GEOMETRY AND STRING THEORY SEMINAR: FALL 2018

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Contents

1. Moduli of flat SL_3-connections and exact WKB: 9/5/18

The first talk this semester was given by Andy Neitzke.

Let C be a thrice-punctured $\mathbb{CP}^3$, say punctured at $\{1, \omega, \omega^2\}$, and let $\mathcal{M}$ denote the moduli space of flat $\mathrm{SL}_3(\mathbb{C})$-connection over C with unipotent holonomy around the punctures; this is an example of a *character variety*. This talk will discuss Andy's work (in progress) with Lotte Hollands on constructing nice coordinate systems on this space, using ideas coming from physics.

Let's start with the simpler case of SL_2, and consider the *Mathieu equation*, a Schrödinger equation with periodic potential. Let $\hbar > 0$; then, the Mathieu equation is

$$\left(-\frac{\hbar^2}{2}\partial_x^2 + \cos x - E\right)\psi(x) = 0. \tag{1.1}$$

Parallel transport (i.e. evolution of solutions) of this equation defines a flat $\mathrm{SL}_2(\mathbb{R})$-connection ∇ on $\mathbb{R}$. You might think it's $\mathrm{GL}_2(\mathbb{R})$, because there are two solutions, but they're related by the Wronskian. Since the potential is periodic, this is a connection on $\mathbb{R}/2\pi\mathbb{Z} = S^1$; now we can ask about its monodromy, or about its eigenvalues (which are easier to write down without making additional choices). In physics, the eigenvalues are known as *quasi-momenta* for a particle moving with respect to this potential.

Let ψ be an eigenfunction with eigenvalue λ. If $E \gg 1$, then $\cos x$ is small, so

$$\psi_\pm(x) \coloneqq \exp\left(\pm i\frac{\sqrt{2E}}{\hbar}x\right) \tag{1.2}$$

is a basis of the solutions. The eigenvalues are

$$\lambda_\pm = \exp\left(\pm 2\pi\frac{\sqrt{2E}}{\hbar}\right), \tag{1.3}$$

and the trace is $2\cos(2\pi\sqrt{2E}/\hbar)$. Then $|\lambda_\pm| = 1$ for all E.

So the trace is periodic in $\sqrt{E}$. If this is close to ± 2, we're in a region called the "gap": ΔE is exponentially small, and so solutions are stable. When the absolute value of the trace is smaller, we're in the "band," where the monodromy is complex. This means that solutions exponentially blow up or exponentially decay.

Remark 1.4. In solid-state physics, one example of periodic potentials are crystals. One can show that bands and gaps correspond to conducting and insulating states. ◂

Because of this application, physicists have developed lots of techniques for studying these systems, which we can adapt to geometry to study the monodromy.

First, let's complexify: let $z = e^{ix}$; then we have a complex Schrödinger equation

$$\left(\hbar^2\partial_z^2 + P(z)\right)\psi = 0, \tag{1.5}$$

where

$$P(z) = \frac{1}{z^3} - \frac{2E - \hbar^2/4}{z^2} + \frac{1}{z}. \tag{1.6}$$

The $\hbar^2/4$ correction isn't that important.

Remark 1.7. You can do this on any Riemann surface as long as P is a holomorphic quadratic differental; this requires choosing a complex projective structure. But the ideas can be gotten across in coordinates. ◂

To understand the monodromy, we need to get at the solutions. The exact WKB method constructs solutions of the form

$$\psi(z) = \exp\left(\frac{1}{\hbar}\int_{z_*}^{z} \lambda \, \mathrm{d}z\right). \tag{1.8}$$

In order to satisfy (1.5), λ must satisfy the *Riccati equation*

$$\lambda^2 + P + \hbar\partial_z\lambda = 0. \tag{1.9}$$

This is easier to solve than the original equation. Namely, to leading order in $\hbar$, $\lambda^2 + P = 0$. We will then plug this back in to get at higher orders in $\hbar$. Specifically, we get

$$\lambda = \sqrt{-P} - \hbar\frac{P'}{4P} + \hbar^2\sqrt{-P}\frac{5(P')^2 - 4PP''}{32P^3} + \cdots \tag{1.10}$$

This naturally lives on the *spectral curve* for the equation, i.e. the Riemann surface for $\sqrt{-P}$, $\Sigma \coloneqq \{y^2+P(z) = 0\}$, a double cover of the original surface.

This isn't the end of the story, though: solutions will have monodromy around the zeros of P. But we also can't have monodromy (TODO: I missed why). Looking more closely at (1.10), it doesn't actually converge: it's just an asymptotic series. But it's still useful; it admits Borel summation for $\hbar > 0$ and away from a locus called the *Stokes graph* $W(P)$.[1]

The Stokes graph cuts the Riemann surface into domains; inside each domain, everything works, and you learn a lot about the solutions. But you can't do anything in a neighborhood of a zero of P, which prevents the paradox we chanced upon earlier. The upshot is that in each domain, there's a canonical basis (up to scaling) of the solution space: the solutions are a line bundle over the spectral curve, together with a connection ∇^{ab} represented by $\hbar\lambda\,\mathrm{d}z$. And there'sa canonical way to glue these line bundles over $W(P)$, to obtain a line bundle $L \to \Sigma$ together with a flat connection. It's almost flat (the monodromy around branch points might be -1).

It's natural to compute the holonomy $X_\gamma \in \mathbb{C}^\times$ around a curve γ, and this has nice properties. As $\hbar \to 0$, the asymptotic series of this is computable, e.g. $X_\gamma \sim \exp(\hbar^{-1}Z_\gamma)$, where $Z_\gamma = \oint_\gamma \sqrt{-P}\,\mathrm{d}z$. In a given example (choose P, draw the spectral network, fix a loop), this is completely concrete. The trace is almost an eigenvalue of the monodromy, but it has to cross one of the lines in $W(P)$, and the formula shows that. Specifically, one gets a term for the cosine and a term responsible for the gaps (and hence can be studied to learn about the gaps).

So any particular picture/problem comes with its own picture and defines a coordinate system.

What changes for SL_3? We need a higher-rank analogue of the Schrödinger equation, which will have two potentials P_2 and P_3:

$$\left(\partial_z^3 + \hbar^{-2}P_2(z)\partial_z + \left(\hbar^{-3}P_3(z) + \frac{1}{2}\hbar^{-2}P_2'(z)\right)\right)\psi(z) = 0. \tag{1.11}$$

There's a higher WKB method to deal with such equations, but let's look at a specific example, in which

$$P_3 = -\frac{u}{(z^3-1)^2} \qquad \text{and} \qquad P_2 = \frac{9\hbar^2}{(z^3-1)^2}. \tag{1.12}$$

The $9\hbar^2$ term in P_2 won't matter for the spectral curve, though we can't completely ignore higher-order terms in $\hbar$.

Now parallel transport of solutions gives us (I think?) a flar SL_3-connection on C. We want to study the connections with $u > 0$. The higher WKB machinery gives you a basis $\{\psi_1, \psi_2, \psi_3\}$ inside a chamber (the

[1]The same locus appears in $\mathcal{N} = 2$ supersymmetry, where it's called a *spectral network*, but its origin is older.

Stokes graph divides the Riemann surface into two chambers), and ther three monodromies around points A, B, and C must satisfy

$$C\psi_1, B^{-1}\psi_2 \in \operatorname{span}\{\psi_1, \psi_2\}, \tag{1.13}$$

along with all cyclic permutations of this condition. This is an algebraic geometry question, and has a cool answer: A, B, and C are unipotent, and in this case there's a continuous family of solutions (not as interesting) plus four exceptional ones, and WKB produces one of these.

2. Vertex algebras towards Higgs branches, I: 9/12/18

Today, David spoke about vertex algebras, providing an introduction and background, albeit an ahistorical one.

You can think of vertex algebras as coming from topological field theory. Consider an oriented 2D TQFT Z, whose space of local operators/observables is $V := Z(S^1)$. The pair-of-pants bordism $S^1 \amalg S^1 \to S^1$ defines a multiplication map $V \otimes V \to V$; you can think of this as taking two small circles inside a larger annulus.[2]

If you favor one of the pant legs, you can think of this bordism as a cylinder together with the insertion of a small circle at some point z in the cylinder, which you can label by any $v \in V$. Once you do this, you get a map $V \to V$ given by the cylinder, and therefore get a map $V \to \operatorname{End} V$, which we call $v \mapsto Y(v, z)$.

We're working with a topological field theory, so $Y(v, z)$ must be locally constant in z. Passing to the annulus, we have two inner discs given by the incoming S^1 and a small disc around z. Though $Y(v, z)$ is locally constant in z, interesting things can happen when you move z around v; the structure given by V and $Y(v, z)$ is called an *E_2-algebra* or a *2-disc algebra*.

If our TQFT is valued in $\mathsf{Vect}_{\mathbb{C}}$, an E_2-algebra is fairly simple to understand: we can move v and z around each other, so it's just a commutative algebra. But there are more operations in what's called the *cohomological setting*, where the TQFT is valued in something like graded complex vector spaces. Local constancy means that we have an action of the homology of the $C_2(\mathbb{R}^2)$, the configuration space of two points in $\mathbb{R}^2$ on V, and since $C_2(\mathbb{R}^2) \simeq S^1$, we get data of a map $H^*(S^1) \otimes V \otimes V \to V$, i.e. a map

$$V \longrightarrow \operatorname{End} V \otimes H^*(S^1). \tag{2.1}$$

The cohomology of S^1 is pretty simple; in degree 0 we get back the commutative multiplication, and in degree -1 we get a graded Lie bracket $\{,\}$. This behaves well with respect to the multiplication, and this structure is called a *graded Poisson algebra*, or in this case also a *Gerstenhaber algebra*.[3]

But we can upgrade this to the *derived* setting, replacing cohomology by cochains, which is what supersymmetry taught us to do. This is the setting people usually refer to when they say E_2-algebra. We have a diagram

$$\begin{tikzcd} & \operatorname{End} V \otimes C^*(D) \arrow[d] \\ V \arrow[ur] \arrow[r] \arrow[dr] & \operatorname{End} V \otimes C^*(S^1) \arrow[d] \\ & \operatorname{End} V \otimes H^1(S^1)[-1]. \end{tikzcd} \tag{2.2}$$

Here D is the disc. The top map from V is an honestly commutative map for every pair of points, the middle one is the E_2-algebra structure, and the bottom map is the Lie operad (since it gave us the bracket). So the E_2-operad is built out of these two operads, and gives a commutative multiplication for pairs of points plus other data.

Example 2.3 (String topology)**.** One simple example of a 2D TQFT is called *string topology* on a manifold M, or the A-model on T^*M. The local operators are $H_*(\operatorname{Map}(S^1, M))$: the homology of the loopspace. Setting up the multiplication map takes some thought, and there are papers working this out.

[2]For these to be the same, we need to be doing oriented TQFT, not framed TQFT.

[3]For an interesting and recent application to physics, see https://arxiv.org/abs/1809.00009 by Beem-Ben-Zvi-Bullimore-Dimofte-Neitzke.

There's also a space of *observables*, which comes as part of the description arising from physics, but doesn't fit into the functorial perspective. This is a pity, because they're how vertex algebras enter the story. Specifically, the observables are an algebra $H_*(\mathrm{Map}_c(D, M))$, or $H_*(\Omega^2 M)$. For example, if $M = BG$ is the classifying space of a compact Lie group G with complex form $G_{\mathbb{C}}$, $\Omega^2 BG$ is the affine Grassmannian $LG_{\mathbb{C}}/LG_{\mathbb{C}^\times}$. ◀

We've been thinking of all of this in terms of 2D TQFT, but all of the algebraic structure appears in higher dimensions too: vertex algebras appear in, e.g., 4D gauge theories where there are two topological directions and one holomorphic direction, and the intuition we've been using will carry over to there too.

Vertex algebras are the analogues of E_2-algebras, but for 2D conformal field theory rather than 2D topological field theory. The analogue of the circle is $\operatorname{Spec}\mathbb{C}[[z]]$, and the vector space of observables on that space is again denoted V. Given a vector v and a point z, we again get a $Y(v, z) \in \operatorname{End} V$, but this time, we want it to depend holomorphically[4] on z in the neighborhood $\operatorname{Spec}\mathbb{C}((z))$. That is, given $w \in W$, $Y(v, z) \in V((z))$. This may seem weird, but it's typical for how Laurent series are used to study local neighborhoods in formal algebraic geometry, and the upshot is $Y(v, z) \in \operatorname{End} V[[z^{\pm 1}]]$. We'll think of $Y(v, z)$ as an operator-valued distribution supported at $z \in D$.

Why should we think of these as distributions? Our model of functions is Laurent polynomials, and the algebraic dual of $\mathbb{C}[[z]]$ is $\mathbb{C}[[z^{\pm 1}]]$, so they can be called (algebraic) distributions.

Example 2.4. For example, the δ-function at z can be represented as

$$\delta_1(z) = \sum_{n=-\infty}^{\infty} z^n \in \mathbb{C}[z^{\pm 1}].$$

If you calculate its residues, you get just one at 1. ◀

We'll also write $Y(z, v)$ as $v(z)$, thinking of v acting at z.

Definition 2.5. A *vertex algebra* is the data of

- a vector space V,
- a *unit* or *vacuum element* $|0\rangle \in V$,
- a *translation* $T \in \operatorname{End} V$, and
- a map $Y\colon V \to \operatorname{End} V[[z^{\pm 1}]]$,

subject to the following axioms:

- $Y(|0\rangle, z) = \mathrm{id}$,
- $Y(v, z) \cdot |0\rangle = v + z(\text{stuff})$,
- T encodes $\frac{\mathrm{d}}{\mathrm{d}z}$-equivariance, in that $[T, Y(v, z)] = \partial_z Y(v, z)$, and
- a locality axiom, that $[a(z), b(w)]$ ought to be supported at $z = w$, i.e. there's an $N \gg 0$ such that
$$(z - w)^N [a(z), b(w)] = 0.$$

Again, $\mathbb{C}[[z^{\pm 1}]]$ is a space of test functions; you can think of this as a place where you can solve algebraic equations, somewhat like $\mathcal{D}$-modules.

The last axiom sometimes is written in different ways recalling associativity, Lie brackets, etc. It might seem surprisng, because we don't have Laurent series supported at points, but we're working with distributions: letting

$$\delta(z - w) \coloneqq \sum_{n=-\infty}^{\infty} \frac{w^n}{z^{n+1}}, \tag{2.6}$$

and $(z - w)\delta(z - w) = 0$, so we're OK. As usual, $\delta(z - w)$ and its derivatives span all distributions supported at $z = w$. Therefore locality is equivalent to asking that

$$[a(z), b(w)] = \sum_{n=0}^{N} (a_{(n)}b)(w) \frac{1}{n!} \partial_w^n \delta(z_w) \tag{2.7}$$

[4]This means algebraically if you're thinking algebro-geometrically, which we're mostly doing. You can formulate vertex algebras in either algebraic or analytic language, but all of the structure ends up being completely formal, in the sense of formal algebraic geometry.

for some coefficients $a_{(n)}b$.

This is it: you might want to add associativity or a Jacobi identity, but you don't actually need to. So next we'll talk about how to think about vertex algebras.

First of all, $[a(z), b(w)]$ isn't *quite* supported at $z = w$: it also depends on n^{th}-order derivatives for $n \le N$, so on the N^{th}-order jets, which requires an N^{th}-order neighborhood of $z = w$. This doesn't change much, though.

Definition 2.8. A vertex algebra is *commutative* if $Y(a, z) \cdot b \in V[[z]]$, rather than just $V((z))$.

This is a very strong assumption: there are no poles. In this case, we can define a new multiplication $\cdot\colon V \otimes V \to V$ by

$$a \cdot b = \lim_{z \to 0} a(z) \cdot b. \tag{2.9}$$

Since $[a(z), b(w)]$ is a Taylor series supported at a single point, it must vanish! And therefore V is just a commutative (associative, unital) algebra with a derivation T (with respect to $\cdot$), i.e. a differential commutative algebra. And conversely, given a differential commutative algebra, you can just define

$$a(z) \coloneqq \sum_{n \gg 0} \frac{z^n}{n!}(T^n a), \tag{2.10}$$

and you can check this is a commutative vertex algebra. Of course, any commutative algebra defines a commutative differential algebra with $T = 0$! But there are also nontrivial examples, thankfully.

We can rewrite Y as a map $V \otimes V \to V((z)) = V \otimes \mathscr{O}(D^\times)$; $\mathscr{O}(D^\times)$ can be thought of as the de Rham complex on S^1, a souped-up version of locally constant functions. Therefore a Vertex algebra is such a map satisfying some axioms.

Y induces a map $Y^-\colon V \otimes V \to V((z))/V[[z]] \cong V \otimes H^1_{\mathrm{loc}}(\{0\}, \mathscr{O})$, and Y^- is a Lie algebra structure (but with a differential; these have different names, such as Lie-$*$ algebras). And if we can restrict to $V[[z]]$, we get a commutative algebra. So much like an E_2-algebra is something like a commutative algebra plus a Lie algebra, a vertex algebra is something like a differential commutative algebra and a differential Lie algebra.

The fact that Y^- has a Lie bracket is saying something about δ-functions, because $\mathbb{C}((z))/\mathbb{C}[[z]] \cong \langle \partial^n \delta_0 \rangle = \mathbb{C}[\delta]\delta_0$. To get the Lie bracket, though, we start with a general story on a vector bundle $V \to X$: given a section s, there's a natural pointwise multiplication $s(z) \cdot v$, which is $\mathscr{O}_X$-linear. But you can also define multiplication depending on the Taylor series at a point, which is local in the physical sense. Since $\mathbb{C}[[z]]^* = \mathbb{C}((z))/\mathbb{C}[[z]]$, then Y^- defines maps $Y_t\colon (V \otimes V)[[t]] \to V$, which gives us the Lie bracket structure.

Like for E_2-algebras, vertex algebras are almost commutative: there's a filtration on either whose associated gradeds are commutative. This means the analogue of a Poisson structure: the description in terms of commutative and Lie algebras splits, and we get both structures.

Therefore vertex algebras are some sort of deformation/quantization of the notion of a differential Poisson algebra.

3. Vertex algebras towards Higgs branches, II: 9/26/18

Today David spoke again, continuing from his previous talk.

Vertex algebras are an algebraic structure capturing the observables in a 2D holomorphic field theory on a Riemann surface Σ, such as $\mathbb{C}$. Given an open $U \subset \Sigma$, we get a vector space $\mathscr{F}(U)$ of observables on U, and this should vary holomorphically in U. If $U = U_1 \amalg U_2$, we want $\mathscr{F}$ to satisfy

$$\mathscr{F}(U) = \mathscr{F}(U_1) \otimes \mathscr{F}(U_2). \tag{3.1}$$

Beilinson-Drinfeld realized how to start from this ansatz and write down the definition of a vertex algebra. Specifically, we only consider "opens" which are formal completions of finite subsets of $\mathbb{C}$: they introduce a *Ran space* of Σ, a space of finite subset built as a colimit from ordered finite subsets in a certain way. Then they give data of a certain quasicoherent sheaf $\mathscr{F}$ on these subsets which satisfies (3.1).

This isn't quite a vertex algebra — it's a related structure called a *factorization algebra*. In a vertex algebra, we say that for all singletons $x \in \Sigma$, $\mathscr{F}(x) = V$, and we need to specify what happens when two points collide, which is the map $Y\colon V \otimes V \to V((z))$ that we described last time. Beilinson-Drinfeld showed this algebraic operation, which depends meromorphically on z, defines gluing data for this geometric perspective on vertex algebras.

We saw that this is an amalgam of two related algebraic structures: the quotient $Y_-\colon V\otimes V\to V((z))/[[z]]$ and the sub $Y_+\colon V\otimes V\to V[[z]]$. If $(V,Y,T,|0\rangle)$ is *holomorphic*, meaning $Y=Y_+$, then V is a commutative ring with derivation by

$$Y_+(a,z)=a(z)=\sum_{n\geq 0}\frac{z^n}{n!}T^n a, \tag{3.2}$$

and conversely, this data defines a vertex algebra. Now, in this lecture, we'll study some examples.

Example 3.3. These are somewhat silly examples, but let R be any commutative ring with the derivation $T=0$. ◂

Example 3.4. More interestingly, choose a commutative ring R and let $V\coloneqq R\langle\partial\rangle$, freely adjoining a derivation. This is an algebraic construction, but has a geometric meaning: suppose $R=\mathbb{C}[X]$, the algebra of functions on a variety X, and $X=\operatorname{Spec}R$. Then we can take the space of jets on X, JX, and $R\langle\partial\rangle=\mathbb{C}[JX]$. Specifically, let $J_nX\coloneqq\operatorname{Map}(\operatorname{Spec}\mathbb{C}[z]/(z^{n+1}),X)$; then $JX\coloneqq\varprojlim_n J_nX$. That is, we're looking at n^{th}-order information near a point in X, for some n. This is a scheme, but isn't of finite type. This is what ∂, ∂^2, etc. are tracking. JX is a scheme, but not a variety, as it's not finite type. ◂

Not all vertex algebras are spaces of jets, since some vertex algebras are noncommutative. But these are really good examples, so you could take as your intuition the idea that vertex algebra is a quantization of the space of jets, replacing commutative vertex algebras with Poisson ones.

Since $(\mathbb{C}[[z]])^*\cong\mathbb{C}((z))/[[z]]$, then $\mathbb{C}[\partial_z]\cdot\delta_0$.

Example 3.5. Let $X=\mathfrak{a}^*$ be a vector space. Then $J\mathfrak{a}^*=\mathfrak{a}^*[[z]]$, because

$$\begin{aligned}J\mathfrak{a}^*&=\operatorname{Spec}\operatorname{Sym}((\mathfrak{a}^*\otimes\mathbb{C}[[z]])^*)\\&=\operatorname{Spec}\operatorname{Sym}(\mathfrak{a}\otimes\mathbb{C}[\partial]\cdot\delta)\\&=\operatorname{Spec}(\operatorname{Sym}\mathfrak{a}((z))/(\operatorname{Sym}\mathfrak{a}((z)))(\mathfrak{a}[[z]])).\end{aligned}$$

This computation is telling us something about the appearance of power series in the definition of the vertex operator. For example, suppose we choose a basis for $\mathfrak{a}^*$, writing $\mathfrak{a}^*=\operatorname{Spec}\mathbb{C}[x_1,\dots,x_N]$. Then

$$JX=\operatorname{Spec}\mathbb{C}[x_{1,n}\dots,x_{N,n}]_{n\leq 0}, \tag{3.6}$$

and the derivation is

$$T(x_{i,n})=-(n-1)x_{i,n-1}. \tag{3.7}$$

If $\{J^a\}$ is a basis for $\mathfrak{a}$, then $\{J^a_n\coloneqq J_az^n\mid n<0\}$ is a basis for $\mathfrak{a}((z))/\mathfrak{a}[[z]]$. Inside $\operatorname{End}V[[z]]$,

$$J^a(z)=Y_+(J^a,z)=\sum_{n<0}J^a_nz^{-n-1} \tag{3.8}$$

and

$$Y(J^a,z)=\sum_{n\in\mathbb{Z}}J^a_nz^{-n-1}. \tag{3.9}$$

This is spelled out in greater detail in the book by Edward Frenkel and David Ben-Zvi. ◂

Another class of examples are *vertex Poisson (Coisson) algebras*. In this case, we start with $X=\operatorname{Spec}R$ a Poisson variety; then $V\coloneqq\mathbb{C}[JX]$ is a vertex Poisson algebra. For example, suppose $\mathfrak{g}$ is a Lie algebra for the group G and $X=\mathfrak{g}^*$. Then $\mathbb{C}[X]=\operatorname{Sym}\mathfrak{g}$ and the *arc group* $JG\coloneqq G(\mathbb{C}[[z]])$ acts on $J\mathfrak{g}^*$, which is in fact the Lie algebra of JG. There's also an action of $\mathfrak{g}[[z]]\coloneqq\mathfrak{g}\otimes\mathbb{C}[[z]]$ on $V=\operatorname{Sym}(\mathfrak{g}((z))/\mathfrak{g}[[z]])$. Now we get similar formulas as in Example 3.5: if J^a is a basis of $\mathfrak{g}$ and $J^a_n=J^at^n\in\mathfrak{g}[[t]]$, (3.8) is the same, but we also have a Y_-, whose formula is

$$Y_-(J^a,z)-\sum_{n\geq 0}J^a_nz^{-n-1}. \tag{3.10}$$

These describes maps $\mathfrak{g}\to\operatorname{End}V((z))/[[z]]$ or $\mathfrak{g}[[z]]\to\operatorname{End}V$, so a vertex Poisson algebra is data of $(V,|0\rangle,T)$ together with a map $Y_+\colon V\to\operatorname{End}V[[z]]$, which gives us a commutative vertex algebra, and a map $Y_-\colon V\to\operatorname{End}V((z))/[[z]]$, which has the structure of a differential Lie algebra and acts on Y_+.

Examples of vertex Poisson algebras are things like jets on a Poisson variety. The Y operator degenerates into two parts: $Y \mapsto Y_+$, and taking $\hbar$-linear terms, you get Y_-. This is like ordinary quantization, where associative algebras can become Poisson algebras.

As an example of this deformation (TODO: I think?), consider $\operatorname{Sym} \mathfrak{g} \to \mathcal{U}\mathfrak{g}$; then $\mathbb{C}[\mathfrak{g}^*]$ passes to the distributions at the identity on G. Then the affine Grassmannian makes an appearance:

$$\mathbb{C}[J\mathfrak{g}^*] = \operatorname{Sym} \mathfrak{g}(K)/(\operatorname{Sym} \mathfrak{g}(K)\mathfrak{g}(G)) = \mathbb{C}[T_e^* G(K)/G(\mathscr{O}). \tag{3.11}$$

Here $K = \mathbb{C}((t))$ is Laurent series and $\mathscr{O} = \mathbb{C}[[t]]$ is Taylor series.

This is the "arc version"; now we'll see the "loop version." If we start with $\mathcal{U}\mathfrak{g}$ instead of $\operatorname{Sym} \mathfrak{g}$, then we get distribtuions on the identity of $G(K)/G(\mathscr{O})$, i.e.

$$U\mathfrak{g}(K)/U\mathfrak{g}(K) \cdot \mathfrak{g}(\mathscr{O}) = \operatorname{Ind}_{\mathfrak{g}(\mathscr{O})}^{\mathfrak{g}(K)} \mathbb{C} = \mathcal{U}\mathfrak{g}(K) \otimes_{\mathcal{U}\mathfrak{g}(\mathscr{O})} \mathbb{C}. \tag{3.12}$$

This is denoted $V_{\mathfrak{g},0}$, and is called the (TODO vacuum?) of the affine Kac-Moody algebra at level 0.

For example, letting $|0\rangle$ be a nonzero vector in $\mathbb{C}$ (before we induce to $\mathfrak{g}(K)$), in the representatation of $\widehat{\mathfrak{g}}_0 \coloneqq \mathfrak{g}(K)$,

$$J^a(z) = Y(J_{-1}^a|0\rangle, z) = J_n^a z^{-n-1}. \tag{3.13}$$

Topologically, the affine Grassmannian $G(K)/G(\mathscr{O}) \simeq \Omega^2 BG$. So we're looking at maps from $\mathbb{C}$ rel $\mathbb{C} \setminus \mathbb{D}$ (where $\mathbb{D}$ denotes a disc) to BG, i.e. G-bundles on $\mathbb{C}$ together with trivializations away from a disc (thought of as not much more than a point).

The affine Grassmannian has the structure of a factorization algebra: given a collection of discs, consider the G-bundles trivialized away from these discs. This satisfies a product axiom, so we get a factorization algebra in ind-schemes. Moreover, we always have the trivial bundle, so this is pointed. Therefore any time you linearize this (so take a $\otimes$-functor to Vect, such as homology or distributions), you get a factorization algebra in vector spaces, and in particular a vertex algebra. This is closely related to the observables in string topology.

Next time, we'll talk about dimensional reduction, from vertex to associative (or Poisson) algebras, and the physics thereof.

4. Vertex algebras towards Higgs branches, III: 10/3/18

> *"I don't want to spend a lot of time on Zhu algebras, but... it looks like I don't have a lot of time, so it works out."*

Today David spoke again, continuing his previous talk, discussing the Higgs branch conjecture studied by Beem, Rastelli, and others.

Let's start with a 4D $\mathcal{N} = 2$ superconformal field theory; whatever this is, we can attach to it its moduli of vacua, which contains a subset called the Higgs branch. It's also possible to associated a vertex algebra to this theory, which by some version of dimensional reduction and/or chiralization, is closely related to the Higgs branch.

Higgs branches. For a 4D $\mathcal{N} = 2$ gauge theory, there are eight supercharges. Let's give a Lagrangian description. The gauge group is a compact Lie group G. The theory also includes data of a hyperKähler manifold, together with an action of G on M by hyperKähler isometries, so we have a Hamiltonian moment map $\mu_{\mathbb{H}} \colon M \to \mathfrak{g}_{\mathbb{H}}$. For example, we could take $M = V \oplus V^*$ for a complex G-representation V.

Heuristically, we'll think of the theory as a σ-model with target the stack M/G. One obvious invariant to extract is $M /\!/\!/ G$, a holomorphic symplectic variety (not necessarily smooth) which is defined as an affine GIT quotient, defined as Spec of the ring of $G_{\mathbb{C}}$-invariant functions on $\mu_{\mathbb{C}}^{-1}(0)$. This is the *Higgs branch* of the theory, and is denoted $\mathcal{M}_{\mathrm{Higgs}}$.

Not all 4D $\mathcal{N} = 2$ theories are Lagrangian, e.g. class S theories, so we'd like to know how to access $\mathcal{M}_{\mathrm{Higgs}}$ intrinsically, without the Lagrangian description. The idea is to look at the observables: observables on a theory define functions on its moduli space, so we're going to try to do that.

In general, the observables in a 4D QFT form a factorization algebra on $\mathbb{R}^4$. Approximately this means that we associate to every open subset of $\mathbb{R}^4$, we get a vector space of observables, and disjoint unions give you tensor products. There should also be a cosheaf property: observables on U are also observables on $V \supset U$.

In general, this is a horrible mess, containing the full structure of operator product expansion, etc. So let's try to simplify it, by using twisting: if our theory is supersymmetric, with an odd symmetry Q squaring to zero, and then look at Q-cohomology.

There's a map $[Q,-] \to T_{\mathbb{C}}\mathbb{R}^n$, and we can ask for this to be surjective: every direction is in the image of Q. In this case we say the twist is *topological*. If $[Q,-]$ surjects onto $T^{0,1}\mathbb{C}^2$, so you can impose Cauchy-Riemann equations, we say the twist is *holomorphic*. You can also ask for a twist to be holomorphic in some directions and topological in others, e.g. splitting $\mathbb{R}^4$ and $\mathbb{R}^2 \oplus \mathbb{C}$, getting something topological in the $\mathbb{R}^2$ directions and holomorphic in the $\mathbb{C}$ directions.

It's possible to classify these using superalgebra; a recent paper of Elliot-Safronov gives a comprehensive list. Topological, holomorphic, and holomorphic-topological twists can all appear for 4D $\mathcal{N} = 2$ theories.

The topological twist in this setting is called the *Donaldson-Witten twist*. The observables are the Q-cohomology of the untwisted observables, which become roughly a topological factorization algebra, or an E_4-algebra.[5] That is, it's a commutative algebra, together with a Poisson bracket of degree -3.[6]

Unfortunately, this is the wrong commutative algebra: it's the algebra of holomorphic functions on the Coulomb branch, not the Higgs branch! The Higgs branch came from studying matter (studying M), but the Coulomb branch comes from the gauge group, and looks roughly like the G-equivariant cohomology of a point. The overall moduli space contains the Higgs and Coulomb branches, but also possibly some other stuff.

The holomorphic-topological twist in this setting is called the *Kapustin twist*. In this case, the observables are a factorization algebra on $\mathbb{R}^2 \times \mathbb{C}$ (so, topological and holomorphic). That is, fixing a point in $\mathbb{R}^2$, you get a holomorphic factorization algebra on $\mathbb{C}$, which is what we said a vertex algebra is. If you fix a $z \in \mathbb{C}$, you'll get an E_2-algebra in the topological directions. Therefore the vertex algebra is locally constant in the topological direction, and in fact the whole thing is commutative (i.e. boring): secretly it's coming from an E_4-algebra. If you do run through the construction, you'll get a vertex Poisson algebra.

This is a hint that there's something noncommutative around, and in fact you can quantize this algebra. This is *different* from what Beem and Rastelli do; it's believed to be equivalent, but there's no proof. Let S^1 act on $\mathbb{R}^2$ by rotation, which we think of as rotating around $\mathbb{C} \subset \mathbb{R}^2 \times \mathbb{C}$. Then we look at the S^1-invariant observables, which are a module over $H^*_{S^1}(\mathrm{pt}) = H^*(BS^1) = \mathbb{C}[\varepsilon]$ with $|\varepsilon| = 2$. So this is a family depending on a parameter ε, and if $\varepsilon = 0$, we get back what we started with. So this is a deformation.

Now observables on $\mathbb{C}$ are suck on $\mathbb{C}$, so the argument above that we got a commutative vertex algebra no longer applies, and the ε-deformation of the original algebra is really a deformation quantization into a noncommutative vertex algebra.

Remark 4.1. For the mathematics of the Higgs branch conjecture, we only need the original vertex algebra and its Poisson algebra, but it is nice to know that this isn't completely bland, and that a noncommutative algebra does appear. ◀

It's not quite clear how to make the Higgs branch conjecture into a conjecture: one side is defined mathematically and the other isn't. In physics, it's just known. Anyways, the conjecture says that *this vertex Poisson algebra V is a "chiral version" of the Higgs branch*, i.e. it is (maybe something closely related to) taking the ring of functions on the jets on $\mathcal{M}_{\mathrm{Higgs}}$, akin to what we discussed last time.

More precisely, we'll construct from V another Poisson algebra $V/C_2(V)$ (also called R_V), which was introduced by Yongchang Zhu, and then $\mathbb{C}[\mathcal{M}_{\mathrm{Higgs}}] \cong C_2(V)$. Zhu defined two different constructions in his thesis given a vertex algebra: one produces an associative algebra, and another (R) produces a Poisson algebra. There's a related construction giving a Poisson algebra from a vertex Poisson algebra, also called R.

Definition 4.2. Given a vertex algebra V, the Poisson algebra R_V is defined as $R_V \coloneqq V/\operatorname{span}\{a_{-2}b\}$, with the product $\overline{a} \cdot \overline{b} = \overline{a_{-1}b}$ (where a is a representative for $\overline{a}$ in the quotient, and so on), and $\{\overline{a}, \overline{b}\} = \overline{a_0 b}$.

A vertex algebra has infinitely many products $a \cdot_n b = a_n b$, but they generally don't have nice structure. In this case it's solved by modding out by the (-2)-product.

Associated to V we therefore get a Poisson variety $X_V \coloneqq \operatorname{Spec} R_V$. Is $X_V = \mathcal{M}_{\mathrm{Higgs}}$? We do have a tautological map $JR_V \twoheadrightarrow V$, and there's research about how close this is to an isomorphism for various kinds of vertex algebras. The upshot is that for the vertex algebras appearing in the Higgs branch conjecture, they're very close (only differing by nilpotents), which is good: X_V is almost the same thing as $\mathcal{M}_{\mathrm{Higgs}}$.

[5] There's a subtlety that we're not going to worry about, but the details can be found in Elliot-Safronov.

[6] Curiously, in all known examples, this Poisson bracket is zero!

Zhu's constructions have a physical explanation: an associative algebra is exactly the algebra of observables of a 1D topological field theory, and his process of getting an associative algebra from an associative algebra is a dimensional reduction from $\mathbb{C} \times \mathbb{R}^2 \to \mathbb{R} \times \mathbb{R}^2$. A topological factorization algebra on $\mathbb{R} \times \mathbb{R}^2$ is an E_3-algebra, which is precisely a Poisson algebra with even grading. Physically, the Kapustin twist passes to a Rozansky-Witten theory.

In topology, so using E_2-algebras instead of vertex algebras, there's a natural way to do dimensional reduction. There's an S^1-action on the E_2-algebra $Z(S^1)$. Looking at the target M, we can consider $H^*(LM)^{S^1}$, and do equivariant localization (this is a Tate construction), and recover

$$H^*((LM)^{S^1}) = H^*(M) \otimes \mathbb{C}[\varepsilon, \varepsilon^{-1}], \tag{4.3}$$

where ε is the equivariant parameter. So dimensional reduction means looking at loops in your manifold but you can get back to M by rotation.

TODO: not sure what happened after that, with $H_*(\Omega^2 M)$ on a cylinder to get $H_*(L\Omega M)$ on a cylinder, which we squeeze into a line. This is something like Hochschild homology, leading to the associative algebra of observables on a 1D σ-model into M: $H_*(\Omega M) \otimes \mathbb{C}[\varepsilon, \varepsilon^{-1}]$. The Zhu construction is the holomorphic analogue of this, reducing a holomorphic dimension.