SUMMER 2016 HOMOTOPY THEORY SEMINAR

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Contents

1. Simplicial Localizations and Homotopy Theory: 5/24/16

"It may be a little dry, but it's been raining recently, so perhaps dryness will be good to have."

Today's lecture was given by Ernie Fontes.

The point of this seminar is to study simplicial localizations. This is a somewhat dry topic; today we're going to frame it, suggesting an outline for talks and some motivation. Thus, today we'll discuss homotopy theory in broad strokes.

A good first question: *what is homotopy theory?* Relatedly, *when can we do it?* In general, homotopy theory happens whenever we have a pair of categories $(\mathbf{C}, \mathbf{W})$, where $\mathbf{W}$ is a subcategory of $\mathbf{C}$. The idea is that $\mathbf{W}$ contains morphisms that we'd like to be isomorphisms. If $\mathbf{W}$ contains all of the objects of $\mathbf{C}$, then the pair $(\mathbf{C}, \mathbf{W})$ is called a *relative category.*

Example 1.1.

(1) Often, we choose $\mathbf{C} = \mathbf{Top}$, and make $\mathbf{W}$ the category of a nice class of morphisms, e.g. π_*-isomorphisms or homotopy equivalences.
(2) Another choice is to let $\mathbf{C} = \mathbf{ch}(R)$, the category of chain complexes of R-modules, where $\mathbf{W}$ is the category of *quasi-isomorphisms* (maps which induce an isomorphism on homology).

One nice property that $\mathbf{W}$ could have is the *two-out-of-six property*: that for all triples of morphisms $f : X \to Y$, $g : Y \to Z$, and $h : Z \to Z'$ in $\mathbf{C}$, if gf and hg are in $\mathbf{W}$, then so are f, g, h, and hgf. This implies the *two-out-of-three property*, that if any two of f, g, and h are in $\mathbf{W}$, then so is the third.

Definition 1.2. If $\mathbf{W}$ satisfies the two-out-of-six property, it is called a *homotopical category.*

In either setting, we can form the *homotopy category* $\mathrm{Ho}(\mathbf{C}) = \mathbf{C}[\mathbf{W}^{-1}]$, localizing $\mathbf{C}$ at $\mathbf{W}$. This is the initial category among those categories $\mathbf{D}$ and functors $\mathbf{C} \to \mathbf{D}$ sending the arrows in $\mathbf{W}$ to isomorphisms.

Most questions in homotopy theory can be framed in terms of the homotopy category: two spaces are homotopic iff they're isomorphic in $\mathrm{Ho}(\mathbf{C})$, and the homotopy classes of maps $X \to Y$ are the hom-set $\mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}(X, Y)$ in the homotopy category.

One question which does require a little more sophistication is understanding homotopy (co)limits. Since we've inverted a lot of arrows, taking limits or colimits in a homotopy category behaves very poorly. For example, there's no pushout of the degree-2 map $S^1 \to S^1$ along with the map $S^1 \to \mathrm{pt}$, since it "should be" $\mathbb{RP}^2$ but this doesn't satisfy it. $\mathbb{RP}^2$ is the homotopy pushout, however.

Often, one obtains more structure from a homotopy category, e.g. there are some ∞-categorical notions hiding in the background here. More concretely, one often obtains a natural model category structure, where in addition to the relative category $(\mathbf{C}, \mathbf{W})$, we have classes of cofibrant and fibrant morphisms satisfying a bunch of axioms. This provides tools for computing homotopy limits and colimits, etc., but it's a lot of data; even the definition is redundant (the cofibrations and fibrations determine each other). In fact, the punchline of the three papers we're reading is that only the structure of the relative category $(\mathbf{C}, \mathbf{W})$ is necessary to recover the entire model-categorical structure! For this reason, one makes the analogy that if homotopy theory is to linear algebra, picking a model-categorical structure is akin to picking a basis.

Definition 1.3. A *simplicial set* is a simplicial object in **Set**. That is, it's a collection of sets $\{X_i\}_{i\geq 0}$ and a bunch of maps $d_{ij} : X_i \to X_{i-1}$ for $0 \leq j \leq i$ and $s_{ij} : X_i \to X_{i+1}$ for $0 \leq j \leq i$ satisfying some relations that look like the boundary and inclusion relations for an i-simplex inside an $(i+1)$-simplex.

This is a vague definition, and we'll have a better one next lecture. These are akin to a "better" version of topological spaces, in that they model topological spaces very well, and can be described purely combinatorially.

Here's how the three papers of Dwyer and Kan break this information down.

(1) The first paper [DK1] constructs $\mathbf{C}[\mathbf{W}^{-1}]$, first as "just" a category, and then as a simplicially enriched category $L\mathbf{C}$, meaning that for all $X, Y \in \mathbf{C}$, $L\mathbf{C}(X, Y) \in \mathbf{sSet}$: that is, it's a simplicial set. In particular, we recover $\mathbf{C}[\mathbf{W}^{-1}]$ as the path components of this set: $\mathbf{C}[\mathbf{W}^{-1}](X, Y) = \pi_0 L\mathbf{C}(X, Y)$. There's a lot of comonadic computations here that may be confusing, but are applicable in many parts of algebra.

(2) In [DK2], Dwyer and Kan define a variant called the *hammock localization* $L^H\mathbf{C}(X, Y)_k$. The hammocks in question are commutative diagrams

$$\begin{tikzcd}
& C_{0,1} \arrow[d, "\wr"] & C_{0,2} \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\wr"] & \cdots & C_{0,n-1} \arrow[l, "\sim"'] \arrow[d, "\wr"] \arrow[dr] & \\
X \arrow[ur] \arrow[r] \arrow[dr] & \vdots \arrow[d, "\wr"] & \vdots \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\wr"] & \cdots \arrow[r, "\sim"] & \vdots \arrow[r] \arrow[d, "\wr"] & Y \\
& C_{k,1} \arrow[r, "\sim"] & C_{k,2} \arrow[r, "\sim"] & \cdots & C_{k,n-1} \arrow[l, "\sim"'] \arrow[ur] &
\end{tikzcd}$$

This might not seem like the best construction, but it expresses $L^H\mathbf{C}(X, Y)$ as a colimit of nerves of categories, which are easy to compute, and therefore this is surprisingly easy to work with when it comes to actually computing things. In particular, when certain weak (yet technical) properties hold, $L^H\mathbf{C}(X, Y) \simeq L\mathbf{C}(X, Y)$. The calculations in this paper are much more technical than the first, and it's worth going through more slowly.

(3) The third paper [DK3] establishes a relationship between (simplicially enriched) model categories and $L^H\mathbf{C}(X, Y)$. The takeaway is that the weak equivalences are all that you need to define a model categorical structure.

In the unlikely event we have time, there's an interesting relationship between this and algebraic K-theory: in a similar way, the algebraic K-theory of a model category actually only depends on the hammock localization, due to a paper [BM] of Blumberg and Mandell; this was a cool and surprising result.

Here's the list of planned talks; we can and should deviate from this in order to make sure we understand everything better.

(1) Simplicial sets, especially nerves and classifying spaces. This should definitely include a definition and some important constructions.
(2) Model categories; there's a lot we could talk about here, but we should talk about the definition, how to construct homotopy limits and colimits, mapping spaces, and fibrant and cofibrant replacement. This is intended to be an overview, rather than discussing complicated examples. This will be helpful to see all the structure we don't need!
(3) We then need to talk about localization in general, including the universal property for localizing rings, and discuss the discrete localization of categories. The hard version of this talk would also talk about Bousfield localization.
(4) Now, the first part of [DK1]: localization of $(\mathbf{C}, \mathbf{W})$, comonadic resolutions, and bar constructions, which detail how one constructs things. This is mostly all in the paper, and needs to be teased apart.
(5) Perhaps also it will be useful to discuss the rest of the model structure on small simple categories. Here Julie Bergner's thesis is a useful reference, as she treats this more clearly and in greater generality, though we may or may not need to refer to this.

(6) Moving to [DK2], introduce hammock localization. This is important to understand very closely; don't leave anything out of the talks.
(7) Then, we need homotopy calculus of fractions, which is useful for ensuring hammocks are small.
(8) We then need the theory of simplicial model categories; these have more structure and are more excellent than ordinary model categories. The key is understanding the axiom SM7 for a simplicial model category.
(9) Finally, we should treat the main theorem in [DK3], that $L^H\mathbf{C}(X, Y)$ models the simplicial derived mapping space in a model category.

At that point, the summer will be over, and we will be done.

2. Simplicial Sets: 5/31/16

These are Arun's prepared notes for this talk.

Simplicial sets are a combinatorial analogue of topological spaces that are often simpler to work with, yet in a sense contain the same information from the perspective of homotopy theory. At the same time, they also behave like a nonlinear analogue of chain complexes.

2.1. Two Definitions of Simplicial Sets.

One definition is formal, and easier to write down; the other is more geometric, but requires more words.

Definition 2.1. The *simplex category* $\mathbf{\Delta}$ is the category whose objects are the ordered sets $[n] = \{0, 1, \dots, n\}$ for $n \geq 0$, and whose morphisms are order-preserving functions.

Definition 2.2. A *simplicial set* is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$. With natural transformations as morphisms, these form the category $\mathbf{sSet}$. More generally, for any category $\mathbf{C}$, a *simplicial object* in $\mathbf{C}$ is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{C}$.

That is, a simplicial set X is a set X_n for each $[n]$ (called the set of *n-simplices*) with compatible actions by the morphisms in $\mathbf{\Delta}$. A morphism of simplicial sets $X \to Y$ is a collection of maps $X_n \to Y_n$ for each n that commutes with those actions.

This doesn't seem very topological or geometric; here's another definition.

Definition 2.3. A *simplicial set* X is a collection of sets X_n for each $n \geq 0$, along with functions $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ for $0 \leq i \leq n$, called the *face maps* and *degeneracy maps*, respectively, satisfying the relations

$$\begin{aligned} d_i \circ d_j &= d_{j-1} \circ d_i, \quad i < j \\ s_i \circ s_j &= s_{j+1} \circ s_i, \quad i \leq j \\ d_i \circ s_j &= \begin{cases} 1, & i = j \text{ or } i = j+1 \\ s_{j-1} \circ d_i, & i < j \\ s_j \circ d_{i-1} & i > j+1. \end{cases} \end{aligned} \tag{2.4}$$

A morphism of simplicial sets $f : X \to Y$ is a collection of maps $f_n : X_n \to Y_n$ that commute with the face and degeneracy maps.

We can think of the object $[n] \in \mathbf{\Delta}$ as the standard n-simplex (triangle, tetrahedron, ...); in this case, the face map d_i is induced from the inclusion of the i^{th} face (which is a copy of $[n-1]$) and the degeneracy map s_i is induced from the projection onto the i^{th} face. If you play with this picture, you end up writing down the definitions in (2.4).

Example 2.5.

(1) The *standard n-simplex* is the $\Delta^n = \mathrm{Hom}_{\mathbf{\Delta}}(-, [n])$. Thus, by the Yoneda lemma, for any simplicial set X, $\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, X) = X_n$. Geometrically, think of standard n-simplex as, well, the n-dimensional simplex: the i^{th} face map is the assignment to the i^{th} face of this simplex, and the i^{th} degeneracy map realizes Δ^n as a degenerate $(n+1)$-simplex where vertices i and $i+1$ coincide. See Figure 2 for a depiction of the standard 3-simplex.

By the Yoneda lemma, Δ^n corepresents the functor $X \mapsto X_n$. That is, there is a natural isomorphism (of sets)

$$\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, X) \cong X_n. \tag{2.6}$$

(2) Given a simplicial set X, we can form its *k-skeleton* in much the same way as for CW complexes, by preserving $X_0, \dots, X_k$ and the maps between them, but making all higher simplices degenerate.

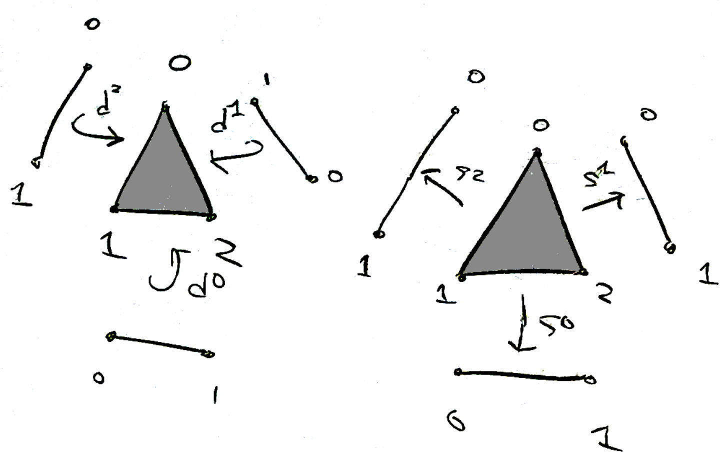

FIGURE 1. Examples of generating maps in Δ that induce the face and degeneracy maps of a simplicial set.

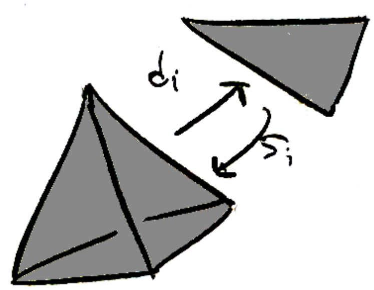

FIGURE 2. The standard 3-simplex, with example face and degeneracy maps.

(3) The *simplicial n-sphere*, denoted $\partial\Delta^n$, is the $(n-1)$-skeleton of Δ^n. Geometrically, this is the n-simplex minus its interior, which is a reasonable thing to call a sphere (to homotopy theorists, at least). Another equivalent formulation is to take $\Delta^n \setminus \{\mathrm{id}\}$ (regarding it as a functor), or the union (or colimit) of all of the faces of Δ^n across the morphisms gluing *their* faces (which are $(n-2)$-simplices).
(4) The *simplicial horn* Λ^n_k is the union (or colimit) of all faces of Δ^n except for the k^{th} face. The notation Λ is suggestive of the geometry. If X is a simplicial set, a *horn in* X is a map of simplicial sets $\Lambda^n_k \to X$.

Definition 2.7. A simplicial set X is a *Kan complex* if every horn $\Lambda^n_k \to X$ can be extended to a map $\Delta^n \to X$, i.e. it factors through the inclusion $\Lambda^n_k \hookrightarrow \Delta^n$. If this is only true for *inner horns*, i.e. $\Lambda^n_k \hookrightarrow X$ where $0 < k < n$, then X is called a *weak Kan complex.*

One says that "every horn has a filler."

Like **Top**, the category **sSet** is complete and cocomplete: all limits and colimits exist, and in fact can be constructed levelwise. In particular, products exist.

Another nice property of this category is that we can build a simplicial set of the morphisms between two simplicial sets, rather than just a set.

Definition 2.8. Given $X, Y \in \mathbf{sSet}$, their *function complex* is the simplicial set $\mathbf{sSet}(X, Y)$ whose n-simplices are the set $\mathbf{sSet}(X, Y)_n = \mathrm{Hom}_{\mathbf{sSet}}(X \times \Delta^n, Y)$, with face and degeneracy maps induced from those on Δ^n.

For any simplicial set Y, there is an adjunction $(- \times Y, \mathbf{sSet}(Y, -))$; one says that **sSet** is *Cartesian closed.* Other Cartesian closed categories include **Set** and the category of compactly generated spaces.

Definition 2.9. A *simplicially enriched category* **C** is defined in exactly the same way as a category, but for every $X, Y \in \mathbf{C}$, there is a simplicial set (sometimes called the *function complex*) $\mathbf{C}(X, Y)$ of morphisms between them, instead of a set.

We require an associative composition law as usual; the identity is a distinguished 0-simplex in $\mathbf{C}(X, X)$ satisfying the same properties as usual.

Sometimes these are called "simplicial categories," but that term is also used to refer to simplicial objects in **Cat**; here **Cat** is the category of small categories with functors as morphisms. However, we can identify simplicially enriched categories with the simplicial categories whose objects are the same in every dimension.

If X and Y are simplicial sets, we defined their function complex $\mathbf{sSet}(X, Y)$, so **sSet** is a simplicially enriched category. The categories **Set** and **Top** can also be simplicially enriched, e.g. $\mathbf{Top}(X, Y)_n = \mathrm{Hom}_{\mathbf{Top}}(X \times |\Delta^n|, Y)$.

2.2. **Geometric Realization and the Total Singular Complex.** Simplicial sets are closely related to topological spaces: they're built out of n-simplices, which are manifestly topological objects. As such, there is an adjunction

$$|-| : \mathbf{sSet} \rightleftarrows \mathbf{Top} : S \tag{2.10}$$

relating simplicial sets and topological spaces.

The left adjoint is called *geometric realization*, and does in fact geometrically realize a simplicial set as a topological space: start with a concrete n-simplex in **Top** for every (abstract) n-simplex in X. Then, the face and degeneracy maps identify some of the faces of these n-simplices, so glue the corresponding concrete simplices together along those edges. Rigorously, "gluing" means a colimit. A simplicial set is essentially the data of n-simplices glued together in a specific way, and in particular

$$X \cong \varinjlim_{\Delta^n \to X} \Delta^n,$$

where the colimit is taken across all maps $\Delta^n \to X$ directed under arrows $\theta : \Delta^n \to \Delta^m$ that commute with the maps to X. We already know how to realize Δ^n as the standard n-simplex $|\Delta^n|$, so the geometric realization can be defined in parallel:

$$|X| = \varinjlim_{\Delta^n \to X} |\Delta^n|.$$

The geometric realization of a simplicial set is a CW complex.

The right adjoint is called the *total singular complex*, and belongs to the analogy

simplicial sets : chain complexes :: total singular complex : singular chain complex.

If Y is a topological space, we've already defined a chain complex of maps from the standard n-simplices into Y; this is a refinement. The total singular complex SY is defined by setting $SY_n = \mathrm{Hom}_{\mathbf{Top}}(\Delta^n, Y)$, the set of all continuous maps of the standard n-simplex (as a topological space) into Y. Given a map $f : \Delta^n \to Y$, we can restrict it to the i^{th} face; this is exactly what the i^{th} face map does. Applying the degeneracy map is given by collapsing Δ^{n+1} onto Δ^n at the i^{th} vertex, then composing with f, giving a map $\Delta^{n+1} \to Y$ as desired.

From this definition, the adjunction isn't too hard to see.

Proposition 2.11. *The functors $|-|$ and S defined above are adjoint, as in* (2.10).

Proof. We want to show for all spaces Y and simplicial sets X, there's a natural isomorphism $\mathrm{Hom}_{\mathbf{Top}}(|X|, Y) \cong \mathrm{Hom}_{\mathbf{sSet}}(X, SY)$. First, it's true when $X = \Delta^n$: $SY_n = \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, Y)$ by definition, and $SY_n = \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, SY)$ by (2.6). Since $\mathrm{Hom}_{\mathbf{C}}(A, -)$ sends colimits to limits,

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Top}}(|X|, Y) &\cong \varprojlim_{\Delta^n \to X} \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, Y) \\
&\cong \varprojlim_{\Delta^n \to X} \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, SY) \\
&\cong \mathrm{Hom}_{\mathbf{sSet}}(X, SY).
\end{aligned}$$

□

One can recover the singular chain complex $C_\bullet(X)$ from the total singular complex by setting $C_n(X)$ to be the free abelian group on X_n with the boundary map

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i. \tag{2.12}$$

Fact. The total singular complex SY is a Kan complex. There's a sense in which this adjunction defines an equivalence of the homotopy theories of **sSet** and **Top**.

2.3. **The Nerve of a Category.** For any small category $\mathbf{C}$, we can build a simplicial set $N\mathbf{C}$, called the *nerve* of $\mathbf{C}$; this is functorial in $\mathbf{C}$, defining a functor $\mathbf{Cat} \to \mathbf{sSet}$ (here $\mathbf{Cat}$ is the category of small categories, with functors as morphisms).

The construction is as follows:

- $N\mathbf{C}_0$ be the set of objects in $\mathbf{C}$.
- $N\mathbf{C}_1$ is the morphisms of $\mathbf{C}$.
- $N\mathbf{C}_2$ is the set of pairs of composable morphisms $X \to Y \to Z$.
- If $n \geq 2$, $N\mathbf{C}_n$ is the set of n-tuples of composable morphisms $X_0 \to X_1 \to \cdots \to X_n$.

In other words, if $[n]$ is regarded as a poset category (so there's a unique map $i \to j$ iff $i \leq j$), $N\mathbf{C}_n$ is the set of functors $[n] \to \mathbf{C}$.

The degeneracy map $s_i : N\mathbf{C}_n \to N\mathbf{C}_{n+1}$ takes a string of arrows and inserts the identity at the i^{th} position. The face map $d_i : N\mathbf{C}_n \to N\mathbf{C}_{n-1}$ replaces the i^{th} and $(i+1)^{\text{th}}$ arrows with their composition, unless $i = 0$ or $i = n$, in which case it just drops the first or last arrow, respectively.

Fact. The nerve of a category is a weak Kan complex.

Example 2.13 (Classifying spaces)**.** If one interprets a group G as a category with a single object, its nerve will correspond to the classifying space BG.

More precisely, a discrete group G defines a category $\mathbf{G}$ with a single object $\bullet$ and $\mathrm{Hom}_{\mathbf{G}}(\bullet, \bullet) = G$, with group multiplication as composition. Its nerve $N\mathbf{G}$ is the simplicial set whose set of n-simplices is G^n: the i^{th} degeneracy map includes e at index i, and the i^{th} face map $d_i : G^n \to G^{n-1}$ multiplies indices i and $i+1$ together (unless $i = 0$ or n, in which case that index is dropped).

Define another simplicial set X whose n-simplices are $X_n = G^{n+1}$ with the same degeneracy maps and face maps, except for d_n, which sends $(g_1, \dots, g_n, g_{n+1}) \mapsto (g_1, \dots, g_n g_{n+1})$ instead of dropping the last index. Then, projection onto the first n coordinates defines maps $p_n : X_n \to N\mathbf{G}_n$ commuting with the face and degeneracy maps, so we obtain a map of simplicial sets $p : X \to N\mathbf{G}$.

Multiplication on the last coordinate defines a right action of G on X: if $h \in G$, $(g_1, \dots, g_{n+1}) \cdot h = (g_1, \dots, g_n, g_{n+1}h)$. This commutes with the face and degeneracy maps of X, making it a simplicial G-set, and the fibers of p are G-torsors.

Now, we geometrically realize, suggestively defining $EG = |X|$ and $BG = |N\mathbf{G}|$. Projection $\pi = |p| : EG \to BG$ is a fiber bundle whose fibers are G-torsors, so $\pi : EG \to BG$ is a principal G-bundle. It's true, albeit harder to show, that EG is contractible, and therefore BG is a model for the classifying space of G.[1] Since G is discrete, BG is also a concrete model for $K(G, 1)$.

Example 2.14 (Bar construction)**.** We can generalize Example 2.13 and obtain a surprisingly useful class of simplicial objects.

Let $\mathbf{C}$ be a monoidal category, M be a monoid in $\mathbf{C}$, and $X, Y \in \mathbf{C}$ be acted on by M from the right and left, respectively.

- If $\mathbf{C} = \mathbf{Top}$, this is the notion of a continuous monoid action (from the right or the left), akin to that of a continuous group action.
- If $\mathbf{C} = \mathbf{Mod}_R$ for a commutative ring R a monoid S in $\mathbf{C}$ is an R-algebra, X is a right S-module and Y is a left S-module.

We'll build a simplicial object in $\mathbf{C}$ called the *bar construction* $B(X, M, Y)$, reminiscent of the nerve:

- The n-simplices $B_n(X, M, Y) = X \otimes M^{\otimes n} \otimes Y$ (here, $\otimes$ denotes the monoidal product; for $\mathbf{C} = \mathbf{Top}$ or $\mathbf{C} = \mathbf{Set}$, this is just Cartesian product).
- If $0 < i < n$, the i^{th} face map multiplies together the i^{th} and $(i+1)^{\text{th}}$ indices:
$$d_i : (x, m_1, \dots, m_n, y) \mapsto (x, m_1, \dots, m_i m_{i+1}, \dots, m_n, y).$$
- The 0^{th} face map sends $(x, m_1, \dots, m_n, y) \mapsto (x \cdot m_1, m_2, \dots, m_n, y)$, and correspondingly the n^{th} face map sends $(x, m_1, \dots, m_n, y) \mapsto (x, m_1, \dots, m_{n-1}, m_n \cdot y)$.
- The i^{th} degeneracy map s_i inserts the identity $e \in M$ at the i^{th} index.

[1] You might be wondering what happens if G isn't discrete, the case where classifying spaces are more interesting. Nearly the same story applies: we regard G as a single-object category enriched over **Top**, so $N\mathbf{G}$ is a *simplicial space* (i.e. simplicial object in **Top**). Geometric realization of simplicial spaces goes through to define the principal G-bundle $EG \to BG$ in the same way.

If we know how to geometrically realize simplicial **C**-objects, then this produces a genuine object of **C**.

- Suppose $\mathbf{C} = \mathbf{Top}$ and $M = G$ is a group. Then, $|B(*, G, *)| = BG$ and $|B(*, G, G)| = EG$ are exactly the constructions we gave in Example 2.13.
- Suppose $\mathbf{C} = \mathbf{Mod}_R$, so the monoid $M = S$ is an R-algebra. Then, $B(X, S, Y)$ is a simplicial R-module, so we can define a chain complex $K_\bullet(X, S, Y)$ of R-modules by letting the boundary map be as in (2.12). This chain complex is the usual resolution for computing $\mathrm{Tor}_S(X, Y)$!

2.4. **Simplicial Homotopies.** Homotopies of topological spaces are defined via unit interval $[0, 1]$; for simplicial sets, Δ^1 plays the analogous role. Everything in the next two sections comes from [GJ].

Definition 2.15. Let $f, g : X \rightrightarrows Y$ be two morphisms of simplicial sets. A *homotopy* $\eta : f \Rightarrow g$ is a morphism $\eta : X \times \Delta^1 \to Y$ such that the diagram

$$\begin{tikzcd}
X \cong X \times \Delta^0 \arrow[r, "(\mathrm{id}, d^1)"] \arrow[dr, "f"'] & X \times \Delta^1 \arrow[d, "\eta"] & X \times \Delta^0 \cong X \arrow[l, "(\mathrm{id}, d^0)"'] \arrow[dl, "g"] \\
& Y &
\end{tikzcd}$$

commutes. Here, $d^0, d^1 : \Delta^0 \rightrightarrows \Delta^1$ are the maps realizing Δ^0 as the zeroth and first vertices of Δ^1, respectively.

This is probably more or less what you were expecting. However, what's more surprising is that homotopy is not an equivalence relation! The 1-simplex Δ^1 defines a homotopy from d^0 to d^1 as maps $\Delta^0 \to \Delta^1$, but since $1 > 0$, there's no 1-simplex which can produce a homotopy from d^1 to d^0. This is awkward, but we do have the following result.

Proposition 2.16. *Homotopy is an equivalence relation on maps $X \to Y$ iff Y is a Kan complex.*

As such, we define simplicial homotopy groups for Kan complexes. Instead of picking an arbitrary basepoint, as we did in **Top**, we choose a vertex $x : \Delta^0 \to X$.

Definition 2.17. If X is a Kan complex, $x : \Delta^0 \to X$ is a vertex, and $n > 0$, we define the n^{th} *simplicial homotopy group* based at x to be the set of homotopy classes of maps $\Delta^n \to X$ that fix the boundary $\partial\Delta^n$, such that $\partial\Delta^n$ maps to x. That is, we consider maps $f : \Delta^n \to X$ fitting into a commutative diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & \Delta^0 \arrow[d, "x"] \\
\Delta^n \arrow[r, "f"] & X,
\end{tikzcd}$$

where two maps are equivalent if there is a homotopy between them fixing $\partial\Delta^n$. To define the group structure, let $f, g : \Delta^n \rightrightarrows X$ represent two elements of $\pi_n(X, x)$. Let

$$v_i = \begin{cases} x, & 0 \le i \le n-2 \\ f, & i = n-1 \\ g, & i = n+1. \end{cases}$$

Then, the assignment $i \mapsto v_i$ defines a map $\tilde{h} : \Lambda_n^{n+1} \to X$, so since X is a Kan extension, this extends to a map $h : \Delta^{n+1} \to X$. Then, we define $[a] \cdot [b] = [d_n h]$,[2] which one can show makes $\pi_n(X, x)$ into a group (the constant map to v is the identity), and an abelian group if $n \ge 2$.

We define $\pi_0(X)$, the set of *path components* of X, to be the set of homotopy classes of vertices of X.

Definition 2.18. Let X and Y be Kan complexes. A map $f : X \to Y$ is a *weak equivalence* if for all vertices x of X and $n \ge 1$, the induced map $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism, and $f_* : \pi_0(X) \to \pi_0(Y)$ is a bijection.

[2] There's a lot to check here: why is $d_n h$ constant on $\partial\Delta^n$? Why is this independent of choice of representative for $[a]$ and $[b]$?

2.5. **Bisimplicial Sets.** Dwyer and Kan mention in [DK1] that they "will often use, explicitly or implicitly," a result about bisimplicial sets (Proposition 2.21, below). As such, it's probably a good idea to at least explain what they're saying.

Bisimplicial sets fit into the analogy

$$\text{simplicial sets} : \text{chain complexes} :: \text{bisimplicial sets} : \text{double complexes}.$$

As double complexes are important in the genesis of spectral sequences, you might guess bisimplicial objects are too, and you'd be right.

Definition 2.19. A *bisimplicial set* is a simplicial object in **sSet**; equivalently, it is a functor $\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} = (\Delta \times \Delta)^{\mathrm{op}} \to$ **Set**. Replacing sets with another category $\mathbf{C}$ defines the notion of a *bisimplicial object* in $\mathbf{C}$.

Given a bisimplicial set viewed as a functor $K : (\Delta \times \Delta)^{\mathrm{op}} \to \mathbf{Set}$, $K([m],[n])$ is written $K_{m,n}$ and called the *degree-*(m,n) *bisimplices* of K. The face and degeneracy maps are bigraded, denoted d_{ij} and s_{ij}.

Definition 2.20. If K is a bisimplicial set, its *diagonal* $\operatorname{diag} K$ is the simplicial set with n-simplices $(\operatorname{diag} K)_n = K_{n,n}$ and whose face and degeneracy maps are the diagonal maps d_{ii} and s_{ii}.

That is, if K is the functor $\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathbf{Set}$ and $\mathrm{Diag} : \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$ is the diagonal functor, then $\operatorname{diag} K = K \circ \mathrm{Diag}$.

Alternatively, thinking of K as a simplicial object in **sSet**, its n-simplices are a simplicial set $K_{n,\bullet}$. These are called the *vertical simplicial sets* associated to K.

Proposition 2.21 [GJ, Prop. IV.1.9]. *If $K \to L$ is a map of bisimplicial sets such that, for every integer $n \geq 0$, the restriction $K_{n,\bullet} \to L_{n,\bullet}$ is a homotopy equivalence, then its diagonal* $\operatorname{diag} K \to \operatorname{diag} L$ *is also a weak homotopy equivalence.*

3. Model Categories: 6/7/16

"This isn't German, so '∞-category' isn't one word."[3]

Today's talk, also titled "I wish I had known this when I started learning about model categories," was given by Adrian. He will demystify the formalism of model categories, which you can read about in any book, by explaining that they correct two serious deficiencies of relative categories, that

- localization in relative categories is not well-behaved, and
- relative functors between relative categories do not pass well to derived functors after localization.

Today, the word "simplicial category" means a simplicially enriched category, as in the papers of Dwyer and Kan.

3.1. Introduction: Relative Categories.

Definition 3.1. A *relative category* $(\mathbf{C}, \mathbf{W})$ is a category $\mathbf{C}$ and a subcategory $\mathbf{W} \subseteq \mathbf{C}$ containing all objects and all isomorphisms.

For example, if R is a commutative ring, the category $\mathbf{Ch}_{\geq 0,R}$ of chain complexes of R-modules in nonnegative degree is a relative category, with $\mathbf{W}$ the (full) subcategory of weak equivalences.

Somehow related to this is the notion of an ∞*-category*, which is, vaguely speaking, a category enriched in homotopy types.

Definition 3.2. Let $\mathbf{C}$ be a simplicial category. Then, its *homotopy category* $\mathbf{ho}(\mathbf{C})$ or $\pi_0\mathbf{C}$ is the category with the same objects as $\mathbf{C}$ and whose morphisms are $(\pi_0\mathbf{C})(X,Y) = \pi_0\mathbf{C}(X,Y)$.

That is, we've identified a morphism with all others in its connected component. Given a functor $F : \mathbf{C} \to \mathbf{C}'$, one obtains a functor $\pi_0 F : \pi_0\mathbf{C} \to \pi_0\mathbf{C}'$.

Definition 3.3. Let $\mathbf{C}$ and $\mathbf{C}'$ be simplicial categories. A functor $F : \mathbf{C} \to \mathbf{C}'$ is a *Dwyer-Kan equivalence*, or *DK-equivalence*, if $\pi_0 F : \pi_0\mathbf{C} \to \pi_0\mathbf{C}'$ is essentially surjective (so for all $X, Y \in \mathbf{C}$, $F : \mathbf{C}(X,Y) \to \mathbf{C}'(FX,FY)$ is an isomorphism).

[3] In German, the word is *Unendlichkategorie.*

That is, a Dwyer-Kan equivalence is not an equivalence of categories, but is an equivalence of homotopy types.

The hammock localization we're going to study will be an assignment from relative categories to simplicial categories. If in addition we work with small categories, there's a theorem of Barwick and Kan that shows that relative categories under DK equivalence are the same as simplicial categories under DK equivalence, with sameness in a precise sense.

Since simplicial categories present ∞-categories, this also means that relative categories present ∞-categories. One has obtained quite a lot of structure from very little information. The analogy is that $\mathbf{C}$ is the generators and $\mathbf{W}$ is the relations, akin to generators and relations of most other algebraic structures.

Classically, the usual notion of localization produces an ordinary category from a relative category, very akin to the localization of a ring or module at a multiplicative subset. The zigzags in this equivalence relation are akin to the zigzags in the hammock localization, and in fact, after taking π_0 of a hammock localization, one obtains the ordinary localization. This story continues in [GZ].

All this is great, but relative categories have problems, which is why we'll introduce model categories.

(1) A big (meaning objects do not form a set) relative category need not be localizable. Consider a category with a proper class of objects, and two distinguished objects $\bullet_1$ and $\bullet_2$, and an isomorphism from $\bullet_1$ and $\bullet_2$ to every other object. Then, if we localize at everything except $\bullet_1$ and $\bullet_2$, the morphisms between them do not form a set, which is bad.
(2) Even when we can localize, it's quite hard to detect.
(3) Even when we know we can localize, it's often hard to describe the localization.[4]
(4) Another drawback is that relative functors are often not sufficient for discussing ∞-functors in practice. This comes up in practice when dealing with homotopy (co)limits: in nice situations one can recover them from regular (co)limits and the relative-categorical structure, but other times one cannot. Model categories increase the range of situations where we may do this.

3.2. **Derived Functors.**

Definition 3.4. Let $(\mathbf{C}, \mathbf{W})$ be a relative category, $\mathbf{D}$ be a category, and $F : \mathbf{C} \to \mathbf{C}$ be a functor. Suppose that $(\mathbf{C}, \mathbf{W})$ is localizable. Then, the *left derived functor* of F, denoted $LF : L\mathbf{C} \to \mathbf{D}$, is the right Kan extension of the diagram

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "F"] \arrow[d] \arrow[dr, phantom, "\Nwarrow"] & \mathbf{D} \\
L\mathbf{C}. \arrow[ur, "\mathrm{RKan}\, F"'] &
\end{tikzcd}$$

The right Kan extension is the indicated functor and natural transformation, such that any other functor (with natural transformation) mapping to the diagram has a unique natural transformation to RKan F. There is an analogous notion of a left Kan extension.

Definition 3.5. If $(\mathbf{C}, \mathbf{W})$ and $(\mathbf{C}', \mathbf{W}')$ are (localizable) relative categories, $F : \mathbf{C} \to \mathbf{C}'$ is a functor, and $\varphi : \mathbf{C}' \to L\mathbf{C}'$ is the localization functor, then the *total left derived functor* of F is the left derived functor of $\varphi \circ F$, and maps $L\mathbf{C} \to L\mathbf{C}'$.

This also exists in a right-hand variant.

Definition 3.6. If $(\mathbf{C}, \mathbf{W})$ is a relative category, then a *left deformation* (Q, ε) of $(\mathbf{C}, \mathbf{W})$ is a functor $Q : \mathbf{C} \to \mathbf{C}$ and a natural transformation $\varepsilon : Q \Rightarrow \mathrm{id}$.

Theorem 3.7 [DHKS]. *Let $(\mathbf{C}, \mathbf{W})$ and $(\mathbf{C}', \mathbf{W}')$ be relative categories, (Q, ε) be a deformation on $\mathbf{C}$, and $F : \mathbf{C} \to \mathbf{C}'$ be a functor. If $F|_{Q\mathbf{C}}$ is relative, then the following diagram commutes.*

$$\begin{tikzcd}
 & \mathbf{C} \arrow[r, "F"] \arrow[dd] & \mathbf{C}' \arrow[dd] \\
Q\mathbf{C} \arrow[ur, hook, "Q"] \arrow[dd] & & \\
 & L\mathbf{C} \arrow[r, "\text{t.l. } F"] & L\mathbf{C}' \\
LQ\mathbf{C}. \arrow[ur, hook, "\simeq"] \arrow[urr, "LF|_{Q\mathbf{C}}"'] & &
\end{tikzcd}$$

[4] If you only want to solve this problem, you can work with simpler structures than model categories: it suffices to have a relative category with left (or right) calculus of fractions. See [GZ] again.

As an example, if we take the relative category of chain complexes, we could take Q to be projective replacement, and this can actually tell us something concrete. I guess the point is that deformations can allow us to enlarge the amount of functors that present ∞-functors. For more on derived functors, check out Kahn-Maltsineotis, who provide a considerably more general theory of derived functors.

There are plenty of theorems that tell us how to localize categories, and there are plenty that inform us how to take derived functors. The advantage of model categories is that they allow us to do both.

3.3. **Model Categories.**

Definition 3.8. Consider a commutative square of morphisms in a category

$$\begin{tikzcd} \bullet \arrow[r, "a"] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "b"'] & \bullet, \end{tikzcd}$$

then f has the *left lifting property* (LLP) with respect to g, and g has the *right lifting property* (RLP) with respect to f if for all a, b making the diagram commute, there exists a φ making the following diagram commute:

$$\begin{tikzcd} \bullet \arrow[r, "a"] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "b"'] \arrow[ur, "\varphi"] & \bullet, \end{tikzcd}$$

In this case, one writes that $f \boxslash g$.

If M is a subcategory of $\mathbf{C}$ that contains all objects, denoted $M \subseteq \operatorname{mor} \mathbf{C}$, we define

$$\begin{aligned} {}^{\boxslash}M &= \{f \in \operatorname{mor} \mathbf{C} \mid \text{ for all } g \in M, f \boxslash g\} \\ M^{\boxslash} &= \{g \in \operatorname{mor} \mathbf{C} \mid \text{ for all } f \in M, f \boxslash g\}. \end{aligned}$$

Proposition 3.9. *$M^{\boxslash}$ is a category (i.e. it's closed under composition), contains all isomorphisms, and is closed under retracts and base change.*

Proposition 3.10. *Suppose $F : \mathbf{C} \rightleftarrows \mathbf{D} : G$ is an adjunction, $L \subseteq \operatorname{mor} \mathbf{C}$, and $R \subseteq \operatorname{mor} \mathbf{D}$. Then, $F(L) \boxslash R$ iff $L \boxslash G(R)$.*

These are both not difficult to prove, and highlight the essentially algebraic nature of factorization systems.

Definition 3.11. A *weak factorization system* is a pair $A, B \subseteq \operatorname{mor} \mathbf{C}$ such that

- for all $\mathbf{C}$-morphisms $f : X \to Y$, there exist morphisms $u \in A$ and $p \in B$ such that $f = p \circ u$, and
- $A = B^{\boxslash}$ and $A^{\boxslash} = B$.

Definition 3.12. Let $(\mathbf{M}, \mathbf{W})$ be a relative category. Then, a *model structure* on $(\mathbf{M}, \mathbf{W})$ is a pair of classes of morphisms $C, F \subseteq \operatorname{mor} \mathbf{M}$ such that:

- $\mathbf{M}$ contains all finite limits and colimits.
- $\mathbf{W}$ has the two-out-of-three property: if $f = g \circ h$ and any two of f, g, and h are in $\mathbf{W}$, so is the third.
- $(C \cap \mathbf{W}, F)$ and $(C, F \cap \mathbf{W})$ are weak factorization systems.

In this case, we have a bunch of words.

- The morphisms in C are called *cofibrations* and denoted $\twoheadrightarrow$, and those in F are called *fibrations*, denoted $\hookrightarrow$.[5]
- A morphism in $C \cap \mathbf{W}$ is called a *acyclic cofibration*, and one in $F \cap \mathbf{W}$ is called an *acyclic fibration*.
- If $*$ denotes the final object in $\mathbf{M}$ and $\varnothing$ denotes the initial object, then an $X \in \mathbf{M}$ is *fibrant* if $X \to *$ is a fibration, and similarly X is *cofibrant* if $\varnothing \to X$ is a cofibration. The fibrant objects are denoted $\mathbf{M}_f$, the cofibrant ones are denoted $\mathbf{M}_c$, and the *fibrant-cofibrant* (i.e. both fibrant and cofibrant) objects are denoted $\mathbf{M}_{fc}$.
- The weak factorization systems imply that for any $X \in \mathbf{M}$, the map $X \to *$ factors as $X \overset{\sim}{\hookrightarrow} X_f \twoheadrightarrow *$ (i.e. the first arrow is an acyclic fibration). This X_f is fibrant, and is called the *fibrant replacement* of X.

[5]It's nice to know that fibrations and cofibrations often behave like the ones in topology, though not always: one can take the model structure on an opposite category, and in that case the intuition is reversed.

- In the same way, the map $\varnothing \to X$ factors as $\varnothing \hookrightarrow X_c \xrightarrow{\sim}\!\!\!\!\rightarrow X$; X_c is cofibrant and is called the *cofibrant replacement* of X.

Example 3.13. Consider again $\mathbf{Ch}_{\geq 0,R}$, the bounded-below chain complexes of R-modules; we already know the weak equivalences to be the quasi-isomorphisms. We can place a model structure on this category, where:

- the cofibrations are the monomorphisms $X \hookrightarrow Y$ such that Y/X is levelwise projective, and
- the fibrations are the epimorphisms.

Thus, all objects are fibrant, and levelwise projective objects are fibrant. Cofibrant replacement is an epimorphism onto a levelwise projective module, which is projective replacement.

Proposition 3.14. *Let $F : \mathbf{M} \rightleftarrows \mathbf{M}' : G$ be an adjunction between the model categories $\mathbf{M}$ and $\mathbf{M}'$. Then, the following are equivalent:*

(1) F preserves cofibrations and acyclic cofibrations.
(2) G preserves fibrations and acyclic fibrations.
(3) F preserves cofibrations and G preserves fibrations.

In this case, (F, G) are called a *Quillen adjunction.* We'll see that Quillen adjunctions induce adjunctions between the localized categories (one must show that the relevant localizations exist, but this is fortunately true).

Proposition 3.15. *With notation as in Proposition 3.14, if (F, G) form a Quillen adjunction, the following are equivalent.*

(1) (F, G) induce an equivalence between $L\mathbf{M}$ and $L\mathbf{M}'$.
(2) For all fibrant $A \in \mathbf{M}$ and cofibrant $X \in \mathbf{M}'$, $f : FA \to X$ is a weak equivalence iff its adjoint $f^* : A \to GX$ is.

In this case, (F, G) are called a *Quillen equivalence.*

One should justify why these localizations exist. This is ultimately due to a particular corollary of Ken Brown's lemma.

Lemma 3.16 (Ken Brown)**.** *With the notation of Proposition 3.14, if (F, G) is a Quillen adjunction and if fibrant and cofibrant replacement are functorial on $\mathbf{M}$ and $\mathbf{M}'$, then F and G are localizable.*

The takeaway is: functorial fibrant and cofibrant replacement allow us to compute left and right derived functors by taking cofibrant and fibrant replacement, respectively, exactly how we computed Ext and Tor in homological algebra (by taking projective and injective replacements).

3.4. Model Categories are Localizable.

Definition 3.17. Let $\mathbf{M}$ be a model category and $X \in M$.

- A *cylinder object* for X in $\mathbf{M}$, denoted $\mathrm{Cyl}\, X$, is an object such that the co-diagonal map $X \amalg X \to X$ factors through $X \amalg X \hookrightarrow \mathrm{Cyl}\, X \xrightarrow{\sim} X$.
- A *path space object* for X, denoted PX, is an object such that the diagonal map $X \to X \times X$ factors as $X \xrightarrow{\sim} PX \twoheadrightarrow X \times X$.

The names are suggestive: in the model structure on $\mathbf{Top}$, $\mathrm{Cyl}\, X = I \times X$ and $PX = \mathbf{Top}(I, X)$, exactly what we would call cylinders and path spaces. And just like for topological spaces, we can use these to do homotopy theory.

Definition 3.18. Let $f, g : A \rightrightarrows X$ be two morphisms in a model category $\mathbf{M}$.

- A *left homotopy* from f to g is a map $H : \mathrm{Cyl}\, A \to X$ such that the diagram

$$\begin{tikzcd} & \mathrm{Cyl}\, A \arrow[dr, "H"] & \\ A \amalg A \arrow[ur] \arrow[rr, "f \amalg g"] & & X \end{tikzcd}$$

  commutes.
- A *right homotopy* from f to g is a map $H : A \to PX$ such that the diagram

$$\begin{tikzcd} & PX \arrow[dr] & \\ A \arrow[ur, "H"] \arrow[rr, "f \times g"] & & X \times X \end{tikzcd}$$

  commutes.

Two different notions might be surprising; it's true in **Top** that left and right homotopy are the same, but this is not true in general. More worryingly, these are not always equivalence relations! However, they are well-behaved on nice objects.

Theorem 3.19. *Left and right homotopy agree on* $\mathbf{M}_{fc}$*, and are equivalence relations there.*

This can be stated in greater generality. In any case, when homotopy is an equivalence relation, it will be denoted $\sim$.

Definition 3.20. A morphism $f : A \to X$ in $\mathbf{M}_{fc}$ is a *homotopy equivalence* if there's a $g : X \to A$ in $\mathbf{M}_{fc}$ such that $g \circ f \sim \mathrm{id}_A$ and $f \circ g \sim \mathrm{id}_X$.

These can be defined more generally, but here is where they are nice.

Theorem 3.21. *The homotopy equivalences and weak equivalences of* $\mathbf{M}_{fc}$ *coincide.*

Moreover, weak equivalence is the same as homotopy equivalence (for nice objects)!

Theorem 3.22. *Let* $A, X \in \mathbf{M}_{fc}$*. Then,* $\mathbf{M}(A, X)/{\sim} \cong L\mathbf{M}(A, X)$.

This tells us, for example, that a zigzag of weak equivalences in $\mathbf{M}$ is represented by a single morphism $A \to X$, which is pretty great. Moreover, even for non-fibrant-cofibrant objects, using (co)fibrant replacement allows one to do the same thing. Thus, model categories are good at both dealing with derived functors and dealing with localizations.

One example of this is that there's a model structure on $\mathbf{Cat}(\bullet \leftarrow \bullet \rightarrow \bullet, \mathbf{M})$ and a Quillen adjunction of this to $\mathbf{M}$, where $\mathbf{M}$ is any model category. This means that one can compute homotopy pushouts by deriving the usual notion of pushout.

Again, if this seems like way too much structure, there are weaker notions, such as categories of fibrant objects, but in these categories one can't take both left and right derived functors. This is something one often would like to do, so we'd want the notion of a model category.

4. Localization: Classical and Bousfield: 6/17/16

> *"So once I had a Russian professor… yes, this is funny already."*

Today Nicky spoke about localizations of categories.

4.1. Classical localizations.

You're probably familiar with the classical localization: if $\mathbf{C}$ is a category and S is a subset of the morphisms of $\mathbf{C}$, then we can form the naïve localization $\mathbf{C}[S^{-1}]$, whose objects are the same objects as $\mathbf{C}$ and whose morphisms are arbitrary zigzags of morphisms in $\mathbf{C}$, morphisms in S, and formal inverses of morphisms in S. This has set-theoretic issues, which was mentioned last time: in a large category, the morphisms between two objects might not be a set, so restrictions have to be placed on S.

We'd like localization to satisfy a universal property: that for any functor $\mathbf{C} \to \mathbf{D}$ taking all arrows in S to isomorphisms factors uniquely through the map $\mathbf{C} \hookrightarrow \mathbf{C}[S^{-1}]$.

One solution, in the form of an assumption on S, is that it has a *left calculus of fractions.*[6]

Definition 4.1. If the following axioms hold, S forms a *left multiplicative system.*

LMS1. S is a subcategory, i.e. it's closed under composition and contains all identity arrows.

LMS2. All *solid diagrams* can be completed, i.e. for any $f : X \to Y$ in $\mathbf{C}$ and $t : X \to Z$ in S, there's a $W \in \mathbf{C}$ and arrows $h : Z \to W$ in $\mathbf{C}$ and $s : Y \to W$ in S such that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "t"] & Y \arrow[d, "s"] \\ Z \arrow[r, "g"] & W \end{tikzcd}$$

commutes.

[6]The right-handed analogue of this story, with spans instead of cospans, is very similar. In many cases it satisfies the same universal property, hence is equivalent.

LMS3. For all pairs $f, g : X \rightrightarrows Y$ of arrows in $\mathbf{C}$ equalized by an arrow $t : A \to X$ in S, i.e. diagrams of the form

$$\begin{tikzcd} A \arrow[r, "t\in S"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y, \end{tikzcd}$$

f and g can be coequailized, i.e. there's an $s : Y \to B$ in S such that $sf = sg$.

Localization of categories generalizes localization of rings; the last axiom generalizes the notion that we don't like to localize by zero divisors.

Dual to the notion of a left multiplicative system is that of a *right multiplicative system*, whose axioms **RMS1**, **RMS2**, and **RMS3** are the mirror images of those for a left multiplicative system. If **LMS***i* and **RMS***i* both hold, one says axiom **MS***i* holds.

Definition 4.2. Suppose S is a left multiplicative system. Then, the *left calculus of fractions* is the category $S^{-1}\mathbf{C}$ given by the following data.

Objects: The objects of $S^{-1}\mathbf{C}$ are the same as the objects of $\mathbf{C}$.

Morphisms: The morphisms $\mathrm{Hom}_{S^{-1}\mathbf{C}}(X, Y)$ are the equivalence classes of cospans, i.e. pairs $(f : X \to Y', s : Y \to Y')$ where $s \in S$, under the equivalence relation where $(f_1 : X \to Y_1, s_1 : Y \to Y_1)$ is equivalent to $(f_2 : X \to Y_2, s_2 : Y \to Y_2)$ if there's a zigzag $(f_3 : X \to Y_3, s_3 : Y \to Y_3)$ that they both map to, i.e. there exist $u : Y_1 \to Y_3$ and $v : Y_2 \to Y_3$ making the following diagram commute:

$$\begin{tikzcd} & Y_1 \arrow[d, "u"] & \\ X \arrow[ur, "f_1"] \arrow[r, "f_3"] \arrow[dr, "f_2"'] & Y_3 & Y \arrow[ul, "s_1"'] \arrow[l, "s_3"'] \arrow[dl, "s_2"] \\ & Y_2. \arrow[u, "v"'] & \end{tikzcd}$$

Composition: Given $(f : X \to Y', s : Y \to Y')$ and $(g : Y \to Z', t : Z \to Z')$, axiom **LMS2** tells us the diagram

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "s"] & Z' \arrow[d, dashed, "u"] \\ Y' \arrow[r, dashed, "h"] & Z' \end{tikzcd}$$

can be filled in for some $h \in \mathbf{C}$ and $u \in S$. Then, we let $g \circ f$ be the class of $(h \circ f : X \to Z'', u \circ t : Z \to Z'')$.

Lemma 4.3. *This data makes $S^{-1}\mathbf{C}$ into a category.*

This is a chore to prove, so we won't do this. There's some set-theoretic stuff hiding in the background, but not crucially.

The cospan $X \xrightarrow{f} Y' \xleftarrow{s} Y$ will be denoted by $s^{-1}f$. There are a bunch of reasons that this is good notation, which we won't delve into.

A lot of proofs hinge on the following important fact, which is analogous to the fact that any finite number of fractions have a common denominator when one localizes a ring.

Proposition 4.4. *If $\{g_1, \dots, g_n\}$ is a finite set of arrows in $S^{-1}\mathbf{C}$, there exist $s : Y \to Y'$ in S and $f_i : X_i \to Y'$ such that $g_i = (f_i : X_i \to Y', s : Y \to Y')$.*

Remark. Given an object $Y \in \mathbf{C}$, one can form the category Y/S, the category of morphisms $sY \to Y'$ such that $s \in S$ (so the morphisms are commutative diagrams). This is a filtered category, which is also quite useful.[7] Filtered categories are nice for taking colimits, and in fact one can show that

$$\mathrm{Hom}_{S^{-1}\mathbf{C}}(X, Y) = \operatorname*{colim}_{s:Y\to Y'\in Y/S} \mathrm{Hom}_{\mathbf{C}}(X, Y').$$

An important part of the universal property for this localization was the data of a map from $\mathbf{C}$ into its localization.

[7] Recall that a category is *filtered* if for all pairs of objects, there's a common object they map to, and for any pair of arrows between two objects, there's an arrow coequalizing them.

Lemma 4.5. *Let S be a left multiplicative system.*

(1) *The assignment $X \mapsto X$ and $(f : X \to Y) \mapsto (f, \mathrm{id}_Y)$ defines a functor $Q : \mathbf{C} \to S^{-1}\mathbf{C}$.*
(2) *For all $s \in S$, $Q(s)$ is an isomorphism.*
(3) *If $G : \mathbf{C} \to \mathbf{D}$ maps all $s \in S$ to isomorphisms, then it factors through Q: there's a unique $G' : S^{-1}\mathbf{C} \to \mathbf{D}$ such that $G = G' \circ Q$.*
(4) *Q preserves finite colimits.*

There's also a right calculus of fractions and right localization; everything can be dualized: given a right multiplicative system, one can take the overcategory S/X rather than the undercategory, which is cofiltered rather than filtered, and so forth. In this case, the notation is also dualized, to give us fs^{-1} rather than $s^{-1}f$.

Since each satisfies the same universal property, given a multiplicative system, there's a natural isomorphism between the left and right localizations: $S^{-1}\mathbf{C} = \mathbf{C}S^{-1}$.

Definition 4.6. A multiplicative set S is a *saturated multiplicative set* if whenever $fg, gh \in S$, then $g \in S$.

One particular consequence is that a saturated multiplicative set contains all isomorphisms.

Proposition 4.7. *If S is a multiplicative system, then $Q(f)$ is an isomorphism iff $f \in S$.*

4.2. **Bousfield localizations.** There's a ton of things that could be said about Bousfield localizations. In a model category, the idea is to create a new model structure on a model category with the same cofibrations, but more weak equivalences (so the fibrant structure changes). For example, one might want to study the model category of rational, or p-local, homotopy types. Fibrant replacement becomes localization: in the rational homotopy category, fibrant replacement is $X \mapsto X_{\mathbb{Q}}$, and so forth.

In triangulated categories, it's possible to take quotients: if the cone of a morphism lies in S, we localize by that morphism. Even in a more general sense, it can be useful to think of Bousfield localization as a quotient.

Let $\mathbf{C}$ be a category, $L : \mathbf{C} \to \mathbf{C}$ be an endofunctor, and $\iota : \mathbb{1}_{\mathbf{C}} \to L$ be a natural transformation.

Definition 4.8. The pair (L, ι) is a *Bousfield localization* if "$L\iota = \iota_L$ with the common value an isomorphism:" concretely, for all objects $X \in \mathbf{C}$, $L(\iota_X : X \to LX)$ is the same map as $\iota_{LX} : LX \to LLX$, and this common map is an isomorphism.

Definition 4.9.

- If σ is a $\mathbf{C}$-morphism such that $L(\sigma)$ is an isomorphism, f is called an *L-equivalence.* The set of L-equivalences is denoted $\mathbf{W} \subset \mathbf{C}$.
- For any subclass $\mathbf{W}$ of the morphisms of $\mathbf{C}$, an object $Z \in \mathbf{C}$ is $\mathbf{W}$*-local* if for all $f : X \to Y$ in $\mathbf{W}$, the induced map $f^* : \mathrm{Hom}_{\mathbf{C}}(Y, Z) \to \mathrm{Hom}_{\mathbf{C}}(X, Y)$ is an isomorphism.[8] The full subcategory of $\mathbf{W}$-local objects will be denoted $\mathbf{C}_{\mathbf{W}}$.

In the model-categorical case, we're trying to produce more weak equivalences, rather than more isomorphisms, so in this case, we'll replace the condition for W-locality to be weak equivalence (e.g. of simplicial sets for a simplicial model category).

Proposition 4.10. *If (L, ι) is a Bousfield localization, then Z is $\mathbf{W}$-local iff ι_Z is an isomorphism.*

Proof. By the definition of a Bousfield localization, if $Z \in \mathbf{C}_{\mathbf{W}}$, then $\iota_Z : Z \to LZ$ is in $\mathbf{W}$, and therefore $\iota_Z^* : \mathrm{Hom}(LZ, Z) \to \mathrm{Hom}(Z, Z)$ is an isomorphism, more or less by the Yoneda lemma: $\iota_Z^{-1}(\mathrm{id}_Z)$ is an inverse of ι_Z.

Conversely, suppose ι_Z is an isomorphism. Proving things are $\mathbf{W}$-local tends to involve diagram chases; so be it. Let $f : X \to Y$ be in $\mathbf{W}$ and $g : X \to Z$ be arbitrary. We need to show that g comes from a unique $h \in \mathrm{Hom}(X, Z)$, i.e. $f^*(h) = h \circ f = g$. The answer is to set $h = \iota_Z^{-1}(Lg)(Lf^{-1})\iota_Y$; recall Lf is invertible because $f \in \mathbf{W}$.

Why does this work? By naturality, $\iota_Z^{-1}(Lg)(Lf)^{-1}\iota_Y f = \iota_Z^{-1} Lg(Lf^{-1})Lf\iota_X = \iota_Z^{-1}\iota_Z g = g$. Magic! Uniqueness is a similar diagram chase. ☐

Proposition 4.11. *If (L, ι) is a Bousfield localization and $\mathbf{W}$ is its L-equivalences, then $\mathbf{C}_{\mathbf{W}} = \mathbf{C}[\mathbf{W}^{-1}]$.*

Proof. We'd like to find a morphism $\mathbf{C} \to \mathbf{C}_{\mathbf{W}}$ factoring L; by Proposition 4.10, L maps $\mathbf{C}$ to $\mathbf{C}_{\mathbf{W}}$. By definition, it takes maps in $\mathbf{W}$ to isomorphisms, so we know L should factor through the localization. We have no assumptions on $\mathbf{W}$,

[8] Using pushforward instead of pullback produces the dual notion of $\mathbf{W}$-*colocal.*

so we have to use the naïve localization $\mathbf{C}[\mathbf{W}^{-1}]$; in any case, this means there's a map $G : \mathbf{C}[\mathbf{W}^{-1}] \to \mathbf{C}_{\mathbf{W}}$ that L factors through.

The final claim is that G is an equivalence of categories, which we prove by constructing an inverse functor $F : \mathbf{C}_{\mathbf{W}} \hookrightarrow \mathbf{C} \to \mathbf{C}[\mathbf{W}^{-1}]$, where the second arrow is the localization functor. The idea is that $LX = FGX$, and $\iota_X : X \to LX$ is an isomorphism. ☐

Now, we can use this to construct Bousfield localizations: if we're not handed (L, ι), but instead were handed $\mathbf{W}$, we can check whether localizing at $\mathbf{W}$ produces a Bousfield localization, and determine conditions for when this is true.

There's also a precise sense in which the Bousfield localization is *not* terrible.

Proposition 4.12. *If* $\mathbf{W}$ *comes from a Bousfield localization, then it is a left multiplicative system.*

Proof. **LMS1** follows from functoriality of L. To get **LMS2**, we can complete $Y \xleftarrow{\beta} X \xrightarrow{\gamma} Z$ with LZ, which is $\mathbf{W}$-local. Hence, $\beta^* : \mathrm{Hom}(Y, LZ) \to \mathrm{Hom}(X, LZ)$ must be an isomorphism, and the preimage of $\iota_Z \circ \gamma : X \to LZ$ is the map δ that makes the requisite diagram commute:

$$\begin{tikzcd} X \arrow[r, "\gamma"] \arrow[d, "\beta"] & Z \arrow[d, "\iota_Z"] \\ Y \arrow[r, "\exists!", "\delta"'] & LZ. \end{tikzcd}$$

Finally, for **LMS3**, $\iota_Z : Z \to Z$ coequalizes anything in $\mathbf{W}$ that equalizes two arrows. ☐

So now, given any $\mathbf{W}$ contained in the arrows of $\mathbf{C}$, we still have definitions of $\mathbf{W}$-local objects and $\mathbf{C}_{\mathbf{W}}$. If we have a nice way to make $\mathbf{W}$-localizations, we can extract a Bousfield localization.

Definition 4.13. A $\sigma : A \to B$ is called a $\mathbf{W}$-*localization* of A if $\sigma \in \mathbf{W}$ and $B \in \mathbf{C}_{\mathbf{W}}$. The category of $\mathbf{W}$-localizable objects in $\mathbf{C}$ is written $\mathbf{C}^{\mathbf{W}}$.

Choosing a localization for each object in $\mathbf{C}^{\mathbf{W}}$ defines a functor $L : \mathbf{C}^{\mathbf{W}} \to \mathbf{C}_{\mathbf{W}}$, because any two localizations of an $A \in \mathbf{C}^{\mathbf{W}}$ are uniquely isomorphic. There's a little work to here, but not too much. But once you've done this, the natural transformation $\mathbb{1}_{\mathbf{C}} \to L$ (when $\mathbf{C}^{\mathbf{W}} = \mathbf{C}$) just drops out.

Proposition 4.14. *If all objects in* $\mathbf{C}$ *are localizable and* $\mathbf{W}$ *is weakly closed,*[9] *then* (L, ι) *is a Bousfield localization and* $\mathbf{W}$ *is exactly the class of* L*-equivalences.*

Proof. We defined (L, ι) by making choices; it suffices to show that $L\iota = \iota_L$ and these are isomorphisms. Since ι is a natural transformation, then for any $A \in \mathbf{C}$, the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\iota_A"] \arrow[d, "\iota_A"] & LA \arrow[d, "\iota_{LA}"] \\ LA \arrow[r, "L\iota_A"] & LLA. \end{tikzcd}$$

Since $LLA \in \mathbf{C}_{\mathbf{W}}$, then applying this to ι_A shows $L\iota_A = \iota_{LA}$. This map is an isomorphism because the diagonal $A \to LLA$ is localization, but the localization is unique, so $L\iota_A$ must be an isomorphism. Showing that this recovers $\mathbf{W}$ is similar. ☐

If $\mathbf{C}$ is a model category, then its homotopy category has a triangulated structure; taking the homotopy category of a Bousfield localization of $\mathbf{C}$ descends to a *Verdier quotient* on the homotopy category. Thus, for practical purposes, one very frequently thinks of Bousfield localization as a quotient. In this case, if $\mathbf{C}$ is pointed, $\mathbf{C}^{\mathbf{W}}$ sometimes denotes the collection of $X \in \mathbf{C}$ such that $* \to X$ is a $\mathbf{W}$-equivalence. In algebraic K-theory, one can obtain a fiber sequence $K(\mathbf{C}^{\mathbf{W}}) \to K(\mathbf{C}) \to K(L\mathbf{C})$, another sense in which this behaves like a kernel.

So in a model category, we're not thinking about inverting morphisms, but rather inverting them up to homotopy, or after passing to left and right derived functors. In particular, for a model category, we should replace isomorphism with weak equivalence, so being $\mathbf{W}$-local will require $f^* : \mathbf{C}(Y, Z) \to \mathbf{C}(X, Z)$ to be a weak equivalence of simplicial

[9] $\mathbf{W}$ is *weakly closed* if it satisfies **LMS1** and the two-out-of-three property. This is similar to, but not the same as, being saturated.

sets. In this case, the Bousfield localization is a new model structure on the same category where the weak equivalences are $\mathbf{W}$-local equivalences and the cofibrations are the same; in this case, the identity functor $\mathbf{C} \to \mathbf{C}[\mathbf{W}^{-1}]$ is part of a left Quillen adjunction. There's also right Bousfield localizations and conditions on when they exist…

One cool fact is that taking $\mathbf{W} = \{f\}$ for any morphism f of topological spaces, it's possible to take the Bousfield localization on $\mathbf{W}$!

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