GROMOV-WITTEN THEORY LEARNING SEMINAR

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CONTENTS

1. AN OVERVIEW OF GROMOV-WITTEN THEORY: 1/29/18

Today, Jonathan spoke, delivering an overview of Gromov-Witten theory and how associativity of quantum cohomology leads to applications in enumerative geometry. Today we always work over $\mathbb{C}$, and follow Fulton-Pandharipande's notes [FP96].

Classically, if X is a nonsingular projective variety and $\beta \in H_2(X;\mathbb{Z})$, we want to know how many algebraic curves in X represent the class β. This relates to very classical questions, such as: if you have $3d - 1$ points in $\mathbb{P}^n$, how many degree-d curves pass through them?

Definition 1.1. To simplify notation, let $A_d(X) := H_{2d}(X;\mathbb{Z})$, and similarly $A^d(X) := H^{2d}(X;\mathbb{Z})$.

The moduli space of stable maps. Another important ingredient, whose construction we will punt on, is the *moduli space of stable maps*. Here we summarize its definition. Let X be a smooth projective variety and $\beta \in A_1(X)$. The moduli space of stable maps, denoted $\mathcal{M}_{g,n}(X,\beta)$ is the moduli space of isomorphism classes of pointed maps

$$(1.2) \qquad\qquad u\colon (C, p_1, \ldots, p_n) \longrightarrow X$$

where C is a projective nonsingular curve of genus g, $p_1, \ldots, p_n$ are distinct marked points in C, and $u_*([c]) = \beta$. We must impose a stability condition which ensures these maps have finitely many automorphisms, where an automorphism $(C, p_1, \ldots, p_n) \to (C', p_1], \ldots, p'_n)$ must send $p_i \mapsto p'_i$ and commute with the maps to X.

This is all right, but we really want something compact, and therefore will have to consider stable maps which are slightly worse. The compactification $\overline{\mathcal{M}}_{g,n}(X,\beta) \supset \mathcal{M}_{g,n}(X,\beta)$ is the space of stable maps as in (1.2), subject to the following conditions.

- C is a projective, connected, reduced, genus-g curve with at worst nodal singularities, and the p_j are distinct smooth points.
- Stability: for every irreducible compact $E \subset C$ such that if $E \simeq \mathbb{P}^1$ and $u(E) = \{\mathrm{pt}\}$, then E contains at least 3 of the points p_i.
- If E is genus 1 and $u(E) = \{\mathrm{pt}\}$, then E contains at least one of the points p_i.

Why is this a compactification? The idea is that if $u\colon (C, p_1, \ldots, p_n) \to X$ is a smooth curve, we can let two points collide. In the compactified moduli space, the collision is avoided by adding another $\mathbb{P}^1$ to C intersecting near the collision point; then, the two points can live in distinct irreducible components.

The next question is: what's the dimension of $\overline{\mathcal{M}}_{g,n}(X,\beta)$ be? Naïvely, the expected dimension is

$$(1.3) \qquad\qquad n + \int_\beta c_1(X) + (\dim X - 3)(1 - g) = n + 3g - 3 + \chi(TX|_C).$$

Here c_1 is the first Chern class, and $\int_\beta c_1(X)$ represents the cap product pairing $A_1(X) \otimes A^1(X) \to \mathbb{Z}$, and $\chi(TX|_C)$ denotes its Euler characteristic:

$$(1.4) \qquad\qquad \chi(TX|_C) := h^0(C; TX|_C) - h^1(C; TX_|C).$$

This does not depend on the choice of C representing β, which is a fun fact about characteristic classes. Why is (1.3) a reasonable guess? Here's what's going on.

- The $3g - 3$ represents the dimension of the moduli space of the curve C, hence representing how C can change on its own.
- The $\chi(TX|_C)$ represents how C can deform in X.
- The n is the extra data corresponding to the marked points.

We said "naïve," and indeed (1.3) is not the dimension of $\overline{\mathcal{M}}_{g,n}(X, \beta)$ in all cases. But it is true in nice cases, and then you can do some cool stuff.

Gromov-Witten invariants. There are natural *evaluation maps* $p_i \colon \overline{\mathcal{M}}_{g,n}(X, \beta) \to X$ sending

$$(1.5) \qquad\qquad (u \colon (C, p_1, \ldots, p_n) \to X) \longmapsto u(p_i).$$

We can pull back cohomology classes along these maps: suppose $\gamma_1, \ldots, \gamma_n \in A^*(X)$. Then, let

$$(1.6) \qquad\qquad I_\beta(\gamma_1, \ldots, \gamma_n) := \int_{[\overline{\mathcal{M}}_{g,n}(X, \beta)]} p_1^*(\gamma_1) \smile \cdots \smile p_n^*(\gamma_n).$$

This is called a *Gromov-Witten invariant* for X. The thing that we're integrating over requires some very technical work to define in general, but for spaces which are "nice" (convex and homogeneous, which we'll discuss later), it's not so bad. $\mathbb{P}^n$ is an example of such a space.

Suppose $\gamma_1, \ldots, \gamma_n$ have the correct dimensions such that (1.6) is a number. Then there's an enumerative interpretation of (1.6) (in the convex case): the number of pointed maps $u \colon \Sigma_g \to X$ such that $u_*([\Sigma_g]) = \beta$ and if Γ_i is a subvariety representing the Poincaré dual to γ_i,[1] then $u(p_i) \in \Gamma_i$. Here Σ_g is a curve of genus g.

Important properties.

Proposition 1.7. *If $\beta = 0$, the only nonzero Gromov-Witten invariants occur when $n = 3$.*

Proof sketch. If $\beta = 0$, there's an identification $\overline{\mathcal{M}}_{0,n}(X, \beta) \cong \overline{\mathcal{M}}_{0,n} \times X$, which carries all of the evaluation maps to projection onto X, where $\overline{\mathcal{M}}_{0,n}$ is the (compactified) moduli space of genus-0 curves with n marked points. Call this map π. Then,

$$\begin{aligned} I_\beta(\gamma_1, \ldots, \gamma_n) &= \int_{\overline{\mathcal{M}}_{0,n}(X,0)} p_1^*(\gamma_1) \smile \cdots \smile p_n^*(\gamma_n) \\ &= \int_{\overline{\mathcal{M}}_{0,n} \times X} \pi^*(\gamma_1 \smile \cdots \smile \gamma_n) \\ &= \int_{\pi_*([\overline{\mathcal{M}}_{0,n} \times X])} \gamma_1 \smile \cdots \smile \gamma_n. \end{aligned}$$

If $n < 3$, $\mathcal{M}_{0,n}$ is empty, because any choice of n points in $\mathbb{P}^1$ doesn't have a finite automorphism group. For $n > 3$, π has positive-dimensional fibers. $\boxtimes$

If $n = 3$, then

$$(1.8) \qquad\qquad I_0(\gamma_1, \gamma_2, \gamma_3) = \int_X \gamma_1 \smile \gamma_2 \smile \gamma_3,$$

so this Gromov-Witten invariant isn't too hard to calculate.

Proposition 1.9. *Suppose $\gamma_1 = 1 \in A^0(X)$. Then, $I_\beta(1, \gamma_2, \ldots, \gamma_n)$ is nonzero only when $\beta = 0$ and $n = 3$.*

Proof sketch. If $\beta \neq 0$, $p_1^*(1) \smile \cdots \smile p_n^*(\gamma_n)$ is the pullback of a class in $\overline{\mathcal{M}}_{0,n-1}(X, \beta)$ along the map

$$(1.10) \qquad\qquad \overline{\mathcal{M}}_{0,n}(X, \beta) \longrightarrow \overline{\mathcal{M}}_{0,n-1}(X, \beta)$$

which forgets the first point. There's a projection formula which then finishes the proof in a similar way to Proposition 1.7. $\boxtimes$

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[1] We can dodge the Steenrod realizability problem because every even-degree homology class of $\mathbb{P}^n$ is represented by a complex subvariety.

In the case $\beta = 0$ and $n = 3$, there's a similar formula to (1.8):

$$(1.11) \qquad I_0(1, \gamma_2, \gamma_3) = \int_X \gamma_2 \smile \gamma_3.$$

Proposition 1.12. *If $\gamma_1 \in A^1(X)$, then*

$$I_\beta(\gamma_1, \ldots, \gamma_n) = \left(\int_\beta \gamma_1 \right) I_\beta(\gamma_2, \ldots, \gamma_n).$$

Since $\int_\beta \gamma_1$ is the number of choices for $p_i \in C$ to map to Γ_1, where Γ_1 is a Poincaré dual to γ_1. The proof idea has something to do with the pushforward map (1.10) again.

Next time we'll talk about the quantum cohomology ring, and show that its associativity provides recursive formulas for enumerative invariants.

REFERENCES

[FP96] W. Fulton and R. Pandharipande. Notes on stable maps and quantum cohomology. 1996. https://arxiv.org/pdf/alg-geom/9608011.pdf. 1