M390C NOTES: GEOMETRIC LANGLANDS

ARUN DEBRAY
AUGUST 25, 2016

These notes were taken in UT Austin's Math 390c (Geometric Langlands) class in Fall 2016, taught by David Ben-Zvi. I live-TeXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to `a.debray@math.utexas.edu`.

Contents

1. The Fourier Transform in Representation Theory: 8/25/16 1

1. The Fourier Transform in Representation Theory: 8/25/16

"One of the traditions we have at UT is we always have to mention Tate."

The initial conception of this class was going to be more akin to a learning seminar about the geometric Langlands program, but this changed: it's now going to be an actual class, but about geometric representation theory and topological field theory. The goal is for this to turn into good lecture notes and even a book, so the class isn't the entire intended audience. As such, feedback is even more helpful than usual.

It's not entirely clear what the prerequisites for this class are; the level of background will grow as the class goes on. The actual amount of technical background needed to state things precisely is huge, and not a reasonable requirement. As such, the class will be more of a sketch and overview of the ideas and how to think about the main characters[1] in this subject. The professor's seminar (Fridays, from 2 to 4, in the same room) is probably a good place to start understanding this material more rigorously.

There will be an introduction to this class this afternoon at geometry seminar.

The Fourier transform. Do you remember Fourier series? The statement is that for L^2 functions $f : S^1 \to \mathbb{C}$,

$$f(\theta) = \sum_{n\in\mathbb{Z}} \widehat{f}(n) e^{2\pi i n\theta}.$$

This is probably the last precise formula we're going to see in this class, which may reassure you or bother you. We also will identify $S^1 \cong \mathrm{U}(1)$. The Fourier coefficients are

$$\widehat{f}(n) = \int_{S^1} f(\theta) e^{-2\pi i n\theta}\,\mathrm{d}\theta.$$

Representation theory starts with this formula.

Relatedly, for an L^2 function $f : \mathbb{R} \to \mathbb{C}$, we have a continuous combination of exponentials with coefficients $\widehat{f}(t)$:

$$f(x) = \int_{\mathbb{R}} \widehat{f}(t) e^{2\pi i x t}\,\mathrm{d}t, \tag{1.1}$$

where

$$\widehat{f}(t) = \int_{\mathbb{R}} f(x) e^{-2\pi i x t}\,\mathrm{d}x.$$

How should we think of these formulas? The exponentials $e^{2\pi i t -}$ are complex-valued functions on $\mathrm{U}(1)$ and $\mathbb{R}$, respectively. But in fact, they land in $\mathbb{C}^\times$, since they don't hit 0, and in fact they have unit norm, so they are maps into $\mathrm{U}(1)$. Since $e^{a+b} = e^a e^b$, these are homomorphisms of groups. Moreover, these are the

[1] Pun intended?

only homomorphisms: if $f(\theta_1+\theta_2) = f(\theta_1)f(\theta_2)$ for an $f : \mathrm{U}(1) \to \mathrm{U}(1)$, then $f(\theta) = e^{2\pi i x\theta}$ for some x, and similarly for functions $\mathbb{R} \to \mathrm{U}(1)$.

In other words, these functions are the *unitary characters* of the domain group: the homomorphisms to $\mathrm{U}(1) \subset \mathrm{GL}(1)$. We can recast these as representations acting through unitary matrices (also *unitary representations*), where an $x \in \mathbb{R}$ acts as multiplication by $e^{2\pi i x t}$ on the (complex) vector space $\mathbb{C}$.

From this viewpoint, we are writing general functions on $\mathrm{U}(1)$ or on $\mathbb{R}$ as linear combinations of characters. This means characters form a "basis." That is, the characters are not strictly a basis, but the space spanned by finite linear combinations of exponentials is dense in any reasonable function space L^2, C^∞, distributions, real analytic functions, L^p spaces, etc. In particular, L^2, smooth, analytic, etc. are conditions on the Fourier coefficients: $f \in L^2(S^1)$ iff $\widehat{f} \in \ell^2$ (the square-integrable sequences of numbers). f is smooth iff its Fourier coefficients are rapidly decreasing (faster than any polynomial).

This is where the analysis of Fourier series takes place: you're interested in different function spaces, and so you're interested in how the coefficients grow. But we're going to ignore it: it's deep and important for analysis, but begins a different track than representation theory. The algebraic content is that algebraic functions (Laurent series) are dense, and we're going to care more about the algebraic side than the analytic side.

Theorem 1.2 (Plancherel)**.** *If $\mathbb{R}$ denotes the x-line and $\widehat{\mathbb{R}}$ denotes the t-line, then the Fourier transform defines a unitary isomorphism $L^2(\mathbb{R}) \xrightarrow{\sim} L^2(\widehat{\mathbb{R}})$.*

This is nice, but doesn't help much for the character-theoretic viewpoint: the exponential $e^{2\pi i x t}$ is not in $L^2(\mathbb{R})$. This is where one uses Schwarz functions.

Definition 1.3. The *Schwarz space* $\mathcal{S}(\mathbb{R})$ is the space of $f \in C^\infty(\mathbb{R})$ such that f and all of its derivatives decrease more rapidly than any polynomial.

The dual space to $\mathcal{S}(\mathbb{R})$, denoted $\mathcal{S}^*$ or $\mathcal{S}'$, is called the space of *tempered distributions*. Our characters $e^{2\pi i x t}$ live in this space, and the Fourier transform extends to a linear homeomorphism $\mathcal{S}'(\mathbb{R}) \cong \mathcal{S}'(\widehat{\mathbb{R}})$.

Thus, it makes sense to define the Fourier transform of the exponential $e^{2\pi i n x}$: we obtain the delta "function" supported at n, δ_n (1 at n and 0 elsewhere), and similarly, the Fourier transform of δ_t is $e^{2\pi i x t}$. That is, the Fourier transform exchanges points and characters; in other words, $\widehat{\mathbb{R}}$ is a sort of moduli space of unitary characters of $\mathbb{R}$.

In some sense, this diagonalizes the group action: if G is either of $\mathbb{R}$ or $\mathrm{U}(1)$, then G acts on itself by translation (both left and right, since G is abelian). Thus, any space of functions on G is acted on by G: an $\alpha \in G$ sends $f \mapsto \alpha * f$ (i.e. $\alpha * f(x) = f(x+\alpha)$). If V is this function space (e.g. $L^2(G)$), then this defines an action of G on V, hence a group homomorphism $G \to \mathrm{End}(V)$. In particular, the exponential $e^{2\pi i x t}$ satisfies

$$\alpha * e^{2\pi i x t} = e^{2\pi i (x+\alpha)t} = \left(e^{2\pi i x\alpha}\right)\left(e^{2\pi i x t}\right).$$

That is, this exponential is an eigenfunction for $\alpha * -$ for all $\alpha \in G$: characters are joint eigenfunctions, and the Fourier transform is a simultaneous diagonalization.

Succinctly, *the Fourier transform exchanges translation and multiplication*: the translation operator $\alpha * -$ is sent to the multiplication operator $\widehat{f} \mapsto \widehat{\alpha}\widehat{f}$, where $\widehat{\alpha}(t) = e^{2\pi i \alpha t}$. From the perspective of Fourier series, we have a $\mathbb{Z} \times \mathbb{Z}$ matrix with respect to the exponential basis, but only the diagonal entries $\widehat{f}(n)e^{2\pi i n\theta}$ are nonzero.

Before we make this more abstract, let's see what happens to differentiation. Since G is a Lie group, it has a Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}$, in this case $\mathbb{R} \cdot \frac{\mathrm{d}}{\mathrm{d}x}$, the infinitesimal translations at a point. The differential $\frac{\mathrm{d}}{\mathrm{d}x}$ is an infinitesimal translation, and the Fourier transform sends it to a multiplication by $(2\pi i)t$.[2]

Pontrjagin duality. We can generalize this to Pontrjagin duality, which is a kind of Fourier transform involving a locally compact abelian topological group (LCA) G, e.g. $\mathbb{R}$, $\mathbb{Z}$, S^1, $\mathbb{Z}/n$, and any finite products of these, including tori, lattices, and finite-dimensional vector spaces. More exotic examples include the p-adics. There will be more interesting examples in the algebraic world.

Definition 1.4. Let G be an LCA group; then, the *(unitary) dual* of G is $\widehat{G} = \mathrm{Hom}_{\mathsf{TopGrp}}(G, \mathrm{U}(1))$, the set of characters of G, with the topology inherited as a subset of the continuous functions $C(G) = \mathrm{Hom}_{\mathsf{Top}}(G, \mathbb{C})$.

[2] To prove this rigorously, one needs to worry about difference quotients.

We saw that if $G = \mathbb{R}$, then $\widehat{G} = \mathbb{R}$ again, and that if $G = \mathrm{U}(1)$, then $\widehat{G} = \mathbb{Z}$. Conversely, if $G = \mathbb{Z}$, then a homomorphism on G is determined by its value at 1, which can be anything in $\mathrm{U}(1)$, so $\widehat{G} = \mathrm{U}(1)$. If V is a finite-dimensional vector space, then $\widehat{V} = V^*$: any linear functional $\xi \in V^*$ defines a character $v \mapsto e^{2\pi i \langle \xi, v\rangle}$. It's a nice exercise to check that these are all the unitary characters. If $G = \Lambda$ is a lattice, then we obtain its *dual torus* T, and correspondingly a torus goes to its *dual lattice*. Lastly, we have finite abelian groups, e.g. $\mathbb{Z}/n$, which is generated by 1, so we must send 1 to an n^{th} root of unity. Thus, $(\mathbb{Z}/n)^\vee = \mu_n$, the group of n^{th} roots of unity. This is isomorphic to $\mathbb{Z}/n$ again, though in algebraic geometry, where we might not have all roots of unity, things can get more interesting, so it's useful to remember μ_n.

The claim is that the Fourier transform looks exactly the same for any LCA group; maybe we haven't defined too many exciting examples, but this is still noteworthy. We want characters on G to correspond to points on $\widehat{G}$. A point $\chi \in \widehat{G}$ defines a function on G, and correspondingly, a point $g \in G$ defines a function $\widehat{g} : \chi \mapsto \chi(g)$ on $\widehat{G}$, which looks like a nascent Fourier transform. If $g, h \in G$, then $\widehat{gh}(\chi) = \chi(gh) = \chi(g)\chi(h) = \widehat{g}\widehat{h}(\chi)$, so this transform that we're building will start from this duality of the group multiplication and the pointwise product.

One important thing to mention: $\widehat{G}$ is also a group, and in fact is locally compact abelian. The group operation is pointwise product $\chi_1 \cdot \chi_2(g) = \chi_1(g)\chi_2(g)$. This agrees with the group operations for the examples we mentioned.

Theorem 1.5 (Pontrjagin duality)**.** *The natural map* $G \mapsto \widehat{\widehat{G}}$ *defined by* $g \mapsto \widehat{g}$ *is an isomorphism of topological groups.*

Hence, this really is a duality. Nonetheless, we'll maintain the distinction between G and $\widehat{G}$: soon we'll try to generalize to nonabelian groups, and then symmetry will break.

Theorem 1.6 (Fourier transform)**.** *If G is an LCA group, then the Fourier transform map*

$$f \longmapsto \widehat{f}(\chi) = \int_G f(g) \cdot \chi(g)\, \mathrm{d}g,$$

where $\mathrm{d}g$ *is the Haar measure on* G,[3] *defined an isomorphism of Hilbert spaces* $L^2(G) \xrightarrow{\sim} L^2(\widehat{G})$.

Notice that, since the characters on $\mathbb{R}$ are the exponentials and the Haar measure on $\mathbb{R}$ is the usual Lesbegue measure, this generalizes (1.1).

This entire story started in Tate's thesis, which applies Pontrjagin duality to more exotic examples such as $\mathbb{Q}_p$ and $\mathbb{Q}_p^\times$ or even the group $\mathbb{A}^\times$ of *adeles*;[4] see the GTM by Ramakrishnan-Valenza for a modern take on this subject, including harmonic analysis on LCA groups.

We'll use this to understand all representations of G (well, nice representations). In general, not all representations of G on a space come from functions on G, but we'll be able to use Pontrjagin duality and the group algebra to do something nice.

Function theory. One important philosophy in representation theory is that the action of G on functions on G (nice functions in whichever context we're working in) is the most important, or universal, representation. We'll talk about functions and convolution from a particular perspective that will be useful several times in the class.

Let X be a finite set. Then, $F(X)$, the set of complex-valued functions on X, is unambiguous. The set of measures on X, $M(X)$, is also clear, but there's a natural bijection between them via the counting measure.

Theorem 1.7 (Finite Riesz representation theorem)**.** *There is a natural identification* $F(X) = \mathrm{Hom}_{\mathbb{C}}(F(X), \mathbb{C})$.

This comes from the inner product on $\mathsf{Fun}(X)$

$$\langle f, g\rangle = \sum_{x \in X} f(x) g(x).$$

The more general Riesz representation theorem is about a Hilbert space of functions on $\mathbb{R}$, and is less trivial.

Now, suppose we have two finite sets X and Y. We can form their product, which looks like Figure 1. It's possible to identify $F(X \times Y) = F(X) \otimes F(Y)$, and via a matrix, or an "integral kernel," this space can be

[3] This is only unique up to a scalar, so we need to pick one.

[4] Not to be confused with the musician.

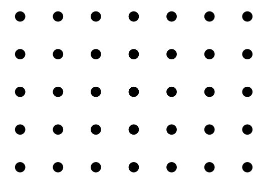

FIGURE 1. The product of two finite sets.

identified with $\mathrm{Hom}_{\mathbb{C}}(F(X), F(Y))$: a kernel $K(x,y) \in F(X \times Y)$ defines an operator $K * - : F(X) \to F(Y)$ defined by

$$K * f(y) = \sum x \in X K(x,y) f(x).$$

In a broader sense, let $\pi_X : X \times Y \to X$ be projection, and define π_Y similarly. Functions can pull back: $\pi_X^* f(x,y) = f(\pi_X(x,y))$, and measures can push forward by integration (or summing, since we're thinking about the counting measure) over the fibers. Thus, we can recast convolution as

$$K * f = \pi_{Y*}(K \cdot \pi_X^* f)(y) = \int_X K(x,y) f(x)\, \mathrm{d}\#.$$

Since $F(X)$ and $F(Y)$ are finite-dimensional vector spaces, K may be identified with a matrix or a linear transform, and this formula is exactly how to multiply a matrix by a vector.

A key desideratum is that, in general, all nice maps between function spaces on X and function spaces on Y come from integral kernels. For example, a map $L^2(\mathbb{R}) \to L^2(\mathrm{pt}) = \mathbb{C}$ is given by a kernel $K \in L^2(\mathbb{R} \times \mathrm{pt}) = L^2(\mathbb{R})$, realized as $f \mapsto \int K \cdot f$, by the Riesz representation theorem for L^2. Another instance of this is the Schwarz kernel theorem.

Theorem 1.8 (Schwatz kernel theorem)**.** *Let X and Y be smooth manifolds. Then,* $\mathrm{Hom}_{\mathsf{Top}}(C_c^\infty(X), \mathrm{Dist}(Y)) \cong \mathrm{Dist}(X \times Y)$.

Here, $\mathrm{Dist}(-)$ is the space of distributions, dual to compactly supported smooth functions on the manifold.

If $X = Y$ (back in the world of finite sets), then we can consider δ_Δ, the δ-function of the diagonal. In a basis, this is just the identity matrix, and convolution with K is the identity operator. More generally, if $g : X \to Y$ is a set map, then $g^* : F(Y) \to F(X)$ is represented by the kernel of the graph $\Gamma_g \subset X \times Y$: $K = \delta_{\Gamma_g}$. If this all seems a little silly, the key is that it's easier to understand over finite sets, but will work for "nice" functions in a great variety of contexts.

We can also use this to understand matrix multiplication. Given three finite sets X, Y, and Z, and kernels (functions) $K_1 : F(X) \to F(Y)$ and $K_2 : F(Y) \to F(Z)$, we can compose them. Consider the projections

$$\begin{tikzcd}
& X \times Y \times Z \arrow[dl, "\pi_{12}"'] \arrow[dr, "\pi_{23}"] \arrow[dd, "\pi_{13}"] & \\
X \times Y & & Y \times Z \\
& X \times Z. &
\end{tikzcd}$$

Exercise 1.9. Show that the formula for $K_2 \circ K_1$ is

$$\pi_{13*}(\pi_{12}^*(K_1) \cdot \pi_{23}^*(K_2)).$$

Relate this to matrix multiplication.

The distinction between functions and measures is irrelevant in the world of finite sets, so we can push-forward and pull back with impunity, but in a continuous setting, it's important to keep them distinct. This equates to choosing a measure (e.g. choosing a Haar measure, as we did above), and even relates to things like Poincaré duality.