MATH 116 NOTES: COMPLEX ANALYSIS

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These notes were taken in Stanford's Math 116 class in Fall 2014, taught by Steve Kerckhoff. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Thanks to Luna Frank-Fischer, Anna Saplitski, and Allan Peng for fixing a few errors.

Contents

1. Complex Differentiability is Very Special: 9/23/14

There are two undergraduate complex analysis classes taught this quarter, 116 and 106, which is more computational (and possibly for other majors than math). The prerequisites for 116 are 51 and 52; having 115 or 171 would be nice, but isn't as important.

In order to talk about functions of a complex variable, we should talk about complex numbers $z \in \mathbb{C}$. These are written as $z = x + iy$, where $i = \sqrt{-1}$. An important operation is the complex conjugate $\overline{z} = x - iy$; then, the size (norm squared) is $\|z\|^2 = x^2 + y^2 = z\overline{z}$. There's also a polar description $z = re^{i\theta}$, where $r = \|z\|$; these are related as set up in Figure 1.

Thus, there is a clear relation between $\mathbb{C}$ and the plane $\mathbb{R}^2$, where $x + iy \longleftrightarrow (x, y)$. Convergence is exactly the same: $z_n \to z$ iff $(x_n, y_n) \to (x, y)$.

A function $f : \mathbb{C} \to \mathbb{C}$ is continuous if whenever $z_n \to z$, $f(z_n) \to f(z)$. Thus, the equivalent function $\hat{f} : \mathbb{R}^2 \to \mathbb{R}^2$ (given by replacing each complex number by its planar representation) is continuous iff f is.

This convergence looks very similar to what we've seen before in real analysis, and the algebraic properties are slightly different. Where things become different is the notion of complex differentiability: the definition of complex differentiability makes a huge difference. There are real-valued functions that are C^1 but not C^2 (or C^{14} but not C^{15}), and C^∞ functions (infinitely many times differentiable) that aren't analytic (given by a Taylor series). However, for a function that is complex differentiable, all derivatives exist and it is analytic — this is pretty magical, even after seeing the proofs.

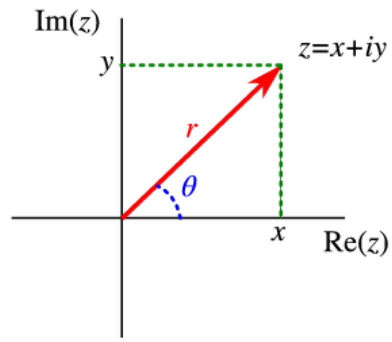

Figure 1. The rectangular and polar forms of a complex number. Source: http://oer.physics.manchester.ac.uk/Math2/Notes/Notes/Notesse2.xht.

The word *domain* will be used to refer to an open $\Omega \subset \mathbb{C}$; that is, for any $z \in \Omega$, there's an $\varepsilon > 0$ such that the open disc of radius ε around z is still in Ω.

Definition. Let Ω be an open domain and $f : \Omega \to \mathbb{C}$; then, f is complex differentiable at a $z \in \Omega$ if

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists, when $h \in \mathbb{C}$. Then, $f'(z)$ is defined to be this limit.

The word *holomorphic* is synonymous with complex-differentiable.

The key is that h is complex-valued, so we look at all values within a small disc around z.

From the definition and the usual relations, the usual algebraic properties are the same as for real differentiable functions. Suppose f and g are holomorphic on Ω; then,

(1) $(f + g)' = f' + g'$.
(2) $(fg)' = f'g + fg'$,
(3) $(f/g)' = (f'g - fg')/g^2$ wherever $g(z) \neq 0$.
(4) $(g \circ f)'(z) = g'(f(z))f'(z)$.

Technically, the last point applies when $f : \Omega \to \Omega'$ and $g : \Omega' \to \mathbb{C}$, so that $g \circ f$ is defined; then, the formula holds as expected.

Example 1.1. Since $f(z) = z$ is holomorphic, then all polynomial functions $f(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, with $a_i \in \mathbb{C}$, are holomorphic.

However, $f(z) = \overline{z}$ (i.e. $f(x + iy) = x - iy$) is not holomorphic: if $h = t \in \mathbb{R}$, then

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \frac{((x+t) - iy) - (x - iy)}{t} = \frac{t}{t} = 1,$$

but if $h = it$ for $t \in \mathbb{R}$, then

$$\begin{aligned}\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} &= \frac{\overline{x + i(y+t)} - \overline{x + iy}}{it} \\ &= \frac{-it}{it} = -1.\end{aligned}$$

Notice this isn't a particularly ugly function; nonetheless, it's not holomorphic. But it's smooth as a map from $\mathbb{R}^2 \to \mathbb{R}^2$, and even linear! So complex differentiability is much stronger of a notion than real differentiability; if f is holomorphic, then $\hat{f}$ is differentiable, but not always the other way around.

Recall that $\hat{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is (real) differentiable at a $p \in \mathbb{R}^2$ if there exists a linear map $D\hat{f}_p$ such that

$$\lim_{H\to 0} \frac{\hat{f}(p+H) - \hat{f}(p) - D\hat{f}_p(H)}{\|H\|} = 0,$$

where $H \in \mathbb{R}^2$.

If $\hat{f}(x, y) = (u(x, y), v(x, y))$, then the derivative matrix is given by

$$Df = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix},$$

which can all be evaluated at a point p. This matrix is usually called the Jacobian. If f is holomorphic, we can say something special about the Jacobian of $\hat{f}$.

Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ be holomorphic. Write $f(z) = w = u + iv$, and $z = x + iy$, so we have the associated $\hat{f}(x, y) = (u, v)$ as before. Let $z_0 = x_0 + iy_0$ and $h = h_1 + ih_2$, and assume

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists (since f is holomorphic). Then, we can walk in either the real or imaginary direction:

$$\begin{aligned}
\frac{f(z_0 + h_1) - f(z_0)}{h} &= \frac{f((x_0 + h) + iy_0) - f(x_0 + iy_0)}{h} \\
&= \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}. \\
\frac{f(z_0 + ih_2) - f(z_0)}{ih_2} &= \frac{f(x_0 + i(y_0 + h_2)) - f(x_0 + iy_0)}{ih_2} \\
&= \frac{1}{i}\frac{\partial f}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.
\end{aligned}$$

Since the limit exists, these must be equal; thus,

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y},$$

That is,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1}$$

This result (1) is known as the *Cauchy-Riemann equations*. It also implies that the Jacobian $D\hat{f}$ is almost skew-symmetric (except that the diagonals are nonzero), which is, again, a very special condition.

Geometrically, a vector in $\mathbb{R}^2$ (as $\mathbb{C}$) is sent from one direction to another direction by f, but then multiplying it by i rotates it through an angle of $\pi/2$, but also maps it image through the same angle. This works for every complex number, which means that holomorphic functions infinitesimally preserve the notion of angle. Thus, holomorphic functions are sometimes known as *conformal* functions. For example, many of these functions look like (especially locally, since $D\hat{f}$ is skew-symmetric) an expansion composed with a rotation.

The converse to this is also true: if $\hat{f}$ satisfies the Cauchy-Riemann equations, then its associated complex-valued function f is holomorphic. This is not hard to prove (the book does it).

Assume $f(z) = u + iv$ is holomorphic, so that (using subscripts to denote partial derivatives) $u_x = v_y$ and $u_y = -v_x$; let's assume further (which will end up being true for all holomorphic functions, though we haven't shown it yet) that $u(x, y)$ and $v(x, y)$ are both C^2, i.e. they have continuous second-order derivatives. This means that the mixed partials of u and v are equal. Since $u_{xy} = u_{yx}$, we can use the Cauchy-Riemann equations to get $u_{xx} = v_{yx} = -u_{yy}$. Thus, $u_{xx} + u_{yy} = 0$, and similarly for v. This means that u and v are *harmonic*: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, and the same for v. These are extremely special and beautiful functions, and come up a lot in physics and other applications — and (as we'll show) all holomorphic functions are analytic, so their coordinates are harmonic!

Here's some useful notation:

$$\begin{aligned}
\frac{\partial}{\partial z} &\overset{\text{def}}{=} \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \\
\frac{\partial}{\partial \overline{z}} &\overset{\text{def}}{=} \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).
\end{aligned}$$

Using the Cauchy-Riemann equations again, f is holomorphic iff $\frac{\partial f}{\partial \overline{z}} = 0$ iff $f'(z) = \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}$.

These simplify the definition of the Laplacian:

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}.$$

Recall that this is 0 for holomorphic functions.

2. Holomorphic Functions: 9/25/14

Recall that last time we looked at functions $f : \Omega \to \mathbb{C}$, where $\Omega \subset \mathbb{C}$ is open, and defined the notion of complex differentiability, that

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists. The word *holomorphic* is also used as a synonym for complex differentiable. We also saw that the algebraic properties of holomorphic functions imply that polynomial functions are holomorphic. Holomorphicity is a very special notion: the Jacobian of a holomorphic function has a certain form, the real and imaginary parts are harmonic, etc.

Today we want to obtain a much larger class of holomorpic functions; specifically, we'll be considering infinite series of the form

$$\sum_{n=0}^{\infty} a_n z^n,$$

where $a_n \in \mathbb{C}$. This defines a function of some $\Omega \to \mathbb{C}$, depending on where it converges, so where do these series converge? Then, the function is defined wherever it does converge, as $f(z) = \sum a_n z^n$.

Definition. If $S_m = \sum_{n=0}^{m} a_n z^n$, then the series $\sum_{n=0}^{\infty} a_n z^n$ *converges* to $\ell \in \mathbb{C}$ if $S_m \to \ell$ as $m \to \infty$.

Notice that this forces $|a_n z^n| \to 0$.

Definition. A series $\sum a_n z^n$ *converges absolutely* if $\sum_{n=0}^{\infty} |a_n||z|^n$ converges.

One can show that absolute convergence implies convergence; the converse, though, is not true. All this is much like real analysis.

Since $|a_n||z|^n \geq 0$, then the partial sums are monotonically increasing, so (an easier thing to check) a series converges iff its partial sums are bounded. Also, if a series converges absolutely for some $z_0 \in \mathbb{C}$, then this implies that it converges absolutely for any z such that $|z| \leq |z_0|$.

The $\limsup$ (said "lim-soup") of a bounded sequence $\{A_k\}$ with $A_k \in \mathbb{R}$ is given by setting $b_n = \sup_{k\geq n} A_k$ (which certainly exists, because $\{A_k\}$ is bounded above). Then, $b_{n+1} \leq b_n$, so $\{b_n\}$ is monotonically decreasing, but bounded below, so its limit exists; then, the $\limsup$ is $\limsup_{k\to\infty} A_k = \lim_{n\to\infty} b_n$.

This is a good replacement for the limit of a sequence; not all sequences have limits, but the lim sup is defined on more sequences.

Lemma 2.1. *Let $\alpha = \limsup_{k\to\infty} |a_k|^{1/k}$; then, $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely on the open disc of radius $R = 1/\alpha$ around the origin; outside of that disc, it diverges.*

This lemma says nothing about the boundary: the behavior can be quite subtle there, converging on some parts of the boundary and diverging on others. One of the homework problems addresses this in more detail.

The R named in Lemma 2.1 is called the *radius of convergence*.

Here's the idea of the proof: suppose $|a_k|^{1/k} \sim \alpha$ (i.e. they're similar for large k), i.e. $|a_k| \sim \alpha^k$. Then, $|a_n||z|^n \sim |\alpha||z|^n$, so this series is approximately geometric, and thus converges absolutely inside of the unit disc and diverges outside of it. Then, after dividing by α, we find the radius of convergence.

Proof of Lemma 2.1. Suppose $|z| < 1/\alpha = R$; then, there exists a $d < 1$ such that $|z| = d/\alpha$.

From the definition of $\limsup$, for any $c < 1$, $|a_k|^{1/k} < \alpha/c$ (since this is greater than the lim sup) for all sufficiently large k, so choose $d < c < 1$. Then,

$$\begin{aligned} |a_k||z|^k &= \left(|a_k|^{1/k}|z|\right)^k \\ &\leq \left(\frac{\alpha}{c}\cdot\frac{d}{\alpha}\right)^k = \left(\frac{d}{c}\right)^k. \end{aligned}$$

Thus, this converges by comparison to the geometric series.

For $|z| > 1/\alpha$, one can show that $a_n z^n \not\to 0$, so it diverges there. ⊠

Now, we can define lots of interesting new functions. For example, the *exponential* is defined as

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Since $e^{|z|} = \sum_{n=0}^{\infty} |z|^n/n!$ converges for all real numbers, then the complex exponential series converges absolutely everywhere, and thus is defined everywhere. The idea is to show that $(1/k!)^{1/k} \to 0$.

Another example, with $a_n = 1$ for all n, is just $\sum a_n$. Then, $\limsup(a_n)^{1/n} = 1$, so this is defined only inside the unit disc (and we can start asking questions about its boundary).

Similarly, we can define $\sin z$ and $\cos z$ by taking the Taylor series and generalizing to complex z; these converge everywhere.

The notation $D(R)$ mean the disc of radius R around the origin. For any $z \in D(R)$, the convergence of the series is *uniform* on a closed disc around z, i.e. for all $\varepsilon > 0$ and all z in that closed disc (the latter being important; this doesn't depend on z), there exists an N such that whenever $n \geq N$ and $z \in D(R)$, the difference between $\sum_{j=1}^{n} a_j z^j$ and $\sum_{j=1}^{\infty} a_j z^j$ is less than ε.

So we have these functions, but the whole point was to get more holomorphic functions. Specifically, if $R = 1/\alpha$ and $\Omega = D(R)$, then is $f(n) = \sum_{n=0}^{\infty} a_n z^n$ holomorphic? Each partial sum is a polynomial and therefore of course holomorphic, so is the limit of these holomorphic functions holomorphic?[1]

Well, if it is differentiable, it would make sense for the derivative to be the limit of the derivatives of the partial sums $S'_m(z) = \sum_{n=1}^{m} n a_n z^{n-1}$. Thus, we will guess that

$$f'(z) = g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

This is another power series; where does it converge? Since $\limsup k^{1/k} = 1$, then $\limsup_{k\to\infty} |k a_k|^{1/k} = \limsup_{k\to\infty} |a_k|^{1/k}$, so the radii of convergence are the same! This is good, though we still don't know that $g(z) = f'(z)$ yet.

There are two ways to prove this; the textbook makes some estimates to formally show using the definition that

$$\left| \frac{f(z+h) - f(z)}{h} - g(z) \right| \to 0.$$

This probably isn't something you want to see right before lunch, so here's an alternative proof that requires a little more analysis.

Suppose $\{f_n\}$ is a sequence of differentiable functions such that $f_n(x) \to f(x)$ (i.e. pointwise convergence) and $f' \to g$ uniformly in some domain Ω. This is a theorem from real analysis, and the same proof works in the complex case, so since we have the uniform convergence in small discs, then f' exists for our power series, and is equal to g.

Since the derivative has the same radius of convergence, then every higher-order derivative can be obtained on the same radius of convergence; in particular, every function given by a power series is not just holomorphic, but C^∞, infinitely many times complex differentiable.

All of the above discussion still works for infinite power series centered around some other $x_0 \in \mathbb{C}$, rather than just the origin, yielding power series of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

which converges absolutely on $D(R, x_0)$ (the disc of radius R around x_0) and diverges elsewhere.

Much of what we've done today applies to real-valued functions, but for real-valued functions, analytic functions (those given by power series) are quite rare; there are many functions that are, for example, thrice differentiable but not four times differentiable. Next week, we will see that one of the miracles of complex analysis is that all holomorphic functions are analytic, so every holomorphic function is automatically C^∞!

The proof of this takes somewhat of a surprising tack, as it actually involves doing line integrals! Thus, we ought to define line integrals for complex-valued functions (not just holomorphic ones). These are sometimes also called line integrals or path integrals (the latter for the physicists in the audience).

Recall the notion of a parameterized smooth path $z : [a, b] \to \mathbb{C}$, which traces a path $z(t)$ in $\mathbb{C}$. This is basically the same as such a path in $\mathbb{R}^2$, though we can feed $z(t)$ to complex-valued functions, obtaining a complex value at each point on the path, $f(z(t))$. However, we need to be aware of the speed of the parameterization, too.

Definition. Given a function $f : \Omega \to \mathbb{C}$ for an open $\Omega \subset \mathbb{C}$ and a smooth paramterized path $\gamma \subset \mathbb{C}$ given by $z : [a, b] \to \Omega$, define

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(z(t)) z'(t)\,\mathrm{d}t.$$

[1] Of course, in the real differentiable case, there are limits of differentiable and even smooth functions that aren't differentiable.

We don't want this to depend on the parameterization; thus, let's reparameterize it and see what happens. Let $t(s) : [c, d] \to [a, b]$ be a reparameterization of γ, so we get $\tilde{z}(s) = z(t(s))$. Let's make sure we get the same answer:

$$\begin{aligned}\int_\gamma f(z)\,\mathrm{d}z &= \int_a^b f(z(t))z'(t)\,\mathrm{d}t \\ &= \int_c^d f(zt(s))z'(t(s))t'(s)\,\mathrm{d}s \\ &= \int_c^d f(\tilde{z}(s))\tilde{z}'(s)\,\mathrm{d}s\end{aligned}$$

by the Chain Rule, so we're good: it's independent of parameterization.

Akin to the equivalent statement for line integrals in the plane, we have a statement like the Fundamental Theorem of Calculus.

Definition. Suppose there exists a holomorphic $F(z)$ on Ω such that $F'(z) = f(z)$ on Ω; then, F is called an *primitive* for f.

Theorem 2.2 (Fundamental Theorem of Line Integrals)**.** *If F is a primitive for f on Ω and γ is a curve with endpoints w_0 and w_1, then*

$$\int_\gamma f(z)\,\mathrm{d}z = F(w_1) - F(w_0).$$

In particular, if γ is a closed curve, so that $w_0 = w_1$, then

$$\oint_\gamma f(z)\,\mathrm{d}z = 0.$$

That is, one just evaluates the primitive at the endpoints.

We'll actually be able to get a stronger result: on any simply connected domain, every holomorphic function has a primitive (and thus line integrals can be evaluated with these endpoints). This is known as Cauchy's Integral Theorem, from which we will derive Cauchy's Integral Formula, and then use it to prove that every holomorphic function is analytic.

3. Cauchy's Integral Theorem: 9/30/14

"Complex analysis is like Disneyland." – Vishesh Jain

Recall that last time we defined line integrals: if $\Omega \subset \mathbb{C}$, $f : \Omega \to \mathbb{C}$ is continuous, and $\gamma = z(t) : [a, b] \to \mathbb{C}$ is smooth, then we defined $\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(z(t))z'(t)\,\mathrm{d}t$, and that this is independent of parameterization.

If there's an *antiderivative* (or *primitive*) for $f(z)$ on Ω, i.e. an $F(z)$ such that $F'(z) = f(z)$, then

$$\int_\gamma f(z)\,\mathrm{d}z = F(z(b)) - F(z(a)).$$

In particular, if γ is a closed curve, this becomes zero.

Today, we want to show the following.

Theorem 3.1 (Cauchy's Integral Theorem)**.** *If $\Omega = D$, i.e. the open unit disc, and f is holomorphic, then $\int_\gamma f(z)\,\mathrm{d}z = 0$ for all closed loops $\gamma \subset D$. In fact, there exists a primitive $F(z)$ for $f(z)$.*

This is so important that we'll prove it in two ways: the first requires additional assumptions, but is more intuitive and uses concepts from Math 52.

Proof 1 of Theorem 3.1. For this proof, we'll have to assume that $f'(z)$ is continuous, i.e. if $f(z) = u(z) + iv(z)$, then all of the partials $\frac{\partial u}{\partial x}$, etc., are continuous. But then we get to use Green's Theorem from multivariable calculus, which is nice.

Recall that if $\mathbf{V}(x, y) = (G(x, y), H(x, y))$ is a vector field and $\gamma : [a, b] \to \mathbb{R}^2$ is smooth, then we define the path integral

$$\int_\gamma \mathbf{V} \cdot \mathrm{d}\gamma = \int_a^b (G(\gamma(t)), H(\gamma(t))) \cdot \gamma'(t)\,\mathrm{d}t.$$

This is given with many different notations, including $\int_a^b (G, H) \cdot (x', y')\,\mathrm{d}x$ or $\int_\gamma G\,\mathrm{d}x + H\,\mathrm{d}y$.

With this setup, we can state Green's Theorem.

Theorem 3.2 (Green). *Suppose $A \subseteq \mathbb{R}^2$ is a region and $\gamma = \partial A$ (the boundary), with A on the left. Then, if $\mathbf{V}(x, y) = (G(x, y), H(x, y))$ is a vector field such that G and H are continuously differentiable, then*

$$\int_\gamma \mathbf{V} \cdot \mathrm{d}\gamma = \iint_A \left(\frac{\partial H}{\partial x} - \frac{\partial G}{\partial y} \right) \mathrm{d}x\, \mathrm{d}y.$$

That G and H are C^1 is required so that the integral on the right-hand side is defined.

This already looks like a complex integral: if $f(z) = u(z) + iv(z)$ and $\gamma = z(t) = z_1(t) + iz_2(t)$, then $z'(t) = z_1'(t) + iz_2'(t)$, so $f(z)z'(t) = u(z)z_1'(t) - v(z)z_2'(t) + i(v(z)z_1'(t) + u(z)_2'(t))$. We can break this up into its real and imaginary parts: if $\mathbf{W} = (u, -v)$ and $\mathbf{W}' = (v, u)$, then

$$\int_\gamma f(z)\, \mathrm{d}z = \int_a^b f(z(t))z'(t)\, \mathrm{d}t = \int_\gamma \mathbf{W} \cdot \mathrm{d}\gamma + i \int_\gamma \mathbf{W}' \cdot \mathrm{d}\gamma.$$

This is always true for integrable f, but now we can assume f is holomorphic and apply the Cauchy-Riemann equations: $u_x = v_y$ and $u_y = -v_x$. Thus, if $\mathbf{W} = (G, H) = (u, -v)$, then $H_x = -v_x - u_y = -(u_x + u_y) = 0$; thus, the real part of this integral is zero. Similarly, $\mathbf{W}' = (G, H) = (v, u)$, so $H_x - G_y = u_x - v_y = 0$. Thus, by Green's theorem again, the imaginary part is zero as long as γ is closed (i.e. is the boundary of a region A), so $\int_\gamma f(z)\, \mathrm{d}z = 0$. ⊠

Geometrically, this implies that the Cauchy-Riemann equations imply that holomorphic functions define conservative vector fields.

The next argument is more general, but a little more magical: it is less clear why it should be true. We'll start with an intermediate result.

Theorem 3.3 (Goursat). *Suppose $\Omega \subset \mathbb{C}$ is a domain and $f(z)$ is holomorphic on Ω. Let T be a triangle such that T, $\mathrm{Int}(T) \subset \Omega$. Then,*

$$\int_T f(z)\, \mathrm{d}z = 0.$$

This is a particularly special case of Cauchy's Integral Theorem.

Proof of Theorem 3.3. We'll have a whole sequence of triangles, starting with $T = T^{(0)}$. Then, divide it into four triangles (by drawing lines between the midpoints of the edges, with the same counterclockwise orientation) $T_1^{(1)}, \dots, T_4^{(1)}$. Thus,

$$\int_T f(z)\, \mathrm{d}z = \sum_{j=1}^{4} \int_{T_j^{(1)}} f(z)\, \mathrm{d}z.$$

Thus, there must be some j such that

$$\left| \int_T f(z)\, \mathrm{d}z \right| \le 4 \left| \int_{T_j^{(1)}} f(z)\, \mathrm{d}z \right|. \tag{2}$$

Now, keep repeating: let $T^{(1)} = T_j^{(1)}$ and break it into four triangles, and choose $T^{(2)}$ to be such that the same sort of inequality as in (2) holds.

In this way, we get an infinite sequence of triangles and interiors to those triangles. Each one is similar to the original triangle, though each time, the length of the perimeter is halved. Let $\mathcal{T}^{(j)}$ denote the interior of $T^{(j)}$, along with $T^{(j)}$ (filled in). Thus, $\mathcal{T} = \mathcal{T}^{(0)} \supset \mathcal{T}^{(1)} \supset \cdots$, and if p_j is the perimeter of $T^{(j)}$, then $p_j = p_0/2^j$, and thus

$$\left| \int_T f(z)\, \mathrm{d}z \right| \le 4^j \left| \int_{T^{(j)}} f(z)\, \mathrm{d}z \right|$$

for any $j \in \mathbb{N}$.

We know there's a $z_0 \in \bigcup_{j=0}^{\infty} \mathcal{T}^{(j)}$.[2] Since $f(z)$ is differentiable at z_0, then $f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0)$ where $|\psi(z)| \to 0$ as $z \to z_0$.

In particular, this implies that over any region γ,

$$\int_\gamma f(z)\, \mathrm{d}z = \int_\gamma (f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0))\, \mathrm{d}z.$$

[2]Actually, it's unique, but we don't need that.

Since $f(z_0)$ is constant and $f'(z_0)(z-z_0)$ is linear, then they both have primitives, so if γ is a closed curve (which, after all, is the case we care about), then their integrals are already known to vanish, so

$$\begin{aligned}\left|\int_{\mathcal{T}^{(j)}} f(z)\,\mathrm{d}z\right| &= \left|\int_{\mathcal{T}^{(j)}} (f(z_0)+f'(z_0)(z-z_0)+\psi(z)(z-z_0))\,\mathrm{d}z\right| \\ &= \left|\int_{\mathcal{T}^{(j)}} \psi(z)(z-z_0)\,\mathrm{d}z\right| \\ &\le \left|\int_{\mathcal{T}^{(j)}} \frac{\varepsilon d_0}{2^j}\,\mathrm{d}z\right|,\end{aligned}$$

where $\varepsilon = \sup_{z\in\mathcal{T}^{(i)}}|\psi(z)|$, and if $z, z_0 \in \mathcal{T}^{(i)}$, then

$$|z-z_0| < \frac{d_0}{2^j} \le \frac{\varepsilon d_0}{2^j} p_0 = \frac{\varepsilon d_0 p_0}{4^j}.$$

Thus, as $\varepsilon \to 0$,

$$4^j\left|\int_{\mathcal{T}^{(j)}} f(z)\,\mathrm{d}z\right| \le \frac{4^j}{4^j}\varepsilon d_0 p_0 \to 0. \qquad \boxtimes$$

This looks pretty magical: it follows because $2+2=4$? Well, the trick is that the primitives already exist for most of the expansion of the derivative, which makes life nicer.

Notice also that the same theorem holds for rectangles $R \subset \Omega$ as well, because it's easy to subdivide a rectangle into four smaller rectangles. But how can we generalize this to all closed curves?

Proof 2 of Theorem 3.1. Let $f(z)$ be holomorphic on the disc $D \subset \mathbb{C}$; we'll construct a primitve $F(z)$ such that $F'(z) = f(z)$ on D. In fact, we can just write down the definition, though then we'll have to prove stuff about it.

Let $z_0 \in \Omega$; then, since Ω is a disc, for any $z \in \Omega$, there's a path γ_z from z_0 to z that consists first of only horizontal motion, then only vertical motion (real, then imaginary); then, let

$$F(z) = \int_{\gamma_z} f(z)\,\mathrm{d}z.$$

This means nice things geometrically about $F(z+h)-F(z)$, because integrals over paths in opposite directions cancel: this is just the integral around a trapezoid (it helps to look at some pictures here; the book has some good ones), or even around a triangle containing z and $z+h$. In particular, $F(z+h)-F(z) = \int_P f(z)\,\mathrm{d}z$, where P is the line directly from z to $z+h$.

Now, parameterize P as $P(t) = z+th$, with $t \in [0,1]$, so

$$\frac{F(z+h)-F(z)}{h} = \frac{1}{h}\int_0^1 f(z+th)h\,\mathrm{d}t,$$

and as $h \to 0$, this becomes

$$\int_0^1 f(z+th)\,\mathrm{d}t,$$

which converges to a constant: thus, it just becomes $f(z)$. $\boxtimes$

The key is that the disc is path-connected, which isn't a big deal, but also that the difference between two paths bounds some region whose integral goes to zero (which uses Theorem 3.3). There isn't even anything all that special about the disc; the proof works identically for any convex or even simply connected region.

However, it's not true for every region: the canonical and very important counterexample is $f(z) = 1/z$ on $\mathbb{C} - \{0\}$. Let $\gamma = z(t) = e^{it}$ on $0 \le t \le 2\pi$ (so, just the unit circle), and $z'(t) = ie^{it} = iz(t)$. Thus,

$$\int_\gamma \frac{\mathrm{d}z}{z} = \int_0^{2\pi} \frac{z'(t)}{z(t)}\,\mathrm{d}t = \int_0^{2\pi} i\,\mathrm{d}t = 2\pi i \ne 0.$$

The point is, no matter what we do to the curve, it's still wound around the missing origin.

It turns out the answer is the same for a circle of any radius: you could just calculate it out, but since the origin is the only problem, then we create a closed curve that approximates the difference of the two integrals on a simply connected region ($\mathbb{C}$ minus the positive real numbers), which becomes zero, so they have the same value. This generalizes to many kinds of curves.

Alternatively if f is C^1, as in this case, use Green's Theorem to calculate the double integral in the area between them, which also works for many kinds of curves. Anything that wraps around once gives a value of $2\pi i$.

4. Cauchy's Integral Formula: 10/2/14

"There's a slip missing from the keyhole, and that's the key to it."

Recall that last time, we proved that for any holomorphic function f on $\Omega = D$, $\int_\gamma f(z)\,\mathrm{d}z = 0$ for any smooth closed curve γ in D. There were two proofs, the latter using Goursat's theorem that established this result for triangles and rectangles. Then, we explicitly constructed a primitive $F(z)$ for $f(z)$ on D by integrating from a basepoint z_0 to z, and then showing this formula gave a holomorphic function.

We used $\Omega = D$, but the main property of Ω we actually needed for the construction of $F(z)$ was a rule for constructing a path γ_z for a given point $z \in \Omega$ such that the difference between γ_z and γ_{z+h} is a sum of rectangles and triangles (so that Goursat's theorem applies). This holds for quite a large collection of regions, including the so-called "keyhole contour" in Figure 2. The condition is in complete generality a bit beyond the scope of the class; it's a topological criterion called simply connected.

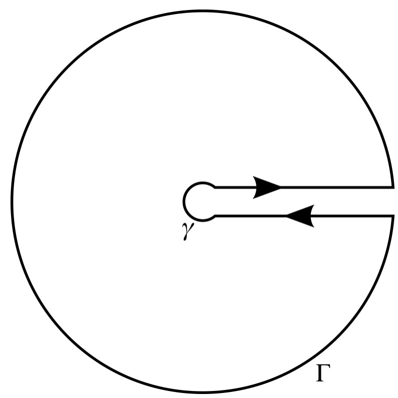

Figure 2. A keyhole contour, one of many useful regions for which Cauchy's Integral Theorem applies. Source: http://en.wikipedia.org/wiki/Methods_of_contour_integration.

However, if $z_0 \in D$, $\Omega = D - \{z_0\}$, the punctured disc, does not have this property, since we saw that $f(z) = 1/(z - z_0)$ is holomorphic on Ω, but if γ winds around the origin,

$$\int_\gamma \frac{\mathrm{d}z}{z - z_0} = 2\pi i.$$

The antiderivative of $f(z) = 1/z$ ought to be $F(z) = \log z$, like for the real numbers, and it makes sense for $z = re^{i\theta}$ to have $\log(z) = \log(r) + i\theta$, but this isn't well-defined, since the same point may have different values of θ (e.g. π and 3π). Thus, the complex logarithm is only actually defined on regions for which θ takes on restricted values.

Cauchy's Integral Theorem leads to Cauchy's Integral Formula.

Theorem 4.1 (Cauchy's Integral Formula). *Suppose f is holomorphic on Ω and D is a disk such that $\overline{D} \subset \Omega$ (i.e. the boundary C is also in Ω). If $z \in D$, then*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

Proof. Consider a keyhole region around z (as in Figure 2, with the inner disc a small disc of radius ε around z, and the slot of width δ). Call the boundary of this keyhole $\gamma_{\delta,\varepsilon}$ (we'll eventually let $\delta, \varepsilon \to 0$). Since z isn't in this keyhole, then $f(\zeta)/(\zeta - z)$ is holomorphic on $D - \{z\}$. Thus, by Cauchy's Integral Theorem,

$$\int_{\gamma_{\delta,\varepsilon}} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = 0.$$

Then, let $\delta \to 0$. In the limit, $\gamma_{\delta,\varepsilon}$ becomes two paths, C and a small disc γ_ε of radius ε around z. Thus,

$$\int_C \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta + \int_{\gamma_\varepsilon} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = 0,$$

so the integral around C is negative that of the integral around $\gamma)_\varepsilon$.

Write

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta) - f(z)}{\zeta - z} + \frac{f(z)}{\zeta - z},$$

but the first term becomes $f'(z)$ as $\zeta \to z$. In particular, this happens as $\varepsilon \to 0$. We actually only need it to be bounded here, which is a slight generalization of the theorem statement, so

$$\int_{-\gamma_\varepsilon} \frac{f(\zeta) - f(z)}{\zeta - z}\,\mathrm{d}\zeta + \int_{-\gamma_\varepsilon} \frac{f(z)}{\zeta - z}\,\mathrm{d}\zeta.$$

The first term goes to zero, since it approaches $f'(z)$ times some length which goes to 0, and the second term is nicer:

$$\int_{-\gamma_\varepsilon} \frac{f(z)}{\zeta - z}\,\mathrm{d}\zeta = f(z)\int_{-\gamma_e} \frac{\mathrm{d}\zeta}{\zeta - z} = 2\pi i f(z).$$

Thus, when we equate everything,

$$\int_C \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = 2\pi i f(z). \qquad \boxtimes$$

A lot of useful things are going to follow from this proof (e.g. holomorphic implying C^∞, and so forth).

It seems like spooky action at a distance: we know the value at $f(z)$ from information collected far away from it. Well, it's something to do with the fact that $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are harmonic, and similar results hold for more general harmonic functions.

Let's stop and appreciate a technique we're going to use over and over and over again: we want to know the integral around a curve, so we approximate the curve with a curve we can control and use the Cauchy Integral Theorem for. This is useful for computational techniques as well as theoretical ones.

Example 4.2. This example is in the text: it shows how real-valued integration can be done with the Cauchy Integral Formula: suppose that we want to calculate

$$\int_0^\infty \frac{1-\cos x}{x^2}\,\mathrm{d}x,$$

a nice, normal, real-valued integral. Strictly speaking, we are actually calculating

$$\lim_{\substack{\varepsilon\to 0\\ A\to\infty}} \int_\varepsilon^A \frac{1-\cos x}{x^2}\,\mathrm{d}x.$$

Complex analysis comes in because we'll construct a holomorphic function which has this as its real part, and thus we can use the Cauchy Integral Theorem to determine the integral around a curve that helps answer the question.

Let $f(z) = (1 - e^{iz})/z^2$, which is holomorphic when $z \neq 0$. When $z = x$ is real, then $\mathrm{Re}(e^{ix}) = \cos x$, because $e^{iz} = \cos z + i\sin z$.

We'll use a nearly semicircular contour, as depicted in Figure 3: let ε be the radius of the inner circle, R be that of the outer circle, and $\gamma_{R,\varepsilon}$ be the name of the resulting curve. Then, since f is holomorphic on the interior of this region, then $\int_{\gamma_{R,\varepsilon}} f(z)\,\mathrm{d}z = 0$.

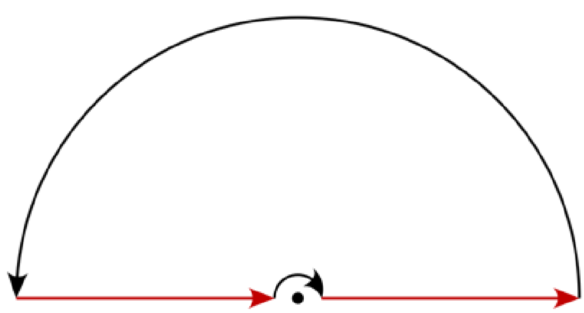

Figure 3. The contour $\gamma_{R,\varepsilon}$ for Example 4.2. Source: StackExchange.

If γ_R is the part of the curve corresponding to the outer semicircle, we want to understand each of the parts of this integral:

- First, that $\int_{\gamma_R} f(z)\,\mathrm{d}z = 0$
- Then, that $\int_{-\gamma_\varepsilon} f(z)\,\mathrm{d}z = \pi$.
- Finally, this implies that $\lim_{R\to\infty} \int_{-R}^R f(z)\,\mathrm{d}z = \pi$, so since it's an even function, this will imply that

$$\int_0^\infty \frac{1-\cos x}{x^2}\,\mathrm{d}x = \frac{\pi}{2}.$$

But first we have to actually show all of this stuff. Here's the first part.

On γ_R, $z = Re^{i\theta} = R(\cos\theta + i\sin\theta)$, so

$$|e^{iz}| = |e^{iR(\cos\theta + i\sin\theta)}| = e^{-R\sin\theta} \le 1$$

when $0 \le \theta \le \pi$, as is the case on γ_R. Thus, $|1 - e^{iz}| \le 2$ on γ_R. Additionally, $|z^2| = R^2$, on γ_R. Thus,

$$\left|\int_{\gamma_R} f(z)\,\mathrm{d}z\right| \le \frac{2}{R^2}\,\mathrm{length}(\gamma_R) = \frac{\pi R}{R^2},$$

which goes to 0 as $R \to \infty$.

For the inner semicircle, we have $(1 - e^{iz})/z^2 = -iz/z^2 + \psi(z)$, for a bounded $\psi(z)$ as $\varepsilon \to 0$. Thus, we can throw away some terms: as $\varepsilon \to 0$, $\int_{-\gamma_\varepsilon} f(z)\,\mathrm{d}z \to \int_0^\pi -i/z\,\mathrm{d}z = \pi$.

Now, we want to go back to the Cauchy Integral Formula: we can use this to compute $f'(z)$ from the definition, if $z \in D$.

$$\begin{aligned} f'(z) &= \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \\ &= \lim_{h\to 0} \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{h}\left(\frac{1}{\zeta - z - h} - \frac{1}{\zeta - z}\right)\mathrm{d}\zeta. \end{aligned}$$

For concision, let $A = 1/(\zeta - z - h)$ and $B = 1/(\zeta - z)$. Then,

$$A - B = \frac{h}{(\zeta - z - h)(\zeta - z)},$$

so

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z - h)(z - h)}\,\mathrm{d}\zeta \xrightarrow{h\to 0} \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^2}\,\mathrm{d}\zeta.$$

This is the desired formula for the first derivative. The astounding implication is not only that the n^{th} derivative exists, but that the formula for it is

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,\mathrm{d}\zeta.$$

The $n!$ term might be surprising, the factorial actually deserving the exclamation mark for once, but it comes from the $(n+1)^{\text{th}}$ power in the denominator, and shows its head in the inductive step.

Thus, if $f(z)$ is holomorphic, then $f(z)$ is C^∞; we can take as many derivatives as we want. This is not at all like real analysis. In fact, f is analytic: we can (and will Tuesday) prove that it's equal to its Taylor series. The only thing we required in any of this is that f is holomorphic in a neighborhood of z.

This relates to something called the Cauchy inequalities.

Corollary 4.3 (Cauchy inequalities)**.** *Suppose $f(z)$ is holomorphic on Ω and $\overline{D} \subset \Omega$ for a disc D of radius R and centered at z_0. Let C be the boundary of D; then,*

$$\|f^{(n)}(z_0)\| \le \frac{n!\,\sup_{z\in C} f}{R^n}.$$

This is true because

$$\begin{aligned} |f^{(n)}(z_0)| &= \frac{n!}{2\pi}\left|\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,\mathrm{d}\zeta\right| \\ &= \frac{n!}{2\pi}\left|\int_C \frac{f(z_0 + Re^i\theta)}{R^{n+1}}\,\mathrm{d}z\right| \\ &\le n!\int_0^{2\pi} \frac{\sup_{z\in C}}{R^{n+1}} R. \end{aligned}$$

Sometimes, $\|f\|_C$ is used to denote $\sup_{z\in C} f(z)$.

Here's an interesting implication: suppose f is *entire*, i.e. holomorphic on all of $\mathbb{C}$, and suppose that f is bounded. Then, the derivatives all go to 0, so f is constant by the Cauchy inequalities. This has a name.

Theorem 4.4 (Liouville)**.** *A bounded, entire function is constant.*

Surprisingly, this theorem wasn't named after Cauchy.

Recall from last time that if Ω is an open, simply connected region of $\mathbb{C}$ and D is a disc such that $\overline{D} \subseteq \Omega$ (i.e. D and its boundary C), then the Cauchy Integral Formula states that if f is holomorphic on Ω, then

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta.$$

Furthermore, by differentiating this, we found the formula for all derivatives:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, \mathrm{d}\zeta.$$

The miraculous conclusion is that all holomorphic functions are C^∞. Now we can ask, is such an f equal to its Taylor series (called *analytic*)? The answer is, once again, yes.

Theorem 5.1. *Suppose f is holomorphic on Ω, D is a disc such that $\overline{D} \subset \Omega$, and z_0 is the center of D. Let C be the boundary of D. Then,*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, \mathrm{d}\zeta.$$

In particular, f is analytic on Ω.

Proof. We certainly know that

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta, \tag{3}$$

so for z fixed, there's an r with $0 < r < 1$ such that $|(z - z_0)/(\zeta - z_0)| < r$, and then we can write

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - (z - z_0)/(\zeta - z_0)}.$$

Since $|(z - z_0)/(\zeta - z_0)| < r < 1$, then the series

$$\frac{1}{1 - (z - z_0)/(\zeta - z_0)} = \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^n$$

converges uniformly for $\zeta \in C$.

When we plug this into (3), we get

$$f(z) = \frac{1}{2\pi i} \int_C \left(\frac{f(\zeta)}{\zeta - z_0}\right) \left(\sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^n}\right) \mathrm{d}\zeta.$$

But since this series converges uniformly, we can interchange the infinte sum and the integral, so

$$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \left(\int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, \mathrm{d}\zeta\right)(z - z_0)^n,$$

which is exactly what we wanted to prove. ⊠

The key step is turning the denominator into a geometric series which converges uniformly. Note also that the proof works for all $z \in D$, for all D such that $\overline{D} \subset \Omega$; it doesn't use that D is a disc.

Last time, we used Cauchy's Integral Formula to derive the Cauchy inequalities, Corollary 4.3, and therefore deduced Liouville's theorem: if a function is holomorphic on all of $\mathbb{C}$ (also called *entire*) and is bounded, then $f(z)$ is constant (since the inequalities provide a bound on $f'(z)$).

This provides a nice proof of the Fundamental Theorem of Algebra, which states that any complex polynomial can be factored completely.

Theorem 5.2 (Fundamental Theorem of Algebra)**.** *Suppose $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$, with $n \geq 1$ and $a_n \neq 0$, is a polynomial. Then, there exists an $\omega \in \mathbb{C}$ such that $P(\omega) = 0$.*

Proof. Suppose not; then, $1/P(z)$ is also entire, since P is and is never zero. If we can show that it's bounded, then Liouville's theorem implies it's constant, so P is as well, which forces a contradiction.

Write

$$\frac{P(z)}{z^n} = a_n + \left(\frac{a_n}{z} + \frac{a_{n-1}}{z^2} + \cdots + \frac{a_0}{z^n}\right).$$

This means there exists an R such that $|P(z)/z^n| \geq c$ when $|z| \geq R$, where $c = |a_n|/2$. Thus, $|P(z)| \geq c|z|^n$. When $|z| > R$, since $P(z) \neq 0$, then $|P(z)|$ is bounded below, so $1/|P(z)|$ is bounded above, and thus Liouville's theorem applies. ⊠

Corollary 5.3. *Every n^{th}-degree complex polynomial has n linear factors.*

Proof. Inductively apply the theorem: we know there's a w_1 such that $P(w_1) = 0$, and, writing $z = (z - w_1) + w_1$,

$$P(z) = b_n(z - w_1)^n + b_{n-1}(z - w_1)^{n-1} + \cdots + b_0,$$

but since $P(w_1) = 0$, then $b_0 = 0$, so we can factor out a $(z - w_1)$; in particular, $P(z) = (z - w_1)Q(z)$, and $Q(z)$ has degree $n - 1$. Then, apply the result to $Q(z)$, so by induction, P has n factors. ⊠

This is a nice, cute application of Liouville's theorem, which is a surprisingly deep result from these estimates.

In the real world, analytic functions are noticeably different from C^∞ functions; in particular, we'll see that for an analytic function, its value on a small set can determine it on a much larger, connected set. This is not true for functions that are merely C^∞.

We'll actually prove something stronger.

Theorem 5.4. *Suppose f is holomorphic on a connected[3] $\Omega \subset \mathbb{C}$. Suppose there exists a sequence $\{z_k\} \subset \Omega$ such that all of the z_k are distinct, $f(z_k) = 0$ for all k, and $z_k \to z \in \Omega$. Then, $f(z) = 0$ for all $z \in \Omega$.*

Contrast this with the counterexample on the first problem set: if $f : \mathbb{R} \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} e^{-1/x^2}, & x > 0 \\ 0, & x \leq 0, \end{cases}$$

then f is C^∞, though it's not equal to its Taylor series, and Theorem 5.4 certainly does not hold (since it's zero on one part and nonzero on another).

Proof of Theorem 5.4. Suppose $w_k \to z_0$ in Ω and $f(w_k) = 0$, and expand $f(z)$ near z_0 as

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n.$$

Assume $f(z) \not\equiv 0$, and let m be the smallest integer such that $a_m \neq 0$; thus, we can also assume that $a_m > 1$ eventually. Thus, we can write $f(z) = a_m(z - z_0)^m(1 + g(z))$, with $g(z) \to 0$ as $z \to z_0$. But $w_k \neq z_0$, $w_k \to z_0$, and $f(w_k) = 0$, and $a_m(w_k - z_0)^m \neq 0$, yet $1 + g(w_k) \neq 0$ when w_k is close to z_0, so we get a contradiction. Thus, $f(z)$ is identically 0 on an open set containing Ω.

Consider the set $U \subset \Omega$ given by $U = \{z \in \Omega \mid f(z) = 0\}$; then, we've just shown that U is open and closed, so $U = \Omega$. ⊠

Corollary 5.5 (Analytic continuation)**.** *If f and g are holomorphic functions in a region Ω and $f(z) = g(z)$ on an open $U \subset \Omega$ (or, equivalently, on a convergent sequence in Ω), then $f(z) = g(z)$ on Ω.*

This didn't actually depend on the complex numbers, just analyticity. The local information of a function is known as its *germ*, so this says that the germs of analytic functions describe them globally as well.

Here are a few other nice little results, somewhat less computational, for what we have done these few days.

Morera's theorem is in some sense a converse to Cauchy's Integral Theorem.

Theorem 5.6 (Morera)**.** *Suppose $f(z)$ is a continuous function on a disc D such that for all triangles $T \subset D$,*

$$\int_T f(z)\,\mathrm{d}z = 0,$$

then $f(z)$ is holomorphic.

[3] We'll use the fact that Ω is path-connected, but since it's open, the two are equivalent.

Proof. Recall that in the proof of Cauchy's Integral Theorem from Goursat's Theorem, we were able to define a holomorphic primitive F for f based on integrating over paths from a basepoint, and this didn't depend on the holomorphicity of f, just that it was integrable.

Since f is continuous, then it is integrable, so there's a holomorphic primitive F for it. But we've shown that F is infinitely differentiable: $F^{(n)}$ is holomorphic for every n, so in particular $F' = f$ is too. ⊠

This is pretty, but it's not so practical, because it's not easy to check all triangles for this condition. But it might be useful in another, more practical, theoretical result.

Theorem 5.7. *Suppose we have a sequence of functions $\{f_i\}$ on D converging uniformly*[4] *to f. If each of the f_i are holomorphic, then so is f.*

Compare again to the real-valued case, where the uniform limit of continuous functions is certainly continuous, but not differentiable!

Proof of Theorem 5.7. By Goursat's Theorem, for all triangles $T \subset \Omega$, $\int_T f_j(z)\,\mathrm{d}z = 0$ for all j. But since the convergence is uniform, we can exchange the limits and the integrals, so $\int_T f(z)\,\mathrm{d}z = 0$ for all triangles T as well. But by Theorem 5.6, f must be holomorphic. ⊠

In the same vein, if $f_j \to f$ uniformly, then

$$\frac{n!}{2\pi i}\int_C \frac{f_j(\zeta)}{(\zeta - z)^{n+1}}\,\mathrm{d}\zeta \longrightarrow \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

Thus, not only do the functions converge uniformly (on compact sets), but all of the sequences of their derivatives do as well. There are of course lots of examples in the real world for which this fails.

6. The Symmetry Principle: 10/9/14

"This is some real analysis theorem from Mars."

Last time, we proved that if f is holomorphic, then not only is it holomorphic, but it's analytic: for all $z_0 \in \Omega$, there's a $D \subset \Omega$ on which

$$f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n.$$

Furthermore, if we have an accumulation point of zeros of f within a connected region, then $f(z)$ is identically 0. This implies that knowing not very many values of a holomorphic function, especially local germs of holomorphic functions, can tell you everything about them, an idea called analytic continuation.

The conclusion is that holomorphic functions are rigid: unlike other classes of functions (such as $C^\infty(\mathbb{R}^n)$), where we might have $f : U_1 \to \mathbb{R}^n$ and $g : U_2 \to \mathbb{R}^n$ such that $U_1 \cap U_2$ is nonempty and be able to join them if they agree on their intersection. This is known as gluing functions, and is very useful in topology. However, for holomorphic functions, we can't glue different ones: if f agrees with g, then g is actually just f: it can't agree with different g, and in this sense is not flexible enough.

Suppose Ω is a connected region symmetric about the real axis, i.e. $z \in \Omega$ iff $\overline{z} \in \Omega$. Then, we will write $\Omega^+ = \{z \in \Omega : \mathrm{Im}(z) > 0\}$, $I = \{z \in \Omega : \mathrm{Im}(z) = 0\}$, and $\Omega^- = \{z \in \Omega : \mathrm{Im}(z) < 0\}$. Thus, $\Omega = \Omega^+ \cup I \cup \Omega^-$, and $I = \Omega \cap \mathbb{R}$.

Theorem 6.1 (Symmetry Principle)**.** *Suppose $f^+(z)$ is holomorphic on Ω^+ and $f^-(z)$ is holomorphic on Ω^-, and that both f^+ and f^- extend to I such that $f^+(x) = f^-(x)$ there; then, the function*

$$f(x) = \begin{cases} f^+(z), & z \in \Omega^+ \\ f^-(z), & z \in \Omega^- \\ f^+(z) = f^-(z), & z \in I \end{cases}$$

is holomorphic on Ω.

Proof. We already know $f(z)$ is continuous on Ω and holomorphic on Ω^+ and Ω^-, so we need to show it's holomorphic at each $z \in I$.

Let $D \subset \Omega$ be such that $z \in D$. Then, by Morera's Theorem (Theorem 5.6), it suffices to show that $\int_T f(z)\,\mathrm{d}z = 0$ for all $T \subset D$. Clearly, if T doesn't cross I, this is true, so the only case we have to worry about is when T intersects R, i.e. $T \cap I \neq \emptyset$.

[4] This means uniform convergence on compact subsets of D; uniform convergence is very hard on open sets, since the boundary interferes.

The first case to consider is when an edge of T is entirely contained within I. Thus, if we move T ever so slightly off of I, e.g. by an $\varepsilon > 0$, then the resulting triangle T_ε satisfies $T_\varepsilon \cap I = \emptyset$, so

$$\int_{T_\varepsilon} f(z)\,\mathrm{d}z = 0.$$

Thus, as $\varepsilon \to 0$, using uniform convergence, we get that $\int_T f(z)\,\mathrm{d}z = 0$ as well.

If T intersects but not just by an edge (or a vertex, for which the above argument still works), then it can be divided into three triangles which do only intersect I at the edge or a vertex, so the integrals of these smaller triangles are 0, and thus same for T. ⊠

The proof isn't too bad, but the theorem isn't trivial. It leads into a related theorem.

Theorem 6.2 (Schwarz Reflection). *With the same notation as before, let f be holomorphic on Ω^+, real on I, and continuous on $\Omega^+ \cup I$. Then, f extends holomophically to all of Ω.*

Proof. Let

$$F(z) = \begin{cases} f(z), & z \in \Omega^+ \cup I \\ \overline{f(\overline{z})}, & z \in \Omega^-. \end{cases}$$

Thus, F and f agree on Ω^+ and on I.[5] Thus, F is holomorphic on Ω^+ and Ω^-, and is well-defined on I, so we want to invoke Theorem 6.1 to prove that F is holomorphic on Ω.

We've shown everything we need to for this, except that F is holomorphic on Ω^-. For any $z_0 \in \Omega^-$, there's an open disc D around it such that $D \subset \Omega^-$, and thus $\overline{D} \subset \Omega^+$ and contains $\overline{z_0}$. Thus, F is holomorphic in D, so it has a Taylor series

$$F(\overline{z}) = \sum_{n=0}^{\infty} (\overline{z} - \overline{z}_0)^n,$$

and thus

$$\begin{aligned} F(z) &= \overline{F(\overline{z})} \\ &= \overline{\sum_{n=0}^{\infty} a_n(\overline{z} - \overline{z}_0)} \\ &= \sum_{n=0}^{\infty} \overline{a_n}(z - z_0)^n. \end{aligned}$$

Thus, F is analytic at z_0. ⊠

This is a surprisingly useful theorem to have around, even if it doesn't appear to be so at first.

The next theorem we'll talk about today isn't usually covered in undergraduate complex analysis courses, since it's a little off-topic, but it's a very remarkable theorem, so it's worth mentioning. The idea is that if $f(z)$ is analytic, then for some D with center z_0, $f(z) = \sum a_n(z - z_0)^n$ so for all compact $K \subset D$, f is uniformly approximated by polynomials (proven by taking partial sums). On the other hand, there's a really nice theorem from real analysis relevant to this.

Theorem 6.3 (Weierstrass Approximation Theorem). *Let $I \subseteq \mathbb{R}$ be a closed interval and $f : I \to \mathbb{R}$ be continuous. Then, f can be uniformly approximated on I by polynomials, i.e. there's a sequence of polynomials whose values uniformly converge to f.*

Of course, the degrees of these polynomials are going to get very high very quickly.

Runge's Theorem is a generalization of these two ideas.

Theorem 6.4 (Runge). *Let $K \subset \mathbb{C}$ be compact and f is holomorphic on some open domain containing K. Then, on K, f can be uniformly approximated by rational functions whose singularities are contained on K^c. In fact, if K^c is connected, then f can be uniformly approximated by polynomials on K.*

Recall that a rational function is a ratio of two polynomials: $R(z) = P(z)/Q(z)$.

Observing when the complement is or isn't connected is a bit relevant to what we were doing. A good example for why we need rational functions is where $K = C$ is the unit circle and $f(z) = 1/z$ is holomorphic on $\Omega \supset C$. f can't be approximated by polynomials on the circle, because if $P(z)$ is any polynomial, then $\int_C P(z)\,\mathrm{d}z = 0$, but $\int_C f(z)\,\mathrm{d}z \neq 0$. In this case, K^c is both the interior and the outside of the circle, so it's not connected.

There's a near converse to Runge's theorem.

[5] You can see that if $f(z) \notin \mathbb{R}$ on I, then F wouldn't even be continuous there, which is why that condition was necessary.

Theorem 6.5. *If Ω is an open region and $K \subset \Omega$ is compact such that $\Omega \setminus K$ isn't connected, then there exists some holomorphic $f : \Omega \to \mathbb{C}$ such that f cannot be approximated by polynomials on K (i.e. one must use rational functions).*

This is actually an exercise in the textbook (chapter 2, problem 4[6]). This is not very easy to prove, but there's a nice hint there.

We have seen that the zeros of a nonzero holomorphic function are *isolated*: for any z_0 such that $f(z_0) = 0$, there's a disc $D_r(z_0)$ around z such that $f(z) \neq 0$ on $D_r(z_0) \setminus \{z_0\}$ (if not, then we have an accumulation point of zeros, so the entire thing is zero). These are known as *isolated zeros*.

In fact, around each zero z_0 there exists an open U such that when $z \in U$,

$$f(z) = \sum_{k=1}^{\infty} a_k (z - z_0)^k,$$

i.e. $a_0 = 0$ in the usual Taylor series expansion (since otherwise, we plug z_0 into f and get 1 from the a_0 term and 0 from the rest). Thus, there's a first nonzero term: let n be such that $a_{n-1} = 0$ and $a_n \neq 0$.

Definition. This n found above is called the *order* of the zero.

We're going to be interested in isolated singularities of holomorphic functions.

Definition. z_0 is an *isolated singularity* of $f(z)$ if $f(z)$ is holomorphic on $D_r(z_0)$, except at z_0 itself.

This disc $D_r(z_0) \setminus \{z_0\}$ is sometimes called the *deleted disc* of radius r around z_0. The notion of isolation means there are no accumulation points of these singularities.

Next week, when we discuss these in more detail, we'll show there are three types of isolated singularities.

(1) The first type, called a *removable singularity*, isn't really a singularity at all: one could just choose not to define a holomorphic function at a point, and this satisfies the definition of an isolated singularity, yet f can be extended continuously and holomorphically to that point.
(2) Another kind is called a *pole*. These are singularities such as $1/z$, $1/z^2$, and so on. The definition of a pole at z_0 is that $1/f(z) = F(z)$ extends holomorphically to z_0 and this $F(z_0) = 0$. Intuitively, we're "dividing by zero" as in $1/z$.
(3) The last kind is an *essential singularity*. These are the singularities that don't fit into any other category. They're fairly wild objects, e.g. $f(z) = e^{1/z}$ at 0, which can be checked to not fit into either of the other types. Generally, this function will take on all possible values as one gets closer to this singularity.

7. Singularities: 10/14/14

Recall that last time we defined the notion of an *isolated singularity* as follows: if $f(z)$ is holomorphic on Ω except at $z_0 \in \Omega$, then f has an isolated singularity at z_0. Then, we talked about the three types of isolated singularities: *removable singularities* (for which $f(z)$ extends to a holomorphic $\widehat{f}$ on all of Ω), *poles*, and *essential singularities*.

The idea that "f extends to $\widehat{f}$" means that $\widehat{f}$ is defined on a superset of where f is, and they agree wherever f is defined. We also had the notion of the *deleted disc* around z_0, which, given a disc D centered at z_0, is just $D \setminus \{z_0\}$.

With these notions, an isolated singularity is a pole if there's a deleted disc around z_0 on which $1/f(z)$ is holomorphic, and extends to an F on z_0 as well, and $F(z_0) = 0$. This is a longwinded way of saying that f sort of looks like $1/z$ near z_0. This will be the most important class of singularities; we'll spend some time on poles today.

An essential singularity is one that doesn't fit into the other two cases.

Poles. Assume that $f(z)$ has a pole at z_0, and let D be a disc centered at z_0. Then, we can extend $1/f$ to a function $F(z)$ using the Taylor expansion; specifically, $F(z) = (z - z_0)^n g(z)$, where $g(z) \neq 0$ on D. n is equal to the order of the zero of $F(z_0)$.

On $D \setminus \{z_0\}$, $f(z) = (z - z_0)^{-n} h(z)$, where $h(z) = 1/g(z)$, so $h \neq 0$ on D.

Definition. n is called the *order* of the pole of f at z_0. The pole is called *simple* if $n = 1$.

We can rewrite f again as

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-n+1}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{(z - z_0)} + G(z), \tag{4}$$

where G is holomorphic on D, using $h(z) = A_0 + A_1(z - z_0) + \cdots$.

[6]This is distinct from exercise 4, which is unrelated and much easier.

Definition.

$$P(z) = f(z) - G(z) = \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{(z-z_0)}$$

is called the *principal part* of $f(z)$ at z_0, and a_{-1} is called the *residue* of f at z_0, denoted $\operatorname{Res}_{z_0} f(z)$.

If z_0 is a simple pole, then

$$\operatorname*{Res}_{z_0} f(z) = a_{-1} = \lim_{z\to z_0} (z-z_0)f(z),$$

and more generally, we need to do this n times:

$$\operatorname*{Res}_{z_0} f(z) = a_{-1} = \lim_{z\to z_0} \frac{1}{(n-1)!}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n-1}((z-z_0)^n f(z)).$$

Thankfully, most of the poles we'll be dealing with are simple poles.

Theorem 7.1. *Suppose $f(z)$ is holomorphic on Ω except for a pole at $z_0 \in \Omega$, and let C be the boundary of a disc centered at z_0. Then,*

$$\int_C f(z)\,\mathrm{d}z = 2\pi i \operatorname*{Res}_{z_0} f(z).$$

Proof. We can just integrate each term in (4). By the Cauchy Integral Formula,

$$\int_C \frac{a_{-1}}{(z-z_0)}\,\mathrm{d}z = (2\pi i)a_{-1},$$

and but in a neighborhood of C, $a_{-k}/(z-z_0)^k$ is the derivative of a holomorphic function, specifically

$$F(z) = -\frac{1}{k-1}\frac{a_{-k}}{(z-z_0)^{k-1}},$$

so $\int_C a_{-k}/(z-z_0)^k\,\mathrm{d}z = 0$ when $k > 1$. And G is holomorphic, so $\int_C G(z)\,\mathrm{d}z = 0$ too. Thus, adding this all together, we get the desired formula. ⊠

This is why the residue is the most important part of the principal formula; it corresponds to the only term in the (Laurent series of) f that doesn't have a primitive.

Theorem 7.1 leads to a whole raft of computational examples and theorems. The following one is pretty typical.

Example 7.2. Once again, we'll be solving real integrals with complex analysis, specifically

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,\mathrm{d}x = \lim_{R\to\infty}\int_{-R}^{R} \frac{\cos x}{1+x^2}\,\mathrm{d}x.$$

We'll extend $f(x)$ to a related $f(z)$ that is holomorphic on Ω, but possibly with some poles. We'll let Ω be the upper semicircle with radius R, γ_R be its boundary, and c_R be the circular arc part of γ_R.

A reasonable first guess is $\cos z/(1+z^2)$. Then, with $z = x+iy$,

$$\cos z = \frac{e^{iz}+e^{-iz}}{2} = \frac{e^{i}e^{-y}+e^{-ix}e^{y}}{2},$$

but that doesn't really lead anywhere, since e^y rockets off to infinity when $R\to\infty$. So maybe we should just try $f(z) = e^{iz}/(1+z^2)$, which does have a cosine in it. Notice that this has exactly one pole at $z = \pm i$, though $-i \notin \Omega$. Thus,

$$\int_{\gamma_R} f(z)\,\mathrm{d}z = 2\pi i \operatorname*{Res}_{z=i} f(z),$$

so let's figure out that residue. $f(z) = e^{iz}/((z-i)(z+i))$, so

$$\begin{aligned}\operatorname*{Res}_{z=i} f(z) &= \lim_{z\to i}(z-i)f(z)\\ &= \lim_{z\to i}\frac{e^{iz}}{z+i} = \frac{e^{-1}}{2i},\end{aligned}$$

so $\int_{\gamma_R} f(z)\,\mathrm{d}z = \pi/e$. Sounds reasonable.

Now, we need to show that $\int_{c_R} f(z)\,\mathrm{d}z \to 0$ as $R\to\infty$, so that we can definitively answer the question. Parameterize $z = Re^{i\theta}$, with $0 \le \theta \le \pi$ and, if $z = x+iy$, then $y \ge 0$. Thus,

$$\left|\frac{e^{iz}}{1+z^2}\right| = \left|\frac{e^{ix}e^{y}}{1+z^2}\right| = \frac{e^{-y}}{|1+z^2|} \le \frac{1}{R^2-1},$$

and therefore as $R \to \infty$,

$$\left|\int_{c_R} f(z)\,\mathrm{d}z\right| \le \int_{c_R} |f(z)|\,\mathrm{d}z \le \frac{\pi R}{R^2-1} \longrightarrow 0.$$

Thus,

$$\int_{-\infty}^{\infty} \frac{\cos x\,\mathrm{d}x}{1+x^2} = \frac{\pi}{e}.$$

Example 7.3. Let's compute

$$\int_0^{\infty} \frac{x^{1/3}}{1+x^2}\,\mathrm{d}x.$$

We'll let $f(z) = z^{1/3}/(1+z^2)$. The cube root is not in general unique on $\mathbb{C}$, so we'll make an arbitrary choice: if $z = re^{i\theta}$, then $z^{1/3} = r^{1/3}e^{i\theta/3}$, for $-\pi/2 \le \theta \le 3\pi/2$ (if we did this around the entire circle, it wouldn't work, but on this region, it's fine).

This is pretty fine, but the cube root of z isn't holomorphic at the origin, so we look at the punctured semicircle with radius R and inner radius $1/R$, and call it Ω. Let γ_R be the boundary of this region, and c_R be the outer arc. Thus $f(z)$ is holomorphic inside γ_R, except for a pole at $z = i$. Thus,

$$\begin{aligned}
\int_{\gamma_R} f(z)\,\mathrm{d}z &= \operatorname*{Res}_{z=i} f(z)\,\mathrm{d}z = 2\pi i \lim_{z\to i}(z-i)f(z) \\
&= 2\pi i \lim_{z\to i} \frac{z^{1/3}}{z+i} \\
&= 2\pi i \frac{i^{1/3}}{2i} = \pi i^{1/3} \\
&= \pi\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right).
\end{aligned}$$

On C_R, $|f(z)| \le R^{1/3}/(R^2-1)$, so

$$\left|\int_{c_R} f(z)\,\mathrm{d}z\right| \to 0$$

as $R \to \infty$, just as in the previous example. And as $R \to \infty$, it's even easier to show that $\int_{C_{1/R}} f(z)\,\mathrm{d}z \to 0$. Thus, we get the integral over the whole real line as $R \to \infty$, which we would expect to fall to be zero, since on the real line $\sqrt[3]{x}/(1+x^2)$ is an odd function. But this complex cube root is more interesting. Specifically, we want

$$\int_{1/R}^{R} f(z)\,\mathrm{d}z = \int_{1/R}^{R} \frac{x^{1/3}}{1+x^2}\,\mathrm{d}x \longrightarrow \int_0^{\infty} f(x)\,\mathrm{d}x = \mathcal{I}.$$

But what about the other part? Let $z = -t$, so $\mathrm{d}z = -\mathrm{d}t$. Then,

$$\begin{aligned}
\int_{-R}^{-1/R} \frac{z^{1/3}\,\mathrm{d}z}{1+z^2} &= -\int_{R}^{1/R} \frac{(-t)^{1/3}}{1+t^2}\,\mathrm{d}t \\
&= \int_{1/R}^{R} \frac{(-t)^{1/3}}{1+t^2}\,\mathrm{d}t.
\end{aligned}$$

From the definition of $z^{1/3}$, we get $(-t)^{1/3} = t^{1/3}e^{\pi i/3}$, so we get

$$= e^{\pi i/3}\int_{1/R}^{R} \frac{t^{1/3}}{1+t^2}\,\mathrm{d}t.$$

Thus,

$$\mathcal{I}(1+e^{\pi i/3}) = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \pi\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right).$$

Since $1 + e^{\pi i/3} = (1/2)(3 + i\sqrt{3})$, then divide by this; thus, $\mathcal{I} = \pi/3$. Magic!

Notice that we don't end up using the first most obvious or ideal guess, but by adjusting it a little bit, we were able to make it happen.

Example 7.4. The textbook also goes over the following example; it would be good to look at it.

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,\mathrm{d}x,$$

where $0 < a < 1$. We'll take $f(z) = e^{az}/(1+e^z)$, but the region may be a bit more of a surprise: we consider the rectangle in $\mathbb{R}^2$ given by $[-R, R] \times [0, 2\pi i]$, and then eventually let $R \to \infty$ (so we get an infinite strip). Then, there is a pole at i.

The exact height might seem arbitrary, but it allows us to take advantage of symmetry in f, so the computation is nicer. The upper edge is a constant multiple of the thing we want, and the lower edge is exactly what we want, so we can compute as before.

8. Singularities II: 10/16/14

Recall that last time, we talked about isolated singularities, including *poles*. A $z_0 \in \Omega$ is a pole if $f(z)$ is holomorphic on $\Omega \setminus \{z_0\}$ and there exists a disc $D \subset \Omega$ such that $1/f(z)$ is holomorphic on $D \setminus \{z_0\}$ and extends holomorphically to an $F(z)$ on all of D such that $F(z_0) = 0$. Then, we have that

$$f(z) = \underbrace{\frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0}}_{(*)} + G(z),$$

where $G(z)$ is holomorphic on Ω. $(*)$ is called the *principal part* of f.

We have also discussed removable singularities, for which there is a nice criterion.

Theorem 8.1. *If f has a singularity at z_0 and $f(z)$ is bounded near z_0, then z_0 is removable.*

The converse is pretty obvious: if it's removable, then we can extend, so the extension is continuous and therefore bounded.

Proof. Since this is a local statement, restrict to a disc $D \subset \Omega$, with $z_0 \in D$, and let C be the boundary of D. Then, let

$$g(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta.$$

We'll show that this is *holomorphic* on D. If $z \neq z_0$, then we will be able to show that $g(z) = f(z)$. We can do this by using a double keyhole around z and z_0, and if γ is the boundary of this keyhole, then Cauchy's theorem implies that

$$\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta = 0.$$

But we can shrink the keyhole, so now we just have the outer boundary c, and circles of radius ε around z and z', which are called c_ε and c'_ε respectively. Thus,

$$\frac{1}{2\pi i}\left(\int_c \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta + \int_{-c_\varepsilon} \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta + \int_{c'_\varepsilon} \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta\right) = 0.$$

Next, we can rewrite

$$\begin{aligned} g(z) &= \frac{1}{2\pi i} \int_c \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta \\ &= \frac{1}{2\pi i}\left(\int_{c_\varepsilon} \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta + \int_{c'_\varepsilon} \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta\right) \\ &= f(z) + \frac{1}{2\pi i} \int_{c'_\varepsilon} \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta, \end{aligned}$$

but since the function is bounded above, then this is bounded by $f(z) + \int_{c'_\varepsilon} B\, \mathrm{d}\zeta$, which goes to 0 as $\varepsilon \to 0$. ⊠

In general, if we try to extend a function, we can ask what it should look like, and that's often a useful guiding light in a proof.

So now we know that if z_0 is a nonremovable singularity, then $|f(z)|$ is not bounded near $z = z_0$. Recall that we say $|f(z)| \to \infty$ as $z \to z_0$ to mean that for all $K \gg 0$, there exists an $\varepsilon > 0$ such that if $0 < d(z, z_0) < \varepsilon$, then $|f(z)| \geq K$. It's not just some parts of that disc that are out of bounds; all of f on that region is.

Theorem 8.2. *If z_0 is a singularity of f, then it's a pole iff $|f(z)| \to \infty$ as $z \to z_0$.*

Proof. If z_0 has a pole, then $|1/f(z)| \to 0$ as $z \to z_0$, so for all $\delta > 0$ there exists an $\varepsilon > 0$ such that $|1/f(z)| < \delta$ for z such that $0 < d(z, z_0) < \varepsilon$, and therefore in this region, $|f(z)| > 1/\delta$.

Conversely, if $|f(z)| \to \infty$, then $|1/f(z)|$ is bounded near z_0, so there's a holomorphic extension $F(z)$ of $1/f(z)$, which must go to 0. Thus, $F(z) = 0$, by continuity of F. ⊠

We also talked about essential singularities, which are a little weirder, e.g. $f(z) = e^{1/z}$. If $z = r > 0$ and $r \to 0$, then $e^{1/r} \to \infty$, but if $z = -r < 0$ and $r \to 0$, then $e^{-1/r} \to 0$. Huh. So these are very different from removable singularities or poles.

But it gets weirder: if $z = iy$ and $y \to 0$, $e^{1/z} = e^{1/iy} = e^{-i/y}$, which becomes $\cos(-1/y) + i\sin(1/y)$, and as $y \to 0$, this limit doesn't even exist; it just oscillates!

Impressively, this is the generic situation.[7]

Theorem 8.3. *Suppose $f(z)$ has an essential singularity at z_0. Then, for any D centered at z_0, the set $\{f(z) : z \in D \setminus \{z_0\}\}$ is dense in $\mathbb{C}$.*

Proof. Suppose not; then, there exists some $w \in \mathbb{C}$ and a $\delta > 0$ such that $|f(z) - w| \geq \delta$ for all $z \in D \setminus \{z_0\}$. Thus,

$$\left|\frac{1}{f(z) - w}\right| \leq \frac{1}{\delta},$$

so $1/(f(z) - w)$ is bounded on $D \setminus \{z_0\}$. Thus, it extends holomorphically to z_0 via some function $F(z)$, so if $F(z_0) \neq 0$, then $|f(z) - w|$ is bounded, implying a removable singuarity at z_0. Alternatively, if $F(z_0) = 0$, then f has a pole at z_0.

Thus, since z_0 is an essential singularity, then neither of these happen, so no such δ exists. ⊠

This is surprisingly easy for such a wild statement. And it gets even better, though a bit beyond the scope of this class.

Theorem 8.4 (Picard)**.** *If f has an essential singularity on z_0, then on any disc D centered at z_0, $f(z)$ hits* every single possible value *in $\mathbb{C}$, except possibly one value, infinitely many times.*

This is pretty out there.

A restatement is that in $D \setminus \{z_0\}$, $f(z) = w$ has infinitely many solutions for all but possibly one $w \in \mathbb{C}$.

For example, if $f(z) = e^{1/z} = w = re^{i\theta}$, then $1/z = x + iy$, and split into polar form: $x = \ln|r|$ and $y = \theta$, which gives a solution as long as $r \neq 0$. And there are infinitely many solutions $\theta + 2\pi k$ for any nonzero point; thus, this function hits every point in $\mathbb{C}$ infinitely many times in any punctured disc centered at the origin, except for 0, which it never touches.

Definition. $f(z)$ is *meromorphic* on Ω if there is a countable set $Z \subset \Omega$ containing no points of accumulation[8], f is holomorphic on $\Omega \setminus Z$, and f has a pole at each $z \in Z$.

We want to extend these ideas about singularities and such to a "point at infinity," which will be made rigorous in the following manner:

Definition. Suppose $f(z)$ is defined for all z such that $|z| \geq R$.

- $f(z)$ has an *isolated singularity* at infinity if $F(z) = f(1/z)$ has an isolated singularity at 0.
- Similarly, f has a *removable singularity* at infinity if F has one at 0, and similarly with *poles* and *essential singularities*.
- $f(z)$ is *meromorphic* on the extended complex plane $\widehat{\mathbb{C}}$ if it is meromorphic on $\mathbb{C}$ and has a pole or a removable singularity at infinity.

This definition will imply that the point at infinity isn't an accumulation point of the poles: an infinite sequence of poles going to infinity implies that these singularities on $F(1/z)$ aren't isolated near 0.

What kinds of functions are meromorphic? Some, but not all, entire functions (e^z has an essential singularity at infinity, for example).

Theorem 8.5. *If $f(z)$ is meromorphic on the extended complex plane, then it is rational, i.e. $f = p/q$, where p and q are polynomials.*

Intuitively, there can be only so many poles, and therefore zeroes of q, and this is a nice first step.

Proof. Since $f(z)$ has a removable discontinuity or a pole at infinity, then $F(z) = f(1/z)$ also has a removable singularity or a pole at $z = 0$. Thus, there are no other poles of $F(z)$ near $z = 0$, i.e. there's an R such that there are no poles for $|z| \geq R$.

Thus, there are a finite number of poles on $\mathbb{C}$, and they can be named $z_1, \dots, z_n$. At each of these points, f has a principal part and a holomorphic part; near z_k, write $f(z) = P_k(z) + g_k(z)$, where P_k is the principal part (i.e. polynomial in $1/(z - z_k)$) and g_k is holomorphic. At infinity, letting $w = 1/z$, so $F(w) = f(z)$, and since F has a nonessential singularity at 0, then write $F(w) = P_\infty(w) + g_\infty(w)$, where P_∞ is polynomial in $1/w$, so in z. Write $p_\infty(w) = f_\infty(z)$.

[7] Apparently these aren't that funny. I readily disagree.

[8] Except for eventually constant sequences, since $z_n = z$ converges to z.

Define $H(z) = f(z) - \sum_{j=1}^{N} p_k(z) - f_\infty(z)$. This function is entire, since I've removed all the poles, and it's bounded near ∞ (i.e. as $|z|$ gets arbitrarily large), because $F(w)$ is bounded near 0. Thus, by Liouville's theorem, H is constant, so write $H(z) = c$.

Thus, we can write

$$f(z) = c + \sum_{k=1}^{N} P_k(z) + f_\infty(z),$$

and each of the terms is a rational function in z, so f must be rational as well. ☒

The crucial idea in the proof is that since there's a finite number of poles, we can get rid of them and then make life easier.

9. The Logarithm and the Argument Principle: 10/21/14

Recall that we discussed that if $f : \Omega \to \mathbb{C}$ is meromrophic (intuitively, holomorphic with poles), then we can compute $\int_\gamma f(z)\,\mathrm{d}z$ with a sum of residues, where γ is a closed curve contained in Ω.

In particular, we'll be interested in things such as $\log f(z)$ when $f(z)$ is holomorphic and nonzero. We want to be able to define $\log z = \log r + i\theta$ so long as $r \neq 0$, when $= re^{i\theta}$, but the argument is only defined up to a multiple of 2π, so this isn't well-defined. But its derivative is $1/z$, which is well-defined when $z \neq 0$, so this represents the total change in argument in a circle; it measures the number of times a curve wraps around a circle.

For holomorphic $f_1(z), f_2(z)$, the equation $\log(f_1(z)f_2(z)) \neq \log(f_1(z)) + \log(f_2(z))$ in general, because of the indeterminacy in θ, but once we take the derivative, we get f_1'/f_1 and f_2'/f_2, so $(f_1f_2)'/f_1f_2 = (f_1'f_2+f_1f_2')/f_1f_2 = f_1'/f_1+f_2'/f_2$, so while log isn't so well-behaved, its derivative is fine.

Suppose $f(z)$ has a zero of order n at $z = z_0$, so $f(z) = (z - z_0)^n g(z)$, where $g(z) \neq 0$ near z_0 and is holomorphic. Thus,

$$\frac{f'(z)}{f(z)} = \frac{n(z-z_0)^{n-1}g(z) + (z-z_0)^n g'(z)}{(z-z_0)^n g(z)} = \frac{n}{z-z_0} + \frac{g'(z)}{g(z)}.$$

Thus, it has a simple pole with residue equal to n, which is an interesting way of gaining information about the function.

Similarly, if f has a pole of order n, then $f(z) = (z - z_0)^{-n} g(z)$, where g is as above holomorphic and nonzero in a neighborhood of z_0, and one can calculate that

$$\frac{f'(z)}{f(z)} = \frac{-n}{z-z_0} + \frac{g'(z)}{g(z)},$$

so this is a simple pole with residue $-n$ at $z = z_0$. This leads to the following theorem.

Theorem 9.1 (Argument Principle)**.** *If f is meromorphic on Ω and D is a disc such that $D \subset \Omega$ and $C = \partial D \subset \Omega$, then suppose f has no zeroes or poles on C. Then, let $\mathcal{Z}_D(f)$ denote the number of zeros of f inside D, with multiplicity, and $\mathcal{P}_D(f)$ denote the number of poles of f inside D, with multiplicity. Then,*

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,\mathrm{d}z = \mathcal{Z}_D(f) - \mathcal{P}_D(f).$$

The proof is basically a direct application of the residue theorem with what was just talked about.

Geometrically, we're doing a u-substitution, with $u = f(z)$ and $\mathrm{d}u = f'(z)\,\mathrm{d}z$, and thus we're computing

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,\mathrm{d}z = \frac{1}{2\pi i}\int_{f(C)} \frac{\mathrm{d}w}{w},$$

so this integral computes the number of times the curve $f(C)$ wraps around the origin, with sign.

For example, if $f(z) = z^2$, and $z_0 = 0$, then f sends $e^{i\theta} \mapsto e^{2i\theta}$, so $f(C)$ is the unit circle wrapped around itself twice, and

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,\mathrm{d}z = 2.$$

But we can perturb it a little bit: if we consider $f(z) = (z - 1/4)^2$, then the zero is moved slightly. Again, it wraps around twice, but since the edge of the circle is closer to the origin, it's not mapped twice onto itself, as the radius increases. However, it still wraps around the origin twice, so this is still fine. The point is, this is stable under perturbation. There's a theorem that describes this more explicitly.

Theorem 9.2 (Rouché)**.** *If $f(z)$ and $g(z)$ are holomorphic in a disc D and $|f(z)| > |g(z)|$ on $C = \partial D$, then $f(z)$ and $(f + g)(z)$ have the same number of zeros inside C.*

In the example, we had $g(z) = -z/2 + 16$. Notice, however, that in the theorem statement we didn't say anything about zeroes on C, because $|f(z)| > |g(z)|$ on C, so $f \neq 0$, and thus also $f + g \neq 0$.

Proof of Theorem 9.2. Consider $f_t(z) = f(z) + tg(z)$, for $0 \leq t \leq 1$; then, $f_0(z) = f(z)$ and $f_1(z) = (f + g)(z)$. Additionally, $f_t(z) \neq 0$ on C, so if

$$\frac{1}{2\pi i} \int_C \frac{f_t'(z)}{f_t(z)} \, \mathrm{d}z = n_t,$$

then n_t is the number of zeroes of f_t inside D, so $n_t \in \mathbb{Z}$. But n_t must depend continuously on t, so this forces it to be constant. ⊠

Note that the kinds of zeroes may change, e.g. a zero of multiplicity 2 may split into two zeroes of multiplicity 1. But the sum with multiplicities is invariant.

Another way of saying this is that if $|f(z)| > |g(z) - f(z)|$ on C, then f and g have the same number of zeroes on D. And yet another way of thinking about it (which requires additional proof) is that if $|f(z) - g(z)| < |f(z)| + |g(z)|$ on C, then f and g have the same number of zeroes.

Theorem 9.2 can be used to compute, or in some cases estimate, the number of zeroes in a region.

Example 9.3. Let $g(z) = z^7 + 5z^3 - z - 2$. Let $f(z) = 5z^3$, so that $|g(z) - f(z)| = |z^7 - z - 2| \leq 4 < 5 = |5z^3|$ on C, so since f has three zeroes inside C, then so must g as well.

Definition. A continuous function $f : \mathbb{C} \to \mathbb{C}$ is an *open mapping* if the image of every open set is open.

Recall that the *inverse* image of a continuous function is always open, but not always the other way around, which is why this definition exists.

Theorem 9.4 (Open Mapping)**.** *If f is nonconstant and holomorphic on Ω, then f is an open mapping.*

Proof. Let $z_0 \in \Omega$ and $w_0 = f(z_0)$. We want to show that there's a $\delta > 0$ such that $f(D_\delta(z_0))$ is a neighborhood of w_0, so if w is near w_0, let $g(z) = f(z) - w$. If $F(z) = f(z) - w_0$ and $G(z) = w_0 - w$, then $g = F + G$.

Assume $D_\delta(z_0) \subset \Omega$ and $w_0 \notin f(C)$ (which is easy, because the zeros of F are isolated, so we can avoid any zeros we do have). Since C is compact, then $f(C)$ doesn't hit w_0, so there's an $\varepsilon > 0$ such that $|f(z) - w_0| < \varepsilon$ for all $z \in C$. Thus, if $w \in D_\varepsilon(w_0)$, then $|F(z)| > \varepsilon > |G(z)|$, so $g = F + G$ has the same number of zeroes as F inside C. But F has at least one zero, so g must as well, and thus there's a z near z_0 such that $f(z) - w$; thus, f does preserve open sets. ⊠

This theorem is extremely powerful, and has two fairly quick corollaries, which are themselves a little bit magical.

Corollary 9.5 (Maximum Modulus Theorem)**.** *Suppose $f : \Omega \to \mathbb{C}$ is holomorphic and nonconstant. Then, $|f(z)|$ does not attain its maximum inside Ω.*

This means that there may be a supremum, but it's not attained at any point: we just get closer and closer.

Proof. Suppose f does attain a maximum w there, realized by a $z_0 \in \Omega$; then, since f is an open mapping, then any w' sufficiently close to w, even one with greater modulus, is realized by some z' near z, so it wasn't a maximum after all. ⊠

This corollary is also very powerful.

Corollary 9.6. *Suppose $f(z)$ is holomorphic on Ω and $\overline{\Omega}$ is compact. Suppose further that $f(z)$ extends continuously to $\overline{\Omega}$; then,*

$$\sup_{z \in \Omega} |f(z)| \leq \sup_{z \in \overline{\Omega} \setminus \Omega} |f(z)|.$$

We can weaken the restriction that f is nonconstant, because it happens to be true for what we want to prove.

Proof. We just saw that $f(z)$ is continuous iff $|f(z)|$ is continuous on $\overline{\Omega}$, so since $\overline{\Omega}$ is compact, it realizes the maximum of $|f(z)|$, and since the maximum isn't in Ω, then it must be on the boundary. ⊠

10. Homotopy and Simply Connected Regions: 10/23/14

Last time, we used $f'(z)/f(z)$ to compute the difference between the number of zeroes and the number of poles, which had nice applications to the Open Mapping Theorem and the Maximum Modulus Theorem. Since $f'(z)/f(z) = \mathrm{d}(\log f(z))$, then if $f(z) = z$, this becomes $1/z$, so we raised the question of which regions $\Omega \subset \mathbb{C}$ are such that $\log z$ is well-defined on Ω. On $\mathbb{C} \setminus \{0\}$, it's multiply defined, and so we have to be careful. Equivalently, we want to know for which Ω the function $f(z) = 1/z$ has a primitive.

More generally, on what kinds of regions does *every* holomorphic function f have a primitive, so that $\int_\gamma f(z) \, \mathrm{d}z = 0$? We saw this was true for the unit disc, with an explicit construction of the primitive by an integral.

Definition. A *path* from α to β in Ω is a continuous $\gamma : [a, b] \to \Omega$ such that $f(a) = \alpha$ and $f(b) = \beta$.

A *homotopy* of paths is a continuous function $H : [0, 1] \times [0, 1] \to \Omega$, such that for all $s \in [0, 1]$, $H(s, a) = \alpha$ and $H(s, b) = \beta$.

For s fixed, $H(s, t)$ is often denoted $\gamma_s(t)$, which is a path from α to β. A homotopy can be considered a one-parameter family of paths continuously moving from γ_0 to γ_1.

Definition. A path-connected region Ω is *simply connected* if for all $\alpha, \beta \in \Omega$, every pair of paths $\gamma_0(t)$ and $\gamma_1(t)$ from α to β are homotopic.

For example, the unit disc is simply connected. But $D \setminus \{0\}$ is not; intuitively, the constant path $\gamma_0(t) = a$ and a path that winds once around the origin are not homotopic, because homotopy cannot move past the origin. Another way to realize this is the following theorem.

Theorem 10.1. *If $\Omega \subset \mathbb{C}$ is simply connected, then every holomorphic f on Ω has a primitive there, i.e. there's an F on Ω such that $F' = f$.*

To prove this, we'll actually have to invoke a different theorem, after which it's not too hard.

Theorem 10.2. *Suppose f is holomorphic on Ω and $\gamma_0(t)$ and $\gamma_1(t)$ are homotopic paths in Ω. Then,*

$$\int_{\gamma_0} f(\zeta)\,\mathrm{d}\zeta = \int_{\gamma_1} f(\zeta)\,\mathrm{d}\zeta.$$

Proof of Theorem 10.1. First, we'll assume Theorem 10.2 to prove Theorem 10.1; let f be holomorphic on Ω, and we'll construct a primitive much like in Cauchy's Integral Theorem: pick a basepoint $z_0 \in \Omega$; then, for any $z \in \Omega$ let γ_z be a path from z_0 to z, and define

$$F(z) = \int_{\gamma_z} f(\zeta)\,\mathrm{d}\zeta.$$

Since Ω is simply connected, then by Theorem 10.2 this doesn't depend on the choice of γ_z, so this is a well-defined, and we can compute with any path we want.

Now, we need to show that

$$\lim_{h\to 0} \frac{F(z+h) - F(z)}{h} = f(z).$$

Let γ_h be a path from z to $z + h$; then, a path from z_0 to $z + h$ can be given by a path from z_0 to z and then γ_n. Next, parameterize γ_h as $\zeta(t) = z + th$, with $0 \le t \le 1$, so

$$F(z+h) - F(z) = \int_{\gamma_h} f(\zeta)\,\mathrm{d}\zeta = \int_0^1 f(z+th)\,\mathrm{d}t,$$

but as $t \to 0$, this goes to $f(z)$. ⊠

Recall how this looks similar to the proof for the unit disc, but is a little nicer and more general. But now we have to clean up the other theorem, which will ultimately also be similar to the proof we did in the unit disc.

Proof of Theorem 10.2. Suppose γ_0 and γ_1 are homotopic, so we have a homotopy $H : [0, 1] \times [a, b] \to \Omega$, with $\gamma_0(t) = H(0, t)$ and $\gamma_1(t) = H(1, t)$. The book makes a precise argument for how carefully to make this work, but the idea is to compare $\gamma_s(t)$ for a finite number of different $s \in [0, 1]$, with t fixed.

Since the image of H is compact, then there's a $\delta > 0$ such that $|s_j - s_{j-1}| < \delta$, and the image $[s_{j-1}, s_j] \times [a, b] \to \Omega$ can be covered by discs $D_0, \dots, D_n$, such that $D_j \cap D_{j+1} \neq \emptyset$, and since the intersection is convex, then the intersection is connected. Thus, we can find values $t_0, t_1, \dots, t_{n+1}$ such that $w_j, w_{j+1}, z_j, z_{j+1} \in D_j$ for $z_j = \gamma_{s_2}(t_j)$ and $w_j = \gamma_{s_1}(t_j)$ (so a sequence of values along the two paths, contained within small discs).

On each disc a primitive exists, and on $D_j \cap D_{j-1}$, the two primitives differ by a constant c_j, so

$$\int_{\gamma_{s_2}} f(\zeta)\,\mathrm{d}\zeta = \sum_{k=1}^{n} (F_k(z_{k+1}) - F_k(z_k)) + \sum_{j=0}^{n-1} c_j,$$

and similarly for γ_{s_1} and the w_i, but since the intersections of the discs are connected, then the constants on the end are the same! But since we're starting at α and ending at β, then the first sum is also the same between the two, so the integrals are the same. ⊠

Really, we need the homotopy to be piecewise smooth, but this is a somewhat minor part of the whole proof.

Finally, we can apply this to the logarithm, showing it's well-defined on simply connected domains and answering the question we asked one-half of a lecture ago.

Theorem 10.3. *Suppose Ω is simply connected, $1 \in \Omega$, and $0 \notin \Omega$. Then, there's a branch $F(z)$ of $\log z$ (i.e. a choice of the multiple values) such that:*

(1) $F(z)$ is holomorphic on Ω,
(2) $e^{F(z)} = z$, and
(3) If $r \in \Omega$ is real near $r = 1$, then $F(r) = \log r$.

Proof. For any $z \in \Omega$, let γ_z be a path joining 1 and z; then, let
$$F(z) = \int_{\gamma_z} \frac{\mathrm{d}w}{w},$$
which, since Ω is simply connected, is well-defined independent of a choice of γ_z. Thus, $F'(z) = 1/z$, showing (1), so to show (2) we want to show that $ze^{-F(z)} = 1$. Since
$$(ze^{-F(z)})' = e^{-F(z)} - zF'(z)e^{-F(z)} = e^{-F(z)} - e^{-F(z)} = 0,$$
but since $ze^{-F(z)} = 1$ when $z = 1$, and this is constant, then this is equal to 1 everywhere; thus, (2) is satisfied.

For (3), let r near 1, so there's a path from 1 to r within $\Omega \cap \mathbb{R}$, so $F(r) = \int_1^r 1/x \, \mathrm{d}x = \log r$. ⊠

The restriction of real values to near 1 is so that there's that real-valued path within Ω; if Ω spirals around the origin, then it's possible for it to contain real values whose paths to 1 most pass through non-real numbers.

Useful implications are that if Ω is simply connected, then z^α is well-defined on Ω for any $\alpha \in \mathbb{C}$, given by $z^\alpha = e^{\alpha \log z}$. Thus, we have more examples (and useful ones, such as square roots, cube roots, and so forth) of multiply valued functions which we can define uniquely on simply connected regions.

We saw that the punctured disc isn't simply connected, because the logarithm isn't well-defined there, but this is a topological criterion; can we characterize simply connected regions topologically?

Proposition 10.4. *Suppose $\Omega \subset \mathbb{C}$ is bounded; then, Ω is simply connected iff Ω^c is connected.*

This is a nice, quick way to check simple connectedness for the disc and for the punctured disc. We won't give a proof, but it's not too hard to show.

This can be generalized considerably (which is the domain of classes such as Math 148, or if you're feeling fancy, Math 215B); there are computational tools to compute the *fundamental group* of a region Ω with a basepoint z_0. Specifically, one considers all closed paths from z_0 to itself, identifying two of them as the equivalent if they are homotopic. The fundamental group is denoted $\pi_1(\Omega, z_0)$.

By concatenation of loops, $\pi_1(\Omega, z_0)$ has a group structure, which isn't too hard to show, and we get the following result (which is often taken as the definition in more general settings).

Theorem 10.5. *Ω is simply connected iff $\pi_1(\Omega, z_0) = \{e\}$.*

11. Laurent Series: 10/28/14

Laurent[9] series are an extension of Taylor series.

Definition. A *Laurent series* around a $z_0 \in \mathbb{C}$ is a sum of the form
$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$
It is said to *converge* if both of
$$\sum_{k=0}^{\infty} a_k (z - z_0)^k \qquad \text{and} \qquad \sum_{j=1}^{\infty} a_{-j} (z - z_0)^{-j}$$
converge, and the value it converges to is the sum of the values of those two series.

The notions of absolute convergence and uniform convergence of Laurent series are the same as before, though they won't figure as heavily into today's discussion. For positive terms, $\sum_{k=0}^\infty a_k(z - z - 0)^k$ converges absolutely on $|z - z_0| < r_2$, where $\limsup_{k \to 0} |a_k|^{1/k} = 1/r_2$, and for negative terms, $\sum_{j=1}^\infty a_{-j}(z - z_0)^{-j}$ converges when $1/|z - z_0| < 1/\beta$, i.e. $|z - z_0| > \beta = r_1$, where $\limsup_{j \to \infty} |a_{-j}|^{1/j} = \beta$. That is, the positive part converges inside some region, and the negative part outside some region, and therefore the series as a whole converges on some annulus.

Theorem 11.1. *Suppose $f(z)$ is holomorphic on a region Ω containing a closed annulus centered around z_0 given by $A = \{z : r_1 \le |z - z_0| \le r_2\}$. Then, f is given by a Laurent series on the interior of A.*

[9]Laurent was French, so his name rhymes with "naw."

Proof. This follows because of the Cauchy Integral Formula: for any $z \in A$,

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta,$$

where γ_1 is a small circle around z_0. By Green's Theorem, or the Cauchy Integral Theorem, if γ_2 is the outer boundary of the annulus, then

$$\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta - \int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta - \int_{\gamma} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = 0,$$

so rearranging the terms,

$$f(z) = \frac{1}{2\pi i}\left(\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta - \int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta\right).$$

Then, this follows from the Taylor expansion for holomorphic functions. Without loss of generality, let $z_0 = 0$ (so there's less to write); then,

$$\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \int_{\gamma_2} \frac{f(\zeta)}{\zeta}\left(\frac{1}{1 - z/\zeta}\right)\mathrm{d}\zeta.$$

Since $|z| < |\zeta|$, then $|z/\zeta| < 1$, so we can rewrite this as the infinite series

$$\begin{aligned}
&= \int_{\gamma_2} \frac{f(\zeta)}{\zeta}\left(\sum_{k=0}^{\infty}\left(\frac{z}{\zeta}\right)^k\right)\mathrm{d}\zeta \\
&= \sum_{k=0}^{\infty}\int_{\gamma_2} \frac{f(\zeta)}{\zeta}\left(\frac{z}{\zeta}\right)^k\mathrm{d}\zeta \\
&= \sum_{k=0}^{\infty}\left(\int_{\gamma_2} \frac{f(\zeta)}{\zeta^{k+1}}\,\mathrm{d}\zeta\right)z^k
\end{aligned}$$

as desired, i.e. if $k \geq 0$, then $a_k = \int_{\gamma_2} f(\zeta)/\zeta^{k+1}\,\mathrm{d}\zeta$.

We'll do the same thing for the integral around γ_1.

$$\begin{aligned}
-\int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta &= \int_{\gamma_1} \frac{f(\zeta)}{z(1 - \zeta/z)}\,\mathrm{d}\zeta \\
&= \int_{\gamma_1} \frac{f(\zeta)}{z}\sum_{k=0}^{\infty}\left(\frac{\zeta}{z}\right)^k \\
&= \sum_{k=0}^{\infty}\left(\int_{\gamma_1} f(\zeta)\zeta^k\,\mathrm{d}\zeta\right)z^{-(k+1)}.
\end{aligned}$$

Thus, $a_{-j} = \int_{\gamma_1} f(\zeta)\zeta^{k-1}\,\mathrm{d}\zeta$. ⊠

Notice how we obtained this: the positive-indexed coefficients come from integrating a curve around the whole disc, and the negative ones from a curve close to z. In particular, one can use any curve γ_r around 0 to compute the Laurent coefficients.

We'll be using the coefficient formulas over and over, so let's unify them.

$$2\pi i a_n = \int_{\gamma_r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,\mathrm{d}\zeta. \tag{5}$$

This formula will provide another way of looking at isolated singularities. Suppose $f(z)$ is holomorphic on $D_R(z_0) \setminus \{z_0\}$, and has an isolated singularity at z_0. Then, for any $\varepsilon > 0$, we have a Laurent series $L(z) = \sum_{-\infty}^{\infty} a_n z^n$ on the annulus $\{\varepsilon < |z - z_0| < R\}$; we want to identify the type of the singularity by properties of the Laurent series.

Proposition 11.2. *f has a removable singularity at z_0 iff $a_n = 0$ whenever $n < 0$, i.e. the Laurent series only has nonnegative terms, like a Taylor series.*

Proof. If near z_0, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then as $z \to z_0$, $f(z) \to a_0$, so in particular f is bounded near z, and therefore the singularity is removable.

Conversely, if f has a removable singularity at z_0, then it's bounded near z_0, so if $k \geq 1$, then

$$a_{-k} = \frac{1}{2\pi i}\int_{\gamma_r} f(\zeta)\zeta^{k+1}\,\mathrm{d}\zeta,$$

which goes to 0 as $r \to 0$, since f is bounded near there. Thus, $a_{-k} = 0$ for all such k. ⊠

Proposition 11.3. *f has a pole at z_0 iff there are a finite number of negative terms in its Laurent series, i.e. there is a $k > 0$ such that $a_{-k} \neq 0$, but $a_{-n} = 0$ if $n > k$.*

Proof. Having a pole at z_0 is equivalent to f being of the form $f(z) = P(z) + g(z)$, where g is holomorphic on $D_r(z_0)$ and

$$P(z) = \frac{a_{-k}}{(z-z_0)^k} + \frac{a_{-k+1}}{(z-z_0)^{k-1}} + \cdots + \frac{a_{-1}}{z-z_0}.$$

Thus, if there is a pole, P gives a finite number of negative terms to add to the Taylor series of g (since g is holomorphic), and therefore we get a Laurent series with a finite number of negative terms.

Conversely, if f has a Laurent series with a finite number of nonnegative terms, then take the negative part:

$$P(z) = (z-z_0)^{-k}(a_{-k} + a_{-k+1}(z-z_0) + \cdots + a_{-1}(z-z_0)^{k-1}),$$

and therefore when $|z - z_0| < \varepsilon$, then $|P(z)| = |z - z_0|^k(|a_{-k}| - c\varepsilon)$, and therefore $|P(z)| \geq R^k(|a_{-k}| - c\varepsilon)$, where $R = 1/|z - z_0|$. Then, as $z \to z_0$ and $\varepsilon \to 0$, this shows $|P(z)| \to \infty$, i.e. f has a pole. ⊠

For essential singularities, we have only one remaining case.

Corollary 11.4. *f has an essential singularity at z_0 iff its Laurent series in an annular neighborhood around z_0 has infinitely many negative terms.*

Though we get this as a corollary, it's a very useful way to think of essential singularities, as those where f is approximated by arbitrarily large negative exponents.

These perspectives on singularities, while not mentioned by our own textbook, are occasionally useful, and even serve as the definitions for the three types of singularities in some books.

The Infinite Product Expansion. Suppose f is entire, and has some zeros. One way to find these zeros is to decompose f as an infinite product of functions; then, if these functions are easier to test for zeros, then we can recover the roots of f from those of its components. This comes from Chapter 5 in the textbook, which we'll briefly look over.

This helps answer some questions, e.g. where the zeros of an entire function can be. We know they can't accumulate, but if we take any discrete set of points in $\mathbb{C}$, do we have a holomorphic function whose zero set is that set of points? To what degree does this determine that function uniquely?

First, we should rigorously talk about the notion of an infinite product. The notation

$$f(z) = \prod_{n=1}^{\infty} F_n(z)$$

means that the partial products $\prod_{n=1}^{N} F_n(z)$ converge to $f(z)$ as $N \to \infty$.

Just as for infinite sums, it will be useful to have a condition on the components to converge.

Proposition 11.5. *Suppose $\{a_n\} \subset \mathbb{C}$ is a sequence such that $\sum_{n=1}^{\infty}|a_n|$ is finite. Then,*

$$\prod_{n=1}^{\infty}(1 + a_n)$$

converges.

Proof. Since $\sum|a_n|$ is finite, so there's an N such that if $n \geq N$, then $|a_n| < 1/2$. Renumber the terms so that we start with a_n; the finite product of the terms before that is obviously finite, so we can just tack it on once we're done with the rest of the proof.

Since $\{1 + a_n\}$ is contained within $D_{1/2}(1)$, then $\log(1 + a_n)$ is well-defined, and $e^{\log(1+a_n)} = 1 + a_n$. In particular, for any $K \in \mathbb{N}$,

$$\prod_{n=1}^{k}(1 + a_n) = \prod_{n=1}^{K} e^{\log(1+a_n)} = \exp\left(\sum_{n=1}^{K} b_n\right).$$

But if $|z| < 1/2$, then $|\log(1 + z)| \leq 2|z|$, so

$$\exp\left(\sum_{n=1}^{K} b_n\right) \leq \exp\left(\sum_{n=1}^{K}|a_n|\right),$$

and the right-hand side is bounded as $K \to \infty$, so the left-hand side is as well. ⊠

The key trick was using the logarithm to turn the infinite product into an infinite sum.

"What do meromorphic functions have in common with elections?" "You can learn a lot about them by looking at polls."

Though we mentioned it a little last time, today we'll go into greater detail about how to write entire functions as infinite products, and use this to analyze zero sets of such functions.

Recall Proposition 11.5, which says that if $\sum a_n$ is absolutely convergent, then $\prod(1+a_n)$ converges. We'll use this to prove the following result.

Proposition 12.1. *Suppose $\{F_n(z)\}$ is a sequence of holomorphic functions on Ω. If there exists a sequence $\{c_n\}$ such that $c_n > 0$, $\sum c_n$ is finite, and $|1 - F_n(z)| \leq c_n$ for all $z \in \Omega$, then:*

(1)

$$F(z) = \prod_{n=1}^{\infty} F_n(z)$$

exists and is holomorphic on Ω.

(2) If $F_n(z) \neq 0$ for all $z \in \Omega$ and $n \in \mathbb{N}$, then

$$\sum_{n=1}^{\infty} \frac{F_n'(z)}{F_n(z)} = \frac{F'(z)}{F(z)}.$$

Proof. Since $|F_n(z) - 1| \leq c_n$ for each $z \in \Omega$, then $\prod_{n=1}^{\infty} F_n(z) \to F(z)$ uniformly on Ω, and uniform limits of holomorphic functions are still holomorphic, so (1) follows.

For part (2), let

$$G_N(z) = \prod_{n=1}^{N} F_n(z),$$

so that $G_N(z) \to F(z)$ uniformly, and therefore also $G_N'(z) \to F'(z)$ uniformly as well, at least on compact subsets $K \subset \Omega$, which follows from the Cauchy Integral Formula for G_N' and F'. Thus, if $G_N(z) \neq 0$ for all N and z, then $|G_N(z)|$ is bounded away from zero, so

$$\frac{G_N'(z)}{G_N(z)} \longrightarrow \frac{F'(z)}{F(z)}$$

uniformly on compact subsets of Ω.

As it happens, $G_N'(z)/G_N(z) = \sum F_N'(z)/F_N(z)$, which we didn't touch on, but is proven in the textbook. ⊠

Example 12.2.

$$g(z) = \frac{\sin \pi z}{\pi} = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) = f(z).$$

Proof. Let $f_n(z)$ be the n^{th} term in the product; then, we can show that

$$\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)}, \tag{6}$$

and since

$$\left(\frac{f(z)}{g(z)}\right)' = \frac{f'(z)g(z) - g'(z)f(z)}{g(z)^2},$$

then rearranging terms,

$$\frac{f(z)}{g(z)}\left(\frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)}\right) = 0,$$

which means that $f(z) = cg(z)$ for some constant c. Once this is shown, divide f and g by z, and as $z \to 0$,

$$\frac{f}{g} = \frac{\prod_{n=1}^{\infty}(1 - z^2/n^2)}{\sin(\pi z)/(\pi z)} \longrightarrow 1,$$

so $c = 1$.

Thus, it remains to show that the logarithmic derivatives[10] are equal, as in (6).

$$\begin{aligned}
\frac{g'(z)}{g(z)} &= \pi\left(\frac{\cos(\pi z)}{\sin(\pi z)}\right) = \pi\cot(\pi z).\\
\frac{f'(z)}{f(z)} &= \frac{1}{z} + \sum_{n=1}^{\infty}\frac{-2z}{n^2-z^2} = \sum_{n=-\infty}^{\infty}\frac{1}{z+n}\\
&= \lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{z+n}.
\end{aligned}$$

Call this $F(z)$ and let $G(z) = \pi\cos(\pi z)$; then, we'll show that $F(z) = G(z)$ in a somewhat synthetic way by showing they have the same properties. To wit,

(1) $G(z)$ is periodic, with period 1,
(2) $G(z) = 1/z + G_0(z)$, where $G_0(z)$ is analytic at $z = 0$, and
(3) $G(z)$ has simple poles at $z \in \mathbb{Z}$, and no singularities anywhere else.

We want these properties to also hold for F.

(1) For the first property, the partial sum $\sum_{n=-N}^{N} 1/(z+n)$ is almost periodic, in that when evaluated at $z+1$, we get

$$\sum_{n=-N}^{N}\frac{1}{(n+1)+z} = \underbrace{\frac{1}{z+N+1} - \frac{1}{z-N}}_{(*)} + \sum_{n=-N}^{N}\frac{1}{z+n},$$

but the terms in $(*)$ go to 0 as $N\to\infty$, so f is indeed periodic, with period 1.
(2) The second part is clear because we can write

$$F(z) = \frac{1}{z} + \sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}, \tag{7}$$

which is clearly analytic at the origin.
(3) Again, looking at (7), it is clear that F has a simple pole at each $z \in \mathbb{Z}$, and has no other singularities.

Consider $\Delta(z) = G(z) - F(z)$, which must be periodic (since F and G are, with the same period), and in fact is entire holomorphic (since they have the same poles, all of which are simple, and the same residues there by periodicity, so the limits go to the same value, and thus Δ can be extended to fill in these removable singularities).

Now, we want to show that it's bounded. By periodicity, it suffices to show that Δ is bounded on $\{z = x+iy \mid |x| \le 1/2\}$. On the rectangle $[-1/2,-1/2]\times[1,1]$, which is compact, Δ must be bounded, since it's continuous. If $|y| \ge 1$, we'll bound both F and G, so that Δ must also be bounded.

First, for G, let's look at $\cot(\pi z)$, which is certainly bounded iff G is.

$$\cot(\pi z) = i\left(\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}}\right) = i\left(\frac{e^{\pi i x}e^{-\pi y}+e^{-\pi i x}e^{\pi y}}{e^{\pi i x}e^{-\pi y}+e^{-\pi i x}e^{\pi y}}\right).$$

Without loss of generality, assume $y \ge 1$; the calculations are very similar in the other case.

$$= i\left(\frac{e^{-2\pi y}+e^{-2\pi i x}}{e^{-2\pi y}-e^{-2\pi i x}}\right),$$

which is bounded above by $(e^{-2\pi}+1)/(1-e^{-2\pi})$. Thus, G is bounded.

Now, why is F bounded? We want to show that

$$\left|\frac{1}{z} + \sum_{n=1}^{\infty}\frac{2z}{z^2+n^2}\right| \le c + c\left(\sum_{n=1}^{\infty}\frac{y^2}{y^2+n^2}\right), \tag{8}$$

where $z = x+iy$. (8) is true because, when $|x| \le 1/2$,

$$\left|\frac{2z}{z^2-n^2}\right| = \left|\frac{2(x+iy)}{x^2-y^2-n^2+2ixy}\right| \le \frac{2y+1}{n^2+(y-1/2)^2+1/2} \le c\frac{y}{n^2+y^2}.$$

[10] We used logarithmic derivatives here because they convert infinite products to infinite sums, which are considerably nicer to play with.

But now we need to show that the infinite series in the right-hand side of (8) is bounded! We're almost there. Let's use an integral test:

$$\sum_{n=1}^\infty \frac{y^2}{y^2+n^2} \le \int_0^\infty \frac{y^2}{y^2+x^2}\,\mathrm{d}x = \int_0^\infty \frac{y^2}{y^2(1+z^2)}\,\mathrm{d}z$$

when $z = x/y$, so $\mathrm{d}z = \mathrm{d}x/y$. This last integral is independent of y, which is deeply, deeply magical, but for the purposes of the proof, all we need is that it converges.

Now, F and G are bounded, so Δ must also be bounded, and therefore since it's entire, then it's constant by Liouville's theorem. It only remains to check that $\Delta(z) = 0$ for some $z \in \mathbb{C}$. ⊠

Thus, we have two useful identities (applicable to both number theory and the homework):

$$\frac{\sin(\pi z)}{\pi} = z\prod_{n=1}^\infty\left(1 - \frac{z^2}{n^2}\right)$$
$$\pi\cot(\pi z) = \sum_{n=-\infty}^\infty \frac{1}{z+n}.$$

This trick with logarithmic derivatives and so forth is not just limited to this case.

Let's see how much we can generalize this idea. Let $\{a_n\}$ be any set with no limit points in $\mathbb{C}$. If there an entire function $f(z)$ such that $\{a_n\}$ is exactly the zero set of f, with multiplicities? The answer will turn out to be yes.

A reasonable first guess is

$$f(z) = z^m\prod_{n=1}^\infty\left(1 - \frac{z}{a^n}\right),$$

though this doesn't always work, since it may not always converge. Thus, we'll have to multiply it by stuff until it converges; specifically, we'll define some *canonical factors*, e.g. $E_0(w) = 1 - w$, and more generally,

$$E_k(w) = (1-w)e^{w+w^2/2+w^3/3+\dots+w^k/k}.$$

The specific statement we want to show is that

$$f(z) = z^m\prod_{n=1}^\infty E_n\left(\frac{z}{a_n}\right)$$

has zeros in the right places, and converges. Why's that?

Well, if $|z| \le R$, then divide $\{a_n\}$ into two sets, $A_1 = \{a_n : |a_n| \le 2R\}$ and $A_2 = \{a_n : |a_n| > 2R\}$. Notice that A_1 must be finite, because $\{a_n\}$ has no points of accumulation, and so the finite product obviously converges, so we only need to look at the terms in A_2, which may indeed be infinite.

We won't prove the following lemma, though it's proven in the textbook.

Lemma 12.3. *If $|w| \le 1/2$, then $|1 - E_k(w)| \le c|w|^{k+1}$.*

We can invoke this for every $a \in A_2$, because if $|z| < R$ and $|a_n| \ge 2R$, then $|z/a_n| \le 1/2$. Then, assuming the lemma, the series is bounded by a geometric series, which therefore converges.

13. The Hadamard Product Theorem and Conformal Mappings: 11/11/14

Recall that we're trying to construct an entire function with a prescribed set of zeroes (some countable collection of zeroes a_n without an accumulation point). Write

$$f(z) = z\prod_{n=1}^\infty E_n\left(\frac{z}{a_n}\right),$$

where $E_n(w) = (1-w)e^{w+w^2/2+\dots+w^n/n}$ and $E_0(w) = 1 - w$. This has zeroes in the correct places, and the E_n ensure the infinite product converges.

How unique is this function? Suppose f_1 and f_2 are entire and have the same zeroes, with multiplicity. Then, $h(z) = f_1(z)/f_2(z)$ has removable singularities, so we can go ahead and remove them; thus, we get an entire function which never vanishes. In particular, this means it can be written as the exponential of another function: $h(z) = e^{g(z)}$.

This is something we need to show: let

$$g(z) = \int_{z_0}^z \frac{h'(\zeta)}{h(\zeta)} + c_0,$$

where $e^{c_0} = h(z_0)$, so let's consider $h(z)e^{-g(z)}$, which we can show is equal to 1.

$$\begin{aligned}\left(h(z)e^{-g(z)}\right)' &= h'(z)e^{-g(z)} - h(z)g'(z)e^{-g(z)} \\ &= (h'(z) - h(z)g'(z))e^{-g'(z)} = 0,\end{aligned}$$

because $g'(z) = h'(z)/h(z)$. Thus, $h(z)e^{-g(z)}$ is constant, but is 1 at z_0, and therefore is equal to 1 everywhere. Thus, the general solution to finding an entire function with a prescribed set $\{a_n\}$ of zeroes is

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_n\left(\frac{z}{a_n}\right). \tag{9}$$

This can be refined slightly into the following theorem, which we won't prove.

Theorem 13.1 (Hadamard Factorization)**.** *In the general solution* (9)*, one can choose degrees n of the E_n to be bounded above by k and $g(z)$ to be a polynomial of degree at most k.*

The k in the above theorem is related to the growth of f (and the growth of the number of zeroes inside $D_R(0)$ for R increasing).

Definition. If f is entire, then it *has growth at most ρ* if there exist $A, B > 0$ such that $|f(z)| \le Ae^{B|z|^\rho}$. The *order* ρ_0 of f is the infimum of such ρ.

Then, in Theorem 13.1, k satisfies $k \le \rho_0 < k+1$. Additionally, if $n(r)$ is the number of zeroes a_n of f such that $|a_n| < r$, then $n(r) \le Cr^{\rho_0}$ for some constant C, when r is sufficiently large.

These are interesting statements, but would be too much of a detour to prove in this class; consult the textbook for detailed proofs of these theorems. This is a much deeper theorem than that of Weierstrass.

Example 13.2. If $f(z) = \sin \pi z$ again, then $f(z) = (e^{\pi i z} - e^{-\pi i z})/2i$, and therefore $|f(z)| \le Ae^{\pi|z|}$, so $\rho_0 = 1$. Since $\{a_n\} = \mathbb{Z}$, then $|n(r)| = 2N + 1$ when $N < r < N + 1$.

Conformal Mappings. Moving into a new topic covered in chapter 9, we want to know for regions $U, V \subset \mathbb{C}$ when they're holomorphically equivalent, in some sense.

Definition. Two regions $U, V \subset \mathbb{C}$ are *conformally equivalent* (*holomorphically equivalent*) if there exists a holomorphic bijection $f : U \to V$ with holomorphic inverse $g : V \to U$.

Proposition 13.3. *It suffices for f to be a holomorphic bijection. That is, if $f : U \to \mathbb{C}$ is holomorphic and one-to-one, then $f'(z) \ne 0$ for all $z \in U$ and $f^{-1} : f(U) \to U$ is holomorphic.*

Proof. Suppose $f'(z_0) = 0$ for some $z_0 \in U$; then, if $\omega_0 = f(z_0)$, then for z near z_0, $f(z) = \omega_0 + a_n(z - z_0)^n$ for some $n \ge 2$, i.e. $f(z) - \omega_0$ has a zero of order $n \ge 2$. Thus, for ω near ω_0 and z near z_0, $f(z) - \omega$ has n zeroes, which we proved on the midterm using Rouché's Theorem. In particular, there's an isolated zero of $f'(z)$ at $z = z_0$, so $f'(z) \ne 0$ for z near z_0. But since it's one-to-one near other points, it isn't at z_0, which forces a contradiction.

Let $g(z) = f^{-1}(z)$ and $V = f(U)$, so that $g : V \to U$. Let $w_0 = f(z_0)$ and $w = f(z)$, so $g(w_0) = z_0$ and $g(w) = z$. Then,

$$\begin{aligned}\lim_{w\to w_0} \frac{g(w) - g(w_0)}{w - w_0} &= \lim_{z\to z_0} \frac{z - z_0}{f(z) - f(z_0)} \\ &= \lim_{z\to z_0} \frac{1}{\frac{f(z)-f(z_0)}{z-z_0}} = \frac{1}{f'(z_0)},\end{aligned}$$

so $g = f^{-1}(z)$ is holomorphic. ⊠

The key idea is that at a zero of order n, locally the function looks like $(z - z_0)^n$, so nearby values have n preimages (and thus injectivity doesn't hold).

Let $\mathbb{H} = \{z = x + iy : y > 0\}$, which is standard notation for this region, called the *upper half-plane*.

Example 13.4.

(1) Let U be the sector $\{z = re^{i\theta} : 0 < r < \pi/n\}$. Under the map $z \mapsto z^n$, $\theta \mapsto n\theta$, so this sector maps to the upper half-plane $\mathbb{H}$; thus, U is holomorphically equivalent to $\mathbb{H}$.

(2) Let $U = \mathbb{H}$ and consider $z \mapsto \log z = \log r + i\theta$ (where $0 < \theta < \pi$). This sends the positive real line to the real line, and the negative real line to the line $y = \pi$. This is a little more surprising: $\mathbb{H}$ is holomorphically equivalent to the infinite strip $\{x + iy : 0 < y < \pi\}$, and even by a quite simple function!

(3) Let $\mathbb{D}$ denote the unit disc, and consider $F : \mathbb{H} \to \mathbb{D}$ and $G : \mathbb{D} \to \mathbb{H}$ given by

$$F(z) = \frac{i - z}{i + z} \qquad \text{and} \qquad G(w) = i\left(\frac{1 - w}{1 + w}\right).$$

Why do these actually work? For f, the idea is that $\mathbb{H}$ is characterized by the set of points that are closer to i than to $-i$, so $|F(z)| < 1$. Thus, $F(z) \subseteq \mathbb{D}$.

We can also show that if $w = u + iv$ and $u^2 + v^2 < 1$, then we can show that $\mathrm{Im}(G(w)) > 0$. Specifically,

$$\mathrm{Im}(G(w)) = \mathrm{Re}\left(\frac{1 - u - iv}{1 + u + iv}\right) = \mathrm{Re}\,\frac{(1 - u - iv)(1 + u + iv)}{(1 + u)^2 + v^2} = \frac{(1 - u)^2 - v^2}{(1 + u)^2 + v^2} > 0.$$

We'll show F and G are inverses to each other in a little bit; the textbook crunches the explicit computation, but it's possible to be smarter than that.

The book goes through many more examples, which were omitted for brevity.

Why the term "conformally equivalent?" The idea is that if $f'(z) \neq 0$, then f infinitesimally looks like multiplication by some complex number $c \neq 0$. But multiplication by a nonzero complex number is a scaling and a rotation, i.e. f locally scales and rotates. These two operations preserve angles, so f infinitesimally preserves angles. In mathematics at large, a *conformal* map is one that preserves angles, so the term conformal mapping is used in this case.

It turns out we can represent these conformal mappings by matrices. Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a complex-valued matrix (i.e. $a, b, c, d \in \mathbb{C}$) and $\det A \neq 0$. Then, define the *fractional linear transformation* associated with A to be

$$f_A : z \longmapsto \frac{az + b}{cz + d}.$$

Then, if $\lambda \in \mathbb{C} \setminus \{0\}$, clearly $f_A = f_{\lambda A}$, but more interestingly, $f_{AB} = f_A \circ f_B$ (which is a calculation we don't need to get into).

Now, we can answer the question from Example 13.4: if A is the matrix corresponding to F and B corresponds to G, then $A = \begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix}$ and $B = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}$, so $G \circ F$ corresponds to

$$B \cdot A = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix} = \begin{pmatrix} 2i & 0 \\ 0 & 2i \end{pmatrix},$$

but this is a scalar multiple of the identity matrix, so it gives the same transformation, which is $f_I(z) = z$. Thus, G and F are indeed inverses.

It turns out we can connect this to the somewhat vague notion we defined of a "point at infinity" earlier in the class: consider the sphere S of radius $1/2$ centered at $(0, 0, 1/2)$, as shown in Figure 4, and call its *north pole* $N = (0, 0, 1)$. Then, we'll map $S \to \mathbb{C} \cup \{\infty\}$ by *stereographic projection*: send $N \to \infty$ and for any $W \in S$, the line connecting N and x intersects $\mathbb{C}$ at one point W; then, send $W \mapsto w$.

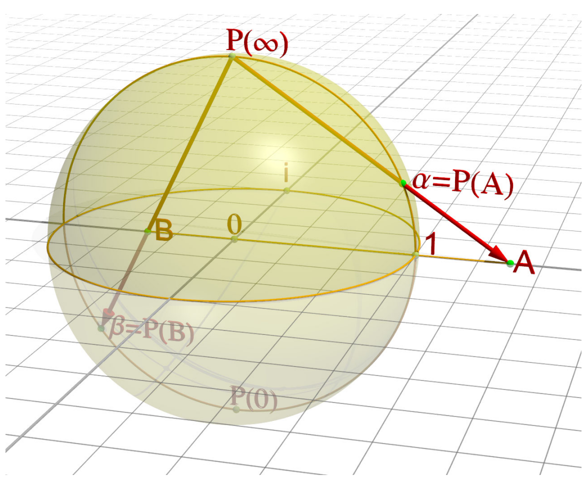

Figure 4. A depiction of the Riemann sphere and projections of points on it to $\mathbb{C}$. Source: Wikipedia.

Geometrically, if $W = (X, Y, Z)$ and $w = (x, y)$, then $x = X/(1-Z)$ and $y = Y/(1-z)$, and $X = x/(1+x^2+y^2)$, $Y = y/(1+x^2+y^2)$, and $Z = (x^2+y^2)/(1+x^2+y^2)$. The unit circle maps to the equator, and so $\mathbb{D}$ maps to the lower hemisphere and $\mathbb{H}$ to the back half of the sphere. It's interesting how distance increases as one gets closer to N.

Another nice property is that under stereographic projection, circles are sent to circles (or straight lines, if the circles go through N), and moreover angles are preserved. We can also show that conformal maps send circles to circles (or lines through the "point at infinity"); for example, in our map $G : \mathbb{D} \to \mathbb{H}$, the straight line $y = 0$ is sent to the unit circle, and the point at infinity is mapped to -1.

The upshot is that there is some beautiful geometry behind these matrix multiplications and conformal maps, and we'll learn about that in upcoming lectures.

14. The Schwarz Lemma and Aut($\mathbb{D}$): 11/13/14

"Isn't this function just its own inverse?"

"You weren't supposed to say that yet! Do you give away the endings of movies, too?"

The Schwarz lemma is quite simple to state and prove, but is ubiquitous throughout complex analysis. We'll use it in one way to talk about conformal maps, but it has plenty of other applications.

Lemma 14.1 (Schwarz)**.** *Suppose $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and $f(0) = 0$. Then,*

(1) for all $z \in \mathbb{D}$, $|f(z)| \le |z|$, and
(2) if there exists a $z_0 \ne 0$ such that $|f(z_0)| = |z_0|$, then f is a rotation: there exists a θ such that $f(z) = e^{i\theta} z$.
(3) Furthermore, $|f'(0)| \le 1$, and if $|f'(0)| = 1$, then f is a rotation.

Proof. Since $f(0) = 0$, then if we expand $f(z)$ at 0, $f(z) = a_1 z + a_2 z^2 + \cdots$, since there's no constant term. In particular, $h(z) = f(z)/z$ has a removable singularity at $z = 0$, so let's just remove it and say that h is holomorphic on $\mathbb{D}$.

Choose a $z \in \mathbb{D}$, so that $|z| = r < 1$ and $|f(z)| < 1$. Thus, $|h(z)| = |f(z)|/|z| < 1/r$. Since h is holomorphic, then the maximum modulus principle tells us that as $r \to 1$, $|h(z)| < 1$ for all $z \in \mathbb{D}$, and therefore $|f(z)| \le |z|$.

For (2), if $|h(z)|$ attains it maximum 1 at an interior point, then it must be constant, i.e. $h(z) = f(z)/z = c$ for some $c \in \mathbb{C}$. But $|h(z)| = |c| = 1$ at that interior point, so $|c| = 1$, i.e. $c = e^{i\theta}$ for some θ, and $f(z) = e^{i\theta} z$, so f is a rotation.

Now, for (3), we want to relate h to f' somehow. Since $|h(z)| \le 1$ and $f(0) = 0$, then

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{f(z)}{z} = h(0).$$

Thus, $|f'(0)| = |h(0)| \le 1$, and if it's equal to 1, then h must be constant, so the same argument as above shows f is a rotation. ⊠

We will use this to study holomorphic (conformal) equivalence; specifically, which open sets $U \subset \mathbb{C}$ are holomorphically equivalent to $\mathbb{D}$? If $f : U \to \mathbb{D}$ is a conformal mapping and $\psi : \mathbb{D} \to \mathbb{D}$ is a conformal mapping (an *automorphism*), then we get another conformal mapping $\psi \circ f : U \to \mathbb{D}$. Thus, understanding the automorphisms of $\mathbb{D}$ is a way to investigate the non-uniqueness of conformal maps $U \to \mathbb{D}$.

Definition. The set of automorphisms of $\mathbb{D}$, i.e. holomorphic bijections $f : \mathbb{D} \to \mathbb{D}$, is denoted Aut($\mathbb{D}$).

$\mathcal{G} = \mathrm{Aut}(\mathbb{D})$ carries a group structure.[11]

Example 14.2. Here are some examples of elements of $\mathcal{G}$.

- The map $z \xrightarrow{f} e^{i\theta} z$, rotation by the angle θ.
- For an $\alpha \in \mathbb{D}$, there's a $\psi_\alpha : \mathbb{D} \to \mathbb{D}$ sending $0 \mapsto \alpha$, given by

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}.$$

Since $|\alpha| < 1$ and $|z| < 1$., then $|\overline{\alpha} z| < 1$, and thus the denominator is never 0 on $\mathbb{D}$, so ψ_α is certainly holomorphic. Additionally, since ψ_α extends to $|z| = 1$, we can see what it does on the unit circle.

$$\psi_\alpha(e^{i\theta}) = \frac{\alpha - e^{i\theta}}{1 - \overline{\alpha} e^{i\theta}} = \frac{1}{e^{i\theta}} \left(\frac{\alpha - e^{i\theta}}{e^{-i\theta} - \overline{\alpha}} \right) = -e^{-i\theta} \left(\frac{\overline{w}}{w} \right),$$

where $w = e^{-i\theta} - \overline{\alpha}$. Thus, $|\psi_\alpha(e^{i\theta})| = 1$, and thus by the maximum modulus principle, when $|z| < 1$, then $|\psi_\alpha(z)| < 1$ as well. Thus, $\psi_\alpha : \mathbb{D} \to \mathbb{D}$.

It turns out this is a bit easier to show if we notice that ψ_α is its own inverse, which can be computed by brute force (and additionally shows ψ_α must be a bijection). But we can save some work by writing ψ_α as a linear

[11] That is, if $f, g \in \mathcal{G}$, then $f \circ g \in \mathcal{G}$, $f^{-1} \in \mathcal{G}$, and the identity ID $\in \mathcal{G}$.

fractional transformation, so that it corresponds to the matrix $\begin{pmatrix} -1 & \alpha \\ -\overline{\alpha} & 1 \end{pmatrix}$, and therefore $\psi_\alpha \circ \psi_\alpha$ corresponds to the square of this matrix:

$$\begin{pmatrix} -1 & \alpha \\ -\overline{\alpha} & 1 \end{pmatrix}\begin{pmatrix} -1 & \alpha \\ -\overline{\alpha} & 1 \end{pmatrix} = \begin{pmatrix} 1-|\alpha|^2 & 0 \\ 0 & 1-|\alpha|^2 \end{pmatrix},$$

but since $1 - |\alpha|^2 \neq 0$, then this is equivalent to the identity map.

We've seen two relatively examples. It turns out these generate all of $\mathrm{Aut}(\mathbb{D})$!

Theorem 14.3. *Suppose $f \in \mathrm{Aut}(\mathbb{D})$. Then, there exits a $\theta \in [0, 2\pi)$ and an $\alpha \in \mathbb{D}$ such that $f(z) = e^{i\theta}\psi_\alpha(z)$.*

Proof. Since f is a bijection, then there's a unique $\alpha \in \mathbb{D}$ such that $f(\alpha) = 0$. Thus, consider $g = f \circ \psi_\alpha$; g is a holomorphic bijection and $g(0) = 0$, so we want to invoke Lemma 14.1. Immediately it tells us that $|g(z)| \leq |z|$, but since g is invertible and $g^{-1}(0) = 0$, then the Schwarz lemma again tells us that $|g^{-1}(z)| \leq |z|$, so therefore $|g(z)| = |z|$ for all $z \in \mathbb{D}$. Thus, using the Schwarz lemma again, this means g is a rotation through an angle θ, so $e^{i\theta} = \circ\psi_\alpha$, and therefore $f = e^{i\theta}\psi_\alpha^{-1} = e^{i\theta}\psi_\alpha$. ⊠

Corollary 14.4. *$\mathcal{G}$ acts* transitively *on $\mathbb{D}$, i.e. for any $\alpha, \beta \in \mathbb{D}$, there's an $f \in \mathrm{Aut}(\mathbb{D})$ such that $f(\alpha) = \beta$.*

We can build this just from the ψ_α; consider $\psi_\beta^{-1} \circ \psi_\alpha$, which is in $\mathrm{Aut}(\mathbb{D})$ and sends $\alpha \mapsto \beta$.

Corollary 14.5. *If $f \in \mathrm{Aut}(\mathbb{D})$ and $f(0) = 0$, then $f(z) = e^{i\theta}z$ for some θ.*

This is true because one can check that $\psi_0(z) = z$.

If $f(z) = e^{i\theta}(\alpha - z)/(1 - \overline{\alpha}z)$, let $\xi^2 = e^{i\theta}$, so $\xi = e^{i\theta/2}$. Then, the matrix corresponding to this fractional linear transformation is

$$\begin{pmatrix} -\xi^2 & \xi^2\alpha \\ -\overline{\alpha} & 1 \end{pmatrix} \longleftrightarrow \frac{1}{\sqrt{1-|\alpha|^2}}\begin{pmatrix} \xi & -\xi\alpha \\ \overline{\alpha}/\xi & 1/\xi \end{pmatrix} \longleftrightarrow \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}.$$

The group of matrices of the latter form is called $\mathrm{SU}(1,1)$, which is for reasons a bit beyond the scope of the class; but nonetheless, we have this elegant isomorphism $\mathrm{Aut}(\mathbb{D}) \cong \mathrm{SU}(1,1)$.

Now, we can talk about $\mathrm{Aut}(\mathbb{H})$, which is similarly defined as the set of holomorphic bijections $\psi : \mathbb{H} \to \mathbb{H}$. Recall that we also have conformal maps $F : \mathbb{H} \to \mathbb{D}$ and $F^{-1} = G : \mathbb{D} \to \mathbb{H}$, so if $\psi \in \mathrm{Aut}(\mathbb{H})$, then the following diagram commutes.

$$\begin{tikzcd} \mathbb{H} \arrow[r, "\psi"] & \mathbb{H} \arrow[d, "F"] \\ \mathbb{D} \arrow[u, "F^{-1}"] \arrow[r] & \mathbb{D} \end{tikzcd}$$

The lowermost arrow is $\varphi = F \circ \varphi F^{-1} \in \mathrm{Aut}(\mathbb{D})$. Conversely, given a $\varphi \in \mathrm{Aut}(\mathbb{D})$, we obtain a commutative diagram

$$\begin{tikzcd} \mathbb{D} \arrow[r, "\varphi"] & \mathbb{D} \arrow[d, "F^{-1}"] \\ \mathbb{H} \arrow[u, "F"] \arrow[r, "\psi"] & \mathbb{H}, \end{tikzcd}$$

where $\psi = F^{-1} \circ \varphi \circ F \in \mathrm{Aut}(\mathbb{H})$. Thus, we have a one-to-one correspondence that (with a little more work) preserves products, so it's a group isomorphism $\Gamma : \mathrm{Aut}(\mathbb{D}) \to \mathrm{Aut}(\mathbb{H})$ sends $\varphi \to F^{-1} \circ \varphi \circ F$, i.e. $\Gamma(\varphi_1 \circ \varphi_2) = \Gamma(\varphi_1) \circ \Gamma(\varphi_2)$.

Since $\mathrm{Aut}(\mathbb{D})$ is given by fractional linear coefficients, then one might imagine $\mathrm{Aut}(\mathbb{H})$ is too. This turns out to be true: $f(z) = (az + b)/(cz + d)$ for $a, b, c, d \in \mathbb{R}$ (remember the real numbers?) such that $ad - bc = 1$. Thus, we can associate f with the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $\det(A) = 1$. The group of such matrices is

$$\mathrm{SL}(2, \mathbb{R}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \det(A) = 1, a, b, c, d \in \mathbb{R} \right\}.$$

Then, $\mathrm{Aut}(\mathbb{H}) = \mathrm{SL}(2, \mathbb{R})/\pm\mathrm{ID}$; that is, multiplying scalars preserves f again, but this time the determinant condition means that the only possible scalars are ± 1.

One might wonder whether $\mathrm{Aut}(\mathbb{H})$ can be explicitly described from the explicit description of $\mathrm{Aut}(\mathbb{D})$ and through Γ; this is an excellent question, and will appear on the homework.

We've said a lot of things, and probably should prove some of them. Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$, so that $z \overset{f_A}{\mapsto} (az+b)/(cz+d)$. If $z \in \mathbb{H}$, then $\mathrm{Im}(z) > 0$, and we want to show that $\mathrm{Im}(f_A(z)) > 0$. Observe that

$$\left(\frac{az+b}{cz+d}\right)\left(\frac{c\overline{z}+d}{c\overline{z}+d}\right) = \frac{acz\overline{z} + bd + adz + cb\overline{z}}{(cz+d)^2},$$

and thus

$$\mathrm{Im}(f_A(z)) = \frac{(ad-bc)\,\mathrm{Im}(z)}{(cz+d)^2} > 0.$$

So that ends up working out, which is fortunate.

Next, we want to prove the more attractive idea that if $f \in \mathrm{Aut}(\mathbb{H})$, then $f = f_A$ for some $A \in \mathrm{SL}(2,\mathbb{R})$. Given a $z \in \mathbb{H}$, we can first find an A such that $f_A(z) = i$: first, find a C such that $|\mathrm{Im}(f_C(z)| = 1$, which is $C = \begin{pmatrix} 0 & -c^{-1} \\ c & 0 \end{pmatrix}$ where $\mathrm{Im}(z)/|cz|^2 = 1$. The next step is to translate $f_C(z)$ to i, which is just horizontal movement, because we've normalized $f_C(z)$ to lie on the line $y = 1$. This translation by $b = i - f_C(z)$ is given by $f_B(z) = z + b$, with $B = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. Thus, for any $z \in \mathbb{H}$, there's an $f \in \mathrm{Aut}(\mathbb{H})$ such that $f(z) = i$, so we know $\mathrm{Aut}(\mathbb{H})$ acts transitively on $\mathbb{H}$.

What about the functions $f(i) = i$? If $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, then we can show that $f_A(i) = i$ (since this corresponds in some sense to a rotation by -2θ around 0 when we map back to $\mathbb{D}$). It turns out these will be all of the things that fix i, which requires a little more work to show.

15. The Riemann Mapping Theorem: 11/18/14

Today we're going to prove one of the deepest theorems in the course, and in doing so discover what kinds of regions in $\mathbb{C}$ are conformally equivalent to $\mathbb{D}$. That means, as we discussed, there's a holomorphic bijection $f : \Omega \to \mathbb{D}$, which implies that Ω must be connected and simply connected.

We can rule out one example: if $\Omega = \mathbb{C}$, then $f : \mathbb{C} \to \mathbb{D}$ is a bounded entire function and therefore constant, so not a bijection.

But this is the only counterexample.

Theorem 15.1 (Riemann Mapping)**.** *Let $\Omega \subsetneq \mathbb{C}$ be a proper, connected, and simply connected open subset of $\mathbb{C}$. Then, for any $z_0 \in \Omega$, there exists a unique conformal equivalence $f : \Omega \to \mathbb{D}$ such that $f(z_0) = 0$ and $f'(z_0) > 0$ (i.e. it's real and positive).*

This is pretty incredible result, since it's so vastly general.

Proof. Uniqueness of f follows from the structure of $\mathrm{Aut}(\mathbb{D})$: if $F, G : \Omega \to \mathbb{D}$ both satisfy the conditions in the theorem, then $\varphi = F \circ G^{-1} \in \mathrm{Aut}(\mathbb{D})$, $\varphi(0) = 0$, and $\varphi'(0) = F'(z_0)(G^{-1})'(0) > 0$.

Since $\varphi(0) = 0$, then $\varphi(z) = e^{i\theta} z$ must be a rotation, and $|e^{i\theta}| = 1$, so $H'(0) = e^{i\theta} > 0$, so together they imply that $e^{i\theta} = 1$, or $\theta = 0$. Thus, φ is the identity, so $F = G$.

The rest of the proof will take at least the rest of class. Here's an outline of the proof idea:

- First, consider all one-to-one maps $f : \Omega \hookrightarrow \mathbb{D}$ such that $f(z_0) = 0$.
- Find the f that maximizes $|f'(z_0)|$, and show that it's surjective. This uses an analytic tool called Montel's Theorem which will allow for some approximations.

Definition. Let $\mathcal{F}$ be a family of functions $f : \Omega \to \mathbb{C}$. Then,

- $\mathcal{F}$ is called a *normal family* if for every sequence $\{f_n\} \subset \mathcal{F}$ there exists a subsequence $\{f_{n_j}\}$ that converges uniformly to an $f : \Omega \to \mathbb{C}$ on compact subsets of Ω.
- $\mathcal{F}$ is *uniformly bounded* if there exists a B such that $|f(z)| \le B$ for all $f \in \mathcal{F}$ and $z \in \Omega$.
- $\mathcal{F}$ is *equicontinuous* on compact subsets of Ω if for any compact $K \subset \Omega$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that if $f \in \mathcal{F}$, $z, w \in \Omega$, and $|z - w| < \delta$, then $|f(z) - f(w)| < \varepsilon$.

Equicontinuity is a particularly strong version of uniform continuity. In general, the latter two conditions imply the first.

Theorem 15.2 (Arzelà-Ascoli)**.** *If $\mathcal{F}$ is a uniformly bounded, equicontinuous family of functions, then it is normal.*

This is slightly different from the Arzelà-Ascoli Theorem that often appears on Math 171 WIM assignments, but is clearly related.

However, we're going to use (and prove) a stronger result specifically for families of holomorphic functions. This is certainly not true in real analysis, and is another great example of how incredible complex differentiability is.

Theorem 15.3 (Montel)**.** *Suppose $\mathcal{F}$ is a family of holomorphic maps. If $\mathcal{F}$ is uniformly bounded, then it is equicontinuous on compact subsets and is normal.*

Proof. Since the second step follows from Theorem 15.2, it's more of a question of real analysis, so we'll focus on the first part, showing equicontinuity, which is actually relevant to holomorphicity.

Suppose $K \subset \Omega$ is compact and $\varepsilon > 0$. Then, there exists an $r > 0$ such that for any $z \in K$, $D_{3r}(z) \subset \Omega$ for each $z \in K$ (since K is compact and Ω is open). Given a $w \in K$, consider the boundary γ of $D_{2r}(w)$ and $z \in D_r(w)$ (so γ has two connected components). Since all of this is within Ω, we can evaluate functions there.

In particular, we're going to use the Cauchy Integral Formula. Every $f \in \mathcal{F}$ is holomorphic, so

$$\begin{aligned}
|f(z) - f(w)| &= \left|\frac{1}{2\pi i}\int_\gamma \left(\frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta - w}\right)\mathrm{d}\zeta\right| \\
&= \left|\frac{1}{2\pi i}\int_\gamma f(\zeta)\left(\frac{z - w}{(\zeta - z)(\zeta - w)}\,\mathrm{d}\zeta\right)\right| \\
&\le \frac{1}{2\pi}\frac{4\pi r}{r^2}B,
\end{aligned}$$

and since $|\zeta - z|, |\zeta - w| > r$, then equicontinuity follows.

We'll tackle the rest of the proof (of normality) later, but assume it for now.

Proposition 15.4. *Let $\Omega \subset \mathbb{C}$ be connected and $f_n : \Omega \hookrightarrow \mathbb{C}$ be a sequence of holomorphic, injective functions converging uniformly on compact subsets of Ω to an $f : \Omega \to \mathbb{C}$. Then, f is holomorphic, and either injective or constant.*

$f_n(z) = z^n/n$ is an example where the limit of a sequence of injective functions isn't injective.

Proof. This proof will invoke the power of winding numbers or counting zeroes.

Suppose f is not injective, so that there exist $z_1, z_2 \in \Omega$ such that $f(z_1) = f(z_2)$. To turn this into a statement about zeroes, let $g_n(z) = f_n(z) - f_n(z_1)$; since each f_n is injective, then each g_n must be as well, so $g_n(z)$ has a unique zero at $z = z_1$. But since the f_n converge uniformly, then $g_n \to g = f - f(z_1)$ uniformly as well.

Assume that f is nonconstant, so that z_1 is an isolated zero of $f(z)$. Let γ be the boundary of a circle around z_2 such that g has no zeroes on γ or inside of it, except at z_2. Thus, $1/g_n(z) \to 1/g(z)$ on γ, and by using the Cauchy Derivative Formula, $g_n'(z) \to g'(z)$. Since there are no zeroes of g_n inside γ, then

$$\frac{1}{2\pi i}\int_\gamma \frac{g_n'(\zeta)}{g_n(\zeta)}\,\mathrm{d}\zeta = 0, \tag{10}$$

but since these g_n uniformly converge to g, (10) also holds true for g, but this can't be right, because g has one zero on the interior of γ. Thus, this is a contradiction.

Well, we have yet some work to do, but it's a little less hard analysis. Recall that we want to find a one-to-one and onto $F : \Omega \to \mathbb{D}$.

First, we need to show there exists an $f : \Omega \hookrightarrow \mathbb{D}$ such that $f(z_0) \ne 0$. This is particularly interesting because $\mathbb{D}$ is bounded, but Ω isn't; the main step is to find an $f_1 : \Omega \to \Omega_1$, where Ω_1 is bounded; then, Ω_1 can be shrunk and translated into $\mathbb{D}$.

This is the only part of the proof that uses the fact that there exists an $\alpha \notin \Omega$: then, $z - \alpha \ne 0$ for all $z \in \Omega$, so, since Ω is simply connected, $\log(z - \alpha)$ exists for all $z \in \Omega$, and therefore there exists an $f : \Omega \to \mathbb{C}$ such that $e^{f(z)} = z - \alpha$ for all $z \in \Omega$. Thus, f is injective, because if $f(z) = f(w)$, then $e^{f(z)} = e^{f(w)}$, so $z - \alpha = w - \alpha$.

For any $w \in \Omega$, we can also prove that $f(z) \ne f(w) + 2\pi i$ for all $z \in \Omega$, in the same way: after taking the exponent, this would imply that $z - \alpha = w - \alpha$, so $z = w$ but $f(z) \ne f(w)$, which would be a contradiction.

In fact, $|f(z) - (f(w) + 2\pi i)|$ is bounded from 0 (i.e. there's an $\varepsilon > 0$ such that it's at least ε), because otherwise there would be a sequence of z_n such that $f(z_n) \to f(w) + 2\pi i$, but then, since f is injective and continuous, then $z_n \to w$, which is a contradiction.

Let $G(z) = 1/(f(z) - (f(w) + 2\pi i))$, which is therefore holomorphic and bounded above, so $G : \Omega \hookrightarrow \Omega_1$, where $\Omega_1 \subset \mathbb{C}$ is bounded. Then, Ω_1 can be rescaled and shifted so that there's an injection $\eta : \Omega_1 \hookrightarrow \mathbb{D}$, and therefore we have a map $\Omega \hookrightarrow \mathbb{D}$.

We've now shown that any such Ω as in the problem statement is conformally equivalent to an open, connected, and simply connected subset of $\mathbb{D}$, so we might as well restrict ourselves to that case.

Consider the family $\mathcal{F} = \{f : \Omega \hookrightarrow \mathbb{D}, f(0) = 0\}$, so that the identity is in $\mathcal{F}$. Then, we want to maximize $|f'(0)|$ over $f \in \mathcal{F}$, which is where Montel's Theorem will come into play. Since $\mathcal{F}$ is uniformly bounded (since $\Omega \subset \mathbb{D}$), then $|f'(0)|$ is uniformly bounded, by the Cauchy Integral Formula. Thus, it has a supremum $S = \sup\{|f'(0)| : f \in \mathcal{F}\}$. Since the derivative of the identity is 1 everywhere and the identity is in $\mathcal{F}$, then $S \geq 1$. Consider a sequence $\{f_n\} \subset \mathcal{F}$ such that $|f_n'(0)| \to S$, so by Theorem 15.3, $\{f_n\}$ has a subsequence converging uniformly to an $f : \Omega \to \mathbb{C}$ such that $|f'(0)| = s \geq 1$. In particular, this implies f is not constant, so by Proposition 15.4, it's injective: $f : \Omega \hookrightarrow \overline{\mathbb{D}}$. But by the Maximum Modulus Principle, since Ω is open, then $f : \Omega \hookrightarrow \mathbb{D}$.

This is where we'll have to stop today; tune in next time to see why this f is surjective.

16. The Riemann Mapping Theorem II: 11/20/14

Today, we will continue the proof of Theorem 15.1, which says that any proper, connected, simply connected open region $\Omega \subsetneq \mathbb{C}$ is conformally equivalent to $\mathbb{D}$, and for any $z_0 \in \Omega$, there exists a conformal mapping $f : \Omega \to \mathbb{D}$ between them such that $f(z_0) = 0$ and $f'(z_0) > 0$ (in particular, it's real).

Contintuation of the Proof of Theorem 15.1. Thus far, we've found a $g : \Omega \hookrightarrow \mathbb{D}$, so we can assume $\Omega \subset \mathbb{D}$, and used Theorem 15.3 to consider the family $\mathcal{F} = \{f : \Omega \hookrightarrow \mathbb{D}, f(0) = 0\}$, and show that there's an $f \in \mathcal{F}$ such that $|f'(0)| = \sup_{g\in\mathcal{F}} |g'(0)| = s$, and that $s \geq 1$. Furthermore, we have that the identity is in $\mathcal{F}$.

Thus, it remains to show that f is onto, and a small amount of the proof of Theorem 15.3; we will today do both of these.

Suppose f isn't onto $\mathbb{D}$, so that there's an $\alpha \in \mathbb{D}$, but $\alpha \notin f(\Omega)$. Consider $\psi_\alpha \in \mathrm{Aut}(\mathbb{D})$, so that $0 \notin (\psi_\alpha \circ f)(\Omega)$. But $(\psi_\alpha \circ f)(\Omega)$ is also simply connected, so we can define the square root function $h(z) = z^{1/2}$ there.

Let $F = \psi_{g(\alpha)} \circ g \circ \psi_\alpha \circ f$, so that $F : \Omega \hookrightarrow \mathbb{D}$ and $F(0) = 1$, so $F \in \mathcal{F}$. But we can show that $F'(0) > f'(0)$, which will cause a contradiction. Write $f = \psi_\alpha^{-1} \circ \eta \circ \psi_{g(\alpha)}^{-1} \circ F = \Phi \circ F$, where $\eta(w) = w^2$ and $\Phi = \psi_\alpha^{-1} \circ \eta \circ \psi_{g(\alpha)}^{-1}$. In particular, $\Phi : \mathbb{D} \to \mathbb{D}$ and $\Phi(0) = 0$, so by the Schwarz lemma, since Φ is not one-to-one (since η isn't), then $|\Phi'(0)| < 1$ and $|f'(0)| = |\Phi'(0)||F'(0)| < |F'(0)|$, which is a contradiction.

Thus, $f : \Omega \twoheadrightarrow \mathbb{D}$, and in particular, it's a conformal equivalence (so we use g to create a conformal equivalence with any proper Ω). To get that $f'(z_0) \neq 0$, it's equal to some complex number, so we can rotate $f(z)$ to make it real-valued and positive, and the result remains a holomorphic bijection. ⊠

So now we're done up to finishing the proof of Montel's theorem. We've shown that if $\mathcal{F}$ is a family of functions $\Omega \to \mathbb{C}$ that are uniformly bounded on compact subsets $K \subset \Omega$, then $\mathcal{F}$ is uniformly equicontinuous on compact subsets of Ω, but we need to show that $\mathcal{F}$ is normal, i.e. for all sequences $\{f_n\} \subset \mathcal{F}$, there's a subsequence converging uniformly on compact subsets $K \subset \Omega$.

Continuation of the Proof of Theorem 15.3. Since we've already done part of the proof, we may assume that $\mathcal{F}$ is uniformly equicontinuous (on compact subsets of Ω).

Definition. If X is a metric space and $Y \subset X$, then an *exhaustion* of Y by compact subsets is a choice of compact subsets $K_1 \subset K_2 \subset \cdots \subset Y$, so that for each ℓ, $K_\ell \subset K_{\ell+1}^0$ (i.e. the interior), $\bigcup_{\ell=1}^\infty K_\ell = Y$, and any compact $K \subset Y$ is contained in K_ℓ for some ℓ.

The idea is that we approximate Y by compact subsets that swallow up any other compact subset.

It turns out that Ω has an exhaustion by compact sets, given by $K_\ell = \{z \in \Omega : |z| \leq \ell, d(z, \partial\Omega) \geq 1/\ell\}$. Then, one should check the definition of an exhaustion, but it's fairly evident.

Choose a countable dense set $\{w_i\} \subset \Omega$, and fix a compact $K \subset \Omega$. Then, given a sequence of $f_n \in \Omega$, we want to find a convergent subsequence. Consider $f_n(w_1)$; since $\mathcal{F}$ is uniformly bounded on K, then $f_n(w_1)$ is bounded, so it has a convergent subsequence $f_{n,1}$. Then, since $\mathcal{F}$ is still uniformly bounded, the sequence $\{f_{n,1}(w_2)\}$ has a convergent subsequence $f_{n,2}$, and so on. Take the *diagonal sequence* $g_n(z) = f_{n,n}(z)$, so that $g_n(w_j)$ converges for all $j \in \mathbb{N}$.

Well, we haven't used uniform equicontinuity on K yet, so pick an $\varepsilon > 0$, so that there exists a $\delta > 0$ such that $|f(z) - f(w)| < \varepsilon$ for all $f \in \mathcal{F}$ and $z, w \in K$ such that $|z - w| < \varepsilon$. Since K is compact, we can cover it by a finite number of disks of radius δ from around the $\{w_j\}$; call their centers $w_1, \ldots, w_J$.

In particular, since there are only finitely many such w_j, then one can find an N such that if $m, n \geq N$, then $|g_n(w_j) - g_m(w_j)| < \varepsilon$ for $1 \leq j \leq J$. If $z \in K$, then $z \in D_\delta(w_j)$ for some $j \in \{1, \ldots, J\}$, so

$$|g_n(z) - g_m(z)| \leq |g_n(z) - g_n(w_j)| + |g_n(w_j) - g_m(w_j)| + |g_m(w_j) - g_m(z)| \leq \varepsilon + \varepsilon + \varepsilon = 3\varepsilon$$

by the triangle inequality, and as $\varepsilon \to 0$, this can be made as small as you like. The first and third terms are bounded by uniform equicontinuity, and the second term because this sequence is Cauchy,[12] so the sequence $\{g_n(z)\}$ is *uniformly* Cauchy, so $g_n(z) \to g(z)$ uniformly.

[12] Once again, Cauchy is everywhere in complex analysis...

We're almost there, save for one final diagonal argument: we have our exhaustion $K_1 \subset K_2 \subset \cdots \subset \Omega$ by compact sets, and for the sequence $f_n \in \mathcal{F}$, there's a uniformly convergent subsequence $f_{1,n}$ on K_1, and this has a uniformly convergent subsequence $f_{2,n}$ on K_2, and so on. Then, take the diagonal subsequence $g_n = g_{n,n}$; this converges uniformly on all of the K_ℓ, and therefore on all compact subsets of Ω. ⊠

Notice that this part of the proof had nothing whatsoever to do with complex analysis, and can be generalized somewhat.

Remark. The Riemann Mapping Theorem has topological implications. In particular, it tells us that if $\Omega \subset \mathbb{C}$ is simply connected (including being connected), then that actually implies that Ω is homeomorphic (and for that matter, diffeomorphic[13]) to $\mathbb{D}$. However, this is false in higher dimensions (both this strong statement and several more plausible-looking, general ones). The use of complex analysis in this is pretty essential.

The Dirichlet Problem. This is a question relating to harmonic functions, so one can actually state it on much more general domains (e.g. higher dimensions), but we'll just worry about the complex-analytic case.

Specifically, given a $\mathbb{D} \subset \mathbb{C}$ and a continuous $u(e^{i\theta})$ on $S^1 = \partial\mathbb{D}$, can one find a harmonic[14] function $\widetilde{u}(x, y) : \mathbb{D} \to \mathbb{R}$ that extends continuously to u on the boundary?

This comes up here because we've shown on the homework that on $\mathbb{D}$ (or indeed any simply connected domain), u is harmonic iff $u = \operatorname{Re}(f)$ for some holomorphic f. We had already seen from the Cauchy-Riemann equations that the real part of any holomorphic function is harmonic, so this is a partial converse. This suggests there should be a nice relationship between holomorphic and harmonic functions.

Specifically, we proved that if $u(x, y) = u(z)$ is a harmonic function $\mathbb{D} \to \mathbb{R}$ that extends continuously to S^1, then it's the real part of the holomorphic function given by convoluting with the *Poisson kernel*:

$$\widetilde{u}(z) = \frac{1}{2\pi}\int_0^\pi P_r(\theta - \varphi)u(e^{i\varphi})\,\mathrm{d}\varphi,$$

where P_r is the Poisson kernel

$$P_r(\gamma) = \frac{1 - r^2}{1 - 2r\cos\gamma + r^2}.$$

In some sense, we're weighting or measuring the boundary of $\mathbb{D}$ to obtain this function.

We want to go the other way. One nice result is to use conformal equivalence to translate these questions between different domains.

Proposition 16.1. *Suppose $F : V \to U$ is holomorphic and $u : U \to \mathbb{R}$ is harmonic. Then, $u \circ F : V \to \mathbb{R}$ is harmonic.*

Proof. This boils down to the fact that u being harmonic is a local statement. Near any $z \in U$, there exists a disc on which $u = \operatorname{Re}(G)$ for some holomorphic G, so $H = F \circ G$ is a holomorphic map $V \to \mathbb{C}$. Then, one can check that $\operatorname{Re}(H) = u \circ F$, which is therefore harmonic. ⊠

Now, we can try to solve a more general Dirichlet problem: suppose $\Omega \subsetneq \mathbb{C}$ is a simply connected region and $f : \partial\Omega \to \mathbb{R}$ is continuous. Then, let $F : \mathbb{D} \to \Omega$ be a conformal equivalence. If F extends to a continuous $\widehat{F} : \partial\mathbb{D} \to \partial\Omega$, then $\widetilde{f} = f \circ \widehat{F} : \partial\mathbb{D} \to \mathbb{R}$ has a Dirichlet problem we already know how to solve, and then we can map the solution back to Ω using F^{-1}.

This is a somewhat big step: the Riemann Mapping Theorem provides a map between the interiors of regions, not the boundaries, but it turns out that in many specific cases this ends up working; we'll talk about polygons after the break, and the book also solves the Dirichlet problem in the infinite strip $\{z : 0 \le \operatorname{Im}(z) \le 1\}$. The interior of this region is conformally equivalent to $\mathbb{D}$, and the boundary data takes the form of f_0, f_1 on $\{y = 0\}$ and $\{y = 1\}$, respectively, such that $|f_1(z)|, |f_2(z)| \to 0$ as $|z| \to \infty$. Then, one can write down a conformal map that sends the interior to $\mathbb{D}$, and maps the boundary to $\partial\mathbb{D}$ excluding ± 1, which are hit by the points at infinity (which are distinct in this framework, because we're not considering the entire Riemann sphere); specifically, $1 \mapsto -\infty$ and $-1 \mapsto \infty$. This is why we need $f_1, f_2 \to 0$ as $|z| \to \infty$. Nonetheless, the textbook describes how to translate the problem between these two domains.

[13] This means there's a differentiable bijection $f : \Omega \to \mathbb{D}$ whose inverse is also differentiable, and is a sort of smooth equivalence.

[14] Recall that a *harmonic* function is one such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Recall from before the break that if U is a proper (i.e. not all of $\mathbb{C}$), simply connected domain, then there's a conformal equivalence between U and $\mathbb{D}$; this result was called the Riemann Mapping Theorem.

Furthermore, if the conformal maps $F : \mathbb{D} \to U$ (and therefore $G : U \to \mathbb{D}$) extend continuously to $\partial\mathbb{D}$ and ∂U as homeomorphisms, then one can solve the Dirichlet problem of finding a harmonic function within U with a certain boundary condition. This relies on more than just the existence of a conformal map, which is a bit more interesting than what we've been proving.

Today, we'll focus on the case where U is a *polygon* $\mathfrak{p}$, i.e. its boundary is piecewise linear, and it's simply connected. Furthermore, we'll consider maps $F : \mathbb{H} \to \mathfrak{p}$, so the boundary is still linear; these are called *Schwarz-Christoffel mappings*. These have applications to airflow and heatflow on regions, and so having an explicit answer, even if it's messy or without an elementary antiderivative, is quite useful for actual applications.

So now, how can we create this "bend" or kink in the line, to get the polygonal boundary? Notice that $f(z) = z^\alpha$ sends $\mathbb{H}$ to an infinite wedge $\{re^{i\theta} : 0 \le \theta \le \alpha\pi\}$ (for $0 < \alpha < 2$). We can rewrite this as $z^\alpha = \alpha \int_0^z \zeta^{\alpha-1}\,\mathrm{d}\zeta = \alpha \int_0^z \zeta^{-\beta}\,\mathrm{d}\zeta$, where $\alpha + \beta = 1$. This is just notation for now.

Another example is

$$F(z) = \int_0^z \frac{\mathrm{d}\zeta}{(1-\zeta^2)^{1/2}}.$$

Notice that $1 - \zeta^2 = (1-\zeta)(1+\zeta)$. There are exciting questions about branch cuts to consider; we need to choose a branch of the square root such that the quantity under the integral sign is real and positive when $|\zeta| < 1$, $\zeta \in \mathbb{R}$. For $\zeta > 1$, this quantity is purely imaginary, and in fact in $\mathbb{H}$. In other words, this maps the boundary of $\mathbb{H}$ to two corners. The upshot is, each time one passes a singularity, there's a rotation. See Figure 5 for an illustration.

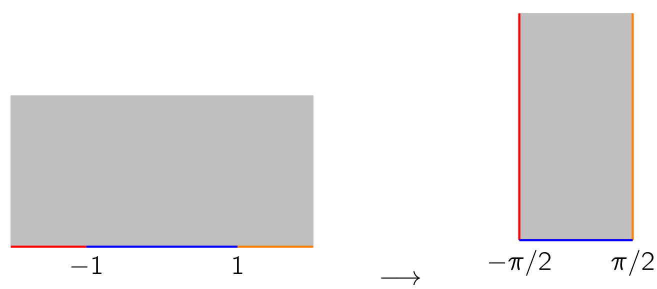

Figure 5. Conformally mapping $\mathbb{H}$ to the region $(-\pi/2, \pi/2) \times (0, \infty)$.

Another instructive example is

$$F(z) = \int_0^z \frac{\mathrm{d}\zeta}{((1-\zeta^2)(1-k^2\zeta^2))^{1/2}},$$

for $0 < k < 1$. This produces a rectangle with vertices $\pm k$ and $\pm k + ik'$; for example, one may calculate that on the segment $[1/k, \infty)$, a change of variables sends this to the corner we previously discussed.

Here's another useful example; we want to choose the branch of $(A-z)^\beta$ to be real and positive if $z = x < A$, and otherwise on the real line, set it as $|A-z|^\beta e^{\pi i \beta}$. Thus, we can define $(A-z)^\beta$ on the region $\mathbb{C} \setminus \{A + iy, y < 0\}$. Then, let

$$f(z) = \prod_{j=1}^k (A_j - z)^{-\beta_j},$$

where $A_1 < A_2 < \cdots < A_n$, which is defined on the complement of the rays $\{A_k + iy, y < 0\}$, which as we will show on the homework is simply connected. Thus, define $S(z) = \int_0^z f(\zeta)\,\mathrm{d}\zeta$, so that $S'(z) = f(z)$. Thus, on the real interval (A_k, A_{k+1}), $\arg(S'(z))$ is constant, and at A_k, the argument changes by adding an angle $\beta_k\pi$. Then, the exterior angle at the k^{th} point, a_k, is $\beta_k\pi$, so the interior angle is $\alpha_k\pi$, where $\alpha_k + \beta_k = 1$.

Assume $0 < \beta_k < 1$ and that $1 < \sum_{k=1}^N \beta_k \le 2$. The first assumption forces $\int_x^{A_k} f(\zeta)\,\mathrm{d}\zeta$ to be finite, whenever $x \in (A_{k-1}, A_k)$. If $a_k = S(A_k)$ (the image of the point), then what happen as $z \to \infty$? This is where the second condition comes in: when $|\zeta|$ is large,

$$\left|\prod_{k=1}^N (A_k - \zeta)^{-\beta_k}\right| \le c|\zeta|^{-\beta_1 - \cdots - \beta_N} \le c|\zeta|,$$

so this implies $S(re^{i\theta})$ converges to some finite value as $r \to \infty$, for any fixed θ.

If we have two angles θ_1 and θ_2, we can consider the pie-slice-shaped region $\{re^{i\theta} : \theta_1 < \theta < \theta_2, r < R\}$ for a given R. The Cauchy Integral Formula implies this must be zero, so we get the same limit for θ_1 and θ_2; in particular, we get some limit a_∞ as $r \to \infty$ for all θ. In some sense, the point at infinity is mapped well-definedly to a_∞.

Proposition 17.1. *If $0 < \beta_k < 1$ and their sum is less than or equal to 2, then the image of $\mathbb{R}$ under $S(z)$ is a polygonal path with vertices (in order) $a_1, a_2, \ldots, a_N$ (and a_∞ if the sum of the β_k is strictly less than 2).*

Proof. The first case is where the sum of the β_k is exactly 2. This means the sum of the interior angles is exactly 2π, even if we ignore a_∞; specifically, the line connecting a_N to a_1 contains a_∞, which isn't a vertex. That this line connects a_k to a_1, which is the whole point, is clear because the angles add up, so it can't go anywhere else.

If $\sum \beta_k < 2$, then the interior angles sum to less than 2π, so there's a missing angle

$$\beta_\infty = 2 - \sum_{k=1}^{N} \beta_k.$$

Thus, a_∞ is a vertex. ⊠

Notice that this explains neatly why we need the sum of the β_k to be bounded by 2: if they're higher, then we get a boundary that wraps around itself twice, which isn't the boundary of a polygon.

This is all good, but we were just mapping $\mathbb{R}$ to boundaries; we need to discuss what happens to $\mathbb{H}$. Furthermore, is this the general formula for such polygonal mappings? The answer is (essentially) yes.

Theorem 17.2. *Suppose $F : \mathbb{H} \to \mathfrak{p}$ is a conformal equivalence with a polygon $\mathfrak{p}$, then $F(z) = c_1 S(z) + c_2$ for an appropriate choice of A_k and β_k, for $c_1, c_2 \in \mathbb{C}$.*

The idea is, S always sends $0 \mapsto 0$; we may need to rotate, scale, or translate the result.

We won't prove this, but here's a key part: if $F : \mathbb{D} \to \mathfrak{p}$ is a conformal equivalence, then F always extends continuously to the boundary: $\partial\mathbb{D} \to \partial\mathfrak{p}$. This fairly hardcore, analytic argument is given in detail in the book, and in the next lecture we'll spend more time on geometry. However, it's worth noticing that the argument only really uses information $\partial\mathbb{D}$ and $\partial\mathfrak{p}$; specifically, that a sufficiently small circle around any $z \in \partial\mathfrak{p}$ intersects $\mathfrak{p}$ at a single arc (and the same for $\partial\mathbb{D}$). This is a pretty weak condition; it's hard to come up with regions that don't have this property, as all smooth or piecewise-smooth regions have conformal equivalences that extend continuously to the boundary. In general, there won't be a nice, explicit formula as with the polygons, but the upshot is that for regions with piecewise smooth boundary, the Dirichlet problem is solvable! This is theoretically pretty.

18. Conformal Maps on Polygonal Regions II: 12/4/14

"Remember, two wrongs don't make a right — but three lefts do."

Last time, we defined

$$S(z) = \int_0^z \frac{d\zeta}{(\zeta - A_1)^{\beta_1}(\zeta - A_2)^{\beta_2} \cdots (\zeta - A_n)^{\beta_n}},$$

though the signs may be flipped (the book is inconsistent about this, and sometimes this is confusing). This maps $\mathbb{R}$ to a polygonal path, assuming that each $\beta_k < 1$ and their sum is at most 2. Then, the vertices of this polygonal path are $a_k = S(A_k)$ (and $a_\infty = S(\infty)$ in some cases explained yesterday), with exterior angles $\beta_k\pi$ (so the interior angles are $\alpha_k\pi$, with $\alpha_k + \beta_k = 1$).

Last time, we assumed the β_k were positive, but then never used that; we can still take $-1 < \beta_k < 1$, and everything is still all right.

Last time, we mentioned that any conformal equivalence $F : \mathbb{D} \to \mathfrak{p}$ for a polygonal region $\mathfrak{p}$ extends continuously to the boundaries: $\overline{F} : \partial\mathbb{D} \to \partial\mathfrak{p}$. The takeaway is that a map $F : \mathbb{H} \to \mathfrak{p}$ extends continously to $\mathbb{H} \cup \mathbb{R} \cup \{\infty\}$.

Today, we want to prove Theorem 17.2, that we can characterize F as $F(z) = c_1 S(z) + c_2$ for a function S as given above and $c_1, c_2 \in \mathbb{C}$. Thus, these kinds of conformal mappings essentially describe all of them.

Proof of Theorem 17.2. We'll start with the case when $F(\infty)$ is not a vertex. Notice that

$$S'(z) = \prod_{k=1}^{n} (z - A_k)^{-\beta_k},$$

and that

$$\frac{d}{dz}(\log S'(z)) = \frac{S''(z)}{S'(z)} = -\sum_{k=1}^{\infty} \frac{\beta_k}{z - A_k}.$$

We want to show that if $F : \mathbb{H} \to \mathfrak{p}$ is any other conformal mapping, then its logarithmic derivative is identical; then, the result is recovered by doing two integrations.

We know that F must send some $A_k \in \mathbb{R}$ to the k^{th} vertex a_k of $\mathfrak{p}$. This turns a straight line into a bend, so F can't be holomorphic at A_k, but we can show it "looks like" $z \mapsto (z - A_k)^{\alpha_k}$ (where α_k is the k^{th} interior angle), and once again we have $\alpha_k + \beta_k = 1$.

Let's look at the strip above (A_{k-1}, A_{k+1}) in $\mathbb{H}$, and let's straighten out the k^{th} vertex by $h_k : w \mapsto (F(z) - a_k)^{1/\alpha_k}$. This does unfold the bend in the line: this sends the strip above (A_{k-1}, A_{k+1}) to some region, but sends (A_{k-1}, A_{k+1}) to a straight line L again. Thus, one can use the Schwarz reflection principle to extend h_k to a holomorphic map on the bi-infinite strip $\{x + iy : x \in (A_{k-1}, A_{k+1})\}$.

Now that we have a holomorphic map, we can start messing around with it. We have $(h_k)^{\alpha_k} = F(z) - a_k$. Let's take the logarithmic derivative:

$$\frac{F'(z)}{F(z) - a_k} = \alpha_k \frac{h_k'}{h_k}.$$

$F'(z) \neq 0$ on the upper strip, since there it's a conformal map, but what happens when $\mathrm{Im}(z) \leq 0$? The logarithmic derivative tells us that $h_k'(z) \neq 0$ as well, so since $h_k^{\alpha_k}$ is one-to-one on open strips because F is, then $h_k'(z) \neq 0$ on (A_{k-1}, A_{k+1}).

Well, $(h_k^{\alpha_k})' = \alpha_k h_k^{-\beta_k} h_k'$, so the logarithmic derivative of this is

$$\mathrm{dlog}(h_k^{\alpha_k})' = \frac{-\beta_k h'}{h} + \frac{h_k''}{h_k'}.$$

Thus, we can conclude that

$$\frac{F''}{F'} = \frac{-\beta_k}{2 - A_k} + E_k,$$

where E_k is holomorphic. This is true on the strip, so now we need to generalize. We can of course repeat it for every other strip, and use the reflection principle to extend F outside $D_R(0)$. Furthermore, since $F(z)$ is bounded at ∞, then the extended $F(z)$ is holomorphic at ∞, so F''/F' is holomorphic at ∞, and it goes to 0 at ∞. We may have some more poles, so F''/F' is meromorphic on $\mathbb{C}$, so in particular

$$\frac{F''}{F'} + \sum_{k=1}^{\infty} \frac{\beta_k}{z - A_k}$$

is holomorphic on $\mathbb{C}$ and bounded, so by Liouville's theorem, it's constant and therefore equal to 0 (since they do agree somewhere).

Thus,

$$\frac{F''}{F'} = -\sum_{l=1}^{n} \frac{\beta_k}{z - A_k},$$

i.e. F' and S' have the same logarithmic derivatives:

$$\mathrm{dlog}\, F' = \mathrm{dlog} \underbrace{\prod_{k=1}^{n} (z - A_k)^{-\beta}}_{Q}.$$

Thus, $\frac{\mathrm{d}}{\mathrm{d}z}(F'/Q) = 0$, so $F' = c_1 Q$, and therefore $F = c_1 S + c_2$, where $c_1, c_2 \in \mathbb{C}$.

There's one more case to deal with, and that's if a_∞ *is* a vertex. We can translate $\mathbb{H} \to \mathbb{H}$ such that $A_k \neq 0$ for all k. Now, choose some A_n^* that isn't a vertex and let Φ be a conformal mapping $\mathbb{H} \to \mathbb{H}$ (one can explicitly write down the linear fractional transformation) such that $\Phi(0) = \infty$ and $\Phi(\infty) = A_n^*$, and then compose with F, after which everything should fall out nicely.

The specific map we want for Φ is $\Phi(z) = A_n^* - 1/z$, which has the matrix $\begin{pmatrix} A_n^* & -1 \\ 1 & 0 \end{pmatrix}$. ⊠

This proof, while in some ways less elegant than other proofs, manages to weave in a lot of what we've done over the last ten weeks.

One interesting consequence of this, which the textbook declines to mention, is that if one composes $F(z) = c_1 S(z) + c_2$ with any $\varphi \in \mathrm{Aut}(\mathbb{H})$, then the resulting function is of the form $c_1' S(z) + c_2'$, which is kind of nice.

We might also care about the reverse question: we know how to get from paths to paths, and if S is well-behaved, then the path is polygonal. Thus, we might wonder whether it's conformal. The answer turns out to be "yes," but the book doesn't really explain this. If one allows $\beta_k < 0$, though, this corresponds to making right turns rather than left turns when traveling counterclockwise around the boundary $\partial\mathfrak{p}$ (since the exterior angle is inverted), and suddenly we're allows to have nonconvex polygons. Here, the image of $\mathbb{R}$ may not be a simple curve, and it's quite difficult to tell in general whether it's simple or not: even a path with only 90° turns can self-intersect in nuanced and exotic ways, even if the total rotation is 2π. And sometimes these do end up simple; the lengths of the edges do matter, and these aren't necessarily easy to compute.

Assuming that the $\beta_k > 0$, then $\mathfrak{p}$ ends up convex, because all of the turns in the boundary are in the same direction, and there's no way to self-intersect, and with the conditions we imposed on S, the image of $\mathbb{R}$ is simple.

Theorem 18.1. *If the image of $\mathbb{R}$ under S is a simple, closed path, then $S : \mathbb{H} \to \mathfrak{p}$ is a conformal equivalence.*

Proof sketch. When the professor said the proof was an easy argument, apparently he meant it invokes the Argument Principle.

The idea is, suppose $w \in \mathfrak{p}$; then, I want to hit w exactly once. Then,

$$\frac{1}{2\pi i}\int_C \frac{\mathrm{d}\zeta}{F(\zeta) - w}$$

counts the number of solutions to $F(z) - w = 0$ inside $\mathbb{H}$. But one can take the image of the straight line of height ε above the real line, and this curve wraps around exactly once, so inside here there is exactly one solution.

Most textbooks contain a good proof of this, so it's unclear why this one doesn't. But it can be fleshed out into the most rigorous version without too much difficulty.