ADAMS SPECTRAL SEQUENCES FOR NON-VECTOR-BUNDLE THOM SPECTRA

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Abstract. When R is one of the spectra ku, ko, tmf, $MTSpin^c$, $MTSpin$, or $MTString$, there is a standard approach to computing twisted R-homology groups of a space X with the Adams spectral sequence, by using a change-of-rings isomorphism to simplify the E_2-page. This approach requires the assumption that the twist comes from a vector bundle, i.e. the twist map $X \to B\mathrm{GL}_1(R)$ factors through $B\mathrm{O}$. We show this assumption is unnecessary by working with Baker-Lazarev's Adams spectral sequence of R-modules and computing its E_2-page for a large class of twists of these spectra. We then work through two example computations motivated by anomaly cancellation for supergravity theories.

Contents

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0. Introduction

There is a standard formula for computing Steenrod squares in the cohomology of the Thom space or spectrum of a vector bundle $V \to X$: if U is the Thom class,

$$\mathrm{Sq}^n(Ux) = \sum_{i+j=n} Uw_i(V)\mathrm{Sq}^j(x). \tag{0.1}$$

The ubiquity of the Steenrod algebra in computational questions in algebraic topology means this formula has been applied to questions in topology and geometry, and recently even in physics, where it is used to run the Atiyah-Hirzebruch and Adams spectral sequences computing groups of invertible field theories.

It is possible to build Thom spectra using more general data than vector bundles, and recently these Thom spectra have appeared in questions motivated by anomaly cancellation in supergravity theories [DY24b, Deb24]. Motivated by these applications (which we discuss more in §3), our goal in this paper is to understand the analogue of (0.1) for non-vector-bundle twists of commonly studied generalized cohomology theories. We found that the most direct generalization of (0.1) is true; in a sense, for the theories we study, these more general Thom spectra behave just like vector bundle Thom spectra for the purpose of computing their homotopy groups with the Adams spectral sequence.

Statement of results. Now for a little more detail: our main theorem and the language needed to define it. We use Ando-Blumberg-Gepner-Hopkins-Rezk's approach to twisted generalized cohomology theories [ABG$^+$14a, ABG$^+$14b], which generalizes the notion of a local system. Twists of $\mathbb{Z}$-valued cohomology on a pointed, connected space X are specified by *local systems* with fiber $\mathbb{Z}$, which are equivalent data to homomorphisms $\pi_1(X) \to \mathrm{Aut}(\mathbb{Z})$, or, since $\mathbb{Z}$ is discrete, to maps $X \to B\mathrm{Aut}(\mathbb{Z})$.

Ando-Blumberg-Gepner-Hopkins-Rezk generalize this to E_∞-ring spectra.[1,2] If R is an E_∞-ring spectrum, Ando-Blumberg-Gepner-Hopkins-Rezk define a notion of local system of free rank-1 R-module spectra that is classified by maps to an object called $B\mathrm{GL}_1(R)$, making $B\mathrm{GL}_1(R)$ the classifying space for twists of R-homology. Given a twist $f\colon X \to B\mathrm{GL}_1(R)$, they then define a *Thom spectrum* Mf, and the homotopy groups of Mf are the f-twisted R-homology groups of X. This construction simultaneously generalizes twisted ordinary homology, twisted K-theory, and the vector bundle twists mentioned above.

We are interested in twisted R-homology for several E_∞-ring spectra, so our first step is to give examples of twists. Most of these examples are known, but by using a result of May-Quinn-Ray [May77, Lemma IV.2.6], one can produce them in a unified way.

Theorem.

(1) *(Proposition 1.21 and Lemma 1.31) There is a map $K(\mathbb{Z}/2,1)\times K(\mathbb{Z},3) \to B\mathrm{GL}_1(MTSpin^c)$, meaning spin^c bordism can be twisted on a space X by $H^1(X;\mathbb{Z}/2)\times H^3(X;\mathbb{Z})$. The induced maps to $B\mathrm{GL}_1(ku)$ and $B\mathrm{GL}_1(KU)$ recover the usual notion of K-theory twisted by $H^1(X;\mathbb{Z}/2)\times H^3(X;\mathbb{Z})$.*

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[1] An *E_∞-ring spectrum* is the avatar in stable homotopy theory of a generalized cohomology theory with a commutative ring structure. Examples include ordinary cohomology, real and complex K-theory, and many cobordism theories.

[2] In fact, much of Ando-Blumberg-Gepner-Hopkins-Rezk's theory works in greater generality, but we only need E_∞-ring spectra in this article.

(2) *(Proposition 1.37 and Lemma 1.45) There is a map $K(\mathbb{Z}/2,1)\times K(\mathbb{Z}/2,2)\to B\mathrm{GL}_1(\mathit{MTSpin})$, meaning spin bordism can be twisted on a space X by $H^1(X;\mathbb{Z}/2)\times H^2(X;\mathbb{Z}/2)$. The induced maps to $B\mathrm{GL}_1(\mathit{ko})$ and $B\mathrm{GL}_1(\mathit{KO})$ recover the usual notion of KO-theory twisted by $H^1(X;\mathbb{Z}/2)\times H^2(X;\mathbb{Z}/2)$.*

(3) *(Corollary 1.58 and Lemma 1.64) Let $SK(4)$ be a classifying space for degree-4 supercohomology classes. Then there is a map $K(\mathbb{Z}/2,1)\times SK(4)\to B\mathrm{GL}_1(\mathit{MTString})$, meaning string bordism can be twisted on a space X by $H^1(X;\mathbb{Z}/2)\times SH^4(X)$. The induced map*

$$K(\mathbb{Z},4)\longrightarrow SK(4)\longrightarrow B\mathrm{GL}_1(\mathit{MTString})\longrightarrow B\mathrm{GL}_1(\mathit{tmf}) \tag{0.2}$$

recovers the Ando-Blumberg-Gepner twist of tmf (and Tmf and TMF) by degree-4 cohomology classes.

Supercohomology refers to a generalized cohomology theory SH introduced by Freed [Fre08, §1] and Gu-Wen [GW14]: $\pi_{-2}SH=\mathbb{Z}/2$ and $\pi_0 SH=\mathbb{Z}$, with the unique nontrivial k-invariant, and no other nonzero homotopy groups. We explicitly define $SK(4)$ in (1.53).

Though twists of *tmf* by degree-4 cohomology classes are relatively well-studied, this supercohomology generalization appears to only be suggested at in the literature by various authors including [FHT10, JF20, BLM23, TY23a, TY23b, TY25], and sees more of the homotopy type of $B\mathrm{GL}_1(\mathit{tmf})$. It would be interesting to study instances of this twist.

We call the twists in the above theorem *fake vector bundle twists*: when the twist is given by a vector bundle V, these cohomology classes appear as characteristic classes of V, but these twists exist whether or not there is a vector bundle with the prescribed characteristic classes.

If R is one of the spectra mentioned in the above theorem, the Thom spectrum Mf of a fake vector bundle twist $f\colon X\to B\mathrm{GL}_1(R)$ is an R-module spectrum. This grants us access to Baker-Lazarev's variant of the Adams spectral sequence [BL01].

Theorem (Baker-Lazarev [BL01])**.** *Let p be a prime number and R be an E_∞-ring spectrum such that $\pi_0(R)$ surjects onto $\mathbb{Z}/p$, so that $H := H\mathbb{Z}/p$ acquires the structure of an R-algebra. For R-module spectra M and N, let $N_R^*M := \pi_{-*}\mathrm{Map}_R(M,N)$. Then there is an Adams-type spectral sequence with signature*

$$E_2^{s,t}=\mathrm{Ext}^{s,t}_{H_R^*H}(H_R^*(M),\mathbb{Z}/p)\Longrightarrow \pi_{t-s}(M)_p^\wedge, \tag{0.3}$$

which converges for all M and all E_∞-ring spectra R we consider in this paper.

What Baker-Lazarev prove is more general than what we state here: we stated only the generality we need.

For $H\mathbb{Z}$, *ko*, and *ku* ($p=2$), and *tmf* ($p=2$ and $p=3$), H_R^*H is known due to work of various authors: let $\mathcal{A}(n)$ be the subalgebra of the mod 2 Steenrod algebra generated by $\mathrm{Sq}^1,\dots,\mathrm{Sq}^{2^n}$. Then, at $p=2$,

(1) $H^*_{H\mathbb{Z}}H\cong\mathcal{A}(0)$,
(2) $H^*_{ku}H\cong\mathcal{E}(1) := \langle \mathrm{Sq}^1, \mathrm{Sq}^2\mathrm{Sq}^1+\mathrm{Sq}^1\mathrm{Sq}^2\rangle$,
(3) $H^*_{ko}H\cong\mathcal{A}(1)$, and
(4) $H^*_{tmf}H\cong\mathcal{A}(2)$.

See (2.3) and the surrounding text. For *tmf* at $p=3$, see Example 2.27. These algebras are small enough for computations to be tractable, so if we can compute the H_R^*H-module structure on $H_R^*(Mf)$ for f a fake vector bundle twist, we can run the Adams spectral sequence and hope to compute $\pi_*(Mf)$. This is the content of our main theorem, Theorem 2.39.

The first step is to understand $H^*_R(Mf)$ as a vector space. In Lemma 2.24, we establish a Thom isomorphism

$$H^*_R(Mf) \xrightarrow{\cong} H^*(X;\mathbb{Z}/2)\cdot U, \tag{0.4}$$

where $U \in H^0_R(Mf)$ is the Thom class. Using this, we can state our main theorem:

Theorem (Theorem 2.39)**.** *Let X be a topological space.*

(1) *Given $a \in H^1(X;\mathbb{Z}/2)$ and $c \in H^3(X;\mathbb{Z})$, let $f_{a,c} : X \to B\mathrm{GL}_1(ku)$ be the corresponding fake vector bundle twist. $H^*_{ku}(M^{ku}f_{a,c})$ is a $\mathcal{E}(1)$-module with Q_0-and Q_1-actions by*

$$\begin{aligned} Q_0(Ux) &\coloneqq Uax + UQ_0(x)\\ Q_1(Ux) &\coloneqq U(c \bmod 2 + a^3)x + UQ_1(x). \end{aligned}$$

(2) *Given $a \in H^1(X;\mathbb{Z}/2)$ and $b \in H^2(X;\mathbb{Z}/2)$, let $f_{a,b}\colon X \to B\mathrm{GL}_1(ko)$ be the corresponding fake vector bundle twist. $H^*_{ko}(M^{ko}f_{a,b})$ is an $\mathcal{A}(1)$-module with Sq^1-and Sq^2-actions*

$$\begin{aligned} \mathrm{Sq}^1(Ux) &\coloneqq U(ax + \mathrm{Sq}^1(x))\\ \mathrm{Sq}^2(Ux) &\coloneqq U(bx + a\mathrm{Sq}^1(x) + \mathrm{Sq}^2(x)). \end{aligned}$$

(3) *Given $a \in H^1(X;\mathbb{Z}/2)$, and $d \in SH^4(X)$, let $f_{a,d}\colon X \to B\mathrm{GL}_1(tmf)$ be the corresponding fake vector bundle twist. $H^*_{tmf}(M^{tmf}f_{a,d})$ is an $\mathcal{A}(2)$-module with Sq^1-and Sq^2-action the same as (2) above, and Sq^4-action*

$$\mathrm{Sq}^4(Ux) = U(\delta x + (t(d)a + \mathrm{Sq}^1(t(d)))\mathrm{Sq}^1(x) + t(d)\mathrm{Sq}^2(x) + a\mathrm{Sq}^3(x) + \mathrm{Sq}^4(x)).$$

Furthermore, $H^*_{tmf}(M^{tmf}f_{0,d};\mathbb{Z}/3)$ is an $\mathcal{A}^{tmf}$-module with β and $\mathcal{P}^1$ actions

$$\begin{aligned} \beta(Ux) &\coloneqq U\beta(x)\\ \mathcal{P}^1(Ux) &\coloneqq U((d \bmod 3)x + \mathcal{P}^1(x)). \end{aligned}$$

This theorem computes the inputs to the Baker-Lazarev Adams spectral sequences for $H\mathbb{Z}$, *ku*, *ko*, and *tmf* for the Thom spectra we study. We find three avatars of this fact:

(1) In Corollary 2.45, we describe in all degrees the E_2-page of the Baker-Lazarev Adams spectral sequence for a fake vector bundle twist of $H\mathbb{Z}$, *ku*, *ko*, or *tmf*.
(2) In Theorem 2.57, we describe in low degrees the E_2-page of the Baker-Lazarev Adams spectral sequence for a fake vector bundle twist of *MTSO*, *MTSpin*^{c}, *MTSpin*, or *MTString*.
(3) In Corollary 2.59, we describe variants of the Baker-Lazarev Adams spectral sequence for fake vector bundle twists of *MTSO*, *MTSpin*^{c}, and *MTSpin*, and compute the E_2-pages in all degrees.

We then give three examples of applications of our techniques.

(1) In §3.1, we use Theorem 2.57 to compute low-dimensional G-bordism groups for $G = \mathrm{Spin}\times_{\{\pm 1\}}\mathrm{SU}_8$. These are the twisted spin bordism groups for a twist over $B(\mathrm{SU}_8/\{\pm 1\})$ which is not a vector bundle twist. In [DY24b], we discussed an application of Ω_5^G to an anomaly cancellation question in 4-dimensional $\mathcal{N} = 8$ supergravity; using Theorem 2.57, we can give a much simpler calculation of Ω_5^G than appears in [DY24b, Theorem 4.26]. See Kuroda [Kur25] for more computations of twisted spin bordism groups using Theorem 2.39.
(2) In §3.2, we study twisted string bordism groups for a non-vector bundle twist over $B((E_8 \times E_8)\rtimes \mathbb{Z}/2)$, where $\mathbb{Z}/2$ acts on $E_8\times E_8$ by swapping the factors. These bordism groups have

applications in the study of the $E_8 \times E_8$ heterotic string; see [Deb24] for more information. Here, we work through the 3-primary calculation, simplifying a computation in [Deb24].

(3) In §3.3, we reprove a result of Devalapurkar [Dev23, Remark 2.3.16] describing $H\mathbb{Z}/2$ as a *ku*-module Thom spectrum; Devalapurkar's proof uses different methods.

Our theorems proceed similarly for several different families of spectra. One naturally wonders if there are more families out there. Specifically, there is a spectrum for which many but not all of the ingredients of our proofs were present at the time we wrote the first version of this paper.

Question 0.5 (Remark 1.18). Let $\mathit{tmf}_1(3)$ denote the connective spectrum of topological modular forms with a level structure for the congruence subgroup $\Gamma_1(3) \subset \mathrm{SL}_2(\mathbb{Z})$ [HL16]. Is there a tangential structure $\xi\colon B \to B\mathrm{O}$ such that $MT\xi$ is an E_∞-ring spectrum with an E_∞-ring map $MT\xi \to \mathit{tmf}_1(3)$ which is an isomorphism on low-degree homotopy groups after 2-completion?

If such a spectrum exists, then one could use our approach to run the Baker-Lazarev Adams spectral sequence to compute twisted $\mathit{tmf}_1(3)$-homology; the needed change-of-rings formula for $\mathit{tmf}_1(3)$ is due to Mathew [Mat16, Theorem 1.2].

Devalapurkar [Dev22] constructed a tangential structure called a *string^h structure* and an E_∞-ring map $\sigma_1(3)\colon \mathit{MTString}^h \to \mathit{tmf}_1(3)_{(2)}$, answering most of Question 0.5; that any such orientation is an isomorphism on low-degree homotopy groups after 2-completion was shown in [DY24a, Corollary 3.53]. We plan to study the Baker-Lazarev Adams spectral sequence for $\mathit{tmf}_1(3)$-module Thom spectra in future work.

Outline. §1 is about twists and Thom spectra. First, in §1.1, we review Ando-Blumberg-Gepner-Hopkins-Rezk's theory of Thom spectra [ABG+14a, ABG+14b] and discuss some constructions and lemmas we need later in the paper. Then, in §1.2, we construct fake vector bundle twists for the four families of ring spectra that we study in this paper: *MTSO* and $H\mathbb{Z}$ in §1.2.1; MTSpin^c, *ku*, and *KU* in §1.2.2; *MTSpin*, *ko*, and *KO* in §1.2.3; and *MTString*, *tmf*, *Tmf*, and *TMF* in §1.2.4.

In §2 we study the Adams spectral sequence for the Thom spectra of these twists. We begin in §2.1 by reviewing how the change-of-rings story simplifies Adams computations for vector bundle Thom spectra. Then, in §2.2, we introduce Baker-Lazarev's R-module Adams spectral sequence [BL01]. In §2.3 we prove Theorem 2.39 computing the input to the Baker-Lazarev Adams spectral sequence for the Thom spectra of our fake vector bundle twists.

We conclude in §3 with some applications and examples of computations using the main theorem: a twisted spin bordism example in §3.1 and an application to U-duality anomaly cancellation; a twisted string bordism example in §3.2 motivated by anomaly cancellation in heterotic string theory; and a twisted *ku*-homology example in §3.3 exhibiting $H\mathbb{Z}/2$ as the 2-completion of a *ku*-module Thom spectrum.

1. Thom spectra and twists à la Ando-Blumberg-Gepner-Hopkins-Rezk

1.1. The Ando-Blumberg-Gepner-Hopkins-Rezk approach to Thom spectra.

In this subsection we introduce Ando-Blumberg-Gepner-Hopkins-Rezk's theory of Thom spectra [ABG+14a, ABG+14b] and recall the key facts we need for our theorems.[3] In this paper, we only need to work

[3]Here and throughout the paper, we work with the symmetric monoidal ∞-category of spectra constructed by Lurie [Lur17, §1.4], where by "∞-category" we always mean quasicategory. In §2, we use work of Baker-Lazarev [BL01], who work with a different model of spectra, the $\mathbb{S}$-modules of Elmendorf-Kriz-Mandell-May [EKMM97, S 2.1]. The equivalence between the ∞-category presented by the model category of $\mathbb{S}$-modules and Lurie's ∞-category of spectra follows from work of Mandell-May-Schwede-Shipley [MMSS01], Schwede [Sch01], and Mandell-May [MM02].

with E_∞-ring spectra, and we will state some theorems in only the generality we need, which is less general than what Ando-Blumberg-Gepner-Hopkins-Rezk prove.

By an ∞-*group* we mean a grouplike E_1-space, which is a homotopically invariant version of topological group. By an *abelian ∞-group* we mean a grouplike E_∞-space.

Definition 1.1 (May [May77, §III.2])**.** Let R be an E_∞-ring spectrum. The *group of units* of R is the abelian ∞-group $\mathrm{GL}_1(R)$ defined to be the following pullback:

$$\begin{tikzcd} \mathrm{GL}_1(R) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega^\infty R \arrow[d] \\ \pi_0(R)^\times \arrow[r] & \pi_0(R)\,. \end{tikzcd} \tag{1.2}$$

The pullback (1.2) takes place in the ∞-category of abelian ∞-groups. As the three legs of the pullback diagram (1.2) are functorial in R, $\mathrm{GL}_1(R)$ is also functorial in R.

Since $\mathrm{GL}_1(R)$ is an ∞-group, it has a classifying space $B\mathrm{GL}_1(R)$; we refer to a map $X \to B\mathrm{GL}_1(R)$ as a *twist* of R over X. There is a sense in which $B\mathrm{GL}_1(R)$ carries the universal local system of R-*lines*, or free R-module spectra of rank 1: see [ABG+14a, Corollary 2.14].

Example 1.3. If A is a commutative ring and $R = HA$, then the equivalence of abelian ∞-groups $\pi_0\colon \Omega^\infty HA \xrightarrow{\cong} A$ induces an equivalence of abelian ∞-groups $\mathrm{GL}_1(R) \simeq A^\times$.

Let $\mathcal{M}od_R$ denote the ∞-category of R-module spectra and $\mathcal{L}ine_R$ denote the ∞-category of R-lines, and let $\pi_{\le\infty}(X)$ denote the fundamental ∞-groupoid of a space X. The identification $|\mathcal{L}ine_R| \xrightarrow{\cong} B\mathrm{GL}_1(R)$ [ABG+14a, Corollary 2.14] allows us to reformulate the inclusion $\mathcal{L}ine_R \hookrightarrow \mathcal{M}od_R$ as a functor $M\colon \pi_{\le\infty}(B\mathrm{GL}_1(R)) \to \mathcal{M}od_R$, which one can think of as sending a point in $B\mathrm{GL}_1(R)$ to the R-line which is the fiber of the universal local system of R-lines on $B\mathrm{GL}_1(R)$. In the rest of this paper, we will simply write X for $\pi_{\le\infty}(X)$, as we will never be in a situation where this causes ambiguity.

Definition 1.4 ([ABG+14a, Definition 2.20])**.** Let R be an E_∞-ring spectrum and $f\colon X \to B\mathrm{GL}_1(R)$ be a twist of R. The *Thom spectrum* $M^R f$ of the map f is the colimit of the X-shaped diagram

$$X \xrightarrow{f} B\mathrm{GL}_1(R) \longrightarrow \mathcal{M}od_R\,. \tag{1.5}$$

When R is clear from context, we will write Mf for $M^R f$.

By construction, Mf is an R-module spectrum. If the reader is familiar with the definition of a Thom spectrum associated to a virtual vector bundle, this definition is related but more general.

Example 1.6 (Thom spectra from vector bundles)**.** Let $V \to X$ be a virtual stable vector bundle of rank zero; V is classified by a map $f_V\colon X \to B\mathrm{O}$. There is a map of abelian ∞-groups $J\colon B\mathrm{O} \to B\mathrm{GL}_1(\mathbb{S})$ called the *J-homomorphism*, where $B\mathrm{O}$ has the abelian ∞-group structure induced by direct sum of (rank-zero virtual) vector bundles [Whi42]. Theorems of Lewis [LMSM86, Chapter IX] and Ando-Blumberg-Gepner-Hopkins-Rezk [ABG+14a, Corollary 3.24] together imply that the Thom spectrum X^V in the usual sense is naturally equivalent to the Thom spectrum $M(J \circ f_V)$ in the Ando-Blumberg-Gepner-Hopkins-Rezk sense.

Likewise, these papers show that commutative algebras in the category of $\mathbb{S}$-modules correspond to E_∞-rings in the ∞-category of spectra.

Example 1.7 (Trivial twists)**.** Suppose that the map $f\colon X \to B\mathrm{GL}_1(R)$ is null-homotopic. Then by definition, the colimit of (1.5) is $R \wedge X_+$; more precisely, a null-homotopy of f induces an equivalence of R-module spectra $Mf \simeq R \wedge X_+$.

We will need the following fact a few times. We believe it is known, but were unable to locate a proof in this generality in the literature.

Lemma 1.8. *Let $g\colon R_1 \to R_2$ be a map of E_∞-ring spectra and $f\colon X \to B\mathrm{GL}_1(R_1)$ be a twist. Then there is an equivalence of R_2-module spectra*

$$M^{R_2}(g \circ f) \xrightarrow{\simeq} M^{R_1} f \wedge_{R_1} R_2. \tag{1.9}$$

When $R_1 = \mathbb{S}$, Ando-Blumberg-Gepner-Hopkins-Rezk [ABG+14b, §1.2] mention that this lemma is a straightforward consequence of a different, equivalent definition of the Thom spectrum [ABG+14b, Definition 3.13].

Proof. We will show that the diagram

$$\begin{tikzcd} B\mathrm{GL}_1(R_1) \arrow[r, "g"] \arrow[d, "M^{R_1}"'] & B\mathrm{GL}_1(R_2) \arrow[d, "M^{R_2}"] \\ \mathcal{M}od_{R_1} \arrow[r, "- \wedge_{R_1} R_2"'] & \mathcal{M}od_{R_2} \end{tikzcd} \tag{1.10}$$

is (homotopy) commutative, where just as above we identify the spaces $B\mathrm{GL}_1(R_i)$ with their fundamental ∞-groupoids. Once we know this, the lemma is immediate from the colimit definition of $M^{R_2}(g \circ f)$ in Definition 1.4: replace $M^{R_2} \circ g \circ f$ with $(- \wedge_{R_1} R_2) \circ M^{R_1} \circ f$.

The key obstacle in establishing commutativity of (1.10) is that $g\colon B\mathrm{GL}_1(R_1) \to B\mathrm{GL}_1(R_2)$ comes from maps of spectra via (1.2), but $- \wedge_{R_1} R_2$ has a more module-theoretic flavor. The resolution, which is the same as in the proof of [ABG+14a, Proposition 2.9], is that the three other pieces of the pullback (1.2) defining GL_1, namely Ω^∞, π_0, and $\pi_0(-)^\times$, have module-theoretic interpretations: there are homotopy equivalences of abelian ∞-groups $\Omega^\infty R \xrightarrow{\simeq} \mathrm{End}_R(R)$, and likewise $\pi_0(R) \xrightarrow{\simeq} \pi_0(\mathrm{End}_R(R))$ and $\pi_0(R)^\times \xrightarrow{\simeq} \pi_0(\mathrm{End}_R(R))^\times$. And all of these identifications are compatible with the tensor product functor $\mathcal{M}od_{R_1} \to \mathcal{M}od_{R_2}$, thus also likewise for their classifying spaces, establishing commutativity of (1.10). ☐

The usual Thom diagonal for a Thom space X^V gives $H^*(X^V; \mathbb{Z}/2)$ the structure of a module over $H^*(X; \mathbb{Z}/2)$. One can generalize this for R-module Thom spectra as follows.

Definition 1.11 (Thom diagonal [ABG+14b, §3.3])**.** Let R be an E_∞-ring spectrum and $f\colon X \to B\mathrm{GL}_1(R)$ be a twist. The *Thom diagonal* for Mf is an R-module map

$$Mf \xrightarrow{\Delta^t} Mf \wedge R \wedge X_+ \tag{1.12}$$

defined by applying the Thom spectrum functor to the maps $f\colon X \to B\mathrm{GL}_1(R)$ and $(f, 0)\colon X \times X \to B\mathrm{GL}_1(R)$: if $\Delta\colon X \to X \times X$ is the diagonal map, then $f = \Delta^*(f, 0)$, so Δ induces the desired map Δ^t of R-module Thom spectra in (1.12).

See Beardsley [Bea23, §4.3] for a nice coalgebraic interpretation of the Thom diagonal.

1.2. **Constructing non-vector-bundle twists.** Let X and Y be E_∞-spaces and $f_1\colon X \to Y$ and $f_2\colon Y \to B\mathrm{GL}_1(\mathbb{S})$ be E_∞-maps. Ando-Blumberg-Gepner [ABG18, Theorem 1.7] show that the E_∞-structure on $f_2 \circ f_1$ induces an E_∞-ring structure on $M(f_2 \circ f_1)$.

Theorem 1.13 (May-Quinn-Ray [May77, Lemma IV.2.6])**.** *Let R be an E_∞-ring spectrum. The data of an E_∞-ring map $\rho\colon M(f_2\circ f_1)\to R$ induces a map $T_{f_1,f_2}\colon Y/X\to B\mathrm{GL}_1(R)$ of abelian ∞-groups.*[4]

An E_∞-ring map ρ of this kind is often called an $M(f_2\circ f_1)$*-orientation* of R.

Remark 1.14. May-Quinn-Ray state this result only for $R = M(f_2\circ f_1)$ and $\rho = \mathrm{id}$; Beardsley [Bea17, Theorem 1] provides another, quite different, proof of this case and uses it to obtain many commonly-studied twists of various cohomology theories. We will usually apply it for maps to $B\mathrm{O}$ and implicitly compose with the E_∞-map $J\colon B\mathrm{O}\to B\mathrm{GL}_1(\mathbb S)$, like in Example 1.6.

The more general version of May-Quinn-Ray's theorem appearing in Theorem 1.13 follows immediately from the version in [May77]: the abelian ∞-group $B\mathrm{GL}_1(R)$ is natural in the E_∞-ring spectrum R, so given $\rho\colon M(f_2\circ f_1)\to R$ as in the statement of Theorem 1.13, we may compose May-Quinn-Ray's map $Y/X\to B\mathrm{GL}_1(M(f_2\circ f_1))$ with the base change map $B\mathrm{GL}_1(M(f_2\circ f_1))\to B\mathrm{GL}_1(R)$ to finish.

Remark 1.15. Just as the map $J\colon B\mathrm{O}\to B\mathrm{GL}_1(\mathbb S)$ is related to the classical J-homomorphism $\pi_*(\mathrm{O})\to\pi_*(\mathbb S)$, the map $T_{f_1,f_2}\colon Y/X\to B\mathrm{GL}_1(M(f_2\circ f_1))$ from Theorem 1.13 is related to Harris' generalized J-homomorphisms [Har69]: see May-Quinn-Ray [May77, §IV.2]. These generalized J-homomorphisms also appear in work of Ray [Ray71, Ray74], Gozman [Goz77], and Bier-Ray [BR78].

Theorem 1.16 (Beardsley [Bea17, Theorem 1])**.** *For $R = M(f_2\circ f_1)$, there is a natural equivalence $M^R T_{f_1,f_2}\xrightarrow{\simeq} M^{\mathbb S} f_2$.*

In this paper we consider twisted R-(co)homology for several different ring spectra R. These spectra are organized into several families: in each family there is a Thom spectrum Mf, another ring spectrum R, and a map of ring spectra $Mf\to R$ which is an isomorphism on homotopy groups in low degrees. In the context of a specific family, we will refer to Mf as the *big sibling* and R as the *little sibling*. The four families we consider in this paper are $(\mathit{MTSO}, H\mathbb Z)$, $(\mathit{MTSpin}, \mathit{ko})$, $(\mathit{MTSpin}^c, \mathit{ku})$, and $(\mathit{MTString}, \mathit{tmf})$:[5]

- The map $\Omega_0^{\mathrm{SO}}\xrightarrow{\cong}\mathbb Z$ counting the number of points refines to a map of E_∞-ring spectra $\mathit{MTSO}\to H\mathbb Z$. Work of Thom [Tho54, Théorème IV.13] shows this map is an isomorphism on homotopy groups in degrees 3 and below.
- The Atiyah-Bott-Shapiro map $\mathit{MTSpin}^c\to \mathit{ku}$ [ABS64] was shown to be a map of E_∞-ring spectra by Joachim [Joa04], and Anderson-Brown-Peterson [ABP67] showed this map is an isomorphism on homotopy groups in degrees 3 and below.
- Joachim [Joa04] also showed the real Atiyah-Bott-Shapiro map $\mathit{MTSpin}\to \mathit{ko}$ [ABS64] is a map of E_∞-ring spectra, and Milnor [Mil63] showed this map is an isomorphism on homotopy groups in degrees 7 and below.

[4] May-Quinn-Ray proves that T_{f_1,f_2} is a map of $\mathcal{I}_*$*-functors*. This implies it is a map of E_∞-spaces, as discussed in [May77, §I.1].

[5] In the homotopy theory literature, it is common to refer to bordism spectra *MSO*, *MSpin*, etc., corresponding to the bordism groups of manifolds with orientations, resp. spin structures, on the stable normal bundle. In the mathematical physics literature, one sees *MTSO*, *MTSpin*, etc., corresponding to the same structures on the stable *tangent* bundle. If $\xi\colon B\to B\mathrm{O}$ is a tangential structure such that the map ξ is a map of abelian ∞-groups, as is the case for O, SO, Spin^c, Spin, and String, there is a canonical equivalence $M\xi\xrightarrow{\simeq} MT\xi$. For other tangential structures, this is not necessarily true: in particular, $\mathit{MPin}^\pm\simeq \mathit{MTPin}^\mp$.

- Ando-Hopkins-Rezk [AHR10] produced a map of E_∞-ring spectra $\sigma\colon \mathit{MTString} \to \mathit{tmf}$, which Hill [Hil09, Theorem 2.1] showed is an isomorphism on homotopy groups in degrees 15 and below.

For all of these cases but *MTString*, one can 2-locally decompose the big sibling into a sum of modules over the little sibling: Wall [Wal60] produced a 2-local equivalence

$$\mathit{MTSO}_{(2)} \xrightarrow{\simeq} H\mathbb{Z}_{(2)} \vee \Sigma^4 H\mathbb{Z}_{(2)} \vee \Sigma^5 H\mathbb{Z}/2 \vee \cdots , \tag{1.17a}$$

and Anderson-Brown-Peterson [ABP67] produced 2-local equivalences

$$\mathit{MTSpin}_{(2)} \xrightarrow{\simeq} \mathit{ko}_{(2)} \vee \Sigma^8 \mathit{ko}_{(2)} \vee \Sigma^{10}(\mathit{ko} \wedge J)_{(2)} \vee \ldots \tag{1.17b}$$

$$\mathit{MTSpin}^c_{(2)} \xrightarrow{\simeq} \mathit{ku}_{(2)} \vee \Sigma^4 \mathit{ku}_{(2)} \vee \Sigma^8 \mathit{ku}_{(2)} \vee \Sigma^8 \mathit{ku}_{(2)} \vee \cdots , \tag{1.17c}$$

where J is a certain spectrum such that $\Sigma^2 \mathit{ko} \wedge J$ is the Postnikov 2-connected cover of *ko*.[6] It is not known whether *tmf* is a summand of *MTString* (see, e.g., [Lau04, Dev19, LS19, Pet19, Dev24]) so we do not know if there is a splitting like in the three other cases.

Remark 1.18 (String bordism with level structures?). Associated to congruence subgroups $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ there are "topological modular forms with level structure:" Hill-Lawson [HL16] and Meier [Mei23] construct E_∞-ring spectra $\mathit{TMF}(\Gamma)$, $\mathit{Tmf}(\Gamma)$, and $\mathit{tmf}(\Gamma)$ with maps between them like for vanilla *tmf*.[7] The case $\Gamma = \Gamma_1(3)$ is especially interesting, as many of the several ingredients we need for the proof of our main theorem are known to be true for $\mathit{tmf}(\Gamma_1(3))$ (usually written $\mathit{tmf}_1(3)$): by work of Mathew [Mat16, Theorem 1.2], there is a change-of-rings theorem allowing one to simplify 2-primary Adams spectral sequence computations to an easier subalgebra (see §2.1), but at the time we originally wrote this paper, it was not yet known how to construct an E_∞-ring Thom spectrum M with an orientation $M \to \mathit{tmf}_1(3)$ that is an isomorphism on low-degree homotopy groups.[8] The existence of such a spectrum M would lead to generalizations of our main theorems to twists of $\mathit{tmf}_1(3)$-homology.[9]

Since we finished the first version of this paper, we learned of a tangential structure, called a "string^h structure," whose Thom spectrum has an E_∞-ring map to $\mathit{tmf}_1(3)$, as shown by Devalapurkar [Dev22] (see also [DY24a] for another construction). In future work, we (the authors of this paper) will use this orientation to study $\mathit{tmf}_1(3)$-module Thom spectra.

1.2.1. *Twists of MTSO and $H\mathbb{Z}$.* We walk through the implications of Theorems 1.13 and 1.16 in a relatively simple setting, addressing

- what cohomology classes define twists of *MTSO* and $H\mathbb{Z}$ by way of Theorem 1.13,
- what the corresponding twisted bordism and cohomology groups are, and
- what Theorem 1.16 implies the Thom spectrum of the universal twist is.

Letting $f_2\colon X \to B\mathrm{O}$ be the identity and $f_1\colon X \to Y$ be $B\mathrm{SO} \to B\mathrm{O}$, we obtain twists of *MTSO*-oriented ring spectra, notably *MTSO* and $H\mathbb{Z}$, by maps to $B\mathrm{SO}/B\mathrm{O} \simeq K(\mathbb{Z}/2, 1)$, recovering a perspective of Hebestreit-Sagave [HS20, §1]. The map $B\mathrm{O} \to B\mathrm{O}/B\mathrm{SO}$ admits a section defined by regarding a map to $K(\mathbb{Z}/2, 1)$ as a real line bundle, so these twists are given by real line bundles in the

[6] The spin^c decomposition is implicit in [ABP67]; see Bahri-Gilkey [BG87b] for an explicit reference.

[7] Before the general constructions of Hill-Laswon and Meier, various examples of $\mathit{TMF}(\Gamma)$, $\mathit{Tmf}(\Gamma)$, and $\mathit{tmf}(\Gamma)$ were constructed by Behrens [Beh06, Beh07], Mahowald-Rezk [MR09], and Stojanoska [Sto12].

[8] See Wilson [Wil15] for results on closely related questions.

[9] A theorem of Meier [Mei23, Theorem 1.4] suggests this may also apply to twists of $\mathit{tmf}_1(n)$-homology for other values of n.

sense of Example 1.6. Specifically, a class $a\in H^1(X;\mathbb Z/2)$ defines a twist $f_a\colon X\to B\mathrm{GL}_1(\mathit{MTSO})$ by interpreting a as a map $X\to B\mathrm{O}/B\mathrm{SO}$ and invoking Theorem 1.13, and a defines a second twist g_a by choosing a real line bundle L_a with $w_1(L_a)=a$ (a contractible choice) and making the vector bundle twist as in Example 1.6, but $f_a\simeq g_a$ and so $M^{\mathit{MTSO}}f_a\simeq \mathit{MTSO}\wedge X^{L_a-1}$. Thus in a sense this example is redundant, as the main theorems of this paper are long known for vector bundle twists, but we include this example because we found it a useful parallel to to other families we study.

Let $\Omega_*^{\mathrm{SO}}(X,a)\coloneqq \pi_*(M^{\mathit{MTSO}}f_a)$. Using the vector bundle interpretation of this twist, $\Omega_*^{\mathrm{SO}}(X,a)$ has an interpretation as twisted oriented bordism groups, specifically the bordism groups of manifolds M with a map $h\colon M\to X$ and an orientation on $TM\oplus h^*L_a$. Alternatively, one could think of this as the bordism groups of manifolds M with a map $h\colon M\to X$ and a trivialization of the class $w_1(M)-h^*a$; this perspective will be useful in later examples of non-vector-bundle twists.

Theorem 1.16 then implies the Thom spectrum of

$$K(\mathbb Z/2,1)\overset{\simeq}{\to} B\mathrm{O}_1\xrightarrow{\sigma} B\mathrm{O}\longrightarrow B\mathrm{GL}_1(\mathbb S)\longrightarrow B\mathrm{GL}_1(\mathit{MTSO}),\tag{1.19}$$

is equivalent to MTO. Lemma 1.8 implies the Thom spectrum of (1.19) is $\mathit{MTSO}\wedge(B\mathrm{O}_1)^{\sigma-1}$, so we have reproved a theorem of Atiyah: $\mathit{MTSO}\wedge(B\mathrm{O}_1)^{\sigma-1}\simeq \mathit{MTO}$ [Ati61, Proposition 4.1].

The twist of $H\mathbb Z$ defined by a recovers the usual notion of integral cohomology twisted by a class in $H^1(X;\mathbb Z/2)$.

1.2.2. *Twists of MTSpin^c, ku, and KU.* Our next family of examples includes spin^c bordism and complex K-theory following the perspective of Hebestreit-Sagave [HS20].[10] In Proposition 1.21 we use Theorem 1.13 to construct a map $K(\mathbb Z/2,1)\times K(\mathbb Z,3)\to B\mathrm{GL}_1(\mathit{MTSpin}^c)$, defining twists of MTSpin^c, ku, and KU by classes in $H^1(–;\mathbb Z/2)$ and $H^3(–;\mathbb Z)$. These recover the usual twists of K-theory by these cohomology classes studied by [DK70, Ros89, AS04, ABG10] (Lemma 1.31), and in Lemma 1.26 we use work of Hebestreit-Joachim [HJ20, Proposition 3.3.6] to describe the homotopy groups of the corresponding MTSpin^c-module Thom spectra as bordism groups of manifolds with certain kinds of twisted spin^c structures.

The Atiyah-Bott-Shapiro orientation [ABS64, Joa04] defines ring homomorphisms $\mathit{Td}\colon \mathit{MTSpin}^c\to ku\to KU$, so by Theorem 1.13 there are maps

$$B\mathrm{O}/B\mathrm{Spin}^c\longrightarrow B\mathrm{GL}_1(\mathit{MTSpin}^c)\xrightarrow{\mathit{Td}} B\mathrm{GL}_1(ku)\longrightarrow B\mathrm{GL}_1(KU),\tag{1.20}$$

i.e. twists of MTSpin^c, ku, and KU by maps to $B\mathrm{O}/B\mathrm{Spin}^c$.[11]

Proposition 1.21. *The map $K(\mathbb Z/2,1)\to B\mathrm{O}$ defined by the tautological line bundle induces a homotopy equivalence of spaces*

$$B\mathrm{O}/B\mathrm{Spin}^c\xrightarrow{\simeq} K(\mathbb Z/2,1)\times K(\mathbb Z,3),\tag{1.22}$$

implying that MTSpin^c, ku, and KU can be twisted over a space X by classes $a\in H^1(X;\mathbb Z/2)$ and $c\in H^3(X;\mathbb Z)$.

---

[10]Hebestreit-Sagave use a different model for parametrized homotopy theory than Ando-Blumberg-Gepner-Hopkins-Rezk's; these two perspectives are shown to be equivalent by Hebestreit-Sagave-Schlichtkrull [HSS20, Theorems 1.6 and 1.8].

[11]The map (1.20) is nowhere near a homotopy equivalence; for example, it misses the "higher twists" of KU studied in, e.g., [DP15, Pen16]. However, Gómez [Góm10] has proven that if G is a compact Lie group, the map $[BG, B\mathrm{O}/B\mathrm{Spin}^c]\to[BG, B\mathrm{GL}_1(KU)]$ induced by (1.20) is an equivalence: there are no higher twists of KU over BG.

See Beardsley-Luecke-Morava [BLM23, Propositions 4.1 and 5.15] for a closely related splitting result.

Proof. We want to apply the third isomorphism theorem to the sequence of maps of abelian ∞-groups $B\mathrm{Spin}^c \to B\mathrm{SO} \to B\mathrm{O}$ to obtain a short exact sequence

$$1 \longrightarrow B\mathrm{SO}/B\mathrm{Spin}^c \longrightarrow B\mathrm{O}/B\mathrm{Spin}^c \longrightarrow B\mathrm{O}/B\mathrm{SO} \longrightarrow 1. \tag{1.23}$$

It is not immediate how to do this in the ∞-categorical setting, but we can do it. Instead of a short exact sequence, we obtain a cofiber sequence, and in a stable ∞-category, the third isomorphism theorem for cofiber sequences is a consequence of the octahedral axiom. The ∞-category of abelian ∞-groups is not stable, as it is equivalent to the ∞-category of connective spectra, but this ∞-category embeds in the stable ∞-category $\mathcal{S}p$ of all spectra, allowing us to make use of stability in certain settings: specifically, cofiber sequences $A \to B \to C$ of abelian ∞-groups for which the induced map $\pi_0(B) \to \pi_0(C)$ is surjective; these cofiber diagrams map to cofiber diagrams in $\mathcal{S}p$, so we may invoke the octahedral axiom in $\mathcal{S}p$. All cofiber sequences of abelian ∞-groups we discuss in this paper satisfy this π_0-surjectivity property, so we will not discuss it further. In particular, we obtain the cofiber sequence (1.23). Throughout this paper, whenever we write a short exact sequence of abelian ∞-groups, we mean a cofiber sequence.

A similar argument allows one to deduce that fiber and cofiber sequences coincide for abelian ∞-groups from the analogous fact for stable ∞-categories, assuming the same π_0-surjectivity hypothesis. Since $B\mathrm{Spin}^c$ is the fiber of $\beta w_2\colon B\mathrm{SO} \to K(\mathbb{Z}, 3)$, which is a map of abelian ∞-groups since βw_2 satisfies the Whitney sum formula for oriented vector bundles, the cofiber $B\mathrm{SO}/B\mathrm{Spin}^c$ is equivalent, as abelian ∞-groups, to $K(\mathbb{Z}, 3)$.[12] Here, $\beta\colon H^k(-;\mathbb{Z}/2) \to H^{k+1}(-;\mathbb{Z})$ is the Bockstein. Likewise, $B\mathrm{SO}$ is the fiber of $w_1\colon B\mathrm{O} \to K(\mathbb{Z}/2, 1)$, which is a map of abelian ∞-groups, so $B\mathrm{O}/B\mathrm{SO} \simeq K(\mathbb{Z}/2, 1)$.

The quotient $B\mathrm{O} \to B\mathrm{O}/B\mathrm{SO} \simeq K(\mathbb{Z}/2, 1)$ admits a section given by the tautological real line bundle $K(\mathbb{Z}/2, 1) \simeq B\mathrm{O}_1 \to B\mathrm{O}$; composing $K(\mathbb{Z}/2, 1) \to B\mathrm{O}$ with the quotient $B\mathrm{O} \to B\mathrm{O}/B\mathrm{Spin}^c$ we obtain a section of (1.23). That section splits (1.23), which implies the proposition statement. $\square$

Definition 1.24. Given classes $a \in H^1(X;\mathbb{Z}/2)$ and $c \in H^3(X;\mathbb{Z})$, we call the twist $f_{a,c}\colon X \to B\mathrm{GL}_1(\mathit{MTSpin}^c)$ that Proposition 1.21 associates to a and c the *fake vector bundle twist* for a and c, and likewise for the induced twists of ku and KU.

The twist $f_{a,c}$ arises from a vector bundle twist if there is a vector bundle $V \to X$ such that $w_1(V) = a$ and $\beta(w_2(V)) = c$, but there are choices of X, a, and c for which no such vector bundle exists, e.g. if c is not 2-torsion.

Now that we have defined these twists, we get to the business of interpreting them.

Definition 1.25. Given X, a, and c as above, let $\Omega_*^{\mathrm{Spin}^c}(X, a, c)$ denote the groups of bordism classes of manifolds M with a map $f\colon M \to X$ and trivializations of $w_1(M) - f^*(a)$ and $\beta(w_2(M)) - f^*(c)$.

This notion of twisted spin^c bordism, in the special case $a = 0$, was first studied by Douglas [Dou06, §5], and implicitly appears in Freed-Witten's work [FW99] on anomaly cancellation.

Lemma 1.26 (Hebestreit-Joachim [HJ20, Corollary 3.3.8])**.** *There is a natural isomorphism* $\pi_*(M^{\mathit{MTSpin}^c} f_{a,c}) \xrightarrow{\cong} \Omega_*^{\mathrm{Spin}^c}(X, a, c)$.

[12] This approach to the $K(\mathbb{Z}, 3)$ twist appears in Hebestreit-Sagave [HS20, §1].

Remark 1.27. Hebestreit-Joachim [HJ20] use a different framework for twists based on May-Sigurdsson's parametrized homotopy theory [MS06]; Ando-Blumberg-Gepner [ABG18, Appendix B] prove a comparison theorem that allows us to pass between May-Sigurdsson's framework and Ando-Blumberg-Gepner-Hopkins-Rezk's. Additionally, Hebestreit-Joachim work with twisted spin bordism and KO-theory, but for the complex case the arguments are essentially the same.

Remark 1.28. Though Hebestreit-Joachim [HJ20, Corollary 3.3.8] state their results as isomorphisms of bordism groups, their proof actually proves an equivalence of spectra. Focusing on the spin^c case, given X, a, and c as above, let $\xi_{a,c}\colon B(a,c)\to B\mathrm{O}$ be the fiber of the map

$$(w_1 - a, \beta(w_2 - c))\colon B\mathrm{O}\times X \to K(\mathbb{Z}/2,1)\times K(\mathbb{Z},3), \tag{1.29}$$

so that $\Omega_*^{\xi_{a,c}} \cong \Omega_*^{\mathrm{Spin}^c}(X,a,c)$. In the twisted spin^c setting, Hebestreit-Joachim's techniques in fact prove that there is a canonical MTSpin^c-module equivalence

$$M^{\mathit{MTSpin}^c} f_{a,c} \xrightarrow{\simeq} MT\xi_{a,c}, \tag{1.30}$$

where the MTSpin^c-module structure on $MT\xi_{a,c}$ arises from the canonical $\xi_{a,c}$-structure on the sum of a spin^c vector bundle and a $\xi_{a,c}$-structured vector bundle.

Similar considerations are true for the twisted spin and string structures we study later in this section.

Lemma 1.31 (Hebestreit-Sagave [HS20])**.** *With X, a, and c as above, the homotopy groups of $M^{KU}f_{a,c}$ are naturally isomorphic to the twisted K-theory groups of* [DK70, Ros89, AS04, ABG10]*.*

Example 1.32. Theorem 1.16 computes a few example of MTSpin^c-module Thom spectra for us.

(1) Letting $X = Y = B\mathrm{O}/B\mathrm{Spin}^c$ and $f_1 = \mathrm{id}$, Theorem 1.16 implies that the Thom spectrum of the universal twist $B\mathrm{O}/B\mathrm{Spin}^c \to B\mathrm{GL}_1(\mathit{MTSpin}^c)$ is MTO. From a bordism point of view, this is the fact that since a and c pull back from $K(\mathbb{Z}/2,1)\times K(\mathbb{Z},3)$, they can be arbitrary classes, so the required trivializations of $w_1(M) - f^*(a)$ and $\beta(w_2(M)) - f^*(c)$ are uniquely specified by $a = w_1(M)$ and $c = \beta(w_2(M))$, so this notion of twisted spin^c structure is no structure at all.

(2) Let Y be as in the previous example and let $f_1\colon X\to Y$ be the map $K(\mathbb{Z},3)\simeq B\mathrm{SO}/B\mathrm{Spin}^c \to B\mathrm{O}/B\mathrm{Spin}^c$. Theorem 1.16 says the Thom spectrum of

$$K(\mathbb{Z},3) \longrightarrow B\mathrm{O}/B\mathrm{Spin}^c \longrightarrow B\mathrm{GL}_1(\mathit{MTSpin}^c) \tag{1.33}$$

is equivalent to MTSO. We stress that this twist by $K(\mathbb{Z},3)$ does not come from a vector bundle because all vector bundle twists of MTSpin^c are torsion and of the form $\beta(w_2(M))$, but the universal twist over $K(\mathbb{Z},3)$ is not.

Lemma 1.34. *The equivalence of spaces $B\mathrm{O}/B\mathrm{Spin}^c \simeq K(\mathbb{Z}/2,1)\times K(\mathbb{Z},3)$ from* (1.22) *is not an equivalence of ∞-groups.*

Beardsley-Luecke-Morava [BLM23, Corollary 4.9] prove a closely related result.

Proof. Suppose that this is an equivalence of ∞-groups. Then the inclusion $K(\mathbb{Z}/2,1)\to K(\mathbb{Z}/2,1)\times K(\mathbb{Z},3)\to B\mathrm{O}/B\mathrm{Spin}^c$ is a map of ∞-groups, so the composition

$$\varphi\colon K(\mathbb{Z}/2,1) \longrightarrow K(\mathbb{Z}/2,1)\times K(\mathbb{Z},3) \longrightarrow B\mathrm{O}/B\mathrm{Spin}^c \longrightarrow B\mathrm{GL}_1(\mathit{MTSpin}^c) \tag{1.35}$$

is a map of ∞-groups. By work of Ando-Blumberg-Gepner [ABG18, Theorem 1.7], this implies the Thom spectrum $M\varphi$ is an E_1-ring spectrum. We will explicitly identify $M\varphi$ and show this is not the case.

We saw above that the map $K(\mathbb{Z}/2,1) \to B\mathrm{O}/B\mathrm{Spin}^c$ factors through the map $K(\mathbb{Z}/2,1) \to B\mathrm{O}$ defined by the tautological line bundle $\sigma \to B\mathrm{O}_1 \simeq K(\mathbb{Z}/2,1)$, meaning that the twist (1.35) is the vector bundle twist of $MTSpin^c$ for the tautological line bundle $\sigma \to B\mathrm{O}_1$. Applying Lemma 1.8 with $R_1 = \mathbb{S}$ and $R_2 = MTSpin^c$, we conclude $M\varphi \simeq MTSpin^c \wedge (B\mathrm{O}_1)^{\sigma-1}$. Bahri-Gilkey [BG87a, BG87b] identify this spectrum with $MTPin^c$, which is known to not be an E_1-ring spectrum: for example, a E_1-ring structure induces a graded ring structure on homotopy groups, making $\pi_k(MTPin^c)$ into a $\pi_0(MTPin^c)$-module for all k, but $\pi_0 MTPin^c \cong \mathbb{Z}/2$ and $\pi_2(MTPin^c) \cong \mathbb{Z}/4$ [BG87b, Theorem 2]. ☐

1.2.3. *Twists of MTSpin, ko, and KO.* The real analogue of §1.2.2 is very similar; we summarize the story here, highlighting the differences. Once again this perspective is due to Hebestreit-Sagave [HS20]. Again there are E_∞ ring spectrum maps $MTSpin \xrightarrow{\widehat{A}} ko \to KO$ [ABS64, Joa04, AHR10], allowing us to use Theorem 1.13 to produce a sequence of maps

$$B\mathrm{O}/B\mathrm{Spin} \longrightarrow B\mathrm{GL}_1(MTSpin) \xrightarrow{\widehat{A}} B\mathrm{GL}_1(ko) \longrightarrow B\mathrm{GL}_1(KO). \tag{1.36}$$

Hebestreit-Sagave [HS20] and Freed-Hopkins [FH21, §10] use the ∞-group $B\mathrm{O}/B\mathrm{Spin}$ to study twists of spin bordism; Freed-Hopkins call it $\mathbf{P}$.

Proposition 1.37. *The map $K(\mathbb{Z}/2,1) \to B\mathrm{O}$ defined by the tautological line bundle induces a homotopy equivalence of spaces*

$$B\mathrm{O}/B\mathrm{Spin} \xrightarrow{\simeq} K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,2), \tag{1.38}$$

implying MTSpin, ko, and KO can be twisted over a space X by classes $a \in H^1(X;\mathbb{Z}/2)$ and $b \in H^2(X;\mathbb{Z}/2)$.

The proof is nearly the same as the proof of Proposition 1.21: fit $B\mathrm{O}/B\mathrm{Spin}$ into a split cofiber sequence with $B\mathrm{SO}/B\mathrm{Spin} \simeq K(\mathbb{Z}/2,2)$ (because $B\mathrm{Spin} \to B\mathrm{O}$ is the fiber of $w_2\colon B\mathrm{SO} \to K(\mathbb{Z}/2,2)$) and $B\mathrm{O}/B\mathrm{SO} \simeq K(\mathbb{Z}/2,1)$. See also Beardsley-Luecke-Morava [BLM23, Propositions 4.1 and 5.19], who prove a closely related splitting result, and Carmeli-Luecke [CL24, Theorem C] for an analogous splitting result in $B\mathrm{GL}_1(K(\mathbb{Z}))$.

Definition 1.39. We call the twist $f_{a,b}\colon X \to B\mathrm{GL}_1(MTSpin)$ associated to a and b the *fake vector bundle twist* for a and b, and likewise for the induced twists of ko and KO.

Remark 1.40. The space of homotopy self-equivalences of $K(\mathbb{Z}/2,1)\times K(\mathbb{Z}/2,2)$ is not connected: for example, if a denotes the tautological class in $H^1(K(\mathbb{Z}/2,1);\mathbb{Z}/2)$ and b is the tautological class in $H^2(K(\mathbb{Z}/2,2);\mathbb{Z}/2)$, the homotopy class of maps $\Phi\colon K(\mathbb{Z}/2,1)\times K(\mathbb{Z}/2,2) \to K(\mathbb{Z}/2,1)\times K(\mathbb{Z}/2,2)$ defined by the classes (a, a^2+b) is not the identity and is invertible. The choice of identification we made in (1.38) matters: if one uses a different identification, one obtains a different notion of fake vector bundle twist and a different formula in Definition 2.31 to make Theorem 2.39 true.

Lemma 1.41. (1.38) *is not an equivalence of ∞-groups.*

This is closely related to a theorem of Beardsley-Luecke-Morava [BLM23, Proposition 4.4].

One can prove Lemma 1.41 in the same way as Lemma 1.34, by pulling back along the section $K(\mathbb{Z}/2,1) \to B\mathrm{O}/B\mathrm{Spin}$ and observing that the Thom spectrum $MTSpin \wedge (B\mathrm{O}_1)^{\sigma-1}$ is not a ring spectrum in much the same way:[13] using the equivalence $MTSpin \wedge (BO_1)^{\sigma-1} \simeq MTPin^-$ [Pet68, §7] and the groups $\pi_0(MTPin^-) \cong \mathbb{Z}/2$ and $\pi_2(MTPin^-) \cong \mathbb{Z}/8$ [ABP69, KT90] to show $MTPin^-$ is not a ring spectrum. There is also another nice proof, which we give below.

Proof. If X is a space and Y is an ∞-group, the set $[X, Y]$ has a natural group structure. Therefore it suffices to find a space such that $[X, B\mathrm{O}/B\mathrm{Spin}]$ and $[X, K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2)]$ are non-isomorphic groups.

To calculate the addition in $[–, B\mathrm{O}/B\mathrm{Spin}]$, we use the fact that if two maps $f, g\colon X \to \mathrm{O}/B\mathrm{Spin}$ factor through $B\mathrm{O}$, meaning they are represented by rank-zero virtual vector bundles $V_f, V_g \to X$, then $f + g$ is the image of $V_f \oplus V_g$ under $B\mathrm{O} \to B\mathrm{O}/B\mathrm{Spin}$. This implies that for classes in the image of that quotient map, if we use (1.38) to identify two classes $\phi_1, \phi_2 \in [X, B\mathrm{O}/B\mathrm{Spin}]$ with pairs $\phi_i = (a_i \in H^1(X; \mathbb{Z}/2), b_i \in H^2(X; \mathbb{Z}/2))$, then addition follows the Whitney sum formula:

$$(a_1, b_1) \oplus (a_2, b_2) = (a_1 + a_2, b_1 + b_2 + a_1 a_2). \tag{1.42}$$

This is different from the componentwise addition on $K(\mathbb{Z}/2, 1)\times K(\mathbb{Z}/2, 2)$: for example, $[B\mathbb{Z}/2, K(\mathbb{Z}/2, 1)\times K(\mathbb{Z}/2, 2)] \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, but the map $[B\mathbb{Z}/2, B\mathrm{O}] \to [B\mathbb{Z}/2, B\mathrm{O}/B\mathrm{Spin}]$ is surjective, so using (1.42), one can show that $[B\mathbb{Z}/2, B\mathrm{O}/B\mathrm{Spin}] \cong \mathbb{Z}/4$. $\square$

Definition 1.43. Given X, a, and b as above, let $\Omega_*^{\mathrm{Spin}}(X, a, b)$ denote the groups of bordism classes of manifolds M with a map $f\colon M \to X$ and trivializations of $w_1(M) - f^*(a)$ and $w_2(M) - f^*(b)$.

B.L. Wang [Wan08, Definition 8.2] first studied these twists of spin bordism in the case $a = 0$.

Lemma 1.44 (Hebestreit-Joachim [HJ20, Corollary 3.3.8])**.** *There is a natural isomorphism* $\pi_*(M^{MTSpin} f_{a,b}) \xrightarrow{\cong} \Omega_*^{\mathrm{Spin}}(X, a, b)$.

Lemma 1.45 (Hebestreit-Sagave [HS20])**.** *With X, a, and b as above, the homotopy groups of $M^{KO} f_{a,b}$ are naturally isomorphic to the twisted KO-theory groups of* [DK70, HJ20].

Example 1.46. Theorem 1.16 implies the Thom spectrum of the universal twist of $MTSpin$ over $B\mathrm{O}/B\mathrm{Spin}$ is MTO, and of the universal twist over $K(\mathbb{Z}/2, 2) \simeq B\mathrm{SO}/B\mathrm{Spin}$ is $MTSO$. The former equivalence is due to Hebestreit-Joachim [HJ20, Observation 3.3.5], and latter equivalence is due to Beardsley [Bea17, §3].

1.2.4. *Twists of MTString, tmf, Tmf, and TMF.* The final family we consider in this paper is string bordism and topological modular forms. The story has a similar shape: we obtain twists by $B\mathrm{O}/B\mathrm{String}$, and we simplify $B\mathrm{O}/B\mathrm{String}$ to define fake vector bundle twists. However, in Proposition 1.50 we learn that $B\mathrm{O}/B\mathrm{String}$ is not homotopy equivalent to a product of Eilenberg-Mac Lane spaces. For this reason, the fake vector bundle twist uses a generalized cohomology theory called *supercohomology* and denoted *SH* (Definition 1.52); we finish this subsubsection by studying cohomology classes associated to a degree-4 supercohomology class, which we will need in the proof of Theorem 2.39.

If $V \to X$ is a spin vector bundle, it has a characteristic class $\lambda(V) \in H^4(X; \mathbb{Z})$ such that $2\lambda(V) = p_1(V)$; a *string structure* on V is a trivialization of λ. It is not hard to check that λ is additive in direct sums, so defines a map of abelian ∞-groups $\lambda\colon B\mathrm{Spin} \to K(\mathbb{Z}, 4)$. The fiber of this map is an ∞-group $B\mathrm{String}$, which is the classifying space for string structures.

[13] This is the first place where the choice of identification (1.38) has explicit consequences, as promised in Remark 1.40: if we compose with the identification of $K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2)$ given by the classes$(a, a^2 + b)$ described in that remark, we would instead obtain $MTSpin \wedge (BO_1)^{3\sigma-3}$. This is not a ring spectrum either, as it can be identified with $MTPin^+$ [Sto88, §8], and $\pi_0(MTPin^+) \cong \mathbb{Z}/2$ and $\pi_4(MTPin^+) \cong \mathbb{Z}/16$ [Gia73b].

Unlike for K-theory, there are three different kinds of topological modular forms: a connective spectrum tmf, a periodic spectrum TMF, and a third spectrum Tmf which is neither connective nor periodic. All three are E_∞-ring spectra, and there are ring spectrum maps $\mathit{tmf} \to \mathit{Tmf} \to \mathit{TMF}$. Ando-Hopkins-Rezk [AHR10] constructed a ring spectrum map $\sigma\colon \mathit{MTString} \to \mathit{tmf}$, so Theorem 1.13 gives us twists of tmf, Tmf, and TMF from $B\mathrm{O}/B\mathrm{String}$:

$$B\mathrm{O}/B\mathrm{String} \to B\mathrm{GL}_1(\mathit{MTString}) \xrightarrow{\sigma} B\mathrm{GL}_1(\mathit{tmf}) \to B\mathrm{GL}_1(\mathit{Tmf}) \to B\mathrm{GL}_1(\mathit{TMF}). \tag{1.47}$$

Like in §1.2.2 and §1.2.3, the section $B\mathrm{O}/B\mathrm{SO} \to B\mathrm{O}$ defines a homotopy equivalence of spaces

$$B\mathrm{O}/B\mathrm{String} \xrightarrow{\simeq} K(\mathbb{Z}/2, 1) \times B\mathrm{SO}/B\mathrm{String}, \tag{1.48}$$

and there is a short exact sequence of abelian ∞-groups

$$1 \longrightarrow \underbrace{B\mathrm{Spin}/B\mathrm{String}}_{K(\mathbb{Z},4)} \xrightarrow{\iota} B\mathrm{SO}/B\mathrm{String} \longrightarrow \underbrace{B\mathrm{SO}/B\mathrm{Spin}}_{K(\mathbb{Z}/2,2)} \longrightarrow 1, \tag{1.49}$$

but now something new happens.

Proposition 1.50. (1.49) *is not split.*

Proof. A splitting of (1.49) defines a section $s\colon B\mathrm{SO}/B\mathrm{String} \to B\mathrm{Spin}/B\mathrm{String}$, meaning $s \circ \iota = \mathrm{id}$. Therefore the map $\lambda\colon B\mathrm{Spin} \to B\mathrm{Spin}/B\mathrm{String} \xrightarrow{\cong} K(\mathbb{Z}, 4)$ factors through $B\mathrm{SO}$:

$$\begin{tikzcd} B\mathrm{Spin} \arrow[r] \arrow[d] \arrow[rr, bend left, "\lambda"] & B\mathrm{Spin}/B\mathrm{String} \arrow[r, "\simeq"'] \arrow[d, shift right, "\iota"'] & K(\mathbb{Z}, 4). \\ B\mathrm{SO} \arrow[r] & B\mathrm{SO}/B\mathrm{String} \arrow[u, shift right, "s"'] & \end{tikzcd} \tag{1.51}$$

We let μ denote the extension of λ to $B\mathrm{SO}$. Brown [Bro82, Theorem 1.5] shows that $H^4(B\mathrm{SO}; \mathbb{Z}) \cong \mathbb{Z}$ with generator p_1, so for any class $x \in H^4(B\mathrm{SO}; \mathbb{Z})$, the pullback of x to $B\mathrm{Spin}$ is some integer multiple of p_1. But the pullback of μ is λ, which is not an integer multiple of p_1, so we have found a contradiction. ☐

We want an analogue of the fake vector bundle twists from §1.2.2 and §1.2.3 for $\mathit{MTString}$, tmf, Tmf, and TMF, but since we just saw that $B\mathrm{SO}/B\mathrm{String}$ is not a product of Eilenberg-Mac Lane spaces, we have to figure out what exactly it is. The answer turns out to be the analogue of an Eilenberg-Mac Lane space for a relatively simple generalized cohomology theory.

Postnikov theory implies that if E is a spectrum with only two nonzero homotopy groups $\pi_m(E) = A$ and $\pi_n(E) = B$ (assume $m < n$ without loss of generality), then E is classified by the data of m, n, A, B, and the *k-invariant* $k_E \in [\Sigma^m HA, \Sigma^{n+1} HB]$, a stable cohomology operation.

Definition 1.52 (Freed [Fre08, §1], Gu-Wen [GW14])**.** Let SH be the spectrum with $\pi_{-2}(SH) = \mathbb{Z}/2$, $\pi_0(SH) = \mathbb{Z}$, and the k-invariant $k_{SH} = \beta \circ \mathrm{Sq}^2\colon H^*(-; \mathbb{Z}/2) \to H^{*+3}(-; \mathbb{Z})$. The generalized cohomology theory defined by SH is called *(restricted) supercohomology.*[14]

Just as the Eilenberg-Mac Lane spectrum $H\mathbb{Z}$ is assembled from Eilenberg-Mac Lane spaces $K(\mathbb{Z}, n)$ and there is a natural isomorphism $H^n(X, \mathbb{Z}) \xrightarrow{\cong} [X, K(\mathbb{Z}, n)]$, if one defines $SK(n)$ to be

[14]The adjective "restricted" is to contrast this theory with "extended" supercohomology of Kapustin-Thorngren [KT17] and Wang-Gu [WG20]. See [GJF19, §5.3, 5.4].

the abelian ∞-group which is the extension

$$0 \longrightarrow K(\mathbb{Z}, n) \longrightarrow SK(n) \longrightarrow K(\mathbb{Z}/2, n-2) \longrightarrow 0 \tag{1.53}$$

classified by $\beta(\mathrm{Sq}^2(T)) \in H^{n+1}(K(\mathbb{Z}/2, n-2); \mathbb{Z})$, where $T \in H^{n-2}(K(\mathbb{Z}/2, n-2); \mathbb{Z}/2)$ is the tautological class and β is the integral Bockstein, then the spaces $SK(n)$ assemble into a model for the spectrum SH and there is a natural isomorphism $SH^n(X) \xrightarrow{\cong} [X, SK(n)]$.

Like Eilenberg-Mac Lane spaces, the spaces $SK(n)$ are related by loops.

Lemma 1.54. *If $n \geq 3$, there is a canonical homotopy class of homotopy equivalences $\Omega SK(n) \xrightarrow{\simeq} SK(n-1)$ compatible with the identifications $\Omega K(A, n) \xrightarrow{\simeq} K(A, n-1)$ and the maps in (1.53).*

Proof. This follows by applying Ω to the cofiber sequence (1.53), then observing that this preserves the k-invariant $\beta \circ \mathrm{Sq}^2$. $\square$

Proposition 1.55. *There is an equivalence of abelian ∞-groups $B\mathrm{SO}/B\mathrm{String} \xrightarrow{\simeq} SK(4)$. Moreover, the space of such equivalences is connected. Therefore there is a natural isomorphism of abelian groups $[X, B\mathrm{SO}/B\mathrm{String}] \cong SH^4(X)$.*

The point of the last sentence in Proposition 1.55 is that in our proof, we do not specify an isomorphism, so a priori there could be ambiguity like in Remark 1.40. But since the space of such identifications is connected, there is a unique identification in the homotopy category, which suffices for the calculations we make in this paper.

Proof of Proposition 1.55. We are trying to identify the extension (1.49) of abelian ∞-groups to relate it to SH. Because $B\mathrm{SO}/B\mathrm{String}$ is an abelian ∞-group, this extension, a priori classified by $H^5(K(\mathbb{Z}/2, 2), \mathbb{Z})$, actually is classified by the stabilization $[\Sigma^2 H\mathbb{Z}/2, \Sigma^5 H\mathbb{Z}]$: this extension is equivalent data to a fiber sequence of connective spectra, so we get to use stable Postnikov theory. Our first step is to understand $[\Sigma^2 H\mathbb{Z}/2, \Sigma^5 H\mathbb{Z}]$.

Lemma 1.56. *For all $k \in \mathbb{Z}$, $[H\mathbb{Z}/2, \Sigma^k H\mathbb{Z}] \cong [H\mathbb{Z}, \Sigma^{k-1} H\mathbb{Z}/2]$.*

Proof. This follows by using the universal coefficient theorem to relate both groups to homology groups: the short exact sequences in the universal coefficient theorem simplify to identify the two groups in the lemma statement with $H_{k-1}(H\mathbb{Z}; \mathbb{Z}/2)$, resp. $H_{k-1}(H\mathbb{Z}/2; \mathbb{Z})$ (the latter because the homology of $H\mathbb{Z}/2$ is torsion). Both of these groups are isomorphic to $\pi_{k-1}(H\mathbb{Z} \wedge H\mathbb{Z}/2)$, so the lemma follows. $\square$

Corollary 1.57. $[\Sigma^2 H\mathbb{Z}/2, \Sigma^4 H\mathbb{Z}] = 0$ *and* $[\Sigma^2 H\mathbb{Z}/2, \Sigma^5 H\mathbb{Z}] \cong \mathbb{Z}/2$.

Proof. By Lemma 1.56, we need to compute $[H\mathbb{Z}, \Sigma^i H\mathbb{Z}/2] = H^i(H\mathbb{Z}; \mathbb{Z}/2)$ for $i = 1, 2$. Let $\mathcal{A}$ denote the mod 2 Steenrod algebra; then $H^*(H\mathbb{Z}; \mathbb{Z}/2) \cong \mathcal{A} \otimes_{\mathcal{A}(0)} \mathbb{Z}/2$ [Wal60, §9]. This vanishes in degree 1 and is isomorphic to $\mathbb{Z}/2$ in degree 2. $\square$

Proposition 1.50 implies (1.49) is classified by a nonzero element of $[\Sigma^2 H\mathbb{Z}/2, \Sigma^5 H\mathbb{Z}]$. And by definition, $SK(4)$ is an extension of $K(\mathbb{Z}/2, 2)$ by $K(\mathbb{Z}, 4)$ classified by $\beta \circ \mathrm{Sq}^2$, which is a nonzero element of $[\Sigma^2 H\mathbb{Z}/2, \Sigma^5 H\mathbb{Z}]$. Since this group is isomorphic to $\mathbb{Z}/2$ by Corollary 1.57, these two nonzero elements must coincide, so there is an equivalence of abelian ∞-groups $B\mathrm{SO}/B\mathrm{String} \simeq SK(4)$. There is a homotopy type of such equivalences, and π_0 of that homotopy type is a torsor over $[\Sigma^2 H\mathbb{Z}/2, \Sigma^4 H\mathbb{Z}]$, which vanishes by Corollary 1.57, so the space of identifications is connected. $\square$

Corollary 1.58. *The map $K(\mathbb{Z}/2,1) \to B\mathrm{O}$ defined by the tautological line bundle induces a homotopy equivalence of spaces*

$$B\mathrm{O}/B\mathrm{String} \xrightarrow{\simeq} K(\mathbb{Z}/2,1) \times SK(4), \tag{1.59}$$

implying that MTString, tmf, and TMF can be twisted over a space X by classes $a \in H^1(X;\mathbb{Z}/2)$ and $d \in SH^4(X)$.

Definition 1.60. We call the twists associated to a and d in Corollary 1.58 the *fake vector bundle twists* for *MTString*, *tmf*, *Tmf*, and *TMF*.

Remark 1.61. Another consequence of Proposition 1.55, applied to the proof strategy of Proposition 1.50, is that, even though $\lambda \in H^4(B\mathrm{Spin};\mathbb{Z})$ does not pull back from $B\mathrm{SO}$, its image in $SH^4(B\mathrm{Spin})$ *does* pull back from a class $\lambda \in SH^4(B\mathrm{SO})$. This is a theorem of Freed [Fre08, Proposition 1.9(i)], with additional proofs given by Jenquin [Jen05, Proposition 4.6] and Johnson-Freyd and Treumann [JFT20, §1.4].

The map $K(\mathbb{Z},4) \simeq B\mathrm{Spin}/B\mathrm{String} \to B\mathrm{SO}/B\mathrm{String}$ means degree-4 ordinary cohomology classes also define degree-4 twists of string bordism and topological modular forms. Twists of this sort have already been studied, so we compare our twists to the literature.

Definition 1.62. Given X, a, and d as in Corollary 1.58, let $\Omega_*^{\mathrm{String}}(X,a,d)$ denote the groups of bordism classes of manifolds M equipped with maps $f\colon M \to X$ and trivializations of $w_1(M) - f^*(a) \in H^1(M;\mathbb{Z}/2)$ and $\lambda(M) - f^*(d) \in SH^4(M)$.

A priori we only defined λ as a characteristic class of oriented vector bundles; for an unoriented vector bundle V, $\lambda(V)$ is be defined to be $\lambda(V \oplus \mathrm{Det}(V))$, as the latter bundle is canonically oriented. Definition 1.62 first appears in work of B.L. Wang [Wan08, Definition 8.4] in the special case when $a = 0$ and d comes from ordinary cohomology.

Lemma 1.63. *There is a natural isomorphism $\pi_*(M^{MTString}f_{a,d}) \xrightarrow{\cong} \Omega_*^{\mathrm{String}}(X,a,d)$.*

This follows from work of Hebestreit-Joachim [HJ20], much like Lemmas 1.26 and 1.44. Though they do not discuss the *MTString* case explicitly, their proof can be adapted to our setting. See [HJ20, Remark 2.2.3].

We can also compare with preexisting twists of *tmf*.

Lemma 1.64. *The fake vector bundle twist defined by $K(\mathbb{Z},4) \to SK(4) \to B\mathrm{GL}_1(tmf)$ is homotopy equivalent to the twist $K(\mathbb{Z},4) \to B\mathrm{GL}_1(tmf)$ constructed by Ando-Blumberg-Gepner* [ABG10, Proposition 8.2]*.*

Proof sketch. This equivalence is not obvious, because Ando-Blumberg-Gepner construct their twist in a different way: beginning with a map $\phi\colon \Sigma^\infty_+ K(\mathbb{Z},3) \to tmf$ and using the adjunction [ABG+14b, (1.4), (1.7)] between Σ^∞_+ and GL_1. However, their argument builds ϕ out of the map $\lambda\colon B\mathrm{Spin} \to B\mathrm{Spin}/B\mathrm{String} \simeq K(\mathbb{Z},4)$, allowing one to pass our construction through their argument and conclude that our twist, as a class in $[K(\mathbb{Z},4), B\mathrm{GL}_1(tmf)]$, coincides with Ando-Blumberg-Gepner's. □

Though these twists by degree-4 cohomology are relatively well-studied, there are not so many examples of lower-degree twists of string bordism or topological modular forms in the literature. See Freed-Hopkins-Teleman [FHT10, §2], Johnson-Freyd [JF20, §2.3], Beardsley-Luecke-Morava [BLM23, Example 5.25], Tachikawa-Yamashita [TY23a, TY23b], Tachikawa-Yonekura [TY25], and [DY24a, Remark 2.16] for some examples.

Example 1.65. Just as in Examples 1.32 and 1.46, Theorem 1.16 calculates some *MTString*-module Thom spectra for us: over $B\mathrm{O}/B\mathrm{String}$ we get MTO; over $B\mathrm{SO}/B\mathrm{String}$ we get $MTSO$, and over $K(\mathbb{Z},4)$ we get $MTSpin$. The last example is due to Beardsley [Bea17, §3].

Remark 1.66. Like in Lemmas 1.34 and 1.41, (1.48) is not an equivalence of ∞-groups. The same two proofs are available to us: pulling back to $K(\mathbb{Z}/2,1)$ and showing we do not obtain an E_1-ring spectrum, and comparing the group structures on $[\mathbb{RP}^\infty, B\mathrm{O}/B\mathrm{String}]$ and $[\mathbb{RP}^\infty, K(\mathbb{Z}/2,1)\times B\mathrm{SO}/B\mathrm{String}]$. For the second proof, one observes that $[\mathbb{RP}^\infty, B\mathrm{O}/B\mathrm{String}]\cong\mathbb{Z}/8$ but $[\mathbb{RP}^\infty, K(\mathbb{Z}/2,1)\times B\mathrm{SO}/B\mathrm{String}]$ has at least four elements of order 4, then concludes.

For the first proof, we obtain $MTString\wedge(B\mathrm{O}_1)^{\sigma-1}$ like before; to our knowledge, this notion of bordism has not been studied.[15] However, since this is a vector bundle Thom spectrum, the change-of-rings trick shows that in topological degrees 15 and below, the E_2-page of the Adams spectral sequence computing $\Omega_*^{\mathrm{String}}((B\mathrm{O}_1)^{\sigma-1})_2^\wedge$ is isomorphic to $\mathrm{Ext}_{\mathcal{A}(2)}^{s,t}(H^*((B\mathbb{Z}/2)^{\sigma-1};\mathbb{Z}/2),\mathbb{Z}/2)$ (see §2.1 for notation and an explanation). Davis-Mahowald [DM78, Table 3.2] have computed these Ext groups, and from their computation it directly follows using the Adams spectral sequence that $\pi_0\cong\mathbb{Z}/2$ and $\pi_3\cong\mathbb{Z}/8$, so just like for $MTPin^c$ and $MTPin^-$, $MTString\wedge(B\mathrm{O}_1)^{\sigma-1}$ does not admit an E_1-ring spectrum structure.

In the proof of Theorem 2.39 we will need to understand the mod 2 cohomology classes naturally associated to a degree-4 supercohomology class d. The quotient $t\colon SH\to\Sigma^{-2}H\mathbb{Z}/2$ gives us a degree-2 class $t(d)$, sometimes called the *Gu-Wen layer* of d.

To proceed further, we study the Serre spectral sequence associated to the fibration $K(\mathbb{Z},4)\to SK(4)\to K(\mathbb{Z}/2,2)$. Let $\overline{\delta}\in H^4(K(\mathbb{Z},4);\mathbb{Z}/2)$ be the mod 2 reduction of the tautological class; this defines a class in $E_2^{0,4}$ of our Serre spectral sequence, which we also call $\overline{\delta}$.

Lemma 1.67. *The class $\overline{\delta}\in E_2^{0,4}$ survives to the E_∞-page.*

Proof. The only possible differential that could be nonzero on $\overline{\delta}$ is the transgressing d_5, which pulls back from the transgressing d_5 on $\overline{\delta}$ in the Serre spectral sequence for the universal fibration with fiber $K(\mathbb{Z},4)$, namely $K(\mathbb{Z},4)\to E(K(\mathbb{Z},4))\to B(K(\mathbb{Z},4))\simeq K(\mathbb{Z},5)$. In the universal fibration, $d_5(\overline{\delta})$ is the mod 2 tautological class $\epsilon\in H^5(K(\mathbb{Z},5);\mathbb{Z}/2)$, so in the fibration with total space $SK(4)$, $d_5(\overline{\delta})$ is the pullback of ϵ by the classifying map $\beta\circ\mathrm{Sq}^2\colon K(\mathbb{Z}/2,2)\to K(\mathbb{Z},5)$. Thus $\epsilon\mapsto(\beta\mathrm{Sq}^2(B))\bmod 2=\mathrm{Sq}^1\mathrm{Sq}^2(B)$, where $B\in H^2(K(\mathbb{Z}/2,2);\mathbb{Z}/2)$ is the tautological class, but $\mathrm{Sq}^1\mathrm{Sq}^2(B)=\mathrm{Sq}^3(B)=0$, as B has degree 2. Thus $d_5(\overline{\delta})=0$. □

Remark 1.68. This is an unstable phenomenon: for $n>4$, a similar argument shows the transgressing differential on the mod 2 tautological class of $K(\mathbb{Z},n)$ is nonzero, so no analogue of $\overline{\delta}$ exists in the cohomology of $SK(n)$.

We want to lift $\overline{\delta}\in E_\infty^{0,4}$ to an element δ of $H^4(SK(4);\mathbb{Z}/2)$. If B is the tautological class of $K(\mathbb{Z}/2,2)$, then there is an ambiguity between δ and $\delta+B^2$. To resolve this ambiguity, pull back across the map $\lambda\colon B\mathrm{SO}\to SK(4)$. By comparing the Serre spectral sequences for the fibrations $K(\mathbb{Z},4)\to SK(4)\to K(\mathbb{Z}/2,2)$ and $B\mathrm{Spin}\to B\mathrm{SO}\to K(\mathbb{Z}/2,2)$, one learns that $\lambda^*(\delta)$ is either w_4 or $w_4+w_2^2$. Choosing the former allows us to uniquely define δ.

Corollary 1.69. *There is a unique class $\delta\in H^4(SK(4);\mathbb{Z}/2)$ such that $\lambda^*(\delta)=w_4$.*

[15] By analogy with SO and O and Spin and Pin^-, one could call this tring^- bordism. We hope there is a better name for this spectrum.

Phrased differently, associated to every $d \in SH^4(X)$ is a class $\delta \in H^4(X;\mathbb{Z}/2)$, such that if there is an oriented vector bundle $V \to X$ with $d = \lambda(V)$, then $\delta = w_4(V)$. The same line of reasoning also shows that $\lambda^*(t(d)) = w_2$.

2. Computing the input to Baker-Lazarev's Adams spectral sequence

2.1. Review: the change-of-rings theorem for vector bundle Thom spectra. We begin by reviewing how the story goes for vector bundle Thom spectra, where we can take advantage of a general change-of-rings theorem. This is a standard technique dating back to work of Anderson-Brown-Peterson [ABP69] and Giambalvo [Gia73a, Gia73b, Gia76]; see Beaudry-Campbell [BC18, §4.5] for a nice introduction.

Lemma 2.1 (Change of rings)**.** *Let $\mathcal{B}$ be a graded Hopf algebra and $\mathcal{C} \subset \mathcal{B}$ be a graded Hopf subalgebra. If M is a graded $\mathcal{C}$-module and N is a graded $\mathcal{B}$-module, then there is a natural isomorphism*

$$\mathrm{Ext}_{\mathcal{B}}^{s,t}(\mathcal{B} \otimes_{\mathcal{C}} M, N) \xrightarrow{\cong} \mathrm{Ext}_{\mathcal{C}}^{s,t}(M, N) \tag{2.2}$$

For the little siblings we consider, we have the following isomorphisms of $\mathcal{A}$-modules:

$$H^*(H\mathbb{Z};\mathbb{Z}/2) \cong \mathcal{A} \otimes_{\mathcal{A}(0)} \mathbb{Z}/2 \tag{2.3a}$$

$$H^*(ku;\mathbb{Z}/2) \cong \mathcal{A} \otimes_{\mathcal{E}(1)} \mathbb{Z}/2 \tag{2.3b}$$

$$H^*(ko;\mathbb{Z}/2) \cong \mathcal{A} \otimes_{\mathcal{A}(1)} \mathbb{Z}/2 \tag{2.3c}$$

$$H^*(tmf;\mathbb{Z}/2) \cong \mathcal{A} \otimes_{\mathcal{A}(2)} \mathbb{Z}/2. \tag{2.3d}$$

Here $\mathcal{A}(n)$ is the subalgebra of $\mathcal{A}$ generated by $\{\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4, \dots, \mathrm{Sq}^{2^n}\}$ and $\mathcal{E}(1) = \langle Q_0, Q_1\rangle$, where $Q_0 = \mathrm{Sq}^1$ and $Q_1 = \mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^1$. The isomorphisms in (2.3) were proven by Wall [Wal60, §9] ($H\mathbb{Z}$), Adams [Ada61] (ku), Stong [Sto63] (ko), and Hopkins-Mahowald [HM14] (tmf).

To use Lemma 2.1, we need to make $\mathcal{A}(0)$, $\mathcal{A}(1)$, $\mathcal{A}(2)$, and $\mathcal{E}(1)$ into Hopf subalgebras of $\mathcal{A}$. This is equivalent to specifying how these algebras interplay with the cup product, which the Cartan formula answers. For the Steenrod squares, this is standard; we also have $Q_i(ab) = aQ_i(b) + Q_i(a)b$ for $i = 0, 1$.

Lemma 2.1, paired with (2.3), greatly simplifies many computations: for any spectrum which splits as $X = R \wedge Y$ where R is one of $H\mathbb{Z}$, ku, ko, or tmf, the E_2-page of the Adams spectral sequence computing the 2-completed homotopy groups of X (or the R-homology of Y) is identified with Ext groups over $\mathcal{A}(0)$, $\mathcal{E}(1)$, $\mathcal{A}(1)$, or $\mathcal{A}(2)$, respectively. These algebras are much smaller than the entire 2-primary Steenrod algebra, so the Ext groups are easier to calculate; thus one often hears the slogan that *ko*-, *ku*-, and *tmf*-homology groups are relatively easy to compute with the Adams spectral sequence,[16] and by (1.17) and (2.3), those computations also compute spin^c, spin, and string bordism (the latter in dimensions 15 and below). See Douglas-Henriques-Hill [DHH11] for a nice related computation of vector bundle twists of string bordism.

Remark 2.4. Another way to phrase this is that, though (2.3) is about the little siblings only, combining it with (1.17) allows us to write down change-of-rings results for the Adams spectral sequences of the big siblings. Specifically, there is an $\mathcal{A}(0)$-module W_1, an $\mathcal{E}(1)$-module W_2, and an

[16] The Adams spectral sequence computing $H\mathbb{Z}$-homology is essentially a repackaging of the Bockstein spectral sequence; see May-Milgram [MM81].

$\mathcal{A}(1)$-module W_3 such that

$$H^*(MTSO;\mathbb{Z}/2) \cong \mathcal{A} \otimes_{\mathcal{A}(0)} W_1 \tag{2.5a}$$

$$H^*(MTSpin^c;\mathbb{Z}/2) \cong \mathcal{A} \otimes_{\mathcal{E}(1)} W_2 \tag{2.5b}$$

$$H^*(MTSpin;\mathbb{Z}/2) \cong \mathcal{A} \otimes_{\mathcal{A}(1)} W_3, \tag{2.5c}$$

so that the E_2-pages of the Adams spectral sequences computing the 2-completions of Ω_*^{SO}, $\Omega_*^{\mathrm{Spin}^c}$, and Ω_*^{Spin} are the Ext groups of W_1, W_2, and W_3, respectively, over $\mathcal{E}(1)$, $\mathcal{A}(1)$, and $\mathcal{A}(2)$ respectively. Explicitly, these modules begin in low degrees with (compare (1.17))

$$W_1 \cong \mathbb{Z}/2 \oplus \Sigma^4\mathbb{Z}/2 \oplus \Sigma^5\mathcal{A}(0) \oplus \Sigma^8\mathbb{Z}/2 \oplus \Sigma^8\mathbb{Z}/2 \oplus \cdots \tag{2.6a}$$

$$W_2 \cong \mathbb{Z}/2 \oplus \Sigma^4\mathbb{Z}/2 \oplus \Sigma^8\mathbb{Z}/2 \oplus \Sigma^8\mathbb{Z}/2 \oplus \Sigma^{10}\mathcal{E}(1) \oplus \cdots \tag{2.6b}$$

$$W_3 \cong \mathbb{Z}/2 \oplus \Sigma^8\mathbb{Z}/2 \oplus \Sigma^{10}\mathcal{A}(1)/\mathrm{Sq}^3 \oplus \cdots \tag{2.6c}$$

Often, though, what one wants is twisted. For vector bundle twists in the sense of Example 1.6, this is not a problem: if $f\colon X \to B\mathrm{GL}_1(R)$ is a vector bundle twist specified by a rank-r virtual vector bundle $V \to X$, or strictly speaking by the rank-0 virtual vector bundle $V - r := V - \underline{\mathbb{R}}^r$, then f factors through $B\mathrm{GL}_1(\mathbb{S})$, so Lemma 1.8 provides a natural homotopy equivalence[17]

$$Mf \xrightarrow{\simeq} R \wedge X^{V-r}. \tag{2.7}$$

Thus, for the ring spectra R we discussed above, one can also use the change-of-rings isomorphism to simplify the computation of twisted R-homology for vector bundle twists: for ko, the E_2-page is

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}(1)}^{s,t}(H^*(X^{V-r};\mathbb{Z}/2),\mathbb{Z}/2) \Longrightarrow ko_{t-s}(X)_2^\wedge, \tag{2.8}$$

and the other choices of R are analogous. The $\mathcal{A}$-action (and hence also the $\mathcal{A}(n)$ and $\mathcal{E}(1)$-actions) on $H^*(X^{V-r};\mathbb{Z}/2)$ is easy to compute: the Thom isomorphism tells us the cohomology as a vector space, and the Stiefel-Whitney classes of V twist the Steenrod squares as described in [BC18, Remark 3.3.5].

This is a powerful generalization: many bordism spectra of interest arise as twists in this way, including pin$^\pm$ bordism and all of the bordism spectra studied in [BG97, Cam17, WW19, WWZ20, Deb21, FH21].

2.2. **Baker-Lazarev's R-module Adams spectral sequence.** For R an E_∞-ring spectrum,[18] Baker-Lazarev [BL01] develop an R-module spectrum generalization of the Adams spectral sequence which reduces to the usual Adams spectral sequence when $R = \mathbb{S}$.

Definition 2.9. For R-modules H and M, the *R-module H-homology* of M is

$$H_*^R(M) := \pi_*(H \wedge_R M), \tag{2.10a}$$

and the *R-module H-cohomology* of M is

$$H_R^*(M) := \pi_{-*}\mathrm{Map}_R(M, H). \tag{2.10b}$$

For the purposes of this paper, R will be one of the little siblings. For each such R, there is a canonical isomorphism $\pi_0(R) \xrightarrow{\cong} \mathbb{Z}$, which lifts to identify the Postnikov quotient $\tau_{\le 0}R \xrightarrow{\simeq} H\mathbb{Z}$; as $\tau_{\le 0}R$ is an E_∞ R-algebra spectrum via the quotient map $R \to \tau_{\le 0}R$ (see [Kri93, Bas99]), this

[17] For a different, less abstract proof of this splitting, see [FH21, §10] or [DDHM24, §10.4].

[18] Baker-Lazarev work with commutative algebras in Elmendorf-Kriz-Mandell-May's $\mathbb{S}$-modules; as we discussed in Footnote 3, we may equivalently work with E_∞-ring spectra.

data provides a canonical E_∞ R-algebra structure on $H\mathbb{Z}$. Composing with the mod n reduction map $H\mathbb{Z} \to H\mathbb{Z}/n$, we also obtain canonical E_∞ R-algebra structures on $H\mathbb{Z}/n$ for all n. This data makes both H_R^*H and H_*^RH into Hopf algebras, analogously to how the Steenrod algebra $H^*(H\mathbb{Z}/p;\mathbb{Z}/p)$ and its dual are Hopf algebras (see [Mil58]). For $n = 2$ we have the following isomorphisms of "Hopf algebras of R-module cohomology operations:"

Theorem 2.11. *Let R be one of the little siblings and $H = H\mathbb{Z}/2$ with the R-algebra structure defined above. Then there are Hopf algebra isomorphisms*

$$R = H\mathbb{Z}\,, \quad H_R^*H \cong \mathcal{A}(0) \tag{2.12a}$$

$$R = ko\,, \quad H_R^*H \cong \mathcal{A}(1) \tag{2.12b}$$

$$R = ku\,, \quad H_R^*H \cong \mathcal{E}(1) \tag{2.12c}$$

$$R = tmf\,, \quad H_R^*H \cong \mathcal{A}(2), \tag{2.12d}$$

and dualizing gives the corresponding Hopf algebras of homology operations, e.g. $H_*^{H\mathbb{Z}}H \cong \mathcal{A}(0)_*$.

This theorem was proven in pieces: the part for $H\mathbb{Z}$ is standard; for ko and ku this is due to Baker [Bak20, Theorem 5.1]; and for tmf it is due to Henriques [DFHH14].

In the setting of Theorem 2.11, for any R-module spectrum M, $H_R^*(M)$ is naturally an H_R^*H-module and $H_*^R(H)$ is naturally an H_*^RH-comodule, analogously to the mod 2 cohomology and homology of a spectrum with respect to the Steenrod algebra and its dual.

At this point, we detour briefly to compare H_*^R, for R one of the little siblings, with the $H_*(R;\mathbb{Z}/2)$-*module indecomposables* functor [Sto92, §5].

Definition 2.13 (Stolz [Sto92, §5])**.** Let $\mathcal{A}_*$ denote the 2-primary dual Steenrod algebra, $\mathcal{B}_*$ be a sub-Hopf algebra of $\mathcal{A}_*$, and R be an E_∞-ring spectrum such that there is an isomorphism of both $\mathcal{A}_*$-comodules and $\mathbb{Z}/2$-algebras

$$H_*(R;\mathbb{Z}/2) \xrightarrow{\cong} \mathcal{A}_* \,\square_{\mathcal{B}_*}\, \mathbb{Z}/2, \tag{2.14}$$

where $\square_{\mathcal{B}_*}$ denotes the cotensor product of $\mathcal{B}_*$-comodules. If N is a bounded-below, finite-type R-module spectrum, the $H_*(R;\mathbb{Z}/2)$-*module indecomposables* of N is the $\mathcal{B}_*$-comodule

$$\overline{H_*(N)} \coloneqq H_*(N;\mathbb{Z}/2) \otimes_{H_*(R;\mathbb{Z}/2)} \mathbb{Z}/2. \tag{2.15}$$

Stolz [Sto92, Proposition 5.4] showed that if N is as in Definition 2.13, there is a natural isomorphism

$$H_*(N;\mathbb{Z}/2) \xrightarrow{\cong} \mathcal{A}_* \,\square_{\mathcal{B}_*}\, \overline{H_*(N)}. \tag{2.16}$$

See Stolz [Sto92, §5] for further discussion with $R = ko$, Führing [Füh22, §5] for $R = H\mathbb{Z}$, and Granath [Gra23, §2.9] for $R = ku$.

The little siblings $H\mathbb{Z}$, ku, ko, and tmf all satisfy (2.14) with $\mathcal{B}_*$ equal to $\mathcal{A}(0)_*$, $\mathcal{E}(1)_*$, $\mathcal{A}(1)_*$, and $\mathcal{A}(2)_*$ respectively; this follows formally by dualizing (2.3).

Proposition 2.17. *Let R be one of $H\mathbb{Z}$, ku, ko, or tmf, so that $\mathcal{B}_* \cong H_*^RH$ by Theorem 2.11. The functors H_*^R and $\overline{H^*(-)}$, from R-module spectra to H_*^RH-comodules, are naturally isomorphic.*

Proof. In this proof, all cohomology has $\mathbb{Z}/2$ coefficients. Consider the Künneth spectral sequence (see [EKMM97, Theorem IV.4.1] and [Til16])[19]

$$E^2_{*,*} = \mathrm{Tor}^{H_*(R)}_{*,*}(\mathbb{Z}/2, H_*(N)) \Longrightarrow \pi_*(H \wedge_R N) = H^R_*(N). \tag{2.18}$$

To prove the proposition, it suffices to show that these Tor groups vanish in positive homological degrees: then the spectral sequence collapses for degree reasons to imply an isomorphism

$$\mathbb{Z}/2 \otimes_{H_*(R)} H_*(N) = \mathrm{Tor}^{H_*(R)}_{0,*}(\mathbb{Z}/2, H_*(N)) \xrightarrow{\cong} H^R_*(N). \tag{2.19}$$

Proving the claimed higher Tor vanishing is not so hard: the natural isomorphism $H_*(N) \cong H_*(R) \otimes \overline{H_*(N)}$ [Sto92, §5] simplifies the E^2-page of (2.18):

$$E^2_{*,*} \cong \mathrm{Tor}^{H_*(R)}_{*,*}(\mathbb{Z}/2, H_*(R) \otimes \overline{H_*(N)}) \cong \mathrm{Tor}^{\mathbb{Z}/2}_{*,*}(\mathbb{Z}/2, \overline{H_*(N)}), \tag{2.20}$$

and Tor over a field vanishes in positive homological degrees. □

Remark 2.21. Despite the equivalence in Proposition 2.17, the two homology theories H^R_* and $\overline{H_*(-)}$ have different strengths. The definition of H^R_* makes it easier to use for the applications we have in mind, and $\overline{H_*(-)}$ is more generally applicable to "homology R-modules" (see Stolz [Sto94, §2]). As we do not need this generality, we stick with H^R_* and H^*_R.

We now present the spectral sequence; let M and N be R-modules, and let H be a commutative R-ring spectrum.

Theorem 2.22 (Baker-Lazarev [BL01]). *Let M and N be R-modules and H be an E_∞ R-algebra, and suppose that H^R_*H is a flat $\pi_*(H)$-module. Then there is a spectral sequence of Adams type, natural in M, N, H, and R, with E_2-page*

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{H^*_R H}(H^*_R M, H^*_R N), \tag{2.23}$$

and if N is connective and M is a cellular R-module spectrum with finitely many cells in each degree,[20] *then this spectral sequence converges to the homotopy groups of the (R-module) H-nilpotent completion of $N^R_* M$.*

Without the flatness assumption, one in general only has a description of the E_1-page, and it is more complicated,[21] though see also recent work of Burklund-Pstrągowski [BP25]. For example, this issue occurs when $R = \mathbb{S}$ and $H = ku$, ko, or tmf; see [Mah81, Dav87, LM87, BOSS19, BBB+20, BBB+21]. However, if p is a prime number, $\pi_*(H\mathbb{Z}/p) \cong \mathbb{Z}/p$ is a field, so the flatness assumption is satisfied for all R; as this is the only case we consider in this paper, we say no more about the flatness assumption in Theorem 2.22.

The notion of the *H-nilpotent completion* of a spectrum is due to Bousfield [Bou79, §5]. When $H = H\mathbb{Z}/p$, p prime, this is the usual p-completion [Rav84, Example 1.16].[22] Thus if the homotopy groups of $N \wedge_R M$ are finitely generated abelian groups, this as usual detects free and p^k-torsion summands, but not torsion for other primes.

[19] This version of the Künneth spectral sequence is associated to the smash product $(H \wedge H) \wedge_{H \wedge R} (H \wedge N) \simeq H \wedge_R N$. See Lawson [Law18, Proposition 2.7.5] or Senger [Sen24, §3].

[20] This condition on M is the analogue in $\mathcal{M}od_R$ of the notion of a CW spectrum with finitely many cells in each degree. If M is the R-module Thom spectrum associated to a map $f\colon X \to B\mathrm{GL}_1(R)$, which is the only case we consider in this paper, then this condition on M is met if X is a CW complex with finitely many cells in each dimension.

[21] In applications, this may be less bad than it seems: for example, McNamara-Reece [MR22, §6.2] interpret the E_1-page of the classical Adams spectral sequence in the context of quantum gravity.

[22] Ravenel assumes $R = \mathbb{S}$, but his result is true in the generality we work in.

When $R = \mathbb{S}$, Theorem 2.22 reduces to the classical H-based Adams spectral sequence, with its standard convergence results. We will apply Theorem 2.22 when R is one of the little siblings, $H = H\mathbb{Z}/p$ for p prime, and $N = R$: there is a canonical homotopy equivalence $R \wedge_R M \simeq M$, so in this setting Baker-Lazarev's spectral sequence takes as input $\mathrm{Ext}_{H_R^*H}(H_R^*(M), \mathbb{Z}/2)$, and converges to the p-completed homotopy groups of M.

For Thom spectra H_R^* is easy.

Lemma 2.24 (R-module Thom isomorphisms)**.** *For any E_∞-ring spectrum R such that $H \coloneqq H\mathbb{Z}/2$ is an R-algebra and any map $f\colon X \to B\mathrm{GL}_1(R)$, there are isomorphisms*

$$H_*(X; \mathbb{Z}/2) \xrightarrow{\cong} H_*^R(Mf) \tag{2.25a}$$

$$H^*(X; \mathbb{Z}/2) \xrightarrow{\cong} H_R^*(Mf). \tag{2.25b}$$

This means that $H_R^*(Mf)$ is a free $H^*(X; \mathbb{Z}/2)$-module on a class $U \in H_R^0(Mf)$, which is the Thom class in this setting.

Proof. Apply Lemma 1.8 with $R_1 = R$ and $R_2 = H\mathbb{Z}/2$ to learn that $Mf \wedge_R H\mathbb{Z}/2$, the object whose homotopy groups are $H_*^R(Mf)$, is the Thom spectrum of a twist $f'\colon X \to B\mathrm{GL}_1(H\mathbb{Z}/2)$. By Example 1.3, $B\mathrm{GL}_1(H\mathbb{Z}/2)$ is contractible, so f' is null-homotopic, so by Example 1.7, $Mf \wedge_R H\mathbb{Z}/2 \simeq X_+ \wedge H\mathbb{Z}/2$. Take homotopy groups to obtain (2.25a).

For cohomology,[23] we have a chain of equivalences of spectra

$$\begin{aligned} \mathrm{Map}_R(Mf, H\mathbb{Z}/2) &\simeq \mathrm{Map}_{H\mathbb{Z}/2}(Mf \wedge_R H\mathbb{Z}/2, H\mathbb{Z}/2) \\ &\simeq \mathrm{Map}_{H\mathbb{Z}/2}((X_+) \wedge H\mathbb{Z}/2, H\mathbb{Z}/2) \\ &\simeq \mathrm{Map}_{\mathbb{S}}(X_+, H\mathbb{Z}/2), \end{aligned} \tag{2.26}$$

and the claim follows by taking homotopy groups. The first and third equivalences in (2.26) are instances of the natural isomorphism $\mathrm{Map}_A(B, C) \simeq \mathrm{Map}_E(B \wedge_A E, C)$ for an E_∞-ring spectrum A, an E_∞ A-algebra spectrum E, and A-modules B and C, and the middle equivalence in (2.26) is the homology Thom isomorphism from the first part of the proof. ☐

For most of our applications we will take $H = H\mathbb{Z}/2$.

Example 2.27 (*tmf* at the prime 3)**.** We will also work with an interesting odd-primary example, where $H = H\mathbb{Z}/3$ and $R = \mathit{tmf}$. Let $\mathcal{A}_3 \coloneqq H^*H$, which is the mod 3 Steenrod algebra, and let $\mathcal{A}^{\mathit{tmf}} \coloneqq H_{\mathit{tmf}}^*H$; Henriques and Hill, using the work of Behrens [Beh06] and unpublished work of Hopkins-Mahowald, showed that

$$\mathcal{A}^{\mathit{tmf}} \cong \mathbb{Z}/3\langle \beta, \mathcal{P}^1\rangle/(\beta^2, \beta(\mathcal{P}^1)^2\beta - (\beta\mathcal{P}^1)^2 - (\mathcal{P}^1\beta)^2, (\mathcal{P}^1)^3). \tag{2.28}$$

Curiously, Rezk showed that $H^*(\mathit{tmf}; \mathbb{Z}/3)$ is not isomorphic to $\mathcal{A}_3 \otimes_{\mathcal{A}^{\mathit{tmf}}} \mathbb{Z}/3$: see [Cul21, §2].

The map $\phi\colon H_{\mathit{tmf}}^*H \to H^*H$ sends β to the Bockstein of $0 \to \mathbb{Z}/3 \to \mathbb{Z}/9 \to \mathbb{Z}/3 \to 0$ and $\mathcal{P}^1$ to the first Steenrod power. However, unlike in the previous examples we studied, ϕ is not injective! The relation $\beta(\mathcal{P}^1)^2 + \mathcal{P}^1\beta\mathcal{P}^1 + (\mathcal{P}^1)^2\beta = 0$ is present in $\mathcal{A}_3$ but not in $\mathcal{A}^{\mathit{tmf}}$ (see, e.g., [BR21, Corollary 13.7]).

Baker-Lazarev's Theorem 2.22 implies that for any *tmf*-module spectrum M, $H_{\mathit{tmf}}^*(M)$ carries a natural $\mathcal{A}^{\mathit{tmf}}$-module action, and there is an Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}^{\mathit{tmf}}}^{s,t}(H_{\mathit{tmf}}^*(M), \mathbb{Z}/3) \Longrightarrow \pi_{t-s}(M)_3^\wedge. \tag{2.29}$$

[23] We thank an anonymous referee for a suggestion to simplify this part of the proof.

In general, we will let $H^*_{tmf}(M)$ refer to the mod 2 *tmf*-module cohomology and denote the mod 3 *tmf*-module cohomology by $H^*_{tmf}(M;\mathbb Z/3)$. Because $(\mathbb Z/3)^\times$ is nontrivial, $B\mathrm{GL}_1(H\mathbb Z/3)$ is not contractible, so the proof of Lemma 2.24 does not directly generalize to this setting; however, as $B\mathrm{GL}_1(H\mathbb Z/3)\cong B(\mathbb Z/3)^\times$ (see Example 1.3), for any twist $f\colon X\to B\mathrm{GL}_1(tmf)$ factoring through a simply connected space, the induced twist of $H\mathbb Z/3$ is trivial and the argument goes through to show $H^*_{tmf}(M^{tmf}f;\mathbb Z/3)\cong H^*(X;\mathbb Z/3)$. As $SK(4)$ is simply connected, this includes the fake vector bundle twists of *tmf* whose components in $H^1(\text{–};\mathbb Z/2)$ vanish.

Like for the mod 2 subalgebras of the Steenrod algebra that we discussed, we will want to know how $\mathcal A^{tmf}$ acts on products. The map $\mathcal A^{tmf}\to\mathcal A_3$ is a map of Hopf algebras [BR21, §13.1], allowing us to use the Cartan formula and multiplicativity of the Bockstein in $\mathcal A_3$ to conclude that in $\mathcal A^{tmf}$,

$$\mathcal P^1(ab) = \mathcal P^1(a)b + a\mathcal P^1(b)\,, \tag{2.30a}$$

$$\beta(ab) = \beta(a)b + (-1)^{|a|}a\beta(b). \tag{2.30b}$$

When R is one of the little siblings, Theorem 2.11 implies that for any R-module spectrum M, Baker-Lazarev's spectral sequence calculates $\pi_*(M)^\wedge_2$ as the Ext of *something* over an algebra much smaller than $\mathcal A$ – one of $\mathcal A(0)$, $\mathcal E(1)$, $\mathcal A(1)$, or $\mathcal A(2)$. Thus the change-of-rings approach to computing $\pi_*(R\wedge Y)^\wedge_2$ that we described in §2.1 generalizes to other R-modules M, in particular when M is an R-module Thom spectrum – we just have to figure out $H^*_R(M)$. This will be the main result of the next section.

2.3. **Proof of the main theorem.** At this point, we know from the previous section that even for non-vector-bundle Thom spectra M^Rf over $R=H\mathbb Z$, *ku*, *ko* and *tmf*, we can work over $\mathcal A(0)$, $\mathcal E(1)$, $\mathcal A(1)$, and $\mathcal A(2)$ to compute the E_2-page of Baker-Lazarev's Adams spectral sequence, implying that a change-of-rings formula for these Thom spectra exists. Our next step is to determine the $\mathcal A(0)$-, $\mathcal E(1)$-, $\mathcal A(1)$-, and $\mathcal A(2)$-modules $H^*_R(M^Rf)$. We describe the actions of the generators of $\mathcal A(0)$, $\mathcal E(1)$, $\mathcal A(1)$, and $\mathcal A(2)$ below in Definition 2.31; however, it is not yet clear that they satisfy the Adem relations, so we describe these modules over freer algebras, then later in the proof of Theorem 2.39 we show they are compatible with the Adem relations, hence are in fact H^*_RH-modules.

Definition 2.31. Let X be a space.

(1) Given $a\in H^1(X;\mathbb Z/2)$, let $M_{H\mathbb Z}(a,X)$ be the $\mathbb Z/2[s_1]$-module which is a free $H^*(X;\mathbb Z/2)$-module on a single generator U, and with s_1-action

$$s_1(Ux) \coloneqq U(ax+\mathrm{Sq}^1(x)). \tag{2.32}$$

(2) Given $a\in H^1(X;\mathbb Z/2)$ and $c\in H^3(X;\mathbb Z)$, let $M_{ku}(a,c,X)$ be the $\mathbb Z/2\langle q_0,q_1\rangle$-module which is a free $H^*(X;\mathbb Z/2)$-module on a single generator U, and with q_0- and q_1-actions given by

$$\begin{aligned} q_0(Ux) &\coloneqq U(ax+Q_0(x))\\ q_1(Ux) &\coloneqq U((c \bmod 2 + a^3)x + Q_1(x)). \end{aligned} \tag{2.33}$$

(3) Given $a\in H^1(X;\mathbb Z/2)$ and $b\in H^2(X;\mathbb Z/2)$, let $M_{ko}(a,b,X)$ be the $\mathbb Z/2\langle s_1,s_2\rangle$-module which is a free $H^*(X;\mathbb Z/2)$-module on a single generator U, and with s_1- and s_2-actions

$$\begin{aligned} s_1(Ux) &\coloneqq U(ax+\mathrm{Sq}^1(x))\\ s_2(Ux) &\coloneqq U(bx + a\mathrm{Sq}^1(x)+\mathrm{Sq}^2(x)). \end{aligned} \tag{2.34}$$

(4) Given $a\in H^1(X;\mathbb Z/2)$, and $d\in SH^4(X)$, let $M_{tmf}(a,d,X)$ be the $\mathbb Z/2\langle s_1,s_2,s_4\rangle$-module which is a free $H^*(X;\mathbb Z/2)$-module on a single generator U, with s_1- and s_2-actions given

by (2.34) with $b = t(d)$, and s_4-action given by

$$s_4(Ux) = U(\delta x + t(d)a + \mathrm{Sq}^1(t(d)))\mathrm{Sq}^1(x) + t(d)\mathrm{Sq}^2(x) + a\mathrm{Sq}^3(x) + \mathrm{Sq}^4(x)). \tag{2.35}$$

(5) Given $d \in SH^4(X)$, let $M'_{tmf}(d, X)$ be the $\mathbb{Z}/3\langle\beta, p_1\rangle/(\beta^2)$-module which is a free $H^*(X;\mathbb{Z}/2)$-module on a single generator U and β- and p_1-actions specified by

$$\begin{aligned} \beta(Ux) &\coloneqq U\beta(x) \\ p_1(Ux) &\coloneqq U((d \bmod 3)x + \mathcal{P}^1(x)). \end{aligned} \tag{2.36}$$

The mod 3 reduction of the supercohomology class d is defined as usual as the image of d after passing to the mod 3 Moore spectrum $\mathbb{S}/3$:

$$[X, \Sigma^4 SH] \longrightarrow [X, \Sigma^4 SH \wedge \mathbb{S}/3] \xrightarrow{\cong} [X, \Sigma^4 H\mathbb{Z}/3], \tag{2.37}$$

because $H\mathbb{Z}/2 \wedge \mathbb{S}/3 \simeq 0$. Thus $d \bmod 3$ is well-defined as a class in $H^4(X;\mathbb{Z}/3)$.

Lemma 2.38. *Keep the notation from Definition 2.31.*

(1) *The action of s_1 on $M_{H\mathbb{Z}}(a, X)$ squares to 0, so the $\mathbb{Z}/2[s_1]$-module structure on $M_{H\mathbb{Z}}(a, X)$ refines to an $\mathcal{A}(0)$-module structure with $\mathrm{Sq}^1(x) \coloneqq s_1(x)$.*

(2) *The actions of q_0 and q_1 on $M_{ku}(a, c, X)$ commute and both square to 0, so the $\mathbb{Z}/2\langle q_0, q_1\rangle$-module structure on $M_{ku}(a, c, X)$ refines to an $\mathcal{E}(1)$-module structure, where for $i = 0, 1$, $Q_i(x) \coloneqq q_i(x)$.*

(3) *The actions of s_1 and s_2 on $M_{ko}(a, b, X)$, and of s_1, s_2, and s_4 on $M_{tmf}(a, b, c, X)$, satisfy the Adem relations with s_i in place of Sq^i, hence refine to an $\mathcal{A}(1)$-module structure on $M_{ko}(a, b, X)$ and an $\mathcal{A}(2)$-module structure on $M_{tmf}(a, c, d, X)$.*

(4) *The actions of β and p^1 on $M'_{tmf}(c, X)$ satisfy the relations in (2.28), hence refine the $\mathbb{Z}/3\langle\beta, p^1\rangle/(\beta^2)$-module structure on $M'_{tmf}(c, X)$ to an $\mathcal{A}^{tmf}$-module structure, where the Bockstein acts as β and $\mathcal{P}^1$ acts as p^1.*

Rather than prove this directly, we will obtain it as a corollary of Theorem 2.39. This theorem says that the modules defined in Definition 2.31 are H_R^* of the Thom spectra for the corresponding twists.

Theorem 2.39. *Let X be a topological space.*

(1) *Given $a \in H^1(X;\mathbb{Z}/2)$, let $f_a\colon X \to B\mathrm{GL}_1(H\mathbb{Z})$ be the corresponding fake vector bundle twist. Then there is an isomorphism of $\mathcal{A}(0)$-modules*

$$H^*_{H\mathbb{Z}}(M^{H\mathbb{Z}} f_a) \xrightarrow{\cong} M_{H\mathbb{Z}}(a, X). \tag{2.40}$$

(2) *Given $a \in H^1(X;\mathbb{Z}/2)$ and $c \in H^3(X;\mathbb{Z})$, let $f_{a,c}\colon X \to B\mathrm{GL}_1(ku)$ be the corresponding fake vector bundle twist. Then there is an isomorphism of $\mathcal{E}(1)$-modules*

$$H^*_{ku}(M^{ku} f_{a,c}) \xrightarrow{\cong} M_{ku}(a, c, X). \tag{2.41}$$

(3) *Given $a \in H^1(X;\mathbb{Z}/2)$ and $b \in H^2(X;\mathbb{Z}/2)$, let $f_{a,b}\colon X \to B\mathrm{GL}_1(ko)$ be the corresponding fake vector bundle twist. Then there is an isomorphism of $\mathcal{A}(1)$-modules*

$$H^*_{ko}(M^{ko} f_{a,b}) \xrightarrow{\cong} M_{ko}(a, b, X). \tag{2.42}$$

(4) *Given $a \in H^1(X;\mathbb{Z}/2)$, and $d \in SH^4(X)$, let $f_{a,d}\colon X \to B\mathrm{GL}_1(tmf)$ be the corresponding fake vector bundle twist. Then there is an isomorphism of $\mathcal{A}(2)$-modules*

$$H^*_{tmf}(M^{tmf} f_{a,d}) \xrightarrow{\cong} M_{tmf}(a, d, X), \tag{2.43}$$

and an isomorphism of $\mathcal{A}^{tmf}$-modules

$$H^*_{tmf}(M^{tmf}f_{0,d};\mathbb{Z}/3) \xrightarrow{\cong} M'_{tmf}(d,X). \tag{2.44}$$

In the last isomorphism, we turn off degree-1 twists so that we have a Thom isomorphism for mod 3 cohomology.

Corollary 2.45. *Keep the notation from Theorem 2.39.*

Twisted $\mathbb{Z}$-homology: *The E_2-page of Baker-Lazarev's Adams spectral sequence computing $\pi_*(M^{H\mathbb{Z}}f_a)^\wedge_2$ is isomorphic as $\mathrm{Ext}^{*,*}_{\mathcal{A}(0)}(\mathbb{Z}/2,\mathbb{Z}/2)$-modules to $\mathrm{Ext}^{s,t}_{\mathcal{A}(0)}(M_{H\mathbb{Z}}(a,X),\mathbb{Z}/2)$.*

Twisted *ku*-homology: *The E_2-page of Baker-Lazarev's Adams spectral sequence computing $\pi_*(M^{ku}f_{a,c})^\wedge_2$ is isomorphic as $\mathrm{Ext}^{*,*}_{\mathcal{E}(1)}(\mathbb{Z}/2,\mathbb{Z}/2)$-modules to $\mathrm{Ext}^{s,t}_{\mathcal{E}(1)}(M_{ku}(a,c,X),\mathbb{Z}/2)$.*

Twisted *ko*-homology: *The E_2-page of Baker-Lazarev's Adams spectral sequence computing $\pi_*(M^{ko}f_{a,b})^\wedge_2$ is isomorphic as $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)}(\mathbb{Z}/2,\mathbb{Z}/2)$-modules to $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(M_{ko}(a,b,X),\mathbb{Z}/2)$.*

Twisted *tmf*-homology:

(1) *The E_2-page of Baker-Lazarev's Adams spectral sequence computing $\pi_*(M^{tmf}f_{a,d})^\wedge_2$ is isomorphic as $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)}(\mathbb{Z}/2,\mathbb{Z}/2)$-modules to $\mathrm{Ext}^{s,t}_{\mathcal{A}(2)}(M_{tmf}(a,d,X),\mathbb{Z}/2)$.*

(2) *The E_2-page of Baker-Lazarev's Adams spectral sequence computing $\pi_*(M^{tmf}f_{0,d})^\wedge_3$ is isomorphic as $\mathrm{Ext}^{*,*}_{\mathcal{A}^{tmf}}(\mathbb{Z}/3,\mathbb{Z}/3)$-modules to $\mathrm{Ext}^{s,t}_{\mathcal{A}^{tmf}}(M'_{tmf}(d,X),\mathbb{Z}/3)$.*

Remark 2.46. In §1.2.1 we saw that the twists of $H\mathbb{Z}$ discussed above are all vector bundle twists, so that the $H\mathbb{Z}$ part of Corollary 2.45 follows from the standard change-of-rings argument; the same is true for the twists of *MTSO* appearing below in Corollary 2.59. In both cases, the other calculations are new.

Remark 2.47. The analogue of Corollary 2.45 is true for a few standard variants of the Adams spectral sequence. For example, one could switch the order of $H^*_R(Mf)$ and $\mathbb{Z}/2$ in $\mathrm{Ext}_{H^*_RH}$ and obtain the E_2-page of Baker-Lazarev's Adams spectral sequence computing twisted R-cohomology for twists over a finite type space. One could also work out a version of Corollary 2.45 in terms of R-module H-homology with its H^R_*H-comodule structure.

Now, given a big sibling and little sibling pair $M \to R$, we lift to M. While it would be nice to completely describe the M-module Baker-Lazarev Adams spectral sequences for $M = \mathit{MTSO}$, MTSpin^c, *MTSpin*, and *MTString*, this ranges between very complicated and intractable. This is because these Adams spectral sequences would in principle determine the ring structures on M_* for these spectra M, which are not presently known for MTSpin^c, *MTSpin*, and *MTString* and which is intricate for *MTSO*.[24] Thus we provide two different lifts of Corollary 2.45:

(1) In Theorem 2.57, we use the connectivity of the orientations from the big to the little siblings to *partially* calculate the Baker-Lazarev Adams spectral sequence for each of the big siblings.
(2) In Corollary 2.59, we use the splittings of M that we reviewed in (1.17) to noncanonically describe M-module Thom spectra as sums of R-module Thom spectra, and therefore obtain an R-module Baker-Lazarev Adams spectral sequence that computes spectrum-level information about M-module Thom spectra. This does not work for *MTString*, which has not been split at 2.[25]

[24] See Abdallah-Salch [AS24] for recent progress in the spin^c case.
[25] While there is substantial evidence suggesting that, 2-locally, *MTString* splits as a wedge sum of *tmf*, $\Sigma^{16}\mathit{tmf}_0(3)$, and other pieces, for example in [Pen83, MG95, MH02, MR09, Lau04, Lau16, LO16, LO18, LS19, Dev19, Abs21,

Before we compare the Baker-Lazarev Adams spectral sequences for the big and little siblings, we need a few facts in homological algebra.

Lemma 2.48. *Let k be a field and A_1 and A_2 be $\mathbb{Z}$-graded k-algebras concentrated in nonnegative degrees. Suppose that we have the following data for some positive integer n:*

(1) *A k-algebra homomorphism $\phi\colon A_1 \to A_2$ which is a k-vector space isomorphism in degrees $\le n$.*
(2) *A_i-modules M_i concentrated in nonnegative degrees, and an A_1-module homomorphism $\psi\colon M_1 \to M_2$ which is an isomorphism in degrees $\le n$.*

Then for $i = 1, 2$, there are free A_i-modules F_i and surjective A_i-module homomorphisms $\chi_i\colon F_i \to M_i$, together with an A_1-module map $\theta\colon F_1 \to F_2$ which is an isomorphism in degrees $\le n$, and such that the following diagram commutes:

$$\begin{tikzcd} F_1 \arrow[r, "\chi_1"] \arrow[d, "\theta"'] & M_1 \arrow[d, "\psi"'] \\ F_2 \arrow[r, "\chi_2"] & M_2 \end{tikzcd} \tag{2.49}$$

Here the A_1-module structures on M_2 and F_2 are the ones induced across ϕ.

Proof. Present M_2 as an A_2-module, and let $\chi_2\colon F_2 \to M_2$ be the quotient map sending the free A_2-module on the generating set S of M_2 to their images in M_2. Let F_1 be the free A_1-module on S, so that ϕ induces the A_1-module map $\theta\colon F_1 \to F_2$. Since F_1 and F_2 are concentrated in nonnegative degrees and ϕ is an isomorphism in degrees n and below, the same is true of θ.

Now to build χ_1. Let $s \in S$, which we regard as a generator of F_1. If $\deg(s) > n$, let $\chi_1(s) = 0$. If $\deg(s) \le n$, $\chi_2(\theta(s)) \in M_2$ has a unique preimage under ψ, because ψ is an isomorphism in degrees n and below; define $\chi_1(s) \coloneqq \psi^{-1}(\chi_2(\theta(s)))$. At this point, we have now verified the entire lemma statement except for surjectivity of χ_1; if χ_1 as constructed has cokernel, add free A_1-module summands to F_1 so that χ_1 is surjective. Then define θ on these summands by the requirement that (2.49) commutes: since the A_1-module structure on M_2 is induced from its A_2-module via ϕ, the A_1-module map $\psi \circ \chi_1$ factors through $F_1 \to F_1 \otimes_{A_1} A_2$, which is a free A_2-module, so we may extend θ. □

Corollary 2.50. *Suppose for $i = 1, 2$, A_i, M_i, ϕ, ψ, and n are as in Lemma 2.48. Then there are free resolutions $P_\bullet^{(i)} \to M_i$ of A_i-modules and a map $\theta_\bullet\colon P_\bullet^{(1)} \to P_\bullet^{(2)}$ of chain complexes of A_1-modules such that for all homological degrees s, θ_s is an isomorphism in grading-degree n and below.*

Proof. Use Lemma 2.48 to build $d_0^{(i)}\colon P_1^{(i)} \to M_i$ for each i and the map $\theta_1\colon P_1^{(1)} \to P_1^{(2)}$. Then there is an induced A_1-module map $\psi'\colon \ker(d_0^{(1)}) \to \ker(d_0^{(2)})$ which is an isomorphism in degrees $\le n$, so we can build the next step in the free resolution by applying Lemma 2.48 to $\ker(d_0^{(i)})$ and ψ', and so on. □

Recall the notion of a *minimal resolution* of a module over an augmented algebra from, e.g., [BC18, §4.4].

Dev24, Tok24], it is not a foregone conclusion that a splitting exists. For example, Kochman [Koc93, Part 1, Theorem 5.4] proved that the symplectic bordism spectrum *MTSp* is indecomposable at the prime 2.

Lemma 2.51. *Keeping the notation and assumptions from Corollary 2.50, now assume in addition that A_1 and A_2 are augmented algebras, such that each augmentation $A_i \to k$ is an isomorphism when restricted to degree-0 elements. Assume also that ϕ is a homomorphism of augmented algebras. Then the resolutions $P_\bullet^{(i)} \to M_i$ may be chosen to be minimal resolutions such that θ_s is an isomorphism in grading-degrees $n+s$ and below.*

Proof. Building the minimal resolutions is exactly as in the proof of Corollary 2.50, thanks to the observation that we can choose the surjections in Lemma 2.48 to satisfy the minimality property.

The assumption that the augmentations $A_i \to k$ are isomorphisms in degree 0 implies that $P_s^{(i)}$ is concentrated in degrees s and above. Therefore we may shift the grading on $P_s^{(i)}$ down by s before applying Lemma 2.48 in the inductive step of Corollary 2.50, then shift it back up, to obtain an isomorphism in degrees $\leq n+s$, as required. $\square$

Lemma 2.52. *Suppose $H = H\mathbb{Z}/p$ for a prime p and we have connective, E_∞-ring spectra R_1 and R_2 with E_∞-ring maps $f\colon R_1 \to R_2$ and $g\colon R_2 \to H$ such that f is n-connected for some $n \geq 1$. Suppose we have connective R_i-module spectra N_i and an n-connected R_1-module map $\varphi\colon N_1 \to N_2$. Then the induced maps*

$$H_*^{R_1}(N_1) \longrightarrow H_*^{R_2}(N_2) \tag{2.53a}$$

$$H^*_{R_1}(N_1) \longrightarrow H^*_{R_2}(N_2) \tag{2.53b}$$

are isomorphisms in degrees $* \leq n$.

Proof. First (2.53a). For $i = 1, 2$, consider the Künneth spectral sequences

$$E^2_{*,*} = \mathrm{Tor}_{*,*}^{\pi_*(R_i)}(\mathbb{Z}/p, \pi_*(N_i)) \Longrightarrow H_*^{R_i}(N_i). \tag{2.54}$$

The Künneth spectral sequence is natural in the data of the map $R_i \to H$ and N_i, so f induces a map of spectral sequences, i.e. a map on each page which commutes with differentials, and which converges to the map $H_*^{R_1}(N_1) \to H_*^{R_2}(N_2)$. Corollary 2.50 implies that the induced map on E^2-pages is an isomorphism in grading degree n and below, and therefore also in total degree n and below. This immediately implies the result for (2.53a) (there may be differentials from total degree $n+1$ to total degree n, but n-connectivity implies surjectivity in degree $n+1$, so those differentials are carried from the first spectral sequence to the second, therefore also implying the isomorphism in degree n).

For (2.53b), use the equivalence $\mathrm{Map}_{R_i}(N_i, H) \xrightarrow{\simeq} \mathrm{Map}_H(H \wedge_{R_i} N_i, H)$ to reduce to (2.53a), similarly to the proof of Lemma 2.24. $\square$

Corollary 2.55. *With notation as in Lemma 2.52, assume also that $H^0_{R_i} H \cong \mathbb{Z}/p$. Then the induced map*

$$\mathrm{Ext}^{s,t}_{H^*_{R_1} H}(H^*_{R_1}(N_1), \mathbb{Z}/p) \longrightarrow \mathrm{Ext}^{s,t}_{H^*_{R_2} H}(H^*_{R_2}(N_2), \mathbb{Z}/p) \tag{2.56}$$

is an isomorphism in topological degree $t - s \leq n$.

Proof. The condition on $H^0_{R_i} H$ implies that $H^*_{R_i} H$ is canonically augmented by the algebra map quotienting by all positive-degree elements. Therefore we may use Lemma 2.51 with $A_i = H^*_{R_i} H$, $M_i = \pi_*(N_i)$, ϕ the map on $H^*_{(-)} H$, and $\psi = \pi_*(\varphi)$. $\square$

Now we provide our first lift of Corollary 2.45 to the big siblings: a computation of the Baker-Lazarev Adams spectral sequence, but only in a range.

Theorem 2.57. *Keep the notation from Theorem 2.39.*

Twisted oriented bordism: *In topological degrees $t-s \le 3$, the E_2-page of Baker-Lazarev's Adams spectral sequence computing $(\Omega_*^{\mathrm{SO}}(X,a))_2^\wedge$ is isomorphic as $\mathrm{Ext}_{\mathcal{A}(0)}^{*,*}(\mathbb{Z}/2,\mathbb{Z}/2)$-modules to $\mathrm{Ext}_{\mathcal{A}(0)}^{s,t}(M_{H\mathbb{Z}}(a,X),\mathbb{Z}/2)$.*

Twisted spin^c bordism: *In topological degrees $t-s \le 3$, the E_2-page of Baker-Lazarev's Adams spectral sequence computing $(\Omega_*^{\mathrm{Spin}^c}(X,a,c))_2^\wedge$ is isomorphic as $\mathrm{Ext}_{\mathcal{E}(1)}^{*,*}(\mathbb{Z}/2,\mathbb{Z}/2)$-modules to $\mathrm{Ext}_{\mathcal{E}(1)}^{s,t}(M_{ku}(a,c,X),\mathbb{Z}/2)$.*

Twisted spin bordism: *In topological degrees $t-s \le 7$, the E_2-page of Baker-Lazarev's Adams spectral sequence computing $(\Omega_*^{\mathrm{Spin}}(X,a,b))_2^\wedge$ is isomorphic as $\mathrm{Ext}_{\mathcal{A}(1)}^{*,*}(\mathbb{Z}/2,\mathbb{Z}/2)$-modules to $\mathrm{Ext}_{\mathcal{A}(1)}^{s,t}(M_{ko}(a,b,X),\mathbb{Z}/2)$. $\mathrm{Ext}_{\mathcal{A}(1)}^{s,t}(M_{ko}(a,b,X),\mathbb{Z}/2)$.*

Twisted string bordism: *In topological degrees $t-s \le 15$, the E_2-page of Baker-Lazarev's Adams spectral sequence computing $(\Omega_*^{\mathrm{String}}(X,a,d))_2^\wedge$, resp. $(\Omega_*^{\mathrm{String}}(X,0,d))_3^\wedge$, are isomorphic to $\mathrm{Ext}_{\mathcal{A}(2)}^{s,t}(M_{tmf}(a,d,X),\mathbb{Z}/2)$, resp. $\mathrm{Ext}_{\mathcal{A}^{tmf}}^{s,t}(M'_{tmf}(d,X),\mathbb{Z}/3)$, as modules over $\mathrm{Ext}_{\mathcal{A}(2)}^{*,*}(\mathbb{Z}/2,\mathbb{Z}/2)$, resp. $\mathrm{Ext}_{\mathcal{A}^{tmf}}^{*,*}(\mathbb{Z}/3,\mathbb{Z}/3)$.*

Proof. Each of these is a consequence of Corollary 2.55, where R_1 is the big sibling, R_2 is the little sibling, N_i is the R_i-module Thom spectrum for the fake vector bundle twist in question, f is the orientation $R_1 \to R_2$ introduced in §1.2, and φ is the induced map of Thom spectra. Here n is 3 for oriented and spin^c bordism, $n = 7$ for spin bordism, and $n = 15$ for string bordism. The only hypothesis we have yet to confirm is that φ is n-connected, which we do now. By Lemma 1.8, $N_2 \simeq R_2 \wedge_{R_1} N_1$ and $\varphi \simeq f \wedge \mathrm{id}\colon N_1 \simeq R_1 \wedge_{R_1} N_1 \to R_2 \wedge_{R_1} N_1 \simeq N_2$. Thus we get an induced map between the following two Künneth spectral sequences:

$$E^2_{*,*} = \mathrm{Tor}_{*,*}^{\pi_*(R_1)}(\pi_*(R_1),\pi_*(N_1)) \Longrightarrow \pi_*(R_1 \wedge_{R_1} N_1) = \pi_*(N_1) \tag{2.58a}$$

$$E^2_{*,*} = \mathrm{Tor}_{*,*}^{\pi_*(R_1)}(\pi_*(R_2),\pi_*(N_1)) \Longrightarrow \pi_*(R_2 \wedge_{R_1} N_1) = \pi_*(N_2), \tag{2.58b}$$

which we apply Corollary 2.50 to, similarly to the proof of Lemma 2.52. Thus for each of the four cases in the theorem statement, we have verified the hypotheses of Corollary 2.55; the conclusion of that corollary finishes the proof of this theorem. ☐

Because Theorem 2.57 only calculates the Baker-Lazarev Adams spectral sequence in a range of degrees, we also provide a version in all degrees for $MTSO$, $MTSpin^c$, and $MTSpin$, which heuristically records the fact that the Wall, resp. Anderson-Brown-Peterson splittings of these Thom spectra fiber over BO/BH. Thus these splittings are compatible with fake vector bundle twists.

Recall the modules W_1, W_2, and W_3 from Remark 2.4.

Corollary 2.59. *Keep the notation from Theorem 2.39.*

Twisted oriented bordism: *There is a strongly convergent spectral sequence of Adams type with signature*

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}(0)}^{s,t}(M_{H\mathbb{Z}}(a,X)\otimes W_1,\mathbb{Z}/2) \Longrightarrow \Omega_*^{\mathrm{SO}}(X,a)_2^\wedge. \tag{2.60}$$

Twisted spin^c bordism: *There is a strongly convergent spectral sequence of Adams type with signature*

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{E}(1)}^{s,t}(M_{ku}(a,c,X)\otimes W_2,\mathbb{Z}/2) \Longrightarrow \Omega_*^{\mathrm{Spin}^c}(X,a,c)_2^\wedge. \tag{2.61}$$

Twisted spin bordism: *There is a strongly convergent spectral sequence of Adams type with signature*

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(M_{ko}(a,b,X)\otimes W_3, \mathbb{Z}/2) \Longrightarrow \Omega_*^{\mathrm{Spin}}(X,a,b)^\wedge_2. \tag{2.62}$$

All tensor products are taken over $\mathbb{Z}/2$ and given an $\mathcal{A}(0)$-, $\mathcal{E}(1)$-, or $\mathcal{A}(1)$-module structure using the Hopf algebra structure on $\mathcal{A}(0)$, $\mathcal{E}(1)$, and $\mathcal{A}(1)$, respectively.

Proof. Throughout this proof, implicitly 2-localize. We give the proof for twisted spin bordism; the remaining cases are analogous. The input is a theorem of Hebestreit-Joachim [HJ20] that the Anderson-Brown-Peterson decomposition of $MTSpin$ as a sum of ko-modules upgrades to a splitting of local systems of spectra over $B\mathrm{O}/B\mathrm{Spin}$. Therefore, given a fake vector bundle twist $f_{a,b}\colon X\to B\mathrm{O}/B\mathrm{Spin}$, there is an equivalence of spectra

$$M^{MTSpin}f_{a,b} \simeq \bigvee_i \Sigma^{\ell_i} M^{ko}f_{a,b} \vee \bigvee_j \Sigma^{m_j} M^{ko}f_{a,b}\wedge_{ko} ko\langle 2\rangle \vee \bigvee_k \Sigma^{n_k} M^{ko}f_{a,b}\wedge_{ko} H\mathbb{Z}/2, \tag{2.63}$$

where the indices i, j, k, n_i, n_j, and n_k represent the indices and shifts in the original Anderson-Brown-Peterson decomposition [ABP67] and $ko\langle 2\rangle$ is the 1-connected cover of ko.

The right-hand side of (2.63) is manifestly a ko-module; use this equivalence to define a ko-module structure on $M^{MTSpin}f_{a,b}$. Then the spectral sequence in the corollary statement is the Baker-Lazarev ko-module Adams spectral sequence for $M^{MTSpin}f_{a,b}$; W_3 appears because it is the direct sum of H^*_{ko} of the summands in the Anderson-Brown-Peterson decomposition.

Hebestreit-Joachim's proof goes through in exactly the same way for $MTSpin^c$ and ku [HJ20]. For $MTSO$ and $H\mathbb{Z}$, we use the fact that the map $B\mathrm{O}/B\mathrm{SO}\to B\mathrm{GL}_1(MTSO)$ factors through $B\mathrm{GL}_1(\mathbb{S})$, so every equivalence of spectra fibers over it. ☐

Proof of Theorem 2.39. All five parts of the theorem have similar proofs, so we walk through the full proof in two cases — $R = ku$, whose proof carries through for $H\mathbb{Z}$, ko, and tmf at $p=3$ with minor changes; and $R = tmf$ at $p=2$, where the presence of supercohomology means the proof is slightly different.

Now we specialize to $R = ku$ and a fake vector bundle twist $f_{a,c}\colon X\to B\mathrm{GL}_1(ku)$. To begin, use Lemma 2.24 to learn that $H^*_{ku}(M^{ku}f_{a,c})\cong H^*(X;\mathbb{Z}/2)$ as $\mathbb{Z}/2$-vector spaces. (In the more familiar case where the twist is given by a vector bundle, this is the Thom isomorphism.) Next, the Thom diagonal (Definition 1.11) and the Cartan formula provide a formula for $Q_i(Ux)$, $i=0,1$, in terms of $Q_i(U)$ and $Q_i(x)$. In particular, this formula implies that if we can show $Q_0(U) = Ua$ and $Q_1(U) = U(a^3+c)$, then the $\mathcal{E}(1)$-module action defined on $M_{ku}(X,a,c)$ in Definition 2.31 is identified with $H^*_{ku}(Mf_{a,c})$. By the naturality of cohomology operations, it suffices to compute $Q_0(U)$ and $Q_1(U)$ for the the universal twist over $B\mathrm{O}/B\mathrm{Spin}^c$. Theorem 1.16 then allows us to infer what the cohomology operations on the Thom class have to be in order to recover the correct $\mathcal{A}$-module structure on the Thom spectrum after applying the universal twist.

Let $f\colon B\mathrm{O}/B\mathrm{Spin}^c\to B\mathrm{GL}_1(MTSpin^c)$ be the universal fake vector bundle twist, $M^{MTSpin^c}f$ be its associated Thom spectrum, and $M^{ku}f$ be the ku-module Thom spectrum obtained by composing f with the map $B\mathrm{GL}_1(MTSpin^c)\to B\mathrm{GL}_1(ku)$ induced by the Atiyah-Bott-Shapiro map. The Atiyah-Bott-Shapiro map is 3-connected, so the map $M^{MTSpin^c}f\to M^{ku}f$ is also 3-connected. Thus, for example, $\pi_0(M^{ku}f)\cong\pi_0(M^{MTSpin^c}f)$; by Example 1.32 $M^{MTSpin^c}f\simeq MTO$, so $\pi_0(M^{ku}f)\cong\Omega_0^{\mathrm{O}}\cong\mathbb{Z}/2$. This and similar ideas will determine $Q_0(U)$ and $Q_1(U)$ for us: in particular we will find $\mathrm{Sq}^1(U) = Ua$ and $Q_1(U) = U(c \bmod 2 + a^3)$ because this is the unique choice that is compatible with the known homotopy groups of the Thom spectra of the universal twists from §1.2.2: MTO over $K(\mathbb{Z}/2,1)\times K(\mathbb{Z},3)$, $MTSO$ over $K(\mathbb{Z},3)$, and $MTPin^c$ over $K(\mathbb{Z}/2,1)$.

We first consider Q_0: $Q_0(U)$ is either 0 or Ua. For either of the two options for $Q_0(U)$, one can explicitly write the $\mathcal{E}(1)$-module structure on $H^*_{ku}(M^{ku}f)$ in low degrees. Then, using Baker-Lazarev's Adams spectral sequence, one finds that if $Q_0(U)=0$, $\pi_0(M^{ku}f)^\wedge_2\cong\pi_0(M^{MTSpin^c}f)$ has at least 4 elements, but since $Mf\simeq MTO$, we know this group is $\Omega_0^{\mathrm{O}}\cong\mathbb{Z}/2$. Thus $Q_0(U)=Ua$.

There are three options for $Q_1(U)$: 0, $Uc \bmod 2$, and $U(c \bmod 2+a^3)$. In order to verify the Q_1 action, we pull back to $K(\mathbb{Z},3)$ and $K(\mathbb{Z}/2,1)$ separately, and then argue in a similar way.

- For $f\colon K(\mathbb{Z},3)\to B\mathrm{GL}_1(MTSpin^c)$, $M^{MTSpin^c}f\simeq MTSO$, which is incompatible with $Q_1(U)=0$; the argument is similar to that for Q_0.
- For $f\colon K(\mathbb{Z}/2,1)\to B\mathrm{GL}_1(MTSpin^c)$, $M^{MTSpin^c}f\simeq MTPin^c$. In $H^*_{ku}(M^{ku}f)$, $Q_1(U)\neq 0$, which one can show by pulling back further along

$$M^{ku}f\wedge H\mathbb{Z}/2\longrightarrow M^{ku}\wedge_{ku}H\mathbb{Z}/2. \tag{2.64}$$

Thus $Q_1(U)=U(c \bmod 2+a^3)$. Using the fact that $\mathcal{E}(1)=\langle Q_0,Q_1\rangle$ and applying the Cartan formula recovers the actions in (2.33).

Because the fake vector bundle twist for tmf uses supercohomology, its part of the proof is different enough that we go into the details. The reduction to the computation of $\mathrm{Sq}^1(U)$, $\mathrm{Sq}^2(U)$, and $\mathrm{Sq}^4(U)$ in the case of the universal twist proceeds in the same way as for ku. In §1.2.4 we computed $H^*(B\mathrm{SO}/B\mathrm{String};\mathbb{Z}/2)$ in low degrees; this and the Künneth formula imply that in the mod 2 cohomology of $K(\mathbb{Z}/2,1)\times B\mathrm{SO}/B\mathrm{String}$, H^1 is spanned by a, H^2 is spanned by $\{a^2,t(d)\}$, and H^4 is spanned by $\{a^4,a^2t(d),a\mathrm{Sq}^1t(d),\delta,t(d)^2\}$. Therefore there are $\lambda_1,\dots,\lambda_8\in\mathbb{Z}/2$ such that

$$\mathrm{Sq}^1(U)=U\lambda_1 a \tag{2.65a}$$

$$\mathrm{Sq}^2(U)=U(\lambda_2a^2+\lambda_3t(d)) \tag{2.65b}$$

$$\mathrm{Sq}^4(U)=U(\lambda_4a^4+\lambda_5a^2t(d)+\lambda_6a\mathrm{Sq}^1t(d)+\lambda_7\delta+\lambda_8t(d)^2). \tag{2.65c}$$

We finish the proof by indicating how to find λ_1 through λ_8. To find λ_7, consider the twist pulled back to $f\colon K(\mathbb{Z},4)\simeq B\mathrm{Spin}/B\mathrm{String}\to B\mathrm{O}/B\mathrm{String}$. Like in the proof for twists of ku, the action of Sq^4 on the Thom class can be detected on either $M^{MTString}f$ or $M^{tmf}f$; as we discussed in Example 1.65, $M^{MTString}f\simeq MTSpin$, so $\pi_3(M^{MTString}f)\cong\Omega_3^{\mathrm{Spin}}=0$, and since the map $M^{MTString}f\to M^{tmf}f$ is sufficiently connected, $\pi_3(M^{tmf}f)=0$ as well. In $H^*_{tmf}(M^{tmf}f)$, the only options for $\mathrm{Sq}^4(U)$ are 0 or U times the tautological class. One can run the Baker-Lazarev Adams spectral sequence for these two options and see that only the latter choice is compatible with $\pi_3(M^{tmf}f)=0$.[26] Thus $\lambda_7=1$.

For the other coefficients, we pull back to vector bundle twists for various vector bundles $V\to X$, where we know $\mathrm{Sq}^k(U)=Uw_k(V)$, $a\mapsto w_1(V)$, $t(d)\mapsto w_2(V)$, and $\delta\mapsto w_4(V)$. Choosing vector bundles with auspicious values of w_1, w_2, and w_4 quickly determines the remaining coefficients.

- Pulling back the twist to $K(\mathbb{Z}/2,1)\simeq B\mathrm{O}_1$ gives the Thom spectrum $tmf\wedge(B\mathrm{O}_1)^{\sigma-1}$, where $\sigma\to B\mathrm{O}_1$ is the tautological line bundle. As $w_1(\sigma)\neq0$ but $w_2(\sigma)=0$ and $w_4(\sigma)=0$,

[26] To do so, it will be helpful to know $\mathrm{Ext}_{\mathcal{A}(2)}(C\nu,\mathbb{Z}/2)$, where $C\nu$ is the $\mathcal{A}(2)$-module with two $\mathbb{Z}/2$ summands in degrees 0 and 4, joined by a Sq^4. These Ext groups have been computed by Bruner-Rognes [BR21, Corollary 4.16, Figure 4.3].

we can plug these Stiefel-Whitney classes into (2.65b) (with $w_1(V)$ in place of a, $w_2(V)$ in place of $t(d)$, and $w_4(V)$ in place of δ as usual) to conclude $\lambda_1 = 1$, $\lambda_2 = 0$, and $\lambda_4 = 0$.

- Let $V \coloneqq \mathcal{O}(1) \oplus \mathcal{O}(2) \to \mathbb{CP}^2$. If $\alpha \in H^2(\mathbb{CP}^2; \mathbb{Z}/2) \cong \mathbb{Z}/2$ is the unique nonzero element, then $w_1(V) = 0$, $w_2(V) = \alpha$, and $w_4(V) = 0$. Plugging this into (2.65b), we find $\mathrm{Sq}^2(U) = U\alpha = U\lambda_3\alpha$, so $\lambda_3 = 1$. And plugging $w_1(V)$, $w_2(V)$, and $w_4(V)$ into (2.65c), we obtain $\mathrm{Sq}^4(U) = 0 = U\lambda_8\alpha^2$, so $\lambda_8 = 0$.
- Let x, resp. y be the nonzero classes in $H^1(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{Z}/2)$ pulled back from the first, resp. second copy of $\mathbb{RP}^2$, and let $\sigma_x, \sigma_y \to \mathbb{RP}^2 \times \mathbb{RP}^2$ be the real line bundles satisfying $w_1(\sigma_x) = x$ and $w_1(\sigma_y) = y$. Now let $V \coloneqq \sigma_x \oplus \sigma_y^{\oplus 3}$; then $w_1(V) = x + y$, $w_2(V) = xy + y^2$, and $w_4(V) = 0$. Plugging into (2.65c), we have $\mathrm{Sq}^4(U) = 0 = U\lambda_5 x^2y^2$, so $\lambda_5 = 0$.
- Repeat the preceding example, but with $\mathbb{RP}^1 \times \mathbb{RP}^3$ in place of $\mathbb{RP}^2 \times \mathbb{RP}^2$; this time, $w_1(V) = x + y$, $w_2(V) = xy + y^2$, and $w_4(V) = xy^3$. Plugging into (2.65c), we have $\mathrm{Sq}^4(U) = Uxy^3 = U(1 + \lambda_6)xy^3$, so $\lambda_6 = 0$. $\square$

3. Applications

In this section, we give examples in which we use Corollaries 2.45 and 2.59 to make computations of twisted (co)homology groups.

3.1. U-duality and related twists of spin bordism.

Let G be a topological group and

$$1 \longrightarrow \{\pm 1\} \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1 \tag{3.1a}$$

be a central extension classified by $\beta \in H^2(BG; \{\pm 1\})$. Then the central extension

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin} \times_{\{\pm 1\}} \widetilde{G} \xrightarrow{\ p\ } \mathrm{SO} \times G \longrightarrow 1 \tag{3.1b}$$

is classified by $w_2 + \beta \in H^2(B(\mathrm{SO} \times G); \mathbb{Z}/2)$. One can prove this is the extension by pulling back along $\mathrm{SO} \to \mathrm{SO} \times G$ and $G \to \mathrm{SO} \times G$ and observing that both pulled-back extensions are non-split. Therefore given an oriented vector bundle $E \to X$ and a principal G-bundle $P \to X$, i.e. the data of an $\mathrm{SO} \times G$ structure on E, a lift of this data to a $\mathrm{Spin} \times_{\{\pm 1\}} \widetilde{G}$-structure is a trivialization of $w_2(E) + f_P^*(\beta)$, where $f_P \colon X \to BG$ is the classifying map of $P \to X$. That is, if ξ denotes the composition

$$\xi \colon B(\mathrm{Spin} \times_{\{\pm 1\}} \widetilde{G}) \xrightarrow{Bp} B\mathrm{SO} \times BG \to B\mathrm{SO} \to B\mathrm{O}, \tag{3.2}$$

then a ξ-structure on E is equivalent to a (BG, β)-twisted spin structure, meaning that by Lemma 1.44 the Thom spectrum $MT\xi$ is canonically equivalent to the MTSpin-module Thom spectrum $Mf_{0,\beta}$ associated to the fake vector bundle twist $f_{0,\beta} \colon BG \to B\mathrm{GL}_1(\mathit{MTSpin})$ (see Remark 1.28 for the spectrum-level statement). $MT\xi$ may or may not split as $\mathit{MTSpin} \wedge X$ for a spectrum X: a sufficient condition is the existence of a vector bundle $V \to BG$ such that $w_2(V) = \beta$, as we discussed in §2.1. But as we will see soon, there are choices of (G, β), even when G is a compact, connected Lie group, for which no such V exists. For these G and β, Theorem 2.39 significantly simplifies the calculation of ξ-bordism.

As an example, consider $G = \mathrm{SU}_8/\{\pm 1\}$ and β the nonzero element of $H^2(BG; \mathbb{Z}/2) \cong \mathrm{Hom}(\pi_1(G), \mathbb{Z}/2) \cong \mathbb{Z}/2$, corresponding to the central extension

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{SU}_8 \longrightarrow \mathrm{SU}_8/\{\pm 1\} \longrightarrow 1. \tag{3.3}$$

In [DY24b], we studied $\Omega_*^{\mathrm{Spin}\times_{\{\pm 1\}}\mathrm{SU}_8}$ as part of an argument that the $E_{7(7)}(\mathbb{R})$ U-duality symmetry of four-dimensional $\mathcal{N}=8$ supergravity is anomaly-free. Speyer [Spe22] shows that all representations of G are spin, so $\beta \neq w_2(V)$ for any vector bundle $V \to BG$ induced from a representation of G, and this can be upgraded to show $Mf_{0,\beta} \not\simeq \mathit{MTSpin} \wedge X$ for any spectrum X (see [DY24b, Footnote 6]). This precludes the standard shearing/change-of-rings argument for computing $\mathrm{Spin}\times_{\{\pm 1\}}\mathrm{SU}_8$ bordism, and indeed in [DY24b, §4.3] we had to give a more complicated workaround. However, thanks to Theorem 2.39, we can now argue over $\mathcal{A}(1)$. We need as input the low-degree cohomology of $B(\mathrm{SU}_8/\{\pm 1\})$.

Proposition 3.4 ([DY24b, Theorem 4.4])**.** *$H^*(B(\mathrm{SU}_8/\{\pm 1\});\mathbb{Z}/2) \cong \mathbb{Z}/2[\beta, b, c, d, e, \dots]/(\dots)$ with $|\beta| = 2$, $|b| = 3$, $|c| = 4$, $|d| = 5$, and $|e| = 6$; there are no other generators below degree 7 and no relations below degree 7. The Steenrod squares are*

$$\begin{aligned}\mathrm{Sq}(\beta) &= \beta + b + \beta^2\\ \mathrm{Sq}(b) &= b + d + b^2\\ \mathrm{Sq}(c) &= c + e + \mathrm{Sq}^3(c) + c^2\\ \mathrm{Sq}(d) &= d + b^2 + \mathrm{Sq}^3(d) + \mathrm{Sq}^4(d) + d^2.\end{aligned} \tag{3.5}$$

By Theorem 2.57, to understand the *MTSpin*-module Baker-Lazarev Adams spectral sequence for $M^{\mathit{MTSpin}}f_{0,\beta}$ in the degrees we care about (i.e. 5 and below), it is equivalent to consider the *ko*-module analogue for $M^{ko}f_{0,\beta}$, Theorem 2.39 tells us how $\mathcal{A}(1)$ acts on $H^*_{ko}(M^{ko}f_{0,\beta})$: $\mathrm{Sq}^1(U) = 0$ and $\mathrm{Sq}^2(U) = U\beta$; to make more computations, use the Cartan formula and the Steenrod squares in Proposition 3.4. Then using the information from (3.5) yields

$$\begin{aligned}\mathrm{Sq}^1(U\beta) &= U\mathrm{Sq}^1(\beta) + \mathrm{Sq}^1(U)\beta = Ub\\ \mathrm{Sq}^2(U\beta) &= U\mathrm{Sq}^2(\beta) + \mathrm{Sq}^1(U)\mathrm{Sq}^1(\beta) + \mathrm{Sq}^2(U)\beta = U(2\beta^2) = 0\end{aligned} \tag{3.6a}$$

$$\begin{aligned}\mathrm{Sq}^1(Ub) &= U\mathrm{Sq}^1(b) + \mathrm{Sq}^1(U)b = 0\\ \mathrm{Sq}^2(Ub) &= U\mathrm{Sq}^2(b) + \mathrm{Sq}^1(U)\mathrm{Sq}^1(b) + \mathrm{Sq}^2(U)b = U(d + b\beta)\end{aligned} \tag{3.6b}$$

$$\begin{aligned}\mathrm{Sq}^1(U(d+b\beta)) &= U\mathrm{Sq}^1(d+b\beta) + \mathrm{Sq}^1(U)(d+b\beta) = U(2b^2) = 0\\ \mathrm{Sq}^2(U(d+b\beta)) &= U\mathrm{Sq}^2(d+b\beta) + \mathrm{Sq}^1(U)\mathrm{Sq}^1(d+b\beta) + \mathrm{Sq}^2(U)(d+b\beta) = 0.\end{aligned} \tag{3.6c}$$

See the lower left (red) piece of Figure 1, left, for a picture of this data. This calculation implies the vector space generated by $\{U, U\beta, Ub, U(d+b\beta)\}$ is an $\mathcal{A}(1)$-submodule of $H^*_{ko}(M^{ko}f_{0,\beta})$; specifically, it is isomorphic to the "seagull" $\mathcal{A}(1)$-module $M_0 \coloneqq \mathcal{A}(1) \otimes_{\mathcal{A}(0)} \mathbb{Z}/2$.[27] This is an $\mathcal{A}(1)$-module whose $\mathcal{A}(1)$-action does not compatibly extend to an $\mathcal{A}$-action. Continuing to compute Sq^1- and Sq^2-actions as in (3.6), we learn that there is an isomorphism of $\mathcal{A}(1)$-modules

$$H^*_{ko}(M^{ko}f_{0,\beta}) \cong M_0 \oplus \Sigma^4 M_0 \oplus \Sigma^4 M_1 \oplus \mathcal{A}(1) \oplus P, \tag{3.7}$$

where P is concentrated in degrees 6 and above (so we can and will ignore it), and M_1 is an $\mathcal{A}(1)$-module which is isomorphic to either M_0 or $C\eta \coloneqq \mathcal{A}(1) \otimes_{\mathcal{E}(1)} \mathbb{Z}/2$. We draw the decomposition (3.7) in Figure 1, left.

The change-of-rings isomorphism (Lemma 2.1) and Koszul duality [BC18, Remark 4.5.4] allow us to compute $\mathrm{Ext}_{\mathcal{A}(1)}(M_0) \cong \mathbb{Z}/2[h_0]$ and $\mathrm{Ext}_{\mathcal{A}(1)}(C\eta) \cong \mathbb{Z}/2[h_0, v_1]$ with h_0 in bidegree $(t-s, s) = (0, 1)$ and v_1 in bidegree $(t-s, s) = (2, 1)$ [BC18, Examples 4.5.5 and 4.5.6]. Therefore we can draw

[27] Adamyk [Ada23] introduced the name "seagull" for M_0.

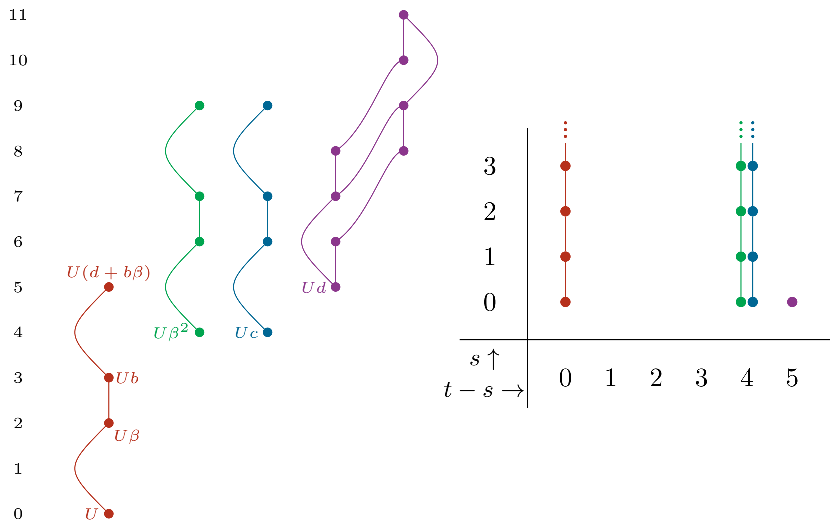

FIGURE 1. Left: the $\mathcal{A}(1)$-module structure on $H^*_{ko}(M^{ko}f_{0,\beta})$ in low degrees, where $\beta \in H^2(B(\mathrm{SU}_8/\{\pm 1\}); \mathbb{Z}/2)$ is the generator. The pictured submodule contains all elements in degrees 5 and below. We have not determined $\mathrm{Sq}^3(Uc)$ – it may be 0, in which case the blue summand would vanish in degrees 7 and above. In either case, the pictured $\mathcal{A}(1)$-module cannot arise as the restriction of an $\mathcal{A}$-action to $\mathcal{A}(1)$, indicating that the fake vector bundle twist $f_{0,\beta}$ of ko cannot arise from a vector bundle. Right: the E_2-page of the corresponding ko-module Adams spectral sequence, which as discussed in §3.1 also computes the 2-completion of $\Omega_*^{\mathrm{Spin}\times_{\{\pm1\}}\mathrm{SU}_8}$ in degrees 7 and below.

the E_2-page of the Adams spectral sequence computing the twisted ko-homology associated to the fake vector bundle twist $f_{0,\beta}\colon B(\mathrm{SU}_8/\{\pm 1\}) \to B\mathrm{GL}_1(ko)$ in Figure 1, right. By Theorem 2.57, this also computes $\pi_*(M^{MTSpin}f_{0,\beta})^\wedge_2$ in low degrees, and by Lemma 1.44, this is isomorphic to the 2-completion of the corresponding twisted spin bordism groups, which we saw above are $\Omega_*^{\mathrm{Spin}\times_{\{\pm1\}}\mathrm{SU}_8}$. This spectral sequence collapses on the E_2-page in degrees 5 and below, using h_0-linearity of differentials, so we have made the following computation.

Theorem 3.8 ([DY24b, Theorem 4.26])**.**

$$\begin{aligned}
\Omega_0^{\mathrm{Spin}\times_{\{\pm1\}}\mathrm{SU}_8} &\cong \mathbb{Z}\\
\Omega_1^{\mathrm{Spin}\times_{\{\pm1\}}\mathrm{SU}_8} &\cong 0\\
\Omega_2^{\mathrm{Spin}\times_{\{\pm1\}}\mathrm{SU}_8} &\cong 0\\
\Omega_3^{\mathrm{Spin}\times_{\{\pm1\}}\mathrm{SU}_8} &\cong 0\\
\Omega_4^{\mathrm{Spin}\times_{\{\pm1\}}\mathrm{SU}_8} &\cong \mathbb{Z}^2\\
\Omega_5^{\mathrm{Spin}\times_{\{\pm1\}}\mathrm{SU}_8} &\cong \mathbb{Z}/2.
\end{aligned} \tag{3.9}$$

There are a few other choices of compact Lie groups G and classes $\beta \in H^2(BG; \mathbb{Z}/2)$ such that β is not equal to w_2 of any representation, including

- $\mathrm{SU}_{4n}/\{\pm 1\}$ for $n > 1$, where β corresponds to the double cover $\mathrm{SU}_{4n} \to \mathrm{SU}_{4n}/\{\pm 1\}$ [Spe22],

- PSO_{8n}, where β corresponds to the double cover $\mathrm{SO}_{8n} \to \mathrm{PSO}_{8n}$ [JF19],
- PSp_n and the double cover $\mathrm{Sp}_n \to \mathrm{PSp}_n$ for $n > 1$, and
- $E_7/\{\pm 1\}$ and the double cover $E_7 \to E_7/\{\pm 1\}$.

For the last two items, the proof is analogous to [DY24b, Footnote 6] for $\mathrm{SU}_8/\{\pm 1\}$: compute the low-degree mod 2 cohomology of BG and use this to show that if β is w_2 of a representation V, the $\mathcal{A}$-action on the cohomology of the corresponding Thom spectrum violates the Adem relations.

For all of these choices of G and β, one can define (at a physics level of rigor) unitary quantum field theories with fermions and a background $\widetilde{G}$ symmetry, such that $-1 \in \widetilde{G}$ acts by -1 on fermions and by 1 on bosons. Then, as described in [WWW19, SW16], these theories can be defined on manifolds with differential $\mathrm{Spin}_n \times_{\{\pm 1\}} \widetilde{G}$ structures, so by work of Freed-Hopkins [FH21], the anomaly field theories of these QFTs are classified using the bordism groups $\Omega_*^{\mathrm{Spin}\times_{\{\pm 1\}}\widetilde{G}}$, and computations such as Theorem 3.8 are greatly simplified using Theorem 2.39.

Kuroda [Kur25] makes some of these computations, using similar methods to the ones we used here to determine the $\mathrm{Spin} \times_{\{\pm 1\}} \mathrm{Sp}_4$, $\mathrm{Spin} \times_{\{\pm 1\}} \mathrm{SU}_8$, and $\mathrm{Spin} \times_{\{\pm 1\}} \mathrm{Spin}_{16}$ bordism groups in degrees 7 and below.

Remark 3.10. Though we focused on invertible field theories in this section, there are other applications of twisted spin bordism groups. For example, Kreck's modified surgery [Kre99] uses twisted spin bordism to classify closed, smooth 4-manifolds whose universal covers are spin up to stable diffeomorphism: given such a manifold M, one shows that $w_1(M)$ and $w_2(M)$ pull back from $B\pi_1(M)$, then considers twisted spin bordism for the fake vector bundle twist over $B\pi_1(M)$ given by $w_1(M)$ and $w_2(M)$. Often one computes these bordism groups with Teichner's *James spectral sequence* [Tei93, §II], a version of the Atiyah-Hirzebruch spectral sequence for spin bordism that can handle non-vector-bundle twists. However, extension questions in this spectral sequence can be difficult, and it is helpful to have the Adams spectral sequence to resolve them (see [Ped17] for an example for a vector bundle twist). Therefore Corollary 2.59 could be a useful tool for studying stable diffeomorphism classes of 4-manifolds, since not all of the relevant twists come from vector bundles.

3.2. **Twists of string bordism.** A story very similar to that of §3.1 takes place one level up in the Whitehead tower for $B\mathrm{O}$. Many supergravity theories require spacetime manifolds M to satisfy a *Green-Schwarz condition* specified by a Lie group G and a class $c \in H^4(BG;\mathbb{Z})$, which Sati-Schreiber-Stasheff [SSS12] characterize as data of a spin structure on M, a principal G-bundle $P \to M$ and a trivialization of $\lambda(M) - c(M)$, i.e. the data of a (BG, c)-twisted string structure on M (see also [Sat10, Sat11, SS19]). In many example theories of interest, this twist does not come from a vector bundle, including the $E_8 \times E_8$ heterotic string and the CHL string [Deb24, Lemma 2.2]. The corresponding twisted string bordism groups are used to study anomalies and defects for these theories; anomalies were touched on in §3.1, and the use of bordism groups to learn about defects is through the McNamara-Vafa cobordism conjecture [MV19].

Theorem 2.57 allows us to use the Baker-Lazarev Adams spectral sequence at $p = 2$ and $p = 3$ to calculate these twisted string bordism groups in dimensions 15 and below, which suffices for applications to superstring theory. (Calculations at primes greater than 3 are easier and can be taken care of with other methods.) We will show an example computation, relevant for the $E_8 \times E_8$ heterotic string at $p = 3$; for applications of Theorem 2.57 to twisted string bordism at $p = 2$, see [Deb24, §2.2, §2.4.1] and [BDDM24], and for more $p = 3$ calculations, see [BDDM24].

Because E_8 is a connected, simply connected, simple Lie group, there is an isomorphism $c\colon H^4(BE_8;\mathbb Z)\xrightarrow{\cong}\mathbb Z$ uniquely specified by making the Chern-Weil class of the Killing form positive; let c be the preimage of 1 under this isomorphism. Bott-Samelson [BS58, Theorems IV, V(e)] showed that, interpreted as a map $BE_8\to K(\mathbb Z,4)$, c is 15-connected.

For $i=1,2$, let $c_i\in H^4(BE_8\times BE_8;\mathbb Z)$ be the copy of c coming from the i^{th} copy of E_8. Let $\mathbb Z/2$ act on $E_8\times E_8$ by switching the two factors; then in the Serre spectral sequence for the fibration of classifying spaces induced by the short exact sequence

$$1 \longrightarrow E_8\times E_8 \longrightarrow (E_8\times E_8)\rtimes\mathbb Z/2 \longrightarrow \mathbb Z/2 \longrightarrow 1, \tag{3.11}$$

the class $c_1+c_2\in E_2^{0,4}=H^4(BE_8\times BE_8;\mathbb Z)$ survives to the E_∞-page and lifts uniquely to define a class $c_1+c_2\in H^4(B((E_8\times E_8)\rtimes\mathbb Z/2);\mathbb Z)$. The Green-Schwarz condition for the $E_8\times E_8$ heterotic string asks for an $(E_8\times E_8)\rtimes\mathbb Z/2$-bundle $P\to M$ and a trivialization of $\lambda(M)-(c_1+c_2)(P)$, so we want to compute $\Omega_*^{\text{String}}(B((E_8\times E_8)\rtimes\mathbb Z/2),c_1+c_2)$. Theorem 2.57 allows us to use the change-of-rings theorem to simplify the Adams spectral sequence at $p=2,3$ for this computation in degrees 15 and below; we will give the 3-primary computation here and point the interested reader to [Deb24, §2.2] for the longer 2-primary computation.

Theorem 3.12 ([Deb24, Theorem 2.65])**.** *The $(B((E_8\times E_8)\rtimes\mathbb Z/2),c_1+c_2)$-twisted string bordism groups lack 3-primary torsion in degrees 11 and below.*

Just like for $\mathrm{Spin}\times_{\{\pm1\}}\mathrm{SU}_8$ bordism and [DY24b] in §3.1, the computation in [Deb24] does not take advantage of the change-of-rings theorem, works over the entire Steenrod algebra, and is significantly harder than our proof here.

Proof. Recall the notation $\mathcal A^{tmf}$, β, and $\mathcal P^1$ from Example 2.27. By Lemma 1.63 (see also Remark 1.28), the Thom spectrum for $(B((E_8\times E_8)\rtimes\mathbb Z/2),c_1+c_2)$-twisted string bordism is identified with the $MTString$-module Thom spectrum $M^{MTString}f_{0,c_1+c_2}$, where f_{0,c_1+c_2} is the fake vector bundle twist defined by the image of the class $c_1+c_2\in H^4(B((E_8\times E_8)\rtimes\mathbb Z/2);\mathbb Z)$ in supercohomology. Let $M^{tmf}f_{0,c_1+c_2}$ be the tmf-module Thom spectrum induced by the Ando-Hopkins-Rezk map $\sigma\colon MTString\to tmf$. As a consequence of Theorem 2.57, in topological degrees 15 and below, the $MTString$-module Baker-Lazarev Adams spectral sequence for $M^{MTString}f_{0,c_1+c_2}$ coincides with the tmf-module Baker-Lazarev Adams spectral sequence for $M^{tmf}f_{0,c_1+c_2}$. Theorem 2.39 describes the $\mathcal A^{tmf}$-module structure on $H^*_{tmf}(M^{tmf}f_{0,c_1+c_2};\mathbb Z/3)$, and hence the input to the tmf-module Baker-Lazarev Adams spectral sequence, in terms of the $\mathcal A_3$-module structure on $H^*(B(E_8\times E_8)\rtimes\mathbb Z/2;\mathbb Z/3)$.

Lemma 3.13. *Let $x\coloneqq(c_1+c_2)\bmod 3$ and $y\coloneqq c_1c_2\bmod 3$. Then $H^*(B(E_8\times E_8)\rtimes\mathbb Z/2;\mathbb Z/3)\cong\mathbb Z/3[x,\mathcal P^1(x),\beta\mathcal P^1(x),y,\dots]/(\dots)$; there are no other generators below degree 12, nor any relations below degree 12.*

The actions of $\mathcal P^1$ and β are as specified via the names of the generators.

Proof. Because $H^*(B\mathbb Z/2;\mathbb Z/3)$ vanishes in positive degrees, the Serre spectral sequence for (3.11) collapses at E_2 to yield an isomorphism to the ring of invariants

$$H^*(B(E_8\times E_8)\rtimes\mathbb Z/2;\mathbb Z/3)\xrightarrow{\cong}(H^*(BE_8\times BE_8;\mathbb Z/3))^{\mathbb Z/2}. \tag{3.14}$$

The lemma thus follows once we know $H^*(BE_8;\mathbb Z/3)\cong\mathbb Z/3[c\bmod 3,\mathcal P^1(c\bmod 3),\beta\mathcal P^1(c\bmod 3),\dots]/(\dots)$, where we have given all generators and relations in degrees 11 and below. Because

$c\colon BE_8 \to K(\mathbb{Z},4)$ is 15-connected [BS58, Theorems IV, V(e)], we may replace BE_8 with $K(\mathbb{Z},4)$, and the mod 3 cohomology of $K(\mathbb{Z},4)$ was computed by Cartan [Car54] and Serre [Ser52]; see Hill [Hil09, Corollary 2.9] for an explicit description. □

To compute $H^*_{tmf}(M^{tmf}f_{0,c_1+c_2})$, we also need to know $\mathcal{P}^1(U)$, and Theorem 2.39 tells us $\mathcal{P}^1(U) = Ux$. Then as usual we compute on all classes in degrees 11 and below using the Cartan formula.

Corollary 3.15. *Let $N_1 \coloneqq \mathcal{A}^{tmf}/(\beta, (\mathcal{P}^1)^2, \beta\mathcal{P}^1\beta)$ and $N_2 \coloneqq \mathcal{A}^{tmf}/(\beta, \beta\mathcal{P}^1, \mathcal{P}^1\beta(\mathcal{P}^1)^2)$. Then there is a map of $\mathcal{A}^{tmf}$-modules*

$$H^*_{tmf}(M^{tmf}f_{0,c_1+c_2}) \longrightarrow N_2 \oplus \Sigma^8 N_1 \oplus \Sigma^8 N_1 \tag{3.16}$$

which is an isomorphism in degrees 11 and below.

We draw the decomposition (3.16) in Figure 6, left. The next step is to compute the Ext groups of N_1 and N_2 over $\mathcal{A}^{tmf}$. To do so, we will repeatedly use the fact that a short exact sequence of $\mathcal{A}^{tmf}$-modules induces a long exact sequence in Ext; see [BC18, §4.6] for more information on this technique, including how to depict the long exact sequence in an Adams chart along with some examples. Let $C\nu$ denote the $\mathcal{A}^{tmf}$-module consisting of two $\mathbb{Z}/3$ summands in degrees 0 and 4 linked by a nontrivial $\mathcal{P}^1$-action. Then there are short exact sequences

$$0 \longrightarrow \Sigma^4\mathbb{Z}/3 \longrightarrow C\nu \longrightarrow \mathbb{Z}/3 \longrightarrow 0, \tag{3.17a}$$

$$0 \longrightarrow \Sigma^5\mathbb{Z}/3 \longrightarrow N_1 \longrightarrow C\nu \longrightarrow 0, \tag{3.17b}$$

$$0 \longrightarrow \Sigma^4 N_1 \longrightarrow N_2 \longrightarrow \mathbb{Z}/3 \longrightarrow 0. \tag{3.17c}$$

We will address (3.17a) in Figure 2, (3.17b) in Figure 3, and (3.17c) in Figure 5. As input to our computations, we need $\mathrm{Ext}_{\mathcal{A}^{tmf}}(\mathbb{Z}/3) \coloneqq \mathrm{Ext}^{*,*}_{\mathcal{A}^{tmf}}(\mathbb{Z}/3,\mathbb{Z}/3)$; this acts on $\mathrm{Ext}_{\mathcal{A}^{tmf}}(V) \coloneqq \mathrm{Ext}^{*,*}_{\mathcal{A}^{tmf}}(V,\mathbb{Z}/3)$ for any $\mathcal{A}^{tmf}$-module V by the Yoneda product (see [BC18, §4.2]). The boundary maps in the long exact sequences of Ext groups induced by short exact sequences of $\mathcal{A}^{tmf}$-modules are linear for this $\mathrm{Ext}_{\mathcal{A}^{tmf}}(\mathbb{Z}/3)$-action, which we will use in Lemmas 3.20, 3.22 and 3.24. Throughout this subsection, if we do not specify the base, Ext means $\mathrm{Ext}_{\mathcal{A}^{tmf}}$.

Theorem 3.18 (Henriques-Hill [Hil07, DFHH14])**.** *$\mathrm{Ext}_{\mathcal{A}^{tmf}}(\mathbb{Z}/3)$ is generated by the classes $h_0 \in \mathrm{Ext}^{1,1}$, $\alpha \in \mathrm{Ext}^{1,4}$, $c_4 \in \mathrm{Ext}^{2,10}$, $\beta \in \mathrm{Ext}^{2,12}$, $c_6 \in \mathrm{Ext}^{3,15}$, and $\Delta \in \mathrm{Ext}^{3,27}$, modulo the relations $\alpha^2 = 0$, $h_0\alpha = 0$, $h_0\beta = 0$, $\alpha c_4 = 0$, $\beta c_4 = 0$, $\alpha c_6 = 0$, $\beta c_6 = 0$, and $c_4^3 - c_6^2 = h_0^3\Delta$.*

Remark 3.19. Our notation differs from that of some authors who study $\mathrm{Ext}_{\mathcal{A}^{tmf}}(\mathbb{Z}/3)$. Compared with Hill [Hil07, §2], our names for generators agree except that what we call h_0 Hill calls v_0. Comparing with Bruner-Rognes [BR21, Chapter 13]: our h_0 is their a_0, our α is their h_0, and our β is their b_0, and other names of generators agree.

The action of h_0 on the E_∞-page of this Adams spectral sequence lifts to multiplying by 3 on the twisted *tmf*-homology groups that the spectral sequence converges to.

In the long exact sequence in Ext corresponding to (3.17a), let $x \in \mathrm{Ext}^{0,0}$ be either generator of $\mathrm{Ext}_{\mathcal{A}^{tmf}}(\mathbb{Z}/3)$ and $y \in \mathrm{Ext}^{0,4}$ be either generator of $\mathrm{Ext}_{\mathcal{A}^{tmf}}(\Sigma^4\mathbb{Z}/3)$, both as modules over $\mathrm{Ext}_{\mathcal{A}^{tmf}}(\mathbb{Z}/3)$. In both cases, there are exactly two generators and they differ by a sign.

Lemma 3.20. *In the long exact sequence in* Ext *associated to* (3.17a)*,* $\partial(y) = \pm\alpha x$*,* $\partial(\beta y) = \pm\alpha\beta x$*, and the boundary map vanishes on all other elements in degrees* 14 *and below (except for* $-c$ *where* c *was a class already listed).*

We draw this in Figure 2, bottom left.

Proof. Apart from on $\pm y$ and $\pm\beta y$, the boundary map vanishes for degree reasons; since ∂ commutes with the action of $\mathrm{Ext}_{\mathcal{A}^{tmf}}(\mathbb{Z}/3)$, once we show $\partial(y) = \pm\alpha x$, $\partial(\beta y) = \pm\alpha\beta x$ follows. Since $\mathrm{Ext}^{1,4}(\mathbb{Z}/3) \cong \mathbb{Z}/3$, if we show $\partial(y) \neq 0$ the only options for ∂y are $\pm\alpha x$.

Since y and $-y$ are the only nonzero elements in $\mathrm{Ext}^{4,0}$ of both $\mathbb{Z}/3$ and $\Sigma^4\mathbb{Z}/3$, $\partial(y) = 0$ if and only if $\mathrm{Ext}^{0,4}_{\mathcal{A}^{tmf}}(C\nu) = 0$. And this Ext group is $\mathrm{Hom}_{\mathcal{A}^{tmf}}(C\nu, \Sigma^4\mathbb{Z}/3) = 0$. □

Remark 3.21. In $\mathrm{Ext}_{\mathcal{A}^{tmf}}(C\nu)$, $\alpha(\alpha y) = \beta x$,[28] but this is not detected by the long exact sequence in Ext. This action is denoted with a dashed gray line in Figure 2, bottom right. We do not need this hidden α-action, so we will not prove it;[29] one way to check it is to compute $\mathrm{Ext}_{\mathcal{A}_3}(C\nu)$ using the software developed by Bruner [Bru22] or by Beauvais-Feisthauer, Chatham, and Chua [BFCC24], obtain the hidden α-action in $\mathrm{Ext}_{\mathcal{A}_3}(C\nu)$, and chase it across the map of Ext groups induced by $\mathcal{A}^{tmf} \to \mathcal{A}_3$.

Thus we obtain $\mathrm{Ext}_{\mathcal{A}^{tmf}}(C\nu)$ in Figure 2, bottom right.

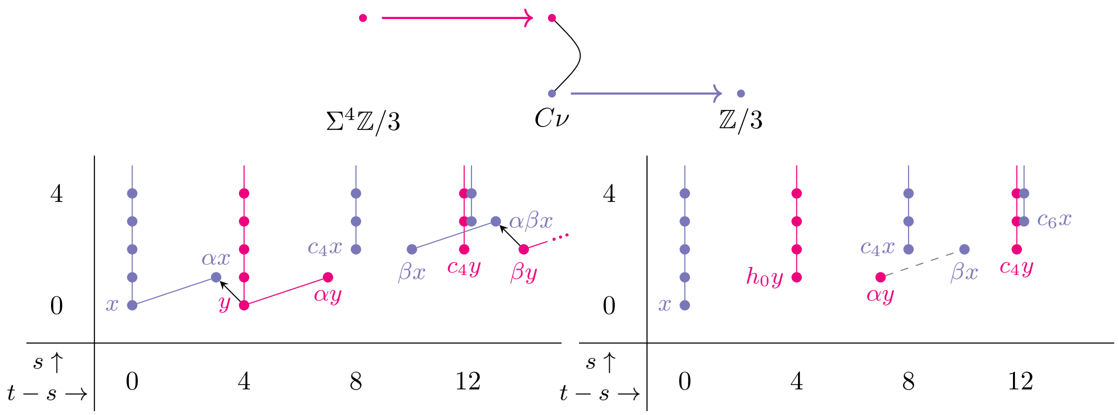

Figure 2. Top: the short exact sequence (3.17a) of $\mathcal{A}^{tmf}$-modules. Lower left: the induced long exact sequence in Ext; we compute the pictured boundary maps in Lemma 3.20. Lower right: $\mathrm{Ext}_{\mathcal{A}^{tmf}}(C\nu)$ as computed by the long exact sequence. The dashed line is a nonzero α-action not visible to this computation; see Remark 3.21.

Now we turn to (3.17b) and its long exact sequence in Ext, depicted in Figure 3. We keep the notation for elements of $\mathrm{Ext}(C\nu)$ from above, so elements are specified by products of classes in $\mathrm{Ext}(\mathbb{Z}/3)$ with x or y. In the long exact sequence induced by (3.17b), let $z \in \mathrm{Ext}^{0,5}$ be a generator of $\mathrm{Ext}(\Sigma^5\mathbb{Z}/3)$ as a module over $\mathrm{Ext}(\mathbb{Z}/3)$ (again, there is exactly one other generator, which is $-z$).

[28] This does not contradict the relation $\alpha^2 = 0$ from Theorem 3.18: since y was killed in the long exact sequence computing $\mathrm{Ext}(C\nu)$, the class $\alpha y \in \mathrm{Ext}(C\nu)$ is not α times anything, so $\alpha(\alpha y)$ need not vanish.

[29] We do use this α-action in the proof of Lemma 3.24, but only to determine Ext groups that will be in too high of a degree to matter in the final computation, so that part of the proof can be left out.

Lemma 3.22. *In the long exact sequence in Ext associated to (3.17b), $\partial(h_0^i z) = \pm h_0^i y$, $\partial(h_0 c_4 z) = \pm h_0^i c_4 y$, and the boundary map vanishes on all other elements in degrees 14 and below (except for $-c$ where c was a class already listed).*

We draw this in Figure 3, bottom left.

Proof. The proof is essentially the same as for Lemma 3.20: all boundary maps other than the ones in the theorem statement vanish for degree reasons; then, $\mathrm{Ext}(\mathbb{Z}/3)$-linearity of boundary maps reduces the theorem statement to the computation of $\partial(z)$, which must be $\pm h_0 y$ because $\mathrm{Ext}^{0,5}_{\mathcal{A}^{tmf}}(N_1) = \mathrm{Hom}_{\mathcal{A}^{tmf}}(N_1, \Sigma^5\mathbb{Z}/3) = 0$. □

Remark 3.23. Like in Remark 3.21, the long exact sequence does not fully specify the $\mathrm{Ext}(\mathbb{Z}/3)$-action on $\mathrm{Ext}(N_1)$. One can show that $h_0 \cdot \alpha z = \pm c_4 x$, but this is missed by our long exact sequence calculation. We do not need this relation in our proof of Theorem 3.12, so we do not prove it; one way to see $h_0 \cdot \alpha z = \pm c_4 x$ would be to deduce it from the analogous h_0-action in $\mathrm{Ext}(N_2)$ via the long exact sequence in Ext induced from (3.17c). To see the corresponding h_0-action in $\mathrm{Ext}(N_2)$, let N_3 be a nonsplit $\mathcal{A}^{tmf}$-module extension of $C\nu$ by $\Sigma^8\mathbb{Z}/3$; this characterizes N_3 up to isomorphism. Then there is a short exact sequence $\Sigma^9\mathbb{Z}/3 \to N_2 \to N_3$, and the h_0-action we want to detect is visible to the corresponding long exact sequence in Ext.

Thus we have $\mathrm{Ext}(N_1)$ in Figure 3, bottom right.

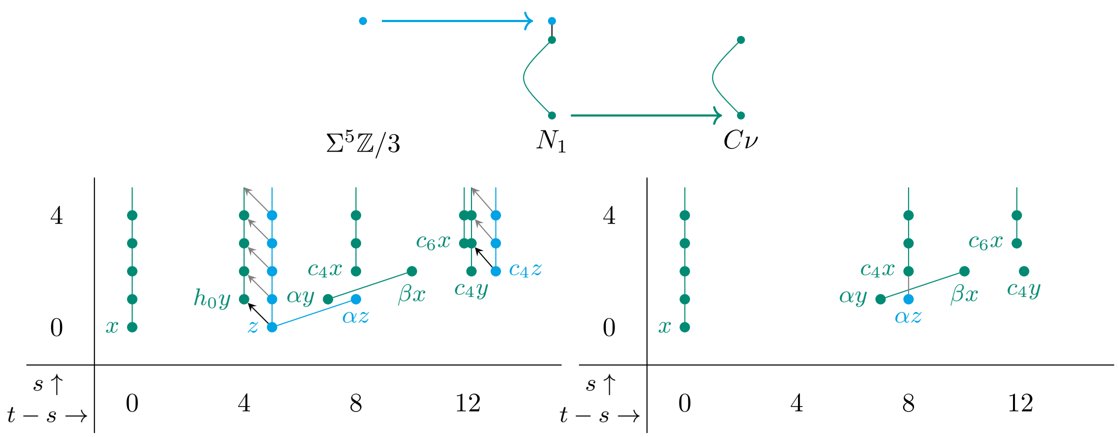

FIGURE 3. Top: the short exact sequence (3.17b) of $\mathcal{A}^{tmf}$-modules. Lower left: the induced long exact sequence in Ext. We compute the pictured boundary maps in Lemma 3.22. Lower right: $\mathrm{Ext}_{\mathcal{A}^{tmf}}(N_1)$ as computed by the long exact sequence. The gray line joining αz and $c_4 x$ indicates a nonzero h_0-action not visible to this computation; see Remark 3.23.

The last long exact sequence we have to run is the one induced by (3.17c). We keep the notation for elements of $\mathrm{Ext}(N_1)$ from above — classes in $\mathrm{Ext}(\mathbb{Z}/3)$ times x, y, or z. We let w denote a generator of $\mathrm{Ext}(\mathbb{Z}/3)$ as an $\mathrm{Ext}(\mathbb{Z}/3)$-module; like before, the two generators are w and $-w$.

Lemma 3.24. *In the long exact sequence in Ext associated to (3.17c), the boundary map takes the values $\partial(x) = \pm\alpha w$, $\partial(\alpha y) = \pm\beta w$, and $\partial(\beta x) = \pm\alpha\beta w$, and vanishes on all other classes in degrees 14 and below (except for $-c$ where c was a class already listed).*

We draw this in Figure 5, bottom left.

Proof. As in Lemmas 3.20 and 3.22, apart from $\partial(\pm x)$, $\partial(\pm\alpha y)$, and $\partial(\pm\beta x)$, the boundary map vanishes for degree reasons, and we infer $\partial(x) = \pm\alpha w$ because this is the only way for $\mathrm{Ext}^{0,4}(N_2) = \mathrm{Hom}(N_2, \Sigma^4\mathbb{Z}/3)$ to vanish. And since $\alpha(\alpha y) = \beta x$, as we discussed in Remark 3.21, it remains only to prove $\partial(\alpha y) = \pm\beta w$; then $\partial(\beta x) = \alpha\beta w$ follows from $\mathrm{Ext}(\mathbb{Z}/3)$-linearity; and since $\mathrm{Ext}^{2,12}_{\mathcal{A}^{tmf}}(\mathbb{Z}/3)$ is one-dimensional, to show $\partial(\alpha y) = \pm\beta w$ it suffices to show $\partial(\alpha y)$ is nonzero.

To compute $\partial(\alpha y)$, we use the characterization of $\mathrm{Ext}^{1,t}_{\mathcal{A}^{tmf}}(M, N)$ as a set of equivalence classes of $\mathcal{A}^{tmf}$-module extensions $0 \to \Sigma^t N \to L \to M \to 0$. We will represent αy as an explicit extension of $\Sigma^4 N_1$ by $\Sigma^{12}\mathbb{Z}/3$ and then show this extension cannot be the pullback of an extension of N_2 by $\Sigma^{12}\mathbb{Z}/3$, which implies $\partial(\alpha y) \neq 0$ by exactness. Up to isomorphism, there is only one non-split extension of $\Sigma^4 N_1$ by $\Sigma^{12}\mathbb{Z}/3$, with αy and $-\alpha y$ distinguished by a sign in the extension maps; we draw this extension in Figure 4, left. In Figure 4, right, we illustrate what goes wrong if we try to obtain this extension as the pullback of an extension of N_2: the relation $(\mathcal{P}^1)^3 = 0$ in $\mathcal{A}^{tmf}$ is violated. Thus $\partial(\alpha y) \neq 0$. ∎

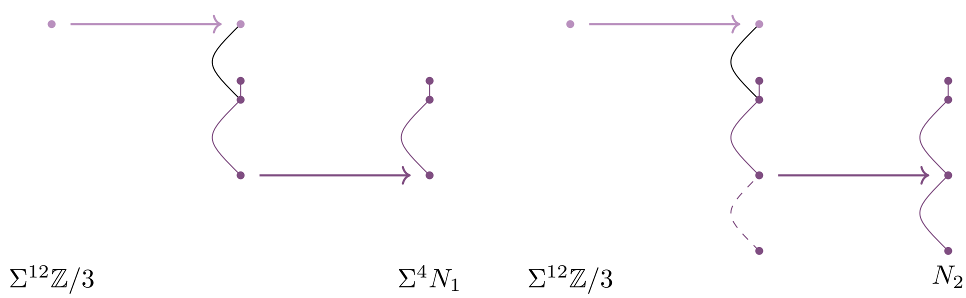

FIGURE 4. Left: an extension of $\mathcal{A}(1)$-modules representing the class $\alpha y \in \mathrm{Ext}^{1,12}_{\mathcal{A}^{tmf}}(\Sigma^4 N_1)$. Right: if we try to form an analogous extension of N_2, we are obstructed by the fact that $(\mathcal{P}^1)^3 = 0$ in $\mathcal{A}^{tmf}$. This is part of the proof of Lemma 3.24.

Now that we know the Ext groups of all $\mathcal{A}^{tmf}$-modules appearing in (3.16), we can draw the E_2-page of the Adams spectral sequence computing $\pi_*(M^{tmf} f_{0,c_1+c_2})^\wedge_3$ in Figure 6, right (and hence, as noted above, the corresponding twisted string bordism groups in degrees 15 and below). For degree reasons, this spectral sequence collapses at E_2 in degrees $t - s \leq 11$; since h_0-actions lift to multiplication by 3, there is no 3-torsion in this range, and we conclude. ∎

Remark 3.25. Other examples of twisted string structures appear in the math and physics literature; see Dierigl-Oehlmann-Schimmanek [DOS23, §3.4] for another 3-primary example.

Remark 3.26. Just as in Remark 3.10, Kreck's modified surgery gives a classification of some closed, smooth 8-manifolds up to stable diffeomorphism in terms of twisted string bordism. There is work applying this in examples corresponding to vector bundle twists [FK96, Fan99, FW10, WW12, CN25]; it would be interesting to apply the *tmf*-module Adams spectral sequence to classes of manifolds where the twist is not given by a vector bundle.

3.3. **$H\mathbb{Z}/2$ as a *ku*-module Thom spectrum.** Devalapurkar uses methods from chromatic homotopy theory to prove the following result. We will reprove it using the tools in this paper.

Theorem 3.27 (Devalapurkar [Dev23, Remark 2.3.16])**.** *There is a map of E_1-spaces $f\colon \mathrm{U}_2 \to B\mathrm{GL}_1(ku)$ and a 2-local equivalence of E_1-ring spectra $Mf \simeq H\mathbb{Z}/2$.*

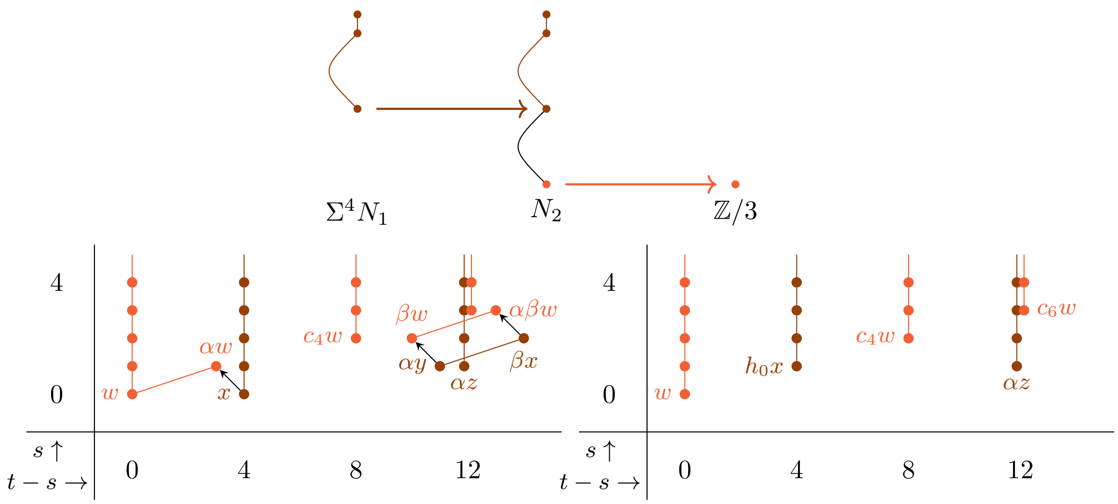

FIGURE 5. Top: the short exact sequence (3.17c) of $\mathcal{A}^{tmf}$-modules. Lower left: the induced long exact sequence in Ext. We compute the boundary maps in Lemma 3.24. Lower right: $\mathrm{Ext}_{\mathcal{A}^{tmf}}(N_2)$ as computed by the long exact sequence.

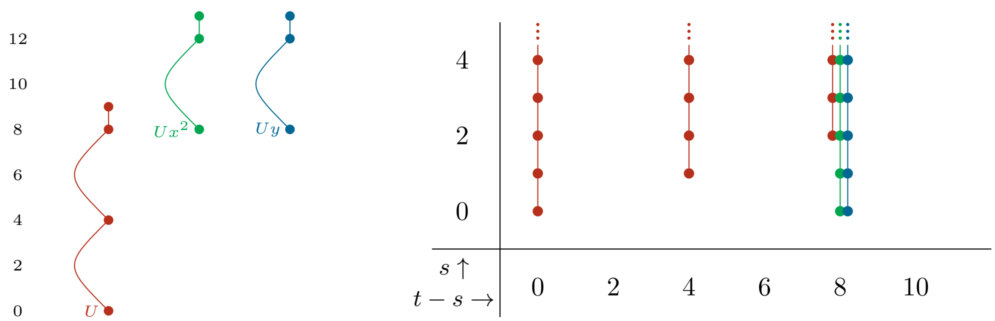

FIGURE 6. Left: the $\mathcal{A}^{tmf}$-module structure on $H^*_{tmf}(M)$ in low degrees; the pictured submodule contains all elements in degrees 11 and below. Right: the E_2-page of the Adams spectral sequence computing $\pi_*(M)^\wedge_3$, which as we discuss in the proof of Theorem 3.12 is isomorphic to the 3-completion of the twisted string bordism groups relevant for $E_8 \times E_8$ heterotic string theory.

We will prove Theorem 3.27 in a sequence of steps. First, we establish an additive equivalence:

Proposition 3.28. *There is a map $f\colon \mathrm{U}_2 \to B\mathrm{GL}_1(ku)$ and a 2-local equivalence $Mf \simeq H\mathbb{Z}/2$.*

Theorem 3.29 (Borel [Bor54, Théorèmes 8.2 et 8.3])**.** *Let A be $\mathbb{Z}$ or $\mathbb{Z}/2$.*

(1) $H^*(B\mathrm{U}_n; A) \cong A[c_1, \ldots, c_n]$, *with* $|c_i| = 2i$.
(2) $H^*(\mathrm{U}_n; A) \cong \Lambda_A(b_1, \ldots, b_n)$ *with* $|b_i| = 2i - 1$.
(3) *The same is true with* SU_n *in place of* U_n, *except that we leave out* c_1.

The inclusion maps $\mathrm{U}_{n-1} \hookrightarrow \mathrm{U}_n$ *and* $B\mathrm{U}_{n-1} \to B\mathrm{U}_n$ *send* $b_i \mapsto b_i$, *resp.* $c_i \mapsto c_i$ *(and likewise with* SU *in place of* U*). Moreover, these statements are true for* $n = \infty$ *(so for* $B\mathrm{U}$, U, $B\mathrm{SU}$, *and* SU*).*

Here $\Lambda_A(\ldots)$ denotes an exterior A-algebra on the specified generators. Below, we will use b_i and c_i to denote the $\mathbb{Z}$-cohomology classes and $\overline{b}_i$ and $\overline{c}_i$ to denote the $\mathbb{Z}/2$-cohomology classes.

Proof of Proposition 3.28. Let $f\colon \mathrm{U}_2 \to B\mathrm{GL}_1(ku)$ be the fake vector bundle twist given by $(\bar{b}_1, b_3)$ (see §1.2.2 for the definition of this class of twists). Borel's theorems that we cited in Theorem 3.29 can be used to show that $\mathcal{A}$ acts trivially on $H^*(\mathrm{U}_2;\mathbb{Z}/2)$.[30] Theorem 2.39 shows that $H^*_{ku}(Mf)$ is isomorphic to $H^*(\mathrm{U}_2;\mathbb{Z}/2)$ as $\mathbb{Z}/2$-vector spaces, and that the $\mathcal{E}(1)$-action is twisted by $Q_0(U) = U\bar{b}_1$ and $Q_1(U) = U\bar{b}_3$. This and the Cartan rule imply $H^*_{ku}(Mf) \cong \mathcal{E}(1)$ as $\mathcal{E}(1)$-modules, so $\mathrm{Ext}_{\mathcal{E}(1)}(H^*_{ku}(Mf), \mathbb{Z}/2)$ consists of a single $\mathbb{Z}/2$ in bidegree $(0,0)$ and vanishes elsewhere. Thus the *ku*-module Adams spectral sequence immediately collapses, and we learn $\pi_0(Mf)^\wedge_2 \cong \mathbb{Z}/2$ and all other homotopy groups vanish. This property characterizes $H\mathbb{Z}/2$ up to 2-local equivalence (e.g. it implies $H^0(Mf;\mathbb{Z}/2) \cong \mathbb{Z}/2$, giving a map $Mf \to H\mathbb{Z}/2$ which is an isomorphism on 2-completed homotopy groups, allowing us to conclude by Whitehead). ☐

The rest of the proof is:

Proposition 3.30. *There is a map $F\colon B\mathrm{U}_2 \to B(B\mathrm{O}/B\mathrm{Spin}^c)$ such that $\Omega F \simeq f$.*

Before we prove Proposition 3.30 we must identify the space $B(B\mathrm{O}/B\mathrm{Spin}^c)$. Recall the space $SK(4)$ from (1.53), which represents SH^4 (degree-4 supercohomology, defined in Definition 1.52).

Proposition 3.31. *There is a homotopy equivalence $B(B\mathrm{O}/B\mathrm{Spin}^c) \xrightarrow{\simeq} SK(4)$ of spaces.*

Proof. Since $B\mathrm{O}/B\mathrm{Spin}^c$ is an abelian ∞-group, $B(B\mathrm{O}/B\mathrm{Spin}^c) \simeq \Sigma(B\mathrm{O}/B\mathrm{Spin}^c)$. Thus, to obtain the homotopy groups of $B(B\mathrm{O}/B\mathrm{Spin}^c)$, we shift up the homotopy groups of $B\mathrm{O}/B\mathrm{Spin}^c$ that we obtained from Proposition 1.21. Thus $B(B\mathrm{O}/B\mathrm{Spin}^c)$ has two nonzero homotopy groups, $\pi_2 \cong \mathbb{Z}/2$ and $\pi_4 \cong \mathbb{Z}$. By definition, $SK(4)$ also has $\pi_2 \cong \mathbb{Z}/2$ and $\pi_4 \cong \mathbb{Z}$, so to establish that $B(B\mathrm{O}/B\mathrm{Spin}^c) \simeq SK(4)$, it suffices to show their k-invariants are equal. In the text around (1.53), we chose the k-invariant of $SK(4)$ to be $\beta \circ \mathrm{Sq}^2$, where $\beta\colon H^*(–;\mathbb{Z}/2) \to H^{*+1}(–;\mathbb{Z})$ is the Bockstein, and by [BLM23, Corollary 4.9], the k-invariant of $B\mathrm{O}/B\mathrm{Spin}^c$ is also $\beta \circ \mathrm{Sq}^2$.[31] ☐

By applying the loop space functor and Lemma 1.54, we also get:

Corollary 3.32. *There is a homotopy equivalence of spaces $B\mathrm{O}/B\mathrm{Spin}^c \simeq SK(3)$.*

Remark 3.33. Using the equivalence of ∞-categories between infinite loop spaces and connective spectra, one can prove that on the sub-∞-category of *connected* abelian ∞-groups, the functor $\Sigma\Omega$ is naturally isomorphic to the identity. Thus if $f\colon X \to Y$ is a map of connected abelian ∞-groups, $\Omega f\colon \Omega X \to \Omega Y$ is the unique homotopy class of maps whose suspension is f.

Lemma 3.34. *Regard $\bar{b}_1 \in H^1(\mathrm{U}_2;\mathbb{Z}/2)$ as a map $\bar{b}_1\colon \mathrm{U}_2 \to K(\mathbb{Z}/2,1)$, and likewise for $\bar{c}_1\colon B\mathrm{U}_2 \to K(\mathbb{Z}/2,2)$. Then $\Omega\bar{c}_1 \simeq \bar{b}_1$.*

Proof. By Theorem 3.29, the pullback maps $H^1(\mathrm{U}_2;\mathbb{Z}/2) \to H^1(\mathrm{U}_1;\mathbb{Z}/2)$ and $H^2(B\mathrm{U}_2;\mathbb{Z}/2) \to H^2(B\mathrm{U}_1;\mathbb{Z}/2)$ are isomorphisms, so it suffices to prove this result with U_2 replaced with U_1. The map $\bar{c}_1\colon B\mathrm{U}_1 \to K(\mathbb{Z}/2,2)$ is a map of abelian ∞-groups (heuristically, the characteristic class $\bar{c}_1$ is additive in tensor products of line bundles), which implies the lemma by Remark 3.33. ☐

Lemma 3.35. *Regard $b_3 \in H^3(\mathrm{U}_2;\mathbb{Z}/2)$ as a map $b_3\colon \mathrm{U}_2 \to K(\mathbb{Z},3)$, and likewise for $c_2\colon B\mathrm{U}_2 \to K(\mathbb{Z},4)$. Then there is some $\lambda \in \mathbb{Z}$ such that $\Omega(\pm c_2 + \lambda c_1^2) \simeq b_3$.*

[30] In fact, Miller [Mil85] showed that the triviality of $H^*(\mathrm{U}_2;\mathbb{Z}/2)$ as an $\mathcal{A}$-algebra lifts to a wedge sum decomposition of $\Sigma^\infty\mathrm{U}_2$ itself; see also [Jam59, Cra87].

[31] Beardsley-Luecke-Morava phrase their results in terms of the *Picard spectrum* $\mathrm{Pic}(KU)$; the relation to $BO/B\mathrm{Spin}^c$ appears in (*ibid.*, §5.2) for twisted spin and string structures, and the story for twisted spin^c structures is analogous.

Proof. Let $i\colon \mathrm{U}_2 \hookrightarrow \mathrm{U}$ and $j\colon \mathrm{SU} \hookrightarrow \mathrm{U}$ be the usual inclusions. Then we have a commutative diagram

$$\begin{tikzcd} H^4(B\mathrm{U}_2;\mathbb{Z}) \arrow[d, "\Omega"'] & H^4(B\mathrm{U};\mathbb{Z}) \arrow[l, "(Bi)^*"'] \arrow[r, "(Bj)^*"] \arrow[d, "\Omega"'] & H^4(B\mathrm{SU};\mathbb{Z}) \arrow[d, "\Omega"'] \\ H^3(\mathrm{U}_2;\mathbb{Z}) & H^3(\mathrm{U};\mathbb{Z}) \arrow[l, "i^*"'] \arrow[r, "j^*"] & H^3(\mathrm{SU};\mathbb{Z}), \end{tikzcd} \tag{3.36}$$

and by Theorem 3.29, i^* and $(Bi)^*$ are isomorphisms and j^* and $(Bj)^*$ are surjective. Specifically, we learn that if $x \in H^4(B\mathrm{U};\mathbb{Z})$ is such that $(Bj)^*(x) = c_2$, then $x = c_2 + \lambda c_1^2$ for some $\lambda \in \mathbb{Z}$. Passing through the isomorphisms i^* and $(Bi)^*$, we have the analogous fact for $B\mathrm{U}_2$ in place of $B\mathrm{U}$.

Since $B\mathrm{SU}$ has the direct sum abelian ∞-group structure, the Whitney sum formula shows that $c_2\colon B\mathrm{SU} \to K(\mathbb{Z},4)$ is a morphism of connected ∞-groups. Alternatively, one may identify this map with the cofiber of the forgetful map $B\mathrm{U}\langle 6\rangle \to B\mathrm{SU}$, which is a map of abelian ∞-groups (see also the text around (1.23)).

By Remark 3.33, $c_2\colon B\mathrm{SU} \to K(\mathbb{Z},4)$ loops to a generator of $H^3(\mathrm{SU};\mathbb{Z})$, which must be $\pm b_3$. Chase this fact across (3.36) to finish the proof. $\square$

Proof of Proposition 3.30. We claim that here is a commutative diagram of long exact sequences

$$\begin{tikzcd}[column sep=small] \cdots \arrow[r, "\beta\circ\mathrm{Sq}^2"] & H^4(B\mathrm{U}_2;\mathbb{Z}) \arrow[r] \arrow[d, "\Omega"'] & SH^4(B\mathrm{U}_2) \arrow[r, "t"] \arrow[d, "\Omega"'] & H^2(B\mathrm{U}_2;\mathbb{Z}/2) \arrow[r, "\beta\circ\mathrm{Sq}^2"] \arrow[d, "\Omega"'] & H^5(B\mathrm{U}_2;\mathbb{Z}) \arrow[r] \arrow[d, "\Omega"'] & \cdots \\ \cdots \arrow[r, "\beta\circ\mathrm{Sq}^2"] & H^3(\mathrm{U}_2;\mathbb{Z}) \arrow[r] & SH^3(\mathrm{U}_2) \arrow[r, "t"] & H^1(\mathrm{U}_2;\mathbb{Z}/2) \arrow[r, "\beta\circ\mathrm{Sq}^2"] & H^4(\mathrm{U}_2;\mathbb{Z}) \arrow[r] & \cdots \end{tikzcd} \tag{3.37}$$

where the vertical arrows are the loop space functor. Specifically, the interpretation of the loop space functor as a map $\Omega\colon H^n(X;A) \to H^{n-1}(\Omega X;A)$ is just as in Lemmas 3.34 and 3.35; the interpretation on supercohomology is completely analogous, using that $\Omega SK(n) \simeq SK(n-1)$ (Lemma 1.54).

The commutative diagram in (3.37) exists essentially because the loop space functor, applied to a cofiber sequence of connected infinite loop spaces, returns a cofiber sequence of infinite loop spaces.

We claim that all four maps labeled $\beta\circ\mathrm{Sq}^2$ in (3.37) vanish. For all of them except $\beta\circ\mathrm{Sq}^2\colon H^2(B\mathrm{U}_2;\mathbb{Z}/2) \to H^5(B\mathrm{U}_2;\mathbb{Z})$, this follows because Sq^2 vanishes on classes in degrees less than 2. To see that the remaining $\beta\circ\mathrm{Sq}^2$ vanishes, check on the generator $\bar{c}_1 \coloneqq c_1 \bmod 2$ (Theorem 3.29): $\mathrm{Sq}^2(\bar{c}_1) = \bar{c}_1^2$ for degree reasons, but $\bar{c}_1^2 = c_1^2 \bmod 2$, so $\beta(\bar{c}_2^2) = 0$. Thus (3.37) simplifies to a map of short exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & H^4(B\mathrm{U}_2;\mathbb{Z}) \arrow[r] \arrow[d, "\Omega_1"'] & SH^4(B\mathrm{U}_2) \arrow[r, "t"] \arrow[d, "\Omega_2"'] & H^2(B\mathrm{U}_2;\mathbb{Z}/2) \arrow[r] \arrow[d, "\Omega_3"'] & 0 \\ 0 \arrow[r] & H^3(\mathrm{U}_2;\mathbb{Z}) \arrow[r] & SH^3(\mathrm{U}_2) \arrow[r, "t"] & H^1(\mathrm{U}_2;\mathbb{Z}/2) \arrow[r] & 0 \end{tikzcd} \tag{3.38}$$

Here we give the loop space functor maps different names Ω_i to distinguish them. By Lemma 3.34, Ω_3 is an isomorphism, and by Lemma 3.35, Ω_1 is surjective (since the generator of $H^3(\mathrm{U}_2;\mathbb{Z}) \cong \mathbb{Z}$ is in the image of Ω_1). Therefore by the four lemma, Ω_2 is surjective. Reinterpreting this fact as in

Proposition 3.31 and Corollary 3.32, we have that the map

$$\Omega\colon [B\mathrm{U}_2, B(B\mathrm{O}/B\mathrm{Spin}^c)] \longrightarrow [\mathrm{U}_2, B\mathrm{O}/B\mathrm{Spin}^c] \tag{3.39}$$

is surjective, which suffices to prove the proposition. ☐

Proof of Theorem 3.27. By Theorem 1.13, the map $T\colon B\mathrm{O}/B\mathrm{Spin}^c \to B\mathrm{GL}_1(ku)$ is a map of abelian ∞-groups, and by the recognition principle, since $f \simeq \Omega F$ by Proposition 3.30, f is a map of E_1-spaces. Thus $T \circ f$ is also E_1, which by [ABG18, Theorem 1.7] implies that its Thom spectrum is an E_1-*ku*-algebra. We identified this Thom spectrum as $H\mathbb{Z}/2$, which has a unique E_1-ring structure, in Proposition 3.28. ☐

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