GEOMETRY AND STRING THEORY SEMINAR: SPRING 2019

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These notes were taken in UT Austin's geometry and string theory seminar in Spring 2019. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

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1. Anomalies and extended conformal manifolds: 1/23/19

These are Arun's prepared notes for his talk, on the paper "Anomalies of duality groups and extended conformal manifolds" [STY18] by Seiberg, Tachikawa, and Yonekura.

1.1. **Generalities on anomalies.** As we've seen previously in this seminar, if you ask four people what an anomaly in QFT is, you'll probably get four different answers. Here are some of them.

- An anomaly means the action isn't invariant under the gauge group.
- An anomaly is an obstruction to coupling the theory to a background G-symmetry, or to gauging such a symmetry.
- An anomaly is realized in the nonvanishing of an anomaly polynomial.
- An anomaly as a *relative field theory* as advocated by Freed-Teleman [**?**]: within the framework of functorial QFT, consider an invertible $(n+1)$-dimensional QFT $\alpha\colon \mathsf{Bord}_n \to \mathsf{C}$; a QFT relative to α is a morphism $Z\colon \mathbf{1} \to \tau_{\le n}\alpha$ (i.e., truncate α). The upshot is that the partition function of on a closed n-manifold X isn't a number, but rather an element of the line $\alpha(X)$, and so on.
- Building on this is the idea that every quantum field theory has an anomaly, and if the anomaly is trivial, trivializing it is data that manifests in choices in studying the theory.

Seiberg, Tachikawa, and Yonekura introduce another perspective! But fortunately they relate it to most of the perspectives above. The idea is to consider a QFT with a parameter space $\mathcal{M}$, or a family of QFTs over $\mathcal{M}$. For example, you might have a parameter in a space $\widehat{\mathcal{M}}$ acted on by a group G, and then $\mathcal{M} = \widehat{\mathcal{M}}/G$. Fixing a spacetime manifold X, one expects the partition function to be a function on $\mathcal{M}$, but for an anomalous theory, this doesn't quite work (e.g. if the partition function as a function on $\widehat{\mathcal{M}}$ isn't G-invariant).

One way to fix this is to consider a space $\mathcal{F}$ of counterterms; the total space $\mathcal{N}$ will then be a fiber bundle over $\mathcal{M}$ with fiber $\mathcal{F}$. If constructed correctly, the partition function is then a function on $\mathcal{N}$, and the anomaly manifests in the fact that $\mathcal{N} \to \mathcal{M}$ isn't the trivial $\mathcal{F}$-bundle, and the function doesn't descend to $\mathcal{M}$. Alternatively, one can descend it as a section of a line bundle on $\mathcal{M}$ rather than a function.

The point is: these perspectives are all related. Today, we'll follow Seiberg-Tachikawa-Yonekura as they discuss an $\mathrm{SL}_2(\mathbb{Z})$-anomaly on 4D Maxwell theory from several of these perspectives.

1.2. **An anomalous $\mathrm{SL}_2(\mathbb{Z})$-symmetry in 4D Maxwell theory.** Now, we'll study these ideas as applied specifically to Maxwell theory in dimension 4. Throughout, we will restrict to spin 4-manifolds; everything still works when generalized to oriented manifolds, but the details are more complicated. Consult Seiberg-Tachikawa-Yonekura to learn what changes.

Maxwell theory is pure 4D U_1 gauge theory. The action on a 4-manifold X is

$$S = \frac{1}{g^2}\int_X F\wedge\star F + \frac{i\theta}{8\pi^2}\int_X F\wedge F, \tag{1.1}$$

where F is the curvature of the U_1 gauge field. Here g and θ are real-valued parameters, though θ is 2π-periodic. In dimension 4 only, we can rewrite this in terms of the self-dual and anti-self-dual pieces of F:

$$S = \frac{i\overline{\tau}}{4\pi}\int_X \|F_+\|^2 - \frac{i\tau}{4\pi}\int_X \|F_+\|^2, \tag{1.2}$$

where

$$\tau \coloneqq \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}. \tag{1.3}$$

That is, our single parameter τ is valued in $\mathbb{H}$, the upper half-plane.

1.2.1. *The anomaly as variance under a symmetry.* Now $\mathrm{SL}_2(\mathbb{Z}) = \langle S, T \mid S^4 = 1, (ST)^3 = S^2\rangle$ acts on $\mathbb{H}$ by $S\tau = -1/\tau$ and $T\tau = \tau + 1$. We'd like to quotient by this action and obtain a parameter space $\mathcal{M} \coloneqq \mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$. This action has stabilizer, though: at $\tau = i$, the stabilizer is $\mathbb{Z}/2$, and at $e^{i\pi/3}$, it's $\mathbb{Z}/3$. Thus it's helpful to think of $\mathcal{M}$ as the quotient *stack*, which means just that we remember the $\mathbb{Z}/2$ at i and the $\mathbb{Z}/3$ at $e^{i\pi/3}$.[1]

TODO: picture of the stack.

However, the action (1.2) is not invariant: Witten [Wit95] uses physics arguments to show that

$$Z_{T\cdot\tau}(X) = Z_\tau(X) \tag{1.4a}$$

$$Z_{S\cdot\tau}(X) = \tau^u\overline{\tau}^v Z_\tau(X), \tag{1.4b}$$

where $u = (1/4)(\chi(X) + \sigma(X))$ and $v = (1/4)(\chi(X) - \sigma(X))$.

Remark 1.5. Because Maxwell theory is a free QFT, making the argument rigorous is probably easier than for general QFTs (this does not mean "easy"!). ◂

In other words, the theory is anomalous for this $\mathrm{SL}_2(\mathbb{Z})$-action: on $\mathcal{M}$, the partition function isn't a well-defined function.

One might attempt to remedy this by throwing counterterms into the action. Our two options are the signature and the Euler characteristic, so such a counterterm would look like

$$f(\tau,\overline{\tau})\chi(X) + g(\tau,\overline{\tau})\sigma(X).\text{[2]} \tag{1.6}$$

This does not save us: consider $\tau = e^{i\pi/3}$, which has a $\mathbb{Z}/3$ stabilizer generated by ST^{-1}. This acts on the partition function by $e^{i\pi\sigma(X)/3}$, but does not change the counterterm (1.6), so this factor cannot be canceled. This is a nice application of the stacky perspective on $\mathcal{M}$.

However, a counterterm can simplify the anomaly. Letting $\eta\colon \mathbb{H} \to \mathbb{C}$ be the Dedekind η-function, $f(\tau,\overline{\tau}) \coloneqq \mathrm{Re}\log\eta(\tau)$ and $g(\tau,\overline{\tau}) \coloneqq i\,\mathrm{Im}\log\eta(\tau)$. The new partition function satisfies

$$Z'_\tau(X) = \eta(\tau)^{-(\chi(X)+\sigma(X))/2}\eta(-\overline{\tau})^{-(\chi(X)-\sigma(X))/2}Z_\tau(X), \tag{1.7}$$

and it transforms under the $\mathrm{SL}_2(\mathbb{Z})$-action as $Z'_{T\cdot\tau}(X) = Z'_\tau(X)$ and $Z'_{S\cdot\tau}(X) = \exp(-i\pi\sigma(X)/3)Z'_\tau(X)$.

This is the first description of this anomaly, as expressing how the partition function changes under $\mathrm{SL}_2(\mathbb{Z})$. It involves the signature, which is a "gravitational" term (meaning an invariant of the underlying manifold), and $\mathrm{SL}_2(\mathbb{Z})$, so it's a mixed anomaly. There could also be a pure $\mathrm{SL}_2(\mathbb{Z})$ anomaly, but to investigate it one should couple the theory to a background principal $\mathrm{SL}_2(\mathbb{Z})$-bundle, which the paper doesn't do.

1.2.2. *Extending the parameter space.* Next we'll describe a fiber bundle $\mathcal{N} \to \mathcal{M}$ such that the partition function is a function on $\mathcal{N}$, and interpret it as a section of a line bundle over $\mathcal{M}$.

The gravitational term $\theta_{\mathrm{grav}} \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$; we'll consider an extended space of parameters, namely $\mathbb{H} \times S^1$, and define an action of $\mathrm{SL}_2(\mathbb{Z})$ on both of them in a way which gets rid of the anomaly.

Definition 1.8. The character group of $\mathrm{SL}_2(\mathbb{Z})$ is cyclic of order 12: given a $k \in \mathbb{Z}/12$, define the character $\chi_k\colon \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{U}_1$ by $\chi_k(S) \coloneqq e^{-i\pi k/2}$ and $\chi_k(T) \coloneqq e^{-i\pi k/6}$.

[1] Really this is the quotient stack $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$; the reason we're using all of $\mathrm{SL}_2(\mathbb{Z})$ is that its center will act nontrivially later.

[2] The notation $(\tau,\overline{\tau})$ indicates these functions need not be holomorphic.

Of course, one should check these satisfy the relations of $\mathrm{SL}_2(\mathbb{Z})$, hence actually define a character. We'll single out χ_8: on T it's 1 and on S it's $e^{2i\pi/3}$, which looks a lot like the anomaly we saw above.

Now define the $\mathrm{SL}_2(\mathbb{Z})$-action on $\mathbb{H} \times S^1$ by

$$g \cdot (\tau, \theta_{\mathrm{grav}}) \coloneqq (g \cdot \tau, \chi_8(g) \cdot \theta_{\mathrm{grav}}) \tag{1.9}$$

(where U_1 acts on S^1 by the standard representation). The partition function is

$$Z'_{\tau,\theta_{\mathrm{grav}}}(X) = Z'_\tau(X) e^{i\theta_{\mathrm{grav}}\sigma/16}, \tag{1.10}$$

so if $g \in \mathrm{SL}_2(\mathbb{Z})$,

$$Z'_{g\cdot(\tau,\theta_{\mathrm{grav}})}(X) = Z'_{(\tau,\theta_{\mathrm{grav}})}(X); \tag{1.11}$$

the factors of $e^{i\pi/3}$ and $e^{2i\pi/3}$ cancel out. The partition function is invariant, so descends to a function on the *extended conformal manifold* $\mathcal{N} \coloneqq (\mathbb{H} \times S^1)/\mathrm{SL}_2(\mathbb{Z})$, which is an S^1-bundle over $\mathcal{M}$. This is a key idea of their paper: the partition function of an anomalous theory is only a function on this extended parameter space $\mathcal{N}$.

An alternative perspective is to use a line bundle to encode a twist. A function on $\mathcal{N}$ is a section of the trivial line bundle. We can ask whether it descends to $\mathcal{M}$, not necessarily as a function, but as a section of a line bundle. This is governed by *descent data*: rotating the S^1 defines a U_1-action on $\mathcal{N}$ whose quotient is $\mathcal{M}$. Then, an equivariant line bundle L' on $\mathcal{N}$ descends to a line bundle L on $\mathcal{M}$ (nonequivariant – in taking the quotient we've "used up" the equivariance), and an equivariant section of L' descends to a section of L.

So let's give the trivial line bundle $\underline{\mathbb{C}} \to \mathcal{N}$ a nontrivial U_1-action: given $(\tau, \theta_{\mathrm{grav}}, w)$ with $(\tau, \theta_{\mathrm{grav}}) \in \mathcal{N}$ and $w \in \underline{\mathbb{C}}_{(\tau,\theta_{\mathrm{grav}})}$, and given a $z \in \mathrm{U}_1$, define

$$z \cdot (\tau, \theta_{\mathrm{grav}}, w) \coloneqq \left(\tau, z \cdot \theta_{\mathrm{grav}}, \exp\left(\frac{\sigma(X)}{16}\right) zw\right). \tag{1.12}$$

This defines an equivariant line bundle $L' \to \mathcal{N}$ such that a function Z on $\mathcal{N}$ such that $Z(\tau, z \cdot \theta_{\mathrm{grav}}) = e^{\sigma(X)/16} Z(\tau, \theta_{\mathrm{grav}})$, such as the partition function, is an equivariant section of this line bundle. Descending, we obtain a nonequivariant line bundle $L \to \mathcal{M}$, and the partition function is a section.

Ok, so which line bundle do we get? We know the U_1-equivariant line bundle on $\mathcal{N}$ that we began with, hence a $\mathrm{U}_1 \times \mathrm{SL}_2(\mathbb{Z})$-bundle on $\mathbb{H} \times S^1$, and then we can quotient by U_1 to obtain an $\mathrm{SL}_2(\mathbb{Z})$-equivariant line bundle on $\mathbb{H}$, then quotient by $\mathrm{SL}_2(\mathbb{Z})$ to get back $L \to \mathcal{M}$. The point is, passing through $\mathbb{H} \twoheadrightarrow \mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ may be easier to think about, and we can compare to known line bundles.

Definition 1.13. The *Hodge bundle* $L_H \to \mathcal{M}$ is the quotient of the $\mathrm{SL}_2(\mathbb{Z})$-equivariant line bundle $L'_H \to \mathbb{H}$ which is nonequivariantly trivial and whose $\mathrm{SL}_2(\mathbb{Z})$-action is defined by $g \cdot (\tau, z) = (g \cdot \tau, \chi_1(g) \cdot z)$.

The 12^{th} tensor power of L_H is trivial, ultimately because the abelianization of $\mathrm{SL}_2(\mathbb{Z})$ is $\mathbb{Z}/12$.

If you run this argument, you get that the line bundle L arising from Maxwell theory on X is $L_H^{\otimes(\sigma/2)}$. The intuition is that we have $\exp(\sigma/16)$ in the U_1-action, and $\mathrm{SL}_2(\mathbb{Z})$ acts on U_1 by eight times the generator, giving us $\sigma/2$.

1.2.3. *Anomaly polynomials.* Seiberg-Tachikawa-Yonekura also discuss anomaly polynomials. In general, suppose we have a family of $2k$-dimensional manifolds $\mathcal{X} \to \mathcal{M}$, and write the fiber at $m \in \mathcal{M}$ as X_m.[3] The partition function $Z(X_m)$ is a section of a line bundle $L \to \mathcal{M}$. The goal of the anomaly polynomial is to determine L, or equivalently its first Chern class. Therefore the anomaly polynomial $\mathbb{A}_{2k+2} \in H^{2k+2}(\mathcal{X})$ is defined to satisfy

$$c_1(L) = \int_{X_m} \mathbb{A}_{2k+2}, \tag{1.14}$$

here denoting integration along the fiber, the pushforward map $H^*(\mathcal{X}) \to H^{*-2k}(\mathcal{M})$.[4]

Seiberg-Tachikawa-Yonekura compute $\mathbb{A}_6$ for Maxwell theory in an interesting way: they realize it as the dimensional reduction of a 6D theory Z_6 along a torus T. This theory also has an anomaly polynomial $\mathbb{A}_8 \in H^8(T \times \mathcal{X})$, and integrating over the fibers $T \times X_m$ produces $c_1(L)$ again.

[3]Despite the similar notation, this time X is varying and the QFT is constant; previously, it was the other way around.

[4]To do this, we need the relative tangent bundle of the map $\mathcal{X} \to \mathcal{M}$ to be oriented.

Therefore we can compute $c_1(L)$ in two ways: first integrating along the fiber of $T \times \mathcal{X} \to \mathcal{X}$, then $\mathcal{X} \to \mathcal{M}$ as above, or by first integrating along the fiber of $T \times \mathcal{X} \to T \times \mathcal{M}$, then $T \times \mathcal{M} \to \mathcal{M}$.

Since I don't have a whole lot of time, and because I didn't fully understand the arguments in this section, I'm going to skip over the computations, which is unfortunate, because they look mathematically interesting. The summary is that knowing the 6D anomaly polynomial, and knowing the anomaly polynomial of the 2D theory $Z_6(- \times X_m)$, allows one to pin down the anomaly polynomial of the 4D theory in terms of the central charge of the 2D theory. In the case of Maxwell theory, the central charge is $c = \sigma(X)$, and using arguments from 2D CFT that I could not follow, the corresponding line bundle $L \to \mathcal{M}$ is $L_H^{c/2}$, which agrees with what we saw above.

1.2.4. *Relative field theory.* This is the most topological perspective on anomalies, so I love it.

So with that in mind, we expect Z to really be a QFT relative to an invertible TFT α in dimension 5, and with the same background fields. That is, the symmetry type is

- spin 5-manifolds, since we began with spin 4-manifolds; together with
- an $\mathrm{SL}_2(\mathbb{Z})$-bundle, since we're considering an $\mathrm{SL}_2(\mathbb{Z})$ symmetry: even though we didn't couple to a background $\mathrm{SL}_2(\mathbb{Z})$-bundle, this $\mathrm{SL}_2(\mathbb{Z})$-symmetry still appears in the anomaly.

The anomaly theory α is topological, because it is a finite-order, unitary invertible field theory.[5] Therefore it cannot see τ, θ, or θ_{grav}, so this is the entire symmetry type; moreover, its partition function is a bordism invariant, an element of

$$\mathrm{Hom}(\Omega_5^{\mathrm{Spin}}(B\mathrm{SL}_2(\mathbb{Z})), \mathrm{U}_1). \tag{1.15}$$

Seiberg-Tachikawa-Yonekura [STY18] show this is abstractly isomorphic to $\mathbb{Z}/36$, but they don't produce an isomorphism, so it's difficult to get one's hands on this.

TODO: their results

Here are some other things we can say about the computation of α.

(1) First, we can only expect to know the answer modulo "pure $\mathrm{SL}_2(\mathbb{Z})$" theories; let's discuss what that means. The action of α can include terms which are characteristic classes for $\mathrm{SL}_2(\mathbb{Z})$-bundles as well as "gravitational" terms which depend on the underlying manifold itself. A theory whose action has no gravitational terms is called a pure $\mathrm{SL}_2(\mathbb{Z})$-theory.

We haven't studied Maxwell theory coupled to a background principal $\mathrm{SL}_2(\mathbb{Z})$-bundle, so we're not going to be able to distinguish any of the pure $\mathrm{SL}_2(\mathbb{Z})$-theories. However, these are a subgroup of the group of all 5D invertible TFTs with symmetry type $\mathrm{Spin} \times \mathrm{SL}_2(\mathbb{Z})$, so we can ask whether we can identify the anomaly in the quotient.

Seiberg-Tachikawa-Yonekura show that the pure $\mathrm{SL}_2(\mathbb{Z})$ theories form a $\mathbb{Z}/6$ inside this $\mathbb{Z}/36$. They make this argument using the Atiyah-Hirzebruch spectral sequence

$$E^2_{p,q} = H_p(B\mathrm{SL}_2(\mathbb{Z}); \Omega_q^{\mathrm{Spin}}(\mathrm{pt})) \Longrightarrow \Omega_{p+q}^{\mathrm{Spin}}(B\mathrm{SL}_2(\mathbb{Z})). \tag{1.16}$$

The pure $\mathrm{SL}_2(\mathbb{Z})$-theories are those on the line $q = 0$, so they only see the homology of $B\mathrm{SL}_2(\mathbb{Z})$ and not the spin bordism groups. The argument that $E^\infty_{5,0} = \mathbb{Z}/6$ is a fun but elaborate spectral sequence proof, chaining together three instances of the Atiyah-Hirzebruch spectral sequence and playing them off of each other.

Let A denote the quotient of $\mathrm{Hom}(\Omega_5^{\mathrm{Spin}}(B\mathrm{SL}_2(\mathbb{Z})), \mathrm{U}_1)$ by the pure $\mathrm{SL}_2(\mathbb{Z})$-theories, so that $A \cong \mathbb{Z}/6$. We've seen that when we don't couple to principal $\mathrm{SL}_2(\mathbb{Z})$-bundles, the anomaly has order 3; this means it has order 3 in A.

(2) If we had an explicit description of the elements of $\mathrm{Hom}(\Omega_5^{\mathrm{Spin}}(B\mathrm{SL}_2(\mathbb{Z})), \mathrm{U}_1)$, we could do more, using the anomaly TFT to determine information about the line bundle $L \to \mathcal{M}$. The idea of a QFT relative to some invertible TFT α is that the partition function of X is an element of the line $\alpha(X)$; applying this to the family of theories Z_τ in $\mathcal{M}$, $\alpha(X)$ defines a line bundle over $\mathcal{M}$, and the partition function is a section of this line bundle.

At $\tau = e^{i\pi/3}$, we have a $\mathbb{Z}/3$-symmetry on L_τ which we've determined just by studying Maxwell theory, and we can compare this with another $\mathbb{Z}/3$-symmetry on $\alpha(X)$ that can be determined just by studying α, and these must match, which could help determine α in general.

[5] This is a quite nontrivial theorem of Freed-Hopkins [?].

More explicitly, consider $\tau = e^{i\pi/3} \in \mathcal{M}$ with its $\mathbb{Z}/3$ stabilizer; this is a $\mathrm{pt}/(\mathbb{Z}/3)$. A strong form of $\mathbb{Z}/3$ symmetry is being able to extend to a family over this space, so let's try to do this. Take a spin 4-manifold X and the trivial $\mathrm{SL}_2(\mathbb{Z})$-bundle $P_{\mathrm{triv}} \to X$ and extend to (X, P) over $\mathrm{pt}/(\mathbb{Z}/3)$, where the monodromy of P around an $a \in \mathbb{Z}/3 \subset \mathrm{SL}_2(\mathbb{Z})$ is right multiplication by a^{-1} on $\mathrm{SL}_2(\mathbb{Z})$. We can then evaluate α on this family of manifolds, to produce a line bundle over $\mathrm{pt}/\mathbb{Z}/3$ (namely, a line with a $\mathbb{Z}/3$-action), and we can compare this with the fiber of L over $e^{i\pi/3}$ with its $\mathbb{Z}/3$-action. These should be isomorphic, and given a concrete description of these invertible TFTs, one could use this to learn more information about the anomaly of Maxwell theory even though we haven't coupled to $\mathrm{SL}_2(\mathbb{Z})$-bundles. We can also do this with the $\mathbb{Z}/2$ stabilizer at $\tau = i$.

References

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