M392C NOTES: SPIN GEOMETRY

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These notes were taken in UT Austin's M392C (Spin Geometry) class in Fall 2016, taught by Eric Korman. I live-TeXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Thanks to Adrian Clough for fixing a few typos.

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Lecture 1.

Lie Groups: 8/25/16

There is a course website, located at https://www.ma.utexas.edu/users/ekorman/teaching/spingeometry/. There's a list of references there, none of which we'll exactly follow.

We'll assume some prerequisites for this class: definitely smooth manifolds and some basic algebraic topology. We'll use cohomology, which isn't part of our algebraic topology prelim course, but we'll review it before using it.

Introduction and motivation. Recall that a *Riemannian manifold* is a pair (M, g) where M is an n-dimensional smooth manifold and g is a *Riemannian metric* on M, i.e. a smoothly varying, positive definite inner product on each tangent space T_xM over all $x \in M$.

Definition 1.1. A *local frame* on M is a set of (locally defined) tangent vectors that give a positive basis for M, i.e. a smoothly varying set of tangent vectors that are a basis at each tangent space.

A Riemannian metric allows us to talk about *orthonormal frames*, which are those that are orthonormal with respect to the metric at all points.

Recall that the special orthogonal group is $\mathrm{SO}(n) = \{A \in M_n \mid AA^{\mathrm{T}} = I, \det A = 1\}$. This acts transitively on orthonormal, oriented bases, and therefore also acts transitively on orthonormal frames (as a frame determines an orientation). Conversely, specifying which frames are orthonormal determines the metric g.

In summary, the data of a Riemannian structure on a smooth manifold is equivalent to specifying a subset of all frames which is acted on simply transitively[1] by the group $\mathrm{SO}(n)$. This set of all frames is a *principal* $\mathrm{SO}(n)$*-bundle* over M.

By replacing $\mathrm{SO}(n)$ with another group, one obtains other kinds of geometry: using $\mathrm{GL}(n, \mathbb{C})$ instead, we get almost complex geometry, and using $\mathrm{Sp}(n)$, we get almost symplectic geometry (geometry with a specified skew-symmetric, nondegenerate form).

Remark. Let G be a Lie group and M be a manifold. Suppose we have a principal G-bundle $E \to M$ and a representation[2] $\rho : G \to V$, we naturally get a vector bundle over M.

A more surprising fact is that all[3] representations of $\mathrm{SO}(n)$ are contained in tensor products of the *defining representation* of $\mathrm{SO}(n)$ (i.e. acting on $\mathbb{R}^n$ by orientation-preserving rotations). Thus, all of the natural vector bundles are subbundles of tensor powers of the tangent bundles. That is, when we do geometry in this way, we obtain no exotic vector bundles.

[1] Recall that a group action on X is *transitive* if for all $x, y \in X$, there's a group element g such that $g \cdot x = y$, and is *simple* if this g is unique.
[2] A representation of a group G is a homomorphism $G \to \mathrm{GL}(V)$ for a vector space V. We'll talk more about representations later.
[3] We're only considering smooth, finite-dimensional representations.

If $n \geq 3$, then $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2$, so its double cover is its universal cover. Lie theory tells us this space is naturally a compact Lie group, called the *Spin group* $\mathrm{Spin}(n)$. In many ways, it's more natural to look at representations of this group. The covering map $\mathrm{Spin}(n) \twoheadrightarrow \mathrm{SO}(n)$ precomposes with any representation of $\mathrm{SO}(n)$, so any representation of $\mathrm{SO}(n)$ induces a representation of $\mathrm{Spin}(n)$. However, there are representations of the spin group that don't arise this way, so if we can refine the orthonormal frame bundle to a principal $\mathrm{Spin}(n)$-bundle, then we can create new vector bundles that don't arise as tensor powers of the tangent bundle.

Spin geometry is more or less the study of these bundles, called *bundles of spinors*; these bundles have a natural first-order differential operator called the *Dirac operator*, which relates to a powerful theorem coming out of spin geometry, the Atiyah-Singer index theorem: this is vastly more general, but has a particularly nice form for Dirac operators, and the most famous proof reduces the general case to the Dirac case. Broadly speaking, the index theorem computes the dimension of the kernel of an operator, which in various contexts is a powerful invariant. Here are a few special cases, even of just the Dirac case of the Atiyah-Singer theorem.

- The Gauss-Bonnet-Chern theorem gives an integral formula for the Euler characteristic of a manifold, which is entirely topological. In this case, the index is the Euler characteristic.
- The Hirzebruch signature theorem gives an integral formula for the signature of a manifold.
- The Grothendieck-Riemann-Roch theorem, which gives an integral formula for the Euler characteristic of a holomorphic vector bundle over a complex manifold.

Lie groups and Lie algebras.

Definition 1.2. A *Lie group* G is a smooth manifold with a group structure such that the multiplication map $G \times G \to G$ sending $g_1, g_2 \mapsto g_1 g_2$ and the inversion map $G \to G$ sending $g \mapsto g^{-1}$ are smooth.

Example 1.3.

- The *general linear group* $\mathrm{GL}(n, \mathbb{R})$ is the group of $n \times n$ invertible matrices with coefficients in $\mathbb{R}$. Similarly, $\mathrm{GL}(n, \mathbb{C})$ is the group of $n \times n$ invertible complex matrices. Most of the matrices we consider will be subgroups of these groups.
- Restricting to matrices of determinant 1 defines $\mathrm{SL}(n, \mathbb{R})$ and $\mathrm{SL}(n, \mathbb{C})$, the *special linear groups*.
- The *special unitary group* $\mathrm{SU}(n) = \{A \in \mathrm{GL}(n, \mathbb{C}) \mid A\overline{A}^{\mathrm{T}} = 1, \det A = 1\}$.
- The special orthogonal group $\mathrm{SO}(n)$, mentioned above.

Definition 1.4. A *Lie algebra* is a vector space $\mathfrak{g}$ with an anti-symmetric, bilinear pairing $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the *Jacobi identity*

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

for all $X, Y, Z \in \mathfrak{g}$.

Example 1.5. The basic and important example: if A is an algebra,[4] then A becomes a Lie algebra with the commutator bracket $[a, b] = ab - ba$. Because this algebra is associative, the Jacobi identity holds.

The Jacobi identity might seem a little vague, but here's another way to look at it: if $\mathfrak{g}$ is a Lie algebra and $X \in \mathfrak{g}$, then there's a map $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ sending $Y \mapsto [X, Y]$. The map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ sending $X \mapsto \mathrm{ad}_X$ is called the *adjoint representation* of X. The Jacobi identity says that ad intertwines the bracket of $\mathfrak{g}$ and the bracket induced from the algebra structure on $\mathrm{End}(\mathfrak{g})$ (where multiplication is composition): $\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y]$. In other words, the adjoint representation is a homomorphism of Lie algebras.

Lie groups and Lie algebras are very related: to any Lie group G, let $\mathfrak{g}$ be the set of left-invariant vector fields on G, i.e. if $L_g : G \to G$ is the map sending $h \mapsto gh$ (the *left multiplication* map), then $\mathfrak{g} = \{X \in \Gamma(TG) \mid \mathrm{d}L_g X = X \text{ for all } g \in G\}$. This is actually finite-dimensional, and has the same dimension as G.

Proposition 1.6. *If e denotes the identity of G, then the map $\mathfrak{g} \to T_e G$ sending $X \mapsto X(e)$ is an isomorphism (of vector spaces).*

The idea is that given the data at the identity, we can translate it by g to determine what its value must be everywhere. Vector fields have a Lie bracket, and the Lie bracket of two left-invariant vector fields is again left-invariant, so $\mathfrak{g}$ is naturally a Lie algebra! We will often use Proposition 1.6 to identify $\mathfrak{g}$ with the tangent space at the identity.

[4] By an *algebra* we mean a ring with a compatible vector space structure.

Example 1.7. Let's look at $\mathrm{GL}(n,\mathbb{R})$. This is an open submanifold of the vector space M_n, an n^2-dimensional vector space, as $\det A \neq 0$ is an open condition. Thus, the tangent bundle of $\mathrm{GL}(n,\mathbb{R})$ is trivial, so we can canonically identify $T_I\mathrm{GL}(n,\mathbb{R}) = M_n$. With the inherited Lie algebra structure, this space is denoted $\mathfrak{gl}(n,\mathbb{R})$.

The $n \times n$ matrices are also isomorphic to $\mathrm{End}(\mathbb{R}^n)$, since they act by linear transformations. The algebra structure defines another Lie bracket on this space.

Proposition 1.8. *Under the above identifications, these two brackets are identical, hence define the same Lie algebra structure on* $\mathfrak{gl}(n,\mathbb{R})$.

Remark. This proposition generalizes to all real matrix Lie groups (Lie subgroups of $\mathrm{GL}(n,\mathbb{R})$): the proof relies on a Lie subgroup's Lie algebra being a subalgebra of $\mathfrak{gl}(n,\mathbb{R})$.

So we can go from Lie groups to Lie algebras. What about in the other direction?

Theorem 1.9. *The correspondence sending a connected, simply-connected Lie group to its Lie algebra extends to an equivalence of categories between the category of simply connected Lie groups and finite dimensional Lie algebras over* $\mathbb{R}$.

Suppose G is any connected Lie group, not necessarily simply connected, and $\mathfrak{g}$ is its Lie algebra. If $\widetilde{G}$ denotes the universal cover of G, then $G = \widetilde{G}/\pi_1(G)$. Since $\widetilde{G}$ is simply connected, the correspondence above identifies $\mathfrak{g}$ with it, and then taking the quotient by the discrete central subgroup $\pi_1(G)$ recovers G.

The special orthogonal group. We specialize to $\mathrm{SO}(n)$, the orthogonal matrices with determinant 1. We'll usually work over $\mathbb{R}$, but sometimes $\mathbb{C}$. This is a connected Lie group.[5]

Proposition 1.10. *If* $\mathfrak{so}(n)$ *denotes the Lie algebra of* $\mathrm{SO}(n)$*, then* $\mathfrak{so}(n) = \{X \in \mathfrak{gl}(n,\mathbb{R}) \mid X + X^{\mathrm{T}} = 0\}$.

That is, $\mathfrak{so}(n)$ is the Lie algebra of skew-symmetric matrices.

Proof. If $F : M_n \to M_n$ is the function $A \mapsto A^{\mathrm{T}}A - I$, then the orthogonal group is $\mathrm{O}(n) = F^{-1}(0)$. Since $\mathrm{SO}(n)$ is the connected component of $\mathrm{O}(n)$ containing the identity, then it suffices to calculate $T_e\mathrm{O}(n)$: if 0 is a regular value of F, we can push forward by its derivative. This is in fact the case:

$$\mathrm{d}F_A(B) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F(A + tB) = A^{\mathrm{T}}B + B^{\mathrm{T}}A,$$

which is surjective for $A \in \mathrm{O}(n)$, so $\mathfrak{so}(n) = T_I\mathrm{SO}(n) = \ker(\mathrm{d}F_I) = \{B \in M_n \mid B + B^{\mathrm{T}} = 0\}$. ⊠

The spin group. We'll end by computing the fundamental group of $\mathrm{SO}(n)$; then, by general principles of Lie groups, each $\mathrm{SO}(n)$ has a unique, simply connected double cover, which is also a Lie group. Next time, we'll provide an *a priori* construction of this cover.

Proposition 1.11.

$$\pi_1(\mathrm{SO}(n)) = \begin{cases} \mathbb{Z}, & n = 2 \\ \mathbb{Z}/2, & n \geq 3. \end{cases}$$

Proof. If $n = 2$, $\mathrm{SO}(n) \cong S^1$ through the identification

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \longmapsto e^{i\theta},$$

and we know $\pi_1(S^1) = \mathbb{Z}$.

For $n \geq 3$, we can use a long exact sequence associated to a certain fibration, so it suffices to calculate $\pi_1(\mathrm{SO}(3))$. Specifically, we will define a Lie group structure on S^3 and a double cover map $S^3 \twoheadrightarrow \mathrm{SO}(3)$; since S^3 is simply connected, this will show $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}/2$.

We can identify S^3 with the unit sphere in the quaternions, which is naturally a group (since the product of quaternions is a polynomial, hence smooth).[6] Realize $\mathbb{R}^3$ inside the quaternions as $\mathrm{span}_{\mathbb{R}}\{i,j,k\}$ (the *imaginary quaternions*); then, we'll define $\varphi : S^3 \twoheadrightarrow \mathrm{SO}(3)$: $\varphi(q)$ for $q \in \mathbb{H}$ is the linear transformation $p \mapsto qpq^{-1} \in \mathrm{GL}(3,\mathbb{R})$,

[5]If we only took orthogonal matrices with arbitrary determinant, we'd obtain the *orthogonal group* $\mathrm{O}(n)$, which has two connected components.

[6]This is important, because when we try to generalize to Spin_n for higher n, we'll be using Clifford algebras, which are generalizations of the quaternions.

where p is an imaginary quaternion. We need to check that $\varphi(q)$ lies in $\mathrm{SO}(3)$, which was left as an exercise. We also need to check this is two-to-one, which is equivalent to $|\ker\varphi| = 2$, and that φ is surjective (hint: since these groups are connected, general Lie theory shows it suffices to show that the differential is an isomorphism). ⊠

Lecture 2.

Spin Groups and Clifford Algebras: 8/30/16

Last time, we gave a rushed construction of the double cover of $\mathrm{SO}(3)$, so let's investigate it more carefully. Recall that $\mathrm{SO}(n)$ is the Lie group of special orthogonal matrices, those matrices A such that $AA^{\mathrm{t}} = I$ and $\det A = 1$, i.e. those linear transformations preserving the inner product and orientation. This is a connected Lie group; we'd like to prove that for $n \geq 3$, $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2$. (For $n = 2$, $\mathrm{SO}(2) \cong S^1$, which has fundamental group $\mathbb{Z}$).

We'll prove this by explicitly constructing the double cover of $\mathrm{SO}(3)$, then bootstrapping it using a long exact sequence of homotopy groups to all $\mathrm{SO}(n)$, using the following fact.

Proposition 2.1. *Let G and H be connected Lie groups and $\varphi : G \to H$ be a Lie group homomorphism. Then, φ is a covering map iff $\mathrm{d}\varphi|_e : \mathfrak{g} \to \mathfrak{h}$ is an isomorphism.*[7]

Here $\mathfrak{g}$ is the Lie algebra of G, and $\mathfrak{h}$ is that of H. Facts like these may be found in Ziller's online notes;[8] the intuitive idea is that the condition on $\mathrm{d}\varphi|_e$ ensures an isomorphism in a neighborhood of the identity, which multiplication carries to a local isomorphism in the neighborhood of any point in G.

Now, we construct a double cover of $\mathrm{SO}(3)$. Recall that the *quaternions* are the noncommutative algebra $\mathbb{H} = \mathrm{span}_{\mathbb{R}}\{1, i, j, k\}$, where $i^2 = j^2 = k^2 = ijk = 1$. We can identify $\mathbb{R}^3$ with the imaginary quaternions, the span of $\{i, j, k\}$, and therefore the unit sphere S^3 goes to $\{q \in \mathbb{H} \mid |q|^2 = 1 = q\overline{q}\}$, where the conjugate exchanges i and $-i$, but also j and $-j$, and k and $-k$. This embedding means that if $v, w \in \mathbb{R}^3$, their product as quaternions is

$$vw = -\langle v, w\rangle + v \times w.$$

and in particular

$$vw + wv = -2\langle v, w\rangle. \tag{2.2}$$

If $q \in S^3$ and $v \in \mathbb{R}^3$, then $qvq^{-1} = qv\overline{q}$, i.e. $\overline{qvq^{-1}} = q\overline{v}\overline{q} = -qv\overline{q}$. That is, conjugation by something in S^3 is a linear transformation in $\mathbb{R}^3$, defining a smooth map $\varphi : S^3 \to \mathrm{GL}(3, \mathbb{R})$; we'd like to show the image lands in $\mathrm{SO}(3)$. Let $q \in S^3$; then, we can use (2.2) to get

$$\begin{aligned}
\langle \varphi(q)v, \varphi(q)w\rangle &= -\frac{1}{2}(\varphi(q)v\varphi(q)w + \varphi(q)w + \varphi(q)v) \\
&= -\frac{1}{2}\left(qvwq^{-1} + qwvq^{-1}\right) \\
&= -\frac{1}{2}\left(q(vw + wv)q^{-1}\right) = \langle v, w\rangle,
\end{aligned}$$

using (2.2) again, and the fact that $\mathbb{R} = Z(\mathbb{H})$. Thus, $\mathrm{Im}(\varphi) \subset \mathrm{O}(3)$, but since S^3 is connected, its image must be connected, and its image contains the identity (since φ is a group homomorphism), so $\mathrm{Im}(\varphi)$ lies in the connected component containing the identity, which is $\mathrm{SO}(3)$.

To understand $\mathrm{d}\varphi|_1$, let's look at the Lie algebras of S^3 and $\mathrm{SO}(3)$. The embedding $S^3 \hookrightarrow \mathbb{H}$ allows us to identify $T_{!}S^3$ with the imaginary quaternions. If p and v are imaginary quaternions, so $\overline{p} = -p$, then

$$\begin{aligned}
\mathrm{d}\varphi|_p(v) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg| |_{t=0}\varphi(e^{tp})v \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} e^{tp}ve^{-tp} \\
&= pv - vp.
\end{aligned}$$

Thus, $\ker \mathrm{d}\varphi|_1 = \{p \in \mathbb{R}^3 \mid pv - vp = 0 \text{ for all imaginary quaternions } v\}$. But if something commutes with all imaginary quaternions, it commutes with all quaternions, since the imaginary quaternions and the reals (which

[7]This isomorphism is as Lie algebras, but it's always a Lie algebra homomorphism, so it suffices to know that it's an isomorphism of vector spaces.

[8]https://www.math.upenn.edu/ wziller/math650/LieGroupsReps.pdf.

are the center of $\mathbb{H}$) span to all of $\mathbb{H}$. Thus, the kernel is the imaginary quaternions in the center of $\mathbb{H}$, which is just $\{0\}$; hence, $\mathrm{d}\varphi|_1$ is injective, and since T_1S^3 and $\mathfrak{so}(3)$ have the same dimension, it is an isomorphism. By Proposition 2.1, φ is a covering map, and $\mathrm{SO}(3) = S^3/\ker(\varphi)$.

We'll compute $|\ker\varphi|$, which will be the index of the cover. The kernel is the set of unit quaternions q such that $qvq^{-1} = v$ for all imaginary quaternions v; just as above, this must be the intersection of the real line with S^3, which is just $\{\pm 1\}$. Thus, φ is a double cover map of SO(3); since S^3 is simply connected, $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}/2$.

Exercise 2.3. The Lie group structure on S^3 is isomorphic to SU(2), the group of 2×2 special unitary matrices.

Now, what about $\pi_1(\mathrm{SO}(n))$, for $n \geq 4$? For this we use a fibration. $\mathrm{SO}(n)$ acts on $S^{n-1}\subset\mathbb{R}^n$, and the stabilizer of a point in S^n is all the rotations fixing the line containing that point, which is a copy of $\mathrm{SO}(n-1)$. This defines a fibration

$$\mathrm{SO}(n-1) \longrightarrow \mathrm{SO}(n) \longrightarrow S^{n-1}.$$

More precisely, let's fix the north pole $p = (0,0,\dots,0,1)\in S^{n-1}$; then, the map $\mathrm{SO}(n)\to S^{n-1}$ sends $A\mapsto Ap$; since A is orthogonal, Ap is a unit vector. The action of $\mathrm{SO}(n)$ is transitive, so this map is surjective. The stabilizer of p is the set of all orthogonal matrices with positive determinant such that the last column is $(0,0,\dots,0,1)$. Orthogonality forces these matrices to have block form

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix},$$

where $A\in\mathrm{SO}(n-1)$; thus, the stabilizer is isomorphic to $\mathrm{SO}(n-1)$.

Now, we can use the long exact sequence in homotopy associated to a fibration:

$$\pi_2(S^{n-1}) \xrightarrow{\delta} \pi_1(\mathrm{SO}(n-1)) \longrightarrow \pi_1(\mathrm{SO}(n)) \longrightarrow \pi_1(S^{n-1}).$$

If $n\geq 4$, $\pi_2(S^{n-1})$ and $\pi + 1(S^{n-1})$ are trivial, so $\pi_1(\mathrm{SO}(n)) = \pi_1(\mathrm{SO}(n-1))$ for $n\geq 4$, so they all agree with $\mathbb{Z}?2$, so $\pi_1(\mathrm{SO}(n))\cong\mathbb{Z}/2$ for all $n\geq 4$.

By general Lie theory, the universal cover of a Lie group is also a Lie group.

Definition 2.4. For $n\geq 3$, the *spin group* $\mathrm{Spin}(n)$ is the unique simply-connected Lie group with Lie algebra $\mathfrak{so}(n)$. For $n = 2$, the spin group Spin(2) is the unique (up to isomorphism) connected double covering group of SO(2).

In particular, there is a double cover $\mathrm{Spin}(n)\to\mathrm{SO}(n)$, and $\mathrm{Spin}(3)\cong\mathrm{SU}(2)$.

Right now, we do not have a concrete description of these groups; since $\mathrm{SO}(n)$ is compact, so is $\mathrm{Spin}(n)$, so we must be able to realize it as a matrix group, and we use Clifford algebras to do this.

Clifford algebras. Our goal is to replace $\mathbb{H}$ with some other algebra to realize $\mathrm{Spin}(n)$ as a subgroup of its group of units.

Recall from (2.2) that for $v,w\in\mathbb{R}^3\hookrightarrow\mathbb{H}$, $vw + wv = -2\langle v,w\rangle$. We'll define a universal algebra for this kind of definition.

Definition 2.5. Let $(V,\langle\cdot,\cdot\rangle)$ be an inner product space. Its *Clifford algebra* is

$$\mathrm{C}\ell(V) = T(V)/(v\otimes v + \langle v,v\rangle 1).$$

Here, $T(V)$ is the tensor algebra, and we quotient by the ideal generated by the given relation.

That is, we've forced (2.2) for a vector paired with itself. That's actually sufficient to imply it for all pairs of vectors.

Remark. Though we only defined the Clifford algebra for nondegenerate inner products, the same definition can be made for all bilinear pairings. If one chooses $\langle\cdot,\cdot\rangle = 0$, one obtains the exterior algebra $\Lambda(V)$, and we'll see that Clifford algebras sometimes behave like exterior algebras.

Recall that the tensor algebra is defined by the following universal property: if A is any algebra,[9] and $f : V\to A$ is linear, then there exists a unique homomorphism of algebras $\widetilde{f} : T(V)\to A$ such that the following diagram

[9] Here, an algebra is a unital ring with a compatible real vector space structure.

commutes:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d] & A \\ T(V). \arrow[ur, "\widetilde{f}"'] & \end{tikzcd}$$

That is, as soon as I know what happens to elements of f, I know what to do to tensors.

This implies a universal property for the Clifford algebra.

Proposition 2.6. *Let A be an algebra and $f : V \to A$ be a linear map. Then, $f(v)^2 = -\langle v, v\rangle 1_A$ iff f extends uniquely to a map $\widetilde{f} : \mathrm{C}\ell(V) \to A$ such that the following diagram commutes:*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d] & A \\ \mathrm{C}\ell(V). \arrow[ur, "\widetilde{f}"'] & \end{tikzcd}$$

The map $V \to \mathrm{C}\ell(V)$ is the composition $V \hookrightarrow T(V) \twoheadrightarrow \mathrm{C}\ell(V)$, where the last map is projection onto the quotient.

We'll end up putting lots of structure on Clifford algebras: a $\mathbb{Z}/2$-grading, a $\mathbb{Z}$-filtration, a canonical vector-space isomorphism with the exterior algebra, and so forth.

Important Example 2.7. Let $\Lambda^\bullet V$ denote the exterior algebra on V, the graded algebra whose k^{th} graded piece is wedges of k vectors: $\Lambda^k(V) = \{v_1, \dots, v_k \mid v_j \in V\}$, with the relations $v \wedge w = -w \wedge v$.

Given a $v \in V$, we can define two maps, *exterior multiplication* $\varepsilon(v) : \Lambda^\bullet(V) \to \Lambda^{\bullet-1}(V)$ defined by $\mu \mapsto v \wedge \mu$, and *interior multiplication* $i(v) : \Lambda^\bullet(V) \to \Lambda^{\bullet-1}(V)$ sending

$$v_1 \wedge \cdots \wedge v_k \mapsto \sum_{i=1}^{k} (-1)^{i-1} \langle v, v_i\rangle v_1 \wedge \cdots \wedge \widehat{v}_i \wedge \cdots \wedge v_k,$$

where $\widehat{v}_i$ means the absence of the i^{th} term.

This has a few important properties:

(1) Both of these maps are idempotents: $\varepsilon(v)^2 = i(v)^2 = 0$.

(2) If $\mu_1, \mu_2 \in \Lambda^\bullet(V)$, then

$$i(v)(\mu_1 \wedge \mu_2) = (i(v)\mu_1) \wedge \mu_2 + (-1)^{\deg \mu_1} \mu_1 \wedge i(v)\mu_2.$$

In particular,

$$\varepsilon(v)i(v) + i(v)\varepsilon(v) = \langle v, v\rangle. \tag{2.8}$$

We can use this to define a representation of the Clifford algebra onto the exterior algebra: define a map $c : V \to \mathrm{End}(\Lambda^\bullet(V))$ by $c(v) = \varepsilon(v) - i(v)$. Then, $c(v)^2 = -(\varepsilon(v)i(v) + i(v)\varepsilon(v)) = \langle v, v\rangle$, so by the universal property, c extends to a homomorphism $c : \mathrm{C}\ell(V) \to \mathrm{End}(\Lambda^* V)$.

Given an inner product on V, there is an induced inner product on $\Lambda^\bullet V$: choose an orthonormal basis $\{e_1, \dots, e_n\}$ for V, and then declare the basis $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \mid 1 \le i_1 < \cdots < i_k \le n\}$ to be orthonormal; then, use the dot product associated to that orthonormal basis. This is coordinate-invariant, however.

Theorem 2.9. *Suppose $\{e_1, \dots, e_n\}$ is a basis for V. Then, $\{e_{i_1} e_{i_2} \cdots e_{i_k} \mid i_1 < i_2 < \cdots i_k\}$ (where the product is in the Clifford algebra) is a vector-space basis for $\mathrm{C}\ell(V)$.*

Today, we'll focus on examples, and perhaps prove this later. This tells us that v and w anticommute iff $v \perp w$, and the relations are

$$e_j e_j = \begin{cases} -e_j e_i, & i \neq j \\ -1, & i = j. \end{cases}$$

This is just like the exterior algebra, but deformed: if $i = j$, we get 1 rather than 0. Theorem **??** also tells us that $\dim \mathrm{C}\ell(V) = 2^{\dim V}$.

Example 2.10. $\mathrm{C}\ell(\mathbb{R}^2) \cong \mathbb{H}$ as $\mathbb{R}$-algebras: $\mathrm{C}\ell(\mathbb{R}^2)$ is generated by 1, e_1, and e_2 such that $e_1e_2 = -e_2e_1$ and $e_1^2 = e_2^2 = -1$. Thus, $\{1, e_1, e_2, e_1e_2\}$ is a basis for $\mathrm{C}\ell(\mathbb{R}^2)$, and $(e_1e_2)^2 = e_1e_2e_1e_2 = -e_1^2e_2^2 = -1$.

Thus, the isomorphism $\mathrm{C}\ell(\mathbb{R}^2) \to \mathbb{H}$ extends from $1 \mapsto 1$, $e_1 \mapsto i$, $e_2 \mapsto j$, and $e_1e_2 \mapsto k$.

Example 2.11. Even simpler is $\mathrm{C}\ell(\mathbb{R}) \cong \mathbb{C}$, generated by 1 and e_1 such that $e_1^2 = -1$.

Example 2.12. If we consider the Clifford algebra of $\mathbb{C}$ as a complex vector space, $\mathbb{C}$ is in the center, so $\mathrm{C}\ell_{\mathbb{C}}(\mathbb{C})$ is generated by 1 and e_1 with $ie_1 = e_1i$.