SPRING 2017 GEOMETRIC LANGLANDS SEMINAR

ARUN DEBRAY
FEBRUARY 6, 2017

These notes were taken in David Ben-Zvi's student seminar in Spring 2017, with lectures given by David Ben-Zvi. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to `a.debray@math.utexas.edu`. Thanks to Roc Gregoric for a few corrections.

Contents

1. A categorified version of the Fourier transform: 1/20/17

We've seen that for two-dimensional gauge theories with group G, there's a relationship with the Fourier transform for G. Today, we're going to talk about a categorified version of this, and in a few weeks we'll connect it to three-dimensional gauge theory.

Let's recall some facets of the Fourier transform. Let G be a locally compact abelian (LCA) group, and let $\widehat{G} = \mathrm{Hom}_{\mathsf{TopGrp}}(G, \mathrm{U}(1))$ be its Pontrjagin dual. This is a dual in that $\widehat{\widehat{G}} \cong G$.

The Fourier transform is an isomorphism $L^2(G) \overset{\cong}{\to} L^2(\widehat{G})$ sending pointwise multiplication to convolution and vice versa. There's a nice dictionary between the two sides:

- A representation of G is sent to a family of vector spaces on $\widehat{G}$.
- Finite groups are sent to finite groups.
- Lattices are sent to tori.
- A vector space is sent to its dual vector space.

Today, we're going to talk about Cartier duality, an algebraic analogue of this.

Let G be an algebraic group: this is the notion of a group in algebraic geometry just as Lie groups are the correct notion of groups in differential geometry. One can think of algebraic groups as functors from rings to groups; this is the functor-of-points perspective.

We have no analogue of $\mathrm{U}(1)$ in this setting, so we consider all characters $\chi : G \to \mathbb{G}_m = \mathrm{GL}_1$; the codomain is defined by the group of units functor $\mathsf{Ring} \to \mathsf{Grp}$ sending $R \mapsto R^\times$. As a scheme, this is $\mathbb{A}^1 \setminus 0$ or $\operatorname{Spec} k[x, x^{-1}]$.

The *Cartier dual* of G is $\widehat{G} = \mathrm{Hom}_{\mathsf{AlgGrp}}(G, \mathbb{G}_m)$. That is, for any ring R, $G(R) = \mathrm{Hom}_{\mathsf{Grp}}(G(R), R^\times)$. For "nice G," we'd like $G \cong \widehat{\widehat{G}}$. But what kinds of groups meet this condition?

G had better be abelian (since $\widehat{G}$ always is), and in fact we'll need it to be a *finite flat group scheme*. This idea might be new if you're used to thinking of algebraic geometry over $\mathbb{C}$, where these are exactly the finite abelian groups, but over other fields, it might be different.

Example 1.1. Let $G = \mathbb{Z}/n$. Then, its dual is $\widehat{\mathbb{Z}/n} = \mathrm{Hom}(\mathbb{Z}/n, \mathbb{G}_m)$, which can be identified with the group of n^{th} roots of unity, μ_n. Over $\mathbb{C}$, this is $\langle e^{2\pi i/n}\rangle$ and therefore identified with $\mathbb{Z}/n$, but over fields with characteristic dividing n, there are fewer n^{th} roots of unity. We're not going to worry too much about this. ◀

Akin to Pontrjagin duality, if we let $G = \mathbb{G}_m$, we get $\widehat{G} = \mathbb{Z}$, and if G is a torus, $\widehat{G}$ is the dual lattice in it.

For the Fourier transform, we want to look at vector spaces, e.g. the *additive group* $\mathbb{G}_a = \mathbb{A}^1$. We want to understand homomorphisms $\mathbb{G}_a \to \mathbb{G}_m$. We know that these would be given by $x \mapsto e^{xt}$, but this doesn't make sense unless t is nilpotent, so that the exponential

$$e^{xt} = \sum \frac{(xt)^n}{n!}$$

is a finite sum! That is, we want the dual of the x-line $\mathbb{G}_a$ to be the t-line, but we don't get very far along t. Since we don't know what order t is, we obtain the *formal completion*

$$\widehat{\mathbb{G}}_a = \varinjlim_n \operatorname{Spec} k[t]/(t^n),$$

heuristically a union of n^{th}-order thickenings of 0. Here, the hat is completion, not dual.

More generally, let V be a vector space. Then, its Cartier dual is the formal completion of the dual vector space: we want to take $e^{\langle v,v^*\rangle}$, but we need v^* to be nilpotent.

Alternatively, since Cartier duality is symmetric, the Cartier dual of the formal completion of the additive group is $\mathbb{G}_a$. That is, if x is nilpotent, e^{xt} makes sense for arbitrary t.

Since we're doing algebraic geometry, it's good to think of this in terms of functions. If G is a group, $\mathscr{O}(G)$ is not just a ring, but also has a *comultiplication* pulling functions back along multiplication: $\mu^* : \mathscr{O}(G) \to \mathscr{O}(G) \otimes \mathscr{O}(G)$. This makes $\mathscr{O}(G)$ into a *coalgebra*, and it's cocommutative iff G is commutative.

If G is finite, then you can dualize explicitly: $\mathscr{O}(G)$ is a finite-dimensional vector space, so $\mathscr{O}(G)^\vee$ has a convolution operator induced from the comultiplication. This is the same as convolution of distributions. In fact, it's possible to prove that the Cartier dual is $\widehat{G} = \operatorname{Spec}(\mathscr{O}(G)^\vee, *)$. Functions on $\widehat{G}$, with multiplication, are the same as distributions on G, with convolution. This is what we had in the analytic setting, albeit with a little more care to functions versus distributions.

A point of $\widehat{G}$ defines an algebraic function on G: it's a character $\chi : G \to \mathbb{G}_m$, so composing with the inclusion $\mathbb{G}_m \hookrightarrow \mathbb{A}^1$, we get a map $G \to \mathbb{A}^1$. We can assemble this into a diagram

$$\begin{tikzcd} & G \times \widehat{G} \arrow[dl] \arrow[dr] & \\ G & & \widehat{G}, \end{tikzcd}$$

and there's a tautological function on $G \times \widehat{G}$, which is evaluation: $(g, \chi) \mapsto \chi(g) \in \mathbb{A}^1$. This is akin to the exponential $(x,t) \mapsto e^{xt}$.

If G is infinite, you have to be more careful with topology. For example, $\mathscr{O}(\mathbb{G}_m) = k[x, x^{-1}]$, which sort of looks like the group algebra $k[\mathbb{Z}]$ over the integers, but there we have to restrict to finite expressions.

A sheaf-theoretic perspective. Rather than looking at functions, which don't behave very well in this context, let's look at sheaves.

There are three tensor categories associated to any group G.

1. Since $R = \mathscr{O}(G)$ is a commutative ring, we can use $\mathsf{Mod}_{\mathscr{O}(G)}$ to generate the category $\mathsf{QC}(G)$ of quasicoherent sheaves on G.[1] The commutative tensor product $\otimes_R$ on Mod_R extends to a symmetric monoidal structure on $\mathsf{QC}(G)$. This does not require G to be a group.
2. Since G is a group, $\mathscr{O}(G)$ is a bialgebra (actually a Hopf algebra), so $\mathsf{Mod}_{\mathscr{O}(G)}$ has a monoidal structure given by tensoring over the base field k rather than over R. That is, if M and N are $\mathscr{O}(G)$-modules, $M \otimes_k N$ has an $R \otimes R$-module structure, and then we can induce along the map $R \to R \otimes R$ to obtain an R-module structure.

   This monoidal structure is a convolution:

$$\begin{tikzcd} & G \times G \arrow[dl] \arrow[dr] \arrow[dd, "\mu"] & \\ G & & G. \\ & G & \end{tikzcd}$$

[1] If G is an affine scheme, the categories are the same.

Here, we take M and N over G and realize them over $G \times G$ using the exterior product $M \boxtimes N$, and then pushforward along the multiplication map. This is the same category $\mathsf{QC}(G)$, but with a completely different structure, and this is one of the advantages of sheaves: instead of having to keep functions and distributions apart, sheaves can both pull back and push forward.

(3) The third approach is to take the category of representations of G, which can be tensored together. How can you say this geometrically? G-representations are $\mathscr{O}(G)$-comodules, vector spaces V with a coaction map $V \to V \otimes \mathscr{O}(G)$ satisfying *coassociativity*, i.e. that the following diagram is an equalizer diagram:

$$V \longrightarrow V \otimes \mathscr{O}(G) \rightrightarrows V \otimes \mathscr{O}(G) \otimes \mathscr{O}(G).$$

In a sense, this encodes the notion that representations are modules over the group algebra, but we don't have distributions, so the arrows go the other way. This is a symmetric monoidal category, where the tensor product has the coalgebra structure defined by composing the maps

$$V \otimes W \longrightarrow V \otimes W \otimes \mathscr{O}(G) \otimes \mathscr{O}(G)$$

and $\mathscr{O}(G) \otimes \mathscr{O}(G) \to \mathscr{O}(G)$.

This is not a category of quasicoherent sheaves on G; rather, it's $\mathsf{QC}(\bullet/G)$, where $\bullet/G$ is the classifying stack (or groupoid) of G. This comes from the pushout diagram $\bullet/G \leftarrow \bullet \leftleftarrows G$.

Cartier duality allows these categories to interact with each other. Namely, suppose G and $\widehat{G}$ are dual (so G is abelian, etc.). Then, Cartier duality establishes an equivalence of categories $\mathsf{Rep}_G \cong \mathsf{QC}(\widehat{G})$, and $\mathscr{O}(G)$-comodules become $\mathscr{O}(G)^\vee$-modules. This is just as in ordinary Pontrjagin duality: representations of G become families of functions on $\widehat{G}$.

(By the way, if you're holding out for examples, we'll soon see a whole bunch of them.)

In fact, the tensor structure is also in play: the duality is between the tensor product structure on Rep_G (or $\mathsf{QC}(\bullet/G)$) and the convolution structure on $\mathsf{QC}(\widehat{G})$.

We're going to abstract G away to a different duality operation $\mathsf{QC}(\mathcal{G}) \xrightarrow{\cong} \mathsf{QC}(\mathcal{G}^\vee)$. In our case, $\mathcal{G} = \bullet/G$ and $\mathcal{G}^\vee = \widehat{G}$. The classifying space $\bullet/G$ (also called BG) classifies G-bundles, and since G is abelian, you can tensor G-bundles. That is, if $\mathcal{P}_1$ and $\mathcal{P}_2$ are G-bundles, the relative tensor product $\mathcal{P}_1 \times_G \mathcal{P}_2$ is again a G-bundle, meaning $\bullet/G$ is an abelian group under the tensor product of G-bundles?

What does this actually mean? We're thinking of varieties (and generalizations such as stacks) as functors $\mathsf{Ring} \to \mathsf{Set}$; that $\bullet/G$ is an abelian group means that the assignment from a ring R to the (groupoid of) G-bundles on $\operatorname{Spec} R$ naturally factors through the category of abelian groups. That is, $\bullet/G$ is an abelian group object in the world of stacks.

Now, we define the *Fourier-Mukai dual* $\mathcal{G}^\vee = \operatorname{Hom}_{\mathsf{Grp}}(\mathcal{G}, B\mathbb{G}_m)$. Here $B\mathbb{G}_m$ classifies line bundles, so this is a version of the Picard group. However, since we've restricted to group homomorphisms, we only get what's known as multiplicative line bundles.

Definition 1.2. Let $\mathscr{L} \to G$ be a line bundle over a group G and $\mu : G \times G \to G$ be multiplication. If $\mu^*\mathscr{L} \cong \mathscr{L} \boxtimes \mathscr{L}$, then $\mathscr{L}$ is called a *multiplicative* line bundle.

The idea is that over $x, y \in G$, $\mathscr{L}_x \otimes \mathscr{L}_y \cong \mathscr{L}_{xy}$.

In a sense, we've shifted the Cartier duality operation: $(\bullet/G)^\vee = \operatorname{Hom}_{\mathsf{Grp}}(\bullet/G, \bullet/\mathbb{G}_m) = \operatorname{Hom}_{\mathsf{Grp}}(G, \mathbb{G}_m) = \widehat{G}$ as before. So why categorify? In this stacky version, instead of a universal function on $G \times \widehat{G}$, there's a universal line bundle $\mathscr{L} \to \mathcal{G} \times \mathcal{G}^\vee$:

$$\begin{tikzcd} & \mathcal{G} \times \mathcal{G}^\vee \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{G} & & \mathcal{G}^\vee. \end{tikzcd}$$

This bundle $\mathscr{L}$ is called the *Poincaré line bundle*. And it allows us to define a Fourier transform: given a sheaf $\mathscr{F}$ on $\mathcal{G}$, we can pullback and pushforward to obtain $\pi_{2*}(\pi_1^*\mathscr{F} \otimes \mathscr{L}) \in \mathsf{QC}(\mathcal{G}^\vee)$. This actually defines an equivalence of categories, which is known as *Cartier duality* or *Laumon-Fourier-Mukai duality.*

Example 1.3. The most interesting example is where $\mathcal{G} = A$ is an abelian variety and $\mathcal{G}^\vee = A^\vee$ is the dual variety. Then, the integral transform with the Poincaré sheaf defines an equivalence of the derived categories $D(A) \cong D(A^\vee)$, which is the classical *Fourier-Mukai transform.* ◀

Example 1.4. We could also take $\mathcal{G} = \mathbb{G}_m$ and $\mathcal{G}^\vee = B\mathbb{Z}$. Then, this duality tells us that $\mathbb{Z}$-graded vector spaces are the same things as representations of $\mathbb{G}_m$. ◀

2. The Fourier-Mukai transform: 2/3/17

Today we're going to talk about the Fourier-Mukai transform, which is a categorical analogue of the Fourier transform.

Recall that if we have geometric spaces X and Y, an *integral transform* is a function $\Phi\colon \mathsf{Fun}(X) \to \mathsf{Fun}(Y)$ represented by a *kernel*, a function $K \in \mathsf{Fun}(X \times Y)$ such that Φ is defined by a pullback-pushforward

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y, \end{tikzcd}$$

in that $\Phi(f) = \pi_{2*}(\pi_1^* f \cdot K)$. The map $x \mapsto f_x(y) \coloneqq K(x, y)$ is $\Phi(\delta_x)$, so this can be thought of as a map $X \to \mathsf{Fun}Y$. If Φ is an isomorphism, then since $\{\delta_x\}$ is a basis for $\mathsf{Fun}X$, then $\{f_x\}$ is a basis for $\mathsf{Fun}Y$. These are the exponentials in the ordinary Fourier transform.

Now suppose X and Y are algebraic varieties, so integral transforms look like functors $\Phi\colon \mathsf{QC}(X) \to \mathsf{QC}(Y)$. If $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$, then $\Phi : \mathsf{Mod}_R \to \mathsf{Mod}_S$, and the Eilenberg-Watts theorem says that Φ must be tensoring with an (R, S)-bimodule ${}_RK_S$, which is the kernel. In particular, $K \in \mathsf{QC}(X \times Y) = \mathsf{Mod}_{R \otimes S}$. Thus, if M is a bimodule,

$$\Phi(M) = \pi_{2*}(\pi_1^* M \otimes_R K_S).$$

The map π_{2*} forgets the R-structure, hence is exact; if we want Φ to be exact, we must assume K is flat over R. In great generality, functors $\Phi\colon \mathsf{QC}(X) \to \mathsf{QC}(Y)$ are given by kernels $K \in \mathsf{QC}(X \times Y)$ satisfying a push-pull formula. However, if K isn't flat, $- \otimes K$ must be taken in a derived sense, and if X isn't affine, π_{2*} (global sections in the X-direction) isn't exact, and must again be taken in a derived sense, taking cohomology. Sometimes, functors like Φ are called *Fourier-Mukai functors*, but there's nothing particulary "Fourier" about them yet.

Suppose $x \in X$; we can identify it with the skyscraper sheaf $\mathscr{O}_x$ at x, which Φ maps to $\mathscr{F}_x \coloneqq \Phi(\mathscr{O}_x) \in \mathsf{QC}(Y)$, and $\mathscr{F}_x = K|_{\pi_1^{-1}(x)}$. This is an assignment of a sheaf on Y to every point in X, therefore defining a map from X to some moduli space of sheaves on Y. This map might not be interesting, but it is sometimes, and it always exists.

In fact, let's suppose $X = \mathcal{M}$ is a moduli space of sheaves on Y. There are natural transforms $\mathsf{QC}(\mathcal{M}) \to \mathsf{QC}(Y)$, e.g. the tautological construction whose kernel on $\pi_1^{-1}(x)$ is the sheaf defined by $x \in \mathcal{M}$. More concretely, let $X = \operatorname{Pic} Y$, the moduli space of line bundles. There's a canonical bundle $\mathcal{P} \to \operatorname{Pic} Y \times Y$ such that $\mathcal{P}|_{(\mathscr{L}, y)} = \mathscr{L}|_y$, and this gives an interesting transform. (There are uninteresting transforms: the moduli space of skyscraper sheaves on Y is just Y itself, and the kernel is the identity matrix).

When is Φ an equivalence of categories, either in the usual or derived sense? The "orthonormal basis" $\mathscr{O}_x$ is mapped to $\mathscr{F}_x$. It's orthogonal in the sense that

$$\operatorname{Hom}(\mathscr{O}_x, \mathscr{O}_y) = \begin{cases} 0, & x \neq y \\ k, & x = y. \end{cases}$$

If $x \in X$ is smooth, the derived analogue is $\operatorname{Ext}(\mathscr{O}_x, \mathscr{O}_x) = \Lambda^\bullet T_x$. The "basis" part is that if $\mathscr{F}$ is coherent, $\operatorname{Hom}(\mathscr{F}, \mathscr{O}_x) = 0$ for all x iff $\mathscr{F} = 0$. So if Φ is to be an equivalence, we need $\operatorname{Hom}(\mathscr{F}_x, \mathscr{F}_y) = 0$ unless $x = y$, in which case you get the same algebra, and you need the same conditions: if $\mathscr{G}$ is coherent and $\operatorname{Hom}(\mathscr{G}, \mathscr{F}_x) = 0$ for all x, then $\mathscr{G} = 0$.

Let G be an abelian group (in schemes or in grouoids), and $Y = G^\vee = \operatorname{Pic}^\mu G$, the space of *multiplicative* line bundles on G. A line bundle $\mathscr{L}$ is multiplicative if there's a coherwnt isomorphism $\mathscr{L}_x \otimes \mathscr{L}_y \xrightarrow{\cong} \mathscr{L}_{x+y}$ (this is data, not a condition!), equivalent to an isomorphism $\mu^*\mathscr{L} \cong \mathscr{L} \boxtimes \mathscr{L}$, where $\mu : G \times G \to G$ is multiplication. $\operatorname{Pic}^\mu G$ can be identified with $\operatorname{Hom}_{\mathsf{Grp}}(G, B\mathbb{G}_m)$, where $B\mathbb{G}_m$ is the moduli of lines.

There's a tautological line bundle $\mathcal{P} \to G \times G^\vee$, which at $(g, \mathscr{L})$ is $\mathscr{L}_g$. This is a kernel, and hence defines a kernel transform.

Theorem 2.1 (Laumon-Fourier-Mukai)**.** *In many situations, this kernel transform is an equivalence, and exchanges tensor product with convolution.*

We'll see plenty of examples, making the "many situations" less vague, and these examples encompass some interesting dualities.

Example 2.2. Let $G = \mathbb{G}_m$. What's $\operatorname{Pic}\mathbb{G}_m$? There's only one line bundle, but it has a lot of automorphisms, so we get $\operatorname{Pic}\mathbb{G}_m = \bullet/\mathscr{O}^*(\mathbb{G}_m)$. The trivial bundle is multiplicative, but asking for automorphisms to preserve this structure rigidifies it: $G^\vee$ is $\bullet$ modulo the homomorphisms $\mathbb{G}_m \to \mathbb{G}_m$, i.e. the characters of $\mathbb{G}_m$. These are given by integers ($x \mapsto x^n$), so $G^\vee = \bullet/\mathbb{Z}$, also denoted $B\mathbb{Z}$.

A quasicoherent sheaf on $\mathbb{G}_m$ is equivalent data to a $\mathbb{C}[z, z^{-1}]$-module, hence the data of a vector space and an invertible map, which is the same thing as a $\mathbb{Z}$-representation, and $\mathsf{Rep}_\mathbb{Z} \cong \mathsf{QC}(B\mathbb{Z})$. This is the duality function; there's nothing derived going on here.

On $\mathsf{QC}(\mathbb{G}_m)$, the tensor product is the usual tensor product, and the convolution is $M * N := M \otimes_\mathbb{C} N$, which is a $\mathbb{C}[z, z^{-1}]$-module via the coproduct map $\mathbb{C}[z, z^{-1}] \to \mathbb{C}[w, w^{-1}] \otimes \mathbb{C}[t, t^{-1}]$ sending $z \mapsto w \otimes t$. This is mapped to the tensor product on $\mathsf{Rep}_\mathbb{Z}$, and the tensor product is mapped to its convolution. A similar story can be told for any group.

This is an example of homological mirror symmetry! We think of $B\mathbb{Z}$ as $S^1 = K(\mathbb{Z}, 1)$ (in some suitable homotopical sense), and so a quasicoherent sheaf on $B\mathbb{Z}$ is the same thing as a locally constant sheaf (local system) on S^1: a $\mathbb{Z}$-representation is determined by what 1 does, and this is the monodromy as you go around S^1. Fukaya categories are meant to make this work: the wrapped Fukaya category attached to T^*S^1 is $\mathsf{QC}(B\mathbb{Z})$, the local systems on S^1.

Mirror symmetry says that the B-model on a space X should be equivalent to the A-model on the mirror $X^\vee$; the mirror of $\mathbb{C}^*$ is $\mathbb{C}^*$. The boundary conditions on the B-model encode $\mathsf{QC}(\mathbb{C}^*)$, and this should map to the Fukaya category of its mirror. Fukaya categories in general are nightmarish, but in this case everything is nice. ◀

Example 2.3. Suppose G is an algebraic torus, so a product of copies of $\mathbb{G}_m$: $G = (\mathbb{G}_m)^n$. Then, $G^\vee = B\Lambda$, where Λ is the character lattice $\Lambda := \operatorname{Hom}_{\mathsf{Grp}}(T, \mathbb{G}_m)$. This can be identified with the dual of the compact torus $T_c^\vee \cong (S^1)^n = K(\Lambda, 1)$. Then, $\mathsf{QC}(T)$ is identified with the Fukaya category on the cotangent space of the compact torus. In some sense, this is the base case of mirror symmetry that people want to reduce everything down to. ◀

Example 2.4. Moving away from mirror symmetry, suppose $G = \mathbb{Z}$. Then, $G^\bullet$ is a point modulo the characters of G, so $\bullet/\mathbb{G}_m = B\mathbb{G}_m$. A sheaf on $\mathbb{Z}$ is a vector space for each integer, so a $\mathbb{Z}$-graded vector space, and a $\mathbb{Z}$-graded vector space is the same thing as a $\mathbb{C}^*$-representation! (The grading is given by the different eigenvalues.) You can generalize this: if G is a lattice, $G^\vee$ is the classifying space of the dual torus. ◀

Example 2.5. If $G = \mathbb{A}^1$, then $G^\vee$ is a point modulo the characters of $\mathbb{A}^1$; last time, we talked about how these are the formal completion of $\mathbb{A}^1$: $G^\vee = \bullet/\widehat{\mathbb{A}}^1$. A quasicoherent sheaf on $\mathbb{A}^1$ is the same thing as a $\mathbb{C}[x]$-module, which is equivalent to a vector space with an endomorphism, and this is the same as a representation of the Lie algebra $\mathbb{C}$. We want to exponentiate, but can only do so in a small neighborhood, so this is the same thing as a representation of the formal group $\widehat{\mathbb{A}}^1$.

More generally, if V is a vector space, $V^\vee = \bullet/\widehat{V}^*$. ◀

Example 2.6. Dually, if $G = \widehat{\mathbb{A}}^1$, then its characters are just $\mathbb{A}^1$ again, so $G^\vee = \bullet/\mathbb{A}^1$. A quasicoherent sheaf on $\widehat{\mathbb{A}}^1$ is a module over $\mathbb{C}[[x]]$, hence a vector space with a nilpotent endomorphism. A representation of the additive group is a representation of its Lie algebra, but we can exponentiate to any order, and therefore the action of the Lie algebra $\mathbb{C}$ must be nilpotent.[2] ◀

These examples are all tautological, in a sense; the following, due to Mukai is not.

Example 2.7. Let $G = A$ be an abelian variety, so it's a compact, connected abelian algebraic group (hence a torus $\mathbb{C}^n/\Lambda$). Let $A^\vee$ be the dual variety: literally the dual vector space modulo the dual torus. This is $\operatorname{Pic}^0 A$, the space of degree-0 line bundles trivialized at the identity. This is the same thing as multiplicative line bundles.

[2] The passage between Lie algebras and formal groups requires some characteristic 0 properties, but a lot of this still works over other fields.

You can think of these not just as line bundles on A, but extensions of A: $A^\vee = \mathrm{Ext}^1_{\mathsf{Grp}}(A, \mathbb{G}_m) = \mathrm{Hom}(A, B\mathbb{G}_m)$: we have a fiber bundle $\mathbb{C}^* \to \mathscr{L}^\times \to A$, and we've identified $\mathscr{L}^\times|_{\mathrm{id}} \cong \mathbb{C}^*$, so what you have is an extension. There's a proof of this in Langlands' book, or Polishchuk's book on abelian varieties.

The Poincaré line bundle $\mathcal{P} \to A \times A^\vee$ applies as usual, but the pushforward in the kernel transform has to be derived:

$$\mathscr{F} \longmapsto \mathbf{R}\pi_{2*}(\pi_1^*\mathscr{F} \otimes \mathcal{P}).$$

This defines an equivalence of derived categories: $D(A) \overset{\cong}{\to} D(A^\vee)$, and was one of the first equivalences of derived categories that anyone considered. In fact, it was the first equivalence of derived categories between non-isomorphic varieties (with no stacky stuff).

More poetically, this says that any sheaf on A can be written as an "integral" of line bundles, or line bundles form a "basis" for sheaves on an abelian variety (as do skyscrapers). If you're interested in studying abelian varieties, this is very useful.

For example, if $A = \mathrm{Jac}\, C$, then it's canonically self-dual, and the transform is an interesting self-duality on $D(\mathrm{Jac}\, A)$. This is the space of degree 0 line bundles; alternatively, you can look at $\mathrm{Bun}^0_T\, C$, the space of degree-0 T-bundles on C (here T is a torus). In this case, the dual is $A^\vee = \mathrm{Bun}^0_{T^\vee}\, C$, the space of dual torus bundles. ◀

The geometric Langlands program is in some sense a fancy generalization of this example.

Example 2.8 (de Rham spaces)**.** We want to quotient $\mathbb{A}^1$ by a normal subgroup. There's not a lot of options, but we can choose the formal completion, and let $G = \mathbb{A}^1/\widehat{\mathbb{A}}^1$ (a sum of points very close to 0 stays close to 0, and everything is abelian). You can do this for any group G: let $\widehat{G}$ denote its formal completion at the identity. Then, translating by some $g \in G$, we get $\widehat{G} \cdot g = \widehat{G}_g$, the completion at g.

The quotient $G/\widehat{G}$ doesn't quite make sense as a variety, but you can define it as a functor $\mathsf{Ring} \to \mathsf{Grp}$, sending $R \mapsto G(R)/\widehat{G}(R)$. (Here, $\widehat{G}(R)$ is the group of maps $\mathrm{Spec}\, R \to G$ that send the reduced part of R to 1.) Consider the groupoid (equivalence relation) $\widehat{G \times G}|_\Delta$, meaning we've identified things that are arbitrarily close to the diagonal; then, modding out by this is the same thing as modding by $\widehat{G}$.

The advantage of this is that you don't need a group structure: for any space X, its *de Rham space* is $X_{\mathrm{dR}} \coloneqq X/\widehat{X \times X}|_\Delta$, so X modulo $x \sim y$ when x is arbitrary close to y. From a functor-of-points perspective, $X_{\mathrm{dR}}(R) \coloneqq X(R^{\mathrm{red}})$: "$X$ modulo calculus." For groups, this is particularly nice: $g, h \in G$ are close iff $h^{-1}g$ is very close to the identity.

Why does this get to be called the de Rham space? The functions are $\mathscr{O}(X_{\mathrm{dR}})$, the functions on X invariant under infinitesimal translation, so must have constant Taylor series. In other words, these functions are the kernel of the de Rham differential $\mathrm{d} : \mathscr{O}(X) \to \Omega^1$. And when you see this, you imagine the rest of the de Rham complex: the derived notion of functions on X_{dR} is the de Rham cohomology of X! So it's almost never representable, but it's still useful for studying de Rham cohomology. The functor $X \mapsto X_{\mathrm{dR}}$ is adjoint to taking reductions: $\mathrm{Hom}(S, X_{\mathrm{dR}}) = \mathrm{Hom}(S_{\mathrm{red}}, R)$. Gaitsgory calls it a "prestack," but there's nothing stacky, as we're quotienting by an equivalence relation.

Great, so what about the sheaves $\mathsf{QC}(X_{\mathrm{dR}})$? These are the sheaves on X where $\mathscr{F}_x \cong \mathscr{F}_y$ if x and y are infinitesimally close. That is, $\mathscr{F}$ is trivialized on formal neighborhoods of a point. This is equivalent to $\mathscr{F}$ being a *crystal* or $\mathcal{D}$-module, or a sheaf with a flat connection (at least in characteristic 0). The idea is this is a sheaf with some kind of locally constant csections, which vanish when you apply the connection $\nabla \colon \mathscr{F} \to \mathscr{F} \otimes \Omega^1$. ◀

This could be considered a roundabout way to introduce $\mathcal{D}$-modules. Suppose $\mathcal{G} \rightrightarrows X$ is a groupoid acting on X. A $\mathcal{G}$-equivariant sheaf is a module for the groupoid algebra of distributions (or measures) on G. Functions on $\mathcal{G}$ form a coalgebra (just as for a group), and a $\mathcal{G}$-equivariant sheaf is a comodule for $\mathscr{O}(\mathscr{G})$. The functions on $\widehat{X \times X}|_\Delta$ is the jets of functions $\mathcal{J}$, functions vanishing to some order.

If you dualize over one of the factors of $X \times X$, the dual is

$$\mathcal{J}^* = \bigcup_n \mathrm{Hom}_{\mathbb{C}}(\mathscr{O}_X, \mathscr{O}_X)^{I_\Delta^{n+1}},$$

where I_Δ is the *ideal of the diagonal*, generated by expressions of the form $f(x) - f(y)$ for $f \in \mathscr{O}(X)$. For $n = 0$, these are the functions $\varphi : \mathscr{O}_X \to \mathscr{O}_X$ that are $\mathscr{O}$-linear. For $n = 1$, we ask for $\varphi f - f\varphi$ to be

$\mathscr{O}$-linear, which is Grothendieck's definition of a differential operator of order at most 1; in general, the n^{th} term is $\mathcal{D}_{\leq n}$, the differential operators of degree at most n. The expression $\varphi f - f\varphi$ is an abstract expression of the Leibniz rule. The *ring of differential operators*, denoted $\mathcal{D}$, is the groupoid algebra of the de Rham groupoid.[3]

Modules over $\mathcal{D}_X$ are what physicists call local operators: you can do whatever you want, as long as it only depends on the Talyor series (jet) at a point. And modules over $\mathcal{D}_X$ are identified with sheaves on X_{dR}. For example, this means integral transforms are disallowed. These sheaves are the input into crystalline cohomology; in characteristic p, where this is most useful, there are different notions of the de Rham groupoid. (Crystalline and de Rham cohomology are closely related, though there are complications in positive characteristic or over non-smooth spaces.) In fact, you can define de Rham cohomology with coefficients in a sheaf $\mathscr{F}$ to be

$$H_{\mathrm{dR}}(X;\mathscr{F}) \coloneqq \mathbf{R}\Gamma(X_{\mathrm{dR}};\underline{\mathscr{F}}).$$

So the point of all this is, if you have a group G, then a $\mathcal{D}$-module on G is identified with a sheaf on G_{dR}, hence a $\widehat{G}$-equviariant sheaf on G, i.e. a sheaf on $G/\widehat{G}$.

This is what we were talking about earlier, sheaves on $G = \mathbb{A}^1/\widehat{\mathbb{A}}^1 = \mathbb{A}^1_{\mathrm{dR}}$ (or more generally using a vector space and its formal completion). We saw the duality sent $\mathbb{A}^1$ to $\bullet/\widehat{\mathbb{A}}^1$ and $\widehat{\mathbb{A}}^1$ to $\bullet/\mathbb{A}^1$, so this duality exchanges vector spaces and formal groups. if you blur your eyes a little bit, you get that $\mathbb{A}^1/\widehat{\mathbb{A}}^1$ is self-dual: $G^\vee = \mathbb{A}^1_{\mathrm{dR}}$.

If you have a vector space V, $V^*/\widehat{V}^* = V^*_{\mathrm{dR}}$. This is an example of the same Cartier duality.

Anyways, Fourier-Mukai duality defines an interesting automorphism $\mathbb{F}$ on $\mathsf{QC}(\mathbb{A}^1/\widehat{\mathbb{A}}^1)$, which is $\mathcal{D}_{\mathbb{A}^1}$-modules. And we know $\mathcal{D}_{\mathbb{A}^1} = \mathbb{C}\langle z, \partial_z\rangle/(\partial_z z - z\partial_z = 1)$. So the duality $\mathcal{D}_{\mathbb{A}^1} \to \mathcal{D}_{\mathbb{A}^1}$ sends $z \mapsto \partial_z$ and $\partial_z \mapsto z$, which does look nostalgically familiar.

In fact, it *is* the Fourier transform on $\mathbb{R}$. Let f be a (generalized) function on $\mathbb{R}$ (e.g. a tempered distribution). Then, f defines a $\mathcal{D}$-module $M_f = \mathcal{D}\cdot f$, the (left) action of all differential operators on f. Let $\widehat{f}$ denote the Fourier transform of f; then, the claim is that $\mathbb{F}(M_f) = M_{\widehat{f}}$, which is another way of expressing that the Fourier transform exchanges multiplication and differentiation.

If you set this up as a kernel transform, you get $M_{e^{xt}} \to \mathbb{A}^1_{\mathrm{dR}} \times \mathbb{A}^1_{\mathrm{dR}}$, the ideal generated by $\mathcal{D}_{\mathbb{A}^1\times\mathbb{A}^1}/(\partial x - t, \partial_t - x)$, so x acts by differentiating t and ∂t acts by differentiating x (this ideal is a differential equation specifying this behavior, which is why we got e^{tx}), and $M_{e^{xt}}$ is a line bundle: $e^{\lambda z} \mapsto \mathcal{D}/\mathcal{D}(\partial_z - \lambda) \cong \mathbb{C}[z]$ as $\mathbb{C}[z]$-modules, so this is even a trivial line bundle! Of course, this is a very longwinded way to get the usual Fourier transform, but once you say it this way, you have a whole lot of generalizations.

Example 2.9. We won't need this example, but it's cool. Consider $\mathbb{A}^1/\mathbb{Z}$, where $\mathbb{Z}$ acts by shifting. Then, the dual replaces $\mathbb{Z}$ with $\mathbb{G}_m$ and $\mathbb{A}^1$ with $\widehat{\mathbb{A}}^1$: the dual is $\mathbb{G}_m/\widehat{\mathbb{A}}^1 = (\mathbb{G}_m)_{\mathrm{dR}}$. The ring controlling difference equations on $\mathbb{A}^1/\mathbb{Z}$ is $\mathbb{C}[t]\langle\sigma,\sigma^{-1}\rangle$, and the ring controlling differential equations on $\mathbb{G}_m/\widehat{\mathbb{A}}^1$ is $\mathbb{C}[z,z^{-1}]/\langle z\partial_z\rangle$, and these two rings are isomorphic. In this context, the transform is called the *Mellin transform.* ◂

References

[3]This definition is due to Grothendieck, but was worded differently (and not just because it was in French.)