FURUTA'S 10/8 THEOREM

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These notes were taken in a learning seminar on Furuta's 10/8 theorem in Spring 2019. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

Contents

1. Introduction to Seiberg-Witten theory: 1/23/19

Riccardo gave the first, introductory talk.

In 1982, Matsumoto conjectured that if M is a closed spin manifold, $b_2(M) \ge (11/8)|\sigma(M)|$. Here $b_2(M)$ is the second Betti number and $\sigma(M)$ is the signature. Equality holds for the K3 surface, so this is the best one can do.

In this seminar we'll study a theorem of Furuta which makes major progress on this conjecture.

Theorem 1.1 (10/8 theorem [Fur01])**.** *If the intersection form of M is indefinite, $b_2(M) \ge (10/8)|\sigma(M)| + 2$.*

If the intersection form is definite, work of Donaldson [Don83] says that, up to a change of orientation, the intersection form is diagonalizable, so that case is dealt with.

Furuta's proof uses both Seiberg-Witten theory and equivariant homotopy theory. It can be pushed a little bit farther, but not enough to prove the $11/8^{\text{ths}}$ conjecture, as shown recently by Hopkins-Lin-Shi-Xu [HLSX18].

Today we'll discuss some background for the proof.

Definition 1.2. Let $V \to M$ be a rank-n real oriented vector bundle. A *spin structure* on V is data $\mathfrak{s} = (P_{\mathrm{Spin}}(V), \tau)$, where $P_{\mathrm{Spin}}(V) \to M$ is a principal Spin_n-bundle and τ is an isomorphism

$$\tau\colon P_{\mathrm{Spin}}(V) \times_{\mathrm{Spin}_n} \mathbb{R}^n \xrightarrow{\cong} V.$$

A spin structure on a manifold M is a spin structure on TM.

Remark 1.3. There are other equivalent definitions of spin structures – for example, just as an orientation is a trivialization of V over the 1-skeleton of M, a spin structure is equivalent to a trivialization over the 2-skeleton. ◀

Here's a cool theorem about spin manifolds.

Theorem 1.4 (Rokhlin [Roh52])**.** *If M is a spin manifold, $\sigma(M) \equiv 0 \bmod 16$.*

The signature makes sense when $4 \mid \dim M$. Smoothness is crucial here; there are topological spin 4-manifolds, whatever that means, that do not satisfy this theorem. Freedman's E_8 manifold is an example.

Suppose M is a spin 4-manifold. The representation theory of Spin_4, in particular the fact that the spin representation S splits as $S^+ \oplus S^-$, leads to two quaternionic line bundles $\mathbb{S}^+, \mathbb{S}^- \to M$ with Hermitian metrics. Physics cares about these bundles, and will lead to powerful theorems in manifold topology.

These bundles have more structure: in particular, they are Clifford bundles.

Definition 1.5. Let $S \to M$ be a real vector bundle with a Euclidean metric $\langle \cdot, \cdot \rangle$. A *Clifford bundle* structure is data of, for each $x \in M$, the data of a Clifford algebra action $C\ell(T_x M)$ on S_x that varies smoothly in x, such that the Clifford action is skew-adjoint, meaning

$$\langle v \cdot s_1, s_2 \rangle = -\langle s_1, v \cdot s_2 \rangle.$$

We also require the existence of a connection which is compatible with the Levi-Civita connection on TM.

Given the data of a Clifford bundle, there's an operator called the *Dirac operator* D, which is the following composition:

$$C^\infty(S) \xrightarrow{\nabla^{C\ell}} C^\infty(T^*M \otimes S) \xrightarrow{\langle \cdot, \cdot \rangle} C^\infty(TM \otimes S) \xrightarrow{\text{Clifford action}} C^\infty(S). \tag{1.6}$$

This operator is denoted $\not{\partial}$, a convention due to Feynman. It is a first-order, elliptic differential operator; ellipticity means that its analysis is nice.

Thus we can consider the *Seiberg-Witten equations* on a spin 4-manifold. Let $(a, \varphi) \in \Omega^1_M(i\mathbb{R}) \times \Gamma(\mathbb{S}^+)$; then the equations are

$$\not{\partial}\varphi + \rho(a)(\varphi) = 0 \tag{1.7a}$$

$$\rho(\mathrm{d}^+ a) - \varphi \otimes \varphi^* + \frac{1}{2}|\varphi^2|\mathrm{id} = 0 \tag{1.7b}$$

$$\mathrm{d}^* a = 0. \tag{1.7c}$$

On a non-spin manifold, the equations are a little more complicated.

2. The monopole equations: 1/28/19

Today, Kai spoke about the monopole equations and some of their important properties, foreshadowing compactness next week. We begin with some motivation.

Recall that if M is a closed, oriented 4-manifold (in either the topological or smooth category), the intersection form $H_2(M) \times H_2(M) \to \mathbb{Z}$ is a unimodular, symmetric bilinear form.

Question 2.1. Which unimodular, symmetric bilinear forms arise as the intersection forms of smooth or topological manifolds?

For example, the intersection form of $S^2 \times S^2$ is $H \coloneqq \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The intersection form of $\mathbb{CP}^2$ is (1). There's an interesting bilinear form called the *E8 form*

$$\mathrm{E8} = \begin{pmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{pmatrix}. \tag{2.2}$$

Can this be realized as the intersection form of a smooth 4-manifold? Rokhlin's theorem tells us the answer is no, because such a manifold would have to be spin, and $16 \nmid \sigma(\mathrm{E8})$. However, Freedman found a topological manifold M_{E8} whose intersection form is E8!

The direct sum of two copies of E8 satisfies Rokhlin's theorem, and this form is realized by the topological 4-manifold $M_{\mathrm{E8}} \# M_{\mathrm{E8}}$. However, Donaldson showed this manifold is not smoothable: specifically, the intersection forms of smooth 4-manifolds can be diagonalized over $\mathbb{Z}$, and E8 cannot.

There's still more interesting example: consider the *K3 surface* $\{z_1^4 + z_2^4 + z_3^4 + z_4^4 = 0\} \subset \mathbb{CP}^3$; its intersection form is $-2\mathrm{E8} \oplus 3H$. So does it split as a connect sum of 3 copies of $S^2 \times S^2$ and two copies of $M_{\mathbb{E}8}$ (with the opposite orientation)? Freedman showed this is true topologically. Smoothly, of course, it can't hold, but we might still get something.

Question 2.3. Is there a smooth, oriented 4-manifold N such that, in the smooth category, $K3 \cong N \# S^2 \times s^2$?

This was a longstanding question.

Seiberg-Witten invariants allow us to answer questions such as this – though in this semester, we're more interested in the monopole map. In any case, let's define the Seiberg-Witten equations.

Let M be a smooth, oriented 4-manifold with b_2^+ odd and a Riemannian metric g, and let $\mathfrak{s}$ be a spin^c structure on M, which determines a *basic class* $K \in H^2(X)$, i.e. an integer cohomology class such that $K \equiv w_2(M) \bmod 2$. The spin^c structure $\mathfrak{s}$ defines for us spinor bundles $\mathbb{S}^+$ and $\mathbb{S}^-$. Let $\mathcal{A}_L$ denote the space of U_1-connections, $A \in \mathcal{A}_L$, and $\psi \in \Gamma(X, \mathbb{S}^+)$ (this is called a *spinor*). The Seiberg-Witten equations are

$$D_A\psi = 0 \tag{2.4a}$$

$$F_A^+ + i\delta = i\sigma(\psi). \tag{2.4b}$$

These equations have a gauge symmetry: if G denotes the group $\mathrm{Map}(X, S^1)$ with pointwise multiplication, G acts on $\mathcal{A}_L \times \Gamma(X, \mathbb{S}^+)$ on the first factor. Let B_K^+ denote the quotient minus the locus of spinors which are identically zero; then $B_K^+ \simeq \mathbb{CP}^\infty$, so we know its cohomology is isomorphic to $\mathbb{Z}[x]$, with $|x| = 2$.

Let $\mathcal{M}_K^\delta(g) \subset B_K^\times$ denote the space of solutions to the Seiberg-Witten equations. This space has dimension

$$d \coloneqq \frac{1}{4}\big(K^2 - (3\sigma(M) + 2\chi(M))\big), \tag{2.5}$$

and, crucially, defines a class $[\mathcal{M}_K^\delta(g)] \in H_d(B_K^\times)$ which does not depend on g for generic choices of the metric. The *Seiberg-Witten invariants* are

$$SW_X(K) \coloneqq \langle x^{d/2}, [\mathcal{M}_K^\delta(g)]\rangle \in \mathbb{Z}. \tag{2.6}$$

The fact that $b_2^+(M) = 0$ implies d is even.

This defines a map SW from the basic classes to $\mathbb{Z}$. Taubes showed two important results.

Theorem 2.7 (Vanishing theorem (Taubes)). *If M is diffeomorphic to a connect sum of two closed, oriented 4-manifolds $X_1 \# X_2$, $b_2^+(X_1) > 0$, and $b_2^+(X_2) > 0$, then the Seiberg-Witten equations of M vanish.*

Theorem 2.8 (Nonvanishing theorem (Taubes)). *If $\mathfrak{s}$ is the canonical spin^c structure associated to a complex structure on M and $b_2^+(M)$ is positive and off, then $SW(\pm c_1(M)) = \pm 1$.*

Corollary 2.9. *K3 cannot split smoothly as a connect sum.*

This leads to an interesting generalization: there are *exotic K3 surfaces*, homeomorphic but not diffeomorphic to the standard K3. They don't all admit complex structures, and many of them are not symplectic. Nonetheless, they also don't split off an $S^2 \times S^2$: this is a consequence of Furuta's 10/8 theorem, because if $K3 \cong N \# (S^2 \times S^2)$, then $b_2(N) = 20$ and $\sigma(N) = -16$, but

$$20 \ngeq \frac{10}{8}|-16| + 2. \tag{2.10}$$

Now let's discuss the monopole map. We now assume M is a spin manifold, with spin structure $\mathfrak{s}$ and spinor bundles $\mathbb{S}^\pm$. Let A denote a spin connection and consider the spaces

$$\widetilde{\mathcal{A}} \coloneqq \{A + i\ker \mathrm{d}\} \times (\Gamma(\mathbb{S}^+) \oplus \Omega^1(X)) \tag{2.11}$$

$$\widetilde{C} \coloneqq \{A + i\ker \mathrm{d}\} \times \big(\Gamma(\mathbb{S}^-) \oplus \Omega^0(X) \oplus H^1(X;\mathbb{R}) \oplus \Omega^+(X)\big). \tag{2.12}$$

Both of these fiber over $H^1(X;\mathbb{R})$: for $\widetilde{\mathcal{A}}$, $A + \alpha \mapsto [\alpha]$, and there is a map $\widetilde{\mu}\colon \widetilde{\mathcal{A}} \to C$ defined by

$$(A, \phi, a) \longmapsto (A, D_A\phi + ia\phi, \mathrm{d}^*a, a_{\mathrm{harm}}, \mathrm{d}^+a - \sigma(\phi)). \tag{2.13}$$

Here

- D_A is the *Dirac operator* $D_A\colon \Gamma(\mathbb{S}^+) \to \Gamma(\mathbb{S}^-)$.
- $a\phi$ denotes Clifford multiplication.
- d^* is the adjoint of d, which sends k-forms to $(k-1)$-forms, and satisfies the equation

$$\mathrm{d}^* = \star\mathrm{d}\star. \tag{2.14}$$

  (This is in dimension 4; the sign convention is different in other dimensions.)
- a_{harm} is the harmonic part of a: it's a general fact that any one-form in dimension 4 splits as $a = a_{\mathrm{harm}} + \mathrm{d}^*\alpha + \mathrm{d}\beta$ for some 0-form β. A form is *harmonic* if the Laplacian $\Delta \coloneqq \mathrm{dd}^* + \mathrm{d}^*\mathrm{d}$ vanishes on it.

- d^+a denotes the self-dual part of $\mathrm{d}a$.
- $\sigma(\phi)$ denotes the trace form of the endomorphism $\phi \otimes \phi^* - (1/2)\|\phi\|^2\mathrm{id}$.

Again the group G acts on $\Gamma(\mathbb{S}^{\pm})$ by pointwise multiplication, using $S^1 \cong \mathrm{U}_1 \subset \mathbb{C}$. If $u \in G$, $u\colon X \to S^1$ also acts on the space of spin^c connections by $d \mapsto udu^{-1}$. Let G act trivially on forms.

Then, the map $\widetilde{\mu}$ defined in (2.13) is G-equivariant. Let G_0 denote the maps which vanish at some specified basepoint p, and let $\mathcal{A} \coloneqq \widetilde{A}/G_0$, $C \coloneqq \widetilde{C}/G_0$, and $\mu \coloneqq \widetilde{\mu}/G_0$; thus we get a map $\mu\colon A \to C$.

Now, both A and C fiber over the Picard group

$$\mathrm{Pic}^g(X) \coloneqq H^1(X;\mathbb{R})/H^1(X;\mathbb{Z}) = H^1(X;\mathbb{R})/G_0. \tag{2.15}$$

Then $S^1 = G/G_0$ acts on $\mu^{-1}(A,0,0,0,0)$, and this is the space we're interested in.

We would like to study this space, and to do so we'll need to consider Sobolev spaces. For a fixed integer $k > 2$, let A_k be the fiberwise completion of A within L^2_k and C_{k-1} be the fiberwise completion of C within L^2_{k-1}. Then, the monopole map μ is a map $A_k \to C_{k-1}$.

Claim 2.16. This monopole map μ is S^1-equivariant, and is a compact perturbation of a linear Fredholm map.

The S^1-equivariance involves chasing through the definition but isn't bad; the rest is harder. What we can do is start by listing the terms that define a linear Fredholm map, and then check that the rest is compact. In the definition of $\widetilde{\mu}$, the terms A, $D_A\phi$, d^*a, a_{harm}, and d^+a are linear and Fredholm; thus we just have to check that $a(\phi)$ and $\sigma(\phi)$ are compact. For the first, we can use the fact that Clifford multiplication is compact, then compose with the map $C_k \to C_{k-1}$, which is also compact.

Proposition 2.17. *Let $T = \ell + c$ be a compact perturbation of a linear Fredholm map ℓ between Hilbert spaces. The restriction of T to any closed, bounded subset Ω is proper.*

Proof. Let p denote projection onto $\ker(\ell)$ and consider the commutative diagram

$$\begin{tikzcd} \Omega \arrow[r, "{(\ell,c,p)}"] \arrow[rrr, bend right, "\ell+c"'] & M \times \overline{c(\Omega)} \times \overline{p(\Omega)} \arrow[r, "{(u,s,e)\mapsto(u+a,s,e)}", "\cong"'] & M \times \overline{c(A)} \times \overline{p(A)} \arrow[r, "\mathrm{proj}"] & M. \end{tikzcd} \tag{2.18}$$

Because the map (ℓ, c, p) is injective, TODO.

⊠

3. Compactness of the moduli space of Seiberg-Witten solutions: 2/3/19

These are Riccardo's notes on the lecture he gave, on the compactness of the moduli space of solutions to the Seiberg-Witten equations. This is a crucial step in Furuta's construction of finite-dimensional approximations, and relies on some functional analysis.

3.1. A closer look at the Seiberg-Witten monopole map.

Let X be a oriented closed spin 4-manifold. Let $\mathfrak{s}$ be a spin structure for it. Let $\mathbb{S}^{\pm}$ be the positive and negative spinor bundles associated to it. Fix a spin connection A on them.

Recall the Seiberg-Witten equations can be thought as a fiber-preserving S^1-equivariant map between these two S^1-Hilbert bundles over $H^1(X;\mathbb{R})$:

$$\widetilde{\mathcal{A}} = (A + i\ker(\mathrm{d})) \times \left(\Gamma(\mathbb{S}^+) \oplus \Omega^1(X)\right) \tag{3.1a}$$

$$\widetilde{\mathcal{C}} = (A + i\ker(\mathrm{d})) \times \left(\Gamma(\mathbb{S}^-) \oplus \Omega^0(X) \oplus H^1(X;\mathbb{R}) \oplus \Omega^+(X)\right). \tag{3.1b}$$

The map $\widetilde{\mu}\colon \widetilde{\mathcal{A}} \to \widetilde{\mathcal{C}}$ is defined by

$$(A, \phi, a) \longmapsto \left(A, D_A\phi + ia\phi, \mathrm{d}^*a, a_{\mathrm{harm}}, \mathrm{d}^+a - \sigma(\phi)\right). \tag{3.2}$$

As explained in the previous seminar, $\sigma(\phi)$ denotes the trace-free endomorphism $i(\phi \otimes \phi^* - \frac{1}{2}\|\phi\|^2\mathrm{id})$ of $\mathbb{S}^+$, considered via the map ρ as a self-dual 2-form on X.

The gauge group $\mathcal{G} = \mathrm{Aut}_{\mathrm{id}}(\mathfrak{s}) \cong \mathrm{Map}(X, S^1)$ acts on spinors on the 4-manifold via multiplication with $u\colon X \to S^1$ and on Spin^c connections via addition of $ud(u^{-1})$. It acts trivially on forms.

The map $\widetilde{\mu}$ is equivariant with respect to the action of $\mathcal{G}$. Dividing by the free action of the pointed gauge group we obtain the monopole map

$$\mu = \widetilde{\mu}/\mathcal{G}_0 : \mathcal{A} \to \mathcal{C}$$

as a fiber preserving map between the bundles $\mathcal{A} = \widetilde{\mathcal{A}}/\mathcal{G}_0$ and $\mathcal{C} = \widetilde{\mathcal{C}}/\mathcal{G}_0$ over $\mathrm{Pic}^{\mathfrak{s}}(X)$. The preimage of the section $(A, 0, 0, 0, 0)$ of $\mathcal{C}$, divided by the residual S^1-action, is called the*moduli space of monopoles.*

For a fixed $k > 2$, consider the fiberwise L^2_k Sobolev completion $\mathcal{A}_k$ and the fiberwise L^2_{k-1} Sobolev completion $\mathcal{C}_{k-1}$ of $\mathcal{A}$ and $\mathcal{C}$. The monopole map extends to a continuous map $\mathcal{A}_k \to \mathcal{C}_{k-1}$ over $Pic^{\mathfrak{s}}(X)$, which will also be denoted by μ.

We will use the following properties of the monopole map.

- It is S^1-equivariant.
- Fiberwise, it is the sum $\mu = l + c$ of a linear Fredholm map l and a nonlinear compact operator c.
- Preimages of bounded sets are bounded.

Claim 3.3. The moment map is S^1-equivariant.

Proof. Equivariance is immediate. The action is the residual action of the subgroup S^1 of gauge transformations which are constant functions on X. This group acts by complex multiplication on the spaces $\Gamma(\mathbb{S}^{\pm})$ of sections of complex vector bundles and trivially on forms. ⊠

Claim 3.4. Fiberwise, the moment map is the sum $\mu = l + c$ of a linear Fredholm map ℓ and a nonlinear compact operator c.

Proof. Restricted to a fiber, the monopole map is a sum of the linear Fredholm operator ℓ, consisting of the elliptic operators D_A and $d^* + d^+$, complemented by projections to and inclusions of harmonic forms. The nonlinear part of μ is built from the bilinear terms $a\phi$ and $\sigma(\phi)$. Multiplication $\mathcal{A}_k \times \mathcal{A}_k \to \mathcal{C}_k$ is continuous for $k > 2$. Combined with the compact restriction map $\mathcal{C}_k \to \mathcal{C}_{k-1}$ (Rellich lemma, see page 2 Lecture 19 in [**?**]) we gain the claimed compactness for c: Images of bounded sets are contained in compact sets. ⊠

Now let us show the following very useful property of compact perturbations of Fredholm operators.

Claim 3.5. The restriction of a compact perturbation $l + c\colon \mathcal{U}' \to \mathcal{U}$ of a linear Fredholm map ℓ between Hilbert spaces to any bounded, closed subset is proper.

Proof. Let p denote a projection to the kernel of ℓ. Let A be a bounded closed subset of $\mathcal{U}'$. It's easy to see that we have the following commutative diagram

$$\begin{tikzcd}
A \arrow[r, "(\ell,c,p)"] \arrow[ddr, "\ell+c"'] & \mathcal{U} \times \overline{c(A)} \times \overline{p(A)} \arrow[d, "\cong"', "+"] \\
& \mathcal{U} \times \overline{c(A)} \times \overline{p(A)} \arrow[d, "\pi"] \\
& \mathcal{U}
\end{tikzcd}$$

We observe that the map $h\colon A \to \mathcal{U} \times \overline{c(A)} \times \overline{p(A)}$ given by $a \mapsto (\ell(a), c(a), p(a))$ is injective and closed. Injectivity is clear since we are projecting on the kernel.

Closedness is a little bit more involved: let $\{(\ell_n, c_n, p_n)\}_n \subset \mathrm{Im}(h)$ converge to $(\ell_\infty, c_\infty, p_\infty)$. In particular there is a sequence $\{a_n\}_n \subset A$ such that $(\ell_n, c_n, p_n) = (\ell(a_n), c(a_n), p(a_n))$. We want to prove that $(\ell_\infty, c_\infty, \rho_\infty) \in h(A)$. Since ℓ is Fredholm we have the following property: *every bounded sequence $\{x_i\}_i$ in the domain whose image is convergent admits a convergent subsequence $\{x_{i_j}\}_j$*. Since A is closed and bounded (and any other closed subset of it would be bounded as well hence we can directly work with A), $\{a_n\}_n$ is bounded. Since ℓ is Fredholm we can extract a convergent subsequence $\{a'_n\}_n$ converging to $a \in A$ (since A is closed). By the uniqueness of the limit, it's easy to prove

$$(\ell_\infty, c_\infty, \rho_\infty) = (\ell(a), c(a), p(a)) \tag{3.6}$$

which proves the closedness of $h(A)$. This implies that h is proper, since h is an homeomorphism onto its image.

The addition map $+\colon (u, s, e) \mapsto (u + s, s, e)$ is an homeomorphism hence proper. The projection to $\mathcal{U}$ is proper since the other two factors are compact. ⊠

3.2. **A collection of results.** We will list here some results needed for the seminar.

Let U be an open subset of $\mathbb{R}^n$. We can consider the space $C_c^\infty(U;\mathbb{R}^r)$ of compactly supported $\mathbb{R}^r$-valued functions. Fix a real number $p>1$ and an integer $k\geq 0$. The Sobolev L_k^p norm is defined by

$$\|f\|_{p,k} := \sum_{|\alpha|<k} \sup_U \|D^\alpha f\|_p. \tag{3.7}$$

The Sobolev space $L_k^p(E)$ is defined to be the completion of $\Gamma(E)$ in the L_k^p norm.

Here are the basic facts about Sobolev spaces.

Sobolev inequality: If $k\leq \ell$ then there exists a constant C such that

$$\|\cdot\|_{p,k} \leq C\|\cdot\|_{p,\ell}, \tag{3.8}$$

and hence we have a bounded inclusion of Sobolev spaces $L_k^p(E)\hookrightarrow L_\ell^p(E)$.

Rellich lemma: The inclusion $L_{k+1}^p(E)\hookrightarrow L_k^p(E)$ is a compact operator.

Morrey inequality: Suppose $\ell\geq 0$ is an integer such that $\ell<k-n/p$; then there is a constant C such that

$$\|\cdot\|_{C^\ell} \leq C\|\cdot\|_{p,k}, \tag{3.9}$$

i.e. there is a bounded inclusion

$$L_k^p(E)\hookrightarrow C^\ell(E). \tag{3.10}$$

Smoothness: One has

$$\bigcap_{k\geq k_0} L_k^p(E) = C^\infty(E). \tag{3.11}$$

Lemma 3.12. *Over a closed Riemannian 4-manifold, multiplication of smooth functions extends to a bounded map*

$$L_k^2(X)\otimes L_\ell^2(X)\to L_\ell^2(X) \tag{3.13}$$

provided that $k\geq 3$ and $k\geq \ell$. In particular, $L_k^2(X)$ is an algebra for $k\geq 3$.

There are also bounded multiplication maps for the lower regularity Sobolev spaces in 4 dimensions, but these bring in Sobolev spaces with $p>2$.

Let now $D\colon \Gamma(E)\to\Gamma(F)$ be a differential operator of order m over a closed, oriented, Riemannian manifold (M,g). The basic point is that D extends to a bounded linear map between Hilbert spaces:

$$D\colon L_{k+m}^2(E)\to L_k^2(F). \tag{3.14}$$

Theorem 3.15 (Elliptic estimate)**.** *If D is elliptic of order m, one has estimates on the L_k^2-Sobolev norms for each $k\geq 0$:*

$$\|s\|_{2,k+m} \leq C_k(\|Ds\|_{2,k}+\|s\|_{2,k}). \tag{3.16}$$

Moreover,

$$\|s\|_{2,k+m} \leq C_k\|Ds\|_{2,k} \tag{3.17}$$

for $s\in(\ker D)^\perp$ (here $^\perp$ denotes the L^2-orthogonal complement).

There is an analogue for $L^{p,k+m}$ bounds.

As a consequence of this important theorem we have the following:

Corollary 3.18. *An elliptic operator D of order m defines a Fredholm map $L_{k+m}^2(E)\to L_k^2(F)$ for any $k\geq 0$. Its index is independent of k. Moreover, its index depends only on the symbol of D.*

Let (M,g) be an oriented Riemannian manifold. Let ∇ be an orthogonal covariant derivative in a real, Euclidean vector bundle $E\to M$. We know that ∇ has a formal adjoint ∇^*.

Proposition 3.19 (The Lichnérowicz formula)**.** *One has*

$$D^2 = \widetilde{\nabla}^*\widetilde{\nabla} + \frac{1}{4}\mathrm{scal}_g\cdot \mathrm{id}_{\mathbb{S}} + \frac{1}{2}\rho(F^\circ). \tag{3.20}$$

Lemma 3.21.

$$\frac{1}{2}\mathrm{d}^*\mathrm{d}(|s|^2) = \langle \nabla^*\nabla s, s\rangle - |\nabla s|^2. \tag{3.22}$$

Proof sketch. See Lemma1.1. L19 in [**?**]. The idea is to study the integral

$$\int_M f\langle \nabla^*\nabla s, s\rangle \,\mathrm{vol} \tag{3.23}$$

where f has compact support. ☒

It's important to remember that the one above is a pointwise equality.
Working locally one has the following result.

Lemma 3.24. *For a smooth function $f\colon M \to \mathbb{R}$ with compact support, if p is a local maximum, then* $(\mathrm{d}^*\mathrm{d}f)(p) \geq 0$.

The following lemma is an easy calculation.

Lemma 3.25. *For $\phi \in \Gamma(\mathbb{S}^+)$, one has*

$$((\phi\phi^*)_0\chi, \chi) = (\phi, \chi)^2 - \frac{1}{2}|\chi|^2|\phi|^2. \tag{3.26}$$

In particular,

$$((\phi\phi^*)_0\phi, \phi) = \frac{1}{2}|\phi|^4. \tag{3.27}$$

Proof. We have

$$\begin{aligned}
((\phi\phi^*)_0\chi, \chi) &= ((\phi\phi^*)\chi, \chi) - \frac{1}{2}(|\phi|^2\chi, \chi) \\
&= ((\phi, \chi)\phi, \chi) - \frac{1}{2}|\phi|^2|\chi|^2 \\
&= (\phi, \chi)^2 - \frac{1}{2}|\phi|^2|\chi|^2.
\end{aligned}$$

☒

Lemma 3.28. *For $\eta \in \Omega^2_X$ and $\phi \in \Gamma(\mathbb{S})$, one has $(\rho(\eta)\phi, \phi) \leq |\eta||\phi|^2$.*

Proof. It suffices to take $\eta = e \wedge f$ for orthogonal unit vectors e and f. One then has

$$\begin{aligned}
(\rho(\eta)\phi, \phi) &= (\rho(e \wedge f)\phi, \phi) && (3.29)\\
&= \frac{1}{2}([\rho(e), \rho(f)]\phi, \phi) && (3.30)\\
&= -\frac{1}{2}(\rho(f)\phi, \rho(e)\phi) && (3.31)\\
&\leq |\rho(e)\phi| \cdot |\rho(f)\phi|, && (3.32)
\end{aligned}$$

where in (3.31) we used the fact that ρ has image in the anti-skew-Hermitian matrices. Now since $|e| = 1$ then $|\rho(e)| = 1$ (similarly for f), and therefore we conclude. ☒

Lemma 3.33. *Let A be a Clifford connection for the spinor bundle of a spin^c structure of X. Let $a \in \Omega^1_X(i\mathbb{R})$; then*

$$D_{A+a}\phi = D_A\phi + a \cdot \phi, \tag{3.34}$$

where the last term is the Clifford multiplication between a and ϕ.

Proof. Let's work in local orthonormal coordinates of TX given by $\{e_1,\dots,e_n\}$. We have

$$\begin{aligned} D_{A+a}\phi &= \sum_i e_i\cdot(A+a)_{e_i}\phi \\ &= \sum_i e_i\cdot A_{e_i}\phi + \sum_i e_i\cdot a(e_i)\phi \\ &= D_A\phi + \sum_i e_i\cdot a(e_i)\phi \\ &= D_A\phi + a\sum_i e_i\phi \\ &= D_A\phi + a\cdot\phi. \end{aligned}$$

Notice that here we used that $a\in\Omega^1_X(i\mathbb{R})$ hence all the coefficients $a(e_i)$ are equal to each other, and without loss of generality we named then a. ⊠

3.3. **Compactness of the moduli space.** If the bundles $\mathcal{A}$ and $\mathcal{C}$ were finite-dimensional, then the boundedness property would be equivalent to properness. In this infinite-dimensional setting, the argument above can be used the same way as Heine-Borel in the finite-dimensional case to show that the boundedness condition implies properness. It turns out that the ingredients of the compactness proof for the moduli space also prove the stronger boundedness property.

Proposition 3.35. *Preimages $\mu^{-1}(B)\subset\mathcal{A}_k$ of bounded disk bundles $B\subset\mathcal{C}_{k-1}$ are contained in bounded disk bundles.*

Proof. It is sufficient to prove this fiberwise for the Sobolev completions of the restriction of the monopole map to the space $\{A\}\times(\Gamma(\mathbb{S}^+)\oplus\ker(d^*))$, which maps to $\{A\}\times\big(\Gamma(\mathbb{S}^-)\oplus\Omega^2_+(X)\oplus H^1(X;\mathbb{R})\big)$. We start by defining the following scalar product: using the elliptic operator $D=D_A+\mathrm{d}^+$ and its adjoint, define the L^2_k-norm via the scalar product on the respective function spaces through

$$(\cdot,\cdot)_i = (\cdot,\cdot)_0 + (D\cdot,D\cdot)_{i-1} \text{ for } 0<i\le k \tag{3.36a}$$

$$(\cdot,\cdot)_0 = \int_X\langle\cdot,\cdot\rangle. \tag{3.36b}$$

Using the elliptic estimates and continuity (i.e. boundedness) of D it's easy to see that this norm is equivalent to the classic Sobolev one. A similar definition can be extended to norms for the L^p_k-spaces. Let us take $\mu(A,\phi,a)=(A,\varphi,b,a_{\mathrm{harm}})\in\mathcal{C}_{k-1}$ with the norm of the latter bounded by some constant R. The Lichnérowicz formula (Proposition 3.19) for a connection $A+a=A'$ reads

$$D^*_{A'}D_{A'} = A'\circ A' + \frac{1}{4}s\cdot\mathrm{id}_{\mathbb{S}} + \frac{1}{2}\rho(F^\circ_{A'}) \tag{3.37}$$

with s denoting the scalar curvature of X. As a consequence we have a pointwise estimate:

$$\begin{aligned} d^*d|\phi|^2 &= 2\langle\nabla^*_{A'}\nabla_{A'}\phi,\phi\rangle - 2\langle\nabla_{A'}\phi,\nabla_{A'}\phi\rangle && (3.38)\\ &\le 2\langle\nabla^*_{A'}\nabla_{A'}\phi,\phi\rangle && (3.39)\\ &\le 2\langle D^*_{A'}D_{A'}\phi - \frac{s}{4}\phi - \frac{1}{2}\rho(F^\circ_{A'})\phi,\phi\rangle && (3.40)\\ &\le \langle 2D^*_{A'}\varphi - \frac{s}{2}\phi - (\sigma(\phi)+b)\phi,\phi\rangle && (3.41) \end{aligned}$$

Where in 3.38 We used Lemma 3.21. In 3.39 we removed the negative quantity on the left to obtain an inequality. In 3.41 we substituted in the second S-W equation.
Now we move some terms to the left and use the equality $D_{A+a}=D_A+a$ together with the fact that the

Dirac operator is self-adjoint to get

$$d^*d|\phi|^2 + \frac{s}{2}|\phi|^2 + \langle\sigma(\phi), \phi\rangle \leq \langle 2D^*_{A'}\varphi, \phi\rangle - \langle b\phi, \phi\rangle \tag{3.42}$$

$$d^*d|\phi|^2 + \frac{s}{2}|\phi|^2 + \frac{1}{2}|\phi|^4 \leq \langle 2D^*_{A}\varphi, \phi\rangle + 2\langle a\cdot\varphi, \phi\rangle - b|\phi|^2 \tag{3.43}$$

$$\leq 2\left(\|D^*_A\varphi\|_\infty + \|a\|_\infty\|\varphi\|_\infty\right)\cdot|\phi| + \|b\|_\infty \cdot |\phi|^2 \tag{3.44}$$

$$\leq c_1\left((1+\|a\|_\infty)\|\varphi\|_{L^2_{k-1}}\cdot|\phi| + \|b\|_{L^2_{k-1}}\cdot|\phi|^2\right) \tag{3.45}$$

Where in 3.43 we are using Lemma 3.25 to bound $\sigma(\phi)$. In 3.45 we used Sobolev embeddings theorem (Morrey's inequality) to bound the L^∞-norm with the Sobolev norm.
Now we need to estimate $\|a\|_\infty$. First thing, for $p > 4$ we get a Sobolev estimate $\|a\|_\infty \leq c_2\|a\|_{L^p_1}$ and then use the elliptic estimate:

$$\|a\|_{L^p_1} = \|a_{\text{harm}} + a'\|_{L^p_1} \leq \|a_{\text{harm}}\|_{L^p_1} + \|a'\|_{L^p_1} \tag{3.46}$$

$$\leq \|a_{\text{harm}}\|_{L^p_0} + \|d^+a\|_{L^p_0} \tag{3.47}$$

where in 3.46 we used the Hodge decomposition of a, in 3.47 we applied the elliptic estimate to both component. Recall that $d^+(a_{\text{harm}}) = 0$ and $d^+a = d^+a'$.
Combination with the equality $d^+a = b + \sigma(\phi)$ then leads to an estimate

$$\|a\|_\infty \leq c_4\left(\|a_{\text{harm}}\|_{L^p_0} + \|b\|_{L^p_0} + \|\sigma(\phi)\|_{L^p_0}\right) \tag{3.48}$$

$$\leq c_5\left(\|a_{\text{harm}}\|_{L^p_0} + \|b\|_{L^2_{k-1}} + \|\phi\|^2_\infty\right) \tag{3.49}$$

In the last passage we control the L^p_0-norm with the L^2_{k-1}-one, since $p > 4$.
Putting these two estimates together, we get something of the form

$$d^*d|\phi|^2 + \frac{1}{2}\|s\|_\infty\|\phi\|^2_\infty + \frac{1}{2}\|\phi\|^4_\infty \leq c\left(1 + c_5\left(\|a_{\text{harm}}\|_{L^p_0} + \|b\|_{L^2_{k-1}} + \|\phi\|^2_\infty\right)\right)\|\varphi\|_{L^2_{k-1}}\cdot\|\phi\|_\infty + \|b\|_{L^2_{k-1}}\cdot\|\phi\|^2_\infty \tag{3.50}$$

$$\leq K\|\phi\|^3_\infty + R\|\phi\|^2_\infty \tag{3.51}$$

Where in 3.51 we applied the bounds we had by assumption on the elements in the image.
So our inequality is now:

$$d^*d|\phi|^2 + \frac{1}{2}\|\phi\|^4_\infty \leq K\|\phi\|^3_\infty + R\|\phi\|^2_\infty - \frac{1}{2}\|s\|_\infty\|\phi\|^2_\infty \tag{3.52}$$

$$\leq K\|\phi\|^3_\infty + R\|\phi\|^2_\infty \tag{3.53}$$

Now this inequality must hold in particular when ϕ achieves its maximum, and on that point the Laplacian is positive, hence we can forget about it and get:

$$\frac{1}{2}\|\phi\|^4_\infty \leq K\|\phi\|^3_\infty + R\|\phi\|^2_\infty \tag{3.54}$$

In particular we bound the 4-th power of a quantity with a polynomial in that quantity of degree 3. This implies that $\|\phi\|_\infty$ must be bounded. Therefore we can bound the L^p_0-norm of (ϕ, a) for every $p \geq 1$.
Now comes bootstrapping: for $i \leq k$, assume inductively L^2_{i-1}-bounds on (ϕ, a). To obtain L^2_i-bounds, compute:

$$\|(\phi, a)\|^2_{L^2_i} - \|(\phi, a)\|^2_{L^2_0} = \|(D_A\phi, d^+a)\|^2_{L^2_{i-1}} \tag{3.55}$$

$$= \|(\phi + ia\phi, b - \sigma(\phi)\|^2_{L^2} \tag{3.56}$$

$$= \|(\phi, b)\|^2_{L^2_{i-1}} + \|(ia\phi, \sigma(\phi))\|^2_{L^2_{i-1}} \tag{3.57}$$

The first equality holds by our definition of the Sobolev norm. The last equality holds as $D_{A'} = D_A + a$. The summands in the last expression are bounded by the assumed L^2_{i-1}-bounds on (ϕ, a) together with the Sobolev multiplication properties. Note that the steps for $i = 2$ and 3 require special care (see [**?**] page 4 L21) or use Sobolev embedding together with the fact that we have control on the L^p-norms of (ϕ, a) for every p, which gives us control on the respective Sobolev norms for $p = 2$. ☒

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