M392C NOTES: INDEX THEORY

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These notes were taken in UT Austin's M392C (Index theory) class in Spring 2018, taught by Dan Freed. I live-TeXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Any mistakes in the notes are my own. Thanks to Rok Gregoric for fixing a few errors, and to Riccardo Pedrotti for providing the notes for §3.

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Lecture 1.

Overview, History, and some Linear Algebra: 1/17/18

"This formula should look fake if you haven't seen it before."

We'll start with and overview and some history of index theory. The overview will use a little bit of complex geometry, but if you don't know it that's okay; the rest of the class will not depend on it.

One of the earliest manifestations of index theory was in the theory of algebraic curves. Let M be a compact smooth connected complex curve, i.e. a Riemann surface, and let D be a divisor on M, a finite formal sum of points of M with integer coefficients. For example, if $p_1, p_2, p_3 \in M$, one divisor is $4p_1 - 2p_2 + 7p_3$.

Definition 1.1. Let f be a meromorphic function on M; then, its divisor $\operatorname{div}(f)$ is the zeros of f minus the poles of f, where both are counted with multiplicity. For $f = 0$, we let $\operatorname{div}(0) = 0$.

For example, if $M = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, then a meromorphic function on M is a rational function. If we took $f(z) = (z-1)^2/(z+2)$, then $\operatorname{div}(f) = 2 \cdot 1 - 1 \cdot (-2) - 1 \cdot \infty$: f has a double zero at 1 and a single pole at -2, and at ∞ there is a simple pole.[1]

A divisor has a *degree* which is the sum of its terms.

Theorem 1.2. *The degree of the divisor of a meromorphic function is zero.*

[1] To see this, use the change-of-variables $z = 1/w$ and evaluate f at $w = 0$.

This is a consequence of the Cauchy integral formula.

A divisor specifies the zeros and poles of a meromorphic function, and it's a classical problem to, given a degree-zero divisor D on a Riemann surface, construct a function whose divisor is D. More generally, let $\mathscr{L}(D)$ denote the set of meromorphic f such that $\operatorname{div}(f)+D \geq 0$.[2] $\mathscr{L}(D)$ is a vector space, and if $\deg(D) < 0$, $\mathscr{L}(D) = 0$; we also have $\mathscr{L}(0) = \mathbb{C}$, given by constant functions.

Another classical question is to compute $\dim \mathscr{L}(D)$. Riemann provided an estimate:

$$\dim \mathscr{L}(D) \geq 1 - g + \deg(D), \tag{1.3}$$

where g is the *genus* of M, defined to be

$$g := \frac{1}{2} \operatorname{rank} H_1(X). \tag{1.4}$$

The next natural question is to identify the discrepancy, and Riemann's student Roch found the answer.

Theorem 1.5 (Riemann-Roch)**.** *here is a canonical divisor K_M such that*

$$\dim \mathscr{L}(D) - \dim \mathscr{L}(K_X - D) = 1 - g + \deg D. \tag{1.6}$$

We won't say much about K_M, though $\deg(K_M) = 2g - 2$.

Corollary 1.7. *The genus is an integer.*

A more modern interpretation of this story is that D determines a holomorphic line bundle $L \to M$, and $\mathscr{L}(D)$ is the vector space of holomorphic sections of L, i.e. $\mathscr{L}(D) \cong H^0(M;L)$. If s is any smooth section of L, s is holomorphic iff $\overline{\partial} s = 0$. That is, in local coordinates $z = x + iy$, and

$$\overline{\partial} = \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right). \tag{1.8}$$

Thus, $\overline{\partial} x = 0$ is a first-order differential equation, and computing $\dim \mathscr{L}(D)$ is asking for the dimension of the space of solutions to the equation. Thus one way you might prove Theorem 1.5 is to analyze the differential operator $\overline{\partial}$, which is a linear operator

$$\overline{\partial} \colon \Omega^{0,0}(M;L) \longrightarrow \Omega^{0,1}(M;L).$$

Then, $\mathscr{L}(D) = \ker(\overline{\partial})$ and $\mathscr{L}(K_M - D) \cong \operatorname{coker}(\overline{\partial})$.

Definition 1.9. The *index* of $\overline{\partial}$ is $\operatorname{ind}(\overline{\partial}) := \dim\ker(\overline{\partial}) - \dim\operatorname{coker}(\overline{\partial})$.

Broadly speaking, this course will be about indices of this sort, and their applications: for example, the Riemann-Roch theorem from this perspective is about computing the index of $\overline{\partial}$.

For a simpler case, let V and W be finite-dimensional vector space and $T \colon V \to W$ be a linear map. Then, $\ker(T) \subset V$ and $\operatorname{coker} T := W/T(V)$. Computing the index is a fundamental theorem in linear algebra.

Theorem 1.10.

$$\operatorname{ind}(T) := \dim(\ker T) - \dim(\operatorname{coker} T) = \dim V - \dim W.$$

In particular, it's independent of T! One way you might prove this is to observe that it's true when $T = 0$ and then try to prove that it's locally constant.

In this class, we're interested in operators between infinite-dimensional vector spaces, such as $\Omega^{p,q}(M;L)$, whose kernels and cokernels are finite-dimensional (such that the definition of an index makes sense). There will be no nice formula like Theorem 1.10, but some aspects stay the same: though the dimension of the kernel or cokernel may jump along a continuous path, their difference is constant.

Another classical subject that relates to index theory is that of the Euler number of a compact smooth n-manifold M. Betti defined *Betti numbers* $b_0, \dots, b_n$ associated to M, and Noether realized they can be identified with ranks of abelian groups (or dimensions of certain real vector spaces).[3]

[2] This is missing a zero element, so one needs to adjoint 0 for everything to work.

[3] These days, this would be called *categorification*: it can often be useful to identify a number as the dimension of some vector space attached to your object.

Definition 1.11. The *Euler characteristic* of M is

$$\chi(M) := \sum_{i=0}^{n} (-1)^i b_i.$$

The Betti numbers are defined via simplices, and how M is built out of cells. Since M is a smooth manifold, one might want to compute them in another way, using the smooth structure of the manifold. To do this, one introduces the *de Rham complex*

$$0 \longrightarrow \Omega^0(M) \xrightarrow{\ \mathrm{d}\ } \Omega^1(M) \xrightarrow{\ \mathrm{d}\ } \Omega^2(M) \longrightarrow \cdots \longrightarrow \Omega^n(M) \longrightarrow 0, \tag{1.12}$$

with linear maps d such that $\mathrm{d}^2 = 0$. Unlike in the previous example, this is built out of real functions and real differential forms.

Definition 1.13. The *de Rham cohomology* of M is the sequence of real vector spaces

$$H^i_{\mathrm{dR}}(M) := \frac{\ker(\mathrm{d}\colon \Omega^i(M) \to \Omega^{i+1}(M))}{\mathrm{Im}(\mathrm{d}\colon \Omega^{i-1}(M) \to \Omega^i(M))}.$$

Theorem 1.14 (de Rham)**.** *There is an isomorphism $H^i_{\mathrm{dR}}(M) \cong H^i(M;\mathbb{R})$, and therefore $\dim H^i_{\mathrm{dR}}(M) = b_i$.*

From this perspective, the Euler characteristic looks more like an index, where we stack together the pieces of the de Rham complex:

$$\bigoplus_{i \text{ even}} \Omega^i(M) \longrightarrow \bigoplus_{i \text{ odd}} \Omega^i(M). \tag{1.15}$$

However, the index of this is *not* the Euler characteristic! The issue is that the de Rham cohomology groups are a subquotient, not just a subspace or just a quotient. To compute the Euler characteristic as an index, we'll need some way of turning them into pure subspaces or quotients. One way to do this is to use an inner product and take orthogonal complements.

Let M be a Riemannian manifold. Then, there is a Laplace operator $\Delta\colon \Omega^i(M) \to \Omega^i(M)$, which is a linear second-order elliptic differential operator.

Remark. There are three basic kinds of differential operators studied in a typical differential equations course: elliptic, parabolic, and hyperbolic. The Laplacian is the basic example of an elliptic operator; the heat operator is the basic example of a parabolic operator; and the Schrödinger operator is the basic example of a hyperbolic operator. We will focus on elliptic operators in this course, but both the heat equation and the Schrödinger equation will appear. ◀

Example 1.16. Let $\mathbb{E}^n$ denote n-dimensional Euclidean space with coordinates $x^1, \dots, x^n$. Then, the Laplacian on $\mathbb{E}^n$ is

$$\Delta = \left(\frac{\partial}{\partial x^1}\right)^2 + \cdots + \left(\frac{\partial}{\partial x^n}\right)^2.$$

◀

For more general Riemannian manifolds, the definition of the Laplacian is more complicated, but not much more so.

Definition 1.17. If M is a Riemannian manifold, there is an L^2 inner product on $\Omega^i(M)$ defined by

$$\langle \alpha, \beta \rangle_{L^2} := \int_M \langle \alpha(M), \beta(M) \rangle \, \mathrm{dvol}_m.$$

Using these inner products, we can let $\mathrm{d}^*\colon \Omega^{i+1}(M) \to \Omega^i(M)$ be the formal adjoint to d.

Fact. d^* exists and is a first-order differential operator. ◀

Definition 1.18. The *Laplace operator* on M is $\Delta := \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$.

A form in the kernel of Δ is called *harmonic*, and the space of harmonic forms is denoted $\mathscr{H}^i(M) \subset \Delta^i(M)$.

Theorem 1.19 (Hodge theorem)**.** *The natural map $\mathscr{H}^i(M) \to H^i_{\mathrm{dR}}(M)$ is an isomorphism. In particular,* $\dim \mathscr{H}^i(M) = b_i$.

This is how index theory enters the picture: if we can access the space of harmonic forms as kernels and cokenels of operators, we could compute the Euler characteristic as an index. And indeed, we can fix (1.15) as folows:

$$\bigoplus_{i \text{ even}} \Omega^i(M) \xrightarrow{\mathrm{d}+\mathrm{d}^*} \bigoplus_{i \text{ odd}} \Omega^i(M). \tag{1.20}$$

The index of this operator is the Euler characteristic.

A third example of index theory is the higher-dimensional Riemann-Roch theorem. Let M be a compact complex manifold; then, the $\overline{\partial}$ operator defines a *Dolbeault complex* analogous to the de Rham complex. If M is 2-(complex-)dimensional, the Euler characteristic satisfies a formula

$$\chi(M) = \frac{1}{12}\big(c_1^2(M) + c_2(M)\big)[M]. \tag{1.21}$$

Here c_1 and c_2 are examples of *characteristic classes*, which we'll start on in the next few lectures. In particular, the right-hand side is an integer. In higher dimensions, there are similar expressions with larger denominators and more characteristic classes.

These were studied by Todd and his student Egger, by Weyl, and others. But the general forms remained conjectures until 1954, when Hirzebruch proved these generalizations of the Riemann-Roch theorem, and an additional, similar result called the signature theorem. He wove together two very new pieces of mathematics: the cobordism theory of René Thom (published only earlier that year!) and the theory of sheaves.

Hirzebruch and others in this field introduced a rational combination of different characteristic numbers, called *Pontrjagin numbers*, called the *$\widehat{A}$-genus* (said "A-hat genus"). This is defined on closed oriented manifolds, and on a spin manifold is an integer.

That $\widehat{A}(M)$ is an integer is a suggestion that it's a dimension of something, and when Singer went to visit Oxford in 1963, Atiyah asked him what object has the $\widehat{A}$-genus as its dimension, and this is the problem that they solved: they constructed a differential operator called the Dirac operator on a spin manifold, and showed that its index is the $\widehat{A}$-genus.

The Dirac operator

$$D := \gamma^\mu \frac{\partial}{\partial x^\mu}$$

for some γ^μ (this notation means the index μ is implicitly summed over) is a first-order linear differential operator. We'd like this to be a square root of the Laplacian operator.

Exercise 1.22. Show that $D^2 = \Delta$ iff

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2\delta^{\mu\nu}. \tag{1.23}$$

Here, $\delta^{\mu\nu}$ means 1 if $\mu = \nu$ and 0 otherwise.

So the operator has to satisfy n^2 equations. If you try to solve this for functions on $\mathbb{E}^n$, you can show that no such γ^μ exist, but one could instead ask for vector-valued functions which satisfy (1.23), and indeed we will spend some time studying the abstract theory of matrices which satisfy this condition, rephrased as the algebraic theory of Clifford modules. In particular, we will be able to show that a spin structure is precisely what one needs to be able to construct the Dirac operator on a Riemannian manifold.

Before Atiyah and Singer told this story, Grothendieck took the Hirzebruch-Riemann-Roch theorem and generalized it still further, and Atiyah and Hirzenbruch saw how to translate his ideas from algebraic geometry to topology, and replace sheaves with vector bundles. They then defined K-theory and rapidly developed it from 1958 to 1962. When Atiyah asked Singer his question, it was in this context.

At the same time, parallel work was undertaken in the Soviet Union under Gelfand and his students. He observed that the index sometimes can be computed topologically, and asked whether this is true in general, and Atiyah-Singer's answer also incorporates this question.

Subsequently, in the 1970s, Gilkey, Patodi, and others were able to provide more rigid, simpler proofs with analytic methods, and in the 1980s Getzler made another important simplifying step to what's now called the heat equation proof of the index theorem, which we'll follow.

We'll use John Roe's book in this course. It's analytic in flavor, but also treats many other nice results, and if we go quickly enough, we'll get to see some of them, including Witten's physical treatment of Morse theory, the Lefschetz theorem, the Hodge theorem, and more.

In this class, the students will give lectures, two each week, and we hope to go through two chapters a week. You don't have to use all three hours!

On the course website (https://www.ma.utexas.edu/users/dafr/M392C/), there will be some useful information, including some old course notes, some historical background, and more to come. These will be there so that you do not forget the beauty of the material amongst all the details in the lectures.

Not everybody may know all of the prerequisites for this course, since it draws in lots of different parts of mathematics. One can ask the professor for references or talk to other students in the course.

The second half of the first day is on the first chapter of the book, reviewing some of the basics of Riemannian geometry.

Let's first start with some linear algebra and differential forms. Let V be an n-dimensional real vector space. Eventually, V will be a tangent space at a point to a manifold, and if the manifold has a Riemannian metric, V picks up an inner product.

Associated to V are several canonical vector spaces built from it: its wedge powers $\Lambda^2 V, \dots, \Lambda^n V$, and $\Lambda^0 V$, which is canonically $\mathbb{R}$. The top exterior power is also called the *determinant line*, $\operatorname{Det} V := \Lambda^n V$. Dually, there are the exterior powers of the dual space V^*: $\mathbb{R}$, V^*, $\Lambda^2 V^*, \dots, \operatorname{Det} V^*$.

An inner product on V canonically induces inner products on all of these exterior powers. One way to see this is to let $e_1, \dots, e_n$ be an orthonormal basis of V; then, there is a dual basis $e^1, \dots, e^n$ of V^*, defined by the relation

$$e^\mu(e_\nu) = \delta^\mu_\nu, \tag{1.24}$$

i.e. 1 if $\mu = \nu$ and 0 otherwise.

We specify the inner product on V^* by declaring this dual basis orthonormal, which suffices, though you have to check that if you change the orthonormal basis of V you started with, you'll end up with the same inner product nontheless.

We also obtain bases for the exterior powers of V and V^*: for $\Lambda^q V$, the basis is

$$\{e_{i_1} \wedge \cdots \wedge e_{i_q} : 1 \le i_1 < \cdots < i_q \le n\}, \tag{1.25}$$

and for $\Lambda^q V^*$, it's

$$\{e^{i_1} \wedge \cdots \wedge e^{i_q} : 1 \le i_1 < \cdots < i_q \le n\}. \tag{1.26}$$

Again we define the inner products on $\Lambda^q V$ and $\Lambda^q V^*$ by asking for these bases to be orthonormal, and again the inner product in question does not depend on the specific choice of orthonormal basis of V.

Definition 1.27. An *orientation* of V is an orientation of its determinant line. That is, $\operatorname{Det} V \setminus 0$ has two components, and an orientation is a choice of one of them.

Given n vectors $e_1, \dots, e_n \in V$, we can wedge them together to an $e_1 \wedge \cdots \wedge e_n \in \operatorname{Det} V$; $\{e_1, \dots, e_n\}$ is a basis iff $e_1 \wedge \cdots \wedge e_n \neq 0$. Thus a basis singles out one of the two rays in $\operatorname{Det} V \setminus 0$, hence defines an orientation. Since $(\operatorname{Det} V)^* = \operatorname{Det}(V^*)$ canonically, then this also defines an orientation on $(\operatorname{Det} V)^*$: the duality pairing implies there's a single $\theta \in \operatorname{Det} V^*$ which sends $e_1 \wedge \cdots \wedge e_n \mapsto 1$; we call it the *volume form* and denote it vol.

On an oriented Riemannian n-manifold, this is a differential n-form, hence can integrate it to determine the volume of the manifold. If it's not oriented, there are two at each point, which may twist globally into something called a density. Nonetheless, this can be integrated, and the volume of, e.g. $\mathbb{RP}^2$ still makes sense.

The pairing $\Lambda^q V^* \otimes \Lambda^{n-q} V^* \to \operatorname{Det} V^*$ defined by

$$\alpha, \beta \longmapsto \alpha \wedge \beta \tag{1.28}$$

is nondegenerate. An orientation of V defines a trivialization of $\operatorname{Det} V^*$ (where vol $= 1$), so this pairing is $\mathbb{R}$-valued. Therefore we obtain an isomorphism $\Lambda^q V^* \cong \Lambda^{n-q}(V)$, though it depends on the inner product and the orientation.

Example 1.29. In three dimensions, we use this frequently, to shift from the perspective of vector fields and scalars and div, ∇, and curl to differential forms. ◀

There's also an isomorphism $\star\colon \Lambda^q V^* \to \Lambda^{n-q} V^*$ which only uses the inner product; this is called the *Hodge star*. Putting everything together, the Hodge star is defined uniquely by the stipulation that

$$\alpha_1 \wedge \star\alpha_2 = \langle \alpha_1, \alpha_2\rangle \mathrm{vol} \tag{1.30}$$

for any $\alpha_1, \alpha_2 \in \Lambda^q V^*$.

Exercise 1.31. For example, check that $\star(e_{i_1} \wedge \cdots \wedge e_{i_q})$ is the wedge of all of the e_j not in $(i_1, \dots, i_q)$, possibly multiplied by -1.

Exercise 1.32. Show that $\star^2 = (-1)^{q(n-q)}$.

Here, "inner product" means nondegenerate inner product; much of this story still goes through for a Lorentz-signature metric, but not all of it.

Exercise 1.33. Show that on a closed, oriented Riemannian manifold M, $\mathrm{d}^* = \pm \star \mathrm{d}\star$, and determine the sign (which depends on n and q).

You can type-check that the right-hand side is a first-order differential operator which lowers the degree by 1. Solving the exercise boils down to checking that

$$\int_M \langle \mathrm{d}\alpha, \beta\rangle \,\mathrm{vol} = \pm \int_M \langle \alpha, \star\mathrm{d}\star\beta\rangle \,\mathrm{vol}.$$

You'll end up using Stokes' theorem.

Now let's think about parallelism. Let $\mathbb{A}^n$ be n-dimensional affine space (no distinguished origin), where we learn calculus. This has parallel transport: if $\xi \in \mathbb{R}^n$ is a tangent vector at some point, we can translate it everywhere to a vector field. This allows us to define differentiation: if $f : U \to \mathbb{R}$, where $U \subset \mathbb{A}^n$ is open, then we define the derivative of f at p in the direction of ξ to be

$$\xi_p f := \lim_{h\to 0} \frac{f(p + h\xi) - f(p)}{h}. \tag{1.34}$$

This uses parallelism in the expression $p + h\xi$.

More generally, if M is a smooth manifold, we don't always have a canonical parallel transport between tangent spaces for different points of the manifold, so we can't compare tangent vectors in different places and differentiate.

For example, if $\gamma\colon [a,b] \to M$ is a curve, its tangent vectors at two different points can't be compared (without extra structure), so there's no way to make the subtraction in (1.34). We'll introduce the structure that allows us to do this.

Definition 1.35. Let $V \to M$ be a vector bundle and $C^\infty(M;V)$ denote its space of smooth sections, which is a real vector space. A *covariant derivative* is a bilinear operator

$$\nabla\colon C^\infty(M;TM) \times C^\infty(M;V) \longrightarrow C^\infty(M;V),$$

denoted

$$X, s \longmapsto \nabla_X s,$$

such that

(1) $\nabla_{fX} s = f\nabla_X s$, and
(2) $\nabla_X(fs) = (X\cdot f)s + f\nabla_X s$,

where $(X \cdot f)$ is the usual directional derivative associated to a vector field.

For $V = TM$, we have the usual Lie bracket

$$[-,-]\colon C^\infty(M;TM) \times C^\infty(M;TM) \longrightarrow C^\infty(M;TM)$$

sending $X, Y \mapsto [X,Y]$; if $f, g\colon M \to \mathbb{R}$ are functions, then

$$[fX, gY] = fg[X,Y] + f(X\cdot g)Y - g(Y\cdot f)X.$$

This operator is the commutator of an infinitesimal flow of X and an infinitesimal flow of Y.

Definition 1.36. Let ∇ be a covariant derivative for the tangent bundle. Its *torsion* is

$$\tau(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y].$$

Exercise 1.37. Show that $\tau(fX, gY) = fg\tau(X,Y)$ and $\tau(X,Y) = -\tau(Y,X)$.

Let's write this out in local coordinates. There are two things we could mean – coordinates on M or on V. Since V is a vector bundle, we can use for its coordinates the coordinates of M and a (local) basis of sections $s_1, \dots, s_r$. (Global nonvanishing sections might not exist at all, e.g. $TS^2 \to S^2$). In this case, you can differentiate s_j, obtaining some linear combination of the sections depending on x in a neighborhood U:

$$\nabla_X s_j = \Gamma_j^i(x) s_i.$$

This is just parameterized linear algebra. These Γ_j^i are 1-forms on U. We can also obtain coordinates for these 1-forms: if we let

$$\nabla_{\partial/\partial x^\mu} s_j = \Gamma_{j\mu}^i s_i,$$

then $\Gamma_j^i = \Gamma_{j\mu}^i \, \mathrm{d}x^\mu$.

If $V \to M$ has an inner product (metric), a positive definite pairing $C^\infty(M;V) \times C^\infty(M;V) \to C^\infty(M)$ sending $s_1, s_2, \mapsto \langle s_1, s_2 \rangle$, we can ask how a covariant derivative interacts with it.

Definition 1.38. A covariant derivative is *compatible with the metric* if for all $X \in C^\infty(M;TM)$ and $s_1, s_2 \in C^\infty(M;V)$,

$$X \cdot \langle s_1, s_2 \rangle = \langle \nabla_X s_1, s_2 \rangle + \langle s_1, \nabla_X s_2 \rangle.$$

Definition 1.39. A section $s \in C^\infty(M;V)$ is *parallel* if $\nabla_X s = 0$ for all X.

Parallel sections exist in $\mathbb{A}^n$ but not in general; the obstruction is called the curvature.

Definition 1.40. The *curvature* of a covariant derivative ∇ is

$$K(X,Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]},$$

i.e.

$$K(X,Y)(s) := \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s.$$

If M is Riemannian, there's a beautiful theorem about how all of these structures interact.

Theorem 1.41 (Levi-Civita)**.** *Let M be a Riemannian manifold. Then, there is a unique connection on $TM \to M$ which is torsion-free and compatible with the metric.*

Exercise 1.42. Prove this theorem. The way you do this is to compute $\langle \nabla_X Y, Z \rangle$, because if you know this for all Z, you know $\nabla_X Y$. Using the torsion-free and metric compatibility conditions, you can expand it out, and after some number of steps, you'll get the answer.

This local but non-global parallelism is an important property of Riemannian manifolds.

Next we will write a local formula for this connection. Suppose we have local coordinates $x^1, \dots, x^n$ on an open set $U \subset M$; then, we obtain the symbols $\Gamma_{jk}^i \colon U \to \mathbb{R}$. If we define the inner product and the Lie bracket, we can write down formulas for them. Namely, if we let

$$g_{ij} := \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle,$$

and since

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^i} \right] = 0,$$

then we can determine equations that the Γ_{jk}^i must satisfy. These can be encoded in the Riemann curvature tensor $R(X,Y)Z$, and in coordinates, on elets

$$R_{jk\ell}^i \frac{\partial}{\partial x^i} = R\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell} \right) \frac{\partial}{\partial x^j}.$$

This tensor has a bunch of important symmetries. The curvature is a 2-form on the manifold, but valued in $\mathrm{End}(TM)$: X and Y are the two directions you're testing, and are the 2 components of the 2-form.

The symmetry $R_{jk\ell}^i = -R_{j\ell k}^i$ means that $R(X,Y)$– is a skew-symmetric endomorphism of TM.

You can also lower an index by defining

$$R_{ijk\ell} = \langle R(\partial_k, \partial_\ell)\partial_j, \partial_i \rangle, \tag{1.43}$$

and skew-symmetry means

$$R_{ijk\ell} = -R_{jik\ell}.$$

These are the two "easier" symmetries, in that they don't use much specifically about R. A more interesting one is

$$R_{ijk\ell} + R_{ik\ell j} + R_{i\ell jk} = 0,$$

and the fourth identity, which follows from the other three, is

$$R_{ijk\ell} = R_{k\ell ij}.$$

Exercise 1.44. Compute the dimension of the vector space of tensors which satisfy these identities, as a subspace of $(V^*)^{\otimes 4}$.

Lecture 2.

Principal G-bundles: 1/24/18

The first part of today's talk was given by George Torres, corresponding to the first part of Chapter 2 of Roe's book.

Definition 2.1. A *Lie group* G is a group that is also a smooth manifold, and such that multiplication $m\colon G \times G \to G$ and inversion $i\colon G \to G$ are smooth maps.

Associated to any Lie group G is its *Lie algebra* $\mathfrak{g} := T_eG$. There is a *Lie bracket* operation

$$[-,-]\colon \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

defined like last time (use multiplication on G to extend tangent vectors to G-invariant vector fields, then take their commutators).

Definition 2.2. Let G be a Lie group and M be a smooth manifold. A *principal G-bundle* is a map of smooth manifolds $\pi\colon E \to M$ together with a smooth right action of G on E whose orbits are fibers of π, such that G acts freely and transitively on each fiber.

This implies that for each $m \in M$, $\pi^{-1}(M) \cong G$ noncanonically, and $E/G \cong M$.

Connections on principal bundles are analogous to those on vector bundles; the goal is to define a horizontal subspace of the bundle, and use that and its G-translates to define parallel transport. Then, one must show that connections always exist, but this turns out to be true. This definition of connections is sometimes called an *Ehresmann connection*.

In order to define connections, we'll need a few preliminary definitions.

Definition 2.3. Let G be a Lie group. The *adjoint representation* of G is the map $\mathrm{Ad}\colon G \to \mathrm{End}(\mathfrak{g})$ which sends a $g \in G$ to $d|_e\psi_g$, where $\psi_g\colon G \to G$ is conjugation by G and $d|_e$ is differentiating at the identity.

Definition 2.4. Let $\pi\colon E \to M$ be a principal G-bundle and $\rho\colon G \to \mathrm{Aut}(F)$ be a (real, finite-dimensional) representation of G. The *associated vector bundle* to E and F is the $E \times_G F := E \times F/\sim$, where $(e\cdot g, f) \sim (e, g\cdot f)$ for all $e \in E$, $f \in F$, and $g \in G$.

Exercise 2.5. Show that $E \times_G F \to M$ is indeed a vector bundle.

Definition 2.6. Let $\pi\colon E \to M$ be a principal G-bundle. A *vertical vector field* is a vector field v on E such that for all $p \in M$, $v_p = d_eA_p(u)$ for some $u \in \mathfrak{g}$, where $A_p\colon G \to E$ is an identification of the fiber E_p with G.[4]

An equivalent, more intuitive, definition is that a vertical vector field is contained within the *vertical subbundle* of E, i.e. the kernel of $d\pi$. A third equivalent definition is that V is vertical if for all $p \in M$, there is some $u \in \mathfrak{g}$ such that

$$v_p = \frac{\mathrm{d}}{\mathrm{d}t}(p\exp(tu))\bigg|_{t=0}. \tag{2.7}$$

Let $R_g\colon E \to E$ denote the right action of G, sending $e \mapsto e\cdot g$. We can pushforward by this map: let

$$(Rg)_*v := \frac{\mathrm{d}}{\mathrm{d}t}(p\exp(tu)g)\bigg|_{t=0}.$$

[4] TODO: I might have gotten this wrong.

In particular, the pushforward of a vertical vector field is still vertical.

Horizontal differential forms are dual to vertical vector fields.

Definition 2.8. Let $\alpha \in \Omega^p(E)$. Then α is *horizontal* if for all vertical vector fields $X_1, \dots, X_p$, $\alpha(X_1, \dots, X_p) = 0$. If in addition $(Rg)^*\omega = \omega$, we say α is *invariant* under the G-action.

Example 2.9. For any $\beta \in \Omega^p(M)$, $\pi^*\beta$ is an invariant horizontal form:

$$(R_g)^*\pi^*\beta = (\pi \circ R_g)^*\beta = \pi^*\beta. \qquad \blacktriangleleft$$

More generally, we can consider G-equivariant forms.

Definition 2.10. Let $\rho : G \to \mathrm{Aut}(F)$ be a representation and $f : E \to F$ be a smooth map. Then, f is *ρ-equivariant* if for all $e \in E$ and $g \in G$,

$$f(eg) = \rho(g^{-1})f(e).$$

Invariance is the same thing as equivariance for the trivial representation.

Lemma 2.11 (Correspondence lemma)**.** *Let $f : E \to F$ be as in the previous definition. There is a bijective correspondence between ρ-equivariant maps $f : E \to F$ and sections of $E \times_G F$.*

Proof. Let f be a ρ-equivariant map; then, we define a section s_f to send $x \in M$ to $(x, f(x)) \in E \times F$. To check that this is indeed a section, we need it to commute with the G-action on $E \times F$, and this follows because

$$\begin{aligned} s_f(x)\cdot g = (x, f(x))\cdot g &= (xg, \rho(g^{-1})f(x)) \\ &= (xg, f(xg)) \\ &= s_f(x \cdot g). \end{aligned}$$

Conversely, let $s \colon M \to E \times_G F$ be a section, and consider the diagram

$$\begin{tikzcd} E \arrow[r, "\sigma"] \arrow[d, "\pi"] & E \times F \arrow[r, "\pi_1"] \arrow[d, "p"] & E \arrow[d, "\pi"] \\ M \arrow[r, "s"] & E \times_G F \arrow[r] & M, \end{tikzcd} \tag{2.12}$$

where p is the quotient map, π_1 is projection onto the first factor, and σ is defined such that the composition of the maps across the top is the identity. The key observations are

(1) the right-hand square is a pullback square, and
(2) $f = \pi_2\sigma$ is ρ-equivariant.

The first property is true because for any $g \in G$,

$$\begin{aligned} \pi_1(\sigma(xg)) &= xg \\ &= \pi_1(\sigma(x))\cdot g = \pi_1(\sigma(x \cdot g)), \end{aligned}$$

and along the other corner,

$$\begin{aligned} p(\sigma(xg)) = s\pi(xg) &= s\pi(x) \\ &= p\sigma(x) \\ &= p\sigma(x \cdot g), \end{aligned}$$

so $\sigma(xg) = \sigma(x)\cdot g$. TODO: I missed the last part, that f is ρ-equivariant (and why these two properties suffice). ⊠

Exercise 2.13. Finish the proof by checking that these assignments are mutual inverses.

With these definitions in mind, we can define connections.

Definition 2.14. Let $\pi \colon E \to M$ be a principal G-bundle. A *connection* on E is a subbundle $H \subset TE$ (H for "horizontal") such that

- (splitting) there is another subbundle $V \subset TE$ such that for all $u \in E$, $T_uE = V_u \oplus H_u$, and
- (G-invariance) $d|_e R_g(H_u) = H_{u\cdot g}$.

Using the splitting lemma for vector bundles, the first condition is equivalent to the existence of a split short exact sequence

$$0 \longrightarrow V \longrightarrow TE \longrightarrow \pi^*TM \longrightarrow 0.$$

The splitting is determined by a section $\pi^*TM \to TE$ (which defines what we call "horizontal") or by a section $TE \to V$. This leads to an equivalent definition of a connection on a principal bundle, which is also useful: a connection on E is a $\mathfrak{g}$-valued 1-form ω on E, called the *connection one-form,* such that

- (G-invariance) for any $\xi \in T_u E$, $\omega(\xi \cdot g) = \mathrm{Ad}(g^{-1})\omega(\xi)$, and
- (splitting) for any $v \in \mathfrak{g}$, $\omega(X_v) = v$.

The splitting lemma guarantees you can always split short exact sequences of vector bundles. But to show that connections exist, we need to address G-invariance, which is not as immediate.

Lemma 2.15. *If G is a Lie group and $\pi\colon E \to M$ is a principal G-bundle, then there is a connection on E.*

Proof. Let $\underline{\mathfrak{g}} := E \times \mathfrak{g}$ and $V := E \times_G \mathfrak{g}$, where G acts on $\mathfrak{g}$ by the adjoint action. Then consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & \underline{\mathfrak{g}} \arrow[r] \arrow[d] & TE \arrow[r] \arrow[d] & \pi^*TM \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & V \arrow[r] & Q \arrow[r] & TM \arrow[r] & 0. \end{tikzcd} \tag{2.16}$$

Proving that this commutes takes a while, so we won't delve into the details; one reference is Atiyah, "Complex analytic connections on fibre bundles."

The point of introducing (2.16) is that if we can lift a splitting from the bottom row to the top row, it will be a G-invariant splitting, hence a connection. So choose a splitting $\sigma\colon TM \to Q$, which splits the bottom row of (2.16). Then we have a diagram

$$\begin{tikzcd} \pi^*TM \arrow[r] \arrow[d] & TM \arrow[d, "\sigma"] \\ TE \arrow[r] \arrow[d] & Q \arrow[d] \\ E \arrow[r, "\pi"] & M. \end{tikzcd} \tag{2.17}$$

The bottom rectangle is a pullback, and the total rectangle (π^*TM, TM, E, and M) is a pullback. Therefore by the universal property of pullbacks, the top rectangle also is a pullback, and this implies that σ lifts across it to something G-invariant. ⊠

There is also a more geometric proof.

Next we'll talk about exterior derivatives.

Definition 2.18. Let $\pi\colon E \to M$ be a principal G-bundle with connection H and let $p_\omega\colon \Omega^*(E) \to \Omega^*(H)$ denote projection onto the horizontal subspace. The *exterior covariant derivative* is the composition

$$\Omega^p(E) \xrightarrow{\mathrm{d}} \Omega^{p+1}(E) \xrightarrow{p_\omega} \Omega^{p+1}(H).$$

Proposition 2.19. *Let $\rho\colon G \to \mathrm{GL}(F)$ be a representation, $\alpha \in \Omega_E^p(F)$ be an F-valued,ρ-equivariant horizontal p-form on E, and ω denote the ($\mathfrak{g}$-valued) connection one-form on E. Then*

$$p_\omega(\mathrm{d}\alpha) = \mathrm{d}\alpha + \rho_*\omega \wedge \alpha.$$

Definition 2.20. Let ω be a differential 1-form. Then, the *curvature* Ω of ω is the 2-form

$$\Omega(X_1, X_2) = \mathrm{d}\omega(X_1, X_2) + [\Omega(X_1), \Omega(X_2)].$$

Exercise 2.21. With notation as above,

$$(p_\omega \mathrm{d})^2\alpha = \rho_*\Omega \wedge \alpha.$$

The point is that acting on α is the same as wedging with $\rho_*\Omega$, and this tells you something about what Ω is doing.

Remark. In some of these formulas, it's important to be careful about what the wedge products are doing. For example, we once or twice saw $\omega \wedge \alpha$, where $\omega \in \Omega^1_E(\mathfrak{g})$ and $\alpha \in \Omega^1_E(F)$. If $\mathfrak{g}$ is a matrix algebra, we can organize the components of ω into a matrix of differential forms, and F is a representation of the matrix algebra $\mathfrak{g}$, so α is a vector. In this case, the wedge product is a combination of matrix multiplication and the wedge product:

$$\begin{pmatrix} \omega^1_1 & \omega^1_2 \\ \omega^2_1 & \omega^2_2 \end{pmatrix} \wedge \begin{pmatrix} \alpha^1 \\ \alpha^2 \end{pmatrix} = \begin{pmatrix} \omega^1_1 \wedge \alpha^1 + \omega^1_2 \wedge \alpha^2 \\ \omega^2_1 \wedge \alpha^1 + \omega^2_2 \wedge \alpha^2 \end{pmatrix}. \tag{2.22}$$

The same is true for computing $\omega \wedge \omega$; in particular, this is not automatically zero.

Therefore one sometimes sees the formula for curvature written

$$\Omega := \mathrm{d}\omega + [\omega \wedge \omega]. \tag{2.23}$$

What does this mean? We have $\omega \wedge \omega \in \Omega^2_E(\mathfrak{g} \otimes \mathfrak{g})$, and the Lie bracket $[-,-]\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$. We implement this on differential forms by

$$(\omega^i \wedge \omega^j) e_i \otimes e_j \longmapsto (\omega^i \wedge \omega^j)[e_i, e_j].$$

In particular,

$$\begin{aligned} [\omega \wedge \omega](X_1, X_2) &= [\omega(X_1), \omega(X_2)] - [\omega(X_2), \omega(X_1)] \\ &= 2[\omega(X_1), \omega(X_2)], \end{aligned}$$

which is why (2.23) has an extra 1/2 in it compared to the first definition. ◄

Remark. One can even define differential forms valued in vector bundles: $\Omega^\bullet_M(E \times_G F)$ consists of sections of the exterior powers of $E \times_G F$. Alternatively, you can think of these as valued in $(E \times_G F)_p$ at a point p; the vector spaces changes as p moves, but that's okay. The quotient map $\pi\colon E \to G$ defines a pullback $\Omega^*_M(E \times_G F) \to \Omega^*_E(F)$. This provides yet another interpretation of the definition of a connection.

- Invariance is that $\alpha \in \operatorname{Im}(\pi^*)$ iff $R^*_g \alpha = \rho(g)^{-1}\alpha$ for all $g \in G$.
- Splitting comes from the fact that $\iota_\zeta \alpha = 0$ when ζ is vertical. (This denotes *contraction*: $\iota_\zeta \alpha(X) := \alpha(\zeta, X)$).

It's a good exercise to check, to get practice manipulating these vector- or bundle-valued forms. But principal bundles make some of these computations easier, by turning some bundle-valued forms into constant vector space-valued forms. ◄

Lecture 3.

Characteristic classes: 1/24/18

The second part of today's lecture was given by Riccardo Pedrotti, on characteristic classes from a geometric perspective.

The theory of characteristic classes comes from the simple question: how can we tell two vector bundles apart? For instance, how do we know that the tangent bundle to the 2-sphere is non-trivial? Characteristic classes gives a systematic approach.

Definition 3.1. A *characteristic class* c is a natural transformation which to each vector bundle V over a manifold M associates an element $c(V)$ of the cohomology group $H^*(M)$, with property that if $V_1 \simeq V_2$ then $c(V_1) = c(V_2)$.

The idea of Chern-Weil theory is the following: suppose that our bundle V is equipped with a connection. In some sense, the curvature of this connection measures the local deviation of V from flatness. Now if V is flat, and the base manifold M is simply connected, then V is trivial. This suggests that there may be a link between curvature and characteristic classes, which measure the global deviation of V from triviality. Such a link is provided by the theory of invariant polynomials.

By polynomial function we mean the following:

Definition 3.2. Let $\mathfrak{gl}_m(\mathbb{C})$ denote the Lie algebra of $m \times m$ matrices over $\mathbb{C}$. A *homogeneous polynomial function* P on $\mathfrak{gl}_m(\mathbb{C})$ is a function such that there exists a $\widetilde{P} \in \mathrm{Sym}^k(\mathbb{C}^m)^*$ such that $P(A) = \widetilde{P}(A, A, \dots, A)$. A *polynomial function* is a sum of homogeneous ones.

Definition 3.3. An *invariant* polynomial on $\mathfrak{gl}_m(\mathbb{C})$ is a polynomial function $P\colon \mathfrak{gl}_m(\mathbb{C}) \to \mathbb{C}$ such that for all $X, Y \in \mathfrak{gl}_m(\mathbb{C})$, $P(XY) = P(YX)$. An *invariant formal power series* is a formal power series over $\mathfrak{gl}_m(\mathbb{C})$ each of whose homogeneous components is an invariant polynomial.

For example, the determinant and the trace are invariant polynomials.

Lemma 3.4. *The ring of invariant polynomials on $\mathfrak{gl}_m(\mathbb{C})$ is a polynomial ring generated by the polynomials*

$$c_k(X) = (-2\pi i)^{-k} \operatorname{tr}(\Lambda^k X),$$

where $\Lambda^k X$ denotes the transformation induced by X on $\Lambda^k \mathbb{C}^m$.

Proof. Let P be any invariant polynomial. Restricting P to diagonal matrices, we see that P must be a polynomial function of the diagonal entries. Since these diagonal entries can be interchanged by conjugation, P must in fact be a *symmetric* polynomial function. Now since P is invariant under conjugation, it must be a symmetric polynomial function of the eigenvalues for all matrices with distinct eigenvalues, since by elementary linear algebra such matrices are conjugate to diagonal matrices. The set of such matrices is dense in $\mathfrak{gl}_m(\mathbb{C})$, so a continuity argument shows that P is just a symmetric polynomial function in the eigenvalues. Now it is easy to see that $\operatorname{tr}(\Lambda^k X)$ is the k^{th} elementary symmetric function in the eigenvalues of X. The main theorem on symmetric polynomials states that the ring of symmetric polynomials is itself a polynomial ring generated by the elementary symmetric functions, and this now completes the proof. ⊠

Example 3.5. To make the idea of the proof more concrete, let $m = 4$. A 4×4 diagonal matrix is of the form

$$X = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$$

for some $a, b, c, d \in \mathbb{R}$. Then, $\Lambda^2 X$ acts on $e_1 \wedge e_2$ by $Xe_1 \wedge Xe_2 = abe_1 \wedge e_2$. Therefore $\Lambda^2 X$ is a 6×6 diagonal matrix with diagonal terms ab, ac, ad, bc, bd, and cd, and therefore its trace is

$$\operatorname{tr}(\Lambda^2 X) = ab + ac + ad + bc + bd + cd.$$

This is a quadratic polynomial, and is symmetric; it's an example of an *elementary symmetric polynomial*. There's a theorem that the ring of all symmetric polynomials are generated by these elementary symmetric polynomials. ◂

Now let V be a complex vector bundle over M with connection ∇ and curvature $K \in \Omega^2_M(\operatorname{End}(V))$. Choosing a local framing for V, we may locally identify K with a matrix of ordinary 2-forms. Hence, if P is an invariant polynomial, we can apply P to this matrix to get an even-dimensional differential form $P(K)$. *A priori*, this depends on the choice of local framing, but since P is invariant, $P(K)$ doesn't depend on the choice, and is therefore globally defined.

In terms of the principal $\mathrm{GL}_m(\mathbb{C})$-bundle E associated to V, this construction may be phrased as follows. Let Ω be the curvature form of the induced connection on E; Ω is a horizontal, equivariant 2-form on E with values in $\mathfrak{gl}_m(\mathbb{C})$, so $P(\Omega)$ is a horizontal invariant form on E. Such a form is the lift to E of a form on M, and this form is $P(K)$.

Since 2-forms are nilpotent elements in the exterior algebra $\Omega^*_M(\mathfrak{gl}_m(\mathbb{C}))$, all formal power series with 2-form-valued variables in fact converge. Thus, this construction makes good sense if P is merely an invariant formal power series.

Proposition 3.6. *For any invariant polynomial (or formal power series) P, the differential form $P(K)$ is closed, and its de Rham cohomology class is independent of the choice of connection ∇ on V.*

Proof. For the purposes of this proof call an invariant formal power series P as *respectable* if the conclusion of the proposition holds for P. Clearly the sum and product of respectable formal power series are respectable. Thus, it is enough to prove that the generators defined in Lemma 3.4 are respectable. Equivalently, since

$$\det(1 + qK) = \sum q^k \operatorname{tr}(\Lambda^k K),$$

it is enough to prove that $\det(1 + qK)$, considered as a formal power series depending on the parameter q, is respectable.

If P is a respectable formal power series with constant term a, and g is a function holomorphic in a neighborhood of a, then $g \circ P$ is also a respectable formal power series. Hence, $\det(1+qK)$ is respectable if and only if $\log\det(1+qK)$ is respectable. We will now prove directly that $\log\det(1+qK)$ is respectable.

For this purpose we will work in the associated principal $\mathrm{GL}_n(\mathbb{C})$-bundle E of frames for V, with matrix-valued connection 1-form ω and corresponding curvature 2-form Ω. Recall the formula

$$\Omega = d\omega + \omega \wedge \omega$$

where the product in the ring of matrix-valued forms is obtained by tensoring exterior product and matrix multiplication as in (2.22).

Now suppose that ω depends on a parameter t; then Ω also depends on t, and if we use a dot to denote differentiation with respect to t, then

$$\dot\Omega = \mathrm{d}\dot\omega + \omega \wedge \dot\omega + \dot\omega \wedge \omega.$$

Consider

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\det(1+q\Omega) = q\,\mathrm{tr}\big(\dot\Omega(1+q\Omega)^{-1}\big) \tag{3.7}$$

$$= \sum_{\ell=0}^{\infty}(-1)^{\ell}q^{\ell+1}\,\mathrm{tr}\big(\Omega^{\ell}(\mathrm{d}\dot\omega + \omega\wedge\dot\omega + \dot\omega\wedge\omega)\big) \tag{3.8}$$

where (3.7) is justified by the formula

$$\frac{\mathrm{d}}{\mathrm{d}t}\det A(t) = (\det A(t))\cdot \mathrm{tr}(\dot A(t)A(t)^{-1}),$$

and (3.8) is justified by the power series expansion

$$\frac{1}{1+z} = \sum_{i=0}^{\infty}(-1)^i z^i.$$

We also need the *second Bianchi identity*

$$\mathrm{d}\Omega = \Omega\wedge\omega - \omega\wedge\Omega,$$

which can be proven directly from the definition of the exterior derivative. Using this, plus the fact that trace is symmetric, we have that

$$\begin{aligned}\mathrm{tr}\big(\Omega^{\ell}(\omega\wedge\dot\omega + \dot\omega\wedge\omega)\big) &= \mathrm{tr}\big(\Omega^{\ell}\wedge\omega\wedge\dot\omega - \omega\wedge\Omega^{\ell}\wedge\dot\omega\big)\\ &= \mathrm{tr}\big((\mathrm{d}\Omega^{\ell})\wedge\dot\omega\big).\end{aligned}$$

Therefore

$$\begin{aligned}\mathrm{tr}\big(\Omega^{\ell}\wedge(\mathrm{d}\dot\omega + \omega\wedge\dot\omega + \dot\omega\wedge\omega)\big) &= \mathrm{tr}\big((\mathrm{d}\Omega^{\ell})\wedge\dot\omega + \Omega^{\ell}\wedge\mathrm{d}\dot\omega\big)\\ &= \mathrm{d}\,\mathrm{tr}\big(\Omega^{\ell}\wedge\dot\omega\big),\end{aligned}$$

so (3.7) simplifies to

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\det(1+q\Omega) = \mathrm{d}\sum_{\ell=0}^{\infty}(-1)^{\ell}q^{\ell+1}\,\mathrm{tr}\big(\Omega^{\ell}\wedge\dot\omega\big), \tag{3.9}$$

and in particular is an exact form on E.

In fact, it is the exterior derivative of a horizontal and invariant form on E: $\dot\omega$ is horizontal and G-equivariant since it is a 1-form on M (it follows from Lemma 2.11 that the space of connections is an affine space modeled on the vector space of V-valued forms on M), Ω is horizontal and G-equivariant as well. Hence $\mathrm{tr}(\Omega^{\ell}\wedge\dot\omega)$ is invariant since the trace is an invariant polynomial, and is horizontal since $\Omega^{\ell}\wedge\dot\omega$ is.

Therefore, the projection to the base manifold

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\det(1+qK)$$

is also exact. Now the result follows; for since any connection can be deformed locally to flatness (i.e. $K=0$), we see that $\log\det(1+qK)$ is locally exact, hence closed, and since any two connections can be connected by a smooth path, the cohomology class of $\log\det(1+qK)$ is independent of the choice of connection, since their difference is an exact form. ⊠

It follows from the proposition that any invariant formal power series P defines a characteristic class for complex vector bundles, by the recipe "pick any connection and apply P to the curvature."

Definition 3.10. The k^{th} *Chern class* is the characteristic class corresponding to the generators c_k defined in Lemma 3.4.

Remark. We immediately see from the definition of Chern classes that if a complex vector bundle has rank m, then $c_k = 0$ for $k > m$: $\Lambda^k K$ is the linear transformation induced by K on $\Lambda^k \mathbb{C}^m$, and for $k > m$, the latter is trivial. Naturality comes from the fact that if on a local patch U_i, E has the local connection form ω_i, then on $f^{-1}(U_i)$, the curvature is $f^*\Omega_i$. ◄

Lemma 3.11. *Let V be a real vector bundle and $V_{\mathbb{C}}$ denote its complexification. Then, $c_{2k+1}(V_{\mathbb{C}}) = 0$.*

Proof. We can give V a metric and compatible connection. The curvature of such a connection is skew (i.e. $\mathfrak{o}(m)$-valued), so

$$\operatorname{tr}(\Lambda^k F) = (-1)^k \operatorname{tr}(\Lambda^k F).$$

To see this, recall that the coefficients of the characteristic equation for F are exactly $\operatorname{tr}(\Lambda^k F)$ up to a sign.[5] If λ is an eigenvalue of a skew-symmetric matrix, then $-\lambda$ is too, and on $\mathbb{C}$ this means that the characteristic polynomial is up to a constant the product of polynomials $(z^2 - \lambda^2)$, so there are no coefficients of odd index, hence proving that for k odd, $\Lambda^k F$ is traceless. ⊠

Genera. Holomorphic functions can be used to build important combinations of characteristic classes. Let $f(z)$ be any function holomorphic near $z = 0$. We can use f to construct an invariant formal power series Π_f by defining

$$\Pi_f(X) := \det\left(f\left(-\frac{1}{2\pi i}X\right)\right).$$

Again, to make sense of this, we need to sidestep convergence issues! But since we'll just be applying this to differential forms, which are nilpotent, this is okay.

The associated characteristic class is called the *Chern f-genus*. It has a few nice properties.

Lemma 3.12. *If $L \to M$ is a complex line bundle, $\Pi_f(L) = f(c_1(L))$.*

Proof. This comes from the fact that in this case the curvature is a $\mathfrak{gl}_1(\mathbb{C})$-valued 2-form, so

$$\begin{aligned}
\Pi_f(L) = \Pi_f(K_L) &= \det\left(f\left(-\frac{1}{2\pi i}K_L\right)\right) \\
&= f\left(-\frac{1}{2\pi i}K_L\right) \\
&= f\left(\operatorname{tr}\left(-\frac{1}{2\pi i}K_L\right)\right) = f(c_1(L)).
\end{aligned}$$ ⊠

Lemma 3.13. *For any complex vector bundles V_1 and V_2, $\Pi_f(V_1 \oplus V_2) = \Pi_f(V_1)\Pi_f(V_2)$.*

Proof sketch. Compute using a direct sum connection, which gives rise to a curvature matrix which is a block matrix. ⊠

Now observe a very useful property: if the eigenvalues of the matrix $(-1/2\pi i)X$ are $\{x_j\}$, then

$$\Pi_f(X) = \prod f(x_j) \tag{3.14}$$

is a symmetric formal power series in the x_j, and can therefore be expressed in terms of the elementary symmetric functions of the x_j. But these elementary symmetric functions are just the Chern classes. Thus in the literature the genus $\Pi_f(V)$ is often written just as in (3.14), where $x_1, \dots, x_m$ are formal variables subject to the relations

$$\begin{aligned}
x_1 + x_2 + \cdots + x_m &= c_1, \\
x_1 x_2 + \cdots + x_{m_1} x_{m-2} &= c_2,
\end{aligned}$$

and so on.

[5] There's a sign convention here; this is true using our definition $\det(1 - qK)$. An alternative choice is to use $\det(q - K)$, in which case one must swap the indices to preserve evenness.

Example 3.15. The genus associated to $f(z) = 1 + z$ is the *total Chern class*

$$c(V) := 1 + c_1(V) + c_2(V) + c_3(V) + \cdots .$$

To see this, consider the power expansion of the determinant:

$$\det\left(1 - \frac{1}{2\pi}X\right) = \sum_k \left(-\frac{1}{2\pi}\right)^k \mathrm{tr}(\Lambda^k X) = \sum_k c_k(X).$$

From this we immediately get that $c(V_1 \oplus V_2) = c(V_1)c(V_2)$. ◀

Definition 3.16. Let $V \to M$ be a real vector bundle and g be a holomorphic function near 0, with $g(0) = 1$. Let $V_{\mathbb{C}}$ be the complexification of V. Denote by f be the branch of $z \mapsto (g(z^2))^{1/2}$ which has $f(0) = 1$; we call the genus associated to f the *Pontrjagin g-genus* of V.

Since f is an even function of z, the associated genus involves only the even Chern classes.

Lemma 3.17. *Let g be as above. Then for a real vector bundle V, the Pontrjagin g-genus is equal to*

$$\prod_j g(y_j)$$

for some formal variables y_j.

Definition 3.18. Let V be a real vector bundle. Its k^{th} *Pontrjagin class* $p_k(V)$ is the k^{th} elementary symmetric function in the formal variables y_j.

Proof of Lemma 3.17. Regard this as an identity between invariant polynomials over $\mathfrak{o}(n)$. Any matrix in $\mathfrak{o}(n)$ is similar to one in block diagonal form, where the blocks are 2×2 and are of the form

$$X = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$$

with eigenvalues $\pm i\lambda$. Since both sides of the desired identity are multiplicative for direct sums, it is enough to prove it for this block X, whose first two elementary symmetric functions are

$$c_1(X) = \left(-\frac{1}{2\pi i}\right)\mathrm{tr}(X),$$
$$c_2(X) = \left(-\frac{1}{2\pi i}\right)^2 \mathrm{tr}(\Lambda^2 X).$$

Since X is skew, then its trace vanishes, so $c_1(X) = 0$. By looking at the characteristic polynomial of X we see that $\mathrm{tr}(\Lambda^2 X) = \lambda^2$, giving

$$c_2(X) = -\frac{\lambda^2}{4\pi^2}.$$

Thus

$$y = p_1(X) = \frac{\lambda^2}{4\pi^2}.$$

On the other hand, X is similar over $\mathbb{C}$ to

$$\begin{pmatrix} -i\lambda & 0 \\ 0 & i\lambda \end{pmatrix},$$

so

$$\Pi_f(X) = f\left(-\frac{\lambda}{2\pi}\right) f\left(\frac{\lambda}{2\pi}\right) = g\left(\frac{\lambda^2}{4\pi^2}\right) = g(y)$$

as required. ⊠

Two important examples are the $\widehat{A}$-genus, which is the Pontrjagin genus associated to the holomorphic function

$$z \longmapsto \frac{\sqrt{z}/2}{\sinh(\sqrt{z}/2)}$$

and the Hirzebruch L-genus, which is the Pontrjagin genus associated with the holomorphic function

$$z \longmapsto \frac{\sqrt{z}}{\tanh(\sqrt{z})}.$$

Lecture 4.

Clifford algebras, Clifford bundles, and Dirac operators: 1/31/18

Ricky spoke today about Clifford algebras and Clifford bundles.

Let k be a field with characteristic not equal to 2. If V is a vector space over k, its *tensor algebra* is

$$T(V) := \bigoplus_{k \geq 0} V^{\otimes k}, \tag{4.1}$$

where $V^{\otimes 0} := k$.

Definition 4.2. Let V be a k-vector space with a quadratic form $Q\colon V \times V \to k$. Let $I_Q \subset T(V)$ denote the two-sided ideal generated by elements of the form $v \otimes v + Q(v)$ for $v \in V$. Then, the quotient algebra

$$C\ell(V, Q) := T(V)/I_Q$$

is called the *Clifford algebra* of V and Q.

Example 4.3. The zero function is a quadratic form, so $C\ell(V, 0)$ is $T(V)/(v \otimes v = 0)$, which is just the exterior algebra $\Lambda(V)$ of V. ◀

There is a natural map $i\colon V \to C\ell(V, Q)$ which is the composition

$$V = V^{\otimes 1} \hookrightarrow T(V) \xrightarrow{\pi_Q} C\ell(V, Q).$$

Lemma 4.4. *$i\colon V \to C\ell(V, Q)$ is injective.*

This is not too hard to check.

Moreover, 1 and V generate $C\ell(V, Q)$, subject to the relations $v^2 = q(v, v)$. To get a smaller set of generators, we can choose a basis of V. *From now on, we assume Q is positive definite and choose an orthonormal basis $e_1, \dots, e_n$ of V.* In this case, $C\ell(V, Q)$ is generated by $1, e_1, \dots, e_n$ subject to the relations

$$\begin{aligned} e_i^2 &= 1 \\ e_j \cdot e_j &= -e_j \cdot e_i, \end{aligned} \tag{4.5}$$

because $e_i^2 = -Q(e_i) = -1$. This implies the following fact.

Proposition 4.6. *The set*

$$\{e_{i_1} \cdots e_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n, 1 \leq k \leq n\}$$

is a basis for $C\ell(V, Q)$ as a vector space, and hence

$$\dim C\ell(V, Q) = 2^n = \sum_{k=0}^{n} \binom{n}{k}.$$

Example 4.7. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{R}^n$; then, $C\ell_n := C\ell(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$. In low dimensions these are familiar.

- $C\ell_1 = \langle 1 \rangle \oplus \langle e \rangle$ with $e^2 = 1$, hence $C\ell_1 \cong \mathbb{C}$.
- $C\ell_2 = \langle 1 \rangle \oplus \langle e_1 \rangle \oplus \langle e_2 \rangle \oplus \langle e_1 \cdot e_2 \rangle$ with $e_1^2 = e_2^2 = (e_1 e_2)^2 = -1$, so as $\mathbb{R}$-algebras, $C\ell_2 \cong \mathbb{H}$, the quaternions.

There is a sense in which real Clifford algebras are 8-fold periodict, which is an instance of *Bott periodicity*. We won't delve into this, but see Atiyah-Bott-Shapiro, "Clifford modules," for more information. ◀

Clifford algebras satisfy a universal property.

Proposition 4.8. *Let A be a k-algebra and $\varphi\colon V \to A$ be a map of vector spaces such that $\varphi(v)^2 = -Q(v) \cdot 1_A$. Then, there is a unique algebra map $\widehat{\varphi}\colon C\ell(V, Q) \to A$ making the following diagram commute:*

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[dr, hook, "i"'] & A. \\ & C\ell(V, Q) \arrow[u, "\widehat{\varphi}", "\exists!"'] \end{tikzcd}$$

Proof. By the universal property of $T(V)$, there's a unique map $\psi\colon T(V)\to A$ sending

$$v_1\otimes\cdots\otimes v_k \longmapsto \varphi(v_1)\varphi(v_2)\cdots\varphi(v_k). \tag{4.9}$$

The claim follows because $\psi(I_Q)=0$, hence factors through the quotient, which is $C\ell(V,Q)$. ⊠

Let QVect_k denote the category of *quadratic spaces* over k, i.e. vector spaces together with quadratic forms; the morphisms $\varphi\colon (V_1,Q_1)\to(V_2,Q_2)$ are data of a linear map $\varphi\colon V_1\to V_2$ such that for all $v,w\in V_1$,

$$Q_1(v,w)=Q_2(\varphi(v),\varphi(w)). \tag{4.10}$$

Using Proposition 4.8, one can show that $C\ell\colon \mathsf{QVect}_k\to\mathsf{Alg}_k$ is a functor.
Another use of the universal property is to define a representation

$$\rho_{C\ell}\colon \mathrm{O}_n\to\mathrm{Aut}(C\ell_n);$$

a map $\mathbb{R}^n\to\mathbb{R}^n$ respecting the inner product defines a map $C\ell_n\to C\ell_n$, and the space of these maps is O_n.

Definition 4.11. Let (V,Q) be a quadratic space. A *Clifford module* over (V,Q) is a k-vector space S together with a k-linear map $\varphi\colon C\ell(V,Q)\to\mathrm{End}_k(S)$.

So it's just a module over the algebra $C\ell(V,Q)$.

Remark. When S is a complex vector space and V is a real vector space, then we will instead ask for the action map to be an $\mathbb{R}$-algebra homomorphism $C\ell(V,Q)\to\mathrm{End}_{\mathbb{C}}(S)$. This is equivalent to having a module over $C\ell(V,Q)\otimes_{\mathbb{R}}\mathbb{C}$. ◀

Remark. By the universal property, it suffices to specify a map $\varphi\colon V\to\mathrm{End}_k(S)$ with $\varphi(v)^2=-Q(v)\cdot\mathrm{id}$. ◀

Example 4.12.

(1) Let's consider $C\ell_1$ as a module over itself. This is the same data as a map $\varphi\colon C\ell_1\to\mathrm{End}_{\mathbb{C}}(\mathbb{C})\cong\mathbb{C}$; one choice is $1\mapsto\mathrm{id}$ and $e\mapsto i$.
(2) We can also make $\mathbb{R}^4$ into a $C\ell_2$-module by having it act on itself by left multiplication. For example, e_1 acts by the matrix

$$\begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0\end{pmatrix}.$$ ◀

Now, we apply this to geometry. Let (M,g) be a Riemannian manifold, so for each $p\in M$, $C\ell(T_pM,g_p)\cong C\ell_n$.

Definition 4.13. The *Clifford tangent bundle* is $C\ell(TM):=\mathscr{P}_{\mathrm{O}}(M)\times_{\mathrm{O}_n}C\ell_n$, where $\mathscr{P}_{\mathrm{O}}(M)$ is the principal O_n-bundle of orthonormal frames on M.

More generally, if $S\to M$ is any complex vector bundle, we can equip S with a Clifford action $c\colon C\ell(TM)\to\mathrm{End}_{\mathbb{C}}(S)$ in a similar way.

Definition 4.14. Let $S\to M$ be a complex vector bundle with a Hermitian metric $\langle\cdot,\cdot\rangle$ and a connection $\nabla\colon C^\infty(TM)\otimes C^\infty(S)\to C^\infty(S)$. This data $(S,\langle\cdot,\cdot\rangle,\nabla)$ defines a *Clifford bundle* if

- for all $X\in C^\infty(TM)$ of unit norm and $s_1,s_2\in C^\infty(S)$, $\langle X\cdot s_1,X\cdot s_2\rangle=\langle s_1,s_2\rangle$ iff $\langle X\cdot s_1,s_2\rangle+\langle s_1,X\cdot s_2\rangle=0$ and
- for all $X,Y\in C^\infty(TM)$ and $s\in C^\infty(S)$, $\nabla^L_X(Y\cdot s)=(\nabla_XY)\cdot s+Y\cdot\nabla_Xs$, where ∇^L denotes the Levi-Civita connection.[6]

Now we need to take a brief detour into something called synchronous frames.

Definition 4.15. Let (M,g) be an n-dimensional Riemannian manifold and $\varphi\colon U\to\mathbb{R}^n$ be a chart for M containing some $y\in U$. Let $e_1,\dots,e_n$ be an orthonormal basis of $T_yM\dots$
TODO: I couldn't figure out what happened here. Sorry. I'll have to fix this later.

Remark. The exponential map gives a canonical choice for a local neighborhood on a Riemannian manifold. ◀

[6]TODO: I am not completely sure I wrote this down correctly.

Dirac operators.

Definition 4.16. Let $S \to M$ be a Clifford bundle. The *Dirac operator* $D\colon C^\infty(S) \to C^\infty(S)$ is the composition

$$C^\infty(S) \xrightarrow{\nabla} C^\infty(T^*M \otimes S) \xrightarrow{g} C^\infty(TM \otimes S) \xrightarrow{C\ell} C^\infty(S).$$

In a neighborhood of a point $x \in M$, choose a local orthonormal frame $e_1, \dots, e_n$. Let $e^i := g(e_i, -)$ be the dual frame. Then, the Dirac operator in coordinates looks like

$$s \longmapsto \nabla_{(\cdot)} s = \sum e^i \otimes \nabla_{e_i} s \longmapsto \sum e_i \otimes \nabla_{e_i} s \longmapsto \sum_i e_i \cdot \nabla_{e_i} s. \tag{4.17}$$

Example 4.18. Let $\mathbb{E}^n$ denote *Euclidean space*, i.e. $\mathbb{R}^n$ with the usual flat metric. If V is a complex vector space, it canonically defines a complex vector bundle $\underline{V} \to \mathbb{E}^n$ by translation. Let $e_1, \dots, e_n$ be the standard orthonormal frame on $T\mathbb{R}^n$.

Let $\gamma := c(e_i)$, where $c\colon C\ell(TM) \to \mathrm{End}_{\mathbb{C}}(\underline{V})$ denotes the Clifford bundle action. Then, the Dirac operator $D\colon C^\infty(\underline{V}) \to C^\infty(\underline{V})$ is

$$D = \sum_i \gamma_i \cdot \partial_i, \tag{4.19}$$

where ∂_i is the usual partial derivative operator. The γ_i satisfy the *anticommutation relations*

$$\{\gamma_i, \gamma_j\} := \gamma_i \cdot \gamma_j + \gamma_j \cdot \gamma_i = -2\delta_{ij}. \tag{4.20}$$

Specifically, if $V = \mathbb{C} = C\ell_1$, then $e_1 \mapsto i$, so $\gamma = i$. Therefore $D = i\frac{\partial}{\partial x}$. ◀

The Dirac operator is self-adjoint. Well, it's *formally self-adjoint*: $C^\infty(S)$ is not a Hilbert space, so we can't talk about self-adjointness strictly speaking. One way to abrogate this problem is to take some kind of L^2 completion, but then it's an unbounded operator, so things are still a little complicated. Anyways, we'll talk about this in a bit.

Definition 4.21. Let M be a closed Riemannian manifold and V be its volume form. Then, there is an inner product on $C^\infty(S)$ defined by

$$\langle s_1, s_2 \rangle := \int_M \langle s_1(x), s_2(x) \rangle \, \mathrm{d}V.$$

Theorem 4.22. *The Dirac operator on a closed manifold is formally self-adjoint.*

Proof. That is, we want to prove that

$$\langle Ds_1, s_2 \rangle = \langle s_1, Ds_2 \rangle. \tag{4.23}$$

We will compute this locally in a synchronous frame $e_1, \dots, e_n$ for a chart in X. Then

$$\begin{aligned}
(Ds_1, s_2) - (s_1, Ds_2) &= \sum_i ((e_i \nabla_i s_1, s_2) - (s_1, e_i \nabla_i s_2)) \\
&= \sum_i \left(\nabla_i (e_i \cdot s_1) - \underbrace{(\nabla_i e_i)}_{=0}|_x \cdot s_1, s_2 \right) - (s_1, e_i \nabla_i s_2) \\
&= \sum_i \nabla_i (e_i \cdot s_1, s_2) \\
&= \sum \partial_{e_i} (e_i \cdot s_1, s_2) \\
&= \mathrm{d}^* \omega,
\end{aligned}$$

where

$$\omega_x := -\sum (e_i \cdot s_1, s_2) e^i.$$

Hence the difference is $\langle 1, \mathrm{d}^* \omega \rangle = \langle d1, \omega \rangle = 0$. ⊠

There's a local-vs.-neighborhood argument to make here, but this is the idea.

Lecture 5.

The Weitzenbock formula: 1/31/18

The next talk was by Ivan, on more Clifford bundles and Dirac operators.

Definition 5.1. Let $S \to M$ be a Clifford bundle on a Riemannian manifold (M, g), $A \in \Omega^2_M(\operatorname{End} S)$, and $\{e_i\}$ be a local synchronous orthonormal frame for M. The *Clifford contraction* of A is $\mathbb{A} \in \Omega^0_M(\operatorname{End} S)$ defined by the local formula

$$\mathbb{A} \cdot s := \sum_{i<j} c(e_i)c(e_j)A(e_i, e_j) \cdot s.$$

One should check this is independent of the choice of frame, but that is true.

Let $\mathbb{K}$ denote the Clifford contraction of the curvature for ∇ on S.

Theorem 5.2 (Weitzenbock formula). *Let D denote the Dirac operator of $S \to M$. Then*

$$D^2 = \nabla^*\nabla + \mathbb{K}.$$

$\nabla^*\nabla$ is called the *covariant Laplacian*, and D^2 the *Dirac Laplacian*.

Proof. Let $p \in M$. Then

$$\begin{aligned} D^2 s|_p &= \sum_{i,j} e_i \cdot \nabla_{e_i}\Big(e_j \cdot \nabla_{e_j} s\Big)\Big|_p \\ &= \sum_{i,j} e_i \cdot e_j \cdot \nabla_{e_i}\nabla_{e_j} s|_p. \end{aligned}$$

Splitting this into the cases $i = j$ and $i \neq j$, we get

$$\begin{aligned} &= -\sum_i \nabla^2_{e_i} s|_p + \sum_{i<j} e_i e_j \Big(\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i}\Big)|_p \\ &= \nabla^*\nabla s|_p + \mathbb{K}s|_p. \end{aligned}$$

This uses the fact that we're on a synchronous frame, so $\nabla_{[e_i,e_j]} = 0$, and therefore the curvature simplifies to $\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i}$ as we used above. ⊠

This formula will be crucial for us, allowing us to supplant some of the general theory of elliptic operators in the proof of the index theorem.

Corollary 5.3 (Bochner theorem). *Let $S \to M$ and $\mathbb{K}$ be as above. If $(\mathbb{K}s, s) > 0$ at some point then there are no nontrivial solutions to $D^2 s = 0$.*

Positivity makes sense because $\mathbb{K}$ is a Hermitian operator on a bundle which is fiberwise Hermitian.

Proof. Suppose that $D^2 s = 0$ and $s \neq 0$. Then, $\nabla^*\nabla s + \mathbb{K}s = 0$, so

$$0 = \underbrace{\|\nabla s\|^2}_{\geq 0} + \underbrace{\int_M \langle \mathbb{K}s, s\rangle \mathrm{d}V}_{>0}, \tag{5.4}$$

which is a contradiction. ⊠

Theorem 5.5. *Let $S \to M$ be as above, and K denote the curvature of ∇ on S. Then, $K = R^s + F^s$, where*

$$\begin{aligned} R^s(X, Y) &:= \frac{1}{4}\sum_{i,j} c(e_i)c(e_j)\langle R(X, Y)e_i, e_j\rangle \\ [F^s(X, Y), c(Z)] &= 0. \end{aligned}$$

R^s is usually called the *Riemann endomorphism*, and only depends on the Riemannian metric of the base manifold. F^s is called the *twisting curvature*.

We'll prove Theorem 5.5 in a series of lemmas. First, we find an obstruction for k being a Clifford module endomorphism.

Lemma 5.6.

$$[K(X,Y),c(Z)] = c(R(X,Y)Z).$$

Proof. Let $\{e_i\}$ denote a synchronous frame at a $p \in M$. Then it suffices to prove the lemma for $X = e_i$, $Y = e_j$, and $Z = e_k$. Since we're in a synchronous frame, the $\nabla_{[e_i,e_j]}$ component of the curvature vanishes, so

$$\begin{aligned} K(e_i,e_j)e_k \cdot s|_p &= \left(\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i}\right)(e_k \cdot s)\Big|_p \\ &= \left(R(e_i,e_j)e_k\right) \cdot s + e_k \cdot K(e_i,e_j)s\Big|_p . \end{aligned}$$

The result follows because[7]

$$\nabla_{e_i}(e_k)\nabla_{e_j} s|_p = 0. \qquad \boxtimes$$

Lemma 5.7.

$$[R^s(X,Y),c(Z)] = c(R(X,Y)Z).$$

Proof. Again let $\{e_i\}$ be an orthonormal frame, $X = e_i$, $Y = e_j$, and $Z = e_k$. Then

$$R^s(e_i,e_j)e_k \cdot s = \frac{1}{4}\sum_{\ell,m} c(e_\ell e_m e_k)\underbrace{\langle R(e_i,e_j)e_\ell, e_m\rangle}_{R_{m\ell ij}} s,$$

and similarly

$$c(e_k)R^s(e_i,e_j)s = \frac{1}{4}\sum_{\ell,m} c(e_k e_\ell e_m)R_{m\ell ij}s.$$

Hence when we put these together, we get

$$[R^s(e_i,e_j),c(e_k)] = \frac{1}{4}\sum_{\ell,m} c([e_\ell e_m, e_k])R_{m\ell ij}s. \tag{5.8}$$

If $\ell = m$ and ℓ, m, and k are distinct, then $[e_\ell e_m, e_k] = 0$, so we only care about the cases $\ell = k \neq m$ and $k = m \neq \ell$.

Both the commutator and $R_{m\ell ij}$ are antisymmetric under the exchange of m and ℓ, so (5.8) reduces to

$$\begin{aligned} [R^s(e_i,e_j),c(e_k)] &= \frac{1}{2}\sum_\ell c([e_\ell e_k, e_k])R_{k\ell ij}s \\ &= \sum_\ell c(e_\ell)R_{\ell kij}s \\ &= c\left(\sum_\ell R_{\ell kij}s\right). \end{aligned}$$

Since we're working in an orthonormal frame, the metric looks like the identity matrix in coordinates, so

$$\begin{aligned} &= c\left(\sum_\ell R^\ell_{kij}s\right) \\ &= c(R(e_i,e_j)e_k)s. \qquad \boxtimes \end{aligned}$$

These two lemmas suffice to prove Theorem 5.5.

Remark. Before we go on, let's review Ricci and scalar curvature, which we'll need. Let (M,g) be a Riemannian manifold. Its *Ricci curvature* is the map $\mathrm{Ric}\colon TX \times TX \to \mathbb{R}$ defined by

$$\mathrm{Ric}(X,Y) := \mathrm{tr}(Z \longmapsto R(Z,X)Y).$$

Why this trace? You could try others, but they all vanish or give you the Ricci curvature up to a sign!

Raising an index, define $\mathscr{R}\mathrm{ic}\colon TX \to TX$ by

$$\mathrm{Ric}(X,Y) = g(X, \mathscr{R}\mathrm{ic}(Y)).$$

Then, the *scalar curvature* of (M,g) is $\kappa := \mathrm{tr}(\mathscr{R}\mathrm{ic})$. In an orthonormal frame, it has the formula

$$\kappa = \sum_j \mathrm{Ric}_{jj}\,. \qquad \blacktriangleleft$$

[7] TODO: maybe I missed something.

Theorem 5.9 (Improvement on Theorem 5.2)**.** *With notation as in Theorem 5.2, let $\mathbb{F}^s$ denote the Clifford contraction of F^s and κ denote the scalar curvature of (M,g). Then*

$$D^2 = \nabla^*\nabla + \mathbb{F}^s + \frac{\kappa}{4}\mathbf{1}_{\operatorname{End} S}.$$

Proof. By Theorem 5.5, $\mathbb{K} = \mathbf{R}^s + \mathbb{F}^s$, where $\mathbf{R}^s$ is the Clifford-contracted Riemann endomorphism. So all we have to show is that $\mathbf{R}^s = (\kappa/4)\mathbf{1}_{\operatorname{End} S}$. Again we compute in an orthonormal basis:

$$\begin{aligned}
\mathbf{R}^s &= \sum_{i<j} c(e_i)c(e_j)R^s(e_i,e_j)s\\
&= \frac{1}{2}\sum_{i,j} c(e_ie_j)R^s(e_i,e_j)s\\
&= \frac{1}{8}\sum_{i,j,k,\ell} c(e_ie_je_ke_\ell)\langle R(e_i,e_j)e_k, e_\ell\rangle s\\
&= \frac{1}{8}\sum_{i,j,k,\ell} c(e_ic_je_ke_\ell)R_{\ell kij}s.
\end{aligned}$$

If you decompose this into parts where various subsets of $\{i,j,k,\ell\}$ are equal to each other, the Bianchi identities allow you to simplify this sum:

$$\begin{aligned}
&= \frac{1}{4}\sum_{\ell,i,j} c(e_ic_je_ie_\ell)R_{\ell iij}s\\
&= \frac{1}{4}\sum_{\ell,j} c(e_je_\ell)\underbrace{\left(\sum_i R^i_{\ell ij}\right)}_{\operatorname{Ric}_{\ell j}}\cdot s.
\end{aligned}$$

If $\ell \neq j$, the Ricci tensor piece is antisymmetric, so does not contribute to the sum. Hence we only get the case where $k=\ell$:

$$= \frac{1}{4}\sum_j \operatorname{Ric}_{jj}\mathbf{1}_{\operatorname{End} S},$$

and this is indeed the scalar curvature. ⊠

Now we'll give an example of a Clifford bundle on a non-flat space.

Example 5.10. Let (M,g) be a closed Riemannian manifold and $S := \Lambda^*(T^*M)\otimes\mathbb{C}$. The Riemannian metric on M induces a Riemannian metric on Λ^*T^*M, hence a Hermitian metric on its complexification; similarly, the Levi-Civita connection induces a connection on Λ^*T^*M and therefore also on its complexification. Since the Levi-Civita connection is compatible with the metric on M, our induced connection is compatible with the Hermitian metric on S.

We define the Clifford action $c\colon TM \to \operatorname{End} S$ to satisfy

$$c(e)^2 = -g(e,e)\mathbf{1}_{\operatorname{End} S}, \tag{5.11}$$

which characterizes it uniquely. Namely, if ω is a k-form and $e\in\Gamma_M(TM)$,

$$c(e)\cdot\omega = \widetilde{e}\wedge\omega - e\lrcorner\,\omega,$$

where the first term is a $(k+1)$-form and the second is a $(k-1)$-form. Then

$$\begin{aligned}
c(e)^2\cdot\omega &= \underbrace{\widetilde{e}\wedge\widetilde{e}\wedge\omega}_{=0} - \widetilde{e}\wedge(e\lrcorner\,\omega) - e\lrcorner\,(\widetilde{e}\wedge\omega) + \underbrace{e\lrcorner\,(e\lrcorner\,\omega)}_{=0}\\
&= \underbrace{-e\wedge(e\lrcorner\,\omega)}_{=0} - (\underbrace{e\lrcorner\,\widetilde{e}}_{g(e,e)})\wedge\omega + \underbrace{\widetilde{e}\wedge(e\lrcorner\,\omega)}_{=0}\\
&= -g(e,e)\omega.
\end{aligned}$$

There are more things to check, including

$$g(e\cdot\omega_1,\omega_2) + g(\omega_1, e\cdot\omega_2) = 0, \tag{5.12}$$

which is left as an exercise, and the fact that

$$\nabla_X(e\cdot\omega) = (\nabla_X e)\cdot\omega + e\cdot(\nabla_X\omega). \tag{5.13}$$

One relatively quick way to prove it is to establish that

$$e \lrcorner \omega = (-1)^? \star(\widetilde{e}\wedge\star\omega). \tag{5.14}$$

This implies

$$\begin{aligned}\nabla_X(e\lrcorner\omega) &= (-1)^? \star\nabla_X(\widetilde{e}\wedge\star\omega)\\ &= (-1)^? \star(\nabla_X(\widetilde{e})\wedge\star\omega + \widetilde{e}\wedge\star\nabla_X\omega)\\ &= (\nabla_X e)\lrcorner\omega + e\lrcorner(\nabla_X\omega).\end{aligned}$$

Hence

$$\begin{aligned}\nabla_X(\widetilde{e}\wedge\omega - e\lrcorner\omega) &= (\nabla_X\widetilde{e})\wedge\omega + \widetilde{e}\wedge\nabla_X\omega - \nabla_X(e\lrcorner\omega)\\ &= \nabla_X\widetilde{e}\wedge\omega + e\wedge\nabla_X\omega - (\nabla_X e)\lrcorner\omega - e\lrcorner\nabla_X\omega\\ &= (\nabla_X e)\cdot\omega + e\cdot(\nabla_X\omega),\end{aligned}$$

proving (5.13). Neither side of (5.13) depends on the metric, and in fact there should be a proof that doesn't use it either.

Now we compute the Dirac operator. Let $\{e_i\}$ be a synchronous frame at p. Then

$$\begin{aligned}D\omega|_p &= \sum_i e_i\cdot\nabla_{e_i}\omega\\ &= \sum_i e^i\wedge\nabla_{e_i}\omega\Big|_p - e_i\lrcorner\nabla_{e_i}\omega\big|_p\\ &= \mathrm{d}\omega|_p - \sum_i *-1)^? \star(e^i\wedge\nabla_{e_i}\star\omega)|_p\\ &= \mathrm{d}\omega|_p + \mathrm{d}^*\omega|_p.\end{aligned}$$

This implies $D = \mathrm{d}+\mathrm{d}^*$, which you can again show in a more abstract way. The Laplacian is

$$\Delta := D^2 = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d},$$

called the *Hodge Laplacian*. This in particular exists on any Riemannian manifold, without any reference to Clifford bundles. ◀

There are a few more theorems proven in similar ways to the other ones above.

Theorem 5.15. *Restricted to* $\Lambda^1 T^*M$, $\Delta = \nabla^*\nabla + \mathrm{Ric}$.

Theorem 5.16 (Brchner vanishing theorem). *If* (M,g) *is a Riemannian manifold,* $\mathrm{Ric}\ge 0$*, and* $\mathrm{Ric} > 0$ *at some point, then* $H^1_{\mathrm{dR}}(M) = 0$.

This one uses some Hodge theory (e.g. identifying harmonic representatives of de Rham cohomology classes).

Lecture 6.

Spin groups: 2/7/18

"The moment you might begin to think it's wrong is when you see 'the proof is obvious.' "

Today, Sebastian spoke about superalgebra and spin groups.

Superalgebra. Superalgebra, or more generally supermathematics, is the process of adding the prefix "super" to things to denote their $\mathbb{Z}/2$-graded counterparts with the Koszul sign rule.

Definition 6.1. An algebra A (over $\mathbb{R}$ or $\mathbb{C}$) is called a *superalgebra* if there is a decomposition as vector spaces $A = A_0 \oplus A_1$ such that $A_i \cdot A_j \subset A_{i+j \bmod 2}$. We call A_0 the *even elements*, A_1 the *odd elements*, and $A_0 \cup A_1$ the *homogeneous elements*.

The degree of a homogeneous element $x \in A_0 \cup A_1$ is denoted $|x| \in \mathbb{Z}/2$.

Remark. This is equivalent data to an involutive automorphism $\varepsilon\colon A \to A$ such that $\varepsilon(a_0 + a_1) = a_0 - a_1$; A_0 and A_1 are the ± 1-eigenspaces of ε. ◀

Definition 6.2. Let A be a superalgebra.

- The *supercommutator* of $x, y \in A$ is
$$[x, y]_s := xy - (-1)^{|x||y|} yx.$$
- The *super center* of A is
$$Z_s(A) := \{x \in A \mid [x, y]_s = 0 \text{ for all } y \in A\}.$$

Exercise 6.3. Let (V, Q) be a quadratic space. Check that $C\ell(V, Q)$ is a superalgebra with
$$\begin{aligned} C\ell(V,Q)_0 &:= \operatorname{span}\{e_{i_1} \cdots e_{i_k} \mid k \text{ even}\} \\ C\ell(V,Q)_1 &:= \operatorname{span}\{e_{i_1} \cdots e_{i_k} \mid k \text{ odd}\} \end{aligned}$$

Proposition 6.4. *If V is a real inner product space, $Z_s(C\ell(V)) \cong \mathbb{R}$ and $Z_s(C\ell(V) \otimes \mathbb{C}) \cong \mathbb{C}$.*

Proof. Suppose $x = a + e_1 b$, where $a = a_0 + \cdots + a_{k-1}$ and $b = b_0 + \cdots + b_{k-1}$. Suppose that $x \in Z_s(C\ell(V))$. Then
$$\begin{aligned} 0 &= [x, e_1]_s \\ &= xe_1 - \sum_{i=1}^{k-1} (-1)^i e_i (a_i - e_1 b_i) \\ &= xe_1 - \sum_{i=0}^{k-1} (-1)^i e_1 a - \sum_{i=0}^{k-1} (-1)^i b_i \\ &= \underbrace{\sum_{i=0}^{k-1} (-1)^i e_1 a_i - \sum_{i=0}^{k-1} (-1)^i e_1 a_i}_{=0} + \sum_{i=0}^{k-1} (-1)^{i+1} b_i - \sum_{i=0}^{k-1} (-1)^{i+1} b_i \\ &= -2 \sum_{i=0}^{k-1} (-1)^i b_i. \end{aligned}$$
This forces $b = 0$, so x cannot have any e_1-component. But e_1 was arbitrary. ⊠

The proof boils down to checking minus signs, but is not hard *per se*.

Exercise 6.5. Let V be an oriented inner product space with a positively oriented orthonormal basis $\{e_1, \dots, e_k\}$. Let
$$\omega := e_1 \cdots e_k.$$
(1) Show that ω does not depend on the choice of basis.
(2) Show that $\omega^2 = (-1)^{k(k+1)/2}$.
(3) Show that for $v \in V$, $\omega v = (-1)^{k-1} v \omega$.

Hence if k is odd, $\omega \in Z(C\ell(V))$ (the ordinary center, not the super center). If $k = 2m$, then $\omega^{-1} = (-1)^m \omega$. Hence
$$\varepsilon(x) = \omega x \omega^{-1} = -x \omega \omega^{-1}. \tag{6.6}$$
Hence ε is an inner automorphism.

Pin and spin groups. From now on, fix $C\ell_k$ to be the Clifford algebra of $\mathbb{R}^k$ with its usual inner product, and let $\{e_1, \dots, e_k\}$ be the standard basis for $\mathbb{R}^k$. Any $v \in \mathbb{R}^k$ satisfies $v \cdot v = -\|v\|^2$, so $V \setminus 0 \subset C\ell_k^\times$.

Definition 6.7.

- The *pin⁻ group* $\mathrm{Pin}_k^- \subset C\ell_k^\times$ is the Lie subgroup generated by the norm-1 elements of $\mathbb{R}^k$.
- The *spin group* is $\mathrm{Spin}_k := \mathrm{Pin}_k^- \cap (C\ell_k)_0$.

Example 6.8. Let $k = 1$.

- $C\ell(1) = \mathbb{R} \oplus \mathbb{R} \cdot e_1$ where $e_1^2 = -1$, so $C\ell_1 \cong \mathbb{C}$.
- $\mathrm{Pin}_1^- = \{\pm 1, \pm i\} \subset \mathbb{C}^\times$.
- $\mathrm{Spin}_1 = \{\pm 1\} \subset \mathbb{C}^\times$.

◀

We want to study the representation theory of these groups. One way to produce representations would be to find actions of $C\ell_k$ on itself that preserve $\mathbb{R}^k$.

Recall that $vw + wv = -2\langle v, w\rangle$, and therefore if $\|v\| = 1$, $v^{-1} = -v$. Hence

$$\begin{aligned} -vxv^{-1} &= vxv \\ &= (-xv - 2\langle x, v\rangle)v \\ &= x - 2\langle x, v\rangle v. \end{aligned}$$

Geometrically, this is the reflection of x through the hyperplane $\langle v\rangle^\perp$, as you may remember from linear algebra. This extends to a group representation

$$\begin{aligned} \rho\colon \mathrm{Pin}_k^- &\longrightarrow \mathrm{GL}_k(\mathbb{R}) \\ \rho(y)(x) &:= yx\varepsilon(y^{-1}). \end{aligned} \tag{6.9}$$

This is called the *twisted adjoint representation*. We will also let ρ denote the restriction of this representation to Spin_k. If $y = u_1 \cdots u_\ell$ for $u_1, \dots, u_\ell \in \mathbb{R}^k$, then

$$yx\varepsilon(y^{-1}) = \pm(u_1 \cdots u_\ell)x(u_\ell^{-1} \cdots u_1^{-1})$$

is also a composition of hyperplane reflections in a similar way as above. This means in particular that if $y \in \mathrm{Spin}_k$, this is a composition of an even number of hyperplane reflections, so $\rho\colon \mathrm{Spin}_k \to \mathrm{SO}_k$, not just O_k.

Proposition 6.10. *There is a short exact sequence*

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}_k \xrightarrow{\ \rho\ } \mathrm{SO}_k \longrightarrow 1. \tag{6.11}$$

Proof. First, ρ is surjective, because every element of SO_k can be expressed as an even number of hyperplane reflections.

Next, let $y \in \ker(\rho)$, so that $yx\varepsilon(y^{-1}) = x$ for all x. Hence $y \in Z_s(C\ell(k))$, so by Proposition 6.4, $y \in \mathrm{Spin}_k \cap \mathbb{R}$, which is $\{\pm 1\}$. ⊠

This implies that Spin_k is a compact simple Lie group, and a double cover of SO_k. We want to know which one it is — in particular, is it a trivial double cover? For $k > 1$, no.

Proposition 6.12. *For $k \ge 2$, Spin_k is connected. For $k \ge 3$, Spin_k is simply connected.*

Proof. The short exact sequence (6.11) is in particular a fiber sequence, hence induces a long exact sequence on homotopy groups.

$$\underbrace{\pi_1(\mathbb{Z}/2)}_{1} \longrightarrow \pi_1(\mathrm{Spin}_k) \longrightarrow \pi_1(\mathrm{SO}_k) \longrightarrow \underbrace{\pi_0(\mathbb{Z}/2)}_{\mathbb{Z}/2} \longrightarrow \pi_0(\mathrm{Spin}_k) \longrightarrow \underbrace{\pi_0(\mathrm{SO}_k)}_{1}$$

The first assertion follows because ± 1 are connected in Spin_k, as $t \mapsto \cos t + e_1 e_2 \sin t$ connects them. Then, the second assertion follows because when $k \ge 3$, $\pi_1(\mathrm{SO}_k) \cong \mathbb{Z}/2$, which can be checked with another long exact sequence of a fibration. ⊠

Since $\mathrm{Spin}_k \twoheadrightarrow \mathrm{SO}_k$ is a finite cover, it induces an isomorphism of the Lie algebras $\mathfrak{spin}_k \cong \mathfrak{so}_k = \{A \in M_k(\mathbb{R}) \mid A = -A^T\}$.

Lemma 6.13. $\mathfrak{spin}_k$ *can be identified with the subspace* $\operatorname{span}\{e_ie_j \mid i \neq j\} \subset C\ell_k$ *via the map*

$$A = (a_{ij}) \longmapsto \frac{1}{4}\sum_{i,j} a_{ij}e_ie_j.$$

Representation theory. Let $E_k < \mathrm{Pin}_k^-$ denote the finite subgroup $\{\pm e_1^{i_1}\cdots e_k^{i_k} \mid i_j \in \{0,1\}\}$. Then $|E_k| = 2^{k+1}$. Let $\nu := -1 \in E_k$.

Proposition 6.14. *There's a one-to-one correspondence between the irreducible representations of* $C\ell_k$ *and the irreducible representations of* E_k *in which* ν *acts by* -1.

Remark. Warning: Roe's book has a stronger version of this theorem, which is **not** true! ◀

Since ν is involutive and central, it acts by ± 1. If it acts by ν on some representation V, then V is a representation for $E_k/\langle\nu\rangle$. This corresponds to ignoring signs, and in particular is an abelian group of order 2^k. Therefore it has 2^k conjugacy classes, hence 2^k irreducible representations.

Therefore the conjugacy classes in E_k must be either of the form $\{g\}$, where g is central, or $\{g, g\nu\}$. Since there are 2^k conjugacy classes, you can figure out how many of each there have to be.

Lemma 6.15.

- *If* k *is even,* $Z(E_k) = \{1, \nu\}$.
- *If* k *is odd,* $Z(E_k) = \{1, \nu, \omega, \nu\omega\}$.

Recall that $\omega := e_1\cdots e_k$. This in particular implies

- If k is even, there are $2^k + 1$ irreducible representations of E_k, hence 1 of $C\ell_k$.
- If k is odd, there are $2^k + 2$ irreducible representations of E_k, hence 2 of $C\ell_k$.

For now, assume $k = 2m$ is even, so there's a unique irreducible Δ. Since

$$|E_k| = 2^{k+1} = 2^k\cdot 1 + (\dim\Delta)^2, \tag{6.16}$$

then $\dim\Delta = 2^m$. Explicitly, it arises through a $C\ell_k$-action on an exterior algebra.

So any finite-dimensional $C\ell_k$-representation W is of the form $W\otimes_{\mathbb{C}} V$, where V is some "coefficient" vector space, or a bunch of copies of the trivial representation. It's possible to recover V from W: using Schur's lemma,[8]

$$V \cong \operatorname{Hom}_{C\ell_k\otimes\mathbb{C}}(\Delta,\Delta) \cong (\Delta^*\otimes_{C\ell_k\otimes\mathbb{C}}\Delta)\otimes_{\mathbb{C}} V \cong \Delta^*\otimes_{C\ell_k\otimes\mathbb{C}} W \cong \operatorname{Hom}_{C\ell_k\otimes\mathbb{C}}(\Delta, W). \tag{6.17}$$

A similar calculation constructs a natural isomorphism $\beta\colon \operatorname{End}_{C\ell_k\otimes\mathbb{C}}(W) \xrightarrow{\cong} \operatorname{End}_{\mathbb{C}}(V)$.

Definition 6.18. The *relative trace* $\operatorname{tr}^{W/\Delta} F$ of an $F|in\operatorname{End}_{C\ell(k)\otimes\mathbb{C}}(W)$ is the trace of $\beta(F)\in\operatorname{End}_{\mathbb{C}}(V)$.

Since the elements of Pin_k^- generate $C\ell_k$, then Δ is also an irreducible representation of Pin_k^-. One can use this to show that Spin_k is an index-2 subgroup of Pin_k, hence is automatically normal. Moreover, $\mathrm{Pin}_k^- \twoheadrightarrow \mathrm{O}_k$ is a double cover map.

Since $\mathrm{Spin}_k \lhd \mathrm{Pin}_k^-$, then Δ is also a spin representation. Since Spin_k is index 2, then either Δ is irreducible or splits as $\Delta = \Delta^+\oplus\Delta^-$.

Proposition 6.19. *As a* Spin_k*-representation* Δ *splits as* $\Delta = \Delta^+\oplus\Delta^-$.

Proof. Recall that $\omega = e_1\cdots e_{2m}$, so $\omega^2 = (-1)^m$ and $\omega x = \varepsilon(x)\omega$. Clearly $\omega\in\mathrm{Pin}_k^-$, and since k is even, it's also in Spin_k. Suppose $\omega v = \lambda v$. Then

$$\omega^2 v = (-1)^m v = \lambda^2 v. \tag{6.20}$$

Consider $i^m\omega\in\mathfrak{spin}_k\otimes\mathbb{C}$. Then $(i^m\omega)^2 v = v = \lambda^2 v$, so $\lambda\in\{\pm1\}$. Let $\Delta_\pm$ denote the ±1-eigenspace of $i^m\omega$, so $\Delta = \Delta^+\oplus\Delta^-$ as vector spaces.

Let's consider how $C\ell_k\otimes\mathbb{C}$ acts on this splitting. Since

$$(i^m\omega)xv_\pm = \varepsilon(x)i^m\omega v_\pm = \pm\varepsilon(x)v_\pm,$$

then if x is even, $xv_\pm\in\Delta_\pm$, and if x is odd, $xv_\pm\in\Delta_\mp$. Therefore Spin_k preserves Δ_+ and Δ_-, so they're Spin_k-representations of dimension 2^{m-1}. ⊠

Δ^+ (resp. Δ^-) is called the *positive* (resp. *negative*) *half-spin representation* of Spin_n. In particular, $\Delta = \Delta^+\oplus\Delta_-$ is a $\mathbb{Z}/2$-graded representation of $C\ell_{2m}$.

[8]This requires semisimplicity of $C\ell_n$, which is true but we haven't proven yet.

Lecture 7.

Spin geometry: 2/7/18

These are Arun's notes for his lecture on spin structures on manifolds. Some errors have been corrected.

Tangential structures. Let $\rho\colon H \to G$ be a homomorphism of Lie groups and $\pi\colon P \to M$ be a principal G-bundle. Recall that a *reduction of the structure group* of P to H is data $(\pi'\colon Q \to M, \theta)$ such that

- $\pi'\colon Q \to M$ is a principal H-bundle, and
- $\theta\colon Q \times_H G \to P$ is an isomorphism of principal G-bundles, where H acts on G through ρ.

An equivalence of reductions $(Q_1, \theta_1) \to (Q_2, \theta_2)$ is a map $\psi\colon Q_1 \to Q_2$ intertwining θ_1 and θ_2.

Definition 7.1. Let M be a smooth n-manifold and $\rho\colon H \to \mathrm{GL}_n(\mathbb{R})$ be a homomorphism of Lie groups. If $\mathscr{B}(M) \to M$ denotes the principal $\mathrm{GL}_n(\mathbb{R})$-bundle of frames on M, an *H-structure* on M is an equivalence class of reductions of the structure group of $\mathscr{B}(M)$ to H.

Example 7.2. Let $\rho\colon \mathrm{O}_n \hookrightarrow \mathrm{GL}_n(\mathbb{R})$ be inclusion. A reduction of the structure group of $\mathscr{B}(M)$ to O_n is a smoothly varying choice of which bases of T_xM are orthonormal, i.e. a smoothly varying inner product on T_xM. Hence it's equivalent data to a Riemannian metric. The space of Riemannian metrics on M is connected, which implies that all reductions are equivalent; a manifold has a single O_n-structure. ◀

Example 7.3. Let $\rho\colon \mathrm{SO}_n \hookrightarrow \mathrm{GL}_n(\mathbb{R})$ be inclusion. In this case, a reduction of the structure group of $\mathscr{B}(M)$ to SO_n specifies which bases of T_xM are oriented at every point, and therefore defines an orientation on M. Two reductions are equivalent iff they define the same orientation. Therefore an SO_n-structure on M is equivalent data to an orientation. ◀

In particular: an H-structure is data, and it need not always exist.

Definition 7.4. A *spin structure* on a manifold M is an H-structure for $H = \mathrm{Spin}_n$ along the map $\rho\colon \mathrm{Spin}_n \twoheadrightarrow \mathrm{SO}_n \hookrightarrow \mathrm{GL}_n(\mathbb{R})$. A *spin manifold* is a manifold with a specified spin structure.

Example 7.3 immediately implies that a spin structure determines an orientation.

Example 7.5 (Spin structures on the circle)**.** The map $\mathrm{Spin}_1 \to \mathrm{GL}_1(\mathbb{R})$ factors through $\mathrm{SO}_1 = \{e\}$, so a trivialization of $\mathscr{B}(S^1)$ defines an orientation of S^1, and a spin structure is a lift of the trivial $\{e\}$-bundle to a principal $\mathbb{Z}/2$-bundle. This is the same as a double cover, and there are two isomorphism classes of these.

The connected double cover defines a spin structure which extends over the disc, and is hence called the *bounding spin structure* S^1_b. In physics, it's also called the *Ramond circle*.

The disconnected double cover defines a spin structure which does not extend over the disc, and is hence called the *nonbounding spin structure* S^1_{nb}. In physics, this is sometimes called the *Neveu-Schwarz circle*. ◀

The spinor bundle. Recall from Sebastian's talk the spin representation $\rho\colon \mathrm{Spin}_n \to \mathrm{GL}(\Delta)$. Throughout this section, (M, g) is a Riemannian manifold with a spin structure implemented by a principal Spin_n-bundle of frames $\mathscr{B}_{\mathrm{Spin}}(M) \to M$.

Definition 7.6. The *spin bundle* or *spinor bundle* of M is $\mathscr{B}_{\mathrm{Spin}}(M) \times_{\mathrm{Spin}_n} \Delta \to M$, which is a complex vector bundle of rank n.

The spinor bundle has a lot of structure.

- The spin representation is $\mathbb{Z}/2$-graded, and therefore the spinor bundle is as well: $S = S^0 \oplus S^1$.
- The spinor bundle has a canonical connection D on it: the Levi-Civita connection lifts from $\mathscr{B}(M)$ to $\mathscr{B}_{\mathrm{Spin}}(M)$, and therefore passes to the associated bundle.
- The spin representation factors through the unitary group $\mathrm{U}(\Delta) \hookrightarrow \mathrm{GL}(\Delta)$, and therefore there is an induced Hermitian metric on S.

Proposition 7.7. *The spin connection D is compatible with h, and therefore the spinor bundle is a Clifford bundle.*

Proof. If $(E, \nabla, g) \to M$ is a real vector bundle with connection and metric, compatibility with the metric means that the connection one-forms are valued in $\mathfrak{o}(E, g)$; if E is complex with a Hermitian metric h, compatibility with the metric means the connection one-forms are valued in $\mathfrak{u}(E, g)$. So this is a fun exercise in what "induced connection" actually means.

Since the Levi-Civita connection is metric-compatible, its connection one-form lives in $\Omega^1_{\mathscr{B}(M)}(\mathfrak{o}_n)$. The pullback connection on $\mathscr{B}_{\mathrm{Spin}}(M)$ has connection one-form in $\Omega^1_{\mathscr{B}_{\mathrm{Spin}}(M)}(\mathfrak{spin}_n)$. Since the spinor representation factors through $\mathrm{U}(\Delta) \hookrightarrow \mathrm{GL}(\Delta)$, the connection on the associated bundle factors through $\mathfrak{u}(\Delta) \hookrightarrow \mathfrak{gl}(\Delta)$; in particular, its connection one-form lives in $\Omega_S(\mathfrak{u}(\Delta))$, so D is compatible with the metric. ⊠

Proposition 7.8. *The twisting curvature of the spinor bundle is zero.*

Proof. Recall from Sebastian's talk that there's a natural identification of $\mathfrak{spin}_n$ with the vector subspace of $C\ell_n$ spanned by products e_ie_j for $i \neq j$ by the assignment

$$a_{ij} \longmapsto \frac{1}{4}\sum a_{ij}e_ie_j. \tag{7.9}$$

Let $\{e_k\}$ be a local orthonormal frame for TM. Therefore the curvature of the Levi-Civita connection is an $\mathfrak{so}_n$-valued 2-form whose (k,ℓ) entry is $\langle Re_k, e_\ell\rangle$, where R is the Riemann curvature tensor. By (7.9), the curvature 2-form of the spin connection is

$$K_{\mathrm{Spin}} = \frac{1}{4}\sum_{k,\ell}\langle Re_k, e_\ell\rangle e_k e_\ell. \tag{7.10}$$

This acts on the spinor representation through $e_i \mapsto c(e_i)$, i.e. through

$$\frac{1}{4}\sum_{k,\ell}\langle Re_k, e_\ell\rangle c(e_k)c(e_\ell). \tag{7.11}$$

With notation as last time, this is exactly $R^s(e_j, e_i)$, so since the twisting curvature F^s satisfies $F^s = K_{\mathrm{Spin}} - R^s$, $F^s = 0$. ⊠

Spin^c structures.

Definition 7.12. The Lie group Spin_n^c is defined to be $\mathrm{Spin}_n \times_{\mathbb{Z}/2} \mathrm{U}_1$, where $\mathbb{Z}/2 \hookrightarrow \mathrm{Spin}_n$ as $\{\pm 1\} \subset C\ell_n$ and $\mathbb{Z}/2 \hookrightarrow \mathrm{U}_1$ as $\{\pm 1\} \subset \mathbb{C}$.

This means in particular there is a short exact sequence

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathrm{Spin}_n^c \longrightarrow \mathrm{SO}_n \times \mathrm{U}_1 \longrightarrow 1. \tag{7.13}$$

A *spin^c structure* on a manifold M is an H-structure for $H = \mathrm{Spin}_n^c$ along the map to SO_n. This comes with some extra structure.

- The map $\mathrm{Spin}_n^c \to \mathrm{SO}_n \times \mathrm{U}_1$ means that a Spin_n^c-structure on M determines an orientation and a principal U_1-bundle $P \to M$. The associated complex line bundle $L := P \times_{\mathrm{U}_1} \mathbb{C} \to M$ is called the *fundamental line bundle* associated to the spin^c structure; its Chern class is called the *Chern class* of the spin^c structure.
- Since the spin representation is a representation of the complexified Clifford algebra, it also induces a representation of Spin_n^c. In the same manner as above, a spin^c structure on M defines an associated real vector bundle $E^c \to M$, again called the *spinor bundle S^c*. Since the spin representation is unitary, the spinor bundle acquires a Hermitian metric.

However, we don't get a connection for free: instead we have to choose a connection ∇_L on L; together with the Levi-Civita connection on $\mathscr{B}_{\mathrm{SO}}(M)$, this induces a connection ∇^c on $\mathscr{B}_{\mathrm{Spin}_n^c}(M) \to M$.

Proposition 7.14. *In this situation, the twisting curvature of ∇^c is $(1/2)K_L$, where K_L is the curvature of ∇_L.*

Characteristic classes. We can use characteristic classes to determine whether a manifold has a spin or spin^c structure.

First we mention a general form of the *splitting principle*, a way of making computations about characteristic classes for principal G-bundles.

Let G be a compact Lie group, T be a maximal torus for G, and $P \to X$ be a principal G-bundle over a paracompact base X. The quotient map induces a map $f : P/T \to X$, and we can pull P back along f:

$$\begin{tikzcd} f^*P \arrow[r] \arrow[d, "G"] & P \arrow[d, "G"] \\ P/T \arrow[r, "f"] & X. \end{tikzcd} \tag{7.15}$$

Then, there is a canonical reduction of the structure group of f^*P along $T \hookrightarrow G$. In general, T splits as a product of copies of S^1, so f^*P similarly splits. The key fact is that the map $f^*\colon H^*(X) \to H^*(P/T)$ is injective. Therefore any question about characteristic classes of P may be solved for f^*P, which splits, with no information lost.

Definition 7.16. If E is a real vector bundle, $\mathfrak{S}(E)$ will denote the Pontrjagin genus of E associated to

$$g(z) = \cosh\left(\frac{1}{2}\sqrt{z}\right).$$

Proposition 7.17. *The Chern character of the spinor bundle on an even-dimensional spin manifold M is*

$$\mathrm{ch}(S) = 2^m \mathfrak{S}(TM).$$

Proof. We will prove a stronger result. A *spin structure* on a real vector bundle $E \to M$ is an (equivalence class of) reduction of the structure group of the principal $\mathrm{GL}_n(\mathbb{R})$-bundle of local frames of E across $\mathrm{Spin}_n \to \mathrm{GL}_n(\mathbb{R})$. Given a spin structure on E, one can define its spinor bundle S_E in the same manner as before. We will show that if E has even rank, then

$$\mathrm{ch}(S_E) = 2^{\dim E/2}\mathfrak{S}(E). \tag{7.18}$$

If $E_1, E_2 \to X$ are even-dimensional spin vector bundles, $S_{E_1 \oplus E_2} \cong S_{E_1} \otimes S_{E_2}$, both sides of (7.18) are multiplicative under direct sum. Therefore we may apply the splitting principle for principal Spin_n-bundles to E. One choice of maximal torus in Spin_n is

- $\mathrm{Spin}_2 \times \cdots \times \mathrm{Spin}_2 \hookrightarrow \mathrm{Spin}_n$ if n is even, or
- $\mathrm{Spin}_2 \times \cdots \times \mathrm{Spin}_2 \times \{1\} \hookrightarrow \mathrm{Spin}_n$ if n is odd.

Thus over E/T, $\mathscr{B}_{\mathrm{Spin}}(E)$ splits as a direct sum of principal Spin_2-bundles, so E splits as a sum of rank-2 vector bundles with spin structure. Since (7.18) is additive, it suffices to assume $\operatorname{rank} E = 2$.

Recall that a *classifying space* BG for a Lie group G is a space with a principal G-bundle $EG \to BG$ whose total space is contractible. This determines BG up to homotopy, and also implies that every principal G-bundle $P \to X$ is the pullback of EG along a map $X \to BG$. Since characteristic classes are natural under pullback, verifying (7.18) over $B\mathrm{Spin}_2$ will prove it everywhere.

Given a representation V of G, we get a vector bundle $EG \times_G V \to BG$. The universal choice of $E \to \mathrm{Spin}_2$ is the one induced from the representation $\mathrm{Spin}_2 \twoheadrightarrow \mathrm{SO}_2 \hookrightarrow \mathrm{GL}_2(\mathbb{R})$, and its spinor bundle is the complex vector bundle $\Delta \to B\mathrm{Spin}_2$ associated to the spinor representation.

Since the map $\mathrm{Spin}_2 \twoheadrightarrow \mathrm{SO}_2$ is isomorphic to the map $\mathbb{T} \to \mathbb{T}$ which is multiplication by 2, the map $H^*(B\mathrm{SO}_2;\mathbb{Q}) \to H^*(B\mathrm{Spin}_2;\mathbb{Q})$ is also multiplication by 2. Therefore if $x \in H^1(B\mathrm{SO}_2;\mathbb{Q})$ denotes the generator, which is the Euler class of the defining representation, $H^*(B\mathrm{SO}_2;\mathbb{Q}) \cong \mathbb{Q}[x]$ and $H^*(B\mathrm{Spin}_2;\mathbb{Q}) \cong \mathbb{Q}[x/2]$. Since $E \to \mathrm{Spin}_2$ comes from the defining representation of SO_2, $c_1(E_{\mathbb{C}}) = x$.

Next we compute $\mathrm{ch}(\Delta)$. This spinor representation is $\mathrm{Spin}_2 \hookrightarrow C\ell_2 \otimes \mathbb{C} \cong M_2(\mathbb{C}) \cong \mathrm{End}(\mathbb{C}^{1|1})$, which acts on $\mathbb{C}^{1|1}$ and therefore induces a representation of the spin group. Since $\mathrm{Spin}_2 \cong \mathrm{U}_1$ abstractly, and irreducible representations of U_1 are $z \mapsto z^n$ for an $n \in \mathbb{Z}$, we will decompose Δ into characters and compare them to E.

The identification $C\ell_2 \otimes \mathbb{C} \cong M_2(\mathbb{C})$ sends

$$e_1 \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{7.19a}$$

$$e_2 \longmapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{7.19b}$$

and therefore

$$e_1e_2 \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.19c}$$

This is the direct sum of Δ^+ (first entry) and Δ^- (second entry). In particular, the Δ^+ is the character $z \mapsto z$, and Δ^- is the character $z \mapsto z^{-1}$. Via the identification $H^*(B\mathrm{Spin}_2;\mathbb{Q}) \cong \mathbb{Q}[x/2]$, $c_1(\Delta^\pm) = \pm x/2$, so $\mathrm{ch}(\Delta^\pm) = e^{\pm x/2}$. Therefore

$$\mathrm{ch}(\Delta) = \mathrm{ch}(\Delta^+) - \mathrm{ch}(\Delta^-) = e^{x/2} - e^{-x/2} = 2\cosh\left(\frac{x}{2}\right). \tag{7.20}$$

We also have $p_1(E) = e(E)^2 = x^2$, so

$$\mathfrak{S}(E) = \cosh\left(\frac{1}{2}\sqrt{p_1(E)}\right) = \cosh\left(\frac{x}{2}\right). \tag{7.21}$$ ⊠

Definition 7.22. Let $B\mathrm{SO}_n$ denote the classifying space of SO_n; there's a natural bijection between isomorphism classes of principal SO_n-bundles $P \to X$ and homotopy classes of maps $X \to B\mathrm{SO}_n$. One can show that when $n \geq 2$, $H^2(B\mathrm{SO}_n;\mathbb{Z}/2) \cong \mathbb{Z}/2$, with generator denoted w_2. Given a principal SO_n-bundle $P \to X$, the pullback of w_2 by the classifying map $X \to B\mathrm{SO}_2$ is called the *second Stiefel-Whitney class* of P, denoted $w_2(P) \in H^2(X;\mathbb{Z}/2)$.

Proposition 7.23. *Let M be a closed, oriented manifold. Then, M has a spin structure iff $w_2(M) = 0$. In this case, the spin structures of M extending its given orientation are a torsor over $H^1(M;\mathbb{Z}/2)$.*

Proof. We follow the proof in Roe's exercises. We'll assume $n \geq 2$; otherwise small modifications must be made, but the result is still true. Let M be a closed, oriented, manifold and $\pi\colon \mathscr{B}_{\mathrm{SO}}(M) \to M$ be its principal SO_n-bundle of frames. This is a fiber bundle with fiber SO_n, hence has an associated spectral sequence

$$E_2^{p,q} = H^p(M; H^q(\mathrm{SO}_n;\mathbb{Z}/2)) \Longrightarrow H^{p+q}(\mathscr{B}_{\mathrm{SO}}(M);\mathbb{Z}/2). \tag{7.24}$$

Associated a first-quadrant spectral sequence converging to $H^n(A)$ is its *five-term exact sequence*

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1(A) \longrightarrow E_2^{0,1} \longrightarrow E_2^{2,0} \longrightarrow H^2(A), \tag{7.25}$$

which in this case specializes to
(7.26)

$$0 \longrightarrow H^1(M;\mathbb{Z}/2) \xrightarrow{\pi^*} H^1(\mathscr{B}_{\mathrm{SO}}(M);\mathbb{Z}/2) \xrightarrow{i^*} H^1(\mathrm{SO}_n;\mathbb{Z}/2) \xrightarrow{\delta} H^2(M;\mathbb{Z}/2) \longrightarrow H^2(\mathscr{B}_{\mathrm{SO}}(M);\mathbb{Z}/2).$$

We claim, and will prove in a bit, that δ sends the generator of $H^1(\mathrm{SO}_n;\mathbb{Z}/2)$ to $w_2(M)$.

A spin structure lifting the specified orientation of M is the data of an isomorphism class of double covers $\mathscr{B}_{\mathrm{Spin}}(M) \to \mathscr{B}_{\mathrm{SO}}(M)$ which induce the connected double cover $\mathrm{Spin}_n \to \mathrm{SO}_n$ when restricted to each fiber. Isomorphism classes of covers of $\mathscr{B}_{\mathrm{SO}}(M)$ are naturally identified with $H^1(\mathscr{B}_{\mathrm{O}}(M);\mathbb{Z}/2)$, and their restriction to a fiber is the map i^* in (7.26), so the set of spin structures on M is in bijection with the subset of $H^1(\mathscr{B}_{\mathrm{SO}}(M);\mathbb{Z}/2)$ which is *not* in $\ker(i^*)$.

Suppose that $w_2(M)$ vanishes, meaning $\delta = 0$, so i^* is a surjective map onto $\mathbb{Z}/2$. Therefore $|(i^*)^{-1}(0)| = |(i^*)^{-1}(1)|$ and π^* is an injection whose image is $(i^*)^{-1}(0)$. Thus we have a noncanonical bijection $H^1(M;\mathbb{Z}/2) \to (i^*)^{-1}(1)$, the set of spin structures on M. The torsor structure arises because $(i^*)^{-1}(0) + (i^*)^{-1}(1) \subset (i^*)^{-1}(1)$.

If instead $w_2(M) \neq 0$, then δ is injective. Then $i^* = 0$, so there can be no spin structures on M.

To finish this, we had better check that δ sends the generator of $H^1(\mathrm{SO}_n;\mathbb{Z}/2)$ to w_2. It suffices to check this for the universal bundle $E\mathrm{SO}_n \to B\mathrm{SO}_n$, as all other principal SO_n-bundles are pullbacks of this one, but since $E\mathrm{SO}_n$ is contractible, (7.26) simplifies to

$$0 \longrightarrow H^1(B\mathrm{SO}_n;\mathbb{Z}/2) \longrightarrow 0 \longrightarrow H^1(\mathrm{SO}_n;\mathbb{Z}/2) \xrightarrow{\delta} H^2(B\mathrm{SO}_n;\mathbb{Z}/2) \longrightarrow 0. \tag{7.27}$$

so δ has to send the nonzero element to the generator of $H^2(B\mathrm{SO}_n;\mathbb{Z}/2)$, which is w_2, as desired. ⊠

Proposition 7.28. *Let M be a closed manifold. Then, M has a spin^c structure iff $w_2(M)$ is the reduction of an integral class. In this case, the spin^c structures of M are a torsor over $H^2(M;\mathbb{Z})$.*

Corollary 7.29. *A spin structure determines a spin^c structure; the converse is not true: $\mathbb{CP}^2$ is a spin^c manifold which is not spin.*

Partial proof. We'll prove the last part, by showing $w_2(\mathbb{CP}^2) \neq 0$. It's equal to the mod 2 reduction of $c_1(\mathbb{CP}^2)$, and $c(\mathbb{CP}^2) = (1+x)^3$, where x is the generator dual to a $\mathbb{CP}^1$, so $c_1(\mathbb{CP}^2)$ is odd and therefore $w_2(\mathbb{CP}^2) \neq 0$, so $\mathbb{CP}^2$ is not spin, but it is spin^c. ⊠

Lecture 8.

Analytic properties of the Dirac operator: 2/14/18

Today, Gill and Kenny spoke about analytic questions related to the Dirac operator $\not{D}\colon C^\infty(S) \to C^\infty(S)$, where $S \to M$ is the spinor bundle on a compact spin Riemannian manifold M.

Since $C^\infty(S)$ is in general an infinite-dimensional vector space, we would like to place a good topology on it. The default is the Fréchet topology, but this is kind of unpleasant to work with: it's not a Hilbert space, for example. Maybe we can relax the regularity of things we feed to $\not{D}$, but since we're interested in the Dirac Laplacian $\not{D}^2$, which is a second-order differential operator, we want everything to be at least twice differentiable.

Relatedly, one asks how smooth solutions to $\not{\partial} x = y$ can be. For example, in $\mathbb{R}^2$, harmonic functions, those functions $u\colon \mathbb{R}^2 \to \mathbb{C}$ satisfying

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{8.1}$$

are smooth and in fact analytic, because $f := u + i\overline{u}$ is holomorphic. This is an instance of *elliptic regularity*, a very general phenomenon.

Thus the general place to start would be to use the induced Hermitian metric h on S and take the space $L^2(S)$, the Hilbert space of L^2 sections of $S \to M$ with respect to h. Since smooth sections are dense in $L^2(S)$, we know how to make sense of $\not{\partial}: L^2(S) \to L^2(S)$.

Exercise 8.2. Show that $\not{\partial}: L^2(S) \to L^2(S)$ is unbounded, i.e. there is no $C \in \mathbb{R}$ such that for all $X \in L^2(S)$, $\|DX\|_{L^2} \le C\|X\|_{L^2}$.

This is a little unfortunate; we would like something whose functional analysis is more tractable. For this reason, we'll change the Hilbert space we're working with slightly.

Recall that if $f, g \in C^\infty(S)$, we defined

$$\langle f, g\rangle_{L^2} := \int_M h(f(x), g(x))\, \mathrm{d}V, \tag{8.3}$$

where $\mathrm{d}V$ is the volume form on M. Then $L^2(S)$ is defined to be the completion of $\{f \in C^\infty(S) \mid \langle f, f\rangle < \infty\}$ in the norm induced by this inner product. Therefore by changing the inner product we can change what Hilbert space we're using.

Definition 8.4. The k^{th} *Sobolev inner product* of smooth sections f and g is

$$\langle f, g\rangle_{H^k} := \int_M \sum_{i=0}^{k} h(\nabla^i f, \nabla^i g)\, \mathrm{d}V,$$

where $\nabla^0 := \mathrm{id}$, and h denotes both the Hermitian metric on S and the induced metrics on $(T^*M)^{\otimes i} \otimes S$ using h and the Riemannian metric on T^*M defined by taking the product of all the metrics.

The completion of $C^\infty(S)$ under $\langle -, -\rangle_{H^k}$ is denoted $H^k(S)$ and called a *Sobolev space*.

Proposition 8.5. *Any choices of connection and metric on M induce equivalent Sobolev k-norms.*

That is, the underlying topological vector spaces of these Hilbert spaces are isomorphic.

Let T^n be the n-torus. It suffices to understand the proposition on all charts U_α of M for an atlas which trivializes S, and we can map these charts onto T^n, thus reducing the question to T^n.

On T^n we use Fourier theory: for any L^2 $f: T^n \to \mathbb{C}$, its *Fourier series* is

$$f(x) = \sum_{\nu \in \mathbb{Z}^n} \widehat{f}(\nu) e^{i\nu x}, \tag{8.6}$$

where

$$\widehat{f}(\nu) := \frac{1}{(2\pi)^n} \int_{T^n}. \tag{8.7}$$

It's a standard result that sending $f \mapsto \widehat{f}$ converts differentiation into multiplication: if f is C^1, then

$$\left(\frac{\partial}{\partial x_j}\right)^{\wedge}(\nu) = \frac{1}{(2\pi)^n} \int_{T^n} \left(\frac{\partial}{\partial x_j} f\right)(x) e^{-i\nu x}\, \mathrm{d}x. \tag{8.8}$$

Hence for an arbitrary L^2 f, we can define $\partial_j f$ through its Fourier series:

$$\frac{\partial f}{\partial x_j} := \sum_{\nu \in \mathbb{Z}^n} i\nu \widehat{f}(\nu) e^{i\nu x}. \tag{8.9}$$

Definition 8.10. Let $H^k(T^n)$, $k \in \mathbb{R}$. The k^{th} *Sobolev space* on T^n, denoted $H^k(T^n)$, is the completion of the subspace of $L^2(T^n)$ for which the norm

$$\|f\|_k^2 := \sum_{\nu \in \mathbb{Z}^n} |\widehat{f}(\nu)|^2 \left(1 + |\nu|^2\right)^k$$

is finite, in this norm. Here $|\nu| = \sum \nu_i^2$.

For example,

$$\|f\|_1^2 = \|f\|_{L^2}^2 + \sum_j \left\| \frac{\partial f}{\partial x_j} \right\|_{L^2}^2, \tag{8.11}$$

which uses the fact that the Fourier series functional is an isometry.

Proposition 8.12. *There exist constants $c_1, c_2 \in \mathbb{R}$ such that*

$$c_1 \sum_{|\boldsymbol{\alpha}| \le k} \left\| \frac{\partial f}{\partial x_{\boldsymbol{\alpha}}} \right\|_{L^2} \le \|f\|_k \le c_2 \sum_{|\boldsymbol{\alpha}| \le k} \left\| \frac{\partial f}{\partial x_{\boldsymbol{\alpha}}} \right\|_{L^2}.$$

This implies in particular that if D is a linear ℓ^{th}-order differential operator, then it's a linear map $D \colon H^k \to H^{k-\ell}$.

Proposition 8.13. *Let $k_1 \le k_2$. There are continuous embeddings*
- $H^{k_1}(S) \hookrightarrow H^{k_2}(S)$ *and*
- $C^k(T^n) \hookrightarrow H^k(T^n)$.

Most of these propositions are true in general Sobolev spaces, since they're all proven locally.

Theorem 8.14 (Sobolev embedding theorem)**.** *If $p > n/2$, $H^{k+p} \hookrightarrow C^k$ continuously.*

Recall that a map of topological vector spaces is *compact* if it carries closures of bounded spaces to compact spaces. (By linearity, it suffices to consider the closed unit ball.)

Theorem 8.15 (Rellich)**.** *If $k_1 \le k_2$, the embedding $H^{k_1} \hookrightarrow H^{k_2}$ is compact.*

With these preliminaries out of the way, we proceed to the actual analysis of the Dirac operator. The estimates we discuss apply more generally, to any first-order operator D on $C^\infty(S)$ such that

$$D^2 = \nabla^*\nabla + B \tag{8.16}$$

for some first-order operator B on S. Since M is compact, $C^\infty(S) \hookrightarrow L^2(S)$, so D is a bounded map $H^1(S) \to L^2(S)$, i.e. there's a $C \in \mathbb{R}$ such that for all $s \in C^\infty(S)$, $\|Ds\|_0 \le C\|s\|_1$. We will call such a D a *generalized Dirac operator*.

Proposition 8.17 (Gårding inequality)**.** *Let D be a generalized Dirac operator. Then there is a C such that for any $s \in C^\infty(S)$,*

$$\|s\|_1 \le C(\|s\|_0 + \|Ds\|_0).$$

This is an analogue of the smoothness of harmonic functions we discussed above, buying us unexpected extra regularity.

Proof. The proof follows fairly standard methods in PDE. It suffices to assume that S is the trivial bundle, because we can work locally: let $\mathfrak{U}$ be an open cover of M which trivializes S and $\{\psi_U : U \in \mathfrak{U}\}$ be a partition of unity subordinate to it. Then

$$\|s\|_0^2 = \sum_{U \in \mathfrak{U}} \int_U \left\langle \sqrt{\psi_U} s, \sqrt{\psi_U} s \right\rangle \mathrm{d}x. \tag{8.18}$$

Since M is compact we can take $\mathfrak{U}$ to be finite, and therefore take the maximum of the constants we got on each $U \in \mathfrak{U}$.

Since

$$\langle D^2 s, s\rangle = \langle \nabla^*\nabla s, s\rangle + \langle Bs, s\rangle, \tag{8.19}$$

then

$$\|Ds\|_0^2 = \|\nabla s\|_0^2 + \langle Bs, s\rangle, \tag{8.20}$$

so by the Cauchy-Schwarz inequality, and the fact that B is bounded in H^1,

$$\|\nabla s\|_0^2 \le c_1\big(\|s\|_0 \|s\|_1 + \|Ds\|_0^2\big). \tag{8.21}$$

We will bound the left side of (8.21) below. Since we're working locally, $\nabla_i s = \partial_i s + T_i s$, where $\nabla s := \mathrm{d}x^i \otimes \nabla_i s$, and therefore

$$\begin{aligned}\langle \nabla s, \nabla s\rangle &= \left\langle \sum_i \mathrm{d}x^i \otimes \nabla_i s, \sum_j \mathrm{d}x^j \otimes \nabla_j s\right\rangle\\ &= \sum_{i,j} \underbrace{\langle \mathrm{d}x^i, \mathrm{d}x^j\rangle}_{g^{ij}} \langle \nabla_i s, \nabla_j s\rangle\\ &= \int \sum_{i,j} g^{ij}\big(\langle \partial_i s, \partial_j s\rangle + 2\operatorname{Re}\big(\partial_i s, T_j s\big) + \big(T_i s, T_j s\big)\big)\,\mathrm{d}x.\end{aligned}$$

Since

$$\|Ds\|_0^2 := \int \langle \nabla s, \nabla s\rangle \,\mathrm{d}x \ge C_2\|s\|_1^2 - C_3\|s\|_0\|s\|_1 \tag{8.22}$$

(the reasoning for which is a little unclear), then

$$C_2\|s\|_1^2 - C_3\|s\|_0\|s\|_1 \le C_1\big(\|s\|_0\|s\|_1 + \|Ds\|_0^2\big), \tag{8.23}$$

so there are constants C_4, C_5 such that

$$\|Ds\|_0^2 \ge C_4\|s\|_1^2 - C_5\|s\|_0\|s\|_1. \tag{8.24}$$

It's a general fact (proven maybe by a convexity argument?) that for any $\varepsilon > 0$, there's a $K > 0$ such that for all $a, b > 0$,

$$ab \le \varepsilon a^2 + Kb^2. \tag{8.25}$$

Therefore there's a C_6 such that

$$C_5\|s\|_0\|s\|_1 \le \frac{1}{2}C_4\|s\|_1^2 + C_6\|s\|_0^2, \tag{8.26}$$

so

$$\|Ds\|_0^2 \ge \frac{1}{2}C_4\|s\|_1^2 - C_6\|s\|_0^2, \tag{8.27}$$

which can be rearranged into what we wanted to show. ⊠

The Gårding inequality says that we can control all of the first derivatives of s if we can control Ds (we need the extra term in case D has kernel – if the first term weren't there, this would've been a proof that D were injective). Something elliptic is going on here, and we can bootstrap it into stronger estimates.

Theorem 8.28 (Elliptic estimate). *For any $k > 0$, there's a constant C_k such that for all $s \in C^\infty(S)$,*

$$\|s\|_{k+1} \le C_k(\|s\|_k + \|Ds\|_k).$$

We now turn to spectral theory. This is particularly nice for Hilbert spaces, so when one wants to study the spectral theory of an operator, part of the story is to find a Hilbert space on which its spectrum is nice.

Definition 8.29. By an *unbounded operator* on a Hilbert space H we mean a linear operator $f : U \to H$, where U is a dense subset of H.

We make no assumptions on continuity; hence unbounded operators are a (in general strict) superset of bounded operators.

Example 8.30. We defined $L^2(S)$ to be the completion of $C^\infty(S)$ under the L^2 norm, and hence $D\colon C^\infty(S) \to L^2(S)$ is an unbounded operator. ◀

Proposition 8.31. *Let D be a generalized Dirac operator. Then the closure of the graph of D is the also graph of an unbounded operator $\overline{D}$.*

Proof. Recall that D has a formal adjoint, i.e. an operator $D^\dagger$ such that $\langle Dx, y\rangle = \langle x, D^\dagger y\rangle$. Let Γ denote the graph of D and suppose that $\overline{\Gamma}$ isn't the graph of a function; then, $\overline{\Gamma}$ fails the "vertical line test," so there are $x, y_1, y_2 \in H$ such that $(x, y_1), (x, y_2) \in \overline{\Gamma}$. Since D is linear, then $(0, y_1 - y_2) \in \overline{\Gamma}$, and $y_1 - y_2 \neq 0$.

Hence there are $x_i \in C^\infty(S)$ such that $x_i \to 0$ and $Dx_i \to z \neq 0$ in L^2. Hence for all $s \in C^\infty(S)$, $\langle Dx_i, s\rangle \to \langle z, s\rangle$ and $\langle x_i, D^\dagger s\rangle \to 0$, but since $\langle Dx_i, s\rangle = \langle x_i, D^\dagger s\rangle$, $\langle z, s\rangle = 0$ for all s. Hence $z = 0$.

From this and the fact that $\overline{\Gamma}$ is a linear subspace of $H \oplus H$, one can infer that it's the graph of a *linear* operator. ⊠

What's the domain of $\overline{D}$? It's all $x \in L^2(S)$ for which there exists a sequence $x_i \to x$ converging in L^2 such that $Dx_i \to \overline{D}x$ in L^2. Hence by Proposition 8.17, the domain contains $H^1(S)$, and one can show that the domain is $H^1(S)$.

Definition 8.32. Let $x, y \in L^2(S)$. We say $Dx = y$ *weakly* if for all $s \in C^\infty(S)$, $\langle x, D^\dagger s\rangle = \langle y, s\rangle$ (equivalently, $\langle Dx, s\rangle = \langle y, s\rangle$).

Denote by $S \boxtimes S^* \to M \times M$ the bundle $\pi_1^* S \otimes \pi_2^* S^*$, where $\pi_i \colon M \times M \to M$ is projection onto the i^{th} factor.

Definition 8.33. Let $A \colon H \to H$ be a bounded operator. Then, A is a *smoothing operator* if there is a smooth section $k(p, q)$ of $S \boxtimes S^*$, called the *kernel* of A, such that

$$A(s)(p) = \int_M k(p, q) s(q)\, \mathrm{d}q. \tag{8.34}$$

The assertion that the integral in (8.34) converges depends both on the compactness of M and a nontrivial element.

Definition 8.35. A *Friedrichs' mollifier* for a section s is a family F_ε of self-adjoint smoothing operators on $L^2(S)$, indexed by $\varepsilon \in (0, 1)$, such that

(1) $\{F_\varepsilon\}$ is bounded in $B(L^2(S))$,
(2) for any first-order differential operator B, $\{[B, F_\varepsilon]\}$ extends to a bounded family on $B(L^2(S))$, and
(3) $F_\varepsilon \rightharpoonup 1$ as $\varepsilon \to 0$ (i.e. in the weak topology on $B(L^2(S))$).

Exercise 8.36. Friedrichs' mollifiers exist.

Theorem 8.37. *Suppose that $x, y \in L^2(X)$ and $Dx = y$ weakly. Then $x \in H^1$ and $\overline{D}x = y$.*

Proof. Let $\{F_\varepsilon\}$ be a Friendrichs' mollifier and $x_\varepsilon := F_\varepsilon x$. Then for any $s \in C^\infty(S)$,

$$\begin{aligned}
\langle Dx_\varepsilon, s\rangle &= \langle x_\varepsilon, D^\dagger s\rangle \\
&= \langle x, F_\varepsilon D^\dagger s\rangle \\
&= \langle x, D^\dagger F_\varepsilon s\rangle + \underbrace{\langle x, [F_\varepsilon, D^\dagger] s\rangle}_{\le \|x\|_0 \|[F_\varepsilon, D^\dagger]\|_{B(L^2(S))} \|s\|_0} \\
&= \langle y, F_\varepsilon s\rangle \le C' \|y\| \|F_\varepsilon\|_B \|s\|_0 \\
&\le K \|s\|_0
\end{aligned}$$

for some K. Using the first two properties of Friedrichs' mollifiers, $|\langle Dx_\varepsilon, s\rangle| \le C' \|s\|_0$; since $\|Dx_\varepsilon\|_0 \le C$, then by Proposition 8.17, $\|x_\varepsilon\|_{H^1} \le C'$.

This implies there exists a subsequence $\varepsilon_j \to 0$ such that $x_{\varepsilon_j} \rightharpoonup z$ in H^1.

Exercise 8.38. Show that compact operators send weakly convergent sequences to strongly convergent sequences.

Thus we can conclude that $x = z$: $\langle x_{\varepsilon_j}, s\rangle \to \langle x, s\rangle = \langle z, s\rangle$, so $\langle x, D^\dagger s\rangle = \langle y, D^\dagger s\rangle$, and after an argument that I missed, this implies that $\overline{D}x = y$. ⊠

The next theorem tells us that eigenvalues of a (generalized) Dirac operator have better regularity than one might expect.

Theorem 8.39. *Let D be a generalized Dirac operator. Then, the kernel of $\overline{D}$ consists of smooth sections.*

Proof. Let $s \in \ker(\overline{D})$; we'll show that $s \in H^k$ for all k, which implies by Theorem 8.14 that it's smooth. We'll induct by assuming $s \in H^{k-1}$. Let $\{F_\varepsilon\}$ be a Friedrichs' mollifier, so that $\{F_\varepsilon\}$ and $\{[D, F_\varepsilon]\}$ are bounded families of operators on H^{k-1}.

Using the elliptic estimate,

$$\|F_\varepsilon s\|_k \le C_k(\|F_\varepsilon s\|_{k-1} + \|DF_\varepsilon s\|_{k-1}) = C_k(\|F_\varepsilon s\|_{k-1} + \|[D, F_\varepsilon]s\|_{k-1}). \tag{8.40}$$

Since $Ds = 0$, this is bounded above. ⊠

Lecture 9.

Spectral theory: 2/21/18

"There's a reason to erase the boards at the beginning of a lecture rather than the end: you have an incentive to do it well."

This part of today's lecture was given by Dan.

Let V be a finite-dimensional complex inner product space and $T : V \to V$ be a self-adjoint operator. Then we can diagonalize T: there is an orthogonal direct sum

$$V = \bigoplus_{\lambda \in \mathbb{R}} V_\lambda \tag{9.1}$$

where $T|_{V_\lambda} = \lambda$. Obviously only finitely many V_λ are nonzero.

Geometrically, you could think of this as a family of vector spaces over a base $\mathbb{R}$, and in fact it's a sheaf (a finite sum of skyscraper sheaves).

Remark. If you don't have an inner product and T isn't self-adjoint, you can still get something useful, the Jordan decomposition. In this case the sheaf has nilpotent information if there are nontrivial Jordan blocks, but we're not going to encounter this in this course. ◂

Given a function $f : \mathbb{R} \to \mathbb{R}$, one obtains a new operator $f(T)$ defined by $f(T)|_{V_\lambda} := f(\lambda)$. This is an excellent way to make new operators out of old. For example, the exponential of a matrix is defined through its diagonalization:

$$\exp\begin{pmatrix} 4 & 2 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} \exp\begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} -1/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^3 & 0 \\ 0 & e^6 \end{pmatrix} \begin{pmatrix} -1/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}. \tag{9.2}$$

We are interested in generalizing this story to infinite dimensions. Let $\mathscr{H}$ be a separable[9] complex Hilbert space and $T : \mathscr{H} \to \mathscr{H}$ be a self-adjoint, positive, compact operator.[10]

Theorem 9.3 (Spectral theorem). *There is an orthogonal basis $\{e_n\}_{n=1}^\infty$ of $\mathscr{H}$ and $\mu_n \in \mathbb{R}_{>0}$ such that*
- $\mu_1 \ge \mu_2 \ge \mu_3 \ge \cdots$,
- $Te_n = \mu e_n$, *and*
- $\lim\limits_{n\to\infty} \mu_n = 0$.

Suppose $s \in \mathscr{H}$ is such that $\langle s, e_n\rangle = 0$ for all n. Then $s = 0$. Hence $\mathscr{H}$ decomposes as

$$\mathscr{H} = \overline{\bigoplus_{\lambda \in \mathbb{R}} \mathscr{H}_\lambda}, \tag{9.4}$$

where $\dim \mathscr{H}_\lambda$ is finite. This is again an orthogonal direct sum, and again $\mathscr{H}_\lambda$ defines a sheaf of Hilbert spaces over $\mathbb{R}$. Because $\{\mu_i\}$ has an accumulation point at 0, this is not a finite sum of skyscraper sheaves.

Another way to think about that it defines a measure of sorts on $\mathbb{R}$. For $\lambda \in \mathbb{R}$, let $\pi_\lambda : \mathscr{H} \to \mathscr{H}_\lambda \hookrightarrow \mathscr{H}$ be orthogonal projection onto $\mathscr{H}_\lambda$ followed by inclusion; hence $\pi_\lambda^2 = \pi_\lambda$. Then

$$T = \sum_{\lambda \in \mathbb{R}} \lambda \pi_\lambda, \tag{9.5}$$

though one must make sense of this uncountable sum! In particular, spectral calculus is the same: given $f : \mathbb{R} \to \mathbb{R}$, one defines

$$f(T) := \sum f(\lambda)\pi_\lambda, \tag{9.6}$$

[9] If you don't know what this is, don't worry about it; its a point-set topological axiom. All Hilbert spaces we will encounter in this class are separable.

[10] By writing that the domain of T is $\mathscr{H}$, and not a dense subset of it, we imply that T is already a bounded operator.

where again we have to make sense of this sum, but this is possible.

We won't get into the guts of the proof of Theorem 9.3, but the idea is to study the quadratic form $Q\colon S(\mathscr{H}) \to \mathbb{R}$ sending $\xi \mapsto \langle \xi, T\xi \rangle$, where $S(\mathscr{H})$ is the unit sphere inside $\mathscr{H}$.

Remark. Theorem 9.3 is the only spectral theorem we'll need in this class, but is not the only one you'll need in your life. ⋖

We will apply this theory to the Dirac operator $D\colon C_M^\infty(S) \to C_M^\infty(S)$ associated to a Clifford bundle $S \to M$ over a closed Riemannian manifold M. (More generally, we could take an operator with $D^2 = \nabla^*\nabla + K$, as discussed last time.)

Example 9.7. The first example is $M = S^1$ with length 2π, $S = \underline{\mathbb{C}}$, and $D = -i\frac{\mathrm{d}}{\mathrm{d}x}\colon C^\infty(S^1) \to C^\infty(S^1)$. This operator is diagonalized by Fourier series: for any $n \in \mathbb{Z}$,

$$-i\frac{\mathrm{d}}{\mathrm{d}x}e^{inx} = ne^{inx}. \tag{9.8}$$

Unfortunately, this is not a positive operator. $D^2 = -\frac{\mathrm{d}}{\mathrm{d}x^2}$ isn't either, because it has 0 for an eigenvalue (its eigenvalues are n^2 for $n \ge 0$). So maybe we should replace it with D^2+1, which is positive, but where is $(D^2+1)^{-1}$ defined? This, and our desire to live in the world of Hilbert spaces, are why we have to think about Sobolev spaces. ⋖

Recall that we defined a Sobolev space $W^k(S)$ for $k = 0, 1, 2, \dots$, denoting the space of L^2 functions which have k derivatives that are also in L^2. (Well, this is not the precise definition, but that's the point.) Then $W^0(S) = L^2(S)$.

Remark. One can more generally use L^p norms for any $1 \le p \le \infty$, and define $L_k^p(\mathbb{R}^n)$ to be the Sobolev space of L^p functions on $\mathbb{R}^k$ which have k derivatives in L^p. ⋖

The Sobolev embedding theorem (Theorem 8.14) shows that $L_k^p(\mathbb{R}^n) \hookrightarrow C^\ell$ iff $k - n/p > \ell$. One can remember this inequality by taking a Sobolev function and scaling it by $x \mapsto \lambda x$. This is the inequality which makes sense as $\lambda \to 0$ or $\lambda \to \infty$.

It's also true that $\bigcap_k W^k = C^\infty$. This produces another interpretation of Sobolev spaces: we want to study smooth functions, but their topology is not the best (it's Fréchet, not Hilbert), so we weaken our regularity a little bit to get a better function space.

If $D = \nabla^*\nabla + K$ is a generalized Dirac operator, then

$$\int |Ds|^2 = \int |\nabla s|^2 + \int \langle Ks, s \rangle, \tag{9.9}$$

so already this looks like a Sobolev norm, which is another hint that Dirac operators naturally live on Sobolev spaces.

We then discussed the Rellich theorem (Theorem 8.15), that if $k' < k$, $W_k \hookrightarrow W_{k'}$ is compact. So W_k has a finer topology, but compactness is a really convenient thing to have. This implies that $D\colon W^1(S) \to W^0(S)$ is bounded, and that $1 + D^2\colon W^2(S) \to W^1(S)$ is bounded. If we could in addition find an inverse R to $1 + D^2$, then $T\colon W^0 \to W^2 \hookrightarrow W^0$ using R, then inclusing, T would be a compact operator, and as the inverse of a positive, self-adjoint operator, T is also self-adjoint. Therefore Theorem 9.3 applies, and by diagonalizing T we're almost all of the way to diagonalizing D.

Roughly, to invert $1 + D^2$, we need to show that

$$\|s\|_{W^2} \le \|(1 + D^2)s\|_{W^0}, \tag{9.10}$$

which shoes $1 + D^2$ is injective and has closed image. Then, it only remains to prove it's surjective.

To prove (9.10), we used Proposition 8.17 a few times. The way Roe does surjectivity is to look at the graph of D as an operator. This allows surjectivity to relate to a direct construction of R. Since W^1 is a dense subspace of $L^@$, the graph Γ of $D\colon W^1 \to L^2$ is a subspace of $L^2 \times L^2$, and we may take its closure. Using self-adjointness of D, we proved Proposition 8.31, that $\overline{\Gamma}$ is also a graph of some operator $\overline{D}$. We can use this to define a map $Q\colon L^2 \to L^2$ to its orthogonal projection onto $\overline{\Gamma}$ (all inside $L^2 \times L^2$), then projecting that back onto the domain L^2. Then for all Y,

$$\begin{aligned} 0 &= \langle Qx - x, y \rangle + \langle \overline{D}Qx, \overline{D}y \rangle \\ &= \langle Qx - x + \overline{D}^2 Qx, y \rangle, \end{aligned}$$

and therefore $(D^2+1)^{-1}Qx = x$, so we have not just surjectivity, but an inverse to D.

We want the eigenvectors of the Dirac operator to be smooth, but *a priori* they're just W^2. This is where the argument involving elliptic estimates and elliptic regularity came in. The upshot is the following spectral theorem for Dirac operators.

Theorem 9.11. *There exists an orthonormal basis $\{s_n\}_{n=1}^\infty$ for $L^2(S)$ and $\lambda_n \in \mathbb{R}_{\ge 0}$ such that*

- $0 \le \lambda_1 \le \lambda_2 \le \cdots$,
- $D^2 s_n = \lambda_n s_n$,
- $\lambda_n \to \infty$, *and*
- $s_n \in C^\infty(S)$.

Later we will prove a theorem about how fast λ_n grows; for the Dirac operator on the trivial bundle over S^1, $\lambda_n \approx n^2$.

For any section s, we have the projection formula

$$s = \sum_{n=1}^\infty \langle e_n, s\rangle e_n = \sum_\lambda s_\lambda, \tag{9.12}$$

where $s_\lambda := \pi_\lambda s$. On the circle, this is just Fourier series: large eigenvalues tell you about large-scale behavior, and small eigenvalues tell you about small-scale behavior. In order for this to converge, larger-scale behavior has to be tamed, and in particular, as you might remember from Fourier theory, better local behavior or regularity is converted to faster decrease of $\|s_\lambda\|$ as $\lambda \to \infty$.

One way to think of this is that $D^2 = -\frac{\mathrm{d}}{\mathrm{d}x^2}$ has units of $1/L^2$, where L is length. Therefore the units of the eigenvalues are also $1/L^2$, hence small at large scales.

Theorem 9.13. *An L^2 section s is smooth iff $\|s_\lambda\| = O(\lambda^\ell)$ for all $\ell \in \mathbb{Z}_{>0}$.*

Proof. Since

$$(D^2)^\ell s = \sum_\lambda \lambda^\ell s_\lambda, \tag{9.14}$$

then

$$\left\|(D^2)^\ell s\right\|_0 \le \sum_\lambda \lambda^\ell \|s_\lambda\| \le \sum_\lambda \frac{C}{\lambda^2} < \infty. \tag{9.15}$$

TODO: I missed something. ⊠

In a similar vein, there's a result for spectral calculus.

Theorem 9.16. *Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(\lambda) = O(\lambda^{-\ell})$ for all $\ell \in \mathbb{Z}_{>0}$.Then $\operatorname{Im}(f(D))$ consists of smooth sections.*

That is, to get smooth sections, we use functions which vanish more quickly than $1/p$ where p is any polynomial.

The proof idea is to show that $f(D)$ is an integral kernel, i.e.

$$f(D)s(x) = \int_M K(x,y)s(y)\,\mathrm{d}y \tag{9.17}$$

for all $x \in M$ and $s \in L^2(S)$. This is smooth iff $K \in C^\infty_{M\times M}(S \boxtimes S^*)$, and we can use elliptic estimates to prove this. We need an explicit formula for $K(x,y)$ in order to do this, and the answer turns out to be

$$K(x,y) = \sum_\lambda f(\lambda)\langle s_\lambda(y), -\rangle s_\lambda(x) \colon S_y \to S_x. \tag{9.18}$$

This is explicit enough that you can do analysis with it, showing that it's bounded in every Sobolev norm, and hence is smooth.

Lecture 10.

Dirac complexes and the Hodge theorem: 2/21/18

’ In this part of the lecture, Adrian spoke about Dirac complexes and the Hodge theorem. Everything will be worked out in finite dimensions first, where things are really easy, and then cover the infinite-dimensional case, whose proof looks similar at large scales, but requires checking a few more things.

A Dirac complex can be viewed as a refinement of a Clifford bundle.

Definition 10.1. Let M be a compact, oriented Riemannian manifold. A *Dirac complex* on M consists of

- a sequence $S^0, \dots, S^n$ of Hermitian vector bundles over M,
- differential operators $d^i\colon C^\infty(S^i) \to C^\infty(S^{i+1})$ for $i = 0, \dots, n-1$, and
- a Clifford bundle structure on
$$S := \bigoplus_{i=0}^{n} S^i,$$

such that $D = d + d^*$, where

- D is the Dirac operator on S and
- d^* is the adjoint of $d\colon S \to S$ with respect to the inner product

$$\langle s, s' \rangle := \int_M \langle s, s' \rangle \,\mathrm{dvol}. \tag{10.2}$$

Now let’s do some finite-dimensional Hodge theory. Let $d\colon V^0 \to V^1$ be a map between finite-dimensional inner product spaces (real or complex) and $d^*\colon V^1 \to V^0$ denote its adjoint.

Lemma 10.3. $\ker(d) = \ker(d^*d)$.

Proof. That $\ker(d) \subseteq \ker(d^*d)$ is obvious, so suppose $v \in \ker(d^*d)$, meaning

$$0 = \langle v, d^*dv \rangle = \langle dv, dv \rangle, \tag{10.4}$$

and therefore $dv = 0$. ⊠

The operators $d^*d\colon V^0 \to V^0$ and $dd^*\colon V^1 \to V^1$ are self-adjoint, so they have well-behaved spectral theory. Let $\lambda_0, \dots, \lambda_n$ be the eigenvalues of d^*d and $v_0, \dots, v_n$ be eigenvectors for them. Then, $\lambda_0, \dots, \lambda_n$ are the eigenvalues of dd^*, and the eigenvectors are $dv_0, \dots, dv_n$.

More generally, consider a finite sequence of finite-dimensional inner product spaces together with adjoint operators

$$0 \longrightarrow V^0 \underset{d^*}{\overset{d}{\rightleftarrows}} V^1 \underset{d^*}{\overset{d}{\rightleftarrows}} V^2 \underset{d^*}{\overset{d}{\rightleftarrows}} \cdots \underset{d^*}{\overset{d}{\rightleftarrows}} V^{n-1} \underset{d^*}{\overset{d}{\rightleftarrows}} V^n \longrightarrow 0. \tag{10.5}$$

Lemma 10.3 generalizes to another, almost as easy, lemma.

Lemma 10.6. *In the setting of* (10.5), $\ker(d^*) \cap \ker(d) = \ker(dd^* + d^*d)$.

Proof. Again the forward inclusion is obvious, so suppose $v \in \ker(dd^* + d^*d)$, so $\langle d^*dv + dd^*v, v \rangle = 0$.

TODO: something I didn’t understand happened here, using eigenvectors. ⊠

In particular, this leads to a canonical identification of $\ker(dd^* + d^*d)$ with the homology of the complex (10.5).

General Hodge theory. In this case we have a complex of vector bundles S^i, and let $V^i := C^\infty(S^i)$, so we have an induced complex of topological vector spaces.[11]

...something happened here regarding how the finite-dimensional statement changes because certain projections exist...

Theorem 10.7. *We have a projection* $C^\infty(S^i) \twoheadrightarrow \mathscr{H}^i$ *such that* $H^i(C^\infty(S^\bullet)) \cong \mathscr{H}^i$.

In fact, you can promote this into an isomorphism of the cohomology of $C^\infty(S^\bullet)$ with $\mathscr{H}^\bullet$, where the latter’s differentials are all the zero map.

[11] TODO: is this actually a Dirac complex?

Proof. There exists an orthogonal projection $P\colon L^2(S^i)\to\mathscr{H}^i$ because $\mathscr{H}^i$ is finite-dimensional. Let $i\colon\mathscr{H}^i\hookrightarrow L^2(S^i)$ denote inclusion. Let δ_{ij} denote the Kronecker delta and $f(\lambda)=1-\delta_{\lambda 0}$. Then $iP=1\cdot f(0)$.

If

$$g(\lambda):=\begin{cases}\lambda^{-2}, & \lambda\neq 0\\ 0, & \lambda=0,\end{cases}\tag{10.8}$$

then g is bounded on D because of (TODO) something I didn't understand. If $G:=g(D)$ and $H:=d^*G$, then

$$(d^*d+dd^*)G=f(D)=Hd+dH=1-(1-f(D))=1-iP\tag{10.9}$$

which means it's a chain homotopy between $(C^\infty(S^\bullet),d)$ and $(\mathscr{H}^\bullet,0)$, so their cohomology groups agree. ⊠

The case we're interested in is the Dirac complex of differential forms: $S^i:=\Lambda^i(T^*M)\otimes\mathbb{C}$ and $d^i:=\mathrm{d}$, with the Clifford action defined as follows: if $v\in TM$, $v\cdot\varphi:=v^\vee\wedge\varphi-v\lrcorner\varphi$ and extending to the rest of the Clifford algebra.

The inner product is the usual inner product on differential forms induced from the Riemannian metric, which we've seen a few times. The whole story follows from a few formal facts.

- The Hodge star is characterized by $\varphi\wedge\star\psi=\langle\varphi,\psi\rangle\,\mathrm{vol}$ for any k-forms φ,ψ.
- $\star\star\varphi=(-1)^{k(n-1)}\varphi$.
- Contraction is a derivation:

$$\lrcorner(\varphi\wedge\psi)=\lrcorner\varphi\wedge\psi+(-1)^{|\varphi|}\varphi\wedge\lrcorner\psi.\tag{10.10}$$

- The relation

$$v\lrcorner\varphi=(-1)^{k(n-1)}\star(\star\varphi\wedge v^\vee).\tag{10.11}$$

- ∇ commutes with the Hodge star.
- The relation

$$\nabla_X(v\lrcorner\varphi)=(\nabla_X v)\lrcorner\varphi+X\lrcorner\nabla_v\varphi.\tag{10.12}$$

You can check all the properties for a Dirac bundle using these, and using asynchronous frames, you can also show that the Dirac operator is the Hodge operator.

Lecture 11.

Applications of Hodge theory: 2/21/18

Rok spoke in this part of the talk. Throughout M is a closed, oriented Riemannian manifold of dimension n.

Corollary 11.1. *If S is a Dirac complex, the cohomology groups $H^i(S)$ are finite.*

This follows because we constructed an isomorphism above. Compactness of M is crucial here, entering in the form of ellipticity.

Theorem 11.2 (Poincaré duality)**.** *The integration pairing*

$$H^k(M;\mathbb{C})\otimes_{\mathbb{C}}H^{n-k}(M;\mathbb{C})\xrightarrow{\smile}H^n(M;\mathbb{C})\xrightarrow[\cong]{\int_M}\mathbb{C}\tag{11.3}$$

is a perfect pairing, i.e. it defines an isomorphism $H^{n-k}(M;\mathbb{C})\cong H^k(M;\mathbb{C})^*$.

Proof. Stokes' theorem implies that, since M is closed, the pairing is well-defined. Fix an $[\alpha]\in H^k(M;\mathbb{C})$ and suppose $\int_M\alpha\wedge\beta=0$ for all β. By the Hodge theorem, there is a unique harmonic representative for $[\alpha]$, which we'll also call α, and $\mathrm{d}\alpha=0$ and $\mathrm{d}^*\alpha=0$. Hence $(-1)^2\star\mathrm{d}\star\alpha=0$, so $\mathrm{d}(\star\alpha)=0$. Taking $\beta=\star\alpha$, we get

$$0=\int_M\alpha\wedge\star\alpha=\|\alpha\|^2_{L^2},\tag{11.4}$$

i.e. $\alpha=0$. ⊠

You can use this to set up intersection theory, in the form of finding Poincaré duals to closed, oriented submanifolds.

Let $C \subset M$ be a closed, oriented submanifold of dimension k. Then integrating k-forms restricted to C defines a linear map $\varphi\colon H^k(M;\mathbb{C}) \to \mathbb{C}$, well-defined by Stokes' theorem. By the Hodge theorem, $H^k(M;\mathbb{C}) \cong \mathscr{H}^k(M)$, so we may complete and extend $\widetilde{\varphi} := \varphi \circ P$, where $P\colon \Omega^k_{L^2}(M) \to \mathscr{H}^k(M)$ is projection onto the closed subspace of harmonic forms. In particular, $\widetilde{\varphi}$ is a continuous linear functional, so by the Riesz representation theorem, $\widetilde{\varphi} = \langle -, \beta\rangle$ for some $\beta \in \Omega^k_{L^2}(M)$.

Since P is a projector, it's self-adjoint, and

$$\begin{aligned}\langle \alpha, \beta\rangle = \int_C P\alpha = \int_C P^2\alpha = \langle P\alpha, \beta\rangle \\ = \langle \alpha, P^*\beta\rangle = \langle \alpha, P\beta\rangle.\end{aligned}$$

Therefore in particular β is harmonic!

Definition 11.5. The *Poincaré dual* to C is $P_C := [\star\beta] \in H^{n-k}(M;\mathbb{C})$.

There are other ways to define this, but the cleanliness of the Hodge star is nice.

We can therefore do intersection theory: if C and C' are closed, oriented, transverse submanifolds of M such that $\dim C + \dim C' = n$, then their intersection number is

$$I(C, C') := \int_C P_{C'} = (-1)^{(\dim C)(\dim C')} \int_M P_{C'} \wedge P_C = (-1)^{(\dim C)(\dim C')} I(C', C). \tag{11.6}$$

So this is commutative up to a sign, just as in cohomology.

Example 11.7. For any point $p \in M$, $P_p \in \Omega^n(M)$ must be λ times the volume form for some $\lambda \in \mathbb{C}$. Then

$$f(p) = \int_M f \cdot P - p = \lambda \int_M f \,\mathrm{vol}. \tag{11.8}$$

This seems quite strange untill you realize that it means $\lambda = 1/\mathrm{vol}(M)$, after which (11.8) is a tautology. ◀

We'll use this to provide another proof of Corollary 5.3, that a compact manifold with nonnegative Ricci curvature, and positive Ricci curvature somewhere, has vanishing first cohomology.

Proof of Corollary 5.3. Recall that $D^2\alpha = 0$ has no nonzero solutions, and by the Hodge theorem, $H^1(M;\mathbb{C}) \cong \mathscr{H}^1(M) = 0$. ⊠

Lecture 12.

The heat and wave equations: 2/28/18

In this part of the lecture, Ravi Mohan spoke about the heat and wave equations on a compact Riemannian manifold (M, g).

Let $S \to M$ be a Clifford bundle on M and s be a smooth section of S varying smoothly in time t. The *heat equation* is the second-order differential equation

$$\frac{\partial s}{\partial t} + D^2 s = 0. \tag{12.1}$$

In physics, this is known as the *diffusion equation*: it takes a point distribution and spreads its mass out over M, modeling heat flow.

The *wave equation* is the first-order equation

$$\frac{\partial s}{\partial t} - iDs = 0. \tag{12.2}$$

Given a wave as initial data, this propagates the wave.

Proposition 12.3. *Let $s_0 \in \Gamma(M, S)$ be a smooth section.*

(1) *The heat equation (12.1) has a unique solution s_t corresponding to the initial data s_0, which exists for all $t \geq 0$. Moreover, $\|s_t\|^2 \leq \|s_0\|^2$.*

(2) *The wave equation (12.2) has a unique solution s_t corresponding to the initial data s_0, which exists for all $t \in \mathbb{R}$. Moreover, $\|s_2\|^2 = \|s_0\|^2$.*

Proof. First we prove (1). For existence, consider $e^{-tD^2} \colon C^\infty(S) \to C^\infty(S)$, and let

$$s_t := e^{-tD^2} s_0. \tag{12.4}$$

Then

$$\frac{\partial}{\partial t} s_t = -D^2 s_t \tag{12.5}$$

$$\Longrightarrow \left(\frac{\partial}{\partial t} + D^2\right) s_t = 0. \tag{12.6}$$

For the norm, let $\{e_i\}$ be an orthonormal basis of eigenfunctions for D^2, so that $a_0 = \sum_i a_i e_i$ and $D^2 e_i = n_i e_i$. Then

$$\begin{aligned} s_t &= e^{-tD^2} s_0 \\ &= e^{-tD^2} \sum_i a_i e_i \\ &= \sum_i a_i e^{-tn_i} e_i. \end{aligned}$$

Therefore

$$\|s_t\|^2 = \sum_i \|a_i\|^2 e^{-2tn_i}, \tag{12.7}$$

which decays with t.

For the wave equation, the proof is very similar, except we use e^{iDt} in place of e^{-tD^2}. For (12.7), $|e^{itn_i}| = 1$, allowing us to get $=$ instead of $\le$. ⊠

Remark. Looking at the heat and wave equations in the eigenbasis allows us to understand what they're doing. For example, the heat equation dampens Fourier modes, but dampens the larger Fourier modes faster. In fact, even if your initial condition s_0 isn't smooth, s_ε is smooth for any $\varepsilon > 0$, because the large Fourier modes which cause s_0 to not be smooth are dampened to something acceptable. ◀

There exists a smooth family of smooth sections k_t, $t > 0$, of $S \boxtimes S^* \to M \times M$ such that for all smooth sections s,

$$e^{-tD^2} s(p) = \int_M k_t(p,q) s(q)\,\mathrm{d}q. \tag{12.8}$$

This k_t is called the *heat kernel* for S.

Proposition 12.9. *We have that*

$$\left(\frac{\partial}{\partial t} = D_p^2\right) k_t(p,q) = 0. \tag{12.10}$$

$$\lim_{t\to 0} k_t = \delta(p-q). \tag{12.11}$$

Proof of part (12.10). It suffices to compute: since

$$\left(\frac{\partial}{\partial t} + D_p^2\right) e^{-tD^2} S(p) = \left(\frac{\partial}{\partial t} + D_p^2\right) \int_M k_t(p,q) s(q)\,\mathrm{d}q, \tag{12.12}$$

then

$$0 = \int \left(\frac{\partial}{\partial t} + D_p^2\right) k_t(p,q) s(q)\,\mathrm{d}q. \tag{12.13}$$

Since this is true for all s, (12.10) follows. ⊠

The heat kernel is for smooth sections. If we have weaker regularity, we need something slightly different.

Definition 12.14. An *approximate heat kernel of order m* is a time-dependent section $k_t'(p,q)$ of $S \boxtimes S^* \to M \times M$ which approximately satisfies the heat equation, in that there is a C^m section r_t^m of $S \boxtimes S^* \to M \times M$ such that

$$\left(\frac{\partial}{\partial t} + D_p^2\right) k_t'(p,q) = t^m r_t^m(p,q).$$

Proposition 12.15. *Let s_t be a continuous family of C^2 sections of S. Then, there is a unique smooth family of smooth sections $\widetilde{s}_t$ of S with $\widetilde{s}_0 = 0$ and satisfying the heat equation*

$$\left(\frac{\partial}{\partial t} + D^2\right)\widetilde{s}_t = s_t.$$

Lemma 12.16. *The asymptotic behavior of the heat kernel in Euclidean space $\mathbb{E}^n$ is*

$$k_t(x,y) \sim \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right). \tag{12.17}$$

On a Riemannian n-manifold (M,g), if p and q are given in geodesic coordinates on M, the asymptotic behavior of the heat kernel is

$$k_t(p,q) \sim h := \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{d(p,q)^2}{4t}\right). \tag{12.18}$$

Here $d(p,q)$ is the geodesic distance, i.e. the infimum of the lengths of geodesics between p and q.

Lemma 12.19. *Let h be as in (12.18). Then*

$$\nabla h = -\frac{h}{2t} r \frac{\partial}{\partial r} \tag{12.20}$$

$$\frac{\partial h}{\partial t} + \Delta h = \frac{rh}{4gt}\frac{\partial g}{\partial r}. \tag{12.21}$$

TODO: proof was given, and I missed it.

Definition 12.22. Let $f : \mathbb{R} \to \mathbb{C}$ be a function and $S := \sum c_n t^n$ be a formal power series. Then S is an *asymptotic series* for f if for all N,

$$\lim_{t\to 0} t^{-N} \left| f(t) - \sum_{n=0}^{N} c_n t^n \right| = 0.$$

Remark. We know that as $t \to \infty$, the heat operator converges very strongly to projection onto the kernel. So when we use it to understand the index, we'll find a formula for the index in terms of the heat kernel at all $t \geq 0$. At infinity, it simplifies to where we'll recognize it as an index; then at 0, we'll be able to identify it with something else.

If $x \neq y$, (12.17) has exponential decay as $t \to 0$. On the diagonal, it's just the function $1/(4\pi t)^{n/2}$, which doesn't converge, but we'll be able to use an asymptotic series to compare what happens on a curved manifold with what happens on $\mathbb{E}^n$. Even though these series don't converge, they're still useful for computing something! The way this is accomplished is by taking polynomials instead of formal power series. ◀

Lecture 13.

Estimates with the heat kernel: 2/28/18

In this part of the lecture, Cameron Darwin spoke.

First, we'll generalize Definition 12.22.

Definition 13.1. Let E be a Banach space and $f : \mathbb{R}_+ \to E$ be a function. Let $S = \sum_{k=0}^{\infty} a_k(t)$ be a power series valued in E (i.e. each a_t is a function $\mathbb{R}_+ \to E$). We say that S is an *asymptotic expansion* for f if for all n, there exists an ℓ_n such that if $\ell \geq \ell_n$, there's a constant $C_{\ell,n}$ such that for t sufficiently small,

$$\left\| f(t) - \sum_{k=0}^{\ell} a_k(t) \right\| \leq C_{\ell,n} |t|^n.$$

Proposition 13.2. *Let k_t be the heat kernel and h_t be as in (12.17).*

1. *There is an asymptotic expansion of for k_t of the form*

$$k_t(p,q) \sim h_t(p,q)\big(\Theta_0 + t\Theta_1 + t^2\Theta_2 + \cdots\big), \tag{13.3}$$

where the Θ_n are smooth sections of $S \boxtimes S^* \to M \times M$.

2. *(13.3) is valid in the Banach space of C^r sections of $S \boxtimes S^*$ for all $r \geq 0$, and may be formally differentiated in p, q, and t to obtain asymptotic expansions for derivaives of k_t.*

(3) *The values of $\Theta_j(p,p)$ can be computed from algebraic expressions in the metric and connection coefficients of S, and the first term is $\Theta_0(p,p) = 1$.*

If k'_t is an approximate heat kernel of order m', then there's a C such that for small t,

$$\|k_t - k'_t\|_{H^{m'}} \le C t^{m'+1}, \tag{13.4}$$

so by the Sobolev embedding theorem, if $m' > m + n/2$, we get a bound for the C^m norm:

$$\|k_t - k'_t\|_{C^m} \le \|k_t - k'_t\|_{H^{m'}} \le C t^{m'+1}. \tag{13.5}$$

A similar analysis can be performed with

$$\|\nabla_{x_1} \cdots \nabla_{x_n} k_t - \nabla_{y_1} \cdots \nabla_{y_n} k_t^j\| \tag{13.6}$$

to bound it above.

Let $\Theta \in \Gamma(M \times M, S \boxtimes S^*)$ and $s \in \Gamma(M,S)$. Then we will use the notation

$$(\theta \vdash s)(p) := \int_M \theta(p,q)s(q)\,\mathrm{d}q. \tag{13.7}$$

Lemma 13.8. *For all sections $\Theta_0, \ldots, \Theta_j$ of $S \boxtimes S^*$,*

$$\big(h_t(\Theta_0 + t\Theta_1 + \cdots + t^j\Theta_j) \vdash s\big)(p) \xrightarrow{t\to 0} \Theta_0(p,p)s(p) \tag{13.9}$$

and the convergence is uniform in p.

Lemma 13.10. *For all $\varepsilon, \delta > 0$, there's a t_0 such that for $0 < t < t_0$ and all $p \in \mathbb{R}^N$,*

$$1 - \varepsilon < \int_{B_\delta(p)} h_t(p,q)\,\mathrm{d}q < 1. \tag{13.11}$$

This says that h_t looks like a delta function near 0.

Corollary 13.12. *For all $\alpha, \beta, \delta > 0$, there's a t_0 such that for $0 < t < t_0$,*

$$\left| \int_{B_\delta(p)} h_t(p,q)\,\mathrm{d}q - 1 \right| < \alpha$$

$$\left| \int_{B_\delta(p)^c} h_t(p,q)\,\mathrm{d}q \right| < \beta.$$

Given a continuous section $s \in \Gamma(M,S)$, we'll define a section $\widetilde{s}$ of $S \times M \to M \times M$ by

$$|\widetilde{s}(p,q)| = |\Theta(p,q)s(q) - \Theta(p,p)s(p)|. \tag{13.13}$$

TODO: what happened here? Why are we defining $\widetilde{s}$ via its norm?

Anyways, by Lemma 13.8, $\widetilde{s}$ is uniformly continuous in p and q. In particular, for all $\omega > 0$, there's a $\delta > 0$ such that if $d(p,q) < \delta$, then $|\widetilde{s}(p,q)| < \omega$. This implies that for all $\varepsilon > 0$, there's a t_0 independent of p such that for $t \in (0, t_0)$,

$$\left| \int_{B_\delta(p)^c} h_t(p,q)\Theta(p,q)s(q)\,\mathrm{d}q \right| < \frac{\varepsilon}{2}. \tag{13.14}$$

Finally, for all $\varepsilon > 0$, and independently of p and R, if

$$\left| \int_{B_R(p)} h_t(p,q)\,\mathrm{d}q - 1 \right| < \alpha \tag{13.15a}$$

and

$$|\Theta(p,q) - \Theta(p,p)s(p)| < \varepsilon, \tag{13.15b}$$

then

$$\left| \int_{B_R(p)} h_t(p,q)\Theta(p,q)s(q)\,\mathrm{d}q - \Theta(p,p)s(p) \right| < \frac{\varepsilon}{2}. \tag{13.16}$$

The upshot of all of this is that we know what conditions to put on Θ_0 such that $h_t(\Theta_0 + t\Theta_1 + \cdots + t^j\Theta_j)$ to converge to a delta function. We'll next discuss what else we need to ask of the Θ_i in order to ensure we get an approximate heat kernel.

Lemma 13.17. *Suppose we have two finite sequences $\Theta_0, \Theta_1, \dots,$ and $\Theta_0', \Theta_1', \dots$ and that there's an open neighborhood U of the diagonal in $M \times M$ such that for all i, $\Theta_i|_U = \Theta_i'|_U$. Then $k_t \sim h_t(\Theta_0 + t\Theta_1 + \cdots)$ iff $k_t \sim h_t(\Theta_0' + t\Theta_1' + \cdots)$.*

Proof sketch. Since M is compact, we can assume U is the set of points $(p,q) \in M \times M$ such that $d(p,q) < \varepsilon$. Therefore

$$h_t(p,q) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{d(p,q)}{4t}\right) \tag{13.18}$$

decays faster than any polynomial on U^c, so only the stuff inside U matters. ⊠

Therefore it suffices to construct these Θ_i in a neighborhood of the diagonal.

Definition 13.19. Let (M, g) be a compact Riemannian manifold.

- At a $p \in M$, the *injectivity radius* is the supremum of the set of radii r such that the exponential map $\exp\colon T_pM \to M$ is injective when restricted to $B_r(0) \subset T_pM$.
- The *injectivity radius of M* is the infimum of the injectivity radii at all $p \in M$.

Let R be less than the injectivity radius of M; then the neighborhood $N_R(\Delta)$ of $\Delta \subset M \times M$ admits a cover by charts which are in geodesic coordinates. Using a system of curoff functions for these charts, we can reduce the problem to working in a single chart in geodesic coordinates.

TODO: something happened here that I missed and didn't understand.

Using these estimates, we get

$$\begin{aligned}\frac{1}{h}\left(\frac{\partial}{\partial t} + D^2\right)(hs) &= \frac{1}{h}\left(h\frac{\partial s}{\partial t} hD^2 s + \left(\frac{\partial h}{\partial t} + \Delta h\right)s - 2\nabla_{\nabla h} s\right)\\ &= \frac{1}{h}\left(h\frac{\partial s}{\partial t} + hD^2 s + \frac{rh}{4gt}\frac{\partial g}{\partial r}s + \frac{h}{t}\nabla_{r\partial/\partial r} s\right)\\ &= \frac{\partial s}{\partial t} + D^2 s + \frac{r}{4gt}\frac{\partial g}{\partial r}s + \frac{1}{r}\nabla_{r\partial/\partial r} s.\end{aligned}$$

Then TODO: ???, and we're trying to determine u, which is TODO. If

$$u \sim u_0 + tu_1 + t^2 u_2 + \cdots, \tag{13.20}$$

then TODO the estimate was erased before I could follow it or write it down. I'll pick up taking notes where I can next follow what's going on...

Let $\varphi := g^{1/4} D^2 u_j$ and

$$\psi(r,s) := \int_0^r \varphi(\rho, s)\,\mathrm{d}\rho. \tag{13.21}$$

Then, the function [TODO: erased before I could write it down] is smooth, which is somewhat mysterious: Roe asserts it, but it's not immediately clear, and there doesn't seem to be a way around it.

Remark. So now we have sections Θ_j in a neighborhood of the diagonal, constructed from an inductive sequenece of ODEs. The coefficients of the ODEs are smooth everywhere, including on the diagonal, so the theorem we're relying on asserts that we can solve a family of ODEs with parameters and obtain smooth solutions. In fact, since the family is linear, we can write down the solutions explicitly with integration.

There's a geometric picture of this, which relates asymptotic expansions to genuine Taylor expansions, but this is harder. ◀

The last thing we'll do is show that these asymptotic sums are actually approximate heat kernels. We can compute

(13.22)

$$\left(\frac{\partial}{\partial t} + D^2\right)\big(h_t(p,q)(\Theta_0(p,q) + \cdots + t^j\Theta_j(p,q))\big) = h\left(\sum_{i=0}^{j} i t^{i-1} u_i + t^i D^2 u_i + t^{i-1}\frac{r}{4g}\frac{\partial g}{\partial r}u_i + t^{i-1}\nabla_{r\partial/\partial r} u_i\right)$$

which is equal to TODO and therefore we can conclude that it's an approximate heat kernel.

The last thing we asserted is that on Δ, $\Theta_i(p,p)$ can be computed using algebraic expressions in the metric, connection coefficients, and finitely many derivatives of these things. This is obvious for $\Theta_0 = \mathbf{1}$, and only slightly less obvious for Θ_1: in synchronous coordinates, it's also possible to see it explicitly. For general Θ_j, the argument is inductive.

Suppose that x^p and x^q are in geodesic coordinates; then, the Taylor expansion for the metric is

$$g \sim 1 + \frac{1}{3}\sum_{i,p,q} R_{ipqi} x^p x^q + O(|x|^3). \tag{13.23}$$

Since $K = \sum_{i,p} R_{ippi}$, we have

$$g^{1/4} \sim 1 - \frac{1}{12}\sum_{i,p,q} R_{ipqi} x^p x^q - \frac{1}{6}\sum_{i,p} K + \cdots \tag{13.24}$$

and therefore can give a few examples of the Θ_j terms.

$$\Theta_0 = \mathbf{1} \tag{13.25a}$$

$$\Theta_1(p,p) = \frac{1}{6}\kappa(p) - \mathbb{K}(p). \tag{13.25b}$$

Here $\mathbb{K}$ is the Clifford-contracted curvature.

Lecture 14.

Asymptotic expansion, I: 3/20/18

Dan spoke about the asymptotic expansion, in order to provide an additional perspective.

Let M be a closed Riemannian manifold. Then the heat equation for $t > 0$ and $x, y \in M$ is

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \Delta_x\right) p_t(x,y) = 0. \tag{14.1}$$

The kernel $p_t(x,y)$ is called the *heat kernel*. As $t \to 0$, $p_t(-,y) \to \delta_y$, so it forms an approximation to the identity:

$$\lim_{t\to 0} \int_M f(x) p_t(x,y)\,|\mathrm{d}x| = f(y). \tag{14.2}$$

There's something strange about (14.1): we're mixing units of time and of length squared. In physics, this is solved by inserting a constant, the *specific heat*, which has units that make everything type-check. For us, this says that, approximately, $T \sim L^2$ in scaling. A similar phenomenon occurs in special relativity, where the constant is the speed of light.

Example 14.3. Let $\mathbb{E}^n$ denote n-dimensional Euclidean space. Then the heat kernel is

$$p_t(x,y) = (4\pi t)^{-n/2} e^{-d_{\mathbb{E}}(x,y)^2/4t}. \tag{14.4}$$

Any exponentiated quantity should be dimensionless, so length squared must be comparable to time. ◀

For a Dirac operator $\Delta = D^2$, where D acts on sections of a spinor bundle $S \to M$, the heat kernel is section-valued, a one-parameter family of sections of $\mathrm{Hom}(S,S)$.

If M is compact, we're going to scale it (which is why we care about which units are comparable). In a neighborhood of a point y that we're interested in, we'll consider an $\varepsilon > 0$ and scale distances near y by $1/\varepsilon$, and correspondingly time by $1/\sqrt{\varepsilon}$. This has the effect of pushing everything else in M far away as $\varepsilon \to 0$, so the geometry only sees things near y.

To make this precise, we can choose a neighborhood of y on which the exponential map is an isometry, then replace its image with a much larger ball around $0 \in T_yM$ and glue. This defines a family of manifolds over $(0,\infty)$: topologically, the fiber is just M, but its Riemannian metric has changed.

Example 14.5. For a simple example, suppose $M = S^1$; then, as $\varepsilon \to 0$ we get longer and longer circles. ◀

When $\varepsilon \to 0$, the limit is precisely T_yM, with Euclidean structure induced from the Riemannian metric on M. The bundle has a section given by y, and this means we obtain an origin on T_yM, so it's an inner product space, not just a Euclidean space.

We will then consider heat flow of a specific $x \in M$ on each fiber of this bundle. This generates a path that blows up as $\varepsilon \to 0$ (since we don't see any $x \neq y$ at 0), so we have a flow $p_\varepsilon^t(x, y)$. If we look at the diagonal and let $t = 1$, this turns out to be a smooth function, hence has a Taylor expansion. In fact, this function is even, so the Taylor expansion is only in terms of even powers of ε.

Then, after scaling back, we'll recover the asymptotic expansion of the heat kernel.

Then, we can make the problem slightly fancier: we don't just want y to be fixed. Instead, we consider the family $\mathscr{Y} \to [0, \infty) \times M$, where over $(-, y)$ we blow the metric up near y. There's a canonical section of this bundle sending $(\varepsilon, y) \mapsto y$ in the blowup by ε around y.

The map $\mathscr{Y} \to [0, \infty) \times M$ is not a fiber bundle, because the fiber over 0 isn't diffeomorphic (or even homeomorphic) to the fiber at ε for any $\varepsilon > 0$. However, it is a surjective submersion, which is still nice. Nonetheless, this class of families of manifolds contains a little weirdness: there's a surjective submersion $X \to \mathbb{R}$ whose fiber is S^1 over a negative number and $S^1 \amalg S^1$ over a positive number. Over 0, it's two lines.

Thus surjective submersions allow you to connect manifolds via paths. This perspective is very important in algebraic geometry; there's a conjecture that Calabi-Yau 3-folds are connected under a similar kind of paths. There's a process called *deformation to the normal cone* which is analogous to studying how a small circle around y blows up as $\varepsilon \to 0$; this isa useful technique to prove analogues of index theorems in algebraic geometry.

Remark. Here we sketch a construction of the bundle $\mathscr{X} \to \mathbb{R}$ for a fixed y. Let Σ_y denote the one-point compactification of T_yM; then, there's a map $F : M \to \Sigma_y$ which is the inverse to the exponential map inside a small neighborhood of y and sends things far from y to the added point ∞. Then, we define $\mathscr{X} \subset \mathbb{R} \times \Sigma_y \times M$ with coordinates (ε, a, x) to be cut out by the equation $F(x) = \varepsilon a$ away from $\{0\} \times \{\infty\} \times M$.

One must show this is a regular value, but it is, so this is a smooth manifold. We have metrics on M and $\mathbb{R}$, and get one on Σ_y induced from the inner product on T_yM, so $\mathscr{X}$ inherits a Riemannian metric — and in particular, so do its fibers. ◀

To show that the function is smooth and even, one has to bring in analysis, making estimates such as

$$\left| k_t^1(a) - |\varepsilon|^{-n} k_{t/\varepsilon^2}^\varepsilon(a/\varepsilon) \right| \le C|\varepsilon|^{-n} e^{-c/4t}. \tag{14.6}$$

This also gives you some concrete information about the asymptotic expansion, which is the case $k_t^1(0)$. Specifically,

$$k_t^1(0) = |\varepsilon|^{-n} k_{t/\varepsilon^2}^\varepsilon(0) + O\big(|e|^{-n} e^{-c/t}\big). \tag{14.7}$$

Now set $\varepsilon = t^{1/2}$; in particular,

$$k_t^1(0) = t^{-n/2} k_t^{t^{1/2}}(0) + O\big(t^{-n/2} e^{-c/t}\big). \tag{14.8}$$

This is the thing whose Taylor expansion we want.

TODO: after this I ceased to follow.

Lecture 15.

Asymptotic expansion, II: 3/20/18

This part of the lecture was by Cameron. I'm really out of it right now, so these notes might not be so great. Sorry about that.

Ultimately, to understand index theory on noncompact manifolds, we'll need to study the heat equation as it relates to the wave equation. Therefore we're interested in the same kind of analysis we've been doing, but for the wave equation, and will prove a similar localization result.

Proposition 15.1. *Let $\mathscr{R}(\mathbb{R})$ denote the space of functions vanishing at infinity and D be a Dirac operator. Then the function from $\mathscr{R}(\mathbb{R})$ to the space of C^∞ integral kernels sending $f \mapsto f(D)$ is continuous.*

The topology on the codomain is a Fréchet topology induced by all of the C^k norms.

Proof. Let λ be an eigenvalue for D and P_λ be projection onto its eigenspace. If K_λ denotes the smoothings kernel for P_λ, then the kernel for $f(D)$ is given by

$$\sum_\lambda f(\lambda)K_\lambda. \tag{15.2}$$

We next claim that for all k, there's a $c_k > 0$ and an $\ell(k) \in \mathbb{Z}^{>0}$ such that for all λ,

$$\|K_\lambda\|_k \le c_k(1+|\lambda|)^{\ell(k)}, \tag{15.3}$$

and $\ell(k) \ge \dim M/2 + k$. The proof of this was not particularly enlightening and wasn't presented.

Something more enlightening to prove: if $\{\psi_n\}$ is an orthonormal basis for $L^2(S)$ such that $D^2\psi_n = \lambda_n\psi_n$, then if $\ell > \dim M/2$, then

$$\sum_n \frac{1}{(1+\lambda n)^\ell} < \infty. \tag{15.4}$$

To prove this claim, we fix an $N > 0$. Then

$$\begin{aligned}\sum_{n\le N} \frac{1}{(1+\lambda n)^\ell} &= \sum_{n\le N} \frac{\|\psi_n\|^2}{(1+\lambda n)^\ell}\\ &= \sum_{n\le N} \frac{1}{(1+\lambda n)^\ell} \int_M \|\psi_n(x)\|^2\,\mathrm{d}x\\ &= \int_M \sum_i \sum_{n\le N} \frac{|\langle \psi_n(x), \sigma_i(x)\rangle|^2}{(1+\lambda n)^2}\,\mathrm{d}x,\end{aligned}$$

where at each x, $\{\sigma_i(x)\}$ is an orthonormal basis for S_x.

Now suppose $\sum a_n\psi_n \in W^\ell$. By the Sobolev embedding theorem, $W^\ell \hookrightarrow C^0$ with the operator norm. An elliptic estimate tells us $\|\psi_n\|_{W^\ell} \le (1+\lambda n)^\ell$, and so

$$\left|\sum_n a_n\psi_n(x)\right|_x \le \left\|\sum_n a_n\psi_n\right\|_{C^0} \le C\left(\sum_n |a_n|^2(1+\lambda n)^\ell\right)^{1/2}. \tag{15.5}$$

Let $\sigma_1, \dots, \sigma_m$ be an orthonormal basis for S_x, and fix an $N > 0$. Then let

$$a_n := \begin{cases} \dfrac{\langle \psi_n(x), \sigma_i\rangle_x}{(1+\lambda n)^\ell}, & n \le N\\ 0, & n > N.\end{cases} \tag{15.6}$$

Then

$$\begin{aligned}\left\langle \sigma_i, \sum_{n\le N} a_n\psi_n(x)\right\rangle_x &= \sum_{n\le N} a_n\langle \sigma_i, \psi_n(x)\rangle\\ &= \sum_{n\le N} \frac{|\langle \sigma_i, \psi_n(x)\rangle_n|^2}{(1+\lambda n)^\ell}.\end{aligned} \tag{15.7}$$

Because $\|\sigma_i\| = 1$, the Cauchy-Schwarz inequality buys us

$$\left|\left\langle \sigma_i, \sum_{n\le N} a_i\psi_n(x)\right\rangle_x\right| \le \left|\sum_{n\le N} a_n\psi_n(x)\right|, \tag{15.8}$$

and because $|a_n|^2 = |\langle \sigma_i, \psi_n(x)\rangle|^2/(1+\lambda n)^{2\ell}$, then one of our previous estimates (TODO: which? probably (15.5)) implies that

$$\left|\sum_{n\le N} a_n\psi_n(x)\right| \le C\left(\sum_{n\le N} \frac{|\langle \sigma_i, \psi_n(x)\rangle_x|^2}{(1+\lambda n)^\ell}\right). \tag{15.9}$$

Therefore

$$\left|\left\langle \sigma_i, \sum_{n\le N} a_n\psi_n(x)\right\rangle_x\right| = \sum_{n\le N} \frac{|\langle \sigma_i, \psi_n(x)\rangle_x|^2}{(1+\lambda n)^\ell} \le c\sqrt{\sum_{n\le N} \frac{|\langle \sigma_i, \psi_n(x)\rangle_x|^2}{(1+\lambda n)^\ell}}. \tag{15.10}$$

Therefore

$$\frac{1}{C}\sqrt{\sum_{n\le N}\frac{|\langle\sigma_i,\psi_n(x)\rangle_x|^2}{(1+\lambda n)^2}}\le 1, \tag{15.11}$$

so summing over i,

$$\sum_i\sum_{n\le N}A_{i,n}(x)\le(\operatorname{rank}S)C^2, \tag{15.12}$$

and therefore the infinity sum is bounded by $(\operatorname{vol}M)(\operatorname{rank}S)C^2$. ⊠

Next we'll discuss finite propagation speed for the wave equation. We need to choose a sign convention in the wave equation, and we choose (TODO: erased before I could write it down).

Remark. For the wave equation, we need to equate units of length and time. You can either judiciously choose units, or end up with a constant with units length over time. This is the speed of the wave, and the proposition below says that the waves travel at that finite speed.

The heat equation, by contrast, flows compactly supported functions into noncompactly supported functions after any $\varepsilon>0$. The upshot is that, according to the heat equation, heat flow travels infinitely fast (though it doesdecay exponentially away from the heat source at small times). ◂

Proposition 15.13. *For any $s\in C_c^\infty(S)$, the support of $e^{itD}s$ lies within distance $|t|$ from the support of s.*

So we've normalized constants such that the speed of the wave is 1.

To prove Proposition 15.13, we'll need an energy estimate. Recall that if ω is the 1-form defined by $\omega(X)=(c(X)s_t,s_t)$, where $c(\cdot)$ is the Clifford action, then

$$(iDs_t,s_t)+(s_t,iDs_t)=-i\mathrm{d}^*\omega. \tag{15.14}$$

Lemma 15.15 (Energy estimate)**.** *Let $m\in M$, R be less than the injectivity radius of M, and s_t be a solution of the wave equation. Then for $t\ge 0$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_{R-t}(m)}|s_t|^2\le 0.$$

Proof. First observe that at $x\in B_R(m)$,

$$\begin{aligned}\frac{\partial}{\partial t}(|s_t|^2)&=\left(\frac{\partial}{\partial t}s_t,s_t\right)+\left(s_t,\frac{\partial}{\partial t}s_t\right)\\&=(iDs_t,s_t)+(s_t,iDs_t)=-i\mathrm{d}^*\omega.\end{aligned}$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_{R-t}(m)}|s_t|^2=-\int_{B_{R-t}(m)}i\mathrm{d}^*\omega-\int_{\partial B_{R-t}(m)}(s_t,s_t)\,i_{n*}\mathrm{vol}, \tag{15.16}$$

where i_{n_*} denotes contracting with the normal vector field on $\partial B_{R-t}(m)$. This is an instance of Stokes' theorem. Applying the Divergence theorem, this is equal to

$$=\int_{\partial B_{R-t}(m)}|-i(c(N)s_t,s_t)|\,i_{n_*}\mathrm{vol}-\int_{\partial B_{R-t}(m)}(s_t,s_t). \tag{15.17}$$

Cauchy-Schwarz shows the first term is bounded above by $|s_t|^2$, so the entire expression is nonpositive, as desired. ⊠

Proof of Proposition 15.13. Since $\exp(i(t_1+t_2)D)=\exp(it_1D)\exp(it_2D)$, it suffices to prove this for small $t_0>0$; specifically, we'll choose t_0 less than the injectivity radius of M, and suppose that $d(m,\operatorname{supp}(s))>t_0$. Then $\int_{B_{t_0}(m)}|s_t|^2\,\mathrm{dvol}=0$, so by Lemma 15.15, $\int_{B_{t_0-t}(m)}s_t=0$ for all $t\in(0,t_0)$. ⊠

Definition 15.18. A C^∞ function $f:\mathbb{R}\to\mathbb{R}$ is called *Schwarz* if f and all of its derivatives are rapidly decreasing. The Fréchet space of Schwarz functions is denoted $\mathscr{S}(\mathbb{R})$.

Proposition 15.19. *The Fourier transform of a Schwarz function is Schwarz, and in fact the Fourier transform is an invertible continuous linear map $\mathscr{S}(\mathbb{R})\to\mathscr{S}(\mathbb{R})$.*

There are conventions in the definition of the Fourier transform; if we define

$$\widehat{f}(\xi) := \int_{\mathbb{R}} e^{-ix\xi} f(x)\,\mathrm{d}x, \tag{15.20a}$$

then the inverse to the Fourier transform is

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\widehat{f}(\xi)\,\mathrm{d}\xi. \tag{15.20b}$$

Proposition 15.21. *Let $f \in \mathscr{S}(\mathbb{R})$. Then for all $x, y \in L^2(S)$, we have*

$$\langle f(D)x, y\rangle = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(x)\langle e^{i\xi D}x, y\rangle\,\mathrm{d}\xi$$

and

$$f(D) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(\xi)e^{i\xi D}\,\mathrm{d}\xi,$$

where we interpret these in a distributional sense.

Proof. Let s be an eigenfunction of D with eigenvalue λ. Then

$$\begin{aligned}\langle f(D)x, y\rangle &= f(\lambda)\langle x, y\rangle\\ &= \langle x, y\rangle\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(\xi)e^{i\lambda\xi}\,\mathrm{d}\xi\\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(\xi)\langle e^{i\xi D}x, y\rangle\,\mathrm{d}\xi.\end{aligned}$$

More generally, we could have a section $s = \sum_\lambda s_\lambda$, and have to justify interchanging the sum and the integral. The trick to do this is to integrate over small regions of the manifold, making everything finite enough to show that the integral converges uniformly in λ, allowing us to switch the sum and the integral. ⊠

Here's the main result.

Theorem 15.22. *Suppose $f \in \mathscr{S}(\mathbb{R})$ and $\widehat{f}$ is supported in $[-c, c]$. Then $\langle f(D)x, y\rangle = 0$ for sections x, y of S whenever $d(\mathrm{supp}(x), \mathrm{supp}(y)) > c$. Consequently, the smoothing kernel of $f(D)$ is contained in a c-neighborhood of the diagonal in $M \times M$. Consequently, the smoothing kernel of $f(D)$ is contained in a*