# FALL 2016 HOMOTOPY THEORY SEMINAR

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CONTENTS



## 1. $s$-COBORDISMS AND WALDHAUSEN'S MAIN THEOREM: 9/7/16

Today, Professor Blumberg gave an overview of Waldhausen's main theorem and its context; this semester, we'll be working though Lurie's proof of it as outlined in his course on the algebraic topology of manifolds.

We'll start from the $h$-cobordism theorem.

**Definition 1.1.** An *$h$-cobordism* is a cobordism $W$ between manifolds $M$ and $N$ such that the equivalences $M \hookrightarrow W$ and $N \hookrightarrow W$ are both homotopy equivalences.

The canonical example is $M \times [0,1]$, which is an $h$-cobordism between $M$ and itself. This is called a *trivial $h$-cobordism*.

We're going to be deliberately vague about what category of manifolds we're dealing with: when we say "isomorphic," we mean as toplogical manifolds, PL manifolds, or smooth manifolds. We're not going to belabor the point right now, though it will be quite important for us later.

**Theorem 1.2** ($h$-cobordism (Smale))**.** *If* $\dim(M) \geq 5$ *and* $\pi_1(M) = 0$*, then every $h$-cobordism is trivial, i.e. suitably isomorphic relative to the boundary to the trivial $h$-cobordism.*

This is a big theorem — a somewhat easy consequence is the Poincaré conjecture in dimensions $\geq 5$! When Smale proved this part of the Poincaré conjecture, he really was attacking this theorem. The proof proceeds via a handlebody decomposition, which illustrates what is easier in dimension 5 than in dimensions 3 and 4: handlebodies can slide past each others using, for example, the *Whitney trick*, which simply doesn't work in dimensions 3 or 4.

We're not interested in the Poincaré conjecture *per se*, but can we generalize Theorem 1.2? If we try to lower the dimension of $M$, we're basically screwed, so can we work with $M$ not simply connected?

**Theorem 1.3** ($s$-cobordism (Barden, Mazur, Stallings))**.** *The set of isomorphism classes of $h$-cobordisms $M \hookrightarrow W \hookleftarrow N$ is in bijection with a certain quotient of $K_1(\mathbb{Z}[\pi_1(M)])$.*

We'll eventually define $K_1$, which is an algebraic gadget that's a ring invariant. It's evident that $K_1$ of the group algebra is a homotopy invariant, but it's less obvious that the set of isomorphism classes of $h$-cobordisms is. This group $K_1(\mathbb{Z}[\pi_1(M)])$ is also the home of *Whitehead torsion*, an invariant of manifolds.

**Question 1.4.** Is this a $\pi_0$ statement? In other words, can we describe a space of $s$-cobordisms such that Theorem 1.3 is recovered on passage to $\pi_0$?

This is a natural question following recent developments in homotopy theory. It may allow us to attach spaces or spectra to these invariants.

The answer, due to names such as Hatcher, Igusa, and Waldhausen, is yes! On the left-hand side, we have something called the stable pseudo-isotopy space, akin to a stabilized form of $B\mathrm{Diff}$, the isomorphisms of a manifold relative to its boundary. This arises as a result of an action on bundles of $s$-cobordisms, which is how classifying spaces appear. Things will be more concretely defined, albeit not at this level of narrative. The point is that *a priori* this isn't a homotopy-invariant, so we have to stabilize in a geometric way, by taking repeated products $M \times I^n$ with an interval.

On the other side, one realizes $\pi_1(M) \cong \pi_0(\Omega M)$, so maybe we can try to construct something like $\mathbb{Z}[\Omega M]$. This works, but it's better to take the $K$-theory *spectrum* associated to something called the spherical group ring $K(S[\Omega M])$, which is (or is equivalent to) $A(M) = K(\Sigma^\infty_+ \Omega M)$, which is an $A_\infty$ ring spectrum.

Waldhausen's theorem is precisely that there is a stable $s$-cobordism theorem: that $A(M) \simeq \Sigma^\infty M \vee \Omega^2 \operatorname{Wh}(M)$; $\operatorname{Wh}(M)$ is something called the *Whitehead spectrum* associated to $M$, and its double loop space is the pseudo-isotopy space we want to construct. This splitting arises from an assembly map, which is a purely formal statement about (topologically or simplicially) enriched functor: $F(X) \vee Y \to F(X \vee Y)$.

So we have an algebraic invariant, which we can hope to calculate, and it tells us geometric information.

We can start with the sphere spectrum $K(S) \simeq S \vee \Omega^\bullet \operatorname{Wh}(*)$. This is already hard and unsolved; solving it will solve several questions in geometric topology, including some on exotic differential structure.

Depending on who you are, you might have different motivations for things like this: May and others were naturally led to ring spectra when considering generalized orientations, but you might also invent them to make this theorem true!

It turns out that $K(S)$ is controlled by $K(\mathbb{Z})$. There's a commutative square

$$\begin{tikzcd} K(S) \arrow[r] \arrow[d] & \mathrm{TC}(S) \arrow[d] \\ K(\mathbb{Z}) \arrow[r] & \mathrm{TC}(\mathbb{Z}). \end{tikzcd}$$

Here, TC is topological cyclic homology. One of the big theorems is that this is a pullback diagram, so we can understand $K(S)$ from topological cyclic homology, the vertical maps, called *trace maps* (which actually generalize the trace of a matrix), and how they fit together. At primes, this is pretty simple, but that still leaves the individual players.

Understanding $\mathrm{TC}(S)$ is, well, slightly harder than computing the stable homotopy groups of the spheres. This is a bad thing, but also a good thing: we can compute some of it, and if important questions depend on a particular element in it, that element can be computed and identified. Conversely, $K(\mathbb{Z})$ is a mess, but an interesting mess — it contains a lot of number-theoretic information, some still unknown. This diagram illustrates that number-theoretic information controls geometric topology.

Waldhausen's proof of his main theorem in [2] is famously complicated, and wasn't available until relatively recently. The book by Waldhausen-Jahren-Rognes [3] provides an exposition, but it's rough going. Certainly, the introduction will be useful.

One reason we might be interested is how Waldhausen proved this. He gave a direct proof in the PL category, and then applied a reduction to prove the smooth case. Is it possible to give a direct proof in the smooth case? Waldhausen and Igusa tried to do this, but didn't succeed. For reasons that are ultimately Floer-homotopy-theoretic, it would be useful to have such a proof in the smooth case. Lurie's proof in [1] follows the broad stokes of Waldhausen's proof, but uses different machinery, and could be useful as a guide for a direct proof in the smooth case.

There will be three kinds of lectures:

(1) Statements of the theorem. Along with this, what is algebraic $K$-theory? What is a Whitehead spectrum?
(2) Background: what is Wall finiteness? What is Whitehead torsion? These are geometric questions, yet are invariants of algebraic $K$-theory, and are good to know for culture.
(3) Then, there's the technology of the proof, uisng something called simple homotopy theory. We can think of $K(\Sigma^\infty_+ \Omega M)$ as the $K$-theory of a category of spaces parameterized over $M$, and this leads to simple homotopy equivalences, related to a prescribed set of equivalences and blowups. This is more geometric, and hence harder. Lurie's proof presents a different approach to this, focusing on constructible sheaves, and this is definitely one of the most worthwhile lessons from his proof.

The key is the construction of the assembly map: Lurie approaches it with some very natural functors from constructible sheaves. This is something we can get to, but we'd have to cover a lot of ground to get there.

Another thing to keep in mind: there will be many different constructions of these objects, all equivalent or equivalent up to a shift. We'll end up constructing a parameterized spectrum of space of cobordisms. It might be interesting to compare these to other cobordism categories.

## 2. The Wall finiteness obstruction: 9/14/16

Today, Nicky spoke.

The Wall finiteness obstruction is an invariant that's pretty easy to write down abstractly; it provides an obstruction for a CW complex to be homotopic to a finite CW complex. Specifically, given a CW complex $X$, let $G = \pi_1(X)$ (relative to any basepoint); we'll construct this obstruction as a class $w(X) \in \widetilde{K}(\mathbb{Z}[G])$.

**Definition 2.1.** Let $R$ be a ring, not necessarily commutative. Then, the *K-theory* of $R$, denoted $K_0(R)$, is the abelian group generated by isomorphism classes of finitely generated[1] projective (left) $R$-modules modulo the relations that for every short exact sequence of $R$-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

we impose a relation $[M] = [M'] + [M'']$.

There's a map $\mathbb{Z} \to K_0(R)$ defined by $z \mapsto z[R]$; if $z > 0$, this is the class $[R^{\oplus z}]$. The *reduced K-theory* mods out by this: $\widetilde{K}_0(R) = K_0(R)/\mathbb{Z}$.

This is a very algebraic object, but we'll use it to discover topological information.

There are some other tools we'll use. Relative homotopy invariants are associated to relative homology groups $H_*(X, Z; R)$, which we can define whenever we're given a map $f : Z \to R$. Using a mapping cylinder $C_f$, this is homotopic to an inclusion, and $H_*(X, Z; R)$ is the homology of the quotient chain complex (of the singular chains).

**Definition 2.2.** A *local system* on a space $X$ is a representation of $\pi_1(X)$ over $\mathbb{Z}$.[2] That is, it's an abelian group with a compatible $\pi_1(X)$-action.

Recall that $G = \pi_1(X)$ acts on the universal cover $\widetilde{X}$ by deck transformations. Thus, the singular complex $C_\bullet(\widetilde{X})$ of $\widetilde{X}$ is a $\mathbb{Z}[G]$-module. Thus, given a local system $V$, we can create new chain complexes, such as $C_\bullet(\widetilde{X}) \otimes_{\mathbb{Z}[G]} V$ or $\mathrm{Hom}_{\mathbb{Z}[G]}(C_\bullet(\widetilde{X}), V)$.[3] These define homology, resp. cohomology theories on $X$, called $H_*(X, V)$, resp. $H^*(X, V)$.

**Definition 2.3.** A space $X$ is *finitely dominated* if there exists a finite CW complex $Z$, an inclusion $i : X \to Z$, and a section $r : Z \to X$ such that $r \circ i \simeq \mathrm{id}_X$.

This basically means $X$ includes into a finite CW complex which retracts onto it, but with a homotopy. We'll hope to show that some properties of finitely dominated spaces actually characterize them.

*Fact.* Let $X$ be a finitely dominated space.

(1) First, $\pi_0(X)$ must be finite (since it factors as a subset of $\pi_0(Z)$).
(2) $\pi_1(X)$ must be finitely presented, because $i_* : \pi_1(X) \to \pi_1(Z)$ has a left inverse, so it's split injective into a finitely generated group.
(3) For local systems $V$, the assignment $V \mapsto H_*(X, V)$ commutes with filtered direct limits.
(4) $X$ has finite *homotopical dimension*, which means there's an $m \geq 0$ such that for all local systems $V$ and $i > m$, $H_i(X, V) = 0$. This will be at most the dimension of the space $Z$ which dominates $X$.

The following theorem is important, but hard; [1, Lec. 2] sketches the proof.

**Theorem 2.4.** *A space satisfying conditions (1), (2), and (3) is finitely dominated.*

**Proposition 2.5.** *Suppose $X$ satisfies conditions (1), (2), and (3). Then, for all $n > 0$, there's a finite CW complex $Z$ of dimension less than $n$ and an $(n-1)$-connected map $Z \to X$.*[4]

This follows from a more general fact.

**Proposition 2.6.** *Suppose $X$ satisfies conditions (1), (2), and (3), and suppose we are given an $(n-1)$-connected map $f : Z \to X$, where $Z$ is a finite CW complex. Then, there exists a space $Z'$, obtained from $Z$ by adjoining finitely many $n$-cells, such that $f$ factors through an $n$-connected map $Z' \to X$.*

This allows us to inductively prove Proposition 2.5.

[1] One can define this for the category of all projective $R$-modules, but this is always zero, thanks to the Eilenberg swindle.

[2] Sometimes, the base ring is different, but for our purposes, we'll prefer $\mathbb{Z}$.

[3] Since $\mathbb{Z}[G]$ is in general noncommutative, there's something to say here about left versus right actions.

[4] For a map to be $(n-1)$-connected means that its homotopy fiber is $(n-1)$-connected as a space, which implies that the induced map on $\pi_k$ is an isomorphism for $k < n-1$ and is a surjection for $k = n-1$.

**Lemma 2.7.** *Let $X$ be a space satisfying* (1), (2), (3), *and* (4). *Let $Z$ be a finite CW complex of dimension at most $n-1$ and $f : Z \to X$ be an $(n-1)$-connected map. Then, $H_n(X, Z; \mathbb{Z}[G])$ is a finitely generated projective $\mathbb{Z}[G]$-module.*

This is where $K$-theory shows up.

*Proof.* Since $Z$ is $(n-1)$-dimensional, then $H^i(Z, V) = 0$ for all $i \geq n$ and all local systems $V$. We have a long exact sequence for relative homology

$$\cdots \longrightarrow H^{i-1}(Z; V) \longrightarrow H^i(X, Z; V) \longrightarrow H^i(X, V) \longrightarrow \cdots,$$

and given a short exact sequence of local systems

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0,$$

and applying $H^*(X, Z; -)$ induces another long exact sequence

$$\cdots \longrightarrow H^n(X, Z; V') \longrightarrow H^n(X, Z; V) \longrightarrow H^n(X, Z; V'') \longrightarrow H^{n+1}(X, Z; V') \longrightarrow \cdots.$$

Using the universal coefficients theorem and the Hurewicz theorem, we have a natural isomorphism

$$H^n(X, Z; -) \cong \mathrm{Hom}_{\mathbb{Z}[G]}(H_n(X, Z; \mathbb{Z}[G]), -).$$

The former is right exact, and therefore so is the latter, so $H_n(X, Z; \mathbb{Z}[G])$ is projective. ⊠

Thus, $H_n(X, Z; \mathbb{Z}[G])$ has a class in $K$-theory.

**Definition 2.8.** The *Wall finiteness obstruction* of $X$ is $w(X) = (-1)^n [H_n(X, Z; \mathbb{Z}[G])] \in K_0(\mathbb{Z}[G])$.

We have a lot to show: that this is independent of $n$ and $Z$, but also that it's at all related to finiteness.

**Proposition 2.9.** *The following are equivalent:*

*(1) $X$ has the homotopy type of a finite CW complex.*
*(2) $H_n(X, Z; \mathbb{Z}[G])$ is stably free (and hence trivial in $K_0(\mathbb{Z}[G])$).*

In the reverse direction, the idea of the proof is to kill generators: if $H_n(X, Z; \mathbb{Z}[G]) \oplus \mathbb{Z}[G]^{\oplus r}$ is free, then it's equal to $H_n(X, Z \vee (S^n)^{\vee r}; \mathbb{Z}[G])$; then, one uses the Whitehead theorem and the relative Hurewicz theorem to kill homotopy groups. The point is that we have a map $Z \vee (S^n)^{\vee r} \to X$, where the domain is a finite CW complex; if we can show that this map induces an isomorphism on all homotopy groups, Whitehead's theorem proves that the map is a homotopy equivalence.

In the other direction, we can compute $H_*(X, Z; \mathbb{Z}[G])$ cellularly, and therefore get an exact sequence of free modules, which forces it to be stably free.

We'll skip the proofs of independence of $Z$ and $n$, which are reasonably pretty, but quite long.

Another interesting fact is that if $G$ is any group, we know $G = \pi_1(X)$ for some space $X$, but it's also true that any class in $K_0(\mathbb{Z}[G])$ is a Wall finiteness obstruction for some space $X$ with $\pi_1(X) = G$.

## References

[1] Jacob Lurie. Algebraic $K$-theory and manifold topology (lecture notes), 2014.
[2] Friedhelm Waldhausen. Algebraic $K$-theory of spaces. *Lecture Notes in Mathematics*, 1126:318–419, 1985.
[3] Friedhelm Waldhausen, Bjørn Jahren, and John Rognes. *Spaces of PL Manifolds and Categories of Simple Maps*. Number 186 in Annals of Mathematics Studies. Princeton University Press, 2013.