SUMMER 2016 ALGEBRAIC GEOMETRY SEMINAR

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Contents

1. Separability, Varieties and Rational Maps: 5/16/16

Today's lecture was given by Tom Oldfield, on the first half of chapter 10.

This seminar has a website, located at
https://www.ma.utexas.edu/users/toldfield/Seminars/Algebraicgeometryreading.html.

The first half of Chapter 10 is about separated morphisms and varieties; it only took us 10 chapters! Vakil writes that he was very conflicted about leaving a proper treatment of algebraic varieties, a cornerstone of classical algebraic geometry, to so late in the notes. But from a modern perspective, our hands are tied: varieties are defined in terms of properties, which means building those properties out of other properties and out of the large amount of technology you need for modern algebraic geometry. With that technology out of the way, here we are.

One of these properties is separability. Let $\pi : X \to Y$ be a morphism of schemes; then, the **diagonal** is the induced morphism $\delta_\pi : X \to X \times_Y X$ defined by $x \mapsto (x, x)$; this maps into the fiber product because it fits into the diagram

$$\begin{tikzcd}
X \arrow[dr, "\delta_\pi"] \arrow[drr, bend left, "1_X"] \arrow[ddr, bend right, "1_X"'] & & \\
& X \times_Y X \arrow[r, "p_2"'] \arrow[d, "p_1"] & X \arrow[d, "\pi"'] \\
& X \arrow[r, "\pi"] & Y.
\end{tikzcd} \tag{1.1}$$

Here, p_1 nad p_2 are the projections onto the first and second components, respectively, and 1_X is the identity map on X.

The diagonal has a few nice properties. Suppose $V \subset Y$ is open, and $U, U' \subset \pi^{-1}(V)$ are open subsets of X. Then, $U \times_V U' = p_1^{-1}(U) \cap p_2^{-1}(U')$: we constructed fiber products such that they send open embeddings to intersections. In particular, if $U \cong \operatorname{Spec} A$, $U' \cong \operatorname{Spec} A'$, and $V \cong \operatorname{Spec} B$ are affine, $U \times_V U' \cong \operatorname{Spec}(A \otimes_B A')$. Therefore $\delta_\pi^{-1}(U \times_V U') = \delta_\pi^{-1}(p_1^{-1}(U) \cap p_2^{-1}(U')) = U \cap U'$. That is, the diagonal turns intersections into fiber products.

This argument feels like it takes place in Set, but goes through word-for-word for schemes.

Definition 1.2. A morphism $\pi : X \to Y$ of schemes is a **locally closed embedding** if it factors as $\pi = \pi_1 \circ \pi_2$, where π_2 is a closed embedding and π_1 is an open embedding.

Proposition 1.3. *For any $\pi : X \to Y$, δ_π is locally closed.*

Proof. Let $\{V_i\}$ be an affine open cover of Y, so $V_i \cong \operatorname{Spec} B_i$ for each B, and $\mathfrak{U}_i = \{U_{ij}\}$ be an affine open cover of $\pi^{-1}(V_i)$ for each i. Then, $\{U_{ij} \times_{V_i} U_{ij'} : i, j, j'\}$ covers $X \times_Y X$. More interestingly, $\{U_{ij} \times_{V_i} U_{ij} : i, j\}$ covers $\operatorname{Im}(\delta_\pi)$: this is because if $x \in U_{ij}$, then $\delta_\pi(x) \in p_1^{-1}(U_{ij})$ and in $p_2^{-1}(U_{ij})$, and $p_1^{-1}(U_{ij}) \cap p_2^{-1}(U_{ij}) = U_{ij} \times_{V_i} U_{ij}$.

Now, it suffices to show that $\delta_\pi : \delta_\pi^{-1}(U_{ij} \times_{V_i} U_{ij}) \to U_{ij} \times_{V_i} U_{ij}$ is closed, since the property of being a closed embedding is affine-local. Since each $U_{ij} \cong \operatorname{Spec} A_{ij}$ is affine, then it suffices to understand what's happening ring-theoretically: the diagonal map corresponds to the ring morphism $A_{ij} \otimes_{V_i} A_{ij} \to A_{ij}$ sending $a \otimes a' \mapsto aa'$. This is clearly surjective, which is exactly the criterion for a morphism of schemes to be a closed embedding. ☐

Corollary 1.4. *If X and Y are affine schemes, then δ_π is a closed embedding.*

Corollary 1.5. *If Δ denotes $\operatorname{Im}(\delta_\pi)$, then for any open $V \subset Y$ and $U \subset \pi^{-1}(V)$, $\Delta \cap (U \times_V U') \cong U \cap U'$ is a homeomorphism of topological spaces.*

This follows because a locally closed embedding is homeomorphic onto its image.

These will all be super useful once we define separability, which we'll do now.

Definition 1.6. A morphism $\pi : X \to Y$ is **separated** if $\delta_\pi : X \to X \times_Y X$ is a closed embedding.

This is weird upon first glance: why do we look at the diagonal to understand things about a morphism? The answer is that the diagonal has nice category-theoretic properties, so we can prove some useful properties by doing a few diagram chases.

More geometrically, separability corresponds to the Hausdorff property in topological spaces, and there's a criterion for this in terms of the diagonal.

Proposition 1.7. *If T is a topological space, then T is Hausdorff iff the diagonal morphism $T \to T \times T$ is a closed embedding.*

Equivalently, the image $\Delta \subset T \times T$ is a closed subspace.

Remark. Since schemes are topological spaces, you might think this proves separated schemes are Hausdorff, but this is untrue: fiber products of schemes are generally not fiber products of underlying spaces, and therefore closed embeddings of schemes are not the same as closed embeddings of their underlying spaces.

Separability is a nice property, and is good to have. But like Hausdorfness, we generally won't need to use schemes that aren't separated.

Example 1.8.

(1) By Corollary 1.4, all morphisms of affine schemes are separated.
(2) If we can cover $X \times_Y X$ by the sets $U_{ij} \times_{V_i} U_{ij}$ (with these sets as in the proof of Proposition 1.3), then π is separated.
(3) For a counterexample, let $X = \mathbb{A}^1_{(0,0)}$ be the "line with two origins" over a field k. This isn't a separated scheme: the diagonal is a "line with four origins," and these cannot be separated topologically: every open set containing one contains all of them. So take one affine piece of X, which contains exactly one origin, and therefore its image ought to contain all four, but it doesn't, so $X \to \operatorname{Spec} k$ isn't closed. This might feel a little imprecise, but one can make it fully rigorous.

We want separated morphisms to be nice: we'd like them to be preserved under base change and composition, and we'd like locally closed embeddings to be separated.

Proposition 1.9. *Locally closed embeddings are separated.*

This is the only example of a hands-on proof of a property; it's not hard, but the rest will be less abstract and easier. First, though, let's reframe it:

Proposition 1.10. *Any monomorphism of schemes is separated.*[1]

[1]More is true in general; all you need is that $p_1 = p_2$ in the diagram (1.1), which is analogous to an injectivity condition on π. Hence, it suffices that π is injective as a map of sets, but this is a weird notion for schemes, so we generally phrase it in terms of monomorphisms.

Proof. By point (2) of Example 1.8, it suffices to prove that fiber products $U_{ij} \times_{V_i} U_{ij}$ cover $X \times_Y X$ for our affine covers. So let's look at the fiber diagram (1.1) again; it tells us that $\pi \circ p_1 = \pi \circ p_2$. But since π is a monomorphism, then $p_1 = p_2$, so for any $z \in X \times_Y Z$, $p_1(z) = p_2(z)$; call this point x_z. Then, if $x_z \in U_{ij}$, $z \in p^{-1}(U_{ij})$ and $z \in p_2^{-1}(U_{ij})$, and their intersection is the fiber product. $\square$

Since locally closed embeddings are monomorphisms, Proposition 1.9 follows as a corollary.

At this point, we can define varieties, and Vakil does so, but can't do anything with them, so we'll come back to them in a little bit.

Proposition 1.11. *If A is a ring, $\mathbb{P}^n_A \to \operatorname{Spec} A$ is separated.*

The idea of the proof is to compute: we already know a cover of $\mathbb{P}^n_A$ by $n+1$ affine schemes, and can check that the induced map on rings is surjective.

The following proposition gives us an important geometric property of separability.

Proposition 1.12. *If A is a ring and $X \to \operatorname{Spec} A$ is separated, then for any affine open subsets $U, V \subset X$, $U \cap V$ is also affine.*

Proof. The diagonal is a closed embedding, so $\delta : U \times V \to U \times_A V$ is also a closed embedding. Therefore $U \times V$ is isomorphic to a closed subscheme of an affine scheme, and therefore is affine. $\square$

It's surprising how useful these arguments with the diagonal are: we got a useful and nontrivial result in one line! In general, you can prove a weirdly large amount of things by factoring them through the diagonal. In fact, le'ts use it to define another property.

Definition 1.13. A morphism $\pi : X \to Y$ is **quasiseparated** if δ_π is quasicompact.

This isn't the same as the other definition we were given, that for all affine $V \subset Y$ and $U, U' \subset \pi^{-1}(V)$, $U \cap U'$ is quasicompact. But it turns out to be equivalent.

Proposition 1.14. *$\pi : X \to Y$ is quasiseparated in the sense of Definition 1.13 iff it's quasiseparated in the sense we defined previously.*

The proof is a diagram chase involving the "magic diagram" for fiber products. This states that if $X_1, X_2 \to Y \to Z$ are maps in some category and the relevant fiber products exist, the diagram

$$\begin{tikzcd} X_1 \times_Y X_2 \arrow[r] \arrow[d] & X_1 \times_Z X_2 \arrow[d] \\ Y \arrow[r, "\delta"] & Y \times_Z Y \end{tikzcd}$$

is a fiber diagram; the proof is a diagram chase following from the associativity of products, or checking the universal property. This diagram is also very ubiquitous for proofs like these.

Proposition 1.15. *Separability and quasiseparability are preserved under base change.*

Proof. Suppose $\pi : X \to Y$ is separated and $\varphi : S \to Y$ is another map of schemes, so there's an induced morphism $\pi' : Z = X \times_Y S \to S$ fitting into the diagram

$$\begin{tikzcd} Z \arrow[r, "\pi'"] \arrow[d, "p_1"] & S \arrow[d, "\varphi"] \\ X \arrow[r, "\pi"] & Y. \end{tikzcd}$$

The magic diagram for this is the fiber diagram

$$\begin{tikzcd} Z \arrow[r, "\delta_{\pi'}"] \arrow[d] & Z \times_S Z \arrow[d] \\ X \arrow[r, "\delta_\pi"] & X \times_Y X. \end{tikzcd}$$

If π is separated, δ_π is closed, and therefore $\delta_{\pi'}$ is closed (since closed embeddings are preserved under base change), so π' is separated. The same argument works with π quasiseparated and δ_π quasicompact. $\square$

There are a few related properties that we won't prove, but whose proofs are very similar to the previous one.

Proposition 1.16. *Separability and quasiseparability are*

1. *local on the target,*
2. *closed under composition, and*
3. *closed under taking products: if $\pi : X \to Y$ and $\pi' : X' \to Y'$ are separated morphisms of schemes over a scheme S, then $\pi \times \pi' : X \times_S X' \to Y \times_S Y'$ is separated; if π and π' are merely quasiseparated, so is $\pi \times \pi'$.*

Each of these is a diagram chase with the right diagram, and not a particularly hard one; the last one follows as a general categorical consequence of the others.

Now, though, we can define varieties.

Definition 1.17. Let k be a field. A **k-variety** is a k-scheme $X \to \operatorname{Spec} k$ that is reduced, separated, and of finite type. A **subvariety** of a given variety X is a reduced, locally closed subscheme.

Reducedness is a property of X, but the others are properties of the structure morphism $X \to \operatorname{Spec} k$. Notice that the affine line with doubled origin is reduced and of finite type, so separability is important for avoiding pathologies.

It's nontrivial that a subvariety $Y \subset X$ is itself a variety. X is finite type over $\operatorname{Spec} k$, so it's covered by finitely many affine opens that are schemes of finitely generated k-algebras, which are Noetherian, so X is Noetherian. Hence, $Y \hookrightarrow X$ is a finite-type morphsism into a Noetherian scheme, so Y is finite type; but we do need separability to be preserved under composition, which we just saw how to prove.

We did not require varieties to be irreducible; irreducibility doesn't behave as well as we would like, unless k is particularly nice.

Proposition 1.18. *The product of irreducible varieties over an algebraically closed field k is an irreducible k-variety.*

This follows from the nontrivial fact that if A and B are k-algebras that are integral domains, then $A \otimes_k B$ is an integral domain.

The last important thing we'll discuss today is a big meta-theorem about classes of morphisms.

Theorem 1.19 (Cancellation theorem)**.** *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "\tau"'] & & Y \arrow[dl, "\rho"] \\ & Z, & \end{tikzcd}$$

i.e. $\tau = \rho \circ \pi$, and let P be a property of morphisms preserved under base change and composition. If τ has P and δ_ρ has P, then π also has P.

The name is because we're "cancelling" ρ out of the composition.

The proof uses the notion of the graph of a morphism.

Definition 1.20. Let X and Y be schemes over a scheme S, and $\pi : X \to Y$ be a map of S-schemes. Then, the **graph** of π is the morphism $\Gamma_\pi : X \to X \times_S Y$ defined by $\Gamma_\pi; (1_X, \pi)$.

That is, this sends a point to its image on the graph. We use this because any morphism factors through its graph. Then, since δ_ρ has P, so must Γ_π, which is useful. It seems weirdly abstract and pointless, but the idea is that the nice properties of the diagonal, including locally closed embeddings, can be canceled off. In fact, if Y is separated, we can cancel off properties of closed embeddings, and if Y is quasiseparated, we can cancell off properties of quasicompact morphisms.

Rational Maps. Let's talk about rational maps, which are rational maps defined almost everywhere, and up to almost everywhere agreement. Rational maps are usually only defined on reduced varieties, since it's nearly impossible to get a hold on them otherwise; they're inherently geometric, and geometry tends to involve varieties.

Definition 1.21. A **rational map** $\pi : X \dashrightarrow Y$ is an equivalence class of morphisms $f : U \to Y$, where $U \subset X$ is a dense open subset; (f, U) and (f', U') are considered equivalent if there's a dense open set $V \subset U \cap U'$ if $f|_V = f'|_V$. One says π is **dominant** if its image is dense, or equivalently, for all nonempty opens $V \subseteq Y$, $\pi^{-1}(V) \neq \varnothing$.

Notice that dominance is well-defined, as it's independent of choice of representative.

Proposition 1.22. *Let X and Y are irreducible schemes, then $\pi : X \dashrightarrow Y$ is dominant iff the generic point of X maps to the generic point of Y.*

Proof. In the reverse direction, the generic point η_Y of Y is contained in every open subset of Y, so the preimage contains the generic point η_X of X, and in particular is nonempty.

In the other direction, suppose $\pi(\eta_X) \neq \eta_Y$; let $U = Y \setminus \overline{\pi(\eta_X)}$, which is an open subset. Thus, $\eta_X \notin \pi^{-1}(U)$, which is an open set. Since η_X is dense, it meets every nonempty open, so $\pi^{-1}(U)$ is empty, and therefore π isn't dominant. ☐

This is a pretty useful characterization of dominance. But why do we care about dominance? Because of composition.

Remark. Let $\pi : X \dashrightarrow Y$ and $\rho : Y \dashrightarrow Z$ be rational maps. If π is dominant and X is irreducible, it's possible to make sense of $\rho \circ \pi : X \dashrightarrow Z$ as a rational map, which is dominant iff ρ is.

This is nontrivial: if π isn't dominant, one might discover that the domain of ρ doesn't intersect the image of π; if they do, however, π^{-1} of the domain of definition of ρ is a nonempty open of X; since X is irreducible, it must be dense.

Definition 1.23. A rational map $\pi : X \dashrightarrow Y$ is **birational** if it's dominant and there exists a dominant $\psi : Y \dashrightarrow X$ such that as rational maps, $\pi \circ \psi \sim 1_X$ and $\psi \circ \pi \circ 1_Y$. In this case, one says π and ψ are **birational(ly equivalent)**.

Proposition 1.24. *Let X and Y be reduced schemes; then, X and Y are birational iff there exist dense open subschemes $U \subset X$ and $V \subset Y$ such that $U \cong V$.*

The idea is that we can let U and V be the domains of definition for our rational maps.

The notion of rationality is very specific to algebraic geometry; in the differentiable category, it's complete nonsense. Since any manifold can be triangulated, any two manifolds of the same dimension are birationally equivalent: remove the edges of the triangles, and you get a dense open set; clearly, any two triangles are birational. However, there exist algebraic varieties of the same dimension that aren't birationally equivalent.

Definition 1.25. A variety X over k is **rational** if it's birational to $\mathbb{A}_k^n$ for some n.

For example, $\mathbb{P}_k^n$ is rational. Rationality loses some information, but what it keeps is interesting.

Finally, let's see what dominance means in terms of ring morphisms.

Definition 1.26. Let $\varphi : \operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism of affine schemes and $\varphi^\sharp : B \to A$ be the induced map on global sections. Then, φ is dominant (i.e. as a rational map) iff $\ker(\varphi^\sharp) \subset \mathfrak{N}(A)$.

Here, $\mathfrak{N}(A)$ denotes its nilradical, the intersection of all prime ideals of A (equivalently, the ideal of nilpotent elements). That is, if A and B are reduced, dominance is equivalent to injectivity! Interestingly, this also corresponds to an inclusion of function fields, i.e. a field extension! We've reduced a geometric problem to a problem about algebra. Often, we can go in the other direction, e.g. for varieties. In this setting, birationality means isomorphism on the function fields.

2. Proper Morphisms: 5/19/16

These are Arun's lecture notes on rational maps to separated schemes and proper morphisms, corresponding to sections 10.2 and 10.3 in Vakil's notes. I'm planning on talking about the following topics:

- Rational maps to separated schemes, including the reduced-to-separated theorem and some corollaries.
- The definition of proper morphisms, and that they form a nice class of morphisms. Projective A-schemes are proper over A.

Throughout this lecture, S is a scheme, which will often be the base scheme.

Rational Maps to Separated Schemes. If X and Y are spaces and $\pi, \pi' : X \rightrightarrows Y$ are continuous, it's sometimes useful to talk about the locus where they agree, $\{x \in X : \pi(x) = \pi'(x)\}$. Categorically, this is the equalizer $\mathrm{Eq}(\pi, \pi') \hookrightarrow X$, which is characterized by the property that if $\varphi : W \to X$ is a continuous map such that $\pi \circ \varphi = \pi' \circ \varphi$, then it factors through $\mathrm{Eq}(\pi, \pi')$, i.e. there's a unique $h : W \to \mathrm{Eq}(\pi, \pi')$ such that the following diagram commutes.

$$\begin{tikzcd} W \arrow[d, dashed, "\exists!\, h"'] \arrow[dr, "\varphi"] & & \\ \mathrm{Eq}(\pi,\pi') \arrow[r, hook] & X \arrow[r, shift left, "\pi"] \arrow[r, shift right, "\pi'"'] & Y. \end{tikzcd}$$

So if we can do this for schemes, we'll have a subscheme where two morphisms agree, rather than just a set. The universal property for the equalizer is the same as for the fiber product

$$\begin{tikzcd} \mathrm{Eq}(\pi,\pi') \arrow[r] \arrow[d, hook, "i"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\delta"] \\ X \arrow[r, "(\pi,\pi')"] & Y \times Y, \end{tikzcd} \tag{2.1}$$

where δ is the diagonal morphism. We know fiber products of schemes exist, so equalizers do too.

Lemma 2.2 (Vakil ex. 10.2.A)**.** *If $\pi, \pi' : X \rightrightarrows Y$ are two morphisms of schemes over S, then $i : \mathrm{Eq}(\pi, \pi') \hookrightarrow X$ is a locally closed subscheme of X. If Y is separated over S, $\mathrm{Eq}(\pi, \pi')$ is a closed subscheme.*

Proof. Since we're over S, the product in (2.1) should be replaced with $Y \times_S Y$, the product in Sch_S. Since δ is a locally closed embedding, and this is a property preserved under base change, then i is too. If $Y \to S$ is separated, then δ is a closed embedding, and this is also preserved by pullbacks. $\square$

Remark. The locus where two maps agree does not need to be reduced, e.g. if $\pi, \pi' : \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ are defined by $\pi(x) = 0$ and $\pi'(x) = x^2$, then they agree "to first order" at 0, and $\mathrm{Eq}(\pi, \pi') = \operatorname{Spec} \mathbb{C}[x]/(x^2)$.

The central result about these is the reduced-to-separated theorem.

Theorem 2.3 (Reduced-to-separated theorem (Vakil Thm. 10.2.2))**.** *Let $\pi, \pi' : X \rightrightarrows Y$ be two morphisms of S-schemes. If X is reduced, Y is separated over S, and π and π' agree on a dense open subset, then $\pi = \pi'$.*

This is equality in the sense of morphisms of schemes, which is stronger than pointwise equality.

Proof. By Lemma 2.2, $\mathrm{Eq}(\pi, \pi') \hookrightarrow X$ is a closed subscheme, but it contains a dense open set. Since X is reduced, its only closed subscheme containing a dense open set is itself. $\square$

Corollary 2.4. *If X is reduced, Y is separated, and $\pi : X \dashrightarrow Y$ is a rational map, then there is a maximal $U \subset X$ such that $\pi|_U : U \to Y$ is an honest morphism. In particular, this is true for rational functions on reduced schemes.*

This U is called the **domain of definition** of π; its complement is sometimes called the **locus of indeterminacy**.

Proof. We can choose U to be the union of all domains of representatives of π. If $f_1 : V_1 \to Y$ and $f_2 : V_2 \to Y$ are two morphisms representing π, then f_1 and f_2 agree on a dense open subset of $V_1 \cap V_2$, so by the reduced-to-separated theorem agree on all of $V_1 \cap V_2$. Thus, we can glue representing morphisms on their intersection and therefore define π on all of U. $\square$

Next, we need to digress slightly to understand the image of a locally closed embedding. This is from section 8.3 of the notes.

If $\pi : X \to Y$ is a morphism of schemes, it's in particular a continuous function, so its image $\pi(X) \subset Y$ is a subspace. This will be referred to as the **set-theoretic image**. As usual, the topological version of a thing tends to be less well-behaved than the scheme-theoretic one, so we'll define an image of π that's a subscheme of Y. Schemes are locally cut out by equations, so it seems reasonable to say that a closed subscheme $i : Z \hookrightarrow Y$ **contains the image** of π if functions in $\mathscr{O}_Y$ that vanish on Z also vanish when pulled back to X. That is, the composition $\mathscr{I}_{Z/Y} \to \mathscr{O}_Y \to \pi_* \mathscr{O}_X$ is zero, where $\mathscr{I}_{Z/Y} = \ker(i^\sharp : \mathscr{O}_Y \to i_* \mathscr{O}_Z)$ is the sheaf of ideals associated to the closed embedding of Z into Y.

Definition 2.5. The **scheme-theoretic image** $\operatorname{Im}(\pi)$ of π is the intersection of all closed subschemes containing the image of π.[2] If π is a locally closed embedding, $\operatorname{Im}(\pi)$ is also called the **scheme-theoretic closure** of π.

That is, $\operatorname{Im}(\pi)$ is the smallest closed subscheme of Y such that locally vanishing on $\operatorname{Im}(\pi)$ implies locally vanishing when pulled back to X.

Theorem 2.6 (Vakil cor. 8.3.5)**.** *Let $\pi : X \to Y$ be a morphism of schemes. If X is reduced or Y is quasicompact, the closure of the set-theoretic image of π is the underlying set of* $\operatorname{Im}(\pi)$*.*

We lack the time to prove this, but it follows from the defining properties of closed embeddings.

Just like we defined the graph of a morphism of S-schemes $\pi : X \to Y$ to be $\Gamma_\pi = (\mathrm{id}, \pi) : X \to X \times_S Y$, we can define the graph of a rational map in nice situations.

Definition 2.7. Let $\pi : X \dashrightarrow Y$ be a rational map over S, where X is reduced and Y is separated over S. For any representative morphism $f : U \to Y$ of π, the **graph of the rational map** π, denoted Γ_π, is the scheme-theoretic closure of the map $\Gamma_f \hookrightarrow U \times_S Y \hookrightarrow X \times_S Y$. (The first map is a closed embedding, and the second is an open embedding.)

The following diagram might make this definition clearer.

$$\begin{tikzcd}
\Gamma_\pi \arrow[r, hook, "\text{cl.}"] & X \times_S Y \arrow[dl] \arrow[dr] & \\
X \arrow[u, dashed] \arrow[rr, dashed, "\pi"] & & Y.
\end{tikzcd}$$

A priori this definition depends on the choice of representative, but fortunately, this isn't actually the case.

Proposition 2.8 (Vakil ex. 10.2.E)**.** *The graph of a rational map π is independent of choice of representative.*

Proof. Let $\xi' : U \to Y$ and $\xi : V \to Y$ be two representatives of π. Without loss of generality, we can assume V is the maximal domain of definition for π, so $U \subset V$ and $\xi' = \xi|_U$. Thus, we have a bunch of embeddings fitting into the diagram

$$\begin{tikzcd}
\Gamma_{\xi'} \arrow[r, hook, "\text{cl.}"] \arrow[d, hook] & U \times Y \arrow[r, hook, "\text{op.}"] \arrow[d, hook, "\text{op.}"] & X \times Y \\
\Gamma_\xi \arrow[r, hook, "\text{cl.}"] & V \times Y. \arrow[ur, hook, "\text{op.}"'] &
\end{tikzcd}$$

Thus, $\Gamma_{\xi'}$ factors as a subset of a closed subset of $V \times Y$, so its scheme-theoretic closure, which is just the closure of its underlying set by Theorem 2.6, must factor through this. In particular, the graph of π as defined with respect to ξ' embeds into $V \times Y$. Thus, we can assume $V = X$, since everything takes place inside V. In this case, Γ_π as defined by ξ is just Γ_ξ, and $\Gamma_\xi \cong X$ by projection onto the first factor. This projection restricts to an isomorphism $\Gamma_{\xi'} \cong U$, and carries the embedding $\Gamma_\xi \hookrightarrow \Gamma_{\xi'}$ to the embedding $U \hookrightarrow X$. Finally, to form the graph of π with respect to ξ', we take the closure, and since U is a dense open subset, we get X, or all of Γ_ξ. ☐

Finally, we discuss one application to effective Cartier divisors. (This is actually an excuse to introduce effective Cartier divisors, since they show up again and again.)

Definition 2.9. A closed embedding $\pi : X \hookrightarrow Y$ is an **effective Cartier divisor** if $\mathscr{I}_{X/Y}$ is locally generated by a single non-zerodivisor. That is, there's an affine open cover $\mathfrak{U}$ of Y such that for each $U_i = \operatorname{Spec} A_i \in \mathfrak{U}$, there's a $t_i \in A$ that is not a zerodivisor and such that $\mathscr{I}_{X/Y}(U) = A_i/(t_i)$.

Proposition 2.10 (Vakil ex. 10.2.G)**.** *Let X be a reduced S-scheme and Y be a separated S-scheme. If $i : D \hookrightarrow X$ is an effective Cartier divisor, there is at most one way to extend an S-morphism $\pi : X \setminus D \to Y$ to all of X.*

[2] There's something to prove here, that containing the image of π is well-behaved under intersections.

Proof. This is true if we know it on an affine cover, so without loss of generality assume $X = \operatorname{Spec} A$ is affine and $D = V(t)$ for some $t \in A$ that isn't a zerodivisor. If $D(t) = X \setminus D$ is dense in X, then we're done by Theorem 2.3. Since X is reduced, then by Theorem 2.6 this is equivalent to the scheme-theoretic closure of $D(t)$ being all of X. Given a closed subscheme $Z \hookrightarrow X$, we want to understand when functions vanishing on Z pull back to the zero function on $D(t)$. The map $\Gamma(X, \mathscr{O}_X) \to \Gamma(D(t), \mathscr{O}_X)$ is also $A \to A_t$; since t isn't a zerodivisor, this is injective, so a function pulls back to 0 on $D(t)$ iff it vanishes on all of X. Hence, $\operatorname{Im}(D(t) \hookrightarrow X) = X$ as desired. ☐

Proper Morphisms. The next topological notion we introduce to algebraic geometry is that of a proper map. Recall that a continuous map of topological spaces is proper if the preimage of any compact set is compact. Compactness doesn't really behave the same way in algebraic geometry, so we'll have to define properness in a different way, which will satisfy similar properties.

Proper maps are closed maps, meaning the image of a closed set is closed. This would be a reasonable starting point, except that closed maps are not preserved by fiber products. It turns out the right way to fix this is just to pick the ones that behave well.

Definition 2.11. A morphism $\pi : X \to Y$ of schemes is **universally closed** if for all morphisms $Z \to Y$, the pullback $Z \times_Y X \to Z$ is a closed map.

That is, it remains closed under arbitrary base change.

Lemma 2.12. *Universal closure is a "nice" property of schemes, i.e. local on the target, closed under composition, and preserved by base change.*

Proof. Clearly, universal closure is closed under composition, and by definition, it's preserved by fiber products. Being a closed map is local on the target, and therefore so is universal closure. ☐

We use universal closure to define the property we really care about.

Definition 2.13. A morphism $\pi : X \to Y$ is **proper** if it's separated, finite type, and universally closed. If A is a ring, an A-scheme X is said to be **proper over** A if the structure morphism $X \to \operatorname{Spec} A$ is proper.

Example 2.14. Closed embeddings are our first example of proper morphisms: they're affine, and therefore separated. Closed embeddings are closed maps, and since the pullback of a closed embedding is a closed embedding, a closed embedding is universally closed. Finally, closed morphisms are finite type (which boils down the fact that if $B \twoheadrightarrow A$ is a surjective ring map, A is a finitely generated B-algebra).[3]

This agrees with our intuition for topological spaces, which is good.

Proposition 2.15 (Vakil prop. 10.3.4)**.**

(1) *Properness is a "nice" property of schemes (in the sense of Lemma 2.12).*
(2) *Properness is closed under products: if $\pi : X \to Y$ and $\pi' : X' \to Y'$ are proper morphisms of S-schemes, then $\pi \times \pi' : X \times_S X' \to Y \times_S Y'$ is proper.*
(3) *Given a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "\tau"'] & & Y \arrow[dl, "\rho"] \\ & Z, & \end{tikzcd}$$

if τ is proper and ρ is separated, then π is proper.

For example, by (3), any morphism from a proper k-scheme to a separated k-scheme is proper (let $Z = \operatorname{Spec} k$).

Proof. Everything in this proposition comes nearly for free. We already knew finite type and separability to be nice properties of schemes, and by Lemma 2.12, so is universal closure; since properness is having all three at once, it too must be a nice property. (2) is a formal consequence of (1), which is proven for any nice class

[3]The same line of reasoning shows that finite morphisms are proper, which is a generalization: they're affine, hence separated, and closed maps; since they're preserved under base change, they must also be universally closed. Finally, finite morphisms are finite type.

of morphisms in Vakil's ex. 9.4.F. Finally, since closed embeddings are proper, the cancellation theorem from last lecture applies to prove (3). $\square$

According to Vakil, the next example is the most important example of proper morphisms.

Theorem 2.16 (Vakil thm. 10.3.5)**.** *If A is a ring and X is a projective A-scheme, $X \to \operatorname{Spec} A$ is proper.*

Proof. Since X is projective, the structure morphism factors as $X \hookrightarrow \mathbb{P}^n_A \to A$, a closed embedding followed by the structure map for $\mathbb{P}^n_A$. Since closed embeddings are proper (Example 2.14), it suffices to show $\mathbb{P}^n_A \to \operatorname{Spec} A$ is proper, because proper morphisms are closed under composition. Projective schemes are finite type, and we proved last time that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is separated, so it remains to check universal closure.

If $\varphi : X \to \operatorname{Spec} A$ is an arbitrary morphism, we would like for the map $\mathbb{P}^n_A \times_A X \to X$ to be closed. Since $\mathbb{P}^n_A = \mathbb{P}^n_{\mathbb{Z}} \times_A \operatorname{Spec} \mathbb{Z}$, then we have the following commutative diagram, in which both squares are pullback squares:

$$\begin{tikzcd}
\mathbb{P}^n_A \times_A X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\varphi"] \\
\mathbb{P}^n_A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Spec} A \arrow[d] \\
\mathbb{P}^n_{\mathbb{Z}} \arrow[r] & \operatorname{Spec} \mathbb{Z}.
\end{tikzcd}$$

By checking the universal property, we see that the outer rectangle is a pullback square too: in other words, $\mathbb{P}^n_A \times_A X = \mathbb{P}^n_X$, so it suffices to show that the structure map $\mathbb{P}^n_X \to X$ is closed for arbitrary X. Being a closed map is a local condition, so we can check on an affine cover of $\mathbb{P}^n_X$; pulling back by $\operatorname{Spec} B \hookrightarrow X$ gives us $\mathbb{P}^n_B \to \operatorname{Spec} B$, so it suffices to know that the structure map is closed for all rings B. This is precisely the fundamental theorem of elimination theory (Thm. 7.4.7 in Vakil's notes), so we're done. $\square$

Perhaps surprisingly, the converse is almost true: it's difficult to come up with examples of schemes that are proper, but not projective.

The last thing we'll prove about proper schemes is another analogue of compactness. Recall that if M is a compact, connected complex manifold, all holomorphic functions on M are constant. We'll be able to prove a scheme-theoretic analogue of this.

Proposition 2.17 (Vakil 10.3.7)**.** *Let k be an algebraically closed field and X be a connected, reduced, proper k-scheme. Then $\Gamma(X, \mathscr{O}_X) \cong k$.*

Proof. First, we can naturally identify $\Gamma(X, \mathscr{O}_X)$ with the ring of k-scheme maps $X \to \mathbb{A}^1_k$: using the $(\Gamma, \operatorname{Spec})$ adjunction, $\operatorname{Hom}_{\mathsf{Sch}_k}(X, \mathbb{A}^1_k) = \operatorname{Hom}_{\mathsf{Alg}_k}(k[t], \Gamma(X, \mathscr{O}_X)) = \Gamma(X, \mathscr{O}_X)$, so functions on X are actually a ring of functions, which is nice.

Let $f \in \Gamma(X, \mathscr{O}_X)$, so f corresponds to a morphism $\pi : X \to \mathbb{A}^1_k$. If $i : \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$ is the usual open embedding, let $\pi' = i \circ \pi$. Since X is proper and $\mathbb{P}^1_k$ is separated over k, then π' must be proper, by Proposition 2.15, part 3 (let $Z = \operatorname{Spec} k$). Thus, π' is closed, so the set-theoretic image of π' is a closed, connected subset of $\mathbb{P}^1_k$. Since $\mathbb{P}^1_k$ has the cofinite topology, then $\operatorname{Im}(\pi)$ must be a single closed point p or all of $\mathbb{P}^1_k$, but if the latter, it can't factor through i. Since π' factors through $\mathbb{A}^1_k$, p is a closed point in $\mathbb{A}^1_k$, hence identified with an element of k.

The underlying set of the scheme-theoretic image of π is the closure of the set-theoretic image, so it's just p again; since X is reduced, so is its scheme-theoretic image. Thus, $\pi : X \to \mathbb{A}^1_k$ is a constant map of schemes $x \mapsto p$, and tracing through the adjunction, this corresponds to the constant function $f = p \in \Gamma(X, \mathscr{O}_X)$. $\square$

3. Dimension: 5/23/16

Today's lecture was given by Gill Grindstaff, on the first half of Chapter 11. This section relies on a lot of commutative algebra, which can make it difficult.

There are equiuvalent topological and algebraic formulations of the definition of the dimension of a scheme. We'd like this to agree with the intuitive notions of dimension: $\mathbb{A}^n$ should be n-dimensional, for example.

To motivate these definitions, recall that the dimension of a vector space V is the cardinality of some (and therefore any) basis for V. However, it's equivalent to say that the dimension of V is the supremum of lengths of nested chains of subspaces $0 \subsetneq W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_n = V$ (so we don't count 0). The scheme-theoretic definition will resemble this.

Definition 3.1.

- The **Krull dimension of a topological space** X is the supremum of lengths of chains $X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_n = X$ in which each X_i is a closed, irreducible subset of X.
- The **Krull dimension of a ring** A is the supremum of lengths of chains $0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ of prime ideals in A.

Fact. If A is a ring, then $\dim \operatorname{Spec} A = \dim A$. This is because $\mathfrak{p}_i \mapsto V(\mathfrak{p}_i)$ defines an inclusion-reversing bijection between the poset of prime ideals of A and the poset of closed irreducible subspaces of $\operatorname{Spec} A$, and in particular sends chains of nested subsets to chains of nested subsets (in the other direction).

Example 3.2.

(1) Every prime ideal of $\mathbb{Z}$ is of the form $\mathfrak{p} = (p)$ for a prime number p, or is the zero ideal. Hence, the longest chain we can make is $(p) \supset (0)$, so $\dim \mathbb{Z} = 1$.
(2) Similarly, for any field k, in $k[t]$ the longest chains we can make are $(f(t)) \supset (0)$ for f irreducible, so $\dim \mathbb{A}^1_k = 1$.
(3) In $k[x]/(x^2)$, (0) is the only prime ideal, so $\dim k[x]/(x^2) = 0$.

Dimension is *not* local, unlike the dimension of manifolds: consider the space $Z \subset \mathbb{A}^3$ consisting of the union of the xy-plane and the z-axis, which is not irreducible. Then, the dimension of the xy-plane is 2 but the dimension of the z-axis is 1. We do know that $\dim(Z)$ is the maximum of the dimensions of its irreducible subsets, however.

Definition 3.3. A scheme X is **equidimensional** if each of its irreducible components has the same dimension.

An equidimensional scheme of dimension 1 is called a **curve**; an equidimensional scheme of dimension 2 is called a **surface**; and so forth.

In order to get a handle on dimension, we'll need to do come commutative algebra.

Theorem 3.4. *Let $\pi : \operatorname{Spec} A \to \operatorname{Spec} B$ be induced from an integral extension $A \to B$ of rings. Then,* $\dim \operatorname{Spec} A = \dim \operatorname{Spec} B$.

This follows from an algebraic result.

Theorem 3.5 (Going-up theorem)**.** *Let $A \hookrightarrow B$ be an integral extension, $\mathfrak{p}_1 \subset \mathfrak{p}_2$ be prime ideals of A, and $\mathfrak{q}_1$ be a prime ideal of B such that $\mathfrak{q}_1 \cap A = \mathfrak{p}_1$.*[4] *Then, there is a prime ideal $\mathfrak{q}_2 \subset B$ such that $\mathfrak{q}_1 \subset \mathfrak{q}_2$ and $\mathfrak{q}_2 \cap A = \mathfrak{p}_2$.*

This can be inductively extended to chains of prime ideals, which proves one half of Theorem 3.4.

The proof of Theorem 3.5 depends on the following lemma.

Lemma 3.6. *Let $A \hookrightarrow B$ be an integral extension. If B is a field, then A is a field.*

The following exercise is another nice property.

Exercise 3.7. Let $\nu : \widetilde{X} \to X$ be the normalization. Then, $\dim \widetilde{X} = \dim X$.

The **normalization** of X replaces rings with their integral closures on an affine cover; after checking that this behaves well, it defines a nice scheme that X embeds into as an open dense subset. The key to the proof is that the dimension of a ring is the same as the dimension of its integral closure, which follows from Theorem 3.4.

The next thing we'd like to define is codimension, but there are some weird pathologies: recalling Z, the union of the xy-plane and the z-axis, what's the codimension of the z-axis in Z? Should it be 0, since there's nothing above it? Or is it 1, since Z is 2-dimensional and the z-axis is 1-dimensional? There's no good answer, and as a result one only defines codimension inside irreducible schemes.

Definition 3.8. Let X be an *irreducible* topological space and $Y \subset X$. Then, the **codimension** $\operatorname{codim}_X Y$ is the supremum of lengths of chains of irreducible closed subsets $\overline{Y} \subsetneq Z_1 \subsetneq Z_2 \subsetneq \cdots \subsetneq Z_n = X$.

[4]One often says that $\mathfrak{q}_1$ **lies over** $\mathfrak{p}_1$.

Since Y might not be closed, we must start with $\overline{Y}$. This is satisfactory: a closed point has codimension 2 inside $\mathbb{A}^2$.

There's also an algebraic analogue of codimension.

Definition 3.9. The **codimension of a prime ideal** $\mathfrak{p}$ in a ring A, written $\operatorname{codim}_R \mathfrak{p}$, is the supremum of lengths of *decreasing* chains of prime ideals $\mathfrak{p} \supsetneq \mathfrak{q}_1 \supsetneq \mathfrak{q}_2 \supsetneq \cdots \supsetneq \mathfrak{q}_n \supsetneq (0)$.

In particular, this implies that $\operatorname{codim}_R \mathfrak{p} = \dim R_{\mathfrak{p}}$.

Here are some useful results about dimension.

Proposition 3.10. *Let R be a UFD and $\mathfrak{p} \subset R$ be a codimension-one prime ideal. Then, $\mathfrak{p}$ is principal.*

Theorem 3.11. *Let A be a finitely generated k-algebra that's an integral domain. Then,* $\dim \operatorname{Spec} A = \operatorname{tr.deg.} K(A)/k$.

Here, $K(A)$ denotes the field of fractions of A, and $\operatorname{tr.deg.} K/k$ is the **transcendence degree** of a field extension K/k; the idea is: how many transcendental elements do you need to adjoin to get K? This intuition turns out to be correct.

Lemma 3.12 (Noether normalization)**.** *Let S be a finitely generated k-algebra that's an integral domain, such that* $\operatorname{tr.deg.} K(A)/k = n$. *Then, there exist $x_1, \ldots, x_n \in A$, algebraically independent*[5] *over k, such that A is a finite extension of $k[x_1, \ldots, x_n]$.*

These are very useful because the transcendence degree is much easier to understand than all prime ideals of a ring.

Just as we have a going-up theorem, there's also one in the pther dimension.

Theorem 3.13 (Going-down theorem)**.** *Let $\phi : B \hookrightarrow A$ be a finite extension of rings (i.e. A is finitely-generated as a B-*module*), where B is an integrally closed domain and A is an integral domain. Suppose $\mathfrak{q}_1 \subset \mathfrak{q}_2$ are prime ideals of B and $\mathfrak{p}_2 \subset A$ is a prime ideal such that $\phi^{-1}(\mathfrak{p}_2) = \mathfrak{q}_2$ (so it lies over $\mathfrak{q}_2$). Then, there exists a prime $\mathfrak{p}_1 \subset \mathfrak{p}_2$ such that $\mathfrak{p}_1$ lies over $\mathfrak{q}_1$.*

This requires more assumptions and is harder to prove.

At this point, we talked about a few exercises.

Lemma 3.14. *Let X be a topological space and $U \subset X$ be open. Then, there's a bijection between the irreducible closed subsets of U and the irreducible closed subsets of X meeting U.*

This resembles a theorem from commutative algebra establishing a bijection between prime ideals of B/I and prime ideals of B containing I (where $I \subset B$ is an ideal).

The proof of Lemma 3.14 sets up the bijection by sending an irreducible closed subset $F \subset U$ to $\overline{F} \subset X$, and sends an irreducible $F' \subset X$ to $F \cap U \subset U$.

Exercise 3.15 (Vakil ex. 11.1.B)**.** Show that a scheme has dimension n iff it can be covered by affine open subsets of dimension at most n, where equality is achieved for some affine scheme in the cover.

4. Codimension One: 5/26/16

Today, Richard spoke about sections 11.3 and 11.4, on codimension 1 miracles. We'll skip the last section of chapter 11, because it provides solely algebraic proofs of some of these theorems, and doesn't assist one's geometric intuition.

Definition 4.1. A scheme X is **locally Noetherian** if it has a cover by affine opens $\operatorname{Spec} A_i$ such that each A_i is a local ring. If in addition X is quasicompact, it's called **Noetherian**.

All varieties are locally Noetherian and even Noetherian

One of the codimension 1 miracles is Krull's principal ideal theorem. There are a couple versions.

Theorem 4.2 ((Geometric) Krull's principal ideal theorem)**.** *Let X be a locally Noetherian scheme and $f \in \Gamma(X, \mathscr{O}_X)$. Then, the irreducible components of $V(f)$ are codimension 0 or 1 in X.*

[5] Just as linear independence means not satisfying any nonzero linear relation, **algebraic independence** means not satisfying any nonzero polynomial relation.

Recall that $V(f)$ is the set of points $x \in X$ such that the stalk $[f] \in \mathscr{O}_{X,x}$ is equal to 0.

Theorem 4.2 follows from the algebraic version.

Theorem 4.3 ((Algebraic) Krull's principal ideal theorem)**.** *Let A be a Noetherian ring and $f \in A$. Then, every prime ideal $\mathfrak{p} \subset A$ minimal among those containing f has codimension at most 1. If f isn't a zerodivisor, the codimension is exactly 1.*

Since we can pass between (co)dimension of prime ideals and (co)dimension of schemes, these two formulations of the theorem are equivalent.

Definition 4.4. Let $X \hookrightarrow Y$ be a closed embedding. Then, X is **locally principal** if there is an affine open cover $\mathfrak{U}$ of Y such that for every $\operatorname{Spec} A \in \mathfrak{U}$, $X \cap \operatorname{Spec} A$ is cut out by a principal ideal of A.

That is, X is locally cut out by a single equation.

Corollary 4.5. *A locally principal closed subscheme has codimension 0 or 1.*

There are a lot of interesting exercises that derive further consequences of this theorem: here are a few.

Proposition 4.6 (Vakil ex. 11.3.C)**.** *Let X be a closed subset of $\mathbb{P}^n_k$ of dimension at least 1. Then, every nonempty hypersurface intersects X.*

This tells us, for example, that there are no parallel hypersurfaces in projective space; in particular, this is not true of affine space. Finding nice hypersurfaces is often a good way to reduce the dimensionality of a question.

Proposition 4.7 (Vakil ex. 11.3.E)**.** *Let $X, Y \subset \mathbb{A}^d_k$ be equidimensional subvarieties of codimensions m and n, respectively. Then, $X \cap Y$ has codimension at most $m + n$.*

Proposition 4.8 (Vakil ex. 11.3.G)**.** *Let A be a Noetherian ring and $f \in A$ be such that f isn't contained in any prime ideal of codimension 1. Then, f is invertible.*

The idea is to consider the dimension of the quotient $A/\mathfrak{p}$ if $f \in \mathfrak{p}$.

Example 4.9. Sometimes, codimension behaves pathologically. Let k be a field and $A = k[x]_{(x)}[t]$: elements of A are expressions of the form

$$\Phi = \sum_{i=1}^{n} \frac{f_i(x)}{g_i(x)} t^i,$$

where $x \nmid g_i(x)$. The ideal $\mathfrak{p} = (xt - 1)$ is prime, and $A/(xt-1) = k[x]_{(x)}[1/x] \cong k(x)$, so $(xt-1)$ is maximal, and hence has dimension 0. By Theorem 4.3, since $xt - 1$ is not a zerodivisor, then $\operatorname{codim}_A \mathfrak{p} = 1$.

Naïvely, we might expect this implies $\dim A = 1$, but in fact there's an irreducible chain of length 2: $(0) \subsetneq (t) \subsetneq (x, t)$, so $\dim A \geq 2$ (and in fact is exactly 2). So codimension is not just the difference in dimension.

Another cool application of dimension is to characterize UFDs (at least among Noetherian rings).

Proposition 4.10 (Vakil 11.3.5)**.** *Let A be a Noetherian integral domain. Then, A is a UFD iff all codimension 1 prime ideals are principal.*

Proof. The forward direction is Proposition 3.10: if $\mathfrak{p}$ is codimension 1, then for any $f \in \mathfrak{p}$, if g is an irreducible prime factor of f, then $(g) \subset \mathfrak{p}$, but since $\operatorname{codim} \mathfrak{p} = 1$, this forces $(g) = \mathfrak{p}$.

Conversely, we want to show that an $a \in A$ is irreducible iff it's prime. If a is irreducible, then by Theorem 4.2, $V(a)$ has a codimension 1 point $[(p)]$, so $a = a'p$ for some a'. Thus, a' must be a unit, so $(a) = (p)$, and hence a is prime. The other direction uses the Noetherian hypothesis. ☐

The next great property of codimension 1 is a generalization of Krull's principal ideal theorems.

Theorem 4.11 (Krull height theorem)**.** *Let $X = \operatorname{Spec} A$, where A is a Noetherian ring and $Z = V(r_1, \ldots, r_\ell)$ be an irreducible subset. Then, $\operatorname{codim}_X Z \leq \ell$.*

Though this looks like it should follow inductively from Theorem 4.2, it's more subtle.

Another nice result is algebraic Hartogs' lemma, analogous to Hartogs' lemma in several complex variables, which is about poles or singularities of holomorphic functions.

Theorem 4.12 (Algebraic Hartogs' lemma)**.** *Let A be an integrally closed Noetherian integral domain. Then, if P denotes the set of prime ideals of A of codimension 1, then $A = \bigcap_{\mathfrak{p}\in P} A_\mathfrak{p}$.*

This intersection is understood to take place in the fraction field $K(A)$. The relation to the complex-analytic version is that if $f \in K(A)$, it can be interpreted as a rational function. If $f \notin A_\mathfrak{p}$, it's thought of as having a pole at $\mathfrak{p}$, and if it's in $\mathfrak{p}A_\mathfrak{p}$, it has a zero at $\mathfrak{p}$. Hartogs' lemma states that we can extend over singularities of codimension 2 or higher, but not necessarily codimension 1.

Dimension of fibers of morphisms of varieties. Recall that the fundamental theorem of elimination theory (which we used to prove Proposition 2.16) states that for every ring A, $\mathbb{P}^n_A \to \operatorname{Spec} A$ is a closed map. This tells us that closed subsets of projective space are cut out by inhomogeneous equations in $n+1$ variables over A.

One therefore wonders about the locus where the solution of the system of $n+1$ inhomogeneous equations is dimension at least d, for some d. This is a closed condition on the coefficients (just as in linear algebra), and therefore this locus is closed (just like in linear algebra).

Proposition 4.13 (Vakil ex. 11.4.A)**.** *Let $\pi : X \to Y$ be a morphism of locally Noetherian schemes, $p \in X$, and $q = \pi(p)$. Then,* $\operatorname{codim}_X p \le \operatorname{codim}_Y q + \operatorname{codim}_{\pi^{-1}(q)} p$.

Example 4.14. For this, it's good to have a picture. Suppose X is the union of the xy-plane and the z-axis inside $\mathbb{A}^3$, and $Y = \mathbb{A}^2$. Let $\pi : X \to Y$ crush the z-axis down to 0, $p = (0,0,1)$, and $q = (0,0)$. Then, $\operatorname{codim}_X p = 1$, $\operatorname{codim}_Y q = 2$, and $\operatorname{codim}_{\pi^{-1}(q)} p = 1$ (since $\pi^{-1}(q)$ is the z-axis). Indeed, $1 \le 2+1$.

Now, we have a result akin to the regular value theorem in differential topology: if $f : X \to Y$ is a smooth map of manifolds and $y \in Y$ is a regular value, then $f^{-1}(y) \subset X$ has codimension equal to the difference of their dimensions, or is empty.

Theorem 4.15 (Vakil 11.4.1)**.** *Let $\pi : X \to Y$ be a morphism of finite type k-schemes,* $\dim X = m$, *and* $\dim Y = n$. *Then, there is an open $U \subseteq Y$ such that for all $q \in U$, the fiber over q has pure dimension $m-n$ or is empty.*

Fiber dimension in general is discontinuous, but curiously, it obeys **upper semicontinuity** (we say that for all $\varepsilon > 0$, there's a $\delta > 0$ such that if $|x_0 - x| < \delta$, then $f(x) \le f(x_0) + \varepsilon$). The intuition is that the value can jump, but then the "upper part" is closed. This is exactly as in real analysis.

Proposition 4.16 (Vakil 11.4.2)**.** *Let $\pi : X \to Y$ be a morphism of finite type k-schemes.*

(1) *The dimension of the fiber of π at a $p \in X$ (specifically, of the largest component of $\pi^{-1}(\pi(p))$ containing p) is upper semicontinuous on X.*

(2) *If in addition π is proper*[6]*, then the dimension of the fiber above a $y \in Y$ is upper semicontinuous on Y.*

Though it's surprising that upper semicontinuity exists, it appears in other places in algebraic geometry. It tells us that the dimension can increase when one takes limits. The dimension of the fiber is never smaller than what you think, but can be bigger, e.g. when we collapsed the z-axis onto the xy-plane in Example 4.14.

5. Regularity: 5/30/16

Tody, Jay Hathaway spoke about sections 12.1–12.3, on regularity, to the sonorous sounds of high schoolers warming up for the Texas State Solo and Ensemble Festival.

First, we'll talk about the Zariski cotangent space. Recall that in differential geometry, a manifold is a locally ringed space (M, C^∞_M), with C^∞_M the sheaf of smooth functions. The cotangent space at an $x \in M$ is linear functionals on germs of smooth functions on x to first order: that is, if $\mathfrak{m}_x$ is the maximal ideal of the local ring $C^\infty_{M,x}$, then the cotangent space is $T^*_x M = \mathfrak{m}_x/\mathfrak{m}_x^2$ (the $\mathfrak{m}_x^2$ term contains all of the higher-order information). This motivates the algebraic definition of a cotangent space.

Definition 5.1. Let $(A, \mathfrak{m})$ be a local ring. Then, the **Zariski cotangent space** of A is the $(A/\mathfrak{m})$-vector space $\mathfrak{m}/\mathfrak{m}^2$. If X is a scheme, the **Zariski cotangent space** at a $p \in X$ is $\mathfrak{m}_p/\mathfrak{m}_p^2$, where $\mathfrak{m}_p$ is the maximal ideal of the local ring $\mathscr{O}_{X,p}$.

[6] Equivalently, it's a closed map, since we already have the other hypotheses.

Just like in differential geometry, tangent vectors correspond to derivations.

Proposition 5.2 (Vakil ex. 12.1.A)**.** *Let X be a scheme and k be the residue field of $\mathscr{O}_{X,p}$ at a $p \in X$. Then, $(\mathfrak{m}_p/\mathfrak{m}_p^2)^\vee \cong \mathrm{Der}_k(\mathscr{O}_{X,p}, \mathscr{O}_{X,p})$.*

Recall that $(\mathfrak{m}_p/\mathfrak{m}_p^2)^\vee = \mathrm{Hom}_k(\mathfrak{m}_p/\mathfrak{m}_p^2, k)$; since this is the dual of the cotangent space, it's reasonable to call it the tangent space.

Partial proof. Let $\nabla : \mathscr{O}_{X,p} \to \mathscr{O}_{X,p}$ be a derivation, and write $f' = \nabla f$ for a germ $f \in \mathscr{O}_{X,p}$. The Leibniz rule tells us that $(fg)' = f'(p)g(p) + f(p)g'(p)$, so the map $f \mapsto f'(p)$ makes sense on $\mathfrak{m}_p$ (the functions that vanish at p) and vanishes on $\mathfrak{m}_p^2$ (functions vanishing to second order at p), so it defines a map $\phi : \mathfrak{m}_p/\mathfrak{m}_p^2 \to k$. $\square$

The other direction is more fiddly, and we'll see a better proof later in the notes. The idea is that we can write $\mathscr{O}_{X,p} = k \oplus \mathfrak{m}_p$ as a split square-zero extension (which is the tricky part, because it's not natural in any sense); then, given a morphism $\phi : \mathfrak{m}_p/\mathfrak{m}_p^2 \to k$ we define a derivation to be 0 on k and ϕ on $\mathfrak{m}_p$, more or less, and this satisfies the Leibniz rule. In a later chapter this is done in greater generality.

Suppose $\pi : X \to Y$ is a map of schemes, $p \in X$, and $q = \pi(p)$. Then, pullback gives us a map of stalks $\pi^\sharp : \mathscr{O}_{Y,q} \to \mathscr{O}_{X,p}$: since a map of schemes is a map of locally ringed spaces, this carries the maximal ideal $\mathfrak{n}_q \subset \mathscr{O}_{Y,q}$ to the maximal ideal $\mathfrak{m}_p \subset \mathscr{O}_{X,p}$. Thus, it descends to a pullback map on the cotangent spaces $\mathfrak{n}_q/\mathfrak{n}_q^2 \to \mathfrak{m}_p/\mathfrak{m}_p^2$. This works for general locally ringed spaces; if you do this for smooth manifolds, the dual of this map is the usual derivative Df of a smooth function f.

Proposition 5.3 (Vakil ex. 12.1.G)**.** *Let $X = \operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_n)$ for $f_1, \dots, f_n \in k[x_1, \dots, x_n]$. Then, $T_p^* X = \ker \mathrm{Jac}_p(f_1, \dots, f_r)$, which is the Jacobian of $f_1, \dots, f_r$ evaluated at p.*

This is a thing you can sit down and compute; much later in the notes, the sheaf of Kähler differentials can be employed to understand this more cleanly. The idea is that we take the ideal $(x_1, \dots, x_n)$, then mod out by all degree 2 monomials. After this, the f_i decompose into their first-order components.

The cotangent space is the beginning of our understanding of smoothness.

Theorem 5.4. *Let $(A, \mathfrak{m})$ be a Noetherian local ring and $k = A/\mathfrak{m}$ be its residue field. Then,*

$$\dim A \leq \dim_k \mathfrak{m}/\mathfrak{m}^2. \tag{5.5}$$

The proof involves Nakayama's lemma.

Definition 5.6. If equality holds in (5.5), one says A is **regular**.

Definition 5.7. Let X be a locally Noetherian scheme.

- If $p \in X$ and $\mathscr{O}_{X,p}$ is a regular local ring, then X is **regular at** p.
- X is **regular** if it's regular at all $p \in X$.
- If X is not regular at p, it's called **singular at** p, and X is called **singular**.

6. An Algebraic Interlude: 6/1/16

Today, Tom reviewed §§7.2 and 7.3, discussing some commutative algebra that is necessary for understanding smoothness, and some finiteness conditions on morphisms.

Definition 6.1. Let $\varphi : B \to A$ be a ring homomorphism.

- An $a \in A$ is **integral** over B if there exists a *monic* polynomial $f \in B[x]$ such that $f(a) = 0$, i.e. there exist $b_0, \dots, b_{n-1} \in B$ such that
$$a^n + \varphi(b_{n-1})a^{n-1} + \cdots + \varphi(b_0) = 0. \tag{6.2}$$
- A is **integral over** B if all $a \in A$ are integral over B. In this case, φ is called **integral**.
- If φ is integral and injective, φ is called an **integral extension**.

Integrality is a generalization of the algebraicity of a field extension.

Definition 6.3. Let $f : X \to Y$ be a morphism of schemes. Then, f is **integral** if for all affine opens $\operatorname{Spec} B \subset Y$ and affine opens $\operatorname{Spec} A \subset f^{-1}(\operatorname{Spec} B)$, the induced map on global sections $f^\sharp : B \to A$ is integral.

This is how we define almost all fancy properties of schemes: take some ring-theoretic property and require it to hold affine-locally.

Definition 6.4.

- A ring homomorphism of schemses $\varphi : B \to A$ is **finite** if it induces a finitely generated B-module structure on A. Often, one says that "A is a finite B-module."
- A morphism $f : X \to Y$ of schemes is **finite** if for all affine opens $\operatorname{Spec} B \subset Y$, the preimage $f^{-1}(\operatorname{Spec} B) \cong \operatorname{Spec} A$ is affine, and the induced map on global sections $f^\sharp : B \to A$ is finite.

Integral and finite morphisms of schemes have the nice properties we require of properties of morphisms.

Proposition 6.5 (Vakil ex. 7.2.A)**.** *Integrality and finiteness can be checked affine-locally.*

The proofs use the same trick that we always use to check this: reduce to the affine communication lemma. It doesn't come for free: one must check that the ring-theoretic statement is true for the ring A iff it's true for each A_{f_i}, where $(f_1, \dots, f_i) = 1$. This is analogous to checking on an open cover. It's good to work this out, albeit not more than once.

Integrality plays well under quotients and localization; intuitively, integrality of morphisms of schemes is well-behaved locally.

Proposition 6.6 (Vakil ex. 7.2.B)**.** *Let $\varphi : B \to A$ be an integral morphism.*

1. *If $S \subset B$ is a multiplicative set, $S^{-1}\varphi : S^{-1}B \to S^{-1}A$ is integral.*
2. *If $J \subseteq A$ is an ideal and $I = \varphi^{-1}(J)$, then $B/I \to A/J$ is integral.*
3. *If $I' \subseteq B$ is an ideal, then $B/I' \to A/I'A$ is integral (here, $I'A$ is the ideal generated by $\varphi(I')$).*

Moreover, (1) *and* (2) *preserve the property that φ is an integral extension.*

Surjective ring maps are tautologically integral, but we can do even better: they're finite, and we'll show finiteness implies integrality.

Partial proof. For the first part, suppose $a/s \in S^{-1}A$, so we know there exist b_i satisfying (6.2). When we multiply by s^n, this shows

$$\left(\frac{a}{s}\right)^n + \frac{b_{n-1}}{s}\left(\frac{a}{s}\right)^{n-1} + \cdots + \frac{b_0}{s^n} = 0,$$

so a/s is integral over $S^{-1}B$.

The second part follows from taking the integrality condition (6.2) mod J.

If φ is an integral extension, we just have to check injectivity. For part (1), this follows because localization is an exact functor: you can check that if $0/1 = \varphi(b)/\varphi(s)$ inside $S^{-1}A$, then there's a $t \in S$ such that $\varphi(t) = 0 = \varphi(tb)$, so $tb = 0$, and therefore $b/s = bt/st = 0$, so φ is injective. For (2), injectivity follows more or less by definition of the quotient. ∎

Lemma 6.7. *Let $\varphi : B \to A$ be a ring homomorphism. Then, $a \in A$ is integral over B iff there's a subalgebra M of A containing a that is finitely generated as a B-module.*

Again, this is a property of algebraicity, and it invites an interesting question: given a monic polynomial satisfying a, and a monic polynomial satisfying a', how do we write down one satisfying $a + a'$? This is tricky, and there is one that exists, but the point of the lemma is that you need not do it directly.

Proof. In the forward direction, suppose a is integral over B. Then, $B[a] \subset A$ is a finitely generated B-module, because it's generated by $1, a, \dots, a^{n-1}$ over B. Conversely, suppose $M = \langle m_1, \dots, m_k\rangle_B$ is a finitely generated B-submodule of A containing a. That is, there are $\lambda_{ij} \in B$ such that

$$am_i = \sum_{j=1}^k \lambda_{ij} m_j.$$

If $\Lambda = (\lambda_{ij})$ is the matrix of these coefficients and $\vec{m} = (m_1, \dots, m_k)^{\mathrm{T}}$, then this says that, as matrices over B, $(aI - \Lambda)\vec{m} = 0$. We'd like to invert this, but we're not over a field. Using the adjugate matrix, which does exist over rings, we have that $\det(aI - \Lambda)$ annihilates $\vec{m}$, and so since A' contains 1 and $(m_1, \dots, m_k)$ generates M, $\det(aI - \Lambda) \cdot 1 = 0$. This is great, because $\det(aI - \Lambda)$ is a monic, degree-k polynomial with coefficients in $\varphi(B)$. ∎

Extending Lemma 6.7, one can show that $a \in A$ is integral over B iff $B[a]$ is a finitely generated B-module.

Corollary 6.8 (Vakil cor. 7.2.2)**.** *If $\varphi : B \to A$ is a finite ring homomorphism, then it's integral.*

The converse is not true, e.g. the inclusion $\mathbb{Q} \hookrightarrow \overline{\mathbb{Q}}$. Corollary 6.8 is a generalization of the field-theoretic statement that finite extensions are algebraic, but the converse is not true.

Proposition 6.9. *A composition of integral ring homomorphisms is integral.*

Proof. Let $\varphi : B \to A$ and $\psi : C \to B$ be integral ring homomorphisms. Let $a \in A$, so $a \in B[\varphi(b_1), \dots, \varphi(b_n)] \subset A$. We can write each $\varphi(b_i)$ as a polynomial in finitely many $\varphi(\psi(\gamma_{ij}))$, and so a is generated over C by these finitely many λ_{ij}. $\square$

Proposition 6.10 (Vakil ex. 7.2.D)**.** *Let $\varphi : B \to A$ be a ring homomorphism. Then, the elements of A that are integral over B form a B-subalgebra $\overline{B} \subset A$, called the **integral closure** of B in A.*

If the ambient ring A is absent, an integral closure usually refers to the integral closure in the field of fractions.

The idea of the proof is that it reduces to checking that if a and a' are integral over B, then so are $a + a'$ and aa'. We want to look at $B[a + a']$ and $B[aa']$, which are subextensions of $B[a][a']$. We know $B[a][a']$ is integral over $B[a]$, and $B[a]$ is integral over B, so by Proposition 6.9, $B[a][a']$ is integral over A.

Using these smaller results, we can understand a bigger theorem, the lying over theorem.

Theorem 6.11 (Lying over, Vakil thm. 7.2.5)**.** *Let $\varphi : B \hookrightarrow A$ be an integral extension. Then, for any prime ideal $\mathfrak{p} \subset B$, there is a prime ideal $\mathfrak{q} \subset A$ **lying over** $\mathfrak{p}$, i.e. $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$.*

What this also says is that if $\pi : \operatorname{Spec} A \to \operatorname{Spec} B$ is an integral map of schemes, then φ is surjective as a map of sets. This can be useful.

We can extend this to a statement about chains of ideals.

Theorem 6.12 (Going up)**.** *Let $\varphi : B \to A$ be an integral ring homomorphism, $n > m \geq 1$, $\mathfrak{q}_1 \subsetneqq \cdots \subsetneqq \mathfrak{q}_m$ be a chain of strictly increasing prime ideals of A, and $\mathfrak{p}_1 \subsetneqq \cdots \subsetneqq \mathfrak{p}_n$ be a chain of strictly increasing prime ideals of B. If each $\mathfrak{q}_i$ lies over $\mathfrak{p}_i$ for $1 \leq i \leq m$, then we can extend the chain: there exist $\mathfrak{q}_{m+1} \subsetneqq \cdots \subsetneqq \mathfrak{q}_n$ such that $\mathfrak{q}_i$ lies over $\mathfrak{p}_i$ for $1 \leq i \leq n$.*

Geometrically, this states that if $\varphi : \operatorname{Spec} A \to \operatorname{Spec} B$ is an integral morphism of schemes and I have a chain of irreducible subsets $Y_1 \supsetneqq Y_2 \supsetneqq \cdots \supsetneqq Y_m$ of $\operatorname{Spec} A$, a chain of irreducible subsets $X_1 \supsetneqq X_2 \supsetneqq \cdots \supsetneqq X_n$ of $\operatorname{Spec} B$, and $\varphi(Y_i) = X_i$, then we can find irreducible subsets mapping to $X_{m+1}, \dots, X_n$ and preserving the chain relations.

The proof idea is to apply the lying over theorem many times. Interestingly, it tells you how to define the scheme-theoretic fiber, and therefore in some sense motivates the definition of the fiber product: the primes lying over $\mathfrak{p}$ are the fiber $\varphi^{-1}(\mathfrak{p})$ as a set: as a scheme, we take the fiber product with $\operatorname{Spec} k_{\mathfrak{p}}$, where $k_{\mathfrak{p}}$ is the residue field $k_{\mathfrak{p}} = B_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$; this is because the prime ideals lying over $\mathfrak{p}$ are the prime ideals lying over 0 in the residue field. Ring-theoretically, we get the pushout $A \otimes_B k_{\mathfrak{p}}$.

The point is, the setwise fiber has the same underlying set as the scheme-theoretic fiber, which admits the more abstract definition through a universal property. Equating these two viewpoints is nice to know. It also makes it easier to compute stalks of points in a fiber.

Now, let's talk about the various theorems called Nakayama's lemma. Like the cupcakes at the front of the class today, some are better than others.

Lemma 6.13 (Nakayama's lemma 1, Vakil 7.2.8)**.** *Let A be a ring, $I \subseteq A$ be an ideal, and M be a finitely generated A-module such that $M = IM$. Then, there's an $a \in A$ such that $a = 1 \pmod{I}$ and $aM = 0$.*

Proof. Choose a generating set $m_1, \dots, m_k$ for M; since $M = IM$, $M = \langle m_1, \dots, m_k \rangle_I$. That is, there exist $\lambda_{ij} \in I$ for $1 \leq i, j \leq k$ such that

$$m_i = \sum_{j=1}^{k} \lambda_{ij} m_j.$$

If $\vec{m} = (m_1, \dots, m_k)^{\mathrm{T}}$ and $\Lambda = (\lambda_{ij})$ as before, then $(1 - \Lambda)\vec{m} = 0$, and therefore we can choose $a = \det(1 - \Lambda)$, so $am_i = 0$ for each i, and $a = 1 \pmod{I}$ (since Λ is I-valued). $\square$

Recall that the **Jacobson radical** $\mathrm{Jac}(A)$ of a ring A is the intersection of its maximal ideals.

Lemma 6.14 (Nakayama's lemma 2, Vakil 7.2.9)**.** *Let A be a ring, $I \subseteq A$ be an ideal contained in $\operatorname{Jac} A$, and M be a finitely generated A-module such that $M = IM$. Then, $M = 0$.*

Proof. Using Lemma 6.13, we have an $a \in A$ such that $a = 1 \pmod{I}$ (so $a = 1 + i$ for some $i \in I$) and $aM = 0$. For all maximal ideals $\mathfrak{m} \subset A$, $a \notin \mathfrak{m}$, because $a = 1 + i$ for some $i \in I \subset \mathfrak{m}$. Thus, a is a unit, so $M = aM = 0$. ☐

This is slick, but doesn't show you why M must be finitely generated. A more explicit proof chooses (using Zorn's lemma) a maximal submodule $N \subsetneq M$ and $x \in M \setminus N$. Then, we can define a map $\varphi : A \twoheadrightarrow M/N$ sending $a \mapsto a \cdot [x]$. Hence, $A/\ker\varphi \cong M/N$ as A-modules (there's no good ring structure here), forcing $I \subseteq \ker\theta$. This implies $IM \subseteq N \subsetneq M$, but $IM = M$, so no such N exists, and therefore $M = 0$. So we don't need M to be finitely generated, which is pretty cool.

All the other versions of Nakayama's lemma follow from these two.

Lemma 6.15 (Nakayama's lemma 3)**.** *Let A be a ring, $I \subset A$ be an ideal contained in $\operatorname{Jac} A$, M be an A-module, and $N \subset M$ be a submodule. If $N/IN \to M/IM$ is surjective, then $N = M$.*

These are useful for proving various submodules are the whole model, etc., which is useful for showing that xactness is a local condition, e.g. if M is an A-module, $M = 0$ iff $M_{\mathfrak{p}} = 0$ for all prime ideals $\mathfrak{p} \subset A$ iff $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m} \subset A$.