Riemannian Geometry

UT Austin, Spring 2017

M392C NOTES: RIEMANNIAN GEOMETRY

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These notes were taken in UT Austin's M392C (Riemannian Geometry) class in Spring 2017, taught by Dan Freed. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

The cover image is the Cosmic Horseshoe (LRG 3-757), a gravitationally lensed system of two galaxies. Einstein's theory of general relativity, written in the language of Riemannian geometry, predicts that matter bends light, so if two galaxies are in the same line of sight from the Earth, the foreground galaxy's gravity should bend the background galaxy's light into a ring, as in the picture. The discovery of this and other gravitational lenses corroborates Einstein's theories. Source: https://apod.nasa.gov/apod/ap111221.html.

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"Do you have all these equations?"

Before we begin with Riemannian manifolds, it'll be useful to do a little geometry in flat space.

Definition 1.1. Let V be a real vector space; then, an *affine space over* V is a set A with a simply transitive right V-action.

That this action is simply transitive means for any $a, b \in A$, there's a unique $\xi \in V$ such that $a \cdot \xi = b$.

Definition 1.2. A set with a simply transitive (right) V-action is called a *(right) V-torsor*.

V-torsors look like copies of V without a distinguished identity.

One of the distinct features of affine space is *global parallelism*: if I have a vector ξ at a point a, I immediately get a vector at every point, which defines a vector field on the entire space.

What is the analogue of a basis for an affine space? This is a collection of points $a_0, \dots, a_n$ such that any $a \in A$ is uniquely written as

$$a = \lambda^0 a_0 + \lambda^1 a_1 + \cdots + \lambda^n a_n \tag{1.3}$$

for some $\lambda^i \in \mathbb{R}$ with $\lambda^0 + \cdots + \lambda^n = 1$.

Equation (1.3) may be written more concisely with *index notation*: any variable written as both a superscript and a subscript is implicitly summed over. That is, we may rewrite (1.3) as

$$a = \lambda^i a_i.$$

Note that in an affine space, we don't know how to add vectors (since we don't have an origin), but we can take weighted averages.

Theorem 1.4 (Giovanni Ceva, 1678)**.** *Let A be an affine plane and $a, b, c \in A$ be a triangle (i.e. three distinct, noncollinear points). Suppose $p \in \overline{bc}$, $q \in \overline{ca}$, and $r \in \overline{ca}$. Then, $\overline{ap}$, $\overline{bq}$, and $\overline{cr}$ are coincident iff*

$$[ar : rb][bp : pc][cq : ca] = 1.$$

Typically, this is thought of as a ratio of lengths, but we don't necessarily have lengths: instead, we can use barycentric coordinates. There is a unique λ such that if $r = (1-\lambda)a + \lambda b$, then $[ar : rb] = \lambda/(1-\lambda)$.

Proof. Let

$$r := (1-\lambda)a + \lambda b$$
$$p := (1-\mu)b + \mu c$$
$$q := (1-\nu)c + \nu a.$$

Set

$$x := \alpha a + \beta b + \gamma c, \tag{1.5}$$

where $\alpha + \beta + \gamma = 1$. Since $x \in \overline{ap}$, then

$$x = \alpha a + C((1-\mu)b + \mu c). \tag{1.6}$$

Comparing (1.5) and (1.6), $\mu/(1-\mu) = \gamma/\beta$.

$\boxtimes$

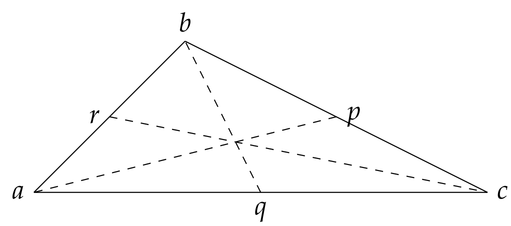

FIGURE 1. Depiction of Ceva's theorem (Theorem 1.4).

Standard affine space $\mathbb{A}^n := \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^i \in \mathbb{R}\}$. You may complain this is the same as $\mathbb{R}^n$, but $\mathbb{A}^n$ only comes with an affine structure, not a vector-space structure.

Definition 1.7. Let A be an affine space modeled on V and B be an affine space modeled on W. Then, a map $f : A \to B$ is *affine* if there exists a linear map $T : V \to W$ such that $f(a + \xi) = f(a) + T\xi$ for all $a \in A$ and $\xi \in V$.

In other words, an affine map is a linear map plus some constant, which is not uniquely defined.

Definition 1.8. An *affine coordinate system* on A is an affine isomorphism $x = (x^1, \ldots, x^n) : A \to \mathbb{A}^n$.

Them, the differentials $\mathrm{d}x_a^1, \ldots, \mathrm{d}x_a^N$ are independent of basepoint a and form a basis for V^*, the dual vector space and dual basis to V and $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$, the tangent space to any $a \in A$.

But affine space is not the only flat geometry we could consider: more generally, we consider a structure on a vector space V which can be promoted to a translationally invariant structure on A. This leads to metric geometry, symplectic geometry, etc.

Definition 1.9. An *inner product* on a (finite-dimensional) vector space V is a bilinear map $\langle -, - \rangle : V \times V \to \mathbb{R}$ which is symmetric and positive definite, i.e. for all $\xi, \eta \in V$, $\langle \xi, \eta \rangle = \langle \eta, \xi \rangle$, $\langle \xi, \xi \rangle \geq 0$, and $\langle \xi, \xi \rangle = 0$ iff $\xi = 0$.

Since $\langle -, - \rangle$ is bilinear, then this can be determined in terms of n^2 numbers: let $v_1, \ldots, v_n$ be a basis for V and define $g_{ij} := \langle v_i, v_j \rangle$ for $i, j = 1, \ldots, n$. Of course, these numbers areen't independent: $g_{ij} = g_{ji}$, so there are really only $n(n+1)$ choices of information.

Definition 1.10. A basis $e_1, \ldots, e_n$ for V is *orthonormal* if

$$\langle e_i, e_j \rangle = \delta_{ij} := \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}.$$

Our first major result of flat Euclidean geometry is that these exist.

Theorem 1.11. *There exist orthonormal bases.*

Proof. Let $v_1, \dots, v_n$ be any basis of V. Let
$$e_1 = \frac{v_1}{\langle v_1, v_1\rangle^{1/2}},$$
and for $i = 2, \dots, n$, let
$$v_i' = v_i - \langle v_i, e_1\rangle e_1.$$
Then, $\langle e_1, e_1\rangle = 1$ and $\langle e_1, v_i'\rangle = 0$. Then, repeat with $v_2', \dots, v_n'$. ⊠

This explicit algorithm is called the *Gram-Schmidt process*.

In an inner product space, we get some familiar geometric constructions: the *length* of a vector $\xi \in V$ is $|\xi| = \langle \xi, \xi\rangle^{1/2}$, and the *angle* between $\xi, \eta \in V \setminus 0$ is the θ such that
$$\cos\theta = \frac{\langle \xi, \eta\rangle}{|\xi||\eta|}.$$

Definition 1.12. A *Euclidean space* E is an affine space over an inner product space V.

This has a notion of distance: $d_E : E \times E \to \mathbb{R}^{\geq 0}$, where $a, b \mapsto |\xi|$, where $b = a + \xi$. This generalizes to notions of area, volume, etc.

Theorem 1.13 (Napoleon, 1820)**.** *Let abc be a triangle in a plane and attach an equilateral triangle to each edge. The centers of these three triangles form an equilateral triangle.*

Exercise 1.14. Prove this.

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We want to understand curved analogues of this classical material, and will pick up where differential topology left off. We work on smooth manifolds: a *smooth manifold* is a space X together with an atlas of charts $U \subset X$ with homeomorphisms $x : U \to \mathbb{A}^n$ such that every point is contained in the domain of some chart and the transition maps are smooth. We do not require a manifold to have a global dimension: the different connected components may have different dimensions, e.g. $S^1 \amalg S^2$.[1]

A chart map $x : U \to \mathbb{A}^n$ is a set of n continuous maps $(x^1, \dots, x^n)$. If p is in the domain of both x and y, we can consider $x \circ y^{-1} : \mathbb{A}^n \to \mathbb{A}^n$; calculus as usual tells us what it means for this transition map to be smooth.

At any $x \in X$, we have a tangent space $T_x X$ and a cotangent space $T_x X$: a chart defines a basis of the tangent space $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ and a basis of the contangent space $\mathrm{d}x_x^1, \dots, \mathrm{d}x_x^n$. This depends strongly on x: unlike for flat space, we may not be able to parallel-transport these globally, even on something as simple as S^2.

In this course, we will study what happens when we go from a curved analogue of affine space to a curved analogue of Euclidean space, whence the following central definition.

Definition 1.15. A *Riemannian metric* on a smooth manifold X is a choice of inner product $\langle -,-\rangle_x$ on $T_x X$ for all $x \in X$ which varies smoothly in x.

Now, we can compute lengths of tangent vectors and the angle that two smooth curves intersect at (or rather, the angle their tangent vectors intersct at). We also obtain a notion of distance between points, and can develop analogues of Euclidean geometry on manifolds.

What does "varying smoothly" mean, exactly? Suppose $x^1, \dots, x^n$ is a set of local coordinates on $U \subset X$; then, for $i, j = 1, \dots, n$, define
$$g_{ij} := \left\langle \left.\frac{\partial}{\partial x^i}\right|_x, \left.\frac{\partial}{\partial x^j}\right|_x \right\rangle_{T_x X}.$$
One can check that if the metric is smoothly varying in one chart, then it's smoothly varying in all charts.

We'll write the metric as
$$g = g_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j.$$
This again uses the summation convention, and it's useful to think about where exactly this lives: it identifies the metric as a tensor.

[1] This is important for, e.g. a space of solutions of certain PDEs.

Many manifolds arise as embedded submanifolds of Euclidean space, and the Whitney embedding theorem shows that all may be emnbedded. Many authors say it's best to meet manifolds as embedded submanifolds first, but there are some which arise without a natural embedding, e.g. the Grassmanian $\mathrm{Gr}_2(\mathbb{R}^4)$, the space of two-dimensional subspaces of $\mathbb{R}^4$.

In any case, if $X \subset \mathbb{E}^N$ is embedded, then X inherits a metric, since $T_xX \subset \mathbb{R}^n$ is also a subspace, and we can restrict the inner product. Classical Riemannian geometry is the study of *plane curves* (one-dimensional submanifolds of $\mathbb{R}^2$), *space curves* (one-dimensional submanifolds of $\mathbb{R}^3$), and *surfaces* (two-dimensional submanifolds of $\mathbb{R}^3$).

To study Riemannian manifolds, we should begin with the simplest cases. The zero-dimensional manifolds are disjoint unions of points with zero-dimensional tangent spaces and the trivial Riemannian metric. In the one-dimensional case, there is a little more to tell. A smooth map $X \to Y$ of Riemannian manifolds is an *isometry* if it's a map that preserves the inner product on each tangent space. This automatically implies it's injective.

Theorem 1.16. *Let C be a (complete) Riemannian 1-manifold which is diffeomorphic to $\mathbb{R}$. Then, C is isometric to $\mathbb{E}^1$.*

Before we prove this, we need a change-of-coordinates lemma. (We'll address completeness later, to avoid finite intervals.)

Remark. Let $x^1, \dots, x^n$ and $y^1, \dots, y^n$ be coordinate systems and suppose a metric can be written as

$$g = g_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j = h_{ab}\,\mathrm{d}y^a \otimes \mathrm{d}y^b.$$

Then,

$$g_{ij} = h_{ab}\frac{\partial y^a}{\partial x^i}\frac{\partial y^b}{\partial x^j}. \tag{1.17}$$

This is n^2 equations: there is no implicit summation here. ◄

Proof of Theorem 1.16. Let $x : C \to \mathbb{R}$ be a diffeomorphism, which defines a global coordinate on C. Let $g(x) = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \rangle$. We seek a new coordinate $y : C \to \mathbb{R}$ such that $h(y) = \langle \frac{\partial}{\partial y}, \frac{\partial}{\partial y} \rangle = 1$ everywhere. By (1.17),

$$g = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2, \tag{1.18}$$

so fix an $x_0 \in C$ and define

$$y(x) = \int_{x_0}^{x} \sqrt{g(t)}\,\mathrm{d}t.$$

This y satisfies (1.18) and therefore is an isometry. ⊠

The analogue to Theorem 1.16 in n dimensions (where $n > 1$) is as follows: if $x^1, \dots, x^n$ is a local coordinate system and g_{ij} is the Riemannian metric in these coordinates, is there a local change of coordinates $y^a(x^1, \dots, x^n)$ such that $h_{ab} = \delta_{ab}$? This is the analogue in Riemannian geometry to finding orthonormal coordinates, guaranteed by Theorem 1.11.

This requires solving an analogue to (1.17), but this time it's a PDE

$$g_{ij} = \sum_a \frac{\partial y^a}{\partial x^i}\frac{\partial y^a}{\partial x^j}.$$

This time, we need to ask whether there are solutions. The only thing we know how to do is differentiate:

$$\frac{\partial g_{ij}}{\partial x^k} = \sum_a \frac{\partial^2 y^a}{\partial x^k \partial x^i}\frac{\partial g^a}{\partial x^j} + \frac{\partial y^a}{\partial x^i}\frac{\partial^2 y^a}{\partial x^k \partial x^j}. \tag{1.19a}$$

By permuting indices, we obtain

$$\frac{\partial g_{ik}}{\partial x^j} = \sum_a \frac{\partial^2 y^a}{\partial x^j \partial x^i}\frac{\partial g^a}{\partial x^k} + \frac{\partial y^a}{\partial x^i}\frac{\partial^2 y^a}{\partial x^j \partial x^k} \tag{1.19b}$$

$$\frac{\partial g_{jk}}{\partial x^i} = \sum_a \frac{\partial^2 y^a}{\partial x^i \partial x^j}\frac{\partial g^a}{\partial x^k} + \frac{\partial y^a}{\partial x^j}\frac{\partial^2 y^a}{\partial x^i \partial x^k}. \tag{1.19c}$$

Taking (1.19a) + (1.19b) − (1.19c), we obtain

$$\frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right) = \sum_a \frac{\partial y^a}{\partial x^i}\frac{\partial^2 y^a}{\partial x^j \partial x^k}.$$

Now we multiply by $\frac{\partial y^b}{\partial x^\ell} g^{\ell i}$, concluding

$$\frac{\partial y^b}{\partial x^\ell}\underbrace{\frac{g^{\ell i}}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right)}_{\Gamma^\ell_{jk}} = \sum_a \frac{\partial y^a}{\partial x^i}\frac{\partial^2 y^a}{\partial x^j \partial x^k} g^{\ell i}\frac{\partial y^b}{\partial x^\ell}.$$

These Γ^ℓ_{jk} symbols therefore satisfy

$$\frac{\partial^2 y^b}{\partial x^j \partial y^k} = \Gamma^i_{jk}\frac{\partial y^b}{\partial x^i}.$$

If we differentiate once again (with respect to x^ℓ), we get

$$\begin{aligned}\frac{\partial^3 y^b}{\partial x^\ell \partial x^j \partial x^k} &= \frac{\partial \Gamma^i_{jk}}{\partial x^\ell}\frac{\partial y^b}{\partial x^i} + \Gamma^i_{jk}\frac{\partial^2 y^b}{\partial x^\ell x^i} \\ &= \left(\frac{\partial \Gamma^i_{jk}}{\partial x^\ell} + \Gamma^m_{jk}\Gamma^i_{m\ell}\right)\frac{\partial y^b}{\partial x^i}.\end{aligned}$$

Since mixed partials commute, then one discovers that if such an isometry exists, the *Riemannian curvature tensor*

$$R^i_{jk\ell} := \frac{\partial \Gamma^i_{j\ell}}{\partial x^\ell} - \frac{\partial \Gamma^i_{jk}}{\partial x^\ell} + \Gamma^m_{jk}\Gamma^i_{m\ell} - \Gamma^m_{j\ell}\Gamma^i_{mk}$$

must vanish. In simple cases, one can calculate that it's not always zero, so we don't always have global parallelism.

Riemann derived this in the middle of the 1800s. It's possible to see the glimmer of special relativity in them, though of course this was discovered later.

There's no text, though there is a website: `http://www.ma.utexas.edu/users/dafr/M392C/index.html`. There are problem sets, so undergraduates have to do some problem sets, and graduate students should. Feel free to talk to the professor about the problems, and especially to establish groups to work on the problem sets. Office hours are Wednesdays 2 to 3.