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Contents

1. A categorified version of the Fourier transform: 1/20/17

We've seen that for two-dimensional gauge theories with group G, there's a relationship with the Fourier transform for G. Today, we're going to talk about a categorified version of this, and in a few weeks we'll connect it to three-dimensional gauge theory.

Let's recall some facets of the Fourier transform. Let G be a locally compact abelian (LCA) group, and let $\widehat{G} = \mathrm{Hom}_{\mathsf{TopGrp}}(G, \mathrm{U}(1))$ be its Pontrjagin dual. This is a dual in that $\widehat{\widehat{G}} \cong G$.

The Fourier transform is an isomorphism $L^2(G) \xrightarrow{\cong} L^2(\widehat{G})$ sending pointwise multiplication to convolution and vice versa. There's a nice dictionary between the two sides:

- A representation of G is sent to a family of vector spaces on $\widehat{G}$.
- Finite groups are sent to finite groups.
- Lattices are sent to tori.
- A vector space is sent to its dual vector space.

Today, we're going to talk about Cartier duality, an algebraic analogue of this.

Let G be an algebraic group: this is the notion of a group in algebraic geometry just as Lie groups are the correct notion of groups in differential geometry. One can think of algebraic groups as functors from rings to groups; this is the functor-of-points perspective.

We have no analogue of $\mathrm{U}(1)$ in this setting, so we consider all characters $\chi : G \to \mathbb{G}_m = \mathrm{GL}_1$; the codomain is defined by the group of units functor $\mathsf{Ring} \to \mathsf{Grp}$ sending $R \mapsto R^\times$. As a scheme, this is $\mathbb{A}^1 \setminus 0$ or $\operatorname{Spec} k[x, x^{-1}]$.

The *Cartier dual* of G is $\widehat{G} = \mathrm{Hom}_{\mathsf{AlgGrp}}(G, \mathbb{G}_m)$. That is, for any ring R, $G(R) = \mathrm{Hom}_{\mathsf{Grp}}(G(R), R^\times)$. For "nice G," we'd like $G \cong \widehat{\widehat{G}}$. But what kinds of groups meet this condition?

G had better be abelian (since $\widehat{G}$ always is), and in fact we'll need it to be a *finite flat group scheme*. This idea might be new if you're used to thinking of algebraic geometry over $\mathbb{C}$, where these are exactly the finite abelian groups, but over other fields, it might be different.

Example 1.1. Let $G = \mathbb{Z}/n$. Then, its dual is $\widehat{\mathbb{Z}/n} = \mathrm{Hom}(\mathbb{Z}/n, \mathbb{G}_m)$, which can be identified with the group of n^{th} roots of unity, μ_n. Over $\mathbb{C}$, this is $\langle e^{2\pi i/n}\rangle$ and therefore identified with $\mathbb{Z}/n$, but over fields with characteristic dividing n, there are fewer n^{th} roots of unity. We're not going to worry too much about this. ◀

Akin to Pontrjagin duality, if we let $G = \mathbb{G}_m$, we get $\widehat{G} = \mathbb{Z}$, and if G is a torus, $\widehat{G}$ is the dual lattice in it.

For the Fourier transform, we want to look at vector spaces, e.g. the *additive group* $\mathbb{G}_a = \mathbb{A}^1$. We want to understand homomorphisms $\mathbb{G}_a \to \mathbb{G}_m$. We know that these would be given by $x \mapsto e^{xt}$, but this doesn't make sense unless t is nilpotent, so that the exponential

$$e^{xt} = \sum \frac{(xt)^n}{n!}$$

is a finite sum! That is, we want the dual of the x-line $\mathbb{G}_a$ to be the t-line, but we don't get very far along t. Since we don't know what order t is, we obtain the *formal completion*

$$\widehat{\mathbb{G}}_a = \varinjlim_n \operatorname{Spec} k[t]/(t^n),$$

heuristically a union of n^{th}-order thickenings of 0. Here, the hat is completion, not dual.

More generally, let V be a vector space. Then, its Cartier dual is the formal completion of the dual vector space: we want to take $e^{\langle v, v^*\rangle}$, but we need v^* to be nilpotent.

Alternatively, since Cartier duality is symmetric, the Cartier dual of the formal completion of the additive group is $\mathbb{G}_a$. That is, if x is nilpotent, e^{xt} makes sense for arbitrary t.

Since we're doing algebraic geometry, it's good to think of this in terms of functions. If G is a group, $\mathcal{O}(G)$ is not just a ring, but also has a *comultiplication* pulling functions back along multiplication: $\mu^* : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$. This makes $\mathcal{O}(G)$ into a *coalgebra*, and it's cocommutative iff G is commutative.

If G is finite, then you can dualize explicitly: $\mathcal{O}(G)$ is a finite-dimensional vector space, so $\mathcal{O}(G)^\vee$ has a convolution operator induced from the comultiplication. This is the same as convolution of distributions. In fact, it's possible to prove that the Cartier dual is $\widehat{G} = \operatorname{Spec}(\mathcal{O}(G)^\vee, *)$. Functions on $\widehat{G}$, with multiplication, are the same as distributions on G, with convolution. This is what we had in the analytic setting, albeit with a little more care to functions versus distributions.

A point of $\widehat{G}$ defines an algebraic function on G: it's a character $\chi : G \to \mathbb{G}_m$, so composing with the inclusion $\mathbb{G}_m \hookrightarrow \mathbb{A}^1$, we get a map $G \to \mathbb{A}^1$. We can assemble this into a diagram

$$\begin{tikzcd} & G \times \widehat{G} \arrow[dl] \arrow[dr] & \\ G & & \widehat{G}, \end{tikzcd}$$

and there's a tautological function on $G \times \widehat{G}$, which is evaluation: $(g, \chi) \mapsto \chi(g) \in \mathbb{A}^1$. This is akin to the exponential $(x, t) \mapsto e^{xt}$.

If G is infinite, you have to be more careful with topology. For example, $\mathcal{O}(\mathbb{G}_m) = k[x, x^{-1}]$, which sort of looks like the group algebra $k[\mathbb{Z}]$ over the integers, but there we have to restrict to finite expressions.

A sheaf-theoretic perspective. Rather than looking at functions, which don't behave very well in this context, let's look at sheaves.

There are three tensor categories associated to any group G.

(1) Since $R = \mathcal{O}(G)$ is a commutative ring, we can use $\mathsf{Mod}_{\mathcal{O}(G)}$ to generate the category $\mathsf{QC}(G)$ of quasicoherent sheaves on G.[1] The commutative tensor product $\otimes_R$ on Mod_R extends to a symmetric monoidal structure on $\mathsf{QC}(G)$. This does not require G to be a group.

(2) Since G is a group, $\mathcal{O}(G)$ is a bialgebra (actually a Hopf algebra), so $\mathsf{Mod}_{\mathcal{O}(G)}$ has a monoidal structure given by tensoring over the base field k rather than over R. That is, if M and N are $\mathcal{O}(G)$-modules, $M \otimes_k N$ has an $R \otimes R$-module structure, and then we can induce along the map $R \to R \otimes R$ to obtain an R-module structure.

[1] If G is an affine scheme, the categories are the same.

This monoidal structure is a convolution:

$$\begin{tikzcd} & G \times G \arrow[dl] \arrow[dr] \arrow[dd, "\mu"] & \\ G & & G. \\ & G & \end{tikzcd}$$

Here, we take M and N over G and realize them over $G \times G$ using the exterior product $M \boxtimes N$, and then pushforward along the multiplication map. This is the same category $\mathsf{QC}(G)$, but with a completely different structure, and this is one of the advantages of sheaves: instead of having to keep functions and distributions apart, sheaves can both pull back and push forward.

(3) The third approach is to take the category of representations of G, which can be tensored together. How can you say this geometrically? G-representations are $\mathcal{O}(G)$-comodules, vector spaces V with a coaction map $V \to V \otimes \mathcal{O}(G)$ satisfying *coassociativity*, i.e. that the following diagram is an equalizer diagram:

$$V \longrightarrow V \otimes \mathcal{O}(G) \rightrightarrows V \otimes \mathcal{O}(G) \otimes \mathcal{O}(G).$$

In a sense, this encodes the notion that representations are modules over the group algebra, but we don't have distributions, so the arrows go the other way. This is a symmetric monoidal category, where the tensor product has the coalgebra structure defined by composing the maps

$$V \otimes W \longrightarrow V \otimes W \otimes \mathcal{O}(G) \otimes \mathcal{O}(G)$$

and $\mathcal{O}(G) \otimes \mathcal{O}(G) \to \mathcal{O}(G)$.

This is not a category of quasicoherent sheaves on G; rather, it's $\mathsf{QC}(\bullet/G)$, where $\bullet/G$ is the classifying stack (or groupoid) of G. This comes from the pushout diagram $\bullet/G \leftarrow \bullet \leftleftarrows G$.

Cartier duality allows these categories to interact with each other. Namely, suppose G and $\widehat{G}$ are dual (so G is abelian, etc.). Then, Cartier duality establishes an equivalence of categories $\mathsf{Rep}_G \cong \mathsf{QC}(\widehat{G})$, and $\mathcal{O}(G)$-comodules become $\mathcal{O}(G)^\vee$-modules. This is just as in ordinary Pontrjagin duality: representations of G become families of functions on $\widehat{G}$.

(By the way, if you're holding out for examples, we'll soon see a whole bunch of them.)

In fact, the tensor structure is also in play: the duality is between the tensor product structure on Rep_G (or $\mathsf{QC}(\bullet/G)$) and the convolution structure on $\mathsf{QC}(\widehat{G})$.

We're going to abstract G away to a different duality operation $\mathsf{QC}(\mathcal{G}) \overset{\cong}{\to} \mathsf{QC}(\mathcal{G}^\vee)$. In our case, $\mathcal{G} = \bullet/G$ and $\mathcal{G}^\vee = \widehat{G}$. The classifying space $\bullet/G$ (also called BG) classifies G-bundles, and since G is abelian, you can tensor G-bundles. That is, if $\mathcal{P}_1$ and $\mathcal{P}_2$ are G-bundles, the relative tensor product $\mathcal{P}_1 \times_G \mathcal{P}_2$ is again a G-bundle, meaning $\bullet/G$ is an abelian group under the tensor product of G-bundles?

What does this actually mean? We're thinking of varieties (and generalizations such as stacks) as functors $\mathsf{Ring} \to \mathsf{Set}$; that $\bullet/G$ is an abelian group means that the assignment from a ring R to the (groupoid of) G-bundles on $\operatorname{Spec} R$ naturally factors through the category of abelian groups. That is, $\bullet/G$ is an abelian group object in the world of stacks.

Now, we define the *Fourier-Mukai dual* $\mathcal{G}^\vee = \operatorname{Hom}_{\mathsf{Grp}}(\mathcal{G}, B\mathbb{G}_m)$. Here $B\mathbb{G}_m$ classifies line bundles, so this is a version of the Picard group. However, since we've restricted to group homomorphisms, we only get what's known as multiplicative line bundles.

Definition 1.2. Let $\mathcal{L} \to G$ be a line bundle over a group G and $\mu : G \times G \to G$ be multiplication. If $\mu^*\mathcal{L} \cong \mathcal{L} \boxtimes \mathcal{L}$, then $\mathcal{L}$ is called a *multiplicative* line bundle.

The idea is that over $x, y \in G$, $\mathcal{L}_x \otimes \mathcal{L}_y \cong \mathcal{L}_{xy}$.

In a sense, we've shifted the Cartier duality operation: $(\bullet/G)^\vee = \operatorname{Hom}_{\mathsf{Grp}}(\bullet/G, \bullet/\mathbb{G}_m) = \operatorname{Hom}_{\mathsf{Grp}}(G, \mathbb{G}_m) = \widehat{G}$ as before. So why categorify? In this stacky version, instead of a universal function on $G \times \widehat{G}$, there's a universal line bundle $\mathcal{L} \to \mathcal{G} \times \mathcal{G}^\vee$:

$$\begin{tikzcd} & \mathcal{G} \times \mathcal{G}^\vee \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{G} & & \mathcal{G}^\vee. \end{tikzcd}$$

This bundle $\mathcal{L}$ is called the *Poincaré line bundle*. And it allows us to define a Fourier transform: given a sheaf $\mathcal{F}$on $\mathcal{G}$, we can pullback and pushforward to obtain $\pi_{2*}(\pi_1^*\mathcal{F} \otimes \mathcal{L}) \in \mathsf{QC}(\mathcal{G}^\vee)$. This actually defines an equivalence of categories, which is known as *Cartier duality* or *Laumon-Fourier-Mukai duality*.

Example 1.3. The most interesting example is where $\mathcal{G} = A$ is an abelian variety and $\mathcal{G}^\vee = A^\vee$ is the dual variety. Then, the integral transform with the Poincaré sheaf defines an equivalence of the derived categories $D(A) \cong D(A^\vee)$, which is the classical *Fourier-Mukai transform*. ◀

Example 1.4. We could also take $\mathcal{G} = \mathbb{G}_m$ and $\mathcal{G}^\vee = B\mathbb{Z}$. Then, this duality tells us that $\mathbb{Z}$-graded vector spaces are the same things as representations of $\mathbb{G}_m$. ◀

2. The Fourier-Mukai transform: 2/3/17

Today we're going to talk about the Fourier-Mukai transform, which is a categorical analogue of the Fourier transform.

Recall that if we have geometric spaces X and Y, an *integral transform* is a function $\Phi\colon \mathsf{Fun}(X) \to \mathsf{Fun}(Y)$ represented by a *kernel*, a function $K \in \mathsf{Fun}(X \times Y)$ such that Φ is defined by a pullback-pushforward

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y, \end{tikzcd}$$

in that $\Phi(f) = \pi_{2*}(\pi_1^* f \cdot K)$. The map $x \mapsto f_x(y) \coloneqq K(x,y)$ is $\Phi(\delta_x)$, so this can be thought of as a map $X \to \mathsf{Fun}Y$. If Φ is an isomorphism, then since $\{\delta_x\}$ is a basis for $\mathsf{Fun}X$, then $\{f_x\}$ is a basis for $\mathsf{Fun}Y$. These are the exponentials in the ordinary Fourier transform.

Now suppose X and Y are algebraic varieties, so integral transforms look like functors $\Phi\colon \mathsf{QC}(X) \to \mathsf{QC}(Y)$. If $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$, then $\Phi : \mathsf{Mod}_R \to \mathsf{Mod}_S$, and the Eilenberg-Watts theorem says that Φ must be tensoring with an (R, S)-bimodule ${}_RK_S$, which is the kernel. In particular, $K \in \mathsf{QC}(X \times Y) = \mathsf{Mod}_{R\otimes S}$. Thus, if M is a bimodule,

$$\Phi(M) = \pi_{2*}(\pi_1^* M \otimes_R K_S).$$

The map π_{2*} forgets the R-structure, hence is exact; if we want Φ to be exact, we must assume K is flat over R. In great generality, functors $\Phi\colon \mathsf{QC}(X) \to \mathsf{QC}(Y)$ are given by kernels $K \in \mathsf{QC}(X \times Y)$ satisfying a push-pull formula. However, if K isn't flat, $- \otimes K$ must be taken in a derived sense, and if X isn't affine, π_{2*} (global sections in the X-direction) isn't exact, and must again be taken in a derived sense, taking cohomology. Sometimes, functors like Φ are called *Fourier-Mukai functors*, but there's nothing particulary "Fourier" about them yet.

Suppose $x \in X$; we can identify it with the skyscraper sheaf $\mathcal{O}_x$ at x, which Φ maps to $\mathcal{F}_x \coloneqq \Phi(\mathcal{O}_x) \in \mathsf{QC}(Y)$, and $\mathcal{F}_x = K|_{\pi_1^{-1}(x)}$. This is an assignment of a sheaf on Y to every point in X, therefore defining a map from X to some moduli space of sheaves on Y. This map might not be interesting, but it is sometimes, and it always exists.

In fact, let's suppose $X = \mathcal{M}$ is a moduli space of sheaves on Y. There are natural transforms $\mathsf{QC}(\mathcal{M}) \to \mathsf{QC}(Y)$, e.g. the tautological construction whose kernel on $\pi_1^{-1}(x)$ is the sheaf defined by $x \in \mathcal{M}$. More concretely, let $X = \operatorname{Pic} Y$, the moduli space of line bundles. There's a canonical bundle $\mathcal{P} \to \operatorname{Pic} Y \times Y$ such that $\mathcal{P}|_{(\mathcal{L},y)} = \mathcal{L}|_y$, and this gives an interesting transform. (There are uninteresting transforms: the moduli space of skyscraper sheaves on Y is just Y itself, and the kernel is the identity matrix).

When is Φ an equivalence of categories, either in the usual or derived sense? The "orthonormal basis" $\mathcal{O}_x$ is mapped to $\mathcal{F}_x$. It's orthogonal in the sense that

$$\operatorname{Hom}(\mathcal{O}_x, \mathcal{O}_y) = \begin{cases} 0, & x \neq y \\ k, & x = y. \end{cases}$$

If $x \in X$ is smooth, the derived analogue is $\operatorname{Ext}(\mathcal{O}_x, \mathcal{O}_x) = \Lambda^\bullet T_x$. The "basis" part is that if $\mathcal{F}$ is coherent, $\operatorname{Hom}(\mathcal{F}, \mathcal{O}_x) = 0$ for all x iff $\mathcal{F} = 0$. So if Φ is to be an equivalence, we need $\operatorname{Hom}(\mathcal{F}_x, \mathcal{F}_y) = 0$ unless $x = y$, in which case you get the same algebra, and you need the same conditions: if $\mathcal{G}$ is coherent and $\operatorname{Hom}(\mathcal{G}, \mathcal{F}_x) = 0$ for all x, then $\mathcal{G} = 0$.

Let G be an abelian group (in schemes or in groupoids), and $Y = G^\vee = \operatorname{Pic}^\mu G$, the space of *multiplicative* line bundles on G. A line bundle $\mathcal{L}$ is multiplicative if there's a coherent isomorphism $\mathcal{L}_x \otimes \mathcal{L}_y \xrightarrow{\cong} \mathcal{L}_{x+y}$ (this is data, not a condition!), equivalent to an isomorphism $\mu^*\mathcal{L} \cong \mathcal{L} \boxtimes \mathcal{L}$, where $\mu : G \times G \to G$ is multiplication. $\operatorname{Pic}^\mu G$ can be identified with $\operatorname{Hom}_{\mathsf{Grp}}(G, B\mathbb{G}_m)$, where $B\mathbb{G}_m$ is the moduli of lines.

There's a tautological line bundle $\mathcal{P} \to G \times G^\vee$, which at $(g, \mathcal{L})$ is $\mathcal{L}_g$. This is a kernel, and hence defines a kernel transform.

Theorem 2.1 (Laumon-Fourier-Mukai)**.** *In many situations, this kernel transform is an equivalence, and exchanges tensor product with convolution.*

We'll see plenty of examples, making the "many situations" less vague, and these examples encompass some interesting dualities.

Example 2.2. Let $G = \mathbb{G}_m$. What's $\operatorname{Pic} \mathbb{G}_m$? There's only one line bundle, but it has a lot of automorphisms, so we get $\operatorname{Pic} \mathbb{G}_m = \bullet/\mathcal{O}^*(\mathbb{G}_m)$. The trivial bundle is multiplicative, but asking for automorphisms to preserve this structure rigidifies it: $G^\vee$ is $\bullet$ modulo the homomorphisms $\mathbb{G}_m \to \mathbb{G}_m$, i.e. the characters of $\mathbb{G}_m$. These are given by integers ($x \mapsto x^n$), so $G^\vee = \bullet/\mathbb{Z}$, also denoted $B\mathbb{Z}$.

A quasicoherent sheaf on $\mathbb{G}_m$ is equivalent data to a $\mathbb{C}[z, z^{-1}]$-module, hence the data of a vector space and an invertible map, which is the same thing as a $\mathbb{Z}$-representation, and $\mathsf{Rep}_\mathbb{Z} \cong \mathsf{QC}(B\mathbb{Z})$. This is the duality function; there's nothing derived going on here.

On $\mathsf{QC}(\mathbb{G}_m)$, the tensor product is the usual tensor product, and the convolution is $M * N \coloneqq M \otimes_\mathbb{C} N$, which is a $\mathbb{C}[z, z^{-1}]$-module via the coproduct map $\mathbb{C}[z, z^{-1}] \to \mathbb{C}[w, w^{-1}] \otimes \mathbb{C}[t, t^{-1}]$ sending $z \mapsto w \otimes t$. This is mapped to the tensor product on $\mathsf{Rep}_\mathbb{Z}$, and the tensor product is mapped to its convolution. A similar story can be told for any group.

This is an example of homological mirror symmetry! We think of $B\mathbb{Z}$ as $S^1 = K(\mathbb{Z}, 1)$ (in some suitable homotopical sense), and so a quasicoherent sheaf on $B\mathbb{Z}$ is the same thing as a locally constant sheaf (local system) on S^1: a $\mathbb{Z}$-representation is determined by what 1 does, and this is the monodromy as you go around S^1. Fukaya categories are meant to make this work: the wrapped Fukaya category attached to T^*S^1 is $\mathsf{QC}(B\mathbb{Z})$, the local systems on S^1.

Mirror symmetry says that the B-model on a space X should be equivalent to the A-model on the mirror $X^\vee$; the mirror of $\mathbb{C}^*$ is $\mathbb{C}^*$. The boundary conditions on the B-model encode $\mathsf{QC}(\mathbb{C}^*)$, and this should map to the Fukaya category of its mirror. Fukaya categories in general are nightmarish, but in this case everything is nice. ◀

Example 2.3. Suppose G is an algebraic torus, so a product of copies of $\mathbb{G}_m$: $G = (\mathbb{G}_m)^n$. Then, $G^\vee = B\Lambda$, where Λ is the character lattice $\Lambda \coloneqq \operatorname{Hom}_{\mathsf{Grp}}(T, \mathbb{G}_m)$. This can be identified with the dual of the compact torus $T_c^\vee \cong (S^1)^n = K(\Lambda, 1)$. Then, $\mathsf{QC}(T)$ is identified with the Fukaya category on the cotangent space of the compact torus. In some sense, this is the base case of mirror symmetry that people want to reduce everything down to. ◀

Example 2.4. Moving away from mirror symmetry, suppose $G = \mathbb{Z}$. Then, $G^\bullet$ is a point modulo the characters of G, so $\bullet/\mathbb{G}_m = B\mathbb{G}_m$. A sheaf on $\mathbb{Z}$ is a vector space for each integer, so a $\mathbb{Z}$-graded vector space, and a $\mathbb{Z}$-graded vector space is the same thing as a $\mathbb{C}^*$-representation! (The grading is given by the different eigenvalues.) You can generalize this: if G is a lattice, $G^\vee$ is the classifying space of the dual torus. ◀

Example 2.5. If $G = \mathbb{A}^1$, then $G^\vee$ is a point modulo the characters of $\mathbb{A}^1$; last time, we talked about how these are the formal completion of $\mathbb{A}^1$: $G^\vee = \bullet/\widehat{\mathbb{A}}^1$. A quasicoherent sheaf on $\mathbb{A}^1$ is the same thing as a $\mathbb{C}[x]$-module, which is equivalent to a vector space with an endomorphism, and this is the same as a representation of the Lie algebra $\mathbb{C}$. We want to exponentiate, but can only do so in a small neighborhood, so this is the same thing as a representation of the formal group $\widehat{\mathbb{A}}^1$.

More generally, if V is a vector space, $V^\vee = \bullet/\widehat{V}^*$. ◀

Example 2.6. Dually, if $G = \widehat{\mathbb{A}}^1$, then its characters are just $\mathbb{A}^1$ again, so $G^\vee = \bullet/\mathbb{A}^1$. A quasicoherent sheaf on $\widehat{\mathbb{A}}^1$ is a module over $\mathbb{C}[[x]]$, hence a vector space with a nilpotent endomorphism. A representation of the additive group is a representation of its Lie algebra, but we can exponentiate to any order, and therefore the action of the Lie algebra $\mathbb{C}$ must be nilpotent.[2] ◂

These examples are all tautological, in a sense; the following, due to Mukai is not.

Example 2.7. Let $G = A$ be an abelian variety, so it's a compact, connected abelian algebraic group (hence a torus $\mathbb{C}^n/\Lambda$). Let $A^\vee$ be the dual variety: literally the dual vector space modulo the dual torus. This is $\operatorname{Pic}^0 A$, the space of degree-0 line bundles trivialized at the identity. This is the same thing as multiplicative line bundles.

You can think of these not just as line bundles on A, but extensions of A: $A^\vee = \operatorname{Ext}^1_{\mathsf{Grp}}(A, \mathbb{G}_m) = \operatorname{Hom}(A, B\mathbb{G}_m)$: we have a fiber bundle $\mathbb{C}^* \to \mathcal{L}^\times \to A$, and we've identified $\mathcal{L}^\times|_{\mathrm{id}} \cong \mathbb{C}^*$, so what you have is an extension. There's a proof of this in Langlands' book, or Polishchuk's book on abelian varieties.

The Poincaré line bundle $\mathcal{P} \to A \times A^\vee$ applies as usual, but the pushforward in the kernel transform has to be derived:

$$\mathcal{F} \longmapsto \mathbf{R}\pi_{2*}(\pi_1^*\mathcal{F} \otimes \mathcal{P}).$$

This defines an equivalence of derived categories: $D(A) \overset{\cong}{\to} D(A^\vee)$, and was one of the first equivalences of derived categories that anyone considered. In fact, it was the first equivalence of derived categories between non-isomorphic varieties (with no stacky stuff).

More poetically, this says that any sheaf on A can be written as an "integral" of line bundles, or line bundles form a "basis" for sheaves on an abelian variety (as do skyscrapers). If you're interested in studying abelian varieties, this is very useful.

For example, if $A = \operatorname{Jac} C$, then it's canonically self-dual, and the transform is an interesting self-duality on $D(\operatorname{Jac} A)$. This is the space of degree 0 line bundles; alternatively, you can look at $\operatorname{Bun}^0_T C$, the space of degree-0 T-bundles on C (here T is a torus). In this case, the dual is $A^\vee = \operatorname{Bun}^0_{T^\vee} C$, the space of dual torus bundles. ◂

The geometric Langlands program is in some sense a fancy generalization of this example.

Example 2.8 (de Rham spaces)**.** We want to quotient $\mathbb{A}^1$ by a normal subgroup. There's not a lot of options, but we can choose the formal completion, and let $G = \mathbb{A}^1/\widehat{\mathbb{A}}^1$ (a sum of points very close to 0 stays close to 0, and everything is abelian). You can do this for any group G: let $\widehat{G}$ denote its formal completion at the identity. Then, translating by some $g \in G$, we get $\widehat{G} \cdot g = \widehat{G}_g$, the completion at g.

The quotient $G/\widehat{G}$ doesn't quite make sense as a variety, but you can define it as a functor $\mathsf{Ring} \to \mathsf{Grp}$, sending $R \mapsto G(R)/\widehat{G}(R)$. (Here, $\widehat{G}(R)$ is the group of maps $\operatorname{Spec} R \to G$ that send the reduced part of R to 1.) Consider the groupoid (equivalence relation) $\widehat{G \times G}|_\Delta$, meaning we've identified things that are arbitrarily close to the diagonal; then, modding out by this is the same thing as modding by $\widehat{G}$.

The advantage of this is that you don't need a group structure: for any space X, its *de Rham space* is $X_{\mathrm{dR}} \coloneqq X/\widehat{X \times X}|_\Delta$, so X modulo $x \sim y$ when x is arbitrary close to y. From a functor-of-points perspective, $X_{\mathrm{dR}}(R) \coloneqq X(R^{\mathrm{red}})$: "$X$ modulo calculus." For groups, this is particularly nice: $g, h \in G$ are close iff $h^{-1}g$ is very close to the identity.

Why does this get to be called the de Rham space? The functions are $\mathcal{O}(X_{\mathrm{dR}})$, the functions on X invariant under infinitesimal translation, so must have constant Taylor series. In other words, these functions are the kernel of the de Rham differential $\mathrm{d} : \mathcal{O}(X) \to \Omega^1$. And when you see this, you imagine the rest of the de Rham complex: the derived notion of functions on X_{dR} is the de Rham cohomology of X! So it's almost never representable, but it's still useful for studying de Rham cohomology. The functor $X \mapsto X_{\mathrm{dR}}$ is adjoint to taking reductions: $\operatorname{Hom}(S, X_{\mathrm{dR}}) = \operatorname{Hom}(S_{\mathrm{red}}, R)$. Gaitsgory calls it a "prestack," but there's nothing stacky, as we're quotienting by an equivalence relation.

Great, so what about the sheaves $\mathsf{QC}(X_{\mathrm{dR}})$? These are the sheaves on X where $\mathcal{F}_x \cong \mathcal{F}_y$ if x and y are infinitesimally close. That is, $\mathcal{F}$ is trivialized on formal neighborhoods of a point. This is equivalent to $\mathcal{F}$ being a *crystal* or $\mathcal{D}$-module, or a sheaf with a flat connection (at least in characteristic 0). The idea is this is a sheaf with some kind of locally constant csections, which vanish when you apply the connection $\nabla \colon \mathcal{F} \to \mathcal{F} \otimes \Omega^1$. ◂

[2]The passage between Lie algebras and formal groups requires some characteristic 0 properties, but a lot of this still works over other fields.

This could be considered a roundabout way to introduce $\mathcal{D}$-modules. Suppose $\mathcal{G} \rightrightarrows X$ is a groupoid acting on X. A $\mathcal{G}$-equivariant sheaf is a module for the groupoid algebra of distributions (or measures) on G. Functions on $\mathcal{G}$ form a coalgebra (just as for a group), and a $\mathcal{G}$-equivariant sheaf is a comodule for $\mathcal{O}(\mathcal{G})$. The functions on $\widehat{X \times X}|_\Delta$ is the jets of functions $\mathcal{J}$, functions vanishing to some order.

If you dualize over one of the factors of $X \times X$, the dual is

$$\mathcal{J}^* = \bigcup_n \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)^{I_\Delta^{n+1}},$$

where I_Δ is the *ideal of the diagonal*, generated by expressions of the form $f(x) - f(y)$ for $f \in \mathcal{O}(X)$. For $n = 0$, these are the functions $\varphi : \mathcal{O}_X \to \mathcal{O}_X$ that are $\mathcal{O}$-linear. For $n = 1$, we ask for $\varphi f - f\varphi$ to be $\mathcal{O}$-linear, which is Grothendieck's definition of a differential operator of order at most 1; in general, the n^{th} term is $\mathcal{D}_{\leq n}$, the differential operators of degree at most n. The expression $\varphi f - f\varphi$ is an abstract expression of the Leibniz rule. The *ring of differential operators*, denoted $\mathcal{D}$, is the groupoid algebra of the de Rham groupoid.[3]

Modules over $\mathcal{D}_X$ are what physicists call local operators: you can do whatever you want, as long as it only depends on the Talyor series (jet) at a point. And modules over $\mathcal{D}_X$ are identified with sheaves on X_{dR}. For example, this means integral transforms are disallowed. These sheaves are the input into crystalline cohomology; in characteristic p, where this is most useful, there are different notions of the de Rham groupoid. (Crystalline and de Rham cohomology are closely related, though there are complications in positive characteristic or over non-smooth spaces.) In fact, you can define de Rham cohomology with coefficients in a sheaf $\mathcal{F}$ to be

$$H_{\mathrm{dR}}(X; \mathcal{F}) \coloneqq \mathbf{R}\Gamma(X_{\mathrm{dR}}; \underline{\mathcal{F}}).$$

So the point of all this is, if you have a group G, then a $\mathcal{D}$-module on G is identified with a sheaf on G_{dR}, hence a $\widehat{G}$-equviariant sheaf on G, i.e. a sheaf on $G/\widehat{G}$.

This is what we were talking about earlier, sheaves on $G = \mathbb{A}^1/\widehat{\mathbb{A}}^1 = \mathbb{A}^1_{\mathrm{dR}}$ (or more generally using a vector space and its formal completion). We saw the duality sent $\mathbb{A}^1$ to $\bullet/\widehat{\mathbb{A}}^1$ and $\widehat{\mathbb{A}}^1$ to $\bullet/\mathbb{A}^1$, so this duality exchanges vector spaces and formal groups. if you blur your eyes a little bit, you get that $\mathbb{A}^1/\widehat{\mathbb{A}}^1$ is self-dual: $G^\vee = \mathbb{A}^1_{\mathrm{dR}}$.

If you have a vector space V, $V^*/\widehat{V}^* = V^*_{\mathrm{dR}}$. This is an example of the same Cartier duality.

Anyways, Fourier-Mukai duality defines an interesting automorphism $\mathbb{F}$ on $\mathsf{QC}(\mathbb{A}^1/\widehat{\mathbb{A}}^1)$, which is $\mathcal{D}_{\mathbb{A}^1}$-modules. And we know $\mathcal{D}_{\mathbb{A}^1} = \mathbb{C}\langle z, \partial_z\rangle/(\partial_z z - z\partial_z = 1)$. So the duality $\mathcal{D}_{\mathbb{A}^1} \to \mathcal{D}_{\mathbb{A}^1}$ sends $z \mapsto \partial_z$ and $\partial_z \mapsto z$, which does look nostalgically familiar.

In fact, it *is* the Fourier transform on $\mathbb{R}$. Let f be a (generalized) function on $\mathbb{R}$ (e.g. a tempered distribution). Then, f defines a $\mathcal{D}$-module $M_f = \mathcal{D} \cdot f$, the (left) action of all differential operators on f. Let $\widehat{f}$ denote the Fourier transform of f; then, the claim is that $\mathbb{F}(M_f) = M_{\widehat{f}}$, which is another way of expressing that the Fourier transform exchanges multiplication and differentiation.

If you set this up as a kernel transform, you get $M_{e^{xt}} \to \mathbb{A}^1_{\mathrm{dR}} \times \mathbb{A}^1_{\mathrm{dR}}$, the ideal generated by $\mathcal{D}_{\mathbb{A}^1 \times \mathbb{A}^1}/(\partial x - t, \partial_t - x)$, so x acts by differentiating t and ∂t acts by differentiating x (this ideal is a differential equation specifying this behavior, which is why we got e^{tx}), and $M_{e^{xt}}$ is a line bundle: $e^{\lambda z} \mapsto \mathcal{D}/\mathcal{D}(\partial_z - \lambda) \cong \mathbb{C}[z]$ as $\mathbb{C}[z]$-modules, so this is even a trivial line bundle! Of course, this is a very longwinded way to get the usual Fourier transform, but once you say it this way, you have a whole lot of generalizations.

Example 2.9. We won't need this example, but it's cool. Consider $\mathbb{A}^1/\mathbb{Z}$, where $\mathbb{Z}$ acts by shifting. Then, the dual replaces $\mathbb{Z}$ with $\mathbb{G}_m$ and $\mathbb{A}^1$ with $\widehat{\mathbb{A}}^1$: the dual is $\mathbb{G}_m/\widehat{\mathbb{A}}^1 = (\mathbb{G}_m)_{\mathrm{dR}}$. The ring controlling difference equations on $\mathbb{A}^1/\mathbb{Z}$ is $\mathbb{C}[t]\langle\sigma, \sigma^{-1}\rangle$, and the ring controlling differential equations on $\mathbb{G}_m/\widehat{\mathbb{A}}^1$ is $\mathbb{C}[z, z^{-1}]/\langle z\partial_z\rangle$, and these two rings are isomorphic. In this context, the transform is called the *Mellin transform.* $\blacktriangleleft$

3. Representations of categories and 3D TFTs: 2:10/17

Today, we're going to talk about topological field theories in three dimensions. Recall that an n-dimensional TFT assigns a number to an n-manifold and a vector space to an $(n-1)$-manifold. Then, it should assign a (linear) category to an $(n-2)$-manifold, but this is complicated, so often we specialize to assigning algebras up to Morita equivalence, which is same as a linear category with only one object.

[3]This definition is due to Grothendieck, but was worded differently (and not just because it was in French.)

We need one more piece of information, which is what to attach in codimension 3. This should be some sort of 2-category; again, it would be easier to think about a 2-category with one object: one object, some morphisms, and some morphisms between the morphisms. Since you can compose 1-morphisms, this isn't exactly the same as a 1-category; instead, composition defines a *monoidal structure* on it, making it a *monoidal (linear) category*. That is, we have a functor $\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ and a unit $\mathbf{1}$ that is associative and unital up to natural isomorphism.

Definition 3.1. Let C be a monoidal category. A category M is an C-*module* if it has an *action map* $\mu : \mathsf{C} \otimes \mathsf{M} \to \mathsf{M}$, together with data expressing its compatibility with $\otimes$ and $\mathbf{1}$, etc.

A 2-category with a single object and a single 1-morphism looks like an algebra, but it's an algebra in algebras of vector spaces, meaning it has the coherence structure of an $\mathbb{E}_2$-algebra.

Anyways, we're working with monoidal linear categories, and we'd like to use these to define TFTs. Recall that if we have a finite groupoid $\mathcal{X} = [\mathcal{G} \rightrightarrows X]$, we can define a $3-2-1$ field theory.

- If M is a 3-manifold,
$$Z(M) = \#[M, \mathcal{X}] = \sum_{x \in \pi_0([M,\mathcal{X}])} \# \operatorname{Loc}_{\pi_1([M,\mathcal{X}],x)} M$$
counts the number of local systems in $\mathcal{X}$ on M, weighted in the groupoid cardinality.
- If N is a 2-manifold,
$$Z(N) = \mathbb{C}[[N, \mathcal{X}]] = \bigoplus_{x \in \pi_0([N,\mathcal{X}])} \mathbb{C}[\operatorname{Loc}_{\pi_1([N,\mathcal{X}],x)} N],$$
the functions on the groupoid of local systems in $\mathcal{X}$ on N.
- If P is a 1-manifold, we attach the category
$$Z(P) = \mathsf{Vect}([P, \mathcal{X}]) = \bigoplus_{x \in \pi_0([P,\mathcal{X}])} \mathsf{Vect}(\bullet/\pi_1([P,\mathcal{X}], x)) = \bigoplus_{x \in \pi_0([P,\mathcal{X}])} \mathsf{Rep}_{\pi_1(\mathcal{X},x)}.$$

In all of these examples, $[X, \mathcal{G}]$ is the groupoid of maps $\pi_{\leq 1} X \to \mathcal{G}$.

If $\mathcal{Y} \to \mathcal{X}$ is a map of finite groupoids, then we get a map $\pi : [N, \mathcal{Y}] \to [N, \mathcal{X}]$ which defines a map $Z_{\mathcal{Y}}(N) \to Z_{\mathcal{X}}(N)$ sending $1 \mapsto \pi_* 1$, and similarly a map $Z_{\mathcal{Y}}(P) \to Z_{\mathcal{X}}(P)$ sending $\underline{\mathbb{C}} \mapsto \pi_* \underline{\mathbb{C}}$.

The prototypical example is $\mathcal{X} = \bullet/G$, for which this defines (untwisted) Dijkgraaf-Witten theory. The space of functions attached to a surface N is defined by a character variety:
$$[N, \mathcal{X}] = \operatorname{Loc}_G N = \operatorname{Hom}_{\mathsf{Grp}}(\pi_1(N), G)/G.$$
On the circle, we obtain the category
$$[S^1, \mathcal{X}] = \operatorname{Loc}_G S^1 = G/G.$$
If Y is a G-set and $\mathcal{Y} = Y/G$, then the projection map $Y \to \bullet$ is G-equivariant, defining a groupoid homomorphism $\mathcal{Y} \to \bullet/G$. The induced map $[S^1, \mathcal{Y}] \to [S^1, \mathcal{X}]$ is the map sending
$$\{g \in G, y \in Y^g\}/G \to G/G = \coprod_{[g]} \bullet/Z_G(g),$$
so the trivial bundle is sent to a vector bundle on G/G whose fiber over $g \in G$ is a $\mathbb{C}[Y^g]$ as a $Z_G(g)$-representation.

For the torus T, $\operatorname{Loc}_G T = [G, G] = \{g, h \in G \mid gh = hg\}/G$, so given a G-set Y and $\mathcal{Y} = Y/G$ as before, the map $[T, \mathcal{Y}] \to [T, \bullet/G]$ solves a counting problem $g, h \mapsto \# Y^{g,h}$.

Before getting too categorical,[4] the algebra $Z(T) = \mathbb{C}[[G, G]]$ is what's known as a *fusion algebra*. It has a lot of structure; for example, $\mathrm{MCG}(T) = \mathrm{SL}_2(\mathbb{Z})$ acts on it through its action on T. There's also a Frobenius algebra structure hiding inside it: if Z_{S^1} is the *dimensional reduction* of Z by S^1, i.e. $Z_{S^1}(X) = Z(S^1 \times X)$ for all X, then $Z(T) = Z_{S^1}(S^1)$. Since Z_{S^1} is a 2D oriented TFT, then $Z_{S^1}(S^1)$ is a commutative Frobenius algebra. However, we can't state this in an invariant way: you have to break symmetry and choose an isomorphism $T \cong S^1 \times S^1$, in effect choosing coordinates, and therefore the Frobenius algebra structure is absolutely not $\mathrm{SL}_2(\mathbb{Z})$-invariant.

[4] Is this the same as 2-categorical?

This lack of invariance is actually pretty interesting, and is the genesis of Lusztig's Fourier transform. There is a convolution structure on $Z(T^2) = \mathbb{C}[G,G]$, because

$$\mathbb{C}[G,G] = \bigoplus_{[g,h]=1} \mathbb{C}_{g,h} = \bigoplus_{[h]\in G/G} \mathbb{C}[Z_G(h)/Z_G(h)],$$

with the usual convolution structure on each $\mathbb{C}[Z_G(h)/Z_G(h)]$ (since it's the class functions for a finite group), and no convolution relation between different components. In other words, this is a direct sum over the conjugacy classes (components of G/G).

In physics, this is called *diagonalizing the fusion rules*: we started with a basis for this algebra, and obtained a ring structure where the multiplication is "diagonalized," i.e. only interesting within each conjugacy class. The matrix transitioning between the standard basis and this new basis is called the *S-matrix*, or the action of $\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ in either basis.

In other words, for every commutative Frobenius algebra structure, you get a basis of its idempotents, and expressing this basis in the natural basis for the Fourier transform gives you an interesting matrix. This theory shows up in the representation theory of finite groups.

Now let's return to extended TFT and try to determine what Z attaches to a point. We've gone from numbers involving $\mathcal{X}$, to functions on $\mathcal{X}$, to vector bundles on $\mathcal{X}$, so the next step should be categories on $\mathcal{X}$, and for any map $\mathcal{Y} \to \mathcal{X}$, the boundary condition should be $\mathsf{Vect}\mathcal{Y}$ as a category over $\mathcal{X}$.

Example 3.2. If $\mathcal{X} = X$ is a finite set (a discrete groupoid), then a category over X is a category $\mathcal{M}_x$ over each $x \in X$. You can think of this as a parent $\mathbb{C}$-linear category $\mathcal{M} = \bigoplus_{x\in X} \mathcal{M}_x$ together with its direct-sum decomposition.

We want to define this *a priori*, so let's look for analogies one level down. There's an equivalence $\mathsf{Vect}X$ with $\mathsf{Mod}_{\mathbb{C}[X]}$, so we can define $\mathbb{C}[X]$-modules "by hand" as a vector space over each point in X. Now, $\mathcal{M}$ is a $(\mathsf{Vect}X, \otimes)$-module, in the sense of a module of a symmetric monoidal category defined earlier. Moreover, the decomposition as a category over each $x \in X$ comes from the action of the "orthogonal idempotents" in $\mathsf{Vect}X$, which are the skyscraper sheaves $\underline{\mathbb{C}}_x$ for each $x \in X$: $\underline{\mathbb{C}}_x \otimes \underline{\mathbb{C}}_y = 0$ unless $x = y$, in which case it's just $\underline{\mathbb{C}}_x$ again. This action gives back the direct-sum decomposition of $\mathcal{M}$. ◂

Example 3.3. Affine schemes provide a more interesting example, so let $X = \operatorname{Spec} R$ for a ring R. Now, a category over X should be a module category for the monoidal category $(\mathsf{QC}(X), \otimes) = (\mathsf{Mod}_R, \otimes)$. In particular, if $\mathcal{M}$ is such a $\mathsf{QC}(X)$-module category and $M \in \mathcal{M}$, the functor $\mathcal{O}_X * -$ is isomorphic to the identity functor, but it has endomorphisms equal to $\operatorname{End}_R(R) = R$. In particular, R acts on $\mathrm{id}_{\mathcal{M}}$, so we have maps $R \to \operatorname{End} M$ for each $M \in \mathcal{M}$, and this is functorial.[5] That is, $\mathcal{M}$ is an R-linear category: all the hom-spaces are R-modules, and composition is R-linear.

In particular, you can localize objects and morphisms over $X = \operatorname{Spec} R$: for every open subset $U \subset X$, we can define $\mathcal{M}_U = \mathcal{M} \otimes_{\mathsf{Mod}_R} \mathsf{Mod}_{\mathcal{O}_X(U)}$. That is, it has the same objects as $\mathcal{M}$, but its hom-sets are

$$\operatorname{Hom}_{\mathcal{M}_U}(M,N) = \operatorname{Hom}_{\mathcal{M}}(M,N) \otimes_R \mathcal{O}_X(U).$$

These glue, so you end up with a *quasicoherent sheaf of categories* on X. Since X is affine, this is a sheaf of categories coming from a $\mathsf{QC}(X)$-module.[6]

For example, if $\pi : Y \to X = \operatorname{Spec} R$ is a map of affine schemes, then $\mathsf{QC}(Y)$ is a $\mathsf{QC}(X)$-module category, or equivalently is R-linear. Then, the assignment

$$(U \subset X) \longmapsto \mathsf{QC}(\pi^{-1}(U))$$

is a quasicoherent sheaf of categories on X. ◂

This is an excuse to introduce an awesome theorem.

[5]Here, R is commutative, so this is all good, but in the derived setting, we need this to be an $\mathbb{E}_2$-map. Similarly, $Z(\mathcal{M})$ will be an $\mathbb{E}_2$-algebra in the derived setting.

[6]Even though we have "the same objects," the isomorphism classes may be different: objects can become isomorphic.

Theorem 3.4 (Gaitsgory's 1-affineness theorem)**.** *Let X be a scheme.*[7] *Let* $\mathsf{ShvCat}(X)$ *denote the 2-category of quasicoherent sheaves of categories on X; then, there is an equivalence of 2-categories*[8]

$$\mathsf{ShvCat}(X) \cong \mathsf{Mod}_{\mathsf{QC}(X)}.$$

This is a hard theorem to prove.

What's cool about this is that one category level lower, this is not true: we're used to $\mathsf{QC}(X) = \mathsf{Mod}_{\Gamma(\mathcal{O}_X)}$ only in the case when X is affine; it's far from true in general. But for sheaves of categories, the algebra is more flexible, and the relationship between sheaves and modules is nicer.

The 1 in 1-affineness is a category number: 0-affine is the same as ordinary affine. You can define n-affineness for higher n, and if you're n-affine, then you're $(n+1)$-affine. There are many examples of 1-affine schemes that aren't 0-affine; there are examples of 2-affine schemes that aren't 1-affine, but this is harder.

Anyways, if X is a finite set or a nice scheme, we have the monoidal category $\mathsf{Vect}X$ or $\mathsf{QC}(X)$ with $\otimes$, and this defines a 3D TFT which to a point attaches $(\mathsf{QC}(X), \otimes)$, or equivalently the 2-category of $\mathsf{QC}(X)$-modules, or by Theorem 3.4, the 2-category $\mathsf{ShvCat}(X)$.

Definition 3.5. This theory is called the *Rozansky-Witten theory* of T^*X.

This is an example of a $3D$ σ-model, the theory of maps into X. (Usually, this is defined for the symplectic manifold T^*X).

3D gauge theories. The other kind of 3D theories we'll talk about are gauge theories, which are theories of bundles. In this case, the groupoid is $\mathcal{X} = \bullet/G$, and we know in codimension ≤ 2 we attach numbers/functions/categories built out of local systems. Now we want some kind of categories over $\bullet/G$, which we'll call *categorical representations of* G or G*-categories.*

Before we define these explicitly, let's think about the examples we want: if Y is a G-space and $\mathcal{Y} = Y/G$, then we get a map $\mathcal{Y} \to \mathcal{X}$. We'd like $\mathsf{Vect}(\mathcal{Y})$ to be a G-category: for every $g \in G$, we should obtain a functor $f^* : \mathsf{Vect}\mathcal{Y} \to \mathsf{Vect}\mathcal{Y}$ such that $g^*h^* \cong (hg)^*$, and these isomorphisms will satisfy some associativity coherence condition.

You can write down what this means exactly: it is a map $G \to \mathrm{Aut}(\mathsf{Vect}Y)$, where this is up to some kind of natural isomorphism. This is a notion of a G-category, though not the most flexible. Just like a representation of a group G on a vector space V is the same data as a $\mathbb{C}[G]$-module action on V, a group G acting on a $\mathbb{C}$-linear category $\mathcal{M}$ is the same data as a group action plus "scalar multiplication" (tensoring with $\mathbb{C}$-vector spaces). Thus, a G-category $\mathcal{M}$ is the same data as a module category for the monoidal category $\mathsf{Vect}G$, where the monoidal product is the convolution product

$$\begin{tikzcd} & G \times G \arrow[dl] \arrow[dr] \arrow[dd, "\mu"] & \\ G & & G \\ & G. & \end{tikzcd}$$

That is, $\mathcal{F} * \mathcal{H} = \mu_*(\mathcal{F} \boxtimes \mathcal{H})$. This is just like the convolution structure on the group algebra $\mathbb{C}[G]$, but one category level higher.

For example, with $\mathcal{Y} = Y/G$ as before, G acts on Y through an action map a, and the induced map of $\mathsf{Vect}G$ on $\mathsf{Vect}Y$ is the push-pull map associated to

$$\begin{tikzcd} & G \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, "a"] & \\ G & & Y \\ & X, & \end{tikzcd}$$

i.e. $\mathcal{F} * \mathcal{H} = a_*(\pi_1^*\mathcal{F} * \pi_2^*\mathcal{H})$.

[7] This holds in much greater generality, including even suitably nice derived stacks, where "suitably nice" means anything you might reasonably run into on the street, specifically a quasicompact stack with an affine diagonal.

[8] You might need to restrict to only invertible natural transformations.

This is good, because it means that you can do this for any group and kind of sheaves where these push-pull diagrams make sense. For example, suppose G is an affine algebraic group. Then, we can push-pull with quasicoherent sheaves, using the Hopf algebra structure on $\mathcal{O}(G)$. Here, G-categories are again modules over $(\mathsf{QC}(G), *)$, where $*$ is the same convolution. If G acts on an algebra A, then Mod_A is a G-category, where a $g \in G$ acts by $M \mapsto M^g$, the A-module with the action twisted by g. When $A = \mathcal{U}(\mathfrak{g})$, the enveloping algebra of a Lie algebra, this has interesting echoes in representation theory, in particular defining a G-category structure on $\mathsf{Mod}_{\mathfrak{g}}$.[9]

From this perspective, you could try to develop an analogue of representation theory from scratch. One of the first things you do is understand the equivalence between an abelian group and its Pontrjagin dual; in this context, if G is abelian, we discussed a Fourier-Mukai transform $(\mathsf{QC}(G), *) \overset{\cong}{\leftrightarrow} (\mathsf{QC}(G^\vee), \otimes)$. In other words, G-categories are exchanged with categories over $G^\vee$, where the monoidal structure is pointwise. Thus, the Fourier-Mukai transform exchanges gauge theories (for G-categories) and σ-models (for categories over $G^\vee$).

Example 3.6 (Teleman)**.** Let $G = B\mathbb{Z} = S^1$ (here, we're thinking of S^1 as only a homotopy type; no manifold structure will come into play). Then, $\mathsf{QC}(G) = \operatorname{Loc} S^1 = \mathsf{Rep}_{\mathbb{Z}}$. Local systems are our notion of sheaves, and the push-pull diagrams exist, so this is some sort of "locally-constant group algebra for S^1." A $\operatorname{Loc} S^1$-module category is the same thing as a category $\mathcal{M}$ with a locally constant action of S^1.

That is, we have an endofunctor of $\mathcal{M}$ for every $z \in S^1$, such that a homotopy $z_1 \to z_2$ defines an identification of the functors for z_1 and z_2. Alternatively, this is an action of S^1 trivial on its contractible subsets. These do appear in nature: for example, if G acts on a space X, then $H_*(G)$ acts on $H_*(X)$ is locally constant. The same is true for the action of G on local systems on X: if you move elements of G a little bit, it won't affect the action. Thus, it defines an action of $\operatorname{Loc} G$ on $\operatorname{Loc} X$.

Anyways, there's an example in mirror symmetry: instead of local systems on X, let's consider the wrapped Fukaya category on a symplectic manifold, $\mathsf{Fuk}_{\mathrm{wr}}(T^*X)$. If $Y = T^*X$ is a Hamiltonian G-space, then $\operatorname{Loc}(G)$ acts on $\mathsf{Fuk}_{\mathrm{wr}}(Y)$. This means that the boundary conditions for our 3D gauge theory are A-models for $G = S^1$. That is, we have a 3D theory whose boundary conditions are A-models; this is called a *3D $\mathcal{N} = 4$ (A-twisted) supersymmetric Yang-Mills* theory.

The Fourier-Mukai transform exchanges S^1-categories (i.e. $\mathsf{Mod}_{\mathsf{QC}(B\mathbb{Z})}$) and categories over $\mathbb{C}^\times$ (i.e. $\mathsf{Mod}_{(\mathsf{QC}(\mathbb{C}^\times), \otimes)}$). That is, it exchanges A-models for S^1-spaces X and B-modules of spaces $X^\vee \to \mathbb{C}^\times$, where $X^\vee$ is the mirror dual to X. Mirror symmetry is defined so the A-model on X should be the B-model on $X^\vee$, meaning we want $\mathsf{Fuk}_{\mathrm{wr}}(X) \cong \mathsf{QC}(X^\vee)$. The left-hand side is complicated, with a G-action, but the right-hand side is simpler, with just maps into $\mathbb{C}^\times$. You can use this to actually build mirrors to some spaces in a process called Hamiltonian reduction. In the other direction, if X has a holomorphic $\mathbb{C}^\times$-action, then its mirror comes with a map to S^1.

The physicists, of course, say the same thing using different words: they say that U(1) gauge fields in 3D are dual to a scalar. That is, we start with a U(1)-gauge field (the left-hand side, or the A-model), and we ended up with a σ-model to $\mathbb{C}^\times$, which is a scalar field (a function). The physics derivation starts with a U(1)-gauge field and a connection $\mathrm{d} + A$ (so A is an endomorphism-valued 1-form), and the field strength is the 2-form $\mathrm{d}A$. Then, the Hodge star establishes a duality between $\mathrm{d}A$ and $\star(\mathrm{d}A) = \mathrm{d}\varphi$, an exact 1-form, and φ is the scalar field in question.

In four dimensions, the Hodge star exchanges 2-forms and 2-forms, hence exchanges gauge theories and gauge theories. This is electric-magnetic duality, and will lead us to the geometric Langlands program. ◀

This was just the abelian case — the nonabelian case is also very interesting, though harder. Gauge transformations mean that there's no guarantee that the Hodge star acts in a gauge-invariant way, unlike in abelian theories.

In two dimensions, the dual of a 2-form is a function, which isn't exact, and in physics words, this means that in 2D, the U(1)-gauge field has no dynamical degrees of freedom. This relates to the fact that in 2D, the moduli space of vacua $\widehat{G}$ (where G is compact) is discrete, so we're looking at vector bundles on a discrete set, and there's no fields on it.

Let's think of the trivial representation of a group G: geometrically, this is the action on functions on a point with the trivial G-action, so we get an action of $\mathsf{QC}(G)$ on Vect, which is the categorified notion of the

[9]Modules over $\mathsf{Vect}_{\mathbb{C}}$ are sometimes called 2-*vector spaces.* If G is continuous, then to be precise we should be talking about dg categories.

trivial representation. This is good, because it means we can define invariants: if $\mathcal{M}$ is a $\mathsf{QC}(G)$-category, its category of *invariants* is

$$\mathcal{M}^G = \mathrm{Hom}_{\mathsf{QC}(G)}(\mathsf{Vect}, \mathcal{M}).$$

Since G acts trivially on Vect, $\mathrm{Hom}_{\mathsf{QC}(G)}(\mathsf{Vect}, \mathcal{M})$ has the objects $M \in \mathcal{M}$ with actions $G * M \overset{\cong}{\to} M$. That is, these are the *equivariant objects* in $\mathcal{M}$. For example, if $\mathcal{M} = \mathsf{Vect}Y$ for a G-space Y, then $\mathcal{M}^G$ is the category of equivariant vector bundles on Y. In general, $\mathcal{M}^G$ is acted on by Rep_G: for example, the map $Y/G \to \bullet/G$ pulls back to a action map $\mathsf{Vect}(\bullet/G) \to \mathsf{Vect}(Y/G)$, inducing the action on $(\mathsf{Vect}Y)^G$.

This is something you don't see one level down: if G acts on a vector space V, then nothing acts on the invariants V^G; it's just a vector space. Well, functions on $\bullet/G$ act on V^G, but this isn't very interesting, and in particular, you can't usually recover a representation from its invariants. One category level higher,

$$(-)^G : \mathsf{Mod}_{\mathsf{QC}(G)} \longrightarrow \mathsf{Mod}_{\mathsf{Rep}_G, \otimes}.$$

is much nicer, and if G is affine,[10] Theorem 3.4 for $\bullet/G$ implies $(-)^G$ is an equivalence! This is because $\mathsf{Mod}_{\mathsf{QC}(G)}$ is $\mathsf{ShvCat}(\bullet/G)$ and $\mathsf{Mod}_{(\mathsf{Rep}_G,\otimes)}$ is $\mathsf{Mod}_{\mathsf{QC}(\bullet/G)}$, and Gaitsgory's theorem says these two are the same. Thus, $(\mathsf{QC}(G), *)$ and $(\mathsf{Rep}_G, \otimes)$ are *Morita equivalent monoidal categories*, meaning that they have the same module categories. You can prove this directly for finite G, where it's already very interesting. This is one of the great things about categorical representation theory: the trivial representation already sees everything!

4. Rozansky-Witten theory: 2/17/17

Today we're going to review the structures we found on 2D theories and discover their analogues in 3D TFT.

Let Z be an oriented 2D TFT. Then, we thought of $Z(S^1)$ in two ways: as a center or operators, coming from the annulus, and as the trace or states, coming from the cylinder. $Z(S^1)$ has a multiplication, and is "commutative" (i.e. E_2), and is the center of the algebra or category $Z(\bullet)$. Moreover, maps $Z(S^1) \to \mathrm{End}(\mathrm{id}_{\mathsf{C}})$ correspond to boundary conditions marked by the category C, and $\mathrm{End}(\mathrm{id}_{\mathsf{C}})$ maps to $\mathrm{End}(c)$ for any object $c \in \mathsf{C}$. This comes from the "eye cobordism" from the identity (a line segment) to the identity (a line segment) containing a single hole.[11]

For example, in Dijkgraaf-Witten theory, $Z(\bullet) = \mathbb{C}G$, and $Z(S^1) = \mathbb{C}[G/G]$, the algebra of class functions, which is the center of $\mathbb{C}G$. For the A-model, if $X = \operatorname{Spec} A$ is an affine scheme, $Z(S^1) = HH^*(A, A) = \mathrm{Ext}_{A\otimes A}(A, A) = \Lambda^\bullet T_X$.

Dually, the cylinder gives us the trace or Hochschild homology of the algebra or category assigned to a point. This is the home for characters: a $B \in Z(\bullet)$ is mapped to $\chi_B \in Z(S^1)$: for Dijkgraaf-Witten theory, this takes a representation V or G and produces its character χ_V. If X is a smooth variety, the trace is the Hochschild homology, isomorphic to $\Omega^\bullet$ with the de Rham differential; if $\mathcal{F} \in D^b(X)$, then we obtain its *Chern character* $\chi_{\mathcal{F}}$.

The action of the circle is not what you expect: it's a topological action, where two homotopic points define the same action. For example, suppose S^1 acts on a Riemannian manifold M. Then, topological quantum mechanics on M has for its Hilbert space $\mathcal{H} = (\Omega_M^\bullet, \mathrm{d})$, but the action of S^1 on its cohomology $H^*_{\mathrm{dR}}(M)$ is trivial.

Another way to think of this, which is more geometric, is that if ξ is a vector field on M, then its Lie derivative $\mathcal{L}_\xi$ is an operator on $\Omega_M^\bullet$, and the *Cartan homotopy formula* says that if ι_ξ is contraction with ξ, then $\mathcal{L}_\xi = [\mathrm{d}, \iota_\xi]$. That is, this presents a chain homotopy from $\mathcal{L}$ to 0, so the Lie derivative is trivial on de Rham cohomology!

A *Lie algebra action* of a Lie algebra $\mathfrak{g}$ on a manifold M is an action through vector fields, i.e. a Lie algebra homomorphism $\mathfrak{g} \to \mathcal{X}(M)$. From a physics perspective, such an action defines an action of the Lie superalgebra $\mathfrak{g} \oplus \mathfrak{g}[1]$, where $(\xi, 0)$ acts by $\mathcal{L}_\xi$ and $(0, \xi)$ acts by ι_ξ, and these supercommute.

[10] This is definitely not true when G isn't affine. For example, it fails for $\mathbb{Z}$.

[11] Boundary conditions are a special case of domain walls: consider a TQFT on bipartite manifolds, partitioned into two pieces by a codimension-1 submanifold, with bordisms respecting this structure. Then, one can set up one field theory on one side and another field theory on the other side. If the left side is empty, you're left with a boundary condition, which in some sense is an object of C on the codimension-1 submanifold. This can be used to derive the Frobenius character formula, where the bulk theory is Dijkgraaf-Witten theory and the boundary theory is maps to $\mathcal{Y} = B\backslash G/B$. It can also be used to prove the Atiyah-Bott fixed-point formula in a similar way.

Anyways, the action of S^1 on M defines an action of its chains on those of M, and this passes to cohomology, inducing an action of $H^*(S^1)$ on $H^*(M)$. That is, we get an action of $\mathbb{R}[\eta]/(\eta^2)$ on $H^*(M)$, given by an operator η of degree -1 which squares to 0, just like the de Rham differential. This is what we mean by "an action of S^1:" by the time we pass to TFT, all that's left is this homological action, a shadow of the original action. In some sense, η is the generator of the Lie algebra for S^1.

Now we'll lift this structure into three dimensions. The example to keep in mind is untwisted Dijkgraaf-Witten theory: fix a finite group G and let $Z(\bullet)$ be the category of $\mathsf{C} := (\mathsf{Vect}G, *)$-modules. Then, $S^1 \mapsto \mathsf{Vect}(G/G) = \mathsf{Vect}(\mathrm{Loc}_G(S^1))$. Again, $Z(S^1)$ has two roles, as a center and as a trace.

Definition 4.1. Let $(\mathsf{C}, *)$ be a monoidal category. Then, its *Drinfeld center* is the subcategory $Z(\mathsf{C})$ whose objects are pairs (F, ψ) where $\psi : F * - \xrightarrow{\cong} - * F$ is a natural isomorphism, and whose morphisms are the morphisms intertwining these ψ. In particular, if $\overline{\mathsf{C}}$ denotes the same category with the opposite monoidal structure, then $Z(\mathsf{C}) = \mathrm{End}_{\mathsf{C}\otimes\overline{\mathsf{C}}}(\mathsf{C}) = \mathrm{End}(\mathrm{id}_{\mathsf{Mod}_\mathsf{C}})$.

This is the analogus of Hochschild cohomology one dimension down, and has a similar-looking definition. In particular, if Z is the field theory sending $\bullet \mapsto (\mathsf{C}, *)$, then the Drinfeld center is $Z(S^1)$.

Example 4.2. Let $\mathsf{C} = (\mathsf{Vect}G, *)$, where G is a finite group, and suppose F is central. Then, $F * \mathbb{C}_g \xrightarrow{\cong} \mathbb{C}_g * F$, and since $\mathbb{C}_g$ is invertible with inverse $\mathbb{C}_{g^{-1}}$, then $F \xrightarrow{\cong} \mathbb{C}_g * F * \mathbb{C}_{g^{-1}}$. That is, $F \in (\mathsf{Vect}G)^G$, where G acts by conjugation, and this is $\mathsf{Vect}G/G$, which is indeed $Z(S^1)$. ◂

Exercise 4.3. Dually, if $\mathsf{C} = (\mathsf{Rep}_G, \bullet)$, show that every $V \in \mathsf{Rep}_G$ has a central structure (which is in general nonunique), and that a central structure is equivalent to a pullback from $V \to \bullet/G$ to a vector bundle on G/G.

This is good because we said that $(\mathsf{Rep}_G, \otimes)$ and $(\mathsf{Vect}G, *)$ are Morita equivalent as monoidal categories, and therefore must have the same center.

The Drinfeld center is naturally braided monoidal: we don't know that $\psi_G \circ \psi_F : F * G \to G * F \to F * G$ is the identity. But it does satisfy the braid relations, and you can see this from the field-theoretic picture.

Dual to the center, we'll think of traces. Let M be a C-module (here we need C to be dualizable); then, there will be a trace of C, akin to the Hochschild homology of C, and M will have a character in $\mathrm{Tr}(\mathsf{C})$. The trace, like the Hochschild homology, is defined as $\mathrm{Tr}(\mathsf{C}) = \mathsf{C} \otimes_{\mathsf{C}\otimes\overline{\mathsf{C}}} \mathsf{C}$, though this requires specifying what the tensor product of monoidal categories is.

Rather than immediately formalizing this, let's work out an example. Let $\mathsf{C} = (\mathsf{Vect}G, *)$. A G-action on Y defines a C-action on $\mathsf{M} = \mathsf{Vect}Y$. If $g \in G$, it defines an endofunctor $F_g : \mathsf{Vect}Y \to \mathsf{Vect}Y$. Identifying $\mathrm{End}\,\mathsf{M} \cong \mathsf{M} \otimes \mathsf{M}^{\mathsf{op}} \cong \mathsf{Vect}(Y \times Y)$, and in this last category F_g is identified with $\underline{\mathbb{C}}_{\Gamma_g}$, where Γ denotes taking the graph. Taking the trace on $\mathsf{M} \otimes \mathsf{M}^{\mathsf{op}}$ defines a map to Vect. Concretely, $g \mapsto \mathbb{C}[Y^g]$, as $Y^g = \Gamma_g \Delta$. As V varies, this defines a vector bundle, and it's invariant under conjugation, because Y^g is, so this defines an object of $\mathsf{Vect}G/G = Z(S^1)$.

If $\mathcal{Y} = Y/G$, we get a map $\mathcal{Y}^{S^1} \to \mathcal{X}^{S^1}$, i.e. a map $\{g \in G, y \in Y^g\} \to G/G$ which sends $\underline{\mathbb{C}}$ to the vector bundle we saw; this is a categorified version of the Frobenius character formula. The map itself is a finite-group version of the *Grothendieck-Springer resolution*.

A fun consequence of this notion of characters is that if a group G acts on a category M, e.g. $\mathsf{Vect}Y$, we can first take the character $\chi_\mathsf{M} \in \mathsf{Vect}(G/G) = Z(S^1)$. Then you can also do it again and wind up in $Z(S^1 \times S^1) = \mathbb{C}[\mathrm{Loc}_G\, S^1 \times S^1]- = \mathbb{C}[[G, G]]$. If g and h commute in G, then given an $\mathcal{F} \in \mathsf{Vect}(G/G)$, you can take its fiber at $[g] \in G/G$, which is a representation of $Z_G(g)$. Since $h \in Z_G(g)$, then we can take the trace of the action h for this representation. We'll let $2\chi_\mathsf{M}$ denote this second character.

Claim 4.4. There's an action of $\mathrm{SL}_2(\mathbb{Z})$ on this through the pair (g, h), and $2\chi_\mathsf{M}$ is invariant under this action.

The idea is that 2χ arises naturally as a boundary condition labeled by M, therefore defining a state in $Z(T^2)$, and this has to be invariant.

There's a lot of literature about Drinfeld centers out there, but not so much about traces: some references include David Ben-Zvi's paper with John Francis, and Ben-Zvi-Nadler's paper "Secondary traces." The trace of a monoidal category appears in Khovanov homology, where it's called *trace decategorification*, and these secondary characters appear in chromatic homotopy theory. There's a paper of Ganter-Kapranov [2] that discusses some of this, defining 2χ, but they don't discuss the $\mathrm{SL}_2(\mathbb{Z})$-invariance.

We can also do something completely different: if X is a smooth variety, let $\mathsf{C} = (\mathsf{QC}(X), \otimes)$.[12] Then, the "sigma model" for X will give us most of a 3D TFT: we don't get numbers for 3-manifolds, though. We'd like $Z(S^1)$ to be the center (HH^*) or trace (HH_*) of $(\mathsf{C}, *)$, which is like a notion of sheaves on $[S^1, X]$. Here, S^1 is $\bullet \rightrightarrows \bullet$, so this space is $X \times_{X\times X} X$, or so the functions we get are those on the diagonal of X intersect itself.

In the two-dimensional theory, if $X = \operatorname{Spec} R$, then $S^1 \mapsto HH_*(R) = \Omega^{-*} = R \otimes^{\mathbb{L}}_{R\times R} R = \operatorname{Sym} \Omega^1[1]$. (This is the Hochschild-Konstant-Rosenberg theorem.)

The three-dimensional case is similar: consider $\mathsf{QC}(X) \otimes_{\mathsf{QC}(X\times X)} \mathsf{QC}(X) = \operatorname{Hom}_{\mathsf{QC}(X\times X)}(\mathsf{QC}(X), \mathsf{QC}(X))$, and both of these are identified with $\Omega_X^{-\bullet}$-modules.[13] So this is what we get for the circle, and it's not too surprising that it's a categorification of what we obtain for the 2D theorem. It's a little odd that the negative degrees appear, but this is because it came out of homology.

Since there's no S^1-action in the picture (yet), there's no differential. Alternatively, you could think of these as dg $\Omega_X^{-\bullet}$-modules where the differential is 0. The monoidal structure is some kind of convolution, tensored over $\mathcal{O}_X$, not differential forms. The multiplication is by operator product expansion again; in particular, the unit is $\mathcal{O}_X$.

This is a little strange, so let's try to identify $Z(S^1)$ in another way. Specifically, we want $(\mathsf{C}, *)$ to be identifed with $(\mathsf{Mod}_R, \otimes)$ for some commutative object R, so that we obtain a tensor category. This forces our hand somewhat: the unit of Mod_R is R, and $\operatorname{End}(1_{\mathsf{Mod}_R}) = \operatorname{End}(R) = R$. For a general monoidal category C, $\operatorname{End}(1_{\mathsf{C}})$ is always commutative, and in the derived setting it's always E_2. Thus, given any monoidal category, we get a ring $R = \operatorname{End}(1_{\mathsf{C}})$, and the category Mod_R sits naturally inside $(\mathsf{C}, *)$ as the things generated by the unit. You might not see all of C, but no information is lost.

This example works really well if your category is Mod_R; it's less great if $\mathsf{C} = \mathsf{Rep}_G$ for a reductive group G; since this is a semisimple category, $\operatorname{End} 1 = \mathbb{C}$. We've been talking about $\mathsf{QC}(X)$, whose unit is $\mathcal{O}_X$; thus, $\operatorname{End}(\mathcal{O}_X) = \operatorname{Hom}(\mathcal{O}_X, \mathcal{O}_X) = \Gamma(\mathcal{O}_X) = \mathcal{O}(X)$, the global functions. Thus, restricting to $\mathcal{O}(X)$-modules is affinization, keeping only the global data of functions.

This has a very nice interpretation from 3D TFT. The monoidal category is $(Z(S^1), *)$, and the unit is $Z(D^2)$ (filling in one of the holes in operator product expansion). Then, $\operatorname{End}(1) = Z(S^2)$, the local operators in the TFT, generalizing the line operators we discussed last semester.

To recap, suppose X is a smooth variety.

- To a point, we assign $(\mathsf{QC}(X), \otimes)$.
- To S^1, we get $\Omega^{-\bullet}$-modules, with $\mathcal{O}_X$ as the unit. We want to know $\operatorname{End}_{\mathsf{Mod}_{\Omega^{-\bullet}}}(\mathcal{O}_X)$; this is an exterior algebra with an augmentation, and we want to know the endomorphisms of the augmentation.

This is a somewhat complicated question, so let's ask a simple question.

Example 4.5. The best exterior algebra is $\Lambda = \mathbb{C}[\eta]/(\eta^2) = H_*(S^1)$, where η has degree -1. Now, Λ-modules are "topological/homotopic representations of S^1," i.e. things with an action of $H_*(S^1)$. As with other representations, we can identify this with a category of sheaves on BS^1.

Anyways, we care about the augmentation module $\Lambda/\eta \cong \mathbb{C}_0$. This is the trivial representation, and is induced by homology by the circle action on a point. We want to calculate $\operatorname{Ext}_\Lambda(\mathbb{C}_0, \mathbb{C}_0)$, which has a nice topological interpretation — maps out of the trivial representation are invariants, so this is the invariants of the circle action on $\mathbb{C}_0$. A more formal name for this is the S^1-equivariant cohomology of a point, namely $H^1(BS^1) = H^1(\mathbb{CP}^\infty) = \mathbb{C}[u]$, where $|u| = 2$. ◂

Returning to our example and running the same calculation, the endomorphisms of the augmentation are $\operatorname{Sym} T[-2]$, which is pretty cool. In particular, it says that this field theory assigns to S^2 the space $\Gamma(X, \operatorname{Sym} T_X[-2])$, and this says that (assuming some finiteness conditions), $Z(S^1) = \mathsf{Mod}_{\Omega^{-\bullet}}$ is also the category of $\operatorname{Sym} T_X[-2]$-modules, with the naïve tensor product! This is nice, because it's what you might have guessed if you knew $Z(S^2)$, and indeed it is.

[12] Since X need not be finite, we'll begin taking everything derived. For example, C is a dg category.

[13] We emphasize that everything is derived.

Ignoring thse shifts by -2, this is functions on T^*X, so the commutative ring of local operators in this theory is functions on T^*X, and therefore to S^1 we've attached (up to grading and finiteness conditions) $\mathsf{QC}(X)$ with the usual tensor product!

Now, can we go backwards? Given a TFT, if it can be put into this form, it's a σ-model for T^*X (up to grading), and therefore deserves to be called Rozansky-Witten theory for T^*X. This tells us that if M is a holomorphic symplectic manifold (i.e. the symplectic form is a $(2,0)$-form, e.g. T^*X where X is any variety), then Rozansky-Witten theory attaches a 3D TFT, and if M is compact, this extends to defining partition functions for 3-manifolds.

How do you recognize this theory on the street? Its main feature is that it's a 3D lift of the B-model: if you take the dimensional reduction $Z_{S^1}(-) = Z(S^1\times -)$, you get the B-model, because $Z_{S^1}(\bullet) = Z(S^1) = \mathsf{QC}(M)$. Rozansky-Witten theory is the theory of maps into M, and the local operators are functions on M. Anything of the form $S^1 \times X$ comes from the B-model, e.g. $Z(S^1 \times S^1) = \Gamma(\Omega_M^{-\bullet})$; if M is compact, this looks like $H_{\mathrm{dR}}(M)$, perhaps up to a grading.

What does Rozansky-Witten theory assign to a point? We don't know, and there are a few ideas, but we're in a good position to answer. One issue is the dirty secret: Rozansky-Witten theory is $\mathbb{Z}/2$-graded where it "should be" $\mathbb{Z}$-graded: if you pretend that there's no grading, things are off by even shifts, so you get the $\mathbb{Z}/2$-graded version of the B-model. Ideally, understanding $Z(\bullet)$ would address this degree issue. It's currently guessed to be something like sheaves of categories (namely $\mathsf{QC}(L)$-modules) over holomorphic Lagrangians L of M.

More explicitly, given Lagrangian submanifolds $L_1, L_2 \subset M$, we can attach categories, e.g. $\mathrm{QCoh}(L_i)$. Locally $M \simeq T^*L_1$. The homomorphisms between $\mathsf{QC}(L_1)$ and $\mathsf{QC}(L_2)$ are expected to be matrix factorizations.

Everything here should be graded; if you think of this from first principles, you have a 3D TFT Z and want to construct a holomorphic symplectic manifold M such that Z is Rozansky-Witten theory for M. This M would be the moduli space of vacua. A first approximation is $\operatorname{Spec} Z(S^2)$, which is akin to the affine version of the moduli space, and this has a Poisson structure, which is how symplectic geometry enters the picture. $Z(S^2)$ is an E_3-algebra in the category of graded vector spaces: its operations are parameterized by pairs of balls in 3-space. This is nothing so fancy: it's a commutative multiplication plus a bracket $\{-,-\}$ of degree -2, arising from $C_2(\mathbb{R}^3) \simeq S^2$, and $H_*(S^2)$ has two generators, the multiplication in degree 0, and the Poisson bracket in degree -2. That is, the fundamental class of the 2-sphere gives you the bracket. Another way to think about is is two circles moving around each other in a Hopf link, which gives you the generator of $H_2(C_2(\mathbb{R}^3))$. So $\operatorname{Spec} Z(S^2)$ is a Poisson variety, but in a graded sense: $\{-,-\}$ has degree -2, exactly why the shifted version of the contangent bundle $T^*X[2]$. Functions on this shifted bundle are an E_3-algebra. This perspective is why we get 2-shifted symplectic structures.

5. The Moduli of Vacua and Categorical Representations: 2/23/17

"Precise but sketchy… does that make sense?"

Today's lecture was in UT's geometry seminar, rather than the usual Friday afternoon. David spoke about a project with Sam Gunningham and David Nadler that's been going on for a long time, but is now finishing up. The general theme is just like in the class last semester: describing the dual of a group describes the moduli space of a gauge theory.

Let K be a finite group. Then, its dual is $\widehat{K}$, the irreducible $\mathbb{C}$-representations of K. The Fourier transform is an identification of algebras between $\mathbb{C}[\widehat{K}]$ under convolution and $Z(\mathbb{C}K)$ under pointwise multiplication. Here, $\widehat{K}$ is the moduli space: the entire representation theory of K decomposes over $\widehat{K}$.

Example 5.1 (2D Dijkgraaf-Witten theory)**.** Dijkgraaf-Witten theory is a 2-dimensional TFT whose partition function on a closed surface Σ is $Z(\Sigma) := \#\operatorname{Loc}_K \Sigma$, the number of K-local systems on Σ. The algebra of local operators is $Z(S^1)$: K-local systems on a circle are identified with class functions $Z(\mathbb{C}K)$, and the algebra structure is identified with convolution. The category of boundary conditions is $Z(\bullet) = \mathsf{Rep}K$.

Everything here sheafifies, i.e. decomposes as a sum, over $\widehat{K}$: the characters χ_λ for irreducible representations $\lambda \in V$ are orthogonal idempotents for $Z(\mathbb{C}K)$, and so

$$Z(S^1) = (Z(\mathbb{C}K), *) = \bigoplus_{\lambda\in\widehat{K}} \mathbb{C}\cdot\chi_\lambda.$$

The partition function is a sum as in Mendykh's formula:

$$Z(\Sigma) = \sum_{\lambda\in\widehat{K}} \left(\frac{\dim V}{|K|}\right)^{2-2g}, \tag{5.2}$$

where g is the genus of Σ. Finally, the boundary conditions are K-representations, which are direct sums of irreducibles, and therefore can be identified

$$Z(\bullet) = \mathsf{Rep}K = \bigoplus_K \mathsf{Vect}.$$ ◂

The goal is to generalize this to the case where K is a compact, connected Lie group, equivalent to a connected complex reductive group $G = K_{\mathbb{C}}$, and to pass to a 3D TFT. In this case, $\widehat{K}$ is an infinite discrete set, identified with the integral points in $C = \mathfrak{h}^*/W$ (where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$). There's an analogous story for topological 2D Yang-Mills theory, where the gauge group G need not be finite, but otherwise things look a lot like Dijkgraaf-Witten theory. This causes some infinities, in particular in (5.2), but these can be addressed: don't divide by $\dim K$, and then do a little bit of magic. If Σ has negative Euler characteristic, then what you get is well-defined, and is identified with the *symplectic volume*.

We're going to focus on $C = \operatorname{Spec} Z(\mathcal{U}(\mathfrak{g}))$; it's a space of infinitesimal characters: all the continuous parameters in the representation theory of $\mathfrak{g}$ (or $G_{\mathbb{R}}$) are controlled by C. This certainly isn't everything, but the representation theory of G is split into continuous and discrete parts, and C controls the continuous part.

The second piece of the generalization is to replace the 2D TFT with a 3D one. Thus the goal is to describe an analogue of $C = \mathfrak{h}^*/W$ for *categorical* representations of G. By this we mean a quantum Hamiltonian G-space. By a *quantum G-space* we mean an associative algeba A together with a G-action; this is the noncommutative analogue of a G-space, and A should be though of as the algebra of functions on that space. That the action is *Hamiltonian* means that it factors through the adjoint action of G on $\mathcal{U}(\mathfrak{g})$ and a G-morphism $\mathcal{U}(\mathfrak{g}) \to A$ called the *moment map*.

Alternatively, we can replace A with its category of modules, which carries an action of G. We'd like this to be *Hamiltonian*, in that $\mathfrak{g}$ acts internally, through the action of A. This is equivalent to the $\exp(\mathfrak{g})$-action being trivialized.

Definition 5.3. A category isomorphic (as categories with a G-action) to modules over a quantum Hamiltonian G-space is called a *(de Rham) G-category*.

Example 5.4.

(1) The category of $\mathcal{U}(\mathfrak{g})$-modules (equivalently, of G-representations) vacuously has a Hamiltonian action.
(2) If X is a G-space, then modules over its its (commutative) algebra of functions are a G-category. This is reassuring, since it's what we meant to generalize.
(3) With X as above, the category of $\mathcal{D}$-modules on X is also a G-category. ◂

This last example is a generalization of the algebra of class functions: let K be a finite group, so K-representations are the same things as $\mathbb{C}K$-modules. In our setting, $(\mathcal{D}\text{-}\mathsf{Mod}(G), *)$ is a monoidal ∞-category: if $\mu : G \times G \to G$ is the multiplication map, then $\mathcal{F}, \mathcal{G} \mapsto \mu_*(\mathcal{F} \boxtimes \mathcal{G})$.

We'd like to do categorical representation theory: understand how G acts by symmetries on G-categories and develop a version of highest weight theory, etc.

An interpretation from physics. Our goal is to understand the moduli space of a 3D topological gauge theory,[14] and it's a general principle that to understand a field theory one should understand its boundary conditions.

The boundary conditions to a 2D gauge theory with gauge group G form a 2D QFT with G-symmetry. In this case, de Rham G-categories are equivalent to the topological B-model on quantized Hamiltonian G-spaces.

Ben-Zvi-Gunnignham-Nadler have constructed this 3D TFT χ, called *character theory*.

[14]Specifically, the theory is twisted maximally supersymmetric (i.e. $\mathcal{N} = 8$) super-Yang-Mills theory in three dimensions. This is the 3D analogue of the 4D theory that Kapustin-Witten described as a physical setting for the geometric Langlands program.

- To a surface Σ, $\chi(\Sigma) = H_*^{\mathrm{BM}}(\mathrm{Loc}_G \Sigma)$.[15]
- The line operators are the category $\chi(S^1) = \mathcal{D}\text{-}\mathsf{Mod}(G/G)$.
- The boundary conditions are $\chi(\bullet)$, the category of de Rham G-categories.

The theory doesn't produce partition functions on most 3-manifolds, or at least not without a lot of work.

Dijkgraaf-Witten theory decomposes over the dual of the group; does something similar happen here? We'll need an analogue of C. One naïve guess is $C = \mathfrak{h}^*/W = \operatorname{Spec} Z(\mathcal{U}(\mathfrak{g}))$, because $\mathcal{U}(\mathfrak{g})$-modules are the only G-category we really understand in detail. And this makes sense, because Schur's lemma tells us the center acts in a very particular way: for any $\lambda \in C$, one can restrict to the category $\mathsf{Mod}_{\mathcal{U}(\mathfrak{g}),\lambda}$ where λ acts in a particular way, and this is a G-category.

If Rep_G denotes the category of finite-dimensional representations of G, then both G and Rep_G act on the category of $\mathcal{U}(\mathfrak{g})$-modules, and from this one can obtain a *translation functor* $\mathsf{Mod}_{\mathcal{U}(\mathfrak{g}),\lambda} \to \mathsf{Mod}_{\mathcal{U}(\mathfrak{g}),\lambda+1}$.

If $G = H$, then the category of $\mathcal{U}(\mathfrak{h})$-modules is equivalent to that of $\mathbb{C}[\mathfrak{h}^*]$-modules or $\mathsf{QC}(\mathfrak{h}^*)$, so Rep_H is determined by the lattice of characters. But intrinsically, we can only see things up to translation by characters, so the space we get is $\mathfrak{h}^*/\Lambda$, which analytically is $H^\vee$.

One version of the Mellin transform is an equivalence of tensor categories between $\mathcal{D}\text{-}\mathsf{Mod}(H)$ under convolution and $\mathsf{QC}(\mathfrak{h}^*/\Lambda)$ under $\otimes$. When $H = \mathbb{C}$ (more generally, when it's a torus), this describes an equivalence between the categories of modules of $\mathbb{C}[z, z^{-1}]\langle z\frac{\partial}{\partial z}\rangle$ and of $\mathbb{C}[t]\langle \sigma, \sigma^{-1}\rangle$. Here, $\sigma : t \mapsto t+1$ is the translation operator.

This suggests that $C = \mathfrak{h}^*/W$ is what we're looking for. For example, if G acts on a space X, a Harish-Chandra system defines a lot of structure: $Z(\mathcal{U}(\mathfrak{g}))$ maps to commuting, G-invariant differential operators Δ_i on X, and we have maps $\mathfrak{g} \to \mathsf{Vect}X$ and $\mathcal{U}(\mathfrak{g}) \to \mathcal{D}_X$. This example is useful in harmonic analysis.

We want this to sheafify over C, which means simultaneously diagonalizing all of these operators. Let $M_\lambda = \{\Delta_i f = \lambda(\Delta_i) f\}$, where the f are functions on X. There are isomorphisms $M_\lambda \cong M_{\lambda+\nu}$.

Example 5.5. Suppose $G = X = \mathbb{C}^*$ and $\lambda \in C = \mathbb{C}$; let $\nabla = \frac{\mathrm{d}}{\mathrm{d}z} - \lambda\frac{\mathrm{d}}{\mathrm{d}z}$. Then, if $z\frac{\mathrm{d}}{\mathrm{d}z} f = \lambda f$, then the only solutions we get are $[\lambda] \in \mathbb{C}/\mathbb{Z}$, where we obtain $e^{2\pi i\lambda}$-monodromy. $\blacktriangleleft$

For this reason we have to instead look at $\mathfrak{h}^*/W^{\mathrm{aff}}$, where $W^{\mathrm{aff}} = W \ltimes \Lambda$.

Theorem 5.6 (Ben-Zvi-Gunnignham-Nadler)**.** *There is a particular symmetric monoidal category* K *a central action of symmetric monoidal categories of* $(\mathsf{K}, \otimes)$ *on* $(\mathcal{D}(G), *)$.

Here, K is the *Kostant category*, which looks like $\mathsf{QC}(\mathfrak{h}^*/W^{\mathrm{aff}})$, but has slightly different equivariance data. This implies that any G-category carries an action of K which is commutative with G; this can also be stated in terms of maps from the moduli space of χ to $\mathfrak{h}^*/W^{\mathrm{aff}} \cong H^\vee/W$, and this describes a decomposition of the data of the character TFT over the dual space, which is what we wanted.

To be less mysterious about K, it's the Whittaker Hecke category (which Sam Raskin talked about in this seminar a few weeks ago), which is a piece of $\mathcal{D}(G)$. This is the category of modules on a quantized Toda lattice. It can also be described in terms of loop groups or as modules over $H_*^{G^\vee \times S^1}(\mathrm{Gr}_{G^\vee})$. You can also describe it in terms of an analogue for the Weyl group: K is the category of modules for the nil-Hecke algebra of the affine Kac-Moody group of $G^\vee$, where $G^\vee$ is the Langlands dual group of G.

Theorem 5.6 is a quantum theorem, and has a classical limits, which is a conceptual construction of Ngô's integration of Hitchin integrable systems in his proof of the fundamental lemma.

The proof idea for Theorem 5.6 is somewhat simpler than the theorem statement. Let's start with a groupoid $\mathcal{G} \rightrightarrows X$: a space X and a set of symmetries on X. For example, a group acting on a point $K \rightrightarrows \bullet$ or an equivalence relation $X \times X \rightrightarrows X$ (meaning there's a unique pair of arrows between any two equivalent points).

Given a groupoid, it's possible to construct a groupoid algebra $H = H_*(\mathcal{G})$: we'd like functions on $\mathcal{G}$, but can't literally do that unless $\mathcal{G}$ is finite. If $\mathcal{G}$ is finite, $H_*(\mathcal{G})$ is functions on $\mathcal{G}$; if $\mathcal{G} = \bullet/K$, you get $\mathbb{C}K$, and for an equivalence relation, this algebra is the algebra of matrices that are block diagonal, where the blocks are the equivalence classes for X.

The category of H-modules is the category of $\mathcal{G}$-equivariant sheaves on X, and the tensor product of equivariant sheaves is equivariant. For example, if $\mathcal{G} = \bullet/K$, $\mathsf{Mod}_H = \mathsf{Rep}_K$, and modules over H for $\mathcal{G} = X \times X \rightrightarrows X$ are sheaves on the (discrete set) of equivalence classes of Y.

[15] This is the Borel-Moore homology, or since $\mathrm{Loc}_G \Sigma$ is a stack, this is the equivariant cohomology $H_G^*(\{\pi_1\Sigma \to G\})$ of a representation variety.

Another construction you can make is the *Hecke category* $\mathcal{H} = \mathsf{Shv}G$ whose objects are certain sheaves, and which has a convolution induced from the usual push-pull: if $\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is the composition map, then $\mathcal{F} * \mathcal{G} = \mu_*(\mathcal{F} \boxtimes \mathcal{G})$. For example, if $\mathcal{G} = \Gamma \rightrightarrows \bullet$, then $\mathsf{Shv}\mathcal{G} = \mathsf{Vect}\Gamma$ under convolution. This is not always a symmetric monoidal category!

There's a natural central extension $\mathsf{Mod}_H \to Z(\mathcal{H})$, which expresses the idea that convolution is linear over the quotient $X/\mathcal{G}$ (e.g. for an equivalence relation, $\mathbb{C}[X \times_{X/\sim} X]$ has a convolution which is $\mathbb{C}[X/\sim]$-linear).

This theory is quite general, and for judicious choices of $\mathcal{G}$ and X and the quotient $Y = \mathcal{G}/X$, previously known results prove the theorem. Namely, let $\mathcal{G} = (S^1 \ltimes LG_+)\backslash(LG^\vee \rtimes S^1)/(LG_+ \rtimes S^1)$, where LG is the loop group of G and $G^\vee$ is the Langlands dual group; let $X = \bullet/LG_+^\vee$; and let $Y = \bullet/LG^\vee$. This does require a major theorem of Bezrukavnikov-Finkelberg called the renormalized Satake correspondence.

This construction is quite surprising from the perspective of 3D gauge theory, but is more natural from the 4D perspective — which makes the appearance of Langlands dual groups and loop groups less unexpected.

6. Geometric class field theory: 3/3/17

Today we'll discuss geometric class field theory, the case of the geometric Langlands program where the group is abelian.

The story starts with Abel and Jacobi. Let X be a compact Riemann surface; from it, we can construct two abelian varieties, the Jacobian $\operatorname{Jac} X$ and the Albanese variety $\operatorname{Alb} X$, which we'll describe.

Let $\operatorname{Pic}(X)$ denote the *Picard group* of X, the moduli stack of line bundles on X. The degree is a homomorphism $\deg : \operatorname{Pic} X \to \mathbb{Z}$, and given an $x \in X$, $n \mapsto \mathcal{O}(n \cdot x)$ is a section. Thus, this singles out a splitting $\operatorname{Pic} X \cong \operatorname{Pic}^0 X \times \mathbb{Z}$, where $\operatorname{Pic}^0 X$ is the group of degree-0 line bundles. The stackiness emerges when you consider automorphisms of a line bundle $\mathcal{L}$: you cna multiply it by invertible functions, and since X is compact, these are constants. Thus, $\operatorname{Aut} \mathcal{L} \cong \mathbb{C}^\times$. A trivialization of a line bundle at a point eliminates the automorphisms, so if you consider the moduli space of line bundles with a trivialization at a point, as Grothendieck did, you obtain a scheme.

The *Jacobi variety* is the variety of degree-zero line bundles with trivializations at a specified point x, so there's a decomposition $\operatorname{Pic} X \cong \operatorname{Jac} X \times \mathbb{Z} \times \bullet/\mathbb{G}_m$: we've isolated the stackiness apart from everything else. If g is the genus of X, $\operatorname{Jac} X \cong \mathbb{C}^g/\Lambda$, where Λ is a full-rank lattice, i.e. $\Lambda \cong \mathbb{Z}^{2g}$. Thus, $\operatorname{Jac} X \cong H^1(X; \mathcal{O}_X)/H^1(X; \mathbb{Z})$.

Abel and Jacobi also studied the Albanese variety, albeit under a different name. The *Abelnese variety* of X is $\operatorname{Alb} X := H^0(X; \Omega)^*/H_1(X; \mathbb{Z})$. You can think of this as functions on differential forms where we modded out periods. If you fix an $x \in X$, there's an *Abel-Jacobi map* $\mathrm{AJ}_x : X \to \operatorname{Alb} X$ sending

$$y \longmapsto \left(\omega \longmapsto \int_x^y \omega\right),$$

where $\omega \in H^0(X; \Omega)$. This map satisfies a surprising universal property: it's initial among maps from X to abelian varieties. That is, if A is an abelian variety and $f : X \to A$ is a morphism of varieties, there's a unique map $\widetilde{f} : \operatorname{Alb} X \to A$ making the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\mathrm{AJ}_x"'] & & A. \\ & \operatorname{Alb} X \arrow[ur, "\widetilde{f}"'] & \end{tikzcd}$$

Everything so far has been true for a much larger class of varieties than compact Riemann surfaces. But it's possible to extend to 0-cycles modulo rational equivalence, also known as the 0^{th} *Chow group* $\mathrm{CH}_0(X)$. Here, $\sum_{y_i \subset X} n_i y_i \mapsto \sum n_i \mathrm{AJ}_x(y_i) \in \operatorname{Alb} X$.

Abel and Jacobi proved that this map is an equivalence. Let $\mathbb{C}(X)$ denote the principal divisors of X.

Theorem 6.1 (Abel-Jacobi)**.** *The diagram*

$$\begin{tikzcd}
\left(\bigoplus_{y\in X}\mathbb{Z}\right)_0 \arrow[rr] \arrow[dr] & & \operatorname{Alb} X. \\
& \left(\bigoplus_{y\in X}\mathbb{Z}\right)_0/\mathbb{C}(X) \arrow[ur] &
\end{tikzcd}$$

and the lower right arrow is an isomorphism.

The point is that the Albanese variety wants to be the free abelian group on X, but X isn't a discrete set, which is encoded in the principal divisors.

The Albanese variety is a torus, so topologically $(S^1)^{2g}$, so the Abel-Jacobi map induces a map $\pi_1(X) \to \pi_1(\operatorname{Alb} X) \cong H_1(\operatorname{Alb} X;\mathbb{Z})$, since $\pi_1(\operatorname{Alb} X)$ is abelian. Therefore passing to $\operatorname{Alb} X$ abelianizes π_1: the universal property of $\operatorname{Alb} X$ implies $H_1(X;\mathbb{Z}) \cong H_1(\operatorname{Alb} X;\mathbb{Z})$, so $\pi_1(X)^{\mathrm{ab}} = \pi_1(\operatorname{Alb} X)$. Thus, the Albanese variety's theory of covering spaces is closely related to that of X: the finite abelian covers of X are in bijection with the finite covers of $\operatorname{Alb} X$ (all of which are abelian).

Galois-theoretically, this provides an identification between finite abelian extensions of $\mathbb{C}(X)$ and finite extensions of $\mathbb{C}(\operatorname{Alb} X)$ (all of which will be abelian). This is very close to ordinary class field theory, which considers finite abelian extensions of a number field.

More generally, the Albanese variety understands everything abelian about X: if G is an abelian group, maps $\pi_1(X) \to G$ factor through $\pi_1(X)^{\mathrm{ab}} = \pi_1(\operatorname{Alb} X)$, so abelian local systems on X are in natural bijection with local systems on $\operatorname{Alb} X$, all of which are abelian. In particular, there's a bijection between the line bundles on X and those on $\operatorname{Alb} X$.

Let $\mathcal{P} \to X \times \operatorname{Pic} X$ be the tautological line bundle, where $\mathcal{P}|_{(x,\mathcal{L})} = \mathcal{L}|_x$. You can extend $\mathcal{P}$ to a bundle over $\operatorname{Alb} X \times \operatorname{Pic} X$, which exhibits $\operatorname{Alb} X$ and $\operatorname{Pic} X$ as dual abelian varieties.[16] Recall that for an abelian variety A, its dual $A^\vee$ is the space of multiplicative line bundles on A, i.e. those line bundles $\mathcal{L}$ such that $\mathcal{L}_x \otimes \mathcal{L}_y \cong \mathcal{L}_{x+y}$. Globally, this is data of $\mu^*\mathcal{L} \cong \mathcal{L} \boxtimes \mathcal{L}$, where $\mu : A \times A \to A$ is the multiplication map.

For Riemann surfaces, the Albanese variety is isomorphic to $\operatorname{Jac} X$, but this is not true for higher-dimensional varieties. This is why it wasn't named after Abel and Jacobi: they only considered the case where it was isomorphic to something else that was named after Jacobi.

At a $y \in X$, the stalk of meromorphic functions is denoted $\mathcal{K}_y$, isomorphic to $\mathbb{C}((t))$, and the stalk of holomorphic function sis $\mathcal{O}_y \cong \mathbb{C}[[t]]$. Thus, $\mathbb{Z} \cong \mathcal{K}_y^*/\mathcal{O}_y^*$ is the order of zeros and poles of meromorphic functions at y.

Now, given an $\mathcal{L} \in \operatorname{Pic} X$, trivialize $\mathcal{L}$ near all $y \in X$ and trivialize it generically. This defines an element of $\prod'_{y\in X}\mathcal{K}_y^*$, where the $'$ means that it's a restricted product: at all but finitely many points, we have a Taylor series rather than a Laurent series. If we mod out by the trivializations, which is a copy of $\mathcal{O}_y^*$ at all y, and the generic trivialization, which is defined by a rational function, we get

$$\mathbb{C}(X)\backslash \prod_{y\in X}\mathcal{K}_y^* / \prod_{y\in X}\mathcal{O}_y^*.$$

This was probably first written down by Weil, though Chevalley also thought a lot about this. $\prod_{y\in X}\mathcal{K}_y^*$ is also called the *idèles* of X, and $\prod_y \mathcal{O}_y^*$ is also called the *integral idèles*.

Anyways, you can also write this as

$$\mathbb{C}(X)\backslash \prod_y \mathcal{K}_y^*/\mathcal{O}_y^* = \mathbb{C}(X)\backslash \bigoplus \mathbb{Z},$$

or divisors modulo principla divisors. This defined an isomorphism $\operatorname{Alb} X \cong \operatorname{Pic} X$. The duality comes from a self-duality on $\operatorname{Pic} X \cong \operatorname{Jac} X \times \mathbb{Z} \times \bullet/\mathbb{G}_m$ — this is a duality on $\operatorname{Jac} X$ and switches $\mathbb{Z}$ and $\bullet/\mathbb{G}_m$.[17]

[16] All of this is literally true for stacks, but you may need a little more for schemes.

[17] You can generalize line bundles to torus bundles, and there the isomorphism is $\operatorname{Bun}_T X \cong (\operatorname{Bun}_{T^\vee} X)^\vee$.

Anyways, we wanted to exhibit this by extending the canonical line bundle $\mathcal{P} \to X \times \operatorname{Pic} X$ to a universal mutliplicative line bundle $\widetilde{\mathcal{P}} \to \operatorname{Alb} X \times \operatorname{Jac} X$. Let $\mathcal{L} \in \operatorname{Jac} X$ be a (degree-0) line bundle on X. Then, we can define a multiplicative line bundle $\mathcal{F}_{\mathcal{L}}$ on $\operatorname{Alb} X$ by $\mathcal{F}_{\mathcal{L}} = \widetilde{\mathcal{P}}|_{\operatorname{Alb} X \times \{\mathcal{L}\}}$.

If $\mu : \operatorname{Alb} X \times \operatorname{Alb} X \to \operatorname{Alb} X$ is the multiplication map, we want $\mu^*\mathcal{F}_{\mathcal{L}} \cong \mathcal{F}_{\mathcal{L}} \boxtimes \mathcal{F}_{\mathcal{L}}$, and we can restrict to $\mathcal{L} \boxtimes \mathcal{F}_{\mathcal{L}}$, where we know what it is. This forces what the value is at the divisor $\sum n_i y_i + z$ for a $z \in X$, we we require

$$\mathcal{F}_{\mathcal{L}}|_{\sum n_i y_i + z} \cong \mathcal{L}_z \otimes \mathcal{F}_{\mathcal{L}}|_{\sum n_i y_i}.$$

That is, $\mathcal{F}_{\mathcal{L}}$ is an eigensheaf for the operation $(+z) : \operatorname{Alb} X \to \operatorname{Alb} X$ with eigenvalue $\mathcal{L}_z$; this is an example of a *Hecke eigenproperty*.

Since X generates $\operatorname{Alb} X$, we're forced to conclude that

$$\mathcal{F}_{\mathcal{L}}|_{\sum n_i y_i} \cong \bigoplus_i (\mathcal{L}_{y_i})^{\otimes n_i}.$$

This doesn't completely uniquely determine everything, e.g. everything could have been tensored with a fixed other vector space, but this is a candidate, and we have a good form of uniqueness. However, we haven't shown existence: there are relations in $\operatorname{Alb} X$ by rational equivalence, and so we have to check that this is singly defined in the quotient.

To see this, let's look at the Fourier-Mukai transform. Since $\operatorname{Alb} X \cong (\operatorname{Jac} X)^\vee$, the Fourier-Mukai theorem provides an equivalence $(D^b(\operatorname{Alb} X),) \cong D^b(\operatorname{Jac} X, \otimes)$. In particular, the line bundle $\mathcal{F}_{\mathcal{L}}$ is dual to the the skyscraper $\mathcal{O}_{\mathcal{L}}$ over $\mathcal{L} \in \operatorname{Jac} X$. The Fourier-Mukai transform sends translation by $z \in X \hookrightarrow \operatorname{Alb} X$, or convolution by $\mathcal{O}_z$, to tensoring by a line bundle $\mathcal{W}_z$. The eigenproperty that $\mathcal{O}_z * \mathcal{F}_{\mathcal{L}} \cong \mathcal{L}_z \otimes \mathcal{F}_{\mathcal{L}}$ is mapped to the statement on $\operatorname{Jac} X$ that skyscrapers are eigenobjects for the tensor product: if $\mathcal{M} \in \mathsf{QC}(X)$, then $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_x \cong \mathcal{M}|_x \otimes_{\mathbb{C}} \mathcal{O}_x \cong (\mathcal{O}_x)^{\oplus n}$. So we're finding the eigenobjects for translation, which is particularly nice because those are the group characters in the usual setting.

The eigencondition is a twisted version of taking invariants: we're used to asking $g \cdot f = f$ for all $g \in G$, but we could also take $g \cdot f = \chi(g) \cdot f$, and this forces χ to be a character of G, a homomorphism $G \to \mathbb{C}^*$. Similarly, given a sheaf instead of a function one could ask for $g^*\mathcal{F} \cong \mathcal{F}$, or $a^*\mathcal{F} \cong \mathcal{O}_X \boxtimes \mathcal{F}$, where $a : G \times Z \to Z$ is the action map, and the eigenproperty is the twisted version, $g^*\mathcal{F} \cong \mathcal{X}_g \otimes \mathcal{F}$, or $a^*\mathcal{F} \cong \mathcal{X} \boxtimes \mathcal{F}$, where $\mathcal{X}$ is a sheaf on G, and to make sense of this, $\mathcal{X}$ has to be multiplicative.

In our situation, $Z = \operatorname{Alb} X$ and G is either the Albanese variety again or $\mathbb{Z}$, and $\mathcal{X} = \mathcal{F}_{\mathcal{L}}$. The notion of an eigensheaf may be abstruse at first glance, but is just equivariance against a character.

More consequences of the $\operatorname{Alb} X$-$\operatorname{Jac} X$ duality. We mentioned the equivalence between (degree-0) line bundles on X, which correspond to skyscrapers on $\operatorname{Jac} X$, which Fourier-Mukai sends to $\operatorname{Alb} X$, where we get degree-0 line bundles on $\operatorname{Alb} X$. Thus, this bijection is captured by the equivalence $D^b(\operatorname{Alb} X) \cong D^b(\operatorname{Jac} X)$.

There are at least three variants of this, which make the connection with the geometric Langlands program more evident. The idea behind these versions is to replace $\operatorname{Pic} X$ with a different moduli space.

The *Dolbeault space* is $\operatorname{Higgs}_{\mathrm{GL}_1}(X) := \{\mathcal{L} \in \operatorname{Pic} X, \eta \in H^0(X; \operatorname{End}(\mathcal{L}) \otimes \Omega_X^1)\}$, the moduli space of rank-1 Higgs bundles on X. This is naturally identified with $\operatorname{Pic} X \times H^0(X; \Omega) \cong T^* \operatorname{Pic} X$. The projection map $T^* \operatorname{Pic} X \to H^0(X; \Omega_X^1)$ is called the *Hitchin map*, and its fibers are all $\operatorname{Pic} X$.

The Fourier-Mukai transform is a non-identity automorphism $D^b(\operatorname{Higgs}_{\mathrm{GL}_1} X) \cong D^b(\operatorname{Higgs}_{\mathrm{GL}_1} X)$.[18]

The other two are a little more interesting. Let $\operatorname{Conn}_{\mathrm{GL}_1}(X)$ denote the space of line bundles $\mathcal{L}$ with connections $\nabla : \mathcal{L} \to \mathcal{L} \otimes \Omega^1$. Since there are no holomorphic 2-forms on a Riemann surface, $\nabla^2 = 0$, so these line bundles are automatically flat. In this case, called the *de Rham case*, the duality is a little more interesting: the Fourier-Mukai transform defines an equivalence $\mathsf{QC}(\operatorname{Conn}_{\mathrm{GL}_1} X) \cong \mathcal{D}\text{-}\mathsf{Mod}(\operatorname{Pic} X)$.

The *Betti case* considers $\operatorname{Loc}_{\mathrm{GL}_1}(X)$, the space of GL_1-local systems on X. Since GL_1 is abelian, this factors through the homology, so if g is the genus of X, $\operatorname{Loc}_{\mathrm{GL}_1}(X) \cong (\mathbb{C}^*)^{2g} \times \bullet/\mathbb{G}_m$. This is by far the simplest of the three spaces, and the analogue of the Fourier-Mukai transform doesn't even need derived categories: $\mathsf{QC}(\operatorname{Loc}_{\mathrm{GL}_1} X) \cong \operatorname{Loc}(\operatorname{Pic} X)$, which is an equivalence of abelian categories. That is, a representation $\pi(X) \to \mathrm{GL}_1$, which necessarily is determined by its abelianization, is the same data as a local system on $\operatorname{Pic} X$, which captures the abelian data for X. A skyscraper sheaf on $\operatorname{Loc}_{\mathrm{GL}_1} X$ is the data of a rank-1 local system on X, and this is sent to a rank-1 local system on $\operatorname{Pic} X$.

[18] If you work instead with a torus T, the equivalence is $D^b(\operatorname{Higgs}_T X) \cong D^b(\operatorname{Higgs}_{T^\vee} X)$.

The left-hand side considers modules for functions on $\mathrm{Loc}_{\mathrm{GL}_1} X \cong (\mathbb{C}^*)^{2g}$, i.e. $\mathbb{C}[z, z^{-1}]^{\otimes 2g}$. The right-hand side is the representations of $\pi_1(\mathrm{Jac}\, X)$ or modules for $\mathbb{C}[\pi_1(\mathrm{Jac}\, X)] = \mathbb{C}\mathbb{Z}^{2g} \cong \mathbb{C}[z, z^{-1}]^{\otimes 2g}$.

There are two more perspectives: the space of flat connections has a natural symplectic structure, and you can try to do a deformation quantization. This leads to two noncommutative versions of the de Rham or Betti equivalences called the *quantum geometric Langlands correspondence*. The whole package of all five equivalences is what you might call the geometric Langlands correspondence for GL_1.

The Betti case is easier as usual, so let's discuss the quantum Betti correspondence for genus 1. Here, $\mathrm{Loc}_{\mathrm{GL}_1} T^* \cong \mathbb{C}^* \times \mathbb{C}^*$ (plus a stacky factor we'll ignore). This has a natural symplectic structure where the symplectic form is $\omega = \mathrm{d}z_1/z_1 \wedge \mathrm{d}z_2/z_2$, which is invariant. This is the same symplectic structure as the one induced on $H^1(T^2; \mathbb{C}^*) \cong \mathbb{C}^* \times \mathbb{C}^*$ by the cup product $H^1(T^2; \mathbb{C}^*) \otimes H^1(T^2; \mathbb{C}^*) \to H^2(T^2; \mathbb{C}^*) \cong \mathbb{C}$.

If you quantize this symplectic structure, you get something nice: $\mathbb{C}[\mathrm{Loc}_{\mathrm{GL}_2} T^2] = \mathbb{C}[x, x^{-1}] \otimes \mathbb{C}[y, y^{-1}]$, which nicely deforms to the associative algebra

$$\mathbb{C}\langle x, y, x^{-1}, y^{-1}\rangle / (xy = qyx),$$

where $q \in \mathbb{C}^*$. This is a natural one-parameter deformation of $\mathbb{C}[x, x^{-1}] \otimes \mathbb{C}[y, y^{-1}]$ called the *Weyl algebra*, and you can check this is a deformation quantization of $\mathrm{Loc}_{\mathrm{GL}_1}(X)$. The quantum version of the Betti equivalence is very easy; the quantum version of the de Rham case is easy once you state in in the right language, which we'll discuss next time.

The point is, this all arises from a 4-dimensional field theory. Why 4? Well, we're attaching categories to Riemann surfaces, so they ought to have codimension 2. Next time, we'll spell this out for the abelian case, and then move towards the nonabelian case.

7. Geometric class field theory II: 3/31/17

Today, we're going to say some more about geometric class field theory. One way to think about geometric class field theory is that you take a Riemann surface (or algebraic curve) C, and you're going to relate abelian structures on C of various kinds and structures on $\mathrm{Jac}\, C$ (or $\mathrm{Pic}\, C$).

The Jacobian is a sort of "abelianization" of C, in that the abelianization of $\pi_1(C)$ is isomorphic to $\pi_1(\mathrm{Pic}\, C)$, and the Abel-Jacobi map implements this isomorphism. This is the map $C \to \mathrm{Pic}^1(C)$ sending $x \mapsto \mathcal{O}(x)$, a skyscraper at x.

Remark 7.1. Class field theory is about number fields, but working in a geometric context, where the number field is replaced with a function field over $\mathbb{C}$. Unlike the Langlands program, the purely abelian story of class field theory can be generalized to higher dimensions. ◀

The Abel-Jacobi map carries abelian local systems on C to local systems on $\mathrm{Pic}\, C$, which are automatically abelian.

Deligne has a great reframing of this: $\mathrm{Pic}^1 C$ is a group, so you can add things together. For example, given a point in $\mathrm{Sym}^2(C)$, we get a degree-2 bundle $\mathcal{O}(x + y) \in \mathrm{Pic}^2(C)$.[19] This generalizes to a map $\mathrm{Sym}^n C \to \mathrm{Pic}^n C$, sending a configuration to a skyscraper over the divisor it defines. If $n > 2g - 2$, then this is a fiber bundle whose fibers are projective spaces (in some sense, all the interesting stuff happens for small n). In particular, the fibers are simply connected.

Let L be a rank-1 local system. Then, we also get a local system $L^{(n)} \to \mathrm{Sym}^n C$, where

$$L^{(n)}_{x_1, \dots, x_n} := \bigotimes_{i=1}^n L_{x_i}.$$

Since the map $\mathrm{Sym}^n C \to \mathrm{Pic}^n C$ has simply connected fibers for $n \gg 0$, then this local system descends to a local system $E_L \to \mathrm{Pic}^n C$ for sufficiently large n. This satisfies the Hecke property: if $\mathcal{L} \in \mathrm{Pic}\, C$, then

$$E_L|_{\mathcal{L}(x)} \simeq E_L|_{\mathcal{L}} \otimes L_x.$$

Here $x \in C$, and $\mathcal{L}(x) := \mathcal{L} \otimes \mathcal{O}(x)$. This property is a kind of multiplicative axiom.

Deligne proved that a sheaf defined on $\mathrm{Pic}^{n \gg 0} C$ with the Hecke property extends uniquely to a sheaf on all of $\mathrm{Pic}\, C$, and the extension still satisfies the Hecke property.

[19]By $\mathcal{O}(x + y)$, we mean the skyscraper on the divisor $x + y$, which is $\mathcal{O}(x) \otimes \mathcal{O}(y)$.

In some sense, we're thinking about the map $\varphi = -\otimes \mathcal{O}(x)\colon \operatorname{Pic} C \to \operatorname{Pic} C$. Its graph is $\mathcal{H}ecke^1_x = \{\mathcal{L}, \mathcal{L}' \mid \mathcal{L}' \cong \mathcal{L}(x)\}$. And E_L is an *eigensheaf* for φ with eigenvalue L under the kernel transform

$$\begin{tikzcd} & \mathcal{H}ecke^1_x \arrow[dl] \arrow[dr] & \\ \operatorname{Pic} C & & \operatorname{Pic} C. \end{tikzcd}$$

That is, you obtain an isomorphism $E_L \cong L_x \otimes E_L$. If you vary x, you get a correspondence

$$\begin{tikzcd} & \mathcal{H}ecke^1 \arrow[dl] \arrow[dr] & \\ C \times \operatorname{Pic} C & & \operatorname{Pic} C, \end{tikzcd}$$

which is the graph of a map $(x, \mathcal{L}) \mapsto \mathcal{L}(x)$. Thus, $\mathcal{H}ecke^1 = \{(x, \mathcal{L}, \mathcal{L}') \mid \mathcal{L}' \cong \mathcal{L}(x)\}$, and the eigenproperty for all x at once is that applying the kernel transform gives

$$\mathcal{H}ecke^1 * E_L \cong L \boxtimes E_L$$

as sheaves over $C \times \operatorname{Pic} C$. You can expand this even more: this map is a piece of something bigger. Through the Abel-Jacobi map, $C \times \operatorname{Pic} C \subset \operatorname{Pic} C \times \operatorname{Pic} C$, and since $\operatorname{Pic} C$ is a group, we obtain a multiplication map $\mu\colon \operatorname{Pic} C \times \operatorname{Pic} C \to \operatorname{Pic} C$. The eigenproperty in this case says that $\mu^* E_L \cong E_L \boxtimes E_L$.

There's a sense in which $\operatorname{Pic} C$ is the free abelian group on C in the category of schemes. The free abelian monoid on C is $\operatorname{Sym}^\bullet C$: we're taking formal commutative sums of elements in C, but we need to add inverses. This defines a map $\operatorname{Sym}^\bullet C \to \operatorname{Pic} C$, and this map is a group completion map. There is a precise statement locally, but maybe not globally. The takeaway is that by freeness, something multiplicative on $\operatorname{Pic} C$ is determined by its restriction to C. This is the algebraic analogue of the Dold-Thom construction: $\operatorname{Sym}^\bullet C$ plays the role of the infinite symmetric product, and $\operatorname{Pic} C$ plays the role of homology.

There's already something interesting in the local setting, Contou-Carrère's local self-duality of the Jacobian. This relates homology, built out of a free abelian group, with some compactly supported cohomology.

If S is a set, there's a nice construction of the free abelian group $F(S)$ on S: by the universal property,

$$\operatorname{Hom}_{\mathsf{Ab}}(F(S), \mathbb{G}_m) \cong \operatorname{Hom}_{\mathsf{Set}}(S, \mathbb{G}_m) = \mathcal{O}^*(S),$$

the invertible functions on the set. Now, we can do something awesome, which is Cartier duality: by duality, $F(S)$ is the Cartier dual of $\mathcal{O}^*(S)$: $F(S) = (\mathcal{O}^*(S))^\vee$. This insight is courtesy of Ivan Mirković. For sets, this is true on the nose, and agrees with more naïve constructions, but we can use to *define* the free abelian group on a geometric object.

We'll replace S with the *formal disc* $D \coloneqq \operatorname{Spf} \mathbb{C}[[x]] \subset C$. This is a subset of C obtained by completing at x, and should be thought of as a disc neighborhood of x (this is the sense in which this description is local). Thus, we let $F(D) \coloneqq (\mathcal{O}^*(D))^\vee$, the Cartier (or Pontrjagin) dual of $\mathbb{C}[[x]]^\vee$. The main ingredient in setting this up is the *Weil pairing* or *Weil symbol*

$$\langle f, g\rangle = \frac{f^{\deg g}}{g^{\deg f}},$$

which is useful, e.g. for understanding H^1 of an elliptic curve.

If $K \coloneqq \mathbb{C}((x))$, then $\mathcal{K}^* = \mathbb{C}((x))^*$ and $(\mathcal{O}^*)^\vee = \mathcal{K}^*/\mathcal{O}^*$. This is something that maps to $\mathbb{Z}$, but whose kernel is some infinite-dimensional formal abelian group, looking something like a formal completion of $\mathbb{A}^\infty$. The reason is that this duality is at the level of group schemes or functors: $\mathcal{K}^*/\mathcal{O}^*$ is the functor that, given any $\mathbb{C}$-algebra R, return $R((x))^\times/R[[x]]^\times$. If $R = \mathbb{C}$, we get $\mathbb{Z}$, so $\mathbb{Z}$ parameterizes the complex points of $\mathcal{K}^*/\mathcal{O}^*$, but in general we get Taylor series

$$\cdots + a_{-N-1}x^{-N} + a_{-N}x^N + a_{-N+1}x^{-N+1} + \cdots,$$

with "leading term $-N$," meaning that all terms below $a_{-N}x^{-N}$ are nilpotent. Thus, $\mathcal{K}^*/\mathcal{O}^*$ is a group whose connected components are parameterized by $\mathbb{Z}$, but each component has an infinite-dimensional "fuzz."

Definition 7.2. $\mathcal{K}^*/\mathcal{O}^*$ is called the *affine Grassmanian* for GL_1, i.e. $\mathcal{K}^*.\mathcal{O}^* = \mathrm{GL}_1(K)/\mathrm{GL}_1(\mathcal{O})$, which is written $\mathrm{Gr}_{\mathrm{GL}_1}$.[20]

What we're saying is that the affine Grassmannian is the free abelian group on the disc.

Geometrically, $\mathcal{K}^*/\mathcal{O}^*$ is the moduli of line bundles on C together with a trivialization away from x. In this interpretation, $\mathcal{O}^*$ detects the trivialization on D, and $\mathcal{K}^* = \mathrm{GL}_1(K)$ is the line bundles plus a trivialization on $C \setminus x$ and on D near x.

These are the compactly supported line bundles $\mathrm{Gr}_{\mathrm{GL}_1} = H^1_x(C, \mathcal{O}^*)$, the line bundles supported at x. The idea is that instead of requiring functions to vanish outside of a set, we ask line bundles to be trivial outside of a point. Contou-Carrère duality says that the homology of D, which is identifed with $F(D) = (\mathcal{O}^*(D))^\vee$, is $\mathcal{K}^*/\mathcal{O}^* = H^1_x * C, \mathcal{O}^*)$.

The affine Grassmannian is a group, and it acts on $\operatorname{Pic} C$ given an $x \in C$, modifying a line bundle by x. We wrote $\operatorname{Pic} C$ in terms of transition functions last time, as

$$\operatorname{Pic} C = \mathbb{C}(X) \backslash \prod_{x \in X}{}' \mathcal{K}^* / \prod_{x \in C} \mathcal{O}^*,$$

and the Grassmannian is a factor $\mathcal{K}^*/\mathcal{O}^*$ at the index $x \in X$. The forgetful map $\mathrm{Gr}_{\mathrm{GL}_1} \to \operatorname{Pic} C$ contains the map $D \to C$, which is surjective on tangent spaces (formally), i.e. we obtain a sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{K}/\mathcal{O} \longrightarrow H^1(C, \mathcal{O}) \longrightarrow H^1(C, \mathcal{K}) = 0.$$

This is not surjective as groups: one must pass to tangent spaces.

Returning to the Hecke correspondence, $\mathcal{H}ecke_x \cong \mathrm{Gr}_{\mathrm{GL}_1} \times \operatorname{Pic} C$, and the correspondence is the action map $\mathrm{Gr}_{\mathrm{GL}_1} \times \operatorname{Pic} C \to \operatorname{Pic} C$. And the name $\mathcal{H}ecke$ is not a coincidence: it's the same idea as the Hecke algebras we saw last semester. Namely, if a group G acts on a space X and $K \subset G$ is a subgroup, then the groupoid $K \backslash G / K = \bullet/K \times_{\bullet/G} \bullet/K$ acts on $K \backslash X$. Here, everything is exactly the same: $\mathrm{Gr}_{\mathrm{GL}_1} = K \backslash G/K$, where $G = \mathcal{K}^*$ and $K = \mathcal{O}^*$. And $\operatorname{Pic} C = K \backslash X$, where $X = (\operatorname{Pic}(C, x))^\vee$, the space of line bundles on C trivialized on D, which $\mathcal{K}^*$ acts on. Since we're in the abelian case, things are much cleaner: the affine Grassmannian is the free abelian group on the disc. Mirković is trying to generalize this to other groups.

Geometric class field theory as a Fourier transform. There are at least five ways to view geometric class field theory as a duality (hence a kind of Fourier transform). We'll set them up, but might not get to all of them.

The idea is that geometric class field theory is a way of relating abelian geometry on C with geometry on $\operatorname{Pic} C$. Here "geometry" means "categories of sheaves;" there are three kinds. Let X be a smooth variety, which will play the role of both C and $\operatorname{Pic} C$.

- **Dolbeault:** A *Higgs sheaf* $\mathcal{F}$ on X is a quasicoherent sheaf of modules for $\operatorname{Sym} T_X$, where T_X is the tangent sheaf. This is the same thing as saying that $\mathcal{F} = \pi_* \widetilde{\mathcal{F}}$ under the map $\pi\colon T^*X \to X$, where $\widetilde{\mathcal{F}} \in \mathsf{QC}(T^*X)$. This is because T^*X is the relative Spec of the symmetric algebra over X. The data of the T_X-action on $\mathcal{F}$ is equivalent to the data of a *Higgs field* $\theta\colon \mathcal{F} \to \mathcal{F} \otimes \Theta^1$ such that $\theta^2 = 0$. Here, $\theta \in H^0(X, \operatorname{End} F \otimes \Omega^1)$.
- **de Rham:** We look at flat connections or $\mathcal{D}$-modules on X, but otherwise the story looks very similar. Here we consider an $\mathcal{F} \in \mathsf{QC}(X)$ together with an action of $\mathcal{D}_X$ (which is a noncommutative version of the symmetric algebra). You can think of this as the pushforward from a noncommutative version of T^*X, but we don't literally have that description. Since $\mathcal{D}_{\le 1} = \mathcal{O} \oplus T_X$, and this generates $\mathcal{D}$, then the action map for T_X determines the $\mathcal{D}$-module structure on $\mathcal{F}$. However, this isn't a linear action: we get a *connection*, a map $\nabla\colon \mathcal{F} \to \mathcal{F} \otimes \Omega^1$, and it's a *flat connection*, meaning $\nabla^2 = 0$.
- **Betti:** This is the easiest version: we look at locally constant sheaves on X, or rank-1 local systems. Roughly, this is the same as representations of $\pi_1(X)$ (you may need to think about the fundamental groupoid instead).

Hodge theory relates these three things. A Higgs line bundle is defined by a line bundle $\mathcal{L}$ plus a 1-form $\theta \in \operatorname{End}(\mathcal{L}) \otimes \Omega^1 = \Omega^1$, so lives in $H^1(X, \mathcal{O}) \oplus H^0(X, \Omega^1) = H^1_{\mathrm{Dol}}(X)$. Similarly, the de Rham case realizes a flat line bundle in $H^1_{\mathrm{dR}}(X; \mathbb{C})$, and the Betti case realizes a rank-1 local system in the Betti cohomology $H^1_{\mathrm{Betti}}(X; \mathbb{C})$. Hodge theory tells us these three H^1s are isomorphic.

[20] A lot of what we're saying in this section generalizes to affine Grassmannians for other groups.

Let $\mathrm{Higgs}_{\mathrm{GL}_1}$ denote the moduli space for Higgs line bundles, $\mathrm{Conn}_{\mathrm{GL}_1}$ denote the moduli space for flat line bundles, and $\mathrm{Loc}_{\mathrm{GL}_1}$ be the moduli space for rank-1 local systems. Then, Hodge theory defines a diffeomorphism $\mathrm{Higgs}_{\mathrm{GL}_1}(X) \xrightarrow{\cong} \mathrm{Conn}_{\mathrm{GL}_1}(X)$, though they have different algebraic structures, and the Riemann-Hilbert correspondence establishes a diffeomorphism $\mathrm{Conn}_{\mathrm{GL}_1}(X) \to \mathrm{Loc}_{\mathrm{GL}_1}(X)$.

If $X = C$ is a curve, then $\mathrm{Higgs}_{\mathrm{GL}_1}(C) = \mathrm{Pic}\, C \times H^0(C, \Omega^1) = T^* \mathrm{Pic}\, C$. Geometric class field theory relates these three kinds of sheaves to corresponding sheaves on Pic.

- **Betti:** Again the Betti version is the easiest case: there's an equivalence of categories $\mathsf{QC}(\mathrm{Loc}_{\mathrm{GL}_1}(\mathbb{C})) \xrightarrow{\cong} \mathrm{Loc}(\mathrm{Pic}\, C)$. It's even simple to describe: since $\mathrm{Loc}_{\mathrm{GL}_1}(C)$ is an affine variety, we can think about rings: $\mathbb{C}[\mathrm{Loc}_{\mathrm{GL}_1}(C)] = \mathbb{C}[\mathrm{Hom}_{\pi_1(C)}, \mathbb{C}^*]$, but a map $\pi_1(C) \to \mathbb{C}^*$ factors through the abelianization $H_1(C, \mathbb{Z})$, and this is a lattice, so we get an functions on an algebraic torus $\mathbb{C}[H^1(C, \mathbb{Z}) \otimes \mathbb{C}^*]$. Thus, we can also write it as $\mathbb{C}[\Lambda]$ or $\mathbb{C}[\pi_1 \mathrm{Pic}\, C]$: functions on $\mathrm{Loc}_{\mathrm{GL}_1}(\mathbb{C})$ are identified with functions on $\pi_1(\mathrm{Pic}(C)) = H_1(C, \mathbb{Z})$, and therefore the categories of modules over these rings are equivalent. Explicitly, if $L \in \mathrm{Loc}_{\mathrm{GL}_1}(C)$, its skyscraper $\mathcal{O}_{\{L\}} \mapsto E_L$: a local system on the curve defines a local system on $\mathrm{Pic}\, C$.
- **Dolbeault:** In this case, we have a derived equivalence. $\mathrm{Higgs}_{\mathrm{GL}_1}(C) = T^* \mathrm{Pic}(C) = \mathrm{Pic}\, C \times H^0(C, \Omega^1)$. The derived equivalence is $\mathsf{QC}(\mathrm{Higgs}_{\mathrm{GL}_1}(C)) \cong \mathrm{Higgs}(\mathrm{Pic}\, C)$. Projecting onto the second factor $\mathrm{Higgs}_{\mathrm{GL}_1}(C) \to H^0(C, \Omega^*)$ is a bundle of Cs, and deserves to be called the *Hitchin map*. But we can also think of it as a projection from $T^* \mathrm{Pic}\, C$. You can think of the equivalence as doing the Fourier-Mukai transform on each fiber: given a Higgs line bundle (L, θ), the skyscraper $\mathcal{O}_{\{L,\theta\}}$, we get an element of the cotangent space at $\mathcal{L} \in \mathrm{Pic}\, C$. You can also think of the forgetful map $\mathsf{QC}(\mathrm{Higgs}_{\mathrm{GL}_1}(C)) \to \mathsf{QC}(\mathrm{Pic}\, C)$, and ask under the Fourier-Mukai transform $\mathsf{QC}(\mathrm{Pic}\, C) \to \mathsf{QC}(\mathrm{Pic}\, C)$, where does the extra structure (the Higgs field) go? Alternatively, you could think of this as $\mathsf{QC}(\mathrm{Pic}\, C \times H^0(C, \Omega^1))$, where the Fourier-Mukai transform only acts on the first component.
- **de Rham:** This sets us up nicely for the de Rham case: we have an affine bundle $H^0(C, \Omega^1) \to \mathrm{Conn}_{\mathrm{GL}_1}(C) \to \mathrm{Pic}\, C$, which is a twisted version of the cotangent bundle. It's almost a torsor (but there are a few empty sets). The extension $\mathrm{Conn}_{\mathrm{GL}_1}(C)$ is an extension of $\mathrm{Pic}\, C$ by $H^0(C, \Omega^1)$, the vector space of connections on $\mathcal{O}$. You can use Cartier duality on this: the dual of a vector space V is $V^\vee = \bullet/\widehat{V}^*$, and $(\mathrm{Conn}_{\mathrm{GL}_1 C})^\vee = (\mathrm{Pic}\, C)^\vee/(H^1(C, \mathcal{O}))^\vee$, the de Rham stack of $\mathrm{Pic}\, C$.

  As in the Dolbeault case, there's a forgetful map $\mathsf{QC}(\mathrm{Conn}_{\mathrm{GL}_1}(C)) \to \mathsf{QC}(\mathrm{Pic}\, C)$, and you can trace through what this structure is under the Fourier-Mukai transform, which ends up being a flat connection: we obtain $\mathcal{D}(\mathrm{Pic}\, C)$. If you're familiar with ordinary class field theory, this is no surprise: there's an identification of the space of flat connections on C and the space of flat connections on $\mathrm{Pic}\, C$.

This description of geometric class field theory as the identification $\mathrm{Conn}_{\mathrm{GL}_1} C \cong \mathrm{Conn}_{\mathrm{GL}_1}(\mathrm{Pic}\, C)$ has to do with the description in terms of the Poincaré bundle we've discussed before: we had a kernel transform

$$\begin{tikzcd} & \mathrm{Pic}\, C \times \mathrm{Pic}\, C \arrow[dl] \arrow[dr] & \\ \mathrm{Pic}\, C & & \mathrm{Pic}\, C, \end{tikzcd}$$

and instead we want to throw in $\mathrm{Conn}_{\mathrm{GL}_1}(\mathrm{Pic}\, C)$: the general theory of moduli spaces defines a tautological de Rham sheaf $\mathcal{P} \to \mathrm{Conn}_{\mathrm{GL}_1}(X) \times X$, which defines a kernel transform

$$\begin{tikzcd} & \mathrm{Conn}_{\mathrm{GL}_1}(X) \times X \arrow[dl] \arrow[dr] & \\ \mathrm{Conn}_{\mathrm{GL}_1}(X) & & X \end{tikzcd}$$

(pullback, tensor with $\mathcal{P}$, then pushforward). This defines an equivalence $\mathsf{QC}(\mathrm{Conn}_{\mathrm{GL}_1} X) \to \mathcal{D}(X)$, which sends $\mathcal{O}_{\mathcal{L},\nabla} \mapsto \mathcal{L}, \nabla$.

There are Betti, Dolbeault, and de Rham versions of the geometric Langlands theorem for GL_1, which are all some version of "$\mathrm{Conn}_{\mathrm{GL}_1} C \cong \mathrm{Conn}_{\mathrm{GL}_1}(\mathrm{Pic}\, C)$." They are much easier than for any other group.

There will be two more versions of this, which we'll talk about next time, which are quantum deformations and have cool formulas:

$$\mathsf{QC}_q(\mathrm{Loc}_{\mathrm{GL}_1} C) \cong \mathrm{Loc}_q(\mathrm{Pic}\, C),$$

where we take deformation quantizations, which provide some kind of noncommutativity. Just like the Betti case wasn't too hard, neither will be the quantum case.

For example, $\mathrm{Loc}_{\mathrm{GL}_1}(T^2) = \mathbb{C}^* \times \mathbb{C}^*$, and functions on it are the algebra $\mathbb{C}[x, x^{-1}, y, y^{-1}]$, and the quantum version deforms the commutativity of x and y:

$$\mathbb{C}[\mathrm{Loc}_{\mathrm{GL}_1} T_q^2] = \mathbb{C}\langle x, x^{-1}, y, y^{-1}\rangle/(xy = qyx).$$

This is called the *q-Weyl algebra*. These will relate to twisted local systems, i.e. local systems on nontrivial $\mathbb{C}^\times$-bundles over T^2.

8. Quantum geometric Langlands: 5/5/17

Continuing where we were last time, more or less, let M be a smooth variety and $\mathcal{L} \to M$ be a manifold. In this generality, you can construct a two-parameter family of (sheaves of) algebras in parameters ε_1 and ε_2.[21]

We'll construct these as two separate deformations of $\mathrm{Sym}(T_M) = \mathcal{O}^*(T^*M)$ (which will be the algebra when $\varepsilon_1 = \varepsilon_2 = 0$), and then combine them.

(1) When $\varepsilon_1 \neq 0$, we'll take the *differential operators* $\mathcal{D}_{\varepsilon_1}$, which is the *Rees construction*

$$\mathcal{D}_{\varepsilon_1} := \bigoplus_{n\in\mathbb{N}} \varepsilon^n \mathcal{D}_{\le n},$$

where $\mathcal{D}$ is the algebra of differential operators. This is a $\mathbb{C}[\varepsilon_1]$-algebra, and this is homogeneous in rescaling (in fact, the whole 2-parameter family will be invariant under rescaling both ε_1 and ε_2). When $\varepsilon = 0$, we just get the associated graded, which is $\mathrm{Sym}(T_M)$ again.

(2) For ε_2, we'll again apply the Rees construction to obtain a filtered *commutative* algebra. Let $T^*_{\mathcal{L}}M$ denote the *twisted cotangent bundle*, a symplectic T^*M-torsor (or affine bundle) which is locally isomorphic to T^*M. You can think of using $\mathcal{L}$ to twist T^*M, such that if $\mathcal{L}$ were trivial, you would just get T^*M back. You can also identify this with the space of connections on $\mathcal{L}$ through the identification of $\alpha \in \Omega^1_M$ to the connection $\mathrm{d} + \alpha$. Now, when $\varepsilon_2 = 1$, we'll get $\mathcal{O}(T^*_{\mathcal{L}}M)$, which is filtered as

$$\mathcal{O}(T^*_{\mathcal{L}}M) = \bigcup_{n\in\mathbb{N}} \mathcal{O}_{\le n}(T^*_{\mathcal{L}}M).$$

Thus, for $\varepsilon_2 = i$, we get

$$\bigoplus \varepsilon_2^i \mathcal{O}_{\le i}(T^*_{\mathcal{L}}M).$$

You could think of this term as $\mathcal{O}_{\le i}(T^*_{\mathcal{L}^{\varepsilon_2}}M)$, but this isn't entirely true — when $\varepsilon_2 \in \mathbb{Z}$, you can take $\mathcal{L}^{\varepsilon_2}$ and this works, but we want to consider $\varepsilon_2 \in \mathbb{C}$.

The standard (amazing but very dense) reference for twisted cotangent bundles and twisted differential operators is Beilinson-Bernstein [1].

Now we want to put things in the middle: they'll look like differential operators and depend on the line bundle. Namely, we'll twist the differential operators by the line bundle: at $(\varepsilon_1, \varepsilon_2)$, we'll assign $\mathcal{D}_{\varepsilon_1}(\mathcal{L}^{\varepsilon_2/\varepsilon_1})$. This will be homogeneous under $q \cdot (\varepsilon_1, \varepsilon_2) = (q\varepsilon_1, q\varepsilon_2)$ for all $z \in \mathbb{C}$.

Let's interpret this. Both ε_1 and ε_2 have the same units in the physics analogy, so $k = \varepsilon_2/\varepsilon_1$ is a dimensionless quantity. If $k \in \mathbb{Z}$, then the twisted differential operators $\mathcal{D}(\mathcal{L}^k)$ are the sheaf of differential operators acting on sections of $\mathcal{L}^k$, so we can look at $\mathcal{L}^k \otimes \mathcal{D} \otimes \mathcal{L}^{-k}$. We'll also get a short exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{D}_{\le 1}(\mathcal{L}^k) \longrightarrow T \longrightarrow 0.$$

Under rescaling by q, $\mathcal{D}_{q\varepsilon_1}(\mathcal{L}^{q\varepsilon_2/q\varepsilon_1})$ converges to $\mathcal{O}(T^*_{\mathcal{L}^{\varepsilon_2}}M)$, so this is a flat two-parameter family of algebras. The Rees construction gives you a filtration, and the associated graded is always $\mathcal{O}(T^*M)$. The limit on a horizontal line (sending $\varepsilon_1 \to 0$) is not a Rees construction, and the algebras in question are distinct! But any vertical line is a Rees construction.

So clearly this is not symmetric if you exchange ε_1 and ε_2. This is geometric Langlands, in a sense.

[21] This notation comes from physics, so physicists might recognize where these ε_1 and ε_2 are coming from.

There's a paper of Ben-Zvi and Frankel which uses different coordinates for a two-parameter family over $\mathbb{A}^1\times\mathbb{P}^1$ in terms of $(\lambda, k) \mapsto \mathcal{D}_\lambda(\mathcal{L}^k)$, but the vertical axis has less going on, and you can't see the homogeneity. The physics picture we're contemplating today is a kind of blow-down of the other construction.

With this we can state a glorified version of geometric class field theory: we've seen a bunch of different equivalences of categories, and this perspective will allow us to stitch them all together. Let C be a projective curve over $\mathbb{C}$, $M = \operatorname{Pic} C$, and $\mathcal{L} = \Theta$ be the determinant line bundle, as we discussed last time. This is the line bundle that expresses the self-duality of $\operatorname{Pic}(C)$.[22]

In this case, we can look at the two-parameter family of algebras, or instead look at the corresponding two-parameter family of categories of modules for these algebras.

- For $\varepsilon_1 = \varepsilon_2 = 0$, we're looking at $\mathsf{QC}(T^* \operatorname{Pic}(C))$, the *Higgs sheaves* on $\operatorname{Pic} C$.
- For $\varepsilon_1 = 1$ and $\varepsilon_2 = 0$, we have the category of $\mathcal{D}$-modules on $\operatorname{Pic} C$.
- For $\varepsilon_1 = 0$ and $\varepsilon_2 = 1$, we have $\mathsf{QC}(T^*_\Theta \operatorname{Pic} C)$.
- In general, let $k = \varepsilon_2/\varepsilon_1$; then, we get the category of $\mathcal{D}(\Theta^k)$-modules on $\operatorname{Pic} C$.

The twisted cotangent bundle has a nice interpretation: $T^*_\Theta \operatorname{Pic} C \cong \operatorname{Conn}_{\mathrm{GL}_1} \operatorname{Pic} C$, which in this case is the same thing as $\operatorname{Conn}_{\mathrm{GL}_1} C$, which is class field theory; the identification with the twisted cotangent bundle is new.[23] You can also write this as $\underline{\operatorname{Conn}}_{\operatorname{Pic} C}\Theta$: (suitably local) connections on the line bundle Θ is the same moduli space as the line bundles with connection on $\operatorname{Pic} C$ itself. This is not a tautology, but there are many ways to prove it, e.g. using the fact that generically (i.e. away from a divisor), you can find a trivialization of the twisted cotangent bundle, and then use that to produce a connection. You can also prove it with the Fourier-Mukai transform.

What this means is that we can update the y-axis of the family of categories: we'll replace it with $\mathsf{QC}(\operatorname{Conn}_{\mathrm{GL}_1} C)$, and at $(0,0)$, we get $\mathsf{QC}(\operatorname{Higgs}_{\mathrm{GL}_1}(C))$.

We've seen that $\mathsf{QC}(\operatorname{Higgs}_{\mathrm{GL}_1} C) \cong \operatorname{Higgs}(\operatorname{Pic} C)$ through a Fourier-Mukai transform (the Dolbeault statement), and the isomorphism is nontrivial.[24] There is also the de Rham version, that $\mathsf{QC}(\operatorname{Conn}_{\mathrm{GL}_1} C) \cong \mathcal{D}(\operatorname{Pic} C)$. These are both more or less classical, in that at least one side is modules over a commutative algebra.

The quantum de Rham correspondence is an equivalence between two categories of modules over noncommutative algebras. However, it follows from the same Fourier-Mukai formalism as the previous two.

Theorem 8.1 (Geometric class field theory, quantum de Rham version)**.** *There's an equivalence of categories between $\mathcal{D}(\Theta^k)$-modules and $\mathcal{D}(\Theta^{-1/k})$-modules.*

We want a unifying result from which all of these follow: a single symmetry and a single family of categories.

Theorem 8.2. *There's a self-equivalence called the* Fourier transform *on our flat family of categories over $\mathbb{A}^2$ sending $\varepsilon_1 \mapsto \varepsilon_2$ and $\varepsilon_2 \mapsto -\varepsilon_1$.*

This exchanges the x- and y-axes (up to a sign, which we can ignore by rescaling).

- At the origin, this is a self-equivalence of $\mathsf{QC}(\operatorname{Higgs}_{\mathrm{GL}_1} C) \cong \mathsf{QC}(T^* \operatorname{Pic} C)$, giving us the Dolbeault version.
- On the x- and y-axes, this is an equivalence $\mathcal{D}(\operatorname{Pic} C) \cong \mathsf{QC}(\operatorname{Conn}_{\mathrm{GL}_1} C)$, which is the (classical) de Rham version.
- Off of the axes, this is an equivalence between $\mathcal{D}(\Theta^k)$-modules and $\mathcal{D}(\Theta^{-1/k})$-modules, yielding the quantum de Rham version.

These are not the only symmetries; we've just seen the symmetry given by

$$\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\end{pmatrix} \longmapsto \underbrace{\begin{pmatrix} & -1\\ 1 & \end{pmatrix}}_{S} \begin{pmatrix}\varepsilon_1\\ \varepsilon_2\end{pmatrix},$$

[22] More abstractly, you could think of $(M, \mathcal{L})$ as a principally polarized abelian variety, and that's all the structure we're using.

[23] If A is any principally polarized abelian variety, $T^*_\Theta A \cong \operatorname{Conn}_{\mathrm{GL}_1} A$.

[24] More generally, if you replace GL_1 with a torus T, you'll get the dual torus $T^\vee$ on the other side.

but we can also act by

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}_{T} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_1 + \varepsilon_2 \end{pmatrix},$$

which sends $\mathcal{L}^k \mapsto \mathcal{L}^{k+1}$. This spells out a Morita equivalence $\mathsf{Mod}_{\mathcal{D}(\mathcal{L}^k)} \cong \mathsf{Mod}_{\mathcal{D}(\mathcal{L}^{k+1})}$; the underlying sheaves aren't equivalent, but their categories of modules are. The action isn't that complicated: when $k = 0$, it's $M \mapsto M \otimes \mathcal{L}$.

Now, we wrote down these two actions because these two matrices S and T are the generators of $\mathrm{SL}_2(\mathbb{Z})$. So we get a whole $\mathrm{SL}_2(\mathbb{Z})$-action on the two-parameter family, which is the most broad version of (unramified) geometric class field theory,

For the general geometric Langlands program, there's a similar formula, albeit with G and $G^\vee$ in different places, and line bundles replaced with rank-n vector bundles.

How this arises from topological field theory. We'll work upwards from two-dimensional field theory to four-dimensional field theory.

The space of states of a two-dimensional theory is $Z(S^1)$ (if you're keeping track of framings, this is the cylinder, not the annulus). It carries an S^1-action by symmetries, which means we can take an ε-deformation $Z(S^1)^{S^1}$, which is a module over $H^*_{S^1}(\bullet)$: the S^1-equivariant cohomology i $H^*(B\S^1)$, and therefore is $\mathbb{C}[\varepsilon]$, where $|\varepsilon| = 2$.[25] Then, we define $Z(S^1)_\varepsilon$ for $\varepsilon \neq 0$ to be the orbit corresponding to nonzero $\varepsilon \in \mathbb{C}$, so we obtain a one-parameter deformation, even though all the $Z(S^1)_\varepsilon$ are isomorphic for $\varepsilon \neq 0$. For $\varepsilon = 0$, we as usual get the Hochschild homology of the category $Z(\bullet)$ of boundary conditions, and the S^1-action are the same.

For example, if Z is the B-model on a smooth complex variety X, then $Z(S^1) \cong \mathbf{R}\Gamma\Omega_X^{-\bullet}$. The ε-deformation is

$$Z(S^1)_\varepsilon = \mathbf{R}\Gamma(\Omega_X^{-\bullet}, \varepsilon \mathrm{d}).$$

This has the correct degree (d acts by -1, then ε by 2, so we get 1). When $\varepsilon = 1$, this relates to the de Rham cohomology: $Z(S^1)_1 \cong H^*_{\mathrm{dR}}(X) \otimes \mathbb{C}[\varepsilon, \varepsilon^{-1}]$. Thus we think of de Rham cohomology as a deformation of Dolbeault cohomology. It's not the fanciest deformation: there are only two isomorphism classes, and the Hodge theorem says when X is projective, it's a trivial deformation. In general, the difference between the two is expressed by the Hodge-to-de Rham spectral sequence. The coordinates ε_1 and ε_2 will come from circle invariants.

Passing to 3-dimensional TFT, $Z(S^1)$ is a category, which we called the category of line defects or line operators. We think of this as the category of boundary conditions in the dimensional reduction $Z(- \times S^1)$, a two-dimensional TFT.

Our main example was Rozansky-Witten theory, which takes as input a holomorphic-symplectic manifold X, e.g. the cotangent bundle T^*M for some complex variety M. One of its distinguishing features is that its dimensional reduction is the B-model on X. In particular, $Z(S^1)$ will be the boundary conditions for the B-model of X, i.e. $\mathsf{QC}(X)$, the derived category of coherent sheaves on X.[26] And since this is the value of a theory on a circle, it admits an S^1-action, hence admits a one-parameter deformation $\mathsf{QC}_\varepsilon(X)$ by passing to (the action of) equivariant cohomology.

How can we describe this? Since X is holomorphic-symplectic, $\mathcal{O}(X)$ has a Poisson bracket, which is commutative. Then, we'll perform deformation quantization to obtain the deformation.

We haven't set up Rozansky-Witten theory in general, but when $X = T^*M$, this begins to look familiar: when $\varepsilon = 0$, we have $\mathsf{QC}(T^*M)$, or modules over $\mathrm{Sym}\, T_M$, and the deformation applies the Rees construction, producing $\mathcal{D}_\varepsilon$-modules. There are some shifts here, e.g. you actually get $\mathrm{Sym}\, T_M[-2]$-modules, equivalent to $\Omega^{-\bullet}[1]$-modules, but it's tricky to worry about the shifts. For this reason, much of the literature only constructs Rozansky-Witten theory as a $\mathbb{Z}/2$-graded theory, but you can promote it to a $\mathbb{Z}$-graded theory. This is a categorification of the passage from Dolbeault cohomology to de Rham cohomology, passing from $\mathsf{QC}(T^*M)$-modules to $\mathcal{D}_\varepsilon$-modules, which are equivalently $(\Omega^{-\bullet}, \mathrm{d})$-modules. In some sense, you're just adding "modules" to everything, but you have to check that it works!

[25]You can think of $H^*_{S^1}(\bullet)$ as the S^1-invariants of $H^*(\bullet)$, but you have to clarify what the S^1-action means to say this.

[26]We're not being exact about coherence/quasicoherence assumptions in the derived setting.

That's the physics origin for the deformation from $\mathsf{QC}(T^*M)$ to $\mathcal{D}_\varepsilon$-modules: you're turning on an S^1-action. Consequently, mathematicians call it "working S^1-equivariantly," and physicists call it the *Nekrasov Ω-background*, which sounds much cooler.

We can get the ε_1-axis from the Rozansky-Witten theory for $T^*\operatorname{Pic} C$, Z'_{RW}: it gives you a deformation from$\mathsf{QC}(T^*\operatorname{Pic} C) = Z'_{RW}(S^1)$ to $Z'_{\mathrm{RW}}(S^1)_\varepsilon$, which is $\mathcal{D}$-modules on $\operatorname{Pic} C$.

But we want to get the other axis out of this, so we need another circle action. Thus we really want a 4-dimensional topological field theory Z (implicitly depending on C and GL_1) such that Z'_{RW} is the dimensional reduction of Z. Thus $Z(S^1 \times S^1) = Z'_{\mathrm{RW}}(S^1) = \mathsf{QC}(T^*\operatorname{Pic} C)$, and therefore we have another circle action. The $\mathrm{SL}_2(\mathbb{Z})$-action we described is the action of the mapping class group of T^2 on $Z(T^2)$.

This theory exists, and is called the *class-S theory* associated to C and $G = \mathrm{GL}_1$. Class S is a family of four-dimensional supersymmetric $\mathcal{N} = 2$ quantum field theories, which in particular give four-dimensional topological field theories through a Donaldson twist, due to Gaiotto-Moore-Neitzke. These accept any group G as input, though are very mysterious and not understood explicitly when $G \neq \mathrm{GL}_1$.

The goal for geometric Langlands in general would be to not depend on C, which brings us to Theory $\mathcal{X}$, a six-dimensional theory[27] depending on the group G (or its Lie algebra), but not the Riemann surface. Then, this four-dimensional theory is the dimensional reduction of Theory $\mathcal{X}$ with C instead of S^1. If you use $C = T^2$, you get $\mathcal{N} = 4$ super-Yang-Mills as contemplated by Kapustin and Witten [3]; their approach has no ε-deformations in it.

Another way to say this is that Kapustin-Witten theory exploits relationships between different Riemann surfaces, unlike class-S theory, which has the ε-parameters. Of course, it would be best if we could figure out theory $\mathcal{X}$, but that's going to take a while. Also, the physics is agnostic about Betti versus de Rham geometric Langlands; it only sees the analytic phenomena.

So you can draw a diagram of dimensional reductions.

$$\begin{tikzcd}
& \mathcal{X} \arrow[dl, "T^2"'] \arrow[dr, "C"] & \\
\text{4D KW} \arrow[dr, "C"] & & \text{4D class } S \arrow[dl, "T^2"] \\
& Z_{\mathrm{KW}}(C). &
\end{tikzcd}$$

This is a very common technique in physics: compactify in two different orders, and since the result cannot depend on the order you compactify, so you get dualities. There are some parameters here, which control what kinds of sheaves you get on the moduli space of Higgs bundle for $Z_{\mathrm{KW}}(C)$.

So the point is, we have a 4-dimensional class-S theory Z and evaluating it on T^2, giving us the 3-dimensional theory on S^1, which is $\mathsf{QC}(T^*\operatorname{Pic} C)$. But the value of a 4-dimensional theory on T^2 suddenly has a whole lot of structure: in particular, it admits an action by $\operatorname{Diff}(T^2) \simeq T^2 \rtimes \mathrm{MCG}(T^2)$. The mapping class group is isomorphic to $\mathrm{SL}_2(\mathbb{Z})$, where S flips the two axes and T is a Dehn twist.

One of our geometric class field theory statements was the interesting auto-equivalence induced by S. The equivalence induced by T is less interesting, namely tensoring with Θ, but the fact that they together generate an $\mathrm{SL}_2(\mathbb{Z})$-action is interesting.

We also have the T^2-component of $\operatorname{Diff}(T^2)$. Any particular group element doesn't give you much, since this is a homotopical action, but we can take T^2-invariants: $Z(T^2)^{T^2}$ is a module over $H^*_{S^1\times S^1}(\bullet) \cong \mathbb{C}[\varepsilon_1, \varepsilon_2]$, where $|\varepsilon_1| = |\varepsilon_2| = 2$. That is, $Z(T^2)$ has a canonical deformation into a two-parameter family of categories.

The $\mathrm{SL}_2(\mathbb{Z})$-action on this two-parameter family is told to us by the semidirect product, coming from the $\mathrm{SL}_2(\mathbb{Z})$-action on the torus. So it acts by the same matrices: S switches the axes and T is a shear transformation, as we mentioned. In particular, we get a lot of derived equivalences, e.g. between deformations over the x-axis and over the y-axis, the origin and itself, etc.

So this deformation and these auto-equivalences always exist for $Z(T^4)$, in any four-dimensional topological field theory. But what are they in our setting?

We know one direction gives you the noncommutative deformation $\mathsf{QC}(T^*\operatorname{Pic} C)$ to $\mathcal{D}_{\operatorname{Pic} C}$-modules, which came from Rozansky-Witten theory. But rotating the other circle is going to change which 3-dimensional theory you're looking at, and actually ε_2 parametrizes a family of 3-dimensional field theories $Z(S^1 \times -)^{S^1}$.

[27]Well, there's an anomaly, so it's really a relative field theory. But that's a digression we won't make today.

This family is Rozansky-Witten theory of the twisted cotangent bundle $T^*_{\Theta^{\varepsilon_2}} \operatorname{Pic} C$, which is a perfectly nice holomorphic symplectic manifold, so this makes sense. However, there's some worries about losing the $\mathbb{C}^\times$-action or the $\mathbb{Z}$-grading, but these end up canceling each other out, and everything works.

The upshot is that if you look at $Z(S^1 \times S^1)$, where $\varepsilon_1 = 0$ but $\varepsilon_2 \neq 0$, you get $\mathsf{QC}(T^*_{\Theta^{\varepsilon_2}} \operatorname{Pic} C)$, as hoped for. And we know what it means to turn on ε_1 in this theory: it's deformation quantization, so we get the deformation quantization of $T^*_{\Theta^{\varepsilon_2}} \operatorname{Pic} C$, which we wrote down last time to be $\mathcal{D}(\Theta^{\varepsilon_2/\varepsilon_1})$-modules. This is related to the fact that $\mathcal{O}(\operatorname{Conn} \mathcal{L})$ has for a deformation quantization $\mathcal{D}(\mathcal{L}^{1/\hbar})$ (so $\hbar \to \infty$ turns $\mathcal{D}$-modules into connections, and we get commutativity).

Now, replacing GL_1 with G, and some details, you have geometric Langlands.

References

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