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These notes were taken in David Ben-Zvi's student seminar in Spring 2017, with lectures given by David Ben-Zvi. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to `a.debray@math.utexas.edu`.

Contents

1. A categorified version of the Fourier transform: 1/20/17

We've seen that for two-dimensional gauge theories with group G, there's a relationship with the Fourier transform for G. Today, we're going to talk about a categorified version of this, and in a few weeks we'll connect it to three-dimensional gauge theory.

Let's recall some facets of the Fourier transform. Let G be a locally compact abelian (LCA) group, and let $\widehat{G} = \mathrm{Hom}_{\mathsf{TopGrp}}(G, \mathrm{U}(1))$ be its Pontrjagin dual. This is a dual in that $\widehat{\widehat{G}} \cong G$.

The Fourier transform is an isomorphism $L^2(G) \overset{\cong}{\to} L^2(\widehat{G})$ sending pointwise multiplication to convolution and vice versa. There's a nice dictionary between the two sides:

- A representation of G is sent to a family of vector spaces on $\widehat{G}$.
- Finite groups are sent to finite groups.
- Lattices are sent to tori.
- A vector space is sent to its dual vector space.

Today, we're going to talk about Cartier duality, an algebraic analogue of this.

Let G be an algebraic group: this is the notion of a group in algebraic geometry just as Lie groups are the correct notion of groups in differential geometry. One can think of algebraic groups as functors from rings to groups; this is the functor-of-points perspective.

We have no analogue of $\mathrm{U}(1)$ in this setting, so we consider all characters $\chi : G \to \mathbb{G}_m = \mathrm{GL}_1$; the codomain is defined by the group of units functor $\mathsf{Ring} \to \mathsf{Grp}$ sending $R \mapsto R^\times$. As a scheme, this is $\mathbb{A}^1 \setminus 0$ or $\operatorname{Spec} k[x, x^{-1}]$.

The *Cartier dual* of G is $\widehat{G} = \mathrm{Hom}_{\mathsf{AlgGrp}}(G, \mathbb{G}_m)$. That is, for any ring R, $G(R) = \mathrm{Hom}_{\mathsf{Grp}}(G(R), R^\times)$. For "nice G," we'd like $G \cong \widehat{\widehat{G}}$. But what kinds of groups meet this condition?

G had better be abelian (since $\widehat{G}$ always is), and in fact we'll need it to be a *finite flat group scheme*. This idea might be new if you're used to thinking of algebraic geometry over $\mathbb{C}$, where these are exactly the finite abelian groups, but over other fields, it might be different.

Example 1.1. Let $G = \mathbb{Z}/n$. Then, its dual is $\widehat{\mathbb{Z}/n} = \mathrm{Hom}(\mathbb{Z}/n, \mathbb{G}_m)$, which can be identified with the group of n^{th} roots of unity, μ_n. Over $\mathbb{C}$, this is $\langle e^{2\pi i/n} \rangle$ and therefore identified with $\mathbb{Z}/n$, but over fields with characteristic dividing n, there are fewer n^{th} roots of unity. We're not going to worry too much about this. ◀

Akin to Pontrjagin duality, if we let $G = \mathbb{G}_m$, we get $\widehat{G} = \mathbb{Z}$, and if G is a torus, $\widehat{G}$ is the dual lattice in it.

For the Fourier transform, we want to look at vector spaces, e.g. the *additive group* $\mathbb{G}_a = \mathbb{A}^1$. We want to understand homomorphisms $\mathbb{G}_a \to \mathbb{G}_m$. We know that these would be given by $x \mapsto e^{xt}$, but this doesn't make sense unless t is nilpotent, so that the exponential

$$e^{xt} = \sum \frac{(xt)^n}{n!}$$

is a finite sum! That is, we want the dual of the x-line $\mathbb{G}_a$ to be the t-line, but we don't get very far along t. Since we don't know what order t is, we obtain the *formal completion*

$$\widehat{\mathbb{G}}_a = \varinjlim_n \operatorname{Spec} k[t]/(t^n),$$

heuristically a union of n^{th}-order thickenings of 0. Here, the hat is completion, not dual.

More generally, let V be a vector space. Then, its Cartier dual is the formal completion of the dual vector space: we want to take $e^{\langle v,v^*\rangle}$, but we need v^* to be nilpotent.

Alternatively, since Cartier duality is symmetric, the Cartier dual of the formal completion of the additive group is $\mathbb{G}_a$. That is, if x is nilpotent, e^{xt} makes sense for arbitrary t.

Since we're doing algebraic geometry, it's good to think of this in terms of functions. If G is a group, $\mathscr{O}(G)$ is not just a ring, but also has a *comultiplication* pulling functions back along multiplication: $\mu^* : \mathscr{O}(G) \to \mathscr{O}(G) \otimes \mathscr{O}(G)$. This makes $\mathscr{O}(G)$ into a *coalgebra*, and it's cocommutative iff G is commutative.

If G is finite, then you can dualize explicitly: $\mathscr{O}(G)$ is a finite-dimensional vector space, so $\mathscr{O}(G)^\vee$ has a convolution operator induced from the comultiplication. This is the same as convolution of distributions. In fact, it's possible to prove that the Cartier dual is $\widehat{G} = \operatorname{Spec}(\mathscr{O}(G)^\vee, *)$. Functions on $\widehat{G}$, with multiplication, are the same as distributions on G, with convolution. This is what we had in the analytic setting, albeit with a little more care to functions versus distributions.

A point of $\widehat{G}$ defines an algebraic function on G: it's a character $\chi : G \to \mathbb{G}_m$, so composing with the inclusion $\mathbb{G}_m \hookrightarrow \mathbb{A}^1$, we get a map $G \to \mathbb{A}^1$. We can assemble this into a diagram

$$\begin{tikzcd} & G \times \widehat{G} \arrow[dl] \arrow[dr] & \\ G & & \widehat{G}, \end{tikzcd}$$

and there's a tautological function on $G \times \widehat{G}$, which is evaluation: $(g, \chi) \mapsto \chi(g) \in \mathbb{A}^1$. This is akin to the exponential $(x,t) \mapsto e^{xt}$.

If G is infinite, you have to be more careful with topology. For example, $\mathscr{O}(\mathbb{G}_m) = k[x, x^{-1}]$, which sort of looks like the group algebra $k[\mathbb{Z}]$ over the integers, but there we have to restrict to finite expressions.

A sheaf-theoretic perspective. Rather than looking at functions, which don't behave very well in this context, let's look at sheaves.

There are three tensor categories associated to any group G.

(1) Since $R = \mathscr{O}(G)$ is a commutative ring, we can use $\mathsf{Mod}_{\mathscr{O}(G)}$ to generate the category $\mathsf{QC}(G)$ of quasicoherent sheaves on G.[1] The commutative tensor product $\otimes_R$ on Mod_R extends to a symmetric monoidal structure on $\mathsf{QC}(G)$. This does not require G to be a group.

(2) Since G is a group, $\mathscr{O}(G)$ is a bialgebra (actually a Hopf algebra), so $\mathsf{Mod}_{\mathscr{O}(G)}$ has a monoidal structure given by tensoring over the base field k rather than over R. That is, if M and N are $\mathscr{O}(G)$-modules, $M \otimes_k N$ has an $R \otimes R$-module structure, and then we can induce along the map $R \to R \otimes R$ to obtain an R-module structure.

This monoidal structure is a convolution:

$$\begin{tikzcd} & G \times G \arrow[dl] \arrow[dr] \arrow[dd, "\mu"] & \\ G & & G. \\ & G & \end{tikzcd}$$

[1] If G is an affine scheme, the categories are the same.

Here, we take M and N over G and realize them over $G \times G$ using the exterior product $M \boxtimes N$, and then pushforward along the multiplication map. This is the same category $\mathsf{QC}(G)$, but with a completely different structure, and this is one of the advantages of sheaves: instead of having to keep functions and distributions apart, sheaves can both pull back and push forward.

(3) The third approach is to take the category of representations of G, which can be tensored together. How can you say this geometrically? G-representations are $\mathscr{O}(G)$-comodules, vector spaces V with a coaction map $V \to V \otimes \mathscr{O}(G)$ satisfying *coassociativity*, i.e. that the following diagram is an equalizer diagram:

$$V \longrightarrow V \otimes \mathscr{O}(G) \rightrightarrows V \otimes \mathscr{O}(G) \otimes \mathscr{O}(G).$$

In a sense, this encodes the notion that representations are modules over the group algebra, but we don't have distributions, so the arrows go the other way. This is a symmetric monoidal category, where the tensor product has the coalgebra structure defined by composing the maps

$$V \otimes W \longrightarrow V \otimes W \otimes \mathscr{O}(G) \otimes \mathscr{O}(G)$$

and $\mathscr{O}(G) \otimes \mathscr{O}(G) \to \mathscr{O}(G)$.

This is not a category of quasicoherent sheaves on G; rather, it's $\mathsf{QC}(\bullet/G)$, where $\bullet/G$ is the classifying stack (or groupoid) of G. This comes from the pushout diagram $\bullet/G \leftarrow \bullet \leftleftarrows G$.

Cartier duality allows these categories to interact with each other. Namely, suppose G and $\widehat{G}$ are dual (so G is abelian, etc.). Then, Cartier duality establishes an equivalence of categories $\mathsf{Rep}_G \cong \mathsf{QC}(\widehat{G})$, and $\mathscr{O}(G)$-comodules become $\mathscr{O}(G)^\vee$-modules. This is just as in ordinary Pontrjagin duality: representations of G become families of functions on $\widehat{G}$.

(By the way, if you're holding out for examples, we'll soon see a whole bunch of them.)

In fact, the tensor structure is also in play: the duality is between the tensor product structure on Rep_G (or $\mathsf{QC}(\bullet/G)$) and the convolution structure on $\mathsf{QC}(\widehat{G})$.

We're going to abstract G away to a different duality operation $\mathsf{QC}(\mathcal{G}) \xrightarrow{\cong} \mathsf{QC}(\mathcal{G}^\vee)$. In our case, $\mathcal{G} = \bullet/G$ and $\mathcal{G}^\vee = \widehat{G}$. The classifying space $\bullet/G$ (also called BG) classifies G-bundles, and since G is abelian, you can tensor G-bundles. That is, if $\mathcal{P}_1$ and $\mathcal{P}_2$ are G-bundles, the relative tensor product $\mathcal{P}_1 \times_G \mathcal{P}_2$ is again a G-bundle, meaning $\bullet/G$ is an abelian group under the tensor product of G-bundles?

What does this actually mean? We're thinking of varieties (and generalizations such as stacks) as functors $\mathsf{Ring} \to \mathsf{Set}$; that $\bullet/G$ is an abelian group means that the assignment from a ring R to the (groupoid of) G-bundles on $\operatorname{Spec} R$ naturally factors through the category of abelian groups. That is, $\bullet/G$ is an abelian group object in the world of stacks.

Now, we define the *Fourier-Mukai dual* $\mathcal{G}^\vee = \mathrm{Hom}_{\mathsf{Grp}}(\mathcal{G}, B\mathbb{G}_m)$. Here $B\mathbb{G}_m$ classifies line bundles, so this is a version of the Picard group. However, since we've restricted to group homomorphisms, we only get what's known as multiplicative line bundles.

Definition 1.2. Let $\mathscr{L} \to G$ be a line bundle over a group G and $\mu : G \times G \to G$ be multiplication. If $\mu^*\mathscr{L} \cong \mathscr{L} \boxtimes \mathscr{L}$, then $\mathscr{L}$ is called a *multiplicative* line bundle.

The idea is that over $x, y \in G$, $\mathscr{L}_x \otimes \mathscr{L}_y \cong \mathscr{L}_{xy}$.

In a sense, we've shifted the Cartier duality operation: $(\bullet/G)^\vee = \mathrm{Hom}_{\mathsf{Grp}}(\bullet/G, \bullet/\mathbb{G}_m) = \mathrm{Hom}_{\mathsf{Grp}}(G, \mathbb{G}_m) = \widehat{G}$ as before. So why categorify? In this stacky version, instead of a universal function on $G \times \widehat{G}$, there's a universal line bundle $\mathscr{L} \to \mathcal{G} \times \mathcal{G}^\vee$:

$$\begin{tikzcd} & \mathcal{G} \times \mathcal{G}^\vee \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{G} & & \mathcal{G}^\vee. \end{tikzcd}$$

This bundle $\mathscr{L}$ is called the *Poincaré line bundle*. And it allows us to define a Fourier transform: given a sheaf $\mathscr{F}$on $\mathcal{G}$, we can pullback and pushforward to obtain $\pi_{2*}(\pi_1^*\mathscr{F} \otimes \mathscr{L}) \in \mathsf{QC}(\mathcal{G}^\vee)$. This actually defines an equivalence of categories, which is known as *Cartier duality* or *Laumon-Fourier-Mukai duality*.

Example 1.3. The most interesting example is where $\mathcal{G} = A$ is an abelian variety and $\mathcal{G}^\vee = A^\vee$ is the dual variety. Then, the integral transform with the Poincaré sheaf defines an equivalence of the derived categories $D(A) \cong D(A^\vee)$, which is the classical *Fourier-Mukai transform*. ◀

Example 1.4. We could also take $\mathcal{G} = \mathbb{G}_m$ and $\mathcal{G}^\vee = B\mathbb{Z}$. Then, this duality tells us that $\mathbb{Z}$-graded vector spaces are the same things as representations of $\mathbb{G}_m$. ◂

References