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Contents

1. The AGT correspondence with surface operators: 5/22/18

Today Shehper spoke about the AGT correspondence, following earlier lectures I wasn't in town for. The reference papers are [AGG+10] and [FGT16].

The AGT correspondence is a correspondence between

- the instanton partition function of a class-S field theory on a Riemann surface $\Sigma_{g,m}$ with gauge group a product of copies of SU_2, and
- conformal blocks of Liouville theory on $\Sigma_{g,m}$.

This requires choosing a pair-of-pants decomposition of $\Sigma_{g,m}$.

This arises from a compactification of the A_1 $(2,0)$ 6D theory; the gauge group (specifically, the number of copies of SU_2) depends on the number of punctures and the genus in a way which can be seen from the Dynkin diagram. The 4D theory (i.e. the class-S theory) also has an SU_2^n flavor symmetry.

But today, we're going to focus on the AGT correspondence when surface operators are inserted. When you write down the 6D $(2,0)$ supersymmetry algebra, it has central charges that suggest the possibility of 2D and 4D defects. That is, instead of scalar central charges, which correspond to worldlines of particles, these central charges live in higher-dimensional representations of the Poincaré algebra. It will also be possible to introduce these defects in the 4D theory — there's no *a priori* reason to do it, just that it's possible and interesting. The 2D defect will be realized by an M5-M2 brane system, and the 4D defect by an M5-M5 brane system.

Since an Md-brane is a $(d+1)$-dimensional object, we can get a line operator in the 4D theory by taking the M2-brane to intersect C (in $C\times\mathbb{R}^4$, where the 6D theory is formulated) in a loop. Similarly, we can get a surface operator by having it intersect only at a single point; we'll call these operators of type A. For the 4D defects, have the two M5-branes intersect on $C\times\mathbb{R}^2\subset C\times\mathbb{R}^4$; thus we obtain another kind of surface operator, called type B.

From string theory considerations that I'm not familiar with, one can deduce that type A operators correspond to type 2D-4D coupled systems, and type B operators to singularities in coupled fields.

- First, what's a 2D-4D coupled system? For concreteness, let's suppose the 4D theory is $\mathcal{N}=2$ pure super-Yang-Mills, and suppose the surface operator $D\simeq\mathbb{R}\cdot\{x_0,x_1\}\subset\mathbb{R}^{1,3}$ (i.e. Minkowski space). Then, consider a 2D theory on D with flavor symmetry SU_2; the coupling is the idea that this is the same as the gauge symmetry in the bulk[1] Specifically, the "coupling" of the 2D-4D coupled system arises from adding a background connection for the 4D gauge group.

  The boundary theory can be any theory which makes this work; since SU_2 acts on $\mathbb{CP}^1$, we can take the σ-model with target $\mathbb{CP}^1$ on D, for example. There is a subtlety, though; you need to integrate out the x_2- and x_3-directions of the SU_2-connection. Conceptually, this seems reasonable, but there are details that have to be justified: why is it that when you restrict the connection to D, you get the global SU_2-symmetry of the boundary theory? Keep in mind that the SU_2 symmetry is a background symmetry, and is not gauged; for example, for the σ-model, the gauge group is U_1.

[1] There is a generalization where the flavor symmetry is replaced with another global symmetry.

- Next, the type B defects, which we claimed are singularities in 4D fields. Choosing $D = \mathrm{span}_{\mathbb{R}}\{x_0, x_1\}$ again, let's introduce polar coordinates (r, θ) on $\mathrm{span}_{\mathbb{R}}\{x_2, x_3\}$. If A is a connection of the form $A = \alpha \mathrm{d}\theta + \cdots$, then there will be a singularity at the origin, as

$$\mathrm{d}\theta \sim \frac{1}{r}\,\mathrm{d}x \pm \frac{1}{r}\,\mathrm{d}y. \tag{1.1}$$

Explicitly, the curvature is a δ-function: $F = 2\pi\alpha\delta_D$.

If A is an SU_2-gauge field, then we take $\alpha \in \mathrm{U}_1$, for reasons which are unclear.

Our goal is to establish the AGT correspondence for both cases, and to understand if there's a relationship between the 4D theories with these two kinds of defects.

To do this, we'll have to modify the instanton partition function. For the type A defects, we'll take the 2D theory to be a σ-model with target $\mathbb{CP}^1$. Therefore we want to consider maps $\phi\colon \mathbb{R}^2 \to \mathbb{CP}^1$ which vanish at infinity,[2] hence extend to maps from the one-point compactification: $\phi\colon S^2 \to \mathbb{CP}^1$ which are 0 at the basepoint. When we take homotopy classes, we get $[S^2, \mathbb{CP}^1] \cong \mathbb{Z}$, which sends a map to its degree $m \in H_2(\mathbb{CP}^1;\mathbb{Z}) \cong \mathbb{Z}$. Let $\mathcal{M}_{k,m}$ denote the moduli space of solutions (to the instanton equations?) with instanton number k in 4D and soliton number m in 2D.

This theory arises as the low-energy theory of a U_1-gauge theory with two chiral superfields of type $\mathcal{N} = (2,2)$ in 2D, and monopoles in the UV flow down to maps $S^1_\infty \to \mathrm{U}_1$ with the prescribed winding number. It's stated in a somewhat different way in the physics, but the idea is to look at what happens to a large circle.

From this perspective, the correct instanton partition function is

$$Z = \sum_{k=0}^{\infty} \sum_{m\in\mathbb{Z}} q^k e^{itm} \int_{\mathcal{M}_{k,m}} \mathrm{dvol}. \tag{1.2}$$

The surface operators are inserted at a point $z \in \Sigma$, and z is related to t in some way. t is called an *FI parameter*, and appears as a term in the action:

$$S_{\mathrm{GLSM}} = \cdots + \int \mathrm{d}^2\widetilde{\theta}\,(-tS). \tag{1.3}$$

This has something to do with a weakly coupled system, and when it flows to the IR, the M5-branes flow to just a single M5-brane wrapping around the Seiberg-Witten curve in $\mathbb{R}^4$. This is not completely understood, but there is a lot of evidence for this argument.

The other side of the AGT correspondence, on conformal blocks of the Liouville theory, now has an insertion of a vertex operator $e^{-b/2\phi(z)}$ inserted. In thus case the four-point function

$$\langle V_1(0)V_2(1)V_3(q)V_4(\infty)\rangle \tag{1.4}$$

is replaced with

$$\langle V_1(0)V_2(1)V_b(z)V_3(q)V_4(\infty)\rangle, \tag{1.5}$$

where $V_3(z) \coloneqq e^{-b/2\phi(z)}$.

Now let's discuss what happens for the operators of type B. Suppose $A = \alpha\mathrm{d}\theta + \cdots$, so $F = 2\pi\alpha\delta_D + \cdots$. Then $\oint F \in 2\pi\mathbb{Z}$, and the instanton partition function is

$$Z^{\mathrm{inst}} \coloneqq \sum_{m\in\mathbb{Z}} \sum_{k=0}^{\infty} q_1^l k q_2^m \int_{\widetilde{\mathcal{M}}_{k,m}} \mathrm{dvol}. \tag{1.6}$$

There is a WZW model conformal field theory associated to the Kac-Moody algebra called affine $\mathfrak{sl}_2$; the AGT correspondence says that the instanton partition function should correspond to conformal blocks of this CFT. There are a bunch of subtleties going into this.

Anyways, we now have two kinds of surface defects, and get AGT correspondences with two different CFTs. One can ask whether these CFTs are related, or dually, whether these 4D theories are related.

The answer is yes: if $Z^{\mathrm{WZW}}(x,\tau)$ denotes the WZW conformal block, where $x \in \mathrm{Bun}_{\mathrm{SU}_2}(\Sigma)$ and $\tau \in \mathbb{H}$, and Z^L denotes the Liouville conformal block, then

$$Z^{\mathrm{WZW}}(x,\tau) = \int \mathrm{d}u\,\kappa(x,y) Z^L(u,\tau). \tag{1.7}$$

[2] Or more accurately, which have a finite limit at infinity.

This is an example of separation of variables! See the referenced papers for details; since conformal blocks are mathematically understood, there's a good chance this is rigorously proven!

For the relations between the 4D theories, we can rewrite the instanton partition function in terms of an *effective twisted superpotential* $\widetilde{W}$:

$$Z^{\mathrm{inst}} = \exp\left(-\frac{F}{\epsilon_1\epsilon_2} - \frac{\widetilde{W}}{\epsilon_1} + \cdots\right). \tag{1.8}$$

For the two theories we have two superpotentials $\widetilde{W}^L$ and $\widetilde{W}^{\mathrm{WZW}}$, and they're related by

$$\widetilde{W}^{\mathrm{WZW}}(a, u, \tau) = \widetilde{W}^L(a, u, \tau) + \widetilde{W}^{\mathrm{SOV}}(x, u, \tau). \tag{1.9}$$

In the IR, these two describe the same physics, hence one says they have IR duality.

References

[AGG+10] Luis F. Alday, Davide Gaiotto, Sergei Gukov, Yuji Tachikawa, and Herman Verlinde. Loop and surface operators in $\mathcal{N} = 2$ gauge theory and Liouville modular geometry. *Journal of High Energy Physics*, 2010(1):113, Jan 2010. https://arxiv.org/pdf/0909.0945.pdf. 1

[FGT16] Edward Frenkel, Sergei Gukov, and Jörg Teschner. Surface operators and separation of variables. *Journal of High Energy Physics*, 2016(1):179, Jan 2016. https://arxiv.org/pdf/1506.07508.pdf. 1