

FURUTA’S 10/8 THEOREM

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MARCH 4, 2019

These notes were taken in a learning seminar on Furuta’s 10/8 theorem in Spring 2019. I live-TeXed them using vim, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Thanks to Riccardo Pedrotti for some useful comments and for the notes for §3.

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1. INTRODUCTION TO SEIBERG-WITTEN THEORY: 1/23/19

Riccardo gave the first, introductory talk.

In 1982, Matsumoto conjectured that if M is a closed spin manifold, $b_2(M) \geq (11/8)|\sigma(M)|$. Here $b_2(M)$ is the second Betti number and $\sigma(M)$ is the signature. Equality holds for the K3 surface, so this is the best one can do.

In this seminar we’ll study a theorem of Furuta which makes major progress on this conjecture.

Theorem 1.1 (10/8 theorem [Fur01]). *If the intersection form of M is indefinite, $b_2(M) \geq (10/8)|\sigma(M)| + 2$.*

If the intersection form is definite, work of Donaldson [Don83] says that, up to a change of orientation, the intersection form is diagonalizable, so that case is dealt with.

Furuta’s proof uses both Seiberg-Witten theory and equivariant homotopy theory. It can be pushed a little bit farther, but not enough to prove the $11/8^{\text{th}}$ conjecture, as shown recently by Hopkins-Lin-Shi-Xu [HLSX18].

Today we’ll discuss some background for the proof.

Definition 1.2. Let $V \rightarrow M$ be a rank- n real oriented vector bundle. A *spin structure* on V is data $\mathfrak{s} = (P_{\text{Spin}}(V), \tau)$, where $P_{\text{Spin}}(V) \rightarrow M$ is a principal Spin_n -bundle and τ is an isomorphism

$$\tau: P_{\text{Spin}}(V) \times_{\text{Spin}_n} \mathbb{R}^n \xrightarrow{\cong} V.$$

A spin structure on a manifold M is a spin structure on TM .

Remark 1.3. There are other equivalent definitions of spin structures – for example, just as an orientation is a trivialization of V over the 1-skeleton of M , a spin structure is equivalent to a trivialization over the 2-skeleton. ◀

Here’s a cool theorem about spin manifolds.

Theorem 1.4 (Rokhlin [Roh52]). *If M is a spin manifold, $\sigma(M) \equiv 0 \pmod{16}$.*

The signature makes sense when $4 \mid \dim M$. Smoothness is crucial here; there are topological spin 4-manifolds, whatever that means, that do not satisfy this theorem. Freedman's E_8 manifold is an example.

Suppose M is a spin 4-manifold. The representation theory of Spin_4 , in particular the fact that the spin representation S splits as $S^+ \oplus S^-$, leads to two quaternionic line bundles $\mathbb{S}^+, \mathbb{S}^- \rightarrow M$ with Hermitian metrics. Physics cares about these bundles, and will lead to powerful theorems in manifold topology.

These bundles have more structure: in particular, they are Clifford bundles.

Definition 1.5. Let $S \rightarrow M$ be a real vector bundle with a Euclidean metric $\langle \cdot, \cdot \rangle$. A *Clifford bundle* structure is data of, for each $x \in M$, the data of a Clifford algebra action $\text{Cl}(T_x M)$ on S_x that varies smoothly in x , such that the Clifford action is skew-adjoint, meaning

$$\langle v \cdot s_1, s_2 \rangle = -\langle s_1, v \cdot s_2 \rangle.$$

We also require the existence of a connection which is compatible with the Levi-Civita connection on TM .

Given the data of a Clifford bundle, there's an operator called the *Dirac operator* D , which is the following composition:

$$(1.6) \quad C^\infty(S) \xrightarrow{\nabla^{C\ell}} C^\infty(T^*M \otimes S) \xrightarrow{\langle \cdot, \cdot \rangle} C^\infty(TM \otimes S) \xrightarrow{\text{Clifford action}} C^\infty(S).$$

This operator is denoted \not{D} , a convention due to Feynman. It is a first-order, elliptic differential operator; ellipticity means that its analysis is nice.

Thus we can consider the *Seiberg-Witten equations* on a spin 4-manifold. Let $(a, \varphi) \in \Omega_M^1(i\mathbb{R}) \times \Gamma(\mathbb{S}^+)$; then the equations are

$$(1.7a) \quad \not{D}\varphi + \rho(a)(\varphi) = 0$$

$$(1.7b) \quad \rho(d^+a) - \varphi \otimes \varphi^* + \frac{1}{2}|\varphi|^2 \text{id} = 0$$

$$(1.7c) \quad d^*a = 0.$$

On a non-spin manifold, the equations are a little more complicated.

2. THE MONOPOLE EQUATIONS: 1/28/19

Today, Kai spoke about the monopole equations and some of their important properties, foreshadowing compactness next week. We begin with some motivation.

Recall that if M is a closed, oriented 4-manifold (in either the topological or smooth category), the intersection form $H_2(M) \times H_2(M) \rightarrow \mathbb{Z}$ is a unimodular, symmetric bilinear form.

Question 2.1. Which unimodular, symmetric bilinear forms arise as the intersection forms of smooth or topological manifolds?

For example, the intersection form of $S^2 \times S^2$ is $H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The intersection form of \mathbb{CP}^2 is (1). There's an interesting bilinear form called the *E8 form*

$$(2.2) \quad \text{E8} = \begin{pmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & & & 1 & 2 \end{pmatrix}.$$

Can this be realized as the intersection form of a smooth 4-manifold? Rokhlin's theorem tells us the answer is no, because such a manifold would have to be spin, and $16 \nmid \sigma(\text{E8})$. However, Freedman found a topological manifold M_{E8} whose intersection form is E8!

The direct sum of two copies of E8 satisfies Rokhlin's theorem, and this form is realized by the topological 4-manifold $M_{\text{E8}} \# M_{\text{E8}}$. However, Donaldson showed this manifold is not smoothable: specifically, the intersection forms of smooth 4-manifolds can be diagonalized over \mathbb{Z} , and E8 cannot.

There's still more interesting example: consider the *K3 surface* $\{z_1^4 + z_2^4 + z_3^4 + z_4^4 = 0\} \subset \mathbb{CP}^3$; its intersection form is $-2E8 \oplus 3H$. So does it split as a connect sum of 3 copies of $S^2 \times S^2$ and two copies of M_{E8} (with the opposite orientation)? Freedman showed this is true topologically. Smoothly, of course, it can't hold, but we might still get something.

Question 2.3. Is there a smooth, oriented 4-manifold N such that, in the smooth category, $K3 \cong N \# S^2 \times S^2$?

This was a longstanding question.

Seiberg-Witten invariants allow us to answer questions such as this – though in this semester, we're more interested in the monopole map. In any case, let's define the Seiberg-Witten equations.

Let M be a smooth, oriented 4-manifold with b_2^+ odd and a Riemannian metric g , and let \mathfrak{s} be a spin^c structure on M , which determines a *basic class* $K \in H^2(X)$, i.e. an integer cohomology class such that $K \equiv w_2(M) \bmod 2$. The spin^c structure \mathfrak{s} defines for us spinor bundles \mathbb{S}^+ and \mathbb{S}^- . Let \mathcal{A}_L denote the space of U_1 -connections, $A \in \mathcal{A}_L$, and $\psi \in \Gamma(X, \mathbb{S}^+)$ (this is called a *spinor*). The Seiberg-Witten equations are

$$(2.4a) \quad D_A \psi = 0$$

$$(2.4b) \quad F_A^+ + i\delta = i\sigma(\psi).$$

These equations have a gauge symmetry: if G denotes the group $\text{Map}(X, S^1)$ with pointwise multiplication, G acts on $\mathcal{A}_L \times \Gamma(X, \mathbb{S}^+)$ on the first factor. Let B_K^+ denote the quotient minus the locus of spinors which are identically zero; then $B_K^+ \simeq \mathbb{CP}^\infty$, so we know its cohomology is isomorphic to $\mathbb{Z}[x]$, with $|x| = 2$.

Let $\mathcal{M}_K^\delta(g) \subset B_K^\times$ denote the space of solutions to the Seiberg-Witten equations. This space has dimension

$$(2.5) \quad d := \frac{1}{4}(K^2 - (3\sigma(M) + 2\chi(M))),$$

and, crucially, defines a class $[\mathcal{M}_K^\delta(g)] \in H_d(B_K^\times)$ which does not depend on g for generic choices of the metric. The *Seiberg-Witten invariants* are

$$(2.6) \quad SW_X(K) := \langle x^{d/2}, [\mathcal{M}_K^\delta(g)] \rangle \in \mathbb{Z}.$$

The fact that $b_2^+(M) = 0$ implies d is even.

This defines a map SW from the basic classes to \mathbb{Z} . Taubes showed two important results.

Theorem 2.7 (Vanishing theorem (Taubes)). *If M is diffeomorphic to a connect sum of two closed, oriented 4-manifolds $X_1 \# X_2$, $b_2^+(X_1) > 0$, and $b_2^+(X_2) > 0$, then the Seiberg-Witten equations of M vanish.*

Theorem 2.8 (Nonvanishing theorem (Taubes)). *If \mathfrak{s} is the canonical spin^c structure associated to a complex structure on M and $b_2^+(M)$ is positive and odd, then $SW(\pm c_1(M)) = \pm 1$.*

Corollary 2.9. *K3 cannot split smoothly as a connect sum.*

This leads to an interesting generalization: there are *exotic K3 surfaces*, homeomorphic but not diffeomorphic to the standard K3. They don't all admit complex structures, and many of them are not symplectic. Nonetheless, they also don't split off an $S^2 \times S^2$: this is a consequence of Furuta's 10/8 theorem, because if $K3 \cong N \# (S^2 \times S^2)$, then $b_2(N) = 20$ and $\sigma(N) = -16$, but

$$(2.10) \quad 20 \not\geq \frac{10}{8}|-16| + 2.$$

Now let's discuss the monopole map. We now assume M is a spin manifold, with spin structure \mathfrak{s} and spinor bundles \mathbb{S}^\pm . Let A denote a spin connection and consider the spaces

$$(2.11) \quad \tilde{\mathcal{A}} := \{A + i \ker d\} \times (\Gamma(\mathbb{S}^+) \oplus \Omega^1(X))$$

$$(2.12) \quad \tilde{C} := \{A + i \ker d\} \times (\Gamma(\mathbb{S}^-) \oplus \Omega^0(X) \oplus H^1(X; \mathbb{R}) \oplus \Omega^+(X)).$$

Both of these fiber over $H^1(X; \mathbb{R})$: for $\tilde{\mathcal{A}}$, $A + \alpha \mapsto [\alpha]$, and there is a map $\tilde{\mu}: \tilde{\mathcal{A}} \rightarrow \tilde{C}$ defined by

$$(2.13) \quad (A, \phi, a) \mapsto (A, D_A \phi + ia\phi, d^*a, a_{\text{harm}}, d^+a - \sigma(\phi)).$$

Here

- D_A is the *Dirac operator* $D_A: \Gamma(\mathbb{S}^+) \rightarrow \Gamma(\mathbb{S}^-)$.
- $a\phi$ denotes Clifford multiplication.

- d^* is the adjoint of d , which sends k -forms to $(k-1)$ -forms, and satisfies the equation

$$(2.14) \quad d^* = \star d \star.$$

(This is in dimension 4; the sign convention is different in other dimensions.)

- a_{harm} is the harmonic part of a : it's a general fact that any one-form in dimension 4 splits as $a = a_{\text{harm}} + d^* \alpha + d \beta$ for some 0-form β . A form is *harmonic* if the Laplacian $\Delta := dd^* + d^*d$ vanishes on it.
- $d^+ a$ denotes the self-dual part of da .
- $\sigma(\phi)$ denotes the trace form of the endomorphism $\phi \otimes \phi^* - (1/2)\|\phi\|^2 \text{id}$.

Again the group G acts on $\Gamma(\mathbb{S}^\pm)$ by pointwise multiplication, using $S^1 \cong \text{U}_1 \subset \mathbb{C}$. If $u \in G$, $u: X \rightarrow S^1$ also acts on the space of spin^c connections by $d \mapsto udu^{-1}$. Let G act trivially on forms.

Then, the map $\tilde{\mu}$ defined in (2.13) is G -equivariant. Let G_0 denote the maps which vanish at some specified basepoint p , and let $\mathcal{A} := \tilde{A}/G_0$, $\mathcal{C} := \tilde{C}/G_0$, and $\mu := \tilde{\mu}/G_0$; thus we get a map $\mu: \mathcal{A} \rightarrow \mathcal{C}$.

Now, both \mathcal{A} and \mathcal{C} fiber over the Picard group

$$(2.15) \quad \text{Pic}^g(X) := H^1(X; \mathbb{R})/H^1(X; \mathbb{Z}) = H^1(X; \mathbb{R})/G_0.$$

Then $S^1 = G/G_0$ acts on $\mu^{-1}(A, 0, 0, 0, 0)$, and this is the space we're interested in.

We would like to study this space, and to do so we'll need to consider Sobolev spaces. For a fixed integer $k > 2$, let A_k be the fiberwise completion of A within L_k^2 and C_{k-1} be the fiberwise completion of C within L_{k-1}^2 . Then, the monopole map μ is a map $A_k \rightarrow C_{k-1}$.

Claim 2.16. This monopole map μ is S^1 -equivariant, and is a compact perturbation of a linear Fredholm map.

The S^1 -equivariance involves chasing through the definition but isn't bad; the rest is harder. What we can do is start by listing the terms that define a linear Fredholm map, and then check that the rest is compact. In the definition of $\tilde{\mu}$, the terms A , $D_A \phi$, $d^* a$, a_{harm} , and $d^+ a$ are linear and Fredholm; thus we just have to check that $a(\phi)$ and $\sigma(\phi)$ are compact. For the first, we can use the fact that Clifford multiplication is compact, then compose with the map $C_k \rightarrow C_{k-1}$, which is also compact.

Proposition 2.17. Let $T = \ell + c$ be a compact perturbation of a linear Fredholm map ℓ between Hilbert spaces. The restriction of T to any closed, bounded subset Ω is proper.

This will be restated as Claim 3.5 in the next lecture, and will be proven there.

3. COMPACTNESS OF THE MODULI SPACE OF SEIBERG-WITTEN SOLUTIONS: 2/3/19

These are Riccardo's notes on the lecture he gave, on the compactness of the moduli space of solutions to the Seiberg-Witten equations. This is a crucial step in Furuta's construction of finite-dimensional approximations, and relies on some functional analysis.

3.1. A closer look at the Seiberg-Witten monopole map. Let X be a oriented closed spin 4-manifold. Let \mathfrak{s} be a spin structure for it. Let \mathbb{S}^\pm be the positive and negative spinor bundles associated to it. Fix a spin connection A on them.

Recall the Seiberg-Witten equations can be thought as a fiber-preserving S^1 -equivariant map between these two S^1 -Hilbert bundles over $H^1(X; \mathbb{R})$:

$$(3.1a) \quad \tilde{\mathcal{A}} = (A + i \ker(d)) \times (\Gamma(\mathbb{S}^+) \oplus \Omega^1(X))$$

$$(3.1b) \quad \tilde{\mathcal{C}} = (A + i \ker(d)) \times (\Gamma(\mathbb{S}^-) \oplus \Omega^0(X) \oplus H^1(X; \mathbb{R}) \oplus \Omega^+(X)).$$

The map $\tilde{\mu}: \tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{C}}$ is defined by

$$(3.2) \quad (A, \phi, a) \mapsto (A, D_A \phi + ia\phi, d^* a, a_{\text{harm}}, d^+ a - \sigma(\phi)).$$

As explained in the previous seminar, $\sigma(\phi)$ denotes the trace-free endomorphism $i(\phi \otimes \phi^* - \frac{1}{2}\|\phi\|^2 \text{id})$ of \mathbb{S}^+ , considered via the map ρ as a self-dual 2-form on X .

The gauge group $\mathcal{G} = \text{Aut}_{\text{id}}(\mathfrak{s}) \cong \text{Map}(X, S^1)$ acts on spinors on the 4-manifold via multiplication with $u: X \rightarrow S^1$ and on spin^c connections via addition of $ud(u^{-1})$. It acts trivially on forms.

The map $\tilde{\mu}$ is equivariant with respect to the action of \mathcal{G} . Dividing by the free action of the pointed gauge group we obtain the monopole map

$$\mu = \tilde{\mu}/\mathcal{G}_0 : \mathcal{A} \rightarrow \mathcal{C}$$

as a fiber preserving map between the bundles $\mathcal{A} = \tilde{\mathcal{A}}/\mathcal{G}_0$ and $\mathcal{C} = \tilde{\mathcal{C}}/\mathcal{G}_0$ over $\text{Pic}^s(X)$. The preimage of the section $(A, 0, 0, 0, 0)$ of \mathcal{C} , divided by the residual S^1 -action, is called the *moduli space of monopoles*.

For a fixed $k > 2$, consider the fiberwise L_k^2 Sobolev completion \mathcal{A}_k and the fiberwise L_{k-1}^2 Sobolev completion \mathcal{C}_{k-1} of \mathcal{A} and \mathcal{C} . The monopole map extends to a continuous map $\mathcal{A}_k \rightarrow \mathcal{C}_{k-1}$ over $\text{Pic}^s(X)$, which will also be denoted by μ .

We will use the following properties of the monopole map.

- It is S^1 -equivariant.
- Fiberwise, it is the sum $\mu = l + c$ of a linear Fredholm map l and a nonlinear compact operator c .
- Preimages of bounded sets are bounded.

Claim 3.3. The moment map is S^1 -equivariant.

Proof. Equivariance is immediate. The action is the residual action of the subgroup S^1 of gauge transformations which are constant functions on X . This group acts by complex multiplication on the spaces $\Gamma(\mathbb{S}^\pm)$ of sections of complex vector bundles and trivially on forms. \square

Claim 3.4. Fiberwise, the moment map is the sum $\mu = l + c$ of a linear Fredholm map l and a nonlinear compact operator c .

Proof. Restricted to a fiber, the monopole map is a sum of the linear Fredholm operator l , consisting of the elliptic operators D_A and $d^* + d^+$, complemented by projections to and inclusions of harmonic forms. The nonlinear part of μ is built from the bilinear terms $a\phi$ and $\sigma(\phi)$. Multiplication $\mathcal{A}_k \times \mathcal{A}_k \rightarrow \mathcal{C}_k$ is continuous for $k > 2$. Combined with the compact restriction map $\mathcal{C}_k \rightarrow \mathcal{C}_{k-1}$ (Rellich lemma, see [Per18, Lecture 19, p. 2]) we gain the claimed compactness for c : Images of bounded sets are contained in compact sets. \square

Now let us show the following very useful property of compact perturbations of Fredholm operators.

Claim 3.5. The restriction of a compact perturbation $l + c : \mathcal{U}' \rightarrow \mathcal{U}$ of a linear Fredholm map l between Hilbert spaces to any bounded, closed subset is proper.

Proof. Let p denote a projection to the kernel of l . Let A be a bounded closed subset of \mathcal{U}' . It's easy to see that we have the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{(\ell, c, p)} & \mathcal{U} \times \overline{c(A)} \times \overline{p(A)} \\ & \searrow \ell + c & \downarrow \cong + \\ & & \mathcal{U} \times \overline{c(A)} \times \overline{p(A)} \\ & & \downarrow \pi \\ & & \mathcal{U} \end{array}$$

We observe that the map $h : A \rightarrow \mathcal{U} \times \overline{c(A)} \times \overline{p(A)}$ given by $a \mapsto (\ell(a), c(a), p(a))$ is injective and closed. Injectivity is clear since we are projecting on the kernel.

Closedness is a little bit more involved: let $\{(\ell_n, c_n, p_n)\}_n \subset \text{Im}(h)$ converge to $(\ell_\infty, c_\infty, p_\infty)$. In particular there is a sequence $\{a_n\}_n \subset A$ such that $(\ell_n, c_n, p_n) = (\ell(a_n), c(a_n), p(a_n))$. We want to prove that $(\ell_\infty, c_\infty, p_\infty) \in h(A)$. Since l is Fredholm we have the following property: *every bounded sequence $\{x_i\}_i$ in the domain whose image is convergent admits a convergent subsequence $\{x_{i_j}\}_j$* . Since A is closed and bounded (and any other closed subset of it would be bounded as well hence we can directly work with A), $\{a_n\}_n$ is bounded. Since l is Fredholm we can extract a convergent subsequence $\{a'_n\}_n$ converging to $a \in A$ (since A is closed). By the uniqueness of the limit, it's easy to prove

$$(3.6) \quad (\ell_\infty, c_\infty, p_\infty) = (\ell(a), c(a), p(a))$$

which proves the closedness of $h(A)$. This implies that h is proper, since h is an homeomorphism onto its image.

The addition map $+: (u, s, e) \mapsto (u + s, s, e)$ is an homeomorphism hence proper. The projection to \mathcal{U} is proper since the other two factors are compact. \square

3.2. A collection of results. We will list here some results needed for the seminar.

Let U be an open subset of \mathbb{R}^n . We can consider the space $C_c^\infty(U; \mathbb{R}^r)$ of compactly supported \mathbb{R}^r -valued functions. Fix a real number $p > 1$ and an integer $k \geq 0$. The Sobolev L_k^p norm is defined by

$$(3.7) \quad \|f\|_{p,k} := \sum_{|\alpha| < k} \sup_U \|D^\alpha f\|_p.$$

The Sobolev space $L_k^p(E)$ is defined to be the completion of $\Gamma(E)$ in the L_k^p norm.

Here are the basic facts about Sobolev spaces.

Sobolev inequality: If $k \leq \ell$ then there exists a constant C such that

$$(3.8) \quad \|\cdot\|_{p,k} \leq C \|\cdot\|_{p,\ell},$$

and hence we have a bounded inclusion of Sobolev spaces $L_k^p(E) \hookrightarrow L_\ell^p(E)$.

Rellich lemma: The inclusion $L_{k+1}^p(E) \hookrightarrow L_k^p(E)$ is a compact operator.

Morrey inequality: Suppose $\ell \geq 0$ is an integer such that $\ell < k - n/p$; then there is a constant C such that

$$(3.9) \quad \|\cdot\|_{C^\ell} \leq C \|\cdot\|_{p,k},$$

i.e. there is a bounded inclusion

$$(3.10) \quad L_k^p(E) \hookrightarrow C^\ell(E).$$

Smoothness: One has

$$(3.11) \quad \bigcap_{k \geq k_0} L_k^p(E) = C^\infty(E).$$

Lemma 3.12. *Over a closed Riemannian 4-manifold, multiplication of smooth functions extends to a bounded map*

$$(3.13) \quad L_k^2(X) \otimes L_\ell^2(X) \rightarrow L_\ell^2(X)$$

provided that $k \geq 3$ and $k \geq \ell$. In particular, $L_k^2(X)$ is an algebra for $k \geq 3$.

There are also bounded multiplication maps for the lower regularity Sobolev spaces in 4 dimensions, but these bring in Sobolev spaces with $p > 2$.

Let now $D: \Gamma(E) \rightarrow \Gamma(F)$ be a differential operator of order m over a closed, oriented, Riemannian manifold (M, g) . The basic point is that D extends to a bounded linear map between Hilbert spaces:

$$(3.14) \quad D: L_{k+m}^2(E) \rightarrow L_k^2(F).$$

Theorem 3.15 (Elliptic estimate). *If D is elliptic of order m , one has estimates on the L_k^2 -Sobolev norms for each $k \geq 0$:*

$$(3.16) \quad \|s\|_{2,k+m} \leq C_k (\|Ds\|_{2,k} + \|s\|_{2,k}).$$

Moreover,

$$(3.17) \quad \|s\|_{2,k+m} \leq C_k \|Ds\|_{2,k}$$

for $s \in (\ker D)^\perp$ (here $^\perp$ denotes the L^2 -orthogonal complement).

There is an analogue for $L^{p,k+m}$ bounds.

As a consequence of this important theorem we have the following:

Corollary 3.18. *An elliptic operator D of order m defines a Fredholm map $L_{k+m}^2(E) \rightarrow L_k^2(F)$ for any $k \geq 0$. Its index is independent of k . Moreover, its index depends only on the symbol of D .*

Let (M, g) be an oriented Riemannian manifold. Let ∇ be an orthogonal covariant derivative in a real, Euclidean vector bundle $E \rightarrow M$. We know that ∇ has a formal adjoint ∇^* .

Proposition 3.19 (The Lichnérowicz formula). *One has*

$$(3.20) \quad D^2 = \tilde{\nabla}^* \tilde{\nabla} + \frac{1}{4} \text{scal}_g \cdot \text{id}_\mathbb{S} + \frac{1}{2} \rho(F^\circ).$$

Lemma 3.21.

$$(3.22) \quad \frac{1}{2}d^*d(|s|^2) = \langle \nabla^* \nabla s, s \rangle - |\nabla s|^2.$$

Proof sketch. See [Per18, Lecture 19, Lemma 1.1]. The idea is to study the integral

$$(3.23) \quad \int_M f \langle \nabla^* \nabla s, s \rangle \text{vol}$$

where f has compact support. □

It's important to remember that the one above is a pointwise equality. Working locally one has the following result.

Lemma 3.24. *For a smooth function $f: M \rightarrow \mathbb{R}$ with compact support, if p is a local maximum, then $(d^*df)(p) \geq 0$.*

The following lemma is an easy calculation.

Lemma 3.25. *For $\phi \in \Gamma(\mathbb{S}^+)$, one has*

$$(3.26) \quad ((\phi\phi^*)_0\chi, \chi) = (\phi, \chi)^2 - \frac{1}{2}|\chi|^2|\phi|^2.$$

In particular,

$$(3.27) \quad ((\phi\phi^*)_0\phi, \phi) = \frac{1}{2}|\phi|^4.$$

Proof. We have

$$\begin{aligned} ((\phi\phi^*)_0\chi, \chi) &= ((\phi\phi^*)\chi, \chi) - \frac{1}{2}(|\phi|^2\chi, \chi) \\ &= ((\phi, \chi)\phi, \chi) - \frac{1}{2}|\phi|^2|\chi|^2 \\ &= (\phi, \chi)^2 - \frac{1}{2}|\phi|^2|\chi|^2. \end{aligned} \quad \square$$

Lemma 3.28. *For $\eta \in \Omega_X^2$ and $\phi \in \Gamma(\mathbb{S})$, one has $(\rho(\eta)\phi, \phi) \leq |\eta||\phi|^2$.*

Proof. It suffices to take $\eta = e \wedge f$ for orthogonal unit vectors e and f . One then has

$$(3.29) \quad (\rho(\eta)\phi, \phi) = (\rho(e \wedge f)\phi, \phi)$$

$$(3.30) \quad = \frac{1}{2}([\rho(e), \rho(f)]\phi, \phi)$$

$$(3.31) \quad = -\frac{1}{2}(\rho(f)\phi, \rho(e)\phi)$$

$$(3.32) \quad \leq |\rho(e)\phi| \cdot |\rho(f)\phi|,$$

where in (3.31) we used the fact that ρ has image in the anti-skew-Hermitian matrices. Now since $|e| = 1$ then $|\rho(e)| = 1$ (similarly for f), and therefore we conclude. □

Lemma 3.33. *Let A be a Clifford connection for the spinor bundle of a spin^c structure of X . Let $a \in \Omega_X^1(i\mathbb{R})$; then*

$$(3.34) \quad D_{A+a}\phi = D_A\phi + a \cdot \phi,$$

where the last term is the Clifford multiplication between a and ϕ .

Proof. Let's work in local orthonormal coordinates of TX given by $\{e_1, \dots, e_n\}$. We have

$$\begin{aligned}
D_{A+a}\phi &= \sum_i e_i \cdot (A + a)_{e_i} \phi \\
&= \sum_i e_i \cdot A_{e_i} \phi + \sum_i e_i \cdot a(e_i) \phi \\
&= D_A \phi + \sum_i e_i \cdot a(e_i) \phi \\
&= D_A \phi + a \sum_i e_i \phi \\
&= D_A \phi + a \cdot \phi.
\end{aligned}$$

Notice that here we used that $a \in \Omega_X^1(i\mathbb{R})$ hence all the coefficients $a(e_i)$ are equal to each other, and without loss of generality we named then a . \square

3.3. Compactness of the moduli space. If the bundles \mathcal{A} and \mathcal{C} were finite-dimensional, then the boundedness property would be equivalent to properness. In this infinite-dimensional setting, the argument above can be used the same way as Heine-Borel in the finite-dimensional case to show that the boundedness condition implies properness. It turns out that the ingredients of the compactness proof for the moduli space also prove the stronger boundedness property.

Proposition 3.35. *Preimages $\mu^{-1}(B) \subset \mathcal{A}_k$ of bounded disk bundles $B \subset \mathcal{C}_{k-1}$ are contained in bounded disk bundles.*

Proof. It is sufficient to prove this fiberwise for the Sobolev completions of the restriction of the monopole map to the space $\{A\} \times (\Gamma(\mathbb{S}^+) \oplus \ker(d^*))$, which maps to $\{A\} \times (\Gamma(\mathbb{S}^-) \oplus \Omega_+^2(X) \oplus H^1(X; \mathbb{R}))$. We start by defining the following scalar product: using the elliptic operator $D = D_A + d^+$ and its adjoint, define the L_k^2 -norm via the scalar product on the respective function spaces through

$$(3.36a) \quad (\cdot, \cdot)_i = (\cdot, \cdot)_0 + (D\cdot, D\cdot)_{i-1} \text{ for } 0 < i \leq k$$

$$(3.36b) \quad (\cdot, \cdot)_0 = \int_X \langle \cdot, \cdot \rangle.$$

Using the elliptic estimates and continuity (i.e. boundedness) of D it's easy to see that this norm is equivalent to the classic Sobolev one. A similar definition can be extended to norms for the L_k^p -spaces. Let us take $\mu(A, \phi, a) = (A, \varphi, b, a_{\text{harm}}) \in \mathcal{C}_{k-1}$ with the norm of the latter bounded by some constant R . The Lichnerowicz formula (Proposition 3.19) for a connection $A + a = A'$ reads

$$(3.37) \quad D_{A'}^* D_{A'} = A' \circ A' + \frac{1}{4} s \cdot \text{id}_{\mathbb{S}} + \frac{1}{2} \rho(F_{A'}^\circ)$$

with s denoting the scalar curvature of X . As a consequence we have a pointwise estimate: using Lemma 3.21,

$$(3.38) \quad d^* d|\phi|^2 = 2\langle \nabla_{A'}^* \nabla_{A'} \phi, \phi \rangle - 2\langle \nabla_{A'} \phi, \nabla_{A'} \phi \rangle.$$

Then, removing the negative quantity on the left to obtain an inequality,

$$(3.39) \quad \leq 2\langle \nabla_{A'}^* \nabla_{A'} \phi, \phi \rangle$$

$$(3.40) \quad \leq 2\langle D_{A'}^* D_{A'} \phi - \frac{s}{4} \phi - \frac{1}{2} \rho(F_{A'}^\circ) \phi, \phi \rangle.$$

Substituting in the second Seiberg-Witten equation,

$$(3.41) \quad \leq \langle 2D_{A'}^* \varphi - \frac{s}{2} \phi - (\sigma(\phi) + b)\phi, \phi \rangle$$

Now we move some terms to the left and use the equality $D_{A+a} = D_A + a$ together with the fact that the Dirac operator is self-adjoint to get

$$(3.42) \quad d^* d|\phi|^2 + \frac{s}{2} |\phi|^2 + \langle \sigma(\phi), \phi \rangle \leq \langle 2D_{A'}^* \varphi, \phi \rangle - \langle b\phi, \phi \rangle.$$

Next, use Lemma 3.25 to bound $\sigma(\phi)$ and obtain

$$(3.43) \quad d^*d|\phi|^2 + \frac{s}{2}|\phi|^2 + \frac{1}{2}|\phi|^4 \leq \langle 2D_A^*\varphi, \phi \rangle + 2\langle a \cdot \varphi, \phi \rangle - b|\phi|^2$$

$$(3.44) \quad \leq 2(\|D_A^*\varphi\|_\infty + \|a\|_\infty\|\varphi\|_\infty) \cdot |\phi| + \|b\|_\infty \cdot |\phi|^2.$$

$$(3.45) \quad \leq c_1 \left((1 + \|a\|_\infty) \|\varphi\|_{L_{k-1}^2} \cdot |\phi| + \|b\|_{L_{k-1}^2} \cdot |\phi|^2 \right),$$

using the Sobolev embedding theorem (Morrey's inequality) to bound the L^∞ -norm with the Sobolev norm.

Now we need to estimate $\|a\|_\infty$. First thing, for $p > 4$ we get a Sobolev estimate $\|a\|_\infty \leq c_2\|a\|_{L_1^p}$ and then use the elliptic estimate:

$$(3.46) \quad \|a\|_{L_1^p} = \|a_{\text{harm}} + a'\|_{L_1^p} \leq \|a_{\text{harm}}\|_{L_1^p} + \|a'\|_{L_1^p}$$

$$(3.47) \quad \leq \|a_{\text{harm}}\|_{L_0^p} + \|d^+a\|_{L_0^p}$$

where in (3.46) we used the Hodge decomposition of a and in (3.47) we applied the elliptic estimate to both component. Recall that $d^+(a_{\text{harm}}) = 0$ and $d^+a = d^+a'$.

Combining with the equality $d^+a = b + \sigma(\phi)$ then leads to an estimate

$$(3.48) \quad \|a\|_\infty \leq c_4 \left(\|a_{\text{harm}}\|_{L_0^p} + \|b\|_{L_0^p} + \|\sigma(\phi)\|_{L_0^p} \right)$$

$$(3.49) \quad \leq c_5 \left(\|a_{\text{harm}}\|_{L_0^p} + \|b\|_{L_{k-1}^2} + \|\phi\|_\infty^2 \right)$$

In the last passage we control the L_0^p -norm with the L_{k-1}^2 -one, since $p > 4$.

Putting these two estimates together, we get something of the form

$$(3.50)$$

$$(3.51) \quad \begin{aligned} d^*d|\phi|^2 + \frac{1}{2}\|s\|_\infty\|\phi\|_\infty^2 + \frac{1}{2}\|\phi\|_\infty^4 &\leq c \left(1 + c_5 \left(\|a_{\text{harm}}\|_{L_0^p} + \|b\|_{L_{k-1}^2} + \|\phi\|_\infty^2 \right) \right) \|\varphi\|_{L_{k-1}^2} \cdot \|\phi\|_\infty + \|b\|_{L_{k-1}^2} \cdot \|\phi\|_\infty^2 \\ &\leq K\|\phi\|_\infty^3 + R\|\phi\|_\infty^2, \end{aligned}$$

where in (3.51) we applied the bounds we had by assumption on the elements in the image.

So our inequality is now:

$$(3.52) \quad d^*d|\phi|^2 + \frac{1}{2}\|\phi\|_\infty^4 \leq K\|\phi\|_\infty^3 + R\|\phi\|_\infty^2 - \frac{1}{2}\|s\|_\infty\|\phi\|_\infty^2$$

$$(3.53) \quad \leq K\|\phi\|_\infty^3 + R\|\phi\|_\infty^2.$$

Now this inequality must hold in particular when ϕ achieves its maximum, and on that point the Laplacian is positive, hence we can forget about it and get

$$(3.54) \quad \frac{1}{2}\|\phi\|_\infty^4 \leq K\|\phi\|_\infty^3 + R\|\phi\|_\infty^2.$$

In particular we bound the 4th power of a quantity with a polynomial in that quantity of degree 3. This implies that $\|\phi\|_\infty$ must be bounded. Therefore we can bound the L_0^p -norm of (ϕ, a) for every $p \geq 1$.

Now comes bootstrapping: for $i \leq k$, assume inductively L_{i-1}^2 -bounds on (ϕ, a) . To obtain L_i^2 -bounds, compute:

$$(3.55) \quad \|(\phi, a)\|_{L_i^2}^2 - \|(\phi, a)\|_{L_0^2}^2 = \|(D_A\phi, d^+a)\|_{L_{i-1}^2}^2$$

$$(3.56) \quad = \|(\phi + ia\phi, b - \sigma(\phi))\|_{L^2}^2$$

$$(3.57) \quad = \|(\phi, b)\|_{L_{i-1}^2}^2 + \|(ia\phi, \sigma(\phi))\|_{L_{i-1}^2}^2.$$

The first equality holds by our definition of the Sobolev norm. The last equality holds as $D_{A'} = D_A + a$. The summands in the last expression are bounded by the assumed L_{i-1}^2 -bounds on (ϕ, a) together with the Sobolev multiplication properties. Note that the steps for $i = 2$ and 3 require special care (see [Per18, Lecture 21, p. 4]) or use Sobolev embedding together with the fact that we have control on the L^p -norms of (ϕ, a) for every p , which gives us control on the respective Sobolev norms for $p = 2$. \square

4. THE Pin_2^- -SYMMETRY: 2/11/19

These are Arun's prepared lecture notes on the group Pin_2^- , its representations, and the Pin_2^- symmetry in the Seiberg-Witten equations associated to a spin 4-manifold.

4.1. Some avatars of Pin_2^- . In the first part of the talk, I'll tell you some basic facts about Pin_2^- . In Seiberg-Witten theory, this group is often just called $\text{Pin}(2)$, but that could be confusing: there's also Pin_2^+ , which is different.

Definition 4.1. Recall that given a vector space V (over \mathbb{R} or \mathbb{C}) and a quadratic form Q , we can form the Clifford algebra $Cl(V, Q) := TV/(v \otimes v - Q(v)1)$. That is, we take the tensor algebra and introduce the relation $v^2 = Q(v)$. This is a $\mathbb{Z}/2$ -graded algebra with the grading given by the length of a tensor mod 2; let α denote the *grading operator*, which acts on the even subspace as 1 and on the odd subspace as -1 . It is common to think of V as sitting inside of $Cl(V, Q)$ as the length-1 tensors.

The *Clifford group* $\Gamma(V, Q)$ is the group of $x \in Cl(V, Q)^\times$ such that $\alpha(x)yx^{-1} \in V \subset Cl(V, Q)$ for all $y \in V$.

Consider the involution $\beta: Cl(V, Q) \rightarrow Cl(V, Q)$ sending $v_1 \otimes \cdots \otimes v_n \mapsto v_n \otimes \cdots \otimes v_1$. The *Clifford norm* is $N(v) := \beta(v) \cdot v$, which is a scalar on $\Gamma(V, Q)$.

The *pin group* $\text{Pin}(V, Q)$ is the kernel of the Clifford norm inside $\Gamma(V, Q)$. The *spin group* $\text{Spin}(V, Q)$ is the subgroup of even elements of $\text{Pin}(V, Q)$. The following shorthand is standard:

- If $V = \mathbb{R}^n$ and $Q(x) = \langle x, x \rangle$, $Cl(V, Q)$ is denoted Cl_n and $\text{Pin}(V, Q)$ is denoted Pin_n^+ ; if $Q(x) = -\langle x, x \rangle$, they're denoted Cl_{-n} and Pin_n^- .
- The spin groups in these cases are canonically isomorphic, and denoted Spin_n .
- If $V = \mathbb{C}^n$ and $Q(x) = \langle x, x \rangle$, $\text{Pin}(V, Q)$ is denoted Pin_n^c , and $\text{Spin}(V, Q)$ is denoted Spin_n^c .

These are all compact, real Lie groups; there's a map $\text{Spin}_n \rightarrow \text{SO}_n$ which is a double cover, connected if $n \geq 2$ and universal if $n \geq 3$. Correspondingly there's a double cover $\text{Pin}_n^\pm \rightarrow \text{O}_n$. Pin_n^\pm has two components if $n > 1$; $\text{Pin}_1^+ \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ and $\text{Pin}_1^- \cong \mathbb{Z}/4$.

Remark 4.2. Why would you want pin groups anyways? *A posteriori*, of course, we're going to find a Pin_2^- symmetry in the Seiberg-Witten equations of a spin 4-manifold, but there are other reasons to care. One rough answer is that there are many places in geometry and physics (index theory, fermionic QFT, ...) where one wants spin or spin^c structures, but if you want to try to study the same story on unoriented manifolds, the analogues are pin and pin^c structures. \blacktriangleleft

Now we focus specifically on Pin_2^- , with the hope of getting some intuition for what it is. We know it contains Spin_2 as an index-2 subgroup, and topologically is two circles.

We can get our hands on it by embedding it in Spin_3 , which we do understand. Consider the map $Cl_{-2} \hookrightarrow Cl_{-3}^0$ (i.e. into the even part of Cl_{-3}) sending $e_1 \mapsto e_1 e_3$ and $e_2 \mapsto e_2 e_3$. This also sends $1 \mapsto 1$ and $e_1 e_2 \mapsto e_1 e_2$.

There's an identification $Cl_{-3}^0 \cong \mathbb{H}$ via $e_1 e_3 \mapsto i$, $e_2 e_3 \mapsto j$, and $e_1 e_2 \mapsto k$, which restricts to the (possibly familiar) isomorphism $\text{Spin}_3 \cong \text{Sp}_1$ (which is also $\text{SU}(2)$). This then restricts to an identification

$$(4.3) \quad \text{Pin}_2^- \cong \{e^{i\theta}\} \cup \{je^{i\theta}\} \subset \text{Sp}_1,$$

which is sometimes taken as a definition in this area, and which we will use heavily. The first thing it gives us is a representation of Pin_2^- on \mathbb{H} . We will also let \mathbb{R} denote the real representation of Pin_2^- which is trivial on Spin_2 , and such that j acts by -1 .

4.2. Appearance in the Seiberg-Witten equations. Furuta produces the Pin_2^- symmetry in the Seiberg-Witten equations in a very elegant way, doing everything over a point, where it's close to obvious, and using the associated bundle construction to move to the tangent and spinor bundles.

Definition 4.4. Here's some notation for some representations of $\text{Spin}_4 \cong \text{Sp}_1 \times \text{Sp}_1$.

- Let ${}_-\mathbb{H}$ denote the left action of $\text{Sp}_1 \times \text{Sp}_1$ on the quaternions \mathbb{H} by the first factor (${}_-\mathbb{H}$) or the second factor (${}_+\mathbb{H}$). These are the spinor representations.
- Let ${}_-\mathbb{H}_+$ denote the action of $\text{Sp}_1 \times \text{Sp}_1$ on \mathbb{H} by $(p, q) \cdot v = pvq^{-1}$. For Spin_4 , this is the representation $\text{Spin}_4 \twoheadrightarrow \text{SO}_4 \hookrightarrow \text{GL}_4(\mathbb{R})$.
- Let ${}_+\mathbb{H}_+$ denote the action of $\text{Sp}_1 \times \text{Sp}_1$ by $(p, q) \cdot v = qvq^{-1}$.

Given any representation or equivariant vector bundle V , we'll let $\tilde{V} := V \otimes \tilde{\mathbb{R}}$.

If (X, \mathfrak{s}) is a 4-manifold with associated principal Spin_4 -bundle $P_{\mathfrak{s}} \rightarrow X$, then we have the associated bundles

$$(4.5a) \quad \mathbb{S}^{\pm} \cong P_{\mathfrak{s}} \times_{\text{Spin}_4} \pm \mathbb{H} \rightarrow X$$

$$(4.5b) \quad TX \cong P_{\mathfrak{s}} \times_{\text{Spin}_4} -\mathbb{H}_+ \rightarrow X$$

$$(4.5c) \quad \Lambda := \mathbb{R} \oplus \Lambda_+^2 T^*X \cong P_{\mathfrak{s}} \times_{\text{Spin}_4} +\mathbb{H}_+ \rightarrow X.$$

Now we throw in a Pin_2^- -action and extend $\pm \mathbb{H}$ and $+\mathbb{H}_{\pm}$ to $\text{Spin}_4 \times \text{Pin}_2^-$ -representations:

- Using the inclusion $\text{Pin}_2^- \hookrightarrow \text{Sp}_1$, we define the action of $g \in \text{Pin}_2^-$ on $\pm \mathbb{H}$ to be right multiplication by g^{-1} .
- Let Pin_2^- act trivially on $\pm \mathbb{H}_{\pm}$.

We need these to commute with the Spin_4 -actions but that's easy, and therefore using (4.5), we have actions of Pin_2^- on the fibers of TX , \mathbb{S}^{\pm} , and Λ .

Proposition 4.6. *The monopole map is equivariant with respect to these Pin_2^- -actions.*

Proof. (1) You can check in one line that the multiplication map $-\mathbb{H}_+ \times +\mathbb{H} \rightarrow -\mathbb{H}$ is $\text{Spin}_4 \times \text{Pin}_2^-$ -equivariant. Passing to associated bundles, this says Clifford multiplication $C: \mathbb{S}^+ \rightarrow \mathbb{S}^-$ is Pin_2^- -equivariant.

- (2) It's just as easy to check that the map $-\mathbb{H}_+ \times -\tilde{\mathbb{H}}_+ \rightarrow -\tilde{\mathbb{H}}_+$ sending $a, b \mapsto \bar{a}b$ is $\text{Spin}_4 \times \text{Pin}_2^-$ -equivariant, so the map

$$(4.7) \quad \begin{aligned} \tilde{C}: T^*X \times \tilde{T}^*X &\longrightarrow \tilde{\Lambda} \\ a, b &\longmapsto (\langle a, b \rangle, (a \wedge b)_+), \end{aligned}$$

which Furuta calls “twisted Clifford multiplication,” is Pin_2^- -equivariant. (Here we passed from TX to T^*X , of course using the metric to do so.)

- (3) All named Pin_2^- -representations have been unitary (orthogonal for $\tilde{\mathbb{R}}$), so the actions of Pin_2^- on \mathbb{S}^{\pm} are unitary (with respect to the Hermitian metric induced from the Riemannian metric on X), and on T^*X , \tilde{T}^*X , Λ , and $\tilde{\Lambda}$ are orthogonal. Therefore the covariant derivatives associated to these bundles are also Pin_2^- -equivariant, hence so are the Dirac operators

$$(4.8a) \quad D_1 := C \circ \nabla: \Gamma(\mathbb{S}^+) \longrightarrow \Gamma(\mathbb{S}^-)$$

$$(4.8b) \quad D_2 := \tilde{C} \circ \nabla: \Gamma(\tilde{T}^*X) \longrightarrow \Gamma(\tilde{\Lambda}).$$

(Here D_2 can be identified with $d^* + d^+$.) Therefore $D := D_1 \oplus D_2$ is also Pin_2^- -equivariant.

- (4) Now consider the map

$$(4.9) \quad \begin{aligned} +\mathbb{H} \times -\tilde{\mathbb{H}}_+ &\longrightarrow -\mathbb{H} \times +\tilde{\mathbb{H}}_+ \\ \phi, a &\longmapsto (a\phi i, \phi i \bar{\phi}). \end{aligned}$$

In a similar way, one can check this is a (nonlinear) $\text{Spin}_4 \times \text{Pin}_2^-$ -equivariant map. It passes to a map of associated bundles $Q: \Gamma(\mathbb{S}^+ \oplus \tilde{T}^*M) \rightarrow \Gamma(\mathbb{S}^- \oplus \tilde{\Lambda})$, which is Pin_2^- -equivariant.¹

Therefore the monopole map $SW = D + Q$ is Pin_2^- -equivariant. Because the Pin_2^- -action is continuous, it doesn't matter what regularity we impose on sections: this fact is true both for smooth sections and their Sobolev completions. \square

4.3. Some computations with the representation ring. The proof of the $10/8^{\text{th}}$ theorem requires a few more pure representation-theoretic results, and since we have time, I'll go over them now. Let's start by listing some representations of Pin_2^- .

Example 4.10. The first representations you'd write down are the trivial representation 1 and the *sign representation* $\sigma := \mathbb{C}$.

We can next define some irreducible two-dimensional representations h_d , indexed by $d \in \mathbb{Z}$, as follows: $\text{Pin}_2^- = \{e^{i\theta}\} \cup \{je^{i\theta}\}$, so let the underlying complex vector space of h_d be $\mathbb{H} = \mathbb{C}^2$, with j acting in the usual

¹In fact, since the second factor is purely imaginary, we know the image isn't just in $\mathbb{S}^- \oplus \tilde{\Lambda}$, but in $\mathbb{S}^- \oplus \Lambda_+^2 T^*X$.

way and $e^{i\theta}$ acting by $(e^{id\theta}, e^{-id\theta})$. You can prove these are irreducible by just choosing a nonzero quaternion and pushing it around with elements of Pin_2^- until you get a basis, and this isn't hard.

As a particular example, h_1 is \mathbb{H} with the Pin_2^- -action restricted from the usual $\text{Spin}_3 = \text{Sp}_1$ -action. \blacktriangleleft

Theorem 4.11. *The above is a complete list of isomorphism classes of irreducible representations of Pin_2^- .*

I don't know how one proves this: it's asserted by both Furuta and Bryan without proof.

Definition 4.12. The *representation ring* of a group G , denoted $RU(G)$, is the Grothendieck ring of the category of complex representations of G . That is, it is the abelian group freely generated by isomorphism classes of finite-dimensional complex representations of G modulo the relations $[V] = [V'] + [V'']$ whenever there is a short exact sequence $0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0$. The ring structure is defined by $[V] \cdot [W] := [V \otimes W]$.

Let's begin with a simple example.

Proposition 4.13. *The representation ring of $\text{Spin}_2 = U_1$ is $\mathbb{Z}[t, t^{-1}]$, where $t: U_1 \rightarrow U_1$ is the identity map.*

Proof. We can compute by taking the irreducible representations as generators and computing their relations. The irreducible representations of U_1 are indexed by \mathbb{Z} , with the d^{th} one χ_d sending $z \mapsto z^d$. The tensor product of one-dimensional matrices is the ordinary product in \mathbb{C} , so $\chi_d \otimes \chi_{d'} = \chi_{d+d'}$. Therefore $\chi_1 \mapsto t$ gives us $\mathbb{Z}[t, t^{-1}]$. \square

Lemma 4.14. *There's an isomorphism $h_{d_1} \otimes h_{d_2} \cong h_{d_1+d_2} \oplus h_{d_1-d_2}$.*

Proof. Inside $h_{d_1} \otimes h_{d_2} \cong \mathbb{H} \otimes_{\mathbb{C}} \mathbb{H}$, the subspace $V := \text{span}_{\mathbb{C}}\{1 \otimes 1, j \otimes j\}$ is preserved by j and $e^{i\theta}$, hence is a subrepresentation. The same applies to $W := \text{span}_{\mathbb{C}}\{1 \otimes j, j \otimes 1\}$. The vector space isomorphism $V \xrightarrow{\cong} h_{d_1+d_2}$ sending $1 \otimes 1 \mapsto e_1$ and $j \otimes j \mapsto e_2$ is Pin_2^- -equivariant, which you can quickly check by hand; the same idea applies to $W \cong h_{d_1-d_2}$. \square

Corollary 4.15.

$$RU(\text{Pin}_2^-) \cong \mathbb{Z}[\sigma, h_d \mid d \in \mathbb{Z}] / (\sigma^2, \sigma h_d = h_{-d}, h_{d_1} h_{d_2} = h_{d_1+d_2} + h_{d_1-d_2}).$$

The last thing we need to do is compute the image of the restriction map $RU(\text{Pin}_2^-) \rightarrow RU(\text{Spin}_2)$.

Corollary 4.16. *Under the above identifications, the map $RU(\text{Pin}_2^-) \rightarrow RU(\text{Spin}_2)$ sends $\sigma \mapsto 1$ and $h_d \mapsto t^d + t^{-d}$.*

5. FINITE-DIMENSIONAL APPROXIMATIONS, I: 2/18/19

"These vector spaces are indexed by your favorite barnyard animals."

Today, Cameron spoke about finite-dimensional approximations to the monopole map. The idea is that taking the one-point compactification of its domain and codomain defines a Pin_2^- -equivariant map between infinite-dimensional spheres. This is pretty cool, except that infinite-dimensional spheres are contractible, even equivariantly, so we need to take some finite-dimensional approximation in order to obtain homotopically interesting information.

The following theorem is the goal of the next two lectures.

Theorem 5.1. *Let M be a closed, spin 4-manifold such that $b_1(M) = 0$, $b_2^+(M) > 0$, and $\sigma(M) < 0$. Then there are finite-dimensional Pin_2^- -representations V_λ and \overline{W}_λ and Pin_2^- -equivariant maps $D_\lambda: V_\lambda \rightarrow \overline{W}_\lambda$ (linear) and $Q_\lambda: V_\lambda \rightarrow \overline{W}_\lambda$ (quadratic) such that*

- (1) *as Pin_2^- -representations, $V_\lambda \cong \mathbb{H}^{k+m} \oplus \widetilde{\mathbb{R}}^n$ for some k, m , and n ; and*
- (2) *there are Pin_2^- -equivariant metrics on V_λ and \overline{W}_λ and an $R > 0$ such that $(D_\lambda + Q_\lambda)(v) \neq 0$ for all $v \in S_R(0)$.*

Recall that Pin_2^- acts on \mathbb{H} through the inclusion $\text{Pin}_2^- \hookrightarrow \text{Spin}_3 = \text{Sp}_1$ that we discussed last time, and on $\widetilde{\mathbb{R}}$ as the sign representation on \mathbb{R} , which is zero on $\text{Spin}_2 \subset \text{Pin}_2^-$, but such that the element we called j acts by -1 .

Today we will prove (2), leaving the determination of the representations for next week.

There's still plenty to say about the statement of Theorem 5.1 – what's λ ? How do we determine V_λ , \overline{W}_λ , D_λ , and Q_λ ? What are k , m , and n ? These will all be answered during the proof.

Let $S^\pm \rightarrow M$ be the spinor bundles, so we have a Clifford multiplication map $C: T^*M \otimes S^+ \rightarrow S^-$ and a Dirac operator $D_1: \Gamma(S^+) \rightarrow \Gamma(S^-)$, and a twisted Clifford multiplication map $\tilde{C}: T^*M \otimes \tilde{T}^*M \rightarrow \tilde{\Lambda}$ which defines another Dirac operator $D_2: \Gamma(\tilde{T}^*M) \rightarrow \Gamma(\tilde{\Lambda})$. Then $D = D_1 + D_2$, as we discussed last time.

Theorem 5.2 (Weitzenböck). *D^*D is equal to $\nabla^*\nabla$ up to a zeroth-order term.*

Corollary 5.3 (Gårding's inequality). *There is some $k \geq 0$ such that*

$$\langle D^*D\psi, \psi \rangle_{L^2} + k\langle \psi, \psi \rangle_{L^2} \geq \|\psi\|_{L_1^2}^2,$$

where L_1^2 denote the Sobolev norm.

Along the way we'll need another Sobolev space.

Definition 5.4. The L_{-1}^2 norm of an $f \in C^\infty(E)$ is the smallest $C \in \mathbb{R}$ such that $\langle f, \psi \rangle_{L^2} \leq C\|\psi\|_{L_1^2}$ for $\psi \in L_1^2(E)$, if it exists. The completion of $C^\infty(E)$ under this norm is denoted $L_{-1}^2(E)$ (or just L_{-1}^2 if E is clear from context).

Hence there is an embedding $L^2 \hookrightarrow L_{-1}^2$.

Fact. The L^2 inner product defines a continuous nondegenerate pairing $L_{-1}^2 \otimes \overline{L_1^2} \rightarrow \mathbb{C}$, hence identifies L_{-1}^2 as the continuous dual of L_1^2 , i.e. the space of continuous linear functionals on L_1^2 . \blacktriangleleft

Therefore we can restate Gårding's inequality as

$$(5.5) \quad \|(D^*D + k)\psi\|_{L_{-1}^2} \geq \|\psi\|_{L_1^2}.$$

Now, D^*D is a second-order differential operator, hence is a map $L_1^2 \rightarrow L_{-1}^2$. Thus (5.5) implies $D^*D + k$ is a continuous injection $L_1^2 \hookrightarrow L_{-1}^2$.

Lemma 5.6. *In fact, $D^*D + k$ is onto.*

Proof. The map $\langle \langle \cdot, \cdot \rangle \rangle: L^2 \times L^2 \rightarrow \mathbb{C}$ defined by

$$(5.7) \quad \langle \langle \varphi, \psi \rangle \rangle := \langle (D^*D + k)\varphi, \psi \rangle_{L^2}$$

defines an inner product on L^2 , and L^2 is complete with respect to $\langle \langle \cdot, \cdot \rangle \rangle$. Therefore, by the Riesz representation theorem, if $f \in L_{-1}^2$, then on L_1^2 , $\langle f, \cdot \rangle_{L^2} = \langle \langle \varphi, \cdot \rangle \rangle$ for some $\varphi \in L_1^2$. In particular,

$$(5.8) \quad \langle (D^*D + k)\varphi, \psi \rangle = \langle f, \psi \rangle$$

for all $\psi \in L_1^2$, so $(D^*D + k)\varphi = f$. \square

Therefore we can invert $D^*D + k$. Consider the composition

$$(5.9) \quad T: L^2 \hookrightarrow L_{-1}^2 \xrightarrow{(D^*D+k)^{-1}} L_1^2 \xrightarrow{\cong} L^2.$$

All three maps are continuous linear maps, and the third is compact, by the Kondrachov theorem. Therefore T is compact, and since

$$(5.10) \quad \langle (D^*D + k)\varphi, \psi \rangle = \langle \varphi, (D^*D + k)\psi \rangle,$$

then T is self-adjoint. Therefore we can throw the nicest spectral theorem at T .

Theorem 5.11 (Spectral theorem for compact, self-adjoint operators). *Let \mathcal{H} be a separable Hilbert space and $T: \mathcal{H} \rightarrow \mathcal{H}$ be a compact self-adjoint operator. Then there is an orthonormal basis² $\{e_n\}$ for \mathcal{H} such that $Te_n = \mu e_n$ for a $\mu \in \mathbb{R}_{\geq 0}$, and $\mu_1 \geq \mu_2 \geq \dots$ with $\mu_n \geq 0$ as $n \rightarrow \infty$.*

²This is a Hilbert basis, not an algebraic basis, in that elements of \mathcal{H} can be infinite sums of basis elements.

In our case, we conclude that (TODO: I missed this calculation). Therefore if you fix a $\lambda > 0$, then there are finitely many eigenvalues $\lambda_n \leq \lambda$, meaning that if V_λ denotes the span of the eigenvectors of λ_n , then V_λ is finite-dimensional.

The operator DD^* has the same spectrum as D^*D , and the eigenspace for a given eigenvalue is the same. So we can do exactly the same thing to define a finite-dimensional subspace $W_\lambda \subset W$, and $D(V_\lambda) \subset W_\lambda$; let $D_\lambda := D|_{V_\lambda}$.

TODO: what happened next (studying Q) was erased before I could write it down. Sorry about that.

Anyways, we have our operator $Q_\lambda: V_\lambda \rightarrow W \xrightarrow{p_\lambda} W_\lambda$. Let $p^\lambda := 1 - p_\lambda$. Then,

- (1) as $\lambda \rightarrow \infty$, $\lim_{\lambda \rightarrow \infty} \|p^\lambda w\|_W \rightarrow 0$, and
- (2) because every bounded sequence in a Hilbert space has a weakly convergent subsequence, then for any bounded sequence $\{v_n\} \in V$, there's a subsequence $\{v_{n_i}\}$ weakly converging to some v_∞ . Because Q is compact, $\{Qv_{n_i}\}$ converges strongly to some w_∞ , and $Qv_\infty = w_\infty$.

Now, since the zero set of the monopole map $D + Q$ is compact, there is some R , as promised, such that $D + Q$ does not vnaish on the sphere of radius R . To prove part (2) of Theorem 5.1, we need to bring this down to $D_\lambda + Q_\lambda$.

Lemma 5.12. *There is an $\varepsilon > 0$ such that if $\|u\|_V = R$, then $\|(D + Q)v\|_W \geq \varepsilon$.*

Proof. Assume otherwise; then there's a sequence $\{v_n\} \in S_R(0)$ such that $(D + Q)v_i \rightarrow 0$. With $\{v_{n_i}\}$, v_∞ , and w_∞ as above, we know Qv_{n_i} converges strongly to Qv_∞ , so $Dv_{n_i} \rightarrow -Qv_\infty$, and therefore $v_{n_i} \rightarrow v_\infty$. In particular $v_\infty \in S_R(0)$ but $(D + Q)(v_\infty) = 0$, which is a contradiction. \square

Lemma 5.13. *For λ sufficiently large, if $\|v\|_V = R$, then $\|p^\lambda Qv\|_W < \varepsilon$.*

Proof. Assume not; then there are sequences $\{\lambda_n\}$ and $\{v_n\}$ such that $\lambda_n \rightarrow \infty$, $v_n \in S_R(0)$, and $v_i \rightarrow v_\infty$, such that $\|p^{\lambda_n} Qv_n\| \geq \varepsilon$. Let $w_\infty := Qv_\infty$ as before.

Restricting to a subsequence $\{v_{n_i}\}$ as before, again $Qv_{n_i} \rightarrow w_\infty$, so $\lim_{n \rightarrow \infty} \|p^\lambda w_\infty\| = 0$. Therefore there's some N such that

$$(5.14) \quad \|p^{\lambda_{n_i}} Qv_\infty\| = \|p^{\lambda_{n_i}} w_\infty\| \leq \frac{\varepsilon}{2}$$

whenever $i \geq N$, and therefore for $i \gg 0$,

$$(5.15) \quad \|p^{\lambda_{n_i}} (Qv_{n_i} - Qv_\infty)\| \leq \frac{\varepsilon}{2}.$$

\square

Therefore, since $Q_\lambda = p^\lambda \circ Q$, if $\|v\| = R$, TODO missed the last part.

6. FINITE-DIMENSIONAL APPROXIMATIONS, II: 2/25/19

“Remember, these are not just vector spaces but Pin_2^- -representations, and we can pin down what they are... oh no.”

Today Leon spoke, continuing the description of the finite-dimensional approximations of the monopole map, and with the specific goal of proving part (1) of Theorem 5.1.

Today we will heavily use the Dirac operators: Furuta uses $D^1: \Gamma(S^+) \rightarrow \Gamma(S^-)$ to denote the usual Dirac operator, and uses $D^2: \Gamma(\tilde{T}^*M) \rightarrow \Gamma(\tilde{\Lambda})$ to denote the operator $d^* \oplus d^+: \Omega^i(M) \rightarrow \Omega^0(M) \oplus \Omega_+^2(M)$. Then $D := D^1 \oplus D^2$; we'll discuss the quadratic part later.

Last time, we defined the finite-dimensional approximations V_λ^i and W_λ^i : given $\lambda \in \mathbb{R}$, V_λ^i is the space spanned by the eigenspaces of $(D^i)^*D^i$ corresponding to eigenvalues less than λ , and W_λ^i is defined similarly. Because D and Q are Pin_2^- -equivariant, these are in fact Pin_2^- -representations. We'd like to know what representations they are, and as a starting point, we'd like to know their dimensions. This is hard, but we will be able to compute the relative dimension.

Lemma 6.1. *For $i = 1, 2$, $\dim_{\mathbb{C}} V_\lambda^i - \dim_{\mathbb{C}} W_\lambda^i = \text{ind}_{\mathbb{C}} D^i$.*

Proof. For $\lambda' < \lambda$, let $V_{=\lambda'}^i$ denote the λ' -eigenspace of D^i , so that $V_{=\lambda}^i$ is a direct sum over all such $V_{=\lambda'}^i$; then define $W_{=\lambda'}^i$ similarly. Then

$$(6.2) \quad \begin{aligned} \operatorname{ind}_{\mathbb{C}} D^i &= \sum_{\lambda' < \lambda} \operatorname{ind}_{\mathbb{C}}(D^i|_{V_{=\lambda'}^i} : V_{=\lambda'}^i \rightarrow W_{=\lambda'}^i) \\ &= \sum_{\lambda' < \lambda} \dim_{\mathbb{C}} V_{=\lambda'}^i - \dim_{\mathbb{C}} W_{=\lambda'}^i = \dim_{\mathbb{C}} V_{=\lambda}^i - \dim_{\mathbb{C}} W_{=\lambda}^i. \end{aligned} \quad \square$$

Now let's recall some of the Pin_2^- -representations we discussed in earlier lectures: the action on \mathbb{H} via the inclusion $\operatorname{Pin}_2^- \hookrightarrow \operatorname{Spin}_3 = \operatorname{Sp}_1$, and the sign action on \mathbb{R} , which is denoted $\widetilde{\mathbb{R}}$; this is trivial on $\operatorname{Spin}_2 \subset \operatorname{Pin}_2^-$ but $j \in \operatorname{Pin}_2^-$ acts by -1 .

Lemma 6.3. *There exist $m, m', n, n' \in \mathbb{N}$ such that $V_{=\lambda}^1 \cong \mathbb{H}^m$, $W_{=\lambda}^1 \cong \mathbb{H}^{m'}$, $V_{=\lambda}^2 \cong \widetilde{\mathbb{R}}^n$, and $W_{=\lambda}^2 \cong \widetilde{\mathbb{R}}^{n'}$.*

Proof. Because $V_{=\lambda}^1$ is a finite-dimensional subspace of $\Gamma(\mathbb{S}^+)$, there exist $p_1, \dots, p_\ell \in M$ such that the evaluation map

$$(6.4) \quad \operatorname{ev} : \Gamma(\mathbb{S}^+) \longrightarrow \bigoplus_{i=1}^{\ell} (\mathbb{S}^+)_{p_i}.$$

is injective when restricted to $V_{=\lambda}^1$. Since ev is Pin_2^- -equivariant and $(\mathbb{S}_p)^+ \cong \mathbb{H}$, then we have a Pin_2^- -equivariant inclusion $V_{=\lambda}^1 \hookrightarrow \mathbb{H}^M$ for some M , and since \mathbb{H} is an irreducible Pin_2^- -representation, this forces $V_{=\lambda}^1 \cong \mathbb{H}^m$ for some m .

The other three identities proceed in the same way. \square

Now we'll throw the Atiyah-Singer index theorem at D_1 . This tells us that

$$(6.5) \quad \operatorname{ind}_{\mathbb{R}}(D^1) = 2\langle \widehat{A}(M), [M] \rangle,$$

where $\widehat{A}(M)$ is the \widehat{A} -genus (pronounced “A-hat”) $\widehat{A}(M) \in H^*(M)$, a characteristic class.³ Bordism arguments tell us that the \widehat{A} -genus is a linear combination of Pontrjagin classes, and in dimension 4 we get that

$$(6.6) \quad = -\left\langle \frac{p_1(M)}{12}, [M] \right\rangle,$$

which by the Hirzebruch signature theorem is

$$(6.7) \quad = -\frac{\sigma(M)}{4}.$$

Because this is an index, it's an integer, and since D^1 is quaternionic, it's divisible by 4.

Corollary 6.8 (Rokhlin's theorem). *If M is a closed spin 4-manifold, $\sigma(M)$ is divisible by 16.*

Let $k := -\sigma(M)/16$. Because this is the index of D^1 over \mathbb{H} , we know there's an $m \in \mathbb{N}$ such that $V_{=\lambda}^1 \cong \mathbb{H}^{m+k}$ and $W_{=\lambda}^1 \cong \mathbb{H}^m$.

Now let's compute the index of D^2 . This is

$$(6.9) \quad -\operatorname{ind} D^2 = \dim \operatorname{coker}(d^* : \Omega^1 \rightarrow \Omega^0) + \dim \ker(d^+ : \Omega^1 \rightarrow \Omega^+).$$

For the first factor,

$$(6.10) \quad \dim \operatorname{coker} d^* = \dim \ker(d : \Omega^0 \rightarrow \Omega^1) = b_0(M) = 1,$$

since M is connected. The other piece will be a little harder.

Lemma 6.11.

- (1) *If $\alpha \in \Omega^1(M)$, then $d^+\alpha = 0$ iff $d\alpha = 0$.*
- (2) *There's a space $\mathcal{H}^+ \subset \Omega_2^+(M)$ such that $\Omega_2^+(M) = \operatorname{Im}(d^+) \oplus \mathcal{H}^+$, and*
- (3) *$\dim \mathcal{H}^+ = b_2^+(M)$.*

We'll come back to this later, but as a corollary, we know $\operatorname{ind} D^2 = -1 - b_2^+$. Therefore we know that in Lemma 6.3, $n' = n + b_2^+ + 1$. Crucially, the indices did not depend on λ , so we're interested in k and b_2^+ , moreso than m and n .

Now we turn to Q , the quadratic part of the monopole map. Recall that Q is a direct sum of two operators:

- $Q_1: \Gamma(\mathbb{S}^+ \oplus \widetilde{T}^*M) \rightarrow \Gamma(\mathbb{S}^-)$ sends $\phi, a \mapsto ia\phi$, and
- $Q_2: \Gamma(\mathbb{S}^+ \oplus \widetilde{T}^*M) \rightarrow \Gamma(\widetilde{\Lambda}) = \Omega^0(M) \oplus \Omega_+^2(M)$ sends $\phi, a \mapsto \phi a \bar{\phi}$.

TODO: I missed this part, sorry. There's W_λ and then you need one more dimension, which lands in a space called \widetilde{W}_λ .

In summary, we have the following theorem.

Theorem 6.12. *For $\lambda \gg 0$, the monopole map defines a Pin_2^- -equivariant map $D_\lambda + Q_\lambda: V_\lambda \rightarrow \widetilde{W}_\lambda$ between finite-dimensional Pin_2^- -equivariant representations such that $V_\lambda \cong \mathbb{H}^{m+k} \oplus \widetilde{\mathbb{R}}^n$ and $\widetilde{W}_\lambda \cong \mathbb{H}^m \oplus \widetilde{\mathbb{R}}^{n+b_2^+}$, where m and n may depend on λ , but k doesn't. Moreover, there are Pin_2^- -invariant inner products on V_λ and W_λ , and an $R > 0$ not depending on λ , such that the image of the sphere of radius R in V_λ is nonzero.*

In particular, we can restrict to $S_R(V_\lambda)$ and obtain a map between spheres with Pin_2^- -actions. That is, existence of M tells us that some Pin_2^- -equivariant homotopy class of maps between spheres exists; we will use some equivariant homotopy theory to show nonexistence of certain such maps, and therefore obtain the bound on the ratio of the signature and $b_2(M)$.

Proof of Lemma 6.11. For the first part, suppose $d\alpha \in \Omega^-$, which means that $\star d\alpha = -d\alpha$. Since $d^* = -\star d\star$, then $\star d\alpha = -d^*(\star^{-1}\alpha)$, and $\star^{-1}\alpha \in \Omega^3$. Using the splitting

$$(6.13) \quad \Omega^2 = \text{Im}(d) \oplus \text{Im}(d^*) \oplus \mathcal{H}^2,$$

we get that $d^*(\star^{-1}\alpha) = d\alpha$.

For the second part, we use (6.13) again. Let $\pi: \Omega^2 \rightarrow \Omega^+$ be defined by $(1 + \star)/2$. We know $\Omega^+ = \text{Im}(d^+) \oplus \text{Im}((d^*)^+) \oplus \mathcal{H}^+$, so it suffices to show that $\text{Im } d^+ \perp \mathcal{H}^+$ and $\text{Im } d^+ = \text{Im}((d^*)^+)$. For the first one, if $d^+\alpha \in \text{Im}(d^+)$ and $\beta \in \mathcal{H}^+$, we have

$$(6.14) \quad \langle d^+\alpha, \beta \rangle = \langle d\alpha, \beta \rangle = \langle \alpha, d^*\beta \rangle = 0.$$

For the second part, let $\alpha \in \Omega^3$; then

$$(6.15) \quad \pi(d^*\alpha) = \pi(-\star d\star\alpha) = -\pi d(\star\alpha) = -d^+(\star\alpha).$$

For the last part, recall that b_2^+ is the maximal dimension of a positive-definite subspace of \mathcal{H} with respect to the pairing $\mathcal{H}^2 \otimes \mathcal{H}^2 \rightarrow \mathbb{R}$. Also recall that $\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \text{dvol}$, so if $\alpha \in \mathcal{H}^2$ is self-dual, then

$$(6.16) \quad \int \langle \alpha \wedge \alpha \rangle \text{dvol} = \int \alpha \wedge \star\alpha = \int \langle \alpha, \alpha \rangle \text{dvol} \geq 0.$$

with equality iff $\alpha = 0$. Therefore $\dim \mathcal{H}^+ \leq b_2^+$, and because the same argument applies to \mathcal{H}^- , we get $\dim \mathcal{H}^- \leq b_2^-$, so using the orthogonal splitting $\mathcal{H}^2 = \mathcal{H}_+^2 \oplus \mathcal{H}_-^2$, and therefore $\dim \mathcal{H}^+ = b_2^+$. \square

7. EQUIVARIANT K -THEORY: 3/4/19

Today, Richard spoke about equivariant K -theory, following Bryan's paper [?], which is a little simpler.

As usual, suppose X is a smooth, spin 4-manifold with $b_1 = 0$ and nonpositive signature. Bryan is interested in a generalization in which there is a spin diffeomorphism $\tau: X \rightarrow X$ of order 2^p or 2^{p+1} , where p is prime.

Definition 7.1. Let X be a spin n -manifold and $\pi: P \rightarrow X$ be the principal Spin_n -bundle of frames. A *spin diffeomorphism* is data of a diffeomorphism $\tau: X \rightarrow X$ and a Spin_n -equivariant map $\tilde{\tau}: P \rightarrow P$ which covers τ (i.e. $\tau \circ \pi = \pi \circ \tilde{\tau}$), and such that $\tau^*P - P = 0 \in H^1(X; \mathbb{Z}/2)$ (recalling that spin structures are a torsor over $H^1(X; \mathbb{Z}/2)$).

Example 7.2. On any spin manifold, there is a spin diffeomorphism called the *spin flip* which is the identity on the manifold but is the nontrivial automorphism of the double cover $P_{\text{Spin}} \rightarrow P_{\text{SO}}$. \blacktriangleleft

Since τ has order 2^p or 2^{p+1} , then τ^{2^p} is either the identity or the spin flip. In the former case, the symmetry group is $G_{\text{ev}} := \text{Pin}_2^- \times \mathbb{Z}/2^p$, and in the latter case, it's $G_{\text{odd}} := \text{Pin}_2^- \times_{\mathbb{Z}/2} \mathbb{Z}/2^{p-1}$, where $\times_{\mathbb{Z}/2}$ means modding out by the common $\mathbb{Z}/2$ subgroup.

Last time, we built a nonlinear Pin_2^- -equivariant map $D_\lambda + Q_\lambda: V_\lambda \rightarrow \widetilde{W}_\lambda$. Complexifying, we obtain a map $f: V_{\lambda, \mathbb{C}} \rightarrow \widetilde{W}_{\lambda, \mathbb{C}}$ given by

$$(7.3) \quad f(u \otimes 1 + v \otimes i) := (D_\lambda + Q_\lambda)u \otimes 1 + (D_\lambda + Q_\lambda)v \otimes i.$$

Let $BV_{\lambda,\mathbb{C}}$ denote the unit ball in $V_{\lambda,\mathbb{C}}$ and $SV_{\lambda,\mathbb{C}}$ denote its boundary. Then, $f(SV_{\lambda,\mathbb{C}})$ does not contain zero, so we can compose with the projection

$$(7.4) \quad \widetilde{W}_{\lambda,\mathbb{C}} \longrightarrow S\widetilde{W}_{\lambda,\mathbb{C}} = (\widetilde{W}_{\lambda,\mathbb{C}} \setminus 0)/\mathbb{R}_+$$

and obtain a Pin_2^- -equivariant map $\bar{f}: SV_{\lambda,\mathbb{C}} \rightarrow S\widetilde{W}_{\lambda,\mathbb{C}}$, i.e. we get a map $\tilde{f}: BV_{\lambda,\mathbb{C}} \rightarrow B\widetilde{W}_{\lambda,\mathbb{C}}$ which sends the boundary to the boundary. Therefore it is a Pin_2^- -equivariant map of pairs

$$(7.5) \quad \hat{f}: (BV_{\lambda,\mathbb{C}}, SV_{\lambda,\mathbb{C}}) \longrightarrow (B\widetilde{W}_{\lambda,\mathbb{C}}, S\widetilde{W}_{\lambda,\mathbb{C}}).$$

This is a Pin_2^- -equivariant map of spheres, and we will study it using K -theory and index theorem.

Remark 7.6. The disc mod sphere construction may remind you of the Thom space construction. Given a rank- n vector bundle $E \rightarrow X$ with a Euclidean metric, let $D(E)$ denote the unit disc bundle and $S(E)$ denote the unit sphere. Then the *Thom space* of E , denoted X^E , is the quotient $D(E)/S(E)$.

If E is oriented (automatic if it's a complex vector bundle), then there is a *Thom class* $\tau_E \in H^n(X^E)$ such that the map $H^*(X) \rightarrow H^{*+n}(X^E)$ sending $a \mapsto a \smile \tau_E$ is an isomorphism, called the *Thom isomorphism*. \blacktriangleleft

Definition 7.7. The *complex K -group* $K^0(X)$ of a space X (also denoted $KU^0(X)$) is the Grothendieck group of the category of vector bundles on X under direct sum, i.e. it is the group generated by isomorphism classes of vector bundles with the relation $[V \oplus W] = [V] + [W]$.

Remark 7.8. If $E \rightarrow X$ is a complex vector bundle (or more generally is even-dimensional and has a spin^c structure), there is a Thom isomorphism in K -theory $KU^0(X) \rightarrow KU^0(X^E)$. Said another way, $KU^0(E)$ is a free $KU^0(X)$ -module on one generator. \blacktriangleleft

Now we add equivariance to the mix.

Definition 7.9. Let G be a compact Lie group acting on a topological space X . The *G -equivariant complex K -group* $K_G^0(X)$ (or $KU_G^0(X)$) is the Grothendieck group of G -equivariant vector bundles on X under direct sum.

Theorem 7.10 (Atiyah). *If $E \rightarrow X$ is a G -equivariant complex vector bundle, there is a Thom isomorphism $K_G^0(X) \xrightarrow{\cong} K_G^0(X^E)$.*

For example, $K_G^0(\text{pt}) \cong RU(G)$, the representation ring of G , and therefore the G -equivariant K -group of a sphere (the Thom space of a vector space thought of as a vector bundle over a point) is also isomorphic to $RU(G)$.

Applied to our map $f: V_{\lambda,\mathbb{C}} \rightarrow W_{\lambda,\mathbb{C}}$, we get a pullback map

$$(7.11) \quad f^*: K_G^0(B\widetilde{W}_{\lambda,\mathbb{C}}, S\widetilde{W}_{\lambda,\mathbb{C}}) \longrightarrow K_G^0(BV_{\lambda,\mathbb{C}}, SV_{\lambda,\mathbb{C}}),$$

where $G = \text{Pin}_2^-$, which is a map $RU(G) \rightarrow RU(G)$. If λ_V and λ_W denote the Bott classes in $K_G^0(V)$, resp. $K_G^0(W)$, then $f^*(\lambda_W) = \alpha_f \lambda_V$ for some (possibly virtual) representation α_f .⁴ We want to understand α_f , so we will take its character, the class function $g \mapsto \text{tr}(\alpha(g))$.⁵

Now let $C \subset G$ be a *topological cyclic subgroup* of G , i.e. either a finite cyclic subgroup or an S^1 , and let c be a generator (if C is finite) or a nonzero element (if $C \cong S^1$). We will try to study the action of f^* on C -fixed points, which is a common technique in equivariant homotopy theory. Specifically, we have a commutative diagram

$$(7.12) \quad \begin{array}{ccc} K_G^0(B\widetilde{W}_{\lambda,\mathbb{C}}, S\widetilde{W}_{\lambda,\mathbb{C}}) & \xrightarrow{f^*} & K_G^0(BV_{\lambda,\mathbb{C}}, SV_{\lambda,\mathbb{C}}) \\ \downarrow \text{res}_C & & \downarrow \text{res}_C \\ K_C^0(B\widetilde{W}_{\lambda,\mathbb{C}}^C, S\widetilde{W}_{\lambda,\mathbb{C}}^C) & \xrightarrow{(f^C)^*} & K_C^0(BV_{\lambda,\mathbb{C}}^C, SV_{\lambda,\mathbb{C}}^C). \end{array}$$

⁴Is α_f in general an actual representation? Probably not, but we're not sure.

⁵The character formula only makes sense literally for actual representations, and α_f might be a difference of two representations, in which case we take the difference of their characters. This does not depend on how we write α_f as a difference of two actual representations.

Now C acts trivially on $V_{\lambda, \mathbb{C}}^C$ and $\widetilde{W}_{\lambda, \mathbb{C}}^C$, so

$$(7.13) \quad (f^C)^* \lambda_{\widetilde{W}_{\lambda, \mathbb{C}}^C} = d(f^C) \lambda_{V_{\lambda, \mathbb{C}}^C},$$

where $d(f^C) = 0$ if $\dim \widetilde{W}_{\lambda, \mathbb{C}}^C \neq \dim V_{\lambda, \mathbb{C}}^C$.

Remark 7.14. α_f is called the *K-theoretic degree* of f and d is the *topological degree*. ◀

Then, using (7.12),

$$(7.15) \quad \text{res}_C(\lambda_{\widetilde{W}_{\lambda, \mathbb{C}}^C}) = \lambda_{-1}((\widetilde{W}_{\lambda, \mathbb{C}}^C)^\perp) \lambda_{\widetilde{W}_{\lambda, \mathbb{C}}^C},$$

where

$$(7.16) \quad \lambda_{-1} \beta = \sum (-1)^i \beta_i.$$

Then we compute, following tom Dieck,

$$\begin{aligned} (f^C)^*(\text{res}_C \lambda_W) &= \text{res}_C(f^*(\lambda_W)) \\ (f^C)^*((\lambda_{-1}(W^C)^\perp) \lambda_{W^C}) &= \text{res}_C(\alpha_f \lambda_V) = \text{res}_C(\alpha_f) \lambda_V \\ (f^C)^*(\lambda_{-1}(W^C)^\perp) d(f^C) \lambda_{V^C} &= \text{res}_C(\alpha_f) \lambda_{-1}((V^C)^\perp) \lambda_{V^C}. \end{aligned}$$

If g is the generator above, then $\lambda_{-1}((V^C)^\perp)(g) \neq 0$, so

$$(7.17) \quad \alpha_f(c) = d(f^C) \lambda_{-1}((W^C)^\perp - (V^C)^\perp)(g)$$

$$(7.18) \quad \text{tr}_g(\alpha_f) = d(f^g) \text{tr}_g(\lambda_{-1}(W_g^\perp - V_g^\perp)).$$

Here $\lambda_{-1}(r) = 1 - r$. This second equation is very useful: we will be able to throw index theory at the degree, and representation theory at the trace, and there are good formulas for it.

REFERENCES

- [Don83] S. K. Donaldson. An application of gauge theory to four-dimensional topology. *J. Differential Geom.*, 18(2):279–315, 1983. https://projecteuclid.org/download/pdf_1/euclid.jdg/1214437665. 1
- [Fur01] M. Furuta. Monopole equation and the $\frac{11}{8}$ -conjecture. *Math. Res. Lett.*, 8(3):279–291, 2001. http://intlpress.com/site/pub/files/_fulltext/journals/mrl/2001/0008/0003/MRL-2001-0008-0003-a005.pdf. 1
- [HLSX18] Michael J. Hopkins, Jianfeng Lin, XiaoLin Danny Shi, and Zhouli Xu. Intersection forms of spin 4-manifolds and the $\text{Pin}(2)$ -equivariant Mahowald invariant. 2018. <http://arxiv.org/abs/1812.04052>. 1
- [Per18] Tim Perutz. A course on Seiberg-Witten theory and 4-manifold topology: lecture notes. 2018. <https://web.ma.utexas.edu/users/perutz/GaugeTheory.html>. 5, 7, 9
- [Roh52] V. A. Rohlin. New results in the theory of four-dimensional manifolds. *Doklady Akad. Nauk SSSR (N.S.)*, 84:221–224, 1952. 1