FURUTA'S 10/8 THEOREM

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These notes were taken in a learning seminar on Furuta's 10/8 theorem in Spring 2019. I live-TEXed them using vim, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

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1. Introduction to Seiberg-Witten theory: 1/23/19

Riccardo gave the first, introductory talk.

In 1982, Matsumoto conjectured that if M is a closed spin manifold, $b_2(M) \geq (11/8)|\sigma(M)|$. Here $b_2(M)$ is the second Betti number and $\sigma(M)$ is the signature. Equality holds for the K3 surface, so this is the best one can do.

In this seminar we'll study a theorem of Furuta which makes major progress on this conjecture.

Theorem 1.1 (10/8 theorem [Fur01]). *If the intersection form of M is indefinite, $b_2(M) \geq (10/8)|\sigma(M)| + 2$.*

If the intersection form is definite, work of Donaldson [Don83] says that, up to a change of orientation, the intersection form is diagonalizable, so that case is dealt with.

Furuta's proof uses both Seiberg-Witten theory and equivariant homotopy theory. It can be pushed a little bit farther, but not enough to prove the $11/8^{\mathrm{ths}}$ conjecture, as shown recently by Hopkins-Lin-Shi-Xu [HLSX18].

Today we'll discuss some background for the proof.

Definition 1.2. Let $V \to M$ be a rank-n real oriented vector bundle. A *spin structure* on V is data $\mathfrak{s} = (P_{\mathrm{Spin}}(V), \tau)$, where $P_{\mathrm{Spin}}(V) \to M$ is a principal Spin_n-bundle and τ is an isomorphism

$$\tau\colon P_{\mathrm{Spin}}(V) \times_{\mathrm{Spin}_n} \mathbb{R}^n \xrightarrow{\cong} V.$$

A spin structure on a manifold M is a spin structure on TM.

Remark 1.3. There are other equivalent definitions of spin structures – for example, just as an orientation is a trivialization of V over the 1-skeleton of M, a spin structure is equivalent to a trivialization over the 2-skeleton. ◀

Here's a cool theorem about spin manifolds.

Theorem 1.4 (Rokhlin [Roh52]). *If M is a spin manifold, $\sigma(M) \equiv 0 \bmod 16$.*

The signature makes sense when $4 \mid \dim M$. Smoothness is crucial here; there are topological spin 4-manifolds, whatever that means, that do not satisfy this theorem. Freedman's E_8 manifold is an example.

Suppose M is a spin 4-manifold. The representation theory of Spin_4, in particular the fact that the spin representation S splits as $S^+ \oplus S^-$, leads to two quaternionic line bundles $\mathbb{S}^+, \mathbb{S}^- \to M$ with Hermitian metrics. Physics cares about these bundles, and will lead to powerful theorems in manifold topology.

These bundles have more structure: in particular, they are Clifford bundles.

Definition 1.5. Let $S \to M$ be a real vector bundle with a Euclidean metric $\langle\cdot,\cdot\rangle$. A *Clifford bundle* structure is data of, for each $x \in M$, the data of a Clifford algebra action $C\ell(T_xM)$ on S_x that varies smoothly in x, such that the Clifford action is skew-adjoint, meaning

$$\langle v \cdot s_1, s_2\rangle = -\langle s_1, v \cdot s_2\rangle.$$

We also require the existence of a connection which is compatible with the Levi-Civita connection on TM.

Given the data of a Clifford bundle, there's an operator called the *Dirac operator* D, which is the following composition:

$$C^\infty(S) \xrightarrow{\nabla^{C\ell}} C^\infty(T^*M \otimes S) \xrightarrow{\langle\cdot,\cdot\rangle} C^\infty(TM \otimes S) \xrightarrow{\text{Clifford action}} C^\infty(S). \tag{1.6}$$

This operator is denoted $\not\partial$, a convention due to Feynman. It is a first-order, elliptic differential operator; ellipticity means that its analysis is nice.

Thus we can consider the *Seiberg-Witten equations* on a spin 4-manifold. Let $(a,\varphi) \in \Omega^1_M(i\mathbb{R}) \times \Gamma(\mathbb{S}^+)$; then the equations are

$$\not\partial\varphi + \rho(a)(\varphi) = 0 \tag{1.7a}$$

$$\rho(\mathrm{d}^+a) - \varphi\otimes\varphi^* + \frac{1}{2}|\varphi^2|\mathrm{id} = 0 \tag{1.7b}$$

$$\mathrm{d}^*a = 0. \tag{1.7c}$$

On a non-spin manifold, the equations are a little more complicated.

2. The monopole equations: 1/28/19

Today, Kai spoke about the monopole equations and some of their important properties, foreshadowing compactness next week. We begin with some motivation.

Recall that if M is a closed, oriented 4-manifold (in either the topological or smooth category), the intersection form $H_2(M) \times H_2(M) \to \mathbb{Z}$ is a unimodular, symmetric bilinear form.

Question 2.1. Which unimodular, symmetric bilinear forms arise as the intersection forms of smooth or topological manifolds?

For example, the intersection form of $S^2 \times S^2$ is $H := \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. The intersection form of $\mathbb{CP}^2$ is (1). There's an interesting bilinear form called the *E8 form*

$$\mathrm{E8} = \begin{pmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{pmatrix}. \tag{2.2}$$

Can this be realized as the intersection form of a smooth 4-manifold? Rokhlin's theorem tells us the answer is no, because such a manifold would have to be spin, and $16 \nmid \sigma(\mathrm{E8})$. However, Freedman found a topological manifold M_{E8} whose intersection form is E8!

The direct sum of two copies of E8 satisfies Rokhlin's theorem, and this form is realized by the topological 4-manifold $M_{\mathrm{E8}} \# M_{\mathrm{E8}}$. However, Donaldson showed this manifold is not smoothable: specifically, the intersection forms of smooth 4-manifolds can be diagonalized over $\mathbb{Z}$, and E8 cannot.

There's still more interesting example: consider the *K3 surface* $\{z_1^4 + z_2^4 + z_3^4 + z_4^4 = 0\} \subset \mathbb{CP}^3$; its intersection form is $-2\mathrm{E8} \oplus 3H$. So does it split as a connect sum of 3 copies of $S^2 \times S^2$ and two copies of $M_{\mathbb{E}8}$ (with the opposite orientation)? Freedman showed this is true topologically. Smoothly, of course, it can't hold, but we might still get something.

Question 2.3. Is there a smooth, oriented 4-manifold N such that, in the smooth category, $K3 \cong N \# S^2 \times s^2$?

This was a longstanding question.

Seiberg-Witten invariants allow us to answer questions such as this – though in this semester, we're more interested in the monopole map. In any case, let's define the Seiberg-Witten equations.

Let M be a smooth, oriented 4-manifold with b_2^+ odd and a Riemannian metric g, and let $\mathfrak{s}$ be a spin^c structure on M, which determines a *basic class* $K \in H^2(X)$, i.e. an integer cohomology class such that $K \equiv w_2(M) \bmod 2$. The spin^c structure $\mathfrak{s}$ defines for us spinor bundles $\mathbb{S}^+$ and $\mathbb{S}^-$. Let $\mathcal{A}_L$ denote the space of U_1-connections, $A \in \mathcal{A}_L$, and $\psi \in \Gamma(X, \mathbb{S}^+)$ (this is called a *spinor*). The Seiberg-Witten equations are

$$D_A\psi = 0 \tag{2.4a}$$

$$F_A^+ + i\delta = i\sigma(\psi). \tag{2.4b}$$

These equations have a gauge symmetry: if G denotes the group $\mathrm{Map}(X, S^1)$ with pointwise multiplication, G acts on $\mathcal{A}_L \times \Gamma(X, \mathbb{S}^+)$ on the first factor. Let B_K^+ denote the quotient minus the locus of spinors which are identically zero; then $B_K^+ \simeq \mathbb{CP}^\infty$, so we know its cohomology is isomorphic to $\mathbb{Z}[x]$, with $|x| = 2$.

Let $\mathcal{M}_K^\delta(g) \subset B_K^\times$ denote the space of solutions to the Seiberg-Witten equations. This space has dimension

$$d := \frac{1}{4}\big(K^2 - (3\sigma(M) + 2\chi(M))\big), \tag{2.5}$$

and, crucially, defines a class $[\mathcal{M}_K^\delta(g)] \in H_d(B_K^\times)$ which does not depend on g for generic choices of the metric. The *Seiberg-Witten invariants* are

$$SW_X(K) := \langle x^{d/2}, [\mathcal{M}_K^\delta(g)]\rangle \in \mathbb{Z}. \tag{2.6}$$

The fact that $b_2^+(M) = 0$ implies d is even.

This defines a map SW from the basic classes to $\mathbb{Z}$. Taubes showed two important results.

Theorem 2.7 (Vanishing theorem (Taubes))**.** *If M is diffeomorphic to a connect sum of two closed, oriented 4-manifolds $X_1 \# X_2$, $b_2^+(X_1) > 0$, and $b_2^+(X_2) > 0$, then the Seiberg-Witten equations of M vanish.*

Theorem 2.8 (Nonvanishing theorem (Taubes))**.** *If $\mathfrak{s}$ is the canonical spin^c structure associated to a complex structure on M and $b_2^+(M)$ is positive and off, then $SW(\pm c_1(M)) = \pm 1$.*

Corollary 2.9. *K3 cannot split smoothly as a connect sum.*

This leads to an interesting generalization: there are *exotic K3 surfaces*, homeomorphic but not diffeomorphic to the standard K3. They don't all admit complex structures, and many of them are not symplectic. Nonetheless, they also don't split off an $S^2 \times S^2$: this is a consequence of Furuta's 10/8 theorem, because if $K3 \cong N \# (S^2 \times S^2)$, then $b_2(N) = 20$ and $\sigma(N) = -16$, but

$$20 \not\geq \frac{10}{8}|-16| + 2. \tag{2.10}$$

Now let's discuss the monopole map. We now assume M is a spin manifold, with spin structure $\mathfrak{s}$ and spinor bundles $\mathbb{S}^\pm$. Let A denote a spin connection and consider the spaces

$$\widetilde{\mathcal{A}} := \{A + i\ker \mathrm{d}\} \times (\Gamma(\mathbb{S}^+) \oplus \Omega^1(X)) \tag{2.11}$$

$$\widetilde{C} := \{A + i\ker \mathrm{d}\} \times \big(\Gamma(\mathbb{S}^-) \oplus \Omega^0(X) \oplus H^1(X;\mathbb{R}) \oplus \Omega^+(X)\big). \tag{2.12}$$

Both of these fiber over $H^1(X;\mathbb{R})$: for $\widetilde{\mathcal{A}}$, $A + \alpha \mapsto [\alpha]$, and there is a map $\widetilde{\mu}\colon \widetilde{\mathcal{A}} \to C$ defined by

$$(A, \phi, a) \longmapsto (A, D_A\phi + ia\phi, \mathrm{d}^*a, a_{\mathrm{harm}}, \mathrm{d}^+a - \sigma(\phi)). \tag{2.13}$$

Here

- D_A is the *Dirac operator* $D_A\colon \Gamma(\mathbb{S}^+) \to \Gamma(\mathbb{S}^-)$.
- $a\phi$ denotes Clifford multiplication.
- d^* is the adjoint of d, which sends k-forms to $(k-1)$-forms, and satisfies the equation

$$\mathrm{d}^* = \star\mathrm{d}\star. \tag{2.14}$$

  (This is in dimension 4; the sign convention is different in other dimensions.)
- a_{harm} is the harmonic part of a: it's a general fact that any one-form in dimension 4 splits as $a = a_{\mathrm{harm}} + \mathrm{d}^*\alpha + \mathrm{d}\beta$ for some 0-form β. A form is *harmonic* if the Laplacian $\Delta := \mathrm{dd}^* + \mathrm{d}^*\mathrm{d}$ vanishes on it.

- $\mathrm{d}^+ a$ denotes the self-dual part of $\mathrm{d}a$.
- $\sigma(\phi)$ denotes the trace form of the endomorphism $\phi \otimes \phi^* - (1/2)\|\phi\|^2 \mathrm{id}$.

Again the group G acts on $\Gamma(\mathbb{S}^{\pm})$ by pointwise multiplication, using $S^1 \cong \mathrm{U}_1 \subset \mathbb{C}$. If $u \in G$, $u\colon X \to S^1$ also acts on the space of spin^c connections by $d \mapsto udu^{-1}$. Let G act trivially on forms.

Then, the map $\widetilde{\mu}$ defined in (2.13) is G-equivariant. Let G_0 denote the maps which vanish at some specified basepoint p, and let $\mathcal{A} \coloneqq \widetilde{A}/G_0$, $C \coloneqq \widetilde{C}/G_0$, and $\mu \coloneqq \widetilde{\mu}/G_0$; thus we get a map $\mu\colon A \to C$.

Now, both A and C fiber over the Picard group

$$\mathrm{Pic}^g(X) \coloneqq H^1(X;\mathbb{R})/H^1(X;\mathbb{Z}) = H^1(X;\mathbb{R})/G_0. \tag{2.15}$$

Then $S^1 = G/G_0$ acts on $\mu^{-1}(A,0,0,0,0)$, and this is the space we're interested in.

We would like to study this space, and to do so we'll need to consider Sobolev spaces. For a fixed integer $k > 2$, let A_k be the fiberwise completion of A within L^2_k and C_{k-1} be the fiberwise completion of C within L^2_{k-1}. Then, the monopole map μ is a map $A_k \to C_{k-1}$.

Claim 2.16. This monopole map μ is S^1-equivariant, and is a compact perturbation of a linear Fredholm map.

The S^1-equivariance involves chasing through the definition but isn't bad; the rest is harder. What we can do is start by listing the terms that define a linear Fredholm map, and then check that the rest is compact. In the definition of $\widetilde{\mu}$, the terms A, $D_A\phi$, $\mathrm{d}^* a$, a_{harm}, and $\mathrm{d}^+ a$ are linear and Fredholm; thus we just have to check that $a(\phi)$ and $\sigma(\phi)$ are compact. For the first, we can use the fact that Clifford multiplication is compact, then compose with the map $C_k \to C_{k-1}$, which is also compact.

Proposition 2.17. *Let $T = \ell + c$ be a compact perturbation of a linear Fredholm map ℓ between Hilbert spaces. The restriction of T to any closed, bounded subset Ω is proper.*

Proof. Let p denote projection onto $\ker(\ell)$ and consider the commutative diagram

$$\begin{tikzcd} \Omega \arrow[r, "{(\ell,c,p)}"] \arrow[rrr, bend right, "\ell+c"'] & M \times \overline{c(\Omega)} \times \overline{p(\Omega)} \arrow[r, "{(u,s,e)\mapsto(u+a,s,e)}", "\cong"'] & M \times \overline{c(A)} \times \overline{p(A)} \arrow[r, "\mathrm{proj}"] & M. \end{tikzcd} \tag{2.18}$$

Because the map (ℓ, c, p) is injective, TODO.

⊠

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