M392C NOTES: TOPICS IN ALGEBRAIC TOPOLOGY

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These notes were taken in UT Austin's M392C (Topics in Algebraic Topology) class in Spring 2017, taught by Andrew Blumberg. I live-TEXed them using vim, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Alternatively, these notes are hosted on Github at https://github.com/adebray/equivariant_homotopy_theory, and you can submit a pull request. Thanks to Rustam Antia-Riedel, Gill Grindstaff, Yixian Wu, and an anonymous reader for catching a few errors; to Ernie Fontes, Tom Gannon, Richard Wong, and Valentin Zakharevich for some clarifications; and to Yuri Sulyma for adding some remarks and references.

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Lecture 1.

G-spaces: 1/17/17

This class will be an overview of equivariant stable homotopy theory. We're in the uncomfortable position where this is a big subject, a hard subject, and one that is poorly served by its textbooks. Algebraic topology is like this in general, but it's particularly acute here. Nonetheless, here are some references:

- Adams, "Prerequisites (on equivariant stable homotopy) for Carlsson's lecture." [Ada84]. This is old, and some parts of it don't reflect how we do things now.
- The Alaska notes [May96], edited by May, is newer, and is written by many authors. Some of it is a grab bag, and some parts (e.g. the rational equivariant bits) aren't entirely right. It's also not a textbook.
- Appendix A of Hill-Hopkins-Ravenel [HHR16]. This is a paper which resolved an old conjecture on manifolds using equivariant stable homotopy theory, but let this be a lesson on referee reports: the authors were asked to provide more background, and so wrote a 150-page appendix on this material. Their suffering is your gain: the appendix is a well-written introduction to equivariant stable homotopy theory, albeit again not a textbook.

There are arguably two very serious modern applications of equivariant stable homotopy theory:

- The first is trace methods in algebraic K-theory: Hochschild homology and its topological cousins are equipped with natural S^1-actions (the same S^1-action coming from field theory). This is how people other than Quillen compute algebraic K-theory.
- The other major application is Hill-Hopkins-Ravenel's settling of the Kervaire invariant 1 conjecture in [HHR16].

The nice thing is, however you feel about the applications, both applications require developing new theory in equivariant stable homotopy theory. Hill-Hopkins-Ravenel in particular required a clarification of the foundations of this subject which has been enlightening.

In this class, we hope to cover the foundations of equivariant stable homotopy theory. On the one hand, this will be a modern take, insofar as we emphasize the norm and the presheaf on orbit categories (these will be explained in due time), the modern emerging consensus on how to think of these things, different than what's written in textbooks. The former is old, but has gained more attention recently; the latter is new. Moreover, there's an increasing sense that a lot of the foundations here are best done in ∞-categories. We will not take this appoach in order to avoid getting bogged down in ∞-categories; moreover, this class is supposed to be rigorous. It will sometimes be clear to some people that ∞-categories lie in the background, but we won't talk very much about them.

We'll cover some old topics such as Smith theory and the Segal conjecture, and newer ones such as trace methods and Hill-Hopkins-Ravenel, depending on student interest. We will not have time to discuss many topics, including equivariant cobordism or equivariant surgery theory.

Prerequisites. If you don't know these prerequisites, that's okay; it means you're willing to read about it on your own.

- Foundations of unstable homotopy theory at the level of May's *A Concise Course in Algebraic Topology* [May99]. For example, we'll discuss equivariant CW complexes, so it will help to know what a CW complex is.
- A little bit of category theory, e.g. found in Mac Lane [Mac78] or Riehl [Rie16].
- This class will not require much in the way of simplicial methods (simply because it's hard to reconcile simplicial methods with non-discrete Lie groups), but you will want to know the bar construction. An excellent source for this is [Rie14, Chapter 4].
- A bit of abstract homotopy theory, e.g. what a model structure is. Good sources for model categories are [Rie14, Part III] and [Hov99].

If you don't know these, feel free to ask the professor for references. His advisor suggested that a foundation for the stable category is Lewis-May-Steinberger's account [LMS86] of the equivariant category and let $G = *$, but perhaps this isn't necessarily a good reference for nontrivial groups.

Unstable equivariant questions are very natural, and somewhat reasonable. But stable questions are harder; they ultimately arise from reasonable questions, but the formulation and answers are hard: even discussing the equivariant analogue of $\pi_0 S^0$ requires some representation theory — and yet of course it should. Thus there's a lot of foundations behind hard calculations. There will be problem sets; if you want to learn the material (or are an undergrad), you should do the problem sets.

Categories of topological spaces. The category of topological spaces we consider is Top, the category of compactly generated, weak Hausdorff spaces (and continuous maps); we'll also consider Top_*, the category of based, compactly generated, weak Hausdorff spaces and continuous, based maps. This is an important and old trick which eliminates some pathological behavior in quotients. It's reasonable to imagine that point-set topology shouldn't be at the heart of foundational issues, but there are various ways to motivate this, e.g. to make Top more resemble a topos or the category of simplicial sets.

Definition 1.1. Let X be a topological space.

- A subset $A \subseteq X$ is **compactly closed** if $f^{-1}(A)$ is closed for every $f : Y \to X$, where Y is compact and Hausdorff.
- X is **compactly generated** if every compactly closed subset of X is closed.
- X is **weak Hausdorff** if the diagonal map $\Delta : X \to X \times X$ is closed when $X \times X$ has the compactly generated topology.

The intuition behind compact generation is that the topology is determined by compact Hausdorff spaces. The weak Hausdorff topology is strictly stronger than T_1 (points are closed), but strictly weaker than Hausdorff spaces. Any space you can think of without trying to be pathological will meet these criteria.

There is a functor k from all spaces to compactly generated spaces which adds the necessary closed sets. This has the unfortunate name of **k-ification** or **kaonification**; by putting the compactly generated topology on $X \times X$, we mean taking $k(X \times X)$. There's also a "weak Hausdorffification" functor w which makes a space weakly Hausdorff, which is some kind of quotient.[1]

When computing limits and colimits, it's often possible to compute it in the category of spaces and then apply k and w to return to Top. This is fine for limits, but for colimits, w is particularly badly behaved: you cannot compute the colimit in Top by computing it in Set and figuring out the topology; more generally, it will be some kind of quotient.

Nonetheless, there are nice theorems which make things work out anyways.

Proposition 1.2. *Let $Z = \operatorname{colim}(X_0 \to X_1 \to X_2 \to \dots)$ be a sequential colimit (sometimes called a **telescope**); if each X_i is weak Hausdorff, then so is Z.*

Proposition 1.3. *Consider a diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \\ C & \end{tikzcd}$$

where f is a closed inclusion. If A, B, and C are weakly Hausdorff, then $B \amalg_A C$ is weakly Hausdorff.

These are the two kinds of colimits people tend to compute, so this is reassuring.

One reason we require regularity on our topological spaces is the following, which is not true for topological spaces in general.

Lemma 1.4. *Let X, Y, and Z be in* Top*; then, the natural map*

$$\operatorname{Map}(X \times Y, Z) \longmapsto \operatorname{Map}(X, \operatorname{Map}(Y, Z))$$

is a homeomorphism.

Enrichments. The categories Top and Top_* are enriched over themselves (as will categories of G-spaces, which we'll see later). This means a brief digression into enriched categories.

Definition 1.5. Let $(\mathsf{V}, \otimes, \mathbf{1})$ be a symmetric monoidal category.[2] Then, an **enrichment** of a category C over V means

- for every $x, y \in \mathsf{C}$, there is a hom-object $\underline{\mathsf{C}}(x, y)$, which is an object in V,
- for every $x \in \mathsf{C}$, there is a unit $\mathbf{1} \to \underline{\mathsf{C}}(x, x)$,
- composition $\underline{\mathsf{C}}(x, y) \otimes \underline{\mathsf{C}}(y, z) \to \underline{\mathsf{C}}(x, z)$ is associative and unital, and
- the underlying category is recovered as $\mathsf{C}(x, y) = \operatorname{Map}(\mathbf{1}, \underline{\mathsf{C}}(x, y))$.

A great deal of category theory can be generalized to enriched categories, including V-enriched functors, V-enriched natural transformations, V-enriched limits and colimits, and more. The canonical reference is Kelley [Kel84], available free and legally online. It covers just about everything we need except for the Day convolution, which can be read from Day's thesis [Day70]. Another good source, with a view towards homotopy theory, is [Rie14, Chapter 3].

Definition 1.6. Let C and D be enriched over V. Then, an **enriched functor** $F : \mathsf{C} \to \mathsf{D}$ is an assignment of objects in C to objects in D and maps $\underline{\mathsf{C}}(x, y) \to \underline{\mathsf{D}}(Fx, Fy)$ that are V-morphisms, and commute with composition.

A category enriched over Top is called a **topological category**.

Exercise 1.7. Work out the definition of enriched natural transformations.

This brings us to the beginning.

Let G be a group. We'll generally restrict to finite groups or compact Lie groups; this is not because these are the only interesting groups, but rather because they are the only ones we really understand. If you can come up with a good equivariant homotopy theory for discrete infinite groups, you will be famous. Throughout, keep in

[1] The k functor is right adjoint to the forgetful map, which tells you what it does to limits.

[2] Briefly, this means V has a tensor product $\otimes$ and a unit $\mathbf{1}$; there are certain axioms these must satisfy.

mind the examples C_p (the cyclic group of order p, sometimes also denoted $\mathbb{Z}/p$), C_{p^n}, the symmetric group S_n, and the circle group S^1.

There's a monad M_G on Top which sends $X \mapsto G \times X$, and analogously M_G^* on Top_* sending $X \to G_+ \wedge X$; then, one can define the category of **G-spaces** $G\mathsf{Top}$ (resp. **based G-spaces** $G\mathsf{Top}_*$) to be the category of algebras over M_G (resp. M_G^*). This is probably not the most explicit way to define G-spaces, but it makes it evident that $G\mathsf{Top}$ and $G\mathsf{Top}_*$ are complete and cocomplete.

More explicitly, $G\mathsf{Top}$ is the category of spaces $X \in \mathsf{Top}$ equipped with a continuous action $\mu\colon G \times X \to X$. That is, μ must be associative and unital. Associativity is encoded in the commutativity of the diagram

$$\begin{tikzcd} G \times G \times X \arrow[r, "1\times\mu"] \arrow[d, "m"] & G \times X \arrow[d, "\mu"] \\ G \times X \arrow[r, "\mu"] & X. \end{tikzcd}$$

The morphisms in $G\mathsf{Top}$ are the **G-equivariant** maps $f : X \to Y$, i.e. those commuting with μ:

$$\begin{tikzcd} G \times X \arrow[r] \arrow[d, "\mu_X"] & G \times Y \arrow[d, "\mu_Y"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

It's possible (but not the right idea) to let $\underline{G}$ denote[3] the category with an object $*$ such that $\underline{G}(*,*) = G$. Then, $G\mathsf{Top}$ is also the category of functors $\underline{G} \to \mathsf{Top}$, with morphisms as natural transformations. This realizes $G\mathsf{Top}$ as a **presheaf category**; it will eventually be useful to do something like this, but not in this specific way.

When we write $\mathrm{Map}(X,Y)$ in $G\mathsf{Top}$ or $G\mathsf{Top}_*$, we could mean three things:

(1) The set of G-equivariant maps $X \to Y$.
(2) The space of G-equivariant maps $X \to Y$ in the subspace topology of all maps from $X \to Y$. As this suggests, $G\mathsf{Top}$ admits an enrichment over Top (resp. $G\mathsf{Top}_*$ admits an enrichment over Top_*).
(3) The G-space of all maps $X \to Y$, where G acts by conjugation: $f \mapsto g^{-1}f(g\cdot)$. This means $G\mathsf{Top}$ is enriched in itself, as is $G\mathsf{Top}_*$.

Each of these is useful in its own way: for constructions it may be important to be self-enriched, or to only look at G-equivariant maps. We will let $\mathrm{Map}^G(X,Y)$ or $\mathrm{Map}(X,Y)$ denote (2) or its underlying set (1), and $G\,\mathrm{Map}(X,Y)$ denote (3).

It turns out you can recover Map^G from Map: the equivariant maps are the fixed points under conjugation of all maps. This is written $\mathrm{Map}(X,Y)^G = \mathrm{Map}^G(X,Y)$.

Throughout this class, "subgroup" will mean "closed subgroup" unless specified otherwise.

Definition 1.8. Let X be a G-set and $H \subseteq G$ be a subgroup. Then, the **H-fixed points** of X is the space $X^H := \{x \in X \mid hx = x \text{ for all } h \in H\}$. This is naturally a WH-space, where $WH = NH/H$ (here NH is the normalizer of H in G).[4]

Definition 1.9. The **isotropy group** of an $x \in X$ is $G_x := \{h \in G \mid hx = x\}$.

These are useful in the following two ways.

(1) Often, it will be helpful to reduce questions from $G\mathsf{Top}$ to Top using $(-)^H$.
(2) It's also useful to induct over isotropy types.

Now, we'll see some examples of G-spaces.

Example 1.10. Let H be a subgroup of G; then, the **orbit space** G/H is a useful example, because it corepresents the fixed points by H. That is, $X^H \cong G\,\mathrm{Map}(G/H, X)$. These spaces will play the role that points did when we build things such as equivariant CW complexes. ◀

Example 1.11. Let $H \subset G$ as usual and $U\colon G\mathsf{Top} \to H\mathsf{Top}$ be the forgetful functor. Then, U has both left and right adjoints:

[3] There isn't really a standard notation for this category, but the closest is BG. This notation emphasizes the fact that groupoids are Quillen equivalent to 1-truncated spaces.

[4] If $H \trianglelefteq G$, then X^H is also a G/H-space.

- The left adjoint sends X to the **balanced product** $G \times_H X \coloneqq G \times X/\sim$, where $(gh, x) \sim (g, hx)$ for all $g \in G$, $h \in H$, and $x \in X$. Despite the notation, this is *not* a pullback! (In the based case, the balanced product is $G_+ \wedge_H X$.) G acts via the left action on G. This is called the **induced G-action** on $G \times_H X$.
- The right adjoint is $F_H(G, X)$ (or $F_H(G_+, X)$ in the based case), the space of H-maps $G \to X$, with G-action $(gf)(g') = f(g'g)$. This is called the **coinduced G-action** on $F_H(G, X)$.[5] ◀

Remark. Here is a categorical perspective on "change of group." Quite generally, a group homomorphism $G \xrightarrow{f} H$ induces adjunctions

$$\begin{tikzcd} G\mathsf{Top} \arrow[r, shift left=2.5ex, "f_!"] \arrow[r, shift right=2.5ex, "f_*"'] & H\mathsf{Top} \arrow[l, "f^*"'] \end{tikzcd}.$$

These are given by $f_!(X) = H \times_G X$ and $f_*(X) = F_G(H, X)$ for a G-space X, where H is given the structure of a G-space by f. When $H = *$, an H-space is just a space, and $f_!(X) = X_G$ is the space of orbits while $f_*(X) = X^G$ is the space of fixed points. Observe that similar statements hold for categories of modules, given a ring homomorphism $R \xrightarrow{f} S$.

In fact, these are both cases of very general abstract nonsense. Let BG denote the category with one object $*$ with $\mathrm{Hom}(*, *) = G$; as we have said above, we can (naïvely) write $G\mathsf{Top}$ as the functor category Top^{BG}. A group homomorphism $G \xrightarrow{f} H$ induces a functor $BG \xrightarrow{F} BH$ (it is not quite true that the two are equivalent—think about why this is). Now $f^*\colon H\mathsf{Top} \to G\mathsf{Top}$ is just restriction along F:

$$\begin{tikzcd} BG \arrow[r, dotted, "f^*(Y)"] \arrow[d, "F"'] & \mathsf{Top} \\ BH \arrow[ur, "Y"'] \end{tikzcd}$$

According to abstract nonsense, restriction along F has a left and right adjoint, called *left and right Kan extension along F*:

$$\begin{tikzcd} BG \arrow[r, "X"] \arrow[d, "F"'] & \mathsf{Top} \\ BH \arrow[ur, dotted, "f_!(X) = \mathrm{Lan}_F X"'] \arrow[u, phantom, shift left=3ex, "\Downarrow\eta"] \end{tikzcd} \qquad \begin{tikzcd} BG \arrow[r, "X"] \arrow[d, "F"'] & \mathsf{Top} \\ BH \arrow[ur, dotted, "f_*(X) = \mathrm{Ran}_F X"'] \arrow[u, phantom, shift left=3ex, "\Uparrow\epsilon"] \end{tikzcd}$$

These diagrams do not commute, but there are natural maps $X \xRightarrow{\eta} f^*f_!(X)$ and $f^*f_*(X) \xRightarrow{\epsilon} X$. When H is the trivial group, BH is the trivial category, and it is known that left/right Kan extensions of a functor X along a functor to the trivial category pick out the colimit/limit of X. That is, still viewing a G-space X as a functor $BG \to \mathsf{Top}$, we have $X_G = \operatorname{colim}_{BG} X$ and $X^G = \lim_{BG} X$.

For an introduction to Kan extensions, we recommend [Rie16, Chapter 6] (which is almost the same as [Rie14, Chapter 1] but with some more amusing examples). Like much of category theory, this is ultimately all trivial, but it may be highly non-trivial to understand why it is trivial. ◀

Example 1.12. Let V be a finite-dimensional real representation of G, i.e. a real inner product space on which G acts in a way compatible with the inner product. (This is specified by a group homomorphism $G \to \mathrm{O}(V)$.) The one-point compactification of V, denoted S^V, is a based G-space; the unit disc $D(V)$ and unit sphere $S(V)$ are unbased spaces, but we have a quotient sequence

$$S(V)_+ \longrightarrow D(V)_+ \longrightarrow S^V.$$

If $V = \mathbb{R}^n$ with the trivial G-action, S^V is S^n with the trivial G-action, so these generalize the usual spheres; thus, these S^V are called **representation spheres**. ◀

We will let S^n denote $S^{\mathbb{R}^n}$, namely our preferred model for the n-sphere with trivial G-action.

[5] This actually *is* a group action, since if $a, b, g \in G$, then $(a(bf))(g) = (bf)(ga) = f(gab) = (ab(f))(g)$.

Beginnings of homotopy theory.

Definition 1.13. A G-**homotopy** is a map $h\colon X \times I \to Y$ in $G\mathsf{Top}$, where G acts trivially on I. We generally think of it, as usual, as interpolating between $h(-,0)$ and $h(-,1)$. This is the same data as a path in $G\operatorname{Map}(X,Y)$. A G-**homotopy equivalence** between X and Y is a map $f : X \to Y$ such that there exists a $g : Y \to X$ such that there are G-homotopies $gf \sim \mathrm{id}_X$ and $fg \sim \mathrm{id}_Y$.

The (well, a) natural question that might arise: what are G-weak equivalences and G-CW complexes? This closely relates to obstruction theory — CW complexes are test objects.

To define G-CW complexes, we need cells. One choice is $G/H \times D^{n+1}$ and $G/H \times S^n$, where the actions on D^{n+1} and S^n are trivial. This is a plausible choice (and in fact, will be the right choice), but it's not clear why — why not $G \times_H D(V)$ or $G \times_H S(V)$ for some H-representation V? Ultimately, this comes from a (quite nontrivial) theorem that these can be triangulated in terms of the cells $G/H \times D^{n+1}$ and $G/H \times S^n$.[6] This is one of several triangulation results proven in the 1970s, which are now assumed without comment, but if you like this kind of math then it's a very interesting story.

Definition 1.14. A G-**CW complex** is a sequential colimit of spaces X_n, where X_{n+1} is a pushout

$$\begin{tikzcd}
\bigvee G/H \times S^n \arrow[r] \arrow[d] & X_n \arrow[d] \\
\bigvee G/H \times D^{n+1} \arrow[r] & X_{n+1},
\end{tikzcd}$$

where H varies over all closed subgroups of G.

That is, it's formed by attaching cells just as usual, though now we have more cells.

This immediately tells you what the homotopy groups have to be: $[G/H \times S^n, X]$, which by an adjunction game is isomorphic to $\pi_n(X^H)$. We let $\pi_n^H(X) := \pi_n(X^H)$. Thus, we can define weak equivalences.

Definition 1.15. A map $f : X \to Y$ of G-spaces is a **weak equivalence** if for all subgroups $H \subset G$, $f_*\colon \pi_n^H(X) \to \pi_n^H(Y)$ is an isomorphism.

These homotopy groups have a more complicated algebraic structure: they're indexed by the lattice of subgroups of G and the integers. This is fine (you can do homological algebra), but some things get more complicated, including asking what the analogue of connectedness is!

One quick question: do we need all subgroups H? What if we only want finite-index ones? The answer, in a very precise sense, is that if you're willing to use fewer subgroups, you get fewer cells $G/H \times S^n$, and that's fine, and you get a different kind of homotopy theory.

Finally, the Whitehead theorem is true for G-CW complexes. This follows for the same reason as in May's course: it follows word-for-word after proving the equivariant HELP lemma (homotopy extension lifting property), which is true by the same argument.

We'll next talk about presheaves on the orbit category, leading to Bredon cohomology.

Lecture 2.

Homotopy theory of G-spaces: 1/19/17

"It's nice to write down, but oh so false."

Last time, we saw the definition of a G-CW complex, but no examples were provided. Today, we'll start with some examples.

Recall that a G-CW complex is a sequential colimit $X = \operatorname{colim}_n X_n$, where X_n is formed by attaching cells $G/H \times D^n$ along maps $G/H \times S^{n-1} \to X_{n-1}$: just like the CW complexes we know and love, but with new cells G/H indexed by the closed subgroups $H \subset G$. The idea is that you're building up a space by attaching different spaces with different isotropy groups (G/H has isotropy group H, just by construction).

Example 2.1 (Zero-dimensional complexes)**.** The zero-dimensional complexes are G/H or disjoint unions $\amalg_i G/H_i$. This is an instance of the slogan that "orbits are points." Keep in mind that if G is a compact Lie group, this might not be zero-dimensional in other, more familiar kinds of dimension. ◄

[6] Illman's thesis [Ill72] is a reference, albeit not the most accessible one.

Example 2.2. Let S^1 act on $\mathbb{R}^2$ by rotation along the origin. This also induces a C_n-action, as $C_n \subseteq S^1$ as the n^{th} roots of unity. Let V denote this C_n-space.

Let $D(V)$ denote the unit disc in V, and S^V denote its one-point compactification, a representation sphere. Then, $D(V)$ looks like wedges of pie, as the origin is fixed. On S^V, the point at infinity is also fixed, so we obtain a beachball.

Now let's consider V as an S^1-space, and write down the CW structure on S^V. There are two fixed points, and each one is a 0-cell $S^1/S^1 \times *$, but there is one 1-cell $S^1 \times I$ attached to the endpoints (thought of as a meridian rotated around the sphere).

Now let's consider the beachball for C_2 on S^V, where there are two hemispheres and C_2 rotates by a half-turn. What's the G-CW structure on this?

- There are two 0-cells $C_2/C_2 \times *$, corresponding to the two fixed points, the north and south poles.
- There is a single free 1-cell $C_2 \times I$, corresponding to the boundary of the hemispheres.
- There is a single 2-cell $C_2 \times D^2$. ◄

Last time, we discussed other prospective cells $G \times_H S(V)$ and $G \times_H D(V)$; these can be decomposed in terms of the actual cells we use. One point to observe about these cells is that G does not act on them by permuting non-equivariant cells around, but rather in a more complicated way; there is virtue in the simplicity of the G-cells we have chosen to work with.

Exercise 2.3. C_2 also acts on S^2 by the antipodal map, which has no fixed points. Write a C_2-CW cell structure for this C_2-space.

Example 2.4. The torus $S^1 \times S^1$ has an S^1-action given by $z(z_1, z_2) = (zz_1, z_2)$. With this action, the torus can be viewed as an S^1-CW complex with one 0-cell $S^1/e \times *$ and one 1-cell $S^1 \times [0,1]$, with the attaching map sending 0 and 1 to $*$. Note that the largest cell we used here was a 1-cell, whereas in the nonequivariant construction of the torus, we are required to use a 2-cell. Check out Figure 1 for a picture. ◄

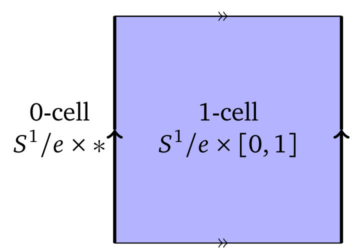

FIGURE 1. The S^1-CW structure on the torus in Example 2.4. There is one 0-cell and one 1-cell.

Example 2.5. Let T be a solid equilateral triangle in the plane, so $D_6 = \langle r, s \mid r^3 = s^2 = 1, srs = r^{-1} \rangle$ acts on it by rotations and reflections.

- There are three 0-cells: the center is a fixed point, so a $D_6/D_6 \times *$. The three vertices are an orbit, with the stabilizer of each vertex conjugate to $\langle s \rangle$ inside D_6, so they form a 0-cell of the form $D_6/\langle s \rangle \times *$. Similarly, the midpoints of each edge are an orbit with each stabilizer conjugate to $\langle s \rangle$, so they're also a 0-cell of the form $D_6/\langle s \rangle \times *$.
- There are three 1-cells: the three line segments from the center to a vertex are an orbit for $D_6/\langle s \rangle$, so form a $D_6/\langle s \rangle \times [0,1]$. Similarly, the three line segments from the center to the midpoint of an edge form a $D_6/\langle s \rangle \times [0,1]$. The six line segments from a vertex to the center of a midpoint are a free orbit of D_6, hence form a $D_6/e \times [0,1]$.
- The triangle minus these 0- and 1-cells is a free orbit, a $D_6/e \times D^2$.

See Figure 2 for a picture. ◄

There will be additional examples of G-CW complexes on the homework, some with richer structure.

Remark. At this point in class, the professor mentioned that these notes are hosted on Github at https://github.com/adebray/equivariant_homotopy_theory. Since there aren't very many sources for learning this material, and existing ones tend to have few examples, the hope is that these notes can be turned into

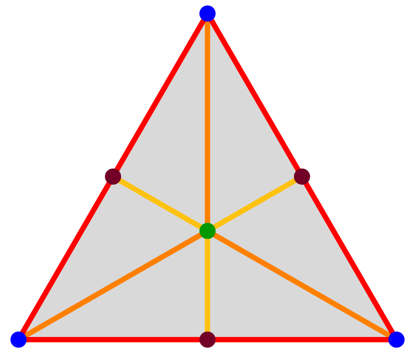

FIGURE 2. The D_6-equivariant structure on a solid triangle, as in Example 2.5. Each cell is depicted in a different color. The three 0-cells are purple ($D_6/\langle s\rangle \times *$), blue ($D_6/\langle s\rangle \times *$), and green ($D_6/D_6 \times *$); the three 1-cells are yellow ($D_6/\langle s\rangle \times [0,1]$), orange ($D_6/\langle s\rangle \times [0,1]$), and red ($D_6/e \times [0,1]$); the one 2-cell is gray ($D_6/e \times D^2$).

a good source of lecture notes for learning this material. So as you're learning this material, feel free to add examples, insert comments (e.g. "this section is confusing/unmotivated"), and let me know if you want access to the repository. ◀

Remark.

(1) There is a technical issue of a G-CW structure on a product of G-CW complexes; namely, there are technical difficulties in cleanly putting a G-CW structure on $G/H_1 \times G/H_2$ involving triangulation. We won't digress into this: it's straightforward for finite groups, but a theorem for compact Lie groups, and required revisiting the foundations. Similarly, if $H \subset G$, we'd like the forgetful functor $G\mathsf{Top} \to H\mathsf{Top}$ to send G-CW complexes to H-CW complexes. This is again possible, yet involves technicalities.

(2) A nicer fact is that computing the fixed points of a G-CW complex is straightforward. Recall that $(-)^H$ is a right adjoint, which can be seen by realizing it as the limit of the diagram

$$\begin{tikzcd} \bullet \arrow[loop above, "H"] \arrow[r] & \mathsf{Top}. \end{tikzcd}$$

Thus, we don't expect it to commute with colimits in general. However, it does commute with many important ones, as in the following proposition. ◀

Proposition 2.6. *The fixed point functor $(-)^H$ commutes with*

(1) *pushouts where one leg is a closed inclusion, and*
(2) *sequential colimits along closed inclusions.*

This is great, because it means we can commute $(-)^H$ through the construction of a G-CW complex! In particular, on each cell,

$$(G/K \times D^n)^H \cong (G/K)^H \times D^n,$$

so we need to understand $(G/K)^H \cong \mathrm{Map}^G(G/H, G/K)$. We will return to this important point.

Two approaches to the Whitehead theorem. We'll now discuss some homotopy theory of G-spaces and the Whitehead theorem. The first will be a hands-on proof using the HELP lemma. This is an elegant approach to unstable homotopy theory due to Peter May in which one lemma gives quick proofs of several theorems. In the equivariant case, it allows a quick reduction to the non-equivariant case; it will be useful to see a proof of this nature. Ultimately, we will take a different approach involving model categories, and this will be the second perspective.

Definition 2.7. Let $X, Y \in \mathsf{Top}$ and $f : X \to Y$ be continuous. Then, f is n-**connected** if $\pi_q(f)\colon \pi_q(X) \to \pi_q(Y)$ is an isomorphism when $q < n$ and surjective when $q = n$.

We wish to generalize this to the equivariant case.

Definition 2.8. Let $\theta\colon \{\text{conjugacy classes of subgroups of } G\} \to \{x \in \mathbb{Z} \mid x \geq -1\}$.

- A map $f : X \to Y$ of G-spaces is θ**-connected** if for all $H \subset G$, f^H is $\theta(H)$-connected.
- A G-CW complex is θ**-dimensional** if all cells of orbit type G/H have (nonequivariant) dimension at most $\theta(H)$.

Theorem 2.9 (Equivariant HELP lemma)**.** *Let A, X, Y, and Z be G-CW complexes such that $A \subseteq X$ is θ-dimensional and let $e\colon Y \to Z$ be a θ-connected G-map. Given $g : A \to Y$, $h\colon A \times I \to Z$, and $f : X \to Z$ such that $eg = hi_0$ and $fi = hi_1$, there exist maps $\tilde{g} : X \to Y$ and $\tilde{h}\colon X \times I \to Z$ that make the following diagram commute:*

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd] & & A \times I \arrow[dl, "h"'] \arrow[dd] & & A \arrow[ll, "i_1"'] \arrow[dl, "g"'] \arrow[dd] \\
& Z & & Y \arrow[ll, "e"'] & \\
X \arrow[rr, "i_0"'] \arrow[ur, "f"] & & X \times I \arrow[ul, dashed, "\tilde{h}"] & & X \arrow[ll, "i_1"] \arrow[ul, dashed, "\tilde{g}"]
\end{tikzcd}$$

This is a massive elaboration of the idea of a Hurewicz cofibration. The best way to understand this is to prove it (though it's not an easy proof).

In the non-equivariant case, one reduces to working one cell at a time, inductively extending over the cells of X not in A.[7] In this case, look at $S^{n-1} \subseteq D^n$. Now you just do it: at this point, there's no way to avoid writing down explicit homotopies.

Exercise 2.10. Think about this argument, and then read the proof in [May99].

The equivariant case is very similar: in the same way, one can reduce to inductively attaching a single cell in the case where X is a finite CW complex. This comes via a map $G/H \times S^{n-1} \to G/H \times D^n$, but the only interesting content is in the nonequivariant part, so we can reduce again to $S^{n-1} \to D^n$ with trivial G-action! This allows us to finish the proof in the same way. It also says that the homotopy theory of G-spaces is lifted from ordinary homotopy theory, in a sense that model categories will allow us to make precise.

The first consequence of Theorem 2.9 is:

Theorem 2.11. *Let $e\colon Y \to Z$ be a θ-connected map and $e_*\colon [X,Y] \to [X,Z]$ be the map induced by composition.*

- *If X has dimension less than θ, e_* is a bijection.*[8]
- *If X has dimension θ, e_* is a surjection.*

The proof is an exercise; filling in the details is a great way to get your hands on what the HELP lemma is actually doing. Hint: consider the pairs $\varnothing \to X$ and $X \times S^0 \to X \times I$, and apply the HELP lemma.

Corollary 2.12 (Equivariant Whitehead theorem)**.** *Let $e\colon Y \to Z$ be a weak equivalence of G-CW complexes. Then, e is a G-homotopy equivalence.*

Proof. This is also a standard argument: using Theorem 2.11, e_* is a bijection, so we can pull back $\mathrm{id}_Z \in [Z,Z]$ to an inverse $(e_*)^{-1}(\mathrm{id}_Z) \in [Z,Y]$, which is a homotopy inverse to e. ⊠

One can continue and prove the cellular approximation theorem in this way, and so forth. We won't do this, because we'll approach it from a model-categorical perspective.

One thing that's useful, not so much for this class as for enriching your life, is to learn how to approach this from the perspective of abstract homotopy theory, learning about disc complexes and so forth. You can prove theorems such as the HELP lemma and its consequences in a general setting, and then specialize them to the cases you need. This is a great way to "just do it" without needing model categories.

Anyways, we'll now define a model structure on $G\mathsf{Top}$ and $G\mathsf{Top}_*$. If you don't know what a model category is, now is a good time to review.

Proposition 2.13. *There is a model structure on $G\mathsf{Top}$ (and on $G\mathsf{Top}_*$) defined by the following data.*

[7] This requires reducing to the case where X is a finite CW complex, but taking a sequential colimit recovers the theorem for all CW complexes X.

[8] We say that X has dimension less than θ if for all closed subgroups $H \subset G$, all cells of orbit type G/H have (nonequivariant) dimension at most n for some $n \le \theta(H)$.

Cofibrations: *The maps $f : X \to Y$ such that for all $H \subset G$, $f^H : X^H \to Y^H$ is a cofibration.*
Weak equivalences: *The maps $f : X \to Y$ such that for all $H \subset G$, $f^H : X^H \to Y^H$ is a weak equivalence.*

So we once again parametrize everything over subgroups of G and use fixed points. This is a cofibrantly generated model category; the cofibrations are specified by generators of acyclic cofibrations in a similar manner to Top. That is, in Top, one can choose generators $I = \{S^{n-1} \to D^n\}$ and $J = \{D^n \to D^n \times I\}$; in GTop, we instead take $I_G = \{G/H \times I\}$ and $J_G = \{G/H \times J\}$.

These are cells that we used to define G-CW complexes, and this is no coincidence: it's a general fact about cofibrantly generated model categories that follows from the small object argument[9] that cofibrant objects are retracts of "cell complexes" built from the things in I, and cofibrations are retracts of cellular inclusions of cell complexes. In this sense, CW complexes are inevitable.

The Whitehead theorem (Corollary 2.12) now falls out of the general theory of model categories.

Theorem 2.14 (Whitehead theorem for model categories)**.** *Let $f : X \to Y$ be a weak equivalence of cofibrant-fibrant objects in a model category. Then, f is a homotopy equivalence.*

In Top and GTop, all objects are fibrant, so this is particularly applicable.

The orbit category. We'll begin talking about the orbit category in the rest of today's lecture, and discuss the bar construction next class.

Definition 2.15. The **orbit category** $\mathscr{O}_G$ is the full subcategory of GTop on the objects G/H.

That is, its objects are the spaces G/H, where $H \subset G$ is closed, and its morphisms are $\mathrm{Map}^G(G/H, G/K) \cong (G/K)^H$. These maps are the same thing as subconjugacy relations, i.e. those of the form

$$gHg^{-1} \subseteq K, \tag{2.16}$$

since for all $h \in H, h(gK) = gK$ if and only if $K = g^{-1}hgK$ if and only if $gHg^{-1} \subseteq K$. A G-map $f : G/H \to G/K$ is completely specified by what it does to the identity coset $f(eH) = gK$, and this g implies the subconjugacy relation (2.16), since, as above, $h(gK) = gK$ for all $h \in H$.

There's another description of the orbit category.

Proposition 2.17. *Let G be a finite group. Then, the orbit category $\mathscr{O}_G$ is equivalent to the category of finite transitive G-sets and G-maps.*

The observation that ignites the proof is that if $x \in X$ has isotropy group H, then its orbit space is isomorphic to G/H.

Definition 2.18. Given a G-space X, we obtain a presheaf on the orbit category, namely a functor $X^{(-)} : \mathscr{O}_G^{\mathrm{op}} \to \mathsf{Top}$, by sending $G/H \to X^H$. This assignment itself is a functor $\psi : G\mathsf{Top} \to \mathsf{Fun}(\mathscr{O}_G^{\mathrm{op}}, \mathsf{Top})$.

Proposition 2.19. $\mathsf{Fun}(\mathscr{O}_G^{\mathrm{op}}, \mathsf{Top})$ *has a projective model structure where the weak equivalences and fibrations are taken pointwise.*

The point is the following result, a revisionist interpretation of Elmendorf's theorem.[10]

Theorem 2.20. *ψ is the right adjoint in a Quillen equivalence; the left adjoint θ is evaluation at G/e.*

The point is, these two model categories have the same homotopy theory.

Exercise 2.21. Check that evaluation at G/e is a left adjoint to ψ.

[9]The small object argument is a beautiful piece of basic mathematics that everybody should know. If you don't know it, your homework is to read enough about model categories to get to that point. In general, there may be large objects and transfinite induction, but for the case we care about large cardinals won't arise.

[10]Elmendorf proved that these two categories have the same homotopy theory, but his proof was more explicit.

Lecture 3.

Elmendorf's theorem: 1/24/17

"What's bad about this proof?"
"It appeals to machinery we didn't develop in this class?"
"No, that's perfectly fine."

We'll start by reviewing the connection between the orbit category and G-sets.

Let X be a finite G-set. Then, X is the coproduct (disjoint union) of a bunch of orbits:

$$X \cong \coprod_i G/H_i.$$

The way you see this is that for any $x \in X$, its orbit is isomorphic to G/G_x. This is yet another manifestation of the slogan that "orbits are points." But it also implies that, rather than just presheaves on $\mathscr{O}_G$, one could work with certain presheaves on the category of finite G-sets, and this perspective will turn out to be useful. By "certain" we mean a compatibility with orbits.

Last time, we talked about Elmendorf's theorem in the form of Theorem 2.20. It's also possible to state it in a more general form.

Theorem 3.1 (Elmendorf). *The functor $G\mathsf{Top} \to \mathsf{Fun}(\mathscr{O}_G^{\mathrm{op}}, \mathsf{Top})$ determined by $X \mapsto \left(G/H \mapsto X^H\right)$ induces an equivalence of $(\infty, 1)$-categories, where the weak equivalences on the left and right are specified by a family $\mathscr{F}$.*

Without delving into $(\infty, 1)$-categories, this means

- the homotopy categories are equivalent, and
- homotopy limits and colimits behave identically.

In other words, from the perspective of abstract homotopy theory, these are the same.

Definition 3.2. By a **family** of subgroups $\mathscr{F}$ of G, we mean a collection of subgroups of G closed under conjugation and taking subgroups.

Examples include the set of all subgroups, the set of just the identity, and the set of finite subgroups. The latter is useful for some S^1-equivariant spaces, where one tends to lose control of the S^1-fixed points, but the finite subgroups behave better.

Definition 3.3. Let $\mathscr{F}$ be a specified family of subgroups of G.

- In $G\mathsf{Top}$, the weak equivalences specified by $\mathscr{F}$ are the maps $f : X \to Y$ such that $f^H : X^H \to Y^H$ is a weak equivalence for all $H \in \mathscr{F}$.
- For $\mathsf{Fun}(\mathscr{O}_G^{\mathrm{op}}, \mathsf{Top})$, a weak equivalence specified by $\mathscr{F}$ is a pointwise weak equivalence at G/H for all $H \in \mathscr{F}$.

We'll give two proofs of Theorem 3.1. The first will be model-categorical.

Recall[11] if $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ is a Quillen adjunction, then the left and right derived functors $(\mathbf{L}F, \mathbf{R}G)$ is an adjunction on the homotopy categories $(\operatorname{Ho}\mathsf{C}, \operatorname{Ho}\mathsf{D})$. If K denotes fibrant replacement in D and Q denotes cofibrant replacement in C, then the derived functors are $\mathbf{L}F = FQ$ and $\mathbf{R}G = GK$.[12]

Definition 3.4. That (F, G) is a **Quillen equivalence** means that for any cofibrant $X \in \mathsf{C}$ and fibrant $Y \in \mathsf{D}$, then $FX \to Y$ is a weak equivalence iff its adjoint $X \to GY$ is.

This is equivalent to asking that $(\mathbf{L}F, \mathbf{R}G)$ are equivalences of categories.

This is a kind of curious way to look at an equivalence of categories. One says that $G \colon \mathsf{D} \to \mathsf{C}$ **creates the weak equivalences** of D if for every morphism f of D, f is a weak equivalence iff Gf is.

Lemma 3.5. *If G creates the weak equivalences of D and for all cofibrant X the unit map $X \to GFX$ is a weak equivalence, then (F, G) is a Quillen equivalence.*

---

[11] If this is not review to you, then exercise: learn this material!

[12] This does require cofibrant and fibrant replacement to be functorial, which is not true in every model category, but will be true for pretty much everything we study.

This is a useful tool for extending model categories along free-forgetful adjunctions; for example, if you have a model category and want to understand abelian group or ring objects in this category, often their weak equivalences are detected by the forgetful functor.

Proof sketch of Theorem 3.1. We want to apply Lemma 3.5 to the adjunction

$$\theta : \mathsf{Fun}(\mathscr{O}_G^{\mathsf{op}}, \mathsf{Top}) \rightleftarrows G\mathsf{Top} : \psi,$$

where $\theta\colon X \mapsto X(G/e)$ is evaluation at G/e and $\psi\colon Y \mapsto \{Y^H\}$. The first condition, that ψ detects the weak equivalences, is straightforward, so we need to check that $X \mapsto \{X(G/e)^H\}$ is a weak equivalence for all cofibrant X.

Cellular objects model the generating cofibrations, so cofibrant objects are retracts of cellular objects. Since weak equivalences are preserved under retracts, then we can check on cellular objects. Here it's easier, since $(-)^H$ commutes with the relevant colimits and is suitably cellular. ⊠

The missing steps in this proofs can be filled in by explicitly identifying the cofibrant objects in $\mathsf{Fun}(\mathscr{O}_G^{\mathsf{op}}, \mathsf{Top})$. These are free diagrams on the orbit category; not hard to write down, but messy enough to avoid on the chalkboard.

Remark. Elmendorf's original proof of his theorem was in the 1980s did not use model categories, even though Quillen had already introduced them at the time. Until the mid-1990s (30 years after Quillen introduced them), many homotopy theorists avoided them, thinking of them as formal gobbledygook. However, about the time EKMM introduced a symmetric monoidal category of spectra, people began realizing they were unavoidable. ◀

You might not like the given proof of Elmendorf's theorem because it's extremely inexplicit: cofibrant replacement is an infinite process, and many of the steps involved are quite abstract. The next proof will be more explicit, building a (homotopical) right adjoint to ψ.

This proof will go through the **bar construction**, a categorical tool that's extremely useful. References for it include May's "Geometry of iterated loop spaces" [May72], Riehl's monograph [Rie14], and Vogt's "Tensor products of functors."

Second proof of Theorem 3.1. Let $M\colon \mathscr{O}_G \to \mathsf{Top}$ realize orbits as spaces: G/H is sent to the topological space G/H, and an equivariant map f is forgotten to a continuous map f.

Given an $X \in \mathsf{Fun}(\mathscr{O}_G^{\mathsf{op}}, \mathsf{Top})$, let

$$\Phi(X) := |B_\bullet(X, \mathscr{O}_G, M)|$$

denote the geometric realization of the simplicial bar construction. Let's be a little more explicit about this. $B_\bullet(X, \mathscr{O}_G, M)$ is a simplicial space that sends

$$[n] \longmapsto \coprod_{G/H_{n-1}\to\cdots\to G/H_0} X(G/H_0) \times M(G/H_{n-1}).$$

As usual, the face maps are defined by composition, and the degeneracies by inserting the identity map. Since G acts on $M(-)$ simplicially (i.e., in a way compatible with the face and degeneracy maps), then $|B_\bullet(X, \mathscr{O}_G, M)|$ is a G-space (passing through the coend formula for the geometric realization).

If $H \subseteq G$, we want to understand $\Phi(X)^H$. Because the G-action passed through geometric realization,

$$\Phi(X)^H \cong |B_\bullet(X, \mathscr{O}_G, M^H)| \cong |B_\bullet(X, \mathscr{O}_G, \mathrm{Map}_{\mathscr{O}_G}(G/H, -))|.$$

Let $X(G/H)$ denote the constant simplicial space $[n] \mapsto X(G/H)$. Then, by general theory of the bar construction for any corepresented functor, there's a simplicial map

$$B_\bullet(X, \mathscr{O}_G, \mathrm{Map}_{\mathscr{O}_G}(G/H, -)) \longrightarrow X(G/H) \tag{3.6}$$

defined by composing and applying X, and this is a simplicial homotopy equivalence (you can write down a retraction).[13] Thus, $\Phi(X)^H \cong X(G/H)$. In other words, Φ is a homotopy inverse, since taking H-fixed points of $\Phi(X)$ gives back what you started with. ⊠

$\Phi(X)$ is still an infinite-dimensional object, but it's much more explicit, and you can work with it.

[13] This is called an **extra degeneracy argument** in the literature. There's an observation probably due to John Moore which approximately says that if you have a simplicial object with an extra degeneracy condition playing well with the preexisting ones, then it must be contractible; this argument is applied to the fiber of (3.6).

Applications of this perspective. We'll be able to use Elmendorf's theorem to make some constructions that would be hard to imagine without the orbit category.

Definition 3.7. Let $\mathscr{F}$ be a family of subgroups of G. Then, the **classifying space** for $\mathscr{F}$ is specified by the universal property that if Z has $\mathscr{F}$-isotropy, then $[Z, E\mathscr{F}]$ has a unique element. An explicit construction is to let $\widetilde{E}\mathscr{F}$ denote the presheaf on the orbit category where

$$\widetilde{E}\mathscr{F}(G/H) := \begin{cases} *, & H \in \mathscr{F} \\ \varnothing, & H \notin \mathscr{F}, \end{cases}$$

and let $E\mathscr{F} := \Phi(\widetilde{E}\mathscr{F})$.

If you unwind the definition, this is the bar construction applied to G in the category of G-spaces with weak equivalences given by $\mathscr{F}$, meaning it deserves to be called a classifying space.

Another useful notion is the G-connected components.

Definition 3.8. Let X be a G-space and $x \in X^G$. Let Y_x be the presheaf on the orbit category sending H to the connected component containing $x \in X^H$. Then, the **G-connected component** of x is $\Phi(Y_x)$.

The third useful application is defining Eilenberg-Mac Lane spaces. This will lead us to cohomology (and then to Smith theory and other things). These will be constructed by working pointwise, then applying Φ.

Definition 3.9. Let G be a finite group, A **coefficient system** is a presheaf $X \in \mathsf{Fun}(\mathscr{O}_G^{\mathsf{op}}, \mathsf{Ab})$.[14]

Elmendorf's theorem says that for any coefficient system, we have an Eilenberg-Mac Lane G-space. You could say here that (Bredon) cohomology is completely determined: cohomology is the things represented by Eilenberg-Mac Lane spaces. But it will be good to see it explicitly. Bredon cohomology is explicit, but there are serious drawbacks: it has poor formal properties, and you need a lot of geometric insight to compute things. We'll later see that this abelian category (meaning we can do homological algebra) is the wrong one; we'll later see this is a $\mathbb{Z}$-graded cohomology theory (or rather graded on subgroups of $\mathbb{Z}$); this will be the wrong answer, especially if you want Poincaré duality, and the right answer uses a grading by the representation ring. But we'll get there.

Remark. Another application of Elmendorf's theorem, which we will not discuss in detail (unless we get to the slice filtration), is Postnikov towers. They're constructed in the same way, by either using the small object argument or killing homotopy groups. ◂

Here are some examples of coefficient systems (which are often denoted with underlines).

Example 3.10.

(1) For a G-space X, the coefficient system $\underline{\pi}_n(X)$ $(n \geq 2)$ sends $G/H \mapsto \{\pi_n X^H\}$. This is an example of a general formula: given a functor $\mathsf{Top} \to \mathsf{Ab}$, we can compose to obtain a map $\mathsf{Fun}(\mathscr{O}_G^{\mathsf{op}}, \mathsf{Top}) \to \mathsf{Fun}(\mathscr{O}_G^{\mathsf{op}}, \mathsf{Ab})$.

(2) In the same way, $\underline{H}_n(X)$ sends $G/H \mapsto \{H_n(X^H; \mathbb{Z})\}$. ◂

We will now define Bredon cohomology, which is what you think it is.

Definition 3.11. Let X be a G-CW complex and X_n denote its n-skeleton. Let

$$\underline{C}_n(X) := \underline{H}_n(X_n, X_{n-1}; \mathbb{Z}),$$

i.e. this coefficient system sends $G/H \mapsto H_n((X^H)_n, (X^H)_{n-1}; \mathbb{Z})$. The connecting homomorphism comes as usual from the triple $((X^H)_n, (X^H)_{n-1}, (X^H)_{n-2})$.[15]

The **Bredon cohomology** for the coefficient system M is

$$H_G^n(X; M) := H^n\Big(\mathrm{Hom}_{\mathsf{Fun}(\mathscr{O}_G^{\mathsf{op}}, \mathsf{Ab})}(\underline{C}_\bullet(X), M)\Big).$$

That is, it's the cohomology of the cochain complex of natural transformations from $\underline{C}_\bullet(X)$ to M.

[14] For G a compact Lie group, the definition is almost the same, but we need to use $\mathsf{Fun}(h\mathscr{O}_G^{\mathsf{op}}, \mathsf{Ab})$, taking presheaves on the homotopy category. For finite groups these definitions coincide.

[15] This requires knowing how to obtain a CW structure on X^H given a G-CW structure on X. If G is finite, this is easy to see; for general compact Lie groups, though, this requires a triangulation argument. One wants the resulting coefficient system to be independent of the choice of triangulation, but as in the nonequivariant case, this is proven via an axiomatic characterization of cohomology.

Example 3.12. Give S^2 the C_2-action that rotates by π, and call it Z, and let M denote the constant coefficient system at $\mathbb{Z}$. First, we compute the cells:

- $\underline{C}_2(Z)(C_2/e) = \mathbb{Z} \oplus \mathbb{Z}$ and $\underline{C}_2(Z)(C_2/C_2) = 0$.
- $\underline{C}_1(Z)(C_2/e) = \mathbb{Z} \oplus \mathbb{Z}$ and $\underline{C}_1(Z)(C_2/C_2) = 0$.
- $\underline{C}_0(Z)(C_2/e) = \mathbb{Z} \oplus \mathbb{Z}$ and $\underline{C}_0(Z)(C_2/C_2) = \mathbb{Z} \oplus \mathbb{Z}$. ◀

Exercise 3.13. Figure out the differentials in the above example.

We'll continue to discuss Bredon cohomology next lecture, and introduce an axiomatic viewpoint. This is basic and fundamental, but not too relevant to the rest of the class. It's also good to calculate; there are surprisingly few examples out in the world.

Lecture 4.

Bredon cohomology: 1/26/17

"Smith's theorem was proven by Smith, hence the name."

Today I was running about two minutes late. Sorry, everybody! (What I missed: Andrew discussed holding a "Q and A" session next week on a weeknight, perhaps with pizza. If you're interested in this, email him to let him know which nights work for you.)

Today we're going to talk more about Bredon cohomology, including an analogue of the Eilenberg-Steenrod axioms for it. Then, we'll turn to the circle of ideas around Smith theory, including the Sullivan conjecture (which we won't prove, because it's hard). Smith theory discusses when one can recover $H_*(X^G)$ from a G-action on X and on $H_*(X)$.

Recall that if X is a G-CW complex, we defined Bredon cohomology as follows: we set up a chain complex of coefficient systems (i.e. functors $\mathscr{O}_G^{\mathsf{op}} \to \mathsf{Ab}$) $\underline{C}_*(X)$, where

$$\underline{C}_n(G/H) := H^n((X^n)^H, (X^{n-1})^H; \mathbb{Z}).$$

The differential $\partial : \underline{C}_n(X) \to \underline{C}_{n-1}(X)$ is induced by the connecting morphism in the long exact sequence for the triple (X^n, X^{n-1}, X^{n-2}); that $\partial^2 = 0$ is something you have to check, though it's not very difficult.

With this you can define two things:

- The Bredon cohomology with coefficients in a coefficient system M is

$$H_G^n(X; M) := H^n(\mathrm{Hom}_{\mathsf{Fun}(\mathscr{O}_G^{\mathsf{op}}, \mathsf{Ab})}(\underline{C}_*(X), M)).$$

  We defined this last time.
- For homology to be covariant, we need the coefficient system M to be a functor $\mathscr{O}_G \to \mathsf{Ab}$ rather than $\mathscr{O}_G^{\mathsf{op}} \to \mathsf{Ab}$ (e.g. $\underline{H}^*(X)$ for a G-space X, which sends $G/H \mapsto \{H^n(X^H)\}$). With M such a coefficient system, the **Bredon homology** with coefficients in M is

$$H_n^G(X; M) := H_n(\underline{C}_n(X) \otimes_{\mathscr{O}_G} M).$$

  By this tensor product, we mean a coend:

$$\begin{aligned} \underline{C}_n(X) \otimes_{\mathscr{O}_G} M &= \int^{G/H \in \mathscr{O}_G} \underline{C}_n(X)(G/H) \otimes M(G/H) \\ &= \coprod_{G/H} \underline{C}_n(X)(G/H) \otimes M(G/H) \,/ \sim, \end{aligned}$$

  where if $f \in \mathrm{Map}_{\mathscr{O}_G}(G/H, G/K)$, $(f^* y, z) \sim (y, f_* z)$.[16]

The whole philosophy of Bredon (co)homology is that you understand the cohomology or homology through the fixed-point sets and the lattice of subgroups of G.

[16] Tensor products are particular instances of coends; instead of inducing an equivalence $mr \otimes n \sim m \otimes rn$, you flip a map across the two objects. One might write $y \cdot f$ for $f^* y$ and $f \cdot z$ for $f_* z$ to emphasize this point of view.

Example 4.1. Let's return to Example 3.12. We had a C_2-action on S^2. There were two 0-cells, one 1-cell $C_2 \times I$, and one 2-cell $C_2 \times D^2$, and the chain complexes $\underline{C}_n(C_2/H)$ as we wrote down last lecture. We aim to compute the Bredon cohomology for the constant coefficient system $M = \underline{\mathbb{Z}}$ (so $M(C_2/H) = \mathbb{Z}$ for all H).

The orbit category for C_2 is particularly simple:

$$\begin{tikzcd} C_2/e \arrow[loop above] \arrow[d] \\ C_2/C_2. \end{tikzcd}$$

So a coefficient system is determined by a map $M_1 \to M_2$ and an involution on M_2. In our case, $\underline{C}_k(X)(C_2/e) = \mathbb{Z}\oplus\mathbb{Z}$ for $k = 0, 1, 2$, and the involution flips the two factors. Therefore we obtain a chain complex of coefficient systems which is

$$0 \longrightarrow \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{f_1} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{f_2} \mathbb{Z}\oplus\mathbb{Z} \longrightarrow 0$$

at C_2/e and

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}\oplus\mathbb{Z} \longrightarrow 0$$

at C_2/C_2. Next we need to determine the differentials. By definition, these are determined by the cellular boundary maps, i.e. the attaching maps. Let (x, y) denote the standard basis of $\mathbb{Z}\oplus\mathbb{Z}$.

- f_1 sends $(x, y) \mapsto (x + y, x + y)$.
- f_2 sends $(x, y) \mapsto (x - y, y - x)$,

Next, we compute the Hom from this coefficient system to the constant coefficient system $\mathbb{Z}$. These groups are evidently determined by what happens at the C_2/e spot. For the first two terms, we are looking at maps $\mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}$ that are equivariant (where $g \in C_2$ acts by transposition on the left and trivially on the right). Such a map is determined by where it sends $(1, 0)$, and so these two terms are isomorphic to $\mathbb{Z}$. The last term is simply all homomorphisms $\mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z}$, which is isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$. Therefore, we have the cochain complex

$$0 \longrightarrow \mathbb{Z}\oplus\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Computing the differentials, we find that the first map is $(a, b) \mapsto a + b$ and the second map is 0.

Taking the cohomology of this complex then produces

$$H^n(S^V; \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z}, & n = 0, 2 \\ 0, & \text{otherwise.} \end{cases}$$

◀

Remark. We're used to reading off properties of a space from its cohomology, and this is still true here, if harder. For example, H^0 tells us the number of connected components of S^2/C_2. ◀

We want cohomology to have nice formal properties analogous to the Eilenberg-Steenrod axioms. We'll think of H_G as a general G-equivariant cohomology theory of pairs $H^n_G(X, A; M)$, but in our case this will just be $H^n_G(X/A; M)$.

(1) H^*_G should be invariant under weak equivalences: a weak equivalence $X \to Y$ should induce an isomorphism $H^n_G(Y; M) \xrightarrow{\cong} H^n_G(X; M)$.
(2) Given a pair $A \subseteq X$, we get a long exact sequence

$$\cdots \longrightarrow H^n_G(X, A; M) \longrightarrow H^n_G(X; M) \longrightarrow H^n_G(A; M) \xrightarrow{\delta} H^{n+1}(X, A; M) \longrightarrow \cdots$$

(3) The excision axiom: if $X = A \cup B$, then

$$H^n_G(X/A; M) \cong H^n_G(B/(A \cap B); M).$$

(4) The Milnor axiom:

$$H^n_G\left(\bigvee_i X_i; M\right) = \prod_i H^n_G(X_i; M).$$

(5) Finally, for now we impose the dimension axiom: our points are orbits G/J, so we ask that $H^n(G/H; M)$ is concentrated in degree 0.

Some of these are easier than others: Bredon cohomology is manifestly homotopy-invariant in the same ways as ordinary cohomology, so invariance under weak equivalence and the Milnor axiom are immediate, and excision follows because if all spaces involved are CW complexes, $X/A \cong B/(A\cap B)$.

What takes more work is the dimension axiom and the long exact sequence. We'll show that $\underline{C}_n(X)$ is a projective object, and hitting projective objects with Hom produces a long exact sequence by homological algebra. (Recall that an object P in a category where you can do homological algebra is **projective** if $\mathrm{Hom}(P,-)$ is exact, which is equivalent to maps to P lifting across surjections $M \twoheadrightarrow P$.)

Proof of the dimension and long exact sequence axioms. Observe that $\underline{C}_n(X)$ splits as a direct sum of pieces $H_n(G/H_+ \wedge S^n) \cong \widetilde{H}_0(G/H)$ indexed by the cells G/H of X. At G/K, $H_0(G/H) = \mathbb{Z}[\pi_0(G/H)^K]$. This is a free abelian group, and we'll directly use the lifting criterion to prove this is projective. That is, we'll write down an isomorphism

$$\varphi\colon \mathrm{Hom}_{\mathsf{Fun}(\mathscr{O}_G^{\mathrm{op}},\mathsf{Ab})}(\underline{H}_0(G/H), M) \xrightarrow{\cong} M(G/H).$$

This immediately proves $\underline{H}_0(G/H)$ is projective: evaluating a coefficient system on an exact sequence produces an exact sequence, and we've shown $\mathrm{Hom}(\underline{H}_0(G/H),-)$ is evaluation of a coefficient system.

The map φ takes a homomorphism θ and applies it to $\mathrm{id}_{G/H}$, which produces something in $M(G/H)$. Why is this an isomorphism? The Yoneda lemma is a fancy answer, but you can prove it in a more elementary manner.

Exercise 4.2. Calculate that any $\theta \in \mathrm{Hom}_{\mathsf{Fun}(\mathscr{O}_G^{\mathrm{op}},\mathsf{Ab})}(H_0(G/H), M)$ is determined by where the identity $\mathrm{id} \in \mathrm{Map}_{\mathscr{O}_G}(G/H, G/H)$ is sent, implying φ is an isomorphism.

Thus, we've effectively calculated the value at G/H, proving the dimension axiom as well. ⊠

The Eilenberg-Steenrod axioms hold for Bredon homology, and the proof is the same.

Remark. Let's foreshadow a little bit. In ordinary homotopy theory, one can show that the Eilenberg-Steenroad axioms plus the value on points determine a cohomology theory, and this is still true in the equivariant case. But then one wonders about Brown representability and what happens when you remove the dimension axiom — and indeed there are lots of interesting examples of generalized equivariant cohomology theories. ◀

Exercise 4.3. We constructed a functor $\underline{H}\colon \mathsf{Fun}(\mathscr{O}_G^{\mathrm{op}}, \mathsf{Top}) \to \mathsf{Fun}(\mathscr{O}_G^{\mathrm{op}}, \mathsf{Ab})$. Show that $\underline{H}M$ represents $H_G^n(-;M)$.

This slick definition of $\underline{H}$ is one of the advantages of working with presheaves on the orbit category.

Remark. You might also want to have a universal coefficient sequence, but it's more complicated. The short exact sequence in ordinary homotopy theory depends on the existence of short projective resolutions. Here, we have enough projectives and injectives, but resolutions are longer. Thus, taking an injective resolution of M and filtering the resulting double complex, one obtains a spectral sequence

$$\mathrm{Ext}^{p,q}(\underline{C}_*(X), M) \Longrightarrow H_G^{p+q}(X;M).$$

Warning: indexing might be slighly off.

There's a corresponding Tor spectral sequence. ◀

Since the category is more complicated, one expects to have to do more work. But sometimes there are nice results nonetheless; Smith theory is an example. This theorem is very old, from the 1940s, so none of the cohomology in the statement is equivariant.

Theorem 4.4 (Smith)**.** *Let G be a finite p-group and X be a finite G-CW complex such that (the underlying topological space of) X is an $\mathbb{F}_p$-cohomology sphere.*[17] *Then, X^G is either empty or an $\mathbb{F}_p$-cohomology sphere of smaller dimension.*

There are sharper statements, but we can prove this one. It's the start of a long program to understand $H_*(X^G)$ using algebraic data calculated from X and the action of G on X. One useful tool in this is the **Borel construction** (well, *a* Borel construction) $EG \times_G X$ (where this is the usual balanced product, not a pullback). This is a "fattened up" version of X/G.

Definition 4.5. The **Borel cohomology** of X is $H_G^*(X) := H^*(EG \times_G X)$.

[17] A space X is an $\mathbb{F}_p$**-cohomology sphere** if there is an isomorphism of graded abelian groups $H^*(X;\mathbb{F}_p) \cong H^*(S^n;\mathbb{F}_p)$ for some n.

Warning! This notation is potentially confusing. Outside of the field of equivariant homotopy theory, "equivariant cohomology" generally means Borel cohomology, not Bredon cohomology. In equivariant homotopy theory, H_G^* can be used to denote both. We will always make the coefficient system for Bredon cohomology explicit, so that $H_G^*(X)$ unambiguously refers to Borel cohomology, as is standard in the literature.[18]

The finiteness in Theorem 4.4 is key: Elmendorf's theorem lets us build a C_p-complex with non-equivariant homotopy type S^n and any set of fixed points. Thus, in the infinite-dimensional case, we should be looking at a different thing than the fixed points, namely the homotopy fixed points.

Definition 4.6. The **homotopy fixed points** of a G-space X is $X^{hG} := \mathrm{Map}(EG, X)^G$.

This can also be defined as a homotopy limit. This is "easy" to compute (relative to the rest of equivariant homotopy theory, that is), as there are spectral sequences, nice models for EG, and other tools.

The terminal map $EG \to *$ induces a map $X^G \to X^{hG}$. The Sullivan conjecture is all about when this happens: when you p-complete, things become really nice. (There will be a precise statement next lecture.)

Returning to Theorem 4.4, we can use Bredon cohomology to give an easy, modern proof. The idea is to adroitly choose coefficient systems such that we recover $H^n(X)$, $H^n(X^G)$, and $H^n((X/G)/G)$ from Bredon cohomology. They'll fit into exact sequences, and using tools like Mayer-Vietoris, we'll get inequalities on the ranks of groups. This will also use the fact that a short exact sequence of coefficient systems induces a long exact sequence on H_G^*. The proof will be beautiful and short, unlike Smith's original proof!

Theorem 4.4 is sufficiently classical that there are several different proofs. Ours illuminates Bredon cohomology at the expense of obscuring the overarching goal of Smith theory. There's another proof by Dwyer and Wilkerson in "Smith theory revisited" [DW88], a short, beautiful paper which is highly recommended. It uses the unstable Steenrod algebra to prove Theorem 4.4 and more.

Over the next day or so, there will be a problem set and a course webpage. Doing the problem sets is recommended!

Lecture 5.

Smith theory: 1/31/17

"It seems that the people registered for the class and the people showing up for class are disjoint."

Today, we will give one or two proofs of Theorem 4.4: let G be a finite p-group, X be a finite-dimensional G-CW complex such that as a topological space, X is an $\mathbb{F}_p$-cohomology sphere (i.e. as graded abelian groups, $H^*(X;\mathbb{F}_p) \cong H^*(S^n;\mathbb{F}_p)$). Then, either $X^G = \varnothing$, or $H^*(X^G;\mathbb{F}_p) \cong H^*(S^m;\mathbb{F}_p)$ as graded abelian groups, for some $m \le \dim X$. We'll try to be consistent with the notation in [May87, May96].

Proof of Theorem 4.4. First, we can quickly reduce to the case where $G = \mathbb{Z}/p$: if $H \subset G$ is a normal subgroup, then $X^G \cong (X^H)^{G/H}$, so you can induct on the order of the group using the Sylow theorems. Thus, we will assume $G = \mathbb{Z}/p$, whose orbit category is simple:

$$\begin{tikzcd} \bullet \arrow[loop above, "g"] \arrow[d] \\ \bullet \end{tikzcd}$$

There is a cofiber sequence

$$X_+^G \longrightarrow X_+ \longrightarrow X/X^G,$$

which is not particularly deep. We're going to construct three special coefficient systems L, M, and N such that

$$\begin{aligned} H_G^*(X;L) &\cong \widetilde{H}^*((X/X^G)/G;\mathbb{F}_p) \\ H_G^*(X;M) &\cong H^*(X;\mathbb{F}_p) \\ H_G^*(X;N) &\cong H^*(X^G;\mathbb{F}_p). \end{aligned}$$

[18] The nLab uses a different convention: Borel cohomology is $H_G^*(X;A)$, and Bredon cohomology is $\mathbf{H}^*(X_G)$.

These will fit into exact sequences which will imply the inequalities we wanted.[19]

How to we construct custom coefficient systems? Since these constructions commute with colimits, it suffices to determine them via computation at $n=0$ and $X=G/H$ for subgroups $H\subset G$, meaning just e and G.

For L, we want to recover $\widetilde{H}^0((X/X^G)/G;\mathbb{F}_p)$ for $X=G/e$ and $X=G/G$.

- For $X=G/e$, we want $\widetilde{H}^0((G/\varnothing)/G;\mathbb{F}_p)\cong\widetilde{H}^0(G_+/G)\cong\mathbb{F}_p$.
- For $X=G/G$, we get $\widetilde{H}^0((*/*)/G;\mathbb{F}_p)=0$.

So we conclude $L(G/e)=\mathbb{F}_p$ and $L(G/G)=0$.

For M, a similar calculation shows we need $H^0(G/e)\cong\mathbb{F}_p[G]$[20] and $H^0(G/G)=\mathbb{F}_p$, and for N, we need $N(G/e)=0$ and $N(G/G)\cong\mathbb{F}_p$.

Remark. Almost everything in this proof generalizes; we will only need X to be an $\mathbb{F}_p$-cohomology sphere in order to know dimensions of a few things. But this technique of customized coefficient systems can be used elsewhere. ◀

Let I denote the **augmentation ideal** of $\mathbb{F}_p[G]$, i.e. the kernel of the map $\mathbb{F}_p[G]\to\mathbb{F}_p$ sending all $g\mapsto 1$. We will let I^n refer to the coefficient system which assigns I^n to G/e and 0 to G/G.

As coefficient systems, $M/I\cong\mathbb{F}_p$, and therefore there is a short exact sequence of coefficient systems

$$0\longrightarrow I\longrightarrow M\longrightarrow N\oplus L\longrightarrow 0.$$

This implies a long exact sequence in Bredon cohomology:

$$\cdots\longrightarrow H^q_G(X;I)\longrightarrow H^q_G(X;M)\longrightarrow H^q_G(X;N\oplus L)\longrightarrow H^{q+1}_G(X;I)\longrightarrow\cdots$$

Exactness at $H^q(X;N\oplus L)$ implies

$$\operatorname{rank} H^q_G(X;N)+\operatorname{rank} H^q_G(X;L)\le\operatorname{rank} H^q_G(X;M)+\operatorname{rank} H^{q+1}_G(X;I).$$

That is,

$$\operatorname{rank} H^q(X^G;\mathbb{F}_p)+\operatorname{rank}\widetilde{H}^q((X/X^G)/G;\mathbb{F}_p)\le\operatorname{rank} H^q(X;\mathbb{F}_p)+\operatorname{rank} H^{q+1}_G(X;I).\tag{5.1}$$

We'll use this to strongly constrain rank $H^q(X^G;\mathbb{F}_p)$, but first we need another inequality coming from another exact sequence. Namely, the following sequence of coefficient systems is exact:

$$0\longrightarrow L\longrightarrow M\longrightarrow I\oplus N\longrightarrow 0.$$

This is because $I^p=0$ and for $0\le n\le p-1$, $I^n/I^{n+1}\cong\mathbb{F}_p$. In particular, $I^{p-1}\cong\mathbb{F}_p\cong L$, so we can think of M/L as M/I^{p-1}. Now we play the same game: the induced long exact sequence is

$$\cdots\longrightarrow H^q_G(X;L)\longrightarrow H^q_G(X;M)\longrightarrow H^q_G(X;I\oplus N)\longrightarrow H^{q+1}_G(X;L)\longrightarrow\cdots$$

which implies

$$\operatorname{rank} H^q_G(X;N)+\operatorname{rank} H^q_G(X;I)\le\operatorname{rank} H^q_G(X;M)+\operatorname{rank} H^{q+1}_G(X;L),$$

i.e.

$$\operatorname{rank} H^q(X^G;\mathbb{F}_p)+\operatorname{rank} H^q_G(X;I)\le\operatorname{rank} H^q(X;\mathbb{F}_p)+\operatorname{rank} H^{q+1}((X/X^G)/G;\mathbb{F}_p).\tag{5.2}$$

Let's use this to prove

$$\operatorname{rank}\widetilde{H}^q((X/X^G)/G;\mathbb{F}_p)+\sum_{i=q}^{q+r}\operatorname{rank} H^i(X^G;\mathbb{F}_p)\le\operatorname{rank}\widetilde{H}^{q+r+1}+\sum_{i=q}^{q+r}\operatorname{rank} H^i(X;\mathbb{F}_p).\tag{5.3}$$

[19] TODO: M is H^0 applied to some spaces and map. What are they?

[20] Here, $\mathbb{F}_p[G]$ is the **group ring**, the $\mathbb{F}_p$-algebra of functions $G\to\mathbb{F}_p$ with addition taken pointwise and multiplication determined by requiring $\delta_g\cdot\delta_h=\delta_{gh}$, where δ_g is a delta function, equal to 1 at g and 0 elsewhere.

Let

$$\begin{aligned} a_q &:= \operatorname{rank} H^q(X^G;\mathbb{F}_p)\\ b_q &:= \operatorname{rank} H^q(X;\mathbb{F}_p)\\ c_q &:= \operatorname{rank} H^q((X/X^G)/G;\mathbb{F}_p)\\ d_q &:= \operatorname{rank} H^q_G(X;I). \end{aligned}$$

Then, (5.1) and (5.2) say

$$a_q + c_q \le b_q + d_{q+1} \qquad \text{and} \qquad a_q + d_q \le b_q + c_{q+1}.$$

Now, adding (5.1) for q even and (5.2) for q odd proves (5.3).

When $q = 0$ and r is large, the finite-dimensionality of X implies that

$$\sum_i \operatorname{rank} H^i(X^G;\mathbb{F}_p) \le \sum_i \operatorname{rank} H^i(X;\mathbb{F}_p). \tag{5.4}$$

This is already an interesting bound, especially relative to the amount of work we've put in.

Specializing to X an $\mathbb{F}_p$-cohomology sphere, (5.4) means

$$\sum_i \operatorname{rank} H^i(X^G;\mathbb{F}_p) \le 2.$$

We want to show this sum isn't 1 (so that we get the cohomology of a sphere) and that the top nonzero rank is at most n. We will do this with another short exact sequence of coefficient systems:

$$0 \longrightarrow I^{n+1} \longrightarrow I^n \longrightarrow L \longrightarrow 0.$$

From this, we get another long exact sequence. Applying the Euler characteristic, we obtain that

$$\chi(X) = \chi(X^G) + p\widetilde{\chi}((X/X^G)/G). \tag{5.5}$$

Here

$$\widetilde{\chi}(Y) := \sum_i (-1)^i \operatorname{rank} \widetilde{H}^i(Y)$$

is the **reduced Euler characteristic**.

Equation 5.5 already implies that $\chi(X) \equiv \chi(X^G) \bmod p$, so $\sum \operatorname{rank} H^*(X^G;\mathbb{F}_p) \ne 1$ in our case.

Exercise 5.6.

(1) Think about choices of q and r that allow you to deduce $m \le n$, finishing the proof.

(2) Small changes need to be made to this argument when $p = 2$; what are they? ⊠

This is an appealing proof: some fairly simple calculations and a dash of formal theory very effectively led to the result. We'll give another proof with different advantages and disadvantages.

Smith theory naturally leads to questions about how to recover $H^*(X^G)$ algebraically from some equivariant cohomology theory on X. Last time, we introduced Borel cohomology $H^*_G(X) := H^*(EG \times_G X)$, where EG is a free G-space that's nonequivariantly contractible (which is simple to construct with the bar construction or through Elmendorf's theorem).

In the following, all cohomology is understood to have coefficients in $\mathbb{F}_p$. Recall that the Bredon cohomology of X is defined to be $H^*_G(X) := H^*(EG \times_G X)$. For a subgroup $H \subset G$, let $S_H \subset H^*(BG)$ be the multiplicative set generated by the classes in $H^2(BG)$ that are images of the Bockstein homomorphism $H^1(BG) \to H^2(BG)$ of the elements that are nontrivial in $H^1(BH)$. This uses the fact that Borel cohomology is an $H^*(BG)$-module: $H^*(EG \times_G *) \cong H^*(EG/G) = H^*(BG)$, and using the terminal map $X \to *$ we get a map $H^*(BG) \to H^*(EG \times_G X)$.

Theorem 5.7. *Let G be a finite p-group and $H \subset G$ be a subgroup. Then, there is an isomorphism* $S_H^{-1} H^*_G(X) \xrightarrow{\cong} S_H^{-1} H^*_G(X^H)$.

There's a rich theory of unstable modules over the Steenrod algebra $\mathscr{A}_p$, which could fill a whole semester. There's a functor Un which produces unstable $\mathscr{A}_p$-modules, in a sense by only keeping the unstable part.

Theorem 5.8 (Dwyer-Wilkerson [DW88]).

$$H^*(X^G) \cong \mathbb{F}_p \otimes_{H^*(BG)} \operatorname{Un}(S_G^{-1} H^*(EG \times_G X)).$$

The proof uses arguments that were hard to think of, but easy to follow.

We'll use these theorems to prove Smith's theorem using the Serre spectral sequence for

$$X \longrightarrow EG \times_G X \longrightarrow BG.$$

This will be the nicest kind of spectral sequence argument: everything degenerates.

Theorem 5.7 is an example of a general class of **localization theorems** in equivariant cohomology. In these theorems, one considers the fiber sequence $X_+^G \to X_+ \to X/X^G$, and wants to show that for some functor E, $E(X_+^G) \cong E(X_+)$. This boils down to showing $E(X/X^G)$ vanishes, which will always follow from showing that E vanishes on G-spaces whose G-actions are free away from the basepoint. In general, this will reduce to considering cells, so one considers $E(G/H_+ \wedge \bigvee_\alpha S^{q_\alpha})$ for some wedge of spheres.

In our case, $E(G/H_+ \wedge \bigvee_\alpha S^{q_\alpha})$ is

$$S_H^{-1}H_G^*\left(G/H_+ \wedge \bigvee_\alpha S^{q_\alpha}\right) = S_H^{-1}H^*\left(EG \times_G \left(G/H_+ \wedge \bigvee_\alpha S^{q_\alpha}\right)\right)$$

Now, the term $EG \times_G G/H \cong EG/H \cong BH$, so we have a piece that looks like $H^*(BH)$, which is how BH inserts itself into the argument.

Exercise 5.9. Finish the Serre spectral sequence proof of Theorem 4.4. Hint: there's a simple geometric reason why the spectral sequence collapses, which is what makes this whole thing go.

Soon we'll start talking about the stable category. As preparation for this, if you don't already know it, it's good to remind yourself of it.

The Q&A session will be Thursday at 8:30 PM.

Lecture 6.

The localization theorem and the Sullivan conjecture: 2/2/17

"The number of children he had was a monotonically increasing function."

There are three goals in today's lecture.

(1) We'll discuss the localization theorem (Theorem 5.7) again, and prove it, providing a bit more context for where it came from.
(2) We'll talk about the Sullivan conjecture. This is really Sullivan's attack on Adams' conjecture, and is a very important story. We won't prove the conjecture, because it's hard, but the context around it was a major motivation for a lot of the work in algebraic and geometric topology in the past 40 years. Sullivan wrote up some notes for a class of his at MIT, which have been published as [Sul05], and you should read them: they are enlightening and contain all of the jokes he told in class!
(3) We'll introduce the stable category, discussing Spanier-Whitehead duality and the Pontrjagin-Thom construction. If you want points (orbits) to have Spanier-Whitehead duals, you're inexorably forced to construct this stable category.

The localization theorem. Recall that the Borel cohomology $H^*(EG \times_G X)$ is an $H^*(BG)$-module, through the map $EG \times_G X \to EG \times_G * = BG$. Thus, we should compute $H^*(BG)$ as a ring. In the case we care about, $G = (\mathbb{Z}/p)^n$; let's start with $n = 1$.

There's a nice geometric model for $B\mathbb{Z}/p$ as an "infinite-dimensional lens space:" let S^∞ denote the unit sphere in an infinite-dimensional complex Hilbert space,[21] and give it a C_p-action by $z \mapsto e^{2\pi i/p}z$. Then, the infinite-dimensional lens space is S^∞/C_p.

The quotient $S^\infty \twoheadrightarrow S^\infty/C_p$ is a covering map, and S^∞ is contractible. The proof goes through some version of the Eilenberg swindle:

- There is a homotopy from id_{S^∞} to $s : (x_1, x_2, \dots) \mapsto (0, x_1, x_2, \dots)$. In fact, it's a straight-line homotopy.
- There is a straight-line homotopy from s to $(0, x_1, \dots) \mapsto (1, 0, 0, \dots)$.

[21] Alternatively, you could choose S^∞ to be the colimit of S^n for all n, through the inclusion $S^n \hookrightarrow S^{n+1}$ at the equator. These two choices are not homeomorphic, but produce homotopy equivalent models of $B\mathbb{Z}/p$.

Thus S^∞/C_p is the quotient of a contractible space by a free G-action, so it deserves to be called $B\mathbb{Z}/p$.

You can set up a cell structure on $B\mathbb{Z}/p$ as with finite-dimensional lens spaces, and therefore compute that

$$H^k(B\mathbb{Z}/p) \cong \begin{cases} \mathbb{Z}/p, & k \text{ even} \\ 0, & k \text{ odd.} \end{cases}$$

Using the universal coefficient theorem, you can then deduce that

$$H^k(B\mathbb{Z}/p;\mathbb{Z}/p) \cong \mathbb{Z}/p$$

for all k. Now we want to deduce the ring structure.

Recall that if

$$0 \longrightarrow M \longrightarrow L \longrightarrow N \longrightarrow 0$$

is a short exact sequence, it induces a long exact sequence in cohomology:

$$\cdots \longrightarrow H^k(-;M) \longrightarrow H^k(-;L) \longrightarrow H^k(-;N) \xrightarrow{\beta} H^{k+1}(-;M) \longrightarrow \cdots$$

Let's apply this to the short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot p"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}/p \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z}/p \arrow[r, "\cdot p"] & \mathbb{Z}/p^2 \arrow[r] & \mathbb{Z}/p \arrow[r] & 0.
\end{tikzcd}$$

This is a map of short exact sequences, inducing a map of their long exact sequences.

$$\begin{tikzcd}
\cdots \arrow[r] & H^q(X;\mathbb{Z}) \arrow[r] \arrow[d] & H^q(X;\mathbb{Z}) \arrow[r] \arrow[d] & H^q(X;\mathbb{Z}/p) \arrow[r] \arrow[d] & H^{q+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^q(X;\mathbb{Z}/p) \arrow[r] & H^q(X;\mathbb{Z}/p^2) \arrow[r] & H^q*(X;\mathbb{Z}/p) \arrow[r, "\beta"] & H^{q+1}(X;\mathbb{Z}/p) \arrow[r] & \cdots
\end{tikzcd} \tag{6.1}$$

The map $\beta : H^q(X;\mathbb{Z}/p) \to H^{q+1}(X;\mathbb{Z}/p)$ will be called the **Bockstein homomorphism**, and is a simple example of a cohomology operation.

We now assume p is odd.

Lemma 6.2. *If n is odd, the Bockstein for $B\mathbb{Z}/p$ is an isomorphism; if n is even, it's* 0.

Proof. First, observe that the diagram

$$\begin{tikzcd}
H^n(B\mathbb{Z}/p;\mathbb{Z}) \arrow[r, "f"] & H^n(B\mathbb{Z}/p;\mathbb{Z}/p) \arrow[r, "g"] \arrow[dr, "\beta"'] & H^{n+1}(B\mathbb{Z}/p;\mathbb{Z}) \arrow[r] \arrow[d, "\pi"] & H^{n+1}(B\mathbb{Z}/p;\mathbb{Z}) \\
 & & H^{n+1}(B\mathbb{Z}/p;\mathbb{Z}/p) &
\end{tikzcd}$$

commutes, and the top row is exact.

- If n is even, f is an isomorphism, so $g = 0$, so $\beta = 0$.
- if n is odd, g and π are surjections, so β is a surjection between two $\mathbb{F}_p$-vector spaces of the same rank, so β is an isomorphism. ⊠

Next, a nice way to compute the cup product. The map $B\mathbb{Z}/p \to \mathbb{CP}^\infty$ is cellular, and is a homeomorphism when restricted to even-dimensional cells. As the cell structure determines the cup product structure, the cup products on $H^{\mathrm{even}}(B\mathbb{Z}/p;\mathbb{Z}/p)$ and $H^{\mathrm{even}}(\mathbb{CP}^\infty;B\mathbb{Z}/p) \cong \mathbb{Z}/p[y_1]$ agree. On the odd-dimensional cells, the cup product is graded commutative rather than strictly commutative, so we get an exterior algebra. Therefore we conclude that

$$H^*(B\mathbb{Z}/p;\mathbb{Z}/p) \cong \Lambda(x_1) \otimes \mathbb{Z}/p[y_1],$$

where $|x_1| = 1$, $|y_1| = 2$, and $\beta x_1 = y_1$.

By essentially the same argument, one can compute the cohomology ring for $B(\mathbb{Z}/p)^n$.

Proposition 6.3. *As rings, there is an isomorphism*

$$H^*(B(\mathbb{Z}/p)^n;\mathbb{Z}/p)\cong\Lambda(x_1,\dots,x_n)\otimes\mathbb{Z}/p[y_1,\dots,y_n],$$

where $\beta x_i = y_i$.

Exercise 6.4. Figure out the slight changes needed for $p=2$.

Recall that the localization theorem asserted that if G is a finite abelian p-group, $S_H^{-1}H^*(EG\times_G X)\cong S_H^{-1}H^*(EG\times_G X^G)$, where S is the multiplicative system generated by images of the Bockstein homomorphism in $H^*(BG;\mathbb{Z}/p)$.

We mentioned last time that it's possible to inductively reduce to considering $H^*(BH)\otimes H^*\big(\bigvee S^{q_i}\big)$. It's now clear[22] that something in S restricts to 0 in $H^*(BH)$, completing the proof (in the abelian case).

The localization fails terribly for infinite nonabelian compact Lie groups G. For example, for any topological space K, there exists a G-CW complex X such that X is nonequivariantly contractible, X is finite-dimensional, and $X^G\simeq K$.

The Sullivan conjecture.

Theorem 6.5 (Sullivan conjecture)**.** *Let G be a finite, abelian p-group. Then, $X^{hG}\to H^G$ is an equivalence on p-completions.*

Recall that $X^{hG}:=\mathrm{Map}(EG,X)^G$, so this asserts a weak equivalence (after p-completing) $\mathrm{Map}(EG,X)^G\to\mathrm{Map}(*,X)^G$.

By p**-completion**, we mean Bousfield localization at $\mathbb{F}_p$ cohomology. This produces the category of spaces where equivalences are detected by $H^*(-;\mathbb{F}_p)$. The most familiar example of completion is **rationalization**, a localization where equivalences are detected by rational homotopy groups, and one of Sullivan's biggest achievements was providing a completely algebraic description of the rational homotopy category in [Sul77]. More broadly, he had the insight that to study a problem in homotopy theory, one could localize at $\mathbb{Q}$ and at each $\mathbb{F}_p$ and study each piece, which has been a very fruitful approach.

p-completion falls into the collection of basic life skills for homotopy theorists, so if you haven't seen it before, you should read about it. The standard reference is [BK72], but this is 500 pages and hard to read.

Sullivan's conjecture is an algebro-geometric attack on the Adams conjecture. This was within Sullivan's program to find algebraic models of manifolds. This is still being done today, and is what led Sullivan to think about string topology and related things.

Let X be a manifold. We first have the homotopical data of X, $C^*(X;\mathbb{Q})$ and $C^*(X;\mathbb{F}_p)$, which are "commutative rings."[23] That X is a manifold means we can see Poincaré duality, which doesn't appear in all $\mathbb{E}_\infty$ dg algebras. But we still need some way to encode additional geometry obstructions, e.g. a way to encode the tangent and normal bundles.

It's been an interesting, but as yet unsuccessful attack, to use a Frobenius algebra structure to try to obtain this geometric data. It's neat to think about what the $\mathbb{E}_\infty$ analogue of a Frobenius algebra is, and this is intimately related to Lurie's approach to the cobordism hypothesis [Lur09], thinking about fully dualizable objects in a symmetric monoidal ∞-category.

This circle of ideas is also related to surgery theory; there's been lots of cool work by smart people in it, and L-theory was invented basically as an algebraic home for these geometric objects.

Sullivan was interested in the Adams conjecture because it says that one can identify the tangent bundle inside K-theory $K(X)$ with its Adams operations ψ^k, as fiberwise homotopy types.

Sullivan's idea, motivated by Quillen, was to use the theory of étale homotopy types. This translates some questions about scheme theory into homotopy theory. For example, if X is a variety (more generally a scheme), one can assign some profinite topological object, built out of something like a system of hypercovers.[24] So if you take the profinite completion of the complex points $X(\mathbb{C})^\wedge$, it has an action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ — and another crazy interpretation of the Adams conjecture is that the profinite completion of $K(X)$ can be interpreted in this way, and has an action of $(\widehat{\mathbb{Z}})^*$ ($\widehat{\mathbb{Z}}$ is the Galois group of the maximal abelian extension over $\mathbb{Q}$). The conjecture is that this action is by the Adams operations.

[22] TODO: I didn't follow this proof in class.

[23] They're not literally commutative; instead, they're $\mathbb{E}_\infty$ dg algebras. They also have more structure as modules over the Steenrod algebra.

[24] If you don't know what this is, it's an example of an extremely interesting construction which you should look up sometime.

There's a deep and inadequately understood story (which could be an opportunity for you) connecting p-adically completed complex K-theory $KU_p^\wedge$ to number theory, specifically the Iwasawa algebra. Adams noticed this, but it's too interesting to be a coincidence.

Anyways, stable fiberwise homotopy types are invariant under this $(\widehat{\mathbb{Z}})^*$-action, which led Sullivan to ask questions about $(X(\mathbb{C})^\wedge)^{hC_2}$ versus $X(\mathbb{R})^\wedge$. The references [Sul05, Sul74] are both excellent for this.

For reasons of scope, we can't go into too much more detail, but you should definitely look this stuff up. The takeaway is that equivariant homotopy theory has been motivated by seemingly unrelated questions about manifolds. There's been a lot of interesting interplay between algebraic and geometric topology in the last half century, and this is one of the cites of contact.

The stable category and stable phenomena. There are lots of ways to think about where stabilization comes from.

(1) The Freudenthal suspension theorem says that if X is nondegenerately based (meaning the based inclusion map $* \hookrightarrow X$ is a cofibration) and $n-1$-connected, then $\pi_q(X) \to \pi_{q+1}(\Sigma X)$ is an isomorphism for $q < 2n-1$ and a surjection when $q = 2n-1$.[25] It's easier to see that cohomology groups are stable under suspension, but this tells us that homotopy groups stabilize in a range that increases at about twice the rate that the connectivity of X does. Since $\Sigma^n X$ is at least n-connected, this suggests you could replace X by the sequence $X, \Sigma X, \dots, \Sigma^n X, \dots$, and keep track of that instead, regarding it as a repository for the **stable homotopy groups** $\pi_n^S(X) := \operatorname{colim}_k \pi_{n+k}(\Sigma^k X)$. One way to think of this is as formally making homotopy theory into a homology theory (which it isn't *a prioiri*); you end up taking the same kind of colimit.

You could do this equivariantly: we have representation spheres. But it's not entirely clear what to do.

(2) Another perspective is that the stable category is the result of inverting the canonical map[26]

$$\bigvee_{i=1}^{k} X \to \prod_{i=1}^{k} X. \tag{6.6}$$

Again, this is something you can think about making precise; the stable category is the initial triangulated category constructed from Top in which (6.6) is an isomorphism. In particular, this forces the homotopy category to be additive.

Again, we could do this equivariantly.

(3) Suppose we have a functor F from finite CW complexes to Top such that $F(*) = *$, and suppose F commutes with filtered colimits and preserves weak equivalences (e.g. if it's topologically or simplicially enriched, for formal reasons). By taking colimits, we can obtain a functor $\widehat{F}$ from CW complexes to Top. We say F is **excisive** if it takes pushouts to pullbacks; this is an old perspective, which was used to show the Dold-Thom theorem, that the infinite symmetric product SP^∞ is a cohomology theory. In any case, if F is excisive, $\{F(S^n)\}$ represents a cohomology theory. Namely, the homotopy pushout

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & D^{n+1} \arrow[d] \\ D^{n+1} \arrow[r] & S^{n+1} \end{tikzcd}$$

creates suspension, and via F, becomes a pullback creating Ω. Asking what the excisive functors are leads to the stable category, and is also something you could do equivariantly.

(4) Another perspective: what is the Spanier-Whitehead dual of a point? Taking shifts, what's the Spanier-Whitehead dual of S^n? Equivariantly, one wants to know the Spanier-Whitehead duals of G/H_+. This is important for defining Poincaré duality, etc. Nonequivariantly, the best way to answer this is the Pontrjagin-Thom construction, which not only answers this, but provides a deep understanding for what the stable homotopy groups of the spheres are. We'll do this equivariantly next time, and it will tell us what the spheres are.

[25] The map comes from the loop-suspension adjunction, which gives us a unit $X \to \Omega\Sigma X$, hence a map $\Omega^q X \to \Omega^{q+1}\Sigma X$, and the map on homotopy groups is π_0 of that map. This is the based version of the mapping space and Cartesian product adjunction: $\Sigma X := S^1 \wedge X$ and $\Omega X := \mathrm{Map}(S^1, X)$ are adjoint functors.

[26] The existence of this map follows from the universal property of the product.

There's a choice here: you could just say "let's take the category of orthogonal G-spectra," but having some motivation for why we're doing and what we're doing is important.

Lecture 7.

Question-and-answer session: 2/2/17

Question 7.1. So, why does anyone care about Spanier-Whitehead duality?

One answer is that Spanier-Whitehead duality formally implies Poincaré duality. Poincaré duality is a remarkable fact about the cohomology of manifolds, which is one good reason to care.

Another is that it's the real setting for the Euler characteristic — really. The Euler characteristic arises as a trace, and therefore some kind of categorical duality should appear.

Being more specific, let's consider a closed symmetric monoidal category C with unit $\mathbb{S}$ (so secretly we're thinking of the stable homotopy category). Spanier-Whitehead duality consists of two maps $\varepsilon\colon X \wedge Y \to \mathbb{S}$ and $M\colon \mathbb{S} \to X \wedge Y$, such that the composition

$$X \cong \mathbb{S} \wedge X \xrightarrow{M\wedge\mathrm{id}} X \wedge Y \wedge X \xrightarrow{\mathrm{id}\wedge(\varepsilon\circ\tau)} X \wedge \mathbb{S} \cong X$$

is the identity.

Using an adjunction, you can show $X \cong F(Y,\mathbb{S})$ and $Y \cong F(X,\mathbb{S})$, and so X is the dual of its dual. This has lots of formal consequences, includin Poincaré duality, the construction of things such as the Euler characteristic, and more. The Spanier-Whitehead dual of X is often denoted DX.

Suppose that we're in an algebraic category, say $H\mathbb{Q}$-module spectra (the rational stable homotopy category). Then, $H\mathbb{Q}$ is the unit, so $DX = C^*(X;\mathbb{Q})$ (maps from X to $\mathbb{Q}$). That is, the Spanier-Whitehead dual embeds the unstable homotopy theory of X, as long as you remember the $\mathbb{E}_\infty$-ring structure on DX (which is bizarre, e.g. it's not connective). There's a quite nontrivial theorem that $\mathbb{E}_\infty$ maps between DX and DY correspond to maps of *spaces* between X and Y.

This sort of data is useful for algebraicizing manifolds, and that would be nice for classifying manifolds, a goal with fairly broad applications outside of homotopy theory. And Spanier-Whitehead duality has important consequences on this: Sullivan began it by showing [Sul77] that the $\mathbb{E}_\infty$-ring structure on $C^*(X;\mathbb{Q})$ controls the rationalization $X_{\mathbb{Q}}$, and Mandell ended it by showing that the $\mathbb{E}_\infty$ structure on $C^*(X;\mathbb{F}_p)$ controls $X_p^\vee$.

Question 7.2. What are some of the obstacles to extending equivariant homotopy theory to groups that aren't compact Lie groups?

One cornerstone of homotopy theory is the Pontrjagin-Thom construction, which depends on the weak Whitney embedding theorem: that a manifold can be embedded in some high-dimensional $\mathbb{R}^N$. This still works equivariantly *but only for compact Lie groups* — a G-manifold X can be embedded in some high-dimensional G-representation. This is again an important piece in the equivariant Pontrjagin-Thom construction, and its failure for noncompact groups is one major reason things don't work. There are other things that require compactness (e.g. descending chain arguments). People have tried and not gotten anywhere.

In stable homotopy theory, there's a modification of the orbit category into something called the Burnside category, and we'll see that it controls a lot of the stable homotopy theory of G-spaces. In fact, if you have something that strongly resembles the Burnside category, you have something that looks like equivariant stable homotopy theory. Clark Barwick and his collaborators have been working on studying presheaves on things that look like Burnside categories.

A lot of it boils down to the fact that compact Lie groups have a tractable representation theory.

Question 7.3. Recall $E\mathscr{F}$, the classifying space for a family of subgroups. What is it used for?

If you want to focus attention on a family of subgroups, you play with $E\mathscr{F}$. One common example is S^1-spaces, in which there are many constructions that are fixed by the finite subgroups of S^1, so having that family $\mathscr{F}$ is helpful, and for this one can smash with $E\mathscr{F}$.

There are various other applications. One is called **isotropy separation**, which splits up $\mathscr{F}$ into pieces that can be detected with different kinds of isotropy subgroups, and one can induct on this in nice cases.

Question 7.4. What does it mean that K-theory and the Adams operations determine the tangent bundle?

It's a somewhat implicit statement: any possible algebraic statement about the tangent bundle can be translated into an equivalent statement in $K(X)$ that uses the Adams operations.

Question 7.5. How many notions of G-spectra are there?

One way you talk about equivariant stable homotopy theory is a **universe**, a countably infinite-dimensional inner product space containing the irreducible representations you care about infinitely often. Then, you have Spanier-Whitehead duality for G/H iff G/H embeds in the universe, and you get different flavors of homotopy theory depending on which universes you use. There are lots of models here — depending on what you mean, spectra objects or diagrams on $\mathscr{O}_G$ might not be the right thing for naïve spectra; instead, you need transfers, so you have to consider sheaves on the Burnside category.

The choice of universe also affects which suspensions you invert: if V is in your universe, you can invert Σ^V.

Of course, there are probably many different point-set models for stable homotopy, but they'll give you the same answer.

Question 7.6. Can you go over how to design a coefficient system such that its cohomology is something obtained from X?

For example, let's try to determine M such that $H_G^*(X;M) \cong H^*(X)$. The reason it suffices to determine what $M(G/H)$ is on all closed subgroups $H \subset G$ is the dimension axiom!

So if $G = \mathbb{Z}/p$, you can assign $M(G/e) := H^0(G/e)$ and $M(G/G) := H^0(G/G)$, and the map between them is H^0 of the map $G/e \to G/G$.

Question 7.7. So far, we've mostly seen G-equivariant homotopy theory where G is a finite cyclic group. Is there anything interesting for nonabelian groups, etc?

Part of the problem is that computing examples is not easy, and it gets much harder when you have a complicated lattice of subgroups. [LMS86] is 500 pages, and there are scarcely any examples, because the computations for nontrivial examples are so hard!

Peter May invented this stuff and set a bunch of grad students to work on it, but there wasn't a lot of buzz until Carlsson proved the Segal conjecture, and then again more recently with Hill-Hopkins-Ravenel. So there haven't been a lot of computations, period. If you do make an interesting computation with, say, the monster group, by all means write it up!

So for the most part people have studied cyclic groups and S^1. There's been a little discussion of dihedral actions, and some stuff with the symmetric groups.

There's also some stuff done for profinite groups. This is in some sense easy to set up formally, especially because you mostly care about finite-index subgroups. People who study Galois actions (e.g. Carlsson's program to lift the Quillen-Lichtenbaum conjecture into this context) care about this.

Question 7.8. So we've spent some time looking at EG, a contractible space with a free G-action, and its quotient BG. What things do people do with these objects?

BG is a classifying space for principal G-bundles (and therefore for $G = \mathrm{O}_n$ or U_n, also vector bundles of rank n). That is, homotopy classes of maps $X \to BG$ are identified with isomorphism classes of principal G-bundles on X. There's a book by May, "Classifying spaces and fibrations," which is excellent and goes into great detail on this stuff. Because $B\mathrm{U}_n$ classifies complex vector bundles of rank n, it's used to construct complex K-theory (and same for $B\mathrm{O}_n$ and real K-theory).

In our case, smashing with EG is often a way to localize.

A third reason to care about this is group cohomology and other purely algebraic stuff. The group cohomology of G is the cohomology of BG, and there are plenty of applications of group cohomology.

So though BG may be infinite-dimensional, it's very simple homotopically.

Question 7.9. What's going on with the construction to the right adjoint to the functor ψ in the proof of Elmendorf's theorem? What's a coend?

Adrian said some stuff here; I wasn't able to get it down. He motivated the bar construction as the thing whose homotopy colimits are ordinary colimits, I think?

Anyways, our setup is that we have a presheaf on the orbit category $X \in \mathsf{Fun}(\mathscr{O}_G^{\mathsf{op}}, \mathsf{Top})$ and want to produce a G-space. The right adjoint[27] to ψ is the geometric realization of the bar construction $B_\bullet(X, \mathscr{O}_G, M)$. The

[27] Recall that the left adjoint was evaluation at G/e.

bar construction is a generalization of the bar resolution to compute the derived tensor product: the functor $M\colon \mathscr{O}_G \to \mathsf{Top}$ sending G/H to the space G/H is thought of as a "right $\mathscr{O}_G$-module;" instead of tensoring a bunch of elements togther, we get a bunch of arrows, meaning we can replace with a coproduct. In some sense, it would be nice to take a tensor product with X, but we have to do so in a derived sense, hence the bar construction.

A coend is a functor that behaves like geometric realization: there's two functors with opposite variance, and you want to glue along their common edge, just like in geometric realization.

Question 7.10. How did Elmendorf formalize his proof, given that it was done before model categories were available?

He didn't: Elmendorf's paper is eminently readable, and simply provides an equivalence of homotopy categories. It was not lifted into model categories unti much later. This stuff was all put into use fairly recently: for example, not that long ago, it was known to experts but not written down that a left Quillen adjoint that's part of a Quillen equivalence preserves homotopy limits.

Here, someone asked about the Freudenthal suspension theorem, and this led to a digression.

Remark. Modern cryptography depends on some hardness assumptions, that some functions, such as the discrete log in a finite field, are hard to compute (but easy to check answers to). There's a paper by Impagliazzo which asks what cryptography and security would look like if certain assumptions were false or true, with cute names for different worlds. Imagine doing that for the Freudenthal suspension theorem — what if the stable range were at about n instead of about $2n$? What if it were $n/2$? ◀

Question 7.11. In non-equivariant rational homotopy theory, there's a standard, completely algebraic description of the rational homotopy category. Does this also work for the rational equivariant homotopy category?

This is very hard — someone was working on this, but the work actually depended on an incorrect calculation, and some of it had to be redone. There are people working on an algebraic model for rational G-spectra, but the algebraic models are *very* complicated.

Question 7.12. We've seen that representations of G play a huge role in equivariant homotopy theory; these could be thought of as relating to K-theory of BG (e.g. the representation ring $\mathrm{RO}(G)$ is $KO^0(BG)$). More generally, do $KO(BG)$ or $MO(BG)$ play a special role "controlling" the G-equivariant stable category?

People certainly care about computing K-theory or bordism of BG. As to whether those control the G-equivariant stable category—that might be true, but I don't think anything like that has been said.

Question 7.13. This question comes from GitHub: suppose you're defining a coefficient system M over $G = C_p$. This is determined by $M(C_p/e)$, $M(C_p/C_p)$, and a map between them. How do you get the map?

The map is determined by functoriality: in the cases we used to prove Theorem 4.4, which is where this question arose, $M(C_p/e)$ and $M(C_p/C_p)$ are both defined to be H^0 of something, so the map between them is H^0 of the map $C_p/e \to C_p/C_p$.

Question 7.14. Do you want to talk about the unstable Steenrod functor Un?

You don't want to know.

Question 7.15. Does Un appear in the theorem as a technical necessity or for deep reasons?

It's very deep, and there's a whole theory of unstable modules over the Steenrod algebra. This involves some extra restrictions.

Question 7.16. So how many kids did Denis Sullivan have?

About 7 or 8 by the end of [Sul05]; one is now also a math professor. There are lots of other anecdotes in the notes: once he was frustrated with some calculations he was doing, so he hurled the book into the Atlantic ocean. While he was on a boat to Oxford.

Question 7.17. Do we have any previews of the Clark school, besides what few things they've posted on the arXiv?

They've been doing some really hard stuff: the idea is that G-spectra should be presheaves of spectra on Burnside categories, or statements like G-spectra being the initial stable G-symmetric monoidal category... you know what the theorems should be, but the proofs are technical. Riehl-Verity technology should make some of this easier, hopefully.

Question 7.18. What does it mean that "the Spanier-Whitehead dual of a point involves taking shifts?"

We've been a little careless about basepoints: this is really the pointed point, so S^0; thus, we really should have said the Spanier-Whitehead duals of spheres.

Question 7.19. The Fruedenthal suspension theorem requires inclusion of the basepoint to be a cofibration. If you add a disjoint basepoint, is that the case?

Yes. You can think of a cofibration as a slightly weaker version of a closed inclusion. This is no big technical obstacle: the **whisker construction** replaces the basepoint with a line segment, so the homotopy theory is unchanged and the other end of the line segment can be a basepoint that's cofibrantly included.

Question 7.20. What's a natural example of a genuine G-spectrum of interest?

One is the equivariant sphere spectrum. Thinking of cohomology theories, there's equivariant K-theory KU_G, different kinds of equivariant bordism, and KR (a genuine equivariant spectrum that's built from K-theory).

Question 7.21. Is there an equivariant approach to de Rham theory, e.g. starting with manifolds and equivariant differential forms?

Rationally, this can work, but integrally there are issues. Relatedly, defining a G-manifold is not too hard, and there are nice things like the equivariant tubular neighborhood theorem, but there are some subtleties.

Question 7.22. One of the interpretations of the homotopy category of a model category is localization at the weak equivalences. This suggests that one only needs to look at weak equivalences. So if you only specify weak equivalences, do you get a unique model category?

No, you might not get one at all, and if you do, it won't be unique. What it means is (modulo some scary set-theoretic issues) that the homotopy theory only depends on the weak equivalences. If you have a category with weak equivalences, you can form the **Dwyer-Kan localization**, an associated simplicial category, and then take the associated quasicategory or ∞-category. Under some hypotheses, you can then retreat to a model-categorical structure. One commonly heard analogy is that a model category is like choosing a basis for a vector space (where the basis-independent answer reflects the underlying ∞-category).

Another perspective is that model categories are like axiomatic obstruction theory, axiomatizing the cellular inclusions (cofibrations) or extension questions common when considering CW theory.

Question 7.23. One way to define the stable category is to invert the canonical map from a finite wedge of things to a product of those things. How can we make this precise?

The goal is to localize with respect to something. There's some set-theoretical issues that Bousfield was good at addressing, but if you have a set of morphisms, you can localize with respect to them, and that's what we're doing here. The Stacks project is a good resource for learning about localization.

The fact that finite wedges and finite products are the same is another way that spectra resemble abelian groups, for which finite products and finite coproducts agree.

Question 7.24. Where's the best place to learn non-equivariant stable homotopy theory?

Adams' book [Ada74], with good calculations, but its construction of the stable category isn't so pleasant. It might be better to consider [MMSS01]. It assumes you already know why you care, then takes the historical constructions and shows how they all relate together. It may seem scary, but looking at the constructions shows they're all not so bad.

Adams' infinite loop spaces book [Ada78] is good bedtime reading.

Question 7.25. We talked about the notion of an excisive functor, and there's a notion of an excisive pair in unstable homotopy theory. Are the two related?

Both deserve the name "excisive" because they satisfy some sort of Mayer-Vietoris principle: excisive functors take pushouts to pullbacks, for example.

Question 7.26. By $\mathbb{RP}^\infty$, do we just mean the colimit of $\mathbb{RP}^n$ over all n?

Yes, and also for $\mathbb{CP}^\infty$. However, there are models for them based on the unit sphere in an infinite-dimensional Hilbert space modulo a $\mathbb{Z}/2$ (resp. S^1) action. This is homotopic to, but not homeomorphic to, the colimit realizations of $\mathbb{RP}^\infty$ and $\mathbb{CP}^\infty$.

Lecture 8.

Dualities: Alexander, Spanier-Whitehead, Atiyah, Poincaré: 2/7/17

"The sun, it burns!"

Pursuant to promises we made last lecture, today's lecture is about duality. This is a reinvention of the original construction of the stable category — Spanier's original construction of spectra was motivated by answering questions on duality, and we'll proceed similarly, if a bit ahistorically. Since we're in the equivariant setting, the answers will be slightly different.

Alexander duality is a tale as old as time, from what could be called the prehistory of algebraic topology.[28]

Theorem 8.1 (Alexander duality). *Let $K \subset S^n$ be compact, locally contractible, and nonempty.[29] Then, K and $S^n \setminus K$ are **Alexander dual** in that there is an isomorphism*

$$\widetilde{H}^{n-i-1}(K;\mathbb{Z}) \cong \widetilde{H}_i(S^n \setminus K;\mathbb{Z}).$$

The proof isn't too hard, e.g. Hatcher does it. This is closely related to considering embeddings in Euclidean spaces, after you take the one-point compactification.

The good part of this proof is that it doesn't depend on the embedding. But there are a few drawbacks:

(1) K does not determine the homotopy type of $S^n \setminus K$. Knot theory is full of examples, and they tell you that the issues arise for the fundamental group.
(2) n can vary, and if you embed $S^n \hookrightarrow S^{n+1}$ as the equator, you get different statements.

Motivated by the second issue, Spanier defined the S-category in the 1950s.[30]

Definition 8.2. The ***S*-category** S is the category whose objects are the objects in Top and whose morphisms are

$$\mathrm{Map}_{\mathsf{S}}(X,Y) := \operatorname*{colim}_n \mathrm{Map}_{\mathsf{Top}}(\Sigma^n X, \Sigma^n Y).$$

By the Freudenthal suspension theorem, the hom-sets stabilize at some finite n. Spanier then defined ***S*-duality**, which we might call **Spanier-Whitehead duality**, by specifying that X and Y are S-dual if $Y \cong S^n \setminus X$ in the S-category.

Remark. The S-category has some issues: it's neither complete nor cocomplete. We like gluing stuff together, so this is unfortunate. ◀

Spanier proved that to every $X \to S^n$, you can assign a dual $D_n X$, that $\Sigma D_n = D_{n+1}$, and $D_{n+1}\Sigma = D_n$. That is, duality commutes with suspension, so in S, every $X \to S^n$ has a unique S-dual: the duals inside S^n and S^{n+1} are the same in the S-category for sufficiently large n.

Spanier and Whitehead then asked one of their graduate students, Elon Lima, to formalize this S-category, leading to the first notion of the category of spectra.

An axiomatic setting for duality. The formal setting for duality is a **closed symmetric monoidal category** C. We're not going to spell out the whole definition, but here are some important parts.

- C is symmetric monoidal, meaning there's a tensor product $\wedge\colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}$, which is (up to natural isomorphism) associative and commutative, and has a unit S. Commutativity is ensured by the **flip map** $\tau\colon X \wedge Y \xrightarrow{\cong} Y \wedge X$.
- There is an internal **mapping object** $F(X,Y) \in \mathsf{C}$ for any $X, Y \in \mathsf{C}$.
- The functors $- \wedge X$ and $F(X,-)$ are adjoint (just like the tensor-hom adjunction).

The unit and counit of the tensor-hom adjunction are used to define duality.

Definition 8.3. The **evaluation map** is the unit $X \wedge F(X,Y) \to Y$, and the **coevaluation map** is the counit $X \to F(Y, Y \wedge X)$. The **dual** of X is $DX = F(X,S)$.

You also get a natural map $\nu\colon F(X,Y) \wedge Z \to F(X, YZ)$.

Exercise 8.4. Check that $X \cong F(S,X)$, which follows directly from the axioms.

[28] If you read between the lines, you can find it in Poincaré's works, but it's from the 1940s or 50s stated explicitly.

[29] Classically, one works simplicially, picking a triangulation of S^n and letting K be a subpolyhedron.

[30] The name "S-category" is somewhat misleading: in those days, suspension was sometimes denoted S instead of Σ to make typeseting easier, and the S in S-category stood for suspension, not spheres.

The adjoint of ε is a map $X \to DDX$.

There are a few good references for this: Dold-Puppe [DP61] is one, and [LMS86] is another, though it presents a somewhat old way of doing things.

Definition 8.5. $X \in \mathsf{C}$ is **strongly dualizable** if there exists an $\eta\colon S \to X \wedge DX$ such that the following diagram commutes.

$$\begin{tikzcd} S \arrow[r, "\eta"] \arrow[d] & X \wedge DX \arrow[d, "\tau"] \\ F(X,X) & DX \wedge X. \arrow[l, "\nu"'] \end{tikzcd} \tag{8.6}$$

Here, the left-hand map comes from an adjunction to the identity $\mathrm{id}_X : X \cong X \wedge S \to X$. The lower map is more explicitly $\nu\colon F(X,S) \wedge X \to F(X,X)$.

Example 8.7. Let R be a commutative ring and $\mathsf{C} = \mathsf{Mod}_R$, and let X be a free R-module. If $\{v_i\}$ is a basis for X and $\{f_i\}$ is the dual basis, then the map $\eta\colon R \to X \otimes_R \mathrm{Hom}_R(X,R)$ is the map sending

$$1 \longmapsto \sum v_i \otimes f_i.$$

If you unravel what (8.6) is saying, it says that the map

$$x \longmapsto \sum_i f_i(x) v_i$$

must be the identity. Thus, X is strongly dualizable iff X is finitely generated and projective. That is, *X is strongly dualizable iff it's a retract of a free module*, which is a perspective that will be useful later. ◀

Another way to think of this is that X is strongly dualizable with dual Y iff there exist maps $\varepsilon\colon X \wedge Y \to S$ and $\eta\colon S \to X \wedge Y$ such that the compositions

$$X \cong S \wedge X \xrightarrow{\eta \wedge \mathrm{id}_X} X \wedge Y \wedge X \xrightarrow{\mathrm{id}_X \wedge \varepsilon} X \wedge S \cong X$$

and

$$Y \cong Y \wedge S \xrightarrow{\mathrm{id}_Y \wedge \eta} Y \wedge X \wedge Y \xrightarrow{\varepsilon \wedge \mathrm{id}_Y} S \wedge Y \cong Y$$

are the identity.[31] From this and a diagram chase, you get some nice results.

Proposition 8.8. *If X and Y are dual, then there are isomorphisms $Y \xrightarrow{\cong} F(X,S)$ and $X \xrightarrow{\cong} DDX$.*

To paraphrase Lang, the best way to learn this is to prove all the statements without looking at the proofs, like all diagram chases.

If you like string calculus, you can think of these in terms of S- or Z-shaped diagrams. In this form, these results are sometimes known as the **Zorro lemmas**.

Another consequence of this formulation is that $- \wedge DX$ is right adjoint to $- \wedge X$, so by uniqueness of adjoints there's a natural isomorphism $- \wedge DX \cong F(X,-)$.

Atiyah duality. The Whitney theorem tells us that for any manifold M and sufficiently large n, there's an embedding $M \hookrightarrow \mathbb{R}^n$. This means we can compute the Spanier-Whitehead dual of a manifold, which is the setting of Atiyah duality. We'll assume M is compact.

By the tubular neighborhood theorem, there's an $\varepsilon > 0$ and a tubular neighborhood M_ε such that M_ε is the disc bundle of the normal bundle $\nu \to M$.[32]

Thom's thesis [Tho54] supplied an amazing connection between cobordism groups and the stable homotopy groups of the spheres by way of the **Pontrjagin-Thom map** $S^n \to \mathbb{R}^n/(\mathbb{R}^n \setminus M_\varepsilon)$ (heuristically, crushing everything outside M_ε), and $\mathbb{R}^n/(\mathbb{R}^n \setminus M_\varepsilon) \cong D(V)/S(V)$, the **Thom space** $T\nu$ of the normal bundle to M.

Thus, we get a sequence of maps

$$S^n \longrightarrow T\nu \longrightarrow T\nu \wedge M_+,$$

[31] In some presentations, this is how duality in a symmetric monoidal category is defined; the two approaches are equivalent.

[32] Let E be a vector bundle, and choose a metric on E; then, the **disc bundle** $D(E)$ is the subset of vectors with $\|x\| \le 1$, and the **sphere bundle** $S(E)$ is the subset of vectors with $\|x\| = 1$. These are fiber bundles with fiber D^n and S^{n-1}, respectively.

whose composition is called the **Thom diagonal**. This should look remarkably like the duality map η. To construct ε, let $s\colon M \to \nu$ be the zero section; then, the composite

$$M \xrightarrow{\Delta} M \times M \xrightarrow{s\times \mathrm{id}} \nu \times M.$$

has trivial normal bundle, and the Pontrjagin-Thom construction yields a map

$$T\nu \wedge M_+ \longrightarrow \Sigma^n M_+,$$

and projecting M to a point, we get

$$\varepsilon\colon T\nu \wedge M_+ \longrightarrow S^n.$$

The following theorem is nice, but quite nontrivial, at least from this approach.

Theorem 8.9 (Atiyah duality)**.** *η and ε exhibit $T\nu$ and M_+ as n-dual in the S-category.*

We'd like $T\nu$ and $\Sigma^{-n}M_+$ to be dual, but we don't know how to show that yet. More accurately, let $\Sigma^\infty\colon \mathsf{Top} \to \mathsf{S}$ be the functor that sends spaces and maps to themselves, so it makes a little more sense to say that $\Sigma^\infty M_+$ and $\Sigma^{-n}\Sigma^\infty T\nu$ are dual in the S-category.

One surprising consequence is that the tangent bundle and normal bundle define stable homotopy invariants through the Thom spectrum, which raises the question of what $T\nu$ looks like for particular examples.

Another is that we can immediately prove Poincaré duality, assuming the Thom isomorphism theorem. Namely, we can establish an isomorphism

$$[DX, H\mathbb{Z}] \cong [S, X_+ \wedge H\mathbb{Z}]. \tag{8.10}$$

Here, we're using the corepresentability of cohomology: we know $H^n(-,\mathbb{Z})$ is represented by $K(\mathbb{Z},n)$, and stitch these together (somehow) into $H\mathbb{Z}$. So the left-hand side is $H^*(DX;\mathbb{Z})$, and the right-hand side is $H_*(X;\mathbb{Z})$.

Applying Theorem 8.9 to (8.10),

$$[\Sigma^{-n}T\nu, H\mathbb{Z}] \cong H^{m-*}(X;\mathbb{Z}).$$

If X is orientable, then the cohomology of the Thom spectrum is the same as the cohomology of X. The Thom isomorphism theorem establishes a degree shift that gets rid of the dependency on n, the dimension of ambient space.

The equivariant setting. We'd like to establish Atiyah duality for G/H.

Definition 8.11. A ***G*-manifold** is a manifold M with a G-action by smooth maps.

The theorems of differential topology that we need hold for G-manifolds. Namely, there is an equivariant tubular neighborhood theorem, etc. In the smooth case, this goes back to the work of Andrew Gleason in the 1930s, and in the PL case to Lashof in the 1950s. If you like manifold topology, these are really nice proofs to read, avoiding triangulation arguments.

We need one key fact.

Theorem 8.12 (Equivariant Whitney's theorem)**.** *Let M be a compact G-manifold. Then, there is a G-equivariant embedding $M \hookrightarrow V$, where V is some finite-dimensional real G-representation.*

Now we proceed as before: all of the arguments are exactly the same, including the equivariant Pontrjagin-Thom construction.

But this means that suspension has to be smashing with S^V, the representation sphere for V. So in order for manifolds to have duals (meaning, in order to establish Poincaré duality), we need an S-category whose morphisms are

$$\mathrm{Map}_S(X,Y) := \operatorname*{colim}_V \mathrm{Map}(\Sigma^V X, \Sigma^V Y),$$

where $\Sigma^V X := S^V \wedge X$. This is not sequential, but there is an equivariant Freudenthal suspension theorem that stabilizes it.

Poincaré duality is one of the oldest results in algebraic topology, and if you don't have it in your theory, what are you even doing? So to obtain Poincaré duality, *we are inexorably forced to smash with representation spheres that G-manifolds embed in*, namely finite-dimensional real representations. So many treatments of equivariant homotopy theory start with defining Σ^V and then they're off to the races, but this is why they're doing this. If you don't need Poincaré duality, then you could do something different.

As before, we want a diagram category akin to the orbit category, but we want Atiyah duality to manifest in it.

Question 8.13. In this setting, what replaces the orbit category?

The naïve choice is the functor category from $\mathscr{O}_G^{\mathrm{op}}$ to the equivariant S-category, but this doesn't see enough: it's like only choosing the trivial representation. This is nontrivial to see, though.

So we need extra structure in the orbit category, and this will come through extra structure in $\mathscr{O}_G$. We'll add extra maps between G/H and G/K called **transfer maps**.

Let M be a G-manifold embedded in some G-representation V, ν be its normal bundle, and τ be its tangent bundle. Then, $\tau \oplus \nu$ is trivial, so the Thom construction applied to it is just smashing with S^V. Thus, we obtain a sequence of maps

$$S^V \longrightarrow T\nu \longrightarrow T(\nu \oplus \tau) \longrightarrow S^V \wedge M_+. \tag{8.14}$$

Exercise 8.15. If you project $M \to *$, you get a map $S^V \to S^V$. Show that the degree of this map is the Euler characteristic $\chi(M)$.

If $K \subset H$ are subgroups of G, then (8.14) defines a map $S^V \to S^V \wedge (H/K)_+$, and we can induce this to get a map

$$(G/H)_+ \wedge S^V \longrightarrow (G/K)_+ \wedge S^V.$$

These are the additional maps we need. We'll go over them again next lecture, but these define the Burnside category, which suffices to define the equivariant stable category! What's really nice is how this derives from some of the oldest constructions and questions in homotopy theory: Thom's thesis can be considered the beginning of modern homotopy theory.

Lecture 9.

Transfers and the Burnside category: 2/9/17

Today, we're going to talk about the Burnside category; there are at least five definitions of it, and we'll provide one. We'll also talk about spectra; some things will be precise, and others will be made precise next time.

Recall that if M is a G-manifold, we can equivariantly embed it in a finite-dimensional real G-representation V, so that we obtain a sequence of maps $S^V \to T\nu \to T(\nu \oplus \tau) \cong S^V \wedge M_+ = \Sigma^V M_+$. (Here ν is the normal bundle of M and τ is the tangent bundle.) Thinking of this as a transfer $M \to *$, we'd like to do something similar for G/H. Namely, if $K \subset H$ are subgroups of G, we'd like to define a transfer map $G/H \to G/K$, a stable map (i.e. one that lives in $\operatorname{colim} \mathrm{Map}(\Sigma^V G/H, \Sigma^V G/K)$). This will be a "wrong-way" map.

Applying $G \times_H -$ to $H/K \to *$, we get a map $G/K \to G/H$. The transfer for $G/K \to G/H$ is via the transfer $H/K \to *$, induced up from a map $H/K \hookrightarrow V$. Here, V is a G-representation, and the embedding is H-equivariant, where H acts on V by restriction. Thus, we obtain a map $S^V \to S^V \wedge H/K$.

We want to understand how induction works on H-spaces whose structure has been defined by restriction from G-spaces.

Exercise 9.1. Show that if X is a G-space and $i_H^* X$ is the H-space defined by restriction, $G \wedge_H i_H^* X \cong G/H_+ \wedge X$.

So our map $G \wedge_H S^V \to G \wedge_H (S^V \wedge H/K_+)$ is identified with a map $G/H_+ \wedge S^V \to G/K_+ \wedge S^V$.

Consider the maps in $\mathrm{Hom}_{\mathscr{O}_G}(G/H_1, G/H_2)$ that are determined by subconjugacy, i.e. $G/H_1 \xrightarrow{\cong} G/g^{-1}H_1 g \to G/H_2$ (we assume $g^{-1}H_1 g \subset H_2$). There's a covariant functor from $\mathscr{O}_G$ to the (equivariant) S-category which sends a map f to its representative in the colimit, and we also have a contravariant functor: $G/H_1 \to G/H_2$ takes the transfer map, then the inverse of the conjugation isomorphism.

Definition 9.2. The **Burnside category** B_G is the full subcategory of the S-category spanned by the orbits G/H.

It's a theorem that every map in B_G is a composition of two maps in the images of the two functors $\mathscr{O}_G \to \mathsf{S}$ described above, which is pretty neat.

The Burnside category is enriched in Top, because the S-category is.

Definition 9.3. The **algebraic Burnside category** is $\pi_0(B_G)$, which is enriched in Ab. A **Mackey functor** is a functor $\pi_0 B_G \to \mathsf{Ab}$.

Mackey functors will be our replacement for coefficient systems. They arise in other contexts, so you could care about Mackey functors without caring about equivariant stable homotopy theory (well, you probably secretly do anyways). For that reason, there are definitions of Mackey functors that are less homotopical.

We'd like to define the category of G-spectra to be "spectral Mackey functors," i.e. functors from B_G to spectra. We need to develop tools for this first: we haven't developed a good model for spectra (in particular, a point-set model, not a model of the stable homotopy category), and we want these functors to be enriched, which will require some more work. We'll now begin to develop the tools to surmount these problems.

Recollections about transfers. One concrete example of a transfer map comes from a finite cover $\widetilde{X} \to X$: summing over the fibers defines a transfer map $H^*(\widetilde{X};\mathbb{Q}) \to H^*(X;\mathbb{Q})$.

Transfers also arise in group cohomology $H^*(G;M)$, where M is a G-module. This is computed by resolving $\mathbb{Z}$ as a $\mathbb{Z}[G]$-module by some resolution $P_\bullet$, then computing $H^*(\mathrm{Hom}_G(P_\bullet, M))$. Now suppose $H \subset G$; there are two adjoints to the restriction functor $\mathsf{Mod}_G \to \mathsf{Mod}_H$.

- The left adjoint sends an H-module $N \mapsto \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N$. This is called the **induced G-module** $\mathrm{Ind}_H^G N$.
- The right adjoint sends $N \mapsto \mathrm{Map}_H(\mathbb{Z}[G], N)$. This is called the **coinduced G-module**, written $\mathrm{CoInd}_H^G N$.

These definitions, and their names, should look familiar.

Now suppose $H \subset G$ is finite-index, e.g. when G is a finite group. Then

$$\mathrm{Ind}_H^G N \cong \bigoplus_{g\in G/H} gN \qquad \text{and } \mathrm{CoInd}_H^G N \cong \prod_{g\in G/H} gN.$$

In an additive category, finite sums and finite products are isomorphic, so $\mathrm{Ind}_H^G N \cong \mathrm{CoInd}_H^G N$. When G is finite, the analogous statement about TODO certain sums and products uniquely characterizes the equivariant stable category.[33]

So if $H \subset G$, we'd like to build a transfer map $H^*(H;M) \to H^*(G;M)$, where M is a G-module made an H-module by restricting. Induction is left adjoint to the restriction functor i_H^*, so the counit of the adjunction is a map $\eta\colon \mathrm{Ind}_H^G i_H^* M \to M$. In particular, $H^*(H;M) \cong H^*(G;\mathrm{Ind}_H^G M) \xrightarrow{\eta} H^*(G;M)$, and this is the desired transfer map.

As group cohomology $H^*G;M)$ is the cohomology of BG in the local system defined by M, you could ask whether the transfer map is induced from a map $BG \to BH$ (in Top). This is true, and the map is a fibration.

Diagram spectra. This section is motivated by [MMSS01], an excellent paper that constructs the point-set stable category using diagram spectra. You should absolutely read this paper; it's a masterwork of exposition and making things look simple and clear in retrospect.

The approach of diagram spectra is different from, but equivalent to, the approaches taken in [LMS86, May96]. Our goal is to define a complete, cocomplete, symmetric monoidal category Sp such that

- the S-category is a full subcategory of Sp, and
- there is a symmetric monoidal functor $\Sigma^\infty\colon \mathsf{Top} \to \mathsf{Sp}$, which is left adjoint to a right adjoint $\Omega^\infty\colon \mathsf{Sp} \to \mathsf{Top}$.

There's a sense in which Sp is the smallest category satisfying these hypotheses, or that you get it by adding limits and colimits to S. In particular, we are constructing a stable analogue of the category of topological spaces, *not* its homotopy category.

Remark. Historically, Boardman constructed the stable homotopy category as a formal completion of the S-category. Then, people tried to find "point-set models," stable model categories whose homotopy categories are isomorphic to Boardman's category. There are several options, but explicit proofs that their homotopy categories are equivalent to Boardman's are rare in the literature. ◂

Definition 9.4. By a **diagram** D we mean a small category, which we assume is enriched in Top and symmetric monoidal. The category of D**-spaces** is the category $\mathsf{Fun}(\mathsf{D}, \mathsf{Top})$ of enriched functors.

The category of D-spaces is symmetric monoidal under **Day convolution**. The idea is to build a symmetric monoidal product via left Kan extension: if F and G are D-spaces, the functor $F \overline{\wedge} G\colon (d_1, d_2) \mapsto F(d_1) \wedge G(d_2)$ is a

[33] This is an instance of some life advice from Mike Hill: in order to avoid being confused by constructions in equivariant stable homotopy theory, try computing the analogous construction in group cohomology first. Group cohomology is very concrete, and so it's definitely worth thinking this through in this case.

$(\mathsf{D}\times\mathsf{D})$-space. To produce a D-space from this, let $\boxtimes\colon \mathsf{D}\to\mathsf{D}$ be the monoidal product on D, and consider the left Kan extension

$$\begin{tikzcd} \mathsf{D}\times\mathsf{D} \arrow[r, "F\overline{\wedge}G"] \arrow[d, "\boxtimes"'] & \mathsf{Top} \\ \mathsf{D} \arrow[ur, dashed, "F\wedge G"'] & \end{tikzcd}$$

This is our symmetric monoidal product $F\wedge G$. More explicitly,

$$\begin{aligned}(F\wedge G)(z) &= \operatorname*{colim}_{x\boxtimes y=z} F(x)\wedge G(y)\\ &:= \int^{x,y\in\mathsf{D}} F(x)\wedge G(y)\wedge \mathsf{D}(x\boxtimes y, z).\end{aligned}$$

This looks like the usual convolution, and the analogy with harmonic analysis can be taken further, e.g. in an unpublished paper of Isaacson-Behrens.

For any $d\in\mathsf{D}$, there's an **evaluation** functor $\mathrm{Ev}_d\colon \mathsf{Fun}(\mathsf{D},\mathsf{Top}_*)\to\mathsf{Top}_*$ sending $X\mapsto X(d)$. It's adjoint to $F_d\colon \mathsf{Top}_*\to\mathsf{Fun}(\mathsf{D},\mathsf{Top}_*)$ defined by

$$(F_dA)(e) := \mathrm{Map}_{\mathsf{D}}(d,e)\wedge A_+.$$

The unit for the symmetric monoidal structure on $\mathsf{Fun}(\mathsf{D},\mathsf{Top})$ is F_0S^0.

Let R be a **commutative monoid object** in $\mathsf{Fun}(\mathsf{D},\mathsf{Top})$, which approximately means there are maps $F_0S^0\to R$, $S^0\to R(0)$, and a unital, associative, commutative map $R(d)\wedge R(e)\to R(d\boxtimes e)$. For example, the unital condition is that the composition

$$R(d)\wedge S^0 \longrightarrow R(d)\wedge R(0)\longrightarrow R(d\boxtimes 0)\cong R(d)$$

must be the identity.

In this case, we can define the category Mod_R of R**-modules** in $\mathsf{Fun}(\mathsf{D},\mathsf{Top})$, those D-spaces M with an action map $\mu\colon R\wedge M\to M$ (satisfying the usual conditions). This is also a symmetric monoidal category (this requires R to be commutative), defined in the same way as the tensor product of modules over a ring: $M\wedge_R N$ is the coequalizer

$$M\wedge R\wedge N \rightrightarrows M\wedge N\longrightarrow M\wedge_R N.$$

Example 9.5 (Prespectra)**.** Let $\mathsf{D}=\mathbb{N}$, with only the identity maps. This is symmetric monoidal under addition: $[m]\boxtimes[n]:=[m+n]$. The assignment $S_{\mathbb{N}}\colon [n]\mapsto S^n$ is a monoid in $\mathbb{N}$-spaces, and the category of $S_{\mathbb{N}}$-modules is classically called **prespectra**; the monoidal structure is the identification of $S^m\wedge S^n\cong S^{m+n}$.

Warning! $S_{\mathbb{N}}$ is *not* a commutative monoid! $S^n\wedge S^m \ncong S^m\wedge S^n$.

This was the cause of thirty years of pain and suffering in the community — they didn't know they were unhappy. People knew what the smash product should be on the homotopy category, and wanted a point-set model that's symmetric monoidal, unlike this example. ◀

Symmetric spectra are one answer, which we won't use in this class. If all of this had been stated in terms of the Day convolution from the get-go, people probably would have figured out symmetric spectra as early as the 1960s, but hindsight is always clearer, and here we are.

Example 9.6 (Orthogonal spectra)**.** Let $\mathscr{I}$ denote the category whose objects are finite-dimensional real inner product spaces V, and whose morphisms $\mathscr{I}(V,W)$ are the linear isometric isomorphisms $V\to W$.

In this category, $V\oplus W$ and $W\oplus V$ aren't equal, but are isomorphic, and the isomorphism between them is reflected in the flip between $S^n\wedge S^m$ and $S^m\wedge S^n$. In particular, the assignment $S_{\mathscr{I}}\colon V\mapsto S^V$ (the one-point compactification of V) is a *commutative* monoid, so the category of $S_{\mathscr{I}}$-modules is a symmetric monoidal category, called the category of **orthogonal spectra**. This is the model of the stable category that we will use. ◀

Example 9.7 ($\mathscr{W}$-spaces)**.** Let $\mathscr{W}$ be the category of finite CW complexes (with either all maps or cellular maps; it doesn't really matter). $\mathscr{W}$-spaces are already like spectra, in a sense, in that they're modules over the identity functor $i\colon \mathscr{W}\hookrightarrow\mathsf{Top}$. There's a map $\varphi\colon A\to\mathrm{Map}(B,A\wedge B)$ sending $a\mapsto(b\mapsto a\wedge b)$, so if F is a $\mathscr{W}$-space, we have a sequence of maps

$$A\xrightarrow{\varphi}\mathrm{Map}(B,A\wedge B)\longrightarrow \mathrm{Map}(F(B),F(A\wedge B)).$$

Taking its adjoint defines a map
$$A \wedge F(B) \longrightarrow F(A \wedge B),$$
so F is a module over i. ◀

The assignment $n \mapsto \mathbb{R}^n$ defines a functor $\mathbb{N} \to \mathscr{I}$, and therefore a functor from prespectra to orthogonal spectra; with the right model structures, this induces an equivalence of their homotopy category. Similarly, the assignment $V \mapsto S^V$ defines a functor $\mathscr{I} \to \mathscr{W}$, hence a functor from orthogonal spectra to $\mathscr{W}$-spaces, and this also will induce a homotopy equivalence.

Definition 9.8. A prespectrum is an **Ω-prespectrum** if for all n, $X_n \xrightarrow{\cong} \Omega^m X_{m+n}$.

Definition 9.9. If X is a prespectrum and $q \in \mathbb{Z}$, the q^{th} **homotopy group** of X is
$$\pi_q(X) := \operatorname*{colim}_n \pi_{n+q} X(n).$$
A **π_*-isomorphism** of prespectra is a map that induces an isomorphism on all homotopy groups.

Notice that negative homotopy groups exist, and may be nontrivial.

Remark. This is one approach to defining the stable category, and is not the only one. In [Ada74] (which is an excellent book), Adams uses a more naïve viewpoint of "cells first, maps later" which doesn't require such abstraction, but it would be a huge mess to prove that his model is complete or cocomplete. The diagram spectra approach rigidly separates point-set techniques (easy, but not as useful) from operations on the homotopy category (more useful, but harder), and this separation is often useful. The ∞-categorical perspective mashes it all together, which can be confusing, but is the only setting in which you can prove things such as the stable category being initial. ◀

We'll reintroduce G-actions soon, and this is pretty slick using orthogonal spectra: we can replace $\mathscr{I}$ with the category of finite-dimensional G-representations with invariant inner products. Orthogonal spectra also have a really nice homotopy theory relative to symmetric spectra (which have other advantages that don't apply as much to us).

Lecture 10.

Homotopy theory of spectra: 2/14/17

Today, we're going to discuss the homotopy theory of spectra in the nonequivariant setting. Last time, we discussed D-spaces, where D is a small topological category, and saw that the category of D-spaces is symmetric monoidal. We discussed a few examples:

- Consider $\mathsf{D} = \mathbb{N}$ with all maps isomorphisms, as in Example 9.5. There's a *non*commutative monoid $S_{\mathbb{N}} \colon n \mapsto S^n$, and the category of $S_{\mathbb{N}}$-modules is called prespectra $\mathsf{Sp}^{\mathbb{N}}$;[34] it's a monoidal category, but is *not* symmetric.
- In Example 9.6, we considered $\mathsf{D} = \mathscr{I}$, the category whose objects are real finite-dimensional inner product spaces and whose morphisms are the linear isometric isomorphisms. Here, the monoid $S_{\mathscr{I}} \colon V \mapsto S^V$ is commutative, and the category of $S_{\mathscr{I}}$-modules, called orthogonal spectra $\mathsf{Sp}^{\mathscr{I}}$, is symmetric monoidal.

Those are the main examples for us, but there are others:

- When D is the category of finite CW complexes, as in Example 9.7, we obtain $\mathscr{W}$-spaces.
- When D is the category of finite, based sets and based maps, as in Example 10.16, we recover Segal's Γ-spaces.

Because these diagram categories are presheaves on nice categories, they inherit some good properties from Top_*; in particular, they are complete and cocomplete, and limits and colimits may be taken pointwise. This is also true for categories of modules over monoids in D-spaces, though it requires more work to prove: computing colimits is a bit harder, just like how the free product of groups is more complicated than the direct prodict. Categories of rings are *not* bicomplete, though.

Definition 10.1. Let $f : X \to Y$ be a map of D-spaces (or prespectra or orthogonal spectra).

- f is a **level equivalence** if for all $d \in \mathsf{D}$, $f(d) \colon X(d) \xrightarrow{\simeq} Y(d)$ is a weak equivalence.

[34] There are many different choices of notation for diagram spectra, as seen in [MMSS01] and [MM02], with various fancy decorations.

- f is a **level fibration** if for all $d \in \mathsf{D}$, $f(d)$ is a fibration.

That is, f is a natural transformation, and it acts through weak equivalences (resp. fibrations).

Theorem 10.2. *The category of* D*-spaces has a model structure, called the* ***level model structure****, in which the weak equivalences are the level equivalences and the fibrations are level fibrations. Moreover, this model category is cofibrantly generated.*

Exercise 10.3. Starting with the usual model structure on Top_*, construct the level model structure.

The "cofibrantly generated" part means that cofibrant objects behave like CW complexes, and in particular there is a theory of cellular objects. If $F_d : \mathsf{Top}_* \to \mathsf{Fun}(\mathsf{D}, \mathsf{Top}_*)$ is the left adjoint to Ev_d we constructed last time, then the **generating cofibrations** are the maps $F_d(S^{n-1}_+ \to D^n_+)$ for each $n \geq 1$, and the **acyclic cofibrations** are $F_d(D^n_+ \to (D_n \times I)_+)$ for each $n \geq 1$.

Since all spaces are fibrant, all D-spaces are fibrant in the level model structure. The cofibrant objects are the retracts of **cellular objects**, which are built by iterated pullbacks

$$\begin{tikzcd} \bigvee F_d S^{n-1}_+ \arrow[r] \arrow[d] & X_n \arrow[d] \\ \bigvee F_d D^n_+ \arrow[r] & X_{n+1}. \end{tikzcd}$$

While this is all nice, it's not what we're looking for, as it contains no information about stable phenomena. It's like the category of spaces, just with more of them. For example, it's not even true that $X \to \Omega\Sigma X$ is a weak equivalence, which is important if you want Ω and Σ to be homotopy inverses.

Recall that we defined a π_*-isomorphim of prespectra to be a map $f : X \to Y$ such that $\pi_q f : \pi_q X \to \pi_q Y$ is an isomorphism for all q. We'll extend this to orthogonal spectra: let $U : \mathsf{Sp}^{\mathscr{I}} \to \mathsf{Sp}^{\mathbb{N}}$ be pullback by the map $[n] \mapsto \mathbb{R}^n$, i.e. $UX([n]) = X(\mathbb{R}^n)$. U is right adjoint to a left Kan extension $P : \mathsf{Sp}^{\mathbb{N}} \to \mathsf{Sp}^{\mathscr{I}}$.

Definition 10.4. A map of orthogonal spectra $f : X \to Y$ is a π_***-isomorphism** if $Uf : UX \to UY$ is a π_*-isomorphism of prespectra.

We also defined an Ω-spectrum in prespectra, or an Ω-prespectrum, to be a prespectrum where the adjoints to the structure maps $X_n \xrightarrow{\cong} \Omega^m X_{n+m}$ are homeomorphisms. This is a pretty rigid condition, and so Ω-prespectra have nice properties.

Definition 10.5. Similarly, we define an Ω**-spectrum** in orthogonal spectra to be an orthogonal spectrum X such that the adjoints to the structure maps $X(U) \xrightarrow{\cong} \Omega^V X(U \oplus V)$ are homeomorphisms.

Classically, there were prespectra and then there were spectra (or Ω-spectra), and you would use some "spectrification" functor that took a prespectrum and produced a spectrum of the same homotopy type. Turning the adjoint maps into homeomorphisms looks difficult and is, as it involves some categorical and point-set wizardry. If you like this stuff, check out the appendix of [LMS86]. The first point-set symmetric monoidal model for the stable category [EKMM97] relies on this and even more magic, both clever and surprising.

Since D-spaces are enriched over spaces, it's possible to tensor with the interval and therefore define homotopies of D-spaces in the same way as for spaces. As usual, $[X, Y]$ will denote the set of homotopy classes of maps $X \to Y$.

Definition 10.6. A **stable equivalence** of prespectra is a map $f : X \to Y$ such that for all Ω-prespectra Z, the induced map $[Y, Z] \to [X, Z]$ is an isomorphism.

Theorem 10.7. *There are* ***stable model structures*** *on the categories of* D*-spaces, prespectra, and orthogonal spectra in which the weak equivalences are stable equivalences. For* $\mathsf{Sp}^{\mathbb{N}}$ *and* $\mathsf{Sp}^{\mathscr{I}}$*, the stable equivalences are the same as the* π_**-isomorphisms.*

Remark. You can make the same construction for symmetric spectra, but the stable equivalences are not the same as π_*-isomorphisms, and homotopy groups are consequently finickier. This ultimately comes from the fact that quotients $\mathrm{O}(n+k)/\mathrm{O}(n)$ get more highly connected as n and k grow in a way that quotients of symmetric groups don't. In any case, since symmetric spectra don't behave so well in the equivariant case, we won't use them. ◁

The stable model structure does in fact manifest stable phenomena: the map $X \to \Omega\Sigma X$ is a weak equivalence.[35]

If X and Y are Ω-prespectra and $f : X \to Y$ is a level equivalence, then f is a π_*-isomorphism. That is, the weak equivalences in the stable model structure contain the weak equivalences in the level model structure, and it's possible to use **Bousfield localization** to obtain the stable model structure from the level model structure.

Let C be a model category and S be a set of maps in C.[36] Bousfield localization produces a new model structure $L_S\mathsf{C}$ on C in which the morphisms in S are weak equivalences. The homotopy category and homotopy (co)limits change, but all point-set phenomena remain the same. Localization is given by fibrant replacement.

Definition 10.8. Let C be a topologically enriched model category and S be as above.

- An **S-local object** in C is an object X such that for all $f : Y \to Z$ in S, the induced map
$$\mathrm{Map}_{\mathsf{C}}(Z, X) \xrightarrow{\simeq} \mathrm{Map}_{\mathsf{C}}(Y, X)$$
is a weak equivalence.
- A map $f : X \to Y$ is an **S-local equivalence** if for all S-local objects Z, the induced map
$$\mathrm{Map}_{\mathsf{C}}(Y, Z) \xrightarrow{\simeq} \mathrm{Map}_{\mathsf{C}}(X, Z)$$
is a weak equivalence.

The following theorem is due to many people, but Hirschhorn's formulation is particularly nice.

Theorem 10.9 ([CITE ME: Hirschhorn])**.** *Let C be a cofibrantly generated, Top-enriched model category. Then, the Bousfield localization $L_S\mathsf{C}$ always exists. Moreover, the weak equivalences are the S-local equivalences, the fibrant objects are the S-local objects, and the cofibrations are exactly those in C.*

Example 10.10. One way this is used is to localize the category of spectra such that the S-local equivalences are detected by $- \wedge H\mathbb{Q}$ or $- \wedge H\mathbb{F}_p$. This is a slick way to construct the rationalization or p-completion, respectively, and in particular makes the localization map functorial. Thus one obtains the rational (or p-completed) stable homotopy category. ◀

The stable model structure is obtained by Bousfield localization at the stable equivalences (Definition 10.6). It follows immediately that Ω-prespectra are local objects and stable equivaiences are weak equivalences.

Proposition 10.11. *The stable model structure is stable, i.e. $X \to \Omega\Sigma X$ is a π_*-isomorphism.*

Proof. On homotopy groups, this is asking for $\mathrm{colim}_n\, \pi_{q+n}X_n \to \mathrm{colim}_n\, \pi_{q+n}\Omega\Sigma X_n$ to be an isomorphism. But by the Freudenthal suspension theorem, these colimits stabilize to the same stable homotopy group. ⊠

This implies that when X is an Ω-prespectrum, $\pi_q X = \pi_q X_0$, and for $q < 0$, we can define $\pi_q X = \pi_0(X_{-q})$.

Theorem 10.12. *The adjunction $P : \mathsf{Sp}^{\mathbb{N}} \rightleftarrows \mathsf{Sp}^{\mathscr{I}} : U$ is a Quillen equivalence, and therefore $\mathrm{Ho}(\mathsf{Sp}^{\mathbb{N}}) \cong \mathrm{Ho}(\mathsf{Sp}^{\mathscr{I}})$ as triangulated categories.*

This homotopy category is called the stable category. In a sense, it's the original triangulated category. Suspension Σ is the shift functor, turning a sequence $X \xrightarrow{f} Y \to Cf$ (where Cf is the homotopy cofiber) into a fiber sequence $Ff \to X \to Y$.[37]

Another sense in which the stable category is stable is that both fiber and cofiber sequences induce long exact sequences of homotopy groups, instead of just fiber sequences.

We'd like to construct a Quillen adjunction $\Sigma^\infty : \mathsf{Top}_* \rightleftarrows \mathsf{Sp}^{\mathscr{I}} : \Omega^\infty$. Ω^∞ is just Ev_0, evaluating at the zero space. If F_d is the adjoint to Ev_d (so that $(F_dA)(e) = \mathrm{Map}_{\mathsf{D}}(d, e)_+ \wedge A$), then we can define $\Sigma^\infty A := F_0 A \wedge S$, where $S = S_{\mathbb{N}}$ for prespectra and $S = S_{\mathscr{I}}$ for orthogonal spectra.

If R is a monoid in D-spaces, the category of R-modules is equivalent to a category of diagram spaces over a more complicated diagram D_R. This is useful because diagrams are nice, and some things become less complicated. The recipe is that D_R is the category whose objects are the same as D and whose morphisms are

$$\mathrm{Map}_{\mathsf{D}_R}(d, e) = \mathrm{Map}_{\mathsf{Mod}_R}(F_d S^0 \wedge R, F_e S^0 \wedge R). \tag{10.13}$$

[35] If you like ∞-categories, you can say that the level model category is an ∞-category of presheaves, and this is different from the ∞-category of spectra, which is presented by the stable model category.

[36] We want S to contain the weak equivalences in C, but there are important set-theoretic issues. Often, one specifies that S contains a generating set (under filtered colimits) of the weak equivalences of C.

[37] TODO: I think I got something wrong.

That is, we take the suspension spectrum of the sphere shifted by d and that of the sphere shifted by e.

Exercise 10.14. Show that for $\mathsf{D} = \mathbb{N}$, the structure maps for prespectra come out of (10.13) for $R = S_{\mathbb{N}}$. (This is an adjunction game.) The same is true for orthogonal spectra and $S_{\mathscr{I}}$.

This feels like a Spanier-Whitehead trick, but constructs the right category. In particular, in D_R-spaces, Σ^∞ is the left adjoint to evaluating at 0.

It's possible to bootstrap this to define model categories of ring spectra, i.e. algebras over $S_{\mathscr{I}}$.

Definition 10.15. Let $\mathbb{T}\colon \mathsf{Sp}^{\mathscr{I}} \to \mathsf{Sp}^{\mathscr{I}}$ denote the free associative algebra monad, i.e.

$$\mathbb{T}X := \bigvee_{n\ge 0} X^{\wedge n},$$

so that the category of $\mathbb{T}$-algebras $\mathsf{Sp}^{\mathscr{I}}[\mathbb{T}]$ is the category of associative monoids in $\mathsf{Sp}^{\mathscr{I}}$. Similarly, let $\mathbb{P}\colon \mathsf{Sp}^{\mathscr{I}} \to \mathsf{Sp}^{\mathscr{I}}$ denote the free commutative algebra monad, so

$$\mathbb{P}X := \bigvee_{n\ge 0} X^{\wedge n}/\Sigma_n,$$

where Σ_n denotes the action of the symmetric group by permutations; thus, the category of $\mathbb{P}$-algebras $\mathsf{Sp}^{\mathscr{I}}[\mathbb{P}]$ is the category of commutative monoids in $\mathsf{Sp}^{\mathscr{I}}$ (i.e. commutative ring spectra).

The stable model structure on $\mathsf{Sp}^{\mathscr{I}}$ induces one on $\mathsf{Sp}^{\mathscr{I}}[\mathbb{T}]$, in which the weak equivalences and fibrations are detected by those in $\mathsf{Sp}^{\mathscr{I}}$. For $\mathsf{Sp}^{\mathscr{I}}[\mathbb{P}]$, the quotient makes things harder: it has a model structure where the weak equivalences are **positive equivalences** (so those which are detected by **positive Ω-spectra**, i.e. those where $X_0 \to \Omega X_1$ need not be an equivalence). But this still doesn't behave very well. Namely, if you try to set the theory up for $\mathbb{P}$ to be the same as that for $\mathbb{T}$, then you get that $\Omega^\infty\Sigma^\infty S^0$ is a commutative topological monoid. It's a fact that any commutative topological monoid is a product of Eilenberg-Mac Lane spaces, and that $\Omega^\infty\Sigma^\infty S^0$ has nontrivial k-invariants. There's a short paper by Lewis which addresses this [Lew91], showing that there are five very reasonable axioms for the stable category that can't all be true! So people decided to forget about letting Σ^∞ be left adjoint to evaluation at 0, and it's okay, if not perfect.

Example 10.16 (Γ-spaces)**.** Let D be the category of finite based sets and based maps, e.g. $n_+ = \{0, 1, \dots, n\}$ with 0 as the basepoint. D-spaces are called **Γ-spaces**, and agree with Segal's notion of Γ-spaces, which are defined differently. The multiplication comes from the map $\psi\colon 2_+ \to 1_+$ sending $1, 2 \mapsto 1$.

Let $d_i : n_+ \to 1_+$ send $j \mapsto \delta_{ij}$ (i.e. 1 if $i = j$, and 0 otherwise). A Γ-space is **special** if the induced map

$$X(n_+) \xrightarrow{\varphi_n} \prod_n X(1_+)$$

is a weak equivalence; it's **very special** if in addition the composition

$$X(1_+) \times X(1_+) \xleftarrow[\simeq]{\varphi_2} X(2_+) \xrightarrow{\psi_*} X(1_+)$$

induces a commutative monoid structure on $\pi_0 X(1_+)$.

Kan extension defines a functor from D to the category of finite CW complexes, and working with π_*-equivalences of these, one obtains a model structure on the category of Γ-spaces. This is Quillen equivalent to the category of **connective spectra**, i.e. those whose negative homotopy groups vanish. ◀

In the equivariant case, there's even more structure, and notions of "extra special" Γ-spaces. We'll spend most of our time on orthogonal G-spectra, which have nuances of their own, as we'll see starting next time.

Lecture 11.

The equivariant stable category: 2/16/17

"May your first talk be more peaceful."

Today, we'll discuss a model for the equivariant stable category, but first we'll say something about algebras and monads that went by quickly last time.

Last time, we introduced two monads $\mathbb{T}$ and $\mathbb{P}$ on $\mathsf{Sp}^{\mathscr{I}}$, defined as

$$\mathbb{T}X := \bigvee_{k\ge 0} X^{\wedge k} \qquad \mathbb{P}X := \bigvee_{k\ge 0} X^{\wedge k}/\Sigma_k.$$

Let's delve into this a little more.

Definition 11.1. A **monad** M on a category C is an endofunctor of C which is a monoid in the functor category $\mathsf{Fun}(\mathsf{C},\mathsf{C})$.

That is, there's a natural transformation $\mu\colon M^2 \to M$ and a unit, and μ is associative and unital in that the relevant diagrams commute.

There are lots of examples: we've already seen that if G is a group, the assignment $X \to G\times X$ is a monad. More generally, algebraic structures can usually be obtained monadically.

Definition 11.2. Let M be a monad on C. Then, the category $\mathsf{C}[M]$ of **algebras over** M is the category whose objects are pairs $X \in \mathsf{C}$ and structure maps $m\colon MX \to X$ satisfying associativity and unitality for M, and whose morphisms are the C-morphisms that are compatible with the structure maps.

The associativity diagram, for example, is

$$\begin{tikzcd} M^2X \arrow[r, "m"] \arrow[d, "\mu"] & MX \arrow[d, "m"] \\ MX \arrow[r, "m"] & X. \end{tikzcd}$$

We require this to commute.

Lots of structures are monadic, e.g. groups are algebras over the free group monad in Set, and similarly for abelian groups, rings, etc. Monads very generally come from free structures in algebra; they also arise from adjunctions: an adjunction $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ defines a monad GF, with the structure map defined by the unit map $G(FG)F \to GF$. Many monads arise in this way.

There is always a free-forgetful adjunction $F\colon \mathsf{C} \rightleftarrows \mathsf{C}[M] : U$, which has to do with the Barr-Beck theorem. Suppose C is a model category. We'd like to lift this to a model structure on $\mathsf{C}[M]$ — when does $\mathsf{C}[M]$ have a model structure where the weak equivalences are determined by the forgetful functor U?[38] There are two issues.

(1) Since C is complete, so is $\mathsf{C}[M]$: the arrows point the right way. But it's not always cocomplete.
(2) How do we define the model structure?

In order for $\mathsf{C}[M]$ to be cocomplete, we'll use a criterion about preserving certain colimits. Since monads tend to arise from adjunctions, the criterion definitely won't be true in general! There are seventeen versions of this criterion in Mac Lane, but we'll only need one, following Hopkins and McClure.

Definition 11.3. Let $f,g\colon X \rightrightarrows Y$ be two maps. A **reflexive coequalizer** is a coequalizer for f and g together with a simultaneous section $s\colon Y \to X$ for both f and g.

Exercise 11.4. Prove that if M preserves reflexive coequalizers, then $\mathsf{C}[M]$ is cocomplete. (This is hard, but worthwhile.)

Proposition 11.5 (Hopkins-McClure)**.** *Under very mild hypotheses, $\mathbb{T}$ and $\mathbb{P}$ preserve reflexive coequalizers.*

See [EKMM97] for a proof. $\mathsf{Sp}^{\mathscr{I}}$ satisfies these hypotheses, so we've addressed caveat (1).

Theorem 11.6 ([CITE ME: Schwede-Shipley])**.** *Under mild hypotheses, $\mathsf{C}[M]$ inherits a model structure from C, where the fibrations are detected by U, and the generating cofibrations are M applied to the generating cofibrations of C.*

This is a very general theorem; the hard step is constructing a nice enough filtration on pushouts.

Warning! These hypotheses are met for the associative monad $\mathbb{T}$, but are *not* met by the commutative monad $\mathbb{P}$! This is another formulation of Lewis' paradox: for commutative ring spectra, you have to change the underlying homotopy theory.

Remark. You might be used to thinking of these as the associative and commutative operads. Every operad determines a monad, and the operadic algebras become monadic algebras, but for $\mathbb{P}$ and $\mathbb{T}$, the explicit form of the monad makes it easier to analyze from the monadic viewpoint. ◀

[38]This is not how we defined the model structure on G-spaces, but we'll use it to define model structures on rings and module spectra.

Returning to diagram spectra, we've been putting a huge emphasis on strict symmetric monoidal structures, rather than just commutativity in the homotopy category. This is useful because it lets you do algebra: if R is a ring in the homotopy category, it's very hard to control the category of modules, e.g. the cofiber of a map of modules may not even be a module, the cyclic bar construction isn't a simplicial object, etc. Some people have tried to use operads to fix this, and this is extremely hard: operadic ring spectra are fine, but their modules are not. In a sense, Lurie's ∞-categorical machinery is designed to do this in a more modern way.

A lot of modern homotopy theory has been importing algebraic constructions about rings into homotopy theory, replacing tensor products with smash products. This has been very useful, yet can be hard, and everything is much more tractable with a point-set symmetric monoidal structure.

G-spectra. In this context, we must talk about universes, making choices of which orbits G/H are dualizable. This is closely related to choosing representations V such that $G/H \hookrightarrow V$ equivariantly.

Definition 11.7. A **universe** is a countably infinite-dimensional real inner product space with a G-action through isometries. There is some collection R of irreducible representations such that U contains countably many copies of each irreducible in R, and R always has all of the trivial representations.[39]

Sometimes we'll ask for more structure, such as R being closed under tensor products. The inclusion of the trivial representation is what makes this a strict generalization of ordinary stable homotopy theory: for example, if $U = \mathbb{R}^\infty$ with the trivial G-action, the homotopy groups are the nonequivariant homotopy groups.[40] Another common choice of U is a **complete universe**, containing all irreducible representations infinitely often, so all orbits are dualizable.

G-prespectra were defined classically in a manner similar to prespectra. They mix point-set and homotopical data, which is a little odd.

Definition 11.8.

- A **G-prespectrum** X is the data of a G-space $X(V)$ for every finite-dimensional subspace $V \subset U$ and G-maps $S^W \wedge X(V) \to X(V \oplus W)$ for each pair V and W, such that the structure maps are associative.
- A G-prespectrum is an **Ω-prespectrum** if for all V and W, the adjoint to the structure map is a homeomorphism: $X(V) \xrightarrow{\cong} \Omega^W X(V \oplus W)$.
- The **homotopy groups** of a G-prespectrum are
$$\pi_q^H(X) := \operatorname*{colim}_V \pi_q^H \Omega^V X(V)$$
$$\pi_{-q}^H(X) := \operatorname*{colim}_{\substack{V \supset \mathbb{R}^m \\ m>0}} \pi_0^H \Omega^{V-\mathbb{R}^m} X(V).$$
Here $q \geq 0$, $\mathbb{R}^m$ denotes the trivial representation of dimension m, which we specified was in U, and $V - \mathbb{R}^n$ denotes the orthogonal complement.

Proposition 11.9. *There is a model structure on G-prespectra where the weak equivalences are the stable equivalences (equivalently, π_*-isomorphisms).*

The proof is identical to the nonequivariant case. One cool aspect of this is that homotopy groups are determined by trivial representations, and we'll see that for orthogonal G-spectra, this is also true at the point-set level. It's nice to not have to carry around the entire universe, just the trivial representations.

Definition 11.10. The **equivariant stable category** (**structured by** U) is the homotopy category of G-prespectra.

This is the category in which we have Poincaré duality for the orbits G/H that embed in U. The functors $\Sigma^V := S^V \wedge -$ and $\Omega^V := \mathrm{Map}(S^V, -)$ are inverse equivalences.

There's an equivariant triangulated structure on the homotopy theory; this hasn't been written down, because it's messy and not super useful. The idea is that a cofiber sequence induces a long exact sequence of equivariant homotopy groups, where the shift is any choice of Σ^V.

[39] Ideally, U should not contain other representations. Sometimes people are vague about this, where there may be other representations and you ignore them, but it's better to just not have them.

[40] More explicitly, this is a stabilization of the category of presheaves on BG. Depending on your definitions, this is more or less a tautology — stabilization in ∞-categories is defined by taking spectrum objects.

Orthogonal G-spectra. Let's build a point-set model of the equivariant stable category. Let V and V' be finite-dimensional irreducible G-representations (by which we mean finite-dimensional subspaces of U), and let $I_G(V,V')$ be the linear, G-equivariant isometries.[41]

Definition 11.11. The **complement bundle** $E(V,V')$ is the subbundle of the product bundle $I_G(V,V')\times V'$ consisting of pairs (f,x) such that $x\in V'-f(V)$. Let $\widetilde{I}_G(V,V')$ denote the Thom space of this bundle.

We're trying to create spheres: $\widetilde{I}_G(V,V')$ should topologize $S^{V'-f(V)}$. For example, if $V\subset V'$, this is isomorphic to $\mathrm{O}(V')\wedge_{\mathrm{O}(V'-V)} S^{V'-V}$.

Let $\widetilde{I}_G$ denote the category whose objects are finite-dimensional subspaces V of U and whose morphisms are $\widetilde{I}_G(V,V')$.

Definition 11.12. An **orthogonal G-spectrum** is a Top-enriched functor $\widetilde{I}_G\to G\mathsf{Top}$. The category of orthogonal G-spectra is denoted Sp^G, or Sp^G_U if the universe needs to be explicit.

For the nonequivariant case, we defined D-spaces and spectra to be modules over a certain monoid in them, and then showed that you could think of them as diagram spaces for a more complicated diagram. Here, we've done this all at once — it's simpler to define $\widetilde{I}_G$ than find a module in a simpler category of diagram spaces.

There is a forgetful functor U from Sp^G to the category of G-prespectra; we define a map f in Sp^G to be a **π_*-isomorphism** if Uf is a π_*-isomorphism. Sp^G is symmetric monoidal under Day convolution, just as before.

Theorem 11.13. Sp^G *has a stable model structure in which the weak equivalences are π_*-isomorphisms, the fibrations are levelwise fibrations, and the generating cofibrations are*

$$F_V\big((G/H\times S^{n-1})_+\longrightarrow (G/H\times D^n)_+\big).$$

Here, F_V is adjoint to evaluation at V, as before.

Moreover, the smash product is compatible with the model structure. If X is cofibrant, then $X\wedge Y$ computes Tor $X\wedge^{\mathbf{L}} Y$. Another way to say this is that if X is cofibrant, $X\wedge -$ preserves weak equivalences. The proof is where cofibrantly generated model categories: $X\wedge -$ is a colimit, so it commutes with colimits, and cofibrant objects are retracts of cell complexes, so you can reduce to the case where X is a single cell, which admits a direct proof. Once again the stable homotopy category behaves like homological algebra.

Functors on Sp^G. Let $H\subset G$ be a subgroup, so restriction defines a functor $i_H\colon \mathsf{Sp}^G\to\mathsf{Sp}^H$. Just as for G-spaces, i_H has a left adjoint $G_+\wedge_H -$ and a right adjoint $F_H(G_+,-)$, and these are constructed space-wise.

Proposition 11.14. *$G_+\wedge_H -$ and $F_H(G_+,-)$ are left (resp. right) Quillen functors.*

Remark. You might worry what universe you end up in after applying i_H — naïvely you get the H-representations that arise from restriction of G-representations, but you might want all H-representations. The way to address this is to use a change-of-universe functor, which we'll mention shortly. This is a good thing to not be sloppy about, but is annoying, and is why people try to get the universe out of the point-set category. In any case, as in group cohomology, induction and coinduction are isomorphic through the **Wirthmüller isomorphism**. ◂

One thing that's nice to do with G-spectra is take fixed points. There are *three* different ways to do this, namely homotopy fixed points, categorical fixed points, and geometric fixed points, and keeping them separate in your head is important.

Sp^G is a *closed* symmetric monoidal category, meaning it has internal function objects. This is true for D-spaces in general, with internal function objects

$$F_{\mathsf{D}}(X,Y)(n):=F(X,Y(n)).$$

This is exactly like the mapping complex of two chain complexes: you're looking at chain maps from X to shifts of Y.

Sp^G is tensored and cotensored over G-spaces, meaning we can smash G-spectra with G-spaces, and take function objects from G-spaces to G-spectra: if A is a G-space, tensoring with A is $A_+\wedge -$ applied levelwise, and cotensoring with A is $F(A_+,-)$ applied levelwise.

Definition 11.15.

[41] Unlike the nonequivariant case, we're not taking isometric isomorphisms, so maps in $I_G(V,V')$ must be injective, but may have nontrivial cokernel.

- The **homotopy fixed points** is the functor $X \mapsto F(EG_+, X)^G$. (Here, EG_+ is a G-space, so we're using the cotensor of G-spaces and G-spectra.)
- The **categorical fixed points** is the functor $(-)^H : X \mapsto F(G/H_+, X)$. As the notation suggests, this is analogous to $(-)^H$ on G-spaces.

Warning! You might hope that $(\Sigma^\infty A_+)^H = \Sigma^\infty A_+^H$, so that fixed points of spaces lead to fixed points of spectra, but this is *not* true. It has a more complicated description called **tom Dieck splitting** as a wedge of other pieces.

Definition 11.16. The **geometric fixed points**, denoted Φ^H or also X^{gH}, is the unique functor such that

(1) it's (derived) symmetric monoidal: if X and Y are cofibrant, $\Phi^H(X) \wedge \Phi^H(Y) \xrightarrow{\cong} \Phi^H(X \wedge Y)$.
(2) Φ^H preserves homotopy colimits.
(3) $\Phi^H \Sigma^\infty X \cong \Sigma^\infty X^H$.

Of course, uniqueness (up to a contractible space of choices) is a theorem, but it's true. We'll also give a preferred construction.

We'll prove that categorical and geometric fixed points detect weak equivalences.

Theorem 11.17.

(1) *A map $X \to Y$ in Sp^G is a weak equivalence iff $X^H \to Y^H$ is a weak equivalence for all subgroups $H \subset G$.*
(2) *A map $X \to Y$ in Sp^G is a weak equivalence iff $\Phi^H X \to \Phi^H Y$ is a weak equivalence for all subgroups $H \subset G$.*

This leads one to consider diagrams on categorical or geometric fixed points: because of this perspective, one can define Tate spectra using this philosophy, and this is particularly useful when applied to S^1-spectra. It was originally developed by John Greenlees, and has recently been repopularized and is being mined for applications, such as Nikolaus-Scholze's description of topological cyclic homology.

We're going to try to understand the equivariant stable category from a few other perspectives: the Wirthmüller isomorphism is an analogue of the isomorphism between finite products and coproducts in spectra, and we'll also learn more about tom Dieck splitting. You can also ask how the transfer maps we talked about characterize the equivariant stable category.

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