M393C NOTES: PARTIAL DIFFERENTIAL EQUATIONS

ARUN DEBRAY
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These notes were taken in UT Austin's M393C (Partial Differential Equations) class in Fall 2019, taught by Francesco Maggi. I live-TeXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Any mistakes in the notes are my own.

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Lecture 1.

Why elliptic equations?: 8/29/19

The course textbook is Evans, but we will probably diverge (hah) from it much of the time. If you're registered for the course, at the end of the course you'll have to give a presentation on a paper near the end of the course.

This course will require some familiarity with Sobolev spaces and weak solutions; if you haven't seen these before, one good reference is Evans, chapters 5 and 6.1.1, respectively.

We're going to be proving some esoteric things soon enough, so let's back up and ask: why elliptic equations?

Our first example is a simple model called the elastic membrane model. Let $\Omega \subseteq \mathbb{R}^n$ be a domain. Think of it as an elastic membrane whose boundary is fixed. The displacement of the membrane defines a function $u\colon \Omega \to \mathbb{R}$, and since the boundary is fixed, the boundary condition $u_0\colon \partial\Omega \to \mathbb{R}$ is given. The theory of *nonlinear elasticity*, understanding how this story behaves under deformations, is part of mathematical physics. For example, one can prove that if $|\nabla u|$ is small, then u will minimize the *Dirichlet energy*

$$\int_\Omega |\nabla u|^2 \tag{1.1}$$

such that $u|_{\partial\Omega} = u_0$.

The theory of Soblev spaces guarantees that there exists a unique minimizer $u \in W^{1,2}(\Omega)$, but $W^{1,2}$ functions can fail to be $L^\infty_{\ell oc}$ in dimension $n \geq 2$ – that is, they can be unbounded on every open set! The typical example is

$$u(x) = \sum_{k=1}^\infty \frac{1}{2^k} |\log|\log|x - x_k|||, \tag{1.2}$$

where $\{x_k\}_{k=1}^\infty$ is a countable dense subset of Ω (e.g. its points with rational coordinates). Sobolev spaces look simple: we have gradients and can pretend that we're acting on smooth functions, but don't forget that these pathologies sneak in.

Of course, we don't expect a minimizer of a nice physical problem to be so silly. So how can we prove strong enough regularity to avoid these pathologies? We will exploit minimality to obtain better information. That is, consider the Dirichlet energy $E(u + t\varphi)$, where φ is a test function, i.e. $\varphi \in C_c^\infty(\Omega)$. Thus $u + t\varphi$ obeys the same boundary condition, because $\varphi|_{\partial\Omega} = 0$. Thus $E(u + t\varphi)$ has a minimum at $t = 0$.

There are two consequences.

(1) The first-order consequence that, no matter what φ is,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} E(u + t\varphi) = 0. \tag{1.3}$$

(2) There's also a second-order consequence: for every φ,

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} E(u+t\varphi) \geq 0. \tag{1.4}$$

For this specific problem, the second one is trivial, and the first one is useful. For $E(u) = \int |\nabla u|^2$, we can explicitly compute the variation

$$E(u+t\varphi) = \int_\Omega |\nabla(u+t\varphi)|^2 = E(u) + 2t\int_\Omega \nabla u \cdot \nabla\varphi + t^2 E(\varphi). \tag{1.5}$$

Since $E(\varphi) \geq 0$, (1.4) contains no new information. But the first condition is an orthogonality of gradients:

$$\int_\Omega \nabla u \cdot \nabla \varphi = 0 \tag{1.6}$$

for *all* test functions φ. This seems a little bizarre, but let's integrate by parts: since $\varphi|_{\partial\Omega} = 0$,

$$\int_\Omega \nabla u \cdot \nabla\varphi = -\int_\Omega \varphi \operatorname{div}(\nabla u). \tag{1.7}$$

However, the right-hand side takes two (weak) derivatives of u, and *a priori* we only have one. So we can't always do this. Recall that $\operatorname{div}(\nabla u)$ is the Laplacian:

$$\operatorname{div}(\nabla u) = \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right) = \sum_{i=1}^n \frac{\partial^2 u}{\partial {x_i}^2} = \Delta u. \tag{1.8}$$

So, assuming u is $W^{2,2}$, this is telling us that u is harmonic: $\int \varphi \Delta u = 0$ for all φ, hence $\Delta u = 0$.

Example 1.9. When $n = 2$, this is telling us $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. The second derivatives tell us about concavity: at any point, if you bend some amount in the x-direction, you must bend the same amount in the opposite concavity in the y-direction. For example, such a function cannot have a local maximum in the interior of Ω! ◀

This rules out pathological examples such as (1.2): minimizers can't be just anything, but their second-order derivatives have to average to zero.

Theorem 1.10. *If $u \in W^{1,2}_{\ell oc}(\Omega)$ such that $\int_\Omega \nabla u \cdot \nabla \varphi$ for all $\varphi \in C_c^\infty(\Omega)$, then $u \in C^\infty(\Omega)$. Moreover, if u minimizes the Dirichlet energy subject to the boundary condition $u|_{\partial\Omega} = u_0$ and moreover, if u_0 has small oscillations, then $|\nabla u|$ is small, and scales linearly in the oscillation of u_0.*

The first step of the proof will be to establish something called the Caccioppoli inequality.

We know that $\int_\Omega \nabla u \cdot \nabla \varphi = 0$ for all $\varphi \in C_c^\infty(\Omega)$, and hence also for all $\varphi \in W_0^{1,2}(\Omega)$, where $W_0^{1,2}(\Omega)$ denotes the closure of $C_c^\infty(\Omega)$ in $W^{1,2}(\Omega)$ in the $W^{1,2}$-norm.

Choose a $\zeta \in C_c^\infty(\Omega)$ and $\varphi = \zeta^2 u$, which vanishes on $\partial\Omega$ and is in $W_0^{1,2}(\Omega)$. Then $\nabla\varphi = 2\zeta u \nabla\zeta + \zeta^2 \nabla u$, and therefore

$$0 = \int_\Omega \nabla u \cdot \nabla \varphi = 2\int_\Omega \zeta u \nabla\zeta \cdot \nabla u + \underbrace{\int_\Omega \zeta^2 |\nabla u^2|}_{(*)}. \tag{1.11}$$

If ζ is some sort of bump function near $x_0 \in \Omega$, then $(*)$ tells you the Dirichlet energy near x_0.[1]

Rewriting (1.11),

$$\int_\Omega \zeta^2 |\nabla u|^2 = -2\int_\Omega (u\nabla\zeta)\cdot(\zeta\nabla u). \tag{1.12}$$

u is a simpler object than ∇u, and so it will be easier to control u than ∇u. We can take the Cauchy-Schwarz inequality $ab \leq a^2/2 + b^2/2$ and throw in an epsilon expressing that we're willing to pay a lot of a and only a little b:

$$\frac{a}{\varepsilon}\cdot \varepsilon b \leq \frac{|a|^2}{2\varepsilon^2} + \frac{\varepsilon^2}{2}|b|^2. \tag{1.13}$$

[1] If ζ isn't normalized, maybe we get some multiple of this, but the distinction is not important.

Apply this to (1.12):

$$\int_\Omega \zeta^2|\nabla u|^2 = 2\int_\Omega \underbrace{(-u\nabla\zeta)}_{a}\cdot\underbrace{(\zeta\nabla u)}_{b} \le \frac{1}{\varepsilon^2}\int_\Omega u^2|\nabla\zeta|^2 + \varepsilon^2\int_\Omega \zeta^2|\nabla u|^2. \tag{1.14}$$

Plugging in $\varepsilon = 1/2$, we have:

Corollary 1.15 (Caccioppoli inequality, preliminary version)**.** *For all $\zeta \in C_c^\infty(\Omega)$,*

$$\frac{1}{2}\int_\Omega \zeta^2|\nabla u|^2 \le 2\int_\Omega u^2|\nabla\zeta|^2. \tag{1.16}$$

This is good: we have better control over $\nabla\zeta$ than ∇u.

Pick an $x_0 \in \Omega$ and some radius $R > 0$. Let's choose as our test function ζ something which is identically 1 on $B_{R/2}(x_0)$, identically 0 on the complement of $B_R(x_0)$, and always between 0 and 1. Let's additionally ask that $\nabla\zeta$ doesn't do anything stupid: we know that if ρ is the radial direction, then

$$\frac{\mathrm{d}\zeta}{\mathrm{d}\rho} > \frac{1-0}{R-R/2} = \frac{2}{R}, \tag{1.17}$$

where equality would be attained by just decreasing linearly in ρ. This isn't smooth, but we can smooth it, and therefore we can (and do) ask that $|\nabla\zeta| \le 3/R$. Therefore

$$\int_{B_{R/2}(x_0)} |\nabla u|^2 \le \int_\Omega \zeta^2|\nabla u|^2 \le \int_\Omega u^2|\nabla\zeta|^2 \le \frac{C}{R^2}\int_{B_R(x_0)}\int u^2. \tag{1.18}$$

Here $C = 9$, but that's not so important.

Corollary 1.19 (Caccioppoli inequality)**.** *If $x_0 \in \Omega$ and R is a radius such that $B_R(x_0) \subset \Omega$, and if $u \in W^{1,2}(\Omega)$ is such that $\int_\Omega \nabla u\cdot\nabla\varphi = 0$ for all $\varphi \in C_c^\infty(\Omega)$, then there's a $C > 0$ such that*

$$\int_{B_{R/2}(x_0)} |\nabla u|^2 \le \frac{C}{R^2}\int_{B_R(x_0)} u^2. \tag{1.20}$$

This is kind of a complicated statement, but the essential fact is that the square of the gradient is controlled by the square of the function, albeit on a larger region. This is very false for general functions, and is only true because of the orthogonality condition we placed on ∇u and $\nabla\varphi$.

The next thing to think about is dimentional analysis. ∇u and u have different dimensions: the former has the dimensionality of the latter divided by length. Letting $[x]$ denote the dimension of a quantity x, this means

$$\left[\int_{B_{R/2}(x_0)} |\nabla u|^2\right] = \frac{[u]^2}{(\mathrm{length})^2}(\mathrm{length})^n. \tag{1.21}$$

Therefore both sides of (1.20) have dimensionality $[u]^2(\mathrm{length})^{n-2}$, which is important: you cannot get an inequality between objects of different dimensions. If you did, you made a mistake.

We'd like to conclude that if the oscillation at the boundary of u is small, then the gradient of u is small, and we have proven something related: roughly, if u is small at the boundary, its gradient is small in the interior.

Next, if u solves $\int \nabla u\cdot\nabla\varphi = 0$, so does $u + k$ for any $k \in \mathbb{R}$, so we can choose k to minimize the right-hand side of (1.20). The best possible value of k is $⨍_{B_R(x_0)} u$ (i.e. the average of u over $B_R(x_0)$). That is,

$$\int_{B_{R/2}(x_0)} |\nabla u|^2 \le \frac{C}{R^2}\int_{B_R(x_0)} |u - (u)_{B_R(x_0)}|^2. \tag{1.22}$$

This is nice, but how are we going to get C^∞ regularity? We'll leverage the Caccioppoli inequality.

Let $\rho_\varepsilon(x) := (1/\varepsilon^n)\rho(x/\varepsilon)$ bw the mollifier of ρ, where $\rho \in C_c^\infty(B_1(x_0))$ be nonnegative and have total integral 1. Then let $u_\varepsilon := u * \rho_\varepsilon$, i.e.

$$u_\varepsilon(x) = \int u(y)\rho_\varepsilon(x-y)\,\mathrm{d}y. \tag{1.23}$$

So we really need ρ defined on the set of points of distance less than ε from Ω, but that's easy to arrange.

The equation $\int_\Omega \nabla u\cdot\nabla\varphi = 0$ is linear, so if u and v both satisfy it for all $\varphi \in C_c^\infty(\Omega)$, then so does $u + v$. This allows us to prove that $\int_\Omega \nabla u_\varepsilon\cdot\nabla\varphi = 0$ for all φ — and u_ε is smooth, so this implies $\Delta u_\varepsilon = 0$.

Since u_ε is harmonic, so is an arbitrary partial derivative w of u_ε of arbitrary order.[2] Thus $\int_\Omega \nabla w \cdot \nabla\varphi = 0$ for all $\varphi \in C_c^\infty(\Omega)$ and all w.

Now we can apply the Caccioppoli inequality to w!

$$\int_{B_{R/2}(x_0)} |\nabla w|^2 \leq \frac{C}{R^2}\int_{B_R(x_0)} |w|^2. \tag{1.24}$$

Letting w range over the k^{th}-order partial derivatives of order ε, this implies

$$\int_{B_{R/2}(x_0)} |\nabla^k u_\varepsilon|^2 \leq \frac{C(n,k)}{R^2}\int_{B_R} |\nabla^{k-1}u_\varepsilon|^2 \leq \frac{C(n,k)}{R^4}\int_{B_{2R}(x_0)} |\nabla^{k-2}u_\varepsilon|^2 < \cdots < \frac{C(n,k)}{R^{2k}}\int_{B_{2^kR}(x_0)} u_\varepsilon^2. \tag{1.25}$$

Here, as usual, $C(n,k)$ means some constant that depends on n and k, and whose value can change between expressions. We also need $\operatorname{dist}(B_R(x_0), \partial\Omega) > \varepsilon$ so that the convolution makes sense.

Now we use a dash of the theory of Sobolev spaces: Morrey's theorem tells us that if $u \in W^{k,2}(\mathbb{R}^n)$ and $k \gg 0$ (the specific value depends on n), then $u \in L^\infty(\mathbb{R}^n)$. That is, if you have control over enough derivatives, you must be bounded. We saw that $W^{1,2}$ isn't enough, but if we pile up a few more, we're good — we can get Hölder continuity, and even better, continuous partial derivatives.

Thus, there is some k (depending on n) such that

$$C(n,k)\int_{B_R(x_0)} u_\varepsilon^2 \geq R^n |u_\varepsilon|_{L^\infty(B_{R/2}(X_0))}. \tag{1.26}$$

That is, the L^2-norm of u_ε controls the L^∞-norm.

Corollary 1.27. *Letting $\varepsilon \to 0$, $u \in W^{k,2}_{\ell oc}(\Omega)$ and*

$$\int_{B_{R/2^k}(x_0)} |\nabla^k u|^2 \leq CR^{-2k}\int_{B_R(x_0)} u^2. \tag{1.28}$$

Thus, $u \in W^{k,2}(\Omega)$ for all k, so $u \in C^\infty(\Omega)$.

In the last few minutes, let's talk about oscillation. The first observation is that if u is a minimizer, so is $\min(u, \sup_{\partial\Omega} u_0)$. This controls the oscillation of the boundary data:

$$\operatorname{osc}(u_0, \partial\Omega) = \operatorname{osc}(u, \Omega). \tag{1.29}$$

Returning to the Caccioppoli inequality, this tells us

$$C(n)R^n \operatorname{osc}(u_0, \partial\Omega)^2 \geq C(n)R^n \operatorname{osc}(u,\Omega)^2 \geq \int_{B_R(x_0)} |u - (u)_{B_R(x_0)}|^2 \geq \frac{R^2}{C}\int_{B_{R/2}(x_0)} |\nabla u|^2. \tag{1.30}$$

We can also apply this to partial derivatives of u in the same way as above:

$$C(n)R^n \operatorname{osc}(u_0, \partial\Omega)^2 \geq \frac{R^{2+n}}{C(n,k)} \|\nabla u\|^2_{L^\infty(B_{R/2^k}(x_0))}. \tag{1.31}$$

[2] There's a small argument to make here.