M392C NOTES: APPLICATIONS OF QUANTUM FIELD THEORY TO GEOMETRY

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These notes were taken in UT Austin's M392C (Applications of Quantum Field Theory to Geometry) class in Fall 2017, taught by Andy Neitzke. I live-TEXed them using vim, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Thanks to Andy Neitzke for a few corrections.

Contents

Lecture 1.

Donaldson invariants and supersymmetric Yang-Mills theory: 8/31/17

"The wind blowing on it, well, that's not the worst thing that could happen to a pond! Now imagine you have a laser..."

The course website is https://www.ma.utexas.edu/users/neitzke/teaching/392C-qft-geometry/. There are also lecture notes which are hosted at https://github.com/neitzke/qft-geometry, and are currently a work in progress; if you have contributions or improvements, feel free to contribute them, as a pull request or otherwise. (I'm also taking notes, of course, and if you find problems or typos in my notes, feel free to let me know.) There's also a Slack channel for course-related discussions, which may be easier to use than office hours.

There will be exercises in this course, and you should do at least one-fourth of them for the best grade. Of course, you also want to do them in order to gain understanding. Some worked-out computations could be useful for submitting to the professor's lecture notes.

This course will be relatively wide-ranging; today's prerequisites involve some gauge theory, but the next few lectures won't as much.

Suppose you want to study the topology of smooth manifolds X. Surprisingly, it's really effective to introduce a geometrical gadget, e.g. a Riemannian metric g. Using it, we can define the *Laplace operator* on differential forms $\Delta\colon \Omega^k(X)\to\Omega^k(X)$, which has the formula

$$\Delta := \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d},$$

where $\mathrm{d}\colon \Omega^k(X)\to\Omega^{k+1}(X)$ is the de Rham differential, and $\mathrm{d}^*\colon \Omega^{k+1}(X)\to\Omega^k(X)$ is its adjoint in the L^2-inner product on differential forms induced by the metric. Thus d is canonical, but d^* depends on the choice of metric.

Next we consider the equation

$$\Delta\omega = 0. \tag{1.1}$$

This is a linear equation, so its space of solutions $\mathscr{H}_{k,g} := \ker(\Delta\colon \Omega^k\to\Omega^k)$, called the *space of harmonic k-forms*, is a vector space. If X is compact, it's even a finite-dimensional vector space, which is a consequence of the ellipticity of the Laplace operator.[1] Hence we can define a nonnegative integer

$$b_k(X) := \dim \mathscr{H}_{k,g},$$

called the k^{th} *Betti number* of X It's a fact that $b_k(X)$ does not depend on the choice of the metric! Thus they are invariants of the smooth manifold X.

[1] For a general differential operator on differential forms, nothing like this is true.

In fact, there's even a categorified version of this. This reflects a recent (last decade or so) trend of replacing numbers with vector spaces, sets with categories, etc.

Theorem 1.2. *If X is compact,*[2] *there is a canonical isomorphism $\mathscr{H}_{k,g} \cong H^k(X;\mathbb{R})$, where the latter is the singular cohomology of X with coefficients in $\mathbb{R}$.*

This shows $b_k(X)$ doesn't depend on the smooth structure of X, and is even a homotopy invariant. This will not be true for the Donaldson invariants that we'll discuss later.

Exercise 1.3. Work out some of these spaces of harmonic forms for a metric on S^1 and S^2.

You have to choose a metric, and there are more or less convenient ones to pick. But no matter how you change the metric, there will be a canonical way to identify them.[3]

If X is oriented and $4n$-dimensional, there's a small refinement of the middle Betti number b_{2n} and space of harmonic forms $\mathscr{H}_{2n}$. The *Hodge star operator*

$$\star\colon \Omega^p(X) \longrightarrow \Omega^{\dim X - p}(X)$$

is an involution on $\Omega^{2n}(X)$.

Remark. Let's recall the Hodge star operator. This is an operator on differential forms defined using the Riemannian metric satisfying $\star^2 = 1$ in even dimension, and $[\star, \Delta] = 0$. Hence it acts on harmonic forms. On $\mathbb{R}^2$ with the usual metric, $\star(1) = \mathrm{d}x \wedge \mathrm{d}y$, and $\star(f\,\mathrm{d}x) = f\,\mathrm{d}y$. ◁

Hence we can decompose $\Omega^{2n}(X)$ into the (± 1)-eigenspaces of $\star$: let $\Omega^{2n,\pm}(X)$ denote the ± 1-eigenspace for $\star$. Similarly, $\mathscr{H}_{2n}(X)$ splits into $\mathscr{H}^{\pm}_{2n}(X)$. Thus b_{2n} also splits:

$$b_{2n}(X) = b^+_{2n}(X) + b^-_{2n}(X).$$

These spaces and numbers are also topological invariants, and can be understood in that way.

Exercise 1.4. In dimension $4n+2$, the Hodge star squares to -1. You can still extract topological information from this; what do you get?

Linear equations seem to behave more or less the same in all dimensions. But nonlinear equations behave very differently in different dimensions. In the 1980s, Donaldson used nonlinear equations to produce new and interesting invariants of 4-manifolds. Let X be a connected, oriented 4-manifold with a Riemannian metric g.

Fix a compact Lie group G. For Donaldson, $G = \mathrm{SU}(2)$, and it's probably fine to assume that for much of this class. Fix a principal G-bundle $P \to X$. We'll consider connections on P.

Remark. If you don't know what a connection is, that's OK. Locally, a connection on P is represented by a Lie algebra-valued 1-form $A \in \Omega^1_X(\mathfrak{g})$, and has a *curvature 2-form* $F \in \Omega^2_X(\mathfrak{g}_P)$, which locally is written

$$F = \mathrm{d}A + A \wedge A.$$

Because SU(2) is nonabelian, $A \wedge A$ isn't automatically zero. ◁

Since F is a 2-form and $\dim X = 4$, we can decompose F into its *self-dual part* F^+ and its *anti-self-dual part* F^-, defined by the splitting of Ω^p by the Hodge star.

Exercise 1.5. Show that if you reverse the orientation of X, F^+ and F^- switch.

Donaldson studied the *anti-self-dual Yang-Mills equation* (ASD YM):

$$F^+ = 0. \tag{1.6}$$

By Exercise 1.5, this is not really different than studing the self-dual Yang-Mills equation; the reason one prefers the ASD version is that it occurs more naturally on certain complex manifolds which were test cases for Donaldson theory.

If G is abelian, e.g. U(1), (1.6) is linear. But if G is nonabelian, e.g. SU(2), then (1.6) is nonlinear.

Definition 1.7. The *instanton moduli space* is the space $\mathscr{M}$ of equations on P obeying (1.6), modulo the action of the *gauge group* $\mathscr{G}$, the bundle automorphisms of P.[4]

[2] Compactness is really necessary for this.

[3] Interesting question: if you change the metric infinitesimally, how does $\mathscr{H}_k$ change?

[4] TODO: not sure if I got this right.

Exercise 1.8. Show that if $G = \mathrm{U}(1)$, then $\mathcal{M} = \mathcal{H}_2^-(X)$.

So in this case we don't find anything new, though the way we found it is still interesting.

When G is nonabelian, this is not a vector space. It still has some reasonable structure. We now fix $G = \mathrm{SU}(2)$. In this case, (topological) isomorphism classes of principal SU(2)-bundles are classified by the integers, given by the formula

$$k := \int_X c_2(P) \in \mathbb{Z},$$

where c_2 denotes the second Chern class.

This means the moduli of instantons is a disjoint union over $\mathbb{Z}$ of spaces $\mathcal{M}_k$.

Theorem 1.9. *If $k > 0$ and g is chosen generically, $\mathcal{M}_k$ is a finite-dimensional manifold.*

Hence one could learn topological information about X by studing topological properties of $\mathcal{M}_k$. The first idea would be the Betti numbers, but these turn out not to depend on the smooth structure.

Proposition 1.10. *Assuming $k > 0$ and g is generic,*

$$\dim \mathcal{M}_k = 8k - 3(1 - b_1(X) + b_2^+(X)).$$

But there's more to $\mathcal{M}_k$ than the dimension. Donaldson introduced an orientation on $\mathcal{M}_k$, which is canonically defined (and a lot of hard work!), and one can produce classs $\tau_\alpha \in \Omega^*(\mathcal{M}_k)$ labeled by classes $\alpha \in H_*(X)$. Using these, the *Donaldson invariants* are the real numbers

$$\langle \mathcal{O}_{\alpha_1} \cdots \mathcal{O}_{\alpha_\ell} \rangle := \int_{\mathcal{M}} \tau_{\alpha_1} \wedge \cdots \wedge \tau_{\alpha_\ell} \in \mathbb{R}. \tag{1.11}$$

Theorem 1.12. *If $b_2^+(X) > 1$, the Donaldson invariants are independent of g.*

Moreover, they really depend on smooth information: it's not possible to reconstruct them out of algebraic or differential topology, unlike the Betti numbers. So these are very powerful. Their study is called *Donaldson theory*. One good refernece is Donaldson and Kronheimer's book.

Unfortunately, Donaldson theory is technically very hard: the ASD YM equation is hard to study: $\mathcal{M}_k$ is usually noncompact, and (1.11) is an integral over a noncompact space, which is no fun.

What does this have to do with quantum field theory? In 1988, Witten, following a suggestion of Atiyah, found an interpretation of the Donaldson invariants in terms of quantum field theory (hence the suggestive notation in (1.11)).

There are many different quantum field theories: the Standard Model describes three of the four fundamental forces of the universe; quantum electrodynamics describes electromagnetism. Witten interpreted the Donaldson invariants in terms of a specific QFT, called "(a topological twist of) $\mathcal{N} = 2$ supersymmetric Yang-Mills theory (SYM) with gauge group SU(2)."

One imagines X to be a "spacetime" or "universe" whose laws of physics are governed by $\mathcal{N} = 2$ supersymmetric Yang-Mills theory, and to compute the Donaldson invariants, one conducts "experimental measurements" (correlation functions). According to the rules of Lagrangian quantum field theory, this means computing an integral over an infinite-dimensional space (which is alarming, but so it goes):

$$\langle \mathcal{O}_\alpha \rangle = \int_{\mathcal{C}} \mathrm{d}\mu\, \Phi_\alpha e^{-S},$$

where

- $\mathcal{C}$ is the *space of fields*, some sort of infinite-dimensional space akin to the space of functions on X or forms on X,
- $S\colon \mathcal{C} \to \mathbb{R}$ is a functional called the *action*,
- $\Phi_\alpha\colon \mathcal{C} \to \mathbb{R}$ is a (set of) *observables*,
- and $\mathrm{d}\mu$ is some measure on $\mathcal{C}$.

In general, computing these correlation functions are very hard,[5] but in $\mathcal{N} = 2$ SYM, Witten found localization, a way to reduce it to Donaldson's integrals over finite-dimensional spaces.

[5]Unless $\dim X = 0$, where $\mathcal{C}$ is finite-dimensional. We'll talk about this in the next few lectures.

This is undoubtedly cool, and brings geometric topology into quantum field theory, but it does not make it much easier to actually compute Donaldson invariants.

The next step was taken in 1995, by Seiberg and Witten, who were interested in a different but related physics problem. They answered a fundamental question about SYM: how it behaves at low energies.

To make an analogy, suppose you have a pond, and you're pond-ering what happens when wind goes across the surface. You're good at physics, so you model the pond as a system of 10^{30} molecules of water and other things, then rent some time on a supercomputer where you model the action on the wind and… somehow this seems wrong. Instead, you model the water and the wind using things like the Navier-Stokes equations. This is not easy, but it's much, much easier.

The idea is there's a "high-energy" description, in terms of 10^{30} particles, but the "low-energy" description[6] involves things like temperature, pressure, liquid, and other things that are hard to define from the high-energy approach. The low-energy picture is very useful for calculations, though if you fire a laser into your pond it wouldn't suffice. Obtaining the description of the low-energy physics from the high-energy physics is typically very hard; in this case, one would have to define temperature and pressure and a lot of things starting from fundamentals. But you just have to do it once, then can apply it to all bodies of water, etc.

Seiberg and Witten applied this to $\mathscr{N} = 2$ SYM with gauge group SU(2), and showed that its low-energy description is (roughly) $\mathscr{N} = 2$ SYM with gauge group U(1), coupled to matter (sometimes called monopoles). Since the gauge group is abelian, this is much easier. Now, one can imagine that there's an easier description of the Donaldson invariants in terms of the low-energy theory (though, again, this was not the original intent of Seiberg and Witten), and this is given by the *Seiberg-Witten equations*. They look more complicated but are actually vastly simpler.[7]

In the Seiberg-Witten equations, the fields are

- a connection Θ in a U(1)-bundle $\mathscr{E}$, or equivalently a determinant line of a spin^c-structure, and
- a section ψ of S^+, a spinor bundle associated to a spin^c-structure.

In this case, there's a *Dirac operator* $\not{D}$ and a pairing

$$q\colon S^+ \otimes S^+ \longrightarrow \Lambda^2_+ T^*X.$$

Then, the Seiberg-Witten equations are

$$F^+ = q(\psi, \overline{\psi}) \tag{1.13a}$$

$$\not{D}\psi = 0. \tag{1.13b}$$

Let $\widetilde{\mathscr{M}}$ denote the moduli space of pairs (Θ, ψ) satisfying (1.13) modulo the action of some group. For generic g, this is a compact manifold, so understanding its topology is much easier, and the correlation functions for the low-energy theory can be written as integrals over $\widetilde{\mathscr{M}}$, and there's a simple formula relating these to the correlation functions for the high-energy theory. Once this was realized, there was very rapid progress of its use in applications, though understanding precisely why it's the same came more slowly, beginning from a physical argument by Moore and Witten and proceeding to a very different-looking mathematical proof much more recently.

This is an application of QFT to geometry, as we will study in this course. Somehow the most powerful applications involve taking a low-energy limit, and many of them also involve localization in supersymmetric QFT (from an infinite-dimensional integral to a finite-dimensional one).

We will start more slowly: first considering QFT where $\dim X = 0$, then $\dim X = 1$ (which is quantum mechanics); in these cases, the physics can be made completely rigorous (though it's not necessarily easy). We'll briefly talk about $\dim X = 2$, then jump into $\dim X = 4$.

[6] The term "low-energy," despite sounding pejorative, is actually a very useful thing to have.

[7] For a reference, check Morgan's book on the subject.