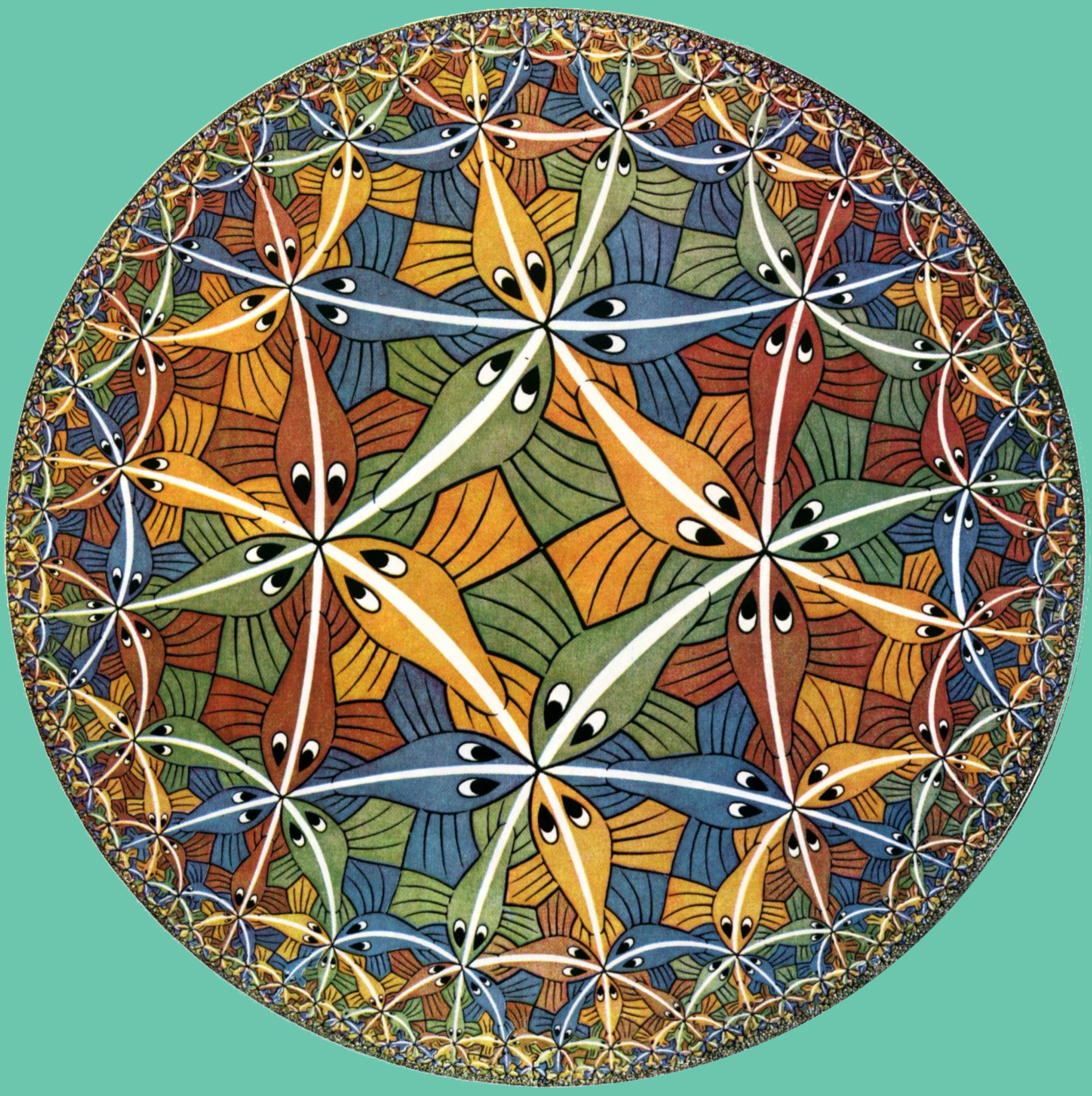
Riemann Surfaces



UT Austin, Spring 2016

M392C NOTES: RIEMANN SURFACES

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JANUARY 29, 2016

These notes were taken in UT Austin's Math 392c (Riemann Surfaces) class in Spring 2016, taught by Tim Perutz. I live-TEXed them using vim, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. The image on the front cover is M.C. Escher's *Circle Limit III* (1959), sourced from http://www.wikiart.org/en/m-c-escher/circle-limit-iii.

Contents

Lecture 1.

Review of Complex Analysis: 1/20/16

Riemann surfaces is a subject that combines the topology of structures with complex analysis: a Riemann surface is a surface endowed with a notion of holomorphic function. This turns out to be an extremely rich idea; it's closely connected to complex analysis but also to algebraic geometry. For example, the data of a compact Riemann surface along with a projective embedding specifies a proper algebraic curve over $\mathbb{C}$, in the domain of algebraic geometry.[1] In fact, the algebraic geometry course that's currently ongoing is very relevant to this one.

The theory of Riemann surfaces ties into many other domains, some of them quite applied: number theory (via modular forms), symplectic topology (pseudo-holomorphic forms), integrable systems, group theory, and so on: so a very broad range of mathematics graduate students should find it interesting.

Moreover, by comparison with algebraic geometry or the theory of complex manifolds, there's very low overhead; we will quickly be able to write down some quite nontrivial examples and prove some deep theorems: by the middle of the semester, hopefully we will prove the analytic Riemann-Roch theorem, the fundamental theorem on compact Riemann surfaces, and use it to prove a classification theorem, called the uniformization theorem.

The course textbook is S.K. Donaldson's *Riemann Surfaces*, and the course website is at http://www.ma.utexas.edu/users/perutz/RiemannSurfaces.html; it currently has notes for this week's material, a rapid review of complex function theory. We will assume a small amount of complex analysis (on the level of Cauchy's theorem; much less than the complex analysis prelim) and topology (specifically, the relationship between the fundamental group and covering spaces). Some experience with calculus on manifolds will be helpful. Some real analysis will be helpful, and midway through the semester there will be a few Hilbert spaces. Thus, though this is a topics course, the demands on your knowledge will more resemble a prelim course.

Let's warm up by (quickly) reviewing basic complex analysis; the notes on the course website will delve into more detail. For the rest of this lecture, G denotes an open set in $\mathbb{C}$.

The following definition is fundamental.

[1] This sentence is packed with jargon you're not assumed to know yet.

Definition. A function $f : G \to \mathbb{C}$ is *holomorphic* if for all $z \in G$, the complex derivative

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists. The set of holomorphic functions $G \to \mathbb{C}$ is denoted $\mathscr{O}(G)$, after the Italian *funzione olomorfa*.

Note that even though it makes sense for the limit to be infinite, this is not allowed.

First, let's establish a few basic properties.

- If $H \subset G$ is open and $f \in \mathscr{O}(G)$, then $f|_H \in \mathscr{O}(H)$.
- The sum, product, quotient, and chain rules hold for holomorphic functions, so $\mathscr{O}(G)$ is a commutative ring (with multiplication given pointwise) and in fact a commutative $\mathbb{C}$-algebra.[2]

In other words, holomorphic functions define a *sheaf* of $\mathbb{C}$-algebras on G.

By a rephrasing of the definition, then if f is holomorphic on G, then it has a *derivative* f' on G, i.e. for all $z \in G$, one can write $f(z+h) = f(z) + f'(z)h + \varepsilon_z(h)$, where $\varepsilon_z(h) \in o(h)$ (that is, $\varepsilon_z(h)/h \to 0$ as $h \to 0$). Thus, a holomorphic function is differentiable in the real sense, as a function $G \to \mathbb{R}^2$. This means that there's an $\mathbb{R}$-linear map $D_z f : \mathbb{C} \to \mathbb{C}$ such that $f(z+h) = f(z) + (D_z f)(h) + o(h)$: here, $D_z f(h) = f'(z)h$.

However, we actually know that $D_z f$ is $\mathbb{C}$-linear. This is known as the *Cauchy-Riemann condition.* Since it's *a priori* $\mathbb{R}$-linear, saying that it's $\mathbb{C}$-linear is equivalent to it commuting with multiplication by i. $D_z f$ is represented by the Jacobian matrix

$$D_z f = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}.$$

A short calculation shows that this commutes with i iff the following equations, called the *Cauchy-Riemann equations*, hold:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1.1}$$

The content of this is exactly that $D_z f$ is complex linear.

Conversely, suppose $f : G \to \mathbb{C}$ is differentiable in the real sense. Then, if it satisfies (1.1), then $D_z f$ is complex linear. But a complex linear map $\mathbb{C} \to \mathbb{C}$ must be multiplication by a complex number $f'(z)$, so f is holomorphic, with derivative f'.

Power Series. The notation $D(c, R)$ means the open disc centered at c with radius R, i.e. all points $z \in \mathbb{C}$ such that $|z - c| < R$.

Definition. Let $A(z) = \sum_{n=0}^\infty a_n (z-c)^n$ be a $\mathbb{C}$-valued power series centered at a $c \in \mathbb{C}$. Then, its *radius of convergence* is $R = \sup\{|z - c| : A(z) \text{ converges}\}$, which may be 0, a positive real number, or ∞.

Theorem 1.1. *Suppose $A(z) = \sum_{n\geq 0} a_n(z-c)^n$ has radius of convergence R. Then:*

(1) $R^{-1} = \limsup |a_n|^{1/n}$*;*
(2) $A(z)$ *converges absolutely on* $D(c, R)$ *to a function* $f(z)$*;*
(3) *the convergence is uniform on smaller discs* $D(c, r)$ *for* $r < R$*;*
(4) *the series* $B(z) = \sum_{n\geq 1} n a_n (z-c)^{n-1}$ *has the same radius of convergence* R*, so converges on* $D(c, R)$ *to a function* $g(z)$*; and*
(5) $f \in \mathscr{O}(D(c, R))$ *and* $f' = g$*.*

These aren't extremely hard to prove: the first few rely on various series convergence tests from calculus, though the last one takes some more effort.

Paths and Cauchy's Theorem. By a *path* we mean a continuous and piecewise C^1 map $[a, b] \to \mathbb{C}$ for some real numbers $a < b$. That is, it breaks up into a finite number of chunks on which it has a continuous derivative. A *loop* is a path γ such that $\gamma(a) = \gamma(b)$.

If γ is a C^1 path in G (so its image is in G) and $f : G \to \mathbb{C}$ is continuous, we define the *integral*

$$\int_\gamma f = \int_\gamma f(z)\,\mathrm{d}z := \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t.$$

[2] A $\mathbb{C}$-*algebra* is a commutative ring A with an injective map $\mathbb{C} \hookrightarrow A$, which in this case is the constant functions.

This is a complex-valued function, because the rightmost integral has real and imaginary parts. This makes sense as a Riemann integral, because these real and imaginary parts are continuous. This is additive on the join of paths, so we can extend the definition to piecewise C^1 paths. Moreover, integrals behave the expected way under reparameterization, and so on.

Theorem 1.2 (Fundamental theorem of calculus)**.** *If $F \in \mathscr{O}(G)$ and $\gamma : [a,b] \to G$ is a path, then*

$$\int_\gamma F'(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)).$$

This is easy to deduce from the standard fundamental theorem of calculus. In particular, if γ is a loop, then the integral of a holomorphic function is 0.

Now, an extremely important theorem.

Definition. A *star-domain* is an open set $G \subset \mathbb{C}$ with a $z^* \in G$ such that for all $z \in G$, the line segment $[z^*, z]$ joining z^* and z is contained in G.

For example, any convex set is a star-domain.

Theorem 1.3 (Cauchy)**.** *If G is a star-domain, γ is a loop in G, and $f \in \mathscr{O}(G)$, then $\int_\gamma f = 0$. Indeed, $f = F'$, where*

$$F(z) = \int_{[z^*,z]} f.$$

The proof is in the notes, but the point is that you can check that this definition of F produces a holomorphic function whose derivative is f; then, you get the result. The idea is to compare $F(z+h)$ and $F(z)$ should be comparable, which depends on an explicit calculation of an integral of a holomorphic function around a triangle, which is not hard.

Cauchy didn't prove Cauchy's theorem this way; instead, he proved Green's theorem, using the Cauchy-Riemann equations. This is short and satisfying, but requires assuming that all holomorphic functions are C^1. This is true (which is great), but the standard (and easiest) way to show this is... Cauchy's theorem.

Lecture 2.

Review of Complex Analysis, II: 1/22/16

Today, we're going to continue not being too ambitious; next week we will begin to geometrify things. Last time, we stopped after Cauchy's theorem for a star domain G: for all f holomorphic on G and loops $\gamma \in G$, $\int_\gamma f = 0$, and in fact one can write down an antiderivative for f, and then apply the fundamental theorem of calculus.

Then one can bootstrap one's way up to a more powerful theorem; the next one is a version of the deformation theorem.

Corollary 2.1 (Deformation theorem)**.** *Let $G \subset \mathbb{C}$ be open and $\gamma_0, \gamma_1 : [a,b] \rightrightarrows G$ be C^1 loops that are C^1 homotopic through loops in G. Then, for all $f \in \mathscr{O}(G)$, $\int_{\gamma_0} f = \int_{\gamma_1} f$.*

Proof sketch. Fix a C^1 homotopy $\Gamma : [a,b] \times [0,1] \to G$ such that $\Gamma(a,s) = \Gamma(b,s)$ for all s, $\gamma_0(t) = \Gamma(t,0)$, and $\gamma_1(t) = \Gamma(t,1)$. Then, it is possible to divide $[a,b] \times [0,1]$ into a grid of rectangles fine enough such that the image of each rectangle is mapped under Γ to a subset of G contained in an open disc in $\mathbb{C}$, as in Figure 1. Now, by Cauchy's theorem in a disc, the integral does not depend on path within each disc, so we can apply Γ in over the rectangles from 0 to 1, showing that the two integrals are the same. ⊠

Corollary 2.2. *Cauchy's theorem holds in any simply connected open $G \subset \mathbb{C}$.*

This is considerably more general than star domains (e.g. the letter **C** is simply connected, but not a star domain). Moreover, on such a domain, any $f \in \mathscr{O}(G)$ has an antiderivative: pick some basepoint $z_0 \in G$, and let $\gamma(z_0, z)$ be a path from z_0 to z. Then,

$$F(z) = \int_{\gamma(z_0,z)} f(z)\,\mathrm{d}z$$

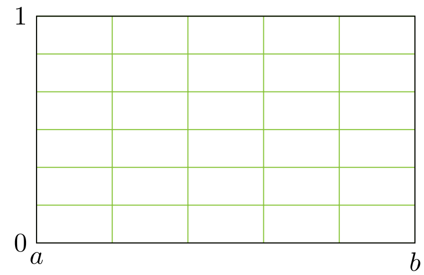

FIGURE 1. Subdividing $[a,b] \times [0,1]$ into rectangles.

is well-defined, because any two choices of path differ by the integral of a holomorphic function on a loop, which is 0.

We can also use this to understand power series representations.

Proposition 2.3 (Cauchy's integral formula)**.** *, Let G be a domain in $\mathbb{C}$ containing the closed disc D. If $f \in \mathscr{O}(G)$, then*

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w-z}\,\mathrm{d}w.$$

Proof idea. Suppose D is centered at z and has radius R, and let $C(z,r)$ denote the circle centered at z and with radius r. We'll also let D^* denote the punctured disc, i.e. D minus its center point. By calculating $\int_\gamma \mathrm{d}z/z = 2\pi i$, one has that

$$\frac{1}{2\pi i} \int \partial D \frac{f(w)}{z-w}\,\mathrm{d}w - f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)-f(z)}{w-z}\,\mathrm{d}w.$$

Using Corollary 2.1, for $r \in (0,R)$,

$$= \frac{1}{2\pi i} \int_{C(z,r)} \frac{f(w)-f(z)}{w-z}\,\mathrm{d}w,$$

and as $r \to 0$, this approaches $f'(z)$, which is bounded, and the integral over smaller and smaller circles of a bounded function tends to zero. ⊠

Theorem 2.4 (Holomorphic implies analytic)**.** *If D is a disc centered at c and $f \in \mathscr{O}(D)$, then on that disc,*

$$f(z) = \sum_{n\geq 0} a_n (z-c)^n, \qquad \textit{where} \qquad a_n = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{(z-c)^{n+1}}\,\mathrm{d}z.$$

Proof sketch. For any $z \in D$, there's a $\delta > 0$ such that the closed disc $\overline{D}(z,\delta)$ of radius δ is contained in D. Hence, by Proposition 2.3,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{C(z,\delta)} \frac{f(w)}{w-z}\,\mathrm{d}w \\ &= \int_{C(c,R')} \frac{f(w)}{w-z}\,\mathrm{d}w \end{aligned}$$

for any $R' \in (0,\delta)$, by Corollary 2.1. We'd like to force a series on this. First, since

$$\frac{1}{w-z} = \frac{1}{(w-c)-(z-c)} = \frac{1}{w-c}\left(\frac{1}{1-\frac{z-c}{w-c}}\right),$$

then

$$\begin{aligned} f(z) &= \frac{1}{3\pi i} \int_{C(c,R')} \frac{f(w)}{w-c} \frac{1}{1-\frac{z-c}{w-c}}\,\mathrm{d}w \\ &= \frac{1}{2\pi i} \oint \frac{f(w)}{w-c} \sum_{n\geq 0} \frac{(z-c)^n}{(w-c)^n}\,\mathrm{d}w. \end{aligned}$$

Since $|(z-c)/(w-c)| < 1$ on $C(c, R')$, then this is well-defined, and since it's a geometric series, it has nice convergence properties, and so we can exchange the sum and integral to obtain

$$= \sum_{n \geq 0} \underbrace{\frac{1}{2\pi i}\left(\oint \frac{f(w)}{(w-c)^{n+1}}\,\mathrm{d}w\right)}_{a_n}(z-c)^n. \qquad \boxtimes$$

One application of this is to understand zeros of holomorphic functions. If $f \in \mathscr{O}(G)$ and $f(c) = 0$, then let $f(z) = \sum a_n(z-c)^n$ be its power series and a_m be the first nonzero coefficient. Then, in a neighborhood of c,

$$f(z) = (z-c)^m \underbrace{\sum_{n \geq m} a_n(z-c)^{n-m}}_{g(z)}.$$

This g is holomorphic and does not vanish on this neighborhood, so the takeaway is $f(z) = (z-c)^m g(z)$ near c, with g holomorphic and nonvanishing. This m is called the *multiplicity*, denoted $\mathrm{mult}(f, c)$. In particular, if $f(c) \neq 0$, then $m = 0$.

Theorem 2.5. *If G is a connected open set and $f \in \mathscr{O}(G)$ is not identically zero, then $f^{-1}(0)$ is discrete in $\mathbb{C}$.*

Proof. If $f(c) = 0$, then there's a disc D on which $f(z) = (z-c)^m g(z)$, where $m \geq 1$ and g is nonvanishing, so the only place f can vanish on D (i.e. near c) is at c itself. $\boxtimes$

Definition. A function $f \in \mathscr{O}(\mathbb{C})$, so holomorphic on the entire plane, is called *entire.*

Theorem 2.6 (Liouville)**.** *A bounded, entire function is constant.*

Proof sketch. We'll show that $f'(z) = 0$ everywhere. By Proposition 2.3, we know

$$f'(z) = \frac{1}{2\pi i}\int C(z,r)\frac{f(w)}{(w-z)^2}\,\mathrm{d}w,$$

and we can deform this loop to $C(0, R)$. Then, one bounds the integral, and the bound ends up being $O(1/R)$, so as $R \to \infty$, this necessarily goes to 0. $\boxtimes$

Lecture 3.

Meromorphic Functions and the Riemann Sphere: 1/25/16

We're still going to be doing classical function theory today, but we're going to begin to geometrify it. Recall that $G \subset \mathbb{C}$ denotes an open set.

We'll begin with the following theorem.

Theorem 3.1 (Morera)**.** *Let $f : G \to \mathbb{C}$ be a continuous function such that for all triangles $T \subset G$, $\int_{\partial T} f = 0$. Then, f is holomorphic.*

This is surprisingly easy to prove, given what we've done.

Proof. Since holomorphy is a local property, we may without loss of generality work on a disc $D(z_0, r) \subset G$. Then, define $F : D(z_0, r) \to \mathbb{C}$ by $F(z) = \int_{[z_0,z]} f$; using the hypothesis on triangles, $F' = f$. Thus, as we showed last time, this means $F \in \mathscr{O}(G)$, and so it's analytic, and therefore it has derivatives of all orders. Thus, $F' = f$ is holomorphic. $\boxtimes$

This is useful, e.g. one may have a function which is defined through an improper integral, or a pointwise limit of holomorphic functions. Then, Morera's theorem allows for an easier, indirect way to show holomorphy. Here's another application.

Definition. If $z_0 \in G$, a function $f \in \mathscr{O}(G \setminus \{z_0\})$ has a *removable singularity* at z_0 if f can be extended holomorphically to G.

Theorem 3.2. *Suppose $f \in \mathscr{O}(G \setminus \{z_0\})$ and $|f|$ is bounded near z_0. Then, f has a removable singularity at z_0.*

There are several ways to prove this quickly.

Proof. We can without loss of generality translate this to the origin, so assume $z_0 = 0$. If $g(z) = zf(z)$, then $g(z) \to 0$ as $z \to 0$, since $|f(z)|$ is bounded in a neighborhood of the origin. Thus, g extends continuously to all of G, with $g(0) = 0$.

Next, one should check that Morera's theorem applies to g; the only nontrivial example is a triangle around the origin. However, since g is holomorphic everywhere except at 0, the deformation theorem allows us to shrink the triangle as much as we want, and since $g \to 0$, the integral goes to 0 as well. If the triangle's edge or vertex touches the origin, one can use the deformation theorem to push it away again.

In particular, g is holomorphic on G and has a zero at 0, so by the discussion on multiplicities last time, $g(z) = z \cdot f(z)$, where f is holomorphic on all of G; this produces our desired extension of f. ⊠

Definition.

- If $z_0 \in G$ and $f \in \mathscr{O}(G \setminus \{z_0\})$, then f has a *pole* at z_0 if there's an $m \in \mathbb{N}$ such that $(z - z_0)^m f(z)$ is bounded near z_0 (and hence has a removable singularity there). The least such m is called the *order* of the pole.
- A *meromorphic* function on G is a pair (Δ, f) consisting of a discrete subset $\Delta \subset G$ and an $f \in \mathscr{O}(G \setminus \Delta)$ such that f has a pole at each $z \in \Delta$.

So, nothing worse than a pole happens for a meromorphic function. There are *essential singularities*, which are singularities which aren't poles, but we will not discuss them extensively; almost everything in sight will be meromorphic.

The Riemann Sphere. In some sense, the Riemann sphere is the most natural setting for meromorphic functions, and the first nontrivial example of a Riemann surface (still to be defined).

Definition. The *Riemann sphere* $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the *one-point compactification* of $\mathbb{C}$, so its topology has as its open sets (1) opens in $\mathbb{C}$, and (2) $(\mathbb{C} \setminus K) \cup \{\infty\}$, where $K \subset \mathbb{C}$ is compact.

There is a homeomorphism $\phi : \widehat{\mathbb{C}} \to S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ given by *stereographic projection*: send $\infty \mapsto (0, 0, 1)$ (the north pole), and then any other $z \in \mathbb{C}$ defines a line from z in the xy-plane to $(0, 0, 1)$ intersecting S^2 at one other point; this is $\phi(z)$. Hence, we will use $\widehat{\mathbb{C}}$ and S^2 interchangeably.

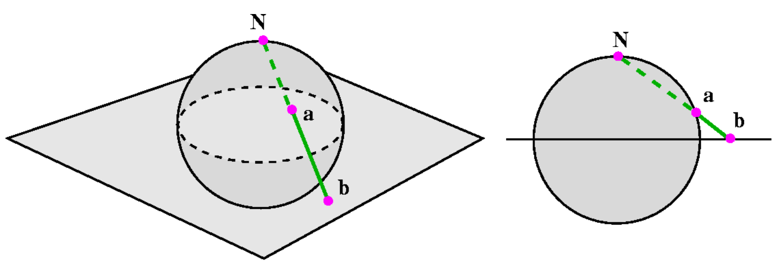

FIGURE 2. Depiction of stereographic projection, where $N = (0, 0, 1)$ is the north pole. Source: http://www.math.rutgers.edu/~greenfie/vnx/math403/diary.html.

Definition. A continuous map $f : G \to S^2$ is *holomorphic* if for all $z \in G$, either

- $f(z) \in \mathbb{C}$ (so it doesn't hit ∞) and $f : G \to \mathbb{C}$ is holomorphic, or
- if $f(z) \in \widehat{\mathbb{C}} \setminus \{0\}$, then $1/f(w) : G \to \mathbb{C}$ is holomorphic, where $1/\infty$ is understood to be 0.

If the image of f contains neither 0 not ∞, then both criteria hold, and are equivalent (since $1/z$ is holomorphic on any neighborhood not containing zero).

Proposition 3.3. *The meromorphic functions on G can be identified with the holomorphic functions $G \to S^2$.*

Proof. Suppose f is meromorphic on G, so that it has a pole of order m at z_0. Then, $f(z) = (1/(z - z_0)^m)g(z)$ for some holomorphic g with a removable singularity at z_0, and $g(z_0) \neq 0$.

By letting $1/0 = \infty$, this realizes f as a continuous map $G \to S^2$, and $1/f = (z - z_0)^m(1/g)$, which is certainly holomorphic near z_0, so f is holomorphic as a map to S^2.

The converse is quite similar, a matter of unwinding the definitions, but has been left as an exercise. ⊠

You can also define a notion of a holomorphic function coming out of S^2, not just into.

Definition. Let $G \subset S^2$ be open. A continuous $f : G \to S^2$ is *holomorphic* if one of the following is true.

- If $\infty \notin G$, then we use the same definition as above.
- If $\infty \in G$, then it's holomorphic on $G \setminus \infty$ and there's a neighborhood N of ∞ in G such that the composition
$$N^{-1} \xrightarrow{z \mapsto 1/z} N \xrightarrow{f} S^2$$
is holomorphic.

If you're used to working with manifolds, this sort of coordinate change is likely very familiar: every time we talk about ∞, we take reciprocals and talk about 0.

Example 3.4. Every rational function $p \in \mathbb{C}(z)$ is meromorphic, and extends to a holomorphic map $S^2 \to S^2$.

Now, we can talk about these geometrically: $z \mapsto z^2$ sends $e^{in\theta} \mapsto e^{2in\theta}$, so it doubles the longitude (modulo 1). In particular, it wraps the sphere twice around itself, preserving 0 and ∞, as in Figure 3.

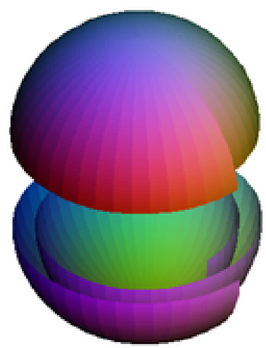

FIGURE 3. A depiction of the map $z \mapsto z^2$ on the Riemann sphere, which fixes the poles. Source: https://en.wikipedia.org/wiki/Degree_of_a_continuous_mapping.

Example 3.5. A *Möbius map* is a map
$$\mu(z) = \frac{az+b}{cz+d},$$
where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. This extends to a holomorphic map $S^2 \to S^2$ with a holomorphic inverse (the Möbius map associated to $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]^{-1}$). Thus, there's a homeomorphism $\mathrm{SL}_2(\mathbb{R})/\{\pm I\}$ to the group of Möbius transformations.

One interesting corollary is that the point at infinity is *not* special, since there's a Möbius map sending it to any other point of S^2, and indeed they act transitively on it. So we don't really have to distinguish the point at infinity from this geometric point of view.

Theorem 3.6. *If $f : S^2 \to S^2$ is holomorphic, then it's a rational function. In particular, the Möbius maps are the only invertible holomorphic maps $S^2 \to S^2$.*

The idea is to eliminate the zeros and poles by multiplying by $(z - z_0)^m$; then, one can apply Liouville's theorem to show that the result is constant.

Lecture 4.

Analytic Continuation: 1/27/16

This corresponds to §1.1 in the textbook, and is one of the classical motivations for Riemann surfaces.

The problem is: if $G \subset \mathbb{C}$ is open and $f \in \mathscr{O}(G)$, then we would like to extend f holomorphically, or maybe meromorphically, to a larger domain $H \supset G$. Such extensions are called *analytic* (resp. *meromorphic*) *continuations* of f.[3]

[3]Though "holomorphic continuation" would make more sense, tradition gives us the term "analytic continuation."

Remark. If H is connected, then there exists at most one meromorphic continuation of f to H, because the difference of two continuations vanishes on the open set G, and hence vanishes everywhere.

Example 4.1. Let $f(z) = \sum_{n\geq 0} z^n$, which converges on the open unit disc, but diverges when $|z| \geq 1$. At first sight, this suggests we'll never get any farther than the disc, but this turns out to merely be an artifact of this presentation of f: we could instead write it as $f(z) = 1/(1-z)$, which meromorphically extends f to the whole of $\mathbb{C}$ (with a single pole at $z = 1$). Thus, this power series representation is not *per se* intrinsic.

One can take this further and define analytic continuations of general functions defined by power series.

Example 4.2. This example is more sophisticated, and will take longer; it reflects a common theme in this subject, that the examples are nontrivial and are worth taking seriously. Define the Γ-*function*

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t$$

on the open set $\operatorname{Re} z > 0$. This integral is doubly improper, since there's a singularity at 0 and it's unbounded on the right, so we really should rewrite it as

$$\Gamma(z) = \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 t^{z-1}e^{-t}\,\mathrm{d}t + \lim_{T\to\infty}\int_1^T t^{z-1}e^{-t}\,\mathrm{d}t.$$

Let $H_a = \{z \mid \operatorname{Re} z > a\}$. We're going to show that Γ extends to the entire plane, but first we need to show that it's holomorphic on the right half-plane.

Proposition 4.3. $\Gamma \in \mathscr{O}(H_0)$.

Proof sketch. Since we need to realize $\Gamma(z)$ as a limit, let

$$g_n(z) = \int_{1/n}^n t^{z-1}e^{-t}\,\mathrm{d}t.$$

This is an integral of a holomorphic function, so $g_n \in \mathscr{O}(\mathbb{C})$ and

$$g_n'(z) = \int_{1/n}^n \frac{\partial}{\partial z}\big(t^{z-1}e^{-t}\big)\,\mathrm{d}t = (z-1)g_n(z-1).$$

If $a > 0$, then g converges uniformly on the strip $a < \operatorname{Re} z < b$ — the goal is to show that g_n is uniformly Cauchy on this strip (the details of which are left to the reader) by comparing to the integral of $e^{-t/2}$ for $t \gg 0$, the point being that $e^{x-1}e^{-t} \leq e^{-t/2}$ for t sufficiently large. For $t < 1$, one should compare it to the integral of t^{x-1}. Then, we need to use the following theorem.

Theorem 4.4. *If $f_n \in \mathscr{O}(G)$ and $f_n(z) \to f(z)$ locally uniformly, then $f \in \mathscr{O}(G)$.*

The proof uses Morera's theorem (Theorem 3.1) and can be found in the review notes (or Rudin, etc.). In any case, this means $\Gamma = \lim_{n\to\infty} g_n$ is holomorphic on the right half-plane. ⊠

Now, we can talk about extending Γ.

Theorem 4.5. *Γ has a meromorphic continuation to $\mathbb{C}$, whose only poles are simple poles*[4] *at 0, -1, -2, and so on.*

Proof. Since the gamma function is given by an integral, let Γ_0 be that integral from 0 to 1, and Γ_∞ be the integral from 1 to ∞. Then, the argument above shows that $\Gamma_\infty \in \mathscr{O}(\mathbb{C})$, so the only extension that we actually need to make is of

$$\Gamma_0(z) = \int_0^1 t^{z-1}e^{-t}\,\mathrm{d}t.$$

The cunning idea is that we're going to look at the n^{th}-order Taylor polynomial for e^{-t}, which provides an integral we can actually do, and then treat everything else separately. Specifically, let

$$e_n(t) = \sum_{j=0}^{n-1}\frac{(-t)^j}{j!},$$

[4] A pole is *simple* if it's degree 1.

so that

$$\Gamma_0(z) = \underbrace{\int_0^1 t^{z-1}\big(e^{-t} - e_n(t)\big)\,\mathrm{d}t}_{\Gamma_n(z)} + \int_0^1 t^{z-1} e_n(t)\,\mathrm{d}t.$$

$$= \Gamma_n(z) + \sum_{j=0}^{n-1} \frac{(-1)^j}{j!(z+j)}.$$

The $(z+j)$ in the denominator on the right gives us simple poles at $0, -1, -2, \ldots, -n+1$. But $e^{-t} - e_n(t)$ has a zero of order n at $t = 0$, so

$$\int_0^1 t^{z-t}\big(e^{-t} - e_n(t)\big)\,\mathrm{d}t$$

exists on H_{-n}, so $\Gamma_n \in \mathscr{O}(H_{-n})$. Thus, we can extend Γ meromorphically to all of $\mathbb{C}$, because any $z \in \mathbb{C}$ is in some H_{-n}, so we can work this with Γ_n. ⊠

It goes without saying that Γ is one of the most prominent functions in analytic number theory.

These two successful examples of meromorphic continuation are in some sense atypical; in general, there is a problem of multi-valuedness or monodromy.

Example 4.6. For an algebraic example of this problem, consider

$$f(z) = \sum_{n \geq 0} \binom{1/2}{n} z^n,$$

where

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}.$$

By a generalized binomial theorem (or checking that it satisfies the right differential equation), one can show that f converges on $D(0,1)$ to a branch of $\sqrt{1+z}$. We can extend holomorphically to the *cut plane* $\mathbb{C} \setminus (-\infty, -1]$ by writing $f(z) = \exp((1/2)\log(1+z))$, where we can choose a branch of $\log(1+z)$ in this cut plane, such as $\log(re^{i\theta}) = \log r + i\theta$, with $\theta \in (-\pi, \pi)$.

There's nothing particularly special about this branch cut. Plenty of other branch cuts (paths from -1 to $-\infty$ whose complements are simply connected) work just as fine — but we cannot extend further, because as we go around a loop around -1, $f(z)$ flips $-f(z)$ (the other branch of $\sqrt{1+z}$), since the logarithm changes by $2\pi i$. This is a little unsatisfactory, since we can't go further.

A similar story holds for just about any algebraic function, since one has to take a branch cut to resolve the ambiguity of multiple roots.

The Riemann surfaces way to approach this is instead of making arbitrary branch cuts, it's more canonical instead to study the equation $w^2 - (1-z) = 0$, which implicitly defines w as a square root of $1+z$. Then, we consider the set

$$X = \{(z,w) \in \mathbb{C}^2 \mid P(z,w) = 0\},$$

where $P(z,w) = w^2 - (1+z)$. Soon, we will see that this X is a Riemann surface. We can play exactly the same game with any $P(z,w) : \mathbb{C}^2 \to \mathbb{C}$ that is holomorphic in each variable separately, includin any polynomial in z and w. This defines for us its zero set $X = \{P(z,w) = 0\}$.

Then, we have an implicit function theorem, which is a major classical motivation for the theory of Riemann surfaces, just as the implicit function theorem on $\mathbb{R}^n$ is a major classical motivation for defining abstract manifolds.

Theorem 4.7 (Implicit function theorem)**.** *If $(z_0, w_0) \in X$ and $\frac{\partial P}{\partial w}(z_0, w_0) \neq 0$, then there's a disc $D_1 \subset \mathbb{C}$ centered at $z_0 \in \mathbb{C}$ and a disc $D_2 \subset \mathbb{C}$ centered at w_0, and a holomorphic $\phi : D_1 \to D_2$ such that $\phi(z_0) = w_0$ and $X \cap (D_1 \times D_2)$ is the graph of ϕ, i.e. $\{(z, \phi(z)) \mid z \in D_1\}$.*

An analogue of this function holds for C^1 real functions (or C^∞ ones), and this version can be extracted from that, but it has a simpler, direct proof.

Proof. This proof hinges on a theorem called the *argument principle*, that if $f \in \mathscr{O}(G)$ and $\overline{D}$ is a closed disc in G with $f(z) \neq 0$ on $\partial\overline{D}$, then

$$\frac{1}{2\pi i}\int_{\partial\overline{D}} \frac{f'(w)}{f(w)}\,\mathrm{d}w = \sum_{\substack{z\in D\\ f(z)=0}} \mathrm{mult}(f;z). \tag{4.1}$$

That is, integrating the logarithmic derivative counts the zeros inside D, with multiplicity. There's also the related formula

$$\frac{1}{2\pi i}\int_{\partial\overline{D}} \frac{wf'(w)}{f(w)}\,\mathrm{d}w = \sum_{\substack{z\in D\\ f(z)=0}} z\,\mathrm{mult}(f;z). \tag{4.2}$$

These are nice exercises in residue calculus.

Returning to the implicit function theorem, let $f_z = P(z,\cdot)$, so $f_{z_0}(w_0) = 0$, but $f'_{z_0}(w_0) \neq 0$. Thus, $\mathrm{mult}(f_{z_0}; w_0) = 1$, and therefore by isolation of zeros, there's a disc D_2 centered at w_0 such that w_0 is the only zero of f_{z_0} in $\overline{D}_2$. Hence, by (4.1),

$$\frac{1}{2\pi i}\int_{\partial D_2} \frac{f'_{z_0}}{f_{z_0}} = 1.$$

Since $f_{z_0} \neq 0$ on the boundary, then there's a $\delta > 0$ such that $|f_{z_0}| > 2\delta > 0$ on ∂D_2. Thus, there's a disc D_1 centered at z_0 such that for all $z \in D_1$, $|f_z| > \delta$ on ∂D_2 because P is continuous. Hence,

$$\frac{1}{2\pi i}\int_{\partial D_2} \frac{f'_z}{f_z} = 1,$$

or, by (4.1), there's a unique solution $w = \phi(z)$ to $P(z,w) = 0$ with $z \in D_1$ and $w \in D_2$. Thus, we need only to show that ϕ is holomorphic. By (4.2),

$$\phi(z) = \frac{1}{2\pi i}\int_{\partial D_2} \frac{wf'_z(w)}{f_z(w)}\,\mathrm{d}w = \frac{1}{2\pi i}\int_{\partial D_2} \frac{w}{P(z,w)}\frac{\partial P}{\partial w}(z,w)\,\mathrm{d}w.$$

Hence, ϕ is holomorphic in z (since its derivative is given by differentiating under the integral sign). ⊠

Thus, even working just with zero sets of algebraic functions, Riemann surfaces show up very nicely.

Lecture 5.

Analytic Continuation Along Paths: 1/29/16

Today, we're going to talk about analytic continuation along paths and the interesting things that result. There's also a more classical Weierstrass way to look at this.

Definition. If ϕ is a holomorphic function defined on a neighborhood U of a $z_0 \in \mathbb{C}$ and $\gamma : [0,1] \to \mathbb{C}$ is a path with $\gamma(0) = z_0$, then an *analytic continuation* of ϕ along γ consists of a pair (U_t, ϕ_t) for all $t \in [0,1]$, where U_t is a neighborhood of $\gamma(t)$ and $\phi_t \in \mathscr{O}(U_t)$ such that:

- $\phi_0 = \phi$ on $U_0 \cap U$, and
- the different ϕ_t should agree, in the sense that for all $s \in [0,1]$, there's a $\delta > 0$ such that if $|t - s| < \delta$, then ϕ_s and ϕ_t agree on $U_s \cap U_t$.

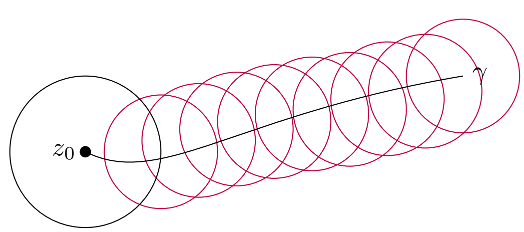

FIGURE 4. Analytic continuation along a path; on sufficiently close circles, the extensions must agree.

Note, however, that if γ intersects itself, then there's no requirement for the extensions to agree on those overlaps (if δ is sufficiently small, for example). Weierstrass said this is how one should think of complex analytic functions, and this confused a lot of people, but did lead to Weyl's work that we'll discuss in a few lectures.

Example 5.1. The logarithm is a very good example. Start with a branch of log defined on some open set U_0, so $\log(re^{i\theta}) = \log r + i\theta$, or $\log z = \log|z| + i\arg z$, for some continuous, real-valued $\arg : U_0 \to \mathbb{R}/2\pi\mathbb{Z}$.

Then, for any $\gamma : [0,1] \to \mathbb{C}^*$ with $\gamma(0) = z_0 \in U_0$, we can uniquly lift $\arg\circ\gamma : [0,1] \to \mathbb{R}/2\pi\mathbb{Z}$ to $\mathbb{R}$ consistent with $\arg(z_0)$; this lift will be called $\arg_\gamma$.[5] Then, define $\log_{\gamma_t}(z) = \log|z| + \arg_{\gamma(t)}(z)$, which defines a continuation of the logarithm around γ.

Example 5.2. For a more algebraic example, let

$$\phi(z) = \sum_{j\ge 0} \binom{1/2}{j} z^j$$

on the unit disc $D(0,1)$, so $\phi(z)^2 = z+1$. Then, one can continue along any γ with image in $\mathbb{C}\setminus\{-1\}$ by setting $\phi_t(z) = \exp((1/2)\log_{\gamma_t}(1+z))$. However, if $\gamma(t) = -1 + e^{2\pi i t}$, then γ winds around -1, and when it returns to a point, the extension of ϕ has a different value!

Example 5.3. Analytic continuation along paths works particularly well with differential equations: let p and q be meromorphic functions. Then, we want to find a $u(z)$ such that $u'' + p(z)u' + q(z)u = 0$, which we'll call $\boxed{p,q}$.[6] If you think differential equations are boring, questions like these are still motivated by study of $\mathcal{D}$-modules and the like in algebraic geometry.

Let's work near a point z_0 where p and q are holomorphic, so z_0 is a *regular point*, and without loss of generality make $z_0 = 0$. We're going to look for *series solutions*: set $p(z) = \sum_{n\ge 0} p_n z^n$ and $q(z) = \sum q_n z^n$ on $D(0,R)$ for some R, and we want to find $u(z) = \sum_{n\ge 0} u_n z^n$. Equating the coefficients of z_n in $\boxed{p,q}$, one obtains the recurrence relation

$$(n+1)(n+2)u_{n+1} + \sum_{i=0}^{n}(n+i-1)p_i u_{n+1-i} + \sum_{j=0}^{n} q_j u_{n-j} = 0.$$

By induction, one shows that all of the u_j are determined by a choice of $(u_0, u_1) \in \mathbb{C}^2$.

Proposition 5.4. *$\sum u_n z^n$ converges in the same disc $D(0,R)$.*

The detailed proof is a homework assignment, and depends on the following lemma, due to an idea of Cauchy.

Lemma 5.5 (Majorization)**.** *Say $|p_n| \le P_n$ and $|q_n| \le Q_n$. Then, let $P(z) = \sum P_n z^n$ and $Q(z) = \sum Q_n z^n$. If $u = \sum u_n z^n$ is a solution to $\boxed{p,q}$ and $U_n = \sum U_n z^n$ is a solution to $\boxed{P,Q}$, and if $U_0 = |u_0|$ and $U_1 = |u_1|$, then $|u_n| \le |U_n|$.*

The proof involves some straightforward estimates after the recurrence formula.

Proof sketch of Proposition 5.4. Let's work on $\overline{D(0,r)}$ where $r < R$. Then, we have estimates like $|p_n| \le M/r^n$ and $|q_n| \le M/r^n$, where $M = \sup_{z\in\overline{D(0,r)}}\{|p(z)|, |q(z)|\}$, which follows from Cauchy's estimates (which themselves are corollaries of the Cauchy integral formula, Proposition 2.3).

Now, using the majorization lemma, we can compare $\boxed{p,q}$ to

$$\boxed{\sum_{n\ge 0}|p_n|z^n, \sum_{n\ge 0}|q_n|z^n} \qquad \text{and} \qquad \boxed{\sum \frac{M}{r^n}z^n, \sum\frac{M}{r^n}z^n}.$$

It makes sense to compare this to $\boxed{M/(1-z/r), M/(1-z/r)^2}$, i.e. the equation

$$u'' + \frac{Mu'}{1-z/r} + \frac{Mu}{(1-z/r)^2} = 0.$$

[5] One can think of this in terms of the theory of covering spaces, which is one reason this function lifts.

[6] "If you're typing notes, feel free to call it something else, like $L_{p.q}$."

This last equation has an explicit solution $\mu/(1 - z/r)$ for some μ, and its Taylor series converges on $D(0, r)$; now, using the majorization lemma, the coefficients of our original series are smaller, and therefore it converges. ⊠

Thus, we have a 2-dimensional $\mathbb{C}$-vector space V of solutions near z_0. The tie-in to the rest of lecture is the following proposition/exercise.

Exercise. Show that if $p, q \in \mathscr{O}(G)$ and $\gamma : [0, 1] \to G$, then any solution to $\boxed{p, q}$ has a solution along γ through solutions to $\boxed{p, q}$.

Monodromy. If γ is now a loop in G, so $\gamma(0) = \gamma(1) = z_0$, then analytic continuation around γ defines a linear map $M_\gamma : V \to V$ called the *monodromy map*: you go around and end up not where you started, and it's easy to see that this dependence is linear.

Exercise. M_γ depends only on the homotopy class of γ (relative to basepoints).

Thus, this is only interesting if G isn't simply connected, so in general we get interesting examples of monodromy by going around poles of p and q. In particular, there's the oxymoronic-sounding notion of regular singular points. The prototype is the following, simpler equation:

$$u'' + \frac{A}{z}u' + \frac{B}{z^2}u = 0, \tag{5.1}$$

where $A, B \in \mathbb{C}$ are just constants. We seek solutions of the form $u(z) = z^\alpha$, where $\alpha \in \mathbb{C}$; this is defined initially near 1, and then analytically continued along paths in $\mathbb{C}^*$. If you write down the left-hand side, you end up getting

$$u'' + \frac{A}{z}u' + \frac{B}{z^2}u = \underbrace{(\alpha(\alpha - 1) + A\alpha + B)}_{I(\alpha)} z^{\alpha-2}.$$

In other words, to get a solution, we need $I(\alpha) = 0$; this is called the *indicial equation*. Since it's a quadratic, then there's one or two roots: if the roots α_1 and α_2 are distinct, then $(z^{\alpha_1}, z^{\alpha_2})$ is a basis for V (the solutions near 1), and if γ is the unit circle, then the monodromy matrix in this basis is

$$M_\gamma = \begin{bmatrix} e^{2\pi i\alpha_1} & 0 \\ 0 & e^{2\pi i\alpha_2} \end{bmatrix}. \tag{5.2}$$

If α is a related root, the basis we get is $(z^\alpha, z^\alpha \log z)$, and the monodromy matrix is a nontrivial Jordan block:

$$M_\gamma = e^{2\pi i\alpha} \begin{bmatrix} 1 & 2\pi i \\ 0 & 1 \end{bmatrix}.$$

One takeaway is that even an equation as simple as (5.1) has monodromy.

This generalizes quite naturally.

Definition. A $z_0 \in \mathbb{C}$ is a *regular singular point* of $u'' + pu' + qu = 0$ if p has a pole of order at most 1 and q has a pole of order at most 2 at z_0.

One seeks solutions via the *Frobenius method*: since p has a simple pole and q has a double pole, then there are $\widetilde{p}, \widetilde{q}$ holomorphic in a neighborhood of 0 such that $p(z) = A/z + \widetilde{p}(z)$ and $q(z) = B/z^2 + C/z + \widetilde{q}(z)$. Thus, the indicial equation is $\alpha(\alpha - 1) + A\alpha + B = 0$.

Proposition 5.6. *If there are indicial roots α_1 and α_2 such that $\alpha_1 - \alpha_2 \notin \mathbb{Z}$, then there are solutions $u_1, u_2 \in V$ such that $u_1 = z^{\alpha_1}w_1$ and $u_2 = z^{\alpha_2}w_2$, and the monodromy matrix is as in (5.2).*