^*\colon \mathcal{QC}(Y) \to \mathcal{QC}(X)$ and it is symmetric monoidal: $\mathcal{O}_Y$ pulls back to $\mathcal{O}_X$, and tensor products pull back to tensor products. In fact, this is universal: $\mathcal{QC}(X)$ satisfies the universal property that whenever we have a map $f\colon \mathrm{Spec}(R) \to X$, there is a functor $f^*\colon \mathcal{QC}(X) \to \mathcal{M}od_R$.

To summarize, just as we had $\mathcal{O}\colon \mathcal{S}ch^{op} \to \mathcal{C}omm\mathcal{R}ing$ and a nice correspondence, we have $\mathcal{QC}\colon \mathcal{S}ch^{op} \to \otimes\text{-}\mathcal{C}at$. Moreover, one can generalize to quite general kinds of stacks, defining $\mathcal{QC}(X)$ for stacks X. We'll say what kinds of stacks we allow in a moment — Artin stacks are good examples, and you can in general think about quotients of schemes by group actions.

This gives us a great source of symmetric monoidal categories: there are other ways to obtain them, but this is our home base, the categories that feel most like geometry and are most our friends. So let's say, given an arbitrary symmetric monoidal category, can we reconstruct it as $\mathcal{QC}(X)$ or $\mathcal{M}od_R$ for some X or R? We want to construct a right adjoint, just as we did for $\mathcal{O}$, which means this categorified Spec satisfies the property that

$$\mathrm{Hom}_{\mathcal{S}tack}(X, \operatorname{Spec}\mathcal{C}) = \mathrm{Hom}_{\otimes\text{-}\mathcal{C}at}(\mathcal{C}, \mathcal{QC}(X)). \tag{14.3}$$

This really boils down to asking that maps $\operatorname{Spec} R \to \operatorname{Spec}\mathcal{C}$ (the former is the usual spectrum of a ring, the latter is our categorified notion) are in natural biijection with symmetric monoidal functors[15] $\mathcal{C} \to \mathcal{M}od_R$.

The unit (TODO: counit?) of the $(\mathcal{O}, \operatorname{Spec})$ adjunction gives you a canonical *affinization* map for any scheme: $X \mapsto \operatorname{Spec}(\mathcal{O}(X))$. For example, for $X = \mathbb{P}^n$, this is the usual map $\mathbb{P}^n \to k$. Likewise, the unit (TODO: counit?) of the adjunction (14.3) gives us a "Tannakization" map $X \to \operatorname{Spec}(\mathcal{QC}(X))$.

TODO: I missed something here, but if S is a commutative ring and $\mathcal{C} = \mathcal{M}od_S$, then $\mathcal{C}$ is *generated by the unit*, meaning that any S-module has a free resolution. That is, you can build all S-modules inductively by starting with S and taking direct sums, kernels, and cokernels. You could also study the functor $\mathrm{Hom}_{\mathcal{M}od_S}(S, -)\colon \mathcal{M}od_S \to \mathcal{A}b$, which has very nice properties that also characterize "generated by the unit." This tells us (TODO: I didn't follow this yet) that $\operatorname{Spec}(\mathcal{M}od_S) = \operatorname{Spec}(S)$, which is reassuring!

But what's really great about this categorified Spec is that we can understand many non-affine things, even stacks. These geometric objects do not have enough functions, but they *do* have enough quasicoherent sheaves. Schemes have lots of sheaves, such as skyscrapers, even though we don't have the decategorified analogue, δ-functions.

Theorem 14.4 (Tannakian reconstruction)**.** *Let X be a geometric stack. Then the "Tannakization" map $X \to \operatorname{Spec}\mathcal{QC}(X)$ is an isomorphism.*

For references, see Lurie [Lur04] and Bhatt and Halpern-Leistner [BHL17]. Anyways, the upshot is that geometric stacks are "categorically affine," and the power of the theorem is that just about anything you might reasonably encounter is a geometric stack. For example, this includes all quascompact schemes. The full definition of a geometric stack is a quasicompact stack with affine diagonal, which means something quite concrete in practice: if X is such a stack, we can build it by taking an affine variety U and an affine groupoid $\mathcal{G}$ acting on U, and $X \cong U/\!\!/\mathcal{G}$.

Remark 14.5. The fact that we have "categorified δ-functions" is not all that surprising: sheaves seem to categorify both functions and distributions/measures. This is related to the fact that they can both pull back and push forward. ◀

We will spend the rest of the lecture on examples.

Example 14.6. For our first example, suppose X is a quasicompact scheme. We can detect maps $\operatorname{Spec} R \to X$ by looking for pullback functors $f^*\colon \mathcal{QC}(X) \to \mathcal{M}od_R$, and Theorem 14.4 tells us that these are equivalent. For example, k-points of X, i.e. maps $\operatorname{Spec} k \to X$, are equivalent to pullbacks $f^*\colon \mathcal{QC}(X) \to \mathcal{QC}(\operatorname{Spec} k) = \mathcal{V}ect_k$. That is, a point is a tensor functor to $\mathcal{V}ect$. ◀

The quintessential example is in representation theory. Let G be a group; then $\mathcal{R}ep_G$ is a symmetric monoidal category (where we consider representations over a chosen field k). The unit is the trivial one-dimensional representation, and the tensor product is the tensor product of vector spaces with diagonal G-action. This is a symmetric monoidal category naturally in G.

Now we plug this into the Tannakian machine: is $\operatorname{Spec}(\mathcal{R}ep_G)$ the category of quasicoherent sheaves on something? To think about this, we need a little more data. Representations are a vector space plus a G-action, meaning we can forget the action, defining a faithful symmetric monoidal functor $\mathcal{R}ep(G) \to \mathcal{V}ect$. In general, a faithful symmetric monoidal functor to $\mathcal{V}ect$ is called a *fiber functor*. This is special: we never have this for $\mathcal{QC}(X)$ for a variety X, because this would be equivalent to a map $i\colon \operatorname{Spec} k \to X$, but pulling back by this map is never faithful unless X itself is a point.

So having a fiber functor is sort of perpendicular to being a scheme. You can think of it as saying that $\operatorname{Spec}\mathcal{C}$ admits a faithfully flat map from a point, which is intuitively saying $\operatorname{Spec}\mathcal{C}$ is a quotient of a point. Stackiness is pretty important here, so that this can be nontrivial.

[15] We want to ask a little more of our morphisms than that they are symmetric monoidal functors. For the precise conditions we need for our Tannakian reconstruction theorem, see Lurie [Lur04] or Bhatt and Halpern-Leistner [BHL17].

In particular (and this is what's classically called Tannakian reconstruction), if G is an *affine algebraic group*, meaning a group object in the category of affine varieties (so, G is an affine variety and multiplication and inversion are maps of varieties, as for $\mathrm{GL}_n(k)$), then $\mathrm{Spec}(\mathcal{R}ep_G) = BG = \mathrm{pt}/G$, and the fiber functor is identified with the quotient $\mathrm{pt} \to \mathrm{pt}/G$. We can recover G as the relations we imposed.

The way this is usually phrased is that, given $\mathcal{C}$ and a fiber functor f, one can construct an affine algebraic group structue on $\mathrm{Aut}(f)$, and $\mathcal{C} \simeq \mathcal{R}ep_G$.

There are lots of weird groups out there that have no finite-dimensional algebraic representations, i.e. no maps to $\mathrm{GL}_n(k)$. We won't be able to see those from this perspective, and in general $\mathrm{Spec}(\mathcal{C})$ is the proalgebraic completion of G, which is some technical thing you have to do to make it more algebraic. In a sense, this proalgebraic completion is the universal algebraic group you can build out of G.

Now let's go out in nature and look for symmetric monoidal categories with fiber functors. (Our examples will be biased towards algebraic geometry, because our test categories are categories of R-modules. If you worked with modules over C^*-algebras, or something like that, your examples will feel more topological.) We can build groups by building categories with fiber functors, i.e. vector spaces with some structure.

Example 14.7. Consider the category of $\mathbb{Z}$-graded vector spaces; forgetting the grading defines a faithful forgetful functor to $\mathcal{V}ect$. The graded tensor product

$$(M \otimes N)_n = \bigoplus_{i+j=n} M_i \otimes N_j, \tag{14.8}$$

We do not use the Koszul sign rule here. Thus the forgetful functor is symmetric monoidal, and the Tannakian reconstruction theorem tells us there's some group G with $\mathcal{V}ect_{\mathbb{Z}} \simeq \mathcal{R}ep_G$. In this case, we know the answer: we've seen tht a $\mathbb{Z}$-graded vector space is the same thing as a representation of the multiplicative group $\mathbb{G}_m$, by taking isotypic components: an irreducible $\mathbb{G}_m$-representation is indexed by $n \in \mathbb{Z}$, with $z \in \mathbb{G}_m$ acting on k by z^n. This is yet another form of the spectral decomposition/Pontrjagin duality, where $\mathbb{Z}$-gradings are dual to $\mathbb{G}_m$-actions. Therefore $\mathcal{V}ect_{\mathbb{Z}} \simeq \mathcal{QC}(\mathbb{G}_m)$. ◀

Pontrjagin duality says that when G is a compact group, $G^\vee$ is discrete, and compact groups have semisimple representation categories; in the algebraic case, semisimple representation categories mean *reductive* groups. G is reductive iff $\mathcal{QC}(\mathrm{pt}/G) = \mathcal{R}ep_G$ is semisimple. You can carry over a fair amount of the story of Pontrjagin duality from compact groups to this setting.

Example 14.9. This example is the reason for the name "fiber functor." Suppose X is a connected (and locally path-connected) topological space, and consider the symmetric monoidal category $\mathrm{Loc}(X)$ of local systems of vector spaces on X; the symmetric monoidal structure is pointwise tensor product.

Choose a basepoint $x \in X$; taking the fiber at x defines a functor $\mathrm{Loc}(X) \to \mathcal{V}ect$; because tensor product is pointwise, this is a symmetric monoidal functor, and because X is connected, this functor is faithful. So taking the fiber is a fiber functor.

We therefore obtain a group G such that $\mathrm{Loc}(X) = \mathcal{R}ep_G$. This is essentially $\pi_1(X, x)$, because representations of π_1 on k-vector spaces are idntified via monodromy with local systems of k-vector spaes on X. However, any group can be a π_1, but not all groups have finite-dimensional algebraic representations, hence "essentially" above — in general, you will obtain that G is the proalgebraic completion of $\pi_1(X)$. ◀

Example 14.10. Let k be a field and consider $\mathcal{C} := \mathcal{V}ect_k$. Choose an algebraic closure $\overline{k}$ of k; then $-\otimes_k \overline{k}$ is a faithful symmetric monoidal functor, identifying $\mathcal{C}$ with $\mathcal{R}ep_G$ for some group G, and in fact $G = \mathrm{Gal}(\overline{k}/k)$. This is a form of Galois descent: Galois groups come from automorphism groups of field extensions, and we're seeing that here. ◀

Even though we're based in algebraic geometry, these examples came from elsewhere, and we attached algebro-geometric objects to them. This is an idea we will see again.

Next time, we will combine this with Fourier-Mukai theory: one nice source of commutative rings is group algebras of abelian groups G. We will have a pushforward by multiplication

$$\mu_* \colon \mathcal{S}h(G) \times \mathcal{S}h(G) \longrightarrow \mathcal{S}h(G), \tag{14.11}$$

which is convolution. This makes sense for any group, and defines a symmetric monoidal structure when G is abelian. We're being nonspecific about what kinds of sheaves appear here — since this is a source of symmetric monoidal categories, it doesn't matter if they're quascoherent or whatever. Then we can define the *Fourier-Mukai dual* or *1-shifted Cartier dual* of G to be

$$G^\vee[1] := \mathrm{Spec}(\mathcal{S}h(G), *). \tag{14.12}$$

For example, if $G = \mathbb{Z}$, a sheaf on $\mathbb{Z}$ is a $\mathbb{Z}$-graded vector space, and convolution is the graded tensor product (14.8). So this is a categorical groupa algebra, and its 1-shifted Cartier dual is $B\mathbb{G}_m$. Here the shift is manifest: the usual Cartier dual of $\mathbb{Z}$ is $\mathbb{G}_m$, but we've shifted and obtained $B\mathbb{G}_m$ instead, even though we followed a similar line of reasoning, just categorified.

Next time we'll discuss 1-shifted Cartier duality more, in a way that will make it feel a lot like Fourier theory.

Lecture 15.

1-Shifted Cartier duality: 3/23/21

Since we just came back from spring break, let's review what we were doing when we last met, which was to interpret Tannakian reconstruction as a categorified analogue of the adjunction between taking functions on a variety and taking Spec of a commutative ring. From this perspective, we recover $\mathcal{R}ep_G$ as the category of quasicoherent sheaves on the stack pt/G, for G an affine algebraic group.

We further interpreted this via a spectral decomposition (in the context of Fourier-Mukai duality, or 1-shifted Cartier duality): if G is an abelian group and $\mu\colon G \times G \to G$ is its multiplication map, then pushforward by multiplication defines a convolution symmetric monoidal product on $\mathcal{QC}(G)$. For example, take $(QC)(\mathbb{Z})$ (or $\mathbb{Z}$-graded vector spaces); its spectrum is $B\mathbb{G}_m$, because this category is equivalent, as symmetric monoidal categories, to the category of $\mathbb{G}_m$-representations. The "shift" is because we started with $\mathbb{Z}$ and obtained $B\mathbb{G}_m$, rather than $\mathbb{G}_m$.

More generally, suppose G is a finite abelian group scheme, and let $G^\vee := \mathrm{Spec}(\mathbb{C}[G], *)$. Functions on G under convolution are therefore identified with functions on $G^\vee$ under pointwise multiplication. And one level up, $\mathcal{R}ep_G \simeq \mathcal{QC}(G^\vee)$. The shift appears again, because $\mathcal{R}ep_G \simeq \mathcal{QC}(BG)$: passing to the categorical level, one of the sides has to be moved down.

Let's step back and ask, what *is* $G^\vee[1]$, the Fourier-Mukai dual? What are its points? We defined it via its functor of points; for a field k, the k-points of $G^\vee[1]$ are symmetric monoidal functors

$$(\mathcal{S}hv(G), *) \simeq (\mathcal{QC}(G), *) \longrightarrow \mathcal{V}ect_k. \tag{15.1}$$

You can think of these as "categorical one-dimensional representations of $(\mathcal{QC}(G), *)$" — an uncategorified representation is a map $G \to \mathrm{GL}_1(k)$ or, equivalently, an algebra homomorphism $\mathbb{C}[G] \to \mathrm{End}(k)$; here we have a symmetric monoidal functor to $\mathrm{End}(\mathcal{V}ect_k) = \mathcal{V}ect_k$.

Therefore G should map to $\mathrm{Aut}(\mathcal{V}ect_k)$, which is the symmetric monoidal category of lines ($\otimes$-invertible vector spaces, or equivalently one-dimensional vector spaces) with tensor product. This object, which is equivalent to $B\mathbb{G}_m$, is a "group," or more precisely a group object in stacks. That is, its functor of points is valued not in abelian groups (like for a group scheme), nor just in groupoids (like for an ordinary stack), but in *Picard groupoids*: symmetric monoidal groupoids in which all objects are $\otimes$-invertible.

Spelling this out a little more, the maps $X \to B\mathbb{G}_m$ are the symmetric monoidal category of line bundles on X, with tensor product. These are all invertible under tensor product. Also, why $B\mathbb{G}_m$? The idea is that all invertible k-vector spaces are isomorphic, but they have a $\mathbb{G}_m$ of isomorphisms, so this stack is equivalent to $\mathrm{pt}/\mathbb{G}_m$, which was our original definition of $B\mathbb{G}_m$.

And part of the upshot is that the points of $\mathrm{Spec}(\mathcal{S}hv(G), *)$ are categorified characters or *character sheaves*: group (stack) homomorphisms $G \to B\mathbb{G}_m$. These are equivalent to multiplicative line bundles $\mathcal{L} \to G$, i.e. $\mathcal{L}_g * \mathcal{L}_h \cong \mathcal{L}_{gh}$ or $\mu^*\mathcal{L} = \mathcal{L} \boxtimes \mathcal{L}$. So

$$G^\vee[1] \cong \mathrm{Hom}_{\mathcal{G}rp}(G, B\mathbb{G}_m) = \{\text{multiplicative line bundles}\}. \tag{15.2}$$

And this is completely analogous to what we did with Cartier duality before: we defined the dual using $\mathrm{Spec}(\mathbb{C}[G], *)$ (just functions, not categorified), and observed this is a group algebra.

Remark 15.3. This shifted duality is happening in the derived category of sheaves of abelian groups on the category of affine schemes — the usual derived category, but in a more universal sense. Group schemes and group stacks define objects in this category. Because this category is triangulated, it makes sense to take a shift here, and shifted Cartier duality and what we called $G^\vee[1]$ is shifted in this triangulated sense. ◀

Yet another way to think about this is tat

$$\mathrm{Hom}_{\mathcal{G}rp}(G, B\mathbb{G}_m) = \mathrm{Ext}^1(G, \mathbb{G}_m), \tag{15.4}$$

so multiplicative line bundles $\mathcal{L} \to G$ should correspond to central extensions of G by $\mathbb{G}_m$. Here we just take the total space of $\mathcal{L}$, which fits into a short exact sequence of abelian group schemes

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \mathcal{L} \longrightarrow G \longrightarrow 0. \tag{15.5}$$

There is a universal bundle $\mathcal{L} \to G \times G^\vee[1]$, and we can therefore write the Fourier-Mukai transform as an integral transform:

$$\begin{tikzcd} & G \times G^\vee[1] \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_2"] & \\ G & & G^\vee[1] \end{tikzcd} \tag{15.6}$$

Specifically, given a sheaf $\mathcal{F}$ on G, its Fourier-Mukai dual is

$$\pi_{2*}(\pi_1^*\mathcal{F} \otimes \mathcal{L}). \tag{15.7}$$

Mukai originally considered connected abelian varieties A, such as elliptic curves or Jacobians of curves. There is a notion of a dual abelian variety A^*, which is just the dual torus, but there is an equivalence of A^* with $A^\vee[1]$, i.e. A^* is the abelian variety of multiplicative line bundles on A.[16] Thus one obtains another kernel $\mathcal{L} \to A \times A^*$ and a kernel transform analogous to (15.6), and it gives an equivalence of *derived* categories. This was the original example of the Fourier-Mukai; there's a great paper of examples by Laumon [Lau91].[17] (TODO: this is the correct paper, right?).

We have to pass to derived stuff at some point, and that means some background that we'll have to discuss.

Remark 15.8. If you care about more general representations, rather than characters, we can categorify the idea that representations of G are sheaves on $G^\vee$. Here, though, categorified representations are quasicoherent sheaves of categories, which is a categorified spectral decomposition, but one has to think about how to make this precise – what is a sheaf of categories, exactly? ◀

Suppose C is a Riemann surface; then $\mathrm{Jac}(C) \simeq \mathrm{Jac}(C)^*$, and $\mathcal{P}ic(C)$, the stack of line bundles on C, is equivalent to $\mathrm{Pic}(C) \times B\mathbb{G}_m$ (here $\mathrm{Pic}(C)$ is the usual, non-stacky thing we called $\mathrm{Pic}(C)$). The non-stacky $\mathrm{Pic}(C)$ splits as $\mathbb{Z} \times \mathrm{Jac}(C) \times B\mathbb{G}_m$; when you dualize, $\mathbb{Z} \leftrightarrow B\mathbb{G}_m$ and $\mathrm{Jac}(C)$ is self-dual, so $\mathrm{Pic}(C)$ is also self-dual, but in an interesting way: the degree and the $\mathbb{G}_m$-stabilizer are exchanged.

Example 15.9 (Betti class field theory)**.** This is related to "Betti class field theory," the version of class field theory that we discussed, sending $\mathrm{Pic}(C)$ to the symmetric monoidal category $(\mathcal{L}oc(\mathrm{Pic}(C)), *)$, and $\mathrm{Spec}(\mathcal{L}oc(\mathrm{Pic}(C)), *)$ is isomorphic to $\mathrm{Loc}_1(C)$. That is, we have a Fourier-Mukai-like duality

$$(\mathcal{L}oc(\mathrm{Pic}(C)), *) \xrightarrow{\simeq} (\mathcal{QC}(\mathrm{Loc}_1(C)), \otimes). \tag{15.10}$$

A character local system on the left χ_L (analogous to an exponential) is exchanged with a local system L on the right (analogous to a δ distribution). This transform is an equivalence of categories.

One way to summarize this would be that the 1-shifted Cartier dual of Loc is Pic.

You can write this as an integral transform with kernel $\chi \to \mathrm{Pic}(C) \times \mathrm{Loc}_1(C)$, but χ is a local system in the $\mathrm{Pic}(C)$ direction and a quasicoherent sheaf in the $\mathrm{Loc}_1(C)$ direction. This happens because we have two different kinds of sheaf theory on the two sides of this correspondence. So it's a little wonky but the theory still works fine.

Choosing a point $x \in C$, which gives us a skyscraper sheaf and hence a map $\mathbb{Z} \to \mathrm{Pic}(C)$, which under this duality passes to a map $W_x \colon \mathrm{Loc}_1 \to B\mathbb{G}_m$. More generally, Hecke modifications $\mathcal{L} \mapsto \mathcal{L}(x)$ act by $\mathbb{Z} \cdot x$ on $\mathrm{Pic}(C)$, and under duality these are identified with the operators which tensor with the line bundle W_x. You can think of W_x as an eigensheaf for this Hecke modification. ◀

[16] Often this is defined just using degree-0 bundles, and one can show this is enough.

[17] Sometimes people use "Fourier-Mukai transform" to mean any integral transform, but all decent operators on function spaces are integral transforms, and likewise all decent functors on derived categories are integral transforms. In this class we will reserve the word "Fourier-Mukai transform" for transforms which interchange convolution and multiplication.

Example 15.11 (de Rham class field theory)**.** We can also do this in the de Rham setting. This has a different flavor: the two sides of the duality are more symmetric (yay!) but more complicated (darn).

Given a variety X, we will introduce the *de Rham space* X_{dR} as the quotient of X by the equivalence relation $x \sim y$ if x and y are infinitesimally close. Formally, this is the quotient of X by a formal neighborhood of X in the diagonal. This is not a scheme, but its functor of points is well-defined enough. Why introduce this object? It is useful for seting up differential operators in algebraic geometry.

Quasicoherent sheaves on the formal neighborhood of the diagonal are quasicoherent sheaves on X with data $\mathcal{F}_x \cong \mathcal{F}_y$ when x and y are infinitesimally close. It turns out that $\mathcal{QC}(X_{\mathrm{dR}}) \simeq \mathcal{M}od_{\mathcal{D}_X}$, the category of *$\mathcal{D}$-modules* on X. On affine opens U, these are data of a quasicoherent sheaf $\mathcal{F}$ together with an action of the ring of differential operators $\mathcal{D}(U)$ on $\mathcal{F}(U)$.

Differential operators are generated by functions and vector fields, and we already have an $\mathcal{O}_X$-action on $\mathcal{F}$, so a $\mathcal{D}$-module structure is equivalent to specifying how vector fields act. This can be specified as a map $T_X \to \mathrm{End}(\mathcal{F})$ satisfing *flatness*, that $[\xi_1, \xi_2]$ acts by the commutator of $(\xi_1\cdot)$ and $(\xi_2\cdot)$, and a Leibniz rule

$$\xi f = f\xi + f'. \tag{15.12}$$

So these are the same thing as flat connections on X.

All of that happened very quickly, but the point today is towards de Rham class field theory. If G is an abelian group, its de Rham space $G_{\mathrm{dR}} = G/\widehat{G}$, i.e. quotienting by the formal neighborhood of the identity in G (the formal group associated to G). That is, g_1 is infinitesimally close to g_2 in G iff $g_1 g_2^{-1}$ is infinitesimally close to the identity in G (or, $g_1 g_2^{-1} \in \widehat{G}$).

With this formalism we can go back and add some spice to our dualities. For example, if we did ordinary Cartier duality for a vector space V, we obtained on the dual side the formal completion of V^*, which was a fact that we don't have exponentials in algebraic geometry unless we work formally.

When we pass to 1-shifted Cartier duality, $V^\vee[1] \simeq B\widehat{V}^*$, which is a little annoying: that formal issue is still there. Similarly, $\widehat{V}^\vee[1] \simeq BV^*$.

Now let's do a trick: apply the de Rham construction to V: $V_{\mathrm{dR}} = V?\widehat{V}$, which in the derived category is homotopy equivalent to the chain complex $\widehat{V} \to V$. Now doing shifted Cartier duality, $V_{\mathrm{dR}}^\vee[1]$ is the shifted Cartier dual of $\widehat{V} \to V$, and some things happen:

- $\widehat{V}$ maps to V^*,
- V maps to $\widehat{V}^*$,
- and the duality is contravariant, so we obtain a map $V^* \leftarrow \widehat{V}^*$,

or in other words the 1-shifted Cartier dual of V_{dR} is V_{dR}^*.

Another example we can look at is

$$\mathcal{D}_{\mathbb{A}^1} = \mathbb{C}\langle x, \partial_x\rangle/(\partial_x x - x\partial_x = 1) \tag{15.13a}$$

$$\mathcal{D}_{(\mathbb{A}^1)^*} = \mathbb{C}\langle y, \partial_y\rangle/(\partial_y y - y\partial_y = 1). \tag{15.13b}$$

Duality sends $x \mapsto \partial_y$ and $\partial_x \mapsto -y$, or in other words $\mathcal{D}_{\mathbb{A}^1}$ has an order-four automorphism, the Fourier transform! It's not quite an involution, because it squares to -1.

This really is the Fourier transform: if φ is a distribution, it defines a $\mathcal{D}_{\mathbb{A}^1}$-module

$$M_\varphi := \{L \cdot \varphi \mid L \in \mathcal{D}_{\mathbb{A}^1}\}. \tag{15.14}$$

Then this transform applied to M_φ gives you the $\mathcal{D}_{(\mathbb{A}^1)^*}$-module $M_{\widehat{\varphi}}$.

Laumon-Rothstein (TODO: cite) showed you can do this with Jacobians: if $A = \mathrm{Jac}(C)$, then $A_{\mathrm{dR}}^\vee[1]$ is the space of rank-1 flat connections on C. The dual is canonically an $\mathbb{A}^1$-bundle over $\mathrm{Jac}(C)$. ◄

This is the last abelian example; Thursday we will dive into the nonabelian world.

Lecture 16.

Betti and de Rham: 3/25/21

Before we generalize to nonabelian groups, let's discuss the difference between the Betti and de Rham versions of this duality story, which may help clear some things up. Recall that we had as our paradigm a duality between the $\mathcal{A}$-side, which is about topology, and the $\mathcal{B}$-side, which is about algebraic geometry (the fields form an algebraic variety). On the $\mathcal{A}$-side, we often take something like locally constant functions.

There are at least three ways to formalize the idea of "locally constant functions," and they lead to different implementations of the duality paradigm.

(1) First, we could take actually locally constant functions, functions which are constant along paths. When we pass to the derived setting, we get the singular cohomology of the space. This is the "Betti setting."
(2) The "de Rham setting" assumes our space is smooth, and says that locally constant means having vanishing derivative. When we pass to the derived setting, we get de Rham cohomology. You can identify this notion of locally constant functions on X as functions on the space X_{dR} (recall X_{dR} is not a space, but some sort of stack, where we identify points which are infinitesimally nearby). On reasonable spaces, singular and de Rham cohomology are isomorphic, of course.
(3) We're not going to say as much about this example, but in symplectic topology we relate the topology of X to the symplectic topology of T^*X, specifically Floer theory. The use of the letter $\mathcal{A}$ tends to be associated with symplectic topology (e.g. the $\mathcal{A}$-model), even though there are also the Betti and de Rham approaches to this duality. We will only care about cotangent bundles in this lass.

This gets subtler when we categorify (then again, what doesn't?). In codimension 2, we want to assign a category which is something like a category of sheaves on the space of fields. On the $\mathcal{B}$-side, the fields are an algebraic variety, and there's no question about what kinds of sheaves we should take: quasicoherent sheaves.[18] On the $\mathcal{A}$-side, the different notions of functions above lift to different notions of sheaves.

(3) In the symplectic setting, you get some sort of Fukaya category, though there is more than one way of making this precise. We're not going to worry very much about this case.

For both the Betti and de Rham settings, there will be a common core, since these approaches are both topological in some sense.

(1) In the Betti setting, the core is locally constant finite-rank sheaves, i.e. sheaves which are locally isomorphic to $U \times \mathbb{C}^r$. We have parallel transport along paths, well-defined up to homotopy. This category is called $\mathcal{L}oc^{fd}(X)$.
(2) In the de Rham setting, we cosnider flat vector bundles. These are also locally trivial, and this is a version of saying the derivative vanishes. This is a sort of equivariance for infinitesimally nearby things (TODO: ??). This category is called $\mathcal{C}onn^\flat(X)$.

These two kinds of sheaves are equivalent, which is a baby case of the Riemann-Hilbert correspondence. But they sit inside quite different larger categories of sheaves.

(1) There are a few ways to think of this on the Betti side, but we can take $\mathcal{L}oc(X)$, the category of all locally constant sheaves on X. One the level of abelian categories, this is equivalent to the category of representations of the fundamental groupoid of X; we're specifying the locally constant condition on paths.[19]
(2) On the de Rham side, we consider the category of $\mathcal{D}_X$-modules, quasicoherent sheaves with flat connection. This is equivalent to $\mathcal{QC}(X_{\mathrm{dR}})$.

These two categories are quite different. Let's look at this in a simple example, $X = \mathbb{C}^\times$ (topologically, you can think about the circle).

(1) Topologically, locally constant sheaves on $\mathbb{C}^\times$ are equivalent to locally constant sheaves on S^1 are equivalent to locally constant sheaves on $B\mathbb{Z}$ are equivalent to representations of $\mathbb{Z}$. That is, to know a locally constant sheaf on $\mathbb{C}^\times$, you just have to understand what's going on when you walk around the nontrivial loop.

    This is equivalent to $\mathcal{M}od_{\mathbb{C}[\mathbb{Z}]} = \mathcal{M}od_{\mathbb{C}[z,z^{-1}]} = \mathcal{QC}(\mathbb{G}_m)$. This $\mathbb{G}_m$ is dual to the $\mathbb{C}^\times$ we started with, in the sense of Cartier duality; if you began with a torus, you'd end up with quasicoherent sheaves on the dual torus.

    Some examples of locally constant sheaves are *character sheaves* L_μ, rank-one local systems with monodromy μ. Under Cartier duality these are exchanged with skyscraper sheaves on $\mathbb{G}_m$.

[18] Later in the class we will generalize to *ind-coherent sheaves*, $\mathcal{QC}^!(\mathcal{F})$, which is better at handling singularities. If the space of fields is smooth there is no difference.

[19] There are even bigger category of spaces that we will consider later, construcible sheaves or perverse sheaves, which are useful for singularities.

(2) The de Rham story, a version of the *Mellin transform*, is quite different. The Mellin transform is the Fourier transform for the group $\mathbb{R}_+$ — of course, this group is isomorphic to $\mathbb{R}$ under the exponential map, so this is the same thing as the usual Fourier transform in multiplicative notation. If f is a function on $\mathbb{R}_+$, then

$$\widehat{f}(s) = \int_0^\infty f(z) z^s \frac{\mathrm{d}z}{z}. \tag{16.1}$$

Here z^s, $s \in \mathbb{C}$, are the characters of $\mathbb{R}^+$; back on $\mathbb{R}$, these pass to e^{zs}.

The ring of differential operators on $\mathbb{C}^\times$ has a derivation $\partial \coloneqq t\partial_t$, where t is the coordinate:

$$\mathcal{D}_{\mathbb{C}^\times} = \mathbb{C}\langle t, t^{-1}, \partial\rangle/(\partial t = t\partial + t). \tag{16.2}$$

We can switch the order of these generators and write $\mathcal{D}_{\mathbb{C}^\times} \cong \mathbb{C}\langle s, \sigma, \sigma^{-1}\rangle/(s\sigma = \sigma s + \sigma)$. This looks like functions on $\mathbb{C}$ together with a shift operator σ acting by translation! That is, $\mathbb{Z}$ acts on $\mathbb{C}$ by translation; these *finite difference operators* has the same relation as for $\mathcal{D}_{\mathbb{C}^\times}$). Let $\Delta_{\mathbb{C}}$ denote this ring of differential operators.

When you pass to modules, $\mathcal{D}_{\mathbb{C}^\times}$-modules are equivalent to $\Delta_{\mathbb{C}}$-modules, which are the same thing as quasicoherent sheaves on $\mathbb{C}$ with a shift action, or equivalently $\mathcal{QC}(\mathbb{C}/\mathbb{Z})$.

As an example, there is the module M_λ which corresponds (TODO: I missed something) to the trivial line bundle on $\mathbb{C}^\times$ with the flat connection

$$\nabla = \mathrm{d} - \lambda\frac{\mathrm{d}t}{t}. \tag{16.3}$$

Thus ∂ acts by λ; ∂ is a grading operator, so any solution f to this ODE is homogeneous of degree λ, such as z^λ.

Now for $n \in \mathbb{Z}$, $M_\lambda \cong M_{\lambda+z}$, because M_λ only depends on its monodromy, which is $\exp(\lambda)$.

TODO: I missed something about $\mathcal{D}$-modules on $\mathbb{C}^\times$, which have all sorts of stuff like δ-functions. The difference between Betti and de Rham corresponds to the difference between $\mathbb{C}/\mathbb{Z}$ and $\mathbb{C}^\times$, which are the same analytically but have quite different structures on algebraic varieties. This is the difference between the two models (Betti and de Rham) on the $\mathcal{B}$-side; on the $\mathcal{A}$-side, these correspond to the two different notions of sheaves.

Let's return to class field theory. We said that there's a correspondence between abelian local systems on C and on $\mathrm{Jac}(C)$; and that rank-one local systems on C are equivalent to character local systems on $\mathrm{Pic}(C)$. When we look at categories, these look quite different.

On the $\mathcal{A}$-side, the space is $\mathrm{Pic}(C) = \mathbb{Z} \times \mathrm{Jac}(C)$ (and a $B\mathbb{G}_m$ factor which will be irrelevant until we pass to the derived setting). There are two ways to take sheaves (linearize): the Betti version takes local systems, and the de Rham version takes $\mathcal{D}$-modules. They both contain a common core, but are actually different: $\mathcal{L}oc(\mathrm{Pic}(C))$ is equivalent to the category of $\mathbb{Z}$-graded representations of $\pi_1(\mathrm{Jac}(C)) = H_1(C;\mathbb{Z}) \cong \mathbb{Z}^{2g}$, where g is the genus of C. This contains finite-dimensional monodromy representations as well as free ones (TODO: aren't these also finite-rank?), such as the universal cover representation.

In the de Rham version, we have $\mathcal{D}$-modules, but because the Jacobian isn't an affine variety, this is more annoying to describe: it's not quite a category of modules. Instead we take the category of objects in $\mathcal{QC}(\mathrm{Pic}(C))$ together with an action of a vector field. This is different from the Betti story but, again, has common objects.

And the class field theory we get differs between these two perspectives: in both cases we have Fourier-Mukai dualities. (TODO: missed one, sorry about that. In general, I've been pretty confused today about when objects are supposed to be topological spaces vs. algebraic varieties, so when it matters when something is $\mathbb{C}^\times$ vs. $\mathbb{G}_m$ or $\mathbb{C}/\mathbb{Z}$ or so on. This probably made the notes pretty unclear. Sorry about that.)

That is, character sheaves $\chi_L \in \mathcal{L}oc(\mathrm{Pic}(C))$ are exchanged under the Fourier-Mukai transform in the Betit case with skyscraper sheaves $\mathcal{O}_L \in \mathcal{QC}(\mathrm{Loc}_1(C))$ at L.

That was the Betti side — on the de Rham side, we begin with what Mukai did: an equivalence of derived categories

$$(\mathcal{QC}(\mathrm{Jac}(C)), *) \overset{\simeq}{\leftrightarrows} (\mathcal{QC}(\mathrm{Jac}(C)), \otimes). \tag{16.4}$$

We want to embiggen these categories: we had $\mathcal{D}$-modules, for example. These are quasicoherent sheaves equivariant for the formal group of $\mathrm{Jac}(C)$, or equivalently, there is an action of the algebra of vector fields on $\mathrm{Jac}(C)$. There's some monadicity/the Barr-Beck theorem hiding in here related to this extra algebrac structure.

On the other side, we enlarge to $\mathcal{QC}(\mathrm{Conn}_1^\flat(C))$, which means the quasicoherent sheaves on $\mathrm{Jac}(C)$ with an action of functions on the fibers. One checks that the Fourier-Mukai transform in fact exchanges these two kinds of sheaves.

There is a forgetful map $\mathrm{Conn}_1^\flat(C) \to \mathrm{Jac}(C)$, which forgets the connection, and the kernel is

$$\Gamma(\mathrm{Jac}(C), \Omega^1) \cong T_e^*\mathrm{Jac}(C) \cong H^0(C; \Omega^1), \tag{16.5}$$

i.e. the forgetful map is an affine bundle.

The Betti story is much more elementary than the de Rham story: you can check it by hand, etc., but you've lost something: in the de Rham story, both sides live in algebraic geometry. This means the de Rham theory has an additional symmetry which was broken in the Betti story.

How do we enhance (16.4)? We can pass from $\mathrm{Jac}(C)$ to its cotangent bundle. Because $\mathrm{Jac}(C)$ is a group, its cotangent bundle is trivial, and

$$T^*\mathrm{Jac}(C) \cong \mathrm{Jac}(C) \times H^0(C; \Omega). \tag{16.6}$$

The fiber $H^0(C;\Omega)$ can be called the *rank-1 Hitchin space* if you want to be fancy: this is a special abelian case of a general and interesting object. There is a projection map onto $H^0(C;\Omega)$ from both the $T^*\mathrm{Jac}(C)$ and $\mathrm{Jac}(C) \times H^0(C;\Omega)$ perspectives, and their fibers are equivalent, and this is a special case of the self-duality of the Hitchin integrable system.

Now let's deformation quantize the equivalence (16.4). There's a natural deformation from $\mathcal{O}(T^*X)$ to differential operators $\mathcal{D}_X = \mathbb{C}\langle x_i, \partial_i\rangle/\sim$. We can summarize this by adding a parameter $\hbar$ and considering

$$\mathbb{C}\langle x_i, \partial_i\rangle/(\partial_i\, x_j - x_j\partial_i = \delta_{ij}\hbar). \tag{16.7}$$

When $\hbar = 1$, this is $\mathcal{D}_X$, and when $\hbar = 0$, this is $\mathcal{O}(T^*X)$; this is the sense in which $\mathcal{D}$-modules are a deformation quantization of functions on the cotangent bundle.

On the $\mathcal{B}$-side, deformation quantization deforms $\mathcal{QC}(T^*\mathrm{Jac}(C))$ to $\mathcal{QC}(\mathrm{Conn}_1^\flat(C))$.

The nonabelian setting is less explicit, which makes things harder, and it may be helpful to use the abelian case as scaffolding/intuition when we generalize.

Lecture 17.

The moduli stack of principal G-bundles on a curve: 3/30/21

We want to generalize this abelian duality that we've seen in the previous lectures to the case where G is a nonabelian group. There is an early, difficult problem: $\mathbb{C}[G]$ is noncommutative, so there's no obvious dual $\mathrm{Spec}(\mathbb{C}[G], *)$. So to sidestep this issue and obtain a duality, we need to find something else which is commutative. We will use topological field theory to do this.

There are two places that you could start this story, from the problem or from the solution (TFT), and we'll start with the problem. We specifically want to replace what appeared in class field theory, $\mathrm{Pic}(C)$, with a nonabelian counterpart: the theory of automorphic forms.

Let C be a smooth projective curve over a field k, which is either $\mathbb{F}_q$ or $\mathbb{C}$ (in the latter case, C is a Riemann surface). In the abelian case, we studied functions on C ($k = \mathbb{F}_q$) or sheaves (more general k, in particular $k = \mathbb{C}$) on $\mathrm{Pic}(C)$, the moduli space of line bundles on C. We will replace "line bundles" here with vector bundles or, more generally, principal G-bundles.

Let G be an algebraic group over k, affine and (usually) reductive. Rather than define reductive right now, we'll say that this includes the usual suspects such as GL_n, SO_n, and Sp_n. A principal G-bundle is a space P with a map $P \to C$ and a right G-action on C such that this locally on C, $P|_U \cong U \times G$, where G acts on G but not on U. When $k = \mathbb{F}_q$, we have to be specific about what "locally" means: we mean étale-locally. For $G = \mathrm{GL}_n$, this is equivalent data to a rank-n vector bundle. Principal bundles on a space form a groupoid: if you write down the notion of a map of principal bundles, analogously to the definition of a map of vector bundles, you will find all such maps are isomorphisms. In particular, principal bundles have automorphisms.

Let $\mathrm{Bun}_G(C)$ be the moduli space (stack) of principal G-bundles on C. The functor of points of $\mathrm{Bun}_G(C)$ sends a ring R to the groupoid of principal G-bundles on $C \times \mathrm{Spec}\, R$. When $G = \mathrm{GL}_1 = \mathbb{G}_m$, then there is an equivalence

$$\mathrm{Bun}_{\mathbb{G}_m}(C) \cong \underline{\mathrm{Pic}}(C) := \mathrm{Pic}(C) \times \mathrm{pt}/\mathbb{G}_m. \tag{17.1}$$

That is, a principal $\mathbb{G}_m$-bundle is equivalent data up to isomorphism as a line bundle, but all principal $\mathbb{G}_m$-bundles have automorphism group $\mathbb{G}_m$.

Say $k = \mathbb{F}_q$; then throw away the stabilizers in $\mathrm{Bun}_G(C)(\mathbb{F}_q)$, yielding a discrete set of isomorphism classes of principal G-bundles — it's not in general finite, but it doesn't have any geometry. Call this discrete set $|\mathrm{Bun}_G(C)(\mathbb{F}_q)|$.

Definition 17.2. An *(unramified) automorphic function* for G on C is a $\mathbb{C}$-valued function on $|\mathrm{Bun}_G(C)(\mathbb{F}_q)|$.[20]

In general, when this set is infinite, we impose some sort of growth condition on these functions, e.g. compact support.

In the abelian case, we gave a Fourier decomposition of functions on $\mathrm{Pic}(C)$; now, we will try to spectrally decompose unramified automorphic forms. We can use Weil's description of $|\mathrm{Bun}_G(C)(\mathbb{F}_q)|$, which is (relatively!) concrete, and is analogous to our description of $\mathrm{Pic}(C)$.

First, any principal G-bundle on C is generically trivial. Some thought is required to see why this is true; it unpacks to finding a rational section of $P \to C$, meaning a section on C minus finitely many points. And a principal bundle with a section is trivialized.

Next, any G-bundle is trivial on a formal neighborhood of a point. So we can trivialize almost everywhere, i.e. on C minus finitely many points x_i; trivialize in formal neighborhoods of each x_i; and then describe the transition functions between these descriptions. We need as an important ingredient the Beauville-Laszlo theorem, which says that we can actually glue a G-bundle from formal transition data.

For each x_i, we have the complete local ring $\mathcal{O}_{x_i}$ of Taylor series, contained within the local field K_{x_i} of Laurent series. On the overlap of $\mathrm{Spec}(\mathcal{O}_{x_i})$ with $C \setminus x$, we get a difference of trivializations, which lives in $G(K_{x_i})$. This data collects into an element of the restricted product

$$\prod_{x\in C}{}' \, G(K_x), \tag{17.3}$$

meaning all but finitely many of these are zero. Weil then identified the cosets.

Theorem 17.4 (Weil)**.** *Suppose $k = \mathbb{F}_q$. Then*

$$|\mathrm{Bun}_G(C)(k)| \cong G(F_C)\backslash \prod_{x\in C}{}' \, G(K_x) / \prod_{x\in C}{}' \, G(\mathcal{O}_x) \tag{17.5a}$$

$$\cong G(F_c)\backslash G(\mathbb{A}_C)/G(\mathcal{O}_{\mathbb{A}}). \tag{17.5b}$$

Recall that we had a similar-looking double coset decomposition in the abelian case — the main difference is that the quotient of an abelian group by a subgroup is always a group, but that is not necessarily true here. In the abelian case, the local factor

$$\mathrm{GL}_1(K_x)/\mathrm{GL}_1(\mathcal{O}_x) = K_x^\times/\mathcal{O}_x^\times \cong \mathbb{Z}, \tag{17.6}$$

which was the source of the description of divisors as line bundles. But in the nonabelian case, we have to grapple with something more difficult:

$$\mathrm{Gr}_G := G(K_x)/G(\mathcal{O}_x) \tag{17.7}$$

is called the *affine Grassmannian* of G. It is not a group: $G(\mathcal{O}_x)$ is not normal in $G(K_x)$. You can think of Gr_G as the space of data of a principal G-bundle on C and a trivialization on $C \setminus x$.

(17.5b) has a nice analogue in the number field setting. Here G is the analogue of the idele class group and $F/\mathbb{Q}$ is a finite extention (most of the general features are already present for $F = \mathbb{Q}$). Consider the object

$$G(F)\backslash G(\mathbb{A}_F)/G(\mathcal{O}_{\mathbb{A}_F}), \tag{17.8}$$

where $\mathbb{A}_F$ is the *ring of adeles*, the product of the completions of $\mathcal{O}_F$ at all places. For $\mathbb{Q}$, $\mathbb{A}_{\mathbb{Q}}$ is the restricted product of $\mathbb{Q}_p$ for each p with $\mathbb{R}$. The $\mathbb{Q}_p$ factors are called the "finite adeles," and is isomorphic to $\widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$. The ring of functions is $\prod_p \mathbb{Z}_p = \widehat{\mathbb{Z}}$. That is, the adeles have a "familiar" finite part ($\mathbb{Q}_p$) and a "weird"

[20] You can use $\overline{\mathbb{Q}}_\ell$-valued functions if you want.

Archimedean part ($\mathbb{R}$), which may be the reverse from what you're used to, if you're not coming from number theory.

So $G(\mathbb{A}_{\mathbb{Q}})$ is the product of $G(\mathbb{A}_{\mathrm{fin}})$ and a real Lie group $G(\mathbb{R})$. We quotient out by $G(\mathbb{Q})$, which has the nice property that it has the discrete topology – a convenient fact about the adeles. So we have

$$G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}})/G(\widehat{Z}) \times K_\infty, \tag{17.9}$$

where K_∞ is a compact subgroup of $G(\mathbb{R})$, such as its maximal compact. (17.9) is what this analogy suggests plays the role of "$\mathrm{Bun}_G(\mathbb{Q})$," though it's not really as topological.

The space

$$G(\mathbb{Q}) \cap \prod_p G(\mathbb{Z}_p)G(\mathbb{R})/K_\infty \tag{17.10}$$

sits inside (17.9). The intersection on the left is rational numbers with no p-powers in the denominator for any p, so this is $G(\mathbb{Z})$. That is, we have $G(\mathbb{Z})\backslash G(\mathbb{R})/K_\infty$, and the claim is that this is a connected component, one of finitely many, and for some particularly nice groups (e.g. simply connected reductive), this is a bijection! The point is, this is relatively concrete, though it priveleges one place ($\mathbb{R}$) over the others. It relates the theory of modular forms to that of automorphic forms.

Example 17.11. Say $G = \mathrm{GL}_2$, in which case this relates to something much more classical. We're asking approximately what is the moduli space of rank 2 vector bundles on $\mathrm{Spec}(\mathbb{Z})$, or some sort of completion of $\mathrm{Spec}(\mathbb{Z})$?

Very naïvely, this should be something like a rank 2 free $\mathbb{Z}$-module, but with a trivialization at infinity. Let $\mathcal{B}$ be the space of $\mathbb{R}$-bases of $\mathbb{C}$, which can be identified with the space of isomorphisms $\mathbb{Z}^2 \otimes \mathbb{R} \to \mathbb{R}^2$; fixing a favorite basis identifies $\mathcal{B} \cong \mathrm{GL}_2(\mathbb{R})$.

We can describe $\mathcal{B}$ concretely as $\mathbb{C}^\times \times \mathbb{C} \setminus \mathbb{R}$ by choosing the first basis vector, then the second. If $\mathcal{L}$ is the space of lattices in $\mathbb{C}$, i.e. free rank-2 $\mathbb{Z}$-modules Λ with an isomorphism $\Lambda \otimes \mathbb{R} \cong \mathbb{C}$, then $\mathcal{L}$ is close to the space $\mathcal{B}$ of bases, but we're identifying bases under invertible *integer*-valued transformations:

$$\mathcal{L} \cong \mathrm{GL}_2(\mathbb{Z})\backslash \mathcal{B}. \tag{17.12}$$

Because we're using $\mathbb{C}$, this space has a natural complex structure. This does not generalize to higher rank.

There are several variants of this, e.g. $\mathcal{L}(N)$, lattice where we also fix a basis mod N, i.e. an identification $\mathcal{L}/N \cong (\mathbb{Z}/N)^{\oplus 2}$. You can think of this as a rank-2 bundle and a trivialization on $(N) \subset \mathrm{Spec}\,\mathbb{Z}$. This uses multplicity, e.g. (p) and (p^2) are different, trivializing on higher-order neighborhoods of the point $p \in \mathrm{Spec}\,\mathbb{Z}$.

Another variant is $\overline{\mathcal{L}}$, the space of lattices up to $\mathbb{C}$-homothety,[21] which is a rescaling operation. This space is

$$\overline{\mathcal{L}} \cong \mathrm{GL}_2(\mathbb{Z})\backslash \mathcal{B}/\mathbb{C}^\times \cong \mathrm{PSL}_2(\mathbb{Z})\backslash \mathbb{H}. \tag{17.13}$$

The following lemma tells us we can identify the space of lattices adelically, but with no factor at infinity.

Lemma 17.14.

$$\mathcal{L} \cong \mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{GL}_2(\mathbb{R}) \cong \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/\mathrm{GL}_2(\mathcal{O}_{\mathbb{A}}).$$

This again uses the fact that $\mathrm{GL}_2(\mathbb{Z}) = \mathrm{GL}_2(\mathbb{Q}) \cap \mathrm{GL}_2(\widehat{\mathbb{Z}})$; you also need that $\mathrm{GL}_2(\mathbb{A}_{\mathrm{fin}})$ is generated by $\mathrm{GL}_2(\mathcal{O}_{\mathbb{A}})$ and $\mathrm{GL}_2(\mathbb{Q})$. This allows us to pass from this automorphic description, which treats all places equally, to the real homogeneous space description, which singles out $\mathbb{R}$. You can get the quotient by $\mathbb{C}^\times$ also by quotienting out by the maximal compact O_2.

This is where modular forms arise classically: taking holomorphic sections of a line bundle on $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ (albeit with a growth condition). These are contained within the space of functions on $\mathrm{GL}_2(\mathbb{Z})\backslash\mathcal{L}$. We can therefore write down the definition of an automorphic form for GL_2 over $\mathbb{Q}$, which is a function on

$$\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/\mathrm{GL}_2(\mathcal{O}_{\mathbb{A}}). \tag{17.15}$$

There's also the level N version, where we replace $\mathrm{GL}_2(\mathcal{O}_{\mathbb{A}})$ with $\mathrm{GL}_2(\mathcal{O}_{\mathbb{A}}^{(N)})$. But let's not worry about that for now, and go back to (17.15), the level 1 version. We want our functions to be smooth on $\mathrm{GL}_2(\mathbb{R})$ and to have a certain growth condition at the infinite place ($\mathbb{R}$). ◀

[21] Exercise: how is "homothety" pronounced?

An analogous approach works for other groups. For example, when $G = \mathrm{O}_n$, one forms a moduli space of orthogonal bundles: fix an even lattice Λ and look at free $\mathbb{Z}$-modules V equipped with a quadratic function $q\colon V \to \mathbb{Z}$. Given this data, we can make sense of when an automorphism of V is orthogonal, i.e. when it preserves q, so we get the orthogonal group $\mathrm{O}(V, q)$. Then we get the double quotient

$$\mathrm{O}_q(\mathbb{Q})\backslash \mathrm{O}_q(\mathbb{A})/\mathrm{O}_q(\mathcal{O}_{\mathbb{A}}). \tag{17.16}$$

This has a very classical meaning: the (isomorphism classes of) quadratic forms in the *genus* of q, meaning that they identify with q mod every N. Things like Θ-series are certain functions on these collections of quadratic forms.

And we can play this game for any group G over a number field F, defining "Bun_G" as

$$G(F)\backslash G(\mathbb{A}_F)/(G(\mathcal{O}_{A_F}) \times K_\infty), \tag{17.17}$$

where K_∞ is some compact subgroup of $G(F \otimes_{\mathbb{Q}} \mathbb{R})$ ($F \otimes_{\mathbb{Q}} \mathbb{R}$ is called the *Archimedean completion* of F, and is some product of copies of $\mathbb{R}$ and $\mathbb{C}$). This is the data at infinity for general F. (17.17) is an *arithmetic locally symmetric space.* $G(F \otimes_{\mathbb{Q}} \mathbb{R})$ factors as $G(\mathbb{R})^{\times s} \times G(\mathbb{C})^{\times t}$, and K_∞ has to have components in all of these.

Sometimes one wants to consider *cohomological automorphic forms*, where we replace functions on $G(F)\backslash G(\mathbb{A}_F)/(G(\mathcal{O}_{\mathbb{A}_F}) \times K_\infty)$ with cohomology. Since $G_{\mathbb{R}}/K_\infty$ is contractibe, this amounts to the group cohomology

$$H^*(G(\mathbb{Z}); \mathbb{C}). \tag{17.18}$$

More generally, of $G(\mathbb{Z})$ acts on a complex vector space V, we can consider $H^*(G(\mathbb{Z}); V)$, which is the cohomology of our arithmetic locally symmetric space twisted by some local system. There is a nonobvious fact called the *Eichler-Shimura isomorphism* (using Hodge theory on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$) which means that classical modular forms sit inside cohomological modular forms for $G = \mathrm{SL}_2$. Rather than worry about holomorphic functions with growth conditions, we can take cohomology.

In the function field case, functions on the discrete set $\mathrm{Bun}_G(C)(\mathbb{F}_q)$ can be identified with $H^*(\mathrm{Bun}_G(C)(\mathbb{F}_q))$, and this is one model for the theory of automorphic forms. In general, the "same" automorphic forms appear in very different places under quite different kinds of functions, and this is a quirk of the general theory. This will lead to a notion of "geometric automorphic forms," analogous to "geometric class field theory." We take sheaves: if k is a field and C a curve over k as before, we will consider sheaves on $\mathrm{Bun}_G(C)$, "automorphic sheaves." This is analogous to what we did in the abelian case, where we sent Σ to sheaves on $\mathrm{Pic}(\Sigma)$. We hope these are attached to surfaces in a topological field theory.

Another potential source of motivation is to look at number fields, where arithmetic locally symmetric spaces appear in classical number theory, so their geometric analogues are hopefully the right thing to study.

The basic problem we run into is that $\mathrm{Bun}_G(C)$ is not a group, and does not have a natural group action we can use to decompose things into eigenthings. Instead, Hecke operators will act, and excitingly they form a commutative algebra, so we can do spectral decomposition. We will get into this more next time. This is where ideas from field theory shine.

Lecture 18.

Hecke algebras: 4/1/21

Last time, we discussed among other things *cohomological automorphic forms*, such as $H^*(G(\mathbb{Z})\backslash G_{\mathbb{R}}/K)$, which is fortunate cases is isomorphc to the group cohomology of $G(\mathbb{Z})$. The function-field analogue is simpler: functions on a discrete set $\mathrm{Bun}_G(C)(\mathbb{F}_q)$.

One important question you might ask here is: this is supposed to be related to some 4d TFT $\mathcal{A}_G$. What does this theory attach to a 3-manifold Y? When G is a torus, we attached to Y the homology of the space of G-connections on Y. But in general, it's not clear what we should attach to a 3-manifold: this is an open problem. This doesn't doom the program to view geometric Langlands in terms of topological field theory: the conjectures themselves are about the categories attached to surfaces, and these we do know how to define.

Recall

$$\mathrm{Bun}_G(C)(\mathbb{F}_q) \cong G(F_c)\backslash G(A_F)/G(\mathcal{O}_{\mathbb{A}_F}) = G(F_c)\backslash {\prod}' G(K_x)/\prod G(\mathcal{O}_X), \tag{18.1}$$

where we take the restricted product over the points $x \in C$, and C is our curve over $\mathbb{F}_q$. These are our automorphic forms. Following our noses from the abelian case, we need an action of something to diagonalize and make a spectral decomposition, but (18.1) is not a group, and so it's not immediately clear how to proceed.

But we do have a space of the form X/K with a group action G acting on X (this might not be the G we took before), where $K \subset G$. So, the general question is, *what acts on functions on a quotient, when the quotient came from a larger symmetry group?* In (18.1), we have a huge amount of symmetry: the group of Laurent series at every point, but we have only quotiented out by part of this, the Taylor series at every point. If we didn't quotient on the right, we would have

$$G(F_c)\backslash \prod_{x\in C}{}' G(K_x), \tag{18.2}$$

and this has an action of $\prod G(K_x)$. This is many copies of $G(K_x)$, and they all commute! Each individual group is nonabelian, though. When we quotient out by $\prod' G(\mathcal{O}_x)$, we still have this symmetry. This is an example of a Hecke algebra symmetry; for the rest of today's lecture, we'll discuss Hecke algebras in several examples.

As one example, suppose a finite group G acts on a set X and $K \subset G$, and take $\mathbb{C}[X/K]$. The coset space X/K has an action of $N_G(K)/K$, but this isn't always very helpful: for some choices of G and K, this is trivial. But functions on X/K has a better action.

We know that the algebra $\mathbb{C}[G]$ acts on the vector space $\mathbb{C}[X]$, and $\mathbb{C}[X/K] = \mathbb{C}[X]^K$, the K-invariants. So what acts on the K-invariants of a G-representation? Taking K-invariants is a functor $(\text{–})^K : \mathcal{R}ep_G \to \mathcal{V}ect$, and it is canonically representable:

$$(\text{–})^K = \operatorname{Hom}_{\mathbb{C}[G]}(V_{G,K}, \text{–}). \tag{18.3}$$

This is because, as K-representations, $V^K = \operatorname{Hom}_{\mathbb{C}[K]}(\mathbb{C}, V|_K)$, and we use the Hom-tensor adjunction

$$\operatorname{Hom}_{\mathbb{C}[K]}(\mathbb{C}, V|_K) = \operatorname{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G] \otimes_{\mathbb{C}[K]} \mathbb{C}, V). \tag{18.4}$$

This is called *Frobenius reciprocity.*[22] Since G is finite, we can identify functions and measures on G:

$$V_{G,K} = \mathbb{C}[G] \otimes_{\mathbb{C}[K]} \mathbb{C} = \{f : G \to \mathbb{C} \mid f(gh) = f(g)\}. \tag{18.5}$$

So these are functions on $\mathbb{C}[G/K] = \mathbb{C}[G]^K$. When G is infinite we would get measures instead of functions.

Definition 18.6. The *Hecke algebra* $\mathcal{H}_{G,K}$ is the algebra of endomorphisms of the K-invariants functor.

Since taking K-invariants is representable,

$$\mathcal{H}_{G,K} = \operatorname{End}_{\mathcal{R}ep_G} V_{G,K} = \operatorname{Hom}_{\mathbb{C}[G]}(V_{G,K}, V_{G,K}) = (V_{G,K})^K = \mathbb{C}[G/K]^K = \mathbb{C}[K\backslash G/K]. \tag{18.7}$$

For the most part, this is why functions on double cosets appear in math. A priori, functions on double cosets don't have an algebra structure, but this construction produces a canonical one.

A piece of abstract nonsense called the Barr-Beck theorem allows us to make the following argument: the K-invariants functor $\mathcal{R}ep_G \to \mathcal{V}ect$ factors through $\mathcal{R}ep_G \to \mathcal{M}od_{\mathcal{H}_{G,K}}$: K-invariants canonically have this extra structure. This forgetful functor forgets less than one might imagine: inside $\mathcal{R}ep_G$, consider the full subcategory whose K-invariants are "big enough," meaning that they are generated as G-representations by their K-invariants. For example, for irreducible representations, this is asking just that there is some nonzero K-invariant vector. Just for fun, we will call these representations *spherical.*

The conclusion is that the forgetful functor from spherical representations to $\mathcal{H}_{G,K}$-modules is an equivalence of categories! There's nothing particularly representation-theoretic going on here; it's all tautologies about adjoint functors. Another way to say this fact is that the spherical category is generated by $V_{G,K}$.

Much of this holds in greater generality, with the exception of a few times we use that functions and measures on a finite set coincide. But the application of the Barr-Beck theorem carries forth.

Example 18.8. The theory of spherical harmonics takes the SO_3-action on S^2 and linearizes, considering the SO_3-actions on $L^2(S^2)$. This leads to the pictures of s, p, d, and f orbitals that we saw in high school chemistry classes.

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[22] $\mathbb{C}[G] \otimes_{\mathbb{C}[K]} V$ is often called the *induced representation* of V, but in general this is the *coinduced representation.* They coincide for finite groups, which can cause confusion later.

We can write S^2 as the coset space $\mathrm{SO}_3/\mathrm{SO}_2$: $G = \mathrm{SO}_3$ and $K = \mathrm{SO}_2$. Then a representation of G is *spherical* if it appears as a summand in $L^2(S^2)$, which is equivalent to it having SO_2-invariants. All SO_3-representations are spherical, but if we used SU_2, we'd find that only half of them are. This is the origin of the word "spherical." $\blacktriangleleft$

Let's describe the algebra structure on $\mathbb{C}[K\backslash G/K]$ more explicitly. For $\mathbb{C}[G]$, we had a push-pull description:

$$\begin{tikzcd} & G \times G \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, "\mu"] & \\ G & & G \\ & G. & \end{tikzcd} \tag{18.9}$$

The product of $f, g \in \mathbb{C}[G]$ is given by

$$f \boxtimes h = \pi_1^* \cdot f \pi_2^* h, \tag{18.10}$$

and to define multiplication (convolution), we push forward by μ. And crucially, this is all invariant for the left K-action on the first factor of G and the right K-action on the second copy of G:

$$\begin{tikzcd} & K\backslash G \times G/K \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, "\mu"] & \\ K\backslash G & & G/K \\ & K\backslash G/K. & \end{tikzcd} \tag{18.11}$$

There is an extra "middle" copy of K acting on $K\backslash G \times G/K$, so we could take the "balanced product" by setting $(g \cdot k, g') = (g, k \cdot g')$, denoted $- \times_K -$. And this diagram is invariant under modding out by this additional copy of K:

$$\begin{tikzcd} & K\backslash G \times_K G/K \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, "\mu"] & \\ K\backslash G/K & & K\backslash G/K \\ & K\backslash G/K. & \end{tikzcd} \tag{18.12}$$

The multiplication is the same: convolution. We can think of this as the subalgebra $\mathbb{C}[G]^{K\times K} \subset \mathbb{C}[G]$ (one of the Ks acts on the right). Some version of this generalizes to sheaves, etc.

Another way to think of the double coset space is

$$K\backslash G/K \cong (G/K \times G/K)/G, \tag{18.13}$$

where G acts diagonally. Therefore

$$\mathbb{C}[K\backslash G/K] = \mathbb{C}[G \times K \times G/K]^G, \tag{18.14}$$

which sits inside $\mathbb{C}[G/K \times G/K]$. For any set X, functions on $X \times X$ have a canonical algebra structure as matrices, or $\mathrm{End}(\mathbb{C}[X])$. Matrix multiplication also has a push-pull description

$$\begin{tikzcd} & X \times X \times X \arrow[dl, "\pi_{12}"] \arrow[dr, "\pi_{23}"] \arrow[dd, "\pi_{13}"] & \\ X \times X & & X \times X \\ & X \times X, & \end{tikzcd} \tag{18.15}$$

by pulling back along the outer two projections, taking the product, then pushing forward. This gives another way to think about the group algebra under convolution, as G-invariant $G \times G$ matrices, where G acts diagonally. The description of (18.14).

This example is the basic one behind all Hecke algebras; if you ever encounter something called a Hecke algebra, this is the example to meditate on.

All right, groupoid time. This is essentially the same example said more abstractly, but it may be helpful for the stackier among us. The groupoid pt/G is $\mathrm{Bun}_G(\mathrm{pt})$: there is a single principal G-bundle, and it has G automorphisms. This can be spelled out in the fiber product diagram

$$\begin{tikzcd} \mathrm{pt} \times_{\mathrm{pt}/G} \mathrm{pt} \arrow[d] \arrow[r] & \mathrm{pt} \\ \mathrm{pt} \arrow[r] & \mathrm{pt}/G. \end{tikzcd} \tag{18.16}$$

Now choose a subgroup $K \subset G$. The map $K \to G$ defines a map $\mathrm{pt}/K \to \mathrm{pt}/G$, and the pullback

$$\begin{tikzcd} {} \arrow[d] \arrow[r] & \mathrm{pt}/K \arrow[d, "G/K"] \\ \mathrm{pt}/K \arrow[r, "G/K"] & \mathrm{pt}/G \end{tikzcd} \tag{18.17}$$

exists, and is precisely $K\backslash G/K$. That is,

$$K\backslash G/K = \mathrm{pt}/K \times_{\mathrm{pt}/G} \mathrm{pt}/K. \tag{18.18}$$

In general, given a map $\pi\colon X \to Y$ (TODO: must this be surjective?), functions on $X \times_Y X$ can be thought of as block diagonal matrices. A similar description gives you the algebra structure on $\mathbb{C}[K\backslash G/K]$. This means that the Hecke algebra is an example of a *groupoid algebra* (a generalization of the group algebra).

Specifying a G-action on a space X is equivalent to knowing the map of stacks $X/G \to \mathrm{pt}/G$. (If stacks aren't your thing, you can say X is a set and X/G and pt/G are both groupoids.) You can then recover X as the pullback

$$\begin{tikzcd} X \arrow[d] \arrow[r] & X/G \arrow[d] \\ \mathrm{pt} \arrow[r] & \mathrm{pt}/G. \end{tikzcd} \tag{18.19}$$

So, recall that we asked what acts on X/K for a G-space X? We saw what the answer was for functions, and we saw that for groups the answer is $N_G(K)/K$. But we can actually see a groupoid action: there is a pullback diagram

$$\begin{tikzcd} X/K \arrow[d] \arrow[r] & \mathrm{pt}/K \arrow[d] \\ X/G \arrow[r] & \mathrm{pt}/G. \end{tikzcd} \tag{18.20}$$

That is, $X/K \to \mathrm{pt}/K$ (equivalent to X with its K-action, as we saw above), is pulled back from $X\mathrm{pt}/G$.

$$\begin{tikzcd} & X \times_{\mathrm{pt}/K} K\backslash G/K \arrow[dl] \arrow[dr] \arrow[dd] & \\ X/K & & X/K \\ & X \times_K G/K. & \end{tikzcd} \tag{18.21}$$

This unwinds to say that over X/K we have a principal G-bundle, and we're doing an associated bundle construction (TODO: I think I missed some things here).

Example 18.22. If $K = \{1\}$, then $K\backslash G/K = G$ and the Hecke algebra is $\mathbb{C}[G]$. ◀

Example 18.23. If $K = G$, $K\backslash G/K$ is not trivial: we have two G-actions and one copy of G, so we get pt/G. But the Hecke algebra is trivial: $\mathcal{H}_{G,G} \cong \mathbb{C}$. ◀

Example 18.24. Pick a Lie group G and a Borel subgroup B and let $K = B(\mathbb{F}_q)$ inside $G(\mathbb{F}_q)$. For example, if $G = \mathrm{GL}_n$, B can be chosen to be the subgroup of upper triangular matrices.

Now $\mathcal{H}_{G,K}$ is the *finite Hecke algebra*, the algebra of $B(\mathbb{F}_q)$-invariant functions on the *flag variety* $G(\mathbb{F}_q)/B(\mathbb{F}_q)$. As a vector space, the Hecke algebra is $\mathbb{C}[W]$, where W is the Weyl group of G (e.g. S_n for $G = \mathrm{GL}_n$), but *the algebra structure is different* — it depends on the size q of the finite field, and in a sense is a q-deformation of $\mathbb{C}[W]$ with its usual algebra structure.

Write $\mathcal{B} = G(\mathbb{F}_q)/B(\mathbb{F}_q)$ and write the double coset spac as $G(\mathbb{F}_q)\backslash\mathcal{B}\times\mathcal{B}$, giving us a relative version of flags. If $G = \mathrm{SL}_2$, so $\mathcal{B} = \mathbb{P}^1_{\mathbb{F}_q}$, then in a coordinate-independent way we can only ask whether two lines are equal or not, so we should expect the $B(\mathbb{F}_q)$-orbits on the pair of flag varieties $\mathcal{B}\times\mathcal{B}$ to just have two elements: the orbit where the two flags are equal, and the orbit where they're unequal. This expectation is correct. Therefore the Hecke algebra is two-dimensional, spanned by $\{1, T\}$; on the equal orbit, $T * T = q$, and on the unequal orbit, $T * T = q - 1$. That is, you ask how many things are different from the flags you had: if they're equal, there's q points, and if they're not, there are only $q - 1$. Notice tht $q = \#(\mathbb{A}^1_{\mathbb{F}_q})$, and $q - 1 = \#(\mathbb{G}_{m\mathbb{F}_q})$. Collating this together,

$$T^2 = (q-1)T + q, \tag{18.25}$$

or equivalently $(T + 1)(T - q) = 0$. If you set $q = 1$, you get $T^2 = 1$, the group algebra of the Weyl group $\mathbb{Z}/2$. This is a little spooky (what's the field with one element?), but motivates the idea that in general, this Hecke algebra is a q-deformation of $\mathbb{C}[\mathbb{Z}/2]$. As we've constructed it, the Hecke algebra is a $\mathbb{C}$-algebra, but the relations are finite Laurent series in q, so $\mathcal{H}_q$ is actually a $\mathbb{Z}[q, q^{-1}]$-algebra, so we can evaluate at $q = 1$. So maybe $G(\mathbb{F}_1) = W$, though there's no way to make that make sense precisely. ◀

This demonstrates an important point: the Hecke algebra in this case is a q-deformation of the Weyl group, and therefore is at least as noncommutative as the Weyl group. And the Weyl group can be plenty noncommutative, e.g. for GL_n we get $\mathbb{C}[S_n]$ for the symmetric group S_n. Hecke algebras are not commutative in general.

But in some cases they are commutative. For an arbitrary group K, $G := K \times K$ where $K \hookrightarrow G$ diagonally, $\mathcal{H}_{G,K}$ is commutative! The double coset space is

$$K\backslash K \times K/K = K/K, \tag{18.26}$$

where K acts on K by conjugation. When you take functions, the algebra structure you get is that of class functions, i.e. functions on K/K with pointwise multiplication, and this is commutative. In fact, this is

$$\mathcal{H}_{K\times K,K} = \mathrm{End}_{\mathbb{C}[K]\otimes\mathbb{C}[K]^{\mathrm{op}}} \mathbb{C}[K] = Z(\mathbb{C}[K]). \tag{18.27}$$

This is the center of the algebra, also known as the zeroth Hochschild homology of $\mathbb{C}[K]$. The identification of the center with $\mathrm{End}_{A\otimes A^{\mathrm{op}}}(A)$ is not a special fact about group algebras; it holds for associative algebras in general. So this Hecke algebra is not just commutative, it's commutative in a very robust way.

We will see that the Hecke algebras that act on automorphic forms, which are large and interesting because we began with a huge amount of symmetry, are commutative algebras! This is the birth of the Langlands program. In fact, they are commutative for structural reasons, but this is less obvious. We will use topological field theory to demonstrate this.

Lecture 19.

Hecke algebras on arithmetic locally symmetric spaces: 4/6/21

Last time, we discussed Hecke algebras: if G acts on a space X and $K \subset G$, the quotient space X/K doesn't have a K-action, but functions on X/K have an action by functions on $K\backslash G/K$, a groupoid algebra. We in particular care about examples such as $\mathrm{Bun}_G(C)$ or its analogue over a number field. We presented these as double coset spaces $\Gamma\backslash G_{\mathbb{R}}/K$, such as $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2$. Let's fit this into the Hecke story.

In this case, $X = \Gamma\backslash G_{\mathbb{R}}$, so the Hecke algebra should be functions on $K\backslash G_{\mathbb{R}}/K$. For example, if we took $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2$, what is the double coset space $\mathrm{SO}_2\backslash\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2$? First, $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2$ is the upper half-plane, and SO_2 acts on the left by rotation around i: the left SO_2 is the stabilizer in $\mathrm{SL}_2(\mathbb{R})$ of $i \in \mathbb{H}$. The quotient is homeomorphic to $\mathbb{R}_+$, the set of all distances (in the hyperbolic metric) from i in $\mathbb{H}$.

Hecke symmetry tells us that from double cosets we get operators on functions on $G_{\mathbb{R}}/K$ which commute with the $G_{\mathbb{R}}$-action. Fix $r \in \mathbb{R}_+$; the operator we get on a function $f\colon \mathbb{H} \to \mathbb{C}$ averages f on a circle of radius r:

$$(\mathcal{O}_r * f)(z) = \oint_{|w-z|-r} f(w). \tag{19.1}$$

These operators all commute, and commute with the $\mathrm{SL}_2(\mathbb{R})$-action. The regularity of these functions isn't the point but we want these integrals to converge, so L^2 works. In particular, because these commute with the $\mathrm{SL}_2(\mathbb{R})$-action, they descend to quotients $\Gamma\backslash\mathbb{H}$, where $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ is a discrete subgroup.

Another manifestation of this is that

$$K\backslash G/K = G\backslash(G/K \times G/K), \tag{19.2}$$

where G acts diagonally. This leads us to think about differential operators — a *differential operator* on a space X is an endomorphism of functions on X which is local. We mean local strongly: if $\mathcal{O}$ is a differential operator, $(\mathcal{O} * f)(x)$ depends only on the Taylor series of f at x. This is the opposite of what we had before: averaging at a fixed distance is very nonlocal. This definition of differential operator is due to Grothendieck; it is a good way to make sense of differential operators in algebraic geometry. From the kernel transform point of view, the integral kernel of a differential operator is supported within an infinitesimal neighborhood of the diagonal in $X \times X$.

Theorem 19.3 (Harish-Chandra)**.** *Let $G_{\mathbb{R}}$ be a real Lie group and K be its maximal compact subgroup. Then the algebra of $G_{\mathbb{R}}$-equivariant differential operators on $G_{\mathbb{R}}/K$ is a commutative ring, and in fact is a polynomial algebra $\mathbb{C}[\mathfrak{a}]^W$.*

Here $\mathfrak{a}$ is the Cartan Lie algebra of $G_{\mathbb{R}}/K$, and W is the little Weyl group.

Example 19.4. For $\mathbb{H} = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2$, the algebra of differential operators is $\mathbb{C}[\Delta_{\mathbb{H}}]$, where $\Delta_{\mathbb{H}}$ is the Laplacian for the hyperbolic metric. ◀

In higher dimensions there are more operators, but they all commute.

This algebra of differential operators, because its kernels are supported in a neighborhood of the diagonal, is a Hecke algebra, now acting by differentiation. $\Gamma\backslash G_{\mathbb{R}}/K$ carries a natural comutative algebra of symmetries, namely $\mathrm{Diff}_{G_{\mathbb{R}}}(G_{\mathbb{R}}/K)$.

In the setting where we took the restricted product over all points of a curve

$$G(F)\backslash {\prod}' G(K_x)/G(\mathcal{O}_x), \tag{19.5}$$

we have a "local" Hecke algebra action for each x, and they commute, giivng us a *huge* algebra action by tensoring all of these local Hecke algebra actions. The local Hecke algebra is associated to the double coset space $G(\mathcal{O}_x)\backslash G(K_x)/G(\mathcal{O}_x)$.

Theorem 19.6 (Satake)**.** *The Hecke algebra $\mathbb{C}_c[G(\mathcal{O}_x)\backslash G(K_x)/G(\mathcal{O}_x)]$ is also a polynomial algebra.*

In the number field setting, we have something like $K_x = \mathbb{Q}_p$ and $\mathcal{O}_x = \mathbb{Z}_p$, and we're looking at $G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ as p varies. In the function field setting, $K_x = \mathbb{F}_q((t))$ and $\mathcal{O}_x = \mathbb{F}_q[[t]]$, so Laurent series double quotiented by Taylor series.

Example 19.7. When $G = \mathrm{PGL}_2$, what do these Hecke operators look like? In this case the double quotient was the space of radii in $\mathbb{H}$; the picture in the local-field setting turns out to be ambivalent about number fields versus function fields.

Fix $K = \mathbb{Q}_p$; the case $K = \mathbb{F}_p((t))$ will be surprisingly analogous. Then $G(K)/G(\mathcal{O}) = \mathrm{PGL}_2(\mathbb{Q}_p)/\mathrm{PGL}_2(\mathbb{Z}_p)$, which is in bijection with the space of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^2$ up to homothety (we saw a similar global example a few lectures back). To be precise, such a lattice is a $\mathbb{Z}_p$-submodule $\Lambda \subset \mathbb{Q}_p^2$, such that when we invert p, we get all of $\mathbb{Q}_p^2$: that is,

$$\Lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \mathbb{Q}_p^2. \tag{19.8}$$

Any lattice is taken to the standard lattice by an action of $\mathrm{GL}_2(\mathbb{Q}_p)$; "homothety" means identifying lattice up to rescaling.

What are some examples of lattices? One is $\Lambda_0 := \mathbb{Z}_p^2 \subset \mathbb{Q}_p^2$, spanned by the standard basis vectors $\{e_1, e_2\}$. We could also take $p\Lambda_0 := \mathrm{span}\{pe_1, pe_2\}$. Or we could have taken the span of $\{pe_1, e_2\}$.

We will draw a graph whose vertices are lattices in $\mathbb{Q}_p^2$ up to homothety (points of the coset space), and connect Λ and Λ' by an edge if, up to homothety, there are inclusions

$$\Lambda \subset \Lambda' \subset p\Lambda. \tag{19.9}$$

This is an equivalence relation — to get symmetry, for example, rescale Λ' to $p^{-1}\Lambda'$.

The resulting graph is known as the *building of* PGL_2 *over* $\mathbb{Q}_p$; it is in fact a tree.[23] The lattices adjacent to Λ_0 are equivalent to lines in $\Lambda_0/p\Lambda_0 \cong \mathbb{F}_p^2$, or equivalently to points in $\mathbb{P}^1(\mathbb{F}_p)$. So Λ_0 has $p+1$ neighbors. You can repeat this analysis for any Λ, and conclude that this graph is an infinite 3-regular tree. See Figure 1 for a picture.

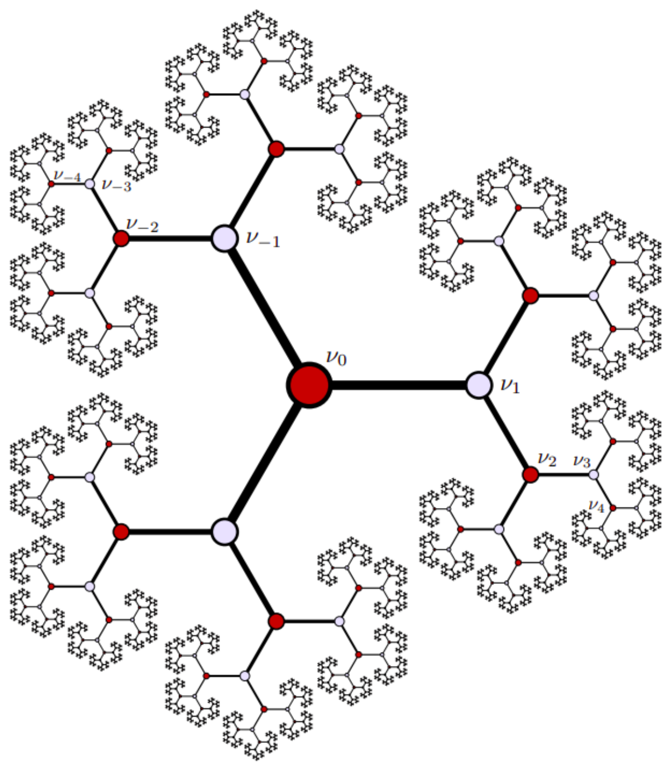


FIGURE 1. The Bruhat-Tits tree for SL_2. Source: `https://ncatlab.org/nlab/files/CasselmanOnBruhatTitsTree2014.pdf` (TODO: add citation).

This tree looks a lot like a p-adic analogue of the unit disc, and in fact this is our analogue of hyperbolic space in the number field setting. The vertices $G(K)/G(\mathcal{O})$ are the space called the *affine Grassmannian* of G, and the double coset space $G(\mathcal{O})\backslash G(K)/G(\mathcal{O})$ is the space of $G(\mathcal{O})$-orbits on the building. In our case, $G(\mathcal{O}) = \mathrm{PGL}_2(\mathbb{Z}_p)$. The orbits are analogous to the case of $\mathbb{H}$: lattice are identified in the double quotient iff they have the same distance from Λ_0, so the double coset space is the space of distances from Λ_0 in the tree metric.

And the differential operator that generates the algebra of equivariant differential operators is the "tree Laplacian" T_{p^n} (a special case of the graph Laplacian in spectral graph theory): if f is a function on the set of vertices, $G(K)/G(\mathcal{O})$,

$$T_{p^n} f(v) = \sum_{d(w,v)=n} f(w). \tag{19.10}$$

The algebra of differential operators is $\mathbb{C}[T_p]$.

[23] "Building" is a p-adic analogue of the notion of a symmetric space.

Now globalize: we have one of these for *every* point on the curve, and they're not just commutative, they also commute with each other. This absolutely enormous commutative algebra action is part of the reason the Langlands program has so much punch. ◂

The fact that local Hecke algebras at different points commute is not the most surprising thing; what's really a miracle is that each one individually is commutative. Once we have that, we should take its spectrum and see what we can learn. We will spend the next few weeks on these two ideas.

Remark 19.11. How does Theorem 19.3 relate to the usual Harish-Chandra isomorphism? Let G be a real reductive group; then Theorem 19.3 tells us that $\mathrm{Diff}_{G\times G}(G)$ is a polynomial ring. If we only cared about left-invariant differential operators, we'd obtain the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$; the right G-action falls down to a G-action on $\mathcal{U}(\mathfrak{g})$, so

$$\mathrm{Diff}_{G\times G}(G) = (\mathcal{U}(\mathfrak{g}))^G, \tag{19.12}$$

which is in fact the center of $\mathcal{U}(\mathfrak{g})$. The usual Harish-Chandra isomorphism identifies

$$Z(\mathcal{U}(\mathfrak{g})) \cong \mathbb{C}[\mathfrak{h}^*]^W, \tag{19.13}$$

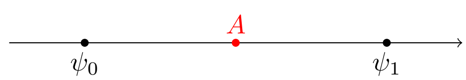
where $\mathfrak{h}$ is the Cartan subalgebra and W is the Weyl group. The Weyl group is always a reflection group, and there is a theorem of Chevalley that invariants under a reflection group are always a polynomial algebra. In particular, we get an algebra $\mathbb{C}[p_1, \dots, p_\ell]$, where $\ell \coloneqq \dim \mathfrak{h}$, which is called the *rank* of G.

Knop [Kno94] upgraded the Harish-Chndra isomorphism to the case of a *spherical variety* X acted on by G. In this case, the algebra of G-equivariant $\mathcal{D}$-modules on X is identified with $\mathbb{C}[\mathfrak{a}_X]^{W_X}$. ◂

Hecke algebras do not always commute in all situations — something nice is happening here. We will use field theory to provide a conceptual reason that these algebras should be commutative. Objects in a field theory often come with large and interesting algebras of operators.

Let's talk about one-dimensional quantum field theory, also known as quantum mechanics. In this case there is a Hilbert space $\mathcal{H}$ of states and an associative algebra A of observables, all the operators on $\mathcal{H}$. We envision operators acting at a specific time.

(19.14)



In this picture, we think of the operator A acting on ψ_0 to produce the different state ψ_1. Time flows to the right.

In Atiyah-Segal style field theory, this is encoded in bordism. If V is the state space of a topological field theory on a manifold M and V' is the linear dual of V, the cylinder $M \times [0,1]$, interpreted as a bordism $M \amalg M \to \varnothing$, gives a map

$$e\colon V \otimes V' \longrightarrow Z(\varnothing) = \mathbb{C}, \tag{19.15}$$

which we call *evaluation*. Likewise, the cylinder is a bordism $\varnothing toM \amalg M'$, giving us a *coevaluation* map $c\colon \mathbb{C} \to V \otimes V'$. These two maps satisfy a relation called the *mark of Zorro* (TODO: figure).

Abstractly, an object in a symmetric monoidal category equipped with evaluation and coevaluation maps, and satisfying the mark of Zorro relation, is called *dualizable*; we've shown that V is dualizable and V' is its dual. This implies that V and V' are finite-dimensional vector spaces; then evaluation is the usual evaluation map $V \otimes V^* \to \mathbb{C}$ given by $v, \ell \mapsto \ell(v)$, and coevaluation is the map

$$c\colon 1 \longmapsto \sum_j e_j \otimes e^j, \tag{19.16}$$

where $\{e_j\}$ is a basis of V and $\{e^j\}$ is the corresponding dual basis. The coevaluation map turns out to not depend on choice of basis; it is the part that makes the requirement for finite-dimensionality really clear.

TODO: operators come out of bordism (picture of the boundary of the pair-of-chaps as a one-dimensional bordism...). I missed this, sorry.

Now let's talk about interfaces. If V and W are finite-dimensional vector spaces, we can form one-dimensional TFTs Z_V and Z_W, whose state spaces (values on a point) are V, resp. W. (TODO: I see the notation $Z(\rightarrow) = V$, $Z(\rightarrow) = W$. May be a source of confusion.) TFTs are functors, so the naïve notion of morphism between them is a natural transformation, but these aren't so helpful. Instead, we will find it useful to consider morphisms akin to the Morita category, where a morphism between algebras A and B is a (B, A)-bimodule.

We want to define a "bipartite field theory," defined on manifolds which have a (possibly empty) separating codimension-one submanifold, and such that one side is colored red and the other is colored blue. We think physically of putting the theory Z_V on the red component and Z_W on the blue component, and that on the codimension-one submanifold between them is some data of an "interface" between them, and this data is a morphism of TFTs.

TODO: pictures.

One can prove that bipartite field theories ("interfaces from V to W") are equivalent to linear maps $V \to W$. So one-dimensional TFTs with interfaces are equivalent to vector spaces. For example, an interface between Z_V and itself is an element of $\mathrm{End}(V)$ — we think of putting a defect or singularity on the interval, and this defect is labeled by an operator in $\mathrm{End}(V)$.

From this point of view, a state, namely an element $\psi \in V$, is equivalent to a map $\mathbb{C} \to V$, which you can think of as an interface from the trivial TFT to Z_V. We can erase the red part of the bipartite manifold, and think of this data as allowing us to extract numbers out of compact 1-manifolds with boundary whose boundary points are labeled by ψ. This is a bit silly, but will get much less silly when we step up to dimension 2 and then to dimensiomn 4: Hecke arise very naturally from these kinds of pictures.

Lecture 20.

Topological field theory, interfaces, and boundary conditions: 4/8/21

TODO: I was a little late (four minutes).

We're going to talk about operators and defects from the perspective of topological field theory. We begin by defining the *(oriented) bordism 2-category* $\mathcal{B}ord_2^{\mathrm{or}}$, whose objects are closed, oriented 0-manifolds with two-dimensional collar neighborhoods; whose 1-manifolds are bordisms (compact, oriented, with one-dimensional collars); and whose 2-morphisms are compact, oriented 2-manifolds with corners which are bordisms between these 1-morphisms. The symmetric monoidal structure is given by disjoint union.

Bordisms come in families, or alternatively have classifying spaces. For example, if Σ_g is a closed, connected, oriented surface of genus g, which we can think of as a bordism from the empty 1-manifold (itself a bordism from the empty 0-manifold to itself) to itself, families of surfaces diffeomorphic to Σ_g are classified by homotopy classes of maps to the classifying space $B\,\mathrm{Diff}(\Sigma_g)$, which is homeomorphic to the moduli space $\mathcal{M}_g$ of genus-g curves.

A two-dimensional topological field theory is then a symmetric monoidal functor

$$Z\colon \mathcal{B}ord_2^{\mathrm{or}} \longrightarrow (\mathcal{C}, \otimes, \mathbf{1}) \tag{20.1}$$

to some symmetric monoidal 2-category. We would like this target 2-category to be $\mathbb{C}$-linear (roughly, hom-categories are $\mathbb{C}$-linear categories), and we would like this to extend the normal notion of TFT, so $\Omega\mathcal{C} \simeq \mathcal{V}ect$. Here $\Omega(-) := \mathrm{Hom}_{(-)}(\mathbf{1}, \mathbf{1})$. The monoidal unit in $\mathcal{B}ord_2^{\mathrm{or}}$ is the empty 0-manifold, so $\Omega\mathcal{B}ord_2^{\mathrm{or}}$ is the usual (nonextended) bordism category of oriented 2-manifolds, and an extended TFT restricts to a nonextended TFT.

A "universal" choice of target $\mathcal{C}$ is the 2-category $\mathcal{C}at_{\mathbb{C}}$ of $\mathbb{C}$-linear categories: if $\mathcal{C}$ is some other category satisfying our desiderata, then (TODO: how?) we can extract a $\mathbb{C}$-linear category from $Z(\mathrm{pt}_+)$, so the map $\mathcal{B}ord_2^{\mathrm{or}} \to \mathcal{C}$ then extends to $\mathcal{C} \to \mathcal{C}at_{\mathbb{C}}$. In practice, $\mathcal{C}$ is often equivalent, or nearly so, to $\mathcal{C}at_{\mathbb{C}}$.

Example 20.2. Another thing you could use as a target is the *Morita 2-category* $\mathcal{A}lg_{\mathbb{C}}$ whose objects are associative algebras over $\mathbb{C}$; whose morphisms $A \to B$ are (B, A)-bimodules; and whose 2-morphisms are bimodule homomorphisms.

There is a symmetric monoidal functor $\mathcal{A}lg_{\mathbb{C}} \to \mathcal{C}at_{\mathbb{C}}$ sending an algebra to its category of representations, and this embeds $\mathcal{A}lg_{\mathbb{C}}$ as a full subcategory of $\mathcal{C}at_{\mathbb{C}}$, and so these two choices really aren't so different. ◀

Let's interpret this 2-category stuff in TFT language. Last time we defined *interfaces* between TFTs; one defines a bordism category of *bipartite 2-manifolds* (surfaces decomposed across a codimension-one locus into two dfferent submanifolds, colored red and blue). Given two TFTs Z and W, we can define a functor from part of this bipartite bordism category into $\mathcal{C}at_{\mathbb{C}}$ by assigning Z to purely red surfaces and W to purely blue ones, but we need more data to define on surfaces with both parts. A theorem called the *cobordism* *hypothesis with singularities*, proven by Lurie, shows that this extra data is precisely an interface, a map $Z(\mathrm{pt}) \to W(\mathrm{pt})$ in $\mathcal{C}$.[24]

This is a longwinded way of motivating the following idea: if we let W be the trivial theory (constant functor valued in the unit), interfaces from W to Z are the same thing as objects of $\mathrm{Hom}(\mathbf{1}, Z(\mathrm{pt}_+))$, which are the same thing as objects of $Z(\mathrm{pt}_+)$. In TFT language, this defines a "boundary theory" for Z: we can extend Z to the bordism category of 2-manifolds with marked boundary components. Physically, one can think of field theories that live on the boundary of the "bulk theory," as some sort of coupled system. And these are objects of $Z(\mathrm{pt}_+)$.

This stuff is a bit confusing — boundary conditions in 2d TFT will naturally form a category, even if we haven't fixed $\mathcal{C} = \mathcal{C}at_{\mathbb{C}}$, and this could be thought of as an "underlying category" operation $\mathcal{C} \to \mathcal{C}at_{\mathbb{C}}$. But sometimes this map isn't very interesting or is trivial. In practice, for example choices of $\mathcal{C}$ we care about, this ends up being good, though.

One can also generalize to *codimension-k defect theories*, where we generalize Z to n-manifolds with an embedded, marked submanifold of codimension k. When $k = 1$, this isn't exactly the same as a boundary condition or an interface, because not all codimension-1 submanifolds are separating; we have more flexibility because we can change the TFT on the two sides of a separating submanifold. A codimension-1 defect is given by the data of an endomorphism of Z (an interface from Z to itself is a map $Z \to Z$). The difference here is that S^0 is disconnected!

Remark 20.3. Bass-Sullivan theory, in homotopy theory, includes a theorem that all generalized cohomology theories can be realized as bordism theories of manifolds with certain kinds of singularities. (Some don't need singularities, e.g. the sphere spectrum, representing stable cohomotopy, is obtained from framed bordism.)

The cobordism hypothesis with singularities is sort of a categorified version of this: we allow certain kinds of "singularities" (defects), and obtain all (small enough, meaning fully dualizable) objects in an (∞, n)-category, morphisms between them from interfaces, and so on. ◂

Let's see how boundary conditions in a 2d TFT Z form a category. By abstract nonsense we know this is true, but we will see it geometrically: there can be interfaces between interfaces! For example, you can choose two different boundary conditions B and R for Z; an interface from B to R allows us to define the TFT on pictures such as Figure 2.

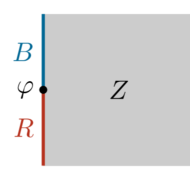


Figure 2. A bordism that can be evalued given an interface φ between the boundary conditions B and R.

That figure isn't closed, but we can close it off and obtain a bordism with corners between two 1-manifolds with marked boundaries, which are assigned vector spaces. So the collection of interfaces from B to R forms a vector space, $Z([0, 1])$, and we will define $\mathrm{Hom}(B, R)$ to be this vector space.

These interfaces compose, thanks to the pair-of-chaps bordism (TODO) defining a linear map from $\mathrm{Hom}(B, R) \otimes \mathrm{Hom}(R, G) \to \mathrm{Hom}(B, G)$, where G is a third interface.

The same idea works in any dimension n: we can build an $(n - 1)$-category of boundary conditions. This is where higher category theory appears physically: one studies defects between defects between defects.

Example 20.4 (Finite gauge theory)**.** This theory has many names, including *(untwisted) Dijkgraaf-Witten theory* and *finite-group Yang-Mills theory.*

Pick a finite group G; we will define a 2d TFT $Z_G \colon \mathcal{B}ord_2^{\mathrm{or}} \to \mathcal{C}at_{\mathbb{C}}$ by linearizing the space of G-local systems on manifolds. This is in a sense a Lagrangian field theory, in that we will assign spaces (well, groupoids) of fields to manifolds, and then linearize by takin sheaves, and obtain categories. This was how we

[24]Here it's important, at least conceptually, to draw all objects and morphisms in this bipartite category as with their collared neighborhoods. Therefore, for example, there's a point with collar red to the left and blue to the right, which is how we get the morphism $Z(\mathrm{pt}_+) \to W(\mathrm{pt}_+)$ below.

motivated assigning categories in codimension 2: we'd want to obtain sheaves on the space of fields, rather than just some abstract functor, and here we really do get sheaves on the space (groupoid) of fields.

Let's start with a closed manifold Σ. Then $Z_G(\Sigma)$ is the cardinality of $\mathrm{Loc}_G(\Sigma)$. The correct way to make this count is to weight by automorphisms: $\mathrm{Loc}_G(\Sigma)$ is a groupoid with finite automorphism groups, so we can define the sum

$$Z_G(\Sigma) = \sum_{[P]\in\pi_0(\mathrm{Loc}_G(\Sigma))} \frac{1}{\#\operatorname{Aut}(P)}. \tag{20.5}$$

We won't be so interested in these numbers, but rather in what goes on in lower codimension. For N a closed 1-manifold, we attach the vetor space of functions on $\mathrm{Loc}_G(N)$:

$$Z_G(N) \coloneqq \mathbb{C}[\mathrm{Loc}_G(N)], \tag{20.6}$$

For example, local systems on S^1 are identifed with the groupoid G/G (here G acts on itself by conjugaton), and functions on G/G are identified with class functions $\mathbb{C}[G]^G$.

For a 0-manifold P, we attach the category of sheaves on $\mathrm{Loc}_G(P)$.

What about on bordisms? A bordism M from N_1 to N_2 gives us a diagram

$$\begin{tikzcd}[column sep=tiny] & \mathrm{Loc}_G(M) \arrow[dl] \arrow[dr] & \\ \mathrm{Loc}_G(N_1), & & \mathrm{Loc}_G(N_2) \end{tikzcd} \tag{20.7}$$

and since G is finite, we can do a push-pull operation to functions or sheaves and obtain the map on bordisms.

From this perspective, what are representations of G? $Z_G(\mathrm{pt}) = \mathcal{V}ect(\mathrm{pt}/G) = \mathcal{R}ep_G$, so representations are boundary conditions. Explicitly, given V a G-representation, we get the data of an extension of the theory to manifolds with a marked red boundary component. For example, the interval $[0,1]$ with a single red boundary component is evaluated to the object $V \in Z_G(\mathrm{pt})$, and the cylinder $[0,1]\times S^1$ defines an element of $Z_G(S^1)$, and this turns out to be precisely the character of V.[25]

Physically, we think of $V \in \mathcal{R}ep_G$ as the state space for some (topological) quantum mechanics. The G-action is extra data, describing how to couple it to the bulk 2d theory Z_G. For example, the regular representation ($V = \mathbb{C}[G]$) is called the *Dirichlet boundary condition* for Z_G, and $V = \mathbb{C}$ is called the *Neumann boundary condition* for Z_G.

Given two boundary conditions (G-representations) V and W, what interfaces can be put between them? We know this is $\operatorname{Hom}_{\mathcal{R}ep_G}(V, W)$. Composition is realized by collision of defects (put them close together, and then think of them as one defect).

We can also think about this in terms of local systems: if G acts on a space X, it also acts on $\mathbb{C}[X]$. The space of fields on a 2-manifold Σ with red boundary labeled by X is the space of G-local systems together with a section of the associated X-bundle on the boundary. Said a little differently, in the bulk we have a map to the groupoid pt/G, and on the boundary we lift that map across $X/G \to \mathrm{pt}/G$. Thus the Dirichlet boundary condition $X = G$ gives the space of fields as local systems with a trivialization on the boundary. ◂

Next time we'll discuss this example further, e.g. the boundary condition of Z_G given by G/K for a subgroup K of G. The double coset space $K\backslash G/K$ has a natural interpretation in this framework via interfaces from $\mathbb{C}[G/K]$ to itself: the fields on the corresponding bordism are G-local systems with a lift to K-local systems on the boundary, and this means mapping to $K\backslash G/K$.

Lecture 21.

Commutative algebras of local operators: 4/13/21

We'd like to interpret the Hecke algebra story we found in terms of boundary conditions. Let's think of a Lagrangian field theory which is a theory of maps $\Sigma \to Y$, where Σ is spacetime and Y is called the *target*. In the language of quantum mechanics, we're linearizing the space of maps $\Sigma \to Y$.

From this perspective, a natural class of boundary conditions pops out: allow Σ to have boundary, and fix a submanifold $Z \subset Y$. Then, we restrict to maps $\phi\colon \Sigma \to Y$ such that $\phi(\partial\Sigma) \subset Z$. It's important to distinguish between a "boundary condition" and a "boundary theory," which is data: for example, there's no

[25]If you're used to the idea in TFT that taking the product with S^1 is a kind of trace, then this makes a lot of sense.

need for Z to be a submanifold of Y; instead, we could take a map $Z \to Y$ and ask for a lift of $\partial\Sigma \to Y$ to $\partial\Sigma \to Z$. This is a typical feature of boundary theories in sigma models (field theories where one of the fields is a map to a target space).

Let's apply this to our finite-group Yang-Mills example. In this case, the target is $Y \coloneqq \mathrm{pt}/G$ (as a groupoid or stack): a map $\varphi\colon \Sigma \to \mathrm{pt}/G$ is a principal G-bundle on Σ, or (since G is finite) a G-local system on Σ. A map $Z \to Y$ is equivalent to a space with a G-action (either: the total space of the principal bundle $X \to Z$ or the pullback of $Z \to \mathrm{pt}/G$ and $\mathrm{pt} \to \mathrm{pt}/G$), so given a G-space X, we obtain boundary data. The space of fields on Σ is the space of G-local systems on X together with a lift of the map $\partial\Sigma \to \mathrm{pt}/G$ across $X/G \to \mathrm{pt}/G$. More geometrically, given a principal G-bundle $P \to \partial\Sigma$, we can form the associated X-bundle

$$P \times_G X \coloneqq \{(p, x) \mid (p \cdot g, x) \sim (p, g \cdot x)\} \longrightarrow \partial\Sigma. \tag{21.1}$$

Then, a field is a principal G-bundle $P \to \Sigma$ together with a section of the associated X-bundle on $\partial\Sigma$. Equivalently, this is a G-equivariant map from the total space of $P|_{\partial\Sigma}$ to X. In physics words, one says that we've coupled the sigma-model to X on the boundary with a bulk G-gauge theory.

For example, if $X = G/H$, a section of the associated X-bundle is a reduction of structure group to a principal H-bundle. When $H = \{1\}$, this is a trivialization of the principal bundle; this is the *Dirichlet boundary condition*, akin to asking a function to be harmonic on the boundary.[26]

In our 2d TFT Z_G, we saw that the category of boundary conditions is $\mathcal{R}ep_G$. The Dirichlet boundary condition corresponds to the regular representation, and the Neumann boundary condition corresponds to the other canonical object, the trivial representation. We can also consider interfaces between boundary theories, which are given by the data of G-equivariant maps between representations. In the sigma-model example, the boundary data X_1/G and $X_2 \to G$ with their maps to pt/G have a canonical interface, the fiber product $S \coloneqq X_1/G \times_{\mathrm{pt}/G} X_2/G$. On a surface with boundary, we have a lift to X_1/G on one part of the boundary and a lift to X_2/G on another part of the boundary, then on the intersection we want both, hence map to the fiber product.

For example, say $X_1 = G/H$ and $X_2 = G/K$, so $X_1/G \cong \mathrm{pt}/H$ and $X_2/G \cong \mathrm{pt}/K$. The pullback of $\mathrm{pt}/H \to \mathrm{pt}/G$ and $\mathrm{pt}/K \to \mathrm{pt}/G$ is $H\backslash G/K$, so geometric or Lagrangian interfaces between these two boundary conditions are given by a lift to the double coset space. Self-interfaces of pt/K are given by lifts to $K\backslash G/K$. Our fields are therefore principal G-bundles with a reduction to a principal H-bundle on one part of the boundary and principal K-bundles on the other, and a lift to the double coset space on the intersection, meaning we have data of both an H-bundle and a K-bundle, together with an isomorphism of the induced G-bundles.

The Hecke algebra structure happens when we compose interfaces. Suppose I_1 is an interface between X_1 and X_2 and I_2 is an interface between X_2 and X_3. Physically, we shrink the region we use for X_2, and then from far enough away it looks like a single defect; this sort of composition is called *operator product expansion*. Mathematically, composition is the map attached to the pair-of-chaps bordism (TODO: picture). When you write down what this is on vector spaces, you get the map

$$\mathbb{C}[K\backslash G/K] \otimes \mathbb{C}[K\backslash G/K] \longrightarrow \mathbb{C}[K\backslash G/K] \tag{21.2}$$

which is exactly multiplication for the Hecke algebra.

For example, if $K = \{1\}$, $K\backslash G/K = G$, and the Hecke algebra is the group algebra. That is, the algebra of endomorphisms of the Dirichlet boundary condition is $\mathbb{C}[G]$. This is a lot of structure for the representation theory of finite groups, but will be very useful in higher dimensions.

Remark 21.3. There is a version of this where G is any compact Lie group, but everything is derived: our space of states isn't functions on pt/G, but instead we get $H^*(\mathrm{pt}/G) = H^*_G(\mathrm{pt}) = \mathbb{C}[\mathfrak{h}]^W$, where $\mathfrak{h}$ is the Cartan subalgebra of the Lie algebra of G and W is the Weyl group. A Borel subgroup gives you an interesting boundary condition. ◀

However, these Hecke algebras, arising as algebras of endomorphisms (as representations) of things like $\mathbb{C}[G/K]$, are not commutative. This is a big obstacle for doing spectral theory. But sometimes Hecke algebras are commutative, e.g. $\mathcal{H}_{G\times G,G} = \mathrm{End}_{G\times G}\,\mathbb{C}[G]$, which is something you can do tautologically for any group. This is always commutative.

[26]There is also a Neumann boundary condition, where $X = G/G$.

Let's dig into this example a little more. A principal $(G \times G)$-bundle is data of two principal G-bundles, and a reduction to the diagonal subgroup is an isomorphism of these two bundles. So we're looking at data of two principal bundles on, say, an interval, with identifications of these two bundles away from the boundary points. It may be worth thinking of this as a principal G-bundle on the "ravioli," the non-Hausdorff space $I \amalg_{I \setminus \mathrm{pt}} I$. Equivalently, this is a G-local system on the circle: we glue two G-bundles on two charts for S^1, which means we need identifications of these bundles on the overlaps. That is, this Hecke algebra has something to say about the G-theory on the circle, not just the $G \times G$ theory. Said yet another way, we're doing Yang-Mills theory for $G \times G$, and we're looking at self-interfaces of the diagonal boundary theory. A boundary condition for $G \times G$ Yang-Mills is equivalent to a self-interface of the G-theory. (This last one is more general: a boundary for the $G \times H$ theory is equivalent to an interface from the G-theory to the H-theory.) A $\mathbb{C}[G \times G]$-module is equivalent to a $\mathbb{C}[G] \otimes \mathbb{C}[G]$-module is equivalent to a $\mathbb{C}[G]^{\mathrm{op}} \otimes \mathbb{C}[G]$-module is equivalent to a $\mathbb{C}[G]$-*bi*module.

This is a general aspect of field theory: once you understand boundary conditions, you understand interfaces, and vice versa.

So we return to an important point: why is the Hecke algebra $\mathcal{H}_{G \times G, G} = \mathbb{C}[G/G]$ commutative? It has no reason to from the Hecke perspective. But from the field theory perspective, there *is* a reason: $\mathcal{H}_{G \times G, G}$ is the algebra of trivial self-interfaces of Z_G. Mathematically, this is the algebra of endomorphisms of the identity functor for $\mathcal{R}ep_G$. For any category $\mathcal{C}$, $\mathrm{End}(\mathrm{id}_{\mathcal{C}})$ is called the *Bernstein center* of $\mathcal{C}$ and denoted $Z(\mathcal{C})$.

This is where commutativity comes from. There are two different directions we can compose, and we can use that to rotate from $A \circ B$ to $B \circ A$; and this is not present for other Hecke algebras. This is closely related to the proof that homotopy groups in degrees 2 and above are commutative; these are sometimes called *Eckmann-Hilton arguments*: the Bernstein center has two compatible monoid structures: composition and tensor product, and they commute. It's a standard and not too hard exercise to show that a set with two compatible monoid structures is a commutative monoid (the two multiplications coincide and commute). Another related argument shows that if G is a topological group, $\pi_1(G)$ is abelian: we have two multiplications, in G and composing loops, and these commute.

Suppose A is an associative algebra; then the Bernstein center $Z(\mathcal{M}od_A) = \mathrm{End}(A)$ (regarding A as an (A, A)-bimodule), because the identity functor on $\mathcal{M}od_A$ is equivalent to tensoring with the (A, A)-bimodule A. The endomorphisms of A, as an (A, A)-bimodule, are things which commute with both left and right multiplication, and this is the center of A. This is why the Bernstein center is called the center: it is a generalization of the center of an algebra.

So for any category $\mathcal{C}$, we have a commutative algebra $Z(\mathcal{C})$, so we should spectrally decompose it; let $\mathcal{M}_{\mathcal{C}} := \mathrm{Spec}(Z\mathcal{C})$; what can we learn from $\mathcal{M}_{\mathcal{C}}$ from this space? For example, for $\mathcal{C} = \mathcal{M}od_A$, this is $\mathrm{Spec}(Z(A))$. This is not always helpful: sometimes $Z(A) = 0$ (e.g. for the Weyl algebra of differential operators).

In general, higher-dimensional QFT will have intrinsically commutative objects in a similar way, arising from local operators and defects. In QFT, a local operator is some sort of measurement or observable localized near a point in spacetime M. One can consider a sphere of radius ε around that point and remove the ball it bounds from spacetime, so we get a bordism from S_ε^{n-1} to the empty set. So we would get functionals on the state space of S_ε^{n-1}. Then, we shrink $\varepsilon \to 0$; this allows fields to be singular at the point where we measure, since ε never equals 0. The expectation value of this operator $\mathcal{O}$ is a path integral over the space of fields:

$$\langle \mathcal{O}(x) \rangle = \int \mathcal{O}_x(\varphi) e^{-S(\varphi)}\, \mathcal{D}\varphi. \tag{21.4}$$

We integrate (heuristically) over the space of fields on $M \setminus x$. So in physics a local operator is something which gives us expectation values as wel as correlation functions (where we have multiple local operators at different points and insert them all into the path integral).

In topological field theory (and more generally, conformal field theory), we don't have to worry about the size ε of the sphere, so don't have to worry about the limit. There's just a single vector space associated to a sphere, and this is the space of local operators. This is called a *state-operator correspondence*. The expectation value of an operator is determined by excising a small sphere, then using that the resulting manifold is a bordism from S^{n-1} to $\varnothing$, so defines a function from local operators to $\mathbb{C}$. This is the expectation value map.

Explicitly, if we do this for 2d topological Yang-Mills theory (so G is finite), the local operators are $Z_G(S^1) = \mathbb{C}[G/G]$. So the δ-functions on G/G are a basis; each one specifies the possible monodromy of a principal G-bundle around a loop. So we interpret the expectation value of the local operator as counting the number of G-local systems on $M \setminus x$ with this specified monodromy around x.

The key point is that $\mathcal{H}_{G\times G,G}$ is the algebra of local operators in Z_G, and that operator product expansion (two points colliding) is the pair-of-pants bordism or an annulus bordism. These all express the same idea: that there's an algebra structure on the space of local operators, and for topological reasons this algebra is commutative. And commutative means we can spectrally decompose. Commutativity of Hecke algebras in number theory is similar: from a Hecke-algebraic perspective, there's no reason for them to be commutative, but we will interpret them as algebras of local operators, explaining their commutativity.

Lecture 22.

Spectral decomposition in topological field theory: 4/15/21

Recall that we began discussing local operators, which we think of as measuring the fields at (or near) a point. The main structure on local operators is operator product expansion, which is a multiplication. Given two points x, y that are nearby, we evaluate the local operator in topological field theory by excising small spheres around these two points and getting a bordism from two spheres to the empty set. But we could also take a larger sphere around both x and y, giving us a single local operator that we can evaluate in the same way: we get a bordism $S^{n-1} \to \varnothing$, and our local operator defines a state in $Z(S^{n-1})$.

If our theory were not conformal, this is a little trickier: to define a single local operator, we shrink the sphere around the point, taking the limit as the radius goes to zero. To perform operator product expansion, we must also bring x and y closer to each other.

Back to the topological case — operator product expansion takes place entirely within the ball B bounded by the larger sphere. Remove the two smaller balls around x and y inside B; then B minus these spheres defines a bordism

$$S^{n-1} \amalg S^{n-1} \longrightarrow S^{n-1}, \tag{22.1a}$$

which the TFT sends to

$$*\colon Z(S^{n-1}) \otimes Z(S^{n-1}) \longrightarrow Z(S^{n-1}). \tag{22.1b}$$

This is the operator product expansion map, and hopefully it looks like a multiplication. More precisely, for any pair of two spheres inside a larger sphere, we obtain such a map, and the general structure is a family of maps on the configuration space of two small $(n-1)$-spheres inside a larger space. Because this theory is topological, this family must be locally constant.

Example 22.2. When $n = 1$, we are looking at pairs of intervals inside a larger line. This configuration space is disconnected: π_0 knows whether x is to the left of y or not. Therefore operator product expansion is not in general commutative for 1-dimensional TFT: $\mathcal{O}_y * \mathcal{O}'_x \neq \mathcal{O}_x * \mathcal{O}'_y$.

However, it is associative: the configuration space of three intervals in a larger interval with a specified order is connected. ◀

When $n > 1$, the configuration space of pairs of discs in a larger disc is connected, and therefore $Z(S^{n-1})$ is an associative, commutative algebra. There is enough room to move operators around — this is subtle in non-topological QFTs, but is particularly nice in the topological case. And in either case, this commutativity is completely absent in quantum mechanics ($n = 1$, for which OPE is only associative).

Remark 22.3. Next time, we will work in the derived setting, and then this story gets more complicated. We will see more information about the configuration space in the derived notion of commutativity that we obtain on $Z(S^{n-1})$. ◀

Given a commutative algebra, we want to spectrally decompose it, so let $\mathcal{M}^{\mathrm{aff}} := \mathrm{Spec}(Z(S^{n-1}))$, which is called the (TODO: I think) *moduli of vacua* of the theory. Given a local operator $\mathcal{O}$ and a "vacuum" $|0\rangle$ (meaning a point on $\mathcal{M}^{\mathrm{aff}}$), we can define a *vacuum expectation value* (VEV) of $\mathcal{O}$, denoted $\langle 0 \mid \mathcal{O} \mid 0\rangle$: an operator is a function on $\mathcal{M}^{\mathrm{aff}}$. In TFT, we will use this as a definition of a vacuum.

The slogan is that a topological field theory (or, its local operators) spectrally decomposes over $\mathcal{M}^{\mathrm{aff}}$: everything you can calculate in a TFT is linear over $Z(S^{n-1})$, so decomposes over its spectrum.

And we mean everything. For example, let M be an $(n-1)$-manifold. We will construct a $Z(S^{n-1})$-action on $Z(M)$. Specifically, consider the cylinder $M \times [0,1]$ and excise a small ball around $t = 1/2$ and some point in M. This modified manifold is a bordism $M \amalg S^{n-1} \to M$, hence defines a map $Z(S^{n-1}) \otimes Z(M) \to Z(M)$ which is our action map. It depends on the choice of point we chose, but again it's locally constant, which is good enough for us (again, things are more nuanced in the derived setting). And another bordism shows this is compatible with OPE, so we obtain a module structure. Passing to the spectrum, $Z(M)$ defines a quasicoherent sheaf on $\mathcal{M}^{\mathrm{aff}}$.

Another way to think about this, which is particularly helpful in the derived setting where "locally constant" matters more, is that the tensor prodiuct of $Z(S^{n-1})$ over all points in M acts on $Z(M)$. This is highly redundant, though, and factors through a quotient denoted

$$\int_M Z(S^{n-1}), \tag{22.4}$$

called the *factorization homology* of $Z(S^{n-1})$ on M. We will return to this later.

We will first look at this sheafification in 2d finite gauge theory; the prize in mind is in 4d, where this construction will be in some sense a source of the geometric Langlands correspondence.

Example 22.5 (2d TFT)**.** Let $\mathcal{C} := Z(\mathrm{pt}_+)$ for Z a fully extended TFT, so $\mathcal{C}$ is the category of boundary conditions of Z. Let $\mathrm{End}(\mathrm{id}_{\mathcal{C}})$ be the *Bernstein center*; for example, if $\mathcal{C} = \mathcal{R}ep_G$, this is the algebra of class functions $\mathbb{C}[G]^G$, whose commutative multiplication is not pointwise, but is instead convolution.

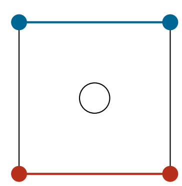
Then $\mathcal{M}^{\mathrm{aff}}$ for this TFT Z_G is $\operatorname{Spec} Z_G(S^1) = \operatorname{Spec} \mathbb{C}[G/G]$, which we defied to be the dual $\widehat{G}$ of G. When G is abelian, this realizes the Pontrjagin dual. But for arbitrary finite G, $\widehat{G}$ is in bijection with the set of isomorphism classes of irreducible representations of G.

This recovers an idea you may have seen before: that when studying representations of a finite group, one wants to study how they decompose into irreducibles (sheafify over $\widehat{G}$). The whole category splits, in fact:

$$\mathcal{R}ep_G \simeq \bigoplus_{\text{irreps } V_\lambda} \mathcal{V}ect \otimes V_\lambda. \tag{22.6}$$

That is, $\mathcal{R}ep_G \simeq \mathcal{V}ect_{\widehat{G}}$.

Now let's discuss the $Z(S^1)$-action on state spaces. Without more data this isn't super exciting, but by choosing two objects in $\mathcal{R}ep_G$ we can define state spaces for compact 1-manifolds whose boundaries are colored red and blue. For example, coloring $[0,1]$ with 0 red and 1 blue, the bordism



(22.7)

defines a $Z(S^1)$-action on $Z([0,1]) = \mathrm{Hom}(V, W)$. That is, the state spaces and morphisms in this TFT are enriched in $Z(S^1)$-modules. ◂

In some sense, the Bernstein center $\mathcal{Z}(\mathcal{C})$ is universal for this structure. $\mathcal{C}$ forms a sheaf of categories over $\mathcal{M} := \operatorname{Spec}(\mathcal{Z}(\mathcal{C}))$.[27] If R is a commutative ring, an R-linear structure on $\mathcal{C}$ is equivalent to data of a ring homomorphism $R \to \mathcal{Z}(\mathcal{C})$: R acts on the Hom-spaces in compatible ways. Thus the universal such R is $\mathcal{Z}(\mathcal{C})$.

If you're in a big enough setting ($\mathcal{C}$ has enough colimits), R-linearity is equivalent to the idea that $\mathcal{C}$ is a module category for $\mathcal{M}od_R$. We're now doing spectral decomposition of categories, albeit only over affine spaces.

There are three different ways to define the idea of a sheaf of categories over $\operatorname{Spec} R$.

(1) Sheafifying $\mathcal{C}$ over R is data of a map $R \to \mathcal{Z}(\mathcal{C})$.
(2) Sheafifying $\mathcal{C}$ over R means $\mathcal{C}$ is enriched in R-modules (equivalently, in quasicoherent sheaves on $\operatorname{Spec} R$).
(3) Sheafifiying $\mathcal{C}$ over R means an action of the symmetric monoidal category $(\mathcal{M}od_R, \otimes_R)$ on $\mathcal{C}$.

[27]Our notation Z for TFTs should not be confused with our notation $\mathcal{Z}$ for the center of an algebra, category, etc.

For the last perspective, we know that if M is a free R-module and $C \in \mathcal{C}$, $M \otimes C \cong C^{\oplus \operatorname{rank}(M)}$. In general, we take a free resolution of M, and therefore obtain a sequence of objects in $\mathcal{C}$. If $\mathcal{C}$ has enough colimits, we can take the colimit of this sequence and obtain something that we call $M \otimes C$. This action is a categorical analogue of linearity.

We can also localize: if $S \subset R$ is a multiplicative subset, we want to be able to localize $\mathcal{C}$ over $\operatorname{Spec} R[S^{-1}]$, much like we can localize a quasicoherent sheaf over $\operatorname{Spec} R[S^{-1}]$. We define the sheaf

$$\underline{\mathcal{C}}(\operatorname{Spec} R[S^{-1}]) = \mathcal{C} \otimes_{\mathcal{M}od_R} \mathcal{M}od_{R[S^{-1}]}. \tag{22.8}$$

Explicitly, this takes $\mathcal{C}$ and inverts all morphisms coming from $S \subset R$. We will not worry right now about the precise definition of the tensor product of categories. But it may help to see that the Hom-sets in $\mathcal{C}$ are already sheaves on R, and localizing $\mathcal{C}$ reduces to localizing sheaves/modules for Hom-sets.

Now say A is an associative algebra, such as $\mathbb{C}[G]$. Then $\mathcal{M}od_A$ sheafifies over $\operatorname{Spec} \mathcal{Z}(A)$, and in particular $\mathcal{R}ep_G$ sheafifies over $\widehat{G}$. Therefore, in fact, $\mathcal{R}ep_G \simeq \mathcal{QC}(\widehat{G})$ — but this is too cool to be really true! This is a **noncanonical** equivalence.

In fact, you should already be suspicious: it's tautologically true that $\mathcal{R}ep_G$ sheafifies over $\widehat{G}$. For S a subset of $\widehat{G}$, the restriction of the sheaf $\mathcal{R}ep_G$ over S is the summand whose characters all lie in S. Letting S be a point we see what the problem is: to get an identification $\mathcal{R}ep_G \simeq \mathcal{QC}(\widehat{G})$, we need to pick isomorphism classes of G-representations.

Said differently, Wedderburn's theorem identifies

$$\mathbb{C}[G] \simeq \bigoplus_{\text{irreps } V_\lambda} \operatorname{End}(V_\lambda), \tag{22.9}$$

a direct sum of matrix algebras over division rings (all algebras of rings). And we've decomposed $\mathbb{C}[G]$ as a direct sum of associative algebras over $\widehat{G}$. But Wedderburn's theorem is abstract, and does not provide an isomorphism.

Lest you think this is pointless category-wrangling, the analogous statement for representations over $\mathbb{R}$ is *false*! $\mathcal{R}ep_G(\mathbb{R})$ is a valid category, and $\widehat{G}$ is defined over $\mathbb{R}$, so it seems like we can be optimistic, but there are three kinds of reality conditions for irreducible real-valued representations V: Schur's lemma tells us $\operatorname{End}(V)$ is a division algebra over a field extension of $\mathbb{R}$, so our options are $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$. The complex representations ($\operatorname{End}(V) \cong \mathbb{C}$) correspond to $\mathbb{C}$-points of $\widehat{G}$ which are not $\mathbb{R}$-points, or said differently that characters are not real-valued. And even for the real points, we don't always get $\operatorname{End}(V) \cong \mathbb{R}$: if $\operatorname{End}(V) \cong \mathbb{H}$, we explicitly obtain something not isomorphic to $\mathcal{QC}(\widehat{G})$. The structure we have is that $\mathcal{R}ep_G(\mathbb{R})$ is an invertible $\mathcal{QC}(\widehat{G})$-module category, but over a field that isn't algebraically closed, this module is not always trivial. It is in a sense a "categorical line bundle" for $\mathcal{QC}(\widehat{G})$, which is known by the shorter name of a *gerbe*. Said differently, $\mathcal{R}ep_G(\mathbb{R})$ is an element of the *Brauer group* of $\mathcal{QC}(\widehat{G})$, much like $\mathbb{H}$ is a nonzero element of the Brauer group of $\mathbb{R}$.

This is closely related to anomalies in physics. $\mathcal{C}$ sheafifies over $\mathcal{M}$, but that doesn't mean it's the simplest thing which sheafifies over $\mathcal{M}$ (which is $\mathcal{QC}(\mathcal{M})$). It could be a gerbe, or even something more complicated. Any random associative algebra might not even have a nonzero center, in which case what we get is very far from $\mathcal{QC}(\mathcal{Z}(A))$.

Remark 22.10. Let Σ_g be a closed, connected, oriented surface of genus g. There is a formula due to Mednykh which computes $Z_G(\Sigma_g)$, the count of local systems on Σ_g weighted by automorphisms, in terms of representation theory of G:

$$Z(\Sigma_g) = \#\operatorname{Loc}_G(\Sigma_g) = \sum_{\text{irreps } V_\lambda} \left(\frac{|G|}{\dim(V_\lambda)}\right)^{2g-2}. \tag{22.11}$$

This is sheafifying the number $Z(\Sigma_g)$ over the space $\widehat{G}$. In general, spreading out quantities in TFT over the algebra of local operators will be even more powerful in higher dimensions. ◂

Lecture 23.

Defects and defect operators: 4/20/21

Let Z be an n-dimensional topological field theory. The operator-state correspondence shows that local operators are equivalent to states in $Z(S^{n-1})$, and under operator product expansion, this vector space is a commutative algebra provided $n \geq 2$. Moreover, for any closed $(n-1)$-manifold M, $Z(M)$ is naturally a $Z(S^{n-1})$-module. And for a closed $(n-2)$-manifold N, $Z(N)$ is not just a $\mathbb{C}$-linear category, but is naturally linear over $Z(S^{n-1})$: by drilling out a small ball, we get a way for Z its boundary, S^{n-1}, to act on the Hom space.

We then let $\mathcal{M}_Z^0$ (what we called $\mathcal{M}_Z^{\mathrm{aff}}$), the *(affine) moduli space of vacua*, be $\operatorname{Spec} Z(S^{n-1})$. What Z assigns to manifolds sheafifies over $\mathcal{M}_Z^0$. We would like to relate Z to a Lagrangian field theory of maps into $\mathcal{M}_Z^0$: for example, for 2d finite gauge theory with gauge group G, $\mathcal{M}_Z^0 = \widehat{G}$, allowing us to decompose Z into irreducibles.

In order to do this more abstractly, it is helpful to have what is in some sense the easiest theory to write down. Let R be a commutative ring, and let $X = \operatorname{Spec} R$. There is then a topological field theory of any dimension, called the *B-model*, based on this data, though it's truncated (meaning we don't assign numbers to top-dimensional manifolds, just invariants to objects in lower dimensions). The idea is that this is a Lagrangian field theory, and the fields are maps $M \to X$. This is a little weird — M is a smooth manifold and X is an affine variety, so we'll do something naïve and ask that these maps be locally constant. So these are equivalent to maps $\pi_0(M) \to X$, This is a toy model for more interesting versions of this theory that will come later. Said differently, $\operatorname{Map}(\pi_0(M), X) = \operatorname{Spec} R \otimes \pi_0(M)$. This doesn't depend on dimension or anything, which is either great (flexibility!) or boring.

Now we have to linearize, defining a truncated TFT B_X^n in a chosen dimension n. For a closed $(n-1)$-manifold M, we assign $\mathcal{O}(\operatorname{Map}(\pi_0(M), X)) = R \otimes \pi_0(M)$. For a closed $(n-2)$-manifold N, we assign $\mathcal{QC}(\operatorname{Map}(\pi_0(N), X)) = \mathcal{M}od_{R\otimes\pi_0(N)}$. You can define push-pull operations on bordisms.

Now let Z be any TFT in dimension 2 or above, and conider the B-model on its moduli space of vacua, $B^n_{\mathcal{M}_Z^0}$. We'd like to guess that Z is well-approximated by this B-model. This is very far from true in general, but it is true for 2d finite gauge theory: $Z_G(S^1) = \mathbb{C}[G/G] = \mathbb{C}[\widehat{G}]$, which is what the B-model attaches, for example, and $Z_G(\mathrm{pt}) = \mathcal{R}ep_G$ is equivalent to what the B-model attahces, which is $\mathcal{QC}(\widehat{G})$.

This is a very naïve construction, but done in more sophisticated ways, it leads to mirror symmetry! This is closest to Gross-Siebert's intrinsic mirror symmetry. One begins with an interesting field theory, the A-model, and construct a variety (mirror) as the spectrum of local operators, and then mirror symmetry identifies the A-model and the B-model. The details are nontrivial: one has to work derived, and sometimes modify the mirror, but it can be done. However, there's plenty to fix even already: last time we saw that this does not work for the 2d TFT defined by the category $\mathcal{R}ep_G(\mathbb{R})$.

Instead, what we get is that we have a $Z(S^{n-1})$-module structure on $Z(M)$ for any point in M, so we know where to drill out the ball. A path between points produces an isomorphism between their module structures, but the upshot is that this condition is only constant, so $Z(M)$ is naturally just a module for $\pi_0(M) \otimes Z(S^{n-1})$, i.e. over $\mathcal{O}(\operatorname{Map}(\pi_0(M), \mathcal{M}_Z^0))$. That is, $Z(M)$ is a sheaf on this mapping space, i.e. an object in what the B-model in one dimension higher assigns to M: $Z(M) \in B^{n+1}_{\mathcal{M}_Z^0}(M)$. Likewise, the invariant attached to an $(n-2)$-manifold is $Z(S^{n-1})$-linear category for any choice of a point in N, and in general this is only locally constant, so we again have to tensor with $\pi_0(N)$. In particular, we obtain an object of the 2-category $B_{\mathcal{M}_Z^0}(N)$, i.e. the category of sheaves of categories on $\operatorname{Map}(\pi_0(N), \mathcal{M}_Z^0)$, or a category linear over $\pi_0(N) \otimes Z(S^{n-1})$.

In some sense, this is the general state-observable correspondence: states are the output of Z, and observables are the output of the higher-dimensional B-model. Said succinctly, Z is canonically a boundary condition for $B^{n+1}_{\mathcal{M}_Z^0})$, and this structurally organizes all of the data given by local operators and their spectral information.

In dimension 4, we studied the "A-model" for $G = \mathrm{GL}_1$, topological Maxwell theory. We showed that $\mathcal{A}_{\mathrm{GL}_1}(Y)$, where Y is a 3-manfold, is the space of locally constant functions on the space of U_1-connections on Y mod gauge equivalence. We'd like to understand the local operators in this theory. To do so, let's recall that the space of fields factored as a product of $H^2(Y;\mathbb{Z})$ and a torus. For local operators, we look at $Y = S^3$, for which H^1 and H^2 both vanish — so there isn't anything interesting. If you're a little more careful, you actually have $B\mathrm{U}_1$, but there aren't any interesting locally constant functions $S^2 \to B\mathrm{U}_1$.

So it seems like the solution is to pass to the derived world, and there you get $H^*(B\mathrm{U}_1) = \mathbb{C}[u]$, which is not the most interesting algebra in the world. Unfortunate – but when we generalize from local operators to more interesting observables (defects), things will get more interesting.

This brings us to defects and defect operators, which we will approach from the perspective of the cobordism hypothesis. This is a theorem which relates notions in topological field theory to notions in category theory. You can use it to build TFTs out of category-theoretic data, or vice versa, use it to make precise the way that pictures are used to reason about higher category theories. The cobordism hypothesis encodes all of the different pictorial calculi used to reason about different kinds of categories, and explains why they should exist.

Let $\square$ be a *tangential structure*, meaning a kind of topological structure we can place on the tangent bundles of manifolds — no structure, an orientation, a spin structure, a framing, etc. You can precisely define this as a group homomorphism $H_n \to \mathrm{O}_n$ and a reduction of structure group of the frame bundle to a principal O_n-bundle. There is a notion of bordism of $\square$-structures, so we obtain a symmetric monoidal bordism n-category $\mathcal{B}ord_n^{\square}$. (On manifolds of dimension less than n, we ask for a $\square$-structure on an n-dimensioal collar neighborhood.) We have defined TFTs of $\square$-manifolds as symmetric monoidal functors

$$Z\colon \mathcal{B}ord_n^{\square} \longrightarrow \mathcal{C}, \tag{23.1}$$

where $\mathcal{C}$ is some other symmetric monoidal n-category.

The cobordism hypothesis says that such a TFT is completely determined by the object $Z(\mathrm{pt}) \in \mathcal{C}$ (where pt has its canonical $\square$-structure), subject to

(1) a finiteness condition (dualizability) and
(2) a self-duality structure of some sort, specified by $\square$.

The finiteness condition can be thought of as germination: grow your field theory upwards from what you hope to attach to a point, and you need a strong finiteness hypothesis to be able to go all the way up.

The cobordism hypothesis was originally conjectured by Baez-Dolan [BD95], then proven by Hopkins-Lurie in dimension 1 (TODO: 1 or 2?) and Lurie [Lur09] in all dimensions.[28]

Baez-Dolan [BD95] generalized this to another conjecture called the tangle hypothesis, and Lurie [Lur09, §4.3] generalized further and proved something called the *cobordism hypothesis with singularities*, where we describe field theories on manifolds which have prescribed singularities of various sorts. For example, we could fix a manifold M and allow cones on M; therefore our bordism category looks like $\square$-manifolds with a singularity whose link is M (TODO: I may have misunderstood this).

Then, the cobordism hypothesis with singularities says that a symmetric monoidal functor from this modified bordism category is equivalent data to a TFT Z as before, together with an object of $Z(M)$, or more precisely a map $1 \to Z(M)$ (which makes sense no matter the dimension of M). This is what the symmetric monoidal functor assigns to the cone of M, which is a bordism from pt to M. There are finiteness and self-duality conditions on this as well.

We are most interested in the cases where $M = S^k$. For example, for $k = n-1$, our manifolds with singularities are n-manifolds with marked points, and the cobordism hypothesis with singularities assigns to this data a TFT Z as usual together with an element of $Z(S^{n-1})$. That is, we've picked out a local operator we can place at the marked points. More generally, if we care about manifolds with an embedded $(k-1)$-dimensional submanifold, the link is S^{n-k}, so the cobordism hypothesis with singularities says that we can evaluate these kinds of manifolds given Z and an object in $Z(S^{n-k})$. These are what we call defects or operators. For example, *line defects* or *line operators* are labeled by objects of the category $Z(S^{n-2})$. That is, $Z(S^{n-2})$ is the category of line operators — by fiat, this is a category, but you can also see this topologically: the morphisms between two line operators are the interfaces between them (what we can place at the junction of a red line segment and a blue line segment).

[28] There are additional partial proofs by Schommer-Pries and Ayala-Francis.

In the same way, $Z(S^{n-3})$ is the 2-category of surface defects: there are surface defects and morphisms between them, which are placed on 1-dimensional junctions, and these can have morphisms between them, which are placed on 0-dimensional junctions between junctions. Wheels within wheels...

Now these embedded submanifolds can be knotted. For example, if Z is a 3d TFT and we choose a line operator, we can evaluate Z on a 3-manifold with an embedded knot, and obtain a number. This (and its generalizations to higher-dimensional defects) is the kind of thing the cobordism hypothesis with singularities allows us to do.

With more operators in our toolbox, let's see what additional structure we have. We used the algebra of local operators to define $\mathcal{M}_Z^0$ and spectrally decompose objects in the theory. For line operators, there is again a notion of operator product expansion, and it makes $Z(S^{n-2})$ into a monoidal category. In TFT, this arises from the bordism given by a ball with two smaller balls removed, which is a bordism

$$S^{n-2} \amalg S^{n-2} \longrightarrow S^{n-2}. \tag{23.2}$$

We think of this as happening at a single point in our line defect, as the link of the line at this point. Again, there are different choices of how to embed the smaller balls in the larger one, so we really get this data in a locally constant way over the configuration space. That is, $Z(S^{n-2})$ has many multiplications, labeled by the configuration space of pairs of discs inside a larger disc. Formally, this is the structure of an algebra over the little $(n-1)$-discs operad, or an E_{n-1}-structure, though that's not terribly helpful until we pass to the derived perspective.

For example, if $n = 2$, a line operator is a self-interface, and operator product expansion is associative but not commutative: (23.2) is a bordism of pairs of intervals in a bigger interval, and the configuration space for this data is not connected. Therefore the products $L_1 * L_2$ and $L_2 * L_1$ cannot be connected by a path in the configuration space, so are not equal in general, but $L_1 * (L_2 * L_3)$ and $(L_1 * L_2) * L_3$ can be connected, so this algebra is associative. In general we have an E_1-algebra (in categories).

If $n = 3$, multiplication is locally constant in the space of small discs in the larger disc. To be a little more precise, this means that TODO. The configuration space has nontrivial π_1, so let ℓ be a nontrivial loop. This defines a map $L_1 * L_2$ to $L_2 * L_1$, but we don't get much more information than that (e.g. no guarantee it squares to the identity). This gives $Z(S^{n-2})$ the structure of a *braided monoidal category*, or an E_2-algebra in categories.

For $n \geq 4$, the configuration space is simply connected, and therefore there is a natural isomorphism $L_1 * L_2 \to L_2 * L_1$ — this is an E_3-category, which is symmetric monoidal (in fact an E_∞-category). So in 4d TFT (and higher dimensions, which we aren't concerned with), we can define $\mathcal{M}_Z^1$ to be Spec of the symmetric monoidal category $Z(S^2)$ with operator product expansion. This is a refinement of $\mathcal{M}_Z^0$, which is its affinization. We have better geometry in higher dimensions, which leads to better spectral decompositions, and this extra structure is important for the TFT approach to the Langlands program. That is what we will do next time, e.g. seeing that there are lots of line operators in 4d Maxwell theory.

Lecture 24.

Defect operators and Maxwell theory: 4/22/21

TODO: I missed a lot in the notes today, sorry about that.

Today we will get to one of the main punchlines of the class. Recall that we've been discussing defect operators in TFT: $Z(S^{n-1})$ is the algebra of local operators, $Z(S^{n-2})$ is the monoidal category of line operators, $Z(S^{n-3})$ is the monoidal 2-category of surface operators, and so on. When n is large enough and the dimension of the defect is small enough, these monoidal structures are commutative. For example, when $n \geq 2$ the algebra of local operators is commutative. For line operators, there are three levels of commutativity: monoidal ($n = 2$), braided monoidal ($n = 3$), and symmetric monoidal ($n \geq 4$). This is related to Baez-Dolan's periodic table of higher categories. We are particularly interested in the case of line operators in $n = 4$, where we look at the symmetric monoidal category $Z(S^2)$.

There is a relationship between local and line operators. There is a trivial defect, which is the unit of the symmetric monoidal category of line operators. So we can put a local operator on a point inside a line, and put the trivial defect on the line. That is, local operators are self-interfaces of trivial line operators.

If you don't like that version of the picture, you can see it a different way: this is a picture of suspensions of spheres. For example, S^{n-2} is the link of a line, and cutting S^{n-1} into two hemispheres factors S^{n-1} as a bordism

$$\varnothing \longrightarrow S^{n-1} \longrightarrow \varnothing, \tag{24.1}$$

therefore putting an element of $Z(S^{n-1})$ on the equator defines TODO. Keeping track of framings or orientations is important, and that's something we're not worrying about right now; the way it affects things is to keep track of duals. (TODO: Missed quite a bit here. Sorry about that.)

Another way to think about this is the "ravioli" or "UFO" picture, which realizes the identification

$$D^{n-1} \amalg_{D^{n-1}\setminus \mathrm{pt}} D^{n-1} \cong S^{n-1}. \tag{24.2}$$

The key takeaway is that $Z(S^{n-1}) = \mathrm{End}(\mathbf{1}_{Z(S^{n-2})})$: local operators are endomorphisms of the trivial line operator. This is not specific to codimension 1 and 2 — you can run the same argument in higher codimension. But returning to local and line operators, recall that $\mathcal{M}_Z^0 \coloneqq \operatorname{Spec} Z(S^{n-1})$ and $\mathcal{M}_Z^1 \coloneqq \operatorname{Spec} Z(S^{n-2})$. So reformulating things slightly, the unit in $\mathcal{QC}(\mathcal{M}_Z^1, \otimes)$ is the structure sheaf of $\mathcal{M}_Z^1$. Said again, $Z(S^{n-1}) = \mathcal{O}(\mathcal{M}_Z^0) = \mathrm{End}_{\mathcal{QC}(\mathcal{M}_Z^1)}(\mathcal{O}) = \mathrm{Hom}(\mathcal{O},\mathcal{O}) = \Gamma(\mathcal{O}_{\mathcal{M}_Z^1})$.

That is, functions on $\mathcal{M}_Z^0$ are the same thing as global functions on $\mathcal{M}_Z^1$. That is, $\mathcal{M}_Z^0$ is the affinization of $\mathcal{M}_Z^1$! (Recall that the affinization of X is $\operatorname{Spec}(\mathcal{O}_X)$.) This is the algebro-geometric incarnation of the fact that $Z(S^{n-1})$ is endomorphisms of the unit in $Z(S^{n-2})$. And in topological field theory, endomorphisms of the unit correspond to suspension.

And again, this does not stop at codimension 2: if n is high enough, there is a space $\mathcal{M}_Z^2$ and a map to $\mathcal{M}_Z^1$. Since we care about dimension 4, codimension 2 is the sweet spot: $Z(S^1)$ is not commutative enough.

TODO: something about stacks. Maybe the moduli space of vacua is upgraded to a moduli stack.

Upshot: for a 4d TFT Z, we've constructed a moduli space of vacua; in algebro-geometric terms this should be called the "spectrum of Z," but spectrum means enough things in physics that it's more common to call it the moduli of vacua. There are a bunch of things we could use specifically, but we're going to use $\mathcal{M}_Z^1$.

Like last time, one of the big utilities of the spectrum of Z is to approximate Z by simple TFTs using algebraic geometry, specifically the "B-models" from last lecture. You can ask yourself, how close is Z to these B-models? These B-models are Lagrangian field theories, with fields maps to targets X, which is a physics interpretation of spectral decomposition.

Recall that given X a variety or stack and any dimension n, we built a truncated n-dimensional TFT (i.e. we assign everything up to dimension $n-1$). $B_X^n(M^{n-1}) = \mathcal{O}(\mathrm{Map}(M,X))$ — we just have to interpret what Map means. Last time we said this means locally constant maps, because it's tricky to refine this when M is a manifold and X is something algebro-geometric.

Varieties are certain functors from rings to sets, and stacks replace sets with groupoids. So given a manifold M and a stack X, we can interpret "locally constant" maps a little more richly: $\mathrm{Map}(\pi_{\le 1}M, X)$. Here $\pi_{\le 1}X$ is the fundamental groupoid of M. In some sense, M also defines a functor from rings to groupoids, but a more trivial one, sending every ring R to $\pi_{\le 1}M$.

Our favorite stack has been pt/G. For this choice of X, $\mathrm{Map}(M,X)$ is exactly the data of a G-local system on M! So these B-models can be more interesting than last lecture.

Now you can extend the B-model further, e.g. on an $(n-2)$-manifold you get $\mathcal{QC}(\mathrm{Map}(\pi_{\le 1}M, X))$, and on higher-codimension manifolds you get sheaves of (higher) categories, and so on.

In a sense, the best approximation to Z within algebraic geometry is $B^n_{\mathcal{M}_Z^1}$, and Z itself is linear over $\mathcal{M}_Z^1$. Last time, we mentioned that this can be used to make Z a boundary condition for $B^{n+1}_{\mathcal{M}_Z^1}$. If N is an $(n-2)$-manifold, $Z(N)$ is a module for $Z(S^{n-2})$ by drilling out a small ball around a point, so we get an action of $\mathcal{QC}(\mathrm{Map}(\pi_{\le 1}N, X))$ on $Z(N)$ for all points on N, and this action is locally constant. Likewise, we have a locally constant action of $Z(S^{n-1})$ on the state space of a closed $(n-1)$-manifold, and this is locally constant in M, so we really have an action of $Z(S^{n-1})^{\#\pi_0(M)}$, which means we do not need to choose a basepoint. That is, without making choices,

$$\mathrm{Map}(M, \mathcal{M}_Z^0) = \operatorname{Spec} R^{\otimes \#\pi_0(M)} = (\mathcal{M}_Z^0)^{\times \pi_0(M)}, \tag{24.3}$$

so it's really a sheaf over $\pi_0(M)$ many components of $\mathcal{M}_Z^0$. This is local operators; you can do the same thing with line operators. This is a little more interesting because the category number is higher. If you have a path from x to y, you get an isomorphism of these actions, and get an action of $\mathcal{AC}(\mathrm{Map}(N, \mathcal{M}_Z^1))$ on $Z(S^{n-2})$. Thinking further about these sorts of things leads to factorization homology. For example, if R is a commutative ring and M is a space,

$$\int_M R = R^{\otimes \# \pi_0(M)} = \mathcal{O}(\mathrm{Map}(M, \operatorname{Spec} R)). \tag{24.4}$$

This is "boring" because R is discrete. If we increase the category number, letting $\mathcal{C}$ be a monoidal category, we can get more interesting things:

$$\int_M \mathcal{C} = \mathcal{O}(\mathrm{Map}(M, \operatorname{Spec} \mathcal{C})). \tag{24.5}$$

At every point we have an action by line operators, and there is a natural way to assemble these into a coherent object.

Given $x \in N$, we have an evaluation map $\mathrm{Map}(N, \mathcal{M}) \to \mathcal{M}$, so obtain a pullbac map $ev_x^* \colon \mathcal{QC}(\mathcal{M}) \to \mathcal{QC}(\mathrm{Map}(N, \mathcal{M}))$. In particular, we can pull back the action of $\mathcal{QC}(M)$ to an action of $\mathcal{QC}(\mathrm{Map}(N, \mathcal{M}))$, and because these actions are locally constant, these fit together correctly. However, to do this we need to know that $\mathcal{C}$ is Tannakian: it's a category of sheaves on its spectrum. (TODO: did I hear this correctly? A little unsure.)

Let's return to topological Maxwell theory $\mathcal{A}_{\mathrm{GL}_1}$. Let M be a closed 3-manifold, so $\mathcal{A}_{\mathrm{GL}_1}(\mathcal{M})$ is the vector space of locally constant functions on the stack of connections. If Σ is a closed 2-manifold, we attach locally constat sheaves on $\mathrm{Pic}(\Sigma)$.

So at each $x \in \Sigma$, we get an action on $\mathcal{A}_{\mathrm{GL}_1}(\Sigma)$ of $\mathcal{A}_{\mathrm{GL}_1}(S^2) = \mathrm{Loc}(\mathrm{Pic}(\mathbb{P}^1))$. But line bundles on $\mathbb{P}^1$ is just $\mathbb{Z}$ up to isomorphism — there is some stacky structure that affects this at the derived level, and we are going to not worry about this right now. Moreover, the tensor product passes to addition. So we get $\mathrm{Loc}(\mathbb{Z})$, which is the category of $\mathbb{Z}$-graded vector spaces with tensor product. This is a version of Gauss' law; we have two small balls inside a larger ball, and their "fluxes" add up, which are the Chern classes of these line bundles. And Chern classes add.

We saw that $\mathcal{M}_{\mathcal{A}}^0$ is trivial, but $\mathcal{M}_{\mathcal{A}}^1$ is $\mathrm{pt}/\mathbb{G}_m$, the moduli stack of line bundles. In general, $\mathrm{Map}(N, \mathcal{M}_{\mathcal{A}}^1) = \mathrm{Loc}_1(N)$, so we find that $\mathcal{A}(\Sigma)$ is acted on by $\mathcal{QC}(\mathrm{Loc}_1(\Sigma))$, i.e. by $\mathcal{QC}(\mathcal{B}(\Sigma))$. So without knowing anything about electric-magnetic duality, we popped out the B-model from the A-model. No need to think about the Fourier transform if you have enough field theory.

Now you say, "acted on" is not what we promised, which was an equivalence, but it turns out that one is a rank-one free module over the other, so we just have to specify a generator.

Next time, we'll discuss this in the nonabelian setting, starting with $A_G(\Sigma) = \mathcal{S}h(\mathrm{Bun}_G(\Sigma))$ and building the spectral side; the geometric Langlands conjecture is that the two sides are equivalent.

Lecture 25.

: 4/27/21

Today we will discuss line operators in the nonabelian setting. Recall that if Z is a 4d TFT, then $Z(\Sigma^2)$ is the category of line operators, which acts on categories assigned to surfaces via drilling out a small ball in the cylinder bordism. Operator product expansion defines a symmetric monoidal structure on $Z(\Sigma^2)$.

Let R be a commutative algebra in a symmetric monoidal higher category $\mathcal{C}$. Assume $\mathcal{C}$ has colimits (we only need some kinds, though), and let Σ be a topological space, presented combinatorially (i.e. a simplicial set). There is then a commutative algebra in $\mathcal{C}$ denoted $R \otimes \Sigma$, built from $R^{\otimes v(\Sigma)}$ (the vertices), with 1-simplices adding relations, 2-simplices adding relations between relations, and so on. This construction $R \otimes \Sigma$ has another name, the *factorization homology* of R on Σ, then denoted $\int_\Sigma R$, but we won't need that very general perspective.

In nice situations, $R \otimes \Sigma$ has a very nice description. For example, if R is a commutative ring,

$$R \otimes \Sigma \cong \mathcal{O}(\mathrm{Map}(\Sigma, \operatorname{Spec} R)), \tag{25.1}$$

and if R is a symmetric monoidal category,

$$R \otimes \Sigma \cong \mathcal{QC}(\mathrm{Map}(\Sigma, \operatorname{Spec} R)). \tag{25.2}$$

Here, maps from Σ to $\operatorname{Spec} R$ are locally constant maps, using $\pi_{\le 1}\Sigma$ as before. So we're given copies of R for points of Σ, and paths give identifications. Then 2-simplices give identifications between identifications, and so on, though we don't see that for discrete rings.

Last time, we also discussed topological Maxwell theory $\mathcal{A}_{\mathrm{GL}_1}$. It has no nontrivial local operators, because $\mathcal{A}_{\mathrm{GL}_1}(S^3)$ is boring. However, it has nontrivial line operators, or equivalently codimension-3 defects. This is because S^2 has nontrivial principal GL_1-bundles, but S^3 doesn't. More explicitly, $\mathcal{A}_{\mathrm{GL}_1}(S^2)$ is the category of sheaves on $\mathrm{Pic}(S^2) = \mathbb{Z}$, so we get the category of $\mathbb{Z}$-graded vector spaces. The $\mathbb{Z}$-grading is the Chern class of the principal U_1-bundle or the monopole number of the operator. There is a graded dimension operator to $\mathbb{C}[\mathbb{Z}]$, and this tells us what we can put on knots in manifolds (TODO: which dimension?).

Remark 25.3. This generalizes to dimensions 3 and 5: Yang-Mills theory has interesting singularities. This is because the Yang-Mills equations have interesting solutions on S^2, so we obtain defects by specifying the topological type of the principal bundle on the linking S^2. These are all called *'t Hooft operators.* ◀

And now these actions assemple: for a surface Σ and $x \in \Sigma$, we have an action of the symmetric monoidal category of $\mathbb{Z}$-graded vector spaces on the category $\mathcal{A}_{\mathrm{GL}_1}(\Sigma)$. These fit together into a tensor product $\Sigma \mathcal{V}ect_{\mathbb{Z}\text{-gr}}$. To identify this, let's use Cartier duality: $\mathbb{Z}$-graded vector spaces are equivalent to $\mathbb{G}_m$-representations, which are quasicoherent sheaves on $\mathrm{pt}/\mathbb{G}_m$. Therefore we want to know the symmetric monoidal category

$$\Sigma \otimes \mathcal{QC}(\mathrm{pt}/\mathbb{G}_m) \simeq \mathcal{QC}(\mathrm{Map}(\Sigma, \mathrm{pt}/\mathbb{G}_m)) \simeq \mathcal{QC}(\mathrm{Loc}_{\mathbb{G}_m}(\Sigma)). \tag{25.4}$$

This is a little odd: the B-side acts on the A-side. We were expecting something different, that they were equivalent — and they are, and in fact this action realizes $\mathcal{A}_{\mathrm{GL}_1}$ as a free rank-1 module over $\mathcal{B}_{\mathbb{G}_m}$. This is reminiscent of something you can already see in the Fourier transform: $\mathrm{Fun}(G)$ with pointwise multiplication is identified with $\mathrm{Fun}(\widehat{G})$ with convolution, but the latter acts on the former by convolution of characters. In fact, $\mathcal{B}_{\mathrm{GL}_1}(\Sigma) = \mathcal{QC}(\mathrm{Loc}_1(\Sigma)) = \Sigma \otimes \mathcal{A}_{\mathrm{GL}_1}(S^2)$, and the action of $\mathcal{B}_{\mathrm{GL}_1}(\Sigma)$ on $\mathcal{A}_{\mathrm{GL}_1}(\Sigma)$ induced by the action of $\mathcal{A}_{\mathrm{GL}_1}(S^2)$ on $\mathcal{A}_{\mathrm{GL}_1}(\Sigma)$ realizes $\mathcal{A}_{\mathrm{GL}_1}(\Sigma)$ as a free rank-one module category over $\mathcal{B}_{\mathbb{G}_m}$.

If Σ is a Riemann surface, $\mathcal{A}_{\mathrm{GL}_1}(\Sigma) = \mathrm{Loc}(\mathrm{Pic}(\Sigma))$ (secretly it's $\mathcal{D}$-modules on $\mathrm{Pic}(\Sigma)$). We want to say the unit is a skyscraper sheaf, but instead we have to take the universal cover $\widetilde{\mathrm{Pic}}(\Sigma) \to \mathrm{Pic}(\Sigma)$, which is contractible. This is probably a bit weird.

Example 25.5. $\mathrm{Loc}(S^1) = \mathcal{R}ep_{\mathbb{Z}} = \mathcal{QC}(\mathbb{G}_m)$. The unit object in $\mathcal{QC}(\mathbb{G}_m)$ is $\mathcal{O}_1$, the skyscraper at the identity, and under the correspondence this is sent to (TODO: not sure about this) the universal cover $\mathbb{R} \to S^1$ as a $\mathbb{Z}$-local system. ◀

In the nonabelian story, something similar will happen; this object is called the *Whittaker sheaf.*

Let G be a complex reductive group, such as $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{SO}_n(\mathbb{C})$, $\mathrm{Sp}_n(\mathbb{C})$, or the complex form of something like E_8. The theory $\mathcal{A}_G$ linearizes Bun_G, and studies its topology. The number-field analogue assigns to a number field $\operatorname{Spec} \mathcal{O}_G$ some space of cohomological automorphic forms, and likewise for a curve C over a finite field $\mathbb{F}_q$, the $\mathcal{A}_G$-theory linearizes spaces of G-bundles: $\mathbb{C}_c[\mathrm{Bun}_G(C)(\mathbb{F}_q)]$.

The $\mathcal{A}_G$ theory is well-defined in the sense of physics, meaning there is a particular construction of 4d $\mathcal{N} = 4$ super-Yang-Mills with gauge group the compact form of G, in the A-twist, studied in the seminal paper of Kapustin-Witten (TODOcite). They do not determine the state spaces on closed 3-manifolds, and we will not discuss that either, but instead we will focus on surfaces.

Let Σ be a Riemann surface. We will then study the moduli stack $\mathrm{Bun}_G(\Sigma)$ and attach to it a category of "topological" sheaves on $\mathrm{Bun}_G(\Sigma)$. There are a few different ways to go about this, but all of them are internal to topology, not algebraic geometry.

Here's an important point which might clarify why we're interested in this: we're going to study the line operators in this TFT.[29] There are lots of interesting line operators, including *'t Hooft monopoles*, which are codimension-3 singularities in Yang-Mills theory.

We first want to know what $\mathcal{A}_G(S^2)$ is, which means we need to study $\mathrm{Bun}_G(\mathbb{P}^1)$. This is known, by Grothendieck-Birkhoff: for example, for $G = \mathrm{GL}_n$, a principal GL_n-bundle is equivalent to a rank-n vector bundle on $\mathbb{P}^1$, and the Grothendieck-Birkhoff theorem shows that V is a direct sum of line bundles, and this decomposition is unique up to permutation. Therefore $\mathrm{Bun}_n(\mathbb{P}^1) \simeq \mathbb{Z}^n/S_n$: the coweight lattice of GL_n modded out by permutations.

[29]As with $G = \mathrm{GL}_1$, there will be no nontrivial local operators, except in the derived sense. And in the derived sense, they correspond to a G-equivariant cohomology ring and ultimately aren't interesting for us.

So lots of principal bundles, which means lots of line operators. These tell you singularities inside the linking S^2: nontrivial characteristic numbers of these line bundles obstruct the ability to extend the fields into the ball the linking S^2 bounds.

Another way to draw this is to collapse time: form a ravioli (raviolo?) $\Sigma \amalg_{\Sigma\setminus x} \Sigma$, where x is the center of the linking S^2 inside Σ. We know what G-bundles on this ravioli are:

$$\mathrm{Bun}_G\big(\Sigma \amalg_{\Sigma\setminus x} \Sigma\big) \simeq \mathrm{Bun}_G(\Sigma) \times_{\mathrm{Bun}_G(\Sigma\setminus x)} \mathrm{Bun}_G(\Sigma). \tag{25.6}$$

That is, a principal G-bundle on the ravioli is data of two bundles on Σ and data of an identification on $\Sigma \setminus x$. In physics, you might think of this as starting with one G-bundle P_1 and ending with another bundle P_2, where we inserted a line operator in the middle of $\Sigma \times [0,1]$ at $(x, 1/2)$; but we've collapsed time to make the ravioli. Therefore P_0 and P_1 must coincide except at x. You can do nothing ($P_0 = P_1$ and the identification is the identity), or you can do a small nothing of nothing.

In particular, we have a correspondence diagram

$$\begin{tikzcd}
 & \mathrm{Bun}_G(\Sigma) \times_{\mathrm{Bun}_G(\Sigma\setminus x)} \mathrm{Bun}_G(\Sigma) & \\
\mathrm{Bun}_G(\Sigma) \arrow[ur] & & \mathrm{Bun}_G(\Sigma), \arrow[ul]
\end{tikzcd} \tag{25.7}$$

and we linearize it by taking sheaves of various sorts, then define the map induced by a particular local operator using a push-pull construction. Let $\mathcal{H}\mathit{ecke}_x \coloneqq \mathrm{Bun}_G(\Sigma) \times_{\mathrm{Bun}_G(\Sigma\setminus x)} \mathrm{Bun}_G(\Sigma)$. We can also realize it by excising a disc and gluing in a non-Hausdorff "doubled disc":

$$\mathcal{H}\mathit{ecke}_x \simeq \mathrm{Bun}_G(\Sigma) \times_{\mathrm{Bun}_G D_\varepsilon(x)} \big(\mathrm{Bun}_G(D_\varepsilon(x)) \times_{\mathrm{Bun}_G(D_e(x)\setminus x)} \mathrm{Bun}_G(D_\varepsilon(x))\big). \tag{25.8}$$

So we get the space of principal G-bundles with a trivialization on a disc, times modifications of the trivial G-bundle on a disc. This is realizing Bun_G "adelically," as a double quotient as we did in the number-field and function-field settings. This description is the Hecke correspondence we discussed before: $\mathcal{H}\mathit{ecke}_x$ is identified with

$$\mathrm{Bun}_G(\Sigma, x) \times (G(\mathcal{O}_x)\backslash G(K_x)/G(\mathcal{O}_x)). \tag{25.9}$$

It is the total space of the associated $G(K_x)/G(\mathcal{O}_x)$-bundle to the tautological $G(\mathcal{O}_x)$-bundle on $\mathrm{Bun}_G(\Sigma, x)$.

The physics punchline is that $\mathcal{A}_G(S^2)$, sheaves on $\mathrm{Bun}_G(S^2)$, acts by line operators, and this is full of interesting operators. The group theory picture is that this is sheaves on $G(\mathcal{O}_x)\backslash G(K_x)/G(\mathcal{O}_x)$, or sheaves on the ravioli, acts. The relationship is that if you add enough filling to a ravioli, it looks like S^2 (though we note that whether or not you do this, the ravioli is delicious).

This category, no matter which avatar it appears in, deserves a special name. We will call it the *spherical Hecke category* and denoted it $\mathcal{S}\mathit{ph}$. The physics/TFT perspective says that line operators in a 4d theory commute — they form a symmetric monoidal category. The group-theoretic perspective does not offer a clear explanation for why they commute, and the physics clarifies the math, even in the arithmetic setting.

Lecture 26.

The affine Grassmannian: 4/29/21

Today we will discuss the geometric Satake correspondence, albeit a little briefly. TODO: I was more confused than usual today, so if something doesn't make sense, there's a good chance I wrote it down incorrectly. Sorry about that.

We've proceeded along a general theme that if G is a group acting on a space X, and K is a subgroup of G, then the quotient X/K has a groupoid action, leading to a Hecke algebra action on things like functions on X/K.

Last time, we introduced a correspondence

$$\begin{tikzcd}
 & \mathcal{H}\mathit{ecke}_x \arrow[dl] \arrow[dr] & \\
\mathrm{Bun}_G(C) & & \mathrm{Bun}_G(C),
\end{tikzcd} \tag{26.1}$$

for a curve C and point $x \in C$. The left map is a fibration with fibers

$$\mathrm{Gr}_G \coloneqq G(K_x)/G(\mathcal{O}_x). \tag{26.2}$$

This space is called the *affine Grassmannian* for G. The space $\mathcal{H}ecke_x$ is the space of pairs of principal bundles on C with data of an identification on $C \setminus x$. As a fibration over $\mathrm{Bun}_G(C)$, $\mathcal{H}ecke_x$ is the associated bundle to an LG-bundle; here $LG = G(K_x)$ and $LG_+ = G(\mathcal{O}_x)$; these play the role of G and K, respectively. Important in this story is the Beauville-Laszlo theorem (TODO: spelling?) telling us how to think G-bundles on a disc.

You can think of Gr_G as classifying principal G-bundles on C with a trivialization on $C \setminus x$, or equivalently principal G-bundles on a disc with a trivialization away from the origin. Given $P \in \mathrm{Bun}_G(C)$, we can trivialize it on a disc, giving P and its trivialization in $\mathrm{Bun}_G(C, x)$; then, modifying the trivial bundle on D, we obtain $(P, P') \in \mathrm{Bun}_G(C, x) \times \mathrm{Gr}_G$. Now mod out by the diagonal action of $G(\mathcal{O}_x)$, which changes the trivializations of both P and P' — this quotient is the affine Grassmannian, and realizes it as an associated bundle.

It's hard to directly specify points in $\mathcal{H}ecke_x$: locally it looks like $\mathrm{Gr}_G \times \mathrm{Bun}_G(C)$, but it is a twisted product globally. You can specify points in a non-invariant way by TODO. Let $\mathbb{O} \subset \mathrm{Gr}_G$ be a coset, equivalently an element of $L\mathrm{Gr}\backslash LG/LG_+$. Then, $\mathbb{O}$ defines a subset of $\mathcal{H}ecke_x$, and in fact correspondences preserve this subset. That is, we have a diagram

$$\begin{tikzcd} & \mathbb{O} \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_G(C) & & \mathrm{Bun}_G(C). \end{tikzcd} \tag{26.3}$$

So in a sense, these orbits give us lots of possible smaller correspondence operations. This is good because we don't have too many group operations in $\mathcal{H}ecke_x$, but these orbits give us more options.

Example 26.4. Let's look at $G = \mathrm{GL}_n$. Concretely, $\mathrm{Gr}_{\mathrm{GL}_n}$ is modifications of the trivial rank-n vector bundle on the disc: we pick $V \in \mathcal{V}ect(D)$ together with a trivialization of V on the punctured disc $D^\times$. Then $\mathrm{Gr}_{\mathrm{GL}_n}$ contains a subspace in which V is trivializable on all of D; this gives us $V(D^\times) \cong K^n$, the Laurent series (TODO: that doesn't make any sense but I think it is what I heard? Sorry about that.) TODO: something about lattices.

The poles in $V(D)$ are bounded: $t^{-N}\mathcal{O}^n \subset V(D) \subset t^N\mathcal{O}^n \subset K^n$. This shows how to realize $\mathrm{Gr}_{\mathrm{GL}_n}$ as a union of projective varieties inside (TODO: something). Therefore $\mathrm{Gr}_{\mathrm{GL}_1}$ is an *ind-proper ind-scheme*. For G a more general reductive group, we can embed $G \hookrightarrow \mathrm{GL}_n$ for some n, and therefore these nice properties of $\mathrm{Gr}_{\mathrm{GL}_n}$ generalize to Gr_G. ◀

Any geometry that's invariant under $G(\mathcal{O})$ makes sense on $\mathrm{Bun}_G(C)$. For example, $G(\mathcal{O})$-equivariant sheaves on Gr_G descend to sheaves on $\mathcal{H}ecke_x$. This includes constant sheaves on orbits. And since $\mathcal{H}ecke_x$ is a groupoid over $\mathrm{Bun}_G(C)$, we get a map from sheaves on $\mathcal{H}ecke_x$ to endomorphisms of sheaves on $\mathrm{Bun}_G(C)$. In particular, the category of $G(\mathcal{O})$-equivariant sheaves on Gr_G is equivalent to the category of sheaves on $G(\mathcal{O})\backslash G(K)/G(\mathcal{O})$, which we called the spherical Hecke category last time. In particular, the spherical Hecke category acts monoidally on the category of sheaves on $\mathrm{Bun}_G(C)$.

Last time, we argued that the monoidal structure on the spherical Hecke category is in fact symmetric monoidal, via a field theory argument: there are three directions that we can move operators in, which is enough to fit a symmetric monoidal structure.

We can extract from this a way to try to prove commutativity directly, trying to see what "moving around these local operators" means. This is what Beilinson-Drinfeld did using the theory of factorization algebras. This uses some extra algebraic structure on Gr_G, which involves varying the point $x \in C$.

Another perspective comes from homotopy theory: when X is a space, $\pi_2(X)$ is an abelian group, and this is for broadly similar reasons. So let's dig into the homotopy theory of Gr_G. In the abelian case, we obtained something homotopy equivalent to $\mathbb{Z}$. In general, Gr_G can be modeled by G-bundles on $\mathbb{A}^1$ with a trivialization on $\mathbb{A}^1 \setminus 0$. Since we only care about homotopy in this paragraph, we can replace $\mathbb{A}^1$ with $\mathbb{R}^2$, and $\mathbb{A}^1 \setminus 0$ with $\mathbb{R}^2 \setminus D_\varepsilon(0)$: replace our formal neighborhood of the origin with an actual neighborhood. Next, we replace G with its maximal compact G_c: we are looking at principal G_c-bundles on $\mathbb{R}^2$ with trivializations outside of a small disc. A principal G_c-bundle means a map to BG_c, so up to homotopy,

$$\mathrm{Gr}_G \simeq \mathrm{Map}((\mathbb{R}^2, \mathbb{R}^2 \setminus D_\varepsilon(0)), (BG_c, *)) = \Omega^2 BG_c = \Omega BG_c. \tag{26.5}$$

It is always true that the inclusion $G_c \hookrightarrow G$ induces a homotopy equivalence $BG_c \xrightarrow{\cong} BG$.

Anyways, this description of the affine Grassmannian is useful for topological information: for example, $\pi_0(\mathrm{Gr}_G) = \pi_1(G) = \pi_2(BG)$. For example, $\mathrm{Gr}_{\mathrm{PGL}_n}$ is connected, and $\mathrm{Gr}_{\mathrm{SL}_n}$ is not.

In general in homotopy theory, realizing a space as Ω^k of some other space is a way of providing a partially-commutative multiplication: you can compose loops, but there are k directions you can compse in. So since Gr_G is, up to homotopy, Ω^2 of something, then Gr_G has (up to homotopy) a "weakly commutative" group structure.

Homotopy theorists formalize this idea of "weakly commutative multiplication" on $\Omega^k X$ using the *little k-discs operad* or *E_k operad*. This is an operad such that, whatever it is, algebras over it in $\mathcal{T}op$ are the same thing as k-fold loop spaces. That is, an E_k-structure on a space is giving you the ability to, given m copies of D^k inside a larger D^k, or a point in $\mathrm{Conf}_m(D^k)$, we get a multiplication on X.

In fact, $\pi_k(X)$ has an E_k-structure, where we think of a map from a disc as sending the boundary to the basepoint. Then composition à la operator product expansion is the same as composition in $\pi_k(X)$. For $k \geq 2$, an E_k-structure on a set is an abelian group structure, so for $k \geq 2$, π_k is abelian.

As we saw above, Gr_G has an E_2-structure, and so if we linearize it in any way (functions, sheaves, ...), we will obtain E_2-objects. For example, sheaves on the affine Grassmannian are an E_2-category, so "weakly commutative" monoidal (actually braided monoidal). This is precisely the structure we obtain on the category of line operators in 3d.

The E_2-structure on Gr_G can be thought of as given by moving $x \in C$ in two different directions on C, which is two-dimensional. But there's one more dimension of commutativity in $G(\mathcal{O}_x)\backslash\mathrm{Gr}_G$ (as you might expect: we have line operators in a 4d theory, not a 3d theory, so we expect a little more commutativity): we have composition in addition to the E_2-structure from Gr_G, so we obtain an E_3-structure, as expected for line operators in 4d.

Remark 26.6. The affine Grassmannian is an ind-scheme, and therefore we have to worry about whether it is reduced. When G is semisimple, Gr_G is reduced, but even for GL_1, the affine Grassmannian is really $\mathbb{Z}$ times an infinite-dimensional formal group! This is not important for what we are doing, which is topology: taking constructible sheaves or $\mathcal{D}$-modules or the like. It does matter for the tangent space, though: $T\mathrm{Gr}_G = K/\mathcal{O}$; for example, for GL_1, we get $\mathbb{C}((t))/\mathbb{C}[[t]]$, which is an infinite-dimensional Lie algebra! Exponentiating, we obtain an infinite-dimensional formal group.

So that's one way to see the non-reducedness. In any case, we'll mostly be sticking to semisimple groups. ◀

Pressley-Segal (TODO: cite) is another good reference on the affine Grassmannian, from a different perspective.

There is a Morse-theoretic perspective on Gr_G due to Białynicki-Birula, giving a cell decomposition indexed by one-parameter subgroups $\mathbb{G}_m \to T$, where T is a maximal torus of G; these one-parameter subgroups are called *coweights*. Coweights form a lattice denoted Λ. This cell decomposition is an infinite-dimensional version of the Schubert decomposition. There is a $\mathbb{G}_m$-action on Gr_G given by rotating the (punctured) disc. We can also pick a generic coweight (meaning the centralizer of the image is T itself) $\rho^\vee$; since $G \subset LG_+$ acts on Gr_G, $\rho^\vee$ defines another $\mathbb{G}_m$-action on Gr_G.

Let's combine these actions and look for fixed points. These turn out to be given by cocharacters $\varphi\colon \mathbb{G}_m \to G$, which map to (TODO). For example, for $G = \mathrm{GL}_n$, we are looking for diagonal matrices with diagonal entires $(t^{i_1}, \dots, t^{i_n})$, which is an element of $\mathrm{GL}_n(\mathbb{C}((t)))$. This is a natural thing to start from: Morse-theoretic arguments in general say that isolated critical points tell you a cell decomposition with one fixed point at the center of each cell (the cells are the stable manifolds).

The cocharacter lattice Λ (TODO: wasn't this the notation for coweights?) is isomorphic to $T(K)/T(\mathcal{O}) \cong \mathrm{Gr}_T$. This is good, because we understand the affine Grassmannian for tori: $\Lambda \cong \mathrm{Gr}_T$, and this injects into Gr_G. We will use this to understand categories of sheaves on Gr_G using abelian data. The next step would be to find a Cartan decomposition; this is not quite the same decomposition as the decomposition as $G(\mathcal{O})$-orbits, but every $G(\mathcal{O})$-orbit is a union of cells, and in fact $G(\mathcal{O})\backslash\mathrm{Gr}_G$ is isomorphic to Λ/W, where W is the Weyl group. The proof of this fact is linear algebra. So the cells we're interested in are in bijection not quite with cocharacters, but cocharacters mod the Weyl group action.

Next time we combine this perspective with the commutative multiplication to obtain the geometric Satake theorem.

Lecture 27.

The geometric Satake correspondence: 5/4/21

The geometric Satake correspondence is a nonabelian counterpart to the theory of Fourier series. Well, there are different ways to do that, as we've seen. This is a generalization of the fact that $\mathbb{Z}$-graded vector spaces are equivalent to $\mathbb{G}_m$-representations. We saw last time that $\mathbb{Z}$-graded vector spaces are equivalent to sheaves on the affine Grassmannian for GL_1. The other side, $\mathbb{G}_m$-representations, can be interpresented as quasicoherent sheaves on $\mathrm{Loc}_{\mathbb{G}_m}(\mathbb{P}^1)$.

We will generalize this, asking what sheaves on Gr_G are for nonabelian G. In the nonabelian case, double cosets are trickier: they're not the same as single cosets in general. We got to the affine Grassmannian by considering principal G-bundles on the "ravioli," thinking about bundles on a disc with an identification away from the origin. Specifically, $LG_+\backslash\mathrm{Gr}_G = LG_+\backslash LG/LG_+$. Relatedly,

$$LG_-\backslash LG/LG_+ \cong \mathrm{Bun}_G(\mathbb{P}^1). \tag{27.1}$$

The LG_+-orbits on Gr_G are a poset under "is contained in the closure of" order. The orbits are all finite-dimensional, and all orbit closures contain finitely many orbits.

Example 27.2. Say $G = \mathrm{GL}_2$. Here are some rank-two vector bundles on $\mathbb{P}^1$, which have principal GL_2-bundles as their bundles of frames: $\mathcal{O}\oplus\mathcal{O}(1)$, $\mathcal{O}\oplus\mathcal{O}$, $\mathcal{O}(2)\oplus\mathcal{O}(2)$, and so on, and the Grothendieck-Birkhoff theorem says all principal GL_2-bundles arise in this way, up to isomorphism.

The degree of a bundle (equally, $\int_{\mathbb{P}^1} c_1(P)$) defined a map to $\mathbb{Z}$; the preimage of 0 is precisely the principal SL_2-bundles; this is a connected compact space. It contains bundles of the form $\mathcal{O}(n)\oplus\mathcal{O}(-n)$, and each n is its own orbit. The closure of the orbit for $\mathcal{O}(n)\oplus\mathcal{O}(-n)$ contains the orbit of $\mathcal{O}(n+1)\oplus\mathcal{O}(n+1)$: you can think of varying the clutching function in families, and it might degenerate by having a zero somewhere. So the poset is $\mathbb{N}$, and all orbits are finite-dimensional. ◀

In general, $LG_- = G(\mathbb{C}[t^{-1}])$. Gr_G contains a lattice Gr_T sitting at the centers for cells (TODO: this is weird, I may have misheard something), which are the fixed points for the $\mathbb{G}_m$-action. And $LG_+\backslash\mathrm{Gr}_G$ is identified with (TODO: this also doesn't make much sense to me) Λ/W. This is the coweight lattice. Abstractly, this is a weight lattice, and by the classification of semisimple groups, we know this is telling us the representation theory of some other reductive group $G^\vee$.

We would like to package this all up nicely: finding a group $G^\vee$ with maximal torus $T^\vee$ and the same Weyl group, such that $\Lambda = \mathrm{Hom}(T^\vee, \mathbb{G}_m)$. That is: the set of orbits looks a lot like the irreducible representations of some reductive group, so let's go find that group. In fact, we will *construct* $G^\vee$ from the spherical Hecke category $\mathcal{S}hv_{LG_+}(\mathrm{Gr}_G)$. Here "sheaves" can mean either $\mathcal{D}$-modules or something called perverse sheaves (certain sheaves which are locally constant on orbits/are vector bundles with flat connections on orbits, or equivalently, local systems on orbits shifted by the dimension of the orbit). For this category of sheaves, simple objects are in bijection with irreducible local systems on orbits, and LG_+ orbits are all simply connected (they are all affine bundles or partial flag manifolds over G), so we understand these local systems. In general categories of sheaves or $\mathcal{D}$-modules are complicated, but this one will be semisimple! (As we would expect if we want it to be a category of representations for $G^\vee$ — but it is still a surprise.) This is something special with the geometry of the Grassmannian, and has to do with parity of dimensions. It turns out that on each connected component of Gr_G, the complex dimensions of the orbits mod 2 is constant. This limits the possible interactions between systems on orbits and their closures — the extra stuff is codimension 2, and this forces some Ext^1 terms to vanish. (The full proof is a little more complicated, but this is the basic idea.)

So we have that the spherical Hecke category is semisimple abelian, but it has more: it is symmetric monoidal. Physics told us this is because it's a category of line operators in a 4d quantum field theory, but there are further reasons. In some sense, there are two structures going on here: because this is sheaves on a double coset space $LG_+\backslash LG/LG_+$, there is always a monoidal structure, though it's not automatically commutative. But there's also a braided multiplication which arises from a completely different story: that Gr_G is a double loop space, so sheaves on it pick up an E_2-algebra structure.

Beilinson-Drinfeld found a purely algebro-geometric analogue of the notion of an E_2-structure, an adelic version, and showed that Gr_G has that structure. It is in a similar way about points on a curve colliding, but they formalized it using something called the *BD Grassmannian*, which doesn't quite have products, but encodes multiplication nonetheless. We fix a curve C, such as $\mathbb{A}^1$, and we consider a fiber bundle $\mathrm{Gr}_{G,C^n} \to C^n$.

For $n = 1$, the fiber of $\mathrm{Gr}_{G,C}$ at x is the space of principal G-bundles on C with a trivialization on $C \setminus x$. To define this precisely one should do it in families or describe the functor of points.

For $n = 2$, we have a family over two copies of the curve, so the fiber at $(x, y) \in C^2$ is the space of principal G-bundles on C with a trivialization on $C \setminus \{x, y\}$. This has two strata, $x \neq y$ and $x = y$. For $x \neq y$, $\mathrm{Gr}_{G,C^2}|_{x,y} \cong \mathrm{Gr}_{G,C}|_x \times \mathrm{Gr}_{G,C}|_y$: a bundle trivialized away from x and y is equivalent data to a bundle trivialized on x and a bundle trivialized on y. So away from the diagonal we have just a product. And when $x = y$, $\mathrm{Gr}_{G,C^2}|_{\{x,x\}} \cong \mathrm{Gr}_{G,C}|_x$: there's no extra data.

This gives us something funny: we've glued two copies of something to a single copy of something. For example, for $G = \mathrm{GL}_1$, this is $\mathbb{Z} \times \mathbb{Z}$ away from the diagonal in C^2 but just $\mathbb{Z}$ on the diagonal. This is weird: in general, dimensions in algebraic geometry can get bigger when you specialize; here, it's twice as big away from the diagonal! The resolution is that you need finite-dimensionality to prove that fact, but everything here is infinite-dimensional.

In general, Gr_{G,C^n} has two key structures. First, its fiber at $(x_1, \dots, x_n)$ only depends on the subset $\{x_1, \dots, x_n\}$, and not on the ordering. This means that when you restrict to diagonal subspaces, you get Gr_{G,C^k} for the expected $k < n$. So we can (TODO: I think) glue all these together, taking a colimit, and define this over the *Ran space* $\mathrm{Ran}(C)$, the space of finite subsets of C. You can also define it as $\mathrm{colim}_{I \to J} C^I$. This is a little weird as a space, but is often useful to think of as a filtered system: a sheaf on the Ran space is sheaves on each C^I compatible with pullback by $I \to J$.

The second important fact is *factorization*: this construction is multiplicative with respect to disjoint union: the fiber at $\{x_i\} \amalg \{y_i\}$ is the product of the fibers at $\{x_i\}$ and at $\{y_i\}$. This is structure, not data, and is a version of the E_2-algebra structure, but more algebro-geometrically. This doesn't give actual multiplication maps like the spaces of little discs, but because there are ways to glue two copies of the Grassmannian to one copy of the Grassmannian, we have something like multiplication in families. This is an example of a *factorization space*, and Gr_G has this structure, which is a way of encoding the somewhat unusual algebraic structure which in the homotopy category *is* the E_2-structure. In algebraic geometry, we don't have this E_2-structure, but we do obtain multiplication some of the time: it defines multiplication on objects on the Grassmannian that we can take limits of.

For example, let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on Gr_G. We will multiply them with operator product expansion: think of $\mathcal{F}$ at a point x and $\mathcal{G}$ at a point y. We therefore have $\mathcal{F} \boxtimes \mathcal{G}$ on $\{x, y\}$, and can do so when we vary x, bringing it closer to y. We would like to extend this over the diagonal, and here it matters what kind of sheaf we have: there is a problem for coherent sheaves, but for more topological things, there is a way to do this, using *nearby cycles*, a topological version of the limit. So we can use this to define $\mathcal{F} * \mathcal{G}$ as the nearby cycles of $\mathcal{F} \boxtimes \mathcal{G}$ on the diagonal. The common theme is that things close to topology have some sort of limit/extension.

This isn't always the right thing to do — under this multiplication, sheaves on Gr_G looks like a braided monoidal structure, but it's not associative! The issue is that we need to constrain the geometry more: if you restrict to LG_+-equivariant sheaves, then it works, giving an associative braided monoidal structure on $\mathcal{S}ph$. (The missing ingredient was a compactness assumption — in a sense, the Grassmannian is sort of an E_2-space, and we think of sheaves on it as having an E_2-structure, but this in fact only works for small enough subcategories of sheaves, such as LG_+-equivariant ones.)

You can think of stacking two sheaves on Gr_G in two ways: you can stack them vertically, which is the Hecke operation for $LG_+ \backslash LG / LG_+$, and is associative; and there is also the factorization operation, letting points collide, which is braided. These two multiplications are associative and commute with each other, and therefore Eckmann-Hilton tells us we get a symmetric monoidal structure, encoded algebro-geometric. We've broken the symmetry of three dimensions into $2 + 1$: two directions in the plane (the curve), and one away from it (the stacking/Hecke operation).

The notion of factorization, algebro-geometrically formalizing something like an E_2-structure, is one of the most important things to come out of the geometric Langlands correspondence. It came out of physics, which is characteristic zero, but now we can take the same notion of factorization and bring them to curves over finite fields, or (as Scholze did) working over p-adic fields.

We will obtain $G^\vee$ by Tannakian reconstruction. We have produced a symmetric monoidal category $(\mathcal{C}, \otimes)$, and we therefore obtain a stack $\mathrm{Spec}(\mathcal{C}, \otimes)$; for decent $\mathcal{C}$, we can recover $\mathcal{C}$ as $\mathcal{QC}(\mathrm{Spec}(\mathcal{C}, \otimes))$. As a special case, which is the more familiar notion of Tannakian reconstruction, we can specify a *fiber functor*, a faithful exact symmetric monoidal functor to $(\mathcal{V}ect_k, \otimes)$. This extra data tells us that $\operatorname{Spec}(\mathcal{C}, \otimes)$ has a k-point, a map from $\operatorname{Spec}(\mathcal{V}ect_k, \otimes) = \operatorname{Spec} k$, and that this map is a flat cover. Therefore $\operatorname{Spec}(\mathcal{C}, \otimes) \cong \mathrm{pt}/BG^\vee$ for some group $G^\vee$ called the *Galois group* or *Tannakian group* of $(\mathcal{C}, \otimes)$, and $G^\vee$ is the group of automorphisms of the fiber functor.

In our case, we have $\mathcal{C}$ the category of sheaves on Gr_G, and we have our nice symmetric monoidal structure. What we need next is a fiber functor, a forgetful functor to $\mathcal{V}ect_k$. In other words, what's the "underlying vector space" of a sheaf? What's a good way to measure it, which is both faithful and symmetric monoidal? There are various ways of thinking about it; the shortest way to say it is: take cohomology. But that might not be the most insightful way to say it; we will think of the fiber functor differently, producing a map

$$\mathcal{S}hv_{LG_+}(\mathrm{Gr}_G) \longrightarrow \mathcal{S}hv(\mathrm{Gr}_T). \tag{27.3}$$

Since Gr_T is a lattice, we get a functor to $\mathcal{R}ep(T^\vee)$, where $T^\vee := \mathrm{Hom}(\Lambda, \mathbb{G}_m)$. Now $\Lambda \subset \mathrm{Gr}_G$, so we can restrict our sheaf to a discrete set. There is a choice to be made about how to measure once you've restricted, and indeed the naïve functor you might write down is wrong, but it's still easier/more clarifying to think about it this way.

Next time we will discuss the answer (parabolic restriction) in more detail, but the quick summary of the data is that in between G and T, there is a Borel B, e.g. the upper triangular matrices when $G = \mathrm{GL}_n$. This gives us as a correspondence, because T is not just a subgroup of G, but a quotient of B:

$$\begin{tikzcd} & \mathrm{Gr}_B \arrow[dl] \arrow[dr] & \\ \mathrm{Gr}_G & & \mathrm{Gr}_T \cong \Lambda. \end{tikzcd} \tag{27.4}$$

Then our fiber functor will be (what else?) a push-pull operator on sheaves, giving us a sheaf on Λ, equivalently a representation of $T^\vee$. Our fiber functor forgets this action, but remembering it gives us additional data: the spherical Hecke category has a map from $\mathrm{pt}/G^\vee$.

Lecture 28.

Parabolic induction and restriction: 5/6/21

We've been discussing the geometric Satake theorem. This theorem is about the category of $G(\mathcal{O})$-equivariant sheaves on the affine Grassmannian (or sheaves on $G(\mathcal{O})\backslash G(K)/G(\mathcal{O})$), which is a semisimple abelian category with a symmetric monoidal structure. With a fiber functor to $\mathcal{V}ect$, we would have a Tannakian category and therefore obtain a category of representations of a reductive group. This is what we will discuss today, in the framework of parabolic induction and restriction. This is a framework for relating representations of G and a maximal torus T.

Let G be a complex reductive group, such as $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{SL}_n(\mathbb{C})$, and so on. We study finite-dimensional G-representations through highest weight theory. A *Borel subgroup* $B \subset G$ is a maximal subgroup such that G/B is a projective varietiy. Choose a Borel subgroup B; G/B is called the *flag variety* of G. The standard choice of B for $G = \mathrm{GL}_n(\mathbb{C})$ is the subgroup of upper triangular matrices. The maximal torus, defined as a subgroup of G, is also a quotient of the Borel by a subgroup N (for the standard Borel, the subgroup of strictly upper triangular matrices). So we have a correspondence

$$\begin{tikzcd} & \mathrm{pt}/B \arrow[dl] \arrow[dr] & \\ \mathrm{pt}/G & & \mathrm{pt}/T. \end{tikzcd} \tag{28.1}$$

We can therefore try to write down push-pull maps. Given $V \in \mathcal{R}ep_G$, we can pull back to pt/B, which is just restricting the representation to B. For the pushforward, we need a little more information: that the pullback of $\mathrm{pt}/B \to \mathrm{pt}/T$ by $\mathrm{pt}/ \to \mathrm{pt}/T$ is pt/N. So given $V|_B$, take its N-invariants, which is now a representation of $B/N = T$. The N-invariants is the usual notion of highest weight vectors, and the T-action is that highest weight. This process of producing a T-representation from a G-representation, which is not the usual way of restricting via $T \subset G$, is called *parabolic restriction*.

Parabolic induction is the adjoint map. Begin with an irreducible T-representation, which is a character $\mathbb{C}_\lambda$, and pull it back to pt/B. Since

$$\mathrm{pt}/B \simeq G\backslash G/B, \tag{28.2}$$

the pullback of $\mathbb{C}_\lambda$ to pt/B is identified with a G-equivariant vector bundle on G/B. The pushforward from the flag variety to pt, keeping track of G-actions, gives us a G-representation, and this is the construction of a highest weight representation.

In physics, this process defines an interface between gauge theory with gauge group G and gauge theory with gauge group T. This more or less means mapping into the diagram (28.1). For example, if you know about the Springer resolution of the nilpotent cone, you can phrase it as mapping into (28.1).

Now we want to upgrade this into parabolic induction and restriction for the spherical category. The analogue for (28.1) is

$$\begin{tikzcd} & \mathrm{Gr}_B \arrow[dl] \arrow[dr] & \\ \mathrm{Gr}_G. & & \mathrm{Gr}_T, \end{tikzcd} \tag{28.3}$$

and we saw $\mathrm{Gr}_T \cong \Lambda$, a lattice. Abelian duality then identified $\mathcal{S}ph_T \cong \mathcal{R}ep_{T^\vee}$, where $T^\vee$ is the dual torus. We haven't seen Gr_B before, and since B is not reductive (it is instead solvable), there's something interesting here. We're also more interested in $\underline{\mathrm{Gr}}_B$, where we incorporate B-equivariance:

$$\underline{\mathrm{Gr}}_B := B\backslash \mathrm{Gr}_B = B(\mathcal{O})\backslash B(K)/B(\mathcal{O}). \tag{28.4}$$

The space Gr_B parametrizes principal B-bundles on a curve C together with a trivialization away from a point. A trivialization of a G-bundle is much stronger than providing a reduction to a B-bundle (which is data of a full flag). Therefore we have a map from $C \setminus \mathrm{pt}$ to the thing that parametrizes flags in the trivial bundle, which is TODO. So a point on the affine Grassmannian for G, a G-bundle with a trivialization on $C \setminus \mathrm{pt}$, gives us a section of the associated G/B-bundle over $C \setminus \mathrm{pt}$. And since G/B is projective, the valuative criterion of properness allows us to extend to all of C. This is not true in families: it only works for field-valued points.

Thus, the map $\mathrm{Gr}_B \to \mathrm{Gr}_G$ is a bijection on k-points. But it is not a homeomorphism — what's actually going on is quite pretty: you can tear apart Gr_B into a disjoint union of spaces $\mathrm{Gr}_{B,\lambda}$ indexed by $\lambda \in \Lambda = \mathrm{Gr}_T$, via the map $\mathrm{Gr}_B \to \mathrm{Gr}_T$. TODO: I think I may have misunderstood something here.

You can consider the orbits of Gr_G acted on by $LN = N(K)$: N is exactly what gets killed by going from B to T. The orbits are both infinite-dimensional and have infinite codimension, and are called *semi-infinite orbits*. This has to do with the kind of pheneomenon that also occurs in Floer theory, where there's infinite-dimensional Morse theory stratifying a space into pieces that are infinite-codimension *and* infinite-dimensional.

The point is that we have a functor $\mathcal{S}ph_G \to \mathcal{S}ph_T \simeq \mathcal{R}ep_{T^\vee}$ described by taking a sheaf $\mathcal{F}$, doing shriek-restriction to $N(K)$-orbits, and taking cohomology. This is push-pull across (28.3). We therefore obtain a Λ-graded vector space. Forgetting the grading, we obtain a functor $\mathcal{S}ph_G \to \mathcal{V}ect$, and this is the fiber functor. The extra structure that's been floating around, such as the Λ-grading, allows one to show the fiber functor is symmetric monoidal; it's also pretty manifestly faithful, which can be seen by looking at orbits. So we can run Tannakian representation.

Theorem 28.5 (Geometric Satake)**.** *As a symmetric monoidal category, $(\mathcal{S}ph_G, *)$ is equivalent to $(\mathcal{R}ep_{G^\vee}, \otimes)$, where $G^\vee$ is the automorphisms of the fiber functor, and is precisely the Langlands dual group of G.*

The history is convoluted: Lusztig first proved a result like this, and Ginzburg built further upon it. Drinfeld produced the factorization picture, which is crucial, and Mirković-Vilonen assembled the pieces into this theorem.

The Langlands dual group $G^\vee$ is a reductive group whose maximal torus is $T^\vee$, the dual to T. This is because we produced a functor $\mathcal{S}ph_G \to \mathcal{R}ep_{T^\vee}$: Tannakian reconstruction tells us that this produces a canonical subgroup $T^\vee \subset G^\vee$: we didn't have to choose it this time. One can check that the Weyl group for $G^\vee$ is isomorphic to that of G, and the root data for G and $G^\vee$ are dual. It is also true that $Z(G^\vee)$ is identified with the Cartier dual of $\pi_1(G) = \pi_0(\mathrm{Gr}_G)$: on connected components, this construction recovers Cartier duality.

There's the question of what field we're working over. The answer is that we obtain $G^\vee$ as a group over the field of coefficients of our sheaves in the spherical category — we have been working over $\mathbb{C}$, but we didn't need to: Mirković-Vilonen showed you can take arbitrary coefficients, even $\mathbb{Z}$, and $G^\vee$ becomes defined over that coefficient ring R, because you're doing Tannakian reconstruction over R. This is a big deal: representations of reductive groups over fields which are not algebraically closed can be very complicated. We previously thought of the geometric Satake theorem as using representation theory (easier over algebraically closed fields of characteristic zero) to understand the spherical category (harder), but over other coefficient rings (e.g. in modular characteristic), the theorem is still true even if the categories aren't semisimple, and people use the spherical category to make progress on understanding $\mathcal{R}ep_G$. The flow goes the other way!

Remark 28.6. "Any coefficients?" What about over $\mathbb{S}$? The spherical category still makes sense in homotopy theory, where we just take sheaves of spectra, but there's a big issue, where this category is no longer symmetric monoidal: it's only E_3, and in derived settings, E_3 is *not* E_∞. It is known what can be done for a derived geometric Satake theorem, but a lot of things are weird: you can't even define a good determinant map, hence SL_n, over $\mathbb{S}$. ◀

Lecture 29.

More on the geometric Satake correspondence: 5/20/21

It's been a little while, so let's remember what we did last time we met, which was the geometric Satake theorem, interpreted in terms of the symmetric monoidal category of line operators in the QFT $\mathcal{A}_G$. The category of line operators is $\mathcal{A}_G(S^2)$, and because we're in dimension 4 this acquires a symmetric monoidal structure due to operator product expansion. Specifically, this is the spherical Hecke category $\mathcal{S}ph$, which is a category of equivariant sheaves on the affine Grassmannian.

Using Tannakian reconstruction, we saw that $\mathcal{S}ph$ with its fiber functor is the category of $G^\vee$-representations, where $G^\vee$ is a reductive algebraic group over k, the field (or ring) of coefficients which we took sheaves in. This is the Langlands dual group of G. There is an isomorphism $\pi_1(G) \cong Z(G^\vee)^\vee$ (the character dual of the center of $G^\vee$), so in particular if G is simply connected, then $G^\vee$ is adjoint, and vice versa. Moreover, choosing a maximal torus $T \subset G$ gives us a canonical dual torus $T^\vee \subset G^\vee$, which is also maximal. Moreover, $G^\vee$ comes with more data, called a *pinning*, which is actually quite rigid, e.g. it determines a Borel, and there is a canonical cocharacter

$$2\rho^\vee \colon \mathbb{G}_m \hookrightarrow T^\vee. \tag{29.1}$$

This is because our fiber functor isn't just to vector spaces, but to $\mathbb{Z}$-graded vector spaces (we identified it with cohomology). The Borel is then given as things with positive cohomological degree. You also get a choice of principal nilpotent! This becomes very relevant in the derived version of the Satake theorem. The determinant line bundle $\mathcal{L} \to \mathrm{Gr}_G$ is part of this structure; this is related to work of Ginzburg.

One of the philosopical reasons you might want or celebrate all this extra structure is that if you're trying to study groups, it helps to pin your representation-theoretic data down uniquely, and this is what pinning does.

Examples of Langlands duals:

- If T is a torus, its Langlands dual is the dual torus $T^\vee$.
- GL_n is Langlands self-dual. That is, for vector bundles we don't see a new group.
- In type A, SL_n is simply connected but has center, so now $G^\vee$ will be different, and indeed we get PGL_n.
- Type D, SO_{2n}, is Langlands self-dual.
- Types B and C are Langlands dual: SO_{2n+1} and Sp_{2n}. This is somewhat spooky: these two groups have nothing to do with each other at first glance, and there doesn't seem to be a clear way to make this work from group actions. This is an important heuristic point: you can't really expect naïve or simple Langlands dualities, cooked up by playing with actions. This B–C duality is saying that Langlands dualities tend to be deep.

Let C be a curve and $x \in C$. Then we obtain an action of the spherical Hecke category on $\mathcal{S}hv(\mathrm{Bun}_G(C))$: an object $V \in \mathcal{S}ph$ acts by a push-pull via the correspondence

$$\begin{tikzcd} & \mathcal{H}ecke_x \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G. \end{tikzcd} \tag{29.2}$$

For $G = \mathrm{GL}_n$, the space of k-dimensional subspaces of the fiber at x gives us an embedding $\mathrm{Gr}(n,k) \subset \mathrm{Gr}_{\mathrm{GL}_n}$. Then GL_n acts on the cohomology of the Grassmannian. This sounds less surprising than it is: the Grassmannian is a homogeneous space, and so it makes sense that GL_n acts on it. But this is not actually how the action arises! It's really an action by $(\mathrm{GL}_n)^\vee$, and this is the k^{th} fundamental representation.

Since $\mathcal{S}ph$ is a "commutative ring" (symmetric monoidal category), we would like to spectrally decompose its action on $\mathcal{S}hv(\mathrm{Bun}_G(C))$. This entails looking for eigenobjects for a $V \in \mathcal{S}ph$: sheaves $\mathcal{F} \in \mathcal{S}hv(\mathrm{Bun}_G)$ together with *data* of an isomorphism

$$V * \mathcal{F} \xrightarrow{\cong} E_x(V) \otimes \mathcal{F}. \tag{29.3}$$

Here $E_x(V)$ is a skyscraper on a vector space V, and this is our notion of scalar for eigensheaves. Unfolding this a little, E_x must be a functor $\mathcal{R}ep(G^\vee) \to \mathcal{V}ect$, and compatibility with tensor product means it must be symmetric monoidal. We can therefore classify the possible "eigenvalues" E_x: such a map is equivalent to a map in the other direction, $\mathrm{pt} \to \mathrm{pt}/G$. There's one of these up to isomorphism, and if we keep track of isomorphisms E_x is equivalent data to a $G^\vee$-torsor. Given a $G^\vee$-torsor E_x and $V \in \mathbb{S} = \mathcal{R}ep_{G^\vee}$, the vector space we obtain is the associated bundle (over a point) $E_x \times_G V$. This structure is a little confusing because everything is noncanonically isomorphic.

So let's vary x, looking at a finite collection of points $S \subset C$. and considering the action of $\bigotimes_{x\in S} \mathcal{R}ep_{G^\vee}$ on $\mathcal{S}hv(\mathrm{Bun}_G(C))$. We've been motivated the whole time by the idea that this data is all part of a topological field theory, and this action is an action of line operators on $\mathcal{A}_G(C)$. The TFT structure buys us the idea that the actions of these line operators are locally constant. This implies it factors through an action of the *factorization homology* $\int_C \mathcal{R}ep_{G^\vee}$, which encodes this local constancy without choosing S. In fact,

$$\operatorname{Spec} \int_C \mathcal{R}ep_{G^\vee} = \mathrm{Map}(C, \mathrm{pt}/G^\vee) = \mathrm{Loc}_{G^\vee}(C). \tag{29.4}$$

We're identifying points with paths between them, so we only see the fundamental groupoid of C – which is why we see local systems.

Remark 29.5. The space of local systems, at least in the Betti setting, is extremely simple: just maps $\pi_1(C) \to G^\vee$ modulo conjugation. And $\pi_1(C)$ has a simple presentation in terms of A- and B-cycles, so a $G^\vee$-local system on C is a choice of $2g$ elements of the group $A_1, \dots, A_g, B_1, \dots, B_g$ (where g is the genus of C, which we assume is connected), satisfying the relation

$$\prod_i [A_i, B_i] = 1. \tag{29.6}$$

In particular, $\mathrm{Loc}_{G^\vee}(C)$ is quite explicit and simple: it is the quotient of an affine variety by $G^\vee$.

The analogue of this in the de Rham setting is false, unfortunately; $\mathrm{Loc}_{G^\vee}(C)$ is more complicated. ◀

Anyways, we were discussing that if you believe $\mathcal{A}_G(C)$ is a topological field theory, you recover that the action by operators at a finite subset of C factors through the factorization homology, and this is not all that complicated: it is $\mathcal{QC}(\mathrm{Loc}_{G^\vee}(C))$. This is one of the main outputs you obtain from topological field theory: the categories we attach to surfaces obtain actions by something nice.

These ideas are in many cases theorems, depending on your notion of sheaves on $\mathrm{Bun}_G(C)$. These are deep and hard theorems.

Theorem 29.7 (Gaitsgory's vanishing theorem)**.** *This holds in the de Rham setting:* $\mathcal{QC}(\mathrm{Loc}_{G^\vee}^{\mathrm{dR}}(C))$ *acts on* $\mathcal{D}(\mathrm{Bun}_G(C))$.

This is hard and one of the most important results in the geometric Langlands program. De Rham local systems means principal G-bundles with a flat connection, and by "sheaves" we mean $\mathcal{D}$-modules.

Theorem 29.8 (Nadler-Yun)**.** *This holds in the Betti setting, where $\mathcal{S}hv(\mathrm{Bun}_G(C))$ means the category of sheaves nilpotent on $\mathrm{Bun}_G(C)$.*

This category of nilpotent sheaves is what you obtain by starting with locally constant sheaves on thing such as $\mathrm{Bun}_T(C)$ or $\mathrm{Bun}_L(C)$ (for L a Levi subgroup) and inducing up to G. The idea is that we wanted locally constant sheaves, because they're the simplest, but we need the category to be closed under some change-of-groups operations, notably parabolic induction. This is related to work of Braverman-Gaitsgory on geometric Eisenstein series.

Theorem 29.9 (Arinkin-Gaitsgory-Kazhdan-Raskin-Rozenblyum-Varshovsky [AGK+20b, AGK+20a, AGK+21])**.** *This holds in the restricted setting over a finite field.*

This setting is in some sense the intersection of de Rham and Betti (the "core geometric Langlands program"): the things common to any formulation of geometric Langlands. Therefore the restricted question makes sense over any field.

Returning to our story, $\mathcal{S}hv(\mathrm{Bun}_G(C))$ spectrally decomposes over $\mathrm{Loc}_{G^\vee}(C)$, and the action by $\mathcal{QC}(\mathrm{Loc}_{G^\vee}(C))$ is equivalent data to a sheaf of categories over $\mathrm{Loc}_{G^\vee}$. The simplest question you could ask about this is: what is the fiber at a local system E? This is an *E-Hecke eigensheaf*: data of $\mathcal{F}\in\mathcal{S}hv(\mathrm{Bun}_G)$ together with for every Hecke operator $V\in\mathcal{R}ep_{G^\vee}$, an isomorphism

$$V_x * \mathcal{F} \xrightarrow{\cong} (V_x)_{E_x}\otimes\mathcal{F}, \tag{29.10}$$

in a coherent way. That is, we've specified a $G^\vee$-torsor over each point $x\in C$, and we want these to fit together into a principal $G^\vee$-bundle E. These are analogues of special functions: sort of how $e^{\lambda x}$ appears as an eigenfunction with eigenvalue λ, and now we replace λ with a local system and take eigenobjects. And so part of the hope of the "naïve geometric Langlands conjecture" is that there's enough of these objects, and they span, much like how exponentials span L^2, which means that you can write down a Fourier transform. More precisely, the naïve conjecture suggests that $\mathcal{A}_G(C)$ is equivalent to $\mathcal{QC}(\mathrm{Loc}_{G^\vee}(C))$, in a way exchanging eigensheaves with skyscrapers. Another way to say this conjecture is that it asks whether the action of $\mathcal{QC}(\mathrm{Loc}_{G^\vee})$ on $\mathcal{S}hv(\mathrm{Bun}_G(C))$ is free cyclic of rank one. The generator is called the *Whittaker sheaf* $\mathbb{W}$, and it plays the role of the skyscraper at the identity. However you say it, this conjecture says that everything you want to know about $\mathcal{A}_G(C)$ can be understood in terms of spectral decomposition.

Well, the naïve geometric Langlands conjecture is false. This isn't terribly terribly deep, and people were aware this was probably false before it was proven to be false, but this is a guiding light for what kind of result we might want when we fix the conjecture into something that's actually correct. And the naïve form of the conjecture isn't terribly terribly wrong. This has to do with the fact that $\mathrm{Loc}_{G^\vee}$ is singular. One way to fix this is to just restruct to the smooth locus, irreducible local systems. For $G=\mathrm{GL}_n$ this means asking for irreducible flat connections; in general it means throwing out connections that admit a reduction of structure group to a parabolic (TODO: I think I got this right but am not completely sure.) The locus of irreducible local systems, $\mathrm{Loc}_{G^\vee}^{\mathrm{irr}}$, is a smooth stack, and is almost a variety. In fact, if G is simply connected, so $G^\vee$ is centerless, then the automorphisms go away, and $\mathrm{Loc}_{G^\vee}^{\mathrm{irr}}$ *is* a variety. Therefore one can attempt to prove the analogue of the naïve geometric Langlands conjecture for this smooth variety. For SL_2 this is due to Drinfeld, and for GL_n (for some n? TODO) this is due to Gaitsgory. Because irreducible local systems have no reduction to a parabolic $P^\vee\subset G^\vee$, they don't talk to geometric Eisenstein series; these are sometimes called *cuspidal local systems*.

Cusp forms are the most important and deepest part of the Langlands correspondence, the forms that don't come by the obvious construction (parabolic induction). It's nice that this is the smooth locus on the dual side, and you might wonder, why bother with anything else? Well, in part because they're the deepest, it's very hard to get your hands on them: you can't obtain them inductively from smaller G. So the "boring part" is the most difficult technically: in the arithmetic Langlands program, this is where all the analysis enters. But the only way to make coherent predictions is to use the entire space, so you need both pieces.

In addition to the naïve geometric Langlands conjecture, you can formulate a naïve version of electromagnetic duality (Montonen-Olive or S-duality): that the entire theory $\mathcal{A}_G$ is equivalent to $\mathcal{B}_{G^\vee}^n$ (some naïve approximation of the actual spectral side $\mathcal{B}_{G^\vee}$). This naïve B-side theory is what we called the 4d B-model on the stack $\mathrm{pt}/G^\vee$. This would mean $\mathcal{B}_{G^\vee}^n(Y)=\mathcal{O}(\mathrm{Loc}_{G^\vee}(Y))$, where Y is a closed 3-manifold; $\mathcal{B}_{G^\vee}^n(\Sigma)=\mathcal{QC}(\mathrm{Loc}_{G^\vee}(\Sigma))$ for a closed surface Σ, and perhaps even $\mathcal{B}_{G^\vee}^n(N)=\mathcal{S}hv\mathcal{C}at(\mathrm{Loc}_{G^\vee}(N))$ for a closed 1-manifold N! Again, this isn't quite true, but it's a good first approximation to the truth. (Well, we could have done something even sillier, guessing what $\mathcal{B}_{G^\vee}$ is by using local operators instead of line operators, but the algebra of local operators in $\mathcal{A}_G$ is trivial, so there's nothing to say there.) And our approximation, built just using line operators, is surprisingly close to the actual truth. We may be able to learn even more from surface operators, though not a lot is known. Passing to the derived setting also provides insights that help fix the naïve duality: you can't just use the derived category of $\mathcal{R}ep_{G^\vee}$, but you have to use something more interesting (TODO: presumably whatever we eventually replace $\mathcal{R}ep_{G^\vee}$ comes with a t-structure whose heart is $\mathcal{R}ep_{G^\vee}$, but I should make sure this is in fact true).

Another question you could ask is, what is "the" automorphic sheaf on $\mathrm{Bun}_G(C)$ corresponding to the trivial $G^\vee$-local system E? This turns out to be the most singular/stacky point on $\mathrm{Loc}_{G^\vee}(C)$; the trivial local system is induced from the trivial $T^\vee$-local system or trivial $L^\vee$-local system for any Levi subgroup $L^\vee \subset G^\vee$, so you get a formula for the sheaf you want on the A-side, in fact one for each of these subgroups. The problem is, they don't agree.

You could try the most obvious sheaf on $\mathrm{Bun}_G(C)$, namely the constant sheaf $\underline{k}$. Then ask: is this a Hecke eigensheaf? This question is more nuanced than you might expect: it depends on what exactly you mean. You can define $V_* * \underline{k}$ as the pushforward of the pullback of $\underline{k}$ restricted to a $G(\mathcal{O})$-orbit, which gives you the cohomology of $\mathcal{H}_V \in \mathcal{S}hv(\mathrm{Gr}_G)$, tensored with $\underline{k}$. This is concerning: the cohomology is not concentrated in degree zero. So it's not quite a Hecke eigensheaf, though it would be OK if you allowed eigenvalues to be graded vector spaces. These are called *Arthur eigensheaves*: they have nontrivial Arthur parameters, meaning the cohomological grading (or Lefschetz SL_2-action) is interesting. So this isn't terrible, but it is the wrong object if you were looking for the skyscraper at the trivial local system. This is the kind of thing we're missing from our naïve approximation to $\mathcal{B}_{G^\vee}$.

For example, the $\mathcal{A}_G$ theory has interesting line operators given by 't Hooft line operators, and these should be exchanged with Wilson line operators (the easier-to-define ones) in $\mathcal{B}_{G^\vee}$. These 't Hooft operators measure singularities (e.g. monopoles) around a point. In some sense, you might want to build the B-side out of the A-side, from the Langlands perspective, but the physics perspective says that there are two twists, A and B, from the beginning, with the same line operators that behave differently on the two sides (this is part of the S-duality conjecture). On the level of a Riemann surface, all this is saying is that the equivalence of sheaves on $\mathrm{Bun}_G(C)$ and sheaves on $\mathrm{Loc}_{G^\vee}(C)$ should be compatible with the geometric Satake identification of $\mathcal{S}ph$ with $\mathcal{R}ep_{G^\vee}$, and these two symmetric monoidal categories acting on $\mathcal{S}hv(\mathrm{Bun}_G(C))$ and on $\mathcal{QC}(\mathrm{Loc}_{G^\vee})$.

Next, we'll discuss decategorification and taking traces, which will allow us to relate this to the classical Satake theorem. The plan is to meet at 1:30 pm next Thursday, though see Slack for details. After that we can talk about the derived version. Maybe we can even get to surface operators (local geometric Langlands) and boundary conditions after that.

Lecture 30.

Categorification: 5/27/21

Today we're going to talk about categorification, which is not actually a precise, well-defined notion: it is a collection of ideas which are inverse to decategorification, which is well-defined.

Given a number, you might ask when it comes from something with more structure. For example, is your number, the quantity you calculated, in fact the dimension of a vector space? Of course, we know which numbers are dimensions, but we're looking for a vector space which has some interesting meaning in context. The only options are positive integers — or we could take a graded dimension or Euler characteristic of a $\mathbb{Z}$-graded vector space, allowing negative integers. This process of getting numbers out in this way is decategorification, and going backwards is an example of categorification.

The purpose of categorifying is to add structure that is useful. For example, if your vector space has an operator U acting on it, we can decategorify and take the trace, obtaining a number. A priori this number has no relationship to the number which is the dimension of the space, but once we've categorified, there is a relationship. Likewise, describing a number as a sum of other numbers is not terribly special, but describing a vector space as a direct sum of vector spaces is quite useful. A similar thing you might be interested in is categorifying characters of group rings to actual representations.

Categorification can sometimes indicate shadows of deeper structure: if a number is unexpectedly a positive integer, perhaps it's the dimension of some vector space that's hanging around.

It is helpful to do this again: categorify vector spaces into categories (often with extra structure). To make sense of this we need a notion of decategorification, and we can use the Grothendieck group K_0 to do so. For example, if you have the algebra $\mathbb{C}[G/\!\!/G]$ of class functions, you might want to categorify it into the category of its representations, and $\mathbb{C}[G/\!\!/G] \cong K_0(\mathcal{R}ep_G)$. If you can realize a vector space as K_0 of a category, you can obtain extra structure, perhaps including a canonical basis from some notion of simple or projective objects in the category. One common example of this is categorifying vector spaces of functions into categories of sheaves.

Topological field theory has a lot to say about categorification, and vice versa. Given a TFT Z, there is another TFT Z' defined by *dimensionally reducing on* S^1: $Z'(X) := Z(S^1 \times X)$. The idea is that this process is a general form of decategorification: $Z'(X)$ is a decategorification of $Z(X)$. For example, if X is a closed $(n-1)$-manifold, $Z'(X)$ is, as a complex number, the dimension of $Z(X)$; in particular, it is a positive integer.

More generally, if $\mathcal{C}$ is a symmetric monoidal category and $Z\colon \mathcal{B}ord_n \to \mathcal{C}$ is a TFT, then we can ask about the relationship between $Z(M)$, an object of $\mathcal{C}$, and $Z(S^1 \times M)$, an element of $\mathrm{End}_{\mathcal{C}}(1)$. The formula is the same: decompose S^1 into two intervals, coevaluation followed by evaluation.

Definition 30.1. Let $\mathcal{C}$ be a symmetric monoidal category and $V \in \mathcal{C}$. Then V is *dualizable* if there exists an object $V^\vee \in \mathcal{C}$ and maps $c\colon 1 \to V \otimes V^\vee$ and $e\colon V^\vee \otimes V \to 1$, called *coevaluation* and *evaluation*, respectively, such that the *mark of Zorro axiom* is satified: TODO diagram.

If V is dualizable, then it has a *dimension*, which is defined to be $e \circ c \in \mathrm{End}_{\mathcal{C}}(1)$ (not necessarily a number!). This makes sense in any symmetric monoidal category.

Example 30.2. For $\mathcal{C} = \mathcal{V}ect_k$, $V^\vee$ is the usual dual: maps into k. Evaluation is the natural pairing $V^\vee \otimes V \to k$ given by $(\varphi, v) \mapsto \varphi(v)$. Coevaluation uses the canonical identification $V \otimes V^\vee \cong \mathrm{End}(V)$ and sends $1 \mapsto \mathrm{id}_V$. This crucially uses finite-dimensionality: that you can write the identity matrix in terms of a basis and dual basis. In particular, $\dim V = \mathrm{tr}(\mathrm{id}_V)$: under the identification $\mathrm{End}(V) \cong V \otimes V^\vee$, evaluation $\mathrm{End}(V) \to k$ is the trace. ◂

Motivated by this, we define a general trace map from $V \otimes V^\vee = \mathrm{End}(V) \to 1$ as evaluation.

This is a lot of category theory for thinking about the dimension of a vector space, but the point is that we learn quite a bit about the possible values in a TFT, as well as how decategorification arises.

Example 30.3. Let $\mathcal{C}$ be the symmetric monoidal category of chain complexes over k. Then the notion of dimension in $\mathcal{C}$ coincides with the Euler characteristic. ◂

Example 30.4. Let's soup this up a bit: k-algebras belong to a symmetric monoidal 2-category $\mathcal{A}lg_k$ whose objects are k-algebras, whose morphisms are bimodules, and whose 2-morphisms are bimodule homomorphisms. Composition of bimodules is tensor product. What is dimension in this setting?

First of all, every k-algebra A is dualizable: the dual is A^{op}. We need evaluation and coevaluation maps, which are $(k, A \otimes A^{\mathrm{op}})$- and $(A \otimes A^{\mathrm{op}}, k)$-bimodules, respectively. These are both equivalent to the data of a left and a right A-action, and there's a natural choice of such a bimodule, namely A itself. You can check this satisfies the mark of Zorro condition, so it makes sense to ask about the dimension of A in this sense. This will be an element of $\mathrm{End}_{\mathcal{A}lg_k}(k)$, the category of (k,k)-bimodules (i.e. vector spaces). Specifically, coevaluating then evaluating, we get

$$A \otimes_{A \otimes_k A^{\mathrm{op}}} A, \tag{30.5}$$

which is known as the *Hochschild homology* $HH_*(A)$. There is a natural map $\tau\colon A \to HH_*(A)$ given by sending $a \mapsto a \otimes 1$, and $\tau(ab) = \tau(ba)$, so τ is a trace. In fact, Hochschild homology has a universal property that this is the universal trace map: any map $A \to B$ which is a trace map (it factors through the coequalizer of $a, b \mapsto ab$ and $a, b \mapsto ba$) factors through$HH_*(A)$ and τ.

If you think about this in a nonderived setting, you just see HH_0, which is the *cocenter* of A: the center is the universal thing mapping in for which $ab = ba$, and the cocenter is the natural thing mapping out.

So HH_* is a form of decategorification. Really, we're decategorifying $\mathcal{M}od_A$: there is an embedding of 2-categories $\mathcal{A}lg_k \to \mathcal{C}at_k$, sending $A \mapsto \mathcal{M}od_A$. So for categories of modules, we have a notion of dimension (and in fact, it is possible to directly define Hochschild homology of dualizable categories in $\mathcal{C}at_k$). But categories of modules are nice, so it's good to have this perspective. ◂

We now have said two different ways to decategorify a category: K-theory K_* and Hochschild homology HH_*. The latter is weaker, yet much easier to calculate. K-theory, especially higher K-theory, is not a linear object and can be very difficult to calculate. In the kinds of categories that one encounters in representation theory, $HH_*(\mathcal{C}) \cong k \otimes_{\mathbb{Z}} K_*(\mathcal{C})$; for example, this is true (TODO: at least for K_0) for the category of representations of a finite group.

Another important reason to like HH_*, even though it's weaker than K-theory, is that Hochschild homology has a home in field theory, and K-theory doesn't.

One advantage of working with ∞-categories is that we can always forget noninvertible morphisms, so keeping track of the category number is less important: we can do some of the same things for defining the dimension of an object in any category level. This is not at all true in the underived setting, where nonassociativity of composition can be a such bigger problem. This is also useful for generalizing some of the story from vector spaces: if the theory is set up correctly, $A \otimes_k A^{op} \simeq \mathrm{End}(\mathcal{M}od_A)$. Therefore the evaluation map from endofunctors to k deserves to be called trace; it spits out a vector space, and $\mathrm{tr}(\mathrm{id}_{\mathcal{M}od} A) = HH_*(A)$. Classically, and more concretely, this means that to any (A, A)-bimodule M, we attach a trace

$$\mathrm{tr}(M) := M \otimes_{A \otimes A^{\mathrm{op}}} A, \tag{30.6}$$

which is conventionally called $HH_*(A, M)$; then $\dim A = HH_*(A, A) = \mathrm{tr}(\mathrm{id}_{\mathcal{M}od_A})$.

Returning to field theory, we defined the notion of dimension purely abstractly, just using duality data. This data is preserved by symmetric monoidal functors, just by how it's defined, and therefore $Z(\dim(X)) = \dim(Z(X))$. This is guaranteed to exist, because all objects in $\mathcal{B}ord_n$ are dualizable! The dual is the manifold with opposite orientation, and the evaluation and coevaluation maps are cylinders, with boundary regarded as purely incoming or purely outgoing, to get maps between $\varnothing$ and $M \amalg (-M)$. The dimension of M in $\mathcal{B}ord_n$ is $S^1 \times M$. You can do this in any category level, as long as M isn't top dimension (so that we can build $M \times S^1$ out of bordisms).

In many cases, our field theory Z factors through a functor taking the space of fields, valued in a category of correspondences of spaces, then linearizing to the target $\mathcal{C}$ by taking functions, sheaves, etc. For example, we sent M to $\mathrm{Loc}_{G^\vee}(M)$ to $\mathcal{QC}(\mathrm{Loc}_{G^\vee}(M))$. As both of these functors are symmetric monoidal, we can ask what this looks like for dimensions.

But before that, let's talk about traces in $\mathcal{B}ord_n$. You take traces of self-maps, i.e. bordisms $M \to M$. For example, you can get these by taking mapping cylinders of diffeomorphisms. In any case, given a bordism $X \colon M \to M$, we can identify the two boundary copies of M and obtain a closed n-manifold; this is the trace of X. If X is the mapping cylinder of a diffeomorphism, its trace is the mapping torus.

In the world of correspondences, we also have traces, which are sometimes called nonlinear spaces. Let $X \leftarrow Z \to Y$ be some diagram of geometric objects (space? variety? stack? objects of a topos? the precise details don't matter right now). Any X in this correspondence category is not just dualizable, but is self-dual: that is, we have a specified isomorphism $X \overset{\cong}{\to} X^\vee$. This is because the diagonal gives us a correspondence $\mathrm{pt} \leftarrow X \overset{\Delta}{\to} X \times X$, which is coevaluation, and evaluation is analogous.

So X is dualizable; what is its dimension? This is a self-correspondence of a point. Composition of correspondences is a pullback, so we end up with

$$\begin{tikzcd} & & X \times_{X \times X} X \arrow[dl] \arrow[dr] & & \\ & X \arrow[dl] \arrow[dr, "\Delta"] & & X \arrow[dl, "\Delta"'] \arrow[dr] & \\ \mathrm{pt} & & X \times X & & \mathrm{pt}. \end{tikzcd} \tag{30.7}$$

The formula for the pullback looks a lot like the formula for Hochschild homology, which is no coincidence; we don't have to worry about $(–)^{\mathrm{op}}$ because X is self-dual. What we have here, interpreted in a derived-geometric setting, is the intersection of the diagonal with itself in $X \times X$. This is called the *loop object* or *inertia object* $\mathcal{L}X$.

Why should this be a loop space? If X is the space of fields on M, often of the form $\mathrm{Map}(M, ??)$, $\dim X$ is the space of fields on $S^1 \times M$, i.e. $\mathrm{Map}(S^1 \times M, ??)$, which is loops on $\mathrm{Map}(M, ??)$.

Now, let's generalize to the trace of a map $f\colon X \to X$ in correspondences. This corresponds to a correspondence $X \overset{\mathrm{id}}{\leftarrow} X \overset{f}{\to} X$. Then the trace is $X|_{\Delta \hookrightarrow X\times X}$, which is the intersection of the graph of f with the diagonal. This makes contact with the usual notion of trace: given a finite set X and a map $f\colon X \to X$, $\mathrm{tr}(f_*\colon k[X] \to k[X])$ is the sum of the diagonal entries, i.e. the number of fixed points. This is the same reason traces appear in the Lefschetz fixed-point formula. So in our three worlds: traces in $\mathcal{B}ord_n$ are mapping tori; traces in $\mathcal{C}orr$ are fixed points; and traces in $\mathcal{C}at$ or $\mathcal{S}h$ are traces or Hochschild homology.

One way to linearize is to take sheaves, defining a symmetric monoidal functor $\mathcal{C}orr \to \mathcal{C}at$; given a correspondence $X \overset{p}{\leftarrow} Z \overset{q}{\to} Y$, the map $\mathcal{S}hv(X) \to \mathcal{S}hv(Y)$ is the push-pull $q_* p^!$. We would like to say that this commutes with taking dimensions or traces, but this requires a highly nontrivial fact that

$$\mathcal{S}hv(X \times X) \overset{?}{=} \mathcal{S}hv(X) \otimes \mathcal{S}hv(X). \tag{30.8}$$

This is so nontrivial that it is false in several important settings, and one must work around it. What we discussed in BZR seminar this past semester is related to this: people put hard work into proving the trace conjecture, rather than observing the trace tautology, in large part because of this.

The geometric Satake theorem is a categorification of the original Satake theorem. Categorification was sort of first uttered in the 1990s, with work of Khovanov, Baez, and many others, but it has antecedents going back as far as Grothendieck, as well as classical ideas such as the Lefschetz fixed-point formula. This formula says that you can count the number of fixed points of a map $f\colon X \to X$ by computing the trace of the induced map on cohomology. (This is not literally true as stated, but the point is that these are closely related, and can be made precise by sprinking in some adjectives.) The formula is the trace on H^i times $(-1)^i$ — but if you make sense of trace in the symmetric monoidal category of chain complexes, this signed graded trace is precisely what you get, so that's good: this really is the trace on cohomology.

The number of fixed points is the count $\#(\Delta \cap \Gamma(f))$ (or the dimension of $H^*(\Delta \cap \Gamma(f))$), i.e. the trace of f in a category of correspondences. So in some sense the Lefschetz theorem is a symmetric monoidality property of taking cohomology.

This was the classical Lefschetz fized-point formula. There is then the *Grothendieck-Lefschetz fixed-points formula*: let X be a variety over $\mathbb{F}_q$; then you can determine the point count of X as a trace on cohomology. Specifically,

$$\#(X(\mathbb{F}_{q^r})) = \mathrm{tr}\Big(\varphi^r|_{H_c^*(X;\overline{\mathbb{Q}}_\ell)}\Big). \tag{30.9}$$

To be precise, we take compactly supported ℓ-adic étale cohomology of the r^{th} power of the Frobenius, but the point for us is: the point count is a trace. In some sense, $H_c^*(X)$ with its Frobenius action categorifies the point count into a vector space with an endomorphism, and we decategorify by taking the trace. And compactly supported cohomology is the pushforward $\pi_!\underline{\overline{\mathbb{Q}}_\ell}$ to a point; this is the genesis of something called the function-sheaf correspondence, one of the deeper manifestations of categorification, and which came out of the Weil conjectures.

To get more into the function-sheaf correspondence, we consider functions on schemes (stacks) over $\mathbb{F}_q$: once we have this, we can take $\mathbb{F}_{q^r}$ points for any r. (Compactly supported) functions have an integration map, producing a number; and sheaves on these schemes have compactly supported cohomology, which gives a vector space with a Frobenius endomorphism, decategorifying into a number.

The function-sheaf correspondence asks for a way to fill in the last edge in the square: a trace on sheaves which produces a function, such that the integral of the function is the trace of the Frobenius on the cohomology of the sheaf we began with:

$$\mathrm{tr}(\varphi, \mathcal{F})(x) = \mathrm{tr}(\varphi, i_x^*\mathcal{F}). \tag{30.10}$$

For this to make sense, $\mathcal{F}$ has to be a Weil sheaf: it is weakly equivariant under the Frobenius φ; and $x \in X(\mathbb{F}_q)$. This is similar to the world of constructible sheaves, where we can get a number at every point by taking the Euler characteristic at every stalk. So a single sheaf gives you infinitely many functions: doing this on $\mathbb{F}_{q^r}$-points for all r. And categorification suggests that interesting functions should come from sheaves, and we should be able to get some additional interesting structure that we wouldn't have seen just at the function level. This idea really took flight in representation theory with work of Lusztig, Deligne-Lusztig, and Kazhdan-Lusztig: one slogan takeaway is that if G is a reductive group (e.g. GL_n or PSL_n), the representation theory of the finite group $G(\mathbb{F}_q)$ (this is a way to get lots of interesting finite groups) is "geometric," meaning it has a categorification: it comes from applying the trace of the Frobenius to sheaves. There are different manifestations of this idea, but it is a general principle that representation theory of these groups has a geometric origin in terms of sheaves on coset spaces. In particular, almost all finite simple groups are $G(\mathbb{F}_q)$ for some q and G — all but the alternating groups (whose representation theory we understand) and the sporadic groups. Lusztig used this to give a description of the character tables of these groups — a quite significant and impressive feat.

For example, Lusztig's character sheaves show that characters of $\mathbb{C}$-valued $G(\mathbb{F}_q)$-representations are traces of sheaves on G/G equivariant for conjugation; conjugation-equivariance implies the trace is a class function. The correspondence isn't exact, in that the simple objects aren't necessarily the same, but it's quite powerful.

Another example is Kazhdan-Lusztig theory: the finite Hecke algebra is

$$H_{G(\mathbb{F}_q),B(\mathbb{F}_q)} = \overline{\mathbb{Q}}_\ell[B(\mathbb{F}_q)\backslash G(\mathbb{F}_q)/B(\mathbb{F}_q)]. \tag{30.11}$$

They realize this as endomorphisms for $\overline{\mathbb{Q}}_\ell[G(\mathbb{F}_q), B(\mathbb{F}_q)]$, and modules for the Hecke algebra appear as (TODO: ??). The point is that we don't get all $G(\mathbb{F}_q)$-representations, but we do get all of those which come from functions on the flag variety. Kazhdan-Lusztig show that this finite Hecke algebra arises via categorification: it comes from the trace of the Frobenius on $\mathcal{S}hv(B(\mathbb{F}_q)\backslash G(\mathbb{F}_q)/B(\mathbb{F}_q))$ (the Frobenius is actually trivial here, so this is also K_0 of that category of sheaves). The additional structure on the category of sheaves, such as Verdier theory, gives among other things a natural basis for the Hecke algebra, by taking the simple objects in your category. This basis has quite nice properties, and knows quite a bit of the representation theory of the Hecke algebra. Replacing the Hecke algebra with the Hecke category is a key example of categorification, and did not come from nowhere; it is part of the story of studying the representation theory of finite groups.

The geometric Satake theorem is the statement that the Satake correspondence is geometric. Unwinding this, we had a Hecke algebra called the spherical Hecke algebra

$$\mathrm{Sph} := \overline{\mathbb{Q}}_\ell[G(\mathcal{O})\backslash G(K)/G(\mathcal{O})]; \tag{30.12}$$

the structure of this algebra doesn't depend on k mostly, except through its residue characteristic; and k is something like $\mathbb{F}_q((t))$ or a finite extension of $\mathbb{Q}_p$. The point is that $\mathrm{Sph} \cong \mathrm{End}_{G(K)} \mathrm{Fun}(G(K)/G(\mathcal{O}))$, so the spherical Hecke algebra acts on functions on $G(F)\backslash G(\mathbb{A})/G(\mathcal{O}_\mathbb{A})$.

The idea "the geometric Satake theorem means the Satake correspondence is geometric" means that the spherical Hecke algebra is the trace of the spherical Hecke category, which is the category of sheaves on $G(\mathcal{O})\backslash G(K)/G(\mathcal{O})$. Here $K = \mathbb{F}_q((t))$ and $\mathcal{O} = \mathbb{F}_q[[t]]$ — characteristic p matters here. There is a version in mixed characteristic, but it is much harder, due to recent work of Fargues-Scholze.

Now geometric Satake identifies the spherical Hecke category with $\mathcal{R}ep_{G^\vee}$. To decategorify, we take the trace of the Frobenius; the Frobenius is the identity, so this is K_0. Thus the spherical Hecke algebra is identified with the representation ring $K_0(\mathcal{R}ep_{G^\vee})$, which is the original Satake theorem. You can think of this as Weyl-group invariant functions on the torus $\mathcal{O}(G^\vee/G^\vee) = \mathcal{O}(T^\vee)^W$. Now this tells us another reason the spherical Hecke algebra is commutative.

The appearance of the Langlands dual group indicates something deep: $\mathrm{Sph} \cong K_0(\mathcal{R}ep_{G^\vee})$, where you build $G^\vee$ combinatorially from the dual root data to G. This is interesting because $K_0(\mathcal{R}ep_{G^\vee})$ does not determine $G^\vee$ in general: a group is not determined by its representation ring. On the other hand, the geometric Satake theorem does recover $G^\vee$. This is one of several reasons the categorified, geometric version is more powerful. Again Lusztig's name is here: he was one of the first to study perverse sheaves on the affine Grassmannian, which is an important part of the story of the geometric Satake correspondence.

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