PHY392T NOTES: TOPOLOGICAL PHASES OF MATTER

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SEPTEMBER 5, 2019

These notes were taken in UT Austin's PHY392T (Topological phases of matter) class in Fall 2019, taught by Andrew Potter. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Any mistakes in the notes are my own.

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Today we'll describe second quantization as a convenient way to describe many-particle quantum-mechanical systems.

In "first quantization" (only named because it came first) one considers a system of N identical particles, either bosons or fermions. The wavefunction $\psi(r_1, \dots, r_N)$ is redundant: if σ is a permutation of $\{1, \dots, N\}$, then

$$\psi(r_1, \dots, r_n) = (\pm 1)\psi(r_{\sigma(1)}, \dots, r_{\sigma(N)}), \tag{2.1}$$

where the sign depends on whether we have bosons or fermions, and on the parity of σ.

For fermionic systems specifically, $\psi(r_1, \dots, r_N)$ is the determinant of an $N \times N$ matrix, which leads to an exponential amount of information in N. It would be nice to have a more efficient way of understanding many-particle systems which takes advantage of the redundancy (2.1) somehow; this is what second quantization does.

Another advantage of second quantization is that it allows for systems in which the total particle number can change, as in some relativistic systems.

The idea of second quantization is to view every degree of freedom as a quantum harmonic oscillator

$$H \coloneqq \frac{1}{2}\omega^2(p^2 + x^2). \tag{2.2}$$

We set the lowest eigenvalue to zero for convenience. If $a \coloneqq (x + ip)/\sqrt{2}$ and $a^\dagger \coloneqq (x - ip)/\sqrt{2}$, then $\hat{n} \coloneqq a^\dagger a$ computes the eigenvalue of an eigenstate.

Now let's assume our particles are all identical bosons. Then we introduce these operators $a_\sigma(\mathbf{r}), a_\sigma^\dagger(\mathbf{r})$ which behave as annihilation, respectively creation operators, in that they satisfy the commutation relations

$$\begin{gathered} [a_\sigma^\dagger(\mathbf{r}), a_{\sigma'}(\mathbf{r}')] = -\delta_{\sigma\sigma'}\delta(\mathbf{r} - \mathbf{r}') \\ [a^\dagger, a^\dagger] = 0. \end{gathered} \tag{2.3}$$

The Hamiltonian is generally of the form

$$H := \sum_{\sigma,\sigma'} \int_{\mathbf{r},\mathbf{r}'} a^\dagger_\sigma(\mathbf{r}) h_{\sigma\sigma'}(\mathbf{r}-\mathbf{r}') a_{\sigma'}(\mathbf{r}) + V_{\alpha\beta\gamma\delta} a^\dagger_\alpha a^\dagger_\beta a_\gamma a_\delta, \tag{2.4}$$

where the first term is the free part and the second term determines a two-particle interaction.

Letting $n_\sigma(\mathbf{r}) := a^\dagger_\sigma(\mathbf{r}) a_\sigma(\mathbf{r})$, which is called the *number operator* (since it counts the number of particles in state σ), there is a state $|\varnothing\rangle$ called the *vacuum* which satisfies $n_\sigma(\mathbf{r})|\varnothing\rangle = 0$ and $a_\sigma(\mathbf{r})|\varnothing\rangle = 0$. Particle creation operators commute, in that

$$a^\dagger(\mathbf{r}_1) a^\dagger(\mathbf{r}_2)|\varnothing\rangle = a^\dagger(\mathbf{r}_2) a^\dagger(\mathbf{r}_1)|\varnothing\rangle. \tag{2.5}$$

This is encoding that the particles are bosons: we exchange them and nothing changes.

The fermionic story is similar, but things should anticommute rather than commute. Letting α be an index, let f_α, resp. $f^\dagger_\alpha$ be the annihilation, resp. creation operators for a fermion in state α. There's again a vacuum $|\varnothing\rangle$, with $f_\alpha|\varnothing\rangle = 0$ for all α. Now we impose the relation

$$f^\dagger_\alpha f^\dagger_\beta |\varnothing\rangle = -f^\dagger_\beta f^\dagger_\alpha |\varnothing\rangle. \tag{2.6}$$

That is, define the *anticommutator* by

$$\{f^\dagger_\alpha, f^\dagger_\beta\} := f^\dagger_\alpha f^\dagger_\beta + f^\dagger_\beta f^\dagger_\alpha. \tag{2.7}$$

Then we ask that $\{f^\dagger_\alpha, f^\dagger_\beta\} = 0$, and $\{f^\dagger_\alpha, f_\beta\} = \delta_{\alpha\beta}$.

Again we have a number operator $n_\alpha := f^\dagger_\alpha f_\alpha$; it satisfies $n_\alpha f_\alpha = f_\alpha(n_\alpha - 1)$, and measures the number of particles in the state α. Because

$$(f^\dagger_\alpha)^2 = f^\dagger_\alpha f^\dagger_\alpha = -f^\dagger_\alpha f^\dagger_\alpha = 0, \tag{2.8}$$

then n_α is a projector (i.e. $n_\alpha^2 = n_\alpha$), and therefore its eigenvalues can only be 0 or 1. This encodes the Pauli exclusion principle: there can be at most a single fermion in a given state.

We'd like to write our second-quantized systems with quadratic Hamiltonians, largely because these are tractable. Let $(h_{\alpha\beta})$ be a self-adjoint matrix and consider the Hamiltonian

$$H := \sum_{\alpha,\beta} f^+_\alpha h_{\alpha\beta} f_\beta. \tag{2.9}$$

The *number operator* $N := \sum n_\alpha$ commutes with the Hamiltonian, which therefore defines a symmetry of the system. The associated conserved quantity is the particle number. Slightly more explicitly, we have a symmetry of the group U_1 (i.e. the unit complex numbers under multiplication): for $\theta \in [0, 2\pi)$, let

$$u_\theta := \exp(i\theta N). \tag{2.10}$$

Then

$$u^\dagger_\theta H u_\theta = \sum_{\alpha,\beta} \underbrace{u^\dagger_\theta f^\dagger_\alpha u_\theta}_{=e^{-i\theta} f^\dagger_\alpha} h_{\alpha\beta} \underbrace{u^\dagger_\theta f_\beta u_\theta}_{=e^{i\theta} f_\beta} = H. \tag{2.11}$$

When you see a Hamiltonian, you should feel a deep-seated instinct to diagonalize it: we want to find $\lambda_n, v^{(n)}$ such that $h_{\alpha\beta} v^{(n)}_\beta = \lambda_n v^{(n)}_\alpha$ and $vv^\dagger = \mathrm{id}$. Let $v_{n\alpha} := v^{(n)}_\alpha$ and

$$\psi_n := \sum_\alpha v_{n\alpha} f_\alpha. \tag{2.12}$$

Then $\psi^\dagger_n$ and ψ_n satisfy the same creation and annihilation relations as $f^\dagger_\alpha$ and f_α:

$$\begin{aligned}
\{\psi^\dagger_n, \psi_m\} &= \{\sum_\alpha v^*_{n\alpha} f^\dagger_\alpha, \sum_\beta v_{m\beta} f_\beta\} && (2.13)\\
&= \sum_{\alpha,\beta} v^*_{n\alpha} v_{m\beta} \underbrace{\{f^\dagger_\alpha, f_\beta\}}_{=\delta_{\alpha\beta}} && (2.14)\\
&= \sum_\alpha v_{m\alpha} (v^\dagger)_{n\alpha} = \delta_{m,n}. && (2.15)
\end{aligned}$$

Let $\hat{n}_n \coloneqq \psi_n^\dagger \psi_n$. Now the Hamiltonian has the nice diagonal form

$$H = \sum_n \lambda_n \psi_n^\dagger \psi_n, \tag{2.16}$$

and we can explicitly calculate its action on a state:

$$H\psi_{n_1}^\dagger \psi_{n_2}^\dagger \cdots \psi_{n_N}^\dagger |\varnothing\rangle = \underbrace{\left(\sum_m \lambda_m \psi_m^\dagger \psi_m \psi_{n_1}^\dagger\right)}_{(*)} \psi_{n_2}^\dagger \cdots \psi_{n_N}^\dagger |\varnothing\rangle. \tag{2.17}$$

The term $(*)$ is equal to

$$\psi_m^\dagger(\delta_{mn} - \psi_{n_1}^\dagger \psi_m) = \delta_{mn_1}\psi_{n_1}^\dagger + \psi_{n_1}^\dagger \psi_m^\dagger \psi_m. \tag{2.18}$$

Then (2.17) is equal to

$$(2.17) = \lambda_{n_1}\psi_{n_1}^\dagger (\psi_{n_2}^\dagger \cdots \psi_{n_N}^\dagger)|\varnothing\rangle, \tag{2.19}$$

so we've split off a term and can induct. The final answer is

$$= \left(\sum_{i=1}^N \lambda_i\right) \psi_{n_1}^\dagger \cdots \psi_{n_N}^\dagger |\varnothing\rangle. \tag{2.20}$$

Example 2.21 (1d tight binding model)**.** Let's consider the system on a circle with L sites (you might also call this periodic boundary conditions). We have operators which create fermions at each state and also some sort of tunneling operators. The Hamiltonian is

$$H \coloneqq -t\sum_{j=1}^L (f_{j+1}^\dagger f_j + f_j^\dagger f_{j+1}) - \mu \sum_{j=1}^L f_j^\dagger f_j, \tag{2.22}$$

where $j+1$ is interpreted mod L as usual. One of t and N (TODO: which?) can be interpreted as the chemical potential. The eigenstates are the Fourier modes

$$\psi_k \coloneqq \frac{1}{\sqrt{L}} \sum_{j=1}^L e^{ikj} f_j, \tag{2.23}$$

where $k = 2\pi n/L$. Hence in particular $e^{ik(L+1)} = e^{ik}$. Now we can compute

$$\sum_{j=1}^L f_{j+1}^\dagger f_j = \frac{1}{L}\sum_{j,k,k'} e^{ik'(j+1)} e^{-ikj} \psi_{k'}^\dagger \psi_k \tag{2.24}$$

$$= \frac{1}{L}\sum_{k,k'} e^{ik'} \sum_j e^{ij(k-k')} \psi_{k'}^\dagger \psi_k \tag{2.25}$$

$$= \sum_k e^{ik} \psi_k^\dagger \psi_k. \tag{2.26}$$

That is, the diagonalized Hamiltonian is

$$H = \sum_{k=1}^L (-2t\cos k - \mu)\psi_k^\dagger \psi_k. \tag{2.27}$$

You can plot λ_k as a function of k, but really k is defined on the circle $\mathbb{R}/2\pi\mathbb{Z}$, which is referred to as the *Brillouin zone*. The ground state of the system is to fill all states with negative energy:

$$|\mathrm{G.S.}\rangle = \left(\prod_{k:\lambda_k<0} \psi_k^\dagger\right)|\varnothing\rangle. \tag{2.28}$$

If L is fixed, k only takes on L different values, but implicitly we'd like to take some sort of thermodynamic limit $L \to \infty$, giving us the actually smooth function $\lambda_k = -2t\cos k - \mu$. ◀

We said that second quantization is useful when the particle number can change, so let's explore that now. This would involve a Hamiltonian that might look something like

$$H = f_\alpha^\dagger h_{\alpha\beta} f_\beta + \frac{1}{2}\Big(\Delta_{\alpha\beta} f_\alpha^\dagger f_\beta^\dagger + \Delta_{\alpha\beta}^\dagger f_\alpha f_\beta\Big). \tag{2.29}$$

These typically arise in mean-field descriptions of superconductors. This typically arises in situations where electrons are attracted to each other — this is a little bizarre, since electrons have the same charge, but you can imagine an electron moving in a crystalline solid with some positive ions. The electron attracts the ions, but they move more slowly, so the electron keeps moving and we get an accumulation of positive charge, and this can attract additional electrons.

This binds pairs of electrons together at a certain point, and this forms a *condensate*, i.e. a superposition of states with different particle numbers. (2.29) describes a superconducting condensate, in which $\Delta_{\alpha\beta}$ describes pairs of particles appearing or disappearing in the condensate. To learn more, take a solid-state physics class.

Remark 2.30. You have to have pairs of fermionic terms — if you try to include an odd number of fermions, or a single fermionic term, you'll get nonlocal interactions between the lone fermion and others. Thus, even though the particle number is not conserved, its value mod 2, which is called *fermion parity*, is conserved. ◀

If you try to directly diagonalize (2.29), some weird stuff happens, so we'll rewrite the Hamiltonian such that it looks like it's particle-conserving, and then apply our old trick. This approach is due to Nambu. Let

$$\Psi_{\alpha,\tau} \coloneqq \begin{pmatrix} f_\alpha \\ f_\alpha^\dagger \end{pmatrix}, \tag{2.31}$$

where τ denotes the vertical index. We can rewrite the Hamiltonian as

$$H = \frac{1}{2}\begin{pmatrix} f_\alpha^\dagger & f_\alpha \end{pmatrix} \begin{pmatrix} h_{\alpha\beta} & \Delta_{\alpha\beta} \\ \Delta_{\alpha\beta}^\dagger & -h_{\alpha\beta}^\dagger \end{pmatrix} \begin{pmatrix} f_\beta \\ f_\beta^\dagger \end{pmatrix} + (\text{constant}) = \frac{1}{2}\Psi_{\alpha\tau}^\dagger \mathcal{H}_{\alpha\beta\tau\tau'} \Psi_{\beta\tau'}. \tag{2.32}$$

However, Ψ and $\Psi^\dagger$ have some redundancy: if σ^x denotes the Pauli matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, then $\Psi_\tau^\dagger = \sigma_{\tau\tau'}^x \Psi_{\tau'}$.

This is telling us that Ψ and $\Psi^\dagger$ create particles with energies (say) e and $-e$, respectively. Now

$$\Psi^\dagger H \Psi = \Psi^{\mathrm{T}} \sigma^x H \sigma^x (\Psi^\dagger)^{\mathrm{T}} = -\Psi^\dagger \sigma^x H^{\mathrm{T}} \sigma^x \Psi, \tag{2.33}$$

and therefore $\mathcal{H} = -\sigma^x \mathcal{H}^* \sigma^x$. Using this, we can determine the eigenstates: $\mathcal{H}v = Ev$ iff $\mathcal{H}\sigma^x v^* = -E\sigma^x v^*$. Then

$$\mathcal{H}\sigma^x v^* = \sigma^x(\sigma^x \mathcal{H}\sigma^x) v^* = \sigma^x(\sigma^x \mathcal{H}^* \sigma^x v)^* = \sigma^x(-\mathcal{H}v)^* = -E\sigma^x v^* \tag{2.34}$$

and

$$\gamma_E \coloneqq \sum_{\alpha,\tau} v_{\alpha\tau} \Psi_{\alpha\tau} \tag{2.35}$$

satisfies $\gamma_{-E} = \gamma_E^\dagger$. TODO: what are we trying to show here?

This $E \leftrightarrow -E$ symmetry is an instance of what's traditionally called "particle-hole symmetry," but it's a little weird — we can't break this symmetry by introducing additional terms to the Hamiltonian. So it might be more accurate to call it *particle-hole structure*, which conveniently has the same acronym.

TODO: some other stuff I missed. I think $\{\Psi_{\alpha\tau}, \Psi_{\beta\tau'}^\dagger\} = \delta_{\alpha\beta}\delta_{\tau\tau'}$ and $\{\gamma_E, \gamma_{E'}^\dagger\} = \delta_{EE'}$, which tells us these (I think) behave like creation and annihilation operators.

At zero energy, $\gamma_0 = \gamma_0^\dagger$, so we have a fermion which is its own antiparticle. This is called a *Majorana fermion*. It will be helpful to have a slightly different normalization here, which we'll discuss more later.

Lecture 3.

The Majorana chain: 9/5/19

Today we will discuss a one-dimensional system studied by Kitaev [Kit01]. Introduce periodic boundary conditions, so that the sites live on a circle with length L. At each site i, we have a local Hilbert space $\mathcal{H}_i \coloneqq \mathbb{C}\cdot\{|0\rangle, |1\rangle\}$, and the total Hilbert space of states is the tensor product of these over all of the states.

Let c_j and $c_j^\dagger$ denote the annihilation, resp. creation operators at site j. Then the Hamiltonian is

$$H \coloneqq -\sum_{j=1}^{L} t(c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1}) - \mu \sum_{j=1}^{L} c_j^\dagger c_j - \Delta(c_{j+1}^\dagger c_j^\dagger + c_j c_{j+1}). \tag{3.1}$$

Here t, Δ, and μ are parameters; μ is called the *chemical potential*.

To solve this Hamiltonian, we will introduce a different set of creation and annihilation operators: let

$$\tilde{c}_k \coloneqq \frac{1}{\sqrt{L}} \sum_{j=1}^{L} e^{ikj} c_j \tag{3.2a}$$

$$\tilde{c}_k^\dagger \coloneqq \frac{1}{\sqrt{L}} \sum_{j=1}^{L} e^{ikj} c_j^\dagger. \tag{3.2b}$$

Here $k \in 2\pi n/L$, like last time, and we only consider those k in $[0,1)$.[1] Using these, we can rewrite (3.1) as

$$H = \sum_k (-2t\cos k - \mu)\tilde{c}_k^\dagger \tilde{c}_k - \Delta \sum_k \left(e^{ik} \tilde{c}_k^\dagger \tilde{c}_{-k}^\dagger + e^{ik} \tilde{c}_{-k} \tilde{c}_k \right). \tag{3.3}$$

To get the last term, use the fact that $\tilde{c}_k^\dagger \tilde{c}_{-k}^\dagger = -\tilde{c}_{-k}^\dagger \tilde{c}_k^\dagger$, so

$$\frac{1}{2} \sum_k \tilde{c}_k^\dagger \tilde{c}_{-k}^\dagger e^{ik} + \frac{1}{2} \sum_k \left(-e^{-ik} \tilde{c}_k^\dagger \tilde{c}_{-k}^\dagger \right). \tag{3.4}$$

Again introduce the Nambu spinor $\Psi \coloneqq \begin{pmatrix} \tilde{c}_k \\ \tilde{c}_k^\dagger \end{pmatrix}$; then we can rewrite (3.3) as

$$H = \frac{1}{2} \sum_k \Psi_k^\dagger \begin{pmatrix} -2t\cos k - \mu & 2i\Delta \sin k \\ -2i\Delta \sin k & 2t\cos k + \mu \end{pmatrix} \Psi_k. \tag{3.5}$$

So now all we have to do is diagonalize a 2×2 matrix, which isn't so hard. In particular, the eigenvalues (energy levels) are

$$E_k = \pm \frac{1}{2} \sqrt{(2t\cos k + \mu)^2 + (2\Delta \sin k)^2}. \tag{3.6}$$

In particular, we can plot these as k varies and see whether the system is gapped.

- Suppose $\Delta = \mu = 0$. Then there are values of k such that the spectrum isn't gapped, but as soon as you make $\Delta \neq 0$, there is a spectral gap.
- If $\mu = -2t$, we again close the gap at $\Delta = 0$ and $k = 0$, but in general there is a gap.

So the phase diagram in μ appears to have three phases and two transitions between them, and is symmetric about $\mu \mapsto -\mu$. For $\mu \to -\infty$, this is adiabatically connected to a trivial phase, and thus is itself trivial: there are no particles. For $\mu \to \infty$, it is also trivial: every site is occupied in the ground state. The third phase is a topological superconductor (though we have yet to show it).

So the two phase transitions happen at $\mu = \pm 2t$. Suppose $\mu = -2t + M$, where M is close to zero, so that we can study the phase transition. Since $-2t\cos k + 2t = O(k^2)$, we'll ignore it, and therefore replace M with $M - 2t\cos k + 2t$. Similarly,

$$2i\Delta \sin k = 2i\Delta k + O(k^3), \tag{3.7}$$

and we will drop the higher-order terms. Under these approximations, the Hamiltonian now is

$$\begin{aligned} H &\approx \frac{1}{2} \sum_k \Psi + k^\dagger \left(\begin{pmatrix} M & 0 \\ 0 & -M \end{pmatrix} + \begin{pmatrix} 0 & 2 - \Delta \\ -2i\Delta & 0 \end{pmatrix} k \right) \Psi_k \\ &= \frac{1}{2} \sum_k \Psi_k^\dagger (M\sigma^z - 2k\Delta\sigma^y) \Psi_k. \end{aligned} \tag{3.8}$$

[1]To me (Arun), this looks like k is in the Pontrjagin dual $(\mathbb{Z}/L)^\vee$, which would be appropriate if this is a Fourier transform.

Here σ^z and σ^y are the usual Pauli matrices. Now (3.8) looks like a Dirac equation with a mass term: letting $a := -2\Delta$, we get

$$H \approx \frac{1}{2}\sum_k \Psi_k^\dagger (ak\sigma^y + M\sigma^z)\Psi_k \approx \frac{1}{2}\int \mathrm{d}x\, \psi^\dagger(x)(-iv\partial_x\sigma^y + M(x)\sigma^z)\psi(x). \tag{3.9}$$

Let's let M vary in space, so that we have a defect between the two phases at $x = 0$. We'll show that the defect has a bound state.

Consider the $E = 0$ solution to the continuum approximation in (3.9). Then

$$\tilde{\gamma} = \int \mathrm{d}x\, \phi_\alpha(x)\hat{\psi}_\alpha(x), \tag{3.10}$$

and we end up with an ordinary differential equation for the solution:

$$(-iv\partial_x\sigma^y + M(x)\sigma^z)\phi_\alpha(x) = 0, \tag{3.11}$$

and therefore $\frac{\partial}{\partial x}\phi_\pm = \pm(M/v)\phi$ (TODO: some details here are missing). Therefore

$$\phi_\pm(x) = \exp\left(\pm\int_0^x \frac{M(x)}{v}\,\mathrm{d}x\right)\phi_\pm(0). \tag{3.12}$$

One of these blows up at infinity and makes no physical sense, but there is a solution which is largest at $M = 0$ and decays to zero at infinity. This is the bound zero mode ϕ_+. Here are a few more facts about this zero mode.

- As $x \to \pm\infty$, $\phi_+(x) \to \exp(-|M_0||x|)$.
- We didn't use much about $M(x)$, only the fact that it switches sign at $M = 0$. This is the sense in which the zero mode is topological: we can deform $M(x)$ and obtain the same behavior.[2]
- $\gamma = \gamma^\dagger$: in a sense, this mode is both a creation and annihilation operator. For this reason, it's called a *Majorana zero mode*.
- The side of these bound states is determined by the *correlation length* $v/M = 1/3$.

This is not a critical system — besides this zero mode, all other phases are gapped. For $M > 0$, we get a trivial insulator, and for $M < 0$, we have the topological phase, a topological superconductor. The bound state at the defect is what implies that the $M < 0$ phase isn't trivial.

How realistic are the periodic boundary conditions? Well, we can't create an infinite wire in the lab, so maybe we should work on the unit interval of length L, which is large with respect to the correlation length. Then, you maybe can convince yourself that there are two Majorana modes, one at each boundary site, and they overlap a little bit in the bulk, approximately at order $e^{-L/3}$. Call these modes γ_L and γ_R. If you let $\psi = (\gamma_M + i\gamma_R)/2$ and $\psi^\dagger = (\gamma_L - i\gamma_R)/2$, then these satisfy the anticommutation relations of creation and annihilation operators of ordinary fermions: for example, $\{\psi, \psi^\dagger\} = 1$. This is a little bit weird.

Another weird aspect of this system is that if L is large enough, you can't couple to both γ_L and γ_R at the same time. If you tried to perturb the system, say by introducing a bosonic field with an electric potential $V = \phi\gamma_L$, well, that's not allowed, because you would get an odd number of fermions. So these Majorana modes are protected by small perturbations, and in that sense might be useful if you care about quantum memory. The drawback is that you can't put a state with an even number of fermions and a state with an odd number of fermions into superposition, which is unfortunate; the solution is to consider several separate copies of this system.

So let's work with N wires, meaning we have $2N$ Majorana zero modes $\gamma_1, \ldots, \gamma_{2N}$, hence N ordinary fermion creation/annihilation operator pairs $\psi^\dagger, \psi$ as we discussed above. This system has a 2^N-dimensional space of ground states: for any subset $S \subset \{1, \ldots, n\}$, we say that the fermion state ψ is occupied for $i \in S$ and unoccupied for $i \notin S$.

The fermion parity

$$P_f := \prod_{i=1}^N i\gamma_{2i-1}\gamma_{2i} \in \{\pm 1\} \tag{3.13}$$

[2] This is an instance of a very general theorem in mathematics called the Atiyah-Singer index theorem, which can be used to produce zero modes in fermionic systems.

is a conserved quantum number of this system (intuitively, it tells us whether there are an even or odd number of fermions present). Therefore we have 2^{N-1} states available as quantum memory.

These give us different ways to label the ground states, but different labelings interact in complicated ways. For example, if $N = 2$,

$$|P_{14} = 1, P_{23} = 1\rangle = \frac{1}{\sqrt{2}}(|P_{12} = 1, P_{34} = 1\rangle - |P_{12} = -1, P_{34} = -1\rangle). \tag{3.14}$$

You can imagine this as follows: we begin with no particles, and create two fermions on each copy of the chain. This doesn't change the parity, because we created them from nothing. Now, we smush together sites 2 and 3 and measure there, and get zero. Then, this is telling us that the remaining states are maximally uncertain. This was an operator

$$|{+}{+}\rangle \longmapsto \frac{1}{\sqrt{2}}(|{+}{+}\rangle - |{-}{-}\rangle). \tag{3.15}$$

This is a topologically protected operation, which is exciting if you want to make quantum computers. But it's proven to not be universal, i.e. we can't get (or even well-approximate) all quantum gates in the Majorana chain. In fact, what we get can be efficiently approximated by a classical computer, and this isn't even universal for classical computer! But there are other examples of phases which are universal for quantum computing, and Microsoft is researching how these might be actually implemented.

These states can have (quasi)particle modes akin to the Majorana modes here. In general these are called *nonabelian anyons* or *nonabelian defects*. The defining property of these is that there is a topologically protected ground state degeneracy associated to the zero modes, and it grows exponentially in the number of particles present. The process of turning two particles into something else will be called *fusion*; for the Majorana chain we have the relation

$$\gamma \times \gamma = 0 + f, \tag{3.16}$$

as we either get nothing or a fermion. This is akin to the fact that if we collide two spin-1/2 particles, they could annihilate each other or produce a spin-1 particle. The fact that the Hilbert space grows exponentially is reminiscent of the fact that for a spin-s particle, the Hilbert space of states has dimension $2s + 1$ to the number of particles: the local dimension is the number of objects. Here, though, we will encounter examples of nonabelian anyons whose quantum dimensions are irrational.

Next time, we'll argue that the Majorana chain is the only nontrivial topological phase that can occur among 1D superconductors (unless we add additional symmetries to the Hamiltonian). We'll also discuss how to see that the phase is nontrivial in the bulk; after that, we'll discuss some possible physical realizations in real system, such a spin-orbit coupled semiconductor wire, put in contact with a normal superconductor.

References

[Kit01] A. Yu. Kitaev. Unpaired Majorana fermions in quantum wires. *Physics-Uspekhi*, 44(10S):131, 2001. https://arxiv.org/abs/cond-mat/0010440. 4