M392C NOTES: TOPICS IN ALGEBRAIC TOPOLOGY

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These notes were taken in UT Austin's M392C (Topics in Algebraic Topology) class in Spring 2017, taught by Andrew Blumberg. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Alternatively, these notes are hosted on Github at https://github.com/adebray/equivariant_homotopy_theory, and you can submit a pull request. Thanks to Rustam Antia-Riedel, Gill Grindstaff, Yixian Wu, and an anonymous reader for catching a few errors; to Ernie Fontes, Tom Gannon, Richard Wong, and Valentin Zakharevich for some clarifications; and to Yuri Sulyma for adding some remarks and references.

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Lecture 1.

G-spaces: 1/17/17

This class will be an overview of equivariant stable homotopy theory. We're in the uncomfortable position where this is a big subject, a hard subject, and one that is poorly served by its textbooks. Algebraic topology is like this in general, but it's particularly acute here. Nonetheless, here are some references:

- Adams, "Prerequisites (on equivariant stable homotopy) for Carlsson's lecture." [Ada84]. This is old, and some parts of it don't reflect how we do things now.
- The Alaska notes [May96], edited by May, is newer, and is written by many authors. Some of it is a grab bag, and some parts (e.g. the rational equivariant bits) aren't entirely right. It's also not a textbook.
- Appendix A of Hill-Hopkins-Ravenel [HHR16]. This is a paper which resolved an old conjecture on manifolds using equivariant stable homotopy theory, but let this be a lesson on referee reports: the authors were asked to provide more background, and so wrote a 150-page appendix on this material. Their suffering is your gain: the introduction is well-written, albeit again not a textbook.

In the world of the professor, there are two major applications of equivariant stable homotopy theory.

- The first is trace methods in algebraic K-theory: Hochschihd homology and its topological cousins are equipped with natural S^1-actions (the same S^1-action coming from field theory). This is how people other than Quillen compute algebraic K-theory.
- The other major application is Hill-Hopkins-Ravenel's settling of the Kervaire invariant 1 conjecture in [HHR16].

The nice thing is, however you feel about the applications, both applications require developing new theory in equivariant stable homotopy theory. Hill-Hopkins-Ravenel in particular required a clarification of the foundations of this subject which has been enlightening.

In this class, we hope to cover the foundations of equivariant stable homotopy theory. On the one hand, this will be a modern take, insofar as we emphasize the norm and the presheaf on orbit categories (these will be explained in due time), the modern emerging consensus on how to think of these things, different than what's written in textbooks. The former is old, but has gained more attention recently; the latter is new. Moreover, there's an increasing sense that a lot of the foundations here are best done in ∞-categories. We will not take this appoach in order to avoid getting bogged down in ∞-categories; moreover, this class is supposed to be rigorous. It will sometimes be clear to some people that ∞-categories lie in the background, but we won't talk very much about them.

We'll cover some old topics such as Smith theory and the Segal conjecture, and newer ones such as trace methods and Hill-Hopkins-Ravenel, depending on student interest. We will not have time to discuss many topics, including equivariant cobordism or equivariant surgery theory.

Prerequisites. If you don't know these prerequisites, that's okay; it means you're willing to read about it on your own.

- Foundations of unstable homotopy theory at the level of May's *A Concise Course in Algebraic Topology* [May99]. For example, we'll discuss equivariant CW complexes, so it will help to know what a CW complex is.
- A little bit of category theory, e.g. found in Mac Lane [Mac78] or Riehl [Rie16].
- This class will not require much in the way of simplicial methods (simply because it's hard to reconcile simplicial methods with non-discrete Lie groups), but you will want to know the bar construction. An excellent source for this is [Rie14, Chapter 4].
- A bit of abstract homotopy theory, e.g. what a model structure is. Good sources for model categories are [Rie14, Part III] and [Hov99].

If you don't know these, feel free to ask the professor for references. His advisor suggested that a foundation for the stable category is Lewis-May-Steinberger's account [LMS86] of the equivariant category and let $G = *$, but perhaps this isn't necessarily a good reference for nontrivial groups.

Unstable equivariant questions are very natural, and somewhat reasonable. But stable questions are harder; they ultimately arise from reasonable questions, but the formulation and answers are hard: even discussing the equivariant analogue of $\pi_0 S^0$ requires some representation theory — and yet of course it should. Thus there's a lot of foundations behind hard calculations. There will be problem sets; if you want to learn the material (or are an undergrad), you should do the problem sets.

Categories of topological spaces. The category of topological spaces we consider is Top, the category of compactly generated, weak Hausdorff spaces (and continuous maps); we'll also consider Top_*, the category of based, compactly generated, weak Hausdorff spaces and continuous, based maps. This is an important and old trick which eliminates some pathological behavior in quotients. It's reasonable to imagine that point-set topology shouldn't be at the heart of foundational issues, but there are various ways to motivate this, e.g. to make Top more resemble a topos or the category of simplicial sets.

Definition 1.1. Let X be a topological space.

- A subset $A \subseteq X$ is **compactly closed** if $f^{-1}(A)$ is closed for every $f : Y \to X$, where Y is compact and Hausdorff.
- X is **compactly generated** if every compactly closed subset of X is closed.
- X is **weak Hausdorff** if the diagonal map $\Delta : X \to X \times X$ is closed when $X \times X$ has the compactly generated topology.

The intuition behind compact generation is that the topology is determined by compact Hausdorff spaces. The weak Hausdorff topology is strictly stronger than T_1 (points are closed), but strictly weaker than Hausdorff spaces. Any space you can think of without trying to be pathological will meet these criteria.

There is a functor k from all spaces to compactly generated spaces which adds the necessary closed sets. This has the unfortunate name of k**-ification** or **kaonification**; by putting the compactly generated topology on $X \times X$, we mean taking $k(X \times X)$. There's also a "weak Hausdorffification" functor w which makes a space weakly Hausdorff, which is some kind of quotient.[1]

When computing limits and colimits, it's often possible to compute it in the category of spaces and then apply k and w to return to Top. This is fine for limits, but for colimits, w is particularly badly behaved: you cannot compute the colimit in Top by computing it in Set and figuring out the topology; more generally, it will be some kind of quotient.

Nonetheless, there are nice theorems which make things work out anyways.

[1]The k functor is right adjoint to the forgetful map, which tells you what it does to limits.

Proposition 1.2. *Let $Z = \operatorname{colim}(X_0 \to X_1 \to X_2 \to \dots)$ be a sequential colimit (sometimes called a* ***telescope****); if each X_i is weak Hausdorff, then so is Z.*

Proposition 1.3. *Consider a diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \\ C & \end{tikzcd}$$

where f is a closed inclusion. If A, B, and C are weakly Hausdorff, then $B \amalg_A C$ is weakly Hausdorff.

These are the two kinds of colimits people tend to compute, so this is reassuring.

One reason we require regularity on our topological spaces is the following, which is not true for topological spaces in general.

Lemma 1.4. *Let X, Y, and Z be in Top; then, the natural map*

$$\operatorname{Map}(X \times Y, Z) \longmapsto \operatorname{Map}(X, \operatorname{Map}(Y, Z))$$

is a homeomorphism.

Enrichments. The categories Top and Top_* are enriched over themselves (as will categories of G-spaces, which we'll see later). This means a brief digression into enriched categories.

Definition 1.5. Let $(\mathsf{V}, \otimes, \mathbf{1})$ be a symmetric monoidal category.[2] Then, an **enrichment** of a category C over V means

- for every $x, y \in \mathsf{C}$, there is a hom-object $\underline{\mathsf{C}}(x, y)$, which is an object in V,
- for every $x \in \mathsf{C}$, there is a unit $\mathbf{1} \to \underline{\mathsf{C}}(x, x)$,
- composition $\underline{\mathsf{C}}(x, y) \otimes \underline{\mathsf{C}}(y, z) \to \underline{\mathsf{C}}(x, z)$ is associative and unital, and
- the underlying category is recovered as $\mathsf{C}(x, y) = \operatorname{Map}(\mathbf{1}, \underline{\mathsf{C}}(x, y))$.

A great deal of category theory can be generalized to enriched categories, including V-enriched functors, V-enriched natural transformations, V-enriched limits and colimits, and more. The canonical reference is Kelley [Kel84], available free and legally online. It covers just about everything we need except for the Day convolution, which can be read from Day's thesis [Day70]. Another good source, with a view towards homotopy theory, is [Rie14, Chapter 3].

Definition 1.6. Let C and D be enriched over V. Then, an **enriched functor** $F : \mathsf{C} \to \mathsf{D}$ is an assignment of objects in C to objects in D and maps $\underline{\mathsf{C}}(x, y) \to \underline{\mathsf{D}}(Fx, Fy)$ that are V-morphisms, and commute with composition.

Exercise 1.7. Work out the definition of enriched natural transformations.

This brings us to the beginning.

Let G be a group. We'll generally restrict to finite groups or compact Lie groups; this is not because these are the only interesting groups, but rather because they are the only ones we really understand. If you can come up with a good equivariant homotopy theory for discrete infinite groups, you will be famous. Throughout, keep in mind the examples C_p (the cyclic group of order p, sometimes also denoted $\mathbb{Z}/p$), C_{p^n}, the symmetric group S_n, and the circle group S^1.

There's a monad M_G on Top which sends $X \mapsto G \times X$, and analogously M_G^* on Top_* sending $X \to G_+ \wedge X$; then, one can define the category of ***G*-spaces** $G\mathsf{Top}$ (resp. **based *G*-spaces** $G\mathsf{Top}_*$) to be the category of algebras over M_G (resp. M_G^*). This is probably not the most explicit way to define G-spaces, but it makes it evident that $G\mathsf{Top}$ and $G\mathsf{Top}_*$ are complete and cocomplete.

[2] Briefly, this means V has a tensor product $\otimes$ and a unit $\mathbf{1}$; there are certain axioms these must satisfy.

More explicitly, $G\mathsf{Top}$ is the category of spaces $X \in \mathsf{Top}$ equipped with a continuous action $\mu\colon G \times X \to X$. That is, μ must be associative and unital. Associativity is encoded in the commutativity of the diagram

$$\begin{tikzcd} G \times G \times X \arrow[r, "1\times\mu"] \arrow[d, "m"] & G \times X \arrow[d, "\mu"] \\ G \times X \arrow[r, "\mu"] & X. \end{tikzcd}$$

The morphisms in $G\mathsf{Top}$ are the **G-equivariant** maps $f : X \to Y$, i.e. those commuting with μ:

$$\begin{tikzcd} G \times X \arrow[r] \arrow[d, "\mu_X"] & G \times Y \arrow[d, "\mu_Y"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

It's possible (but not the right idea) to let $\underline{G}$ denote[3] the category with an object $*$ such that $\underline{G}(*,*) = G$. Then, $G\mathsf{Top}$ is also the category of functors $\underline{G} \to \mathsf{Top}$, with morphisms as natural transformations. This realizes $G\mathsf{Top}$ as a **presheaf category**; it will eventually be useful to do something like this, but not in this specific way.

When we write $\mathrm{Map}(X,Y)$ in $G\mathsf{Top}$ or $G\mathsf{Top}_*$, we could mean three things:

(1) The set of G-equivariant maps $X \to Y$.
(2) The space of G-equivariant maps $X \to Y$ in the subspace topology of all maps from $X \to Y$. As this suggests, $G\mathsf{Top}$ admits an enrichment over Top (resp. $G\mathsf{Top}_*$ admits an enrichment over Top_*).
(3) The G-space of all maps $X \to Y$, where G acts by conjugation: $f \mapsto g^{-1}f(g\cdot)$. This means $G\mathsf{Top}$ is enriched in itself, as is $G\mathsf{Top}_*$.

Each of these is useful in its own way: for constructions it may be important to be self-enriched, or to only look at G-equivariant maps. We will let $\mathrm{Map}^G(X,Y)$ or $\mathrm{Map}(X,Y)$ denote (2) or its underlying set (1), and $G\,\mathrm{Map}(X,Y)$ denote (3).

It turns out you can recover Map^G from Map: the equivariant maps are the fixed points under conjugation of all maps. This is written $\mathrm{Map}(X,Y)^G = \mathrm{Map}^G(X,Y)$.

Throughout this class, "subgroup" will mean "closed subgroup" unless specified otherwise.

Definition 1.8. Let X be a G-set and $H \subseteq G$ be a subgroup. Then, the **H-fixed points** of X is the space $X^H := \{x \in X \mid hx = x \text{ for all } h \in H\}$. This is naturally a WH-space, where $WH = NH/H$ (here NH is the normalizer of H in G).[4]

Definition 1.9. The **isotropy group** of an $x \in X$ is $G_x := \{h \in G \mid hx = x\}$.

These are useful in the following two ways.

(1) Often, it will be helpful to reduce questions from $G\mathsf{Top}$ to Top using $(-)^H$.
(2) It's also useful to induct over isotropy types.

Now, we'll see some examples of G-spaces.

Example 1.10. Let H be a subgroup of G; then, the **orbit space** G/H is a useful example, because it corepresents the fixed points by H. That is, $X^H \cong G\,\mathrm{Map}(G/H, X)$. These spaces will play the role that points did when we build things such as equivariant CW complexes. ⋖

Example 1.11. Let $H \subset G$ as usual and $U\colon G\mathsf{Top} \to H\mathsf{Top}$ be the forgetful functor. Then, U has both left and right adjoints:

- The left adjoint sends X to the **balanced product** $G \times_H X := G \times X/\sim$, where $(gh, x) \sim (g, hx)$ for all $g \in G$, $h \in H$, and $x \in X$. Despite the notation, this is *not* a pullback! (In the based case, the balanced product is $G_+ \wedge_H X$.) G acts via the left action on G. This is called the **induced G-action** on $G \times_H X$.
- The right adjoint is $F_H(G, X)$ (or $F_H(G_+, X)$ in the based case), the space of H-maps $G \to X$, with G-action $(gf)(g') = f(g'g)$. This is called the **coinduced G-action** on $F_H(G, X)$.[5] ⋖

[3] There isn't really a standard notation for this category, but the closest is BG. This notation emphasizes the fact that groupoids are Quillen equivalent to 1-truncated spaces.

[4] If $H \trianglelefteq G$, then X^H is also a G/H-space.

[5] This actually *is* a group action, since if $a, b, g \in G$, then $(a(bf))(g) = (bf)(ga) = f(gab) = (ab(f))(g)$.

Remark. Here is a categorical perspective on "change of group." Quite generally, a group homomorphism $G \xrightarrow{f} H$ induces adjunctions

$$\begin{tikzcd} G\mathsf{Top} \arrow[r, shift left=4, "f_!"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] \arrow[r, shift right=4, "f_*"'] & H\mathsf{Top}. \arrow[l, "f^*"'] \end{tikzcd}$$

These are given by $f_!(X) = H \times_G X$ and $f_*(X) = F_G(H, X)$ for a G-space X, where H is given the structure of a G-space by f. When $H = *$, an H-space is just a space, and $f_!(X) = X_G$ is the space of orbits while $f_*(X) = X^G$ is the space of fixed points. Observe that similar statements hold for categories of modules, given a ring homomorphism $R \xrightarrow{f} S$.

In fact, these are both cases of very general abstract nonsense. Let BG denote the category with one object $*$ with $\mathrm{Hom}(*, *) = G$; as we have said above, we can (naïvely) write $G\mathsf{Top}$ as the functor category Top^{BG}. A group homomorphism $G \xrightarrow{f} H$ induces a functor $BG \xrightarrow{F} BH$ (it is not quite true that the two are equivalent—think about why this is). Now $f^* \colon H\mathsf{Top} \to G\mathsf{Top}$ is just restriction along F:

$$\begin{tikzcd} BG \arrow[r, dotted, "f^*(Y)"] \arrow[d, "F"'] & \mathsf{Top} \\ BH \arrow[ur, "Y"'] & \end{tikzcd}$$

According to abstract nonsense, restriction along F has a left and right adjoint, called *left and right Kan extension along F*:

$$\begin{tikzcd} BG \arrow[r, "X"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\eta"] & \mathsf{Top} \\ BH \arrow[ur, dotted, "f_!(X)=\mathrm{Lan}_F X"'] & \end{tikzcd} \qquad \begin{tikzcd} BG \arrow[r, "X"] \arrow[d, "F"'] \arrow[dr, phantom, "\Uparrow\epsilon"] & \mathsf{Top} \\ BH \arrow[ur, dotted, "f_*(X)=\mathrm{Ran}_F X"'] & \end{tikzcd}$$

These diagrams do not commute, but there are natural maps $X \xRightarrow{\eta} f^* f_!(X)$ and $f^* f_*(X) \xRightarrow{\epsilon} X$. When H is the trivial group, BH is the trivial category, and it is known that left/right Kan extensions of a functor X along a functor to the trivial category pick out the colimit/limit of X. That is, still viewing a G-space X as a functor $BG \to \mathsf{Top}$, we have $X_G = \operatorname*{colim}_{BG} X$ and $X^G = \lim_{BG} X$.

For an introduction to Kan extensions, we recommend [Rie16, Chapter 6] (which is almost the same as [Rie14, Chapter 1] but with some more amusing examples). Like much of category theory, this is ultimately all trivial, but it may be highly non-trivial to understand why it is trivial. ◂

Example 1.12. Let V be a finite-dimensional real representation of G, i.e. a real inner product space on which G acts in a way compatible with the inner product. (This is specified by a group homomorphism $G \to \mathrm{O}(V)$.) The one-point compactification of V, denoted S^V, is a based G-space; the unit disc $D(V)$ and unit sphere $S(V)$ are unbased spaces, but we have a quotient sequence

$$S(V)_+ \longrightarrow D(V)_+ \longrightarrow S^V.$$

If $V = \mathbb{R}^n$ with the trivial G-action, S^V is S^n with the trivial G-action, so these generalize the usual spheres; thus, these S^V are called **representation spheres**. ◂

We will let S^n denote $S^{\mathbb{R}^n}$, namely our preferred model for the n-sphere with trivial G-action.

Beginnings of homotopy theory.

Definition 1.13. A G**-homotopy** is a map $h \colon X \times I \to Y$ in $G\mathsf{Top}$, where G acts trivially on I. We generally think of it, as usual, as interpolating between $h(-, 0)$ and $h(-, 1)$. This is the same data as a path in $G\,\mathrm{Map}(X, Y)$. A G**-homotopy equivalence** between X and Y is a map $f : X \to Y$ such that there exists a $g : Y \to X$ such that there are G-homotopies $gf \sim \mathrm{id}_X$ and $fg \sim \mathrm{id}_Y$.

The (well, a) natural question that might arise: what are G-weak equivalences and G-CW complexes? This closely relates to obstruction theory — CW complexes are test objects.

To define G-CW complexes, we need cells. One choice is $G/H \times D^{n+1}$ and $G/H \times S^n$, where the actions on D^{n+1} and S^n are trivial. This is a plausible choice (and in fact, will be the right choice), but it's not clear why — why not $G \times_H D(V)$ or $G \times_H S(V)$ for some H-representation V? Ultimately, this comes from a (quite nontrivial) theorem that these can be triangulated in terms of the cells $G/H \times D^{n+1}$ and $G/H \times S^n$.[6] This is one of several triangulation results proven in the 1970s, which are now assumed without comment, but if you like this kind of math then it's a very interesting story.

Definition 1.14. A G**-CW complex** is a sequential colimit of spaces X_n, where X_{n+1} is a pushout

$$\begin{tikzcd}
\bigvee G/H \times S^n \arrow[r] \arrow[d] & X_n \arrow[d] \\
\bigvee G/H \times D^{n+1} \arrow[r] & X_{n+1},
\end{tikzcd}$$

where H varies over all closed subgroups of G.

That is, it's formed by attaching cells just as usual, though now we have more cells.

This immediately tells you what the homotopy groups have to be: $[G/H \times S^n, X]$, which by an adjunction game is isomorphic to $\pi_n(X^H)$. We let $\pi_n^H(X) := \pi_n(X^H)$. Thus, we can define weak equivalences.

Definition 1.15. A map $f : X \to Y$ of G-spaces is a **weak equivalence** if for all subgroups $H \subset G$, $f_* \colon \pi_n^H(X) \to \pi_n^H(Y)$ is an isomorphism.

These homotopy groups have a more complicated algebraic structure: they're indexed by the lattice of subgroups of G and the integers. This is fine (you can do homological algebra), but some things get more complicated, including asking what the analogue of connectedness is!

One quick question: do we need all subgroups H? What if we only want finite-index ones? The answer, in a very precise sense, is that if you're willing to use fewer subgroups, you get fewer cells $G/H \times S^n$, and that's fine, and you get a different kind of homotopy theory.

Finally, the Whitehead theorem is true for G-CW complexes. This follows for the same reason as in May's course: it follows word-for-word after proving the equivariant HELP lemma (homotopy extension lifting property), which is true by the same argument.

We'll next talk about presheaves on the orbit category, leading to Bredon cohomology.

Lecture 2.

Homotopy theory of G-spaces: 1/19/17

"It's nice to write down, but oh so false."

Last time, we saw the definition of a G-CW complex, but no examples were provided. Today, we'll start with some examples.

Recall that a G-CW complex is a sequential colimit $X = \mathrm{colim}_n X_n$, where X_n is formed by attaching cells $G/H \times D^n$ along maps $G/H \times S^{n-1} \to X_{n-1}$: just like the CW complexes we know and love, but with new cells G/H indexed by the closed subgroups $H \subset G$. The idea is that you're building up a space by attaching different spaces with different isotropy groups (G/H has isotropy group H, just by construction).

Example 2.1 (Zero-dimensional complexes)**.** The zero-dimensional complexes are G/H or disjoint unions $\amalg_i G/H_i$. This is an instance of the slogan that "orbits are points." Keep in mind that if G is a compact Lie group, this might not be zero-dimensional in other, more familiar kinds of dimension. ◀

Example 2.2. Let S^1 act on $\mathbb{R}^2$ by rotation along the origin. This also induces a C_n-action, as $C_n \subseteq S^1$ as the n^{th} roots of unity. Let V denote this C_n-space.

Let $D(V)$ denote the unit disc in V, and S^V denote its one-point compactification, a representation sphere. Then, $D(V)$ looks like wedges of pie, as the origin is fixed. On S^V, the point at infinity is also fixed, so we obtain a beachball.

[6] Illman's thesis [Ill72] is a reference, albeit not the most accessible one.

Now let's consider V as an S^1-space, and write down the CW structure on S^V. There are two fixed points, and each one is a 0-cell $S^1/S^1 \times *$, but there is one 1-cell $S^1 \times I$ attached to the endpoints (thought of as a meridian rotated around the sphere).

Now let's consider the beachball for C_2 on S^V, where there are two hemispheres and C_2 rotates by a half-turn. What's the G-CW structure on this?

- There are two 0-cells $C_2/C_2 \times *$, corresponding to the two fixed points, the north and south poles.
- There is a single free 1-cell $C_2 \times I$, corresponding to the boundary of the hemispheres.
- There is a single 2-cell $C_2 \times D^2$.

Last time, we discussed other prospective cells $G \times_H S(V)$ and $G \times_H D(V)$; these can be decomposed in terms of the actual cells we use. It's also worth mentioning that the action of G on our actual G-cells is cellular, unlike for the other cells.[7]

Exercise 2.3. C_2 also acts on S^2 by the antipodal map, which has no fixed points. Write a C_2-CW cell structure for this C_2-space.

Example 2.4. The torus $S^1 \times S^1$ has an S^1-action given by $z(z_1, z_2) = (zz_1, z_2)$. With this action, the torus can be viewed as an S^1-CW complex with one 0-cell $S^1/e \times *$ and one 1-cell $S^1 \times [0,1]$, with the attaching map sending 0 and 1 to $*$. Note that the largest cell we used here was a 1-cell, whereas in the nonequivariant construction of the torus, we are required to use a 2-cell.

There will be additional examples of G-CW complexes on the homework, some with richer structure.

Remark. At this point in class, the professor mentioned that these notes are hosted on Github at `https://github.com/adebray/equivariant_homotopy_theory`. Since there aren't very many sources for learning this material, and existing ones tend to have few examples, the hope is that these notes can be turned into a good source of lecture notes for learning this material. So as you're learning this material, feel free to add examples, insert comments (e.g. "this section is confusing/unmotivated"), and let me know if you want access to the repository.

Remark.

(1) There is a technical issue of a G-CW structure on a product of G-CW complexes; namely, there are technical difficulties in cleanly putting a G-CW structure on $G/H_1 \times G/H_2$ involving triangulation. We won't digress into this: it's straightforward for finite groups, but a theorem for compact Lie groups, and required revisiting the foundations. Similarly, if $H \subset G$, we'd like the forgetful functor $G\mathsf{Top} \to H\mathsf{Top}$ to send G-CW complexes to H-CW complexes. This is again possible, yet involves technicalities.

(2) A nicer fact is that computing the fixed points of a G-CW complex is straightforward. Recall that $(-)^H$ is a right adjoint, which can be seen by realizing it as the limit of the diagram

$$\begin{tikzcd} \bullet \arrow[loop above, "H"] \arrow[r] & \mathsf{Top}. \end{tikzcd}$$

Thus, we don't expect it to commute with colimits in general. However, it does commute with many important ones, as in the following proposition.

Proposition 2.5. *The fixed point functor $(-)^H$ commutes with*

(1) *pushouts where one leg is a closed inclusion, and*
(2) *sequential colimits along closed inclusions.*

This is great, because it means we can commute $(-)^H$ through the construction of a G-CW complex! In particular, on each cell,

$$(G/K \times D^n)^H \cong (G/K)^H \times D^n,$$

so we need to understand $(G/K)^H \cong \mathrm{Map}^G(G/H, G/K)$. We will return to this important point.

[7] TODO: there must have been a different word than "cellular" here; what was it?

Two approaches to the Whitehead theorem. We'll now discuss some homotopy theory of G-spaces and the Whitehead theorem. The first will be a hands-on proof using the HELP lemma. This is an elegant approach to unstable homotopy theory due to Peter May in which one lemma gives quick proofs of several theorems. In the equivariant case, it allows a quick reduction to the non-equivariant case; it will be useful to see a proof of this nature. Ultimately, we will take a different approach involving model categories, and this will be the second perspective.

Definition 2.6. Let $X, Y \in \mathsf{Top}$ and $f : X \to Y$ be continuous. Then, f is n**-connected** if $\pi_q(f) : \pi_q(X) \to \pi_q(Y)$ is an isomorphism when $q < n$ and surjective when $q = n$.

We wish to generalize this to the equivariant case.

Definition 2.7. Let $\theta : \{\text{conjugacy classes of subgroups of } G\} \to \{x \in \mathbb{Z} \mid x \geq -1\}$.

- A map $f : X \to Y$ of G-spaces is θ**-connected** if for all $H \subset G$, f^H is $\theta(H)$-connected.
- A G-CW complex is θ**-dimensional** if all cells of orbit type G/H have (nonequivariant) dimension at most $\theta(H)$.

Theorem 2.8 (Equivariant HELP lemma)**.** *Let A, X, Y, and Z be G-CW complexes such that $A \subseteq X$ is θ-dimensional and let $e : Y \to Z$ be a θ-connected G-map. Given $g : A \to Y$, $h : A \times I \to Z$, and $f : X \to Z$ such that $eg = hi_0$ and $fi = hi_1$, there exist maps $\tilde{g} : X \to Y$ and $\tilde{h} : X \times I \to Z$ that make the following diagram commute:*

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd] & & A \times I \arrow[dl, "h"'] \arrow[dd] & & A \arrow[ll, "i_1"'] \arrow[dl, "g"'] \arrow[dd] \\
& Z & & Y \arrow[ll, "e"'] & \\
X \arrow[rr, "i_0"'] \arrow[ur, "f"] & & X \times I \arrow[ul, dashed, "\tilde{h}"] & & X \arrow[ll, "i_1"] \arrow[ul, dashed, "\tilde{g}"]
\end{tikzcd}$$

This is a massive elaboration of the idea of a Hurewicz cofibration. The best way to understand this is to prove it (though it's not an easy proof).

In the non-equivariant case, one reduces to working one cell at a time, inductively extending over the cells of X not in A.[8] In this case, look at $S^{n-1} \subseteq D^n$. Now you just do it: at this point, there's no way to avoid writing down explicit homotopies.

Exercise 2.9. Think about this argument, and then read the proof in [May99].

The equivariant case is very similar: in the same way, one can reduce to inductively attaching a single cell in the case where X is a finite CW complex. This comes via a map $G/H \times S^{n-1} \to G/H \times D^n$, but the only interesting content is in the nonequivariant part, so we can reduce again to $S^{n-1} \to D^n$ with trivial G-action! This allows us to finish the proof in the same way. It also says that the homotopy theory of G-spaces is lifted from ordinary homotopy theory, in a sense that model categories will allow us to make precise.

The first consequence of Theorem 2.8 is:

Theorem 2.10. *Let $e : Y \to Z$ be a θ-connected map and $e_* : [X, Y] \to [X, Z]$ be the map induced by composition.*

- *If X has dimension less than θ, e_* is a bijection.*[9]
- *If X has dimension θ, e_* is a surjection.*

The proof is an exercise; filling in the details is a great way to get your hands on what the HELP lemma is actually doing. Hint: consider the pairs $\varnothing \to X$ and $X \times S^0 \to X \times I$, and apply the HELP lemma.

Corollary 2.11 (Equivariant Whitehead theorem)**.** *Let $e : Y \to Z$ be a weak equivalence of G-CW complexes. Then, e is a G-homotopy equivalence.*

[8] This requires reducing to the case where X is a finite CW complex, but taking a sequential colimit recovers the theorem for all CW complexes X.

[9] We say that X has dimension less than θ if for all closed subgroups $H \subset G$, all cells of orbit type G/H have (nonequivariant) dimension at most n for some $n \leq \theta(H)$.

Proof. This is also a standard argument: using Theorem 2.10, e_* is a bijection, so we can pull back $\mathrm{id}_Z \in [Z, Z]$ to an inverse $(e_*)^{-1}(\mathrm{id}_Z) \in [Z, Y]$, which is a homotopy inverse to e. ⊠

One can continue and prove the cellular approximation theorem in this way, and so forth. We won't do this, because we'll approach it from a model-categorical perspective.

One thing that's useful, not so much for this class as for enriching your life, is to learn how to approach this from the perspective of abstract homotopy theory, learning about disc complexes and so forth. You can prove theorems such as the HELP lemma and its consequences in a general setting, and then specialize them to the cases you need. This is a great way to "just do it" without needing model categories.

Anyways, we'll now define a model structure on $G\mathsf{Top}$ and $G\mathsf{Top}_*$. If you don't know what a model category is, now is a good time to review.

Proposition 2.12. *There is a model structure on $G\mathsf{Top}$ (and on $G\mathsf{Top}_*$) defined by the following data.*

- **Cofibrations:** *The maps $f : X \to Y$ such that for all $H \subset G$, $f^H : X^H \to Y^H$ is a cofibration.*
- **Weak equivalences:** *The maps $f : X \to Y$ such that for all $H \subset G$, $f^H : X^H \to Y^H$ is a weak equivalence.*

So we once again parametrize everything over subgroups of G and use fixed points. This is a cofibrantly generated model category; the cofibrations are specified by generators of acyclic cofibrations in a similar manner to Top. That is, in Top, one can choose generators $I = \{S^{n-1} \to D^n\}$ and $J = \{D^n \to D^n \times I\}$; in $G\mathsf{Top}$, we instead take $I_G = \{G/H \times I\}$ and $J_G = \{G/H \times J\}$.

These are cells that we used to define G-CW complexes, and this is no coincidence: it's a general fact about cofibrantly generated model categories that follows from the small object argument[10] that cofibrant objects are retracts of "cell complexes" built from the things in I, and cofibrations are retracts of cellular inclusions of cell complexes. In this sense, CW complexes are inevitable.

The Whitehead theorem (Corollary 2.11) now falls out of the general theory of model categories.

Theorem 2.13 (Whitehead theorem for model categories)**.** *Let $f : X \to Y$ be a weak equivalence of cofibrant-fibrant objects in a model category. Then, f is a homotopy equivalence.*

In Top and $G\mathsf{Top}$, all objects are fibrant, so this is particularly applicable.

The orbit category. We'll begin talking about the orbit category in the rest of today's lecture, and discuss the bar construction next class.

Definition 2.14. The **orbit category** $\mathscr{O}_G$ is the full subcategory of $G\mathsf{Top}$ on the objects G/H.

That is, its objects are the spaces G/H, where $H \subset G$ is closed, and its morphisms are $\mathrm{Map}^G(G/H, G/K) \cong (G/K)^H$. These maps are the same thing as subconjugacy relations, i.e. those of the form

$$gHg^{-1} \subseteq K, \tag{2.15}$$

since for all $h \in H, h(gK) = gK$ if and only if $K = g^{-1}hgK$ if and only if $gHg^{-1} \subseteq K$. A G-map $f : G/H \to G/K$ is completely specified by what it does to the identity coset $f(eH) = gK$, and this g implies the subconjugacy relation (2.15), since, as above, $h(gK) = gK$ for all $h \in H$.

There's another description of the orbit category.

Proposition 2.16. *Let G be a finite group. Then, the orbit category $\mathscr{O}_G$ is equivalent to the category of finite transitive G-sets and G-maps.*

The observation that ignites the proof is that if $x \in X$ has isotropy group H, then its orbit space is isomorphic to G/H.

Definition 2.17. Given a G-space X, we obtain a presheaf on the orbit category, namely a functor $X^{(-)} : \mathscr{O}_G^{\mathrm{op}} \to \mathsf{Top}$, by sending $G/H \to X^H$. This assignment itself is a functor $\psi : G\mathsf{Top} \to \mathsf{Fun}(\mathscr{O}_G^{\mathrm{op}}, \mathsf{Top})$.

Proposition 2.18. $\mathsf{Fun}(\mathscr{O}_G^{\mathrm{op}}, \mathsf{Top})$ *has a projective model structure where the weak equivalences and fibrations are taken pointwise.*

[10] The small object argument is a beautiful piece of basic mathematics that everybody should know. If you don't know it, your homework is to read enough about model categories to get to that point. In general, there may be large objects and transfinite induction, but for the case we care about large cardinals won't arise.

The point is the following result, a revisionist interpretation of Elmendorf's theorem.[11]

Theorem 2.19. *ψ is the right adjoint in a Quillen equivalence; the left adjoint θ is evaluation at G/e.*

The point is, these two model categories have the same homotopy theory.

Exercise 2.20. Check that evaluation at G/e is a left adjoint to ψ.

Lecture 3.

Elmendorf's theorem: 1/24/17

"What's bad about this proof?"
"It appeals to machinery we didn't develop in this class?"
"No, that's perfectly fine."

We'll start by reviewing the connection between the orbit category and G-sets.

Let X be a finite G-set. Then, X is the coproduct (disjoint union) of a bunch of orbits:

$$X \cong \coprod_i G/H_i.$$

The way you see this is that for any $x \in X$, its orbit is isomorphic to G/G_x. This is yet another manifestation of the slogan that "orbits are points." But it also implies that, rather than just presheaves on $\mathscr{O}_G$, one could work with certain presheaves on the category of finite G-sets, and this perspective will turn out to be useful. By "certain" we mean a compatibility with orbits.

Last time, we talked about Elmendorf's theorem in the form of Theorem 2.19. It's also possible to state it in a more general form.

Theorem 3.1 (Elmendorf)**.** *The functor $G\mathsf{Top} \to \mathsf{Fun}(\mathscr{O}_G^{\mathrm{op}}, \mathsf{Top})$ determined by $X \mapsto \big(G/H \mapsto X^H\big)$ induces an equivalence of $(\infty, 1)$-categories, where the weak equivalences on the left and right are specified by a family $\mathscr{F}$.*

Without delving into $(\infty, 1)$-categories, this means

- the homotopy categories are equivalent, and
- homotopy limits and colimits behave identically.

In other words, from the perspective of abstract homotopy theory, these are the same.

Definition 3.2. By a **family** of subgroups $\mathscr{F}$ of G, we mean a collection of subgroups of G closed under conjugation and taking subgroups.

Examples include the set of all subgroups, the set of just the identity, and the set of finite subgroups. The latter is useful for some S^1-equivariant spaces, where one tends to lose control of the S^1-fixed points, but the finite subgroups behave better.

Definition 3.3. Let $\mathscr{F}$ be a specified family of subgroups of G.

- In $G\mathsf{Top}$, the weak equivalences specified by $\mathscr{F}$ are the maps $f : X \to Y$ such that $f^H : X^H \to Y^H$ is a weak equivalence for all $H \in \mathscr{F}$.
- For $\mathsf{Fun}(\mathscr{O}_G^{\mathrm{op}}, \mathsf{Top})$, a weak equivalence specified by $\mathscr{F}$ is a pointwise weak equivalence at G/H for all $H \in \mathscr{F}$.

We'll give two proofs of Theorem 3.1. The first will be model-categorical.

Recall[12] if $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ is a Quillen adjunction, then the left and right derived functors $(\mathbf{L}F, \mathbf{R}G)$ is an adjunction on the homotopy categories $(\mathrm{Ho}\,\mathsf{C}, \mathrm{Ho}\,\mathsf{D})$. If K denotes fibrant replacement in D and Q denotes cofibrant replacement in C, then the derived functors are $\mathbf{L}F = FQ$ and $\mathbf{R}G = GK$.[13]

Definition 3.4. That (F, G) is a **Quillen equivalence** means that for any cofibrant $X \in \mathsf{C}$ and fibrant $Y \in \mathsf{D}$, then $FX \to Y$ is a weak equivalence iff its adjoint $X \to GY$ is.

[11] Elmendorf proved that these two categories have the same homotopy theory, but his proof was more explicit.

[12] If this is not review to you, then exercise: learn this material!

[13] This does require cofibrant and fibrant replacement to be functorial, which is not true in every model category, but will be true for pretty much everything we study.

This is equivalent to asking that $(\mathbf{L}F, \mathbf{R}G)$ are equivalences of categories.

This is a kind of curious way to look at an equivalence of categories. One says that $G : \mathsf{D} \to \mathsf{C}$ **creates the weak equivalences** of D if for every morphism f of D, f is a weak equivalence iff Gf is.

Lemma 3.5. *If G creates the weak equivalences of D and for all cofibrant X the unit map $X \to GFX$ is a weak equivalence, then (F, G) is a Quillen equivalence.*

This is a useful tool for extending model categories along free-forgetful adjunctions; for example, if you have a model category and want to understand abelian group or ring objects in this category, often their weak equivalences are detected by the forgetful functor.

Proof sketch of Theorem 3.1. We want to apply Lemma 3.5 to the adjunction

$$\theta : \mathsf{Fun}(\mathscr{O}_G^{\mathsf{op}}, \mathsf{Top}) \rightleftarrows G\mathsf{Top} : \psi,$$

where $\theta : X \mapsto X(G/e)$ is evaluation at G/e and $\psi : Y \mapsto \{Y^H\}$. The first condition, that ψ detects the weak equivalences, is straightforward, so we need to check that $X \mapsto \{X(G/e)^H\}$ is a weak equivalence for all cofibrant X.

Cellular objects model the generating cofibrations, so cofibrant objects are retracts of cellular objects. Since weak equivalences are preserved under retracts, then we can check on cellular objects. Here it's easier, since $(-)^H$ commutes with the relevant colimits and is suitably cellular. ⊠

The missing steps in this proofs can be filled in by explicitly identifying the cofibrant objects in $\mathsf{Fun}(\mathscr{O}_G^{\mathsf{op}}, \mathsf{Top})$. These are free diagrams on the orbit category; not hard to write down, but messy enough to avoid on the chalkboard.

Remark. Elmendorf's original proof of his theorem was in the 1980s did not use model categories, even though Quillen had already introduced them at the time. Until the mid-1990s (30 years after Quillen introduced them), many homotopy theorists avoided them, thinking of them as formal gobbledygook. However, about the time EKMM introduced a symmetric monoidal category of spectra, people began realizing they were unavoidable. ◂

You might not like the given proof of Elmendorf's theorem because it's extremely inexplicit: cofibrant replacement is an infinite process, and many of the steps involved are quite abstract. The next proof will be more explicit, building a (homotopical) right adjoint to ψ.

This proof will go through the **bar construction**, a categorical tool that's extremely useful. References for it include May's "Geometry of iterated loop spaces" [May72], Riehl's monograph [Rie14], and Vogt's "Tensor products of functors."

Second proof of Theorem 3.1. Let $M : \mathscr{O}_G \to \mathsf{Top}$ realize orbits as spaces: G/H is sent to the topological space G/H, and an equivariant map f is forgotten to a continuous map f.

Given an $X \in \mathsf{Fun}(\mathscr{O}_G^{\mathsf{op}}, \mathsf{Top})$, let

$$\Phi(X) := |B_\bullet(X, \mathscr{O}_G, M)|$$

denote the geometric realization of the simplicial bar construction. Let's be a little more explicit about this. $B_\bullet(X, \mathscr{O}_G, M)$ is a simplicial space that sends

$$[n] \longmapsto \coprod_{G/H_{n-1} \to \cdots \to G/H_0} X(G/H_0) \times M(G/H_{n-1}).$$

As usual, the face maps are defined by composition, and the degeneracies by inserting the identity map. Since G acts on $M(-)$ simplicially (i.e., in a way compatible with the face and degeneracy maps), then $|B_\bullet(X, \mathscr{O}_G, M)|$ is a G-space (passing through the coend formula for the geometric realization).

If $H \subseteq G$, we want to understand $\Phi(X)^H$. Because the G-action passed through geometric realization,

$$\Phi(X)^H \cong |B_\bullet(X, \mathscr{O}_G, M^H)| \cong |B_\bullet(X, \mathscr{O}_G, \mathrm{Map}_{\mathscr{O}_G}(G/H, -))|.$$

Let $X(G/H)$ denote the constant simplicial space $[n] \mapsto X(G/H)$. Then, by general theory of the bar construction for any corepresented functor, there's a simplicial map

$$B_\bullet(X, \mathscr{O}_G, \mathrm{Map}_{\mathscr{O}_G}(G/H, -)) \longrightarrow X(G/H) \tag{3.6}$$

defined by composing and applying X, and this is a simplicial homotopy equivalence (you can write down a retraction).[14] Thus, $\Phi(X)^H \cong X(G/H)$. In other words, Φ is a homotopy inverse, since taking H-fixed points of $\Phi(X)$ gives back what you started with. ⊠

$\Phi(X)$ is still an infinite-dimensional object, but it's much more explicit, and you can work with it.

Applications of this perspective. We'll be able to use Elmendorf's theorem to make some constructions that would be hard to imagine without the orbit category.

Definition 3.7. Let $\mathscr{F}$ be a family of subgroups of G. Then, the **classifying space** for $\mathscr{F}$ is specified by the universal property that if Z has $\mathscr{F}$-isotropy, then $[Z, E\mathscr{F}]$ has a unique element. An explicit construction is to let $\widetilde{E}\mathscr{F}$ denote the presheaf on the orbit category where

$$\widetilde{E}\mathscr{F}(G/H) := \begin{cases} *, & H \in \mathscr{F} \\ \varnothing, & H \notin \mathscr{F}, \end{cases}$$

and let $E\mathscr{F} := \Phi(\widetilde{E}\mathscr{F})$.

If you unwind the definition, this is the bar construction applied to G in the category of G-spaces with weak equivalences given by $\mathscr{F}$, meaning it deserves to be called a classifying space.

Another useful notion is the G-connected components.

Definition 3.8. Let X be a G-space and $x \in X^G$. Let Y_x be the presheaf on the orbit category sending H to the connected component containing $x \in X^H$. Then, the **G-connected component** of x is $\Phi(Y_x)$.

The third useful application is defining Eilenberg-Mac Lane spaces. This will lead us to cohomology (and then to Smith theory and other things). These will be constructed by working pointwise, then applying Φ.

Definition 3.9. Let G be a finite group, A **coefficient system** is a presheaf $X \in \mathsf{Fun}(\mathscr{O}_G^{\mathrm{op}}, \mathsf{Ab})$.[15]

Elmendorf's theorem says that for any coefficient system, we have an Eilenberg-Mac Lane G-space. You could say here that (Bredon) cohomology is completely determined: cohomology is the things represented by Eilenberg-Mac Lane spaces. But it will be good to see it explicitly. Bredon cohomology is explicit, but there are serious drawbacks: it has poor formal properties, and you need a lot of geometric insight to compute things. We'll later see that this abelian category (meaning we can do homological algebra) is the wrong one; we'll later see this is a $\mathbb{Z}$-graded cohomology theory (or rather graded on subgroups of $\mathbb{Z}$); this will be the wrong answer, especially if you want Poincaré duality, and the right answer uses a grading by the representation ring. But we'll get there.

Remark. Another application of Elmendorf's theorem, which we will not discuss in detail (unless we get to the slice filtration), is Postnikov towers. They're constructed in the same way, by either using the small object argument or killing homotopy groups. ◂

Here are some examples of coefficient systems (which are often denoted with underlines).

Example 3.10.

(1) For a G-space X, the coefficient system $\underline{\pi}_n(X)$ $(n \geq 2)$ sends $G/H \mapsto \{\pi_n X^H\}$. This is an example of a general formula: given a functor $\mathsf{Top} \to \mathsf{Ab}$, we can compose to obtain a map $\mathsf{Fun}(\mathscr{O}_G^{\mathrm{op}}, \mathsf{Top}) \to \mathsf{Fun}(\mathscr{O}_G^{\mathrm{op}}, \mathsf{Ab})$.

(2) In the same way, $\underline{H}_n(X)$ sends $G/H \mapsto \{H_n(X^H; \mathbb{Z})\}$. ◂

We will now define Bredon cohomology, which is what you think it is.

Definition 3.11. Let X be a G-CW complex and X_n denote its n-skeleton. Let

$$\underline{C}_n(X) := \underline{H}_n(X_n, X_{n-1}; \mathbb{Z}),$$

[14] This is called an **extra degeneracy argument** in the literature. There's an observation probably due to John Moore which approximately says that if you have a simplicial object with an extra degeneracy condition playing well with the preexisting ones, then it must be contractible; this argument is applied to the fiber of (3.6).

[15] For G a compact Lie group, the definition is almost the same, but we need to use $\mathsf{Fun}(h\mathscr{O}_G^{\mathrm{op}}, \mathsf{Ab})$, taking presheaves on the homotopy category. For finite groups these definitions coincide.

i.e. this coefficient system sends $G/H \mapsto H_n((X^H)_n, (X^H)_{n-1}; \mathbb{Z})$. The connecting homomorphism comes as usual from the triple $((X^H)_n, (X^H)_{n-1}, (X^H)_{n-2})$.[16]

The **Bredon cohomology** for the coefficient system M is

$$H^n_G(X; M) := H^n\Big(\mathrm{Hom}_{\mathsf{Fun}(\mathscr{O}_G^{\mathsf{op}}, \mathsf{Ab})}(\underline{C}_\bullet(X), M)\Big).$$

That is, it's the cohomology of the cochain complex of natural transformations from $\underline{C}_\bullet(X)$ to M.

Example 3.12. Give S^2 the C_2-action that rotates by π, and call it Z, and let M denote the constant coefficient system at $\mathbb{Z}$. First, we compute the cells:

- $\underline{C}_2(Z)(C_2/e) = \mathbb{Z} \oplus \mathbb{Z}$ and $\underline{C}_2(Z)(C_2/C_2) = 0$.
- $\underline{C}_1(Z)(C_2/e) = \mathbb{Z} \oplus \mathbb{Z}$ and $\underline{C}_1(Z)(C_2/C_2) = 0$.
- $\underline{C}_0(Z)(C_2/e) = \mathbb{Z} \oplus \mathbb{Z}$ and $\underline{C}_0(Z)(C_2/C_2) = \mathbb{Z} \oplus \mathbb{Z}$. ◂

Exercise 3.13. Figure out the differentials in the above example.

We'll continue to discuss Bredon cohomology next lecture, and introduce an axiomatic viewpoint. This is basic and fundamental, but not too relevant to the rest of the class. It's also good to calculate; there are surprisingly few examples out in the world.

Lecture 4.

Bredon cohomology: 1/26/17

"Smith's theorem was proven by Smith, hence the name."

Today I was running about two minutes late. Sorry, everybody! (What I missed: Andrew discussed holding a "Q and A" session next week on a weeknight, perhaps with pizza. If you're interested in this, email him to let him know which nights work for you.)

Today we're going to talk more about Bredon cohomology, including an analogue of the Eilenberg-Steenrod axioms for it. Then, we'll turn to the circle of ideas around Smith theory, including the Sullivan conjecture (which we won't prove, because it's hard). Smith theory discusses when one can recover $H_*(X^G)$ from a G-action on X and on $H_*(X)$.

Recall that if X is a G-CW complex, we defined Bredon cohomology as follows: we set up a chain complex of coefficient systems (i.e. functors $\mathscr{O}_G^{\mathsf{op}} \to \mathsf{Ab}$) $\underline{C}_*(X)$, where

$$\underline{C}_n(G/H) := H^n((X^n)^H, (X^{n-1})^H; \mathbb{Z}).$$

The differential $\partial : \underline{C}_n(X) \to \underline{C}_{n-1}(X)$ is induced by the connecting morphism in the long exact sequence for the triple (X^n, X^{n-1}, X^{n-2}); that $\partial^2 = 0$ is something you have to check, though it's not very difficult.

With this you can define two things:

- The Bredon cohomology with coefficients in a coefficient system M is
$$H^n_G(X; M) := H^n(\mathrm{Hom}_{\mathsf{Fun}(\mathscr{O}_G^{\mathsf{op}}, \mathsf{Ab})}(\underline{C}_*(X), M)).$$
We defined this last time.
- For homology to be covariant, we need the coefficient system M to be a functor $\mathscr{O}_G \to \mathsf{Ab}$ rather than $\mathscr{O}_G^{\mathsf{op}} \to \mathsf{Ab}$ (e.g. $\underline{H}^*(X)$ for a G-space X, which sends $G/H \mapsto \{H^n(X^H)\}$). With M such a coefficient system, the **Bredon homology** with coefficients in M is
$$H_n^G(X; M) := H_n(\underline{C}_n(X) \otimes_{\mathscr{O}_G} M).$$
By this tensor product, we mean a coend:
$$\underline{C}_*(X) \otimes_{\mathscr{O}_G} M := \coprod_{G/H} \underline{C}_n(X)(G/H) \times M(G/H)/\sim,$$
where if $f \in \mathrm{Map}_{\mathscr{O}_G}(G/H, G/K)$, $(f^*y, z) \sim (y, f_*z)$.[17]

[16] This requires knowing how to obtain a CW structure on X^H given a G-CW structure on X. If G is finite, this is easy to see; for general compact Lie groups, though, this requires a triangulation argument. One wants the resulting coefficient system to be independent of the choice of triangulation, but as in the nonequivariant case, this is proven via an axiomatic characterization of cohomology.

[17] Tensor products are particular instances of coends; instead of inducing an equivalence $mr \otimes n \sim m \otimes rn$, you flip a map across the two objects. One might write $y \cdot f$ for f^*y and $f \cdot z$ for f_*z to emphasize this point of view.

The whole philosophy of Bredon (co)homology is that you understand the cohomology or homology through the fixed-point sets and the lattice of subgroups of G.

Example 4.1. Let's return to Example 3.12. We had a C_2-action on S^2. There were two 0-cells, one 1-cell $C_2 \times I$, and one 2-cell $C_2 \times D^2$, and the chain complexes $\underline{C}_n(C_2/H)$ as we wrote down last lecture. We aim to compute the Bredon cohomology for the constant coefficient system $M = \underline{\mathbb{Z}}$ (so $M(C_2/H) = \mathbb{Z}$ for all H).

The orbit category for C_2 is particularly simple:

$$\begin{tikzcd} C_2/e \arrow[loop above] \arrow[d] \\ C_2/C_2. \end{tikzcd}$$

So a coefficient system is determined by a map $M_1 \to M_2$ and an involution on M_2. In our case, $\underline{C}_k(X)(C_2/e) = \mathbb{Z}\oplus\mathbb{Z}$ for $k = 0, 1, 2$, and the involution flips the two factors. Since $\mathrm{Hom}(\mathbb{Z}\oplus\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z}$, we obtain a cochain complex

$$0 \longrightarrow \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{f_1} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{f_2} \mathbb{Z}\oplus\mathbb{Z} \longrightarrow 0,$$

and we want to determine the differentials. Fortunately, this is controlled by the cellular boundary map, i.e. the attaching map. Let (x, y) denote the standard basis of $\mathbb{Z}\oplus\mathbb{Z}$.

- f_1 sends $(x, y) \mapsto (x - y, x - y)$.
- f_2 sends $(x, y) \mapsto (x - y, y - x)$, because TODO. Question: does this need to commute with the involution? We'll resolve this post-lecture.

In terms of matrices, the differentials are given by

$$0 \longrightarrow \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}} \mathbb{Z}\oplus\mathbb{Z} \longrightarrow 0$$

Taking the cohomology of this complex, as usual, produces

$$H^n(S^2; \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z}, & n = 0, 2 \\ 0, & \text{otherwise.} \end{cases}$$

◀

Remark. We're used to reading off properties of a space from its cohomology, and this is still true here, if harder. For example, H^0 tells us the number of connected components of S^2/C_2. ◀

We want cohomology to have nice formal properties analogous to the Eilenberg-Steenrod axioms. We'll think of H_G as a general G-equivariant cohomology theory of pairs $H^n_G(X, A; M)$, but in our case this will just be $H^n_G(X/A; M)$.

(1) H^*_G should be invariant under weak equivalences: a weak equivalence $X \to Y$ should induce an isomorphism $H^n_G(Y; M) \xrightarrow{\cong} H^n_G(X; M)$.

(2) Given a pair $A \subseteq X$, we get a long exact sequence

$$\cdots \longrightarrow H^n_G(X, A; M) \longrightarrow H^n_G(X; M) \longrightarrow H^n_G(A; M) \xrightarrow{\delta} H^{n+1}(X, A; M) \longrightarrow \cdots$$

(3) The excision axiom: if $X = A \cup B$, then

$$H^n_G(X/A; M) \cong H^n_G(B/(A\cap B); M).$$

(4) The Milnor axiom:

$$H^n_G\left(\bigvee_i X_i; M\right) = \prod_i H^n_G(X_i; M).$$

(5) Finally, for now we impose the dimension axiom: our points are orbits G/J, so we ask that $H^n(G/H; M)$ is concentrated in degree 0.

Some of these are easier than others: Bredon cohomology is manifestly homotopy-invariant in the same ways as ordinary cohomology, so invariance under weak equivalence and the Milnor axiom are immediate, and excision follows because if all spaces involved are CW complexes, $X/A \cong B/(A\cap B)$.

What takes more work is the dimension axiom and the long exact sequence. We'll show that $\underline{C}_n(X; M)$ is a projective object, and hitting projective objects with Hom produces a long exact sequence by homological algebra.

(Recall that an object P in a category where you can do homological algebra is **projective** if $\mathrm{Hom}(P,-)$ is exact, which is equivalent to maps to P lifting across surjections $M \twoheadrightarrow P$.)

Proof of the dimension and long exact sequence axioms. Observe that $\underline{C}_n(X)$ splits as a direct sum of pieces $H_n(G/H_+ \wedge S^n) \cong \widetilde{H}_0(G/H)$. At G/K, $H_0(G/H) = \mathbb{Z}[\pi_0(G/H)^K]$. This is a free abelian group, and we'll directly use the lifting criterion to prove this is projective. That is, we'll write down an isomorphism

$$\varphi\colon \mathrm{Hom}_{\mathsf{Fun}(\mathscr{O}_G^{\mathrm{op}},\mathsf{Ab})}(\underline{H}_0(G/H), M) \xrightarrow{\cong} M(G/H).$$

This immediately proves $\underline{H}_0(G/H)$ is projective: evaluating a coefficient system on an exact sequence produces an exact sequence, and we've shown $\mathrm{Hom}(\underline{H}_0(G/H),-)$ is evaluation of a coefficient system.

The map φ takes a homomorphism θ and applies it to $\mathrm{id}_{G/H}$, which produces something in $M(G/H)$. Why is this an isomorphism? The Yoneda lemma is a fancy answer, but you can prove it in a more elementary manner.

Exercise 4.2. Calculate that any $\theta \in \mathrm{Hom}_{\mathsf{Fun}(\mathscr{O}_G^{\mathrm{op}},\mathsf{Ab})}(H_0(G/H), M)$ is determined by where the identity $\mathrm{id} \in \mathrm{Map}_{\mathscr{O}_G}(G/H, G/H)$ is sent, implying φ is an isomorphism.

Thus, we've effectively calculated the value at G/H, proving the dimension axiom as well. ⊠

The Eilenberg-Steenrod axioms hold for Bredon homology, and the proof is the same.

Remark. Let's foreshadow a little bit. In ordinary homotopy theory, one can show that the Eilenberg-Steenroad axioms plus the value on points determine a cohomology theory, and this is still true in the equivariant case. But then one wonders about Brown representability and what happens when you remove the dimension axiom — and indeed there are lots of interesting examples of generalized equivariant cohomology theories. ◀

Exercise 4.3. We constructed a functor $\underline{H} : \mathsf{Fun}(\mathscr{O}_G^{\mathrm{op}}, \mathsf{Top}) \to \mathsf{Fun}(\mathscr{O}_G^{\mathrm{op}}, \mathsf{Ab})$. Show that $\underline{H}M$ represents $H_G^n(-;M)$.

This slick definition of $\underline{H}$ is one of the advantages of working with presheaves on the orbit category.

Remark. You might also want to have a universal coefficient sequence, but it's more complicated. The short exact sequence in ordinary homotopy theory depends on the existence of short projective resolutions. Here, we have enough projectives and injectives, but resolutions are longer. Thus, taking an injective resolution of M and filtering the resulting double complex, one obtains a spectral sequence

$$\mathrm{Ext}^{p,q}(\underline{C}_*(X), M) \Longrightarrow H_G^{p+q}(X;M).$$

Warning: indexing might be slighly off.

There's a corresponding Tor spectral sequence. ◀

Since the category is more complicated, one expects to have to do more work. But sometimes there are nice results nonetheless; Smith theory is an example. This theorem is very old, from the 1940s, so none of the cohomology in the statement is equivariant.

Theorem 4.4 (Smith)**.** *Let G be a finite p-group and X be a finite G-CW complex such that (the underlying topological space of) X is an $\mathbb{F}_p$-cohomology sphere.*[18] *Then, X^G is either empty or an $\mathbb{F}_p$-cohomology sphere of smaller dimension.*

There are sharper statements, but we can prove this one. It's the start of a long program to understand $H_*(X^G)$ using algebraic data calculated from X and the action of G on X. One useful tool in this is the **Borel construction** (well, *a* Borel construction) $EG \times_G X$ (where this is the usual balanced product, not a pullback). This is a "fattened up" version of X/G.

Definition 4.5. The **Borel cohomology** of X is $H_B^*(X) := H^*(EG \times_G X)$.

The finiteness in Theorem 4.4 is key: Elmendorf's theorem lets us build a C_p-complex with non-equivariant homotopy type S^n and any set of fixed points. Thus, in the infinite-dimensional case, we should be looking at a different thing than the fixed points, namely the homotopy fixed points.

Definition 4.6. The **homotopy fixed points** of a G-space X is $X^{hG} := \mathrm{Map}(EG, X)^G$.

[18] A space X is an $\mathbb{F}_p$**-cohomology sphere** if there is an isomorphism of graded abelian groups $H^*(X;\mathbb{F}_p) \cong H^*(S^n;\mathbb{F}_p)$ for some n.

This can also be defined as a homotopy limit. This is “easy” to compute (relative to the rest of equivariant homotopy theory, that is), as there are spectral sequences, nice models for EG, and other tools.

The terminal map $EG \to *$ induces a map $X^G \to X^{hG}$. The Sullivan conjecture is all about when this happens: when you p-complete, things become really nice. (There will be a precise statement next lecture.)

Returning to Theorem 4.4, we can use Bredon cohomology to give an easy, modern proof. The idea is to adroitly choose coefficient systems such that we recover $H^n(X)$, $H^n(X^G)$, and $H^n((X/G)/G)$ from Bredon cohomology. They'll fit into exact sequences, and using tools like Mayer-Vietoris, we'll get inequalities on the ranks of groups. This will also use the fact that a short exact sequence of coefficient systems induces a long exact sequence on H_G^*. The proof will be beautiful and short, unlike Smith's original proof!

Theorem 4.4 is sufficiently classical that there are several different proofs. Ours illuminates Bredon cohomology at the expense of obscuring the overarching goal of Smith theory. There's another proof by Dwyer and Wilkerson in “Smith theory revisited” [DW88], a short, beautiful paper which is highly recommended. It uses the unstable Steenrod algebra to prove Theorem 4.4 and more.

Over the next day or so, there will be a problem set and a course webpage. Doing the problem sets is recommened!

Lecture 5.

Smith theory: 1/31/17

““It seems that the people registered for the class and the people showing up for class are disjoint.”

Today, we will give one or two proofs of Theorem 4.4: let G be a finite p-group, X be a finite-dimensional G-CW complex such that as a topological space, X is an $\mathbb{F}_p$-cohomology sphere (i.e. as graded abelian groups, $H^*(X;\mathbb{F}_p) \cong H^*(S^n;\mathbb{F}_p)$). Then, either $X^G = \varnothing$, or $H^*(X^G;\mathbb{F}_p) \cong H^*(S^m;\mathbb{F}_p)$ as graded abelian groups, for some $m \le \dim X$. We'll try to be consistent with the notation in [May87, May96].

Proof of Theorem 4.4. First, we can quickly reduce to the case where $G = \mathbb{Z}/p$: if $H \subset G$ is a normal subgroup, then $X^G \cong (X^H)^{G/H}$, so you can induct on the order of the group using the Sylow theorems. Thus, we will assume $G = \mathbb{Z}/p$, whose coefficient system is simple:

$$\begin{tikzcd} \bullet \arrow[loop above, "g"] \\ \bullet \arrow[u] \end{tikzcd}$$

There is a cofiber sequence

$$X_+^G \longrightarrow X_+ \longrightarrow X/X^G,$$

which is not particularly deep. We're going to construct three special coefficient systems L, M, and N such that

$$\begin{aligned} H_G^*(X;L) &\cong \widetilde{H}^*((X/X^G)/G;\mathbb{F}_p) \\ H_G^*(X;M) &\cong H^*(X;\mathbb{F}_p) \\ H_G^*(X;N) &\cong H^*(X^G;\mathbb{F}_p). \end{aligned}$$

These will fit into exact sequences which will imply the inequalities we wanted.

How to we construct custom coefficient systems? Since these constructions commute with colimits, it suffices to determine them via computation at $n = 0$ and $X = G/H$ for subgroups $H \subset G$, meaning just e and G.

For L, we want to recover $\widetilde{H}^0((X/X^G)/G;\mathbb{F}_p)$ for $X = G/e$ and $X = G/G$.

- For $X = G/e$, we want $\widetilde{H}^0((G/\varnothing)/G;\mathbb{F}_p) \cong \widetilde{H}^0(G_+/G) \cong \mathbb{F}_p$.
- For $X = G/G$, we get $\widetilde{H}^0((*/*)/G;\mathbb{F}_p) = 0$.

So we conclude $L(G/e) = \mathbb{F}_p$ and $L(G/G) = 0$.

For M, a similar calculation shows we need $H^0(G/e) \cong \mathbb{F}_p[G]$ and $H^0(G/G) = \mathbb{F}_p$, and for N, we need $N(G/e) = 0$ and $N(G/G) \cong \mathbb{F}_p$.

Remark. Almost everything in this proof generalizes; we will only need X to be an $\mathbb{F}_p$-cohomology sphere in order to know dimensions of a few things. But this technique of customized coefficient systems can be used elsewhere. ◂

Let I denote the **augmentation ideal** of $\mathbb{F}_p[G]$, i.e. the kernel of the map $\mathbb{F}_p[G] \to \mathbb{F}_p$ sending all $g \mapsto 1$. We will let I^n refer to the coefficient system which assigns I^n to G/e and 0 to G/G.

As coefficient systems, $M/I \cong \mathbb{F}_p$, and therefore there is a short exact sequence of coefficient systems

$$0 \longrightarrow I \longrightarrow M \longrightarrow N \oplus L \longrightarrow 0.$$

This implies a long exact sequence in Bredon cohomology:

$$\cdots \longrightarrow H^q_G(X;I) \longrightarrow H^q_G(X;M) \longrightarrow H^q_G(X;N\oplus L) \longrightarrow H^{q+1}_G(X;I) \longrightarrow \cdots$$

Exactness at $H^q(X;N\oplus L)$ implies

$$\operatorname{rank} H^q(X;N) + \operatorname{rank} H^q(X;L) \le \operatorname{rank} H^q(X;M) + \operatorname{rank} H^{q+1}(X;I).$$

That is,

$$\operatorname{rank} H^q(X^G;\mathbb{F}_p) + \operatorname{rank} \widetilde{H}^q((X/X^G)/G;\mathbb{F}_p) \le \operatorname{rank} H^q(X;\mathbb{F}_p) + \operatorname{rank} H^{q+1}(X;I). \tag{5.1}$$

We'll use this to strongly constrain $\operatorname{rank} H^q(X^G;\mathbb{F}_p)$, but first we need another inequality coming from another exact sequence. Namely, the following sequence of coefficient systems is exact:

$$0 \longrightarrow L \longrightarrow M \longrightarrow I \oplus N \longrightarrow 0.$$

This is because $I^p = 0$ and for $0 \le n \le p-1$, $I^n/I^{n+1} \cong \mathbb{F}_p$. In particular, $I^{p-1} \cong \mathbb{F}_p \cong L$, so we can think of M/L as M/I^{p-1}. Now we play the same game: the induced long exact sequence is

$$\cdots \longrightarrow H^q_G(X;L) \longrightarrow H^q_G(X;M) \longrightarrow H^q_G(X;I\oplus N) \longrightarrow H^{q+1}_G(X;L) \longrightarrow \cdots$$

which implies

$$\operatorname{rank} H^q_G(X;N) + \operatorname{rank} H^q_G(X;I) \le \operatorname{rank} H^q_G(X;M) + \operatorname{rank} H^{q+1}_G(X;L),$$

i.e.

$$\operatorname{rank} H^q(X^G;\mathbb{F}_p) + \operatorname{rank} H^q_G(X;I) \le \operatorname{rank} H^q(X;\mathbb{F}_p) + \operatorname{rank} H^{q+1}((X/X^G)/G;\mathbb{F}_p). \tag{5.2}$$

Let's use this to prove

$$\operatorname{rank} \widetilde{H}^q((X/X^G)/G;\mathbb{F}_p) + \sum_{i=q}^{q+r} \operatorname{rank} H^i(X^G;\mathbb{F}_p) \le \operatorname{rank} \widetilde{H}^{q+r+1} + \sum_{i=q}^{q+r} \operatorname{rank} H^i(X;\mathbb{F}_p). \tag{5.3}$$

Let

$$\begin{aligned} a_q &:= \operatorname{rank} H^q(X^G;\mathbb{F}_p)\\ b_q &:= \operatorname{rank} H^q(X;\mathbb{F}_p)\\ c_q &:= \operatorname{rank} H^q((X/X^G)/G;\mathbb{F}_p)\\ d_q &:= \operatorname{rank} H^q_G(X;I). \end{aligned}$$

Then, (5.1) and (5.2) say

$$a_q + c_q \le b_q + d_{q+1} \qquad \text{and} \qquad a_q + d_q \le b_q + c_{q+1}.$$

Now, adding (5.1) for q even and (5.2) for q odd proves (5.3).

When $q = 0$ and r is large, the finite-dimensionality of X implies that

$$\sum_i \operatorname{rank} H^i(X^G;\mathbb{F}_p) \le \sum_i \operatorname{rank} H^i(X;\mathbb{F}_p). \tag{5.4}$$

This is already an interesting bound, especially relative to the amount of work we've put in.

Specializing to X an $\mathbb{F}_p$-cohomology sphere, (5.4) means

$$\sum_i \operatorname{rank} H^i(X^G;\mathbb{F}_p) \le 2.$$

We want to show this sum isn't 1 (so that we get the cohomology of a sphere) and that the top nonzero rank is at most n. We will do this with another short exact sequence of coefficient systems:

$$0 \longrightarrow I^{n+1} \longrightarrow I^n \longrightarrow L \longrightarrow 0.$$

From this, we get another long exact sequence. Applying the Euler characteristic, we obtain that

$$\chi(X) = \chi(X^G) + p\widetilde{\chi}((X/X^G)/G). \tag{5.5}$$

Here

$$\widetilde{\chi}(Y) := \sum_i (-1)^i \operatorname{rank} \widetilde{H}^i(Y)$$

is the **reduced Euler characteristic**.

Equation 5.5 already implies that $\chi(X) \equiv \chi(X^G) \bmod p$, so $\sum \operatorname{rank} H^*(X^G; \mathbb{F}_p) \neq 1$ in our case.

Exercise 5.6.

(1) Think about choices of q and r that allow you to deduce $m \le n$, finishing the proof.
(2) Small changes need to be made to this argument when $p = 2$; what are they? ⊠

This is an appealing proof: some fairly simple calculations and a dash of formal theory very effectively led to the result. We'll give another proof with different advantages and disadvantages.

Smith theory naturally leads to questions about how to recover $H^*(X^G)$ algebraically from some equivariant cohomology theory on X. Last time, we introduced Borel cohomology $H^*_B(X) = H^*(EG \times_G X)$, where EG is a free G-space that's nonequivariantly contractible (which is simple to construct with the bar construction or through Elmendorf's theorem).

In the following, all cohomology is understood to have coefficients in $\mathbb{F}_p$.

Theorem 5.7. *Let G be a finite p-group. Then, there is an isomorphism $S^{-1}H^*(EG \times_G X) \xrightarrow{\cong} S^{-1}H^*(EG \times_G X^H)$. Here, S is the multiplicative set generated by the classes in $H^2(BG)$ that are images of the Bockstein homomorphism $H^1(BG) \to H^2(BG)$ of the elements that are nontrivial in $H^1(BH)$.*

This uses the fact that Borel cohomology is an $H^*(BG)$-module: $H^*(EG \times_G *) \cong H^*(EG/G) = H^*(BG)$, and using the terminal map $X \to *$ we get a map $H^*(BG) \to H^*(EG \times_G X)$.

There's a rich theory of unstable modules over the Steenrod algebra $\mathscr{A}_p$, which could fill a whole semester. There's a functor Un which produces unstable $\mathscr{A}_p$-modules, in a sense by only keeping the unstable part.

Theorem 5.8 (Dwyer-Wilkerson [DW88]).

$$H^*(X^G) \cong \mathbb{F}_p \otimes_{H^*(BG)} \operatorname{Un}(S^{-1}H^*(EG \times_G X)).$$

The proof uses arguments that were hard to think of, but easy to follow.

We'll use these thoerems to prove Smith's theorem using the Serre spectral sequence for

$$X \longrightarrow EG \times_G X \longrightarrow BG.$$

This will be the nicest kind of spectral sequence argument: everything degenerates.

Theorem 5.7 is an example of a general class of **localization theorems** in equivariant cohomology. In these theorems, one considers the fiber sequence $X^G_+ \to X_+ \to X/X^G$, and wants to show that for some functor E, $E(X^G_+) \cong E(X_+)$. This boils down to showing $E(X/X^G)$ vanishes, which will always follow from showing that E vanishes on G-spaces whose G-actions are free away from the basepoint. In general, this will reduce to considering cells, so one considers $E(G/H_+ \wedge \bigvee_\alpha S^{q_\alpha})$ for some wedge of spheres.

In our case, $E(G/H_+ \wedge \bigvee_\alpha S^{q_\alpha})$ is

$$S^{-1}H^*_B\left(G/H_+ \wedge \bigvee_\alpha S^{q_\alpha}\right) = S^{-1}H^*\left(EG \times_G \left(G/H_+ \wedge \bigvee_\alpha S^{q_\alpha}\right)\right)$$

Now, the term $EG \times_G G/H \cong EG/H \cong BH$, so we have a piece that looks like $H^*(BH)$, which is how BH inserts itself into the argument.

Exercise 5.9. Finish the Serre spectral sequence proof of Theorem 4.4. Hint: there's a simple geometric reason why the spectral sequence collapses, which is what makes this whole thing go.

Soon we'll start talking about the stable category. As preparation for this, if you don't already know it, it's good to remind yourself of it.

The Q&A session will be Thursday at 8:30 PM.

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