FALL 2018 HOMOTOPY THEORY SEMINAR

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Contents

1. Overview: 9/5/18

This short overview was given by Richard.

In the beginning, there were homotopy groups $\pi_n(X) := [S^n, X]$. Homotopy theory begins with the study of these groups, which are hard to calculate. Even the homotopy groups of the spheres, $\pi_k(S^n)$, are complicated. However, there are patterns.

Theorem 1.1 (Freudenthal suspension theorem). *For $n \geq k+2$, $\pi_{k+n}(S^n) \cong \pi_{k+n+1}(S^{n+1})$.*

The first few of these stable homotopy groups are $\pi_n(S^n) = \mathbb{Z}$, $\pi_{n+1}(S^n) = \mathbb{Z}/2$, $\pi_{n+2}(S^n) = \mathbb{Z}?2$, $\pi_{n+3}(S^n) = \mathbb{Z}/24$, $\pi_{n+4}(S^n) = \pi_{n+5}(S^n) = 0$, $\pi_{n+6}(S^n) = \mathbb{Z}/2$, and $\pi_{n+7}(S^n) = \mathbb{Z}/120$.

You can encode all of this stability data in one place using spectra. There's an object $\mathbb{S}$ called the *sphere spectrum* built in a precise way from spheres, and the homotopy groups of $\mathbb{S}$ are the stable homotopy groups of the spheres.

These stable homotopy groups are very hard to calculate. However, we can work locally (at primes), which simplifies the problem a little bit.

Theorem 1.2 (Fracture square). *Let X be a space, $X_\mathbb{Q}$ be its rationalization, and for p a prime let X_p denote the p-completion of X. Then the following square is a homotopy pullback:*

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & X_\mathbb{Q} \arrow[d] \\
\prod\limits_{p \text{ prime}} X_p \arrow[r] & \left(\prod\limits_{p \text{ prime}} X_p\right)_\mathbb{Q}.
\end{tikzcd}$$

Here $\pi_*(X_p) = \pi_*(X) \otimes \mathbb{Z}_p$ and $\pi_*(X_\mathbb{Q}) \cong \pi_*(X) \otimes \mathbb{Q}$. The upshot of Theorem 1.2 is that these groups determine the original homotopy groups of X.

The rational homotopy groups of spheres are known, due to an old theorem of Serre. Over p, there are other techniques, such as the Adams and Adams-Novikov spectral sequences. The Adams-Novikov spectral sequences uses a filtration on X_p to produce a spectral sequence with E_2-term

$$E_2^{*,*} = \mathrm{Ext}_{BP_*BP}(BP_*, BP_*(X)), \tag{1.3}$$

and converging to $\pi_*(X)_{(p)}$ (p-local, not p-complete!). Here BP is a spectrum, but you don't actually need to know much about it (yet): BP_* is some algebra, and BP_*BP is a Hopf algebra, and they can be described explicitly. We'll learn more about this spectral sequence in time.

If you look at a picture of the E_∞-page of the Adams-Novikov spectral sequence for any p (maybe just p odd for now), there are strong patterns: a pattern along the bottom, which is the α-family (said to be v_1-periodic), and some periodic things along the diagonal (said to be v_2-periodic), containing the β-family. Both of these are families in the homotopy groups of spheres, providing structue in the complicated story — we don't know the stable homotopy groups of spheres past about 60, so producing families is very helpful for our understanding! In a similar way, one can find v_3-periodic elements, including something called the γ-family, and so forth.

Of course, there's a lot of work to do even from here: how to we get here from the E_2-page? Do the extension problems go away, giving us actual elements of the stable stem? For the α-, β-, and γ-families, these are known, and there are even geometric interpretations for small n (up to 3 or 4) and large p (usually something like $p > 5$ or $p > 7$). Specifically, if $V(0)$ denotes cofiber of the multiplication-by-p map $\mathbb{S} \to \mathbb{S}$, the α-family comes from self-maps $\Sigma^k V(0) \to V(0)$, together with the maps to and from $\Sigma^k \mathbb{S}$ coming from the cofiber sequence. There are less explicit complexes $V(1)$ and $V(2)$ which give you the β- and γ-families, and there is a similar story.

2. Introduction to spectra: 9/12/18

I unfortunately missed Rok's talk, but he gave the last 10 minutes as the first 10 minutes of the second week, so here it is.

Recall that a spectrum X is a sequence of pointed spaces $\{X_n\}_{n\in\mathbb{Z}}$ together with weak equivalences $X_n \simeq \Omega X_{n+1}$. There's a functor Σ^∞ from spaces to spectra which turns several topological concepts into algebraic ones that make Sp behave like the derived category $\mathscr{D}(R)$ of R-modules for R a commutative ring. Here's a dictionary:

- Σ^∞pt is the *zero spectrum*, which corresponds to the *zero complex* of R-modules (zero in every degree).
- $\Sigma^\infty S^0$, denoted $\mathbb{S}$, is the *sphere spectrum*, which corresponds to R as an R-module.
- Suspension of spaces is sent to suspension of spectra, which corresponds to the shift functor $[1]$ of a derived category.
- The (based) loop space functor Ω maps to *de*suspension of spectra, which corresponds to the shift functor $[-1]$ in the derived category.
- Wedge sum of spaces turns into wedge sum of spectra, which can be thought of as a direct sum, and corresponds to the direct sum of complexes of R-modules.
- Smash product of spaces turns into smash product of spectra, which is their tensor product, and corresponds to the derived tensor product ${}^{\mathbf{L}}\!\otimes_R$ of complexes.
- Stable homotopy groups of spaces map to homotopy groups of spectra, which behave like cohomology groups in the derived category.

There's a homotopical reason to believe this analogy between spectra and the derived category: the Eilenberg-Mac Lane functor $H\colon \mathsf{Ab} \to \mathsf{Sp}$ induces an equivalence between the (homotopy or $(\infty,1)$) categories Mod_{HR} of R-module spectra and $\mathscr{D}(R)$ which sends smash product over HR to the derived tensor product over R.

The sphere spectrum is the unit for the smash product, so we can think of spectra as the category of $\mathbb{S}$-modules, which is a very useful, and sometimes literaly, analogy.

Spectra define cohomology theories: if E is a spectrum and X is a space (non-pointed), then the associated cohomology theory is defined by $E^i(X) := [X, \Sigma^i E]$.

3. Spectral sequences: 9/17/18

Here's Ricky's talk on spectral sequences, followed (TODO) by notes from Arun's part of the talk.

Let $C = \bigoplus_{n=0}^\infty C^n$ be a graded R-module and assume it has a decreasing filtration by chain maps

$$C \supseteq \cdots \supseteq F^p C \supseteq F^{p+1} C \supseteq \cdots, \tag{3.1}$$

meaning that d carries $F^p C^{p+q}$ into $F^p C^{p+q+1}$. (Upper indices typically correspond to decreasing filtrations.) Let's assume for now that

- $R = k$ is a field, and
- for each n, $F^\bullet C^n$ is finite.

Then there's a filtration on cohomology, where

$$F^p H^*(C) := \operatorname{Im}(H^*(F^p C \hookrightarrow C)) = \pi(\underbrace{F^p C^{p+q} \cap \ker(d)}_{Z_\infty^{p,q}}), \tag{3.2}$$

where $\pi\colon \ker(d) \to \ker(d)/\operatorname{Im}(d) = H^{p+q}(C)$ is the quotient map. Because

$$F^p H(C)/F^{p+1} H(C) = \pi(Z_\infty^{p,q})/\pi(Z_\infty^{p+1,q-1}) = Z_\infty^{p,q}/(Z_\infty^{p+1,q-1} + B_\infty^{p,q}), \tag{3.3}$$

where $B_\infty^{p,q} := F^pC^{p+q} \cap \mathrm{Im}(d)$.

Let $E_0^{p,q} := F^pC^{p+q}/F^{p+1}C^{p+q}$; then, the differentials induce maps $E_0^{p,q-1} \to E_0^{p,q} \to E_0^{p,q+1}$, and they satisfy $d_0^2 = 0$ because we originally had $d^2 = 0$. Then

$$\frac{\ker(d_0)}{\mathrm{Im}(d_0)} = \frac{F^pC^{p+q} \cap d^{-1}(F^{p+1}C^{p+q+1})}{\underbrace{F^pC^{p+q} \cap d(F^pC^{p+q-1})}_{B_0^{p,q}} + \underbrace{F^{p+1}C^{p+q}}_{Z_0^{p,q-1}}} = \frac{Z_1^{p,q}}{B_0^{p,q} + Z_0^{p,q-1}}. \tag{3.4}$$

Define

$$\begin{aligned} Z_r^{p,q} &:= F^pC^{p+q} \cap d^{-1}(F^{p+r}C^{p+q+1}) \\ B_r^{p,q} &:= F^pC^{p+q} \cap d(F^{p-r}C^{p+q-1}) \\ E_r^{p,q} &:= Z_r^{p,q}/(Z_{r-1}^{p,q-1} + B_{r-1}^{p,q}). \end{aligned} \tag{3.5}$$

The key claim is that

$$H^*(E_r^{p,q}, d_r) = E_{r+1}^{p,q}. \tag{3.6}$$

A spectral sequence is, roughly speaking, something which behaves like this.

Definition 3.7. A *(cohomologically graded) spectral sequence* is a collection $\{E_r^{\bullet,\bullet}, d_r\}$ of differentially bigraded modules such that d_r has bidegree $(r, 1-r)$ and such that $E_{r+1}^{p,q} = H^*(E_r^{p,q}, d_r)$. If $E_r^{p,q}$ is constant in r when p and q are fixed after some finite number of pages r, then we also call it $E_\infty^{p,q}$.

The spectral sequence *converges* to (H^*, F), a filtered graded R-module, if $E_\infty^{p,q}$ is the associated graded of H^*. This implies H^r is a direct sum of $E_\infty^{p,q}$ over all $p+q=r$.[1]

Sometimes spectral sequences have more structure given by multiplication. In this case, we want each $E_r^{\bullet,\bullet}$ to be a *differential bigraded R-algebra*, meaning it has a multiplication map which is additive on bidegrees of homogeneous elements, and that the differential obeys a graded Leibniz rule with respect to total grading:

$$d(xy) = d(x)y + (-1)^{|x|}xd(y). \tag{3.8}$$

Suppose we took the spectral sequence of a filtered R-module above, but it's also an R-algebra. Unfortunately, the higher pages in the spectral sequence aren't R-algebras without some work (TODOI missed this).

The Serre spectral sequence. Here's Arun's example with the Serre spectral sequence.[2]

Definition 3.9. A *(Serre) fibration* $f : E \to X$ of topological spaces is a map such that if Δ^n denotes the n-simplex and one has commuting maps

$$\begin{tikzcd} \Delta^n \times \{0\} \arrow[r] \arrow[d] & E \arrow[d, "f"] \\ \Delta^n \times [0,1] \arrow[r] & X, \end{tikzcd}$$

there exists a map $G \colon \Delta^n \times [0,1] \to E$ that commutes with the maps in the diagram.

We always take X to be path-connected, in which case $f^{-1}(x) \simeq f^{-1}(x')$ for all $x, x' \in X$. This preimage is called the *fiber* of f, and is often denoted F; the triple $F \to E \to X$ is called a *fiber sequence*. We will also assume X is simply connected, which will allow us to obtain stronger results.

Example 3.10. Let M be a manifold of dimension n. Then, $TM \to M$ is a fibration, and the fiber is $\mathbb{R}^n$. ◁

Theorem 3.11 (Serre). *Fix a coefficient ring R; let $f : E \to X$ be a fibration and F be its fiber. Then, there exists a multiplicative spectral sequence, called the* Serre spectral sequence

$$E_2^{p,q} = H^p(X; H^q(F;R)) \Longrightarrow H^{p+q}(E;R).$$

Proof sketch. Let $\{X_i\}$ be the CW filtration of X, and let $E_i := f^{-1}(X_i)$, which induces an exhaustive filtration $\{E_i\}$ of E. Applying $H^q(-;R)$ defines a spectral sequence by the formalism from the first lecture. The multiplicative structure comes from the cup product on X. ⊠

[1] If R isn't a field, then it might instead be an extension that doesn't split.

[2] I learned this example from Ernie Fontes, and this presentation is adapted from his presentation of this example.

Remark 3.12. Let A be a multiplicative generalized cohomology theory (e.g. K-theory). Then, we could have applied A instead of $H^q(-;R)$ and obtained a multiplicative spectral sequence

$$E_2^{p,q} = H^p(X; A^q(F)) \Longrightarrow A^{p+q}(E).$$

Letting $A = H^*(-,R)$, we recover the Serre spectral sequence, and letting $E \to X$ be the identity map $X \to X$, which is a fibration, we recover the Atiyah-Hirzebruch spectral sequence. For this reason this spectral sequence is sometimes called the *Serre-Atiyah-Hirzebruch spectral sequence*. ◄

Example 3.13. Let $PX := \mathsf{Top}_*(I, X)$ denote the *path space*, i.e. the maps from the unit interval to X. Evaluation at 0 defines a map $ev\colon PX \to X$. The path space is made of spaghetti, hence is contractible: shrink each path from time 1 to time t, and let $t \to 0$.

$ev\colon PX \to X$ is a fibration, and the fiber is ΩX, the space of (based) loops in X (i.e. based maps $S^1 \to X$). Recall that a fiber sequence induces a long exact sequence in homotopy groups:

$$\cdots \longrightarrow \pi_n(F) \longrightarrow \pi_n(E) \longrightarrow \pi_n(X) \longrightarrow \pi_{n-1}(F) \longrightarrow \cdots \tag{3.14}$$

Since $\pi_n(PX) = 0$, this implies $\pi_n(X) \cong \pi_{n-1}(\Omega X)$.

Let's apply the Serre spectral sequence to this fibration in the case where $R = \mathbb{Q}$ and $X = S^3$. The Serre spectral sequence takes the form

$$E_2^{p,q} = H^p(S^3; H^q(\Omega S^3; \mathbb{Q})) \Longrightarrow H^{p+q}(PS^3; \mathbb{Q}).$$

We know the E_∞ page already: it's 0 unless $p + q = 0$, in which case it's $\mathbb{Q}$. So we're going to reverse-engineer the spectral sequence, to use the E_∞ page to compute the E_2 page.

We also know $H^*(S^3; \mathbb{Q}) = E_{\mathbb{Q}}(X)$, where $|x| = 3$, an exterior algebra in one variable. This is also isomorphic to $\mathbb{Q}[x]/x^2$, so has a $\mathbb{Q}$ in degrees 0 and 3, and is 0 elsewhere.

We know $H^0(\Omega S^3; \mathbb{Q}) = \mathbb{Q}$, so the E_2 page looks like

3	?			?
2	?			?
1	?			?
0	1			x
	0	1	2	3

with the missing entries equal to 0.

We know that the $(3, 0)$ term has to vanish by the E_∞ page, so it either *supports a differential* (has a nonzero differential mapping out of it) or *receives a differential* (has a nonzero differential mapping into it). Since this is a first-quadrant spectral sequence, all differentials out of x hit 0, so it has to receive a differential. But on the E_2 page, this differential comes from the 0 in position $(1, 1)$, so it's zero, and any differentials in page 4 or above mapping into x come from the fourth quadrant, so there has to be a nonzero differential on the E_3 page mapping into x, so there's some $y \in E_2^{0,2}$, which generates a copy of $\mathbb{Q}$, such that $d_3 y = x$. There can't be more than one generator in $E_2^{0,2}$, because then either it would survive to the E_∞ page (which can't happen), or it gets killed, meaning the difference of it and y is not killed by d_3 and hence survives. Oops. Thus, $E_2^{0,2} = H^2(\Omega S^3; \mathbb{Q}) \cong \mathbb{Q}$.

Hence we know $E_2^{3,2} = H^3(S^3;\mathbb{Q})$ as well, and the spectral sequence looks like

$$\begin{tikzcd}
2 & y \arrow[ddrrr, "d_3"] & & & \mathbb{Q} \\
1 & ? & & & ? \\
0 & 1 & & & x. \\
& 0 & 1 & 2 & 3
\end{tikzcd}$$

We can also immediately determine $E_2^{\bullet,2}$: looking at $E_2^{0,2}$, there are no differentials that hit something nonzero, or map from something nonzero to it. So anything in it survives to the E_∞ page, and hence it must be zero. Thus $H^1(\Omega S^3;\mathbb{Q}) = 0$ and hence $E_2^{1,3} = 0$ too.

The multiplicative structure tells us that the generator of $E_2^{3,2}$ must be $y \cdot x$. Thus, the spectral sequence looks like

$$\begin{tikzcd}
2 & y \arrow[ddrrr, "d_3"] & & & yx \\
1 & & & & \\
0 & 1 & & & x. \\
& 0 & 1 & 2 & 3
\end{tikzcd}$$

But now yx has to die, and the only way that can happen is if it's hit by d_3 of the $E_2^{0,4}$ term, which turns out to be y^2. This is because $d_3 y = x$, so

$$d_3(y^2) = d_3(y)y + (-1)^2 y d_3(y) = 2xy.$$

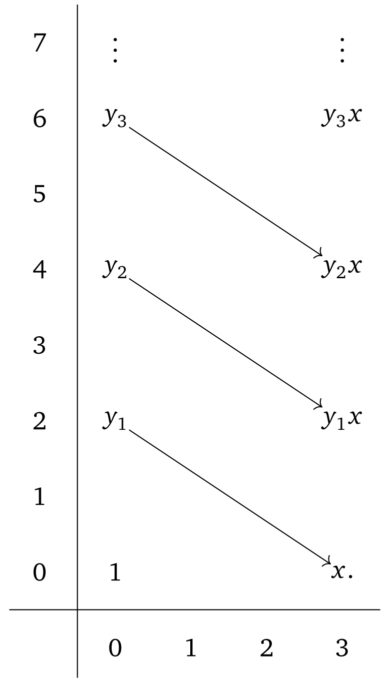
Thus d_3 is multiplication by 2. Hence the spectral sequence looks like

$$\begin{tikzcd}
4 & y^2 \arrow[ddrrr, "\cdot 2"] & & & y^2x \\
3 & & & & \\
2 & y \arrow[ddrrr, "\cdot 1"] & & & yx \\
1 & & & & \\
0 & 1 & & & x. \\
& 0 & 1 & 2 & 3
\end{tikzcd}$$

But now we need y^2x to vanish, and it's hit by $y^3 \in E_2^{0,6}$ via d_3, which is multiplication by 3, and so on. Inductively we can conclude that

$$H^*(\Omega S^3;\mathbb{Q}) \cong \begin{cases} \mathbb{Q}, & n \text{ even} \\ 0, & n \text{ odd}. \end{cases}$$

Much of this argument, but not all of it, works with $\mathbb{Q}$ replaced by $\mathbb{Z}$. The difference is that multiplication by 2, 3, etc. is not an isomorphism, so we have a sequence of generators $y_1, y_2, \dots$:



Now we have to figure out the multiplicative structure. We know $y_1^2 = c_1 y_2$ for some $c_1 \in \mathbb{Z}$, so since d_3 is an isomorphism, let's compute: we know $d_3(y_2) = y_1 x$ by construction, and $d_3(y_1^2) = 2y_1 x$ for the same reason as over $\mathbb{Q}$, so $y_1^2 = 2y_2$.

A similar calculation in general shows that $y_1^n = n! y_n$, as

$$\begin{aligned} d_3(y_1^n) = d_3(y_1 y_1^{n-1}) &= d_3(y_1)y_1^{n-1} + y_1(n-1)!d(y_{n-1}) \\ &= xy_1^{n-1} + y_1(n-1)!xy_{n-2} \\ &= x(n-1)!y_{n-1} + (n-1)y_{n-1}x(n-1)! \\ &= n!xy_{n-1}, \end{aligned}$$

but $d_3(n!y_n) = n!xy_{n-1}$. Hence the ring structure on $H^*(\Omega S^3)$ is a divided power algebra.

Definition 3.15. A *divided power algebra* on a single generator x in degree k, denoted $\Gamma(x)$, is the free algebra generated by $\{x_i\}_{i\geq 1}$ where $|x_i| = ki$, subject to the relations

$$x_i x_+ j = \binom{i+j}{j} x_{i+j} \qquad \text{and} \qquad x_i = \frac{x^i}{i!}.$$

Thus $H^*(\Omega S^3) \cong \Gamma(y)$ with $|y| = 2$. ◄

4. First steps with the Adams spectral sequence: 9/24/18

Today's talk was given by Riccardo and Alberto.

Fix R a commutative ring and M an R-module.

Definition 4.1. A *left exact* functor $F\colon \mathsf{Mod}_R \to \mathsf{Ab}$ is a functor which sends a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \tag{4.2}$$

to an exact sequence

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C), \tag{4.3}$$

which may not necessarily complete to an exact sequence.

The easiest example of a left exact functor which isn't exact is $\mathrm{Hom}_R(-,M)$ for certain choices of M.

Lemma 4.4. *With R and M as above, $\mathrm{Hom}_R(-,M)$ is exact iff M is projective.*

So if we'd like to understand what happens when we hit exact sequences with $\mathrm{Hom}_R(-,M)$ for M not projective, it would be good to approximate M by projectives.

Definition 4.5. A *projective resolution* of M is an exact sequence

$$\cdots \longrightarrow P_j \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

written $P_\bullet \to M$, such that each P_j is projective.

Lemma 4.6 (Fundamental lemma of homological algebra)**.** *Any two projective resolutions of M are chain homotopy equivalent.*

This makes the following definition independent of $P_\bullet \to M$.

Definition 4.7. Let N be another R-module. The i^{th} *Ext group* is $\mathrm{Ext}^i_R(M,N) := H^i(\mathrm{Hom}(P_\bullet, N))$, where $P_\bullet \to M$ is a projective resolution.

Theorem 4.8. *Let R, M, and N be as above.*

(1) $\mathrm{Ext}^0_R(M,N) = \mathrm{Hom}_R(M,N)$.

(2) *A short exact sequence $0 \to A \to B \to C \to 0$ of R-modules induces a long exact sequence*

$$0 \longrightarrow \mathrm{Hom}_R(M,A) \longrightarrow \mathrm{Hom}_R(M,B) \longrightarrow \mathrm{Hom}_R(M,C) \xrightarrow{\delta} \mathrm{Ext}^1_R(M,A) \longrightarrow \dots \tag{4.9}$$

with natural maps $\mathrm{Ext}^i_R(M.C) \to \mathrm{Ext}^{i+1}_R(M,A)$.

Now we assume R is a graded ring and M is a graded R-module. We will use Σ^r to denote shift by R, i.e. $\Sigma^r M$ is the graded R-module with $(\Sigma^r M)^t := M^{t-r}$.

Example 4.10. In this setting $\mathrm{Hom}_R(M,N)$ is also a graded object, with $\mathrm{Hom}^i_R(M,N) := \mathrm{Hom}_R(M,\Sigma^i N)$ (the latter are degree-preservving maps). ◀

This implies Ext is *bi*graded: $\mathrm{Ext}^{r,s}_R(M,N) :=$TODO. There's a pairing called the Yoneda product on Ext groups, which has signature

$$\mathrm{Ext}^{s,t}_R(M,N) \otimes \mathrm{Ext}^{s,t}_R(L,N) \longrightarrow \mathrm{Ext}^{s+s',t+t'}_R(L,N). \tag{4.11}$$

The Adams spectral sequence involves bigraded Ext for a specific choice of R, so let's turn to that choice of R.

Definition 4.12. A *cohomology operation of degree k*[3] is a natural transformation $\gamma\colon H^*(-,\mathbb{F}_2) \to H^{*+k}(-;\mathbb{F}_2)$. If it commutes with the suspension isomorphism, we say γ is *stable*.

Definition 4.13. The *Steenrod algebra $\mathscr{A}$* is the graded, noncommutative, infinitely generated $\mathbb{F}_2$-algebra of stable cohomology operations: in degree k it is the degree-k stable cohomology operations.

Since $H^n(-;\mathbb{F}_2) \cong [-, K(\mathbb{F}_2, n)]$, and these Eilenberg-Mac Lane spaces are the constituents in the Eilenberg-Mac Lane spectrum $H\mathbb{F}_2$, then essentially by the Yoneda lemma, $\mathscr{A} \cong H\mathbb{F}_2^*(H\mathbb{F}_2)$. This implies no stable cohomology operations of negative degre exist (since $H\mathbb{F}_2$ is connective).

Theorem 4.14. *For all $k \geq 0$, there is a stable cohomology operation Sq^k of degree k with the following properties:*

- $\mathrm{Sq}^0 = \mathrm{id}$ *and* Sq^1 *is the* Bockstein*, the natural transformation $H^*(-;\mathbb{Z}/2) \to H^{*+1}(-;\mathbb{Z}/2)$ coming from the connecting morphism in the long exact sequence in cohomology induced from the short exact sequence* $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$.
- *If $x \in H^k(X;\mathbb{Z}/2)$, then $\mathrm{Sq}^k(x) = x^2$.*
- *If $x \in H^i(X;\mathbb{Z}/2)$ and $i < k$, then $\mathrm{Sq}^k(x) = 0$.*
- *(Cartan formula)*
$$\mathrm{Sq}^k(x \smile y) = \sum_{i+j=k} \mathrm{Sq}^i(x)\mathrm{Sq}^j(y).$$

The Steenrod algebra is generated by these elements, and these properties characterize them.

[3]In general one can consider other coefficient groups than $\mathbb{F}_2$.

In fact, these generators have redundancies: $\mathscr{A}$ is generated by Sq^{2^i} for $i \geq 0$.

Example 4.15. We can use this to show the Hopf fibration $\eta\colon S^3 \to S^2$ is nontrivial. This is the quotient of S^3 by the U_1-action on it as the unit sphere in $\mathbb{C}^2$; the quotient is $\mathbb{CP}^1$, also known as S^2. It suffices to know that the cofiber of η, which has the homology of $S^3 \wedge S^2$, isn't homotopic to $S^3 \wedge S^2$, and you can check this by showing its cohomology has a different $\mathscr{A}$-module structure. ◂

This data all enters into a spectral sequence called the *Adams spectral sequence*. Fix spaces (or spectra) X and Y; then, the spectral sequence has E_2-page

$$E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(Y), H^*(X)), \tag{4.16}$$

and which converges to $[X, Y_{(2)}^\vee]_{t-s}$. This means stable homotopy classes of maps between X and the 2-completion of Y. (There are analogues of this, and of the Steenrod algebra, over other primes.) This completion on groups gives you $\varprojlim_n G/2^n$, and does something similar for spaces.

If $X = Y$, the Yoneda product on $\mathrm{Ext}_{\mathscr{A}}^{s,t}$ induces a product on the E_2-page of the Adams spectral sequence.

Since $\mathscr{A}$ isn't finitely generated, the Adams spectral sequence is complicated, but there's a clever simple application using connective *ko*-theory (a version of *KO*-theory with no nonzero negative homotopy groups). One can compute that

$$H^*(ko; \mathbb{Z}/2) \cong \mathscr{A} \otimes_{\mathscr{A}(1)} \mathbb{Z}/2, \tag{4.17}$$

where $\mathscr{A}(1) = \langle \mathrm{Sq}^0, \mathrm{Sq}^1, \mathrm{Sq}^2 \rangle$ inside $\mathscr{A}$. The change-of-rings formula for Hom induces a change-of-rings formula for Ext:

$$\mathrm{Ext}_{\mathscr{A}}^{s,t}(\mathscr{A} \otimes_{\mathscr{A}(1)} \mathbb{Z}/2, \mathbb{Z}/2) \cong \mathrm{Ext}_{\mathscr{A}(1)}^{s,t}(\mathbb{Z}/2, \mathbb{Z}/2). \tag{4.18}$$

This is much nicer: $\mathscr{A}(1)$ is 8-dimensional, making all of the algebra simpler. Moreover, there's a traditional diagrammatic way to describe $\mathscr{A}(1)$-module structures, in which Sq^1-actions are given by straight lines and Sq^2-actions are given by curly lines. For example, $\mathscr{A}(1)$ is drawn in Figure 1.

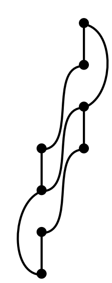


FIGURE 1. The algebra $\mathscr{A}(1)$: the vertical stratification is the degree, the straight lines are Sq^1, and the curvy lines are Sq^2.

For example, we can draw a projective resolution of $\mathbb{Z}/2$ as an $\mathscr{A}(1)$-module (on the board, but not really live-TeXable in time). If you work out a few terms, you'll see that there's a pattern of the kernel, so the terms in the resolution are always of the form $\Sigma^{m_1}\mathscr{A}(1) \oplus \Sigma^{m_2}\mathscr{A}(1)$. Since

$$\mathrm{Hom}^s(\Sigma^r \mathscr{A}(1), \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & r = s \\ 0, & \text{otherwise,} \end{cases} \tag{4.19}$$

passing to the E_2-page is relatively simple once you have the resolution. Looking at a picture of the E_2-page, one sees infinitely many dots for $t - s = 0$ or 4 (or 8, etc.), one dot each in $t - s = 1, 2$, and $9, 10$, etc., and no places where there could be nontrivial differentials. Therefore, if you can resolve an extension problem you've proven Bott periodicity for *ko*-theory.

Recall that we've seen two spectral sequences so far: the Serre spectral sequence, with signature

$$E_2^{p,q} = H^p(B; H^q(F)) \Longrightarrow H^{p+q}(F), \tag{5.1}$$

and the Adams spectral sequence, with signature

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathscr{A}_*}(H^*(Y;\mathbb{F}_2),\mathbb{F}_2) \Longrightarrow \pi_*(Y)^\wedge_{(2)}. \tag{5.2}$$

This converges when Y is connective and of finite type. The algera $\mathscr{A}$, called the *Steenrod algebra*, is $[H\mathbb{F}_2, H\mathbb{F}_2] = H\mathbb{F}_2^* H\mathbb{F}_2$.

The Adams spectral sequence is pretty amazing, and it would be nice to generalize it. There are versions over other primes, using $\mathscr{A}_p := H\mathbb{F}_p^* H\mathbb{F}_p$, but these are also kind of messy. One idea is to dualize everything: let $\mathscr{A}^\vee := \pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p) = H\mathbb{F}_{p*} H\mathbb{F}_p$, and to use the homology of Y instead.

We built the Serre spectral sequence from the Postnikov tower for the total space, which is a "resolution" of the space by spaces in which we've killed off homotopy groups. Dual to that, there's an *Adams tower* which kills off cohomology: starting with Y, define spaces $\cdots \to Y_2 \to Y_1 \to Y$ such that

1. the cofiber Ci_j of the map $i_j \colon Y_{j+1} \to Y_j$ is a wedge of $H\mathbb{F}_p$s, and
2. such that the induced map on cohomology $H^*(Ci_j) \to H^*(Y_j)$ is an epimorphism.

Exercise 5.3. In this setting, the sequence

$$0 \longleftarrow H^*(Y) \longleftarrow H^*(Ci_0) \longleftarrow H^*(\Sigma Ci_1) \longleftarrow H^*(\Sigma^2 Ci_2) \longleftarrow \cdots \tag{5.4}$$

is a resolution of $H^*(Y)$ as $\mathscr{A}$-modules.

Proposition 5.5. *Adams towers exist for all Y.*

Proof. Let $\overline{H\mathbb{F}_p}$ be the fiber of the unit map $\epsilon \colon \mathbb{S} \to H\mathbb{F}_p$. Then we can let

$$\begin{aligned} Y_s &:= (\overline{H\mathbb{F}_p})^{\wedge s} \wedge Y_0 \\ Ci_s &:= H\mathbb{F}_p \wedge Y_s, \end{aligned} \tag{5.6}$$

and check that these satisfy the criteria. ⊠

Remark 5.7. Via some cosimplicial nonsense,[4] Adams towers for Y are equivalent to cosimplicial resolutions of Y, which is a kind of Dold-Kan correspondence. The cosimplicial resolution corresponding to the Adams tower we described above is

$$\mathrm{Tot}_n(CB^\bullet(H\mathbb{F}_p) \wedge Y), \tag{5.8}$$

where $CB^\bullet(H\mathbb{F}_p)$ is a *cobar construction*:

$$H\mathbb{F}_p \rightrightarrows H\mathbb{F}_p \wedge H\mathbb{F}_p \rightrightarrows H\mathbb{F}_p \wedge H\mathbb{F}_p \wedge H\mathbb{F}_p \rightrightarrows \cdots \tag{5.9}$$

◀

Anyways, taking π_* of our cobar resolution, we get a resolution for $H_*(Y)$, which is a good first step for the generalized Adams spectral sequence.

We want to produce an E-based Adams spectral sequence, where E is a commutative ring spectrum, meaning we want resolutions, an E_2-term which uses Ext, and some nice convergence result.

The maps $E \rightrightarrows E \wedge E$ (unit smash identity, identity smash unit) induce on homotopy groups left and right E_*-actions on $E_*E = \pi_*(E \wedge E)$. This is the first step in the sequence

$$E \rightrightarrows E \wedge E \rightrightarrows E \wedge E \wedge E \rightrightarrows \cdots, \tag{5.10}$$

or smashing with a space X,

$$X \wedge E \rightrightarrows X \wedge E \wedge E \rightrightarrows X \wedge E \wedge E \wedge E \rightrightarrows \cdots. \tag{5.11}$$

Suppose that E_*E is a flat E_*-module. Then the map

$$\mathrm{id} \wedge m \wedge \mathrm{id} \colon (E \wedge E) \wedge_E (E \wedge X) \longrightarrow E \wedge E \wedge X \tag{5.12}$$

[4] Not to be confused with simplicial cononsense.

induces on homotopy groups am isomorphism

$$E_*E \otimes_{E_*} E_*X \cong \pi_*(E \wedge E \wedge X), \tag{5.13}$$

and we don't have to take the derived tensor product! Therefore in this situation we can use the Künneth spectral sequence to compute the left-hand side: if M and N are R-modules,

$$E_2^{p,q} = \mathrm{Tor}^{R_*}(M_*, N_*) \Longrightarrow \pi_*(M \otimes_R N). \tag{5.14}$$

This looks closer to what we want the generalized Adams spectral sequence to look like.

Remark 5.15. For $E = H\mathbb{Z}$ or ku, E_*E is not flat over E_*; however, this does work for $H\mathbb{F}_p$, MU, and BP. ⊲

The pair (E_*, E_*E) is a *Hopf algebroid*: it has maps $E_*E \to E_*$ and $E_* \to E_*E$ together with a *comultiplication map*

$$\Delta\colon E_*E \longrightarrow E_*E \otimes_{E_*} E_*E. \tag{5.16}$$

If E is commutative it's even a commutative Hopf algebroid. There's a little more structure (e.g. an antipode).

Definition 5.17. An (E_*, E_*E)-comodule is a left E_*-module M together with an E_*-linear map $M \to E_*E \otimes_{E_*} M$.

The category Comod_{E_*E} has a cotensor product, and is an abelian category, which allows us to make sense of things such as Ext. This leads eventually to the *Adams-Novikov spectral sequence(s)*, a family of spectral sequences using this idea:

$$\mathrm{Ext}^{*,*}_{MU_*MU}(MU_*, MU_*(X)) \Longrightarrow \pi_*(X), \tag{5.18}$$

or working at a prime p,

$$\mathrm{Ext}_{BP_*,BP_*BP}(BP_*, BP_*(X)) \Longrightarrow \pi_*(X)^\wedge_{(p)}. \tag{5.19}$$

The general E-based Adams spectral sequence for computing $\pi_*(L_E Y)$ has nice convergence properties when

(1) E and Y are both connective,
(2) $\pi_0 E \subset \mathbb{Q}$ or is $\mathbb{Z}/n$, and
(3) E_*E is concentrated in even degrees.

Next week we'll discus multiplicative structures.