GEOMETRY AND STRING THEORY SEMINAR: FALL 2019

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These notes were taken in UT Austin's geometry and string theory seminar in Fall 2019. I live-TEXed them using vim, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

Contents

1. A Heisenberg uncertainty principle for fluxes: 9/4/19

Today Dan spoke on a Heisenberg uncertainty principle for fluxes, which will provide background for subsequent talks. The material in this lecture is based on two papers by Freed, Moore, and Segal [FMS07a, FMS07b]; a followup talk will discuss a newer paper of García-Etxebarria, Heidenreich, and Regalado [GEHR19].

First, let's recall the story for electromagnetism. Let Y be a closed oriented Riemannian manifold and give $M := \mathbb{R} \times Y$ the Lorentz metric in which $\mathbb{R}$ is timelike. The *electromagnetic field* is a two-form $F \in \Omega^2(M)$; locally this is given by six functions, three of which tell us the electric field and three of which tell us the magnetic field. We also have electric and magnetic currents $j_E, j_B \in \Omega^3(M)$, which are closed forms.

Maxwell's equations can then be written concisely as

$$\mathrm{d}F = j_B = 0 \tag{1.1a}$$

$$\mathrm{d}{\star}F = j_E. \tag{1.1b}$$

Here the Hodge star is the one on M, not on Y, so keep in mind the Lorentz signature when chasing signs.

We're interested in fluxes. Let $\Sigma \subset Y$ be a closed, oriented siurface. The *magnetic flux* through Σ is

$$\mathcal{E}^{c\ell}_{t,\Sigma}(F) = \int_{\{t\}\times\Sigma} F \tag{1.2}$$

and the *electric flux* through Σ is

$$\mathcal{B}^{c\ell}_{t,\Sigma}(F) = \int_{\{t\}\times\Sigma} {\star}F. \tag{1.3}$$

If $j_B = j_E = 0$, then these fluxes are independent of t and depend only on the homology class of Σ, by Stokes' theorem. Hence if V is the vector space of solutions to (1.1), then $\mathcal{B}^{c\ell}$ and $\mathcal{E}^{c\ell}$ define linear functions $V \to H^2_{\mathrm{dR}}(Y)$.

Along the way to quantizing this theory, we should put it in the Hamiltonian formalism. Let

$$F := B - \mathrm{d}t \wedge E, \tag{1.4}$$

where $B(t) \in \Omega^2(Y)$ and $E(t) \in \Omega^1(Y)$. In this setting, Maxwell's equations are

$$\frac{\partial B}{\partial T} = -\mathrm{d}_Y E \tag{1.5a}$$

$$\frac{\partial \star_Y E}{\partial t} = \mathrm{d}_Y \star_Y B. \tag{1.5b}$$

We would like to express these in terms of a Poisson bracket. The Hamiltonian is

$$H \coloneqq \frac{1}{2}\int_Y \left(\|B\|^2 + \|E\|^2\right) \mathrm{vol}_Y. \tag{1.6}$$

Let $W \coloneqq \Omega^2(Y)_{\mathrm{cl}} \times \Omega^2(Y)_{\mathrm{cl}}$, and define the map $\theta_t \colon V \to W$ by

$$F \longmapsto (F|_{\{t\}\times Y}, \star F|_{\{t\}\times Y}). \tag{1.7}$$

Then W carries a Poisson structure as follows:[1] let $\eta \in \Omega^1(Y)/\mathrm{d}\Omega^0(Y)$ and define

$$\ell_\eta(B, \star_Y E) \coloneqq \int_Y \eta \wedge B \tag{1.8a}$$

$$\ell'_\eta(B, \star_Y E) \coloneqq \int_Y \eta \wedge \star_Y E. \tag{1.8b}$$

These satisfy the Poisson anticommutation relations

$$\{\ell_{\eta_1}, \ell_{\eta_2}\} = 0 \tag{1.9a}$$

$$\{\ell'_{\eta_1}, \ell'_{\eta_2}\} = 0 \tag{1.9b}$$

$$\{\ell_{\eta_1}, \ell'_{\eta_2}\} = \int_Y \mathrm{d}\eta_1 \wedge \eta_2. \tag{1.9c}$$

The electric and magnetic fluxes define a map $\mathcal{B}^{c\ell}, \mathcal{E}^{c\ell} \colon W \to H^2_{\mathrm{dR}}(Y) \times H^2_{\mathrm{dR}}(Y)$ sending $(B, \star_Y E) \mapsto ([B], \star_Y E)$.

A key point is that $\mathcal{B}^{c\ell}$ and $\mathcal{E}^{c\ell}$ commute. TODO: this has physics implications which I missed.

Now, following Dirac, we quantize: there are some units in which both electric and magnetic charges are integers. First, we quantize $[j] \in H^3_{\mathrm{dR}}(Y)$ by requiring it to lie in the image of $H^3(Y;\mathbb{Z}) \to H^3(Y;\mathbb{R}) \cong H^3_{\mathrm{dR}}(Y)$. This is a one-dimensional vector space, but soon enough we will think about more interesting examples.

Next, refine charge to an element of $H^3(X;\mathbb{Z})$. In this setting we haven't changed very much, but in more general settings this allows them to be torsion.[2] For example, we also refine the fluxes to elements of $H^2(Y;\mathbb{Z})$, which can have torsion, e.g. for $\mathbb{RP}^3$ and lens spaces.

The refined magnetic flux fits into a diagram

$$\begin{tikzcd} & \Omega^2(Y)_{\mathrm{cl}} \arrow[d, "\mathcal{B}^{c\ell}"] \\ H^2(Y;\mathbb{Z}) \arrow[r] & H^2_{\mathrm{dR}}(Y). \end{tikzcd} \tag{1.10}$$

So you might think the right place to situate it is the fiber product of these abelian groups, but this is wrong for physics reasons: the resulting abelian group is not local, ultimately because $H^2(X;\mathbb{Z})$ isn't. Instead, one can take a homotopy fiber product in a certain setting, landing in the relevant *differential cohomology* group $\check{H}^2(Y)$, the group of isomorphism classes of principal U_1-bundles with connection:

$$\begin{tikzcd} \check{H}^2(Y) \arrow[r, "\text{curvature}"] \arrow[d, "c_1"] & \Omega^2(Y)_{\mathrm{cl}} \arrow[d, "\mathcal{B}^{c\ell}"] \\ H^2(Y;\mathbb{Z}) \arrow[r] & H^2_{\mathrm{dR}}(Y). \end{tikzcd} \tag{1.11}$$

[1] This Poisson structure doesn't come from a symplectic structure; there is a kernel. In this case there is a foliation with symplectic leaves.

[2] Now that we're no longer working over $\mathbb{R}$, there are other choices for this refinement than $H^*(-;\mathbb{Z})$, such as generalized cohomology theories. They don't appear in the Maxwell-theoretic story, but can appear in string theory.

The differential cohomology group $\check{H}^2(Y)$ can be thought of as a local refinement of the fiber product – for example, a differential cohomology class defines a circle bundle with connection, but an element of the fiber product doesn't in general. Though to be really precise, the element of $\check{H}^2(Y)$ isn't local, but rather the circle bundle with connection that defines it (which forms a groupoid). You can see this by thinking through bundles on the circle (can be nontrivial) versus bundles on the two semicircles (always trivial). So the field in physics comes from the groupoid, not the differential cohomology group.

Remark 1.12. $\check{H}^2(Y)$ is an abelian Lie group:[3] we can tensor together two principal U_1-bundles with connection into a third. It's instructive to think through its homotopy groups — though only π_0 and π_1 are nonzero. ◂

Armed with these fine refinements, let's turn back to Maxwell theory; this is a sort of semiclassical perspective.

Let A be an $\mathbb{R}/\mathbb{Z}$-connection on $M = \mathbb{R} \times Y$ with curvature $F_A \in \Omega^2(M)$. The Lagrangian is

$$L \coloneqq -\frac{1}{2} F_A \wedge \star F_A \tag{1.13}$$

and Maxwell's equations can be understood as the Euler-Lagrange equation for this Lagrangian, namely

$$\mathrm{d}\star F_A = 0, \tag{1.14a}$$

and the Bianchi identity

$$\mathrm{d}F_A = 0. \tag{1.14b}$$

Now, $T(\check{H}^2(Y))$[4] plays the role of W, though it's no longer a vector space and the fluxes define maps $\mathcal{B}^{c\ell}, \mathcal{E}^{c\ell} \colon T(\check{H}^2(Y)) \to H^2_{\mathrm{dR}}(Y)$.

Lemma 1.15. *The fluxes still commute in this semiclassical setting:* $\{\mathcal{B}^{c\ell}, \mathcal{E}^{c\ell}\} = 0$.

Remark 1.16. The (isomorphism classes of) flat connections form a subgroup of $\check{H}^2(Y)$, and this subgroup is isomorphic to $H^1(Y;\mathbb{R}/\mathbb{Z})$. As Y is compact, this is a finite-dimensional Lie group. There is an isomorphism $\beta \colon \pi_0(H^1(Y;\mathbb{R}/\mathbb{Z})) \to \mathrm{Tors}H^2(Y;\mathbb{Z})$ called the *Bockstein homomorphism*.[5] More explicitly, we have a short exact sequence

$$0 \longrightarrow T^1(Y) \longrightarrow H^1(Y;\mathbb{R}/\mathbb{Z}) \xrightarrow{\beta} \mathrm{Tors}H^2(Y;\mathbb{Z}) \longrightarrow 0, \tag{1.17}$$

where $T^1(Y) \coloneqq H^1(Y;\mathbb{R})/H^1(Y;\mathbb{Z})$. ◂

The above story is semiclassical in that we've quantized charges and fluxes, but haven't produced a full Hilbert space on Y. Heuristically, we would like $\mathcal{H}_Y$ to be $L^2(\check{H}^2(Y))$, but since $\check{H}^2(Y)$ is an infinite-dimensional manifold there are some nuances going into that definition.

There are two gradings on $\mathcal{H}_Y$, a *magnetic grading* indexed by $b \in H^2(Y;\mathbb{Z})$ (a decomposition involving connected components of $\check{H}^2(Y)$), and an *electric grading*, produced by an $H^1(Y;\mathbb{R}/\mathbb{Z})$-action coming from a motion in some way on $\check{H}^2(Y)$; then, we decompose into a sum of irreducible representations. Since this group is abelian, this is just $H^1(Y;\mathbb{R}/\mathbb{Z})^\vee \cong H^2(Y;\mathbb{Z})$ by Poincaré duality.

Can we make these gradings simultaneously? No, because the $H^1(Y;\mathbb{R}/\mathbb{Z})$-action reshuffles the components in accordance with the Bockstein, whenever $H^2(Y;\mathbb{Z})$ has torsion.

To try and fix this, we can grade by quotient groups of $H^2(Y;\mathbb{Z})$, such as H^2 modulo torsion, and these two gradings commute and define a bigrading.

Now in a quantum theory, these gradings should arise from the spectra of commuting operators. Given $\omega \in H^1(Y;\mathbb{R}/\mathbb{Z})$, we obtain operators $\mathcal{B}^q(\omega)$ and $\mathcal{E}^q(\omega)$, respectively multiplication by $\exp(2\pi i\langle b, \omega\rangle)$ on $\mathcal{H}^b$

[3] Well, it's not a finite-dimensional manifold, but can be made into an infinite-dimensional manifold, and in that more general sense is an abelian Lie group.

[4] Here $T(\check{H}^2(Y))$ denotes the tangent bundle of the infinite-dimensional manifold $\check{H}^2(Y)$. You can think of this as initial conditions for solutions to the Maxwell equation; as these are linear wave equations, this intuition is well-beahved enough to be accurate.

[5] This is an instance of a general phenomenon: a short exact sequence of chain complexes induces a long exact sequence in cohomology. A *Bockstein homomorphism* is a connecting map in the long exact sequence induced from the short exact sequence of chain complexes corresponding to a short exact sequence of coefficient groups; this one comes from the sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$.

(the summand associated to b in the magnetic grading) and pullback by translation by $\omega \in H^1(Y;\mathbb{R}/\mathbb{Z}) \subset \check{H}^2(Y)$ (thought of as a flat connection).

This is a version of Heisenberg's uncertainty principle: $\mathcal{B}^q$ and $\mathcal{E}^q$ don't commute.

Theorem 1.18. $[\mathcal{B}^q(\omega_1), \mathcal{E}^q(\omega_2)] = \exp(2\pi i \langle \omega_1 \smile \beta\omega_2, [Y]\rangle)\,\mathrm{d}_{\mathcal{H}_Y}$.

Here $[-,-]$ denotes the commutator in a Lie group. The cup product of a $\mathbb{Z}$-cohomology class and an $\mathbb{R}/\mathbb{Z}$-cohomology class is an $\mathbb{R}/\mathbb{Z}$-cohomology class, using the $\mathbb{Z}$-module structure on $\mathbb{R}/\mathbb{Z}$, and then we pair with the fundamental class to obtain an element of $\mathbb{R}/\mathbb{Z}$. Exponentiating, we get a number.

We will quantize, as usual, by making a Heisenberg group extension

$$0 \longrightarrow \mathbb{T} \longrightarrow \mathcal{G} \longrightarrow \mathcal{A} \coloneqq \check{H}^2(Y) \times \check{H}^2(Y) \longrightarrow 0. \tag{1.19}$$

Here $\mathbb{T}$ is central. Such a $\mathcal{G}$ is characterized up to isomorphism by the commutator map $[-,-]\colon \mathcal{A}\times\mathcal{A}\to\mathbb{T}$. The idea is that since $\mathbb{T}$ is central, the map only depends on the equivalence class of $g \in \mathcal{G}$ in $\mathcal{A}$, and what we get is central.

The commutator map is skew but not necessarily alternating, but it does define a $\mathbb{Z}/2$-grading on the Heisenberg group, then gradings on representations, etc.

In our example, the commutator map is

$$s(A_1, A_2) = \exp\left(2\pi i \int_Y A_1 \cdot A_2\right). \tag{1.20}$$

Here $A_1 \cdot A_2$ is the product in $\check{H}^\bullet(Y)$. This is a good general way to deal with abelian gauge fields, but here we can make it more explicit: any closed 3-manifold with two $\mathbb{R}/\mathbb{Z}$-connections bounds a compact 4-manifold with two $\mathbb{R}/\mathbb{Z}$-connections,[6] and then… (TODO: ?).

Next, representation theory. For finite-dimensional Heisenberg groups subject to some nondegeneracy condition, there's a unique representation extending a given representation of $\mathbb{T}$, but in infinite-dimensions, one needs a polarization on $\mathcal{G}$, and this comes from a positve energy assumption in physics. Then $\mathcal{B}^q(\omega_1)$ and $\mathcal{E}^q(\omega_2)$ are just images of elements in the Heisenberg group, and one can compute the commutator there to prove Theorem 1.18.

Remark 1.21. This story applies any time you have an abelian gauge field. The Dirac quantization we saw above first chooses some cohomology theory, which can be determined by things like anomalies or other features of the theory in question. For Maxwell theory, we chose ordinary cohomology over the integers, denoted $H\mathbb{Z}$.

But in Type II string theory on a 9-dimensional manifold Y, something different can happen: there is a *Neveu-Schwarz field* $H \in \Omega^3(Y)$ and a *Ramond field* $F \in \Omega^*(Y)$, either even or odd (TODO: I think this is the IIA/IIB distinction). When $H = 0$, Witten and Sen proposed that the right way to solve all the constraints on string theory is to choose not $H\mathbb{Z}$ but complex K-theory KU. The abelian group of fluxes is $K^0(Y)$ (in IIA) or $K^1(Y)$ (for IIB); since K-theory is 2-periodic, these are the only options. There are examples where $K^0(Y)$ has torsion subgroups, and once again the Heisenberg uncertainty principle outlined above implies that the grading only works modulo torsion.

One then builds the Heisenberg group from *differential K-theory* $\check{K}^0(Y)$, and some features of the story are different, leading to interesting physics. ◀

2. IIB flux noncommutativity and theory $\mathcal{X}$: 9/18/19

Today, Jacques gave the first of two talks on the paper of García-Etxebarria, Heidenreich, and Regalado [GEHR19], which explains how the noncommutativity of fluxes in type IIB string theory is related to the fact that theory $\mathcal{X}$, also known as the 6D $\mathcal{N} = (2,0)$ superconformal field theory, is a relative field theory, in that its partition function is not a number, but rather an element of a vector space.

Last time, Dan told us about noncommutativity of fluxes in free Maxwell theory in dimension 4; today we'll begin by revisiting it in a different way that will be helpful for the string-theoretic story. This fits into two more general perspectives on quantum field theory, that we should study it on manifolds with nontrivial topology, and that we should study nonlocal operators, or defects.

[6] This question is only interesting for bundles; once we know it there, the extension is automatic.

Let's formulate free Maxwell theory on a 4-manifold M. Recall that the electric and magnetic fluxes Φ_E and Φ_M take values in $H^2(M;\mathbb{Z})$; Φ_M is in fact the first Chern class. We can think of these as generators of a 1-form global symmetry whose charged objects are the Wilson and 't Hooft lines. The noncommutativity of fluxes, then, is equivalent data to a mixed 't Hooft anomaly between these two 1-form symmetries. This means you can couple background fields to each such symmetry, but you can't make the fields dynamical and integrate them; in this case the obstruction will be precisely the noncommutativity of these fluxes.

Let $\mathcal{H}_Y$ denote the Hilbert space of the theory on $Y \times \mathbb{R}$, where Y is a 3-manifold. Suppose we want to map this onto a particular flux eigenspace, i.e. an irreducible representation of the one-form symmetry group G (which is necessarily abelian). We can do that by introducing a background field for G, then perform a weighted sum over G-bundles[7] on $Y \times S^1$. Concretely, given such a representation ρ in $\mathcal{H}_Y$, the projector is

$$P_\rho \coloneqq \frac{1}{|G|}\sum_{g\in G}\rho(g), \tag{2.1}$$

which is a map $\mathcal{H}_Y \to \mathcal{H}_Y$. Thus, for the *identity sector* $\mathcal{H}_1$ (the summand of $\mathcal{H}_Y$ on which G acts trivially),

$$\mathrm{tr}_{\mathcal{H}_1}(e^{-\beta H}) = \mathrm{tr}(P_1 e^{-\beta H}) = \frac{1}{|G|}\sum_{P\in \mathcal{B}un_G(Y\times S^1)} Z(Y\times S^1, P). \tag{2.2}$$

Here $Z(Y \times S^1, P)$ denotes the partition function, which depends on the 4-manifold and the principal G-bundle.

In our specific setting, there are two such symmetries G_E and G_M, and so we get two projection operators from the above formula. The mixed 't Hooft anomaly tells us that we can't simultaneously diagonalize them, hence cannot simultaneously gauge them.

Remark 2.3. Things get a little trickier in self-dual (higher) abelian gauge theories, such as the Ramond-Ramond 5-field strength in type IIB string theory. This amounts to something like identifying the electric and magnetic fields in these theories.

In this case, the symmetry group isn't a product in an interesting way, and we still have a 't Hooft anomaly, just not a mixed one. This anomaly obstructs projecting onto a self-dual sector, and one has to make non-covariant choices, which could cause a headache later. ◂

Now we allow Y to be noncompact. Maxwell's equations with current tell us

$$\begin{aligned} \mathrm{d}{\star}F &= j_E\\ \mathrm{d}F &= j_M, \end{aligned} \tag{2.4}$$

where $j_E, j_M \in \check{H}^3_{cs}(Y\times\mathbb{R})$. Here $\check{H}^3$ denotes differential cohomology as usual and H_{cs} means compact support in the space direction (i.e. along Y); we want our sources to be contained in some compact subspace of Y, which allows us to make sense of the flux at infinity. After applying the map $\check{H}^3_{cs}(Y\times\mathbb{R}) \to \check{H}^3(Y\times\mathbb{R})$, we want j_E and j_M to vanish. We also have $F, {\star}F \in \check{C}^2(Y\times\mathbb{R})$, though they might not be closed if the sources don't vanish.

Remark 2.5. There is a similar story in nonabelian gauge theories, and this presumably plays a role in Theory $\mathcal{X}$ in that some of its compactifications are nonabelian gauge theories. Let $\widetilde{G}$ be a simply connected compact Lie group with Lie algebra $\mathfrak{g}$. Let $Z \coloneqq Z(\widetilde{G})$; then (pure) $\widetilde{G}$-gauge theory has a 1-form Z-symmetry under which charged objects are Wilson lines.

If $H \subset Z$ and $G \coloneqq \widetilde{G}/H$, then a pure G-gauge theory has a 1-form symmetry which is a subgroup of $Z\times Z$.[8] This is the analogue of the product $G_E \times G_M$ we saw in Maxwell theory. ◂

Now let's talk about Theory $\mathcal{X}$ on a 6-manifold M. This is believed to arise as the compactification of type IIB string theory along $\mathbb{C}^2/\Gamma$, where Γ is a finite subgroup of SU_2; i.e. Theory $\mathcal{X}$ on M is the IIB theory on $M \times \mathbb{C}^2/\Gamma$. There's an ADE classification of finite subgroups of SU_2; let G_Γ be the compact, simply connected Lie group associated to the ADE type of Γ. Then $Z(G_\Gamma) \cong \Gamma/[\Gamma,\Gamma]$.

Since $\mathbb{C}^2/\Gamma$ is not a manifold, but rather an orbifold, this is a bit funny, but we can try to blow up the singular point to fix this, or work with it as a singular variety. In any case, by smoothing we obtain

[7] Since G is a higher-form symmetry, by "G-bundle" we really mean "G-gerbe," but that's not a crucial point here, so if you prefer to think about principal bundles, feel free to.

[8] For more details on this example, see Aharony, Seiberg, and Tachikawa [AST13].

some hyperKähler space X, which is asymptotically locally Euclidean, and $H_2(X_\Gamma)$ is generated by a set of embedded spheres, whose intersection form is (the negative of) the Cartan matrix of $\mathfrak{g}_\Gamma$, which is encoded in the Dynkin diagram.

The $(2,0)$ theory has a 2-form global symmetry for the group $Z(G_\Gamma)$ whose charged operators are "strings," aka surface operators; this is the same story we had before, just one dimension higher. We can couple to background fields for this symmetry, and there is an 't Hooft anomaly, preventing us from making these fields dynamical. Thus we cannot fix all of the fluxes in this theory.

Theory $\mathcal{X}$ has a moduli space of vacua of dimension $5\,\mathrm{rank}(\mathfrak{g}_\Gamma)$; when we smooth $\mathbb{C}^2/\Gamma$ to X_Γ, we can take the triplet of Kähler forms on X_Γ and integrate them on each S^2 in the generating set for $H_2(X_\Gamma)$; these give several real parameters. We also have the integrals of the Neveu-Schwarz B-field and Ramond-Ramond C-field along these S^2s, giving us U_1-valued parameters; if one counts them, the result is $5\,\mathrm{rank}(\mathfrak{g}_\Gamma)$. This is the parameter space of the theory.

On this moduli space, a D3-brane wrapped on any one of these S^2s is a dynamical string for this theory, and a D3-brane wrapped on a noncompact 2-cycle (such as a generator for the Borel-Moore homology group, which in this case is just $H_2(X_\Gamma, \partial X_\Gamma)$), which is a nondynamical defect, or some kind of surface operator. The difference here is $H_2^{\mathrm{BM}}(X_\Gamma)/H_2(X_\Gamma) \cong Z(G_\Gamma)$, which arises as part of a long exact sequence in homology: using that $\partial X_\Gamma = S^3/\Gamma$,

$$\cdots \longrightarrow \underbrace{H_2(S^3/\Gamma)}_{=0} \longrightarrow H_2(X_\Gamma) \longrightarrow H_2(X_\Gamma, S^3/\Gamma) \longrightarrow H_1(S^3/\Gamma) \longrightarrow \underbrace{H_1(X_\Gamma)}_{=0} \longrightarrow \cdots \tag{2.6}$$

Now $H_1(S^3;\Gamma)$ is the abelianization of $\pi_1(S^3/\Gamma) = \Gamma$, so we recover $Z(G_\Gamma)$. This tells us something about which surface operators can see which other surface operators.

Now assume M has torsion-free cohomology (and is spin); then the torsion subgroup of $H^5(M \times S^3/\Gamma) \cong H^3(M) \otimes Z(G_\Gamma) \cong H^3(M; Z(G_\Gamma))$ by the universal coefficient theorem. This cohomology group is our group of fluxes; there will be an 't Hooft anomaly as before, and the best we can do is choose a Lagrangian inside $H^3(M; Z(G_\Gamma))$ and turn on background gauge fields for that subspace, leading us to the partition function as an element of a vector space (of choices of such Lagrangians), the state space of some seven-dimensional noninvertible TFT.

3. Type IIB and 6D SCFTs: 9/25/19

Today Shehper spoke, continuing the previous talk on [GEHR19] on noncommutativity of fluxes in type IIB string theory and its relationship to 6D $\mathcal{N} = (2,0)$ theories and 4D $\mathcal{N} = 4$ theories.

3.1. **Heisenberg groups and their representations.**

For a reference for this section, see Freed-Moore-Segal [FMS07a, FMS07b].

Let $\mathcal{A}$ denote a topological abelian group and consider a central extension

$$1 \longrightarrow \mathrm{U}_1 \longrightarrow \widetilde{\mathcal{A}} \longrightarrow \mathcal{A} \longrightarrow 1. \tag{3.1}$$

As a topological space, $\widetilde{\mathcal{A}} \cong \mathcal{A} \times \mathrm{U}_1$, but the multiplication is different: for $x, y \in \mathcal{A}$ and $a, b \in \mathrm{U}_1$, define

$$(x, a) \cdot (y, b) \coloneqq (x + y, c(x, y)ab). \tag{3.2}$$

Here c is a function $\mathcal{A} \times \mathcal{A} \to \mathrm{U}_1$ which is a *cocycle*, meaning it satisfies a condition that implies the multiplication in (3.2) is associative.

Given a function $f\colon \mathcal{A} \to \mathrm{U}_1$, not necessarily a group homomorphism, we can modify c by a *coboundary*, sending

$$c(x, y) \longmapsto \frac{f(xy)}{f(x)f(y)} c(x, y). \tag{3.3}$$

If we use this modified cocycle to define a central extension as above, what we get is isomorphic to $\widetilde{\mathcal{A}}$ as central extensions.

We are interested in studying commutators in $\widetilde{\mathcal{A}}$; in general,

$$[(x, a), (y, b)] = (0, s(x, y)), \tag{3.4}$$

where $s(x, y) \coloneqq c(x, y)/c(y, x)$. Thus $s\colon \mathcal{A} \times \mathcal{A} \to \mathrm{U}_1$ is an alternating bimultiplicative form, and $s(x, x) = 1$.

Theorem 3.5. *The map sending $\widetilde{\mathcal{A}}$ to s defines a bijection between isomorphism classes of central extensions of $\mathcal{A}$ by U_1 and alternating bimultiplicative forms on $\mathcal{A}$.*

Definition 3.6. We call $\widetilde{\mathcal{A}}$ a *Heisenberg group* if s is *nondegenerate*, meaning that for all $x \neq 1$ in $\mathcal{A}$, there's a $y \in \mathcal{A}$ such that $s(x, y) = 1$.

Nondegeneracy is equivalent to asking for the map $h\colon \mathcal{A} \to \mathrm{Hom}(\mathcal{A}, \mathrm{U}_1)$ sending $x \mapsto s(x, -)$ to be injective. If it's also surjective, we call s a *perfect pairing.*

Example 3.7. In the original example considered by Heisenberg, $\mathcal{A} = \mathbb{R} \times \mathbb{R}$, which we will write multiplicatively as $(a, b) \mapsto (e^{ax}, e^{bp})$. Define

$$s((e^{ax}, e^{bp}), (e^{cx}, e^{dp})) := e^{i\hbar(ad-bc)}. \tag{3.8}$$

If we set $b = c = 0$ we can recover the commutation relation $[x, p] = i\hbar$ in the Lie algebra. ◂

Theorem 3.9 (Stone-von Neumann)**.** *With $\widetilde{\mathcal{A}}$ as above, assume $\mathcal{A}$ is finite-dimensional and s is a perfect pairing. Then there is a unique unitary irreducible representation $\mathcal{H}$ of $\widetilde{\mathcal{A}}$ such that $\mathrm{U}_1 \subset \widetilde{\mathcal{A}}$ acts by scalar multiplication.*

We briefly sketch the construction. Let $L \subset \mathcal{A}$ be a maximal isotropic subgroup with respect to s; that is, $s|_{L\times L} = 1$, and L is not contained inside a strictly larger subgroup of $\mathcal{A}$ with this property. Let $\widetilde{L}$ denote the preimage of L under the map $\widetilde{\mathcal{A}} \twoheadrightarrow \mathcal{A}$; then $\widetilde{L}$ is abelian, and in fact a maximal torus.

As a central extension of L by U_1, $\widetilde{L}$ splits; we choose a splitting, which is a homomorphism $\widetilde{L} \to \mathrm{U}_1$, which is in particular a unitary representation on which U_1 acts by scalar multiplication. Now we induce up to $\widetilde{\mathcal{A}}$.

3.2. **Fluxes and type IIB string theory.** Recall from a few lectures ago, fluxes in Maxwell theory on $Y \times \mathbb{R}$ (where Y is a 3-manifold) live in $H^2(Y) \times H^2(Y)$, and that the fluxes which might not commute live in $\mathrm{Tors}\, H^2(Y) \times \mathrm{Tors}\, H^2(Y)$. Therefore the Hilbert space can be graded as

$$\mathcal{H}_Y = \bigoplus_{(\overline{e},\overline{m})\in\mathcal{A}} \mathcal{H}_{(\overline{e},\overline{m})}, \tag{3.10}$$

where

$$\mathcal{A} := (H^2(Y)/\,\mathrm{Tor}\, H^2(Y)) \times (H^2(Y)/\,\mathrm{Tor}\, H^2(Y)). \tag{3.11}$$

Recall that there is a *torsion pairing* or *linking pairing*

$$\ell k\colon\ \mathrm{Tor}\, H^2(Y) \times \mathrm{Tor}\, H^2(Y) \longrightarrow \mathbb{Q}/\mathbb{Z}. \tag{3.12}$$

By Poincaré duality, we can use H_1 instead of H^2. The definition is: given a class $\overline{a} \in \mathrm{Tors}\, H^2(Y)$ by some cocycle $a \in Z^2(Y)$, there's some $k \in \mathbb{Z}$ such that $ka = \delta c$, where $c \in Z^1(Y)$. Let $\overline{c}$ denote the cohomology class of c. Then

$$\ell k(\overline{a}, \overline{b}) := \frac{1}{k}(\overline{c} \smile \overline{b}). \tag{3.13}$$

We made choices here, and must check that this is independent of those choices, but it is in fact independent.

Composing ℓk with the inclusion $\mathbb{Q}/\mathbb{Z} \hookrightarrow \mathrm{U}_1$, we obtain a perfect pairing and therefore a Heisenberg group. Now $\mathcal{H}_{(\overline{0},\overline{0})}$ is a direct sum of infinitely many copies of the representation associated to this Heisenberg group by Theorem 3.9.

Now let's consider type IIB string theory on $M_{10} = \mathbb{C}^2/\Gamma \times M_6$, where M_6 is a spin 6-manifold and Γ is a finite subgroup of SU_2. We want to study fluxes on

$$N := \partial M_{10} = (S^3/\Gamma) \times M_6. \tag{3.14}$$

In particular, M_{10} isn't a manifold, but N is.

The Ramond-Ramond fluxes in type IIB string theory on M_{10} live in $K^1(N)$, which is an abelian group, and by Freed-Moore-Segal [FMS07a, FMS07b], the Hilbert space of the theory on N admits a grading by

$$\overline{K} := K^1(N)/\,\mathrm{Tors}\, K^1(N). \tag{3.15}$$

Again, the graded piece corresponding to $\overline{0} \in \overline{K}$ is some number of copies of the Heisenberg representation of the Heisenberg group associated to a pairing

$$\operatorname{Tors} K^1(N) \times \operatorname{Tors} K^1(N) \longrightarrow \mathbb{Q}/\mathbb{Z} \hookrightarrow \mathrm{U}_1, \tag{3.16}$$

which is a generalization of the pairing in (3.12) to K-theory.

If you're not fluent in K-theory, that's OK,[9] because if $H^*(M_6)$ is torsion-free, then $K^1(N)$ can be expressed in terms of ordinary cohomology groups: first, using the Künneth formula for K-theory,

$$K^1(N) \cong K^1(M_6) \otimes K^0(S^3/\Gamma) \oplus K^0(M_6) \otimes K^1(S^3/\Gamma). \tag{3.17}$$

Furthermore, the *Chern character* provides an isomorphism

$$ch\colon K^i(X) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \bigoplus_{n \equiv i \bmod 2} H^n(X;\mathbb{Q}), \tag{3.18}$$

so if $H^*(M_6)$ is torsion-free, we conclude

$$K^i(M_6) \cong \bigoplus_{n \equiv i \bmod 2} H^n(X), \tag{3.19}$$

though there's no canonical isomorphism in general.

Lemma 3.20 ([GEHR19])**.** *$K^0(S^3/\Gamma) \cong \mathbb{Z} \oplus \Gamma^{\mathrm{ab}}$ and $K^1(S^3/\Gamma) \cong \mathbb{Z}$.*

These groups can be computed via a spectral sequence.

Corollary 3.21. *Assuming $H^*(M_6)$ has no torsion.*

$$\operatorname{Tors} K^1(N) \cong \bigoplus_{n \equiv 1 \bmod 2} H^n(M_6) \otimes \Gamma^{\mathrm{ab}}. \tag{3.22}$$

Hence, from now on we assume $H^*(M_6)$ is torsion-free.

The $\mathbb{Z}$ summand of $K^0(S^3/\Gamma)$ should be thought of as coming from H^0, and the Γ^{ab} summand from H^2. The identification $K^1(S^3/\Gamma) \cong \mathbb{Z}$ can be thought of as coming from H^3.

Remark 3.23. In [GEHR19], they also identify how the torsion pairing (3.16) behaves under the isomorphism in Corollary 3.21; they claim that it's just the torsion pairing for cohomology on $\Gamma^{\mathrm{ab}} = \operatorname{Tors} H^2(S^3/\Gamma)$, as the $H^n(M)$ factor is torsion-free. ◀

As before, we obtain a Heisenberg group associated to (3.16):

$$1 \longrightarrow \mathrm{U}_1 \longrightarrow \mathcal{W} \longrightarrow \bigoplus_{n=1,3,5} H^n(M_6) \otimes H^2(S^3/\Gamma) \longrightarrow 1. \tag{3.24}$$

This central extension splits as a direct product $\mathcal{W} = \mathcal{W}_{1,5} \times \mathcal{W}_3$, given by restricting the pairings to $H^1 \oplus H^5$, resp. H^3, and fitting into central extensions

$$1 \longrightarrow \mathrm{U}_1 \longrightarrow \mathcal{W}_{1,5} \longrightarrow \bigoplus_{n=1,5} H^n(M_6) \otimes H^2(S^3/\Gamma) \longrightarrow 1 \tag{3.25a}$$

$$1 \longrightarrow \mathrm{U}_1 \longrightarrow \mathcal{W}_3 \longrightarrow H^3(M_6) \otimes H^2(S^3/\Gamma) \longrightarrow 1. \tag{3.25b}$$

It is a fact that the vector space of partition functions of theory $\mathcal{X}$ is the Heisenberg representation of $\mathcal{W}_3$.

Remark 3.26. What happens to $\mathcal{W}_{1,5}$? This is a mismatch, and indicates that we're missing something subtle. This may have something to do with the fact that there's a mixed anomaly between zero- and four-form symmetries. This might be related to work of Diaconescu-Moore-Witten [DMW02] on the way that the C-field (in M-theory) in $H^3(M_6)$ gives birth to K-theory in type II string theory. ◀

4. GSO projections from invertible field theories: 10/2/19

These are Arun's prepared lecture notes on the recent paper of Kaidi, Parra-Martinez, and Tachikawa [KPMT19], which describes how to study the GSO projection in superstring theory using the classification of invertible field theories on the worldsheet.

[9] No pun intended.

4.1. **Overview: the GSO projection in string theory.** Let's begin with a quick review of the basics of string theory and the GSO projection. This review is as much for me as it is for anyone else, so if it's redundant hold on for just a second.

String theory is a quantum theory in physics which includes a lot of data. Today we will consider superstring theory, as opposed to bosonic string theory. This theory considers maps of surfaces Σ into a 10-manifold X. Here Σ is called the *worldsheet* and X *spacetime*. There's data of various fields and structures on both Σ and X, but not all of them are relevant to today's talk.

The worldsheet has a field which locally is a pair of spin structures inducing opposite orientations. It also has 16 fermions: eight *left-movers* ψ_L^i, defined using the first spin structure, and eight *right-movers* ψ_R^i, defined using the second spin structure.

There are also objects called D-branes in string theory, which carry quantized charges. These charges live in a K-theory group of spacetime; the specific group in question depends on the kind of string theory.

There are a few different kinds of superstring theories, and they modify this field in some way.

Type II: Σ is also oriented. In this case, the field is globally two spin structures $\mathfrak{s}_L, \mathfrak{s}_R$ inducing opposite orientations; without loss of generality, let's say $\mathfrak{s}_L$ is the one compatible with the given orientation.

Type I: No additional data. Globally, what we have is a spin structure on the orientation double cover Σ' of Σ.

Type 0: This is a less-considered variant in which we force $\mathfrak{s}_L = \mathfrak{s}_R^{\mathrm{op}}$. We can ask for it to be a spin structure or a pin structure.

These further split into a few variants, e.g. IIA and IIB. The point of today's talk is to systematically catalog these variants.

These variants arise because the worldsheet theory undergoes *GSO projection*, in which one sums the worldsheet theory over spin structures.[10] We will consider modifying the worldsheet theory before GSO projection by tensoring with an invertible field theory.

4.2. **Invertible field theories.**

Definition 4.1 (Freed-Moore [FM06]). A quantum field theory α is *invertible* if there is some other quantum field theory α' such that $\alpha \otimes \alpha'$ is trivial.

Physically, "$\otimes$" means formulating both α and α' on the same spacetime, but with no or minimal interactions between them. In condensed-matter physics, this is often called *stacking*. For topological field theories, we can upgrade this to a mathematical definition.

The Hilbert space of states of $\alpha \otimes \alpha'$ on some codimension-1 manifold is the tensor product of the Hilbert spaces for α and for α'. Since $\alpha \otimes \alpha'$ is trivial, this is one-dimensional, and therefore the Hilbert spaces for α and for α' must also be one-dimensional — isomorphic to $\mathbb{C}$, but not canonically so. This suggests two ways in which invertible field theories can be useful.

- If F is a noninvertible quantum field theory and α is a nontrivial invertible field theory, the tensor product $Z \otimes \alpha$ is different from Z, but not by very much. So there are settings in physics where you have different variations of your theory indexed by the invertible field theories in that dimension. If you've worked with discrete θ-parameters, they're one example; today's lecture is about another example, using different invertible field theories on the worldsheet to classify different superstring theories.
- Another important application of invertible field theories is to the study of anomalies. In an anomalous quantum field theory, the partition function isn't quite a number — rather, it's an element of a one-dimensional vector space that isn't canonically trivialized. Following Freed-Teleman [FT14], this vector space can be interpreted as the state space of an invertible field theory in one dimension higher, and therefore classifying invertible field theories informs us about which anomalies are possible.

So it would be nice if we had a way to classify invertible field theories. Fortunately, we do!

Theorem 4.2 (Freed-Hopkins [FH16]). *The unitary*[11] *invertible topological field theories in dimension n and symmetry type $H \to \mathrm{O}_n$ are classified by* $\mathrm{Hom}(\mathrm{Tors}\,\Omega_n^H, \mathrm{U}_1)$.

[10] Some of the theories have variants of spin structures, such as pin structures, and we'd instead sum over those variants.

[11] Well, something-something "Wick-rotated", something something "actually reflection positive." The upshot is that these are the invertible theories that could actually happen in unitary quantum field theory.

The classification sends a theory to its partition function; it's not trivial that this is a bordism invariant.

Conjecture 4.3 (*ibid.*)**.** The unitary invertible field theories (possibly non-topological) in dimension n and symmetry type $H \to \mathrm{O}_n$ are noncanonically the above plus the free part of Ω_{n+1}^H.

In all cases of interest today, this part will vanish.

4.3. **Results.** Today's thesis is: in various kinds of string theories, GSO projection sums over symmetry types which are variants of structures on the worldsheet. The classification of 2D invertible field theories of those symmetry types tells us what things we can tensor with before summing, and therefore effects[12] the classifications of those types of string theories. The general recipe is

- $\mho_H^2 := \mathrm{Hom}(\Omega_2^H, \mathrm{U}_1)$ classifies the possible invertible field theories on the worldsheet, and
- $\mho_H^3 := \mathrm{Hom}(\Omega_3^H, \mathrm{U}_1)$ classifies the possible anomalies on the worldsheet. These obstruct the sum over spin structures. However, in all settings we have eight fermions (or eight left-movers and eight right-movers), so we tensor together eight copies of the anomaly theory, landing in $8\mho_H^3$, and the obstructions to gauging live in $\mho_H^3/8\mho_H^3$.[13]

(There are no non-topological invertible field theories in these dimensions, assuming Conjecture 4.3.)

4.3.1. *Type II.* The tangential structure is "two spin structures $\mathfrak{s}_L, \mathfrak{s}_R$ inducing opposite orientations," but we can replace $\mathfrak{s}_R$ with $\mathfrak{s}_R^{\mathrm{op}}$, which induces the same orientation as $\mathfrak{s}_L$; then, since spin structures are an $H^1(\Sigma; \mathbb{Z}/2)$-torsor, it's equivalent to consider $\mathfrak{s}_L$ and a principal $\mathbb{Z}/2$-bundle.

A closed surface with a spin structure has a $\mathbb{Z}/2$-valued invariant called the *Arf invariant*. It can be defined to be the mod 2 index of the spin Dirac operator (and there are several other equivalent definitions if that one isn't friendly to you!).

- $\mho_{\mathrm{Spin}}^2(B\mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, generated by the Arf invariants of $\mathfrak{s}_L$ and $\mathfrak{s}_R$.
- $\mho_{\mathrm{Spin}}^3(B\mathbb{Z}/2) \cong \mathbb{Z}/8$. The generator can be described as follows: on a closed spin 3-manifold, we have two η-invariants associated to the two spin structures; their difference is an eighth root of unity, and is a bordism invariant (both of these facts are nontrivial). But since $8\mho_{\mathrm{Spin}}^3(B\mathbb{Z}/2) = 0$, we do not have to worry about anomalies.

We therefore have four choices for GSO projections, given by their partition functions: 1, $(-1)^{\mathrm{Arf}(\mathfrak{s}_L)}$, $(-1)^{\mathrm{Arf}(\mathfrak{s}_R)}$, and $(-1)^{\mathrm{Arf}(\mathfrak{s}_L)+\mathrm{Arf}(\mathfrak{s}_R)}$. However, some of these will produce the same theory out in the end: it's possible to perform a parity transformation along a single direction in spacetime,[14] and this acts on the worldsheet fermions, sending $(\psi_L^1, \psi_R^1) \mapsto (-\psi_L^1, \psi_R^1)$ and leaving the rest unchanged. What I think is going on here is that a fermion on a spin surface has an anomaly that is trivial, but there are two different ways to trivialize it, and the two trivializations are a $\mho_{\mathrm{Spin}}^2$-torsor: switching from one to the other is equivalent to tensoring with the Arf theory.

A parity transformation on spacetime induces this switch on the worldsheet for both the left-movers and the right-movers, so picks up a factor of $(-1)^{\mathrm{Arf}(\mathfrak{s}_L)+\mathrm{Arf}(\mathfrak{s}_R)}$. But this does not actually change the theory, so the GSO projections using 1 and $(-1)^{\mathrm{Arf}(\mathfrak{s}_L)+\mathrm{Arf}(\mathfrak{s}_R)}$ should yield the same string theory, and the GSO projections using $(-1)^{\mathrm{Arf}(\mathfrak{s}_L)}$ and $(-1)^{\mathrm{Arf}(\mathfrak{s}_R)}$ should yield the same theory.

Indeed, the former case corresponds to type IIB string theory, and the latter to IIA. D-brane charges live in $K^0(X)$ and $K^1(X)$, respectively.

Remark 4.4. T-duality along a spacetime direction acts by switching the sign on ψ_R, but leaves ψ_L unchanged. Therefore, as above, it amounts to tensoring the worldsheet theory by $(-1)^{\mathrm{Arf}(\mathfrak{s}_R)}$ before GSO projection. This recovers the previously-known fact that T-duality exchanges type IIA and type IIB theories. ◀

4.3.2. *Oriented, type 0.* In this case, GSO projection amounts to summing over spin structures. The relevant spin bordism groups and spin bordism invariants are classical.

- $\mho_{\mathrm{Spin}}^2 \cong \mathbb{Z}/2$; the nontrivial bordism invariant is the Arf invariant. Therefore the invertible TFTs we have to consider are the trivial theory and the *Arf theory* [MS06, Gun16].
- $\mho_{\mathrm{Spin}}^3 = 0$, so we don't have to worry about anomalies at all.

[12] See https://xkcd.com/326/.

[13] There's something to be said about the choice of trivialization of the anomaly, but that's out of scope of today's talk.

[14] This assumes something about the global structure of spacetime, so perhaps the situation is more nuanced in general. That's beyond the scope of this talk.

So we have two choices for GSO projections: do nothing, or tensor with the Arf theory. A parity transformation along one direction of spacetime causes an additional factor of $(-1)^{2\,\mathrm{Arf}(\mathfrak{s})} = 1$, so we can't cut these choices down.

- If we do nothing, we get oriented type 0B string theory. In this case, D-brane charges live in $K^0(X) \oplus K^0(X)$.
- If we use the Arf theory, we get oriented type 0A string theory. In this case, D-brane charges live in $K^1(X) \oplus K^1(X)$.

4.3.3. *Unoriented type 0, pin$^+$ case.* There are two different generalizations of the notion of a spin structure to unoriented manifolds, pin$^+$ structures and pin$^-$ structures. This is related to the fact that we can introduce two different kinds of time-reversal symmetry T into a fermionic system, corresponding to $T^2 = 1$ or $T^2 = (-1)^F$. Here the tangential structure is pin$^+$; again the relevant bordism groups and invariants are classical.

- $\mho^2_{\mathrm{Pin}^+} \cong \mathbb{Z}/2$, generated by the Arf invariant of the orientation double cover, which has a spin structure induced from the pin$^+$ structure on Σ.
- $\mho^3_{\mathrm{Pin}^+} \cong \mathbb{Z}/2$, so we can trivialize any anomalies appearing.

Here, a parity transformation acts nontrivially. We can see this as follows: it's either trivial or nontrivial, and it's a local computation (it amounts to tensoring with an invertible field theory). Locally, the pin$^+$ structure is two opposite spin structures, corresponding to the left- and right-movers, and this is equivalent data to an *equivariant* spin structure on the orientation double cover. The Arf theory on (this local piece of) the orientation double cover is equivalent to the tensor product of the Arf theories on the two components, which is what parity acts by. So what we get is the Arf theory on the orientation double cover — in other words, a parity transformation acts nontrivially and we don't get anything new.

4.3.4. *Unoriented type 0, pin$^-$ case.* In this case, the tangential structure is pin$^-$. The bordism groups and invariants are again classical.

- $\mho^2_{\mathrm{Pin}^-} \cong \mathbb{Z}/8$. The generator is called the *Arf-Brown-Kervaire invariant*; like the Arf invariant, it admits a description in terms of how the pin$^-$ structure interacts with the mod 2 intersection pairing.[15]
- $\mho^3_{\mathrm{Pin}^-} = 0$.

For the same reason as above, a parity transformation shifts by the Arf invariant of the double cover — but for pin$^-$ manifolds, this is always 1, and therefore we have eight different GSO projections! The restriction map $\mho^2_{\mathrm{Pin}^-} \to \mho^2_{\mathrm{Spin}}$ is the nontrivial map $\mathbb{Z}/8 \twoheadrightarrow \mathbb{Z}/2$, and therefore four of these refine type 0B and four refine type 0A. One can work out which D-branes appear in which of these eight theories, and the answer matches the prediction that their charges lie in $KO^n(X) \otimes KO^{-n}(X)$, where $n \in \mathbb{Z}/8 \cong \mho^2_{\mathrm{Pin}^-}$.

4.3.5. *Type I.* In this case, the tangential structure is a spin structure on the double cover, which the authors call a dpin structure. Unlike all the previous cases, these bordism groups weren't previously known.

- $\mho^2_{\mathrm{DPin}} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$; the bordism invariants are $(-1)^{\chi(\Sigma)}$ and the Arf invariant of the orientation double cover.
- $\mho^3_{\mathrm{DPin}} \cong \mathbb{Z}/8$. Again we can cancel any anomaly.

For the same reasons as above, a parity transformation picks up a factor of the Arf invariant of the double cover, so we're left with two options: the trivial theory and the $(-1)^{\chi(\Sigma)}$ theory. These give us two slightly different theories called type I and type $\tilde{\mathrm{I}}$; the only difference is that the O9$^\pm$-planes are switched.

4.4. **Methods of computation.** These low-dimensional bordism groups can be computed in a few different ways. The spin bordism groups were first calculated by Milnor; later, Kirby-Taylor [KT90] computed these spin, pin$^+$, and pin$^-$ bordism groups using hands-on techniques from differential topology.

I'll tell you about a unified way to compute these bordism groups, as well as some others that appear in physics, usually relating to some sort of fermionic systems.

[15] This 8 has something to do with the periodicity of real Clifford algebras. This has been mentioned or alluded to in a few places, but the clearest connection I know of was made by Turzillo [Tur18].

(1) Rewrite the tangential structure in question as a *twisted spin structure*. That is, find a (virtual) vector bundle $V \to X$ such that an H-structure on M is equivalent data to a map $f\colon M \to X$ and a spin structure on $f^*V \oplus X$.
(2) Use this twisting to write $MTH \simeq MSpin \wedge X^V$, where X^V is the Thom spectrum of the vector bundle $V \to X$. The upshot is that $\Omega_d^H \cong \widetilde{\Omega}_d^{\mathrm{Spin}}(X^V)$.
(3) We understand spin bordism well, so can use knowledge of Ω_d^{Spin} and $H_*(X^V)$ to infer information about $\Omega_d^{\mathrm{Spin}}(X^V)$ through the Atiyah-Hirzebruch spectral sequence.[16]

Example 4.5. A pin$^-$ structure on a vector bundle V is equivalent data to a spin structure on $V \oplus \operatorname{Det} V$. Therefore we can rewrite a pin$^-$ structure on M as a line bundle ℓ and a spin structure on $\ell \oplus TM$, as an orientation of $\ell \oplus TM$ chooses an equivalence class of isomorphisms $\ell \cong \operatorname{Det}(TM)$.

This (replacing ℓ with the virtual vector bundle $\ell - 1$) leads to a splitting $MTPin^- \simeq MSpin \wedge \Sigma^{-1} MO_1$, which implies the *Smith isomorphism* $\widetilde{\Omega}_d^{\mathrm{Spin}}(B\mathbb{Z}/2) \xrightarrow{\cong} \Omega_{d-1}^{\mathrm{Pin}^-}$. ◂

Similar stories apply for the other tangential structures we saw today; for example, a pin$^+$ structure on V is equivalent to a spin structure on $V \oplus 3 \operatorname{Det} V$, and a dpin structure on V is equivalent to two line bundles ℓ_1 and ℓ_2 and a spin structure on $V \oplus (\ell_1 \otimes \ell_2) \oplus \ell_2^{\oplus 3}$. For the pair of spin structures, i.e. a spin structure and a principal $\mathbb{Z}/2$-bundle, we're trying to compute $\Omega_*^{\mathrm{Spin}}(B\mathbb{Z}/2)$, so can skip directly to (3).

5. Entanglement in quantum field theory, part I: 10/16/19

Today, Ivan spoke on Witten's notes on entanglement in quantum field theory [Wit18], including the Reeh-Schleider theorem and relative entropy in quantum field theory.

Today's setting is a quantum field theory on $M_d \coloneqq \mathbb{R}^{d-1,1}$, where we have only a real scalar field $\phi(x_\mu)$. Let $\mathcal{H}$ denote the Hilbert space of states, $|\Omega\rangle \in \mathcal{H}$ be the vacuum, and $\mathcal{H}_0$ be the *vacuum sector*, i.e. the Hilbert space generated by

$$|\psi_{\vec{f}}\rangle \coloneqq \phi_{f_1} \cdots \phi_{f_n} |\Omega\rangle, \tag{5.1}$$

where $f_i \in C_c^\infty(M_d)$ and $\phi_f \coloneqq \int \mathrm{d}^d x\, \phi(x) f(x)$, a smeared-out local operator.

Definition 5.2. An algebra of operators acting on a Hilbert space is a *von Neumann algebra* if it's closed under adjoints and weak limits.

In particular, suppose A_n converges weakly to A, denoted $A_n \xrightarrow{w} A$ or $A_n \rightharpoonup A$. Then $\langle \psi \mid A_n \psi\rangle$ converges to $\langle \psi \mid A \mid \psi\rangle$ for all $\psi \in \mathcal{H}$.

Given an open $U \subset M_d$, let A_U denote the von Neumann algebra generated by all bounded operators made from ϕ_f with $\operatorname{supp}(f) \subset U$.

Theorem 5.3 (Reeh-Schleider)**.** *Let $\Sigma \subset M_d$ be a complete, spacelike hypersurface. Let $U \subset M_d$ and $V \subset U \cap \Sigma$ be open. Then $\{a|\Omega\rangle \mid a \in A_U\}$ is dense in $\mathcal{H}_0$.*

As a corollary, let's choose $V, V^* \subset \Sigma$ which are disjoint, and $U \subset V \cap \Sigma$, $U^* \subset V^* \cap \Sigma$ which are spacelike separated. Then Theorem 5.3 applies separately to U and U^*, so if $a \in A_U$ an d$a^* \in A_{U^*}$ satisfy $a|\Omega\rangle = 0$ and $aa^*|\Omega\rangle = 0$, then $a|_{\mathcal{H}_0} = 0$.

Definition 5.4. A state $|\psi\rangle \in \mathcal{H}_0$ is *cyclic* for A_U if $\{a|\psi\rangle \mid a \in A_U\}$ is dense in $\mathcal{H}_0$. $|\psi\rangle$ is *separating* for A_U if $a|\psi\rangle = 0$ for $a \in A_U$ implies $a|_{\mathcal{H}_0} = 0$.

Corollary 5.5. *$|\Omega\rangle$ is both cyclic and separating for A_U.*

Example 5.6. Let V, V^*, U, and U^* be as above; in particular, U and U^* are spacelike separated. We define an operator $M \in A_{U^*}$ by asking for $\langle \psi \mid M \mid \psi\rangle$ to be 1 if $|\psi\rangle$ contains the moon,[17] and 0 if otherwise. By Theorem 5.3, there is some $a \in A_U$ such that $\langle \Omega \mid M \mid \Omega\rangle = 0$ and

$$\langle a\Omega \mid M \mid a\Omega\rangle = \langle \Omega \mid M a^\dagger a \mid \Omega\rangle = 1. \tag{5.7}$$

This implies you can't "create the moon" in U^* without applying physical operators in U. This is also related to nonunitary operators violating causality?? TODO. ◂

[16] Alternatively, using the equivalence $\tau_{\le 7} MSpin \simeq ko$, the Adams spectral sequence is also relatively tractable for these bordism groups.

[17] This was not meant to be literal, and I'm unfortunately not sure what it means. TODO

Another perhaps surprising takeaway is that there can still be correlations between operators in A_U and A_{U^*}. This leads us into the notions of modular operators and relative entropy in QFT.

Let $|\psi\rangle \in \mathcal{H}_0$ be cyclic and separating for A_U and $|\phi\rangle \in \mathcal{H}_0$.

Definition 5.8. The *Tomita operator* associated to $(|\psi\rangle, |\phi\rangle, U)$ is the antilinear operator

$$S_{\psi|\phi}(a|\psi\rangle) = a^\dagger|\phi\rangle \tag{5.9}$$

for $a \in A_U$.

What's the domain of this operator? Since $|\psi\rangle$ is separating, we can't have $a|\psi\rangle = 0$ and $a^\dagger|\phi\rangle \neq 0$, so this operator is well-defined. Since $|\psi\rangle$ is cyclic, $S_{\psi|\phi}$ is defined on a dense subset of $\mathcal{H}_0$. It can be unbounded, so we can't extend to all of $\mathcal{H}_0$ in general, though.

Definition 5.10. The *modular operator* associated to $(|\psi\rangle, |\phi\rangle, U)$ is $\Delta_{\psi|\phi} \coloneqq S^\dagger_{\psi|\phi} \mid S_{\psi|\phi}$.

This is linear, positive semidefinite, and Hermitian.

Definition 5.11. The *relative entropy* between $|\psi\rangle$ and $|\phi\rangle$ for measurements taken in U is

$$\mathcal{E}_{\psi|\phi}(U) \coloneqq -\langle\psi \mid \log(\Delta_{\psi|\phi}) \mid \psi\rangle. \tag{5.12}$$

This takes values in $(-\infty, \infty]$; ∞ can occur when $|\phi\rangle$ isn't separating.

We have the definition of $\mathcal{E}_{\psi|\phi}$, but what is its meaning? In particular, how is it related to the definition in quantum information?

Proposition 5.13. *$\mathcal{E}_{\psi|\phi}(U) \geq 0$, and is 0 iff $|\phi\rangle = a'|\psi\rangle$ for $a' \in (A_U)' \cap \mathrm{U}(\mathcal{H}_0)$, i.e. it's both unitary on $\mathcal{H}_0$ and in the commutant of A_U.*

This is not an obvious fact.

Supposing that the conclusion to Proposition 5.13 holds. Then, for every $a \in A_U$, $\langle\phi \mid a \mid \phi\rangle = \langle\psi \mid a \mid \psi\rangle$, meaning $|\psi\rangle$ and $|\phi\rangle$ are indistinguishable by measurements made in U. Therefore $\mathcal{E}_{\psi|\phi}(U)$ measures the degree to which you can distinguish $|\psi\rangle$ and $|\phi\rangle$ in U.

In quantum information, $\mathcal{H}$ is typically finite-dimensional. We choose $\rho, \sigma \in \mathrm{End}(\mathcal{H})$ and define their *relative entropy* to be

$$S(\rho \,\|\, \sigma) \coloneqq \mathrm{tr}_{\mathcal{H}}(\rho(\log\rho - \log\sigma)). \tag{5.14}$$

This is always nonnegative, and is zero iff $\rho = \sigma$.

Now assume $\mathcal{H}$ (still finite-dimensional) factors as $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Let A be the algebra of operators of the form $a \otimes \mathrm{id}_{\mathcal{H}_2}$ and A' be the algebra of operators of the form $\mathrm{id}_{\mathcal{H}_1} \otimes a$. A general $|\psi\rangle \in \mathcal{H}$ can be written as

$$|\psi\rangle = \sum_{k=1}^m c_k \psi_k \otimes \psi'_k \tag{5.15}$$

for $\{\psi_k\}$ and $\{\psi'_k\}$ orthonormal sets (not necessarily bases) for $\mathcal{H}_1$, resp. $\mathcal{H}_2$. Then $|\psi\rangle$ is cyclic and separating for A iff $c_k \neq 0$ for all k and $\dim\mathcal{H}_1 = \dim\mathcal{H}_2$. In this case, $\{\psi_k\}$ and $\{\psi'_k\}$ are bases.

Now choose some other state $|\phi\rangle = \sum_\alpha d_\alpha \phi_\alpha \otimes \phi'_\alpha$. Then

$$\Delta_{\psi|\phi}(\phi_\alpha \otimes \psi'_k) = \frac{|d_\alpha|^2}{|c_k|^2} \phi_\alpha \otimes \psi'_k. \tag{5.16}$$

Let's express this in terms of density matrices. Write $\rho \coloneqq |\psi\rangle\langle\psi|$ and $\sigma \coloneqq |\phi\rangle\langle\phi|$. Let $\rho_i \coloneqq \mathrm{tr}_{\mathcal{H}_i}(\rho)$, and define σ_1, σ_2 similarly. Then

$$\rho_2 = \sum_k |c_k|^2 \psi_k \otimes \psi^*_k \tag{5.17a}$$

$$\sigma_1 = \sum_\alpha |d_\alpha|^2 \phi_\alpha \otimes \phi^*_\alpha, \tag{5.17b}$$

and $\Delta_{\psi|\phi} = \sigma_1 \otimes \rho_2^{-1}$. Then the relative entropy is

$$\mathcal{E}_{\psi|\phi}(A) = -\langle \psi \mid \log(\Delta_{\psi|\phi}) \mid \psi\rangle \tag{5.18}$$

$$= -\operatorname{tr}(\rho \log(\sigma_1 \otimes \rho_2^{-1})) \tag{5.19}$$

$$= -\operatorname{tr}(\rho(\log(\sigma_1) \otimes \mathrm{id}_{\mathcal{H}_2} - \mathrm{id}_{\mathcal{H}_1} \otimes \log(\rho_2))) \tag{5.20}$$

$$= -\operatorname{tr}_{\mathcal{H}_1}(\rho_1 \log(\sigma_1)) + \operatorname{tr}_{\mathcal{H}_2}(\rho_2 \log(\rho)) \tag{5.21}$$

$$= \operatorname{tr}_{\mathcal{H}_1}(\rho_1(\log(\rho_1) - \sigma_1)) = S(\rho_1 \,\|\, \sigma_1) \tag{5.22}$$

as desired.

6. Entanglement in quantum field theory, part II: 10/23/19

Today, Sebastian spoke, continuing the discussion of Witten's notes on entanglement entropy [Wit18].

Recall that we've been considering an open set $U \subset \mathbb{R}^{1,d-1}$, so we have an algebra A_U of local operators acting on the Hilbert space $\mathcal{H}$ of states. The vacuum state is denoted $|\Omega\rangle$. We discussed when this is cyclic, i.e. $\mathcal{A}_U \cdot |\Omega\rangle$ is dense in $\mathcal{H}$, and when it's separating, meaning that if $a|\Omega\rangle = 0$, then $a = 0$. The algebra A_U is a von Neumann algebra, i.e. a $*$-algebra of bounded operators on a Hilbert space which contains the identity and is closed under weak limits.

The point is that the mathematical theory of von Neumann algebras allows us to extend proofs of various useful facts about quantum field theory; specifically, statements which we can usually only prove when $\mathcal{H}$ factors as a tensor product of two Hilbert spaces can be generalized using von Neumann algebras.

Definition 6.1. A *factor* is a von Neumann algebra with trivial center. Here "trivial" means just constant multiples of the identity.

Theorem 6.2 (von Neumann, 1949)**.** *Every von Neumann algebra acting on a separable Hilbert space is a finite integral*[18] *of factors.*

This is a semisimplicity condition, much like semisimple Lie algebras: once you understand the simple pieces (here, the factors), you can just put them together. However, this isn't quite a finite sum, but a finite total measure integral, so it's a little more elaborate.

Murray and von Neumann proved in 1936 that factors can be classified into three types. Type I von Neumann algebras A are those which can act irreducible on a separable Hilbert space $\mathcal{K}$, i.e. A is (isomorphic to) the algebra of bounded operators on $\mathcal{K}$. If $\mathcal{K}$ is d-dimensional, we say A is Type I_d (this includes $d = \infty$). These are the simplest factors.

Definition 6.3. A *trace* on a von Neumann algebra A is a linear functional $\operatorname{tr}\colon A \to \mathbb{C}$ such that $\operatorname{tr}(ab) = \operatorname{tr}(ba)$ and $\operatorname{tr}(a^*a) > 0$ for $a \neq 0$.

Type I_d factors always have traces: these can be identified with matrix algebras, and we get a trace in the usual way. But Type I_∞ factors only have a trace on a subalgebra: if there were a trace on the entire algebra, $\operatorname{tr}(\mathrm{id}) = \infty$, which is no good.

We won't discuss all Type II factors, but instead a nice subclass called Type II_1. These are akin to limits of Type I factors. Consider the Hilbert space V of 2×2 complex matrices with the inner product $\langle v, w\rangle := \operatorname{tr}(v^\dagger w)$. The algebra M of 2×2 complex matrices acts on V on the left and on the right: on the left it's just $a \cdot v := av$ for $a \in M$ and $v \in V$, and on the right $v \cdot a := va^{\mathrm{T}}$. These two actions commute, which follows from associativity of matrix multiplication — and in fact, the algebra acting on one side is precisely the commutant of the algebra acting on the other side.

Write $V = W \otimes W'$, where $W \cong \mathbb{C}^2$ is column vectors, and $W' \cong \mathbb{C}^2$ is row vectors, then the left action is just on W, and the right action is just on W'. This is what's called a *bipartite quantum system.* Let $I_2' := (1/\sqrt{2})\mathrm{id}$, a renormalized identity matrix; then, I_2' is a renormalized, maximally entangled element of V. What this means is that if $|\psi\rangle \in V$ has norm 1 and $\rho := |\psi\rangle\langle\psi| \in V \otimes V^* = \operatorname{End}(V)$, then we can take the partial trace $\rho_1 := \operatorname{tr}_{W'} \rho \in \operatorname{End}(W)$ and compute the entropy $\operatorname{tr}_W(\rho_1 \log \rho_1)$, and this is maximal for $|\psi\rangle = I_2'$.

Now let's tensor countably many copies of V together. This doesn't quite work — what we obtain is an inseparable Hilbert space. So let's consider the subspace of that Hilbert space spanned by tensors of

[18] You can think of "integral" as direct sum, though technically there's some sort of completion going on.

countably many vectors v_i, such that all but finitely many $v_i = I_2'$. Then let $\mathcal{H}$ be the closure of this space. A typical element of this space has a sequence including some elements which aren't I_2', but they must approach I_2' fairly fast.

For the algebra structure, let A_0 denote the span of tensors of countably many elements of M (acting from the left), such that all but finitely many are the identity; then, let A denote the closure of A_0 in $B(\mathcal{H})$. If we did this with the right M-action, we'd obtain an isomorphic algebra A' which is the commutant of A. The vector $|\psi\rangle := I_2' \otimes I_2' \otimes \dots$ is both cyclic and separating, making it a good candidate for a vacuum vector. In order for this to be a good candidate for a space of states in a quantum field theory, we hope that there isn't a trace.

Let $F(a) := \langle \psi \mid a \mid \psi \rangle$. Since ψ is separating, then if $a \neq 0$,

$$F(a^*a) = \langle \psi \mid a^*a \mid \psi \rangle = \| a|\psi\rangle \| > 0. \tag{6.4}$$

Computing F seems to be an infinite problem, but all of those copies of the identity don't come into play. Letting

$$\begin{aligned} a &= a_1 \otimes \cdots \otimes a_k \otimes \mathrm{id} \otimes \mathrm{id} \otimes \cdots \\ b &= b_1 \otimes \cdots \otimes b_n \otimes \mathrm{id} \otimes \mathrm{id} \otimes \cdots, \end{aligned} \tag{6.5}$$

then without loss of generality assum $n \geq k$; then, we can ignore everything after n, and

$$F(ab) = \langle \psi \mid a_1 b_1 \otimes \cdots \otimes a_n b_n \mid \psi \rangle = F(ba). \tag{6.6}$$

By density, this holds on all of A.

In the state $|\psi\rangle$, A and A' have infinte entanglement entropy, but different $x \in \mathcal{H}$ still have the same divergence asymptotics, which implies the universal divergence of entanglement entropy. This trace is wrong if we were hoping for something coming from quantum field theory.

This A is a Type II_1 factor. It has no irreduicble representations, which also suggests this is unphysical. For example, we can find a subrepresentation of $\mathcal{H}$ as the image of the projector

$$\Pi_k' := \underbrace{J_2 \otimes J_2 \otimes \cdots \otimes J_2}_{k} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \dots, \tag{6.7}$$

where $J_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. This subrepresentation also isn't irreducible, and you can play the same game.

On to Type III, which are the von Neumann algebras we actually find in physics. Let

$$K_{2,\lambda} := \frac{!}{(1+\lambda)^{1/2}} \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{1/2} \end{pmatrix}, \tag{6.8}$$

where $\lambda \in (0,1)$. Inside a completed tensor product of countably many copies of $\mathbb{C}^2$, let $\mathcal{H}_{0,\lambda}$ be the span of all tensors such that all but finitely many are $K_{2,\lambda}$, and let $\mathcal{H}_\lambda$ denote its closure. With A_0 as above, let A_λ denote the closure of A_0 inside $B(\mathcal{H}_\lambda)$. Let A' denote the commutant of A.

As before, $|\psi\rangle := K_{2,\lambda} \otimes K_{2,\lambda} \otimes \dots$ is cyclic and separating, but if we define $F(a)$ as above, $F(ab) \neq F(ba)$ in general! In fact, A_λ doesn't admit a trace at all. This is good.

Remark 6.9. Again we get the universal divergence of entanglement entropy, but the action of A_λ on $\mathcal{H}_\lambda$ is not irreducible! This is a little weird. It's a general fact that a hyperfinite Type III factor (such as this one) has only one nontrivial representation up to isomorphism. ◀

Theorem 6.10 (Powers)**.** *If $\lambda \neq \lambda'$, then $A_\lambda \ncong A_{\lambda'}$.*

We can generalize this slightly, by fixing a list $\vec{\lambda} = (\lambda_1, \lambda_2, \dots)$, defining $\mathcal{H}_{0,\vec{\lambda}}$ to be the tensors $v_1 \otimes v_2 \otimes \dots$ where all but finitely many v_i are K_{2,λ_i}. Then we make the same definitions of $\mathcal{H}$ and $A_{\vec{\lambda}}$. These have an interesting classification.

- If $\lambda_n \to \lambda \in (0,1)$, then $A_{\vec{\lambda}} \cong A_\lambda$. This is called Type III_λ.
- If $\lambda_n \to 0$, two things can happen: if this convergence is fast enough, we get Type I_∞, and if it's not fast enough, we get a new type called Type III_0.
- If $\{\lambda_n\}$ has at least two accumulation points λ, λ' such that there are no m, n wuth $\lambda^n = (\lambda')^m$ (which is a generic condition), then we again get something new, called Type III_1.

Now back to quantum field theory. The algebras of local operators that we consider in quantum field theory are Type III, and are believed to be of type III_1. Witten gives a heuristic argument for this, using the spectrum of the modular operator Δ_ψ from last time.

- For Type III_0, we get $\{0, 1\}$.
- For Type III_λ, we get $\{0, \lambda^n \mid n \in \mathbb{Z}\}$.
- For Type III_1, we get $[0, \infty)$.

In quantum field theory, $\Delta_\Omega = \exp(-2\pi k)$, where k is a Lorentz boost, so the spectrum is again $[0, \infty)$. The punchline is that the algebra of bounded local operators in quantum field theory are of Type III; at first, these were completely intractable, and almost 40 years after the initial steps in the theory of von Neumann algebras, Tomita and Takesaki revolutionized the field and made Type III accessible, allowing us to use their theory to gain insights into algebras of local operators.

Here's an example application. Take U, U' to be complementary open sets, e.g. the two cones of a lightcone. If $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, where A_U acts on $\mathcal{H}_1$ and $A_{U'}$ acts on $\mathcal{H}_2$, then you could specify physics independently on U and U', but of course thats's not how the real world works. This is a little unfortunate, because it would simplify many proofs.

Example 6.11. Let ρ be a density matrix in $\mathcal{H}$. Can one cook up another $\chi \in \mathcal{H}$ which is indistinguishable from ρ for measurements in some region U?

If $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, with A_U acting on $\mathcal{H}_1$, then we can let $\rho_1 := \mathrm{tr}_{\mathcal{H}_2} \rho \in \mathrm{End}(\mathcal{H}_1)$; then we can let χ be a purification of ρ_1 in $\mathcal{H}_1 \otimes \mathcal{H}_2$.

But Tomita-Takesaki theory allows us to remove this nonphysical assumption. Let $F\colon A_U \to \mathbb{C}$ send $a \mapsto \mathrm{tr}_{\mathcal{H}}(\rho a)$. The *Gelfand-Neimark-Segal construction* for Type III factors products a Hilbert space $\mathcal{K}$, an action of A_U on $\mathcal{K}$, and a cyclic, separating vector $|\psi\rangle \in \mathcal{K}$. If we let $\mathcal{K}_0 := \mathrm{span}\{a\psi\}$ and $F(a) := \langle \psi \mid a \mid \psi \rangle$, then $F(a^\dagger b) = \langle a\psi \mid b\psi \rangle$ and $\mathcal{K} = \overline{\mathcal{K}_0}$. So by embedding into a larger Hilbert space, we can purify without respect to a particular tensor product. Then, there is an isometric embedding $T\colon \mathcal{K} \hookrightarrow \mathcal{H}$, and can let $\chi := T(\psi)$. This is pretty mind-blowing. ◀

7. Fractons, part I: 10/30/19

"Who cares about symmetries?"

Today, Andrew Potter gave the first of two talks on fractons, following Nandkishore and Hermele [NH19], a review paper which summarizes what's known and includes lots of good references.

One of the major research programs in condensed-matter physics in the last decade or so is to classify topological phases of matter. Gapped phases are much easier to understand than gapless ones — gapped phases are believed to be well approximated by their continuum limits, which should be topological quantum field theories, and these are relatively well understood.

Recent discoveries in 3d have challenged this classification idea. The goal of these talks is to discuss these discoveries, focusing on a specific example.

Example 7.1. Before we go to 3d, though, let's review a fundamental, hopefully familiar example: the 2d toric code.[19] This is a model on a two-dimensional lattice with a spin degree on each edge. The Hamiltonian is

$$H = -\sum_{\text{vertices } v} \prod_{e : \partial e = v} Z_e - \sum_{\text{plaquettes } p} \prod_{e \in \partial p} X_e. \tag{7.2}$$

Here Z_e and X_e are the Pauli Z- and X-matrices acting on the spin degree on the specified edge e. The terms in the Hamiltonian are commuting projectors, which makes life really nice.

This model is gapped, and on $\mathbb{R}^2$ this has a unique ground state. However, if you put it on a closed manifold, there is a ground state degeneracy, i.e. the space of ground states has dimension greater than 1. For example, on the torus, it's 4-dimensional; on a general closed, connected, oriented, genus-g surface Σ, the dimension is

$$2^{|H^1(\Sigma;\mathbb{Z}/2)|} = 4^g. \tag{7.3}$$

Crucially, this doesn't depend on the lattice size.

In this model, one can produce excitations in pairs only: given an open string on the lattice, there are operators creating particles at the ends of the string. There are two such kinds of particles, one where you put Z operators on the string and one where you put X operators on the string. These particles both behave as bosons if you consider their self-statistics (what happens when you exchange 2 particles of the same kind),

[19] If this is not review, that's fine; Kitaev's original paper [Kit03] on it is good.

but their mutual statistics look fermionic. This is an indication that these are anyons, rather than bosons or fermions.

The low-energy limit of this model is $\mathbb{Z}/2$ gauge theory, which is a topological field theory: the space of ground states is identified with the state space of $\mathbb{Z}/2$ gauge theory. The next eigenspace depends on the lattice size, as is generally true. ◀

The toric code is the *Drosophila* of topological phases: a fundamental example, heavily studied. So if we want to study 3d phases, we might as well start by trying to generalize the toric code. There are two ways one might do this.

Example 7.4. Let's consider a system on a 3d cubic lattice where we keep almost everything the same: spins are still on the edges, and the Hamiltonian has a vertex term and a plaquette term, and the plaquettes are still two-dimensional. Now, there are two kinds of excitations, pointlike and linelike (well, looplike). Linelike operators have statistics, e.g. you can link two of them into a Hopf link, or fancier possibilities with more than two loops. It's not fully known what data characterizes these loop braiding properties in general — but what you get in the end is a system where the ground state degeneracy depends only on the underlying manifold, and is independent of the number of sites in the model. For example, on T^3, it's 8-dimensional. Everything again admits a continuum description, and the continuum limit is again a $\mathbb{Z}/2$ gauge theory. ◀

But when you change the system only slightly, the system's behavior changes dramatically.

Example 7.5 (The X-cube model (Vijay-Haah-Fu [VHF15])). Again the Hamiltonian has two terms.

- The vertex term is as follows: a vertex v is contained in three planes. Take the product of Z_e over the edges e in each plane separately, and then sum them up.
- Instead of a plaquette term, we have a term for each cube, taking the product of X_e for all edges in the cube.

A given edge e is contained in four cubes, so acting by a Z operator on that edge creates an excitation in each cube. We can't exactly separate these excitations — you can act on other edges in the cube, but then you'll create a few more excitations, and can't clean them all up. So we get a sheet of excitations, and they are all confined to the same plane: we can't bend them or move them. This is very weird — we've produced pointlike excitations, but we cannot move them without introducing additional excitations.

We also can act with X operators on edges. These also behave weirdly — we get another kind of particle, but again if you try to move it by using another X operator nearby, you create additional excitations. Said differently, motion in a direction causes excitations and costs energy. Moreover, the excitation is confined to a line. In total, we have

- zero-dimensional excitations, which are completely immobile, created by operators on a plaquette;
- one-dimensional excitations, which are confined to a line; and
- two-dimensional excitations, which are bound states of corners of membrane operators, and which are confined to a plane.

Now let's study the space of ground states, where we place the system on a 3-torus (periodic boundary conditions) with length L in each direction. We can compute this by studying the loop operators which commute with the Hamiltonian, just as for the 2d toric code. For example, given a cycle c of T^3 which is a straight line in the lattice, we can drag the one-dimensional excitations around that cycle. Concretely, this acts by a product of X operators on each edge in the cycle. Because these excitations are confined to this line, different cycles yield inequivalent operators.

In a similar vein, one can act by Z on a membrane, and produce ladder-like operators. These anticommute with the X operators, and commute with the Hamiltonian, so for each intersection, we get additional ground state degeneracy! This leads to a count of 2^{6L} ground states — though there is some redundancy in this description, and in the end we get approximately 2^{6L-3}.

Again, this is very weird — why should the space of ground states depend exponentially on the lattice length? The ground states are again indistinguishable by local operators. ◀

This was not the first example discovered with immobile particles, though it's one of the simplest. The original model, which we'll see in a second, is much weirder — excitations live at the boundaries of fractal structures, hence the name "fractons." In the X-cube model, though, there are no fractals, so maybe the name "fractons" isn't as accurate as it could be, but the term has stuck now.

Example 7.6 (Haah [Haa11])**.** In this model, the Hamiltonian has a terms for cubes, with X and Z operators placed as follows.

$$
\begin{tikzcd}
& IX \arrow[rr, dash] \arrow[dd, dash] & & XI \arrow[dd, dash] \\
XI \arrow[rr, dash] \arrow[ur, dash] \arrow[dd, dash] & & II \arrow[ur, dash] \arrow[dd, dash] & \\
& XX \arrow[rr, dash] & & IX \\
IX \arrow[rr, dash] \arrow[ur, dash] & & XI \arrow[ur, dash] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& IZ \arrow[rr, dash] \arrow[dd, dash] & & ZI \arrow[dd, dash] \\
ZI \arrow[rr, dash] \arrow[ur, dash] \arrow[dd, dash] & & ZZ \arrow[ur, dash] \arrow[dd, dash] & \\
& II \arrow[rr, dash] & & IZ \\
IX \arrow[rr, dash] \arrow[ur, dash] & & XI. \arrow[ur, dash] &
\end{tikzcd}
\tag{7.7}
$$

One can then product operators given by tetrahedra appearing in (finite stages of) the 3d version of the Sierpiński gasket (TODO: there was a picture), and then the excitations are localized, but confined to the tetrahedra. In the continuum limit, we get better and better approximations of the Sierpiński fractal, and these are truly fractons. ◀

Just as the toric code can arise from a study of duality in the Ising model, the X-cube model can be studied as an extension of the Ising model by some plaquette terms. In this dual representation, there is a subsystem symmetry, along each plane where the excitations are confined. This suggests that this model is some kind of stack of toric codes (not just the tensor product — these are separated in space as well), with the subsystem symmetries corresponding to the gauge symmetries of the toric code.

This is not quite correct, but it's close — people have discovered that one can describe the X-cube model by stacks of toric codes in all three directions, together with a strong coupling between the toric codes in different directions. This gives 2^{6L} ground states *a priori*, but a few of the ground states don't survive, giving 2^{6L-3} ground states.

Question 7.8. What is the right notion of equivalence for these kinds of phases? Normally we'd say that two gapped phases are equivalent if we can tack on some additional, unentangled degrees of freedom to each and then transform one into the other via a local unitary operator.

Here, though, this notion of equivalence seems to not work. One proposal suggests that, in addition to these degrees of freedom, one can also add additional layers with 2d topological order. This is a nice characterization, and is called *foliated fracton order*.

Lastly, we mention that, though these models are all abelian, there are nonabelian versions, e.g. coupling to a Levin-Wen model. Trying to understand how these fit into our understanding of field theories and topological order is a little bizarre — can these be realized in materials? Do we treat these as pathological counterexamples, or as a need to reshape our definitions?

Next time, we'll discuss some examples of gapless models.

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