M392C NOTES: A COURSE ON SEIBERG-WITTEN THEORY AND 4-MANIFOLD TOPOLOGY

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These notes were taken in UT Austin's M392C (A course on Seiberg-Witten theory and 4-manifold topology) class in Spring 2016, taught by Tim Perutz. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Any mistakes in the notes are my own.

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Lecture 1.

Classification problems in differential topology: 1/18/18

"This is my opinion, but it's the only reasonable opinion on this topic."

This course will be on gauge theory; specifically, it will be about Seiberg-Witten theory and its applications to the topology of 4-manifolds. The course website is https://www.ma.utexas.edu/users/perutz/GaugeTheory.html; consult it for the syllabus, assignments, etc.

The greatest mystery in geometric topology is: *what is the classification of smooth, compact, simply-connected four-manifolds up to diffeomorphism?* The question is wide open, and the thoery behaves very differently than the theory in any other dimension.

There's a fascinating bit of partial information known, mostly via PDEs coming from gauge theory, e.g. the *instanton equation* $F_A^+ = 0$ as studied by Donaldson, Uhlenbeck, Taubes, and others. More recently, people have also studied the *Seiberg-Witten equations*

$$D_A\psi = 0 \tag{1.1a}$$

$$\rho(F_A^+) = (\psi\otimes\psi^*)_0. \tag{1.1b}$$

Even without defining all of this notation, it's evident that the Seiberg-Witten equations are more complicated than the instanton equation, and indeed they were discovered later, by Seiberg and Witten in 1994. However, they're much easier to work with — after their discovery, the results of Donaldson theory were quickly reproven, and more results were found, within the decade after their discovery. This course will focus on results from Seiberg-Witten theory.

In some sense, this is a closed chapter: the stream of results on 4-manifolds has slowed to a trickle. But Seiberg-Witten theory has in the meantime found new applications to 3-manifolds, contact topology (including the remarkable proof of the Weinstein conjecture by Taubes), knots, high-dimensional topology, Heegaard-Floer homology, and more. Throughout this constellation of applications, there are many results whose only known proofs use the Seiberg-Witten equations.

The central problem in differential topology is to classify manifolds up to diffeomorphism. To make the problem more tractable, let's restrict to smooth, compact, and boundaryless. An ideal solution would solve the following four problems for some class of manifolds (e.g. compact of a particular dimension, and maybe with some topological constraints).

(1) Write down a set of "standard manifolds" $\{X_i\}_{i\in I}$ such that each manifold is diffeomorphic to precisely one X_i. For example, a list of diffeomorphism classes of closed oriented connected surfaces is given by the sphere and the n-holed torus for all $n \geq 0$.
(2) Given a description of a manifold M, a way to compute invariants to decide for which $i \in I$ $M \cong X_I$. For example, if M is a closed, connected, oriented surface, we can completely classify it by its Euler characteristic.

    A variant of this problem asks for an explicit algorithm to do this when M is encoded with finite information, e.g. a solution set to polynomial equations in $\mathbb{R}^N$ with rational coefficients.
(3) Given M and M', compute invariants to decide whether M is diffeomorphic to M'; once again, there's an algorithmic variant to that problem.
(4) Understand families (fiber bundles) of manifolds diffeomorphic to M. In some sense, this means understanding the homotopy type of the topological group $\mathrm{Diff}(M)$ of self-diffeomorphisms of M.

This is an ambitious request, but much is known in low dimensions. In dimension 1, the first three questions are trivial, and the last is nontrivial, but solved.

Example 1.2. For compact, orientable, connected surfaces, we have a complete solution: a list of diffeomorphism classes is the sphere and $(T^2)^{\#g}$ for all $g \geq 0$, and the Euler characteristic $\chi := 2 - 2g$ is a complete invariant which is algorithmically computable from any reasonable input data, solving the second and third questions. Here, "reasonable input data" could include a triangulation, a good atlas (meaning nonempty intersections are contractible), or monodromy data for holomorphic a branched covering map $\Sigma \to S^2$, where here we're thinking of surfaces as Riemann surfaces, with chosen complex structures. Here, the Riemann-Hurwitz formula can be used to compute the Euler characteristic.

For the fourth question, let $\mathrm{Diff}^+(\Sigma)$ denote the topological group of orientation-preserving self-diffeomorphisms of Σ.

Theorem 1.3 (Earle-Eells)**.**

- *The inclusion* $\mathrm{SO}_3 \hookrightarrow \mathrm{Diff}^+(S^2)$ *is a homotopy equivalence.*
- *The identification* $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$ *defined a map* $T^2 \hookrightarrow \mathrm{Diff}^+(T^2)$ *as translations; this map is a homotopy equivalence into the connected component of the identity in* $\mathrm{Diff}^+(T^2)$*, and* $\pi_0 \mathrm{Diff}^+(T^2) \cong \mathrm{SL}_2(\mathbb{Z})$.
- *If* $g > 1$*, every connected component of* $\mathrm{Diff}^+(\Sigma_g)$ *is contractible, and the* mapping class group $\mathrm{MCG}(\Sigma_g) := \pi_0 \mathrm{Diff}^+(\Sigma_g)$ *is a finitely presented infinite group which acts with finite stabilizers on a certain contractible manifold called* Teichmüller space.[1]

So all four questions have satisfactory answers, though understanding the mapping class groups of surfaces is still an active area of research. ◀

Example 1.4. The classification of compact, orientable 3-manifolds looks remarkably similar to the classification of surfaces (albeit much harder!), through a vision of Thurston, realized by Hamilton and Perelman. The solution is almost as complete. The proof uses geometry, and nice representatives are quotients by groups acting on hyperbolic space.

As for invariants, the fundamental group is very nearly a complete invariant.[2] ◀

In higher dimensions, there are a few limitations. Generally, the index set I will be uncountable. For example, there are an uncountable number of smooth 4-manifolds homeomorphic to $\mathbb{R}^4$! So there will be no nice list, and no nice moduli space either. But restricted to compact manifolds, there are countably many classes, which follows from triangulation arguments or work of Cheeger in Riemannian geometry.

The next obstacle involves the fundamental group. If M is presented as an *n-handlebody* (roughly, a CW complex with cells of dimension at most n), there is an induced presentation of $\pi_1(M)$, and if M is compact, this is a finite presentation (finitely many generators, and finitely many relations).

[1] Heuristically, but not literally, Teichmüller space is a classifying space for this group.

[2] The fundamental group cannot distinguish lens spaces, and that's pretty much the only exception.

Fact. For each $n \geq 4$, all finite presentations arise from compact n-handlebodies (namely, closed n-manifolds). ◀

This is pretty cool, but throws a wrench in our classification goal.

Theorem 1.5 (Markov)**.** *There is no algorithm that decides whether a given finite group presentation gives the trivial group.*

The proof shows that an algorithm which could solve this problem could be used to construct an algorithm that solves the halting problem for Turing machines.

Corollary 1.6. *There is no algorithm to decide whether a given n-handlebody, $n \geq 4$, is simply connected.*

This means that a general classification algorithm cannot possibly work for $n \geq 4$; thus, we will have to restrict what kinds of manifolds we classify.

A third issue in higher-dimensional topology is that in dimension $n \leq 3$, there are existence and uniqueness theorems of "optimal" Riemannian metrics (e.g. constraints on their isometry groups), but for $n \geq 5$, this is not true for any sense of optimal; some choices fail existence, and others fail uniqueness. This is discussed further (and more precisely) in Shmuel Weinberger's "Computers, Rigidity, and Moduli," which has some very interesting things to say about the utility of Riemannian geometry to classify manifolds (or lack thereof).

So four dimensions is special, but for many reasons, not just one.

Those setbacks notwithstanding, we can still say useful things.

- We will restrict to closed manifolds.
- We will focus on the simply-connected case, eliding Markov's theorem.[3]

With these restrictions, we have good answers to the first three questions.

Example 1.7. There is a countable list of compact, simply-connected 5-manifolds, and invariants (cohomology, characteristic classes) which distinguish any two. ◀

Example 1.8. Kervaire-Milnor produced a classification of homotopy spheres in dimensions $5 \leq n \leq 18$, and a conceptual answer in higher dimensions, and further work has applied this in higher dimensions. ◀

There is a wider range of conceptual answers to all four questions, more or less explicit, through *surgery theory*, when $n \geq 5$ (surgery theory fails radically in dimension 4). This gives an answer to the following questions.

- Given a finite, n-dimensional CW complex X (where $n \geq 5$), when is it the homotopy type of a compact n-manifold?
- Given a simply-connected compact manifold M, what are the diffeomorphism types of manifolds homotopy equivalent to M? (Again, we need $\dim M \geq 5$.)

Here are necessary and sufficient conditions for the existence question.

- X must be an n-dimensional *Poincaré duality space*, i.e. there is a fundamental class $[X] \in H_n(X;\mathbb{Z})$ which implements the Poincaré duality isomorphism. This basic fact about closed manifolds gets you an incredibly long way towards the answer.
- Next, X must have a tangent bundle — but it's not clear what this means for a general Poincaré duality space. Here we mean a rank-n vector bundle $T \to X$ which is associated to the homotopy type in a certain precise sense: the unit sphere bundle of the stablization of T, considered as a spherical fibration, has to be manifest in X in a certain way.
- If $n \equiv 0 \bmod 4$, there's another obstruction — a certain $\mathbb{Z}$-valued invariant must vanish, interpreted as asking that $T \to X$ satisfies the Hirzebruch signature theorem: the signature of the cup product form on $H^{n/2}(X)$ must be determined by the Pontrjagin classes of T.
- If $n \equiv 2 \bmod 4$, the obstruction is a similar $\mathbb{Z}/2$-valued invariant related to the Arf invariant of the intersection form.
- If n is odd, there are no further obstructions.

[3] More generally, one could pick some fixed group G and ask for a classification of closed n-manifolds with $\pi_1(M) \cong G$; people do this, but we won't worry about it.

That's it. Uniqueness is broadly similar — once you specify a tangent bundle, there are only finitely many diffeomorphism types!

Now we turn to dimension 4, the hardest case. We want to classify smooth, closed, simply-connected 4-manifolds. The first basic invariant (even of 4-dimensional Poincaré duality spaces) is the intersection form Q_P, which we'll begin studying in detail next week. You can realize it as a unimodular matrix modulo integral equivalence. That is, it's a symmetric square matrix over $\mathbb{Z}$ with determinant ± 1, and integral equivalence means up to conjugation by elements of $\mathrm{GL}_b(\mathbb{Z})$.

Theorem 1.9 (Milnor)**.** *The intersection form defines a bijection from the set of homotopy classes of 4-dimensional simply-connected Poincaré spaces to the set of unimodular matrices modulo equivalence.*

So this form captures the entire homotopy type! That's pretty cool.

Theorem 1.10 (Freedman)**.** *The intersection form defines a bijection from the set of homeomorphism classes of 4-dimensional simply-connected topological manifolds to the set of unimodular matrices modulo equivalence.*

Thus this completely classifies (closed, simply-connected) topological four-manifolds. This theorem won Freedman a Fields medal.

The next obstruction, having a tangent bundle, is a mild constraint told to us by Rokhlin.

Theorem 1.11 (Rokhlin)**.** *Let X be a closed 4-manifold. If Q_X has even diagonal entries, then its signature is divisble by* 16.

The signature is the number of positive eigenvalues minus the number of negative eigenvalues. Algebra tells us this is already divisible by 8, so this is just a factor-of-2 obstruction, which is not too bad.

But the rest of the story of surgery theory is just wrong in dimension 4. This is where analysis of an instanton moduli space comes in.

Theorem 1.12 (Donaldson's diagonalizability theorem)**.** *Let X be a compact, simply-connected 4-manifold. If Q_X is positive definite, i.e. $xQ_Xx > 0$ for all nonzero $x \in \mathbb{Z}^b$, then Q_X is equivalent to the identity matrix.*

Donaldson proved this theorem as a second-year graduate student!

There's a huge number of unimodular matrices which are positive definite, but not equivalent to the identity; the first example is known as E_8. So this is a strong constraint on their realizability by 4-manifolds.

In subsequent years, Donaldson devised invariants distinguishing infinitely many diffeomorphism types within a single homotopy class. Then, from 1994 onwards, there came new proofs of these results via Seiberg-Witten theory, which tended to be simpler,[4] and to provide sharper, more general results. We will prove several of these in the second half of the class.

Lecture 2.

Review of the algebraic topology of manifolds: 1/23/18

Though today might be review for some students, it's important to make sure we're all on the same page, and we'll get to the good stuff soon enough. We won't do too many examples today, but will see many in the future.

Cup products. Cup products make sense in a more general sense than manifolds. Let X and Y be CW complexes; then, there is a canonical induced CW structure on $X \times Y$: the product of a pair of discs is homeomorphic to a disc, and we take the cells of $X \times Y$ to be the products of cells of X and cells of Y.

Recall that the *cellular chain complex* $C_*(X)$ is the free abelian group on the set of cells, and the *cellular cochain complex* is the dual: $C^*(X) \coloneqq \mathrm{Hom}(C_*(X), \mathbb{Z})$.

Proposition 2.1 (Künneth formula)**.** *Let X and Y be CW complexes. There is a canonical isomorphism*

$$C^*(X \times Y) \cong C^*(X) \otimes C^*(Y). \tag{2.2}$$

[4] That said, Donaldson's original proof of the diagonalizability theorem stands as one of the most beautiful things in gauge theory.

This follows because the cells of $X \times Y$ are the products of those in X and those in Y. There is an analogue of the Künneth formula for pretty much any kind of (ordinary) cohomology theory.

The *diagonal map* $\Delta\colon X \to X \times X$ sending $x \mapsto (x, x)$ is, annoyingly, *not* a cellular map (i.e. it does not preserve the k-skeleton). However, it is homotopic to a cellular map $\delta\colon X \to X \times X$.

Definition 2.3. The *cup product of cochains* is the map $\smile\colon C^*(X) \otimes C^*(X) \to C^*(X)$ which is the composition

$$C^*(X) \otimes C^*(X) \xrightarrow[(2.2)]{\cong} C^*(X \times X) \xrightarrow{\delta^*} C^*(X).$$

We haven't said anything about coboundaries, but the cup product plays well with them, and therefore induces a cup product on cellular cohomology, $\smile\colon H^*(X) \otimes H^*(X) \to H^*(X)$. This is an associative map, and it's *graded*, meaning it sends $H^i(X) \otimes H^j(X)$ into $H^{i+j}(X)$. It's unital and *graded commutative*, meaning

$$x \smile y = (-1)^{|x||y|} y \smile x. \tag{2.4}$$

This turns $H^*(X)$ into a graded commutative ring.

Remark. The cup product is *not* graded commutative on the level of cochains. However, there are coherent homotopies between $x \smile y$ and $(-1)^{|x||y|} y \smile x$. ◀

The fact that we had to choose $\Delta \simeq \delta$ is annoying, since it's non-explicit and non-canonical. The cup product in singular cohomology does not have this problem, as you can just work with Δ itself, but the tradeoff is that the Künneth formula is less explicit.

There are a few other incarnations of the cup product which are more geometrically transparent, and this will be useful for us when studying manifolds. These have other drawbacks, of course.

(1) Čech cohomology is a somewhat unintuitive way to define cohomology, but has the advantage of providing a completely explicit formula for the cup product.
(2) de Rham cohomology provides a model for the cup product which is graded-commutative on cochains, but only works with $\mathbb{R}$ coefficients.
(3) The intersection theory of submanifolds is a beautiful model for the cup product, but is not always available.

We'll discuss these in turn.

Čech cohomology. Let M be a manifold,[5] and $\mathfrak{U} = \{U_i\}_{i \in I}$ be an open cover of M. For $J \subset I$, write

$$U_J := \bigcap_{i \in J} U_i.$$

Definition 2.5. We say that $\mathfrak{U}$ is a *good cover* if it is locally finite and all U_J, $J \neq \varnothing$, are empty or contractible.

In particular, on a compact manifold, a good cover is finite.

Lemma 2.6. *Any manifold admits a good cover.*

There are two standard proofs of this — one chooses small geodesic balls around each point for a Riemannian metric on M, and the other chooses an embedding $M \hookrightarrow \mathbb{R}^N$ and then uses the intersections of small balls in $\mathbb{R}^N$ with M.

There is also a uniqueness (really cofinality) statement.

Lemma 2.7. *Any two good covers of a manifold M admit a good common refinement.*

For $k \in \mathbb{Z}_{\geq 0}$, let $[k] := \{0, \ldots, k\}$.

Definition 2.8. Let $k \in \mathbb{Z}_{\geq 0}$. A *$k$-simplex* of $\mathfrak{U}$ is a way of indexing a k-fold intersection in $\mathfrak{U}$; specifically, it is an injective map $\sigma\colon [k] \hookrightarrow I$ such that $\mathfrak{U}_{\sigma([k])}$ is nonempty. The set of k-simplices of $\mathfrak{U}$ is denoted $S_k(\mathfrak{U})$.

There is a *boundary map* $\partial_i\colon S_k(\mathfrak{U}) \to S_{k-1}(\mathfrak{U})$ which deletes $\sigma(i)$.

[5] Čech cohomology works on a more general class of spaces, but we work with manifolds for simplicity.

Definition 2.9. Let A be a commutative ring. The *Čech cochain complex valued in A* is the cochain complex $\check{C}^*(M, \mathfrak{U}; A)$ defined by
$$C^k(M, \mathfrak{U}; A) := \prod_{S_k} A$$
and with differential $\delta\colon \check{C}^k(M, \mathfrak{U}; A) \to \check{C}^{k+1}(M, \mathfrak{U}; A)$ defined by
$$(\delta\eta)(\sigma) := \sum_{i=0}^{k+1} (-1)^{i+1} \eta(\partial_i \sigma),$$
where η is a cochain and $\sigma\colon [k] \hookrightarrow I$.

One can show that $\delta^2 = 0$, hence define the *Čech cohomology groups* $\check{H}^*(M, \mathfrak{U}; A) := \ker(\delta)/\operatorname{Im}(\delta)$.

Proposition 2.10. *Let $\mathfrak{U}$ and $\mathfrak{V}$ be good covers of a manifold M. Then, there is an isomorphism $\check{H}^*(M, \mathfrak{U}; A) \cong \check{H}^*(M, \mathfrak{V}; A)$.*

Proof idea. By Lemma 2.7, $\mathfrak{U}$ and $\mathfrak{V}$ admit a common refinement $\mathfrak{W}$; then, check that a refinement map of good covers induces an isomorphism in Čech cohomology. ⊠

Thus the Čech cohomology is often denoted $\check{H}^*(M; A)$.

In Čech cohomology, there is a finite, combinatorial model for the cup product: let $\alpha \in \check{C}^i$, $\beta \in \check{C}^j$, and $\sigma\colon [i+j] \hookrightarrow I$. Then, we let
$$(\alpha \smile \beta)(\sigma) := \alpha(\text{beginning of } \sigma) \cdot \beta(\text{end of } \sigma). \tag{2.11}$$
To be sure, this works in a more general setting (and indeed is the definition of cup product in singular cohomology), but the finiteness of Čech cochains on a compact manifold makes it a lot nicer in this setting. However, it's not at all transparent that the cup product is graded commutative on cohomology.

Theorem 2.12. *There is an isomorphism of graded rings $\check{H}^*(M; A) \cong H^*(M; A)$ (where the latter means cellular cohomology).*

de Rham cohomology. Recall that $\Omega^k(M)$ denotes the space of differential k-forms on a manifold M, and
$$\Omega^*(M) := \bigoplus_{k \ge 0} \Omega^k(M).$$
There is an exterior derivative $\mathrm{d}\colon \Omega^*(M) \to \Omega^{*+1}(M)$ with $\mathrm{d}^2 = 0$, so we can define the *de Rham cohomology* $H^*_{\mathrm{dR}}(M) := \ker(\mathrm{d})/\operatorname{Im}(\mathrm{d})$ in the usual way.

In this case, the cup product is induced by the wedge product of differential forms
$$\wedge\colon \Omega^i(M) \otimes \Omega^j(M) \to \Omega^{i+j}(M).$$

Proposition 2.13. *The wedge product is graded commutative on differential forms, hence makes $\Omega^*(M)$ into a DGA (differential graded algebra).*

This is really nice, but can only occur in characteristic zero; if you tried to do this over a field of positive characteristic, you would run into obstructions called Steenrod squares to defining a functorial graded-commutative cochain model for cohomology.

Theorem 2.14 (de Rham)**.** *Let $\mathfrak{U}$ be a good cover of a manifold M. Then there is a natural isomorphism of graded $\mathbb{R}$-algebras $H^*_{\mathrm{dR}}(M) \cong \check{H}^*(M, \mathfrak{U}; \mathbb{R})$.*

There are several different ways of proving this. One is to show that they both satisfy the Eilenberg-Steenrod axioms with $\mathbb{R}$ coefficients, and that up to natural isomorphism there is a single cohomology theory satisfying these isomorphisms. Another is to observe that Čech cohomology is a model for sheaf cohomology, and that both Čech and de Rham cohomology are derived functors of the same functor of sheaves on M applied to the constant sheaf valued in $\mathbb{R}$.

An alternative way to prove it, whose details can be found in Bott-Tu's book, is to form the *Čech-de Rham complex*, a double complex $\check{C}^*(M, \mathfrak{U}; \Omega^\bullet)$. Let D^* denote its *totalization*. Then there are quasi-isomorphisms $\check{C}^* \hookrightarrow D^*$ and $\Omega^*(M) \hookrightarrow D^*$ respecting products, hence inducing isomorphisms $\check{H}^* \cong H^*(D^*) \cong H^*_{\mathrm{dR}}(M)$.

Poincaré duality and the fundamental class Poincaré duality is one of the few (relatively) easy facts about topological manifolds, and one of the only things known until the work of Kirby and Siebenmann in the 1970s. Throughout this section, X denotes a nonempty, connected topological manifold of dimension n. For a reference for this section, see May's *A Concise Course in Algebraic Topology.*

Proposition 2.15.

(1) *If* $k > n$, $H_k(X) = 0$.
(2) $H_n(X) \cong \mathbb{Z}$ *if* X *is compact and orientable, and is* 0 *otherwise.*

If X is compact, a choice of orientation defines a generator $[X] \in H_n(X)$, called the *fundamental class* of X. A homeomorphism $f\colon X \to Y$ sends $[X] \mapsto [Y]$ if f preserves orientation and $[X] \mapsto -[Y]$ if f reverses orientation. If X is a CW complex with no cells of dimension $> n$ and a single cell e_n in dimension n, then in cellular homology, $[X] = \pm[e_n]$.

There is a *trace map* or *evaluation map* $H^n(X; A) \to A$ sending $c \mapsto \mathrm{eval}(c, [X])$ (that is, evaluate c on $[X]$); in the de Rham model on a smooth manifold, this is the integration map

$$\eta \longmapsto \int_X \eta.$$

The graded abelian group $H_{-*}(X)$ is a graded module over the graded ring $H^*(X)$ via a map called the *cap product*

$$\frown\colon H^k(X) \otimes H_i(X) \longrightarrow H_{i-k}(X).$$

Place a CW structure on X, and recall that $\delta\colon X \to X \times X$ was our cellular approximation to the diagonal. Then, we can give a cellular model for the cap product:

$$C^*(X) \otimes C_*(X) \xrightarrow{\mathrm{id}\otimes\delta_*} C^*(X) \otimes C_*(X) \otimes C_*(X) \xrightarrow{\mathrm{eval}\otimes\mathrm{id}} C_*(X).$$

Let X and Y be smooth n-manifolds, where X is closed and oriented. Then, $\text{–} \frown f_*[X]\colon H^n_{\mathrm{dR}}(Y) \to H^n_{\mathrm{dR}}(X)$ has the explicit model

$$\eta \longmapsto \int_X f^*\eta,$$

showing how the cap product relates to the evaluation map.

Theorem 2.16 (Poincaré duality)**.** *For* X *a closed, oriented manifold, the map*

$$D_X \coloneqq \text{–} \frown [X]\colon H^*(X) \to H_{n-*}(X)$$

is an isomorphism.

For a proof, see May. In the case of smooth manifolds, there's a slick proof using Morse theory; but Poincaré duality is true for topological manifolds as well.

We will let $D^X \coloneqq (D_X)^{-1}$.

Intersections of submanifolds. Intersection theory, though not its relation to the cup product, was discussed in the differential topology prelim. Let X, Y, and Z be closed, oriented manifolds of dimensions n, $n-p$, and $n-q$ respectively, and let $f\colon Y \to X$ and $g\colon Z \to X$ be smooth maps. Let $c_Y \coloneqq D^X(f_*[Y]) \in H^p(X)$, and similarly let $c_Z \coloneqq D^X(g_*[Z]) \in H^q(X)$. We will be able to give a nice interpretation of $c_Y \smile c_Z$.

First, let f' be a smooth map homotopic to f and transverse to g; standard theorems in differential topology show that such a map exists. Transversality means that if $y \in Y$ and $z \in Z$ are such that $f'(y) = g(z)$, then

$$T_xX = Df'(T_yY) + D_g(T_zZ).$$

Let $P \coloneqq Y\,{}_{f'}\!\times_g X$, which is exactly the space of pairs (y, z) such that $f'(y) = g(z)$; transversality guarantees this is a smooth manifold of codimension $p+q$ in X. The orientations on X, Y, and Z induce one on P, and there is a canonical map $\phi\colon P \to X$ sending $(y, z) \mapsto f'(y) = g(z)$.

Theorem 2.17. *Let* $c_P \coloneqq D^X(\phi_*([P]))$. *Then,* $c_P = c_Y \smile c_Z$.

If Y and Z are transverse submanifolds of X, P is exactly their intersection. We will use this result frequently.

Intersection of submanifolds gives a geometric realization of the cup product, but only for those classes represented by maps from manifolds; not all homology classes are realized in this way.

Classes of codimension at most 2 always have representatives arising from embedded submanifolds. The idea is that in general, there's a natural isomorphism

$$H^n(X) \cong [X, K(\mathbb{Z}, n)],$$

where brackets denote homotopy classes of maps and $K(\mathbb{Z}, n)$ is an *Eilenberg-Mac Lane space* for $\mathbb{Z}$ in dimension n, i.e. a space whose only nontrivial homotopy group is $\pi_n \cong \mathbb{Z}$. These spaces always exist, and any two models for $K(\mathbb{Z}, n)$ are homotopic.

Usually Eilenberg-Mac Lane spaces are not smooth manifolds, but there are a few exceptions, including $K(\mathbb{Z}, 1) \simeq S^1$. Hence there is a bijection $[X, S^1] \to H^1(X)$. In the de Rham model, this is the map

$$[f] \longmapsto f^*\omega,$$

where $\omega \in H^1(S^1) \cong \mathbb{Z}$ is the generator. Alternatively, you could think of ω as $D^{S_1}[\mathrm{pt}]$, for any choice of $\mathrm{pt} \in S^1$.

Thus, take $f\colon X \to S^1$ to be a smooth map, where X is a closed, oriented manifold. Let $H_t := f^{-1}(t) \subset X$, where $t \in S^1$ is a regular value. Then, H_t comes with a co-orientation, hence an orientation, and $[H_t] = D_X(f^*\omega)$. Thus codimension-1 submanifolds are realizable.

In this course, the case of codimension 2 will be more useful.

Proposition 2.18. *Let $\mathbb{CP}^\infty := \operatorname{colim}_n \mathbb{CP}^n$ (the union via the inclusions $\mathbb{CP}^n \hookrightarrow \mathbb{CP}^{n+1}$). Then, $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$.*

Hence there is a class $c \in H^2(\mathbb{CP}^\infty)$ and a natural bijection $[X, \mathbb{CP}^\infty] \to H^2(X)$ sending $[f] \mapsto f^*(c)$.

$\mathbb{CP}^\infty$ is not a smooth manifold, but its low-dimensional skeleta are, and this leads to codimension-2 realizability. Specifically, the inclusion $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty$ defines the pullback map $H^2(\mathbb{CP}^\infty) \to H^2(\mathbb{CP}^1) \cong H_0(\mathbb{CP}^1) \cong \mathbb{Z}$. This maps the tautological class c to $[\mathrm{pt}]$ for any $\mathrm{pt} \in \mathbb{CP}^1$.

Let $f\colon \mathbb{CP}^\infty$ be a map, where X is a smooth, oriented, closed manifold, which is homotopic to a smooth map $\mathbb{CP}^N$ followed by the inclusion $\mathbb{CP}^N \hookrightarrow \mathbb{CP}^\infty$. Let $D \subset \mathbb{CP}^N$ be a hyperplane and $H_D := g^{-1}(D)$. Assuming $g \pitchfork D$ (which can always be done for some g in the homotopy class of f), then H_D is a codimension-2 oriented submanifold of X, and $[H_D] = D^X(g^*c)$. Thus codimension-2 homology classes are representable. We will most commonly use this in dimension 4, for which any class $a \in H^2(X)$ is represented by an embedding of a closed, oriented surface $\Sigma \hookrightarrow X$.

In general, realizability is controlled by oriented cobordism, which in higher codimension is different from cohomology, governed by maps into Thom spaces rather than Eilenberg-Mac Lane spaces. This was studied in the 1950s by Rene Thom.

Lecture 3.

Unimodular forms: 1/25/18

Today's lecture will be more algebraic in flavor, though with topology in mind; we'll be discussing the algebra that arises in the middle cohomology of even-dimensional manifolds.

Let M be a closed, oriented manifold of dimension $2n$; its middle cohomology $H^n(M)$ carries a bilinear form $\cdot\colon H^n(M) \otimes H^n(M) \to \mathbb{Z}$ sending

$$x, y \longmapsto x \cdot y := \langle x \smile y, [M]\rangle, \tag{3.1}$$

where $\langle -, - \rangle$ denotes evaluation. If n is even (i.e. $4 \mid \dim M$), this is a symmetric form; if n is odd, it's skew-symmetric, which follows directly from the graded-commutativity of the cup product.

Poincaré duality means there are three different ways to think of this product.

- As defined, it's a pairing $H^n \otimes H^n \to \mathbb{Z}$, where the pairing is evaluation and the cup product.
- Using Poincaré duality, we could reinterpret it as a map $H^n \otimes H_n \to \mathbb{Z}$. In this case, the pairing is evaluation. This is because the Poincaré duality map is capping with the fundamental class, so

$$(x \smile y) \frown [M] = x \frown (y \frown [M]) = x \frown D_M(y).$$

- Using Poincaré duality again, it's a pairing $H_n \otimes H_n \to \mathbb{Z}$, which is the intersection product.

For an abelian group A, let $A_{\mathrm{tors}} \subset A$ denote its torsion subgroup and $A' := A/A_{\mathrm{tors}}$. In this case, the form (3.1) descends to a form on $H^n(M)'$, and we usually use this version of the form.

Remark. If M is 4-dimensional, the universal coefficients theorem guarantees a short exact sequence

$$0 \longrightarrow (H_1(M))_{\mathrm{tors}} \longrightarrow H^2(M) \longrightarrow H^2(M)' \longrightarrow 0,$$

and that it splits, but non-canonically. In particular, if $H_1(M) = 0$, $H^2(M)$ is torsion-free. ◀

Let $\{e_i\}$ be a $\mathbb{Z}$-basis for $H^n(M')$ and $Q = (Q_{ij})$ be the matrix with entries $Q_{ij} \coloneqq e_i \cdot e_j$. This is a symmetric matrix of n is even, and is skew-symmetric if n is odd.

The universal coefficients theorem also implies that $H^n(M') \cong \mathrm{Hom}(H_n(M), \mathbb{Z})$, where the map sends a cohomology class y to the evaluation pairing of y and a homology class. This, plus the fact that $\cdot$ is dual to evaluation, implies the following proposition.

Proposition 3.2. *The pairing $\cdot$ is nondegenerate on $H^n(M)'$, i.e. the map $H^n(M') \to \mathrm{Hom}(H^n(M)', \mathbb{Z})$ sending $x \mapsto (y \mapsto x \cdot y)$ is an isomorphism of abelian groups.*

Corollary 3.3. $\det Q \in \{\pm 1\}$.

Now we focus on the case of manifolds which are boundaries. Suppose there is a compact oriented $(2n+1)$-dimensional manifold N such that $M = \partial N$, and let $i\colon N \hookrightarrow M$ denote inclusion.

Proposition 3.4. *Let $L = \mathrm{Im}(i^*) \subset H^n(M; \mathbb{R})$. Then,*

(1) *L is* isotropic*, i.e. for all $x, y \in L$, $x \cdot y = 0$.*
(2) *$\dim L = (1/2) \dim H^n(M; \mathbb{R})$.*

Proof. Perhaps unsurprisingly, this proof uses algebraic topology of manifolds with boundary, namely *Poincaré-Lefschetz duality*, the analogue of Poincaré duality on a compact manifold with boundary.

Part (1) follows from the fact that i^* is a ring homomorphism:

$$\begin{aligned} i^*u \cdot i^*v &= (i^*u \smile i^*v) \frown [M] \\ &= i^*(u \smile v) \frown [M] \\ &= (u \smile v) \frown i_*[M], \end{aligned}$$

but since $M = \partial N$, $[M] = \partial[M, N]$, where $\partial\colon H_{2n+1}(M, N; \mathbb{R}) \to H_{2n}(N; \mathbb{R})$ is the boundary map in the long exact sequence of a pair and $[M, N] \in H_{2n+1}$ is the relative fundamental class. Hence $i_*[M] = 0$, since $i_* \circ \partial = 0$.

For part (2), Poincaré-Lefschetz duality implies the following diagram is commutative with exact rows:

$$\begin{tikzcd} \cdots \arrow[r] & H^n(N;\mathbb{R}) \arrow[r, "i^*"] \arrow[d, "\cong"'] & H^n(M;\mathbb{R}) \arrow[r, "\delta"] \arrow[d, "\cong"', "D_M"] & H^{n+1}(N,M;\mathbb{R}) \arrow[r, "q"] \arrow[d, "\cong"', "D"] & H^{n+1}(N;\mathbb{R}) \arrow[r] \arrow[d, "\cong"', "D"] & \cdots \\ \cdots \arrow[r] & H_{n+1}(N,M;\mathbb{R}) \arrow[r, "\partial"] & H_n(M;\mathbb{R}) \arrow[r, "i_*"] & H_n(M;\mathbb{R}) \arrow[r, "p"] & H_n(N,M;\mathbb{R}) \arrow[r] & \cdots \end{tikzcd} \tag{3.5}$$

Fix a complement K to L in $H^n(M; \mathbb{R})$; it suffices to show that $\dim K = \dim L$. Since $L = \mathrm{Im}(i^*) = \ker(\delta)$, then $K \cong H^n(M; \mathbb{R})/\ker(\delta) \cong \mathrm{Im}(\delta)$. Since the upper row of (3.5) is exact, $\mathrm{Im}(\delta) = \ker(q)$, and by Poincaré-Lefschetz duality this is isomorphic to $\ker(p)$. Since the lower row of (3.5) is exact, this is isomorphic to $\mathrm{Im}(i_*) \subset H_n(N; \mathbb{R})$. However, i_* and i^* are dual (in the sense of $\mathrm{Hom}(–, \mathbb{R})$), and linear algebra tells us that a map and its dual have the same rank.[6] ⊠

Now let's focus on the case when n is even, so $4 \mid \dim(M)$.

Definition 3.6. A *unimodular lattice* (Λ, σ) is a finite-rank free abelian group Λ together with a nondegenerate symmetric bilinear form $\sigma\colon \Lambda \otimes \Lambda \to \mathbb{Z}$.

Therefore a closed, oriented $4m$-manifold defines a unimodular lattice $(H^{2m}, \cdot)$.

Recall that for (V, σ) a symmetric bilinear form on a real vector space V, there is an orthogonal decomposition

$$V = R \oplus V^+ \oplus V^-, \tag{3.7}$$

where R is the subspace orthogonal to everything, V^+ is the subspace on which σ is positive definite, and V^- is the subspace on which σ is negative definite. Clearly (V, σ) determines $\dim V^+$ and $\dim V^-$.

[6]The matrix version of this statement is that a matrix and its transpose have the same rank.

Proposition 3.8 (Sylvester's law of inertia)**.** *If (V,σ) is a symmetric bilinear form on a real vector space, then $\dim R$, $\dim V^+$, and $\dim V^-$ determine (V,σ) up to isomorphism.*

Definition 3.9. The *signature* of (V,σ) is $\tau \coloneqq \dim V^+ - \dim V^-$. If (Λ,σ) is a unimodular lattice, then $\tau(\Lambda,\sigma) \coloneqq \tau(\Lambda\otimes\mathbb{R}, \sigma\otimes \mathrm{id}_{\mathbb{R}})$. If M is a closed, oriented, $4m$-dimensional manifold, its signature is $\tau(M) \coloneqq \tau(H^{2m}(M)', \cdot)$.

Fact. If M is a closed, oriented $4m$-manifold, then M admits an orientation-reversing self-diffeomorphicm iff $\tau(M) = 0$. ◂

Theorem 3.10.

(1) *Let X_1 and X_2 be closed, oriented 4-manifolds and Y be an oriented cobordism between them, i.e. Y is a compact, oriented 5-manifold together with an orientation-preserving diffeomorphism $\partial Y^5 \cong (-X_1) \amalg X_2$.*[7] *Then, $\tau(X_1) = \tau(X_2)$.*

(2) *Conversely, if $\tau(X_1) = \tau(X_2)$, then X_1 and X_2 are cobordant.*

Therefore, in particular, the signature is a complete cobordism invariant.

Partial proof. For part (1), $H^2(-X_1 \amalg X_2)\otimes\mathbb{R}$ admits a middle-dimensional isotropic subspace by Proposition 3.4, hence has signature zero. But

$$\begin{aligned}\tau(-X_1 \amalg X_2) &= \tau((-H^2(X_1)\otimes\mathbb{R}) \oplus (H^2(X_2)\otimes\mathbb{R}))\\ &= \tau(-X_1) + \tau(X_2)\\ &= -\tau(X_1) + \tau(X_2) = 0.\end{aligned}$$ ⊠

We will not give a full proof of part (2). The idea is that cobordism classes of oriented 4-manifolds form an abelian group Ω_4^{SO} under disjoint union, and by part (1), the signature defines a homomorphism

$$\tau\colon \Omega_4^{\mathrm{SO}} \longrightarrow \mathbb{Z}.$$

This map must be surjective, because $\tau(\mathbb{CP}^2) = 1$ (where the orientation is the standard one coming from its complex structure). To prove it's injective, one uses Thom's cobordism theory, which identifies Ω_4^{SO} with a homotopy group of a space called a *Thom space*, then calculates that group using the Hurewicz theorem and calculation of the homology of the Thom space in question.

Next we discuss unimodular lattices mod 2.

Definition 3.11. A *characteristic vector c* for a unimodular lattice (Λ,σ) is a $c\in\Lambda$ such that $\sigma(c,x)\equiv \sigma(x,x) \bmod 2$.

Lemma 3.12. *The characteristic vectors form a coset of 2Λ in Λ.*

Proof. Let $\lambda = \Lambda/2\Lambda$, which is a vector space over $\mathbb{F}_2$. The freshman's dream mod 2 implies that the map $\lambda\to\mathbb{Z}/2$ sending $[x]\mapsto \sigma(x,x) \bmod 2$ is linear! Hence there is a symmetric bilinear form $\overline{\sigma}$ on λ induced by σ with determinant 1, hence is nondegenerate. Hence there exists a unique $\overline{c}\in\Lambda$ such that $\overline{\sigma}(x,x) = \overline{\sigma}(\overline{c},x)\in\mathbb{Z}/2$ for all $x\in\lambda$. The characteristic vectors are exactly the lifts of $\overline{c}$ to Λ, hence are a coset of 2Λ. ⊠

Remark. In the case of a simply connected 4-manifold M, the element $\overline{c}\in H^2(M)/2H^2(M)\cong H^2(M;\mathbb{Z}/2)$ is exactly the second Stiefel-Whitney class $w_2(TM)$. We'll talk about characteristic classes more next lecture. This follows from the *Wu formula*.

Moreover, the characteristic vectors $c\in H^2(M;\mathbb{Z})$ are exactly the first Chern classes of spin^c structures on M, and the Seiberg-Witten invariants arefunctions on the set of characteristic vectors to $\mathbb{Z}$. We'll say more about this later.

Most of this is true even in the case of non-simply-connected manifolds, but is harder. It is not true, however, that $H^2(M)/2H^2(M)\cong H^2(M;\mathbb{Z}/2)$. ◂

Lemma 3.13. *Let c and c' be characteristics for (Λ,σ). Then,*

$$\sigma(c,c)\equiv\sigma(c',c') \bmod 8.$$

[7] Here $-X_1$ denotes X_1 with the opposite orientation.

Proof. Write $c' - c = 2x$ for some $x \in \Lambda$. Then,

$$\sigma(c', c') = \sigma(c + 2x, c + 2x) = \sigma(c, c) + 4\underbrace{(\sigma(c, x) + \sigma(x, x))}_{(*)},$$

and $(*)$ is even. ⊠

Definition 3.14. Let (Λ, σ) be a unimodular form. Its type $t \in \mathbb{Z}/2$ is even if $\sigma(x, x)$ is even for all $x \in \Lambda$, and otherwise, it's odd.

Theorem 3.15 (Hasse-Minkowski theorem on unimodular forms)**.** *An indefinite unimodular form (Λ, σ) is classified up to isomorphism by three invariants:*

- *its rank* $\dim_{\mathbb{R}}(\Lambda \otimes \mathbb{R})$,
- *its signature $\tau \in \mathbb{Z}$, and*
- *its type $t \in \mathbb{Z}/2$.*

For the (quite nontrivial) proof, see Serre's *A Course in Arithmetic*. The proof idea is to solve the quadratic equation $\sigma(x, x) = 0$ for $x \in (\Lambda \otimes \mathbb{Q}) \setminus 0$. This is achieved via a *local-to-global principle* which says it suffices to find solutions $x_\infty \in \Lambda \otimes \mathbb{R}$ and $x_p \in \Lambda \otimes \mathbb{Q}_p$ (the p-adic numbers), and this can be done using unimodularity.

This is completely different from the positive definite (equivalently, negative definite) case, for which there are finitely many isomorphism classes below a given rank r, though this number grows rapidly with r and is only known in relatively few cases.

Example 3.16. Let $I_+ := (\mathbb{Z}, 1)$ with $\sigma(x, y) = xy$ and $I_- := (\mathbb{Z}, -1)$ with $\sigma(x, y) = -xy$. Then, $I_+^{\oplus m} \oplus I_-^{\oplus n}$ has rank $m + n$, signature $m - n$, and odd type.

There is a characteristic vector $c := (1, \dots, 1)$. In particular, $c^2 = m - n - \tau$, and therefore for any characteristic vector c, $c^2 \equiv \tau \bmod 8$. ◂

Example 3.17. Let

$$U := \left(\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$$

with quadratic form $(a, b)^2 := 2ab$. This is an even unimodular lattice with rank 2 and signature 0. ◂

The Hasse-Minkowski principle implies the following.

Corollary 3.18. *Let Λ be a unimodular lattice with signature τ and c be a characteristic vector for Λ. Then, $c \cdot c \equiv \tau \bmod 8$. In particular, if Λ is even, then $\tau \equiv 0 \bmod 8$.*

Proof. Either $\Lambda \oplus I_+$ or $\Lambda \oplus I_-$ is indefinite. It has odd type, and signature $\tau(\Lambda \oplus I_\pm) = \tau(\Lambda) \pm 1$, and if c is a characteristic vector for Λ, a characteristic vector for $\Lambda \oplus I_\pm$ is $c \oplus 1$ with square $c \cdot c = \pm 1$.

By Theorem 3.15, $\Lambda \oplus I_\pm \cong mI_+ \oplus nI_-$, so $c^2 \pm 1 \equiv \tau(\Lambda) \pm 1 \bmod 8$. ⊠

Hence an even, positive-definite, unimodular lattice has rank $8k$ for some k.

Example 3.19 (E_8 lattice)**.** The basic example is the E_8 lattice, associated to the Dynkin diagram for the exceptional simple Lie group E_8. As a matrix, it has the form

$$\begin{pmatrix} 2 & -1 & & & & & & -1 \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ -1 & & & & & & -1 & 2 \end{pmatrix}. \tag{3.20}$$

Hasse-Minkowski implies that any even unimodular lattice Λ is isomorphic to $mU \oplus (n \pm E_8)$. The intersection forms of interesting 4-manifolds tend to have E_8 terms. ◂

For more on the E_8 lattice, see the professor's class notes.

Lecture 4.

The intersection form and characteristic classes: 1/30/18

Let's start by writing down the homology and cohomology of a closed, oriented 4-manifold. Poincaré duality narrows the search space considerably.

$$\begin{aligned}
H_4(X) &\cong H^0(X) = \mathbb{Z}\cdot 1\\
H_3(X) &\cong H^1(X) = \operatorname{Hom}(\pi_1(X),\mathbb{Z})\\
H_2(X) &\cong H^2(X) \cong \operatorname{Hom}(H_2(X),\mathbb{Z}) \oplus H_1(X)_{\mathrm{tors}}\\
\pi_1(X)^{\mathrm{ab}} = H_1(X) &\cong H^3(X)\\
\mathbb{Z}\cdot[\mathrm{pt}] = H_0(X) &\cong H^4(X).
\end{aligned}$$

Additively, the homology and cohomology are determined up to isomorphism by π_1 and H_2. Together with the intersection form Q_X on H_2 (or its torsion-free quotient), we have a lot of information, but we still don't have everything: there could be cup products from H^1 and H^2 to H^3, and various other questions, e.g. what's the Hurewicz map $\pi_2(X)\to H_2(X)$? What about mod p coefficients?

If X is simply connected, the story is much simpler:

$$\begin{aligned}
H_4(X) &\cong H^0(X) = \mathbb{Z}\cdot 1\\
H_3(X) &\cong H^1(X) = 0\\
H_2(X) &\cong H^2(X) \cong \operatorname{Hom}(H_2(X),\mathbb{Z})\\
H_1(X) &\cong H^3(X) = 0\\
\mathbb{Z}\cdot[\mathrm{pt}] = H_0(X) &\cong H^4(X).
\end{aligned}$$

The only real information is H_2, which determines the cohomology additively. The intersection form determines $H^*(X)$ as a graded ring and $H_*(X)$ as a graded $H^*(X)$-module. All mod p cohomology classes are reductions of integral classes, which follows from the universal coefficient theorem. The natural map $\pi_2(X)\to H_2(X)$ is an isomorphism, by the Hurewicz theorem, which applies to any simply connected space. You can think about higher homotopy groups, and next time we'll think about the homotopy type of X, but thus far everything we can see has been determined by Q_X. (In fact, next time we'll see that it determines the homotopy type of X.)

So that knocks out the homotopy type, but we want further invariants. The next step is to investigate the tangent bundle, a distinguished rank-4 vector bundle. In particular, it has characteristic classes.

Remark. Spoiler alert: all characteristic classes we discuss today are determined by Q_X, so they don't define any new invariants.[8] Nonetheless, they provide useful tools for computing the intersection form, and therefore will be useful to us. ◀

We now review some of the theory of characteristic classes.

Example 4.1. Let $V\to X$ be a finite-rank real vector bundle over an arbitrary topological space X. The *Stiefel-Whitney classes* are characteristic classes of V in mod 2 cohomology with the following properties.

- The i^{th} *Stiefel-Whitney class* $w_i(V)\in H^i(X;\mathbb{Z}/2)$ for $i\ge 0$. The *total Stiefel-Whitney class* is
$$w(V) := w_0(V) + w_1(V) + \cdots$$
- $w_0(V) = 1$.
- If $i > \operatorname{rank} V$, $w_i(V) = 0$. Hence the total Stiefel-Whitney class is a finite sum.

It's a theorem that the Stiefel-Whitney classes are characterized by the following properties.

(1) If $f\colon Y\to X$ is continuous, $w_i(f^*V) = f^*w_i(V)$ for all i.
(2) If $i > \operatorname{rank} V$, $w_i(V) = 0$.
(3) The *Whitney sum formula*: if $U, V\to X$ are vector bundles, then $w(U\oplus V) = w(U)w(V)$.[9]

[8] In fact, Q_X determines the homotopy type of the classifying map for TX!

[9] If X is paracompact, every short exact sequence of vector bundles over X splits, so we may replace $U\oplus V$ with an extension of U by V.

(4) If $L \to \mathbb{RP}^1$ denotes the *tautological line bundle* whose fiber over a point $\ell \in \mathbb{RP}^1$ is the line ℓ itself, $w_1(L) \neq 0$ in $H^1(\mathbb{RP}^1; \mathbb{Z}/2) \cong \mathbb{Z}/2$.

For a proof, see Hatcher's notes on vector bundles and K-theory, or Milnor-Stasheff, which provides a somewhat baffling construction in terms of Steenrod algebra on the Thom space. There are various differing constructions which make various facts about Stiefel-Whitney classes easier to prove. ◂

If X is path-connected, the isomorphism $H^1(X; \mathbb{Z}/2) \cong \mathrm{Hom}(\pi_1(X); \mathbb{Z}/2)$ sends $w_1(V)$ to the orientation character of V. In particular, $w_1(V) = 0$ iff V is orientable, so if M is a manifold, M is orientable iff $w_1(TM) = 0$.

Now suppose M is a closed manifold and $V \to M$ is a rank-r vector bundle. Its top Stiefel-Whitney class $w_r(V) \in H^r(M; \mathbb{Z}/2) \cong H_{n-r}(M; \mathbb{Z}/2)$ maps to some homology class, which admits a representation by some codimension-r cycle. This cycle has an explicit description: let s be a section of V transverse to the zero section; then $s^{-1}(0)$ is codimension r and represents the homology class which is Poincaré dual to $w_r(V)$.

If V is an orientable bundle on a closed submanifold, its top Stiefel-Whitney class is the mod 2 reduction of its Euler class.

Proposition 4.2. *Let $H \in H^1(\mathbb{RP}^2; \mathbb{Z}/2) \cong \mathbb{Z}/2$ denote the generator, which is Poincaré dual to a hyperplane $\mathbb{RP}^{n-1} \subset \mathbb{RP}^n$. Then $w(T\mathbb{RP}^n) = (1+H)^{n+1}$.*

For a proof, see Milnor-Stasheff or the professor's notes.

Now we specialize to 4-manifolds.

Theorem 4.3 (Wu)**.** *Let X be a closed 4-manifold and $w := w_1^2(TX) + w_2(TX)$. Then w is the* characteristic element *of $H^2(X; \mathbb{Z}/2)$, i.e. for all $u \in H^2(X; \mathbb{Z}/2)$, $w \smile u = u \smile u$.*

There is a more general statement of Wu's theorem for manifolds in other dimensions. See Milnor-Stasheff.

For simply-connected 4-manifolds, the Stiefel-Whitney classes are determined by information we already have.

- If X is a closed, simply-connected 4-manifold, then $w_1(TX)$ vanishes, since H^1 does.
- Hence $w_2(TX) \cup u = u \smile u$ for all $u \in H^2(X; \mathbb{Z}/2)$. In this case, $H^2(X; \mathbb{Z}/2) = H^2(X)/2H^2(X)$; therefore $w_2(TX)$ is the mod 2 reduction of any characteristic vector for Q_X. Therefore w_2 provides no new information.
- $w_4(TX)$ is Poincaré dual to the zeroes of a vector field, so $w_4(TX) \frown [X]$ is the number of zeroes mod 2 of a generic vector field, i.e. the Euler characteristic mod 2. This is again not new information, since it can be read off $H^*(X)$.
- Finally, since H^3 vanishes, so must w_3. It's a theorem of Hirzenbuch and Hopf that $w_3(TX)$ vanishes for any closed, orientable 4-manifolds, and this is quite relevant for our class.

Remark. The more general version of Wu's theorem shows that on any closed, oriented 4-manifold X, $w_3(TX) = \mathrm{Sq}^1 w_2(TX)$. Here Sq^1 is the first *Steenrod square*, a cohomology operation, which has an explicit identification as the *Bockstein map* which measures whether a mod 2 cohomology class lifts to $\mathbb{Z}$ coefficients. In particular, $w_3(TX) = 0$ iff $w_2(TX)$ is the reduction of an integral class, and such lifts are the first Chern classes of spin^c structures. Thus the Hirzebruch-Hopf theorem is important for us because it implies that all closed, oriented 4-manifolds admit a spin^c structure. Since this theorem is trivial in the simply-connected case, though, we will not prove it. ◂

Example 4.4. *Chern classes* are characteristic classes $c_{2i}(E) \in H^{2i}(X; \mathbb{Z})$ for complex vector bundles $E \to X$; again the *total Chern class*

$$c(E) := c_0(E) + c_1(E) + \cdots$$

and again $c_0(E) = 1$ and $c_i(E) = 0$ for $i > \operatorname{rank} E$. The Chern classes are uniquely characterized by similar axioms:

(1) $c_i(E) = 0$ if $i > \operatorname{rank} E$.
(2) $c(E \oplus F) = c(E)c(F)$.
(3) If $L \to \mathbb{CP}^1$ denotes the tautological line bundle $L \to \mathbb{CP}^1$, then the Poincaré dual of $c_1(L) \in H^2(\mathbb{CP}^1; \mathbb{Z})$ is $-1 \in H_0(\mathbb{CP}^1)$ (where we take as a generator any positively oriented point).

◂

Proposition 4.5. *If $H \in H^2(\mathbb{CP}^n)$ denotes the Poincaré dual to a hyperplane, so $H = -c_1$ of the tautological line bundle over $\mathbb{CP}^n$, then $c(T\mathbb{CP}^n) = (1+H)^{n+1}$. The argument is formally identical to the real case.*

Definition 4.6. Isomorphism classes of complex line bundles on X form a group under tensor product; this is called the *topological Picard group* and denoted $\mathrm{Pic}(X)$.

If M is a closed, oriented manifold and $E \to M$ is a rank-r complex vector bundle, then the Poincaré dual of $c_r(E)$ is the zero locus of a generic section of E (i.e. transverse to the zero section).

Proposition 4.7. *The first Chern class defines a homomorphism $c_1\colon \mathrm{Pic}(X) \to H^2(X;\mathbb{Z})$, and this is an isomorphism.*

In particular, for line bundles L_1 and L_2, $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$. One way to use this is to use the above characterization of the Poincaré dual of the top Chern class; another is to show that $B\mathrm{U}_1$, the classifying space for complex line bundles, is a $K(\mathbb{Z}, 2)$, an Eilenberg-Mac Lane space, hence representing cohomology.

For a reference for the following theorem, see Hatcher's notes on vector bundles and K-theory.

Theorem 4.8. *Let $E \to X$ be a complex vector bundle and $E_{\mathbb{R}}$ denote the underlying real vector bundle.*

- *$w_{2i}(E_{\mathbb{R}})$ is the mod 2 reduction of $c_i(E)$.*
- *$w_{2i+1}(E_{\mathbb{R}}) = 0$.*

In particular, $w_2 = c_1 \bmod 2$.

In order to apply Chern classes to manifolds, we need some sort of complex structure.

Definition 4.9. Let M be an even-dimensional manifold. An *almost complex structure* on M is a $J \in \mathrm{End}(TM)$ such that $J^2 = -\mathrm{id}$.

This structure makes TM into a complex vector bundle, where i acts as J. Then we have access to Chern classes, albeit depending on J.

Remark. As the notation suggests, complex manifolds are almost complex. ◀

Example 4.10. One example of a complex manifold is a complex hypersurface in $\mathbb{CP}^n$: let F be a degree-d homogeneous polynomial in $x_0, \dots, x_d$ and $X \coloneqq \{F = 0\} \subset \mathbb{CP}^n$. Homogeneity means this makes sense in projective space; if we additionally assume that whenever $F = 0$ at least one partial derivative of F is nonzero, then X is smooth.

To study complex manifolds, we can import tools from algebraic geometry. A holomorphic hypersurface D in a complex manifold M defines an invertible sheaf (i.e. complex line bundle) $\mathscr{O}_M(D)$ whose sections over $U \subset M$ are meromorphic functions on U with only simple poles along $D \cap U$. Moreover, if $N_{D/M} \coloneqq TM|_D/TD$ denotes the normal bundle to $D \hookrightarrow M$, there's an isomorphism $\mathscr{O}_M(D)|_D \cong N_{D/M}$.

On $\mathbb{CP}^n$, a holomorphic line bundle E is determined by its degree $d \coloneqq c_1(E) \in H^2(\mathbb{CP}^n)$, because the holomorphic sheaf cohomology $H^1(\mathbb{CP}^n; \mathscr{O}_{\mathbb{CP}^n}) = 0$. If $L \to \mathbb{CP}^n$ denotes the tautological line bundle, then L^* is the positive generator of $\mathrm{Pic}\,\mathbb{CP}^n$, so there's an isomorphism $E \cong (L^*)^{\otimes d}$.

It follows that $\mathscr{O}_{\mathbb{CP}^n}(X) \cong (L^*)^{\otimes d}$, so $c_1(\mathscr{O}_{\mathbb{CP}^n}(X)) = dH$, where $H = c_1(L^*)$ as before. Hence if $h = i^*H \in H^2(X)$, where $i\colon X \hookrightarrow \mathbb{CP}^n$ is inclusion, then

$$c_1(N_{X/\mathbb{CP}^n}) = i^* c_1(\mathscr{O}_{\mathbb{CP}^n}(X)) = dh. \tag{4.11}$$

Using the short exact sequence

$$0 \longrightarrow TX \longrightarrow T\mathbb{CP}^n|_X \longrightarrow N_{X/\mathbb{CP}^n} \longrightarrow 0, \tag{4.12}$$

we get that

$$\begin{aligned} i^* c(T\mathbb{CP}^n) &= c(TX) c(N_{X/\mathbb{CP}^n}) \\ (1+h)^{n+1} &= c(TX)(1+dh), \end{aligned}$$

and therefore

$$c_j(TX) + dh c_{j-1}(TX) = \binom{n+1}{j} h^j. \tag{4.13}$$

It's possible to explicitly solve this when you have a specific X; the base case is

$$c_1(TX) = (n+1-d)h. \tag{4.14}$$

◀

Example 4.15. Let $n = 3$, so X is a degree-d complex surface, hence a closed, oriented 4-manifold. In that case

$$\begin{aligned} c_1(TX) &= (4-d)h \\ c_2(TX) &= (d^2 - 4d + 6)h^2. \end{aligned} \tag{4.16}$$

Since $[X] \in H_4(\mathbb{CP}^3) \cong H^2(\mathbb{CP}^3)$ is identified with dH under Poincaré duality, then

$$c_2(TX) \frown [X] = d(d^2 - 4d + 6). \tag{4.17}$$

Since this is again the number of zeros (with orientation) of a generic vector field, this integer is the the Euler characteristic, so we have an explicit formula for the Euler characteristic of a degree-d complex surface:

$$\chi(X) = d(d^2 - 4d + 6). \tag{4.18}$$

◂

Theorem 4.19 (Lefschetz hyperplane theorem)**.** *Suppose $n \geq 3$ and X is a hypersurface in $\mathbb{CP}^n$. Then X is simply connected.*

There is a more general version of this theorem.

Therefore if X is a hypersurface in $\mathbb{CP}^n$, its first and third Betti numbers vanish, so

$$\chi(X) = 1 + b_2(X) + 1, \tag{4.20}$$

so

$$b_2(X) = d(d^2 - 4d + 6) - 2. \tag{4.21}$$

So we know the dimension of H^2. It's possible to write down bases using this information, though we won't get into this.

Example 4.22. The *Pontrjagin classes* $p_i(V) \in H^{4i}(X)$ of a real vector bundle V are defined by

$$p_i(V) \coloneqq (-1)^i c_{2i}(V \otimes \mathbb{C}).$$

◂

Pontrjagin classes satisfy very similar axioms to Stiefel-Whitney and Chern classes; however, be aware that which ones vanish might be tricky. For example, the complexification of a rank-3 vector bundle is a rank-3 complex vector bundle, hence only has access to c_0 and c_2 for defining Pontrjagin classes.

If X is a closed oriented 4-manifold, TX has only one nonvanishing Pontrjagin class, which is $p_4(TX) \in H^4(X) \cong \mathbb{Z}$.

Lemma 4.23. *Let X be a closed, oriented 4-manifold. Then $\sigma(X) \coloneqq p_1(TX) \frown [X] \in \mathbb{Z}$ is an oriented cobordism invariant.*

The basic idea is that if W is a 5-manifold bounding X, then $TW = TM \oplus \underline{\mathbb{R}}$, which implies $p_1(W) = i^* p_1(M)$, where $i \colon M \hookrightarrow W$ is inclusion.

Lemma 4.24. *If $V \to X$ is a complex tangent bundle, $p_1(V_{\mathbb{R}}) = c_1(V)^2 - 2c_2(V)$.*

For a proof, see the notes.

Recall that this cobordism group $\Omega_4^{\mathrm{SO}} \cong \mathbb{Z}$, and that the signature $\tau \colon \Omega_4^{\mathrm{SO}} \to \mathbb{Z}$ is an isomorphism. Hence $\sigma(X)$ must be proportional to $\tau(X)$.

Theorem 4.25 (Hirzebruch signature theorem)**.** *$\sigma = 3\tau$, i.e. on a closed, oriented 4-manifold X, $p_1(X) \frown [X] = 3\tau(X)$.*

Proof. The proof is corollary of Thom's work: we just have to check on a generator of Ω_4^{SO}, such as $\mathbb{CP}^2$, which has signature 1. Then we use Lemma 4.24: $c_1(T\mathbb{CP}^2) = 3H$ and $c_2(T\mathbb{CP}^2) = \chi(\mathbb{CP}^1) = 3$, so

$$p_1(T\mathbb{CP}^2) = c_1^2(\mathbb{CP}^2) - c_2(\mathbb{CP}^2) = 9 - 2 \cdot 3 = 3. \tag{4.26}$$

⊠

This also implies that if X_d is a degree-d hypersurface, its signature is

$$\begin{aligned} \tau(X_d) &= \frac{1}{3}(c_1^2(TX_d) - c_2(X_d)) \frown [X_d] \\ &= -\frac{1}{3}(d-2)d(d+2). \end{aligned}$$

Certainly $(d-2)d(d+2)$ is divisible by 3, so this produces an interger, even if we didn't already know that. Moreover,

$$\begin{aligned} w_2(TX_d) &= c_1(TX_d) \bmod 2 \\ &= (4-d)h \bmod 2 \\ &= dh \bmod 2. \end{aligned}$$

Hence d is even iff $w_2(TX_d) = 0$ iff X_d has even type (i.e. its intersection form has even type). For $d \geq 2$, the intersection form is indefinite, so Theorem 3.15 tells us there's a unique intersection form in this class.

Example 4.27. For $d = 4$, X_d is a *K3 surface*, and one concludes that $c_1 = 0$, $b_2 = 24$, $\tau = -16$, and the intersection form has even type. Hence by Theorem 3.15 the intersection form is $3U \oplus 2(-E_8)$. ◀

The point is that we can explicitly compute intersection forms in examples of interest, and how characteristic classes made this a bit easier.

Lecture 5.

Tangent bundles of 4-manifolds: 2/1/18

This lecture has two goals. The first is to show that if X is a closed, simply-connected manifold, then its tangent bundle is essentially determined by w_2, τ, and χ. The second is to discuss the following theorem.

Theorem 5.1 (Rokhlin)**.** *Let X be a closed, oriented 4-manifold. Then $16 \mid \tau(M)$.*

There are different proofs of this; today we're going to focus on its equivalence to the following fact about stable homotopy groups of the spheres.

Proposition 5.2. *Let $k \geq 5$. Then $\pi_{3+k}(S^k) \cong \mathbb{Z}/24$.*

Obstruction theory For a reference for this part of the lecture, see Hatcher's notes on vector bundles and K-theory.

Obstruction theory is about the following question: let $\pi\colon E \to X$ be a fiber bundle where X is a CW complex. When is there a section of X?

At first, we will assume X is simply connected; we will be able to weaken this hypothesis later. Let F denote the typical fiber of π, i.e. the fiber over a chosen $x \in X$ (which we'll assume is in the 0-skeleton). We'll construct the section s inductively as a series of sections $s^k\colon X^{(k)} \to E|_{X^{(k)}}$ (where $X^{(k)}$ denotes the k-skeleton). If we're given s^k, when does it *not* extend to s^{k+1}?

Let $\Phi\colon (D^{k+1}, \partial D^{k+1}) \to (X^{(k+1)}, X^{(k)})$ denote the inclusion of a $(k+1)$-cell and $\phi := \Phi|_{\partial D^{k+1}}\colon S^k \to X^{(k)}$ be the attaching map. Then $\phi^*(s^k) := s^k \circ \phi$ is a section of the fiber bundle $\phi^* E \to S^k$. This bundle sits inside of $\Phi^* E \to D^{k+1}$, which is trivially trivial (i.e. canonically trivialized), because D^{k+1} is contractible. Therefore you can think of $\phi^* s^k$ as a map to a fiber: $S^k \to E_{\Phi(0)}$, and canonically up to homotopy, $E_{\Phi(0)} \cong F$ since X is simply connected.

Therefore we have a map $\{(k+1)\text{-cells}\} \to \pi_k F$ sending Φ to the map $\phi^* s^k\colon S^k \to F$. Since $\pi_k(F)$ is an abelian group (here $k > 1$), this map defines a cellular cochain

$$o^{k+1}(E, S^k) \in C^{k+1}(X; \pi_k(F)).$$

Let $\mathfrak{o}^{k+1} \in H^{k+1}(X; \pi_k(F))$ denote the cohomology class of o^{k+1}; this depends only on the homotopy class of s^k, and if $\mathfrak{o}^{k+1}$ vanishes, then s^k extends to a section s^{k+1} on the $(k+1)$-skeleton.[10]

Definition 5.3. Suppose $\pi_i F = 0$ for $i < k$. Then the *primary obstruction* for E is $\mathfrak{o}^{k+1}(X; \pi_k(F))$.

This is an invariant of the fiber bundle, hence is easier to understand than the higher obstructions, which depend on the choices of s^k that ones makes.

Remark. If we relaxed the assumption that X is simply connected, then we'd have to use local coefficients with the action of $\pi_1(X)$ on $\pi_k(F)$. So if $\pi_1(X)$ acts trivially on $\pi_i F$ for $i \leq k$, the story continues in almost the same way. ◀

[10]One may have to replace s^k by a section homotopic to it.

Stiefel-Whitney classes as primary obstructions. Though they are usually presented differently, Stiefel-Whitney classes were historically discovered as obstructions to collections of sections of vector bundles.

Definition 5.4. Let $E \to X$ be a real, rank-n vector bundle with a Euclidean metric. Then $V_k(E) \to E$ denotes the *Stiefel bundle*, the fiber bundle whose fiber at x is $V_k(E_x)$, the *Stiefel manifold* of orthonormal k-frames for E_x; standard fiber-bundle methods construct $V_k(E) \to X$ in a canonical fashion.

The typical fiber of $V_k(E)$ is $V_k(\mathbb{R}^n)$. A section of $V_k(E)$ is a k-tuple of orthonormal, hence linearly independent, sections of $E \to X$.

Stiefel and Whitney applied obstruction theory to $V_k(E)$ to understand when E admits linearly independent sections.

Proposition 5.5. *$V_k(\mathbb{R}^n)$ is $(n-k-1)$-connected, and*

$$\pi_{n-k}(V_k(\mathbb{R}^n)) \cong \begin{cases} \mathbb{Z}, & n-k \text{ even or } k=1, \\ \mathbb{Z}/2, & \text{otherwise.} \end{cases}$$

For example, when $n=1$, this is telling us that the first homotopy group of S^k is $\pi_k S^k = \mathbb{Z}$. We're not going to prove Proposition 5.5; the idea is to use the long exact sequence of homotopy groups associated to a fiber bundle.

Remark. It's worth thinking through what information is needed to identify a homotopy group with $\mathbb{Z}$; there are two choices, so this is something like requiring an orientation. ◀

Hence the primary obstruction to finding a section of $V_k(E) \to X$ is an $\mathfrak{o}^{n-k+1} \in H^{n-k+1}(X; \pi_{n-k}(V_k(\mathbb{R}^n)))$. Thus we have characteristic classes for E,

$$\mathfrak{o}_n^k(E) \in \begin{cases} H^{n-k+1}(X;\mathbb{Z}), & n-k \text{ even or } k=1, \\ H^{n-k+1}(X;\mathbb{Z}/2), & \text{otherwise.} \end{cases}$$

The canonical nature of this construction means these are natural under pullback of vector bundles, hence are indeed characteristic classes.

In either case, there is a mod 2 characteristic class $\overline{\mathfrak{o}}_k^n \in H^{n-k+1}(X;\mathbb{Z}/2)$, and this actually makes sense irrespective of $\pi_1 X$, because $\mathrm{Aut}(\mathbb{Z}/2) = 1$, so all $\mathbb{Z}/2$ local systems are trivial.

Theorem 5.6. $\overline{\mathfrak{o}}_k^n(E) = w_{n-k+1}(E)$.

This perspective on Stiefel-Whitney classes shows that some of them come with canonical integral lifts. For $k=1$, $\mathfrak{o}_1^n(E) \in H^n(X;\mathbb{Z})$ is the Euler class of E, which requires a choice of orientation on E (unless you use coefficients twisted by the orientation bundle of E). You can take this as the definition of the Euler class if you wish.

Applications to 4-manifolds. For a reference, see the classical papers by Dold-Whitney and Hirzebruch-Hopf.

When $E \to X$ is a rank-4 real vector bundle, Theorem 5.6 tells us that $w_2(E)$ is the obstruction to finding 3 linearly independent sections of E over $X^{(2)}$. If ℓ denotes an orthogonal complement to these three sections given a metric on E, then $\ell \to X$ is a line subbundle of E, which is trivial iff $w_1(E) = 0$.

Corollary 5.7. *Let X be a closed, oriented 4-manifold. Then $w_2(TX)$ is the complete obstruction to trivializing TX over $X \setminus \mathrm{pt}$.*

Proof. Choose a metric g for TX and a CW structure for $Y = X \setminus \mathrm{pt}$. Over $Y^{(2)}$, we can find orthonormal vector fields $v_1, v_2, v_3 \in \Gamma(TX|_Y)$ by Theorem 5.6, and also a unit vector $v_4 \in (v_1, v_2, v_3)^\perp$. This is because the line bundle $\ell \subset TX$ is trivial: X is orientable, so $w_1(TX) = 0$.

Let $P \to Y$ denote the principal SO_4-bundle of oriented orthonormal frames for Y; then, (v_1, v_2, v_3, v_4) is a section of $P|_{Y^2} \to Y^2$. The obstruction to extending this to Y^3 lies in $H^3(Y; \pi_2 \mathrm{SO}_4)$. We'll see later that the universal cover for SO_4 is $S^3 \times S^3$, so

$$\pi_2(\mathrm{SO}_4) = \pi_2(S^3 \times S^3) = \pi_2(S^3) \times \pi_2(S^3) = 0. \tag{5.8}$$

(It's actually true that for any Lie group G, $\pi_2(G) = 0$. But here we don't need the full, harder result.) The upshot is, the obstruction vanishes, so $(v_1, \dots, v_4)$ extends to a section on $Y^{(3)}$. And the cohomology of a punctured 4-manifold vanishes above dimension 3, so all further obstructions vanish. ⊠

Theorem 5.9. *Let X be a closed, oriented 4-manifold, and suppose that $T, T' \to X$ are two rank-4 oriented vector bundles such that $w_2(T) = w_2(T') = 0$.*[11] *Then*

(1) $T \oplus \underline{\mathbb{R}} \cong T' \oplus \underline{\mathbb{R}}$ *iff* $p_1(T) = p_1(T')$, *and*
(2) $T \cong T'$ *as oriented vector bundles iff* $p_1(T) = p_1(T')$ *and* $e(T) = e(T')$.

Recall that the Euler class is the Poincaré dual to the zero set of a generic section.
To prove this, we'll need a result that will be useful again.

Lemma 5.10. *There exists a rank-4 vector bundle $E \to S^4$ with $\langle p_1(E), [S^4]\rangle = -2$ and $\langle e(E), [S^4]\rangle = 1$.*

Proof sketch. Though it's possible to give an explicit construction, one can produce this bundle as a representative of the Bott element $\beta \in K^0(S^4)$ ⊠

Proof of Theorem 5.9. We'll use *Pontrjagin-Thom collapse* to construct a map $f\colon X \to S^4$ which is smooth (at least near an $x \in X$) of degree 1 with $D_x f$ an isomorphism such that $f^{-1}(f(x)) = x$. The idea is to product a map $X \to D^4$ which sends everything outside a disc neighborhood of X to ∂D^4, then collapse by the identification $D^4/\partial D^4 \cong S^4$.

Since $w_2(T) = 0$, there's some ball $B \subset X$ containing x such that T is trivial over $X \setminus B$. Therefore $T \cong f^*U$ for some rank-4 oriented bundle $U \to S^4$. Thus $p_1(T) = f^*p_1(U)$ and $e(T) = f^*e(U)$, and of course the same is true for T' and a $U' \to S^4$. Therefore it suffices to prove the theorem on S^4.

Bundles on the 4-sphere aren't so complicated. Let D_+ and D_- denote the two hemispheres, so $S^4 = D_+ \cup_{S^3} D_-$. $U, U' \to D_\pm$ are canonically trivialized, but the two trivializations of U over D_+ and D_- need not agree; instead, they are related by a map $S^3 \to \mathrm{SO}_4$ called the *clutching function*, which defines an element of $\pi_3\mathrm{SO}_4$.

Conversely, given a $[\gamma] \in \pi_3\mathrm{SO}_4$, one can construct a vector bundle $E_\gamma \to \mathrm{SO}_4$ by gluing the trivial bundles on D_+ and D_- by any representative γ for $[\gamma]$. In the same way, rank 5 vector bundles on S^4 correspond to elements of $\pi_3\mathrm{SO}_5$.

Now we prove part (1). By what we've shown so far, $U \oplus \underline{\mathbb{R}} \cong U' \oplus \underline{\mathbb{R}}$ iff they have the same clutching function in $\pi_3\mathrm{SO}_5 \cong \mathbb{Z}$.[12] The first Pontrjagin number defines a homomorphism

$$\begin{aligned}\pi_3\mathrm{SO}(5) &\longrightarrow \mathbb{Z}\\ [\gamma] &\longmapsto \langle p_1(E_\gamma), [S^4]\rangle,\end{aligned}$$

and by Lemma 5.10, there exists a vector bundle on S^4 with nonzero Pontrjagin number. Hence this map is injective.

To prove part (2), we look at $\pi_3\mathrm{SO}_4 \cong \pi_3(S^3 \times S^3) = \pi_3 S^3 \times \pi_3 S^3 = \mathbb{Z}^2$. Hence we can define a map

$$\begin{aligned}\pi_3(\mathrm{SO}_4) \cong \mathbb{Z}^2 &\longrightarrow \mathbb{Z}^2\\ [\gamma] &\longmapsto (\langle p_1(E_\gamma), [S^4]\rangle, \langle e(E_\gamma), [S^4]\rangle).\end{aligned}$$

Again, using Lemma 5.10, there's a bundle E whose image under the above map is $(-2, 1)$. For TS^4, $p_1 = 0$ and $e = 2$, so the above map is given by the matrix

$$\begin{pmatrix} -2 & 0 \\ 1 & 2 \end{pmatrix},$$

hence is injective. ⊠

Remark. The theorem is true more generally assuming that $w_2(T) = w_2(T') \neq 0$: p_1 determines their stable isomorphism class and (p_1, e) determines their unstable isomorphism class. This requires a more sophisticated use of obstruction theory, and the proof is more involved. ◀

Corollary 5.11. *Suppose that X and X' are closed, oriented, simply-connected 4-manifolds and $f\colon X' \to X$ is a homotopy equivalence. Then $f^*TX \cong TX'$ as oriented vector bundles.*

Proof. The intersection form Q_X determines $p_1 = 3\tau$ and $e = \chi$, hence the isomorphism classes of f^*TX and TX' coincide. ⊠

[11] So we could say that T and T' are two spin vector bundles.
[12] Again, if G is any simple Lie group, $\pi_3 G = \mathbb{Z}$.

Thus TX knows no more than the (oriented) homotopy type of a closed, oriented, simply-connected manifold.

Next time we'll turn to Rokhlin's theorem.