FALL 2017 GOODWILLIE CALCULUS SEMINAR

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These notes were taken in Andrew Blumberg's student seminar in Fall 2017. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

Contents

1. Introduction: 9/13/17

Today, Nicky gave an overview of Goodwillie calculus, following Nick Kuhn's notes.

The setting of Goodwillie calculus is to consider two topologically enriched,[1] based model categories C and D and a functor $F\colon \mathsf{C} \to \mathsf{D}$ between them.

Example 1.1.
(1) Top, the category of topological spaces.
(2) Sp, the category of spectra.
(3) If Y is a topological space, we can also consider ${}_{Y\backslash}\mathsf{Top}_{/Y}$, the category of spaces over and under Y, i.e. the diagrams $Y \to X \to Y$ which compose to the identity. ◀

We want F to satisfy some kind of Mayer-Vietoris property, or excision. Hence, we assume C and D are *proper*, in that the pushout of a weak equivalence along a cofibration is also a weak equivalence. We'll also ask that in D, sequential colimits of homotopy Cartesian cubes are again homotopy Cartesian, and we'll elaborate on what this means.

We also place a condition on F: Goodwillie calls it "continuous," meaning that it's an enriched functor: the induced map

$$\mathrm{Map}_{\mathsf{C}}(X,Y) \longrightarrow \mathrm{Map}_{\mathsf{D}}(F(X),F(Y))$$

is a continuous map between topological spaces (or a morphism of simplicial sets; for the rest of this section, we'll let V denote the choice of Top_* or sSet_* that we made). If $X \in \mathsf{C}$ and $K \in \mathsf{V}$, then we have a tensor-hom adjunction

$$\mathsf{C}(X\otimes K, Y) \cong \mathsf{V}(K, \mathsf{C}(X,Y)).$$

From this, F produces the *assembly map*

$$F(X)\otimes K \longrightarrow F(X\otimes K).$$

We'll also require F to be weakly homotopical in that it sends homotopy equivalences to homotopy equivalences.

The idea of Goodwillie calculus is to approximate F by a tower of functors, akin to Postnikov truncations, $\dots \to P_2 \to P_1 \to P_1 \to P_0$. The fiber D_i of P_i, akin to the i^{th} Postnikov section, is like the i^{th} term in a Taylor series:

$$\begin{aligned} P_0(X) &\simeq P_0(*)\\ D_1(X) &\simeq D_1(*)\otimes X\\ D_2(X) &\simeq (D_2(X)\otimes X\otimes X)_{h\Sigma_2}, \end{aligned}$$

[1] As usual, we can take them to be enriched either over Top or over sSet. This has the important consequence that C and D are tensored and cotensored over Top_*, resp. sSet_*.

where Σ_2 acts by switching the two copies of X, and so on. Each P_i will satisfy more and more of the Mayer-Vietoris property. This is akin to the first three terms in a Taylor series for f: $f(a)$, $xf'(a)$, and $x^2 f''(a)/2$.

Weak natural transformations. We'll also need to know what a weak equivalence of functors is. This would allow us to study the homotopy category of $\mathsf{Fun}(\mathsf{C},\mathsf{D})$.

Definition 1.2. A *weak natural transformation* $F \Rightarrow G\colon \mathsf{C} \to \mathsf{D}$ is one of the two zigzags

$$F \xleftarrow{\sim} H \longrightarrow G \qquad\qquad \text{or} \qquad\qquad F \longleftarrow H \xrightarrow{\sim} G,$$

where $F \xrightarrow{\sim} G$ means an objectwise weak equivalence.

Commutativity of a diagram of weak natural transformations is computed in $\mathsf{ho}(\mathsf{D})$.[2] You can also form spectra in D in the usual way (inverting suspension, etc).

Diagrams[3]. Let S be a finite set. We'll let $\mathcal{P}(S)$ denote its power set, made into a poset category under inclusion. Similarly, we'll let $\mathcal{P}_0(S) \coloneqq \mathcal{P}(S) \setminus \{\varnothing\}$ and $\mathcal{P}_1(S) \coloneqq \mathcal{P}(S) \setminus \{S\}$, again regarded as poset categories.

Definition 1.3.

(1) A *d-cube* in C is a functor $\chi\colon \mathcal{P}(S) \to \mathsf{C}$, where $|S| = d$.
(2) A d-cube χ is *Cartesian* if
$$\chi(\varnothing) \xrightarrow{\sim} \operatorname*{holim}_{T \in \mathcal{P}_0(S)} \chi(T).$$
(3) A d-cube χ is *co-Cartesian* if
$$\chi(S) \xrightarrow{\sim} \operatorname*{hocolim}_{T \in \mathcal{P}_1(S)} \chi(T).$$
(4) A d-cube χ is *strongly co-Cartesian* if $\chi|_{\mathcal{P}(T)}\colon \mathcal{P}(T) \to \mathsf{C}$ is co-Cartesian for all $T \in \mathcal{P}(S)$ with $|T| \geq 2$.

Example 1.4.

(1) If $d = 0$, a (Cartesian or co-Cartesian) 0-cube is something weakly equivalent to the initial object.
(2) A (Cartesian or co-Cartesian) 1-cube is an equivalence.
(3) A 2-cube is something of the form
$$\begin{tikzcd} \mathrm{fib}_f \arrow[r] \arrow[d] & \mathrm{fib}_g \arrow[d] \\ A \arrow[r] \arrow[d, "f"] & B \arrow[d, "g"] \\ C \arrow[r] & D. \end{tikzcd}$$
We let $\partial\chi$ denote the *boundary* of χ, the top row; the middle row is $\chi_\top$, and the bottom row is $\chi_\perp$.[3] In this case, Cartesian and co-Cartesian correspond to (homotopy) pushout and pullback squares, which explains their names in the general case. ◀

There's a way to produce co-Cartesian cubes canonically from a finite set. Let $\phi\colon X^{\amalg T} \to X$ denote the fold map.

Definition 1.5. Let T be a finite set and $X \in \mathsf{C}$, and let

$$X \star T \coloneqq \mathrm{cofib}\left(\phi\colon \coprod_T X \to X\right).$$

Now, for $T \subset [d]$, the assignment $T \mapsto X \star T$ defines a co-Cartesian $(d+1)$-cube.

[2]There are more worked-out expositions of the homotopy theory of functors, in case this one looks ad hoc, but we don't need the entire background.

[3]These are also written χ_{top} and χ_{bottom}.

For example, when $d = 1$, this is the homotopy pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d] & CX \simeq * \arrow[d] \\ CX \simeq * \arrow[r] & \Sigma X. \end{tikzcd}$$

This formalism allows us to define the analogue of the Mayer-Vietoris principle that we'll need for the Goodwillie tower.

Definition 1.6. An $F\colon \mathsf{C} \to \mathsf{D}$ with F, C, and D as above is *d-excisive* if for all strongly co-Cartesian $(d+1)$-cubes χ, $F(\chi)$ is a Cartesian $(d+1)$-cube in D.

Example 1.7.

(1) 0-excisive functors are homotopy constant.
(2) 1-excisive functors are those that satisfy the Mayer-Vietoris property. In Sp, $\mathrm{Map}_{\mathsf{Sp}}(C, -)$ and L_E are both 1-excisive. $\blacktriangleleft$

There are some nice properties about how d-excisive functors behave with respect to cofibration sequences, etc., and we will return to them in due time. We will now construct Taylor towers.

Fix an $X \in \mathsf{C}$, and let

$$T_d F(X) \coloneqq \underset{T \in \mathcal{P}_0([d+1])}{\mathrm{holim}} F(X \star T).$$

Remark. There is a natural map $t_d F\colon F \to T_d F$, and by definition, this is an equivalence if F is d-excisive. $\blacktriangleleft$

Set $P_d F\colon \mathsf{C} \to \mathsf{D}$ to be the functor sending

$$X \longmapsto \mathrm{hocolim}\left(F(X) \xrightarrow{t_d F} T_d F(X) \xrightarrow{t_d t_d F} T_d T_d F(X) \longrightarrow \cdots \right).$$

For example, if $F(*) \simeq *$, then $T_1 F(X)$ is the homotopy pullback of

$$\begin{tikzcd} & F(CX) \simeq * \arrow[d] \\ * \simeq F(CX) \arrow[r] & F(\Sigma X), \end{tikzcd}$$

and hence is $\Omega F(\Sigma X)$. In this case

$$P_1 F(X) = \underset{n \to \infty}{\mathrm{hocolim}}\, \Omega^n F \Sigma^n X.$$

For example, if $F = \mathrm{id}$ and $\mathsf{C} = \mathsf{D}$, then $P_1(\mathrm{id}) = \Omega^\infty \Sigma^\infty$, which is cool: the "first derivative" of the identity tells us information of stable homotopy! The calculation of the second derivative will be harder.

2. Interpolating between stable and unstable phenomena: 9/20/17

Today, Adrian gave an overview of what we're going to learn about this semester.

Functors are like functions. We have an analogy between smooth functions and nice functors from Top_* to Top_* or Sp.[4] This analogy sends

- degree-n polynomials to n-excisive functors,
- homogeneous degree-n polynomials to homogeneous n-excisive functors (defined using Cartesian cubes), and
- Taylor series to Taylor towers of functors.

In Higher Algebra, Lurie takes the idea that an ∞-category is like a manifold as an anchor for doing a lot of very interesting mathematics, which is one angle for interpreting this analogy.

Let $\mathsf{Homog}_n(\mathsf{C}, \mathsf{D})$ denote the category of homogeneous n-excisive functors $F\colon \mathsf{C} \to \mathsf{D}$, where C and D are categories with the assumptions we placed on them last time.

[4]Perhaps more generality is possible, but we'll worry about that later.

Theorem 2.1 (Goodwillie, Lurie)**.** *The functor*

$$\Omega^\infty \circ -\colon \mathsf{Homog}_n(\mathsf{Top}_*, \mathsf{Sp}) \longrightarrow \mathsf{Homog}_n(\mathsf{Top}_*, \mathsf{Top})$$

is an equivalence.

Let $\mathsf{Lin}_n(\mathsf{C}, \mathsf{D})$ denote the category of multilinear functors in n variables and FS_{Σ_n} denote the category of *FS-spectra* for Σ_n,[5] the category of spectra together with an action of Σ_n by automorphisms.

Theorem 2.2 (Goodwillie, Lurie)**.** *When* $\mathsf{C} = \mathsf{Top}_*$ *or* Sp*, the functors*

$$\mathsf{FS}_{\Sigma_n} \xrightarrow{A} \mathsf{Lin}_n(\mathsf{C}, \mathsf{C}) \xrightarrow{B} \mathsf{Homog}_n(\mathsf{C}, \mathsf{C})$$

are both equivalences, where

- *A sends C_n to the multilinear functor*
$$(X_1, \dots, X_n) \longrightarrow (C_n \wedge X_1 \wedge \dots \wedge X_n)_{h\Sigma_n},$$
and
- *$B = - \circ \Delta$, where $\Delta\colon X \mapsto (X, \dots, X)$ is the diagonal.*

So there's not really a difference between these different perspectives.

We'd like to push this analogy further: is it true that n-excisive functors are precisely the things you get by extending $(n-1)$-excisive functors by n-homogeneous excisive functors? Fortunately, this is true, for "nice" n-excisive functors (where "nice" isn't too restrictive).

Another thing about polynomials is that they're uniquely determined by $n+1$ points. There's an analogue for functors. Let $\mathsf{Set}_*^{\le n+1}$ denote the full subcategory of Set_* consisting of sets with cardinality at most $n+1$ (including the basepoint) and $i\colon \mathsf{Set}_*^{\le n+1} \hookrightarrow \mathsf{Top}_*$ be the usual inclusion.

Theorem 2.3 (Lurie)**.** *The n-excisive functors $F\colon \mathsf{Top}_* \to \mathsf{Sp}$ are precisely the functors arising as left Kan extension of a functor $\widetilde{F}\colon \mathsf{Set}_*^{\le n+1} \to \mathsf{Sp}$ along i.*

Interpolating between stable and unstable homotopy theory. Unfortunately, I didn't get everything that happened here, but the idea is to consider the Taylor tower of the identity $\mathsf{Top}_* \to \mathsf{Top}_*$. The first homogeneous piece is $\Omega^\infty\Sigma^\infty$, which somehow says that we see stable information, and after that is $\Omega^\infty(C_2 \wedge X \wedge X)_{\Sigma_2}$ and so on. You can get a spectral sequence out of this.

The Blakers-Massey theorem is another manifestation or maybe explanation of the fact that Goodwillie calculus gets stable phenomena out of unstable ones.

Theorem 2.4 (Blakers-Massey)**.** *Consider a diagram indexed on the unit n-cube (the objects are the vertices, interpreted as a poset category using the dictionary order), and assume the map from the space at $(0, \dots, 0)$ to the space at e_i is k_i-connected. Then, the arrow from the homotopy limit of this diagram to the space at $(0, \dots, 0)$ is $(-1 + n + \sum k_i)$-connected.*

So we don't quite have spectra at any finite level, but if you impose higher and higher excisiveness, you can't have bounded connectivity.

Calculus of embeddings. Let M be a manifold, and consider presheaves of topological spaces on it, i.e. functors $F\colon O(M)^{\mathsf{op}} \to \mathsf{Top}$, where $O(M)$ is the poset category of open sets on M, ordered by inclusion. We restrict to the F such that

- if $U \subset V$ is an isotopy equivalence, then $F(U) \to F(V)$ is a homotopy equivalence, and
- $$F\left(\bigcup_i U_i\right) = \operatorname{holim} F(U_i),$$
indexed by the inclusion relations among the U_i.

Definition 2.5. Such an F is an *n-excisive sheaf* if for any closed subsets $A_1, \dots, A_n \subseteq U$, the homotopy colimit of the "cube" diagram of $U \setminus \mathcal{A}$ for all $\mathcal{A} \subset \{A_1, \dots, A_n\}$ is $F(U)$.

For $n = 1$, this is the same as the usual sheaf condition (which is the strongest condition: the least amount of information is needed to determine it from local information).

[5]This term is due to C. Wu. You might also hear *doubly naïve Σ_n-spectra* or *spectra with a Σ_n-action.*