

STACKS

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These notes were taken in Adrian Clough's minicourse in Summer 2018. I live-TeXed them using vim, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

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Day 1. June 18

I missed the morning and afternoon lectures for this day, which focused on the two main themes (descent and building new spaces from old) through examples.

Day 2. June 19

1. MORNING

We'll start with a review of some of yesterday's material.

Let X and R be manifolds, and $g : X \rightarrow R$ be a smooth map. Given a submersion $p : U \rightarrow X$, g lifts to a map $U \rightarrow R$. In this case, the pullback $U \times_X U \rightrightarrows U$ is a smooth manifold, and p is exactly the coequalizer of that diagram. This is a very simple instance of descent: we have an isomorphism between the space of smooth maps from X to R with the space of maps from U to R that coequalize $U \times_X U \rightrightarrows U$ (i.e. after composing with such a map, the two structure maps become equal).

More generally, let \mathcal{U} be an open cover of X by balls and $U := \coprod_{V \in \mathcal{U}} V$. The story goes through with $U \times_X U \rightrightarrows U \rightarrow X$. But descent doesn't always hold: for example, if you try to lift a map $E \rightarrow X$ to U . You need additional data.

It's also useful to replace sets with groupoids. For example, if you want to consider descent data for principal G -bundles, then you can't consider the set of equivalence classes: the pullback map $\mathrm{Bun}_G(X) \rightarrow \mathrm{Bun}_G(U)$ isn't very interesting, because $\pi_0 \mathrm{Bun}_G(U)$ is trivial. Therefore in order to set up descent, we'll need to (1) consider groupoids, and (2) provide additional data on \mathcal{U} that makes interesting things possible. In this case the passage from injective to surjective to bijective is replaced with faithful to fully faithful to an equivalence of categories.

Recall that an *epimorphism* in a category \mathcal{C} is a map $f : A \rightrightarrows B$ such that for any two maps $g, h : B \rightrightarrows C$ such that $g \circ f = h \circ f$, then $g = h$. In Set this corresponds to the usual notion of surjective.

Definition 1.1. An epimorphism $f : U \rightarrow X$ is an *effective epimorphism* if the pullback $U \times_X U$ exists and f is its coequalizer.

You can think of this as a categorified version of an equivalence relation.

It's also possible to go the other direction: given an equivalence relation $R \rightarrow U \times U$ on U and an epimorphism $U \rightrightarrows X$, you can ask whether $R \cong U \times_X U$.

In geometric situations, it seems generally true that one glues by equivalence relations. This seems a little mysterious, but is one motivation for thinking about stacks in this way.

Now we turn to creating new spaces from old. In many situations you're considering a nice class of spaces (schemes, varieties) \mathcal{C} , but some reasonable construction starting with stuff in \mathcal{C} leads you to something which does not exist in \mathcal{C} . For example, studying manifolds leads to asking about classifying spaces of principal G -bundles, which cannot be finite-dimensional; thus, you have to enlarge your category of spaces to get at them. Another example is fine moduli spaces in algebraic geometry, which often aren't varieties and sometimes aren't schemes.

The general approach is to (1) enlarge to a category $\widehat{\mathcal{C}}$ of *all* objects that can reasonably be constructed from objects in \mathcal{C} (i.e. by taking colimits and satisfying a descent condition), and then (2) restricting to a subcategory of nicer spaces. Depending on your application, there are different choices of nice subcategories for a given $\widehat{\mathcal{C}}$.

Example 1.2. Let Cart be the category whose objects are open subsets of \mathbb{R}^n and whose morphisms are smooth maps. Then, inside $\widehat{\text{Cart}}$, one could restrict to smooth manifolds, or more generally Fréchet manifolds, or more generally Frölicher spaces, or more generally diffeological spaces, depending on your application. ◀

To do this, we need some good candidate for $\widehat{\mathcal{C}}$. One, which we denote $\widehat{\mathcal{C}}$, is the collection of *presheaves* on \mathcal{C} , i.e. contravariant functors $\mathcal{C} \rightarrow \text{Set}$. We have the Yoneda embedding $\mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$, and we might want to complete $\widehat{\mathcal{C}}$ under colimits to a map $\mathcal{C} \rightarrow \mathcal{D}$; \mathcal{D} exists and is unique by a universal property. However, often coequalizers of the kinds we've been considering are not preserved by this construction. Here are two possible fixes for a category \mathcal{C} and a collection τ of coequalizers we want to keep:

- (1) Let $U \times_X U \rightrightarrows U \rightarrow X$ be in τ , and apply the Yoneda embedding to it, producing a diagram

$$(1.3) \quad \begin{array}{ccccc} \mathcal{C}(-, U) \times_{\mathcal{C}(-, X)} \mathcal{C}(-, U) & \xrightarrow[f]{g} & \mathcal{C}(-, U) & \longrightarrow & \mathcal{C}(-, X). \\ & & \searrow & & \uparrow (***) \\ & & & & \text{Coeq}(f, g) \end{array}$$

The fix will be to localize the morphisms $(**)$ in $\widehat{\mathcal{C}}$, namely, let \mathcal{W} denote the subcategory generated by these morphisms over all diagrams in τ . The localization is denoted $\widehat{\mathcal{C}}[\mathcal{W}]$, and has an obvious universal property: given a functor $F: \mathcal{C} \rightarrow \mathcal{D}$, such that \mathcal{D} is cocomplete and F preserves colimits and all coequalizers in τ , there's a unique functor $\widehat{\mathcal{C}}[\mathcal{W}] \rightarrow \mathcal{D}$. This is nice, but localization is terrible (see Remark 1.8), so it might be good to try something else.

- (2) The new basic idea is that if $X \in \widehat{\mathcal{C}}$, then we should be able to recover X from $\widehat{\mathcal{C}}(-, X)$ restricted to \mathcal{C} . Thus we identify X and $\widehat{\mathcal{C}}(-, X)$; that is, we want $\widehat{\mathcal{C}} \subseteq \widehat{\widehat{\mathcal{C}}}$.

Now let's think about colimits. Let $F: \mathcal{J} \rightarrow \mathcal{A}$ be a diagram; then $A(\varinjlim F, X) \cong \varprojlim A(F, X)$, or in other words, $\varinjlim X$ represents the functor $\varprojlim A(F, -)$. This means that $A \rightrightarrows B \rightarrow C$ is a coequalizer iff for all $X \in \mathcal{A}$,

$$(1.4) \quad A(\mathcal{C}, X) \longrightarrow A(\mathcal{B}, X) \rightrightarrows A(\mathcal{A}, X)$$

is an *equalizer* diagram (since $A(-, X)$ is contravariant).

Now assume $P \in \widehat{\mathcal{C}} \subseteq \widehat{\widehat{\mathcal{C}}}$ and $R \rightrightarrows U \rightarrow X$ is in τ . Then we can apply $\widehat{\mathcal{C}}(-, P)$ to obtain a diagram

$$(1.5) \quad \begin{array}{ccccc} \widehat{\mathcal{C}}(X, P) & \longrightarrow & \widehat{\mathcal{C}}(U, P) & \rightrightarrows & \widehat{\mathcal{C}}(R, P) \\ \parallel & & \parallel & & \parallel \\ P(X) & \longrightarrow & P(U) & \rightrightarrows & P(R). \end{array}$$

Requiring this to be a coequalizer diagram is exactly a sheaf condition! For example, in the category of smooth manifolds, if \mathcal{U} is an open cover of X and $P: \text{Man}^{\text{op}} \rightarrow \text{Set}$ is a presheaf, then we can apply it to the diagram

$$(1.6) \quad \coprod_{U, V \in \mathcal{U}} U \cap V \rightrightarrows \coprod_{U \in \mathcal{U}} U \longrightarrow X$$

to obtain (assuming $C \rightarrow \tilde{C}$ preserves coproducts)

$$(1.7) \quad \begin{array}{ccccc} P(X) & \longrightarrow & P\left(\coprod_{U \in \mathcal{U}} U\right) & \Longrightarrow & P\left(\coprod_{U, V \in \mathcal{U}} U \cap V\right) \\ \parallel & & \parallel & & \parallel \\ P(X) & \longrightarrow & \prod_{U \in \mathcal{U}} P(U) & \Longrightarrow & \prod_{U, V \in \mathcal{U}} P(U \cap V), \end{array}$$

and asking for this to be a coequalizer diagram is the usual notion of a presheaf being a sheaf (e.g. you can find this diagram in Vakil).

Remark 1.8. Localization of categories is technical and messy: for example, it's easy to write down an example where after the naïve notion of localization, there are collections of morphisms between objects that are so big as not to be sets. This is bad. One of the reasons people like model categories is because they avoid this problem. ◀

2. AFTERNOON

“Geometry is the art of giving funny names to morphisms.”

These two different ways of fixing coequalizers turn out to be the same.

Now let's suppose that \tilde{C} is the category of sheaves on C with respect to the coequalizers in τ (so the second approach).

Theorem 2.1. *The inclusion $\tilde{C} \hookrightarrow \hat{C}$ admits a left adjoint $a: \hat{C} \rightarrow \tilde{C}$, which is precisely the localization functor from the first approach.*

This implies in particular that the localization exists and is nice (e.g. you don't have to consider arbitrarily long zigzags in it, much like in a model category). This theorem is saying something deep.

Remark 2.2. If you do the ∞ -version of this, with simplicial sets and/or ∞ -groupoids, this recovers localization of model categories (or ∞ -categories), producing simplicial sheaves from simplicial presheaves. ◀

Theorem 2.1 holds for arbitrary τ (probably), but we'll prove it under a reasonable assumption on τ , which is also a motivation to learn about sites.

Sites.

Definition 2.3. Let C be a category.¹ Then a *coverage* on C is data of, for all $X \in C$, a family $T_X = \{\{\pi_i: U_i \rightarrow X\}_{i \in I}\}$ of families of maps to X , such that given a map $f: X' \rightarrow X$ in C and a family $\{U_i \rightarrow X\}_{i \in I}$ in T_X , there is a family $\{U'_j \rightarrow X'\}_{j \in J}$ in $T_{X'}$ such that for all $j \in J$, there exists an $i \in I$ and a $g: U'_j \rightarrow U_i$ such that the diagram

$$(2.4) \quad \begin{array}{ccc} U'_j & \xrightarrow{g} & U_i \\ \downarrow \pi'_j & & \downarrow \pi_i \\ X' & \xrightarrow{f} & X \end{array}$$

commutes. A category together with a coverage is called a *site*.

Before we see some examples, we'll define some related notions.

Definition 2.5. Given a coverage \mathbf{T} of C and a presheaf $\mathcal{F} \in \hat{C}$, let $\text{Desc}_{\mathcal{F}}(\{U_i \rightarrow X\}_{i \in I})$ be the set of $(s_i) \in \prod_{i \in I} \mathcal{F}(U_i)$ such that for all $i, j \in I$ and diagrams $U_i \xleftarrow{f} V \xrightarrow{g} U_j$ such that $s_i|_V = s_j|_V$, $\mathcal{F}(f)(s_i) = s_i|_V$ and $\mathcal{F}(g)(s_j) = s_j|_V$.

This is a lot like a limit, but it's not clear how to make it exactly one nicely.

Definition 2.6. Given a category C with a coverage \mathbf{T} , a \mathbf{T} -*sheaf* \mathcal{F} is a presheaf $\mathcal{F}: C^{\text{op}} \rightarrow \text{Set}$ such that for all $U \in C$ and families $\{U_i \rightarrow U\}$ in \mathbf{T} , the natural map $\mathcal{F}(U) \rightarrow \text{Desc}_{\mathcal{F}}(\{U_i \rightarrow U\})$ is an isomorphism.

Example 2.7.

¹We'll assume C is small; this is not necessary for the definition, but will be true for all examples we consider.

- (1) In manifolds, we could choose \mathbf{T} to be coverings in the usual way, or jointly² surjective submersions (more in spirit of what we did yesterday), or *étale covers*, i.e. jointly surjective local homeomorphisms.
- (2) Let \mathbf{C} denote the category of open subsets of \mathbb{R}^n (for all n) and smooth maps between them. Then one can once again consider covering maps, surjective submersions, or *étale covers*.
- (3) Let \mathbf{Cart} denote the category whose objects are \mathbb{R}^n for each n and whose maps are smooth maps between them. In this case we can take surjective submersions but not open covers, because there are no coproducts.
- (4) Let X be a space and \mathbf{C} be the category of open subsets on X . Then we can take classical coverings, i.e. $\{U_i \rightarrow U\}$ is a covering iff $U \subset \bigcup U_i$. A \mathbf{T} -sheaf for this is a sheaf in the usual sense; this is because $\text{Desc}_{\mathcal{F}}(\{U_i \subseteq U\})$ is exactly the collections of sections $\mathcal{F}(U_i)$ that agree on intersections. \blacktriangleleft

Looking into the last example in more detail, if you fix a manifold M , $\text{Top}(M)^{\text{op}} \hookrightarrow \text{Man}^{\text{op}}$ sending $U \mapsto U$, so we get a map from presheaves on Man to presheaves on $\text{Top}(M)$, namely, restrict a presheaf \mathcal{F} to open subsets of M .

Exercise 2.8. Show that a presheaf \mathcal{F} on Man is a \mathbf{T} -sheaf (for the three choices of \mathbf{T} we discussed above) iff for all manifolds M , $\mathcal{F}|_M$ is a classical sheaf.

In fact, in Man , from any covering we can construct the classical descent situation, and we actually get an equivalence of categories $\tilde{\text{Man}} \simeq \tilde{\text{Cart}}$. This is why we want general coverages: in \mathbf{Cart} we can't take coproducts, but we can in Man . This ultimately means that the kinds of spaces discussed earlier, namely Fréchet manifolds, Frölicher spaces, and diffeological spaces, can all be described as sheaves on \mathbf{Cart} .

In general, suppose \mathbf{C} admits pullbacks; then we have a similar story as for $\text{Top}(X)$: descent means that given a map $f : V \rightarrow U$ and covers $\{V_i \rightarrow V\}$, $\{U_i \rightarrow U\}$, we get a diagram

$$(2.9) \quad \begin{array}{ccc} \coprod_{i,j} V_i \times_V V_j & \xrightarrow{f''} & \coprod_{i',j'} U_{i'} \times_U U_{j'} \\ \Downarrow & & \Downarrow \\ \coprod_j V_j & \xrightarrow{f'} & \coprod_{j'} U_{j'} \\ \downarrow & & \downarrow \\ V & \xrightarrow{f} & U. \end{array}$$

Descent means that we should be able to reconstruct f just knowing f'' and f' . This looks a lot like descent for bundles, which seems important.

This ties to the stuff we saw this morning: in $\tilde{\mathbf{C}}$, for $\{U_i \rightarrow U\}$ in T_U , $\coprod \mathbf{C}(-, U_i) \rightarrow \mathbf{C}(-, U)$ becomes an effective epimorphism!

²For a family, this means that the map from the disjoint union of all of them is surjective.