M392c NOTES: ALGEBRAIC GEOMETRY

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These notes were taken in UT Austin's M392c (Algebraic geometry) class in Fall 2018, taught by Sam Raskin. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Any mistakes in the notes are my own. Thanks to Alberto San Miguel, Kenny Schefers, and Yuri Sulyma for correcting some typos.

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Lecture 1.

Some questions in algebraic geometry: 8/29/18

Office hours are Fridays from 11-1, in room 9.164 (at least for now). Today we'll talk about some questions (and some answers, too!) relating to algebraic geometry and why one might find it interesting. We're going to focus on concreteness.

Broadly speaking, algebraic geometry studies zero sets of polynomials. These could be polynomials over $\mathbb{Q}$, or $\mathbb{R}$, or $\mathbb{C}$, or finite fields, or more. The first question you might ask is, *are there solutions*? This is an *arithmetic question*: in arithmetic situations, there might not be solutions.

Example 1.1 (Taylor-Wiles, 1994)**.** If $n \geq 3$, the polynomial $x^n + y^n = 1$ has no solutions over $\mathbb{Q}$ when $x, y \neq 0$. ◄

You might recognize this as a reformulation of Fermat's last theorem.

Another form of the same question is *can you parameterize solutions of the equation*? For example, let's try it with $x^2 + y^2 = 1$, which we know has solutions. In this case, it is possible to parameterize solutions, via the one-parameter family

$$(1.2) \qquad x = \frac{\lambda^2 - 1}{\lambda^2 + 1}, \qquad y = \frac{2\lambda}{\lambda^2 + 1}.$$

These kinds of questions are called *rationality questions*. One can also ask these questions over $\mathbb{C}$ (or over other algebraically closed fields), where they can feel a bit different.

There is a general result that any quadric hypersurface with a rational point is rational. What this means is that if you assume the existence of one solution (x_0, y_0) to a degree-2 polynomial in x and y over, say, $\mathbb{Q}$, then you can use that one solution to parameterize all other solutions. If you plot the solutions in the xy-plane, the parameter of another solution (x_1, y_1) is the slope of the line between (x_0, y_0) and (x_1, y_1). Indeed, in (1.2), the parameter λ is this slope. Because the equation is a quadric, one expects such a line to intersect in exactly two points, the first solution and another one. This is all extremely explicit, to the point that you could explain why you care to a middle schooler.

There are a few other rationality results.

Theorem 1.3 (Segre, 1940s; Manin, 1970s; Kollár, 2000)**.** *Smooth cubics in at least three variables are rational.*

So $x^2 + y^2 = 1$ isn't rational, but $x^2 + y^2 + z^2 = 1$ is. However, this doesn't give you everything.

Theorem 1.4 (Clemens-Griffiths, 1974)**.** *There are cubics in at least four variables which are unirational but not rational, i.e. that one cannot parameterize all solutions in a one-to-one manner.*

This was a hard theorem. How would you prove something like this?

Recent work (2012-15) by many people (Voisin, Colliot-Thèlene, Pirutka, Totaro[1]) generalizes this.

Theorem 1.5. *For cubics in at least five variables, one can also not parameterize solutions in a one-to-one way, even by adding additional "dummy variables."*

For four-variables cubics, this is open.

Schemes. Though this result is stated completely explicitly, it was studied using some very abstract-looking machinery. In this course, we'll also work with this abstract machinery, namely the language of schemes. These are things like solutions to systems of polynomials, but not quite — they encode among other things the equivalence of such systems under changes of coordinates, which doesn't really change the underlying geometry of the solution set. Classification problems with this perspective are a big area of research, and Birkar just won a Fields medal for work in this area from 2006.

Algebraic geometry over $\mathbb{C}$. A third thing you could care about is specific stuff about algebraic geometry over your favorite field (typically $\mathbb{C}$, but not always). In many cases (such as $\mathbb{C}$), you have topology around, and you can ask how it interacts with the algebraic geometry we've been talking about.

For example, if $q \in \mathbb{C}^\times$ isn't a root of unity, then there's a cubic equation $y^2 = x^2 + ax + b$ whose solutions are parameterized by $\mathbb{C}^\times / q\mathbb{Z}$. This may be a bit surprising, and indicates a way in which analytic or topological information can be useful: now we can learn about the universal cover of the solution space, and other topological invariants. Then you might ask whether something like this is true in positive characteristic, which tends to be harder.

More generally, one can study the topology of algebraic varieties over $\mathbb{C}$.

Theorem 1.6. *The odd Betti numbers of smooth proper varieties are even.*

The proof uses the study of the Hodge Laplacian operator on a variety X. This needs a metric, but projective means that X embeds in some $\mathbb{CP}^n$, and we can borrow its metric. There is a purely algebro-geometric proof of this, but first you need to come up with the right notion of Betti numbers (so étale cohomology, which is hard), and then invoke Deligne's proof of the Weil conjectures (also hard). Nonetheless, it's true in characteristic p.

More generally, the cohomology of a complex projective variety has more structure, and is much richer than that of a random manifold.[2]

Conjecture 1.7 (Hodge conjecture, imprecise statement)**.** *The differential topology of a projective algebraic variety over $\mathbb{C}$ knows everything about its algebraic geometry.*

This is a Millennium Prize problem, meaning it comes with a $1 million reward. You can infer that it's hard.

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[1] If you like pictures of cats, check out Totaro's math blog: **https://burttotaro.wordpress.com/**.

[2] This doesn't require smoothness *per se*, but it's more difficult to formulate in the singular case.

Algebraic geometry over $\mathbb{Z}$. If you work over $\mathbb{Z}$ instead of over $\mathbb{C}$, meaning your polynomial has integer coefficients, then you can reduce mod p and solve it there. This is the first thing anyone does in number theory, because it often simplifies the problem to a finite question. This naturally leads one to ask, *how do the systems of equations at different primes p relate to each other?*

There's a lot to say about this, beginning with quadratic reciprocity, which is very classical yet a little weird, and continuing all the way to the Langlands program.

Supposing X encodes the system of solutions to your polynomial with $\mathbb{Z}$ coefficients. Then one can define a zeta function, reminiscent of the Riemann zeta function, as follows:

$$(1.8) \qquad \zeta_X(s) := \prod_{p \text{ prime}} \exp\left(\sum \frac{1}{n}(\text{number of solutions in } \mathbb{F}_{p^n})p^{-ns}\right).$$

For $X = \operatorname{Spec}\mathbb{Z}$, corresponding to solutions to an empty set of polynomials, this recovers the usual Riemann zeta function.

For any particular X, one conjectures this is meromorphic (and almost entire, in some sense), and that the analogue of the Riemann hypothesis holds; for some X, this is known due to Deligne. There are some other related conjectures related to this known as Sato-Tate conjectures.

Cohomology theories. Over $\mathbb{C}$, you have topology, and therefore can invoke algebraic topology to compute cohomology of algebraic varieties. Over other fields or rings, you might not have these techniques, and there are several other approaches.

- Over an algebraically closed field, one has *étale cohomology*, whose ideas are built from covering space theory, has $\mathbb{Z}_\ell$ coefficients, where ℓ is a prime that's not the characteristic of the field.
- Over any field k, there's *de Rham cohomology*, which uses the idea that $\mathrm{d}z/z$ understands $\mathbb{C}^\times$ isn't simply connected (since $\oint \mathrm{d}z/z \neq 0$). This has coefficients in k.

There are others, too. One wants these to all be the same, or at least closely related; if $k = \mathbb{Q}_p$ and $\ell = p$ ($\mathbb{Q}_p$ has characteristic zero!), then these two are related by p-adic Hodge theory. This is related to deep and recent work by Fontaine, Scholze, and others, and relates to Scholze's Fields medal work. In 2016, Bhatt-Morrow-Scholze showed that one can sometimes interpolate between different cohomology theories. See Scholze's ICM address for more on this. The ultimate question in this corner of algebraic geometry is whether there's some universal cohomology theory interpolating between everything we have, and which is also the source of the ζ-functions mentioned above.

Degenerations. We get additional power by studying solutions in families. For example, we can degenerate $x^2 + y^2 = 1$ to $x^2 + y^2 = 0$, which is much simpler. One asks questions such as, *what invariants are preserved under degenerations?* Therefore one might be able to use a degeneration to reduce a harder problem to an easier problem.

Computations. This subfield of algebraic geometry tries to make these abstract invariants concrete, by writing good algorithms to compute these invariants for explicit systems of polynomials.

Geometric complexity theory. This is another way to relate algebraic geometry and computer science. The goal of this field is to approach another Millennium Prize problem, P vs. NP, using algebraic geometry techniques. This roughly involves studying certain varieties and analyzing whether they're as complicated as they seem. Algebraic geometry has lots of techniques which might help, but on the other hand they haven't yet.

Probably the best way to learn algebraic geometry is to have an application or research focus in mind that you can apply the things you learn to. This method of learning tends to produce algebraic geometers.

Lecture 2.

Defining schemes, I: 8/31/18

The goal of today's lecture is to define a scheme, first heuristically and then rigorously.

"Definition" 2.1. A scheme is a "space" that is a Zariski sheaf which admits an "open cover" by affine schemes.

Of course, in order to do this, we need to know what all of these words — spaces, Zariski sheaves, affine schemes, and open covers — mean in this setting.

Remark 2.2. There's another approach to schemes using the formalism of *locally ringed spaces*, which is followed by Hartshorne, Vakil, and many others. It's more concrete, but it makes it harder to think about what a specific scheme, such as projective space, is supposed to be. ◄

The motivation for "Definition" 2.1 is that a scheme should be something which is locally defined by algebraic equations. For example, let's look at the *Fermat equation* $X_n = \{x^n + y^n = z^n\}$. Fermat was interested in solutions in $\mathbb{Z}$, but the set of solutions makes sense in any commutative ring. This suggests our definition of space, which is not the same as a topological space.

Definition 2.3. A *space* is a functor $X\colon \mathsf{CommRing} \to \mathsf{Set}$.

Concretely, this means that for every ring A, we get a set $X(A)$, and for every map of commutative rings $f\colon A \to B$, we get a map of sets $X(f)\colon X(A) \to X(B)$, and these morphisms should compose well (meaning that $X(f \circ g) = X(f) \circ X(g)$ and $X(\mathrm{id}) = \mathrm{id}$). For example, we could let $X_n(A)$ denote the set of solutions to the Fermat equation in the ring A; then, if we've solved it in A, we can map the solution into B via $f\colon A \to B$, and we'll obtain a solution in B, so this defines a space X_n.

We should also say how spaces interact.

Definition 2.4. A *morphism of spaces* $f\colon X \to Y$ is data of, for all commutative rings A, a map $f_A\colon X(A) \to Y(A)$ such that for all ring homomorphisms $g\colon A \to B$, the diagram

$$
\begin{array}{ccc}
X(A) & \xrightarrow{\ f_A\ } & Y(A) \\
{\scriptstyle X(g)}\downarrow & & \downarrow{\scriptstyle Y(g)} \\
X(B) & \xrightarrow{\ f_B\ } & Y(B)
\end{array}
$$

commutes.

Schemes are special examples of spaces, in a way that feels surprisingly down-to-Earth.

Our first example of a space is the solutions to the Fermat equation in A, as discussed above. Here's another example.

Example 2.5. Let A be a commutative ring. We'll define the space $\operatorname{Spec} A$ to be the functor $(\operatorname{Spec} A)(B) = \operatorname{Hom}(A, B)$; given a ring homomorphism $\varphi\colon B \to C$, we use the map $\operatorname{Hom}(A, B) \to \operatorname{Hom}(A, C)$ given by postcomposition with φ. ◄

Definition 2.6. An *affine scheme* is a space of the form $\operatorname{Spec} A$ for some A.

You don't have to be a commutative algebra expert to learn algebraic geometry, but you can see that commutative algebra is built into the definitions of algebraic geometry, so some commutative algebra knowledge is helpful.

Example 2.7. The space X_n sending A to the solutions of the Fermat equation in A is an affine scheme; explicitly,

$$X_n \cong \operatorname{Spec} \mathbb{Z}[x, y, z]/(x^n + y^n - z^n).$$

This is because a ring homomorphism $\mathbb{Z}[x, y, z]/(x^n + y^n - z^n) \to A$ is exactly the data of $x, y, z \in A$ satisfying the relation $x^n + y^n - z^n = 0$. ◄

Lemma 2.8 (Yoneda lemma). *For all spaces X, $\operatorname{Hom}_{\mathsf{Spaces}}(\operatorname{Spec} A, X) \cong X(A)$.*

Proof sketch. First we define a map from $\operatorname{Hom}_{\mathsf{Spaces}}(\operatorname{Spec} A, X)$ to $X(A)$. Specifically, a map $f\colon \operatorname{Spec} A \to X$ is the data of for all commutative rings B, $\operatorname{Spec}(A)(B) \to X(B)$. Take $B = A$; then, $\operatorname{Spec}(A)(A) = \operatorname{Hom}(A)$, so take the image of the identity. It remains to check this is an equivalence. ⊠

Corollary 2.9. $\operatorname{Hom}_{\mathsf{Spaces}}(\operatorname{Spec} A, \operatorname{Spec} B) \cong \operatorname{Hom}_{\mathsf{CommRing}}(B, A)$.

It's interesting that the direction reverses!

Proof. By the Yoneda lemma, $\operatorname{Hom}_{\mathsf{Spaces}}(\operatorname{Spec} A, \operatorname{Spec} B) = \operatorname{Spec}(B)(A) = \operatorname{Hom}(B, A)$. ⊠

This tells you that as long as you make sure to reverse the arrows, anything you can do with commutative rings, you can do with affine schemes, and vice versa.

Fiber products. This is a categorical construction which we're going to use a lot.

Definition 2.10. Let X, Y, and Z be sets and $f\colon X \to Z$ and $g\colon Y \to Z$ be set maps. Then the *fiber product* of X and Y over Z is

$$(2.11) \qquad X \times_Z Y \coloneqq \{(x,y) \in X \times Y \mid f(x) = g(y)\}.$$

If X, Y, and Z are spaces, and f and g are maps of spaces, then the *fiber product* of X and Y over Z is the space defined by

$$(2.12) \qquad (X \times_Z Y)(A) \coloneqq X(A) \times_{Z(A)} Y(A).$$

Technically, the notation should include f and g, but in practice there's usually no ambiguity.

Example 2.13. Suppose we're given commutative rings A, B, and C and maps $\operatorname{Spec} B \to \operatorname{Spec} C$ and $\operatorname{Spec} A \to \operatorname{Spec} C$ (which are equivalent data to maps $\varphi\colon C \to A$ and $\psi\colon C \to B$). Then

$$\operatorname{Spec} A \times_{\operatorname{Spec} C} \operatorname{Spec} B \cong \operatorname{Spec}(A \otimes_C B),$$

where C acts on A, resp. B, through φ, resp. ψ. It's worth working through this one on your own, though it's not extremely hard. ◄

We'll define some properties of affine schemes with geometric names, but the definitions will rest on algebraic properties of rings. One of the real miracles of algebraic geometry is that this really works to define geometry, and even extends geometric intuition to places such as finite fields that are otherwise very hard to reason about.

Definition 2.14. A morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is a *closed embedding* if the induced map $A \to B$ is surjective.

Equivalently, $B = A/I$ for some ideal I of A.

The geometric idea behind defining $\operatorname{Spec} A$ is that geometric objects have a ring of functions on them, e.g. a smooth manifold M has a ring $C^\infty(M)$ of smooth $\mathbb{R}$-valued functions, and a map of manifolds $M \to N$ induces a map in the other direction by pullback: $C^\infty(N) \to C^\infty(M)$. Functional analysis results such as the Gelfand-Naimark theorem tell you what data you need to add to $C^\infty(M)$ to recover M as a topological space, and we're trying to imitate this in a more abstract algebraic setting.

This context allows us to explain why Definition 2.14 deserves to be called a closed embedding: let $I = (f_1, f_2, \dots)$, so

$$(2.15) \qquad \operatorname{Spec} A/I = \{f_i = 0 \text{ for all } i\} = \{f = 0 \text{ for all } f \in I\}.$$

So we think of $\operatorname{Spec} B$ as some kind of closed subspace of $\operatorname{Spec} A$, and I as the ideal of functions on $\operatorname{Spec} A$ which vanish on $\operatorname{Spec} B$. This intuition can be turned into something precise.

Using fiber products, we can extend this to all spaces.

Definition 2.16. A map $X \to Y$ of spaces is a *closed embedding* if for all maps $\operatorname{Spec} A \to Y$, the "pullback" φ in the fiber product diagram

$$(2.17)$$

$$\begin{array}{ccc} \operatorname{Spec} A \times_Y X & \xrightarrow{\ \varphi\ } & \operatorname{Spec} A \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

is a closed embedding of affine schemes. In particular, we require $\operatorname{Spec} A \times_Y X$ to be an affine scheme, which is not always satisfied.

For a quick consistency check, we should ask that Definitions 2.14 and 2.16 agree on affine schemes, and indeed, if $I \subset A$ is an ideal, and $\operatorname{Spec} B \to \operatorname{Spec} A$ is a closed embedding in the sense of Definition 2.14, then (2.17) looks like

$$(2.18)$$

$$\begin{array}{ccc} \operatorname{Spec}(A/I \otimes_A B) & \longrightarrow & \operatorname{Spec} B \\ \downarrow & & \downarrow \\ \operatorname{Spec}(A/I) & \longrightarrow & \operatorname{Spec} A, \end{array}$$

and since $A/I \otimes_A B \cong B/BI$, this is a closed embedding in the more general sense as well.

We'd also like to know what an open embedding is. We'd like to say that it's something whose complement is a closed embedding. Let's make this precise.

Definition 2.19. Let $Z \hookrightarrow X$ be a closed embedding of spaces. The *complement* $X \setminus Z$ of Z in X is the space with $(X \setminus Z)(A)$ the set of $x \in X(A) = \text{Hom}_{\mathsf{Spaces}}(\text{Spec}\, A, X)$ such that the diagram

(2.20)
$$\begin{array}{ccc} \varnothing & \longrightarrow & Z \\ \downarrow & & \downarrow \\ \text{Spec}\, A & \xrightarrow{\ x\ } & X \end{array}$$

is a fiber product diagram. Here $\varnothing = \text{Spec}(0)$, which sends every ring to the empty set.[3]

Definition 2.21. If $X = \text{Spec}\, A$ is an affine scheme, an *open embedding* is a map of spaces $j \colon U \to X$ such that $U = X \setminus Z$ for some closed embedding $Z \hookrightarrow X$.

Example 2.22. Letting $X = \text{Spec}\, A$, if $f \in A$ and $Z = \text{Spec}(A/f)$, the map $A \twoheadrightarrow A/f$ induces a closed embedding $Z \hookrightarrow X$. Its complement is $\text{Spec}\, A[f^{-1}]$, the *localization* of A at f, so $\text{Spec}\, A[f^{-1}] \to \text{Spec}\, A$ is an open embedding. ◄

The intuition is that f generates the ideal of functions that vanish precisely on the closed subset Z. Therefore on the complement of Z, they should be invertible, so we adjoin an inverse to f.

Lemma 2.23. *Let $X = \text{Spec}\, A$ and $Z = \text{Spec}\, A/I$. Then maps $\text{Spec}\, B \to X \setminus Z$ correspond bijectively to maps $A \to B$ such that $B \cdot I = B$.*

Proof. The diagram (2.20) specializes to

(2.24)
$$\begin{array}{ccc} \varnothing & \longrightarrow & \text{Spec}(A/I) \\ \downarrow & & \downarrow \\ \text{Spec}\, B & \longrightarrow & \text{Spec}\, A, \end{array}$$

and this fiber product is $\text{Spec}(B \otimes_A A/I) = \text{Spec}(B/IB) = \varnothing$, which is equivalent to $IB = B$. ☒

Example 2.25. *Affine n-space* over $\mathbb{Z}$ is the affine scheme $\mathbb{A}^n_{\mathbb{Z}} \coloneqq \text{Spec}\, \mathbb{Z}[x_1, \ldots, x_n]$, and $0 \hookrightarrow \mathbb{A}^n_{\mathbb{Z}}$ is the closed embedding corresponding to the ideal $(x_1, \ldots, x_n)$. The complement $\mathbb{A}^n_{\mathbb{Z}} \setminus 0$ is not affine for $n > 1$! We'll prove that later when we have more tools. ◄

Exercise 2.26. Show that $\mathbb{A}^n_{\mathbb{Z}} \setminus 0$ is the space which maps a ring A to the set of n-tuples $(x_1, \ldots, x_n) \in A^n$ such that the equation $\sum x_i y_i = 1$ has a solution.

Lecture 3.

Open covers: 9/5/18

We've been talking about functors as if they were honest geometric objects. And they *are*: the crucial reason is that we're defining open and closed subspaces of affine schemes. You can picture these as akin to open or closed sets in a topological space, and they will allow us to make sense of geometry by giving us notions of locality.

Recall that $Z \hookrightarrow X = \text{Spec}\, A$ is a closed embedding means that this embedding is of the form $\text{Spec}(A/I) \to \text{Spec}\, A$ induced by the map $A \twoheadrightarrow A/I$, and that open embeddings are complements of closed ones. You might think of the complement as $(X \setminus Z)(B) = X(B) \setminus Z(B)$, but **this is wrong**: it's not even functorial! Instead, we want to say $(X \setminus Z)(B) = \{\text{Spec}\, B \to X \setminus Z\}$. What this means is maps $\text{Spec}\, B \to X$ such that the pullback $\text{Spec}\, B \times_X Z = \varnothing$. Geometrically, this fiber product is telling you the intersection of the image of $\text{Spec}\, B$ with X.

Last time, we also talked about $\mathbb{A}^1$ (also $\mathbb{A}^1_{\mathbb{Z}}$ if you want to specify the base), which is by definition $\text{Spec}\, \mathbb{Z}[t]$. It would be nice to think of this as a line, in the sense you can draw; but it behaves more like a complex line

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[3]Caution: this is only true if we work with functors on nonzero rings. However, $\varnothing = \text{Spec}\, 0$ still counts as affine. There are other ways to correct this issue, but this is among the fastest and cheapest.

(that is, a plane). For example, $\mathbb{A}^1$ minus a point is connected. So thinking of it as a complex line is good, but for drawing pictures you'll run out of dimensions, so the picture of a real line is also helpful.

If B is a commutative ring, $\mathbb{A}^1(B) = \{\operatorname{Spec} B \to \mathbb{A}^1\}$, i.e. $\operatorname{Hom}(\mathbb{Z}[t], B) = B$, because the map is determined by where it sends t. This makes precise the notion that the ring of functions on $\operatorname{Spec} B$ is B. This is another avatar of geometry as we know it: functions on a geometric object (say, a complex manifold) are functions to a complex line, and in this setting we replace the complex line by $\mathbb{A}^1$.

Consider the embedding $0 \hookrightarrow \mathbb{A}^1_{\mathbb{Z}}$, where 0 denotes the locus where $t = 0$, i.e. $\operatorname{Spec} \mathbb{Z}[t]/(t)$. As an affine scheme, this is isomorphic to $\operatorname{Spec} \mathbb{Z}$, because $\mathbb{Z}[t]/(t) \cong \mathbb{Z}$, but this defines a particular closed embedding $0 \hookrightarrow \mathbb{A}^1_{\mathbb{Z}}$. Last time, we discussed $\mathbb{A}^1 \setminus 0$. A map $\operatorname{Spec} B \to \mathbb{A}^1 \setminus 0$ is a function that avoids zero, which means that it's invertible.

Exercise 3.1. Show that $(\mathbb{A}^1 \setminus 0)(B) = B^{\times}$, and therefore that $\mathbb{A}^1 \setminus 0 \cong \operatorname{Spec} \mathbb{Z}[t, t^{-1}]$.

If we did this with $\mathbb{A}^2 \setminus 0$ instead of $\mathbb{A}^1 \setminus 0$, we'd obtain a nonaffine scheme.

Open coverings are another important geometric notion, and they exist in this setting too.

Definition 3.2. If $X = \operatorname{Spec} A$ is an affine scheme, a *(Zariski) open covering* of X is a collection of open embeddings $\mathfrak{U} = \{(U, i_U : U \hookrightarrow X)\}$ such that for every nonempty $S = \operatorname{Spec} B$ and $f : S \to X$, there's some $(U, i_U) \in \mathfrak{U}$ such that $U \times_X S \neq \varnothing$.

This is the first notion of open covering in algebraic geometry; there are some others around.

The intuition behind open coverings is that points of X are given by maps $\operatorname{Spec} B \to X$, and we want every point in X to intersect some open embedding in the cover.

Proposition 3.3. *Let $X = \operatorname{Spec} A$ and $\mathfrak{U} = \{(U, i_U : U \to X)\}$ be a collection of open embeddings. The following are equivalent:*

(1) *$\mathfrak{U}$ is an open covering.*
(2) *$\mathfrak{U}$ has a finite subset $\mathfrak{V} \subset \mathfrak{U}$ which is also an open covering of X.*
(3) *For all fields k and maps $x : \operatorname{Spec} k \to X$, there's some $(U, i_U) \in \mathfrak{U}$ such that x factors through i_U.*
(4) *Letting $U = X \setminus Z_U$ for each $U \in \mathfrak{U}$, and writing $Z_U = \operatorname{Spec}(A/I_U)$, then*

$$\sum_{U \in \mathfrak{U}} I_U = A.$$

Point (2) is very weird coming from topology, where the open covering $\{(i-1, i+1) \mid i \in \mathbb{Z}\}$ is an open cover of $\mathbb{R}$ with no finite subcover. In other words, affine schemes feel like compact spaces from the perspective of open coverings!

The idea behind (3) is that points are affine schemes of the form $\operatorname{Spec} k$ for k a field. There are different fields, and therefore different kinds of points. The reason for including (4) is that it's very useful for checking in practice. It has a similar feel to partitions of unity in manifold topology, but if you don't know what that is, that's OK.

Proof. We'll first show (1) $\implies$ (4). Suppose $\mathfrak{U}$ is an embedding for which (4) does not hold. Then let

$$(3.4) \qquad\qquad B := A / \sum_{U \in \mathfrak{U}} I_U.$$

By hypothesis, $B \neq 0$, and we have a closed embedding $\operatorname{Spec} B \hookrightarrow \operatorname{Spec} A$. We'll show that $\operatorname{Spec} B \times_X U = \varnothing$ for all $U \in \mathfrak{U}$.

Lemma 3.5. *Let $Z = \operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A = X$ be a closed embedding and $f : \operatorname{Spec} B \to X$ be a map. Then*

$$(\operatorname{Spec} B) \setminus f^{-1}(Z) = \operatorname{Spec} B \times_X (X \setminus Z).$$

This is more or less a tautology.

Returning to the claim, $\operatorname{Spec} B \times_X U$ is the complement of $Z_U \times_X \operatorname{Spec} B = \operatorname{Spec}(B/BI_U)$. But $B/BI_U = B$, so the complement of $Z_U \times_X \operatorname{Spec} B$ is the empty set.

Next, we'll show (4) $\implies$ (3). Let k be a field, and $x : \operatorname{Spec} k \to X$ be a map. We want to show this map factors through some U. Since $X = \operatorname{Spec} A$, x corresponds to a map $\varphi : A \to k$. We claim there's a $U \in \mathfrak{U}$ with $\varphi(I_U) \neq 0$; otherwise $\varphi(\sum I_U) = 0$, and therefore $\varphi(A) = 0$. However, $\varphi(1) = 1$, so this is impossible. By Lemma 2.23, since $\varphi(I_U) \neq 0$, $k \cdot \varphi(I_U) = k$, and therefore $x : \operatorname{Spec} k \to X$ factors through U.

Next we'll show (3) $\implies$ (1). Let B be as in (3.4) and $f\colon S = \operatorname{Spec} B \to X$ be a map. We want to show that $S \times_X U \neq 0$ for some $U \in \mathfrak{U}$. Since $B \neq 0$, it has a maximal ideal $\mathfrak{m}$, and $B/\mathfrak{m}$ is a field k (TODO: to be continued...)

$\boxtimes$

Lecture 4.

Defining schemes, II: 9/7/18

"I'll let $\operatorname{Fun}(Y)$*, which is such a fun notation, denote..."*

Last time, we talked about open embeddings and open covers for affine schemes; today, we'll generalize this to spaces.

Definition 4.1. Let X be a space.

(1) A map $U \to X$ of spaces is an *open embedding* if for all affine schemes $S = \operatorname{Spec} A$ and maps $f\colon S \to X$ of spaces, the pullback $g\colon U \times_X S \to S$ arising in the diagram

$$
\begin{array}{ccc}
U \times_X S & \longrightarrow & U \\
\big\downarrow{\scriptstyle g} & & \big\downarrow \\
S & \xrightarrow{\ f\ } & X
\end{array}
$$

is an open embedding (since we've already define open embeddings where the target is affine).

(2) A *Zariski open covering* of X is the same as in Definition 3.2, but for open embeddings of spaces, rather than affine schemes.

In this case, Proposition 3.3 need not hold: there are open coverings of some spaces X (such as an infinite disjoint union of points) which have no finite subcoverings.

Definition 4.2. A space X is a *Zariski sheaf* if for all $S = \operatorname{Spec} A$ and open coverings $\mathfrak{U}$ of S, the map

$$\operatorname{Hom}(S, X) \longrightarrow \{(f_U \colon U \to X \text{ for all } U) \in \mathfrak{U} \mid f_U|_{U \cap V} = f_V|_{U \cap V} \text{ for all } U, V \in \mathfrak{U}\}$$

is an isomorphism. (Here $U \cap V = U \times_X V$.)

Not everything is a Zariski sheaf, but the things that aren't are terrible, and you shouldn't worry too much about them.

Now we have all the definitions at hand to define schemes!

Definition 4.3. A *scheme* is a space which is a Zariski sheaf and admits an open cover $\mathfrak{U}$ such that all $U \in \mathfrak{U}$ are affine schemes.

Exercise 4.4. Let X be the space with

$$X(A) = \{t \in A \mid t \in A^\times \text{ or } (1 - t) \in A^\times\}.$$

Show that X is *not* a Zariski sheaf. Also, if you know what sheafification is, show that the sheafification of X is $\mathbb{A}^1$.

Proposition 4.5. *If X is an affine scheme, then it's a scheme.*

Obviously X admits an open cover by affines, given by $\operatorname{id}\colon S \to S$; the meat of the proof (or, if you prefer, tofu) is that it's a Zariski sheaf. Unlike EGA, we will start with a special case and use it to bootstrap to the general case.

Let $X = \mathbb{A}^1$.

Definition 4.6. A *function* on a space Y is a map to $\mathbb{A}^1$. We'll let $\operatorname{Fun}(Y) \coloneqq \operatorname{Hom}(Y, \mathbb{A}^1)$.

We're explicitly trending towards geometric notation and intuition for things: one of the key processes of learning scheme theory is to start thinking geometrically rather than with commutative algebra – except when you need to prove something.

We want to show that for all affine schemes S and open coverings $\mathfrak{U}$ of S, the map

$$(4.7) \qquad \operatorname{Fun}(S) \longrightarrow \{(f_U \in \operatorname{Fun}(U)) \mid f_U|_{U \cap V} = f_V|_{U \cap V}\}$$

is an isomorphism.

First we'll prove this for a nice class of open covers.

Lemma 4.8. *Let A be a commutative ring and $f_1, \dots, f_n \in A$. Let $D(f_i) := \operatorname{Spec} A \setminus \operatorname{Spec}(A/(f_i))$. Then*

(1) $D(f_i) = \operatorname{Spec} A[f_i^{-1}]$, *and*

(2) $\{D(f_i)\}$ *is an open cover iff* $\{f_i\}$ *generates the unit ideal.*

The proof will be left as an exercise.

In the case (2) holds, the open cover $\{D(f_i)\}$ is called a *basic open cover*. It's really nice because it's an affine open cover; we'll see that there are a lot of these coverings, and enough that we will eventually be able to reduce to this case.

One can alternately characterize $D(f_i)$ as the pullback

(4.9)
$$
\begin{array}{ccc}
D(f_i) & \longrightarrow & \operatorname{Spec} A \\
\downarrow & & \downarrow \\
\mathbb{A}^1 \setminus 0 & \longrightarrow & \mathbb{A}^1.
\end{array}
$$

Lemma 4.10. *Let $f \colon M \to N$ be a map of A-modules. Then f is injective (resp. surjective, resp. bijective) iff for all i, the map $M[f_i^{-1}] \to N[f_i^{-1}]$ is injective (resp. surjective, resp. bijective).*

Recall that $M[f_i^{-1}] := M \otimes_A A[f_i^{-1}]$.

Remark 4.11. Let's review some facts about localization. If M is a $\mathbb{Z}[t]$-module, which is equivalent data to an abelian group with an endomorphism $t \colon M \to M$, then we can form $M[t^{-1}] := M \otimes_{\mathbb{Z}[t]} \mathbb{Z}[t, t^{-1}]$. Then:

- This construction is *exact*; that is, it preserves kernels and cokernels.
- There's a natural map $M \to M[t^{-1}]$, and its kernel is the submodule of $m \in M$ with $t^n m = 0$ for some n.

The way you prove all of this is to write $\mathbb{Z}[t, t^{-1}]$ as the union of $t^{-n}\mathbb{Z}[t]$ for all n, or as the colimit of the multiplication-by-t map on $\mathbb{Z}[t]$. These are all free modules, hence flat, and one can prove that filtered colimits of flat modules are flat without too much angst. ◄

Now we can get back to the lemma.

Proof of Lemma 4.10. For injectivity, let's compare $\ker \varphi$ with $\ker(\varphi[f_i^{-1}])$. By Remark 4.11, $(\ker \varphi)[f_i^{-1}] = \ker(\varphi[f_i^{-1}])$, so we can reduce to showing that $M = 0$ iff $M[f_i^{-1}] = 0$ for all i. One direction is immediate, of course; conversely, if $M[f_i^{-1}] = 0$ for all i, then for all $m \in M$ and all f_i, then $f_i^{n_i} m = 0$ for some $n_i \gg 0$. There are finitely many f_i, so we can let N be the biggest one, and then $f_i^N m = 0$ for all i.

Lemma 4.12. $(f_1, \dots, f_n) = A$ *iff* $(f_1^N, \dots, f_n^N) = A$.

Proof. There's a naïve argument which isn't too bad, but the geometric reason is that f_i is a function, and f_i vanishes on the same locus where f_i^N vanishes, and therefore $D(f_i^N) = D(f_i)$. Therefore $\{D(f_i^N)\}$ is also an open cover, which by (4.8) means $(f_1^N, \dots, f_n^N) = A$.

If this proof feels sketchy, here's a more careful one (which unfortunately masks the geometry): if $(f_1^N, \dots, f_n^N) \subsetneq A$, then it's contained in some maximal ideal $\mathfrak{m}$. Therefore for all i, $f_i^N = 0$ in $A/\mathfrak{m}$, and therefore $f_i = 0$ in $\mathfrak{m}$, because $A/\mathfrak{m}$ is a field; hence $f_i \in \mathfrak{m}$, which is a contradiction. ⊠

Now, using Lemma 4.12, there are some $g_1, \dots, g_n$ with $\sum_i g_i f_i^N = 1$, and therefore

(4.13)
$$
M = \sum_i f_i f_i^N \cdot M = 0.
$$

The proof of surjectivity is similar, but using cokernels instead of kernels. ⊠

This lemma is a bridge between the geometry of schemes and the linear algebra of modules. You should think of inverting f_i as restricting to $D(f_i)$; we will return to this idea.

Proof of Proposition 4.5, special case. Now we'll prove that an affine scheme is a Zariski sheaf for basic open covers $\{D(f_i)\}$. We want to show that (4.7) is an isomorphism, and by Lemma 4.10 it suffices to show this after inverting f_i.

Let $g_i \in \operatorname{Fun}(D(f_i))$ be a collection of functions that agree on overlaps... TODO: I missed the last part. ⊠

Lecture 5.

$\mathbb{A}^1$ is a Zariski sheaf: 9/10/18

Today, we're going to continue proving Proposition 4.5, that affine schemes are schemes. We're still working on the special case that $\mathbb{A}^1$ is a scheme; the key piece of the proof is showing that it's a Zariski sheaf.

Definition 5.1. Let S be a space and $\mathfrak{U}$ be an open cover of S. A *refinement* of $\mathfrak{U}$ is an open covering $\mathfrak{V}$ of S such that for all $U \in \mathfrak{U}$, $\mathfrak{V}_U := \{V \in \mathfrak{V} \mid V \subset U\}$ is an open covering of U.

The Zariski sheaf condition for maps $X \to S$ is a constraint on compatible functions on all open covers of S. If we only ask about a specific open cover $\mathfrak{U}$, we say "the Zariski sheaf property for X with respect to $\mathfrak{U}$."

Lemma 5.2. *Let X and S be spaces, $\mathfrak{U}$ be an open cover of S, and $\mathfrak{V}$ be a refinement of $\mathfrak{U}$. Suppose the Zariski sheaf property holds for X with respect to $\mathfrak{V}$, and for each $U \in \mathfrak{U}$ with respect to $\mathfrak{V}_U$, then it holds with respect to $\mathfrak{U}$.*

After you unwind all the definitions, this is a definition check which isn't very hard.

Remark 5.3. One corollary of Lemma 5.2 is that in the definition of the sheaf property, we may replace "for all affine schemes S" with "for all spaces S." All of the definitions were built from the beginning to favor affine schemes as important or special, and this is one consequence. ◄

Definition 5.4. A *big basic open covering* of an affine scheme S is an open covering by sets of the form $D(f_i)$ as in Lemma 4.8, but over a possibly infinite indexing set.

This is only a temporary definition. The Zariski sheaf property for X and every basic open covering of an affine scheme S implies the Zariski sheaf for all big basic open coverings.

Proposition 5.5. *Let S be an affine scheme and $\mathfrak{U}$ be an open cover of S. Then there's a big basic open covering of S refining $\mathfrak{U}$.*

Proof. Write $S = \operatorname{Spec} A$ and for each $U \in \mathfrak{U}$, let $Z_U := S \setminus U$; the inclusion $Z_U \hookrightarrow S$ is a closed embedding, so $Z_U = \operatorname{Spec}(A/I_U)$ for some ideal $I_U \subset A$. Recall from Proposition 3.3 that since $\mathfrak{U}$ is an open covering,

$$\tag{5.6} \sum_{U \in \mathfrak{U}} I_U = A,$$

and this is an equivalent condition. Consider the big basic open cover

$$\tag{5.7} \mathfrak{V} := \{D(f) \mid f \in I_U \setminus 0 \text{ for some } U \in \mathfrak{U}\}.$$

That this is a big basic open cover is because an ideal is generated by its elements. It's also a refinement of $\mathfrak{U}$, which follows from a more general lemma.

Lemma 5.8. *Let $U = S \setminus Z$, where $S = \operatorname{Spec} A$ and $Z = \operatorname{Spec}(A/I)$. Then $\{D(f) \mid f \in I \setminus 0\}$ is an open cover of U.*

Proof. We want to show that for all $T = \operatorname{Spec} B$ and maps $g \colon T \to U$, the set $\mathfrak{V}_g := \{g^{-1}(D(f)) \mid f \in I \setminus 0\}$ is an open cover of T.[4] Then... TODO ☒

Thus we've proven the proposition. ☒

Corollary 5.9. *Let S be an affine scheme with an open covering $\mathfrak{U}$. Then there's a big basic open covering $\mathfrak{V}$ refining $\mathfrak{U}$ and with the property that for all $U \in \mathfrak{U}$, $\{V \in \mathfrak{V} \mid V \subset U\}$ is a big basic open covering of U.*

This is the technical proposition that lets us reduce to algebra.

Remark 5.10. Corollary 5.9 also tells us that a big basic open covering of a space X is an open covering $\mathfrak{U}$ of X such that for all maps of affine schemes to X, the pullback of $\mathfrak{U}$ is also a big basic open covering. ◄

Corollary 5.11. $\mathbb{A}^1$ *is a Zariski sheaf.*

---
[4]The preimage is defined to be $g^{-1}(D(f)) := D(f) \times_U T$.

Proof. We showed that $\mathbb{A}^1$ is a Zariski sheaf with respect to all basic open covers of affine schemes, hence for all big basic open covers of affine schemes, hence by Remark 5.10 with respect to all spaces with big basic open covers, hence by Proposition 5.5 any affine scheme and any open cover, and therefore any space and any open cover.   ⊠

Corollary 5.12. *Let I be a set and let $\mathbb{A}^I \coloneqq \mathrm{Spec}\,\mathbb{Z}[\{x_i \mid i \in I\}]$. Then $\mathbb{A}^I$ is a Zariski sheaf.*

Proof. The sheaf property is preserved under arbitrary products.   ⊠

If I is an n-element set, then $\mathbb{A}^I$ is also written $\mathbb{A}^n$.

Proof sketch of Proposition 4.5. We can use this to show that if $X = \mathrm{Spec}\,A$ is an affine scheme, then it's a Zariski sheaf. Let I be a generating set for A and $J \subset \mathbb{Z}[\{x_i \mid i \in I\}]$ be the ideal of relations; then, the quotient map $\mathbb{Z}[\{x_i \mid i \in I\}] \twoheadrightarrow A$ defines a closed embedding $X \subseteq \mathbb{A}^I$ cut out by $X = \{x \mid f(x) = 0 \text{ for all } f \in J\}$.

One then has to check that the sheaf property is preserved under closed embeddings, which is formal.   ⊠

We'll spend the next lecture giving examples of schemes, but here are a few to start with.

- As we just showed, affine schemes are schemes.
- A *quasi-affine scheme* is an open subset of an affine scheme, such as $\mathbb{A}^2 \setminus 0$. These are indeed schemes (though not always affine): if U is the complement of $\mathrm{Spec}(A/I) \subset A$, then U admits a covering by $\{D(f) \mid f \in I \setminus 0\}$.

We can use this to prove $\mathbb{A}^2 \setminus 0$ isn't affine.

Lecture 6.

Relative algebraic geometry: 9/12/18

One of the advantages of algebraic geometry is the ability to work relative to a given space, which generalizes choosing a base field (or ring).

Definition 6.1. Let S be a space. A *scheme over S* is a space X with a map $X \to S$, often just written X/S, such that for all affine schemes T and maps $T \to S$, $X \times_S T$ is a scheme. A morphism of schemes over S is a morphism of schemes which commutes with the two maps to S.

In the same way one can define affine schemes over S. If $S = \mathrm{Spec}\,A$, for A a ring, we might write X/A instead of X/S and say schemes over A; often A will be a field.

We defined spaces as functors $\mathsf{CommRing} \to \mathsf{Set}$, and there's a similar description for schemes over A.

Proposition 6.2. *Let A be a commutative ring. There's an equivalence of categories between spaces over A and functors $\mathsf{CommAlg}_A \to \mathsf{Set}$ (where we ignore the zero algebra).*

Proof sketch. Given $X \colon \mathsf{CommAlg}_A \to \mathsf{Set}$, we can define a functor on all commutative rings by sending B to the set of pairs of (i, x) where $i \colon A \to B$ is an A-algebra structure on B and $x \in X(B)$. Then the forgetful map $(i, x) \mapsto i$ defines the desired map to $\mathrm{Spec}\,A$.

In the other direction, let $p \colon X \to \mathrm{Spec}\,A$ be a scheme over A. We'll define a functor on commutative A-algebras by sending $(B, i \colon A \to B)$ to the set of maps $\varphi \colon \mathrm{Spec}\,B \to X$ for which the diagram

$$
(6.3) \qquad\qquad
\begin{array}{ccc}
 & & X \\
 & \nearrow{\scriptstyle\varphi} & \downarrow{\scriptstyle p} \\
\mathrm{Spec}\,B & \xrightarrow{\ i^*\ } & \mathrm{Spec}\,A
\end{array}
$$

commutes.   ⊠

Example 6.4. Complex conjugation is $\mathbb{Z}$-linear (and even $\mathbb{R}$-linear) but not $\mathbb{C}$-linear, and therefore induces a map of schemes $\mathrm{Spec}\,\mathbb{C} \to \mathrm{Spec}\,\mathbb{C}$ which is a map of schemes over $\mathbb{R}$, but not of schemes over $\mathbb{C}$.   ◄

Proposition 6.5. *Let X, Y, and Z be schemes together with maps $X \to Z$ and $Y \to Z$. Then $X \times_Z Y$ is a scheme.*

Proof. If $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, and $Z = \operatorname{Spec} C$ are affine, this is certainly true: the pullback is $\operatorname{Spec} A \otimes_C B$. Now we'll show more general cases reduce to this one.

If Y and Z are affine but X isn't, then X admits an open cover $\mathfrak{U}$ by affines, and $\{U \times_Z Y \mid U \in \mathfrak{U}\}$ is an affne open cover of $X \times_Z Y$. In the same way, we may assume only that X and Z are affine.

Therefore if you only assume Z is affine, you can pick affine open covers of X and Y called $\mathfrak{U}$ and $\mathfrak{V}$, respectively. Then $\{U \times_Z V \mid U \in \mathfrak{U}, V \in \mathfrak{V}\}$ is an affine open cover of $X \times_Z Y$.

Next, we assume X and Y are affine, but Z might not be.[5] Let $\mathfrak{W}$ be an affine open cover of Z, and $W \in \mathfrak{W}$. By definition, the map

(6.6) $$X \times_Z W \longrightarrow X$$

is an open embedding, and this implies that $X \times_Z W$ is a scheme (we called these quasi-affine): it's the complement of a closed embedding $\operatorname{Spec} A/I \to X = \operatorname{Spec} A$, and is covered by $\{D(f)\}$ where $\{f\}$ generates I. Anyways, then $X \times_Z Y$ is covered by

(6.7) $$\mathfrak{W}' := \{(X \times_Z W) \times_W (Y \times_Z W) \mid W \in \mathfrak{W}\}.$$

Since W is affine, this is a scheme by one of the earlier cases. Therefore $X \times_Z Y$ is covered by schemes, so it must be a scheme (choose an affine cover of each element of $\mathfrak{W}'$, and check this is an affine open cover of $X \times_Z Y$).

Finally, we assume none of them are affine. This is the same as the case where X and Y are affine, but now we can use the previous step to show that if $\mathfrak{U}$ is an open cover of X, $\mathfrak{V}$ is an open cover of Y, $U \in \mathfrak{U}$, and $V \in \mathfrak{V}$, then $U \times_Z V$ is a scheme.

We've ignored the Zariski sheaf property, but it's relatively simple to show that it's preserved by fiber products.   ⊠

Corollary 6.8. *If S is a scheme, schemes over S are the same thing as schemes with a map to S.*

Proof. We can check the definition on an affine open cover of S; Proposition 6.5 tells us that pulling back to T preserves scheminess.   ⊠

If S is a space that's not a space, Corollary 6.8 isn't necessarily true.

Quasicoherent sheaves and/or linear algebra. In commutative algebra, one often studies a ring by studying its modules; these are linear-algebraic in nature, which can make them easier to reason about. The analogue for schemes is quasicoherent sheaves.

Definition 6.9. Let X be a scheme. A *quasicoherent sheaf* (QC sheaf) $\mathscr{F}$ on X is data of, for all maps $f \colon \operatorname{Spec} A \to X$, an A-module $\mathscr{F}_f$, and for every map $g \colon \operatorname{Spec} B \to \operatorname{Spec} A$, an isomorphism

(6.10) $$\alpha_{f,g} \colon \mathscr{F}_{g \circ f} \overset{\cong}{\to} \mathscr{F}_f \otimes_A B$$

of B-modules, and such that a cocycle condition holds: given a triple

(6.11) $$\operatorname{Spec} C \overset{h}{\longrightarrow} \operatorname{Spec} B \overset{g}{\longrightarrow} \operatorname{Spec} A \overset{f}{\longrightarrow} X,$$

$\alpha_{f, g \circ h} = \alpha_{f \circ g, h}$ as maps $\mathscr{F}_{f \circ g \circ h} \cong (\mathscr{F}_f \otimes_A B) \otimes_B C$, using the natural isomorphism $(\mathscr{F}_f \otimes_A B) \otimes_B C \cong \mathscr{F}_f \otimes_A C$.[6] A morphism of quasicoherent sheaves $\mathscr{F} \to \mathscr{G}$ is data of maps of A-modules $\mathscr{F}_f \to \mathscr{G}_f$ for all $f \colon \operatorname{Spec} A \to X$, such that all induced diagrams commute. The category of QC sheaves on X is denoted $\mathsf{QCoh}(X)$.

Remark 6.12. The word "quasicoherent" isn't really great unless you're playing Scrabble. It grew out of a generalization of coherent sheaves, which originally came from the analytic setting, where the name was more reasonable. You should think of analogues of modules when you hear QC sheaves.   ◄

This is a lot of data! So we're going to find a way to express a quasicoherent sheaf with less data.

Proposition 6.13. *If $X = \operatorname{Spec} A$, the functor $\Gamma \colon \mathsf{QCoh}(X) \to \mathsf{Mod}_A$ sending $\mathscr{F} \mapsto \mathscr{F}_{\mathrm{id}}$ is an equivalence of categories, with inverse sending an A-module M to the sheaf $\mathscr{F}_M$ defined by $(\mathscr{F}_M)_f := M \otimes_A B$ for all $f \colon \operatorname{Spec} B \to X$.*

---

[5]From here, the proof was finished up in Friday's lecture.

[6]The cocycle condition can be expressed more concisely by asking that $\mathscr{F}$ is a functor from the category of affine schemes to abelian groups.

Example 6.14. For any scheme X, there's a quasicoherent sheaf $\mathscr{O}_X$, called the *structure sheaf* of X, defined to send $f\colon \operatorname{Spec} A \to X$ to $(\mathscr{O}_X)_f = A$. The maps are what you think they are. ◄

Lecture 7.

Quasicoherent sheaves: 9/14/18

> *"You know when you're looking for your phone and it was in your hand the whole time?*
> *This proof was like that."*

Here are two exercises we've been sort of implicitly using, and are good to do to get some comfort with this language.

Exercise 7.1.

   (1) Let $U \to X$ be a map of schemes and U has an open cover $\mathfrak{V}$ such that for all $V \in \mathfrak{V}$, $V \to X$ is an open embedding. Then $U \to X$ is an open embedding.

   (2) If $V \to U$ and $U \to X$ are open embeddings, their composition $V \to U$ is an open embedding.

Now back to quasicoherent sheaves. On an affine scheme $X = \operatorname{Spec} A$, these are a lot like A-modules (in fact, exactly like A-modules, according to Proposition 6.13).

Definition 7.2. Let $f\colon X \to Y$ be a map of schemes and $\mathscr{F} \in \mathsf{QCoh}(Y)$. The *pullback* of $\mathscr{F}$, denoted $f^*\mathscr{F} \in \mathsf{QCoh}(X)$, is the quasicoherent sheaf given by the following data: for every map $g\colon \operatorname{Spec} A \to X$, $(f^*\mathscr{F})_g := \mathscr{F}_{f \circ g}$.

One must check the compatibility conditions, but these aren't so bad.

If $S = \operatorname{Spec} A$ is affine, then an A-module M defines a quasicoherent sheaf $\mathscr{M}$ by sending $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ to $\mathscr{M}_f := M \otimes_A B$. The pullback of $\mathscr{M}$ along f is exactly the quasicoherent sheaf defined by the module $M \otimes_A B$.

Since we understand quasicoherent sheaves on affine schemes, let's next see how they behave on open covers. We'll start with a different-looking definition, then show it's equivalent. This second definition will be useful because it involves substantially less data.

Definition 7.3. Let X be a scheme and $\mathfrak{U}$ be an open cover of X. Let $\mathsf{QCoh}(X; \mathfrak{U})$ denote the category of tuples of $\mathscr{F}_U \in \mathsf{QCoh}(U)$ for all $U \in \mathfrak{U}$ together with, for all intersecting $U, V \in \mathfrak{U}$, isomorphisms

$$(7.4) \qquad\qquad \alpha_{UV}\colon \mathscr{F}_U|_{U \cap V} \xrightarrow{\;\cong\;} \mathscr{F}_V|_{U \cap V}$$

satisfying a cocycle condition on triple intersections.

   This is what's sheafy about quasicoherent sheaves: they are determined from compatible local data.

   There's a functor $\Phi\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(X; \mathfrak{U})$ which takes a quasicoherent sheaf and produces its pullback on all $U \in \mathfrak{U}$.

Theorem 7.5 (Serre). *The functor Φ is an equivalence of categories.*

Proof sketch. This will look a lot like what we did before. The first step is to reduce to the case where $X = \operatorname{Spec} A$ is affine and $\mathfrak{U}$ is a basic open cover, using a similar argument to the one from two lectures ago. The second step is similar to the proof that $\mathbb{A}^1$ is a Zariski sheaf.

   Explicitly, after we've reduced to $X = \operatorname{Spec} A$ and $\mathfrak{U} = \{D(f_i) \mid (f_1, \ldots, f_n) = A\}$, then a quasicoherent sheaf on $D(f_i)$ is (equivalent data to) an $A[f_i^{-1}]$-module M_i, together with the natural isomorphisms $\alpha_{ij}\colon M_i[f_j^{-1}] \xrightarrow{\cong} M_j[f_i^{-1}]$ as $A[(f_i f_j)^{-1}]$-modules.

   Given this data, we want to functorially build an A-module. The answer will be

$$(7.6) \qquad\qquad M := \{s_i \in M_i, 1 \le i \le n \mid \text{in } M_i[f_j^{-1}] \cong M_j[f_i^{-1}],\ s_i = s_j\}.$$

Now the proof is the same as in the $\mathbb{A}^1$-setting, though there we only worried about functions, not sections. The other way is simple once one invokes the flatness of $A[f_i^{-1}]$. ⊠

   We might not have defined it yet, but for a field k, $\mathbb{A}_k^2 = \operatorname{Spec} k[x, y]$. This is slightly nicer to work with for some applications than $\mathbb{A}_{\mathbb{Z}}^2$. Let $X := \mathbb{A}_k^2 \setminus 0$, our favorite non-affine scheme, with its open cover $U := \mathbb{A}^1 \times (\mathbb{A}^1 \setminus 0)$ and $V := (\mathbb{A}^1 \setminus 0) \times \mathbb{A}^1$. Then Theorem 7.5 says a quasicoherent sheaf on $\mathbb{A}_k^2 \setminus 0$ is the data of

- a $k[x, x^{-1}, y]$-module M,
- a $k[x, y, y^{-1}]$-module N, and
- an isomorphism $\alpha \colon M[y^{-1}] \cong N[x^{-1}]$ of $k[x, x^{-1}, y, y^{-1}]$-modules.

Modules can be big, so it will be useful to have some finiteness hypotheses.

Definition 7.7. Let X be a scheme and $\mathscr{F} \in \mathsf{QCoh}(X)$. Then $\mathscr{F}$ is *locally finitely generated* (l.f.g.) if for all open embeddings $j \colon \operatorname{Spec} A \to X$, $j^*\mathscr{F}$ is a finitely generated A-module.

Theorem 7.8 (Nakayama's lemma). *Let $\mathscr{F}$ be a locally finitely generated QC sheaf on a scheme X, k be a field, and $x \colon \operatorname{Spec} k \to X$ be such that $x^*\mathscr{F} = 0$. Then there's an open $j \colon U \hookrightarrow X$ containing x (i.e. x factors through j) and such that $j^*\mathscr{F} = 0$.*

Geometrically, this is saying that if an l.f.g. sheaf vanishes at a point, it also vanishes in a neighborhood of that point.

Proof. First we'll reduce to the affine case: we know there's an affine open $V \subseteq X$ such that x factors through V (geometrically, the point x lies in V), so we'll replace X by V (and call it X). Let $X = \operatorname{Spec} A$, so that $\mathscr{F}$ corresponds to a finitely generated A-module M, and x corresponds to a map $\varphi \colon A \to k$. Our hypothesis means that $M \otimes_A k = 0$.

Let's induct on the number of generators of M. If M is generated by zero elements, we're done, so assume we know it for all modules generated by n elements... we'll finish this Monday.   ⊠

Lecture 8.

Quasicoherent sheaves, II: 9/17/18

We're in the middle of proving Nakayama's lemma, Theorem 7.8. We're proving it by induction on the number of generators of the A-module M, and the base case is trivial. So let's assume it's true for all modules generated by n elements.[7]

Remark 8.1. Let's pause to ask what a finitely generated A-module looks like. If it has one generator, it's isomorphic to A/I for some ideal I. If it has two generators, it's an extension of A/I by A/J for some ideals I and J of A. More generally, a module M with m generators is an extension $0 \to N \to M \to A/I \to 0$, where N has $m-1$ generators.

This means some specific subcases of Nakayama's lemma, such as that for local rings, are close to trivial.[8] You could prove Theorem 7.8 by reducing to the local case, though we're using a different approach.

The fact that finitely generated modules have quotients which look like A/I is the catalyst of the proof: it's untrue for modules which aren't finitely generated, such as $\mathbb{Q}$ as a $\mathbb{Z}$-module, which has no quotients of the form $\mathbb{Z}/n$.   ◄

So M is an extension of an A-module N generated by n elements by A/I:

$$(8.2) \qquad\qquad 0 \longrightarrow N \longrightarrow M \longrightarrow A/I \longrightarrow 0.$$

By assumption, $M \otimes_A k = 0$, which means that, since tensor product is right exact,

$$(8.3) \qquad\qquad (A/I) \otimes_A k \cong k/Ik = 0.$$

Recall that we had data of a map $\varphi \colon A \to k$; since k is a field, this and (8.3) imply there's some $f \in I$ wth $\varphi(f) \neq 0$. Let's localize at f; the map $\varphi \colon A \to k$ passes to a map $\widetilde{\varphi} \colon A[f^{-1}] \to k$, and since localization is exact, (8.2) induces a short exact sequence

$$(8.4) \qquad\qquad 0 \longrightarrow N[f^{-1}] \longrightarrow M[f^{-1}] \longrightarrow (A/I)[f^{-1}] \longrightarrow 0,$$

but since $A[f^{-1}] = 0$, $N[f^{-1}] \cong M[f^{-1}]$, which (crucially) is generated by n elements as an $A[f^{-1}]$-module. Since $\varphi(f) \neq 0$, then $x \in D(f)$, so there's an open $U \subset D(f)$ containing x such that $(\mathscr{F}|_{D(f)})|_U = 0$ by the inductive hypothesis, and that's exactly what we wanted to prove.

Definition 8.5. Let M be an A-module. Then its *annihilator* $\operatorname{Ann}(M) \coloneqq \{f \in A \mid f \cdot M = 0\}$, which is an ideal of A.

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[7]The theorem is true for non-affine schemes, but we've already reduced to the affine case.

[8]There are many different things called Nakayama's lemma; ours is not the most general one.

Corollary 8.6. *Let X be a scheme and $\mathscr{F} \in \mathsf{QCoh}(X)$ be locally finitely generated. Then the subset $U_{\mathscr{F}} := \{f\colon \operatorname{Spec} B \to X \mid f^*\mathscr{F} = 0\}$ is an open subscheme of X. In particular, if $X = \operatorname{Spec} A$ is affine, then $U_{\mathscr{F}}$ is the complement of the locus of X on which all $f \in \operatorname{Ann}(\mathscr{F})$ vanish.*

That is, the locus where $\mathscr{F}$ vanishes is open. This fits into your intuition: if you're on $\operatorname{Spec} A$ and $\mathscr{F}$ corresponds to $A/(f)$, then $\mathscr{F}$ vanishes wherever f doesn't.

Proof. It suffices to prove the affine statement, and this is a matter of unwinding its definition: let $X = \operatorname{Spec} A$ and $\mathscr{F}$ be an A-module. Given $\varphi\colon A \to B$, it suffices to prove the following are equivalent: $\operatorname{Ann}(\mathscr{F}) \cdot B = B$ and $\mathscr{F} \otimes_A B = 0$.

First, the forward implication: we know there are $f_i \in \operatorname{Ann}(\mathscr{F})$ and $g_i \in B$ such that

$$(8.7) \qquad \sum_{i=1}^{n} \varphi(f_i) g_i = 1.$$

Therefore 1 acts by 0 on $\mathscr{F} \otimes_A B$, so that module must be the zero module.

The reverse direction is a bit harder. Suppose for a contradiction that $\operatorname{Ann}(\mathscr{F}) \cdot B \subsetneq B$, so it's contained in some maximal ideal $\mathfrak{m}$; let $k := B/\mathfrak{m}$, which is a field. Then

$$(8.8) \qquad \mathscr{F} \otimes_A k = (\mathscr{F} \otimes_A B) \otimes_B k = 0.$$

Hence, by Theorem 7.8, there's a $U \subset \operatorname{Spec} A$ containing $\operatorname{Spec} k$ such that $\mathscr{F}|_U = 0$. We can assume $U = D(f)$ for some $f \in A$, so we're assuming $\mathscr{F}[f^{-1}] = 0$. Because $\mathscr{F}$ is finitely generated, this means $f^N \mathscr{F} = 0$ for some $N \gg 0$, or $f^N \in \operatorname{Ann}(f)$. Since $\varphi(\operatorname{Ann}(\mathscr{F})) \subset \mathfrak{m}$, then $\varphi(f^N) = 0 \bmod \mathfrak{m}$, so $\varphi(f) = 0 \bmod \mathfrak{m}$, which contradicts the assumption that $\operatorname{Spec} k \in U$. $\boxtimes$

You can draw a picture of this: given a locally finitely generated sheaf $\mathscr{F}$, $\operatorname{Ann}(\mathscr{F})$ has a vanishing locus; if $\mathscr{F}$ corresponds to the module A/I (here we should be on an affine scheme), then this is also the closed subset $\operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A$.

Exercise 8.9. Deduce every other version of Nakayama's lemma that you know (e.g. the one in Matsumara) from these versions.

Definition 8.10. A *vector bundle* on a scheme X is a quasicoherent sheaf $\mathscr{E} \in \mathsf{QCoh}(X)$ which is locally finitely generated and *locally projective*, i.e. for some (equivalently any) affine open cover $\mathfrak{U}$ of X, for every $U = \operatorname{Spec} A \in \mathfrak{U}$, the pullback of $\mathscr{E}$ to U is a projective A-module.

Proposition 8.11. *Let $\mathscr{E} \in \mathsf{QCoh}(X)$. The following are equivalent:*

  (1) *$\mathscr{E}$ is a vector bundle.*
  (2) *There is an affine open cover $\mathfrak{U}$ of X such that for all $U \in \mathfrak{U}$, $\mathscr{E}|_U$ is a finitely generated free module.*

Lecture 9.

Vector bundles: 9/19/18

> *"It's not a good exercise; it's an exercise."*

Today we're going to continue talking about vector bundles, which are an absolutely crucial concept in algebraic geometry. First we'll prove Proposition 8.11, equating two definitions of vector bundles: quasicoherent sheaves which are locally projective and those which are locally free of finite rank, i.e. locally isomorphic to $\mathscr{O}_U^{\oplus r}$. This r is called the *rank* of the vector bundle over U.

Remark 9.1. The rank of a vector bundle is locally constant, but doesn't have to be constant. ◄

Lemma 9.2. *Let $X = \operatorname{Spec} A$ and $\mathfrak{U}$ be a collection of open subsets of X. Then $\mathfrak{U}$ is an open cover of X iff for all maximal ideals $\mathfrak{m}$ of A, there's a $U \in \mathfrak{U}$ such that $\operatorname{Spec}(A/\mathfrak{m}) \hookrightarrow \operatorname{Spec} A$ factors through U.*

The idea is that a *closed point* is an embedding $\operatorname{Spec} k \hookrightarrow X$, where k is a field. So a collection of opens is an open cover if it contains every closed point, which is nice. Affineness is important here: there are general schemes with no closed points!

Proof of Proposition 8.11. The thing we want to prove is local, so we can immediately reduce to the case where $X = \operatorname{Spec} A$ is affine, and therefore $\mathscr{E}$ corresponds to an A-module, which we also denote $\mathscr{E}$. For the forward direction, we assume $\mathscr{E}$ is finitely generated and projective.

Let $\mathfrak{m}$ be a maximal ideal of A. Then $\mathscr{E}/\mathfrak{m}$ is a finitely-generated $A/\mathfrak{m}$-module; since $A/\mathfrak{m}$ is a field, $\mathscr{E}/\mathfrak{m}$ is free, so it has a basis $\overline{s}_1, \ldots, \overline{s}_n$. We lift this to $s_1, \ldots, s_n \in \mathscr{E}$, which define a map $\tau\colon A^{\oplus n} \to \mathscr{E}$ which is surjective mod $\mathfrak{m}$. We'd like to show this is an isomorphism on some open U containing $\operatorname{Spec}(A/\mathfrak{m})$.

Since $\mathscr{E}$ is a finitely generated A-module, so too is $\operatorname{coker}(\tau) = \mathscr{E}/\tau(A^{\oplus n})$, and since modding out by $\mathfrak{m}$ is right exact, $\operatorname{coker}(\tau)|_{\operatorname{Spec}(A/\mathfrak{m})} = 0$. Therefore by Theorem 7.8, there is some $U_0 \subset X$ containing $\operatorname{Spec}(A/\mathfrak{m})$ such that $\operatorname{coker}(\tau)|_{U_0} = 0$; without loss of generality, we can take U_0 to be affine, i.e. τ is surjective on U_0. Let's replace X by U_0 and continue.

Because $\mathscr{E}$ is projective, the map $\tau\colon A^{\oplus n} \twoheadrightarrow \mathscr{E}$ splits; let $\sigma\colon \mathscr{E} \to A^{\oplus n}$ be a section. This means $\ker(\tau)$ is finitely generated, and therefore $\ker(\tau)|_{\operatorname{Spec}(A/\mathfrak{m})} = 0$. Now we use Nakayama's lemma again and conclude that τ is an isomorphism on some open U_1 containing $\operatorname{Spec}(A/\mathfrak{m})$.

The converse isn't immediately trivial: if $\mathfrak{U}$ is an affine cover of $\operatorname{Spec} A$ and M is an A-module such that $M|_U$ is projective for all $U \in \mathfrak{U}$, why is M necessarily projective? Since A need not be Noetherian, we also need to show M is finitely generated and presented given that its localizations are. This is not a super important point, so it's left as an exercise. Once M is finitely presented, you can show that for any $f \in A$,

$$(9.3) \qquad \operatorname{Hom}_A(M, N)[f^{-1}] = \operatorname{Hom}_{A[f^{-1}]}(M[f^{-1}], N[f^{-1}]).$$

This is definitely false if you don't assume finite presentation of M! Anyways, using this, you can recover projectivity on $\operatorname{Spec} A$ from projectivity on a basic affine open cover. $\boxtimes$

Next we'll turn to constructions with quasicoherent sheaves, and something not quite as related, affine morphisms.

Definition 9.4. Let $\mathscr{F}$ and $\mathscr{G}$ be quasicoherent sheaves on a scheme X and $\tau\colon \mathscr{F} \to \mathscr{G}$ be a morphism. Then we can define sheaves $\ker(\tau), \operatorname{coker}(\tau) \in \mathsf{QCoh}(X)$, such that for all affine opens $U \subset X$, $\ker(\tau)|_U = \ker(\tau|_U)$ and $\operatorname{coker}(\tau)|_U = \operatorname{coker}(\tau|_U)$.

For this to make sense, we need to invoke Serre's theorem that this data actually defines a quasicoherent sheaf, along with the fact that $A \to A[f^{-1}]$ is flat, which means kernels and cokernels are preserved under pullback by an open embedding, so that gluing works.

Definition 9.5. With notation as before, there is a quasicoherent sheaf $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ such that on every affine open $U = \operatorname{Spec} A \hookrightarrow X$, $(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})|_U = \mathscr{F}|_U \otimes_A \mathscr{G}|_U$, and on any open $V \hookrightarrow X$, $(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})|_V = \mathscr{F}|_V \otimes_{\mathscr{O}_V} \mathscr{G}|_V$.

Checking that this is well-defined is easier than for the kernel and cokernel; you don't have to invoke Serre's theorem.

Definition 9.6. Let X be a space. A map $f\colon Y \to X$ is *affine* if for all affine schemes S and maps $S \to X$, the pullback $S \times_X Y$ is affine.

Example 9.7.
   (1) Any map of affine schemes is affine, which is a rebranding of the theorem that fiber products preserve affine schemes.
   (2) Closed embeddings are also affine.
   (3) Not all open embeddings are affine: the standard counterexample is $\mathbb{A}^2 \setminus 0 \to \mathbb{A}^2$, because its fiber product with the identity map $\mathbb{A}^2 \to \mathbb{A}^2$ gives us back $\mathbb{A}^2 \setminus 0$, which isn't affine. ◄

Affine morphisms are nice because they have nice algebraic descriptions. Specifically, affine maps $Y \to X$, where X is a scheme, correspond to commutative algebras in $\mathsf{QCoh}(X)$. TODO: I had to head out

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Lecture 10.

Affine morphisms and projective space: 9/21/18

Last time, we defined affine morphisms $Y \to X$, which are those such that if you pull back by an affine scheme $\operatorname{Spec} A \to X$, $Y \times_X \operatorname{Spec} A$ is also affine. We claimed these are equivalent to commutative algebras in $\mathsf{QCoh}(X)$, akin to how affine schemes are commutative rings, but working relatively (i.e. over a scheme).

Definition 10.1. Let X be a scheme. A *commutative algebra* in $\mathsf{QCoh}(X)$ is a quasicoherent sheaf together with an associative, commutative multiplication map $m\colon \mathscr{A} \otimes_{\mathscr{O}_X} \mathscr{A} \to \mathscr{A}$ with a unit $\varepsilon\colon \mathscr{O}_X \to \mathscr{A}$.

Definition 10.2. Given a commutative algebra $\mathscr{A} \in \mathsf{QCoh}(X)$, we can define a scheme $\operatorname{Spec}_X(\mathscr{A})$ together with an affine map to X. If B is a commutative ring, we let $(\operatorname{Spec}_X(\mathscr{A}))(B)$ be the set of pairs $x\colon \operatorname{Spec} B \to X$ together with maps $\rho\colon \operatorname{Spec} B \to \operatorname{Spec} x^*(\mathscr{A})$ which are sections of the canonical map arising from the B-algebra structure on $x^*(\mathscr{A})$.

Forgetting the section defines a map to X. This map is affine because if $x\colon \operatorname{Spec} B \to X$ is a map, then

(10.3) $$\operatorname{Spec}_X(\mathscr{A}) \times_X \operatorname{Spec} B = \operatorname{Spec}(x^*(\mathscr{A})).$$

Definition 10.4. Let $\pi\colon Y \to X$ be an affine map of schemes and $\mathscr{F} \in \mathsf{QCoh}(Y)$. We will define a *pushforward* $\pi_*\mathscr{F} \in \mathsf{QCoh}(X)$ as follows: given any map $x\colon \operatorname{Spec} B \to X$, the pullback $Y \times_X \operatorname{Spec} B$ is affine, isomorphic to $y\colon \operatorname{Spec} C \to Y$ for some C. Then we define $x^*(\pi_*(\mathscr{F})) := y^*(\mathscr{F})$: this is *a priori* a C-module, but picks up a B-module structure by the map $B \to C$.

Exercise 10.5. π^* and π_* are adjoint functors, i.e. for any affine map $\pi\colon X \to Y$, $\mathscr{F} \in \mathsf{QCoh}(Y)$, and $\mathscr{G} \in \mathsf{QCoh}(X)$, there's a canonical isomorphism

$$\operatorname{Hom}_{\mathsf{QCoh}(X)}(\mathscr{G}, \pi_*(\mathscr{F})) \cong \operatorname{Hom}_{\mathsf{QCoh}(Y)}(\pi^*(\mathscr{G}), \mathscr{F}).$$

Proposition 10.6. *If $\pi\colon Y \to X$ is affine, then $Y = \operatorname{Spec}_X(\mathscr{A})$ for some algebra $\mathscr{A}$ in $\mathsf{QCoh}(X)$.*

Proof sketch. The key is that $\pi_*(\mathscr{O}_Y)$ is a commutative algebra in $\mathsf{QCoh}(X)$: by Exercise 10.5, the multiplication map is equivalent data to a map

(10.7) $$\pi^*(\pi_*(\mathscr{O}_Y) \otimes_{\mathscr{O}_X} \pi_*(\mathscr{O}_Y)) = \pi^*\pi_*\mathscr{O}_Y \otimes_{\mathscr{O}_Y} \pi^*\pi_*\mathscr{O}_Y \longrightarrow \mathscr{O}_Y.$$

The unit of the adjunction is a map $\pi^*\pi_*\mathscr{O}_Y \to \mathscr{O}_Y$, so we can pass to $\mathscr{O}_Y$ and then multiply.

We then claim that as schemes over X (i.e. with a map to X), $Y \to X$ is isomorphic to $\operatorname{Spec}_X(\pi_*(\mathscr{O}_Y)) \to X$, which one has to check. ⊠

Exercise 10.8. Let X be a scheme. Show that closed subschemes $Z \subseteq X$ are equivalent to *ideal sheaves* $\mathscr{I} \to \mathscr{O}_X$, i.e. quasicoherent sheaves $\mathscr{I}$ with vanishing kernel.

Projective space. Projective space $\mathbb{P}^n$ is a scheme designed to parametrize lines in $\mathbb{A}^{n+1}$. IF you try this in $\mathbb{A}^2_{\mathbb{R}}$, you notice that you get a circle; if you do this in $\mathbb{A}^2_{\mathbb{C}}$, you get a sphere (harder). But in general it looks different.

We'll describe $\mathbb{P}^n$ via its functor of points. The idea is that a map $S = \operatorname{Spec} A \to \mathbb{P}^n$ should be a line $\mathscr{L}$ together with an embedding $\mathscr{L} \to \mathbb{A}^{n+1}$, but we have to make this precise.

Definition 10.9. Let S be a scheme. A *line bundle* on S is a vector bundle of rank 1, i.e. a quasicoherent sheaf $\mathscr{L}$ on S locally isomorphic to $\mathscr{O}_S$.

When S is affine, these correspond to rank-1 projective A-modules. If $S = \operatorname{Spec} k$, this exactly covers 1-dimensional k-vector spaces, and this is the right generalization to commutative rings (or even to schemes).

Now we want to embed the line in $\mathbb{A}^{n+1}$, which we can think of as a map $i\colon \mathscr{L} \to \mathscr{O}_S^{\oplus(n+1)}$; "embedding" means we want $i \neq 0$.

Proposition 10.10. *The following are equivalent for a vector bundle $\mathscr{E} \to S$ and a map $i\colon \mathscr{L} \to \mathscr{E}$ where $\mathscr{L}$ is a line bundle.*

(1) *For all affine schemes T and maps $f\colon T \to S$, $f^*(\mathscr{L}) \to f^*(\mathscr{E})$ is nonzero.*
(2) *(Will be done Monday)*
(3) *(Will be done Monday)*

Definition 10.11. If the conditions in Proposition 10.10 hold, i is called *everywhere nonvanishing*.

Definition 10.12. *Projective n-space* $\mathbb{P}^n$ is the space whose functor of points sends an affine scheme S to the set of isomorphism classes of data $(\mathscr{L}, i)$ where $\mathscr{L}$ is a line bundle on S and $i\colon \mathscr{L} \to \mathscr{O}_S^{\oplus(n+1)}$ is everywhere nonvanishing.

There's something to be said about morphisms, but given a map $f\colon T \to S$, we can pull back $\mathscr{L}$ and i, and obtain a line bundle with an everywhere nonvanishing embedding.

We need to take isomorphism classes to ensure we get a set, not a category. This also ensures that we've modded out by rescaling (since that's an isomorphism $\mathscr{L} \to \mathscr{L}$). We'll show this is a scheme, but is usually not affine.

Lecture 11.

Projective n-space and projectivizations: 9/24/18

"These scare quotes should be less scary than those scare quotes."

Last time, we defined projective n-space, $\mathbb{P}^n$, whose functor of points sends A to the set of isomorphism classes of data $(\mathscr{L}, i)$, where $\mathscr{L} \to \operatorname{Spec} A$ is a line bundle and $i\colon \mathscr{L} \to \mathscr{O}_{\operatorname{Spec} A}^{\oplus(n+1)}$ is an everywhere nonvanishing map.

Definition 11.1. More generally, if X is a scheme and $\mathscr{E}$ is a vector bundle on X, then we can define a space $\mathbb{P}(\mathscr{E})$, the *projectivization* of $\mathscr{E}$: if A is a commutative ring, $\mathbb{P}(\mathscr{E})(A)$ is the set of isomorphism classes of data $(\mathscr{L}, x, i)$, where $\mathscr{L}$ is a line bundle on $\operatorname{Spec} A$, $x\colon \operatorname{Spec} A \to X$ is a map ("an A-valued point"), and $i\colon \mathscr{L} \to x^*(\mathscr{E})$ is everywhere nonvanishing.

To recover $\mathbb{P}^n$, let $X = \operatorname{Spec}\mathbb{Z}$ and $\mathscr{E} = \mathbb{Z}^{\oplus(n+1)}$.

Remark 11.2. Right now, schemes and line and vector bundles probably feel very abstract. That's OK; soon enough we will see many, many examples of line bundles over curves, and make them very concrete.   ◄

Example 11.3. Consider the quasicoherent sheaf given by $k[t]/(t)$ over $\mathbb{A}^1_k := \operatorname{Spec} k[t]$. This is not a vector bundle: it's not locally free, because in a sense it's nonzero over the point 0 (a one-dimensional vector space) but vanishes everywhere else.   ◄

Proposition 11.4. *Let S be a scheme, $\mathscr{L}$ be a line bundle on S, $\mathscr{E}$ be a vector bundle on S, and $i\colon \mathscr{L} \to \mathscr{E}$. The following are equivalent:*

(1) *The induced map $\Theta(i)\colon \Theta(\mathscr{L}) \to \Theta(\mathscr{E})$ is a closed embedding.*
(2) *For all affine schemes T and maps $\alpha\colon T \to S$, $\alpha^*(i)$ is nonzero.*
(3) *For all fields k and maps $\alpha\colon \operatorname{Spec} k \to S$, $\alpha^*(i)$ is nonzero.*
(4) *For all affine open subschemes $U \subseteq S$ such that $\mathscr{L}|_U \cong \mathscr{O}_U$ and $\mathscr{E}|_U \cong \mathscr{O}_U^{\oplus r}$, if the induced map $\mathscr{O}_U \to \mathscr{O}_U^{\oplus r}$ sends $1 \mapsto (f_1, \ldots, f_r)$, then $(f_1, \ldots, f_r)$ generates the unit ideal in $\operatorname{Fun}(U)$.*
(5) *The dual map $i^\vee\colon \mathscr{E}^\vee \to \mathscr{L}^\vee$ is an epimorphism (i.e. its cokernel is zero).*[9]
(6) *$\operatorname{coker}(\mathscr{L} \to \mathscr{E})$ is a vector bundle.*

To make sense of this, we have to define Θ, a way of turning vector bundles into schemes. If $P^\vee$ is unfamiliar for an A-module P, it simply means $\operatorname{Hom}_A(P, A)$.

Exercise 11.5. Let P be a finitely generated, projective A-module.

(1) Show that for all A-modules M, $\operatorname{Hom}_A(P, M) \cong P^\vee \otimes_A M$.
(2) Show that $P^\vee$ is projective.
(3) Describe a natural isomorphism $P \to P^{\vee\vee}$.

Definition 11.6. If $\mathscr{E}$ is a vector bundle over a scheme S, its *dual vector bundle* $\mathscr{E}^\vee$ (sometimes also written $\mathscr{E}^*$) is the quasicoherent sheaf attaching to every affine open $i\colon U \to S$ the dual projective module of $i^*\mathscr{E}$.

By Exercise 11.5, part (1), $\mathscr{E}^\vee$ is indeed a quasicoherent sheaf.

Example 11.7. If $\mathscr{E} = \mathscr{O}_X^{\oplus r}$, then $\mathscr{E}^\vee \cong \mathscr{O}_X^{\oplus r}$ as well, albeit not canonically.   ◄

Duality is contravariantly functorial: a map $f\colon \mathscr{F} \to \mathscr{E}$ of vector bundles induces a dual map $f^\vee\colon \mathscr{E}^\vee \to \mathscr{F}^\vee$ (do this on affines, where it's precomposition for the corresponding modules).

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[9]In many situations, "epimorphism" means "surjective," but these are quasicoherent sheaves, so we don't have elements to ask about preimages of.

Definition 11.8. Let $\mathscr{E}$ be a vector bundle on X. Its *total space* is the scheme

(11.9) $$\Theta(\mathscr{E}) \coloneqq \operatorname{Spec}_X(\operatorname{Sym}_{\mathscr{O}_X}(\mathscr{E}^\vee)).$$

Here, $\operatorname{Sym}_{\mathscr{O}_X}(\mathscr{E})$ is the $\mathscr{O}_X$-module

(11.10) $$\operatorname{Sym}_{\mathscr{O}_X}(\mathscr{E}) \coloneqq \bigoplus_{n \geq 0} \operatorname{Sym}^n_{\mathscr{O}_X}(\mathscr{E}),$$

where $\operatorname{Sym}^n_{\mathscr{O}_X}(\mathscr{E}) \coloneqq (\mathscr{E}^{\otimes_{\mathscr{O}_X} n})_{S_n}$, where the symmetric group S_n acts by permuting the elements; more explicitly, we allow the elements in an n-tensor to be arbitrarily shuffled, which you can do with a quotient.

Example 11.11. Let $X = \operatorname{Spec} k$ and $\mathscr{E} = k^{\oplus n}$ be a k-vector space with basis $e_1, \ldots, e_n$. Let $e^1, \ldots, e^n$ denote the dual basis. Then $\operatorname{Sym}^n V$ is the vector space of degree-n homogeneous polynomials in $e^1, \ldots, e^r$, and $\operatorname{Sym} V = k[e^1, \ldots, e^r]$. ◄

Lecture 12.

More vector bundles: 9/26/18

Today we're in the business of proving Proposition 11.4.

Lemma 12.1. *Let X be a scheme and $\mathscr{A}, \mathscr{B} \in \mathsf{QCoh}(X)$ be commutative algebras together with a map $f\colon \mathscr{A} \to \mathscr{B}$ of algebras. Then $\operatorname{Spec}_X(\mathscr{B}) \to \operatorname{Spec}_X(\mathscr{A})$ is a closed embedding iff f is an epimorphism.*

This is a relative version of the definition of closed embeddings of affine schemes we gave awhile ago.

We also need to define the what taking the total space Θ does to morphisms. Ultimately this is because $(-)^\vee$, $\operatorname{Sym}_{\mathscr{O}_X}$, and Spec_X are all functors, so we know what they do on morphisms; two are contravariant and one is covariant, so we get a covariant functor.

In part (1) of Proposition 11.4, Lemma 12.1 tells us this is equivalent to $\operatorname{Sym}_{\mathscr{O}_X} \mathscr{E}^\vee \to \operatorname{Sym}_{\mathscr{O}_X} \mathscr{L}^\vee$ is an epimorphism. This is a $\mathbb{Z}$-graded sheaf, and it suffices to show this for $\operatorname{Sym}^d_{\mathscr{O}_X}$ for each d.

Exercise 12.2. Show that if $\mathscr{E}^\vee \to \mathscr{L}^\vee$, then $\operatorname{Sym}^d \mathscr{E}^\vee \to \operatorname{Sym}^d \mathscr{L}^\vee$ is too.

Proof of Proposition 11.4, (5) $\iff$ (4). We know condition (5) is equivalent to $\operatorname{coker}(i^\vee) = 0$. This can be checked on an open cover $\mathfrak{U}$, such as an affine open cover which trivializes $\mathscr{E}$ and $\mathscr{L}$, as in (4). In this case, for each U in $\mathfrak{U}$, $i|_U\colon \mathscr{O}_U \to \mathscr{O}_U^{\oplus r}$ is determined by a row vector $(f_1, \ldots, f_r)^{\mathrm{T}}$, and $i^\vee|_U\colon \mathscr{O}_U^{\oplus r} \to \mathscr{O}_U$ is the column vector $(f_1, \ldots, f_r)$. Its image is the ideal generated by $(f_1, \ldots, f_r)$, so the cokernel is 0 iff $(f_1, \ldots, f_r) = \operatorname{Fun}(U)$. ⊠

Proof of Proposition 11.4, (4) $\implies$ (6). We can again assume without loss of generality that $U = \operatorname{Spec} B$ is affine, and now we have maps

(12.3) $$B \xrightarrow{\begin{bmatrix} f_1 \\ \vdots \\ f_r \end{bmatrix}} B^{\oplus r} \xrightarrow{[g_1, \ldots, g_r]} B,$$

where $\sum f_i g_i = 1$. Since the composition is the identity, the map is split, and so the cokernel is a direct summand, in particular is a direct summand of a free module, and must be free. ⊠

Exercise 12.4. Conversely, show that in Proposition 11.4, (6) $\implies$ (4).

Since (2) tautologically implies (3), it suffices to show (3) $\implies$ (5) and (5) $\implies$ (2); (1) is addressed by one of the exercises above, I think (TODO: I probably just missed it.)

Since $\operatorname{coker}(i)$ is locally finitely generated, Nakayama's lemma applies: if V denotes the complement of the support of $\operatorname{coker}(i)$, then V is open, and $S = \operatorname{Spec} A \to X$ factors through V iff $\operatorname{coker}(i)|_S = 0$. So (5) is equivalent to $V = X$.

Proof of Proposition 11.4, (3) $\implies$ (5). To show $V = X$, it suffices to show any map $x\colon \operatorname{Spec} k \to X$ factors through V, since V is open. One can show that x^* commutes with cokernels: since $\operatorname{Spec} k$ is affine, it arises as a tensor product, and tensor products are right exact.[10] Therefore it suffices to show that $x^*(\operatorname{coker}(i^\vee)) = 0$.

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[10]More generally, x^* is a left adjoint, even not on affines; this automatically means it's right exact.

If $x^*(i^\vee)$ is nonzero, then $x^*(\mathscr{E}) \to x^*(\mathscr{L})$ is a nonzero map to a 1-dimensional vector space, hence surjective, and therefore the cokernel is zero. $\boxtimes$

Proof of Proposition 11.4, (5) $\implies$ (2). Let $x\colon T \to X$ be a map, where T if affine. As before, x^* is right exact, hence commutes with cokernels. By assumption, $\operatorname{coker}(i^\vee) = 0$, so $x^*(i^\vee)\colon x^*(\mathscr{E}^\vee) \to x^*(\mathscr{L}^\vee)$ is an epimorphism. This is the dual map to $x^*(i)$, so if $T \neq \varnothing$, this means $x^*(i)\colon x^*(\mathscr{L}) \to x^*(\mathscr{E})$ is nonzero. $\boxtimes$

These equivalent conditions are all examples of the nicest possible maps of vector bundles. It's good to have all of these different perspectives partly because they provide flexibility in the nice case, but also because it will be useful to know what happens when things go bad. When we study projective varieties, several of these conditions will come up. For example, it will be useful for showing $\mathbb{P}^n$ is a scheme!

Lecture 13.

$$: 9/28/18$$

Theorem 13.1. $\mathbb{P}^n$ *is a scheme.*

Proof. First we need to check that $\mathbb{P}^n$ is a Zariski sheaf. The basic idea is that line bundles glue: if you have line bundles on each open of an open cover, together with isomorphisms on intersections satisfying a cocycle condition, you can glue them.

Next we need to cover it by affines. For $i = 0, \ldots, n$, the locus $\{s_i \neq 0\} \subset \mathbb{P}^n$ is isomorphic to $\mathbb{A}^n$. We could say QED here, but let's explain what's going on.

Let $U_i \coloneqq \{s_i = 0\}$ be defined by saying what it means for a map $S \to \mathbb{P}^n$ factors through U_i. Specifically, the map $S \to \mathbb{P}^n$ is equivalent to (an isomorphism class of) data of a line bundle $\mathscr{L}$ on S and a map $(s_0, \ldots, s_{n+1})^{\mathrm{T}} \colon \mathscr{L} \to \mathscr{O}_S^{\oplus(n+1)}$ which is everywhere nonvanishing. Thus the map is described by $s_0^\vee, s_n^\vee$; we say the map factors through U_i if $s_i^\vee$ is nonvanishing.[11]

First let's check that for all i, $U_i \subset \mathbb{P}^n$ is open. Intuitively this makes sense: we're asking for something to not vanish, which is an open condition. More precisely, we want to show that for every affine scheme S with a map $S \to \mathbb{P}^n$, the pullback $U_i \times_{\mathbb{P}^n} S \to S$ is open. So let $S \to \mathbb{P}^n$ be such a map (in particular, S is affine). This is equivalent data to a line bundle $\mathscr{L}$ on S and $s_0, \ldots, s_{n+1} \colon \mathscr{L} \to \mathscr{O}_S$. That is, there are sections of the map $\Theta(\mathscr{L}^\vee) \to S$. Then you can check that $U_i = S \times_{\Theta(\mathscr{L}^\vee)} \Theta(\mathscr{L}^\vee) \setminus S$, where the first map $S \to \Theta(\mathscr{L}^\vee)$ is by s_i and the second map is by the zero section.

The other claim we want to check is that $U_i \cong \mathbb{A}^n$. The proof is that a map $S \to U_i$ is equivalent data to the maps $s_0, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n \colon \mathscr{L} \to \mathscr{O}_S$, since we know $s_i \colon \mathscr{L} \to \mathscr{O}_S$ is an isomorphism. Each s_j, $j \neq i$, is equivalent to a function on S: since s_i is nonvanishing, there are no further conditions on the remaining maps. Thus a map to U_i is equivalent to n functions, and this is natural in S, so $U_i \simeq \mathbb{A}^n$.

Finally, we need to show that $U_0, \ldots, U_n$ is a cover of $\mathbb{P}^n$, i.e. that it pulls back to a Zariski open cover on all affines. So once again let $S \to \mathbb{P}^n$ be a map from an affine, so we have a line bundle $\mathscr{L}$ and $(n+1)$ maps $s_0, \ldots, s_n \colon \mathscr{L} \to \mathscr{O}_S$ which collectively are nonvanishing. Without loss of generality, we can assume $\mathscr{L}$ is trivial, since it's locally trivial, and we can check that it's an open cover locally. So it suffices to show that $S \times_{\mathbb{P}^n} U_i$ is a Zariski open cover of S, which is more or less equivalent to $s_0, \ldots, s_n$ being everywhere nonvanishing, in view of what we did last lecture. $\boxtimes$

Typically, people only use functions, rather than sections of a line bundle, to provide a first naïve definition of $\mathbb{P}^n$. In this case, the Zariski sheaf property fails: you can glue trivial line bundles to obtain something nontrivial.

Corollary 13.2. *Let $\mathscr{E}$ be a vector bundle on a scheme X. Then $P(\mathscr{E})$ is a scheme.*

Proof. X has an affine open cover $\mathfrak{U}$ such that $\mathscr{E}|_U$ is trivial for all $U \in \mathfrak{U}$. Then $\mathbb{P}(\mathscr{E}) \times_X U \cong \mathbb{P}^n \times U$ as schemes. $\boxtimes$

$\mathbb{P}^n$ has an important line bundle called $\mathscr{O}_{\mathbb{P}^n}(1)$ (sometimes just $\mathscr{O}(1)$ if $\mathbb{P}^n$ is implicit).

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[11]That is, the collection $(s_0, \ldots, s_n)^{\mathrm{T}}$ is always nonvanishing, but we're asking specifically about s_i, which is stronger.

Definition 13.3. The line bundle $\mathscr{O}_{\mathbb{P}^n}(1) \in \mathsf{QCoh}(\mathbb{P}^n)$ is the line bundle which, given a map $f\colon S \to \mathbb{P}^n$ which is a line bundle $\mathscr{L}$ on S and the maps $s_0, \ldots, s_n$, defines $f^*(\mathscr{O}_{\mathbb{P}^n}(1)) := \mathscr{L}^\vee$.

For any $n \in \mathbb{Z}$, we define $\mathscr{O}_{\mathbb{P}^n}(m) := \mathscr{O}_{\mathbb{P}^n}(1)^{\otimes m}$. If $m = 0$, we interpret the empty tensor product as $\mathscr{O}_{\mathbb{P}^n}$, the sheaf of functions; if $m < 0$, we interpret this as $((\mathscr{O}_{\mathbb{P}^n}(1))^\vee)^{\otimes(-m)}$.

Definition 13.4. Let X be a scheme and $\mathscr{F} \in \mathsf{QCoh}(X)$. The *sections* of X, denoted $\Gamma(X, \mathscr{F})$, is the abelian group $\mathrm{Hom}_{\mathsf{QCoh}(X)}(\mathscr{O}_X, \mathscr{F})$.

That is, a section is compatible data, for all affines S and maps $x\colon S \to X$, of $\sigma_x \in x^*\mathscr{F}$ regarded as a module.

The sections $\Gamma(X, \mathscr{O}_X) = \mathrm{Fun}(X)$, and if $X = \mathrm{Spec}\, A$, $\Gamma(X, \mathscr{F})$ is canonically the A-module associated to $\mathscr{F}$ (which we've just been calling $\mathscr{F}$ again).

Definition 13.5. If A is a commutative ring, we let $\mathbb{P}^n_A := \mathbb{P}^n \times \mathrm{Spec}\, A$.

Proposition 13.6. *For $r \geq 0$, $\Gamma(\mathbb{P}^n_A, \mathscr{O}(r))$ is the A-module of homogeneous degree-r polynomials with A coefficients in $n+1$ variables.*

For $r = 1$, a map $A^{\oplus(n+1)} \to \Gamma(\mathbb{P}^n_A, \mathscr{O}(1))$ is definedc by $n+1$ maps $\mathscr{O}_{\mathbb{P}^n_A} \to \mathscr{O}_{\mathbb{P}^n_A}(1)$, i.e. for all $f\colon S \to \mathbb{P}^n_A$, i.e. line bundles $\mathscr{L}$ with $s_0, \ldots, s_n$, compatible maps $\mathscr{O}_S \to f^*(\mathscr{O}(1))$ which identify $f^*(\mathscr{O}(1)) \cong \mathscr{L}^\vee$, and the ith section is identified with $s_i \in \mathscr{L}^\vee$. Following your nose with the formal stuff provides a complete proof.

Over the next few days we'll discuss finite conditions: the nicest possible schemes are smooth projective curves, and we'll discuss these soon.