

# THE COBORDISM HYPOTHESIS

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## 1. INTRODUCTION: 9/9/20

Today's talk was given by Araminta Amabel, and was an introduction/overview to the cobordism hypothesis: what is it, and why should you believe it? For today, we assume all manifolds are smooth, compact, and oriented.

**1.1. Modeling field theories.** The cobordism hypothesis is a statement about field theories. So we should begin by discussing how to model field theories mathematically. There are several ways to do this, but most of them take these key features into account:

**Space:** Where are we? Where does the experiment take place?

**Time:** How long does the experiment run for?

In relativity, these are unified into a single concept called *spacetime*. For example, if the theory takes place on a manifold  $X$  representing space, and over the time interval  $[0, 1]$ , then spacetime is  $X \times [0, 1]$ , though one can (and we will) consider example spacetimes which aren't products.

**Fields:** We won't describe the general idea of fields here, but these provide information in your theory and are associated to open subsets  $U$  inside spacetime. For example, there's a field theory called the *particle-in-a-box*. In this theory, space is  $X$  and time is  $[0, 1]$ , and the fields on an open  $U \subset X \times [0, 1]$  are the maps from  $U$  into the "box," thought of as paths the particle can take.

**Rules:** Differential equations governing what paths are allowed. For example, in a theory called the *free massless theory*, paths must be straight lines. Often these are wrapped up into something called the *equations of motion* of the theory, such as the *Euler-Lagrange equations*.

**Observables:** These are the measurements you can make, such as the length of a path. In the Euler-Lagrange formalism, the observables on an open subset  $U$ , these are maps from the space of solutions to the Euler-Lagrange equations to  $\mathbb{R}$ .<sup>1</sup>

**Correlation functions:** These are statistical measurements that, in experimental physics, are what we actually want to compare to real-world experiments. Out of all of these, we will be most interested in something called the *partition function*.

We will work with a specific model of field theory, which is Atiyah's definition — but only of *topological* field theories. We will say what all of the above notions mean, mathematically, in Atiyah's model of TFT, but first we need some definitions.

**Definition 1.1.** Let  $n \in \mathbb{N}$ , and let  $\mathcal{Cob}(n)$  denote the symmetric monoidal category given by the following data.

**Objects:** Closed, oriented,  $(n - 1)$ -manifolds.

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<sup>1</sup>This is for the classical theory; in quantum field theories this is not always true.

**Morphisms:** A morphism  $M_1 \rightarrow M_2$  is a bordism  $X$  from  $M_1$  to  $M_2$ , i.e. a compact, oriented  $n$ -manifold  $X$  and an equivalence class of identifications  $\partial X \cong M_1 \amalg -M_2$ , modulo diffeomorphisms of  $X$  that preserve the boundary. Here  $-M_2$  denotes  $M_2$  with the opposite orientation.

**Composition:** To compose, glue bordisms. To set this up precisely, one needs to specify collar neighborhoods of  $M_1$  and  $M_2$  within  $X$ , but there is a way to make this work.

**Symmetric monoidal structure:** The tensor product is disjoint union, and the unit is the empty set, which is a manifold of every nonnegative-integer dimension. One should specify the associator, etc., but we're not going to delve into these details right now.

Atiyah came up with this definition, building on Segal's definition of a conformal field theory.

Let  $\mathcal{Vect}_k$  denote the category of vector spaces over a field  $k$ , with the symmetric monoidal structure given by tensor product.

**Definition 1.2.** A *topological field theory* (TFT), sometimes also *topological quantum field theory* (TQFT), is a symmetric monoidal functor  $Z: \mathcal{Cob}(n) \rightarrow \mathcal{Vect}_k$ .

So, for example, the empty set maps to  $k$ , and gluing bordisms maps to composition of linear maps.

Now let's revisit the key concepts in field theory.

**Space:** All objects (i.e. closed  $(n-1)$ -manifolds) are thought of as spaces. That is, we study this theory for all possible spaces at once!

**Time:**  $[0, 1]$ .

**Spacetime:** All compact  $n$ -manifolds, possibly with boundary, are thought of as spacetimes. We're working with this theory for all spacetimes at the same time, which is a bit of a perspective shift from what we did before.

**Observables:** If the TFT is denoted  $Z$ , observables are the vector space  $Z(S^{n-1})$ .

We'll return to fields and equations of motion later.

The identity morphism in  $\mathcal{Cob}(n)$  is the cylinder  $M \times [0, 1]$  (with the correct gluing data), and as  $Z$  is a functor,  $Z(M \times [0, 1]) = \text{id}_M$ . But we can do more with these bordisms: regard both  $M$  and  $-M$  as incoming and  $\emptyset$  as outgoing, which results in something macaroni-looking. When you hit this with  $Z$ , you get a map

$$(1.3) \quad e: Z(M) \otimes Z(-M) \longrightarrow k.$$

Conversely, regarding both  $M$  and  $-M$  as outgoing, we get a map

$$(1.4) \quad c: k \longrightarrow Z(M) \otimes Z(-M).$$

**Lemma 1.5** (Zorro's lemma).  *$e$  is a perfect pairing.*

This is a fun exercise to do, playing with bordisms and  $c$  and  $e$ .

**1.2. Classifying TFTs.** A mathematician encounters a concept, and wants to classify the possible examples. This is hard and scary in general, as far as we know right now, so let's start with a pretty simple case.

**Example 1.6** ( $n = 1$ ). Objects of  $\mathcal{Cob}(1)$  are finite oriented sets, i.e. finite sets with each element labeled with  $+$  or  $-$ . Symmetric monoidality implies that if  $Z$  is a one-dimensional TFT, the value of  $Z$  on objects is determined by its values on  $\text{pt}_+$  and  $\text{pt}_-$ .

Let  $V := Z(\text{pt}_+)$ . Then,  $Z(\text{pt}_-) = V^\vee$ , which is ultimately because of Lemma 1.5.  $Z(\text{pt}_+ \amalg \text{pt}_-) = V \otimes V^\vee = \text{End}(V)$ , and in general a disjoint union of  $n$  copies of  $\text{pt}_+$  and  $m$  copies of  $\text{pt}_-$  is sent to  $V^{\otimes n} \otimes (V^\vee)^{\otimes m}$ .

Now what about morphisms? We know the cylinders (well, line segments) go to identity maps. The macaroni bordism  $\text{pt}_+ \amalg \text{pt}_- \rightarrow \emptyset$  is mapped to  $e: V \otimes V^\vee \rightarrow k$ , which can be identified with the evaluation map that takes a covector  $\ell$  and a vector  $v$  and returns  $\ell(v)$ . Under the identification  $V \otimes V^\vee \rightarrow \text{End}(V)$ , this map is taken to the trace map  $\text{End}(V) \rightarrow k$ . The opposite-direction macaroni is sent to the adjoint of this map.

All bordisms in this dimension are made of disjoint unions of these two macaronis and also circles. To determine  $Z(S^1): k \rightarrow k$ , we factor the bordism  $S^1: \emptyset \rightarrow \emptyset$  into two macaronis. This computes  $\text{tr}(\text{id}_V) = \dim V$ . In particular,  $V$  must be finite-dimensional; all such  $V$  determine TFTs, and  $V$  determines the TFT completely.  $\blacktriangleleft$

**Remark 1.7.** The observables of the 1d TFT sending  $\text{pt}_+ \mapsto V$  are  $Z(S^0) = \text{End}(V)$ . This is an associative algebra, and that's not a coincidence — often, the space of observables is an algebra of some sort. As homotopy theorists, we'll be interested in working with  $\infty$ -categories eventually, and the algebraic structures we'll get on observables will be quite interesting.  $\blacktriangleleft$

**Example 1.8** ( $n = 2$ ). Let  $Z: \mathcal{Cob}(2) \rightarrow \mathcal{Vect}_k$  be a TFT. Objects are closed 1-manifolds, which are all isomorphic to finite disjoint unions of  $S^1$ . Morphisms are compact, oriented, 2-manifolds with boundary. When you draw a complicated one, you can factor it as a composition and/or disjoint union of simpler bordisms, including discs with  $S^1$  viewed as incoming or outgoing, and pairs of pants regarded as incoming or outgoing. (And cylinders, but those are identity morphisms, so not as difficult.)

The disc with  $S^1$  incoming is often called a *cap*, and with  $S^1$  outgoing is often called a *cup*.

As  $S^1$  has an orientation-reversing diffeomorphism, we do not need to keep track of the difference between  $S^1$  and  $-S^1$ . The pair-of-pants therefore defines a multiplication-like structure on  $Z(S^1)$ , as a map  $Z(S^1) \otimes Z(S^1) \rightarrow Z(S^1)$ .

In fact, one can show this extends to a commutative algebra structure on  $A := Z(S^1)$ : associativity and commutativity come from finding equivalent bordisms representing, e.g.,  $m(x_1, x_2)$  and  $m(x_2, x_1)$ . Moreover, the pair-of-pants composed with the cap is the macaroni bordism for  $S^1$ , and we already know it's a perfect pairing. So we get a counit map  $\text{tr}: A \rightarrow k$ . Moreover, we have a unit  $e: k \rightarrow A$ , which sends  $1 \mapsto 1_A$ , the unit element in  $A$ .

Let's give this structure a name.

**Definition 1.9.** A *commutative Frobenius algebra* is a finite-dimensional commutative  $k$ -algebra  $A$  with a linear map  $\text{tr}: A \rightarrow k$  such that  $a, b \mapsto \text{tr}(ab)$  is a perfect pairing.

**Theorem 1.10.** The map sending  $Z \mapsto Z(S^1)$  is an equivalence of categories between  $\mathcal{Cob}(2)$  and the category of commutative Frobenius algebras.

This was a folklore theorem for a bit; one reference is Robbert Dijkgraaf's thesis; another is Joachim Kock's book on Frobenius algebras and TFTs.

The observables are, once again,  $Z(S^1) = A$ , which has an algebra structure. It's worth thinking about what  $Z$  assigns to the pants with  $i$  legs. ◀

In higher dimensions, there's way too many things to work with:  $\mathcal{Cob}(3)$  has infinitely many isomorphism classes of connected objects! So in a sense it's not finitely generated. It would be nice if there were a way to simplify this, by using the fact that all closed, connected, oriented 2-manifolds are diffeomorphic to connect-sums of  $T^2$ , and to consider TFTs that "understand" this somehow. And maybe the decompositions we did of surfaces in terms of pants, cups, and caps could apply in this case. But  $\mathcal{Cob}(3)$  as we defined it doesn't know how to cut in lower dimensions — it doesn't even know  $S^1$  exists.

In general, we want to be able to cut up our manifold into simpler manifolds in a way that includes all dimensions down to 0. Why do we want this? One compelling reason is that otherwise this classification question is pretty much unapproachable, and the TFTs we get are still interesting.

The solution: higher categories! There is a higher-categorical version of  $\mathcal{Cob}(n)$  which takes this desideratum into account. But: defining higher categories is hard. Defining a higher-categorical version  $\mathcal{Cob}_n(n)$  of  $\mathcal{Cob}(n)$ , even given a nice formalism of higher categories, is still hard. We'll spend the next few lectures building these tools that we need to consider this kind of TFT. Once we do, though, we can make the following definition.

**Definition 1.11.** An *extended TFT* of dimension  $n$  valued in a symmetric monoidal  $n$ -category  $\mathcal{C}$  is a symmetric monoidal functor between  $n$ -categories  $Z: \mathcal{Cob}_n(n) \rightarrow \mathcal{C}$ .

With this definition in hand (...eventually), we might expect that there's an equivalence of (higher) categories of extended TFTs and  $\mathcal{C}$ . This is wrong in two different ways: first, we need to restrict to small enough objects in  $\mathcal{C}$ , called "fully dualizable" ones, akin to using only finite-dimensional vector spaces in Example 1.6. Second, in dimension  $n = 1$ , framed is the same thing as oriented, and we miss something important: in general asking for a descent to oriented bordisms is extra data. But when we take these into account, we get:

**Theorem 1.12** (Baez-Dolan cobordism hypothesis (Hopkins-Lurie, Lurie)). *There is an equivalence of  $n$ -categories  $\mathcal{P}un^{\otimes}(\mathcal{Cob}_n^{fr}(n), \mathcal{C}) \xrightarrow{\sim} \mathcal{C}^{fd}$ .*

Though Lurie provided a detailed sketch of a proof, there are more complete proofs available in special cases, e.g. in Schommer-Pries' thesis for  $n \leq 2$ , and a nearly complete, very different approach by Ayala-Francis.

## 2. $n$ -FOLD COMPLETE SEGAL SPACES: 9/16/20

Today, Adrian spoke.

**2.1. Complete Segal spaces.** Once we've understood complete Segal spaces, which will occupy us for the bulk of the talk, it won't be too terrible to generalize to  $n$ -fold complete Segal spaces. Complete Segal spaces are a lift of the definition of a category from set theory to homotopy theory.

Recall that the *nerve* functor  $N: \mathcal{Cat} \rightarrow s\mathcal{Set}$  sends a category  $\mathcal{C}$  to the simplicial set  $N_\bullet \mathcal{C}$  whose set of  $k$ -simplices is the set of strings of diagrams in  $\mathcal{C}$   $A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \dots \xrightarrow{f_k} A_k$ , i.e. there are  $k$  morphisms.

If you've taken the nerve of  $\mathcal{C}$ , you can recover the Hom-set  $\mathcal{C}(c, c')$  fairly simply: it fits into a pullback diagram

$$(2.1) \quad \begin{array}{ccc} \mathcal{C}(c, c') & \longrightarrow & C_1 \\ \downarrow & & \downarrow \\ \{(c, c')\} & \longrightarrow & C_0. \end{array}$$

Here the right-hand vertical map sends a map  $f: c \rightarrow c'$  to  $(c, c')$ .

Generalizing, one can ask, given an arbitrary simplicial set  $S_\bullet$ , whether it satisfies the *Segal condition* that for all  $m, n \in \mathbb{N}$ ,<sup>2</sup>

$$(2.2) \quad \begin{array}{ccc} X_{m+n} & \longrightarrow & X_m \\ \downarrow & & \downarrow \\ X_m & \longrightarrow & X_0 \end{array}$$

is a pullback diagram. Here the arrow on top sends  $x_1 \rightarrow \dots \rightarrow x_{m+n}$  to  $x_m \rightarrow \dots \rightarrow x_{m+n}$ ; the arrow on the left sends it to  $x_0 \rightarrow \dots \rightarrow x_m$ , and the arrows to the lower right send these to  $x_m$ .

**Theorem 2.3.** *The essential image of the nerve functor is precisely those simplicial sets which satisfy the Segal condition.*

We will generalize this to homotopy theory, where categories will be generalized to  $\infty$ -categories. We can use Theorem 2.3 to characterize categories as certain simplicial sets, and we will do the same thing for  $\infty$ -categories.

**Definition 2.4.** A commutative square of spaces

$$(2.5) \quad \begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow q \\ Y & \xrightarrow{p} & Z \end{array}$$

is a *homotopy pullback* if the canonical map

$$(2.6) \quad W \longrightarrow X \times_Z Z^{\Delta^1} \times_Z Y = \{(x, \gamma, y) \mid \gamma: p(x) \rightarrow p(y)\}$$

is a weak equivalence.

What is this canonical map? Well,  $W$  maps to the usual pullback  $X \times_Z Y$ , and this maps to the homotopy pullback by sending  $(x, y) \mapsto (x, \text{id}, y)$ , as  $p(x) = p(y)$  if  $(x, y) \in X \times_Z Y$ .

Now we can translate the set-theoretic Segal condition into a homotopy-theoretic definition.

**Definition 2.7.** A simplicial space  $X: \Delta^{op} \rightarrow \mathcal{T}op$  is a *Segal space* if for all  $m, n \in \mathbb{N}$ , (2.2) is a homotopy pullback.

We will create Hom *spaces* for Segal spaces. This isn't all the stuff you need for enrichment, but you should think of it as: categories are tautologically enriched in sets, and Segal spaces are tautologically enriched in spaces. But we'll return to this and shape it up.

**Definition 2.8.** Let  $X$  be a Segal space and  $x, x' \in X$ . Then the *Hom space* from  $x$  to  $x'$  is the space

$$(2.9) \quad X^h(x, x') := \{(\gamma, f, \gamma') \mid \gamma: x \rightarrow d_1(f), \gamma': d_0(f) \rightarrow x'\},$$

*Remark 2.10.* Given a Segal space  $X$ , we can construct a homotopy category, whose objects are the underlying set of  $X$  and whose set of morphisms  $x \rightarrow y$  is the set  $\pi_0 X^h(x, y)$ .  $\blacktriangleleft$

<sup>2</sup>For us,  $0 \in \mathbb{N}$ .

**Example 2.11.** A simplicial space is *homotopically constant* if all of its face and degeneracy maps are weak equivalences. All homotopically constant simplicial spaces are Segal spaces. This in particular includes (the nerves of) categories. ◀

In this way, the homotopy theory of topological spaces embeds into Segal spaces; moreover, Segal spaces do in fact generalize (the nerves of) categories.

In a category, objects are isomorphic or they aren't. In Segal spaces, we have a new phenomenon:  $x, x' \in X_0$  may be equivalent in different ways. They may be *isomorphic*, meaning there is some  $(\gamma, f, \gamma') \in X^h(x, x')$  which passes to an isomorphism in the homotopy category, or they may be in the same path component in  $X$ .

**Example 2.12.** This is just a sketch for now, but of an idea that matters for us. There is (almost) a Segal space  $\mathcal{C}ob_d$  whose space of 0-simplices is the space of closed  $(d-1)$ -dimensional submanifolds of  $\mathbb{R}^\infty := \varinjlim \mathbb{R}^n$ , and whose space of  $n$ -simplices is  $d$ -dimensional submanifolds of  $\mathbb{R}^\infty \times [0, n]$  (together with a transversality requirement at  $\mathbb{R}^\infty \times \{i\}$  for  $i = 0, \dots, d$ ). These can be thought of as the time-slices in a bordism which tell you how this is a  $d$ -fold composition.

If you continue the construction here, though, you won't get degeneracy maps, because there are strictness issues coming from rescaling when you check the relations between face and degeneracy maps. There are at least four different fixes to this problem, e.g. you can realize  $\mathcal{C}ob_d$  as an  $A_\infty$ -category; all four fixes are at least a little awkward.

For all  $M, N \in (\mathcal{C}ob_d)_0$ , there is a map from the path space from  $M$  to  $N$  to  $(\mathcal{C}ob_d)^h(M, N)$ . This was accompanied by a cool picture. This map, thought of as a map  $(\mathcal{C}ob_d)_1 \rightarrow (\mathcal{C}ob_d)_0 \times (\mathcal{C}ob_d)_0$ . ◀

There are two ways to be equivalent: diffeomorphic (so in the same path in  $(\mathcal{C}ob_d)_0$ ) or bordant via an invertible bordism. These are inequivalent — this is because in dimensions 5 (maybe 4) and above, a pseudoisotopy (which arises from an invertible bordism) need not be an isotopy (which arises from a diffeomorphism). Likely this is open in dimension 4; in lower dimensions, pseudoisotopy and isotopy coincide.

In general, given a Segal space  $X$  and  $x, x' \in X_0$ , there is a space of paths  $\text{Path}(x, x')$  from  $x$  to  $x'$ .

**Definition 2.13.** A Segal space  $X$  is *complete* if for all  $x, x' \in X_0$ , the map  $\text{Path}(x, x') \rightarrow X^h(x, x')$  is a weak equivalence.

Let  $\alpha_k : [1] \rightarrow [n]$  be the map sending  $0 \mapsto k-1$  and  $1 \mapsto k$ . Given a Segal space  $X$ , let  $X^{\text{inv}}$  denote the subspace of  $X$  consisting of only the simplices  $c \in X_n$  such that  $\alpha_k c$  is invertible for all (TODO: which? Didn't get down in time)  $k$ .

**Proposition 2.14.**

- (1) If  $X$  is a Segal space, then for all  $x, x' \in X_0$ ,  $(\gamma, f, \gamma') \in X^h(x, x')$  is invertible iff  $(\text{const}_{d_1 f}, f, \text{const}_{d_0 f})$  is invertible.
- (2) TODO: missed.

Rezk showed that the category of simplicial spaces has a model structure whose fibrant objects are precisely the Reedy fibrant Segal spaces, which is nice. Forcing fully faithful essentially surjective leads to a model structure in which the fibrant objects are complete. From a different perspective, this can be thought of as a univalence axiom.

Bordism categories don't quite behave as nicely; one fix is to use *flagged higher categories*, as introduced by Ayala-Francis.

(TODO: I think I missed the definition of a *category object* in a category  $\mathcal{C}$ ).

**Example 2.15.**

- (1) A category object in  $\mathcal{S}et$  is a category.
- (2) A category object in  $\mathcal{T}op$  is a *topological category* – though this is not quite the same as a topologically enriched category.
- (3) A category object in  $\mathcal{C}at$  is called a *double category*, one model for a kind of 2-category. ◀

$\mathcal{S}et$  is a full subcategory of both  $\mathcal{T}op$  and  $\mathcal{C}at$ , and in fact a topologically enriched category is the same as a topological category  $X$  such that  $X_0 \in \mathcal{S}et$ . A 2-category is the same thing as a double category with  $X_0 \in \mathcal{S}et$ .

**2.2.  $n$ -fold complete Segal spaces.** A *2-fold complete Segal spaces* is a simplicial object  $X$  in complete Segal spaces such that

- (1)  $X$  satisfies the (homotopy) Segal condition as a simplicial object (i.e. we use homotopy pullbacks rather than strict pullbacks).<sup>3</sup>
- (2)  $X_{0\bullet}$  is a homotopy constant simplicial space.
- (3) Completeness for  $X_{\bullet 0}$ , which is the most confusing condition of the three. The idea is to extract the underlying Segal space of  $X$ , sending  $n \mapsto X_{n\bullet}^{\text{inv}}$ .

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & & & & & \\
 & & X_{20} & & X_{21} & & X_{22} & & \dots \\
 & & & & & & & & \\
 & & X_{10} & & X_{11} & & X_{12} & & \dots \\
 & & & & & & & & \\
 & & X_{00} & \text{---} & X_{01} & \text{=} & X_{02} & & \dots
 \end{array}$$

(2.16)

The homotopy constancy condition on  $X_{0\bullet}$  tells us the lowest horizontal maps are homotopy complete. The homotopy completeness condition (in the vertical direction) tells us the leftmost vertical maps are invertible.

**TODO:** OK, but then what's the actual condition...?

#### REFERENCES

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<sup>3</sup>The definition of a homotopy pullback in the category of complete Segal spaces is just the levelwise homotopy colimit.