http://dx.doi.org/10.1090/conm/158/01458

Contemporary Mathematics
Volume 158, 1994

On the Classifying Spaces of $SL_3(\mathbb{Z})$, $St_3(\mathbb{Z})$ and Finite Chevalley Groups

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Dedicated to Professor Masahiro Sugawara on his 60th birthday

ABSTRACT. Mod-p local homotopy of the classifying spaces of $SL_3(\mathbb{Z})$, $St_3(\mathbb{Z})$ and Finite Chevalley Groups is studied.

Tists building is used in the last case, which analysis and results of Soulé are used to study the first two cases.

Amongst all, a complete stable splitting of $BSL_3(\mathbb{Z})$ is given. The result suggests that $SL_3(\mathbb{Z})$ behaves like finite groups, from the viewpoint of the stable homotopy theory.

§0. Introduction

In this paper we study classifying spaces of two different types of groups: infinite discrete groups ($SL_3(\mathbb{Z})$ and $St_3(\mathbb{Z})$), and finite groups with a split (B, N)-pair of characteristic p and their central extensions.

The starting point is the paper of Soulé **[S]**, where he completely calculated the cohomology of $SL_3(\mathbb{Z})$ and $St_3(\mathbb{Z})$. In both cases, the constant coefficient positive dimensional cohomology was shown to be torsion, and furthermore only 2-torsion and 3-torsion are present. As far as 3-local problem concerns, Corollary (i) of p. 9 in **[S]** exhibits a map

$$B\Sigma_3 \vee BD_{12} \to BSL_3(\mathbb{Z}),$$

which induces an isomorphism of mod 3 cohomology. I should warn that our notation of the relevant finite groups differs from that of **[S]**. For our notation, see the list at the end of this section. Anyway, since $(BD_{12})_3^\wedge \simeq (B\Sigma_3)_3^\wedge$ and

1991 *Mathematics Subject Classification.* Primary 55R35, 11F06, 20E42; Secondary 20E08, 20H05, 55P42.

Research at MSRI supported in part by NSF Grant #8505550.

This paper was originally distributed as MSRI preprint series 07025-90 (1990).

This paper is in final form and no version of it will be submitted for publication elsewhere

$(BSt_3(\mathbb{Z}))^\wedge_3 \simeq (BSL_3(\mathbb{Z}))^\wedge_3$ ($(?)^\wedge_p$ is the p-adic completion of the space (?) in the sense of [**BK**]), 3-local problem is completely solved (see [**B**] also). Therefore, we concentrate on the more complicated remaining 2-local case. For both $BSL_3(\mathbb{Z})$ and $BSt_3(\mathbb{Z})$, we construct very simple 2-local models as amalgam products of finite groups (Corollary 3.2). Furthermore we have a stable splitting of $BSL_3(\mathbb{Z})$, using summands of finite groups (Therorem 3.4).

Actually, our study on p-split (B, N)-pair and its central extensions was motivated by that of $SL_3(\mathbb{Z})$ and $St_3(\mathbb{Z})$. Our other main theorem (Theorem 2.1) permits us to express p-locally the classifying space using the proper parabolic subgroups. The idea is to use the Tits building and its cohomological property (Solomon's theorem, Theorem 2.2).

In §1, we briefly review the geometric part of [**S**] and present it in a suitable way for our purpose. In §2, we give a p-local approximation of the classifying space of p-split (B, N)-pair and its central extensions. In §3, we give 2-local analysis of $BSL_3(\mathbb{Z})$ and $BSt_3(\mathbb{Z})$, using §1 and (the easiest case of) §2.

The author would like to express his gratitude to N. Yagita for explaining his calculation [**Y**] and thus turning the author's attention to the paper of Soulé [**S**]. Though the calculation of [**Y**] is a very complicated K-theoretic adaptation of Soulé's cohomological calculation [**S**], our Theorem 3.4 gives the additive result instantly and our Corollary 3.2 gives a much faster way of obtaining the multiplicative result. Thanks are also due to S. Priddy for informing the author of his unpublished (cohomology-based) result, i.e. 2-local approximation of $BSL_3(\mathbb{F}_2)$ using BD_8 and $2B\Sigma_4$'s (see the remark at the end of §2). His result became the starting point of our study in §2.

In this paper, we frequently quote the notations and the definitions from [**S**], except our notations of the relevant groups differs from that of [**S**]. Because of this, we list our notations as follows:

NOTATION.
D_n: dihedral group of order n.
Σ_n: symmetric group of n letters.
$\mathbb{Z}_n$: cyclic groups of order n.

For $H \subset SL_3(\mathbb{R})$, $\tilde{H}$ means the inverse image of H by the canonical projection

$$St_3(\mathbb{R}) \to SL_3(\mathbb{R}).$$

Even though this expression is not rigorous, this should not cause any confusion.

§1. Review of Soulé's geometry

As is usual the case of arithmetic groups, we should find and study some Γ-space X so that

$$E\Gamma \times_\Gamma X \to B\Gamma$$

gives a nice approximation to study the classifying space of arithmetic group Γ. Of course, we are thinking of the case $\Gamma = SL_3(\mathbb{Z})$ and $\tilde{\Gamma} = St_3(\mathbb{Z})$ and we denote so throughout this paper. Soulé's [**S**] 2-local analysis of $B\Gamma$ (and $B\tilde{\Gamma}$) consists of two steps.

Soulé first found a contractible 3-dimensional sub Γ-complex X_3 inside the standard $X_1 = SL_3(\mathbb{R})/SO_3(\mathbb{R})$. What he did was a generalization of what Serre did for $SL_2(\mathbb{Z})$ [**Se**], i.e. constructing a tree inside the upper half plane $= SL_2(\mathbb{R})/SO_2(\mathbb{R})$. The key in this difficult first step was to construct a deformation retraction of X_1 onto X_3 (Theorem 1 of [**S**]) (and thus to prove the contractibility of X_3). For this purpose, Soulé used Minkowski's results on $SL_n(\mathbb{Z})$-fundamental domain in $SL_n(\mathbb{R})/SO_n(\mathbb{R})$ [**M**], and the geodesic action of Borel–Serre [**BS**]. Anyway, by this step, we get a homotopy equivalence:

$$E\Gamma \times_\Gamma X_3 \xrightarrow{\sim} B\Gamma.$$

Now X_3 is 3-dimensional and still too big to handle with. If our interest is just the constant coefficient mod 2 cohomology, we can get a better buy; this is Soulé's second step. Actually, Soulé (Theorem 4 of [**S**]) offers the following 1-dimensional Γ-subspace X_4 of X_3:

$$X_4 = \Gamma \cdot \left(\overset{P}{\bullet}\!\!\text{———}\!\!\overset{M'}{\bullet}\!\!\text{———}\!\!\overset{0}{\bullet} \right) \subset X_3.$$

Warning: The notation X_4 was not used in [**S**]. Here $\overset{P}{\bullet}\!\!\text{——}\!\!\overset{M'}{\bullet}\!\!\text{——}\!\!\overset{0}{\bullet}$ is inside a Γ-fundamental domain in X_3, and the isotropy subgroups of the vertices $P, M', 0$ and the edges $PM', M'0$ can be listed as follows (the first column is the vertex or the edge, the second column gives the generator of the stabilizer of the first column, and the third gives the name of the stabilizer):

$$\begin{array}{llll}
P & \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} & \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & \Sigma_4 \\
M' & \begin{pmatrix} 0 & -1 & -1 \\ 0 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} & \Sigma_4 \\
0 & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} & \Sigma_4 \\
PM' & \begin{pmatrix} 0 & -1 & -1 \\ 0 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} & D_8 \\
M'0 & \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} & & \mathbb{Z}_2
\end{array}$$

Again, our notation of finite groups differs from that of [**S**] (see the notation at the end of §0). From this, the isotropy subgroups can be characterized as follows:

$$P \quad M' \quad 0$$

(*)
$$\begin{array}{ccccccc} \Sigma_4 & & \Sigma_4 & & \Sigma_4 \\ i_2 \nwarrow & & \nearrow i_1 \ j_1 \nwarrow & & \nearrow j_2 , \\ & D_8 & & \mathbb{Z}_2 & \end{array}$$

where, in particular, j_1 (resp. j_2) is (resp. is not) an embedding into the alternating group A_4.

Therefore as a Γ-space, X_4 is obtained from

$$\Gamma/\Sigma_4 \amalg (\Gamma/D_8 \times I) \amalg \Gamma/\Sigma_4 \amalg (\Gamma/\mathbb{Z}_2 \times I) \amalg \Gamma/\Sigma_4,$$

(where $I = [0, 1]$ the unit interval), by identifying $\Gamma/D_8 \times \{0\}$ (resp. $\Gamma/D_8 \times \{1\}$) with the left (resp. right) Γ/Σ_4 using i_2 (resp. i_1), and $\Gamma/\mathbb{Z}_2 \times \{0\}$ (resp. $\Gamma/\mathbb{Z}_2 \times \{1\}$) with the middle (resp. right) Γ/Σ_4 using j_1 (resp. j_2). But it is more conceptual to express as

$$X_4 = \xrightarrow{\text{hocolim}} \left(\begin{array}{ccccc} \Gamma/\Sigma_4 & & \Gamma/\Sigma_4 & & \Gamma/\Sigma_4 \\ \bar{i}_2 \nwarrow & & \nearrow \bar{i}_1 \ \bar{j}_1 \nwarrow & & \nearrow \bar{j}_2 \\ & \Gamma/D_8 & & \Gamma/\mathbb{Z}_2 & \end{array} \right).$$

For $\xrightarrow{\text{hocolim}}$, we refer [**BK**] [**JMO**]. Since $\xrightarrow{\text{hocolim}}$ commutes with the Borel construction, we immediately get

$$E\Gamma \times_\Gamma X_4 = \xrightarrow{\text{hocolim}} \left(\begin{array}{ccccc} B\Sigma_4 & & B\Sigma_4 & & B\Sigma_4 \\ Bi_2 \nwarrow & & \nearrow Bi_1 \ Bj_1 \nwarrow & & \nearrow Bj_2 \\ & BD_8 & & B\mathbb{Z}_2 & \end{array} \right).$$

To complete this second step, Soulé showed that the canonical map

$$E\Gamma \times_\Gamma X_4 \to E\Gamma \times_\Gamma X_3 \xrightarrow{\sim} B\Gamma$$

is a constant mod -2 coefficient cohomology isomorphism, using the Leray spectral sequence of sheaf cohomology.

As was noted in the same paper [**S**], the above analysis for $\Gamma = SL_3(\mathbb{Z})$ can be carried over to $\tilde{\Gamma} = St_3(\mathbb{Z})$. Actually $\tilde{\Gamma}$ acts on X_4 and X_3 through the projection $\tilde{\Gamma} \to \Gamma$, which describes $\tilde{\Gamma}$ as a central extension of Γ by $\mathbb{Z}_2$. This time, $\tilde{\Gamma}$-space description of X_4 is given by the following diagram

$$\begin{array}{ccccc} \tilde{\Sigma}_4 & & \tilde{\Sigma}_4 & & \tilde{\Sigma}_4 \\ \tilde{i}_2 \nwarrow & & \nearrow \tilde{i}_1 \ \tilde{j}_1 \nwarrow & & \nearrow \tilde{j}_2 \\ & \tilde{D}_8 & & \tilde{\mathbb{Z}}_2 = \mathbb{Z}_4 & \end{array}$$

which is obtained by lifting $(*)$ through the projection $\tilde{\Gamma} \to \Gamma$.

Anyway, Soulé's 2-local analysis of $B\Gamma = BSL_3(\mathbb{Z})$ and $B\tilde{\Gamma} = BSt_3(\mathbb{Z})$ can be summarized as follows (even though [**S**] does not state in this way):

THEOREM 1.1 ([**S**]). *There exist* mod -2 *(co)homology isomorphisms:*

$$\varphi : \xrightarrow{\text{hocolim}} \left(\begin{array}{ccccccc} B\Sigma_4 & & & B\Sigma_4 & & & B\Sigma_4 \\ Bi_2 \nwarrow & & \nearrow Bi_1 & & Bj_1 \nwarrow & & \nearrow Bj_2 \\ & BD_8 & & & & B\mathbb{Z}_2 & \end{array} \right) \to B\Gamma$$

and

$$\tilde{\varphi} : \xrightarrow{\text{hocolim}} \left(\begin{array}{ccccccc} B\tilde{\Sigma}_4 & & & B\tilde{\Sigma}_4 & & & B\tilde{\Sigma}_4 \\ B\tilde{i}_2 \nwarrow & & \nearrow B\tilde{i}_1 & & B\tilde{j}_1 \nwarrow & & \nearrow B\tilde{j}_2 \\ & B\tilde{D}_8 & & & & B\mathbb{Z}_4 & \end{array} \right) \to B\tilde{\Gamma}.$$

§2. Split (B, N)-pair

Let (G, B, N, S) be a finite split Tits system of characteristic p (see [**C**], for example). Let $\mathcal{C}$ be the poset of proper subsets of S. For each subset $I \subseteq S$, we put

$$W_I = \langle r \mid r \in I \rangle$$
$$P_I = BW_IB$$

as usual. Now we want to work in the general sitaution of the central extensions:

$$0 \to Z \to \hat{G} \xrightarrow{\pi} G \to 1,$$

where Z is some (possibly trivial) center of $\hat{G}$ and π is the quotient homomorphism. We put $\hat{P}_I = \pi^{-1}(P_I)$. The purpose of this section is to prove the following:

THEOREM 2.1. *The canonical map*

$$\xrightarrow[I \in \mathcal{C}]{\text{hocolim}} B\hat{P}_I \to B\hat{G}$$

is a mod p *(co)homology isomorphism.*

For the proof, we need the following well-known fact about Tits building T associated with G [**Ti**]:

THEOREM 2.2 (Solomon [**Sol**]). *As G-representations,*

$$H_*(T) = \begin{cases} Z & * = 0 \\ 0 & 0 < * < |S| - 1 \\ St & * = |S| - 1, \end{cases}$$

where St is the Steinberg representation [**St1**] *and therefore its restriction to B is the regular representation.*

The relevance of Theorem 2.2 to our Theorem 2.1 is manifested by the following:

LEMMA 2.3. *There is a G-homeomorphism between the Tits building T and* $\xrightarrow[I\in\mathcal{C}]{\text{hocolim}}(G/P_I)$.

PROOF. We are going to establish a sequence of G-homeomorphisms, as follows:

$$\begin{aligned} T &= |\text{the poset of simplices in } T| \\ &= |\text{the poset dual of simiplices in } T| \\ &= |\text{the poset of proper parabolic subgroups of } G| \\ &= \xrightarrow[I\in\mathcal{C}]{\text{hocolim}}(G/P_I). \end{aligned}$$

The first equality follows from (1.4) **[Q]**, and the second from (1.1) **[Q]**. The third follows from the construction of the Tits building **[Ti]**.

To explain the last equality, we let $F : \mathcal{C} \to G$-spaces denote the functor which assigns G/P_I to each $I \in \mathcal{C}$. We then form the category $\mathcal{C}_F$ as usual **[JMO]**:

$$\begin{aligned} \operatorname{Ob}\mathcal{C}_F &= \{(x,a) \mid x \in \operatorname{Ob}(\mathcal{C}),\ a \in F(x)\} \\ \operatorname{Mor}_{\mathcal{C}_F}((x,a),(y,b)) &= \{f \in \operatorname{Mor}_{\mathcal{C}}(x,y) \mid F(f)(a) = b\}. \end{aligned}$$

Then $\xrightarrow[I\in\mathcal{C}]{\text{hocolim}}(B/P_I)$ is nothing but the classifying space of the category $\mathcal{C}_F$. Now the upshot is that we can identify $\mathcal{C}_F$ with the category of the poset of the proper parabolic subgroups of G. So the last equality follows. We note that each equality is a G-homeomorphism, so the claim follows. □

PROOF OF THEOREM 2.1. Since

$$\begin{aligned} \xrightarrow[I\in\mathcal{C}]{\text{hocolim}} B\hat{P}_I &= \xrightarrow[I\in\mathcal{C}]{\text{hocolim}} E\hat{G} \times_{\hat{G}} (\hat{G}/\hat{P}_I) \\ &= \xrightarrow[I\in\mathcal{C}]{\text{hocolim}} E\hat{G} \times_{\hat{G}} (G/P_I) = E\hat{G} \times_{\hat{G}} \left(\xrightarrow[I\in\mathcal{C}]{\text{hocolim}} G/P_I\right) \\ &= E\hat{G} \times_{\hat{G}} T, \end{aligned}$$

where the last equality follows from Lemma 2.3, we only have to show that the canonical map

$$E\hat{G} \times_{\hat{G}} T \to E\hat{G} \times_{\hat{G}} * = B\hat{G}$$

is a mod -p (co)homology equivalence. For this, we use the Leray–Serre spectral sequence

$$E^2_{r,q} = H_r(B\hat{G}, H_q(T;\mathbb{F}_p)) \Rightarrow H_{r+q}(E\hat{G} \times_{\hat{G}} T;\mathbb{F}_p),$$

and we will show $E^2_{r,q} = 0$ for $q > 0$. From Theorem 2.2, we are done if we can show $H_r(\hat{G}, St \otimes \mathbb{F}_p) = 0$ for any r. Applying the Hochschild–Serre spectral sequence:

$$\begin{array}{l} E^2_{r,q} = H_r(G, H_q(Z, St \otimes \mathbb{F}_p)) \Rightarrow H_{r+q}(\hat{G}, St \otimes \mathbb{F}_p) \\ \qquad\qquad \| \\ \qquad\quad H_r(G, St \otimes H_q(Z, \mathbb{F}_p)) \\ \qquad\qquad \| \\ \qquad\quad H_r(G, St \otimes \mathbb{F}_p) \otimes H_q(Z, \mathbb{F}_p), \end{array}$$

we have reduced to the case $\hat{G} = G$. Now the vanishing for $r > 0$ follows from the $\mathbb{F}_p[G]$-projectivity of $St \otimes \mathbb{F}_p$, and the vanishing for $r = 0$, i.e. $\mathbb{F}_p \otimes_{\mathbb{F}_p[G]} (St \otimes \mathbb{F}_p) = 0$, is due to the fact that the Steinberg representation is a nontrivial irreducible module. □

REMARK. (i) As is clear from the proof, the map in Theorem 2.1 is actually an isomorphism of $H_*(\;; \mathbb{Z}[|G/B|^{-1}])$, not merely that of mod-p (co)homology.

(ii) For examples of central extensions $\hat{G}$, see **[St2]** for example.

(iii) The simplest case of Theorem 2.1, i.e. $p = 2$, $\hat{G} = G = GL(3, \mathbb{F}_2)$, was known to S. Priddy by cohomological calculations.

§3. $BSL_3(\mathbb{Z})$ and $BSt_3(\mathbb{Z})$

Our unstable main theorem is the following:

THEOREM 3.1. *There exist* mod -2 *(co)homology isomorphisms* $\pi, \tilde{\pi}, \varphi, \tilde{\varphi}$:

$$\xrightarrow{\text{hocolim}} \left(\begin{array}{ccccccc} B\Sigma_4 & & & B\Sigma_4 & & & B\Sigma_4 \\ & Bi_2 \nwarrow & & \nearrow Bi_1 \; Bj_1 \nwarrow & & \nearrow Bj_2 & \\ & & BD_8 & & BZ_2 & & \end{array} \right) \xrightarrow{\varphi} BSL_3(\mathbb{Z})$$

$$\downarrow \pi$$

$$\xrightarrow{\text{hocolim}} \left(\begin{array}{ccc} BSL_3(\mathbb{F}_2) & & B\Sigma_4 \\ & Bj \nwarrow \quad \nearrow Bj_2 & \\ & B\mathbb{Z}_2 & \end{array} \right)$$

$$\xrightarrow{\text{hocolim}} \left(\begin{array}{ccccccc} B\tilde{\Sigma}_4 & & & B\tilde{\Sigma}_4 & & & B\tilde{\Sigma}_4 \\ & B\tilde{i}_2 \nwarrow & & \nearrow B\tilde{i}_1 \; B\tilde{j}_1 \nwarrow & & \nearrow B\tilde{j}_2 & \\ & & B\tilde{D}_8 & & B\mathbb{Z}_4 & & \end{array} \right) \xrightarrow{\tilde{\varphi}} BSt_3(\mathbb{Z})$$

$$\downarrow \tilde{\pi}$$

$$\xrightarrow{\text{hocolim}} \left(\begin{array}{ccc} BSL_2(\mathbb{F}_7) & & B\tilde{\Sigma}_4 \\ & B\tilde{j} \nwarrow \quad \nearrow B\tilde{j}_2 & \\ & B\mathbb{Z}_4 & \end{array} \right)$$

Here φ and $\tilde{\varphi}$ were defined in Theorem 1.1, j (resp. j_2) sends the generator of $\mathbb{Z}_2$ to

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in SL_3(\mathbb{F}_2) (\simeq PSL_2\mathbb{F}_7))$$

(resp. an odd permutation of Σ_4*, say* (12) *up to conjugacy), and* π *is induced by*

$$\sigma : \xrightarrow{\text{hocolim}} \begin{pmatrix} B\Sigma_4 & & B\Sigma_4 \\ Bi_2 \nwarrow & & \nearrow Bi_1 \\ & BD_8 & \end{pmatrix} \to BSL_3(\mathbb{Z}) \to BSL_3(\mathbb{F}_2).$$

Furthermore, $\tilde{j}$ *is the lift of*

$$\mathbb{Z}_2 \xrightarrow{j} SL_3(\mathbb{F}_2) \simeq PSL_2(\mathbb{F}_7),$$

and $\tilde{\pi}$ *is induced by*

$$\tilde{\sigma} : \xrightarrow{\text{hocolim}} \begin{pmatrix} B\tilde{\Sigma}_4 & & B\tilde{\Sigma}_4 \\ B\tilde{i}_2 \nwarrow & & \nearrow B\tilde{i}_1 \\ & B\tilde{D}_8 & \end{pmatrix} \to BSL_2(\mathbb{F}_7),$$

the lift of σ*.*

COROLLARY 3.2.

$$BSL_3(\mathbb{Z})^{\wedge}_2 \simeq \xrightarrow{\text{hocolim}} \begin{pmatrix} BSL_3\mathbb{F}_2 & & B\Sigma_4 \\ Bj \nwarrow & & \nearrow Bj_2 \\ & B\mathbb{Z}_2 & \end{pmatrix}^{\wedge}_2$$

and

$$BSt_3(\mathbb{Z})^{\wedge}_2 \simeq \xrightarrow{\text{hocolim}} \begin{pmatrix} BSL_2\mathbb{F}_7 & & B\tilde{\Sigma}_4 \\ B\tilde{j} \nwarrow & & \nearrow B\tilde{j}_2 \\ & B\mathbb{Z}_4 & \end{pmatrix}^{\wedge}_2,$$

where $(?)^{\wedge}_2$ *is the* 2*-completion of the space* $(?)$ *in the sense of* [**BK**].

PROOF OF THEOREM 3.1. The claim about φ and $\tilde{\varphi}$ is nothing but Theorem 1.1. We have to show the well-definedness of $\tilde{j}$ and $\tilde{\sigma}$, but this will be postponed until after Corollary 3.3.

Now, assuming this, we show σ and $\tilde{\sigma}$ are mod -2 isomorphisms to guarantee the claim about π and $\tilde{\pi}$. The key observation is that the relevant system of subgroups of $SL_3(\mathbb{Z})$:

$$\begin{matrix} \Sigma_4 & & \Sigma_4 \\ i_2 \nwarrow & & \nearrow i_1 \\ & D_8 & \end{matrix},$$

when considered as a system of subgroups of $SL_3(\mathbb{F}_2)$ by the canonical map $SL_3(\mathbb{Z}) \to SL_3(\mathbb{F}_2)$ (we have to check this is possible, but it is trivial) is conjugate to the system of subgroups of $SL_3(\mathbb{F}_2)$:

$$\begin{matrix} P_{\{s_1\}} = \left\{\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}\right\} & & P_{\{s_2\}} = \left\{\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}\right\} \\ \nwarrow & & \nearrow \\ & P_\phi = \left\{\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}\right\} & \end{matrix}$$

by $\begin{pmatrix}1&0&0\\1&1&0\\0&1&1\end{pmatrix} \in SL_3(\mathbb{F}_2)$, where

$$s_1 = \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix} \text{ and } s_2 = \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}.$$

This is a consequence of an easy calculation, and so left to the reader. By this fact, we can apply Theorem 2.1 to find both σ and $\tilde{\sigma}$ are mod -2 isomorphisms.

Anyway we have finished the proof for $SL_3(\mathbb{Z})$, in which case we can go one step more:

COROLLARY 3.3. *We have the following cofibre sequences*

$$\begin{tikzcd} BSL_3(\mathbb{F}_2)^\wedge_2 \arrow[r, "s_1"] & BSL_3(\mathbb{Z})^\wedge_2 \arrow[l, bend right] \arrow[r] & (B\Sigma_4/B\mathbb{Z}_2)^\wedge_2 \arrow[d, equal] \\ (B\mathbb{Z}_2)^\wedge_2 \arrow[r, "s_2"] & (B\Sigma_4)^\wedge_2 \arrow[l, bend right] \arrow[r] & (B\Sigma_4/B\mathbb{Z}_2)^\wedge_2, \end{tikzcd}$$

where s_1 and s_2 are sections, each induced by the canonical map $SL_3(\mathbb{Z}) \to SL_3(\mathbb{F}_2)$ and the sign map $\Sigma_4 \to \mathbb{Z}_2$.

PROOF. The map $BSL_3(\mathbb{F}_2)^\wedge_2 \to BSL_3(\mathbb{Z})^\wedge_2$ is induced by

$$\xrightarrow{\text{hocolim}} \begin{pmatrix} B\Sigma_4 & & B\Sigma_4 \\ Bi_2 \nwarrow & & \nearrow Bi_1 \\ & BD_8 & \end{pmatrix}$$

$$\to \xrightarrow{\text{hocolim}} \begin{pmatrix} B\Sigma_4 & & B\Sigma_4 & & B\Sigma_4 \\ Bi_2 \nwarrow & & \nearrow Bi_1\ Bj_1 \nwarrow & & \nearrow Bj_2 \\ & BD_8 & & B\mathbb{Z}_2 & \end{pmatrix}.$$

Then we only have to show that s_1 is a section. But this follows from the fact σ is a mod -2 isomorphism (see the proof of Theorem 3.1). $\square$

COMPLETION OF THE PROOF OF THEOREM 3.1 FOR $St_3(\mathbb{Z})$. We have to show the well-definedness of $\tilde{j}$ and $\tilde{\sigma}$. Consider the following diagram

$$\begin{tikzcd} \xrightarrow{\text{hocolim}} \begin{pmatrix} B\Sigma_4 & & B\Sigma_4 \\ Bi_2 \nwarrow & & \nearrow Bi_1 \\ & BD_8 & \end{pmatrix}^\wedge_2 \arrow[r] \arrow[d, "\sigma"'] & BSL_3(\mathbb{Z})^\wedge_2 \arrow[d, equal] \\ BSL_3(\mathbb{F}_2)^\wedge_2 \arrow[r] & BSL_3(\mathbb{Z})^\wedge_2. \end{tikzcd}$$

This diagram commutes since the bottom horizontal arrow was defined by inverting σ. Now

$$\begin{matrix} \tilde{\Sigma}_4 & & \tilde{\Sigma}_4 \\ \tilde{i}_2 \nwarrow & & \nearrow \tilde{i}_1, \\ & \tilde{D}_8 & \end{matrix}$$

the system of subgroups of $St_3(\mathbb{Z})$, is the non-trivial central extensions of

$$\begin{array}{ccc} \Sigma_4 & & \Sigma_4 \\ i_2 \nwarrow & & \nearrow i_1 \\ & D_8 & \end{array},$$

by pulling back some element in $H^2(SL_3(\mathbb{Z});\mathbb{Z}_2)$. Then the commutativity of the above-mentioned diagram and $H^2(SL_3(\mathbb{F}_2);\mathbb{Z}_2) = \mathbb{Z}_2$ (see [**MP**] for example) imply that

$$\begin{array}{ccc} \tilde{\Sigma}_4 & & \tilde{\Sigma}_4 \\ \tilde{i}_2 \nwarrow & & \nearrow \tilde{i}_1 \\ & \tilde{D}_8 & \end{array},$$

is also induced by pulling back the non-trivial element in $H^2(SL_3(\mathbb{F}_2);\mathbb{Z}_2) = \mathbb{Z}_2$ by σ. This is nothing but the well-definedness of $\tilde{\sigma}$, which immediately imply that of $\tilde{j}$. □

Finally, we state our main stable splitting result:

THEOREM 3.4. (i) $\Sigma(BSL_3(\mathbb{Z}))_2^\wedge \simeq \Sigma(BSL_3(\mathbb{F}_2))_2^\wedge \vee \Sigma(B\Sigma_4/B\mathbb{Z}_2)_2^\wedge$
(ii) $\Sigma BSL_3(\mathbb{Z}) \simeq \Sigma(BSL_3(\mathbb{F}_2))_{\frac{1}{7}} \vee \Sigma(B\Sigma_4/B\mathbb{Z}_2)$
(iii) $\Sigma^\infty BSL_3(\mathbb{Z})_2^\wedge \simeq \Sigma^\infty BSL_3(\mathbb{F}_2)_2^\wedge \vee \Sigma^\infty BSL_3(\mathbb{F}_2)_2^\wedge \vee L(2)$
(iv) $\Sigma^\infty BSL_3(\mathbb{Z}) \simeq \Sigma^\infty BSL_3(\mathbb{F}_2)_{\frac{1}{7}} \vee \Sigma^\infty BSL_3(\mathbb{F}_2)_{\frac{1}{7}} \vee L(2)$
Here $L(2) = \Sigma^{-2}(sp^4 s^0/sp^2 s^0)_2^\wedge$ [**MP**].

PROOF. (i) is an immediate consequence of Corollary 3.3. (iii) follows from (i) and the Mitchell–Priddy stable splitting of $B\Sigma_4$ [**MP**]:

$$\Sigma^\infty B\Sigma_{4}{}_2^\wedge \simeq \Sigma^\infty BSL_3(\mathbb{F}_2)_2^\wedge \vee L(2) \vee \Sigma^\infty B\mathbb{Z}_2.$$

(ii) (resp. (iv)) follows from (i) (resp. (iii)) and

$$\Sigma BSL_3(\mathbb{Z}) \simeq \Sigma BSL_3(\mathbb{Z})_2^\wedge \vee \Sigma B\Sigma_{3}{}_3^\wedge \vee \Sigma B\Sigma_{3}{}_3^\wedge,$$
$$\Sigma BSL_3(\mathbb{F}_2)_{\frac{1}{7}} \simeq \Sigma BSL_3(\mathbb{F}_2)_2^\wedge \vee \Sigma B\Sigma_{3}{}_3^\wedge,$$

where the first is an immediate consequence of [**S**] or [**B**]. □

REMARK. (i) Corollary 3.4, in particular, shows that $BSL_3(\mathbb{Z})_2^\wedge$ contains $BSL_3(\mathbb{F}_2)_2^\wedge$ as a retract. Its weaker stable-version is previously known and quite easy to show ([**Ta**][**STY**]).

(ii) Even though we obviously get similar cofiber sequences for $BSt_3(\mathbb{Z})_2^\wedge$ like those in Corollary 3.3, s_1 and s_2 cannot be lifted and actually there are no sections corresponding to s_1 or s_2. This can be seen from the cohomology calculation of 2.4 [**S**].

Warning: As was pointed out by M. Tezuka, Theorem 8(ii) of [**S**] contains a mistake, i.e. $v_2^2 = 0$ rather than $v_2 v_3 = 0$. This can be seen much faster by using our Corollary 3.2.

(iii) We can use Corollary 3.2 to calculate $K(BSt_3(\mathbb{Z}))$ (for relevant calculations, see [**FS**] for instance). Here we simply note $K^{-1}(BSt_3(\mathbb{Z}))_{(2)} \neq 0$, which shows $BSt_3(\mathbb{Z})$ is very exotic in view of [**AS**].

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