M392C NOTES: MATHEMATICAL GAUGE THEORY

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These notes were taken in UT Austin's M392C (Mathematical gauge theory) class in Spring 2019, taught by Dan Freed. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Any mistakes in the notes are my own.

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Lecture 1.

Some useful linear algebra: 1/22/19

"Why did the typing stop?"

Today we'll discuss some basic linear algebra which, in addition to being useful on its own, is helpful for studying the self-duality equations. You should think of this as happening pointwise on the tangent space of a smooth manifold.

Let V be a real n-dimensional vector space. The exterior powers of V define more vector spaces: the scalars $\mathbb{R}$, V, $\Lambda^2 V$, and so on, up to $\Lambda^n V = \operatorname{Det} V$. We can also apply this to the dual space, defining $\mathbb{R}$, V^*, $\Lambda^2 V^*$, etc, up to $\Lambda^n V^* = \operatorname{Det} V^*$.

There is a duality pairing

$$\begin{aligned} \theta\colon \Lambda^k V^* \times \Lambda^k V &\longrightarrow \mathbb{R} \\ (v^1 \wedge \cdots \wedge v^k, v_1 \wedge \cdots \wedge v_k) &\longmapsto \det(v^i(v_j))_{i,j}, \end{aligned} \tag{1.1}$$

where $v^i \in V^*$ and $v_j \in V$.

Now fix a $\mu \in \operatorname{Det} V^* \setminus 0$, which we call a *volume form.* Then we get another duality pairing

$$\begin{aligned} \Lambda^k V \times \Lambda^{n-k} V &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto \theta(\mu, x \wedge y). \end{aligned} \tag{1.2}$$

Thus $\Lambda^k V \cong \Lambda^{n-k} V^*$.

Suppose we have additional structure: an inner product and an orientation. Let $e_1, \dots, e_n$ be an oriented, orthonormal basis of V, and $e^1, \dots, e^n$ be the dual basis. Now we can choose $\mu = e^1 \wedge \cdots \wedge e^n$.

Definition 1.3. The *Hodge star operator* is the linear operator $\star\colon \Lambda^k V^* \to \Lambda^{n-k} V^*$ characterized by

$$\alpha \wedge (\star\beta) = \langle \alpha, \beta \rangle_{\Lambda^k V} \cdot \mu. \tag{1.4}$$

The inner product on $\Lambda^k V^*$ is defined by

$$\langle v^1 \wedge \cdots \wedge v^k, w^1 \wedge \cdots \wedge w^k \rangle \coloneqq \det(\langle v^i, w^j \rangle)_{i,j}. \tag{1.5}$$

The Hodge star was named after W.V.D. Hodge, a British mathematician. Notice how we've used both the metric and the orientation – it's possible to work with unoriented vector spaces (and eventually unoriented Riemannian manifolds), but one must keep track of some additional data.

Example 1.6.

- $\star(e^{i_1} \wedge \cdots \wedge e^{i_k}) = e^{j_1} \wedge \cdots \wedge e^{j_{n-k}}$ if the permutation $1, \dots, n$ to $i_1, \dots, i_k, j_1, \dots, j_{n-k}$ of $[n] \coloneqq \{1, \dots, n\}$ is even. Otherwise there's a factor of -1.

- Suppose $n = 4$. Then $\star(e^1 \wedge e^2) = e^3 \wedge e^4$ and $\star(e^1 \wedge e^3) = -e^2 \wedge e^4$, and so on. ◂

Remark 1.7. The Hodge star is natural. First, you can see that we didn't make any choices when defining it, other than an orientation and a volume form, but there's also a functoriality property. Let $T\colon V \to V$ be an automorphism; this induces $(\Lambda^k T^*)^{-1}\colon \Lambda^k T^* \to \Lambda^k T^*$, and if T is an orientation-preserving isometry,

$$\star \circ (\Lambda^k T^*)^{-1} = (\Lambda^{n-k} T^*)^{-1} \circ \star. \tag{1.8}$$

Hence $\star\star\colon \Lambda^k V^* \to \Lambda^k V^*$ is some nonzero scalar multiple of the identity, and we can determine which multiple it is. Certainly we know

$$\star\star(e^1 \wedge \cdots \wedge e^k) = \star(e^{k+1} \wedge \cdots \wedge e^1) = \lambda e^1 \wedge \cdots \wedge e^k, \tag{1.9}$$

and we just have to compute the parity of these permutations: one uses k transpositions, and the other uses $n-k$. Therefore we conclude that

$$\star\star = (-1)^{k(n-k)}\colon \Lambda^k V^* \to \Lambda^k V^*. \tag{1.10}$$

◂

Now suppose $n = 2m$, so we have a middle dimension m, and $\star\star\colon \Lambda^m \to \Lambda^m$ is $(-1)^m$. This induces additional structure on $\Lambda^m V^*$.

- If m is even (so $n \equiv 0 \bmod 4$), the double Hodge star is an endomorphism squaring to 1. This defines a $\mathbb{Z}/2$-grading on $\Lambda^m V^*$, given by the ± 1-eigenspaces, which we'll denote $\Lambda^m_\pm V^*$. The $+1$-eigenspace is called *self-dual* m-forms, and the -1-eigenspace is called the *anti-self-dual* m-forms.
- If m is odd (so $n \equiv 2 \bmod 4$), the double Hodge star squares to -1, so this defines a complex structure on $\Lambda^m V^*$, where i acts by the double Hodge star.

Exercise 1.11. Especially for those interested in physics, work out this linear algebra in indefinite signature (particularly Lorentz). The signs are different, and in Lorentz signature the two bullet points above switch!

Exercise 1.12. Show that if $4 \mid n$, the direct-sum decomposition $\Lambda^m V^* = \Lambda^m_+ V^* \oplus \Lambda^m_- V^*$ is orthogonal. See if you can find the one-line proof that self-dual and anti-self-dual forms are orthogonal.

Next we introduce conformal structures. This allows the sort of geometry which knows angles, but not lengths.

Definition 1.13. A *conformal structure* on a real vector space V is a set C of inner products on V such that any $g_1, g_2 \in C$ are related by $g_1 = \lambda g_2$ for a $\lambda \in \mathbb{R}_+$.

In this setting, one can obtain g_2 from g_1 by pulling back g_1 along the dilation $T_\lambda\colon v \mapsto \lambda v$. This induces an action of $(T_\lambda^*)^{-1}$ on $\Lambda^k V^*$, which is multiplication by λ^{-k}: if μ_i is the volume form induced from g_i, so that

$$\alpha \wedge \star\beta = g_1(\alpha, \beta)\mu_1, \tag{1.14}$$

then

$$\lambda^{-2k}\alpha \wedge \star\beta = g_2(\alpha, \beta)\lambda^{-n}\mu_2. \tag{1.15}$$

Thus pulling back by dilation carries the Hodge star to $\lambda^{n-2k}\star$. Importantly, if $n = 2m$, then $\star\colon \Lambda^m V^* \to \Lambda^m V^*$ is preserved by this dilation, so it only depends on the orientation and the conformal structure.

Remark 1.16. A conformal structure is independent from an orientation. For example, on a one-dimensional vector space, a conformal structure is no information at all (all inner products are multiples of each other), but an orientation is a choice. ◂

Example 1.17. Suppose $n = 2$ and choose an orientation and a conformal structure on V. As we just saw, this is enough to define the Hodge star $\star\colon V^* \to V^*$, which defines a complex structure on V. Pick a square root i of -1 and let $\star$ act by it (there are two choices, acted on by a Galois group).

We get more structure by complexifying: $V^* \otimes \mathbb{C}$ splits as a the $\pm i$-eigenspaces of the Hodge star; we denote the i-eigenspace by $V^{(1,0)}$ (the $(1,0)$-*forms*) and the $-i$-eigenspace by $V^{(0,1)}$ (the $(0,1)$-*forms*).

Now let's globalize this: everything has been completely natural, so given an oriented, conformal 2-manifold X, it picks up a complex structure, hence is a Riemann surface, and the Hodge star is a map $\star\colon \Omega^1_X \to \Omega^1_X$. Moreover, we can do this on the complex differential forms, which split into $(1,0)$-forms and $(0,1)$-forms.

How do 1-forms most naturally appear? They're differentials of functions, so given an $f\colon X \to \mathbb{C}$, we can ask what it means for $\mathrm{d}f \in \Omega_X^{1,0}$. This is the equation

$$\star\,\mathrm{d}f = i\,\mathrm{d}f. \tag{1.18}$$

This is precisely the Cauchy-Riemann equation; its solutions are precisely the holomorphic functions on X. ◀

Remark 1.19. More generally, one can ask about functions to $\mathbb{C}^n$ or even sections of complex vector bundles; the analogue gives you notions of holomorphic sections. In this case, the equations have the notation

$$\overline{\partial} f = \left(\frac{1+i\star}{2}\right)\mathrm{d}f. \tag{1.20}$$ ◀

We'll spend some time in this class understanding a four-dimensional analogue of all of this structure.

Symmetry groups. Symmetry is a powerful perspective on geometry. If we think about V together with some structure (orientation, metric, conformal structure, some combination,...), we can ask about the symmetries of V preserving this structure. Of course, to know this, we must know V, but we can instead look at a model space $\mathbb{R}^n$ to define a *symmetry type*, and ask about its symmetry group G: then an isomorphism $\mathbb{R}^n \to V$ preserving all of the data we're interested in defines an isomorphism from G to the symmetry group of V.

Example 1.21. When $\dim V = 2$, the most general symmetry group is $\mathrm{GL}_2(\mathbb{R})$, the invertible matrices acting on $\mathbb{R}^2$. Adding more structure we get more options.

- If we restrict to orientation-preserving symmetries, we get $\mathrm{GL}_2^+(\mathbb{R})$.
- If we restrict to symmetries preserving a conformal structure, the group is called $\mathrm{CO}_2 = \mathrm{O}_2 \times \mathbb{R}^{>0}$.
- If we ask to preserve an orientation and a complex structure, we get $\mathrm{CO}_2^+ = \mathrm{SO}_2 \times \mathbb{R}^{>0}$. This is isomorphic to $\mathbb{C}^\times = \mathrm{GL}_1(\mathbb{C})$: an element of $\mathrm{SO}_2 \times \mathbb{R}^{>0}$ is rotation through some angle θ and a positive number r; this is sent to $re^{i\theta} \in \mathbb{C}^\times$.

This provides another perspective on why an orientation and a conformal structure give us a complex structure. ◀

Example 1.22. Now suppose $n = 4$, and choose a conformal structure C and an orientation on V. Then orthogonal makes sense, though orthonormal doesn't, and the Hodge star induces a $\mathbb{Z}/2$-grading on $\Lambda^2 V^* = \Lambda^2_+ V^* \oplus \Lambda^2_- V^*$, the self-dual and anti-self-dual 2-forms. The total space $\Lambda^2 V^*$ is six-dimensional, and these two subspaces are each three-dimensional.

Suppose e^1, e^2, e^3, e^4 is an orthonormal basis for some inner product in C. We can use these to define bases of $\Lambda^2_\pm V^*$, given by

$$\begin{aligned} \alpha_1^\pm &\coloneqq e^1 \wedge e^2 \pm e^3 \wedge e^4 \\ \alpha_2^\pm &\coloneqq e^1 \wedge e^3 \mp e^2 \wedge e^4 \\ \alpha_3^\pm &\coloneqq e^1 \wedge e^4 \pm e^2 \wedge e^3. \end{aligned} \tag{1.23}$$

Now, what symmetry groups do we have? Inside $\mathrm{GL}_4(\mathbb{R})$, preserving an orientation lands in the subgroup $\mathrm{GL}_4^+(\mathbb{R})$; preserving a conformal structure lands in $\mathrm{O}_4 \times \mathbb{R}^{>0}$; and preserving both lands in $\mathrm{SO}_4 \times \mathbb{R}^{>0}$. The first three of these act irreducibly on $\Lambda^2(\mathbb{R}^4)^*$, but the action of $\mathrm{SO}_4 \times \mathbb{R}^{>0}$ has two irreducible summands, $\Lambda^2_\pm(\mathbb{R}^4)^+$.

To understand this better, we should learn a little more about SO_4. Recall that Sp_1 is the Lie group of unit quaternions. This is isomorphic to SU_2, the group of determinant-1 unitary transformations of $\mathbb{C}^2$. This group has an irreducible 3-dimensional representation ρ in which Sp_1 acts by conjugation on the imaginary quaternions (since $\mathbb{R} \subset \mathbb{H}$ is preserved by this action).

Remark 1.24. Another way of describing ρ is: let ρ' denote the action of SU_2 on $\mathbb{C}^2$ by matrix multiplication. Then $\rho \cong \mathrm{Sym}^2 \rho'$. ◀

Proposition 1.25. *There is a double cover* $\mathrm{Sp}_1 \times \mathrm{Sp}_1 \to \mathrm{SO}_4$*. Under this cover, the* SO_4*-representation* $\Lambda^4_\pm(\mathbb{R}^4)^*$ *pulls back to a real three-dimensional representation in which one copy of* Sp_1 *acts by* ρ *and the other acts trivially.*

Proof. Let W' and W'' be two-dimensional Hermitian vector spaces with compatible quaternionic structures J', resp. J''.[1] Then, $V := W' \otimes_{\mathbb{C}} W''$ has a real structure $J' \otimes J''$: two minuses make a plus, and compatibility of J' and J'' means the real points of V have an inner product. (These kinds of linear-algebraic spaces are things you should prove once in your life.)

By tensoring symmetries we obtain a homomorphism $\mathrm{Sp}(W') \times \mathrm{Sp}(W'') \to \mathrm{O}(V)$. This factors through $\mathrm{SO}(V) \hookrightarrow \mathrm{O}(V)$, which you can see for two reasons:

- $\mathrm{Sp}(W')$ and $\mathrm{Sp}(W'')$ are connected, so this homomorphism must factor through the identity component of $\mathrm{O}(V)$, which is $\mathrm{SO}(V)$; or
- a complex vector space has a canonical orientation, and using this we know these symmetries are orientation-preserving.

Now we want to claim this map is two-to-one. One can quickly check that $(-1, -1)$ is in the kernel; the rest is an exercise. ◁

Since Spin_n is the double cover of SO_n, this is telling us $\mathrm{Spin}_4 = \mathrm{Sp}_1 \times \mathrm{Sp}_1$. This splitting is the genesis of a lot of what we'll do in the next several lectures.

Consider the 16-dimensional space

$$V^* \otimes V^* = (W')^* \otimes (W')^* \otimes (W'')^* \otimes (W'')^*. \tag{1.26}$$

Because the map

$$\begin{aligned} \omega' \colon W' \times W' &\longrightarrow \mathbb{C} \\ \xi', \eta' &\longmapsto h'(J'\xi', \eta') \end{aligned} \tag{1.27}$$

is skew-symmetric, it lives in $\Lambda^2(W')^* \subset (W')^* \otimes (W')^*$. In particular, the embedding

$$\mathrm{Sym}^2(W')^* \oplus \mathrm{Sym}^2(W'')^* \hookrightarrow (W')^* \otimes (W')^* \otimes (W'')^* \otimes (W'')^* \tag{1.28}$$

is the map sending

$$\alpha, \beta \longmapsto \alpha \otimes w'' + \omega' \otimes \beta. \tag{1.29}$$

Remark 1.30. This story can be interpreted in terms of representations of $\mathrm{Sp}(W') \times \mathrm{Sp}(W'')$. Let $\mathbf{1}$ denote the trivial representation of Sp_1 and $\mathbf{3}$ be the three-dimensional irreducible representation we discussed above. Then (1.26) enhances to

$$V^* \otimes V^* = \mathbf{1}_{\mathrm{Sp}(W')} \otimes \mathbf{3}_{\mathrm{Sp}(W')} \otimes \mathbf{1}_{\mathrm{Sp}(W'')} \otimes \mathbf{3}_{\mathrm{Sp}(W'')}. \tag{1.31}$$

The skew-symmetric part is $\mathbf{3}_{\mathrm{Sp}(W')} \otimes \mathbf{1}_{\mathrm{Sp}(W'')} \oplus \mathbf{1}_{\mathrm{Sp}(W')} \otimes \mathbf{3}_{\mathrm{Sp}(W'')}$, and the "rest" (complement) is symmetric. ◁

The group $\mathrm{Sp}_1 \times \mathrm{Sp}_1 = \mathrm{Spin}_4$ has complex (quaternionic) two-dimensional representations $S^\pm$, the *spin representations*, and $\Lambda^2_\pm V \cong \mathrm{Sym}^2 S^\pm$.

So two-forms have self-dual and anti-self-dual parts, and curvature is a natural source of 2-forms! ◁

[1] That is, J' is an antilinear endomorphism of W' squaring to -1, and similarly for J''. Compatible means with the Hermitian metric: h is a map $\overline{W} \times W \to \mathbb{C}$ and J is a map $W \to \overline{W}$, and if $\xi, \eta \in W'$, we want

$$h(J'\xi, \overline{J'\eta}) = \overline{h(\xi, \eta)} \qquad \text{and} \qquad h(J\xi, \eta) = -h(J\eta, \xi).$$