

FALL 2016 HOMOTOPY THEORY SEMINAR

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1. s -COBORDISMS AND WALDHAUSEN'S MAIN THEOREM: 9/7/16

Today, Professor Blumberg gave an overview of Waldhausen's main theorem and its context; this semester, we'll be working through Lurie's proof of it as outlined in his course on the algebraic topology of manifolds.

We'll start from the h -cobordism theorem.

Definition 1.1. An h -cobordism is a cobordism W between manifolds M and N such that the equivalences $M \hookrightarrow W$ and $N \hookrightarrow W$ are both homotopy equivalences.

The canonical example is $M \times [0, 1]$, which is an h -cobordism between M and itself. This is called a *trivial h -cobordism*.

We're going to be deliberately vague about what category of manifolds we're dealing with: when we say "isomorphic," we mean as topological manifolds, PL manifolds, or smooth manifolds. We're not going to belabor the point right now, though it will be quite important for us later.

Theorem 1.2 (h -cobordism (Smale)). *If $\dim(M) \geq 5$ and $\pi_1(M) = 0$, then every h -cobordism is trivial, i.e. suitably isomorphic relative to the boundary to the trivial h -cobordism.*

This is a big theorem — a somewhat easy consequence is the Poincaré conjecture in dimensions ≥ 5 ! When Smale proved this part of the Poincaré conjecture, he really was attacking this theorem. The proof proceeds via a handlebody decomposition, which illustrates what is easier in dimension 5 than in dimensions 3 and 4: handlebodies can slide past each others using, for example, the *Whitney trick*, which simply doesn't work in dimensions 3 or 4.

We're not interested in the Poincaré conjecture *per se*, but can we generalize Theorem 1.2? If we try to lower the dimension of M , we're basically screwed, so can we work with M not simply connected?

Theorem 1.3 (s -cobordism (Barden, Mazur, Stallings)). *The set of isomorphism classes of h -cobordisms $M \hookrightarrow W \hookleftarrow N$ is in bijection with a certain quotient of $K_1(\mathbb{Z}[\pi_1(M)])$.*

We'll eventually define K_1 , which is an algebraic gadget that's a ring invariant. It's evident that K_1 of the group algebra is a homotopy invariant, but it's less obvious that the set of isomorphism classes of h -cobordisms is. This group $K_1(\mathbb{Z}[\pi_1(M)])$ is also the home of *Whitehead torsion*, an invariant of manifolds.

Question 1.4. Is this a π_0 statement? In other words, can we describe a space of s -cobordisms such that Theorem 1.3 is recovered on passage to π_0 ?

This is a natural question following recent developments in homotopy theory. It may allow us to attach spaces or spectra to these invariants.

The answer, due to names such as Hatcher, Igusa, and Waldhausen, is yes! On the left-hand side, we have something called the stable pseudo-isotopy space, akin to a stabilized form of $B\text{Diff}$, the isomorphisms of a manifold relative to its boundary. This arises as a result of an action on bundles of s -cobordisms, which is how classifying spaces appear.

Things will be more concretely defined, albeit not at this level of narrative. The point is that *a priori* this isn't a homotopy-invariant, so we have to stabilize in a geometric way, by taking repeated products $M \times I^n$ with an interval.

On the other side, one realizes $\pi_1(M) \cong \pi_0(\Omega M)$, so maybe we can try to construct something like $\mathbb{Z}[\Omega M]$. This works, but it's better to take the K -theory *spectrum* associated to something called the spherical group ring $K(S[\Omega M])$, which is (or is equivalent to) $A(M) = K(\Sigma_+^\infty \Omega M)$, which is an A_∞ ring spectrum.

Waldhausen's theorem is precisely that there is a stable s -cobordism theorem: that $A(M) \simeq \Sigma^\infty M \vee \Omega^2 \text{Wh}(M)$; $\text{Wh}(M)$ is something called the *Whitehead spectrum* associated to M , and its double loop space is the pseudo-isotopy space we want to construct. This splitting arises from an assembly map, which is a purely formal statement about (topologically or simplicially) enriched functor: $F(X) \vee Y \rightarrow F(X \vee Y)$.

So we have an algebraic invariant, which we can hope to calculate, and it tells us geometric information.

We can start with the sphere spectrum $K(S) \simeq S \vee \Omega^\bullet \text{Wh}(\ast)$. This is already hard and unsolved; solving it will solve several questions in geometric topology, including some on exotic differential structure.

Depending on who you are, you might have different motivations for things like this: May and others were naturally led to ring spectra when considering generalized orientations, but you might also invent them to make this theorem true!

It turns out that $K(S)$ is controlled by $K(\mathbb{Z})$. There's a commutative square

$$\begin{array}{ccc} K(S) & \longrightarrow & \text{TC}(S) \\ \downarrow & & \downarrow \\ K(\mathbb{Z}) & \longrightarrow & \text{TC}(\mathbb{Z}). \end{array}$$

Here, TC is topological cyclic homology. One of the big theorems is that this is a pullback diagram, so we can understand $K(S)$ from topological cyclic homology, the vertical maps, called *trace maps* (which actually generalize the trace of a matrix), and how they fit together. At primes, this is pretty simple, but that still leaves the individual players.

Understanding $\text{TC}(S)$ is, well, slightly harder than computing the stable homotopy groups of the spheres. This is a bad thing, but also a good thing: we can compute some of it, and if important questions depend on a particular element in it, that element can be computed and identified. Conversely, $K(\mathbb{Z})$ is a mess, but an interesting mess — it contains a lot of number-theoretic information, some still unknown. This diagram illustrates that number-theoretic information controls geometric topology.

Waldhausen's proof of his main theorem in [3] is famously complicated, and wasn't available until relatively recently. The book by Waldhausen-Jahren-Rognes [4] provides an exposition, but it's rough going. Certainly, the introduction will be useful.

One reason we might be interested is how Waldhausen proved this. He gave a direct proof in the PL category, and then applied a reduction to prove the smooth case. Is it possible to give a direct proof in the smooth case? Waldhausen and Igusa tried to do this, but didn't succeed. For reasons that are ultimately Floer-homotopy-theoretic, it would be useful to have such a proof in the smooth case. Lurie's proof in [2] follows the broad strokes of Waldhausen's proof, but uses different machinery, and could be useful as a guide for a direct proof in the smooth case.

There will be three kinds of lectures:

- (1) Statements of the theorem. Along with this, what is algebraic K -theory? What is a Whitehead spectrum?
- (2) Background: what is Wall finiteness? What is Whitehead torsion? These are geometric questions, yet are invariants of algebraic K -theory, and are good to know for culture.
- (3) Then, there's the technology of the proof, using something called simple homotopy theory. We can think of $K(\Sigma_+^\infty \Omega M)$ as the K -theory of a category of spaces parameterized over M , and this leads to simple homotopy equivalences, related to a prescribed set of equivalences and blowups. This is more geometric, and hence harder. Lurie's proof presents a different approach to this, focusing on constructible sheaves, and this is definitely one of the most worthwhile lessons from his proof.

The key is the construction of the assembly map: Lurie approaches it with some very natural functors from constructible sheaves. This is something we can get to, but we'd have to cover a lot of ground to get there.

Another thing to keep in mind: there will be many different constructions of these objects, all equivalent or equivalent up to a shift. We'll end up constructing a parameterized spectrum of space of cobordisms. It might be interesting to compare these to other cobordism categories.

2. THE WALL FINITENESS OBSTRUCTION: 9/14/16

Today, Nicky spoke.

The Wall finiteness obstruction is an invariant that's pretty easy to write down abstractly; it provides an obstruction for a CW complex to be homotopic to a finite CW complex. Specifically, given a CW complex X , let $G = \pi_1(X)$ (relative to any basepoint); we'll construct this obstruction as a class $w(X) \in \tilde{K}(\mathbb{Z}[G])$.

Definition 2.1. Let R be a ring, not necessarily commutative. Then, the K -theory of R , denoted $K_0(R)$, is the abelian group generated by isomorphism classes of finitely generated¹ projective (left) R -modules modulo the relations that for every short exact sequence of R -modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

we impose a relation $[M] = [M'] + [M'']$.

There's a map $\mathbb{Z} \rightarrow K_0(R)$ defined by $z \mapsto z[R]$; if $z > 0$, this is the class $[R^{\oplus z}]$. The *reduced K -theory* mods out by this: $\tilde{K}_0(R) = K_0(R)/\mathbb{Z}$.

This is a very algebraic object, but we'll use it to discover topological information.

There are some other tools we'll use. Relative homotopy invariants are associated to relative homology groups $H_*(X, Z; R)$, which we can define whenever we're given a map $f : Z \rightarrow R$. Using a mapping cylinder C_f , this is homotopic to an inclusion, and $H_*(X, Z; R)$ is the homology of the quotient chain complex (of the singular chains).

Definition 2.2. A *local system* on a space X is a representation of $\pi_1(X)$ over \mathbb{Z} .² That is, it's an abelian group with a compatible $\pi_1(X)$ -action.

Recall that $G = \pi_1(X)$ acts on the universal cover \tilde{X} by deck transformations. Thus, the singular complex $C_\bullet(\tilde{X})$ of \tilde{X} is a $\mathbb{Z}[G]$ -module. Thus, given a local system V , we can create new chain complexes, such as $C_\bullet(\tilde{X}) \otimes_{\mathbb{Z}[G]} V$ or $\text{Hom}_{\mathbb{Z}[G]}(C_\bullet(\tilde{X}), V)$.³ These define homology, resp. cohomology theories on X , called $H_*(X, V)$, resp. $H^*(X, V)$.

Definition 2.3. A space X is *finitely dominated* if there exists a finite CW complex Z , an inclusion $i : X \rightarrow Z$, and a section $r : Z \rightarrow X$ such that $r \circ i \simeq \text{id}_X$.

This basically means X includes into a finite CW complex which retracts onto it, but with a homotopy. We'll hope to show that some properties of finitely dominated spaces actually characterize them.

Fact. Let X be a finitely dominated space.

- (1) First, $\pi_0(X)$ must be finite (since it factors as a subset of $\pi_0(Z)$).
- (2) $\pi_1(X)$ must be finitely presented, because $i_* : \pi_1(X) \rightarrow \pi_1(Z)$ has a left inverse, so it's split injective into a finitely generated group.
- (3) For local systems V , the assignment $V \mapsto H_*(X, V)$ commutes with filtered direct limits.
- (4) X has finite *homotopical dimension*, which means there's an $m \geq 0$ such that for all local systems V and $i > m$, $H_i(X, V) = 0$. This will be at most the dimension of the space Z which dominates X .

The following theorem is important, but hard; [2, Lec. 2] sketches the proof.

Theorem 2.4. A space satisfying conditions (1), (2), and (3) is finitely dominated.

Proposition 2.5. Suppose X satisfies conditions (1), (2), and (3). Then, for all $n > 0$, there's a finite CW complex Z of dimension less than n and an $(n-1)$ -connected map $Z \rightarrow X$.⁴

This follows from a more general fact.

Proposition 2.6. Suppose X satisfies conditions (1), (2), and (3), and suppose we are given an $(n-1)$ -connected map $f : Z \rightarrow X$, where Z is a finite CW complex. Then, there exists a space Z' , obtained from Z by adjoining finitely many n -cells, such that f factors through an n -connected map $Z' \rightarrow X$.

¹One can define this for the category of all projective R -modules, but this is always zero, thanks to the Eilenberg swindle.

²Sometimes, the base ring is different, but for our purposes, we'll prefer \mathbb{Z} .

³Since $\mathbb{Z}[G]$ is in general noncommutative, there's something to say here about left versus right actions.

⁴For a map to be $(n-1)$ -connected means that its homotopy fiber is $(n-1)$ -connected as a space, which implies that the induced map on π_k is an isomorphism for $k < n-1$ and is a surjection for $k = n-1$.

This allows us to inductively prove Proposition 2.5.

Lemma 2.7. *Let X be a space satisfying (1), (2), (3), and (4). Let Z be a finite CW complex of dimension at most $n - 1$ and $f : Z \rightarrow X$ be an $(n - 1)$ -connected map. Then, $H_n(X, Z; \mathbb{Z}[G])$ is a finitely generated projective $\mathbb{Z}[G]$ -module.*

This is where K -theory shows up.

Proof. Since Z is $(n - 1)$ -dimensional, then $H^i(Z, V) = 0$ for all $i \geq n$ and all local systems V . We have a long exact sequence for relative homology

$$\cdots \longrightarrow H^{i-1}(Z; V) \longrightarrow H^i(X, Z; V) \longrightarrow H^i(X, V) \longrightarrow \cdots,$$

and given a short exact sequence of local systems

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0,$$

and applying $H^*(X, Z; -)$ induces another long exact sequence

$$\cdots \longrightarrow H^n(X, Z; V') \longrightarrow H^n(X, Z; V) \longrightarrow H^n(X, Z; V'') \longrightarrow H^{n+1}(X, Z; V') \longrightarrow \cdots.$$

Using the universal coefficients theorem and the Hurewicz theorem, we have a natural isomorphism

$$H^n(X, Z; -) \cong \text{Hom}_{\mathbb{Z}[G]}(H_n(X, Z; \mathbb{Z}[G]), -).$$

The former is right exact, and therefore so is the latter, so $H_n(X, Z; \mathbb{Z}[G])$ is projective. \square

Thus, $H_n(X, Z; \mathbb{Z}[G])$ has a class in K -theory.

Definition 2.8. The *Wall finiteness obstruction* of X is $w(X) = (-1)^n [H_n(X, Z; \mathbb{Z}[G])] \in K_0(\mathbb{Z}[G])$.

We have a lot to show: that this is independent of n and Z , but also that it's at all related to finiteness.

Proposition 2.9. *The following are equivalent:*

- (1) X has the homotopy type of a finite CW complex.
- (2) $H_n(X, Z; \mathbb{Z}[G])$ is stably free (and hence trivial in $K_0(\mathbb{Z}[G])$).

In the reverse direction, the idea of the proof is to kill generators: if $H_n(X, Z; \mathbb{Z}[G]) \oplus \mathbb{Z}[G]^{\oplus r}$ is free, then it's equal to $H_n(X, Z \vee (S^n)^{\vee r}; \mathbb{Z}[G])$; then, one uses the Whitehead theorem and the relative Hurewicz theorem to kill homotopy groups. The point is that we have a map $Z \vee (S^n)^{\vee r} \rightarrow X$, where the domain is a finite CW complex; if we can show that this map induces an isomorphism on all homotopy groups, Whitehead's theorem proves that the map is a homotopy equivalence.

In the other direction, we can compute $H_*(X, Z; \mathbb{Z}[G])$ cellularly, and therefore get an exact sequence of free modules, which forces it to be stably free.

We'll skip the proofs of independence of Z and n , which are reasonably pretty, but quite long.

Another interesting fact is that if G is any group, we know $G = \pi_1(X)$ for some space X , but it's also true that any class in $K_0(\mathbb{Z}[G])$ is a Wall finiteness obstruction for some space X with $\pi_1(X) = G$.

Postscript. (added by Andrew Blumberg) The basic theorem at work in the development of the Wall finiteness obstruction is a result saying that a CW complex that is finitely dominated is finite if and only if a certain relative homology group was stably free.

(Recall that here stably free means that $M \oplus R^n \cong R^m$, for some n and m .)

There is a perspective from which it is now very natural to imagine K_0 entering the picture. Specifically, let's make the following two definitions:

Definition 2.10. Let M and N be objects of Mod_R that are f.g. and projective. Then M and N are *stably isomorphic* if there exists n such that $M \oplus R^n \cong N \oplus R^n$. M and N are *stably equivalent* if there exist m and n such that $M \oplus R^m \cong N \oplus R^n$.

Now, a lemma, which you should prove as an exercise (it's very easy).

Lemma 2.11. $[P] = [Q]$ in $K_0(R)$ if and only if P and Q are stably isomorphic.

Furthermore, if we define $\tilde{K}_0(R)$ to be the quotient of $K_0(R)$ by the image of \mathbb{Z} under the natural map that takes n to $[R^n]$, we have a corresponding result:

Lemma 2.12. $[P] = [Q]$ in $\tilde{K}_0(R)$ if and only if P and Q are stably equivalent.

As a consequence, it seems natural that if what you care about is a module being stably free, you might well look at K_0 or \tilde{K}_0 .

Remark.

- (1) Per Yuri's questions, indeed, finitely dominated CW complexes are finite for simply-connected spaces. Also, check out [1] for a discussion of applications of the Wall finiteness obstruction in surgery theory.
- (2) A natural question to ask is about whether (and when) stably free modules are actually free. Serre proved this as part of his work on the conjecture that all projective modules over a polynomial ring on a field k are free. Quillen and Suslin eventually proved the conjecture.

3. THE ALGEBRAIC K -THEORY OF THE SPHERE SPECTRUM: 9/21/16

Today, we have a guest: Mike Mandell (Indiana University) spoke, about the algebraic K -theory of the sphere spectrum, $K(\mathbb{S})$.

The K -theory of the sphere spectrum is very important; there are two way to explain why, either from the K -theory of rings and ring spectra or through Waldhausen's K -theory of spaces. In both cases, $K(\mathbb{S})$ is important.

If we wanted to talk about K -theory of rings and ring spectra, \mathbb{Z} is important, because it's the initial ring. Ring spectra are generalizations of rings in stable homotopy theory, and there's a simpler one than \mathbb{Z} : the sphere spectrum \mathbb{S} , which is the initial ring spectrum. If one thinks of rings as acting on abelian groups, which are \mathbb{Z} -modules, then one can adopt a similar approach to spectra: modules over a ring spectrum are spectra, which are modules over \mathbb{S} . So if you want to understand ring spectra, it's reasonable to start with \mathbb{S} (or, maybe it's the hardest, but we know it will be universal).

From the perspective of the K -theory of spaces, the simplest space is the one-point set $*$,⁵ and its K -theory $A(*)$ is equal to $K(\mathbb{S})$.

The algebraic K -theory of spaces arose from a lot of work in differential topology. Hatcher and others studied concordances and pseudo-isotopies: given a smooth manifold X , you might want to understand $\text{Diff}(X)$, the space of its diffeomorphisms. This is hard, but we can simplify to understanding the isotopies. An *isotopy* between two diffeomorphisms $f_0, f_1 : X \rightarrow X$ is a homotopy $X \times I \rightarrow X$ from f_0 to f_1 that restricts to a diffeomorphism over every $t \in I$. This weakens to a notion of *pseudoisotopies*, which can be stabilized by iterating the process: $X \times I^n \rightarrow X \times I^n$. Hatcher noticed this is combinatorial, and moreover, depends only on the homotopy type of X . Pseudoisotopies aren't a functor, but stabilized pseudoisotopies define a functor A , and Waldhausen noticed that it resembles Quillen's algebraic K -theory: specifically, $A(X) \cong K(\mathbb{S}[\Omega X])$. Here, ΩX is the loop space, so X ought to have a basepoint!

ΩX isn't quite a group, but it's not far from one: it has a homotopy-associative binary operation. So just as we can form a group ring $\mathbb{Z}[G]$ given a group G , it's possible to take this homotopical grouplike object and adjoin it to the sphere spectrum. If $X = *$, then $\Omega X = *$, so $K(\mathbb{S}[\Omega X]) = K(\mathbb{S})$. This allows one to use K -theory techniques to study stable, high-dimensional differential topology. This is all happening in the 1980s.

Now you ask, what can you say with this? This is where the story starts: we want to understand the homotopy type of $K(\mathbb{S})$, which is a spectrum. Since ring spectra generalize rings, and \mathbb{S} is the initial ring spectrum, there's a unique map $\mathbb{S} \rightarrow \mathbb{Z}$, and applying K to it, we obtain a map $K(\mathbb{S}) \rightarrow K(\mathbb{Z})$.

In the 1990s, Bökstedt, Hsiang, Madsen, and Goodwillie defined *topological cyclic homology* $TC(R)$ for any ring spectrum R , which is a version of cyclic homology, but constructed using purely spectrum-level information, and is purely (equivariant) stable homotopy theory. This arose from another spectrum-level construction called *topological Hochschild homology* using some operations defined on it. Topological cyclic homology is built precisely for homotopy theorists to be able to calculate with it, and, in a similar way to the relation of K -theory to cyclic homology, there's a natural transformation $K \rightarrow TC$.

If R is a connective ring spectrum, there's a map of ring spectra $R \rightarrow \pi_0 R$, and the natural transformation induces a commutative square

$$\begin{array}{ccc} K(R) & \longrightarrow & K(\pi_0(R)) \\ \downarrow & & \downarrow \\ TC(R) & \longrightarrow & TC(\pi_0(R)). \end{array} \tag{3.1}$$

⁵This construction requires basepoints, so we can't consider the empty set.

Applying this to $R = \mathbb{S}$, we obtain a commutative square relating K -theory and topological cyclic homology of \mathbb{S} and $\pi_0 \mathbb{S} = \mathbb{Z}$:

$$\begin{array}{ccc} K(\mathbb{S}) & \longrightarrow & K(\mathbb{Z}) \\ \downarrow & & \downarrow \\ TC(\mathbb{S}) & \longrightarrow & TC(\mathbb{Z}). \end{array}$$

We will now complete at a prime p . For most of this talk, we'll need p to be odd, but for right now, $p = 2$ also works. Recall that a finite abelian group A decomposes as a direct sum

$$A \cong \mathbb{Z}/p^{r_1} \oplus \mathbb{Z}/p^{r_2} \oplus \cdots \oplus \mathbb{Z}/p^{r_n} \oplus A^1,$$

where $|A^1|$ has order coprime to p . We'd like to only consider the parts that p knows about, so p -completing A comes down to throwing out A^1 . This is useful to simplify problems.

p -completion is a functor, though maybe a surprising one. Modding out by any p^n is bad, because you could have some $r^i > n$, so you have to take a limit: p -completion is the functor

$$A_p^\wedge = \varinjlim_n A/p^n,$$

and it comes with a natural transformation $A \rightarrow A_p^\wedge$. We can take the product of all of these functors and map

$$A \mapsto \prod_{p \text{ prime}} A_p^\wedge, \quad (3.2)$$

and for finite abelian groups, this is an isomorphism.

If A is instead finitely generated, this is no longer true: the p -completion of a free factor \mathbb{Z} is the p -adic integers \mathbb{Z}_p , which isn't isomorphic to \mathbb{Z} . But it's still possible to recover a lot of information about a finitely generated abelian group from its image under (3.2). Specifically, there is a density theorem after tensoring with \mathbb{Q} : consider an element in $\widehat{\mathbb{Z}} = \prod_{p \text{ prime}} \mathbb{Z}_p$, and suppose it winds up in \mathbb{Q} after tensoring with \mathbb{Q} . Then, it must have come from an integer repeating. In other words, the following diagram is a pullback diagram:

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \prod_{p \text{ prime}} \mathbb{Z}_p \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{Q} & \longrightarrow & \mathbb{Q} \otimes \prod_{p \text{ prime}} \mathbb{Z}_p. \end{array}$$

This means we can recover $A \otimes \mathbb{Q}$ from the product of all of the p -completions of A .

In the world of spectra, p -completion is instead the homotopy limit $A_p^\wedge = \text{holim}_n A/p^n$; akin to tensoring with \mathbb{Q} is an operation called *rationalization*, which applies $-\otimes \mathbb{Q}$ to homotopy groups, and these combine into a homotopy pullback diagram

$$\begin{array}{ccc} A & \longrightarrow & \prod_{p \text{ prime}} A_p^\wedge \\ \downarrow & & \downarrow \\ A_{\mathbb{Q}} & \longrightarrow & \left(\prod_{p \text{ prime}} A_p^\wedge \right)_{\mathbb{Q}}. \end{array}$$

What does it mean for this to be a homotopy pullback? There are several equivalent characterizations: that the homotopy fibers of the horizontal arrows agree, or their homotopy cofibers, or the homotopy fibers (or cofibers) of the vertical maps agree. And, as with abelian groups, from the collection of p -completions, one can recover the rationalization.

The map of ring spectra $\mathbb{S} \rightarrow \mathbb{Z}$ is an isomorphism on π_0 and a rational equivalence on π_n where $n > 0$; since \mathbb{S} and \mathbb{Z} are connective, we don't have to worry about negative homotopy groups. This also implies $K(\mathbb{S}) \rightarrow K(\mathbb{Z})$ is a

rational equivalence.⁶ Borel calculated $K(\mathbb{Z})_{\mathbb{Q}}$ in the 1970s, so we know the homotopy groups are

$$\pi_n K(\mathbb{S})_{\mathbb{Q}} = \begin{cases} \mathbb{Q}, & n = 0 \\ 0, & n = 1 \\ 0, & n \equiv 2, 3, 0 \pmod{4}, n \neq 0 \\ \mathbb{Q}, & n \equiv 1 \pmod{4}, n > 1. \end{cases}$$

Now, let's fix a prime p and try to understand $K(\mathbb{S})_p^{\wedge}$. There's a theorem of Dundes that says if you p -complete the K -theory and TC square (3.1), the result is a homotopy Cartesian diagram. So if we understand $\mathrm{TC}(\mathbb{S})_p^{\wedge}$, $\mathrm{TC}(\mathbb{Z})_p^{\wedge}$, and $K(\mathbb{Z})_p^{\wedge}$, we can piece them together and determine $K(\mathbb{S})_p^{\wedge}$.

Bökstedt, Hsiang, and Madsen calculated

$$\mathrm{TC}(\mathbb{S})_p^{\wedge} \simeq \mathbb{S}_p^{\wedge} \vee \Sigma(\mathbb{CP}_{-1}^{\infty})_p^{\wedge}.$$

The latter spectrum is related to two-dimensional topological field theories! But this splitting doesn't yet admit a geometric explanation.

If you're familiar with Thom spectra, this isn't very different. Let γ_n be the tautological bundle over \mathbb{CP}^n , and consider the *Thom space* $T(\gamma_n^{\perp})$ (which has for its total space the one-point compactification of γ_n). We want it to be the negative of γ_n , so we desuspend $2n$ times. We can stitch these together over various n to obtain

$$\mathbb{CP}_{-1}^{\infty} = \bigcup_n \Sigma^{-2n} T(\gamma_n^{\perp}).$$

This is an example of a *Madsen-Tillman spectrum*, and it appears in the proof of the Mumford conjecture. It's not simple *per se*, but thanks to the Thom isomorphism, we can compute its homology and cohomology over complex-oriented cohomology theories. For example, its rational homology is free on *Mumford classes* k_0, k_1, \dots .

Bökstedt-Madsen calculated $\mathrm{TC}(\mathbb{Z})_p^{\wedge}$; it's easier to describe its connective cover, which is

$$j \vee \Sigma j \vee \Sigma^3 ku_p^{\wedge}.$$

Let's talk about the components.

- Topological periodic complex K -theory (i.e. starting with complex vector bundles, after Atiyah-Hirzebruch) defines an extraordinary cohomology theory represented by a spectrum KU ; ku is its connective cover.
- j is the p -completion of the *image of j* spectrum. The Adams operations on K -theory define Adams operations $ku \rightarrow ku$; if you choose a particular Adams operation and subtract one, then j arises as the fiber. Alternatively, take the localization of \mathbb{S} at KU , p -complete it, and take its connective cover; this describes j .

The third ingredient is $K(\mathbb{Z})_p^{\wedge}$: this isn't fully understood yet, but it splits as a sum

$$K(\mathbb{Z})_p^{\wedge} = j \vee y_0 \vee \dots \vee y_{p-2}.$$

For i odd, y_i is related to the Bernoulli numbers. For i even, this isn't well understood yet, and understanding it better will require addressing the Vandiver conjecture in number theory. This would imply they break down as suspensions of Adams summands of ku .

What about the maps between these spectra? We don't know much about $\mathrm{TC}(\mathbb{S})_p^{\wedge} \rightarrow \mathrm{TC}(\mathbb{Z})_p^{\wedge}$; because $\Sigma \mathbb{CP}_{-1}^{\infty}$ is a Thom spectrum and ku is complex-oriented, we know what the group of homotopy classes of maps between them is, but we don't know which specific map it is. The map $K(\mathbb{Z})_p^{\wedge} \rightarrow \mathrm{TC}(\mathbb{Z})_p^{\wedge}$ is better understood: the j maps to the j , the Σj in $\mathrm{TC}(\mathbb{Z})_p^{\wedge}$ is missed, and the remaining $p-1$ summands map diagonally, especially if we assume the Vandiver conjecture. This in general is related to p -adic ℓ -functions, which aren't well-understood yet (do they take the value 0? This is open). So at a fundamental level, we don't really understand these maps, even if we can write them in terms of existing names.

⁶A *rational equivalence* of two spaces or two spectra is a map $X \rightarrow Y$ that induces an isomorphism $\pi_n(X) \otimes \mathbb{Q} \rightarrow \pi_n(Y) \otimes \mathbb{Q}$ for all n .

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