# TOPOLOGICAL QUANTUM FIELD THEORY

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These notes were taken in a class given by Katrin Wehrheim at UC Berkeley in Spring 2020. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

## 1. TQFT: definition and Atiyah's examples: 1/19/20

We begin with the definition of a topological quantum field theory due to Atiyah, now over 30 years ago.

**Definition 1.1.** Fix a base field $k$. A $d$-dimensional *topological quantum field theory* (TQFT) consists of data of, for every closed, oriented, smooth $d$-manifold, a finitely generated $k$-vector space $Z(\Sigma)$, and for every compact, oriented, smooth $(d+1)$-manifold $M$, an element $Z(M) \in Z(\partial M)$, satisfying some axioms.

Many interrelated ideas went into this definition: Segal's mathematical formalization of two-dimensional conformal field theory, mathematical perspectives on quantum field theory (fields, Hilbert spaces, etc.).

Later, Atiyah's definition was packaged more concisely into asking for $Z$ to be a symmetric monoidal functor

$$Z\colon \mathcal{C}ob_{n,n-1} \longrightarrow \mathcal{V}ect_k, \tag{1.2}$$

where $\mathcal{V}ect_k$ is the symmetric monoidal category of $k$-vector spaces with tensor product, and $\mathcal{C}ob_{n,n-1}$ is the cobordism category, whose objects are closed, oriented $(n-1)$-manifolds and whose morphisms are (diffeomorphism classes of) oriented bordisms between them. Cylinders in the cobordism category can be thought of as time evolution, but the inclusion of all other bordisms has something to do with a relativistic perspective.

*Remark* 1.3. Atiyah used $d$ to denote the dimension of space, i.e. the dimension of manifolds assigned vector spaces. These days, it's more common to refer to the cobordism category using the top dimension (what we just called $n$), the "spacetime dimension." ◀

There are many different flavors of the cobordism category. Some of these involve technical details that we have to account for: for example, even compact 0-manifolds are too big to form a set, so to more accurately define $\mathcal{C}ob_{1,0}$ (or in any dimension) we should pick a set of representatives of oriented diffeomorphism classes of $(n-1)$-manifolds.

*Remark* 1.4. There are other ways to work around set-theoretic issues: for example, the topological cobordism category of Galatius, Madsen, Tillmann, and Weiss begins with the space $\mathbb{R}^\infty$ and works with manifolds and bordisms explicitly embedded in $\{t\}\times\mathbb{R}^\infty$, resp. $[t_1,t_2]\times\mathbb{R}^\infty$. Then one must quotient out by diffeomorphisms, just as in the abstract cobordism category, but now we don't just have the "internal diffeomorphisms" of an embedded $M$, but also "external diffeomorphisms" of the ambient space that carry $M$ to something diffeomorphic, but embedded via a different map. We will not work with embedded bordisms, at least for now. ◀

There are several other generalizations we won't discuss today, but are worth mentioning.

- There's notions of *topological conformal field theory* (TCFT) and *homological conformal field theory* (HCFT), in which $\mathcal{C}ob_{2,1}$ is upgraded to a category where bordisms carry some additional structure (e.g. a conformal structure), and we only identify conformally equivalent bordisms.
- In *fully extended* topological quantum field theory, $\mathcal{C}ob_{n,n-1}$ becomes an $(\infty,n)$-category, by allowing manifolds in all dimensions $n$ and below.

In both cases, we must replace the target category $\mathcal{V}ect_k$ with something related, but different.

In these, and in any, generalizations, the overarching question is: *what kind of algebraic structure do we get from these field theories?* To address this question, we generally must first fix a target category. But there are a few "holy grail" theorems in some of these settings.

**Theorem 1.5** (Cobordism hypothesis (Lurie))**.** *A fully extended topological field theory is determined by its value on the* 0*-manifold* $\mathrm{pt}_+$*.*

This is more of a slogan than a theorem, but one can pin it down into a precise theorem, e.g. by making precise what kinds of TQFTs one considers. Here, by "$\mathrm{pt}_+$" we might more generally mean looking at generators and relations of the appropriate bordism category.

**Example 1.6.** In (spacetime) dimension $n = 1$, TQFTs are vacuously fully extended (with the caveat that 1-categorical and $(\infty, 1)$-categorical TQFT aren't quite the same). Then, the theorem is that for any symmetric monoidal category $\mathcal{C}$, $\mathcal{F}un^{\otimes}(\mathcal{C}ob_{1,0}, \mathcal{C})$ is equivalent to the groupoid of *dualizable* objects in $\mathcal{C}$.[1]

**Exercise 1.7.** For $\mathcal{C} = \mathcal{V}ect_k$, check that dualizability is equivalent to being finite-dimensional.[2]

So fixing $\mathcal{C} = \mathcal{V}ect_k$ for now, given a one-dimensional *unoriented* (i.e. manifolds and bordisms in $\mathcal{C}ob_{1,0}$ are not oriented) TQFT $Z$ we get a finite-dimensional vector space $V := Z(\mathrm{pt}_+)$, and a biliear pairing $e\colon V \otimes V \to k$. This pairing must be nondegenerate, as one can show via the "$Z$-diagram" being equivalent to an interval (which is the identity $\mathrm{pt} \to \mathrm{pt}$).

Conversely, given a finite-dimensional vector space $V$ and an inner product $e\colon V \otimes V \to k$, we can build a TQFT $Z_{V,e}\colon \mathcal{C}ob \to \mathcal{V}ect_k$, because there aren't that many diffeomorphism classes of 1-manifolds, so we know generators and relations: the interval, regarded as a bordism from $\mathrm{pt} \to \mathrm{pt}$, is sent to $\mathrm{id}_V$; the interval, regarded as a bordism from $\mathrm{pt} \amalg \mathrm{pt} \to \varnothing$, is sent to $e$; and the interval, regarded as a bordism $\emptyset \to \mathrm{pt} \amalg \mathrm{pt}$, is sent to the adjoint of $e$. ◀

*Remark* 1.8. Some things change in the oriented 1-dimensional case. We don't need the inner product: if you keep careful track of the orientations induced on a boundary, the interval is now a bordism between $\mathrm{pt}_+ \amalg \mathrm{pt}_-$ and $\varnothing$, and one can show that $\mathrm{pt}_- \mapsto V^*$. Then these intervals are sent to the evaluation map $V \otimes V^* \to k$ and the coevaluation map $k \to V \otimes V^*$. ◀

In dimension 1, the cobordism hypothesis feels somewhat silly. But in higher dimensions things can quickly get nontrivial, and difficult. For example, for the oriented 2-dimensional cobordism category (before we extend), this is known by the classification of surfaces: the pair of pants, regarded as a bordism $S^1 \amalg S^1 \to S^1$, and, separately, regarded as a morphism $S^1 \to S^1 \amalg S^1$; the disc, both as a bordism $S^1 \to \varnothing$ and as a bordism $\varnothing \to S^1$; and the cylinder $S^1 \to S^1$. In dimension 1 we just have the circle. But if we try to extend down to points, then discovering generators is more complicated — now we have to determine generators and relations using surfaces with corners. The surface theory isn't that bad, and this will get worse when we care about higher-dimensional manifolds.

And we do care about higher-dimensional manifolds: two key questions in this course will be:

(1) how does this (both the axiomatic structure of TQFT and tools such as the cobordism hypothesis) help build invariants for 3- and 4-manifolds, and
(2) how do geometric/PDE-based invariants of 3- and 4-manifolds yield TQFTs?

With regards to question (2) specifically, Atiyah gave a few examples in his original paper on TQFT.

**Example 1.9.** This example, built on work of Floer and Gromov, is a 2-dimensional TQFT. Fix a symplectic manifold $(X, \omega)$; the quantum field theory here will be about maps $S^1 \to X$. We begin with a "classical phase space" $\mathrm{Map}(S^1, X)$; to a closed, oriented 2-manifold $\Sigma$, we should associate the number of pseudoholomorphic maps $u\colon \Sigma \to X$. There's a lot to define here; what is a pseudoholomorphic map? Defining the number of such maps is also nontrivial; in some settings, there are infinitely many, and we must impose point constraints somehow, which makes the theory feel less topological.

The definition of a pseudoholomorphic map involves a PDE, which will be an interesting thing to dig into.

The theory also has a Lagrangian form. In the Lagrangian form, we instead look at paths in $X$, rather than loops, though we ask that they end on prescribed Lagrangian submanifolds of $X$. These are a kind of boundary condition. ◀

[1] *A priori*, the subcategory of dualizable objects in $\mathcal{C}$ is not a groupoid, but we can make it one by throwing out the non-invertible morphisms.

[2] In higher dimensions, "dualizable" generalizes to "fully dualizable," and the fact that "fully dualizable" and "finite-dimensional" have the same initials makes for a good mnemonic.

Atiyah doesn't go into much more detail about this theory, but Schwarz did (assuming $\omega|_{\pi_2(X)}$ vanishes), and we will discuss this example in detail. Ultimately, $Z(S^1)$ will be $H_*(X)$, and the pair-of-pants is sent to a quantum deformation of the cup product which counts pseudoholomorphic curves — Schwarz proves this with Floer theory, but it also makes contact with Gromov-Witten theory.

**Example 1.10** (Chern-Simons theory)**.** There are several different flavors of this next example, a 3-dimensional theory. Pick a Lie group $G$, maybe compact; the classical phase space associated to a closed surface $\Sigma$ is the moduli space of flat $G$-bundles on $\Sigma$. This isn't infinite-dimensional, because we imposed that our connections are flat, though the space of all connections is infinite-dimensional. If $G$ is nonabelian, this is nonlinear (i.e. not a vector space).

The Lagrangian functional for this theory is the Chern-Simons functional associated to a connection. There's been plenty of work on this example, from different perspectives not just including TQFT, including work by Jones, Witten, Casson, Johnson, and Thurston. ◀

**Example 1.11** (Floer theory/Donaldson theory)**.** This is a 4-dimensional example, in which the invariant assigned to a closed 4-manifold $X$ is the Donaldson polynomials on $H_2(M)$ (a tool encoding all of the Donaldson invariants). Atiyah doesn't say what we should do with cobordisms, but fora closed 3-manifold $Y$, following the Hamiltonian perspective in physics, one should do Floer theory for the Chern-Simons functional on $Y$ (for some Lie group that you have to pick — though only $G = \mathrm{SU}_2$ and $G = \mathrm{U}_2$ have really been worked out, which is Donaldson theory).

Unfortunately, this cannot be an oriented theory — Donaldson polynomials depend on more data. ◀

Awesomely, Atiyah ends with the question *why does the Chern-Simons functional appear in both the three- and four-dimensional cases?* There ought to be an answer in terms of extended TQFT: Chern-Simons theory really seems to be about dimensions 4, 3, and 2.

**Example 1.12.** After Atiyah's paper came out, Seiberg-Witten theory appeared, as a variant of Example 1.11, and it should fit into a TQFT framework in the same way. This is again a 4-dimensional theory. ◀

We will begin by digging into Example 1.9. Pseudoholomorphic curves are a huge subject; good references include Salamon's lecture notes and the book of Audin-Damian, which is very detailed but doesn't illustrate the analysts' perspective as well as Salamon. The big book of McDuff-Salamon is also good. The professor also has a survey paper, "Lagrangian boundary conditions for anti-self-dual instantons and the Atiyah-Floer conjecture," which is a good way to get an overview of this perspective.

Before we get into pseudoholomorphic curves, here's an important convention: *when we say "symplectic manifold," we always mean closed* (compact and without boundary).

**Definition 1.13.** A *symplectic manifold* $(X, \omega)$ is a manifold $X$ and a 2-form $\omega \in \Omega^2(X)$ which is closed and *nondegenerate*, i.e. $\omega^{\wedge n}$ is a volume form.

This immediately implies $\dim X = 2n$, and is in particular even; and $[\omega] \neq 0$ in $H^2_{\mathrm{dR}}(X)$, which rules out, e.g., $S^4$.

You can get through a good part of the course thinking of these as even-dimensional manifolds with a particular functional on them. Let $\mathcal{L}X := \mathrm{Map}(S^1, X)$, the unbased loop space of $X$.

**Definition 1.14.** The *symplectic action functional* associated to a symplectic manifold $(X, \omega)$ is the functional $\mathcal{A}\colon \mathcal{L}X \to \mathbb{R}$ sending a loop $\gamma\colon S^1 \to X$ to the number

$$\int_{[0,1]\times S^1} u^*\omega. \tag{1.15}$$

Here $u\colon [0,1] \times S^1 \to X$ a smooth map with $u(0, –)$ a fixed reference loop $u_0$ and $u(1, –) = \gamma$.

Often, $u_0$ is constant, in which case this is choosing a disc whose boundary is $\gamma$. There are issues defining this, so the actual target is $\mathbb{R}$ modulo the possible values of $\omega$ on tori. If you want to study all of $\mathcal{L}X$, you need to fix a basepoint in each connected component (homotopy class), though often people only study the connected component containing the constant loops, as Floer did.

Given a nice functional, one should want to try gradient flow and Morse theory with it, even though $\mathcal{L}X$ is infinite-dimensional; we will see the definition of a pseudoholomorphic curve pop out naturally from this definition. We will also do Morse theory with the Chern-Simons functional. Doing Morse theory with a function valued in a circle is a bit different, but we'll be able to do it. And in fact, it's the reason we work with the Novikov ring.