SPRING 2017 HOMOTOPY THEORY SEMINAR

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Contents

1. Higher K-theory: 1/25/17

Today, Nicky spoke on a few approaches to higher K-theory.

Let C be a pointed ∞-category with finite colimits (as in Lurie's approach) or a category with cofibrations and weak equivalences satisfying certain axioms (as in Waldhausen's approach).

Recall that $K_0(\mathsf{C})$ was defined to be the free abelian group on isomorphism classes of objects of C modulo $[X] = [X'] + [X'']$ whenever we have a pullback

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\
* \arrow[r] & X''.
\end{tikzcd}$$

We want to generalize to a based space W such that $\pi_1(W) = K_0(\mathsf{C})$, and satisfying a universal property for C: every object X in C should determine a path p_X from $*$ to $*$ in W, and for any cofiber sequence $X' \to X \to X''$, we'd like the 2-cell bounded by the paths p_X, $p_{X'}$, and $p_{X''}$ to be contractible in W.

Remark. Given a map $f\colon X \to Y$, we'll let Y/X denote the cofiber of f. Waldhausen is working with maps that are already cofibrant (since he works with categories that already have special classes of maps), but the suitable cofibrant replacement also exists for ∞-categories. This notation implies that in K_0, $[Y] = [X] + [Y/X]$. ◂

Proposition 1.1. *Let $X \to Y \to Z$ be a cofiber sequence. Then, $[Z] = [X] + [Y/X] + [Z/Y]$.*

Proof. One way to prove this is to observe that $X \to Y \to Z$ means that the following two sequences are cofiber sequences:

$$X \longrightarrow Z \longrightarrow Z/X$$
$$Y/X \longrightarrow Z/X \longrightarrow Z/Y.$$

Alternatively, you could observe that that the following two sequences are cofiber sequences:

$$Y \longrightarrow Z \longrightarrow Z/Y$$
$$X \longrightarrow Y \longrightarrow Y/X. \qquad \square$$

These two proofs of this identity are two homotopies between the paths p_Z and $p_X \circ p_{Y/X} \circ p_{Z/Y}$:

$$\begin{tikzcd}
& * \arrow[r, "p_{Y/X}"] \arrow[drr, Rightarrow, "p_Y"] & * \arrow[dr, "p_X"] & \\
* \arrow[ur, "p_{Z/Y}"] \arrow[urr, Rightarrow, "p_{Z/X}"'] \arrow[rrr, "p_Z"'] & & & *.
\end{tikzcd} \tag{1.2}$$

We'd like for these two homotopies to be homotopic: the two proofs of Proposition 1.1 define a map $\partial\Delta^3$ into the diagram (1.2), and we want this to extend to a map from Δ^3. In a similar way, we'd like to have a similar "coherence" statement corresponding to sequences $X_1 \to X_2 \to \cdots \to X_n$.

Waldhausen's $S_\bullet$-construction does this all formally for us. It works by gluing classifying spaces of these sequences together, which feels like a homotopy coherent nerve but isn't quite one. One way to think about is that there are choices made when making a quotient; the $S_\bullet$ construction keeps them around as simplicial data. More explicitly, given the sequence $X_0 \to X_1 \to \cdots \to X_n$, you want the 0^{th} face map to arise from a sequence $\cdots \to X_i/X_1 \to X_{i+1}/X_1 \to \cdots$, but there are choices made in picking these maps.

The formalism of the $S_\bullet$ construction will involve some homotopy theory of posets, but is nicer than last semester's stuff. Let P be a poset, and set

$$P^{(2)} := \{(i,j) \in P \times P \mid i \le j\},$$

which is also $\mathsf{Fun}(\Delta^1, P)$.

Definition 1.3. A *P***-gapped object** in C is a functor $X\colon N(P^{(2)}) \to \mathsf{C}$ such that for all $i \in P$, $X(i,i) = *$ and for all $i \le j \le k$ in P, we have a pushout square

$$\begin{tikzcd} X(i,j) \arrow[r] \arrow[d] & X(i,k) \arrow[d] \\ * = X(j,j) \arrow[r] & X(j,k). \end{tikzcd}$$

This is a functorial notion: if $P \to Q$ is a map of posets, we get a map $N(P) \to N(Q)$, and f takes Q-gapped objects to P-gapped ones. That is, "$(-)$-gapped objects" is a functor from the simplicial indexing functor to ∞-categories. We're going to bundle this up into a simplicial set.

As usual, let $[n]$ denote the poset $\{0 < 1 < 2 < \cdots < n\}$. Let $\mathrm{Gap}_{[n]}(\mathsf{C})$ denote the ∞-category of $[n]$-gapped objects in C. Concretely, this is a diagram category for the diagram

$$\begin{tikzcd} * = X(0,0) \arrow[r] & X(0,1) \arrow[r] \arrow[d] & X(0,2) \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & X(0,n) \arrow[d] \\ & * = X(1,1) \arrow[r] & X(1,2) \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & X(1,n) \arrow[d] \\ & & * = X(2,2) \arrow[r] & \cdots \arrow[r] \arrow[d] & X(2,n) \arrow[d] \\ & & & \vdots & \vdots \arrow[d] \\ & & & & * = X(n,n). \end{tikzcd}$$

There's a nice animation of this available at `https://www.ma.utexas.edu/users/ysulyma/`.

Let $S_n(\mathsf{C})$ denote the underlying Kan complex of $\mathrm{Gap}_{[n]}(\mathsf{C})$: it's not necessarily a groupoid, but we can throw away all non-invertible arrows.[1] Thus, $S_\bullet(\mathsf{C})$ is a simplicial Kan complex,[2] and we're going to geometrically realize it. In low degrees, this recovers things we've seen before:

- $S_0(\mathsf{C}) = \mathrm{Gap}_{[0]}(\mathsf{C})$ is the full subcategory of 0-objects, which is contractible.
- $\mathrm{Gap}_{[1]}(\mathsf{C}) \cong \mathsf{C}$ (diagrams of the form $* \to X \to *$), and $S_1(\mathsf{C})$ is equivalent to the category of isomorphisms in C.

[1] Alternatively, if you're working with categories of weak equivalences, rather than ∞-categories, you're throwing out everything but the weak equivalences.

[2] By a **simplicial Kan complex**, we mean a bisimplicial set such that each $S_n(\mathsf{C})$ is a Kan complex.

- $\mathrm{Gap}_{[2]}(\mathsf{C})$ is equivalent to the ∞-category of cofiber sequences in C.

Now, we can define K-theory.

Definition 1.4. The **algebraic K-theory** of C is

$$K(\mathsf{C}) := \Omega|S_\bullet(\mathsf{C})|.$$

Because $S_\bullet(\mathsf{C})$ is a simplicial Kan complex, we must specify the geometric realization; you can either geometrically realize the diagonal or geometrically realize one axis with topology on the sets of simplices.

The K-**groups** of C are $K_n(\mathsf{C}) := \pi_n K(\mathsf{C}) = \pi_{n+1}|S_\bullet(\mathsf{C})|$.

These agree with the K-groups we defined in low dimensions, but this is a theorem.

Remark.

- Eventually, we will see how to promote this from a space to a spectrum.
- If $S_\bullet(\mathsf{C})$ is contractible, then $|S_\bullet(\mathsf{C})|$ is conncted.
- Let $F\colon \mathsf{C} \to \mathsf{D}$ be a functor between suitably nice[3] ∞-categories; then, we obtain a map $K(\mathsf{C}) \to K(\mathsf{D})$.
- The projections $\mathsf{C} \leftarrow \mathsf{C} \times \mathsf{D} \to \mathsf{D}$ are nice, so

$$K(\mathsf{C} \times \mathsf{D}) \cong K(\mathsf{C}) \times K(\mathsf{D}).$$

- The coproduct functor $\amalg\colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ is sufficiently nice, so there's a multiplication map $m\colon K(\mathsf{C}) \times K(\mathsf{C}) \to K(\mathsf{C})$, which is coherently associative and commutative. In fact, $K(\mathsf{C})$ has an E_∞-structure. ◂

We'd like to compare the new K_0 and the old K_0. $|S_\bullet(\mathsf{C})|$ is a direct limit across the **skeleton functors** sk_i sending X to its i-skeleton:

$$|S_\bullet(\mathsf{C})| = \operatorname{colim}\big(\mathrm{sk}_0|S_\bullet(\mathsf{C})| \longrightarrow \mathrm{sk}_1|S_\bullet(\mathsf{C})| \longrightarrow \mathrm{sk}_2|S_\bullet(\mathsf{C})| \longrightarrow \cdots\big).$$

- We know the 0-skeleton: $\mathrm{sk}_0|S_\bullet(\mathsf{C})| = S_0(\mathsf{C})$ is contractible.
- For the 1-skeleton, $\mathrm{sk}_1|S_\bullet(\mathsf{C})| = \Sigma S_1\mathsf{C} = \Sigma \operatorname{iso}\mathsf{C}$. Thus, we have a map $\Sigma \operatorname{iso}\mathsf{C} \to |S_\bullet(\mathsf{C})|$ whose adjoint begins a sequence

$$\operatorname{iso}\mathsf{C} \longrightarrow \Omega|S_\bullet(\mathsf{C})| \longrightarrow \Omega^2|S_\bullet S_\bullet(\mathsf{C})| \longrightarrow \cdots$$

  These are the maps that will define the K-theory spectrum.

Thus, we know $K_0(\mathsf{C}) = \pi_0(K(\mathsf{C})) = \pi_1(S_\bullet(\mathsf{C}))$, which is generated by isomorphims classes of objects in C. The relations are generated by things in $\mathrm{sk}_2|S_\bullet(\mathsf{C})|$: we've glued in 2-cells in $S_2(\mathsf{C})$ to introduce relations. That is, the relations are defined by $\pi_0(S_2(\mathsf{C}))$, which is the set of cofiber sequences. Thus, $K_0(\mathsf{C})$ is the abelian group generated by objects and modulo cofiber sequences, as desired.

Algebraic K-theory as a spectrum. Since $\mathrm{sk}_1|S_\bullet(\mathsf{C})|$ is obtained from $\mathrm{sk}_0|S_\bullet(\mathsf{C})|$ by attaching $S_1\mathsf{C} \times \Delta^1$ and $\mathrm{sk}_0|S_\bullet(\mathsf{C})|$ is contractible, then $\mathrm{sk}_1|S_\bullet(\mathsf{C})| \simeq \Sigma S_1(\mathsf{C}) \simeq \Sigma \operatorname{iso}\mathsf{C}$.

The 1-skeleton includes into the whole space, so by adjunction, we have an inclusion $\operatorname{iso}\mathsf{C} \hookrightarrow \Omega|S_\bullet(\mathsf{C})|$. Thus we can begin to define a spectrum, in fact an Ω-spectrum.

Definition 1.5. The **algebraic K-theory spectrum** $\widetilde{K}(\mathsf{C})$ is the spectrum assigning

$$n \longmapsto |\operatorname{iso}\underbrace{S_\bullet S_\bullet \cdots S_\bullet}_{n}\mathsf{C}|,$$

with the maps induced from the adjunction above.

Remark. There's a technicality here with basepoints. Waldhausen solved this by requiring exact functors to be based, but typically for ∞-categories, one requires a functor to send a zero object to a zero object. This is an issue for setting up functoriality, which is worth being aware of. One way to solve this is to quotient out by these choices. In practice, however, exact functors tend to strictly preserve the basepoint. ◂

[3] Probably pointed and with finite colimits.

Some things to notice here: the n^{th} term is $(n-1)$-connected (since the 0-skeleton of the $S_\bullet$-construction is contractible). This is an ingredient in the additivity theorem, an important result that will be presented next time. This will imply that the K-theory spectrum is an Ω-spectrum, allowing a more concise definition of K-theory space:

$$K(\mathsf{C}) = \varinjlim_n \Omega^n |S_\bullet^{(n)}(\mathsf{C})| = \Omega^\infty |S_\bullet^{(\infty)}(\mathsf{C})|.$$

This is helpful because it shows that $K(\mathsf{C})$ is an infinite loop space, and this is how we get the E_∞ structure. The point is, the Ω-spectrum structure gives you the infinite loop space structure on the nose; you don't have to take a colimit.

Remark. The $S_\bullet$-construction looks a little bit like a suspension, and there's a way in which this can be made precise. Another way of looking at this is that if you don't shift up and deloop, you have an Ω-spectrum after the 0^{th} level. This relates to the fact that the $S_\bullet$-construction is not a Kan complex, but after one subdivision, it becomes one. The class of simplicial sets with this property is formally interesting. ◂

2. The additivity theorem: 2/1/17

Today, Ernie talked about the additivity theorem. Reference: McCarthy, "Fundamental theorems in algebraic K-theory," which gives the coolest proof, presented today. It's only about four pages long. The running question is: what hypotheses do we even need for this proof? The answer is "not very much," and it can be generalized further than we go today.

Recall that $S_2\mathsf{C}$ was the cofiber sequences in C.

Theorem 2.1 (Additivity, 1-categorical case)**.** *Let C be a Waldhausen category. The exact functor $S_2\mathsf{C} \to \mathsf{C}\times\mathsf{C}$ sending $(a \rightarrowtail c \twoheadrightarrow b) \mapsto (a,b)$ induces a homotopy equivalence $S_\bullet S_2\mathsf{C} \to S_\bullet(\mathsf{C}\times\mathsf{C}) \cong S_\bullet\mathsf{C}\times S_\bullet\mathsf{C}$.*

Lurie's notes state this slightly differently:

Theorem 2.2 (Additivity, ∞-categorical case)**.** *Let C be a pointed ∞-category with finite colimits. Then, the exact functor $\mathsf{Fun}(\Delta^1,\mathsf{C}) \to \mathsf{C}\times\mathsf{C}$ sending $(\alpha\colon a\to c)\mapsto (a, \operatorname{cofib}\alpha)$ induces a homotopy equivalence $S_\bullet S_2\mathsf{C} \to S_\bullet(\mathsf{C}\times\mathsf{C}) \cong S_\bullet\mathsf{C}\times S_\bullet\mathsf{C}$.*

In particular, this induces an equivalence of K-theory spectra.

Corollary 2.3. *The functor $(a,b)\mapsto (a \rightarrowtail a\vee b \twoheadrightarrow b)$ is a homotopy inverse to the functor in Theorem 2.1, and therefore induces an isomorphism on K-theory.*

That is, K-theory splits short exact sequences. The additivity theorem is the main structural theorem about algebraic K-theory, and says that, in a universal-property sense, K-theory splits exact sequences. You may find this extremely emotionally satisfying.

Corollary 2.4. *Let C and D be Waldhausen categories (or pointed ∞-categories with finite colimits) and $F' \rightarrowtail F \twoheadrightarrow F''$ be a* ***cofiber sequence*** *of functors $\mathsf{C}\to\mathsf{D}$, i.e. for any cofibration $A\rightarrowtail B$, the map $FA \amalg_{F'A} F'B \to FB$ is a cofibration.*[4] *Then, they induce cofiber sequences pointwise. Then, on K-theory, $K(F)\simeq K(F')\vee K(F'')$.*

Example 2.5. Consider the cone and suspension functors $\mathrm{Cone}, \Sigma\colon \mathsf{C}\to\mathsf{C}$.

There is a pullback square of functors

$$\begin{tikzcd} \mathrm{id} \arrow[r, tail] \arrow[d] & \mathrm{Cone} \arrow[d, two heads] \\ * \arrow[r] & \Sigma. \end{tikzcd}$$

Thus, on K-theory, the cone functor is homotopic to $\mathrm{id}\vee\Sigma$, but the cone is null-homotopic, so Σ acts by "-1" on K-theory.[5] ◂

[4] This is an example of a **latching condition**; it can often be suppressed in the ∞-categorical world, though perhaps at the expense of more work somewhere else. Conditions like this show up when establishing model-categorical structures.

[5] In fact, Σ acts by precisely -1 on $K_0(\mathsf{C})$.

Not all Waldhausen categories have a cone functor, but this tells us that, when we do have it, K-theory is a stable invariant.

The proof will involve simplicial homotopy theory. Let $F\colon \mathsf{C} \to \mathsf{D}$ be an exact functor of Waldhausen categories. Following McCarthy's notation, let $S_\bullet F|\mathsf{D}$ denote the bisimplicial set whose (m,n)-simplices are the pairs of cofiber sequences

$$\begin{aligned} &0 = c_0 \rightarrowtail c_1 \rightarrowtail \cdots \rightarrowtail c_m \\ &0 = d_0 \rightarrowtail dc_1 \rightarrowtail \cdots \rightarrowtail d_m \rightarrowtail e_1 \rightarrowtail \cdots \rightarrowtail e_n \end{aligned}$$

such that for each i, $F(c_i) = d_i$ and for $0 \le i < m$, the diagram

$$\begin{tikzcd} F(c_i) \arrow[r, tail] \arrow[d, equal] & F(c_{i+1}) \arrow[d, equal] \\ d_i \arrow[r, tail] & d_{i+1}. \end{tikzcd}$$

Such sequences are written with bars:

$$\frac{0 = c_0 \rightarrowtail c_1 \rightarrowtail \cdots \rightarrowtail c_m}{0 = d_0 \rightarrowtail dc_1 \rightarrowtail \cdots \rightarrowtail d_m \rightarrowtail e_1 \rightarrowtail \cdots \rightarrowtail e_n}$$

We use this to obtain an easier condition to prove.

Proposition 2.6. *The following are equivalent:*

(1) *$S_\bullet F\colon S_\bullet \mathsf{C} \to S_\bullet \mathsf{D}$ is a homotopy equivalence.*
(2) *The bisimplicial map $S_\bullet F|\mathsf{D} \to S_\bullet DR$ is a homotopy equivalence.*

Here, if $X \in \mathsf{sSet}$, XR denotes the bisimplicial set with (m,n)-simplices $XR_{m,n} := X_m$, i.e. it's constant in the first entry. The analogous construction XL operates in the other index. We need this because $S_\bullet D$ is merely a simplicial set.

Proof. Given a $\mathbf{c}/\mathbf{de} \in (S_\bullet F|\mathsf{D})_{m,n}$, we can forget $\mathbf{d}$ and $\mathbf{e}$ to obtain the cofiber sequence of the c_i. This defines a map $\pi_F\colon S_\bullet F|\mathsf{D} \to S_\bullet \mathsf{C}L$. Similarly, there is a map $\rho_F\colon S_\bullet F|\mathsf{D} \to S_\bullet \mathsf{D}R$ sending

$$\mathbf{c}/\mathbf{de} \longmapsto 0 = e_0/e_0 \rightarrowtail e_1/e_0 \rightarrowtail \cdots \rightarrowtail e_n/e_0.$$

These fit into a diagram of bisimplicial sets

$$\begin{tikzcd} S_\bullet \mathsf{D}R \arrow[d, equal] & S_\bullet F|\mathsf{D} \arrow[l, "\rho_F"'] \arrow[r, "\pi_F"] \arrow[d, "F"] & S_\bullet \mathsf{C}L \arrow[d, "S_\bullet F"] \\ S_\bullet \mathsf{D}R & S_\bullet \mathrm{id}|\mathsf{D} \arrow[l, "\rho_{\mathrm{id}}"'] \arrow[r, "\pi_{\mathrm{id}}"] & S_\bullet \mathsf{D}L. \end{tikzcd} \tag{2.7}$$

Using the following lemma, one can show that ρ_{id}, π_{id}, and π_F are homotopy equivalences.

Lemma 2.8 (Realization). *Let $f\colon X_{\bullet,\bullet} \to Y_{\bullet,\bullet}$ be a map of bisimplicial sets such that $f_{\bullet,n}\colon X_{\bullet,n} \to Y_{\bullet,n}$ is a weak equivalence for all n. Then, f is a weak equivalence.*

This means that along the outer square of (2.7), if $S_\bullet F$ is a homotopy equivalence, so is ρ, and vice versa: they're connected by a zigzag of homotopy equivalences. ☐

Corollary 2.9. *Let $E_n\colon S_\bullet F|\mathsf{D}_{\bullet,n} \to S_\bullet F|\mathsf{D}_{\bullet,n}$ be the simplicial map sending*

$$\mathbf{c}/\mathbf{de} \longmapsto \mathbf{0}/\mathbf{0}(e_0/e_0 \rightarrowtail \cdots \rightarrowtail e_n/e_0).$$

Suppose that for all n, E_n is a weak equivalence; then, $S_\bullet F\colon S_\bullet \mathsf{C} \to S_\bullet \mathsf{D}$ is a weak equivalence.

Proof. The map $\rho_F\colon S_\bullet F|\mathsf{D}_{\bullet,n} \to S_\bullet \mathsf{D}R_{\bullet,n}$ is split by a map I_n that puts the zeros in front; then, $I_n \circ \rho = E_n$, and the assumptions and Lemma 2.8 finish it off. ☐

Now we can get down to proving the additivity theorem.

Proof of Theorem 2.1. Let $F\colon (a \rightarrowtail c \twoheadrightarrow b) \mapsto (a, b)$ be the functor in question. We'll check that E_n is a homotopy equivalence for all n. Let $\Gamma\colon S_\bullet F|\mathsf{C}^2_{\bullet,n} \to S_\bullet F|\mathsf{C}^2_{\bullet,n}$ be the functor sending the diagram

$$
\begin{array}{c}
\begin{tikzcd}
a_0 \arrow[r, tail] \arrow[d, tail] & a_1 \arrow[r, tail] \arrow[d, tail] & \cdots \arrow[r, tail] & a_m \arrow[d, tail] \\
c_0 \arrow[r, tail] \arrow[d, two heads] & c_1 \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & c_m \arrow[d, two heads] \\
b_0 \arrow[r, tail] & a_1 \arrow[r, tail] & \cdots \arrow[r, tail] & b_m
\end{tikzcd} \\
\hline
\begin{tikzcd}
a_0 \arrow[r, tail] & \cdots \arrow[r, tail] & a_m \arrow[r, tail] & s_0 \arrow[r, tail] & \cdots \arrow[r, tail] & s_m \\
b_0 \arrow[r, tail] & \cdots \arrow[r, tail] & b_m \arrow[r, tail] & t_0 \arrow[r, tail] & \cdots \arrow[r, tail] & t_m
\end{tikzcd}
\end{array}
$$

to the diagram whre a_i has been set to 0, $\mathbf{c}$ with $\mathbf{b}$, and s_i with s_i/s_0.

$$
\begin{array}{c}
\begin{tikzcd}
0 \arrow[r, tail] \arrow[d, tail] & 0 \arrow[r, tail] \arrow[d, tail] & \cdots \arrow[r, tail] & 0 \arrow[d, tail] \\
b_0 \arrow[r, tail] \arrow[d, two heads] & b_1 \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & b_m \arrow[d, two heads] \\
b_0 \arrow[r, tail] & a_1 \arrow[r, tail] & \cdots \arrow[r, tail] & b_m
\end{tikzcd} \\
\hline
\begin{tikzcd}
0 \arrow[r, tail] & \cdots \arrow[r, tail] & 0 \arrow[r, tail] & s_0/s_0 \arrow[r, tail] & \cdots \arrow[r, tail] & s_m/s_0 \\
b_0 \arrow[r, tail] & \cdots \arrow[r, tail] & b_m \arrow[r, tail] & t_0 \arrow[r, tail] & \cdots \arrow[r, tail] & t_m.
\end{tikzcd}
\end{array}
$$

Thus, Γ projects onto the subcomplex $\mathcal{X}$ of these diagrams with all $a_i = 0$.

Proposition 2.10.

(1) *As maps* $S_\bullet F|\mathsf{C}^2_{\bullet,n} \to S_\bullet F|\mathsf{C}^2_{\bullet,n}$, $\Gamma \simeq \mathrm{id}$.
(2) *On* $\mathcal{X}$, $E_n|_{\mathcal{X}} \simeq \mathrm{id}_{\mathcal{X}}$.

Proof sketch. For the second part, on $\mathcal{X}$, Γ behaves akin to the nerve of a category, but with an additional terminal object.[6]

For the first part, we'll need to write an explicit homotopy down. Recall that a simplicial homotopy is data of the form $f, g\colon X \to Y$ and maps $h_i\colon X_q \to Y_{q+1}$ whenever $0 \le i \le q$ plus tons of rules for the face and degeneracy maps that encode what a map $I \times X \to Y$ means. So what's our desired homotopy?

Given $\mathbf{a}$, $\mathbf{c}$, and $\mathbf{s}$, let $X_i := c_i \amalg_{a_i} s_0$, and let $h_i(e)$ produce the diagram

$$
\begin{array}{c}
\begin{tikzcd}
0 = a_0 \arrow[r, tail] \arrow[d, tail] & a_1 \arrow[r, tail] \arrow[d, tail] & \cdots \arrow[r, tail] \arrow[d, tail] & a_i \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\ulcorner", very near end] & s_0 \arrow[r, equal] \arrow[d, tail] & s_0 \arrow[r, equal] \arrow[d, tail] & \cdots \arrow[r, equal] \arrow[d, tail] & s_0 \arrow[d, tail] \\
0 = c_0 \arrow[r, tail] \arrow[d, two heads] & c_1 \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] \arrow[d, two heads] & c_i \arrow[r, tail] \arrow[d, two heads] & X_i \arrow[r, tail] \arrow[d, two heads] & X_{i+1} \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] \arrow[d, two heads] & X_m \arrow[d, two heads] \\
0 = b_0 \arrow[r, tail] & b_1 \arrow[r, tail] & \cdots \arrow[r, tail] & b_i \arrow[r, tail] & b_{i+1} \arrow[r, tail] & b_{i+2} \arrow[r, tail] & \cdots \arrow[r, tail] & b_m.
\end{tikzcd} \\
\hline
\begin{tikzcd}
0 = a_0 \arrow[r, tail] & \cdots \arrow[r, tail] & a_i \arrow[r, tail] & s_0 \arrow[r, equal] & s_0 \arrow[r, equal] & \cdots \arrow[r, equal] & s_0 \\
0 = b_0 \arrow[r, tail] & \cdots \arrow[r, tail] & b_i \arrow[r, equal] & b_i b_{i+1} \arrow[r, tail] & \cdots \arrow[r, tail] & b_m.
\end{tikzcd}
\end{array}
$$

Then, one can check $d_0 h_0 = \Gamma$, $d_{m+1} h_m = \mathrm{id}$, and (though it's laborious to check), the h_i satisfy the needed simplicial identities.

This finishes off the proof of the additivity theorem. ∎

[6] I didn't fully follow this. What happened?

If the simplicial argument is terrifying or murky, look at the proof of Quillen's theorem A, which could be a good warm-up for this argument.

Exercise 2.11. What's needed from your functor F such that $S_\bullet F$ satisfies the additivity theorem? A lot of this can be relaxed, but not all of it.

We're going to skip a lot of Lurie's lectures, since we care more about the assembly map than algebraic K-theory this semester.

3. Algebraic K-theory of spaces: 2/8/17

Today, Zhu talked about the algebraic K-theory of spaces, corresponding to lecture 21 in Lurie's notes.

Today we'll be working in ∞-categorical language, and will use a lot of Kan extensions. X will always denote a topological space, which we'll think of as a Kan complex, and C will denote an ∞-category.

Definition 3.1. A **local system on X with values in C** is a map of ∞-categories (i.e. a simplicial map) $X \to \mathsf{C}$. The category of C-valued local systems on X is denoted C^X.

Exercise 3.2. We're used to thinking of local systems with values in a group or a ring. How does that definition fit into this one?

Example 3.3. For example, if C is an ordinary category, we think of its higher simplices as just being identities, so a local system on X valued in C factors through the fundamental groupoid $\pi_{\le 1}(X)$ of X. Since X is a Kan complex and so everything in this is invertible, these will also factor through the maximal subgroupoid of C — it only sees invertible morphisms. ◀

Example 3.4. If $\mathsf{C} = \mathsf{Top}$ is the ∞-category of spaces, then we get what's called a **local system of spaces**. The collection of these local systems forms the functor category $\mathsf{Fun}(X, \mathsf{Top})$, which is equivalent to the category of spaces over X, $\mathsf{Top}_{/X}$: a space over X such as $p\colon Y \to X$ is sent to the local system of homotopy fibers $x \mapsto p^{-1}(x)$.

If $X = BG$, a space over BG can be pulled back from a map $\varphi\colon Y \to BG$, defining a G-space $Y_* = \varphi^{-1}(EG \to BG)$. This is related to Lurie's straightening and unstraightening construction, and is also an example of a Grothendieck construction; it demonstrates the interplay between the perspective of spaces over BG and G-spaces. (There may be some technical details incorrect here.) ◀

Example 3.5. Let $\mathsf{C} = \mathsf{Top}_*$ be the category of based systems. Then, local systems on X valued in C can be identified with $\mathsf{Top}_{X/\!/X}$, the category of spaces that retract to X, since such a local system defines maps $X \to Y \to X$ (TODO: why?) whose composition is id_X. ◀

Let's consider a local system $\mathcal{L}$ valued in the ∞-category Sp of spectra. Since Sp is complete and cocomplete, we can define $C_*(X;\mathcal{L}) := \operatorname{colim}\mathcal{L}$ and $C^*(X;\mathcal{L}) := \lim\mathcal{L}$. If $\mathcal{L}$ is the constant system valued in E, then $C_*(X;\mathcal{L}) = E \wedge X_+$ and $C^*(X;\mathcal{L}) = E^X$.[7]

Given a map $f\colon X \to Y$ induces a map on local systems $f^*\colon \mathsf{Sp}^Y \to \mathsf{Sp}^X$, and by categorical nonsense, f^* will have a left and a right adjoint: the left adjoint is denoted $f_!$ and read "f lower-shriek," and the right adjoint is denoted f_* and read "f lower-star." They're given by Kan extensions.

Definition 3.6. Let $F\colon \mathsf{A} \to \mathsf{B}$ and $H\colon \mathsf{A} \to \mathsf{C}$ be functors, where A is small and B is complete and cocomplete.

- The **left Kan extension** of F along H, denoted $LK.F$, is defined by $LK.F(c) := \operatorname{colim} F|_{\mathsf{A}\downarrow c}$, and comes with a natural transformation to F.
- The **right Kan extension** of F along H, denoted $RK.F$, is defined by $RK.F(c) := \lim F|_{\mathsf{A}\downarrow c}$, and comes with a natural transformation from F.

This allows us to define the two adjoints: if $\mathscr{F} \in \mathsf{Sp}^X$, $(f!\mathscr{F})_y := C_*(X_y;\mathscr{F}|_{X_y})$ and $(f_*\mathscr{F})_y = C^*(X_y;\mathscr{F}|_{X_y})$. $i_!$ is a colimit, so can be thought of as a total space construction, or something sort of like this.

Proposition 3.7. *The category Sp^X is compactly generated.*

[7] If you like sheafy things, you might like that spectra-valued local systems are locally constant presheaves of spectra on X.

Recall that C is **compactly generated** if it's equivalent to the ind-category of its full subcategory of compact objects. The desired functor $F\colon \mathrm{Ind}(\mathsf{C}_{\mathrm{cpt}}) \to \mathsf{C}$ is a left Kan extension, and is sometimes called **Yoneda extension**.

Proof. Let C denote the category of compact objects in Sp^X and $F\colon \mathrm{Ind}(\mathsf{C}) \to \mathsf{Sp}^X$ be inclusion. By general categorical theory, F has a right adjoint G, and it suffices to prove G is **conservative**, namely that for any morphism α in Sp^X, $G(\alpha)$ is equivalent implies that α is equivalent.

Let $i : x \hookrightarrow X$ be inclusion of some point into X and $i^* : \mathsf{Sp}^X \to \mathsf{Sp}$ be the pullback. This has a left adjoint $i_! : \mathsf{Sp} \to \mathsf{Sp}^X$. As this is a left adjoint of a left adjoint, it preserves compact objects, and so if $E \in \mathsf{Sp}$ is compact and $\alpha\colon \mathscr{L} \to \mathscr{L}'$ is a morphism of local systems, the adjunction tells us

$$\begin{tikzcd} \mathrm{Map}(i_!E, G(\mathscr{L})) \arrow[r, "G(\alpha)"] \arrow[d, "\wr"] & \mathrm{Map}(i_!E, G(\mathscr{L}')) \arrow[d, "\wr"] \\ \mathrm{Map}(E, i^*\mathscr{L}) \arrow[r] & \mathrm{Map}(E, i^*\mathscr{L}'), \end{tikzcd}$$

so if the bottom map is an equivalence, then by compactness, α must also be an equivalence. $\square$

Remark. The theory of compact generation and being determined under filtered colimits by the compact objects is a formal way of saying there's a theory of cell complexes. For example, in Sp, the compact objects (which coincide with dualizable objects) are finite cell complexes, and taking filtered colimits gives you all cell complexes. So if you have control over the homotopy theory, something like this tends to occur; for shape theory and pro-spaces and pro-spectra, this doesn't quite work.

This is related to the fact that cohomology theories are *a priori* only defined on finite objects, and to why Bousfield localization works as generally as it does. $\blacktriangleleft$

Spectra are particularly nicely generted (by a single object!), so Sp^X is generated by $\mathrm{Im}(i_!|_{\mathrm{Comap}(\mathsf{Sp})})$. In particular, if X is connected, then Sp^X has a single compact generator $i_!\mathbb{S}$. By general theory of ∞-categories, this means $\mathsf{Sp}^X \cong \mathsf{Mod}_{\mathrm{End}(i_!\mathbb{S})}$ (the ∞-category of modules over the ring spectrum $\mathrm{End}(i_!\mathbb{S})$). But $\mathrm{End}(i_!\mathbb{S}, i_!\mathbb{S}) \cong \mathrm{Map}(\mathbb{S}, i^*i_!\mathbb{S}) = i^*i_!\mathbb{S}$. Replacing i with a fibration $E\Omega X \to X$ (whose fiber is ΩX), we get that $i^*i_!\mathbb{S} = (i_!)_x = C_*(\Omega X; \mathbb{S})$, which is the same thing as the spherical group ring $\mathbb{S}[\Omega X]$. This is analogous to thinking about the cohomology of local systems, and this can actually be recovered from the general formalism. The theory of transfers can also be derived from this, an idea going back to Becker and Gottleib, and now has a modern derivation using parameterized homotopy theory. This is one of the reasons parameterized spectra are pretty cool (technical details notwithstanding): you can use them to construct maps that you care about in the nonequivariant case.

We can now define the algebraic K-theory of spaces through the K-theory of ∞-categories.

Definition 3.8. The **algebraic K-theory** of X, denoted $A(X)$, is defined to be $A(X) \coloneqq K(\mathsf{C})$, where C is the full subcategory of compact objects in Sp^X.

The point is that C is the same thing as $\mathsf{Mod}_{\mathbb{S}[\Omega X]}$, a big generalization of looking at modules over $\mathbb{Z}[\pi_1(X)]$. There are enough colimits to show that stabilization induces an isomorphism in K-theory, so you can feed unstable information to it without worrying. This will be useful when applying this to geometric topology. Since $A(X)$ is a K-theory construction, it's an infinite loop space, and hence refines to a spectrum. It's not always an $\mathbb{E}_\infty$ ring spectrum, however.

The first few homotopy groups of $A(X)$ are well-understood.

- $\pi_0(A(X)) = K_0(\mathbb{Z}[\pi_1(X)])$, as we defined purely algebraically.
- $\pi_1(A(X))$ is $K_1(\mathbb{Z}[G]) \coloneqq \mathrm{GL}_\infty(\mathbb{Z}[G])^{\mathrm{ab}}$.

Now we can talk about the assembly map. There is a sequence of functors

$$\mathsf{Top}_X \xrightarrow{(-)_+} \mathsf{Top}_*^X \xrightarrow{\Sigma^\infty} \mathsf{Sp}^X,$$

so if $F\colon \mathsf{Top} \to \mathsf{Sp}$ is a functor, consider left Kan extension F_+ of $F|_{\{*\}}$ (i.e. restricted to the one-point space) along the inclusion $\{\hookrightarrow\}\mathsf{Top}$. That is,

$$F_+(X) := \operatorname*{colim}_{*\downarrow X} F(*).$$

The category $* \downarrow X$ is the points of X, so $F_+(X) = F(*) \wedge X_+$, and this maps to $F(X)$ by functoriality.

For $F = A$, this map $A(*) \wedge X_+ \to A(X)$ is called the **assembly map**, and its cofiber is called the **Whitehead spectrum** of X, denoted $\mathcal{W}h(X)$.

One can calculate that $\pi_0(A(*)) \cong \mathbb{Z}$ and $\pi_1(A(*)) \cong \mathbb{Z}/2$, so running the Atiyah-Hirzebruch spectral sequence for a connected space X,

$$H_0(X; A(*)) \cong H_0(X; \mathbb{Z}) \cong \mathbb{Z}$$

and there's a sequence

$$H_0(X; \pi_1(A(*))) \longrightarrow H_1(X; A(*)) \rightleftarrows H_1(X; \mathbb{Z}) \longrightarrow \cdots$$

so $H_1(X; A(*)) = \mathbb{Z}/2 \oplus \pi_1(X)^{\mathrm{ab}}$. Therefore we have a long exact sequence

$$\mathbb{Z}/2 \oplus \pi_1(X)^{\mathrm{ab}} \xrightarrow{\beta} K_1(\mathbb{Z}[\pi_1(X)]) \longrightarrow \pi_1\mathcal{W}h(X) \xrightarrow{0} \mathbb{Z} \rightleftarrows K_0(\mathbb{Z}[\pi_1(X)]) \longrightarrow \pi_0\mathcal{W}h(X) \longrightarrow 0.$$

In particular, $\pi_0\mathcal{W}h(X) \cong \widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$ and $\pi_1\mathcal{W}h(X)$ is the Whitehead group of X. Thus, the Whitehead spectrum generalizes the Whitehead group into stable homotopy theory.

We're now going to study the assembly map in more detail. It exists for pretty much any enriched functor in great generality, and in our case it relates to cool things relating to constructible sheaves and parameterized spectra.

The general construction for a functor F is that the sequence $A \to \mathrm{Map}(B, A \wedge B) \to \mathrm{Map}(F(B), F(A \wedge B))$, by an adjunction, produces a sequrence $A \wedge F(B) \to F(A \wedge B)$, which is where this map comes from.