GEOMETRY AND STRING THEORY SEMINAR: SPRING 2019

ARUN DEBRAY
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These notes were taken in UT Austin's geometry and string theory seminar in Spring 2019. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Thanks to Rok Gregoric, Qianyu Hao, and Charlie Reid for some helpful comments and corrections.

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1. Anomalies and extended conformal manifolds: 1/23/19

These are Arun's prepared notes for his talk, on the paper "Anomalies of duality groups and extended conformal manifolds" [STY18] by Seiberg, Tachikawa, and Yonekura.

1.1. **Generalities on anomalies.** As we've seen previously in this seminar, if you ask four people what an anomaly in QFT is, you'll probably get four different answers. Here are some of them.

- An anomaly means the action isn't invariant under the gauge group.
- An anomaly is an obstruction to coupling the theory to a background G-symmetry, or to gauging such a symmetry.
- An anomaly is realized in the nonvanishing of an anomaly polynomial.
- An anomaly as a *relative field theory* as advocated by Freed-Teleman [FT14]: within the framework of functorial QFT, consider an invertible $(n+1)$-dimensional QFT $\alpha\colon \mathsf{Bord}_n \to \mathsf{C}$; a QFT relative to α is a morphism $Z\colon \mathbf{1} \to \tau_{\le n}\alpha$ (i.e., truncate α). The upshot is that the partition function of on a closed n-manifold X isn't a number, but rather an element of the line $\alpha(X)$, and so on.
- Building on this is the idea that every quantum field theory has an anomaly, and if the anomaly is trivial, trivializing it is data that manifests in choices in studying the theory.

Seiberg, Tachikawa, and Yonekura introduce another perspective! But fortunately they relate it to most of the perspectives above. The idea is to consider a QFT with a parameter space $\mathcal{M}$, or a family of QFTs over $\mathcal{M}$. For example, you might have a parameter in a space $\widehat{\mathcal{M}}$ acted on by a group G, and then $\mathcal{M} = \widehat{\mathcal{M}}/G$. Fixing a spacetime manifold X, one expects the partition function to be a function on $\mathcal{M}$, but for an anomalous theory, this doesn't quite work (e.g. if the partition function as a function on $\widehat{\mathcal{M}}$ isn't G-invariant).

One way to fix this is to consider a space $\mathcal{F}$ of counterterms; the total space $\mathcal{N}$ will then be a fiber bundle over $\mathcal{M}$ with fiber $\mathcal{F}$. If constructed correctly, the partition function is then a function on $\mathcal{N}$, and the anomaly manifests in the fact that $\mathcal{N} \to \mathcal{M}$ isn't the trivial $\mathcal{F}$-bundle, and the function doesn't descend to $\mathcal{M}$. Alternatively, one can descend it as a section of a line bundle on $\mathcal{M}$ rather than a function.

The point is: these perspectives are all related. Today, we'll follow Seiberg-Tachikawa-Yonekura as they discuss an $\mathrm{SL}_2(\mathbb{Z})$-anomaly on 4D Maxwell theory from several of these perspectives.

1.2. **An anomalous $\mathrm{SL}_2(\mathbb{Z})$-symmetry in 4D Maxwell theory.** Now, we'll study these ideas as applied specifically to Maxwell theory in dimension 4. Throughout, we will restrict to spin 4-manifolds; everything still works when generalized to oriented manifolds, but the details are more complicated. Consult Seiberg-Tachikawa-Yonekura to learn what changes.

Maxwell theory is pure 4D U_1 gauge theory. The action on a 4-manifold X is

$$S = \frac{1}{g^2}\int_X F \wedge \star F + \frac{i\theta}{8\pi^2}\int_X F \wedge F, \tag{1.1}$$

where F is the curvature of the U_1 gauge field. Here g and θ are real-valued parameters, though θ is 2π-periodic. In dimension 4 only, we can rewrite this in terms of the self-dual and anti-self-dual pieces of F:

$$S = \frac{i\overline{\tau}}{4\pi}\int_X \|F_+\|^2 - \frac{i\tau}{4\pi}\int_X \|F_+\|^2, \tag{1.2}$$

where

$$\tau \coloneqq \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}. \tag{1.3}$$

That is, our single parameter τ is valued in $\mathbb{H}$, the upper half-plane.

1.2.1. *The anomaly as variance under a symmetry.* Now $\mathrm{SL}_2(\mathbb{Z}) = \langle S, T \mid S^4 = 1, (ST)^3 = S^2\rangle$ acts on $\mathbb{H}$ by $S\tau = -1/\tau$ and $T\tau = \tau + 1$. We'd like to quotient by this action and obtain a parameter space $\mathcal{M} \coloneqq \mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$. This action has stabilizer, though: at $\tau = i$, the stabilizer is $\mathbb{Z}/2$, and at $e^{i\pi/3}$, it's $\mathbb{Z}/3$. Thus it's helpful to think of $\mathcal{M}$ as the quotient *stack*, which means just that we remember the $\mathbb{Z}/2$ at i and the $\mathbb{Z}/3$ at $e^{i\pi/3}$.[1]

TODO: picture of the stack.

However, the action (1.2) is not invariant: Witten [Wit95] uses physics arguments to show that

$$Z_{T\cdot\tau}(X) = Z_\tau(X) \tag{1.4a}$$

$$Z_{S\cdot\tau}(X) = \tau^u\overline{\tau}^v Z_\tau(X), \tag{1.4b}$$

where $u = (1/4)(\chi(X) + \sigma(X))$ and $v = (1/4)(\chi(X) - \sigma(X))$.

Remark 1.5. Because Maxwell theory is a free QFT, making the argument rigorous is probably easier than for general QFTs (this does not mean "easy"!). ◀

In other words, the theory is anomalous for this $\mathrm{SL}_2(\mathbb{Z})$-action: on $\mathcal{M}$, the partition function isn't a well-defined function.

One might attempt to remedy this by throwing counterterms into the action. Our two options are the signature and the Euler characteristic, so such a counterterm would look like

$$f(\tau,\overline{\tau})\chi(X) + g(\tau,\overline{\tau})\sigma(X).^{2} \tag{1.6}$$

This does not save us: consider $\tau = e^{i\pi/3}$, which has a $\mathbb{Z}/3$ stabilizer generated by ST^{-1}. This acts on the partition function by $e^{i\pi\sigma(X)/3}$, but does not change the counterterm (1.6), so this factor cannot be canceled. This is a nice application of the stacky perspective on $\mathcal{M}$.

However, a counterterm can simplify the anomaly. Letting $\eta\colon \mathbb{H} \to \mathbb{C}$ be the Dedekind η-function, $f(\tau,\overline{\tau}) \coloneqq \mathrm{Re}\log\eta(\tau)$ and $g(\tau,\overline{\tau}) \coloneqq i\,\mathrm{Im}\log\eta(\tau)$. The new partition function satisfies

$$Z'_\tau(X) = \eta(\tau)^{-(\chi(X)+\sigma(X))/2}\eta(-\overline{\tau})^{-(\chi(X)-\sigma(X))/2}Z_\tau(X), \tag{1.7}$$

and it transforms under the $\mathrm{SL}_2(\mathbb{Z})$-action as $Z'_{T\cdot\tau}(X) = Z'_\tau(X)$ and $Z'_{S\cdot\tau}(X) = \exp(-i\pi\sigma(X)/3)Z'_\tau(X)$.

[1] Really this is the quotient stack $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$; the reason we're using all of $\mathrm{SL}_2(\mathbb{Z})$ is that its center will act nontrivially later.

[2] The notation $(\tau,\overline{\tau})$ indicates these functions need not be holomorphic.

This is the first description of this anomaly, as expressing how the partition function changes under $\mathrm{SL}_2(\mathbb{Z})$. It involves the signature, which is a "gravitational" term (meaning an invariant of the underlying manifold), and $\mathrm{SL}_2(\mathbb{Z})$, so it's a mixed anomaly. There could also be a pure $\mathrm{SL}_2(\mathbb{Z})$ anomaly, but to investigate it one should couple the theory to a background principal $\mathrm{SL}_2(\mathbb{Z})$-bundle, which the paper doesn't do.

1.2.2. *Extending the parameter space.* Next we'll describe a fiber bundle $\mathcal{N} \to \mathcal{M}$ such that the partition function is a function on $\mathcal{N}$, and interpret it as a section of a line bundle over $\mathcal{M}$.

The gravitational term $\theta_{\mathrm{grav}} \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$; we'll consider an extended space of parameters, namely $\mathbb{H} \times S^1$, and define an action of $\mathrm{SL}_2(\mathbb{Z})$ on both of them in a way which gets rid of the anomaly.

Definition 1.8. The character group of $\mathrm{SL}_2(\mathbb{Z})$ is cyclic of order 12: given a $k \in \mathbb{Z}/12$, define the character $\chi_k \colon \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{U}_1$ by $\chi_k(S) := e^{-i\pi k/2}$ and $\chi_k(T) := e^{-i\pi k/6}$.

Of course, one should check these satisfy the relations of $\mathrm{SL}_2(\mathbb{Z})$, hence actually define a character. We'll single out χ_8: on T it's 1 and on S it's $e^{2i\pi/3}$, which looks a lot like the anomaly we saw above.

Now define the $\mathrm{SL}_2(\mathbb{Z})$-action on $\mathbb{H} \times S^1$ by

$$g \cdot (\tau, \theta_{\mathrm{grav}}) := (g \cdot \tau, \chi_8(g) \cdot \theta_{\mathrm{grav}}) \tag{1.9}$$

(where U_1 acts on S^1 by the standard representation). The partition function is

$$Z'_{\tau,\theta_{\mathrm{grav}}}(X) = Z'_\tau(X) e^{i\theta_{\mathrm{grav}}\sigma/16}, \tag{1.10}$$

so if $g \in \mathrm{SL}_2(\mathbb{Z})$,

$$Z'_{g\cdot(\tau,\theta_{\mathrm{grav}})}(X) = Z'_{(\tau,\theta_{\mathrm{grav}})}(X); \tag{1.11}$$

the factors of $e^{i\pi/3}$ and $e^{2i\pi/3}$ cancel out. The partition function is invariant, so descends to a function on the *extended conformal manifold* $\mathcal{N} := (\mathbb{H} \times S^1)/\mathrm{SL}_2(\mathbb{Z})$, which is an S^1-bundle over $\mathcal{M}$. This is a key idea of their paper: the partition function of an anomalous theory is only a function on this extended parameter space $\mathcal{N}$.

An alternative perspective is to use a line bundle to encode a twist. A function on $\mathcal{N}$ is a section of the trivial line bundle. We can ask whether it descends to $\mathcal{M}$, not necessarily as a function, but as a section of a line bundle. This is governed by *descent data*: rotating the S^1 defines a U_1-action on $\mathcal{N}$ whose quotient is $\mathcal{M}$. Then, an equivariant line bundle L' on $\mathcal{N}$ descends to a line bundle L on $\mathcal{M}$ (nonequivariant – in taking the quotient we've "used up" the equivariance), and an equivariant section of L' descends to a section of L.

So let's give the trivial line bundle $\underline{\mathbb{C}} \to \mathcal{N}$ a nontrivial U_1-action: given $(\tau, \theta_{\mathrm{grav}}, w)$ with $(\tau, \theta_{\mathrm{grav}}) \in \mathcal{N}$ and $w \in \underline{\mathbb{C}}_{(\tau,\theta_{\mathrm{grav}})}$, and given a $z \in \mathrm{U}_1$, define

$$z \cdot (\tau, \theta_{\mathrm{grav}}, w) := \left(\tau, z \cdot \theta_{\mathrm{grav}}, \exp\left(\frac{\sigma(X)}{16}\right) zw\right). \tag{1.12}$$

This defines an equivariant line bundle $L' \to \mathcal{N}$ such that a function Z on $\mathcal{N}$ such that $Z(\tau, z \cdot \theta_{\mathrm{grav}}) = e^{\sigma(X)/16} Z(\tau, \theta_{\mathrm{grav}})$, such as the partition function, is an equivariant section of this line bundle. Descending, we obtain a nonequivariant line bundle $L \to \mathcal{M}$, and the partition function is a section.

Ok, so which line bundle do we get? We know the U_1-equivariant line bundle on $\mathcal{N}$ that we began with, hence a $\mathrm{U}_1 \times \mathrm{SL}_2(\mathbb{Z})$-bundle on $\mathbb{H} \times S^1$, and then we can quotient by U_1 to obtain an $\mathrm{SL}_2(\mathbb{Z})$-equivariant line bundle on $\mathbb{H}$, then quotient by $\mathrm{SL}_2(\mathbb{Z})$ to get back $L \to \mathcal{M}$. The point is, passing through $\mathbb{H} \twoheadrightarrow \mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ may be easier to think about, and we can compare to known line bundles.

Definition 1.13. The *Hodge bundle* $L_H \to \mathcal{M}$ is the quotient of the $\mathrm{SL}_2(\mathbb{Z})$-equivariant line bundle $L'_H \to \mathbb{H}$ which is nonequivariantly trivial and whose $\mathrm{SL}_2(\mathbb{Z})$-action is defined by $g \cdot (\tau, z) = (g \cdot \tau, \chi_1(g) \cdot z)$.

The 12^{th} tensor power of L_H is trivial, ultimately because the abelianization of $\mathrm{SL}_2(\mathbb{Z})$ is $\mathbb{Z}/12$.

If you run this argument, you get that the line bundle L arising from Maxwell theory on X is $L_H^{\otimes(\sigma/2)}$. The intuition is that we have $\exp(\sigma/16)$ in the U_1-action, and $\mathrm{SL}_2(\mathbb{Z})$ acts on U_1 by eight times the generator, giving us $\sigma/2$.

1.2.3. *Anomaly polynomials.* Seiberg-Tachikawa-Yonekura also discuss anomaly polynomials. In general, suppose we have a family of $2k$-dimensional manifolds $\mathcal{X} \to \mathcal{M}$, and write the fiber at $m \in \mathcal{M}$ as X_m.[3] The partition function $Z(X_m)$ is a section of a line bundle $L \to \mathcal{M}$. The goal of the anomaly polynomial is to determine L, or equivalently its first Chern class. Therefore the anomaly polynomial $\mathbb{A}_{2k+2} \in H^{2k+2}(\mathcal{X})$ is defined to satisfy

$$c_1(L) = \int_{X_m} \mathbb{A}_{2k+2}, \tag{1.14}$$

here denoting integration along the fiber, the pushforward map $H^*(\mathcal{X}) \to H^{*-2k}(\mathcal{M})$.[4]

Seiberg-Tachikawa-Yonekura compute $\mathbb{A}_6$ for Maxwell theory in an interesting way: they realize it as the dimensional reduction of a 6D theory Z_6 along a torus T. This theory also has an anomaly polynomial $\mathbb{A}_8 \in H^8(T \times \mathcal{X})$, and integrating over the fibers $T \times X_m$ produces $c_1(L)$ again.

Therefore we can compute $c_1(L)$ in two ways: first integrating along the fiber of $T \times \mathcal{X} \to \mathcal{X}$, then $\mathcal{X} \to \mathcal{M}$ as above, or by first integrating along the fiber of $T \times \mathcal{X} \to T \times \mathcal{M}$, then $T \times \mathcal{M} \to \mathcal{M}$.

Since I don't have a whole lot of time, and because I didn't fully understand the arguments in this section, I'm going to skip over the computations, which is unfortunate, because they look mathematically interesting. The summary is that knowing the 6D anomaly polynomial, and knowing the anomaly polynomial of the 2D theory $Z_6(- \times X_m)$, allows one to pin down the anomaly polynomial of the 4D theory in terms of the central charge of the 2D theory. In the case of Maxwell theory, the central charge is $c = \sigma(X)$, and using arguments from 2D CFT that I could not follow, the corresponding line bundle $L \to \mathcal{M}$ is $L_H^{c/2}$, which agrees with what we saw above.

1.2.4. *Relative field theory.* This is the most topological perspective on anomalies, so I love it.

So with that in mind, we expect Z to really be a QFT relative to an invertible TFT α in dimension 5, and with the same background fields. That is, the symmetry type is

- spin 5-manifolds, since we began with spin 4-manifolds; together with
- an $\mathrm{SL}_2(\mathbb{Z})$-bundle, since we're considering an $\mathrm{SL}_2(\mathbb{Z})$ symmetry: even though we didn't couple to a background $\mathrm{SL}_2(\mathbb{Z})$-bundle, this $\mathrm{SL}_2(\mathbb{Z})$-symmetry still appears in the anomaly.

The anomaly theory α is topological, because it is a finite-order, unitary invertible field theory.[5] Therefore it cannot see τ, θ, or θ_{grav}, so this is the entire symmetry type; moreover, its partition function is a bordism invariant, an element of

$$\mathrm{Hom}(\Omega_5^{\mathrm{Spin}}(B\mathrm{SL}_2(\mathbb{Z})), \mathrm{U}_1). \tag{1.15}$$

Seiberg-Tachikawa-Yonekura [STY18] show this is abstractly isomorphic to $\mathbb{Z}/36$, but they don't produce an isomorphism, so it's difficult to get one's hands on this.

TODO: their results

Here are some other things we can say about the computation of α.

(1) First, we can only expect to know the answer modulo "pure $\mathrm{SL}_2(\mathbb{Z})$" theories; let's discuss what that means. The action of α can include terms which are characteristic classes for $\mathrm{SL}_2(\mathbb{Z})$-bundles as well as "gravitational" terms which depend on the underlying manifold itself. A theory whose action has no gravitational terms is called a pure $\mathrm{SL}_2(\mathbb{Z})$-theory.

 We haven't studied Maxwell theory coupled to a background principal $\mathrm{SL}_2(\mathbb{Z})$-bundle, so we're not going to be able to distinguish any of the pure $\mathrm{SL}_2(\mathbb{Z})$-theories. However, these are a subgroup of the group of all 5D invertible TFTs with symmetry type $\mathrm{Spin} \times \mathrm{SL}_2(\mathbb{Z})$, so we can ask whether we can identify the anomaly in the quotient.

 Seiberg-Tachikawa-Yonekura show that the pure $\mathrm{SL}_2(\mathbb{Z})$ theories form a $\mathbb{Z}/6$ inside this $\mathbb{Z}/36$. They make this argument using the Atiyah-Hirzebruch spectral sequence

$$E^2_{p,q} = H_p(B\mathrm{SL}_2(\mathbb{Z}); \Omega_q^{\mathrm{Spin}}(\mathrm{pt})) \Longrightarrow \Omega_{p+q}^{\mathrm{Spin}}(B\mathrm{SL}_2(\mathbb{Z})). \tag{1.16}$$

 The pure $\mathrm{SL}_2(\mathbb{Z})$-theories are those on the line $q = 0$, so they only see the homology of $B\mathrm{SL}_2(\mathbb{Z})$ and not the spin bordism groups. The argument that $E^\infty_{5,0} = \mathbb{Z}/6$ is a fun but elaborate spectral sequence

[3] Despite the similar notation, this time X is varying and the QFT is constant; previously, it was the other way around.

[4] To do this, we need the relative tangent bundle of the map $\mathcal{X} \to \mathcal{M}$ to be oriented.

[5] This is a quite nontrivial theorem of Freed-Hopkins [FH16].

proof, chaining together three instances of the Atiyah-Hirzebruch spectral sequence and playing them off of each other.

Let A denote the quotient of $\mathrm{Hom}(\Omega_5^{\mathrm{Spin}}(B\mathrm{SL}_2(\mathbb{Z})), \mathrm{U}_1)$ by the pure $\mathrm{SL}_2(\mathbb{Z})$-theories, so that $A \cong \mathbb{Z}/6$. We've seen that when we don't couple to principal $\mathrm{SL}_2(\mathbb{Z})$-bundles, the anomaly has order 3; this means it has order 3 in A.

(2) If we had an explicit description of the elements of $\mathrm{Hom}(\Omega_5^{\mathrm{Spin}}(B\mathrm{SL}_2(\mathbb{Z})), \mathrm{U}_1)$, we could do more, using the anomaly TFT to determine information about the line bundle $L \to \mathcal{M}$. The idea of a QFT relative to some invertible TFT α is that the partition function of X is an element of the line $\alpha(X)$; applying this to the family of theories Z_τ in $\mathcal{M}$, $\alpha(X)$ defines a line bundle over $\mathcal{M}$, and the partition function is a section of this line bundle.

At $\tau = e^{i\pi/3}$, we have a $\mathbb{Z}/3$-symmetry on L_τ which we've determined just by studying Maxwell theory, and we can compare this with another $\mathbb{Z}/3$-symmetry on $\alpha(X)$ that can be determined just by studying α, and these must match, which could help determine α in general.

More explicitly, consider $\tau = e^{i\pi/3} \in \mathcal{M}$ with its $\mathbb{Z}/3$ stabilizer; this is a $\mathrm{pt}/(\mathbb{Z}/3)$. A strong form of $\mathbb{Z}/3$ symmetry is being able to extend to a family over this space, so let's try to do this. Take a spin 4-manifold X and the trivial $\mathrm{SL}_2(\mathbb{Z})$-bundle $P_{\mathrm{triv}} \to X$ and extend to (X, P) over $\mathrm{pt}/(\mathbb{Z}/3)$, where the monodromy of P around an $a \in \mathbb{Z}/3 \subset \mathrm{SL}_2(\mathbb{Z})$ is right multiplication by a^{-1} on $\mathrm{SL}_2(\mathbb{Z})$. We can then evaluate α on this family of manifolds, to produce a line bundle over $\mathrm{pt}/\mathbb{Z}/3$ (namely, a line with a $\mathbb{Z}/3$-action), and we can compare this with the fiber of L over $e^{i\pi/3}$ with its $\mathbb{Z}/3$-action. These should be isomorphic, and given a concrete description of these invertible TFTs, one could use this to learn more information about the anomaly of Maxwell theory even though we haven't coupled to $\mathrm{SL}_2(\mathbb{Z})$-bundles. We can also do this with the $\mathbb{Z}/2$ stabilizer at $\tau = i$.

2. Integrability and 4D gauge theory I: 1/30/19

Today, Sebastian spoke about work of Costello-Witten-Yamazaki [CWY17, CWY18] on integrability and gauge theory. Next week's talk will continue this story, with calculations supporting the ideas presented today.

Integrability is roughly the same thing as being exactly solvable. Across a wide variety of mathematical disciplines (knot theory, differential equations, statistical mechanics, ...), this is understood to mean satisfying the *Yang-Baxter equation*. There's a nice, very readable paper of Perk-Au-Yang [PAY06] which explains this unifying perspective.

Consider some particles in a 2D system, with quantum numbers in some vector space V with basis $\{e_i\}$, and suppose there's a "spectral parameter" $z \in \mathbb{C}$. Consider some particles participating in a collision as in Figure 1: the incoming particles have quantum numbers i and j, and spectral parameters z_1 and z_2, and the outcoming particles have quantum numbers k and ℓ. This defines a map $R(z_1, z_2)\colon V \otimes V \to V \otimes V$; assume $R(z_1, z_2)$ only depends on $z_1 - z_2$.

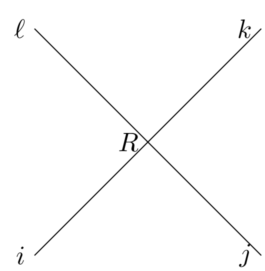

Figure 1. The definition of the R-matrix.

The Yang-Baxter equation encodes the ability to move a line behind such a picture. We'll use the notation $z_{ij} := z_i - z_j$, and $R_{ij}\colon V^{\otimes 3} \to V^{\otimes 3}$ to act by R on the i and j^{th} copies of V and by the identity on the remaining copy. Then the Yang-Baxter equation (YBE) is

$$R_{23}(z_{23}) R_{13}(z_{13}) R_{12}(z_{12}) = R_{12}(z_{12}) R_{13}(z_{13}) R_{23}(z_{23}). \tag{2.1}$$

The picture is in Figure 2.

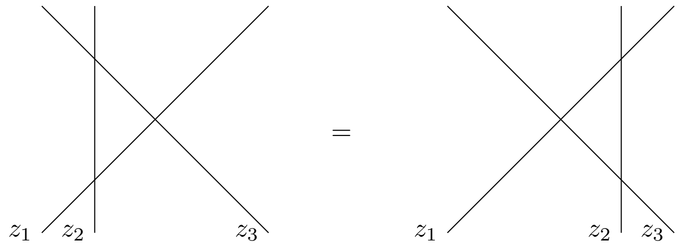

FIGURE 2. The string diagram version of the Yang-Baxter equation.

One in addition often asks for solutions to satisfy *unitarity*, i.e. $R_{12}(z_{12})R_{21}(z_{21}) = \mathrm{id}$. The string diagram perspective is that one can separate two strands which cross twice. Unitary solutions to the Yang-Baxter equation are still too complicated to classify, so people usually introduce more structure.

Another variant is the *quasi-classical Yang-Baxter equation*: one introduces another complex parameter $\hbar$, and near $\hbar = 0$, we ask

$$R_\hbar(z) = \mathbf{1} + \hbar r(z) + O(\hbar^2), \tag{2.2}$$

where $r(z)$ is the *classical R-matrix*, which up to $O(\hbar^2)$ satisfies the *classical Yang-Baxter equation*

$$[r_{12}(z_{12}), r_{13}(z_{13}) + r_{23}(z_{23})] + [r_{13}(z_{13}), r_{23}(z_{23})] = 0. \tag{2.3}$$

Belavin-Drinfeld [BD82] classified solutions associated to a complex Lie algebra $\mathfrak{g}$ associated to a real Lie group G. In this case $r(z) \in \mathfrak{g} \otimes \mathfrak{g}$. Choosing a basis $\mathfrak{g} = \langle t^a \rangle$, we can write

$$r(z) = \sum_{a,b} r_{a,b}(z) t^a \otimes t^b, \tag{2.4}$$

where $\det(r_{a,b}) \neq 0$. The poles of $r(z)$ span a lattice Γ of rank ≤ 2, and three situations emerge.

- If Γ is rank 0, this is the *rational* setting, and we think of solutions as representations of a *Yangian*. In this case $\mathbb{C}/\Gamma = \mathbb{C}$.
- If Γ is rank 1, this is the *trigonometric* setting, and solutions are representations of a *quantum affine algebra*. $\mathbb{C}/\Gamma \cong \mathbb{C}^\times$.
- If Γ is rank 2, this is the *elliptic* setting, and solutions are representations of an *elliptic algebra*. The quotient is an elliptic curve.

The goal of Costello-Witten-Yamazaki is to relate this story to 4D gauge theory. So let's consider 4D gauge theory on $\mathbb{R}^2 \times \mathbb{C}$, with coordinates $(x, y, z, \overline{z})$, and fields

$$A = A_x\,\mathrm{d}x + A_y\,\mathrm{d}y + A_{\overline{z}}\,\mathrm{d}\overline{z}. \tag{2.5}$$

In particular, each A_i is not required to be holomorphic in z. On our Lie algebra $\mathfrak{g}$, choose an invariant nondegenerate quadratic form, such as the Killing form, and let t_a be an orthonormal basis.

The action is

$$S = \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{C}} \mathrm{d}z \wedge CS(A) = -\frac{1}{2\pi}\int z\,\mathrm{tr}(F\wedge F), \tag{2.6}$$

where the Chern-Simons term is

$$CS(A) \coloneqq \mathrm{tr}\left(A \wedge \mathrm{d}A + \frac{2}{3}A^3\right). \tag{2.7}$$

The diffeomorphism group of $\mathbb{R}^2$ preserves this action, where the action is by

$$A_i \longmapsto g^{-1}A_i g + g^{-1}\partial_i g, \tag{2.8}$$

where $i = x, y, \overline{z}$.

The classical equations of motion are $F_{xy} = 0$ and $F_{x\overline{z}} = F_{y\overline{z}} = 0$, meaning solutions are a flat bundle over $\mathbb{R}^2$ and vary holomorphically in $\mathbb{C}$. Therefore all gauge-invariant quantities coming from A must vanish. Correlation functions look like

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}A\,\mathcal{O}\exp(iS/\hbar)}{\int \mathcal{D}A\,\exp(iS/\hbar)}. \tag{2.9}$$

If the length is $\hbar$, the theory is not renormalizable by power counting, though all possible counterterms vanish by the equations of motion. Therefore one can quantize using perturbation theory, and the quantum theory is IR free.

Remark 2.10. This theory has a framing anomaly, similarly to ordinary Chern-Simons theory. We're not going to worry much about this today, though we'll learn more about it next week. ◀

So now we can formulate this QFT on curved manifolds $\Sigma \times C$, where Σ is a topological surface and C is a Riemann surface. Choose a closed holomorphic 1-form ω;[6] the action is

$$S = \frac{1}{2\pi}\int \omega \wedge CS(A). \tag{2.11}$$

We want to study this with perturbation theory; the action depends on $\omega/\hbar$. This causes a problem: zeros of ω correspond to poles of $\hbar$. At least we can understand poles of ω as zeros of $\hbar$, and they spend some time on this. Therefore let's assume ω has no zeros.

Knowledge of the genus of C allows us to use the Riemann-Roch theorem. Since ω has no zeros, this tells us

$$-\#\text{poles} = 2g(C) - 2. \tag{2.12}$$

This cuts down the possibilities pretty drastically. There are only a few options (up to rescaling, etc.):

- $C = \mathbb{C}$ with $\omega = \mathrm{d}z$, so there's a double pole at infinity; this is the rational case.
- $C = \mathbb{C}^\times$ with $\omega = \mathrm{d}z/z$, so there are simple poles at 0 and ∞; this is the trigonometric case.
- C is an elliptic curve with $\omega = \mathrm{d}z$, and there are no poles; this is the elliptic case.

These correspond to the three cases we mentioned earlier.

Remark 2.13. It's no coincidence we get an abelian group in all three cases: since ω has no zeros, we can invert it and get a vector field, and that vector field generates the group action. ◀

The framing anomaly puts constraints on Σ; it must be 2-framed, so it must be a torus.[7]

The equations of motion tell us there are no local observables, so the easiest operators to look at are Wilson lines. Choose a loop $K \subset \Sigma \times C$ and a G-representation ρ; then the Wilson line operator is

$$W_\rho(K) = \mathrm{tr}_\rho\, P\exp\left(\oint_K A_i\, \mathrm{d}x^i\right). \tag{2.14}$$

That is, we get a group element g by walking around K using the connection A, and then compute the trace of $\rho(g)$. However, since A had no $\mathrm{d}z$ term, we can't parallel transport in the z-direction, which means K must be a loop solely in Σ, with a fixed value z_0 in C.

Actually, we can do something different, using a representation $\widehat{\rho}$ of the Lie algebra

$$\mathfrak{g}[[z]] = \bigoplus_{n\geq 0} \mathfrak{g}\otimes\mathbb{C}\cdot z^n. \tag{2.15}$$

(More generally, we could use $\mathfrak{g}[[z-z_0]]$.) If $\mathfrak{g}$ is spanned by t_a, then $\mathfrak{g}[[z]]$ is spanned by $t_{a,n}(z) \coloneqq t_a z^n$, which you can think of as "matrices." Write

$$[t_{a,n}(z), t_{b,m}(z)] = f_{ab}{}^c t_{c,m+n}. \tag{2.16}$$

We'd like finite-dimensional representations with $t_{a,n} = 0$ for $n > n_0$. For example, if $n_0 = 2$, this tells us that $[t_{a,1}, t_{b,1}] = 0$, $\{t_{a,0}\}$ span $\mathfrak{g}$, and $[t_{a,0}, t_{b,1}] = f_{ab}^c t_{c,1}$ means that we just get the adjoint representation of $\mathfrak{g}$.

Because of the IR freeness of the theory, the fields must vanish at infinity, and this allows one to understand the holonomy itself, rather than just its trace, as a meaningful observable.

How does this relate to the Yang-Baxter equations? Looking back at Figure 2, we want this diagram to mean a triple of Wilson lines labeled in representations. When you move one across the others, the $\mathrm{Diff}(\mathbb{R}^2)$-symmetry is broken at the middle, but we still expect the Yang-Baxter equation, and the unitary equation, to be true, since these can be at different heights (so we can really move one behind the others).

[6] Or meromorphic if you want to work on compact manifolds.

[7] This is different from standard Chern-Simons theory, which imposes a weaker constraint.

3. Integrability and 4D gauge theory II: 2/6/19

Today, Ivan spoke, continuing the story of the work of Costello-Witten-Yamazaki [CWY17, CWY18] on how solutions of the Yang-Baxter equation arise from a 4D gauge theory.

Last time, we considered a system of particles in 2D spacetime, with a vector space V of internal states and a spectral parameter $z \in \mathbb{C}$. Particle-particle interactions are governed by the R-matrix $R(z_1, z_2) \in \mathrm{End}(V_1 \otimes V_2)$, and we asked for it to satisfy the Yang-Baxter equation (2.1), which you can think of in terms of the picture Figure 2.

People don't really solve just the Yang-Baxter equation without introducing more structure. For example, one can let the R-matrices depend holomorphically on another parameter $\hbar$, in which case we want to satisfy the quasi-classical Yang-Baxter equation (2.2). Working in a fixed semisimple Lie algebra $\mathfrak{g}$, the solutions fall into three kinds: rational, trigonometric, and elliptic, corresponding to representations of a Yangian $Y_\hbar(\mathfrak{g})$, a quantum group $U_{q,\hbar}(\mathfrak{g})$, or an elliptic algebra $E_{\tau,q,\hbar}(\mathfrak{g})$, respectively.

We will relate this to a 4D gauge theory on $M \coloneqq \Sigma \times C$, where Σ is an oriented smooth surface and C is a Riemann surface. The gauge group G is a semisimple complex Lie group. We took for fields the partial G-connections $A = A_x\,\mathrm{d}x + A_y\,\mathrm{d}y + A_{\overline{z}}\,\mathrm{d}\overline{z}$; $g \in G$ acts on A_i by $gA_ig^{-1} + g\partial_i g^{-1}$ for $i = x, y, \overline{z}$.

The action (2.6) can be generalized by replacing $\mathrm{d}z$ with a meromorphic closed 1-form with no zeros. This constrains C by Riemann-Roch to either $\mathbb{C}$, $\mathbb{C}^\times$, or an elliptic curve, corresponding respectively to the rational, trigonometric, and elliptic situations.

The next question is: how does the R-matrix arise in this theory? The answer is to look at Wilson lines $K \subset \Sigma \times \{z\}$; these correspond in the Yang-Baxter picture to worldlines labeled by $z \in \mathbb{C}$. This still doesn't explain why there's an R-matrix. The answer is that the theory we're looking at it IR-free, so there's a local picture, obtained by a scaling limit in Σ. In this case, the R-matrix picture (two-particle interactions) looks a lot like Feynman diagrams for the perturbation theory in this scaling limit.

Now, let $C = \mathbb{C}$, $\omega = \mathrm{d}z$, and $\Sigma = \mathbb{R}^2$ (so we're in the rational setting). Our goal is to compute the $O(\hbar)$ contribution to the R-matrix. It turns out the angle of the two worldlines doesn't matter.

Remark 3.1. The theory does have a framing anomaly, and higher-order terms in $\hbar$ do have an angle dependence expressed in terms of this anomaly. But just for first-order it's OK. ◂

We end up getting

$$\hbar t_{a,p} \otimes t_{b,p'} \int \mathrm{d}x\,\mathrm{d}y'\, P^{ab}(x - x', y - y', z - z', \overline{z} - \overline{z}'), \tag{3.2}$$

where $\{t_a\}$ is an orthonormal basis for $\mathfrak{g}$ with respect to the Killing form, $t_{a,n} \coloneqq t_a z^n \in \mathfrak{g}[[z]]$, and P denotes the propagator for the free theory. Then the equations of motion are fairly simple:

$$\mathrm{d}z \wedge F_A = 0 \tag{3.3a}$$

$$F_{xy} = F_{x\overline{z}} = F_{y\overline{z}} = 0, \tag{3.3b}$$

and the linearized equations of motion are even simpler:

$$\mathrm{d}z \wedge \mathrm{d}A = 0. \tag{3.4}$$

Fix the gauge

$$\partial_X A_x + \partial_y A_y + 4\partial_z A_{\overline{z}} = 0. \tag{3.5}$$

In this case the propagator P satisfies

$$\frac{i}{2\pi}\,\mathrm{d}z \wedge \mathrm{d}P = \delta^4(x, y, z, \overline{z})\mathbf{1} \tag{3.6}$$

$$(\partial_x i u_x + \partial_y i_y + \partial z i_{\partial\overline{z}})P = 0. \tag{3.7}$$

Therefore

$$P^{ab}(x, y, z, \overline{z}) = \frac{\delta^{ab}}{2\pi}\frac{x\,\mathrm{d}y \wedge \mathrm{d}\overline{z} y\,\mathrm{d}\overline{z} \wedge \mathrm{d}x + z\overline{z}\,\mathrm{d}x \wedge \mathrm{d}y}{(x^2 + y^2 + z\overline{z})^2}. \tag{3.8}$$

Plugging this into (3.2), we get for the $O(\hbar)$ contribtution

$$(3.2) = \hbar t_{a,p} \otimes t_{b,p'} \int \mathrm{d}x\,\mathrm{d}y\, \frac{\delta^{ab} 2\overline{z}_1 - \overline{z}_2}{(x^2+y^2+|z_1-z_2|^2)^2} \tag{3.9}$$

$$= \frac{\hbar(\sum_a t_{a,p} \otimes t_{a,p'})}{z_1 - z_2} \tag{3.10}$$

$$= \frac{\hbar C_{p,p'}}{z_1 - z_2}. \tag{3.11}$$

That is,

$$R_\hbar(z_1 - z_2) = \mathbf{1} + \frac{\hbar C_{p,p'}}{z_1 - z_2} + O(\hbar^2), \tag{3.12}$$

which recovers what we expect in the rational setting.

This is related to the Yangian, in that if V_1 and V_2 are representations of the Yangian, then $R_\hbar(z) = \mathrm{End}(V_1 \otimes V_2)$ should be an intertwiner. We can interpret this using OPEs of Wilson lines. Consider two Wilson lines K, K_ε supported on $\Sigma \times \{0\}$ and $\Sigma \times \{\varepsilon\}$. Letting $\varepsilon \to 0$, one has

$$\lim_{\varepsilon \to 0} \widetilde{W}_\rho(K_\varepsilon) \otimes \widetilde{W}_{\rho'}(K_\varepsilon) = \widetilde{W}_{\rho \otimes \rho'}(K) - \hbar t_{a,p} \otimes t_{b,p'} f_{abc} \int \mathrm{d}x\, m \partial_z A_x^C(x,y,0,0) + O(\hbar^2). \tag{3.13}$$

Last time, we said tht if $\widehat{\rho}$ is a representation of $\mathfrak{g}[[z]]$ with $\widehat{\rho}(z^n t_a) = t_{a,n}$, then

$$\widetilde{W}_{\widehat{\rho}}(K) = P\exp\left(\oint_K \sum_{n=0}^\infty \frac{\partial}{\partial z^n} A_i^c(x,y,0,0) t_{c,n}\, \mathrm{d}x^i\right). \tag{3.14}$$

Therefore it seems that the fused Wilson line is associated to a $\mathfrak{g}[[z]]$-representation with

$$\begin{aligned} t_{a,0} &= t_{a,p} \otimes 1 + 1 \otimes t_{a,p'} \\ t_{c,1} &= -\hbar t_{a,p} \otimes t_{b,p'} f_{abc}. \end{aligned} \tag{3.15}$$

But this is actually not true: there are further identities that don't hold. On $\mathfrak{g}[[z]]$, $[t_{a,1}, t_{b,2}] = f_{abc} t_{c,2}$ plus the Jacobi identity imply that $f_{\mu\nu a}[t_{a,1}, t_{b,1}]$ is equal to a cyclic permutation of $\mu\nu a$ in this. But instead, their sum is $Q_{\mu\nu b}(t_{a,0})$, which is nonzero. The answer is that the fused Wilson line is associated to a representation of a one-parameter deformation of $\mathcal{U}(\mathfrak{g}[[z]])$.

4. From Gaussian measures to factorization algebras: 2/13/19

Today Charlie spoke about the second chapter of Costello-Gwilliam [**?**], with the goal of motivating (pre)factorization algebras via Gaussian integrals on finite-dimensional vector spaces.

Let $Q\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a positive-definite bilinear form and $p\colon \mathbb{R}^n \to \mathbb{R}$ be a polynomial function. We're interested in computing the expectation value

$$\langle p \rangle := \int_{\mathbb{R}^n} p(x) \exp\left(-\frac{1}{2} Q(x,x)\right) \mathrm{d}^n x. \tag{4.1}$$

We're going to try to compute this without having to evaluate any integrals!

Definition 4.2. We will use $\mathrm{Vect}(\mathbb{R}^n)$ to denote the space of vector fields on $\mathbb{R}^n$ whose component functions are polynomials and $P(\mathbb{R}^n)$ to denote the space of polynomial functions on $\mathbb{R}^n$.

Definition 4.3. Given a polynomial vector field v on $\mathbb{R}^n$, its *divergence* $\mathrm{Div}\, v \in P(\mathbb{R}^n)$ is defined to act on a volume form ω by $(\mathrm{Div}\, V)\omega = \mathcal{L}_v \omega = \mathrm{d}\iota_v \omega$ (this last equality follows from Cartan's magic formula).

Here are a few important facts about the divergence.

Lemma 4.4.

(1) *For any polynomial vector field v and volume form ω, $\int (\mathrm{Div}\, v)\omega = 0$.*

(2)

$$\begin{aligned}(\operatorname{Div} v)\omega &= \nabla\biggl(\exp\biggl(-\frac{1}{2}Q(x,x)v\biggr)\biggr)\,\mathrm{d}x^n\\ &= \biggl(v\exp\biggl(-\frac{1}{2}Q(x,x)\biggr) + \nabla\cdot v\exp\biggl(-\frac{1}{2}Q(x,x)\biggr)\biggr)\,\mathrm{d}x^n\\ &= (-Q(v,x) + \nabla\cdot v)\omega.\end{aligned}$$

(3) *$P(\mathbb{R}^n)/\operatorname{Im}(\operatorname{Div})$ is one-dimensional, and we have a canonical identification with $\mathbb{R}$ provided by integration.*

The last piece is particularly crucial: the quotient is still one-dimensional in a certain infinite-dimensional setting, where we'll use an identification with $\mathbb{R}$ as a definition of integration.

Now we'll apply this to physics. Consider the free scalar quantum field theory on a Riemannian manifold (M,g), which is spacetime. The space of fields is $C^\infty(M)$, and the action is

$$S(\phi) := \frac{1}{2}\int_M \phi(\Delta_g + m^2)\phi\,\mathrm{dvol}_g, \tag{4.5}$$

where m is the mass of the theory and Δ_g is the Laplacian. We would like to define correlation functions

$$\langle\phi(x_1)\cdots\phi(x_n)\rangle \text{ “:=” } \int_{C^\infty(M)} \phi(x_1)\cdots\phi(x_n)e^{-S(\phi)}\,\mathrm{d}\phi, \tag{4.6}$$

but of course the measure $\mathrm{d}\phi$ doesn't exist on most M.

Our strategy to abrogate this is to put the observables into a chain complex: for an open $U\subset M$, take

$$\mathrm{Obs}^{-3}(v) \longrightarrow \mathrm{Obs}^{-2}(v) \longrightarrow \mathrm{Vect}(C^\infty(U)) \xrightarrow{\mathrm{Div}} P(C^\infty(U)). \tag{4.7}$$

We would like this to have the following properties:

(1) H^0 of this complex should be the space of physically distinguishable observables on U, and
(2) if U_1 and U_2 are disjoint open subsets of an open $V\subset X$, an observable on U_1 and an observable on U_2 should together define an observable on V, and this should define a map $H^0(\mathrm{Obs}^*(U_1))\otimes H^0(\mathrm{Obs}^*(U_2))\to H^0(\mathrm{Obs}^*(V))$.

To make this into real math, we have to define $P(C^\infty(U))$, $\mathrm{Vect}(C^\infty(U))$, and Div.

Definition 4.8. Given an $F\in C_c^\infty(U^n)$, define a function $F\colon C^\infty(U)\to\mathbb{R}$ by

$$\phi\longmapsto \int_{U^n} F(x_1,\dots,x_n)\phi(x_1)\cdots\phi(x_n)\,\mathrm{dvol}^n. \tag{4.9}$$

The symmetric group S_n acts on these functions by permuting the n inputs. We then define $P(C^\infty(U))$ as

$$P(C^\infty(U)) := \bigoplus_{n=0}^\infty C_c^\infty(U^n)_{S_n}. \tag{4.10}$$

Definition 4.11. An $F\in C_c^\infty(U^{n+1})$ defines a vector field on U^n by the rule

$$\phi\longmapsto \int_{U^n} F(x_1,\dots,x_{n+1})\phi(x_1)\cdots\phi(x_n)\,\mathrm{dvol}^n. \tag{4.12}$$

(Precisely, this is a rule for differentiating functions into one-forms, which is specified by a vector field.) The symmetric group again acts, and we define $\mathrm{Vect}(C^\infty(U))$ by

$$\mathrm{Vect}(C^\infty(U)) := \bigoplus_{n=0}^\infty C_c^\infty(U^{n+1})_{S_n}. \tag{4.13}$$

Defining the divergence is a little trickier.

Definition 4.14. Let $X\in\mathrm{Vect}(C^\infty(U))$ be homogeneous, i.e. a homogeneous element in the direct sum (4.13), and suppose $\deg(X)=n$. Then we define

$$\mathrm{Div}(X) := -\bigl(\Delta_{g,n+1}+m^2\bigr)X(x_1,\dots,x_{n+1}) + \sum_{i=1}^n \int_U X(x_1,\dots,x_i,\dots,x_n,X_i)\,\mathrm{dvol}, \tag{4.15}$$

and extend to inhomogeneous elements by linearity.

Now we have some structure.

- Given $P \in P(C^\infty(U))$ and $X \in \mathrm{Vect}(C^\infty(U))$, we can multiply them to obtain another vector field.
- Given $P \in P(C^\infty(U)))$, if $U \subset V$, we can extend by 0 to obtain $\epsilon_0(P) \in P(C^\infty(V))$.

Lemma 4.16. *Let $P \in P(C^\infty(U))$ and $X \in \mathrm{Vect}(C^\infty(U))$. Then $\mathrm{Div}(PX) = P\,\mathrm{Div}\,X + X(P)$.*

Now we've got everything we need to make (4.7) make sense. Given $U_1, U_2 \subset V$ all opens in M, there is a map

$$\begin{aligned} \widetilde{m}_V^{U_1,U_2}\colon P(C^\infty(U_1)) \otimes P(C^\infty(U_2)) &\longrightarrow P(C^\infty(V)) \\ P_1 \otimes P_2 &\longmapsto \epsilon_0(P_1)\cdot\epsilon_0(P_2). \end{aligned} \tag{4.17}$$

Lemma 4.18. *The map $\widetilde{m}_V^{U_1,U_2}$ descends to a map $m_V^{U_1,U_2}\colon H^0(\mathrm{Obs}^*(U_1)) \otimes H^0(\mathrm{Obs}^*(U_2)) \to H^0(\mathrm{Obs}^*(V))$ if and only if U_1 and U_2 are disjoint.*

Proof. Let $P_W := P(C^\infty(W))$ and consider the diagram

$$\begin{tikzcd} (\mathrm{Im}(\mathrm{Div}_{U_1}) \otimes P_{U_2}) \oplus (P_{U_1} \otimes \mathrm{Im}(\mathrm{Div}_{U_2})) \arrow[r, hook] & P_{U_1} \otimes P_{U_2} \arrow[r] \arrow[d, "\widetilde{m}_V^{U_1,U_2}"] & H^0(\mathrm{Obs}^*(U_1)) \otimes H^0(\mathrm{Obs}^*(U_2)) \\ \mathrm{Im}(\mathrm{Div}_V) \arrow[r] & P_V \arrow[r] & H^0(\mathrm{Obs}^*(V)). \end{tikzcd} \tag{4.19}$$

We would like to move the vertical arrow from the second column to the third column. Since the rows are exact, it suffices to show that $\widetilde{m}_V^{U_1,U_2}$ applied to anything coming from the upper left is zero in cohomology. Well, if $P \in P_{U_1}$ and $X \in \mathsf{Vect}(C^\infty(U_2))$, then using Lemma 4.16,

$$P\,\mathrm{Div}(X) = \mathrm{Div}_V(PX) + X(P). \tag{4.20}$$

U_1 and U_2 are disjoint iff $X(P)$ vanishes for all X and all P, and in this case, $P\,\mathrm{Div}(X) \in \mathrm{Im}(\mathrm{Div}_V)$, so when we pass to $H^0(\mathrm{Obs}^*(V))$ along the bottom row, it's zero by exactness. The case when $P \in P_{U_2}$ and $X \in \mathrm{Vect}(C^\infty(U_1))$ is the same. ⊠

With the remaining time, let's talk a bit about the classical limit. In this case we consider $e^{-S/\hbar}$; to say this more explicitly,

$$\mathrm{Div}_\hbar(X) := -\frac{1}{\hbar}\big(\Delta_{g,n+1} + m^2\big)X + \nabla\cdot X. \tag{4.21}$$

The limit of the image of $\mathrm{Div}_\hbar$ as $\hbar \to 0$ is the ideal in $P(C^\infty(U))$ generated by linear functionals, and therefore $\mathrm{Obs}^0(U)$ becomes functions on the critical locus of the Euler-Lagrange equations. The upshot is that multiplication is commutative, so we don't just get a prefactorization algebra, but in fact have a commutative one.

5. Classical free scalar field theory: 2/20/19

Today, Mario spoke about chapter 2, with a perspective rooted a little more in physics. The goal is to discuss how some of the formalism from last week can be used to recover some physics statements about free theories.

Let's recall some facts about classical field theory on a Riemannian manifold (M, g). There is some action S, which is a function on smooth functions on M, typically expressed as an integral of a Lagrangian density:

$$S(\phi) = \int_M \mathrm{dvol}\,\mathcal{L}(\phi). \tag{5.1}$$

This functional is minimized when ϕ satisfies the *Euler-Lagrange equations*

$$\partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi} - \frac{\delta\mathcal{L}}{\delta\phi} = 0. \tag{5.2}$$

The *classical observables* are the functions on the space of solutions to the Euler-Lagrange equations (the critical locus of S).

In quantum field theory, observables are no longer functions, but are instead operators Φ which act on the Hilbert space. Speaking precisely, these are operator-valued distributions. For example, we could choose a $U \subset M$ and a smooth function f supported in U; then we define

$$\Phi(f) \coloneqq \int_M \mathrm{dvol}\, f(x)\Phi(x). \tag{5.3}$$

Now let's discuss why we want the divergence operator and the other tools from last time. Strictly speaking, you cannot observe $\Phi(f)$, but instead its expectation values on states $\langle \Phi(f_1)\cdots\Phi(f_n)\rangle$. Heuristically, this is defined via the path integral

$$\langle \Phi(f_1)\cdots\Phi(f_n)\rangle \coloneqq \frac{\int \mathcal{D}\phi\, \langle f_1\Phi\rangle \cdots \langle f_n\Phi\rangle e^{-S/\hbar}}{\int \mathcal{D}\phi\, e^{-S/\hbar}}. \tag{5.4}$$

However, this is not a rigorous definition; the motivation for what we did last week is to be able to replace (5.4) with something mathematical.

Recall that we defined $H^0(\mathrm{Obs}^*(U))$ for $U \subset M$, the observables of a (free) quantum field theory, to be the polynomial functions on $C^\infty(U)$ modulo the image of the divergence operator Div_μ, where μ is a measure. Specifically, Div_μ is a map $\mathrm{Vect}(\mathbb{R}^n) \to P(\mathbb{R}^n)$ which, given a vector v, returns $\mathcal{L}_v(\mu) = \mathrm{d}(i_v\mu)$.

If you're in the image of Div_μ, then you have zero expectation value: following directly from the definition,

$$\int \mathrm{Div}_\mu(v)\, \mathrm{d}\mu = 0. \tag{5.5}$$

Therefore observables whose difference is in $\mathrm{Im}(\mathrm{Div}_\mu)$ can't be distinguished by observations, which is why we mod out by this space.

Recall that $P(C^\infty(U))$ is the direct sum over $n \geq 0$ of $C_c^\infty(U^n)_{S_n}$, where S_n acts by permuting the indices. If P is given by $f_1, \ldots, f_n \in \mathrm{Sym}^n(C_c^\infty(U))$, we can evaluate P on Φ by

$$P(\Phi) = \int \mathrm{dvol}\, f_1\Phi \cdots \int \mathrm{dvol}\, f_n\Phi. \tag{5.6}$$

Last time, we also define $\mathrm{Vect}_c(C^\infty(U))$ in a similar way, but this time we take the direct sum of $C^\infty(U^{n+1})_{S_n}$. Then $\mathrm{Div}_\mu\colon \mathrm{Vect}_c(C^\infty(U)) \to P(C^\infty(U))$ by

$$\mathrm{Div}_\mu\left(f_1 \ldots f_n \frac{\partial}{\partial\phi}\right) = -f_1\cdots f_n \frac{\partial S}{\partial \phi} + \sum_i f_1 \ldots \widehat{f_i} \ldots f_n \int \mathrm{dvol}\, f_i \cdot \phi. \tag{5.7}$$

Here the notation $\widehat{f_i}$ means that f_i is missing from the product.

If for example $S(\phi) = (1/2)\int \mathrm{dvol}\, \Phi(\Delta + m^2)\Phi$ is the action for the free scalar field theory with mass m, then

$$\frac{\partial S}{\partial \Phi}(\Phi) = (\Delta + m^2)\Phi. \tag{5.8}$$

Now, we can recover Wick's theorem by looking at the cokernel of the divergence operator. Specifically, let $G(x,y)\colon U \times U \to \mathbb{R}$ be the Green's function for $\Delta_x + m^2$, i.e.

$$(\Delta_x + m^2)G(x,y) = \delta(x-y), \tag{5.9}$$

and given functions f_1 and f_2, let $\widetilde{\Phi} \coloneqq \int \mathrm{d}y\, G(x,y) f_2(y)$. Then

$$\mathrm{Div}\left(f \frac{\partial}{\partial \widetilde{\Phi}}\right) = -f_1(\Delta + m^2)\widetilde{\Phi} + \int \mathrm{d}x\, f_1(x)\widetilde{\Phi}(x) \tag{5.10}$$

$$= -f_1 \cdot f_2 + \int \mathrm{d}x\,\mathrm{d}y\, f_1(x)G(x-y)f_2(y), \tag{5.11}$$

which implies that

$$\langle \Phi(f_1)\Phi(f_2)\rangle = \int \mathrm{d}x\,\mathrm{d}y\, f_1(x)G(x,y)f_2(y). \tag{5.12}$$

Here $\frac{\partial}{\partial\widetilde{\Phi}}(\widetilde{f}) = \int \mathrm{dvol}\,\widetilde{\Phi}\widetilde{f}$. And the conclusion is, the two-point correlation functions are exactly what we want them to be, which is a special case of Wick's theorem. And more generally, if we have four functions f_1, f_2, f_3, f_4, then letting $\widetilde{\Phi} \coloneqq \int \mathrm{d}y\, G(x,y) f_4(x)$, we have the general case of Wick's theorem:

$$\operatorname{Div}\left(f_1 f_2 f_3 \frac{\partial}{\partial\widetilde{\Phi}}\right) = -f_1 f_2 f_3 f_4 + f_1 f_2 \int \mathrm{d}x\, f_3(x) G(x,y) f_4(x) + \cdots \tag{5.13}$$

As we discussed last time, we do not have a map $H^0(\mathrm{Obs}^*(V)) \otimes H^0(\mathrm{Obs}^*(V)) \to H^0(\mathrm{Obs}^*(V))$, which is saying that we don't quite have an algebra; instead, we can pair observables together iff they live on disjoint opens. This leads to the definition of a prefactorization algebra.

6. Prefactorization algebras and some examples: 2/27/19

Today, Ivan spoke about the definition of prefactorization algebras and some examples, including their relationship to associative algebras over $\mathbb{R}$, and how to build a prefactorization algebra out of the universal enveloping algebra of a Lie algebra.

Definition 6.1. Let M be a topological space and C be a category. A *precosheaf* on M valued in C is a covariant functor $\mathscr{F}\colon \mathsf{Top}(M) \to \mathsf{C}$, where $\mathsf{Top}(M)$ denotes the category whose objects are open subsets of M and whose morphisms are given by inclusions.

Definition 6.2. Let M be a topological space and $(\mathsf{C}, \otimes, 1)$ be a symmetric monoidal category. A *prefactorization algebra* (PFA) over M valued in C is a precosheaf $\mathscr{F}\colon \mathsf{Top}(M) \to \mathsf{C}$ together with, for every finite ordered set $\mathfrak{U} \coloneqq \{U_1, \dots, U_n\}$ of disjoint opens contained in an open $V \subset M$, a *structure map*

$$s_{\mathfrak{U}}\colon \mathscr{F}(U_1) \otimes \cdots \otimes \mathscr{F}(U_n) \longrightarrow \mathscr{F}(V). \tag{6.3}$$

We require these maps to be compatible with the symmetric monoidal structure in the following sense: if $\tau\colon \mathscr{F}(U) \otimes \mathscr{F}(V) \to \mathscr{F}(V) \otimes \mathscr{F}(U)$ is the symmetrizer,[8] and $U, V \subset M$ are disjoint, we require the diagram

$$\begin{tikzcd}
\mathscr{F}(U) \otimes \mathscr{F}(V) \arrow[rr, "\tau"] \arrow[dr, "s_{\{U,V\}}"'] & & \mathscr{F}(V) \otimes \mathscr{F}(U) \arrow[dl, "s_{\{V,U\}}"] \\
& \mathscr{F}(U \amalg V) &
\end{tikzcd} \tag{6.4}$$

to commute.

Example 6.5. Let $\mathsf{C} = \mathsf{Vect}^{\mathrm{fd}}$ denote the category of finite-dimensional vector spaces, M be any topological space, and $\mathscr{F}$ be any precosheaf. (In particular, you might take M to be a smooth manifold and $\mathscr{F}(U) = C_c^\infty(U)$, which is covariant via extension by zero.) Then, $\mathrm{Sym}^* \mathscr{F}$ has the structure of a prefactorization algebra: given disjoint $U_1, \dots U_n \subset V$, the maps $\mathscr{F}(U_i) \to \mathscr{F}(V)$ together define a map $\mathscr{F}(U_1) \oplus \cdots \oplus \mathscr{F}(U_n) \to \mathscr{F}(V)$. Since Sym^* sends direct sums to tensor products, we have a map

$$\mathrm{Sym}^*\left(\bigoplus_{i=1}^n \mathscr{F}(U_i)\right) \cong \bigotimes_{i=1}^n \mathrm{Sym}^* \mathscr{F}(U_i) \longrightarrow \mathscr{F}(V). \tag{6.6}$$

and this is compatible with the tensor product on $\mathsf{Vect}^{\mathrm{fd}}$. The reason we can't use all vector spaces is because then we might have to worry about completions, etc. ◀

Example 6.7. Let A be a unital associative k-algebra; we will use it to define a prefactorization algebra A^{fact} over $\mathbb{R}$ (meaning $\mathbb{R}$ is the space M) valued in k-algebras. Namely, for any interval I, let $A^{\mathrm{fact}}(I) \coloneqq A$, and let $A^{\mathrm{fact}}(\varnothing) = k$. If I is a disjoint union of n intervals, we assign to it $A^{\otimes n}$. Then the structure map $A^{\mathrm{fact}}(U) \otimes A^{\mathrm{fact}}(V) \to A^{\mathrm{fact}}(U \amalg V)$ is the identity map $A^{\otimes 2} \to A^{\otimes 2}$, and the higher-order structure maps are similar. ◀

Definition 6.8. A factorization algebra over is *locally constant* if TODO: I missed this.

Proposition 6.9. *A factorization algebra $\mathscr{F}$ over $\mathbb{R}$ is locally constant iff it's isomorphic to A^{fact}, where $A = \mathscr{F}(\mathbb{R})$.*

[8] The *symmetrizer*, a natural isomorphism $\tau\colon X \otimes Y \to Y \otimes X$ for all objects X and Y, is part of the data of a symmetric monoidal category.

TODO: I missed the proof.

But the most important example is that if $\mathfrak{g}$ is a Lie algebra, its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ defines a prefactorization algebra over $\mathbb{R}$. This is a particular case of a general construction, the factorization envelope of a sheaf of Lie algebras, which will be the observables of free theories, and will also appear in other places (Kac-Moody vertex algebras, and, in volume 2 of Costello-Gwilliam, Noether's theorem in QFT). In our case, we take the constant $\mathfrak{g}$-valued sheaf over $\mathbb{R}$. In general, one can build a prefactorization algebra on $\mathbb{R}$ from any dg Lie algebra, and using just a Lie algebra one obtains the locally constant prefactorization algebra associated to $\mathcal{U}(\mathfrak{g})$.

Definition 6.10. A *dg Lie algebra* (dg stands for "differential graded") is data of a cochain complex $(L^\bullet, d_L)$ and a Lie bracket $[-,-]\colon L^\bullet \times L^\bullet \to L^\bullet$ making $L^\bullet$ into a graded Lie algebra, subject to the *graded Jacobi identity*

$$d_L[X,Y] = [d_L X, Y] + (-1)^{|X|}[X, d_L Y] \tag{6.11}$$

for all homogeneous elements X and Y of $L^\bullet$.

For example, if $\mathfrak{g}$ is any Lie algebra and $n \in \mathbb{Z}$, there's a dg Lie algebra $\mathfrak{g}[n]$ which is just $\mathfrak{g}$ living in degree n and 0 in other degrees, with $d_{\mathfrak{g}[n]} = 0$. There are plenty of other examples.

Definition 6.12. Associated to any dg Lie algebra $(L^\bullet, d_L, [-,-])$ is a double complex called the *Chevalley-Eilenberg complex* $C_\bullet(L^\bullet)$. As a vector space, it's $\mathrm{Sym}^*(L[1])$, i.e. in degree k it's $\mathrm{Sym}^k(L[1])$. For example, if $\mathfrak{g}$ is a Lie algebra, then $\mathrm{Sym}(\mathfrak{g}[1]) = \Lambda^*\mathfrak{g}$. Then, there are two differentials: d_L induced from the differential on $L[1]$, and $d_{[-,-]}$ defined by

$$d_{[-,-]}(X^1 \cdots X^n) \coloneqq \sum_{1 \le i \le j \le k} (-1)^{?}[X^i, X^j] \cdot X^1 \cdots \widehat{X}^i \cdots X^j \cdots X^n. \tag{6.13}$$

where as usual $\widehat{X}^i$ means leaving X^i out of the sum.

If we let $\widetilde{C}^{-k,i}(L) \coloneqq (C_k(L)_i$, then d_L is a map $\widetilde{C}^{\bullet,\bullet} \to \widetilde{C}^{\bullet,\bullet+1}$ and $d_{[-,-]}$ is a map $\widetilde{C}^{\bullet,\bullet} \to \widetilde{C}^{\bullet+1,\bullet}$.

We're interested in the *totalization* of this double complex, which is the chain complex

$$\left(T^\bullet(L) \coloneqq \bigoplus_{i+j=\bullet} \widetilde{C}^{i,j}(L), d_P + d_{[-,-]} \right). \tag{6.14}$$

You can check that this differential squares to zero:

$$(d_L + d_{[-,-]})^2 = \underbrace{d_L^2}_{=0} + \underbrace{d_L d_{[-,-]} + d_{[-,-]} d_L}_{=0 \text{ by } (6.11)} + \underbrace{d_{[-,-]}^2}_{=0}. \tag{6.15}$$

The cohomology of this complex is called *Lie algebra cohomology*; if that's what you're after in the end, you can define it faster (if more abstractly) as some derived functor.

Example 6.16. Let $\mathfrak{g}$ be a Lie algebra and $U \subset \mathbb{R}$ be an open set. Then $\Omega_c^\bullet(U) \otimes \mathfrak{g}$ the compactly supported $\mathfrak{g}$-valued forms, has the structure of a dg Lie algebra, where the differential is the de Rham differential and the bracket is

$$[\alpha \otimes A, \beta \otimes B] \coloneqq \alpha \wedge \beta \otimes [A, B]. \tag{6.17}$$

◀

Theorem 6.18. *Let $\mathfrak{g}$ be a Lie algebra and for any open $U \subset \mathbb{R}$, let $\mathfrak{g}^{\mathbb{R}}(U) \coloneqq \Omega_c^\bullet(U) \otimes \mathfrak{g}$. Let $\mathcal{H}$ denote the precosheaf $\mathcal{H}(U) \coloneqq H^*(T^\bullet(\mathfrak{g}^{\mathbb{R}}(U)), \mathrm{d_{dR}} + d_{[-,-]})$. Then $\mathcal{H}$ is a locally constant prefactorization algebra modeled on $\mathcal{H}(\mathbb{R}) = \mathcal{U}(\mathfrak{g})$.*

Proof sketch. If $i\colon U \hookrightarrow V$ is an inclusion of one interval into another, then the induced map $\Omega_c^\bullet(U) \otimes \mathfrak{g} \to \Omega_c^\bullet(V) \otimes \mathfrak{g}$ is a *quasi-isomorphism* of dg Lie algebras, meaning that it's a dg Lie algebra homomorphism which is an isomorphism on cohomology (of the complex, not necessarily Lie algebra cohomology). Therefore $\mathcal{H}$ is locally constant.

Therefore it suffices to identitfy $\mathcal{H}(\mathbb{R}) = H^*(C_\bullet(\mathfrak{g}^{\mathbb{R}}(\mathbb{R})))$. The map $\Omega_c^\bullet(\mathbb{R}) \otimes \mathfrak{g} \to H_c^*(\mathbb{R}) \otimes \mathfrak{g}$ is a quasi-isomorphism of dg Lie algebras, so we have the following identifications of vector spaces:

$$H^*(C_\bullet(\mathfrak{g}^{\mathbb{R}}(\mathbb{R})) \cong H^*(C_\bullet(H_c^*(\mathbb{R}) \otimes \mathfrak{g})) \cong T^\bullet(H_c^*(\mathbb{R}) \otimes \mathfrak{g}).$$

You can check that the double complex is concentrated in degree zero, so this is

$$\begin{aligned}&= T^0(H_c^*(\mathbb{R})\otimes\mathfrak{g})\\&= \mathrm{Sym}^*(H_c^1(\mathbb{R})\otimes\mathfrak{g}) = \mathrm{Sym}^*\mathfrak{g}.\end{aligned}$$

Finally, we have to address the algebra structure. There are identifications

$$T^0(\mathfrak{g}^{\mathbb{R}}(U)) = \mathrm{Sym}^*(\Omega_c^1(U)\otimes\mathfrak{g}) \tag{6.19a}$$

$$T^{-1}(\mathfrak{g}^{\mathbb{R}}(U)) = (\Omega_c^0(U)\otimes\mathfrak{g})\cdot\mathrm{Sym}^*(\Omega_c^1(U)\otimes\mathfrak{g}) \tag{6.19b}$$

$$T^{-2}(\mathfrak{g}^{\mathbb{R}}(U)) = (\Lambda^2\Omega_c^0(U)\otimes\mathfrak{g})\cdot\mathrm{Sym}^*(\Omega_c^1(U)\otimes\mathfrak{g}) \tag{6.19c}$$

and so on. Let $\epsilon\in\Omega_c^1(\mathbb{R})$ have total integral 1; then we get a map $\Phi\colon\mathfrak{g}\to\mathcal{H}(\mathbb{R})$ sending $X\mapsto[\epsilon\otimes X]$. We want to show that $\Phi([X,Y]) = [\Phi(X),\Phi(Y)]$, which would imply the vector space map $\mathcal{U}(\mathfrak{g}) = \mathrm{Sym}\,\mathfrak{g}\to\mathcal{H}(\mathbb{R})$ is an algebra isomorphism, via the universal property of $\mathcal{U}(\mathfrak{g})$.

Write $\epsilon = f_0\,\mathrm{d}x$, where f_0 is supported in $(-\delta,\delta)$, and let $f_t(x) = f_0(x-t)$. Then

$$\Phi(X)\Phi(Y) = [\epsilon\otimes X]\cdot[\epsilon\otimes Y] = [f_0(x)\,\mathrm{d}x\otimes X]\cdot[f_1(x)\,\mathrm{d}x\otimes Y], \tag{6.20a}$$

as long as $\delta<1/2$, and similarly

$$\Phi(Y)\Phi(X) = [f_{-1}(x)\,\mathrm{d}x\otimes Y]\cdot[f_0(x)\,\mathrm{d}x\otimes X]. \tag{6.20b}$$

Since $\int_{\mathbb{R}}(f_{-1}-f_1)\,\mathrm{d}x = 0$, there's an $h\in\Omega_c^0(\mathbb{R})$ such that h is 1 on (δ,δ) and $\mathrm{d}h = (f_{-1}-f_1)\,\mathrm{d}x$. (Explicitly, h is an antiderivative of $f_{-1}-f_1$.) Then

(6.21a)

$$(d_L + d_{[-,-]})((f_0\,\mathrm{d}x\otimes X)\cdot(h\otimes Y)) = (f_0\,\mathrm{d}x\otimes X)(\mathrm{d}h\otimes Y) + f_0h\,\mathrm{d}x\otimes[X,Y]$$

$$= (f_0\,\mathrm{d}x\otimes X)(f_{-1}\,\mathrm{d}x\otimes Y) - (f_0\,\mathrm{d}x\otimes X)(f_1\,\mathrm{d}X\otimes Y) + f_0\,\mathrm{d}x\otimes[X,Y]. \tag{6.21b}$$

The first two terms are the same in homology, and the last term is $[\epsilon\otimes[X,Y]]$. ⊠

7. More examples of prefactorization algebras: 3/6/19

"If you have any questions, I answered them before you came."

Today, Andy spoke about chapter 3 of Costello-Gwilliam.

Recall from last time that if A is a unital associative $\mathbb{C}$-algebra, we can build from it a locally constant prefactorization algebra $\mathscr{F}_A$ over the real line. This assigns $\mathbb{C}$, as a commutative algebra, to the empty set, and to every interval $I\subset\mathbb{R}$, assigns the vector space A. This is also an algebra, of course, and we'll be able to see that structure from $\mathscr{F}_A$. Finally, $\mathscr{F}_A(\mathbb{R}) = A$. That $\mathscr{F}_A$ is locally constant means all inclusions of intervals are isomorphisms – and in this case they're just the identity. We can recover the algebra structure on A from the factorization product: if I_1 and I_2 are intervals, let J be an interval containing I; then we have a product

$$A\otimes A = \mathscr{F}_A(I_1)\otimes\mathscr{F}_A(I_2)\longrightarrow\mathscr{F}_A(I_1\amalg I_2)\longrightarrow\mathscr{F}_A(J) = A, \tag{7.1}$$

and this is the multiplication map.

Conversely, if $\mathscr{F}$ is a locally constant prefactorization algebra over $\mathbb{R}$, it's isomorphic to $\mathscr{F}_A$ for $A\coloneqq\mathscr{F}(\mathbb{R})$.

We also have a little more structure: the inclusion $\varnothing\to I$ induces a map $\mathbb{C}\to A$, and the various axioms imply this sends 1 to the identity of A.

Example 7.2. Out next example is a variant of this. Let A and A' be unital associative algebras over $\mathbb{C}$ and M be a pointed (A,A')-bimodule (meaning it comes with a distinguished element, which could be zero). We will use this to build a "piecewise locally constant" prefactorization algebra $\mathscr{F}_{A,M,A'}$ over $\mathbb{R}$, with a *defect* at $0\in\mathbb{R}$.

Now we have three kinds of intervals: intervals contained in $\mathbb{R}_{<0}$, which we'll denote I_{left}; intervals contained in $\mathbb{R}_{>0}$, which we'll denote I_{right}; and intervals containing zero, which we'll denote I_{mid}. We have $\mathscr{F}_{A,M,A'}(\varnothing) = \mathbb{C}$ again, and then assign

$$\begin{aligned}\mathscr{F}_{A,M,A'}(I_{\mathrm{left}}) &\coloneqq A\\ \mathscr{F}_{A,M,A'}(I_{\mathrm{right}}) &\coloneqq A'\\ \mathscr{F}_{A,M,A'}(I_{\mathrm{mid}}) &= M.\end{aligned} \tag{7.3}$$

The factorization product for two intervals of the form I_{left} or I_{right} is exactly as it was in the previous example. For $I_{\text{left}} \amalg I_{\text{mid}}$, we produce the action of A on M, and for $I_{\text{mid}} \amalg I_{\text{right}}$, we get the action of A' on M. The inclusions $\varnothing \hookrightarrow I_{\text{left}}$ or $\varnothing \hookrightarrow I_{\text{right}}$ pick out the identity elements of A, resp. A', and the inclusion $\varnothing \hookrightarrow I_{\text{mid}}$ picks out the distinguished element of M. ◀

Example 7.4. This example is more reminiscent of quantum mechanics. Given a finite-dimensional complex vector space V and a Hermitian operator $H\colon V \to V$ called the *Hamiltonian*, together with two vectors $\psi_i \in V$ and $\psi_o \in \overline{V}$, we can obtain a prefactorization algebra on the interval $[0, T]$.

Let $A := \operatorname{End} V$. We then make the following assignments:

- $\mathscr{F}((a, b)) = A$,
- $\mathscr{F}([0, a)) = V$,
- $\mathscr{F}((b, T]) = \overline{V}$, and
- $\mathscr{F}([0, T]) = \mathbb{C}$.

The factorization product is time evolution by the Hamiltonian: if $a < b < c < d$, the factorization product $\mathscr{F}((a, b)) \otimes \mathscr{F}((c, d)) \to \mathscr{F}(a, d)$ is identified with the map $A \otimes A \to A$ sending

$$\mathcal{O}_1 \otimes \mathcal{O}_2 \longmapsto \mathcal{O}_1 e^{iH(c-b)} \mathcal{O}_2. \tag{7.5}$$

The map $\mathscr{F}(\varnothing) \to \mathscr{F}((a, b))$ sends 1 to $e^{iH(b-a)}$, and the map $\mathscr{F}(\varnothing) \to \mathscr{F}([0, a))$ sends $1 \mapsto e^{iHa}\psi_i$.

We can also see expectation values: the map

$$\mathscr{F}([0, a) \cup (b, T]) \longrightarrow \mathscr{F}([0, T]) \tag{7.6a}$$

is identified with the map $V \otimes \overline{V} \to \mathbb{C}$ sending

$$\psi_1 \otimes \psi_2 \longmapsto \langle \psi_2, e^{iH(b-a)}\psi_1 \rangle. \tag{7.6b}$$

So in some sense, everything in $[0, a)$ and $(b, T]$ is accounted for by the states you picked, and the factorization algebra handles time evolution in between. ◀

Remark 7.7. You can make a variant of the above example on all of $\mathbb{R}$ without needing the boundary conditions. This is a locally constant prefactorization algebra, and the isomorphism with $\mathscr{F}_A$ involves writing down isomorphisms that intertwine time evolution by the Hamiltonian.

This might be a little weird, because locally constant factorization algebras are in a sense, topological, and there's no dynamics. The reason we had dynamics in this example is that the isomorphism destroys the information about the Hamiltonian. If you ask for more structure, namely equivariance under the group of translations, then the Hamiltonian can be recovered from the abstract isomorphism type of the factorization algebra. ◀

We want more examples, and hence will study *factorization envelopes*, which are a way to product many more examples. The input data will be a fine sheaf $\mathscr{L}$ of dg Lie algebras over a manifold M. Here, "fine" is a technical term whose upshot is that partitions of unit exist and/or it's possible to make sense of compactly supported sections; this implies hihger cohomology vanishes. Theorem 6.18 from last time is a special case, where we fixed a dg Lie algebra $\mathfrak{g}$ and let $\mathscr{L}(U) := \mathfrak{g} \otimes \Omega^*_M(U)$.

Now, we can build a cosheaf $\mathscr{L}_c$ of dg Lie algebras over M whose value on an open $U \subset M$ is $C_c^\infty(U)$. Taking the Chevalley-Eilenberg cochains produces another cosheaf $C_\bullet \mathscr{L}_c$ of dg Lie algebras:

$$(C_\bullet \mathscr{L}_c)(U) = C_\bullet(\mathscr{L}_c(U)) = \operatorname{Sym}^\bullet(\mathscr{L}_c(U)[1]), \tag{7.8}$$

with a differential. Be aware that $\mathscr{L}_c(U)$ is an infinite-dimensional space, so taking its symmetric algebra involves delving into some functional-analytic details; consult the book for details.

Anyways, $C_\bullet \mathscr{L}_c$ is a prefactorization algebra valued in cochain complexes. You can therefore take cohomology, and obtain a locally constant prefactorization algebra $H^*(C_\bullet \mathscr{L}_c)$ on M.

Example 7.9. If $M = \mathbb{R}$ and $\mathscr{L}(U) = \mathfrak{g} \otimes \Omega^*_M(U)$, then the locally constant prefactorization algebra $H^*(C_\bullet \mathscr{L}_c)$ can be identified with $\mathscr{F}_A$ above, where A is the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.

The physical interpretation of $\mathcal{U}(\mathfrak{g})$ is the algebra of Noether charges in quantum mechanics with $\mathfrak{g}$-symmetry. ◀

Example 7.10. If $M = \mathbb{R}^n$, then this is equivalent to the structure of an E_n*-algebra* in vector spaces. This is an algebra over the little n-discs operad, which roughly means that for every arrangement of k D^ns inside D^n, we get an operation $A^{\otimes m} \to A$.

As n increases, this gets more and more commutative. For example, E_n-objects in categories are: monoidal for $n = 1$, braided monoidal for $n = 2$, and symmetric monoidal for $n \geq 3$. In vector spaces, it's less elaborate: an associative algebra for $n = 1$ and a commutative algebra for $n = 2$. If we had an extra differential, perhaps coming from supersymmetry, then we'd land in E_n-algebras in cochain complexes, and there we get a new structure for each n.

Possibly this has the interpretation as the algebra of currents in an n-dimensional quantum field theory with $\mathfrak{g}$-symmetry. ◀

In later chapters, we'll also see a slight generalization called *twisted factorization envelopes.* Let $\widetilde{\mathscr{L}_c}$ be a one-dimensional extension of $\mathscr{L}_c$, fitting into a short exact sequence

$$0 \longrightarrow \underline{\mathbb{C}}[k] \longrightarrow \widetilde{\mathscr{L}_c} \longrightarrow \mathscr{L}_c \longrightarrow 0, \tag{7.11}$$

where $\underline{\mathbb{C}}[k]$ denotes the constant cosheaf valued in $\mathbb{C}$, shifted by k. We can then run the same construction for $\widetilde{\mathscr{L}_c}$, and obtain a prefactorization algebra again, but this time we get a little extra structure: it's a prefactorization algebra in modules for the algebra $\mathbb{C}[c]$, where $\deg c = -k - 1$.

Example 7.12. Now, let $\mathfrak{g}$ be a Lie algebra and Σ be a Riemann surface, and let $\mathscr{L} \coloneqq \mathfrak{g} \otimes \Omega^{0,\bullet}_\Sigma$, so the differential is $\overline{\partial}$. Fix an invariant bilinear pairing $\langle -, - \rangle$ on $\mathfrak{g}$. Then we can define a one-dimensional central extension of $\mathscr{L}_c$ by

$$[\alpha \otimes X, \beta \otimes Y] \coloneqq \alpha \wedge \beta \otimes [X, Y] + \int_\Sigma (\alpha \wedge \partial\beta) \langle X, Y \rangle c, \tag{7.13}$$

where c is the generator of $\underline{\mathbb{C}}[k]$ (since (7.11) splits as vector spaces). We call $H^*(C_\bullet \widetilde{\mathscr{L}_c})$ the *Kac-Moody prefactorization algebra* over Σ; c will be the level, and $k = -1$. Later on, we'll prove that one can recover the Kac-Moody vertex algebra from this structure, akin to how we recovered $\mathcal{U}(\mathfrak{g})$ in Example 7.9. ◀

There's also a version of this over higher-dimensional complex manifolds, which appear in topologically twisted supersymmetric gauge theories.

There's also an equivariant notion of a prefactorization algebra over a manifold with a G-action. For every $g \in G$, we have an isomorphism $\sigma_g \colon \mathscr{F}(U) \to \mathscr{F}(gU)$, and we'd like for this to be compatible with the $\mathfrak{g}$-action of derivations. This recovers some things we know in physics, e.g. that the Poincaré algebra acts on spaces of observables.

8. Free field theories: 3/13/19

These are Arun's prepared notes on (the first part of) chapter 4 of Costello-Gwilliam, about free theories and their observables. First, a brief outline of the chapter.

(1) The first main idea (§2) is the definition of a free field theory and its prefactorization algebra of quantum observables. If you've been thinking that the thing we called $H^0(\mathrm{Obs}^q)$ should actually be H^0 of something called Obs^q, you'll be pleased by this section.
(2) On $\mathbb{R}$, this recovers the Weyl algebra (§3), and this allows for a description of canonical quantization (§4). I hope, but don't expect, to get all the way to here.
(3) In §5, they consider an extended example: abelian Chern-Simons theory. Interesting as this is, we had to leave it out. So I ask the audience: what do you all think of having another talk on this section? Presumably that would be whoever's slated to speak after Jacques.
(4) In §6, they provide another description of quantization, which allows (in §7) for a definition of the correlation functions of the free theory.
(5) §8 is about translation-invariant PFAs, which allows for some fun physics goodies in §9, such as states and vacua. I think Jacques will be speaking about §§6 through 9.

As you read this section, you'll notice the derived geometry ramps up a little bit. It's not a huge roadblock; if you know what a cochain complex is then things are fine, but they do take the opportunity to reword some ideas in derived geometric language (e.g. "observables as functions on the derived critical locus"), so I'd like to say a few brief words about why derived geometry should pop up in physics anyways.

Strictly speaking, I think this approach isn't necessary for the free theories on the board today, but is introduced here to make it easier to digest, rather than appearing simultaneously with other significant concepts.[9]

Often in some quantum field theory, we have a space X of fields and a functional $S\colon X \to \mathbb{R}$ whose action we'd like to extremize, cutting out some subspace of X. Hopefully this is a finite-dimensional manifold, but not always – if the equations aren't transverse, there can be settings where we get a manifold of the wrong dimension, or not even a manifold at all. One way to solve this is to perturb the equations a bit, but one of the major motivating ideas of derived geometry is to be able to handle non-transversality directly.

8.1. **The free scalar theory.** In this section, corresponding to §4.2, we'll study the prefactorization algebra associated to a free scalar field theory, and briefly discuss the general definition of a free theory.

Let (M, g) be a Riemannian manifold, and consider the free scalar field theory on M. As usual, the action is

$$S(\phi) = \int_M \phi\Delta\phi. \tag{8.1}$$

The ordinary space of solutions to the equations of motion on an open $U \subset M$ is the space of harmonic functions, but Costello and Gwilliam consider the derived space of solutions, which is the cochain complex

$$\mathscr{E}(U) := \left(C^\infty(U) \xrightarrow{\Delta} C^\infty(U)[-1] \right). \tag{8.2}$$

We want to say that the observables on U are the polynomial functions on $\mathscr{E}(U)$. Thus far, we've interpreted this as the symmetric algebra on the dual space. Making sense of that in this setting is a little trickier; Costello and Gwilliam develop the theory of "differentiable vector spaces" with nice analytic and linear-algebraic properties and make sense of $\mathscr{E}(U)$ as a cochain complex of differentiable vector spaces. This allows us to define the space of *homogeneous polynomials of degree* n on $\mathscr{E}(U)$ as

$$P_n(\mathscr{E}(U)) := \mathrm{Hom}_{\mathsf{DVS}}(\mathscr{E}(M) \times \cdots \times \mathscr{E}(M), \mathbb{R})_{S_n}, \tag{8.3}$$

and then $P(\mathscr{E}(U)) := \bigoplus_n P_n(\mathscr{E}(U))$.

Definition 8.4. The *classical observables with support in* U is the symmetric algebra of the cochain complex $\mathscr{E}(U)^\vee$.

Explicitly, since $\mathscr{E}(U)$ is concentrated in degrees 0 and 1, $\mathscr{E}(U)^\vee$ is concentrated in degrees -1 and 0, with two copies of $C^\infty(U)^\vee = \mathcal{D}_c(U)$, the compactly supported distributions on U.

Remark 8.5. We'd like to say that the observables on U are $\mathrm{Sym}^\bullet(\mathscr{E}(U)^\vee)$. This leads to difficulties defining the quantum observables. In an interacting theory, this will be one of the problems that renormalization solves, but in this case, there's a simpler solution.

Let $\mathscr{E}_c^!(U) := \mathscr{E}(U)^\vee \otimes \mathrm{Dens}_M$, which is given by two copies of $C_c^\infty(M)$ in degrees -1 and 0. A compactly supported function defines a distribution, so there's a map $\mathscr{E}_c^!(U) \to \mathscr{E}(U)^\vee$, and it's a homotopy equivalence (this is not immediate). The upshot is that up to a chain homotopy equivalence, we can replace a distributional observable with a "smeared" version, given by a compactly supported smooth function that approximates it. ◀

Therefore the classical observables are

$$\mathrm{Obs}^{c\ell}(U) := \mathrm{Sym}^\bullet(\mathscr{E}_c^!(U)). \tag{8.6}$$

Concretely, $\mathrm{Sym}^n(\mathscr{E}_c^!(U))$ is the subspace of $P_n(\mathscr{E}(U))$ of distributions given by compactly supported smooth functions. Costello-Gwilliam then prove that $\mathrm{Obs}^{c\ell}(U) \to P(\mathscr{E}(U))$ is a chain homotopy equivalence, which is analogous (and whose proof wasn't clear to me), and similarly means that we can replace distributional observables with equivalent, "smeared-out" versions, given by smooth polynomial observables.

Remark 8.7. There is an identification of graded vector spaces $\mathrm{Obs}^{c\ell}(U) \overset{\cong}{\to} \mathrm{PV}_c(C^\infty(U))$. Here $\mathrm{PV}_c(C^\infty(U))$ refers to polynomial vector fields along $T_c C^\infty(U)$, i.e. first-order variations of fields which vanish outside of a compact set. Speaking approximately, this carries the differential to "$- \vee \mathrm{d}S$", where $S(\phi) = (1/2)\int \phi\Delta\phi$. That is, we want to contract with $\mathrm{d}S$. There are convergence issues: $S(\phi)$ might not converge. But a variation

[9] Yeah, I know, technically it depends how fast you're moving relative to the textbook.

by a compactly supported tangent vector is fine, which means $-\vee\,\mathrm{d}S$ makes sense along $\mathrm{PV}_c(C^\infty(U))$, as we're only allowing compactly supported (poly)vectors. ◂

I'll say just a bit about the general definition. Unfortunately, much of the motivation is punted to the second volume, and there are some technical details under the hood, so I don't see the point of giving the precise definition, but enough to get the general idea.

Definition 8.8. Let M be a manifold. A *free field theory* on M is the following data.

(1) A graded vector bundle $E \to M$. We will let $\mathscr{E}$ denote its sheaf of sections and $\mathscr{E}_c$ denote its sheaf of compactly supported sections.
(2) A differential $d\colon E \to E$ of cohomological degree 1, such that $d^2 = 0$, and which makes $\mathscr{E}$ into an elliptic complex.
(3) Let $E^! := E^\vee \otimes \mathrm{Dens}_M$ and $\mathscr{E}^!$ and $\mathscr{E}_c^!$ denote its sheaf of sections (resp. compactly supported sections). There's integration pairing $\mathscr{E}_c(U) \otimes \mathscr{E}^!(U) \to \mathbb{R}$ compatible with differentials.
    The final datum is an isomorphism $E \cong E^![-1]$ respecting d. This induces a pairing $\langle -, - \rangle\colon \mathscr{E}_c(U) \otimes \mathscr{E}_c(U) \to \mathbb{R}$, and we ask that it's antisymmetric.

In this case, $\mathscr{E}(U)$ is thought of as the derived space of solutions to the equations of motion on U, even though we can't (yet?) see what the equations of motion are.

Example 8.9 (Free scalar field theory)**.** For the free scalar field theory on M with mass m, we can take

$$\mathscr{E}(U) := \left(C^\infty(U) \xrightarrow{\Delta + m} C^\infty(U) \right) \tag{8.10}$$

as above. ◂

Definition 8.11. Given a free field theory as above, the prefactorization algebra of *classical observables* is the following data.

- On any open $U \subset M$, $\mathrm{Obs}^{c\ell}(U) := \mathrm{Sym}^\bullet(\mathscr{E}_c^!(U))$.
- Given an inclusion $U \hookrightarrow V$ of opens, we have an extension-by-zero map $\mathscr{E}_c^!(U) \to \mathscr{E}_c^!(V)$, which induces a map $i_V^U\colon \mathrm{Obs}^{c\ell}(U) \to \mathrm{Obs}^{c\ell}(V)$.
- Given disjoint $U_1, U_2 \subset V$, the factorization product of $\alpha_1 \in \mathrm{Obs}^{c\ell}(U_1)$ and $\alpha_2 \in \mathrm{Obs}^{c\ell}(U_2)$ is the product of $i_V^{U_1}\alpha_1$ and $i_V^{U_2}\alpha_2$ in $\mathrm{Obs}^{c\ell}(V)$, which is a symmetric algebra. If ϕ is a field (I think this means a smooth section of E?), the product observable takes the value $\alpha_1(\phi|_{U_1})\alpha_2(\phi|_{U_2})$.
    The generalization to higher-order factorization products looks the same.

Next the prefactorization algebra Obs^q of quantum observables. First, a really succinct definition: remember that last time, we discussed the prefactorization envelope of a sheaf of dg Lie algebras. Let's define a sheaf of dg Lie algebras by

$$\widehat{\mathscr{E}}_c(U) := \mathscr{E}_c(U) \oplus \mathbb{R}\cdot\hbar, \tag{8.12}$$

with Lie bracket

$$[\alpha, \beta] = \hbar\langle\alpha, \beta\rangle. \tag{8.13}$$

More explicitly, $\mathrm{Obs}^q(U)$ is the Chevalley-Eilenberg complex $C_\bullet(\widehat{\mathscr{E}}_c(U))$ of this sheaf of Lie algebras.[10]

Here are a few interesting facts about Obs^q.

(1) In a previous talk, we saw a prefactorization algebra $H^0(\mathrm{Obs}^q)$ (valued in vector spaces) of quantum observables. As the notation suggests, this is indeed by the zeroth cohomology of Obs^q.
(2) As we discussed in Remark 8.7, $\mathrm{Obs}^{c\ell}(U)$ can be identified with a complex of polyvector fields $(\mathrm{PV}_c(C^\infty(U)), -\vee\mathrm{d}S)$. This identification carries over to an identification of $\mathrm{Obs}^q(U)$ with $(\mathrm{PV}_c(C^\infty(U)), -\vee\,\mathrm{d}S + \hbar\,\mathrm{Div})$.
(3) There is an identification $\mathrm{Obs}^q(U) \cong \mathrm{Obs}^{c\ell}(U)[\hbar]$ as graded vector spaces, but the differentials are different. However, modulo $\hbar$, the differentials coincide.

[10] Some of these vector spaces are infinite-dimensional, which means the tensor products appearing in the definition of the Chevalley-Eilenberg complex have to be completed in a certain sense.

8.2. **Quantum mechanics and the Weyl algebra.** One cool thing we can do with this technology is recover the Weyl algebra from the free scalar field on $\mathbb{R}$. More specifically, the claim is that

(1) the cohomology of the prefactorization algebra for the free scalar field on $\mathbb{R}$ (with mass m) is locally constant, hence is equivalent data to an associative algebra A_m; and
(2) for any m, A_m is the Weyl algebra $\mathbb{C}\langle p, q, \hbar\rangle/([p,q] = \hbar, [p,\hbar] = [q,\hbar] = 0)$.

Proof of (1). $\mathrm{Obs}^q(U)$ is the Chevalley-Eilenberg cochain complex on the Heisenberg (dg) Lie algebra$\mathcal{H}(U) = \left(C_c^\infty(U) \xrightarrow{\Delta+m^2} C_c^\infty(U)[-1]\right)$, so $\mathrm{Obs}^q(U) = \mathrm{Sym}^\bullet(\mathcal{H}(U)[1])$. Therefore we can filter it as

$$F^{\le k}\mathrm{Obs}^q(U) \coloneqq \mathrm{Sym}^{\le k}(\mathcal{H}(U)[1]). \tag{8.14}$$

The associated graded is $\mathrm{Obs}^{c\ell}(U)[\hbar]$. So if we can show $H^*(\mathrm{Obs}^{c\ell})$ is locally constant, we'll be done: associated to an inclusion $U \hookrightarrow V$ we'll have a map of spectral sequences for the filtration (8.14). That the cohomology of the classical observables is locally constant implies the map on the E_0-page is an isomorphism, which implies the entire map is an isomorphism, and in particular the map $H^*(\mathrm{Obs}^q(U)) \to H^*(\mathrm{Obs}^q(V))$ is an isomorphism.

Then one must actually calculate $H^*(\mathrm{Obs}^{c\ell}((a,b))$. The way they go about this is to explicitly construct a chain homotopy from $\Delta + m^2 \colon C_c^\infty((-\varepsilon,\varepsilon)) \to C_c^\infty((-\varepsilon,\varepsilon))$ to $\mathbb{R}^2 \to 0$, with the $\mathbb{R}^2$ spanned by (for $m \ne 0$)

$$\phi_q(x) \coloneqq \frac{e^{mx} + e^{-mx}}{2} \qquad\qquad \phi_p(x) \coloneqq \frac{e^{mx} - e^{-mx}}{2m}, \tag{8.15}$$

ultimately because these are a basis for the kernel of $\partial_x^2 + m^2$. Thus $\mathrm{Obs}^{c\ell}((a,b))$ is the symmetric algebra of this, which is a polynomial algebra in two variables. The key observation, however, is that this is independent of ε, so the maps induced by inclusion are isomorphisms, and therefore $H^*(\mathrm{Obs}^{c\ell})$ is locally constant. ⊠

Proof sketch of (2). Let $I_t \coloneqq (t - 1/2, t + 1/2)$. We'll build position and momentum observables out of ϕ_q and ϕ_p. First, let $f \in C_c^\infty(I_0)$ be an even function with $\int_{-\infty}^\infty f(x)\phi_q(x)\,\mathrm{d}x = 1$, which implies $\int_{-\infty}^\infty f(x)\phi_p(x)\,\mathrm{d}x = 0$, and let $f_t(x) \coloneqq f(x-t) \in C_c^\infty(I_t)$. Define observables Q_t and P_t by

$$\begin{aligned} Q_t &= f_t \\ P_t &= -f_t', \end{aligned} \tag{8.16}$$

These have cohomological degree zero. We can think of them as functionals on $C^\infty(I_t) \oplus C^\infty(I_t)[-1]$ which measure average position and momentum of a field near t.

Let p denote the cohomology class of P_0, and q denote that of Q_0. Since p and q generate $H^*(\mathrm{Obs}^{c\ell}(\mathbb{R}))$, they and $\hbar$ generate $H^*(\mathrm{Obs}^{c\ell}(\mathbb{R})[\hbar])$, which is the associated graded of $H^*(\mathrm{Obs}^q(\mathbb{R}))$, and therefore p, q, and $\hbar$ generate $H^*(\mathrm{Obs}^q(\mathbb{R}))$.

Now the commutation relations. That $[p,\hbar] = [q,\hbar] = 0$ is relatively straightforward, so let's focus on showing $[p,q] = \hbar$, which is more involved. This more or less involves a direct computation of $qp - pq$, but first doing some work to simplify it.

The first step is to replace P_t and Q_t with observables $\mathcal{P}_t$ and $\mathcal{Q}_t$ which agree with P_t and Q_t when $t = 0$, and whose cohomology classes are independent of t. First, suppose $\widetilde{Q}_t$ is the observable in degree -1 given by $f_t \in C_c^\infty(I_t)[1]$; then, for any function $a(t)$ with $\ddot{a}(t) = m^2 a(t)$,

$$\frac{\partial}{\partial t}(a(t)P_t - \dot{a}(t)Q_t) = a(t)\frac{\partial^2}{\partial x^2} f_t(x) - m^2 a(t) f_t(x) = -\mathrm{d}(a(t)\widetilde{Q}_t), \tag{8.17}$$

where $\mathrm{d} = \Delta + m^2$ is the differential for observables. (There are a few more steps that I skipped.) In particular, consider

$$\begin{aligned} \mathcal{P}_t &\coloneqq \phi_q(t)P_t - \dot{\phi}_p(t)Q_t \\ \mathcal{Q}_t &\coloneqq \phi_p(t)P_t - \dot{\phi}_q(t)Q_t. \end{aligned} \tag{8.18}$$

Because $\phi_p, \phi_q \in \ker(-\partial_x^2 + m^2)$, $\frac{\partial}{\partial t}\mathcal{P}_t$ and $\frac{\partial}{\partial t}\mathcal{Q}_t$ are exact, and therefore the cohomology classes of these observables do not depend on t. As promised, $\mathcal{P}_0 = P_0$ and $\mathcal{Q}_0 = Q_0$. We also have that if

$$h_{s,t} \coloneqq \int_s^t \phi_q(u)\widetilde{Q}_u(x)\,\mathrm{d}u, \tag{8.19}$$

then $\mathrm{d}h_{s,t} = \mathcal{P}_s - \mathcal{P}_t$ (again, this is the differential on observables, where we regard $h_{s,t}$ in degree -1).

Now let's recall how the associative algebra structure $\star$ on $H^0(\mathrm{Obs}^q(\mathbb{R}))$ is defined. If $|t| > 1$, then $\mathcal{P}_t$ has disjoint support from Q_0, so from the prefactorization structure map

$$\mathrm{Obs}^q(I_0) \otimes \mathrm{Obs}^q(I_t) \longrightarrow \mathrm{Obs}^q(\mathbb{R}), \tag{8.20}$$

we get an observable $Q_0 \cdot \mathcal{P}_t \in \mathrm{Obs}^q(\mathbb{R})$. Then by definition,

$$[Q_0] \star [P_0] \coloneqq [Q_0 \cdot \mathcal{P}_t] \text{ when } t > 1. \tag{8.21}$$

Similarly, $[P_0] \star [Q_0] \coloneqq [Q_0 \cdot \mathcal{P}_{-t}]$ as long as $t > 1$. We have reduced to showing that

$$[Q_0 \cdot \mathcal{P}_t] - [Q_0 \cdot \mathcal{P}_{-t}] = \hbar, \tag{8.22}$$

which we will accomplish by writing down an observable whose differential is $Q_0 \cdot (\mathcal{P}_t - \mathcal{P}_{-t}) - \hbar$. Specifically, let

$$S \coloneqq f(x) h_{-t,t}(y) \in C_c^\infty(\mathbb{R}) \otimes C_c^\infty(\mathbb{R})[1], \tag{8.23}$$

so we regard f in degree 1 and $h_{-t,t}$ in degree -1. To compute $\mathrm{d}S$, remember that there's the $\Delta + m^2$ term and the bracket term with an $\hbar$. We get that

$$\mathrm{d}S = f(x)(-\partial_x^2 + m^2) h_{-t,t}(y) + \hbar \int_{-\infty}^{\infty} h_{-t,t}(x) f(x)\,\mathrm{d}x \tag{8.24a}$$

$$= Q_0 \cdot (\mathcal{P}_t - \mathcal{P}_{-t}) + \hbar \int_{-\infty}^{\infty} h_{-t,t}(x) f(x)\,\mathrm{d}x. \tag{8.24b}$$

We're almost there – now we need to show this integral is equal to 1. We can rewrite it as

$$\int_{-t}^{t} \int_{-\infty}^{\infty} f(x) f(x-u) \phi_q(u)\,\mathrm{d}x\,\mathrm{d}u. \tag{8.25}$$

Since f is compactly supported, this is independent of t for $t \gg 0$, so we can send $t \to \infty$. Since f is even, $f(x-u) = f(u-x)$. Thus under the change of coordinates $u \mapsto u - x$, the integrand is $f(x)f(u)\phi_q(u+x)$.

Recall that $\phi_q(u+x) = (1/2)(e^{m(x+u)} + e^{-x(m+u)})$, and that

$$\int f(x) e^{mx}\,\mathrm{d}x = \int f(x) e^{-mx}\,\mathrm{d}x = 1. \tag{8.26}$$

Therefore the inner integral is

$$\int_{-\infty}^{\infty} \phi_q(u+x) f(u)\,\mathrm{d}u = \phi_q(x), \tag{8.27}$$

so the outer integral is

$$\int_{-\infty}^{\infty} \phi_q(x) f(x)\,\mathrm{d}x = 1. \tag{8.28}$$

⊠

One more fun fact that I didn't discuss: one can also construct a Hamiltonian operator H on $\mathrm{Obs}^q(\mathbb{R})$, and the Hamiltonian knows the mass of the theory. This Hamiltonian is defined to be the derivation on A_m induced from the derivation $-\frac{\partial}{\partial x}$ on Obs^q, and they calculate that it's

$$H(a) = \frac{1}{2\hbar}[p^2 - m^2 q^2, a]. \tag{8.29}$$

8.3. **Canonical quantization and state spaces.** Now, I'll briefly talk about canonical quantization: if we formulate a free theory on a product $N \times \mathbb{R}$, we can push forward the PFA of observables to $\mathbb{R}$. There, we can relate it to an infinite tensor product of PFAs for quantum mechanics.

Let $\pi\colon N \times \mathbb{R} \to \mathbb{R}$ be projection. Then we can define the pushforward prefactorization algebra $\pi_*\mathrm{Obs}^q$ by $\pi_*\mathrm{Obs}^q(U) \coloneqq \mathrm{Obs}^q(\pi^{-1}(U))$, just as for sheaves. Canonical quantization relates this to the associative algebra

$$A_N \coloneqq \bigotimes_{\substack{\mathbb{R}[\hbar] \\ \lambda \in \mathrm{Spec}(\Delta + m^2)}} A_{\sqrt{\lambda}}, \tag{8.30}$$

where $A_{\sqrt{\lambda}}$ is the Weyl algebra for quantum mechanics with mass $\sqrt{\lambda}$. (This doesn't depend on λ, but the Hamiltonian does.)

This infinite tensor product is defined to be the colimit of its finite components: if $A_N^{(n)}$ denotes the tensor product over the first n λ, then $A_N := \varinjlim_n A_N^{(n)}$.

Anyways, Costello-Gwilliam prove the following theorem.

Theorem 8.31. *There is a dense sub-prefactorization algebra B of $\pi_*\mathrm{Obs}^q$ which is locally constant, and such that $H^*(B)(\mathbb{R}) \cong A_N$.*

That is: $H^*(B)$ is equivalent data to an associative algebra, as we've seen, and that associative algebra is A_N. Here "dense" is in the sense of a dense subspace of a topological vector space.

What's going on here physics-wise? Taking the pushforward is the PFA analogue of compactifying along N to obtain a theory on $\mathbb{R}$. The algebra $A_{\sqrt{\lambda}}$ represents the algebra of observables for a quantum-mechanical particle moving on N with mass m and energy λ, and the idea is that a general observable splits as a combination of these observables at different energy levels. Because this is a free theory, different observables at different energy levels don't interact.

The identification in the above theorem can be upgraded: the Hamiltonians on each $A_{\sqrt{\lambda}}$ define a Hamiltonian on A_N, and there is a Hamiltonian on $\pi_*\mathrm{Obs}^q$ which is the derivation coming from infinitesimal translation along $\mathbb{R}$. The isomorphism above intertwines these Hamiltonians.

9. Two kinds of quantization of free theories: 3/27/19

10. Abelian Chern-Simons theory

> *"If you didn't know what quantum field theory is, this definition might be the first one you'd think of."*

Today, Charlie spoke about abelian Chern-Simons theory from the Costello-Gwilliam perspective. We consider the gauge group U_1, but in some ways it seems to actually be $\mathbb{R}^+$.

Let's recall how one might typically think of abelian Chern-Simons theory. Let M be a closed 3-manifold and $L \to M$ be a trivial line bundle. A connection on L is given as $\mathrm{d} + \alpha\colon \Gamma(L) \to \Gamma(L) \otimes \Omega^1(M)$, where $\alpha \in \Omega^1(M)$, so the space of connections is $\Omega^1(M)$. An $f \in \Omega^0(M)$ defines a gauge transformation as follows:

$$e^{-f}(\mathrm{d}+\alpha)(e^f s) = (\mathrm{d}+\alpha+\mathrm{d}f)s. \tag{10.1}$$

The action is

$$S(\alpha) = \frac{1}{2}\int_M \mathrm{d}\alpha \wedge \alpha, \tag{10.2}$$

so that

$$\langle \mathrm{d}S(\alpha), \beta\rangle = \int_M \mathrm{d}\alpha \wedge \beta. \tag{10.3}$$

Typically we take the space of fields to be connections modulo gauge transformations, but in this derived approach to Chern-Simons theory this is represented by the complex

$$\mathcal{F} := \left(\Omega^0(M)[1] \xrightarrow{\mathrm{d}} \Omega^1(M)\right). \tag{10.4}$$

Then the action is a map $\mathrm{d}S\colon \mathcal{F} \to \mathcal{F}^*$, where $\mathcal{F}^*$ is the dual complex

$$\mathcal{F}^* := \left(\Omega^1(M)^* \xrightarrow{\mathrm{d}^*} \Omega^0(M)^*[-1]\right) \simeq \left(\Omega^2(M) \xrightarrow{\mathrm{d}} \Omega^3(M)[-1]\right). \tag{10.5}$$

Ordinarily we would define the critical locus to be the kernel of the action, but in this case we want the homotopy kernel, which glues together $\mathcal{F}$ and $\mathcal{F}^*$ to produce

$$\mathrm{hKer}(\mathrm{d}S) = \left(\Omega^1(M)[1] \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \Omega^2(M)[-1] \xrightarrow{\mathrm{d}} \Omega^3(M)[-2]\right) = \Omega^*(M)[1]. \tag{10.6}$$

This leads Costello-Gwilliam to make the following definition.

Definition 10.7. Chern-Simons theory with gauge group $\mathbb{R}^+$ is the free field theory specified by the following data.

- The cochain complex of fields on an open set U is $\Omega^*(U)[1]$ (this glues to a sheaf of cochain complexes). As required, this is an elliptic complex.
- The integration pairing is $\langle\alpha,\beta\rangle \coloneqq (-1)^{|\beta|}\int_M \alpha\wedge\beta$.

The classical observables on an open $U\subset M$ is the dg commutative algebra $\mathrm{Sym}^\bullet(\Omega_c^*(U)[2])$. Taking cohomology commutes with taking the symmetric algebra, so

$$H^*\mathrm{Obs}^{c\ell}(U)\cong \mathrm{Sym}^*(H_c^*(U)[2]). \tag{10.8}$$

This is already something interesting: the cohomology of the classical observables is polynomials on compactly supported cohomology.

Let d_1 denote the classical differential on $\mathrm{Obs}^{c\ell}(U)$. The quantum observables, as in a general free field theory, are the Heisenberg Lie algebra (well, sheaf of dg Lie algebras), given explicitly by the Chevalley-Eilenberg complex. We take the underlying graded vector space to be $\mathrm{Obs}^{c\ell}(U)[\hbar]$, with a differential $d_{CW}=d_1+d_2$, where $d_2\colon \mathrm{Sym}^k\to\mathrm{Sym}^{k-2}$ is new.

We can compute $H^*(\mathrm{Obs}^q(U))$ via a spectral sequence argument: there's a filtration on $\mathrm{Sym}^\bullet(\Omega_c^*(U)[2])[\hbar]$ whose k^{th} piece is $\mathrm{Sym}^{\le k}(\Omega_c^*(U)[2])$. Hence we get an induced spectral sequence, which is kind of like perturbation theory in $\hbar$!

1. The E_1-page is $H^*(\mathrm{Obs}^{c\ell}(U))$ with zero differentials.
2. Hence the E_2-page is also $H^*(\mathrm{Obs}^{c\ell}(U))$, but now there is a differential $\widetilde{d}_2$.
3. The E_3-page is the homology of $\widetilde{d}_2$. After this (TODO: I may have written this down wrong) the spectral sequence collapses, and all further pages are the same.

Lemma 10.9. *Suppose M is a closed, oriented 3-manifold. Let ν be a generator of $H^3(M)$ and $\alpha_1,\dots,\alpha_{b^1}$ be a basis of $H^1(M)$ (here $b^1\coloneqq\dim H^1(M)$). Then $H^*(\mathrm{Obs}^q(M))[\hbar^{-1}]\cong\mathbb{R}[\hbar][1-b^1]$, and it's generated by $\nu\alpha_1\cdots\alpha_{b^1}$.*

The proof uses the spectral sequence, though apparently it's a little annoying.

Remark 10.10. Costello-Gwilliam used H^2 (and b^2) instead of H^1 (and b^1), but this doesn't make sense degree-wise. Since M is closed and oriented, using H^1 means we get the same stuff but in the correct grading. ◀

Now we consider the case of $M=\Sigma\times\mathbb{R}$, where Σ is a closed, connected, oriented surface. Let $\pi\colon\Sigma\times\mathbb{R}\to\mathbb{R}$ be projection; as we discussed last time, we can study the Hilbert space of states on Σ.

Let $\{\alpha_i,\beta_i\}$ be a symplectic basis for $H^1(\Sigma)$ – while the homology of a closed, oriented 3-manifold was shifted symplectic, this is just symplectic. Let $\mu\in H^0(\Sigma)$ be a generator and $\nu\in H^2(\Sigma)$ be its dual.

Proposition 10.11. *$H^*\pi_*\mathrm{Obs}^q$ is the locally constant prefactorization algebra associated to the Weyl algebra A_Σ generated by α_i,β_i,μ, and ν, subject to the relations $[\alpha_i,\beta_j]=\delta_{ij}$, $[\mu,\nu]=\hbar$, and the remaining brackets vanish.*

Ultimately the reason this is true is that

$$\mathrm{Obs}^q(\Sigma\times\mathbb{R}) = C_*(\Omega_c^*(\mathbb{R})\otimes\Omega^*(\Sigma)[1]\oplus\hbar\mathbb{R})\simeq C_*(\Omega_c^*(\mathbb{R})\otimes H^*(\Sigma)[1]\oplus\hbar\mathbb{R}). \tag{10.12}$$

The next case is when $\overline{M}$ is a compact oriented 3-manifold with boundary Σ. Let $M=\overline{M}\setminus\Sigma$. We can consider a map $p\colon M\to(0,1]$ such that $p^{-1}((0,1))\cong\Sigma\times\mathbb{R}$, which follows from the tubular neighborhood theorem (then collapse everything else onto 1).

Pushing forward by p, we will obtain a PFA on $(0,1)$.

Proposition 10.13. *The PFA on $(0,1)$ $H^*(p_*\mathrm{Obs}^q)$ is the prefactorization algebra associated to A_Σ from Proposition 10.11. We also have $H^*(p_*\mathrm{Obs}^q((a,1]))\cong H^*(\mathrm{Obs}^q(M))$, and the structure maps of $H^*(p_*(\mathrm{Obs}^q((0,1]))$ make $H^*\mathrm{Obs}^q(M)$ into an A_Σ-module.*

This is an instance of something we expected: (locally constant) PFAs on $\mathbb{R}$ give us algebras, and on clopen connected intervals give us modules.

We can also insert operators into this theory. Let $K\subset M$ be a link; then, we get an operator $O_K(A)\coloneqq\int_K A$, where A is a connection… TODO: I didn't follow what happened next, but it involves trying to realize this operatr in $\mathrm{Obs}^q(U)$, where $K\subset U$.

11. Holomorphic translation-invariant PFAs and vertex algebras: 4/10/19

Today, Rok spoke about the first half of chapter 5, focusing on the formalism. Next week there will be some examples. Our goal today is to discuss how holomorphic translation-invariant prefactorization algebras give rise to vertex algebras, but we havem't done a lot with translation-invariant prefactorization algebras (or indeed equivariant prefactorization algebras in general), so let's begin with those.

Let a Lie group G act smoothly on a manifold M, and $\mathscr{F}$ be a prefactorization algebra on M. To say that $\mathscr{F}$ is *G-equivariant* is additional data.

- For every open $U \subset M$, we want a G-action $\tau_g\colon \mathscr{F}(U) \xrightarrow{\cong} \mathscr{F}(g(U))$, which is compatible with the factorization product and smooth in the variable g, and of course we want $\tau_g \circ \tau_h = \tau_{gh}$.
- There is additional data of a compatible action of $\mathfrak{g}$ on $\mathscr{F}(U)$ by derivations. One has to say precisely what derivations are in general, but we have sums and products, and it really is the same notion. "Compatibility" will be explained below, in (11.4).

Given this, we can make the following definition.

Definition 11.1. Let $V \subset M$ be open and $U_1, \dots, U_n$ be open subsets of V. Then define $\mathrm{Disj}_G(U_1, \dots, U_n \mid V)$ to be the sort of tuples $(g_1, \dots, g_n) \in G^n$ such that each g_iU_i is an open subset of V, and the g_iU_i are pairwise disjoint.

The fact that τ is compatible with the factorization product m means that for any $(g_1, \dots, g_n) \in G^n$, we obtain a map

$$m_{(g_1,\dots,g_n)}\colon \mathscr{F}(U_1) \otimes \dots \otimes \mathscr{F}(U_n) \xrightarrow[\sim]{\tau} \mathscr{F}(g_1U_1) \otimes \dots \otimes \mathscr{F}(g_nU_n) \xrightarrow{m} \mathscr{F}(V). \tag{11.2}$$

This amounts to an assignment

$$m\colon \mathscr{F}(U_1) \otimes \dots \otimes \mathscr{F}(U_n) \longrightarrow C^\infty(\mathrm{Disj}_G(U_1, \dots, U_n \mid V), \mathscr{F}(V)). \tag{11.3}$$

There is a way to make sense of this as a coalgebra structure over a colored cooperad, which is cool, but if you don't know what that is, it's not a problem.

Now, as promised, the compatibility of the Lie algebra action. This is the condition that, for all $X \in \mathfrak{g}$, if $\frac{\partial}{\partial X}$ denotes the action of X on $\mathscr{F}(U)$, $(g_1, \dots, g_n) \in \mathrm{Disj}_G(U_1, \dots, U_n \mid V)$, and $\alpha_i \in \mathscr{F}(U_i)$, then

$$\frac{\partial}{\partial X_i} m_{(g_1,\dots,g_n)}(\alpha_1, \dots, \alpha_n) = m_{(g_1,\dots,g_n)}\biggl(\alpha_1, \dots, \frac{\partial}{\partial X}\alpha_i, \dots, \alpha_n\biggr). \tag{11.4}$$

We're going to use this later. In physics-speak, this is has something to do with the stress-energy tensor, and making the BRST complex exact.

For the rest of this talk, $M = \mathbb{C}$. Let $\mathbb{C}$ act on itself by translation.

Definition 11.5. A *holomorphic translation-invariant prefactorization algebra* is a $\mathbb{C}$-equivariant prefactorization algebra on $\mathbb{C}$ together with a degree-(-1) derivation $\eta\colon \mathscr{F} \to \mathscr{F}$ such that $\mathrm{d}\eta = \frac{\partial}{\partial \bar z}$ in the dg Lie algebra of derivations on $\mathscr{F}$. Moreover, one requires that $[\eta, \frac{\partial}{\partial \bar z}] = 0$.

Because of translation-invariance, it suffices to restrict our attention to the disc $U = D_r(z)$ of radius r around some point $z \in \mathbb{C}$. Let $\mathscr{F}(r) := \mathscr{F}(D_r(0))$. Let

$$\mathrm{Disk}(r_1, \dots, r_n \mid s) := \mathrm{Disj}_{\mathbb{C}}(D_{r_1}(0), \dots, D_{r_n}(0) \mid D_s(0)). \tag{11.6}$$

As we shrink s, we'll see the configuration space of n points in a disc. Anyways, (11.3) specializes to a map

$$\mathscr{F}(r_1) \otimes \dots \otimes \mathscr{F}(r_n) \longrightarrow C^\infty(\mathrm{Disk}(r_1, \dots, r_n \mid s), \mathscr{F}(s)). \tag{11.7}$$

If X is a complex manifold, let $\Omega^{0,\bullet}(X, V) := \Omega^{0,\bullet}(X) \otimes_{C^\infty(X)} C^\infty(X, V)$. You can think of this, in the derived way, as holomorphic sections. The map (11.7) factors as

$$\mathscr{F}(r_1) \otimes \dots \otimes \mathscr{F}(r_n) \longrightarrow \Omega^{0,\bullet}(\mathrm{Disk}(r_1, \dots, r_n \mid s), \mathscr{F}(s)) \longrightarrow C^\infty(\mathrm{Disk}(r_1, \dots, r_n \mid s), \mathscr{F}(s)). \tag{11.8}$$

We also let $\mathrm{Hol}(X, V) := \ker(\overline{\partial}\colon C^\infty(X, V) \to \Omega^{0,1}(X, V))$.

Now we relate this to vertex algebras. We briefly recall the definition, since we discussed it in more detail last semester.

Definition 11.9. A *vertex algebra* is a vector space V (thought of as a space of states), a *vacuum* $|0\rangle \in V \setminus 0$, a *translation* operator $T\colon V \to V$, and a map $Y(-, z)\colon V \to \mathrm{End}(V)[[z^{\pm 1}]]$ satisfying some axioms:

- The vaccum axiom: $Y(|0\rangle, z) = \mathrm{id}_v$ and $Y(v,z)|0\rangle = v + zV[[z]]$.
- The translation axiom: $T|0\rangle = 0$ and

$$[T, Y(v,z)] = \frac{\partial}{\partial z} Y(v,z). \tag{11.10}$$

- For all $v_1, v_2 \in V$, there's some $N > 0$ such that $(z-w)^N[Y(v_1,z), Y(v_2,w)] = 0$ in $\mathrm{End}(V)[[z^{\pm 1}, w^{\pm 1}]$.

Given a U(1)-equivariant holomorphic translation-invariant prefactorization algebra on $\mathbb{C}$, it's also a $\mathrm{U}(1) \ltimes \mathbb{C}$-equivariant prefactorization algebra on $\mathbb{C}$: $\mathrm{U}(1) \ltimes \mathbb{C}$ is the isometries of $\mathbb{C}$. The Lie algebra is $\mathbb{C}\{\partial_z, \partial_{\overline{z}}, \partial_\theta\}$, where ∂_θ is the U(1)-generator, and the compatibility of this Lie algebra action extends to a dg Lie algebra action of $\mathbb{C}\{\partial_z, \partial_{\overline{z}}, \partial_\theta, \eta\}$, where $\mathrm{d}\eta = \overline{\partial}_{\overline{z}}$, $[\partial_\theta, \eta] = -\eta$, and $|\eta| = -1$.

Theorem 11.11. *Let $\mathscr{F}$ be a holomorphic* U(1)*-equivariant translation-invariant prefactorization algebra on* $\mathbb{C}$*, and let $\mathscr{F}_k(r) \subset \mathscr{F}(r)$ be the weight-k eigenspace for the* U(1)*-action. Assume that*

1. *for all $r' < r$, the map $\mathscr{F}(r') \to \mathscr{F}(r)$ is an isomorphism,*
2. *for $k \gg 0$, $H^*(\mathscr{F}_k(r)) = 0$, and*
3. *$H^*(\mathscr{F}_k(r))$ isn't too big (in some precise, but pretty technical, sense).*

In this case, let $V_k := H^*(\mathscr{F}_k(r))$ and $V := \bigoplus_{k\in\mathbb{Z}} V_k$. Then V inherits a vertex algebra structure from $\mathscr{F}$.

We explicate some of the structure of the vertex algebra in terms of some maps we can write down on V. For example, consider

$$V \longrightarrow H^*(\mathscr{F}(r)) \longrightarrow \prod_k V_k =: \overline{V}, \tag{11.12}$$

induced by the inclusion $\mathscr{F}_k(r) \hookrightarrow \mathscr{F}(r)$ followed by the projection $\mathscr{F}(r) \twoheadrightarrow \mathscr{F}_k(r)$. We also have another map $V \to \overline{V}$ defined by

$$V \longrightarrow \varprojlim_{r\to 0} H^*(\mathscr{F}(r)) \longrightarrow H^*(\mathscr{F}(r)) \longrightarrow \overline{V}. \tag{11.13}$$

We also get a map associated to any configuration of k points in the plane, which I wasn't able to write down in time. But letting $k = 2$, we do see some of the structure maps.

12. Beta-gamma systems: 4/17/19

Today, Ivan spoke, continuing the discussion of holomorphic field theories and vertex algebras. The goal of today's talk is to compute the prefactorization algebra of a β-γ system, which is a holomorphically translation-invariant prefactorization algebra over $\mathbb{C}$ (valued in differentiable cochain complexes). Using Theorem 11.11, we get a vertex algebra from this data, and we'll see that this is isomorphic to the usual vertex algebra of a β-γ system.

The β-γ system is a theory on an arbitrary Riemann surface, but we'll work over $\mathbb{C}$. The fields on an open $U \subset \mathbb{C}$ are $\Omega^{0,0}(U) \oplus \Omega^{0,1}(U)$; the $(0,0)$-forms are called *γ-fields* and the $(0,1)$-forms are called the *β-fields*. The action is

$$S(\gamma, \beta) = \int_{\mathbb{C}} \beta \wedge \overline{\partial}\gamma. \tag{12.1}$$

so the equations of motion impose $\overline{\partial}\beta = 0$ and $\overline{\partial}\gamma = 0$, i.e. β and γ are holomorphic.

This tells us what the derived space of fields on U should be:

$$\mathscr{E}(U) := \left(\Omega^{0,\bullet}(U) \oplus \Omega^{1,\bullet}(U), \overline{\partial}\right), \tag{12.2}$$

where $\bullet = 0, 1$. The pairing $\langle -,- \rangle \colon \mathscr{E}_c \otimes \mathscr{E}_c \to \mathbb{C}$ is the only thing it can be:

$$\langle \gamma_0 + \beta_0, \gamma_1 + \beta_1 \rangle = \int_{\mathbb{C}} \gamma_0 \wedge \beta_1 + \beta_0 \wedge \gamma_1. \tag{12.3}$$

The quantum observables on U are the Chevalley-Eilenberg cochains on the Heisenberg dg Lie algebra

$$\mathcal{H}(U) = \mathscr{E}_c(U) \oplus \mathbb{C} \cdot \hbar[-1], \tag{12.4}$$

with bracket $[u,v] := \hbar\langle u, v\rangle$ for $u, v \in \mathscr{E}_c(U)$. Thus

$$\mathrm{Obs}^q(U) = C_*(\mathcal{H}(U)) = (\mathrm{Sym}^\bullet(\mathscr{E}_c(U)[1])[\hbar], \overline{\partial} + d_{[-,-]}). \tag{12.5}$$

The β-γ vertex algebra can be recovered from the following data:[11] $W = \mathbb{C}[a_n, a_m^*]$ for $n < 0$ and $m \le 0$, with $|0\rangle = 1 \in W$. The translation operator is $T(a_i) = -ia_{i-1}$ and $T(a_i^*) = -(i-1)a_{i-1}^*$, and

$$Y(a_{-1}, z) = \sum_{n<0} a_n z^{-1-n} - \sum_{n\ge 0} \frac{\partial z^{-1-n}}{\partial a_{-n}^*} \tag{12.6a}$$

$$Y(a_0^*, z) = \sum_{n<0} a_n^* z^{-n} - \sum_{n>0} \frac{\partial z^{-n}}{\partial a_{-n}}. \tag{12.6b}$$

To apply Theorem 11.11, we want Obs^q to be holomorphically translation-invariant. That it's translation-invariant is clear. To check that it's holomorphically so, let $\eta \in \mathrm{Der}(\mathrm{Obs}^q)$ be of degree -1, such that $\mathrm{d}\eta = \frac{\partial}{\partial \overline{z}}$ (recall that $\mathrm{d}\eta \coloneqq [\overline{\partial} + d_{[-,-]}, \eta)$ and $[\eta, \frac{\partial}{\partial \overline{z}}] = 0$. Then we compute

$$(\overline{\partial} + d_{[-,-]})(\eta)(\gamma_0 + \beta_0)(\gamma_1 + \beta_1) = -(\overline{\partial} + d_{[-,-]})(\gamma_0 + \beta_0)(\eta(\gamma_1 + \beta_1)) \tag{12.7a}$$

$$= -(\overline{\partial}(\gamma_0 + \beta_0))(\eta(\gamma_1 + \beta_1)) + (\gamma_0 + \beta_0)\frac{\partial}{\partial \overline{z}}(\gamma_1 + \beta_1) \tag{12.7b}$$

and

$$\eta(\overline{\partial} + d_{[-,-]})(\gamma_0 + \beta_0)(\gamma_1 + \beta_1) = \eta((\overline{\partial}(\gamma_0 + \beta_1))(\gamma_1 + \beta_1) + \hbar\langle \gamma_0 + \beta_0, \gamma_1 + \beta_1\rangle) \tag{12.8a}$$

$$= \left(\frac{\partial}{\partial \overline{z}}(\gamma_0 + \beta_0)\right)(\gamma_1 + \beta_1) + \overline{\partial}(\gamma_0 + \beta_0)\eta(\gamma_1 + \beta_1), \tag{12.8b}$$

so the terms like up, and this is indeed holomorphic.

Now we want to check the cohomological conditions in Theorem 11.11. But because we have a Green's function G for $\overline{\partial}$, we have an isomorphism of cochain complexes from the classical to quantum observables: the map $\alpha \mapsto \exp(\hbar \overline{\partial}_G)\alpha$ identifies the cochain complexes $(\mathrm{Obs}^{c\ell}(U)[\hbar], \overline{\partial})$ and $(\mathrm{Obs}^q(U), \overline{\partial} + d_{[-,-]})$. Therefore we can work with $H^*(\mathrm{Obs}^{c\ell}(U))$.

Let $D(x, r)$ denote the disc of radius r around $x \in \mathbb{C}$. For $n \ge 0$, we have an observable $c_n(x)\colon \Omega^{0,0}(D(x, r)) \to \mathbb{C}$ given by

$$c_n(x)(\gamma) \coloneqq \frac{1}{n!}\gamma^{(n)}(x). \tag{12.9}$$

For $n > 0$, we have an observable $\beta_n(x)\colon \Omega^{1,0}(D(x, r)) \to \mathbb{C}$ with the formula

$$b_n(x)(\beta(z)\,\mathrm{d}z) = \frac{1}{(n-1)!}\beta^{(n-1)}(x). \tag{12.10}$$

These define observables in $H^0(\mathrm{Obs}^{c\ell}(D(0, r))$.

Let $c_n \coloneqq c_n(0)$ and $b_n \coloneqq b_n(0)$, $\boldsymbol{k}$ denote a multi-index $(k_1, \dots, k_n)$, and $c_{\boldsymbol{k}}(x) \coloneqq c_{k_1}(x) \cdots c_{k_n}(x)$ and similarly for $b_{\boldsymbol{k}}$.

Theorem 12.11. *If $* > 0$, then $H^*(\mathrm{Obs}_k^{c\ell}(D(0, r))) = 0$. For $* = 0$, $H^0(\mathrm{Obs}_{-k}^{c\ell}(D(0, r)))$ has for a basis $\{b_{\boldsymbol{k}} c_{\boldsymbol{\ell}}\}_{|\boldsymbol{k}|+|\boldsymbol{\ell}|=k}$.*

Here $|\boldsymbol{k}| \coloneqq k_1 + \cdots + k_n$. Here, "basis" means every element is a finite sum of basis elements.[12]

Anyways, this makes it relatively explicit that the conditions in Theorem 11.11 hold, and therefore

$$V \coloneqq \bigoplus_{k\in\mathbb{Z}} H^0(\mathrm{Obs}_k^q(D(0, r))) \tag{12.12}$$

is a vertex algebra. If we rewrite $V_\hbar = \mathbb{C}[b_n, c_n]$, we'll identify this with W from above, which will send $T \mapsto \frac{\partial}{\partial \overline{z}}$. Specifically, we'll get an isomorphism of commutative algebras $V_\hbar \cong W$ sending $b_n \mapsto a_n$ and $c_n \mapsto a_n^*$.

[11]This fact is a nontrivial reconstruction theorem.

[12]The theorem in Costello-Gwilliam uses the language of differentiable vector spaces, which are certain sheaves on the site of smooth manifolds. So that's a slightly "smeared-out" version of this result.

Next, we need to compute $Y(b,z)(\beta)$ in $V_\hbar$. Again, we can multiply by $e^{\hbar\partial_G}$ to identify the cochain complexes of classical and quantum observables. Suppose U and V are disjoint opens in $D(0,r)$, $\alpha \in \mathrm{Obs}^{c\ell}(U)[\hbar]$ and $\beta \in \mathrm{Obs}^{c\ell}(V)[\hbar]$, so we have

$$\alpha \star_\hbar \beta e^{-\hbar\partial_G}\left(e^{\hbar\partial_G}\alpha \cdot e^{\hbar\partial_G}\beta\right). \tag{12.13}$$

Now we compute: the theorem tells us that $Y(\alpha,z)(\phi) = \mathcal{L}_z(b_1(z)\star_\hbar \phi)$, and

$$b_1(z) \star_\hbar \phi = e^{-\hbar\partial_G}\left(e^{\hbar\overline{\partial}_G}b_1(z)\cdot e^{\hbar\partial_G}\phi\right) \tag{12.14a}$$

$$= e^{-\hbar\overline{\partial}_G}(b_1(z)\cdot e^{\hbar\partial_G}\phi). \tag{12.14b}$$

Since $[[\partial_G, b_1(z)],\partial_G] = 0$,

$$= b_1(z)\cdot\phi - \hbar[\partial_G, b_1(z)]\phi. \tag{12.14c}$$

Now $[\partial_G, b_1(z)]$ is a first-order differential, and we can explicitly evaluate it on the generators:

$$[\partial_G, b_1(z)]c_n = \partial_G(b_1(z)c_n) \tag{12.15a}$$

$$= \frac{1}{2\pi i}\frac{1}{n!}\frac{\partial^n}{\partial w^n} = \frac{1}{z-w}\bigg|_{w=0} = \frac{1}{2\pi i}z, \tag{12.15b}$$

(12.15c)

and it vanishes on the other generators (TODO: I think). Thus we have

$$[\partial_G, b_1(z)] = \frac{1}{2\pi i}\sum_{n\geq 0} z^{-1-n}\frac{\partial}{\partial c_n} \tag{12.16}$$

and

$$b_1(z)(\gamma+\beta\,\mathrm{d}z) = \beta(z) = \sum_{n=0}^\infty \frac{\beta^{(n)}(0)}{n!}z^n = \sum_{n=0}^\infty b_{n+1}(\beta)z^n. \tag{12.17}$$

Thus

$$Y(b_1,z)(\phi) = \mathcal{L}_z b_1(z)\star\phi = \left(\sum_{n=0}^\infty b_{n+1}z^n - \frac{\hbar}{2\pi i}\sum_{n\geq 0} z^{-1-n}\frac{\partial}{\partial c_n}\right)\phi, \tag{12.18}$$

which matches the data specified in the vertex algebra. In fact,

Theorem 12.19. *At $\hbar = 2\pi i$, $V_{2\pi i}$ is isomorphic to W as a vertex algebra.*

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