M392C NOTES: REPRESENTATION THEORY

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These notes were taken in UT Austin's M392C (Representation Theory) class in Spring 2017, taught by Sam Gunningham. I live-TEXed them using vim, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Thanks to Sam Gunningham, Jay Hathaway, and Surya Raghavendran for correcting a few errors.

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Lecture 1.

Lie groups and smooth actions: 1/18/17

"I've never even seen this many people in a graduate class... I hope it's good."

Today we won't get too far into the math, since it's the first day, but we'll sketch what exactly we'll be talking about this semester.

This class is about representation theory, which is a wide subject: previous incarnations of the subject might not intersect much with what we'll do, which is the representation theory of Lie groups, algebraic groups, and Lie algebras. There are other courses which cover Lie theory, and we're not going to spend much time on the basics of differential geometry or topology. The basics of manifolds, topological spaces, and algebra, as covered in a first-year graduate class, will be assumed.

In fact, the class will focus on the reductive semisimple case (these words will be explained later). There will be some problem sets, maybe 2 or 3 in total. The problem sets won't be graded, but maybe we'll devote a class midsemester to going over solutions. If you're a first-year graduate student, an undergraduate, or a student in another department, you should turn something in, as per usual.

Time for math.

We have to start somewhere, so let's define Lie groups.

Definition 1.1. A *Lie group* G is a group object in the category of smooth manifolds. That is, it's a smooth manifold G that is also a group, with an operation $m : G \times G \to G$, a C^∞ map satisfying the usual group axioms (e.g. a C^∞ inversion map, associativity).

Though in the early stages of group theory we focus on finite or at least discrete groups, such as the dihedral groups, which describe the symmetries of a polygon. These have discrete symmetries. Lie groups are the objects that describe continuous symmetries; if you're interested in these, especially if you come from physics, these are much more fundamental.

Example 1.2. The group of $n \times n$ invertible matrices (those with nonzero determinant) is called the *general linear group* $\mathrm{GL}_n(\mathbb{R})$. Since the determinant is multiplicative, this is a group; since $\det(A) \neq 0$ is an open condition, as the determinant is continuous, $\mathrm{GL}_n(\mathbb{R})$ is a manifold, and you can check that multiplication is continuous. ◀

Example 1.3. The *special linear group* $\mathrm{SL}_n(\mathbb{R})$ is the group of $n \times n$ matrices with determinant 1. This is again a group, and to check that it's a manifold, one has to show that 1 is a regular value of $\det : M_n(\mathbb{R}) \to \mathbb{R}$. But this is true, so $\mathrm{SL}_n(\mathbb{R})$ is a Lie group. $\blacktriangleleft$

Example 1.4. The *orthogonal group* $\mathrm{O}(n) = \mathrm{O}(n, \mathbb{R})$ is the group of orthogonal matrices, those matrices A for which $A^t = A^{-1}$. Again, there's an argument here to show this is a Lie group. $\blacktriangleleft$

You'll notice most of these are groups of matrices, and this is a very common way for Lie groups to arise, especially in representation theory.

We can also consider matrices with complex coefficients.

Example 1.5. The *complex general linear group* $\mathrm{GL}_n(\mathbb{C})$ is the group of $n \times n$ invertible complex matrices. This has several structures.

- For the same reason as $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$ is a Lie group.
- $\mathrm{GL}_n(\mathbb{C})$ is also a *complex Lie group*: it's a complex manifold, and multiplication and inversion are not just smooth, but holomorphic.
- It's also a *algebraic group* over $\mathbb{C}$: a group object in the category of algebraic varieties. This perspective will be particularly useful for us.

We can also define the *unitary group* $\mathrm{U}(n)$, the group of $n \times n$ complex matrices such that $A^\dagger = A^{-1}$: their inverses are their transposes. One caveat is that this is *not* a complex Lie group, as this equation isn't holomorphic. For example, $\mathrm{U}(1) = \{z \in \mathbb{C} \text{ such that } |z| = 1\}$ is topologically S^1, and therefore is one-dimensional as a real manifold! This is also $\mathrm{SO}(2)$ (the circle acts by rotating $\mathbb{R}^2$). More generally, a *torus* is a finite product of copies of $\mathrm{U}(1)$. $\blacktriangleleft$

There are other examples that don't look like this, *exceptional groups* such as G_2, E_6, and F_4 which are matrix groups, yet not obviously so. We'll figure out how to get these when we discuss the classification of simple Lie algebras.

Here's an example of interest to physicists:

Example 1.6. Let q be a quadratic form of signature $(1, 3)$ (corresponding to Minkowski space). Then, $\mathrm{SO}(1, 3)$ denotes the group of matrices fixing q (origin-fixing isometries of Minkowski space), and is called the *Lorentz group*.

Smooth actions. If one wants to add translations, one obtains the *Poincaré group* $\mathrm{SO}(1, 3) \ltimes \mathbb{R}^{1,3}$. $\blacktriangleleft$

In a first course on group theory, one sees actions of a group G on a set X, usually written $G \curvearrowright X$ and specified by a map $G \times X \to X$, written $(g, x) \mapsto g \cdot x$. Sometimes we impose additional structure; in particular, we can let X be a smooth manifold, and require G to be a Lie group and the action to be smooth, or Riemannian manifolds and isometries, etc.[1]

It's possible to specify this action by a continuous group homomorphism $G \to \mathrm{Diff}(X)$ (or even smooth: $\mathrm{Diff}(X)$ has an infinite-dimensional smooth structure, but being precise about this is technical).

Example 1.7. $\mathrm{SO}(3) := \mathrm{SL}_3(\mathbb{R}) \cap \mathrm{O}(3)$ denotes the group of rotations of three-dimensional space. Rotating the unit sphere defines an action of $\mathrm{SO}(3)$ on S^2, and this is an *action by isometries*, i.e. for all $g \in \mathrm{SO}(3)$, the map $S^2 \to S^2$ defined by $x \mapsto g \cdot x$ is an isometry. $\blacktriangleleft$

Example 1.8. Let $\mathbb{H} := \{x + iy \mid y > 0\}$ denote the upper half-plane. Then, $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ by Möbius transformations. $\blacktriangleleft$

Smooth group actions arise in physics: if S is a physical system, then the symmetries of S often form a Lie group, and this group acts on the space of configurations of the system.

Where do representations come into this? Suppose a Lie group G acts on a space X. Then, G acts on the complex vector space of functions $X \to \mathbb{C}$, and G acts by linear maps, i.e. for each $g \in G$, $f \mapsto f(g \cdot -)$ is a linear map. This is what is meant by a representation, and for many people, choosing certain kinds of functions on X (smooth, continuous, L^2) is a source of important representations in representation theory. Representations on $L^2(X)$ are particularly important, as $L^2(X)$ is a Hilbert space, and shows up as the state space in quantum mechanics, where some of this may seem familiar.

[1] What if X has singular points? It turns out the axioms of a Lie group action place strong constraints on where singularities can appear in interesting situations, though it's not completely ruled out.

Representations.

Definition 1.9. A *(linear) representation* of a group G is a vector space V together with an action of G on V by linear maps, i.e. a map $G \times V \to V$ written $(g, v) \mapsto g \cdot v$ such that for all $g \in G$, the map $v \mapsto g \cdot v$ is linear.

This is equivalent to specifying a group homomorphism $G \to \mathrm{GL}(V)$.[2] Sometimes we will abuse notation and write V to mean V with this extra structure.

If G is in addition a Lie group, one might want the representation to reflect its smooth structure, i.e. requiring that the map $G \to \mathrm{GL}(V)$ be a homomorphism of Lie groups.

The following definition, codifying the idea of a representation that's as small as can be, is key.

Definition 1.10. A representation V is *irreducible* if it has no nontrivial invariant subspaces. That is, if $W \subseteq V$ is a subspace such that for all $w \in W$ and $g \in G$, $g \cdot w \in W$, then either $W = 0$ or $W = V$.

We can now outline some of the goals of this course:

- Classify the irreducible representations of a given group.
- Classify all representations of a given group.
- Express arbitrary representations in terms of irreducibles.

These are not easy questions, especially in applications where the representations may be infinite-dimensional.

Example 1.11 (Spherical harmonics)**.** Here's an example of this philosophy in action.[3]

Let's start with the Laplacian on $\mathbb{R}^3$, a second-order differential operator

$$\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2,$$

which acts on $C^\infty(\mathbb{R}^3)$. After rewriting in spherical coordinates, the Laplacian turns out to be a sum

$$\Delta = \frac{1}{r^2}\Delta_{\mathrm{sph}} + \Delta_{\mathrm{rad}},$$

of spherical and radial parts independent of each other, so Δ_{sph} acts on functions on the sphere. We're interested in the eigenfunctions for this spherical Laplacian for a few reasons, e.g. they relate to solutions to the *Schrödinger equation*

$$\dot{\psi} = \widehat{H}(\psi),$$

where the Hamiltonian is

$$\widehat{H} = -\Delta + V(r),$$

where V is a potential.

The action of $\mathrm{SO}(3)$ on the sphere by rotation defines a representation of $\mathrm{SO}(3)$ on $C^\infty(S^2)$, and we'll see that finding the eigenfunctions of the spherical Laplacian boils down to computing the irreducible components inside this representation:

$$V_0 \oplus V_2 \oplus V_4 \oplus \cdots \overset{\text{dense}}{\subseteq} C^\infty(S^2),$$

where the V_{2k} run through each isomorphism class of irreducible representations of $\mathrm{SO}(3)$. They are also the eigenspaces for the spherical Laplacian, where the eigenvalue for V_{2k} is $\pm k(k+1)$, and this is not a coincidence since the spherical Laplacian is what's known as a Casimir operator for the Lie algebra $\mathfrak{so}(3)$. We'll see more things like this later, once we have more background. ◂

Lecture 2.

Representations theory of compact groups: 1/20/17

First, we'll discuss some course logistics. There are course notes (namely, the ones you're reading now) and a website, https://www.ma.utexas.edu/users/gunningham/reptheory_spring17.html. We won't stick to one textbook, as indicated on the website, but the textbook of Kirrilov is a good reference, and is available online. The class' office hours will be Monday from 2 to 4 pm, at least for the time being.

The course will fall into two parts.

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[2]This general linear group $\mathrm{GL}(V)$ is the group of invertible linear maps $V \to V$.

[3]No pun intended.

(1) First, we'll study finite-dimensional representations of things such as compact Lie groups (e.g. $\mathrm{U}(n)$ and $\mathrm{SU}(2)$) and their complexified Lie algebras, reductive Lie algebras (e.g. $\mathfrak{gl}_n(\mathbb{C})$ and $\mathfrak{sl}_2(\mathbb{C})$). There is a nice dictionary between the finite-dimensional representation theories of these objects. The algebra $\mathfrak{sl}_2(\mathbb{C})$ is semisimple, which is stronger than reductive. Every reductive Lie algebra decomposes into a sum of a semisimple Lie algebra and an abelian Lie algebra, and abelian Lie algebras are relatively easy to understand, so we'll dedicate some time to semisimple Lie algebras.

We'll also spend some time understanding the representation theory of reductive algebraic groups over $\mathbb{C}$, e.g. $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{SL}_2(\mathbb{C})$. Again, there is a dictionary between the finite-dimensional representations here and those of the Lie groups and reductive Lie algebras we discussed.

All together, these form a very classical and standard subject that appears everywhere in algebra, analysis, and physics.

(2) We'll then spend some time on the typically infinite-dimensional representations of noncompact Lie groups, such as $\mathrm{SL}_2(\mathbb{R})$ or the Lorentz group $\mathrm{SO}(1,3)$. These groups have interesting infinite-dimensional, but irreducible representations; the classification of these representations is intricate, with analytic issues, yet is still very useful, tying into among other things the Langlands program.

All these words will be defined when they appear in this course.

We'll now begin more slowly, with some basics of representations of compact groups.

Example 2.1. Here are some examples of compact topological groups.

- Finite groups.
- Compact Lie groups such as $\mathrm{U}(n)$.
- The p-adics $\mathbb{Z}_p$ with their standard topology: two numbers are close if their difference is divisible by a large power of p. $\mathbb{Z}_p$ is also a profinite group. ◀

Definition 2.2. Let G be a compact group. A *(finite-dimensional) (continuous) (complex) representation* of G is a finite-dimensional complex vector space V together with a continuous homomorphism $\rho : G \to \mathrm{GL}(V)$.

If you pick a basis, $V \cong \mathbb{C}^n$, so $\mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{C})$. These are the $n \times n$ invertible matrices over the complex numbers, so ρ assigns a matrix to each $g \in G$ in a continuous manner, where $\rho(g_1g_2) = \rho(g_1)\rho(g_2)$, so group multiplication is sent to matrix multiplication. Sometimes it's more natural to write this through the *action map* $G \times V \to V$ sending $(g, v) \mapsto \rho(g) \cdot v$.

The plethora of parentheses in Definition 2.2 comes from the fact that representations may exist over other fields, or be infinite-dimensional, or be discontinuous, but in this part of the class, when we say a representation of a compact group, we mean a finite-dimensional, complex, continuous one.

Example 2.3. Let S_3 denote the *symmetric group on* 3 *letters*, the group of bijections $\{1,2,3\} \to \{1,2,3\}$ under composition. Its elements are written in *cycle notation*: $(1\ 2)$ is the permutation exchanging 1 and 2, and $(1\ 2\ 3)$ sends $1 \mapsto 2$, $2 \mapsto 3$, and $3 \mapsto 1$. There are six elements of S_3: $S_3 = \{e, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$.

For representation theory, it can be helpful to have a description in terms of generators and relations. Let $s = (1\ 2)$ and $t = (2\ 3)$, so $(1\ 3) = sts = tst$, $(1\ 2\ 3) = st$, and $(1\ 3\ 2) = ts$. Thus we obtain the presentation

$$S_3 = \langle s, t \mid s^2 = t^2 = e, sts = tst\rangle. \tag{2.4}$$

The relation $sts = tst$ is an example of a *braid relation*; there exist similar presentations for all the symmetric groups, and this leads into the theory of Coxeter groups.

There's a representation you can always build for any group called the *trivial representation*, in which $V = \mathbb{C}$ and $\rho_{\mathrm{triv}} : G \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$ sends every $g \in G$ to the identity map ($1 \in \mathbb{C}^\times$).

To get another representation, let's remember that we wanted to build representations out of functions on spaces. S_3 is a discrete space, so let's consider the space $X = \{x_1, x_2, x_3\}$ (with the discrete topology). Then, S_3 acts on X by permuting the indices; we want to linearize this.

Let $V = \mathbb{C}[X] = \mathbb{C}x_1 \oplus \mathbb{C}x_2 \oplus \mathbb{C}x_3$, a complex vector space with basis $\{x_1, x_2, x_3\}$. We'll have S_3 act on V by permuting the basis; this is an example of a *permutation representation*. This basis defines an isomorphism $\mathrm{GL}(V) \xrightarrow{\sim} \mathrm{GL}_3(\mathbb{C})$, which we'll use to define a representation. Since s swaps x_1 and x_2, but fixes x_3, it should map to the matrix

$$s \longmapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly,

$$t \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(You should check that these assignments satisfy the relations in (2.4).)

Now something interesting happens. When you think of X as an S_3-space, it has no invariant subsets (save itself and the empty set). But this linearization is no longer irreducible: the element $v = x_1 + x_2 + x_3 \in V$ is fixed by all permutations acting on X: $\rho(g) \cdot v = v$ for all $g \in S_3$.

More formally, let W be the subspace of V spanned by v; then, W is a subrepresentation of V. ◀

Let's take a break from this example and introduce some terminology.

Definition 2.5. Let G be a group and V be a G-representation. A *subrepresentation* or *G-invariant subspace* of V is a subspace $W \subseteq V$ such that for all $g \in G$ and $w \in W$, $g \cdot w \in W$. If V has no nontrivial (i.e. not 0 or V) subrepresentations, V is called *irreducible*.

This means the same G-action defines a representation on W.

If $W \subseteq V$ is a subrepresentation, the quotient vector space V/W is a G-representation, called the *quotient representation*, and the G-action is what you think it is: in coset notation, $g \cdot (v + W) = (gv + W)$. This is well-defined because W is G-invariant.

We can always take quotients, but unlike for vector spaces in general, it's more intricate to try to find a complement: does the quotient split to identify V/W with a subrepresentation of V?

Returning to Example 2.3, we found a three-dimensional representation V and a one-dimensional subrepresentation W. Let's try to find another subrepresentation U of V such that, as S_3-representations, $V \cong W \oplus U$. The answer turns out to be $U = \operatorname{span}_{\mathbb{C}}\{x_1 - x_2, x_2 - x_3\}$.

Claim. U is a subrepresentation, and $V = U \oplus W$.

This isn't as obvious, because neither $x_1 - x_2$ or $x_2 - x_3$ is fixed by all elements of S_3. However, for any $g \in S_3$, $g \cdot (x_1 - x_2)$ is contained in U, and similarly for $x_2 - x_3$. Let's set $U \cong \mathbb{C}^2$ with $x_1 - x_2 \mapsto \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $x_2 - x_3 \mapsto \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. Then, we can explicitly describe U in terms of the matrices for s and t in $\mathrm{GL}(U) \cong \mathrm{GL}_2(\mathbb{C})$, where the identification uses this basis.

Since $s = (1\ 2)$, it sends $x_1 - x_2 \mapsto x_2 - x_1 = -(x_1 - x_2)$ and $x_2 - x_3 \mapsto x_1 - x_3 = (x_1 - x_2) + (x_2 - x_3)$, so

$$s \longmapsto \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}.$$

In the same way,

$$t \longmapsto \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}.$$

The general theme of finding interesting representations inside of naturally arising representations will occur again and again in this class.

Lecture 3.

Operations on representations: 1/23/17

"I want to become a representation theorist!"
"Are you Schur?"

Last time, we discussed representations of groups and what it means for a representation to be irreducible; today, we'll talk about some other things one can do with representations. For the time being, G can be any group; we will specialize later.

The first operation on a representation is very important.

Definition 3.1. A *homomorphism* of G-representations $V \to W$ is a linear map $\varphi : V \to W$ such that for all $g \in G$, the diagram

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "g\cdot"] & W \arrow[d, "g\cdot"] \\ V \arrow[r, "\varphi"] & W. \end{tikzcd}$$

This is also called an *intertwiner*, a *G-homomorphism*, or a *G-equivariant* map.

An *isomorphism* of representations is a homomorphism that's also a bijection.[4]

More explicitly, this means φ commutes with the G-action, in the sense that $\varphi(g \cdot v) = f \cdot \varphi(v)$. This is one advantage of dropping the ρ-notation: it makes this definition cleaner.

Remark. If $\varphi : V \to W$ is a G-homomorphism, then $\ker(\varphi) \subseteq V$ is a subrepresentation, and similarly for $\mathrm{Im}(\varphi) \subseteq W$. ◀

The set of G-homomorphisms from V to W is a complex vector space,[5] denote $\mathrm{Hom}_G(V, W)$.

Several constructions carry over from the world of vector spaces to the world of G-representations. Suppose V and W are G-representations.

- The direct sum $V \oplus W$ is a G-representation with the action $g \cdot (v, w) = (g \cdot v, g \cdot w)$. This has dimension $\dim V + \dim W$.
- The tensor product $V \oplus W$ is a G-representation: since it's generated by pure tensors, it suffices to define $g \cdot (v \otimes w)$ to be $(gv) \otimes (gw)$ and check that this is compatible with the relations.
- The dual space $V^* := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is a G-representation: if $\alpha \in V^*$, we define $(g \cdot \alpha)(v) := \alpha(g^{-1}v)$. This might be surprising: you would expect $\alpha(gv)$, but this doesn't work: you want $g \cdot (h\alpha)$ to be $(gh) \cdot \alpha$, but you'd get $(hg) \cdot \alpha$. This is why you need the g^{-1}.
- Since $\mathrm{Hom}_{\mathbb{C}}(V, W)$ is naturally isomorphic to $V^* \otimes W$, it inherits a G-representation structure.

Definition 3.2. Given a G-representation V, the *space of G-invariants* is the space

$$V^G := \{v \in V \mid g \cdot v = v\}.$$

This can be naturally identified with $\mathrm{Hom}_G(\mathbb{C}_{\mathrm{triv}}, V)$, where $\mathbb{C}_{\mathrm{triv}}$ is the trivial representation with action $g \cdot z = z$ for all $z \in \mathbb{C}$. The identification comes by asking where 1 goes to.

These are also good reasons for using the action notation rather than writing $\rho : G \to \mathrm{Aut}(V)$, which would require more complicated formulas.

There are a couple of different ways of stating Schur's lemma, but here's a good one.

Lemma 3.3 (Schur)**.** *Let V and W be irreducible G-representations. Then,*

$$\mathrm{Hom}_G(V, W) = \begin{cases} 0, & \text{if } V \not\cong W \\ \mathbb{C}, & \text{if } V \cong W. \end{cases}$$

"Irreducible" is the key word here.

Remark. Schur's lemma requires us to work over $\mathbb{C}$ (more generally, over any algebraically closed field). It also assumes that V and W are finite-dimensional. There are no assumptions on G; this holds in much greater generality (e.g. over $\mathbb{C}$-linear categories). ◀

In general, there's a distinction between "isomorphic" and "equal" (or at least naturally isomorphic); in the latter case, there's a canonical isomorphism, namely the identity. In this case, the second piece of Lemma 3.3 can be restated as saying for any irreducible G-representation V,

$$\mathrm{Hom}_G(V, V) = \mathbb{C} \cdot \mathrm{id}_V.$$

Thus, any G-homomorphism $\varphi : V \to V$ is $\lambda \cdot \mathrm{id}_V$ for some $\lambda \in \mathbb{C}$, and in a basis is a diagonal matrix with every diagonal element equal to λ.

[4] If $f : V \to W$ is an isomorphism of representations, then $f^{-1} : W \to V$ is also a G-homomorphism, making this a reasonable definition. This is a useful thing to check, and doesn't take too long.

[5] Recall that we're focusing exclusively on complex representations. If we look at representations over another field k, we'll get a k-vector space.

Proof of Lemma 3.3. Suppose $\varphi : V \to W$ is a nonzero G-homomorphism. Thus, $\ker(\varphi) \subset V$ isn't be all of V, so since V is irreducible it must be 0, so φ is injective. Similarly, since $\mathrm{Im}(\varphi) \subset W$ isn't 0, it must be all of W, since W is irreducible, Thus, φ is an isomorphism, so if $V \ncong W$, the only G-homomorphism is the zero map.

Now, suppose $\varphi : V \to V$ is a G-homomorphism. Since $\mathbb{C}$ is algebraically closed, φ has an eigenvector: there's a $\lambda \in \mathbb{C}$ and a $v \in V$ such that $\varphi(v) = \lambda \cdot v$. Since φ and $\lambda \cdot \mathrm{id}_V$ are G-homomorphisms, so is $\varphi - \lambda \mathrm{id}_V : V \to V$, so its kernel, the λ-eigenspace of φ, is a subrepresentation of V. Since it's nonzero, then it must be all of V, so $V = \ker(\varphi - \lambda \mathrm{id}_V)$, and therefore $\varphi = \lambda \mathrm{id}_V$. ☒

This is the cornerstone of representation theory, and is one of the reasons that the theory is so much nicer over $\mathbb{C}$.

Corollary 3.4. *If G is an abelian group, then any irreducible representation of G is one-dimensional.*

Proof. Let V be an irreducible G-representation and $g \in G$. Since G is abelian, $v \mapsto gv$ is a G-homomorphism: $g \cdot (hv) = h(g \cdot v)$. By Schur's lemma, the action of g is $\lambda \cdot \mathrm{id}_V$ for some $\lambda \in \mathbb{C}$, so any $W \subseteq V$ is G-invariant. Since V is irreducible, this can only happen when V is 1-dimensional. ☒

Example 3.5. Let's talk about the irreducible representations of $\mathbb{Z}$. This isn't a compact group, but we'll survive. A representation of $\mathbb{Z}$ is a homomorphism $\mathbb{Z} \to \mathrm{GL}(V)$ for some vector space V; since $\mathbb{Z}$ is a free group, this is determined by what 1 goes to, which can be chosen freely. That is, representations of $\mathbb{Z}$ are in bijection with invertible matrices.

By Corollary 3.4, irreducible $\mathbb{Z}$-representations are the 1-dimensional invertible matrices, which are identified with $\mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$, the nonzero complex numbers. ◀

Our greater-scope goal is to understand all representations by using irreducible ones as building blocks. In the best possible case, your representation is a direct-sum of irreducibles; we'll talk about that case next time.

Lecture 4.

Complete reducibility: 1/25/17

We've discussed what it means for a representation to be irreducible, and irreducible representations are the smallest representations; we hope to build all representations out of irreduicbles. The nicest possible case is complete reducibility, which we'll discuss today.

Suppose G is a group (veyr generally), V is a representation, and $W \subseteq V$ is a subrepresentation. If $i : W \hookrightarrow V$ denotes the inclusion map, then there's a projection map onto the quotient $V \twoheadrightarrow U := V/W$. This is encoded in the notion of a *short exact sequence*:

$$0 \longrightarrow W \xrightarrow{i} V \xrightarrow{j} U \longrightarrow 0,$$

which means exactly that i is injective, j is surjective, and $\mathrm{Im}(i) = \ker(j)$. The nicest short exact sequence is

$$0 \longrightarrow W \longrightarrow W \oplus U \longrightarrow U \longrightarrow 0,$$

where the first map is inclusion into the first factor and the second is projection onto the second factor. In this case, one says the short exact sequence *splits*. This is equivalent to specifying a projection $V \twoheadrightarrow W$ or an inclusion $U \hookrightarrow V$. Since direct sums are easier to understand, this is the case we'd like to know better.

Example 4.1. We saw last time that a representation of $\mathbb{Z}$ is given by the data of an invertible matrix which specifies the action of 1, akin to a discrete translational symmetry.

Consider the $\mathbb{Z}$-representation V on $\mathbb{C}^2$ given by the matrix

$$A := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

This is not an irreducible representation, because $\binom{1}{0}$ is an eigenvector for A with eigenvalue 1, so

$$W := \left\{ \begin{pmatrix} a \\ 0 \end{pmatrix} \mid a \in \mathbb{C} \right\}$$

is a subrepresentation of V. Since $\binom{1}{0}$ has eigenvalue 1, W is the trivial representation $\mathbb{C}_{\mathrm{triv}}$. Moreover, the quotient V/W is also the trivial representation, so V sits in a short exact sequence

$$0 \longrightarrow \mathbb{C}_{\mathrm{triv}} \longrightarrow V \longrightarrow \mathbb{C}_{\mathrm{triv}} \longrightarrow 0.$$

However, V itself is not trivial, or it would have been specified by a diagonalizable matrix. Thus, V is not a direct sum of subrepresentations (one says it's *indecomposable*), but it's not irreducible! (There's a little more to flesh out in this argument.) ◀

We want to avoid these kinds of technicalities on our first trek through representation theory, and fortunately, we can.

Definition 4.2. A representation V of G is *completely reducible* or *semisimple* if every subrepresentation $W \subseteq V$ has a *complement*, i.e. another subrepresentation U such that $V \cong U \oplus W$.

Remark. There are ways to make this more general, e.g. for infinite-dimensional representations, one may want closed subrepresentations. But for finite-dimensional representations, this definition suffices. ◀

A finite-dimensional semisimple representation V is a direct sum of its irreducible subrepresentations. The idea is that its subrepresentations must also be semisimple, so you can use induction.

The terminology "semisimple" arises because *simple* is a synonym for irreducible, in the context of representation theory.

So semisimple representations are nice. You might ask, *for which groups G are all representations semisimple*? To answer this question, we'll need a few more concepts.

Definition 4.3. A representation V of G is called *unitary* if it admits a G-invariant inner product, i.e. a map $B : V \times V \to \mathbb{C}$ that is:

- linear in the first factor and antilinear in the other,[6]
- *antisymmetric*, i.e. $B(v,w)) = \overline{B(w,v)}$,
- *positive definite*, i.e. $B(v,v) \geq 0$ and $B(v,v) = 0$ iff $v = 0$, and
- *G-invariant*, meaning $B(g \cdot v, g \cdot w) = B(v,w)$ for all $g \in G$.

The reason for the name is that if V is a unitary representation with form B, then as a map $G \to \mathrm{GL}(V) \cong \mathrm{GL}(\mathbb{C}^n)$, this representation factors through $\mathrm{U}(n)$, the *unitary matrices*, which preserve the standard Hermitian inner product on $\mathbb{C}^n$. So unitary representations are representations of G into some unitary group.

Proposition 4.4. *Unitary representations are completely reducible.*

Proof sketch. How would you find a complement to a subspace? The usual way to do this is to take an orthogonal complement, and an invariant inner product is what guarantees that the orthogonal complement is a subrepresentation.

Let V be a unitary representation and $B(\text{–},\text{–})$ be its invariant inner product. Let $W \subseteq V$ be a subrepresentation, and let

$$U = W^\perp := \{v \in V \mid B(v,w) = 0 \text{ for all } w \in W\}.$$

Then, U is a subrepresentation (which you should check), and $V = W \oplus U$. ⊠

Classifying unitary representations is a hard problem in general. Building invariant Hermitian forms isn't too bad, but making them positive definite is for some reason much harder. In any case, for compact groups there is no trouble.

Proposition 4.5.

(1) *Given an irreducible unitary representation, the invariant form $B(\text{–},\text{–})$ is unique up to multiplication by a positive scalar.*

(2) *If $W_1, W_2 \subseteq V$, where V is unitary and W_1 and W_2 are nonisomorphic irreducible subrepresentations, then W_1 is orthogonal to W_2, i.e. $B(w_1, w_2) = 0$ for all $w_1 \in W_1$ and $w_2 \in W_2$.*

[6] Some people have the opposite convention, defining the first factor to be antilinear and the second to be linear. Of course, the theory is equivalent. Such a form is called a *Hermitian form*, and if it's antisymmetric and positive definite, it's called a *Hermitian inner product*.

Proof. Part (1) is due to Schur's lemma (Theorem 3.3). Consider the map $B : V \to \overline{V}^*$ defined by $v \mapsto B(v, -)$ (here, $\overline{V}$ is the *conjugate space*, where the action of $a + bi$ on $\overline{V}$ is the action of $a - bi$ on V); in fact, you could use this map to define a unitary structure on a representation. B is a G-isomorphism, so by Schur's lemma, every such isomorphism, derived from every possible choice of Hermitian form, must be a scalar multiple of this one. Since B must be positive definite, this scalar had better be positive. ⊠

One particular corollary of part (1) is that if V is a unitary representation, $V^* \cong \overline{V}$.

So if you care about compact groups (in particular finite groups), this is all you need.

Theorem 4.6 (Maschke)**.** *Any representation of a compact group admits a unitary structure.*

We'll give a first proof for finite groups, then later one for Lie groups; we probably won't prove the most general case.

Proof for finite groups. Let G be a finite group and V be a G-representation. The first step to finding a G-invariant inner product is to find any Hermitian inner prodict, e.g. picking a basis and declaring it to be orthonormal, yielding an inner product B_0 that's probably not G-invariant.

We want to average B_0 over G to obtain something G-invariant, which is why we need finiteness.[7] That is, let

$$B(v, w) \coloneqq \frac{1}{|G|} \sum_{g \in G} B_0(g \cdot v, g \cdot w).$$

Then, B is a unitary structure: it's still positive definite and Hermitian, and since multiplication by an $h \in H$ is a bijection on G, this is G-invariant. ⊠

[7] More generally, we could take a compact group, replacing the sum with an integral. Compactness is what guarantees that the integral converges.