THE STABLE PONTRJAGIN-THOM THEOREM

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FEBRUARY 23, 2016

ABSTRACT. The stable Pontrjagin-Thom theorem provides an isomorphism between the stable homotopy groups of spheres and the framed bordism classes of manifolds, allowing one to understand a major object in homotopy theory geometrically. In this talk, I will define each of these rings and provide a proof of the theorem.

1. THE STABLE HOMOTOPY GROUPS OF SPHERES

Definition. The n^{th} *stable homotopy group of the spheres* is $\pi_n^S = \pi_{n+k}(S^k)$ for $k \gg 0$.

By the Freudenthal suspension theorem, this is well-defined, and $\pi_n^S = \varinjlim_k \pi_{n+k}(S^k)$. This limit is taken along the sequence $S^1 \hookrightarrow S^2 \hookrightarrow S^3 \hookrightarrow \cdots$, where $S^k \hookrightarrow S^{k+1}$ as the equator.

For example, $\pi_0^S = \mathbb{Z}$, $\pi_1^S = \mathbb{Z}$, and $\pi_2^S = \mathbb{Z}/2$. For large n, calculating these groups is open.

We can place a graded ring structure on these groups, too.[1]

Definition. Let

$$\pi_\bullet^S = \bigoplus_{n=0}^{\infty} \pi_n^S.$$

This is a $\mathbb{Z}$-graded ring[2] (where if $n < 0$, $\pi_n^S = 0$), with multiplication given by smash product: given maps $f : S^k \to S^m$ and $g : S^\ell \to S^n$, we define $[f] \cdot [g]$ to be the homotopy class of $f \wedge g : S^k \wedge S^\ell \to S^m \wedge S^n$, since $S^k \wedge S^\ell \cong S^{k+\ell}$. The fact is this is independent of choice of homotopy class and of stabilization, so it turns $\pi_\bullet^S$ into a $\mathbb{Z}$-graded ring, called the *stable homotopy groups of spheres*.

2. FRAMED BORDISM

If you're feeling bord, here's the geometric part.

Definition. Let Y_0 and Y_1 be compact, n-dimensional smooth manifolds. Then, a *bordism* between Y_0 and Y_1 is an $(n+1)$-dimensional manifold with boundary X such that $\partial X \cong Y_0 \amalg Y_1$.

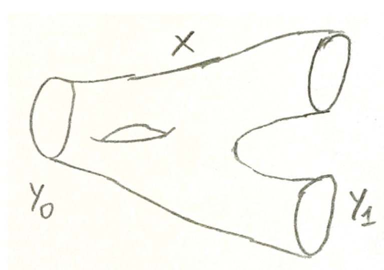

FIGURE 1. A bordism between Y_0 and Y_1.

[1] Recall that a $\mathbb{Z}$-*graded ring* R is a ring that is a direct sum, indexed by $\mathbb{Z}$, of abelian subgroups R_i, and such that multiplication is a map $R_i \times R_j \to R_{i+j}$. The simplest example is a ring of polynomials, with the grading given by the degree. A morphism of $\mathbb{Z}$-graded rings is just a ring homomorphism that preserves this grading.

[2] "If you like it, then you better put a ring on it."

This is an equivalence relation, and the set of bordism classes of compact manifolds also has a $\mathbb{Z}$-graded ring structure: addition is disjoint union, multiplication is Cartesian product, and the grading is given by the dimension. This ring is denoted Ω^{O}.

More generally, we can consider bordisms respecting some sort of structure on Y_0 and Y_1, such as orientation, and in general we get other $\mathbb{Z}$-graded rings. Today, the structure we're going to look at is stable framing.

Definition. A *stable framing*[3] of a manifold X is a stable trivialization of its tangent bundle, i.e. a morphism of vector bundles

$$\begin{tikzcd} TX \oplus \underline{\mathbb{R}}^m \arrow[rr, "\cong"] \arrow[dr] & & \underline{\mathbb{R}}^{m+\dim X} \arrow[dl] \\ & X. & \end{tikzcd}$$

A nontrivial example of a stable tangential framing is for the unit sphere S^n inside $\mathbb{R}^{n+1}$. The normal bundle ν_{S^n} can be trivialized, so $\underline{\mathbb{R}}^n = TS^n \oplus \nu_{S^n} = TS^n \oplus \underline{\mathbb{R}}$ is a stable framing of S^n.

If X is a bordism from Y_0 to Y_1, then a stable framing on X restricts to one on Y_0 and on Y_1, so it makes sense to refer to a bordism that preserves stable framings, and the bordism classes under this relation form a $\mathbb{Z}$-graded ring, denoted $\Omega_\bullet^{\mathrm{fr}}$, the *framed bordism ring*.

3. The Stable Pontrjagin-Thom Theorem

The big result is that these two constructions are the same.

Theorem 1 (Stable Pontrjagin-Thom)**.** *There is an isomorphism $p : \pi_\bullet^S \to \Omega_\bullet^{\mathrm{fr}}$ of $\mathbb{Z}$-graded rings.*

This is a quite different, but equivalent, perspective on $\pi_\bullet^S$, and as such is quite useful: it's completely geometric, unlike the very abstract stable homotopy groups.

The proof has five main steps.

1. First, we'll reduce to the unstable case by defining framed bordisms $\Omega^{\mathrm{fr}}_{m;S^n}$ within S^n.
2. Next, we'll define a map $\phi : \pi_{n+k}(S^n) \to \Omega^{\mathrm{fr}}_{k;S^n}$.
3. Then, we'll define a map $\psi : \Omega^{\mathrm{fr}}_{k;S^n} \to \pi_{n+k}(S^n)$.
4. We'll show that ϕ and ψ are inverses.
5. Finally, we'll deal with the graded ring structure.

3.1. Reducing to the Unstable Case.

Definition. Let M be a smooth, m-dimensional manifold.

- A *framing* of an n-dimensional submanifold $N \subset M$ is a continuous choice of basis for its normal bundle ν_N; that is, we have $\operatorname{codim}_M N$ sections of ν_N that are a basis on each fiber.
- Two n-dimensional submanifolds $N_1, N_2 \subset M$ are *framed bordant* if they are the restriction of a framed $(n+1)$-submanifold of $[0,1] \times M$ to its boundary. The set of these bordism classes is denoted $\Omega^{\mathrm{fr}}_{m-n;M}$.

A framed bordism is equivalent to the existence of a connected, oriented manifold Q and a smooth map $f : [0,1] \times M \to Q$ such that $\{0\} \times N_1$ and $\{1\} \times N_2$ are the preimages of two regular values of f.

In order to reduce from $\Omega_\bullet^{\mathrm{fr}}$ to $\Omega^{\mathrm{fr}}_{k;S^n}$, we'll need to pass from stable tangential framings to normal framings. A *stable normal framing* is, in the same manner as stable tangential framing, a trivialization of $\nu_N \oplus \underline{\mathbb{R}}^k$ for some k.

Lemma 2. *If M is an m-dimensional submanifold of S^n, then homotopy classes of stable normal and stable tangential framings are in correspondence.*

Proof sketch. Given a stable tangential framing $TM \oplus \underline{\mathbb{R}}^k \to \underline{\mathbb{R}}^{k+m}$, the usual tangent bundle-normal bundle split exact sequence induces

$$0 \longrightarrow TM \oplus \underline{\mathbb{R}}^k \longrightarrow TS^n|_M \oplus \underline{\mathbb{R}}^k \longrightarrow \nu_M \longrightarrow 0,$$

[3] Why is this called a stable framing, you may ask? The inclusion of S^1 into $\mathbb{R}^2$ defines a stable framing of S^1, because $TS^1 \oplus \nu_{S^1} = T\mathbb{R}^2$, which is trivial. If you draw your tangent and normal vectors large enough, it looks like an innocent S^1 is trapped in a cell, saying, "help, I've been framed!"

but the middle entry is trivial, as we already saw, and splitting the sequence gives us a stable normal framing. The proof in the other direction is very similar. $\square$

Now, we can restate the Pontrjagin-Thom isomorphism in terms of one at a finite level.

- Let M be an n-manifold with a stable tangential framing, so M defines a class in $\Omega_\bullet^{\mathrm{fr}}$. Using the Whitney embedding theorem, M embeds into S^k for some k, by Lemma 2 its stable tangential framing induces a stable normal framing $\nu_M \oplus \underline{\mathbb{R}}^{k+\ell} \cong \underline{\mathbb{R}}^{k+\ell+n}$. The inclusion $S^k \hookrightarrow S^{k+1}$ as the equator makes this a stable normal framing of one dimension lower, and doing this ℓ times gives us a genuine normal framing for $M \hookrightarrow S^{k+\ell}$, defining a class in $\Omega^{\mathrm{fr}}_{k+\ell-n;S^{k+\ell}}$. This is independent of choice of embedding, because any two such embeddings will be framed bordant in some sufficiently high-dimensional sphere.
- In the reverse direction, a normal framing of a submanifold $M \subset S^k$ is a stable normal framing, and in particular induces a stable tangential framing by Lemma 2, and this is invariant under the inclusion $S^k \hookrightarrow S^{k+1}$.

Both of these send disjoint unions to disjoint unions, so we have an isomorphism of abelian groups $\Omega_\bullet^{\mathrm{fr}} \cong \bigoplus_n \Omega^{\mathrm{fr}}_{n;S^{n+k}}$. If we can show that $\pi_n^S \cong \Omega^{\mathrm{fr}}_{n;S^{n+k}}$ for large k, then $\pi_\bullet^S \cong \Omega_\bullet^{\mathrm{fr}}$ as abelian groups. Then, the last step will be to understand why the graded ring structures are isomorphic.

3.2. **Forward Map: Preimage of a Regular Value.** We'll define $\phi : \pi_{n+k}(S^n) \to \Omega^{\mathrm{fr}}_{n;S^{n+k}}$ as "preimage of a regular value." First, though, to translate homotopy theory to geometry, we'll need that every class in $\pi_{n+k}(S^k)$ can be represented by a smooth map $S^{n+k} \to S^k$, which follows from the Whitney approximation theorem.

Given an $f : S^{n+k} \to S^n$ representing a class in $\pi_{n+k}(S^n)$, by Sard's theorem there's a regular value $x \in S^n$; define $\phi([f]) = [f^{-1}(p)] \in \Omega^{\mathrm{fr}}_{n;S^{n+k}}$. This is well-defined, because if $f_0, f_1 : S^{n+k} \to S^n$ are homotopic, let $F : [0,1] \times S^{n+k} \to S^n$ be a smooth homotopy between them. Then, the preimage of a regular value of F is a framed bordism, after making things transverse, etc. (and this is independent of choice of regular value because any one can be rotated into another).

Finally, what happens with the group structure? Addition $f + g$ in $\pi_{n+k}(S^n)$ is given by thinking of the spheres as cubes and joining faces, and up to homotopy, the intersection can be made arbitrarily small, so the preimages of f and g can be made disjoint. Since we care only about their bordism class, this is good enough.

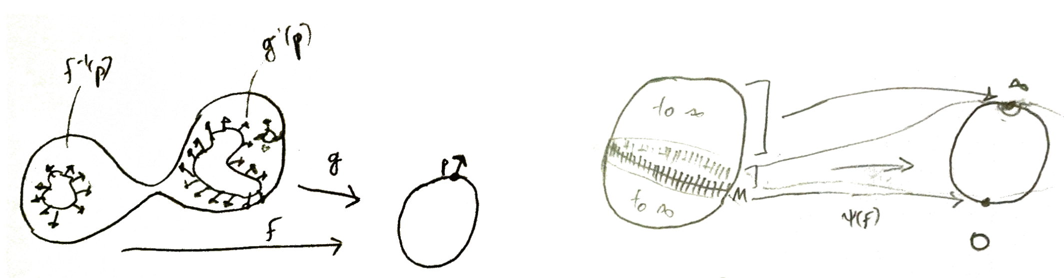

FIGURE 2. Left: ϕ is a group homomorphism. Right: Pontrjagin-Thom collapse.

3.3. **Reverse Map: Pontrjagin-Thom Collapse.** Suppose I have a framed k-dimensional submanifold $M \subset S^{n+k}$. Let U be a tubular neighborhood of M; the framing on M induces a map $h : U \to M \times \mathbb{R}^n$ sending an element to its representation in that basis. Then, pick a *cutoff function* $\rho : [0,\infty) \to (0,1)$, a smooth, decreasing function such that $\rho(0) = 1$ and that tends asymptotically to zero.

Now, regarding $S^n = \mathbb{R}^n \setminus \{\infty\}$, let $f_M : S^{n+k} \to S^n$ be defined by

$$f_M(x) = \begin{cases} \dfrac{h(x)}{\rho(|h(x)|)}, & x \in U \\ \infty, & x \notin U. \end{cases} \tag{1}$$

This is continuous, so it defines an element in $\pi_{n+k}(S^n)$; we define $\psi([M]) = [f_M] \in \pi_{n+k}(S^n)$.

Of course, we need to know that this is well-defined. First of all, if you choose a different cutoff function, you get a homotopic map. Now, suppose $X \subset [0,1] \times M$ is a framed bordism. This is a submanifold with boundary that has a well-defined normal bundle, so it has a tubular neighborhood and we can define the Pontrjagin-Thom collapse map for it, and it turns out this is a smooth homotopy between the collapse maps on the boundaries.

3.4. **These Maps Are Inverses.** $\phi \circ \psi$ is the identity because $0 \in S^n$ is a regular value of the Pontrjagin-Thom collapse function, and its preimage is the manifold we started with.

For $\psi \circ \phi$, let $f_0 : S^{n+k} \to S^n$ be smooth (recall that all homotopy classes of maps have smooth representatives) and $f_1 = \psi \circ \phi(f_0)$.[4] We'd like to show that $f_0 \simeq f_1$. Without loss of generality, assume $0 \in S^n$ is a regular value of f_0, since we can rotate if we really need to; then, let $Y = f_0^{-1}(0)$; by the construction of Pontrjagin-Thom collapse, 0 is also a regular value of f_1, and $f_1^{-1}(0) = Y$. Moreover, $\mathrm{d}f_0|_Y = \mathrm{d}f_1|_Y$. (This amounts to "stare at (1) for a bit.") This is enough to show that there's a homotopy between f_0 and f_1.

The next step is to show that f_0 and f_1 are homotopic on a neighborhood of Y. This is done by choosing a tubular neighborhood for it, which is identified with $Y \times \mathbb{R}^n$ by the framing. We can assume that the images of f_0 and f_1 on this tubular neighborhood don't hit $\infty \in S^n$, so we can average them within $\mathbb{R}^n \subset S^n$. Then, if we weight the average with the cutoff function (1), by defining the smooth homotopy $G : [0,1] \times Y \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$G(t, y, \xi) = g_0(y, \xi) + t\rho(|\xi|)(g_1(y, \xi) - g_0(y, \xi)),$$

the map $h = G(1, \cdot, \cdot)$ can be extended to agree with f_0 outside of U since the cutoff function goes to 0 as we leave U, and this function agrees with f_2 on a neighborhood $V \supset Y$, so $f_0 \simeq h$ and h agrees with f_1 on a neighborhood of Y, as intended.

The last step is to extend this to all of S^{n+k}. Now, we think of 0 as our point at infinity: on the complement of V, neither h nor f_1 hits it, so we can think of their images as lying in a copy of $\mathbb{R}^n$: we can move whatever we want outside of V, and on V they agree. Thus, $f_0 \simeq h \simeq f_1$.

3.5. **The $\mathbb{Z}$-graded Ring Structure.** The $\mathbb{Z}$-grading will be preserved because at each finite level, we have a map from n^{th}-degree elements to n^{th}-degree elements, so we just have to check that the ring structure is preserved.

Let $f_1 : S^k \to S^m$ and $f_2 : S^\ell \to S^n$ be two smooth maps representing classes in $\pi_\bullet^S$. f_1 maps to the preimage of a regular value p_1 and f_2 maps to the preimage of a regular value p_2, and so $(p_1, p_2) \in S^m \wedge S^n$ is a regular value, and its preimage is $f_1^{-1}(p_1) \times f_2^{-1}(p_2)$, which is exactly what we were hoping for.

[4]This technically defines f_1 only up to homotopy, but we choose the representative f_1 obtained from the geometric constructions for ϕ and ψ from f_0.