

M392C NOTES: MATHEMATICAL GAUGE THEORY

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Lecture 1.

Some useful linear algebra: 1/22/19

“Why did the typing stop?”

Today we'll discuss some basic linear algebra which, in addition to being useful on its own, is helpful for studying the self-duality equations. You should think of this as happening pointwise on the tangent space of a smooth manifold.

Let V be a real n -dimensional vector space. The exterior powers of V define more vector spaces: the scalars \mathbb{R} , V , $\Lambda^2 V$, and so on, up to $\Lambda^n V = \text{Det } V$. We can also apply this to the dual space, defining \mathbb{R} , V^* , $\Lambda^2 V^*$, etc, up to $\Lambda^n V^* = \text{Det } V^*$.

There is a duality pairing

$$(1.1) \quad \theta: \Lambda^k V^* \times \Lambda^k V \longrightarrow \mathbb{R}$$

$$(v^1 \wedge \cdots \wedge v^k, v_1 \wedge \cdots \wedge v_k) \longmapsto \det(v^i(v_j))_{i,j},$$

where $v^i \in V^*$ and $v_j \in V$.

Now fix a $\mu \in \text{Det } V^* \setminus 0$, which we call a *volume form*. Then we get another duality pairing

$$(1.2) \quad \Lambda^k V \times \Lambda^{n-k} V \longrightarrow \mathbb{R}$$

$$(x, y) \longmapsto \theta(\mu, x \wedge y).$$

Thus $\Lambda^k V \cong \Lambda^{n-k} V^*$.

Suppose we have additional structure: an inner product and an orientation. Let e_1, \dots, e_n be an oriented, orthonormal basis of V , and e^1, \dots, e^n be the dual basis. Now we can choose $\mu = e^1 \wedge \cdots \wedge e^n$.

Definition 1.3. The *Hodge star operator* is the linear operator $\star: \Lambda^k V^* \rightarrow \Lambda^{n-k} V^*$ characterized by

$$(1.4) \quad \alpha \wedge (\star \beta) = \langle \alpha, \beta \rangle_{\Lambda^k V} \cdot \mu.$$

The inner product on $\Lambda^k V^*$ is defined by

$$(1.5) \quad \langle v^1 \wedge \cdots \wedge v^k, w^1 \wedge \cdots \wedge w^k \rangle := \det(\langle v^i, w^j \rangle)_{i,j}.$$

The Hodge star was named after W.V.D. Hodge, a British mathematician. Notice how we've used both the metric and the orientation – it's possible to work with unoriented vector spaces (and eventually unoriented Riemannian manifolds), but one must keep track of some additional data.

Example 1.6.

- $\star(e^{i_1} \wedge \cdots \wedge e^{i_k}) = e^{j_1} \wedge \cdots \wedge e^{j_{n-k}}$ if the permutation $1, \dots, n$ to $i_1, \dots, i_k, j_1, \dots, j_{n-k}$ of $[n] := \{1, \dots, n\}$ is even. Otherwise there's a factor of -1 .
- Suppose $n = 4$. Then $\star(e^1 \wedge e^2) = e^3 \wedge e^4$ and $\star(e^1 \wedge e^3) = -e^2 \wedge e^4$, and so on. \blacktriangleleft

Remark 1.7. The Hodge star is natural. First, you can see that we didn't make any choices when defining it, other than an orientation and a volume form, but there's also a functoriality property. Let $T: V \rightarrow V$ be an automorphism; this induces $(\Lambda^k T^*)^{-1}: \Lambda^k T^* \rightarrow \Lambda^k T^*$, and if T is an orientation-preserving isometry,

$$(1.8) \quad \star \circ (\Lambda^k T^*)^{-1} = (\Lambda^{n-k} T^*)^{-1} \circ \star.$$

Hence $\star\star: \Lambda^k V^* \rightarrow \Lambda^k V^*$ is some nonzero scalar multiple of the identity, and we can determine which multiple it is. Certainly we know

$$(1.9) \quad \star\star(e^1 \wedge \cdots \wedge e^k) = \star(e^{k+1} \wedge \cdots \wedge e^1) = \lambda e^1 \wedge \cdots \wedge e^k,$$

and we just have to compute the parity of these permutations: one uses k transpositions, and the other uses $n - k$. Therefore we conclude that

$$(1.10) \quad \star\star = (-1)^{k(n-k)}: \Lambda^k V^* \rightarrow \Lambda^k V^*. \quad \blacktriangleleft$$

Now suppose $n = 2m$, so we have a middle dimension m , and $\star\star: \Lambda^m \rightarrow \Lambda^m$ is $(-1)^m$. This induces additional structure on $\Lambda^m V^*$.

- If m is even (so $n \equiv 0 \pmod{4}$), the double Hodge star is an endomorphism squaring to 1. This defines a $\mathbb{Z}/2$ -grading on $\Lambda^m V^*$, given by the ± 1 -eigenspaces, which we'll denote $\Lambda_{\pm}^m V^*$. The $+1$ -eigenspace is called *self-dual m -forms*, and the -1 -eigenspace is called the *anti-self-dual m -forms*.
- If m is odd (so $n \equiv 2 \pmod{4}$), the double Hodge star squares to -1 , so this defines a complex structure on $\Lambda^m V^*$, where i acts by the double Hodge star.

Exercise 1.11. Especially for those interested in physics, work out this linear algebra in indefinite signature (particularly Lorentz). The signs are different, and in Lorentz signature the two bullet points above switch!

Exercise 1.12. Show that if $4 \mid n$, the direct-sum decomposition $\Lambda^m V^* = \Lambda_+^m V^* \oplus \Lambda_-^m V^*$ is orthogonal. See if you can find the one-line proof that self-dual and anti-self-dual forms are orthogonal.

Next we introduce conformal structures. This allows the sort of geometry which knows angles, but not lengths.

Definition 1.13. A *conformal structure* on a real vector space V is a set C of inner products on V such that any $g_1, g_2 \in C$ are related by $g_1 = \lambda g_2$ for a $\lambda \in \mathbb{R}_+$.

In this setting, one can obtain g_2 from g_1 by pulling back g_1 along the dilation $T_\lambda: v \mapsto \lambda v$. This induces an action of $(T_\lambda^*)^{-1}$ on $\Lambda^k V^*$, which is multiplication by λ^{-k} : if μ_i is the volume form induced from g_i , so that

$$(1.14) \quad \alpha \wedge \star \beta = g_1(\alpha, \beta) \mu_1,$$

then

$$(1.15) \quad \lambda^{-2k} \alpha \wedge \star \beta = g_2(\alpha, \beta) \lambda^{-n} \mu_2.$$

Thus pulling back by dilation carries the Hodge star to $\lambda^{n-2k} \star$. Importantly, if $n = 2m$, then $\star: \Lambda^m V^* \rightarrow \Lambda^m V^*$ is preserved by this dilation, so it only depends on the orientation and the conformal structure.

Remark 1.16. A conformal structure is independent from an orientation. For example, on a one-dimensional vector space, a conformal structure is no information at all (all inner products are multiples of each other), but an orientation is a choice. \blacktriangleleft

Example 1.17. Suppose $n = 2$ and choose an orientation and a conformal structure on V . As we just saw, this is enough to define the Hodge star $\star: V^* \rightarrow V^*$, which defines a complex structure on V . Pick a square root i of -1 and let \star act by it (there are two choices, acted on by a Galois group).

We get more structure by complexifying: $V^* \otimes \mathbb{C}$ splits as the $\pm i$ -eigenspaces of the Hodge star; we denote the i -eigenspace by $V^{(1,0)}$ (the $(1,0)$ -forms) and the $-i$ -eigenspace by $V^{(0,1)}$ (the $(0,1)$ -forms).

Now let's globalize this: everything has been completely natural, so given an oriented, conformal 2-manifold X , it picks up a complex structure, hence is a Riemann surface, and the Hodge star is a map $\star: \Omega_X^1 \rightarrow \Omega_X^1$. Moreover, we can do this on the complex differential forms, which split into $(1,0)$ -forms and $(0,1)$ -forms.

How do 1-forms most naturally appear? They're differentials of functions, so given an $f: X \rightarrow \mathbb{C}$, we can ask what it means for $df \in \Omega_X^{1,0}$. This is the equation

$$(1.18) \quad \star df = i df.$$

This is precisely the Cauchy-Riemann equation; its solutions are precisely the holomorphic functions on X . \blacktriangleleft

Remark 1.19. More generally, one can ask about functions to \mathbb{C}^n or even sections of complex vector bundles; the analogue gives you notions of holomorphic sections. In this case, the equations have the notation

$$(1.20) \quad \bar{\partial}f = \left(\frac{1 + i\star}{2} \right) df. \quad \blacktriangleleft$$

We'll spend some time in this class understanding a four-dimensional analogue of all of this structure.

Symmetry groups. Symmetry is a powerful perspective on geometry. If we think about V together with some structure (orientation, metric, conformal structure, some combination, ...), we can ask about the symmetries of V preserving this structure. Of course, to know this, we must know V , but we can instead look at a model space \mathbb{R}^n to define a *symmetry type*, and ask about its symmetry group G : then an isomorphism $\mathbb{R}^n \rightarrow V$ preserving all of the data we're interested in defines an isomorphism from G to the symmetry group of V .

Example 1.21. When $\dim V = 2$, the most general symmetry group is $\mathrm{GL}_2(\mathbb{R})$, the invertible matrices acting on \mathbb{R}^2 . Adding more structure we get more options.

- If we restrict to orientation-preserving symmetries, we get $\mathrm{GL}_2^+(\mathbb{R})$.
- If we restrict to symmetries preserving a conformal structure, the group is called $\mathrm{CO}_2 = \mathrm{O}_2 \times \mathbb{R}^{>0}$.
- If we ask to preserve an orientation and a complex structure, we get $\mathrm{CO}_2^+ = \mathrm{SO}_2 \times \mathbb{R}^{>0}$. This is isomorphic to $\mathbb{C}^\times = \mathrm{GL}_1(\mathbb{C})$: an element of $\mathrm{SO}_2 \times \mathbb{R}^{>0}$ is rotation through some angle θ and a positive number r ; this is sent to $re^{i\theta} \in \mathbb{C}^\times$.

This provides another perspective on why an orientation and a conformal structure give us a complex structure. \blacktriangleleft

Example 1.22. Now suppose $n = 4$, and choose a conformal structure C and an orientation on V . Then orthogonal makes sense, though orthonormal doesn't, and the Hodge star induces a $\mathbb{Z}/2$ -grading on $\Lambda^2 V^* = \Lambda_+^2 V^* \oplus \Lambda_-^2 V^*$, the self-dual and anti-self-dual 2-forms. The total space $\Lambda^2 V^*$ is six-dimensional, and these two subspaces are each three-dimensional.

Suppose e^1, e^2, e^3, e^4 is an orthonormal basis for some inner product in C . We can use these to define bases of $\Lambda_\pm^2 V^*$, given by

$$(1.23) \quad \begin{aligned} \alpha_1^\pm &:= e^1 \wedge e^2 \pm e^3 \wedge e^4 \\ \alpha_2^\pm &:= e^1 \wedge e^3 \mp e^2 \wedge e^4 \\ \alpha_3^\pm &:= e^1 \wedge e^4 \pm e^2 \wedge e^3. \end{aligned}$$

Now, what symmetry groups do we have? Inside $\mathrm{GL}_4(\mathbb{R})$, preserving an orientation lands in the subgroup $\mathrm{GL}_4^+(\mathbb{R})$; preserving a conformal structure lands in $\mathrm{O}_4 \times \mathbb{R}^{>0}$; and preserving both lands in $\mathrm{SO}_4 \times \mathbb{R}^{>0}$. The first three of these act irreducibly on $\Lambda^2(\mathbb{R}^4)^*$, but the action of $\mathrm{SO}_4 \times \mathbb{R}^{>0}$ has two irreducible summands, $\Lambda_\pm^2(\mathbb{R}^4)^*$.

To understand this better, we should learn a little more about SO_4 . Recall that Sp_1 is the Lie group of unit quaternions. This is isomorphic to SU_2 , the group of determinant-1 unitary transformations of \mathbb{C}^2 . This

group has an irreducible 3-dimensional representation ρ in which Sp_1 acts by conjugation on the imaginary quaternions (since $\mathbb{R} \subset \mathbb{H}$ is preserved by this action).

Remark 1.24. Another way of describing ρ is: let ρ' denote the action of SU_2 on \mathbb{C}^2 by matrix multiplication. Then $\rho \cong \mathrm{Sym}^2 \rho'$. \blacktriangleleft

Proposition 1.25. *There is a double cover $\mathrm{Sp}_1 \times \mathrm{Sp}_1 \rightarrow \mathrm{SO}_4$. Under this cover, the SO_4 -representation $\Lambda_{\pm}^4(\mathbb{R}^4)^*$ pulls back to a real three-dimensional representation in which one copy of Sp_1 acts by ρ and the other acts trivially.*

Proof. Let W' and W'' be two-dimensional Hermitian vector spaces with compatible quaternionic structures J' , resp. J'' .¹ Then, $V := W' \otimes_{\mathbb{C}} W''$ has a real structure $J' \otimes J''$: two minuses make a plus, and compatibility of J' and J'' means the real points of V have an inner product. (These kinds of linear-algebraic spaces are things you should prove once in your life.)

By tensoring symmetries we obtain a homomorphism $\mathrm{Sp}(W') \times \mathrm{Sp}(W'') \rightarrow \mathrm{O}(V)$. This factors through $\mathrm{SO}(V) \hookrightarrow \mathrm{O}(V)$, which you can see for two reasons:

- $\mathrm{Sp}(W')$ and $\mathrm{Sp}(W'')$ are connected, so this homomorphism must factor through the identity component of $\mathrm{O}(V)$, which is $\mathrm{SO}(V)$; or
- a complex vector space has a canonical orientation, and using this we know these symmetries are orientation-preserving.

Now we want to claim this map is two-to-one. One can quickly check that $(-1, -1)$ is in the kernel; the rest is an exercise. \blacktriangleleft

Since Spin_n is the double cover of SO_n , this is telling us $\mathrm{Spin}_4 = \mathrm{Sp}_1 \times \mathrm{Sp}_1$. This splitting is the genesis of a lot of what we'll do in the next several lectures.

Consider the 16-dimensional space

$$(1.26) \quad V^* \otimes V^* = (W')^* \otimes (W')^* \otimes (W'')^* \otimes (W'')^*.$$

Because the map

$$(1.27) \quad \begin{aligned} \omega' : W' \times W' &\longrightarrow \mathbb{C} \\ \xi', \eta' &\longmapsto h'(J'\xi', \eta') \end{aligned}$$

is skew-symmetric, it lives in $\Lambda^2(W')^* \subset (W')^* \otimes (W')^*$. In particular, the embedding

$$(1.28) \quad \mathrm{Sym}^2(W')^* \oplus \mathrm{Sym}^2(W'')^* \hookrightarrow (W')^* \otimes (W')^* \otimes (W'')^* \otimes (W'')^*$$

is the map sending

$$(1.29) \quad \alpha, \beta \longmapsto \alpha \otimes w'' + \omega' \otimes \beta.$$

Remark 1.30. This story can be interpreted in terms of representations of $\mathrm{Sp}(W') \times \mathrm{Sp}(W'')$. Let $\mathbf{1}$ denote the trivial representation of Sp_1 and $\mathbf{3}$ be the three-dimensional irreducible representation we discussed above. Then (1.26) enhances to

$$(1.31) \quad V^* \otimes V^* = \mathbf{1}_{\mathrm{Sp}(W')} \otimes \mathbf{3}_{\mathrm{Sp}(W')} \otimes \mathbf{1}_{\mathrm{Sp}(W'')} \otimes \mathbf{3}_{\mathrm{Sp}(W'')}.$$

The skew-symmetric part is $\mathbf{3}_{\mathrm{Sp}(W')} \otimes \mathbf{1}_{\mathrm{Sp}(W'')} \oplus \mathbf{1}_{\mathrm{Sp}(W')} \otimes \mathbf{3}_{\mathrm{Sp}(W'')}$, and the “rest” (complement) is symmetric. \blacktriangleleft

The group $\mathrm{Sp}_1 \times \mathrm{Sp}_1 = \mathrm{Spin}_4$ has complex (quaternionic) two-dimensional representations S^{\pm} , the *spin representations*, and $\Lambda_{\pm}^2 V \cong \mathrm{Sym}^2 S^{\pm}$.

So two-forms have self-dual and anti-self-dual parts, and curvature is a natural source of 2-forms! \blacktriangleleft

¹That is, J' is an antilinear endomorphism of W' squaring to -1 , and similarly for J'' . Compatible means with the Hermitian metric: h is a map $\overline{W} \times W \rightarrow \mathbb{C}$ and J is a map $W \rightarrow \overline{W}$, and if $\xi, \eta \in W'$, we want

$$h(J'\xi, \overline{J'\eta}) = \overline{h(\xi, \eta)} \quad \text{and} \quad h(J\xi, \eta) = -h(J\eta, \xi).$$

Lecture 2.

Fantastic 2-forms and where to find them: 1/24/19*“I’ve taught this before, so I know it’s true.”*

Last time, we discussed some linear algebra which is a local model for phenomena we will study in differential geometry. For example, we saw that on an oriented even-dimensional vector space with an inner product, the Hodge star defines a self-map of the middle-dimensional part of the exterior algebra, which induces extra structure, such a splitting into self-dual and anti-self-dual pieces in dimensions divisible by 4. This therefore generalizes to a $4k$ -dimensional manifold with a metric and an orientation: the space of $2k$ -forms splits as an orthogonal direct sum of self-dual and anti-self-dual forms. (We also discussed other examples, such as how 1-forms on an oriented 2-manifold split into holomorphic and antiholomorphic pieces.)

We’re particularly interested in the case $k = 1$, where this splitting depends only on a conformal structure, and applies to 2-forms. To study its consequences we’ll discuss where one can find 2-forms in differential geometry.

Definition 2.1. A *fiber bundle* is the data of a smooth map $\pi: E \rightarrow X$ of smooth manifolds if for all $x \in X$ there’s an open neighborhood U of x and a diffeomorphism $\varphi: U \times \pi^{-1}(x) \rightarrow \pi^{-1}(U)$ such that the diagram

$$(2.2) \quad \begin{array}{ccc} U \times \pi^{-1}(x) & \xrightarrow{\varphi} & \pi^{-1}(U) \\ & \searrow \text{proj}_1 & \swarrow \pi \\ & U & \end{array}$$

commutes. In this case we call X the *base space* and E the *total space*. If there is a manifold F such that in the above definition we can replace $\pi^{-1}(x)$ with F , we call π a *fiber bundle with fiber F* .² The map φ is called the *local trivialization*.

Example 2.3. The *trivial bundle* with fiber F is the projection map $X \times F \rightarrow X$. ◀

Remark 2.4. Fiber bundles were first defined by Steenrod in the 1940s, albeit in a different-looking way. His key insight was local triviality. There are variants depending on what kind of space you care about: for example, you can replace manifolds with spaces and smooth maps with continuous maps.

Keep in mind that a fiber bundle is data (π) and a condition. Often people say “ E is a fiber bundle” when they really mean “ π is a fiber bundle”; specifying E doesn’t uniquely specify π . ◀

If F has more structure, such as a Lie group, torsor, vector space, algebra, Lie algebra, etc., we ask that $\varphi|_{\pi^{-1}(x)}: F \rightarrow \pi^{-1}(x)$ preserve this structure. For example, in a fiber bundle whose fibers are vector spaces, we want φ to be linear; in this case we call it a *vector bundle*.

Definition 2.5. If $\pi: E \rightarrow X$ is a vector bundle, the space of *k -forms valued in E* , denoted $\Omega_X^k(E)$, is the space of C^∞ sections of $\Lambda^k T^*X \otimes E \rightarrow X$.

For ordinary differential forms (so when E is a trivial bundle), we have the de Rham differential $d: \Omega_X^k \rightarrow \Omega_X^{k+1}$, but we do not have this in general.

Definition 2.6. Let X be a smooth manifold.

- (1) A *distribution* on X is the subbundle $E \subset TX$.
- (2) A vector field ξ on X *belongs to E* if $\xi_x \in E_x \subset T_x X$ for all x .
- (3) A submanifold $Y \subset X$ is an *integral submanifold* for E if for all $y \in Y$, $T_y Y = E_y$ inside $T_y X$.

Do integral submanifolds exist? This is a local question and a global question (the latter about maximal integral submanifolds). In general, the answer is “no,” as in the next example.

Example 2.7. Consider a distribution on \mathbb{A}^3 with coordinates (x, y, z) given by

$$(2.8) \quad E_{(x,y,z)} = \text{span} \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} + x \frac{\partial}{\partial z} \right\}.$$

There is no integral surface for this distribution. **TODO:** I missed the argument, sorry. ◀

²Not all fiber bundles have a fiber in this sense, e.g. a fiber bundle with different fibers over different connected components.

This is the basic example that illustrates curvature. It turns out that the existence of an integral submanifold is determined completely by the (non)vanishing of a tensor.

Definition 2.9. Let $E \subset TX$ be a distribution. The *Frobenius tensor* $\phi_E: E \times E \rightarrow TX/E$ given by

$$\xi_1, \xi_2 \mapsto [\xi_1, \xi_2] \bmod E.$$

Let's think about this: the Lie bracket is defined for vector fields, not vectors. So we have to extend ξ_1 and ξ_2 to vector fields (well, sections of E , since they're in E), which is a choice, and then check that what we obtain is independent of this choice. It suffices to know that this is linear over functions: that

$$(2.10) \quad [f_1 \xi_1, f_2 \xi_2] \stackrel{?}{=} f_1 f_2 [\xi_1, \xi_2].$$

Of course, this is not what the Lie bracket does: it differentiates in both variables, so we have the extra terms $f_1(\xi \cdot f_2)\xi_2$ and $f_2(\xi \cdot f_1)\xi_1$. But both of these are sections of E , so vanish mod E , and therefore we do get a well-defined, skew-symmetric form, a section of $\Lambda^2 E^* \otimes TX/E$ – not quite a differential form.

Frobenius did many important things in mathematics, across group theory and representation theory and this theorem, which is about differential equations!

Theorem 2.11 (Frobenius theorem). *An integral submanifold of E exists locally iff $\phi_E = 0$.*

This is a nonlinear ODE. As such, our proof will rely on some facts from a course on ODEs.

Lemma 2.12. *Let X be a smooth manifold, ξ be a vector field on X , and $x \in X$ be a point where ξ doesn't vanish. Then there are local coordinates x^1, \dots, x^n around x such that $\xi = \partial x^1$ in this neighborhood.*

Proof. Let φ_t be the local flow generated by ξ , and choose coordinates y^1, \dots, y^n near x such that $\xi_x = \frac{\partial}{\partial y^1} \Big|_x$. Define a map $U: \mathbb{R}^n \rightarrow X$ by

$$(2.13) \quad x^1, \dots, x^n \mapsto \varphi_{x^1}(0, x^2, \dots, x^n).$$

The right-hand side is expressed in y -coordinates. Now we need to check this is a coordinate chart, which follows from the inverse function theorem, because the differential of φ is invertible at 0 (in fact, it's the identity). The theorem then follows because x^1 is the time direction for flow along ξ in this coordinate system. \square

Lemma 2.14. *With notation as above, let ξ_1, \dots, ξ_k be vector fields which are linearly independent at x and such that $[\xi_i, \xi_j] = 0$ for all $1 \leq i, j \leq k$. Then there exist local coordinates x^1, \dots, x^n such that for $1 \leq i \leq k$, $\xi_i = \frac{\partial}{\partial x^i}$.*

In fact, the converse is also true, but trivially so: it's the theorem in multivariable calculus that mixed partials commute.

Proof. Let $\varphi_1, \dots, \varphi_k$ be the local flows for ξ_1, \dots, ξ_k . Because the pairwise Lie brackets commute, $\varphi_i \varphi_j = \varphi_j \varphi_i$. Since these vector fields are linearly independent at x , we can choose local coordinates y^1, \dots, y^n around x such that $\xi_i|_x = \frac{\partial}{\partial y^i} \Big|_x$. Then, as above, define

$$(2.15) \quad x^1, \dots, x^n \mapsto (\varphi_1)_{x_1}(\varphi_2)_{x_2} \cdots (\varphi_k)_{x_k}(0, \dots, 0, x^{k+1}, \dots, x^n).$$

You can check that $d\varphi$ is invertible, so this is a change of coordinates, and then, using the fact that the flows commute, you can see that the lemma follows. \square

These lemmas are important theorems in their own right.

Proof of Theorem 2.11. Since the theorem statement is local, we can work in affine space \mathbb{A}^n . Let $\pi: \mathbb{A}^n \rightarrow \mathbb{A}^k$ be an affine surjection such that $d\pi_0$ restricts to an isomorphism $E_0 \rightarrow \mathbb{R}^k$. Restrict to a neighborhood U of 0 in \mathbb{A}^n such that $d\pi_p|_{E_p}: E_p \rightarrow \mathbb{R}^k$ is an isomorphism for all $p \in U$, and choose $\xi_i|_p \in E_p$ such that $d\pi_p(\xi_p) = \frac{\partial}{\partial y^i}$. Then, $[\xi_i, \xi_j] = 0$: we know it's in E , and

$$(2.16) \quad d\pi[\xi_i, \xi_j] = [d\pi(\xi_i), d\pi(\xi_j)] = \left[\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right] = 0.$$

Now apply Lemma 2.14; then $\{y^{k+1} = \dots = y^n = 0\}$ gives the desired integral submanifold. \square

The idea of the theorem is that it's a local normal form for an involutive distribution (one whose Frobenius tensor vanishes): locally it looks like the splitting of \mathbb{R}^n into the first k coordinates and the last $(n - k)$ coordinates. And in that local model, we know what the integral manifolds are.

Consider a fiber bundle with a discrete fiber (i.e. the inverse image of every point has the discrete topology). This is also known as a *covering space*. On a “nearby fiber,” whatever that means (without more data, we don't have a metric on the base space), we have some sort of parallel transport. The precise statement is that there's a neighborhood of any x on the base space such that any path in that neighborhood lifts to a path on the total space, unique if you specify a point in the fiber. More generally, you can lift families of paths, which illustrates a homotopy-theoretic generalization of a fiber bundle called a *fibration*. But globally, given an element of $\pi_1(X)$, it might lift to a nontrivial automorphism of the fiber.

We'd like to do this for more general fiber bundles $\pi: E \rightarrow X$, in which case we'll need more data. The kernel of $d\pi$ is a distribution, and consists of the “vertical” vectors (projection down to X kills them). A complement is “horizontal”.

Without any choice, we get a short exact sequence at every $e \in E$:

$$(2.17) \quad 0 \longrightarrow \ker(d\pi_e) \longrightarrow T_e E \xrightarrow{d\pi_e} T_x X \longrightarrow 0,$$

and a splitting is exactly the choice of a complement $H_e: T_x X \rightarrow T_e E$. We would like to do this over the whole base, which motivates the next definition.

Definition 2.18. Let $\pi: E \rightarrow X$ be a fiber bundle. A *horizontal distribution* is a subbundle $H \subset TE$ transverse to $\ker(d\pi)$, or equivalently a section of the (surjective) map $TE \rightarrow \pi^*TX$ of vector bundles on E .

We must address existence and uniqueness. At e the space of splittings is an affine space modeled on $\text{Hom}(T_x X, \ker(d\pi_e))$, because **TODO** something with a short exact sequence.

Therefore existence and uniqueness of a horizontal distribution is a question about existence and uniqueness of a section of an affine bundle over X . Using partitions of unity, we can construct many of these: existence is good, but uniqueness fails.

What about path lifting? Suppose $\gamma: [0, 1] \rightarrow X$ is a path in X beginning at x_0 and terminating at x_1 . We can pull back both E and H by γ , to obtain a rank-1 distribution γ^*H in γ^*TE , and the projection map to $T[0, 1]$ is a fiberwise isomorphism. Therefore given a vector at $x_0 = \gamma(0)$ we get a unique horizontal lift along $[0, 1]$ to a vector field, and therefore get a unique integral curve above γ .

Note that you cannot always lift higher-dimensional submanifolds, and again the obstruction is the Frobenius tensor, because that's the obstruction to the existence of an integral submanifold. In this context the Frobenius tensor is called *curvature* – right now it's on the total space, but in some settings we can descend it to the base.

Lecture 3.

Principal bundles, associated bundles, and the curvature 2-form: 1/29/19

“For whatever reason I'm being a little impressionistic. . .”

Last time, we discussed a way in which 2-forms appear in geometry: as the obstruction to integrability of a distribution $E \subset TX$. That is, a distribution contains vectors, and we can ask whether integral curves of those vectors have tangent vectors contained within E . Associated to E we defined a Frobenius tensor $\phi_E: \Lambda^2 E \rightarrow TX/E$ sending

$$(3.1) \quad \xi_1, \xi_2 \mapsto [\tilde{\xi}_1, \tilde{\xi}_2] \text{ mod } E,$$

where $\tilde{\xi}_i$ is a vector field extending ξ (and we showed this doesn't depend on the choice of extension). In Theorem 2.11, we saw that ϕ_E is exactly the local obstruction to integrability; we can then move to global questions.

More generally, suppose that $\pi: E \rightarrow X$ is a fiber bundle. Then TE fits into a short exact sequence (2.17), and we can ask for a horizontal lift from TX to TE , which is a section H of (2.17). Then, given a vector $e \in T_x X$ and a path $\gamma: [0, 1] \rightarrow X$ with $\gamma(0) = x$, we can pull back³ π and H to obtain a distribution in γ^*E .

³In general, we can form the pullback of $[0, 1] \rightarrow X \leftarrow E$ in the category of sets or spaces, but we want to put a smooth manifold structure on it. We can do it when these two maps are transverse – and since $\pi: E \rightarrow X$ is a submersion, this is always satisfied.

The Frobenius tensor vanishes, because $[0, 1]$ is one-dimensional, so we can extend to an integral curve and therefore parallel-transport along γ . However, if we choose different paths in a ball, there's no guarantee that parallel transport along nearby paths agree at all; the Frobenius tensor may still be nonzero on X .

Steenrod's elegant perspective on fiber bundles (see his book *The Topology of Fiber Bundles*) considered in the spirit of Felix Klein symmetry groups associated to fiber bundles. This leads to the definition of a *principal G -bundle* as a fiber bundle of right G -torsors.

Definition 3.2. Let G be a Lie group and recall that a *right G -torsor* is a smooth manifold T and a smooth right T -action on T such that the action map $T \times G \rightarrow T \times T$ sending $(t, g) \mapsto (t, t \cdot g)$ is an isomorphism.

Example 3.3. The prime example of a torsor is to let V be a real vector space; then, the manifold $\mathcal{B}(V)$ of bases of V is a $\mathrm{GL}_n(\mathbb{R})$ -torsor: $\mathrm{GL}_n(\mathbb{R})$ acts by precomposition. This also works over \mathbb{C} and \mathbb{H} . \blacktriangleleft

Example 3.4. Now let X be a smooth manifold. Our first example of a principal bundle spreads Example 3.3 over X : let $\mathcal{B}(X)$ be the smooth manifold⁴ of pairs (x, b) where $x \in X$ and b is a basis of $T_x X$, i.e. an isomorphism $b: \mathbb{R}^n \xrightarrow{\cong} T_x X$. There's a natural forgetful map $\pi: \mathcal{B}(X) \rightarrow X$ sending $(x, b) \mapsto x$.

This fiber bundle is a principal $\mathrm{GL}_n(\mathbb{R})$ -bundle: given $g \in \mathrm{GL}_n(\mathbb{R})$ and a basis $b: \mathbb{R}^n \rightarrow T_x X$, we let $b \cdot g := b \circ g: \mathbb{R}^n \rightarrow T_x X$, using the standard action of $\mathrm{GL}_n(\mathbb{R})$ on \mathbb{R}^n . This is called the *frame bundle* of X . \blacktriangleleft

This principal bundle controls a lot of the geometry of X , via its associated fiber bundles.

Definition 3.5. Let $\pi: P \rightarrow X$ be a principal G -bundle and F be a smooth (left) G -manifold. The *associated fiber bundle* with fiber F is the quotient $P \times_G F := (P \times F)/G$, which is a fiber bundle over X with fiber F . Here, G acts on $P \times F$ on the right by $(p, f) \cdot g = (p \cdot g, g^{-1} \cdot f)$.

One has to check this is a fiber bundle, and in particular that the total space is a smooth manifold. Since G acts freely on P , it acts freely on $P \times F$, but for G noncompact there's more to say.

Example 3.6. $\mathrm{GL}_n(\mathbb{R})$ acts linearly on \mathbb{R}^n . Because \mathbb{R}^n carries the additional structure of a vector space, the associated bundle $\mathcal{B}(X) \times_{\mathrm{GL}_n(\mathbb{R})} \mathbb{R}^n$ has additional structure: it's a vector bundle. In general, additional structure on F manifests in additional structure on $P \times_G F$.

Anyways, what vector bundle do we get? (Can you guess?) An element of the fiber of $\mathcal{B}(X) \times_{\mathrm{GL}_n(\mathbb{R})} \mathbb{R}^n$ is an equivalence class of an element $v \in \mathbb{R}^n$ and a basis $p: \mathbb{R}^n \rightarrow T_x X$; let $\xi := p(v) \in T_x X$. Another representative of this equivalence class are represented by $g^{-1}v$ and $p \circ g$ for some $g \in \mathrm{GL}_n(\mathbb{R})$, so this pair defines the same tangent vector ξ . Therefore we recover the tangent bundle. \blacktriangleleft

In general, a principal bundle is telling you some internal coordinates. You know these coordinates up to some symmetry G , and the principal bundle tracks that: you have to make a choice to get coordinates, and it tells you how different choices are related.

We want to show local triviality of a principal G -bundle $\pi: P \rightarrow X$, which will follow from local triviality as a fiber bundle. Consider a local section $s: U \rightarrow \pi^{-1}(U)$, where $U \subset X$; we would like to exhibit an isomorphism of fiber bundles $U \times G \rightarrow \pi^{-1}(U)$ over U . The map is exactly

$$(3.7) \quad x, g \mapsto s(x) \cdot g.$$

This exhibits $U \times G \cong \pi^{-1}(U)$ as principal G -bundles, so we have local trivialization. Then in every associated bundle to P , we also obtain local triviality, hence local coordinates. For example, if the bundle of frames is trivialized over U , we get local coordinates (i.e. a local trivialization of TU).

Definition 3.8. A *connection* on a principal G -bundle $\pi: P \rightarrow X$ is a G -invariant horizontal distribution.

Specifically, given $g \in G$, we have the right action map $R_g: P \rightarrow P$, and can therefore define $H_{p \cdot g} := (R_g)_*(H_p)$ for a distribution H .

In this setting, the Frobenius tensor is going to do something nice: it's a map

$$(3.9) \quad \phi_H: H \wedge H \rightarrow TP/H \cong \ker(\pi_*),$$

⁴You have to put a smooth manifold structure on this set! The way to do this is the only tool we have right now: work in an atlas \mathcal{U} of X which trivializes TX , do this locally, and check that the transition maps are smooth. This will also show that the map $\pi: \mathcal{B}(X) \rightarrow X$ is a fiber bundle.

so given two horizontal vectors, we get a vertical vector. Since H is G -invariant, the Frobenius tensor is also G -invariant, so we ought to be able to descend it to the base: there's only one piece of information on each fiber. That is, given vectors ξ_1, ξ_2 on X , we can lift them to P and compute the Frobenius tensor there, and G -invariance means it doesn't matter how we lift.

If \mathfrak{g} is the Lie algebra of the Lie group G , we have an isomorphism $\underline{\mathfrak{g}} \xrightarrow{\cong} \ker(\pi_*)$ as vector bundles on P . Specifically, let $\xi \in \mathfrak{g}$, and consider the exponential map $\exp: \mathfrak{g} \rightarrow G$. Given $p \in P$ with $\pi(p) = x$, we get a curve in P given by $t \mapsto p \cdot \exp(t\xi)$ sending $0 \mapsto p$, and this curve is contained entirely within P_x . Therefore its tangent vector at p is in $\ker(\pi_*)$.

So the Frobenius tensor is a map $\phi_H: H \wedge H \rightarrow \underline{\mathfrak{g}}$. Now let's descend to the base. We'd like to claim that what we get in \mathfrak{g} is invariant, but that's just not true: if $g \in G$, the action of g on $p \cdot \exp(t\xi)$ is not the same as $p \cdot g \cdot \exp(t\xi)$: the issue is that $g \exp(t\xi)$ and $\exp(t\xi)g$ may not agree. This will make it slightly more interesting to descend to the base.

First, extend ϕ_H to a map

$$(3.10) \quad \tilde{\phi}_H: TP \wedge TP \longrightarrow \underline{\mathfrak{g}}$$

by projecting $p_H: TP \twoheadrightarrow H$, which has kernel $\ker(\pi_*)$. That is, $\tilde{\phi}_H(\eta_1 \wedge \eta_2) := p_H \eta_1 \wedge p_H \eta_2$. Thus $\tilde{\phi}_H \in \Omega_P^2(\underline{\mathfrak{g}})$.

Lemma 3.11. *Let $g \in G$. Then in $\Omega_P^2(\underline{\mathfrak{g}})$, $R_g^* \tilde{\phi}_H = \text{Ad}_{g^{-1}} \tilde{\phi}_H$.*

So once we choose a basis for \mathfrak{g} , we can think of elements of $\Omega_P^2(\underline{\mathfrak{g}})$ as matrix-valued differential forms.⁵ The proof of Lemma 3.11 comes from the observation above that to get from $p \cdot g \cdot \exp(t\xi)$ to

$$(3.12) \quad p \cdot \exp(t\xi)g = p \cdot g \cdot (g^{-1} \exp(t\xi)g) = p \cdot g \cdot \text{Ad}_{g^{-1}}(\xi).$$

So this is exactly an example of an associated bundle to P , where the G -manifold F is \mathfrak{g} with the adjoint G -bundle. So associated to P is the *adjoint bundle* $\mathfrak{g}_P \rightarrow X$ defined as $P \times_G \mathfrak{g}$. This is a vector bundle, in fact a bundle of Lie algebras because the adjoint action preserves the Lie bracket.

A section of \mathfrak{g}_P is a function upstairs valued in \mathfrak{g} , which is exactly what $\tilde{\phi}_H$ is.

Corollary 3.13. *$\tilde{\phi}_H$ descends to a 2-form $-\Omega_H \in \Omega_X^2(\mathfrak{g}_P)$.*

In this case Ω_H is called the *curvature* of H . In particular, if X is a 4-manifold with a conformal structure, we can ask for this to be self-dual or anti-self-dual.

In the short exact sequence

$$(3.14) \quad 0 \longrightarrow \underline{\mathfrak{g}} \longrightarrow TP \xrightarrow{\pi_*} \pi^*TX \longrightarrow 0,$$

a section $H: \pi^*TX \rightarrow TP$ is equivalent to a section $\Theta: TP \rightarrow \underline{\mathfrak{g}}$, i.e. a form $\Theta \in \Omega_P^1(\underline{\mathfrak{g}})$. This is called the *connection form*, and $H = \ker(\Theta)$. It has to satisfy some properties.

- Θ must be G -invariant: $R_g^* \Theta = \text{Ad}_{g^{-1}} \Theta$. This is a linear equation inside the infinite-dimensional vector space $\Omega_P^1(\underline{\mathfrak{g}})$.
- The other constraint is affine: $\Theta|_{\text{vertical}} = \text{id}$.

So the space \mathcal{A}_P of one-forms Θ satisfying these conditions is affine. This is the space of connections, and in particular tells us that there are lots of connections.

We can also interpret the Frobenius tensor in terms of Θ . Let ζ_1 and ζ_2 be horizontal vectors, and extend them to vector fields $\tilde{\zeta}_1$ and $\tilde{\zeta}_2$. Then $\zeta \cdot \Theta(\tilde{\zeta}_{1-i}) = 0$, so

$$(3.15) \quad \begin{aligned} d\Theta(\zeta_1, \zeta_2) &= \zeta_1 \Theta(\tilde{\zeta}_2) - \zeta_2 \Theta(\tilde{\zeta}_1) - \Theta([\tilde{\zeta}_1, \tilde{\zeta}_2]) \\ &= -\Theta([\tilde{\zeta}_1, \tilde{\zeta}_2]) = -\phi_H(\zeta_1, \zeta_2). \end{aligned}$$

Thus we have proved

Proposition 3.16. $\pi^* \Omega_H = -\tilde{\phi}_H = d\Theta + (1/2)[\Theta \wedge \Theta]$.

The notation $[\Theta \wedge \Theta]$ means: $\Theta \wedge \Theta \in \Omega_P^2(\mathfrak{g} \otimes \mathfrak{g})$, and this has a Lie bracket map $[\cdot]: \Omega_P^2(\mathfrak{g} \otimes \mathfrak{g}) \rightarrow \Omega_P^2(\mathfrak{g})$.

⁵Here I suppose we need to use a Lie group G that admits a faithful finite-dimensional representation, but all compact Lie groups, and most noncompact Lie groups that you'll encounter, have this property.

Corollary 3.17 (Bianchi identity). $d\Omega + [\Theta \wedge \Omega] = 0$.

Proof.

$$\begin{aligned}
 d\Omega_H &= [d\Theta \wedge \Theta] \\
 (3.18) \quad &= \left[\Omega - \frac{1}{2}[\Theta \wedge \Theta] \wedge \Theta \right] \\
 &= [\Omega \wedge \Theta]
 \end{aligned}$$

by the Jacobi identity. □

This has been more theory than examples of principal bundles, but we will see plenty of examples when we delve into gauge theory.

Now given a principal G -bundle $\pi: P \rightarrow X$ with a connection, and any associated bundle F_P with fiber F , we get a horizontal distribution. There's a hands-on way to construct this, or you could think of it in terms of path lifting: given an $x \in X$ and a lift $p \in P$, the connection lifts a path $\gamma: [0, 1] \rightarrow X$ based at x to a path $\tilde{\gamma}: [0, 1] \rightarrow P$ based at p , so given an $f \in F$, we can define the path $\hat{\gamma}: [0, 1] \rightarrow F_P$ by $t \mapsto (\tilde{\gamma}(t), f)$.

Suppose V is a G -representation, so its associated vector bundle $V_P \rightarrow X$ is a vector bundle. Then the horizontal distribution we obtain on V_P is tangent to the zero section of V_P . Let $\psi: X \rightarrow V_P$ be a section and $\xi \in T_x X$; we would like to differentiate ψ in the direction ξ . If ψ were valued in a fixed vector space, we could do this as usual: extend ξ to a curve $\gamma: (-\varepsilon, \varepsilon) \rightarrow M$, and then define

$$(3.19) \quad \nabla_\xi \psi := \left. \frac{d}{dt} \right|_{t=0} \psi(\gamma(t)).$$

This is precisely the directional derivative. In V_P , the fibers are different vector spaces, which seems like a problem except that the connection on P defines parallel transport τ_t along γ for the fibers of V_P , and therefore we can define the directional derivative of ψ as

$$(3.20) \quad \nabla_\xi \psi := \left. \frac{d}{dt} \right|_{t=0} \tau_{-t} \psi(\gamma(t)).$$

This is called the *covariant derivative*.

Exercise 3.21. Show that this satisfies the Leibniz rule: if f is a function on X , then

$$(3.22) \quad \nabla_\xi (f \cdot \psi) = (\xi \cdot f) \psi + f(x) \nabla_\xi \psi.$$

In other words, the existence of the horizontal distribution is somehow telling us about the Leibniz rule, though this is a somewhat mysterious fact.

Lecture 4.

Harmonic forms and (anti)-self-dual connections: 1/31/19

“The key to humor is..... timing!”

Last time, we discussed connections on principal bundles, and what they induce on associated vector bundles. We also briefly saw the covariant derivative associated to a connection. We begin with more on covariant derivatives.

Definition 4.1. Let $E \rightarrow X$ be a vector bundle. A *covariant derivative* is a linear map $\nabla: \Omega_X^0(E) \rightarrow \Omega_X^1(E)$ satisfying the *Leibniz rule*

$$(4.2) \quad \nabla(fs) = df \cdot s + f \nabla s,$$

where f is a smooth function on X and s is a smooth section of E .

If E is a trivial bundle with constant fiber V , the usual directional derivative is a covariant derivative, but there can be others.

We can extend ∇ to a sequence of first-order differential operators

$$(4.3) \quad 0 \longrightarrow \Omega_X^0(E) \xrightarrow{d\nabla} \Omega_X^1(E) \xrightarrow{d\nabla} \Omega_X^2(E) \xrightarrow{d\nabla} \dots$$

defined by

$$(4.4) \quad d_{\nabla}(\omega \cdot s) := d\omega \cdot s + (-1)^k \omega \wedge \nabla s,$$

where $\omega \in \Omega_X^k$ and $s \in \Omega_X^0(E)$. Thus the first map $d_{\nabla}: \Omega_X^0(E) \rightarrow \Omega_X^1(E)$ is just ∇ .

Exercise 4.5. Show that $d_{\nabla}^2(fs) = fd_{\nabla}^2(s)$.

In other words, this says the *symbol* of d_{∇}^2 vanishes; this second-order operator is really a first-order operator. Therefore there exists an $F_{\nabla} \in \Omega_X^2(\text{End } E)$, called the *curvature*, such that $d_{\nabla}^2(s) = F_{\nabla} \cdot s$.

Digression 4.6. We recall what the symbol of an operator is. Let $E, F \rightarrow X$ be vector bundles and $D: \Omega_X^0(E) \rightarrow \Omega_X^0(F)$ be a differential operator. By definition, D is first-order if for every function f and section s ,

$$(4.7) \quad D(fs) = \sigma(df)s + fDs$$

for some $\sigma: T^*X \otimes E \rightarrow F$, which is called the *symbol* of D . ◀

Exercise 4.8. Compute d_{∇}^3 . (Answer: it's zero.)

Now we have two notions of curvature: the curvature associated to a covariant derivative as above, and the curvature associated to a principal bundle with connection and an associated vector bundle.

Exercise 4.9. Let G be a Lie group, $\pi: P \rightarrow X$ be a principal G -bundle with connection $\Theta \in \Omega_P^1(\mathfrak{g})$, and $\rho: G \rightarrow \text{Aut}(\mathbb{E})$ be a linear representation of G . Let $E := \mathbb{E}_P = P \times_G \mathbb{E} \rightarrow X$ be the associated bundle, which carries a covariant derivative $\nabla: \Omega_X^0(E) \rightarrow \Omega_X^1(E)$. Compute d_{∇}^2 in terms of $\Omega = d\Theta + (1/2)[\Theta \wedge \Theta]$.

Example 4.10. Let's think about connections on a principal \mathbb{T} -bundle.⁶ Consider \mathbb{C}^2 with coordinates z^0, z^1 and metric

$$(4.11) \quad \langle (z^0, z^1), (w^0, w^1) \rangle := \overline{z^0}w^0 + \overline{z^1}w^1.$$

The circle group \mathbb{T} acts on $S^3 \subset \mathbb{C}^2$ on the right by $(z^0, z^1) \cdot \lambda := (z^0\lambda, z^1\lambda)$. This is a free action, so its quotient is a smooth manifold, specifically $\mathbb{CP}^1 \cong S^2$, the manifold of complex lines through the origin in \mathbb{C}^2 . Thus we obtain a principal \mathbb{T} -bundle $\pi: S^3 \rightarrow \mathbb{CP}^1$, called the⁷ *Hopf bundle*.

Now let's put a connection on π . We want a horizontal distribution on the total space S^3 . Inside $T_{(z^0, z^1)}S^3$, there's a one-dimensional subspace of vectors in the direction of the fiber $\{(z^0, z^1) \cdot \lambda\}$. The standard Riemannian metric on $\mathbb{C}^2 = \mathbb{R}^4$ allows us to choose a complementary line at each point, which is a horizontal distribution. Because \mathbb{T} acts by isometries, this is an invariant distribution, hence a connection.

This is all pretty and geometric, but we need to compute the connection form $\Theta \in \Omega_{S^3}^1(i\mathbb{R})$ (the Lie algebra of \mathbb{T} is a line with trivial bracket, and is more canonically $i\mathbb{R}$). Specifically,

$$(4.12) \quad \Theta = \text{Im}(\overline{z^0} dz^0 + \overline{z^1} dz^1).$$

In the vertical direction, $\Theta = \text{id}$, $(z^0 e^{it}, z^1 e^{it}) = (iz^0, iz^1)$. Looking inside the complexified tangent bundle (a four-dimensional complex vector bundle), which has basis $\{\partial_{z^0}, \partial_{\overline{z^0}}, \partial_{z^1}, \partial_{\overline{z^1}}\}$, we get

$$(4.13) \quad iz^0 \frac{\partial}{\partial z^0} - iz^{\overline{0}} \frac{\partial}{\partial \overline{z^0}} + iz^1 \frac{\partial}{\partial z^1} - iz^{\overline{1}} \frac{\partial}{\partial \overline{z^1}}.$$

So on vertical vectors, this is the identity. One (you) can check that on a vector normal to S^3 , this vanishes – this is just linear algebra over the complex numbers, so nothing too intimidating.

Next we'd like to see

$$(4.14) \quad \Omega = d\Theta = \text{Im}(\overline{dz^0} \wedge dz^0 + \overline{dz^1} \wedge dz^1),$$

though this is already imaginary, so we can remove the 'Im' in front. You can check this descends to \mathbb{CP}^1 . It's a 2-form on \mathbb{C}^2 , visibly of type $(1, 1)$, and we restrict it to S^3 ; the claim is that there's a form on \mathbb{CP}^1 whose pullback by π is $\Omega|_{S^3}$. This involves verifying two things: that Ω is \mathbb{T} -invariant, and that it's trivial in the vertical direction. This is a good practice computation.

⁶Here $\mathbb{T} \subset \mathbb{C}^\times$ is the group of unit-magnitude complex numbers, sometimes also denoted U_1 or S^1 .

⁷Well, there's more than one Hopf bundle, and we'll see some others later, but this is the first example.

Let Ω also denote the form on \mathbb{CP}^1 : $\Omega \in \Omega_{\mathbb{CP}^1}^2(i\mathbb{R})$. We claim

$$(4.15) \quad \int_{\mathbb{CP}^1} \frac{1}{2\pi} i\Omega = 1.$$

To compute this, we need some coordinates on \mathbb{CP}^1 . We'll construct a section s of π over $\mathbb{CP}^1 \setminus \infty \cong \mathbb{C}$. Specifically, given $z \in \mathbb{C}$, which we think of as $[z : 1] \in \mathbb{CP}^1$, let

$$(4.16) \quad s(z) = \frac{(z, 1)}{\sqrt{1 + |z|^2}}.$$

The term in the denominator means that the function decays at infinity in \mathbb{C} , so we expect this integral to converge. (But you should still do it!) \blacktriangleleft

Consider a more general principal \mathbb{T} -bundle $\pi: P \rightarrow X$, where X is a smooth manifold. Is it a pullback of the Hopf bundle by a map $X \rightarrow \mathbb{CP}^1$? This need not be true, but something weaker is. Consider the generalized Hopf bundle $S^{2N+1} \rightarrow \mathbb{CP}^N$, defined in the same way as the Hopf bundle.

Theorem 4.17. *Every principal \mathbb{T} -bundle P over a smooth manifold X arises as a pullback of a Hopf bundle $S^{2N+1} \rightarrow \mathbb{CP}^N$ for some N .*

We can choose N independent of P , but it will depend on X . So in general you can think of pulling back from \mathbb{CP}^∞ .

Proof sketch. A pullback is a \mathbb{T} -equivariant map $\varphi: P \rightarrow S^{2N+1}$; the quotient by \mathbb{T} defines a map $X \rightarrow \mathbb{CP}^N$ satisfying the theorem. But this is equivalent data to a section of the associated bundle $S_P^{2N+1} \rightarrow X$. This is good: there are tools in topology for constructing sections. First, using an approximation theorem, one shows that it suffices to find a continuous section. Then, one uses obstruction theory: choose a CW structure on X and a q -cell $D \rightarrow X$. We'd like to extend a section over this cell; since D is contractible, it's equivalent to ask that the map $S^{q-1} = \partial D \rightarrow S_P^{2N+1}$ is trivial (up to homotopy). This is a question about homotopy groups, and for N large enough, the relevant homotopy group vanishes. \square

So the next question is: can we construct universal connections Θ^{univ} on these Hopf bundles such that every connection arises as a pullback? This is finickier. Supposing it exists, and $\varphi: (P, \Theta) \rightarrow (S^{2N+1}, \Theta^{\text{univ}})$, then since connections form an affine space, there's an $\alpha \in \Omega_X^1(i\mathbb{R})$ such that

$$(4.18) \quad \varphi^* \Theta^{\text{univ}} - \Theta = \pi^* \alpha,$$

and hence

$$(4.19) \quad \bar{\varphi}^* \Omega^{\text{univ}} - \Omega = d\alpha.$$

This therefore implies $d\Omega = 0$, where $\Omega \in \Omega_X^2(i\mathbb{R})$, so it has a de Rham cohomology class $[i\Omega/2\pi] \in H_{\text{dR}}^2(X)$. This is the pullback of a class $(c_1)_{\mathbb{R}} \in H_{\text{dR}}^2(\mathbb{CP}^N)$. We can see this class explicitly; \mathbb{CP}^N has a very simple CW structure with one cell in each even dimension. Therefore the cochain complex for CW cohomology with \mathbb{Z} coefficients looks like $\mathbb{Z} \rightarrow 0 \rightarrow \mathbb{Z} \rightarrow 0 \rightarrow \mathbb{Z} \rightarrow \dots$, and we claim c_1 is the generator⁸ of $H^2(\mathbb{CP}^N; \mathbb{Z})$. Then there's an argument for why these two agree, namely just calculate on \mathbb{CP}^N , and this is the beginning of Chern-Weil theory, relating curvature and characteristic classes.

Remark 4.20. There's a similar story for higher Chern classes, but it's sufficiently complicated enough that it's generally easier to calculate using the splitting principle to split a vector bundle as a direct sum of line bundles. \blacktriangleleft

Let's come back to 4-manifolds and self-duality: we let X be an oriented 4-manifold with a conformal structure $[g]$. This is enough to define the Hodge star $\star: \Omega_X^2 \rightarrow \Omega_X^2$, which squares to the identity. Tensoring with a vector bundle allows us to define $\star: \Omega_X^2(E) \rightarrow \Omega_X^2(E)$ for any vector bundle $E \rightarrow X$, which also squares to the identity; therefore we can also define self-dual and anti-self-dual forms valued in E in the same way.

Definition 4.21. Let $P \rightarrow X$ be a principal G -bundle with connection Θ and $\Omega \in \Omega_X^2(\mathfrak{g}_P)$ be the associated connection form. We say Θ is *self-dual* (resp. *anti-self-dual*) if $\star\Omega = \Omega$ (resp. $\star\Omega = -\Omega$).

⁸We need to pick a sign, but this is determined by the canonical orientation of \mathbb{CP}^N coming from the complex structure.

As we discussed in the first lecture, this is the four-dimensional analogue of a two-dimensional question on oriented, conformal surfaces: whether a function (form, ...) is holomorphic or antiholomorphic. The sign isn't all that intrinsic: changing the orientation on X changes it.

Anti-self-dual connections are of interest to physicists, since the 1970s, beginning with work of Polyakov and others looking at flat space. Uhlenbeck produced a condition guaranteeing that solutions to $\star\Omega = -\Omega$ extend over S^4 , and later Atiyah, Bott, Hitchin, and Singer claimed there are more solutions, and used algebraic geometry to produce them. We will study more of this story in this class, but first some examples.

The simplest case is $G = \mathbb{T}$. Often this is called “the” abelian case, though there are certainly other abelian Lie groups, such as \mathbb{T}^2 . Anyways, in this case Ω lives in $\Omega_X^2(i\mathbb{R})$, $d\Omega = 0$, and if $\star\Omega = \pm\Omega$, then $d\star\Omega = 0$ iff $d^*\Omega = 0$. Together these imply that Ω is a harmonic form if X is closed.

Digression 4.22. Let M be a Riemannian manifold (though for just dimension 4, we're only going to need the conformal class of the metric.) For example, we could take $M = \mathbb{E}^n$, which denotes \mathbb{R}^n with the standard Riemannian metric. Then the *Laplacian* is

$$(4.23) \quad \Delta := -\left(\frac{\partial^2}{\partial(x^1)^2} + \cdots + \frac{\partial^2}{\partial(x^n)^2}\right).$$

Why the minus sign? This has a discrete spectrum, and we'd like it to be nonnegative rather than nonpositive.

The de Rham derivative has the form

$$(4.24) \quad d = \varepsilon(dx^i) \frac{\partial}{\partial x^i},$$

where ε denotes exterior multiplication (which is its symbol). Using the metric, the formal adjoint is

$$(4.25) \quad d^* = -\iota(dx^i) \frac{\partial}{\partial x^i}.$$

(whose symbol is $-\iota$; here ι is interior multiplication). Then you can check that $\Delta := dd^* + d^*d$.

Now we can bring this to any Riemannian manifold M : we know what d is, and can define d^* by integrating by parts to construct the formal adjoint of d , or construct it locally. But, for the same reason that interior multiplication requires a metric, d^* depends on the metric. And therefore we can define the Laplacian Δ on M to be $dd^* + d^*d$. This means the analogue of (4.23) on M in local coordinates (x^1, \dots, x^n) is

$$(4.26) \quad \Delta = - \sum_{1 \leq i \leq j \leq n} -g^{ij} \frac{\partial^2}{\partial x^i \partial x^j}.$$

Here $g_{ij} := \langle \partial_i, \partial_j \rangle$, and g^{ij} is the (components of the) inverse to the matrix $(g_{ij})_{i,j}$. If you haven't seen this before, it's good to work it out.

Now suppose M is closed and $\Delta\omega = 0$. Then

$$\begin{aligned} 0 &= dd^*\omega + d^*d\omega \\ &= \langle dd^*\omega, \omega \rangle + \langle d^*d\omega, \omega \rangle \\ &= \int_M (\langle dd^*\omega, \omega \rangle + \langle d^*d\omega, \omega \rangle) d\text{vol} \\ &= \int_M (\langle d^*\omega, d^*\omega \rangle + \langle d\omega, d\omega \rangle) d\text{vol}. \end{aligned}$$

In fact, the converse is true.

Theorem 4.27. *On a closed Riemannian manifold, $d\omega = 0$ and $d^*\omega = 0$ iff $\Delta\omega = 0$.*

Such a form ω is called *harmonic*. The space of harmonic k -forms is denoted $\mathcal{H}_M^k(g) \subset \Omega_M^k$. Elliptic theory shows this is finite-dimensional,⁹ and in fact more is true.

Theorem 4.28 (Hodge decomposition). *There is a splitting*

$$\Omega_M^k \cong \underbrace{\mathcal{H}_M^k(g)}_{\text{closed}} \oplus \text{Im}(d) \oplus \text{Im}(d^*).$$

⁹We'll use some elliptic theory later this semester, and will therefore go over some of the ingredients that you'd use to prove this.

Since harmonic forms are closed, there's a projection $\mathcal{H}_M^k(g) \rightarrow H_{\text{dR}}^k(M)$, and in fact this is an isomorphism! So every cohomology class has a unique harmonic representative. ◀

And now back to 4-manifolds. If X is an oriented Riemannian 4-manifold, we have $\mathcal{H}_X^2(g) \cong H^2(X; \mathbb{R})$, and $\mathcal{H}_X^2(g)$ has two distinguished subspaces: the self-dual forms $\mathcal{H}_+^2(g)$ and the anti-self-dual forms $\mathcal{H}_-^2(g)$. These are distinct subspaces, so every harmonic 2-form ω decomposes as a sum $\omega = \omega_+ + \omega_-$, where $\omega_{\pm} \in \mathcal{H}_{\pm}^2(g)$. Explicitly,

$$(4.29) \quad \omega_{\pm} = \frac{\omega \pm \star \omega}{2}.$$

All of this depended on the metric, so we can ask how this changes as the metric moves, which involves some Sard-Smale theory, as we discussed in Morse theory last semester. But Chern-Weil theory tells us that if Ω comes from a connection on a principal \mathbb{T} -bundle, then $i\Omega/2\pi$ defines an integer-valued cohomology class. Therefore self-dual or anti-self-dual connections are the intersection of an integer lattice in H^2 with two lines $\mathcal{H}_{\pm}^2(g)$. Generically, this has no solutions, unless one of $\mathcal{H}_{\pm}^2(g)$ is zero (so all forms are self-dual, or are anti-self-dual). Perhaps that's a little disappointing.

To study this, we'll look at the intersection form, a symmetric bilinear 2-form on $H^2(X; \mathbb{Z})$ sending $c_1, c_2 \mapsto \langle c_1 \smile c_2, [X] \rangle$. Let b_+^2 (resp. b_-^2) denote the dimension of the largest subspace on which this form is positive (resp. negative). Then $b_+^2 + b_-^2 = b^2(M)$, and their difference is the signature. We'll put conditions on b_{\pm}^2 which make it possible to find (anti)-self-dual connections.

Example 4.30.

- (1) On S^4 , $b^2 = 0$, so $b_{\pm}^2 = 0$. So no self-dual forms here.
- (2) On \mathbb{CP}^2 , $b_+^2 = 1$ and $b_-^2 = 0$. In this case, self-dual forms exist! Hooray.
- (3) But on a K3 surface, $b_-^2 = 19$ and $b_+^2 = 3$, so no self-dual forms generically. ◀

This is a little annoying. Maybe we should work with a different Lie group.

The next simplest example is $\text{SU}_2 = \text{Sp}_1$. Associated to it is another Hopf bundle: Sp_1 acts on $S^7 \subset \mathbb{H}^2$, as (right) multiplication by unit quaternions, and the quotient is $\mathbb{HP}^1 \cong S^4$. We can use this to follow the same story as above, defining a connection geometrically and so on.

Lecture 5.

The Yang-Mills functional: 2/5/19

Let X be an oriented, conformal 4-manifold, $P \rightarrow X$ be a principal G -bundle, and $\Theta \in \Omega_P^1(\mathfrak{g})$ be a connection. We will study gauge theory in this situation; sometimes we will use a Riemannian metric in the conformal class for X .

In gauge theory, people typically use slightly different notation.

- The connection Θ is usually denoted A .
- Its curvature is denoted $F = F_A = dA + (1/2)[A \wedge A] \in \Omega_P^2(\mathfrak{g})$ – but F_A is also used to denote the curvature form on X , $F_A \in \Omega_X^2(\mathfrak{g}_P)$.

Definition 5.1. We say that A is *self-dual*, resp. *anti-self-dual*, if $\star F_A = F_A$, resp. $\star F_A = -F_A$.

These are first-order nonlinear PDEs in A . Let's say something about where they come from.

Hodge theory and minimization. Let (M, g) be a closed, oriented Riemannian n -manifold.¹⁰ Given a cohomology class $c \in H^k(M; \mathbb{R})$, what's the “best” differential form representative for c ? That is, what's the “best” $\omega \in \Omega_M^k$ with $[\omega] = c$?

Well, what does “best” mean? Maybe smallest-norm: let's ask for an ω which minimizes

$$(5.2) \quad f: \omega \mapsto \int_M \|\omega\|^2 \, \text{dvol}_g = \int_M \omega \wedge \star \omega$$

such that $[\omega] = c$.

¹⁰One can generalize to open manifolds, but then one needs some vanishing or growth conditions at infinity, or a boundary condition. We're not going to worry about this in this motivational section.

Remark 5.3. If M is not oriented, then we don't have a volume form, and $\omega \wedge \star\omega$ is a density. Asking to minimize this norm still makes sense. \blacktriangleleft

Fix an ω_0 such that $[\omega_0] = c$. On the affine line $\omega_0 + d\Omega_M^{k-1}$, consider the function

$$(5.4) \quad f(\omega_0 + d\eta) = \int_M (\omega_0 + d\eta) \wedge \star(\omega_0 + d\eta).$$

This is a quadratic function on a real affine line. We know what those look like – parabolas. So we can find the unique minimum where the derivative of f is zero. The derivative is

$$(5.5) \quad df_{\omega_0}(d\eta) = 2 \int_M d\eta \wedge \star\omega_0 = \pm 2 \int_M \eta \wedge d\star\omega_0,$$

using Stokes' theorem, since M is closed. The output equations are

$$(5.6) \quad \begin{aligned} d\star\omega_0 &= 0 \\ d\omega_0 &= 0. \end{aligned}$$

These are the Euler-Lagrange equations for this problem.¹¹ They're satisfied iff $\Delta\omega_0 = 0$, where $\Delta := dd^* + d^*d$ is the *Laplacian*. Solutions to $\Delta\omega_0 = 0$ are called *harmonic* forms.

Lemma 5.7. *On a closed manifold M , ω is harmonic iff $d\omega = 0$ and $d\star\omega = 0$.*

Proof sketch. Suppose ω is harmonic. Then

$$\begin{aligned} (5.8) \quad 0 &= \int_M \Delta\omega \wedge \star\omega \\ (5.9) \quad &= \int_M (dd^*\omega \wedge \star\omega + dd^*\omega \wedge \star\omega) \\ (5.10) \quad &= \int_M (\|d^*\omega\|^2 + \|d\omega\|^2) d\text{vol}_g. \end{aligned}$$

Since this is the integral of a nonnegative function, that function must be 0 everywhere, so we conclude that $d^*\omega = 0$ and $d\omega = 0$. To get from d^* to $d\star$, use the fact that the formal adjoint of d , namely d^* , is also $\pm d\star$, where the sign depends on the degree of ω and the dimension of M .

The other direction is up to you. \boxtimes

Now let's apply this to connections on 4-manifolds. Let \mathcal{A}_P denote the affine space of connections on the principal G -bundle $P \rightarrow X$.

Definition 5.11. The *Yang-Mills functional* $Y: \mathcal{A}_P \rightarrow \mathbb{R}$ is

$$(5.12) \quad Y(A) := \int_X \|F_A\|^2 d\text{vol}_g.$$

This looks nice and all that, but we haven't yet defined everything: we need to make sense of the norm on $\Omega_P^2(\mathfrak{g})$. We'll come back to this.

Example 5.13. Suppose $G = \mathbb{T}$, so $\mathfrak{g} = i\mathbb{R}$. Then $F_A \in \Omega_X^2(i\mathbb{R})$, and we know how to take the norm of these kinds of differential forms: the Yang-Mills functional is

$$(5.14) \quad Y(A) = - \int_X F_A \wedge \star F_A.$$

¹¹Well, this is a little silly in this setting, since all we did is take a derivative. But in general they're more involved.

We can decompose F_A into its self-dual and anti-self-dual pieces F_A^\pm : $F_A = F_A^+ + F_A^-$, and then $\star F_A = F_A^+ - F_A^-$. Thus we can rewrite the Yang-Mills functional as¹²

$$(5.15) \quad Y(A) = \int_X \underbrace{F_A^- \wedge F_A^-}_{\geq 0} - \underbrace{F_A^+ \wedge F_A^+}_{\leq 0}$$

$$(5.16) \quad \geq \int_X F_A^- \wedge F_A^- + F_A^+ \wedge F_A^+$$

$$(5.17) \quad = \int_X F_A \wedge F_A$$

$$(5.18) \quad = 4\pi^2 \langle c_1(P)^2, [X] \rangle.$$

Here $c_1(P) = (i/2\pi)[F_A] \in H_{\text{dR}}^2(M)$ is the first (well only) Chern class of the principal \mathbb{T} -bundle P . So we obtain a lower bound on the Yang-Mills functional in terms of characteristic classes. Equality is achieved exactly when $F_A^+ \wedge F_A^+ = 0$, i.e. A is anti-self-dual.

Next, what are the critical points of Y ? The differential is

$$(5.19) \quad dY_A(\dot{A}) = -2 \int_X \dot{A} \wedge d\star F_A,$$

where $A \in \mathcal{A}_P$ and $\dot{A} \in \Omega_X^1(i\mathbb{R})$ (a variation of A), so the critical points are those such that $d\star F_A = 0$. The Bianchi identity says that, in addition, $dF_A = 0$, so the critical points have harmonic curvature forms. These two PDEs are second-order (curvature is a derivative, and then we take one more derivative), but linear.

Definition 5.20. The *Yang-Mills equations* are $d\star F_A = 0$ and $dF_A = 0$.

Suppose $b_2^+(M) > 0$. Then for generic g , the minimum of Y is *not* realized: we're trying to intersect a line and a lattice. But we can get arbitrarily close, producing a sequence of connections approaching the minimum. \blacktriangleleft

What changes for G more general? First we need a G -invariant inner product $\langle -, - \rangle_{\mathfrak{g}}$ on \mathfrak{g} ; this will induce an inner product $\langle -, - \rangle_{\mathfrak{g}_P}$ on $\Omega_X^2(\mathfrak{g}_P)$: we think of $\omega, \eta \mapsto \omega \wedge \eta \in \Omega_X^4(\mathfrak{g} \otimes \mathfrak{g})$, then use $\langle -, - \rangle_{\mathfrak{g}}$ to get from $\mathfrak{g} \otimes \mathfrak{g}$ to \mathbb{R} , and then integrate to get a number.

G acts on \mathfrak{g} by the adjoint action, so we want something invariant under this action. Such a form always exists: you can take for example the Killing form. Then the Yang-Mills functional is

$$(5.21) \quad Y(A) = \int_X \langle F_A \wedge \star F_A \rangle_{\mathfrak{g}_P},$$

and the above calculation generalizes to

$$(5.22) \quad Y(A) = \int_X \underbrace{\langle F_A^- \wedge F_A^- \rangle_{\mathfrak{g}_P}}_{\geq 0} - \underbrace{\langle F_A^+ \wedge F_A^+ \rangle_{\mathfrak{g}_P}}_{\leq 0}$$

$$(5.23) \quad \geq \int_X \langle F_A^- \wedge F_A^- \rangle_{\mathfrak{g}_P} + \langle F_A^+ \wedge F_A^+ \rangle_{\mathfrak{g}_P}$$

$$(5.24) \quad = \int_X \langle F_A \wedge F_A \rangle_{\mathfrak{g}_P}$$

$$(5.25) \quad = 4\pi^2 \langle c(P), [X] \rangle,$$

where $c(P)$ is some degree-4 characteristic class for principal G -bundles. In particular, this is constant, so we once again obtain a lower bound.

Exercise 5.26. Show that

$$dY_A(\dot{A}) = -2 \int_X \langle \dot{A} \wedge d_A \wedge \star F_A \rangle_{\mathfrak{g}_P},$$

where $\dot{A} \in \Omega_X^1(\mathfrak{g})$ is a variation of A .

¹²If the signs look weird, keep in mind F_A is imaginary. But there may yet be sign errors.

Proof. First, let's differentiate the curvature operator $F: \mathcal{A}_P \rightarrow \Omega_X^2(\mathfrak{g}_P)$.

$$(5.27) \quad dF_A(\dot{A}) = \left. \frac{d}{dt} \right|_{t=0} F(A + t\dot{A})$$

$$(5.28) \quad = \left. \frac{d}{dt} \right|_{t=0} d(A + t\dot{A}) + \frac{1}{2} [(A + t\dot{A}) \wedge (A + t\dot{A})]$$

$$(5.29) \quad = d\dot{A} + [A \wedge \dot{A}] = d_A \dot{A}.$$

We'll use this to differentiate the Yang-Mills functional, starting with (5.21). Then

$$(5.30) \quad dY_A(\dot{A}) = 2 \int_X \langle d_A \dot{A} \wedge \star F_A \rangle$$

$$(5.31) \quad = -2 \int_X \langle \dot{A} \wedge d_A \star F_A \rangle,$$

using the Leibniz rule

$$(5.32) \quad d\langle \omega \wedge \eta \rangle = \langle d_A \omega \wedge \eta \rangle \pm \langle \omega \wedge d_A \eta \rangle$$

where $\omega, \eta \in \Omega_X^*(\mathfrak{g}_P)$. □

Finding the critical points of the Yang-Mills functional is now a nonlinear second-order PDE. This is part of a very general story, including work of many people. One doesn't have to restrict to dimension 4; for example, Atiyah and Bott studied the Yang-Mills functional on Riemann surfaces and the topology of (certain equivalence classes of) the space of connections and its relationship with the algebraic geometry of the Riemann surface.

For the rest of today's lecture, we'll discuss some more classical geometric preliminaries, beginning with some cool facts about the conformal group. Let W be a finite-dimensional real vector space,¹³ and suppose $\langle -, - \rangle: W \times W \rightarrow \mathbb{R}$ is a nondegenerate, symmetric bilinear form. We aren't assuming it's positive definite, e.g. it could be Lorentz. Inside W , we have the *null cone* N_W of vectors $\xi \in W$ with $\langle \xi, \xi \rangle = 0$. This is a cone because if $\xi \in N_W$, then $t\xi \in N_W$ too. If the form is definite, the null cone is just $\{0\}$.

Let Q_W denote the image of $N_W \setminus 0$ under the quotient $W \setminus 0 \rightarrow \mathbb{P}W$ by \mathbb{R}^\times . Then Q_W is a compact real manifold, a real quadric, and carries a natural conformal structure. Suppose $\ell \in Q_W$, meaning it's a line in the null cone. Then $T_\ell Q_W \subset T_\ell \mathbb{P}W = \text{Hom}(\ell, W/\ell)$, and under this identification, $T_\ell Q_W = \text{Hom}(\ell, \ell^\perp/\ell)$: since ℓ is null, $\ell \subset \ell^\perp$, and in fact ℓ^\perp is a codimension-one subspace of W : it's cut out by one equation.

The bilinear form $\langle -, - \rangle$ descends to $\ell^\perp/\ell \times \ell^\perp/\ell \rightarrow \mathbb{R}$, and therefore defines a conformal structure on $\text{Hom}(\ell, \ell^\perp/\ell) \cong \ell^\perp/\ell \otimes \ell^*$. This obviously varies smoothly in ℓ , hence defines a conformal structure on Q_W .

Suppose, for example, $W = V \oplus H$ is an orthogonal direct sum, where $\langle -, - \rangle_V$ is an inner product and H is a hyperbolic plane, so it has signature $(1, 1)$ and exactly two lines. Then $\mathbb{P}H$ is a circle, and the null cone is two points inside it. The entire quadric cone splits as $Q_W = V \amalg N_V \amalg Q_V$. How does this work? Inside $\mathbb{P}W = \{[v : s : t]\}$, $V = \{[v : -\|v\|^2 : 1]\}$, $N_V = \{[v : 1 : 0]\}$, and $Q_V = \{[v] : 0 : 0\}$.

If V is n -dimensional, then N_V is $(n-1)$ -dimensional and Q_V is $(n-2)$ -dimensional. Therefore Q_W is a compactification of V : we've added strata of codimensions one and two to it, and picked up a conformal structure.

Example 5.33. Suppose $\langle -, - \rangle$ is an inner product. Then N_V is a point and Q_V is the empty set, so Q_W is diffeomorphic to S^n with the conformal structure induced from the usual metric.¹⁴ In this case, the symmetry group is $O(V)$, and with an orientation we can get $SO(V)$. ◀

Example 5.34. More generally, let $V = \mathbb{R}^{p,q}$, where $p+q = n$, with the indefinite-signature form

$$(5.35) \quad \langle \xi, \eta \rangle = \sum_{i=1}^p \xi^i \eta^i - \sum_{i=p+1}^q \xi^i \eta^i.$$

For example, $H = \mathbb{R}^{1,1}$. In this case, $W \cong \mathbb{R}^{p+1,q+1} = \mathbb{R}^{p+1} \oplus \mathbb{R}^{q+1}$ (though the inner product on the second has a minus sign put in front). If $\xi \in \mathbb{R}^{p+1}$ and $\eta \in \mathbb{R}^{q+1}$, then $(\xi, \eta) \in N_W$ iff $\|\xi\| = \|\eta\| = 1$, so Q_W is diffeomorphic to $(S^p \times S^q)/\{\pm 1\}$, and $\text{PO}(W)$ acts via conformal transformations on Q_W . Inside this we have

¹³We could work with complex vector spaces, or infinite-dimensional vector spaces.

¹⁴**TODO:** I would like to double-check this.

$O(V) \cong O_{p,q}$. So this says that the conformal symmetries in signature (p, q) are the orthogonal symmetries in signature $(p+1, q+1)$. \blacktriangleleft

For example, the group of conformal symmetries of \mathbb{R}^4 with positive-definite inner product is $O_{5,1}$, which acts on S^4 , the conformal compactification of \mathbb{R}^4 . The conformal group of \mathbb{R}^2 , sitting inside its compactification S^2 , is $O_{3,1}$; with an orientation we get $SO_{3,1} \cong \text{PSL}_2(\mathbb{C})$, one of the low-dimensional exceptional isomorphisms of Lie groups. These act by Möbius transformations. Analogously, there's a special isomorphism $SO_{5,1} \cong \text{PSL}_2(\mathbb{H})$, allowing us to get at conformal symmetries in dimension 4 via the quaternions. Next time we'll apply this to the self-duality equations in dimension 4.

Lecture 6.

Spinors in low dimensions and special isomorphisms of Lie groups: 2/7/19

"I'll tell a story for three minutes and then you can go."

Today we'll continue discussing preliminaries to the ADHM construction, sometimes using them to introduce interesting nearby ideas in their own right. Today we'll discuss exceptional isomorphisms of Lie groups in low dimensions, which are applicable in other cases. For example, if you care about fermions in, e.g. supersymmetric quantum field theories in dimensions 6 or below, these ideas appear.

Fix a 4-dimensional complex vector space \mathbb{S} , which could be \mathbb{C}^4 . We can take exterior powers $\Lambda^2\mathbb{S}$, $\Lambda^3\mathbb{S}$, and $\Lambda^4\mathbb{S} = \text{Det}\mathbb{S}$, and similarly for \mathbb{S}^* . Choose a volume form $\mu \in \text{Det}\mathbb{S}^* \setminus 0$; we want to consider the symmetries of (\mathbb{S}, μ) . A linear map $T: \mathbb{S} \rightarrow \mathbb{S}$ has an associated volume $\det T$, so we're essentially asking for automorphisms with determinant 1. This is literally true for $\mathbb{S} = \mathbb{C}^4$, in which case $\text{Aut}(\mathbb{S}, \mu) = \text{SL}_4(\mathbb{C})$.

Now given such an automorphism T , we obtain an automorphism $\Lambda^2 T: \Lambda^2\mathbb{S} \rightarrow \Lambda^2\mathbb{S}$. The condition that T preserves the volume form is mapped to the bilinear pairing $B: \Lambda^2\mathbb{S} \times \Lambda^2\mathbb{S} \rightarrow \mathbb{C}$ sending

$$(6.1) \quad x, y \longmapsto \langle \mu, x \wedge y \rangle_{\text{Det}\mathbb{S}^*, \text{Det}\mathbb{S}}.$$

This is symmetric and nondegenerate, hence an inner product, so $\Lambda^2 T$ preserves this inner product! Therefore the map $\phi = \Lambda^2: \text{Aut}(\mathbb{S}, \mu) \rightarrow \text{Aut}(\Lambda^2\mathbb{S}, B)$ amounts to a homomorphism $\text{SL}_4(\mathbb{C}) \rightarrow \text{O}_6(\mathbb{C})$.

You can directly check that $\{\pm 1\} \subset \ker(\phi)$. Since $\text{SL}_4(\mathbb{C})$ is connected, the image of ϕ is contained in $\text{SO}_6(\mathbb{C})$.

Claim 6.2. $\phi: \text{SL}_4(\mathbb{C}) \rightarrow \text{SO}_6(\mathbb{C})$ is surjective.

The proof would amount to checking that it's an isomorphism on Lie algebras, so the image is open, and that the image is closed, hence all of $\text{SO}_6(\mathbb{C})$. One corollary is that we've identified the spinor representation as \mathbb{S} .

Moreover, since $\text{SL}_4(\mathbb{C})$ is simply connected, the map $\phi: \text{SL}_4(\mathbb{C}) \rightarrow \text{SO}_6(\mathbb{C})$ is the nontrivial double cover map. Therefore we have produced an isomorphism $\text{SL}_4(\mathbb{C}) \cong \text{Spin}_6(\mathbb{C})$.

Remark 6.3. We can define the complex spin groups in the same way as the real ones: for $n \geq 3$, $\pi_1 \text{SO}_n(\mathbb{C}) = \mathbb{Z}/2$, and for $n = 2$, $\pi_1 \text{SO}_2(\mathbb{C}) = \mathbb{Z}$, so we can ask for the connected double cover of $\text{SO}_n(\mathbb{C})$, which has a canonical Lie group structure, and define it to be $\text{Spin}_n(\mathbb{C})$. \blacktriangleleft

There can be two different realizations of this double cover, but you can prove there's a unique Lie group homomorphism between them respecting the covering map, and it's an isomorphism.

There are several other exceptional isomorphisms, and they all follow from this one.

Example 6.4. Let $J: \mathbb{S} \rightarrow \mathbb{S}$ be a *quaternionic structure*, i.e. an antilinear map such that $J^2 = -\text{id}_{\mathbb{S}}$. Then $\Lambda^k J: \Lambda^k \mathbb{S} \rightarrow \Lambda^k \mathbb{S}$ is also antilinear, and squares to $(-1)^k \text{id}_{\mathbb{S}}$. So on $\Lambda^2\mathbb{S}$ it defines a real structure, and on $\Lambda^3\mathbb{S}$ it's quaternionic.

We can impose another constraint, that $(\det J)^* \mu = \mu$, i.e. μ is real. That is, it's in the subspace of $\text{Det}\mathbb{S}^*$ fixed by $\det J$, which is a one-dimensional real vector space. Therefore we obtain a map

$$(6.5) \quad \phi: \text{Aut}(\mathbb{S}, J, \mu) \longrightarrow \text{Aut}(\Lambda^2\mathbb{S}, B, \Lambda^2 J).$$

Now suppose $\mathbb{S} = \mathbb{C}^4$. Then the codomain is O_{p+q} for some $p+q = 6$, but B might not be positive definite. The domain is $\text{SL}_2(\mathbb{H})$ – though you have to be careful with what this means. There's no determinant map $\text{GL}_n(\mathbb{H}) \rightarrow \mathbb{H}^\times$, but we can take the determinant to be a complex number (via regarding quaternionic matrices as complex matrices), and ask for it to be 1.

Working out the details, the kernel will once again be $\{\pm 1\}$, and this will be a double cover, so this will identify $\mathrm{SL}_2(\mathbb{H}) \cong \mathrm{Spin}_{p,q}$, an isomorphism of 15-dimensional real Lie groups. Then \mathbb{S} is the spinor representation again – but it’s quaternionic, so we know we can’t get Spin_6 .¹⁵

To determine the signature, let $\{e_1, Je_1, e_2, Je_2\}$ be a basis for \mathbb{S} . Then we can write down a basis for $(\Lambda^{\otimes 2}\mathbb{S})_{\mathbb{R}}$ as follows:

$$(6.6) \quad \begin{array}{ll} e_1 \wedge Je_1 & e_2 \wedge Je_2 \\ e_1 \wedge Je_2 + e_2 \wedge Je_1 & i(e_1 \wedge Je_2 - e_2 \wedge Je_1) \\ e_1 \wedge e_2 + Je_1 \wedge Je_2 & i(e_1 \wedge e_2 - Je_1 \wedge Je_2). \end{array}$$

You can check these are orthogonal. The first two form a hyperbolic pair (signature $(1, 1)$), and the last four all self-pair to -1 . Therefore the signature is $(1, 5)$, and we conclude $\mathrm{SL}_2(\mathbb{H}) \cong \mathrm{Spin}_{1,5}$.

Taking the quotient, we also get $\mathrm{PSL}_2(\mathbb{H}) \cong \mathrm{SO}_{1,5}^0$ ¹⁶ – in indefinite signature, special orthogonal groups aren’t connected. Now, $\mathrm{SO}_{1,5}^0$ acts as the group of automorphisms of the conformal compactification of \mathbb{R}^4 with a definite metric, which gives you S^4 : $\mathrm{PSL}_2(\mathbb{H})$ is the conformal group of the 4-sphere. We can also identify $\mathrm{PSL}_2(\mathbb{H}) \cong \mathrm{PGL}_2(\mathbb{H})$ via the diffeomorphism $S^4 \cong \mathbb{HP}^1$.

So in summary, we have $\mathrm{SL}_2(\mathbb{H}) \cong \mathrm{Spin}_{1,5}$ and $\mathrm{PSL}_2(\mathbb{H}) \cong \mathrm{PGL}_2(\mathbb{H}) \cong \mathrm{SO}_{1,5}^0$. \blacktriangleleft

Remark 6.7. If you want to imitate the above story but get Spin_6 , you’ll want to get the maximal compact in $\mathrm{SL}_4(\mathbb{C})$, which is SU_4 . So throw out J and instead ask that your automorphisms fix a Hermitian inner product. \blacktriangleleft

Example 6.8. Now let’s introduce a symplectic form $\omega \in \Lambda^2 \mathbb{S}^*$, and let’s say that $\mu = (1/2)\omega \wedge \omega$. We’ll ask about the automorphisms of \mathbb{S} that fix μ ; if $\mathbb{S} = \mathbb{C}^4$, this group is called $\mathrm{Sp}_4(\mathbb{C})$.

Passing up to $\Lambda^2 \mathbb{S}$, such an automorphism must preserve $\ker(\omega)$, which has codimension 1, and the bilinear form B from before. The automorphisms of $\ker(\omega)$ and B form the group $\mathrm{O}_5(\mathbb{C})$. As in the previous case, the map $\mathrm{Sp}_4(\mathbb{C}) \rightarrow \mathrm{O}_5(\mathbb{C})$ has image the identity component $\mathrm{SO}_5(\mathbb{C}) \subset \mathrm{O}_5(\mathbb{C})$, and is a double cover onto it. Therefore we obtain an isomorphism $\mathrm{Spin}_5(\mathbb{C}) \cong \mathrm{Sp}_4(\mathbb{C})$. \blacktriangleleft

Example 6.9. If you ask for automorphisms of \mathbb{S} which preserve both J and ω ? In this case we get a map $\mathrm{Aut}(\mathbb{S}, J, \omega) \rightarrow \mathrm{Aut}(\ker(\omega), \Lambda^2 J, B)$. The basis of $\ker(\omega)$ contains the last four vectors in (6.6), but instead of the first two we have the vector $e_1 \wedge Je_1 - e_2 \wedge Je_2$, which self-pairs to -1 . Therefore we’re in definite signature, so this map is identified with the map $\mathrm{Sp}_2 \rightarrow \mathrm{O}_5$. Here, Sp_2 is the symmetries of \mathbb{H}^2 with its standard inner product.

As usual, this only sees $\mathrm{SO}_5 \subset \mathrm{O}_5$, and is a double cover, providing for us an isomorphism $\mathrm{Spin}_5 \cong \mathrm{Sp}_2$. \blacktriangleleft

Let’s digress from the linear algebra a little bit and talk about quaternions. A general quaternion is of the form $x = x^0 + x^1 i + x^2 j + x^3 k$.

Definition 6.10. The *conjugate* of a quaternion x as above is

$$\bar{x} := x^0 - x^1 i - x^2 j - x^3 k.$$

The *imaginary part* of x is $\mathrm{Im}(x) := x^1 i + x^2 j + x^3 k$.

Therefore $x\bar{x} = (x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$; this is the norm squared of x . We have $dx = dx^0 + dx^1 i + dx^2 j + dx^3 k$, and therefore

$$(6.11) \quad dx \wedge d\bar{x} = -(dx^0 \wedge dx^1 + dx^2 \wedge dx^3)i - (dx^0 \wedge dx^2 - dx^1 \wedge dx^3)j - (dx^0 \wedge dx^3 + dx^1 \wedge dx^2)k.$$

You’ll notice this is self-dual, and in fact the coefficients are a basis for the space of self-dual 2-forms. Similarly, $d\bar{x} \wedge dx$ is anti-self-dual, and its coefficients are a basis for the space of anti-self-dual forms.

Example 6.12. Continuing Example 6.9, consider $S^7 \subset \mathbb{H}^2$, which is preserved by $\mathrm{Sp}_2 = \mathrm{Spin}_5$. Restricting to $\mathbb{H}^2 \setminus 0$, we have a projection down to $\mathbb{HP}^1 \cong S^4$, the quaternionic projective line.¹⁷ This restricts to a submersion $\pi: S^7 \rightarrow S^4$. Two points in S^7 have the same image iff they are acted on by a unit-norm quaternion. The group of unit-norm quaternions is $\mathrm{Sp}_1 = \mathrm{SU}_3 = \mathrm{Spin}_2$. Since $S^7 = \mathrm{Sp}_2/\mathrm{Sp}_1$, $S^4 \cong \mathrm{Sp}_2/\mathrm{Sp}_1 \times \mathrm{Sp}_1$. In

¹⁵You can also check that $Z(\mathrm{SL}_2(\mathbb{H})) \cong \mathbb{Z}/2$ but $Z(\mathrm{Spin}_6) \cong \mathbb{Z}/4$.

¹⁶ $\mathrm{PSL}_2(\mathbb{H})$ is defined to be $\mathrm{SL}_2(\mathbb{H})$ modulo its center $\{\pm 1\}$.

¹⁷Because \mathbb{H} isn’t commutative, we have to specify that we’re taking the quotient of the right action.

other words, $\pi: S^7 \rightarrow S^4$ is a principal Sp_1 -bundle, and it is homogeneous for the left Sp_2 -action. This is another example of a Hopf fibration.¹⁸

Now who acts on S^4 ? Well, SO_5 acts by isometries. The whole situation is invariant under the isometry group, so we can move the total space around by isometries of the base, and therefore SO_5 lifts to the Spin_5 -action we already identified on S^7 . Inside the isometries, there's a bigger group $\mathrm{SO}_{1,5}^0$ of conformal transformations. We'd like to ask whether these lift to conformal isometries on S^7 . The identification $S^4 \cong \mathbb{HP}^1$ carries SO_5 to PSP_2 and $\mathrm{SO}_{1,5}^0 \cong \mathrm{PSL}_2(\mathbb{H})$, and this lifts to the $\mathrm{SL}_2(\mathbb{H})$ -action on \mathbb{H}^2 .

Now we introduce connections, mimicking the basic story we already saw over \mathbb{C} in §4. We want to consider a connection Θ for $S^7 \rightarrow S^4$ regarded as an Sp_1 -bundle, an Sp_1 -invariant horizontal distribution. Specifically, you can check that

$$(6.13) \quad \Theta := -\mathrm{Im}\left(q^0 \overline{dq^0} + q^1 \overline{dq^1}\right)$$

is a connection, by checking it's orthogonal to orbits everywhere. Then $\Omega_{0,1}$, restricted to the horizontal distribution, is $-\mathrm{Im}(dq^0 \wedge \overline{dq^0})$. But from the calculation in (6.11), we already know this is imaginary, so $\Omega_{0,1}|_{\mathrm{horiz.}} = -dq^0 \wedge \overline{dq^0}$; we also know it's self-dual. ◀

So we've found one solution to the self-dual equations; we can discover others by transforming by conformal transformations. Therefore $\mathrm{SL}_2(\mathbb{H}) = \mathrm{Spin}_{1,5}$ acts on Θ to produce self-dual connections. This is an $\mathrm{SL}_2(\mathbb{H})$ -orbit inside \mathcal{A}_P ; we know it's a homogeneous manifold, so if we want to know what it is, we should compute the stabilizer of Θ . This is precisely the isometries, which are Spin_5 , so the orbit is $\mathrm{Spin}_{1,5}/\mathrm{Spin}_5 = \mathrm{SO}_{1,5}/\mathrm{SO}_5$, which is a hyperbolic 5-ball M , and Θ is actually the center. As you get closer to the "edge" S^4 , which we think of as the base S^4 , the curvature concentrates more and more at the boundary point, and we could think of the connections at infinity as having curvature in a δ -function (this doesn't actually work, of course). The only connection which has no privileged concentration of curvature is Θ , at the center.

Theorem 6.14 (Atiyah-Drinfeld-Hitchin-Manin). *$\mathrm{SO}_{1,5}/\mathrm{SO}_5$ is the moduli space of self-dual Sp_1 -instantons with Chern class $c_2 = 1$.*

This is the only characteristic class data; for an $\mathrm{Sp}_1 = \mathrm{SU}_2$ -bundle, $c_1 = 0$, and there are no more Chern classes. On S^4 , $c_2 \in H^4(S^4)$, and the orientation identifies this cohomology group with \mathbb{Z} , so we can make sense of $c_4 = 1$. This is an instance of fixing discrete data in a moduli problem, which is common.

This is a basic case where we can picture what's going on, and illustrates a good part of the general story. But how do you prove Theorem 6.14? How do we know that every connection is one of these? Stay tuned; we'll prove this later. Then there are other questions involving other Chern classes, other manifolds, and so on.

This kind of question was first investigated by physicists, Polyakov and others. Atiyah and Singer then checked the dimension by linearizing the equations and found that the physicists had missed some. When Simon Donaldson was a graduate student, he had the brilliant idea of taking this example, but generalizing to 4-manifolds where the intersection form is positive definite. Once again, the moduli space is five-dimensional, and you can take connections with curvature concentrated near a point, and extend by zero. Taubes show you can wiggle this a little bit and actually get a solution – and therefore you again get a copy of the manifold at infinity. Through this (and of course, plenty more) Donaldson was able to prove his first theorem – this exhibits a bordism from this manifold to something else.

¹⁸There are also Hopf fibrations over the reals, including the double cover $\mathbb{Z}/2 \rightarrow S^1 \rightarrow \mathbb{RP}^1$. There's also one over the octonions.