FALL 2016 HOMOTOPY THEORY SEMINAR

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CONTENTS

1. s-COBORDISMS AND WALDHAUSEN'S MAIN THEOREM: 9/7/16

Today, Professor Blumberg gave an overview of Waldhausen's main theorem and its context; this semester, we'll be working though Lurie's proof of it as outlined in his course on the algebraic topology of manifolds.

We'll start from the h-cobordism theorem.

Definition 1.1. An *h-cobordism* is a cobordism W between manifolds M and N such that the equivalences $M \hookrightarrow W$ and $N \hookrightarrow W$ are both homotopy equivalences.

The canonical example is $M \times [0,1]$, which is an h-cobordism between M and itself. This is called a *trivial h-cobordism*.

We're going to be deliberately vague about what category of manifolds we're dealing with: when we say "isomorphic," we mean as toplogical manifolds, PL manifolds, or smooth manifolds. We're not going to belabor the point right now, though it will be quite important for us later.

Theorem 1.2 (h-cobordism (Smale)). *If $\dim(M) \geq 5$ and $\pi_1(M) = 0$, then every h-cobordism is trivial, i.e. suitably isomorphic relative to the boundary to the trivial h-cobordism.*

This is a big theorem — a somewhat easy consequence is the Poincaré conjecture in dimensions ≥ 5! When Smale proved this part of the Poincaré conjecture, he really was attacking this theorem. The proof proceeds via a handlebody decomposition, which illustrates what is easier in dimension 5 than in dimensions 3 and 4: handlebodies can slide past each others using, for example, the *Whitney trick*, which simply doesn't work in dimensions 3 or 4.

We're not interested in the Poincaré conjecture *per se*, but can we generalize Theorem 1.2? If we try to lower the dimension of M, we're basically screwed, so can we work with M not simply connected?

Theorem 1.3 (s-cobordism (Barden, Mazur, Stallings)). *The set of isomorphism classes of h-cobordisms $M \hookrightarrow W \hookleftarrow N$ is in bijection with a certain quotient of $K_1(\mathbb{Z}[\pi_1(M)])$.*

We'll eventually define K_1, which is an algebraic gadget that's a ring invariant. It's evident that K_1 of the group algebra is a homotopy invariant, but it's less obvious that the set of isomorphism classes of h-cobordisms is. This group $K_1(\mathbb{Z}[\pi_1(M)])$ is also the home of *Whitehead torsion*, an invariant of manifolds.

Question 1.4. Is this a π_0 statement? In other words, can we describe a space of s-cobordisms such that Theorem 1.3 is recovered on passage to π_0?

This is a natural question following recent developments in homotopy theory. It may allow us to attach spaces or spectra to these invariants.

The answer, due to names such as Hatcher, Igusa, and Waldhausen, is yes! On the left-hand side, we have something called the stable pseudo-isotopy space, akin to a stabilized form of $B\mathrm{Diff}$, the isomorphisms of a manifold relative to its boundary. This arises as a result of an action on bundles of s-cobordisms, which is how classifying spaces appear. Things will be more concretely defined, albeit not at this level of narrative. The point is that *a priori* this isn't a homotopy-invariant, so we have to stabilize in a geometric way, by taking repeated products $M \times I^n$ with an interval.

On the other side, one realizes $\pi_1(M) \cong \pi_0(\Omega M)$, so maybe we can try to construct something like $\mathbb{Z}[\Omega M]$. This works, but it's better to take the K-theory *spectrum* associated to something called the spherical group ring $K(S[\Omega M])$, which is (or is equivalent to) $A(M) = K(\Sigma^\infty_+ \Omega M)$, which is an A_∞ ring spectrum.

Waldhausen's theorem is precisely that there is a stable s-cobordism theorem: that $A(M) \simeq \Sigma^\infty M \vee \Omega^2 \mathrm{Wh}(M)$; $\mathrm{Wh}(M)$ is something called the *Whitehead spectrum* associated to M, and its double loop space is the pseudo-isotopy space we want to construct. This splitting arises from an assembly map, which is a purely formal statement about (topologically or simplicially) enriched functor: $F(X) \vee Y \to F(X \vee Y)$.

So we have an algebraic invariant, which we can hope to calculate, and it tells us geometric information.

We can start with the sphere spectrum $K(S) \simeq S \vee \Omega^\bullet \mathrm{Wh}(*)$. This is already hard and unsolved; solving it will solve several questions in geometric topology, including some on exotic differential structure.

Depending on who you are, you might have different motivations for things like this: May and others were naturally led to ring spectra when considering generalized orientations, but you might also invent them to make this theorem true!

It turns out that $K(S)$ is controlled by $K(\mathbb{Z})$. There's a commutative square

$$\begin{tikzcd} K(S) \arrow[r] \arrow[d] & \mathrm{TC}(S) \arrow[d] \\ K(\mathbb{Z}) \arrow[r] & \mathrm{TC}(\mathbb{Z}). \end{tikzcd}$$

Here, TC is topological cyclic homology. One of the big theorems is that this is a pullback diagram, so we can understand $K(S)$ from topological cyclic homology, the vertical maps, called *trace maps* (which actually generalize the trace of a matrix), and how they fit together. At primes, this is pretty simple, but that still leaves the individual players.

Understanding $\mathrm{TC}(S)$ is, well, slightly harder than computing the stable homotopy groups of the spheres. This is a bad thing, but also a good thing: we can compute some of it, and if important questions depend on a particular element in it, that element can be computed and identified. Conversely, $K(\mathbb{Z})$ is a mess, but an interesting mess — it contains a lot of number-theoretic information, some still unknown. This diagram illustrates that number-theoretic information controls geometric topology.

Waldhausen's proof of his main theorem in [3] is famously complicated, and wasn't available until relatively recently. The book by Waldhausen-Jahren-Rognes [4] provides an exposition, but it's rough going. Certainly, the introduction will be useful.

One reason we might be interested is how Waldhausen proved this. He gave a direct proof in the PL category, and then applied a reduction to prove the smooth case. Is it possible to give a direct proof in the smooth case? Waldhausen and Igusa tried to do this, but didn't succeed. For reasons that are ultimately Floer-homotopy-theoretic, it would be useful to have such a proof in the smooth case. Lurie's proof in [2] follows the broad stokes of Waldhausen's proof, but uses different machinery, and could be useful as a guide for a direct proof in the smooth case.

There will be three kinds of lectures:

(1) Statements of the theorem. Along with this, what is algebraic K-theory? What is a Whitehead spectrum?
(2) Background: what is Wall finiteness? What is Whitehead torsion? These are geometric questions, yet are invariants of algebraic K-theory, and are good to know for culture.
(3) Then, there's the technology of the proof, uisng something called simple homotopy theory. We can think of $K(\Sigma^\infty_+ \Omega M)$ as the K-theory of a category of spaces parameterized over M, and this leads to simple homotopy equivalences, related to a prescribed set of equivalences and blowups. This is more geometric, and hence

harder. Lurie's proof presents a different approach to this, focusing on constructible sheaves, and this is definitely one of the most worthwhile lessons from his proof.

The key is the construction of the assembly map: Lurie approaches it with some very natural functors from constructible sheaves. This is something we can get to, but we'd have to cover a lot of ground to get there.

Another thing to keep in mind: there will be many different constructions of these objects, all equivalent or equivalent up to a shift. We'll end up constructing a parameterized spectrum of space of cobordisms. It might be interesting to compare these to other cobordism categories.

2. The Wall finiteness obstruction: 9/14/16

Today, Nicky spoke.

The Wall finiteness obstruction is an invariant that's pretty easy to write down abstractly; it provides an obstruction for a CW complex to be homotopic to a finite CW complex. Specifically, given a CW complex X, let $G = \pi_1(X)$ (relative to any basepoint); we'll construct this obstruction as a class $w(X) \in \widetilde{K}(\mathbb{Z}[G])$.

Definition 2.1. Let R be a ring, not necessarily commutative. Then, the *K-theory* of R, denoted $K_0(R)$, is the abelian group generated by isomorphism classes of finitely generated[1] projective (left) R-modules modulo the relations that for every short exact sequence of R-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

we impose a relation $[M] = [M'] + [M'']$.

There's a map $\mathbb{Z} \to K_0(R)$ defined by $z \mapsto z[R]$; if $z > 0$, this is the class $[R^{\oplus z}]$. The *reduced K-theory* mods out by this: $\widetilde{K}_0(R) = K_0(R)/\mathbb{Z}$.

This is a very algebraic object, but we'll use it to discover topological information.

There are some other tools we'll use. Relative homotopy invariants are associated to relative homology groups $H_*(X, Z; R)$, which we can define whenever we're given a map $f : Z \to R$. Using a mapping cylinder C_f, this is homotopic to an inclusion, and $H_*(X, Z; R)$ is the homology of the quotient chain complex (of the singular chains).

Definition 2.2. A *local system* on a space X is a representation of $\pi_1(X)$ over $\mathbb{Z}$.[2] That is, it's an abelian group with a compatible $\pi_1(X)$-action.

Recall that $G = \pi_1(X)$ acts on the universal cover $\widetilde{X}$ by deck transformations. Thus, the singular complex $C_\bullet(\widetilde{X})$ of $\widetilde{X}$ is a $\mathbb{Z}[G]$-module. Thus, given a local system V, we can create new chain complexes, such as $C_\bullet(\widetilde{X}) \otimes_{\mathbb{Z}[G]} V$ or $\mathrm{Hom}_{\mathbb{Z}[G]}(C_\bullet(\widetilde{X}), V)$.[3] These define homology, resp. cohomology theories on X, called $H_*(X, V)$, resp. $H^*(X, V)$.

Definition 2.3. A space X is *finitely dominated* if there exists a finite CW complex Z, an inclusion $i : X \to Z$, and a section $r : Z \to X$ such that $r \circ i \simeq \mathrm{id}_X$.

This basically means X includes into a finite CW complex which retracts onto it, but with a homotopy. We'll hope to show that some properties of finitely dominated spaces actually characterize them.

Fact. Let X be a finitely dominated space.

(1) First, $\pi_0(X)$ must be finite (since it factors as a subset of $\pi_0(Z)$).
(2) $\pi_1(X)$ must be finitely presented, because $i_* : \pi_1(X) \to \pi_1(Z)$ has a left inverse, so it's split injective into a finitely generated group.
(3) For local systems V, the assignment $V \mapsto H_*(X, V)$ commutes with filtered direct limits.
(4) X has finite *homotopical dimension*, which means there's an $m \geq 0$ such that for all local systems V and $i > m$, $H_i(X, V) = 0$. This will be at most the dimension of the space Z which dominates X.

The following theorem is important, but hard; [2, Lec. 2] sketches the proof.

Theorem 2.4. *A space satisfying conditions (1), (2), and (3) is finitely dominated.*

[1] One can define this for the category of all projective R-modules, but this is always zero, thanks to the Eilenberg swindle.

[2] Sometimes, the base ring is different, but for our purposes, we'll prefer $\mathbb{Z}$.

[3] Since $\mathbb{Z}[G]$ is in general noncommutative, there's something to say here about left versus right actions.

Proposition 2.5. *Suppose X satisfies conditions (1), (2), and (3). Then, for all $n > 0$, there's a finite CW complex Z of dimension less than n and an $(n-1)$-connected map $Z \to X$.*[4]

This follows from a more general fact.

Proposition 2.6. *Suppose X satisfies conditions (1), (2), and (3), and suppose we are given an $(n-1)$-connected map $f : Z \to X$, where Z is a finite CW complex. Then, there exists a space Z', obtained from Z by adjoining finitely many n-cells, such that f factors through an n-connected map $Z' \to X$.*

This allows us to inductively prove Proposition 2.5.

Lemma 2.7. *Let X be a space satisfying (1), (2), (3), and (4). Let Z be a finite CW complex of dimension at most $n-1$ and $f : Z \to X$ be an $(n-1)$-connected map. Then, $H_n(X, Z; \mathbb{Z}[G])$ is a finitely generated projective $\mathbb{Z}[G]$-module.*

This is where K-theory shows up.

Proof. Since Z is $(n-1)$-dimensional, then $H^i(Z, V) = 0$ for all $i \geq n$ and all local systems V. We have a long exact sequence for relative homology

$$\cdots \longrightarrow H^{i-1}(Z; V) \longrightarrow H^i(X, Z; V) \longrightarrow H^i(X, V) \longrightarrow \cdots,$$

and given a short exact sequence of local systems

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0,$$

and applying $H^*(X, Z; –)$ induces another long exact sequence

$$\cdots \longrightarrow H^n(X, Z; V') \longrightarrow H^n(X, Z; V) \longrightarrow H^n(X, Z; V'') \longrightarrow H^{n+1}(X, Z; V') \longrightarrow \cdots.$$

Using the universal coefficients theorem and the Hurewicz theorem, we have a natural isomorphism

$$H^n(X, Z; –) \cong \mathrm{Hom}_{\mathbb{Z}[G]}(H_n(X, Z; \mathbb{Z}[G]), –).$$

The former is right exact, and therefore so is the latter, so $H_n(X, Z; \mathbb{Z}[G])$ is projective. ⊠

Thus, $H_n(X, Z; \mathbb{Z}[G])$ has a class in K-theory.

Definition 2.8. The *Wall finiteness obstruction* of X is $w(X) = (-1)^n [H_n(X, Z; \mathbb{Z}[G])] \in K_0(\mathbb{Z}[G])$.

We have a lot to show: that this is independent of n and Z, but also that it's at all related to finiteness.

Proposition 2.9. *The following are equivalent:*

(1) X has the homotopy type of a finite CW complex.
(2) $H_n(X, Z; \mathbb{Z}[G])$ is stably free (and hence trivial in $K_0(\mathbb{Z}[G])$).

In the reverse direction, the idea of the proof is to kill generators: if $H_n(X, Z; \mathbb{Z}[G]) \oplus \mathbb{Z}[G]^{\oplus r}$ is free, then it's equal to $H_n(X, Z \vee (S^n)^{\vee r}; \mathbb{Z}[G])$; then, one uses the Whitehead theorem and the relative Hurewicz theorem to kill homotopy groups. The point is that we have a map $Z \vee (S^n)^{\vee r} \to X$, where the domain is a finite CW complex; if we can show that this map induces an isomorphism on all homotopy groups, Whitehead's theorem proves that the map is a homotopy equivalence.

In the other direction, we can compute $H_*(X, Z; \mathbb{Z}[G])$ cellularly, and therefore get an exact sequence of free modules, which forces it to be stably free.

We'll skip the proofs of independence of Z and n, which are reasonably pretty, but quite long.

Another interesting fact is that if G is any group, we know $G = \pi_1(X)$ for some space X, but it's also true that any class in $K_0(\mathbb{Z}[G])$ is a Wall finiteness obstruction for some space X with $\pi_1(X) = G$.

[4]For a map to be $(n-1)$-connected means that its homotopy fiber is $(n-1)$-connected as a space, which implies that the induced map on π_k is an isomorphism for $k < n-1$ and is a surjection for $k = n-1$.

Postscript. (added by Andrew Blumberg) The basic theorem at work in the development of the Wall finiteness obstruction is a result saying that a CW complex that is finitely dominated is finite if and only if a certain relative homology group was stably free.

(Recall that here stably free means that $M \oplus R^n \cong R^m$, for some n and m.)

There is a perspective from which it is now very natural to imagine K_0 entering the picture. Specifically, let's make the following two definitions:

Definition 2.10. Let M and N be objects of Mod_R that are f.g. and projective. Then M and N are *stably isomorphic* if there exists n such that $M \oplus R^n \cong N \oplus R^n$. M and N are *stably equivalent* if there exist m and n such that $M \oplus R^m \cong N \oplus R^n$.

Now, a lemma, which you should prove as an exercise (it's very easy).

Lemma 2.11. *$[P] = [Q]$ in $K_0(R)$ if and only if P and Q are stably isomorphic.*

Furthermore, if we define $\widetilde{K}_0(R)$ to be the quotient of $K_0(R)$ by the image of $\mathbb{Z}$ under the natural map that takes n to $[R^n]$, we have a corresponding result:

Lemma 2.12. *$[P] = [Q]$ in $\widetilde{K}_0(R)$ if and only if P and Q are stably equivalent.*

As a consequence, it seems natural that if what you care about is a module being stably free, you might well look at K_0 or $\widetilde{K}_0$.

Remark.

(1) Per Yuri's questions, indeed, finitely dominated CW complexes are finite for simply-connected spaces. Also, check out [1] for a discussion of applications of the Wall finiteness obstruction in surgery theory.
(2) A natural question to ask is about whether (and when) stably free modules are actually free. Serre proved this as part of his work on the conjecture that all projective modules over a polynomial ring on a field k are free. Quillen and Suslin eventually proved the conjecture.

3. The Algebraic K-theory of the Sphere Spectrum: 9/21/16

Today, we have a guest: Mike Mandell (Indiana University) spoke, about the algebraic K-theory of the sphere spectrum, $K(\mathbb{S})$.

The K-theory of the sphere spectrum is very important; there are two way to explain why, either from the K-theory of rings and ring spectra or through Waldhausen's K-theory of spaces. In both cases, $K(\mathbb{S})$ is important.

If we wanted to talk about K-theory of rings and ring spectra, $\mathbb{Z}$ is important, because it's the initial ring. Ring spectra are generalizations of rings in stable homotopy theory, and there's a simpler one than $\mathbb{Z}$: the sphere spectrum $\mathbb{S}$, which is the initial ring spectrum. If one thinks of rings as acting on abelian groups, which are $\mathbb{Z}$-modules, then one can adopt a similar approach to spectra: modules over a ring spectrum are spectra, which are modules over $\mathbb{S}$. So if you want to understand ring spectra, it's reasonable to start with $\mathbb{S}$ (or, maybe it's the hardest, but we know it will be universal).

From the perspective of the K-theory of spaces, the simplest space is the one-point set $*$,[5] and its K-theory $A(*)$ is equal to $K(\mathbb{S})$.

The algebraic K-theory of spaces arose from a lot of work in differential topology. Hatcher and others studied concordances and pseudo-isotopies: given a smooth manifold X, you might want to understand $\mathrm{Diff}(X)$, the space of its diffeomorphisms. This is hard, but we can simplify to understanding the isotopies. An *isotopy* between two diffeomorphisms $f_0, f_1 : X \rightrightarrows X$ is a homotopy $X \times I \to X$ from f_0 to f_1 that restricts to a diffeomorphism over every $t \in I$. This weakens to a notion of *pseudoisotopies*, which can be stabilized by iterating the process: $X \times I^n \to X \times I^n$. Hatcher noticed this is combinatorial, and moreover, depends only on the homotopy type of X. Pseudoisotopies aren't a functor, but stabilized pseudoisotopies define a functor A, and Waldhausen noticed that it resembles Quillen's algebraic K-theory: specifically, $A(X) \cong K(\mathbb{S}[\Omega X])$. Here, ΩX is the loopspace, so X ought to have a basepoint!

ΩX isn't quite a group, but it's not far from one: it has a homotopy-associative binary operation. So just as we can form a group ring $\mathbb{Z}[G]$ given a group G, it's possible to take this homotopical grouplike object and adjoin it to the sphere spectrum. If $X = *$, then $\Omega X = *$, so $K(\mathbb{S}[\Omega X]) = K(\mathbb{S})$. This allows one to use K-theory techniques to study stable, high-dimensional differential topology. This is all happening in the 1980s.

[5]This construction requires basepoints, so we can't consider the empty set.

Now you ask, what can you say with this? This is where the story starts: we want to understand the homotopy type of $K(\mathbb{S})$, which is a spectrum. Since ring spectra generalize rings, and $\mathbb{S}$ is the initial ring spectrum, there's a unique map $\mathbb{S} \to \mathbb{Z}$, and applying K to it, we obtain a map $K(\mathbb{S}) \to K(\mathbb{Z})$.

In the 1990s, Bökstedt, Hsiang, Madsen, and Goodwillie defined *topological cyclic homology* $\mathrm{TC}(R)$ for any ring spectrum R, which is a version of cyclic homology, but constructed using purely spectrum-level information, and is purely (equivariant) stable homotopy theory. This arose from another spectrum-level construction called *topological Hochschild homology* using some operations defined on it. Topological cyclic homology is built precisely for homotopy theorists to be able to calculate with it, and, in a similar way to the relation of K-theory to cyclic homology, there's a natural transformation $K \to \mathrm{TC}$.

If R is a connective ring spectrum, there's a map of ring spectra $R \to \pi_0 R$, and the natural transformation induces a commutative square

$$\begin{tikzcd} K(R) \arrow[r] \arrow[d] & K(\pi_0(R)) \arrow[d] \\ \mathrm{TC}(R) \arrow[r] & \mathrm{TC}(\pi_0(R)). \end{tikzcd} \tag{3.1}$$

Applying this to $R = \mathbb{S}$, we obtain a commutative square relating K-theory and topological cyclic homology of $\mathbb{S}$ and $\pi_0 \mathbb{S} = \mathbb{Z}$:

$$\begin{tikzcd} K(\mathbb{S}) \arrow[r] \arrow[d] & K(\mathbb{Z}) \arrow[d] \\ \mathrm{TC}(\mathbb{S}) \arrow[r] & \mathrm{TC}(\mathbb{Z}). \end{tikzcd}$$

We will now complete at a prime p. For most of this talk, we'll need p to be odd, but for right now, $p = 2$ also works.

Recall that a finite abelian group A decomposes as a direct sum

$$A \cong \mathbb{Z}/p^{r_1} \oplus \mathbb{Z}/p^{r_2} \oplus \cdots \oplus \mathbb{Z}/p^{r_n} \oplus A^1,$$

where $|A^1|$ has order coprime to p. We'd like to only consider the parts that p knows about, so p-completing A comes down to throwing out A^1. This is useful to simplify problems.

p-completion is a functor, though maybe a surprising one. Modding out by any p^n is bad, because you could have some $r^i > n$, so you have to take a limit: p-completion is the functor

$$A_p^\wedge = \varinjlim_n A/p^n,$$

and it comes with a natural transformation $A \to A_p^\wedge$. We can take the product of all of these functors and map

$$A \longmapsto \prod_{p \text{ prime}} A_p^\wedge, \tag{3.2}$$

and for finite abelian groups, this is an isomorphism.

If A is instead finitely generated, this is no longer true: the p-completion of a free factor $\mathbb{Z}$ is the *p-adic integers* $\mathbb{Z}_p$, which isn't isomorphic to $\mathbb{Z}$. But it's still possible to recover a lot of information about a finitely generated abelian group from its image under (3.2). Specifically, there is a density theorem after tensoring with $\mathbb{Q}$: consider an element in $\widehat{\mathbb{Z}} = \prod_{p \text{ prime}} \mathbb{Z}_p$, and suppose it winds up in $\mathbb{Q}$ after tensoring with $\mathbb{Q}$. Then, it must have come from an integer repeating. In other words, the following diagram is a pullback diagram:

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \displaystyle\prod_{p \text{ prime}} \mathbb{Z}_p \arrow[d] \\ \mathbb{Q} \arrow[r] & \mathbb{Q} \otimes \displaystyle\prod_{p \text{ prime}} \mathbb{Z}_p. \end{tikzcd}$$

This means we can recover $A \otimes \mathbb{Q}$ from the product of all of the p-completions of A.

In the world of spectra, p-completion is instead the homotopy limit $A_p^\wedge = \operatorname{holim}_n A/p^n$; akin to tensoring with $\mathbb{Q}$ is an operation called *rationalization*, which applies $-\otimes\mathbb{Q}$ to homotopy groups, and these combine into a homotopy pullback diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \prod_{p \text{ prime}} A_p^\wedge \arrow[d] \\ A_{\mathbb{Q}} \arrow[r] & \left(\prod_{p \text{ prime}} A_p^\wedge\right)_{\mathbb{Q}}. \end{tikzcd}$$

What does it mean for this to be a homotopy pullback? There are several equivalent characterizations: that the homotopy fibers of the horizontal arrows agree, or their homotopy cofibers, or the homotopy fibers (or cofibers) of the vertical maps agree. And, as with abelian groups, from the collection of p-completions, one can recover the rationalization.

The map of ring spectra $\mathbb{S}\to\mathbb{Z}$ is an isomorphism on π_0 and a rational equivalence on π_n where $n>0$; since $\mathbb{S}$ and $\mathbb{Z}$ are connective, we don't have to worry about negative homotopy groups. This also implies $K(\mathbb{S})\to K(\mathbb{Z})$ is a rational equivalence.[6] Borel calculated $K(\mathbb{Z})_{\mathbb{Q}}$ in the 1970s, so we know the homotopy groups are

$$\pi_n K(\mathbb{S})_{\mathbb{Q}} = \begin{cases} \mathbb{Q}, & n=0 \\ 0, & n=1 \\ 0, & n\equiv 2,3,0 \bmod 4, n\neq 0 \\ \mathbb{Q}, & n\equiv 1 \bmod 4, n>1. \end{cases}$$

Now, let's fix a prime p and try to understand $K(\mathbb{S})_p^\wedge$. There's a theorem of Dundes that says if you p-complete the K-theory and TC square (3.1), the result is a homotopy Cartesian diagram. So if we understand $\mathrm{TC}(\mathbb{S})_p^\wedge$, $\mathrm{TC}(\mathbb{Z})_p^\wedge$, and $K(\mathbb{Z})_p^\wedge$, we can piece them together and determine $K(\mathbb{S})_p^\wedge$.

Bökstedt, Hsiang, and Madsen calculated

$$\mathrm{TC}(\mathbb{S})_p^\wedge \simeq \mathbb{S}_p^\wedge \vee \Sigma(\mathbb{CP}^\infty_{-1})_p^\wedge.$$

The latter spectrum is related to two-dimensional topological field theories! But this splitting doesn't yet admit a geometric explanation.

If you're familiar with Thom spectra, this isn't very different. Let γ_n be the tautological bundle over $\mathbb{CP}^n$, and consider the *Thom space* $T(\gamma_n^\perp)$ (which has for its total space the one-point compactification of γ_n). We want it to be the negative of γ_n, so we desuspend $2n$ times. We can stich these together over various n to obtain

$$\mathbb{CP}^\infty_{-1} = \bigcup_n \Sigma^{-2n} T(\gamma_n^\perp).$$

This is an example of a *Madsen-Tillman spectrum*, and it appears in the proof of the Mumford conjecture. It's not simple *per se*, but thanks to the Thom isomorphism, we can compute its homology and cohomology over complex-oriented cohomology theories. For example, its rational homology is free on *Mumford classes* $k_0, k_1, \dots$.

Bökstedt-Madsen calculated $\mathrm{TC}(\mathbb{Z})_p^\wedge$; it's easier to describe its connective cover, which is

$$j \vee \Sigma j \vee \Sigma^3 ku_p^\wedge.$$

Let's talk about the components.

- Topological periodic complex K-theory (i.e. starting with complex vector bundles, after Atiyah-Hirzebruch) defines an extraordinary cohomology theory represented by a spectrum KU; ku is its connective cover.
- j is the p-completion of the *image of* j spectrum. The Adams operations on K-theory define Adams operations $ku\to ku$; if you choose a particular Adams operation and subtract one, then j arises as the fiber. Alternatively, take the localization of $\mathbb{S}$ at KU, p-complete it, and take its connective cover; this describes j.

[6] A *rational equivalence* of two spaces or two spectra is a map $X\to Y$ that induces an isomorphism $\pi_n(X)\otimes\mathbb{Q}\to\pi_n(Y)\otimes\mathbb{Q}$ for all n.

The third ingredient is $K(\mathbb{Z})_p^\wedge$: this isn't fully understood yet, but it splits as a sum

$$K(\mathbb{Z})_p^\wedge = j \vee y_0 \vee \cdots \vee y_{p-2}.$$

For i odd, y_i is related to the Bernoulli numbers. For i even, this isn't well understood yet, and understanding it better will require addressing the Vandiver conjecture in number theory. This would imply they break down as suspensions of Adams summands of *ku*.

What about the maps between these spectra? We don't know much about $\mathrm{TC}(\mathbb{S})_p^\wedge \to \mathrm{TC}(\mathbb{Z})_p^\wedge$; because $\Sigma\mathbb{CP}_{-1}^\infty$ is a Thom spectrum and *ku* is complex-oriented, we know what the group of homotopy classes of maps between them is, but we don't know which specific map it is. The map $K(\mathbb{Z})_p^\wedge \to \mathrm{TC}(\mathbb{Z})_p^\wedge$ is better understood: the j maps to the j, the Σj in $\mathrm{TC}(\mathbb{Z})_p^\wedge$ is missed, and the remaining $p-1$ summands map diagonally, especially if we assume the Vandiver conjecture. This in general is related to p-adic ℓ-functions, which aren't well-understood yet (do they take the value 0? This is open). So at a fundamental level, we don't really understand these maps, even if we can write them in terms of existing names.

4. Whitehead torsion: 9/28/16

"Does the word 'adjunction' make you happy or sad?"

Today, Richard talked about K_1 and Whitehead torsion, which is an invariant of a finite CW complex X that lives in $K_1(\mathbb{Z}[\pi_1(X)])$. To wit, suppose X and Y are finite CW complexes and $f : X \to Y$ is a homotopy equivalence; when may f be witnessed in the finite CW category? That is, is it a simple homotopy equivalence? The answer comes in an obstruction $\tau(f)$ in a quotient of $K_1(\mathbb{Z}[\pi_1(X)])$.[7]

Let's elaborate this definition of "finite CW complexes." We mean CW complexes with specified cell decompositions, e.g. S^n comes with its hemisphere decomposition $S^n = D_-^n \amalg_{S^{n-1}} D_+^n$.

Given a finite CW complex Y and a map $(D_-^{n-1}, S^{n-2}) \to (Y^{(n-1)}, Y^{(n-2)})$, we can attach a D^n along D_-^{n-1} and take the pushout

$$\begin{tikzcd}
D_-^{n-1} \arrow[r] \arrow[d] & Y \arrow[d] \\
D^n \arrow[r] & Y \amalg_{S^{n-1}} D^n.
\end{tikzcd}$$

This is known as an *elementary expansion*. The dual operation is *elementary collapse*, formed by pushing out along the retract of a disc D^n onto the lower hemisphere D_-^{n-1} of its boundary S^{n-1}.

Definition 4.1. A *simple homotopy equivalence* is a homotopy equivalence of finite CW complexes that factors as a finite sequence of elementary expansions and elementary collapses.

We'll show that not all homotopy equivalences are simple homotopy equivalences.

Example 4.2. Let $f : X \to Y$ be continuous. Then, we can form the *mapping cylinder* $M(f) = Y \cup e \times (0,1) \cup e \times \{0\}$ for all cells $e \in X$. Then, there is a simple homotopy equivalence $M(f) \to Y$, and $f : X \to Y$ is homotopic to an inclusion $X \hookrightarrow M(f)$. Thus, for simple homotopy equivalences it suffices to consider inclusions.

Simple homotopy equivalences are a combinatorial model for homotopy equivalences, and in particular are homotopy equivalences with geometric content. This is useful for geometric topology, because it's often useful to turn questions from geometric topology into homotopy theory, and this step often passes through homotopies with good geometric models, which is why simple homotopy theory appears in the context of the s- and h-cobordism theorems.

Simple homotopy equivalence is equivalent to having contractible fibers, which is another nice reason to think about it.

We'll define Whitehead torsion, which is an obstruction to a map being a simple homotopy equivalence. First, though, we need to define K_1.

[7] If G is a (finite) group, the *group ring* $\mathbb{Z}[G]$ is the ring of formal finite sums

$$\sum_{g \in G} a_i g_i,$$

(so all but finitely many a_i are zero). Addition and multiplication are just like that of polynomials.

Definition 4.3. Let R be a ring, not necessarily commutative. We can form $\mathrm{GL}_n(R)$, the group of $n\times n$ matrices with coefficients in R, and include $\mathrm{GL}_n(R)\hookrightarrow \mathrm{GL}_{n+1}(R)$ by sending

$$A\longmapsto \begin{pmatrix} A & 0\\ 0 & 1\end{pmatrix}.$$

This defines a directed system

$$\cdots \longrightarrow \mathrm{GL}_n(R)\longrightarrow \mathrm{GL}_{n+1}(R)\longrightarrow \mathrm{GL}_{n+2}(R)\longrightarrow\cdots, \tag{4.4}$$

and the colimit is denoted $\mathrm{GL}(R)$, the *infinite general linear group*. Concretely, elements of $\mathrm{GL}(R)$ are invertible matrices of any dimension, where we multiply them by expanding the matrices with only 1s on the diagonal until they have the same size.

The *first K-group* is $K_1(R)=\mathrm{GL}(R)^{\mathrm{ab}}$. Explicitly, this is invertible matrices modulo *elementary matrices*, which are matrices that differ from the identity by a single off-diagonal entry.

Definition 4.5. Over a ring R, we can define a *based chain complex* $(F_\bullet,d)$ to be a finite chain complex of finite-dimensional free R-modules

$$F_0\xrightarrow{d}F_1\longrightarrow\cdots\longrightarrow F_{n-1}\xrightarrow{d}F_n,$$

where we specify as data a basis for each F_i. The *Euler characteristic* is

$$\chi(F_\bullet,d)=\sum_{i=0}^{n}(-1)^i\dim F_i,$$

which may behave badly if R is noncommutative. We say $(F_\bullet,d)$ is *acyclic* if its homology vanishes.

Lemma 4.6. *If a based chain complex $(F_\bullet,d)$ is acyclic, then $\mathrm{id}_{F_\bullet}$ is chain homotopic to the zero map.*

Recall that two maps of chain complexes $f,g:F_\bullet\rightrightarrows G_\bullet$ are *chain homotopic* if there are maps $h:F_i\to F_{n-1}$ such that $d_Gh+hd_F=f-g$.

Proof. Since $(F_\bullet,d)$ is acyclic, then $F_n=\ker(d_n)\oplus\mathrm{Im}(d_{n+1})$. Thus, we can define $h:F_n\to F_{n+1}$ to be zero on $\ker(d_n)$ and a right inverse for d_{n+1} on $\mathrm{Im}(d_{n+1})$, which satisfies $dh+hd=\mathrm{id}$. (TODO: this doesn't look right; I must've missed something). ⊠

We let

$$F_{\mathrm{even}}=\bigoplus_n F_{2n}\cong R^a\qquad\text{and}\qquad F_{\mathrm{odd}}=\bigoplus_n F_{2n+1}\cong R^b$$

for some $a,b\in\mathbb{N}$.

Claim. $d+h:F_{\mathrm{even}}\to F_{\mathrm{odd}}$ is an isomorphism.

Proof. $(d+h)^2=d^2+dh+hd+h^2=\mathrm{id}+h^2$, whose inverse is $\mathrm{id}-h^2+h^4-h^6+\cdots$, but since $F_\bullet$ is a finite chain complex, this must terminate, producing an actual inverse to $(d+h)^2$. ⊠

Thus, we can consider $d+h\in\mathrm{GL}_a(R)$ and therefore its image in $K_1(R)$. A different choice of h may change the sign of $[d+h]\in K_1(R)$, so we consider the class in $\widetilde{K}_1(R)=K_1(R)/(\pm1)$.

Definition 4.7. The *torsion* $\tau(F_\bullet,d)$ of the based chain complex $(F_\bullet,d)$ is the class $[d+h]\in\widetilde{K}_1(R)$.

Let $f:(X_\bullet,d)\to(Y_\bullet,d)$ be a chain map. Then, its *mapping cone* $(C(f),d)$ is the chain complex whose n^{th}-degree term is

$$C(f)_n=X_{n-1}\oplus Y_n$$

and whose differential is

$$d(x,y)=(-dx,f(x)+dy),$$

which is chosen so that $d^2=0$. Then, we can define the *torsion* of f to be $\tau(f)=\tau(C(f),d)$.

We'd like to use this algebraic machinery to say something about topology. We'll start with cellular maps; by the cellular approximation theorem, any map of finite CW complexes is homotopic to a cellular map, so we'll define the Whitehead torsion first for cellular maps and then show it's invariant under homotopy, allowing us to define it for all maps.

Thus, let $f : X \to Y$ be a cellular map and $\widetilde{Y}$ be the universal cover of Y. Let $\widetilde{X}$ be the pullback

$$\begin{tikzcd}
\widetilde{X} = X \times_Y \widetilde{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{Y} \arrow[d] \\
X \arrow[r, "f"] & Y.
\end{tikzcd}$$

If $G = \pi_1(Y)$, then G acts on $\widetilde{X}$ and $\widetilde{Y}$ by deck transformations. Since f is cellular, it induces a map $\lambda : C_\bullet(\widetilde{X};\mathbb{Z}) \to C_\bullet(\widetilde{Y};\mathbb{Z})$, and the action of G allows us to interpret both of these as finite chain complexes of $\mathbb{Z}[G]$-modules (e.g. $C_k(\widetilde{X};\mathbb{Z})$ is generated by the k-cells of X).

We'd like to define the torsion of f to be $\tau(\lambda)$, but this is ambiguous: we need to define a basepoint in $\widetilde{X}$ given one in X, which is an ambiguity up to G.

Definition 4.8. The *Whitehead group* of a group G is $\mathrm{Wh}(G) = K_1(\mathbb{Z}[G])/(\pm g, g \in G)$.

This group is in general nontrivial: the determinant $\det : \mathrm{GL}_n(R) \to R^\times$ is surjective, and is compatible with the directed system (4.4), so it passes to a surjective map $\mathrm{GL}(R) \to R^\times$. This factors through elements $\pm g$, so the determinant defines a surjection $\mathrm{GL}(R)/(\pm g) \twoheadrightarrow \mathrm{Wh}(G)$.

Example 4.9. If $G = \mathbb{Z}/5$, then $\mathbb{Z}[G] = \mathbb{Z}[t]/(t^5 - 1)$, and $1 - t^2 - t^3$ is not generated by $(\pm g)$ for $g \in G$. Thus, $\mathrm{Wh}(\mathbb{Z}/5)$ is nontrivial.

Thus, we can define the *Whitehead torsion* $\tau(f)$ to be the image of $\tau(\lambda)$ in $\mathrm{Wh}(G)$, which doesn't depend on any choices.

Proposition 4.10. *Whitehead torsion is a homotopy invariant: if $f, g : X \rightrightarrows Y$ are homotopic cellular maps, then their Whitehead torsion is the same.*

Thus, it makes sense to define Whitehead torsion for any map using cellular approximation.

Finally, the fact tying it to simple homotopy theory:

Proposition 4.11. *Suppose f is a simple homotopy equivalence. Then, $\tau(f) = 1$.*

Proof sketch. Since Whitehead torsion is a homotopy invariant, we may assume f factors as a finite sequence of elementary expansions and elementary collapses. Moreover, it's possible to show that $\tau(f \circ g) = \tau(f)\tau(g)$.

Thus, it suffices to compute the Whitehead torsion of each. For an elementary expansion, the attaching map has for its mapping cone the chain complex

$$0 \longrightarrow \mathbb{Z}[G] \xrightarrow{\pm g} \mathbb{Z}[G] \longrightarrow 0,$$

so in $\mathrm{Wh}(G)$, this is zero. The elementary collapse case is similar.

Proposition 4.12. *Conversely, if $f : X \to Y$ is a homotopy equivalence with trivial Whitehead torsion, then it is a simple homotopy equivalence.*

Proof sketch. The proof idea is to use the mapping cylinder to assume $f : X \hookrightarrow Y$ is an inclusion, and then do "cell trading," where a simple expansion or collapse on X produces a simple-homotopy-equivalent CW complex X' such that the mapping cylinder has lower dimension. After finitely many iterations, Y is just a union of X and e^n and e^{n+1} cells, at which point one must construct the simple homotopy equivalence explicitly.

5. Higher Simple Homotopy Theory: 10/5/16

Today, Adrian talked about higher simple homotopy theory. The proofs are convoluted and confusing, so this will more of a guide to [2, Lec. 5, 6] than a complete exposition.

5.1. **Motivation.** It would be really nice if we could understand the classifying space of all fibrations $\coprod_X B\operatorname{Aut}(X)$ over all homotopy types X. This is too pie-in-the-sky of a goal, so we'll restrict to finite homotopy types, or even to an easier moduli space $\mathscr{M}$ with a map

$$\mathscr{M} \to \coprod_X B\operatorname{Aut} X.$$

Here, for any polyhedron B, the maps $B \to \mathscr{M}$ are in bijection with fibrations $E \to B$ where E is a polyhedron, and homotopies $f \Rightarrow g : B \rightrightarrows M$ are in bijection with (parametrized) *concordances*

$$\begin{tikzcd} E \arrow[rr, "\sim"] \arrow[dr] & & E' \arrow[dl] \\ & B. & \end{tikzcd}$$

In particular, if $B = *$, the homotopy classes of maps $* \to M$ should be in bijection with unparameterized concordance classes between polyhedra.

Theorem 5.1. *Every concordance between polyhedra is a simple homotopy equivalence.*

(The converse is also true, but we won't go into that.)

There are many different constructions of $\mathscr{M}$; [2] discusses several of them. Lurie also mentions that if you take the homotopy pullback of

$$\begin{tikzcd} \mathscr{M} \arrow[r] & \coprod_X B\operatorname{Aut}(X) \\ & {[Y]}, \arrow[u, hook] \end{tikzcd}$$

the result is a shifted loopspace of a Whitehead spectrum $\Omega^{\infty+1}\operatorname{Wh}(Y)$. This mysterious diagram has something to do with relating general homotopy equivalences to simple homotopy equivalences.

5.2. **Definitions and disambiguations.**

Definition 5.2 (Polyhedra)**.** A subset $K \subset \mathbb{R}^n$ is a *polyhedron* if it admits a *triangulation* $S = \{\sigma \subseteq K\}$ (a set of simplices) such that

(1) for every simplex $\sigma \in S$, all faces of σ are in S,
(2) any nonempty intersection between two simplices is a simplex in S, and
(3) the union of the simplices in S is K.

This is the most rigid version of the realization of an abstract simplicial complex; finite simplicial complexes (and every finite homotopy type) may be realized by polyhedra.

Definition 5.3. A *piecewise linear map* between polyhedra is a map $f : X \to Y$ such that if S is a triangulation for S, then for all $\sigma \in S$, $f|_\sigma$ is affine.

Definition 5.4 (Simple homotopy structures)**.** A *simple homotopy structure* on a topological space X is an equivalence class of homotopy equivalences[8] $Y \to X$ where Y is a CW complex; we say that $Y \to X$ is equivalent to $Y' \to X$ if there is a simple homotopy equivalence

$$\begin{tikzcd} Y \arrow[rr, "\sim"] \arrow[dr] & & Y' \arrow[dl] \\ & X. & \end{tikzcd}$$

Theorem 5.5. *Every polyhedron has a canonical simple homotopy structure.*

Definition 5.6 (Concordances)**.** A *concordance* between finite polyhedra X and Y is a *PL fibration* $E \to [0,1]$, i.e. a fibration of topological spaces that's piecewise linear as a map of polyhedra, such that $E|_0 \cong X$ and $E|_1 \cong Y$.

There is also a parameterized version of this.

[8]Here, we mean Hurewicz equivalences, *not* weak homotopy equivalences.

5.3. **Statement of Results.** A point of a lot of this is to develop recognition criteria for simple homotopy equivalences. Last time, we saw that Whitehead torsion is one example.

Theorem 5.7. *Any homeomorphism between finite CW complexes is a simple homotopy equivalence.*

This is proven using infinite-dimensional topology; we'll only need a weaker version, which is easier to prove. One says that a continuous map is *cell-like* if its fibers are trivial in a suitable sense.

Theorem 5.8. *Cell-like maps between finite CW complexes are simple homotopy equivalences.*

5.4. **Geography.** There's a simpler version of Theorem 5.7:

Theorem 5.9. *Let X and Y be finite CW complexes and consider a homeomorphism $f : X \xrightarrow{\cong} Y$ such that for all cells e of Y, $f^{-1}(e)$ is a union of cells. Then, f is a simple homotopy equivalence.*

The idea of the proof is to use Whitehead torsion, which is a proof by induction on the skeletons of the CW complexes.

Definition 5.10. A topological space X has *trivial shape* if all continuous maps $X \to Y$, where Y is a CW complex, are contractible.

This is in opposition to the notion of a homotopy type: maps into a space X (by spheres) determine contractibility, and this is the dual picture, where we test a space by the maps out of it. This is a subject called *shape theory*. One example is that the *Warsaw circle* (which you get by taking the graph of $\sin(1/x)$ for $x > 0$ and wrap it back around akin to a circle) is simply connected, but has highly nontrivial shape.

Proposition 5.11. *Suppose X has trivial shape. Then,*

1. *X is connected,*
2. *every locally constant sheaf on X is constant, and*
3. *for every abelian group A, the sheaf cohomology $H^i(X; \underline{A})$ in the constant sheaf $\underline{A}$ is equal to A in degree 0 and 0 in all other degrees.*

Definition 5.12. Let $f : X \to Y$ be a continuous map of topological spaces. Then, f is *cell-like* if

1. f is a closed map,
2. all fibers of f are compact, and
3. all fibers of f have trivial shape.

The weaker version of Theorem 5.8:

Theorem 5.13. *Any cell-like, cellular map $f : X \to Y$, where X and Y are finite CW complexes, such that for every cell $e \subset Y$, $f^{-1}(e)$ is a union of cells, is a simple homotopy equivalence.*

The proof goes through the following lemma.

Lemma 5.14. *Any cell-like map between finite CW complexes is a homotopy equivalence.*

The proof of the lemma proceeds in two steps: First we show that a cell-like map $f : X \to Y$ induces an equivalence of fundamental groupoids by showing that f induces an equivalence between locally constant sheaves on X and Y. Then we show that for any local system $\mathscr{A}$ on Y we get isomorphisms $H^*(Y; \mathscr{A}) \to H^*(X; f^*\mathscr{A})$, so that we obtain a homotopy equivalence by Whitehead's theorem.

The proof of Theorem 5.13 then is almost identical to that of Theorem 5.9, as we have similar starting conditions.

6. Fibrations of Polyhedra: 10/12/16

"These... things give you an equivalence of things."

Today Yuri spoke about polyehdral (or PL) fibrations.

Last time, we talked about simple homotopy equivalences of finite polyhedra $X \xrightarrow{\simeq} Y$, and that this is the same thing as giving a PL fibration $E \to [0,1]$ together with identifications of the fibers $X \simeq E_0$ and $Y \simeq E_1$. There's a general theme here, that fibrations allow one to understand functors in more general categories — indeed, $[0,1]$ is a category (regarded as the simplicial set $\Delta^1 = (0 \to 1)$), and a fibration over $[0,1]$ is a functor from $[0,1]$ to the category whose objects are finite polyhedra and whose morphisms are simple homotopy equivalences.

As we are homotopy theorists, we'd like to construct a universal PL fibration over general polyhedra. This doesn't necessarily exist in the world of finite polyhedra, so we'll need to expand our horizons and work with simplicial sets. We won't define simplicial sets explicitly in this lecture, but the idea is to think of a simplicial set X as a collection of n-simplices X_n for every $n \geq 0$, along with maps between $X_n \to X_{n-1}$ (identifying the faces of an n-simplex) and $X_n \to X_{n+1}$ (identifying an n-simplex as a degenerate $(n-1)$-simplex). Simplicial sets behave combinatorially, and model both topological spaces and (∞-)categories. For example, the simplicial set $\bullet \rightrightarrows \bullet$ is an analogue to the circle S^1 (each arrow is a semicircle) and looks like a small commutative diagram.

Since simplicial sets model categories, we can sometimes realize moduli spaces for simplicial sets as the category of objects we're considering. Accordingly, let $\mathscr{M}$ be the simplicial set whose set of n-simplices $\mathscr{M}_n$ are the set of all finite polyhedra $E \subset \Delta^n \times \mathbb{R}^\infty$ such that $E \to \Delta^n$ is a PL fibration. Thus, $\Delta^n \to \mathscr{M}_n$ classifies PL fibrations over Δ^n.[9] Even though $\mathscr{M}$ is completely determined by finite simplicial sets, it's not a finite simplicial set, which is typical for moduli spaces.

Definition 6.1. A simplicial set X is *nonsingular* if every nondegenerate simplex $\Delta^n \to X$ is injective.

For example, we can consider S^1 as a map from a 0-simplex to itself. This is singular, because it's realized as a nondegenerate map $\Delta^1 \to S^1$, but this identifies its two endpoints. The realization as two maps $S^1 = \bullet \rightrightarrows \bullet$ is nonsingular, however.

Nonsingular simplicial sets correspond to Δ-complexes.

Proposition 6.2. *If X is a nonsingular, finite simplicial set (i.e. there are finitely many nondegenerate simplices), then $|X|$ is a finite polyhedron.*

Proposition 6.3. *If B is a finite, nonsingular simplicial set, then the set $\mathrm{Hom}_{\mathsf{sSet}}(B, \mathscr{M})$ is naturally identified with the set of finite polyhedra $E \subset |B| \times \mathbb{R}^\infty$ such that $E \to |B|$ is a PL fibration.*

Thus, for nonsingular simplicial sets at least, $\mathscr{M}$ is the moduli space we wanted.

Corollary 6.4. *$\mathscr{M}$ is a Kan complex.*

From the perspective of homotopy theory, this means it's basically a space.

We've already defined concordance of two polyhedra, and this can be thought of as a family of polyhedra over an interval $[0,1]$. Similarly, a single polyhedron is a family over a point. We can generalize this to obtain concordance over any polyhedra.

Definition 6.5. Let $f : X \twoheadrightarrow B$ and $g : Y \twoheadrightarrow B$ be PL fibrations, so X, Y, and B are finite polyhedra. Then, X and Y (really f and g) are *concordant* if there's a PL fibration $h : Z \to B \times [0,1]$ such that the following diagram commutes.

$$\begin{tikzcd}
X \cong h^{-1}(0) \arrow[r, hook] \arrow[dr, two heads, "f"'] & Z \arrow[d, two heads, "h"] & h^{-1}(1) \cong Y \arrow[l, hook'] \arrow[dl, two heads, "g"] \\
& B \times [0,1]. &
\end{tikzcd}$$

Proposition 6.6. *If B is a finite and nonsingular simplicial set, then $[B, \mathscr{M}]$ is naturally identified with the set of PL fibrations over $|B|$ modulo concordance.*

This provides a nice relation between homotopy in the land of simplicial sets and concordance in the land of finite polyhedra. Some of this niceness has something to do with $\mathscr{M}$ being a Kan complex.

Thus, $\mathscr{M}$ is the moduli space we wanted.

Lecture 8 asks when a map between polyhedra is a PL fibration. This is kind of awful, so we'll ask the question in the land of simplicial sets. For the rest of the lecture, all simplicial sets except $\mathscr{M}$ are finite and nonsingular.

Given such a simplicial set X, let $\Sigma(X)$ denote the poset of nondegenerate simplicies of X, ordered under inclusion. The *(barycentric) subdivision* of X, denoted $\mathrm{Sd}(X)$, is the nerve of this poset.

For example, $\Delta^1 = (0 \to 1)$, so $\Sigma(X) = \{0, 1, 01\}$, with $0, 1 \leq 01$. $\Sigma(\Delta^2)$ looks like the actual barycentric subdivision of the triangle: 012 is the vertex at the center, 01, 02, and 12 are the vertices at the midpoint of each edge, and 0, 1, and 2 remain where they were.

Proposition 6.7. *There is a natural map $w : X \to \mathrm{Sd}(X)$, and its geometric realization is a homeomorphism.*

[9] We use $\mathbb{R}^\infty$ as an explicit model in order to make it evident that $\mathscr{M}_n$ is really a set, and there are no issues caused by Russell's paradox.

Here's a better way to say some of this: for each $\sigma \in \Sigma(X)$, pick a point $v_\sigma \in |X|$ belonging to the interior of $|\sigma|$. Then, for each inclusion (face map) $\sigma_1 \hookrightarrow \sigma_2$, we add a 1-simplex, for each each pair of composable inclusions fill in a 2-simplex, etc. After this, there is a *unique* PL homeomorphism $|X| \to |\mathrm{Sd}(X)|$ such that σ maps to v_σ. Moreover, this homeomorphism is functorial for embeddings of finite simplicial sets.

Recall that if $f : X \to Y$ is a continuous map of topological spaces, it's called cell-like if it's closed and all of its fibers are compact and have trivial shape. In the same way, if f is a map of finite simplicial sets, we call it cell-like if its geometric realization is cell-like. We can characterize this without using geometric realization.

Proposition 6.8. *The following are equivalent for a map $f : X \to Y$ of simplicial sets.*

(1) f is cell-like.
(2) For all $\sigma \in \Sigma(Y)$, $f^{-1}(\sigma)$ is weakly contractible.
(3) For all $\sigma \in \Sigma(Y)$, the simplicial set $(\sigma \downarrow f) = \{\tau \in \Sigma(X) \mid \sigma \subset f(\tau)\}$ is weakly contractible.

The trickiest part of this is turning weak contractibility into trivial shape.

Now we have to do something slightly confusing. Let $f : P \to Q$ be a map of posets. Then, the assignment $f^* : q \mapsto f^{-1}(q)$ is a functor $Q^{\mathsf{op}} \to \mathsf{Poset}$.

Definition 6.9. Let $f : P \to Q$ be a poset and L be the poset of lifts $\{0 \to 1\} \to P$ in the following diagram

$$\begin{tikzcd} \{1\} \arrow[r] \arrow[d, hook] & P \arrow[d, "f"] \\ \{0 \to 1\} \arrow[r] \arrow[ur, dashed] & Q. \end{tikzcd}$$

If L has a maximal element, then f is called a *Cartesian fibration*.

Equivalently, for all $q, q' \in Q$ such that $q \leq q'$, and for all $p \in P$ such that $f(p) = q$, f is a Cartesian fibration if the set $\{a \in P \mid a \leq p, f(a) \leq q'\}$ has a maximal element p' such that $f(p') = q'$.

The general definition of a Cartesian fibration is more complicated, but useful. It's one of several things called the Grothendieck construction.

Anyways, we use this because if $f : X \to Y$ is a map of (finite, nonsingular) simplicial sets, $\Sigma(f) : \Sigma(X) \to \Sigma(Y)$ is a Cartesian fibration. Specifically, for any $\sigma \in \Sigma(X)$ and $\tau \subseteq f(\sigma)$, the maximal element is $\sigma \cap f^{-1}(\tau)$. This is some kind of right adjoint.

If these simplicial sets model finite polyhedra, when is the realization of f a PL fibration? It's easier to understand this if these simplicial sets are nerves of posets, so we can work with $\mathrm{Sd}(f) : \mathrm{Sd}(X) \to \mathrm{Sd}(Y)$, since this is a nerve, and has the same geometric realization.

Corollary 6.10. *Let $f : P \to Q$ be a Cartesian fibration of posets. The following are equivalent:*

(1) For all $q \in Q$, the fiber $P_q = f^{-1}(q)$ is weakly contractible (meaning its nerve is a weakly contratible simplicial set).
(2) $N(f) : N(P) \to N(Q)$ is cell-like.

Fact. If $f : P \to Q$ is a Cartesian fibration, then the nerve of P is the homotopy limit of $(q \mapsto N(P_q))$ indexed by $q \in Q$.

Proposition 6.11. *Let $f : X \to Y$ be a map of (finite, nonsingular) simplicial sets. The following are equivalent:*

(1) $|f| : |X| \to |Y|$ is a PL fibration.
(2) For all $\sigma' \subset \sigma$ in $\Sigma(Y)$ and every $\tau' \in \Sigma(X)$ such that $f(\tau') = \sigma'$, the simplicial set $\{\tau \in \Sigma(X) \mid f(\tau) = \sigma$ and $\tau' = \tau \cap f^{-1}(\sigma')\}$ has weakly contractible fibers.

7. Combinatorial Models for $\mathscr{M}$: 10/26/16

Today, Richard talked about combinatorial models for $\mathscr{M}$, the moduli space of PL fibrations of simplicial sets.

Recall that we defined $\mathscr{M}$ to be the simplicial set whose n-simplices are the set of finite polyhedra E in $\Delta^n \times \mathbb{R}^\infty$ such that $E \to \Delta^n$ is a PL fibration. We also had a criterion for when the realization of a PL map of simplicial sets is a PL fibration, Proposition 6.11, which says this is true exactly when certain preimages have weakly contractible fibers. It's worth comparing this to Quillen's theorem B.

The combinatorics is encoded in (geometric realizations of) nerves of posets.

Theorem 7.1. *If $f : P \to Q$ is a Cartesian fibration of posets (see Definition 6.9), then The following are equivalent:*

(1) $|N(f)| : |N(P)| \to |N(q)|$ is a Cartesian fibration.
(2) For all $q' \leq q$ inside Q, the induced map of fibers $P_q \to P_{q'}$ is left cofinal*, i.e. for every $\tau \in f^{-1}(q)$, the poset $\{\tau \mid \sigma = f(\tau)\}$ is weakly contractible.*

The power of simplicial sets is that we have a moduli *space*, rather than a moduli set, just defined combinatorially; we'll use Theorem 7.1 to construct some more combinatorial models. From posets we obtain homotopy types.

Proof. Consider $\mathrm{Chain}(P) = \Sigma(N(P))$, and fix a $\sigma \in \mathrm{Chain}(P)$ with image $\tau_0 \in \mathrm{Chain}(Q)$. For any $\tau \geq \tau_0$, let

$$S_\tau = \{\sigma \in \mathrm{Chain}(P) \mid f(\sigma) = \tau \text{ and } \sigma \cap f^{-1}(\tau_0) = \sigma_0\}.$$

Then, Proposition 6.11 shows S_τ is weakly contractible for all τ.

Now, suppose $q' \leq q$; we want to show that $P_q \to P_{q'}$ is left cofinal. If we fix a $p' \in P_{q'}$, then $T = \{p \in P_q \mid p' \leq p\}$ is weakly contractible. Take $\sigma_0 = \{p'\}$ and $\tau_0 = \{q'\}$, so $\tau = \{q, q'\}$. Then, $S_\tau = \mathrm{Chain}(T)$, so $N(S_\tau) = N(\Sigma(N(T)) = \mathrm{Sd}(N(T))$, which is homotopy equivalent to $N(T)$.

Conversely, if $\sigma_0 \in \mathrm{Chain}(P)$ and $\tau \in \mathrm{Chain}(Q)$, we want to show that S_τ is weakly contractible. I didn't understand what happened after this. TODO ⊠

Now, we're going to use Theorem 7.1 to construct some categories whose nerves are weakly equivalent to $\mathscr{M}$.

Definition 7.2. The category $\mathsf{C}_{\mathsf{Cof}}$ is defined by the following data:

Objects: Finite posets.
Morphisms: Maps of posets $f : P \to Q$ such that for all $q' \leq q$ in Q, the map on the fibers $f^{-1}(q') \to f^{-1}(q')$ is cofinal.

Exercise 7.3. Cofinality is a statement about preservation of (homotopy) colimits. What is the relationship between this and other notions of cofinality?

Anyways, we're going to look at $N(\mathsf{C}^{\mathsf{op}}_{\mathsf{Cof}})$. An n-simplex $\sigma \in N(\mathsf{C}^{\mathsf{op}}_{\mathsf{Cof}})$ is a chain of elements $P_0 \leftarrow P_1 \leftarrow \cdots \leftarrow P_n$; let

$$P(\sigma) = \bigcup_{i=0}^{n} P_i,$$

so that $P(\sigma) \to [n]$ is a Cartesian fibration satisfying the second criterion in Theorem 7.1, so we obtain a fibration $|N(P(\sigma))| \to \Delta^n$.

We can also consider the augmented version, $\widetilde{N(\mathsf{C}^{\mathsf{op}}_{\mathsf{Cof}})}$, whose n-simplices are the pairs (f, η), where f is an n-simplex of $N(\mathsf{C}^{\mathsf{op}}_{\mathsf{Cof}})$ and $\eta : |N(P(\sigma))| \to \mathbb{R}^\infty$ is a PL embedding. Forgetting η defines a projection map $\phi : \widetilde{N(\mathsf{C}^{\mathsf{op}}_{\mathsf{Cof}})} \to N(\mathsf{C}^{\mathsf{op}}_{\mathsf{Cof}})$, and sending the map $P_0 \leftarrow \cdots \leftarrow P_n$ to the image $|P(\sigma)| \subset \mathbb{R}^\infty$ defines a map $\psi : \widetilde{N(\mathsf{C}^{\mathsf{op}}_{\mathsf{Cof}})} \to \mathscr{M}$.

The map $\psi \circ \varphi^{-1} : N(\mathsf{C}^{\mathsf{op}}_{\mathsf{Cof}}) \to \mathscr{M}^{\mathsf{op}} \simeq \mathscr{M}$ can be "turned around," and is equivalent to a map $\gamma : N(\mathsf{C}_{\mathsf{Cof}}) \to \mathscr{M}$.

Definition 7.4. Let $\mathsf{Set}^{\mathsf{ns}}_\Delta$ denote the category defined by the following data.

Objects: Finite nonsingular simplicial sets.
Morphisms: Cell-like maps (closed, compact maps with trivial shape).

Since $f : X \to Y$ is a cell-like map iff $\Sigma(X) \to \Sigma(Y)$ is left cofinal, then it defines a functor $\mathsf{Set}^{\mathsf{ns}}_\Delta \to \mathsf{C}_{\mathsf{Cof}}$.

Definition 7.5. Let $\mathsf{C}_{\mathsf{Cell}}$ denote the category defined by the following data.

Objects: Finite posets.
Morphisms: Cartesian fibrations with weakly contractible fibers.

Theorem 7.6. *In the sequence of maps*

$$N(\mathsf{Set}^{\mathsf{ns}}_\Delta) \xrightarrow{\alpha} N(\mathsf{C}_{\mathsf{Cell}}) \xrightarrow{\beta} N(\mathsf{C}_{\mathsf{Cof}}) \xrightarrow{\gamma} \mathscr{M},$$

α, β, and γ are weak homotopy equivalences.

As we slowly march towards manifold topology, it will eventually be very useful to have this combinatorial model for something that relates to stable pesudoisotopy theory, but doesn't look much like it.

Partial proof sketch. For α, suppose $f : P \to Q$ is a Cartesian fibration with weakly contractible fibers, so that it induces a cell-like map $N(P) \to N(Q)$. This defines a functor $F : \mathsf{C}_{\mathsf{Cell}} \to \mathsf{Set}_\Delta^{\mathrm{ns}}$; let $\alpha' = N(F)$. Then, $\alpha' \circ \alpha$ is induced by $N(\Sigma(-)) = \mathrm{Sd}$. We have a map $\mathrm{Sd}(X) \to X$ sending $\sigma_0 \subset \cdots \subset \sigma_n$ to the map $(\Delta^n \to \sigma_n \to X)$, and it's cell-like, so it's weakly homotopic to the identity (there is more to say here).

In the other direction, $\alpha \circ \alpha'$ is given by $P \mapsto \mathrm{Chain}(P)$, so if we let

$$T(P) = \{(p, \sigma) \in P \times \mathrm{Chain}(P) : p' \in \sigma, p' \leq p\},$$

then we get a span $P \leftarrow T(P) \to \mathrm{Chain}(P)$. Both maps are Cartesian fibrations with weakly contractible fibers.

Taking care of β is harder. There's a canonical map $\mathrm{Sd}(X) \to X$ (in essence because $\mathrm{Sd}(X)$ is a colimit of the subdivisions of the simplicies of X, glued along X), and when X is nonsingular, it will be a weak homotopy equivalence. We'll define a map $\delta : \mathrm{Sd}(N(\mathsf{C}_{\mathsf{Cof}}^{\mathrm{op}})) \to N(\mathsf{C}_{\mathsf{Cell}}^{\mathrm{op}})$, which is induced by a functor $v : \Sigma(\Delta^n)^{\mathrm{op}} \to \mathsf{C}_{\mathsf{Cell}}$.

Given a simplex e, defined by a diagram $P_0 \leftarrow \cdots \leftarrow P_n$, and an n-dimensional facet of Δ^n, $\tau = \{i_0 < \cdots < i_m\} \subseteq [n]$, we define

$$v(\tau) = \{(\sigma_0, \ldots, \sigma_m) \in \mathrm{Chain}(P_{i_0}) \times \cdots \times \mathrm{Chain}(P_{i_m}) \mid \sigma_0 \leq \cdots \leq \sigma_n\}.$$

Then, quite a lot of work is needed to show that δ is a homotopy inverse to β. ⊠

The next lecture proves that γ is a weak homotopy equivalence, and the one after that shows that, instead of considering nonsingular simplicial sets, one can use finite simplicial sets.

8. Proof of the Combinatorial Equivalence: 11/2/16

Let's first recall some things that were defined last time.

- $\mathsf{C}_{\mathsf{Cof}}$ is the category whose objects are finite posets and whose morphisms are left cofinal maps. Recall that a map of posets $f : P \to Q$ is left cofinal if for all $q \in Q$, the poset $\{p \in P \mid q \leq f(p)\}$ has a weakly contractible nerve.
- We define a map $\gamma : N(\mathsf{C}_{\mathsf{Cof}}^{\mathrm{op}}) \to \mathscr{M}$. An n-simplex $\sigma \in N(\mathsf{C}_{\mathsf{Cof}}^{\mathrm{op}})$ is a collection of left cofinal maps of posets.

  $$P_0 \longleftarrow P_1 \longleftarrow P_2 \longleftarrow \cdots \longleftarrow P_n$$

  Let $P(\sigma)$ denote the disjoint union of these posets. Let S be the simplicial set whose n-simplices are

  $$S_n = \{(\sigma, e) \mid \sigma \in N(\mathsf{C}_{\mathsf{Cof}}^{\mathrm{op}}), e : |N(P(\sigma))| \hookrightarrow \mathbb{R}^\infty \text{ is a PL embedding}\}.$$

  Forgetting the embedding defines a projection $\phi : S \to N(\mathsf{C}_{\mathsf{Cof}}^{\mathrm{op}})$, and forgetting σ and only remembering $\mathrm{Im}(|P(\sigma)|)$ defines a map $\psi : S \to \mathscr{M}$. Choose a section of ϕ, compose it with ψ, and identify $\mathscr{M} \cong \mathscr{M}^{\mathrm{op}}$ to obtain a map $\gamma : N(\mathsf{C}_{\mathsf{Cof}}^{\mathrm{op}}) \to \mathscr{M}$, which is well-defined up to homotopy.
- A poset is weakly contractible means that its nerve is weakly contractible.

Today, we're going to prove (part of) the following theorem.

Theorem 8.1. *The map $\gamma : N(\mathsf{C}_{\mathsf{Cof}}^{\mathrm{op}}) \to \mathscr{M}$ is a weak homotopy equivalence.*

If X is a finite polyhedron with a specified triangulation, γ induces a map of homotopy groups based at X:

$$\gamma_* : \pi_n(|N(\mathsf{C}_{\mathsf{Cof}}^{\mathrm{op}})|, [\Sigma(X)]) \longrightarrow \pi_n(|\mathscr{M}|, [X])$$

for all $n \geq 0$. By Whitehead's theorem, it suffices to show that γ_* is an isomorphism for all X and n.

One tool we'll need a few times is a homotopy between two maps of simplicial sets with different domains.

Definition 8.2. Let $f : A \to X$ and $f' : A' \to X$ be maps of simplicial sets. We say f and f' are *homotopic* if there exists a simplicial set $\overline{A}$, a map $\overline{f} : \overline{A} \to X$, and trivial cofibrations $i : A \to \overline{A}$ and $i' : A' \to \overline{A}$ such that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[dr, hook, "i", "\sim"'] \arrow[drr, bend left, "f"] & & \\
& \overline{A} \arrow[r, "\overline{f}"] & X \\
A' \arrow[ur, hook, "i'"', "\sim"] \arrow[urr, bend right, "f'"'] & &
\end{tikzcd}$$

Definition 8.3. Let $f : P \to Q$ be a Cartesian fibration of finite posets.[10] Then, f is called *good* if it satisfies any of the following equivalent criteria.

(1) For each inequality $q \le q'$ in Q, the induced map on fibers $P_{q'} \to P_q$ is left cofinal.
(2) For every $p \in P$ and $q \ge f(p)$ in Q, the poset $\{a \in f^{-1}(q) \mid a \ge p\}$ is weakly contractible.
(3) The induced map of spaces $|N(f)| : |N(P)| \to |N(Q)|$ is a fibration.

If $f : P \to Q$ is a good Cartesian fibration, the assignment $q \mapsto P_q$ defines a functor $Q \to \mathsf{C}^{\mathrm{op}}_{\mathsf{Cof}}$, and therefore a map $\chi_f : N(Q) \to N(\mathsf{C}^{\mathrm{op}}_{\mathsf{Cof}})$. Then, the composite $\gamma \circ \chi_f : N(Q) \to \mathscr{M}$ classifies the PL fibration $|N(P)| \to |N(Q)|$. This will be useful.

Definition 8.4. Let $f : P \to Q$ be a Cartesian fibration of posets. Then, $(p, p') \in P \times P$ is *Cartesian* if p is a maximal element of $\{a \in P \mid a \le p', f(a) \le f(p)\}$.

Given any map of posets $f : P \to P'$, we let $P' \amalg_f P$ denote the poset whose underlying set is $P' \amalg P$. The ordering keeps the orderings on P' and P, and defines $p' \le p$ (for $p' \in P'$ and $p \in P$) iff $p' \le f(p)$ in P'. Let's prove some properties about this thing.

Proposition 8.5. *Suppose we're given a commutative diagram of posets*

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "u"] & P' \arrow[d, "u'"] \\ Q \arrow[r, "g"] & Q'. \end{tikzcd}$$

Then, the induced map of sets $h : P' \amalg_f P \to Q' \amalg_f Q$ is order-preserving. Moreover, h is Cartesian iff f and g are and f carries Cartesian pairs in P to Cartesian pairs in P'.

The idea is that a lift to $P' \amalg_f P$ determines and is determined by lifts to P and P', but we need the lift to be a map of posets, yielding the extra condition.

Proposition 8.6. *With notation as in Proposition 8.5, h is good iff f and g are and for each $q \in Q$, the induced map of fibers $P_q \to P'_{g(q)}$ is left cofinal.*

With the same notation as in Proposition 8.5, suppose that u and u' are good Cartesian fibrations between finite posets, and suppose g is a weak homotopy equivalence, so that $N(Q) \hookrightarrow N(Q' \amalg_g Q)$ is a weak homotopy equivalence, and same for $N(Q')$. The map $P' \amalg_f P \to Q' \amalg_g Q$ is a Cartesian fibration, hence classified by a map $\overline{u} : N(Q' \amalg_g Q) \to N(\mathsf{C}^{\mathrm{op}}_{\mathsf{Cof}})$; this map, together with the weak equivalences above, define a homotopy (in the sense of Definition 8.2) between χ_u and $\chi_{u'}$.

Now, we can make some progress.

Proposition 8.7. *Let $u : P \to Q$ be a good Cartesian fibration of finite posets and $e : N(Q) \to N(\mathsf{C}^{\mathrm{op}}_{\mathsf{Cof}})$ be its classifying map. Then, $\gamma(e)$ classifies the PL fibration $q : |N(P)| \to |N(Q)|$, so we can form $\chi_q : N(\Sigma(N(Q))) = N(\mathrm{Chain}(Q)) \to N(\mathsf{C}^{\mathrm{op}}_{\mathsf{Cof}})$. Then, χ_q and e are homotopic.*

Proof. For a poset P, let $T(P) = \{(p, \sigma) \subset P \times \mathrm{Chain}(P) \mid p \le \min(\sigma)\}$. If $u : P \to Q$ is a Cartesian fibration, then so is $T(u) : T(P) \to T(Q)$, and if u is good, so is $T(u)$. In particular, the Cartesian pairs of $T(u)$ are pairs $((p, \sigma), (p', \sigma'))$ such that (p, p') and (σ, σ') are Cartesian pairs of u and $\mathrm{Chain}(u)$, respectively. Over each $(p, \sigma) \in T(Q)$, there's a correspondence

$$\begin{tikzcd} & T(P)_{(q,\sigma)} \arrow[dl] \arrow[dr] & \\ \mathrm{Chain}(P)_\sigma & & P_q \end{tikzcd}$$

[10] Recall that a Cartesian fibration of posets is a fibration, in the sense that it has a lift across $\{0\} \hookrightarrow \{0 < 1\}$, and is Cartesian in the sense that the poset of lifts has a maximal element.

where the maps are projections. Both projections are Cartesian fibrations whose fibers are weakly contractible, so are left cofinal. u induces a diagram

$$\begin{tikzcd}
\mathrm{Chain}(P) \arrow[d, "\mathrm{Chain}(u)"] & T(P) \arrow[l] \arrow[r] \arrow[d, "T(u)"] & P \arrow[d, "u"] \\
\mathrm{Chain}(Q) & T(Q) \arrow[l] \arrow[r] & Q.
\end{tikzcd}$$

Each horizontal arrow is a good Cartesian fibration, and therefore, as described above, we have homotopies $\chi_q = \chi_{\mathrm{Chain}(u)} \simeq \chi_{T(u)} \simeq \chi_u = e$. ⊠

Next, we'll show that the χ_q construction isn't too sensitive to our choices of triangulations.

Proposition 8.8. *Suppose we have a commutative diagram of triangulated polyhedra*

$$\begin{tikzcd}
E' \arrow[r] \arrow[d, "q'"] & E \arrow[d, "q"] \\
B' \arrow[r] & B
\end{tikzcd}$$

where the horizontal maps are PL homeomorphisms and the vertical maps are fibrations respecting the triangulations. Then, $\chi_q : N(\Sigma(B)) \to N(\mathsf{C}^{\mathrm{op}}_{\mathrm{Cof}})$ *and* $\chi_{q'} : N(\Sigma(B')) \to N(\mathsf{C}^{\mathrm{op}}_{\mathrm{Cof}})$ *are homotopic.*

Proof. Without loss of generality, assume the triangulations on E' and B' refine those on E and B, respectively.

Let

$$P = \{(\sigma, \sigma') \in \Sigma(E) \times \Sigma(E') \mid \sigma' \subseteq \sigma\}$$
$$Q = \{(\sigma, \sigma') \in \Sigma(B) \times \Sigma(B') \mid \sigma' \subseteq \sigma\}.$$

q and q' induce a map $a : P \to Q$, which is a Cartesian fibration.

(1) First, we will show that a is a good Cartesian fibration. It's a Cartesian fibration because $\Sigma(E) \to \Sigma(B)$ and $\Sigma(E') \to \Sigma(B')$ are; to show that it's good, we must prove that for every pair $(\sigma_0', \sigma_0) \in P$ and $(\tau', \tau) \in Q$ such that $\tau' \supset \sigma_0'$ and $\tau \supset \sigma_0$, then the set
$$C = \{(\sigma', \sigma) \in P \mid q(\sigma') = \tau', q(\sigma) = \tau, \sigma_0' \subset \sigma', \sigma_0 \subset \sigma\}$$
is weakly contractible. Because the map $\Sigma(E) \to \Sigma(B)$ is good, then $D = \{\sigma \in \Sigma(E) \mid q(\sigma) = \tau, \sigma_0 \subset \sigma\}$ is weakly contractible. Projection defines a map $z : C^{\mathrm{op}} \to D^{\mathrm{op}}$, which is a Cartesian fibration, so to show C is contractible, it suffices to show that the fibers of z are weakly contractible. The fiber over a $\sigma \in D$ is
$$z^{-1}(\sigma) = \{\sigma' \in \Sigma(E') \mid \sigma_0' \subset \sigma' \subset \sigma, q(\sigma') = \tau'\}.$$
Since q is a fibration, the combinatorial criterion we gave a few lectures ago proves $z^{-1}(\sigma)$ is contractible.

(2) Projections define the horizontal arrows in the diagram

$$\begin{tikzcd}
\Sigma(E) \arrow[d, "b"] & P \arrow[l] \arrow[r] \arrow[d] & \Sigma(E') \arrow[d, "b'"] \\
\Sigma(B) & Q \arrow[l] \arrow[r] & \Sigma(B').
\end{tikzcd}$$

We'll show these satisfy the hypotheses of Propositions 8.5 and 8.6.

- First we need to show that for every pair $(\tau', \tau) \in Q$, the map $h : P_{(\tau', \tau)} \to \Sigma(E)_\tau$ is left cofinal. That is, for every $\sigma_0 \in q^{-1}(\tau) \subset \Sigma(E)$, we want the poset
$$C = \{(\sigma', \sigma) \in P \mid q(\sigma') = \tau', q(\sigma) = \tau, \sigma_0 \subseteq \sigma\}$$
to be weakly contractible.
The related poset $D = \{\sigma \in \Sigma(E) \mid q(\sigma) = \tau, \sigma_0 \subseteq \sigma\}$ has a minimal element σ_0, and hence is weakly contractible. Forgetting the first factor defines a map $z : C^{\mathrm{op}} \to D^{\mathrm{op}}$ which is a Cartesian fibration, and hence it suffices to show that the fibers of z are weakly contractible. The fiber over $\sigma \in D$ is $\{\sigma' \in \Sigma(E') \mid \sigma' \subset \sigma, f(\sigma') \subset \tau'\}$. Two talks ago, we had a combinatorial criterion for fibrations

where this poset is weakly contractible iff $E' \to B'$ is a fibration, but we knew that, so $z^{-1}(\sigma)$ is weakly contractible as needed.

- Then we must show that for every pair $(\tau', \tau) \in Q$, the other projection map $h' : P_{(\tau',\tau)} \to \Sigma(E')_{\tau'}$ is left cofinal. This is a Cartesian fibration, so it suffices to show its fibers are weakly contractible. Explicitly, given a $\sigma' \in \Sigma(E')$ such that $q(\sigma') = \tau'$, we need $(h')^{-1}(\sigma') = \{\sigma \in \Sigma(E) \mid q(\sigma) = \tau, \sigma' \subset \sigma\}$ to be weakly contractible. There is a smallest simplex σ_0 of E that contains σ', which means $(h')^{-1}(\sigma') = \{\sigma \in \Sigma(E) \mid q(\sigma) = \tau, \sigma_0 \subseteq \sigma\}$. So we can use the combinatorial criterion again to show this poset is weakly contractible.

Then, Proposition 8.6 defines homotopies $\chi_b \simeq \chi_a \simeq \chi_{b'}$, just as in the proof of Proposition 8.7. ⊠

The proof of Theorem 8.1 almost follows from these two propositions; we'll need one more result which keeps track of basepoints. With notation as in Proposition 8.8, the homeomorphism $E \to E'$ defines a path p from $[\Sigma(E)] \to [\Sigma(E')]$ inside $N(\mathsf{C}^{\mathrm{op}}_{\mathsf{Cof}})$.

Proposition 8.9. *The homotopy in Proposition 8.8 may be chosen such that $\gamma(p)$ is homotopic to the constant loop in $\mathscr{M}$.*

Proof of Theorem 8.1. We saw that it suffices to show γ_* is an isomorphism.

Injectivity: Let $\eta \in \pi_n(|N(\mathsf{C}^{\mathrm{op}}_{\mathsf{Cof}})|, [\Sigma(X)])$ be such that $\gamma_*(\eta)$ vanishes. We want to represent it by a map of simplicial sets $\partial\Delta^{n+1} \to N(\mathsf{C}^{\mathrm{op}}_{\mathsf{Cof}})$; this is possible, but we must subdivide $\partial\Delta^{n+1}$ some finite number of times. In particular, there is some finite poset Q such that $|N(Q)| \cong S^n$ and η is represented by a map $f : N(Q) \to N(\mathsf{C}^{\mathrm{op}}_{\mathsf{Cof}})$, which is equivalent to data of a functor $Q \to \mathsf{C}^{\mathrm{op}}_{\mathsf{Cof}}$, hence a PL fibration $q : |N(P)| \to |N(Q)| \cong S^n$.

Since $\gamma_*(\eta)$ vanishes, this fibration extends to a fibration $\overline{q} : E \to B$ for $B \cong D^{n+1}$, which is contractible. We can choose triangulations of E and B such that $\overline{q}$ respects them, such that the images of $|N(P)|$ and $|N(Q)|$ are subcomplexes $E' \subset E$, resp. $B' \subset B$, and such that the triangulations on E' and B' are refinements of those on $N(P)$ and $N(Q)$, respectively. Thus we have a map of simplicial sets $\chi_{\overline{q}} : N(\Sigma(B)) \to N(\mathsf{C}^{\mathrm{op}}_{\mathsf{Cof}})$. Using Propositions 8.8 and 8.7, one can show that the restriction of $\chi_{\overline{q}}$ to $N(\Sigma(B_0))$ is homotopic to f, and hence $\chi_{\overline{q}}$ is a homotopy from f to a constant map, so $\eta = 0$ and γ_* is injective.

Surjectivity: Let $\eta \in \pi_n(|\mathscr{M}|, [X])$, and let $f : B = \partial\Delta^{n+1} \to \mathscr{M}$ represent it. This is the classifying map for a PL fibration $q : E \to |B|$ with fiber X. Let $b \in |B|$ be a basepoint.

Choose triangulations of E and $|B|$ such that q respects them (and such that b is a vertex of $|B|$), so we obtain a map $\chi_q : N(\Sigma(|B|)) \to N(\mathsf{C}^{\mathrm{op}}_{\mathsf{Cof}})$ such that $\gamma(\chi_q) \simeq f$. However, *a priori* we don't know about basepoints; Proposition 8.9 provides for us a path $[\Sigma(E_b)]$ to $[\Sigma(X)]$ whose image in $\mathscr{M}$ is homotopic to the constant path, allowing us to replace the homotopy with a based homotopy. Hence, $\eta \in \mathrm{Im}(\gamma_*)$. ⊠

9. Looking Forward and ∞-categories: 11/9/16

This week, Rok Gregoric spoke, about the introduction to the second part of Lurie's notes: connecting what we've been doing, and especially the moduli space $\mathscr{M}$ of simple homotopy types, to Waldhausen's K-theory and the Wall finiteness obstruction.

That $\mathscr{M}$ is the moduli space of simple homotopy types means that its n-simplices are the set of finite polyhedra in $\Delta^n \times \mathbb{R}^\infty$ together with projections $X \to \Delta^n$. We'd like to go from simple homotopy types to some kind of "underlying" homotopy types.

Let Q be some "big" contractible space, e.g. the Hilbert cube.[11] Then, let $\mathscr{M}^h_n$ denote the set of finitely dominated $X \subseteq \Delta^n \times Q$ together with fibrations $X \to \Delta^n$; these sets stitch together into a simplicial set $\mathscr{M}^h$.

An inclusion $\mathbb{R}^\infty \hookrightarrow Q$ defines a map $\mathscr{M} \to \mathscr{M}^h$. Given a fibration $p : E \to X$, we can take the homotopy fiber product

$$\begin{tikzcd} \mathfrak{S}(X) \arrow[r] \arrow[d] & \mathscr{M} \arrow[d] \\ * \arrow[r, "{[X]}"'] & \mathscr{M}^h, \end{tikzcd}$$

[11] What does "big" mean? The hope is that everything you want to care about can be embedded into it. In a similar way, the bordism category is constructed by starting with $\mathbb{R}^\infty$, in which every bordism can be embedded.

which in a sense controls the original question. We'd thus like to study $\mathfrak{S}(X)$. If it's nonempty, its connected components are $\pi_0\mathfrak{S}(X) \cong \mathrm{Wh}(X)$ (recall the Whitehead group is $\mathrm{Wh}(X) = \widetilde{K}_1(\mathbb{Z}[G])/\mathrm{Im}(G)$). In particular, $\pi_0\mathfrak{S}(X)$ is a group. But there's also more structure: explicitly, its n-simplices $\mathfrak{S}(X)_n$ are the set of equivalence classes of pairs (E,φ) where E is a finite polyhedron, $\varphi : \Delta^n \times X \xrightarrow{\sim} E$, where we say $(E,\varphi) \cong (E',\varphi')$ iff the induced homotopy equivalence $E \cong E'$ is simple.

There's more structure: there's a map $\mu : \mathfrak{S}(X)_n \times \mathfrak{S}(X)_n \to \mathfrak{S}(X)_n$ defined by

$$(\Delta^n \times X \to E, \Delta^n \times X \to E') \longmapsto \Delta^n \times X \to E \amalg_{\Delta^n \times X} E'.$$

There's something to check here (e.g. why is $E \amalg_{\Delta^n\times X} E'$ a finite polyhedron?), but in the end μ is associative, commutative, and unital, up to coherent homotopy. In particular, it makes $\mathfrak{S}(X)$ into an $\mathbb{E}_\infty$ space.[12] In particular, we can get a spectrum from it. The goal is now to understand this spectrum better. Namely, we want to find a description for it that doesn't depend on the gigantic spaces $\mathscr{M}$ or $\mathscr{M}^h$.

Algebraic K-theory of spaces answers our cry: though it's not the most computable thing in the world, it's very helpful here. Specifically, there will be a functor $A : \mathsf{Top} \to \mathsf{Spectra}$ such that if $x \in X$, it defines a map $A(*) \to A(X)$ which is "continuous" with respect to x, and then an assembly map $\alpha : A(*) \wedge X_+ \to A(X)$. This map will take some time to construct, but is a powerful tool to understand $\mathfrak{S}(X)$. Specifically, there is a homotopy cofiber diagram

$$\begin{tikzcd}
\mathfrak{S}(X) \arrow[r] \arrow[d] & \Omega^\infty(A(*)\wedge X_+) \arrow[d, "\Omega^\infty\alpha"] \\
* \arrow[r] & \Omega^\infty(A(X)).
\end{tikzcd}$$

Definition 9.1. The *Whitehead spectrum* of X, denoted $\underline{\mathrm{Wh}}(X)$, is the cofiber of α.

This is useful because, in the ∞-category of spectra, fiber and cofiber sequences are the same (since it's a stable ∞-category). In particular, we have a sequence $A(*) \wedge X_+ \to A(X) \to \underline{\mathrm{Wh}}(X)$, and this implies $\mathfrak{S}(X) \simeq \Omega^{\infty+1}\underline{\mathrm{Wh}}(X)$.

And now for something completely different. To continue further, we're going to need to use some ∞-categories. The intuition is that, instead of just having objects and morphisms, there should be 2-morphisms between morphisms, and 3-morphisms between those 2-morphisms, and so on (there's a lot of work that goes into that "and so on").

The simplest case is when all morphisms, 2-morphisms, 3-morphisms, etc., are invertible. Such an ∞-category is called an ∞*-groupoid*. Since the morphisms are invertible, one can think of them as simplices that fit together like in simplicial sets, and in particular, ∞-groupoids should be the same thing as topological spaces.

Suppose that all 2-morphisms, 3-morphisms, etc., are invertible. This structure is called an $(\infty,1)$*-category*. These are the ∞-categories we'd like to study. There are many ways to formalize this; the one we use was popularized by Lurie and Joyal.

Definition 9.2 (Quasicategories)**.** By an ∞*-category* we mean a simplicial set C satisfying a *horn-filling condition*: for $0 \le i \le n$, let Λ^n_i denote Δ^n minus its i^{th} face. Then, we ask that for $0 < i < n$, any map $f : \Lambda^n_i \to \mathsf{C}$ factors through the inclusion to Δ^n as follows:

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "f"] \arrow[d, hook] & \mathsf{C}. \\
\Delta^n \arrow[ur, dashed] &
\end{tikzcd}$$

Example 9.3. Let C be an ordinary category. Then, its nerve $N(\mathsf{C})$ is a simplicial set whose n-simplices are the set of composable strings of morphisms in C with length n. In this case, the inner horns may always be filled: we want to fill in things of the form $\bullet \xrightarrow{g} \bullet \xrightarrow{f} \bullet$ with $g \circ f$, but the outer horns are of the form $\bullet \to \bullet \leftarrow \bullet$, which cannot be filled in unless all morphisms are invertible.

Thus, $N(\mathsf{C})$ satisfies the horn-filling condition in a unique way, hence is an ∞-category.

[12]This is more motivational than useful: we're going to give a different realization of this multiplication eventually using algebraic K-theory and actually use that.

Example 9.4. Let X be a topological space and $\mathrm{Sing}_\bullet(X)$ denote its singular complex, the simplicial set whose n-simplices are the set of maps $|\Delta^n| \to X$. Then, $\mathrm{Sing}_\bullet(X)$ satisfies the horn-filling condition: a map $|\Lambda^n_i| \to X$ can be filled in in a degenerate way, but sometimes also in nontrivial ways. Thus, in this case, $\mathrm{Sing}_\bullet(X)$ satisfies the horn-filling condition, but not in a unique way.

Definition 9.5. An ∞-category C such that every map from an inner horn C can be filled in a unique way is called a *Kan complex*.

Just about anything you do with ordinary categories can also be done with ∞-categories: there are analogues of limits and colimits, commutative diagrams, initial objects, and much more.

Definition 9.6. One new construction is the *homotopy category* $h\mathsf{C}$ associated to an ∞-category C. This is an ordinary category described by the following data.

Objects: the set C_0 of 0-simplices of C.

Morphisms: are the set C_1 modulo an equivalence relation: given a 2-simplex with edges x, y, and z (so that $y = z \circ x$), we set $x \sim y$.

The homotopy category is left adjoin to the nerve functor, which is another way to define it.

∞-categories generalize both categories and spaces. As an example of the latter, it's possible to define a space of maps associated to $C, C' \in \mathsf{C}$, $\mathrm{Hom}_{\mathsf{C}}(C, C') \in \mathsf{Top}$, the ∞-category of topological spaces.

We'd like to define the algebraic K-theory of an ∞-category. We'll start with K_0.

Definition 9.7. An ∞-category C is *pointed* if it has a *zero object*, i.e. an object that is both initial and terminal.

One fun subtlety is that initial and terminal objects aren't unique, even in ordinary categories. For ordinary categories, they're unique up to unique isomorphism, which is good, but here we have to relate the isomorphisms, etc. It's more precise to say that initial, terminal, and zero objects are unique up to a contractible choice of data, if they exist.

Definition 9.8. Let $f : X \to Y$ be a map in a pointed ∞-category C, where $* \in \mathsf{C}$ is the basepoint, and form the pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ * \arrow[r] & \mathrm{cofib}(f). \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

The pushout $\mathrm{cofib}(X)$ is called the *cofiber* of X.

- The *suspension* of X, denoted ΣX, is the cofiber of the map $X \to *$.
- The *loop space* of X, denoted ΩX, is the fiber of the map $* \to X$.

A sequence $X \xrightarrow{f} Y \to \mathrm{cofib}(f)$ a *cofiber sequence*.

Now, we have enough information to define K_0. The idea is that cofiber sequences are the analogue of short exact sequences, and, like in topological K-theory, we'll split them.

Definition 9.9. Let C be a pointed ∞-category that contains finite pushouts. Then, $K_0(\mathsf{C})$ is the abelian group with generators $[X]$ for every $X \in \mathsf{C}$ and relations $[X] = [X'] + [X'']$ for all cofiber sequences $X' \to X \to X''$.

If $F : \mathsf{C} \to \mathsf{D}$ is a functor that preserves finite colimits, then it defines a homomorphism $F_* : K_0(\mathsf{C}) \to K_0(\mathsf{D})$. Notice also that since $* \to * \to *$ is a cofiber sequence, $[*] = 2[*] = 0$. Since $X \to * \to \Sigma X$ is a cofiber sequence, then $[\Sigma X] = -[X]$, so $\Sigma : \mathsf{C} \to \mathsf{C}$ maps to the homomorphism $K_0(\mathsf{C}) \to K_0(\mathsf{C})$ that multiplies by -1.

K_0 isn't interesting for large ∞-categories, meaning those with all countable coproducts. Specifically, for any X, there's a cofiber sequence

$$\coprod_{i \ge 1} X \longrightarrow \coprod_{i \ge 0} \longrightarrow X,$$

and therefore $[X] = 0$ (if Y is the infinite coproduct, then we have $[Y] + [X] = [Y]$). This is known as the *Eilenberg swindle*, and relates to the allegory of Hilbert's hotel. Thus, for such an ∞-category C, $K_0(\mathsf{C}) = 0$.

It's also possible to define K-theory of ordinary categories, as long as you have a good notion of cofiber sequences: splitting them defines an abelian group just as in Definition 9.9. Triangulated categories provide an example.

Since suspension is mapped to an invertible morphism, it may be useful to consider ∞-categories for which suspension is an isomorphism.

Definition 9.10. An ∞-category is *stable* if it has a zero object and finite pushouts, and $\Sigma : \mathsf{C} \to \mathsf{C}$ is an equivalence of categories.

Equivalently, fiber and cofiber sequences coincide. This is very related to the classical notion of taking suspensions to understand stable phenomena in algebraic topology.

Every ∞-category has a *stabilization* $\mathsf{C} \to \mathrm{St}(\mathsf{C})$, heuristically a colimit over iterated applications of suspension. The stabilization is a stable ∞-category. Lurie calls this $\mathscr{SW}(\mathsf{C})$, after Spanier and Whitehead.

The final observation is that K_0 of a category is the same as K_0 of its stabilization, which is interesting.

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