M392C NOTES: MATHEMATICAL GAUGE THEORY

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These notes were taken in UT Austin's M392C (Mathematical gauge theory) class in Spring 2019, taught by Dan Freed. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Any mistakes in the notes are my own.

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Lecture 1.

Some useful linear algebra: 1/22/19

"Why did the typing stop?"

Today we'll discuss some basic linear algebra which, in addition to being useful on its own, is helpful for studying the self-duality equations. You should think of this as happening pointwise on the tangent space of a smooth manifold.

Let V be a real n-dimensional vector space. The exterior powers of V define more vector spaces: the scalars $\mathbb{R}$, V, $\Lambda^2 V$, and so on, up to $\Lambda^n V = \operatorname{Det} V$. We can also apply this to the dual space, defining $\mathbb{R}$, V^*, $\Lambda^2 V^*$, etc, up to $\Lambda^n V^* = \operatorname{Det} V^*$.

There is a duality pairing

$$\begin{aligned} \theta\colon \Lambda^k V^* \times \Lambda^k V &\longrightarrow \mathbb{R} \\ (v^1 \wedge \cdots \wedge v^k, v_1 \wedge \cdots \wedge v_k) &\longmapsto \det(v^i(v_j))_{i,j}, \end{aligned} \tag{1.1}$$

where $v^i \in V^*$ and $v_j \in V$.

Now fix a $\mu \in \operatorname{Det} V^* \setminus 0$, which we call a *volume form*. Then we get another duality pairing

$$\begin{aligned} \Lambda^k V \times \Lambda^{n-k} V &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto \theta(\mu, x \wedge y). \end{aligned} \tag{1.2}$$

Thus $\Lambda^k V \cong \Lambda^{n-k} V^*$.

Suppose we have additional structure: an inner product and an orientation. Let $e_1, \dots, e_n$ be an oriented, orthonormal basis of V, and $e^1, \dots, e^n$ be the dual basis. Now we can choose $\mu = e^1 \wedge \cdots \wedge e^n$.

Definition 1.3. The *Hodge star operator* is the linear operator $\star\colon \Lambda^k V^* \to \Lambda^{n-k} V^*$ characterized by

$$\alpha \wedge (\star\beta) = \langle \alpha, \beta \rangle_{\Lambda^k V} \cdot \mu. \tag{1.4}$$

The inner product on $\Lambda^k V^*$ is defined by

$$\langle v^1 \wedge \cdots \wedge v^k, w^1 \wedge \cdots \wedge w^k \rangle \coloneqq \det(\langle v^i, w^j \rangle)_{i,j}. \tag{1.5}$$

The Hodge star was named after W.V.D. Hodge, a British mathematician. Notice how we've used both the metric and the orientation – it's possible to work with unoriented vector spaces (and eventually unoriented Riemannian manifolds), but one must keep track of some additional data.

Example 1.6.

- $\star(e^{i_1} \wedge \cdots \wedge e^{i_k}) = e^{j_1} \wedge \cdots \wedge e^{j_{n-k}}$ if the permutation $1, \ldots, n$ to $i_1, \ldots, i_k, j_1, \ldots, j_{n-k}$ of $[n] := \{1, \ldots, n\}$ is even. Otherwise there's a factor of -1.
- Suppose $n = 4$. Then $\star(e^1 \wedge e^2) = e^3 \wedge e^4$ and $\star(e^1 \wedge e^3) = -e^2 \wedge e^4$, and so on. ◀

Remark 1.7. The Hodge star is natural. First, you can see that we didn't make any choices when defining it, other than an orientation and a volume form, but there's also a functoriality property. Let $T\colon V \to V$ be an automorphism; this induces $(\Lambda^k T^*)^{-1}\colon \Lambda^k T^* \to \Lambda^k T^*$, and if T is an orientation-preserving isometry,

$$\star \circ (\Lambda^k T^*)^{-1} = (\Lambda^{n-k} T^*)^{-1} \circ \star. \tag{1.8}$$

Hence $\star\star\colon \Lambda^k V^* \to \Lambda^k V^*$ is some nonzero scalar multiple of the identity, and we can determine which multiple it is. Certainly we know

$$\star\star(e^1 \wedge \cdots \wedge e^k) = \star(e^{k+1} \wedge \cdots \wedge e^1) = \lambda e^1 \wedge \cdots \wedge e^k, \tag{1.9}$$

and we just have to compute the parity of these permutations: one uses k transpositions, and the other uses $n - k$. Therefore we conclude that

$$\star\star = (-1)^{k(n-k)}\colon \Lambda^k V^* \to \Lambda^k V^*. \tag{1.10}$$

◀

Now suppose $n = 2m$, so we have a middle dimension m, and $\star\star\colon \Lambda^m \to \Lambda^m$ is $(-1)^m$. This induces additional structure on $\Lambda^m V^*$.

- If m is even (so $n \equiv 0 \bmod 4$), the double Hodge star is an endomorphism squaring to 1. This defines a $\mathbb{Z}/2$-grading on $\Lambda^m V^*$, given by the ± 1-eigenspaces, which we'll denote $\Lambda^m_{\pm} V^*$. The $+1$-eigenspace is called *self-dual* m-forms, and the -1-eigenspace is called the *anti-self-dual* m-forms.
- If m is odd (so $n \equiv 2 \bmod 4$), the double Hodge star squares to -1, so this defines a complex structure on $\Lambda^m V^*$, where i acts by the double Hodge star.

Exercise 1.11. Especially for those interested in physics, work out this linear algebra in indefinite signature (particularly Lorentz). The signs are different, and in Lorentz signature the two bullet points above switch!

Exercise 1.12. Show that if $4 \mid n$, the direct-sum decomposition $\Lambda^m V^* = \Lambda^m_+ V^* \oplus \Lambda^m_- V^*$ is orthogonal. See if you can find the one-line proof that self-dual and anti-self-dual forms are orthogonal.

Next we introduce conformal structures. This allows the sort of geometry which knows angles, but not lengths.

Definition 1.13. A *conformal structure* on a real vector space V is a set C of inner products on V such that any $g_1, g_2 \in C$ are related by $g_1 = \lambda g_2$ for a $\lambda \in \mathbb{R}_+$.

In this setting, one can obtain g_2 from g_1 by pulling back g_1 along the dilation $T_\lambda\colon v \mapsto \lambda v$. This induces an action of $(T_\lambda^*)^{-1}$ on $\Lambda^k V^*$, which is multiplication by λ^{-k}: if μ_i is the volume form induced from g_i, so that

$$\alpha \wedge \star\beta = g_1(\alpha, \beta)\mu_1, \tag{1.14}$$

then

$$\lambda^{-2k} \alpha \wedge \star\beta = g_2(\alpha, \beta)\lambda^{-n}\mu_2. \tag{1.15}$$

Thus pulling back by dilation carries the Hodge star to $\lambda^{n-2k}\star$. Importantly, if $n = 2m$, then $\star\colon \Lambda^m V^* \to \Lambda^m V^*$ is preserved by this dilation, so it only depends on the orientation and the conformal structure.

Remark 1.16. A conformal structure is independent from an orientation. For example, on a one-dimensional vector space, a conformal structure is no information at all (all inner products are multiples of each other), but an orientation is a choice. ◀

Example 1.17. Suppose $n = 2$ and choose an orientation and a conformal structure on V. As we just saw, this is enough to define the Hodge star $\star\colon V^* \to V^*$, which defines a complex structure on V. Pick a square root i of -1 and let $\star$ act by it (there are two choices, acted on by a Galois group).

We get more structure by complexifying: $V^* \otimes \mathbb{C}$ splits as a the $\pm i$-eigenspaces of the Hodge star; we denote the i-eigenspace by $V^{(1,0)}$ (the $(1,0)$-*forms*) and the $-i$-eigenspace by $V^{(0,1)}$ (the $(0,1)$-*forms*).

Now let's globalize this: everything has been completely natural, so given an oriented, conformal 2-manifold X, it picks up a complex structure, hence is a Riemann surface, and the Hodge star is a map $\star\colon \Omega^1_X \to \Omega^1_X$. Moreover, we can do this on the complex differential forms, which split into $(1,0)$-forms and $(0,1)$-forms.

How do 1-forms most naturally appear? They're differentials of functions, so given an $f\colon X \to \mathbb{C}$, we can ask what it means for $\mathrm{d}f \in \Omega^{1,0}_X$. This is the equation

$$\star\,\mathrm{d}f = i\,\mathrm{d}f. \tag{1.18}$$

This is precisely the Cauchy-Riemann equation; its solutions are precisely the holomorphic functions on X. ◂

Remark 1.19. More generally, one can ask about functions to $\mathbb{C}^n$ or even sections of complex vector bundles; the analogue gives you notions of holomorphic sections. In this case, the equations have the notation

$$\overline{\partial} f = \left(\frac{1+i\star}{2}\right)\mathrm{d}f. \tag{1.20}$$ ◂

We'll spend some time in this class understanding a four-dimensional analogue of all of this structure.

Symmetry groups. Symmetry is a powerful perspective on geometry. If we think about V together with some structure (orientation, metric, conformal structure, some combination,...), we can ask about the symmetries of V preserving this structure. Of course, to know this, we must know V, but we can instead look at a model space $\mathbb{R}^n$ to define a *symmetry type*, and ask about its symmetry group G: then an isomorphism $\mathbb{R}^n \to V$ preserving all of the data we're interested in defines an isomorphism from G to the symmetry group of V.

Example 1.21. When $\dim V = 2$, the most general symmetry group is $\mathrm{GL}_2(\mathbb{R})$, the invertible matrices acting on $\mathbb{R}^2$. Adding more structure we get more options.

- If we restrict to orientation-preserving symmetries, we get $\mathrm{GL}_2^+(\mathbb{R})$.
- If we restrict to symmetries preserving a conformal structure, the group is called $\mathrm{CO}_2 = \mathrm{O}_2 \times \mathbb{R}^{>0}$.
- If we ask to preserve an orientation and a complex structure, we get $\mathrm{CO}_2^+ = \mathrm{SO}_2 \times \mathbb{R}^{>0}$. This is isomorphic to $\mathbb{C}^\times = \mathrm{GL}_1(\mathbb{C})$: an element of $\mathrm{SO}_2 \times \mathbb{R}^{>0}$ is rotation through some angle θ and a positive number r; this is sent to $re^{i\theta} \in \mathbb{C}^\times$.

This provides another perspective on why an orientation and a conformal structure give us a complex structure. ◂

Example 1.22. Now suppose $n = 4$, and choose a conformal structure C and an orientation on V. Then orthogonal makes sense, though orthonormal doesn't, and the Hodge star induces a $\mathbb{Z}/2$-grading on $\Lambda^2 V^* = \Lambda^2_+ V^* \oplus \Lambda^2_- V^*$, the self-dual and anti-self-dual 2-forms. The total space $\Lambda^2 V^*$ is six-dimensional, and these two subspaces are each three-dimensional.

Suppose e^1, e^2, e^3, e^4 is an orthonormal basis for some inner product in C. We can use these to define bases of $\Lambda^2_\pm V^*$, given by

$$\begin{aligned}
\alpha_1^\pm &\coloneqq e^1 \wedge e^2 \pm e^3 \wedge e^4\\
\alpha_2^\pm &\coloneqq e^1 \wedge e^3 \mp e^2 \wedge e^4\\
\alpha_3^\pm &\coloneqq e^1 \wedge e^4 \pm e^2 \wedge e^3.
\end{aligned} \tag{1.23}$$

Now, what symmetry groups do we have? Inside $\mathrm{GL}_4(\mathbb{R})$, preserving an orientation lands in the subgroup $\mathrm{GL}_4^+(\mathbb{R})$; preserving a conformal structure lands in $\mathrm{O}_4 \times \mathbb{R}^{>0}$; and preserving both lands in $\mathrm{SO}_4 \times \mathbb{R}^{>0}$. The first three of these act irreducibly on $\Lambda^2(\mathbb{R}^4)^*$, but the action of $\mathrm{SO}_4 \times \mathbb{R}^{>0}$ has two irreducible summands, $\Lambda^2_\pm(\mathbb{R}^4)^+$.

To understand this better, we should learn a little more about SO_4. Recall that Sp_1 is the Lie group of unit quaternions. This is isomorphic to SU_2, the group of determinant-1 unitary transformations of $\mathbb{C}^2$. This group has an irreducible 3-dimensional representation ρ in which Sp_1 acts by conjugation on the imaginary quaternions (since $\mathbb{R} \subset \mathbb{H}$ is preserved by this action).

Remark 1.24. Another way of describing ρ is: let ρ' denote the action of SU_2 on $\mathbb{C}^2$ by matrix multiplication. Then $\rho \cong \mathrm{Sym}^2\,\rho'$. ◂

Proposition 1.25. *There is a double cover* $\mathrm{Sp}_1 \times \mathrm{Sp}_1 \to \mathrm{SO}_4$*. Under this cover, the* SO_4*-representation* $\Lambda^4_\pm(\mathbb{R}^4)^*$ *pulls back to a real three-dimensional representation in which one copy of* Sp_1 *acts by* ρ *and the other acts trivially.*

Proof. Let W' and W'' be two-dimensional Hermitian vector spaces with compatible quaternionic structures J', resp. J''.[1] Then, $V \coloneqq W' \otimes_{\mathbb{C}} W''$ has a real structure $J' \otimes J''$: two minuses make a plus, and compatibility of J' and J'' means the real points of V have an inner product. (These kinds of linear-algebraic spaces are things you should prove once in your life.)

By tensoring symmetries we obtain a homomorphism $\mathrm{Sp}(W') \times \mathrm{Sp}(W'') \to \mathrm{O}(V)$. This factors through $\mathrm{SO}(V) \hookrightarrow \mathrm{O}(V)$, which you can see for two reasons:

- $\mathrm{Sp}(W')$ and $\mathrm{Sp}(W'')$ are connected, so this homomorphism must factor through the identity component of $\mathrm{O}(V)$, which is $\mathrm{SO}(V)$; or
- a complex vector space has a canonical orientation, and using this we know these symmetries are orientation-preserving.

Now we want to claim this map is two-to-one. One can quickly check that $(-1, -1)$ is in the kernel; the rest is an exercise. ◀

Since Spin_n is the double cover of SO_n, this is telling us $\mathrm{Spin}_4 = \mathrm{Sp}_1 \times \mathrm{Sp}_1$. This splitting is the genesis of a lot of what we'll do in the next several lectures.

Consider the 16-dimensional space

$$V^* \otimes V^* = (W')^* \otimes (W')^* \otimes (W'')^* \otimes (W'')^*. \tag{1.26}$$

Because the map

$$\begin{aligned} \omega' \colon W' \times W' &\longrightarrow \mathbb{C} \\ \xi', \eta' &\longmapsto h'(J'\xi', \eta') \end{aligned} \tag{1.27}$$

is skew-symmetric, it lives in $\Lambda^2(W')^* \subset (W')^* \otimes (W')^*$. In particular, the embedding

$$\mathrm{Sym}^2(W')^* \oplus \mathrm{Sym}^2(W'')^* \hookrightarrow (W')^* \otimes (W')^* \otimes (W'')^* \otimes (W'')^* \tag{1.28}$$

is the map sending

$$\alpha, \beta \longmapsto \alpha \otimes w'' + \omega' \otimes \beta. \tag{1.29}$$

Remark 1.30. This story can be interpreted in terms of representations of $\mathrm{Sp}(W') \times \mathrm{Sp}(W'')$. Let $\mathbf{1}$ denote the trivial representation of Sp_1 and $\mathbf{3}$ be the three-dimensional irreducible representation we discussed above. Then (1.26) enhances to

$$V^* \otimes V^* = \mathbf{1}_{\mathrm{Sp}(W')} \otimes \mathbf{3}_{\mathrm{Sp}(W')} \otimes \mathbf{1}_{\mathrm{Sp}(W'')} \otimes \mathbf{3}_{\mathrm{Sp}(W'')}. \tag{1.31}$$

The skew-symmetric part is $\mathbf{3}_{\mathrm{Sp}(W')} \otimes \mathbf{1}_{\mathrm{Sp}(W'')} \oplus \mathbf{1}_{\mathrm{Sp}(W')} \otimes \mathbf{3}_{\mathrm{Sp}(W'')}$, and the "rest" (complement) is symmetric. ◀

The group $\mathrm{Sp}_1 \times \mathrm{Sp}_1 = \mathrm{Spin}_4$ has complex (quaternionic) two-dimensional representations $S^\pm$, the *spin representations*, and $\Lambda^2_\pm V \cong \mathrm{Sym}^2 S^\pm$.

So two-forms have self-dual and anti-self-dual parts, and curvature is a natural source of 2-forms! ◀

Lecture 2.

Fantastic 2-forms and where to find them: 1/24/19

"I've taught this before, so I know it's true."

[1] That is, J' is an antilinear endomorphism of W' squaring to -1, and similarly for J''. Compatible means with the Hermitian metric: h is a map $\overline{W} \times W \to \mathbb{C}$ and J is a map $W \to \overline{W}$, and if $\xi, \eta \in W'$, we want

$$h(J'\xi, \overline{J'\eta}) = \overline{h(\xi, \eta)} \qquad \text{and} \qquad h(J\xi, \eta) = -h(J\eta, \xi).$$

Last time, we discussed some linear algebra which is a local model for phenomena we will study in differential geometry. For example, we saw that on an oriented even-dimensional vector space with an inner product, the Hodge star defines a self-map of the middle-dimensional part of the exterior algebra, which induces extra structure, such a splitting into self-dual and anti-self-dual pieces in dimensions divisible by 4. This therefore generalizes to a $4k$-dimensional manifold with a metric and an orientation: the space of $2k$-forms splits as an orthogonal direct sum of self-dual and anti-self-dual forms. (We also discussed other examples, such as how 1-forms on an oriented 2-manifold split into holomorphic and antiholomorphic pieces.)

We're particularly interested in the case $k = 1$, where this splitting depends only on a conformal structure, and applies to 2-forms. To study its consequences we'll discuss where one can find 2-forms in differential geometry.

Definition 2.1. A *fiber bundle* is the data of a smooth map $\pi\colon E \to X$ of smooth manifolds if for all $x \in X$ there's an open neighborhood U of x and a diffeomorphism $\varphi\colon U \times \pi^{-1}(x) \to \pi^{-1}(U)$ such that the diagram

$$\begin{tikzcd} U \times \pi^{-1}(x) \arrow[rr, "\varphi"] \arrow[dr, "\mathrm{proj}_1"'] & & \pi^{-1}(U) \arrow[dl, "\pi"] \\ & U & \end{tikzcd} \tag{2.2}$$

commutes. In this case we call X the *base space* and E the *total space*. If there is a manifold F such that in the above definition we can replace $\pi^{-1}(x)$ with F, we call π a *fiber bundle with fiber* F.[2] The map φ is called the *local trivialization*.

Example 2.3. The *trivial bundle* with fiber F is the projection map $X \times F \to X$. ◀

Remark 2.4. Fiber bundles were first defined by Steenrod in the 1940s, albeit in a different-looking way. His key insight was local triviality. There are variants depending on what kind of space you care about: for example, you can replace manifolds with spaces and smooth maps with continuous maps.

Keep in mind that a fiber bundle is data (π) and a condition. Often people say "E is a fiber bundle" when they really mean "π is a fiber bundle"; specifying E doesn't uniquely specify π. ◀

If F has more structure, such as a Lie group, torsor, vector space, algebra, Lie algebra, etc., we ask that $\varphi|_{\pi^{-1}(x)}\colon F \to \pi^{-1}(x)$ preserve this structure. For example, in a fiber bundle whose fibers are vector spaces, we want φ to be linear; in this case we call it a *vector bundle*.

Definition 2.5. If $\pi\colon E \to X$ is a vector bundle, the space of *k-forms valued in E*, denoted $\Omega^k_X(E)$, is the space of C^∞ sections of $\Lambda^k T^*X \otimes E \to X$.

For ordinary differential forms (so when E is a trivial bundle), we have the de Rham differential $\mathrm{d}\colon \Omega^k_X \to \Omega^{k+1}_X$, but we do not have this in general.

Definition 2.6. Let X be a smooth manifold.

(1) A *distribution* on X is the subbundle $E \subset TX$.
(2) A vector field ξ on X *belongs to* E if $\xi_x \in E_x \subset T_xX$ for all X.
(3) A submanifold $Y \subset X$ is an *integral submanifold* for E if for all $y \in Y$, $T_yY = E_y$ inside T_yX.

Do integral submanifolds exist? This is a local question and a global question (the latter about maximal integral submanifolds). In general, the answer is "no," as in the next example.

Example 2.7. Consider a distribution on $\mathbb{A}^3$ with coordinates (x, y, z) given by

$$E_{(x,y,z)} = \mathrm{span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}\right\}. \tag{2.8}$$

There is no integral surface for this distribution. TODO: I missed the argument, sorry. ◀

This is the basic example that illustrates curvature. It turns out that the existence of an integral submanifold is determined completely by the (non)vanishing of a tensor.

Definition 2.9. Let $E \subset TX$ be a distribution. The *Frobenius tensor* $\phi_E\colon E \times E \to TX/E$ given by

$$\xi_1, \xi_2 \longmapsto [\xi_1, \xi_2] \bmod E.$$

[2]Not all fiber bundles have a fiber in this sense, e.g. a fiber bundle with different fibers over different connected components.

Let's think about this: the Lie bracket is defined for vector fields, not vectors. So we have to extend ξ_1 and ξ_2 to vector fields (well, sections of E, since they're in E), which is a choice, and then check that what we obtain is independent of this choice. It suffices to know that this is linear over functions: that

$$[f_1\xi_1, f_2\xi_2] \stackrel{?}{=} f_1 f_2[\xi_1, \xi_2]. \tag{2.10}$$

Of course, this is not what the Lie bracket does: it differentiates in both variables, so we have the extra terms $f_1(\xi \cdot f_2)\xi_2$ and $f_2(\xi \cdot f_1)\xi_1$. But both of these are sections of E, so vanish mod E, and therefore we do get a well-defined, skew-symmetric form, a section of $\Lambda^2 E^* \otimes TX/E$ – not quite a differential form.

Frobenius did many important things in mathematics, across group theory and representation theory and this theorem, which is about differential equations!

Theorem 2.11 (Frobenius theorem)**.** *An integral submanifold of E exists locally iff $\phi_E = 0$.*

This is a nonlinear ODE. As such, our proof will rely on some facts from a course on ODEs.

Lemma 2.12. *Let X be a smooth manifold, ξ be a vector field on X, and $x \in X$ be a point where ξ doesn't vanish. Then there are local coordinates $x^1, \dots, x^n$ around x such that $\xi = \partial x^1$ in this neighborhood.*

Proof. Let φ_t be the local flow generated by ξ, and choose coordinates $y^1, \dots, y^n$ near x such that $\xi_x = \frac{\partial}{\partial y^1}\Big|_x$. Define a map $U\colon \mathbb{R}^n \to X$ by

$$x^1, \dots, x^n \longmapsto \varphi_{x^1}(0, x^2, \dots, x^n). \tag{2.13}$$

The right-hand side is expressed in y-coordinates. Now we need to check this is a coordinate chart, which follows from the inverse function theorem, because the differential of φ is invertible at 0 (in fact, it's the identity). The theorem then follows because x^1 is the time direction for flow along ξ in this coordinate system. ⊠

Lemma 2.14. *With notation as above, let $\xi_1, \dots, \xi_k$ be vector fields which are linearly independent at x and suc that $[\xi_i, \xi_j] = 0$ for all $1 \le i, j \le k$. Then there exist local coordinates $x^1, \dots, x^n$ such that for $1 \le i \le k$, $\xi_i = \frac{\partial}{\partial x^i}$.*

In fact, the converse is also true, but trivially so: it's the theorem in multivariable calculus that mixed partials commute.

Proof. Let $\varphi_1, \dots, \varphi_k$ be the local flows for $\xi_1, \dots, \xi_k$. Because the pairwise Lie brackets commute, $\varphi_i\varphi_j = \varphi_j\varphi_i$. Since these vector fields are linearly independent at x, we can choose local coordinates $y^1, \dots, y^n$ around x such that $\xi_i|_x = \frac{\partial}{\partial y^i}\Big|_x$. Then, as above, define

$$x^1, \dots, x^n \longmapsto (\varphi_1)_{x_1}(\varphi_2)_{x_2} \cdots (\varphi_k)_{x_k}(0, \dots, 0, x^{k+1}, \dots, x^n). \tag{2.15}$$

You can check that $\mathrm{d}\varphi$ is invertible, so this is a change of coordinates, and then, using the fact that the flows commute, you can see that the lemma follows. ⊠

These lemmas are important theorems in their own right.

Proof of Theorem 2.11. Since the theorem statement is local, we can work in affine space $\mathbb{A}^n$. Let $\pi\colon \mathbb{A}^n \to \mathbb{A}^k$ be an affine surjection such that $\mathrm{d}\pi_0$ restricts to an isomorphism $E_0 \to \mathbb{R}^k$. Restrict to a neighborhood U of 0 in $\mathbb{A}^n$ such that $\mathrm{d}\pi_p|_{E_p}\colon E_p \to \mathbb{R}^n$ is an isomorphism for all $p \in U$, and choose $\xi_i|_p \in E_p$ such that $\mathrm{d}\pi_p(\xi_p) = \frac{\partial}{\partial y^i}$. Then, $[\xi_i, \xi_j] = 0$: we know it's in E, and

$$\mathrm{d}\pi[\xi_i, \xi_j] = [\mathrm{d}\pi(\xi_i), \mathrm{d}\pi(\xi_j)] = \left[\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right] = 0. \tag{2.16}$$

Now apply Lemma 2.14; then $\{y^{k+1} = \cdots = y^n = 0\}$ gives the desired integral submanifold. ⊠

The idea of the theorem is that it's a local normal form for an involutive distribution (one whose Frobenius tensor vanishes): locally it looks like the splitting of $\mathbb{R}^n$ into the first k coordinates and the last $(n-k)$ coordinates. And in that local model, we know what the integral manifolds are.

Consider a fiber bundle with a discrete fiber (i.e. the inverse image of every point has the discrete topology). This is also known as a *covering space*. On a "nearby fiber," whatever that means (without more data, we don't have a metric on the base space), we have some sort of parallel transport. The precise statement is that there's a neighborhood of any x on the base space such that any path in that neighborhood lifts to a path on the total space, unique if you specify a point in the fiber. More generally, you can lift families of paths, which illustrates a homotopy-theoretic generalization of a fiber bundle called a *fibration.* But globally, given an element of $\pi_1(X)$, it might lift to a nontrivial automorphism of the fiber.

We'd like to do this for more general fiber bundles $\pi\colon E \to X$, in which case we'll need more data. The kernel of $\mathrm{d}\pi$ is a distribution, and consists of the "vertical" vectors (projection down to X kills them). A complement is "horizontal".

Without any choice, we get a short exact sequence at every $e \in E$:

$$0 \longrightarrow \ker(\mathrm{d}\pi_e) \longrightarrow T_eE \xrightarrow{\mathrm{d}\pi_e} T_xX \longrightarrow 0, \tag{2.17}$$

and a splitting is exactly the choice of a complement $H_e\colon T_xX \to T_eE$. We would like to do this over the whole base, which motivates the next definition.

Definition 2.18. Let $\pi\colon E \to X$ be a fiber bundle. A *horizontal distribution* is a subbundle $H \subset TE$ transverse to $\ker(\mathrm{d}\pi)$, or equivalently a section of the (surjective) map $TE \to \pi^*TX$ of vector bundles on E.

We must address existence and uniqueness. At e the space of splittings is an affine space modeled on $\mathrm{Hom}(T_xX, \ker(\mathrm{d}\pi_e))$, because TODOsomething with a short exact sequence.

Therefore existence and uniqueness of a horizontal distribution is a question about existence and uniqueness of a section of an affine bundle over X. Using partitions of unity, we can construct many of these: existence is good, but uniqueness fails.

What about path lifting? Suppose $\gamma\colon [0,1] \to X$ is a path in X beginning at x_0 and terminating at x_1. We can pull back both E and H by γ, to obtain a rank-1 distribution γ^*H in γ^*TE, and the projection map to $T[0,1]$ is a fiberwise isomorphism. Therefore given a vector at $x_0 = \gamma(0)$ we get a unique horizontal lift along $[0,1]$ to a vector field, and therefore get a unique integral curve above γ.

Note that you cannot always lift higher-dimensional submanifolds, and again the obstruction is the Frobenius tensor, because that's the obstruction to the existence of an integral submanifold. In this context the Frobenius tensor is called *curvature* – right now it's on the total space, but in some settings we can descend it to the base.

Lecture 3.

Principal bundles, associated bundles, and the curvature 2-form: 1/29/19

"For whatever reason I'm being a little impressionistic..."

Last time, we discussed a way in which 2-forms appear in geometry: as the obstruction to integrability of a distribution $E \subset TX$. That is, a distribution contains vectors, and we can ask whether integral curves of those vectors have tangent vectors contained within E. Associated to E we defined a Frobenius tensor $\phi_E\colon \Lambda^2E \to TX/E$ sending

$$\xi_1, \xi_2 \longmapsto [\widetilde{\xi_1}, \widetilde{\xi_2}] \bmod E, \tag{3.1}$$

where $\widetilde{\xi_i}$ is a vector field extending ξ (and we showed this doesn't depend on the choice of extension). In Theorem 2.11, we saw that ϕ_E is exactly the local obstruction to integrability; we can then move to global questions.

More generally, suppose that $\pi\colon E \to X$ is a fiber bundle. Then TE fits into a short exact sequence (2.17), and we can ask for a horizontal lift from TX to TE, which is a section H of (2.17). Then, given a vector $e \in T_xX$ and a path $\gamma\colon [0,1] \to X$ with $\gamma(0) = x$, we can pull back[3] π and H to obtain a distribution in γ^*E. The Frobenius tensor vanishes, because $[0,1]$ is one-dimensional, so we can extend to an integral curve and therefore parallel-transport along γ. However, if we choose different paths in a ball, there's no guarantee that parallel transport along nearby paths agree at all; the Frobenius tensor may still be nonzero on X.

[3]In general, we can form the pullback of $[0,1] \to X \leftarrow E$ in the category of sets or spaces, but we want to put a smooth manifold structure on it. We can do it when these two maps are transverse – and since $\pi\colon E \to X$ is a submersion, this is always satisfied.

Steenrod's elegant perspective on fiber bundles (see his book *The Topology of Fiber Bundles*) considered in the spirit of Felix Klein symmetry groups associated to fiber bundles. This leads to the definition of a *principal G-bundle* as a fiber bundle of right G-torsors.

Definition 3.2. Let G be a Lie group and recall that a *right G-torsor* is a smooth manifold T and a smooth right T-action on T such that the action map $T \times G \to T \times T$ sending $(t, g) \mapsto (t, t \cdot g)$ is an isomorphism.

Example 3.3. The prime example of a torsor is to let V be a real vector space; then, the manifold $\mathcal{B}(V)$ of bases of V is a $\mathrm{GL}_n(\mathbb{R})$-torsor: $\mathrm{GL}_n(\mathbb{R})$ acts by precomposition. This also works over $\mathbb{C}$ and $\mathbb{H}$. ◂

Example 3.4. Now let X be a smooth manifold. Our first example of a principal bundle spreads Example 3.3 over X: let $\mathcal{B}(X)$ be the smooth manifold[4] of pairs (x, b) where $x \in X$ and b is a basis of T_xX, i.e. an isomorphism $b\colon \mathbb{R}^n \xrightarrow{\cong} T_xX$. There's a natural forgetful map $\pi\colon \mathcal{B}(X) \to X$ sending $(x, b) \mapsto x$.

This fiber bundle is a principal $\mathrm{GL}_n(\mathbb{R})$-bundle: given $g \in \mathrm{GL}_n(\mathbb{R})$ and a basis $b\colon \mathbb{R}^n \to T_xX$, we let $b \cdot g := b \circ g\colon \mathbb{R}^n \to T_xX$, using the standard action of $\mathrm{GL}_n(\mathbb{R})$ on $\mathbb{R}^n$. This is called the *frame bundle* of X. ◂

This principal bundle controls a lot of the geometry of X, via its associated fiber bundles.

Definition 3.5. Let $\pi\colon P \to X$ be a principal G-bundle and F be a smooth (left) G-manifold. The *associated fiber bundle* with fiber F is the quotient $P \times_G F := (P \times F)/G$, which is a fiber bundle over X with fiber F. Here, G acts on $P \times F$ on the right by $(p, f) \cdot g = (p \cdot g, g^{-1} \cdot f)$.

One has to check this is a fiber bundle, and in particular that the total space is a smooth manifold. Since G acts freely on P, it acts freely on $P \times F$, but for G noncompact there's more to say.

Example 3.6. $\mathrm{GL}_n(\mathbb{R})$ acts linearly on $\mathbb{R}^n$. Because $\mathbb{R}^n$ carries the additional structure of a vector space, the associated bundle $\mathcal{B}(X) \times_{\mathrm{GL}_n(\mathbb{R})} \mathbb{R}^n$ has additional structure: it's a vector bundle. In general, additional structure on F manifests in additional structure on $P \times_G F$.

Anyways, what vector bundle do we get? (Can you guess?) An element of the fiber of $\mathcal{B}(X) \times_{\mathrm{GL}_n(\mathbb{R})} \mathbb{R}^n$ is an equivalence class of an element $v \in \mathbb{R}^n$ and a basis $p\colon \mathbb{R}^n \to T_xX$; let $\xi := p(v) \in T_xX$. Another representative of this equivalence class are represented by $g^{-1}v$ and $p \circ g$ for some $g \in \mathrm{GL}_n(\mathbb{R})$, so this pair defines the same tangent vector ξ. Therefore we recover the tangent bundle. ◂

In general, a principal bundle is telling you some internal coordinates. You know these coordinates up to some symmetry G, and the principal bundle tracks that: you have to make a choice to get coordinates, and it tells you how different choices are related.

We want to show local triviality of a principal G-bundle $\pi\colon P \to X$, which will follow from local triviality as a fiber bundle. Consider a local section $s\colon U \to \pi^{-1}(U)$, where $U \subset X$; we would like to exhibit an isomorphism of fiber bundles $U \times G \to \pi^{-1}(U)$ over U. The map is exactly

$$x, g \longmapsto s(x) \cdot g. \tag{3.7}$$

This exhibits $U \times G \cong \pi^{-1}(U)$ as principal G-bundles, so we have local trivialization. Then in every associated bundle to P, we also obtain local triviality, hence local coordinates. For example, if the bundle of frames is trivialized over U, we get local coordinates (i.e. a local trivialization of TU).

Definition 3.8. A *connection* on a principal G-bundle $\pi\colon P \to X$ is a G-invariant horizontal distribution.

Specifically, given $g \in G$, we have the right action map $R_g\colon P \to P$, and can therefore define $H_{p\cdot g} := (R_g)_*(H_p)$ for a distribution H.

In this setting, the Frobenius tensor is going to do something nice: it's a map

$$\phi_H\colon H \wedge H \to TP/H \cong \ker(\pi_*), \tag{3.9}$$

so given two horizontal vectors, we get a vertical vector. Since H is G-invariant, the Frobenius tensor is also G-invariant, so we ought to be able to descend it to the base: there's only one piece of information on each fiber. That is, given vectors ξ_1, ξ_2 on X, we can lift them to P and compute the Frobenius tensor there, and G-invariance means it doesn't matter how we lift.

[4]You have to put a smooth manifold structure on this set! The way to do this is the only tool we have right now: work in an atlas $\mathfrak{U}$ of X which trivializes TX, do this locally, and check that the transition maps are smooth. This will also show that the map $\pi\colon \mathcal{B}(X) \to X$ is a fiber bundle.

If $\mathfrak{g}$ is the Lie algebra of the Lie group G, we have an isomorphism $\underline{\mathfrak{g}} \xrightarrow{\cong} \ker(\pi_*)$ as vector bundles on P. Specifically, let $\xi \in \mathfrak{g}$, and consider the exponential map $\exp\colon \mathfrak{g} \to G$. Given $p \in P$ with $\pi(p) = x$, we get a curve in P given by $t \mapsto p \cdot \exp(t\xi)$ sending $0 \mapsto p$, and this curve is contained entirely within P_x. Therefore its tangent vector at p is in $\ker(\pi_*)$.

So the Frobenius tensor is a map $\phi_H\colon H \wedge H \to \underline{\mathfrak{g}}$. Now let's descend to the base. We'd like to claim that what we get in $\mathfrak{g}$ is invariant, but that's just not true: if $g \in G$, the action of g on $p \cdot \exp(t\xi)$ is not the same as $p \cdot g \cdot \exp(t\xi)$: the issue is that $g\exp(t\xi)$ and $\exp(t\xi)g$ may not agree. This will make it slightly more interesting to descend to the base.

First, extend ϕ_H to a map

$$\widetilde{\phi}_H\colon TP \wedge TP \longrightarrow \underline{\mathfrak{g}} \tag{3.10}$$

by projecting $p_H\colon TP \twoheadrightarrow H$, which has kernel $\ker(\pi_*)$. That is, $\widetilde{\phi}_H(\eta_1 \wedge \eta_2) \coloneqq p_H\eta_1 \wedge p_H\eta_2)$. Thus $\widetilde{\phi}_H \in \Omega^2_P(\mathfrak{g})$.

Lemma 3.11. *Let $g \in G$. Then in $\Omega^2_P(\mathfrak{g})$, $R_g^*\widetilde{\phi}_H = \mathrm{Ad}_{g^{-1}}\widetilde{\phi}_H$.*

So once we choose a basis for $\mathfrak{g}$, we can think of elements of $\Omega^2_P(\mathfrak{g})$ as matrix-valued differential forms.[5] The proof of Lemma 3.11 comes from the observation above that to get from $p \cdot g \cdot \exp(t\xi)$ to

$$p \cdot \exp(t\xi)g = p \cdot g \cdot (g^{-1}\exp(t\xi)g) = p \cdot g \cdot \mathrm{Ad}_{g^{-1}}(\xi). \tag{3.12}$$

So this is exactly an example of an associated bundle to P, where the G-manifold F is $\mathfrak{g}$ with the adjoint G-bundle. So associated to P is the *adjoint bundle* $\mathfrak{g}_P \to X$ defined as $P \times_G \mathfrak{g}$. This is a vector bundle, in fact a bundle of Lie algebras because the adjoint action preserves the Lie bracket.

A section of $\mathfrak{g}_P$ is a function upstairs valued in $\mathfrak{g}$, which is exactly what $\widetilde{\phi}_H$ is.

Corollary 3.13. *$\widetilde{\phi}_H$ descends to a 2-form $-\Omega_H \in \Omega^2_X(\mathfrak{g}_P)$.*

In this case Ω_H is called the *curvature* of H. In particular, if X is a 4-manifold with a conformal structure, we can ask for this to be self-dual or anti-self-dual.

In the short exact sequence

$$0 \longrightarrow \underline{\mathfrak{g}} \longrightarrow TP \xrightarrow{\pi_*} \pi^*TX \longrightarrow 0, \tag{3.14}$$

a section $H\colon \pi^*TX \to TP$ is equivalent to a section $\Theta\colon TP \to \underline{\mathfrak{g}}$, i.e. a form $\Theta \in \Omega^1_P(\mathfrak{g})$. This is called the *connection form*, and $H = \ker(\Theta)$. It has to satisfy some properties.

- Θ must be G-invariant: $R_g^*\Theta = \mathrm{Ad}_{g^{-1}}\Theta$. This is a linear equation inside the infinite-dimensional vector space $\Omega^1_P(\mathfrak{g})$.
- The other constraint is affine: $\Theta|_{\mathrm{vertical}} = \mathrm{id}$.

So the space $\mathcal{A}_P$ of one-forms Θ satisfying these conditions is affine. This is the space of connections, and in particular tells us that there are lots of connections.

We can also interpret the Frobenius tensor in terms of Θ. Let ζ_1 and ζ_2 be horizontal vectors, and extend them to vector fields $\widetilde{\zeta}_1$ and $\widetilde{\zeta}$. Then $\zeta \cdot \Theta(\widetilde{\zeta}_{1-i}) = 0$, so

$$\begin{aligned} \mathrm{d}\Theta(\zeta_1, \zeta_2) &= \zeta_1\Theta(\widetilde{\zeta}_2) - \zeta_2\Theta(\widetilde{\zeta}_1) - \Theta([\widetilde{\zeta}_1, \widetilde{\zeta}_2]) \\ &= -\Theta([\widetilde{\zeta}_1, \widetilde{\zeta}_2]) = -\phi_H(\zeta_1, \zeta_2). \end{aligned} \tag{3.15}$$

Thus we have proved

Proposition 3.16. $\pi^*\Omega_H = -\widetilde{\phi}_H = \mathrm{d}\Theta + (1/2)[\Theta \wedge \Theta]$.

The notation $[\Theta \wedge \Theta]$ means: $\Theta \wedge \Theta \in \Omega^2_P(\mathfrak{g} \otimes \mathfrak{g})$, and this has a Lie bracket map $[\cdot]\colon \Omega^2_P(\mathfrak{g} \otimes \mathfrak{g}) \to \Omega^2_P(\mathfrak{g})$.

Corollary 3.17 (Bianchi identity)**.** $\mathrm{d}\Omega + [\Theta \wedge \Omega] = 0$.

[5]Here I suppose we need to use a Lie group G that admits a faithful finite-dimensional representation, but all compact Lie groups, and most noncompact Lie groups that you'll encounter, have this property.

Proof.

$$\begin{aligned} \mathrm{d}\Omega_H &= [\mathrm{d}\Theta \wedge \Theta] \\ &= \left[\Omega - \frac{1}{2}[\Theta\wedge\Theta]\wedge\Theta\right] \\ &= [\Omega \wedge \Theta] \end{aligned} \tag{3.18}$$

by the Jacobi identity. ⊠

This has been more theory than examples of principal bundles, but we will see plenty of examples when we delve into gauge theory.

Now given a principal G-bundle $\pi\colon P \to X$ with a connection, and any associated bundle F_P with fiber F, we get a horizontal distribution. There's a hands-on way to construct this, ot you could think of it in terms of path lifting: given an $x \in X$ and a lift $p \in P$, the connection lifts a path $\gamma\colon [0,1] \to X$ based at x to a path $\widetilde{\gamma}\colon [0,1] \to P$ based at p, so given an $f \in F$, we can define the path $\widehat{\gamma}\colon [0,1] \to F_P$ by $t \mapsto (\widetilde{\gamma}(t), f)$.

Suppose V is a G-representation, so its associated vector bundle $V_P \to X$ is a vector bundle. Then the horizontal distribution we obtain on V_P is tangent to the zero section of V_P. Let $\psi\colon X \to V_P$ be a section and $\xi \in T_xX$; we would like to differentiate ψ in the direction ξ. If ψ were valued in a fixed vector space, we could do this as usual: extend ξ to a curve $\gamma\colon (-\varepsilon, \varepsilon) \to M$, and then define

$$\nabla_\xi \psi \coloneqq \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \psi(\gamma(t)). \tag{3.19}$$

This is precisely the directional derivative. In V_P, the fibers are different vector spaces, which seems like a problem except that the connection on P defines parallel transport τ_t along γ for the fibers of V_P, and therefore we can define the directional derivative of ψ as

$$\nabla_\xi \psi \coloneqq \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \tau_{-t}\psi(\gamma(t)). \tag{3.20}$$

This is called the *covariant derivative.*

Exercise 3.21. Show that this satisfies the Leibniz rule: if f is a function on X, then

$$\nabla_\xi(f\cdot\psi) = (\xi \cdot f)\psi + f(x)\nabla_\xi\psi. \tag{3.22}$$

In other words, the existence of the horizontal distribution is somehow telling us about the Leibniz rule, though this is a somewhat mysterious fact.

Lecture 4.

Harmonic forms and (anti)-self-dual connections: 1/31/19

"The key to humor is......... timing!"

Last time, we discussed connections on principal bundles, and what they induce on associated vector bundles. We also briefly saw the covariant derivative associated to a connection. We begin with more on covariant derivatives.

Definition 4.1. Let $E \to X$ be a vector bundle. A *covariant derivative* is a linear map $\nabla\colon \Omega^0_X(E) \to \Omega^1_X(E)$ satisfying the *Leibniz rule*

$$\nabla(fs) = \mathrm{d}f\cdot s + f\nabla s, \tag{4.2}$$

where f is a smooth function on X and s is a smooth section of E.

If E is a trivial bundle with constant fiber V, the usual directional derivative is a covariant derivative, but there can be others.

We can extend ∇ to a sequence of first-order differential operators

$$0 \longrightarrow \Omega^0_X(E) \xrightarrow{\mathrm{d}_\nabla} \Omega^1_X(E) \xrightarrow{\mathrm{d}_\nabla} \Omega^2_X(E) \xrightarrow{\mathrm{d}_\nabla} \cdots \tag{4.3}$$

defined by

$$\mathrm{d}_\nabla(\omega\cdot s) \coloneqq \mathrm{d}\omega\cdot s + (-1)^k\omega\wedge\nabla s, \tag{4.4}$$

where $\omega \in \Omega^k_X$ and $s \in \Omega^0_X(E)$. Thus the first map $\mathrm{d}_\nabla\colon \Omega^0_X(E) \to \Omega^1_X(E)$ is just ∇.

Exercise 4.5. Show that $\mathrm{d}^2_\nabla(fs) = f\mathrm{d}^2_\nabla(s)$.

In other words, this says the *symbol* of d^2_∇ vanishes; this second-order operator is really a first-order operator. Therefore there exists an $F_\nabla \in \Omega^2_X(\operatorname{End} E)$, called the *curvature*, such that $\mathrm{d}^2_\nabla(s) = F_\nabla \cdot s$.

Digression 4.6. We recall what the symbol of an operator is. Let $E, F \to X$ be vector bundles and $D\colon \Omega^0_X(E) \to \Omega^0_X(F)$ be a differential operator. By definition, D is first-order if for every function f and section s,

$$D(fs) = \sigma(\mathrm{d}f)s + fDs \tag{4.7}$$

for some $\sigma\colon T^*X \otimes E \to F$, which is called the *symbol* of D. ◄

Exercise 4.8. Compute d^3_∇. (Answer: it's zero.)

Now we have two notions of curvature: the curvature associated to a covariant derivative as above, and the curvature associated to a principal bundle with connection and an associated vector bundle.

Exercise 4.9. Let G be a Lie group, $\pi\colon P \to X$ be a principal G-bundle with connection $\Theta \in \Omega^1_P(\mathfrak{g})$, and $\rho\colon G \to \operatorname{Aut}(\mathbb{E})$ be a linear representation of G. Let $E := \mathbb{E}_P = P \times_G \mathbb{E} \to X$ be the associated bundle, which carries a covariant derivative $\nabla\colon \Omega^0_X(E) \to \Omega^1_X(E)$. Compute d^2_∇ in terms of $\Omega = \mathrm{d}\Theta + (1/2)[\Theta \wedge \Theta]$.

Example 4.10. Let's think about connections on a principal $\mathbb{T}$-bundle.[6] Consider $\mathbb{C}^2$ with coordinates z^0, z^1 and metric

$$\langle (z^0, z^1), (w^0, w^1)\rangle := \overline{z^0}w^0 + \overline{z^1}w^1. \tag{4.11}$$

The circle group $\mathbb{T}$ acts on $S^3 \subset \mathbb{C}^2$ on the right by $(z^0, z^1)\cdot\lambda := (z^0\lambda, z^1\lambda)$. This is a free action, so its quotient is a smooth manifold, specifically $\mathbb{CP}^1 \cong S^2$, the manifold of complex lines through the origin in $\mathbb{C}^2$. Thus we obtain a principal $\mathbb{T}$-bundle $\pi\colon S^3 \to \mathbb{CP}^1$, called the[7] *Hopf bundle*.

Now let's put a connection on π. We want a horizontal distribution on the total space S^3. Inside $T_{(z^0,z^1)}S^3$, there's a one-dimensional subspace of vectors in the direction of the fiber $\{(z^0, z^1)\cdot\lambda\}$. The standard Riemannian metric on $\mathbb{C}^2 = \mathbb{R}^4$ allows us to choose a complementary line at each point, which is a horizontal distribution. Because $\mathbb{T}$ acts by isometries, this is an invariant distribution, hence a connection.

This is all pretty and geometric, but we need to compute the connection form $\Theta \in \Omega^1_{S^3}(i\mathbb{R})$ (the Lie algebra of $\mathbb{T}$ is a line with trivial bracket, and is more canonically $i\mathbb{R}$). Specifically,

$$\Theta = \operatorname{Im}\Big(\overline{z^0}\,\mathrm{d}z^0 + \overline{z^1}\,\mathrm{d}z^1\Big). \tag{4.12}$$

In the vertical direction, $\Theta = \mathrm{id}$, $(z^0e^{it}, z^1e^{it}) = (iz^0, iz^1)$. Looking inside the complexified tangent bundle (a four-dimensional complex vector bundle), which has basis $\{\partial_{z^0}, \partial_{\overline{z^0}}, \partial_{z^1}, \partial_{\overline{z^1}}\}$, we get

$$iz^0\frac{\partial}{\partial z^0} - i\overline{z^0}\frac{\partial}{\partial\overline{z^0}} + iz^1\frac{\partial}{\partial z^1} - i\overline{z^1}\frac{\partial}{\partial\overline{z^1}}. \tag{4.13}$$

So on vertical vectors, this is the identity. One (you) can check that on a vector normal to S^3, this vanishes – this is just linear algebra over the complex numbers, so nothing too intimidating.

Next we'd like to see

$$\Omega = \mathrm{d}\Theta = \operatorname{Im}\Big(\overline{\mathrm{d}z^0} \wedge \mathrm{d}z^0 + \overline{\mathrm{d}z^1} \wedge \mathrm{d}z^1\Big), \tag{4.14}$$

though this is already imaginary, so we can remove the 'Im' in front. You can check this descends to $\mathbb{CP}^1$. It's a 2-form on $\mathbb{C}^2$, visibly of type $(1,1)$, and we restrict it to S^3; the claim is that there's a form on $\mathbb{CP}^1$ whose pullback by π is $\Omega|_{S^3}$. This involves verifying two things: that Ω is $\mathbb{T}$-invariant, and that it's trivial in the vertical direction. This is a good practice computation.

Let Ω also denote the form on $\mathbb{CP}^1$: $\Omega \in \Omega^2_{\mathbb{CP}^1}(i\mathbb{R})$. We claim

$$\int_{\mathbb{CP}^1} \frac{1}{2\pi} i\Omega = 1. \tag{4.15}$$

[6]Here $\mathbb{T} \subset \mathbb{C}^\times$ is the group of unit-magnitude complex numbers, sometimes also denoted U_1 or S^1.

[7]Well, there's more than one Hopf bundle, and we'll see some others later, but this is the first example.

To compute this, we need some coordinates on $\mathbb{CP}^1$. We'll construct a section s of π over $\mathbb{CP}^1 \setminus \infty \cong \mathbb{C}$. Specifically, given $z \in \mathbb{C}$, which we think of as $[z : 1] \in \mathbb{CP}^1$, let

$$s(z) = \frac{(z, 1)}{\sqrt{1 + |z|^2}}. \tag{4.16}$$

The term in the denominator means that the function decays at infinity in $\mathbb{C}$, so we expect this integral to converge. (But you should still do it!) ◀

Consider a more general principal $\mathbb{T}$-bundle $\pi\colon P \to X$, where X is a smooth manifold. Is it a pullback of the Hopf bundle by a map $X \to \mathbb{CP}^1$? This need not be true, but something weaker is. Consider the generalized Hopf bundle $S^{2N+1} \to \mathbb{CP}^N$, defined in the same way as the Hopf bundle.

Theorem 4.17. *Every principal $\mathbb{T}$-bundle P over a smooth manifold X arises as a pullback of a Hopf bundle $S^{2N+1} \to \mathbb{CP}^N$ for some N.*

We can choose N independent of P, but it will depend on X. So in general you can think of pulling back from $\mathbb{CP}^\infty$.

Proof sketch. A pullback is a $\mathbb{T}$-equivariant map $\varphi\colon P \to S^{2N+1}$; the quotient by $\mathbb{T}$ defines a map $X \to \mathbb{CP}^N$ satisfying the theorem. But this is equivalent data to a section of the associated bundle $S_P^{2N+1} \to X$. This is good: there are tools in topology for constructing sections. First, using an approximation theorem, one shows that it suffices to find a continuous section. Then, one uses obstruction theory: choose a CW structure on X and a q-cell $D \to X$. We'd like to extend a section over this cell; since D is contractible, it's equivalent to ask that the map $S^{q-1} = \partial D \to S_P^{2N+1}$ is trivial (up to homotopy). This is a question about homotopy groups, and for N large enough, the relevant homotopy group vanishes. ⊠

So the next question is: can we construct universal connections Θ^{univ} on these Hopf bundles such that every connection arises as a pullback? This is finickier. Supposing it exists, and $\varphi\colon (P, \Theta) \to (S^{2N+1}, \Theta^{\mathrm{univ}})$, then since connections form an affine space, there's an $\alpha \in \Omega_X^1(i\mathbb{R})$ such that

$$\varphi^*\Theta^{\mathrm{univ}} - \Theta = \pi^*\alpha, \tag{4.18}$$

and hence

$$\overline{\varphi}^*\Omega^{\mathrm{univ}} - \Omega = \mathrm{d}\alpha. \tag{4.19}$$

This therefore implis $\mathrm{d}\Omega = 0$, where $\Omega \in \Omega_X^2(i\mathbb{R})$, so it has a de Rham cohomology class $[i\Omega/2\pi] \in H^2_{\mathrm{dR}}(X)$. This is the pullback of a class $(c_1)_\mathbb{R} \in H^2_{\mathrm{dR}}(\mathbb{CP}^N)$. We can see this class explicitly; $\mathbb{CP}^N$ has a very simple CW structure with one cell in each even dimension. Therefore the cochain complex for CW cohomology with $\mathbb{Z}$ coefficients looks like $\mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to \cdots$, and we claim c_1 is the generator[8] of $H^2(\mathbb{CP}^N; \mathbb{Z})$. Then there's an argument for why these two agree, namely just calculate on $\mathbb{CP}^N$, and this is the beginning of Chern-Weil theory, relating curvature and characteristic classes.

Remark 4.20. There's a similar story for higher Chern classes, but it's sufficiently complicated enough that it's generally easier to calculate using the splitting principle to split a vector bundle as a direct sum of line bundles. ◀

Let's come back to 4-manifolds and self-duality: we let X be an oriented 4-manifold with a conformal structure $[g]$. This is enough to define the Hodge star $\star\colon \Omega_X^2 \to \Omega_X^2$, which squares to the identity. Tensoring with a vector bundle allows us to define $\star\colon \Omega_X^2(E) \to \Omega_X^2(E)$ for any vector bundle $E \to X$, which also squares to the identity; therefore we can also define self-dual and anti-self-dual forms valued in E in the same way.

Definition 4.21. Let $P \to X$ be a principal G-bundle with connection Θ and $\Omega \in \Omega_X^2(\mathfrak{g}_P)$ be the associated connection form. We say Θ is *self-dual* (resp. *anti-self-dual*) if $\star\Omega = \Omega$ (resp. $\star\Omega = -\Omega$).

As we discussed in the first lecture, this is the four-dimensional analogue of a two-dimensional question on oriented, conformal surfaces: whether a function (form, ...) is holomorphic or antiholomorphic. The sign isn't all that intrinsic: changing the orientation on X changes it.

[8] We need to pick a sign, but this is determined by the canonical orientation of $\mathbb{CP}^N$ coming from the complex structure.

Anti-self-dual connections are of interest to physicists, since the 1970s, beginning with work of Polyakov and others looking at flat space. Uhlenbeck produced a condition guaranteeing that solutions to $\star\Omega = -\Omega$ extend over S^4, and later Atiyah, Bott, Hitchin, and Singer claimed there are more solutions, and used algebraic geometry to produce them. We will study more of this story in this class, but first some examples.

The simplest case is $G = \mathbb{T}$. Often this is called "the" abelian case, though there are certainly other abelian Lie groups, such as $\mathbb{T}^2$. Anyways, in this case Ω lives in $\Omega^2_X(i\mathbb{R})$, $\mathrm{d}\Omega = 0$, and if $\star\Omega = \pm\Omega$, then $\mathrm{d}\star\Omega = 0$ iff $\mathrm{d}^*\Omega = 0$. Together these imply that Ω is a harmonic form if X is closed.

Digression 4.22. Let M be a Riemannian manifold (though for just dimension 4, we're only going to need the conformal class of the metric.) For example, we could take $M = \mathbb{E}^n$, which denotes $\mathbb{R}^n$ with the standard Riemannian metric. Then the *Laplacian* is

$$\Delta \coloneqq -\left(\frac{\partial^2}{\partial(x^1)^2} + \cdots + \frac{\partial^2}{\partial(x^n)^2}\right). \tag{4.23}$$

Why the minus sign? This has a discrete spectrum, and we'd like it to be nonnegative rather than nonpositive.

The de Rham derivative has the form

$$\mathrm{d} = \varepsilon(\mathrm{d}x^i)\frac{\partial}{\partial x^i}, \tag{4.24}$$

where ε denotes exterior multiplication (which is its symbol). Using the metric, the formal adjoint is

$$\mathrm{d}^* = -\iota(\mathrm{d}x^i)\frac{\partial}{\partial x^i}. \tag{4.25}$$

(whose symbol is $-\iota$; here ι is interior multiplication). Then you can check that $\Delta \coloneqq \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$.

Now we can bring this to any Riemannian manifold M: we know what d is, and can define d^* by integrating by parts to construct the formal adjoint of d, or construct it locally. But, for the same reason that interior multiplication requires a metric, d^* depends on the metric. And therefore we can define the Laplacian Δ on M to be $\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$. This means the analogue of (4.23) on M in local coordinates $(x^1, \dots, x^n)$ is

$$\Delta = -\sum_{1\le i\le j\le n} -g^{ij}\frac{\partial^2}{\partial x^i \partial x^j}. \tag{4.26}$$

Here $g_{ij} \coloneqq \langle \partial_i, \partial_j\rangle$, and g^{ij} is the (components of the) inverse to the matrix $(g_{ij})_{i,j}$. If you haven't seen this before, it's good to work it out.

Now suppose M is closed and $\Delta\omega = 0$. Then

$$\begin{aligned}
0 &= \mathrm{d}\mathrm{d}^*\omega + \mathrm{d}^*\mathrm{d}\omega\\
&= \langle \mathrm{d}\mathrm{d}^*\omega, \omega\rangle + \langle \mathrm{d}^*\mathrm{d}\omega, \omega\rangle\\
&= \int_M (\langle \mathrm{d}\mathrm{d}^*\omega, \omega\rangle + \langle \mathrm{d}^*\mathrm{d}\omega, \omega\rangle)\,\mathrm{dvol}\\
&= \int_M (\langle \mathrm{d}^*\omega, \mathrm{d}^*\omega\rangle + \langle \mathrm{d}\omega, \mathrm{d}\omega\rangle)\,\mathrm{dvol}.
\end{aligned}$$

In fact, the converse is true.

Theorem 4.27. *On a closed Riemannian manifold,* $\mathrm{d}\omega = 0$ *and* $\mathrm{d}^*\omega = 0$ *iff* $\Delta\omega = 0$.

Such a form ω is called *harmonic*. The space of harmonic k-forms is denoted $\mathcal{H}^k_M(g) \subset \Omega^k_M$. Elliptic theory shows this is finite-dimensional,[9] and in fact more is true.

Theorem 4.28 (Hodge decomposition). *There is a splitting*

$$\Omega^k_M \cong \underbrace{\mathcal{H}^k_M(g) \oplus \mathrm{Im}(\mathrm{d})}_{\text{closed}} \oplus \mathrm{Im}(\mathrm{d}^*).$$

Since harmonic forms are closed, there's a projection $\mathcal{H}^k_M(g) \to H^k_{\mathrm{dR}}(M)$, and in fact this is an isomorphism! So every cohomology class has a unique harmonic representative. ◀

[9] We'll use some elliptic theory later this semester, and will therefore go over some of the ingredients that you'd use to prove this.

And now back to 4-manifolds. If X is an oriented Riemannian 4-manifold, we have $\mathcal{H}^2_X(g) \cong H^2(X;\mathbb{R})$, and $\mathcal{H}^2_X(g)$ has two distinguished subspaces: the self-dual forms $\mathcal{H}^2_+(g)$ and the anti-self-dual forms $\mathcal{H}^2_-(g)$. These are distinct subspaces, so every harmonic 2-form ω decomposes as a sum $\omega = \omega_+ + \omega_-$, where $\omega_\pm \in \mathcal{H}^2_\pm(g)$. Explicitly,

$$\omega_\pm = \frac{\omega \pm \star\omega}{2}. \tag{4.29}$$

All of this depended on the metric, so we can ask how this changes as the metric moves, which involves some Sard-Smale theory, as we discussed in Morse theory last semester. But Chern-Weil theory tells us that if Ω comes from a connection on a principal $\mathbb{T}$-bundle, then $i\Omega/2\pi$ defines an integer-valued cohomology class. Therefore self-dual or anti-self-dual connections are the intersection of an integer lattice in H^2 with two lines $\mathcal{H}^2_\pm(g)$. Generically, this has no solutions, unless one of $\mathcal{H}^2_\pm(g)$ is zero (so all forms are self-dual, or are anti-self-dual). Perhaps that's a little disappointing.

To study this, we'll look at the intersection form, a symmetric bilinear 2-form on $H^2(X;\mathbb{Z})$ sending $c_1, c_2 \mapsto \langle c_1 \smile c_2, [X]\rangle$. Let b^2_+ (resp. b^2_-) denote the dimension of the largest subspace on which this form is positive (resp. negative). Then $b^2_+ + b^2_- = b^2(M)$, and their difference is the sigature. We'll put conditions on $b^2_\pm$ which make it possible to find (anti)-self-dual connections.

Example 4.30.

(1) On S^4, $b^2 = 0$, so $b^2_\pm = 0$. So no self-dual forms here.
(2) On $\mathbb{CP}^2$, $b^2_+ = 1$ and $b^2_- = 0$. In this case, self-dual forms exist! Hooray.
(3) But on a K3 surface, $b^2_- = 19$ and $b^2_+ = 3$, so no self-dual forms generically. ◀

This is a little annoying. Maybe we should work with a different Lie group.

The next simplest example is $\mathrm{SU}_2 = \mathrm{Sp}_1$. Associated to it is another Hopf bundle: Sp_1 acts on $S^7 \subset \mathbb{H}^2$, as (right) multiplication by unit quaternions, and the quotient is $\mathbb{HP}^1 \cong S^4$. We can use this to follow the same story as above, defining a connection geometrically and so on.